Semigroup of positive maps for qudit states and entanglement in tomographic probability representation

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Abstract

Stochastic and bistochastic matrices providing positive maps for spin states (for qudits) are shown to form semigroups with dense intersection with the Lie groups $IGL(n, \mathbb{R})$ and $GL(n, \mathbb{R})$ respectively. The density matrix of a qudit state is shown to be described by a spin tomogram determined by an orbit of the bistochastic semigroup acting on a simplex. A class of positive maps acting transitively on quantum states is introduced by relating stochastic and quantum stochastic maps in the tomographic setting. Finally, the entangled states of two qubits and Bell inequalities are given in the framework of the tomographic probability representation using the stochastic semigroup properties.

Key words Quantum tomograms, semigroups, positive maps, entangled states.

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1 Introduction

The description of a physical system admitting a probabilistic interpretation, be it classical or quantum, requires two collections of objects called states and observables, say $S$ and $O$ respectively, along with a pairing $\mu$ associating with any state $\rho$ and observable $A$ a Borel probability measure on the real line $\mathbb{R}$. If $A$ is measured while the system is in a state $\rho$, $\mu_{A,\rho}$ represents the probability distribution for the observed values of $A$. Thus if $E \subseteq \mathbb{R}$ is a Borel set, $\mu_{A,\rho}(E) \in [0,1]$ is the probability that the measured value of $A$ will be in the set $E$ when the system is known to be in the state $\rho$. From a general point of view, above
properties seem to be the minimal features that any physical system should possess.

This approach has been studied by several authors, for instance one can find a nice discussion by G. Mackey\cite{1}. The set $S$ describes the basic mathematical structure we are dealing with, while $\mu_{A,\rho}$ provides us with a physical interpretation. This probabilistic point of view is compatible with convex combinations on the space of states, indeed if $\rho_1$ and $\rho_2$ give rise to probability distributions, by setting $\mu_{A,\lambda\rho_1+(1-\lambda)\rho_2} = \lambda \mu_{A,\rho_1} + (1-\lambda) \mu_{A,\rho_2}$ we define a new probability distribution when $0 \leq \lambda \leq 1$. Usually one requires some additional structure telling us how the system changes from time $s$ to a later time $t$, i.e. requires the existence of a family of mappings $U_{t,s} : S \to S$ representing the dynamics and called evolution operator. The requirement that a state at a given time determines the state at a later time forces us to postulate the semigroup property

$$U_{t_2,t_1} = U_{t_2,s} \circ U_{s,t_1}$$

with $U_{t,t}$ the identity. Within this setting

$$\mu : S \times O \to \{\text{Borel probability measures on } \mathbb{R}\}$$

A subset of observables is said to be a tomographic set $\tau$ if it allows to identify the state $\rho$ (to “reconstruct” the state) when $\{\mu_{A,\rho}\}_{A \in \tau}$ is known. Very often $\tau$ is generated by acting with a group $G$ on some fiducial observable $A_0$, i.e. it is the orbit of $G$ in $O$ through $A_0$. According to the group we use and the fiducial observable we start with, we deal with symplectic tomography, photon-number tomography and so on. While this approach is general enough to allow us to deal both with classical and quantum systems, here, to be more definite, we shall consider a quantum system with a finite number of levels.

States for quantum systems with a finite number of levels will be thought of as the spin states (or qudits) they can be described by density matrices which are hermitian nonnegative $(2j+1) \times (2j+1)$ matrices with unit trace. The linear maps of the spin states, positive maps, can be described by $(2j+1)^2 \times (2j+1)^2$ matrices with special properties\cite{2}. Recently it was shown\cite{3, 4, 5} that qudit states can be described by probability distributions of random spin projection (called tomogram) depending on the direction of the quantization axis. In view of this the geometry of qudit states can be associated with the geometry of a simplex and the set of positive maps of qudit states can be associated with stochastic and bistochastic matrices moving points on the simplex. The aim of this work is to find the connection of spin tomograms with a unitary matrix containing eigenvectors of the density matrix of a qudit state and a point on the simplex which has the eigenvalues of the density matrix as its coordinates.

Another aim of the work is to define positive maps of qudit states through the transitive actions of both the unitary group on the eigenvectors of the density matrix and the stochastic matrix semigroup on the eigenvalues of the density matrix regarded as points of the simplex.

The qudit states of multipartite systems can be either separable or entangled. We formulate the properties of a qudit tomogram, which is the joint probability
distribution of two spin projections on their own quantization axes, able to distinguish separable and entangled states. We consider the Bell inequalities\textsuperscript{6, 7} in the context of the properties of stochastic matrices constructed by using spin tomograms. The Cirelson\textsuperscript{8} bound $2\sqrt{2}$ for the Bell-CHSH inequality of two qubits will be connected with some properties of a universal stochastic matrix obtained from the tomographic probability distribution describing maximally entangled two spin-1/2 states. The connection of positive maps with the semigroup of stochastic matrices provides the possibility to find a new relation of the maps with the Lie group of the general linear real transformations $GL(n, \mathbb{R})$ for bistochastic matrices and with the inhomogeneous group $IGL(n, \mathbb{R})$ for stochastic matrices.

This connection (which seems to have been unknown) provides a possibility to construct unitary representations of stochastic and bistochastic semigroups by reducing known infinite dimensional unitary irreducible representations of the Lie groups to the subsets of the Lie groups which are the semigroups under consideration.

The paper is organized as follows: in section 2 we review the spin tomography approach for one and two qudits. Examples of a qutrit and two qubit states in tomographic probability representation are studied in section 3. The relation of stochastic and bistochastic semigroups with Lie groups is discussed in section 4, mainly in the case a qutrit. In section 5 a class of positive maps acting transitively on quantum states is introduced by relating stochastic and quantum stochastic maps in the tomographic setting. The relation of stochastic matrices with Bell inequality violation for entangled states of two qubits is discussed in section 6. Some conclusions and perspectives are finally drawn in section 7.

2 Spin tomograms and unitary group

As it was shown in \textsuperscript{3, 9, 10} the qudit state described by a $(2j + 1) \times (2j + 1)$–matrix $\rho$ can be also described by a tomographic probability distribution function, or tomogram, $W(m, U) \geq 0$ where $m$ is the spin projection: $m = -j, -j + 1, \ldots, j - 1, j$; and $U$ is a unitary $(2j + 1) \times (2j + 1)$–matrix. This matrix can be considered as a matrix of an irreducible representation of the rotation group depending on two Euler angles $\phi, \theta$ determining the direction of quantization (or a point on the Bloch sphere $S^2$). The physical meaning of the tomogram $W(m, U)$ is that, in the spin state with the given density matrix $\rho$, it gives the probability to obtain $m$ as spin projection on the direction determined by the two angles $\phi, \theta$. It corresponds to choose $\{U^\dagger J_z U\}$ as tomographic set of isospectral observables, where $J_z = \sum_{m = -j}^{j} m \left\langle m \right| \left. m \right\rangle$ is one of the generators of the irreducible representation of the rotation group, so that $W(m, U)$ is nothing but the value of the concentrated measure $\mu_{U^\dagger J_z U, \rho}$ at the spectral point $m$:

$$W(m, U) = \mu_{U^\dagger J_z U, \rho}(m) = \text{Tr} U^\dagger |m\rangle \langle m| U \rho = \langle m | U \rho U^\dagger | m\rangle.$$  \hspace{1cm} (3)
The probability distribution is obviously nonnegative and normalized, i.e.

\[ \sum_{m=-j}^{j} \mathcal{W}(m, U) = 1 \]  

for any direction of the quantization axis. The spin tomogram can be also regarded as the diagonal matrix element of the rotated density matrix \( U \rho U^\dagger \) in the natural basis \( |m\rangle \).

The relation is invertible and knowing the tomogram \( \mathcal{W}(m, U) \) for the matrices \( U(\phi, \theta) \) of an irreducible representation of \( SU(2) \) one obtains the density matrix \( \rho \) by means of a linear transform[9, 10] which is the analog of the integral Radon transform but in the space of qudit states. Thus the quantum state of a qudit (a spin-\( j \) state) is known if the probability distribution \( \mathcal{W}(m, U) \) of random spin projection as a function of the unitary matrix \( U \) is known. The tomogram \( \mathcal{W}(m, U) \) can be used, consequently, in alternative to spinors (wave functions) or density matrices for describing spin states. The information on the spin state contained in the tomogram is redundant since it is sufficient to know the tomogram only for several directions determined by a set of angles \( \{ \phi_k, \theta_k \} \), whose number corresponds to the number of parameters determining the density matrix, equal to \( (2j+1)^2 - 1 \). But at the same time the dependence of the tomogram \( \mathcal{W}(m, U) \) on the parameters of the unitary matrix \( U \) provides some advantage in considering the spins, \( j = 0, 1/2, 1, \ldots \); and also the quantum states of several spins in an unified approach. For two qudits (spin \( j_1 \) and \( j_2 \)) the tomogram of the quantum state with the \( (2j_1+1)(2j_2+1) \times (2j_1+1)(2j_2+1) \) density matrix \( \rho \) is the normalized joint probability distribution

\[ \mathcal{W}(m_1, m_2, U) = \langle m_1 m_2 | U \rho U^\dagger | m_1 m_2 \rangle \]  

of two random spin projections \( m_1 = -j_1, -j_1 + 1, \ldots, j_1 - 1, j_1 \) and \( m_2 = -j_2, -j_2 + 1, \ldots, j_2 - 1, j_2 \) onto the corresponding directions determined by two pairs of Euler angles, \( \phi_1, \theta_1 \) and \( \phi_2, \theta_2 \). The information contained in the tomogram with a dependence on the matrix \( U \) of such a form is sufficient to reconstruct the density matrix \( \rho \). But we define the tomogram by Eq.(3) to use the redundant information on the quantum state of bipartite systems in studying the entanglement properties of the system states. We remark that in Eq.(3) we could also use the full unitary group instead of \( SU(2) \) and this we will do sometimes in the following.

The tomographic probability distribution of a qudit state \( \mathcal{W}(m, U) \) can be considered as a column vector \( \vec{\mathcal{W}}(U) \) with components

\[ \mathcal{W}_1(U) = \mathcal{W}(j, U), \mathcal{W}_2(U) = \mathcal{W}(j - 1, U), \ldots, \mathcal{W}_{2j+1}(U) = \mathcal{W}(-j, U). \]  

Since all the components are nonnegative and the normalization condition \( \sum_{m=-j}^{j} \mathcal{W}(m, U) = 1 \) holds, from a geometrical point of view the components \( \{ \mathcal{W}_k \} \) of the tomographic probability vector determine the coordinates \( \{ x_k \} \) of points belonging to a simplex. For a qubit such a simplex is the segment \( \{ x_1 + x_2 = 1; 0 \leq \)
\( x_1, x_2 \leq 1 \) in the plane \( x_1, x_2 \). For a generic qudit the simplex is a polyhedron in a \((2j + 1)\)-dimensional space determined by equations:

\[
\sum_{k=1}^{2j+1} x_k = 1; \ 0 \leq x_1, x_2, \ldots, x_{2j+1} \leq 1 .
\]  

(7)

Thus, the spin tomogram is a function of a unitary group element \( U \) with values in a simplex. The linear maps of probability vectors

\[
\vec{W}^\prime(U) = M \vec{W}(U)
\]  

are expressed in terms of matrices \( M \) which are known to form the semigroup of stochastic matrices. A stochastic matrix is a matrix with nonnegative entries such that the sum of the elements of each column is one. If in addition the sum of the elements of each row is one, the matrix is bistochastic. Also the bistochastic matrices form a semigroup. In the next section we study the properties of the tomograms and stochastic and bistochastic maps on the example of qutrit states.

3 Qutrit density matrix in tomographic probability representation

To present the semigroup approach to describe generic qudit states we start from qutrit states (spin-1 states). The density matrix of a qutrit state is a nonnegative hermitian \( 3 \times 3 \) matrix with unit trace and it can be presented in the product form

\[
\rho = U_0 \tilde{\rho} U_0^\dagger
\]  

(9)

where the \( 3 \times 3 \) matrix \( \tilde{\rho} \) is diagonal:

\[
\tilde{\rho} = \begin{pmatrix}
\tilde{\rho}_1 & 0 & 0 \\
0 & \tilde{\rho}_2 & 0 \\
0 & 0 & \tilde{\rho}_3
\end{pmatrix}
\]  

(10)

with nonnegative eigenvalues \( \tilde{\rho}_k \), \( k = 1, 2, 3 \), satisfying the normalization condition

\[
\sum_{k=1}^{3} \tilde{\rho}_k = 1 .
\]  

(11)

The columns of the \( 3 \times 3 \) unitary matrix \( U_0 \) are the components of the normalized eigenvectors of the density matrix \( \rho \), i.e.

\[
U_0 = \begin{pmatrix}
u_{11} & u_{12} & u_{13} \\
u_{21} & u_{22} & u_{23} \\
u_{31} & u_{32} & u_{33}
\end{pmatrix} \equiv |\vec{u}_1, \vec{u}_2, \vec{u}_3|
\]  

such that

\[
\rho \vec{u}_1 = \tilde{\rho}_1 \vec{u}_1 , \ \rho \vec{u}_2 = \tilde{\rho}_2 \vec{u}_2 , \ \rho \vec{u}_3 = \tilde{\rho}_3 \vec{u}_3 .
\]  

(12)
The eigenvalues $\tilde{\rho}_k$ can be considered as the components of a probability vector

$$\tilde{\rho} = \begin{bmatrix} \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{bmatrix}$$

(13)

corresponding to a point on the triangle with vertices in $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, which is the two dimensional simplex given by Eq.(11) with the constraints $0 \leq \tilde{\rho}_k \leq 1$, $k = 1, 2, 3$. The triangle (qutrit simplex) is embedded in the plane determined by Eq.(11), where the coordinates $\tilde{\rho}_k$ are arbitrary real numbers.

The linear maps of the qutrit probability vector $\tilde{\rho}$ are determined by the $3 \times 3$ stochastic matrices $M$, i.e.

$$\tilde{\rho} \rightarrow \tilde{\rho}' = M \tilde{\rho}$$

(14)

where the matrix $M$ has real nonnegative entries and the sum of the elements of each column is unity. The product of two stochastic matrices is again a stochastic matrix. Thus stochastic matrices form a semigroup. Also, any convex sum of stochastic matrices is a stochastic matrix. But one can check that the inverse matrix $M^{-1}$ if it exists is not stochastic because some entries are negative numbers. Thus the triangle (qutrit simplex) is invariant under the action of a linear stochastic map and the point on the plane (11) given by the vector $M \tilde{\rho}$ belongs to the same triangle.

The tomogram $W(m, U)$ of the qutrit state with density matrix $\rho$ given by Eq.(9) is the probability vector with components

$$W_1(U) = W(+1, U) = \left\langle 1 \left| U U_0 \tilde{\rho} U_0^\dagger U^\dagger \right| 1 \right\rangle,$$

$$W_2(U) = W(0, U) = \left\langle 0 \left| U U_0 \tilde{\rho} U_0^\dagger U^\dagger \right| 0 \right\rangle,$$

$$W_3(U) = W(-1, U) = \left\langle -1 \left| U U_0 \tilde{\rho} U_0^\dagger U^\dagger \right| -1 \right\rangle.$$  

(15)

Direct calculation shows that the above formulae can be written in the following form:

$$W_k(U) = \sum_{h=1}^{3} \left| (UU_0)_{kh} \right|^2 \tilde{\rho}_h$$

(16)

or in vector form:

$$\tilde{W}(U) = M \tilde{\rho}$$

(17)

where the elements of the matrix $M$ read

$$M_{kh} = \left| (UU_0)_{kh} \right|^2.$$  

(18)

As a product of unitary matrices, the matrix $UU_0$ is unitary and the sum of the (nonnegative) elements of each column and each row of $M$ is one. So, $M$ is orthostochastic, a particular bistochastic matrix. The result of Eq.s(17), (18) means that the qutrit state is determined by a bistochastic map acting on the probability 3–vector whose components are the eigenvalues of the density matrix. The bistochastic matrix $M$ of the map has elements which are the square
moduli of the elements of a unitary matrix. In turn, the unitary matrix is the product of two unitary matrices, one rotating the basis in the space of spin states and the other one having columns formed by the eigenvectors of the density matrix. From a geometrical point of view, this means that qutrit states are the orbit of the unitary group acting on the points of the triangle (qutrit simplex), but the action of the unitary group is made via the action of an orthostochastic map.

In the case of two qubits that construction yields the description of the quantum states of the corresponding two spin−1/2 system by the tomographic probability 4−vector \( \vec{W}(U) \), where the the unitary matrix \( U \) belongs to the group \( U(4) \). The spin tomographic probability vector \( \vec{W}(U) \) has the following components

\[
W_1(U) = \mathcal{W}(\frac{1}{2}, \frac{1}{2}, U) = \left\langle \frac{1}{2} \frac{1}{2} | U U_0 \tilde{\rho} U_0^\dagger U | \frac{1}{2} \frac{1}{2} \right\rangle ,
\]

\[
W_2(U) = \mathcal{W}(\frac{1}{2}, -\frac{1}{2}, U) = \left\langle \frac{1}{2} \frac{-1}{2} | U U_0 \tilde{\rho} U_0^\dagger U | \frac{1}{2} \frac{-1}{2} \right\rangle ,
\]

\[
W_3(U) = \mathcal{W}(-\frac{1}{2}, \frac{1}{2}, U) = \left\langle \frac{-1}{2} \frac{1}{2} | U U_0 \tilde{\rho} U_0^\dagger U | \frac{-1}{2} \frac{1}{2} \right\rangle ,
\]

\[
W_4(U) = \mathcal{W}(-\frac{1}{2}, -\frac{1}{2}, U) = \left\langle \frac{-1}{2} \frac{-1}{2} | U U_0 \tilde{\rho} U_0^\dagger U | \frac{-1}{2} \frac{-1}{2} \right\rangle .
\]

Here the columns of the unitary 4 × 4−matrix \( U_0 \) are the eigenvectors and \( \tilde{\rho} \) is the diagonal form of the density matrix \( \rho \) of the two qubits state. One can easily check that the tomogram \( \vec{W}(U) \) can be written in the form of the Eq.s (17),(18) if the eigenvalues of \( \rho \) are organized as a 4−vector \( \tilde{\vec{\rho}} \). This vector belongs to a 3−dimensional simplex, a polyhedron, in a 4−dimensional space. The points in the simplex belong to the orbit of the group \( U(4) \) in its Cartan subalgebra whose elements are labelled by nonnegative numbers.

4 The relation to general linear and inhomogeneous general linear groups

In this section we find a relation of stochastic and bistochastic maps of the tomographic probability vectors to Lie groups. If we restrict to invertible stochastic and bistochastic matrices and leave out the nonnegativity of their entries, we obtain a group since the invertible matrices have inverse matrices of the same kind. To prove this let us formulate the stochasticity property in terms of stability of a vector \( \vec{e}_0 \) with all its \( n \) components equal to 1. The (column) stochasticity property of an \( n \times n \)−matrix \( M \) results by the requirement:

\[
\vec{e}_0^T \vec{e}_0^T \vec{e}_0 = \vec{e}_0^T \vec{e}_0
\]

where \( \vec{e}_0^T = (1, 1, \ldots, 1) \) is the transpose of \( \vec{e}_0 \). This requirement is equivalent to demand that the sum of the elements of any column of \( M \) is 1. The identity
matrix $I$ obviously satisfies the above equation. Since $I = MM^{-1}$ one has
\[ e_0^T = e_0^T MM^{-1} = e_0^T M^{-1} \]  
(21)
and this proves that Eq. (20) is satisfied also by the inverse matrix $M^{-1}$, when it exists.

The stochastic matrix $M$ is bistochastic if in addition to Eq. (20) it satisfies the condition
\[ M\vec{e}_0 = \vec{e}_0. \]  
(22)

The same previous argument shows that also the inverse matrix $M^{-1}$ satisfies Eq. (22), when it exists.

The invertible matrices $M$ satisfying equations (20) and (22) with real entries of arbitrary sign form a Lie group $G_{BS}$. The subset of $G_{BS}$ of invertible matrices $M$ with real nonnegative entries is an open dense subsemigroup of the bistochastic matrices semigroup.

Analogously, the invertible matrices $M$ satisfying equation (20) with real entries of arbitrary sign form a Lie group $G_{S}$. The subset of $G_{S}$ of invertible stochastic matrices is an open dense subsemigroup of the stochastic matrices semigroup.

It is obvious that one can also consider complex matrices, rather than real ones, satisfying the above equations. Then one obtains other Lie groups.

It is clear from Eq. (22) that the Lie group $G_{BS}$ is $GL(n-1, \mathbb{R})$ for the $n$–dimensional case, and $GL(2, \mathbb{R})$ for the qutrit case.

Let us discuss the qutrit case in detail to recognize $G_{BS}$ as $GL(2, \mathbb{R})$. We choose a rotation $O$, $OO^T = O^T O = I$, in the 3–dimensional space such that
\[ \frac{1}{\sqrt{3}} O \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]  
(23)

For instance, we choose:
\[ O = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \sqrt{3} \end{pmatrix}. \]  
(24)

Let us take the bistochastic matrix $M$ of the form
\[ M = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \]  
(25)
with
\[ \sum_k x_k = \sum_k y_k = \sum_k z_k = 1, \]  
(26)
and
\[ x_k + y_k + z_k = 1; \ k = 1, 2, 3. \]  
(27)
One can check that
\[
\tilde{M} = \mathcal{O} \mathcal{M} \mathcal{O}^T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] (28)
where
\[
a = \frac{1}{2} (x_1 - x_2 - y_1 + y_2) ; \\
b = \frac{3}{2\sqrt{2}} (z_2 - z_1) ; \\
c = \frac{3}{2\sqrt{2}} (y_3 - x_3) ; \\
d = \frac{1}{2} (2z_3 - x_3 - y_3) .
\] (29)
The 4–parameter group of Eq. (28) is the noncompact Lie group \(GL(2, \mathbb{R})\). As the determinant of the orthogonal matrix \(\mathcal{O}\) is nonzero, the set of matrices \(\tilde{M}\) is isomorphic to the group of bistochastic matrices \(M\).

Besides, if one considers the stochastic matrices Lie group \(G_S\), that is matrices \(M\) of the form (25) satisfying Eq. (26) but not Eq. (27), one can check that
\[
\tilde{M} = \mathcal{O} \mathcal{M} \mathcal{O}^T = \begin{pmatrix} A & B & m \\ C & D & n \\ 0 & 0 & 1 \end{pmatrix}
\] (30)
where
\[
A = \frac{1}{2} (x_1 - x_2 - y_1 + y_2) ; \\
B = \frac{1}{2\sqrt{2}} (x_1 + y_1 - 2z_1 - x_2 - y_2 + 2z_2) ; \\
C = \frac{3}{2\sqrt{2}} (y_3 - x_3) ; \\
D = \frac{1}{2} (2z_3 - x_3 - y_3) ; \\
m = \frac{1}{\sqrt{6}} (x_1 + y_1 + z_1 - x_2 - y_2 - z_2) ; \\
n = \frac{1}{\sqrt{2}} (1 - x_3 - y_3 - z_3) .
\] (31)
or equivalently
\[ x_1 = \frac{1}{6} \left( 3A + \sqrt{2}B + \sqrt{2}C + D + \sqrt{6}m + \sqrt{2}n + 2 \right), \]  \hspace{1cm} (32) \\
\[ x_2 = \frac{1}{6} \left( -3A - \sqrt{2}B + \sqrt{2}C + D - \sqrt{6}m + \sqrt{2}n + 2 \right), \]  \\
\[ y_1 = \frac{1}{6} \left( -3A + \sqrt{2}B - \sqrt{2}C + D + \sqrt{6}m + \sqrt{2}n + 2 \right), \]  \\
\[ y_2 = \frac{1}{6} \left( 3A - \sqrt{2}B - \sqrt{2}C - D - \sqrt{6}m + \sqrt{2}n + 2 \right), \]  \\
\[ z_1 = \frac{1}{6} \left( -2\sqrt{2}B - 2D + \sqrt{6}m + \sqrt{2}n + 2 \right), \]  \\
\[ z_2 = \frac{1}{6} \left( 2\sqrt{2}B - 2D - \sqrt{6}m + \sqrt{2}n + 2 \right). \]

From previous formulae one can readily see that pure translations (\( A = D = 1 \) and \( B = C = 0 \)) and pure homogeneous linear transformations (\( m = n = 0 \)) do not correspond to any stochastic matrix.

The group of matrices of Eq. (30) is just \( IGL(2,\mathbb{R}) \), isomorphic to the direct product of dilation group times \( ISL(2,\mathbb{R}) \), which in this particular dimension is isomorphic with the Poincaré group in two plus one dimensions. In this way we have established that the stochastic invertible semigroup of \( 3 \times 3 \) matrices is in one-to-one correspondence with a subset of the group \( IGL(2,\mathbb{R}) \).

Finally, it is clear that this argument holds true for arbitrary qudits (arbitrary spin-\( j \) states).

Now we are able to describe briefly the action of the stochastic and bistochastic groups on the carrier space \( \mathbb{R}^n \) of their above representation. The fiducial vector \( e_0^T \) determines an invariant foliation given by

\[ e_0^T (Mv) = e_0^T (v) = \text{const} ; \quad v \in \mathbb{R}^n \]  \hspace{1cm} (33)

so that the simplex of probability vectors belong to the leaf \( e_0^T (v) = 1 \). Moreover that simplex is invariant under the action of the whole semigroup of stochastic matrices, and we recall that the orbit of the semigroup of stochastic or even bistochastic matrices starting from a vertex is the whole simplex. While the action of the stochastic semigroup on the simplex is transitive, the orbit of the bistochastic semigroup starting from a point \( \vec{v} = (v_1, ..., v_n)^T \) is the convex hull of all the vectors \( \vec{v}_\pi \) whose components are some permutation \((v_{\pi_1}, ..., v_{\pi_n})\) of the components of the given vector \( \vec{v} \).

In particular, any point of the simplex can be also reached from a given vertex with an invertible stochastic matrix. The same holds true for invertible bistochastic matrix, if some lower dimensional set of points of the simplex is dropped, in any case the point \( \vec{e}_0/n \) cannot be reached.

From the point of view of matrix analysis (for a general reference of the following discussion see, e.g., [11]), a generic stochastic matrix \( M \) is a positive matrix, i.e. all its entries are positive. Then, by Perron’s theorem, we know that its spectral radius, which is 1, is an algebraic simple eigenvalue of maximum
modulus (Perron root of the matrix), with an eigenvector $\vec{p}$ (Perron vector) which may be chosen to be positive, i.e. a probability vector.

$$M\vec{p} = \vec{p}; \quad p_i > 0 \ \forall i = 1, \ldots, n; \quad \sum_i p_i = 1$$  \hspace{1cm} (34)

The fiducial vector $\vec{e}_0$ is just the left Perron vector of the stochastic matrix $M$ belonging to the same eigenvalue 1, and the following limit does exist:

$$\lim_{k \to \infty} M^k = L := \vec{p}e_0^T = ||\vec{p}, \vec{p}, \ldots, \vec{p}||. \hspace{1cm} (35)$$

In other words, the matrix $M^k$ approaches a limit which is a rank-one stochastic matrix whose columns are the Perron vector of $M$. As a result, because $L\vec{v} = \vec{p}$ for any vector of the leaf $e_0^T(\vec{v}) = 1$, we have that

$$\lim_{k \to \infty} M^k\vec{v} = \vec{p}$$  \hspace{1cm} (36)

independently of the starting point $\vec{v}$. In particular, if $M$ is bistochastic, $\vec{p} = \vec{e}_0/n$.

We do not insist on the possibility of characterizing that limit in terms of a stochastic Markov process.

When the stochastic matrix $M$ is nonnegative, rather than positive, again a nonnegative probability eigenvector $\vec{p}$ of the eigenvalue 1 exists, and we recover the same limit $L$ if the spectral radius of $M$ is the only eigenvalue of maximum modulus.

More generally, if the matrix $I - M + L$ is invertible, we get the same limit $L$ by a Cesaro summation:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} M^k = L := \vec{p}e_0^T = ||\vec{p}, \vec{p}, \ldots, \vec{p}||. \hspace{1cm} (37)$$

That hypothesis is crucial as the following example shows: consider the permutation bistochastic matrix

$$M = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \hspace{1cm} (38)$$

it gives a Cesaro limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} M^k = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

which does not have the form of $L$. In any case the columns of the Cesaro limit, when it exists, are eigenvectors of $M$ belonging to the eigenvalue 1.


## 5 Positive maps

Having discussed the transformations of probability vectors we are now in the position of describing transformations of density states by means of the tomographic representation of them.

The problem of constructing positive maps has a long history and the dynamical maps providing a density matrix which could appear in a process of quantum evolution was studied in [12], see also [12 13]. For finite level systems the space of states is a convex body in the dual space of the infinitesimal generators of the unitary group, say $u^\ast(H)$. For closed quantum systems the dynamics is conventionally represented by transformations associated with one-parameter groups of unitary transformations in Hilbert spaces, this formalism however is not appropriate to deal with irreversible behaviours like decay of metastable particles, approach to thermodynamic equilibrium and more generally to deal with decoherence phenomena.

For finite level quantum systems, probability measures are fully described by probability vectors, say $n$-vectors $\vec{v}$ with nonnegative components whose sum is 1:

$$\left\{ \vec{v} : v_m \geq 0 \forall m = 1, \ldots, n; \sum_m v_m = 1 \right\}.$$  \hfill (39)

For those systems, states are identified with $n \times n$-density matrices $\rho$ which are positive semi-definite and satisfy

$$\rho^\dagger = \rho, \quad \text{Tr}\rho = 1$$ \hfill (40)

Dynamical maps would assign to each $\rho$ another density state $\rho(t) = M(t)\rho_0$, with $M$ some $n^2 \times n^2$-matrix satisfying appropriate constraints to guarantee that $\rho(t)$ is still a density state. A dynamical map on states induces a dynamical map on probability vectors which may be described by stochastic matrices. These maps, on states are called quantum stochastic maps.

Our aim is to relate stochastic maps acting on probability vectors to quantum stochastic maps acting on density states, by using the tomographic setting. This approach is different from the standard one[14 15 16] which describes quantum stochastic maps as projection of isometries.

We may parametrize density states by a pair

$$\rho \rightarrow (U_0, \vec{\tilde{\rho}})$$ \hfill (41)

where $U_0$, as a unitary matrix, provides us with the eigenvectors of $\rho$ while the components of $\vec{\tilde{\rho}}$ are the corresponding eigenvalues. Symbolically we could write

$$\rho U_0 = \vec{\tilde{\rho}} U_0$$ \hfill (42)

meaning that $\rho$ acting on the $k$-th column of $U_0$ will provide the same column multiplied by the corresponding eigenvalue $\rho_k$. Equivalently, the columns of $U_0$ are an orthonormal frame in the Hilbert space and therefore define a family of orthonormal projectors while $\vec{\tilde{\rho}}$ gives the weights to attribute to each corresponding rank-one projector of the decomposition of $\rho$. 

It should be clear from this description that $U_0$ is determined up to $(U(1))^n$,

$$\rho = \sum_{m=1}^{n} \rho_m |e_m\rangle \langle e_m|$$

(43)

in a given chosen basis of orthonormal vectors. However if the eigenvalues are degenerate, the ambiguity increases, replacing each $U(1)$ subgroup with a $U(k)$ subgroup depending on the degeneracy of each eigenvalue. Thus, in order to define unambiguously the density matrix map of Eq.(41) we have to choose a "gauge" by fixing the phase factors of $U_0$ and an ordering of both the components of $\tilde{\rho}$ and the columns of $U_0$ so that Eq.(42) holds true.

This particular parametrization allows to deal more easily with quantum stochastic maps $\rho \rightarrow \rho'$ parametrized in terms of unitary maps and stochastic maps, providing therefore a different parametrization with respect to those normally used in the literature. The density matrix $\rho$ is mapped onto another density matrix $\rho'$ in such a way that the map is convex linear. This means that if two density states are mapped onto other density states

$$\rho_1 \rightarrow \rho'_1 ; \rho_2 \rightarrow \rho'_2 ,$$

(44)

any their convex sum is mapped onto the same convex sum of the images

$$\lambda_1 \rho_1 + \lambda_2 \rho_2 \rightarrow \lambda_1 \rho'_1 + \lambda_2 \rho'_2 .$$

(45)

In the tomographic framework one has the tomographic map of the density matrix $\rho_1$ onto the probability vector of the qudit state

$$\rho_1 \rightarrow \tilde{\mathcal{W}}_1(U) = |UU_{01}|^2 \tilde{\rho}_1$$

(46)

and of the density density matrix $\rho_2$ onto another probability vector of the qudit state

$$\rho_2 \rightarrow \tilde{\mathcal{W}}_2(U) = |UU_{02}|^2 \tilde{\rho}_2$$

(47)

We introduce positive maps of density matrices, parametrized by an unitary matrix $V$ and a stochastic matrix $M$, using this tomographic setting in the following way:

$$|UU_{01}|^2 \tilde{\rho}_1 \rightarrow |UU'_{01}|^2 \tilde{\rho}'_1 = |UVU_{01}|^2 M \tilde{\rho}_1 ,$$

$$|UU_{02}|^2 \tilde{\rho}_2 \rightarrow |UU'_{02}|^2 \tilde{\rho}'_2 = |UVU_{02}|^2 M \tilde{\rho}_2 .$$

(48)

Thus our map is equivalent to the map of the pair $(U_0, \tilde{\rho})$ onto another analogous pair $(U'_0, \tilde{\rho}')$ obtained by acting with an unitary matrix $V$ and a stochastic matrix $M$ as:

$$(U_0, \tilde{\rho}) \rightarrow (U'_0, \tilde{\rho}') = (VU_0, M \tilde{\rho}) .$$

(49)

Since any density matrix $\rho$ is completely determined by the corresponding pair $(U_0, \tilde{\rho})$ and as the left actions on the unitary group and on the simplex by
stochastic maps are transitive, we have described a class of positive maps which act transitively on density matrices.

To illustrate that the above map is convex linear we discuss in detail the simple qubit case. To do this first we find explicit formulae for the maps of density matrix $\rho$ and pair $(U_0, \tilde{\rho})$

$$\rho \rightarrow (U_0, \tilde{\rho})$$  \hspace{1cm} (50)

and *vice versa*

$$(U_0, \tilde{\rho}) \rightarrow \rho.$$  \hspace{1cm} (51)

As we have already remarked the map of Eq.(50) requires a “gauge” choice because we are going from a three dimensional manifold to a five dimensional one.

Let the qubit density matrix be

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad 0 \leq \det \rho \leq \frac{1}{4}. \hspace{1cm} (52)$$

Then its eigenvalues read

$$\tilde{\rho}_1 = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \det \rho} \hspace{1cm} (53)$$

$$\tilde{\rho}_2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \det \rho}. \hspace{1cm} (54)$$

The eigenvector corresponding to $\tilde{\rho}_1$ is

$$\tilde{u}_1 = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \hspace{1cm} (55)$$

where

$$u_{11} = \frac{2 \rho_{12}}{1 + \sqrt{1 - 4 \det \rho - 2 \rho_{11}}} y_1 \hspace{1cm} (56)$$

$$u_{21} = y_1 \left( \left| \frac{2 \rho_{12}}{1 + \sqrt{1 - 4 \det \rho - 2 \rho_{11}}} \right|^2 + 1 \right)^{-\frac{1}{2}}. \hspace{1cm} (57)$$

The eigenvector corresponding to $\tilde{\rho}_2$ is

$$\tilde{u}_2 = \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} \hspace{1cm} (58)$$

where

$$u_{12} = \frac{2 \rho_{12}}{1 - \sqrt{1 - 4 \det \rho - 2 \rho_{11}}} y_2 \hspace{1cm} (59)$$

$$u_{22} = y_2 \left( \left| \frac{2 \rho_{12}}{1 - \sqrt{1 - 4 \det \rho - 2 \rho_{11}}} \right|^2 + 1 \right)^{-\frac{1}{2}}. \hspace{1cm} (60)$$
and the phases have been chosen to be zero. Thus given the qubit matrix \( \rho \) of Eq. (52) one has the corresponding pair of unitary matrix
\[
U_0 = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}
\] (61)

with components (56), (57), (59), (60) and the point on simplex given by the probability vector
\[
\tilde{\rho} = \begin{pmatrix} \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{pmatrix}
\] (62)

with components (53), (54). One can check that the matrix \( \rho \) of Eq. (52) has the representation
\[
\rho = U_0 \begin{pmatrix} \tilde{\rho}_1 \\ 0 \\ 0 \end{pmatrix} U_0^\dagger.
\] (63)

Let us consider the inverse problem. Namely, given a unitary matrix \( V_0 \) of the form (61) and a probability vector \( \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \), \( \chi_1 + \chi_2 = 1 \) (64)
describing a point on the simplex, let us construct the matrix
\[
\chi = V_0 \begin{pmatrix} \chi_1 \\ 0 \\ \chi_2 \end{pmatrix} V_0^\dagger = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix}
\] (65)

whose matrix elements explicitly are:
\[
\begin{align*}
\chi_{11} &= |V_{11}|^2 \chi_1 + |V_{12}|^2 \chi_2 = 1 - \chi_{22} \\
\chi_{12} &= V_{11} V_{21}^\ast \chi_1 + V_{12} V_{22}^\ast \chi_2 = \chi_{21}.
\end{align*}
\] (66)

Thus, given the pair \((V_0, \chi)\), one has a density matrix \( \chi \) with the previous matrix elements. One can see that the map \( \rho \rightarrow (U_0, \tilde{\rho}) \) is nonlinear. The formulae (66), (67) provide a polynomial dependence of the elements of density matrix on unitary matrix and probability vector components. The dependence of density matrix elements on probability vector components is linear and on unitary matrix elements \( V_{jk} \) is quadratic. It is obvious that the inverse transform \( \rho \rightarrow (U_0, \tilde{\rho}) \) is also nonlinear as formulae (53), (54), (55), (57), (59), (60) show. Moreover formula (66) shows how two \( U(1) \) elements disappear in defining \( \chi_{jk} \).

Let us discuss now the representation of a convex sum of two density matrices \( \rho^{(1)} \) and \( \rho^{(2)} \) given by
\[
\rho = \lambda_1 \rho^{(1)} + \lambda_2 \rho^{(2)}, \quad \lambda_1 + \lambda_2 = 1, \quad 0 \leq \lambda_j \leq 1
\] (68)
The matrix elements of this matrix read
\[
\rho_{jk} = \lambda_1 \rho^{(1)}_{jk} + \lambda_2 \rho^{(2)}_{jk}
\] (69)
The two eigenvectors of this matrix are
\[ \tilde{\rho}_{1,2} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4\lambda_1^2 \det \rho^{(1)} - 4\lambda_2^2 \det \rho^{(2)} - 4\lambda_1 \lambda_2 \gamma} \] (70)
where
\[ \gamma = \rho_{11}^{(1)} \rho_{22}^{(2)} + \rho_{11}^{(2)} \rho_{22}^{(1)} - \rho_{12}^{(1)} \rho_{21}^{(2)} - \rho_{12}^{(2)} \rho_{21}^{(1)} \] (71)
One can check the following equality of probability vectors:
\[ \left| UU_0(\tilde{\lambda}) \right|^2 \tilde{\rho}(\tilde{\lambda}) = \lambda_1 \left| UU_0^{(1)} \right|^2 \tilde{\rho}_1 + \lambda_2 \left| UU_0^{(2)} \right|^2 \tilde{\rho}_2 \] (72)
where \( \tilde{\rho}(\tilde{\lambda}) \) is the probability vector with components given by Eq.s (70), (71).

The unitary matrix \( U_0(\tilde{\lambda}) \) is obtained by formula (61) with entries (56), (57), (59), (60) after the replacements
\[ \rho_{12} \rightarrow \lambda_1 \rho_{12}^{(1)} + \lambda_2 \rho_{12}^{(2)} \]
\[ \rho_{11} \rightarrow \lambda_1 \rho_{11}^{(1)} + \lambda_2 \rho_{11}^{(2)} \]
\[ \det \rho \rightarrow \lambda_1^2 \det \rho^{(1)} + \lambda_2^2 \det \rho^{(2)} + \lambda_1 \lambda_2 \gamma \]
where \( \gamma \) is given by Eq.(71). Thus we checked directly the convex superposition formula for two tomographic probability vectors corresponding to a convex mixture of two density matrix \( \rho^{(1)} \) and \( \rho^{(2)} \) for two qubits. From Eq. (72) one can readily see that
\[ \tilde{\rho}(\tilde{\lambda}) = \lambda_1 \left| U_0^{-1}(\tilde{\lambda})U_0^{(1)} \right|^2 \tilde{\rho}_1 + \lambda_2 \left| U_0^{-1}(\tilde{\lambda})U_0^{(2)} \right|^2 \tilde{\rho}_2. \] (74)
Thus we have illustrated in detail that the map of pairs \( (U_0, \tilde{\rho}) \rightarrow (Vu_0, M\tilde{\rho}) \) by means of left action on unitary group and stochastic matrix map on simplex is convex linear. It is worth to note that the map \( (U_0, \tilde{\rho}) \rightarrow (U_0^\ast, \tilde{\rho}) \), which is just the transposition of the density matrix map, is not equivalent to left multiplication by a matrix. So, we can extend the map using automorphism groups of both the unitary matrices and simplex points.

6 Entanglement

Stochastic maps may be used successfully to characterize the entanglement of states for a composite physical system in the tomographic scheme as the following example shows.

To study entangled states we consider two qubit states with state vector
\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}\rangle - |\frac{1}{2}\rangle + |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \right). \] (75)
The density matrix of this state reads
\[ \rho = |\psi\rangle \langle \psi| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \] (76)
so its four eigenvalues yield the probability vector
\[
\vec{\rho} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},
\]
while the corresponding eigenvectors form \( U_0 \):
\[
U_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
In view of this, the tomographic probability vector of the two qubit state has the following form:
\[
\vec{W}(U) = \frac{1}{2} \begin{bmatrix} |u_{12} + u_{13}|^2 \\ |u_{22} + u_{23}|^2 \\ |u_{32} + u_{33}|^2 \\ |u_{42} + u_{43}|^2 \end{bmatrix},
\]
where the \( u \)'s are the matrix elements of \( U \). One can recognize the entanglement of the state with this tomogram, calculating the stochastic matrix \( M \) whose columns are four probability vectors \( \vec{W}(U_{ab}), \vec{W}(U_{ac}), \vec{W}(U_{db}), \vec{W}(U_{dc}) \), where each unitary matrix \( U_{hk} \) is a tensor product of two \( 2 \times 2 \) unitary matrices \( U_h \otimes U_k \):
\[
U_{hk} = U_h \otimes U_k, \ (h = a, d ; \ k = b, c).
\]
Eventually, the stochastic matrix \( M \) has the form
\[
M = \begin{pmatrix} x_{ab} & x_{ac} & x_{db} & x_{dc} \\ \frac{1}{2} - x_{ab} & \frac{1}{2} - x_{ac} & \frac{1}{2} - x_{db} & \frac{1}{2} - x_{dc} \\ \frac{1}{2} - x_{ab} & \frac{1}{2} - x_{ac} & \frac{1}{2} - x_{db} & \frac{1}{2} - x_{dc} \\ x_{ab} & x_{ac} & x_{db} & x_{dc} \end{pmatrix}.
\]
where \( x_{hk} \) is a trigonometric function of the Euler angles determining \( U_h, U_k \).

To evaluate the Bell number \( B \) satisfying Bell’s inequality \([6, 7]\)
\[
B \leq 2
\]
one has to calculate [17] a trace:
\[
B = \text{Tr}(ME)
\]
where \( E \) is the following matrix:
\[
E = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.
\]
The elements of the matrix $M$ are functions of four directions. The Cirelson bound for the Bell number is $2\sqrt{2}$ and can be achieved in entangled states only. One can check that the bound is achieved when

$$x_{ab} = x_{ac} = x_{db} = x ; x_{dc} = 1 - x .$$

In that case one has

$$2\sqrt{2} = 4(4x - 1) ; x = \frac{2 + \sqrt{2}}{8} .$$

Thus the universal stochastic matrix corresponding to an arbitrary, maximally entangled state of two qubits is

$$M = \begin{pmatrix}
\frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} \\
\frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} \\
\frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} \\
\frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8}
\end{pmatrix} .$$

Thus we have clarified what is the relation of stochastic matrices with Bell inequality violation for entangled states of two qubits.

\section{Conclusions}

We point out the main results of this paper. We have established a connection of spin tomograms with stochastic maps acting on a simplex and unitary group elements. We have shown that stochastic and bistochastic $n \times n$ matrices have a dense intersection with the Lie groups $IGL(n - 1, \mathbb{R})$ and $GL(n - 1, \mathbb{R})$ respectively. We have constructed positive maps of density states as maps determined by pairs of unitary matrices and stochastic matrices. We have demonstrated that for entangled two qubit states the Cirelson bound for the Bell number is associated with a universal stochastic matrix.

To conclude, we observe that in description of quantum states an important role is played by unitary irreducible representations of Lie groups. One can see that starting with unitary representations of the $GL(2, \mathbb{R})$ group which are infinite-dimensional and restricting the representations to the semigroup subset one obtains unitary representations of invertible bistochastic matrix semigroup. Analogously the unitary representations of the stochastic semigroup can be obtained by restricting to the semigroup subset the infinite-dimensional unitary representations of the inhomogeneous linear group (either real or complex). The problem of irreducibility or reducibility of the semigroup representations has to be analyzed further.

Finally, we hope to clarify the relations between the constructed positive map and other existing approaches in a future work.

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