On Mean Field Limits for Dynamical Systems

Niklas Boers, Peter Pickl, Detlef Dürr

May 7, 2014

Abstract

Recently, a new strategy has been invented to prove propagation of molecular chaos in mean-field situations, establishing mean-field equations (Hatree or Gross-Pitaevskii) for microscopic many-particle quantum dynamics. It relies on a Gronwall estimate for a suitably defined measure of independence. We extend the method to classical systems and apply it to prove Vlasov-type limits for classical many-particle systems. We prove $L^1$-convergence and obtain from that existence and uniqueness of strong solutions of the Vlasov-type equations.

Contents

1 Introduction 1

2 A simple example 4

3 The $\alpha$-measure 4

4 The main result 8

4.1 Skeleton of the proof . . . . . . . . . . . . . . . . . . . . . . . 8

4.2 Estimates on the Flux $\Phi^N$ and $\varphi$. . . . . . . . . . 9

4.3 The weighted Lipschitz norm . . . . . . . . . . . . . . . . . 12

4.4 The law of large numbers . . . . . . . . . . . . . . . . . . . . 17

4.5 Properties of the summands $G$. . . . . . . . . . . . . . . . 18

4.6 A Gronwall type estimate for $\alpha$ . . . . . . . . . . . . . . 26

4.7 Proof of Theorem 1.1 . . . . . . . . . . . . . . . . . . . . . . . 29

1 Introduction

Consider a system consisting of $N$ interacting identical particles. The dynamics is given by a flow $(\Phi^N_{t,s})_{t,s \in \mathbb{R}} : \mathbb{R}^{6N} \to \mathbb{R}^{6N}$, which is assumed to be symmetric under permutation of coordinates. The goal is to compare the microscopic $N$-particle time evolution with an effective one-particle description given by a flow $(\varphi_{t,s})_{t,s \in \mathbb{R}} : \mathbb{R}^6 \to \mathbb{R}^6$ and to prove convergence of $\Phi^N_{t,s}$ to the product of $\varphi_{t,s}$ in the limit $N \to \infty$. As an example, one can think of a system of $N$ Newtonian particles with pair interaction. In this case, $\varphi_t$ is the classical Vlasov flow. The strategy which we shall present in the following is designed for stochastic initial conditions. We are interested in a proof which is based on typicality as this
allows for greater freedom and offers the chance to prove Vlasov like results for more complicated dynamics, as for example the Vlasov-Maxwell system (for a recent result see for example [Elskens et al., 2009] and [Golse, 2012]) or for systems involving other field degrees of freedom. The key idea is to show that during time evolution, the $N$-particle density maintains its product structure in a suitable sense. This is usually referred to as propagation of molecular chaos. Such results ([Braun and Hepp, 1977] [Dobrushin, 1979]) have recently regained interest, see for example [Hauray and Jabin, 2007] and references therein, where molecular chaos has also been established, but in contrast to our approach, it is derived from a prior result for deterministic initial conditions. We extend a method which has first been invented to prove mean field equations in quantum mechanics ([Pickl, 2010] [Pickl, 2011]).

We first state our assumptions on $\Phi^N_t$ and $\varphi$:

**Assumption 1.1.** (a) Let $\Phi^N_{t,s}$ be generated by the time dependent vector field $V : \mathbb{R}^{6N} \times \mathbb{R} \to \mathbb{R}^{6N}$

\[
\frac{d}{dt} \Phi^N_{t,s}(X) = V_t(\Phi^N_{t,s}(X)) .
\]  

where we used the the coordinate notation $X = (x_1, \ldots, x_N)$.

(b) Let $N_j$ be the set given by

\[
\omega \in N_j \iff \omega \subset \{1, 2, \ldots, N\} \setminus \{j\} \quad |\omega| = d - 1 , .
\]

We assume that only a fraction $N > d \geq 2$ of particles “interact simultaneously”, i.e. we assume that there exists a $v : \mathbb{R}^{6d} \times \mathbb{R} \to \mathbb{R}^6$ such that

\[
(V_t(X))_j = \left(\frac{N - 1}{d - 1}\right)^{-1} \sum_{\{j_2, j_3, \ldots, j_d\} \in N_j} v_t(x_j, x_{j_2}, \ldots, x_{j_d}) ,
\]

where $v$ is symmetric under the exchange of any two coordinates $x_j$ and $x_k$ with $2 \leq j, k \leq d$ and $(\cdot)_j \in \mathbb{R}^6$ stands for the $j^{th}$ component of the vector, in particular $(X)_j = x_j$.

(c) $v_l(x_1, x_2, \ldots, x_d)$ is Lipschitz-continuous in the following sense: there exists a constant $1 \leq L < \infty$ such that for any $X, Y \in \mathbb{R}^{6d}$ (note the slight abuse of notation of coordinates, the dimensionality of which is defined by being the argument of the respective function) and any $t, s \in \mathbb{R}$

\[
\|v_l(X) - v_s(Y)\| \leq L (\|X - Y\| + |t - s|)
\]

and

\[
\|v_l(X)\| \leq L(1 + \|x_1\|)
\]

where $\|\cdot\|$ is the Euclidean norm and $x_1$ is the first argument of $v_t$.

(d) $\Phi^N_{t,s}$ is volume preserving, i.e. for any $A \subset \mathbb{R}^{6N}$ and any $s, t \in \mathbb{R}$ it holds $|A| = |\Phi^N_{s,0}(A)| = |\Phi^N_{t,s}(A)|$.

**Notation 1.** Note that $v_l(x_1, \ldots, x_d)$ is not symmetric under exchange of two variables. In the conditions we imposed, the variable $x_1$ plays a special role. $V_t$, however, is symmetric.
We introduce now for any $f: \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}_+$, $f_t \in L^1(\mathbb{R}^6, \mathbb{R}_+)$, the effective one particle flow $(\varphi_{t,s}^f)_{t \geq 0}$:

$$\frac{d}{dt} \varphi_{t,s}^f(x) = v_t * f_t(\varphi_{t,s}^f(x))$$

$$:= \int v_t(\varphi_{t,s}^f(x), x_2, x_3, \ldots, x_d) \prod_{j=2}^d f_t(x_j) d^6 x_2 \ldots d^6 x_d.$$

**Definition 1.1.** Let $s$-lim denote the strong $L^1$-limit given by $s$-lim $\lim_{N \to \infty} f_N$ if and only if $\lim_{N \to \infty} \|f_N - f_t\|_1 = 0$ uniformly in $t$ on any compact interval.

**Theorem 1.1.** Let $f_0: \mathbb{R}^6 \to \mathbb{R}_+$ be a probability density with $|f_0(x)| + \|\nabla f_0(x)\| \leq C(1 + \|x\|)^{-10}$ for some $C < \infty$ and any $x \in \mathbb{R}^6$. Let $F_N^t: \mathbb{R}^{6N} \to \mathbb{R}_+$ denote the time-dependent probability density of $N$ particles with initial condition $F_N^0$, $F_N^t = F_N^0 \circ \Phi_{0,t}^N$, where $\Phi_{0,t}^N$ satisfies assumptions (7). Let $s^*F_t := \int F_t^N dx_{s+1} \ldots dx_N$ be the $s$-marginal of $F_t$. Then, if $F_N^t = f_0^\otimes N$,

(a) the limit $s$-lim $\lim_{N \to \infty} F_N^t$ exists.

(b) the respective limiting function $f := \lim_{N \to \infty} F_N^t$, solves

$$f_t = f_0 \circ \varphi_{0,t}^f$$

for each $t$.

(c) for any probability density $f_0$ there exists a unique solution $f_t: \mathbb{R} \to L^1$ of (1).

(d) the $s$-marginals of $F_N$ converge strongly

$$s$-lim $\lim_{N \to \infty} F_N^t = f^\otimes s \quad \forall s \in \mathbb{N}$$

as $N \to \infty$.

**Corollary 1.1.** Let $\mu_t^f(dx) = f_t(x)dx$ and $\mu_N^{X(t)}$ be the discrete measure concentrated at $X(t) = \Phi_{0,t}^N(X)$. Then the strong convergence

$$s$-lim $\lim_{N \to \infty} F_N^t = f \quad \text{and} \quad s$-lim $\lim_{N \to \infty} F_N^t = f \otimes f$$

implies for the bounded Lipschitz distance $d_{BL}$

$$\mu_N^{X(t)} \to \mu_t^f \quad \text{in mean with respect to} \ \mathbb{P}^{F_N^t}.$$
2 A simple example

As an example, we consider a Newtonian system consisting of $N$ identical particles of mass $m = 1$ interacting via a spherically symmetric pair potential $A_N = \frac{1}{2}A$ with $A \in C^2_b(\mathbb{R}^3)$. Here, $C^2_b(\mathbb{R}^3)$ denotes the set of functions on $\mathbb{R}^3$ with bounded and continuous second derivatives.

In this case, $d = 2$ and $v_t(q_1, p_1, q_2, p_2) = (p_1, \nabla_1 A(q_1 - q_2))$, i.e. the equations of motion are

$$\frac{dq_i}{dt} = p_i \tag{6}$$
$$\frac{dp_i}{dt} = \frac{d^2 q_i}{dt^2} = -\frac{1}{N} \sum_{j=1}^{N} \nabla_{q_i} A(q_i - q_j).$$

Corollary 2.1. Under the assumption that $\nabla A$ is bounded and Lipschitz continuous, $v_t(q_1, p_1, q_2, p_2)$ satisfies all the assumptions 1.1. The respective effective flow is given by the Vlasov flow, i.e. the macroscopic probability density solves the Vlasov equation with initial condition $f$:

$$\partial_t f_t + \nabla q f_t \cdot \dot{q} - \nabla p f_t \cdot f_t \ast \nabla A = 0, \tag{7}$$

where $\ast$ denotes convolution.

3 The $\alpha$-measure

Equation (6) is the propagation of molecular chaos: if the initial $N$-particle phase space distribution is a product, i.e. all particles are distributed independently and identically according to the single particle distribution $f_0$, then the product structure essentially survives the time evolution in the limit $N \to \infty$.

For showing this, one needs good control of the typical configurations of the $N$ particles. Clearly, in view of assumption 1.1 (b), for finite $N$ all particles become correlated for any time $t > 0$, i.e. the initial independence of the particles’ positions and momenta is immediately lost. But if $N$ is large and the mean field picture dominates, dependences become weak. The heart of the problem is to find a good measure of how weak this dependence is, i.e. how close a state is to a product state. What we are about to describe is on the one hand very reminiscent of de Finetti’s representation of symmetric distributions, which states that any symmetric (exchangeable) distribution (which we of course also have, since the particles are indistinguishable) is a convex mixture of product distributions. This in itself is however far too general since we want the true $N$ particle distribution $F_t^N$ to be of a specific almost product form, where most of the factors are approximately given by some $f_t$ (more details on the choice of $f_t$ shall be given below). So we need to modify the representation in such a way that it meets our needs. We shall call such a decomposition a good mixture. This optimization is basic to the measure of independence we are after. Before we define the measure we shall employ, we wish to characterize it by how it is put to use.

The measure will be a counting device $\alpha$, a functional of $N$-particle and 1-particle probability densities, which puts weights according to the degree of dependences in the mixture. All particles independent means that the measure...
is zero. Independence will get lost over time, correspondingly the measure will typically increase with time. Let us have a closer look at this loss of independence. Assume first that at time $t$ all particles are independent. The time derivative of $(\Phi^N_t, X)_{1}$ is given by

$$\frac{d}{dt}(\Phi^N_t, X)_{1} = (V_t(\Phi^N_t, X))_{1} = \left(\frac{N-1}{d-1}\right) - \sum_{(j_2, \ldots, j_d) \in \mathcal{N}_1} v_t(x_1, x_2, \ldots, x_d).$$

For $f_t$-typical distributions of the particles $2, \ldots, N$, the law of large numbers implies that $(V_t(X))_{1}$ can be approximated by

$$(V_t(X))_{1} \approx \int v_t(x_1, x_2, \ldots, x_d) \prod_{j=2}^{d} f_t(x_j) d^3 x_2 \ldots d^3 x_d + O(N^{-1/2}).$$

Therefore, $(V_t(X))_{1}$ is in this case close to an effective one-particle dynamics so that independence can persist as $N$ gets large.

Suppose now that at time $t$ the measure $\alpha(\mathcal{F}^N_t, f_t)$ indicates that $k$ particles are correlated. Those particles will disturb the mean field by an order $k/N \sim \alpha$. Then, in the next time-step, the uncorrelated particles will get correlated with rate $\alpha$, i.e. the measure will grow according to a Gronwall type equation

$$\dot{\alpha} \leq \text{const.} \cdot \alpha + o_N(1) \quad (8)$$

From Gronwall’s lemma it would then follow that, if initially $\alpha(\mathcal{F}^N_0, f_0) = 0$, we would have $\lim_{N \to \infty} \alpha(\mathcal{F}^N_t, f_t) = 0$ for all times $t$ and independence would dominate. We shall show below that the propagation of molecular chaos follows indeed.

To summarize, we want the measure $\alpha$ to fulfill the following desiderata for densities $\mathcal{F}^N$ on $\mathbb{R}^{6N}$ and $f$ on $\mathbb{R}^6$:

(i) $\alpha(\mathcal{F}^N, f)$ measures the distance between $\mathcal{F}^N$ and the product of $f$’s in the sense that if $\alpha(\mathcal{F}^N, f)$ is small most of the particles are close to being independently and identically distributed with one-particle distribution close to $f$.

(ii) Choosing for $\mathcal{F}^N_t$ a solution of the microscopic dynamical equation and a suitable $f_t$, $\alpha(\mathcal{F}^N_t, f_t)$ satisfies a Gronwall estimate.

The following (preliminary) measure is good for the first desideratum:

$$\hat{\alpha}(\mathcal{F}^N, f) = \inf_{\mathcal{G} = \sum_{i=1}^{n} \lambda_i \mathcal{G}_i} \sum_{i=1}^{n} \lambda_i (m(k_i) + \|f - g_i\|_1) + \|\mathcal{G} - \mathcal{F}^N\|_1.$$

We explain: Instead of directly looking at the $N$-particle density $\mathcal{F}^N$, we consider an approximation by a probability distribution $\mathcal{G}$ which can be more easily decomposed into a good mixture. The appearance of the $L_1$-norm $\|\mathcal{G} - \mathcal{F}^N\|_1$ and taking the infimum assures that the approximation is good, i.e. that $\mathcal{G}$ is decomposed into a good mixture – a convex sum of probability densities $\mathcal{G}_i$, $i = 1, \ldots, n$, $\sum \lambda_i = 1$, where each $\mathcal{G}_i$ is a density with $0 \leq k_i \leq N$.
bad particles and the $G_i$ are chosen such that $\alpha$ is as small as possible. More specifically, let $\chi_i$ be non-negative and such that
\[
\int \chi_i(x_1, \ldots, x_N)dx_1 \ldots dx_{k_i} = 1,
\]
then $G_i$ is the symmetrized distribution
\[
G_i = \text{Symm}\{\chi_i(x_1, \ldots, x_N) \prod_{n=k_i+1}^{N} g_i(x_n)\}.
\]
Hence, for probability densities $g_i : \mathbb{R}^6 \to \mathbb{R}$, $G_i$ is a symmetric probability density composed of $N - k_i$ factors $g_i$ and a non-negative factor $\chi_i$. The appearance of $\|f - g_i\|_1$ in the counting measure ensures that the $g_i$ are close to $f$. The term $m(k_i)$ makes $\alpha$ a weighted counting measure: for the time being one may think of $m(k_i) = \frac{k_i}{N}$. It gives weights according to the relative number of particles which are dependent. If all are independent, then $k_i = 0$ and the sum is zero; if on the other hand all are dependent, then only $G_N$ is non-zero and the sum will be 1. The choice of the weighting factor will become clear in the proof of lemma 3.1 below.

$\tilde{\alpha}$ must still be adjusted to also fulfill the second desideratum. We wish that the measure changes in time in such a way that we can build a Gronwall inequality. There is a small caveat which needs to be taken care of.

To be able to control the influence of fluctuations of the mean-field force on the density (cf. right hand side of (2)), we need to introduce densities which are stable under small shifts of arguments.

A possible way to do so is by virtue of the weighted Lipschitz norm $\|f\|_{\varphi}$, which is given by
\[
\|f\|_{\varphi} := \sup_{\|a\| \leq \|b\|} (1 + \|a\|) \frac{|f(a) - f(b)|}{\|a - b\|},
\]
We say more about this functional in section 4.3.

The $\alpha$-measure is now defined as follows: Recalling (9) and (10), we set for $K > 0$
\[
\alpha_K(\mathcal{F}^N, f) = \inf_{S_K} \sum_{i=1}^{n} \lambda_i (m(k_i) + \|f - g_i\|_1 + \|G - \mathcal{F}^N\|_1),
\]
\[
S_K = \left\{\mathcal{G} = \sum_{i=1}^{n} \lambda_i G_i, \|g_i\|_{\varphi} \leq K \forall i\right\}.
\]

Remark 3.1. We note for later use that testing $\alpha$ with $\mathcal{G} = \mathcal{F}^N$ yields
\[
\alpha_K(\mathcal{F}^N, f) \leq m(N) + \inf_{\|g\|_{\varphi} \leq K} \|f - g\|_1.
\]

Remark 3.2. The counting measure is continuous in $\mathcal{F}^N$ and $f$ in the sense that
\[
\alpha_K(\mathcal{F}^N, f) \leq \alpha_K(\mathcal{H}, h) + \|f - h\|_1 + \|\mathcal{F}^N - \mathcal{H}\|_1.
\]
To see this note that by definition for any $\varepsilon > 0$ there exists $\lambda_i, G_i$ and $g_i$ such that

$$\alpha_K(H, h) \geq \sum_{i=1}^{n} \lambda_i (m(k_i) + \|h - g_i\|_1) + \|G - H\|_1 - \varepsilon .$$  \hfill (15)

On the other hand, $\lambda_i, G_i$ and $g_i$ can be used to estimate $\alpha(\mathcal{F}^N, f)$ from above

$$\alpha_K(H^N, f) \leq \sum_{i=1}^{n} \lambda_i (m(k_i) + \|h - g_i\|_1) + \|G - H\|_1$$

$$\leq \alpha_K(H, h) + \|h - f\|_1 + \|\mathcal{F}^N - \mathcal{H}\|_1 + \varepsilon ,$$

where in the second line we used triangle inequality and in the third line (15).

**Remark 3.3.** The counting measure satisfies a convexity inequality in the following sense: For any $\sum_{i=1}^{n} \lambda_i = 1$ of non-negative $\lambda_i$

$$\alpha(\sum_{i=1}^{n} \lambda_i, \mathcal{F}^N) \leq \sum_{i=1}^{n} \lambda_i \alpha(\mathcal{F}^N, f) .$$  \hfill (16)

We begin with showing that the measure is powerful enough to show $L^1$-convergence of the 1-particle marginal and that the respective limit solves (2)

**Lemma 3.1.** Let $m(k) \geq \frac{s}{N}$. If

$$\alpha_K(\mathcal{F}^N, f) \xrightarrow{N \to \infty} 0 ,$$  \hfill (17)

then the marginal densities of order $s$ of $\mathcal{F}^N$ converge:

$$s \mathcal{F}^N \xrightarrow{L^1} f^{\otimes s} \quad \forall s \in \mathbb{N}$$  \hfill (18)

as $N \to \infty$. More precisely

$$\|s \mathcal{F}^N - f^{\otimes s}\|_1 \leq 2s \left( \alpha_K(\mathcal{F}^N, f) + s/N \right) .$$  \hfill (19)

**Proof.** Since for any weight $\tilde{m} \geq m$ the functional $\alpha_K(\mathcal{F}^N, f)$ for $\tilde{m}$ is always larger than $\alpha_K(\mathcal{F}^N, f)$ for $m$ we can without loss of generality consider the weight $m(k) = \frac{s}{N}$.

Let us first consider densities $G = G_i$ with $G_i$ of the form (10), i.e. $N - k_i$ particles are independently distributed according to the product of $g_i$ and $k_i$ of the particles are dependent. Since the distribution is symmetric, the probability for $s$ uniformly randomly chosen particles being independently distributed is then given by

$$\mu = \binom{N - k_i}{s} : \binom{N}{s} = \frac{(N - k_i)!(N - s)!}{(N - s - k_i)!N!} ,$$

where we set $\mu = 0$ if $s > N - k_i$. Observing that $\frac{k_i + s}{N} \leq 1$ we can use Bernoulli’s inequality and find that

$$1 \geq \mu \geq \frac{(N - k_i - s)^s}{N^s} = \left(1 - \frac{k_i + s}{N}\right)^s \geq 1 - s \frac{k_i + s}{N} .$$  \hfill (20)
which we shall use below. We can now write the $s$–marginals of $G$ as convex sums

$$G^s = \mu g_i^\otimes s + (1 - \mu) h$$

with the appropriate $s$-particle probability density $h$. By triangle inequality and observing that all functions are normalized

$$\|G^s - f^\otimes s\|_1 \leq \mu \|f^\otimes s - g_i^\otimes s\|_1 + 2(1 - \mu)$$

Further

$$\|f^\otimes s - g_i^\otimes s\|_1 = \|f^\otimes s - (f + g_i - f) \otimes g_i^{\otimes s-1}\|_1$$

by triangle inequality. Repeating this $s$ times on the remaining $g_i$ factors we get

$$\|G^s - f^\otimes s\|_1 \leq s \|f - g_i\|_1 + 2(1 - \mu) .$$

Using (20) we finally get

$$\|G^s - f^\otimes s\|_1 \leq s \|f - g_i\|_1 + 2s \left( \frac{k_i + s}{N} \right) . \quad (21)$$

Now for $\mathcal{F}^N$ and general $G = \sum \lambda_i G_i$ we get, again applying triangle inequality, observing (12) as well as (21) and the fact that $s^2 > 1$

$$\|s \mathcal{F}^N - f^\otimes s\|_1 \leq \|G^s - f^\otimes s\|_1 + \|s \mathcal{F}^N - G^s\|_1$$

$$\leq \sum_{i=1}^n \lambda_i \left( 2s \frac{k_i + s}{N} + s \|f_t - g_i\|_1 \right) + \|G - \mathcal{F}^N\|_1$$

$$\leq 2s \left( \alpha_K(\mathcal{F}^N, f) + s/N \right) . \quad \square$$

4 The main result

4.1 Skeleton of the proof

Assumption 1.1 includes a Lipschitz condition on the vector field $v$. We shall show that

- Boundedness of the $g_i$’s in weighted Lipschitz norm results in a Lipschitz condition on $\alpha(\mathcal{F}^N, f)$ (as function of $t$) as stated in Lemma 4.12.

- The weighted Lipschitz norm itself stays bounded under time evolution.

The proof is then organized as follows:

(a) In section 4.2 we give some estimates on the time evolutions according to $\varphi$ and $\Phi^N$. In one of the Lemmas we show that the Lipschitz condition on $v$ (see assumption 1.1 (d)) results in a Lipschitz condition on $\varphi$ and $\Phi^N$ (Lemma 4.1).
4.2 Estimates on the Flux $\Phi^N$ and $\varphi$

We shall now give some estimates on the Flux $\Phi^N$ and that the functions of finite weighted Lipschitz norm are dense in $L^1$. The latter is important to finally get the statement in Theorem 1.1 (c) without restrictions on the one particle densities $f$. Then we show that the weighted Lipschitz norm stays finite under time evolution.

(b) In section 4.3 we show that the weighted Lipschitz norm is in fact a norm and that the functions of finite weighted Lipschitz norm are dense in $L^1$. In section 4.4 we formulate the law of large numbers in a version that will be convenient for later reference: We shall need below that for product distributions $g^\otimes M$ large deviations of the vector field $V$ from the mean vector field $v_t \ast d^{-1}$ $g$ have exponentially small probability. This is shown — in a more generalized formulation — in Corollary 4.3.

(c) In section 4.5 we use the results of the previous sections to control the weighted Lipschitz norm stays finite under time evolution.

(d) In section 4.6 we use these estimates to show that $\alpha(\mathcal{F}^N_t, f^N_t)$ is Lipschitz continuous. For $f^N_t$ we choose $f^N_t := f_0 \circ \varphi^N_{0,t}$. We also control the $L^1$-distance of $f^N_t$ and $f^N_t$. Finally, we provide a bound on $\alpha(\mathcal{F}^N_t, f^N_t)$ via a Gronwall argument.

We will show, using estimates from section 4.6, that $f^N_t$ is Cauchy, thus the strong limit $f := \text{s-lim}_{N \to \infty} f^N_t$ exists.

We also prove that $\text{s-lim}_{N \to \infty} f^N = f$. Using continuity of the flow in the sense that $\text{s-lim}_{n \to \infty} h_0 \circ \varphi^N_n = h_0 \circ \varphi_0$ for a function $h_0$ satisfying some technical conditions, we can show that $f = \text{s-lim}_{N \to \infty} f^N = \text{s-lim}_{N \to \infty} f^N = f_0 \circ \varphi^N_0 = f_0 \circ \varphi^N_0$.

Using Lemma 3.1 we shall also verify the other statements of Theorem 1.1.

4.2 Estimates on the Flux $\Phi^N$ and $\varphi$

We shall now give some estimates on the Flux $\Phi^N$ and $\varphi$.

**Lemma 4.1.** Let $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ with $\|f_t\|_1 = \|g_t\|_1 = 1$ for all $s, t \in \mathbb{R}$, let $\Phi^N_t$ be given as above. Then

\[
\begin{align*}
\|\varphi^f_{t,s}(x) - \varphi^g_{t,s}(y)\| &\leq e^{L|t-s|} \left(\|x - y\| + Ld(1 + \|x\|) \int_s^t \|f_\tau - g_\tau\|_1 d\tau\right) \\
\text{in particular} \quad \|\varphi^f_{t,s}(x) - \varphi^g_{t,s}(y)\| &\leq e^{L|t-s|}\|x - y\| \quad (22)
\end{align*}
\]

and

\[
\begin{align*}
\|\varphi^f_{t,s}(x) - \varphi^g_{t,s}(x)\| &\leq e^{L|t-s|} Ld(1 + \|x\|) \int_s^t \|f_\tau - g_\tau\|_1 d\tau \\
\end{align*}
\]

(b) \[1 + \|\varphi^f_{t,s}(x)\| \leq e^{L|t-s|}(1 + \|x\|) \quad (23)\]

for all $x, y \in \mathbb{R}^d$, $t, s \in \mathbb{R}$. 

(c) 
\[
\| \Phi_{t,s}^N(X) - \Phi_{t,s}^N(Y) \| \leq e^{NL|t-s|} (\| X - Y \|) \\
1 + \| \Phi_{t,s}^N(X) \| \leq e^{NL|t-s|} (1 + \| X \|)
\]

for all \( X, Y \in \mathbb{R}^{6^N} \), \( k \in \{1, \ldots, N\} \) and \( t, s \in \mathbb{R} \). 

Proof. Let \( \mathcal{N} \) be the set of all pairs \((j, n_j)\) given by \((j, n_j) \in \mathcal{N} \Leftrightarrow 1 \leq j \leq N, n_j \in \mathcal{N}_j \). It follows that 
\[
|\mathcal{N}| = N \binom{N-1}{d-1} . \tag{24}
\]

For any \( \omega = (j_1, \{j_2, j_3, \ldots, j_d\}) \in \mathcal{N} \) we define the function \( v^\omega_t : \mathbb{R}^{6N} \to \mathbb{R}^6 \) by 
\[
v^\omega_t(X) := \left( \begin{array}{c} N-1 \\ d-1 \end{array} \right) v_t(x_{j_1}, x_{j_2}, \ldots, x_{j_d}) ,
\]

i.e. \((V_t(X))_j = \sum_{n_j \in \mathcal{N}_j} v^\omega_t(j, n_j)(X)\). 

First note that with assumption (c) 
\[
\| v_t \star s^{-1} f(x) - v_s \star s^{-1} f(y) \| \\
= \left\| \int (v_t(x_1, x_2, \ldots, x_d) - v_s(y_1, y_2, \ldots, y_d)) \prod_{j=2}^d f(x_j) d^d x_2, \ldots d^d x_d \right\| \\
\leq \int \| v_t(x_1, x_2, \ldots, x_d) - v_s(y_1, y_2, \ldots, y_d) \| \prod_{j=2}^d f(x_j) d^d x_2, \ldots d^d x_d \\
\leq L (\| x - y \| + |t-s|) . \tag{25}
\]

Similarly 
\[
\| v_t \star s^{-1} f(x) \| \leq L (\| x \| + 1) , \tag{26}
\]

and 
\[
\| V_t(X) - V_s(Y) \| \leq \sum_{\omega \in \mathcal{N}} \| v^\omega_t(X) - v^\omega_s(Y) \| \\
\leq N \binom{N-1}{d-1} \left( \begin{array}{c} N-1 \\ d-1 \end{array} \right) ^{-1} L (\| X - Y \| + |t-s|) \\
= NL (\| X - Y \| + |t-s|) \tag{27}
\]

as well as 
\[
\| (V_t(X))_k \| = \sum_{\{j_2, \ldots, j_d\} \in \mathcal{N}_k} \| v_t^{(k, \{j_2, \ldots, j_d\})}(X) \| \\
\leq L (1 + \| x_k \|) . \tag{28}
\]

(b) Using Minkowski inequality and 
\[
\left| \frac{d}{dt} \left( 1 + \| \varphi_{t,s}(x) \| \right) \right| \leq \left| \frac{d}{dt} \varphi_{t,s}(x) \right| = \left| v_t \star s^{-1} f(\varphi_{t,s}(x)) \right| \\
\leq L \left( 1 + \| \varphi_{t,s}(x) \| \right) . 
\]

It follows with Gronwall’s Lemma that 
\[
1 + \left( \| \varphi_{t,s}(x) \| \right) \leq e^{L|t-s|} (1 + \| x \|) .
\]

10
Similarly as in (a) we get with (27) that

\[ \frac{d}{dt} \left\| (\varphi_{t,s}^f(x) - \varphi_{t,s}^g(y)) \right\| \leq \frac{d}{dt} \left\| (\varphi_{t,s}^f(x) - \varphi_{t,s}^g(y)) \right\| \]

\[ = \left\| v_t \right\| \left\| f_t(\varphi_{t,s}^f(x)) - v_t \right\| g_t(\varphi_{t,s}^g(y)) \right\| \]

\[ \leq \left\| v_t \right\| \left\| f_t(\varphi_{t,s}^f(x)) - v_t \right\| g_t(\varphi_{t,s}^f(x)) \right\| + \left\| v_t \right\| \left\| g_t(\varphi_{t,s}^f(x)) - v_t \right\| g_t(\varphi_{t,s}^g(y)) \right\| \]

Since \( v \) is Lipschitz we get with (23) and (26) that the first summand is controlled by

\[ \left\| v_t \right\| \left\| f_t(\varphi_{t,s}^f(x)) - v_t \right\| g_t(\varphi_{t,s}^f(x)) \right\| \]

\[ \leq L (1 + \left\| \varphi_{t,s}^f(x) \right\|) \left\| f_t - g_t \right\|_{1} \leq L d \left\| f_t - g_t \right\| e^{L|t-s|(1 + \left\| x \right\|)} . \]

For the second summand we use (29) and get

\[ \left\| v_t \right\| \left\| g_t(\varphi_{t,s}^f(x)) - v_t \right\| g_t(\varphi_{t,s}^g(y)) \right\| \leq L \left( \left\| \varphi_{t,s}^f(x) - \varphi_{t,s}^g(y) \right\| + |t - s| \right) . \]

It follows that

\[ \left\| \varphi_{t,s}^f(x) - \varphi_{t,s}^g(y) \right\| \leq \left( \left\| \varphi_{t,s}^f(x) - \varphi_{t,s}^g(y) \right\| + \left\| \varphi_{t,s}^f(x) - \varphi_{t,s}^g(y) \right\| \right) \]

\[ + L d e^{L|t-s|}(1 + \left\| x \right\|) \left\| f_t - g_t \right\|_1 \]

Integrating we get

\[ \left\| \varphi_{t,s}^f(x) - \varphi_{t,s}^g(y) \right\| \leq e^{L|t-s|} \left( \left\| x - y \right\| + L d (1 + \left\| x \right\|) \int_s^t \left\| f_\tau - g_\tau \right\|_1 d\tau \right) \]

(c) Similarly as in (a) we get with (27) that

\[ \left\| \frac{d}{dt} \left( \Phi_{t,s}^N(X) - \Phi_{t,s}^N(Y) \right) \right\| \leq \left\| \frac{d}{dt} \left( \Phi_{t,s}^N(X) - \Phi_{t,s}^N(Y) \right) \right\| 

\[ = \left\| V_t(\Phi_{t,s}^N(X)) - V_t(\Phi_{t,s}^N(Y)) \right\| \leq N L \left\| \Phi_{t,s}^N(X) - \Phi_{t,s}^N(Y) \right\| \]

and with (28)

\[ \left\| \frac{d}{dt} \left( \Phi_{t,s}^N(X) \right) \right\| \leq \left\| \frac{d}{dt} \left( \Phi_{t,s}^N(X) \right) \right\| \leq \sum_{k=1}^{N} \left\| (V_t(\Phi_{t,s}^N(X)))_k \right\| \]

\[ \leq N L (1 + \left\| \Phi_{t,s}^N(X) \right\|) \]

It follows that

\[ \left\| \Phi_{t,s}^N(X) - \Phi_{t,s}^N(Y) \right\| \leq e^{N L|t-s|}(\left\| X - Y \right\|) \]

and

\[ 1 + \left\| \Phi_{t,s}^N(X) \right\| \leq e^{N L|t-s|}(1 + \left\| X \right\|) . \]
Definition 4.1. Let $\Psi_{t,s}(X) := X + (t-s)V_t(X)$. 

Lemma 4.2. Under the assumptions 1.1 it follows that for any $k \in \{1, \ldots, N\}$
\[ \|\Phi^N_{t,s}(X) - \Psi_{t,s}(X)\| \leq N^2L^2(1 + \|X\|)(t-s)^2 \]
for all $X \in \mathbb{R}^{6N}$ and sufficiently small $|t-s|$.

Proof. Using (28)
\[
\frac{d}{dt} \left(1 + \left\|\Phi^N_{t,s}(X)\right\|_k\right) \leq \left\|\frac{d}{dt} (\Phi^N_{t,s}(X))_k\right\| = \left\|\left(V_t(\Phi^N_{t,s}(X))\right)_k\right\|
\leq L(1 + \left\|\Phi^N_{t,s}(X)\right\|_k)
\]
and we get with Gronwall
\[
\left(1 + \left\|\Phi^N_{t,s}(X)\right\|_k\right) \leq e^{L|t-s|}(1 + \|x_k\|).
\]

Now,
\[
\left\|\Phi^N_{t,s}(X) - X\right\|_k = \left\|\int_s^t (V_\tau(\Phi^N_{\tau,s}(X)))_k\,d\tau\right\| \leq \int_s^t L(1 + \left\|\Phi^N_{\tau,s}(X)\right\|_k)d\tau
\leq \int_s^t L(e^{L|\tau-s|}(1 + \|x_k\|))d\tau.
\]

For $t < \frac{1}{L}$ the latter is bounded by $(3L(1 + \|x_k\|))t$.

With assumption 1.1 and using that $NL > 1$
\[
\left\|\Phi^N_{t,s}(X) - X - V_t(X)(t-s)\right\| = \left\|\int_s^t V_\tau(\Phi^N_{\tau,s}(X)) - V_t(X)\,d\tau\right\|
\leq \sum_{k=1}^N \int_s^t NL\left(\left\|\Phi^N_{\tau,s}(X) - X\right\|_k + t - \tau\right)d\tau
\leq \sum_{k=1}^N \int_s^t NL(3L(1 + \|x_k\|)s + t - \tau)d\tau
\leq 2N^2L^2(1 + \|X\|)(t-s)^2.
\]

4.3 The weighted Lipschitz norm

The bounds on the weighted Lipschitz norm we imposed for the densities $g_i$ are used to deduce a Lipschitz condition on $\alpha$ from a Lipschitz condition on the vector field $V$. In the following we shall give some properties of the weighted Lipschitz norm for reference below.

Lemma 4.3. (a) The set
\[ \{g \in L^1(\mathbb{R}^6), \|g\|_{e,s} < \infty\} \]
is dense in $L^1(\mathbb{R}^6)$. 

12
(b) The weighted Lipschitz norm

\[ \|f\|_{\text{ws}} := \sup_{\|a\| \leq \|b\|} (1 + \|a\|^{10}) \frac{|f(a) - f(b)|}{\|a - b\|}, \]

is indeed a norm on this set.

(c) Any function with finite weighted Lipschitz norm decays like \(|g(x)| \leq \|g\|_{\text{ws}} (1 + \|x\|)^{-9}.\)

**Proof.** (a) Let \(f \in L^1\) with \(|f(x)| + \|\nabla f(x)\| \leq C(1 + \|x\|)^{-10}\) for some \(C < \infty\) and any \(x \in \mathbb{R}^6.\)

\[ \|f\|_{\text{ws}} = \sup_{\|a\| \leq \|b\|} (1 + \|a\|^{10}) \frac{|f(a) - f(b)|}{\|a - b\|}. \]

In the case \(\|a - b\| \geq 1\) the term \((1 + \|a\|^{10})\frac{|f(a) - f(b)|}{\|a - b\|}\) is bounded by \((1 + \|a\|)^{-10}\) and \(f(b)\) decays faster than \((1 + \|b\|)^{-10}\). Hence for \(\|a\| \leq \|b\|\) the term \((1 + \|a\|)^{10}\frac{|f(a) - f(b)|}{\|a - b\|}\) is bounded.

In the case \(\|a - b\| < 1\) we get by the mean value theorem

\[ (1 + \|a\|^{10}) \frac{|f(a) - f(b)|}{\|a - b\|} = (1 + \|a\|^{10}) \|\nabla f(x)\| \]

for some \(x\) with \(\|a - x\| < 1\). Since \(\|\nabla f\|\ (\xi)\) decays faster than \((1 + \|\xi\|)^{-10}\) it follows again that \((1 + \|a\|)^{10}\frac{|f(a) - f(b)|}{\|a - b\|}\) is bounded. Hence \(f\) has finite weighted Lipschitz norm. The set of such functions is dense in \(L^1\).

(b) Clearly \(\|g\|_{\text{ws}} \geq 0\) and \(\|\lambda g\|_{\text{ws}} = |\lambda| \|g\|_{\text{ws}}\) for any real number \(\lambda\). For \(g \equiv 0\) the weighted Lipschitz norm is zero. On the other hand \(\|g\|_{\text{ws}} = 0\) implies that \(g\) is constant. The only constant function in \(L^1\) is \(g \equiv 0\). Moreover, it is easy to see that the triangle inequality holds.

(c) Let \(e \in \mathbb{R}^6, \|e\| = 1, \xi \in \mathbb{R}^+\). Since \(g \in L^1\) it follows that \(\lim_{\xi \to \infty} g(\epsilon \xi) = 0\). Further \((1 + \xi^{10}) \frac{d}{d\xi} g(\epsilon \xi) \leq \|g\|_{\text{ws}}.\) By the fundamental theorem of calculus it follows that

\[ |g(\epsilon \xi)| = \left| \int_{\infty}^{\xi} \frac{d}{d\xi} g(\epsilon \xi) d\xi \right| \leq \|g\|_{\text{ws}} \int_{\infty}^{\xi} (1 + \xi)^{-10} d\xi \leq \|g\|_{\text{ws}} (1 + \xi^{-9}). \]

\[ \square \]

**Definition 4.2.** For \(\epsilon > 0\) and \(g \in L^1(\mathbb{R}^6)\) we define

\[ \epsilon g(x) = \inf_{\|e\| \leq 1} g(x + \epsilon e(1 + \|x\|)) \]

and

\[ \epsilon_n g := \frac{\epsilon g}{\|\epsilon g\|_1}. \] (30)

**Lemma 4.4.** Under assumption \([f]_{L^0}\) for any positive \(f\) and \(g\) the following holds
Proof. (a) Writing $a' := \varphi_{t,s}^f(a)$ and $b' := \varphi_{t,s}^f(b)$ one has with Lemma [11] (a) equation (22)
\[
\|a' - b'\| \leq e^{t-L[t-s]} \|a - b\|; \quad 1 + \|a\| \leq e^{L[t-s]}(1 + \|a'\|); \\
1 + \|b\| \leq e^{L[t-s]}(1 + \|b'\|).
\]
Hence if $\|a'\| \leq \|b'\|
\[
\|g \circ \varphi_{t,s}^f\|_{\infty} = \sup_{\|a\| \leq \|b\|} (1 + \|a\|)^{10} \frac{|g(\varphi_{t,s}^f(a)) - g(\varphi_{t,s}^f(b))|}{\|a - b\|} \\
\leq e^{11L[t-s]} \sup_{\|a\| \leq \|b\|} (1 + \|a'\|) \frac{10 |g(a') - g(b')|}{\|a' - b'\|} = e^{11L[t-s]} \|g\|_{\infty}
\]
and if $\|b'\| < \|a'\|
\[
\|g \circ \varphi_{t,s}^f\|_{\infty} \leq \sup_{\|b\| \leq \|a\|} (1 + \|b\|) \frac{10 |g(a) - g(b)|}{\|a - b\|} \\
\leq e^{11L[t-s]} \sup_{\|b'\| \leq \|a'\|} (1 + \|b'\|) \frac{10 |g(a') - g(b')|}{\|a' - b'\|} = e^{11L[t-s]} \|g\|_{\infty}.
\]
(b) Let $g$ be positive. Using the supremum property there exists for any $\delta > 0$ $\mathbb{R}^d$-vectors $\|a\| \leq \|b\|$ such that
\[
\|\varepsilon g\|_{\infty} \leq (1 + \|a\|)^{10} \frac{\varepsilon g(a) - \varepsilon g(b)}{\|a - b\|} + \delta.
\]
Assume that $\varepsilon g(a) \geq \varepsilon g(b)$. Using the infimum property we can find $|\varepsilon|' \leq 1$ such that
\[
g(b + \varepsilon \varepsilon'(1 + \|b\|)) \leq \inf_{|\varepsilon|' \leq 1} g(b + \varepsilon \varepsilon(1 + \|b\|)) + \delta = \varepsilon g(b) + \delta.
\]
It follows that
\[
\|\varepsilon g\|_{\infty} \leq (1 + \|a\|)^{10} \frac{\varepsilon g(a) - \varepsilon g(b + \varepsilon \varepsilon'(1 + \|b\|))}{\|a - b\|} + 2\delta
\]
\[
\leq (1 + \|a\|)^{10} \frac{g(a + \varepsilon \varepsilon'(1 + \|a\|)) - g(b + \varepsilon \varepsilon(1 + \|b\|))}{\|a - b\|} + 2\delta
\]
Setting $a' = a + \varepsilon \varepsilon'(1 + \|a\|)$ and $b' = b + \varepsilon \varepsilon'(1 + \|b\|)$ we get
\[
\|a' - b'\| = \|a + \varepsilon \varepsilon'(1 + \|a\|) - b - \varepsilon \varepsilon'(1 + \|b\|)\|
\leq \|a - b\| + \varepsilon \varepsilon' \|a - b\| \leq (1 + \varepsilon) \|a - b\|
\]
and
\[
(1 + \|a\|) \leq (1 + \|a'\| + \varepsilon) \leq (1 + \|a'\|)(1 + \varepsilon).
\]
Proof. We use the notation \(g_{t,s} := g \circ \varphi_{t,s}^f\). Note first, that for any \(\|e\| = 1\) and \(0 < \varepsilon \leq \frac{1}{2}\) by triangle inequality

\[
1 + \|x + \varepsilon e(1 + \|x\|)\| \geq 1 + \|x\| - \frac{1}{2}(1 + \|x\|) = \frac{1}{2}(\|x\| + 1)
\]

By definition

\[
\|g - \varepsilon g\|_1 = \int g(x) - \inf_{\|e\| \leq 1} g(x + \varepsilon e(1 + \|x\|)) d^d x
\]

\[
= \int \sup_{\|e\| \leq 1} (g(x) - g(x + \varepsilon e(1 + \|x\|))) d^d x
\]

\[
= 2^{10} \varepsilon \int (1 + \|x\|)^{-9} \sup_{\|e\| \leq 1} 2^{-10} (1 + \|x\|)^{10} g(x) - g(x + \varepsilon e(1 + \|x\|)) d^d x
\]

\[
\leq 2^{10} \varepsilon \int (1 + \|x\|)^{-9} \sup_{\|e\| \leq 1} 2^{-10} (1 + \|x\|)^{10} \frac{|g(x) - g(b)|}{\|x - b\|} d^d x
\]

Lemma 4.5. There exists a constant \(C\) such that for any \(f : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^3_0\) with \(\|f_t\|_1 = 1\) for any \(t \in \mathbb{R}\), any \(g \in L^1\), any \(0 < \varepsilon \leq \frac{1}{2}\) and any \(t, s \in \mathbb{R}\)

\[
\|\varepsilon^f g_{t,s} - g_{t,s} \|_1 \leq C \varepsilon \|f\|_1 \|g\|_e,
\]

Proof. We use the notation \(g_{t,s} := g \circ \varphi_{t,s}^f\). Note first, that for any \(\|e\| = 1\) and \(0 < \varepsilon \leq \frac{1}{2}\) by triangle inequality

\[
1 + \|x + \varepsilon e(1 + \|x\|)\| \geq 1 + \|x\| - \frac{1}{2}(1 + \|x\|) = \frac{1}{2}(\|x\| + 1)
\]

By definition

\[
\|g - \varepsilon g\|_1 = \int g(x) - \inf_{\|e\| \leq 1} g(x + \varepsilon e(1 + \|x\|)) d^d x
\]

\[
= \int \sup_{\|e\| \leq 1} (g(x) - g(x + \varepsilon e(1 + \|x\|))) d^d x
\]

\[
= 2^{10} \varepsilon \int (1 + \|x\|)^{-9} \sup_{\|e\| \leq 1} 2^{-10} (1 + \|x\|)^{10} g(x) - g(x + \varepsilon e(1 + \|x\|)) d^d x
\]

\[
\leq 2^{10} \varepsilon \int (1 + \|x\|)^{-9} \sup_{\|e\| \leq 1} 2^{-10} (1 + \|x\|)^{10} \frac{|g(x) - g(b)|}{\|x - b\|} d^d x
\]
Since
\[ 1 + \|b\| = 1 + \|x + \varepsilon(1 + \|x\|)\| \geq \frac{1}{2}(\|x\| + 1) \]
it follows that \( \sup_b 2^{-10}(1 + \|x\|)10 \frac{|g(x) - g(b)|}{\|x - b\|} \leq \|g\|_{\varepsilon^+} \). Hence
\[ \|g - \varepsilon g\|_1 \leq C\varepsilon \|g\|_{\varepsilon^+}. \] (31)
with
\[ C = 2^{10} \int (1 + \|x\|)^{-9} d\varepsilon x < 2^{10} \pi^3 \] (32)

Using triangle inequality
\[ \|\varepsilon g_{t,s}\|_1 \geq \|g_{t,s}\|_1 - \|\varepsilon g_{t,s}\|_1 \geq 1 - C\varepsilon \|g_{t,s}\|_{\varepsilon^+}. \] (33)

Hence
\[ \|\varepsilon g_{t,s} - g_{t,s}\|_1 \leq \|\varepsilon g_{t,s} - \varepsilon g_{t,s}\|_1 + \|g_{t,s} - g_{t,s}\|_1 \]
\[ = (1 - \|\varepsilon g_{t,s}\|_1) + \|g_{t,s} - g_{t,s}\|_1 \]
\[ \leq 2C\varepsilon \|g_{t,s}\|_{\varepsilon^+}, \]
where we used triangle inequality and that \( \varepsilon g_{t,s} \) and \( \varepsilon g_{t,s} \) are linearly dependent.

Using Lemma 4.3 (a) the latter is bounded by \( 2C\varepsilon e^{1L\|t-s\|} \|g\|_{\varepsilon^+}. \)

**Lemma 4.6.** Let \( f, g : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^6_+ \) with \( \|f_t\|_1 = \|g_t\|_1 = 1 \) for all \( t \in \mathbb{R} \), let \( h_0 \) be a function with \( \|h_0\|_{\varepsilon^+} < \infty \). Then
\[ \|h_0 \circ \varphi_{t,s}^f - h_0 \circ \varphi_{t,s}^g\|_1 \leq C \|h_0\|_{\varepsilon^+} e^{L[t-s]} L \left| \int_s^t \|f_\tau - g_\tau\|_1 d\tau \right| \]
with \( C \) given in (32).

In particular, \( h_0 \circ \varphi_{0}^{(s)} \) is continuous in the sense that if \( (f^n)_{n \in \mathbb{N}} \) converges in \( L^1 \) uniformly on the time interval \([s, t]\) then
\[ \text{s-lim}_{n \to \infty} h_0 \circ \varphi_{0}^{f^n} = h_0 \circ \varphi_{0}^{\text{s-lim}_{n \to \infty} f^n}. \]

**Proof.** By Lemma 4.1 (a)
\[ \left| \varphi_{t,s}^f(x) - \varphi_{t,s}^g(x) \right| \leq e^{L[t-s]} L(1 + \|x\|) \left| \int_s^t \|f_\tau - g_\tau\|_1 d\tau \right| =: \delta(t,s)(1 + \|x\|), \]

hence \( \delta(t,s)h_0 \circ \varphi_{t,s}^f(x) \leq h_0 \circ \varphi_{t,s}^g(x) \) and thus
\[ \left| h_0 \circ \varphi_{t,s}^f - h_0 \circ \varphi_{t,s}^g \right|_1 \leq 2 \left| \delta(t,s)h_0 \circ \varphi_{t,s}^f - h_0 \circ \varphi_{t,s}^g \right|_1 \]
\[ = 2 \left| \delta(t,s)h_0 - h_0 \right|_1. \]

Using (31)
\[ \left| h_0 \circ \varphi_{t,s}^f - h_0 \circ \varphi_{t,s}^g \right|_1 \leq C \|h_0\|_{\varepsilon^+} \delta(t,s) \]
and the Lemma follows. \( \square \)
4.4 The law of large numbers

Definition 4.3. For any \( M \in \mathbb{N} \), any \( X \in \mathbb{R}^6 \) and any \( g, h \in L^1(\mathbb{R}^6 \to \mathbb{R}) \) with \( \|g\|_1 = 1 \) and \( \|h\|_{\infty} \leq \infty \) let \( d(X, g, h) \) be given by

\[
d(X, g, h) = \frac{1}{M} \sum_{j=1}^{M} h(x_j) - h \ast g(0) \] 

Lemma 4.7. For any \( g, h \in L^1(\mathbb{R}^6 \to \mathbb{R}) \) with \( \|g\|_1 = 1 \) and \( \|h\|_{\infty} \leq \infty \), any \( \kappa > 0 \) and any \( k \in \mathbb{N} \) there exists a \( C_{\kappa, k} \in \mathbb{R}^+ \) such that

\[
\int_{d(X, g, h) \geq 2M^{-1/2+\kappa}} \prod_{j=1}^{M} g(x_j) d\nu x \leq C_{\kappa, k} M^{-k} .
\]

Proof. The Lemma is a standard result based on the law of large numbers and will be omitted in this manuscript.

Corollary 4.1. Let \( n \in \mathbb{N} \), \( \kappa > 0 \), \( h_n : \mathbb{R}^6 \to \mathbb{R}_0^+ \) with \( \|h_n\|_{\infty} \leq \infty \) and \( S \) be the set given by

\[
X \in S \iff \sum_{\{j_1, j_2, \ldots, j_n\} \subset \{1, \ldots, M\}} h_n(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) - h_n \ast^n g(0) \geq M^{-1/2+\kappa} .
\]

Then there exists for any \( k \in \mathbb{N} \) a \( C_{\kappa, k, n} \) such that for any \( x \in \mathbb{R}^6 \)

\[
\int_{S} \prod_{j=1}^{M} g(x_j) d\nu x \leq C_{\kappa, k, n} M^{-k} \|h_n\|_{\infty} .
\]

Proof. We prove the Corollary by induction over \( n \). Setting \( n = 1 \) the Lemma is Lemma 4.7.

Assume now that the Corollary holds for all \( j \leq n \in \mathcal{M} \). Using triangle inequality \( X \in S \) implies that either

\[
\left| \sum_{\{j_1, j_2, \ldots, j_{n+1}\} \subset \{1, \ldots, M\}} h_{n+1}(x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}}) - \sum_{j=1}^{M} M^{-1}(h_{n+1} \ast^n g)(0, x_{n+1}) \right| \geq \frac{1}{2} M^{-1/2+\kappa} \tag{34}
\]

or

\[
\left| \sum_{j=1}^{M} M^{-1} h_{n+1} \ast^n g(0, x_{n+1}) - h_{n+1} \ast^{n+1} g(0) \right| \geq \frac{1}{2} M^{-1/2+\kappa} . \tag{35}
\]

implies that at least for one \( j_{n+1} \in \{1, \ldots, M\} \)

\[
\left| \sum_{\{j_1, j_2, \ldots, j_n\} \subset \{1, \ldots, M\}} M^{-1} h_{n+1}(x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}}) - M^{-1} h_{n+1} \ast^n g(0, x_{j_{n+1}}) \right| \geq M^{-1/2} M^{-1/2+\kappa} . \tag{36}
\]
In view of (35) and (36) we define the sets $S_\infty$ and $S_k$ by

$$X \in S_\infty \Leftrightarrow \left| \sum_{j=1}^{M} M^{-1} (h_{n+1} \ast^n g)(0, x_{n+1}) - h_{n+1} \ast^{n+1} g(0) \right| \geq \frac{1}{2} M^{-1/2+\kappa}$$

and

$$Y \in S_k \Leftrightarrow \left| \sum_{\{j_1, j_2, \ldots, j_n\} \subset \{1, \ldots, M\}} h_{n+1}(y_{j_1}, y_{j_2}, \ldots, y_k) - h_{n+1} \ast^n g(0, y_k) \right| \geq \frac{1}{2} M^{-1/2+\kappa}.$$ 

It follows that $S \subset \cup_{k=1}^{M} S_k \cup S_\infty$ and thus

$$\int \prod_{j=1}^{M} g(x_j) d^6M x \leq \int_{S_\infty} \prod_{j=1}^{M} g(x_j) d^6M x + \sum_{j=1}^{M} \int_{S_k} \prod_{j=1}^{M} g(x_j) d^6M x .$$

The first summand can be controlled by Lemma 4.7, the second by the induction assumption. It follows that $\int_S \prod_{j=1}^{M} g(x_j) d^6M x$ decays faster than any polynomial in $M$.

### 4.5 Properties of the summands $G$

**Definition 4.4.** For any $g : \mathbb{R}^6 \to \mathbb{R}_+^+$ with $\|g\|_1 = 1$ and any $k \in \{0, 1, \ldots, N\}$ we define the set $M^k_g \subset L^1(\mathbb{R}^{6N} \to \mathbb{R})$ via

$$G \in M^k_g \Leftrightarrow \exists \chi \in L^1(\mathbb{R}^{6N} \to \mathbb{R}) \text{ with } \int \chi(x_1, \ldots, x_N) dx_1 \ldots dx_k = 1$$

such that $G = \text{Symm}\{\chi(x_1, \ldots, x_N)\Pi_{n=k+1}^{N} g(x_n)\}$

and $M_g^{k,+}$ as

$$G \in M_g^{k,+} \Leftrightarrow G \in M^k_g \text{ and } G \geq 0 .$$

**Lemma 4.8.** Let $G \in M_g^k$. Then there exists $G^+ \in M_g^{k,+}$ such that

$$\|G - G^+\|_1 \leq 2 \|G^{neg}\|_1 ,$$

where $G = G^{pos} + G^{neg}$, the positive and negative parts of $G$.

**Proof.** Since $g$ is positive

$$G^{neg} = \text{Symm}\{\chi^{neg}(x_1, \ldots, x_N)\Pi_{n=k+1}^{N} g(x_n)\}$$

and

$$G^{pos} = \text{Symm}\{\chi^{pos}(x_1, \ldots, x_N)\Pi_{n=k+1}^{N} g(x_n)\} .$$

Since $\int \chi(x_1, \ldots, x_N) dx_1 \ldots dx_k = 1$ it follows that

$$\int \chi^{pos}(x_1, \ldots, x_N) dx_1 \ldots dx_k \geq 1 , \text{ thus } \|G^{pos}\|_1 \geq 1 .$$
We now define \( G^+ \in M_g^{k^+} \) by

\[
G^+(x_1, \ldots, x_N) := G^{pos}(x_1, \ldots, x_N) / \|G^{pos}\|_1.
\]

Observing that

\[
\int G^+ d\mu^N x = 1 = \int G d\mu^N x = \int G^{pos} + G^{neg} d\mu^N x
\]

we have that \( \int G^{pos} - G^+ d\mu^N x = - \int G^{neg} d\mu^N x = \|G^{neg}\|_1 \). Furthermore \( G^{pos} \leq G^+ \), hence \( \|G^{pos} - G^+\|_1 = \|G^{neg}\|_1 \). Using now triangle inequality

\[
\|G - G^+\|_1 \leq \|G^{neg}\|_1 + \|G^{pos} - G^+\|_1 = 2 \|G^{neg}\|_1.
\]

\[ \blacksquare \]

We will now consider the effect of the time evolution \( \Phi^N_t \) for a small time interval \( t \). Its effect on the \( N \)-particle distribution \( \mathcal{F}^N \) will be controlled by the approximating \( \widetilde{G} \) which needs to be properly decomposed into the desired good mixture with appropriate \( \alpha \)’s.

The bound on \( \|g_i\|_{\alpha} \) in (12) has the consequence that the densities \( \mathcal{G} \) (and thus finally \( \alpha \)) satisfy a Lipschitz condition with respect to changes in the coordinates \( X \) of the \( N \) particles.

**Lemma 4.9.** Let \( \varepsilon > 0 \) and \( \Phi^N_\varepsilon : \mathbb{R}^{6N} \to \mathbb{R}^{6N} \) be a volume preserving map such that

\[
\| (\Phi^N_\varepsilon(X) - X) \| \leq \varepsilon \|X_k\|
\]

for any \( X \in \mathbb{R}^{6N} \) and any \( k \in \{1, 2, \ldots, N\} \). Then for any \( \mathcal{G} \in M_g^{k^+} \) there exists \( \mathcal{G} \in M_g^{k^+} \) and \( \lambda > 0 \) such that \( \lambda \mathcal{G} \leq \mathcal{G} \circ \Phi^N_\varepsilon \) and

\[
\| \lambda \mathcal{G} - \mathcal{G} \circ \Phi^N_\varepsilon \|_1 \leq CN\varepsilon \|g\|_{\alpha},
\]

with \( C \) given in (32).

**Proof.** Let \( \mathcal{G} \in M_g^{k^+}, \) i.e. there exists a \( \chi \in L^1(\mathbb{R}^{6N} \to \mathbb{R}) \) with

\[
\int \chi(x_1, \ldots, x_N) dx_1 \ldots dx_k = 1
\]

such that \( \mathcal{G} = Symm\{\chi(x_1, \ldots, x_N)\Pi_{n=k+1} g(x_n)\} \). By definition of \( \varepsilon g \) it follows for \( \mathcal{H}(x_1, x_2, \ldots, x_N) = id(x_1, \ldots, x_k) \Pi_{n=k+1} g(x_n) \)

\[
\text{id}(x_1, \ldots, x_k) \prod_{n=k+1}^N \varepsilon g(x_n) \leq \mathcal{H} \circ \Phi^N_\varepsilon(X).
\]

Setting \( \hat{\chi} := \chi \circ \Phi^N_\varepsilon \) the function \( \hat{\mathcal{G}} := Symm\{\hat{\chi}(x_1, \ldots, x_N)\Pi_{n=k+1} \varepsilon g(x_n)\} \) is never larger than \( \mathcal{G} \circ \Phi^N_\varepsilon \). Since by (33) \( \|g\|_1 \geq 1 - C\varepsilon \|g\|_{\alpha} \) we get with Bernoulli that \( \lambda := \| \hat{\mathcal{G}} \|_1 \geq 1 - CN\varepsilon \|g\|_{\alpha} \). Defining \( \tilde{\mathcal{G}} = \lambda^{-1} \hat{\mathcal{G}} \) the Lemma follows.

\[ \blacksquare \]
Corollary 4.2. Let $\Phi_N^N$ be as in Lemma 4.9. Then there exists for any $\mathcal{G} \in \mathcal{M}_{g}^{k,+}$ a $\tilde{\mathcal{G}} \in \mathcal{M}_{g}^{k,+}$ such that

$$
\left\| \tilde{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \leq 3NCg \|g\|_{\infty}.
$$

Proof. Let $\mathcal{G} \in \mathcal{M}_{g}^{k,+}$. Due to Lemma 4.3 there exists a $\overline{\mathcal{G}} \in \mathcal{M}_{g}^{k,+}$ and a $\lambda > 0$ such that $\lambda \overline{\mathcal{G}} \leq \mathcal{G} \circ \Phi_N^N$ and

$$
\left\| \lambda \overline{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \leq CN \|g\|_{\infty}.
$$

$\tilde{\mathcal{G}}$ will be $\overline{\mathcal{G}}$ with all the factors $\epsilon g^n$ replaced by $g$. It is left to estimate

$$
\left\| \tilde{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1,
$$

which is by triangle inequality bounded by

$$
\left\| \tilde{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \leq \left\| \overline{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 + \left\| \tilde{\mathcal{G}} - \overline{\mathcal{G}} \right\|_1.
$$

Since $\overline{\mathcal{G}}$ and $\mathcal{G} \circ \Phi_N^N$ are normalized it follows by triangle inequality that

$$
|\lambda| = \left\| \lambda \overline{\mathcal{G}} \right\|_1 \geq \left\| \mathcal{G} \circ \Phi_N^N \right\|_1 - \left\| \lambda \overline{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \geq 1 - CN \|g\|_{\infty},
$$

hence

$$
\left\| \overline{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \leq 2CN \|g\|_{\infty}.
$$

Since by Lemma 4.3 $\|g^n - g\|_1 \leq Cg \|g\|_{\infty}$ replacing all the factors $\epsilon g^n$ in $\overline{\mathcal{G}}$ by $g$ we get a $\tilde{\mathcal{G}} \in \mathcal{M}_{g}^{k,+}$ with $\left\| \tilde{\mathcal{G}} - \overline{\mathcal{G}} \right\|_1 \leq NCg \|g\|_{\infty}$. Using triangle inequality

$$
\left\| \tilde{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \leq \left\| \tilde{\mathcal{G}} - \overline{\mathcal{G}} \right\|_1 + \left\| \overline{\mathcal{G}} - \mathcal{G} \circ \Phi_N^N \right\|_1 \leq NCg \|g\|_{\infty} + 2NCg \|g\|_{\infty},
$$

and the Corollary follows.

Lemma 4.10. Let $\Phi_{t,s}^N$ be a volume preserving flow with $\|\Phi_{t,s}^N(X) - X\| \leq \tilde{C}_N(t - s)^a(1 + \|X\|)$ for some $a \in \mathbb{N}$ and some $N$-dependent constant $\tilde{C}_N$. Let the function $\mathcal{G}$ be given by $\mathcal{G} = g^{\otimes N}$ for some $g : \mathbb{R}^6 \to \mathbb{R}_0^+$ with $\|g\|_1 = 1$ and finite weighted Lipschitz norm.

Then there exists a $N$-dependent constant $C_N$ such that

$$
\|\mathcal{G} \circ \Phi_{t,s}^N - \mathcal{G}\|_1 \leq C_N \left(|t - s|^{a - \frac{1}{2}} + |t - s|^{\frac{3}{2}}\right).
$$

Proof. Since $\Phi_{t,s}^N$ is volume conserving it follows that

$$
\|\mathcal{G} \circ \Phi_{t,s}^N - \mathcal{G}\|_1 = 2 \left\| (\mathcal{G} \circ \Phi_{t,s}^N - \mathcal{G})^{neg} \right\|_1.
$$

Using Lemma 4.3 (c) it follows that

$$
|\mathcal{G}(X)| \leq \prod_{j=1}^{N} \|g\|_{\infty} \left(1 + \|x_j\|\right)^{-9} \leq \|g\|_{\infty}^N \left(1 + \|X\|\right)^{-9}.
$$

Hence the probability that $\|X\| > \sqrt{N}|t - s|^{-1/2}$ is bounded by $|t - s|^{3/2}$ times some $N$-dependent constant

$$
\int_{\|X\| \geq \sqrt{N}|t - s|^{-1/2}} \mathcal{G}(X)dX \leq C_N|t - s|^{3/2}.
$$
Defining $\Psi_{t,s}(x)$ via
\[
\Psi_{t,s}(x) := \begin{cases} 
\Phi_{t,s}^N(X) & \text{for } \|X\| \leq \sqrt{N}|t-s|^{-1/2} \\
X & \text{else,}
\end{cases}
\]
it follows that $G \circ \Phi_{t,s}^N(X) - G \circ \Psi_{t,s}(x)$ is zero for $\|X\| \leq \sqrt{N}|t-s|^{-1/2}$, hence with (38)
\[
\|G \circ \Phi_{t,s}^N(x) - G \circ \Psi_{t,s}(x)\| \leq \int_{\|X\| \geq \sqrt{N}|t-s|^{-1/2}} G \circ \Psi_{t,s}(x) dX.
\]
Due to Corollary 4.2 we can find a $\tilde{G} \in M_{\theta}^{\alpha'}$ and a $N$-dependent constant $C_N$ such that $\|\tilde{G} \circ \Psi_{t,s}\| \leq C_N|t-s|^\alpha$. With (38) and triangle inequality and the only function in $M_{\theta}^{\alpha'}$ is $G(X) = \prod_{j=1}^N g(x_j)$ the Lemma follows in view of (37).

**Corollary 4.3.** Let $\Psi_{t,s}^N$ be the flow given by Definition (22). Then there exists for any $T > 0$ a constant $C_N$ such that for any $0 < t < T$ and any $\Delta t \in \mathbb{R}$
\[
\| F_0^N \circ \Phi_{0,t+\Delta t}^N - F_0^N \circ \Phi_{0,t}^N \|_1 \leq C_N \Delta t^{3/2} \tag{40}
\]
\[
\| F_0^N \circ \Phi_{0,t+\Delta t}^N - F_0^N \circ \Phi_{0,t}^N \|_2 \leq C_N \Delta t^{1/2}. \tag{41}
\]

**Proof.** By Lemma 4.10
\[
\sup_{X \in \mathbb{R}^n} \| \Psi_{t,t+\Delta t}^N(X) - \Phi_{t,t+\Delta t}^N(X) \| \leq N^2 L^2 (1 + \|X\|/\Delta t)^2.
\]
Using both formulas in Lemma 4.1 (c) it follows, writing $Y = \Phi_{t,t+\Delta t}^N(X)$,
\[
\|X - \Phi_{0,t}^N \circ \Phi_{t,t+\Delta t}^N \| = \| \Phi_{0,t}^N (\Phi_{t,t+\Delta t}^N(Y)) - \Phi_{t,t+\Delta t}^N (Y) \| 
\]
\[
\leq e^{NL^2} N^2 L^2 (1 + \|X\|/\Delta t)^2 \leq e^{2N L^2} N^2 L^2 (1 + \|X\|/\Delta t)^2.
\]
With Lemma 4.10 we can find a $N$-dependent constant $C_N$ such that
\[
\| F_0^N - F_0^N \circ \Phi_{0,t}^N \circ \Phi_{t,t+\Delta t}^N \|_1 \leq C_N \Delta t^{3/2}.
\]
Since $\Phi_{t,t+\Delta t,0}^N$ is volume conserving we get (10).
\(10\) follows in a similar way: Due to (20) $\| \Phi_{t,t+\Delta t}^N(Y) - Y \| \leq L \Delta t e^{L \Delta t} (1 + \|Y\|)$ and with Lemma 4.1 (c)
\[
\|X - \Phi_{0,t}^N \circ \Phi_{t,t+\Delta t,0}^N(X) \| = \| \Phi_{0,t}^N (\Phi_{t,t+\Delta t}^N(Y)) - \Phi_{t,t+\Delta t,0}^N(Y) \| 
\]
\[
\leq L \Delta t e^{(N+1)L \Delta t} (1 + \|X\|) \leq L \Delta t e^{(N+1)L \Delta t} (1 + \|X\|).
\]
With Lemma 4.10 we get (10).
Lemma 4.11. Let $g_0 \in L^1(\mathbb{R}^6 \to \mathbb{R})$ with $\|g_0\|_{\infty} < \infty$, $g_t := g_0 \circ \text{stat}_{\Psi^{N}_t}$. By the index stat we wish to point out that we consider the flux given by a stationary density which is for all times equal to $g_0$. Let $\mathcal{G} \in \mathcal{M}^{k,+}_{g_0}$ for some $k < \frac{N}{2}$ and let $\mathcal{G}_t := \mathcal{G} \circ \Psi^{N}_t$ where $\Psi^{N}_t$ is given in definition 4.2. For some $\kappa > 0$ let $\theta := 2 \left( \frac{k}{N} + N^{-1/2 + \kappa} \right) L$. Then there exist functions $\mathcal{G}_a \in \mathcal{M}^{k,+}_{g_0}$, $\mathcal{G}_b \in \mathcal{M}^{k+1,+}_{g_0}$, a $N$-dependent constant $C_N$ and a $N$-independent constant $C_0$ and a positive $\lambda_a$ with $\lambda_a \geq 1 - N\theta |t - s| \|g_0\|_{\infty}$ such that

(a) $\|\mathcal{G}_t - \lambda_a \mathcal{G}_a\|^0_1 \leq C_0 |t - s| + C_N (t - s)^2$

(b) $\|\mathcal{G}_t - \mathcal{G}_b\|_1 \leq C_N (t - s)^2$.

Proof. (a) Let $\mathcal{G} \in \mathcal{M}^{k,+}_g$, i.e. we can write

$$\mathcal{G} = \text{Symm} \left\{ \chi^k(x_1, \ldots, x_N) \prod_{j=k+1}^N g_0(x_j) \right\}.$$ 

Let

$$\mathcal{G}_0 := \text{Symm} \left\{ \chi^k(x_1, \ldots, x_N) \prod_{j=k+1}^N g_t(x_j) \right\}$$

$$\mathcal{G}_a := \text{Symm} \left\{ \chi^k(x_1, \ldots, x_N) \prod_{j=k+1}^N g_t(x_j) \right\}$$

$$\mathcal{G}_1 := \text{Symm} \left\{ \chi^k(x_1, \ldots, x_N) \prod_{j=k+1}^N g_t(x_j) \right\}$$

with

$$\chi^k(x_1, \ldots, x_N) := \chi^k(x_1 + t(V_t(X))_1, x_2 + t(V_t(X))_2, \ldots, x_k + t(V_t(X))_k, x_{k+1}, \ldots, x_N).$$

Let

$$\lambda_a := \left\| g|t-s| g_t \right\|_{N-k}^{N-k}$$

$$\lambda_1 := \left\| L|t-s| g_t \right\|_{1}^{N-k}.$$ (42)

(43)

For any $X \in \mathbb{R}^{6N}$ it holds $\lambda_1 \mathcal{G}_1(X) \leq \mathcal{G}(X + tV_t(X)) = \mathcal{G}_t(X)$ (see definition 4.2).

Let for any $\kappa > 0$ the set $\mathcal{E} \subset \mathbb{R}^{6N}$ be given by

$$X \in \mathcal{E} \Leftrightarrow \left| (V_t(X))_j - v_t d^{j-1} g_0(x_j) \right| \geq \theta (1 + |x_j|)$$

for at least one $1 \leq j \leq N$.

22
and \( \mathcal{E}_j \subset \mathbb{R}^{6N} \) be given by
\[
X \in \mathcal{E}_j \iff \left| (V_t(X))_j - v_t \star^{d-1} g_0(x_j) \right| \geq \theta(1 + |x_j|).
\]

It follows that \( G_t(X) \geq \lambda_0 G_0(X) \) for all \( X \notin \mathcal{E} \). Using also that \( G_t \geq \lambda_1 G_1 \) and \( G_0 \geq \lambda_0 G_0 \)
\[
\| (G_t - \lambda_0 G_0)^{neg} \|_1 \leq \| 1 \varepsilon (G_t - \lambda_0 G_0)^{neg} \|_1 \leq \| 1 \varepsilon (\lambda_1 G_1 - \lambda_0 G_0)^{neg} \|_1 \leq \| 1 \varepsilon (\lambda_1 G_1 - G_0)^{neg} \|_1.
\]

Writing \( g_0 = L|t-s|g + \delta g \) we have that \( \delta g \) is positive and with (31) and Lemma [33]
\[
\| \delta g \|_1 \leq CL|t-s|\| g \|_{\infty} \leq e^{11L|t-s|}CL|t-s|\| g \|_{\infty} \quad \text{(44)}
\]

It follows that
\[
\| (G_t - \lambda_0 G_0)^{neg} \|_1 \leq (N-k) \left\| 1 \varepsilon \left( \chi_{k}^j(x_1, \ldots, x_N) \delta g(x_{k+1}) \prod_{j=1}^{N} g(x_j) \right) \right\|_1 + O(t-s)^2 \leq (N-k) \sum_{j=1}^{N} \left\| 1 \varepsilon \left( \chi_{k}^j(x_1, \ldots, x_N) \delta g(x_{k+1}) \prod_{j=1}^{N} g(x_j) \right) \right\|_1 + O(t-s)^2.
\]

Defining \( \bar{E}_j \supset \mathcal{E}_j \) by \( X = (x_1, \ldots, x_N) \in \bar{E}_j \) if and only if
\[
\sum_{\{j_1, \ldots, j_d\} \subset \{k+2, \ldots, N\} \setminus \{j\}} \frac{v_t(x_j, x_{j_1}, \ldots, x_{j_d})}{1 + \| x_j \|} - \frac{v_t \star^{d-1} g(x_j)}{1 + \| x_j \|} \geq 2N^{-\frac{1}{2}+\kappa}L
\]

it follows with (44) that
\[
\| (G_t - \lambda_0 G_0)^{neg} \|_1 \leq CL|t-s|\| g \|_{\infty} e^{11L|t-s|} (N-k) N \left\| 1 \varepsilon \prod_{j=1}^{N} g(x_j) \right\|_1 + O(t-s)^2.
\]

Choosing \( h_{d-1}(x_1, \ldots, x_{d-1}) := \left( \frac{N-1}{d-1} \right)^{-1} v_t(\cdot, x_2, \ldots, x_d)/(1 + \| x_1 \|) \) it follows with assumption [31] that \( \| h_{d-1} \|_{\infty} < L \). We get with Corollary [44] setting \( M = N - k - 1 \) that
\[
\| (G_t - \lambda_0 G_0)^{neg} \|_1 \leq CL|t-s|\| g \|_{\infty} e^{11L|t-s|} \frac{(N-k)N}{(N-k-1)^3} C_{\kappa,r,d-1} + O(t-s)^2.
\]

Using (39) and Bernoulli’s inequality
\[
\lambda_0 = \left( \| g^{t-s} \|_{\infty} \right)^{N-k} \geq (1 - C\theta|t-s|a(k)\| g_t \|_{\infty})^{N-k}
\geq 1 - (N-k)C\theta|t-s|a(k)\| g_t \|_{\infty}.
\]

23
(b) As in (a) we write $G = (G^{\text{asym}})^{\text{sym}}$ with
\[ G^{\text{asym}} = \chi^k(x_1, \ldots, x_N) \prod_{j=k+1}^N g(x_j). \]

We split the set $\mathcal{N}$ into two parts $\mathcal{A}$ and $\mathcal{B}$ given by $\mathcal{A} \cap \mathcal{B} = \emptyset$, $\mathcal{A} \cup \mathcal{B} = \mathcal{N}$ and
\[(j, n_j) \in \mathcal{A} \Leftrightarrow k < j \leq N. \]

Let for any $\omega \in \mathbb{N}$ the flux $\phi_{\omega \lambda}^{t,s}$ be given by
\[ \phi_{\omega \lambda}^{t,s}(X) = X + (t - s)v_{\omega \lambda}^{t}(X). \]

We also define for any $S = \{\omega_1, \ldots, \omega_n\} \subset \mathcal{N}$ with $|S| = n$ the flux
\[ \chi_S^{t,s}(X) := \phi_{\omega_n \lambda}^{t} \circ \ldots \circ \phi_{\omega_2 \lambda}^{t} \circ \phi_{\omega_1 \lambda}^{t}(X). \]

We also define for any $\Theta_S^{t,s}(X) := X + \sum_{\omega \in S} (t - s)v_{\omega \lambda}^{t}(X).$

To prove part (b) we shall show that for any $S \subset \mathcal{A}$

(i) \[ \|\chi_S^{t,s}(X) - \Theta_S^{t,s}\| \leq O(t^2)(1 + \|(x_1, \ldots, x_{N-k})\|) \] for all $X \in \mathbb{R}^{6N}$ and sufficiently small $t$.

(ii) there exists a $G^{S}_1 \in M_{k}^{\lambda_S}$, $G^{S}_2 \in M_{k+1}^{\lambda_S}$ and a $0 \leq \lambda_S \leq 1$ such that
\[ \|\text{Symm} \{G^{\text{asym}} \circ \Theta_S^{t,s} \circ \chi_S^{t,s}\} - G_S^1\|_1 = O(t) \] (46)
\[ \|\text{Symm} \{G^{\text{asym}} \circ \Theta_S^{t,s} \circ \chi_S^{t,s}\} - \lambda_S G_S^2\|_1 = O(t^2). \] (47)

Setting $S = \mathcal{A}$ (45) and (47) imply that there exists a $\tilde{G}_b \in M_{k+1}^{\lambda_{S(b)}}$ such that \[ \|G_t - \tilde{G}_b\|_1 \leq C_N(t-s)^2. \] Since $G_t$ and $\tilde{G}_b$ have integral one it follows that the $L^1$-norm of the negative part of $\tilde{G}_b$ is of order $(t-s)^2$. Choosing for $\tilde{G}_b$ the normalized positive part of $\tilde{G}_b$ we get (b).

We now prove (45), (46) and (47) by induction over the cardinality of the set $S$.

$|S| = 0$ For $S = \emptyset$ the left hand side of (45), (46) and (47) are zero.

$|S| = n \Rightarrow |S| = n+1$: Assume that (45), (46) and (47) hold for all sets $S$ with cardinality $|S| = n$. Let $\mathcal{Z} \subset \mathcal{A}$ with $|\mathcal{Z}| = n+1$ and $\mathcal{Z} = S \cup \{\omega_{n+1}\}$ for some $\{\omega_{n+1}\} \in \mathcal{A}$.
It follows using Assumption 1.1 (c) and the induction assumption

\[
\left\| \phi_t^{\omega_{n+1}} \circ \chi_t^S(X) - \phi_t^{\omega_{n+1}} \left( X + \sum_{\omega \in S} tv_t^\omega(X) \right) \right\|
\leq (1 + L|t|) \left\| \chi_t^S(X) - X + \sum_{\omega \in S} tv_t^\omega(X) \right\|
= O(t^2)(1 + \|(x_1, \ldots, x_{N-k})\|),
\]

(48)

With assumption 1.1 (c)

\[
\left\| tv_t \left( \left( X + \sum_{\omega \in S} tv_t^\omega(X) \right) \right) - tv_t ((X)_d) \right\|
\leq tL \left\| \left( \sum_{\omega \in S} tv_t^\omega(X) \right) \right\|
= O(t^2)(1 + \|(x_1, \ldots, x_{N-k})\|),
\]

hence

\[
\left\| \phi_t^{\omega_{n+1}} \left( X + \sum_{\omega \in S} tv_t^\omega(X) \right) - X - \sum_{\omega \in S} tv_t^\omega(X) \right\|
\leq O(t^2)(1 + \|(x_1, \ldots, x_{N-k})\|)
\]

and with (48) and triangle inequality we get (45) for \( Z \).

Using Corollary 4.2 on \( G^S \) we can find a \( G^Z \in M_{g}^{k+1} \) (the respective \( \tilde{G} \) in the Corollary) such that \( \| G^S \circ \phi_t^{\omega_{n+1}} - G^Z \|_1 = O(t) \). Using triangle inequality and that \( \phi_t^{\omega_{n+1}} \) is norm conserving we get

\[
\left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^Z \right\|_1
\leq \left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^S \circ \phi_t^{\omega_{n+1}} \right\|_1 + \left\| G^S \circ \phi_t^{\omega_{n+1}} - G^Z \right\|_1
\leq \left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^S \right\|_1 + \left\| G^S \circ \phi_t^{\omega_{n+1}} - G^Z \right\|_1.
\]

Using the induction assumption, i.e. that (45) holds for any \( S \subset A \) with cardinality \( n \), we get that (40) holds for any \( Z \subset A \) with cardinality \( n + 1 \). In a similar way we can prove (47). Note first, that by (40) and (47) \( |\lambda^S| \) is of order \( |t-s| \). Using Corollary 4.2 on there exists a \( \tilde{G} \in M_{g}^{k+1} \) such that \( \lambda^S \| G^S \circ \phi_t^{\omega_{n+1}} - \tilde{G} \|_1 = O(t^2) \). Setting \( \lambda^Z G^Z := \lambda^S \tilde{G} + G^S \circ \phi_t^{\omega_{n+1}} - \tilde{G} \) we get using triangle inequality

\[
\left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^Z \right\|_1
\leq \left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^S \circ \phi_t^{\omega_{n+1}} \right\|_1 + \lambda^Z \| G^S \circ \phi_t^{\omega_{n+1}} - \tilde{G} \|_1
\]

\[
= \left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^S \circ \phi_t^{\omega_{n+1}} - \lambda^Z \| G^S \circ \phi_t^{\omega_{n+1}} - \tilde{G} \|_1
\]

\[
\leq \left\| Symm \left\{ G^{\text{asym}} \circ \Theta_{t,s}^S \circ \chi_{t,s} \right\} - G^S \circ \phi_t^{\omega_{n+1}} \right\|_1 + \lambda^Z \| G^S \circ \phi_t^{\omega_{n+1}} - \tilde{G} \|_1
\]

25
Lemma 4.12. Let $N$ (in general $1 \leq \alpha \leq 2$)
For the proof we choose now the weight in the definition of $\alpha$ (a).

In view of Corollary 4.3 we can choose the flux $\Psi$ as follows: Let

$$m(\gamma) := \begin{cases} \frac{1}{\gamma} & \text{for } k \leq N \gamma \\ 1 & \text{else} \end{cases} \quad (49)$$

Since $m(\gamma) \geq \frac{1}{N}$ for any $k$, Lemma 4.11 holds.

**Lemma 4.12.** Let $\Delta t > 0$ and for some probability density $f_0 : \mathbb{R}^6 \to \mathbb{R}^+_0$

$$f^N_t := f_0 \circ \varphi^N_{0,t}. \quad (50)$$

There exist $0 < \gamma \leq 1$ and $\delta > 0$ such that for any $M \geq 0$ there exist $C_1, C_2 \in \mathbb{R}_+^+$
and a $N$-dependent $C_N$ such that uniformly in $K \leq M$ and sufficiently small (in general $N$-dependent) $\Delta t$

(a) $\alpha_{K(1+C_1\Delta t)}(F^N_{t+\Delta t}, F^M_{t+\Delta t}) \leq \alpha_K(F^N_{t}, F^N_{t}) + C_2 \left( \alpha_K(F^N_{t}, F^N_{t}) + N^{-\delta} \right) \Delta t + C_N \Delta t^{3/2}$

(b) $\|f^N_{t+\Delta t} - f^M_{t+\Delta t}\|_1 \leq \|f^N_{t} - f^M_{t}\|_1$

+ $C_2 \Delta t \left( \|f^N_{t} - f^M_{t}\|_1 + \alpha_{K,K^t}(F^N_{t}, F^N_{t}) + \alpha_{K,K^t}(F^M_{t}, F^M_{t}) \right)$.

**Proof.** (a) In view of Corollary 4.3 we can choose the flux $\Psi^N$ instead of $\Phi^N$
for the time evolution of $F^N_t$, i.e. redefine $F^N_{t+\Delta t} := F_t \circ \Psi_{0,t+\Delta t}^N$.

There exists a decomposition which is arbitrarily close to the value of $\alpha_K(F^N_{t}, F^N_{t})$, i.e. there exist probability densities $g_i$ with $\|g_i\|_{\infty} \leq K$
and functions $\mathcal{G}_i \in \mathcal{M}^N_{g_i}$ such that for $\mathcal{G} = \sum \lambda_i \mathcal{G}_i$

$$\sum_{i=1}^n \lambda_i \left( m(\gamma)(k_i) + \|f^N_{t} - g_i\|_1 \right) + \|\mathcal{G} - F^N_{t}\|_1 \leq \alpha_K(F^N_{t}, F^N_{t}) + (\Delta t)^2. \quad (51)$$

We shall estimate $\alpha_K(F^N_{t+\Delta t}, F^N_{t+\Delta t})$ for a suitable $C_1 > 0$ by
constructing a suitable decomposition of $F^N_{t+\Delta t}$ which we shall use to test
the infimum.

As in the proof of Lemma 3.1, we assume that only one of the $\lambda_i$ equals one
and all the others are zero. In view of (16) this then extends to the proof of Lemma 4.12. That is, we assume that there exist $k$, $g$, and $\mathcal{G} \in \mathcal{M}^N_{g}$
such that

$$\alpha_K(F^N_{t}, F^N_{t}) = m(\gamma)(k) + \|f^N_{t} - g\|_1 + \|F^N_{t} - \mathcal{G}\|_1 + (\Delta t)^2. \quad (52)$$

4.6 A Gronwall type estimate for $\alpha$

For the proof we choose now the weight in the definition of $\alpha$ as follows: Let
$1/2 < \gamma < 1$, then

Using that $\phi^N_{t+\Delta t}$ is norm conserving and the induction assumption we get, as above, that this is of order $O(t-s)^2$.

By induction we get that (45), (40) and (17) hold in full generality. □
For $k > N^\gamma$ we have $m_\gamma(k) = 1$. Since this is the largest possible value for $m_\gamma$, $\alpha$ has reached its maximum. In more detail: for $k > N^\gamma$

$$a_K(F_t^N, f_t^N) \geq 1 + \| F_t^N - g \|_1^2.$$

Observing (13) and (49) we have that

$$a_K(1+C, \Delta t)(F^N_{t+\Delta t}, f^N_{t+\Delta t}) \leq 1 + \| F_t^N - g \|_1^2$$

and the Lemma follows for $k > N^\gamma$.

It is left to consider the case $0 < k < N^\gamma$.

Let $G_{\Delta t} := G \circ \Psi_{t,t+\Delta t}^N \lambda_a$ and $G_{a,b}$ the functions we get from Lemma 4.11. Let $G_c$ be the respective $G^+$ we get using Lemma 4.8 on $(G_b - \lambda_a G_a)/(1 - \lambda_a)$. It follows that $G_a \in M^{\kappa+1, +}_a, G_b \in M^{k+1, +}_a, G_c \in M^{\kappa+1, +}_a$ and

$$\lambda_a \geq 1 - N\theta \Delta t \| g \| e, \quad \lambda_c = 1 - \lambda_a \leq N\theta \Delta t \| g \| e,$$

and

$$\| G_{\Delta t} - \lambda_a G_a - \lambda_c G_c \|_1 \leq \| G_{\Delta t} - G_b \|_1 + \| G_b - \lambda_a G_a - \lambda_c G_c \|_1$$

$$\leq \| G_{\Delta t} - G_b \|_1 + \| (G_b - \lambda_a G_a) \|_1$$

$$\leq 2 \| G_{\Delta t} - G_b \|_1 + \| (G_{\Delta t} - \lambda_a G_a) \|_1.$$  (53)

By the estimates of Lemma 4.11 we get that

$$\| G_{\Delta t} - \lambda_a G_a - \lambda_c G_c \|_1 \leq \frac{C_0 \Delta t}{N} + C_N(\Delta t)^2.$$  (53)

Using the estimates of Lemma 4.4 (a) and (b) we can find some constant $C$ and some $N$-dependent $C_N$ such that

$$\| g_{\Delta t}(x) \|_e \leq (1 + C \Delta t)K + C_N \Delta t^2.$$  (53)

Choosing $\Delta t$ sufficiently small we find a constant $C_1$ such that

$$\| g_{\Delta t}(x) \|_e \leq (1 + C_1 \Delta t)K.$$  (53)

Using (53)

$$\| g_{\Delta t}(x) \|_1 \leq 1 - \theta \Delta t K$$

and we get

$$\| g_{\Delta t} n g_{\Delta t}(x) \|_e \leq \frac{(1 + C_1 \Delta t)K}{1 - \theta \Delta t K}.$$  (53)

Recall that $0 < k < N^\gamma$, thus $\theta \Delta t < 2(N^{\gamma-1} + N^{-1/2+\kappa})L \Delta t$. Choosing $N$ sufficiently large and $\Delta t$ sufficiently small we can find a constant $C_2$ such that

$$\| g_{\Delta t} n g_{\Delta t}(x) \|_e \leq K + C_2 \Delta t K.$$  (53)

Thus we can use $G_a \in M^{\kappa+1, +}_a, G_b \in M^{k+1, +}_a$ to test the functional

$$a_K(1+L \Delta t)(F^N_{\Delta t}, f^N_{\Delta t}).$$

Note that $f^N_{t+\Delta t} = f^N_t \circ \Psi_{t,t+\Delta t}^N$ and $g_{\Delta t} = g_0 \circ \Psi_{t,t+\Delta t}^N.$
\[ \alpha_{K(1+L\Delta t)}(\mathcal{F}_t^{\Delta t}, f_t^{\Delta t}) \leq \lambda_c m_\gamma(k) + \lambda_c m_\gamma(k + d) \]

\[ + \|f_t^{\Delta t} - \theta \Delta t \circ g_{\Delta t}\|_1 + \|\mathcal{F}_t^{\Delta t} - \lambda_c \mathcal{G}_t - \lambda_c \mathcal{G}_c\|_1 \]

\[ \leq m_\gamma(k) + \lambda_c (m_\gamma(k + 1) - m_\gamma(k)) + \|f_t^{\Delta t} - \varphi_t^{\Delta t} \circ \varphi_{t+\Delta t} - g_{\Delta t}\|_1 \]

\[ + \|\varphi_{t+\Delta t} - \mathcal{G}_\Delta t\|_1 + \|\mathcal{G}_\Delta t - \lambda_c \mathcal{G}_t - \lambda_c \mathcal{G}_c\|_1 \]

where we used triangle inequality in the last step. Using that \( m_\gamma(k + 1) - m_\gamma(k) \leq N^{-\gamma} \) and that \( \Phi^N \) and \( \varphi^N \) are norm preserving we get after reordering

\[ \alpha_{K(1+L\Delta t)}(\mathcal{F}_t^{\Delta t}, f_t^{\Delta t}) \leq m_\gamma(k) + \|f_t^{\Delta t} - g_0\|_1 + \|\mathcal{F}_t^{\Delta t} - \mathcal{G}_t\|_1 \] (54)

\[ + \lambda_c N^{-\gamma} + \|g_0 \circ \varphi_t^{\Delta t} - g_0 \circ \varphi_{t+\Delta t}\|_1 \] (55)

\[ + \|g_{\Delta t} - \theta \Delta t g_{\Delta t}\|_1 + \|\mathcal{G}_\Delta t - \lambda_c \mathcal{G}_t - \lambda_c \mathcal{G}_c\|_1 \] (56)

In view of (52) we get that (54) is bounded by \( \alpha_K(\mathcal{F}_t^N, f_t^N) + \Delta t^2 \).

Choosing \( \kappa = \gamma - 1/2 \) we get that the first summand in (54) is bounded by

\[ \lambda_c N^{-\gamma} \leq N^{1-\gamma} \theta \Delta t \|g_0\|_1 = \frac{k + 1}{N^\gamma} \Delta t \|g_0\|_1 \]

\[ \leq (m_\gamma(k) + N^{-\gamma}) \Delta t \|g_0\|_1 \leq (\alpha_K(\mathcal{F}_t^N, f_t^N) + N^{-\gamma}) \Delta t \|g_0\|_1 . \]

By Lemma 4.6 the second summand in (55) is bounded by

\[ C \|g_0\|_1 e^{L\Delta t} \mathcal{L}_{\mathcal{F}_t^N} g_0 - g_0 \] \( \int_{t}^{t+\Delta t} \|\mathcal{F}_s^{\Delta t} - g_0\|_1 ds \).

Using Corollary 1.3 the latter is bounded by

\[ C \|g_0\|_1 e^{L\Delta t} L \Delta t \left( \|\mathcal{F}_s^{\Delta t} - g_0\|_1 + C_N \Delta t^{1/2} \right) . \]

Since by Lemma 3.1 (equation (12)) \( \|\mathcal{F}_t^{\Delta t} - f_t^N\|_1 \leq 2\alpha_K(\mathcal{F}_t^N, f_t^N) + 2N^{-1} \) and since \( \|f_t^N - g_0\|_1 \leq \alpha_K(\mathcal{F}_t^N, f_t^N) \) the second summand of (55) is bounded by

\[ C \|g_0\|_1 e^{L\Delta t} L \Delta t \left( 3\alpha_K(\mathcal{F}_t^N, f_t^N) + 2N^{-1} + C_N \Delta t^{1/2} \right) . \]

The second summand of (56) is bounded by (52), the first summand in (55) can in view of (51) be controlled by \( C\theta \Delta t \|g_{\Delta t}\|_1 \), which is due to Lemma 4.4 (a) bounded by \( Ce^{\Delta t} \|g_0\|_1 \). It follows that

\[ \alpha_{K(1+L\Delta t)}(\mathcal{F}_t^{\Delta t}, f_t^{\Delta t}) \leq (\alpha_K(F^N, f^N) + N^{-\gamma}) \Delta t \|g_0\|_1 \]

\[ + C \|g_0\|_1 e^{L\Delta t} L \Delta t \left( 3\alpha_K(F^N, f^N) + 2N^{-1} + C_N \Delta t^{1/2} \right) + Ce^{\Delta t} \|g_0\|_1 + C_0 \Delta t + C_N \Delta t^2 . \]
With help of Lemma 4.12 (a) we get good control of \( f^N \) and recall that \( 0 < k < N^\gamma \) and \( \kappa = \gamma - 1/2 \) such that \( \theta \leq 4LN^{\gamma - 1} \), we get (a).

(b) Using triangle inequality, we get that \( \frac{\partial f^N}{\partial t} \) is norm conserving and with Lemma 4.10 and Lemma 1.13 (a) we get

\[
\left\| f_{t+\Delta t}^N - f_t^M \right\| = \left\| f_{t+\Delta t}^N - f_t^M \circ \frac{\partial f^N}{\partial t} \right\|_1 \\
\leq \left\| f_t^N \circ \frac{\partial f^N}{\partial t} - f_t^M \circ \frac{\partial f^N}{\partial t} \right\|_1 \\
+ \left\| f_t^M - f_t^M \circ \frac{\partial f^N}{\partial t} \right\|_1 \\
\leq C \left\| \frac{\partial f^N}{\partial t} \right\|_1 \left\| f_{t+\Delta t}^N - f_t^M \right\|_1 + \left\| f_t^N - f_t^M \right\|_1 \\
\leq C \left\| \frac{\partial f^N}{\partial t} \right\|_1 e^{k\Delta t} L + \left\| f_{t+\Delta t}^N - f_t^M \right\|_1 + \left\| f_t^N - f_t^M \right\|_1 \\
\leq C \left\| \frac{\partial f^N}{\partial t} \right\|_1 e^{k\Delta t} L + \left\| f_{t+\Delta t}^N - f_t^M \right\|_1 + \left\| f_t^N - f_t^M \right\|_1 \\
By triangle inequality and Lemma 4.11
\[
\left\| f_{t+\Delta t}^N - f_t^M \right\|_1 \leq \left\| f_t^N - f_t^M \right\|_1 + \alpha_K e^{C_1 t} (f_t^N, f_t^M) + \alpha_K e^{C_1 t} (f_t^N, f_t^M).
\]

\[\square\]

### 4.7 Proof of Theorem 1.1

With help of Lemma 4.12 (a) we get good control of \( \alpha_K (f_t^N, f_t^M) \) on any compact time interval \([0, T]\) and are able to prove Theorem 1.1.

**Proof.** (a) Since we assumed that the derivative of \( f_0 \) decays, we get with Lemma 4.13 (a) that \( \| f_0 \|_{\ell^\infty} \) is finite, hence there exists a \( K > 0 \) such that

\[
\alpha_K (f_0^N, f_0) = 0.
\]

Let \( C_2 \) be given by Lemma 4.12 and \( K = \| f_0 \|_{\ell^\infty} \). Let \( \beta_t \) be given by \( \beta_t := e^{C_2 t} \alpha_K (f_0^N, f_0) + (e^{C_2 t} - 1) N^{-\delta} \). Then by induction \( \alpha_K e^{C_1 t} (f_n^N, f_n^M) \leq \beta_n \) for any \( n \in \mathbb{N}_0 \) with \( n \Delta t \leq T \); obviously \( \beta_0 = \alpha_K (f_0^N, f_0) \). Furthermore we get since \( K e^{C_1 (n+1) \Delta t} \geq K e^{C_1 n \Delta t} (1 + C_1 \Delta t) \)

\[
\alpha_K e^{C_1 (n+1) \Delta t} (f_n^N, f_n^M) \leq \alpha_K e^{C_1 n \Delta t} (1 + C_2 \Delta t) (f_n^N, f_n^M).
\]

Assuming \( \alpha_K e^{C_1 n \Delta t} (f_n^N, f_n^M) \leq \beta_n \) we get by Lemma 4.12

\[
\alpha_K e^{C_1 (n+1) \Delta t} (f_n^N, f_n^M) \leq \alpha_K e^{C_1 n \Delta t} (f_n^N, f_n^M) + C_2 \left( \alpha_K e^{C_1 n \Delta t} (f_n^N, f_n^M) + N^{-\delta} \right) \Delta t + C_N \Delta t^{3/2}
\leq \beta_n + C_2 \left( \beta_n + N^{-\delta} \right) \Delta t + C_N \Delta t^{3/2} = \beta_{n+1} + \beta_{n+1} \Delta t + C_N \Delta t^{3/2}.
\]

Using that the second time derivative of \( \beta_t \) is positive, the latter is bounded by \( \beta_{(n+1) \Delta t} \). Summarizing we have with 57

\[
\alpha_K e^{C_1 t} (f_t^N, f_t) \leq e^{C_2 t} \alpha_K (f_0^N, f_0) + (e^{C_2 t} - 1) N^{-\delta} = (e^{C_2 t} - 1) N^{-\delta}.
\]

It follows that for any \( t > 0 \)

\[
\lim_{\Delta t \to 0} \lim_{N \to \infty} \alpha_K e^{C_1 t} (f_t^N, f_t) = 0.
\]
By a similar Gronwall argument we can use Lemma 4.4 to show that
\[ \lim_{N \to \infty} \lim_{\Delta t \to 0} \| f_t^N - f_t^M \|_1 = 0. \]
Both convergences are uniform in \( t \) on any compact time interval. Using completeness of \( L^1 \) there exists a function
\[ f = s\text{-lim}_{N \to \infty} f^N. \]
By Lemma 5.4 and (b) of Theorem 1.1 follows.

(b) By Lemma 4.6
\[ f = s\text{-lim}_{N \to \infty} f^N = s\text{-lim}_{N \to \infty} f_0 \circ \varphi_{0,t}^{1,F^N} = f_0 \circ (\varphi_{N-t,t}^{s\text{-lim}_{N \to \infty} F^N}) = f_0 \circ \varphi_f^1. \]

(c) (b) gives existence of solutions of (2) for any \( g \), with \( \| f_0 \|_\infty < \infty \). Since these functions are dense in \( L^1 \) we can use Lemma 4.6 to generalize this result: Let \( f_0 \in L^1 \), \( (f_n^0)_{n \in \mathbb{N}} \subset L^1 \) with \( \lim_{n \to \infty} \| f_n^0 - f_0 \|_1 = 0 \). With Lemma 4.4 it follows that
\[ \| f_t^n - f_t^m \|_1 = \| f_0^n \circ \varphi_{0,t}^{f_n^0} - f_0^m \circ \varphi_{0,t}^{f_m^0} \|_1 \leq \| f_0^n - f_0^m \|_1 + C \| f_0 \|_\infty e^{L|t|L(1 + \| x \|)} \left| \int_0^t \| f_{s}^{n} - f_{s}^{m} \|_1 ds \right| . \]
Since \( \lim_{n,m \to \infty} \| f_0^n - f_0^m \|_1 = 0 \) we get with Gronwall that for any \( t \)
\[ \lim_{n \to \infty} \lim_{m \to \infty} \| f_t^n - f_t^m \|_1 = 0. \]
By completeness we can find a \( f_t : \mathbb{R} \to L^1 \) such that \( \lim_{n \to \infty} f_t^n = f_t \) \( \| f_t \|_1 = 0 \). The convergences are uniform on any compact time interval and it follows with Lemma 4.4 that
\[ f = s\text{-lim}_{n \to \infty} f_0^n \circ \varphi_{0,t}^{f_n^0} = f_0 \circ \varphi_{N-t,t}^{s\text{-lim}_{n \to \infty} F_t^N} = f_0 \circ \varphi_f^1. \]
Assume now that for some initial \( f_0 \) there exist two solutions \( f_t \) and \( g_t \) of (2). Let again \( (f_n^0)_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} f_n^0 = f_0 \). Choosing \( F_t^N = \prod_{j=1}^N f_0^n(x_j) \) it follows that
\[ \lim_{n \to \infty} \lim_{K \to \infty} \lim_{N \to \infty} \alpha_K(F_t^N, f_0) = 0 \]
and with a similar estimate as above we get that
\[ \lim_{n \to \infty} \lim_{K \to \infty} \lim_{N \to \infty} \alpha_K(F_t^N, f_t) = 0 = \lim_{n \to \infty} \lim_{K \to \infty} \lim_{N \to \infty} \alpha_K(F_t^N, g_t). \]
By Lemma 5.3 the reduced one particle density of \( F_t^N \) converges to \( f_t \) and \( g_t \). Thus \( f_t = g_t \) and we get uniqueness of the solutions.

(d) is a direct consequence of Lemma 5.1 and the convergence of \( \alpha \).
References

[Braun and Hepp, 1977] Braun, W. and Hepp, K. (1977). The vlasov dynamics and its fluctuations in the 1/n limit of interacting classical particles. *Communications in Mathematical Physics*, 56(2):101–113.

[Dobrushin, 1979] Dobrushin, R. L. (1979). Vlasov equations. *Functional Analysis and Its Applications*, 13(2):115–123.

[Elskens et al., 2009] Elskens, Y., Kiessling, M.-H., and Ricci, V. (2009). The vlasov limit for a system of particles which interact with a wave field. *Communications in Mathematical Physics*, 285(2):673–712.

[Golse, 2012] Golse, F. (2012). The mean-field limit for a regularized vlasov-maxwell dynamics. *Communications in Mathematical Physics*, 310(3):789–816.

[Hauray and Jabin, 2007] Hauray, M. and Jabin, P.-E. (2007). N-particles approximation of the vlasov equations with singular potential. *Archive for rational mechanics and analysis*, 183(3):489–524.

[Pickl, 2010] Pickl, P. (2010). Derivation of the time dependent gross pitaevskii equation with external fields. *arXiv preprint arXiv:1001.4894*.

[Pickl, 2011] Pickl, P. (2011). A simple derivation of mean field limits for quantum systems. *Letters in Mathematical Physics*, 97(2):151–164.

[Spohn, 1991] Spohn, H. (1991). *Large scale dynamics of interacting particles*, volume 174. Springer-Verlag New York.