Numerical analyses of a Couette-Taylor flow in the presence of a magnetic field

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Abstract. An axisymmetric Couette-Taylor flow of liquid metal in the presence of a magnetic field has been numerically studied. An inner cylinder of a coaxial container is rotating at a constant angular velocity whereas the outer cylindrical wall is at rest. An axial or a toroidal magnetic field is applied to this configuration to investigate the influence of such magnetic fields on the liquid metal Couette-Taylor flow. The toroidal magnetic field can be produced with a straight wire along the central axis in which electric current passes. The governing equations of mass conservation, momentum, Ohm’s law and conservation of electric charge for an axisymmetric cylindrical coordinate system have been numerically solved with a finite difference method using the HSMAC algorithm. In the numerical analyses, since the Joule heating and the induced magnetic field are neglected, the system parameters are the Hartmann number and the Reynolds number. The numerical results reveal significant difference in the Couette-Taylor flow depending on whether the applied magnetic field is axial or toroidal as well as on the Hartmann and Reynolds numbers. The axial magnetic field damps out the secondary flow efficiently and velocity gradient in the direction of the magnetic field tends to diminish while the toroidal magnetic field does not have such an efficient damping.

1. Introduction

The Couette-Taylor flow is induced by the instability of force balance between the centrifugal force and the pressure gradient in radial direction within the gap of co-axial double cylinder. When the rotating Reynolds number is small, the primary azimuthal flow is dominant and the secondary meridional flow is inconspicuous. However, with an increase in the Reynolds number, the secondary flow becomes remarkable. This flow transition at a certain critical value of the Reynolds number is similar to that of the Rayleigh-Bénard convection, which takes its critical Rayleigh number at around 1708. Chandrasekar [1] studied theoretically both the Couette-Taylor flow and the Rayleigh-Bénard convection, and also studied the effect of the vertical magnetic field on the Rayleigh-Bénard convection. The critical Rayleigh number for the onset of convection is proportional to the square of the Hartmann number when the Hartmann number is much larger than unity. In this study, the effect of the magnetic field on the Couette-Taylor flow is numerically investigated for a co-axial double cylinder with the outer wall at rest.

2. Governing equations

We consider an axisymmetric Couette-Taylor flow of liquid metal in the presence of a uniform axial or a toroidal magnetic field. The inner cylinder is rotating at a constant angular velocity while the
outer cylindrical wall and the top and bottom walls are stationary. The liquid metal filling the gap between the inner and outer walls is assumed to be an incompressible Newtonian fluid and the viscous dissipation is neglected. In the present paper, the effects of the induced magnetic field and the Joule heating are neglected since the magnetic Reynolds number is much less than unity in the system considered. The governing equations in a non-dimensional form are written as follows:

<Continuity of mass>
\[
0 = \frac{\partial U}{\partial R} + \frac{U}{R} \frac{\partial Z}{\partial Z} + \frac{\partial W}{\partial Z} = 0
\]  

(Momentum equation)
\[
\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial R} + W \frac{\partial U}{\partial Z} = -\nabla_{RZ} P + \frac{1}{\text{Re}} \left( \nabla_{RZ}^2 U - \frac{U}{R^2} \frac{V}{R^2} e_{\theta} \right) + F
\]

<Ohm’s law>
\[
J = -\nabla_{RZ} \Phi + U \times e_B
\]

<Conservation of electric charge>
\[
\frac{\partial J_R}{\partial R} + \frac{J_R}{R} + \frac{\partial J_Z}{\partial Z} = 0
\]

Where, \( U = U e_R + V e_\theta + W e_Z \), \( \nabla_{RZ} = (\partial / \partial R) e_R + (\partial / \partial Z) e_Z \), \( \nabla_{RZ}^2 = \partial^2 / \partial R^2 + (1 / R) \partial / \partial R + \partial^2 / \partial Z^2 \), \( e_R, e_\theta, e_Z \) are the unit vectors for radial, azimuthal, axial and magnetic directions respectively.

The non-dimensional parameters of the magnetic Couette-Taylor flow are the rotating Reynolds number, \( \text{Re} \) and the Hartmann number, \( \text{Ha} \). The non-dimensional variables and numbers are defined as follows:

\[
(R, Z) = \left( \frac{r}{r_i}, \frac{z}{r_i} \right), \quad (U, V, W) = \left( \frac{u}{\Omega r_i}, \frac{v}{\Omega r_i}, \frac{w}{\Omega r_i} \right), \quad \tau = \frac{t}{\Omega}, \quad P = \frac{p}{\mu \Omega^2 r_i^2}, \quad (J_R, J_\theta, J_Z) = \left( \frac{j_R, j_\theta, j_Z}{\Omega r_i} \right), \quad \Phi = \frac{\phi}{\Omega B_0 r_i}, \quad e_B = \frac{B}{B_0} = \left( \frac{B}{\mu_m i / 2 \pi r_i} \right), \quad \text{Re} = \frac{r_i (r_o - r_i)}{\nu}, \quad \text{Ha} = \frac{\sigma}{\rho \nu} \sqrt{B_0 \mu_m r_i}.
\]

Where, \((r, z)\) represent the radial and axial coordinates respectively, \((u, v, w)\) are the velocity components, \(r_i\) and \(r_o\) are the inner and outer radii of the cylinder respectively, \( \Omega \) is the angular velocity of the inner cylinder, \( t \) is the time, \( p \) is the pressure, \((j_R, j_\theta, j_Z)\) are the components of the electric current density, \( \rho \) is the density of fluid, \( \phi \) is the electric potential, \( B \) is the magnetic flux density vector, \( B_0 \) is the magnitude of the magnetic flux density, \( \mu_m \) is the magnetic permeability of fluid, \( i \) is the electric current passing in the wire, \( \nu \) is the kinematic viscosity, and \( \sigma \) the electric conductivity of fluid.

2.1. Uniform axial magnetic field

When the axial magnetic field is applied to the Couette-Taylor flow of liquid metal, the dimensionless magnetic induction in Ohm’s law is written as
\[
e_B = (0) e_R + (0) e_\theta + (1) e_Z,
\]
and the electric current density is calculated as
\[
J = \left( -\frac{\partial \Phi}{\partial R} + V \right) e_R + (-U) e_\theta + \left( -\frac{\partial \Phi}{\partial Z} \right) e_Z.
\]
Finally the external force included in the momentum equation takes the form
\[
F = \left( \frac{\text{Ha}^2}{\text{Re}} (0) e_R + \frac{V^2}{R} \right) e_R + \left( \frac{\text{Ha}^2}{\text{Re}} \left( \frac{\partial \Phi}{\partial R} - V \right) - \frac{UV}{R} \right) e_\theta + (0) e_Z.
\]

2.2. Toroidal magnetic field
For the toroidal magnetic field, which can be induced with the electric current passing in an axial infinitely long straight wire located at the center of the inner cylinder, the induced toroidal magnetic field has only a circumferential component and it is inversely proportional to the radius. The magnetic induction, the electric current density and the external force are written as follows:

\[
\mathbf{e}_B = (0)\mathbf{e}_R + \left(\frac{1}{R}\right)\mathbf{e}_\theta + (0)\mathbf{e}_Z \tag{8}
\]

\[
\mathbf{J} = \left( -\frac{\partial \Phi}{\partial R} - \frac{W}{R}\right)\mathbf{e}_R + (0)\mathbf{e}_\theta + \left(-\frac{\partial \Phi}{\partial Z} + \frac{U}{R}\right)\mathbf{e}_Z \tag{9}
\]

\[
\mathbf{F} = \left( \frac{Ha^2}{Re} \left( \frac{1}{R} \frac{\partial \Phi}{\partial Z} - \frac{U}{R^2} \right) + \frac{V^2}{R}\right)\mathbf{e}_R + \left(-\frac{UV}{R}\right)\mathbf{e}_\theta + \left(\frac{Ha^2}{Re} \left(-\frac{1}{R} \frac{\partial \Phi}{\partial R} - \frac{W}{R^2} \right)\right)\mathbf{e}_Z \tag{10}
\]

2.3. Boundary conditions and numerical strategy

The general boundary condition for the electric current density can be expressed using the electric conductance of wall \( c \), which is proportional to the electric conductivity of the wall and the wall thickness, as follows:

\[
\mathbf{J} \cdot \mathbf{n} = c\nabla^2 \Phi \tag{11}
\]

Where, \( \mathbf{n} \) denotes the unit vector perpendicular to the wall, and the subscript in the Laplacian means there are two tangential components but no normal one. Usually, the electric conductance of the wall affects the current density inside the liquid metal and therefore the fluid velocity as well. In this paper, we limited ourselves to the case that all walls are electrically insulating \((c = 0)\).

The boundary conditions are summarized as follows:

- \( U = W = 0 \) and \( V = 1 \) at \( R = 1 \) (inner wall is rotating)
- \( U = V = W = 0 \) at \( R = 2 \) (outer wall is at rest)
- \( U = V = W = 0 \) at \( Z = 0, 4 \) (top and bottom walls are at rest)
- \( J_z = 0 \) at \( R = 1, 2 \) (inner and outer walls are insulating)
- \( J_z = 0 \) at \( Z = 0, 4 \) (top and bottom walls are insulating)

The initial conditions are as follows:

- \( U = V = W = P = J_x = J_y = J_z = \Phi = 0 \)

The partial differential equations are transformed to finite difference equations. The computational domain is divided into small regular square meshes. The number of meshes is 64 and 256 in the radial \((R = 1 \text{ to } 2)\) and vertical \((Z = 0 \text{ to } 4)\) directions respectively. The pressure and electric potential terms are solved successfully by using the HSMAC (Highly Simplified Marker and Cell) method [2]. The convective terms are approximated with the UTOPIA (Uniformly Third Order Polynomial Algorithm) scheme [3], which is one of the third-order upwind schemes.

3. Numerical results

3.1. No magnetic field

Figure 1 shows the numerical results for the meridional flow velocity vectors, contours of azimuthal velocity and pressure respectively at \( Ha = 0 \). The left vertical wall indicates the inner wall and the right one the outer wall. In (a), there are two vortices in the upper and the lower parts. The meridional flow is due to the existence of the top and bottom stationary walls. In the vicinity of these walls, the centrifugal force is weaker than that in the middle part and therefore the opposing pressure gradient is dominant and two vortices are generated. As shown in (b) by doubling the rotating speed, four vortices appear instead of two. This is due to transition of flow caused by instability of the force balance. According to the review article by Di Prima and Swinney [4], the critical Reynolds number for radius ratio 0.5 (present system) is 68.2.
3.2. Uniform axial magnetic field

Figure 2 shows the numerical results for the meridional flow velocity vectors, contours of azimuthal velocity, and contours of pressure respectively in the absence of a magnetic field ($Ha = 0$) for (a) $Re = 50$ and (b) $Re = 100$.

Figure 1. Numerical results for the meridional flow velocity vectors, contours of azimuthal velocity, and contours of pressure respectively in the absence of a magnetic field ($Ha = 0$) for (a) $Re = 50$ and (b) $Re = 100$.

3.2. Uniform axial magnetic field

Figure 2 shows the numerical results for the meridional flow velocity vectors, contours of azimuthal velocity, meridional electric current density vectors, contours of azimuthal current density and contours of electric potential respectively at $Re = 100$ for (a) $Ha = 5$ and (b) $Ha = 20$. Due to the imposition of the uniform vertical magnetic field, the Lorentz force acts to reduce the number of vortices. In the figure of meridional electric current vectors, the electric current in the vicinity of the top and bottom walls acts to accelerate the azimuthal flow but in the middle part it acts to decelerate the azimuthal flow. Therefore, the azimuthal velocity tends to be uniform along the direction of magnetic field. In a similar way, as seen in the contours of azimuthal electric current, the electric current in the vicinity of the top and bottom walls, the electric current acts to accelerate the positive radial flow. In this case that the magnetic field is applied along the central axis, the Hartmann layers, which develop in the vicinity of the walls perpendicular to the applied magnetic field, play a significant role. Since all the electric current generated in the middle part must pass in the thin Hartmann layers, the resolution of the layers determines the whole electric current field and therefore flow field as well. In the Hartmann layers, the Lorentz force and viscous force are almost balanced to each other. Hence, the thickness of the Hartmann layer is inversely proportional to the Hartmann number [5]. The vertical uniform magnetic field can stabilize both the radial and azimuthal velocity and the critical Reynolds number for the flow transition increases with increase in the Hartmann number.
3.3. Toroidal magnetic field

Figure 3 shows the results for a toroidal magnetic field. The figures show the meridional flow vectors, contours of azimuthal velocity, meridional electric current density vectors, contours of azimuthal current density, and contours of electric potential respectively at $Re = 100$ and $Ha = 5$. Even with applying the magnetic field as almost the same as in figure 2 (a), the number of cells is still four. This indicates that the toroidal magnetic field does not have a strong damping and stabilizing effect in comparison with the uniform vertical magnetic field. In this case of the magnetic field, the azimuthal current density is zero as indicated in equation (9) since axisymmetric flow is assumed. The direction of the magnetic field is azimuthal and therefore no Lorentz force acts on the azimuthal flow itself.
Figure 4 shows the numerical results for the meridional velocity vectors, contours of azimuthal velocity, and contours of electric potential respectively at $Re = 1000$ for the toroidal magnetic field. The figure (a) shows a snapshot since this case ($Ha = 50$) exhibits an oscillatory flow. On the other hand, the figure (b) exhibits a steady flow ($Ha = 500$). In the case of the toroidal magnetic field, the direction of the magnetic field is the same as the primary flow driven by the rotation of inner cylinder. Therefore, the Hartmann layers, which develop in the vicinity of the walls perpendicular to the magnetic field, do not exist. We do not have to be concerned about the resolution within the Hartmann layer in this toroidal case even for the high Ha number. The figure of contours of electric potential is quite similar to that of the meridional circulation as shown in the figures 3 and 4. This indicates that the electric potential almost cancels out the electromotive force and the resulting electric current is quite weak. Hence, the toroidal magnetic field can only weakly stabilize.
4. Conclusions

A Couette-Taylor flow of liquid metal in the presence of a uniform vertical or a toroidal magnetic field has been numerically studied for a co-axial double cylinder with the rotation of inner wall while the outer wall is at rest. The numerical results with imposition of a vertical magnetic field show that the flow tends to be uniform along the direction of the magnetic field and the damping effect is rather strong. On the other hand, the toroidal magnetic field reveals a slight damping effect on the flow and the distribution of electric potential is quite similar to the meridional circulation, which indicates that liquid metal Couette-Taylor flow can be estimated if the electric potential inside the flow is experimentally measured.

References

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