Noether gauge symmetry for $f(R)$ gravity in Palatini formalism

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Abstract In this study, we consider a flat Friedmann-Robertson-Walker (FRW) universe in the context of Palatini $f(R)$ theory of gravity. Using the dynamical equivalence between $f(R)$ gravity and scalar-tensor theories, we construct a point Lagrangian in the flat FRW spacetime. Applying Noether gauge symmetry approach for this $f(R)$ Lagrangian we find out the form of $f(R)$ and the exact solution for cosmic scale factor. It is shown that the resulting form of $f(R)$ yield a power-law expansion for the scale factor of the universe.

Keywords $f(R)$ gravity; Palatini formalism; Noether gauge symmetry

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1 Introduction

It is well-known that the Einstein-Hilbert action produces general relativity (GR) which leads mostly to observational success. Furthermore the modified gravity theories issued by more general gravitational actions could explain the observational facts. Observations of the supernovae Type Ia (Riess et al. 1998; Perlmutter et al. 1999) and the cosmic microwave background radiation (Netterfield 2002) indicate that current expansion of the universe is accelerating, on the contrary to Friedmann-Robertson-Walker (FRW) solution of GR. To explain such an accelerated expansion, in the framework of GR, many authors introduced mysterious cosmic fluid, the so-called dark energy which can be described by the cosmological constant (Copeland et al. 2006; Durrer and Maartens 2008).

On the other hand, to get a solution of this problem, some modifications in GR theory have been proposed. The $f(R)$ gravity theory is a particular class of modified gravity theories (Sotiriou and Faraoni 2010; Nojiri and Odintsov 2007; De Felice and Tsujikawa 2010). The simplest form of this theory can be constructed by replacing the Ricci scalar $R$ with an arbitrary function $f(R)$ in the Einstein-Hilbert Lagrangian (EHL). In recent literature some different forms of $f(R)$ gravity have been proposed, and discussed in different contexts. For example, it has been shown that early-time inflation and current cosmic acceleration may take place by adding negative and positive powers of curvature into the EHL (Nojiri and Odintsov 2003). Carroll et al. (Carroll et al. 2004) have also proposed that a small correction to the EHL by adding an inverse term of $R$ would lead to cosmic speed-up which originate from purely gravitational effects. Similar modifications of GR have also been proposed to drive inflation (Starobinsky 1980). On the other hand, it is worth noting that the main deficiency of such theories is that they are seriously constrained by solar system tests (Olmo 2005a; Chiba et al. 2007). A number of viable $f(R)$ models that can satisfy both cosmological and local gravity constraints have been proposed in Refs. (Amendola et al. 2007; Starobinsky 2007; Cognola et al. 2008).

There are two kinds of Noether symmetry approach for cosmological studies in the literature: The first one is the so-called Noether symmetry approach in which the Lie derivative of a given Lagrangian vanishes (Capozziello and de Ritis 1993; de Ritis et al. 1996; Demianski et al. 1987; Saurav and Modak 2001; Camci and Kucukakca 2007; Sonza and Kremer 2008) and the second one is the so-called Noether gauge symmetry (NGS) approach (Jamil et al. 2011; Hussain et al. 2011). The latter is a generalization of the former Noether symmetry approach in the sense that Noether symmetry equation...
includes a gauge term. Taking into account a gauge term in Noether symmetry equation gives a more general definition of the Noether symmetry: this is the so-called Noether gauge symmetry (NGS) approach. Thus one may expect extra (more than one) symmetry generators from this definition.

We note that the Noether symmetry approach without gauge term allows one to choose the potential dynamically in the scalar-tensor gravity theory (Sanyal et al. 2003), and very recently, the explicit form of the function \( f(R) \) (Capozziello and de Felice 2008; Vakili 2008). For this form of \( f(R) \), cosmological solutions of FRW metric can describe the accelerated period of the universe. Applying this Noether symmetry approach, the spherically symmetric solutions in \( f(R) \) theories of gravity have been also found in (Capozziello et al. 2007). Using the definition of NGS, some authors (Jamil et al. 2011; Hussain et al. 2011) have discussed \( f(R) \) and \( f(R) \)-Tachyon model in the metric formalism, where they conjectured that the application of NGS to generic \( f(R) \) Lagrangian results in zero gauge function. The Palatini formalism has not been considered by using the NGS approach in the literature yet. In a recent work Roshan and Shojai (Roshan and Shojai 2008) studied Palatini \( f(R) \) cosmology in flat FRW spacetimes following Noether symmetry approach without gauge term for the matter dominated universe. They found out the form of \( f(R) \) as power-law and exact solutions for cosmic scale factor. Some authors (Jamil et al. 2011; Hussain et al. 2011) considered as independent torsionless connection \( \Gamma \) independent of \( g_{ab} \) and \( L_{m} \) is the matter Lagrangian and \( \psi \) represents collectively the matter fields. The matter Lagrangian is chosen as \( L_{m} = -\rho_{m} a^{-3} \) for matter dominated cosmology. In general there are two approach in order to derive the dynamical field equations of motion in \( f(R) \) gravity. The first one is Palatini approach (Flanagan 2004; Olmo 2005; Fay et al. 2007; Baojiu et al. 2007) in which the metric and connection are considered as independent quantities, and the action is varied with respect to both of them. The second approach is the metric formalism in which action is varied with respect to metric tensor. The field equations in the metric formalism are fourth-order differential equations, while for Palatini formalism they are second-order. If \( f(R) \) is linear in \( R \), the two approaches lead to the same equation. In this study we will use the Palatini formalism. Variation of the action \( \mathcal{A} \) with respect to metric tensor yields following field equations (Brax et al. 2008)

\[
f_\kappa \nabla^a g_{ab} - \frac{1}{2} f_{ab} = \kappa T_{ab},
\]

and the variation of the action \( \mathcal{A} \) with respect to the connection gives

\[
\nabla_c (\sqrt{-g} f_{R} g^{ab}) = 0,
\]

where \( f_{R} = df/dR \) and \( T_{ab} \) is the usual stress-energy tensor of the matter. Also, the trace of Eq. \( \mathcal{A} \) is

\[
f_{R} \nabla^a g_{ab} - 2 f = \kappa T.
\]

The \( f(R) \) gravity theory in Palatini formalism is equivalent to \( \omega_{BD} = -3/2 \) a Brans-Dicke (BD) theory. In order to construct a canonically effective point-like Lagrangian, we have to use the dynamical equivalence between Palatini \( f(R) \) formalism and the BD theory of gravity (Sotiriou 2006; Capozziello et al. 2011). Therefore the action \( \mathcal{A} \) can be written as follows

\[
\mathcal{A} = \int d^4x \sqrt{-g} \left[ f(R) + 2\kappa L_m(g_{ab}, \psi) \right].
\]

In four dimensions, the action of the Palatini \( f(R) \) gravity theory with matter is written as

\[
\mathcal{A} = \int d^4x \sqrt{-g} \left[ f(R) + 2\kappa L_m(g_{ab}, \psi) \right].
\]

Here \( f(R) \) is a differentiable function of the Ricci scalar \( R = g^{ab} R_{ab}(\Gamma) \), \( R_{ab}(\Gamma) \) is the Ricci tensor of any independent torsionless connection \( \Gamma \) independent of \( g_{ab} \) and \( L_{m} \) is the matter Lagrangian and \( \psi \) represents collectively the matter fields. The matter Lagrangian is chosen as \( L_{m} = -\rho_{m} a^{-3} \) for matter dominated cosmology. In general there are two approach in order to derive the dynamical field equations of motion in \( f(R) \) gravity. The first one is Palatini approach (Flanagan 2004; Olmo 2005; Fay et al. 2007; Baojiu et al. 2007) in which the metric and connection are considered as independent quantities, and the action is varied with respect to both of them. The second approach is the metric formalism in which action is varied with respect to metric tensor. The field equations in the metric formalism are fourth-order differential equations, while for Palatini formalism they are second-order. If \( f(R) \) is linear in \( R \), the two approaches lead to the same equation. In this study we will use the Palatini formalism. Variation of the action \( \mathcal{A} \) with respect to metric tensor yields following field equations (Brax et al. 2008)

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\[
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\]
where \(a(t)\) is the cosmic scale factor, \(\rho_{m0}\) is an integration constant associated with matter content and the dot indicates the derivation with respect to cosmic time \(t\). It is found that the energy function \(E_L\) associated with the Lagrangian \(L\) vanishes, i.e.

\[
E_L = \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{1}{4}\dot{\phi}^2 - \frac{U}{\dot{\phi}^2} - \frac{\kappa\rho_{m0}}{3a^2} = 0, \tag{7}
\]

which is known as the modified Friedmann equation. The equations of motion can be obtained by varying the Lagrangian \(L\) with respect to \(a\) and \(\phi\), respectively, as follows

\[
\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\dot{\phi}}{\phi} + \frac{2\dot{a}\ddot{\phi}}{2a\phi} + \frac{3\ddot{\phi}^2}{4\dot{\phi}^2} - \frac{U}{2\dot{\phi}} = 0, \tag{8}
\]

\[
\dot{a} + \frac{\dot{a}^2}{2a} + \frac{\dot{\phi}}{2\phi} + \frac{3\ddot{\phi}}{2a\phi} - \frac{\ddot{\phi}^2}{4\dot{\phi}^2} - \frac{U'}{6} = 0. \tag{9}
\]

### 3 The Noether symmetry approach

For most of studies in the context of both \(f(R)\) gravity and the scalar-tensor theory, it has been used definition of Noether symmetry without a gauge term \(\text{(Capozziello and de Ritis 1993; Kamila et al. 2004; Vakili 2008, Roshan and Shojaei 2008).} \) Recently, some work on NGS approach for FRW metric has been appeared \(\text{(Jamil et al. 2011; Hussain et al. 2011).}\) Noether gauge symmetry is defined as follows. Let us consider a vector field \(\text{Brugimov 1999}\)

\[
X = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi}, \tag{10}
\]

where \(\tau, \alpha \) and \(\beta \) are depend on \(t, a\), and \(\phi\). Here, \(t\) is the independent variable, \(a(t)\) and \(\phi(t)\) are the dependent variables. The first prolongation of the above vector field is given by

\[
X^{[1]} = X + \alpha_t \frac{\partial}{\partial a} + \beta_t \frac{\partial}{\partial \phi}. \tag{11}
\]

in which

\[
\alpha_t = D_t \alpha - \dot{a} D_t \tau, \quad \beta_t = D_t \beta - \dot{\phi} D_t \tau, \quad \tag{12}
\]

where \(D_t\) is the operator of total differentiation with respect to \(t\)

\[
D_t = \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial a} + \dot{\phi} \frac{\partial}{\partial \phi}. \tag{13}
\]

The vector field \(X\) is a NGS of a Lagrangian \(L(t, a, \phi, \dot{a}, \dot{\phi})\) if there exists a gauge function, \(B(t, a, \phi)\), such that

\[
X^{[1]}L + LD_t(\tau) = D_t B. \tag{14}
\]

For Noether symmetry without gauge term (i.e. \(B = 0\)) it is required that \(\tau = 0\), and thus the above equation reduces to the form \(E(X)L = 0\) which is the existence condition for Noether symmetry without gauge term. The significance of NGS is clear from the following theorem \(\text{(Noether 1918).}\)

Theorem: If \(X\) is the Noether symmetry corresponding to the Lagrangian \(L(t, a, \phi, \dot{a}, \dot{\phi})\), then

\[
I = \tau L + (\alpha - \tau \dot{a}) \frac{\partial L}{\partial a} + (\beta - \tau \dot{\phi}) \frac{\partial L}{\partial \phi} - B \tag{15}
\]

is a first integral or a conserved quantity associated with \(X\). Now we seek the condition in order that the Lagrangian \(L\) would admit NGS.

For the flat FRW metric, the NGS condition \(\text{Brugimov 1999}\) yields the following set of equations

\[
\tau_a = \tau_\phi = 0, \tag{16}
\]

\[
6a(2\phi \alpha_t + a\beta_t) - B_a = 0, \tag{17}
\]

\[
3a^2 \frac{\partial}{\phi} (2\phi \alpha_t + a\beta_t) - B_\phi = 0, \tag{18}
\]

\[
2\alpha + a (\alpha_a + \beta_\phi - \tau_t) + 2\phi \alpha_\phi + \frac{a^2}{2\phi} \beta_a = 0, \tag{19}
\]

\[
\phi \alpha + a \beta + a (2\alpha_a - \tau_\phi) + 4\phi \alpha_\phi = 0, \tag{20}
\]

\[
3\alpha - \frac{a}{\phi} \beta + a (2\beta_\phi - \tau_t) + 4\phi \alpha_\phi = 0, \tag{21}
\]

\[
a^2(3\alpha + a\tau_t) + a^3 U' \beta + 2k\rho_{m0} \tau_t - B_t = 0. \tag{22}
\]

After some algebraic calculations, the solutions of the above set of differential equations \(\text{Rashid et al. 2011}\) for \(\alpha, \beta, \tau, B\) and the potential \(U(\phi)\) are obtained as follows:

\[
\alpha = c_1 a + c_2 a (a^2 \phi)^{-\frac{\ell + 1}{\ell - 2}}, \tag{23}
\]

\[
\beta = -(2\ell - 1) \phi \left[ \frac{3c_1}{\ell - 2} - c_2 (a^2 \phi)^{-\frac{\ell + 1}{\ell - 2}} \right], \tag{24}
\]

\[
\tau = -\frac{3c_1 (\ell + 1)}{\ell - 2} t + c_3, \quad B = -\frac{6c_1 k \rho_{m0} (\ell + 1)}{\ell - 2} t + c_4 \tag{25}
\]

\[
U(\phi) = \lambda \phi^{-\frac{\ell + 1}{\ell - 2}}, \tag{26}
\]
where \( c_i, \ell \) and \( \lambda \) are constants and \( \ell \neq -1/2, 2 \). Since \( B = 0 \) and \( \tau = 0 \) for Noether symmetry without gauge term, the parameters \( c_1, c_3 \) and \( c_4 \) should vanish. This case has been studied in Ref. \( \text{[Roshan and Shojaei 2008]} \). Taking \( \phi = \varphi^2 \) in the Lagrangian \( \mathcal{L} \) and doing the above calculations again, the obtained form of \( \alpha \) and \( \beta \) generalizes the ones found in Ref. \( \text{[Roshan and Shojaei 2003]} \) (which are represented as \( A \) and \( B \) in this reference).

It is seen from (23)-(25) that the Lagrangian \( \mathcal{L} \) admits three Noether symmetry generators

\[
X_1 = \frac{\partial}{\partial t},
\]

\[
X_2 = (a^2 \phi)^{-\frac{\ell+1}{2}} \left( \frac{\partial}{\partial a} + (2\ell - 1)\phi \frac{\partial}{\partial \phi} \right),
\]

\[
X_3 = -\frac{3(\ell+1)t}{\ell-2} \frac{\partial}{\partial t} + a \frac{\partial}{\partial a} - \frac{3(2\ell-1)\phi}{\ell-2} \frac{\partial}{\partial \phi}.
\]

The corresponding Lie algebra has the following non-vanishing commutators:

\[
[X_1, X_3] = -\frac{3(\ell+1)}{\ell-2} X_1,
\]

\[
[X_2, X_3] = \left( \frac{\ell+1}{\ell-2} \right) X_2.
\]

The first integrals associated with \( X_i \) are

\[
I_1 = 6a^2 \dot{\phi}^2 + 6a\dot{\phi}^2 + \frac{3a^3}{2\phi} \phi^2 - a^3 U - 2\kappa \rho_m t.
\]

\[
I_2 = 3(2\ell+1)a(a^2 \phi)^{\frac{\ell+1}{2}} \frac{d}{dt}(a^2 \phi),
\]

\[
I_3 = \frac{3(\ell+1)}{\ell-2} I_1 - \frac{3(4\ell+1)a}{\ell-2} \frac{d(a^2 \phi)}{dt} + 6(\ell+1)\kappa \rho_m a.\]

Here the constant parameter \( c_4 \) is assumed to be zero in the gauge function \( B \). We note that the first integral \( I_1 \) is related with the energy function \( \mathcal{L} \), so that the first integral \( I_1 \) vanishes.

4 The Cosmological Solution

As an inverse problem of finding \( f(\mathcal{R}) \) Lagrangian, it is only required to give \( U(\varphi) \). Using the algebraic relation

\[
U(\varphi) = \phi \chi(\varphi) - f(\chi(\varphi)),
\]

it is possible to solve \( f(\mathcal{R}) \), where \( \mathcal{R} = \chi(\varphi) \) and \( \phi = f(\mathcal{R}) \). Putting the potential \( \mathcal{L} \) into Eq. (32) yields

\[
f(\mathcal{R}) = \mathcal{L} = \chi(\varphi) - \frac{3}{2\ell+1} \mathcal{R}.
\]

Thus, it is straightforward to get the following two solutions from Eq. (30) for \( f(\mathcal{R}) \)

\[
f(\mathcal{R}) = p(\ell)\mathcal{R}^{\frac{3}{2\ell+1}},
\]

\[
f(\mathcal{R}) = R_0 - \lambda \mathcal{R}^{\frac{3}{2\ell+1}},
\]

where \( p(\ell) = 2(\ell+1) \left[ \frac{3^{2(\ell+1)} - 1}{1-2\ell} \right]^{\frac{2(\ell+1)}{3(2\ell+1)}} \), \( R_0 \) is a constant and \( \ell \neq -1, 1/2 \).

In the following subsections, considering \( f(\mathcal{R}) \) given by \( \mathcal{L} \) and \( \mathcal{L} \), we will search the exact solution for cosmic scale factor. For \( \alpha \) and \( \beta \) given by \( \mathcal{L} \) and \( \mathcal{L} \), it is appeared the singularities for the values of \( \ell = -1/2 \) and \( \ell = -1 \). For these values of \( \ell \), after solving Eq. (30) one has the same form with \( \mathcal{L} \) for \( f(\mathcal{R}) \).

4.1 Case (i). \( f(\mathcal{R}) = p(\ell)\mathcal{R}^{\frac{3}{2\ell+1}} \)

Using the trace relation \( \mathcal{L} \) and \( \phi = f(\mathcal{R}) \) we obtain

\[
\phi = r(\ell)a^{2\ell-1},
\]

where \( r(\ell) = \frac{3p(\ell)}{2(\ell+1)} \left( \frac{2\kappa \rho_m}{p(\ell)} \right)^{\frac{1-2\ell}{2\ell+1}} \), \( \ell \neq -1/4 \). Hence, the Noether first integrals \( \mathcal{L} \) and \( \mathcal{L} \) can be written as

\[
I_2 = 3(2\ell+1)^2 r(\ell) a^{\ell+1} \dot{a},
\]

\[
I_3 = \frac{3(4\ell+1)(2\ell+1)r(\ell)}{\ell-2} a^{(2\ell+1)} \dot{a} + \frac{6\kappa \rho_m (\ell+1)}{\ell-2} a.
\]

Now the Eq. (40) can be used to find out the time dependence of the cosmic scale factor as

\[
a(t) = \left[ \frac{I_2(\ell+1)r(\ell)^{\frac{3}{2\ell+1}}}{3(2\ell+1)^2} t + t_0 \right]^{\frac{2\ell+1}{3}},
\]

where \( t_0 \) is a constant of integration. Considering \( a(t) \) in Eq. (41), the following constraint equations are found

\[
I_2^2(4\ell+1)r(\ell)^{1/(2\ell+1)} - 18\kappa \rho_m (2\ell+1)^3 = 0,
\]

\[
I_2 t_0 (4\ell+1)r(\ell)^{(\ell+1)/(2\ell+1)} + (\ell-2)(2\ell+1)I_3 = 0.
\]
It is explicitly seen here that if \( t_0 = 0 \), then the first integral \( I_3 \) is equal to zero. Also, using Eqs. (43) and (51), we have additional constraint relations as

\[
\frac{I_2^2 (2\ell - 1) r(\ell)^{1/(2\ell + 1)}}{18(2\ell + 1)^3} + \lambda r(\ell)^{-3/(2\ell - 1)} = 0, \quad (45)
\]

\[
\frac{I_2^2 r(\ell)^{1/(2\ell + 1)}}{6(2\ell + 1)^2} - \left[ \lambda r(\ell)^{-3/(2\ell - 1)} + 2\kappa \rho m_0 \right] = 0. \quad (46)
\]

Thus, the constraint Eqs. (43) and (45) give the following simple relation

\[
\kappa \rho m_0 = -\frac{\lambda (4\ell + 1)}{2\ell - 1} r(\ell)^{-3/(2\ell - 1)}. \quad (47)
\]

The deceleration parameter, which is an important quantity in cosmology, is defined by \( q = -\ddot{a}/\dot{a}^2 \), where the positive sign of \( q \) indicates the standard decelerating models whereas the negative sign corresponds to accelerating models and \( q = 0 \) corresponds to expansion with constant velocity. It takes the following form in this model

\[
q = \ell. \quad (48)
\]

The effective equation of state parameter defined by \( w_{\text{eff}} = -1 - \frac{2\ell}{\ell + 1} \) (Capozziello et al. 2003; Allemandi et al. 2004a, 2004b) can be obtained as

\[
w_{\text{eff}} = \frac{2\ell - 1}{3}, \quad (49)
\]

where \( H \) is Hubble parameter. Astrophysical data indicate that \( w \) lies in a very narrow strip close to \( w = -1 \). The case \( w = -1 \) corresponds to the cosmological constant. For \( w < -1 \) the phantom phase is observed, and for \( -1 < w < -1/3 \) the phase is described by quintessence. Thus, in the interval \(-1 < \ell < 0 \) we have quintessence phase. If \(-\infty < \ell < -1 \), then the phantom phase occurs, where the universe is both expanding and accelerating.

4.2 Case (ii). \( f(\mathcal{R}) = R_0 \mathcal{R} - \lambda R_0^{3/(2\ell - 1)} \)

If \( R_0 = 1 \), i.e. \( \phi = 1 \) from the relation \( \phi = f_\mathcal{R} = R_0 \), then the action is reduced to the Einstein-Hilbert action with cosmological constant ( \( f(\mathcal{R}) = \mathcal{R} - \lambda \)). We note here that Palatini and metric formalism are coincide. In this case, the Noether first integrals (43) and (44) can be written as

\[
I_2 = 6(2\ell + 1) a^{2\ell + 1} \dot{a}, \quad (50)
\]

\[
I_3 = -\frac{6(4\ell + 1)}{\ell - 2} a^2 \dot{a} + \frac{6(\ell + 1)\kappa \rho m_0}{\ell - 2} t. \quad (51)
\]

The modified Friedmann equation (7) for this case reduces to the form

\[
\frac{\dot{a}^2}{a^2} - \frac{\lambda}{6} - \frac{\kappa \rho m_0}{3a^3} = 0. \quad (52)
\]

From Eq. (50) the scale factor is solved as

\[
a(t) = \left[ \frac{I_2 (4\ell + 1) + I_3}{6(2\ell + 1)^2 t + t_1} \right]^{2/3}. \quad (53)
\]

Inserting the scale factor (53) into Eq. (51) and Eq. (52) one gets \( \lambda = 0 \), \( \ell = 1/2 \) and as a constraint equations \( \kappa \rho m_0 = I_2^3/48 \) and \( t_1 = I_3/I_2 \). Therefore, for this case the scale factor has the form

\[
a(t) = \left( \frac{I_2}{8t + I_2} \right)^{2/3}. \quad (54)
\]

which is obtained from (53) when \( \ell = 1/2 \). Taking \( \ell = -1/4 \) in Eq. (50), we have \( a(t) = \exp(\frac{4t}{I_2}) \) which gives \( \rho_m = 0 \) and \( \lambda = 2I_2^2/3 \). Using these results for \( \ell = -1/4 \) in the Eq. (51) one can find \( I_3 = 0 \).

5 Concluding remarks

The Palatini approach consider the metric \( g_{ab} \) and the connection \( \Gamma_{bc}^a \) as independent field variables, but the spacetime metric \( g_{ab} \) is only independent variable in metric formalism. The Palatini formalism can be seen as containing two independent metrics \( g_{ab} \) and \( h_{ab} = f_\mathcal{R} g_{ab} \) rather that a metric and independent connection. In Palatini \( f(\mathcal{R}) \) gravity the second metric \( h_{ab} \) determine the geodesic structure with the connection \( \Gamma_{bc}^a \) which is the Levi-Civita connection of new metric \( h_{ab} \). In BD theory of gravity the second metric \( h_{ab} \) is related to the non-minimal coupling of the BD scalar, i.e. \( \phi = f_\mathcal{R} \). In Palatini approach, \( f(\mathcal{R}) \) gravity given by the action (11) is equivalent to a special BD theory with a scalar field potential. The action (6) is clearly that of a BD theory with BD parameter \( w_{BD} = -3/2 \) and a potential \( U(\phi) \), which is considered in this paper. In the metric formalism, \( f(\mathcal{R}) \) gravity is equivalent to the BD theory with \( w_{BD} = 0 \) (see the review paper (Capozziello and Laurentis 2011)). An \( w_{BD} = 0 \) BD theory was originally studied for the aim of getting Yukawa correction to the Newtonian potential in the weak-field limit (O’Hanlon 1972).

In this study, we examined the matter dominated flat FRW universe by considering the Palatini \( f(\mathcal{R}) \) formalism and by following the NGS approach which leads to explicit form for \( f(\mathcal{R}) \). This approach is based on the search for Noether gauge symmetries which allow one to find the form of \( f(\mathcal{R}) \). We have obtained
two type Noether symmetric $f(R)$, \cite{57} which yields case (i) and \cite{58} which gives case(ii), where $f(R)$ functions give rise to a power-law Lagrangian and EHL with cosmological constant, respectively. In case (i), it is found the power-law form of cosmic scale factor by \cite{12}. For case (ii), the cosmic scale factor is obtained by \cite{59} for $\ell = 1/2$ and de Sitter solution for $\ell = -1/4$. We have presented the effective equation of state parameter for Palatini $f(R)$ cosmology. In the first model, case(i), the expansion of Universe is accelerating at the intervals $-1 < \ell < 0$, quintessence phase, and $-\infty < \ell < -1$, phantom phase. Thus, this model can provide a natural gravitational alternative for the dark energy without the necessity to introduce an exotic fluid with a negative equation of state parameter. We note here that while the gauge function turns out to be zero in Refs. \cite{Jamil et al. 2011, Hussain et al. 2011}, but our analysis shows that it depends on the cosmic time.

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