A Generalization of Piatetski–Shapiro Sequences

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Abstract. We consider a generalization of Piatetski–Shapiro sequences in the sense of Beatty sequences, which is of the form \( \lfloor \alpha n^c + \beta \rfloor \) with real numbers \( \alpha \geq 1, \ c > 1 \) and \( \beta \). We show there are infinitely many primes in the generalized Piatetski–Shapiro sequence with \( c \in (1, \frac{14}{13}) \). Moreover, we prove there are infinitely many Carmichael numbers composed entirely of the primes from the generalized Piatetski–Shapiro sequences with \( c \in (1, \frac{64}{63}) \).

1. Introduction

The Piatetski–Shapiro sequences are sequences of the form

\[ \mathcal{N}(c) := (\lfloor n^c \rfloor)_{n=1}^{\infty}, \quad c > 1, \ c \notin \mathbb{N}. \]

Such sequences have been named in honor of Piatetski–Shapiro, who proved \(^{30}\) that \( \mathcal{N}(c) \) contains infinitely many primes if \( c \in (1, \frac{12}{11}) \). More precisely, for such \( c \) he showed that the counting function

\[ \pi^{(c)}(x) := \# \{ \text{prime } p \leq x : p \in \mathcal{N}(c) \} \]

satisfies the asymptotic relation

\[ \pi^{(c)}(x) \sim \frac{x^{1/c}}{\log x} \quad \text{as } x \to \infty. \]

The range for \( c \) in which it is known that \( \mathcal{N}(c) \) contains infinitely many primes has been extended many times over the years \(^{9,18,23,20}\) and the above formula is currently known to hold for all \( c \in (1, \frac{2817}{2426}) \) thanks to Rivat and Sargos \(^{31}\). Rivat and Wu \(^{32}\) also showed that there are infinitely many Piatetski–Shapiro primes for \( c \in (1, \frac{243}{205}) \). The same result is expected to hold for all larger values of \( c \). We remark that if \( c \in (0, 1) \) then \( \mathcal{N}(c) \) contains all natural numbers, hence all primes in particular. More recent research related to Piatetski–Shapiro sequences can be found in \(^{1,4,7,15,24,25,27,29,33}\) and references therein.

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For fixed real numbers $\alpha$, $\beta$ the associated non-homogeneous Beatty sequence is the sequence of integers defined by

$$B_{\alpha, \beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

where $[t]$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences are also called generalized arithmetic progressions. If $\alpha$ is irrational, it follows from a classical exponential sum estimate of Vinogradov [35] that $B_{\alpha, \beta}$ contains infinitely many prime numbers; in fact, one has the asymptotic relation

$$\# \{\text{prime } p \leq x : p \in B_{\alpha, \beta} \} \sim \alpha^{-1} \pi(x), \quad x \to \infty,$$

where $\pi(x)$ is the prime counting function. More recent literatures related to prime numbers and Beatty sequences can be found in [10–12,15–17].

It is interesting to generalize the Piatetski–Shapiro sequences in the sense of Beatty sequences, since both Piatetski–Shapiro sequences and Beatty sequences produce infinitely many primes. Let $\alpha \geq 1$ and $\beta$ be real numbers. We investigate the following generalized Piatetski–Shapiro sequences

$$N_{\alpha, \beta}^{(c)} := (\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}.$$

Note that the special case $N_{1,0}^{(c)}$ is the normal Piatetski–Shapiro sequences. Let

$$\pi(x; d, a) := \#\{p \leq x : p \equiv a \mod d\}$$

and

$$\pi_{\alpha, \beta, c}(x; d, a) := \#\{p \leq x : p \in N_{\alpha, \beta}^{(c)} \text{ and } p \equiv a \mod d\}.$$

We prove the following theorem.

**Theorem 1.1.** Let $\alpha \geq 1$ and $\beta$ be real numbers. Let $c \in (1, 14/13)$.

$$\pi_{\alpha, \beta, c}(x; d, a) = \alpha^{-1/c} c^{-1} x^{1/c - 1} \pi(x; d, a)$$

$$+ \alpha^{-1/c} c^{-1} (1 - c^{-1}) \int_{2}^{x} u^{1/c - 2} \pi(u; d, a) \, du + O\left(x^{3/(5c) + 13/35 + \epsilon}\right).$$

Note that

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

We conclude that

**Corollary 1.2.** Let $\alpha \geq 1$ and $\beta$ be real numbers. Let $c \in (1, 14/13)$. Let

$$\pi_{\alpha, \beta, c}(x) := \#\{p \leq x : p \in N_{\alpha, \beta}^{(c)}\}.$$

Then

$$\pi_{\alpha, \beta, c}(x) = \frac{x^{1/c}}{\alpha^{1/c} \log x} + O\left(\frac{x^{1/c}}{\log^2 x}\right).$$

(1.1)
We clarify that (1.1) can be proved by a similar argument to the proof of Piatetski–Shapiro prime number theorem. The key point is to estimate
\[
\sum_{1 \leq h \leq H} \left| \sum_{N \leq n \leq N_1} \Lambda(n) e(\theta h n^\gamma) \right|
\]
where \(H, N, N_1, \theta, \gamma\) are positive numbers such that \(H \geq 1, N_1 \leq 2N, \theta < 1, \gamma < 1\). Since the function \(\theta h n^\gamma\) is smooth enough to apply the method of exponent pairs, the constant \(\theta\) does not play a big role in the estimation of exponential sums. We expect that all the methods for Piatetski–Shapiro prime number theorem should work for estimating \(\pi_{\alpha, \beta, c}(x)\). However, in this paper, we mean to give a first result. For the sake of simplicity, we do not give more discussion to the prime counting function \(\pi_{\alpha, \beta, c}(x)\).

In the end, we prove a theorem related to Carmichael numbers, which are the composite natural numbers \(N\) with the property that \(N \mid (a^N - a)\) for every integer \(a\). In 1994, Alford, Granville and Pomerance proved there exist infinitely many Carmichael numbers. Their proof relies on the arithmetic properties of primes. Since we show the arithmetic properties of the primes in \(\mathcal{N}_{\alpha, \beta}^{(c)}\), we are able to prove the following result by a similar method of 2.

**Theorem 1.3.** For every \(c \in (1, 64/63)\), there are infinitely many Carmichael numbers composed entirely of the primes from the set \(\mathcal{N}_{\alpha, \beta}^{(c)}\).

2. Preliminaries

2.1. Notation

We denote by \([t]\) and \(\{t\}\) the integer part and the fractional part of \(t\), respectively. As is customary, we put
\[
e(t) := e^{2\pi it}.
\]
We make considerable use of the sawtooth function defined by
\[
\psi(t) := t - [t] - \frac{1}{2} = \{t\} - \frac{1}{2}, \quad t \in \mathbb{R}.
\]
The letter \(p\) always denotes a prime. For the generalized Piatetski–Shapiro sequence \([(\lfloor \alpha n^c + \beta \rfloor)]_{n=1}^\infty\), we denote \(\gamma := c^{-1}\) and \(\theta := \alpha^{-\gamma}\). We use notation of the form \(m \sim M\) as an abbreviation for \(M < m \leq 2M\).

Throughout the paper, implied constants in symbols \(O, \ll\) and \(\gg\) may depend (where obvious) on the parameters \(\alpha, c, \varepsilon\) but are absolute otherwise. For given functions \(F\) and \(G\), the notations \(F \ll G\), \(G \gg F\) and \(F = O(G)\) are all equivalent to the statement that the inequality \(|F| \leq C|G|\) holds with some constant \(C > 0\).
2.2. Technical lemmas

We need the following well-known approximation of Vaaler \[34\].

**Lemma 2.1.** For any \( H \geq 1 \) there are numbers \( a_h, b_h \) such that

\[
\left| \psi(t) - \sum_{0 < |h| \leq H} a_h e(th) \right| \leq \sum_{|h| \leq H} b_h e(th), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.
\]

Next, we recall the following identity for the von Mangoldt function \( \Lambda \), which is due to Vaughan.

**Lemma 2.2.** Let \( U, V \geq 1 \) be real parameters. For any \( n > U \) we have

\[
\Lambda(n) = -\sum_{k|n} a(k) + \sum_{cd=n, \quad \substack{d \leq V \atop c \leq U}} (\log c)\mu(d) - \sum_{ke=n, \quad \substack{k > 1 \atop c < U}} \Lambda(c) b(k),
\]

where

\[
a(k) = \sum_{\substack{cd=k \atop c \leq U}} \Lambda(c)\mu(d) \quad \text{and} \quad b(k) = \sum_{\substack{d|k \atop d \leq V}} \mu(d).
\]

**Proof.** See Davenport \[13, p. 139\].

The Vaughan’s identity gives a decomposition for sums of the form

\[
S(f) := \sum_{X < n \leq X'} \Lambda(n) f(n)
\]

where \( f \) is any complex-valued function, and \( X' \sim X \). Let \( N_1 \leq 2N \). A Type I sum is a sum of the form

\[
S_I(K, L) := \sum_{k \sim K, \quad l \sim L} \sum_{N < kl \leq N_1} a_k f(kl)
\]

where \( |a_k| \ll k^\varepsilon \) for every \( \varepsilon > 0 \). A Type II sum is a sum of the form

\[
S_{II}(K, L) := \sum_{k \sim K, \quad l \sim L} \sum_{N < kl \leq N_1} a_k b_l f(kl)
\]

where \( |a_k| \ll k^\varepsilon \) and \( |b_l| \ll l^\varepsilon \) for every \( \varepsilon > 0 \).

**Lemma 2.3.** Suppose that every Type I sum with \( K \ll X^{3/7} \) satisfies the bound

\[
S_I \ll B(X)
\]
and every Type II sum with \( X^{3/7} \ll K \ll X^{1/2} \) satisfies the bound
\[
S_{II} \ll B(X).
\]

Then
\[
S(f) \ll B(X)X^\varepsilon.
\]

**Proof.** The lemma can be deduced by the Vaughan’s identity (Lemma 2.2) since \( S(f) \) can be written as a linear decomposition of Type I and Type II sums. A detailed proof of a similar lemma can be found in [3, Lemma 2]. \( \square \)

**Lemma 2.4.** For a bounded function \( g \) and \( N' \sim N \) we have
\[
\sum_{N < p \leq N'} g(p) \ll \frac{1}{\log N} \max_{N_1 \leq 2N} \left| \sum_{N < n \leq N_1} \Lambda(n)g(n) \right| + N^{1/2}.
\]

**Proof.** See [14, p. 48]. \( \square \)

**Lemma 2.5.** Let
\[
L(Q) := \sum_{j=1}^{J} C_j Q^{c_j} + \sum_{k=1}^{K} D_k Q^{-d_k},
\]
where \( C_j, c_j, D_k, d_k > 0 \). Then

(1) For any \( Q \geq Q' > 0 \) there exists \( Q_1 \in [Q', Q] \) such that
\[
L(Q_1) \ll \sum_{j=1}^{J} \sum_{k=1}^{K} (C_j D_k^{(c_j + d_k) / (c_j + d_k)}) + \sum_{j=1}^{J} C_j (Q')^{c_j} + \sum_{k=1}^{K} D_k Q^{-d_k}.
\]

(2) For any \( Q > 0 \) there exists \( Q_1 \in (0, Q] \) such that
\[
L(Q_1) \ll \sum_{j=1}^{J} \sum_{k=1}^{K} (C_j D_k^{(c_j + d_k) / (c_j + d_k)}) + \sum_{k=1}^{K} D_k Q^{-d_k}.
\]

**Proof.** The proof of the first assertion is in [14, Lemma 2.4]. The proof of the second assertion is similar. \( \square \)

**Lemma 2.6.** Let \( f \) be twice continuously differentiable on a subinterval \( \mathcal{I} \) of \((N, 2N]\). Suppose that for some \( \lambda > 0 \), the inequalities
\[
\lambda \ll |f''(t)| \ll \lambda, \quad t \in \mathcal{I}
\]
hold, where the implied constants are independent of \( f \) and \( \lambda \). Then
\[
\sum_{n \in \mathcal{I}} e(f(n)) \ll N\lambda^{1/2} + \lambda^{-1/2}.
\]
Proof. See [14, Theorem 2.2].

Lemma 2.7. Suppose $|a_k| \leq 1$ for all $k \sim K$. Fix $\gamma \in (0, 1)$, $\mu, \rho \in \mathbb{R}$, $\mu \neq 0$ and $m \in \mathbb{N}$. Then for any $K \ll N^{3/7}$ the Type I sum

$$S_I = \sum_{k \sim K} \sum_{l \sim L} a_k e(\mu mk^{\gamma}l^{\gamma} + \rho kl)$$

satisfies the bound

$$S_I \ll m^{1/2}N^{3/7+\gamma/2} + m^{-1/2}N^{1-\gamma/2}.$$  

Proof. Writing $f(l) = \mu mk^{\gamma}l^{\gamma} + \rho kl$ we see that

$$|f''(l)| = |\mu m^{\gamma}(\gamma - 1)k^{\gamma}l^{\gamma - 2}| \asymp mK^{\gamma}L^{\gamma/2}.$$  

Using Lemma 2.6 it follows that

$$\sum_{l \sim L \atop N < kl \leq N_1} a_k e(\mu mk^{\gamma}l^{\gamma} + \rho kl) \ll m^{1/2}K^{\gamma/2}L^{\gamma/2} + m^{-1/2}K^{-\gamma/2}L^{1-\gamma/2}.$$  

Since $|a_k| \leq 1$ for all $k \sim K$ we see that

$$S_I = \sum_{k \sim K} \sum_{l \sim L \atop N < kl \leq N_1} a_k e(\mu mk^{\gamma}l^{\gamma} + \rho kl)$$

$$\ll m^{1/2}K^{1+\gamma/2}L^{\gamma/2} + m^{-1/2}K^{1-\gamma/2}L^{1-\gamma/2}$$

$$\ll m^{1/2}N^{3/7+\gamma/2} + m^{-1/2}N^{1-\gamma/2}. \qed$$

We need the following lemma to bound the Type II sum.

Lemma 2.8. Let $1 < Q < L$. If $f$ is a function of the form $f(n) = e(g(n))$, then any Type II sum satisfies

$$|S_{II}|^2 \ll N^2Q^{-1} + NQ^{-1} \sum_{0 <| q |< Q} \sum_{l \sim L} |S(q,l)|,$$

where

$$S(q,l) = \sum_{k \in \mathcal{I}(q,l)} e(g(kl) - g(k(l + q)))$$

for a certain subinterval $\mathcal{I}(q,l)$ of $(K, 2K]$.

Proof. See the proof of [14, Lemma 4.13]. \qed
Lemma 2.9. Suppose $|a_k| \leq 1$ and $|b_l| \leq 1$ for $(k, l) \sim (K, L)$. Fix $\gamma \in (0, 1)$, $\mu, \rho \in \mathbb{R}$, $\mu \neq 0$ and $m \in \mathbb{N}$. For any $K$ in the range $N^{3/7} \ll K \ll N^{1/2}$, the Type II sum

$$S_{II} = \sum_{k \sim K} \sum_{l \sim L} a_k b_l e(\mu k^{\gamma} l^\gamma + \rho k l)$$

satisfies the bound

$$S_{II} \ll m^{1/6} N^{16/21 + \gamma/6} + N^{25/28} + m^{-1/4} N^{1-\gamma/4}.$$ 

Proof. We assume that $KL \asymp N$. By Lemma 2.8 we have

$$|S_{II}|^2 \ll K^2 L^2 Q^{-1} + KLQ^{-1} \sum_{l \sim L} \sum_{0 < |q| \leq Q} |S(q; l)|,$$

where

$$S(q; l) = \sum_{k \in I(q; l)} e(f(k)) \quad \text{and} \quad f(k) = \mu k^{\gamma} (l^\gamma - (l + q)^\gamma) - \rho k q,$$

and each $I(q; n)$ is a certain subinterval in the set of numbers $k \sim K$. Since

$$|f''(k)| = |\mu m \gamma (1 - \gamma) k^{\gamma - 2} (l^\gamma - (l + q)^\gamma)|$$

$$\asymp m K^{\gamma - 2} L^{\gamma - 1} |q|,$$

it follows from Lemma 2.6 that

$$S(q; l) \ll (m K^{\gamma - 2} L^{\gamma - 1} |q|)^{1/2} K + (m K^{\gamma - 2} L^{\gamma - 1} |q|)^{-1/2}.$$ 

Inserting the bound to $|S_{II}|^2$ and summing over $l$ and $q$, we derive that

$$|S_{II}|^2 \ll K^2 L^2 Q^{-1} + m^{1/2} K^{1+\gamma/2} L^{3/2+\gamma/2} Q^{1/2} + m^{-1/2} K^{2-\gamma/2} L^{5/2-\gamma/2} Q^{-1/2}.$$ 

By Lemma 2.5 we have

$$|S_{II}|^2 \ll m^{1/3} K^{4+3\gamma/3} L^{5+3\gamma/3} + K^{3/2} L^2 + K^2 L + m^{-1/2} N^{2-\gamma/2}$$

$$\ll m^{1/3} K^{-1/3} N^{5+3\gamma/3} + K^{-1/2} N^2 + K N + m^{-1/2} N^{2-\gamma/2}.$$ 

We have $N^{3/7} \ll K \ll N^{1/2}$, the proof is done. 

We use the following lemma, which provides a characterization of the numbers that occur in the Piatetski–Shapiro sequence $\mathcal{N}(c)$.

Lemma 2.10. A natural number $m$ has the form $[n^c]$ if and only if $\chi^{(c)}(m) = 1$, where $\chi^{(c)}(m) := [-m^\gamma] - [-(m + 1)^\gamma]$. Moreover,

$$\chi^{(c)}(m) = \gamma m^{\gamma - 1} + \psi(-(m + 1)^\gamma) - \psi(-m^\gamma) + O(m^{\gamma - 2}).$$

In particular, for any $c \in (1, 2817/2426)$ the results of [31] yield the estimate

$$\pi^{(c)}(x) = \sum_{p \leq x} \chi^{(c)}(p) = \frac{x^\gamma}{c \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right).$$
3. Proof of Theorem 1.1

Recall that $\gamma = c^{-1}$ and $\theta = \alpha^{-\gamma}$. A prime $p$ equals $\lfloor \alpha n c + \beta \rfloor$ if and only if

$$p \leq \alpha n c + \beta < p + 1,$$

which is equivalent to

$$\theta(p - \beta) \gamma \leq n < \theta(p + 1 - \beta) \gamma,$$

except for the case that $p = [\beta]$. Then

$$\pi_{\alpha, \beta, c}(x; d, a) = \sum_{p \leq x, p \equiv a \mod d} \left( \lfloor -\theta(p - \beta) \gamma \rfloor - \lfloor -\theta(p + 1 - \beta) \gamma \rfloor \right) = \sum_1 + \sum_2 + O(1),$$

where

$$\sum_1 := \theta_\gamma \sum_{p \leq x, p \equiv a \mod d} (p - \beta)^{\gamma - 1},$$

and

$$\sum_2 := \sum_{p \leq x, p \equiv a \mod d} \left( \psi(-\theta(p - \beta + 1)^{\gamma}) - \psi(-\theta(p - \beta)^{\gamma}) \right).$$

A partial summation gives

$$\sum_1 = \theta_\gamma x^{\gamma - 1} \pi(x; d, a) + \theta_\gamma (1 - \gamma) \int_2^x u^{\gamma - 2} \pi(u; d, a) du + O(x^{\gamma - 1} + 1),$$

which is the main term. Let $N \leq x$ and $N_1 \leq 2N$. We estimate $\sum_2$ by considering

$$S := \sum_{N < n \leq N_1 \atop n \equiv a \mod d} \Lambda(n) \left( \psi(-\theta(n - \beta + 1)^{\gamma}) - \psi(-\theta(n - \beta)^{\gamma}) \right) = S_1 + O(S_2)$$

by Lemmas 2.1 and 2.4, where

$$S_1 := \sum_{N < n \leq N_1 \atop n \equiv a \mod d} \Lambda(n) \sum_{0 < |h| \leq H} a_h \left( e(\theta h(n - \beta + 1)^{\gamma}) - e(\theta h(n - \beta)^{\gamma}) \right)$$

and

$$S_2 := \sum_{N < n \leq N_1 \atop n \equiv a \mod d} \Lambda(n) \sum_{|h| \leq H} b_h \left( e(\theta h(n - \beta + 1)^{\gamma}) + e(\theta h(n - \beta)^{\gamma}) \right)$$

for some $H \geq 1$. We consider the upper bound of $S_1$ firstly. By partial summation (see [14, p. 48]), we have

$$S_1 \ll N^{\gamma - 1} \max_{N_1 \leq 2N} \sum_{1 \leq h \leq H} \left| \sum_{N < n \leq N_1 \atop n \equiv a \mod d} \Lambda(n) e(\theta h n^{\gamma}) \right|.$$
Note that
\[
\sum_{N < n \leq N_1, n \equiv a \mod d} \Lambda(n) e(\theta hn^\gamma) = \frac{1}{d} \sum_{m=1}^{d} \sum_{N < n \leq N_1} \Lambda(n) e\left(\theta hn^\gamma + \frac{(n-a)m}{d}\right).
\]
Hence we need to bound
\[
T := \sum_{N < n \leq N_1} \Lambda(n) e\left(\theta hn^\gamma + \frac{nm}{d}\right).
\]
We apply Lemmas 2.3, 2.7 and 2.9 with \((\mu, m, \rho) \rightarrow (\theta, h, \frac{m}{d})\) and obtain
\[
(3.2) \quad TN^{-\epsilon} \ll h^{1/2}N^{3/7+\gamma/2} + h^{1/6}N^{16/21+\gamma/6} + N^{25/28} + h^{-1/4}N^{1-\gamma/4},
\]
for \(\epsilon\) being a small positive number.

Now we work on the upper bound of \(S_2\). The contribution from \(h = 0\) is
\[
(3.3) \quad 2b_0 \sum_{N < n \leq N_1, n \equiv a \mod d} \Lambda(n) \ll \frac{b_0 N}{\varphi(d)} \ll H^{-1}N,
\]
where the function \(\varphi(d)\) is the Euler’s totient function and \(b_h \ll H^{-1}\). Taking into account that
\[
(n - \beta + 1)^\gamma = n^\gamma + O(n^{\gamma-1})
\]
and \(\gamma - 1 < 0\), the contribution from \(h \neq 0\) is
\[
(3.4) \quad \ll H^{-1} \max_{N_1 \leq 2N} \sum_{0 < h \leq H} \left| \sum_{N < n \leq N_1, n \equiv a \mod d} \Lambda(n) e(\theta hn^\gamma) \right|.
\]
The right-hand side of (3.4) can be estimated by the same method of (3.2). Therefore, inserting (3.2) into (3.1) and (3.4), and combining with (3.3), it follows that
\[
SN^{-\epsilon} \ll S_1 + S_2
\]
\[
\ll H^{3/2}N^{3\gamma/2-4/7} + H^{7/6}N^{7\gamma/6-5/21} + HN^{\gamma-3/28} + H^{3/4}N^{3\gamma/4}
\]
\[
+ H^{1/2}N^{3/7+\gamma/2} + H^{1/6}N^{16/21+\gamma/6} + N^{25/28} + H^{-1/4}N^{1-\gamma/4} + H^{-1}N
\]
holds for any \(H \geq 1\). By Lemma 2.5, we get that
\[
SN^{-\epsilon} \ll N^{3\gamma/5+13/35} + N^{7\gamma/13+3/7} + N^{\gamma/2+25/56} + N^{\gamma/3+13/21} + N^{\gamma/7+13/14} + N^{25/28}.
\]
Note that \(\sum_1 \ll x^\gamma\), so we need that \(S \ll x^{\gamma-\epsilon}\). Hence
\[
\gamma > \max\left(\frac{13}{14}, \frac{25}{28}\right) = \frac{13}{14},
\]
and
\[
S \ll x^{3\gamma/5+13/35+\epsilon}.
\]
4. Sketch of proof of Theorem 1.3

We sketch the proof of Theorem 1.3 because the idea of the proof is close to the proof in [2], the proof of [6, Theorem 7] or the proof of Theorem 1.1. We only give the changes that are necessary for our Theorem 1.3.

We set
\[ \vartheta(x; d, a) := \sum_{p \leq x} \log p \]
and consider a weighted counting function
\[ \vartheta_{\alpha, \beta, c}(x; d, a) := \sum_{p \leq x} \log p \]
\[ \left\lfloor -\theta(p - \beta)^\gamma \right\rfloor - \left\lfloor -\theta(p + 1 - \beta)^\gamma \right\rfloor \log p. \]

By a similar argument as in the proof of Theorem 1.1, we conclude that

**Theorem 4.1.** Let \( \alpha \geq 1 \) and \( \beta \) be real numbers. Let \( c \in (1, 14/13) \). Then
\[ \vartheta_{\alpha, \beta, c}(x; d, a) = \alpha^{-1/c} \gamma x^{\gamma - 1} \vartheta(x; d, a) \]
\[ + \alpha^{-1/c} \gamma (1 - \gamma) \int_2^x u^{\gamma - 2} \vartheta(u; d, a) \, du + O(x^{3\gamma/5 + 13/35 + \varepsilon}). \]

The Brun–Titchmarsh theorem states that for \( d < x^{1-\varepsilon} \), there is some \( C > 0 \) such that
\[ \pi(x; d, a) \leq \frac{Cx}{\varphi(d) \log x}. \]

We also give a Brun–Titchmarsh bound for the primes in the generalized Piatetski–Shapiro sequences.

**Corollary 4.2.** Let \( \alpha \geq 1 \) and \( \beta \) be real numbers. Let \( c \in (1, 14/13) \) and \( A \in (0, -13/35 + 2\gamma/5) \). There is a number \( C = C(\alpha, c, A) > 0 \) such that
\[ \pi_{\alpha, \beta, c}(x; d, a) \leq \frac{Cx^{\gamma}}{\varphi(d) \log x} \]
if \( (a, d) = 1 \) and \( 1 \leq d \leq x^A \).

**Proof.** Let \( \varepsilon > 0 \) be chosen so that
\[ \max \left( 2A\gamma; \frac{3}{5} \gamma + \frac{13}{35} + \varepsilon \right) \leq \gamma - A - \varepsilon. \]
Then by Theorem 1.1 we have

\[
\pi_{\alpha,\beta,c}(x; d, a) \ll \alpha^{-1/c} x^{\gamma-1} \pi(x; d, a) \\
+ \alpha^{-1/\gamma}(1 - \gamma) \int_{2}^{x} u^{\gamma-2} \pi(u; d, a) \, du + x^{\gamma-A-\varepsilon}
\]

where the implied constant depends on \(c, \alpha, A\). If \(1 \leq d \leq x^A\), then

\[
x^{\gamma-A-\varepsilon} \ll \frac{x^{\gamma-A}}{\log x} \leq \frac{x^{\gamma}}{\varphi(d) \log x}.
\]

Applying the Brun–Titchmarsh theorem, we prove Corollary 4.2.

The following statement analogous to [2, Theorem 2.1] and [6, Lemma 28] is important in the construction of Carmichael numbers.

**Lemma 4.3.** Let \(\alpha \geq 1\) and \(\beta\) be real numbers. Let \(c \in (1, 14/13)\) and \(B \in (0, -13/35 + 3\gamma/5)\). There exist numbers \(\eta > 0\), \(x_0\) and \(D\) such that for all \(x \geq x_0\) there is a set \(D(x)\) consisting of at most \(D\) integers such that

\[
\left| \vartheta_{\alpha,\beta,c}(x; d, a) - \frac{x\gamma}{\varphi(d)} \right| \leq \frac{x\gamma}{2\varphi(d)}
\]

provided that

1. \(d\) is not divisible by any element of \(D(x)\);
2. \(1 \leq d \leq x^B\);
3. \(\gcd(a, d) = 1\).

Every number in \(D(x)\) exceeds \(\log x\), and all, but at most one, exceeds \(x^\eta\).

**Sketch of proof.** We set

\[
\vartheta_c(x; d, a) := \sum_{\substack{p \leq x \\ \ \ \ \ \ p \in \mathbb{N}^\prime(c) \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ p \equiv d \mod d}} \log p.
\]

By Theorem 26 in [6], we conclude that

\[
\vartheta_{\alpha,\beta,c}(x; d, a) \sim \theta \vartheta_c(x; d, a).
\]

Replacing the factor \(17/39 + 7\gamma/13 + \varepsilon\) in the proof of Lemma 28 in [6] by \(13/35 + 3\gamma/5 + \varepsilon\) in this case, the proof of Lemma 4.3 in this paper is a straightforward reworking of the proof of Lemma 28 in [6].

By Lemma 4.3, we extend [2, Theorem 3.1] to the setting of the primes in the generalized Piatetski–Shapiro sequence.
Lemma 4.4. Let \( \alpha \geq 1 \) and \( \beta \) be real numbers. Let \( c \in (1, 14/13) \) and let \( A, B, B_1 \) be positive real numbers such that \( B_1 < B < A < -13/35 + 2\gamma/5 \). Let \( C > 0 \) have the property described in Corollary 4.2. There exists a number \( x_2 \) such that if \( x \geq x_2 \) and \( L \) is a squarefree integer not divisible by any prime \( q \) exceeding \( x^{(A-B)/2} \) and for which
\[
\sum_{\text{prime } q | L} \frac{1}{q} \leq \frac{1 - A}{16C},
\]
then there is a positive integer \( k \leq x^{1-B} \) with \( \gcd(k, L) = 1 \) such that
\[
\#\{d | L: dk + 1 \leq x \text{ and } p = dk + 1 \text{ is a prime in } \mathcal{N}_{\alpha,\beta}^{(c)}\} \geq 2 - D - 2(x^{1-B+\gamma-1}) - 1 \log x \sum_{\text{prime } q | L'} \frac{1}{q} \#\{d | L: x^{B_1} \leq d \leq x^B\},
\]
where \( D \) is chosen as in Lemma 4.3.

Sketch of proof. We follow the proof of [6, Lemma 29] and use the notation of [2, Theorem 3.1]. By the same argument we have
\[
\pi_{\alpha,\beta,c}(dx^{1-B}; d, 1) \geq \frac{\theta (dx^{1-B})^\gamma}{2 \phi(d) \log x}, \quad d | L', \ 1 \leq d \leq x^B
\]
and
\[
\pi_{\alpha,\beta,c}(dx^{1-B}; dq, 1) \leq \frac{4\theta C}{q(1 - A)} (dx^{1-B})^\gamma, \quad 1 \leq d \leq x^B
\]
for every prime \( q \) dividing \( L' \). The rest of the proof stays the same as the proof of [6, Lemma 29] by considering primes in \( \mathcal{N}_{\alpha,\beta}^{(c)} \) instead of primes in \( \mathcal{N}^{(c)} \).

Let \( \pi(x, y) \) be the number of those primes for which \( p - 1 \) is free of prime factors exceeding \( y \). Let \( E \) be the set of numbers \( E \) in the range \( 0 < E < 1 \) for which
\[
\pi(x, x^{1-E}) \geq x^{1+o(1)}, \quad x \to \infty,
\]
where the function implied by \( o(1) \) depends only on \( E \). By a similar argument as in [6, pp. 64-66], we conclude the following statement.

Lemma 4.5. Let \( \alpha \geq 1 \) and \( \beta \) be real numbers. Let \( c \in (1, 49/48) \). Let \( B, B_1 \) be positive real numbers such that \( B_1 < B < -13/35 + 2\gamma/5 \). For any \( E \in E \) there is a number \( x_3 \) depending on \( c, B, B_1, E \) and \( \varepsilon \), such that for any \( x \geq x_1 \) there are at least \( x^{EB+(1-B+B_1)(\gamma-1)\varepsilon} \) Carmichael numbers up to \( x \) composed solely of primes from \( \mathcal{N}_{\alpha,\beta}^{(c)} \).

Taking \( B \) and \( B_1 \) arbitrarily close to \(-13/35 + 2\gamma/5 \), Lemma 4.5 implies that there are infinitely many Carmichael numbers composed entirely of the primes from \( \mathcal{N}_{\alpha,\beta}^{(c)} \) with
\[
\left(-\frac{13}{35} + \frac{2}{5}\gamma\right)E + \gamma - 1 > 0.
\]
Taking \( E = 0.7039 \) from [8], we eventually have \( \gamma > 63/64 \).
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