A Hamiltonian treatment of stimulated Brillouin scattering in nanoscale integrated waveguides — Supplementary Material

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S.I. HEISENBERG EQUATIONS FOR ACOUSTIC FIELD

In this section, we derive the equations of motion (8) in section II of the main paper. Using Eqs. (91) and (7) we can derive the relations

\[
\frac{\partial \pi^n(r, t)}{\partial t} = \frac{1}{\mathcal{I}} \left[ \pi^n(r), H \right]
\]

\[
= \frac{1}{\mathcal{I}} \int \frac{\left[ u^n(r, \pi^i(r')) \right] \pi^i(r')}{2\rho(r')} \, dr' + \frac{1}{\mathcal{I}} \int \frac{\pi^i(r') \left[ u^n(r), \pi^i(r') \right]}{2\rho(r')} \, dr'
\]

\[
= \delta^{ni} \int \frac{\delta(r - r') \pi^i(r')}{2\rho(r')} \, dr' + \delta^{ni} \int \frac{\pi^i(r') \delta(r - r')}{} \, dr'
\]

\[
= \frac{\pi^n(r)}{\rho(r)}
\]

and

\[
\frac{\partial \pi^n(r, t)}{\partial t} = \frac{1}{\mathcal{I}} \left[ \pi^n(r), H \right]
\]

\[
= \frac{1}{\mathcal{I}} \frac{1}{2} \int \frac{\partial \left[ \pi^n(r), u^j(r') \right]}{\partial r_j} \frac{\partial u^k(r')}{\partial r_l} \, dr' + \frac{1}{\mathcal{I}} \frac{1}{2} \int \frac{\partial u^i(r')}{\partial r_j} \frac{\partial \left[ \pi^n(r), u^k(r') \right]}{\partial r_l} \, dr'
\]

\[
= -\frac{1}{\mathcal{I}} \delta^{ni} \int \frac{\partial \delta(r - r')}{\partial r_j} \frac{\partial u^k(r')}{\partial r_l} \, dr' - \frac{1}{2} \delta^{nk} \int \frac{\partial u^i(r')}{} \frac{\partial \delta(r - r')}{} \, dr'
\]

\[
= \frac{1}{2} \frac{\partial}{\partial r_j} \left( c^{ijkl}(r) \frac{\partial u^k(r)}{\partial r_l} \right) + \frac{1}{2} \frac{\partial}{\partial r_j} \left( c^{ijkl}(r) \frac{\partial u^k(r)}{\partial r_l} \right)
\]

\[
= \frac{\partial}{\partial r_j} \left( c^{ijkl}(r) S^{kl}(r) \right),
\]

where in the second last line we have used (5).

S.II. OPERATOR $\mathcal{M}^{nk}(r)$ IS HERMITIAN

Here we show that the operator $\mathcal{M}^{nk}(r)$ of (15) is Hermitian. Consider an integral over an appropriate volume and assume that fields are either periodic over the volume or vanish at the surface of the volume. Then for vector functions $\mathbf{C}(r)$ and $\mathbf{D}(r)$, integrating by parts twice gives

\[
\int (D^n(r))^* (\mathcal{M}^{nk}(r) C^k(r)) \, dr = -\int \frac{(D^n(r))^*}{\rho^{1/2}(r)} \frac{\partial}{\partial r_j} \left( c^{ijkl}(r) \frac{\partial \left( C^k(r) \rho^{1/2}(r) \right)}{\partial r_l} \right) \, dr
\]

\[
= \int \left( \frac{\partial}{\partial r_j} \left( \frac{(D^n(r))^*}{\rho^{1/2}(r)} \right) \right) \left( c^{ijkl}(r) \frac{\partial \left( C^k(r) \rho^{1/2}(r) \right)}{\partial r_l} \right) \, dr
\]

\[
= -\int \frac{C^k(r)}{\rho^{1/2}(r)} \frac{\partial}{\partial r_j} \left( c^{ijkl}(r) \frac{\partial \left( (D^n(r))^* \rho^{1/2}(r) \right)}{\partial r_l} \right) \, dr
\]

Now using (5) we put $c^{ijkl}(r) = c^{klnj}(r)$ and switching the dummy indices $j \leftrightarrow l$, we can write this as

\[
\int (D^n(r))^* (\mathcal{M}^{nk}(r) C^k(r)) \, dr = -\int \frac{C^k(r)}{\rho^{1/2}(r)} \frac{\partial}{\partial r_j} \left( c^{klnj}(r) \frac{\partial \left( (D^n(r))^* \rho^{1/2}(r) \right)}{\partial r_l} \right) \, dr
\]

\[
= \left( -\int \frac{(C^k(r))^*}{\rho^{1/2}(r)} \frac{\partial}{\partial r_j} \left( c^{klnj}(r) \frac{\partial \left( (D^n(r))^* \rho^{1/2}(r) \right)}{\partial r_l} \right) \, dr \right)^*
\]

\[
= \left( \int (C^k(r))^* (\mathcal{M}^{kn}(r) D^n(r)) \, dr \right)^*.
\]
and so the differential operator $\mathcal{M}^{nk}(r)$ is Hermitian.

S.III. PROPERTIES OF MODE FUNCTIONS AND PARTNER FUNCTIONS

Here we establish some useful properties of the mode functions $F_\Lambda(r)$ introduced in (17) to (20), that are required to reduce the acoustic Hamiltonian to canonical harmonic oscillator form in S.IV.

Note that since $\mathcal{M}^{nk}(r)$ is real, if $F_\Lambda(r)$ is an eigenfunction then $\bar{F}_\Lambda(r)$ is also an eigenfunction with the same eigenvalue $\omega_\Lambda$. This may happen simply because $F_\Lambda(r)$ is purely real. In fact, as we show in S.XI, it is always possible to choose the set of eigenfunctions $\{F_\Lambda(r)\}$ such that each of them is purely real. But it is often more convenient to work with complex eigenfunctions (traveling waves rather than standing waves, for example). Section S.XI establishes that if we include complex eigenfunctions in the set $\{F_\Lambda(r)\}$, the set can be chosen so that each eigenfunction $F_\Lambda(r)$ is either real or, if not, there is another eigenfunction $\bar{F}_\Lambda(r)$ in the set such that $F_\Lambda(r) = F_\Lambda^*(r)$.

Typically the naturally chosen set of eigenfunctions will make this so; for example, if $F_\Lambda(r)$ is a traveling wave to the right, then $\bar{F}_\Lambda(r)$ is a traveling wave to the left. We refer to $\bar{F}_\Lambda(r)$ as the “partner” of $F_\Lambda(r)$. That is, each complex eigenfunction in the set has a partner that is also in the set. If there are purely real eigenfunctions in the set $\{F_\Lambda(r)\}$, we take them to be their own partners. Then the set of eigenfunctions $\{F_\Lambda(r)\}$ is equivalent to the set of $\{\bar{F}_\Lambda(r)\}$ of partner eigenfunctions, and $F_\Lambda(r) = F_\Lambda^*(r)$ for each $\Lambda$. Since the set of partners is equivalent to the original set, then from Eq. (20) of the main paper we can also write

$$\int F_\Lambda^*(r) \cdot F_\Lambda(r) \, dr = \delta_{\Lambda\overline{\Lambda}},$$

and

$$\sum_{\Lambda} F^n_\Lambda(r) (F^m_\Lambda(r'))^* = \delta^{nm} \delta(r-r').$$

Similarly, from Eq. (18) of the main paper we see that we have

$$\tilde{U}_\Lambda(r) = \tilde{U}_\Lambda^*(r),$$

$$\tilde{\Pi}_\Lambda(r) = -\tilde{\Pi}_\overline{\Lambda}(r).$$

S.IV. ACOUSTIC MODE EXPANSION

Here we show that through use of the partner functions introduced in S.III, the acoustic field operators and Hamiltonian can be expanded in terms of the mode functions as expressed in Eqs. (22) and (23). Recall that the $\{F_\Lambda(r)\}$ are the eigenfunctions of (15) and that the $\{\tilde{U}_\Lambda(r)\}$ and $\{\tilde{\Pi}_\Lambda(r)\}$ are defined as in Eq. (18). Both sets of functions are proportional to the $\{F_\Lambda(r)\}$, so we can take each to constitute a complete set of states. We can then expand

$$\tilde{u}(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} C^{(1)}_\Lambda \tilde{U}_\Lambda(r),$$

$$\tilde{\pi}(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} C^{(2)}_\Lambda \tilde{\Pi}_\Lambda(r),$$

where $C^{(1)}_\Lambda$ and $C^{(2)}_\Lambda$ are operators and the factors $\sqrt{\hbar/2\Omega_\Lambda}$ are added for later convenience.

In the Heisenberg picture, $C^{(1)}_\Lambda$ and $C^{(2)}_\Lambda$ are time-dependent, and therefore so are $\tilde{u}(r)$ and $\tilde{\pi}(r)$. However, since $\tilde{u}(r)$ and $\tilde{\pi}(r)$ are Hermitian the $\{C^{(1)}_\Lambda\}$ are not all independent, nor are the $\{C^{(2)}_\Lambda\}$. Using Eqs. (S.3) we have

$$\tilde{u}^\dagger(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} (C^{(1)}_\Lambda)^\dagger \tilde{U}^\dagger_\Lambda(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} (c^{(1)}_\Lambda)^\dagger \tilde{U}^\dagger_\Lambda(r),$$

$$\tilde{\pi}^\dagger(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} (C^{(2)}_\Lambda)^\dagger \tilde{\Pi}^\dagger_\Lambda(r) = -\sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} (\tilde{\Pi}^{(2)}_\Lambda)^\dagger \tilde{\Pi}^\dagger_\Lambda(r),$$
so that (S.4) may also be written as a sum over partner modes,

\[ \tilde{u} = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_{\Lambda}}} C^{(1)}_{\Lambda} \tilde{U}_{\Lambda}(r), \]

\[ \tilde{\pi} = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_{\Lambda}}} C^{(2)}_{\Lambda} \tilde{\Pi}_{\Lambda}(r), \]

where we have used \( \Omega_{\bar{\Lambda}} = \Omega_{\Lambda} \). Then from the Hermiticity of the canonical fields we see that we require

\[ \left( C^{(1)}_{\Lambda} \right)^\dagger = C^{(1)}_{\bar{\Lambda}}, \]

\[ -\left( C^{(2)}_{\Lambda} \right)^\dagger = C^{(2)}_{\bar{\Lambda}}, \]

which may be satisfied by setting

\[ C^{(1)}_{\Lambda} = b_{\Lambda} + b_{\Lambda}^\dagger, \]

\[ C^{(2)}_{\Lambda} = b_{\Lambda} - b_{\Lambda}^\dagger. \]

For real eigenfunctions \( F_{\Lambda}(r) \) we have \( \bar{\Lambda} = \Lambda \) and this just says that \( C^{(1)}_{\Lambda} \) is (proportional to) a coordinate operator, and \( C^{(2)}_{\Lambda} \) is (proportional to) a momentum operator (as we will see). For partners, \( b_{\Lambda} \) and \( b_{\bar{\Lambda}} \) are independent operators (or independent amplitudes in the classical case). Using (S.5) in (S.4) we have

\[ \tilde{u}(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_{\Lambda}}} \left( b_{\Lambda} + b_{\Lambda}^\dagger \right) F_{\Lambda}(r) \]  

(S.6)

where in the third line we use the fact that the sum is over the whole set of partner functions. Similarly

\[ \tilde{\pi}(r) = \sum_{\Lambda} \left( -i \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} \right) \left( b_{\Lambda} - b_{\Lambda}^\dagger \right) F_{\Lambda}(r) \]  

(S.7)

\[ = -i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b_{\Lambda} F_{\Lambda}(r) + i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b_{\Lambda}^\dagger F_{\bar{\Lambda}}^*(r) \]

\[ = -i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b_{\Lambda} F_{\Lambda}(r) + i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b_{\Lambda}^\dagger F_{\bar{\Lambda}}^*(r) \]

\[ = -i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b_{\Lambda} F_{\Lambda}(r) + \text{h.c.} \]  

(S.8)

in accordance with Eq. (22).

Postulating the commutation relations

\[ [b_{\Lambda}, b_{\Lambda'}] = 0, \]

\[ [b_{\Lambda}, b_{\Lambda'}^\dagger] = \delta_{\Lambda\Lambda'}, \]

(S.9)
we find

\[ [\hat{u}^m(r), \pi^m(r')] = i \sum_{\Lambda, \Lambda'} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} \sqrt{\frac{\hbar\Omega_{\Lambda'}}{2}} \left[ b_\Lambda, b^\dagger_{\Lambda'} \right] F^n_\Lambda(r) (F^m_{\Lambda'}(r'))^* \]

\[ -i \sum_{\Lambda, \Lambda'} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} \sqrt{\frac{\hbar\Omega_{\Lambda'}}{2}} \left[ b^\dagger_{\Lambda'}, b_\Lambda \right] (F^n_\Lambda(r))^* F^m_{\Lambda'}(r') \]

\[ = \frac{i\hbar}{2} \sum_{\Lambda} F^n_\Lambda(r) (F^m_{\Lambda}(r'))^* + \frac{i\hbar}{2} \sum_{\Lambda} (F^n_\Lambda(r))^* F^m_{\Lambda}(r') \]

\[ = \frac{i\hbar}{2} \sum_{\Lambda} F^n_\Lambda(r) (F^m_{\Lambda}(r'))^* + \frac{i\hbar}{2} \sum_{\Lambda} F^n_{\Lambda}(r) (F^m_{\Lambda}(r'))^* \]

\[ = i\hbar \delta^{mn} \delta(r - r'), \tag{S.10} \]

where we have used (20) and (S.2). Since \([\hat{u}^m(r), \pi^m(r)] = [u^m(r), \pi^m(r)]\) we recover the starting commutation relations (2). It is possible but more complicated to show that demanding the result (S.10) one can find that the \(b_\Lambda\) and \(b^\dagger_{\Lambda}\) must satisfy (S.9).

Now we look at the Hamiltonian in terms of the \(b_\Lambda\) and \(b^\dagger_{\Lambda}\). From (16) we have

\[ H^\Lambda = \frac{1}{2} \int \hat{\pi}^i(r) \hat{\pi}^i(r) \, dr + \frac{1}{2} \int \hat{\pi}^i(r) M^{ik}(r) \hat{u}^k \, dr. \]

From the above we have

\[ \hat{\pi}^i(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} b_\Lambda F^i_\Lambda(r) + \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} b^\dagger_{\Lambda} (F^i_{\Lambda}(r))^*, \]

\[ \hat{\pi}^i(r) = -i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b_\Lambda F^i_\Lambda(r) + i \sum_{\Lambda} \sqrt{\frac{\hbar\Omega_{\Lambda}}{2}} b^\dagger_{\Lambda} (F^i_{\Lambda}(r))^*, \]

so

\[ \int \hat{\pi}^i(r) \hat{\pi}^i(r) \, dr = - \sum_{\Lambda, \Lambda'} \frac{\hbar}{2} \sqrt{\Omega_\Lambda \Omega_{\Lambda'}} b_\Lambda b_{\Lambda'} \int F^i_\Lambda(r) F^i_{\Lambda'}(r) \, dr \]

\[ - \sum_{\Lambda, \Lambda'} \frac{\hbar}{2} \sqrt{\Omega_\Lambda \Omega_{\Lambda'}} b^\dagger_{\Lambda'} b^\dagger_{\Lambda} \left( \int F^i_{\Lambda'}(r) F^i_\Lambda(r) \, dr \right)^* \]

\[ + \sum_{\Lambda, \Lambda'} \frac{\hbar}{2} \sqrt{\Omega_\Lambda \Omega_{\Lambda'}} b_\Lambda b_{\Lambda'} \int F^i_\Lambda(r) (F^i_{\Lambda'}(r))^* \, dr \]

\[ + \sum_{\Lambda, \Lambda'} \frac{\hbar}{2} \sqrt{\Omega_\Lambda \Omega_{\Lambda'}} b^\dagger_{\Lambda} b^\dagger_{\Lambda'} \left( \int F^i_{\Lambda'}(r) (F^i_{\Lambda'}(r))^* \, dr \right)^*. \]

In the last two terms, orthogonality gives \(\Lambda' = \Lambda\). In the first two, we replace the sum over \(\Lambda\) by a sum over \(\bar{\Lambda}\) and use the fact that \(F_{\Lambda}(r) = F^\Lambda_{\bar{\Lambda}}(r)\); then orthogonality demands that \(\Lambda' = \Lambda'\). Recalling that \(\Omega_{\Lambda} = \Omega_{\bar{\Lambda}}\) we then have

\[ \int \hat{\pi}^i(r) \hat{\pi}^i(r) \, dr = - \frac{1}{2} \sum_{\Lambda} \hbar \Omega_{\Lambda} b_\Lambda b_{\Lambda} - \frac{1}{2} \sum_{\Lambda} \hbar \Omega_{\Lambda} b^\dagger_{\Lambda} b^\dagger_{\Lambda} \]

\[ + \frac{1}{2} \sum_{\Lambda} \hbar \Omega_{\Lambda} b_\Lambda b_{\Lambda} + \frac{1}{2} \sum_{\Lambda} \hbar \Omega_{\Lambda} b^\dagger_{\Lambda} b^\dagger_{\Lambda}. \]

Then since

\[ M^{ik}(r) \hat{u}^k(r) = \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} \Omega_\Lambda b_\Lambda F^i_\Lambda(r) + \sum_{\Lambda} \sqrt{\frac{\hbar}{2\Omega_\Lambda}} \Omega_\Lambda b^\dagger_{\Lambda} (F^i_{\Lambda}(r))^*, \]
we have

\[
\int \tilde{u}_i(r)M^{ik}(r)\tilde{u}^k(r)\,dr = \sum_{\lambda,\Lambda} \frac{\hbar}{2\sqrt{\Omega_{\lambda}\Omega_{\Lambda}}} \Omega_{\Lambda}^2 b_{\lambda} b_{\Lambda} \int F^i_{\lambda}(r)F^k_{\Lambda}(r)\,dr
+ \sum_{\lambda,\Lambda} \frac{\hbar}{2\sqrt{\Omega_{\lambda}\Omega_{\Lambda}}} \Omega_{\Lambda}^2 b^\dagger_{\lambda} b^\dagger_{\Lambda} \left( \int F^i_{\lambda}(r)F^k_{\Lambda}(r)\,dr \right)^* \\
+ \sum_{\lambda,\Lambda} \frac{\hbar}{2\sqrt{\Omega_{\lambda}\Omega_{\Lambda}}} \Omega_{\Lambda}^2 b_{\lambda} b^\dagger_{\Lambda} \int F^i_{\lambda}(r)(F^k_{\Lambda}(r))^*\,dr
+ \sum_{\lambda,\Lambda} \frac{\hbar}{2\sqrt{\Omega_{\lambda}\Omega_{\Lambda}}} \Omega_{\Lambda}^2 b^\dagger_{\lambda} b^\dagger_{\Lambda} \left( \int F^i_{\lambda}(r)(F^k_{\Lambda}(r))^*\,dr \right)^*.
\]

Using the same strategy as above this gives

\[
\int \tilde{u}_i(r)M^{ik}(r)\tilde{u}^k(r)\,dr = \frac{1}{2} \sum_{\lambda} \hbar \Omega_{\lambda} (b_{\lambda} b^\dagger_{\lambda} + b^\dagger_{\lambda} b_{\lambda}) \\
+ \frac{1}{2} \sum_{\lambda} \hbar \Omega_{\lambda} b_{\lambda} b^\dagger_{\lambda} + \frac{1}{2} \sum_{\lambda} \hbar \Omega_{\lambda} b^\dagger_{\lambda} b_{\lambda}.
\]

Combining (S.11, S.11) we have

\[
H = \frac{1}{2} \sum_{\lambda} \hbar \Omega_{\lambda} \left( b_{\lambda} b^\dagger_{\lambda} + b^\dagger_{\lambda} b_{\lambda} + \frac{1}{2} \right).
\]

**S.V. THE GROUP VELOCITY**

Here we work out the group velocity of the acoustic modes in terms of the modal field providing an explicit expression for Eq. (87a), a result which is needed in the next section. We take the continuous limit of Eq. (26), writing

\[
\mathcal{F}_{aq}(r) = \sqrt{\ell} \mathcal{F}_{aq}(r) = \mathbf{F}_{aq}(x, y)e^{iqz}.
\]  

(S.11)

From (17) we have

\[
\Omega^2_{aq} \mathcal{F}^n_{aq}(r) = M^{nk}(r)\mathcal{F}^k_{aq}(r) = -\frac{1}{\sqrt{\rho(x, y)}} \frac{\partial}{\partial r^j} \left( c_{njkl}(x, y) \frac{\partial}{\partial r^i} \left( \frac{\mathcal{F}^k_{aq}(r)}{\sqrt{\rho(x, y)}} \right) \right).
\]  

(S.12)

It is helpful to re-express $M^{nk}$ in terms of an operator $\mathcal{L}^n_q$ operating on the transverse spatial variables only. Applying $M^{nk}$ to the mode (S.11) gives

\[
[M^{nk} [f^{k}_{aq} e^{iqz}]] e^{-iqz} = -\left[ \frac{1}{\sqrt{\rho(x, y)}} \frac{\partial}{\partial r^j} \left( c_{njkl}(x, y) \frac{\partial}{\partial r^i} \left( \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho(x, y)}} \right) \right) \right] e^{-iqz}
\]  

(S.13)

\[
= -\left[ \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial r^j} \left( c_{njkl}(x, y) \frac{\partial}{\partial r^i} \left( \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} \right) \right) \right] e^{-iqz}
\]

\[
= -\frac{1}{\sqrt{\rho}} \left( \frac{\partial}{\partial r^j} c_{njkl}(x, y) \frac{\partial}{\partial r^i} \left( \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} \right) \right) + c_{njkl}(x, y) \frac{\partial^2}{\partial r^i \partial r^j} \left( \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} \right) + i q c_{njkl}(x, y) \frac{\partial}{\partial r^i} \left( \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} \right)
\]

\[
+ iq \left( \frac{\partial}{\partial r^j} c_{njkl}(x, y) \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} + i q c_{njkl}(x, y) \frac{\partial}{\partial r^j} \left( \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} \right) - q^2 c_{njkl}(x, y) \frac{f^k_{aq} e^{iqz}}{\sqrt{\rho}} \right)
\]

\[
= \mathcal{L}^n_q [f^k_{aq}],
\]
where the last line defines the action of the operator $\mathcal{L}_{aq}^n$ on $f_{aq}(x, y)$. It follows from the Hermiticity of $\mathcal{M}_{aq}^n$ that $\mathcal{L}_{aq}^n$ is Hermitian with respect to integration over the transverse plane. We can then write

$$\Omega_{aq}^2 f_{aq}^n(x, y) = \mathcal{L}_{aq}^n f_{aq}^n(x, y),$$

(S.14)

so that the $f_{aq}^n$ are eigenfunctions of $\mathcal{L}_{aq}^n$ and may be taken as orthogonal.

Taking the inner product with $(f_{aq}^n)^*$ and using the orthogonality of the $f_{aq}$, we have

$$\Omega_{aq}^2 = \int dx dy (f_{aq}^n(x, y))^* \mathcal{L}_{aq}^n f_{aq}^n(x, y).$$

(S.15)

Differentiating with respect to $q$ gives

$$2\Omega_{aq} \frac{d\Omega_{aq}}{dq} = \frac{d}{dq} \int dx dy (f_{aq}^n(x, y))^* \mathcal{L}_{aq}^n f_{aq}^n(x, y).$$

(S.16)

Since $\mathcal{L}_{aq}^n$ is Hermitian, we may invoke the Hellmann-Feynman theorem to simplify the right hand side:

$$2\Omega_{aq} \frac{d\Omega_{aq}}{dq} = \int dx dy \left[ (f_{aq}^n)^* \frac{\partial}{\partial q} \mathcal{L}_{aq}^n f_{aq}^n(x, y) \right] = \int dx dy \left[ \frac{\partial}{\partial q} \left( q c_{nzkj} f_{aq}^n \sqrt{\rho} \right) - i c_{nzkj} \frac{\partial f_{aq}^n}{\partial q} \right] .$$

(S.17)

Then the group velocity

$$v_{aq} = \frac{d\Omega_{aq}}{dq} \bigg|_{q' = q},$$

(S.18)

is given by

$$v_{aq} = \frac{1}{\Omega_{aq}} \int dx dy \left[ (f_{aq}^n)^* c_{nzkj} \frac{f_{aq}^k}{\sqrt{\rho}} - i \frac{\partial}{\partial q} \left( c_{nzkj} \frac{f_{aq}^k}{\sqrt{\rho}} + c_{nzkj} \left( \frac{\partial f_{aq}^n}{\partial q} \right) \right) \right]$$

$$= \frac{1}{\Omega_{aq}} \int dx dy \left( c_{nzkj} \frac{f_{aq}^k}{\sqrt{\rho}} - \frac{i}{2\Omega_{aq}} \int dx dy \left( \frac{\partial}{\partial q} (f_{aq}^n)^* \frac{f_{aq}^k}{\sqrt{\rho}} \right) - \frac{i}{2\Omega_{aq}} \int dx dy \left( \frac{\partial}{\partial q} f_{aq}^n \frac{f_{aq}^k}{\sqrt{\rho}} \right) - i c_{nzkj} \left( \frac{\partial f_{aq}^n}{\partial q} \right) \right).$$

(S.19)

Swapping the dummy indices $n \leftrightarrow k$ in the second term and using (5) gives

$$v_{aq} = \frac{1}{\Omega_{aq}} \int dx dy \left[ (f_{aq}^n)^* c_{nzkj} \frac{f_{aq}^k}{\sqrt{\rho}} + i \frac{\partial}{\partial q} (f_{aq}^n)^* c_{nzkj} \frac{f_{aq}^k}{\sqrt{\rho}} \right]$$

$$= \frac{1}{\Omega_{aq}} \int dx dy \left[ c_{nzkj} \frac{f_{aq}^n}{\sqrt{\rho}} + \text{Re} \left[ \frac{i}{\Omega_{aq}} \int dx dy \left( \frac{\partial}{\partial q} (f_{aq}^n)^* c_{nzkj} \frac{f_{aq}^k}{\sqrt{\rho}} \right) \right] \right]$$

$$= q \Omega_{aq} \int dx dy \left( u_{aq}^n \right)^* c_{nzkj} u_{aq}^k + \text{Re} \left[ i \Omega_{aq} \int dx dy \left( \frac{\partial}{\partial q} (u_{aq}^n)^* c_{nzkj} u_{aq}^k \right) \right].$$

(S.20)

where the final line follows from (28).
S.VI. ACOUSTIC POWER FLOW

With an expression for the group velocity in terms of the modal field established in section S.V, we can now discuss the acoustic power.

Even in the presence of coupling the displacement to the electromagnetic fields, or other forces, we expect the first of (8a) still to hold,

$$\frac{\partial}{\partial t} \mathbf{u}(\mathbf{r}, t) = \frac{\pi(\mathbf{r}, t)}{\rho(\mathbf{r})},$$  \hfill (S.22)

Since in general the power density at a point in the medium in a direction \( \hat{n} \) is classically given by

$$P_n = -\frac{\partial u^i(\mathbf{r})}{\partial t} e^{ijklm}(\mathbf{r}) S^{lm}(\mathbf{r}) n^i,$$  \hfill (S.23)

the power in the waveguide in the \( \hat{z} \) direction, integrated over the \( xy \) plane, is

$$P_{z}(z) = -\int dx dy \frac{\pi^i(\mathbf{r})}{\rho(x,y)} e^{ijklm}(x,y) S^{lm}(\mathbf{r}) \partial_r u^i(\mathbf{r}) \bigg|_{\partial z}$$  \hfill (S.24)

$$= -\int dx dy \frac{\pi^i(\mathbf{r})}{\rho(x,y)} e^{ijklm}(x,y) \partial_r u^i(\mathbf{r}),$$  \hfill (S.25)

where the second line follows from the symmetry properties of the stiffness tensor.

We form the operator corresponding to the classical \( \pi^i(\mathbf{r}) \) and \( \partial u^i(\mathbf{r})/\partial r^m \), we obtain the operator \( P(z) \) by using the symmetrized version of the operators corresponding to \( \pi^i(\mathbf{r}) \) and \( \partial u^i(\mathbf{r})/\partial r^m \):

$$P(z) = -\frac{1}{2} \int dx dy \frac{\pi^i(\mathbf{r})}{\rho(x,y)} \left( \pi^i(\mathbf{r}) \frac{\partial u^l(\mathbf{r})}{\partial r^m} + \frac{\partial u^l(\mathbf{r})}{\partial r^m} \pi^i(\mathbf{r}) \right)$$

$$= -\frac{1}{2} \int dx dy \frac{\pi^i(\mathbf{r})}{\rho(x,y)} K^{ilm}(\mathbf{r}),$$

where we put

$$K^{ilm}(\mathbf{r}) = \pi^i(\mathbf{r}) \frac{\partial u^l(\mathbf{r})}{\partial r^m} + \frac{\partial u^l(\mathbf{r})}{\partial r^m} \pi^i(\mathbf{r}).$$  \hfill (S.26)

Using (31), we see that \( K^{ilm}(\mathbf{r}) \) has the form

$$K^{ilm}(\mathbf{r}) = \frac{\hbar}{4\pi} \sum_{\alpha, \alpha'} \int dq dq' \sqrt{\Omega_{\alpha q} \Omega_{\alpha' q'}} \kappa^{ilm}_{\alpha \alpha'}(q, q'),$$  \hfill (S.27)

where

$$\kappa^{ilm}_{\alpha \alpha'}(q, q') = \tilde{\kappa}^{ilm}_{\alpha \alpha'}(q, q') + \tilde{\kappa}^{ilm}_{\alpha \alpha'}(q, q').$$  \hfill (S.28)

The first term contains parts rapidly-varying in space and time:

$$\tilde{\kappa}^{ilm}_{\alpha \alpha'}(q, q') = b_{\alpha' q'} b_{\alpha q}$$

$$\times \left[ \left( \pi^i_{\alpha' q'}(x, y) e^{iq' z} \right) \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha q}(x, y) e^{iq z} \right) \right) + \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha' q'}(x, y) e^{iq' z} \right) \right) \left( \pi^i_{\alpha q}(x, y) e^{iq z} \right) \right]$$

$$+ b_{\alpha' q'}^{\dagger} b_{\alpha q}$$

$$\times \left[ \left( \pi^i_{\alpha' q'}(x, y) e^{iq' z} \right)^* \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha q}(x, y) e^{iq z} \right) \right)^* + \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha' q'}(x, y) e^{iq' z} \right) \right)^* \left( \pi^i_{\alpha q}(x, y) e^{iq z} \right)^* \right],$$

and \( \tilde{\kappa}^{ilm}_{\alpha \alpha'}(q, q') \) contains the slowly-varying terms,

$$\tilde{\kappa}^{ilm}_{\alpha \alpha'}(q, q') = b_{\alpha' q'} b_{\alpha q}$$

$$\times \left[ \left( \pi^i_{\alpha' q'}(x, y) e^{iq' z} \right)^* \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha q}(x, y) e^{iq z} \right) \right)^* + \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha' q'}(x, y) e^{iq' z} \right) \right)^* \left( \pi^i_{\alpha q}(x, y) e^{iq z} \right)^* \right]$$

$$+ b_{\alpha' q'}^{\dagger} b_{\alpha q}$$

$$\times \left[ \left( \pi^i_{\alpha' q'}(x, y) e^{iq' z} \right) \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha q}(x, y) e^{iq z} \right) \right) + \left( \frac{\partial}{\partial r^m} \left( u^l_{\alpha' q'}(x, y) e^{iq' z} \right) \right) \left( \pi^i_{\alpha q}(x, y) e^{iq z} \right) \right].$$
We write

$$P(z) = P_{rv}(z) + P_{sv}(z),$$

(S.29)

where $P_{rv}(z)$ contains the contributions from $\tilde{\kappa}_{\alpha\alpha'}^{lm}(q, q')$ and $P_{sv}(z)$ those from $\tilde{\kappa}_{\alpha\alpha'}^{lm}(q, q')$. Our interest is in the latter. Since the sums and integrals in (S.27) are over all $\alpha, \alpha', q$, and $q'$, we may switch the dummy indices in the second term on the right-hand-side of (S.29):

$$\tilde{\kappa}_{\alpha\alpha'}^{lm}(q, q') \rightarrow b_{\alpha,q'}^\dagger b_{\alpha,q} \left[ (\pi_{\alpha,q'}^i(x, y)e^{iq'z})^* \left( \frac{\partial}{\partial y_m} u_{\alpha,q}(x, y)e^{iqz} \right) + (\pi_{\alpha,q}^i(x, y)e^{iqz}) \left( \frac{\partial}{\partial y_m} u_{\alpha,q'}(x, y)e^{iq'z} \right) \right],$$

$$+ b_{\alpha,q} b_{\alpha',q'}^\dagger \left[ (\pi_{\alpha,q'}^i(x, y)e^{iqz})^* \left( \frac{\partial}{\partial y_m} u_{\alpha,q'}(x, y)e^{iq'z} \right) + (\pi_{\alpha',q'}^i(x, y)e^{iq'z}) \left( \frac{\partial}{\partial y_m} u_{\alpha,q}(x, y)e^{iqz} \right) \right].$$

Moving to normal-ordering with

$$b_{\alpha,q} b_{\alpha',q'}^\dagger = b_{\alpha',q'}^\dagger b_{\alpha,q} + \delta_{\alpha\alpha'} \delta(q - q'),$$

we have

$$\tilde{\kappa}_{\alpha\alpha'}^{lm}(q, q') = 2L_{\alpha\alpha'}^{lm}(q', q; x, y) b_{\alpha',q'}^\dagger b_{\alpha,q} + T_{\alpha\alpha'}^{lm}(q) \delta_{\alpha\alpha'} \delta(q' - q),$$

(S.31)

where

$$L_{\alpha\alpha'}^{lm}(q', q; x, y) = \left( \pi_{\alpha,q'}^i(x, y)e^{iq'z} \right)^* \left( \frac{\partial}{\partial y_m} u_{\alpha,q}(x, y)e^{iqz} \right) + \left( \pi_{\alpha,q}^i(x, y)e^{iqz} \right) \left( \frac{\partial}{\partial y_m} u_{\alpha,q'}(x, y)e^{iq'z} \right),$$

(S.32)

$$T_{\alpha\alpha'}^{lm}(q) = \left( \pi_{\alpha,q'}^i(x, y)e^{iqz} \right)^* \left( \frac{\partial}{\partial y_m} u_{\alpha,q}(x, y)e^{iq'z} \right) + \left( \pi_{\alpha,q}^i(x, y)e^{iq'z} \right) \left( \frac{\partial}{\partial y_m} u_{\alpha,q'}(x, y)e^{iqz} \right).$$

(S.33)

The term involving $T_{\alpha\alpha'}^{lm}(q)$ in Eq. (S.31) represents vacuum zero-point contributions and should give no net contribution to $P_{sv}(z)$, which is a directed quantity. Indeed, using Eqs. (28) and the property $f_{\alpha,q}(x, y) = (f_{\alpha,-q}(x, y))^*$, which follows from the Hermiticity of $\mathcal{M}_{\alpha\alpha'}^{mk}$ (see S.III), it can be shown that its contribution to Eq. (S.27) vanishes.

The remaining contribution to $P_{sv}$ can be written

$$P_{sv}(z) = \sum_{\alpha, \alpha'} \int \frac{dq dq'}{2\pi} b_{\alpha',q'}^\dagger b_{\alpha,q} e^{i(q-q')z} p_{\alpha\alpha'}^{A}(q', q),$$

(S.34)

where the pairwise term

$$p_{\alpha\alpha'}^{A}(q', q) = \frac{\hbar}{2} \sqrt{\Omega_{\alpha q} \Omega_{\alpha' q'}} \int dx dy \frac{c^{izk}}{\rho} L_{\alpha\alpha'}^{ikl}(q', q; x, y).$$

Evaluating the derivatives in Eq. (S.32),

$$L_{\alpha\alpha'}^{ikl}(q', q; x, y) = \delta_{iz} \left[ -q \left( f_{\alpha,q'}^i \frac{f_{\alpha,q}}{\Omega_{\alpha q}} - q' \left( f_{\alpha,q}^i \frac{f_{\alpha,q'}}{\Omega_{\alpha' q'}} \right) \right) + \left( \pi_{\alpha,q'}^i \right)^* \left( \frac{\partial}{\partial y_l} u_{\alpha q}^k \right) + \left( \frac{\partial}{\partial y_l} u_{\alpha q}^k \right)^* \left( \pi_{\alpha,q}^i \right) \right].$$

Using (28) we have

$$L_{\alpha\alpha'}^{ikl}(q', q; x, y) = \delta_{iz} \left[ -q \left( f_{\alpha,q'}^i \frac{f_{\alpha,q}}{\Omega_{\alpha q}} - \frac{q'}{\Omega_{\alpha' q'}} \left( f_{\alpha,q'}^i \right)^* \left( f_{\alpha,q} \right) \right) \right] + i\sqrt{\rho} \left( f_{\alpha,q'}^i \right)^* \left( \frac{\partial}{\partial y_l} \left( \frac{f_{\alpha,q}}{\Omega_{\alpha q} \sqrt{\rho}} \right) \right) - \left( \frac{\partial}{\partial y_l} \left( \frac{f_{\alpha,q'}^i}{\Omega_{\alpha' q'} \sqrt{\rho}} \right) \right)^* \left( f_{\alpha,q} \right).$$

Consequently,

$$p_{\alpha\alpha'}^{A}(q', q) = \frac{\hbar}{2} \sqrt{\Omega_{\alpha q'} \Omega_{\alpha q}} \int dx dy \frac{c^{izk}}{\rho} \left( f_{\alpha,q'}^i \right)^* \left( f_{\alpha,q} \right) + \frac{\hbar}{2} \sqrt{\Omega_{\alpha q} \Omega_{\alpha' q'}} \int dx dy \frac{c^{izk}}{\rho} \left( f_{\alpha,q} \right)^* \left( f_{\alpha,q'}^i \right) - \frac{i\hbar}{2} \sqrt{\Omega_{\alpha q'} \Omega_{\alpha q}} \int dx dy \frac{c^{izk}}{\sqrt{\rho}} \left( \frac{\partial}{\partial y_l} \left( \frac{f_{\alpha,q}}{\Omega_{\alpha q} \sqrt{\rho}} \right) \right) + \frac{i\hbar}{2} \sqrt{\Omega_{\alpha q} \Omega_{\alpha' q'}} \int dx dy \frac{c^{izk}}{\sqrt{\rho}} \left( \frac{\partial}{\partial y_l} \left( \frac{f_{\alpha,q'}^i}{\Omega_{\alpha' q'} \sqrt{\rho}} \right) \right)^*. $$
so that

\[ p^A_{\alpha\alpha}(q, q) = \frac{\hbar}{2} q \int dz \left( \frac{f_{\alpha q}^*}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} f^k_{\alpha q} \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) + \frac{\hbar}{2} q \int dz \left( \frac{f_{\alpha q}^*}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} f^k_{\alpha q} \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \]

\[ - \frac{i \hbar}{2} \int dz \left( \frac{f_{\alpha q}^*}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} \left( \frac{\partial}{\partial r^l} \left( \frac{f_{\alpha q}^*}{\sqrt{\rho}} \right) \right) + \frac{i \hbar}{2} \int dz \left( \frac{f_{\alpha q}^*}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} \left( \frac{\partial}{\partial r^l} \left( \frac{f_{\alpha q}^*}{\sqrt{\rho}} \right) \right) \ast. \]

In the second term we may exchange \( i \) and \( k \) because the other elements of the stiffness tensor are both the same to obtain

\[ p^A_{\alpha\alpha}(q, q) = \frac{q}{\Omega_{\alpha q}} \int dz \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} f^k_{\alpha q} \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right)^* + \frac{i}{2 \Omega_{\alpha q}} \int dz \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} \left( \frac{\partial}{\partial r^l} \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \right) \ast - \frac{i}{2 \Omega_{\alpha q}} \int dz \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} \left( \frac{\partial}{\partial r^l} \left( \frac{f^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \right) \ast \]

\[ = q \Omega_{\alpha q} \int dz \left( \frac{u^k_{\alpha q} \ast}{\sqrt{\rho}} \right) \cdot e^{i \gamma k} u^k_{\alpha q} + \text{Re} \left[ i \Omega_{\alpha q} \int dz \left( \frac{\partial}{\partial r^l} u^k_{\alpha q} \right) \ast \left( \frac{\partial}{\partial r^l} u^k_{\alpha q} \right) \ast \right], \quad (S.35) \]

which by Eq. (S.19) is simply the group velocity of the acoustic mode. The desired result (89) then follows from (88).

**S.VII. SIMPLIFICATION OF THE OPTO-ACOUSTIC INTERACTION TERM**

Here we show how the interaction (52) may be reduced to the form shown in (56). Inserting the expansion of the strain tensor (54) into (52) gives

\[ V = \frac{1}{\epsilon_0} \sum_{\gamma, \gamma', \alpha} \int dkdq \left( a^l_{k \gamma k \gamma'} b_{\alpha q} \right) \sqrt{\frac{\hbar \omega_{\gamma k}}{4\pi}} \sqrt{\frac{\hbar \omega_{\gamma' k}}{4\pi}} \sqrt{\frac{\hbar \Omega_{\alpha q}}{4\pi}} \int e^{i(k' - k + q)z} dz \]

\[ \times \int \left( d_{\gamma k}(x, y) \right)^* d_{\gamma' k'}(x, y) \left( p^{ijlm}(x, y) s_{ijlm}(x, y) - \delta^{ij} \left( \frac{\partial \beta_{ref}(x, y)}{\partial r^l} \right) \right) u_{\alpha q}(x, y) \] \[ + \frac{1}{\epsilon_0} \sum_{\gamma, \gamma', \alpha} \int dkdq \left( a^l_{k \gamma k \gamma'} b_{\alpha q} \right) \sqrt{\frac{\hbar \omega_{\gamma k}}{4\pi}} \sqrt{\frac{\hbar \omega_{\gamma' k}}{4\pi}} \sqrt{\frac{\hbar \Omega_{\alpha q}}{4\pi}} \int e^{i(k' - k - q)z} dz \]

\[ \times \int \left( d_{\gamma k}(x, y) \right)^* d_{\gamma' k'}(x, y) \left( p^{ijlm}(x, y) s_{ijlm}(x, y) \right)^* - \delta^{ij} \left( \frac{\partial \beta_{ref}(x, y)}{\partial r^l} \right) \right) \ast (u_{\alpha q}(x, y))^* dx dy. \]

Since the inverse dielectric tensor is symmetric even under strain, we have \( p^{ijlm}(x, y) = p^{ijlm}(x, y) \), and swapping the dummy indices \( k, k' \) in the second term gives

\[ V = \frac{1}{\epsilon_0} \sum_{\gamma, \gamma', \alpha} \int dkdq \left( a^l_{k \gamma k \gamma'} b_{\alpha q} \right) \sqrt{\frac{\hbar \omega_{\gamma k}}{4\pi}} \sqrt{\frac{\hbar \omega_{\gamma' k}}{4\pi}} \sqrt{\frac{\hbar \Omega_{\alpha q}}{4\pi}} \int e^{i(k' - k + q)z} dz \]

\[ \times \int \left( d_{\gamma k}(x, y) \right)^* d_{\gamma' k'}(x, y) \left( p^{ijlm}(x, y) s_{ijlm}(x, y) - \delta^{ij} \left( \frac{\partial \beta_{ref}(x, y)}{\partial r^l} \right) \right) u_{\alpha q}(x, y) \] \[ + \frac{1}{\epsilon_0} \sum_{\gamma, \gamma', \alpha} \int dkdq \left( a^l_{k \gamma k \gamma'} b_{\alpha q} \right) \sqrt{\frac{\hbar \omega_{\gamma k}}{4\pi}} \sqrt{\frac{\hbar \omega_{\gamma' k}}{4\pi}} \sqrt{\frac{\hbar \Omega_{\alpha q}}{4\pi}} \int \eta^{(k - k' - q)z} dz \]

\[ \times \int \left( d_{\gamma k}(x, y) \right)^* d_{\gamma' k'}(x, y) \left( p^{ijlm}(x, y) s_{ijlm}(x, y) \right)^* - \delta^{ij} \left( \frac{\partial \beta_{ref}(x, y)}{\partial r^l} \right) \right) \ast (u_{\alpha q}(x, y))^* dx dy. \]

We can now write this as

\[ V = \sum_{\gamma, \gamma', \alpha} \int \frac{dkdq}{(2\pi)^{3/2}} \left( a^l_{k \gamma k \gamma'} b_{\alpha q} \right) \int \Gamma(\gamma k; \gamma' k'; \alpha q) e^{i(k' - k + q)z} dz \]

\[ + \sum_{\gamma, \gamma', \alpha} \int \frac{dkdq}{(2\pi)^{3/2}} \left( b_{\alpha q} a^l_{k \gamma k \gamma'} \right) \int \Gamma^*(\gamma k; \gamma' k'; \alpha q) e^{-i(k' - k + q)z} dz, \quad (S.36) \]
FIG. S.1. Geometry for smoothing of discontinuous fields at waveguide interfaces.

where the coupling is characterized by the slowly-varying coefficients

\[
\Gamma(\gamma k; \gamma' k'; \alpha q) = \frac{1}{\epsilon_0} \sqrt{\frac{\hbar \omega_{\gamma k}}{2}} \sqrt{\frac{\hbar \omega_{\gamma' k'}}{2}} \sqrt{\frac{\hbar \Omega_{\alpha q}}{2}} \\
\times \int dx dy \left( d^*_{\gamma k}(x,y) d^*_{\gamma' k'}(x,y) \left( p^{ijlm}(x,y) s^*_{\alpha q}(x,y) - \delta^{ij} \left( \frac{\partial \beta_{\text{ref}}(x,y)}{\partial r^l} \right) u^l_{\alpha q}(x,y) \right) \right).
\]

S.VIII. SMOOTHING THE SURFACE MATRIX ELEMENT

In the \((x,y)\) plane we can in general identify a number of curves \(C\) that indicate where \(\beta_{\text{ref}}(x,y)\) will change discontinuously from one value to another. These may be straight or curved lines. We only contemplate discontinuous changes in \(\beta_{\text{ref}}(x,y)\), adding up the neighborhoods of all such curves identifies the regions where \(\beta_{\text{ref}}(x,y)\) is assumed to vary in the \(xy\) plane. We write \(\mathbf{R} = (x,y)\), and parameterize such a curve by \(\mathbf{R}_c(s) = (x_c(s), y_c(s))\), and for a given curve let \(s\) range from 0 to 1. As \(s\) increases along the curve we have

\[
d\mathbf{R}_c(s) = \hat{x} \frac{dx_c(s)}{ds} ds + \hat{y} \frac{dy_c(s)}{ds} ds = \hat{u}(s) dR_c,
\]

where the length

\[
dR_c = ds \sqrt{\left( \frac{dx_c(s)}{ds} \right)^2 + \left( \frac{dy_c(s)}{ds} \right)^2}, \tag{S.37}
\]

and the unit vector

\[
\hat{u}(s) = \frac{\hat{x} \frac{dx_c(s)}{ds} + \hat{y} \frac{dy_c(s)}{ds}}{\sqrt{\left( \frac{dx_c(s)}{ds} \right)^2 + \left( \frac{dy_c(s)}{ds} \right)^2}}. \tag{S.38}
\]

We introduce a normal to the curve as

\[
\hat{n}(s) \equiv \hat{z} \times \hat{u}(s) = \frac{-\hat{y} \frac{dx_c(s)}{ds} + \hat{x} \frac{dy_c(s)}{ds}}{\sqrt{\left( \frac{dx_c(s)}{ds} \right)^2 + \left( \frac{dy_c(s)}{ds} \right)^2}}.
\]

and in the neighborhood of the curve we can specify the points in the \(xy\) plane by \((s, \zeta)\), where

\[
\mathbf{R} = \mathbf{R}_c(s) + \zeta \hat{n}(s), \tag{S.39}
\]
or
\[
x = x_c(s) + \zeta (\hat{x} \cdot \hat{n}(s)),
\]
\[
y = y_c(s) + \zeta (\hat{y} \cdot \hat{n}(s)).
\]

For fixed \(s\), \(\beta_{\text{ref}}\) changes as \(\zeta\) passes from \(<0\) to \(>0\); that is, it is only a function of \(\zeta\). We assume now that the change in \(\beta_{\text{ref}}(x, y)\) occurs only in such a small region about \(\mathbf{R} = \mathbf{R}_c(s)\) (we will eventually take that change to be a Dirac delta function there) that the mapping from \((\zeta, s)\) to \((x, y)\) is one-to-one. Then we can write
\[
\left(\frac{\partial \beta_{\text{ref}}(x, y)}{\partial \gamma_k}\right) u_k^\alpha q(x, y) = \left(\frac{d\beta_{\text{ref}}(\zeta)}{d\zeta}\right) \hat{n}(s) \cdot u_{\alpha q}(x(s, \zeta), y(s, \zeta)),
\]
(S.40)
and we have
\[
- \int dy \left( \left( d^\alpha_{\gamma k}(x, y) \right)^* d^\beta_{\gamma' k'}(x, y) \left( \frac{\partial \beta_{\text{ref}}(x, y)}{\partial \gamma_k}\right) u_k^\alpha q(x, y) \right.
\]
\[
\left. - \int |J(s, \zeta)| d\zeta \left( \frac{d\beta_{\text{ref}}(\zeta)}{d\zeta}\right) \hat{n}(s) \cdot u_{\alpha q}(x(s, \zeta), y(s, \zeta)), \right)
\]
where in the second line we understand \(x = x(s, \zeta)\) and \(y = y(s, \zeta)\); that is, we have switched integration variables from \(x\) and \(y\) to \(s\) and \(\zeta\). The Jacobian
\[
J(s, \zeta) = \left| \frac{\partial x}{\partial s} \frac{\partial y}{\partial \zeta} - \frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial s} \right| = \left( \left( \frac{dx_c(s)}{ds} + \zeta \frac{\partial \hat{n}(s)}{\partial s} \right) \hat{y} \cdot \hat{n}(s) - \left( \frac{dy_c(s)}{ds} + \zeta \hat{y} \cdot \frac{\partial \hat{n}(s)}{\partial s} \right) \hat{x} \cdot \hat{n}(s) \right).
\]
(S.42)

Now at each point \((s, \zeta)\) we can use the local reference frame to identify
\[
\left( d^\alpha_{\gamma k}(x, y) \right)^* d^\beta_{\gamma' k'}(x, y) = \left( d^\|_{\gamma k}(x, y) \right)^* d^\|_{\gamma' k'}(x, y) + \left( d^\perp_{\gamma k}(x, y) \right)^* d^\perp_{\gamma' k'}(x, y),
\]
(S.43)
where
\[
d^\perp_{\gamma' k'}(x, y) = \hat{n}(s) \cdot d_{\gamma' k'}(x, y),
\]
\[
d^\perp_{\gamma' k'}(x, y) = \hat{n}(s) \cdot d_{\gamma' k'}(x, y) + \hat{n}(s) \cdot \hat{n}(s) \cdot d_{\gamma' k'}(x, y)
\]
\[
= d^\perp_{\gamma' k'}(x, y) - \hat{n}(s) \cdot \hat{n}(s) \cdot d_{\gamma' k'}(x, y),
\]
again recalling that \(x = x(s, \zeta)\) and \(y = y(s, \zeta)\). Then we can write (S.41) as
\[
- \int dy \left( \left( d^\alpha_{\gamma k}(x, y) \right)^* d^\beta_{\gamma' k'}(x, y) \left( \frac{\partial \beta_{\text{ref}}(x, y)}{\partial \gamma_k}\right) u_k^\alpha q(x, y) \right.
\]
\[
\left. - \int |J(s, \zeta)| d\zeta \left( d^\|_{\gamma k}(x, y) \right)^* d^\|_{\gamma' k'}(x, y) \left( \frac{d\beta_{\text{ref}}(\zeta)}{d\zeta}\right) \hat{n}(s) \cdot u_{\alpha q}(x(s, \zeta), y(s, \zeta)), \right)
\]
\[
- \int |J(s, \zeta)| d\zeta \left( d^\perp_{\gamma k}(x, y) \right)^* d^\perp_{\gamma' k'}(x, y) \left( \frac{d\beta_{\text{ref}}(\zeta)}{d\zeta}\right) \hat{n}(s) \cdot u_{\alpha q}(x(s, \zeta), y(s, \zeta)).
\]
(S.44)

From the relation
\[
\varepsilon_{\text{ref}}(\zeta) = \frac{1}{\beta_{\text{ref}}(\zeta)},
\]
(S.45)
follows
\[
\frac{d\beta_{\text{ref}}(\zeta)}{d\zeta} = -\frac{1}{\varepsilon_{\text{ref}}^2(\zeta)} \frac{d\varepsilon_{\text{ref}}(\zeta)}{d\zeta}.
\]
(S.46)

Now the simplest characterization of the variation of \(\beta_{\text{ref}}(\zeta)\) would be to write
\[
\beta_{\text{ref}}(\zeta) = \beta_- + (\beta_+ - \beta_-) \theta(\zeta),
\]
(S.47)
where $\theta(\zeta)$ is the step function, $\beta_-$ is the value of $\beta_{\text{ref}}(\zeta)$ for negative $\zeta$, and $\beta_+$ is the value of $\beta_{\text{ref}}(\zeta)$ for positive $\zeta$. Similarly, from (S.45) we can write

$$
\varepsilon_{\text{ref}}(\zeta) = \frac{1}{\beta_-} + \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right) \theta(\zeta). \tag{S.48}
$$

To differentiate with respect to $\zeta$ and then integrate in (S.44) we smooth these functions. We introduce a smoothing function $g_l(\zeta)$ which is non-negative, peaked at $\zeta = 0$, satisfies

$$
\int g_l(\zeta) d\zeta = 1, \tag{S.49}
$$

for each $l$, and approaches a Dirac delta function as $l \to 0$. Then for finite $l$ we have smoothed functions

$$
\tilde{\beta}_{\text{ref}}(\zeta) = \int g_l(\zeta - \zeta') \beta_{\text{ref}}(\zeta') d\zeta',
$$

$$
\tilde{\varepsilon}_{\text{ref}}(\zeta) = \int g_l(\zeta - \zeta') \varepsilon_{\text{ref}}(\zeta') d\zeta'.
$$

One strategy for evaluating $\partial \beta_{\text{ref}}(\zeta)/\partial \zeta$ is to take

$$
\frac{d\beta_{\text{ref}}(\zeta)}{d\zeta} \to \frac{d\tilde{\beta}_{\text{ref}}(\zeta)}{d\zeta} = (\beta_+ - \beta_-) g_l(\zeta). \tag{S.50}
$$

Alternately, using (S.46), we could take

$$
\frac{d\beta_{\text{ref}}(\zeta)}{d\zeta} \to -\frac{1}{\tilde{\varepsilon}_{\text{ref}}^2(\zeta)} \frac{d\tilde{\varepsilon}_{\text{ref}}(\zeta)}{d\zeta} \tag{S.51}
$$

$$
= -\frac{1}{\tilde{\varepsilon}_{\text{ref}}^2(\zeta)} \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right) g_l(\zeta).
$$

Using (S.50) and (S.51) in the two right-hand expressions of (S.44) respectively, gives

$$
- \int dxdy \left( d_i^\gamma(x,y) \right)^* d_{\gamma',k'}^\gamma(x,y) \left( \frac{\partial \beta_{\text{ref}}(x,y)}{\partial r_k} \right) u_{\alpha q}(x,y)
$$

$$
= - (\beta_+ - \beta_-) \int |J(s,\zeta)| ds d\zeta \left( d_i^\gamma(x,y) \right)^* d_{\gamma',k'}^\gamma(x,y) g_l(\zeta) \hat{n}(s) \cdot u_{\alpha q}(x,y)
$$

$$
+ \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right) \int |J(s,\zeta)| ds d\zeta \left( d_i^\gamma(x,y) \right)^* d_{\gamma',k'}^\gamma(x,y) \frac{\tilde{\varepsilon}_{\text{ref}}(\zeta)}{\varepsilon_{\text{ref}}^2(\zeta)} g_l(\zeta) \hat{n}(s) \cdot u_{\alpha q}(x,y), \tag{S.52}
$$

where we still understand $x = x(s,\zeta)$ and $y = y(s,\zeta)$. Now we can let $l \to 0$ in both terms, because the rest of the integrands are continuous about $\zeta = 0$. Recalling (S.49), we have

$$
- \int dxdy \left( d_i^\gamma(x,y) \right)^* d_{\gamma',k'}^\gamma(x,y) \left( \frac{\partial \beta_{\text{ref}}(x,y)}{\partial r_k} \right) u_{\alpha q}(x,y)
$$

$$
\to - (\beta_+ - \beta_-) \int |J(s,0)| \left( d_i^\gamma(R_c(s)) \right)^* d_{\gamma',k'}^\gamma(R_c(s)) \hat{n}(s) \cdot u_{\alpha q}(R_c(s)) ds
$$

$$
+ \epsilon_0^2 \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right) \int |J(s,0)| \left( e_i^\gamma(R_c(s)) \right)^* e_{\gamma',k'}^\gamma(R_c(s)) \hat{n}(s) \cdot u_{\alpha q}(R_c(s)) ds. \tag{S.53}
$$

where we have used the fact that $x(s,\zeta) \to x_c(s)$, and $y(s,\zeta) \to y_c(s)$, as $\zeta \to 0$. Finally, we have

$$
|J(s,0)| ds = \left| \frac{dx_c(s)}{ds} \hat{y} \cdot \hat{n}(s) - \frac{dy_c(s)}{ds} \hat{x} \cdot \hat{n}(s) \right| ds
$$

$$
= \sqrt{\left( \frac{dx_c(s)}{ds} \right)^2 + \left( \frac{dy_c(s)}{ds} \right)^2} ds = dR_c(s),
$$
the element of length along the curve. So we can write

\[ - \int \! dx \! dy \left[ d_{\gamma k}(x, y) \right]^* d_{\gamma' k'}(x, y) \left( \frac{\partial \beta_{\text{ref}}(x, y)}{\partial y_k} \right) u_{\alpha q}(x, y) \]

\[ \rightarrow \left( \beta_+ - \beta_- \right) \int \! dR_c(s) \left[ d_{\gamma k}(R_c(s)) \right]^* d_{\gamma' k'}(R_c(s)) \left[ \hat{n}(s) \cdot u_{\alpha q}(R_c(s)) \right] dR_c(s) \]

\[ + \varepsilon_0^2 \frac{1}{\beta_+ - \beta_-} \int \! dR_c(s) \left[ \mathbf{e}_{\gamma k}(R_c(s)) \right]^* \cdot \mathbf{e}_{\gamma' k'}(R_c(s)) \left[ \hat{n}(s) \cdot u_{\alpha q}(R_c(s)) \right] dR_c(s) \]

\[ \left( \frac{1}{\varepsilon_-} - \frac{1}{\varepsilon_+} \right) \int \! dR_c(s) \left[ \mathbf{e}_{\gamma k}(R_c(s)) \right]^* \cdot \mathbf{e}_{\gamma' k'}(R_c(s)) \left[ \hat{n}(s) \cdot u_{\alpha q}(R_c(s)) \right] dR_c(s) \]

from whence (61) follows. The full expression for \( \Gamma_{\text{surf}}(\gamma; \gamma'; \alpha q) \) then involves a sum over all such curves where a transition from one dielectric constant to another occurs. Note there is no ambiguity in evaluating these terms, since \( d_{\gamma k}(r) \) is continuous across a step discontinuity in \( \beta_{\text{ref}}(x, y) \), as is \( \mathbf{e}_{\gamma k}(r) \).

S.IX. REDUCED MATRIX ELEMENTS

Using the normalization conditions (29) and (41) we can write the matrix elements in the form

\[ \Gamma(\gamma; \gamma'; \alpha q) = \frac{1}{2^{5/2} \Omega_{\alpha q} \sqrt{|v_{\gamma k} v_{\gamma' k'} v_{\alpha q}|}} \]

\[ \times \left[ \int \! dx \! dy \left( d_{\gamma k}(x, y) \right)^* d_{\gamma' k'}(x, y) p^{ijlm}(x, y) S^{lm}_{ij}(x, y) \left[ \int \! dx \! dy \rho(x, y) u_{\alpha q}^*(x, y) \cdot u_{\alpha q}(x, y) \right]^{1/2} \]

\[ \left[ \int \! dx \! dy \beta_{\text{ref}}(r) d_{\gamma k}^*(x, y) \cdot d_{\gamma' k'}(x, y) \left[ \int \! dx \! dy \rho(x, y) u_{\alpha q}^*(x, y) \cdot u_{\alpha q}(x, y) \right]^{1/2} \]

\[ \left[ \int \! dx \! dy \beta_{\text{ref}}(r) d_{\gamma k}^*(x, y) \cdot d_{\gamma' k'}(x, y) \left[ \int \! dx \! dy \rho(x, y) u_{\alpha q}^*(x, y) \cdot u_{\alpha q}(x, y) \right]^{1/2} \right] \]

\[ \left[ \int \! dx \! dy \beta_{\text{ref}}(r) d_{\gamma k}^*(x, y) \cdot d_{\gamma' k'}(x, y) \left[ \int \! dx \! dy \rho(x, y) u_{\alpha q}^*(x, y) \cdot u_{\alpha q}(x, y) \right]^{1/2} \right] \]

The advantage of this form is that it can now be used regardless of how the mode fields are normalized. Again, the last two lines should be summed over all curves that contribute.

S.X. ELECTROMAGNETIC POWER FLOW

Here we justify the relations (96) to (98) in the main paper for the optical power transport in terms of the optical field envelope operators.

The operator for the power carried by the field is given by the Poynting vector which we write in the symmetrized form

\[ \mathbf{S}(r, t) = \frac{1}{2} \left[ \mathbf{E}(r, t) \times \mathbf{H}(r, t) - \mathbf{H}(r, t) \times \mathbf{E}(r, t) \right] \]

(S.56)

Following (92), the \( \mathbf{E} \) and \( \mathbf{H} \) field operators are given by

\[ \mathbf{E}(r, t) = \sum_{\gamma j} e^{ik_j z} \int \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\hbar \omega_{\gamma k}}{2}} \mathbf{e}_{\gamma k}(x, y) a_{\gamma k} e^{i(k-k_j)z} + \text{h.c.} \]

(S.57)

\[ \mathbf{H}(r, t) = \sum_{\gamma j} e^{ik_j z} \int \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\hbar \omega_{\gamma k}}{2}} \mathbf{h}_{\gamma k}(x, y) a_{\gamma k} e^{i(k-k_j)z} + \text{h.c.} \]

(S.58)
where the mode functions satisfy

\[
e_{\gamma k}(x, y) = \frac{d_{\gamma k}(x, y)}{\epsilon_0 \epsilon(x, y)} \tag{S.59}
\]

\[
h_{\gamma k}(x, y) = \frac{1}{i \omega_{\gamma k} \mu_0} \left[ \nabla \times (e_{\gamma k}(x, y)e^{ikz}) \right] e^{-ikz}. \tag{S.60}
\]

It also follows from Maxwell’s equations that in lossless systems, for each mode \(\gamma k\), there is a partner mode \(\gamma \bar{k}\) with \(\bar{k} = -k\), \(\omega_{\gamma \bar{k}} = \omega_{\gamma k}\) and

\[
e_{\gamma \bar{k}}(x, y) = e_{\gamma k}^*(x, y) \tag{S.61}
\]

\[
h_{\gamma \bar{k}}(x, y) = -h_{\gamma k}^*(x, y). \tag{S.62}
\]

Using (S.59) and (S.60) in (S.66), the operator describing the total power flow in the waveguide is

\[
P_{\text{EM}}^\text{z}(z) = \int \text{d}x \text{d}y \mathbb{S}(r, t) \cdot \dot{z}
\]

\[
= \frac{1}{2} \int \text{d}r \text{d}y \sum_{\gamma, \gamma', r, s} \frac{dk dk'}{2\pi} \frac{\hbar \omega_{\gamma k}}{2} \frac{\hbar \omega_{\gamma' k'}}{2} \int \text{d}x \text{d}y
\]

\[
\dot{z} \left\{ \left[ e_{\gamma k} \times (h_{\gamma k})^* \right] a_{\gamma k} e^{i(k-k')z} + (e_{\gamma k})^* a_{\gamma k}^\dagger e^{-i(k-k')z} \right. \times \left[ h_{\gamma k}^\dagger a_{\gamma' k} e^{i(k' - k_j)z} + (h_{\gamma k})^* a_{\gamma' k}^\dagger e^{-i(k' - k_j)z} \right] - \left. \left[ h_{\gamma k}^\dagger a_{\gamma' k} e^{i(k' - k_j)z} + (h_{\gamma k})^* a_{\gamma' k}^\dagger e^{-i(k' - k_j)z} \right] \right. \times \left[ e_{\gamma k} a_{\gamma k} e^{i(k-k)z} + (e_{\gamma k})^* a_{\gamma k}^\dagger e^{-i(k-k)z} \right] \right\} \tag{S.63}
\]

The temporally slowly-varying part of this expression is

\[
P_{\text{EM}}^\text{sv} = \frac{1}{2} \sum_{\gamma, \gamma', r, s} \frac{dk dk'}{2\pi} \frac{\hbar \omega_{\gamma k}}{2} \frac{\hbar \omega_{\gamma' k'}}{2} \int \text{d}x \text{d}y
\]

\[
\dot{z} \left[ \left[ e_{\gamma k} \times (h_{\gamma k})^* \right] a_{\gamma k} a_{\gamma k}^\dagger e^{i[(k-k') - (k_j-k')]z} + (e_{\gamma k})^* a_{\gamma k}^\dagger a_{\gamma k} e^{-i[(k-k') - (k_j-k')]z} \right] - \left[ h_{\gamma k}^\dagger a_{\gamma' k} e^{i(k' - k_j)z} + (h_{\gamma k})^* a_{\gamma' k}^\dagger e^{-i(k' - k_j)z} \right] \times \left[ e_{\gamma k} a_{\gamma k} e^{i(k-k)z} + (e_{\gamma k})^* a_{\gamma k}^\dagger e^{-i(k-k)z} \right] - \left[ h_{\gamma k}^\dagger a_{\gamma' k} e^{i(k - k_j)z} + (h_{\gamma k})^* a_{\gamma' k}^\dagger e^{-i(k - k_j)z} \right] \right\} \tag{S.64}
\]
Since the $k$ integral is over all wavenumbers including all partner modes, it is easy to show using (S.61) that the second term in this expression, associated with vacuum contributions, vanishes, as we would expect for a signed quantity.

For the remaining non-vacuum contribution, since the sums and integrals are over all values we may swap the indices $\gamma, \gamma'$ and $k, k'$ in the second term in square brackets to give

\[
P_{av}^{EM} = \sum_{\gamma, \gamma'} \int \frac{dk dk'}{2\pi} \sqrt{\frac{\hbar \omega_{\gamma k}}{2}} \sqrt{\frac{\hbar \omega_{\gamma' k'}}{2}} \int dx dy \hat{z} \cdot \left[ e_{\gamma k} \times (h_{\gamma' k'})^\ast + (e_{\gamma' k'})^\ast \times h_{\gamma k} \right] a^{\dagger}_{\gamma' k'} a_{\gamma k} e^{i[(k-k')-(k_j-k_j')]z}.
\]

where we have introduced the quantity

\[
p_{\gamma' \gamma}(k', k) = \sqrt{\frac{\hbar \omega_{\gamma k}}{2}} \sqrt{\frac{\hbar \omega_{\gamma' k'}}{2}} \int dx dy \hat{z} \cdot (e_{\gamma k} \times (h_{\gamma' k'})^\ast + (e_{\gamma' k'})^\ast \times h_{\gamma k}).
\]  

(S.65)

Finally, if the different modes $\gamma$ have very different center wavenumbers $k_j$, then only the $\gamma = \gamma'$ terms will contribute significantly to (S.65) and we may approximate

\[
P_{av}^{EM} \approx \sum_{\gamma} \int \frac{dk dk'}{2\pi} a^{\dagger}_{\gamma k'} a_{\gamma k} e^{i[(k-k')z} p_{\gamma \gamma}^{EM}(k, k),
\]

(S.67)

with $p_{\gamma \gamma}^{EM}(k, k)$ the power carried by the normalized mode functions $\gamma$ at center wavenumber $k$.

\section*{S.1. Interpretation as the photon number density operator}

To convert the result in (S.67) to a simple expression involving the photon envelope operators we require the group velocity in terms of the fields.

Noting that the basis functions $B_{\gamma k}(r) = b_{\gamma k}(x, y) e^{ikz}$ are eigenmodes of the vector Helmholtz equation (38), the transverse mode functions $b_{\gamma k}$ are eigenfunctions of the equation

\[
\mathcal{O}_k b_{\gamma k} = \frac{\omega_{\gamma k}^2}{c^2} b_{\gamma k},
\]

(S.68)

where the $k$-dependent operator $\mathcal{O}_k$ operates on a vector function $f$ as

\[
\mathcal{O}_k f = \nabla_t \times \left( \frac{1}{n^2} \nabla_t \times f \right) - \frac{k^2}{n^2} \hat{z} \times \hat{z} \times f + ik \left[ \hat{z} \times f + \nabla_t \times \left( \frac{1}{n^2} \hat{z} \times f \right) \right],
\]

(S.69)

and where $\nabla_t = [\partial_x, \partial_y, 0]$. It can be shown that $\mathcal{O}_k$ is Hermitian such that

\[
\int dx dy f_1^* \cdot (\mathcal{O}_k f_2) = \left( \int dx dy f_2 \cdot (\mathcal{O}_k f_1) \right)^*.
\]

(S.70)

From Ampere’s law, we also have that

\[
\nabla_t \times b_{\gamma k} = -im_0 \omega_{\gamma k} d_{\gamma k} - ik \hat{z} \times b_{\gamma k}.
\]

(S.71)

We now take the inner product with $b_{\gamma k}^\ast$ in (S.68) and differentiate both sides with respect to $k$:

\[
\frac{\partial}{\partial k} \int dx dy b_{\gamma k}^\ast \cdot \mathcal{O}_k b_{\gamma k} = \frac{\partial}{\partial k} \left( \frac{\omega_{\gamma k}^2}{c^2} \int dx dy b_{\gamma k}^\ast \cdot b_{\gamma k} \right) = m_0 \frac{\partial}{\partial k} \frac{\omega_{\gamma k}^2}{c^2} = \frac{2m_0 \omega_{\gamma k} \partial \omega_{\gamma k}}{c^2}.
\]  

(S.72)
where we used the normalization \( \int \! dx \! dy \, b_{\gamma k}^* \cdot b_{\gamma k} / \mu_0 = 1 \) which follows from (41) and Maxwell’s equations. By the Hermiticity of \( O_k \), we can invoke the Hellmann-Feynman theorem to write the left hand side as

\[
\int \! dx \! dy \, b_{\gamma k}^* \cdot \left( \frac{\partial}{\partial k} O_k \right) b_{\gamma k} = -2k \int \! dx \! dy \, b_{\gamma k}^* \cdot \hat{z} \times (\hat{z} \times b_{\gamma k}) \left( \frac{1}{n^2} \right)
+ i \int \! dx \! dy \, b_{\gamma k}^* \cdot \left[ \frac{1}{n^2} \nabla \times b_{\gamma k} + b_{\gamma k}^* \cdot \nabla \times \left( \frac{1}{n^2} \hat{z} \times b_{\gamma k} \right) \right]
= -2k \int \! dx \! dy \, b_{\gamma k}^* \cdot \hat{z} \times (\hat{z} \times b_{\gamma k}) \left( \frac{1}{n^2} \right)
+ i \int \! dx \! dy \, b_{\gamma k}^* \cdot \left[ \frac{1}{n^2} (-i\mu_0 \omega_{\gamma k} d_{\gamma k} - ik \hat{z} \times b_{\gamma k}) \right] + (\nabla \times b_{\gamma k}) \cdot \left( \frac{1}{n^2} \hat{z} \times b_{\gamma k} \right)
= -2k \int \! dx \! dy \, b_{\gamma k}^* \cdot \hat{z} \times (\hat{z} \times b_{\gamma k}) \left( \frac{1}{n^2} \right)
+ i \int \! dx \! dy \, b_{\gamma k}^* \cdot \left[ \frac{1}{n^2} (-i\mu_0 \omega_{\gamma k} d_{\gamma k} - ik \hat{z} \times b_{\gamma k}) \right]
+ i \int \! dx \! dy \, (i\mu_0 \omega_{\gamma k} d_{\gamma k} + ik \hat{z} \times b_{\gamma k}) \cdot \left( \frac{1}{n^2} \hat{z} \times b_{\gamma k} \right)
= \frac{\omega_{\gamma k}}{c^2} \int \! dx \! dy \, \left[ b_{\gamma k}^* \cdot \hat{e}_{\gamma k} - e_{\gamma k}^* \cdot \hat{z} \times b_{\gamma k} \right]
= \frac{\mu_0 \omega_{\gamma k}}{c^2} \hat{z} \cdot \int \! dx \! dy \, e_{\gamma k}^* \times h_{\gamma k} + e_{\gamma k}^* \times h_{\gamma k} \tag{S.73}
\]

Comparing (S.72) and (S.73) yields

\[
\frac{\partial \omega_{\gamma k}}{\partial k} = \frac{1}{2} \hat{z} \cdot \int \! dx \! dy \, e_{\gamma k}^* \times h_{\gamma k} + e_{\gamma k}^* \times h_{\gamma k}
= \frac{1}{\hbar \omega_{\gamma k}} p_{\gamma\gamma}^{\text{EM}}(k, k), \tag{S.74}
\]

Finally, from (S.66) we then have \( p_{\gamma\gamma}^{\text{EM}}(k, k) = \hbar \omega_{\gamma k} v_{\gamma k}^2 \), and from (S.67) with (90) we obtain (98)

\[
P_{\gamma\gamma}(z) \approx \sum_{\gamma'j} \hbar \omega_{\gamma'\gamma} \psi_{\gamma' j}^\dagger(z) \psi_{\gamma j}(z), \tag{S.75}
\]

in exact analogy with the acoustic result in (89) but allowing for the sum over electromagnetic modes.

**S.XI. THE ORGANIZATION OF EIGENFUNCTIONS**

This section establishes the basic properties of partner eigenfunctions for a Hermitian operator that are invoked in section S.III.

Consider a Hermitian operator, schematically \( H(x, \frac{\partial}{\partial x}, \ldots) \); the eigenvalue equation is

\[
H(x, \frac{\partial}{\partial x}, \ldots) f(x) = \lambda f(x). \tag{S.76}
\]

Hermiticity guarantees real eigenvalues and the fact that eigenfunctions of different eigenvalues are orthogonal. The inner product of two such functions vanishes, where the inner product of \( g(x) \) with \( f(x) \) is

\[
\int g^*(x) f(x) dx \tag{S.77}
\]

We consider eigenfunctions that are normalized, so

\[
\int f^*(x) f(x) dx = 1. \tag{S.78}
\]
We want to consider first a number of degenerate eigenfunctions, all with the same eigenvalue. Suppose now that besides being Hermitian, $H$ is also real. Then if $f(x)$ is an eigenfunction, $f^*(x)$ will also be an eigenfunction with the same eigenvalue.

$$H(x, \frac{\partial}{\partial x}, \ldots) f^*(x) = \lambda f^*(x)$$  \hspace{1cm} (S.79)

For a given $f(x)$, of course one possibility is that $f^*(x)$ is just a constant phase factor times $f(x)$. Then $f(x)$ could be readjusted to be purely real (or purely imaginary), for example.

Suppose this is not the case. Then $f(x)$ and $f^*(x)$ are linearly independent, and they span a two-dimensional space. Of course, they need not be orthogonal. That is, there is no guarantee that the inner product of $f(x)$ and $f^*(x)$ would just be a phase factor times a real function.

First find

$$\bar{c}(x) = N(f(x) + f^*(x)),$$  \hspace{1cm} (S.81)

where $N$ is a real normalization constant; $c(x)$ does not vanish, because by assumption $f^*(x)$ is not just a multiple of $f(x)$. If we choose $N$ to be real, then $c(x)$ is also purely real. Now take out from $f(x)$ the amount proportional to $c(x)$,

$$\bar{f}(x) = f(x) - c(x) \int c(x') f(x') dx',$$  \hspace{1cm} (S.82)

where we do not need $c^*(x')$ in the integral because $c(x')$ is real. Of course $\bar{f}(x)$ cannot vanish everywhere because otherwise $f(x)$ would just be proportional to $c(x)$ and then $f(x)$ would just be a phase factor times a real function.

Now by construction $c(x)$ is orthogonal to $f(x)$,

$$\int c(x) \bar{f}(x) dx = 0.$$  \hspace{1cm} (S.83)

Perhaps $\bar{f}(x)$ is purely real; if so, normalize it and call the result $s(x)$. Perhaps $\bar{f}(x)$ is purely imaginary; if so, divide by $i$, normalize it and call the result $s(x)$. If $\bar{f}(x)$ is neither, note that from (S.83) we have

$$\int c(x) \bar{f}^*(x) dx = 0,$$  \hspace{1cm} (S.84)

since $c(x)$ is purely real. Then

$$\bar{f}(x) + \bar{f}^*(x)$$  \hspace{1cm} (S.85)

is a real function that is orthogonal to $c(x)$; it cannot vanish everywhere because we have assumed that $\bar{f}(x)$ is not purely imaginary. Now normalize this function and call it $s(x)$.

Whatever route we have taken to get $s(x)$, we now have two real functions $c(x)$ and $s(x)$ that are orthogonal to each other and normalized,

$$\int c^2(x) dx = 1,$$
$$\int s^2(x) dx = 1,$$
$$\int c(x)s(x) dx = 0.$$

They span the space spanned by $f(x)$ and $f^*(x)$. We can then form partner functions for this subspace,

$$f_1(x) = \frac{1}{\sqrt{2}} (c(x) + is(x)),$$
$$f_1^*(x) = \frac{1}{\sqrt{2}} (c(x) - is(x)).$$
These functions are normalized,
\[
\int [f_1(x)]^* f_1(x) dx = 1,
\]
\[
\int [f_1^*(x)]^* f_1^*(x) dx = 1,
\]
and orthogonal,
\[
\int [f_1^*(x)]^* f_1(x) dx = 0,
\]
\[
\int [f_1(x)]^* f_1^*(x) dx = 0.
\]

So we have constructed partner eigenfunctions \(f_1(x)\) and \(f_1^*(x)\) that span the subspace spanned by \(f(x)\) and \(f^*(x)\). Suppose now there are more eigenfunctions with the same eigenvalue, which are orthogonal to \(f(x)\) and \(f^*(x)\). Call one of them \(g(x)\). Then \(g(x)\) must be orthogonal to \(f_1(x)\) and \(f_1^*(x)\) since they span the same subspace as \(f(x)\) and \(f^*(x)\),
\[
\int g^*(x)f_1(x) dx = 0,
\]
\[
\int g(x)f_1^*(x) dx = 0.
\]

Now if \(g(x)\) is an eigenfunction of \(H(x)\), then \(g^*(x)\) is an eigenfunction of \(H(x)\) with the same eigenvalue. Suppose \(g^*(x)\) is not just a constant phase factor times \(g(x)\); then \(g^*(x)\) and \(g(x)\) span a two dimensional subspace that, since from (S.86) we have immediately
\[
\int g(x)f_1(x) dx = 0,
\]
\[
\int g(x)f_1^*(x) dx = 0,
\]
has no overlap with the subspace spanned by \(f_1(x)\) and \(f_1^*(x)\). So from \(g(x)\) and \(g^*(x)\) we can form two partner wave functions \(f_2(x)\) and \(f_2^*(x)\) that are orthogonal to each other and each orthogonal to each of \(f_1(x)\) and \(f_1^*(x)\).

Thus we can proceed and organize our eigenfunctions. As we investigate all the eigenfunctions of a particular eigenvalue we will sometimes find it is possible to immediately make an eigenfunction real (as we could have, for example, if \(g^*(x)\) had simply been proportional to \(g(x)\) with a constant phase factor), or otherwise we can establish partners. So we can imagine listing all our wave functions grouped in the following manner,
\[
\begin{align*}
  f_1(x) & \quad f_1^*(x) \\
  f_2(x) & \quad f_2^*(x) \\
  f_3(x) & \quad f_3^*(x) \\
  & \quad \vdots \\
  f_N(x) & \quad f_N^*(x) \\
  f_I(x) & \quad f_I^*(x) \\
  & \quad \vdots
\end{align*}
\]

Here the Roman numerals indicate real wave functions that “don’t have partners”; we take them to be purely real. Of course, if we have an even number of real wave functions without partners we can start combining them into partners. For example, in the list above we could replace \(f_I(x)\) and \(f_I^*(x)\) by the partners
\[
f_{N+1}(x) = \frac{1}{\sqrt{2}} (f_I(x) + if_{II}(x)),
\]
\[
f_{N+1}^*(x) = \frac{1}{\sqrt{2}} (f_I(x) - if_{II}(x)).
\]
If we have an even number of eigenfunctions of a particular eigenvalue, then we could pair them all up in partnerships. If we have an odd number then there must be at least one “unpartnered” wave function. It is also possible to “divorce” some partners; for example, in place of \( f_3(x) \) and \( f_3^*(x) \) we could choose the real functions

\[
\begin{align*}
c_3(x) &= \frac{1}{\sqrt{2}} (f_3(x) + f_3^*(x)), \\
s_3(x) &= -\frac{i}{\sqrt{2}} (f_3(x) - f_3^*(x)).
\end{align*}
\]

But it is often convenient and natural to have wave functions in partnerships. In any case, we assume that we have eigenfunctions organized according to (S.87). However, we henceforth write \( f_1^*(x) \) as \( f_1^1(x) \), and so on, so the list (S.87) can be given as

\[
\begin{align*}
f_1(x) & f_1^1(x) \\
f_2(x) & f_2^1(x) \\
f_3(x) & f_3^1(x) \\
& \\
f_N(x) & f_N^1(x) \\
f_I(x) & f_I^1(x) \\
& \\
& ...
\end{align*}
\]

Then if we denote a general eigenfunction by \( f_\alpha(x) \), the list of possible \( J \)s is

\[
1, \bar{1}, 2, \bar{2}, 3, \bar{3}, \ldots N, \bar{N}, I, II, III \ldots (S.89)
\]

These eigenfunctions are all orthogonal,

\[
\int f_\alpha^*(x)f_{\alpha'}(x)dx = \delta_{\alpha\alpha'}.
\]

(S.90)

as \( \alpha \) and \( \alpha' \) range over this list. Associated with a list of \( \alpha \)s we introduce a list of \( \bar{\alpha} \)s,

\[
\bar{1}, 1, \bar{2}, 2, 3, \bar{3}, \ldots \bar{N}, N, I, II, III \ldots (S.91)
\]

That is, if \( \alpha \) is one of a partnership, \( \bar{\alpha} \) is the other partner; if \( \alpha \) is a real wave function, \( \bar{\alpha} \) is that wave function itself. Clearly

\[
\sum_{\bar{\alpha}} = \sum_{\alpha},
\]

(S.92)

and

\[
f_{\bar{\alpha}}^*(x) = f_{\alpha}(x),
\]

(S.93)

either because \( \bar{\alpha} \) identifies the partner of \( \alpha \), or because \( f_{\alpha}(x) \) is real, in which case \( f_{\alpha}(x) \) can be considered its own partner.

Now if we consider the eigenfunctions of a whole range of eigenvalues \( \lambda \) we can do the same sort of organization within the subspace of each eigenvalue. Then we can let \( \alpha \) range over the whole list of labels of all eigenfunctions of all eigenvalues. For a given \( \alpha \) we identify the eigenvalue by \( \lambda_\alpha \). Then over this whole range of \( \alpha \)s we have

\[
\int f_{\alpha}^*(x)f_{\alpha'}(x)dx = \delta_{\alpha\alpha'},
\]

(S.94)

where between eigenfunctions associated with different eigenvalues the orthogonality holds because of Hermiticity of the operator, while between eigenfunctions associated with the same eigenvalue the orthogonality holds because of the construction we have adopted. We still have generally

\[
f_{\bar{\alpha}}^*(x) = f_{\alpha}(x),
\]

(S.95)

and of course

\[
\lambda_{\bar{\alpha}} = \lambda_\alpha.
\]

(S.96)