The lattice $SU(2)$ gluodynamics in the maximal abelian projection is reduced to the abelian theory, in which the natural small parameter exists. We show that in the zeroth order of the expansion in this parameter the theory is equivalent to the compact abelian gauge theory coupled to ghosts of charge 2. The theory is a renormalizable and asymptotically free. This theory is represented in the form of the theory of open string with the boundary consisting of the worldline of the quark. The ghosts live on the worldsheet of the string. The naive continuum limit of such string representation gives a simple expression for the chromoelectric string action.
1 Introduction

The most important problem in the nonabelian theories is the nonlinear selfinteraction of gluons. Due to this selfinteraction the physical processes are not understood completely yet both quantitatively and qualitatively. 't Hooft suggested the method to convert the nonabelian theory into the abelian one. We hope that in this theory all physical processes are much more transparent. This method is known as the procedure of abelian projection[1]. There exists a lot of abelian projections, but only in the one of them, in the so-called maximal abelian projection, the monopoles are shown to be related to the dual superconductor mechanism of confinement[2]. This mechanism was proposed by 't Hooft and Mandelstam [3]. The maximal abelian projection makes the link variables as diagonal as possible. It occurs that there exists a small parameter which is the ratio of the contribution of the nondiagonal elements of link matrices into the effective action to the contribution of the diagonal elements[4]. Lattice calculations show that this parameter is less than 0.1 [4].

In this paper we show that in the zeroth order of the expansion in this parameter the lattice $SU(2)$ theory becomes the compact abelian gauge theory coupled to ghosts of charge 2. We can represent the Wilson loop average in this theory as the sum over the worldsheets of the electric string. The boundary of this string consists of the given loop. The ghost worldline is closed and always lies on the string’s worldsheet. The orientation of the string worldsheet is changing on the ghost’s worldline. The form of the string action gives the evidence of the existence of confinement phase in this theory. In the naive continuum limit of the abelian projected theory we come to the asymptotically free renormalizable theory. This shows that it’s lattice version can admit the continuum limit.

There is the longstanding question in the chromodynamics - what is the action of the chromoelectric string. The investigation of the simplest Nambu - Goto string showed, that it can not be the physical string, because of the crumpling. A new form of the string action was recently suggested, which possesses the extended reparametrization invariance [5]. The string representation of the Wilson loop average in this theory satisfies the $SU(\infty)$ - loop equations without contact terms. In
In this paper we note that the string with this action arises as the naive continuum limit of the string representation of the effective abelian gauge theory mentioned above.

In section 2 we consider the gluodynamics in maximal abelian gauge.

In section 3 we derive the effective abelian theory and the string representation of the effective abelian theory.

In section 4 we consider the naive continuum limit of the effective abelian theory.

2 \textit{SU}(2) - gluodynamics in the maximal abelian gauge.

We consider the \textit{SU}(2) - gluodynamics with the Wilson action \( S(U) = \beta \sum_{\text{plaq}} (1 - 1/2 \text{Tr}U_{\text{plaq}}) \). Here the sum is over the plaquettes of the lattice. If the given plaquette consists of the links \([xy], [yz], [zw], [wx]\) then \( U_{\text{plaq}} = U_{[xy]} U_{[yz]} U_{[zw]} U_{[wx]} \). We start from the Wilson loop average

\[
< W_C > = \int DU \exp(-S(U)) \text{Tr} \Pi_{[xy] \in C} U_{[xy]}.
\]

Here the loop \( C \) consists of links \([xy]\), \( \Pi_{[xy] \in C} U_{[xy]} \) is the ordered product of link matrices along the given loop.

The projection diagonalizes the link matrix \( U \) as much as possible. This procedure makes the functional \( Q = \sum_{\text{links}} |U_{11}|^2 \) maximal with respect to the gauge transformations. This gauge condition is invariant under the Cartan subgroup of \( \text{SU}(2) \). This subgroup is created by the matrices \( \text{diag}(e^{i\alpha}, e^{-i\alpha}) \). We parametrize the matrix \( U \) as \( U_{11} = \cos(\rho)e^{i\theta}, U_{12} = \sin(\rho)e^{i(\theta + \lambda)} \). Then the field \( \theta \) plays the role of abelian gauge field. It transforms as \( \theta_{xy} \rightarrow \theta_{xy} + \alpha_x - \alpha_y \) under the action of Cartan subgroup \( U(1) \) of \( \text{SU}(2) \). The infinitesimal elements of \( \text{SU}(2)/U(1) \) can be represented as

\[
g = \begin{pmatrix} 1 & \phi^+ \\ -\phi & 1 \end{pmatrix}
\]

where \( \phi \) is complex - valued.
The differential condition for the maximal abelian gauge is

\[ \frac{\delta}{\delta \phi_x} Q[U] = - \sum_{y,y>x} U_{xy}^{12} U_{xy}^{11} + \sum_{y,y<x} U_{yx}^{12} (U_{yx}^{11})^* = 0 \] (3)

In order to fix the gauge we insert the unity into the functional integral

\[ 1 = \int Dg \delta \left( \frac{\delta}{\delta \phi_x} Q[U^g] \right) \Gamma(U^g) \text{Det} \frac{\delta^2}{\delta \phi_x \delta \phi_y} Q[U^g] \] (4)

where \( g \in SU(2)/U(1) \) and its infinitesimal element is given by (2). The function \( \Gamma \) is integer-valued and shares out the field configurations from the set of solutions of the differential condition mentioned above that are the global maxima of the integral gauge condition.

There exists the small parameter in the theory, namely, the ratio of the contribution into the effective action of nondiagonal part of \( U \) to the contribution of the diagonal part of \( U \).

We can rewrite the Wilson loop average as follows.

\[ <W_C> = \int DUD\bar{\psi}D\psi \exp(-\beta \sum_{\text{plaq}} (1 - 1/2 \text{Tr} U_{\text{plaq}})) \]

\[ + \sum_{[xy]} \bar{\psi}_x \left[ \frac{\delta^2}{\delta \phi_x \delta \phi_y} Q[U^g]|_{g=1} \right] \psi_y \Pi_{[xy] \in C} U_{[xy]} \]

\[ \delta(- \sum_{y,y>x} U_{xy}^{12} U_{xy}^{11} + \sum_{y,y<x} U_{yx}^{12} (U_{yx}^{11})^*) \Gamma(U). \] (5)

Here the field \( \theta \) is the abelian gauge field, \( \psi \) are the ghost anticommuting variables, and the vector field of charge two \( f = \sin 2 \rho \exp(i \lambda) \) is hidden in this expression.

### 3 The effective abelian theory.

As it was mentioned, there exists the small parameter in the maximal abelian projection. Thus there exists the perturbation theory in this small parameter. In the zeroth order of expansion in our small parameter, one can neglect the contribution
of the vector charged field. The direct calculation of \( \frac{\delta^2}{\delta \phi_x \delta \phi_y} Q[U^g] \big|_{g=1} \) then leads us to the expression:

\[
\langle W_C \rangle = \int D\theta D\bar{\psi} D\psi \exp(-\beta \sum_{\text{plaq}} (1 - \cos (d\theta)_{\text{plaq}})
\]
\[
+ \sum_{\{xy\}} (\bar{\psi}_x - e^{2i\theta_{xy}} \bar{\psi}_y)(\psi_x - e^{-2i\theta_{xy}} \psi_y) + i(\theta, C)),
\]

(6)

We write the Wilson loop average in the terms of some abelian theory. We shall refer to it as to effective abelian theory. Here and below we use the notations of differential forms on the lattice (see [6]).

In order to take into account the higher orders, one should consider the full expression for \( \frac{\delta^2}{\delta \phi_x \delta \phi_y} Q[U^g] \) and the function \( \Gamma(U) \) which defines the fundamental modular region [7].

Below we consider the leading approximation in details. For simplicity we choose the Villain form of the action of the effective abelian gauge theory. It corresponds to the choice of modified action for the initial \( SU(2) \) gauge theory. One can easily see that, for the case of the Wilson action, all the results of this section remain valid, with some nonessential formal complications. For the Wilson loop we have

\[
\langle W_C \rangle = \int D\theta D\bar{\psi} D\psi \sum_n \exp(-\beta (d\theta + 2\pi n, d\theta + 2\pi n))
\]
\[
+ \sum_{\{xy\}} (\bar{\psi}_x - e^{2i\theta_{xy}} \bar{\psi}_y)(\psi_x - e^{-2i\theta_{xy}} \psi_y) + i(\theta, C)),
\]

(7)

where \( n \) is the integer - valued 1 - form.

Let us consider the duality transformation

\[
\langle W_C \rangle = \int D\theta D\bar{\psi} D\psi \sum_n \exp(-\beta (d\theta + 2\pi n, d\theta + 2\pi n))
\]
\[
+ \sum_{\{xy\}} (\bar{\psi}_x - e^{2i\theta_{xy}} \bar{\psi}_y)(\psi_x - e^{-2i\theta_{xy}} \psi_y) + i(\theta, C))
\]
\[
\int D\theta DF \sum_{\delta j=0}^n \sum_n (-1)^{k(j)} A[j] \exp(-\beta(F, F) + i(\theta, C + 2j)) \delta(F - d\theta - 2\pi n)
\]

= \int D\theta DF \sum_{\delta j=0}^\sigma \sum \exp(-\beta(F, F) + i(F - d\theta, 2\pi \sigma) + i(\theta, C + 2j))

= \sum_{\delta j=0} (-1)^{k(j)} A[j] \sum_{\delta \sigma = C + 2j} \exp(-\frac{\pi^2}{\beta}(\sigma, \sigma)). \tag{8}
\]

Here \( j \) is the integer - valued 1 - form representing the ghost worldlines, \( k(j) \) is the number of ghost loops, \( A[j] \) is a combinatorial factor, \( \sigma \) is the integer - valued 2 - form representing the electric flux strings. As it follows from the condition \( \delta \sigma = C + 2j \), the boundary of the string’s worldsheets consists of the given loop \( C \) and the ghost worldline \( j \). The ghost worldline creates two strings. This fact we can treat as follows. The boundary of the string’s worldsheets consists of the given loop only. The ghosts live on the string’s worldsheets and the worldsheets change it’s orientation on the ghost’s worldline.

Here we represented the fermion determinant as the sum over closed lines \( j \).

Thus, we can see, that the theory has the phase of confinement of the fundamental charges.

One can rewrite the expression (8) in the other way

\[
< W_C > = \lim_{m \to \infty} \int D\theta D\bar{\psi}D\psi \sum_n \exp(-\beta(d\theta + 2\pi n, d\theta + 2\pi n)
\]

\[-\frac{4\pi^2}{m^2}(dn, dn) + \sum_{[xy]} (\bar{\psi}_x - e^{2i\theta_{xy}}\bar{\psi}_y)(\psi_x - e^{-2i\theta_{xy}}\psi_y) + i(\theta, C))\]

= \lim_{m \to \infty} \sum_{\delta j=0}^\infty (-1)^{k(j)} A[j] \int DBD\epsilon \sum_i \exp(- \frac{1}{4\beta}(dB + 2\pi s[C + 2j], dB + 2\pi s[C + 2j])\]

5
\[-\frac{m^2}{4\beta}(d\epsilon + B + 2\pi l, d\epsilon + B + 2\pi l)). \]  

(9)

This expression defines the dual representation of the effective abelian theory. Here $B$ is the noncompact field, $\epsilon$ is the compact field, $\epsilon \in (-\pi, \pi]$, $s[C + 2j]$ is the surface, which boundary is $C + 2j$, $l$ is an integer valued one-form. We note, that the representation (8) follows from the expression (9) when we apply the BKT transformation [6].

Let us note, that if we omit the summation over ghost worldlines $j$, we come to the noncompact Abelian Higgs theory with infinitely deep potential [6] and external monopole current $C$. In this theory $B$ is the gauge field, $\epsilon$ is the phase of the monopole field: $\phi_{\text{monop}} = r \exp(i\epsilon)$. Here the radial part $r$ of the monopole field is frozen. Due to the condition $m \to \infty$ we have: $r \to \infty$.

Thus we treat the dual representation of the theory as the dual superconductor theory with additional monopole excitations, which are the ghosts of the initial representation. The infiniteness of parameters of the potential is the deficiency of our approximation.

4 The naive continuum limit of the effective theory.

The naive continuum limit of our theory is the QED with ghosts of charge 2. The perturbative expansion of this theory differs from the scalar electrodynamics by the sign ($-$) for the ghost loops. Due to this sign the model is asymptotic free. This gives us the reason to believe, that the lattice effective abelian theory has the continuum limit (as we know, the asymptotically free nonabelian gauge theories have the continuum limit, while the compact QED has not).

The naive continuum limit of the string representation of the theory in the Villain form is the string theory with the action:

\[ S = \text{const} \int d^2\sigma_1 \int d^2\sigma_2 \delta^{(4)}(x(\sigma_1) - x(\sigma_2)) \]  

(10)
\[ \partial_\alpha x^\mu(\sigma_1) \partial_\beta x^\nu(\sigma_1) \epsilon^{\alpha \beta} \partial_\gamma x^\mu(\sigma_2) \partial_\rho x^\nu(\sigma_2) \epsilon^{\gamma \rho} \]

Here the functions \( x^\mu(\sigma) \) represent the space-time coordinates of the string, \( \sigma_1 \) and \( \sigma_2 \) are the coordinates, which parametrize the worldsheet of the string. This string possesses the extended reparametrization invariance [5] and, thus, one can hope that it does not suffer from crumpling. Also, it is known that the string theory with this action solves the \( SU(\infty) \) - loop equations without contact terms. In this paper we have found that the string theory with this action naturally arises from the effective abelian gauge theory. The new feature is the existence of ghosts at the ends of the string. The naive continuum limit of the expression for the Wilson loop average (8) has the form

\[
< W_C > = \int D y (\tau) \int \delta [ x(\sigma) ] = C + 2 [ y(\tau) ] D x \exp (-c o n s t \int d^2 \sigma_1 \int d^2 \sigma_2 \delta^{(4)} (x(\sigma_1) - x(\sigma_2)) \partial_\alpha x^\mu(\sigma_1) \partial_\beta x^\nu(\sigma_1) \epsilon^{\alpha \beta} \partial_\gamma x^\mu(\sigma_2) \partial_\rho x^\nu(\sigma_2) \epsilon^{\gamma \rho} - c o n s t _2 \int d \tau \sqrt{\left( y'(\tau) \right)^2})
\]

Here the functions \( y^\mu(\tau) \) are the space-time coordinates of the worldline of ghosts, coordinate \( \tau \) parametrizes this worldline, \( k([y]) \) is the number of ghost loops. The functional integrals \( \int D y \) and \( \int D x \) are taken over the coordinates of the ghost worldline and over the coordinates of the string worldsheet respectively. The condition \( \delta [ x(\sigma) ] = C + 2 [ y(\tau) ] \) can be treated as in the previous section. The boundary of the worldsheet consists of the given loop \( C \). The ghosts live on the worldsheet and change its orientation. The measure \( D x \) is chosen in the correspondence with the natural metric on the worldsheet of the string \( h_{\alpha \beta}(\sigma) = \partial_\alpha x^\mu(\sigma) \partial_\beta x^\nu(\sigma) \). The measure \( D y \) corresponds to the natural metric on the ghost worldline \( h(\tau) = \frac{dy^\mu}{d\tau} \frac{dy^\mu}{d\tau} \).

The ghosts just change the orientation of the string’s worldsheet. It influence the string’s action only in the case, when one piece of the worldsheet is superimposed on the other one. The entropy of this case seems to be negligible and thus we treat the ghosts as unphysical objects. Hence, for example, there are no reasons to consider the asymptotic states created by them.
5 Conclusions

In this paper we derive the effective abelian theory for the $SU(2)$-gluodynamics and the corresponding action for the chromoelectric string. This theory possesses the asymptotic freedom and, thus, we hope that it’s lattice version has the continuum limit. The string action possesses the additional reparametrization invariance and we believe, that the crumpling problem is absent for it.

The theory obtained is shown to have the phase, in which the fundamental charges are confined. Our approximation is the zeroth order of the expansion in the small parameter existing in the maximal abelian gauge of the theory. We do not know what is the behavior of this parameter when $\beta$ tends to $\infty$ where we have the phase transition to the continuum theory. If this parameter remains small and the effective abelian theory has the continuum limit as $\beta$ tends to $\infty$, than we can state that it is a selfconsistent approximation to gluodynamics, which demonstrates the mechanism of confinement and gives the reasonable expression for the chromoelectric string’s action.

One can easily see that our consideration remains valid with some nonessential complications in the case of $SU(N)$ theory.

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