Qutrit squeezing via semiclassical evolution

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Abstract. We introduce a concept of squeezing in collective qutrit systems through a geometrical picture connected to the deformation of the isotropic fluctuations of $su(3)$ operators when evaluated in a coherent state. This kind of squeezing can be generated by Hamiltonians nonlinear in the generators of $su(3)$ algebra. A simplest model of such a nonlinear evolution is analyzed in terms of semiclassical evolution of the $SU(3)$ Wigner function.

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* Dedicated to the memory of Fred Mueller.
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1. Introduction

The concept of squeezing in different systems has attracted significant attention due to its transparent physical meaning, related to the reduction of quantum fluctuations of a given observable below some specified threshold at the expense of an increase in the fluctuations of another observable. Although most of the applications of squeezing are related to the improvement of precision measurements, squeezing intrinsically reflects the existence of particular correlations between basis states in the Hilbert space of a quantum system. Since the squeezing parameters contain easily measurable first- and second-order moments of collective operators, this entails successful application of squeezing criteria to detect quantum entanglement [1–3].

Historically, much attention has been paid to squeezing of the electromagnetic field modes or squeezing in $SU(2)$—or spin-like—systems. Recently, more complex experiments on quantum systems with higher symmetries have been proposed, particularly in relation to possible applications to quantum information processes [5]. Candidate qutrit or three-level systems described by the group $SU(3)$ include Bose–Einstein condensates [6] and biphoton systems [7], among others.

The definition of squeezing, while universal for harmonic oscillator-like systems, is otherwise far from unique. In spin-like systems there are several approaches used to define a squeezing parameter [1, 3, 4, 8–11]. All parameters compare fluctuations of suitably chosen observables with a threshold given by fluctuations in some reference state (or family of states). The coherent states of the corresponding quantum system are often taken as the family of reference states.

One of the crucial properties of coherent states is the invariance of the fluctuations of some observables under a certain type of continuous transformations. In this paper, we use this property of coherent states to define the so-called standard quantum limit [12] and introduce the concept of squeezing. The main idea consists in defining the full family $\mathcal{K}$ of collective operators (which in practice are some linear combinations of generators of the $su(3)$ algebra) for which the fluctuations evaluated using $SU(3)$ coherent states are invariant under the same group transformation that leaves invariant the fiducial state used to construct the set of coherent states.

We will show that for a Hilbert space carrying an irreducible representation (henceforth irrep) of $SU(3)$ of symmetric type, we can use three continuous parameters $\alpha_3, \beta_3, \chi$ to label a generic element $\mathcal{K}(\alpha_3, \beta_3, \chi) \in \mathcal{K}$, but fluctuations of $\mathcal{K}(\alpha_3, \beta_3, \chi)$, when evaluated using a suitable $SU(3)$ coherent state, are isotropic, i.e. do not depend on $\alpha_3, \beta_3, \chi$. Considering these (invariant) fluctuations as defining our threshold, we introduce squeezing as a reduction of fluctuations below the limit of these isotropic fluctuations in coherent states.

Since our objective is to show how $SU(3)$ squeezing can emerge rather than to propose a general criterion, we will focus on the deformations of probability distributions resulting from the Hamiltonian evolution of an initial coherent state. Geometrically, a group transformation obtained by exponentiating a linear combination of generators and acting on a state produces a simple rigid displacement of the associated probability distribution and is not associated with the introduction of correlations. A deformation of the probability density does mean that quantum correlations between the original basis states are generated; hence quantum correlations that generate the squeezing can only arise from nonlinear interactions.

As the characteristic times needed to produce such correlations are inversely proportional to some power of the dimension of the system, correlations develop very rapidly and the analysis
can be done using semi-classical methods. In this paper we will use the $SU(3)$ Wigner function method \cite{13} to describe a nonlinear evolution of a quantum system with the $SU(3)$ symmetry group.

The paper is organized as follows. In section 2 we briefly recall general ideas on the coherent states for systems with $SU(2)$ and $SU(3)$ symmetries and construct the operators with isotropic fluctuations in the corresponding coherent states. In section 3 we analyze squeezing generated by a simple nonlinear $SU(3)$ Hamiltonian. In section 4, the $SU(3)$ Wigner function formalism is presented and applied to find the evolution of the squeezing parameter under the nonlinear Hamiltonian.

2. Coherent states

Following the general construction \cite{14,15}, a coherent state for a system with a given symmetry group $G$ acting irreducibly in a Hilbert space $\mathbb{H}$ is defined as a fiducial state displaced by a group transformation in $G$. We take this fiducial state to be the highest weight state of the irrep carried by $H$. The highest weight state is invariant (up to a phase) under transformation from the subgroup $H \subset G$, so displacements of this state are labelled by points $\Omega$ on the coset $G/H$. The latter is known to be the classical phase space of the corresponding quantum system \cite{16}.

Below, we briefly review coherent states for the $SU(2)$ and $SU(3)$ groups, focusing only on the symmetric representations. In this case a coherent state can be considered as a composite state, occurring as a direct product of identical 'single-particle' states of systems with two or three energy levels, and invariant under permutation of the 'particle' labels. In other words, coherent states can be conveniently thought of as symmetric (under permutation of particles) factorized states, thus displaying maximal classical correlations. Given any coherent state we can always find an operator written as a linear combination of generators such that the fluctuations of this operator evaluated in the coherent state are invariant with respect to the transformations generated by the stationary subgroup $H$. Moreover, the fluctuations of this operator reach a value determined by the dimension of $\mathbb{H}$.

2.1. $SU(2)$ coherent states

The $su(2)$ algebra is spanned by $\{ \hat{S}_+, \hat{S}_-, \hat{S}_z \}$, with nonzero commutation relations

$$[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_z. \quad (1)$$

A basis for the irrep $j$ of dimension $2j+1$ is spanned by the states $\{|jm\rangle, m = -j, \ldots, j\}$. The basis states satisfy

$$\hat{S}_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle, \quad \hat{S}_z |jm\rangle = m |jm\rangle. \quad (2)$$

The highest weight state is $|jj\rangle$. It is invariant (up to a phase) under the subgroup $H = \{ T(\gamma) \equiv e^{-i\gamma\hat{S}_z} \}$. The parameter $\gamma$ ranges between 0 and $2\pi$ when $2j$ is even, and between 0 and $4\pi$ when $2j$ is odd. We can now define a family $\mathcal{S}$ of observables through

$$\mathcal{S} = \{ T(\chi) \hat{S}_x T^{-1}(\chi) \}, \quad T(\chi) = e^{-ix\hat{S}_z} \in H. \quad (3)$$

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A typical element of the family is
\[ \hat{S}(\chi) \equiv T(\chi) \hat{S}_x T^{-1}(\chi) = \hat{S}_x \cos \chi + \hat{S}_y \sin \chi. \]
(4)
For any \( \hat{S}(\chi) \in \mathcal{S} \), we find, using \(|jj\rangle\), that
\[ (\Delta \hat{S}(\chi))^2 \equiv \langle j, j | \hat{S}(\chi)^2 | j, j \rangle - \langle j, j | \hat{S}(\chi) | j, j \rangle^2 = j, \]
(5)
independent of the element \( T(\chi) \in \mathcal{H} \).

The standard set \( \{ |\vartheta, \varphi\rangle \} \) of \( SU(2) \) coherent states of angular momentum \( j \) is defined as
\[ |\vartheta, \varphi\rangle = D(\vartheta, \varphi) |j, j\rangle, \]
(6)
where \( D(\vartheta, \varphi) = \exp(-i/2(\cos \varphi \hat{S}_x - \sin \varphi \hat{S}_y)). \) The coherent states (6) can be represented as a product of 2\( j \) one-qubit states,
\[ |\vartheta, \varphi\rangle \propto |\vartheta, \varphi\rangle_1 \otimes |\vartheta, \varphi\rangle_2 \otimes \ldots \otimes |\vartheta, \varphi\rangle_{2j}, \]
(7)
\[ |\vartheta, \varphi\rangle_a \equiv e^{i\varphi/2} \sin \frac{1}{2} \vartheta \left| + \frac{1}{2} \right>_a + e^{-i\varphi/2} \cos \frac{1}{2} \vartheta \left| - \frac{1}{2} \right>_a. \]
(8)

\( |\vartheta, \varphi\rangle \) is completely specified geometrically through the direction \( \vec{n} = (n_x, n_y, n_z) \) of the mean spin vector \( \langle \vec{S} \rangle \):
\[ n_x = \sin \vartheta \cos \varphi = \langle \vartheta, \varphi | \hat{S}_x | \vartheta, \varphi \rangle / j, \]
\[ n_y = \sin \vartheta \sin \varphi = \langle \vartheta, \varphi | \hat{S}_y | \vartheta, \varphi \rangle / j, \]
\[ n_z = \cos \vartheta = \langle \vartheta, \varphi | \hat{S}_z | \vartheta, \varphi \rangle / j. \]
(9)

A property of coherent states essential to us is the existence of a special tangent plane orthogonal to the direction \( \vec{n} \). If we define a direction vector \( \vec{n}_\perp(\chi) \) as \( D(\vartheta, \varphi) T(\chi) \hat{x} \) with \( \hat{x} = (1, 0, 0) \), we find \( \vec{n}_\perp(\chi) \cdot \vec{n} = 0 \) for any \( \chi \).

The observable
\[ \hat{S}_\perp(\vartheta, \varphi; \chi) \equiv \vec{n}_\perp(\chi) \cdot \vec{S} = D(\vartheta, \varphi) T(\chi) \hat{S}_x T^{-1}(\chi) D^{-1}(\vartheta, \varphi) \]
(10)
satisfies
\[ (\Delta \hat{S}_\perp(\vartheta, \varphi; \chi))^2 = j. \]
(11)
independent of the angles \( \vartheta, \varphi \) and \( \chi \) when evaluated using \(|\vartheta, \varphi\rangle\).

We will use the condition (11) to fix the threshold of quantum fluctuations and use this to define spin squeezing as was done by many authors [17]: a state of angular momentum \( j \) is squeezed if there is an orientation of \( \vec{n}_\perp(\chi^*) \) in the tangent plane, defined for \( T(\chi^*) \in \mathcal{H} \), for which
\[ (\Delta \hat{S}_\perp(\vartheta, \varphi; \chi^*))^2 \leq j. \]
(12)

2.2. \( SU(3) \) coherent states for \((\lambda, 0)\) irreps

For \( su(3) \) we consider symmetric irreducible representations of the type \((\lambda, 0)\). The algebra is spanned by the six ladder operators \( \hat{C}_{ij} \), where \( i, j = 1, 2, 3 \) with \( i \neq j \) and two Cartan elements \( \hat{h}_1 = 2 \hat{C}_{11} - \hat{C}_{22} - \hat{C}_{33}, \hat{h}_2 = \frac{1}{3}(\hat{C}_{22} - \hat{C}_{33}) \). A convenient realization is given in terms of harmonic oscillator creation and destruction operators for mode \( i \) by \( \hat{C}_{ij} = \tilde{a}_i^\dagger \tilde{a}_j \) acting on
the harmonic oscillator kets $|n_1 n_2 n_3\rangle$ with $n_1 + n_2 + n_3 = \lambda$. One verifies, for instance,

$$[\hat{C}_{ij}, \hat{C}_{kl}] = \hat{C}_{ik}\delta_{jk} - \hat{C}_{kj}\delta_{il},$$

$$\hat{C}_{12} |n_1 n_2 n_3\rangle = \sqrt{(n_1 + 1)n_2} |n_1 + 1, n_2 - 1, n_3\rangle.$$ (13) (14)

$SU(3)$ elements are parametrized following a slight adaptation of [18] by

$$R(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2) = R_{23}(\alpha_1, \beta_1, -\alpha_1)R_{12}(\alpha_2, \beta_2, -\alpha_2) \times R_{23}(\alpha_3, \beta_3, -\alpha_3)e^{-i\gamma_1\hat{h}_1}e^{-i\gamma_2\hat{h}_2},$$

where $R_{ij}(\eta, \theta, \varphi)$ is a transformation of the $SU(2)$ subgroup with subalgebra spanned by $\hat{C}_{ij}, \hat{C}_{ji}, \frac{1}{2} [\hat{C}_{ij}, \hat{C}_{ji}]$.

The highest weight state $|\lambda, 00\rangle$ is invariant (up to a phase) under transformations of the type $R_{23}(\alpha_3, \beta_3, -\alpha_3)e^{-i\gamma_1\hat{h}_1}e^{-i\gamma_2\hat{h}_2}$, which generate a $\mathcal{H} = U(2)$ subgroup. Coherent states are labeled by points on $SU(3)/U(2) \sim S^4$. Thus, using $\omega = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ as the coordinates on $S^4$, we generate the coherent state $|\omega\rangle$ in the standard form [14, 15] as orbit of the highest weight $|\lambda, 00\rangle$ under the action of the displacement operator on $S^4$:

$$|\omega\rangle = D(\omega)|\lambda, 00\rangle \equiv R_{23}(\alpha_1, \beta_1, -\alpha_1)R_{12}(\alpha_2, \beta_2, -\alpha_2)|\lambda, 00\rangle.$$ (16)

This coherent state can also be represented as a product of $\lambda$ one-qutrit states

$$|\omega\rangle \propto |\omega_1\rangle \otimes |\omega_2\rangle \otimes \cdots \otimes |\omega_\lambda\rangle,$$

$$|\omega_a\rangle = \cos \frac{1}{2}\beta_2 |100\rangle_a + e^{i\alpha_2} \cos \frac{1}{2}\beta_1 \sin \frac{1}{2}\beta_2 |010\rangle_a + e^{i(\alpha_1 + \alpha_2)} \sin \frac{1}{2}\beta_1 \sin \frac{1}{2}\beta_2 |001\rangle_a.$$ (17) (18)

$|\omega\rangle$ is completely determined by a ‘mean vector’ $\bar{n}$ with (complex) components

$$\bar{n} = (\langle \hat{C}_{23}, \langle \hat{C}_{32}, \langle \hat{C}_{12}, \langle \hat{C}_{21}, \langle \hat{C}_{13}, \langle \hat{C}_{31}, \langle \hat{h}_1, \langle \hat{h}_2).$$

(A vector with eight real components is obtained using $\langle \hat{C}_{ij} \rangle + \langle \hat{C}_{ji} \rangle$ and $-i(\langle \hat{C}_{ij} \rangle - \langle \hat{C}_{ji} \rangle$.)

With

$$T \equiv R_{23}(\alpha_3, \beta_3, -\alpha_3)e^{-i\gamma_1\hat{h}_1}e^{-i\gamma_2\hat{h}_2} \in \mathcal{H},$$

it is easy to verify that the variance of the observable

$$\hat{K}(\alpha_3, \beta_3, \chi) \equiv T(\hat{C}_{13} + \hat{C}_{31})T^{-1}$$

$$=(\hat{C}_{13} + \hat{C}_{31}) \cos \frac{1}{2}\beta_3 \cos \frac{1}{2}\chi - i(\hat{C}_{13} - \hat{C}_{31}) \cos \frac{1}{2}\beta_3 \sin \frac{1}{2}\chi$$

$$-(\hat{C}_{12} + \hat{C}_{21}) \sin \frac{1}{2}\beta_3 \cos (\alpha_3 - \frac{1}{2}\chi) - i(\hat{C}_{12} - \hat{C}_{21}) \sin \frac{1}{2}\beta_3 \sin (\alpha_3 - \frac{1}{2}\chi),$$ (21) (22)

where $\chi = 6\gamma_1 + \gamma_2$, when evaluated using the highest weight state $|\lambda, 00\rangle$, is $\lambda$ and independent of the angles $(\alpha_3, \beta_3, \gamma_1, \gamma_2)$. Hence, the variance of

$$\hat{K}_\perp(\omega; \alpha_3, \beta_3, \chi) = D(\omega)\hat{K}(\alpha_3, \beta_3, \chi)D^{-1}(\omega),$$

when evaluated using the coherent state $D(\omega)|\lambda, 00\rangle$, is also independent of the ‘direction’ $(\alpha_3, \beta_3, \chi)$ in the ‘tangent hyperplane’ perpendicular to $\bar{n}$, and equal to $\lambda$. Thus, we will use $(\Delta \hat{K}_\perp(\omega; \alpha_3, \beta_3, \chi))^2 = \lambda$ as our squeezing threshold and define an $su(3)$ state $|\psi\rangle$ as squeezed if there is an observable of the form $\hat{K}_\perp(\omega; \alpha_3^*, \beta_3^*, \chi^*)$ for which

$$(\Delta \hat{K}_\perp(\omega; \alpha_3^*, \beta_3^*, \chi^*))^2 < \lambda$$

(23) (24)

when evaluated in $|\psi\rangle$. Other approaches to squeezing criteria are possible [19].
3. Semiclassical squeezing

Squeezing related to a given algebra of observables is understood to reflect correlations (commonly called quantum correlations) between the components of a basis of an irrep. As mentioned before, group transformations, obtained by exponentiation of linear combinations of elements from the algebra, produce rigid displacements of the basis states. Correlations between basis states cannot as a matter of definition be induced by such group transformations. Rather, correlations can be either constructed through a special preparation or obtained as a result of nonlinear (in terms of the algebra of observables) transformations (usually from nonlinear Hamiltonian evolution) applied to initially uncorrelated systems.

In the case of large systems, it is convenient to analyze the evolution using the phase-space approach. The reasons are twofold: we can, on the one hand, represent the initial state as a real-valued function and ‘draw’ it (for some appropriately chosen cuts) in the form of a distribution ‘covering’ some slices of the phase space, but more importantly we can also deduce many qualitative features of the time evolution of this distribution. For a wide class of quantum systems with a symmetry group $G$, the phase-space functions are defined through an invertible map, so that we associate with an operator $\hat{X}$ a phase-space symbol $\hat{X} \mapsto \omega_X$, (25) where the quantization kernel $\hat{w}(\Omega)$ is a Hermitian operator defined on the classical manifold $G/H$ and $\Omega$ denotes the phase-space coordinates.

A feature of this mapping is that the commutator of two elements $\hat{X}$ and $\hat{Y}$ of the Lie algebra $\mathfrak{g}$ corresponding to the group $G$ is mapped to the Poisson brackets of the respective symbols:

$$[\hat{X}, \hat{Y}] \propto \{\omega_X, \omega_Y\}_P. \quad (26)$$

The commutator of two generic operators is, in general, mapped to the so-called Moyal bracket.

For $SU(3)$ irreps of the type $(\lambda, 0)$ and $\lambda \gg 1$, and for sufficiently localized initial states in a class dubbed ‘semiclassical’ [21, 24], the short time dynamics can be well described by the Liouville-type equation for the evolution of the Wigner function:

$$\partial_t \omega_{\rho}(\Omega) = \varepsilon \{\omega_{\rho}(\Omega), \omega_H(\Omega)\}_P + O(\varepsilon^3), \quad (27)$$

where $\omega_{\rho}(\Omega)$ is the Wigner function, i.e. the symbol of the density matrix $\hat{\rho}$ of the system, $\omega_H(\Omega)$ is the symbol of the Hamiltonian, and $\varepsilon$ is the so-called semiclassical parameter. The Poisson bracket is, in fact, the leading term in an expansion of the Moyal bracket in inverse powers of the square root of eigenvalue of one of the Casimir operators in the $SU(3)$ irrep $(\lambda, 0)$; we found that, for the mapping defined in [13] on $SU(3)/U(2)$, the semiclassical parameter $\varepsilon$ is

$$\varepsilon = \frac{1}{2\sqrt{\lambda(\lambda + 3)}}. \quad (28)$$

The solution of (27) can be written in general form as

$$\omega_{\rho}(\Omega|t) = \omega_{\rho}(\Omega(\delta t)), \quad (29)$$

where $\Omega(\delta t)$ denotes classical trajectories on $SU(3)/U(2)$. The approximation of dropping in equation (27) higher order terms in $\varepsilon$ describes well the initial stage of the nonlinear dynamics, when self-interference is negligible. In physical applications, semiclassical states often have the
form of localized states (e.g. coherent states) and their ‘classicality’ depends on non-invariance under the transformations induced by symmetry subgroups of the (nonlinear) Hamiltonian (‘classicality’ is a subtle and delicate question not addressed here) [25, 26].

The method of the Wigner functions allows us to calculate average values of the observables giving drastically better results than the ‘naive’ solution of the Heisenberg equations of motion with decoupled correlators. On the other hand, the quantum phenomena that are due to self-interference (such as Schrödinger cats) are beyond the scope of this semiclassical approximation.

3.1. Phase-space considerations

From the parametrization of the coherent state of equation (16), we deduce a Poisson bracket on \(S^4\), given by

\[
\{f, g\} = \frac{4}{\sin \beta_1 \sin^2 \frac{1}{2} \beta_2} \left( \frac{\partial f}{\partial \alpha_1} \frac{\partial g}{\partial \beta_1} - \frac{\partial g}{\partial \alpha_1} \frac{\partial f}{\partial \beta_1} \right) - \frac{2 \tan \frac{1}{2} \beta_1}{\sin^2 \frac{1}{2} \beta_2} \left( \frac{\partial f}{\partial \alpha_2} \frac{\partial g}{\partial \beta_1} - \frac{\partial g}{\partial \alpha_2} \frac{\partial f}{\partial \beta_1} \right)
\]

\[+ \frac{4}{\sin \beta_2} \left( \frac{\partial f}{\partial \alpha_3} \frac{\partial g}{\partial \beta_2} - \frac{\partial g}{\partial \alpha_3} \frac{\partial f}{\partial \beta_2} \right), \tag{30}\]

where \(f\) and \(g\) are any two functions on \(SU(3)/U(2)\).

Following the prescription of [13], we associate with an operator \(\hat{X}\) a phase-space symbol \(W_X(\Omega)\) according to equation (25). This map is linear on \(\hat{X}\), so we only need to consider the phase-space symbols of a basis set constructed from \(su(3)\) tensors \(T_{(\nu_1 \nu_2 \nu_3)}^{(\sigma, \sigma)}\), which transforms under conjugation by \(g \in \mathcal{G}\) as the state \(|(\sigma, \sigma)_{\nu_1 \nu_2 \nu_3} I\rangle\) in irrep \((\sigma, \sigma)\) transforms under \(g\).

\(T_{(\nu_1 \nu_2 \nu_3)}^{(\sigma, \sigma)}\) takes the general form

\[
T_{(\nu_1 \nu_2 \nu_3)}^{(\sigma, \sigma)} I = \sum_{n_1 n_2 n_3 m_1 m_2 m_3} |n_1 n_2 n_3 \rangle \langle m_1 m_2 m_3| \hat{C}^{(\sigma, \sigma)}_{(\nu_1 \nu_2 \nu_3) I} C_{n_1 n_2 n_3; m_1 m_2 m_3}, \tag{31}\]

with \(|n_1 n_2 n_3\rangle\) a state in the irrep \((\lambda, 0)\), \(|m_1 m_2 m_3\rangle\) an element in the dual representation \((0, \lambda)\) and \(\hat{C}^{(\sigma, \sigma)}_{(\nu_1 \nu_2 \nu_3) I}\) a coefficient closely related to the \(su(3)\) Clebsch–Gordan coefficient occurring in the decomposition of elements in \((\lambda, 0) \otimes (0, \lambda) \rightarrow (\sigma, \sigma)\) [23]. Note that weights in \((\lambda, 0)\) and \((0, \lambda)\) are multiplicity-free so the triples \((n_1 n_2 n_3)\) and \((m_1 m_2 m_3)\) are enough to uniquely identify the states.

For irreps of the type \((\sigma, \sigma)\), some weights occur multiple times and the label \(I\), which specifies transformation properties of the states under \(SU(2)\) transformations generated by \(R_{23}\), is required to fully distinguish states with the same weights. The tensors \(T^{(1, 1)}_{(\nu_1 \nu_2 \nu_3) I}\) are proportional to the generators of the \(su(3)\) algebra.

The \(SU(3)\) quantization kernel \(\hat{w}(\Omega)\) given in [13] can be expanded so as to take the explicit form

\[
\hat{w}(\Omega) = \sum_{\sigma} \sqrt{\frac{\dim(\sigma, \sigma)}{\dim(\lambda, 0)}} \sum_{\nu_1 \nu_2 \nu_3; I} D^{(\sigma, \sigma)}_{(\nu_1 \nu_2 \nu_3) I; (\sigma, \sigma) 0}(\Omega) T^{(\sigma, \sigma)}_{(\nu_1 \nu_2 \nu_3) I} , \tag{32}\]

where \(\dim(\lambda, \mu) = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2)\) the dimension of the irrep \((\lambda, \mu)\) and where \(D\) is an \(SU(3)\) group function defined in the usual way as the overlap of two \(su(3)\) states in the irrep \((\sigma, \sigma)\):

\[
D^{(\sigma, \sigma)}_{(\nu_1 \nu_2 \nu_3) I; (\sigma, \sigma) 0}(\Omega) \equiv \langle (\sigma, \sigma)_{\nu_1 \nu_2 \nu_3}; I | R(\Omega) | (\sigma \sigma \sigma \sigma) 0 \rangle. \tag{33}\]
A tensor $T^{(\sigma,\sigma)}_{\mu\nu}$ is thus mapped to the phase-space function

$$T^{(\sigma,\sigma)}_{\mu\nu} \mapsto W^{(\sigma,\sigma)}_{\mu\nu}(\Omega) = \sqrt{\frac{2(\sigma + 1)^3}{(\lambda + 1)(\lambda + 2)}} D^{(\sigma,\sigma)}_{\mu\nu}(\lambda,00)(\Omega). \quad (34)$$

The Wigner function corresponding to $|\lambda,00\rangle\langle\lambda,00|$ is given by

$$W^{|\lambda,00\rangle\langle\lambda,00|}(\Omega) = \sum_{\sigma=0}^{\lambda} \bar{C}^{(\sigma,\sigma)}_{\lambda,00,00} \sqrt{\frac{2(\sigma + 1)}{(\lambda + 1)(\lambda + 2)}} \left( P_{\sigma+1}(\cos \beta_2) - P_{\sigma}(\cos \beta_2) \right) \cos \beta_2 - 1, \quad (35)$$

with $P_{\ell}$ a Legendre polynomial of order $\ell$. For $\lambda \gg 1$, we have found, with much similarity to the $SU(2)$ case [27], that $W^{|\lambda,00\rangle\langle\lambda,00|}(\beta_2)$ is well approximated by

$$W^{|\lambda,00\rangle\langle\lambda,00|}(\beta_2) \approx A e^{\lambda(\cos \beta_2 - 1)}, \quad (37)$$

where $A = \frac{4\lambda^3}{(\lambda + 1)(\lambda + 2)}$ is a constant obtained so that the normalization condition

$$1 = \frac{(\lambda + 1)(\lambda + 2)}{8\pi^2} \int d\Omega \ W_{\rho_\omega}(\Omega), \quad (38)$$

$$\int d\Omega = \int_0^{2\pi} d\alpha_2 \int_0^{2\pi} d\alpha_1 \int_0^\pi \sin \beta_1 d\beta_1 \int_0^\pi \frac{1 - \cos \beta_2}{4} \sin \beta_2 \quad (39)$$

is satisfied. The approximate expression (37) does not describe very well the tail of the Wigner function, but for our purposes this is not essential.

For the coherent state $|\omega\rangle = R(\omega)|\lambda,00\rangle$, the density operator $\hat{\rho}_\omega = |\omega\rangle\langle\omega|$ is mapped to the Wigner function $W_{\rho_\omega}(\Omega)$

$$W_{\rho_\omega}(\Omega) = W^{|\lambda,00\rangle\langle\lambda,00|}(\omega^{-1}\Omega). \quad (40)$$

### 3.2. Semiclassical evolution

A simple Hamiltonian that leads to squeezing is

$$\hat{H} = \hat{h}_1^2 - \frac{2\lambda + 3}{5} \hat{h}_1, \quad \hat{h}_1 = 2\hat{C}_{11} - \hat{C}_{22} - \hat{C}_{33}, \quad (41)$$

where the factor $\frac{2\lambda + 3}{5}$ is chosen so that no terms in $T^{(1,1)}_{\mu\nu\Psi}$ appear in the expansion of $H$; this guarantees that no rigid motion on the $S^4$ sphere is produced. This choice of $H$ is motivated on the following physical grounds. The operator $\hat{h}_1$ is invariant under the same $U(2)$ transformations that leave the highest weight invariant. Squeezing resulting from its evolution is thus a pure $SU(3)$ effect, distinct from $SU(2)$ correlations that are present in the individual $U(2)$ subspaces contained in $(\lambda,0)$. Pure $SU(2)$ correlations generated by nonlinear Hamiltonians have been analyzed elsewhere [28]. The symbol for this Hamiltonian is (up to a constant factor)

$$W_H = \frac{q}{\delta t} \sqrt{(\lambda - 1)\lambda(\lambda + 3)(\lambda + 4)} (3 + 4 \cos \beta_2 + 5 \cos(2\beta_2)). \quad (42)$$

We choose as initial state a coherent state with coordinates $\omega = (A_1, B_1, A_2, B_2)$, so it ‘sits’ above the minimum of $H$ in (42), i.e. is located at $A_1 = B_1 = A_2 = 0$ and $B_2 = \arccos(-1/5)$. 

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If we write the coset representative of $\omega^{-1}\Omega$ as $(\vec{\alpha}_1, \vec{\beta}_1, \vec{\alpha}_2, \vec{\beta}_2)$, we find for the coherent state and its symbol, respectively,

$$|\omega\rangle = R_{12}(0, B_2, 0)|\lambda 00\rangle,$$

$$W_{\rho_\omega}(\Omega) = W_{|\lambda 00\rangle\langle 00\lambda|}(\vec{\beta}_2),$$

$$\cos \vec{\beta}_2 = -1 + 2 \cos^2(\frac{1}{2} B_2) \cos^2(\frac{1}{2} \beta_2) + 2 \cos^2(\frac{1}{2} \beta_1) \sin^2(\frac{1}{2} B_2) \sin^2(\frac{1}{2} \beta_2) + \cos(\alpha_2) \cos(\frac{1}{2} \beta_1) \sin(\beta_2) \sin(B_2),$$

with $W_{|\lambda 00\rangle\langle 00\lambda|}$ given in equation (36).

Typical squeezing times scale as $t \sim \lambda^{-p}$, $p > 0$ and are much shorter than self-interference times. Hence, using $|\omega\rangle$ as the initial state, we can use equation (27) to obtain the approximate evolution as

$$\frac{\partial W_{\rho_\omega}}{\partial t} = \frac{9}{5} \sqrt{(\lambda - 1)(\lambda + 4)}(1 + 5 \cos \beta_2) \frac{\partial W_{\rho_\omega}}{\partial \alpha_2}.$$  

This in turn implies that the angle $\alpha_2$ evolves in time according to

$$\alpha_2(t) = \alpha_2 - \frac{9}{5} \sqrt{(\lambda - 1)(\lambda + 4)}(1 + 5 \cos \beta_2)t,$$

all other angles having no time dependence. Thus, the time evolution of the system is obtained by the replacement $\alpha_2 \rightarrow \alpha_2(t)$ of equation (47) in the argument $\cos \vec{\beta}_2$ of equation (45) in the Wigner function of equation (44):

$$W_{\rho_\omega}(\vec{\beta}_2(t)) = W_{\rho_\omega}(\vec{\beta}_2).$$

3.3. Semiclassical squeezing

In figure 1, we present as a three-dimensional (3D) plot and as a contour plot the Wigner function for the initial state (43), time-evolved using the exact quantum mechanical evolution equation. The slices are taken at $\alpha_1 = \beta_1 = 0$ and at specific values of $t = 0, 0.008$ and $0.015$ as indicated. (The value of $t = 0.015$ is the time at which the fluctuation of $(\Delta \tilde{K}_\perp(\Omega; \alpha_3, \beta_3, \chi^*)(t))^2$ reaches a minimum, as seen in figure 3.) One observes that the initial coherent state is rapidly deformed from its nearly Gaussian shape in $S^4$. In particular, small negative regions are generated in the vicinity of the main peak. By plotting the Wigner functions for various values of $\alpha_1$ and $\beta_1$, one observes the function spreading beyond the original tangent hyperplane. As these plots do not add to the substance of our analysis of the deformation of the Wigner function, we choose not to present them.

Figure 2 illustrates the 3D and contour plots of slices of the Wigner function time-evolved using semiclassical evolution of the initial state. The times and slices are the same as those for the exact evolution to facilitate comparisons. Obviously, we cannot observe negative regions in the Wigner function.

Fluctuations of the operator $\hat{K}_\perp(\Omega; \alpha_3, \beta_3, \chi)$ of equation (23) are invariant under $U(2)$ transformations $T$ when evaluated using the coherent state $|\omega\rangle$ of equation (43). If the quantum correlations are induced by a nonlinear Hamiltonian leaving stationary the mean vector of equation (19) characterizing $|\omega\rangle$, we can use the same observables $\hat{K}_\perp(\Omega; \alpha_3, \beta_3, \chi)$ to detect squeezing. Operationally, this means the fluctuations of $\hat{K}_\perp(\Omega; \alpha_3, \beta_3, \chi)$ will now depend on the parameters $\alpha_3, \beta_3, \chi = 6\gamma_1 + \gamma_2$ of the transformation $T$ of equation (20) through the

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Figure 1. Slices of the Wigner function for the initial state (43), evolved using the exact evolution equation, for $t = 0, 0.008$ and $0.015$. The slices are taken at $\alpha_1 = \beta_1 = 0$. Note the small negative regions near the central peak at $t > 0$.

Figure 2. Slices of the Wigner function for the initial state (43), evolved using the classical evolution equation, for $t = 0, 0.008$ and $0.015$. The slices are taken at $\alpha_1 = \beta_1 = 0$. There are no regions where the function is negative.

combinations of equation (22), in such a way that there may exist ‘directions’ parameterized by $\alpha_3^*, \beta_3^*, \chi^*$ in the tangent hyperplane where the fluctuations are smaller than in the coherent state $|\omega\rangle$. It remains to select from those directions the one along which the fluctuations are smallest to complete our definition of squeezing.
Figure 3. The time evolution of the smallest fluctuations of a system having as initial state the coherent state (43). The full line is the smallest fluctuation of \( \hat{K}_\perp (\Omega ; \alpha_3, \beta_3, \chi) \) calculated using the quantum evolution of (43), the thick dashed line was obtained using the classical evolution of the exact Wigner function (44) for (43) and the thick dotted line was obtained using the classical evolution of the approximate Wigner function (37) for (43).

Average values and the fluctuations are computed using the standard phase-space techniques, i.e. integrating the symbols of \( \hat{K}_\perp (\Omega ; \alpha_3, \beta_3, \chi) \) and its square with the time-evolved Wigner function. Although the analytical integration can be done, the corresponding expressions for \( (\Delta \hat{K}_\perp (\Omega ; \alpha_3^*, \beta_3^*, \chi^*) (t))^2 \) are formidable; we will only provide numerical results and compare in figure 3 the results of exact quantum mechanical calculations with those obtained from the Wigner function method.

Figure 3 shows the time evolution of the smallest fluctuations of \( \hat{K}_\perp (\Omega ; \alpha_3, \beta_3, \chi) \) for the initial coherent state (43) or its approximation (37) (where \( \beta_2 \rightarrow \bar{\beta}_2 \)) with \( \lambda = 20 \) under the Hamiltonian \( \hat{H} = \hat{h}_1^2 - \frac{1}{2} \hat{h}_1 \). The best squeezing direction \( (\alpha_3^*, \beta_3^*, \chi^*) \) has been found through numerical optimization.

The results are illustrative of a number of calculations performed for irreps of the type \((\lambda, 0)\) with \( \lambda \) ranging between 10 and 30. The differences between the exact quantum evolution and the classical evolutions decrease with \( \lambda \). Through numerical simulations, we have found that the location in time of the minimum of \( (\Delta \hat{K}_\perp (\Omega ; \alpha_3^*, \beta_3^*, \chi^*) (t))^2 \) scales as \( t_{\min} \sim \lambda^{-9/11} \) and the effective squeezing, defined as the ratio of the minimum \( (\Delta \hat{K}_\perp (\Omega ; \alpha_3^*, \beta_3^*, \chi^*) (t))^2 / \lambda \), scales as \( \lambda^{-1/3} \) for large values of \( \lambda \).

4. Conclusion

We have shown that the reduction of fluctuations in the systems with \( SU(3) \) symmetry can be achieved in a manner similar to the reduction in spin-like systems: by correlating initially factorized coherent states via an evolution generated by a Hamiltonian nonlinear on the generators of the \( su(3) \) algebra.

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We constructed the Hamiltonian in such a way that it does not produce a rigid motion of the initial state, so we can use as observables those having uniform fluctuations in a coherent state as the reference to detect squeezing. Although we have not established a general criterion for SU(3) squeezing, we have shown how quantum correlations (in the sense described above) can lead to a reduction of fluctuations, which is reflected through a specific deformation (‘squeezing’) of the initial coherent state. It must be emphasized that, in quantum systems with higher symmetries, different types of squeezing can be identified, and these types can be conceptually different from the so-called one- and two-axis squeezing typically found in spin-like systems. Here we used the Hamiltonian invariant under U(2) transformations, thus producing ‘true’ (i.e. not reducible to the U(2)-type interactions) SU(3) correlations.

It should also be observed that in contrast to spin-like systems, the exact quantum mechanical calculations for physical models with SU(3) symmetries can be extremely cumbersome. The application of phase-space methods is thus very helpful not only for geometrical interpretation and state visualization, but also in estimating the evolution of systems in the limit of large dimensions through the use of semiclassical calculations. In particular, important physical effects such as squeezing, which originate from nontrivial evolutions of collective qutrit fluctuations, can be described in terms of semiclassical evolution of the initial Wigner distribution for suitable initial states. This is ultimately possible because the approximate solutions (37) and (48) describe well the dynamics of initial semiclassical states for times of order \( t \sim 1 \), while the major squeezing effect is achieved for times \( t \sim \lambda^{-p} \), where \( p > 0 \).

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