INDUCED QUASI-ACTIONS: A REMARK

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1. INTRODUCTION

In this note we observe that the notion of an induced representation has an analog for quasi-actions, and give some applications.

We will use the definitions and notation from [KL01].

1.1. Induced quasi-actions and their properties. Let $G$ be a group and $\{X_i\}_{i \in I}$ be a finite collection of unbounded metric spaces.

Definition 1.1. A quasi-action $G \stackrel{\rho}{\curvearrowright} \prod_i X_i$ preserves the product structure if each $g \in G$ acts by a product of quasi-isometries, up to uniformly bounded error. Note that we allow the quasi-isometries $\rho(g)$ to permute the factors, i.e. $\rho(g)$ is uniformly close to a map of the form $(x_i) \mapsto (\phi_{\sigma^{-1}(i)}(x_{\sigma^{-1}(i)}))$ with a permutation $\sigma$ of $I$ and quasi-isometries $\phi_i : X_i \mapsto X_{\sigma(i)}$.

Associated to every quasi-action $G \stackrel{\rho}{\curvearrowright} \prod_i X_i$ preserving product structure is the action $G \stackrel{\rho_I}{\curvearrowright} I$ corresponding to the induced permutation of the factors; this is well-defined because the $X_i$’s are unbounded metric spaces. For each $i \in I$, the stabilizer $G_i$ of $i$ with respect to $\rho_I$ has a quasi-action $G_i \curvearrowright X_i$ by restriction of $\rho$. It is well-defined up to equivalence in the sense of [KL01, Definition 2.3].

If the permutation action $\rho_I$ is transitive, all factors $X_i$ are quasi-isometric to each other, and the restricted quasi-actions $G_i \curvearrowright X_i$ are quasi-conjugate (when identifying different stabilizers $G_i$ by inner automorphisms of $G$). The main result of this note is that in this case any of the quasi-actions $G_i \curvearrowright X_i$ determines $\rho$ up to quasi-conjugacy, and moreover any quasi-conjugacy class may arise as a restricted action.

Theorem 1.2. Let $G$ be a group, $H$ be a finite index subgroup, and $H \stackrel{\alpha}{\curvearrowright} X$ be a quasi-action of $H$ on an unbounded metric space $X$.

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Then there exists a quasi-action \( G \curvearrowright \prod_{i \in G/H} X_i \) preserving product structure, where

1. Each factor \( X_i \) is quasi-isometric to \( X \).
2. The associated action \( G^{\beta_{G/H}} \curvearrowright G/H \) is the natural action by left multiplication.
3. The restriction of \( \beta \) to a quasi-action of \( H \) on \( X_H \) is quasi-conjugate to \( H \curvearrowright X \).

Furthermore, there is a unique such quasi-action \( \beta \) preserving the product structure, up to quasi-conjugacy by a product quasi-isometry. Finally, if \( \alpha \) is an isometric action, then the \( X_i \) may be taken isometric to \( X \) and \( \beta \) may be taken to be an isometric action.

**Definition 1.3.** Let \( G, H \) and \( H \curvearrowright X \) be as in Theorem 1.2. The quasi-action \( \beta \) is called the quasi-action induced by \( H \curvearrowright X \).

As a byproduct of the main construction, we get the following:

**Corollary 1.4.** If \( G^{\rho} \curvearrowright X \) is an \((L, A)\)-quasi-action on an arbitrary metric space \( X \), then \( \rho \) is \((L, 3A)\)-quasi-conjugate to a canonically defined isometric action \( G \curvearrowright X' \).

1.2. **Applications.** The implication of Theorem 1.2 is that in order to quasi-conjugate a quasi-action on a product to an isometric action, it suffices to quasi-conjugate the factor quasi-actions to isometric actions. We begin with a special case:

**Theorem 1.5.** Let \( G^{\rho} \curvearrowright X \) be a cobounded quasi-action on \( X = \prod_i X_i \), where each \( X_i \) is either an irreducible symmetric space of non-compact type, or a thick irreducible Euclidean building of rank at least two, with cocompact Weyl group. Then \( \rho \) is quasi-conjugate to an isometric action on \( X \), after suitable rescaling of the metrics on the factors \( X_i \).

**Remarks**

- Theorem 1.5 was stated incorrectly as Corollary 4.5 in [KL01]. The proof given there was only valid for quasi-actions which do not permute the factors.
- Rescaling of the factors is necessary, in general: if one takes the product of two copies of \( \mathbb{H}^2 \) where the factors are scaled to have different curvature, then a quasi-action which permutes the factors will not be quasi-conjugate to an isometric action.
We now consider a more general situation. Let \( G \curvearrowright \prod_{i \in I} X_i \) be a quasi-action, where each \( X_i \) is one of the following four types of spaces:

1. An irreducible symmetric space of noncompact type.
2. A thick irreducible Euclidean building of rank/dimension \( \geq 2 \), with cocompact Weyl group.
3. A bounded valence bushy tree in the sense of [MSW03]. We recall that a tree is bushy if each of its points lies within uniformly bounded distance from a vertex having at least three unbounded complementary components.
4. A quasi-isometrically rigid Gromov hyperbolic space which is of coarse type I in the sense of [KKL98, sec. 3] (see the remarks below). A space is quasi-isometrically rigid if every \((L, A)\)-quasi-isometry is at distance at most \( D = D(L, A) \) from a unique isometry.

By [KKL98, Theorem B], the quasi-action preserves product structure, and hence we have an induced permutation action \( G \curvearrowright I \). Let \( J \subset I \) be the set of indices \( i \in I \) such that \( X_i \) is either a real hyperbolic space \( \mathbb{H}^k \) for some \( k \geq 4 \), a complex hyperbolic space \( \mathbb{C}H^l \) for some \( l \geq 2 \), or a bounded valence bushy tree. Generalizing Theorem 1.5 we obtain:

**Theorem 1.6.** If the quasi-action \( G_j \curvearrowright X_j \) is cobounded for each \( j \in J \), then \( \alpha \) is quasi-conjugate by a product quasi-isometry to an isometric action \( G_n \curvearrowright \prod_{i \in I} X'_i \), where for every \( i \), \( X'_i \) is quasi-isometric to \( X_i \), and precisely one of the following holds:

1. If \( X_i \) is not a bounded valence bushy tree, then \( X'_i \) is isometric to \( X'_v \), for some \( v \) in the \( G \)-orbit \( G(i) \) of \( i \).
2. If \( X_i \) is a bounded valence bushy tree, then so is \( X'_i \).

As in the previous corollary, it is necessary to permit \( X'_i \) to be non-isometric to \( X_i \). Moreover, there may be factors \( X_i \) and \( X_j \) of type (4) lying in the same \( G \)-orbit, but which are not even homothetic, so it is not sufficient to allow rescaling of factors.

**Proof.** We first assume that the action \( G \curvearrowright I \) is transitive. Pick \( n \in I \). Then the quasi-action \( G_n \curvearrowright X_n \) is quasi-conjugate to an isometric action \( G_n \curvearrowright X'_n \), where \( X'_n \) is isometric to \( X_n \) unless \( X_n \) is a bounded valence bushy tree, in which case \( X'_n \) is a bounded valence bushy tree but not necessarily isometric to \( X_n \); this follows from:

- [Hin90, Gab92, CJ94, Mar06] when \( X_n = \mathbb{H}^2 \). Note that any quasi-action on \( \mathbb{H}^2 \) is quasi-conjugate to an isometric action.
When $X_n$ is a rank 1 symmetric space other than $\mathbb{H}^2$, note that Sullivan’s theorem implies that any quasi-action on $\mathbb{H}^3$ is quasi-conjugate to an isometric action. Also, the proof given in Chow’s paper on the complex hyperbolic case covers arbitrary cobounded quasi-actions, even though it is only stated for discrete cobounded quasi-actions.

- [KL97, Lee00] when $X_n$ is an irreducible symmetric space or Euclidean building of rank at least 2.
- [MSW03] when $X_n$ is a bounded valence bushy tree.

By Theorem 1.2, the associated induced quasi-action of $G$ is quasi-conjugate to the original quasi-action $G \curvearrowright \prod_{i \in I} X_i$ by a product quasi-isometry, and we are done.

In the general case, for each orbit $G(i) \subset I$ of the action $G \curvearrowright I$, we have a well-defined associated quasi-action $G \curvearrowright \prod_{j \in G(i)} X_j$ for which the theorem has already been established, and we obtain the desired isometric action $G \curvearrowright \prod_{i \in I} X'_i$ by taking products.

**Corollary 1.7.** Let $\{X_i\}_{i \in I}$ be as above, and suppose $G$ is a finitely generated group quasi-isometric to the product $\prod_{i \in I} X_i$. Then $G$ admits a discrete, cocompact, isometric action on a product $\prod_{i \in I} X'_i$, where for every $i$, $X'_i$ is quasi-isometric to $X_i$, and precisely one of the following holds:

1. $X_i$ is not a bounded valence bushy tree, and $X'_i$ is isometric to $X'_{i'}$ for some $i'$ in the $G$-orbit $G(i) \subset I$ of $i$.
2. Both $X_i$ and $X'_i$ are bounded valence bushy trees.

**Proof.** Such a group $G$ admits a discrete, cobounded quasi-action on $\prod_{i \in I} X_i$. Theorem 1.6 furnishes the desired isometric action $G \curvearrowright \prod_{i \in I} X'_i$. $\square$

**Remarks.**

- Corollary 1.7 refines earlier results [Ahl02, KL01, MSW03].
- A proper Gromov hyperbolic space with cocompact isometry group is of coarse type I unless it is quasi-isometric to $\mathbb{R}$ [KKL98, Sec. 3].
- The classification of the four different types of spaces above is quasi-isometry invariant, with one exception: a space of type (1) will also be a space of type (4) iff it is a quasi-isometrically...
rigid rank 1 symmetric space (i.e. a quaternionic hyperbolic space or the Cayley hyperbolic plane [Pan89]). See Lemma 3.1.

• Two irreducible symmetric spaces are quasi-isometric iff they are isometric, up to rescaling [Mos73, Pan89, KL97]. Two Euclidean buildings as in (2) above are quasi-isometric iff they are isometric up to rescaling [KL97, Lee00].

2. THE CONSTRUCTION OF INDUCED QUASI-ACTIONS

The construction of induced quasi-actions is a direct imitation of one of the standard constructions of induced representations. We now review this for the convenience of the reader.

Let $H$ be a subgroup of some group $G$, and suppose $\alpha : H \act V$ is a linear representation. Then we have an action $H \act G \times V$ where $(h, (g, v)) = (gh^{-1}, hv)$. Let $E := (G \times V)/H$ be the quotient. There is a natural projection map $\pi : E \to G/H$ whose fibers are copies of $V$; this would be a vector bundle over the discrete space $G/H$ if $V$ were endowed with a topology. The action $G \act G \times V$ by left translation on the first factor descends to $E$, and commutes with the projection map $\pi$. Moreover, it preserves the linear structure on the fibers. Hence there is a representation of $G$ on the vector space of sections $\Gamma(E)$, and this is the representation of $G$ induced by $\alpha$.

We use the terminology of [KL01, Sec. 2]. (However, we replace quasi-isometrically conjugate by the shorter and more accurate term quasi-conjugate.)

We will work with generalized metrics taking values in $[0, +\infty]$. A finite component of a generalized metric space is an equivalence class of points with pairwise finite distances. Clearly, quasi-isometries respect finite components.

Let $\{X_i\}_{i \in I}$ be a finite collection of unbounded metric spaces in the usual sense, i.e. the metric on each $X_i$ takes only finite values. On their product $\prod_{i \in I} X_i$ we consider the natural ($L^2$-)product metric. On their disjoint union $\sqcup_{i \in I} X_i$ we consider the generalized metric which induces the original metric on each component $X_i$ and gives distance $+\infty$ to any pair of points in different components.

We observe that a quasi-isometry $\prod_{i \in I} X_i \to \prod_{i \in I} X'_i$ preserving the product structure gives rise to a quasi-isometry $\sqcup_{i \in I} X_i \to \sqcup_{i \in I} X'_i$, well-defined up to bounded error, and vice versa. Thus equivalence
classes of quasi-actions $\alpha : G \curvearrowright \prod_{i \in I} X_i$ preserving the product structure correspond one-to-one to quasi-actions $\beta : G \curvearrowright \sqcup_{i \in I} X_i$. In what follows we will prove the disjoint union analog of Theorem 1.2. (The index of $H$ can be arbitrary from now on.)

**Lemma 2.1.** Suppose that $Y$ is a generalized metric space and that $G \curvearrowright Y$ is a quasi-action such that $G$ acts transitively on the set of finite components of $Y$. Let $Y_0$ be one of the finite components and $H$ its stabilizer in $G$. Then the restricted action $H \curvearrowright Y_0$ determines the action $G \curvearrowright Y$ up to quasi-conjugacy.

**Proof.** If $G \curvearrowright Y'$ is another quasi-action, $Y_0'$ is a finite component with stabilizer $H$, then any quasi-conjugacy between $H \curvearrowright Y_0$ and $H \curvearrowright Y_0'$ extends in a straightforward way to a quasi-conjugacy between $G \curvearrowright Y$ and $G \curvearrowright Y'$. □

We will now show how to recover the $G$-quasi-action from the $H$-quasi-action by quasifying the construction of induced actions as described above.

**Definition 2.2.** An $(L, A)$-coarse fibration $(Y, \mathcal{F})$ consists of a (generalized) metric space $Y$ and a family $\mathcal{F}$ of subsets $F \subset Y$, the coarse fibers, with the following properties:

1. The union $\bigcup_{F \in \mathcal{F}} F$ of all fibers has Hausdorff distance $\leq A$ from $Y$.
2. For any two fibers $F_1, F_2 \in \mathcal{F}$ holds
   \[ d_H(F_1, F_2) \leq L \cdot d(y_1, F_2) + A \quad \forall y_1 \in F_1. \]

We also say that $\mathcal{F}$ is a coarse fibration of $Y$.

Note that the coarse fibers are not required to be disjoint.

It follows from part (2) of the definition that $d_H(F_1, F_2) < +\infty$ if and only if $F_1$ and $F_2$ meet the same finite component of $Y$. We will equip the “base space” $\mathcal{F}$ with the Hausdorff metric.

**Lemma 2.3.** If $H \curvearrowright Y$ is an $(L, A)$-quasi-action then the collection of quasi-orbits $O_y := H \cdot y$ forms an $(L, 3A)$-coarse fibration of $Y$.

**Proof.** For $h, h_1, h_2 \in H$ and $y_1, y_2 \in Y$ we have
\[
   d(hy_1, (hh_1^{-1}h_2)y_2) \leq d((hh_1^{-1})(h_1y_1), (hh_1^{-1})(h_2y_2)) + 2A \leq L \cdot d(h_1y_1, h_2y_2) + 3A
\]
and so
\[
   d(O_{y_1}, O_{y_2}) \leq L \cdot d(h_1y_1, O_{y_2}) + 3A.
\]
Let \((Y, \mathcal{F})\) and \((Y', \mathcal{F}')\) be coarse fibrations. We say that a map \(\phi : Y \to Y'\) quasi-respects the coarse fibrations if the image of each fiber \(F \in \mathcal{F}\) is uniformly Hausdorff close to a fiber \(F' \in \mathcal{F}'\), \(d_H(\phi(F), F') \leq C\). The map \(\phi\) then induces a map \(\tilde{\phi} : \mathcal{F} \to \mathcal{F}'\) which is well-defined up to bounded error \(\leq 2C\). Observe that if \(\phi\) is an \((L, A)\)-quasi-isometry then \(\tilde{\phi}\) is an \((L, A + 2C)\)-quasi-isometry.

We say that a quasi-action \(\rho : G \curvearrowright Y\) quasi-respects a coarse fibration \(\mathcal{F}\) if all maps \(\rho(g)\) quasi-respect \(\mathcal{F}\) with uniformly bounded error. The quasi-action \(\rho\) then descends to a quasi-action \(\bar{\rho} : G \curvearrowright \mathcal{F}\) which is unique up to equivalence (cf. [KL01, Definition 2.3]).

We apply these general remarks to the following situation in order to obtain our main construction.

Let \(G\) be a group, \(H < G\) a subgroup (of arbitrary index) and \(H \curvearrowright X\) an \((L, A)\)-quasi-action. Let \(Y = G \times X\) where \(G\) is given the metric \(d(g_1, g_2) = +\infty\) unless \(g_1 = g_2\). That is, \(Y\) consists of \(|G|\) finite components each of which is a copy of \(X\). The quasi-action \(\alpha\) gives rise to a product quasi-action \(H \overset{\rho_H}{\curvearrowright} Y\) via

\[
\rho_H(h, (g, x)) = (gh^{-1}, hx).
\]

We denote by \(\mathcal{F}_H\) the coarse fibration of \(Y\) by \(H\)-quasi-orbits. The isometric \(G\)-action given by

\[
\tilde{\rho}_G(g', (g, x)) = (g'g, x)
\]

commutes with \(\rho_H\). As a consequence, \(\tilde{\rho}_G\) descends to an isometric action

\[(2.4) \quad \hat{\beta} := \tilde{\rho}_G : G \curvearrowright \mathcal{F}_H.\]

If \(H = G\) then \(\alpha\) is quasi-conjugate to \(\hat{\beta}\) via the quasi-isometry \(x \mapsto \rho_H(H) \cdot (e, x)\).

In general, the finite components of \(\mathcal{F}_H\) correspond to the left \(H\)-cosets in \(G\). More precisely, \(gH\) corresponds to \(\cup_{x \in X} \rho_H(H) \cdot (g, x)\), that is, to the union of \(\rho_H\)-quasi-orbits contained in \(gH \times X\). \(H\) stabilizes the finite component \(\cup_{x \in X} \rho_H(H) \cdot (e, x)\). The action of \(H\) on this component is quasi-conjugate to \(\alpha\).

As remarked in the beginning of this section, \(\hat{\beta}\) is the unique \(G\)-quasi-action up to quasi-conjugacy such that \(G\) acts transitively on finite components and such that \(H\) is the stabilizer of a finite component and the restricted \(H\)-quasi-action is quasi-isometrically conjugate to \(\alpha\).
Passing back from disjoint unions to products we obtain Theorem 1.2.

3. Quasi-isometries and the classification into types (1)-(4)

We now prove:

**Lemma 3.1.** Suppose $Y$ and $Y'$ are spaces of one of types (1)-(4) as in Theorem 1.6. If $Y$ is quasi-isometric to $Y'$, then they have the same type, unless one is a quasi-isometrically rigid rank 1 symmetric space, and the other is of type (4).

**Proof.** First suppose one of the spaces is not Gromov hyperbolic. Since Gromov hyperbolicity is quasi-isometry invariant, both spaces must be higher rank space of either of type (1) or (2). But by [KL97], two irreducible symmetric spaces or Euclidean buildings of rank at least two are quasi-isometric iff they are homothetic. Thus in this case they must have the same type.

Now assume both spaces are Gromov hyperbolic. Then $\partial Y$ and $\partial Y'$ are homeomorphic.

If $Y$ is a bounded valence bushy tree, then it is well-known that $Y$ is quasi-isometric to a trivalent tree, and $\partial Y$ is homeomorphic to a Cantor set. Therefore $Y$ cannot be quasi-isometric to a space of type (1), since the boundary of a Gromov hyperbolic symmetric space is a sphere. Also, the quasi-isometry group of a trivalent tree $T$ has an induced action on the space of triples in $\partial T$ which is not proper, and hence it cannot be quasi-isometric to a space of type (4).

If $Y$ is a hyperbolic or complex hyperbolic space, then the induced action of $\text{QI}(X)$ on the space of triples in $\partial X$ is not proper, and hence $Y$ cannot be quasi-isometric to a space of type (4).

The lemma follows. □

**References**

[Ahl02] A. R. Ahlin. The large scale geometry of products of trees. *Geom. Dedicata*, 92:179–184, 2002. Dedicated to John Stallings on the occasion of his 65th birthday.

[Cho96] R. Chow. Groups quasi-isometric to complex hyperbolic space. *Trans. Amer. Math. Soc.*, 348(5):1757–1769, 1996.

[CJ94] A. Casson and D. Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.*, 118(3):441–456, 1994.
[Gab92] D. Gabai. Convergence groups are Fuchsian groups. *Ann. of Math. (2)*, 136(3):447–510, 1992.

[Gro] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*., pages 183–213.

[Hin90] A. Hinkkanen. Abelian and nondiscrete convergence groups on the circle. *Trans. Amer. Math. Soc.*, 318(1):87–121, 1990.

[KKL98] M. Kapovich, B. Kleiner, and B. Leeb. Quasi-isometries and the de Rham decomposition. *Topology*, 37(6):1193–1211, 1998.

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197 (1998), 1997.

[KL01] B. Kleiner and B. Leeb. Groups quasi-isometric to symmetric spaces. *Comm. Anal. Geom.*, 9(2):239–260, 2001.

[Lee00] B. Leeb. *A characterization of irreducible symmetric spaces and Euclidean buildings of higher rank by their asymptotic geometry*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 326. Universität Bonn Mathematisches Institut, Bonn, 2000.

[Mar06] V. Markovic. Quasisymmetric groups. *J. Amer. Math. Soc.*, 19(3):673–715 (electronic), 2006.

[Mos73] G. D. Mostow. *Strong rigidity of locally symmetric spaces*. Princeton University Press, Princeton, N.J., 1973. Annals of Mathematics Studies, No. 78.

[MSW03] L. Mosher, M. Sageev, and K. Whyte. Quasiconformal groups. *J. Analyse Math.*, 46:318–346, 1986.

[Tuk86] P. Tukia. On quasi-conformal groups. *J. Analyse Math.*, 46:318–346, 1986.

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