Global Unique Solutions for the Inhomogeneous Navier-Stokes equations with only Bounded Density, in Critical Regularity Spaces

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Abstract: We here aim at proving the global existence and uniqueness of solutions to the inhomogeneous incompressible Navier-Stokes system in the case where the initial density \( \rho_0 \) is discontinuous and the initial velocity \( u_0 \) has critical regularity. Assuming that \( \rho_0 \) is close to a positive constant, we obtain global existence and uniqueness in the two-dimensional case whenever the initial velocity \( u_0 \) belongs to the critical homogeneous Besov space \( \dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2) \) (\( 1 < p < 2 \)) and, in the three-dimensional case, if \( u_0 \) is small in \( \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \) (\( 1 < p < 3 \)). Next, still in a critical functional framework, we establish a uniqueness statement that is valid in the case of large variations of density with, possibly, vacuum. Interestingly, our result implies that the Fujita-Kato type solutions constructed by Zhang (Adv Math 363:107007, 2020) are unique. Our work relies on interpolation results, time weighted estimates and maximal regularity estimates in Lorentz spaces (with respect to the time variable) for the evolutionary Stokes system.

1. Introduction

We are concerned with the initial value problem for the following inhomogeneous incompressible Navier-Stokes system:

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= 0, \\
\rho u_t + \rho u \cdot \nabla u - \mu \Delta u + \nabla P &= 0, \\
\text{div } u &= 0, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0),
\end{align*}
\]

\((INS)\)

where \( \rho = \rho(t, x) \geq 0 \), \( P = P(t, x) \in \mathbb{R} \) and \( u = u(t, x) \in \mathbb{R}^d \) stand for the density, pressure and velocity field of the fluid, respectively. We consider the evolution for positive times \( t \) in the case where the space variable \( x \) belongs to the whole space \( \mathbb{R}^d \) with \( d = 2, 3 \).
It has long been observed that smooth enough solutions obey the following energy balance:

\[
\frac{1}{2} \| \sqrt{\rho(t)} u(t) \|_{L^2}^2 + \mu \int_0^t \| \nabla u(t) \|_{L^2}^2 \, dt = \frac{1}{2} \| \sqrt{\rho_0 u_0} \|_{L^2}^2, \tag{1.1}
\]

and that, as a consequence of the divergence free property of the velocity field, the Lebesgue measure of

\[
\{ x \in \mathbb{R}^d : \alpha \leq \rho(t, x) \leq \beta \}
\]

is independent of \( t \), for any \( 0 \leq \alpha \leq \beta \).

In 1974, by combining these relations with Galerkin approximation and compactness arguments, Kazhikhov [1] established that for any data such that \( \rho_0 \in L_\infty \), \( \text{div} \, u_0 = 0 \) and \( \sqrt{\rho_0} u_0 \in L_2 \), and provided \( \rho_0 \) is bounded away from vacuum (that is \( \inf \rho_0(x) > 0 \)), (INS) has at least one global distributional solution satisfying (1.1) with an inequality. The no vacuum assumption was removed later by Simon in [2], then, by taking advantage of the theory developed in [3], Lions [4] extended the previous results to the case of a density dependent viscosity, proved that the mass equation of (INS) is satisfied in the renormalized meaning, that the velocity field admits a unique generalized flow and, finally, that (1.2) is true. However, from that time whether these weak solutions are unique is an open question, even in dimension two.

By using totally different approaches, a number of authors proved that in the case of smooth enough data, (INS) admits a unique solution at least locally in time. In fact, as for the classical incompressible Navier-Stokes equations (that is (INS) with constant positive density), the general picture is that provided the initial density is sufficiently smooth, bounded and bounded away from zero, there exists a global unique solution if the initial velocity is small in the sense of some ‘critical norm’ if \( d = 3 \). In contrast, if \( d = 2 \), then there is no restriction on the size of the velocity. This general fact has been first pointed out by Ladyženskaja and Solonnikov [5] in the case where the fluid domain \( \Omega \) is a smooth bounded subset of \( \mathbb{R}^d \) (\( d = 2, 3 \)) and the velocity vanishes at the boundary. More precisely, assuming that \( u_0 \) is in the Sobolev–Slobodeckij space \( W^{2-\frac{2}{p}, p} (\Omega) \) with \( p > d \), is divergence free and has null trace on \( \partial \Omega \), and that \( \rho_0 \) is \( C^1 \) and is bounded away from zero, they proved:

- The global well-posedness in dimension \( d = 2 \),
- The local well-posedness in dimension \( d = 3 \) (global well-posedness if \( u_0 \) is small in \( W^{2-\frac{2}{p}, p} (\Omega) \)).

After this pioneering work, results in the same spirit in other functional frameworks have been proved by a number of authors (see the survey paper [6] for references). Still for smooth enough data, the non vacuum assumption has been weakened by Choe and Kim in [7] in the case where \( \rho \) and \( u \) are in \( L_\infty \times L_2 \). A natural question is the minimal regularity requirement for the data ensuring (at least local) existence and uniqueness. It has been observed by Fujita and Kato [8] in the constant density case (and later for a number of evolutionary equations) that this issue is closely linked to the scaling invariance of the system under consideration. Here it is obvious that if \( (\rho, u, P) \) is a solution of (INS) on \( \mathbb{R}_+ \times \mathbb{R}^d \) for data \( (\rho_0, u_0) \) then, for all \( \lambda > 0 \), the rescaled triplet \( (\rho, u, P) \to (\rho_\lambda, u_\lambda, P_\lambda) \) defined by

\[
(\rho_\lambda, u_\lambda, P_\lambda) \overset{\text{def}}{=} (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 P(\lambda^2 t, \lambda x)) \tag{1.3}
\]

is a solution of (INS) on \( \mathbb{R}_+ \times \mathbb{R}^d \), with data \( (\rho_0(\lambda \cdot), \lambda u_0(\lambda \cdot)) \).
A number of works have been dedicated to the well-posedness of (INS) in $\mathbb{R}^d$, in so-called critical framework, that is to say in functional spaces endowed with norms having the above scaling invariance. Restricting our attention to the case where the density tends to some positive constant at infinity (say 1 to simplify the presentation) and setting $a_{\text{def}} = 1/\rho - 1$, System (INS) rewrites in terms of $(a, u, P)$ as follows:

\[
\begin{aligned}
   a_t + u \cdot \nabla a &= 0, \\
   u_t + u \cdot \nabla u - (1 + a)(\mu \Delta u - \nabla P) &= 0, \\
   \text{div } u &= 0, \\
   (a, u)|_{t=0} &= (a_0, u_0).
\end{aligned}
\]  

(1.4)

In [9], the first author established the existence and uniqueness of a solution to (1.4) in critical Besov spaces. More precisely, in the case where $a_0 \in \dot{B}^{d/2}_{2,1}(\mathbb{R}^d)$ and $u_0 \in \dot{B}^{d/2-1}_{2,1}(\mathbb{R}^d)$ with $\text{div } u_0 = 0$, he proved that there exists a constant $c$ depending only on $d$ such that, if

\[
\|a_0\|_{\dot{B}^{d/2}_{2,1}} \leq c,
\]

then (1.4) admits a unique local solution $(a, u, \nabla P)$ with $a \in C_b([0, T); \dot{B}^{d/2}_{2,1})$,

\[
u \in C_b([0, T); \dot{B}^{d/2}_{2,1}) \cap L_1(0, T; \dot{B}^{d/2+1}_{2,1}) \quad \text{and } \nabla P \in L_1(0, T; \dot{B}^{d/2-1}_{2,1})
\]

and that there exists $c' > 0$ such that this solution is global (i.e. one can take $T = \infty$) if $\|u_0\|_{\dot{B}^{d/2-1}_{2,1}} \leq c' \mu$.

Shortly after, these results have been extended by Abidi in [10], then Abidi and Paicu [11] to critical Besov spaces of type $\dot{B}^s_{p,1}$ with $p > 1$.

Still in the critical functional framework, Huang, Paicu and Zhang noticed that, somehow, only $d - 1$ components of $u_0$ need to be small for global existence: in [12], they just required that

\[
(\mu \|a_0\|_{L_{\infty}} + \|u_0^b\|_{\dot{B}^{d/2}_{p,r} - 1 + \frac{d}{p}}) \exp (C_r \mu^{-2r} \|u_0^d\|_{\dot{B}^{d/2-1}_{p,r} - 1 + \frac{d}{p}}^2) \leq c_0 \mu
\]

for some positive constants $c_0$ and $C_r$.

Achieving results in the critical functional framework when the density has large variations requires techniques that are not just based on perturbation arguments. In [13], the first author investigated the problem in Sobolev spaces but failed to reach the critical exponent. Recently, Abidi and Gui [14] proved the global unique solvability of the 2-D incompressible inhomogeneous Navier-Stokes equations whenever $\rho_0 - 1$ is in $\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$ for some $2 \leq p < \infty$, and $u_0$ is in $\dot{B}^0_{2,1}(\mathbb{R}^2)$. This is, to our knowledge, the first global well-posedness result at the critical level of regularity, that does not require any smallness condition (see also the work by Xu in [15] dedicated to the 3D case and based on different techniques).

A number of recent works aimed at proving existence and uniqueness results in the case where the density is only bounded (and not necessarily continuous). In this respect, significant progresses have been done by Paicu, Zhang and Zhang in [16] where the global existence and uniqueness of solution to (INS) is shown in $\mathbb{R}^d$, $d = 2, 3$ assuming only that $\rho_0$ is bounded and that $u_0 \in H^s(\mathbb{R}^2)$ for some $s > 0$ (2D case) or $u_0 \in H^1(\mathbb{R}^3)$ with $\|u_0\|_{L^d} \|
abla u_0\|_{L^2}$ sufficiently small (3D case). This result has been
extended to velocities in $H^s(\mathbb{R}^3)$ with $s > 1/2$ by Chen, Zhang and Zhao in [17]. Finally, the lower bound assumption was totally removed by the first author and Mucha in [18] in the case where the fluid domain is either bounded or the torus. There, it is only needed that $u_0$ is in $H^1(\Omega)$ and that $\rho_0$ is bounded.

Very recently, in the 3D case, Zhang [19] established the global existence of solutions to the 3D inhomogeneous incompressible Navier-Stokes system with initial density in $L_\infty(\mathbb{R}^3)$, bounded away from zero, and initial velocity sufficiently small in the critical Besov space $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. This is the first example of a global existence result within a Besov critical framework for the velocity and no regularity for the density, in the large variations case. In [19], the uniqueness issue is left open however.

The primary goal of our paper is to establish the global existence of solutions of (INS) that are unique in a critical regularity framework, in the case where the initial density is close to a positive constant in $L_\infty(\mathbb{R}^d)$ but has no regularity whatsoever. To our knowledge, no result of this type has been proved before. In accordance with the state-of-the art for the homogeneous Navier-Stokes equations (that is, with constant density), smallness of the velocity will be required if $d = 3$, but not if $d = 2$.

The uniqueness part of our statements will come up as an easy consequence of a much more general result within a critical regularity framework, that allows for density with large variations (that is even allowed to vanish on arbitrary sets if the dimension is 3). As a by-product, we shall obtain that the global solutions constructed by Zhang in [19] (that are allowed to have large density variations), are actually unique.

Our existence results are strongly based on a novel maximal regularity estimate for the Stokes system equation originating from the recent paper [20] by Mucha, Tolksdorf and the first author, where the time regularity is measured in Lorentz spaces. Time weighted estimates will also play an important role (see the end of the next section for more explanation).

2. Tools, Results and Approach

Before stating our main existence results for (INS), introducing a few notations and recalling some results is in order.

First, throughout the text, $A \lesssim B$ means that $A \leq C B$, where $C$ designates various positive real numbers the value of which does not matter.

For any Banach space $X$, index $q$ in $[1, \infty]$ and time $T \in [0, \infty]$, we use the notation $\| z \|_{L^q(0,T;X)} \overset{\text{def}}{=} \| \| z \|_{X} \|_{L^q(0,T)}$. If $T = \infty$, then we just write $\| z \|_{L^q(X)}$. In the case where $z$ has $n$ components $z_k$ in $X$, we keep the notation $\| z \|_{X}$ to mean $\sum_{k \in \{1,\ldots,n\}} \| z_k \|_{X}$.

We shall use the following notation for the convective derivative:

$$\frac{D}{Dt} \overset{\text{def}}{=} \partial_t + u \cdot \nabla \quad \text{and} \quad \dot{u} \overset{\text{def}}{=} u_t + u \cdot \nabla u. \quad (2.1)$$

Next, let us recall the definition of Besov spaces on $\mathbb{R}^d$. Following [21, Chap. 2], we fix two smooth functions $\chi$ and $\varphi$ such that

$$\text{Supp } \varphi \subset \{ \xi \in \mathbb{R}^d, \, 3/4 \leq |\xi| \leq 8/3 \} \quad \text{and} \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,$$

$$\text{Supp } \chi \subset \{ \xi \in \mathbb{R}^d, \, |\xi| \leq 4/3 \} \quad \text{and} \quad \forall \xi \in \mathbb{R}^d, \, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,$$
and set for all \( j \in \mathbb{Z} \) and tempered distribution \( u \),
\[
\dot{\Delta}_j u \stackrel{\text{def}}{=} \mathcal{F}^{-1} (\varphi(2^{-j} \cdot \hat{u})) \stackrel{\text{def}}{=} 2^j \hat{h}(2^j \cdot \ast u) \quad \text{with} \quad \hat{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1} \varphi,
\]
\[
\dot{S}_j u \stackrel{\text{def}}{=} \mathcal{F}^{-1} (\chi(2^{-j} \cdot \hat{u})) \stackrel{\text{def}}{=} 2^j \hat{h}(2^j \cdot \ast u) \quad \text{with} \quad \hat{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1} \chi,
\]
(2.2)
where \( \mathcal{F} u \) and \( \hat{u} \) denote the Fourier transform of \( u \).

**Definition 2.1** (Homogeneous Besov spaces). Let \((p, r) \in [1, \infty]^2 \) and \( s \in \mathbb{R} \). We set
\[
\| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \stackrel{\text{def}}{=} \| 2^{js} \| \dot{\Delta}_j u \|_{L^p(\mathbb{R}^d)} \|_{\ell^r(\mathbb{Z})}
\]
and denote by \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) the set of tempered distributions \( u \) on \( \mathbb{R}^d \) such that \( \| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} < \infty \) and
\[
\lim_{j \to -\infty} \| \dot{S}_j u \|_{L^\infty(\mathbb{R}^d)} = 0. \quad \text{(2.3)}
\]

It is classical that the scaling invariance condition for \( u_0 \) pointed out in (1.3) is satisfied for all elements of \( \dot{B}_{p,1}^{1+d/p}(\mathbb{R}^d) \) with \( 1 \leq p, r \leq \infty \).

Next, we define Lorentz spaces, and recall a useful characterization.

**Definition 2.2.** Given \( f \) a measurable function on a measure space \((X, \nu)\) and \( 1 \leq p, r \leq \infty \), we define
\[
\widetilde{\| f \|}_{L^p(\mathbb{R}^d)} := \begin{cases} \int_0^\infty (t^{1/r} f^*(t))^r \frac{dt}{t} & \text{if } r < \infty, \\ \sup_{t > 0} t^{1/r} f^*(t) & \text{if } r = \infty, \end{cases}
\]
where
\[
f^*(t) := \inf \{ s \geq 0 : \nu(\{|f| > s\}) \leq t \}.
\]
The set of all \( f \) with \( \widetilde{\| f \|}_{L^p(\mathbb{R}^d)} < \infty \) is called the Lorentz space with indices \( p \) and \( r \).

**Remark 2.3.** It is well known that \( L_{p,p}(X, \nu) \) coincides with the Lebesgue space \( L_p(X, \nu) \). Furthermore, according to [22, Prop.1.4.9], the Lorentz spaces may be endowed with the following (equivalent) quasi-norm:
\[
\| f \|_{L^p(\mathbb{R}^d)} := \begin{cases} p^{1/r} \left( \int_0^\infty (s \nu(\{|f| > s\})^{1/r}) \frac{ds}{s} \right)^{1/r} & \text{if } r < \infty, \\ \sup_{s > 0} s \nu(\{|f| > s\})^{1/r} & \text{if } r = \infty. \end{cases}
\]

Our results will strongly rely on a maximal regularity property for the following evolutionary Stokes system:
\[
\begin{cases}
    u_t - \mu \Delta u + \nabla P = f \quad \text{in } \mathbb{R}^d, \\
    \text{div } u = 0 \quad \text{in } \mathbb{R}^d, \\
    u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^d.
\end{cases}
\]
It has been pointed out in [20, Prop. 2.1] that for the free heat equation supplemented with initial data \( u_0 \) in \( \dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d) \), the solution \( u \) is such that \( u_t \) and \( \nabla^2 u \) are in \( L_{r,r}(\mathbb{R}^d; L_p(\mathbb{R}^d)) \) and that, conversely, the Besov regularity \( \dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d) \) corresponds to the regularity of the trace at \( t = 0 \) of functions \( u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \) such that \( u_t, \nabla^2 u \in L_{q,r}(\mathbb{R}^+; L_p(\mathbb{R}^d)) \).

This motivates us to introduce the following function space:

\[
\dot{W}^{2,1}_{p, (q, r)}(\mathbb{R}^d) := \{ u \in C(\mathbb{R}^+; \dot{B}^{2-2/q}_{p,r}(\mathbb{R}^d)) : u_t, \nabla^2 u \in L_{q,r}(\mathbb{R}^+; L_p(\mathbb{R}^d)) \}. \tag{2.5}
\]

Back to (INS), in accordance with (1.3), we need \( 2 - 2/q = -1 + d/p \). Furthermore, for reasons that will be explained later on (in particular the fact that \( \dot{B}^{d/p}_{p,r}(\mathbb{R}^d) \) embeds in \( L_\infty(\mathbb{R}^d) \) if and only if \( r = 1 \)), we shall only consider Besov spaces of type \( \dot{B}^{d/p-1}_{p,1}(\mathbb{R}^d) \).

It is now time to state the main results of the paper. In the two-dimensional case, our global existence result reads:

**Theorem 2.4.** Let \( p \in (1, 2) \) and \( q \) be defined by \( 1/q + 1/p = 3/2 \). Denote by \( s \) and \( m \) the conjugate Lebesgue exponents of \( p \) and \( q \), respectively. Assume that the initial divergence-free velocity \( u_0 \) is in \( \dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^d) \), and that \( \rho_0 \) belongs to \( L_\infty(\mathbb{R}^2) \). There exists a constant \( c > 0 \) such that

\[
\| \rho_0 - 1 \|_{L_\infty(\mathbb{R}^2)} < c, \tag{2.6}
\]

then (INS) has a unique global-in-time solution \( (\rho, u, \nabla P) \) satisfying the energy balance (1.1), \( u \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}^+ \times \mathbb{R}^2), \nabla P \in L_{q,1}(\mathbb{R}^+; L_p(\mathbb{R}^2)) \),

\[
\| \rho - 1 \|_{L_\infty(\mathbb{R}^+ \times \mathbb{R}^2)} = \| \rho_0 - 1 \|_{L_\infty(\mathbb{R}^2)} < c, \tag{2.7}
\]

and the following properties:

- \( \nabla u \in L_1(\mathbb{R}^+; L_\infty(\mathbb{R}^2)) \) and \( u \in L_2(\mathbb{R}^+; L_\infty(\mathbb{R}^2)) \);
- \( tu \in L_\infty(\mathbb{R}^+; \dot{B}^{1+2/m}_{m,1}(\mathbb{R}^2)) \);
- \( (u, (tu)_t, \nabla^2 (tu), \nabla (tP)) \in L_{s,1}(\mathbb{R}^+; L_{m,1}(\mathbb{R}^2)) \);
- \( t\dot{u} \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}^+ \times \mathbb{R}^2) \) and \( t\dot{u} \in L(\mathbb{R}^+; L_\infty(\mathbb{R}^2)) \);
- \( t^{\frac{k+1}{2}} \nabla^k u \in L_\infty(\mathbb{R}^+; L_2(\mathbb{R}^2)) \) and \( t^{\frac{k}{2}} \nabla^{k+1} u \in L_2(\mathbb{R}^+ \times \mathbb{R}^2) \) for \( k = 0, 1, 2 \);
- \( t^{\frac{k+1}{2}} \nabla^k \dot{u} \in L_\infty(\mathbb{R}^+; L_2(\mathbb{R}^2)) \) for \( k = 0, 1, 2 \), \( t^{\frac{k}{2}} \nabla \dot{u} \in L_\infty(\mathbb{R}^+; L_2(\mathbb{R}^2)) \) for \( k = 0, 1, 2 \).

In dimension three, our global existence result reads:

**Theorem 2.5.** Let \( p \in (1, 3) \) and \( q \in (1, \infty) \) such that \( 3/p + 2/q = 3 \). There exist a positive constant \( c \) such that if the initial density is such that

\[
\| \rho_0 - 1 \|_{L_\infty(\mathbb{R}^3)} < c, \tag{2.8}
\]

and if the initial divergence-free velocity satisfies

\[
u_0 \in \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \ (1 < p \leq 2) \quad \text{or} \quad u_0 \in \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \cap L_2(\mathbb{R}^3) \ (2 < p < 3),
\]
with
\[\|u_0\|_{\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)} < c\mu,\]  
then (INS) has a unique global-in-time solution \((\rho, u, \nabla P)\) with
\[\nabla P \in L_{q,1}(\mathbb{R}_+; L_p(\mathbb{R}^3)) \text{ and } u \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}_+ \times \mathbb{R}^3),\]
satisfying the energy balance (1.1) if \(p > 2\),
\[\|\rho - 1\|_{L_\infty(\mathbb{R}_+ \times \mathbb{R}^3)} = \|\rho_0 - 1\|_{L_\infty(\mathbb{R}^3)} < c,\]  
and, furthermore, the following properties:
\begin{itemize}
  \item \(\nabla u \in L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^3))\) and \(u \in L_2(\mathbb{R}_+; L_\infty(\mathbb{R}^3))\);
  \item \((tu) \in W^{2,1}_{m,(s,1)}(\mathbb{R}_+ \times \mathbb{R}^3)\) and \(t\nabla P \in L_{s,1}(\mathbb{R}_+; L_m(\mathbb{R}^3))\) for all \(3 < m < \infty\) and \(q < s < \infty\) such that \(3/m + 2/s = 1\);
  \item \(t\dot{u} \in \dot{W}^{2,1}_{p,(q,1)}(\mathbb{R}_+ \times \mathbb{R}^3)\);
  \item \((u, \dot{u}) \in L_{s,1}(\mathbb{R}_+; L_m(\mathbb{R}^3))\).
\end{itemize}

Remark 2.6. If \(p > 2\), then the (subcritical) assumption \(u_0 \in L_2(\mathbb{R}^3)\) ensures the constructed solution to have finite energy. It is only required for proving uniqueness, and it is not needed if \(p \leq 2\) (of course if one assumes in addition that \(u_0 \in L^2(\mathbb{R}^3)\) then the energy balance holds true in this case, too). Note that the priori estimates leading to global existence are performed in critical spaces, and do not require the energy to be finite.

Like in the two-dimensional case, higher order time weighted energy estimates may be proved. However, since they are not needed for getting uniqueness, we refrain from stating them.

The uniqueness part of the above two theorems is a consequence of the following much more general result.

Theorem 2.7. Let \((\rho_1, u_1, P_1)\) and \((\rho_2, u_2, P_2)\) be two solutions of (INS) on \([0, T] \times \mathbb{R}^d\) corresponding to the same initial data. Assume in addition that:
\begin{itemize}
  \item \(\sqrt{\rho_1}(u_2 - u_1) \in L_\infty(0, T; L_2(\mathbb{R}^d))\);
  \item \(\nabla u_2 - \nabla u_1 \in L_2(0, T \times \mathbb{R}^d)\);
  \item \(\nabla u_2 \in L_1(0, T; L_\infty(\mathbb{R}^d))\);
  \item \(t\dot{u}_2 \in L_2(0, T; L_\infty(\mathbb{R}^d))\);
  \item Case \(d = 2\): \(t\nabla^2 \dot{u}_2 \in L_q(0, T; L_p(\mathbb{R}^2))\) for some \(1 < p, q < 2\) such that \(1/p + 1/q = 3/2\), and \(\rho_0\) is bounded away from zero,
  \item Case \(d = 3\): \(t\nabla^2 \dot{u}_2 \in L_2(0, T; L_3(\mathbb{R}^3))\).
\end{itemize}

Then, \((\rho_1, u_1, P_1) \equiv (\rho_2, u_2, P_2)\) on \([0, T] \times \mathbb{R}^d\).

Remark 2.8. Although the density may have large variations (and even vanish in the three-dimensional case), the regularity requirements in the above uniqueness result are all at the critical level in the sense of (1.3).

In the last section of the paper, we shall present another uniqueness statement in dimension two, that allows for vacuum, but requires a slightly supercritical regularity assumption.
We shall also establish that the Fujita-Kato type global solutions constructed by Zhang in [19] satisfy $\nabla u \in L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^3))$, hence are unique owing to Theorem 2.7. This leads to the following global well-posedness statement.

**Theorem 2.9.** Let $(\rho_0, u_0)$ satisfy

$$0 < c_0 \leq \rho_0 \leq C_0 < \infty \text{ and } u_0 \in \dot{B}_{2,1}^{1/2}(\mathbb{R}^3).$$

Then, there exists a constant $\varepsilon_0 > 0$ depending only on $c_0$, $C_0$ such that if

$$\|u_0\|_{\dot{B}_{2,1}^{1/2}(\mathbb{R}^3)} \leq \varepsilon_0 \mu, \quad (2.11)$$

then System (INS) has a unique global solution $(\rho, u, \nabla P)$ with

$$u \in C(\mathbb{R}_+; \dot{B}_{2,1}^{1/2}(\mathbb{R}^3)) \cap L_2(\mathbb{R}_+; \dot{B}_{2,1}^{3/2}(\mathbb{R}^3))$$

which satisfies

$$c_0 \leq \rho \leq C_0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^3, \quad (2.12)$$

and, for some absolute constant $C$,

$$\|u\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{1/2})} + \sqrt{\mu} \|(u, t\dot{u})\|_{L_2(\mathbb{R}_+; \dot{B}_{2,1}^{3/2})} + \|\sqrt{\mu} t u\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{3/2})}$$

$$+ \|\sqrt{t}(\mu \nabla u, P)\|_{L_2(\mathbb{R}_+; \dot{B}_{6,1}^{1/2})} + \mu \|\sqrt{t} u_t\|_{L_2(\mathbb{R}_+; \dot{B}_{2,1}^{1/2})}$$

$$+ \|\sqrt{t}(\mu \nabla^2 u, \nabla P)\|_{L_2(\mathbb{R}_+; L_3)} + \|tu_t\|_{L_\infty(\mathbb{R}_+; \dot{B}_{2,1}^{3/2})} \leq C \|u_0\|_{\dot{B}_{2,1}^{1/2}}. \quad (2.13)$$

Furthermore, we have $\nabla u$ is in $L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^3))$ with

$$\mu \|\nabla u\|_{L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^3))} \leq C \|u_0\|_{\dot{B}_{2,1}^{1/2}(\mathbb{R}^3)}. \quad (2.14)$$

Let us shortly present the main ingredients leading to the above statements.

The common starting point for proving the existence part in Theorems 2.4 and 2.5 is the maximal regularity result in Lorentz spaces stated in Proposition A.5. In fact, in parabolic spaces, the Besov regularity that is required for the initial velocity exactly corresponds to the trace at $t = 0$ of functions $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ such that $u_t$ and $\nabla^2 u$ are in $L_{q,1}(\mathbb{R}_+; L_p(\mathbb{R}^d))$. Then, proving estimates for (INS) is based on a perturbation argument from the Stokes system (this is the only place where we need the density to be close to some positive constant). In dimension $d = 2$, the space $L_2(\mathbb{R}^2)$ turns out to be critical, and one can combine these estimates with the energy balance (1.1) so as to discard any smallness assumption for the velocity.

Since the first part of (INS) is a transport equation, in order to prove the uniqueness, it is essentially mandatory to have $\nabla u \in L_{1,loc}(\mathbb{R}_+; L_\infty(\mathbb{R}^d))$. This property will be achieved by combining critical estimates for $u$ and $tu$, with an interpolation argument involving, again, Lorentz norms for the time variable.

In our setting, it is not clear whether having only $\nabla u \in L_{1,loc}(\mathbb{R}_+; L_\infty(\mathbb{R}^d))$ is enough to get uniqueness. Here, to conclude, we establish a number of time weighted estimates of energy type (still involving only critical norms). We will in particular get accurate enough information on $\dot{u}$, which will spare us going to Lagrangian coordinates. In fact, in contrast with recent works on similar issues (see e.g. [18, 20]) our proof of uniqueness is performed directly on the original system (INS): we estimate the difference of velocities in the energy space and, by means of a duality argument, the difference of densities in
In dimension $d = 3$, we do not need the density to be positive. In the two-dimensional case, the space $\mathring{H}^1(\mathbb{R}^2)$ fails to be embedded in any Lebesgue space, which complicates the proof, unless the density has a positive lower bound. If it is not the case, then one can combine a suitable logarithmic interpolation inequality with Osgood lemma so as to get a uniqueness result in some cases where the density vanishes. However, we have to strengthen slightly our regularity requirement on the velocity (see the end of Sect. 6).

The rest of the paper unfolds as follows. The a priori estimates leading to global existence for Theorems 2.4 and 2.5 are performed in the next two sections. Section 5 is devoted to the proof of the global existence. Section 6 is dedicated to the proof of various stability estimates and uniqueness statements that, in particular, imply Theorem 2.7 and the uniqueness part of Theorems 2.4, 2.5 and 2.9. For reader’s convenience, we present in Appendix the maximal regularity result in Lorentz spaces of [20] adapted to the Stokes system, recall a few properties of Besov and Lorentz spaces and prove a critical bilinear estimate with a logarithmic loss that is needed for uniqueness in dimension $d = 2$.

3. A Priori Estimates in the 2D Case

This part is devoted to the proof of a priori estimates for (INS) in the 2D case. We shall first establish estimates for $u$ in the critical regularity space $W_{p,q}^{2,1}(\mathbb{R}^+ \times \mathbb{R}^2)$ with $1/q + 1/p = 3/2$ defined in (2.5), which actually suffices to get the global existence of a solution. Then, we will prove time weighted estimates both of energy type and in critical Besov spaces that are needed for uniqueness. The last statement of the section points out higher order time weighted estimates, of independent interest.

For better readability, we drop from now on $\mathbb{R}^2$ in the norms.

**Proposition 3.1.** Let $(\rho, u)$ be a smooth solution of (INS) on $[0, T] \times \mathbb{R}^2$ with sufficiently decaying velocity, and density satisfying

\[
\sup_{t \in [0, T]} \| \rho(t) - 1 \|_{L^\infty} \leq c \ll 1. \tag{3.1}
\]

Then, it holds that

\[
(1 - c) \| u \|_{L^\infty(0, T; L^2)}^2 + 2 \mu \| \nabla u \|_{L^2(0, T \times \mathbb{R}^2)}^2 \leq (1 + c) \| u_0 \|_{L^2}^2 \tag{3.2}
\]

and, for all $1 < p, q < 2$ with $1/p + 1/q = 3/2$,

\[
\mu^{\frac{1}{p} - \frac{1}{2}} \| u \|_{L^\infty(0, T; B_{p,1}^{-1+2/p})} + \| u_1 \|_{L^q(0, T; L^p)} \mu \| \nabla u \|_{L^2} \leq C \mu^{\frac{1}{p} - \frac{1}{2}} \| u_0 \|_{B_{p,1}^{-1+2/p}} + C \mu^{-\frac{1}{2}} \| u_1 \|_{B_{p,1}^{-1+2/p}}, \tag{3.3}
\]

for a constant $C$ independent of $T$ and $\mu$, with $s$ and $m$ being the conjugate exponents of $q$ and $p$, respectively. Furthermore, we have

\[
\| u \|_{L^q(0, T; L^p)} \leq C \mu^{\frac{1}{p} - \frac{1}{2}} \| u_0 \|_{B_{p,1}^{-1+2/p}} + C \mu^{-\frac{1}{2}} \| u_0 \|_{L^2} \tag{3.4}
\]

and

\[
\mu^{\frac{1}{2}} \| u \|_{L^2(0, T; L^\infty)} \leq C \mu^{\frac{1}{p} - \frac{1}{2}} \| u_0 \|_{B_{p,1}^{-1+2/p}} + C \mu^{-\frac{1}{2}} \| u_0 \|_{L^2}^2. \tag{3.5}
\]
Proof. Putting together the energy balance (1.1) and (3.1) yields (3.2). For proving the other inequalities, note that thanks to the following rescaling:

\[
(\tilde{\rho}, \tilde{u}, \tilde{P})(t, x) := \left( \frac{\rho}{\mu}, \frac{u}{\mu} \cdot \frac{P}{t}, x \right), \quad (\tilde{\rho}_0, \tilde{u}_0)(x) := \left( \rho_0, \frac{u_0}{\mu} \right)(x),
\]

one may assume without loss of generality that \( \mu = 1 \).

In order to prove (3.3), let us observe that

\[
u_t - \Delta u + \nabla P = -(\rho - 1)u_t - \rho u \cdot \nabla u, \quad \text{div } u = 0. \tag{3.7}\]

Looking at (3.7) as a Stokes equation with source term, Proposition A.5 gives us

\[
\|u\|_{L_\infty(0,T;\tilde{B}^{-1+2/p}_{p,1})} + \|u_t, \nabla^2 u, \nabla P\|_{L_{q,1}(0,T;L_p)} + \|u\|_{L_{q,1}(0,T;L_m)} \\
\leq C(\|u_0\|_{\tilde{B}^{-1+2/p}_{p,1}} + \|\rho - 1\|u_t + \rho u \cdot \nabla u\|_{L_{q,1}(0,T;L_p)}). \tag{3.8}
\]

By Hölder inequality, we have

\[
\|(\rho - 1)u_t + \rho u \cdot \nabla u\|_{L_{q,1}(0,T;L_p)} \leq \|\rho - 1\|_{L_\infty(0,T;\mathbb{R}^2)} \|u_t\|_{L_{q,1}(0,T;L_p)} + \|\rho\|_{L_\infty(0,T;\mathbb{R}^2)} \|u \cdot \nabla u\|_{L_{q,1}(0,T;L_p)}.
\]

If \( c \) is small enough in (3.1), then the first part in the right-hand side can be absorbed by the left-hand side of (3.8). For the last term, we have by Hölder inequality,

\[
\|u \cdot \nabla u\|_{L_{q,1}(0,T;L_p)} \leq \|u\|_{L_{q,1}(0,T;L_m)} \|\nabla u\|_{L_2(0,T;L_2)}.
\]

Hence, there exists a (small) constant \( \alpha > 0 \) such that if

\[
\|\nabla u\|_{L_2(0,T;\mathbb{R}^2)} \leq \alpha, \tag{3.9}
\]

then (3.8) implies that

\[
\|u\|_{L_\infty(0,T;\tilde{B}^{-1+2/p}_{p,1})} + \|u_t, \nabla^2 u, \nabla P\|_{L_{q,1}(0,T;L_p)} \\
\leq \|u_0\|_{\tilde{B}^{-1+2/p}_{p,1}} + \|\rho\|_{L_\infty(0,T;\mathbb{R}^2)} \|u \cdot \nabla u\|_{L_{q,1}(0,T;L_p)}.
\]

If (3.9) is not satisfied then we follow the method used for proving [20, Theorem 3.1] and split \([0, T]\) into a finite number \( K \) of intervals \([T_{k-1}, T_k]\) with \( T_0 = 0, T_K = T, \) and \( T_1, \ldots, T_{K-1} \) defined by:

\[
\|\nabla u\|_{L_2((T_{k-1}, T_k) \times \mathbb{R}^2)} = \alpha \quad \text{if } 1 \leq k \leq K - 1;
\]

\[
\|\nabla u\|_{L_2((T_{K-1}, T_K) \times \mathbb{R}^2)} \leq \alpha.
\]

For fixed \( \alpha \), we calculate the value of \( K \) by

\[
K \alpha^2 \geq \sum_{k=1}^{K} \|\nabla u\|_{L_2((T_{k-1}, T_k) \times \mathbb{R}^2)}^2 = \|\nabla u\|_{L_2(0,T;\mathbb{R}^2)}^2 \\
> \sum_{k=1}^{K-1} \|\nabla u\|_{L_2((T_{k-1}, T_k) \times \mathbb{R}^2)}^2 = (K - 1) \alpha^2,
\]

whence

\[
K \alpha^2 > (K - 1) \alpha^2.
\]

This implies that \( K \alpha^2 > 0 \) and therefore \( \alpha > 0 \).
which gives

\[ K = \left[ \alpha^{-2} \| \nabla u \|_{L^2(0,T \times \mathbb{R}^2)}^2 \right]. \]  

(3.10)

Then, we adapt (3.8) to each interval \([T_k, T_{k+1})\) getting

\[
\| u \|_{L^\infty(T_k, T_{k+1}; \dot{B}_{p,1}^{-1+2/p})} + \| u_t, \nabla^2 u, \nabla P \|_{L^q,1(T_k, T_{k+1}; L^p)} \\
+ \| u \|_{L^{s,1}(T_k, T_{k+1}; L^m)} \leq C \| u(T_k) \|_{\dot{B}_{p,1}^{-1+2/p}}.
\]

Arguing by induction, taking \( K \) according to (3.10) and using (3.2) so as to bound \( \| \nabla u \|_{L^2(0,T \times \mathbb{R}^2)} \), we conclude that

\[
\| u \|_{L^\infty(0,T; \dot{B}_{p,1}^{-1+2/p})} + \| u_t, \nabla^2 u, \nabla P \|_{L^q,1(0,T; L^p)} \\
+ \| u \|_{L^{s,1}(0,T; L^m)} \leq C \| u_0 \|_{\dot{B}_{p,1}^{-1+2/p}} \exp (C \| u_0 \|_{L^2}^2).
\]

(3.11)

In order to prove (3.4), it suffices to use the fact that

\[
\| \dot{u} \|_{L^q,1(0,T; L^p)} \leq \| u_t \|_{L^q,1(0,T; L^p)} + \| u \cdot \nabla u \|_{L^q,1(0,T; L^p)} \\
\leq \| u_t \|_{L^q,1(0,T; L^p)} + \| u \|_{L^{s,1}(0,T; L^m)} \| \nabla u \|_{L^2(0,T; L^2)}.
\]

Bounding the right-hand side according to (3.2) and (3.11) yields (3.4).

Finally, as a consequence of Gagliardo-Nirenberg inequality and embedding, we have:

\[
\| z \|_{L^\infty} \leq \| z \|_{L^2}^{1-q/2} \| \nabla^2 z \|_{L^p}^{q/2} \leq \| z \|_{\dot{B}_{p,1}^{-1+2/p}}^{1-q/2} \| \nabla^2 z \|_{L^p}^{q/2}.
\]

(3.12)

Hence, using Inequality (3.11), we find that

\[
\int_0^T \| u \|_{L^\infty}^2 \, dt \leq C \int_0^T \| u_t \|_{L^q,1}^{2-q} \| \nabla^2 u \|_{L^p}^q \, dt \\
\leq C \| u_0 \|_{\dot{B}_{p,1}^{-1+2/p}}^{2-q} \| \nabla^2 u \|_{L^q(0,T; L^p)}^q \\
\leq C \| u_0 \|_{\dot{B}_{p,1}^{-1+2/p}}^2 \exp (C \| u_0 \|_{L^2}^2).
\]

(3.13)

This completes the proof of the proposition. \( \square \)

**Proposition 3.2.** Under the assumptions of Proposition 3.1, we have

\[
\mu \| tu \|_{L^\infty(0,T; \dot{B}_{m,1}^{2-2/s})} + \mu \| (tu)_t, t\dot{u}, \mu \nabla^2 (tu), \nabla (t P) \|_{L^{s,1}(0,T; L^m)} \\
\leq C \mathcal{E}_0 \| u_0 \|_{\dot{B}_{p,1}^{-1+2/p}} \exp \left( C \mu^{-1} \| u_0 \|_{\dot{B}_{p,1}^{-1+2/p}}^2 \right).
\]

**Proof.** Again, we use the rescaling (3.6) to reduce the proof to the case \( \mu = 1 \). Now, multiplying both sides of (3.7) by time \( t \) yields

\[
(tu)_t - \Delta (tu) + \nabla (t P) = -(\rho - 1)(tu)_t + \rho u - \rho u \cdot \nabla tu, \quad \text{div} (tu) = 0.
\]

Then, taking advantage of of Proposition A.5 with Lebesgue indices \( m \) and \( s \) gives

\[
\| tu \|_{L^\infty(0,T; \dot{B}_{m,1}^{2-2/s})} + \| (tu)_t, \nabla^2 (tu), \nabla (t P) \|_{L^{s,1}(0,T; L^m)}
\]
\[ \leq \|\rho - 1\|_{L_\infty(0,T \times \mathbb{R}^2)} \| (tu)_t \|_{L_{s,1}(0,T;L_m)} + \|\rho\|_{L_\infty(0,T \times \mathbb{R}^2)} \left( \|u\|_{L_{s,1}(0,T;L_m)} + \|tu \cdot \nabla u\|_{L_{s,1}(0,T;L_m)} \right). \]

Owing to (3.1), the second line may be absorbed by the first one. Next, as \(2/m = 1 - 2/s\), combining Hölder inequality and the following embedding:
\[ \hat{B}_{m,1}^{2/m}(\mathbb{R}^2) \hookrightarrow L_\infty(\mathbb{R}^2) \quad (3.14) \]
yields
\[ \|tu \cdot \nabla u\|_{L_{s,1}(0,T;L_m)} \leq \|t \nabla u\|_{L_\infty(0,T \times \mathbb{R}^2)} \|u\|_{L_{s,1}(0,T;L_m)} \leq \|tu\|_{L_\infty(0,T;\hat{B}_{m,1}^{2/2s})} \|u\|_{L_{s,1}(0,T;L_m)}. \]

Hence, there exists a (small) positive constant \(\beta\) such that, if
\[ \|u\|_{L_{s,1}(0,T;L_m)} \leq \beta, \]
then we have
\[ \|tu\|_{L_\infty(0,T;\hat{B}_{m,1}^{2/2s})} + \|(tu)_t, \nabla^2 (tu), \nabla (tP)\|_{L_{s,1}(0,T;L_m)} \leq C \|u\|_{L_{s,1}(0,T;L_m)}. \]

If \(\|u\|_{L_{s,1}(0,T;L_m)} > \beta\), then one can argue as in the proof of the previous proposition: there exists a finite sequence \(0 = T_0 < T_1 < \cdots < T_{K-1} < T_K = T\) such that
\[ \|u\|_{L_{s,1}((T_{k-1},T_k);L_m)} = \beta \quad \text{if } 1 \leq k \leq K - 1; \]
\[ \|u\|_{L_{s,1}((T_{K-1},T);L_m)} \leq \beta \quad (3.15). \]

Indeed, denoting \(U(t) := \|u(t, \cdot)\|_{L_m}\), from Remark 2.3, we get
\[ \|U(t)\|_{L_{s,1}(0,T)} = s \int_0^\infty \{|t \in (0,T) : |U(t)| > \lambda\}^{1/s} \, d\lambda \]
which, together with Lebesgue dominated theorem gives
\[ \int_0^\infty \{|t \in (T_1, T_2) : |U(t)| > \lambda\}^{1/s} \, d\lambda \to 0 \quad \text{as} \quad T_2 - T_1 \to 0. \]

This allows to construct a family \((T_k)_{0 \leq k \leq K}\) satisfying (3.15). Now, by Hölder inequality (with exponents \(s\) and \(p\)) we have for all \(\lambda > 0\),
\[ \sum_{k=1}^K \{|t \in (T_{k-1}, T_k) : |U(t)| > \lambda\}^{1/s} \leq K^{1/p} \left( \sum_{k=1}^K \{|t \in (T_{k-1}, T_k) : |U(t)| > \lambda\} \right)^{1/s} \]
\[ = K^{1/p} \|\{t \in (0,T) : |U(t)| > \lambda\}\|^{1/s}. \]

Hence, integrating with respect to \(\lambda\) and using (3.15) yields
\[ K \lesssim \beta^{-s} \|u\|_{L_{s,1}(0,T;L_m)}^s. \]

Arguing by induction, we thus obtain
\[ \|tu\|_{L_\infty(0,T;\hat{B}_{m,1}^{2/2s})} + \|(tu)_t, \nabla^2 (tu), \nabla (tP)\|_{L_{s,1}(0,T;L_m)} \leq C \|u\|_{L_{s,1}(0,T;L_m)} \|u\|_{L_{s,1}(0,T;L_m)}^s e^{C \|u\|_{L_{s,1}(0,T;L_m)}^s}. \]
In the end, using (3.3), one concludes that
\[ \|tu\|_{L^\infty(0,T; B_{m,1}^{2-2/s})} + \|tu\|_{L^2} + \nabla^2 (tu), \nabla (tP) \|_{L^s(0,T; L^m)} \leq C \|u_0\|_{B_{p,1}^{1+2/p}} \exp \left( C \|u_0\|^s_{B_{p,1}^{1+2/p}} \exp \left( C \|u_0\|_{L^2}^2 \right) \right). \] (3.16)

To bound \( t\dot{u} \), we just have to observe that \( t\dot{u} = (tu)_t - u + tu \cdot \nabla u \). Hence, by Hölder inequality and (3.14), we get
\[ \|t\dot{u}\|_{L^1(0,T; L^m)} \leq \|tu\|_{L^1(0,T; L^m)} + \|u\|_{L^1(0,T; L^m)} + \|tu\|_{L^\infty(0,T; B_{m,1}^{2-2/s})} \|u\|_{L^1(0,T; L^m)}. \]

At this stage, using Inequalities (3.3) and (3.16) gives the desired result. \( \square \)

**Corollary 3.3.** With the notation of Proposition 3.2, we have:
\[ \mu \int_0^T \|\nabla u\|_{L^\infty} dt \leq C \|u_0\|_{B_{p,1}^{1+2/p}} E_0 e^{C \mu^{-2} \|u_0\|^2_{L^2}}, \] (3.17)
\[ \mu \left( \int_0^T t \|\nabla u\|^2_{L^\infty} dt \right)^{1/2} \leq C \|u_0\|_{B_{p,1}^{1+2/p}} E_0 e^{C \mu^{-2} \|u_0\|^2_{L^2}}, \] (3.18)
\[ \sup_{t \in [0,T]} (\mu t)^{1/2} \|u(t)\|_{L^\infty} \leq C \|u_0\|_{L^2}^{1/2} \|u_0\|_{B_{p,1}^{1+2/p}}^{1/2} E_0. \] (3.19)

**Proof.** Just consider the case \( \mu = 1 \). From the following Gagliardo-Nirenberg inequality
\[ \|z\|_{L^\infty} \lesssim \|\nabla z\|_{L^p}^{1-2/m} \|\nabla z\|_{L^m}^{2/m}, \]
and Hölder estimates in Lorentz spaces (see Proposition A.1), we gather that
\[ \int_0^T \|\nabla u\|_{L^\infty} dt \lesssim \int_0^T t^{-2/m} \|\nabla^2 u\|_{L^p}^{1-2/m} \|\nabla u\|_{L^m}^{2/m} dt \lesssim \|t^{-2/m} \|_{L^{m/2,\infty}(0,T)} \|\nabla^2 u\|_{L^{q,1}(0,T; L^p)} \|\nabla u\|_{L^1(0,T; L^m)}, \]
As \( t \mapsto t^{-2/m} \in L^{m/2,\infty}(\mathbb{R}_+) \) and the other terms of the right-hand side may be bounded by means of Propositions 3.1 and 3.2, we get (3.17). Next, by virtue of (3.14), we have
\[ \int_0^T t \|\nabla u\|^2_{L^\infty} dt \leq \int_0^T \|\nabla u\|_{B_{m,1}^{2-2/s}} \|\nabla u\|_{L^\infty} dt \lesssim \|tu\|_{L^\infty(0,T; B_{m,1}^{2-2/s})} \|\nabla u\|_{L^1(0,T; L^\infty)}, \]
whence the second inequality.

Finally, by interpolation, we have for all \( t \in [0, T], \)
\[ t^{1/2} \|u(t)\|_{L^\infty} \lesssim \|u(t)\|_{L^2}^{1/2} \|tu(t)\|_{B_{m,1}^{1+2/m}}^{1/2} \]
which, in light of (3.2) and of Proposition 3.2 completes the proof. \( \square \)
The rest of this section is devoted to establishing supplementary time weighted estimates of energy type that will be needed to prove the uniqueness of solutions of (INS). For expository purpose, we shall always assume that $\mu = 1$.  

**Proposition 3.4.** Under the assumptions of Proposition 3.1, we have for all $t \in [0, T]$,

$$
\int_{\mathbb{R}^2} t \rho |u(t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^2} \tau (|\rho u|^2 + |\nabla^2 u|^2 + |\nabla P|^2) \, dx \, d\tau
\leq C \|u_0\|^2_{L^2} \exp \left( C \|u_0\|_{L^2} \|u_0\|_{\dot{H}^{-1} L^2} \right).
$$

**Proof.** Let us rewrite the velocity equation as:

$$
\rho \dot{u} = \Delta u - \nabla P \quad \text{with} \quad \dot{u} := u_t + u \cdot \nabla u.
$$

(3.20)

As $\text{div } u = 0$, testing (3.20) by $\dot{u}$ yields

$$
\int_{\mathbb{R}^2} \rho t |\dot{u}|^2 \, dx = t \int_{\mathbb{R}^2} \Delta u \cdot u_t \, dx - t \int_{\mathbb{R}^2} \nabla P \cdot u_t \, dx + t \int_{\mathbb{R}^2} (\Delta u - \nabla P) \cdot (u \cdot \nabla u) \, dx
$$

whence, integrating by parts and using again (3.20),

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} t |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \rho t |\dot{u}|^2 \, dx = \int_{\mathbb{R}^2} \rho \dot{u} \cdot (u \cdot \nabla u) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx.
$$

Performing a time integration, we get for all $0 \leq t \leq T$,

$$
\frac{t}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^2} \tau \rho |\dot{u}|^2 \, dx \, d\tau = \int_0^t \int_{\mathbb{R}^2} \tau \rho \dot{u} \cdot (u \cdot \nabla u) \, dx \, d\tau
$$

$$
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} |\nabla u(\tau)|^2 \, dx \, d\tau.
$$

To bound the right-hand side, we use the fact that

$$
\int_0^t \int_{\mathbb{R}^2} \tau \rho \dot{u} \cdot (u \cdot \nabla u) \, dx \, d\tau \leq \int_0^t \|\sqrt{\rho \tau} \dot{u} \|_{L^2} \|\sqrt{\rho \tau} u \cdot \nabla u\|_{L^2} \, d\tau
$$

$$
\leq \frac{1}{2} \int_0^t \|\sqrt{\rho \tau} \dot{u}\|^2_{L^2} \, d\tau + \frac{\|\rho_0\|_{L^\infty}}{2} \int_0^t \|u\|^2_{L^\infty} \|\sqrt{\tau} \nabla u\|^2_{L^2} \, d\tau.
$$

Observe that, thanks to (3.20), we have for some constant $C$ depending only on $\|\rho_0\|_{L^\infty}$,

$$
\|\nabla^2 u\|^2_{L^2} + \|\nabla P\|^2_{L^2} \leq C \|\sqrt{\rho} \dot{u}\|^2_{L^2}.
$$

(3.21)

Hence, applying Gronwall lemma yields some constant $C$ depending only on $\|\rho_0\|_{L^\infty}$, and such that

$$
\int_{\mathbb{R}^2} |\nabla u(t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^2} \tau \rho |\dot{u}|^2 \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^2} \tau \left( |\nabla^2 u|^2 + |\nabla P|^2 \right) \, dx \, d\tau
$$

$$
\leq C \int_0^t \|\nabla u\|^2_{L^2} \exp \left( C \int_\tau^t \|u\|^2_{L^\infty} \, d\tau \right) \, d\tau.
$$

Putting together with (3.2) and (3.13) completes the proof of the proposition. □
Proposition 3.5. Under the assumptions of Proposition 3.1, there exists a constant $C_0$ depending only on $p$ and on $\|u_0\|_{\dot{B}^{-1+2/p}_p}$, such that for all $t \in [0, T]$,

$$
\int_{\mathbb{R}^2} t^2 \left( \rho |u|^2 + |\dot{u}|^2 \right) \ dx + \int_0^t \int_{\mathbb{R}^2} \tau^2 (|\nabla u|^2 + |\nabla \dot{u}|^2) \ dx \ d\tau \leq C_0.
$$

Proof. From (3.14), the definition of $\dot{u}$ and Hölder inequality, one can write

$$
\|t \dot{u} - tu_t\|_{L^\infty(0,T;L^2)} \leq \|t \nabla u\|_{L^\infty(0,T \times \mathbb{R}^2)} \|u\|_{L^\infty(0,T;L^2)}
$$

$$
\leq C \|tu\|_{L^\infty(0,T;\dot{B}^{1+2/m}_{m,1})} \|u\|_{L^\infty(0,T;L^2)}
$$

and

$$
\|t \nabla \dot{u} - t \nabla u_t\|_{L^2(0,T \times \mathbb{R}^2)} \leq \|t \nabla u\|_{L^2(0,T \times \mathbb{R}^2)} \|u\|_{L^2(0,T \times \mathbb{R}^2)} + \|tu \otimes \nabla^2 u\|_{L^2(0,T \times \mathbb{R}^2)}
$$

$$
\leq \|t \nabla u\|_{L^2(0,T \times \mathbb{R}^2)} \|u\|_{L^2(0,T \times \mathbb{R}^2)} + \|tu \otimes \nabla^2 u\|_{L^2(0,T;L^2)} \|u\|_{L^2(0,T;L^\infty)}
$$

$$
\leq C \|tu\|_{L^\infty(0,T;\dot{B}^{1+2/m}_{m,1})} \|u\|_{L^\infty(0,T;L^2)} + \|tu \otimes \nabla^2 u\|_{L^2(0,T;L^2)} \|u\|_{L^2(0,T;L^\infty)}.
$$

Furthermore, (3.21) implies that

$$
\|t \nabla^2 u\|_{L^2_{x,t}}^2 + \|t \nabla P\|_{L^2_{x,t}}^2 \leq C \|t \sqrt{\rho} \dot{u}\|_{L^2_{x,t}}^2.
$$

Hence, to complete the proof, it is only a matter of showing that

$$
\|tu_t\|_{L^\infty(0,T;L^2)} + \|t \nabla u_t\|_{L^2(0,T \times \mathbb{R}^2)} \leq C_0.
$$

To do so, apply $\partial_t$ to the momentum equation of (INS). We get

$$
\rho u_t + \rho u \cdot \nabla u_t - \Delta u_t + \nabla P_t = - \rho_t \dot{u} - \rho u_t \cdot \nabla u.
$$

As $\text{div } u_t = 0$, by taking the $L^2(\mathbb{R}^2; \mathbb{R}^2)$ scalar product of (3.23) with $t^2 u_t$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |u|^2 \ dx + \int_{\mathbb{R}^2} t^2 |\nabla u_t|^2 \ dx
$$

$$
\leq \int_{\mathbb{R}^2} t \rho |u|^2 \ dx - \int_{\mathbb{R}^2} t^2 \rho_t \cdot u_t \ dx - \int_{\mathbb{R}^2} t^2 \rho (u_t \cdot \nabla u) \cdot u_t \ dx.
$$

Then, integrating with respect to time yields for all $t \in [0, T]$,

$$
\frac{1}{2} \sup_{\tau \leq t} \|t \sqrt{\rho} u_t\|_{L^2_{x,t}}^2 + \|t \nabla u_t\|_{L^2(0,t \times \mathbb{R}^2)}^2 \leq \int_0^t \int_{\mathbb{R}^2} \tau \rho |u_t|^2 \ dx \ d\tau
$$

$$
- \int_0^t \int_{\mathbb{R}^2} \tau^2 \rho \tau \dot{u} \cdot u_t \ dx \ d\tau - \int_0^t \int_{\mathbb{R}^2} \tau^2 \rho (u_t \cdot \nabla u) \cdot u_t \ dx \ d\tau =: I_1 + I_2 + I_3.
$$

For term $I_2$, the mass equation of (INS) and integration by parts yield

$$
I_2 = \int_0^t \int_{\mathbb{R}^2} \tau^2 \text{div} (\rho u) \dot{u} \cdot u_t \ dx \ d\tau
$$

$$
= - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot \nabla \dot{u}) \cdot u_t \ dx \ d\tau - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot \nabla u_t) \cdot \dot{u} \ dx \ d\tau
$$

$$
=: I_{21} + I_{22}.
$$
Since $\dot{u} = u_t + u \cdot \nabla u$, we may write

$$I_{21} = - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot \nabla u_t) \cdot u_t \, dx \, d\tau - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot (\nabla u \cdot u_t)) \cdot u_t \, dx \, d\tau$$

$$= - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot \nabla u_t) \cdot u_t \, dx \, d\tau - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot (\nabla^2 u \cdot u_t)) \, dx \, d\tau$$

$$- \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot (\nabla u \cdot u_t)) \cdot u_t \, dx \, d\tau$$

and

$$I_{22} = - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot \nabla u_t) \cdot u_t \, dx \, d\tau - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u \cdot (u \cdot \nabla u)) \, dx \, d\tau.$$

Applying Young’s inequality and remembering that $\rho$ is bounded gives for all $\varepsilon > 0$,

$$I_{21} \lesssim \int_0^t \|u\|_{L_\infty} \|\tau \nabla u_t\|_{L_2} \|\tau \sqrt{\rho} u_t\|_{L_2} \, d\tau + \int_0^t \|u\|_{L_\infty}^2 \|\nabla^2 u\|_{L_2} \|\tau \sqrt{\rho} u_t\|_{L_2} \, d\tau$$

$$+ \int_0^t \|\tau^{1/2} \nabla u\|_{L_2} \|\tau^{1/2} \nabla u\|_{L_\infty} \|\tau \sqrt{\rho} u_t\|_{L_2} \|u\|_{L_\infty} \, d\tau$$

$$\leq C\varepsilon^{-1} \int_0^t \|u\|_{L_2}^2 \|\tau \sqrt{\rho} u_t\|_{L_2}^2 \, d\tau + C \int_0^t \|\tau^{1/2} \nabla u\|_{L_\infty}^2 \|\tau^{1/2} \nabla^2 u\|_{L_2}^2 \, d\tau$$

$$+ C \int_0^t \|\tau^{1/2} \nabla u\|_{L_2}^2 \|\tau^{1/2} \nabla u\|_{L_\infty}^2 \, d\tau + \varepsilon \int_0^t \|\tau \nabla u_t\|_{L_2}^2 \, d\tau,$$

and

$$I_{22} \lesssim \int_0^t \|\tau \sqrt{\rho} u_t\|_{L_2} \|\tau \nabla u_t\|_{L_2} \|u\|_{L_\infty} \, d\tau + \int_0^t \|\tau \nabla u_t\|_{L_2} \|\tau \nabla u_t\|_{L_2}^2 \|\nabla u\|_{L_2} \, d\tau$$

$$\leq C\varepsilon^{-1} \left( \int_0^t \|\tau \sqrt{\rho} u_t\|_{L_2}^2 \|u\|_{L_\infty}^2 \|\nabla u\|_{L_2} \, d\tau$$

$$+ \int_0^t \|\tau^{1/2} \nabla u\|_{L_\infty} \|\tau^{1/2} \nabla u\|_{L_2} \, d\tau \right) + \varepsilon \int_0^t \|\tau \nabla u_t\|_{L_2}^2 \, d\tau.$$

For $I_3$, one has

$$I_3 = - \int_0^t \int_{\mathbb{R}^2} \tau^2 (\rho u_t \cdot \nabla u) \cdot u_t \, dx \, d\tau \leq \int_0^t \|\tau \sqrt{\rho} u_t\|_{L_2}^2 \|\nabla u\|_{L_\infty} \, d\tau.$$

Taking $\varepsilon$ small enough, then reverting to (3.24) and applying Gronwall inequality gives

$$\sup_{\tau \leq t} \|\tau \sqrt{\rho} u_t\|_{L_2}^2 + \int_0^t \|\tau \nabla u_t\|_{L_2}^2 \, d\tau \leq C \exp \left( \int_0^t \|u\|_{L_\infty}^2 + \|\nabla u\|_{L_\infty} \, d\tau \right)$$

$$\left( \int_0^t \|\tau^{1/2} \nabla u\|_{L_2}^2 \|\tau^{1/2} \nabla u\|_{L_\infty}^2 \, d\tau + \int_0^t \int_{\mathbb{R}^2} \tau \rho |u_t|^2 \, dx \, d\tau$$

$$+ \int_0^t \|u\|_{L_\infty}^2 \|u\|_{L_2}^2 \|\tau^{1/2} \nabla u\|_{L_2}^2 \, d\tau + \int_0^t \|\tau^{1/2} u\|_{L_\infty}^2 \|\tau^{1/2} \nabla u\|_{L_2}^2 \, d\tau \right).$$

Combining with Propositions 3.2 and 3.4, Inequality (3.13) and Corollary 3.3 allows to bound the right-hand side by $C_0$ for all $t \in [0, T]$, and using also (3.22) completes the proof. \hfill \Box
In order to get a higher order time weighted estimate, one has to consider the evolutionary equation for $\dot{u}$. So we take the convective derivative of (3.20), getting
\[
\frac{D}{Dt}(\rho \dot{u}) - \frac{D}{Dt}u + \frac{D}{Dt}P = 0.
\]
Observe that
\[
- \frac{D}{Dt}u = -\Delta \dot{u} + \Delta u \cdot \nabla u + 2\nabla u \cdot \nabla^2 u
\]
with \((\nabla u \cdot \nabla^2 u)^i := \sum_{1 \leq j, k \leq d} \partial_k u^j \partial_j \partial_k u^i\).
\[
\frac{D}{Dt} \nabla P = \nabla \dot{P} - \nabla u \cdot \nabla P,
\]
\[
\frac{D}{Dt}(\rho \dot{u}) = \rho \ddot{u} \quad \text{with} \quad \ddot{u} := \frac{D}{Dt}\dot{u}.
\]
Hence, we have
\[
\rho \ddot{u} - \Delta \dot{u} + \nabla \dot{P} = f \quad \text{with} \quad f := -\Delta u \cdot \nabla u - 2\nabla u \cdot \nabla^2 u + \nabla u \cdot \nabla P.
\tag{3.25}
\]

**Proposition 3.6.** Under the assumptions of Proposition 3.2, it holds that
\[
\|t\dot{u}\|_{L^\infty(0,T;\dot{H}^{1+2/p}_{p,1}(\mathbb{R}^2))} + \|(t\dot{u})_t, t\nabla^2 \dot{u}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^2))}
\]
\[
+\|t\dot{u}\|_{L^2(0,T;L^\infty(\mathbb{R}^2))} \leq C_0.
\]

**Proof.** From (3.25), we get the following equation for $t\dot{u}$:
\[
\rho (t\dot{u})_t - \Delta (t\dot{u}) + \nabla (t\dot{P}) = -t\rho u \cdot \nabla \dot{u} + \rho \ddot{u} + tf.
\tag{3.26}
\]
Since $\text{div} \dot{u} \neq 0$, one cannot apply directly Proposition A.5. Now, let us introduce the Helmholtz projectors on divergence free and gradient like vector-fields, namely,
\[
\mathbb{P} := \text{Id} + \nabla (-\Delta)^{-1} \text{div} \quad \text{and} \quad \mathbb{Q} := -\nabla (-\Delta)^{-1} \text{div}.
\tag{3.27}
\]
We observe that
\[
\nabla (t\dot{P}) = \mathbb{Q} \left( -t\rho u \cdot \nabla \dot{u} + \rho \ddot{u} + tf - \rho (t\dot{u})_t + \Delta (t\dot{u}) \right).
\]
Hence, reverting to (3.26) implies that
\[
\rho (t\dot{u})_t - \Delta (t\dot{u}) = \mathbb{P}(\rho \ddot{u} + tf - t\rho u \cdot \nabla \dot{u}) + \mathbb{Q}(\rho (t\dot{u})_t - \Delta (t\dot{u})).
\tag{3.28}
\]
Using the fact that $\text{div} u = 0$, we easily get
\[
\text{div} \dot{u} = \sum_{1 \leq i, j \leq d} \partial_i u^j \partial_j u^i = \text{Tr}(\nabla u \cdot \nabla u),
\tag{3.29}
\]
whence
\[
\mathbb{Q}(t\Delta \dot{u}) = t\nabla \text{Tr}(\nabla u \cdot \nabla u)
Finally, Inequality (3.12) enables us to conclude that

\[
\mathbb{Q}(\rho(t\dot{u})) = \mathbb{Q}((\rho - 1)(t\dot{u}) + \dot{u} + tu \cdot \nabla u + tu_1 \cdot \nabla u),
\]

we get in the end,

\[
(t\dot{u})_t - \Delta t\dot{u} = \mathbb{P}[(1 - \rho)(t\dot{u})_t - tpu \cdot \nabla \dot{u} + \rho \ddot{u} + tf]
+ \mathbb{Q}(\dot{u} + tu_1 \cdot \nabla u + tu \cdot \nabla u_1) - \nabla \text{Tr}(t\nabla u \cdot \nabla u).
\]  

(3.30)

At this point, we use the maximal regularity estimate for the heat equation stated in [20, Prop. 2.1] as well as the continuity of \(\mathbb{P}\) and \(\mathbb{Q}\) on \(L_p\) to conclude that

\[
\|t\dot{u}\|_{L^\infty(0,T; B^p_{1+2/p})} + \|(t\dot{u})_t, \nabla^2 t\dot{u}\|_{L^q,1(0,T; L_p)}
\leq \|(1 - \rho)(t\dot{u})_t - tpu \cdot \nabla \dot{u} + \rho \ddot{u} + tf\|_{L^q,1(0,T; L_p)}
+ \|\dot{u} + tu_1 \cdot \nabla u + tu \cdot \nabla u_1\|_{L^q,1(0,T; L_p)} + \|t\nabla u \otimes \nabla^2 u\|_{L^q,1(0,T; L_p)}.
\]

As usual, owing to (2.7), the first term in the right-hand side may be absorbed by the left-hand side. Now, using (3.14) and the definition of \(f\) in (3.25), we get

\[
\|tf\|_{L^q,1(0,T; L_p)} \leq C \|t\nabla u\|_{L^\infty(0,T \times \mathbb{R}^2)}(\|\nabla^2 u\|_{L^q,1(0,T; L_p)} + \|\nabla P\|_{L^q,1(0,T; L_p)})
\leq C \|t\nabla u\|_{L^\infty(0,T; B^p_{1+2/m})}(\|\nabla^2 u\|_{L^q,1(0,T; L_p)} + \|\nabla P\|_{L^q,1(0,T; L_p)}).
\]

Next, \(\|\dot{u}\|_{L^q,1(0,T; L_p)}\) may be bounded according to Inequality (3.4). Finally, we have

\[
\|t\dot{u}\|_{L^q,1(0,T; L_p)} \leq C \|t\nabla \dot{u}\|_{L^2(0,T \times \mathbb{R}^2)}\|u\|_{L^{1,1}(0,T; L_m)},
\]

\[
\|t\nabla u\|_{L^q,1(0,T; L_p)} \leq C \|t\nabla u\|_{L^2(0,T \times \mathbb{R}^2)}\|u\|_{L^{1,1}(0,T; L_m)},
\]

\[
\|t\nabla u\|_{L^q,1(0,T; L_p)} \leq C \|t\nabla u\|_{L^{1,1}(0,T; L_m)}\|\nabla u\|_{L^2(0,T \times \mathbb{R}^2)},
\]

\[
\|t\nabla u \otimes \nabla^2 u\|_{L^q,1(0,T; L_p)} \leq C \|t\nabla u\|_{L^\infty(0,T \times \mathbb{R}^2)}\|\nabla^2 u\|_{L^q,1(0,T; L_p)}
\leq C \|t\nabla u\|_{L^\infty(0,T; B^p_{1+2/m})}\|\nabla^2 u\|_{L^q,1(0,T; L_p)}.
\]

Then, putting all together with Proposition 3.1, Inequality (3.4), Proposition 3.2 and Proposition 3.5, we discover that

\[
\|t\dot{u}\|_{L^q,1(0,T; B^p_{1+2/p})} + \|(t\dot{u})_t, t\nabla^2 \dot{u}\|_{L^q,1(0,T; L_p)} \leq C_0.
\]

Finally, Inequality (3.12) enables us to conclude that

\[
\|t\dot{u}\|_{L^2(0,T; L^\infty)} \leq \|t\dot{u}\|_{L^2(0,T; B^p_{1+2/p})}^{2-q} \|t\nabla^2 \dot{u}\|_{L^q,1(0,T; L_p)}^{2} \leq C_0,
\]

which completes the proof. \(\square\)

We end this section proving higher order energy type time weighted estimates (that are not required for proving the uniqueness).

**Proposition 3.7.** Under the assumptions of Proposition 3.2, we have for all \(t \in [0, T],\)

\[
\sup_{\tau \in [0,t]} \|\tau^{3/2}\nabla u\|_{L^2}^{2} + \int_0^t \|\tau^{3/2}\nabla^2 \dot{u}, \tau^{3/2} \nabla \dot{P}, \tau^{3/2} \sqrt{\rho} \tilde{u}\|_{L^2}^2 d\tau \leq C_0,
\]

where \(C_0\) depends only on \(p\) and on \(\|u_0\|_{B^p_{1+2/p}}.\)
Proof. Taking the $L_2(\mathbb{R}^2; \mathbb{R}^2)$ inner product of (3.25) with $t^3 \ddot{u}$ then integrating on $[0, t]$ yields

$$
\frac{t^3}{2} \int_{\mathbb{R}^2} |\nabla \ddot{u}|^2 \, dx + \int_0^t \int_{\mathbb{R}^2} \tau^3 \rho |\ddot{u}|^2 \, dx \, d\tau = \int_0^t \int_{\mathbb{R}^2} \frac{3 \tau^2}{2} |\nabla \ddot{u}|^2 \, dx \, d\tau
+ \int_0^t \int_{\mathbb{R}^2} \Delta \ddot{u} \cdot \tau^3 u \cdot \nabla \ddot{u} \, dx \, d\tau
- \int_0^t \int_{\mathbb{R}^2} \nabla \dot{P} \cdot \left( \tau^3 u \cdot \nabla u \right) \, dx \, d\tau
+ \int_0^t \int_{\mathbb{R}^2} \nabla \dot{P} \cdot \left( \tau^3 u_t \cdot \nabla u \right) \, dx \, d\tau
+ \int_0^t \int_{\mathbb{R}^2} f \cdot \tau^3 \ddot{u} \, dx \, d\tau =: J_k, \quad 1 \leq k \leq 6
$$

(3.31)

In order to bound $J_2, J_3, J_4, J_5$, we proceed as follows:

$$
J_2 = \int_0^t \int_{\mathbb{R}^2} \Delta \ddot{u} \cdot \tau^3 u \cdot \nabla \ddot{u} \, dx \, d\tau
\leq \| \tau^{3/2} \nabla^2 \ddot{u} \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau \nabla \ddot{u} \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau^{1/2} \nabla u \|_{L_\infty(0,t \times \mathbb{R}^2)},
$$

$$
J_3 = -\int_0^t \int_{\mathbb{R}^2} \nabla \dot{P} \cdot \left( \tau^3 u \cdot \nabla \ddot{u} \right) \, dx \, d\tau
\leq \| \tau^{3/2} \nabla \dot{P} \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau \nabla \ddot{u} \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau^{1/2} \nabla u \|_{L_\infty(0,t \times \mathbb{R}^2)},
$$

$$
J_4 = \int_0^t \int_{\mathbb{R}^2} \nabla \dot{P} \cdot \left( \tau^3 u_t \cdot \nabla u \right) \, dx \, d\tau
\leq \| \tau^{3/2} \nabla \dot{P} \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau u_t \|_{L_\infty(0,t; L_2)} \| \tau^{1/2} \nabla u \|_{L_2(0,t; L_\infty)},
$$

$$
J_5 = \int_0^t \int_{\mathbb{R}^2} \nabla \dot{P} \cdot \left( \tau^3 u \cdot \nabla u_t \right) \, dx \, d\tau
\leq \| \tau^{3/2} \nabla \dot{P} \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau \nabla u_t \|_{L_2(0,t \times \mathbb{R}^2)} \| \tau^{1/2} \nabla u \|_{L_\infty(0,t \times \mathbb{R}^2)}.
$$

At this point, we have to explain how to bound $t^{3/2} \nabla^2 \ddot{u}$ and $t^{3/2} \nabla \dot{P}$ in $L_2(0, T \times \mathbb{R}^2)$. Observe that (3.25) and (3.29) ensure that

$$
\nabla \dot{P} = Q f - Q(\rho \ddot{u}) + \nabla \text{Tr}(\nabla u \cdot \nabla u).
$$

(3.32)

Hence, owing to the continuity of $Q$ on $L_2$,

we have for all $\tau \in [0, T]$,

$$
\| \tau^{3/2} \nabla \dot{P}(\tau) \|_{L_2} \lesssim \| \nabla \dot{P}(\tau) \|_{L_2} + \| \tau^{3/2}(\nabla u \otimes \nabla^2 u)(\tau) \|_{L_2} + \| \tau^{3/2} f(\tau) \|_{L_2}.
$$

Hence, since

$$
\tau^{3/2} \Delta \ddot{u} = \tau^{3/2} \nabla \dot{P} + \rho \tau^{3/2} \ddot{u} + \tau^{3/2} \Delta u \cdot \nabla u + 2 \tau^{3/2} \nabla u \cdot \nabla^2 u - \tau^{3/2} \nabla u \cdot \nabla P,
$$

we easily get

$$
\| \tau^{3/2} \nabla^2 \ddot{u} , \tau^{3/2} \nabla \dot{P} \|_{L_2(0,t \times \mathbb{R}^2)} \lesssim \| \tau^{3/2} \sqrt{\rho \ddot{u}} \|_{L_2(0,t \times \mathbb{R}^2)}
$$

$$
+ \| \tau^{3/2} \nabla u \otimes \nabla^2 u \|_{L_2(0,t \times \mathbb{R}^2)} + \| \tau^{3/2} \nabla u \cdot \nabla u \|_{L_2(0,t \times \mathbb{R}^2)}
$$

$$
\lesssim \| \tau^{3/2} \sqrt{\rho \ddot{u}} \|_{L_2(0,t \times \mathbb{R}^2)}
$$

$$
+ \| \nabla^2 u , \tau \nabla P \|_{L_\infty(0,t; L_2)} \| \tau^{1/2} \nabla u \|_{L_2(0,t; L_\infty)}.
$$
Thanks to Corollary 3.3 and Proposition 3.5, we thus end up with
\[ \| \tau^{3/2} \nabla^2 \dot{u}, \tau^{3/2} \nabla \dot{P} \|_{L^2(0,T;\mathbb{R}^2)} \leq \| \tau^{3/2} \sqrt{\rho} \ddot{u} \|_{L^2(0,T;\mathbb{R}^2)} + C_0. \] (3.33)

Reverting to the above inequalities for \( J_2 \) to \( J_5 \) and taking advantage of Corollary 3.3, Propositions 3.4 and 3.5, we conclude that there exists some constant \( C_0 \) depending only on \( p \) and on \( \| u_0 \|_{B^{-1+2/p}_{p,1}} \), and such that
\[ \sum_{k=2}^5 J_k \leq C_0 \left( \| \tau^{3/2} \sqrt{\rho} \ddot{u} \|_{L^2(0,T;\mathbb{R}^2)} + C_0 \right) \]
\[ \leq \frac{1}{4} \| \tau^{3/2} \sqrt{\rho} \ddot{u} \|_{L^2(0,T;\mathbb{R}^2)}^2 + 2C_0^2. \] (3.34)

For \( J_6 \), we write that
\[ J_6 = \int_0^T \int_{\mathbb{R}^2} f \cdot \tau^3 \ddot{u} \, dx \, d\tau \]
\[ = \int_0^T \int_{\mathbb{R}^2} (-\Delta u \cdot \nabla u - 2\nabla u \cdot \nabla^2 u + \nabla u \cdot \nabla P) \cdot \tau^3 \ddot{u} \, dx \, d\tau \]
\[ \lesssim \| \tau^{3/2} \ddot{u} \|_{L^2(0,T;\mathbb{R}^2)} \| \tau^{1/2} \nabla u \|_{L^2(0,T;L^\infty)} \| \tau (\nabla^2 u, \nabla P) \|_{L^\infty(0,T;L^2)}, \]
which along with Proposition 3.5, Corollary 3.3 and (2.7) gives
\[ J_6 \leq C_0 \| \tau^{3/2} \sqrt{\rho} \ddot{u} \|_{L^2(0,T;\mathbb{R}^2)} \leq \frac{1}{4} \| \tau^{3/2} \sqrt{\rho} \ddot{u} \|_{L^2(0,T;\mathbb{R}^2)}^2 + C_0^2. \]

Inserting the above inequality and (3.34) in (3.31), we get
\[ \int_0^T \int_{\mathbb{R}^2} |\nabla \dot{u}|^2 \, dx + \int_0^T \int_{\mathbb{R}^2} \tau^3 \rho |\ddot{u}|^2 \, dx \, d\tau \leq 3 \int_0^T \int_{\mathbb{R}^2} \tau^2 |\nabla \dot{u}|^2 \, dx \, d\tau + 3C_0^2 \]
which, by virtue of Proposition 3.5, completes the proof. \( \square \)

4. Estimates in the Three-Dimensional Case

Here we establish the inequalities that are needed to prove Theorem 2.5. The first two propositions are required for proving the existence of a global solution, while the last one is needed for uniqueness.

**Proposition 4.1.** Let \((\rho, u)\) be a smooth solution of (INS) on \([0,T] \times \mathbb{R}^3\), with \( u \) sufficiently decaying at infinity and \( \rho \) such that
\[ \sup_{t \in [0,T]} \| \rho(t) - 1 \|_{L^\infty(\mathbb{R}^3)} \leq c \ll 1. \] (4.1)

Then, for all indices \( 1 < m, p, q, s < \infty \) satisfying
\[ \frac{3}{p} + \frac{2}{q} = 3 \quad \text{and} \quad \frac{3}{m} + \frac{2}{s} = 1 \quad \text{with} \quad p < m < \infty \quad \text{and} \quad q < s < \infty, \] (4.2)
the following inequalities hold true:
\[ \mu \frac{3}{2} \cdot \frac{1}{2} \| u \|_{L^\infty(0,T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \mu \frac{3}{2} \cdot \frac{1}{2} \| u \|_{L^1_t(0,T; L^m(\mathbb{R}^3))} \]

\[ + \| \dot{u}, u_t, \mu \nabla^2 u, \nabla P \|_{L^q_t(0,T; L^p(\mathbb{R}^3))} \leq C \mu \frac{3}{2} \cdot \frac{1}{2} \| u_0 \|_{\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)}, \]  

(4.3)

and \[ \mu \frac{1}{2} \| u \|_{L^2(0,T; L^\infty(\mathbb{R}^3))} \leq C \| u_0 \|_{\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)}. \]  

(4.4)

**Proof.** For notational simplicity, we omit to specify the dependence of the norms with respect to \( \mathbb{R}^3 \) in the proof. As usual, we only consider the case \( \mu = 1 \). Now, applying Proposition A.5 to System (3.7) yields

\[ \| u \|_{L^\infty(0,T; \dot{B}^{-1+3/p}_{p,1})} + \| u_t, \nabla^2 u, \nabla P \|_{L^q_t(0,T; L^p)} + \| u \|_{L^1_t(0,T; L^m)} \]

\[ \leq C \left( \| u_0 \|_{\dot{B}^{-1+3/p}_{p,1}} + \| (\rho - 1)u_t + \rho u \cdot \nabla u \|_{L^q_t(0,T; L^p)} \right). \]  

(4.5)

By Hölder inequality, we have

\[ \| (\rho - 1)u_t + \rho u \cdot \nabla u \|_{L^q_t(0,T; L^p)} \]

\[ \leq \| \rho - 1 \|_{L^\infty(0,T \times \mathbb{R}^3)} \| u_t \|_{L^q_t(0,T; L^p)} + \| \rho \|_{L^\infty(0,T \times \mathbb{R}^3)} \| u \cdot \nabla u \|_{L^q_t(0,T; L^p)}. \]

Owing to (4.1), the first term can be absorbed by the left-hand side of (4.5). For term \( \| u \cdot \nabla u \|_{L^q_t(0,T; L^p)} \), by embedding

\[ \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \]  

(4.6)

and

\[ \dot{W}^1_p(\mathbb{R}^3) \hookrightarrow L^{p^*}(\mathbb{R}^3) \] with \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3} \),  

(4.7)

we obtain

\[ \| u \cdot \nabla u \|_{L^q_t(0,T; L^p)} \leq \| u \|_{L^\infty(0,T; L^3)} \| \nabla u \|_{L^q_t(0,T; L^{p^*})} \]

\[ \lesssim \| u \|_{L^\infty(0,T; \dot{B}^{-1+3/p}_{p,1})} \| \nabla^2 u \|_{L^q_t(0,T; L^p)}. \]

Denoting \( \Phi_0 := \| u_0 \|_{\dot{B}^{-1+3/p}_{p,1}} \) and

\[ \Phi := \| u \|_{L^\infty(0,T; \dot{B}^{-1+3/p}_{p,1})} + \| u_t, \nabla^2 u, \nabla P \|_{L^q_t(0,T; L^p)} + \| u \|_{L^1_t(0,T; L^m)}, \]

we can conclude that

\[ \Phi \leq C(\Phi_0 + \Phi^2). \]  

Hence, if

\[ 4C \Phi_0 < 1, \]  

(4.8)

then one can assert that

\[ \Phi \leq 2 \Phi_0. \]  

(4.9)

Clearly, \( \dot{u} \) satisfies the same inequality since \( \Phi \) is small and, by Hölder inequality,
Moreover, the following inequality holds true:
\[ \|u\|_{L^q(0,T;L^p)} \leq \|u_t\|_{L^q(0,T;L^p)} + \|u\|_{L^\infty(0,T;\dot{B}_{p,1}^{-1+3/p})} \|\nabla^2 u\|_{L^q(0,T;L^p)} \] 
\[ \leq C \Phi(1 + \Phi). \]

Finally, as a consequence of Gagliardo-Nirenberg inequality and embedding, we have:
\[ \|z\|_{L^\infty} \lesssim \|z\|_{L^1}^{1-q/2} \|\nabla^2 z\|_{L^p}^{q/2} \lesssim \|z\|_{\dot{B}_{p,1}^{-1+3/p}}^{1-q/2} \|\nabla^2 z\|_{L^p}^{q/2}, \tag{4.10} \]
whence
\[ \int_0^T \|u\|_L^2 \, dt \leq C \int_0^T \|u\|_{\dot{B}_{p,1}^{-1+3/p}}^{2-q} \|\nabla^2 u\|_{L^p}^q \, dt \]
\[ \leq C \|u\|_{L^\infty(0,T;\dot{B}_{p,1}^{-1+3/p})} \|\nabla^2 u\|_{L^q(0,T;L^p)} \]
\[ \leq C \Phi^2. \tag{4.11} \]

Owing to (4.9), this yields (4.4).

**Proposition 4.2.** Under the assumptions Proposition 4.1, we have
\[ \mu \|tu\|_{L^\infty(0,T;\dot{B}_{m,1}^{1+3/m}(\mathbb{R}^3))} + \mu \frac{1}{2} \|\nabla (tu), (tu)_t\|_{L^1(0,T;L^m(\mathbb{R}^3))} \]
\[ \leq C \|u_0\|_{\dot{B}_{p,1}^{-1+3/p}(\mathbb{R}^3)}. \]

Moreover, the following inequality holds true:
\[ \mu \int_0^T \|\nabla u\|_{L^\infty(\mathbb{R}^3)} \, dt + \left( \mu \int_0^T t \|\nabla u\|_{L^\infty(\mathbb{R}^3)}^2 \, dt \right)^{1/2} \leq C \|u_0\|_{\dot{B}_{p,1}^{-1+3/p}(\mathbb{R}^3)}. \]

**Proof.** Assume that \( \mu = 1 \). Multiplying both sides of (3.7) by time \( t \) yields
\[(tu)_t - \Delta (tu) + \nabla (tP) = (1 - \rho)(tu)_t + \rho u - \rho u \cdot \nabla tu.\]

Then, taking advantage of Proposition A.5, we get:
\[ \|tu\|_{L^\infty(0,T;\dot{B}_{m,1}^{1+3/m})} + \|(tu)_t, \nabla^2 (tu), \nabla (tP)\|_{L^1(0,T;L^m)} \]
\[ \lesssim \|\rho - 1\|_{L^\infty(0,T \times \mathbb{R}^3)} \|(tu)_t\|_{L^1(0,T;L^m)} \]
\[ + \|\rho\|_{L^\infty(0,T \times \mathbb{R}^3)} \left( \|u\|_{L^1(0,T;L^m)} + \|tu \cdot \nabla u\|_{L^1(0,T;L^m)} \right). \]

Owing to (4.1), the first term of the right-hand side may be bounded by the left-hand side, and we deduce from Hölder inequality and the embedding
\[ \dot{B}_{m,1}^{3/m}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \tag{4.12} \]
that
\[ \|tu \cdot \nabla u\|_{L^1(0,T;L^m)} \leq \|u\|_{L^1(0,T;L^m)} \|t \nabla u\|_{L^\infty(0,T \times \mathbb{R}^3)} \]
\[ \leq C \|u\|_{L^1(0,T;L^m)} \|tu\|_{L^\infty(0,T;\dot{B}_{m,1}^{1+3/m})}. \]

Remember that Proposition 4.1 allows to bound \( u \) in \( L^s,1(0, T; L^m(\mathbb{R}^3)) \) by \( \Phi_0 \). Hence, setting
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\[ \Pi := \|tu\|_{L^\infty(0,T;B^{1+3/m}_{m,1})} + \| (tu)_t, \mu \nabla^2 (tu), \nabla (t P) \|_{L^1(0,T;L^m)}, \]

the above calculations imply that \( \Pi \leq C (1 + \Pi) \Phi_0 \) and, as \( \Phi_0 \) is small, this completes the proof of the first part of the proposition.

Bounding \( \nabla u \) relies on the following interpolation inequality (as (4.2) implies that \( p < 3 < m \):

\[ \|u\|_{L^\infty(\mathbb{R}^3)} \leq \| \nabla u \|_{L_p(\mathbb{R}^3)} \| \nabla u \|_{L_m(\mathbb{R}^3)}. \]

Hence, applying Hölder inequality in Lorentz spaces with exponents:

\[ (p_1, r_1) = \left( \frac{3(m-p)}{m-3-p}, \infty \right), \quad (p_2, r_2) = \left( \frac{3q(m-p)}{p(m-3)}, \frac{p_2}{q} \right), \quad (p_3, r_3) = \left( \frac{3s(m-p)}{m-3-p}, \frac{p_3}{s} \right) \]

using the fact that \( t^{-\alpha} \) with \( \alpha = m(3-p)/(3(m-p)) \) is in \( L_{1/\alpha, \infty}(\mathbb{R}_+) \), (4.3) and the first inequality of Proposition 4.2, we end up with

\[
\int_0^T \| \nabla u \|_{L^\infty} \, dt \leq \int_0^T t^{-\frac{m(3-p)}{3(m-p)}} \| \nabla^2 u \|_{L_p} \| t \nabla^2 u \|_{L_m} \, dt \\
\leq C \| \nabla^2 u \|_{L^{q,1}_r(0,T;L_p)} \| t \nabla^2 u \|_{L^{r,1}_s(0,T;L_m)} \\
\leq C \| u_0 \|_{\tilde{B}^{1+3/p}_{p,1}}.
\]

Furthermore, we deduce from (4.12) that

\[
\int_0^T \| \nabla u \|_{L^\infty}^2 \, dt \leq \int_0^T \| \nabla u \|_{\tilde{B}^{3/m}_{m,1}} \| \nabla u \|_{L^\infty} \, dt \\
\leq \| tu \|_{L^\infty(0,T;\tilde{B}^{1+3/m}_{m,1})} \int_0^T \| \nabla u \|_{L^\infty} \, dt \\
\leq C \| u_0 \|_{\tilde{B}^{1+3/p}_{p,1}}^2,
\]

by virtue of the inequality we proved just before. \( \square \)

To prove the uniqueness, the following time weighted estimate is required.

**Proposition 4.3.** Under the assumptions of Proposition 4.1, it holds that

\[
\mu \frac{1}{2p} \| t \dot{u} \|_{L^\infty(0,T;\tilde{B}^{-1+3/p}_{p,1})} + \| (t \dot{u})_t, \mu t \nabla^2 \dot{u} \|_{L^{q,1}_r(0,T;L_p)} = 0 \quad (4.13)
\]

\[
\mu \frac{1}{2p} \| t \dot{u} \|_{L^{1+3/m}_r(0,T;L_m)} \leq C \mu \frac{1}{2p} \| u_0 \|_{\tilde{B}^{-1+3/p}_{p,1}} \quad (4.14)
\]

Furthermore, we have

\[
\mu \frac{1}{2} \| t \nabla \dot{u} \|_{L^2(0,T;L^3(\mathbb{R}^3))} + \frac{1}{2} \| t \dot{u} \|_{L^2(0,T;L^\infty(\mathbb{R}^3))} \leq C \| u_0 \|_{\tilde{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)} \quad (4.15)
\]
Proof. We know that $t\dot{u}$ satisfies (3.30) and we also observe, owing to $\text{div }u = \text{div }u_t = 0$, that

$$Q(u \cdot \nabla u_t) = Q(u_t \cdot \nabla u).$$

Hence, using the maximal regularity estimates in Lorentz spaces for the heat equation (cf [20, Prop. 2.1]) and the continuity of the Helmholtz projectors on $L_{q,1}(0,T;L_p)$, we get

$$\|t\dot{u}\|_{L_{q,1}(0,T;\dot{B}^{-1+3/p}_{p,1})} + \|(t\dot{u})_t\|_{L_{q,1}(0,T;\dot{B}^{-2}_{p,1})} + \|t\dot{u}\|_{L_{q,1}(0,T;L_m)}$$

$$\lesssim \|(1-\rho)(t\dot{u})_t\|_{L_{q,1}(0,T;L_p)} + \|t\rho u \cdot \nabla\dot{u}\|_{L_{q,1}(0,T;L_p)}$$

$$+ \|t\dot{u}\|_{L_{q,1}(0,T;L_p)} + \|tf\|_{L_{q,1}(0,T;L_p)} + \|t+tu_t \cdot \nabla u\|_{L_{q,1}(0,T;L_p)}$$

$$+ \|t\nabla \text{Tr}(\nabla u \cdot \nabla u)\|_{L_{q,1}(0,T;L_p)}.$$  

By virtue of (4.1), the first term in the right-hand side may be absorbed by the left-hand side, and Proposition 4.1 allows to bound $\|\dot{u}\|_{L_{q,1}(0,T;L_p)}$. Also recall that

$$f = -\Delta u \cdot \nabla u - 2\nabla u \cdot \nabla^2 u + \nabla u \cdot \nabla P.$$  

Hence, thanks to (4.12) and to Propositions 4.1, 4.2,

$$\|tf\|_{L_{q,1}(0,T;L_p)} + \|t\nabla \text{Tr}(\nabla u \cdot \nabla u)\|_{L_{q,1}(0,T;L_p)}$$

$$\lesssim \|t\nabla u\|_{L_{q,1}(0,T;L_p)} \|\nabla^2 u, \nabla P\|_{L_{q,1}(0,T;L_p)}$$

$$\lesssim \|tu\|_{L_{q,1}(0,T;\dot{B}^{1+3/m}_{m,1})} \|\nabla^2 u, \nabla P\|_{L_{q,1}(0,T;L_p)}$$

$$\lesssim \|u_0\|^2_{\dot{B}^{-1+3/p}_{1,p}}.$$  

Using the Hölder inequality in Lorentz spaces, the embeddings (4.7) and (4.6), and Propositions 4.1, 4.2, we obtain

$$\|t\rho u \cdot \nabla\dot{u}\|_{L_{q,1}(0,T;L_p)} \lesssim \|t\nabla\dot{u}\|_{L_{q,1}(0,T;L_p)} \|u\|_{L_{q,1}(0,T;L_3)}$$

$$\lesssim \|t\nabla^2\dot{u}\|_{L_{q,1}(0,T;L_p)} \|u_0\|^2_{\dot{B}^{-1+3/p}_{1,p}},$$

$$\|tu_t \cdot \nabla u\|_{L_{q,1}(0,T;L_p)} \leq \|u_t\|_{L_{q,1}(0,T;L_p)} \|t\nabla u\|_{L_{q,1}(0,T;L_p)}$$

$$\lesssim \|u_t\|_{L_{q,1}(0,T;L_p)} \|tu\|_{L_{q,1}(0,T;L_p)}$$

$$\lesssim \|u_0\|^2_{\dot{B}^{-1+3/p}_{1,p}}.$$  

Putting the above inequalities together, we conclude that

$$\|t\dot{u}\|_{L_{q,1}(0,T;\dot{B}^{-1+3/p}_{p,1})} + \|(t\dot{u})_t\|_{L_{q,1}(0,T;\dot{B}^{-2}_{p,1})} + \|t\dot{u}\|_{L_{q,1}(0,T;L_m)}$$

$$\lesssim \|u_0\|^2_{\dot{B}^{-1+3/p}_{1,p}} + (1 + \|t\nabla^2\dot{u}\|_{L_{q,1}(0,T;L_p)}) \|u_0\|^2_{\dot{B}^{-1+3/p}_{1,p}}.$$  

Since $\|u_0\|_{\dot{B}^{-1+3/p}_{1,p}}$ is small, we have (4.13).

In order to prove Inequality (4.15), let us first consider the case $3/2 < p < 3$ (which implies that $1 < q < 2$). Then, Inequality (4.10) ensures that

$$\|\dot{u}\|_{L_{q,1}} \leq C \|\dot{u}\|_{L_{3}}^{1-q/2} \|\nabla^2\dot{u}\|^q_{L_p} \leq C \|\dot{u}\|_{\dot{B}^{-1+3/p}_{1,p}}^{1-q/2} \|\nabla^2\dot{u}\|^q_{L_p}.$$
Consequently,
\[ \| t \dot{u} \|_{L^2(0,T;L^\infty)} \leq C \| t \dot{u} \|^{1-\frac{q}{2}}_{L^\infty(0,T;\dot{B}^{-1+3/p}_{p,1})} \| \nabla^2 (t \dot{u}) \|^{\frac{q}{2}}_{L^q(0,T;L^p)}. \]

Then, applying (4.13) gives the second part of (4.15).

In order to complete the proof of (4.15), it suffices to apply Proposition A.4 with \( r = 3 \) to \( t \dot{u} \) (keeping in mind that \( -1 + 3/p = 2 - 2/q \)) then Hölder inequality with respect to the time variable. In the end, as \( p \in (3/2, 3) \), we get
\[ \| t \nabla \dot{u} \|_{L^2(0,T;L^3)} \lesssim \| t \dot{u} \|^{\theta}_{L^\infty(0,T;\dot{B}^{-1+3/p}_{p,1})} \| t \nabla^2 \dot{u} \|^{1-\theta}_{L^q(0,T;L^p)} \] with \( \theta = \frac{2p - 3}{3p - 3} \).

Then, applying the first part of the proposition gives the desired result.

The case \( 1 < p \leq 3/2 \) reduces to the case we treated before since \( \dot{B}^{-1+\frac{3}{p}}_{p,1} \hookrightarrow \dot{B}^{-1+\frac{3}{p_1}}_{p_1,1} \) for some \( p_1 \in (3/2, 3) \).

5. Existence

This section is devoted to the proof of existence of a global solution under our assumptions (both in dimensions 2 and 3).

As a first step, we shall smooth out the data so as to apply prior results ensuring the existence of a sequence \(( a^n, u^n, \nabla P^n )_{n \in \mathbb{N}} \) of strong (relatively) smooth solutions to (1.4). The estimates of Sects. 3 and 4 will guarantee that the solution \(( a^n, u^n, \nabla P^n )_{n \in \mathbb{N}} \) is global and uniformly bounded in the expected spaces. In order to pass to the limit, we shall take advantage of compactness arguments. A technical point is that Lorentz spaces \( L_{q,1} \) are nonreflexive, so that one cannot resort to the classical results, like Aubin-Lions’ lemma. To overcome the difficulty, we shall look at the approximate solutions in the slightly larger (but reflexive) space
\[ \dot{W}^{2,1}_{p,r}(\mathbb{R}_+ \times \mathbb{R}^d) := \left\{ u \in C_b(\mathbb{R}_+; \dot{B}^{2-2/r}_{p,r}(\mathbb{R}^d) : u_t, \nabla^2 u \in L_r(\mathbb{R}_+; L^p(\mathbb{R}^d)) \right\} \]
for some \( 1 < r < \infty \), then check afterward that the constructed solution has the desired regularity.

As a first, let us smooth out the initial data \( a_0 \) and \( u_0 \) by means of non-negative mollifiers, to get a sequence \(( a^n_0, u^n_0 )_{n \in \mathbb{N}} \) of smooth data such that
\[ \| a^n_0 \|_{L^\infty} \leq \| a_0 \|_{L^\infty}, \quad \| u^n_0 \|_{\dot{B}^{-1+d/p}_{p,1}} \leq C \| u_0 \|_{\dot{B}^{-1+d/p}_{p,1}} \tag{5.1} \]
with, in addition,
\[ a^n_0 \rightharpoonup a_0 \quad \text{weak} \ast \in L^\infty \quad \text{and} \quad u^n_0 \rightharpoonup u_0 \quad \text{strongly in} \quad \dot{B}^{-1+d/p}_{p,1}. \]

According to e.g. [13], there exists \( T > 0 \) such that System (1.4) supplemented with initial data \(( a^n_0, u^n_0 ) \) admits a unique smooth local solution \(( a^n, u^n, \nabla P^n ) \) on \([0, T] \times \mathbb{R}^d \). In particular, the energy balance is satisfied (in the cases where \( u_0 \) is in \( L^2(\mathbb{R}^d) \)), \( a^n \in C_b([0, T] \times \mathbb{R}^d) \) and \(( u^n, \nabla P^n ) \) is in the following space for all \( r \geq 1 \):
\[ E_T^{p,r} = \{ (u, \nabla P) \mid u \in \dot{W}^{2,1}_{p,r}(0, T \times \mathbb{R}^d) \quad \text{and} \quad \nabla P \in L_r(0, T; L^p) \}. \]
Let us denote by $T^n$ the maximal time of existence of $(a^n, u^n, \nabla P^n)$. Since the calculations of the previous sections just follow from the properties of the heat flow and transport equation, basic functional analysis and integration by parts, each $(a^n, u^n, \nabla P^n)$ satisfies the estimates therein up to time $T^n$, and thus

$$\|a^n(t)\|_{L_\infty} = \|a_0^n\|_{L_\infty} \leq \|a_0\|_{L_\infty} \quad \text{for all} \quad t \in [0, T^n) \tag{5.2}$$

and, due to (5.1),

$$\|u^n\|_{\dot{B}^{2,1}_{p,r}(0,T^n;\mathbb{R}^d)} + \|\nabla P^n\|_{L_q,(0,T^n;L_p)} \leq C\|u_0\|_{\dot{B}^{-1+d/p}_{p,1}}. \tag{5.3}$$

Moreover, taking any $r \in (1, \infty)$ and applying Proposition A.5 with $q = r$ to

$$\partial_t u^n - \Delta u^n + \nabla P^n = -a^n \partial_t u^n - (1 + a^n)u^n \cdot \nabla u^n, \quad \text{div } u^n = 0,$n

yields for all $T < T^n$,

$$\|u^n, \nabla P^n\|_{E_{T}^{p,r}} \lesssim \|u_0^n\|_{\dot{B}^{2,2/r}_{p,r}} + \|u^n\|_{L_r(0,T;L_p)(T^n)}$$

whence, defining $\beta$ by $1/\beta + 1/d - 1/dr = 1/p$,

$$\|u^n, \nabla P^n\|_{E_{T}^{p,r}} \lesssim \|u_0^n\|_{\dot{B}^{2,2/r}_{p,r}} + \|u^n\|_{L_r(0,T;L_p)}$$

Combining with Proposition A.4 and Young’s inequality gives for all $\varepsilon > 0$,

$$\|u^n, \nabla P^n\|_{E_{T}^{p,r}} \leq C\|u_0^n\|_{\dot{B}^{2,2/r}_{p,r}} + \|u^n\|_{L_r(0,T;L_p)} + C\varepsilon \int_0^T \|u^n\|_{L_r}^{2r} \|u^n\|_{\dot{B}^{2,2/r}_{p,r}} dt.$$

Then, taking $\varepsilon$ small enough and using Gronwall’s inequality yields

$$\|u^n, \nabla P^n\|_{E_{T}^{p,r}} \leq C\|u_0^n\|_{\dot{B}^{2,2/r}_{p,r}} \exp \left( C\int_0^T \|u^n\|_{L_r}^{2r} \|u^n\|_{\dot{B}^{2,2/r}_{p,r}} dt \right). \tag{5.4}$$

In the end, Gagliardo-Nirenberg inequality and embedding give

$$\|u^n\|_{L_r}^{1/r} \leq \|u^n\|_{L_\infty}^{\frac{1}{p}} \|u^n\|_{\dot{B}^{-1+d/p}_{p,1}}.$$
which implies that
\[
\int_0^T \| u^n \|_{L^2(\Omega)}^2 \, dt \leq \| u^n \|^2 L_2(0,T;L_\infty) \| u^n \|^{2(r-1)} L_\infty(0,T;B_{p,1}^{-1+1/d/p}).
\] (5.5)

Now, we deduce from Proposition 3.1 (case \( d = 2 \)) or Proposition 4.1 (case \( d = 3 \)) that the two factors of the right-hand side of (5.5) are bounded by \( \| u_0 \|_{B_{p,1}^{-1+1/d/p}} \). Hence, reversing to (5.4) and using a classical continuation argument allows to conclude that the solution is global and belongs to all spaces \( W^{r,1}_p(\mathbb{R}^+ \times \mathbb{R}^d) \) with \( 1 < r < \infty \). Furthermore, since the solution is smooth and (5.1) is satisfied, all the a priori estimates of Sects. 3 and 4 are satisfied uniformly with respect to \( n \).

In particular, \((u^n, \nabla P^n)_{n \in \mathbb{N}}\) is bounded in \( E^{p,q}_T \) for all \( T \geq 0 \). This, together with (5.2) ensures that there exists a subsequence, still denoted by \((a^n, u^n, \nabla P^n)_{n \in \mathbb{N}}\), and \((a, u, \nabla P)\) with
\[
a \in L_\infty(\mathbb{R}^+ \times \mathbb{R}^d), \quad \nabla P \in L_q(\mathbb{R}^+; L_p(\mathbb{R}^d)) \quad \text{and} \quad u \in \dot{W}^{2,1}_p(\mathbb{R}^+ \times \mathbb{R}^d)
\]
such that
\[
a^n \rightharpoonup a \quad \text{weak * in} \quad L_\infty(\mathbb{R}^+ \times \mathbb{R}^d),
\]
\[
u^n \rightharpoonup u \quad \text{weak * in} \quad L_\infty(\mathbb{R}^+; \dot{H}^{2-2/q}_p),
\]
\[
(\partial_t u^n, \nabla^2 u^n) \rightharpoonup (\partial_t u, \nabla^2 u) \quad \text{weakly in} \quad L_q(\mathbb{R}^+; L_p),
\]
\[
\nabla P^n \rightharpoonup \nabla P \quad \text{weakly in} \quad L_q(\mathbb{R}^+; L_p).
\] (5.6)

Furthermore, as all the spaces under consideration in the previous sections have the Fatou property, the estimates proved therein as still valid. For example, one gets
\[
u \in \dot{W}^{2,1}_p(\mathbb{R}^+ \times \mathbb{R}^d) \quad \text{and} \quad \nabla P \in L_{q,1}(\mathbb{R}^+; L_p(\mathbb{R}^d)).
\]

Note that the fact that \((\partial_t u^n)_{n \in \mathbb{N}}\) is bounded in \( L_q(\mathbb{R}^+; L_p)\) enables us to take advantage of Arzelà-Ascoli Theorem in order to get strong convergence results for \( u \) like, for instance, for all small enough \( \varepsilon > 0 \),
\[
u^n \rightarrow u \quad \text{strongly in} \quad L_{\infty,loc}(\mathbb{R}^+; L_{d-\varepsilon,loc}(\mathbb{R}^d)),
\]
\[
\nabla u^n \rightarrow \nabla u \quad \text{strongly in} \quad L_{q,loc}(\mathbb{R}^+; L_{p^* - \varepsilon,loc}) \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}.
\] (5.7)

This allows to pass to the limit in the convection term of the velocity equation of (1.4). In order to pass to the limit in the terms containing \( a^n \) and conclude that \((a, u, \nabla P)\) is a global weak solution, one can argue as in [12,23]. Since \( a \in L_\infty(\mathbb{R}^+ \times \mathbb{R}^d)\), \( \nabla u \in L_{2q}(\mathbb{R}^+; L_{\frac{d}{2q-1}}) \) and \( \text{div} \, u = 0 \), the Di Perna–Lions theory in [3] ensures that \( a \) is the only solution to the mass equation of (INS) and that
\[
a^n \rightharpoonup a \quad \text{strongly in} \quad L_{\alpha,loc}(\mathbb{R}^+ \times \mathbb{R}^d) \quad \text{for all} \quad 1 < \alpha < \infty.
\]

Then, using the uniform bounds for \((a^n, u^n, P^n)_{n \in \mathbb{N}}\) one can pass to the limit in all the terms of the following equations, that are satisfied by construction of \((a^n, u^n, \nabla P^n)\) for all functions \( \phi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}^d) \) and \( \Phi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d) \) with \( \Phi \equiv 0 \):
\[
\int_0^\infty \int_{\mathbb{R}^d} a^n(\partial_t \phi + u^n \cdot \nabla \phi) \, dx \, dt + \int_{\mathbb{R}^d} \phi(0, x) a^n_0(x) \, dx = 0,
\]
\begin{align*}
\int_0^\infty \int_{\mathbb{R}^d} \phi \, \text{div}\, u^n \, dx \, dt &= 0 \quad \text{and} \\
\int_0^\infty \int_{\mathbb{R}^d} \left( (1 + a^n) (u^n \cdot \partial_t \Phi + (u^n \otimes u^n : \nabla \Phi)) + (\Delta u^n - \nabla P^n) \cdot \Phi \right) \, dx \, dt \\
&+ \int_{\mathbb{R}^d} u^n_0 \cdot \Phi(0, x) \, dx = 0.
\end{align*}

This ensures that \((a, u, \nabla P)\) is a distributional solution of (1.4), which completes the proof of the existence parts of Theorems 2.4 and 2.5.

6. Uniqueness Results

The goal of this section is to prove Theorem 2.7, and to show that it implies the uniqueness part of Theorems 2.4, 2.5 and 2.9. Theorem 2.7 will come up as a consequence of the stability estimates of Proposition 6.1 (in the 3D case) and of Propositions 6.2, and 6.3 (2D case).

Throughout this part, we denote
\[ r_0 := \inf_{x \in \mathbb{R}^d} \rho_0(x) \quad \text{and} \quad R_0 := \sup_{x \in \mathbb{R}^d} \rho_0(x). \tag{6.1} \]

6.1. Uniqueness in the three-dimensional case. Let us first state general stability estimates in the 3D case.

**Proposition 6.1.** Let \((\rho_1, u_1, P_1)\) and \((\rho_2, u_2, P_2)\) be two finite energy solutions of (INS) on \([0, T] \times \mathbb{R}^3\) corresponding to the same initial density \(\rho_0\) but, possibly, two different initial velocities \(u_{1,0}\) and \(u_{2,0}\). Let \(\delta \rho := \rho_1 - \rho_2, \delta u := u_1 - u_2, \)
\[ f(t) := \|\dot{\delta} u_2\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \delta u_2\|_{L^1(\mathbb{R}^3)} \quad \text{and} \quad g(t) := \|\nabla u_2\|_{L^\infty(\mathbb{R}^3)}. \]

There exists an absolute constant \(C\) such that the following a priori estimate holds true for all \(t \in [0, T]\):
\[ \sup_{\tau \in (0, t]} \frac{\|\delta \rho(\tau)\|_{H^{-1}(\mathbb{R}^3)}}{\tau} \leq R_0^{1/2} \left( \int \sqrt{\rho_0} \delta u_0 \|_{L^1(\mathbb{R}^3)} \right) e^{C R_0 \int_0^t f^2 \, d\tau \exp(2 \int_0^\tau g \, d\tau)} e^{2 \int_0^t g \, d\tau}, \]
\[ \left\| \sqrt{\rho_1(t)} \delta u(\tau) \right\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \delta u\|_{L^2(\mathbb{R}^3)}^2 \, d\tau \leq \left( \int \sqrt{\rho_0} \delta u_0 \|_{L^1(\mathbb{R}^3)} \right) e^{C R_0 \int_0^t f^2 \, d\tau \exp(2 \int_0^\tau g \, d\tau)} e^{2 \int_0^t g \, d\tau}. \]

**Proof.** The beginning of the proof is independent of the dimension \(d\). In sharp contrast with [6, 18, 20], our stability estimates are performed directly in Eulerian coordinates: we consider the following system that is satisfied by \(\delta \rho, \delta u\) and \(\delta P := P_1 - P_2\), denoting \(\dot{u}_2 := (u_2)_t + u_2 \cdot \nabla u_2, \)
\[ \begin{cases}
(\delta \rho)_t + \delta u \cdot \nabla \rho_1 + u_2 \cdot \nabla \delta \rho = 0, \\
\rho_1 (\delta u)_t + \rho_1 u_1 \cdot \nabla \delta u - \Delta \delta u + \nabla \delta P = -\delta \rho \dot{u}_2 - \rho_1 \delta u \cdot \nabla u_2, \\
\delta \rho|_{t=0} = 0, \quad \delta u|_{t=0} = \delta u_0.
\end{cases} \tag{6.2} \]
Let us set $\phi := -(-\Delta)^{-1} \delta \rho$ (so that $\|\delta \rho\|_{H^{-1}(\mathbb{R}^d)} = \|\nabla \phi\|_{L_2(\mathbb{R}^d)}$). Testing the first equation of (6.2) by $\phi$ yields after integrating by parts and using that $\text{div}~u_1 = \text{div}~u_2 = 0$,

$$\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|^2_{L_2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \nabla u_2 : (\nabla \phi \otimes \nabla \phi) \, dx - \int_{\mathbb{R}^d} \rho_1 \delta u \cdot \nabla \phi \, dx$$

$$\leq \|\nabla u_2\|_{L_\infty(\mathbb{R}^d)} \|\nabla \phi \otimes \nabla \phi\|_{L_1(\mathbb{R}^d)} + R_0^{1/2} \|\sqrt{\rho_1} \delta u\|_{L_2(\mathbb{R}^d)} \|\nabla \phi\|_{L_2(\mathbb{R}^d)}.$$  

After time integration, we find that for all $t \in [0, T]$,

$$\|\nabla \phi(t)\|_{L_2(\mathbb{R}^d)} \leq \int_0^t \|\nabla u_2\|_{L_\infty(\mathbb{R}^d)} \|\nabla \phi\|_{L_2(\mathbb{R}^d)} \, d\tau + \int_0^t R_0^{1/2} \|\sqrt{\rho_1} \delta u\|_{L_2(\mathbb{R}^d)} \, d\tau.$$  

For all $t \in [0, T]$, set

$$X(t) := \sup_{\tau \in [0, t]} \tau^{-1} \|\delta \phi(\tau)\|_{\dot{H}^{-1}(\mathbb{R}^d)}$$

and

$$Y(t) := \left( \sup_{\tau \in [0, t]} \left( \|\sqrt{\rho_1} \delta u(\tau)\|_{L_2(\mathbb{R}^d)}^2 + \|\nabla \delta u\|_{L_2(0, t \times \mathbb{R}^d)}^2 \right) \right)^{1/2}.$$  

From the above inequality and the mass conservation, we end up with

$$X(t) \leq \int_0^t g \, d\tau + R_0^{1/2} Y(t)$$  

and Gronwall lemma thus gives (since $Y$ is nondecreasing)

$$X(t) \leq R_0^{1/2} Y(t) e^{\int_0^t g \, d\tau}.$$  

In order to control $Y$, we test (6.2) by $\delta u$ and find that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \rho_1 |\delta u|^2 \, dx + \int_{\mathbb{R}^d} |\nabla \delta u|^2 \, dx = -\int_{\mathbb{R}^d} \delta \rho \dot{u}_2 \cdot \delta u \, dx - \int_{\mathbb{R}^d} \rho_1 (\delta u \cdot \nabla u_2) \cdot \delta u \, dx.$$  

Bounding the last term is straightforward: we just write

$$-\int_{\mathbb{R}^d} \rho_1 (\delta u \cdot \nabla u_2) \cdot \delta u \, dx \leq \|\nabla u_2\|_{L_\infty(\mathbb{R}^d)} \|\sqrt{\rho_1} \delta u\|_{L_2(\mathbb{R}^d)}^2.$$  

In order to estimate the term with $\delta \rho \dot{u}_2 \cdot \delta u$, we argue by duality, writing that

$$-\int_{\mathbb{R}^d} \delta \rho \dot{u}_2 \cdot \delta u \, dx \leq \|\delta \rho\|_{\dot{H}^{-1}(\mathbb{R}^d)} \|\dot{u}_2 \cdot \delta u\|_{\dot{H}^1(\mathbb{R}^d)}$$

$$\leq X(\|\tau \nabla \dot{u}_2\|_{L_2(\mathbb{R}^d)} + \|\tau \dot{u}_2 \cdot \nabla \delta u\|_{L_2(\mathbb{R}^d)}).$$  

Assuming in the rest of the proof that $d = 3$, and using Hölder inequality and the embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L_6(\mathbb{R}^3)$, we get

$$-\int_{\mathbb{R}^3} \delta \rho \dot{u}_2 \cdot \delta u \, dx \leq X(\|\tau \nabla \dot{u}_2\|_{L_3(\mathbb{R}^3)} \|\delta u\|_{L_6(\mathbb{R}^3)} + \|\tau \dot{u}_2 \|_{L_\infty(\mathbb{R}^3)} \|\nabla \delta u\|_{L_2(\mathbb{R}^3)})$$

$$\leq C X \|\nabla \delta u\|_{L_2(\mathbb{R}^3)} (\tau \|\nabla \dot{u}_2\|_{L_3(\mathbb{R}^3)} + \|\tau \dot{u}_2\|_{L_\infty(\mathbb{R}^3)})$$

$$\leq \frac{1}{2} \|\nabla \delta u\|_{L_2(\mathbb{R}^3)}^2 + \frac{C^2}{2} X^2 (\|\tau \nabla \dot{u}_2\|_{L_3(\mathbb{R}^3)} + \|\tau \dot{u}_2\|_{L_\infty(\mathbb{R}^3)})^2.$$
Hence, plugging (6.7) and the above inequality in (6.6) and using the notation of the statement, we get (for another constant $C$):

$$
\frac{d}{dt} \int_{\mathbb{R}^3} \rho_1 |\delta u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla \delta u|^2 \, dx \leq 2 \|\nabla u_2\|_{L_\infty(\mathbb{R}^d)} \|\sqrt{\rho_1} \delta u\|_{L_2(\mathbb{R}^3)}^2 + C f^2 X^2.
$$

Integrating on $[0, t]$, the above inequality becomes

$$
Y^2(t) \leq Y^2(0) + 2 \int_0^t g Y^2 \, d\tau + C \int_0^t f^2 X^2 \, d\tau.
$$

Hence, Gronwall lemma gives

$$
Y^2(t) \leq e^{2 \int_0^t g \, d\tau} \left( Y^2(0) + C \int_0^t e^{-2 \int_0^t g \, d\tau} f^2 X^2 \, d\tau \right).
$$

Plugging (6.5) in the above inequality, we discover that

$$
Y^2(t) \leq e^{CR_0(\int_0^t f^2 \, d\tau)} \exp(2 \int_0^t g \, d\tau) e^{2 \int_0^t g \, d\tau} Y^2(0).
$$

Inserting this latter inequality in (6.5) completes the proof. 

We claim that the above proposition implies the uniqueness part of Theorem 2.5. As a first, we have to explain why the map $t \mapsto t^{-1} \delta \rho$ belongs to $L_\infty(0, T; \dot{H}^{-1}(\mathbb{R}^3))$. In fact, for all $t \in [0, T]$, integrating the mass equation of (INS) on $[0, t]$ yields

$$
\rho_i(t) - \rho_0 = - \int_0^t \text{div} (\rho_i u_i) \, d\tau, \quad i = 1, 2.
$$

Hence,

$$
\|\rho_i(t) - \rho_0\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq \int_0^t \|\text{div} (\rho_i u_i)\|_{\dot{H}^{-1}(\mathbb{R}^3)} \, d\tau \\
\leq t \|\sqrt{\rho_i} u_i\|_{L_\infty(\mathbb{R}_+; L_2(\mathbb{R}^3))} \|\sqrt{\rho_i}\|_{L_\infty(\mathbb{R}_+ \times \mathbb{R}^3)},
$$

and thus, thanks to the energy balance and the mass equation,

$$
t^{-1} \|\delta \rho(t)\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq \sqrt{R_0} \|\sqrt{\rho_0}\|_{L_2(\mathbb{R}^3)}.
$$

Next, we have to show that $\delta u$ is in the energy space. If $2 < p < 3$, then this is guaranteed by the assumption of Theorem 2.5. If $1 < p \leq 2$, then we argue as follows. By construction, $\partial_t u$ is in $L_{q,1}([0, T]; L_p(\mathbb{R}^3))$ and $u$ is in $C_b([0, T]; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))$, whence

$$
u(t) - u_0 \in C([0, T]; L_p(\mathbb{R}^3)) \cap C([0, T]; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)).$$
Hence \( u(t) - u_0 \in C([0, T]; B_{p,1}^{-1+3/p}(\mathbb{R}^3)) \) (nonhomogeneous Besov space). Owing to the classical embedding

\[
B_{p,1}^{-1+3/p}(\mathbb{R}^3) \hookrightarrow H^{1/2}(\mathbb{R}^3) \quad \text{for } 1 < p \leq 2,
\]

we obtain

\[
u(t) - u_0 \in C([0, T]; L_2(\mathbb{R}^3)).
\]

Taking \((s, m) = (4, 2)\) in (A.3), we see that \(\nabla u_i\) belongs to \(L_{4,1}(0, T; L_2)\), hence to \(L_2(0, T \times \mathbb{R}^d)\). From this, we eventually conclude that \(\partial_t u\) is in \(L_\infty(0, T; L_2) \cap L_2(0, T; H^1)\).

The solution constructed in Theorem 2.5 satisfies \(t \nabla \dot{u} \in L_2(\mathbb{R}^d; L_3(\mathbb{R}^3))\) and \(\nabla u \in L_1(\mathbb{R}^d; L_\infty(\mathbb{R}^d))\). Hence, all the assumptions of Theorem 2.7 are satisfied by the solutions constructed in Theorem 2.5, which are thus unique.

Another corollary of Proposition 6.1 is the uniqueness of Zhang’s solutions constructed in [19]. Indeed, if \((\rho, u)\) stands for a solution of Theorem 2.9 then it satisfies (2.13). Therefore, thanks to the embeddings

\[
\dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L_3(\mathbb{R}^3), \quad \dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L_\infty(\mathbb{R}^3),
\]

one can write that for all \(T > 0\), we have

\[
t \nabla \dot{u} \in L_2(\mathbb{R}^d; L_3(\mathbb{R}^3)) \quad \text{and} \quad t \dot{u} \in L_2(\mathbb{R}^d; L_\infty(\mathbb{R}^3)).
\]

Hence, if we prove in addition that \(\nabla u \in L_1(0, T; L_\infty(\mathbb{R}^3))\) for all \(T > 0\), then Proposition 6.1 will ensure uniqueness.

In order to prove this latter property, let us observe as in [19] that if \((\rho, u, \nabla P)\) is a solution to (INS) on \([0, T] \times \mathbb{R}^3\), and if we look at the following linear Stokes system with convection:

\[
\begin{align*}
\rho \partial_t u_j + \rho u \cdot \nabla u_j - \Delta u_j + \nabla P_j &= 0, \\
\operatorname{div} u_j &= 0, \\
u_j|_{t=0} &= \tilde{\Delta} u_0,
\end{align*}
\]

then, by uniqueness, we have

\[
u = \sum_{j \in \mathbb{Z}} u_j \quad \text{and} \quad \nabla P = \sum_{j \in \mathbb{Z}} \nabla P_j.
\] (6.9)

In [19], under assumptions (2.11) and (2.12), the following time weighted estimates have been proved (see Corollaries 3.1, 3.2 and 4.2 and Inequalities (2.10), (3.8) and (3.23)):

\[
\|\sqrt{t} \nabla^2 u_j\|_{L_2(0,T \times \mathbb{R}^3)} + \|t \nabla \partial_t u_j\|_{L_2(0,T \times \mathbb{R}^3)} + \|t \nabla^2 u_j\|_{L_\infty(0,T;L_2)} \lesssim d_j^{1/2} 2^{\frac{s}{2} - \frac{d}{2}} \|u_0\|_{\dot{B}^{1/2}_{2,1}}, \quad (6.10)
\]

\[
\|\nabla^2 u_j\|_{L_2(0,T \times \mathbb{R}^3)} + \|\sqrt{t} \nabla^2 u_j\|_{L_\infty(0,T;L_2)} + \|t \nabla \partial_t u_j\|_{L_\infty(0,T;L_2)} \lesssim d_j^{1/2} 2^{\frac{s}{2} - \frac{d}{2}} \|u_0\|_{\dot{B}^{1/2}_{2,1}}, \quad (6.11)
\]

with \(\{d_j^1\}_{j \in \mathbb{Z}}\) and \(\{d_j^2\}_{j \in \mathbb{Z}}\) in the unit ball of \(\ell_1(\mathbb{Z})\).
Together with the following interpolation result (see Proposition A.1):
\[ L_{4,1}(0, T; L_2) = \left( L_2(0, T; L_2), L_\infty(0, T; L_2) \right)_{1/2,1} \]
this yields
\[ \| \sqrt{t} \nabla^2 u_j \|_{L_{4,1}(0,T;L_2)} \leq C d_j \| u_0 \|_{\dot{B}^{1/2}_{2,1}} \text{ with } \sum_{j \in \mathbb{Z}} d_j = 1. \] (6.12)

Next, since
\[ -\Delta u_j + \nabla P_j = -\rho \partial_t u_j - \rho u_j \cdot \nabla u_j, \]
we have in light of the standard elliptic regularity result for the Stokes system:
\[ \| \nabla^2 u_j, \nabla P_j \|_{L_2(0,T;L_6)} \leq C \left( \| \partial_t u_j \|_{L_2(0,T;L_6)} + \| u \cdot \nabla u_j \|_{L_2(0,T;L_6)} \right). \]

Hence, as \( \| u_0 \|_{\dot{B}^{1/2}_{2,1}} \) is small, using Hölder inequality, \( \dot{H}^1(\mathbb{R}^3) \hookrightarrow L_6(\mathbb{R}^3) \) and \( \dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L_\infty(\mathbb{R}^3) \) and, eventually, (2.13) and (6.10), we get
\[ \| \nabla^2 u_j, \nabla P_j \|_{L_2(0,T;L_6)} \lesssim \left( \| \partial_t u_j \|_{L_2(0,T;L_6)} + \| u \cdot \nabla u_j \|_{L_2(0,T;L_6)} \right) \lesssim d_j^{1/2} \| u_0 \|_{\dot{B}^{1/2}_{2,1}}. \]

Similarly, we deduce from elliptic regularity, embedding, (2.13) and (6.11) that
\[ \| \nabla^2 u_j, \nabla P_j \|_{L_\infty(0,T;L_6)} \lesssim \| \partial_t u_j \|_{L_\infty(0,T;L_6)} + \| u \cdot \nabla u_j \|_{L_\infty(0,T;L_6)} \lesssim \| \nabla^2 u_j \|_{L_\infty(0,T;L_2)} \| \sqrt{t} \nabla^2 u_j \|_{L_\infty(0,T;L_2)} \lesssim d_j^{1/2} \| u_0 \|_{\dot{B}^{1/2}_{2,1}}. \]

Together with the interpolation property
\[ L_{4,1}(0, T; L_6) = \left( L_2(0, T; L_6), L_\infty(0, T; L_6) \right)_{1/2,1}, \]
this yields
\[ \| t \nabla^2 u_j \|_{L_{4,1}(0,T;L_6)} \leq C d_j \| u_0 \|_{\dot{B}^{1/2}_{2,1}} \text{ with } \sum_{j \in \mathbb{Z}} d_j = 1. \] (6.13)

Summing up on all \( j \in \mathbb{Z} \) we deduce from (6.9), (6.12) and (6.13) that
\[ \| t \nabla^2 u \|_{L_{4,1}(0,T;L_6)} + \| \sqrt{t} \nabla^2 u \|_{L_{4,1}(0,T;L_2)} \lesssim \| u_0 \|_{\dot{B}^{1/2}_{2,1}}. \] (6.14)

It is now easy to bound \( \nabla u \) in \( L_1(0, T; L_\infty) \) for all \( T > 0 \). Recall the following Gagliardo-Nirenberg inequality:
\[ \| z \|_{L_\infty} \lesssim \| \nabla z \|_{L_2}^{1/2} \| \nabla^2 z \|_{L_6}^{1/2}. \]

Then, combining with Proposition A.1 (items (iii), (iv) and (v)) and (6.14) yields
\[ \int_0^T \| \nabla u \|_{L_\infty} dt \leq C \int_0^T \| \nabla^2 u \|_{L_2}^{1/2} \| \nabla^2 u \|_{L_6}^{1/2} dt \]
This completes the proof of the Lipschitz regularity for the velocity. Now, applying Proposition 6.1 yields uniqueness in Theorem 2.9.

6.2. Stability and uniqueness in the two-dimensional case. Let us first present a result that requires the density to be bounded away from zero (that is \( r_0 > 0 \) in (6.1)).

**Proposition 6.2.** Let \((\rho_1, u_1, p_1)\) and \((\rho_2, u_2, p_2)\) be two solutions of (INS) on \([0, T] \times \mathbb{R}^2\) corresponding to the same initial density \(\rho_0\) bounded away from zero and, possibly, two different initial velocities \(u_{1,0}\) and \(u_{2,0}\). Denote \(g(t) := \| \nabla u_2(t) \|_{L^\infty(\mathbb{R}^2)}\) and, for some \(1 < p, q < 2\) such that \(1/p + 1/q = 3/2\),

\[
f_1(t) := \|tu_2\|_{L^2(\mathbb{R}^2)}^2 + \|t\nabla^2 u_2\|_{L^p(\mathbb{R}^2)}^q.
\]

Put \(\delta \rho := \rho_1 - \rho_2, \delta u := u_1 - u_2, X(t) := \sup_{\tau \in [0, t]} \tau^{-1} \|\delta \rho(\tau)\|_{\dot{H}^{-1}(\mathbb{R}^2)}\) and

\[
Y(t) := \left( \sup_{\tau \in [0, t]} \left( \|\nabla (\sqrt{\rho_1} \delta u)(\tau)\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla \delta u\|_{L^2(0, t \times \mathbb{R}^d)}^2 \right) \right)^{1/2}.
\]

There exists a constant \(C\) depending only \(p\) such that for all \(t \in [0, T]\):

\[
\max \left( \frac{\exp \left( -\int_0^t g d\tau \right)}{R_0} X^2(t), Y^2(t) \right) \leq \|\sqrt{\rho_0} \delta u_0\|_{L^2(\mathbb{R}^2)}^2 \exp \left( \int_0^t \left( 2g + C \frac{f_1}{R_0} \right) d\tau \right) \exp \left( C R_0 e^{\int_0^t g d\tau} \int_0^t f_1 d\tau \right).
\]

**Proof.** Let us define the functions \(X\) and \(Y\) according to (6.3). Compared to the three-dimensional case, the only change is in the treatment of the first term of the right-hand side of (6.8). Now, using Hölder inequality, the embedding

\[
\dot{W}_p^1(\mathbb{R}^2) \hookrightarrow L^m(\mathbb{R}^2) \quad \text{with} \quad \frac{1}{m} = \frac{1}{p} - \frac{1}{2},
\]

and Gagliardo-Nirenberg inequality, we get (with \(1/s := 1 - 1/p\)):

\[
\|\nabla u_2 \cdot \delta u\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla u_2\|_{L^m(\mathbb{R}^2)}\|\delta u\|_{L^s(\mathbb{R}^2)}\]

\[
\leq C\|\nabla^2 u_2\|_{L^p(\mathbb{R}^2)}\|\delta u\|_{L^2(\mathbb{R}^2)}^2\|\nabla \delta u\|_{L^2(\mathbb{R}^2)}^2.
\]

Hence, reverting to (6.8) and using Young inequality, we obtain that

\[
- \int_{\mathbb{R}^2} \delta \rho \ u_2 \cdot \delta u \, dx \leq \frac{1}{2} \|\nabla \delta u\|_{L^2(\mathbb{R}^2)}^2 + C \left( X^2 \|tu_2\|_{L^\infty(\mathbb{R}^2)}^2 + X^q \|\nabla^2 u_2\|_{L^p(\mathbb{R}^2)}^q \|\delta u\|_{L^2(\mathbb{R}^2)}^{2-q} \right)
\]

\[
\leq C \left( \max \left( \frac{\exp \left( -\int_0^t g d\tau \right)}{R_0} X^2(t), Y^2(t) \right) \right) \exp \left( \int_0^t \left( 2g + C \frac{f_1}{R_0} \right) d\tau \right) \exp \left( C R_0 e^{\int_0^t g d\tau} \int_0^t f_1 d\tau \right).
\]
Then, substituting (6.7) and the above inequality into (6.6) yields
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \rho_1 |\partial u|^2 \, dx + \int_{\mathbb{R}^2} |\nabla \partial u|^2 \, dx \leq 2g \sqrt{\rho_1 |\partial u|_{L^2(\mathbb{R}^2)}} + \mathcal{C} f_1(X^2 + |\partial u|_{L^2(\mathbb{R}^2)}^2).
\]
As the density is bounded from below by \( r_0 > 0 \), after integrating on \([0, t]\), we get
\[
Y^2(t) \leq \int_0^t (2g(\tau) + \mathcal{C} r_0^{-1} f_1(\tau)) Y^2(\tau) \, d\tau + C \int_0^t X^2(\tau) f_1(\tau) \, d\tau + Y^2(0).
\]
Hence, applying Gronwall’s inequality yields
\[
Y^2(t) \leq e^{\int_0^t (2g + \mathcal{C} r_0^{-1} f_1) \, d\tau} (Y^2(0) + \int_0^t C X^2 f_1 e^{-\int_0^\tau (2g + \mathcal{C} r_0^{-1} f_1) \, d\tau'} \, d\tau),
\]
which together with (6.5) implies
\[
Y^2(t) \leq e^{\int_0^t (2g + \mathcal{C} r_0^{-1} f_1) \, d\tau} (Y^2(0) + \mathcal{C} R_0 \int_0^t Y^2 f_1 e^{-\int_0^\tau \mathcal{C} r_0^{-1} f_1 \, d\tau'} \, d\tau).
\]
Hence, we deduce from Gronwall’s inequality that
\[
Y^2(t) \leq Y^2(0) \exp \left( \int_0^t (2g + \mathcal{C} r_0^{-1} f_1) \, d\tau \right) \exp \left( \mathcal{C} R_0 e^{\int_0^t g \, d\tau} \int_0^t f_1 \, d\tau \right).
\]
Inserting this latter inequality in (6.5) completes the proof. \(\square\)

Proposition 6.2 implies the uniqueness part of Theorem 2.4. Indeed, the density of the solutions constructed therein is bounded away from zero, we have \( \nabla u \in L_1(\mathbb{R}_+; L_\infty(\mathbb{R}^d)) \), we have \( t \nabla^2 \hat{u} \in L_q(\mathbb{R}_+; L_p(\mathbb{R}^2)) \) with \( 1 < p, q < 2 \) such that \( 1/p + 1/q = 3/2 \) and \( t \hat{u} \in L_2(\mathbb{R}_+; L_\infty(\mathbb{R}^2)) \), the solutions have finite energy, the map \( t \mapsto t^{-1} \delta \rho \) is in \( L_\infty(0, T; H^{-1}(\mathbb{R}^3)) \) for all \( T > 0 \) (the proof is exactly the same as in the 3D case).

Having in mind the results in the three-dimensional case, it is natural to address the uniqueness issue without assuming that the density has a positive lower bound. The following result ensures uniqueness in the case of periodic boundary conditions, without making any particular assumption on the density.

**Proposition 6.3.** Let \((\rho_1, u_1, P_1)\) and \((\rho_2, u_2, P_2)\) be two solutions of (INS) on \([0, T] \times \mathbb{T}^2\) corresponding to the same data \((\rho_0, u_0)\) such that \( M := \int_{\mathbb{T}^2} \rho_0 \, dx \) is positive.

If, in addition, \( \nabla u_2 \in L_1(0, T; L_\infty(\mathbb{T}^2)) \) and
\[
[t \mapsto t \log (1 + t^{-1}) \hat{u}_2] \in L_2(0, T; L_\infty(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)),
\]
then \((\rho_1, u_1, P_1) = (\rho_2, u_2, P_2)\) on \([0, T] \times \mathbb{T}^2\).
\textbf{Proof.} Compared to the previous proposition, the only change is in the treatment of the first term of the right-hand side of (6.8). Thanks to Inequality (A.8) adapted to the periodic setting\(^1\), we have
\begin{equation}
- \int_{T^2} \delta \rho \ u_t \cdot \delta u \, dx \leq \| \delta u \|_{H^1(T^2)} \| \delta \rho \|_{H^{-1}(T^2)}
\end{equation}
\begin{equation}
\lesssim \| \delta u \|_{H^1(T^2)} \| \delta \rho \|_{H^{-1}(T^2)} \log^{\frac{1}{2}} \left(1 + \frac{\| \delta \rho \|_{L^2(T^2)}}{\| \delta \rho \|_{H^{-1}(T^2)}}\right) \| \dot{u}_2 \|_{(H^1 \cap L^\infty)(T^2)}. \tag{6.15}
\end{equation}

Note that one cannot bound directly \( \| \delta u \|_{H^1(T^2)} \) from \( \| \nabla \delta u \|_{L^2(T^2)} \) since \( \int_{T^2} \delta u \, dx \) need not be zero and, in the periodic setting,
\begin{equation}
\| \delta u \|_{H^1(T^2)}^2 = \left| \int_{T^2} \delta u \, dx \right|^2 + \| \nabla \delta u \|_{L^2(T^2)}^2.
\end{equation}

To bound the first term we write that by virtue of Cauchy-Schwarz and Poincaré inequalities,
\begin{equation}
\left| M \int_{T^2} \delta u \, dx \right| = \left| \int_{T^2} \rho_1 \delta u \, dx + \int_{T^2} (M - \rho_1) \left( \delta u - \int_{T^2} \delta u \, dx \right) \, dx \right|
\leq \sqrt{M} \| \sqrt{\rho_1} \delta u \|_{L^2(T^2)} + C \| M - \rho_1 \|_{L^2(T^2)} \| \nabla \delta u \|_{L^2(T^2)}.
\end{equation}

Therefore, there exists a constant \( C \) depending only on \( M \) and on \( R_0 \), and such that
\begin{equation}
\| \delta u \|_{H^1(T^2)} \leq C \left( \| \sqrt{\rho_1} \delta u \|_{L^2(T^2)} + \| \nabla \delta u \|_{L^2(T^2)} \right).
\end{equation}

Let us denote
\begin{equation}
g(t) := \| \nabla u_2(t) \|_{L^\infty(T^2)} \quad \text{and} \quad f_2(t) := \| t \dot{u}_2 \|_{L^\infty(T^2) \cap H^1(T^2)}.
\end{equation}

Plugging the above inequality in (6.15) and using (6.16), (6.3) and (6.1), and, finally, Young’s inequality for the second line yields
\begin{equation}
- \int_{T^2} \delta \rho \ u_t \cdot \delta u \lesssim \left( \| \sqrt{\rho_1} \delta u \|_{L^2(T^2)} + \| \nabla \delta u \|_{L^2(T^2)} \right) \chi \log^{\frac{1}{2}} \left(1 + \frac{R_0}{\| \delta \rho \|_{H^{-1}(T^2)}}\right) f_2
\end{equation}
\begin{equation}
\leq \frac{1}{2} \| \nabla \delta u \|_{L^2(T^2)}^2 + Y^2 + CX^2 f_2^2 \log \left(1 + \frac{R_0}{tX}\right).
\end{equation}

Still thanks to (6.1), we see that there exists a constant \( C \) such that
\begin{equation}
\sup_{t \in [0,T]} \| \delta \rho(t) \|_{L^2} \leq C R_0.
\end{equation}

Hence, reverting to (6.6) and using the notation of (6.3) yields
\begin{equation}
\frac{d}{dt} Y^2 \lesssim (1 + g) Y^2 + X^2 f_2^2 \log \left(1 + \frac{R_0}{tX}\right).
\end{equation}

\(^1\) The Littlewood-Paley decomposition that is required for proving (A.8) may be adapted to the periodic setting, see e.g. [24].
Remembering (6.5) and integrating the above inequality yields
\[ Y^2(t) \lesssim \int_0^t (1 + g)Y^2 \, d\tau + R_0 \int_0^t e^{2 \int_0^\tau g \, d\tau'} f_2^2 \, Y^2 \log \left( 1 + \frac{R_0^{1/2}}{Y e^{\int_0^\tau g \, d\tau'}} \right) \, d\tau. \]

Hence, taking advantage of the following basic inequality:
\[ \log (1 + aY^{-1}) \leq \log (1 + a)(1 - \log Y), \quad a \geq 0, \quad Y \in (0, 1), \]
we get
\[ Y^2(t) \lesssim \int_0^t (1 + g)Y^2 \, d\tau + CR_0 \int_0^t \log (1 + R_0^{1/2} \tau^{-1}) e^{2 \int_0^\tau g \, d\tau'} f_2^2 \, Y^2 \left( 1 - \frac{1}{2} \log Y^2 \right) \, d\tau. \]

Our assumptions ensure that both \( g \) and \( \tau \mapsto \log (1 + R_0^{1/2} \tau^{-1}) e^{2 \int_0^\tau g \, d\tau'} f_2^2(\tau) \) are integrable on \([0, T]\). Furthermore, the function \( r \mapsto r (1 - \frac{1}{2} \log r) \) is increasing near \( 0^+ \) and satisfies
\[ \int_0^1 \frac{dr}{r (1 - \frac{1}{2} \log r)} = \infty. \]

Hence one can apply Osgood lemma (see e.g. [21, Lemma 3.4]) so as to conclude that \( Y \equiv 0 \) on \([0, T]\), and thus, owing to (6.5), we have \( X \equiv 0 \), too. \( \square \)

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Appendix A

For the reader’s convenience, we here list some results involving Besov spaces and Lorentz spaces, prove maximal regularity estimates in Lorentz spaces for (2.4), and product estimates that were needed in the last section.

The following properties of Lorentz spaces may be found in [22] (and [25, Th2:1.18.6] for the first item):

Proposition A.1 (Properties of Lorentz spaces). There holds:
(1) Interpolation: For all \( 1 \leq r, q \leq \infty \) and \( \theta \in (0, 1) \), we have
\[
\left( L_{p_1}(\mathbb{R}^d); L_q(\mathbb{R}^d) ; L_{p_2}(\mathbb{R}^d); L_q(\mathbb{R}^d) \right)_{\theta,r} = L_{p,r}(\mathbb{R}^d; L_q(\mathbb{R}^d)),
\]
where \( 1 < p_1 < p < p_2 < \infty \) are such that \( \frac{1}{p} = \frac{(1-\theta)}{p_1} + \frac{\theta}{p_2} \).

(2) Embedding: \( L_{p,r_1} \hookrightarrow L_{p,r_2} \) if \( r_1 \leq r_2 \), and \( L_{p,p} = L_p \).

(3) Hölder inequality: for \( 1 < p, p_1, p_2 < \infty \) and \( 1 \leq r, r_1, r_2 \leq \infty \), we have
\[
\|fg\|_{L_p,r} \lesssim \|f\|_{L_{p_1,r_1}} \|g\|_{L_{p_2,r_2}} \ \text{if} \ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \ \text{and} \ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.
\]

This still holds for couples \((1,1)\) and \((\infty,\infty)\) with the convention \( L_{1,1} = L_1 \) and \( L_{\infty,\infty} = L_\infty \).

(4) For any \( \alpha > 0 \) and nonnegative measurable function \( f \), we have \( \|f^\alpha\|_{L_{p,r}} = \|f\|_{L_{p_\alpha, r_\alpha}} \).

(5) For any \( k > 0 \), we have \( \|x^{-k}1_{\mathbb{R}^d}\|_{L_{1/k, \infty}} = 1 \).

Next, let us state a few classical properties of Besov spaces.

**Proposition A.2** (Besov embedding). There holds:

(1) For any \((p, q)\) in \([1, \infty]^2\) such that \( p \leq q \), we have
\[
\dot{B}^{d/p-d/q}_{p,1}(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d).
\]

(2) Let \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq r_1 \leq r_2 \leq \infty \). Then, for any real number \( s \),
\[
\dot{B}^s_{p_1, r_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1} - \frac{1}{p_2})}_{p_2, r_2}(\mathbb{R}^d).
\]

The interpolation theory in Besov spaces played an important role in our paper. Below are listed some results that we used (see details in [21, Prop. 2.22] or in [25, chapter 2.4.2]).

**Proposition A.3** (Interpolation). A constant \( C \) exists that satisfies the following properties. If \( s_1 \) and \( s_2 \) are real numbers such that \( s_1 < s_2 \) and \( \theta \in ]0, 1[ \), then we have, for any \((p, r)\) in \([1, \infty]^2\) and any tempered distribution \( u \) satisfying (2.3),
\[
\|u\|_{\dot{B}^{\theta s_1 + (1-\theta)s_2}_{p,r}(\mathbb{R}^d)} \leq C \|u\|_{\dot{B}^{s_1}_{p,\infty}(\mathbb{R}^d)}^\theta \|u\|_{\dot{B}^{s_2}_{p,\infty}(\mathbb{R}^d)}^{1-\theta}.
\]
and, for some constant \( C \) depending only on \( \theta \) and \( s_2 - s_1 \),
\[
\|u\|_{\dot{B}^{\theta s_1 + (1-\theta)s_2}_{p,1}(\mathbb{R}^d)} \leq C \|u\|_{\dot{B}^{s_1}_{p,\infty}(\mathbb{R}^d)}^\theta \|u\|_{\dot{B}^{s_2}_{p,\infty}(\mathbb{R}^d)}^{1-\theta}.
\]

Furthermore, we have for all \( s \in (0, 1) \) and \((p, q)\) in \([1, \infty]^2\) :
\[
\dot{B}^s_{p,q}(\mathbb{R}^d) = \left( L_p(\mathbb{R}^d) ; W^1_p(\mathbb{R}^d) \right)_{s,q}.
\]

The following proposition has been used several times.
Proposition A.4. Let \( 1 \leq q < \infty, 1 \leq p < r \leq \infty \) and \( \theta \in (0, 1) \) such that
\[
\frac{1}{r} + \frac{1}{d} - \frac{2\theta}{dq} = \frac{1}{p}. \tag{A.1}
\]
Then, there exists \( C \) so that the following inequality holds true
\[
\| \nabla u \|_{L_r(\mathbb{R}^d)} \leq C \| \nabla^2 u \|_{L_p(\mathbb{R}^d)}^{\theta} \| u \|_{\dot{B}^{-2+\theta/q}_{p,\infty}(\mathbb{R}^d)}^{1-\theta}. \tag{A.2}
\]

Proof. Proposition A.3 tells us in particular that
\[
\| u \|_{\dot{B}^{-2+\theta/q}_{p,\infty}(\mathbb{R}^d)} \lesssim \| \nabla u \|_{L_p(\mathbb{R}^d)}^{\theta}. \tag{A.3}
\]
It is obvious that
\[
\| u \|_{\dot{B}^{-2+\theta/q}_{p,\infty}(\mathbb{R}^d)} \lesssim \| \nabla^2 u \|_{L_p(\mathbb{R}^d)}
\]
and, according to Proposition A.2 and to the definition of \( \theta \), we have
\[
\dot{B}^{-2+\theta/q}_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{-2+\theta/q}_{r,1}(\mathbb{R}^d) = \dot{B}^1_{r,1}(\mathbb{R}^d).
\]
As \( \dot{B}^1_{r,1}(\mathbb{R}^d) \hookrightarrow \dot{W}^1_{p,\theta,q}(\mathbb{R}^d) \), we get the desired inequality. \( \Box \)

The following result that is an easy adaptation of [20, Prop. 2.1] played a key role in Sects. 3 and 4. Note that estimates in the same spirit but for source terms valued in Besov spaces have been proved by Kozono and Shimizu in [26].

Proposition A.5. Let \( 1 < p, q < \infty \) and \( 1 \leq r \leq \infty \). Then, for any \( u_0 \in \dot{B}^{-2+\theta/q}_{p,r}(\mathbb{R}^d) \) with \( \text{div} \ u_0 = 0 \), and any \( f \in L_{q,r}(0, T; L_p(\mathbb{R}^d)) \), the Stokes system (2.4) has a unique solution \((u, \nabla P)\) with \( \nabla P \in L_{q,r}(0, T; L_p(\mathbb{R}^d)) \) and \( u \) in the space \( \dot{W}^2_{p,(q,r)}((0, T) \times \mathbb{R}^d) \) defined by
\[
\{ u \in C([0, T]; \dot{B}^{-2+\theta/q}_{p,r}(\mathbb{R}^d)) : u_t, \nabla^2 u \in L_{q,r}(0, T; L_p(\mathbb{R}^d)) \}.
\]
Furthermore, there exists a constant \( C \) independent of \( T \) such that
\[
\mu^{1-1/q} \| u \|_{L_\infty(0, T; \dot{B}^{-2+\theta/q}_{p,r}(\mathbb{R}^d))} + \| u_t, \mu \nabla^2 u, \nabla P \|_{L_{q,r}(0, T; L_p(\mathbb{R}^d))} \leq C \left( \mu^{1-1/q} \| u_0 \|_{\dot{B}^{-2+\theta/q}_{p,r}(\mathbb{R}^d)} + \| f \|_{L_{q,r}(0, T; L_p(\mathbb{R}^d))} \right). \tag{A.2}
\]

Let \( \tilde{s} > q \) be such that
\[
\frac{1}{q} - \frac{1}{\tilde{s}} \leq \frac{1}{2} \quad \text{and} \quad \frac{d}{2p} + \frac{1}{q} - \frac{1}{\tilde{s}} > \frac{1}{2},
\]
and define \( \tilde{m} \geq p \) by the relation
\[
\frac{d}{2\tilde{m}} + \frac{1}{\tilde{s}} = \frac{d}{2p} + \frac{1}{q} - \frac{1}{2}.
\]

\(^2\) Only weak continuity holds if \( r = \infty \).
Then, the following inequality holds true:

\[
\mu^{1+\frac{1}{\gamma}} \frac{1}{q} \| \nabla u \|_{L_{\gamma,r}(0,T; L_\gamma(\mathbb{R}^d))} \leq C (\mu^{1-1/q} \| u \|_{L_\infty(0,T; \tilde{B}^{2-2/q}_{p,r}(\mathbb{R}^d))} + \| u \|_{L_q(0,T; L_p(\mathbb{R}^d))}) \].

(A.3)

Finally, if \(2/q + d/p > 2\), then for all \(s \in (q, \infty)\) and \(m \in (p, \infty)\) such that

\[
\frac{d}{2m} + \frac{1}{s} = \frac{d}{2p} + \frac{1}{q} - 1,
\]

it holds that

\[
\mu^{1+\frac{1}{\gamma}} \frac{1}{q} \| u \|_{L_{\gamma,r}(0,T; L_\gamma(\mathbb{R}^d))} \leq C (\mu^{1-1/q} \| u \|_{L_\infty(0,T; \tilde{B}^{2-2/q}_{p,r}(\mathbb{R}^d))} + \| u \|_{L_q(0,T; L_p(\mathbb{R}^d))}) \].

(A.4)

Proof. Let \(\mathbb{P}\) and \(\mathbb{Q}\) be the Helmholtz projectors defined in (3.27). As \(u = \mathbb{P} u\), we have

\[
u_t - \mu \Delta u = \mathbb{P} f, \quad u|_{t=0} = u_0.
\]

Applying [20, Prop 2.1] and using that \(\mathbb{P}\) is continuous on \(L_{q,r}(0, T; \tilde{L}_p(\mathbb{R}^d))\) gives (A.2) and (A.4) for \(u\). Since \(\nabla P = \mathbb{Q} f\), and \(\mathbb{Q}\) is also continuous on \(L_{q,r}(0, T; L_p(\mathbb{R}^d))\), \(\nabla P\) satisfies (A.2) too.

In order to prove (A.3), take \(q_0\) and \(q_1\) such that \(1 < q_0 < q < q_1 < \infty\) and \(2/q = 1/q_0 + 1/q_1\). From the mixed derivative theorem we have for all \(\gamma \in (0, 1)\) and \(i = 0, 1\),

\[
\hat{W}^{2,1}_{p,q_i}((0, T) \times \mathbb{R}^d) := \hat{W}^{2,1}_{p,(q_i,q_i)}((0, T) \times \mathbb{R}^d) \hookrightarrow \hat{W}^\gamma_{q_i}(0, T; \hat{W}^{2-2\gamma}_p(\mathbb{R}^d)).
\]

Let \(\gamma := 1/q - 1/\tilde{s}\) (so that \(d/\tilde{m} = d/p + 2\gamma - 1\)). As \(\gamma \in (0, \frac{1}{2}]\) and \(1 - 2\gamma < d/p\), one can use the Sobolev embedding

\[
\hat{W}^\gamma_{q_i}(0, T; \hat{W}^{2-2\gamma}_p(\mathbb{R}^d)) \hookrightarrow \hat{L}^\gamma_{\tilde{s}_i}(0, T; \hat{W}^{1}_m(\mathbb{R}^d)) \text{ with } \frac{1}{\tilde{s}_i} = \frac{1}{q_i} - \gamma.
\]

(A.5)

In the proof of [20, Prop. 2.1], it is pointed out that

\[
\hat{W}^{2,1}_{p,(q,r)}((0, T) \times \mathbb{R}^d) = \left(\hat{W}^{2,1}_{p,q_0}((0, T) \times \mathbb{R}^d); \hat{W}^{2,1}_{p,q_1}((0, T) \times \mathbb{R}^d)\right)_{\frac{1}{2},r}.
\]

Consequently, the embeddings (A.5) with \(i = 0\) and \(i = 1\) imply that

\[
\hat{W}^{2,1}_{p,(q,r)}((0, T) \times \mathbb{R}^d) \hookrightarrow \left(L_{\tilde{s}_0}(0, T; \hat{W}^{1}_m(\mathbb{R}^d)); L_{\tilde{s}_1}(0, T; \hat{W}^{1}_m(\mathbb{R}^d))\right)_{\frac{1}{2},r}.
\]

(A.6)

Note that our definition of \(\gamma, \tilde{s}_0, \tilde{s}_1, q_0\) and \(q_1\) ensures that

\[
\frac{1}{2} \left( \frac{1}{\tilde{s}_0} + \frac{1}{\tilde{s}_1} \right) = \frac{1}{2} \left( \frac{1}{q_0} + \frac{1}{q_1} \right) - \gamma = \frac{1}{\tilde{s}}.
\]

Hence the real interpolation space in the right of (A.6) is nothing but \(L_{q,r}(0, T; \hat{W}^{1}_m(\mathbb{R}^d))\), which completes the proof. \(\square\)
The usual product is continuous in many Besov spaces (see e.g. [9,10,27]). We here present a result that played a key role in the proof of uniqueness in dimension two. In order to prove it, we need to introduce the following so-called Bony decomposition (see [28]):

\[ uv = T_u v + T_v u + R(u, v) \]

with

\[ T_u v \triangleq \sum_{j \geq 1} S_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) \triangleq \sum_{j \geq -1} \sum_{|k-j| \leq 1} \Delta_j u \Delta_k v. \]

Above, we used the notation \( \Delta_j := S_j - S_{j-1} \delta_1 \) for \( j \geq 0 \), \( S_{-1} := 0 \), \( \Delta_j = 0 \) if \( j \leq -2 \) and \( S_j := \sum_{j' \leq j-1} \Delta_j \).

Operators \( T \) and \( R \) are called paraproduct and remainder, respectively. Their general properties of continuity may be found in [21,25,29]. The last inequality is new to the best of our knowledge.

**Proposition A.6.** Let \( 2 \leq p \leq \infty \) and \( 1 \leq r_1, r_2 \leq \infty \) satisfy \( \frac{1}{r_1} + \frac{1}{r_2} = 1 \). Then, the following inequality holds true:

\[ \| R(u, v) \|_{B^{-d}_{p,\infty}(\mathbb{R}^d)} \lesssim \| u \|_{B^d_{p,r_1}(\mathbb{R}^d)} \| v \|_{B^d_{p,r_2}(\mathbb{R}^d)} . \]  

(A.7)

In \( \mathbb{R}^2 \), it holds that

\[ \| uv \|_{H^{-1}(\mathbb{R}^2)} \lesssim \log \left( \frac{1}{2} + \frac{\| v \|_{L^2(\mathbb{R}^2)}}{\| v \|_{H^{-1}(\mathbb{R}^2)}} \right) \| u \|_{H^1(\mathbb{R}^2)} \| v \|_{H^1(\mathbb{R}^2)} \].  

(A.8)

**Proof.** To prove the first statement, we use that, by definition of the homogeneous remainder operator

\[ R(u, v) = \sum_{j \geq -1} \Delta_j u \Delta_j v \quad \text{with} \quad \Delta_j \triangleq \Delta_{j-1} + \Delta_j + \Delta_{j+1}. \]

Hence, owing to the support properties of the dyadic partition of unity, there exists an integer \( N_0 \) such that

\[ \Delta_k R(u, v) = \sum_{j \geq k-N_0} \Delta_k (\Delta_j u \Delta_j v) = \sum_{v \leq N_0} \Delta_k (\Delta_{k-v} u \Delta_{k-v} v). \]  

(A.9)

As \( 2 \leq p \leq \infty \), thanks to Bernstein’s inequality, we have

\[ \| \Delta_k R(u, v) \|_{L^p(\mathbb{R}^d)} \leq 2^{k \frac{d}{p}} \| \Delta_k R(u, v) \|_{L^{p/2}(\mathbb{R}^d)}. \]

Therefore, using convolution inequalities and (A.9), we discover that

\[ 2^{-k \frac{d}{p}} \| \Delta_k R(u, v) \|_{L^p(\mathbb{R}^d)} \lesssim \sum_{v \leq N_0} \| \Delta_{k-v} u \Delta_{k-v} v \|_{L^{p/2}(\mathbb{R}^d)} \]

\[ \lesssim \sum_{v \leq N_0} 2^{(k-v) \frac{d}{p}} \| \Delta_{k-v} u \|_{L^p(\mathbb{R}^d)} 2^{-(k-v) \frac{d}{p}} \| \Delta_{k-v} v \|_{L^p(\mathbb{R}^d)} \]
which gives (A.7).

In order to prove (A.8), we start from the following properties of continuity of the
paraproduct operator (see e.g. [21, Chapter 2]):

\[ \| Tu v \|_{H^{-1}(\mathbb{R}^2)} \lesssim \| u \|_{L^\infty(\mathbb{R}^2)} \| v \|_{H^{-1}(\mathbb{R}^2)}, \quad (A.10) \]

\[ \| Tv u \|_{H^{-1}(\mathbb{R}^2)} \lesssim \| v \|_{H^{-1}(\mathbb{R}^2)} \| u \|_{H^1(\mathbb{R}^2)}, \quad (A.11) \]

Next, we decompose \( R(u, v) \) into low and high frequencies, using (A.7), to get for all \( n \in \mathbb{N} \),

\[ \| R(u, v) \|_{H^{-1}(\mathbb{R}^2)}^2 = \sum_{j \geq -1} 2^{-2j} \| \Delta_j R(u, v) \|_{L^2(\mathbb{R}^2)}^2 \]

\[ = \sum_{-1 \leq j \leq N} 2^{-2j} \| \Delta_j R(u, v) \|_{L^2(\mathbb{R}^2)}^2 + \sum_{j > N} 2^{-2j} \| \Delta_j R(u, v) \|_{L^2(\mathbb{R}^2)}^2 \]

\[ \lesssim N \| R(u, v) \|_{B^{-1}_{2,\infty}(\mathbb{R}^2)}^2 + 2^{-2N} \| R(u, v) \|_{B^{2}_{2,\infty}(\mathbb{R}^2)}^2 \]

\[ \lesssim N \| u \|_{H^1(\mathbb{R}^2)}^2 \| v \|_{H^{-1}(\mathbb{R}^2)}^2 + 2^{-2N} \| u \|_{H^1(\mathbb{R}^2)}^2 \| v \|_{B^{-1}_{\infty,2}(\mathbb{R}^2)}^2. \]

Then, choosing \( N \) to be the closest integer larger than \( \log_2 \left( 1 + \frac{\| u \|_{B^{-1}_{\infty,2}(\mathbb{R}^2)}}{\| v \|_{H^{-1}(\mathbb{R}^2)}} \right) \) leads to

\[ \| R(u, v) \|_{H^{-1}(\mathbb{R}^2)} \lesssim \log_2 \left( 1 + \frac{\| v \|_{B^{-1}_{\infty,2}(\mathbb{R}^2)}}{\| u \|_{H^1(\mathbb{R}^2)}} \right) \| u \|_{H^1(\mathbb{R}^2)} \| v \|_{H^{-1}(\mathbb{R}^2)}. \]

Combining with the embedding \( B^{0}_{2,2}(\mathbb{R}^2) \hookrightarrow B^{-1}_{\infty,2}(\mathbb{R}^2) \), and Inequalities (A.10), (A.11),
this completes the proof of (A.8). \( \square \)

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