Monetary Risk Measures∗

Guangyan Jia† Jianming Xia‡ Rongjie Zhao§

Abstract

In this paper, we study general monetary risk measures (without any convexity or weak convexity). A monetary (respectively, positively homogeneous) risk measure can be characterized as the lower envelope of a family of convex (respectively, coherent) risk measures. The proof does not depend on but easily leads to the classical representation theorems for convex and coherent risk measures. When the law-invariance and the SSD (second-order stochastic dominance)-consistency are involved, it is not the convexity (respectively, coherence) but the comonotonic convexity (respectively, comonotonic coherence) of risk measures that can be used for such kind of lower envelope characterizations in a unified form. The representation of a law-invariant risk measure in terms of VaR is provided.

Key words: Monetary risk measure, (comonotonic) convex risk measure, (comonotonic) coherent risk measure, law-invariance

1 Introduction

It is an important subject to measure the risk of a financial position. VaR (value at risk) has long been a standard risk measure in industry, whether by choice or by regulation. However, VaR has been criticized in both academia and industry, mainly for two shortcomings: (1) VaR focuses on the probability of loss, regardless of the magnitude, and therefore, fails to capture “tail risk”; (2) VaR is not sub-additive, and therefore, violates the principle of diversification.

Recognizing the shortcomings of VaR, in their seminal work, Artzner, Delbaen, Eber and Heath (1999) argued that a good risk measure should satisfy a set of reasonable axioms (monotonicity, translation-invariance, sub-additivity and positive homogeneity), leading to the so-called coherent risk measures. Föllmer and Schied (2002), as well as

∗Supported by National Key R&D Program of China (NO. 2018YFA0703900) and the Major Project of National Social Science Foundation of China (NO.19ZDA091): Research on Dynamic Monitoring of Local Financial Operation and Early Warning of Systemic Risk.
†Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan 250100, China; Email: jigy@sdu.edu.cn.
‡RCSDS, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Email: xia@amss.ac.cn.
§School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China; Email: zrj2010@hotmail.com.

1For representations of coherent risk measures on general probability spaces, see Delbaen (2002).
Frittelli and Rosazza Gianin (2002), argued that, in many situations, the risk of a position might increase in a nonlinear way with the size of the position and suggested to relax the sub-additivity and positive homogeneity to the convexity, leading to the so-called convex risk measures. In addition, Song and Yan (2006, 2009) and Kou, Peng and Heyde (2013) further argued that the sub-additivity and the convexity for all risks might be too restrictive and suggested relaxations to the comonotonic sub-additivity or the comonotonic convexity, leading to the so-called comonotonic sub-additive or comonotonic convex risk measures. For discussions on law-invariance of risk measures, see among others Kusuoka (2001), Frittelli and Rosazza Gianin (2005), Jouini, Schachermayer and Touzi (2006), and Song and Yan (2009).

All kinds of risk measures above satisfy two basic axioms: monotonicity and translation-invariance. A risk measure satisfying the two basic axioms is usually called a monetary risk measure; see, e.g., Föllmer and Schied (2016). In this paper, our focus is on the characterizations and representations of a general monetary risk measure (without any convexity or weak convexity).

A closely related work to ours is Mao and Wang (2020), which is fundamental and inspiring. They argued that a risk measure should be consistent with risk aversion, which can be described by the consistency with respect to SSD (second-order stochastic dominance), leading to the so-called SSD-consistent risk measures. They provided characterizations and representations, as well as some applications, of SSD-consistent risk measures. Although no convexity is imposed a priori on an SSD-consistent risk measure, we will see that an SSD-consistent risk measure still satisfies a kind of weak convexity, which is called ID-convexity (identical-distribution convexity) in this paper; see Theorem 5.5 below.

The most famous non convex risk measure is VaR, which is excluded by Mao and Wang (2020) since it is not SSD-consistent. Every risk measure has its advantages as well as disadvantages. No one can do everything better than another. We do not argue for one risk measure against another, but investigate general monetary risk measures, which include VaR as a special case.

The contribution and the structure of our paper are as follows (the results for positively homogeneous risk measures are similar and hence not listed here for the brevity).

Section 2 collects some preliminaries about risk measures and acceptance sets.

In Section 3, we show that a monetary risk measure $\rho$ is the lower envelope of a family of convex risk measures and has the following representation

$$\rho(X) = \min_{\lambda \in \Lambda} \sup_{Q \in \mathcal{M}_1(P)} (E_Q[-X] - \alpha_\lambda(Q)),$$

where $\mathcal{M}_1(P)$ denotes all probability measures which are absolutely continuous with respect to $P$ and $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ is a family of convex functionals $\alpha_\lambda : \mathcal{M}_1(P) \to (-\infty, \infty]$. As a consequence, we have, for every monetary (similarly, positively homogeneous) risk

---

2Some other relaxations, from the translation-invariance and convexity to the quasi-convexity (and cash-subadditivity), were carried out by El Karoui and Ravanelli (2009) and Cerreia-Vioglio et al. (2011).
measure $\rho$,

$$\rho(X) = \inf\{h(X) \mid h \text{ convex (coherent) and } h \geq \rho\},$$

which extends the following equality in Artzner, Delbaen, Eber and Heath (1999, Proposition 5.2):

$$\text{VaR}_t(X) = \inf\{h(X) \mid h \text{ coherent and } h \geq \text{VaR}_t\}. \quad (1.1)$$

It is interesting that the proof of the representation theorem for general monetary risk measures does not depend on but easily leads to the classical representation theorem for convex and coherent risk measures, as Corollary 3.3 and Remark 3.5 show.

In Section 4 we show that a law-invariant monetary risk measure has the following representation

$$\rho(X) = \min_{\lambda \in \Lambda} \sup_{g \in \mathcal{G}} \left( \int_0^1 \text{VaR}_t(X)g(t)dt - \alpha_\lambda(g) \right),$$

where $\mathcal{G}$ denotes all probability density functions on $(0, 1)$ and $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ is a family of convex functionals $\alpha_\lambda : \mathcal{G} \to (\mathbb{R}, \infty]$.

In Section 5 we investigate SSD-consistent risk measures. Some results of Mao and Wang (2020) are recovered and some other representations of SSD-consistent risk measures are provided. We provide another equivalent formulation of SSD-consistency: SSD-consistency is equivalent to the combination of law-invariance, the Fatou property and ID-convexity. As far as we know, the ID-convexity has not yet been introduced in the literature to study risk measures.

From Sections 3–5 we know each monetary risk measure is the lower envelope of a family of convex risk measures. When the law-invariance is involved, it is a “natural” expectation that each law-invariant monetary risk measure is the lower envelope of a family of law-invariant convex risk measures. This “natural” expectation, however, is not correct since law-invariant convex risk measures are SSD-consistent and hence so is the lower envelope of some of them. For example,

$$\text{VaR}_t(X) \neq \inf\{h(X) \mid h \text{ law-invariant and coherent and } h \geq \text{VaR}_t\}. \quad (1.2)$$

A natural question arises:

*What is the “good” property such that general (respectively, law-invariant, SSD-consistent) monetary risk measures can be represented in a unified form: they are the lower envelopes of a family of general (respectively, law-invariant, SSD-consistent) monetary risk measures having the “good” property?*

Section 6 replies to this question. It turns out that the comonotonic convexity is the desired “good” property.
2 Preliminaries: Risk Measures and Acceptance Sets

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We use $L^\infty(P)$ to denote $L^\infty(\Omega, \mathcal{F}, P)$ for brevity. We study risk measures on $L^\infty(P)$. Some basic definitions and facts are recalled in this section.

**Definition 2.1.** A mapping $\rho : L^\infty(P) \to \mathbb{R}$ is called a monetary risk measure if it satisfies the following two conditions for all $X, Y \in L^\infty(P)$.

- **Monotonicity:** If $X \leq Y$ $P$-a.s., then $\rho(X) \geq \rho(Y)$.
- **Translation-Invariance:** If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

**Definition 2.2.** A monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ is called a positively homogeneous risk measure if it satisfies

- **Positive Homogeneity:** $\rho(\alpha X) = \alpha \rho(X)$ for $X \in L^\infty(P)$ and $\alpha \geq 0$.

**Definition 2.3.** A monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ is called a convex risk measure if it satisfies

- **Convexity:** $\rho(\alpha X + (1 - \alpha)Y) \leq \alpha \rho(X) + (1 - \alpha)\rho(Y)$ for $X, Y \in L^\infty(P)$ and $\alpha \in [0, 1]$.

**Definition 2.4.** A monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ is called a coherent risk measure if it is convex and positively homogeneous.

Given a monetary risk measure $\rho$, its acceptance set $\mathcal{A}_\rho$ is given by

$$\mathcal{A}_\rho \triangleq \{X \in L^\infty(P) \mid \rho(X) \leq 0\}.$$

Given a subset $\mathcal{A} \subseteq L^\infty(P)$, let the mapping $\rho_{\mathcal{A}}$ be given by

$$\rho_{\mathcal{A}}(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}, \quad X \in L^\infty(P).$$

The following proposition summarizes the relation between monetary risk measures and their acceptance sets; see, e.g., Föllmer and Schied (2016, Propositions 4.6–4.7).

**Proposition 2.5.** A mapping $\rho : L^\infty(P) \to \mathbb{R}$ is a monetary risk measure if and only if $\rho = \rho_{\mathcal{A}}$ for a nonempty subset $\mathcal{A} \subseteq L^\infty(P)$ satisfying the following two conditions.

(i) $\inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} > -\infty$;

(ii) $X \in \mathcal{A}$, $Y \in L^\infty(P)$, $Y \geq X$ $P$-a.s. $\Rightarrow Y \in \mathcal{A}$.

If it is the case, the set $\mathcal{A}$ can be chosen as the acceptance set $\mathcal{A}_\rho$ of $\rho$. Moreover, we have

- $\rho$ is a positively homogeneous risk measure if and only if $\mathcal{A}_\rho$ is a cone;
• $\rho$ is a convex risk measure if and only if $\mathcal{A}_\rho$ is a convex set;

• $\rho$ is a coherent risk measure if and only if $\mathcal{A}_\rho$ is a convex cone.

The following easy lemma will be frequently used, whose proof is omitted.

**Lemma 2.6.** Assume $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ is a class of subsets of $L^\infty(P)$ and each $\mathcal{A}_\lambda$ satisfies conditions (i)–(ii) of Proposition 2.5. Let $\mathcal{A} = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$. Then $\rho_{\mathcal{A}}(X) = \inf_{\lambda \in \Lambda} \rho_{\mathcal{A}_\lambda}(X)$ for $X \in L^\infty(P)$.

### 3 Representations of Monetary Risk Measures

Let

$$\mathcal{M}_1(P) = \mathcal{M}_1(\Omega, \mathcal{F}, P)$$

denote the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $P$. Let

$$\mathcal{M}_{1,f}(P) = \mathcal{M}_{1,f}(\Omega, \mathcal{F}, P)$$

denote the set of all finitely additive probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $P$. It is well known that $\mathcal{M}_{1,f}(P)$ is compact under the weak*-topology, which is induced by $L^\infty(P)$.

The following theorem shows that a monetary risk measure is the lower envelope of a family of convex risk measures.

**Theorem 3.1.** For a mapping $\rho : L^\infty(P) \to \mathbb{R}$, the following assertions are equivalent.

(a) $\rho$ is a monetary risk measure.

(b) There exists a family $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ of convex functionals $\alpha_\lambda : \mathcal{M}_1(P) \to (-\infty, \infty]$ such that

$$\rho(X) = \min_{\lambda \in \Lambda} \sup_{Q \in \mathcal{M}_1(P)} (E_Q[-X] - \alpha_\lambda(Q)) \quad \text{for } X \in L^\infty(P).$$

(c) There exists a family $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ of lower semi-continuous (under the weak* topology), convex functionals $\alpha_\lambda : \mathcal{M}_{1,f}(P) \to (-\infty, \infty]$ such that

$$\rho(X) = \min_{\lambda \in \Lambda} \max_{Q \in \mathcal{M}_{1,f}(P)} (E_Q[-X] - \alpha_\lambda(Q)) \quad \text{for } X \in L^\infty(P).$$

(d) There exists a family $\{\rho_\lambda \mid \lambda \in \Lambda\}$ of convex risk measures on $L^\infty(P)$ such that

$$\rho(X) = \min_{\lambda \in \Lambda} \rho_\lambda(X) \quad \text{for } X \in L^\infty(P).$$

(e) For each $X \in L^\infty(P)$,

$$\rho(X) = \inf\{h(X) \mid h \text{ is a convex risk measure and } h \geq \rho\}. \quad (3.1)$$
Proof. We only prove “(a)⇒(b)” and “(a)⇒(c)”, since “(b)⇒(d)⇒(e)⇒(a)” and “(c)⇒(d)” are obvious.

(a)⇒(b): Assume $\rho$ is a monetary risk measure. For any $Z \in \mathcal{A}$, let

$$\mathcal{A}(Z) = \{Y \in L^\infty(P) \mid Y \geq Z \text{ P-a.s.}\}.$$ 

Firstly, each $\mathcal{A}(Z)$ is obviously a convex subset of $L^\infty(P)$ satisfying conditions (i)–(ii) of Proposition 2.5. Then by Proposition 2.5, each $\rho_{\mathcal{A}(Z)}$ is a convex risk measure. Next, $\mathcal{A} = \bigcup_{Z \in \mathcal{A}} \mathcal{A}(Z)$. Then by Lemma 2.6

$$\rho(X) = \rho_{\mathcal{A}}(X) = \inf_{Z \in \mathcal{A}} \rho_{\mathcal{A}}(Z)(X), \quad \forall X \in L^\infty(P). \quad (3.2)$$

Moreover, for each $X \in L^\infty(P)$, we have $Z_0 = X + \rho(X) \in \mathcal{A}$ and

$$\rho_{\mathcal{A}}(Z_0)(X) = \inf\{m \in \mathbb{R} \mid X + m \geq X + \rho(X)\} = \rho(X),$$

which implies that the infimum in (3.2) can be attained and hence

$$\rho(X) = \min_{Z \in \mathcal{A}} \rho_{\mathcal{A}}(Z)(X).$$

Finally, for each $X \in L^\infty(P)$ and $Z \in \mathcal{A}$, we have

$$\rho_{\mathcal{A}}(Z)(X) = \inf\{m \in \mathbb{R} \mid X + m \geq Z \text{ P-a.s.}\}$$

$$= \operatorname{ess sup} (Z - X)$$

$$= \sup_{Q \in \mathcal{M}_1(P)} E_Q[Z - X]$$

$$= \sup_{Q \in \mathcal{M}(P)} (E_Q[-X] - \alpha_Z(Q)),$$

where $\alpha_Z(Q) = E_Q[-Z]$ for each $Q \in \mathcal{M}(P)$. Therefore, $\{\alpha_Z \mid Z \in \mathcal{A}\}$ is a desired family of convex functionals on $\mathcal{M}(P)$.

(a)⇒(c): According the proof of “(a)⇒(b)”, we have

$$\rho_{\mathcal{A}}(Z)(X) = \operatorname{ess sup} (Z - X)$$

$$= \max_{Q \in \mathcal{M}_{1,f}(P)} E_Q[Z - X]$$

$$= \max_{Q \in \mathcal{M}_{1,f}(P)} (E_Q[-X] - \alpha_Z(Q)),$$

where $\alpha_Z(Q) = E_Q[-Z]$ for each $Q \in \mathcal{M}_{1,f}(P)$. Therefore, $\{\alpha_Z \mid Z \in \mathcal{A}\}$ is a desired family of convex functionals on $\mathcal{M}_{1,f}(P)$. \□

Remark 3.2. (a) From the proof of Theorem 3.1, where a similar argument of Mao and Wang (2020) is used, we can see that a monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ has the following representations

$$\rho(X) = \min_{Z \in \mathcal{A}} \sup_{Q \in \mathcal{M}_1(P)} (E_Q[\max\{X,Z\}] - E_Q[\max\{X,-Z\}]$$

$$= \min_{Z \in \mathcal{A}} \max_{Q \in \mathcal{M}_{1,f}(P)} (E_Q[\max\{X,Z\}] - E_Q[\max\{X,-Z\}]), \quad \forall X \in L^\infty(P).$$

6
The characterizations \(3.1\) and \(3.4\) below are extensions of equality \(1.1\), which is from Artzner, Delbaen, Eber and Heath (1999).

For some related applications in decision theory under uncertainty, see Xia (2020).

As the next corollary shows, the proof of Theorem 3.1 does not depend on but leads to the classical representation theorem for convex risk measures of Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002).

**Corollary 3.3.** Assume \(\rho : L^\infty(P) \to \mathbb{R}\) is a convex risk measure. Then there exists a lower-semicontinuous convex functional \(\alpha : \mathcal{M}_{1,f}(P) \to (-\infty, \infty]\) such that

\[
\rho(X) = \max_{Q \in \mathcal{M}_{1,f}(P)} (E_Q[-X] - \alpha(Q)) \text{ for } X \in L^\infty(P).
\]

**Proof.** The convexity of \(\rho\) implies that its acceptance set \(\mathcal{A}_\rho\) is convex. Moreover, \(\mathcal{M}_{1,f}(P)\) is a weak*-compact and convex set. From the representations in Remark 3.2 and by the minmax theorem\(^3\), we have for each \(X \in L^\infty(P)\) that

\[
\rho(X) = \min_{Z \in \mathcal{A}_\rho} \max_{Q \in \mathcal{M}_{1,f}(P)} (E_Q[-X] - E_Q[-Z])
= \max_{Q \in \mathcal{M}_{1,f}(P)} \inf_{Z \in \mathcal{A}_\rho} (E_Q[-X] - E_Q[-Z])
= \max_{Q \in \mathcal{M}_{1,f}(P)} (E_Q[-X] - \alpha(Q)),
\]

where \(\alpha(Q) = \sup_{Z \in \mathcal{A}_\rho} E_Q[-Z]\) defines the desired lower-semicontinuous convex functional on \(\mathcal{M}_{1,f}(P)\).

The following theorem shows that a positively homogeneous risk measure is the lower envelope of a family of coherent risk measures.

**Theorem 3.4.** For a mapping \(\rho : L^\infty(P) \to \mathbb{R}\), the following assertions are equivalent.

(a) \(\rho\) is a positively homogeneous risk measure.

(b) There exists a family \(\{\mathcal{D}_\lambda : \lambda \in \Lambda\}\) of nonempty, weak*-compact, convex subsets of \(\mathcal{M}_{1,f}(P)\) such that

\[
\rho(X) = \min_{\lambda \in \Lambda} \max_{Q \in \mathcal{D}_\lambda} E_Q[-X] \text{ for } X \in L^\infty(P).
\]

(c) There exists a family \(\{\rho_\lambda : \lambda \in \Lambda\}\) of coherent risk measures on \(L^\infty(P)\) such that

\[
\rho(X) = \min_{\lambda \in \Lambda} \rho_\lambda(X) \text{ for } X \in L^\infty(P).
\]

\(^3\)The minmax theorem is frequently used in this paper, for which see, e.g., Mertens, Sorin and Zamir (2015, Theorem I.1.1).
For each \( X \in L^\infty(P) \),
\[
\rho(X) = \inf\{h(X) \mid h \text{ is a coherent risk measure and } h \geq \rho\}. \tag{3.4}
\]

Proof. We only prove “(a) ⇒ (b)”, since “(b) ⇒ (c) ⇒ (d) ⇒ (a)” is obvious.

Assume \( \rho \) is a positively homogeneous risk measure. For any \( Z \in \mathcal{A}_\rho \), let \( \mathcal{A}(Z) \) be given as in the proof of Theorem 3.1. By Proposition 2.5, we know that \( \mathcal{A}_\rho \) is a cone, which combined with Remark 3.2 imply that, for each \( X \in L^\infty(P) \),
\[
\rho(X) = \min_{Z \in \mathcal{A}_\rho} \max_{Q \in \mathcal{M}_1 f(P)} (E_Q[-X] + E_Q[Z])
\]
\[
= \min_{Z \in \mathcal{A}_\rho} \inf_{\alpha \geq 0} \max_{Q \in \mathcal{M}_1 f(P)} (E_Q[-X] + E_Q[\alpha Z])
\]
\[
= \min_{Z \in \mathcal{A}_\rho} \max_{Q \in \mathcal{M}_1 f(P)} (E_Q[-X] + \inf_{\alpha \geq 0} E_Q[tZ])
\]
\[
= \min_{Z \in \mathcal{A}_\rho} \max_{Q \in \mathcal{M}_1 f(P)} E_Q[-X],
\]
where we can apply the minmax theorem since \( \mathcal{M}_1 f(P) \) is weak*-compact. Let \( \mathcal{Q}_Z = \{Q \in \mathcal{M}_1 f(P) \mid E_Q[Z] \geq 0\} \) for each \( Z \in \mathcal{A}_\rho \). Then \( \{\mathcal{Q}_Z \mid Z \in \mathcal{A}_\rho\} \) is a desired family of subsets of \( \mathcal{M}_1 f(P) \).

Remark 3.5. From the proof of Theorem 3.4, we can see that a positively homogeneous risk measure \( \rho : L^\infty(P) \to \mathbb{R} \) has the following representation
\[
\rho(X) = \min_{Z \in \mathcal{A}_\rho} \max_{Q \in \mathcal{M}_1 f(P)} E_Q[-X], \quad \forall X \in L^\infty(P).
\]
Similar to Corollary 3.3, we can see that the proof of Theorem 3.4 does not depend on but leads to the classical representation theorem for coherent risk measures of Artzner, Delbaen, Eber and Heath (1999).

4 Law-Invariant Risk Measures

From now on, we assume the probability space \((\Omega, \mathcal{F}, P)\) is nonatomic, that is, it supports a random variable with a continuous distribution. Now we study the class of all risk measures which are law-invariant.

Definition 4.1. A monetary risk measure \( \rho : L^\infty(P) \to \mathbb{R} \) is called law-invariant if \( \rho(X) = \rho(Y) \) whenever \( X \) and \( Y \) have the same distribution under \( P \).
**Definition 4.2.** Given two random variables $X, Y \in L^\infty(P)$. We say that $X$ first-order stochastically dominates (FSD, in short) $Y$ and write it $X \succeq_1 Y$, if

$$E_P[f(X)] \geq E_P[f(Y)]$$

for all increasing functions $f$.

It is easy to see that a monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ is law-invariant if and only if it is FSD-consistent in the following sense:

- **FSD-Consistency:** $\rho(X) \leq \rho(Y)$ whenever $X \succeq_1 Y$.

For a random variable $X \in L^\infty(P)$, its right-continuous probability distribution function is denoted by $F_X$ and its (upper) quantile function $q_X : [0, 1] \to \mathbb{R}$ is given by

$$q_X(t) \triangleq \inf \{x \in \mathbb{R} | F_X(x) > t\}, \quad t \in [0, 1),
$$

$$q_X(1) \triangleq q_X(1-) = \lim_{t \uparrow 1} q_X(t).$$

For more details about quantile functions, see Appendix A.3 of Föllmer and Schied (2016).

One of the most well known law-invariant risk measures is VaR. The VaR of $X$ at level $t \in [0, 1]$ is given by

$$\text{VaR}_t(X) \triangleq -q_X(t).$$

It is well known that, for any $X, Y \in L^\infty(P)$,

$$X \succeq_1 Y \iff \text{VaR}_t(X) \leq \text{VaR}_t(Y), \quad \forall t \in [0, 1]. \quad (4.1)$$

Now we introduce some notations.

- $\mathcal{G}$ denotes all nonnegative Borel functions $g$ on $(0, 1)$ such that $\int_0^1 g(t)dt = 1$. That is, $\mathcal{G}$ is the set of all probability density functions on $(0, 1)$.

- $\mathcal{B}[0,1]$ denotes all Borel subsets of $[0, 1]$.

- $\ell$ denotes the Lebesgue measure on $[0, 1]$.

- $\mathcal{M}_{1,f}(\ell)$ denotes all finitely additive probability measures on $([0, 1], \mathcal{B}[0,1])$ which are absolutely continuous with respect to $\ell$.

It is well known that $\mathcal{M}_{1,f}(\ell)$ is compact under the weak*-topology, which is induced by $L^\infty([0, 1], \mathcal{B}[0,1], \ell)$.

The following theorem characterizes law-invariant monetary risk measures on $L^\infty(P)$.

**Theorem 4.3.** Assume the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. For a mapping $\rho : L^\infty(P) \to \mathbb{R}$, the following two assertions are equivalent.

---

4Throughout the paper “increasing” means “non-decreasing” and “decreasing” means “non-increasing.”
(a) $\rho$ is a law-invariant monetary risk measure.

(b) There exists a family $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ of convex functionals $\alpha_\lambda : \mathcal{G} \to (-\infty, \infty]$ such that

$$\rho(X) = \min_{\lambda \in \Lambda} \sup_{g \in \mathcal{G}} \left( \int_0^1 \text{VaR}_t(X)g(t)dt - \alpha_\lambda(g) \right) \text{ for } X \in L^\infty(P).$$  \hspace{1cm} (4.2)

(c) There exists a family $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ of lower semi-continuous (under the weak*-topology), convex functionals $\alpha_\lambda : \mathcal{M}_{1,f}(\ell) \to (-\infty, \infty]$ such that

$$\rho(X) = \min_{\lambda \in \Lambda} \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \left( \int_0^1 \text{VaR}_t(X)\mu(dt) - \alpha_\lambda(\mu) \right) \text{ for } X \in L^\infty(P).$$  \hspace{1cm} (4.3)

**Proof.** "(b)⇒(a)" and "(c)⇒(a)" are obvious.

Now we show "(a)⇒(b)". Assume $\rho$ is a law-invariant monetary risk measure. For any $Z \in \mathcal{A}_\rho$, let

$$\mathcal{D}_\lambda(Z) = \{Y \in L^\infty(P) \mid Y \succeq_1 Z\}.$$

By a similar discussion as in the proof of Theorem 3.1, we can see that each $\rho_{\mathcal{D}_1}(Z)$ is a law-invariant monetary risk measure and

$$\rho(X) = \min_{Z \in \mathcal{A}_\rho} \rho_{\mathcal{D}_1}(Z)X.$$

Moreover, for each $X \in L^\infty(P)$ and $Z \in \mathcal{A}_\rho$, we have

$$\rho_{\mathcal{D}_1}(Z)X = \inf\{m \in \mathbb{R} \mid X + m \succeq_1 Z\} = \inf\{m \in \mathbb{R} \mid \text{VaR}_t(X + m) \leq \text{VaR}_t(Z), \forall t \in (0, 1)\} = \inf\{m \in \mathbb{R} \mid \text{VaR}_t(X) - m \leq \text{VaR}_t(Z), \forall t \in (0, 1)\} = \sup_{t \in (0,1)}(\text{VaR}_t(X) - \text{VaR}_t(Z))$$

$$= \sup_{g \in \mathcal{G}} \int_0^1 (\text{VaR}_t(X) - \text{VaR}_t(Z))g(t)dt,$$

$$= \sup_{g \in \mathcal{G}} \left( \int_0^1 \text{VaR}_t(X)g(t)dt - \alpha_Z(g) \right),$$

where $\alpha_Z(g) = \int_0^1 \text{VaR}_t(Z)g(t)dt$ for each $g \in \mathcal{G}$. Therefore, $\{\alpha_Z \mid Z \in \mathcal{A}_\rho\}$ is a desired family of convex functionals on $\mathcal{G}$.

The proof of "(a)⇒(c)" is similar to the corresponding part of Theorem 3.1. □

**Remark 4.4.** From the proof of Theorem 4.3, we can see that a law-invariant monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ has the following representation

$$\rho(X) = \min_{Z \in \mathcal{A}_\rho} \sup_{g \in \mathcal{G}} \left( \int_0^1 \text{VaR}_t(X)g(t)dt - \int_0^1 \text{VaR}_t(Z)g(t)dt \right)$$

$$= \min_{Z \in \mathcal{A}_\rho} \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \left( \int_0^1 \text{VaR}_t(X)\mu(dt) - \int_0^1 \text{VaR}_t(Z)\mu(dt) \right),$$

for all $X \in L^\infty(P)$. 10
The following theorem characterizes law-invariant, positively homogeneous risk measures on $L^\infty(P)$, whose proof, based on Theorem 4.3, is similar to Theorem 3.4 and hence omitted.

**Theorem 4.5.** Assume the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. For a mapping $\rho : L^\infty(P) \to \mathbb{R}$, the following two assertions are equivalent.

(a) $\rho$ is a law-invariant, positively homogeneous risk measure.

(b) There exists a family $\{\mathcal{M}_\lambda \mid \lambda \in \Lambda\}$ of nonempty, weak*-compact, convex subsets of $\mathcal{M}_{1,f}(\ell)$ such that

$$
\rho(X) = \min_{\lambda \in \Lambda} \max_{\mu \in \mathcal{M}_\lambda} \int_0^1 \text{VaR}_t(X) \mu(dt) \quad \text{for} \quad X \in L^\infty(P).
$$

**Remark 4.6.** Similarly to Remark 3.5, we can see that a law-invariant, positively homogeneous risk measure $\rho : L^\infty(P) \to \mathbb{R}$ has the following representation

$$
\rho(X) = \min_{Z \in \mathcal{A}} \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \int_0^1 \text{VaR}_t(Z) \mu(dt) \leq \int_0^1 \text{VaR}_t(X) \mu(dt), \quad \forall X \in L^\infty(P).
$$

5 SSD-Consistent Risk Measures

Now we study SSD-consistent risk measures in this section.

**Definition 5.1.** Given two random variables $X, Y \in L^\infty(P)$. We say that $X$ second-order stochastically dominates $Y$ and write it $X \succeq_2 Y$, if

$$
E_P[f(X)] \geq E_P[f(Y)]
$$

for all increasing and concave functions $f$.

**Definition 5.2.** A monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ is called SSD-consistent if $\rho(X) \leq \rho(Y)$ whenever $X \succeq_2 Y$.

Mao and Wang (2020) comprehensively investigated SSD-consistent risk measures and provided four equivalent conditions of SSD-consistency. Theorem 5.5 below gives another one: SSD-consistency is equivalent to the combination of law-invariance, the Fatou property and a kind of weak convexity, called ID-convexity. As far as we know, the ID-convexity has not yet been introduced in the literature to study risk measures.

**Definition 5.3.** A monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ is called ID-convex (identical-distribution convex) if it satisfies

- **ID-Convexity:** $\rho(\sum_{i=1}^n \alpha_i X_i) \leq \sum_{i=1}^n \alpha_i \rho(X_i)$ whenever $X_1, X_2, \ldots, X_n$ are identically distributed under $P$, each $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$. 

Definition 5.4. We say a monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ has the Fatou property if
\[(X_n) \text{ is a bounded sequence in } L^\infty(P) \implies \liminf_{n \to \infty} \rho(X_n) \geq \rho(X).\]

Theorem 5.5. Assume the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. For a monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$, the following two assertions are equivalent.

(a) $\rho$ is SSD-consistent.

(b) $\rho$ is law-invariant and ID-convex and has the Fatou property.

Proof.

“(a)⇒(b)”: Assume $\rho$ is SSD-consistent. The law-invariance of $\rho$ is clear. The Fatou property is from Mao and Wang (2020, Theorem 3.5). It remains to show the ID-convexity. Actually, let $X_1, X_2, \ldots, X_n$ be identically distributed under $P$, each $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$. Obviously, we have $\sum_{i=1}^n \alpha_i X_i \geq 2 X_1$ and hence
\[\rho \left( \sum_{i=1}^n \alpha_i X_i \right) \leq \rho(X_1) = \sum_{i=1}^n \alpha_i \rho(X_i).\]
Therefore, $\rho$ is ID-convex.

“(b)⇒(a)”: Assume $\rho$ is law-invariant and ID-convex and has the Fatou property. For any $X, Y \in L^\infty(P)$ with $X \geq 2 Y$, we need to show $\rho(X) \leq \rho(Y)$. Actually, by a result of Ryff (1967), see also Carlier and Lachapelle (2011, Lemma 2.3), there exists a sequence $Z_n$ of the form $Z_n = \sum_{i=1}^n \alpha_i^n Y_i^n$ with $\alpha_i^n \geq 0$, $\sum_{i=1}^n \alpha_i^n = 1$ and each $Y_i^n$ has the same distribution of $Y$, such that $Z_n$ converges to $X$ $P$-a.s.. Then by the law-invariance, ID-convexity and the Fatou property of $\rho$, we have
\[\rho(X) \leq \liminf_{n \to \infty} \rho(Z_n) \leq \liminf_{n \to \infty} \sum_{i=1}^n \alpha_i^n \rho(Y_i^n) = \rho(Y).\]

The most well known SSD-consistent risk measure is AVaR (average value at risk).

The AVaR of $X$ at level $t \in [0, 1]$ is given by
\[\text{AVaR}_t(X) \triangleq \frac{1}{t} \int_0^t \text{VaR}_s(X) ds, \quad t \in (0, 1],\]
\[\text{AVaR}_0(X) \triangleq \text{VaR}_0(X).\]

It is well known that, for any $X, Y \in L^\infty(P)$,
\[X \geq_2 Y \iff \text{AVaR}_t(X) \leq \text{AVaR}_t(Y), \ \forall t \in [0, 1]. \quad (5.1)\]

\[\text{AVaR}_0(X) \triangleq \text{VaR}_0(X).\]

Also sometimes termed “expected shortfall” or “conditional value at risk” in the literature.
Let
\[ W \triangleq \{ w : [0, 1] \to [0, 1] \mid 0 \leq w(0) \leq w(1) = 1, \ w \text{ is increasing and right-continuous} \}. \]

Then \( W \) is the set of all probability distribution functions on \([0, 1]\). For each probability distribution function \( w \in W \), the probability of \( \{0\} \) is \( w(0) \) and the probability of \( \{1\} \) is \( 1 - w(1^-) \). It is well known that \( W \) is compact under the weak topology, which is induced by all continuous functions on \([0, 1]\).

The results of following two theorems have been reported in Mao and Wang (2020), which are presented and proved here in the manner of the previous two sections.

**Theorem 5.6.** Assume the probability space \((\Omega, \mathcal{F}, P)\) is nonatomic. For a mapping \( \rho : L^\infty(P) \to \mathbb{R} \), the following assertions are equivalent.

(a) \( \rho \) is an SSD-consistent monetary risk measure.

(b) There exists a family \( \{ \alpha_\lambda \mid \lambda \in \Lambda \} \) of convex functionals \( \alpha_\lambda : \mathcal{G} \to (-\infty, \infty] \) such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \sup_{g \in \mathcal{G}} \left( \int_0^1 \text{AVaR}_t(X)g(t)dt - \alpha_\lambda(g) \right) \quad \text{for} \ X \in L^\infty(P). \quad (5.2)
\]

(c) There exists a family \( \{ \alpha_\lambda \mid \lambda \in \Lambda \} \) of lower semi-continuous (under the weak topology), convex functionals \( \alpha_\lambda : W \to (-\infty, \infty] \) such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \max_{w \in W} \left( \int_{[0,1]} \text{AVaR}_t(X)dw(t) - \alpha_\lambda(w) \right) \quad \text{for} \ X \in L^\infty(P). \quad (5.3)
\]

(d) There exists a family \( \{ \rho_\lambda \mid \lambda \in \Lambda \} \) of law-invariant, convex risk measures on \( L^\infty(P) \) such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \rho_\lambda(X) \quad \text{for} \ X \in L^\infty(P).
\]

(e) For each \( X \in L^\infty(P) \),
\[
\rho(X) = \inf \{ h(X) \mid h \text{ is a law-invariant, convex risk measure and } h \geq \rho \}.
\]

**Proof.** The “(b)⇒(d)⇒(e)⇒(a)” and “(c)⇒(d)” parts are obvious. The proof of “(a)⇒(b)” and “(a)⇒(c)” parts is similar to Theorem 4.3. \( \square \)

**Theorem 5.7.** Assume the probability space \((\Omega, \mathcal{F}, P)\) is nonatomic. For a mapping \( \rho : L^\infty(P) \to \mathbb{R} \), the following assertions are equivalent.

(a) \( \rho \) is an SSD-consistent, positively homogeneous risk measure.

(b) There exists a family \( \{ W_\lambda \mid \lambda \in \Lambda \} \) of nonempty, weakly compact, convex subsets of \( W \) such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \max_{w \in W_\lambda} \int_{[0,1]} \text{AVaR}_t(X)dw(t) \quad \text{for} \ X \in L^\infty(P).
\]
(c) There exists a family \( \{ \rho_\lambda \mid \lambda \in \Lambda \} \) of law-invariant, coherent risk measures on \( L^\infty(P) \) such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \rho_\lambda(X) \quad \text{for} \quad X \in L^\infty(P).
\]

(d) For each \( X \in L^\infty(P) \),
\[
\rho(X) = \inf \{ h(X) \mid h \text{ is a law-invariant, coherent risk measure and } h \geq \rho \}.
\]

**Proof.** It is similar to Theorem 3.4 based on Theorem 5.6. \( \square \)

**Remark 5.8.** Similar to Remarks 4.4 and 4.6, we have the following two assertions.

(a) An SSD-consistent monetary risk measure \( \rho : L^\infty(P) \to \mathbb{R} \) has the following representations
\[
\rho(X) = \min_{Z \in \mathcal{A}_\rho} \sup_{g \in G} \left( \int_0^1 \text{AVaR}_t(X)g(t)dt - \int_0^1 \text{AVaR}_t(Z)g(t)dt \right)
= \min_{Z \in \mathcal{A}_\rho} \max_{w \in \Psi} \int_{[0,1]} \text{AVaR}_t(X)dw(t) - \int_{[0,1]} \text{AVaR}_t(Z)dw(t) \right), \quad \forall X \in L^\infty(P).
\]

(b) An SSD-consistent, positively homogeneous risk measure \( \rho : L^\infty(P) \to \mathbb{R} \) has the following representation
\[
\rho(X) = \min_{Z \in \mathcal{A}_\rho} \max_{w \in \Psi} \int_{[0,1]} \text{AVaR}_t(X)dw(t), \quad \forall X \in L^\infty(P).
\]

The SSD-consistency obviously implies the law-invariance. Then a question arises: what is the representation in terms of VaR for an SSD-consistent monetary risk measure, when it is regarded as a law-invariant monetary risk measure? The same question arises for SSD-consistent, positively homogeneous risk measures as well. They are positively replied by the next corollary, where the following notations will be used.

\[ \mathcal{G}^+ \triangleq \{ g \in \mathcal{G} \mid g \text{ is decreasing on } (0,1) \}, \]
\[ \Psi \triangleq \{ \psi \in \mathcal{W} \mid \psi \text{ is concave on } [0,1] \}. \]

By Föllmer and Schied (2016, Lemma 4.69), the identity
\[
\begin{align*}
\psi'(t) &= \int_{(t,1]} s^{-1}dw(s), \quad t \in [0,1) \\
\psi(0) &= w(0)
\end{align*}
\]

defines a bijection
\[ J : \mathcal{W} \to \Psi, \]

14
where $\psi'$ denotes the right derivative. Under identity (5.4), an application of Fubini theorem implies that
\[
\int_{[0,1]} \text{VaR}_t(X) d\psi(t) = \int_{[0,1]} \text{AVaR}_t(X) dw(t), \quad \forall X \in L^\infty(P). \tag{5.5}
\]
Let $\Psi$ be endowed with the topology induced by the bijection $J$. Then $W$ and $\Psi$ are homeomorphic and $J$ is a homeomorphism. Therefore, $\Psi$ is a compact space and, for every $X \in L^\infty(P)$, the functional
\[
\psi \mapsto \int_{[0,1]} \text{VaR}_t(X) d\psi(t)
\]
is continuous on $\Psi$.

**Corollary 5.9.** Assume the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. For a mapping $\rho : L^\infty(P) \to \mathbb{R}$, we have the following two assertions.

(a) $\rho$ is an SSD-consistent monetary risk measure if and only if one of the following conditions holds.

(i) There exists a family $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ of convex functionals $\alpha_\lambda : \mathcal{G}^1 \to (-\infty, \infty]$ such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \sup_{g \in \mathcal{G}^1} \left( \int_0^1 \text{VaR}_t(X) g(t) dt - \alpha_\lambda(g) \right) \quad \text{for } X \in L^\infty(P). \tag{5.6}
\]

(ii) There exists a family $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ of lower semi-continuous, convex functionals $\alpha_\lambda : \Psi \to (-\infty, \infty]$ such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \max_{\psi \in \Psi} \left( \int_{[0,1]} \text{VaR}_t(X) d\psi(t) - \alpha_\lambda(\psi) \right) \quad \text{for } X \in L^\infty(P).
\]

(b) $\rho$ is an SSD-consistent, positively homogeneous risk measure if and only if there exists a family $\{\Psi_\lambda \mid \lambda \in \Lambda\}$ of nonempty, compact, convex subsets of $\Psi$ such that
\[
\rho(X) = \min_{\lambda \in \Lambda} \max_{\psi \in \Psi_\lambda} \int_{[0,1]} \text{VaR}_t(X) d\psi(t) \quad \text{for } X \in L^\infty(P).
\]

**Proof.** It is an obvious consequence of Theorems 5.6 and 5.7. \hfill \Box

**Remark 5.10.** Similar to Remark 5.8, an SSD-consistent monetary risk measure $\rho : L^\infty(P) \to \mathbb{R}$ has the following representations
\[
\rho(X) = \min_{Z \in \mathcal{Z}_\rho} \sup_{g \in \mathcal{G}_1} \left( \int_0^1 \text{VaR}_t(X) g(t) dt - \int_0^1 \text{VaR}_t(Z) g(t) dt \right)
\]
\[
= \min_{Z \in \mathcal{Z}_\rho} \max_{\psi \in \Psi} \left( \int_{[0,1]} \text{VaR}_t(X) d\psi(t) - \int_{[0,1]} \text{VaR}_t(Z) d\psi(t) \right),
\]
for all $X \in L^\infty(P)$. 15
6 Characterizations in a Unified Form

We have shown that each monetary risk measure \( \rho \) is the lower envelope of a family of convex risk measures; particularly,

\[
\rho(X) = \inf \{ h(X) \mid h \text{ convex and } h \geq \rho \}.
\]

When the law-invariance is involved, it is a “natural” expectation that each law-invariant monetary risk measure is the lower envelope of a family of law-invariant convex risk measures. But each law-invariant convex risk measure is SSD-consistent. Therefore, the lower envelope of a family of law-invariant convex risk measures must be SSD-consistent as well. As a consequence, the previous “natural” expectation is not correct. For example,

\[
\text{VaR}_t(X) \neq \inf \{ h(X) \mid h \text{ law-invariant, convex and } h \geq \text{VaR}_t \}.
\]

Furthermore, as Theorem 5.6 shows, each SSD-consistent monetary risk measure \( \rho \) is the lower envelope of a family of law-invariant convex risk measures; particularly,

\[
\rho(X) = \inf \{ h(X) \mid h \text{ law-invariant, convex and } h \geq \rho \}.
\]

A natural question arises: what is a “good” property such that general/law-invariant/SSD-consistent monetary risk measures can be characterized in a unified form as follows?

- A monetary risk measure is the lower envelope of a family of “good” risk measures.
- A law-invariant monetary risk measure is the lower envelope of a family of law-invariant, “good” risk measures.
- An SSD-consistent monetary risk measure is the lower envelope of a family of SSD-consistent, “good” risk measures.

The convexity is too strict to unify the characterizations, as we have seen. It turns out that the “comonotonic convexity” (CoM-convexity, for short) is the desired “good” property, which is weaker than the convexity and were introduced by Song and Yan (2006, 2009), as well as by Kou, Peng and Heyde (2013), to investigate risk measures. Some basic definitions and facts about CoM-convex (CoM-coherent) risk measures are summarized in Appendix A.

The following theorem replies to the previous question.

Theorem 6.1. For a mapping \( \rho : L^\infty(P) \to \mathbb{R} \), we have the following assertions.

(a) \( \rho \) is a monetary risk measure if and only if it is the lower envelope of a family of CoM-convex risk measures.

(b) Assume \( (\Omega, \mathcal{F}, P) \) is nonatomic, then \( \rho \) is a law-invariant monetary risk measure if and only if it is the lower envelope of a family of law-invariant, CoM-convex risk measures.
(c) Assume $(\Omega, \mathcal{F}, P)$ is nonatomic, then $\rho$ is an SSD-consistent monetary risk measure if and only if it is the lower envelope of a family of SSD-consistent, CoM-convex risk measures.

**Proof.** (a) is an obvious consequence of Theorem 3.1. (c) is a consequence of Theorems 5.6 and A.8. The “if” part of (b) is obvious. Now we show the “only if” part of (b).

Assume $\rho$ is a law-invariant monetary risk measure. Recalling the proof of Theorem 4.3, the set $\mathcal{D}_1(Z)$ is given by

$$\mathcal{D}_1(Z) = \{ Y \in L_\infty(P) \mid Y \succeq 1 \} Z \}.$$

It is easy to see that $\mathcal{D}_1(Z)$ is CoM-convex in the following sense:

$$\alpha X + (1 - \alpha)Y \in \mathcal{D}_1(Z) \text{ whenever } X, Y \in \mathcal{D}_1(Z) \text{ are comonotonic and } \alpha \in [0, 1].$$

Then it is easy to verify that $\rho_{\mathcal{D}_1(Z)}$ is CoM-convex. Therefore, $\{\rho_{\mathcal{D}_1(Z)} \mid Z \in \mathcal{A}_\rho\}$ is a desired family of law-invariant, CoM-convex risk measures, whose lower envelope is $\rho$. □

Similarly, we have the following theorem.

**Theorem 6.2.** For a mapping $\rho : L_\infty(P) \to \mathbb{R}$, we have the following assertions.

(a) $\rho$ is a positively homogeneous risk measure if and only if it is the lower envelope of a family of CoM-coherent risk measures.

(b) Assume $(\Omega, \mathcal{F}, P)$ is nonatomic, then $\rho$ is a law-invariant, positively homogeneous risk measure if and only if it is the lower envelope of a family of law-invariant, CoM-coherent risk measures.

(c) Assume $(\Omega, \mathcal{F}, P)$ is nonatomic, then $\rho$ is an SSD-consistent, positively homogeneous risk measure if and only if it is the lower envelope of a family of SSD-consistent, CoM-coherent risk measures.

The next two corollaries show that the proofs of Theorem 4.3 and Corollary 5.9 lead to another proof of the representation theorems of Song and Yan (2009) for law-invariant/SSD-consistent and CoM-convex/CoM-coherent risk measures.

**Corollary 6.3.** Assume $(\Omega, \mathcal{F}, P)$ is nonatomic and $\rho : L_\infty(P) \to \mathbb{R}$ is a law-invariant CoM-convex risk measure. Then there exists a lower-semicontinuous convex functional $\alpha : \mathcal{M}_{1,f}(\ell) \to (-\infty, \infty]$ such that

$$\rho(X) = \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \left( \int_0^1 \text{VaR}_t(X)\mu(dt) - \alpha(\mu) \right), \text{ for } X \in L_\infty(P).$$

Furthermore, assume $\rho : L_\infty(P) \to \mathbb{R}$ is law-invariant and CoM-coherent. Then there exists a nonempty, weak* compact and convex subset $\mathcal{M} \subseteq \mathcal{M}_{1,f}(\ell)$ such that

$$\rho(X) = \max_{\mu \in \mathcal{M}} \int_0^1 \text{VaR}_t(X)\mu(dt), \text{ for } X \in L_\infty(P).$$

17
Proof. Assume $\rho : L^\infty(P) \to \mathbb{R}$ is a law-invariant CoM-convex risk measure. Let $Q$ denote the set of quantile functions $q_X$ of all random variables $X \in L^\infty(P)$. Consider a subset $Q_\rho$ of $Q$ given by

$$Q_\rho = \{ q \in Q \mid q = q_Z \text{ for some } Z \in \mathcal{A}_\rho \}.$$  

Then $Q_\rho$ is a convex set. Actually, for any $q_1, q_2 \in Q_\rho$, we have $q_i = q_{Z_i}$ for some $Z_i \in \mathcal{A}_\rho$, $i = 1, 2$. Consider random variables $Y_i = q_i(U)$, $i = 1, 2$, where $U$ is a $(0,1)$-uniformly distributed random variable. Obviously, for each $i$, $Z_i$ and $Y_i$ have the same distribution. Moreover, $Y_1$ and $Y_2$ are comonotonic. Then we have for each given $\alpha \in (0,1)$,

$$\alpha q_1 + (1-\alpha)q_2 = q_Y,$$

where $Y = \alpha Y_1 + (1-\alpha)Y_2$. Moreover,

$$\rho(Y) \leq \alpha \rho(Y_1) + (1-\alpha)\rho(Y_2) = \alpha \rho(Z_1) + (1-\alpha)\rho(Z_2) \leq 0,$$

which implies that $Y \in \mathcal{A}_\rho$ and hence $\alpha q_1 + (1-\alpha)q_2 \in Q_\rho$. Therefore, $Q_\rho$ is convex.

The convexity of $Q_\rho$ and the weak*-compactness and convexity of $\mathcal{M}_{1,f}(\ell)$ allows for an application of the minmax theorem to the representations in Remark 4.4, which leads to, for each $X \in L^\infty(P)$,

$$\rho(X) = \min_{q \in Q_\rho} \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \left( \int_0^1 \text{VaR}_t(X) \mu(dt) + \int_0^1 q(t) \mu(dt) \right)$$

$$= \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \inf_{q \in Q_\rho} \left( \int_0^1 \text{VaR}_t(X) \mu(dt) + \int_0^1 q(t) \mu(dt) \right)$$

$$= \max_{\mu \in \mathcal{M}_{1,f}(\ell)} \left( \int_0^1 \text{VaR}_t(X) \mu(dt) - \alpha(\mu) \right),$$

where $\alpha(\mu) = \sup_{q \in Q_\rho} \left( -\int_0^1 q(t) \mu(dt) \right)$ defines the desired lower-semicontinuous convex functional on $\mathcal{M}_{1,f}(\ell)$.

Furthermore, if $\rho$ is law-invariant and CoM-coherent, then it is easy to see that $\mathcal{A}_\rho$ and $Q_\rho$ are cones, which implies that

$$\alpha(\mu) = \sup_{q \in Q_\rho} \left( -\int_0^1 q(t) \mu(dt) \right) = \sup_{q \in Q_\rho, \alpha > 0} \left( -\int_0^1 \alpha q(t) \mu(dt) \right)$$

$$= \begin{cases} 0 & \text{if } \sup_{q \in Q_\rho} \int_0^1 q(t) \mu(dt) \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{M} = \left\{ \mu \in \mathcal{M}_{1,f}(\ell) \mid \sup_{q \in Q_\rho} \int_0^1 q(t) \mu(dt) \geq 0 \right\}.$$  

Then $\mathcal{M}$ is the desired subset of $\mathcal{M}_{1,f}(\ell)$. $\square$

Corollary 6.4. Assume $(\Omega, \mathcal{F}, P)$ is nonatomic and $\rho : L^\infty(P) \to \mathbb{R}$ is an SSD-consistent CoM-convex risk measure. Then there exists a lower-semicontinuous convex functional
\(\alpha : \Psi \rightarrow (-\infty, \infty]\) such that

\[
\rho(X) = \max_{\psi \in \Psi} \left( \int_{[0,1]} \text{VaR}_t(X) \psi(dt) - \alpha(\psi) \right), \quad \text{for } X \in L^\infty(P).
\]

Furthermore, assume \(\rho : L^\infty(P) \rightarrow \mathbb{R}\) is SSD-consistent and CoM-coherent. Then there exists a nonempty, compact and convex subset \(\Psi_1 \subseteq \Psi\) such that

\[
\rho(X) = \max_{\psi \in \Psi_1} \int_{[0,1]} \text{VaR}_t(X) \psi(dt), \quad \text{for } X \in L^\infty(P).
\]

**Proof.** It is similar to Corollary 6.3, based on Remarks 5.10, and hence omitted. \(\square\)

**Appendix**

**A Comonotonic Convex Risk Measures**

**Definition A.1.** Two random variables \(X, Y \in L^\infty(P)\) are called comonotonic if there exists some \(\Omega_0 \in \mathcal{F}\) such that \(P(\Omega_0) = 1\) and

\[
(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \quad \text{for } \omega, \omega' \in \Omega_0.
\]

**Definition A.2.** A monetary risk measure \(\rho : L^\infty(P) \rightarrow \mathbb{R}\) is called comonotonic additive (CoM-additive, in short) if

\[
\rho(X + Y) = \rho(X) + \rho(Y)
\]

whenever \(X, Y \in L^\infty(P)\) are comonotonic.

**Definition A.3.** A monetary risk measure \(\rho : L^\infty(P) \rightarrow \mathbb{R}\) is called comonotonic convex (CoM-convex, in short) if it satisfies

- CoM-Convexity: \(\rho(\alpha X + (1 - \alpha)Y) \leq \alpha \rho(X) + (1 - \alpha)\rho(Y)\) whenever \(X, Y \in L^\infty(P)\) are comonotonic and \(\alpha \in [0, 1]\).

**Definition A.4.** A monetary risk measure \(\rho : L^\infty(P) \rightarrow \mathbb{R}\) is called a comonotonic coherent (CoM-coherent, in short) if it is CoM-convex and positively homogeneous.

**Definition A.5.** A set function \(c : \mathcal{F} \rightarrow [0,1]\) is called a capacity on \((\Omega, \mathcal{F})\) if \(c(\emptyset) = 0\), \(c(\Omega) = 1\), and \(c(A) \leq c(B)\) whenever \(A \subseteq B\). A capacity \(c\) on \((\Omega, \mathcal{F})\) is called absolutely continuous with respect to \(P\) if \(c(A) = c(B)\) whenever \(P(A \triangle B) = 0\).

Let \(\mathcal{C}(P)\) denote all capacities on \((\Omega, \mathcal{F})\) that are absolutely continuous with respect to \(P\). It is well known that a monetary risk measure \(\rho : L^\infty(P) \rightarrow \mathbb{R}\) is CoM-additive if and only if there exists some capacity \(c \in \mathcal{C}(P)\) such that

\[
\rho(X) = \int (-X) \, dc \quad \text{for } X \in L^\infty(P),
\]
where \( \int (-X) \, dc \) is the Choquet integral of \(-X\) with respect to \( c\); see, e.g., Schmeidler (1986), and also Föllmer and Schied (2016).

Obviously, any convex (coherent) risk measure is CoM-convex (CoM-coherent). For the following two theorems, which characterize CoM-convex (CoM-coherent) risk measures, see Song and Yan (2006) as well as Kou, Peng and Heyde (2013).

**Theorem A.6.** A mapping \( \rho : L^\infty(P) \to \mathbb{R} \) is a CoM-convex risk measure if and only if there exists some convex functional \( \alpha : \mathcal{C}(P) \to (\mathbb{R}, \infty] \) such that

\[
\rho(X) = \sup_{c \in \mathcal{C}(P)} \left\{ \int (-X) \, dc - \alpha(c) \right\} \quad \text{for } X \in L^\infty(P).
\]

**Theorem A.7.** A mapping \( \rho : L^\infty(P) \to \mathbb{R} \) is a CoM-coherent risk measure if and only if there exists some nonempty subset \( C \subseteq \mathcal{C}(P) \) such that

\[
\rho(X) = \sup_{c \in C} \left\{ \int (-X) \, dc \right\} \quad \text{for } X \in L^\infty(P).
\]

For the following theorem, which characterizes SSD-consistent, CoM-convex (CoM-coherent) risk measures, see Song and Yan (2009, Theorems 3.2 and 3.6).

**Theorem A.8.** Assume \((\Omega, \mathcal{F}, P)\) is nonatomic. For a monetary risk measure \( \rho \) on \( L^\infty(P) \), we have the following assertions.

(a) \( \rho \) is SSD-consistent and CoM-convex if and only if \( \rho \) is law-invariant and convex.

(b) \( \rho \) is SSD-consistent and CoM-coherent if and only if \( \rho \) is law-invariant and coherent.
References

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999): “Coherent measures of risk,” Mathematical Finance 9, 203–228.

Carlier, G. and A. Lachapelle (2011): “A Numerical Approach for a Class of Risk-Sharing Problems,” Journal of Mathematical Economics 47, 1–13.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and L. Montrucchio (2011): “Risk Measures: Rationality and Diversification,” Mathematical Finance 21, 743–774.

Delbaen, F. (2002): “Coherent measures of risk on general probability spaces.” In: Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann, Springer-Verlag, 1–37.

El Karoui, N. and C. Ravanelli (2009): “Cash subadditive risk measures and interest rate ambiguity,” Mathematical Finance 19, 561–590.

Föllmer, H. and A. Schied (2002): “Convex measures of risk and trading constraints,” Finance and Stochastics 6, 429–447.

Föllmer, H. and A. Schied (2016): Stochastic Finance: An Introduction in Discrete Time (4th Edition). 1st Edition: 2002. Berlin: Walter de Gruyter.

Frittelli, M. and E. Rosazza Gianin (2002): “Putting order in risk measures,” J. Banking & Finance 26, 1473–1486.

Frittelli, M. and E. Rosazza Gianin (2005): “Law-invariant convex risk measures,” Advances in Mathematical Economics 7, 33–46.

Jouini, E., W. Schachermayer, and N. Touzi (2006): “Law invariant risk measures have the Fatou property,” Advances in Mathematical Economics 9, 49–71.

Kou, S. G., X. Peng, and C. C. Heyde (2013): “External Risk Measures and Basel Accords,” Mathematics of Operations Research 38, 393–417.

Kusuoka, S. (2001): “On law invariant coherent risk measures,” Advances in Mathematical Economics 3, 83–95.

Mao, T., and R. Wang (2020): “Risk aversion in regulatory capital principles,” SIAM Journal on Financial Mathematics 11, 169–200.

Mertens, J.-F., S. Sorin, and S. Zamir (2015): Repeated Games. New York: Cambridge University Press.

Ryff, J. V. (1967): “Extreme Points of Some Convex Subsets of $L^1$, ” Proceedings of the American Mathematical Society 116, 1026–1034.

Schmeidler, D. (1986): “Integral Representation without Additivity,” Proceedings of the American Mathematical Society 97, 255–261.

Song, Y., and J.-A. Yan (2006): “The representations of two types of functions on $L^\infty(\Omega, \mathcal{F})$ and $L^\infty(\Omega, \mathcal{F}, P)$,” Science in China Series A: Mathematics 49, 1376–1382.

Song, Y., and J.-A. Yan (2009): “Risk Measures with Comonotonic Subadditivity or Convexity and Respecting Stochastic Orders,” Insurance: Mathematics and Economic 45, 459–465.

Xia, J. (2020): “Decision Making under Uncertainty: A Game of Two Selves,” working paper.