Iterative method for solving nonlinear integral equations describing rolling solutions in String Theory

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Abstract

We consider a nonlinear integral equation with infinitely many derivatives that appears when a system of interacting open and closed strings is investigated if the nonlocality in the closed string sector is neglected. We investigate the properties of this equation, construct an iterative method for solving it, and prove that the method converges.

Nonlinear equations with infinite number of time derivatives have recently became the subject of investigation in both standard and p-adic string theories [1–5, 9–13]. Properties of some of them were systematically studied in [12]. In present work further investigations of such equations will be presented, we will consider equation of the following form which appears in string theory

\[ a\Phi^3(t) + (1 - a)\Phi(t) = \exp \left( a\frac{d^2}{dt^2}\right)\Phi(t), \]

(1)

where \( a \) is a constant, \( a \in (0, 1] \).

Precise meaning to equation (1) will be given below. We note that in the case \( a = 1 \), this equation takes a form of the equation for \( p \)-adic string with \( p = 3 \), which was considered in the papers [3, 9, 12].

Equation (1) arises when studying system of interacting open and closed string in the approximation when nonlocality in the interaction of closed string is neglected [6–8]. Note that resulting equations are still nonlocal in
terms of open string tachyon. It is interesting to note that in the mechanical approximation \([4, 7]\) this equation transforms to equation

\[
\Phi^3(t) - \Phi(t) = \frac{d^2}{dt^2} \Phi(t),
\]

which have well known kink solution

\[
\Phi(t) = \tanh\left(\frac{t}{\sqrt{2}}\right).
\]

Note that kink usually describes the solution depending on spatial coordinates.

In this paper we will investigate the properties of equation (1) and consider boundary value problems for bounded solution. In particular, we will construct rolling solutions interpolating between two vacua.

Equation (1) is a pseudo-differential equation with symbol \(e^{-a\xi^2}\) which for positive \(a\) might be presented as a nonlinear integral equation (in full accordance with the equations considered in \([2, 12, 13]\))

\[
C_a[\Phi](t) = a\Phi^3(t) + (1 - a)\Phi(t),
\]

where constant \(a\): \(0 < a < 1\) and operator \(C_a\) is defined as

\[
C_a[\psi](t) = \int_{-\infty}^{\infty} C_a[(t - \tau)^2] \psi(\tau) d\tau,
\]

with the kernel

\[
C_a[(t - \tau)^2] = \frac{1}{\sqrt{4\pi a}} e^{-\frac{(t-\tau)^2}{4a}}.
\]

We look for solutions of equation (2) in the class of real-valued measurable functions.

**Theorem 1.** If solution \(\Phi(t)\) of equation (2) is bounded, then it satisfies the estimate

\[
|\Phi(t)| \leq 1, \quad t \in \mathbb{R}
\]

**Proof.** Let us suppose that solution \(\Phi(t)\) of equation (2) is bounded, i.e. there exists such a number \(M > 0\) that

\[
\sup_t |\Phi(t)| = M.
\]
It follows from (2) and (3) that

\[
|a\Phi^3(t) + (1 - a)\Phi(t)| = \left| \int_{-\infty}^{+\infty} \Phi(\tau)C_a[(t - \tau)^2]d\tau \right| \leq (6)
\]

\[
\leq \int_{-\infty}^{+\infty} |\Phi(\tau)|C_a[(t - \tau)^2]d\tau \leq \sup_{\tau} |\Phi(\tau)| \int_{-\infty}^{+\infty} C_a[(t - \tau)^2]d\tau = M,
\]

and

\[
\sup_{t} |a\Phi^3(t) + (1 - a)\Phi(t)| = aM^3 + (1 - a)M \leq M,
\]
i.e. \( M \leq 1 \). This proves the theorem.

Remark 1. The theorem 1, as well as theorems 4 and 5 proved below, are similar to the corresponding theorems from papers [2, 12], but are proved here using features of more general equation (2).

Lemma 1. If function \( \Phi(t) \) is bounded, then function \( C_a[\Phi](t) \) is continuous in \( t \).

Proof. By the formulation of lemma the function \( \Phi(t) \) is bounded, i.e. there exists such a number \( M > 0 \), for which \( \sup_{t} |\Phi(t)| = M \). Let us consider the estimate

\[
|C_a[\Phi](t + \delta) - C_a[\Phi](t)| = \frac{1}{\sqrt{4\pi a}} \left| \int_{-\infty}^{+\infty} \left( e^{-\frac{(t+\delta-\tau)^2}{4a}} - e^{-\frac{(t-\tau)^2}{4a}} \right) \Phi(\tau)d\tau \right| \leq
\]

\[
\leq \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} \left| e^{-\frac{(t+\delta-\tau)^2}{4a}} - e^{-\frac{(t-\tau)^2}{4a}} \right| |\Phi(\tau)|d\tau \leq
\]

\[
\leq \sup_{\tau} |\Phi(\tau)| \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} \left| e^{-\frac{(t+\delta-\tau)^2}{4a}} - e^{-\frac{(t-\tau)^2}{4a}} \right| d\tau \leq
\]

\[
= \frac{M}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} \left| e^{-\frac{(t+\delta)^2}{4a}} - e^{-\frac{\tau^2}{4a}} \right| dy
\]

Evaluating the absolute value in the integrand, we get

\[
|C_a[\Phi](t + \delta) - C_a[\Phi](t)| \leq 2M \text{erf} \left( \frac{\delta}{4\sqrt{a}} \right)
\]
(7)
where \( \text{erf}(t) \) is an error function, which is defined by

\[
\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2} d\tau.
\]  (8)

The estimate (7) with (8) taken into account shows that the function \( C_a[\Phi](t) \) is continuous.

**Theorem 2.** Any bounded solution \( \Phi(t) \) of equation (2) is continuous.

**Proof.** By Lemma 1 the functional \( C_a[\Phi](t) \) is continuous. Let us prove that the solution \( \Phi(t) \) itself is continuous. Let us suppose the opposite, i.e. that \( \Phi(t) \) is not continuous; it then follows that functions \( a\Phi^3(t) \) and \( (1-a)\Phi(t) \) are also not continuous. Taking into account that \( a \in [0,1] \), we get that the function \( a\Phi^3(t) + (1-a)\Phi(t) \) is also not continuous, which contradicts the statement that function \( \Phi(t) \) solves equation (2) since the function \( C_a\Phi(t) \) should be continuous. Thus the continuity of the function \( \Phi(t) \) is proved.

**Theorem 3.** If solution \( \Phi(t) \) of equation (2) is positive and bounded, then result of acting with operator \( C_a \) is decreasing, i.e. \( C_a[\Phi](t) \leq \Phi(t) \), and if solution \( \Phi(t) \) of the equation (2) is negative and bounded, then result of acting with operator \( C_a \) is increasing, i.e. \( C_a[\Phi](t) \geq \Phi(t) \).

**Proof.** Let us prove the first statement of the theorem. Following assumption that the solution \( \Phi(t) \) of the equation (2) is positive and bounded, by Theorem 1 we get \( |\Phi(t)| \leq 1 \) and thus

\[
C_a[\Phi](t) = a\Phi^3(t) + (1-a)\Phi(t) \leq \Phi(t)
\]

i.e. \( C_a[\Phi](t) \leq \Phi(t) \) for positive and bounded solutions.

The second statement of the Theorem is proved analogously.

**Theorem 4.** If solution \( \Phi(t) \) of equation (2) has a definite limit as \( t \rightarrow \infty \), then the limit takes one of the following possible values: \(-1, 0, 1\).

**Proof.** Following the formulation of the Theorem solution of equation (2) has a limit \( \lim_{t\to+\infty} \Phi(t) \), let \( \lim_{t\to+\infty} \Phi(t) = b \), calculating \( C_a[\Phi(t)] \)

\[
\lim_{t\to+\infty} C_a[\Phi(t)] \lim_{t\to+\infty} \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} e^{-\frac{(t-\tau)^2}{4a}} \Phi(\tau)d\tau
\]
\[
\frac{1}{\sqrt{4\pi a}} \left[ \lim_{t \to +\infty} \int_{-\infty}^{0} e^{-\frac{(t-\tau)^2}{4a}} \Phi(\tau) d\tau + \lim_{t \to +\infty} \int_{0}^{+\infty} e^{-\frac{(t-\tau)^2}{4a}} \Phi(\tau) d\tau \right]
\]

and making change of variables \(\tau \to t - u\) in the first integral and \(\tau \to t + u\) in the second one we get

\[
\frac{1}{\sqrt{4\pi a}} \left[ \lim_{t \to +\infty} \int_{t}^{+\infty} e^{-\frac{u^2}{4a}} \Phi(t - u) du + \lim_{t \to +\infty} \int_{-t}^{+\infty} e^{-\frac{u^2}{4a}} \Phi(t + u) du \right] =
\]

\[
= \frac{1}{\sqrt{4\pi a}} \left[ \int_{+\infty}^{+\infty} e^{-\frac{w^2}{4a}} b du + \int_{-\infty}^{+\infty} e^{-\frac{w^2}{4a}} b du \right] = \frac{b}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{4a}} du,
\]
i.e. we obtain that \(\lim_{t \to +\infty} C_a[\Phi(t)] = b\).

Taking the limit \(t \to +\infty\) in the equation (2), we obtain the equation

\[
ab^3 + (1 - a)b = b,
\]
which has three roots \(b = -1\), \(b = 0\), and \(b = 1\).

Subsequently we proved that the limit might takes only the following values \(0, -1, 1\).

**Theorem 5.** There exists a unique nonnegative bounded continuous solution \(\Phi(t) \equiv 1\) of equation (2) that satisfies the boundary conditions

\[
\lim_{t \to \pm \infty} \Phi(t) = 1 \quad (9)
\]

**Proof.** Note that \(\Phi(t) \equiv 1\) is a solution of the boundary value problem in (2). Let \(\Phi^*(t), 0 \leq \Phi^*(t) \neq 1\) be another bounded solution of this problem. Then \(0 \leq \Phi^*(t) \leq 1\) and by (9) there exists a \(t_0\) such that

\[
0 \leq \Phi^*(t_0) = \min_t \Phi^*(t) \leq 1. \quad (10)
\]

From equation (2) we have

\[
a\Phi^3(t_0) + (1 - a)\Phi^*(t_0) = \int_{-\infty}^{+\infty} \Phi^*(\tau) C_a[(t_0 - \tau)^2] d\tau \geq \Phi^*(t_0),
\]
this inequality holds if \(\Phi^*(t_0) \geq 1\) and \(\Phi^*(t_0) = 0\). Taking into consideration that \(|\Phi^*(t)| \leq 1\), we obtain that inequality (11) holds only if \(\Phi^*(t_0) = 0\) or \(\Phi^*(t_0) = 1\), but the solution of inequality (11), \(\Phi^*(t_0) = 0\), does not satisfy the boundary problem (9). Hence we consider \(\Phi^*(t) \geq 0\) and \(\exists t_0 : C_a[\Phi^*](t_0) = 0\), then \(\Phi^*(t) \equiv 0\), i.e. there exists a unique nonnegative solution \(\Phi^*(t) \equiv \Phi(t) \equiv 1\) of the boundary value problem (9), (2). \(\square\)
Theorem 6. There exists a continuous solution of equation (2) that satisfies the boundary conditions

\[ \lim_{t \to \infty} \Phi(t) = \begin{cases} 1, & t \to \infty \\ -1, & t \to -\infty \end{cases} \] (12)

Moreover the iterative procedure

\[ C_a \Phi_n = a\Phi_{n+1}^3 + (1 - a)\Phi_{n+1} \] (13)

converges to this solution.

Proof. Taking into account the invariance of the equation (2) under the change of variables \( \phi(t) \to -\phi(-t) \), we will seek for odd solutions. The boundary value problem (2), (12) rewrites on the positive semiaxis \( t \geq 0 \) as

\[ K_a[\phi](t) = a\phi^3(t) + (1 - a)\phi(t), \quad t \geq 0, \] (14)

and

\[ \lim_{t \to +\infty} \phi(t) = 1 \] (15)

where the operator \( K_a \) is given by

\[ K_a[\phi](t) = \int_0^\infty K_a(t, \tau)\phi(\tau)d\tau, \]

with

\[ K_a(t, \tau) = \frac{1}{\sqrt{4\pi a}} \left[ e^{-\frac{(a-\tau)^2}{4a}} - e^{-\frac{(a+\tau)^2}{4a}} \right]. \]

Solution of the original problem \( \Phi(t) \) is given by

\[ \Phi(t) = \begin{cases} \phi(t), & t \geq 0, \\ -\phi(-t), & t < 0. \end{cases} \] (16)

We will seek solution of equation (2), using an iterative procedure (13) which on the semiaxis \( t \geq 0 \) takes the following form

\[ K_a \phi_n = a\phi_{n+1}^3 + (1 - a)\phi_{n+1} \] (17)

Solving the cubic equation (17) for \( \phi_{n+1} \), we obtain

\[ \phi_{n+1} = -v_n(1 - a) + \frac{1}{3av_n}, \] (18a)
Figure 1: a) results of iterative procedure $\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_{50}, \phi_{150}$ from below to up correspondingly; b)–the difference between $\phi_{50}$ and $\phi_{150}$-th iterations.

where

$$v_n = \left( \frac{2}{27a^2B_n + \sqrt{108 (1-a)^3a^3 + 729a^4B_n^2}} \right)^{1/3}, \quad (18b)$$

and

$$B_n = K_a \phi_n. \quad (18c)$$

As initial iteration we take the function

$$\phi_0 = \frac{1}{2} (1 - e^{- (at)^2}).$$

Acting with operator $K_a$, we get

$$K_a \phi_0 = \frac{1}{2} (1 - e^{- \sqrt{(at)^2}}).$$

Evaluating $a \phi_0^3 + (1-a) \phi_0$ we obtain

$$a \phi_0^3 + (1-a) \phi_0 \leq K_a \phi_0,$$

i.e.

$$a \phi_0^3 + (1-a) \phi_0 \leq a \phi_1^3 + (1-a) \phi_1,$$

subsequently $\phi_0 \leq \phi_1$. Taking into account the fact that the kernel is non-negative after integration we obtain $K_a \phi_0 \leq K_a \phi_1$, i.e. $B_0 \leq B_1$.

Using explicit expressions (18b) and (18a) for the functions $v_n$ and $\phi_{n+1}$ we conclude that inequality $B_0 \leq B_1$, implies inequalities $v_0 \geq v_1$, and $\phi_1 \leq \phi_2$. Repeating the above argument $(n-1)$ times we obtain

$$\phi_0 \leq \phi_1 \leq \ldots \leq \phi_n \leq \phi_{n+1}.$$
This illustrates the figure of calculations of the iterations (see fig[1]).

We now prove that
\[ \phi_n \leq 1. \]

Iterative procedure have the form (17). We can see that initial iteration \( \phi_0 \) is bounded, \( \phi_0 < 1 \). Then we have \( K_a \phi_0 \leq \phi_0 < 1 \), and therefore \( a\phi_1 + (1 - a)\phi_1 < 1 \). Let us suppose that there exists \( t_1 \geq 0 \) such that \( \phi_1(t_1) > 1 \), then \( a\phi_1 + (1 - a)\phi_1 > \phi_1 > 1 \), which contradicts the estimate obtained above and therefore \( \phi_1 \leq 1 \). Repeating this argument \( n \) times, we obtain that all the functions \( \phi_{n+1} \) are bounded, \( \phi_{n+1} \leq 1 \).

Let us now prove that functions \( \phi_{n+1} \) are monotonic. It is enough to prove that assumption that \( \phi_n(t) \) is nonnegative monotonically increasing function leads to
\[ \frac{d}{dt} (K_a[\phi_n](t)) \geq 0 \]

We have
\[ \frac{d}{dt} (K_a[\phi_n](t)) = \int_{0}^{\infty} K'_a(t, \tau) \phi_n(\tau) d\tau = \int_{0}^{\infty} \left( K'_1(t, \tau) + K'_2(t, \tau) \right) \phi_n(\tau) d\tau, \]
where
\[ K'_1(t, \tau) = \frac{1}{2a\sqrt{4\pi a}} (\tau - t) e^{-\frac{(\tau - t)^2}{4a}} \]
and
\[ K'_2(t, \tau) = \frac{1}{2a\sqrt{4\pi a}} (\tau + t) e^{-\frac{(\tau + t)^2}{4a}} \]

Because for all \( t, \tau \geq 0 \) the kernel \( K'_2(t, \tau) \geq 0 \) and the function \( \phi_n(\tau) \geq 0 \) and increases it follows that
\[ \frac{d}{dt} (K_a[\phi_n](t)) \geq \int_{0}^{\infty} K'_1(t, \tau) \phi_n(\tau) d\tau = \int_{0}^{t} K'_1(t, \tau) \phi_n(\tau) d\tau + \int_{t}^{\infty} K'_1(t, \tau) \phi_n(\tau) d\tau \geq \int_{0}^{t} K'_1(t, \tau) \phi_n(\tau) d\tau + \int_{t}^{\infty} K'_1(t, \tau) \phi_n(\tau) d\tau = \phi_n(t) \int_{t}^{\infty} K'_1(t, \tau) d\tau \geq 0 \]

Using that \( \phi_0(t) \) is nonnegative monotonically increasing function, we have
\[ K_a[\phi_0](t_0) \leq K_a[\phi_0](t_1), \quad t_0 \leq t_1 \]
or
\[ a\phi_1^3(t_0) + (1 - a)\phi_1(t_0) \leq a\phi_1^3(t_1) + (1 - a)\phi_1(t_1). \]

It follows from the inequality above that \( \phi_1(t_0) \leq \phi_1(t_1) \) for \( t_0 \leq t_1 \). Repeating this argument \( n \) times we find that \( \phi_{n+1}(t_0) \leq \phi_{n+1}(t_1) \), for \( t_0 \leq t_1 \). We thus have shown that the iterations \( \{\phi_{n+1}\} \) are a sequence of monotonically increasing functions, therefore there is a limit [14]

\[ \lim_{n \to \infty} \phi_n(t) = f(t). \]  

(19)

Taking the limit as \( n \to \infty \) in (17) and using the Lebesque theorem [15, 16] we obtain equation

\[ af^3 + (1 - a)f - K_a f = 0, \]  

(20)

where \( f \in L_{\infty}[0, \infty] \). Thus the function \( f \) is a solution of equation (13). The function \( f \) is bounded, \( \phi_n \leq 1 \), hence by theorem 2 the function is continuous. Thus we have proved that our iterative procedure converges to continuous solution \( f \). The function \( f \) is monotonically increasing (since all \( \phi_n \) are monotonically increasing) and is bounded \( 0 \leq f(t) \leq 1 \), so there exists a limit \( \lim_{t \to +\infty} f(t) \). Hence the function \( f \) is bounded from below by initial iteration \( \phi_0 \), and from above by one, \( \phi_0 \leq f \leq 1 \) and \( \lim_{t \to +\infty} \phi_0 = 1/2 \).

Now taking into account theorem 4, \( \lim_{t \to +\infty} f(t) = 1 \). Thus the function \( f(t) \) is the solution of the boundary value problem.

We have thus proved that iteration process (13) converges to a continuous solution of the boundary problem (12) and (2).

In conclusion, we have investigated properties of the integral equation (1) with infinitely many derivative, we have constructed an iterative method for solving it and have proved its convergence.

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