Robust adaptive variable selection in ultra-high dimensional linear regression models

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\section*{ABSTRACT}
We consider the problem of simultaneous variable selection and parameter estimation in an ultra-high dimensional linear regression model. The adaptive penalty functions are used in this regard to achieve the oracle variable selection property with simpler assumptions and lesser computational burden. Noting the non-robust nature of the usual adaptive procedures (e.g. adaptive LASSO) based on the squared error loss function against data contamination, quite frequent with modern large-scale data sets (e.g. noisy gene expression data, spectra and spectral data), in this paper, we present a new adaptive regularization procedure using a robust loss function based on the density power divergence (DPD) measure under a general class of error distributions. We theoretically prove that the proposed adaptive DPD-LASSO estimator of the regression coefficients is highly robust, consistent, asymptotically normal and leads to robust oracle-consistent variable selection under easily verifiable assumptions. Numerical illustrations are provided for the mostly used normal and heavy-tailed error densities. Finally, the proposal is applied to analyse an interesting spectral dataset, in the field of chemometrics, regarding the electron-probe X-ray microanalysis (EPXMA) of archaeological glass vessels from the 16th and 17th centuries.

\section*{1. Introduction}
Let us consider the standard linear regression model (LRM) given by

\[ y = X\beta + \epsilon, \]

where \( y = (y_1, \ldots, y_n)^T \) denotes the vector of observations from a response variable \( Y \), \( X = (x_1, \ldots, x_n)^T \) is the design matrix containing associated values of the explanatory variable \( X = (X_1, \ldots, X_p) \in \mathbb{R}^p \), \( \beta = (\beta_1, \ldots, \beta_p) \) is the regression coefficient vector, and \( \epsilon = (\epsilon_1, \ldots, \epsilon_n)^T \) follows \( n \)-variate normal distribution with mean vector \( 0_n \) and variance–covariance matrix \( \sigma^2 I_p \). High-dimensional statistics under the LRM (1) refers to the situation where the number of unknown parameters \( (p) \) is of much larger order than

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the sample size \(n\). Here, we consider the case where \(p\) grows exponentially with \(n\), i.e. \(p = O(e^{nl})\) for some \(l \in (0, 1)\); such case is often referred to as ultra-high dimensional or of non-polynomial dimensionality.

In recent years, several statistical methods, algorithms and theories have been developed to perform high-dimensional data analysis. Penalized least square methods have become popular for the high-dimensional LRM since Tibshirani [1] introduced the least absolute shrinkage and selection operator (LASSO) estimate. Under the LRM (1), the LASSO estimate of \(\beta\) is defined as

\[
\beta_L = \beta_L(\lambda) = \underset{\beta \in \mathbb{R}^p}{\arg \min} \left( \|y - X\beta\|_2^2 + \lambda \|\beta\|_1 \right),
\]

where \(\|\beta\|_1 = \sum_{i=1}^{p} |\beta_i|\) is the \(\ell_1\)-norm, \(\|y - X\beta\|_2^2 = \sum_{i=1}^{n}(y_i - x_i^T \beta)^2\) is the least-squares loss (\(\ell_2\)-norm), and \(\lambda > 0\) is a (penalty) regularization parameter. This estimator is proved to perform variable selection as some components of \(\hat{\beta}\) become exactly zero depending on the choice of \(\lambda\).

Let \(\beta_0 = (\beta_{01}, \ldots, \beta_{0p})^T\) be the true parameters in the LRM (1). We denote by \(S_0 = \{j : \beta_{0j} \neq 0\}\) the subset of true non-zero coefficients and assume that the cardinality of \(S_0\) is \(s_0 \ll n\) (sparsity condition). Let \(\hat{\beta}\) be an estimator of \(\beta\), we say that a procedure (or estimator) enjoys variable selection consistency, if \(\{j : \hat{\beta}_j \neq 0\} = S_0\) with probability tending to one. It is well known that the LASSO tends to provide biased estimate of regression coefficients and also requires strong conditions (see, e.g. [2]) for model selection consistency, which often do not hold in practice [3,4].

To overcome the drawback with the oracle properties, Fan and Li [3] proposed the use of some general non-concave penalty function, specifically the SCAD penalty, instead of LASSO. However, such non-concave penalties increase the computational burden significantly as the dimension (\(p\)) increases. Alternatively, Zou [4] introduced a computationally faster adaptive version of the LASSO estimator, which assigns different weights to different coefficients within the \(\ell_1\)-penalty. Precisely, the adaptive LASSO estimator of \(\beta\) in the LRM (1) is defined as

\[
\hat{\beta}_{AL} = \hat{\beta}_{AL}(\lambda) = \underset{\beta \in \mathbb{R}^p}{\arg \min} \left( \|y - X\beta\|_2^2 + \lambda \sum_{j=1}^{p} \frac{|\beta_j|}{|\beta_j| + \delta_n I(\beta_j = 0)} \right),
\]

where \(\tilde{\beta}_j\) is any (initial) consistent estimator of \(\beta_j\) for \(j = 1, \ldots, p\), black and \(\delta_n\) is a very small positive number chosen to avoid division by zero. Note that, for any fixed \(\lambda\), the components having non-zero initial estimates get relatively lower penalty than the zero (initial) components (for which the penalty weight goes to infinity). Consequently, the adaptive LASSO estimator is able to reduce the estimation bias and improves variable selection accuracy. For fixed \(p\), Zou [4] proved that the adaptive LASSO has the oracle property and for \(p > n\), Huang et al. [5] established that, under the partial orthogonality and certain other conditions, the adaptive LASSO estimators are consistent and also efficient when the marginal regression estimators are used as the initial estimators. The convexity of adaptive LASSO criterion ensures that these desired properties are global instead of local and further this estimator can be easily computed using the same efficient algorithms that are used for LASSO, namely the least angle regression (LARS Efron et al. [6]).

However, the above methods are based on the squared-error loss function whose lack of robustness is well known. Outlying values of \(x_i\) (leverage point) or extreme values of
(\(x_i, Y_i\)) (influence points) have significant influence in the regularization procedures based on \(\ell_2\) loss function. For this purpose, the penalized M-estimators have been developed, which replaces the loss function by a convex function of the form \(\sum_{i=1}^{n} \rho(y_i - x_i^T \beta)\). Wang et al. [7] proposed the least absolute deviation (LAD) loss with \(\rho(x) = |x|\) along with the LASSO penalty (LAD-LASSO method). This LAD loss does not work well for small errors as it strongly penalizes the small residuals (Owen [8] and Lambert-Lacroix and Zwald [9]). Lambert-Lacroix and Zwald [9] alternatively proposed using the so-called Huber loss, given by

\[
\rho(x) = \begin{cases} 
  x^2, & |x| \leq M \\
  2M |x| - M^2, & |x| > M 
\end{cases}
\]

This function is quadratic in small values of \(x\) but grows linearly for large values of \(x\), and a data-adaptive choice of \(M\) was discussed in Fan et al. [10]. Robust LASSO with Huber’s loss function (Huber [11]) is robust to outliers in the response variable but not for leverage points (outliers in the covariates). Arslan [12] presented a weighted version of the LAD-LASSO method which is also robust with respect to the leverage points. Other approaches that are robust with respect to leverage point were presented in Khan et al. [13], Li et al. [14] and Alfons et al. [15]. In Chang et al. [16], a penalized method is presented using Tukey’s biweight loss that is resistant to outliers in both the response and the covariates. The theory of adaptive LASSO along with the quantile regression loss function has been developed in Fan et al. [17] for the ultra-high dimensional set-up. Qi et al. [18] considered, to gain robustness and efficiency simultaneously, a data-driven convex combination of adaptive LASSO and LAD-LASSO methods.

Recently, Zhang et al. [19] and Ghosh and Majumdar [20] have considered a robust loss function based on the density power divergence (DPD) of Basu et al. [21] along with grouped LASSO penalty and the general class of non-concave penalty functions, respectively. This DPD-based loss function is seen to provide significantly improved performance, with better trade-offs between efficiency and robustness, under classical low-dimensional set-ups (see, e.g. Basu et al. [22], Ghosh and Basu [23], among many others) as well as in high-dimensional set-up with non-concave penalties (Ghosh and Majumdar [20]). In brief, the great properties of the DPD that led us to choose this particular loss function in the present paper include its high-robustness with only a small loss of efficiency, bounded influence functions with decreasing gross-error sensitivities, simple interpretation as an intuitive robust generalization of the classical MLE and the possibility to avoid the complications of non-parametric smoothing while still using a density-based divergence; see Section 2 for some brief descriptions and refer to Basu et al. [22] for further details and examples. Further, it has also been shown that the minimum DPD estimator of the regression coefficient can achieve 50% asymptotic breakdown point under appropriate conditions for the low-dimensional linear regression models and are robust with respect to both outliers and leverage points (Ghosh and Basu [23]). Such strong robustness properties of the DPD-based loss function are also seen to translate for high-dimensional regression models in Zhang et al. [19], Ghosh and Majumdar [20] and Ghosh [24].

Although the general non-concave penalized DPD-based procedure (DPD-ncv) of Ghosh and Majumdar [20] is seen to yield significantly improved robust and efficient performance both in terms of variable selection and parameter estimation and satisfies the oracle variable selection property under appropriate conditions, the method appears
to be computationally challenging in ultra-high dimensional set-ups with growing number of covariates. A typical way to address the computational challenges in ultra-high dimensional scenarios involves employing a two-step process that combines a screening procedure with variable selection. However, this approach sacrifices the inherent advantages of regularization methods, namely the simultaneous robust variable selection and parameter estimation, and may discard truly related variables due to sample contamination in the screening step (suitable robust screening method, e.g. [25], may be used to avoid the effects of contamination in the screening stage but it would add own computational challenges over the usual simple screening). More importantly, screening methods may have a higher bias than regularization methods, affecting the asymptotic properties of the final parameter estimates if the screening procedure is employed before applying the regularized estimation and selection procedure.

Therefore, the main purpose of the present paper is to develop a robust yet simpler theoretically sound regularization procedure for ultra high-dimensional LRM (1) that would be able to produce competitive estimation and variable selection performances as the DPD-ncv procedure with lower computational costs. For this purpose, we would stick with the DPD-based loss function but considering an appropriately adaptive penalty function. This is motivated by the fact that the classical adaptive procedures significantly reduce the computational burden compared to a non-concave penalized procedures. So, in this paper, we combine the advantages from these two fronts, the adaptive penalty and the robust DPD-based loss function, to present a new robust adaptive regularization procedure under the ultra-high dimensional LRM which will have the desired oracle property. In the following, we summarize the main contributions of the present paper, highlighting its difference from the closely related work of [20], as follows:

- Firstly, the present work differs from the work of [20] mainly in the choice of the penalty function, which greatly simplifies the assumptions required for theoretical derivations as well as the practical computations. We use a general class of data-driven adaptive penalty functions instead of non-concave penalties so the stochastic nature of the penalty made our procedures much simple without significance compromise in the model selection and estimation performances. As a particular case of our proposed robust adaptive procedures, we additionally provide a computationally faster approximation to the DPD-ncv method of [20].

- Secondly, although the methodological idea is a simple modification from [20], the theoretical derivations of the oracle consistency, asymptotic distributions and the robustness of the proposed procedure required extensive non-trivial extensions of the existing literature, which are the major contributions of the present paper. The required assumptions in our theoretical derivations are much weaker than those used in [20] for LASSO penalty and, hence, they hold more easily in real-life applications.

- Finally, we have provided extensive empirical illustrations comparing the performance and computational costs (run times) of our proposed procedures with the DPD-ncv from [20] and other existing robust and non-robust model penalized procedures. It has been illustrated in detail that our simple and computationally efficient proposal of adaptively penalized DPD-based estimates of the regression coefficients performs well under data contaminations (competitive to the computationally extensive and complex DPD-ncv method of [20]).
Therefore, in terms of both theoretical and computational aspects, the present work provides a significant improvement over the existing work of [20] by considering (data) adaptive penalty function in defining the objective function along with the robust DPD-based loss function.

At this point, we would like to emphasis that we are considering robustness against pre-specified parametric family of model distributions and not non-parametric robustness with fully unspecified model distribution. That is, we always assume a known model distribution, the error distribution in the LRM (1) in the present paper, that is followed by the majority of the observed data and talk about robust estimation and variable selection against contamination in parts of the data (e.g. outliers). Such parametrically robust procedures provide significantly improved efficiency compared to the fully non-parametric methods when a parametric model can be assumed for the majority of the data except possibly for a part of it being misspecified or containing outliers; we refer readers to Basu et al. [22] for more details about this approach of robust and (highly) efficient parametric inference. Accordingly, our proposed estimation and variable section procedure would be dependent on the pre-assumed form of the error density in the LRM (1); we will first start the description of our proposal by assuming normal error density but subsequently extend the definitions for general error distributions from a location-scale family (where the density still have to be pre-specified) in Section 2. As the DPD loss function is already seen to provide better trade-off between efficiency and robustness compared to other such parametrically robust procedures [19,20,24], in the present paper, we mainly compare the performance of DPD-based loss function with different choices of penalty function to propose and justify a computationally simpler approximation to the general non-concave penalties as mentioned above.

The rest of the paper is organized as follows: We start with the definition of the DPD loss function and the corresponding robust regularized estimator with adaptively weighted penalty functions in the LRM Section 2. The robustness of the proposed adaptive procedure is examined theoretically via its influence function analysis Section 3. Further, large sample oracle model selection property of our proposed estimator is derived along with its consistency and asymptotic normality Section 4. Subsequently, an efficient computational algorithm has been discussed Section 5 and the finite-sample performance of the proposal is illustrated through simulation studies, comparing its performance with that of other existing robust and non-robust regularization methods Section 6. Finally, the proposed method is applied to a real high-dimensional data set from the field of chemometrics, highlighting the advantage of our proposal in the presence of leverage points Section 7. For brevity, the proofs of all the results, details of the computational algorithm and additional numerical results are presented in the Appendix, available in the Online Supplementary Material.

2. The proposed estimation procedure based on the density power divergence loss and the adaptive LASSO penalty

The DPD family represents a rich class of density-based divergences. Given two densities $g$ and $f$ with respect to some common dominating measure, the DPD between them is
defined, as a function of a non-negative tuning parameter $\gamma$, as

$$d_\gamma(g, f) = \begin{cases} \int \left( f^{1+\gamma}(x) - \left( 1 + \frac{1}{\gamma} \right) f^\gamma(x) g(x) + \frac{1}{\gamma} g^{1+\gamma}(x) \right) \, dx, & \text{for } \gamma > 0, \\ \int g(x) \log \frac{g(x)}{f(x)} \, dx, & \text{for } \gamma = 0. \end{cases}$$

(4)

The quantities defined in Equation (4) are genuine divergences in the sense that $d_\gamma(g, f) \geq 0$ for any densities $g$ and $f$ and all $\gamma \geq 0$, and $d_\gamma(g, f)$ is equal to zero if and only if the densities $g$ and $f$ are identically equal.

For a brief background, let us consider the parametric model family of densities $\{\theta : \theta \in \Theta \subset \mathbb{R}^p\}$ as the model for a population having true density $g$ and true distribution function $G$. The minimum DPD functional $T_\gamma(G)$ at $G$ is defined as $d_\gamma(g, f_{T_\gamma(G)}) = \min_{\theta \in \Theta} d_\gamma(g, f_\theta)$. Clearly, the third term $\int g^{1+\gamma}(x) \, dx$ has no role in the minimization of $d_\gamma(g, f_\theta)$ over $\theta \in \Theta$. Thus, the essential objective function for the minimum DPD functional $T_\gamma(G)$ reduces to

$$\int \left( f^{1+\gamma}_\theta(x) - \left( 1 + \frac{1}{\gamma} \right) f^\gamma_\theta(x) g(x) \right) \, dx = \int f^{1+\gamma}_\theta(x) \, dx - \left( 1 + \frac{1}{\gamma} \right) \int f^\gamma_\theta(x) \, dG(x).$$

(5)

Thus, given a random sample $y_1, \ldots, y_n$ from the distribution $G$, we can estimate the above objective function by replacing $G$ with its empirical estimate $G_n$. For a given tuning parameter $\gamma$, therefore, the minimum DPD estimator (MDPDE) $\hat{\theta}_\gamma$ of $\theta$ can be obtained by minimizing the objective function

$$H_{n, \gamma}(\theta) = \int f^{1+\gamma}_\theta(x) \, dx - \left( 1 + \frac{1}{\gamma} \right) \int f^\gamma_\theta(x) \, dG_n(x)$$

$$= \int f^{1+\gamma}_\theta(x) \, dx - \left( 1 + \frac{1}{\gamma} \right) \frac{1}{n} \sum_{i=1}^n f^\gamma_\theta(y_i)$$

(6)

over $\theta \in \Theta$. Importantly, unlike many other divergence-based estimation procedures, the minimization problem associated with the MDPDE does not require the use of a non-parametric density estimate.

Durio and Isaia [26] extended the concept of the MDPDE to the problem of robust estimation in the LRM with $n > p$ (low-dimensional settings). For such classical LRM, $f_\theta(y_i)$ is a normal density with mean $x_i^T \beta$ and variance $\sigma^2$. It is a simple exercise to establish that, in this case the corresponding DPD loss function has the form

$$L_{n, \gamma}(\beta, \sigma) = \frac{1}{(2\pi)^{p/2} \sigma^\gamma} \left( \frac{1}{\sqrt{\gamma + 1}} - \frac{\gamma + 1}{\gamma} \frac{1}{n} \sum_{i=1}^n \exp \left\{ -\gamma \left( \frac{y_i - x_i^T \beta}{2\sigma^2} \right)^2 \right\} \right) + \frac{1}{\gamma}.$$  

(7)

The term $1/\gamma$ has been introduced to get the log-likelihood function as a limiting case as $\gamma \downarrow 0$, and therefore the corresponding MDPDE (at $\gamma = 0$) is nothing but the usual maximum likelihood estimator (MLE). When both $Y$ and $X$ are random but we only assume the parametric model for the conditional distribution of $Y$ given $X$, the loss function $L_{n, \gamma}(\beta, \sigma)$ in (7) can be seen as an empirical estimate of the expectation (with respect to the unknown
covariate distribution) of the DPD objective function (5) between the conditional data and model densities of \(Y\) given \(X\). However, under the fixed design setup with non-stochastic covariates \(x_i, i = 1, \ldots, n\), the only random observations \(y_1, \ldots, y_n\) are independent but non-homogeneous. Ghosh and Basu [23] studied this problem and suggested to minimize the average DPD measure between the data and the model densities for each given covariate value. Interestingly, this approach also leads to the same loss function given in (7). Therefore the loss function \(L_{n, \gamma}(\beta, \sigma)\) in (7), referred to as the DPD loss function (with tuning parameter \(\gamma\)), can be used to obtain robust MDPDE under the LRM (1) with normal errors for both stochastic and fixed design matrices.

In this paper, we first consider the penalized objective function constructed with the DPD-loss function and the standard adaptive lasso penalty from (3), as given by

\[
Q_{n, \gamma, \lambda}(\beta, \sigma) = L_{n, \gamma}(\beta, \sigma) + \lambda \sum_{j=1}^{p} \frac{|\beta_j|}{|\hat{\beta}_j| + \delta_n I(\hat{\beta}_j = 0)},
\]

where \(\lambda\) is the usual regularization parameter and \(\gamma \geq 0\) is a robustness tuning parameter. As we will show later, we need this initial estimator \(\hat{\beta}_j\) to be robust to achieve the robustness of the final estimator, say \((\hat{\beta}_{\gamma, \lambda}, \hat{\sigma}_{\gamma, \lambda})\), obtained by minimizing \(Q_{n, \gamma, \lambda}(\beta, \sigma)\). We will refer to these final estimators \((\hat{\beta}_{\lambda, \gamma}, \hat{\sigma}_{\lambda, \gamma})\) as the adaptive DPD-LASSO (Ad-DPD-LASSO) estimator of \((\beta, \sigma)\).

Note that, we consider the minimization of \(Q_{n, \gamma, \lambda}(\beta, \sigma)\) with respect to both \(\beta\) and \(\sigma\) for their simultaneous estimation; the additional adaptive LASSO penalty on \(\beta\) helps sparse selection of the components of \(\beta\) with oracle (variable selection) consistency. At \(\gamma = 0\), the Ad-DPD-LASSO leads to the non-robust penalized MLEs of \(\beta\) and \(\sigma\). Furthermore, for known \(\sigma\), it leads to the classical adaptive LASSO in (3). Thus, for \(\gamma > 0\), the minimization of \(Q_{n, \gamma, \lambda}(\beta, \sigma)\) provides a generalization of the adaptive LASSO estimator \(\hat{\beta}_{AL}\) with the additional benefit of robustness against data contamination.

Before starting the theoretical analysis of our proposal, let us further extend its definition by considering the more general class of adaptively weighted LASSO penalty and a location-scale family of error distribution, as in Ghosh and Majumdar [20]. Letting the random errors \(\epsilon_i\)'s in the model (1) be independent and identically distributed having a location-scale density of the form \(\frac{1}{\sigma} f \left( \frac{\epsilon}{\sigma} \right)\) with location 0 and scale \(\sigma\) (\(f\) being an univariate distribution with mean 0 and variance 1), the corresponding DPD loss function is given by (generalizing from (7))

\[
L_{n, \gamma}(\beta, \sigma) = \frac{1}{\sigma^{\gamma}} \left( M_f^{\gamma_{\beta}} - \frac{1 + \gamma}{\gamma} \frac{1}{n} \sum_{i=1}^{n} f^{\gamma_{\beta}} \left( \frac{y_i - x_i^T \beta}{\sigma} \right) \right) + \frac{1}{\gamma},
\]

where \(M_f^{\gamma_{\beta}} = \int f(\epsilon)^{1+\gamma} d\epsilon\) is assumed to exist finitely. Then, the generalized adaptively weighted DPD-LASSO (AW-DPD-LASSO) estimators of \((\beta, \sigma)\) is the minimizer of

\[
Q_{n, \gamma, \lambda}(\beta, \sigma) = L_{n, \gamma}(\beta, \sigma) + \lambda \sum_{j=1}^{p} w(|\hat{\beta}_j|) |\beta_j|,
\]

with \(L_{n, \gamma}(\beta, \sigma)\) being now given by (9) and \(w\) is a suitable weight function. Note that, the general objective function (10) coincides with (8) for a hard-thresholding weight function.
\( w(s) = (s + \delta_n I(s = 0))^{-1} \) for some very small positive number \( \delta_n \) and standard normal error density. Although the weights are generally assumed to be stochastic depending on the initial estimators, they could also be non-stochastic; for example, if \( w(s) = 1 \) for all \( s \), the general objective function (10) coincides with that of the DPD-LASSO.

In the rest of the paper, unless otherwise mentioned, we will denote by \( L_{n, \gamma}(\beta, \sigma) \) and \( Q_{n, \gamma, \lambda}(\sigma, \beta) \) the generalized quantities in (9) and (10), respectively, and derive the theoretical results for the general class of AW-DPD-LASSO estimators under the ultra-high dimensional set-up. The simplification of all the results for the Ad-DPD-LASSO estimator (and some others) will also be provided as special cases. All the theoretical results presented here are valid for any general location-scale family of error distributions as long as it satisfies the associated assumptions.

**Remark 2.1 (Connection with the Non-concave penalized DPD estimators):** Given a penalty function \( p_\lambda(\cdot) \), with regularization parameter \( \lambda \), the DPD-ncv estimator is the minimizer of the penalized objective function \( L_{n, \gamma}(\beta, \sigma) + \lambda \sum_{j=1}^{p} p_\lambda(|\beta_j|) \), which is computationally challenging in higher dimensions. However, in view of our general AW-DPD-LASSO estimator and its objective function (10), we can obtain an easily computable approximation of the DPD-ncv estimator. Following the idea from Zou and Li [27] and Fan et al. [17], given a good initial estimator \( \tilde{\beta} \), we can use the following approximation:

\[
p_\lambda(|\beta_j|) \approx p_\lambda(|\tilde{\beta}_j|) + p_\lambda'(|\tilde{\beta}_j|)(|\beta_j| - |\tilde{\beta}_j|).
\]

Therefore, an AW-DPD-LASSO estimator with weight function \( w = p_\lambda' \), which is much easier to compute even in ultra-high dimension, is expected to work as a substitute for the corresponding DPD-ncv estimator. We will verify its performance both theoretically and empirically in the subsequent sections for the popular SCAD penalty function, for which

\[
w(s) = p_\lambda'(s) = I(s \leq \lambda) + \frac{(a\lambda - s)^+}{(a - 1)\lambda} I(s > \lambda), \tag{11}
\]

with \( a > 2 \) being a tuning constant suggested from the common literature as \( a = 3.7 \). We will refer to (11) as the SCAD weight function.

### 3. Robustness: influence function analyses

The influence function (IF) is a classical tool for measuring (local) robustness which indicates the possible asymptotic bias in the estimation functional due to an infinitesimal contamination (Hampel et al. [28]). The concept has been extended rigorously for the penalized estimators by Avella-Medina [29], and this extended IF has been further used in the high-dimensional context by Ghosh and Majumdar [20]. Here, we will derive the IF for our AW-DPD-LASSO estimators \( (\hat{\beta}_{\gamma, \lambda}, \hat{\sigma}_{\gamma, \lambda}) \), to examine their theoretical (local) robustness against data contamination.

In order to define the IF, we first extend the definition of the AW-DPD-LASSO estimator as a statistical functional. Assuming the true joint distribution of \( (Y, X) \), to be \( G(y, x) \), the statistical functional \( T_{\gamma, \lambda}(G) = (T_{\gamma, \lambda}^\beta(G), T_{\gamma, \lambda}^\sigma(G)) \) corresponding to \( (\hat{\beta}_{\gamma, \lambda}, \hat{\sigma}_{\gamma, \lambda}) \) is
where they possibly depend on the functional 

\[ T(\theta) = \left( \theta_1, \ldots, \theta_p \right) \]

with respect to \( \theta = (\beta, \sigma) \), where \( U(G) = (U_1(G), \ldots, U_p(G)) \) is the statistical functional corresponding to the initial estimator \((\tilde{\beta}_j)_{j=1,\ldots,p} \) and

\[ L^*_\gamma((y, x); \beta, \sigma) = \frac{1}{\sigma^\gamma} \left( M_f(y) - \frac{1 + \gamma}{\gamma} f(y - x^T \beta) \right) + \frac{1}{\gamma}. \]

It is straightforward that the objective function in (12) coincides with the empirical objective function (10) when \( G \) is substituted by \( G_n \), and hence \((T_{\gamma, \lambda}^{\beta}(G_n), T_{\gamma, \lambda}^{\sigma}(G_n)) = (\tilde{\beta}_{\gamma, \lambda}, \tilde{\sigma}_{\gamma, \lambda})\).

Note that, by definition, our AW-DPD-LASSO estimator also belongs to the class of M-estimators considered in Avella-Medina [29], with their \( L(Z, \theta) \) function coinciding with our \( L^*_\gamma((Y, X); \theta) \). We will apply their extended definition of IF to derive the IF of our estimator. The major problem for extending the theory of IFs is the non-differentiability of the penalty function at zero (for most common weight function including the one for adaptive LASSO); Following Avella-Medina [29], we can rather consider a sequence of continuous and infinitely differentiable penalties, \( \{p_{m, \lambda}(s, t(G))\}_{m \geq 1} \), which converges to our adaptively weighted LASSO penalty \( \lambda w(t(G)) | s | \) in the Sobolev space as \( m \to \infty \); note that they possibly depend on the functional \( t(G) = U_j(G) \), the \( j \)th component of the initial estimator \( U(G) \). Correspondingly, we define the statistical functionals \( T_{m, \gamma, \lambda}(G) = (T_{m, \gamma, \lambda}^{\beta}(G), T_{m, \gamma, \lambda}^{\sigma}(G)) \), for \( m = 1, 2, \ldots \), as the minimizer of

\[
Q_{m, \gamma, \lambda}(\beta, \sigma) = \int L^*_\gamma((y, x); \beta, \sigma) \, dG(y, x) + \sum_{j=1}^p p_{m, \lambda}(\beta_j, U_j(G)),
\]

with respect to \( \theta = (\beta, \sigma) \). One can then define the IF of the AW-DPD-LASSO estimator, at the contamination points \((y_t, x_t)\), as the limit of the IFs of \( T_{m, \gamma, \lambda}(G) \) as

\[
\mathcal{IF}((y_t, x_t), T_{\gamma, \lambda}, G) = \lim_{m \to \infty} \mathcal{IF}((y_t, x_t), T_{m, \gamma, \lambda}, G).
\]

Now, to compute the IFs of \( T_{m, \gamma, \lambda}(G) \) at the contamination points \((y_t, x_t)\), we consider its estimating equation obtained by equating the derivatives of \( Q_{m, \gamma, \lambda}(\beta, \sigma) \), with respect to the parameters \( \theta = (\beta, \sigma) \), to zero. Then, by some standard calculations, these estimating equations can be simplified as

\[
\begin{align*}
\frac{(1 + \gamma)}{\sigma^\gamma + 1} \int \psi_{1, \gamma}(\frac{y - x^T \beta}{\sigma}) \, dG(y, x) + P_{m, \lambda}^*(\beta, U(G)) &= 0_p, \\
\frac{(1 + \gamma)}{\sigma^\gamma + 1} \int \psi_{2, \gamma}(\frac{y - x^T \beta}{\sigma}) \, dG(y, x) &= 0.
\end{align*}
\]

where \( \psi_{1, \gamma}(s) = u(s)f^\gamma(s), \psi_{2, \gamma}(s) = [su(s) + 1]f^\gamma(s) - \frac{\gamma}{\gamma + 1} M_f((\gamma)) \), \( u = f'/f \) with \( f' \) denoting the derivative of \( f \) and \( P_{m, \lambda}^*(\beta, U(G)) \) is a \( p \)-vector having \( j \)th element as \( \frac{\partial}{\partial \beta_j} p_{m, \lambda}(\beta_j, U_j(G)) \). Now, we substitute the contaminated distribution \( G_e = (1 - \epsilon)G + \)
Theorem 3.1: Consider the above-mentioned set-up with the general error density \( f \) and the true parameter value \( \theta^g = (\beta^g, \sigma^g) = T_{\gamma, \lambda}(G) \), where \( \beta^g \) is sparse with only \( s(< n) \) non-zero components (recall \( p \gg n \)). Without loss of generality, assume \( \beta^g = (\beta_1^g T, 0_{p-s})^T \), where

\[
\epsilon \wedge (y_t, x_t), \text{ with } \epsilon \text{ and } \wedge (y_t, x_t) \text{ being the contamination proportion and the degenerate contamination distribution at } (y_t, x_t), \text{ respectively, in place of } G \text{ in the above estimating Equation (16) and differentiate with respect to } \epsilon \text{ at } \epsilon = 0. \text{ Collecting terms after some algebra, and assuming all the relevant integrals exist finitely, we get the influence function of } T_{m, \gamma, \lambda} \text{ as given by }

\[
\mathcal{I} \mathcal{F}((y_t, x_t), T_{m, \gamma, \lambda}, G) = -(f^*)^{-1} \left[ \frac{(1 + \gamma)}{\sigma^{1 + \gamma}} \psi_{1, \gamma} \left( \frac{y_t - x_t^T \beta}{\sigma} \right) x_t + P_{m, \lambda}^*(\beta, U(G)) \right].
\]

where \( \mathcal{I} \mathcal{F}((y_t, x_t), U, G) \) denote the IF of the initial estimator \( U \), \( P_{m, \lambda}^*(\beta, U(G)) \) is a \( p \times p \) diagonal matrix with \( j \)-th diagonal being the derivative of the \( j \)-th component of \( P_{m, \lambda}^*(\beta, U(G)) \) with respect to its \( k \)-th argument for \( k = 1, 2 \), and the \( (p + 1) \times (p + 1) \) matrix \( J^*_\gamma = J^*_\gamma(G; \beta, \sigma) = J^*_\gamma(G; \beta, \sigma) + \text{diag}(P_{m, \lambda}^*(\beta, U(G)), 0) \) with

\[
J^*_\gamma(G; \beta, \sigma) = E_G \left[ \frac{\partial^2 L^*_\gamma((y, x); \beta, \sigma)}{\partial \beta \partial \beta^T} \right] = -(1 + \gamma) \sigma^{1 + \gamma} \left[ \begin{array}{ccc}
J_{11, \gamma} \left( \frac{y - x^T \beta}{\sigma} \right) xx^T & J_{12, \gamma} \left( \frac{y - x^T \beta}{\sigma} \right) x \\
J_{12, \gamma} \left( \frac{y - x^T \beta}{\sigma} \right) x^T & J_{22, \gamma} \left( \frac{y - x^T \beta}{\sigma} \right)
\end{array} \right],
\]

where \( J_{11, \gamma}(s) = \{y u^2(s) + u'(s)f^\gamma(s), J_{12, \gamma}(s) = \{(1 + \gamma)u(s) - \gamma su^2(s) + su'y(s) \text{ and} J_{22, \gamma}(s) = \{(1 + \gamma)(1 + 2su(s) + s^2u'^2(s)f^\gamma(s) - \gamma M_j^\gamma) \text{. Throughout this paper, we will assume that } f \text{ is such that } f(s) > 0 \text{ for all } s \text{ and } i = 1, 2 \).

It has been shown in Proposition 1 of Avella-Medina [29] that under certain conditions including the existence of the above IF of \( T_{m, \gamma, \lambda} \) compactness of the parameter space \( \Theta \) and the continuity of the relevant functions in \( \theta = (\beta, \sigma) \), the limit of \( \mathcal{I} \mathcal{F}((y_t, x_t), T_{m, \gamma, \lambda}, G) \) exists as \( m \rightarrow \infty \) and the limit is also independent of the choice of the differentiable penalty sequence \( p_{m, \lambda}(s) \). Therefore, we can uniquely define the IF of our AW-DPD-LASSO estimator \( T_{\gamma, \lambda} \) by (15) with any appropriate penalty sequence and the resulting IF will, in fact, be the distributional derivative of \( T_{\gamma, \lambda}(G_{\epsilon}) \) with respect to \( \epsilon \) at \( \epsilon = 0 \). So, we take a particular choice of differentiable penalty functions as \( p_{m, \lambda}(s, U_j(G)) = \lambda w(h_m(U_j(G)))h_m(s) \) where the infinitely differentiable function

\[
h_m(s) = \frac{2}{m} \log(e^{sm} + 1) - s \rightarrow |s|, \text{ as } m \rightarrow \infty.
\]

This particular choice of \( p_m \) satisfies all the required regularity conditions for convergence, stated in Avella-Medina [29], of the sequence of IFs (of \( T_{m, \gamma, \lambda} \)) to the unique limiting IF (of \( T_{\gamma, \lambda} \)). Therefore, calculating the IFs of \( T_{m, \gamma, \lambda} \) with this particular penalty function for each \( m \) and taking limit as \( m \rightarrow \infty \), we have derived the IF of our AW-DPD-LASSO estimator \( T_{\gamma, \lambda} \) which is presented in the following theorem.
$\beta_1^s$ contains all and only $s$-non-zero elements of $\beta^s$. Let us denote $x_1, x_{1, t}$ and $\beta_1$ to be the $s$-vectors of the first $s$ elements of the $p$-vectors $x, x_1$ and $\beta$, respectively, and the corresponding partition of our functional as $T_{y, \lambda}(G) = (T_{y, \lambda}^1(G), T_{y, \lambda}^2(G), T_{y, \lambda}^\sigma(G))^T$. Then, whenever the associated quantities exist finitely, the influence function of $T_{y, \lambda}^1$ is identically zero at the true distribution $G$ and that of $(T_{y, \lambda}^2, T_{y, \lambda}^\sigma)$ at $G$ is given by

$$
\mathcal{I}(y_1, x_t, (T_{y, \lambda}^1, T_{y, \lambda}^\sigma), G) = -S_{y}^{-1} \left[ \frac{(1+\gamma)}{(\sigma^8)^{\gamma+1}} \psi_{1, y} \left( \frac{y_1-x_1^T \beta^s}{\sigma^8} \right) x_{1, t} + \lambda P^*(\beta, U(G)) \right],
$$

where $U^{(1)}$ is the first $s$ elements of $U, P^*(\beta, U(G))$ is an $s-$dimensional vector having $j$th element as $w((U_j(G)) \| \beta_j)$ for $j = 1, \ldots, s$ and $P^*(2)(\beta, U(G))$ is an $s \times s$ diagonal matrix having $j$th diagonal entry as $w'(([U_j(G)] \| \beta_j))$ for $j = 1, \ldots, s$, with $w'(s)$ denoting the derivative of $w(s)$ in $s$, and the $(s+1) \times (s+1)$ matrix $S_{y} = S_y(G; \beta, \sigma)$ is defined as

$$
S_y(G; \beta, \sigma) = -\frac{(1+\gamma)}{\sigma^y+2} E_G \left[ \begin{array}{cc} J_{11, y} \left( \frac{y-x^T \beta}{\sigma} \right) x_1 x_1^T & J_{12, y} \left( \frac{y-x^T \beta}{\sigma} \right) x_1 \\ J_{12, y} \left( \frac{y-x^T \beta}{\sigma} \right) x_1^T & J_{22, y} \left( \frac{y-x^T \beta}{\sigma} \right) \end{array} \right].
$$

For the LRM in (1) with conditional density of $Y$ given $X = x$ given by $\frac{1}{s} f \left( \frac{y-x^T \beta}{\sigma} \right)$, let us denote the corresponding joint distribution $F(\beta, \sigma)$, which also contains the marginal distribution (say $H$) of $x$ along with the above model conditional distribution. In this case, the matrix $S_y(G; \beta, \sigma)$ can be further simplified as

$$
S_y^{(0)}(G; \beta, \sigma) = -\frac{(1+\gamma)}{\sigma^y+2} \left[ \begin{array}{cc} f_{11, y} E_H \left( x_1 x_1^T \right) & f_{12, y} E_H \left( x_1 \right) \\ f_{12, y} E_H \left( x_1 \right)^T & f_{22, y} \end{array} \right],
$$

where

$$
\begin{align*}
f_{11, y} &= \gamma M_{f, 0, 2}^{(y)} + M_{f, 0}^{(y)*}, \\
f_{12, y} &= (1+\gamma) M_{f, 1, 2}^{(y)} - \gamma M_{f, 1, 1}^{(y)} + M_{f, 1}^{(y)*}, \\
\text{and } f_{22, y} &= 2(1+\gamma) M_{f, 1, 1}^{(y)} + M_{f, 1}^{(y)*} + \gamma M_{f, 2, 2}^{(y)} + M_{f}^{(y)}
\end{align*}
$$

with $M_{f, i, j}^{(y)} = \int f(s) s^{i+y} ds$ and $M_{f, i}^{(y)*} = \int f(s) s^{i+y} ds$ for $i, j = 0, 1, 2$.

In particular, if the error density $f$ satisfies $f_{12, y} = 0$ or $E(x) = 0_p$, then we can separately write down the IFs of $T_{y, \lambda}^1$ and $T_{y, \lambda}^\sigma$ at the model $G = F(\beta, \sigma)$ from Theorem 3.1 as given by

$$
\mathcal{I}(y_1, x_t, T_{y, \lambda}^1, F(\beta, \sigma)) = \frac{\sigma}{f_{11, y}^{(0)}} \left[ E_H \left( x_1 x_1^T \right) \right]^{-1} \left[ \psi_{1, y} \left( \frac{y_1-x_1^T \beta_1}{\sigma} \right) x_{1, t} \right]
$$
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\[ + \frac{\lambda \sigma^{\gamma+1}}{(1+\gamma)} \left( P^\ast(\beta, \beta) + P^{n(2)}(\beta, \beta) \mathcal{I}\mathcal{F}(\{y_t, x_t\}, U^{(1)}, F(\beta, \sigma)) \right), \]  

(19)

\[ \mathcal{I}\mathcal{F}(\{y_t, x_t\}, T^\ast_{\gamma, \lambda}, F_\theta) = \frac{\sigma}{J_{22, \gamma}^{(0)}} \psi_{2, \gamma} \left( \frac{y_t - x^T_{t, t} \beta_1}{\sigma} \right), \]  

(20)

since \((\beta^\ast_{1}, \sigma^\ast) = (\beta_1, \sigma)\) at \(G = F(\beta, \sigma)\) and \(U(F(\beta, \sigma)) = \beta\) by its consistency.

It is clear from the above formulas of the IF of the AW-DPD-LASSO estimator that it is expected to be robust having bounded IF if the quantities \(\psi_{1, \gamma}(\{y_t - x^T_{t, t} \beta_1\}/\sigma)x_{1, t}\), \(\psi_{2, \gamma}(\{y_t - x^T_{t, t} \beta_1\}/\sigma)\) and \(\mathcal{I}\mathcal{F}(\{y_t, x_t\}, U^{(1)}, F(\beta, \sigma))\) are bounded in either or both of the contamination point \((y_t, x_t)\). The first two quantities depend directly on the model assumption on the error density \(f\) and the tuning parameter \(\gamma\); for most common densities having exponential structure they can be seen to be bounded in \((y_t, x_t)\) for any \(\gamma > 0\). The last quantity, namely the IF of the initial estimator \(U\), needs also to be bounded and a robust starting point is needed for the generalized adaptive DPD-LASSO. One possible choice could be the DPD-LASSO estimator or other robust M-estimators of regression coefficient from the existing literature.

**Remark 3.1:** If the weights are fixed (non-stochastic) in the definition of AW-DPD-LASSO estimator, there is no involvement of \(U(G)\) in (16). Hence the corresponding IF will not depend on the IF of \(U\) but will coincide with the results of Ghosh and Majumdar [20].

Finally, we simplify Theorem 3.1 to obtain the influence function of the Ad-DPD-LASSO estimator, defined as a minimizer of the simpler objective function in (8), where the error density \(f\) is the standard normal density (so that \(J_{12, \gamma}^{(0)} = 0\) and \(w(s) = (s + \delta_s I(s = 0))\) so that \(w' = -s^{-2}\) whenever \(s \neq 0\). The simplified results are presented in the following corollary; clearly the IF of the Ad-DPD-LASSO estimator is bounded in the contamination points for all \(\gamma > 0\) indicating the claimed robustness of our proposal against (infinitesimal) data contamination in both the response and covariates spaces, provided the initial estimator is chosen robustly.

**Corollary 3.2:** Consider the Ad-DPD-LASSO estimation by the minimization of the objective function in (8) for the LRM in (1) under normal error density. Suppose that true distribution \(F(\beta, \sigma)\) underlying the data satisfies the LRM and the true regression coefficient \(\beta\) is sparse with only \(s < n\) non-zero components (but \(p > n\)). Without loss of generality, assume \(\beta = (\beta^T_1, 0^T_{p-s})^T\), where \(\beta_1\) contains all and only \(s\)-non-zero elements of \(\beta\). Also assume that the initial estimator is consistent in the sense \(U(F(\beta, \sigma)) = \beta\) and denote \(P^0_\ast(\beta) = (|\beta_1|^{-1}, \ldots, |\beta_s|^{-1})\) and \(P^{n(2)}_0(\beta) = \text{diag}(\beta_1^{-2}, \ldots, \beta_s^{-2})\). If the functional corresponding to the Ad-DPD-LASSO estimator is partitioned as \(T^\beta_{\gamma, \lambda}(G) = (T^\beta_{\gamma, \lambda}(G)^T, T^\beta_{\gamma, \lambda}(G)^T, T^\beta_{\gamma, \lambda}(G)^T)^T\), with \(T^\beta_{\gamma, \lambda}\) being of length \(s\), then their influence functions at the model \(F(\beta, \sigma)\) are given by

\[ \mathcal{I}\mathcal{F}(\{y_t, x_t\}, T^\beta_{\gamma, \lambda}, F(\beta, \sigma)) = -\sigma(\gamma + 1)^{3/2} \left[ E_H \left( x_1 x_1^T \right) \right]^{-1} \left( y_t - x^T_{t, t} \beta \right) e^{-\frac{(y_t - x^T_{t, t} \beta)^2}{2\sigma^2}} x_{1, t} \]

\[ + \frac{\lambda \sigma^{2\gamma+3} (2\pi)^{\gamma/2}}{(1+\gamma)} \left( P^\ast_0(\beta) + P^{n(2)}_0(\beta) \mathcal{I}\mathcal{F}(\{y_t, x_t\}, U^{(1)}, F(\beta, \sigma)) \right), \]
\[ \mathcal{I} \mathcal{F}(\{y_t, x_t\}, T_{\gamma, \lambda}^{\beta}, F_{(\beta, \sigma)}) = 0, \]
\[ \mathcal{I} \mathcal{F}(\{y_t, x_t\}, T_{\gamma}^{\sigma}, F_{\theta}) = \frac{\sigma (1 + \gamma)^{5/2}}{2 + \gamma^2} \left[ 1 - \left( \frac{y_t - x_t^T \beta}{\sigma} \right)^2 \right] e^{-\gamma (y_t - x_t^T \beta)^2 / 2 \sigma^2} - \frac{\gamma}{(1 + \gamma)^{1/2}}, \]

whenever the associated quantities exist finitely. Here, all notations are the same as in Theorem 3.1.

**Remark 3.2:** Note that, at \( \gamma = 0 \), the Ad-DPD-LASSO coincides with the usual adaptive LASSO and hence the above results also provide its influence function which is new in the literature of adaptive LASSO. The IF at \( \gamma = 0 \) is unbounded at any contamination point, even if we start with a robust initial estimator, indicating the non-robust nature of the usual adaptive LASSO against data contamination.

### 4. Oracle consistency

We now study the asymptotic properties of the AW-DPD-LASSO estimators under the ultra-high dimensional set-up of non-polynomial order, following Fan et al. [17]. With the notation of Sections 1 and 2, recall that, the number \( p \) of the available covariates is assumed to grow exponentially with the sample size \( n \). However, only few of them are significantly (linearly) associated with the response under the true model so that the true value \( \beta_0 = (\beta_{10}, \ldots, \beta_{p0}) \) of the regression coefficient is sparse having only \( s \ll n \) non-zero entries. Without loss of generality, \( \beta_0 = (\beta_{10}, 0_{p-s})^T \) and then \( S_0 = \{1, 2, \ldots, s\} \). Let us allow \( s = s_n = o(n) \) to slowly diverge with the sample size \( n \), but the subscript will be suppress unless required to avoid any confusion. The oracle property refers to the fact that any estimator can correctly identify this true model \( S_0 \), i.e. the first \( s \) components of the estimated regression coefficient vector are consistent estimators of the components of \( \beta_{10} \) whereas the remaining components are zero asymptotically with probability tending to one. We will now show that, under certain conditions, the general AW-DPD-LASSO estimator of \( \beta \) enjoys the oracle property and subsequently simplify the required conditions for the Ad-DPD-LASSO estimator.

For simplicity, we here assume that the design matrix \( X \) is fixed with each column being standardized to have \( \ell_1 \)-norm \( \sqrt{n} \) and the response is also standardized so that the error variance \( \sigma^2 \) is assumed to be known and equal to one. The case of unknown \( \sigma^2 \) can be tackled by similar arguments with slightly modified assumptions as described later on; note that the objective function is convex in \( \sigma \) and hence its minimizer can be shown to be consistent and asymptotically normal through standard arguments and will be independent of the penalty used (see Ghosh and Majumdar [20]). Further, in consistency with the high-dimensional literature, we will assume that the regularization parameter \( \lambda = \lambda_n \) in our objective function (10) depends of the sample size \( n \) but their explicit relation is given later on following the required assumptions. In the following, given any \( S \subseteq \{1, 2, \ldots, p\} \) and any \( p \)-vector \( v = (v_1, \ldots, v_p)^T \), we will denote \( S^c = \{1, 2, \ldots, p\} \setminus S \), \( v_S = (v_j : j \in S) \) and \( v_{S^c} = (v_j : j \notin S) \) whereas \( \text{Supp}(v) = \{j : v_j \neq 0\} \). Note that, for the true model \( S_0 \), we have \( \beta_{0S_0} = \beta_{10} \), \( \beta_{0S_n} = 0_{p-s} \) and \( \text{Supp}(\beta_0) = S_0 \). Further, let \( X_S \) consists of the \( j \)th column of \( X \) for all \( j \in S \) for any \( S \), and put \( X_1 = X_{S_0} \) and \( X_2 = X_{S_n} \) so that \( X = [X_1 : X_2] \); the corresponding partition of the \( i \)th row of \( X \) would be denoted by \( x_i = (x_{i1}^T, x_{i2}^T)^T \). Also
Since the AW-DPD-LASSO estimator of $\beta$ coincides with the least-squares adaptive LASSO estimator at $\gamma = 0$, we here focus on deriving their asymptotic properties for $\gamma > 0$ only. Under a general error distribution $f$ and a fixed $\gamma > 0$, we need the following basic assumptions on the corresponding DPD loss function $L_{n,\gamma}(\beta) = L_{n,\gamma}(\beta, 1)$ in (9) along with the boundedness of the design matrix $X = ((x_{ij}))_{i=1,...,n;j=1,...,p}$. All our assumptions and results are given in terms of a fixed design matrix, but they can be easily extended for the random design matrix by showing that the required assumptions holds for the random design asymptotically with probability one.

(A1) The error density $f$ is such that $f^\gamma$ is Lipschitz with the Lipschitz constant $L_f$.
(A2) The eigenvalues of the matrix $n^{-1}(X_i^TX_i)$ are bounded below and above by positive constants $c_0$ and $c_0^{-1}$, respectively. Also $\kappa_n := \max_{i,j} |x_{ij}| = o(n^{1/2}s^{-1})$.

Note that Assumption (A1) is implied by the boundedness of the function $\psi_{1,\gamma}(s)$ whenever it is differentiable and this holds for most common exponential family of distributions at any $\gamma > 0$; in particular, it holds for the usual normal error distribution. The second assumption, on the other hand, is the same as used in Fan et al. [17]; the first part is quite standard in high-dimensional literature whereas the second part holds for appropriate fixed design matrix as well as for common stochastic designs with asymptotic probability one (see Fan et al. [17] for some example).

4.1. General AW-DPD-LASSO estimator with fixed non-stochastic weights

We first consider the AW-DPD-LASSO estimators with fixed non-stochastic weights in the corresponding objective function in (10). With $\sigma = 1$, the objective function in $\beta$ can now be re-expressed as

$$Q_{n,\gamma,\lambda}(\beta) = L_{n,\gamma}(\beta) + \lambda_n \sum_{j=1}^p w_j |\beta_j|,$$

where $w_j$ is the fixed weights corresponding to the $j$th penalty term and the DPD loss $L_{n,\gamma}(\beta)$ has the form $L_{n,\gamma}(\beta) = \frac{1}{n} \sum_{i=1}^n \rho_\gamma(x_i^T\beta, y_i)$ with

$$\rho_\gamma(x_i^T\beta, y_i) = M^{(\gamma)}_j - \frac{1+\gamma}{\gamma} f^\gamma(y_i - x_i^T\beta) + \frac{\gamma}{\gamma}.$$

Note that $\nabla L_{n,\gamma}(\beta) = n^{-1}[X_i^TH^{(1)}_{\gamma}(\beta)]$ and $\nabla^2 L_{n,\gamma}(\beta) = n^{-1}[X_i^T H^{(2)}_{\gamma}(\beta) X]$, where $\nabla$ and $\nabla^2$ denote the first- and second-order gradient with respect to $\beta$, respectively. Let us also denote $w = (w_1, \ldots, w_p)$, $w_0 = w_0$, $w_1 = w_1$, and define $\delta_n = \sqrt{s \log n} / \sqrt{n} + \lambda_n ||w_0||_2$.

We first study the properties of an oracle estimator obtained by minimizing the above objective function $Q_{n,\gamma,\lambda}(\beta)$ with fixed weights over the restricted oracle parameter space

$$H^{(1)}_{\gamma}(\beta) = ((1 + \gamma)\psi_{1,\gamma}(y_i - x_i^T\beta) : i = 1, \ldots, n)^T$$

and

$$H^{(2)}_{\gamma}(\beta) = \text{Diag}((1 + \gamma)J_{11,\gamma}(y_i - x_i^T\beta) : i = 1, \ldots, n).$$
\( \Theta^o = \{ \beta = (\beta_1^T, \beta_2^T)^T \in \mathbb{R}^p : \beta_2 = 0_{p-s} \} \equiv \mathbb{R}^s \times \{0\}^{p-s} \); let us denote the corresponding minimizer as \( \hat{\beta}^o = (\hat{\beta}_1^o, 0_{p-s}) \), the oracle estimator for our model.

We would like to point out that the oracle estimator defined above are exactly in line with the classical definitions from Bühlmann and van de Geer [2]; it is also used by the pioneer paper by Fan and Li [3] and the large pool of subsequent works build upon this paper. To see their alignments, note that the classical oracle estimate is the one obtained based on a low-dimensional sub-model of any high-dimensional models which includes only the \( s \) important covariates given to us by an oracle. Once it is assumed that we know the correct sub-model from Oracle, the parameter space can then be restricted only to a particular \( s \)-dimensional subspace of the whole parameter space corresponding to the \( s \) important covariates. Assuming that the first \( s \) covariates are important (without loss of generality), this oracle restricted parameter space is then given by our \( \Theta^o \), and the estimate of the regression coefficient vector over this space can then be called as the oracle estimate, which only estimate the coefficients associated with \( s \) important covariates (first \( s \) in this case) and put zero for the other coefficient, i.e. \( \hat{\beta}^o \) as defined above. For good estimation procedures, such an oracle estimate should be close to the true parameter values (while considering estimating accuracy) and also any general estimate of the regression coefficients obtained over the whole parameter space would asymptotically be very close to this oracle estimator with high-probability (while considering model selection accuracy); the later property is often known as the Oracle property in the literature. In the following, we will also prove the same oracle properties for our proposed adaptive DPD-based procedures under suitable assumptions. In particular, we need the following additional assumption on the DPD loss function; from now on, all expectations are taken with respect to the true model density \( f \) of \( \epsilon_i = y_i - x_i^T \beta_0 \).

(A3) The diagonal elements of \( E[H_y^{(2)}(\beta_0)] \) are all finite and bounded from below by a constant \( c_1 > 0 \).

(A4) Expectation of third-order partial derivatives of \( \rho_y(x_i^T \beta, y_i), i = 1, \ldots, n \), with respect to all components of \( \beta_{S_0} \) are uniformly bounded in a neighbourhood of \( \beta_{10} \).

Note that these two assumptions are very common in the statistical inference using the DPD loss function even in low-dimensional and they hold for most common (regular) models for the error; see Basu et al. [22], Ghosh and Basu [23]. Then, we have the following result about the \( \ell_2 \)-consistency of the oracle estimator \( \hat{\beta}^o \).

**Theorem 4.1:** If Assumptions (A1)–(A4) hold and \( \lambda_n ||w_0||_2 \sqrt{\delta_n} \to 0 \), then, given any constant \( C_1 > 0 \) and \( \delta_n = \sqrt{s} \log \mathbb{N} / n + \lambda_n ||w_0||_2 \), there exists some \( c > 0 \) such that

\[
P \left( \left\| \hat{\beta}_1^o - \beta_{10} \right\|_2 \leq C_1 \delta_n \right) \geq 1 - n^{-c}.
\]

Further, if \( \delta_n^{-1} \min_{1 \leq j \leq s} |\beta_{j0}| \to \infty \), then the sign of each component of \( \hat{\beta}_1^o \) matches with that of \( \beta_{10} \).

Next we will show that the oracle estimator in Theorem 4.1 is indeed an asymptotic global minimizer of the objective function \( Q_{n,y,\lambda}(\beta) \) over the whole parameter space with probability tending to one. For this purpose, we need the following additional assumptions
controlling the correlation between the important and unimportant covariates in the same spirit as Condition 3 in Fan et al. [17].

(A5) \[ |\|n^{-1}(X_n^T E[H_\gamma^{(2)}(\beta_0)]X_1)||_2 \leq \frac{\lambda_n \min(\|w_1\|)}{2C_1 \delta_n}, \]\text{for some constant } C_1 > 0, \text{where we denote } ||A||_{2,\infty} = \sup_{x \in \mathbb{R}^q \setminus \{0\}} \frac{||Ax||_{\infty}}{||x||_2} \text{ for any } p \times q \text{ matrix } A \text{ and } \min(\|w_1\|) = \min_{j > s} |w_j|.

**Theorem 4.2:** Suppose that Assumptions (A1)–(A5) hold with \( \lambda_n > 2(\sqrt{c} + 1) \log p/n \) and \( \min(\|w_1\|) > c_3 \) for some constant \( c, c_3 > 0 \), and

\[
\lambda_n ||w_0||_{2,K_n} \max\{\sqrt{s}, ||w_0||_2\} \rightarrow 0, \quad \delta_n s^{3/2} \kappa_n^2 (\log n)^2 = o(n\lambda_n^2).
\]

Then, with probability at least \( 1 - O(n^{-c_5}) \), there exists a global minimizer \( \hat{\beta} = ((\hat{\beta}_1^o)^T, \hat{\beta}_2^T)^T \) of the AW-DPD-LASSO objective function \( Q_{n,\gamma,\lambda}(\beta) \) such that

\[
||\hat{\beta}_1 - \beta_1||_2 \leq C_1 \delta_n, \quad \text{and} \quad \hat{\beta}_2 = 0_{p-s}.
\]

Note that Theorem 4.2 presents consistency of both the parameter estimates and oracle variable selection of our AW-DPD-LASSO estimators under appropriate assumptions. Next, we will derive the conditions under which these estimators are asymptotically normal after suitable normalizations. Let us define \( V_n = [X_n^T E[H_\gamma^{(2)}(\beta_0)]X_1]^{-1/2}, Z_n = X_1 V_n \) and \( \Omega_n = \text{Var}[H_\gamma^{(1)}(\beta_0)] \), and consider the following assumption.

(A6) \( Z_n^T \Omega_n Z_n \) is positive definite, \( \lambda_n ||w_0||_2 = O(\sqrt{s/n}) \), and \( \sqrt{s/n} \min_{1 \leq j \leq s} |\beta_{j0}| \rightarrow \infty \).

These Assumptions (A5)-(A6) are similar to the conditions used in Fan et al. [17] while developing the properties of the adaptive quantile regressions. Assumption (A5) provides the maximum correlation between the important and unimportant covariates allowed for the proposed method to have oracle model selection consistency. This maximum allowed correlation further depends proportionally on the minimum of the weights attached to the truly zero coefficients. Assumption (A6) put further restrictions on these weights ensuring that all the weights for unimportant covariates are penalized but those for the important covariates to be small enough depending on the rate of convergences of \( \lambda_n \) and \( s \). We now derive the following theorem on asymptotic distribution of the AW-DPD-LASSO.

**Theorem 4.3:** Suppose that the assumptions of Theorem 4.2 hold along with Assumption (A6). Then, with probability tending to one, there exists a global minimizer \( \hat{\beta} = (\hat{\beta}_1^o, \hat{\beta}_2^T)^T \) of the AW-DPD-LASSO objective function \( Q_{n,\gamma,\lambda}(\beta) \) such that \( \hat{\beta}_2 = 0_{p-s} \) and

\[
u^T [Z_n^T \Omega_n Z_n]^{-1/2} V_n^{-1} \left[ (\hat{\beta}_1^o - \beta_1) + n\lambda_n V_n^2 \tilde{w}_0 \right] \xrightarrow{L} N(0, 1),
\]

for any arbitrary s-dimensional vector \( \nu \) satisfying \( \nu^T \nu = 1 \), where \( \tilde{w}_0 \) is an s-dimensional vector with jth element \( w_j \cdot \text{sign}(\beta_{j0}) \).

Assumption (A6) is crucial for the above asymptotic normality result of the AW-DPD-LASSO estimator, since it ensures that the bias term, namely \( n\lambda_n V_n^2 \tilde{w}_0 \), does not diverges asymptotically ensuring a legitimate asymptotic distribution of \( (\hat{\beta}_1^o - \beta_1) \). It is
controlled by the choice of weight and regularization sequence depending on $s$. Further, the asymptotic variance of the AW-DPD-LASSO estimator depends crucially on the tuning parameter $\gamma > 0$; it can be seen that that there is a slight increase in these asymptotic variances with increasing $\gamma > 0$ in consistence with the classical theory of DPD-based estimation Basu et al. [21].

As a special case of the above results, we get the properties of the DPD-LASSO estimators by using the weights $w_j = 1$ for all $j = 1, \ldots, p$. Then, we have $||w_0||_2 = \sqrt{s}$, $\min(|w_j|) = 1 > 0$ and hence $\delta_n = \sqrt{s(log n)/n + \lambda_n \sqrt{s}}$. We can then obtain the consistency and model selection property of the DPD-LASSO estimators, which is presented in the following corollary.

**Corollary 4.4 (Properties of the DPD-LASSO Estimator):** Under the set-up of this section, assume that Assumption (A1)-(A4) hold true. Then, for a given constant $C_1 > 0$, we have the following results for the DPD-LASSO estimator with regularization parameter $\lambda_n = O(n^{-1/2})$.

(a) There exists some $c > 0$ such that, with probability at least $1 - n^{-cs}$, the corresponding oracle estimator $\hat{\beta}_1^o$ satisfies

$$||\hat{\beta}_1^o - \beta_{10}||_2 \leq C_1 \left[ \sqrt{s(log n)/n + \lambda_n \sqrt{s}} \right].$$ (25)

(b) If $\sqrt{s(log n)/n} = o(\min_{1 \leq j \leq s} |\beta_{j0}|)$, then the sign of each component of $\hat{\beta}_1^o$ matches with that of the true $\beta_{10}$.

(c) If Assumption (A5) holds with $\lambda_n > 2\sqrt{(c + 1) \log p/n}$ for some constant $c > 0$ and $n^{1/2}(\log n)^{5/2} = O(log p)$, then, with probability at least $1 - O(n^{-cs})$, there exists a global minimizer $\hat{\beta} = ((\hat{\beta}_1^o)^T, \hat{\beta}_2^T)^T$ of the DPD-LASSO objective function such that $\hat{\beta}_1^o$ satisfies (25) and $\hat{\beta}_2 = 0_p$.

(d) In addition to the assumptions of item (c), if $\sqrt{n/s} \min_{1 \leq j \leq s} |\beta_{j0}| \to \infty$ and $Z_n \Omega_n Z_n$ is positive definite, then the DPD-LASSO estimator $\hat{\beta}_1^o$ obtained in item (c) further satisfies (24), but the associated bias $n\lambda_n V_n \tilde{w}_0$ is non-diminishing.

It is important to note that the good properties of the DPD-LASSO estimator requires $\lambda_n = O(n^{-1/2})$, which is known to be quite low for a thresholding level even for Gaussian noise Fan et al. [17]. This is in consistence with any LASSO estimators that motivated researchers to consider adaptive LASSO. In general, the properties of the AW-DPD-LASSO estimators crucially depends note only on the choice of weights $w$ but also on the structure of the oracle through different assumptions on $w_0$ and $w_1$. These are not practically feasible to satisfy with the non-adaptive fixed weights and we need to estimates the weights adaptively from the data. However, the results developed in this section with fixed weights will indeed be useful in understanding the behaviours of the AW-DPD-LASSO estimators with stochastic (adaptive) weights as discussed in the next subsection.

### 4.2. General AW-DPD-LASSO estimator with adaptive weights

We now consider the AW-DPD-LASSO estimators with adaptive (stochastic) weights as in the objective function in (10), but with $\sigma = 1$. Given an initial estimator of
\[
\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_p)^T, \text{ we can now re-express the objective function in } \beta \text{ as }
\]
\[
\hat{Q}_{n,Y,\lambda}(\beta) = L_{n,Y}(\beta) + \lambda_n \sum_{j=1}^p \hat{w}_j |\beta_j|,
\]
where the loss \(L_{n,Y}(\beta)\) are exactly as in Section 4.1 but the weights are now estimated adaptively as \(\hat{w}_j = w(|\beta_j|)\) for some suitable function \(w(\cdot)\). We first study the property of the resulting estimator for general weight function and initial estimator \(\tilde{\beta}\) and then simplify them for an important special case.

In this section as well, we will continue to use the notation of Section 4.1, and additionally denote \(\hat{w} = (\hat{w}_1, \ldots, \hat{w}_p)^T\) and \(w^* = (w_1^*, \ldots, w_p^*)^T\) with \(w_j^* = w(|\beta_{0j}|)\) and their partitions \(\hat{w}_0 = \hat{w}_{S_0}, \hat{w}_1 = \hat{w}_{S_0}, w_0^* = w_{S_0}^*\) and \(w_1^* = w_{S_0}^*.\) Further, let us define \(\delta_n^* = (\sqrt{s \log n} / n + \lambda_n (||w_0^*||_2^2 + C_2 c_5 \sqrt{s \log p} / n))\), where \(C_2\) and \(c_5\) are defined in the following assumptions.

(A7) The initial estimator \(\tilde{\beta}\) satisfies \(||\tilde{\beta} - \beta_0||_2 \leq C_2 \sqrt{s \log p} / n\) for some constant \(C_2 > 0\), with probability tending to one.

(A8) The weight function \(w(\cdot)\) is non-increasing over \((0, \infty)\) and is Lipschitz continuous with Lipschitz constant \(c_5 > 0\). Further, \(w(C_2 \sqrt{s \log p} / n) > 1/2 w(0^-)\) for large enough \(n\), where \(C_2\) is as in Assumption (A7).

(A9) With \(C_2\) as in Assumption (A7), \(\min_{1 \leq j \leq s} |\beta_{0j}| > 2C_2 \sqrt{s \log p} / n\). Further, the derivative of the weight satisfies \(w'(|b|) = o(s^{-1} \lambda_n^{-1}(n \log p)^{-1/2})\) for any \(|b| > 1/2 \min_{1 \leq j \leq s} |\beta_{0j}|\).

These assumptions are exactly the same as Conditions 4–6 of Fan et al. [17]. The first one put the weaker restriction on the initial estimator to be only consistent in order to achieve the variable selection consistency of the second stage AW-DPD-LASSO estimators as shown in the following theorem; this minor requirement is satisfied by most simple LASSO estimators, including the DPD-LASSO, so that either of them can be used as the initial estimator here. On the other hand, Assumptions (A8) and (A9) put restrictions on the weight functions used where the first one is needed for model selection consistency and both are needed for the asymptotic normality of the resulting estimators.

**Theorem 4.5:** Suppose that the assumptions of Theorem 4.2 hold with \(w = w^*\) and \(\delta_n = \delta_n^*\). Additionally, if Assumptions (A7)-(A8) hold with \(\lambda_n s k \sqrt{s \log p} / n \rightarrow 0\) then, with probability tending to one, there exists a global minimizer \(\hat{\beta} = (\hat{\beta}_1^T, \hat{\beta}_2^T)^T\) of the AW-DPD-LASSO objective function \(Q_{n,Y,\lambda}(\beta)\) in (26), with adaptive weights, such that
\[
||\hat{\beta}_1 - \beta_{10}||_2 \leq C_1 \delta_n, \quad \text{and} \quad \hat{\beta}_2 = 0_{p-s}.
\]

**Theorem 4.6:** Suppose that the assumptions of Theorem 4.3 hold with \(w = w^*\) and \(\delta_n = \delta_n^*\). Additionally, suppose that Assumptions (A7)-(A9) hold. Then, with probability tending to one, there exists a global minimizer \(\hat{\beta} = (\hat{\beta}_1^T, \hat{\beta}_2^T)^T\) of the AW-DPD-LASSO objective function \(Q_{n,Y,\lambda}(\beta)\) in (26), with adaptive weights, having the same asymptotic properties as those described in Theorem 4.3.
It is important to note that these results can be further simplified for any given weight function. For example, if we use the SCAD weight function as defined in (11), it has been verified in Fan et al. [17] that Assumption (A8) is satisfied if \( \lambda_n > \frac{2}{(a+1)} C_2 \sqrt{\frac{s \log p}{n}} \), whereas Assumption (A9) is satisfied if \( \min_{1 \leq j \leq s} |\beta_{0j}| \geq 2a\lambda_n \). Therefore, under appropriately simplified conditions as given in Corollary 1 of Fan et al. [17], the AW-DPD-LASSO estimator with SCAD weight function can be seen to satisfy exactly the same model section oracle property and asymptotic normality as the DPD-ncv estimator with SCAD penalty. However, the Ad-DPD-LASSO estimators may not enjoy both of these properties since the corresponding weight function is not Lipschitz around zero but they are so locally on intervals of the form \((c, \infty)\) for any \( c > 0 \).

5. Computational algorithm

Several computationally efficient algorithms have been developed in the literature to solve the least squares regression problem with different types (e.g. adaptive, grouped, etc.) of LASSO penalties. These techniques often use the local convexity of the objective function. In this section, we develop an appropriate estimating algorithm for the DPD-LASSO, the Ad-DPD-LASSO and the general AW-DPD-LASSO estimators under the assumption of normal error distribution. To minimize the corresponding objective functions, we propose an iterative optimization algorithm. Firstly the objective function is minimized in \( \beta \) with a fixed \( \sigma \), and secondly it is optimized on \( \sigma \) for fixed \( \beta \); these two steps are repeated consecutively until convergence. To perform the first minimization with respect to \( \beta \), we use the approach of MM-algorithm; here, the observed data are weighted to bound the DPD loss function by a quadratic loss, and hence transforming the minimization problem to a least-squares LASSO penalized problem. In the alternative steps, given \( \beta \), \( \sigma \) is updated using coordinate descent algorithm. A detailed step-by-step derivation of the computing algorithm can be found in Section 2 of the Appendix (available in the Online Supplementary Material).

Algorithm 1 (General AW-DPD-LASSO estimator)

1. Set \( m = 0 \). Choose values of the hyper-parameters \( \lambda \) and \( \gamma \), and set robust initial values \( \tilde{\beta} \) and \( (\tilde{\beta}^0, \tilde{\sigma}^0) \) using any suitable robust algorithm.
2. For each \( m = 0, 1, \ldots \), do the following:
   a) (a) Compute \( \hat{\beta}^{(m+1)} \) from \( (\hat{\beta}^{(m)}, \hat{\sigma}^{(m)}) \) as
      \[
      \hat{\beta}^{(m+1)} = \arg \min \left[ \sum_{i=1}^{n} \mu_i^{(m)} \left( \frac{y_i - x_i^T \hat{\beta}^{(m)}}{\hat{\sigma}^{(m)}} \right)^2 + \lambda \sum_{j=1}^{p} \omega(|\tilde{\beta}_j|) \cdot |\beta_j| \right] 
      \]  
      with
      \[
      \mu_i^{(m)} = \exp \left( -\frac{\gamma}{2} \left( \frac{y_i - x_i^T \hat{\beta}^{(m)}}{\hat{\sigma}^{(m)}} \right)^2 \right) \left[ \sum_{i=1}^{n} \exp \left( -\frac{\gamma}{2} \left( \frac{y_i - x_i^T \hat{\beta}^{(m)}}{\hat{\sigma}^{(m)}} \right)^2 \right) \right]^{-1}.
      \]
Compute $\hat{\sigma}^{(m+1)}$ from $(\hat{\beta}^{(m+1)}, \hat{\sigma}^{(m)})$ as

$$(\hat{\sigma}^{(m+1)})^2 = \left[ \frac{1}{n} \sum_{i=1}^{n} w_i^{(m)} - \frac{\gamma}{(\gamma + 1)^{3/2}} \right]^{-1} \frac{1}{n} \sum_{i=1}^{n} w_i^{(m)} \left( y_i - x_i^T \hat{\beta}^{(m+1)} \right)^2,$$

(28)

where $w_i^{(m)} = \exp \left( -\frac{\gamma}{2} \left( \frac{y_i - x_i^T \hat{\beta}^{(m+1)}}{\hat{\sigma}^{(m)}} \right)^2 \right)$.

(3) If $|Q_{n,\gamma,\lambda}(\hat{\beta}^{(m+1)}, \hat{\sigma}^{(m+1)}) - Q_{n,\gamma,\lambda}(\hat{\beta}^{(m)}, \hat{\sigma}^{(m)})| \leq \varepsilon$ (pre-specified): Go to Step 4.

Else: Set $\hat{\beta} = \hat{\beta}^{(m)}$, $m = m + 1$ and return to Step 2.

(4) Output: $\hat{\beta} = \hat{\beta}^{(m+1)}$ and $\hat{\sigma} = \hat{\sigma}^{(m+1)}$ as the AW-DPD-LASSO estimator.

The performance of Algorithm 1 depends on the choice of initial values $\tilde{\beta}$ and $(\hat{\beta}^0, \hat{\sigma}^0)$, as well as on the regularization parameter $\lambda$. We can apply any robust standard regression method, such as RLARS, or DPD-LASSO to get the required initial values of the parameters. Note that $\tilde{\beta}$ and $\hat{\beta}^0$ can be chosen differently or the same but they should both be conservative as not to lose any important covariate. In our analysis, we employed the DPD-LASSO estimator for both $\tilde{\beta}$ and $(\hat{\beta}^0, \hat{\sigma}^0)$, which is robust and tends to be quite conservative in variable selection. Just like the usual LASSO estimator serves as a good initial estimator for the computation of the classical adaptive LASSO estimator, we have seen that the our choice of DPD-LASSO as the initial estimator in computation of AW-DPD-LASSO estimator gives desirable and sufficiently good results; so we stick with this recommendation for the choice of initial estimators. However, other initial estimators may be explored in future although the proposed algorithm appears not to be very sensitive to these choices as long as the initial estimators of the regression coefficients are robust enough and do not miss any important variables.

For the selection of the (best) $\lambda$, we propose to use the high-dimensional Bayesian Information Criterion (HBIC), which has demonstrably better performance compared to the standard BIC under non-polynomial dimensionality (Kim et al. [30]; Wang et al. [7]; Fan and Tang [31]). The HBIC is defined as

$$\text{HBIC}(\lambda) = \log(\hat{\sigma}_\lambda^2) + \frac{\log\log(n) \log p}{n} \| \hat{\beta}_\lambda \|_0. \quad (29)$$

and we select the optimal $\lambda$ minimizing the HBIC values over a pre-determined set (via grid search).

6. Simulation study

6.1. Simulation settings

The data are generated from the LRM in (1) following similar set-up as Zou and Li [27]. We set the sample size $n = 100$, the true error standard deviation $\sigma_0 = 0.5$, and $p = 500, 1000$. Errors are generated independently from $N(0, \sigma_0^2)$ distributions. Two different scenarios are considered for the true sparse regression coefficients $\beta_0$ as follows:
• Setting A: Only three true coefficients are not null. $\beta_0 = (3, 1.5, 0, 0, 2, 0)_{p-5}$.
• Setting B: A more challenging scenario obtained by modifying the precedent Setting A, where we divide the first 60 components into continuous blocks of size 20, and assign the coefficient values $(3, 1.5, 0, 0, 2, 0)_{15}$ to each block. Thus, the true model here has nine important covariates.

Rows of the design matrix $X$ are drawn from the normal distribution $\mathcal{N}(0, \Sigma)$, where $\Sigma$ is a positive definite matrix of Toeplitz structure, with the $(i, j)$th element being $0.5^{|i-j|}$. To study the robustness of the method, we additionally modify these (pure) data by four type of contamination as follows:

• $Y$-outliers : We add random observations, drawn independently from a normal distribution $\mathcal{N}(20, 1)$, to the response variable for a random 10% of each sample.
• $X$-outliers : We add random observations, drawn independently from a normal distribution, $\mathcal{N}(20, 1)$, to the covariate values in 10 columns of $X$ for a random 10% of each sample.
• Heavy-tailed $Y$-outliers : We add random observations centred at 20, drawn independently from a t-student distributions with 3 degrees of freedom, to the response variable for a random 10% of each sample.
• Heavy-tailed $X$-outliers : We add random observations centred at 20, drawn independently from a t-student distributions with 3 degrees of freedom, to the covariate values in 10 columns of $X$ for a random 10% of each sample.

We repeat the simulations over $R = 100$ replications to compute different performance measures. We evaluate the variable selection performances through Model Size (MS), True Positive proportion (TP) and True Negative proportion, defined as

$$MS(\hat{\beta}) = |\text{supp}(\hat{\beta})|, \quad \text{TP}(\hat{\beta}) = \frac{|\text{supp}(\hat{\beta}) \cap \text{supp}(\beta_0)|}{|\text{supp}(\beta_0)|},$$
$$\text{TN}(\hat{\beta}) = \frac{|\text{supp}^c(\hat{\beta}) \cap \text{supp}^c(\beta_0)|}{|\text{supp}^c(\beta_0)|}.$$ 

Additionally, in order to assest the estimation accuracy, we compute the mean square error for the true non-zero coefficients (MSES) and zero coefficients (MSEN) of $\hat{\beta}$ separately, as well as the absolute Estimation Error (EE) of $\hat{\sigma}$:

$$\text{MSES}(\hat{\beta}) = \frac{1}{s} \parallel \hat{\beta}_S - \beta_0S \parallel^2, \quad \text{MSEN}(\hat{\beta}) = \frac{1}{p-s} \parallel \hat{\beta}_{N} \parallel^2, \quad \text{EE}(\hat{\sigma}) = |\hat{\sigma} - \sigma_0|.$$ 

Finally, we also examine the prediction accuracy on an unused test sample of size $n = 100$, generated from the same model distributions as that of the training sample. For this purpose, we use the Absolute Prediction Bias (APrB) defined as

$$\text{APrB}(\hat{\beta}) = \parallel y_{\text{test}} - X_{\text{test}}\hat{\beta} \parallel_1.$$

### 6.2. Competing methods

We compare our adaptive robust methods with the robust least angle regression (RLARS; Khan et al. [13]), sparse least trimmed squares (sLTS; Alfons et al. [15]), random sample
consensus (RANSAC), and the LAD-LASSO (Wang et al. [7]) estimators. Additionally, our Ad-DPD-LASSO and AW-DPD-LASSO methods (with SCAD penalty) are also compared with the related DPD-based methods, namely the DPD-LASSO and the DPD-ncv with SCAD penalty, for the same values of tuning parameters ($\gamma = 0.1, 0.3, 0.5, 0.7, 1$). Moreover, for comparing our methods in terms of efficiency loss, we use three standard non-robust methods, namely the ones based on the least-squares loss along with the LASSO, SCAD and MCP penalties, which we will refer to as the LS-LASSO, LS-SCAD and LS-MCP, respectively. The standard adaptive LASSO (Ad-LS-LASSO) is also considered, with the initial parameters being obtained via LS-LASSO.

For the DPD-LASSO and DPD-ncv, the starting points are taken as the RLARS estimates because of computational efficiency. We use 5-fold cross-validation for the selection of the regularized parameter $\lambda$ in all the above competing methods except LAD-LASSO, DPD-LASSO and DPD-ncv. We use BIC criterion to choose the optimum $\lambda$ in LAD-LASSO, whereas the HBIC is used for DPD-LASSO and DPD-ncv.

### 6.3. Simulation results

For brevity, we present the simulation results (performance measures) for the most challenging case of Setting B with $p = 1000$ in Tables 1–5, for pure data and four types of contaminated data. The results for the remaining cases are provided in Section C of the Appendix available as Supplementary Material.

| Method         | MS($\hat{\beta}$) | TP($\hat{\beta}$) | TN($\hat{\beta}$) | MSES($\hat{\beta}$) $(10^{-2})$ | MSEN($\hat{\beta}$) $(10^{-5})$ | EE($\hat{\sigma}$) $(10^{-2})$ | APB($\hat{\beta}$) $(10^{-2})$ |
|----------------|-------------------|-------------------|-------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| LS-LASSO       | 21.27             | 1.00              | 0.99              | 2.52                          | 1.52                          | 34.91                         | 6.31                          |
| Ad-LS-LASSO    | 9.00              | 1.00              | 1.00              | 1.49                          | 0.00                          | 31.30                         | 5.16                          |
| LS-SCAD        | 9.03              | 1.00              | 1.00              | 0.36                          | 0.00                          | 19.62                         | 4.23                          |
| LS-MCP         | 9.04              | 1.00              | 1.00              | 0.36                          | 0.00                          | 19.60                         | 4.22                          |
| LAD-LASSO      | 16.70             | 1.00              | 0.99              | 7.55                          | 2.63                          | 53.97                         | 9.17                          |
| RLARS          | 12.70             | 1.00              | 1.00              | 0.48                          | 2.82                          | 7.65                          | 4.49                          |
| sLTS           | 61.80             | 0.79              | 0.94              | 219.62                        | 300.30                        | 6.79                          | 46.08                         |
| RANSAC         | 15.10             | 0.52              | 0.99              | 238.20                        | 394.16                        | 143.68                        | 49.35                         |
| DPD-LASSO $\gamma = 0.1$ | 10.37         | 1.00              | 1.00              | 4.59                          | 114.54                        | 58.75                         | 11.10                         |
| DPD-LASSO $\gamma = 0.3$ | 8.73           | 0.90              | 1.00              | 27.95                         | 701.22                        | 211.92                        | 25.86                         |
| DPD-LASSO $\gamma = 0.5$ | 13.13          | 0.99              | 1.00              | 3.13                          | 95.99                         | 35.29                         | 10.61                         |
| DPD-LASSO $\gamma = 0.7$ | 22.12          | 0.73              | 0.98              | 70.05                         | 1201.08                       | 25.81                         | 29.53                         |
| DPD-LASSO $\gamma = 1$   | 14.40             | 0.69              | 0.99              | 73.96                         | 1500.41                       | 163.41                        | 37.38                         |
| DPD-ncv $\gamma = 0.1$   | 9.00              | 1.00              | 1.00              | 0.14                          | 2.64                          | 5.58                          | 4.20                          |
| DPD-ncv $\gamma = 0.3$   | 9.00              | 1.00              | 1.00              | 0.17                          | 3.08                          | 10.28                         | 4.21                          |
| DPD-ncv $\gamma = 0.5$   | 9.00              | 1.00              | 1.00              | 0.19                          | 3.36                          | 13.86                         | 4.23                          |
| DPD-ncv $\gamma = 0.7$   | 9.00              | 1.00              | 1.00              | 0.23                          | 3.82                          | 16.63                         | 4.25                          |
| DPD-ncv $\gamma = 1$    | 9.01              | 1.00              | 1.00              | 0.26                          | 4.12                          | 19.82                         | 4.31                          |
| Ad-DPD-LASSO $\gamma = 0.1$ | 11.34          | 1.00              | 1.00              | 0.49                          | 2.79                          | 6.18                          | 4.24                          |
| Ad-DPD-LASSO $\gamma = 0.3$ | 9.10           | 1.00              | 1.00              | 0.64                          | 0.04                          | 3.04                          | 4.33                          |
| Ad-DPD-LASSO $\gamma = 0.5$ | 9.58           | 1.00              | 1.00              | 0.66                          | 0.45                          | 4.65                          | 4.39                          |
| Ad-DPD-LASSO $\gamma = 0.7$ | 9.16           | 1.00              | 1.00              | 0.79                          | 0.11                          | 5.29                          | 4.31                          |
| Ad-DPD-LASSO $\gamma = 1$   | 9.18              | 1.00              | 1.00              | 1.06                          | 0.38                          | 8.68                          | 4.36                          |
| AW-DPD-LASSO $\gamma = 0.1$ | 10.68           | 1.00              | 1.00              | 0.34                          | 0.28                          | 3.92                          | 3.78                          |
| AW-DPD-LASSO $\gamma = 0.3$ | 9.62           | 1.00              | 1.00              | 0.36                          | 0.00                          | 3.93                          | 3.92                          |
| AW-DPD-LASSO $\gamma = 0.5$ | 10.70           | 1.00              | 1.00              | 0.43                          | 0.30                          | 5.46                          | 4.00                          |
| AW-DPD-LASSO $\gamma = 0.7$ | 9.82           | 1.00              | 1.00              | 0.50                          | 0.20                          | 6.68                          | 4.04                          |
| AW-DPD-LASSO $\gamma = 1$   | 9.60              | 1.00              | 1.00              | 0.66                          | 0.09                          | 9.56                          | 3.99                          |
Table 2. Performance measures obtained by different methods for $p = 1000$, Setting B and $Y$-outliers.

| Method          | $\text{MS}(\hat{\beta})$ | $\text{TP}(\hat{\beta})$ | $\text{TN}(\hat{\beta})$ | $\text{MSES}(\hat{\beta})$ (10$^{-2}$) | $\text{MSEN}(\hat{\beta})$ (10$^{-5}$) | $\text{EE}(\hat{\beta})$ (10$^{-2}$) | $\text{APrB}(\hat{\beta})$ |
|-----------------|---------------------------|---------------------------|---------------------------|------------------------------------------|------------------------------------------|------------------------------------------|---------------------------|
| LS-LASSO        | 7.85                      | 0.46                      | 1.00                      | 323.57                                   | 56.63                                    | 923.77                                   | 53.67                     |
| Ad-LS-LASSO     | 5.24                      | 0.42                      | 1.00                      | 282.04                                   | 196.13                                   | 728.11                                   | 46.04                     |
| LS-SCAD         | 26.32                     | 0.63                      | 0.98                      | 253.30                                   | 369.77                                   | 526.46                                   | 44.85                     |
| LS-MCP          | 11.40                     | 0.52                      | 0.98                      | 256.43                                   | 316.72                                   | 584.97                                   | 44.96                     |
| LAD-LASSO       | 25.93                     | 0.85                      | 0.98                      | 144.66                                   | 226.31                                   | 315.34                                   | 36.80                     |
| RLARS           | 20.25                     | 0.79                      | 0.99                      | 90.67                                    | 320.57                                   | 144.45                                   | 26.38                     |
| sLTS            | 51.28                     | 0.93                      | 0.96                      | 83.66                                    | 131.04                                   | 7.52                                    | 22.84                     |
| RANSAC          | 13.00                     | 0.37                      | 0.99                      | 319.90                                   | 646.28                                   | 296.99                                   | 56.92                     |
| DPD-LASSO $\gamma = 0.1$ | 8.50                     | 0.74                      | 1.00                      | 65.91                                    | 1475.56                                  | 348.96                                   | 70.62                     |
| DPD-LASSO $\gamma = 0.3$ | 6.94                     | 0.62                      | 1.00                      | 80.41                                    | 1681.17                                  | 381.90                                   | 39.62                     |
| DPD-LASSO $\gamma = 0.5$ | 12.22                    | 0.88                      | 1.00                      | 27.28                                    | 618.61                                   | 130.50                                   | 20.97                     |
| DPD-LASSO $\gamma = 0.7$ | 23.58                    | 0.61                      | 0.98                      | 92.40                                    | 1643.77                                  | 30.59                                    | 36.54                     |
| DPD-LASSO $\gamma = 1$ | 14.03                    | 0.53                      | 0.99                      | 105.81                                   | 2131.45                                  | 215.98                                   | 47.96                     |
| DPD-ncv $\gamma = 0.1$ | 9.10                     | 0.91                      | 1.00                      | 23.00                                    | 425.28                                   | 127.54                                   | 16.34                     |
| DPD-ncv $\gamma = 0.3$ | 9.34                     | 0.98                      | 1.00                      | 4.78                                     | 118.05                                   | 19.60                                    | 6.88                      |
| DPD-ncv $\gamma = 0.5$ | 9.12                     | 0.98                      | 1.00                      | 4.65                                     | 110.96                                   | 13.06                                    | 6.67                      |
| DPD-ncv $\gamma = 0.7$ | 9.21                     | 0.98                      | 1.00                      | 4.47                                     | 101.86                                   | 10.88                                    | 6.64                      |
| DPD-ncv $\gamma = 1$ | 9.22                     | 0.99                      | 1.00                      | 4.49                                     | 100.00                                   | 10.20                                    | 6.62                      |
| Ad-DPD-LASSO $\gamma = 0.1$ | 18.71                    | 0.81                      | 0.99                      | 81.49                                    | 835.39                                   | 130.08                                   | 22.43                     |
| Ad-DPD-LASSO $\gamma = 0.3$ | 10.92                    | 0.99                      | 1.00                      | 6.21                                     | 27.67                                    | 10.02                                    | 6.14                      |
| Ad-DPD-LASSO $\gamma = 0.5$ | 9.98                     | 0.99                      | 1.00                      | 5.81                                     | 24.30                                    | 5.94                                     | 5.50                      |
| Ad-DPD-LASSO $\gamma = 0.7$ | 9.48                     | 0.99                      | 1.00                      | 3.87                                     | 11.31                                    | 7.30                                     | 5.11                      |
| Ad-DPD-LASSO $\gamma = 1$ | 9.96                     | 0.97                      | 1.00                      | 15.38                                    | 48.27                                    | 8.59                                     | 6.66                      |
| AW-DPD-LASSO $\gamma = 0.1$ | 12.98                    | 0.80                      | 0.99                      | 92.47                                    | 567.77                                   | 169.98                                   | 23.06                     |
| AW-DPD-LASSO $\gamma = 0.3$ | 10.84                    | 0.99                      | 1.00                      | 6.36                                     | 31.77                                    | 9.71                                     | 5.91                      |
| AW-DPD-LASSO $\gamma = 0.5$ | 12.72                    | 0.99                      | 1.00                      | 3.84                                     | 15.54                                    | 7.87                                     | 4.91                      |
| AW-DPD-LASSO $\gamma = 0.7$ | 10.80                    | 0.99                      | 1.00                      | 3.69                                     | 13.75                                    | 8.37                                     | 5.14                      |
| AW-DPD-LASSO $\gamma = 1$ | 11.51                    | 0.96                      | 1.00                      | 17.81                                    | 57.74                                    | 8.38                                     | 8.38                      |

All these results evidence the significant advantage entailed by DPD-based methods in terms of robustness, accentuated in the high-dimensional context where classical inferential methods are specially sensitive to outliers. Moreover, all other robust methods considered, LAD-LASSO, RLARS, sLTS and RANSAC, perform worse than the proposed DPD-based methods in variable selection and parameter estimation, under all scenarios of contamination as well as under pure data. Then, DPD-based estimation methods represent an appealing robust alternative to classical likelihood-based methods, with the lowest efficiency loss and greatest robustness gain. It is interesting to note that all methods underperform on the second scenario (Setting B), where the true model size is greater. Additionally, as expected, adaptive methods including DPD-based estimators improve the estimation accuracy with respect to their corresponding standard LASSO. Indeed, adaptive methods select more parsimonious models containing all true significant variables.

Furthermore, there exist significant differences between the four DPD-based methods considered. The DPD-ncv (with SCAD penalty) estimator performs generally better than Ad-DPD-LASSO and AW-DPD-LASSO estimators in terms of variable selection and it registers lower mean square error for the true non-zero coefficient (MSES). However, DPD-ncv estimator presents higher mean squared error for the zero coefficients (MSEN) as well as much greater errors for the error variance. On the other hand, DPD-LASSO performs worse in both MSES and MSEN measures compared to any other DPD-based estimator, as it selects a larger number of non-significant variables. Note that DPD loss with the parameter value $\gamma = 0.1$ is the comparatively less robust, and hence produces greater MSES and MSEN in the presence of data contamination (close to the LS-based results). For larger
### Table 3. Performance measures obtained by different methods for $p = 1000$, Setting B and X-outliers

| Method        | MS($\hat{\beta}$) | TP($\hat{\beta}$) | TN($\hat{\beta}$) | MSES($\hat{\beta}$) $(10^{-2})$ | MSEN($\hat{\beta}$) $(10^{-5})$ | EE($\hat{\sigma}$) $(10^{-2})$ | APrB($\hat{\beta}$) $(10^{-2})$ |
|---------------|---------------------|--------------------|-------------------|---------------------------------|---------------------------------|-------------------|-----------------------------|
| LS-LASSO      | 7.85                | 0.46               | 1.00              | 332.57                          | 56.63                           | 923.77            | 53.67                       |
| Ad-LS-LASSO   | 9.00                | 1.00               | 1.00              | 1.52                            | 0.00                            | 31.56             | 4.85                        |
| LS-SCAD       | 26.32               | 0.63               | 0.98              | 253.30                          | 396.77                         | 526.46            | 44.85                       |
| LS-MCP        | 11.40               | 0.52               | 0.99              | 256.43                          | 316.72                         | 584.97            | 49.96                       |
| LAD-LASSO     | 25.93               | 0.85               | 0.98              | 144.66                          | 226.31                         | 315.34            | 36.80                       |
| RLARS         | 20.25               | 0.79               | 0.99              | 90.67                           | 320.57                         | 144.45            | 26.38                       |
| sLTS          | 51.28               | 0.93               | 0.96              | 83.66                           | 131.04                         | 7.52              | 22.84                       |
| RANSAC        | 12.23               | 0.36               | 0.99              | 335.92                          | 696.48                         | 322.69            | 51.78                       |
| DPD-LASSO $\gamma = 0.1$ | 10.33          | 1.00               | 1.00              | 4.58                            | 114.30                         | 58.74             | 10.80                       |
| DPD-LASSO $\gamma = 0.3$ | 8.71            | 0.90               | 1.00              | 27.94                           | 700.75                         | 211.92            | 26.32                       |
| DPD-LASSO $\gamma = 0.5$ | 13.08           | 0.99               | 1.00              | 3.14                            | 96.10                          | 35.47             | 9.83                        |
| DPD-LASSO $\gamma = 0.7$ | 21.04          | 0.79               | 0.99              | 56.38                           | 945.35                         | 31.38             | 24.17                       |
| DPD-LASSO $\gamma = 1$ | 14.30           | 0.69               | 0.99              | 73.45                           | 1515.09                        | 164.59            | 37.40                       |
| DPD-ncv $\gamma = 0.1$ | 9.00             | 1.00               | 1.00              | 0.14                            | 2.64                           | 5.58              | 3.88                        |
| DPD-ncv $\gamma = 0.3$ | 9.00             | 1.00               | 1.00              | 0.17                            | 3.09                           | 10.29             | 3.89                        |
| DPD-ncv $\gamma = 0.5$ | 9.00             | 1.00               | 1.00              | 0.19                            | 3.38                           | 13.85             | 3.94                        |
| DPD-ncv $\gamma = 0.7$ | 9.00             | 1.00               | 1.00              | 0.23                            | 3.81                           | 16.61             | 3.99                        |
| DPD-ncv $\gamma = 1$ | 9.01             | 1.00               | 1.00              | 0.28                            | 4.14                           | 19.80             | 4.10                        |
| Ad-DPD-LASSO $\gamma = 0.1$ | 11.20           | 1.00               | 1.00              | 0.50                            | 2.62                           | 5.91              | 4.20                        |
| Ad-DPD-LASSO $\gamma = 0.3$ | 9.10             | 1.00               | 1.00              | 0.64                            | 0.04                           | 3.04              | 4.15                        |
| Ad-DPD-LASSO $\gamma = 0.5$ | 9.58             | 1.00               | 1.00              | 0.66                            | 0.45                           | 4.65              | 4.13                        |
| Ad-DPD-LASSO $\gamma = 0.7$ | 9.16             | 1.00               | 1.00              | 0.79                            | 0.11                           | 5.30              | 4.17                        |
| Ad-DPD-LASSO $\gamma = 1$ | 9.18             | 1.00               | 1.00              | 1.06                            | 0.39                           | 8.68              | 4.02                        |
| AW-DPD-LASSO $\gamma = 0.1$ | 10.68           | 1.00               | 1.00              | 0.34                            | 0.28                           | 3.92              | 3.58                        |
| AW-DPD-LASSO $\gamma = 0.3$ | 9.42             | 1.00               | 1.00              | 0.36                            | 0.00                           | 3.88              | 3.76                        |
| AW-DPD-LASSO $\gamma = 0.5$ | 10.70           | 1.00               | 1.00              | 0.43                            | 0.30                           | 5.46              | 3.80                        |
| AW-DPD-LASSO $\gamma = 0.7$ | 9.80             | 1.00               | 1.00              | 0.50                            | 0.20                           | 6.67              | 3.94                        |
| AW-DPD-LASSO $\gamma = 1$ | 9.58             | 1.00               | 1.00              | 0.67                            | 0.09                           | 9.56              | 4.02                        |

Values of $\gamma \geq 0.3$, DPD-based estimators provide extremely robust inference against contamination in both the response variable and the covariates, often achieving the true model (TP of 100%), contrarily to non-robust methods. From our empirical results, optimal $\gamma$ value hover around $\gamma \in [0.3, 0.5]$. Additionally, it is also evident from simulations that the AW-DPD-LASSO estimator (with SCAD-based weight) performs very competitively with the DPD-ncv method (with SCAD penalty) in terms of variables selection and even better than the DPD-ncv in terms of estimating $\sigma$ and prediction performance. Therefore, it can serve as a fast yet excellent alternative to DPD-ncv in ultra-high dimensional set-ups.

Further comparisons of the proposed Ad-DPD-LASSO and AW-DPD-LASSO (with SCAD penalty) estimators are provided in Section C of the Appendix available as Supplementary Material. For both of them, the prediction errors are seen to decrease as the number of covariates or the values of tuning parameter $\gamma$ increase. The AW-DPD-LASSO performs slightly better than the Ad-DPD-LASSO at larger values of $\gamma$, whereas the opposite is observed at smaller values of $\gamma$.

### 6.4. Runtime comparisons

To illustrate the computational advantages of the proposed robust adaptive methods, we compare the computation times of the proposed and competitive methods in the simulation Settings A and B presented in the paper (refer to Section 6.1). In particular, we compare the mean running time of the non-concave penalty SCAD and its corresponding
Table 4. Performance measures obtained by different methods for \( p = 1000 \), Setting B and heavy-tailed \( Y \)-outliers.

| Method               | MS(\( \hat{\beta} \)) | TP(\( \hat{\beta} \)) | TN(\( \hat{\beta} \)) | MSES(\( \hat{\beta} \)) \((10^{-2})\) | MSEN(\( \hat{\beta} \)) \((10^{-5})\) | EE(\( \hat{\sigma} \)) \((10^{-2})\) | APB(\( \hat{\beta} \)) \((10^{-2})\) |
|---------------------|------------------------|------------------------|------------------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| LS-LASSO            | 6.71                   | 0.48                   | 1.00                   | 331.65                                   | 39.19                                    | 933.86                                   | 49.59                                    |
| LS-SCAD             | 26.50                  | 0.62                   | 0.98                   | 259.29                                   | 423.82                                   | 531.59                                   | 36.96                                    |
| LS-MCP              | 11.23                  | 0.51                   | 0.99                   | 265.74                                   | 339.76                                   | 595.32                                   | 36.20                                    |
| LAD-LASSO           | 24.15                  | 0.84                   | 0.98                   | 145.58                                   | 174.40                                   | 321.14                                   | 33.35                                    |
| RLARS               | 18.49                  | 0.77                   | 0.99                   | 93.50                                    | 311.20                                   | 152.40                                   | 23.97                                    |
| sLTS                | 50.92                  | 0.93                   | 0.96                   | 85.14                                    | 137.32                                   | 8.24                                     | 22.48                                    |
| RANSAC             | 9.10                   | 0.37                   | 0.99                   | 321.35                                   | 371.72                                   | 521.68                                   | 47.27                                    |
| DPD-LASSO \( \gamma = 0.1 \) | 11.44                  | 0.81                   | 1.00                   | 103.20                                   | 5.27                                     | 213.00                                   | 17.11                                    |
| DPD-LASSO \( \gamma = 0.3 \) | 13.03                  | 0.96                   | 1.00                   | 69.31                                    | 4.74                                     | 135.92                                   | 9.03                                     |
| DPD-LASSO \( \gamma = 0.5 \) | 16.09                  | 0.91                   | 0.99                   | 73.84                                    | 38.48                                    | 80.23                                    | 16.43                                    |
| DPD-LASSO \( \gamma = 0.7 \) | 13.21                  | 0.68                   | 0.99                   | 207.95                                   | 95.24                                    | 219.09                                   | 35.05                                    |
| DPD-LASSO \( \gamma = 1 \) | 8.39                   | 0.90                   | 1.00                   | 23.67                                    | 405.46                                   | 110.48                                   | 11.75                                    |
| DPD-ncv \( \gamma = 0.1 \) | 9.04                   | 0.99                   | 1.00                   | 3.41                                     | 76.12                                    | 21.48                                    | 6.66                                     |
| DPD-ncv \( \gamma = 0.3 \) | 9.19                   | 0.99                   | 1.00                   | 2.86                                     | 87.24                                    | 17.18                                    | 5.64                                     |
| DPD-ncv \( \gamma = 0.5 \) | 8.79                   | 0.92                   | 1.00                   | 12.38                                    | 251.15                                   | 13.92                                    | 7.95                                     |
| DPD-ncv \( \gamma = 1 \) | 6.77                   | 0.64                   | 1.00                   | 53.31                                    | 1131.87                                  | 14.37                                    | 21.74                                    |
| Ad-DPD-LASSO \( \gamma = 0.1 \) | 7.86                   | 0.85                   | 1.00                   | 58.77                                    | 61.17                                    | 129.14                                   | 11.96                                    |
| Ad-DPD-LASSO \( \gamma = 0.3 \) | 8.64                   | 0.96                   | 1.00                   | 18.49                                    | 15.55                                    | 30.00                                    | 6.00                                     |
| Ad-DPD-LASSO \( \gamma = 0.5 \) | 8.64                   | 0.96                   | 1.00                   | 15.42                                    | 2.39                                     | 26.15                                    | 6.18                                     |
| Ad-DPD-LASSO \( \gamma = 0.7 \) | 8.72                   | 0.97                   | 1.00                   | 13.27                                    | 3.34                                     | 20.85                                    | 6.12                                     |
| Ad-DPD-LASSO \( \gamma = 1 \) | 8.64                   | 0.96                   | 1.00                   | 18.30                                    | 3.34                                     | 24.72                                    | 6.63                                     |
| AW-DPD-LASSO \( \gamma = 0.1 \) | 12.18                  | 0.82                   | 1.00                   | 83.09                                    | 491.03                                   | 144.38                                   | 21.07                                    |
| AW-DPD-LASSO \( \gamma = 0.3 \) | 10.80                  | 0.98                   | 1.00                   | 12.56                                    | 90.51                                    | 8.87                                     | 6.14                                     |
| AW-DPD-LASSO \( \gamma = 0.5 \) | 9.84                   | 0.97                   | 1.00                   | 13.18                                    | 78.97                                    | 12.90                                    | 6.47                                     |
| AW-DPD-LASSO \( \gamma = 0.7 \) | 10.23                  | 0.97                   | 1.00                   | 17.79                                    | 73.99                                    | 16.81                                    | 7.30                                     |
| AW-DPD-LASSO \( \gamma = 1 \) | 8.84                   | 0.84                   | 1.00                   | 82.33                                    | 114.36                                   | 85.87                                    | 12.59                                    |

Further, we compare computation times of the different methods for increasing \( p \) under the contamination setting A as specified in Section 6.1. We compare four competing methods based on the DPD loss, namely DPD-LASSO, Ad-DPD-LASSO, AW-DPD-LASSO and DPD-ncv. We fix the value of penalty parameter \( \lambda \) (based on our simulation experiences) so that all methods perform suitably for model selection and parameter estimation. For such purpose, we have taken \( \lambda = 0.025 \) for LASSO-based methods, DPD-LASSO and Ad-DPD-LASSO, and \( \lambda = 0.078 \) for the AW-DPD-LASSO and DPD-ncv methods with the non-concave penalty SCAD. All methods are initialized using RLARS, and the sample size is fixed to \( n = 100 \).
### Table 5. Performance measures obtained by different methods for $p = 1000$, Setting B and heavy-tailed X-outliers.

| Method               | MS($\hat{\beta}$) | TP($\hat{\beta}$) | TN($\hat{\beta}$) | MSES($\hat{\beta}$) | MSEN($\hat{\beta}$) | EE($\hat{\beta}$) | APB($\hat{\beta}$) |
|----------------------|---------------------|---------------------|---------------------|-----------------------|-----------------------|-------------------|---------------------|
| LS-LASSO             | 21.23               | 1.00                | 0.99                | 2.58                  | 1.47                  | 34.90             | 5.24                |
| LS-SCAD              | 9.02                | 1.00                | 1.00                | 0.35                  | 0.00                  | 19.97             | 3.55                |
| LS-MCP               | 9.01                | 1.00                | 1.00                | 0.35                  | 0.00                  | 19.99             | 3.54                |
| LAD-LASSO            | 16.59               | 1.00                | 0.99                | 7.61                  | 3.03                  | 55.06             | 7.78                |
| RLARS                | 13.79               | 1.00                | 1.00                | 0.76                  | 0.00                  | 8.60              | 4.01                |
| LTS                  | 61.45               | 0.79                | 0.95                | 217.87                | 304.40                | 6.04              | 42.31               |
| RANSAC              | 10.44               | 0.54                | 0.99                | 235.33                | 182.41                | 359.73            | 39.38               |
| DPD-LASSO $\gamma = 0.1$ | 12.46               | 1.00                | 1.00                | 4.09                  | 0.71                  | 21.24             | 6.03                |
| DPD-LASSO $\gamma = 0.3$ | 12.46               | 1.00                | 1.00                | 5.54                  | 1.01                  | 25.40             | 6.91                |
| DPD-LASSO $\gamma = 0.5$ | 11.07               | 0.97                | 1.00                | 30.79                 | 2.30                  | 75.72             | 12.66               |
| DPD-LASSO $\gamma = 0.7$ | 14.11               | 0.96                | 0.99                | 44.89                 | 12.47                 | 75.02             | 14.84               |
| DPD-LASSO $\gamma = 1$ | 9.56                | 0.75                | 1.00                | 178.82                | 30.81                 | 258.91            | 33.98               |
| DPD-ncv $\gamma = 0.1$ | 9.00                | 1.00                | 1.00                | 0.13                  | 2.60                  | 5.17              | 3.56                |
| DPD-ncv $\gamma = 0.3$ | 9.00                | 1.00                | 1.00                | 0.16                  | 3.03                  | 10.65             | 3.61                |
| DPD-ncv $\gamma = 0.5$ | 9.00                | 1.00                | 1.00                | 0.19                  | 3.71                  | 15.54             | 3.60                |
| DPD-ncv $\gamma = 0.7$ | 9.01                | 1.00                | 1.00                | 0.24                  | 4.46                  | 19.84             | 3.72                |
| DPD-ncv $\gamma = 1$ | 8.98                | 0.99                | 1.00                | 0.94                  | 25.21                 | 24.32             | 4.19                |
| Ad-DPD-LASSO $\gamma = 0.1$ | 9.00                | 1.00                | 1.00                | 0.64                  | 0.00                  | 2.84              | 3.66                |
| Ad-DPD-LASSO $\gamma = 0.3$ | 9.00                | 1.00                | 1.00                | 0.82                  | 0.00                  | 3.05              | 3.72                |
| Ad-DPD-LASSO $\gamma = 0.5$ | 9.00                | 1.00                | 1.00                | 1.10                  | 0.00                  | 3.42              | 3.87                |
| Ad-DPD-LASSO $\gamma = 0.7$ | 9.00                | 1.00                | 1.00                | 0.88                  | 0.00                  | 5.19              | 3.74                |
| Ad-DPD-LASSO $\gamma = 1$ | 9.00                | 1.00                | 1.00                | 1.56                  | 0.00                  | 6.79              | 4.25                |
| AW-DPD-LASSO $\gamma = 0.1$ | 9.58                | 1.00                | 1.00                | 0.39                  | 1.77                  | 4.99              | 3.53                |
| AW-DPD-LASSO $\gamma = 0.3$ | 9.47                | 1.00                | 1.00                | 0.40                  | 1.34                  | 5.38              | 3.59                |
| AW-DPD-LASSO $\gamma = 0.5$ | 9.15                | 1.00                | 1.00                | 0.45                  | 0.60                  | 5.47              | 3.66                |
| AW-DPD-LASSO $\gamma = 0.7$ | 9.43                | 1.00                | 1.00                | 0.56                  | 1.39                  | 8.76              | 3.85                |
| AW-DPD-LASSO $\gamma = 1$ | 9.43                | 0.98                | 1.00                | 11.60                 | 8.13                  | 17.07             | 5.49                |

Figure 1 shows computation time curves (in seconds) of the four proposed methods under three scenarios of contamination, namely pure data, 10% of $Y$-outliers and 10% of $X$-outliers with $R = 100$ replications. For a better comparison between competing methods, computational time curves for LASSO-based methods are plotted on the left, and computational time curves for AW-DPD-LASSO and DPD-ncv (based on SCAD) are plotted on the right part of Figure 1. These plots show smaller computational times for the proposed robust adaptive methods, in comparison to their non-adaptive competitors, for any given value of the tuning parameter $\gamma$. Furthermore, such computational advantage becomes more significant with large values of $p$. In particular Ad-DPD-LASSO is generally faster than DPD-LASSO for all values of $\gamma$, but in presence of $Y$-outliers its average computation time increases slightly approaching to the runtimes took by classical DPD-LASSO. On the other hand, the AW-DPD-LASSO method, which weights are chosen to linearly approximate the non-concave SCAD penalty, is faster than the DPD-ncv in all scenarios. Therefore, in the cases with large values of the ratio $p/n$, of the number of explanatory variables to the sample size (ultra-high dimensional settings), the proposed AW-DPD-LASSO method is an appealing alternative to the DPD-ncv estimator for a faster computation without efficiency and robustness loss.

### 6.5. On the choice of robustness tuning parameter $\gamma$

We would like to point out that the choice of robustness tuning parameters ($\gamma$ in the present case with DPD) is an important practical issue in the classical (low-dimensional)
robustness literature, since larger values of the tuning parameter produce more robust but less efficient estimators. A moderately large value $\gamma$ around 0.3 to 0.5 generally offers a decent trade-off between efficiency and robustness in most practical use of the DPD-based robust inferential procedures. However, an optimal value of $\gamma$ would depend on the

Figure 1. Computational time curves (in seconds) for different penalized procedures under different simulation settings. (a) Pure data, (b) $Y$-outliers and (c) $X$-outliers.
contamination level in data, which is quite difficult to measure in real applications. Several criteria have been proposed in the literature for choosing optimal values of the DPD tuning parameter for classical (low-dimensional) statistical models. Most popularly, Warwick and Jones [32] introduced a useful data-based procedure for IID data based on the minimization of an estimate of the asymptotic MSE of the minimum DPD estimators based on the given sample data, which is later extended and extensively studied for non-homogeneous models, including general class of parametric regressions, by Ghosh and Basu [23,33]. This method depends on the choice of a pilot estimator which can have a significant impact on the optimal tuning parameter choice, as the pilot invariably draws the final estimator towards itself. Basak et al. [34] improved the method by alleviating the dependency on the initial estimator, by proposing an iterative algorithm which updates the pilot estimator at each step with the optimal estimate obtained till then and the process is repeated until there is no further change in the optimal estimate.

For the present cases with the ultra-high dimensional setting, the computation of optimal $\gamma$ at each step would inevitably involve a high-time cost. This cost would not benefit us much as we have seen through our extensive simulation exercises in this paper. The model section and estimation performances of the proposed Ad-DPD-LASSO and AW-DPD-LASSO does not depend too crucially on the choice of $\gamma$ within the range $[0.3, 0.7]$ (unlike the traditional robustness literature on parameter estimation), and hence, it does not justify the time-cost for selection of $\gamma$ during model selection through these procedures; any moderate value of $\gamma$ around 0.5 should provide reasonably good model selection performances in any practical applications. However, if one wants to bear the high computational cost of choosing a data-driven value of $\gamma$ for a high-dimensional dataset, the procedure of Warwick and Jones [32] or Basak et al. [34] may be extended suitably for the high-dimensional settings or a similar procedure as in [10] may be developed for the present case of DPD-based loss function. Considering the length of the current manuscript, we have left the detailed exploration of such data-driven tuning parameter selection procedures for our future research.

7. Real data analysis

Robust high-dimensional regression is highly important in the field of chemometrics, where hundreds or even thousands of spectra need to be analysed. We apply our proposed methods to a real dataset regarding electron-probe X-ray microanalysis (EPXMA) of archaeological glass vessels from the 16th and 17th centuries, where each of $n = 180$ glass vessels is represented by a spectrum on 1920 frequencies. For each vessel the contents of 13 chemical compounds are registered. These data were first introduced in Janssens et al. [35], where the archaeological glass vessels were investigated through chemical analysis. However, it was realized that some spectra in the dataset had been measured with a different detector efficiency, which in the statistical sense, may lead to bad leverage points (outliers in the covariates space). Besides leverage points, there are also four different material compositions of the glass vessels, further increasing the inhomogeneity of the spectral data. These data have been used to identify multivariate outliers by Filzmoser et al. [36] and subsequently in Serneels et al. [37], Maronna [38] and Smucler and Yohai [39] to illustrate high-dimensional robust regression methods using the content of PbO (lead monoxide) as response variable. However, because the data on PbO (and any of
its Box-Cox transformations) are not normally distributed, we have instead opted to use the seventh variable in the data, which is the content of chlorine (Cl), as response variable. For this response variable (Cl content), the Shapiro-Wilk normality test yielded a p-value of 0.6552, indicating that the data can be assumed to follow a normal distribution and hence our proposed algorithm (in Section 5) for the computation of the AW-DPD-LASSO estimator can be used here. Accordingly, a linear model with normal error distribution is fitted with this response data (Cl content) and considering the frequencies as covariates.

Since the frequencies below 15 and above 500 have mostly the values of zero, we keep frequencies from 15 to 500 in our modelling, so that we have \( p = 486 \) covariates. We estimate the coefficients of the regression model fitted using the adaptive lasso penalties Ad-LS-LASSO, LS-SCAD, and the four DPD-based methods penalized with adaptive lasso penalties, namely the Ad-DPD-LASSO, AW-DPD-LASSO. For comparison purposes with the AW-DPD-LASSO we also all DPD-ncv. To study the performance of the different methods, Maronna [38] used 10% trimmed root mean square error, RMSE(0.9), which is a more robust criterion than the usual RMSE. Using this measure prevents the outliers from inflating the RMSE. To compare the precedent estimating methods, we report the model size (MS), RMSE(0.9), and the minimum and maximum error (MAX and MIN) in Table C17 of the Appendix. The robust DPD-based methods present greater maximum error (MAX) than non-robust ones, as well as higher MSE and MSPE(0.9), as outliers lead to larger residuals in a robust fit. Indeed, it can be noticed that the difference between the RMSEs(0.9), are more pronounced than for MSEs, showing again the great gain in robustness.

Further, to assess the prediction performance of different methods on these data, Smucler and Yohai [39] proposed using the \( \tau \)-scale of the residuals, calculated as in Maronna and Zamar [40]. To define this \( \tau \)-scale for a univariate sample \( x = (x_1, \ldots, x_n) \), we consider the function

\[
W_c(x) = \left( 1 - \left( \frac{x}{c} \right)^2 \right)^2 \mathbb{I}(|x| \leq c),
\]

and put \( w_i = W_{c_1} \left( \frac{x_i - \text{med}(x)}{\sigma_0} \right), \quad i = 1, \ldots, n, \)

where \( \sigma_0 \) is the median absolute deviation of the sample \( x \). Then, the \( \tau \)-scale statistic for \( x \) is defined as \( \sigma(x) = \frac{\sigma_0^2}{n} \sum_i \rho_{c_2} \left( \frac{x_i - \mu(x)}{\sigma_0} \right) \), where \( \rho_{c_2}(x) = \min(x^2, c^2) \) and \( \mu(x) = \frac{\sum x_i w_i}{\sum w_i} \) is a weighted mean. To combine robustness and efficiency, Maronna and Zamar [40] suggested to take \( c_1 = 4.5 \) and \( c_2 = 3 \), which yield approximately 80% efficient univariate location and scale estimation for normal data. In our case, we randomly split the data into a train set \( (n = 120) \) used to fit the model and a test set \( (n = 60) \) used to calculate prediction residuals and their \( \tau \)-scale statistics. We apply the same schema for all the methods adaptive considered, namely the Ad-LS-LASSO, LS-SCAD, Ad-DPD-LASSO, AW-DPD-LASSO and DPD-ncv, and repeat the process \( R = 100 \) times. The box-plots of the \( \tau \)-scale statistics of the prediction residuals, obtained by different methods, across the 100 replications are presented in Figure 2; the median value of the \( \tau \)-scales are reported in Table C18 of the Appendix available as Supplementary Material. It is again evident that DPD-based methods produce lower error than the least-squares-based method in the test set for both adaptive penalties,– adaptive lasso and weighted adaptive lasso. Note that the \( \tau \)-scale measure is generally greater for larger values of \( \gamma \); moderate and low values of \( \gamma \) offer a great trade-off between efficiency and robustness. Moreover, the performance of AW-DPD-LASSO (with SCAD penalty) is sufficiently close (or even better) to that of the
corresponding DPD-ncv estimator, which again justifies the usefulness of our proposed general adaptive DPD-LASSO estimators as a fast alternative to the DPD-ncv approach. Additionally, it is interesting to note that the weighted adaptive lasso penalty performs better than the standard adaptive lasso with our robust DPD loss, but does not do the same with the least-squares loss.

8. Conclusions

In this paper we have presented a new robust adaptive LASSO method based on the DPD-loss function for the LRM under the ultra-high dimensional set-up. This election of the loss grants high robustness against outliers in the data, while the use of adaptive LASSO penalty ensures the oracle property, and hence it performs as well as if the true underlying model was given in advance. Further, the computation of the proposed AW-DPD-LASSO estimator can be solved by using the same efficient algorithms as the DPD-LASSO method, just properly transforming the data. Through an extensive simulation study, it has been shown that the use of AW-DPD-LASSO method improves the accuracy of the parameters estimation (both regression coefficients and error variance) compared to several other existing robust and non-robust methods. This advantage is accentuated for the estimation of the error standard deviation.

We should note that, although the theoretical results are derived for any general error distribution from the location-scale family, the computational algorithm for our proposed estimators is developed only for the common case of normal error distribution. It would be an important future research to appropriately modify the proposed algorithm for the computation of our AW-DPD-LASSO estimator with any other specified (non-normal) error density. Additionally, the simplicity and usefulness of the adaptive DPD framework encourages for its extension to other parametric regression models. In particular, in future works, our interest will be to consider the general adaptively weighted...
DPD-LASSO approach for the binary and multiple logistic regression models, as well as for Poisson regression model, under the ultra-high dimensional set-up. Their extension to the ultra-high dimensional generalized linear models will also be an important future work.

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References
[1] Tenenbein A. A double sampling scheme for estimating from misclassified multinomial data with applications to sampling inspection. Technometrics. 1972;14:187–202. doi: 10.1080/00401706.1972.10488895
[2] Bühlmann P, van de Geer S. Statistics for high-dimensional data: methods, theory and applications. Springer Berlin, Heidelberg: Springer; 2011.
[3] Fan J, Li R. Variable selection via nonconcave penalized likelihood and its oracle properties. J Amer Statist Assoc. 2001;96:1348–1360. doi: 10.1198/016214501753382273
[4] Zou H. The adaptive lasso and its oracle properties. J Amer Statist Assoc. 2006;101:1418–1429. doi: 10.1198/016214506000000735
[5] Huang J, Ma S, Zhang CH. Adaptive lasso for sparse high-dimensional regression models. Statist Sinica. 2008;18:1603–1618.
[6] Efron B, Hastie T, Johnstone I, et al. Least angle regression. Ann Stat. 2004;32:407–499. doi: 10.1214/009053604000000067
[7] Wang H, Li G, Jiang G. Robust regression shrinkage and consistent variable selection through the lad-lasso. J Bus Econ Stat. 2007;25:347–355. doi: 10.1198/073500106000000251
[8] Owen AB. A robust hybrid of lasso and ridge regression. Comtemp Math. 2007;443:59–72.
[9] Lambert-Lacroix L, Zwald. Robust regression through the hubers criterion and adaptive lasso penalty. Electron J Stat. 2011;5:1015–1053. doi: 10.1214/11-EJS635
[10] Li Q, Wang Y, Fan J. Estimation of high dimensional mean regression in the absence of symmetry and light tail assumptions. J R Stat Soc Ser B (Stat Methodol). 2017;79:247–265. doi: 10.1111/rssb.12166

[11] Huber PJ. Robust statistics. New York (NY): John Wiley & Sons; 1981.

[12] Arslan O. Weighted lad-lasso method for robust parameter estimation and variable selection in regression. Comput Stat Data Anal. 2012;56:1952–1965. doi: 10.1016/j.csda.2011.11.022

[13] van Aelst S, Zamar RH, Khan JA. Robust linear model selection based on least angle regression. J Amer Statist Assoc. 2007;102:1289–1299. doi: 10.1198/016214507000000950

[14] Peng H, Zhu L, Li G. Nonconcave penalized m-estimation with a diverging number of parameters. Statist Sinica. 2011;21:391–419.

[15] Alfons A, Croux C, Gelper S. Sparse least trimmed squares regression for analyzing high-dimensional large data sets. Ann Appl Stat. 2013;7:226–248. doi: 10.1214/12-AA1575

[16] Chang L, Roberts S, Welsh A. Regression using tukey’s biweight criterion. Technometrics. 2018;60:36–47. doi: 10.1080/00401706.2017.1305299

[17] Fan Y, Barut E, Fan J. Adaptive robust variable selection. Ann Stat. 2014;42(1):324–351. doi: 10.1214/13-AOS1191

[18] Gallagher C, Kulasekera KB, Qi Z. Adaptive lasso for variable selection. Commun Stat Theory Methods. 2017;46(9):4642–4659. doi: 10.1080/03610926.2015.1019138

[19] Zhang CH. Nearly unbiased variable selection under minimax concave penalty. Ann Stat. 2010;38:894–942. doi: 10.1214/09-AOS729

[20] Ghosh A, Majumdar S. Ultrahigh-dimensional robust and efficient sparse regression using non-concave penalized density power divergence. IEEE Trans Inf Theory. 2020;66(12):7812–7827.

[21] Basu A, Harris R, Hjort N, et al. Robust and efficient estimation by minimising a density power divergence. Biometrika. 1998;85:549–559. doi: 10.1093/biomet/85.3.549

[22] Basu A, Shioya H, Park C. Statistical inference: the minimum distance approach. Boca Raton (FL): Chapman and Hall; 2011.

[23] Fan Y., Tang C.Y. Tuning parameter selection in high dimensional penalized likelihood models. Ann Stat. 2008;36(4):1509.

[24] Hampel FR, Ronchetti E, Rousseeuw PJ, Stahel W.A. Robust statistics: the approach based on influence functions. New York.: John Wiley & Sons; 1986.

[25] Avella-Medina M. Influence functions for penalized m-estimators. Bernoulli. 2017;7(4B):3178–3196.

[26] Kwon S, Choi H, Kim Y. Consistent model selection criteria on high dimensions. J Mach Learn Res. 2012;13:1037–1057.

[27] Li R, Zou H. One-step sparse estimates in nonconcave penalized likelihood models. Ann Stat. 2008;36(4):1509.

[28] Jones MC, Warwick J. Choosing a robustness tuning parameter. J Stat Comput Simul. 2005;75(7):581–588. doi: 10.1080/00949650412331299120

[29] Deraedt I, Schalm O, Veeckman J, et al. Composition of 15–17th century archaeological glass vessels excavated in antwerp, belgium. In: Modern developments and applications in microbeam analysis, Vienna, Springer. 1998; p. 253–267.
[36] Filzmoser P, Maronna R, Werner M. Outlier identification in high dimension. Computational statistics & data analysis. 2008;52(3):1694–1711.

[37] Serneels S, Croux C, Filzmoser P, et al. Partial robust m-regression. Vol. 79(1-2)Chemometrics and Intelligent Laboratory Systems; 2005. p. 55–64.

[38] Maronna RA. Robust ridge regression for high-dimensional data. Technometrics. 2011;53:44–53. doi: 10.1198/TECH.2010.09114

[39] Smucler E, Yohai VJ. Robust and sparse estimators for linear regression models. Comput Stat Data Anal. 2017;111:116–130. doi: 10.1016/j.csda.2017.02.002

[40] Zamar RH, Maronna RA. Robust estimates of location and dispersion for high-dimensional datasets. Technometrics. 2002;44(4):307–317. doi: 10.1198/004017002188618509