Existence of solutions to degenerate parabolic problems with two weights via the Hardy inequality

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Abstract

The paper concentrates on the application of the following Hardy inequality

\[ \int_{\Omega} |\xi(x)|^p \omega_1(x) dx \leq \int_{\Omega} |\nabla \xi(x)|^p \omega_2(x) dx, \]

to the proof of existence of weak solutions to degenerate parabolic problems of the type

\[ \begin{cases} u_t - \text{div}(\omega_2(x)|\nabla u|^{p-2} \nabla u) = \lambda W(x)|u|^{p-2}u & x \in \Omega, \\ u(x, 0) = f(x) & x \in \Omega, \\ u(x, t) = 0 & x \in \partial \Omega, \ t > 0, \end{cases} \]

on an open subset \( \Omega \subseteq \mathbb{R}^n \), not necessarily bounded, where

\[ W(x) \leq \min\{m, \omega_1(x)\}, \quad m \in \mathbb{R}_+. \]

Key words and phrases: existence of solutions, Hardy inequalities, parabolic problems, weighted \( p \)–Laplacian, weighted Sobolev spaces

Mathematics Subject Classification (2010): 35K55, 35A01, 47J35.

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The research of A.Z.-G. has been supported by the Foundation for Polish Science grant no. POMOST BIS/2012-6/3
1 Introduction

We investigate existence and regularity of solutions to a broad class of non-linear parabolic equations

\[ u_t - \text{div}(\omega_2(x)|\nabla u|^{p-2}\nabla u) = \lambda \omega_1(x)|u|^{p-2}u, \quad \text{in } \Omega \]  

(1)

on an open subset \( \Omega \subseteq \mathbb{R}^n \), not necessarily bounded, with certain weights \( \omega_1, \omega_2 \geq 0 \). We explore the meaning of the optimal constant in the Hardy inequality in parabolic problems, see [12] for details. The optimal constant in classical versions of the Hardy inequality indicates the critical \( \lambda \) for blow-up or global existence, as well as the sharp decay rate of the solution, see e.g. [1, 2, 3, 7, 8, 14, 12, 27, 26], Section 2.5 in [20]). In the elliptic case, existence results can also be obtained via the Hardy inequality, e.g. [10, 22, 23].

The inequalities which are crucial in our approach used to be called improved Hardy inequalities, improved Hardy–Sobolev inequalities, or Hardy–Poincaré inequalities. One of the first well-known ‘improvements’ was introduced in [7] as

\[ C_1 \int_{\Omega} \frac{u^2}{|x|^2} \, dx + C_2 \left( \int_{\Omega} u^q \, dx \right)^{\frac{2}{q}} \leq \int_{\Omega} |\nabla u|^2 \, dx \]

and was used therein in investigations on qualitative properties to the equation \( -\Delta u = \lambda f(u) \), with convex and increasing function \( f \).

In [26] Vazquez and Zuazua describe the asymptotic behaviour of the heat equation that reads

\[ u_t = \Delta u + V(x)u \quad \text{and} \quad \Delta u + V(x)u + \mu u = 0, \]

where \( V(x) \) is an inverse–square potential (e.g. \( V(x) = \frac{1}{|x|^2} \)). The authors consider solutions to the Cauchy–Dirichlet problem in a bounded domain and to the Cauchy problem in \( \mathbb{R}^N \) as well. The key tool is an improved form of the Hardy–Poincaré inequality. Furthermore, in [26] the authors generalize the seminal paper by Baras and Goldstein [3]. Nevertheless, the involved Hardy inequalities do not admit a broad class of weights.
In several papers, e.g. [4, 5, 6], dealing with the rate of convergence of solutions to fast diffusion equations

\[ u_t = \Delta u^m, \]

the authors study the estimates for the constants in Hardy-Poincaré-type inequalities. The optimal constant in Hardy-type inequalities used to indicate the critical \( \lambda \) for blow-up or global existence, as well as the sharp decay rate of the solution. However, they usually deal with weights of the form

\[ |x|^{-\alpha} \quad \text{or} \quad (1 + |x|^2)^{-\alpha}. \]

The inspiration of our research was the paper of García Azorero and Peral Alonso [12], who apply the Hardy inequality [12, Lemma 2.1] of the form

\[ \lambda_{N,p} \int_{\mathbb{R}^N} |\xi|^p |x|^{-p} \, dx \leq \int_{\mathbb{R}^N} |\nabla \xi|^p \, dx, \]

where \( \lambda_{N,p} \) is optimal, to analyse positivity of the following nonlinear operator

\[ \mathcal{L}_\lambda u = -\Delta_p u - \frac{\lambda}{|x|^p} |u|^{p-2} u \]

in \( W^{1,p}_0(\Omega) \) and to obtain the existence of weak solutions to the corresponding parabolic problem.

Our proof follows classical methods of Lions [19], as well as the approach of Anh, Ke [2], who consider the initial boundary value problem for a class of quasilinear parabolic equations involving weighted \( p \)-Laplace operator. Our major difficulties are of technical nature and require more advanced setting, i.e. two-weighted Sobolev spaces \( W^{1,p}_{(\omega_1,\omega_2)}(\Omega) \), due to presence of general class of weights both in the leading part of the operator as well as on the right-hand side of (1).

To involve broader class of weights in (1), we need more general Hardy inequalities. We apply the ones derived in [24], having the form

\[ \int_{\Omega} |\xi(x)|^{p_1} \omega_1(x) \, dx \leq \int_{\Omega} |\nabla \xi(x)|^{p_2} \omega_2(x) \, dx, \]
where the involved weights $\omega_1, \omega_2$ depend on a weak solution to PDI:

$$-\Delta_p v \geq \Phi \quad \text{in} \quad \Omega,$$

with a locally integrable function $\Phi$ (see Theorem 2.1). Quite a general function $\Phi$ is allowed. It can be negative or sign changing if only it is, in a certain sense, bounded from below. The weights $\omega_1, \omega_2$ in (1) are assumed to be a pair in the Hardy inequality. This inequality is needed to obtain a priori estimates for solutions to (1).

Moreover, to ensure that the weighted Sobolev spaces $W^{1,p}_{(\omega_1,\omega_2)}(\Omega)$ and $L^p(0,T;W^{1,p}_{(\omega_1,\omega_2)}(\Omega))$, with two different weights $\omega_1, \omega_2$, have proper structure, we impose additional regularity restrictions on the weights, see Subsection 2.3. We suppose $2 \leq p < N$ to deal with embeddings.

In the paper we add yet another restriction, namely we consider the problem

$$\begin{align*}
  u_t - \text{div}(\omega_2(x)|\nabla u|^{p-2}\nabla u) &= \lambda W(x)|u|^{p-2}u \quad x \in \Omega, \\
  u(x,0) &= f(x) \quad x \in \Omega, \\
  u(x,t) &= 0 \quad x \in \partial \Omega, \; t > 0,
\end{align*}$$

on an open subset $\Omega \subseteq \mathbb{R}^n$, not necessarily bounded, where the function $W(x) \leq \min\{m, \omega_1(x)\}$, $m \in \mathbb{R}_+$. The case of unbounded potential on the right-hand side is more difficult and requires more complex arguments. This work is in progress. We decided to add the restriction here in order to make the presentation more transparent.

2 Preliminaries

2.1 Notation

In the sequel we assume that $2 \leq p < N$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\Omega \subseteq \mathbb{R}^N$ is an open subset, not necessarily bounded. For $T > 0$ we denote $\Omega_T = \Omega \times (0, T)$.

We denote the $p$-Laplace operator by

$$\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$$
and the $\omega$-$p$-Laplacian by
\[
\Delta_\omega^p u = \text{div}(\omega|\nabla u|^{p-2}\nabla u),
\] (2)
with a certain weight function $\omega : \Omega \rightarrow \mathbb{R}$.

By $\langle f, g \rangle$ we denote the standard scalar product in $L^2(\Omega)$.

## 2.2 Sobolev spaces

Suppose $\omega$ is a positive, Borel measurable, real function defined on an open set $\Omega \subseteq \mathbb{R}^N$, satisfying the so-called $B_p$ condition, i.e.
\[
\omega^{-1/(p-1)} \in L^1_{loc}(\Omega),
\] (3)
see [18]. This condition is weaker than the $A_p$-condition, see [21].

Whenever $\omega_1, \omega_2$ satisfy (3), we denote
\[
W^{1,p}_{(\omega_1, \omega_2)}(\Omega) := \{ f \in L^p_{\omega_1}(\Omega) : \nabla f \in (L^p_{\omega_2}(\Omega))^N \},
\] (4)
where $\nabla$ stands for the distributional gradient. The space is equipped with the norm
\[
\| f \|_{W^{1,p}_{(\omega_1, \omega_2)}(\Omega)} := \| f \|_{L^p_{\omega_1}(\Omega)} + \| \nabla f \|_{(L^p_{\omega_2}(\Omega))^N}.
\]

### Fact 2.1 (e.g. [18]).
If $p > 1$, $\Omega \subset \mathbb{R}^N$ is an open set, $\omega_1, \omega_2$ satisfy (3), then
- $W^{1,p}_{(\omega_1, \omega_2)}(\Omega)$ defined by (4) equipped with the norm $\| \cdot \|_{W^{1,p}_{(\omega_1, \omega_2)}(\Omega)}$ is a Banach space;
- $L^p_{\omega_1, loc}(\Omega) \subseteq L^1_{loc}(\Omega)$;
- $\overline{Lip_0}(\Omega) = C_0^\infty(\Omega) \subseteq W^{1,p}_{(\omega_1, \omega_2), 0}(\Omega)$, where the closure is in the norm $\| \cdot \|_{W^{1,p}_{(\omega_1, \omega_2)}(\Omega)}$. 

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• if $\omega_1, \omega_2$ are a pair in the Hardy-Poincaré inequality of the form \([10]\), we may consider the Sobolev space $W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)$ equipped with the norm

$$\|f\|_{W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)} = \|\nabla f\|_{L^p_\omega(\Omega)}.$$  

**Fact 2.2.** The operator $\Delta^{\omega_2}$, given by \([2]\), is hemicontinuous, i.e. for all $u, v, w \in W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)$ the mapping $\lambda \mapsto \langle \Delta^{\omega_2}(u + \lambda v), w \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.

We look for solutions in the space $L^p(0, T; W^{1,p}_{(\omega_1, \omega_2)}(\Omega))$, i.e.

$$L^p(0, T; W^{1,p}_{(\omega_1, \omega_2)}(\Omega)) = \{ f \in L^p(0, T; L^p_{\omega_1}(\Omega)) : \nabla f \in (L^p(0, T; L^p_{\omega_2}(\Omega)))^N \}$$

(as before $\nabla$ stands for the distributional gradient with respect to the spacial variables), equipped with the norm

$$\|f\|_{L^p(0,T;W^{1,p}_{(\omega_1, \omega_2)}(\Omega))} := \left( \int_0^T \|f\|_{L^p_{\omega_1}(\Omega)} dt \right)^{\frac{1}{p}} + \left( \int_0^T \|\nabla f\|_{(L^p_{\omega_2}(\Omega))^N} dt \right)^{\frac{1}{p}}.$$

**Dual space**

By $W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega)$ we denote the dual space to $W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)$ and the duality pairing is given by the standard scalar product.

We note that $L^{p'}(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega))$ is the dual space to $L^p(0, T; W^{1,p}_{(\omega_1, \omega_2),0}(\Omega))$.

**Embeddings**

In the framework of weighted Sobolev spaces we impose restrictions on the weights to ensure that we have the Sobolev–type embeddings. Following \([2]\), we introduce the following conditions:

$$(\mathcal{H}_\alpha) \text{ There exists some } \alpha \in (0, p) \quad \lim_{x \to z} \frac{\omega}{|x - z|^\alpha} > 0 \quad \forall z \in \Omega.$$

\(6\)
There exists some $\beta > p + \frac{N}{2} (p - 2)$
\[
\liminf_{x \to z} \frac{\omega}{|x|^{-\beta}} > 0 \quad \forall z \in \Omega.
\]  
We have the following result, which is a direct consequence of [2, Proposition 2.1].

**Proposition 2.1.** Suppose $2 \leq p < \infty$, the function $\omega_2 : \Omega \to \mathbb{R}$ is locally integrable and satisfies condition $(\mathcal{H}_\alpha)$ and, if $\Omega$ is unbounded, we additionally require that $\omega_2$ satisfies also condition $(\mathcal{H}_\alpha^{\infty})$. Assume further that $(\omega_1, \omega_2)$ is a pair of weights in the Hardy inequality $(10)$.

Then for each $r \in \left[1, \frac{p N}{N - p + \alpha}\right)$ we have
\[
W^{1,p}_{(\omega_1,\omega_2),0}(\Omega) \subset \subset L^r(\Omega).  
\]  
If, additionally, for arbitrary $U \subset \subset \Omega$ there exists a constant $c_U$ such that $\omega_2(x) \geq c_U > 0$ in $U$, then we can choose $\alpha = 0$.

In particular, under the above conditions, we have
\[
W^{1,p}_{(\omega_1,\omega_2),0}(\Omega) \subset \subset L^2(\Omega)  
\]  
and
\[
L^p(0,T; W^{1,p}_{(\omega_1,\omega_2),0}(\Omega)) \subset \subset L^2(0,T; L^2(\Omega)) = L^2(\Omega_T).
\]

### 2.3 The weights

In this section we give the restrictions on $\omega_1$ and $\omega_2$ sufficient for the existence of solutions to the problem
\[
 u_t - \text{div}(\omega_2|\nabla u|^{p-2}\nabla u) = \lambda \omega_1 |u|^{p-2}u \quad \text{on} \quad \Omega_T.
\]  
We call the pair of functions $(\omega_1, \omega_2)$ an admissible pair in our framework if the following conditions are satisfied
1. \(\omega_1, \omega_2 : \overline{\Omega} \to \mathbb{R}_+ \cup \{0\}\) and \(\omega_2\) is such that for any \(U \subset \subset \Omega\) there exists a constant \(\omega_2(x) \geq c_U > 0\) in \(U\);

2. \(\omega_1, \omega_2\) satisfy the \(B_p\)-condition \([3]\);

3. \(\omega_2\) satisfies \((\mathcal{H}_{\alpha})\) and if \(\Omega\) is unbounded we additionally require that \(\omega_2\) satisfies also condition \((\mathcal{H}^\infty_{\alpha,\beta})\);

4. \((\omega_1, \omega_2)\) is a pair of weights in the Hardy inequality \([10]\).

**Comments**

We give here the reasons for which we assume the above conditions, respectively:

1. is necessary for the strict monotonicity of the operator;

2. is necessary for \(W_{(\omega_1, \omega_2)}^{1,p}(\Omega)\) to be a Banach space;
   it implies \(\omega_1, \omega_2 \in L^1_{\text{loc}}(\Omega)\);

3. is necessary for the compact embedding \([7]\);
   in particular it provides the existence of the basis of \(W_{(\omega_1, \omega_2),0}^{1,p}(\Omega)\), which
   is orthogonal in \(L^2(\Omega)\);

4. is necessary for a priori estimates for solutions.

**Examples**

We give several examples of weights admissible in our setting. Following \([24, 25]\), we can take

- on \(\mathbb{R}^N \setminus \{0\}\)
  \[\omega_1(x) = |x|^{-\gamma}, \quad \omega_2(x) = |x|^{\gamma}, \quad \text{for } \gamma < p - N, \text{ with the optimal}\]
  \[\text{constant } \lambda_{N,p} = ((p - N - \gamma)/p)^p;\]

- on \(\mathbb{R}^N\)
  \[\omega_1(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}, \quad \omega_2(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma},\]
  \[\text{for } \gamma > 1, \text{ with } K = n \left(\frac{p(\gamma-1)}{p-1}\right)^{p-1} \text{ optimal whenever } \gamma \geq n+1-\frac{2}{p};\]
Let us consider any function $u$ that is superharmonic in $\Omega \subseteq \mathbb{R}^N$ (i.e. $\Delta u \leq 0$) and an arbitrary $\beta > 3$. We can take

$$\omega_1(x) = u^{-\beta-1}(x)|\nabla u(x)|^2 \quad \text{and} \quad \omega_2(x) = u^{-\beta+1}(x),$$

if only each of them satisfies $B_p$ condition (3). Then $K = 3(\beta - 3)$.

**General Hardy inequality**

The following result \cite{24, Theorem 4.1} gives sufficient conditions for the Hardy-type inequality to hold.

**Theorem 2.1.** Let $\Omega$ be any open subset of $\mathbb{R}^N$, $1 < p < \infty$, and nonnegative function $v \in W^{1,p}_{loc}(\Omega)$ such that $-\Delta_p v \in L^1_{loc}(\Omega)$. Suppose that the following condition is satisfied

$$\sigma_0 := \inf \{ \sigma \in \mathbb{R} : -\Delta_p v \cdot v + \sigma |\nabla v|^p \geq 0 \text{ a.e. in } \Omega \cap \{ v > 0 \} \} \in \mathbb{R}.$$

Moreover, let $\beta$ and $\sigma$ be arbitrary real numbers such that $\beta > \min\{0, \sigma\}$.

Then, for every Lipschitz function $\xi$ with compact support in $\Omega$, we have

$$K \int_{\Omega} |\xi|^p \omega_1(x) dx \leq \int_{\Omega} |\nabla \xi|^p \omega_2(x) dx,$$

where

$$K = \left( \frac{\beta - \sigma}{p - 1} \right)^{p-1},$$

$$\omega_1(x) = \left( -\Delta_p v \cdot v + \sigma |\nabla v|^p \right) \cdot v^{-\beta-1} \chi_{\{v > 0\}},$$

$$\omega_2(x) = v^{p-\beta-1} \chi_{\{|\nabla v| \neq 0\}}.$$

**Remark 2.1.** Note that under the assumptions of Theorem 2.1 we have $\omega_1, \omega_2 \in L^1_{loc}(\Omega)$. 

Remark 2.2. By $\lambda_{N,p}$ we denote the greatest possible constant $K$ such that (10) holds (for fixed weights $\omega_1, \omega_2$).

Let us mention several examples of application of [24, Theorem 4.1] leading to the inequalities with the best constants. Namely, they are achieved in the classical Hardy inequality (Section 5.1 in [24]); the Hardy-Poincaré inequality obtained in [25] due to [24], confirming some constants from [13] and [4] and establishing the optimal constants in further cases; the Poincaré inequality concluded from [24], confirmed to hold with best constant in Remark 7.6 in [9]. Moreover, the inequality in Theorem 5.5 in [24] can also be retrieved by the methods of [15] with the same constant, while some inequalities from Proposition 5.2 in [16] are comparable with Theorem 5.8 in [24]. Generalisation of [24, Theorem 4.1] in [11] leads to the optimal result in [17].

3 Existence

Let us start with the definition of a weak solution to the parabolic problem.

We consider

$$W : \Omega \to \mathbb{R}_+$$

such that

$$W(x) \leq \min\{m, \omega_1(x)\}$$

with a certain $m \in \mathbb{R}_+$.

**Definition 3.1.** We call a function $u$ a weak solution to

$$\begin{cases}
    u_t - \Delta_p^{\omega_2} u = \lambda W(x)|u|^{p-2}u & x \in \Omega, \\
    u(x, 0) = f(x) & x \in \Omega, \\
    u(x, t) = 0 & x \in \partial \Omega, \ t > 0,
\end{cases}$$

if

$$u \in L^p(0, T; W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)),$$

$$u_t \in L^{p'}(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega)),$$

and

$$\int_{\Omega_T} (u_t \xi + \omega_2 |\nabla u|^{p-2}\nabla u \nabla \xi + \lambda W(x)|u|^{p-2}u \xi) \, dx \, dt = 0,$$

holds for every $\xi \in L^p(0, T; W^{1,p}_{(\omega_1, \omega_2),0}(\Omega))$. 

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Existence of a solution to the problem is obtained by the Galerkin approximation, where we use the fact that the operator $-\Delta_p^{\omega_2}$ from (2) is monotone (note that it is implied by $\omega_2 > 0$ in $\Omega$).

**Theorem 3.1.** Suppose $2 \leq p < N$, $\Omega \subseteq \mathbb{R}^N$ is an open subset, $f \in L^2(\Omega)$, $m \in \mathbb{R}_+$. Assume that $\omega_1, \omega_2 : \Omega \rightarrow \mathbb{R}_+$ satisfying (3) are given by (11), (12), respectively. Moreover, assume $\omega_2$ satisfies $(H_\alpha)$ and if $\Omega$ is unbounded additionally require that $\omega_2$ satisfies also condition $(H_{\alpha, \beta})$.

There exist $\lambda_0 = \lambda_0(p, N, \omega_1, \omega_2)$ and a weak solution $u$ to (13), such that for all $\lambda \in (0, \lambda_0)$ the solution $u$ is in $L^\infty(0, T; L^2(\Omega))$.

**Proof.** We apply the Galerkin method. Remind that according to (8) we have

$$W^{1,p}_{(\omega_1, \omega_2),0}(\Omega) \subset L^2(\Omega),$$

which is a and the former is a closed subspace of the latter. Thus, each basis of $W^{1,p}_{(\omega_1, \omega_2),0}(\Omega)$ is contained in the $L^2(\Omega)$ and can be orthogonalised with respect to $L^2(\Omega)$ scalar product. Suppose $(e_j)_{j=1}^\infty$ is a basis of $W^{1,p}_{(\omega_1, \omega_2),0}(\Omega) \cap L^2(\Omega)$, which is orthogonal in $L^2(\Omega)$. We construct an approximating sequence $(u^n)_{n=1}^\infty$ given by

$$u^n(t) = \sum_{k=1}^n a^n_k(t)e_k, \quad t \in [0, T]. \quad (14)$$

We determine $a^n_k$ by solving the differential equation, which is a projection of the original problem to the finite-dimensional subspace span$(e_1, \ldots, e_n)$,

$$\begin{align*}
(a^n_k)'(t) &= \\
&= \sum_{j=1}^n \int_\Omega \omega_2 \left| \sum_{l=1}^n a^n_l(t) \nabla e_l \right|^{p-2} a^n_l \nabla e_j \nabla e_k dx + \lambda \int_\Omega W \left| \sum_{l=1}^n a^n_l(t) e_l \right|^{p-2} a^n_l e_j e_k dx \quad (15)
\end{align*}$$

with the initial conditions

$$\langle a^n_k(0), e_k \rangle = \langle f, e_k \rangle, \quad k = 1, \ldots, n,$$
\[ u^n(0) = f_n = \sum_{k=1}^{n} \langle f, e_k \rangle e_k \in \text{span}(e_1, \ldots, e_n). \]

Using the Peano theorem, we get the local existence of the coefficients \( a_k(t) \) on some intervals \([0, t_n]\). Note that \( a_k(t) \) depend on \( \|W\|_{L^\infty(\Omega)} \leq m \).

Let us now establish an a priori estimate for \( u^n \) and \( (u^n)_t \). Because of (15), we have
\[
\frac{1}{2} d\|u^n(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} \omega_2 |\nabla u^n|^p dx = \lambda \int_{\Omega} |W| u^n|^p dx
\]
and moreover
\[
\int_{0}^{T} \frac{1}{2} d\|u^n(t)\|_{L^2(\Omega)}^2 dt = \frac{1}{2} \|u^n(T)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u^n(0)\|_{L^2(\Omega)}^2.
\]

Therefore,
\[
\frac{1}{2} \|u^n(T)\|_{L^2(\Omega)}^2 + \int_{0}^{T} \|\nabla u^n(t)\|_{L^p(\Omega)}^p dt \leq \lambda \int_{0}^{T} \|u^n(t)\|_{L^p(\Omega)}^p dt + \frac{1}{2} \|u^n(0)\|_{L^2(\Omega)}^2.
\]

We estimate the right-hand side using the Hardy-type inequality (10) and we obtain
\[
\frac{1}{2} \|u^n(T)\|_{L^2(\Omega)}^2 + \int_{0}^{T} \|\nabla u^n(t)\|_{L^p(\Omega)}^p dt \leq \frac{\lambda}{K} \int_{0}^{T} \|\nabla u^n(t)\|_{L^p(\Omega)}^p dt + \frac{1}{2} \|u^n(0)\|_{L^2(\Omega)}^2,
\]
which we rearrange to
\[
\frac{1}{2} \|u^n(T)\|_{L^2(\Omega)}^2 + \left(1 - \frac{\lambda}{K}\right) \int_{0}^{T} \|\nabla u^n(t)\|_{L^p(\Omega)}^p dt \leq \frac{1}{2} \|u^n(0)\|_{L^2(\Omega)}^2. \quad (16)
\]

Thus we can assume \( t_n = T \) for each \( n \) and as \( u^n(0) \to f \in L^2(\Omega) \). We conclude that the sequence \((u^n)_{n=1}^\infty\) is bounded in
\[
L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_{(\omega_1, \omega_2, 0}(\Omega)).
\]
Thus, we can choose its subsequence (denoted \((u^n)_{n=1}^\infty\) as well) such that

\[
\begin{align*}
    u^n & \xrightarrow[n \to \infty]{} u \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)), \\
    u^n & \xrightarrow[n \to \infty]{} u \quad \text{in} \quad L^p(0, T; W^{1,p}_{(\omega_1, \omega_2), 0}(\Omega)), \\
    u^n(T) & \xrightarrow[n \to \infty]{} \zeta \quad \text{in} \quad L^2(\Omega).
\end{align*}
\]

The fact that \(\zeta = u(T)\) is a direct consequence of the arguments of [19], Chap. 2, Par. 1.2.2).

Let us show that the limit function \(u\) satisfies (13). As \(u^n\) solves the finite-dimensional projection of the problem (13), for each test function \(w \in L^p(0, T; W^{1,p}_{(\omega_1, \omega_2), 0}(\Omega))\) we have

\[
\int_0^T \int_{\Omega} -\Delta_p u^n w \, dx \, dt = \int_0^T \int_{\Omega} \nabla u^n |\nabla u^n|^{p-2} \nabla u^n \nabla w \omega_2 \, dx \, dt = \int_0^T \int_{\Omega} (\int_{\Omega} \omega_2 |\nabla u^n|^p \, dx)^{\frac{p-1}{p}} (\int_{\Omega} \omega_2 |\nabla w|^p \, dx)^{\frac{1}{p}} \, dt \leq \left( \int_0^T \left\| \nabla u^n \right\|_{L^p(\Omega, \omega_2)}^p \, dt \right)^\frac{p-1}{p} \left( \int_0^T \left\| \nabla w \right\|_{L^p(\Omega, \omega_2)}^p \, dt \right)^\frac{1}{p} = \|u^n\|_{L^p(0, T; W^{1,p}_{(\omega_1, \omega_2), 0}(\Omega))} \|w\|_{L^p(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega)).}
\]

Using boundedness of \((u^n)_{n=1}^\infty\) in \(L^p(0, T; W^{1,p}_{(\omega_1, \omega_2), 0}(\Omega))\) we infer that \((-\Delta_p u^n)_{n=1}^\infty\) is bounded in \(L^{p'}(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega))\). Moreover, we observe that there exists \(\chi \in L^{p'}(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega))\), such that (up to a subsequence) we have

\[-\Delta_p u^n \xrightarrow[n \to \infty]{} \chi \quad \text{in} \quad L^{p'}(0, T; W^{-1,p'}_{(\omega_1, \omega_2)}(\Omega)).\]
As for \((u^n)_t(t)\), for each \(w \in L^p(0,T;W^{1,p}_{(\omega_1,\omega_2),0}(\Omega))\) we have
\[
\int_0^T \int_\Omega (u^n)_t(t) w \, dx \, dt = \int_0^T \int_\Omega \left( -\Delta_p^{\omega_2} u^n(t) + \lambda W|u^n(t)|^{p-2} u^n(t) \right) w \, dx \, dt.
\]

Therefore
\[
(u^n)_t(t) \xrightarrow{n \to \infty} u_t(t) \quad \text{in} \quad L^p(0,T;W^{-1,p'}_{(\omega_1,\omega_2)}(\Omega)).
\]

Our aim is now to show that \(\chi = -\Delta_p^{\omega_2} u\), which finishes the proof. We observe that \(-\Delta_p^{\omega_2}\) is a monotone operator, therefore for each \(w \in L^p(0,T;W^{1,p}_{(\omega_1,\omega_2),0}(\Omega))\) there holds
\[
A^n := \int_0^T \langle -\Delta_p^{\omega_2} u^n(t) + \Delta_p^{\omega_2} w(t), u^n(t) - w(t) \rangle \, dt \geq 0.
\]

Since
\[
\int_0^T \langle -\Delta_p^{\omega_2} u^n(t), u^n(t) \rangle \, dt = \lambda \int_0^T \langle W|u^n|^{p-2} u^n, u^n \rangle \, dt + \frac{1}{2} \|u^n(0)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u^n(T)\|_{L^2(\Omega)}^2,
\]
we have
\[
A^n = \lambda \int_0^T \langle W|u^n|^{p-2} u^n, u^n \rangle \, dt + \frac{1}{2} \|u^n(0)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u^n(T)\|_{L^2(\Omega)}^2 + \int_0^T \langle -\Delta_p^{\omega_2} u^n(t), w(t) \rangle \, dt - \int_0^T \langle -\Delta_p^{\omega_2} w(t), u^n(t) - w(t) \rangle \, dt
\]
\[
= I_1 + I_2 + I_3 + I_4 + I_5.
\]
When \(n \to \infty\), taking into account (17), we observe that
\[
\bullet \quad I_1 \text{ converges (up to a subsequence, since } W \leq \omega_1, (11) \text{, and (16))};
\]
• $I_2, I_4, I_5$ converge;

• in the case of $I_3$, due to weak convergence of $u^n(T)$, we have

$$\lim \inf \|u^n(T)\|^2_{L^2(\Omega)} \geq |u^n(T)|^2.$$

We take upper limit in the above equation to get

$$0 \leq \limsup_{n \to \infty} A^n \leq \lambda \int_0^T W\|u\|^p_{L^p(\Omega)} dt + \frac{1}{2} \|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u(T)\|^2_{L^2(\Omega)} +$$

$$- \int_0^T \langle \chi, v \rangle dt - \int_0^T \langle - \Delta_\omega^2 v(t), u(t) - v(t) \rangle dt. \quad (18)$$

Note that $u + \chi = \lambda W|u|^{p-2}u$, so

$$\lambda \int_0^T W\|u\|^p_{L^p(\Omega)} dt + \frac{1}{2} \|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u(T)\|^2_{L^2(\Omega)} = \int_0^T \langle \chi, u \rangle dt.$$

This, together with $(18)$, implies

$$0 \leq \int_0^T \langle \chi, u \rangle dt - \int_0^T \langle \chi, v \rangle dt - \int_0^T \langle - \Delta_\omega^2 v, u - v \rangle dt = \int_0^T \langle \chi - (\Delta_\omega^2 v), u - v \rangle dt.$$

As we stated in Fact 2.2, hemicontinuity of $\Delta_\omega^2$ implies

$$\int_0^T \langle \chi - (\Delta_\omega^2 (u - \kappa w), w \rangle dt \geq 0,$$

where $v = u - \kappa w$, $w \in L^p(0, T; W_0^{1,p}(\Omega, \omega))$, and $\kappa > 0$ is arbitrary. Letting now $\kappa \to 0$ we get

$$\int_0^T \langle \chi - (\Delta_\omega^2 u), w \rangle dt \geq 0,$$

independently of the sign of $w$. Thus,

$$\chi = -\Delta_\omega^2 u.$$
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