AUTOMORPHISMS OF TWO-GENERATOR FREE GROUPS AND SPACES OF ISOMETRIC ACTIONS ON THE HYPERBOLIC PLANE

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Abstract. The automorphisms of a two-generator free group $F_2$ acting on the space of orientation-preserving isometric actions of $F_2$ on hyperbolic 3-space defines a dynamical system. Those actions which preserve a hyperbolic plane but not an orientation on that plane is an invariant subsystem, which reduces to an action of a group $\Gamma$ on $\mathbb{R}^3$ by polynomial automorphisms preserving the cubic polynomial

$$\kappa(x, y, z) := -x^2 - y^2 + z^2 + xyz - 2$$

and an area form on the level surfaces $\kappa^{-1}(k)$.

The Fricke space of marked hyperbolic structures on the 2-holed projective plane with funnels or cusps identifies with the subset $\mathcal{F}(C_{0,2}) \subset \mathbb{R}^3$ defined by

$$z \leq -2, \quad xy + z \geq 2.$$ 

The generalized Fricke space of marked hyperbolic structures on the 1-holed Klein bottle with a funnel, a cusp, or a conical singularity identifies with the subset $\mathcal{F}'(C_{1,1}) \subset \mathbb{R}^3$ defined by

$$z > 2, \quad xyz \geq x^2 + y^2.$$ 

We show that $\Gamma$ acts properly on the subsets $\Gamma \cdot \mathcal{F}(C_{0,2})$ and $\Gamma \cdot \mathcal{F}'(C_{1,1})$. Furthermore for each $k < 2$, the action of $\Gamma$ is ergodic on the complement of $\Gamma \cdot \mathcal{F}(C_{0,2})$ in $\kappa^{-1}_\phi(k)$ for $k < -14$. In particular, the action is ergodic on all of $\kappa^{-1}_\phi(k)$ for $-14 \leq k < 2$.

For $k > 2$, the orbit $\Gamma \cdot \mathcal{F}(C_{1,1})$ is open and dense in $\kappa^{-1}_\phi(k)$. We conjecture its complement has measure zero.

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This paper concerns moduli spaces of actions of discrete groups on hyperbolic space. Spaces of $\text{PSL}(2,\mathbb{C})$-representations of fundamental groups of surfaces of negative Euler characteristic are natural objects upon which mapping class groups and related automorphism groups act. They arise as deformation spaces of (possibly singular) locally homogeneous geometric structures on surfaces. We relate the interpretation in terms of geometric structures to the dynamics of automorphism groups, see also [24].
The rank two free group \( F_2 \) is the simplest such surface group. It arises as the fundamental group of four non-homeomorphic surfaces: the three-holed sphere \( \Sigma_{(0,3)} \), the one-holed torus \( \Sigma_{1,1} \), the two-holed projective plane (or cross-surface) \( C_{(0,2)} \), and the one-holed Klein bottle \( C_{(1,1)} \). Of these the first two are orientable and the second two are nonorientable. Furthermore \( \Sigma_{1,1} \) enjoys the remarkable property that every automorphism of \( \pi_1(\Sigma_{1,1}) \) is induced by a homeomorphism \( \Sigma_{1,1} \to \Sigma_{1,1} \). Equivalently, every homotopy-equivalence \( \Sigma_{1,1} \to \Sigma_{1,1} \) is homotopic to a homeomorphism. For concreteness, we choose free generators \( X \) and \( Y \) for \( F_2 \). When \( F_2 \) arises as the fundamental group \( \pi_1(\Sigma) \) of a nonorientable surface \( \Sigma \), we require that \( X \) and \( Y \) correspond to orientation-reversing simple closed curves. This defines a homomorphism

\[
\Phi : F_2 \to \{\pm 1\}
\]

where \( \Phi(X) = \Phi(Y) = -1 \). Then a loop \( \gamma \) preserves orientation if and only if \( \Phi(\gamma) = +1 \) and reverses orientation if and only if \( \Phi(\gamma) = -1 \).

\( F_2 \) possesses a rich and mysterious representation theory into the group \( \text{PSL}(2, \mathbb{C}) \) of orientation-preserving isometries of \( H^3 \). Such representations define orientation-preserving isometric actions on \( H^3 \), and intimately relate to geometric structures on surfaces and 3-manifolds having fundamental group isomorphic to \( F_2 \).

The dynamical system arises from an action of the outer automorphism group \( \text{Out}(F_2) := \text{Aut}(F_2)/\text{Inn}(F_2) \). The action on the abelianization \( \mathbb{Z}^2 \) of \( F_2 \) defines an isomorphism \( \text{Out}(F_2) \cong \text{GL}(2, \mathbb{Z}) \). The group \( \text{Out}(F_2) \) acts on the quotient \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \) by conjugation; the quotient space consists of equivalence classes of orientation-preserving isometric actions of \( F_2 \) on hyperbolic 3-space \( H^3 \). By an old result of Vogt [30] (sometimes also attributed to Fricke [7]), the quotient variety \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \) is isomorphic to \( \mathbb{C}^3 \), where the coordinates \( \xi, \eta, \zeta \in \mathbb{C} \) correspond to the traces of matrices representing \( \rho(X), \rho(Y), \rho(XY) \) respectively. We define a group \( \hat{\Gamma} \), arising from \( \text{Out}(F_2) \), which acts effectively and polynomially on \( \mathbb{C}^3 \), preserving the cubic polynomial

\[
\kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2
\]
corresponding to the trace of the commutator \([\rho(X), \rho(Y)]\). This group is commensurable to \( \text{GL}(2, \mathbb{Z}) \). See §3.3 for details.

This paper concerns isometric actions of \( F_2 \) on the hyperbolic plane \( H^2 \subset H^3 \). Isometric actions on \( H^2 \) which preserve orientation correspond to representations into \( \text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C}) \), and are discussed in detail in Goldman [9]. In particular they form a moduli space, parametrized by a quotient space of a subset of \( \mathbb{R}^3 \subset \mathbb{C}^3 \), upon which
Out$(F_2)$ acts, preserving the polynomial $\kappa$, and [9] gives the dynamical decomposition of the $\hat{\Gamma}$-action on the level sets of $\kappa$. (See also [10].)

The present paper concerns orientation-preserving isometric actions on $H^3$ which preserve $H^2$, but do not preserve the orientation on $H^2$. Important cases of such actions arise from hyperbolic structures on the two nonorientable surfaces $C(0,2)$ and $C(1,1)$. As above, we assume that the generators $X,Y$ of their fundamental groups are represented by orientation-reversing isometries of $H^2$, and the effect on orientation is recorded in the homomorphism $\Phi$ defined in (1), which may be interpreted as a Stiefel-Whitney class.

Explicitly they correspond to representations $\rho$ into $\text{PSL}(2,\mathbb{C})$ where $\rho(X)$ and $\rho(Y)$ are represented by purely imaginary matrices. This corresponds the case when the trace coordinates $\xi, \eta$ are purely imaginary, that is, the moduli space is another real form $i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \subset \mathbb{C}^3$

The effect on orientation of $H^2$ is extra information which breaks the symmetry of $\hat{\Gamma}$. This leads to a slightly smaller subgroup $\Gamma \subset \hat{\Gamma}$, also commensurable to $\text{GL}(2,\mathbb{Z})$, which preserves this extra information. See §2 for a detailed description of $\Gamma$ and its action on $\mathbb{R}^3 \cong i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \subset \mathbb{C}^3$.

For now, define $\Gamma$ as the group of automorphisms generated by the three Vieta involutions

$$
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
\eta - \xi \\
\eta \\
\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
\xi - \eta \\
\zeta \\
\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
\xi \\
\eta \\
\zeta - \zeta
\end{bmatrix},
$$

the three sign-changes

$$
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
-\xi \\
-\eta \\
-\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
-\xi \\
-\eta \\
-\zeta
\end{bmatrix},
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} \mapsto \begin{bmatrix}
-\xi \\
-\eta \\
\zeta
\end{bmatrix}
$$

and coordinate permutations preserving $\Phi$.

1.1. Fricke spaces and Fricke orbits. Representations

$$F_2 \xrightarrow{\xi} \text{PGL}(2,\mathbb{R})$$

arise naturally from hyperbolic surfaces with fundamental group isomorphic to $F_2$. If $S$ is such a hyperbolic surface, then the composition of an isomorphism $F_2 \cong \pi_1(S)$ with the holonomy representation $\pi_1(S) \hookrightarrow \text{PGL}(2,\mathbb{R})$ a representation $\rho$. Such representations are characterized as those which are injective with discrete image, and we call such representations discrete embeddings. Moreover $\text{Out}(F_2)$ acts
properly on the space of equivalence classes of discrete embeddings in $\text{PGL}(2, \mathbb{R})$.

If $\rho$ is a discrete embedding, then the hyperbolic surface

$$S := \mathbb{H}^2/\text{Image}(\rho)$$

has fundamental group $\text{Image}(\rho) \cong \mathbb{F}_2$. If $S$ is orientable, then its holonomy representation maps into the group $\text{PSL}(2, \mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}^2$. The corresponding homomorphism $\Phi$ is trivial and is treated in [9]. This paper concerns the case when $S$ is nonorientable and $\Phi$ is nontrivial.

When $S$ is nonorientable, it is homeomorphic to either a two-holed cross-surface $C_{(2,0)}$ or a one-holed Klein bottle $C_{(1,1)}$. (See, for example, Norbury [23] for a lucid description of these surfaces.) Let $S$ be one of these two surfaces. The Fricke space of $S$ is defined as the space of marked complete hyperbolic structures on $\text{int}(S)$. Here the ends are either cusps or funnels, ends of infinite area bounded by closed geodesics. A homotopy equivalence $\Sigma_{1,1} \rightarrow M$ to a hyperbolic surface $M \approx S$ induces a discrete embedding $\mathbb{F}_2 \rightarrow \text{PGL}(2, \mathbb{R})$ which we assume determines the invariant $\Phi$. The set $\mathcal{O}(S)$ of corresponding characters admits an action of $\Gamma$ and is a disjoint union of the copies of the Fricke space $\mathfrak{F}(S)$ of $S$. The mapping class group $\text{Mod}(S)$ acts on $\mathfrak{F}(S)$, and the components of $\mathcal{O}(S)$ bijectively correspond to the cosets $\text{Mod}(S) \setminus \Gamma$. We call $\mathfrak{O}(S)$ the Fricke orbit of $S$.

The generalized Fricke space $\mathfrak{F}'(S)$ of $S$ includes hyperbolic structures $S$ where some of the boundary components are replaced by complements of conical singularities. Define the generalized Fricke orbit $\mathfrak{O}'(S)$ as the orbit of the generalized Fricke space $\mathfrak{F}'(S)$.

Some of the components of the $\mathcal{O}(S)$ have a particularly simple description, and correspond to the Fricke space $\mathfrak{F}(S)$. In his dissertation [25], Stantchev computed the Fricke spaces of these surfaces and proved partial results on the $\Gamma$-action on the level sets of $\kappa$. This paper presents and extends Stantchev’s results. (Compare also [13].)

If $S \approx C_{(0,2)}$, take $X$ and $Y$ to be orientation-reversing simple loops with one intersection, such that $\partial S$ consists of curves in the homotopy classes $Z := XY$ and $Z' := XY^{-1}$. Then the Fricke space $\mathfrak{F}(C_{0,2})$ identifies with the subset

$$\mathfrak{F}(C_{0,2}) := \{(x, y, z) \in \mathbb{R}^3 \mid z \leq -2, \: xy + z \geq 2\}$$

with boundary traces $z$ and

$$z' := -xy - z.$$
Here $X, Y$ are represented by purely imaginary matrices in $\text{SL}(2, \mathbb{C})$, with respective traces $ix, iy \in i\mathbb{R}$.

If $S \approx C_{1,1}$, then, taking $X$ and $Y$ to be disjoint orientation-reversing simple loops, $\partial S$ is represented by a curve in the homotopy class $X^2Y^2$, and the boundary trace is
\[\delta := x^2 + y^2 - xyz + 2.\]

The Fricke space of $C_{1,1}$ identifies with
\[\mathcal{F}(C_{1,1}) := \{(x, y, z) \in \mathbb{R}^3 \mid z < -2, \ xyz > x^2 + y^2 + 4\}.

The generalized Fricke space $\mathcal{F}'(C_{1,1})$ comprises hyperbolic structures on the one-holed Klein bottle $C_{1,1}$ with a boundary component or a cusp, or on the Klein bottle $C_{1,0}$ with a conical singularity of cone angle $0 < \theta < 2\pi$. It identifies with
\[\mathcal{F}'(C_{1,1}) := \{(x, y, z) \in \mathbb{R}^3 \mid z < -2, \ xyz > x^2 + y^2\}.

where the boundary trace satisfies $\delta = -2\cos(\theta/2)$ for $-2 < \delta < 2$.

The generalized Fricke orbit $\mathcal{O}'(C_{1,1})$ is the union of all $\Gamma$-translates of the generalized Fricke space $\mathcal{F}'(C_{1,1})$.

1.2. The orientation-preserving case. Representations $\rho$ with
\[\xi, \eta, \zeta \leq -2\]
correspond to hyperbolic structures on $\Sigma_{0,3}$ whose ends are either collars about closed geodesics or cusps. In particular the Fricke space $\mathcal{F}(\Sigma_{0,3})$ of $\Sigma_{0,3}$ identifies with
\[\mathcal{F}(\Sigma_{0,3}) := (\infty, -2]^3 \subset \mathbb{R}^3 \subset \mathbb{C}^3.

Its stabilizer in $\widehat{\Gamma}$ consists of the symmetric group
\[\mathfrak{S}_3 \subset \widehat{\Gamma}.

Furthermore, if $\phi \in \widehat{\Gamma} \setminus \mathfrak{S}_3$, then
\[\mathcal{F}(\Sigma_{0,3}) \cap \phi\mathcal{F}(\Sigma_{0,3}) = \emptyset.

In particular $\widehat{\Gamma}$ acts properly on $\widehat{\Gamma}\mathcal{F}(\Sigma_{0,3})$.

Equivalence classes of representations $\rho$ with
\[\xi > 2, \eta > 2, \zeta > 2\]
and $\kappa(\xi, \eta, \zeta) \leq -2$ form the Fricke space $\mathcal{F}(\Sigma_{1,1})$ corresponding to marked complete hyperbolic structures on the one-holed torus. The ends are either cusps or funnels, that is, collars about closed geodesics or cusps.

More generally, when $-2 < \kappa(\xi, \eta, \zeta) < 2$ and $(\xi, \eta, \zeta) \in [2, \infty)^3$, the representation $\rho$ corresponds to a hyperbolic structure on a torus with
one singularity. This singularity has a neighborhood isometric to a cone of cone angle $\cos^{-1}(\kappa(\xi, \eta, \zeta)/2)$. We call the collection of marked hyperbolic structures whose ends are either funnels, cusps, or complete to conical singularities the \textit{generalized Fricke space} of $\Sigma$. It identifies in trace coordinates with the subset

$$F'(\Sigma_{1,1}) := \{(\xi, \eta, \zeta) \in (2, \infty)^3 | \kappa(\xi, \eta, \zeta) < 2\} \subset \mathbb{R}^3 \subset \mathbb{C}^3$$

If $\gamma \in \hat{\Gamma}$, then either

$$\gamma \cdot F(\Sigma_{1,1}) = F(\Sigma_{1,1})$$

or $\gamma$ is a nonzero element of $\text{Hom}(F_2, \{-\mathbb{I}\})$. In particular $\hat{\Gamma} \cdot F(\Sigma_{1,1})$ has four components, corresponding to the four elements of $\text{Hom}(F_2, \{-\mathbb{I}\})$. Furthermore $\hat{\Gamma}$ acts properly on $\hat{\Gamma} \cdot F(\Sigma_{1,1})$.

By definition, $\kappa < 2$ on $F(\Sigma_{1,1})$, and $\kappa \geq 18$ on $F(\Sigma_{0,3})$. In particular, $\kappa > 2$.

The main results of [9] may be summarized as follows. The dynamics behavior depends crucially on the commutator trace $k$. The classification divides into three cases, depending on whether $k < 2$, $k = 2$, or $k > 2$, respectively.

\textbf{Theorem.} Suppose $k < 2$. Let

$$U := [-2, 2]^3 \cap \kappa^{-1}(-\infty, 2] \subset \mathbb{R}^3$$

be the subset corresponding to unitary representations.

- $\hat{\Gamma}$ acts properly on $\kappa^{-1}(k) \setminus U$.
- The action of $\hat{\Gamma}$ on $U \cap \kappa^{-1}(k)$ is ergodic.
- If $k < -2$, then

$$U \cap \kappa^{-1}(k) = \emptyset$$

The case $k = 2$ corresponds to reducible representations and the action of $\hat{\Gamma}$ on $\kappa^{-1}(2) \setminus U$ is ergodic.

\textbf{Theorem.} Suppose $k > 2$.

- $\hat{\Gamma}$ acts properly on $\hat{\Gamma} \cdot F(\Sigma_{0,3})$.
- The action of $\hat{\Gamma}$ on $\kappa^{-1}(k) \setminus \hat{\Gamma} \cdot F(\Sigma_{0,3})$ is ergodic.
- When $2 \leq k \leq 18$, the action of $\hat{\Gamma}$ on $\kappa^{-1}(k)$ is ergodic.

1.3. \textbf{The Main Theorem.} The two homeomorphism types of nonorientable surfaces with fundamental group $F_2$, belong to the two-holed \textit{cross-surface} (projective plane) $C_{0,2}$ and the one-holed Klein bottle $C_{1,1}$. (Note that $C_{0,2}$ and $C_{1,1}$ can each be obtained from a three-holed sphere $\Sigma_{0,3}$ by attaching one or two cross-caps, respectively.) In his thesis [25], Stantchev computed the Fricke spaces of these surfaces and
proved partial results on the $\Gamma$-action on the level sets of $\kappa$. This paper presents and extends Stantchev’s results.

In particular the Fricke space $\mathfrak{F}(C_{0,2})$ identifies with the subset

$$\mathfrak{F}(C_{0,2}) := \{(x, y, z) \in \mathbb{R}^3 \mid z \leq -2, \ xy + z \geq 2\}$$

with boundary traces $z$ and

$$z' := -xy - z,$$

both of which satisfy $z, z' \leq -2$. The generalized Fricke space of $C_{1,1}$ identifies with

$$\mathfrak{F}'(C_{1,1}) := \{(x, y, z) \in \mathbb{R}^3 \mid z < -2, \ xyz > x^2 + y^2\},$$

with boundary trace $\delta := x^2 + y^2 - xyz + 2$. Define

$$\mathbb{R}^3 \xrightarrow{\kappa_{\Phi}} \mathbb{R}$$

$$(x, y, z) \mapsto -x^2 - y^2 + z^2 + xyz - 2,$$

corresponding to the trace of the commutator. Again we divide the statements into three parts, depending on whether the commutator trace $k$ satisfies $k < 2$, $k = 2$, or $k > 2$, respectively.

**Theorem.** Suppose that $k < 2$. Then:

- $\Gamma$ acts properly on $\Gamma \cdot \mathfrak{F}(C_{0,2})$.
- The action of $\Gamma$ on the complement $(\kappa_{\Phi})^{-1}(k) \setminus \Gamma \cdot \mathfrak{F}(C_{0,2})$ is ergodic.

When $k = 2$, the action is ergodic.

Suppose $k > 2$. Then:

- $\Gamma$ acts properly on $\Gamma \cdot \mathfrak{F}'(C_{1,1})$.
- interior$\left(\kappa^{-1}(k) \setminus \Gamma \cdot \mathfrak{F}'(C_{1,1})\right) = \emptyset$

In the last case, we conjecture that $\kappa^{-1}(k) \setminus \Gamma \cdot \mathfrak{F}'(C_{1,1})$ has measure zero.

Each of the surfaces $\Sigma_{0,3}$, $\Sigma_{1,1}$, $C_{0,2}$, $C_{1,1}$ has fundamental group isomorphic to $\mathbb{F}_2$. We choose a basis $(X, Y)$ for $\pi_1(\Sigma)$ represented by simple closed curves having geometric intersection number 0 or 1. If $\Sigma$ is nonorientable then we choose $X, Y$ to reverse orientation.

These provide further examples, the first being Goldman [9], Tan-Wong-Zhang [28], of where the $\Gamma$-action is proper although the representations themselves have dense image. Representations of $\mathbb{F}_2$ which are primitive-stable in the sense of Minsky [20] are included among the examples in Tan-Wong-Zhang [26].

A notable feature of this classification is that the dihedral characters corresponding to the triples $(0,0,z)$ where $z < -2$ or $z > 2$ lie in
the closure of the domain of discontinuity. These correspond to strong degenerations of a hyperbolic structure on a Klein bottle with one cone point (as the angle approaches $2\pi$). These degenerations lie on the boundary of the generalized Fricke orbits, but are not generic points on the boundary.

The dynamics of mapping class group actions on character varieties is quite complicated, already in the case of $\text{Rep}(F_2, \text{PSL}(2, \mathbb{C}))$, where the fractal behavior is reminiscent of holomorphic dynamics in one complex variable. However, for $\text{PSL}(2, \mathbb{R})$-representations, this complication is absent [9]. Another notable feature of the present work is the appearance of fractal-type behavior in $\text{Rep}(F_2, \text{PGL}(2, \mathbb{R}))$.

Another new phenomenon is that the closure of the Bowditch set $\mathcal{B}$ contains dihedral characters.

Another motivation for this study is that imaginary characters (for $k = -2$) arise in Bowditch’s original investigation [1] of Markoff triples, in proving that orbits accumulate at the origin in the Markoff surface. Maloni-Palesi-Tan [19] prove related results for 3-generator groups.

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**Notation and terminology**

Denote the cardinality of a set $S$ by $\#(S)$. For a given equivalence relation on a set $S$, denote the equivalence class of an element $s \in S$ by $[s]$. If $S \xrightarrow{A} S$ is a mapping, denote the set of fixed points of $A$ by $\text{Fix}(A)$. Denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$ by $\Sym_n$.

If $G$ is a group, denote the group of automorphisms $G \to G$ by $\text{Aut}(G)$. If $g \in G$, denote the corresponding inner automorphism by:

$$G \xrightarrow{\text{Inn}(g)} G$$

$$x \longmapsto gxg^{-1}.$$  

Denote the cokernel of the homomorphism

$$G \xrightarrow{\text{Inn}} \text{Aut}(G)$$

by

$$\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G).$$

If $G, H$ are groups and $G \to H$ is a homomorphism, denote the corresponding semidirect product by $G \rtimes H$ or $H \ltimes G$. Then $G \triangleleft (G \rtimes H)$ where $\triangleleft$ denotes the relation of normal subgroup. Denote the center of a group $G$ by $\text{Center}(G)$.

Denote the free group of rank two by $F_2$.

The usual Euclidean coordinates on $\mathbb{R}^3$ (respectively $\mathbb{C}^3$) will be denoted $x, y, z$ (respectively $\xi, \eta, \zeta$). The corresponding coordinate vector fields will be denoted $\partial_x, \partial_y, \partial_z$ (respectively $\partial_\xi, \partial_\eta, \partial_\zeta$).

Denote the identity matrix by $I$. Denote the transpose of a matrix $A$ by $A^\dagger$. An anti-involution of a complex algebraic object is an anti-holomorphic self-mapping of order two.

If $A$ is a ring and $n \in \mathbb{N}$ then $\text{GL}(n, A), \text{SL}(n, A)$ have their usual meanings. $\text{SL}_\pm(2, \mathbb{R})$ denotes the subgroup of $\text{GL}(2, \mathbb{R})$ comprising matrices of determinant $\pm 1$. Denoting multiplicative subgroup of units in $A$ by $A^\times$; then $\text{PGL}(n, A)$ denotes the cokernel of the homomorphism

$$A^\times \to \text{GL}(n, A)$$

$$a \mapsto a\mathbb{I}.$$
and $\text{PSL}(n, A)$ denotes the image of $\text{SL}(n, A)$ under the restriction of the quotient epimorphism $\text{GL}(n, A) \twoheadrightarrow \text{PGL}(n, A)$.

The *level two congruence subgroup* $\text{GL}(2, \mathbb{Z})_{(2)}$ is the kernel of the homomorphism defined by reduction modulo $2$:

$$\text{GL}(2, \mathbb{Z}) \longrightarrow \text{GL}(2, \mathbb{Z}/2)$$

and denote the corresponding subgroup of $\text{PGL}(2, \mathbb{Z})$ by:

$$\text{PGL}(2, \mathbb{Z})_{(2)} := \text{GL}(2, \mathbb{Z})_{(2)}/\{\pm I\} \subset \text{PGL}(2, \mathbb{Z}).$$

For any field $k$, denote the projective line over $k$ by $\mathbb{P}^1(k)$. The elements of $\mathbb{P}^1(k)$ are one-dimensional linear subspaces of the plane $k^2$. Denote the one-dimensional subspace of $k^2$ containing $(a_1, a_2)$ by $[a_1 : a_2] \in \mathbb{P}^1(k)$. The scalars $a_1, a_2$ are the *homogeneous coordinates* of the point $[a_1 : a_2] \in \mathbb{P}^1(k)$.

If $S$ is a manifold, denote its boundary by $\partial S$. If $V \subset S$ is a hypersurface, then denote the manifold-with-boundary obtained by splitting $S$ along $V$ by $S|V$. The quotient map $S|V \rightarrow S$ restricts to $q^{-1}(S \setminus V)$ by a homeomorphism, and to $q^{-1}(V)$ by a double covering-space.

## 2. The rank two free group and its automorphisms

In this section and the next we describe the group $\Gamma$ which acts polynomially on

$$\mathbb{R}^3 \overset{\kappa}{\longrightarrow} i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \hookrightarrow \mathbb{C}^3,$$

preserving the function $\kappa_\Phi$ and the Poisson structure $B_\Phi$ defined in (19). Both the function $\kappa_\Phi$ and the bivector field $B_\Phi$ are polynomial tensor fields.

The dynamical system $(\Gamma, \mathbb{R}^3)$ arises from automorphisms of the rank two free group $F_2$ which preserve a nonzero $\{\pm 1\}$-character $\Phi$ corresponding to the change of orientation of the invariant hyperbolic plane $H^2 \subset H^3$. It also contains a normal subgroup $\Sigma \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ of *sign-changes* which “twists” representations $\rho$ by homomorphisms $F_2 \rightarrow \{\pm 1\}$, where $\{\pm 1\}$ is the center of the group $\text{SL}(2, \mathbb{C})$. These transformations do not correspond to automorphisms of $F_2$ and we postpone their discussion to the next section. In this section we describe the group arising from $\text{Aut}(F_2)$ which acts faithfully on the character variety $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$, and its isomorphism to the *modular group* $\text{PGL}(2, \mathbb{Z})$.

### 2.1. The modular group and automorphisms

Let $F_2$ be a free group freely generated by elements $X, Y$. We call such an ordered pair $(X, Y)$ of generators an (ordered) *basis* of $F_2$. Every automorphism
\( \phi \in \text{Aut}(F_2) \) induces an automorphism of the abelianization of \( F_2 \), which is the free abelian group \( \mathbb{Z}^2 \). The corresponding epimorphism

\[
\text{Aut}(F_2) \twoheadrightarrow \text{GL}(2, \mathbb{Z})
\]

has kernel \( \text{Inn}(F_2) \), and induces an isomorphism \( \text{Out}(F_2) \cong \text{GL}(2, \mathbb{Z}) \).

The center of \( \text{GL}(2, \mathbb{Z}) \) is \( \{ \pm I \} \). The quotient \( \text{PGL}(2, \mathbb{Z}) := \text{GL}(2, \mathbb{Z})/\{ \pm I \} \) enjoys the structure of a semidirect product

\[ \mathcal{S}_3 \rtimes (\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2), \]

where the symmetric group \( \mathcal{S}_3 \) acts by permuting the three free factors.

Further analysis of \( \text{Aut}(F_2) \) involves a homomorphism \( \text{Aut}(F_2) \rightarrow \mathcal{S}_3 \) defined by reduction modulo 2. The \( \mathbb{Z}/2 \)-vector space

\[
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong \text{Hom}(F_2, \mathbb{Z}/2).
\]

has three nonzero elements. Since the field \( \mathbb{Z}/2 \) has only one nonzero element, this three-element set identifies with the projective line \( \mathbb{P}^1(\mathbb{Z}/2) \):

\[
\begin{aligned}
[0 : 1] & \leftrightarrow \text{even} \quad \text{odd} \quad \rightarrow 0 \\
[1 : 0] & \leftrightarrow \text{odd} \quad \text{even} \quad \rightarrow \infty \\
[1 : 1] & \leftrightarrow \text{odd} \quad \text{odd} \quad \rightarrow 1.
\end{aligned}
\]

Every permutation of this set is realized by a unique projective transformation, resulting in an epimorphism

\[
\text{PGL}(2, \mathbb{Z}) \twoheadrightarrow \text{PGL}(2, \mathbb{Z}/2) \cong \mathcal{S}_3.
\]

This homomorphism splits. Its kernel is the \textit{level two congruence subgroup} of \( \text{PGL}(2, \mathbb{Z}) \), denoted \( \text{PGL}(2, \mathbb{Z})(2) \). It is freely generated by the three involutions

\[
\mathcal{J}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{J}_1 := \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathcal{J}_2 := \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.
\]

Geometrically these involutions correspond to reflections in the sides of an ideal triangle in the Poincaré upper half-plane \( \mathbb{H}^2 \) with vertices \( 0, 1, \infty \) respectively. Specifically, on \( \mathbb{P}^1(\mathbb{Q}) \),

\[
\text{Fix}(\mathcal{J}_3) = \{ 0, \infty \}, \quad \text{Fix}(\mathcal{J}_2) = \{ \infty, 1 \}, \quad \text{Fix}(\mathcal{J}_1) = \{ 1, 0 \}.
\]

These elements of \( \text{GL}(2, \mathbb{Z}) \) correspond to automorphisms of \( F_2 \) as follows: The central involution \( -I \in \text{GL}(2, \mathbb{Z}) \) corresponds to the \textit{elliptic}
involution

\[ F_2 \overset{\iota}{\to} F_2, \quad X \mapsto X^{-1}, \quad Y \mapsto Y^{-1}. \]

This involution and \( \text{Inn}(F_2) \) generate a normal subgroup

\[ \text{Inn}^e(F_2) := \text{Inn}(F_2) \rtimes \langle e \rangle \trianglelefteq \text{Aut}(F_2), \]

with quotient \( \text{Aut}(F_2)/\text{Inn}^e(F_2) \cong \text{PGL}(2, \mathbb{Z}) \).

The automorphisms

\[
\begin{align*}
X \overset{\bar{3}_1}{\rightarrow} X & \quad X \overset{\bar{3}_1}{\rightarrow} Y^{-1}X^{-1}Y^{-1} & \quad X \overset{\bar{3}_2}{\rightarrow} X^{-1} \\
Y \overset{}{\rightarrow} Y^{-1} & \quad Y \overset{}{\rightarrow} Y & \quad Y \overset{}{\rightarrow} X^2Y
\end{align*}
\]

are involutions of \( F_2 \), and they represent the involutions \( \mathcal{I}_3, \mathcal{I}_1, \mathcal{I}_2 \) respectively.

Reduction modulo 2

\[ \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{P}^1(\mathbb{Z}/2) \]

partitions \( \mathbb{P}^1(\mathbb{Q}) \) into the three \( \text{PGL}(2, \mathbb{Z})_{(2)} \)-orbits, represented by \( \{0, \infty, 1\} \) as in (3). Points in the orbit of

\[ 1 \leftrightarrow \frac{\text{odd}}{\text{odd}} \]

we call \textit{totally odd}; points in the other two orbits we call \textit{partially even}.

Finally observe that the homomorphism

\[ \text{GL}(2, \mathbb{Z}) \overset{\text{Det}}{\to} \{\pm 1\} \]

extends to a homomorphism

\[ \text{Aut}(F_2) \overset{\text{Det}}{\to} \{\pm 1\} \]

factoring through

\[ \mathfrak{S}_3 \overset{\text{sgn}}{\to} \{\pm 1\}. \]

Using the identifications \( F_2 \cong \pi_1(\Sigma_{1,1}) \) and \( \text{Out}(F_2) \cong \text{Mod}(\Sigma_{1,1}) \), this action corresponds to the induced action of \( \text{Mod}(\Sigma_{1,1}) \) on the relative homology group

\[ H_2(\Sigma_{1,1}, \partial\Sigma_{1,1}; \mathbb{Z}) \cong \mathbb{Z}, \]

defined by the effect on orientation on \( \Sigma_{1,1} \).
2.2. **The tree of superbases.** We associate to $F_2$ a natural *a trivalent tree* $T$ upon which $\text{Aut}(F_2)$ acts. (Figure 1 depicts this tree.) The tree $T$ comes equipped with an embedding in the plane, and a tricoloring of its edges. Associated to a character $[\rho]$ is a natural flow $\xrightarrow{\rho}$ on $T$ (that is, a directed tree whose underlying tree is $T$). This dynamical invariant encodes the dynamics of the $\Gamma$-action, and is due to Bowditch [1].

Vertices of the tree correspond to *superbases*, which define coordinate systems $\text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$. The automorphisms $\tilde{f}_j$ defined in (6) correspond to the edges, and generate our dynamical system.

An *(ordered) basis* is an ordered pair $(X,Y)$ where $\{X,Y\}$ freely generates $F_2$. An element of $F_2$ is *primitive* if it lies in a basis. Denote the set of primitive elements of $F_2$ by $\text{Prim}(F_2)$. Primitive elements $X,Y$ are *equivalent* if $X$ is conjugate to either $Y$ or $Y^{-1}$. This equivalence relation on $\text{Prim}(F_2)$ arises from the $\text{Inn}(F_2) \times \mathbb{Z}/2$-action where inversion $Y \mapsto Y^{-1}$ generates the $\mathbb{Z}/2$-action. Geometrically, equivalence classes of primitives correspond to isotopy classes of *unoriented* essential nonperipheral simple closed curves on $\Sigma_{1,1}$.

It is well known that equivalence classes $[Z]$ of primitive elements correspond to points of $\mathbb{P}^1(\mathbb{Q})$: an equivalence class $[Z]$ corresponds to $[p : q]$ (that is, $p/q \in \mathbb{Q}$, assuming $q \neq 0$) if and only if

$$Z \equiv X^p Y^q (\text{mod } [F_2, F_2])$$

in the abelianization

$$F_2/[F_2, F_2] \cong \mathbb{Z}^2$$

and

$$\infty \longleftrightarrow [1 : 0] \in \mathbb{P}^1(\mathbb{Q}).$$

Equivalently, the pair $(p,q) \in \mathbb{Z}^2$ corresponds to the homology class of $Z$ in $H_1(F_2, \mathbb{Z}) \cong \mathbb{Z}^2$, or the projectivized homology class of an *oriented curve* representing the unoriented curve.

Define a *basic triple* to be an ordered triple $(X,Y,Z) \in \text{Prim}(F_2) \times \text{Prim}(F_2) \times \text{Prim}(F_2)$ such that:

- $(X,Y)$ is a basis of $F_2$;
- $XYZ = I$.

Clearly, an ordered basis $(X,Y) \in \text{Prim}(F_2) \times \text{Prim}(F_2)$ of $F_2$ extends uniquely to a basic triple, and bases of $F_2$ correspond bijectively to basic triples.

A *superbasis* of $F_2$ is an equivalence class of basic triples, where the equivalence relation is defined as follows. Basic triples $(X,Y,Z)$ and $(X',Y',Z')$ are *equivalent* if and only if

$$[X'] = [X], \quad [Y'] = [Y], \quad [Z'] = [Z].$$
Alternatively, a superbasis is an $\text{Inn}^* (F_2)$-equivalence class of basic triples, where $\text{Inn}^* (F_2)$ is defined in (5) and $e$ is the elliptic involution defined in (4). That is, two basic triples $(X, Y, Z)$ and $(X', Y', Z')$ represent the same superbasis if and only if
\begin{align*}
X' &= WXW^{-1} & X' &= WX^{-1}W^{-1} \\
Y' &= WYW^{-1} & \text{or} & & Y' &= WY^{-1}W^{-1} \\
Z' &= WZW^{-1} & & Z' &= WYZ^{-1}Y^{-1}W^{-1}
\end{align*}
for some $W \in F_2$. (Compare Charette-Drumm-Goldman [4].) Geometrically, a superbasis corresponds to an ordered triple of isotopy classes of unoriented simple loops on $\Sigma_{1,1}$ with mutual geodesic intersection numbers 1.

The notion of \textit{superbasis} is due to Conway [6] in the context of Markoff triples. Our notion is a slight modification to $F_2$, since Conway consider superbases of $Z^2$.

Now we define the three elementary moves on a superbasis. These correspond to the edges in the tree $T$. Suppose $b$ is a superbasis represented by a basic triple $(X, Y, Z)$ as above. Then $(X, Y)$ is a basis of $F_2$, and the pair $([X], [Y])$ correspond to two (unoriented) simple loops intersecting transversely in one point $p$, which we can conveniently use as a base point to define the fundamental group $\pi_1(\Sigma_{1,1}, p)$. There are exactly two ways to extend $([X], [Y])$ to a superbasis, depending on the the respective orientations of representative elements of $\pi_1(\Sigma_{1,1}, p)$. That is, there is a unique superbasis $(X', Y', Z')$ where
\begin{align*}
[X'] &= [X], & [Y'] &= [Y], & [Z'] &\neq [Z].
\end{align*}
Explicitly, this corresponds to the transformation of basic triples:
\begin{align*}
X' &= X \\
Y' &= Y^{-1} \\
Z' &= YX^{-1},
\end{align*}
the automorphism $\tilde{F}_3$ defined in (6).

Similarly there are unique superbases corresponding to basic triples $(X', Y'', Z'')$ with
\begin{align*}
[X'] &\neq [X], & [Y'] &= [Y], & [Z'] &= [Z]
\end{align*}
and
\begin{align*}
[X'] &= [X], & [Y'] &\neq [Y], & [Z'] &= [Z],
\end{align*}
which respectively correspond to $\tilde{F}_1$ and $\tilde{F}_2$. We call these three superbases the superbases \textit{neighboring} the superbasis $([X], [Y], [Z])$. 
The tree $T$ is defined as follows. Its vertex set $\text{Vert}(T)$ consists of superbases. We denote the vertex corresponding to a superbasis $([X], [Y], [Z])$ by

$$v = v(X, Y, Z) \in \text{Vert}(T).$$

Denote the set of edges by $\text{Edge}(T)$. Edges correspond to pairs of neighboring superbases, or, equivalently, equivalence classes of bases, and we write

$$e = e^{X, Y} \in \text{Edge}(T)$$

for the edge corresponding to the basis $(X, Y)$. This tree is just the Cayley graph of the rank three free Coxeter group

$$\mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2 \cong \text{PGL}(2, \mathbb{Z})_{(2)}.$$
2.3. The tricoloring and the planar embedding. Bowditch begins with the tree $T$ and a planar embedding of $T$. The extra structure of the planar embedding arises naturally as follows. Since $T$ is the Cayley graph of $\mathbb{Z}/2*\mathbb{Z}/2*\mathbb{Z}/2$, its edges correspond to the three free generators $I_1, I_2, I_3$. Thus we may color the edges accordingly, obtaining a tricoloring. Choosing a cyclic ordering of these three generators determines the germ of an embedding into a regular 2-dimensional neighborhood of each vertex, that is, the structure of a fatgraph on $T$; this structure extends to a planar embedding of $T$. This planar embedding (and the set $\Omega$ of components of the complement of $T$) plays a central role in Bowditch’s theory.

We introduce the following notation and terminology for $T$. Let $\text{Edge}(T)$ denote the set of edges in $T$ and $\Omega$ the set of complementary regions in the plane. Then $\Omega$ bijectively corresponds to the set of equivalence classes of primitive elements of $F_2$. If $([X], [Y], [Z])$ is a superbasis, then the three complementary regions around $v$ correspond to $X, Y$ and $Z$. If $e = e^{X,Y}$, then we say that the edge $e$ abuts the regions corresponding to $X$ and $Y$. One of the endpoints of $e$ is $v(X, Y, Z)$ and the other endpoint is $v(X, Y, Z')$ where $([X], [Y], [Z'])$ is the neighboring superbasis. We write

$$e = e^{X,Y}(Z, Z') \in \text{Edge}(T),$$

and say that the edge $e$ ends at the regions corresponding to $Z$ and $Z'$. Compare Figure 2.

Reduction modulo 2 solidifies the role of the tricoloring. Since

$$\Omega := \pi_0(H^2 \setminus T) \longleftrightarrow (\text{Prim}(F_2)/\sim) \leftrightarrow P^1(\mathbb{Q}),$$

the trichotomy

$$P^1(\mathbb{Q}) \twoheadrightarrow P^1(\mathbb{Z}/2) \cong \{\infty, 0, 1\}$$

tricolors $\Omega$. For every basic triple $(X, Y, Z)$, the mod 2 reductions

$$\{[X](\text{mod } 2), [Y](\text{mod } 2), [Z](\text{mod } 2)\} = \{\infty, 0, 1\}.$$

(This follows readily from the fact that the abelianizations of $X, Y, Z$ generate $\mathbb{Z}^2$.) Thus for each vertex $v = v(X, Y, Z)$ as above, the three regions around $v$ have three colors.

For an edge $e = e^{X,Y}(Z, Z')$ as above, then where $Z = Y^{-1}X^{-1}$ and $Z' = Y^{-1}X$, and $Z$ and $Z'$ have the same parity and so determine the same element in $P^1(\mathbb{Z}/2) = \{\infty, 0, 1\}$, which is an invariant of the edge. The resulting map

$$\text{Edge}(T) \twoheadrightarrow \{\infty, 0, 1\}.$$ 

describes a tricoloring of the tree. That is, each edge is colored by one of $\infty$, 0, or 1, as well.
Furthermore each edge $e$ is fixed by a unique involution of $T$ which is conjugate to a unique $I_j$ where $j = 1, 2, 3$ indexes the tricoloring. We write $j = j(e)$.

Hu-Tan-Zhang[15] give an alternate approach to this structure, not using a planar embedding. They define $\Omega$ using alternating geodesics, which correspond to alternating words in the $I_j$, as described in §2.4.

2.4. Paths and Alternating Geodesics.

**Definition 2.4.1.** By a finite path we mean a sequence

$$P = (v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{n-2}} v_{n-1} \xrightarrow{e_{n-1}} v_n)$$

where each $v_i \in \text{Vert}(T)$, each $e_i \in \text{Edge}(T)$ and $\partial e_i = \{v_i, v_{i+1}\}$. An infinite path is an infinite sequence

$$v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{n-2}} v_{n-1} \xrightarrow{e_{n-1}} v_n \xrightarrow{e_n} \ldots$$
such that each
\[ v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{k-2}} v_{k-1} \xrightarrow{e_{k-1}} v_k \]
is a finite path as above, for each \( k > 0 \). A path is a geodesic if and only if \( e_i \neq e_{i+1} \) for all \( i \). When the tree is directed, we shall consider directed geodesics, where \( e_i \) points from \( v_{i-1} \) to \( v_i \).

Associated to a finite path in \( T \) is the word
\[ w(P) := j_{j(e_{n-1})} \cdots j_{j(e_0)} \in \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2 \cong \text{PGL}(2, \mathbb{Z})_{(2)}. \]
If the path is a geodesic, then the word is reduced, that is, the sequence \( j(e_0), \ldots, j(e_n) \) contains no repetitions. If \( [(X_0, Y_0, Z_0)] \) is the superbasis corresponding to \( v_0 \), then the superbasis
\[ (9) \quad \left[ (w(P)(X_0), w(P)(Y_0), w(P)(Z_0)) \right] \]
corresponds to \( v_n \).

Given a complementary region \( Z \in \Omega \), the set \( C(Z) \) of edges abutting \( Z \) forms a geodesic in \( T \). The corresponding reduced word is an alternating sequence of \( j_i \) and \( j_j \) where \( i, j \in \{1, 2, 3\} \). Thus \( \Omega \) defines a collection of preferred geodesics, called alternating geodesics. The set \( \Omega \) bijectively corresponds to the set of alternating geodesics.

2.5. Relation to the one-holed torus. This tree arises from the topology of \( \Sigma_{1,1} \) as follows. The pants graph \( \text{Pants}(\Sigma_{1,1}) \) of \( \Sigma_{1,1} \) is the following graph:

- The vertex set consists of the set of free homotopy classes of unoriented essential, non-boundary parallel simple closed curves on \( \Sigma_{1,1} \) and identifies with \( \Omega \).
- Two vertices are connected by an edge if and only if the geometric intersection number of the corresponding curves is one.

The valence of every vertex is infinite, and every edge is contained in precisely two triangles in the graph. The pants graph identifies with the Farey tessellation on \( \mathbb{H}^2 \), with its vertex set identified with \( \mathbb{P}^1(\mathbb{Q}) \). This identification is fixed once we fix an identification of \( X, Y \) and \( Z = (XY)^{-1} \) with \( 0, \infty \) and \( 1 \), for a fixed basis \( X, Y \) of \( F_2 \). The graph dual to \( \text{Pants}(\Sigma_{1,1}) \) is the tree \( T \), with vertex set \( \text{Vert}(T) \), edge set \( \text{Edge}(T) \) and directed edge set \( \text{Edge}^+(T) \).

2.6. Effect of a \( \{\pm 1\} \)-character. This paper concerns actions \( \rho \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \) which preserve a hyperbolic plane \( \mathbb{H}^2 \subset \mathbb{H}^3 \) but not an orientation on \( \mathbb{H}^2 \). Thus the image of \( \rho \) lies in the stabilizer
$\text{PGL}(2, \mathbb{R})$ of $\mathbb{H}^2$. How $\rho$ fails to preserve an orientation is detected by the composition

$$F_2 \xrightarrow{\rho} \text{PGL}(2, \mathbb{R}) \xrightarrow{} \pi_0(\text{PGL}(2, \mathbb{R})) \cong \{\pm 1\}.$$ 

This defines an invariant of $\rho$, which is a nonzero element

$$\Phi \in \text{Hom}(F_2, \{\pm 1\}).$$

The three nonzero elements of $\text{Hom}(F_2, \{\pm 1\})$ are permuted by $\text{Aut}(F_2)$ via the homomorphism to $\mathfrak{S}_3$. We will choose one of these elements $\Phi$ once and for all, and define the subgroup $\text{PGL}(2, \mathbb{Z})_{\Phi}$ to be the stabilizer of $\Phi$ in $\text{PGL}(2, \mathbb{Z})$. Clearly $\text{PGL}(2, \mathbb{Z})_{\Phi}$ has index three in $\text{PGL}(2, \mathbb{Z})$ and contains $\text{PGL}(2, \mathbb{Z})_{(2)}$ with index two. Indeed the diagram

$$
\begin{array} {cccc}
\text{PGL}(2, \mathbb{Z})_{(2)} & \longrightarrow & \text{PGL}(2, \mathbb{Z})_{\Phi} & \longrightarrow & \text{PGL}(2, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \langle (12) \rangle & \longrightarrow & \mathfrak{S}_3
\end{array}
$$

commutes, where the horizontal arrows are inclusions and the vertical arrows are quotient homomorphisms by the congruence subgroup $\text{PGL}(2, \mathbb{Z})_{(2)} = \text{Ker}(\text{PGL}(2, \mathbb{Z}) \to \mathfrak{S}_3)$.

Since $F_2 = \langle X, Y \rangle$, the kernel $\text{Ker}(\Phi)$ must contain at least one of $X, Y$. We choose $\Phi$ to be nontrivial on both $X$ and $Y$:

\begin{align*}
(10) \quad F_2 & \xrightarrow{\Phi} \{\pm 1\} \\
X & \mapsto -1 \\
Y & \mapsto -1
\end{align*}

$\text{PGL}(2, \mathbb{Z})_{\Phi}$ equals the inverse image of the cyclic group $\langle (12) \rangle \subset \mathfrak{S}_3$ under the homomorphism

$$\hat{\Gamma} \twoheadrightarrow \mathfrak{S}_3$$

and is generated by $\text{PGL}(2, \mathbb{Z})_{(2)}$ and the automorphism corresponding to the transposition $(12) \in \mathfrak{S}_3$:

$$
\begin{array} {cccc}
F_2 & \xrightarrow{\rho_{(12)}} & F_2 \\
X & \mapsto & Y \\
Y & \mapsto & X
\end{array}
$$
3. CHARACTER VARIETIES AND THEIR AUTOMORPHISMS

Our basic object of study is the character variety \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \) of equivalence classes of \( \text{SL}(2, \mathbb{C}) \)-representations of \( F_2 \). Two representations are equivalent if the closures of their \( \text{Inn}(\text{SL}(2, \mathbb{C})) \)-orbits intersect. Here the closures of \( \text{Inn}(\text{SL}(2, \mathbb{C})) \)-orbits are encoded using the trace function \( \text{SL}(2, \mathbb{C}) \xrightarrow{\text{tr}} \mathbb{C} \) applied to the representation.

We begin with the spaces and their geometry; the main result is Vogt’s theorem which identifies \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \) with affine space \( \mathbb{C}^3 \). This identification \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \leftrightarrow \mathbb{C}^3 \) depends on a superbasis. We compute the action of the group \( \Gamma \) by polynomial automorphisms, which we interpret as a group of automorphisms of the tree \( T \). Then we discuss how the \( \{\pm 1\} \)-character \( \Phi \) defines a purely imaginary real form of \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \). Finally we describe the invariant function \( \kappa_\Phi \) and the Poisson bivector \( B \).

3.1. The deformation space. The space \( \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \) of \( \text{SL}(2, \mathbb{C}) \)-representations of \( F_2 \) identifies with the Cartesian product \( \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \) via:

\[
\text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \longrightarrow \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \\
\rho \longmapsto (\rho(X), \rho(Y))
\]

Composition of \( \rho \) with an inner automorphism \( \text{Inn}(g) \) corresponds to the diagonal action by conjugation of \( g \) on \( \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \). The character variety is defined as the categorical quotient of this action.

To describe this action, a more convenient presentation of \( F_2 \) is the following redundant presentation corresponding to a basic triple:

\[
F_2 = \langle X, Y, Z \mid XYZ = \mathbb{I} \rangle,
\]

(so that \( Z = (XY)^{-1} \)). The mapping

\[
\text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \xrightarrow{\text{tr}} \mathbb{C}^3 \\
\rho \longmapsto \begin{bmatrix} \text{tr}(\rho(X)) \\ \text{tr}(\rho(Y)) \\ \text{tr}(\rho(Z)) \end{bmatrix}
\]

is a categorical quotient for the \( \text{Inn}(\text{SL}(2, \mathbb{C})) \)-action, that is:

**Proposition 3.1.1** (Vogt [30]). Let

\[
\text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \xrightarrow{f} \mathbb{C},
\]
be an \( \text{Inn}(\text{SL}(2, \mathbb{C})) \) -invariant regular function. Then

\[
f(\rho) = F\left( \text{tr}(\rho(X)), \text{tr}(\rho(Y)), \text{tr}(\rho(Z)) \right)
\]

for a unique polynomial function

\[
F(\xi, \eta, \zeta) \in \mathbb{C}[\xi, \eta, \zeta].
\]

Furthermore the restriction of \( \mathfrak{X} \) to the subset of irreducible representations is a quotient mapping for the action of \( \text{Inn}(\text{SL}(2, \mathbb{C})) \) on \( \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \). In particular two irreducible representations have the same character if and only if they are \( \text{SL}(2, \mathbb{C}) \)-conjugate.

(See Goldman [12] for a twenty-first century discussion.)

3.2. Sign-changes. The group \( \text{PGL}(2, \mathbb{Z}) \) acts faithfully and polynomially on the \( \text{SL}(2, \mathbb{C}) \)-character variety of \( F_2 \) and encodes the action of \( \text{Aut}(F_2) \). It preserves extra structure coming from the function \( \kappa \) and the Poisson bivector \( B \). However, a small extension of this action preserves all this structure. This extension includes transformations arising from different ways of lifting representations from \( \text{PSL}(2, \mathbb{C}) \) to \( \text{SL}(2, \mathbb{C}) \).

Suppose that \( \rho_1, \rho_2 \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \) both project to the same representation in \( \text{Hom}(F_2, \text{PSL}(2, \mathbb{C})) \). Let \( \gamma \in F_2 \). Since

\[
\text{Ker}\left( \text{SL}(2, \mathbb{C}) \rightarrow \text{PSL}(2, \mathbb{C}) \right) = \{ \pm I \},
\]

the difference \( \rho_1(\gamma)\rho_2(\gamma)^{-1} \in \{ \pm I \} \). Since \( \{ \pm I \} = \text{Center}(\text{SL}(2, \mathbb{C})) \),

\[
F_2 \xrightarrow{\sigma} \{ \pm I \} \\
\gamma \mapsto \rho_1(\gamma)\rho_2(\gamma)^{-1}
\]

is a homomorphism. Conversely, any homomorphism \( F_2 \xrightarrow{\sigma} \{ \pm I \} \) acts on \( \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \), preserving the projection

\[
\text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \rightarrow \text{Hom}(F_2, \text{PSL}(2, \mathbb{C})).
\]

Indeed, this projection is a \( \Sigma \)-principal bundle, where

\[
\Sigma := \text{Hom}(F_2, \{ \pm I \}) \cong \{ \pm 1 \} \times \{ \pm 1 \}
\]

the Klein four-group, with three nontrivial elements \( \sigma_1, \sigma_2, \sigma_3 \), each of order two.

Let \( \rho \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \). Here is how \( \sigma \in \Sigma \) acts on \( \rho \):

(12)

\[
\begin{align*}
X & \xrightarrow{\sigma_1(\rho)} \rho(X) & X & \xrightarrow{\sigma_2(\rho)} -\rho(X) & X & \xrightarrow{\sigma_3(\rho)} -\rho(X) \\
Y & \mapsto -\rho(Y) & Y & \mapsto \rho(Y) & Y & \mapsto -\rho(Y).
\end{align*}
\]
We call $\Sigma$ the group of \textit{sign-changes}.

Both $\text{Aut}(F_2)$ and $\Sigma$ act on $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$, and they generate an action of the semidirect product $\text{Aut}(F_2) \ltimes \Sigma$ on $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$. Furthermore $\text{Inn}^e(F_2)$ remains normal in this group. The quotient group $\hat{\Gamma} := (\text{Aut}(F_2) \ltimes \Sigma)/\text{Inn}^e(F_2) \cong \text{PGL}(2, \mathbb{Z}) \ltimes \Sigma$ is the group which acts faithfully on the $\text{SL}(2, \mathbb{C})$-character variety. Note that the semidirect product is defined by the homomorphism $\text{PGL}(2, \mathbb{Z}) \to S_3$, where $S_3$ acts by permutations of $\Sigma \setminus \{1\}$. In particular $\hat{\Gamma}$ is a split extension $\text{PGL}(2, \mathbb{Z})(2) \times \Sigma \to \hat{\Gamma} \to S_3$.

Observe that the homomorphism $\text{Det}$ defined in (7) extends to a homomorphism
\begin{equation}
\hat{\Gamma} \xrightarrow{\text{Det}} \{\pm 1\}.
\end{equation}

3.3. \textbf{Action of automorphisms}. The sign-change group $\Sigma$ and the automorphism group $\text{Aut}(F_2)$ act on $\text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$ preserving its algebraic structure. They generate an action of the semidirect product $\text{Aut}(F_2) \ltimes \Sigma$ on $\mathbb{C}^3$ by polynomial automorphisms. Here we describe this action explicitly on some of the generators.

Observe that $\text{Aut}(F_2) \ltimes \Sigma$ does not act faithfully on $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$. We describe its kernel. First note that the trace function $\text{tr} : \text{SL}(2, \mathbb{C}) \to \mathbb{C}$ is $\text{Inn}(\text{SL}(2, \mathbb{C}))$-invariant, that is, it is a \textit{class function}. Hence $\text{Inn}(F_2)$ acts trivially on $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$. Therefore the $\text{Aut}(F_2) \ltimes \Sigma$-action factors through $\text{Out}(F_2) \ltimes \Sigma$.

This action is still not faithful, since the elliptic involution $\epsilon$ defined in (4) acts trivially as well. First note that $\text{tr}$ is \textit{inversion-invariant}: if $A \in \text{SL}(2, \mathbb{C})$, then
\begin{equation}
\text{tr}(A^{-1}) = \text{tr}(A).
\end{equation}
If $\rho \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$, then
\[\epsilon(Z) = X^{-1}Y^{-1} = \text{Inn}(X^{-1})(Z^{-1})\]
so:
\[
\text{tr}(\rho(\epsilon(X))) = \text{tr}(\rho(X^{-1})) = \text{tr}(\rho(X))
\]
\[
\text{tr}(\rho(\epsilon(Y))) = \text{tr}(\rho(Y^{-1})) = \text{tr}(\rho(Y))
\]
\[
\text{tr}(\rho(\epsilon(Z))) = \text{tr}(\text{Inn}(\rho(X^{-1}))(\rho(Z)^{-1})) = \text{tr}(\rho(Z)).
\]
Therefore $\varepsilon$ acts trivially on $[\rho]$ as claimed.

Therefore the restriction of the $(\text{Aut}(F_2) \ltimes \Sigma)$-action to the subgroup $\text{Inn}^e(F_2) \subset \text{Aut}(F_2)$ is trivial, and the $(\text{Aut}(F_2) \ltimes \Sigma)$-action factors through

$$\hat{\Gamma} := \left( \text{Aut}(F_2)/\text{Inn}(F_2) \right) \ltimes \Sigma \cong \text{PGL}(2, \mathbb{Z}) \times \Sigma.$$ 

The sign-changes $\Sigma$, defined in (12), act on characters by:

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto -\begin{bmatrix} -\xi \\ -\eta \\ -\zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} -\xi \\ -\eta \\ -\zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto -\begin{bmatrix} -\xi \\ -\eta \\ -\zeta \end{bmatrix}.$$

The involutions $\mathcal{I}_i$, defined in (6), freely generate $\text{PGL}(2, \mathbb{Z}) \times \Sigma$. They act by the following Vieta involutions on $\mathbb{C}^3$:

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \eta \zeta - \xi \\ \eta \\ \zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi \zeta - \eta \\ \xi \\ \zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi \zeta - \eta \\ \xi \\ \zeta \end{bmatrix}.$$

Although $\text{PGL}(2, \mathbb{Z}) \rightarrow \mathfrak{S}_3$ splits, the composition

$$\text{Aut}(F_2) \rightarrow \text{PGL}(2, \mathbb{Z}) \rightarrow \mathfrak{S}_3$$

does not split. However, a left-inverse $\mathfrak{S}_3 \rightarrow \text{PGL}(2, \mathbb{Z})$ determines the usual action of $\mathfrak{S}_3$ on $\mathbb{C}^3$ by permuting the coordinates. For example the transposition $(12) \in \mathfrak{S}_3$ induces the automorphism:

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \eta \\ \xi \\ \zeta \end{bmatrix}.$$

For properties of this action see Magnus [18], Bowditch [1], [12], Cantat-Loray [3] and Cantat [2].

3.4. Real forms of the character variety. Several real structures on $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$ exist. For example, complex-conjugation on $\text{SL}(2, \mathbb{C})$ induces an anti-involution on $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ whose fixed-point set is $\text{Hom}(F_2, \text{SU}(2))$. Another example is the Cartan anti-involution

$$\text{SL}(2, \mathbb{C}) \rightarrow \text{SL}(2, \mathbb{C})$$

$$A \mapsto (A^\dagger)^{-1},$$

which fixes $\text{SU}(2) \subset \text{SL}(2, \mathbb{C})$. This anti-involution induces an anti-involution of $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ whose fixed-point set is $\text{Hom}(F_2, \text{SU}(2))$. They induce the same anti-involution on $\text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$ (given by complex-conjugation on $\mathbb{C}^3$) and whose fixed-point set is $\mathbb{R}^3 \subset \mathbb{C}^3$. 


This coincidence arises from the fact that the two anti-involutions on $\text{SL}(2, \mathbb{C})$ are $\text{Inn}(\text{SL}(2, \mathbb{C}))$-related:

$$(\bar{A}^t)^{-1} = JAJ^{-1} = \text{Inn}(J)\bar{A} \quad \text{where} \quad J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

We are interested in a slight variation of this real structure, when it is twisted by a character $\Phi \in \text{Hom}(F_2, \{\pm 1\})$. Namely, the map associating to a representation $\rho \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ the representation

$$\gamma \mapsto \Phi(\gamma) \bar{\rho}(\gamma)$$

is an anti-involution of $\text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ inducing an anti-involution of $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$. When $\Phi$ is trivial, this is just the above anti-involution. When $\Phi$ is the $\{\pm 1\}$-character defined in (10), then the corresponding anti-involution of $\text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$ fixes

$$i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \subset \mathbb{C}^3.$$ 

3.5. **Real and imaginary characters.** Real characters correspond to representations conjugate to $\text{SU}(2)$-representations or $\text{SL}(2, \mathbb{R})$-representations. In terms of hyperbolic geometry, $\text{SU}(2)$-representations correspond to actions of $F_2$ on $\mathbb{H}^3$ which fix a point in $\mathbb{H}^3$ and $\text{SL}(2, \mathbb{R})$-representations correspond to actions which preserve an oriented plane $\mathbb{H}^2$ in $\mathbb{H}^3$. A point $(\xi, \eta, \zeta) \in \mathbb{R}^3$ corresponds to an $\text{SU}(2)$-representation if and only if

$$-2 \leq \xi, \eta, \zeta \leq 2, \quad \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta \leq 4.$$ 

Equivalence classes of $\text{SL}(2, \mathbb{R})$-representations correspond to points in the complement in $\mathbb{R}^3$ of the interior of this set. (The boundary of this set consists of representations in $\text{SO}(2) = \text{SU}(2) \cap \text{SL}(2, \mathbb{R})$.)

The set of $\{\pm 1\}$-characters equals the four-element group

$$\text{Hom}(F_2, \{\pm 1\}) \cong H^1(F_2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

under the isomorphism

$$\mathbb{Z}/2 \xrightarrow{\cong} \{\pm 1\}$$

$$n \mapsto (-1)^n.$$ 

The group $\text{Out}(F_2) \cong \text{PGL}(2, \mathbb{Z})$ acts on the three-element set of nonzero elements, by the homomorphism

$$\text{Out}(F_2) \cong \text{PGL}(2, \mathbb{Z}) \rightarrow \text{PGL}(2, \mathbb{Z}/2) \cong S_3.$$
This paper concerns actions which preserve $H^2 \subset H^3$ but do not preserve orientation on $H^2$. Since $X, Y$ generate $F_2$, at least one of $\rho(X), \rho(Y)$ reverse orientation. Since $\rho(Z) = \rho(Y)^{-1}\rho(X)^{-1}$, exactly one of $\rho(X), \rho(Y), \rho(Z)$ preserves orientation. Therefore three cases arise, which are all equivalent under the cyclic permutation of $X, Y, Z$ (apparent from the presentation (11)). We reduce to the case that $\rho(Z)$ preserves orientation, so that $\rho(X)$ and $\rho(Y)$ each reverse orientation. This is the case defined in (10). Then the stabilizer of $\Phi$ corresponds to the subgroup comprising matrices
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ab - cd = \pm 1
\]
where $a \equiv 1(\text{mod } 2), c \equiv 0(\text{mod } 2)$.

Whether an element of $F_2$ preserves or reverses orientation is detected by the homomorphism $\Phi$ defined in §2.6. In terms of traces, this means that $\xi = \text{tr}(\rho(X))$ and $\eta = \text{tr}(\rho(Y))$ are purely imaginary and that $\zeta = \text{tr}(\rho(Z))$ is real. Thus we write
\[
\begin{align*}
\text{tr}(\rho(X)) &= ix \\
\text{tr}(\rho(Y)) &= iy \\
\text{tr}(\rho(Z)) &= z
\end{align*}
\]
where $x, y, z \in \mathbb{R}$.

3.6. **Invariants of the action.** The function
\[
\kappa(\xi, \eta, \zeta) := \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2
\]
arises as the trace of the commutator $[X, Y] := XYX^{-1}Y^{-1}$:
\[
\text{tr}(\rho([X, Y])) = \kappa(\xi, \eta, \zeta).
\]
By Nielsen’s theorem [22] that $\text{Aut}(F_2)$ preserves $[X, Y]$ up to conjugacy and inversion (and the fact that $\text{tr}(a) = \text{tr}(a^{-1})$ for $a \in \text{SL}(2, \mathbb{C})$), the function $\kappa$ is $\text{Aut}(F_2)$-invariant.

The action also preserves the exterior 3-form $d\xi \wedge d\eta \wedge d\zeta$ on $\mathbb{C}^3$ and its dual exterior trivector field $\partial_\xi \wedge \partial_\eta \wedge \partial_\zeta$ on $\mathbb{C}^3$. Interior product of this trivector field with the 1-form $dk$ defines a bivector field
\[
(16) \quad B := (2\zeta - \xi\eta) \partial_\xi \wedge \partial_\eta + (2\xi - \eta\zeta) \partial_\eta \wedge \partial_\zeta + (2\eta - \zeta\xi) \partial_\zeta \wedge \partial_\xi
\]
which defines a complex-symplectic structure on each level set $\kappa^{-1}(t)$. This bivector is $\hat{\Gamma}$-invariant in the sense that if $\gamma \in \hat{\Gamma}$, then
\[
(17) \quad \gamma_\ast B = \text{Det}(\gamma)B
\]
where $\text{Det}$ is the homomorphism defined by (13).

We now restrict $(\kappa, B)$ to the real forms of $\text{Rep}(F_2, SL(2, C))$ defined in §3.4. Under the substitution

$$
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} :=
\begin{bmatrix}
ix \\
iy \\
z
\end{bmatrix}
$$

the restriction of $\kappa$ to $i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$ equals:

$$(18) \quad \kappa_\Phi(x, y, z) := \kappa(ix, iy, z) = -x^2 - y^2 + z^2 + xyz - 2$$

where $\kappa$ is defined in (15). Similarly the restriction of $B$ equals:

$$(19) \quad B_\Phi := (2z + xy) \partial_x \wedge \partial_y + (-2x + yz) \partial_y \wedge \partial_z + (-2y + zx) \partial_z \wedge \partial_x.$$ 

The action of $\Gamma$ restricts to a polynomial action on $i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$, which preserves the function $\kappa_\Phi$. This action preserves area on each level set $\kappa_\Phi^{-1}(k)$ with respect to the area form induced by (16), in the sense of (17). It is with respect to these measures that we studied the ergodicity of the group actions in [9].

We first describe the topology of the level sets $\kappa_\Phi^{-1}(k)$ and then relate the dynamical system defined by the $\Gamma$-action to geometric structures on $C_{1,1}$ and $C_{0,2}$.

4. Topology of the imaginary commutator trace

In this section we consider the purely imaginary real form

$$
\text{Rep}(F_2, SL(2, C))^{\Phi} \cong \mathbb{R}^3 \overset{\cong}{\rightarrow} i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \\
\subset \mathbb{C}^3 \cong \text{Rep}(F_2, SL(2, C))
$$

of $\text{Rep}(F_2, SL(2, C))$ associated to the $\{\pm 1\}$-character $\Phi$ as defined in §3.4. The restriction of the commutator trace function is the cubic polynomial $\mathbb{R}^3 \overset{\kappa_\Phi}{\rightarrow} \mathbb{R}$ defined in (18). In this section we analyze the topology of the level sets $\kappa_\Phi^{-1}(k)$ for $k \in \mathbb{R}$:

**Theorem 4.**  
(1) Suppose $k > -2$.

- $\kappa_\Phi^{-1}(k)$ is a smooth surface with two connected components.
- The restriction of the projection

$$
\kappa_\Phi^{-1}(k) \overset{\Pi_{xy}}{\rightarrow} \mathbb{R}^2
$$

to each component is a diffeomorphism onto $\mathbb{R}^2$.
- The components of $\kappa_\Phi^{-1}(k)$ are graphs of functions $\mathbb{R}^2 \overset{\pm}{\rightarrow} \mathbb{R}$.
- The sign-changes $\sigma_1, \sigma_2$ permute the two components.
• When \( k \geq 2 \), the region \( \mathbb{R} \times \mathbb{R} \times [-2,2] \) separates the two components of \( \kappa_{-1}(k) \).

(2) \( \kappa_{-1}(-2) \) is connected, with one singular point \((0,0,0)\), and \( \sigma_1, \sigma_2 \) permute the two components of \( \kappa_{-1}(-2) \setminus \{(0,0,0)\} \).

(3) Finally, when \( k < -2 \), the level set \( \kappa_{-1}(k) \) is smooth and connected. The restriction of the projection \( \Pi_{xy} \) to \( \kappa_{-1}(-2) \) is connected, with one singular point \((0,0,0)\), and \( \sigma_1, \sigma_2 \) permute the two components of \( \kappa_{-1}(-2) \) \{\( (0,0,0) \}\).

The only value of \( k \) for which the level set \( \kappa_{-1}(k) \) admits a rational parametrization is \( k = 2 \); compare §10 for more details. Figure 3 depicts the contours of the function \((x^2 + 4)(y^2 + 4)\).

4.1. Preliminaries. Let \( \Pi_{xy} \) denote projection to the \( xy \)-plane:

\[
\mathbb{R}^3 \xrightarrow{\Pi_{xy}} \mathbb{R}^2
\]

(20) \((x,y,z) \mapsto (x,y)\).

Let \( Q_z(x,y) := x^2 - zxy + y^2 \).

**Lemma 4.1.1.** For \(|z| < 2\), the quadratic form \( Q_z \) is positive definite.

**Proof.** Write

\[
Q_z(x,y) = \frac{2 + z}{4}(x - y)^2 + \frac{2 - z}{4}(x + y)^2.
\]

Since \(|z| < 2\), this is a positive linear combination of two squares. \( \square \)

**Lemma 4.1.2.** \((0,0,0)\) is the only critical point of \( \mathbb{R}^3 \xrightarrow{\kappa_{-1}} \mathbb{R} \).

**Proof.** Since

\[
d\kappa_{-1} = (-2x + yz)dx + (-2y + zx)dy + (2z + xy)dz,
\]

the point \( p = (x, y, z) \in \mathbb{R}^3 \) is critical if and only if

\[
-2x + yz = 0 \quad (23)
\]
\[
-2y + zx = 0 \quad (24)
\]
\[
2z + xy = 0 \quad (25)
\]

First suppose that \( z \neq 0 \). Then (25) implies \( xy \neq 0 \), that is, \( x \neq 0 \) and \( y \neq 0 \). Apply (23) and (24) to obtain:

\[
\frac{x}{y} = \frac{y}{x} = \frac{z}{2},
\]

whence \( x/y = \pm 1 \), that is, \( x = \pm y \) and \( z = \pm 2 \). Now (25) implies \( x^2 = -4 \), contradicting \( x \in \mathbb{R} \).
Thus $z = 0$. By (23), $x = 0$ and by (24), $y = 0$. Thus $p = (0, 0, 0)$ as desired. □

4.2. Projection when $k > -2$. The proof crucially uses:

Lemma 4.2.1. Suppose that $\kappa_\Phi(x, y, z) = k$ and $2z + xy \neq 0$. Then the vector fields

(26) \[ \tilde{\partial}_x := \partial_x + \frac{2x - yz}{2z + xy} \partial_z, \quad \tilde{\partial}_y := \partial_y + \frac{2y - xz}{2z + xy} \partial_z \]

form a basis of the tangent space $T_{(x, y, z)}(\kappa^{-1}_\Phi(k))$. Furthermore $\tilde{\partial}_x, \tilde{\partial}_y$ project under $d\Pi_{xy}$ to the coordinate basis $\partial_x, \partial_y \in T_{(x, y)} \mathbb{R}^2$.

Proof. Since

(27) \[ d\kappa_\Phi = (-2x + yz) \, dx + (-2y + xz) \, dy + (2z + xy) \, dz, \]
the vector fields $\tilde{\partial}_x, \tilde{\partial}_y$ lie in the kernel of $d\kappa_\Phi$. Therefore they are tangent to $\kappa^{-1}_\Phi(k)$. Their projections under $d\Pi_{xy}$ form the coordinate basis of $T_{(x, y)} \mathbb{R}^2 \cong \mathbb{R}^2$. Thus they are linearly independent and base

\[ T_{(x, y, z)}(\kappa^{-1}_\Phi(k)) = \text{Ker}(d\kappa_\Phi) \]

as claimed. □

Proposition 4.2.2. $\kappa^{-1}_\Phi(k)$ is a smooth surface and the restriction of $\Pi_{xy}$ is a local diffeomorphism.

Proof. Since $\kappa_\Phi(0, 0, 0) = -2 < k$, Lemma 4.1.2 implies that $\kappa^{-1}_\Phi(k)$ contains no critical points of $\kappa_\Phi$. Thus $\kappa^{-1}_\Phi(k) \subset \mathbb{R}^3$ is a smooth surface.

If $2z + xy \neq 0$, then Lemma 4.2.1 implies that $\Pi_{xy}$ restricts to a diffeomorphism. Thus it suffices to show $2z + xy \neq 0$.

To this end, suppose $2z + xy = 0$. Then $\kappa_\Phi(x, y, z) = k$ implies:

\[ (2z + xy)^2 = (x^2 + 4)(y^2 + 4) + 4(k - 2). \]

Since $k > -2$,

\[ 0 = (2z + xy)^2 = (x^2 + 4)(y^2 + 4) + 4(k - 2) \geq 16 + 4(k - 2) = 4(k + 2) > 0, \]

a contradiction. □

Proposition 4.2.3. Suppose that $k > -2$. Then the plane $\mathbb{R} \times \mathbb{R} \times \{0\}$ separates $\kappa^{-1}_\Phi(k) \subset \mathbb{R}^3$ into two components.

- Each component projects diffeomorphically onto $\mathbb{R}^2$ under $\Pi_{xy}$.
- Each of the sign-changes $\sigma_1, \sigma_2$ permutes the two components.
When \( k > 2 \), the region \( \mathbb{R} \times \mathbb{R} \times [-2, 2] \) separates the two components of \( \kappa_{\Phi}^{-1}(k) \subset \mathbb{R}^3 \).

The open region \( \mathbb{R} \times \mathbb{R} \times (-2, 2) \) separates the two components of \( \kappa_{\Phi}^{-1}(2) \).

**Proof.** The identity

\[
\kappa_{\Phi}(x, y, z) = z^2 - 2 - Q_z(x, y),
\]

implies (since \( k > -2 \)),

\[
z^2 = k + 2 + Q_z(x, y) > Q_z(x, y).
\]

If \( |z| < 2 \), then Lemma 4.1.1 implies that \( Q_z(x, y) \geq 0 \) and \( z^2 > 0 \).

Thus \( z \neq 0 \). Consequently \( \kappa_{\Phi}^{-1}(k) \subset \mathbb{R}^3 \) is the disjoint union of the two subsets where \( z > 0 \) and \( z < 0 \) respectively. These two subsets are interchanged by \( \sigma_i \).

Suppose that \( k > 2 \). If \( |z| < 2 \), then, as above, Lemma 4.1.1 implies \( Q_z(x, y) \geq 0 \), and

\[
z^2 = k + 2 + Q_z(x, y) \geq k + 2 > 4,
\]

contradicting \( |z| < 2 \). Thus \( \mathbb{R} \times \mathbb{R} \times [-2, 2] \) separates the two components of \( \kappa_{\Phi}^{-1}(k) \). Proposition 4.2.2 implies \( \Pi_{xy} \) restricts to a diffeomorphism, as claimed. The case \( k = 2 \) is handled similarly. \( \square \)

Writing

\[
\kappa_{\Phi}(x, y, z) = \left( z + \frac{xy}{2} \right)^2 - \frac{(x^2 + 4)(y^2 + 4)}{4} + 2,
\]

the two solutions of \( \kappa_{\Phi}(x, y, z) = k \) are:

\[
z = z_{\pm}(x, y) := -\frac{xy \pm \sqrt{(x^2 + 4)(y^2 + 4) + 4(k - 2)}}{2}.
\]

Since \( |z| > 2 \), and

\[
-xy + \sqrt{(x^2 + 4)(y^2 + 4) + 4(k - 2)} > 0,
\]

\[
-xy - \sqrt{(x^2 + 4)(y^2 + 4) + 4(k - 2)} < 0,
\]

the solution \( z = z_{+}(x, y) \) satisfies \( z > 2 \) and the solution \( z = z_{-}(x, y) \) satisfies \( z < -2 \). Therefore \( \kappa_{\Phi}^{-1}(k) \) consists of two components, which are, respectively, the graphs of the functions \( z_{+, -} \) defined in (29).

Solving (28), the level set

\[
\kappa_{\Phi}^{-1}(k) \cap (\mathbb{R} \times \mathbb{R} \times \{z\})
\]

is a (nondegenerate) hyperbola for \( z \neq \pm \sqrt{k + 2} \) and the union of two crossing lines (a degenerate hyperbola) for \( z = \pm \sqrt{k + 2} \).
First suppose that \(-2 < k < 2\). Then
\[
(x^2 + 4)(y^2 + 4) \geq 16 > 4(2 - k)
\]
since \(k > -2\).

Suppose that \((x, y) \in \mathbb{R}^2\). Then the preimages \(\Pi_{xy}^{-1}(x, y)\) are points
\((x, y, z)\) where \(z\) is one of the two solutions \(z = z_\pm(x, y)\) given by (29).
(These two solutions are distinct by (30).)

Thus
\[
\kappa^{-1}_\Phi(k) = \text{graph}(z_+) \coprod \text{graph}(z_-)
\]
and the sign-change
\[
(31)
\]
interchanges these two preimages.

For \(-2 < z < 2\), Lemma 4.1.1 implies that the quadratic form \(Q_z\)
is positive definite. As in the proof of Proposition 4.2.3 again, (28) expresses \(\kappa^{-1}_\Phi(k)\) as a level set of \(Q_z\) for fixed \(z\). Thus the nonempty level sets
\[
\kappa^{-1}_\Phi(k) \cap z^{-1}(z_0)
\]
are ellipses whenever \(-2 < z_0 < 2\).

Now suppose \(-2 < k\) as before. Since \(\sqrt{2 + k} < 2\), the expression
(28) of \(\kappa^{-1}_\Phi(k)\) in terms of \(Q_z\) implies that \(\kappa^{-1}_\Phi(k)\) does not intersect the level set \(z = z_0\) when
\[
\lvert z_0 \rvert < \sqrt{2 + k} < 2.
\]
Thus the region \(\mathbb{R} \times \mathbb{R} \times (-\sqrt{2 + k}, \sqrt{2 + k})\) separates the two components of \(\kappa^{-1}_\Phi(k)\), which are interchanged by the involution \(\sigma_2\) defined in (31).

4.3. The invariant area form. Proposition 4.2.3 implies that the coordinate functions \(x, y\) give global coordinates on \(\kappa^{-1}_\Phi(k)\) when \(k > -2\). In these coordinates the invariant area form has the following simple expression:
\[
(32)
\]
\[
dA_k := \frac{\text{sign}(z) \, dx \wedge dy}{\sqrt{(x^2 + 4)(y^2 + 4) + 4(k - 2)}}
\]

Lemma 4.3.1. The exterior 2-form \(dA_k\) defined in (32) defines a real-analytic \(\Gamma\)-invariant area form on \(\kappa^{-1}_\Phi(k)\).
Proof. Since $k > -2$ and $x, y \in \mathbb{R}$,

$$(x^2 + 4)(y^2 + 4) + 4(k - 2) \geq 4(k + 2) > 0$$

so $dA_k$ is a real-analytic area form.

Since the complex exterior bivector field $B$ on $\text{Rep}(\mathbb{F}_2, \text{SL}(2, \mathbb{C}))$ defined by (16) is $\hat{\Gamma}$-invariant, it suffices to show that, on the real subvariety $\kappa^{-1}_\Phi(k)$, the complex bivector field $B$ restricts to the real bivector field

$$(dA_k)^{-1} := \text{sign}(z) \sqrt{(x^2 + 4)(y^2 + 4) + 4(k - 2)} \partial_x \wedge \partial_y$$

dual to $dA_k$. That is, we prove the embedding

$$\mathbb{R}^2 \xrightarrow{(\Pi_{xy})^{-1}} \kappa^{-1}_\Phi(k) \subset \mathbb{R}^3$$

pushes $(dA_k)^{-1}$ forward to $B_\Phi$.

To this end, (19) implies that $d\Pi_{xy}$ maps the Hamiltonian vector fields

$$\text{Ham}(x) = -(2z + xy) \partial_y + (xz - 2y) \partial_z = -(2z + xy) \tilde{\partial}_y$$

$$\text{Ham}(y) = (2z + xy) \partial_x + (2x - yz) \partial_z = (2z + xy) \tilde{\partial}_x$$

to $-(2z + xy) \partial_y$ and $(2z + xy) \partial_x$ respectively. Now (29) implies

$$2z + xy = \text{sign}(z) \sqrt{(x^2 + 4)(y^2 + 4) + 4(k - 2)},$$

completing the calculation. □

Similar calculations show:

$$(34) \quad \text{Ham}(z) = (2y - xz) \partial_x + (yz - 2x) \partial_y$$

$$= (2y - xz) \tilde{\partial}_x + (yz - 2x) \tilde{\partial}_y.$$  

4.4. The level set for $k \leq -2$. Here more interesting topologies arise.

Proposition 4.4.1. Suppose that $k \leq -2$.

- If $k = -2$, then $\kappa^{-1}_\Phi(-2)$ is connected, with one singular point $(0, 0, 0)$. Each of the two components of $\kappa^{-1}_\Phi(k) \setminus \{(0, 0, 0)\}$ projects diffeomorphically to $\mathbb{R}^2 \setminus \{(0, 0)\}$ under $\Pi_{xy}$.
- If $k < -2$, then $\kappa^{-1}_\Phi(k)$ is homeomorphic to a cylinder.

Proof. When $k = -2$, the preimage $\kappa^{-1}_\Phi(-2)$ is defined by the equation

$$Q_z(x, y) = z^2$$

which is singular only at $(0, 0, 0)$. The intersection $\kappa^{-1}_\Phi(-2) \setminus \mathbb{R}^2 \times \{0\}$ consists of two components, each being the graph of one of the functions $z_+, z_-$ defined in (29). These two components of the complement of the
origin are interchanged by $\sigma_2$. This *purely imaginary real form* of the Markoff surface plays an important role in Bowditch’s paper [1].

Finally consider the case $k < -2$. In this case the image $\Pi_{xy}(\kappa_\Phi^{-1}(k)))$ is the region of $\mathbb{R}^2$ defined by

$$(1 + \left(\frac{x}{2}\right)^2) \left(1 + \left(\frac{y}{2}\right)^2\right) \geq \frac{2 - k}{4} > 1.$$  

(Figure 3 depicts the level sets of this function.) For $-2 < z_0 < 2$, the ellipses defined by $z = z_0$ form a cylinder. As $z_0 \to \pm 2$, their eccentricities tend to $\infty$, and the limits (defined by $z = \pm 2$ are degenerate conics consisting of two parallel lines

$$(x \mp y)^2 = 2 - k.$$  

For $|z_0| > 2$, the level sets are hyperbolas. The entire level set $\kappa_\Phi^{-1}(-2)$ is homeomorphic to a cylinder.

$\square$

5. **Generalized Fricke spaces**

Hyperbolic structures on a surface $\Sigma$ with fundamental group $\mathbb{F}_2$ determine representations of $\mathbb{F}_2$ into the isometry group of $\mathbb{H}^2$. This group identifies with the disconnected group $\text{PGL}(2, \mathbb{R})$, and as before we lift to the double covering which we may identify with

$$\text{SL}(2, \mathbb{R}) \bigcup (i\text{GL}(2, \mathbb{R}) \cap \text{SL}(2, \mathbb{C})).$$

The homomorphism $\Phi$ is the composition

$$\mathbb{F}_2 \xrightarrow{\rho} \text{PGL}(2, \mathbb{R}) \twoheadrightarrow \pi_0(\text{PGL}(2, \mathbb{R})) \cong \mathbb{Z}/2.$$
It is trivial if and only if Σ is orientable. In that case ρ preserves orientation on H^2, and the lift lies in SL(2, R). This case is analyzed in [9]. Otherwise, exactly one of ρ(X), ρ(Y) and ρ(Z) maps to SL(2, R); by applying a permutation in Aut(F_2), we may assume that ρ(Z) ∈ SL(2, R), that is, Φ is the {±1}-character with

\[ \Phi(X) = \Phi(Y) = 1 \pmod{2}, \quad \Phi(Z) = 0 \pmod{2}. \]

In this case, if the structure is complete, the representation is a discrete embedding, that is, an isomorphism of F_2 onto a discrete subgroup of PGL(2, R) and the hyperbolic surface is the quotient H^2/ρ(F_2).

There are two cases for the topology of the surface H^2/ρ(F_2): either it is homeomorphic to a 1-holed Klein bottle C_{1,1}, which corresponds to the case when k > 2, or a 2-holed cross-surface C_{0,2}, which corresponds to the case when k < 2. (See Charette-Drumm-Goldman [4] for more information on the Fricke space of C_{0,2}.) In the former case, we consider more general geometric structures, the boundary component of C_{1,1} is replaced by a cone point; from our viewpoint it is more natural to allow such an isolated singularity. In this particular case the singular hyperbolic surfaces behave dynamically like nonsingular complete hyperbolic structures on C_{1,1}.

| Surface Σ | Peripheral elements | Boundary traces |
|-----------|---------------------|-----------------|
| Σ_{0,3}   | X, Y, Z := (XY)^{-1} | x, y, z         |
| Σ_{1,1}   | K := XYX^{-1}Y^{-1}  | k := x^2 + y^2 + z^2 + xyz - 2 |
| C_{0,2}   | Z := XY, Z' := Y^{-1}X | z, z' := (ix)(iy) - z |
| C_{1,1}   | D := X^2Y^2         | δ := 2 - (ix)^2 - (iy)^2 + (ix)(iy)z |

5.1. **Geometric structures on the two-holed cross-surface.** We begin with analyzing the Fricke space \( \mathfrak{F}(C_{0,2}) \) of marked hyperbolic structures on C_{0,2} in trace coordinates. Later, in §9, we characterize the corresponding characters dynamically as those for which the flow possesses an attractor (see Definition 6.2.1, in the case when k < 2). Compare Goldman [12] and Charette-Drumm-Goldman [5] for further discussion.

**Proposition 5.1.1.** Suppose ρ is the holonomy representation for a hyperbolic structure M on C_{0,2}. Then for some choice of basis for \( \pi_1(M) \), the character of ρ satisfies

\[ z \leq -2 \]

(35)

\[ z' = -xy - z \leq -2. \]
Conversely, if \((ix, iy, z)\) is a purely imaginary character satisfying (35), then it corresponds to a complete hyperbolic structure on \(C_{0,2}\).

**Proof.** Suppose \(M\) is a complete hyperbolic surface diffeomorphic to \(C_{0,2}\). Let \(X\) be a one-sided simple curve on \(M\) and let \(M|X\) be the hyperbolic surface obtained by splitting \(M\) along the closed geodesic corresponding to \(X\). Since \(X\) is one-sided, \(M|X\) is connected and has one more boundary component than \(M\). Furthermore \(\chi(M|X) = \chi(M) = -1\) and

\[
\#\pi_0(\partial(M|X)) = \#\pi_0(\partial M) + 1 = 3.
\]

By the classification of surfaces \(M|X \approx \Sigma_{0,3}\). Thus \(M\) is obtained from the three-holed sphere \(M|X\) by attaching a cross-cap to a boundary component \(C \subset \partial(M|X)\). Let \(A, B\) denote the other two components of \(\partial(M|X)\), which correspond to the components of \(\partial M\).

The Fricke space of \(\Sigma_{0,3}\) is parametrized by the three boundary traces \((a, b, c)\) where \(a = \text{tr}(\rho(A))\), \(b = \text{tr}(\rho(B))\) correspond to the boundary components of \(M\) and \(C\) corresponds to \(X^{-2}\). Since

\[
\text{tr}(\rho(C)) = \text{tr}(\rho(X^2)) = \text{tr}(\rho(X))^2 - \text{tr}(\mathbb{I}) = -x^2 - 2 < -2,
\]

the other two boundary traces \(\text{tr}(\rho(A)), \text{tr}(\rho(A))\) have the same sign. Applying a sign-change automorphism if necessary, we can assume they are both negative.

Let \(Y\) be another one-sided simple curve with \(i(X, Y) = 1\). Then, with appropriate choices of basepoint and representative curves, we may assume that \(A = XY\) and \(B = Y^{-1}X\) (so that \(ABC = I\)). Their traces satisfy:

\[
a + b = (ix)(iy) = -xy.
\]

These traces correspond to the traces \(z, z'\) of \(\rho(Z) = \rho(Y^{-1}X^{-1})\) and \(\rho(Z') = \rho(YX^{-1})\), since

\[
a = \text{tr}(\rho(A)) = \text{tr}(\rho(XY)) = z,
b = \text{tr}(\rho(B)) = \text{tr}(\rho(X^{-1}Y)) = z',
\]

as desired.

Conversely, suppose that \((ix, iy, z)\) is a purely imaginary character satisfying (35). Define

\[
z' := -z - xy.
\]
Since
\[ z \leq -2, \]
\[ z' \leq -2, \]
\[ -x^2 - 2 < -2, \]
there exists a complete hyperbolic surface \( N \) diffeomorphic to \( \Sigma_{0,3} \) with boundary traces \( (-x^2 - 2, z, z') \). The boundary components \( A, B, C \) correspond to traces
\[ a = z, \]
\[ b = z', \]
\[ c = -x^2 - 2. \]

Let \( M \) be the complete hyperbolic surface obtained by attaching a cross-cap to \( C \). Then \( \pi_1(M) \) is obtained from
\[ \pi_1(N) = \langle A, B, C \mid ABC = I \rangle \]
by adjoining a generator \( X \) satisfying \( C = X^{-2} \). Define \( Y := X^{-1}A \), so that
\[ XY = A, \]
\[ Y^{-1}X = B \]
and the holonomy of \( M \) has character \((ix, iy, z)\) as desired. \( \square \)

5.2. Geometric structures on the one-holed Klein bottle. We now determine the Fricke space and generalized Fricke space for \( C_{1,1} \) in trace coordinates. The surface \( C_{1,1} \) is rather special in that it contains a unique isotopy class of nontrivial nonperipheral 2-sided simple closed curve. We consider a marking of \( C_{1,1} \) in which this curve has homotopy class \( Z \), where \((X, Y, Z)\) is a basic triple. Represent \( X \) and \( Y \) by 1-sided simple loops intersecting tangentially at the basepoint (so \( i(X, Y) = 0 \)). With respect to a suitable choice of basepoint and defining loops, \( D = X^2Y^2 \) is an element of \( \pi_1(C_{1,1}) \) generating \( \pi_1(\partial C_{1,1}) \). For a given representation \( \rho \) defining a hyperbolic structure on \( C_{1,1} \), we denote the corresponding trace function by
\[ \delta = \text{tr}(\rho(D)). \]
The basic trace identity implies that
\[ \delta + k = z^2. \]
Suppose that \( \mu \leftrightarrow (ix, iy, z) \in i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \) is a character with \( k > 2 \), which corresponds to the holonomy of a hyperbolic structure on
Proposition 4.2.3 implies that $|z| > 2$. Furthermore both $x, y \neq 0$ since otherwise $0 > z^2 - (k + 2) = Q_z(x, y) > 0$.

**Proposition 5.2.1.** Suppose that $k > 2$ and $|z| < \sqrt{k + 2}$. Let 

$$\delta := z^2 - k < 2.$$ 

Then $\mu$ corresponds to the holonomy of a hyperbolic structure on $C_{1,1}$

- with geodesic boundary if $\delta < -2$, and the boundary has length $2 \cosh^{-1}(-\delta/2)$;
- with a puncture if $\delta = -2$;
- with a conical singularity of cone angle $\cos^{-1}(-\delta/2)$ if $-2 < \delta < 2$.

**Proof.** Depending on the three cases for $\delta$, the character 

$$(−2−x^2,−2−y^2,\delta)$$

corresponds to the holonomy of a hyperbolic surface $S \approx \Sigma_{0,3}$ where the boundary with trace $\delta$ either is a closed geodesic, a puncture, or a conical singularity, respectively, depending on whether $\delta < -2$, $\delta = -2$ or $2 > \delta > -2$ respectively. Attaching a cross-cap to the boundary components corresponding to $X$ and $Y$ yields a hyperbolic surface $S' \approx C_{1,1}$ whose holonomy has character $(ix, iy, z)$. (Compare [12] for a discussion of attaching cross-caps to geodesic boundary components of hyperbolic surfaces.)

**Corollary 5.2.2.** The generalized Fricke space of the one-holed Klein bottle $C_{(1,1)}$ is defined by the inequalities

$$|z| > 2$$

$$x^2 - zxy + y^2 < 0.$$ 

Here the boundary trace $\delta$ and commutator trace $k$ are:

$$\delta := (x^2 - zxy + y^2) + 2$$

$$k := z^2 - \delta$$

The mapping class group of $C_{1,1}$ is realized as the subgroup of $\text{Aut}(F_2)$ generated by the elliptic involution $e$, the transposition $(12) \in \mathfrak{S}_3$ and the involution $I_1$ defined in (6). As $e$ acts trivially on $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$, the action on $\kappa^{-1}_\phi(k)$ of the subgroup of $\Gamma$ corresponding to $\text{Mod}(C_{1,1})$ is generated by the transposition $\mathcal{P}(12)$ and the Vieta involutions $I_1$ and $I_2$ defined in §3.3. Figure 12 depicts these spaces.
5.3. Lines on the Markoff surface. Here we interpret lines on the imaginary real form on the Markoff surface \((k = -2)\) as Fricke spaces of a certain 2-orbifold. Lee and Sakuma \([16, 17]\) studied the complex lines on the Markoff surface. We consider the real lines on the real form and show they contain characters of non-injective representations with discrete image and identify the quotient hyperbolic orbifold.

Recall that the Markoff surface is defined by \((\xi, \eta, \zeta) \in \mathbb{C}^3\) satisfying \(\xi^2 + \eta^2 + \zeta^2 = \xi \eta \zeta\). We consider the lines defined by the vanishing of one of the coordinates, for example \(\xi = 0\). In that case the lines are defined by \(\zeta = \pm i \eta\), which yield imaginary characters \((0, iy, y)\), where \(y \in \mathbb{R}\). We show that, when \(y \geq 2\), these correspond to representations of \(F_2\) with discrete image.

Write \((\xi, \eta, \zeta) = (0, 2i \sinh(\ell/2), \sinh(\ell/2))\), so that \(Y\) corresponds to a glide-reflection of displacement \(\ell\), or a reflection when \(\ell = 0\). Representative matrices are, for example:

\[
\rho(X) = i \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\rho(Y) = i \begin{bmatrix} 2 \sinh(\ell/2) & 1 \\ 1 & 0 \end{bmatrix}, \\
\rho(Z) = \begin{bmatrix} 0 & -1 \\ -1 & 2 \sinh(\ell/2) \end{bmatrix}
\]

Then \(\rho(X)\) represents the reflection and \(\rho(Y)\) represents a glide-reflection (or reflection when \(\ell = 0\)), whose respective fixed-point sets are:

\[
\text{Fix}(\rho(X)) = \{0, \infty\} \\
\text{Fix}(\rho(Y)) = \{e^{\ell/2}, -e^{-\ell/2}\}
\]

If \(\ell > 2 \log(1 + \sqrt{2})\), then \(\rho(Z)\) is a transvection with

\[
\text{Fix}(\rho(Z)) = \left\{ -\sinh(\ell/2) \pm \sqrt{\cosh^2(\ell) - 3/2} \right\}.
\]

When \(\ell = 2 \log(1 + \sqrt{2})\), then \(\rho(Z)\) is a parabolic fixing \(-1\).

In these latter two cases, \(\text{Image}(\rho)\) is discrete, and the quotient hyperbolic orbifold \(H^2/\text{Image}(\rho)\) is a punctured disc with a mirrored arc on its boundary. Its orbifold double covering-space

\[
H^2/\langle\rho(Y), \rho(X)\rho(Y)\rho(X)\rangle
\]

is a two-holed cross-surface \(C_{0,2}\) with reflection symmetry defined by the reflection \(\rho(X)\). When \(\ell = 2 \log(1 + \sqrt{2})\), the corresponding character is \((2i, 2i, 2)\). The closed ray defined by \(\ell \geq 2 \log(1 + \sqrt{2})\) corresponds to the Fricke space of this nonorientable hyperbolic orbifold \(H^2/\text{Image}(\rho)\).
6. **Bowditch theory**

Bowditch [1] introduced objects (which he called *Markoff maps*)

\[ \Omega \xrightarrow{\mu} \mathbb{C} \]

which are equivalent to characters in \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3 \) where the commutator of a pair of free generators has trace \(-2\). (Tan-Wong-Zhang [29] extended this to arbitrary characters in \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3 \), calling them *generalized Markoff maps.*) Bowditch used Markoff maps to study the orbit of a character under the action of the automorphism group \( \text{Aut}(F_2) \). Departing from Bowditch’s terminology, we call these maps *trace labelings*. We apply Bowditch’s approach to the dynamics of the \( \Gamma \)-action on purely imaginary characters

\[(\xi, \eta, \zeta) \in i\mathbb{R} \times i\mathbb{R} \times \mathbb{R} \subset \mathbb{C}^3.\]

This section reviews the theory in the more general setting of \( \text{SL}(2, \mathbb{C}) \)-characters, that is, when \( (\xi, \eta, \zeta) \subset \mathbb{C}^3 \) before specializing to the purely imaginary characters.

Bowditch’s method was used and extended by Tan, Wong and Zhang in [27, 26, 28] and Maloni-Palesi-Tan [19] to study various aspects of the dynamics of this action, and obtain variations of McShane’s identities. See the above papers for more details.

Following Bowditch [1], encode the dynamics of the action of \( \text{Out}(F_2) \) on characters, described by trace labelings \( \mu \in \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \), in terms of a dynamical system on the tree \( \bar{T} \). The character provides a way of directing \( \bar{T} \), and the dynamics of the associated directed tree \( \bar{T}_\mu \) (the *flow*) is Bowditch’s invariant.

**6.1. The trace labeling associated to a character.** Suppose that

\[ F_2 \xrightarrow{\rho} \text{SL}(2, \mathbb{C}) \]

is a representation. Define the corresponding *trace labeling* \( \mu_\rho:

\[ \Omega \xrightarrow{\mu_\rho} \mathbb{C} \]

\[ \omega \mapsto \text{tr}(\rho(\gamma_\omega)) \]

where \( \gamma_\omega \) is an element of \( F_2 \) corresponding to \( \omega \). The trace labeling \( \mu \) associated to a character in \( \text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3 \) satisfies a equation for each edge and an equation for each vertex as follows.

First, fix the parameter

\[ k = \kappa(\mu) := \kappa(\xi, \eta, \zeta) \]
throughout the discussion. Vertex Relation (37) implies that $\kappa$ is constant on vertices of $T$ and Edge Relation (36) provides the basic inductive step for navigating through $T$. In particular, once $\mu$ is defined on the three regions around a vertex, then (36) forces a unique extension $\Omega \to \mathbb{C}$.

Suppose that $e^{X,Y}(Z, Z') \in \text{Edge}(T)$ is an edge. Then

\begin{equation}
\zeta + \zeta' = \xi \eta
\end{equation}

Suppose $v = v(X, Y, Z) \in \text{Vert}(T)$ is a vertex. Then

\begin{equation}
\xi^2 + \eta^2 + \zeta^2 - \xi \eta \zeta - 2 = k
\end{equation}

where the lower case Greek letters represent the trace labeling of the region denoted by the corresponding upper case Roman letters. Figure 4 depicts an example of a trace labeling.

The Edge Relation (36) and the Vertex Relation (37) intimately relate. Fixing a pair $X, Y \in \Omega$ which abut one another, there are exactly two elements of $\Omega$ which abut $X$ and $Y$; these are $Z$ and $Z'$ respectively. Their corresponding traces $\zeta, \zeta'$ are the two roots of the quadratic equation obtained from (37) by fixing $\xi, \eta$. This follows directly from (36).

The trace labeling $\mu$ also intimately relates to the $\Gamma$-action on the character variety $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$. In particular, the triple of trace values about each vertex is the image of $\mu$ under an element of $\Gamma$. This is the action of $\Gamma$ on superbases described in (9) where the superbasis determines a coordinate system $\text{Rep}(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$.

Conversely, for any $\gamma \in \Gamma$, the image $\gamma(\mu)$ is a triple of trace values around some vertex of the tree.

6.2. The flow associated to a character. From the trace labeling $\mu$ associated to a character in $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$, Bowditch [1] associates a directed tree $\vec{T}$ (which Tan-Wong-Zhang [29] call a flow on $T$) as follows.

Suppose that $e = e^{X,Y}(Z, Z') \in \text{Edge}(T)$ is an edge. Say that $e$ is decisive (relative to $\mu$) if

$|\zeta| \neq |\zeta'|$;

in that case direct the edge as follows:

\begin{align*}
e^{X,Y}(Z \to Z') & \iff |\zeta| > |\zeta'| \\
e^{X,Y}(Z \leftarrow Z') & \iff |\zeta| < |\zeta'|.
\end{align*}
Otherwise, say that $e$ is indecisive and assign either direction to the edge $e$. Say that an edge $e$ points decisively towards a vertex $v$ if and only if $e$ is decisive and points towards $v$.

If every edge is decisive, say that $\mu$ is completely decisive. If every edge is indecisive, say that $\mu$ is completely indecisive. For many cases of interest, only finitely many edges are indecisive. The ambiguity at an indecisive edge does not affect the subsequent discussion. In particular, if the arrow points from $Z$ to $Z'$, then $|\zeta| \geq |\zeta'|$. Denote the above directed tree by $\overrightarrow{T}_{(\xi, \eta, \zeta)}$ or $\overrightarrow{T}_{\mu}$.

Sign-change automorphisms do not affect the directed tree $\overrightarrow{T}_{\mu}$ associated to a character $\mu$. 
The vertices $\text{Vert}(\vec{T})$ admit the following classification:

| Number of edges pointing towards $v$ | Type of $v$ |
|-------------------------------------|-------------|
| 0                                  | source      |
| 1                                  | fork        |
| 2                                  | merge       |
| 3                                  | sink        |

Each merge possesses a unique trace-reducing direction since exactly one directed edge incident to $v$ points away from $v$.

The following definition is due to Tan-Wong-Zhang [29], §2.18 and is a crucial idea in Bowditch [1]:

**Definition 6.2.1.** Let $\vec{T}$ be a directed tree. An attractor in $\vec{T}$ is a minimal subtree $\vec{A} \subset \vec{T}$ satisfying the following two properties:

- $\vec{A}$ is finite;
- Every edge $e \notin \vec{A}$ points inward towards $\vec{A}$.

Examples of attractors include sinks and attracting indecisive edges. For imaginary characters, these are the only types of attractors.

**Proposition 6.2.2.** $\Gamma$ acts properly on the set $\mathcal{S}$ of characters $\mu$ for which $\vec{T}_\mu$ has an attractor.

**Proof.** Denote by $\text{Finite}(\text{Vert}(T))$ the set of finite subsets of $\text{Vert}(T)$. Then the map

$$\mathcal{S} \rightarrow \text{Finite}(\text{Vert}(T))$$

which associates to $\mu \in \mathcal{S}$ the attractor of $\vec{T}_\mu$ is $\Gamma$-equivariant. Since $\Gamma$ acts properly on $\text{Finite}(\text{Vert}(T))$, it acts properly on $\mathcal{S}$. \qed
Definition 6.2.3. A descending path in the directed tree $\vec{T}_\mu$ is a path in $T$ as above, where each edge $f_i$ points away from $v_i$ and towards $v_{i+1}$:

(38) \[ \vec{P} = (u_0 \xrightarrow{f_0} v_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} v_n \xrightarrow{f_n} \cdots) \]

where $u_i \in \text{Vert}(T)$ and $f_i \in \text{Edge}(T)$, which forms a geodesic in $T$.

If $P$ is a descending path as above, then the sequence of values $(\xi_n, \eta_n, \zeta_n)$ at $u_n$ are decreasing in absolute value, that is:

(39) \[ |\xi_{n+1}| \leq |\xi_n| \]
\[ |\eta_{n+1}| \leq |\eta_n| \]
\[ |\zeta_{n+1}| \leq |\zeta_n|. \]

This follows since at each step, two of the three coordinates $\xi, \eta, \zeta$ remain constant and the absolute value of the remaining coordinate does not increase.

6.3. Exceptional characters. We state here some of the basic results associated to the trace labelings $\mu$ as well as the directed trees $\vec{T}_\mu$; see [1], [28], and [29] for more details. For ease of exposition, we exclude certain exceptional cases from our discussion below.

To begin, exclude the characters arising from reducible representations, namely characters in $\kappa^{-1}(2)$. The dynamics on the set of reducible characters is discussed in [12]. Next, exclude the $\hat{\Gamma}$-orbit

$\hat{\Gamma} \cdot (\{0\} \times \mathbb{C} \times \mathbb{C})$.

For each $k \in \mathbb{C}$, this set intersects $\kappa^{-1}(k)$ in a set of measure zero, and hence does not affect our claims of ergodicity. Nonetheless this case presents quite interesting dynamics.
Among the exceptional characters are dihedral characters. These are characters of representations where both elements of some basis map to involutions in $\text{SL}(2, \mathbb{C})$. Taking that basis to be $(X, Y)$, the corresponding character equals $(0, 0, \zeta)$ for some $\zeta \in \mathbb{C}$. Therefore dihedral characters lie in the $\Gamma$-orbit of the coordinate line $\{0\} \times \{0\} \times \mathbb{C}$. The orbit of $(0, 0, \zeta)$ equals

$$\{(0, 0, \zeta), (0, 0, -\zeta), (0, \zeta, 0), (0, -\zeta, 0), (\zeta, 0, 0), (-\zeta, 0, 0)\}.$$ 

Dihedral characters are the most degenerate characters since directing the tree $T$ is completely arbitrary: The directed trees corresponding to dihedral characters are completely indecisive.

We call the elements of the set

$$\kappa^{-1}(2) \cup (\bar{\Gamma} \cdot (0, \eta, \zeta)) \subset \mathbb{C}^3$$

exceptional characters, and elements of its complement unexceptional characters. The associated trace labelings and directed trees are called exceptional or unexceptional accordingly. Note that an unexceptional trace labeling $\mu$ takes values only in $\mathbb{C} \setminus \{0\}$.

### 6.4. The Fork Lemma

The following fundamental result is proved in [1] (Lemma 3.2(3)) for characters $(\xi, \eta, \zeta) \in \kappa^{-1}(-2)$ and extended in [29] (Lemma 3.7) and [28] (Proposition 3.3) for general characters. Recall that if the two edges are directed away from a vertex $v$, then that vertex must either be a fork or a source. We will see that such vertices are rare in our applications.

We supply a brief proof for the reader’s convenience.

**Lemma 6.4.1** (The Fork Lemma). Suppose that $\mu$ is a trace labeling with directed tree $\overrightarrow{T}_\mu$. If for the vertex $v(X, Y, Z) \in \text{Vert}(T)$, the edges opposite to $X$ and $Y$ are directed away from $X$ and $Y$ respectively, then $|\zeta| \leq 2$ or $\xi = \eta = 0$.

**Proof.** Write $e^{X,Z}(X \to X')$ and $e^{X,Z}(Y \to Y')$ for the two edges directed away from $v$, with corresponding traces $\xi, \eta, \zeta, \xi', \eta'$. By assumption,

$$|\xi| \geq |\xi'|$$

$$|\eta| \geq |\eta'|,$$

which implies:

$$2|\xi| \geq |\xi| + |\xi'| \geq |\xi + \xi'| = |\eta\zeta|$$

$$2|\eta| \geq |\eta| + |\eta'| \geq |\eta + \eta'| = |\xi\zeta|$$
Summing these two equations yields:

\[ 2 \left( |\xi| + |\eta| \right) \geq |\zeta| \left( |\xi| + |\eta| \right) \]

Thus \( 2 \geq |\zeta| \) or \( \xi = \eta = 0 \). \( \square \)

Since characters vanishing on a pair of two adjacent complementary regions are exceptional, a region \( Y \) adjacent to a source or a fork arising from an unexceptional character must satisfy \( |\mu(Y)| \leq 2 \).

We prove properness of the action using the two \( Q \)-conditions introduced by Bowditch in [1]. These conditions describe an open subset of the relative character variety with proper \( \Gamma \)-action. Tan-Wong-Zhang [29] extended this set to an open subset of the full character variety with proper \( \Gamma \)-action. For our purposes, we relax the conditions slightly, allowing finitely many parabolics among the set of primitive elements as in [27]. The resulting subset is no longer open, but the \( \Gamma \)-action remains proper.

**Definition 6.4.2.** Given a character \( \mu \) and constant \( C > 0 \), define \( \Omega_{\mu}(C) \) to be the union of the set of closures of complementary regions \( Y \in \Omega \) with \( |\mu(Y)| \leq C \).

Lemma 6.4.1 immediately implies:

**Lemma 6.4.3** (Bowditch [1], Theorem 1(2)). Suppose \( C \geq 2 \). For every character \( \mu \), the set \( \Omega_{\mu}(C) \) is connected.

(See also [29], Theorem 3.1(2).)

The following (extended) \( BQ \)-conditions were first defined by Bowditch in [1], and generalized by Tan-Wong-Zhang [29, 27]):

\begin{align*}
(40) & \quad \mu(X) \not\in [-2, 2] \text{ for all } X \in \Omega; \\
(41) & \quad \mu(X) \not\in (-2, 2) \text{ for all } X \in \Omega; \\
(42) & \quad \Omega_{\mu}(2) = \{Y \in \Omega : |\mu(Y)| \leq 2\} \text{ is finite.}
\end{align*}

Condition (40) means that every primitive element of \( F_2 \) maps to a loxodromic element, and Condition (41) means that no primitive element of \( F_2 \) maps to an elliptic. Condition (42) is equivalent to the condition that \( \Omega_{\mu}(C) \) is finite for any \( C \geq 2 \). See [1] and [27] for details.

**Definition 6.4.4.** A character \( \mu \) is a Bowditch character if and only if \( \mu \) is irreducible and satisfies (40) and (42) above. A character \( \mu \) is an extended Bowditch character if and only if \( \mu \) is unexceptional and irreducible and satisfies (41) and (42) above.

The Bowditch set \( \mathcal{B} \) comprises all Bowditch characters, and the extended Bowditch set \( \mathcal{B}' \) comprises all extended Bowditch characters.
Condition (41) allows for finitely many such parabolics among the primitives. The extended Bowditch set $\mathcal{B}'$ is no longer open in $\mathbb{C}^3$ but $\Gamma$ still acts properly on it.

Exceptional characters do not lie in $\mathcal{B}$: in the case of the reducible characters where $k = 2$ by definition and otherwise, Condition (40) is not satisfied if $\mu \in \hat{\Gamma} \cdot (0, \eta, \zeta)$.

Similarly, if $\mu(Z) = \pm \sqrt{k + 2}$ for some $Z \in \Omega$, then the proof of Proposition 6.5.4 implies that $\mu$ cannot satisfy Condition (42).

Bowditch [1] and Tan-Wong-Zhang [29, 27] proved that $\mathcal{B}$ is an open subset of the character variety and, for every $\mu \in \mathcal{B}'$, the flow $\overrightarrow{\Gamma}_\mu$ possesses an attractor. Thus Proposition 6.2.2 implies:

**Theorem 6.4.5** (Bowditch [1], Tan-Wong-Zhang [29]). $\Gamma$ acts properly on the extended Bowditch set $\mathcal{B}'$.

For general $\mu \in \text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$, Tan-Wong-Zhang [28] associate a closed subset $\mathcal{E}(\mu)$ of the projectivized measured lamination space of $\Sigma_{1,1}$, the set of end invariants of $\mu$. (This space identifies with $\mathbb{P}^1(\mathbb{R})$, the completion of $\mathbb{P}^1(\mathbb{Q})$. A character $\mu$ lies in $\mathcal{B}'$ if and only if $\mathcal{E}(\mu) = \emptyset$ ([28], Theorem 1.3). We complete their computation of $\mathcal{E}(\mu)$ for imaginary characters in this paper. We show that, for $k > 2$, either $\mathcal{E}(\mu) = \emptyset$ (when $\mu$ lies in the domain of discontinuity of the $\Gamma$-action) or $\mathcal{E}(\mu)$ consists of only one point.

### 6.5. Alternating geodesics

For a fixed $Z \in \Omega$, the values of a trace labeling $\mu$ on the other regions abutting $C(Z)$ admits an explicit description by inductively iterating (36).

First we establish some notation. Choose a trace labeling $\mu$ associated to a character $[\rho] \in \text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$. Denote the trace $\mu(A)$ associated with $A \in \Omega$ by the corresponding lower-case letter $a$.

Let $\{e_n \mid n \in \mathbb{Z}\}$ be the ordered set of vertices adjacent to $Z$, labeled so that $v_n = (Y_{n-1}, Y_n, Z)$, inductively beginning at $Y_0 = X$ and $Y_1 = Y$. Let $e_n \in \text{Edge}(T)$ be the edge between $v_n$ and $v_{n+1}$. We write $\cdots v_{n-1} e_0 v_0 e_1 v_1 e_2 v_2 \cdots$ for this geodesic. Let $Y_n \in \Omega$ be the region adjacent to $Z$ sharing the edge $e_n$ with $Z$. Finally, let $\psi_n$ be the edge incident to $v_n$ not adjacent
to $Z$, and $Z'_n \in \Omega$ the other region opposite to the edge $\psi_n$. Denote the corresponding traces by $\zeta = \mu(Z)$ and $\eta_n = \mu(Y_n)$. Compare Figure 6.

The indices $j(\epsilon_n) \in \{1, 2, 3\}$ for the tricoloring described in §2.3 alternate between two elements in $\{1, 2, 3\}$, depending on the mod 2 class associated to $Z$. This leads to a simple formula (43) for the set of traces $\eta_n$. In particular the sequence of characters forms a lattice in a hyperbola. Recall that the hyperbola $\{(\xi, \eta) \mid \xi \eta = 1\}$ is naturally an algebraic group (the multiplicative group of the ground field), and a lattice is a discrete subgroup $\Lambda$ with compact quotient; such a lattice is necessarily a cyclic group.

**Proposition 6.5.1.** Suppose $\zeta \neq \pm 2$. Let $\lambda, \lambda^{-1}$ be the two (distinct) solutions of

$$\lambda^2 - \zeta \lambda + 1 = 0$$

so that

$$\zeta = \lambda + \lambda^{-1}.$$  

Then, for $n \in \mathbb{Z}$,

$$\eta_n = A \lambda^n + B \lambda^{-n}$$

where

$$A = \frac{\eta - \lambda^{-1} \xi}{\lambda - \lambda^{-1}},$$

$$B = \frac{\lambda \xi - \eta}{\lambda - \lambda^{-1}}.$$  

Furthermore

$$AB = \frac{\zeta^2 - k - 2}{\zeta^2 - 4}.$$  

**Proof.** (36) implies that $\eta_n$ satisfies the recursion

$$\eta_{n+1} = \zeta \eta_n + \eta_{n-1}.$$  

Rewrite this recursion as:

$$Y_{n+1} = Z Y_n$$

in terms of a vector variable $Y_n$ and matrix variable $Z$ defined by:

$$Y_n := \begin{bmatrix} \eta_n \\ \eta_{n-1} \end{bmatrix}, \quad Z = \begin{bmatrix} \zeta & -1 \\ 1 & 0 \end{bmatrix}.$$  

Diagonalize $Z$ as:

$$Z = \frac{1}{\lambda - \lambda^{-1}} \begin{bmatrix} \lambda^{-1} & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} -1 & \lambda \\ 1 & -\lambda^{-1} \end{bmatrix}.$$
where
\[
\frac{1}{\lambda - \lambda^{-1}} \begin{bmatrix} \lambda^{-1} & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & \lambda \\ 1 & -\lambda^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Then
\[
Y_n = Z^n Y_0 = \frac{1}{\lambda - \lambda^{-1}} \begin{bmatrix} \lambda^{-1} & \lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda^{-n} & 0 \\ 0 & \lambda^n \end{bmatrix} \begin{bmatrix} -1 & \lambda \\ 1 & -\lambda^{-1} \end{bmatrix}
\]
from which the result follows.

When \(\zeta = \pm 2\), the situation is even simpler:

**Proposition 6.5.2.** If \(\zeta = 2\), then
\[
\eta_n = \xi + n(\eta - \xi)
\]
and if \(\zeta = -2\), then
\[
\eta_n = (-1)^n(\xi - n(\xi + \eta))
\]

When one of the coordinates equal \(\pm 2\), the corresponding level set is a union of two parallel lines. For example, \(\zeta = \pm 2\) defines the degenerate conic:
\[
(\xi \mp \eta)^2 = k - 2.
\]
The characters corresponding to vertices \(v(Y_{n-1}, Y_n, Z)\) then form a lattice on one of these lines (in the sense of \(\S 6.5\)).

**Propositions 6.5.1 and 6.5.2** easily imply:

**Corollary 6.5.3.** Suppose that \(k \neq 2\).

(a) If either
- \(\zeta \not\in [-2, 2]\) and \(\zeta^2 \neq k + 2\), or
- \(\zeta = \pm 2\),
then \(\lim_{n \to \pm\infty} |\eta_n| = \infty\).

(b) If \(\zeta \in (-2, 2)\), then \(|\lambda| = 1\). In particular \(|\eta_n|\) is bounded for all \(n \in \mathbb{Z}\).

(c) If \(\zeta = 0\), then \(\eta_{n+2} = -\eta_n\) for all \(n \in \mathbb{Z}\). Every edge abutting \(Z\) is indecisive.

(d) If \(\zeta^2 = k + 2\) and \((\xi, \eta) \neq (0, 0)\), then \(\lim_{n \to \pm\infty} |\eta_n|\) equals either 0 or \(\infty\), and \(\lim_{n \to -\infty} |\eta_n|\) equals either \(\infty\) or 0 respectively.

**Proof.** For (a), note that \(|\lambda| \neq 1\) and \(AB \neq 0\). For (d), note that \(AB = 0\). Now apply (43). \(\square\)

Here is an important consequence of the Fork Lemma 6.4.1.

**Proposition 6.5.4.** Let \(\mu\) be an unexceptional character with \(\kappa(\mu) = k\). Suppose that \(|\zeta| = |\mu(Z)| > 2\) for \(Z \in \Omega\). Let
\[
\tilde{C}(Z) := \{e_n\}_{n \in \mathbb{Z}}
\]
be the sequence of directed edges of $\Gamma_\mu$ surrounding $Z$. Then either:

(a) $\zeta^2 \neq k + 2$, and $\partial Z$ contains a vertex $v$ such that all $\overrightarrow{e}_n$ point towards $v$, or

(b) $\zeta^2 = k + 2$, and $\overrightarrow{e}_n$ all point in one direction.

Proof. Suppose that neither the $\overrightarrow{e}_n$ all point in one direction, nor is there a vertex on $\partial Z$ towards which all the $\overrightarrow{e}_n$ point. Then $\partial Z$ contains a vertex $v$ which is a fork or source, with $Z$ adjacent to two edges directed out of $v$. Since $\mu$ is not dihedral and $|\zeta| > 2$ by assumption, this contradicts Proposition 6.4.1.

Now let $Y_n$, $n \in \mathbb{Z}$ be the (ordered) neighboring regions of $Z$, with $Y_n$ adjacent to $e_n$. By Proposition 6.5.1,

$$\eta_n = A\lambda^n + B\lambda^{-n},$$

where

$$AB = \frac{\zeta^2 - k - 2}{\zeta^2 - 4}, \quad \lambda + \lambda^{-1} = \zeta.$$ 

Note that $|\lambda| \neq 1$ since $\zeta \not\in [-2, 2]$. If $\zeta^2 \neq k + 2$, then $A, B \neq 0$ so that $|\eta_n| \to \infty$ as $n \to \pm \infty$. Thus $\overrightarrow{e}_n$ must eventually point inwards from both directions. Hence all $\overrightarrow{e}_n$ point towards $v$ for a unique vertex $v$ adjacent to $Z$.

If $\zeta^2 = k + 2$, exactly one of $A, B$ equals 0 (since $\mu$ is not dihedral). Then $|\eta_n|$ is monotone in $n$, increasing to $\infty$ in one direction and decreasing to 0 in the other. Therefore all $\overrightarrow{e}_n$ point in the same direction. \hfill \Box

When $\zeta^2 = k + 2$, the $\zeta$-level set on $\kappa^{-1}(k)$ consists of two lines which intersect in the dihedral character $(0, 0, \zeta)$ as follows. Choose $\phi$ so that $\phi^2 = k - 2$. Then $\zeta = \lambda + \lambda^{-1}$ where

$$\lambda := \frac{\zeta + \phi}{2}, \quad \lambda^{-1} := \frac{\zeta - \phi}{2}.$$ 

Then the solutions of $\kappa(\xi, \eta, \zeta) = k$ for fixed $\zeta$ with $\zeta^2 = k + 2$ as above form the degenerate hyperbola

$$(\xi - \lambda\eta)(\xi - \lambda^{-1}\eta) = 0$$

which is the union of the two lines

$$\xi - \lambda\eta = 0, \quad \xi - \lambda^{-1}\eta = 0.$$ 

This is a degenerate conic section of the cubic surface $\kappa^{-1}(k)$. The sequence of vertices $v(Y_{n-1}, Y_n, Z)$ then correspond to a lattice in the group $\mathbb{C}^*$ acting on the complement of $(0, 0, \zeta)$ in one of the above lines.
The case when $\zeta = \pm 2$ is similar to the case where $|\zeta| > 2$ using Proposition 6.5.2: in this case all $\vec{e}_n$ point towards some vertex $v \in \partial Z$.

6.6. **Indecisive edges and orthogonality.** Characters where two loxodromic generators have orthogonal axes play a special role in this theory, especially for imaginary characters. Recall that the natural complex-orthogonal pairing on $\mathfrak{sl}(2, \mathbb{C})$

\[
\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathbb{C}
\]

\[
(X, Y) \mapsto X \cdot Y := \text{tr}(XY)
\]

is invariant under the adjoint representation $\text{SL}(2, \mathbb{C}) \xrightarrow{\text{Ad}} \text{Aut}(\mathfrak{sl}(2, \mathbb{C}))$. The *traceless projection*

\[
\text{SL}(2, \mathbb{C}) \longrightarrow \mathfrak{sl}(2, \mathbb{C})
\]

\[
X \mapsto X - \frac{\text{tr}(X)}{2} \mathbb{I}
\]

is $\text{Ad}$-equivariant. We may normalize to obtain $\hat{X} \in \text{SL}(2, \mathbb{C}) \cap \mathfrak{sl}(2, \mathbb{C})$ by dividing by a square root of

\[
\text{Det}
\left[
X - \frac{\text{tr}(X)}{2} \mathbb{I}
\right]
= \frac{\text{tr}(X)^2 - 4}{4}
\]

to obtain the *involution* fixing the invariant axis $\text{Axis}(X)$ when $X$ is loxodromic; compare [12], §3.2.

**Definition 6.6.1.** *Loxodromic elements* $X, Y \in \text{SL}(2, \mathbb{C})$ are orthogonal if and only if $\text{Axis}(X)$ and $\text{Axis}(Y)$ are orthogonal lines in $\mathbb{H}^3$.

**Proposition 6.6.2.** Let $(\xi, \eta, \zeta) \in \mathbb{C}^3 = \text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$ and consider the effect of a Vieta involution, for example

\[
(\xi, \eta, \zeta) \mapsto (\xi, \eta, \zeta') = (\xi, \eta, \xi \eta - \zeta).
\]

If $\rho \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ is a representation corresponding to $(\xi, \eta, \zeta)$, then $\zeta' = \zeta$ if and only $\rho(X)$ and $\rho(Y)$ are orthogonal.

**Proof.** The traces of $\rho(X), \rho(Y)$ and $\rho(XY) = \rho(X)\rho(Y)$ are, respectively, $\xi, \eta$ and $\zeta$. Now $\rho(X)$ and $\rho(Y)$ are orthogonal if and only if

\[
\rho(X) \cdot \rho(Y) = 0,
\]
or equivalently,
\[
0 = \text{tr} \left( \left( \rho(X) - \frac{\text{tr}(\rho(X))}{2} \mathbb{1} \right) \left( \rho(Y) - \frac{\text{tr}(\rho(Y))}{2} \mathbb{1} \right) \right) \\
= \text{tr}(\rho(X)\rho(Y)) - \frac{\xi\eta}{2} - \frac{\xi\eta}{4} + \frac{\xi\eta}{2} \text{tr}(\mathbb{1}) \\
= \zeta - \frac{\xi\eta}{2} = \frac{\zeta - \zeta'}{2}.
\]
Thus \( \zeta' = \zeta \) if and only if \( \rho(X) \) and \( \rho(Y) \) are orthogonal, as desired. \( \square \)

We now construct indecisive edges in \( \vec{T}_\mu \), and characterize them, at least when \( \mu \) is unexceptional and real or imaginary:

**Corollary 6.6.3.** Suppose that \( \mu \) is a real or purely imaginary unexceptional trace labeling. Then an edge joining \((\xi, \eta, \zeta)\) to \((\xi, \eta, \zeta')\) is indecisive if and only if the corresponding generators \( \rho(X), \rho(Y) \) are orthogonal.

**Proof.** The edge is indecisive if and only if \(|\zeta'| = |\zeta|\). Since \( \zeta, \zeta' \in \mathbb{R} \cup i\mathbb{R} \), this occurs if and only if \( \zeta' = \pm \zeta \). If \( \zeta' = -\zeta \), then \( \xi\eta = 0 \), contradicting \( \mu \) being unexceptional. Thus \( \zeta' = \zeta \) and Proposition 6.6.2 implies \( \text{Axis}(\rho(X)) \perp \text{Axis}(\rho(Y)) \) as claimed. \( \square \)

Attracting indecisive edges and sinks are the only attractors arising for real and imaginary characters. For hyperbolic structures on \( \Sigma_{3,0} \), the only attractors are sinks, but for \( \Sigma_{1,1} \) attracting indecisive edges occur exactly when two simple geodesics intersect orthogonally in one point. For the nonorientable surfaces \( C_{1,1} \) and \( C_{0,2} \), the situation is similar: A hyperbolic structure homeomorphic to \( C_{1,1} \) (with a possible cone point) corresponds to an attracting indecisive edge if and only if the two-sided interior simple geodesic meets a one-sided simple geodesic once, and orthogonally. A hyperbolic structure homeomorphic to \( C_{0,2} \) corresponds to an attracting indecisive edge if and only if the two one-sided simple geodesics intersect once, and orthogonally.

### 7. Imaginary Trace Labelings

We apply the results of the previous section to trace labelings arising from characters of the form \((ix, iy, z) \in \mathbb{C}^3\) where \(x, y, z \in \mathbb{R}\) which is the case of interest in this paper. These are the trace labelings which...
are adapted to the \( \{ \pm 1 \} \)-character \( \Phi \) defined by
\[
X \mapsto -1 \\
Y \mapsto -1 \\
Z \mapsto 1.
\]
Following [1, 29], we call these imaginary trace labelings (or imaginary characters).

When \( k > 2 \), Proposition 7.1.2 characterizes imaginary Bowditch characters in terms of the associated directed tree, which is later related to the generalized Fricke space \( \mathfrak{F}(C_{1,1}) \) (§8.3). The main step is Theorem 8. Similarly, when \( k < 2 \), the Bowditch characters are characterized in terms of the associated directed tree which relate to the Fricke space \( \mathfrak{F}(C_{0,2}) \). Away from \( \mathfrak{F}(C_{0,2}) \) are separating curves mapping to elliptic elements. The existence of simple closed curves mapping to elliptic elements implies ergodicity on the complement of the wandering domains. Existence of elliptic primitives is the main qualitative difference between the cases \( k > 2 \) and \( k < 2 \).

7.1. Well-directed trees.

**Definition 7.1.1.** A directed tree with no forks or sources is said to be well-directed. In this case, the edges are either well-directed towards an attractor or towards a unique end of the tree. If the attractor is a sink, then \( \overrightarrow{T} \) is said to be well-directed towards a sink.

Each case yields a tree well-directed towards a sink or an indecisive edge for Bowditch characters.

We adopt the following conventions. We use the letters \( Z, W \) to denote elements of \( \Omega \) with real trace labels and \( X \) or \( Y \) for elements of \( \Omega \) with purely imaginary trace labels. Furthermore, write:
\[
\begin{align*}
z &:= \mu(Z), \\
w &:= \mu(W), \\
ix &:= \mu(X), \\
iy &:= \mu(Y),
\end{align*}
\]
where \( x, y, z, w \in \mathbb{R} \).

**Proposition 7.1.2.** Let \( \mathcal{W} \) denote the set of unexceptional imaginary trace labelings \( \mu \) for which \( \overrightarrow{T}_\mu \) is well-directed towards a sink \( \nu(X,Y,Z) \) where \( z \in \mathbb{R} \), \( |z| \geq 2 \). Then \( \mathcal{W} \subset \mathcal{B}' \). In particular \( \Gamma \) acts properly on \( \mathcal{W} \).

Conversely, the directed tree associated to an imaginary Bowditch character is well-directed towards such a sink.
We defer the proof to the latter part of this section. Our analysis separates naturally into two cases, depending on whether \( k > 2 \) or \( k < 2 \) where \( k = \kappa(\mu) \). Proposition 7.1.2 will be proved separately for these two cases. The case \( k > 2 \) includes the generalized Fricke space for the one-holed Klein bottle \( C_{1,1} \) and the case \( k < 2 \) includes the Fricke space of the two-holed cross-surface \( C_{0,2} \).

We first discuss how a \( \{\pm 1\} \)-character \( \Phi \) affects the tree \( T \).

7.2. \( \{\pm 1\} \)-characters on \( F_2 \). A nontrivial homomorphism \( F_2 \xrightarrow{\Phi} \mathbb{Z}/2 \) divides the complementary regions in \( \Omega = \text{Prim}(F_2) \) into two classes:

- \( \Omega_{\mathbb{R}} \): Those in the kernel of \( \Phi \) correspond to orientation-preserving curves, and have real trace labels.
- \( \Omega_{i\mathbb{R}} \): Those upon which \( \Phi \) is nonzero correspond to orientation-reversing curves and have purely imaginary trace labels.

Similarly, \( \Phi \) determines a dichotomy of the edges of \( T \) into \textit{real} edges and \textit{imaginary} edges:

**Definition 7.2.1.** Let \( \Phi \) be the \( \{\pm 1\} \)-character and let \( e \in \text{Edge}(T) \) be an edge. Define:

- \( e \) is real with respect to \( \Phi \) if neither of the two complementary regions in \( \Omega \) abutted by \( e \) lie in \( \Omega_{\mathbb{R}} \).
- \( e \) is imaginary with respect to \( \Phi \) if exactly one of the two complementary regions in \( \Omega \) abutted by \( e \) lies in \( \Omega_{\mathbb{R}} \).

In terms of the tricoloring \( \text{Edge}(T) \xrightarrow{j} \{1, 2, 3\} \) defined in §2.3, an edge \( e \) is an \( \mathbb{R} \)-edge if and only if \( j(e) = 3 \) and is an \( i\mathbb{R} \)-edge if and only if \( j(e) = 1 \) or \( j(e) = 2 \).

When \( \Phi \) is understood, we say that \( e \) is an \( \mathbb{R} \)-edge if \( e \) is real with respect to \( \Phi \) and \( e \) is an \( i\mathbb{R} \)-edge if \( e \) is imaginary with respect to \( \Phi \).

![Figure 7. An \( \mathbb{R} \)-edge and an \( i\mathbb{R} \)-edge](image)

An \( \mathbb{R} \)-edge ends at two regions in \( \Omega_{\mathbb{R}} \) and an \( i\mathbb{R} \)-edge ends at two regions in \( \Omega_{i\mathbb{R}} \), in the sense of §2.2. Compare Figure 7.

**Proposition 7.2.2.** For every \( v \in \text{Vert}(T) \), exactly two edges incident to \( v \) are \( i\mathbb{R} \)-edges. The union of \( i\mathbb{R} \)-edges is a disjoint union of
geodesics, namely the alternating geodesics corresponding to $\Omega_{\mathbb{R}}$. Thus the edges composing alternating geodesics of elements of $\Omega_{\mathbb{R}}$ consist entirely of $i\mathbb{R}$-edges. The edges composing alternating geodesics of elements of $\Omega_{i\mathbb{R}}$ alternate between $i\mathbb{R}$-edges and $\mathbb{R}$-edges.

**Proof.** The unique $\mathbb{R}$-edge at $v$ is the edge $e$ with $j(e) = 3$. Thus exactly two $i\mathbb{R}$-edges emanate from $v$. Thus the union of the $i\mathbb{R}$-edges is a disjoint union of geodesics, one through each vertex of $\text{Vert}(T)$: Start at a vertex $v$ as above, and denote it $v_0$. Denote the endpoints of the two $i\mathbb{R}$-edges through $v_0$ by $v_{-1}$ and $v_1$ respectively. Inductively define $v_n$ for integers $|n| > 1$ as follows. If $n > 1$, let $v_n$ be the endpoint of the unique $i\mathbb{R}$-edge through $v_{n-1}$ not containing $v_{n-2}$. If $n < -1$, let $v_n$ be the endpoint of the unique $i\mathbb{R}$-edge through $v_{n+1}$ not containing $v_{n+2}$. Then $v_n$ form the vertices of a geodesic through $v$ all of whose edges are $i\mathbb{R}$-edges. Thus $T$ decomposes into a disjoint union of geodesics, containing all vertices and all $i\mathbb{R}$-edges. □

7.3. **Positive and negative vertices.** Let $v = v(X,Y,Z)$ be a vertex with trace labeling $(ix,iy,z)$ where $x,y,z \in \mathbb{R}$. Define $v$ to be positive (respectively negative) if $xyz > 0$ (respectively $xyz < 0$). If $\mu$ is unexceptional, then every vertex is positive or negative. Furthermore applying a sign-change automorphism to $\mu$ preserves the dichotomy of vertices into positive and negative vertices.

**Lemma 7.3.1.** If $v$ is positive, then the $\mathbb{R}$-edge adjacent to $v$ directs inwards.

**Proof.** Apply the Edge Relation to the $\mathbb{R}$-edge so that $z + z' = -xy$. Since $xy$ has the same sign as $z$,

$$|z'| = |-z - xy| > |z|$$

as desired. □

8. **Imaginary characters with $k > 2$**

We summarize the main results of this section:

**Theorem 8.** Suppose that $\mu$ is an unexceptional imaginary trace labeling with $k > 2$.

- The associated directed tree $\overrightarrow{T}_\mu$ is well-directed.
- $\overrightarrow{T}_\mu$ is well-directed towards a unique sink if and only if

$$|\mu(Z)| < \sqrt{k+2}$$
for some $Z \in \Omega_R$. Equivalently, $\mu$ corresponds to a hyperbolic structure on the 1-holed Klein bottle $C_{1,1}$ with a funnel, cusp, or cone point.

- The Bowditch set $\mathcal{B}$ equals the generalized Fricke orbit $\mathcal{D}'(C_{1,1})$, and intersects the slice $\kappa^{-1}_\Phi(k)$ in a dense open subset of $\kappa^{-1}_\Phi(k)$.

### 8.1. Alternating geodesics when $k > 2$.

The proof begins by analyzing the alternating geodesic $C(Z)$ surrounding a complementary region $Z \in \Omega_R$. This geodesic contains a unique sink if and only if $|z| < \sqrt{k+2}$, in which case the character lies in the generalized Fricke space $\mathcal{F}'(C_{1,1})$. When $|z| > \sqrt{k+2}$, one edge points out of $Z$ towards another $Z' \in \Omega_R$ and we obtain a descending path. If this descending path ends, then $\mu$ lies in $\mathcal{D}'(C_{1,1})$.

First we record a useful observation:

**Lemma 8.1.1.** If $\mu$ is an unexceptional imaginary trace labeling with $k > 2$ and $Z \in \Omega_R$, then $|z| > 2$ where $z = \mu(Z)$. 

**Proof.** Proposition 4.2.3 implies that $\rho(\gamma)$ is hyperbolic if $\gamma$ is primitive and $\Phi(\gamma) = 1$. $\square$

The discussion divides into the three cases:

- $2 < |z| < \sqrt{k+2}$;
- $z = \pm\sqrt{k+2}$;
- $|z| > \sqrt{k+2}$.

We retain the notations of §6.5 for $e_n, v_n, Y_n, \psi_n$ and $Z'_n, n \in \mathbb{Z}$.

**Proposition 8.1.2.** Suppose $2 < |z| < \sqrt{k+2}$.

- Each $v_n$ is positive;
- All $e_n$’s point towards $v_m$, for a unique $m \in \mathbb{Z}$;
- Each $\psi_n$ points towards $Z'$;

$v_m$ is a sink and all the other $v_i$’s where $i \neq m$ are merges.

**Proof.** When $2 < |z| < \sqrt{k+2}$, Proposition 6.5.1 implies:

$$iy_n = A\lambda^n + B\lambda^{-n}$$

where $\lambda + \lambda^{-1} = z$ and

$$AB = \frac{z^2 - k - 2}{z^2 - 4}.$$

Applying a sign-change if necessary, assume

$$2 < z < \sqrt{k+2}, \quad \lambda > 1,$$

and that

$$A = ai, \quad B = bi$$
where $a, b > 0$. Then
\[ y_n = a\lambda^n + b\lambda^{-n} > 0 \]
for all $n \in \mathbb{Z}$ and $y_n \to \infty$ as $n \to \pm\infty$.

Since $z > 0$ and all $y_n > 0$, each $v_n$ is a positive vertex. Lemma 7.3.1 implies that $\psi_n$ points decisively towards $Z$ for all $n$. Proposition 6.5.4 implies all the $e_n$’s point towards a vertex, say $v_0(Z, Y_{-1}, Y_0)$.

Hence $v_0$ is a sink and $v_n$ is a merge for $n \neq 0$. At most one $e_n$ is indecisive. For example, if $y_1 = y_{-1}$, the sink is either $v_0$ or $v_1$ (depending on the direction of $e_0$). Another case occurs when $y_{-2} = y_0$. All other $e_n$’s are decisive.

**Proposition 8.1.3.** Suppose $z = \pm\sqrt{k + 2}$.

- Each vertex $v_n$ is positive;
- All the $e_n$’s point in the same direction.
- Each $\psi_n$ points towards $Z$ for all $n$;

All the vertices $v_n$ are merges.

**Proof.** When $z = \pm\sqrt{k + 2}$, assume
\[ z = \sqrt{k + 2}, \quad \lambda > 1 \]
as in Proposition 8.1.3 above. Since $AB > 0$ and $\mu$ is not dihedral, one of $A, B \neq 0$, we may assume $B = 0$ and $A = ia$ where $a > 0$. Therefore
\[ y_n = a\lambda^n > 0 \]
for all $n$ and increases monotonically as $n \to \infty$. As in Proposition 8.1.2, each $v_n$ is positive, $\psi_n$ points decisively towards $Z$ for all $n$ and $e_n$ points towards $e_{n-1}$ for all $n$. Hence all the $v_n$’s are merges, as claimed.

**Proposition 8.1.4.** Suppose $|z| > \sqrt{k + 2}$.

- $v_m$ is negative for a unique $m \in \mathbb{Z}$;
- Each $e_n$ points towards $v_m$;
- $\psi_m$ points away from $Z$ while all the other $\psi_i$’s, with $i \neq m$, point towards $Z$.

All the $v_n$’s are merges. $v_m$ is the only merge directed away from $Z$.

**Proof.** If $|z| > \sqrt{k + 2}$, assume (as above)
\[ z > \sqrt{k + 2}, \quad \lambda > 1. \]

Since $AB > 0$, further assume
\[ A = ia, \quad B = ib \]
where $a > 0$ and $b < 0$. Hence $y_n$ increases monotonically and 
\[
\lim_{n \to -\infty} y_n = -\infty, \quad \lim_{n \to +\infty} y_n = +\infty.
\]
Re-indexing if necessary, assume $y_n > 0$ for all $n \geq 0$ and $y_n < 0$ for $n < 0$. Then $v_0$ is negative and all other $v_n$ are positive. Edge Relations (36) imply that all the $e_n$'s point decisively towards $v_0$. Lemma 7.3.1 implies that all the $\psi_n$'s point decisively towards $Z$ for $n \neq 0$. For $\psi_0$, 
\[z'_0 = -y_{-1}y_0 - z < 0, \quad -y_{-1}y_0 > 0\]
so $|z| > |z'_0|$. Hence $\psi_0$ points decisively away from $Z$, as claimed. ■

Proposition 8 now follows from Lemma 8.1.1 and Propositions 8.1.2, 8.1.3, 8.1.4: Since every vertex is adjacent to some $Z$ with $|z| > 2$, there are neither forks nor sources. Thus every vertex is a merge or a sink and $\overrightarrow{T}$ is well-directed.

8.2. The Bowditch set. Recall that a peripheral structure on a free group $G$ is a collection of conjugacy classes of cyclic subgroups of $G$ corresponding to the components of $\partial S$, where $S$ is a surface-with boundary with $\pi_1(S) \cong G$. Nielsen proved that an automorphism of $G$ corresponds to a mapping class of a surface $S$ if and only if it preserves a peripheral structure on $\pi_1(S)$. For example, peripheral structures on $F_2$ corresponding to $S \approx \Sigma_{3,0}$ correspond to superbases, and Nielsen’s theorem on $\Sigma_{1,1}$ is equivalent to the statement that every automorphism of $F_2$ preserves the peripheral structure defined by the simple commutators of bases.

The peripheral structures on $F_2$ corresponding to a one-holed Klein bottle $S$ with fundamental group $F_2$ are determined by the equivalence class in $F_2$ corresponding to $\partial S$. In terms of a basis $(X, Y)$ of $F_2$, such an equivalence class is $\text{Aut}(F_2)$-related to that of $X^2Y^2$. Its homology class in $H_1(F_2) \cong \mathbb{Z}^2$ is twice that of $XY$, which is a primitive element of $\mathbb{Z}^2$. Therefore the peripheral structures corresponding to one-holed Klein bottles correspond bijectively to inversion-conjugacy classes of primitive elements in $F_2$, that is, to points in $\mathbb{P}^1(\mathbb{Q})$. When the coloring $\Phi$ is specified, these primitive classes correspond to

\[
1 \leftrightarrow \text{odd} \quad \text{odd} \in \mathbb{P}^1(\mathbb{Q})
\]
under reduction modulo 2; see (3). We refer to such an element of $\mathbb{P}^1(\mathbb{Q})$ (or the corresponding primitive class) as totally odd.

Apply Proposition 5.2.1 to Proposition 8 to obtain:

Corollary 8.2.1. $\overrightarrow{T}_\mu$ is well-directed towards a unique sink if and only if $\mu$ lies in $\mathcal{O}'(C_{1,1}) = \Gamma \cdot \mathfrak{f}'(C_{(1,1)})$. 

\[\text{odd} \quad \text{odd} \in \mathbb{P}^1(\mathbb{Q})\]
We identify this Fricke orbit with the Bowditch set:

**Proposition 8.2.2.** Suppose that $\mu$ is an unexceptional imaginary trace labeling with $k > 2$. Then $\overrightarrow{T}_\mu$ is well-directed towards a sink if and only if $\mu \in \mathcal{B}$.

**Proof.** Suppose that $\overrightarrow{T}$ is well-directed toward a sink $v_0(X_0, Y_0, Z_0)$ where

$$
\mu(X_0) = ix_0, \\
\mu(Y_0) = iy_0, \\
\mu(Z_0) = z_0,
$$

and $x_0, y_0, z_0 \in \mathbb{R}$. We verify the Bowditch conditions.

Since $\mu$ is unexceptional and $|z_0| > 2$, $|x_0|, |y_0|, |z_0| - 2 > \varepsilon$ for some $\varepsilon > 0$.

Propositions 8.1.2, 8.1.3 and 8.1.4 imply that all but possibly one edge of $T_\mu$ point decisively towards $v_0$. Furthermore the difference in the absolute values of the regions at the opposite ends of a decisive edge is $> \varepsilon^2$, except possibly for one $\mathbb{R}$-edge adjacent to $v_0$. Hence, moving away from the sink, each step (except for possibly the first step) increases the absolute value of the new trace label by at least $\varepsilon^2$. Hence $\Omega_\mu(2)$ is finite and $\mu$ satisfies Condition (42).

Similarly, since the trace labeling $\mu$ is unexceptional, no trace value is zero. Now by Proposition 4.2.3, $|\mu(Z)| > 2$ for $Z \in \Omega_\mathbb{R}(T)$. Hence, no trace value lies in $[-2, 2]$ which implies Condition (40). Thus each Bowditch condition is satisfied and $\mu \in \mathcal{B}$, as claimed.

Another proof follows from Proposition 6.2.2. In this case the only attracting finite subtrees are indecisive attracting edges.

**8.3. Planar projection of the Bowditch set.** Now we describe in more detail the structure of the Bowditch set (or Fricke orbit) and its complement in the level surface $\kappa_\Phi^{-1}(k)$. Proposition 4.2.2 implies that the two components of the level surface $\kappa_\Phi^{-1}(k)$ are graphs of the functions $z_\pm$ defined in (29) and the projection $\Pi_{xy}$ maps each component diffeomorphically onto the $xy$-plane.

First observe that the imaginary characters $(ix, iy, \sqrt{k + 2})$ lie on the lines in the plane $z = \sqrt{k + 2}$

$$
y = m^\pm x
$$
having slopes
\begin{equation}
m^\pm = m^\pm(k) := \frac{\sqrt{k+2} \pm \sqrt{k-2}}{2}.
\end{equation}
These lines intersect in the dihedral character $\Delta = (0, 0, \sqrt{k+2})$. Moreover characters $(ix, iy, z)$ with $2 < z < \sqrt{k+2}$ project to the sectors in the first and third quadrant bounded by these lines. Similarly, the characters $(ix, iy, -\sqrt{k+2})$ lie on lines
\[ y = -m^\pm x \]
in the plane $z = -\sqrt{k+2}$. These lines intersect in the dihedral character $-\Delta = (0, 0, -\sqrt{k+2})$. In particular the tangent space at $\pm\Delta$ equals the horizontal plane defined by $z = 0$:
\begin{equation}
T_{\pm\Delta}(\kappa^{-1}_\Phi(k)) = \mathbb{R}^2 \oplus 0 \subset \mathbb{R}^3.
\end{equation}
Characters $(ix, iy, z)$ with $-\sqrt{k+2} < z < -2$ project to sectors in the second and fourth quadrant bounded by these lines. The infinite dihedral group generated by $I_1, I_2$ preserves these sectors.

Each sector identifies with the generalized Fricke space $\mathcal{F}'(C_{1,1})$ of the one-holed Klein bottle. The four components represent the orbit of $\mathcal{F}'(C_{1,1})$ under sign-change automorphisms.

Recall that the group $\Gamma$ is generated by sign-changes $\Sigma$ and $PGL(2, \mathbb{Z})_\Phi \cong (\mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2) \rtimes \mathbb{Z}/2$ and this group acts faithfully on $i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$. The projection $\Pi_{xy}(\mathcal{B})$ is then the projection of the $\Gamma$-orbit of these sectors. Each sector is $\hat{\Gamma}_\Phi$-invariant, and therefore $\hat{\Gamma}_\Phi$ acts on the set of sectors. Thus the pair consisting of $(0, 0, \pm z)$ is $\Gamma$-invariant. These correspond to dihedral characters. Therefore they meet the boundaries of components of $\Pi_{xy}(\mathcal{B})$. All the other components are bounded by curves which are the images of the lines $y = m^\pm x$ under $\Gamma/\Gamma_\Phi$. Except for the lines $y = m^\pm x$, these curves are not straight lines.

The projection $\Pi_{xy}$ maps $\mathcal{B}$ onto countably many connected components in the $xy$-plane, all of whose closures contain the origin.

The components are indexed by the totally odd rationals, that is, rational numbers $p/q$ where both $p, q$ are odd integers. These components bijectively correspond to the subset $\Omega_{\mathbb{R}} \subset \Omega$. Furthermore a cyclic ordering on the set respects the cyclic ordering of the rationals in $\hat{\mathbb{R}}$. The complement of $\Pi_{xy}(\mathcal{B})$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$ consists of uncountably many connected components indexed by $\hat{\mathbb{R}}$ and has empty interior.

These can be understood in terms of the end invariants introduced by Tan-Wong-Zhang [28] which we describe briefly. Recall that the
tree of superbases $T$ can be identified with the dual graph of the Farey triangulation so that the set of complementary regions $\Omega$ of $T$ is identified with $\mathbb{P}^1(\mathbb{Q})$. The trace labeling $\mu$ associated to a character $\rho \in \text{Hom}(F_2, \text{SL}(2, \mathbb{C}))$ is a map from $\Omega$ to $\mathbb{C}$. The equivalence classes of ends of the tree $T$ identifies with $\mathbb{P}^1(\mathbb{R})$, where the identification is two-to-one on the rationals (corresponding to the two ends of the alternating geodesic about $Z \in \Omega$) and one-to-one on the irrationals. We identify the ends of $T$ with $\mathbb{P}^1(\mathbb{R})$. The topology of $\mathbb{P}^1(\mathbb{R})$ induces a topology on the set of ends. The cyclic ordering on $\mathbb{P}^1(\mathbb{R})$ induces a cyclic ordering on the set of ends.

Let $\lambda$ denote an end of $T$. Then, following [28], call $\lambda$ an end invariant of $\rho$ if, for some constant $K > 0$ and infinitely many distinct $Z_n \in \Omega$ adjacent to $\lambda$,

$$|\mu(Z_n)| < K.$$  

Denote the set of end invariants of $\mu$ by $E(\mu) \subset \mathbb{P}^1(\mathbb{R})$.

For characters on $\kappa^{-1}(k)$ when $k > 2$, at most one end invariant exists. This follows from the fact that the tree is well-directed. Note also that, in this case, $\mathcal{B}' = \mathcal{B}$ since for all $Z \in \Omega$ with real trace, $|\mu(Z)| > 2$.

As proved in [28], $E(\mu) = \emptyset$ if and only if $\mu \in \mathcal{B} = \mathcal{B}'$. Otherwise $E(\mu) = \{\lambda\}$ for a unique $\lambda \in \mathbb{P}^1(\mathbb{R})$. End invariants of characters on the boundary of $\mathcal{B}$ correspond to totally odd rationals. In particular end invariants of characters corresponding to points on the lines of slope $m^\pm$ correspond to $1 \in \mathbb{P}^1(\mathbb{Q})$ These lines bound the two sectors $\mathcal{B}_0$ defined by

$$2 < z < \sqrt{k + 2},$$  

or, equivalently,

$$m^- < y/x < m^+$$

The other components of $\mathcal{B}$ correspond to the images of $\mathcal{B}_0$ under $\gamma \in \Gamma$, and are indexed by the totally odd rational numbers $\gamma(1)$. Points on the boundary of these components correspond to characters with end invariant the corresponding totally odd rational numbers $\gamma(1)$.

The complement of $\Pi_{xy}(\mathcal{B})$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ consists of uncountably many connected components easily described in terms of the end invariants: Components with end invariant indexed by totally odd $p/q \in \mathbb{Q}$ ($p, q$ odd) correspond to the boundaries of the components of $\Pi_{xy}(\mathcal{B})$, Components with end invariant indexed by $p/q \in \mathbb{Q}$ with either $p$ or $q$ even correspond to the the exceptional characters $\Gamma_{\mathcal{B}}(0, -iy, z)$. Components with end invariant indexed by an irrational correspond to characters whose trace labeling $\mu$ has directed tree $\overrightarrow{T}$ directed towards a fixed irrational end.
As in §3.3, three involutions \( \mathcal{I}_1, \mathcal{I}_2, \sigma_1 \circ \mathcal{I}_3 \subset \Gamma \) act on \( \mathbb{R}^3 \) by:

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} \overset{\mathcal{I}_1}{\rightarrow} \begin{bmatrix}
  yz - x \\
  y \\
  z
\end{bmatrix}, \quad \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} \overset{\mathcal{I}_2}{\rightarrow} \begin{bmatrix}
  xz - y \\
  x \\
  z
\end{bmatrix}, \quad \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} \overset{\sigma_1 \circ \mathcal{I}_3}{\rightarrow} \begin{bmatrix}
  x - y \\
  xy + z \\
  -y
\end{bmatrix}
\]

(We compose the Vieta involution \( \mathcal{I}_3 \) with the sign-change \( \sigma_1 \) to preserve each of the two components of \( \kappa_\Phi^{-1}(k) \).) Each involution fixes the line \( \{0\} \times \{0\} \times \mathbb{R} \). This line consists of dihedral characters. Their derivatives at \( p = (0, 0, \pm \sqrt{k + 2}) \) determine three linear involutions as follows. Since both \( \{0\} \times \{0\} \times \mathbb{R} \) and \( \kappa_\Phi^{-1}(k) \) are invariant and intersect transversely at \( p \), the tangent spaces decompose as direct sums:

\[
T_\Delta(\mathbb{R}^3) = T_\Delta(\kappa_\Phi^{-1}(k)) \oplus T_\Delta(\{(0, 0)\} \times \mathbb{R})
\]

\[
= (\mathbb{R}^2 \oplus 0) \quad \oplus \quad (0 \oplus 0 \oplus \mathbb{R})
\]

With respect to this decomposition, \( \mathcal{I}_1, \mathcal{I}_2, \sigma_1 \circ \mathcal{I}_3 \) act infinitesimally at \( \Delta \) by their derivatives:

\[
D_\Delta(\mathcal{I}_1) = \begin{bmatrix}
  -1 & \sqrt{k + 2} \\
  0 & 1
\end{bmatrix} \oplus [1]
\]

\[
D_\Delta(\mathcal{I}_2) = \begin{bmatrix}
  1 & 0 \\
  \sqrt{k + 2} & -1
\end{bmatrix} \oplus [1]
\]

\[
D_\Delta(\sigma_1 \circ \mathcal{I}_3) = \begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix} \oplus [1]
\]

Projectivizing these \( 2 \times 2 \) matrices yield reflections of the Poincaré upper halfplane \( \mathbb{H}^2 \):

\[
z \mapsto \sqrt{k + 2} - \bar{z}, \quad z \mapsto \frac{\bar{z}}{\sqrt{k + 2} \, \bar{z} - 1}, \quad z \mapsto -\bar{z},
\]

respectively. These are reflections in the geodesics with pairs of endpoints:

\[
\{\infty, \sqrt{k + 2}/2\}, \quad \{2/\sqrt{k + 2}, 0\}, \quad \{0, \infty\},
\]

respectively. These generate a Fuchsian group whose quotient is doubly covered by a disc with two cusps and geodesic boundary of length \( 2 \cosh^{-1}(k/2) \). (Compare Figure 8.)

The limit set \( \Lambda \) is a Cantor set in \( \mathbb{P}^1(\mathbb{R}) \). Its complement is a countable collection of open intervals, corresponding to the orbit of \( \mathcal{F}'(C_{1,1}) \). The fixed points of the composition

\[
D_\Delta(\mathcal{I}_1) \circ D_\Delta(\mathcal{I}_2)
\]
Figure 8. The Fuchsian group generated by Vieta involutions

exactly correspond to the slopes of the lines \( y = m \pm x \) bounding \( \mathcal{G}'(C_{1,1}) \) (where \( m \pm \) is defined as in (44)). The points of \( \Lambda \) correspond to the components of the complement of the domain of discontinuity.

For an exceptional imaginary trace labeling

\[
\mu \in \hat{\Gamma} \cdot (0, iy, z),
\]

where \( y \neq 0 \) and \( |z| > 2 \), only one region \( X \in \Omega \) with \( \mu(X) = 0 \) exists. Furthermore all edges of \( T_\mu \) not adjacent to \( X \) point towards \( X \). Edges adjacent to \( X \) are indecisive and can therefore be directed arbitrarily.

8.4. Density of the Bowditch set. By Bowditch [1] and Tan-Wong-Zhang [29], \( \mathcal{B} \) is open. Now we prove that it is dense:

Theorem 8.4.1. The interior of the complement of the Bowditch set \( \mathcal{B} \) is empty.

The proof involves the \( \Gamma \)-invariant smooth measure provided by Lemma 4.3.1 arising from the Poisson bivector \( \mathcal{B}_\Phi \) defined in (19). Denote the measure of a Lebesgue-measurable subset \( S \subset \kappa^{-1}_\Phi(k) \) by \( \text{area}(S) \).

Proof. Suppose not; let

\[
U \subset \kappa^{-1}_\Phi(k) \setminus (\mathcal{B} \cup \{(0, 0, z)\})
\]

be a nonempty connected open set.
1.2. Two remarks. The following two observations allow one to completely determine the dynamics solving Goldman’s problem.

Observation 1: The first of these is that one can find a linearization of the dynamics at a “global fixed point” for $K$: for any $k$ the point $(0, 0, p_k + 2)$ is a fixed point of every element of $K$. The linearization is a homomorphism $K: K \to GL(\mathbb{T}(0, 0, p_k + 2) \cong 1(k))$, $f^\ast \mapsto D(0, 0, p_k + 2)f^\ast$. Since the $K$ action preserves an area form on the level set $1(k)$ the image of $K$ is actually contained in $SL$.

One might hope that the linearization determines the dynamics on the whole level set. In fact, we shall use the action of this linearization on the tangent plane and the associated projective line to serve as a model for the dynamics on the rest of the level set.

Observation 2: Let $Z$ denote the infinite cyclic subgroup of $K$ generated by $Q_xQ_y$; note that $(Q_xQ_y)^1 = Q_yQ_x$.

Since $U \cap \mathfrak{B} = \emptyset$, a character $u \in U$ determines a descending path

$$\vec{P}(u) = (u = u_0 \xrightarrow{f_0} u_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} u_n \xrightarrow{f_n} \cdots)$$

where each $f_i$ corresponds to $I_{j(i)}$ for some $j(i) = 1, 2, 3$ and $j(i) \neq j(i + 1)$.

Moreover, $\vec{P}$ depends continuously on the character $u$. Suppose $\mu$ is a character lying two different translates of $\mathfrak{B}_0$. Then there are two different descending paths leading to two different $Z, Z' \in \Omega$ with $|z|, |z'| < \sqrt{k + 2}$. That would imply existence of two sinks, a contradiction. Therefore $\vec{P}$ depends continuously on the character $u$. Since $U$ is connected, $\vec{P}$ is independent of $u \in U$.

Then $\vec{P}$ determines a sequence

$$\gamma_n := \mathcal{I}_{j(n)} \circ \cdots \circ \mathcal{I}_{j(1)} \circ \mathcal{I}_{j(0)} \in \Gamma.$$  

**Lemma 8.4.2.** The $\gamma_n(U)$ lie in finitely many connected components of the complement of $\mathfrak{B}'$. 

*Figure 9. $xy$-projection of domain of discontinuity for $k = 3$*
Proof. We may assume that \( U \) lies in
\[
\Box_R := \left\{ (x, y, z_\pm(x, y)) \in \kappa_\Phi^{-1}(k) \mid |x|, |y| \leq R \right\}
\]
for some \( R > 0 \). Observe that \( \text{area}(\Box_R) < \infty \). By (39), the coordinates \(|x|, |y|\) are non-increasing, so that \( \gamma_n(U) \subset \Box_R \) for all \( n > 0 \).

Suppose first that the \( \gamma_n(U) \) lie in infinitely many components of the complement of \( \mathfrak{B}' \). Choosing a subsequence if necessary, we may assume these components are all distinct. Then the \( \gamma_n(U) \) are all disjoint. Since \( \text{area}(\gamma_n(U)) = \text{area}(U) > 0 \),
\[
\text{area}(\Box_R) \geq \sum_{n=1}^{\infty} \text{area}(\gamma_n(U)) = \infty,
\]
a contradiction. \( \square \)

Lemma 8.4.3. The descending path corresponding to the component of the complement containing \( U \) is infinite.

Proof. Otherwise \( U \) is contained in a component which is the image of either the \( x \) or \( y \) axis under some element of \( \Gamma \). Since these sets have measure zero, this contradicts \( \text{area}(U) > 0 \). \( \square \)

Lemma 8.4.4. The descending path is eventually periodic.

Proof. Since there are only finitely many components, \( \gamma_n(U) \) and \( \gamma_m(U) \) lie in the same component of the complement for some \( n > m \). Replacing \( U \) by \( \gamma_m(U) \), we may assume that \( \gamma_N(U) \) and \( U \) lie in the same component for some \( N > 1 \). Since all elements of the same component of the complement have the same infinite descending path, the infinite descending path for \( U \) must be periodic. \( \square \)

Thus we may write \( \gamma_{mn} = (I_{j_1}I_{j_2} \ldots I_{j_m})^n \) where \( I_{j_1}I_{j_2} \ldots I_{j_m} \) is the period.

Lemma 8.4.5. The descending path cannot consist entirely of imaginary edges. That is, at least one \( j_n \) must equal 3.

Proof. Otherwise the (minimal) period would be \( I_1I_2 \) or \( I_2I_1 \). Then \( U \) would be contained in the two lines which bound the Fricke space. Since these lines have measure zero, this contradicts \( \mu(U) > 0 \). \( \square \)

Thus we may assume that \( j_1 = 3 \), that is, the period begins with \( I_3 \) and the path begins with an \( \mathbb{R} \)-edge. Equivalently the vertex corresponding to \( (x, y, z) \in U \) is negative.

Choose \( \delta > 0 \) sufficiently small so that the lift
\[
\mathfrak{N}_\delta := (\Pi_{xy})^{-1}(\mathfrak{N}_\delta(0,0))
\]
of the Euclidean $\delta$-ball $\mathfrak{M}_\delta(0,0) \subset \mathbb{R}^2$ to $\kappa^{-1}_\varphi(k)$ has area less than $\text{area}(U)/2$. Then
\[
\text{area}(U \setminus \mathfrak{M}_\delta) \geq \text{area}(U) - \text{area}(\mathfrak{M}_\delta) > \text{area}(\mathfrak{M}_\delta).
\]
Thus by replacing $U$ by $U \setminus \mathfrak{M}_\delta$ we may suppose that $U$ is disjoint from $\mathfrak{M}_\delta$ and $\text{area}(\mathfrak{M}_\delta) < \text{area}(U)$.

**Lemma 8.4.6.** $\gamma_n(U) \setminus \mathfrak{M}_\delta = \emptyset$ for $n$ sufficiently large.

**Proof.** First bound $\gamma_n(U) \setminus \mathfrak{M}_\delta$ away from the coordinate axes: The coordinate axes describe four components of the complement of $\mathfrak{B}$, so we may assume the infinite descending path misses these axes. Furthermore $\gamma_n(U) \setminus \mathfrak{M}_\delta$ lie in finitely many closed subsets disjoint from these axes, and inside the compact subset $\square_R$. Thus we may assume that the coordinates of $\gamma_n(U) \setminus \mathfrak{M}_\delta$ are bounded away from 0. That is, there exists $\epsilon > 0$ with the following properties: For each $u_n \in \gamma_n(U)$, either
\[
|x_n| \geq \epsilon, \quad |y_n| \geq \epsilon,
\]
or $u_n \in \mathfrak{M}_\delta$, in which case we are done. Furthermore by Lemma 8.1.1, $|z_n| > 2$ for all $n$.

Now (29) implies that if $|x|, |y| \leq R$, then
\[
|z_{\pm}(x,y)| \leq C
\]
where
\[
C := \frac{1}{2} \left( R^2 + \sqrt{(R^2 + 4)^2 + 4(k - 2)} \right).
\]
The infinite descending path contains infinitely many words containing $\mathfrak{I}_3$. Otherwise eventually it becomes an alternating sequence of $\mathfrak{I}_1$ and $\mathfrak{I}_2$, which corresponds to totally odd rationals. For each $\gamma_n$, the value of the $z$-coordinate either remains constant or changes by:
\[
|z_{n+1}| = |x_n y_n - z_n| \leq |z_n| - \epsilon^2.
\]
Furthermore, since the period contains an $\mathfrak{I}_3$, the value of $z$ changes at least once over each period. The value of $|z_n|$ therefore decreases by at least $\epsilon^2$ over each period. Thus, for $n > mC/\epsilon^2$ where $m$ is the period,
\[
\gamma_n(u) \in \mathfrak{M}_\delta,
\]
since otherwise $|z_n| < 0$, a contradiction. \hfill \Box

**Conclusion of proof of Theorem 8.4.1.** Lemma 8.4.6 implies that
\[
\gamma_n(U) \subset \mathfrak{M}_\delta
\]
for sufficiently large $n$. Thus
\[
\text{area}(U) = \text{area}(\gamma_n(U)) \leq \text{area}(\mathfrak{M}_\delta) < \text{area}(U),
\]
a contradiction. □

**Conjecture 8.4.7.** If \( k > 2 \), then \( \text{area}(\kappa^{-1}_\Phi(k) \setminus \mathcal{O}'(C_{1,1})) = 0 \).

9. **Imaginary characters with \( k < 2 \).**

For an imaginary trace labeling \( \mu \) with \( k < 2 \), possibly \( z = \mu(Z) \in (-2, 2) \) for some \( Z \in \Omega_{\mathbb{R}} \). Then the action of \( \Gamma \) on an open subset of \( \kappa^{-1}_\Phi(k) \) is ergodic. In particular if \(-14 \leq k < 2 \), the \( \Gamma \)-action is ergodic on all of \( \kappa^{-1}_\Phi(k) \).

If \( k < -14 \), then the Fricke orbit is a disjoint union of wandering domains, and the action on the complement is ergodic. For \( k = -14 \), the Fricke orbit is a countable discrete set, upon whose complement the action is ergodic.

9.1. **Existence of elliptics.** The first step will be to show that some primitive element is mapped to an elliptic.

**Proposition 9.1.1.** Suppose that \( k < 2 \). Then there exists \( \epsilon = \epsilon(k) > 0 \) with the following property: For a non-exceptional trace labeling \( \mu \) with

\[
|\mu(Y)| = |iy| < \epsilon
\]

for some

\[
Y \in \Omega \setminus \Omega_{\mathbb{R}},
\]

then \( Y \) has a neighbor \( Z \in \Omega_{\mathbb{R}} \) such that

\[
|\mu(Z)| = |z| < 2.
\]

**Proof.** For \( y \) sufficiently small, some \( |\mu(Z_n)| < 2 \) where

\[
\ldots, Z_{n-1}, X_n, Z_n, X_{n+1}, \ldots
\]

denotes the sequence of regions abutting \( Y \). Since \( \Phi(Z_n) = 0 \) and \( \Phi(X_n) = 1 \), the corresponding sequence of traces alternates between real and purely imaginary:

\[
\mu(X_n) = ix_n
\]
\[
\mu(Z_n) = z_n
\]

where \( x_n, z_n \in \mathbb{R} \). (Compare Figure 6.)

Analogously to (28), for fixed \( y \), the \( y \)-level sets of \( \kappa^{-1}_\Phi(k) \) are hyperbolas

\[
H_{k,y} := \kappa^{-1}_\Phi(k) \cap (\mathbb{R} \times \{y\} \times \mathbb{R})
\]
\[=(Q'_y)^{-1}(y^2 - k + 2)\]
where $Q'_y$ denotes the quadratic form
\begin{equation}
Q'_y(x, z) := z^2 + yxz - x^2.
\end{equation}
Choose $\beta \in \mathbb{R}$ such that the trace $iy = (i\beta) + (i\beta)^{-1}$, that is,
\begin{equation}
y = \beta - \beta^{-1}.
\end{equation}
By Edge Relation (36), the trace sequence
\[\ldots, z_{n-1}, ix_n, z_n, ix_{n+1}, \ldots\]
satisfies:
\begin{equation}
\begin{bmatrix}
ix_{n+1} \\
\hline
z_{n+1}
\end{bmatrix}
= M_y
\begin{bmatrix}
z_n \\
ix_{n+1}
\end{bmatrix}
\end{equation}
\begin{equation}
\begin{bmatrix}
z_n \\
ix_{n+1}
\end{bmatrix}
= M_y
\begin{bmatrix}
ix_n \\
z_n
\end{bmatrix}
\end{equation}
where
\begin{equation}
M_y := \begin{bmatrix}
0 & 1 \\
-1 & iy
\end{bmatrix}.
\end{equation}
(Compare Proposition 6.5.1.) Moreover $M_y$ has eigenvalues $\pm i\beta$ and preserves the quadratic form $Q'_y$ defined in (47). Furthermore $Q'_y$ factors as the product
\begin{equation}
Q'_y(x, z) = l_\beta(x, z)l'_\beta(x, z)
\end{equation}
of homogenous linear functions
\begin{align*}
l_\beta(x, z) &:= z + \beta x \\
l'_\beta(x, z) &:= z - \beta^{-1} x.
\end{align*}
These linear factors are eigen-covectors under $(M_y)^2$:
\begin{align*}
l_\beta \circ (M_y)^2 &= -\beta^{-2}l_\beta \\
l'_\beta \circ (M_y)^2 &= -\beta^2l'_\beta
\end{align*}
The points $(x_n, z_n)$ form a lattice in the hyperbola $H_{k,y}$ (in the sense of §6.5) defined by (46).

For $|y| < \sqrt{k + 2}$, this hyperbola meets the strip defined by $|z| < 2$. (Since $k < 2$, the level value is negative: $2 - k + y^2 < 0$.) The multiplicative increment of the lattice formed by the points $(x_n, z_n) \in H_{k,y}$ equals $\beta$. For $y$ small, that is, when $\beta \sim 1$, the lattice points are so closely spaced that at least one lattice point lies in the strip. □
9.2. **Alternating geodesics for** \( k < 2 \). Suppose that \( \mu \) is an unexceptional imaginary trace labeling with \( k < 2 \) and let \( Z \in \Omega_\mathbb{R} \). We refine Proposition 6.5.4 when \( k < 2 \). Propositions 9.2.1,9.2.2 sharpen Propositions 8.1.2,8.1.3,8.1.4.

**Proposition 9.2.1.** Suppose that \( |z| < 2 \). Then \( Z \) corresponds to an elliptic element and \( |y_n| \) is bounded for all \( n \in \mathbb{Z} \). The points \( (y_n,y_{n+1}) \in \mathbb{R}^2 \) lie on an ellipse in \( \mathbb{R}^2 \). They comprise a dense subset of the ellipse if and only if \( Z \) corresponds to an elliptic element of infinite order, that is, if \( z = 2 \cos(\alpha \pi) \) where \( \alpha \) is irrational.

*Proof.* Apply Corollary 6.5.3(b). \( \square \)

**Proposition 9.2.2.** Suppose that \( |z| \geq 2 \).

- \( v_m \) is negative for a unique \( m \in \mathbb{Z} \);
- Every \( e_n \) points towards \( v_m \);
- Every \( \psi_i \), with \( i \neq m \), points towards \( Z \);
- The edge \( \psi_m \) points away from \( Z \) if
  \[
  |z| > | - y_m y_{m+1} - z |,
  \]
  in which case \( v_m \) is a merge.
- The edge \( \psi_m \) points towards \( Z \) if
  \[
  |z| < | - y_m y_{m+1} - z |,
  \]
  in which case \( v_m \) is a sink.
- If
  \[
  |z| = | - y_m y_{m+1} - z |,
  \]
  then \( \psi_m \) is indecisive, with two possible choices of the sink.

*Proof.* We only consider the case \( |z| > 2 \). The case when \( |z| = 2 \) is similar, using Proposition 6.5.2 instead of Proposition 6.5.1. As in the proof of Proposition 6.5.4 (a) assume

\[
 z > 2, \quad \lambda > 1.
\]

Since \( AB > 0 \), further assume

\[
 A = ia, \quad B = ib
\]

where \( a > 0 \) and \( b < 0 \). Hence \( y_n \) increases monotonically and

\[
 \lim_{n \to -\infty} y_n = -\infty, \quad \lim_{n \to \infty} y_n = \infty.
\]

Re-indexing if necessary, assume \( y_n > 0 \) for all \( n \geq 0 \) and \( y_n < 0 \) for \( n < 0 \). Hence \( v_0 \) is negative and \( v_n \) is positive for all \( n \neq 0 \). Edge Relation (36) imply that all the \( e_n \)'s point decisively towards
Lemma 7.3.1 implies that each $\psi_n$'s point decisively towards $Z$ for $n \neq 0$. For the case of $\psi_0$,

$$z'_0 = -y_{-1}y_0 - z,$$

$$-y_{-1}y_0 > 0.$$  

If $z'_0 < 0$ then $|z| > |z'_0|$ so $\psi_0$ points towards $Z'_0$ and $v_0$ is a merge. Similarly, if $z > z'_0 > 0$, then $\psi_0$ is also directed towards $Z'_0$ and $v_0$ is a merge. If $z'_0 > z > 0$ then $\psi_0$ points towards $Z$ and $v_0$ is a sink. This only happens if $k < -14$, in the case of the Fricke space $\mathfrak{F}(C_{0,2})$ (compare [25]). Finally, if $z'_0 = z$ then $\psi_0$ is indecisive and either vertex at its endpoints can be chosen to be sinks. 

\[ \square \]

9.3. Descending Paths. We use Proposition 9.1.1 to find a simple closed curve $\gamma$ such that $\rho(\gamma)$ is elliptic. Equivalently, we find a complementary region $\omega \in \Omega$ such that $\mu(\omega) \in (-2, 2)$. We accomplish this by following a descending path to such an elliptic region. Otherwise, we fall into a sink. Sinks correspond to the Fricke orbit of $C_{0,2}$. This case is analogous to the case treated in Goldman [9] when $k > 18$, and the Fricke space of $\Sigma_{0,3}$ appears. The following key result implies that an infinite descending path eventually contains an elliptic or meets the Fricke orbit:

**Proposition 9.3.1.** Suppose that $\mu$ is a trace labeling with $\kappa(\mu) < 2$. Let

$$\vec{P} = \langle v_0 \xrightarrow{f_0} v_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} v_n \xrightarrow{f_n} \cdots \rangle$$

be a descending path in $T_\mu$. Then there exists a vertex $v_m = v(X,Y,Z)$ with either:

- $|\mu(Z)| < 2$, or:
- $v(X,Y,Z)$ is a sink with $|\mu(Z)| \geq 2$.

In the latter case, if

$$\mu(X) = ix,$$

$$\mu(Y) = iy,$$

$$\mu(Z) = z,$$

where $x, y, z \in \mathbb{R}$ with $z \geq 2$, then $-xy - z \geq 0$.

The proof breaks into a series of lemmas. Consider an infinite descending path $\vec{P}$ of (38) such that $|z_n| \geq 2$ for every vertex $u_n(X,Y,Z)$ on this path.
Lemma 9.3.2. Infinitely many $f_n$ are $\mathbb{R}$-edges.

Proof. Suppose $P$ contains only finitely many $\mathbb{R}$-edges. Then there exists $N$ such that edge $f_n$ is a $i\mathbb{R}$-edge for $n \geq N$. Furthermore, Proposition 7.2.2 implies that $f_n$ lies in a geodesic bounding a complementary region $Z \in \Omega_\mathbb{R}$ for all $n \geq N$.

By assumption $|z| \geq 2$. By Proposition 9.2.2, each $f_m$ points towards some $u \in \text{Vert}(T)$, a contradiction since $P$ is a descending path. □

Lemma 9.3.3. Choose $\epsilon(k) > 0$ as in Proposition 9.1.1. For each $\mathbb{R}$-edge $f_n$ as above, let $X,Y,Z,W$ be the complementary regions so that $f_n = e^{X,Y}(Z \to W)$ with corresponding trace labels $ix, iy, z, w$. If $|x|, |y| > \epsilon(k)$ and $|z|, |w| \geq 2$, then $|w| < |z| - \epsilon(k)^2$.

Proof. Since $f_n$ is a $\mathbb{R}$-edge directed away from the vertex $v_n(X,Y,Z)$, Proposition 9.2.2 implies $v_n(X,Y,Z)$ is negative. Therefore $xy$ and $z$ have opposite signs.

We claim that $v(X,Y,W) = v_{n+1}$ must be positive. Otherwise, Proposition 9.2.2 applied to the neighbors of $W$ implies that $v(X,Y,W) = v_{n+1}$ is a sink. This contradicts $P$ being an infinite descending path.

Thus $xy$ and $w$ have the same sign, but $z$ has the opposite sign. Applying a sign-change if necessary, assume $z > 0$. Then $xy < 0$ and $w < 0$. Since $w = -xy - z$, and $xy < -\epsilon(k)^2$,

$$|w| = -w = xy + z < -\epsilon(k)^2 + z = -\epsilon(k)^2 + |z|$$

as claimed. □

Lemma 9.3.4. The infinite descending path contains a vertex $v_n$ such that:

- $|z_n| < 2$, or:
- One of $|x_n|, |y_n|$ is $< \epsilon(k)$.

Proof. Suppose, for all vertices $v_n$, that $|x_n|, |y_n| \geq \epsilon(k)$. Lemma 9.3.2 guarantees an infinite set of $\mathbb{R}$-edges in this descending path. Apply Lemma 9.3.3 to this, obtaining a subsequence $(x_n, y_n, z_n)$. Then $|z_n|$ decreases uniformly by $\epsilon(k)^2$. Eventually $|z_n| < 2$ as desired. □

If $|z_n| < 2$, then we're done. Therefore we may assume that $|x_n| < \epsilon(k)$ (the case that $|y_n| < \epsilon(k)$ being completely analogous).

Proposition 9.1.1 implies that $X_n$ has a neighboring region $W$ such that $|w| < 2$. Let $v'(X_n,Y',Z')$ be the vertex on $\partial X_n$ nearest to
$v(X_n, Y_n, Z_n)$, and such that $|z'| < 2$. We claim that the path descends in a unique direction from $v(X_n, Y_n, Z_n)$ to $v'(X_n, Y', Z')$.

First note that by construction, if $v(X, Y, Z)$ is a vertex between these two vertices, then $|z| \geq 2$ and hence $v(X, Y, Z)$ is a merge or a sink. Furthermore, the edge $e^{X_n, Y'}$ which lies on the path between these two vertices points towards $v'(X_n, Y', Z')$, again by construction. Therefore all the edges from $v(X_n, Y_n, Z_n)$ to $v'(X_n, Y', Z')$ point towards $v'(X_n, Y', Z')$ and this descending path is unique from $v(X_n, Y_n, Z_n)$. Finally, this leads to a contradiction as the descending path now meets a region $Z'$ with $|z'| < 2$.

**Proposition 9.3.5.** Suppose that $(ix, iy, z)$ is a purely imaginary character with $k = \kappa(ix, iy, z) < 2$. Consider the corresponding directed tree $\vec{T}$. If both $z$ and $z' := -xy - z$ are $\geq 2$, then the edge $e$ between $Z$ and $Z'$ contains a sink. All the edges adjacent to $e$ are merges and point towards $e$.

**Proof.** By Edge Relation (36), $xy = -(z + z') \leq -4$. Furthermore $k < -2 - zz' \leq -6$. In that case $v(ix, iy, z)$ is a sink. To this end, define $y' := xz - y$. We show that $|y'| > |y|$ as follows. By a sign-change, we can assume that $y > 0$. Since

$$4 \leq z + z' = -xy,$$

$x < 0$ and

$$y' = xz - y < 0.$$

Thus

$$|y'| = -y' = y - xz > y = |y|$$

as claimed.

This argument implies that the directed edge $e = e^{X, Y}(Z, Z')$ points towards $v(X, Y, Z)$. Similarly the other three edges adjacent to the edge connecting $Z$ and $Z'$ point towards $v(X, Y, Z)$. Thus all four edges adjacent to $e$ point inward.

If this edge is decisive, one of these two vertices must be a sink. Otherwise the edge is indecisive, and either of two vertices will be a sink. Changing the direction establishes that other vertex as a sink. \qed

**Conclusion of proof of Proposition 7.1.2 when $k < 2$.**

If $k < 2$ and $|\mu(Z)| \geq 2$ for a sink $v(X, Y, Z)$, Proposition 9.2.2 implies that $\vec{T}$ is well-directed towards $v(X, Y, Z)$ and $\mu \in \mathfrak{B}'$.

Otherwise, Proposition 9.3.1 implies $|\mu(Z)| < 2$ for some $Z \in \Omega_2$, so $\mu \notin \mathfrak{B}$. This completes the proof of Proposition 7.1.2 when $k < 2$. \qed
9.4. Ergodicity. We now complete the proof that the action of $\Gamma$ is ergodic on the complement of the Fricke orbit $\mathcal{O}(C_{0,2})$ when $k < 2$.

**Proposition 9.4.1.** Suppose that $\mu$ is a trace labeling with $k < 2$. Suppose that $Z \in \Omega_R$ with $|z| < 2$. Then $\Gamma$ acts ergodically on the complement

$$\kappa_{\phi}^{-1}(k) \setminus \mathcal{O}(C_{0,2}).$$

The complementary domain $Z$ corresponds to a primitive element in $F_2$ and the corresponding Nielsen move (or Dehn twist) $\nu_Z$ defines an automorphism of $F_2$:

$$F_2 \xrightarrow{\nu_Z} F_2 \xrightarrow{} X \mapsto ZX = Y^{-1} \quad Y \mapsto YZ^{-1} = Y^2 X \quad Z \mapsto Z = Y^{-1}X^{-1}$$

which induces the polynomial automorphism of $\text{Rep}(F_2, \text{SL}(2, \mathbb{C}))$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\nu_Z} \begin{bmatrix} xz - y \\ x \\ z \end{bmatrix}. $$

The induced map $(\nu_Z)_*$ preserves the Hamiltonian flow $\text{Ham}(\mu_Z)$ of the trace function, whose orbits when $|z| < 2$ are ellipses. In terms of the parametrization of these orbits, $(\nu_Z)_*$ acts by translation by $2 \cos^{-1}(\mu_Z/2)$, and when $\cos^{-1}(\mu_Z/2) \notin \mathbb{Q}\pi$, acts by an irrational rotation of $S^1$. In particular this action is ergodic with respect to the natural Lebesgue measure on this orbit. By ergodic decomposition of the invariant area form with respect to this foliation, every $(\nu_Z)_*$-invariant measurable function will be almost everywhere $\text{Ham}(\mu_Z)$-invariant. (This is the same technique used in [8, 9, 14, 11] for related results.)

The orbits of $\text{Ham}(\mu_Z)$ are ellipses on the level plane $z = z_0$ centered at the origin. These ellipses contain the orbit of $\nu_Z$ as a countable dense subset. Thus for each $\epsilon > 0$, for some $N$, the coordinates $(x_n, y_n, z_0)$ of $\nu_Z^n(x, y, z_0)$ satisfy

$$|x_n| < \epsilon, \quad |y_n| < M(k, z_0)$$

where $M = M(k, z_0)$ depends just on $k$ and $z_0$.

The trace function $\mu_Z'$ satisfies

$$|\mu_Z'| = |\mu_X\mu_Y - \mu_Z| = | - x_ny_n - z| < |z| + M\epsilon$$
so for some \( n \) we may assume \( Z' \) is elliptic.

Away from a nullset, the trace function \( \mu \) also defines a Hamiltonian flow whose orbits are ellipses. We show that \( \text{Ham}(\mu_Z) \) and \( \text{Ham}(\mu_{Z'}) \) span the tangent space to the level surface at such a point.

By (19), the Hamiltonian vector field of \( z \) equals

\[
\text{Ham}(\mu_Z) = B_\Phi \cdot dz = (2x - yz)\partial_y + (-2y + zx)\partial_x
\]

and the Hamiltonian vector field of \( z' \) equals

\[
\text{Ham}(\mu_{Z'}) = B_\Phi \cdot dz' = \left( zx + (x^2 + 2)y \right)\partial_x + \left( -zy - (y^2 + 2)x \right)\partial_y + 2(x^2 - y^2)\partial_z
\]

since \( dz' = -y \, dx - x \, dy - dz \).

At a point \( p = (x, y, z) \), the vector field \( \text{Ham}(\mu_Z) \) is zero if and only if

\[
-2y + zx = -2x + yz = 0.
\]

As in the proof of Lemma 4.1.2, if \( z = 0 \), then \( p = (0, 0, 0) \) and \( k = -2 \).

Thus assume \( z \neq 0 \). Then if either \( x = 0 \) or \( y = 0 \), then both \( x = y = 0 \) and \( p \) is the dihedral character \( (0, 0, \pm \sqrt{k + 2}) \).

Thus we can assume \( x \neq 0, y \neq 0, z \neq 0 \). As in the proof of Lemma 4.1.2 again, \( x/y = y/x = 2/z \) and

\[
p \in \left\{ (x, x, 2) \mid x \in \mathbb{R} \right\} \cup \left\{ (x, -x, -2) \mid x \in \mathbb{R} \right\}
\]

which implies \( k = 2 \).

When \( x \neq \pm y \), then \( x^2 - y^2 \neq 0 \), then the coefficient of \( \partial_z \) in \( \text{Ham}(\mu_{Z'}) \) is nonzero, whereas the coefficient of \( \partial_z \) in \( \text{Ham}(\mu_Z) \) is zero. Thus \( \text{Ham}(\mu_Z) \) and \( \text{Ham}(\mu_{Z'}) \) are linearly independent unless \( x = \pm y \), which describes a nullset in \( \kappa_{\Phi}^{-1}(k) \).
10. Imaginary characters with $k = 2$.

When $k = 2$, the level set $\kappa_\Phi^{-1}(2)$ admits a rational parametrization as in Goldman [12]. These characters correspond to reducible representations, which may be taken to be diagonal matrices. The representation

$$
\rho(X) := i \begin{bmatrix} e^{a/2} & 0 \\ 0 & -e^{-a/2} \end{bmatrix},
\rho(Y) := i \begin{bmatrix} e^{b/2} & 0 \\ 0 & -e^{-b/2} \end{bmatrix},
\rho(Z) := \begin{bmatrix} -e^{-(a+b)/2} & 0 \\ 0 & -e^{(a+b)/2} \end{bmatrix}
$$

(50)

corresponds to the character $(ix, iy, z)$ where

$$
x = 2 \sinh(a/2),
\quad y = 2 \sinh(b/2),
\quad z = -2 \cosh((a + b)/2)
$$

(51)

are real and satisfy the defining equation:

$$
x^2 + y^2 - z^2 + xyz = 4.
$$

The resulting mapping

$$
\mathbb{R}^2 \xrightarrow{\Psi} \kappa_\Phi^{-1}(2)
$$

(52)

$$
(a, b) \mapsto (ix, iy, z)
$$

defines a diffeomorphism of $\mathbb{R}^2$ onto the component $\text{graph}(z_-)$ of $\kappa_\Phi^{-1}(2)$. (The composition $\sigma_1 \circ \Psi$ (or $\sigma_2 \circ \Psi$) defines a diffeomorphism of $\mathbb{R}^2$ with the other component $\text{graph}(z_+)$ of $\kappa_\Phi^{-1}(2)$.)

Geometrically, these characters correspond to actions which stabilize a line $\ell \subset H^2$. When $x \neq 0$, then $\rho(X)$ is a glide-reflection about $\ell$. Otherwise $\rho(X)$ is reflection about $\ell$. Similarly $y \neq 0$ (respectively $y = 0$) corresponds to the case that $\rho(Y)$ is a glide-reflection (reflection) about $\ell$. In these cases $\rho(Z)$ is either $I$ or a transvection in $\ell$.

Ergodicity of the $\Gamma$-action on $\kappa_\Phi^{-1}(2)$ follows from Moore’s ergodicity theorem [21] (see also Zimmer [31]) as in Goldman [10] in the orientation-preserving case.

The mapping $\Psi$ is equivariant with respect to an action of $\text{GL}(2, \mathbb{Z})_\Phi$ on $\mathbb{R}^2$. Furthermore $\text{GL}(2, \mathbb{Z})_\Phi$ is a non-uniform lattice in the Lie group $\text{SL}_+(2, \mathbb{R})$, and $\text{SL}_+(2, \mathbb{R})$ acts transitively on $\mathbb{R}^2$ with isotropy subgroup

$$
N := \left\{ \begin{bmatrix} 1 & x \\ 0 & \pm 1 \end{bmatrix} \mid x \in \mathbb{R} \right\}.
$$
Since $N \subset \text{SL}_\pm(2, \mathbb{R})$ is noncompact and $\text{SL}_\pm(2, \mathbb{R})/\text{GL}(2, \mathbb{Z})_{\Phi}$ has finite Haar measure, Moore’s theorem implies the action of $\text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ is ergodic. It follows that $\Gamma$ acts ergodically on $\kappa_{\Phi}^{-1}(2)$. 
Figure 10. Two views of the two-component level set for $k > 2.1$. Each component projects diffeomorphically onto $\mathbb{R}^2$. 
Figure 11. The first figure depicts the connected, but singular, level set, for $k = -2 < 2$. This is the purely imaginary real form of the Markoff surface. The second figure depicts the level set for $k = -10 < 2$, which is connected and intersects the elliptic region $|z| < 2$ in an annulus.
Figure 12. The four components of the generalized Fricke space \( \mathfrak{F}(C_{1,1}) \cap \kappa^{-1}_\Phi(k) \) for \( k = 23 \). These components are permuted by the group of sign-changes. They are bounded by the singular hyperbolae on the planes \( z = \pm 5 \). In the second picture, the four components of the generalized Fricke space \( \mathfrak{F}'(C_{1,1}) \cap \kappa^{-1}_\Phi(k) \) for \( k = 23 \), projected to the \( xy \)-plane, forming four triangles each having a vertex at the origin.
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