Relative Phase Shifts for Metaplectic Isotopies Acting on Mixed Gaussian States

Maurice A. de Gosson*
University of Vienna
Faculty of Mathematics (NuHAG)

Fernando Nicacio†
Universidade Federal de Rio de Janeiro
Instituto de Física

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Abstract

We address in this paper the notion of relative phase shift for mixed quantum systems. We study the Pancharatnam–Sjöqvist phase shift \( \varphi(t) = \text{Arg} \text{Tr}(\hat{U}_t\hat{\rho}) \) for metaplectic isotopies acting on Gaussian mixed states. We complete and generalize previous results obtained by one of us while giving rigorous proofs. This gives us the opportunity to review and complement the theory of the Conley–Zehnder index which plays an essential role in the determination of phase shifts.

1 Introduction

While the postulates of quantum mechanics seem to recognize complex wave functions as mere instruments for calculating probability amplitudes, their phases should definitively not be viewed as secondary objects. There is actually a plethora of examples in which the phase plays the title-role. The arguably most famous example of this is the Aharonov–Bohm effect dealing with questions about the factual significance of electromagnetic potentials [2]. A break point on general phases behavior in quantum mechanics also emerges from Berry’s seminal work [3]. Its main contribution is to recognize

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* maurice.de.gosson@univie.ac.at
† nicacio@if.ufrj.br
that the total phase of a system is composed by two essentially distinct effects: a phase related to the system dynamics, and a geometrical phase, which mirrors the geometry of the underlying Hilbert space of the system of pure states. The same kind of geometrical phenomenon was observed earlier by Pancharatnam [41] in the context of classical optics, where the phase shift is due to the spherical geometry of the polarization states of light. Also, the Hannay angle [29] is an example of phase shifts induced by the space shape in classical mechanics.

A more profound and general knowledge was acquired by Mukunda and Simon [38], where the authors gave precise definitions for the total, the dynamical, and the geometrical phases for pure states only as functions of paths in the Hilbert space.

Consider a quantum system represented at initial time \( t = 0 \) by a function \( \psi \in L^2(\mathbb{R}^n) \). Assuming that the time-evolution of the system is governed by a one-parameter family of unitary operators \( \hat{U}_t \) on \( L^2(\mathbb{R}^n) \) the system will be represented at time \( t \) by the function \( \psi_t = \hat{U}_t \psi \). In [38] Mukunda and Simon (see also [1]) defined the (relative) phase shift of the system when \( (\hat{U}_t \psi | \psi)_{L^2} \neq 0 \) by the formula

\[
\phi(t) = \text{Arg}(\hat{U}_t \psi | \psi)_{L^2},
\]

which was named as the Pancharatnam or total phase.

Suppose now that the system under consideration is in a “mixed state” represented by a density operator \( \hat{\rho} = \sum_j \lambda_j \hat{\Pi} \psi_j \) (\( \hat{\Pi} \psi_j \) the orthogonal projection on the ray \( \mathbb{C} \psi_j \)). The operator \( \hat{\rho} \) is a positive semidefinite (and hence self-adjoint) trace class operator on \( L^2(\mathbb{R}^n) \) with trace \( \text{Tr}(\hat{\rho}) = 1 \). Its time evolution is given by \( \hat{\rho}_t = \hat{U}_t \hat{\rho} \hat{U}_t^\dagger \) and one now defines, following Sjöqvist et al. [46], the phase shift of this quantum state by

\[
\phi(t) = \text{Arg} \text{Tr}(\hat{U}_t \hat{\rho})
\]

when \( \text{Tr}(\hat{U}_t \hat{\rho}) \neq 0 \). It is easy to see that this definition coincides with Pancharatnam’s formula (1) when \( \hat{\rho} = \hat{\Pi} \psi \); since trace class operators form a two-sided ideal in the algebra of bounded operators the product \( \hat{U}_t \hat{\rho} \) is a trace class operator with same rank one as \( \hat{\rho} \); it follows that \( \text{Tr}(\hat{U}_t \hat{\rho}) \) is precisely the only eigenvalue of this operator. The equation \( \hat{U}_t \hat{\rho} \phi = \lambda \phi \) is equivalent to \( (\phi | \psi)_{L^2} \hat{U}_t \psi = \lambda \phi \). Choosing \( \phi = \hat{U}_t \psi \) we get \( (\hat{U}_t \psi | \psi)_{L^2} \hat{U}_t \psi = \lambda \hat{U}_t \psi \) hence \( \lambda = (\hat{U}_t \psi | \psi)_{L^2} \). The generalization (2) relies on the fact that this quantity, as well as the one in (1), can be defined and measured by interferometric techniques as explained in [46] [40].
We will study in this paper the Pancharatnam–Sjöqvist phase shift when the Wigner distribution of \( \hat{\rho} \) is of the type
\[
\rho(z) = (2\pi)^{-n} \sqrt{\det V} e^{-\frac{1}{2}V^{-1}z\cdot z}
\] (3)
and \((\hat{U}_t)\) is the Schrödinger evolution operator determined by a time-dependent quadratic Hamiltonian. Gaussians distributions of the type (3) play a central role in quantum mechanics and optics, and are paradigmatic for all other states. We thus extend the results obtained by one of us in the recent work \[40\]; this allows us in particular to give precise formulas for the harmonic oscillator in \(n\) dimensions.

This work is structured as follows:

- In section 2 we introduce the notion of symplectic isotopy: a symplectic isotopy is a \(C^1\)-path of symplectic matrices passing through the origin at time \(t = 0\). This notion generalizes that of one-parameter group; we show that a symplectic isotopy can always be viewed as a Hamiltonian flow of a (possibly time-dependent) Hamiltonian that is a quadratic form in the position and momentum variables. To every symplectic isotopy is associated in a canonical way a \(C^1\)-path of metaplectic operators; this allows the derivation of Schrödinger’s equation for quadratic Hamiltonians \[17, 21\];

- In section 3 we review the properties of the Weyl symbol of metaplectic operators as developed by one of us \[20, 21, 22\]; these properties will be instrumental for our derivation of phase formulas. This gives us the opportunity to present “in a nutshell” a rather technical topic which is not very well-known outside mathematicians working on symplectic geometry and intersection theory; this section begins by a review of the general notion of Weyl transform;

- In section 4 we review the basic properties of density operators we will need, focusing in particular on the Gaussian case, which is of great practical interest in quantum optics. Among all quantum states, Gaussian states are those whose properties are the best understood from a theoretical point of view; they play a significant role in many areas of quantum mechanics and optics, quantum chemistry, and signal theory.

- Section 5 is devoted to the study of the Pancharatnam–Sjöqvist phase shift when the Hamiltonian flow is determined by a quadratic Hamiltonian and acts on a density operator with Gaussian Wigner distribution.
We prove a general formula for the action of metaplectic operators on Gaussian density matrix and thereafter give detailed calculations for the harmonic oscillator.

- In section ?? we generalize the previous results to the Inhomogeneous metaplectic group, taking into account affine transformations related to displacements in phase space.

To make the paper self-contained we have carefully detailed the construction and the properties of the Conley–Zehnder index, and added two Appendices: in Appendix A we collect the main definitions and properties of the metaplectic group, and in Appendix B we review the theory of the Leray–Maslov index which plays an essential role in the definition of the extended Conley–Zehnder intersection index of symplectic paths without restrictions on the endpoint of these paths.

**Notation and prerequisites** The standard symplectic form on \( \mathbb{R}^{2n} \) is \( \sigma = \sum_{j=1}^{n} dp_j \wedge dx_j \), that is \( \sigma(z, z') = p \cdot x' - p' \cdot x \) if \( z = (x, p) \), \( z' = (x', p') \); in vector notation \( \sigma(z, z') = J z \cdot z' = (z')^T J z \) where \( J = \begin{pmatrix} 0_n \times n & I_n \times n \\ -I_n \times n & 0_n \times n \end{pmatrix} \).

The scalar product on the space \( L^2(\mathbb{R}^n) \) is defined by

\[
(\psi|\phi)_{L^2} = \int \psi(x) \overline{\phi(x)} d^n x.
\]

Let \( Q \) be a real quadratic form on \( \mathbb{R}^m \). The signature \( \text{sign}(Q) \) is the number of \( > 0 \) eigenvalues of the Hessian matrix of \( Q \) minus the number of \( < 0 \) eigenvalues. We will use the generalized Fresnel formula (see Appendix A of Folland [17])

\[
\int e^{-\frac{1}{2\pi} A z \cdot z} d^n z = (2\pi \hbar)^n \det^{-1/2} A
\]

which is valid for all \( A = A^* \) with \( \text{Re} A > 0 \) and where \( (\det A)^{-1/2} = \alpha_1^{-1/2} \cdots \alpha_n^{-1/2} \), the \( \alpha_j^{-1/2} \) being the square roots of \( \alpha_j^{-1} \) with positive real part.

### 2 Symplectic and Metaplectic Isotopies

#### 2.1 Hamiltonian and symplectic isotopies

A symplectomorphism of \( \mathbb{R}^{2n} \) is a \( C^\infty \) diffeomorphism \( f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) such that \( f^* \sigma = \sigma \); equivalently the Jacobian matrix \( Df(z) \in \text{Sp}(n) \) for every \( z \in \mathbb{R}^{2n} \).
$\mathbb{R}^{2n}$. If in addition there exists a Hamiltonian function $H \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$ such that $f = f_t^H \ ((f_t^H)$ the flow determined by the Hamilton equations for $H$) then $f$ is called a Hamiltonian symplectomorphism. A symplectic isotopy is a one-parameter family $V = (S_t)_{t \in I}$ of elements of $\text{Sp}(n)$ depending in a $C^1$ fashion on $t \in I$ where $I$ is some real interval containing 0 and such that $f_0 = I_d$. The interval $I$ can be bounded, or unbounded. If each $f_t$ is a Hamiltonian symplectomorphism, then $\Sigma = (f_t)_{t \in I}$ is called a Hamiltonian isotopy.

It is immediate to check that if $S_t = e^{tX}$ with $X \in \mathfrak{sp}(n)$ (the symplectic Lie algebra) then $(S_t)_{t \in \mathbb{R}}$ is a genuine one-parameter subgroup of $\text{Sp}(n)$, in fact the flow determined by the quadratic Hamiltonian $H = -\frac{1}{2}JXz \cdot z$. It turns out that each symplectic isotopy is a Hamiltonian isotopy determined by some time-dependent $H$ (we are following here the presentation in [25]):

**Proposition 1** Let $(f_t)_{t \in I}$ be a Hamiltonian isotopy. We have $(f_t^H)_{t \in I}$ with

$$H(z,t) = -\int_0^1 \sigma(\dot{f}_t \circ f_t^{-1}(\lambda z), z) d\lambda$$

where $\dot{f}_t = df_t/dt$. Equivalently:

$$H(z,t) = -\int_0^1 \sigma(X_H(f_t^{-1}(\lambda z), z)) d\lambda$$

where $X_H = J\partial_z H$ is the (time-dependent) Hamilton vector field of $H$.

**Proof.** See Wang [48]; on a more conceptual level see Banyaga [5].

In the case of general linear symplectic isotopies we have:

**Corollary 2** Let $\Sigma = (S_t)_{t \in \mathbb{R}}$ be a symplectic isotopy in $\text{Sp}(n)$.

(i) The associated Hamiltonian function is the quadratic form

$$H(z,t) = -\frac{1}{2}J\dot{S}_t S_t^{-1} z \cdot z = \frac{1}{2} \sigma(z, J\dot{S}_t S_t^{-1} z)$$

where $\dot{S}_t = dS_t/dt$.

(ii) Writing $S_t$ in block-matrix form

$$S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$$

that Hamiltonian is explicitly given by

$$H = \frac{1}{2}(\dot{D}_t C_t^T - \dot{C}_t D_t^T)x^2 + (\dot{C}_t B_t^T - \dot{D}_t A_t^T)p \cdot x + \frac{1}{2}(\dot{B}_t A_t^T - \dot{A}_t B_t^T)p^2.$$
Proof. (i) Applying formula (5) we get

\[ H(z, t) = -\int_0^1 \sigma \left( \dot{S}_t S_t^{-1}(\lambda z), z \right) d\lambda \]  

(10)

which yields

\[ H(z, t) = \frac{1}{2} \sigma(z, J\dot{S}_t S_t^{-1} z) \]

hence (7), taking into account the linearity of \( \sigma \) and \( S_t \). (ii) It follows from the identity \( S_t J S_t^T = J \) that

\[ S_t^{-1} = \begin{pmatrix} D_t^T & -B_t^T \\ -C_t^T & A_t^T \end{pmatrix} \]  

(11)

and hence

\[ J\dot{S}_t S_t^{-1} = \begin{pmatrix} \dot{C}_t D_t^T - \dot{D}_t C_t^T & \dot{D}_t A_t^T - \dot{C}_t B_t^T \\ \dot{B}_t C_t^T - \dot{A}_t D_t^T & \dot{A}_t B_t^T - \dot{B}_t A_t^T \end{pmatrix} \]

formula (9) readily follows. ■

2.2 The Conley–Zehnder index of a symplectic isotopy

The Conley–Zehnder index \( i_{CZ} (\Sigma) \) for symplectic paths was introduced in [10] in the context of the study of Hamiltonian periodic orbits in \( \mathbb{R}^{2n} \). Meinrenken [36] has considerably extended this index and applied it to Gutzwiller-type trace formulas [35] (also see the recent papers [12, 43]). In [23] one of us has shown that Conley–Zehnder index can be viewed as a particular case of an index due to Leray which generalizes the Maslov index.

The vocation of \( i_{CZ} (\Sigma) \) is to count the intersections of a symplectic path \( \Sigma = (S_t)_{0 \leq t \leq 1} \) with the manifold

\[ \text{Sp}_0(n) = \{ S \in \text{Sp}(n) : \det(S - I) = 0 \} \]

Let us describe the original construction of the Conley–Zehnder index; we will need for that some preparatory material. Consider the following subsets of \( \text{Sp}(n) \):

\[ \text{Sp}_+(n) = \{ S \in \text{Sp}(n) : \det(S - I) > 0 \} \]

\[ \text{Sp}_-(n) = \{ S \in \text{Sp}(n) : \det(S - I) < 0 \} \]

these sets are connected and contractible and, together with \( \text{Sp}_0(n) \) form a partition of the symplectic group:

\[ \text{Sp}(n) = \text{Sp}_0(n) \cup \text{Sp}_+(n) \cup \text{Sp}_-(n) \]
Consider now the particular symplectic matrices $S_+$ and $S_-$ defined by

$$S_+ = -I \quad \text{and} \quad S_- = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}, \quad L = \text{diag}(2, -1, \ldots, -1);$$

it is straightforward to check that we have $S_+ \in \text{Sp}_+(n)$ and $S_- \in \text{Sp}_-(n)$. Define now an extension $\tilde{\Sigma}$ of the symplectic path $\Sigma$ by

$$\tilde{\Sigma}(t) = \begin{cases} S_t, & 0 \leq t \leq 1 \\ S'_t, & 1 \leq t \leq 2 \end{cases}$$

where the $S'_t \in \text{Sp}(n)$ are defined as follows: if $S \in \text{Sp}_+(n)$ then $(S'_t)_{1 \leq t \leq 2}$ is a continuous path joining $S = S_1$ to $S_+$ in $\text{Sp}_+(n)$, and if $S \in \text{Sp}_-(n)$ then $(S'_t)_{1 \leq t \leq 2}$ is a path joining $S$ to $S_-$ in $\text{Sp}_-(n)$. Recalling [17, 21] that every $S \in \text{Sp}(n)$ has a polar decomposition $S = PR$ where $P = (S^T S)^{1/2} \in \text{Sp}(n)$ is positive definite and $R = (S^T S)^{-1/2} S$ is in the unitary subgroup $U(n)$ of $\text{Sp}(n)$ (i.e. $R^T R = R R^T = I$, see [21]), we define

$$R_t = \left( \tilde{\Sigma}^T(t) \tilde{\Sigma}(t) \right)^{-1/2} \tilde{\Sigma}(t) \in U(n)$$

hence $R_t$ is of the type

$$R_t = \begin{pmatrix} A_t & B_t \\ -B_t & A_t \end{pmatrix}, \quad u_t = A_t + iB_t \in U(n, \mathbb{C}).$$

To the path $\tilde{\Sigma}$ we associate a path $\gamma$ in $\mathbb{C}$ by the formula $\gamma(t) = (\det u_t)^2$. We have $\gamma(0) = \gamma(2) = 1$ and $|\gamma(t)| = 1$ hence $\gamma$ is in fact a loop in $S^1$. The Conley–Zehnder of $\Sigma$ is, by definition, the winding number of that loop:

$$i_{\text{CZ}}(\Sigma) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z}. \quad (12)$$

We note that since $\text{Sp}_+(n)$ and $\text{Sp}_-(n)$ are contractible, the integer $i_{\text{CZ}}(\Sigma)$ does not depend on the choice of the extension $\tilde{\Sigma}$ of the symplectic path $\Sigma$.

The main properties of the Conley–Zehnder index are summarized below. We denote by $\mathcal{C}(\text{Sp}_+(n))$ the set of all symplectic isotopies having their endpoint in $\text{Sp}_+(n) = \text{Sp}_+(n) \cup \text{Sp}_-(n)$. We denote by $\Sigma \ast \Sigma'$ the concatenation of two paths $\Sigma$ and $\Sigma'$.

**Proposition 3** The index $i_{\text{CZ}}$ is the only mapping $\mathcal{C}(\text{Sp}_+(n)) \rightarrow \mathbb{Z}$ having the following properties:
(CZ1) The integer $i_{CZ}(\Sigma)$ only depends on the homotopy class (with fixed endpoints) of $\Sigma$;

(CZ2) For every $\Sigma \in \text{Sp}_\pm(n)$ we have $i_{CZ}(\Sigma^{-1}) = -i_{CZ}(\Sigma)$;

(CZ3) Let $\Sigma \in \text{Sp}_\pm(n)$ have endpoint $S$ and let $\Sigma'$ be a continuous path joining $S$ to $S'$ in the same connected component $\text{Sp}_+(n)$ or $\text{Sp}_-(n)$ as $S$. Then $i_{CZ}(\Sigma * \Sigma') = i_{CZ}(\Sigma)$;

(CZ4) For every $r \in \mathbb{Z}$ we have $i_{CZ}(\Sigma * \alpha^r) = i_{CZ}(\Sigma) + 2r$ ($\alpha$ the generator of $\pi_1[\text{Sp}(n)] \cong \pi_1[U(n)]$ whose image in $\mathbb{Z}$ is $+1$).

**Proof.** See [30] and [21], §4.3.1. Observe that (CZ1) is an immediate consequence of the definition (12) of the Conley–Zehnder index since $\text{Sp}_+(n)$ and $\text{Sp}_-(n)$ are contractible. We moreover have the following important conjugation property:

**Proposition 4** Let $\Sigma = (S_t)_{t \in I}$ be a symplectic isotopy in $\text{Sp}(n)$ with endpoint $S \notin \text{Sp}_0(n)$.

(i) $i_{CZ}(\Sigma)$ is locally constant on the set of all $\Sigma$ with fixed $\dim \text{Ker}(S - I)$;

(ii) For every $R \in \text{Sp}(n)$ we have

$$i_{CZ}(RSR^{-1}) = i_{CZ}(\Sigma)$$

where $RSR^{-1} = (RS_tR^{-1})_{t \in I}$.

**Proof.** See Meinrenken [36] (Proposition 6). Formula (13) follows from (i) since $i_{CZ}(RSR^{-1})$ is invariant if one connects $R$ to the identity in $\text{Sp}(n)$ (alternatively, it readily follows from definition (12)).

One of us has defined [21, 22, 23] the Conley–Zehnder index of symplectic path without any restriction on the endpoint using the properties of the Leray index (see Appendix B), to which it is closely related. Introducing the symplectic form $\sigma^\ominus = \sigma \oplus (-\sigma)$ on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and denoting by $\text{Sp}^\ominus(2n)$ and $\text{Lag}^\ominus(2n)$ the corresponding symplectic group and Grassmannian Lagrangian, the following results identifies the Conley–Zehnder index as previously defined with the Leray index:

**Proposition 5** Let $\Sigma = (S_t)_{t \in I}$ be an arbitrary symplectic isotopy in $\text{Sp}(n)$ with endpoint $S$. Let

$$\nu(\Sigma) = \frac{1}{2} \mu^\ominus_\Delta((I \oplus S)_\infty)$$

where $\mu^\ominus_\Delta$ is the $\Delta$-Maslov index on the universal covering group $\text{Sp}^\ominus_\Delta(2n)$ of $\text{Sp}^\ominus(2n)$ with $\Delta = \{(z, z) : z \in \mathbb{R}^{2n}\}$ and $(I \oplus S)_\infty \in \text{Sp}^\ominus_\Delta(2n)$ the homotopy class of the path

$$I \ni t \mapsto \{(z, S_tz) : z \in \mathbb{R}^{2n}\} \in \text{Lag}^\ominus(2n).$$
We have \( \nu(\Sigma) \in \mathbb{Z} \) and
\[
\nu(\Sigma) = i_{\text{CZ}}(\Sigma)
\] (15)
when \( S \notin \text{Sp}_0(n) \).

**Proof.** That \( \frac{1}{2} \mu^\Delta((I \oplus S)_\infty) \in \mathbb{Z} \) can be seen as follows: in view of the congruence in (B7) we have
\[
\mu^\Delta((I \oplus S)_\infty) \equiv 2n + \dim((I \oplus S)\Delta \cap \Delta) \mod 2.
\]
But
\[
(I \oplus S)\Delta \cap \Delta = \{ z \in \mathbb{R}^{2n} : S z = z \} = \text{Ker}(S - I)
\]
so that
\[
\mu^\Delta((I \oplus S)_\infty) = 2n + \dim \text{Ker}(S - I) \mod 2.
\]
The eigenvalue 1 of a symplectic mapping having even multiplicity the integer \( \dim \text{Ker}(S - I) \) is always even and so is thus \( \mu^\Delta((I \oplus S)_\infty) \). Using the characteristic property (MA) of the relative Maslov index (Appendix B2) it is easy to show that the restriction of \( \nu \) to \( \text{Sp}_+(n) \cup \text{Sp}_-(n) \) satisfies the properties (CZ1)-(CZ4) of the Conley–Zehnder index (for a detailed argument see [21], §4.3.3 or [23]. We thus have \( \nu(\Sigma) = i_{\text{CZ}}(\Sigma) \) for such paths.

To study the Conley–Zehnder index of products of symplectic isotopies we need the notion of symplectic Cayley transform of \( S \in \text{Sp}_0(n) \). It is, by definition [21, 22], the symmetric \( 2n \times 2n \) matrix
\[
M(S) = \frac{1}{2} J(S + I)(S - I)^{-1} = \frac{1}{2} J + J(S - I)^{-1}.
\] (16)
It has the following properties:
\[
M(S^{-1}) = -M(S) \quad \text{and} \quad R^T M(S) R = M(R^{-1} S R)
\] (17)
for \( S \notin \text{Sp}_0(n) \) and \( R \in \text{Sp}(n) \). We will use several times in this paper following addition result:

**Lemma 6** Let \( S, S' \in \text{Sp}_0(n) \). If \( SS' \in \text{Sp}_0(n) \) then \( M = M(S) + M(S') \) is invertible and we have
\[
M = J(S - I)^{-1}(SS' - I)(S' - I)^{-1}.
\] (18)

**Proof.** See [21], §4.3.2, Lemma 4.1.4. ■

In the following result we give an explicit expression for the Conley–Zehnder index of the product of two symplectic isotopies:
Proposition 7 Let $\Sigma = (S_t)_{t \in I}$ and $\Sigma' = (S'_t)_{t \in I}$ be two symplectic isotopies and set $\Sigma \Sigma' = (S_t S'_t)_{t \in I}$ with endpoints $S$ and $S'$ not in $Sp_0(n)$. If $SS' \notin Sp_0(n)$ then

$$\nu(\Sigma \Sigma') = \nu(\Sigma) + \nu(\Sigma') + \frac{1}{2}\text{sign}(M(S) + M(S'))$$

(19)

where $M(S)$ and $M(S')$ are the symplectic Cayley transforms of $S$ and $S'$, and $\text{sign}M$ is the signature of the invertible symmetric matrix $M$.

Proof. See [21] (Proposition 4.20) or [22] pp. 1163–1164 for detailed proofs. Notice that $\text{sign}M = \text{sign}(M(S) + M(S'))$ is an even integer since $M$ is invertible.

2.3 Metaplectic isotopies

The metaplectic group $Mp(n)$ being a twofold covering of the symplectic group (see Appendix A), it follows from the path lifting property for covering groups that every symplectic isotopy $\Sigma = (S_t)_{t \in I}$ in $Sp(n)$ can be lifted in a unique way to a path $\hat{\Sigma} : t \mapsto \hat{S}_t$ ($t \in I$) in $Mp(n)$ such that $\hat{S}_0 = I_d$ (i.e. $\pi^{Mp}(\hat{S}_t) = S_t$). Conversely, every such path (which we call a metaplectic isotopy) covers a symplectic isotopy. The following result is well-known [17, 21]:

Proposition 8 Let $\hat{\Sigma} = (\hat{S}_t)_{t \in I}$ be a metaplectic isotopy and $\Sigma = (S_t)_{t \in I}$ the symplectic isotopy it covers. We have

$$i\hbar \frac{d}{dt} \hat{S}_t = \hat{H} \hat{S}_t$$

where $H$ is the quadratic Hamiltonian function (7) determined by $(S_t)_t$ and $\hat{H}$ is the Weyl quantization of $H$.

This result thus justifies Schrödinger’s equation in the case of quadratic Hamiltonians; for a detailed discussion of the quantum-classical correspondence from the symplectic point of view see de Gosson [25].

We will need the following elementary conjugation result in our calculation of the relative phase shift:

Lemma 9 Let $\Sigma^H = (S^H_t)_{t \in I}$ be the symplectic isotopy determined by a Hamiltonian $H$. (i) For $R \in Sp(n)$ the symplectic isotopy $R \Sigma^H R^{-1} = (RS_t^H R^{-1})_{t \in I}$ is determined by $H \circ R^{-1}$, that is

$$R \Sigma^H R^{-1} = \Sigma^{H \circ R^{-1}}.$$
(ii) Let $\hat{\Sigma}^H = (\hat{S}_t^H)_{t \in I}$ be the metaplectic isotopy induced by $\Sigma^H$. We have $\hat{R} \hat{\Sigma}^H \hat{R}^{-1} = \hat{\Sigma}^{H \circ R^{-1}}$ that is

$$(\hat{R} \hat{S}_t^H \hat{R}^{-1})_{t \in I} = (\hat{S}_t^{H \circ R^{-1}})_{t \in I}$$

where $\pi^{\text{Mp}}(\hat{R}) = R$.

**Proof.** Property (i) follows from formula (7) noting that $JS = (S^T)^{-1}J$ since $S \in \text{Sp}(n)$. Property (ii) follows from (i) using the symplectic covariance of Weyl operators which implies that

$$i \hbar \frac{d}{dt}(\hat{R} \hat{S}_t^H \hat{R}^{-1}) = \hat{R}^{-1}(\hat{H} \hat{S}_t^H) \hat{R} = (\hat{R}^{-1} \hat{H} \hat{R})(\hat{R}^{-1} \hat{S}_t^H \hat{R});$$

noting that $\hat{R} \hat{H} \hat{R}^{-1} = H \circ R^{-1}$ we have $\hat{S}_t^{H \circ R^{-1}} = \hat{R} \hat{S}_t^H \hat{R}^{-1}$ in view of the uniqueness of the solution to Schrödinger’s equation with given initial datum in $S(\mathbb{R}^n)$. 

We define the (extended) Conley–Zehnder index $\nu(\hat{\Sigma})$ of a metaplectic isotopy $\hat{\Sigma} = (\hat{S}_t)_{t \in I}$ on $\text{Mp}(n)$ as being $[\nu(\Sigma)] \mod 4$, the class modulo 4 of the Conley–Zehnder index $\nu(\Sigma)$ of the projected symplectic path $\Sigma = \pi^{\text{Mp}}(\hat{\Sigma})$. When the endpoint $\hat{S}$ of $\hat{\Sigma}$ is such that $S = \pi^{\text{Mp}}(\hat{S}) \not\in \text{Sp}_0(n)$ then, by Proposition 5

$$\nu(\hat{\Sigma}) = [i_{\text{CZ}}(\Sigma)] \mod 4. \quad (20)$$

Following result is important for practical calculations; it shows that the value modulo four of the Conley–Zehnder index $i_{\text{CZ}}(\Sigma)$ of a symplectic isotopy with endpoint a free symplectic matrix $S_W$ is related to the Maslov index of the corresponding metaplectic isotopy:

**Proposition 10** Let $\Sigma = (S_t)_{t \in I}$ be a symplectic isotopy in $\text{Sp}(n)$ with endpoint $S \not\in \text{Sp}_0(n)$ and $SL_P \cap \ell_P = 0$. Thus $S = S_W$ for some generating function $W$ (see (A4)). We have

$$\nu(\hat{\Sigma}) = i_{\text{CZ}}(\hat{\Sigma}) = m - \text{Inert } W_{xx} \mod 4 \quad (21)$$

where $\hat{S} = \hat{S}_{W,m}$ is the endpoint of the metaplectic isotopy $\hat{\Sigma} = (\hat{S}_t)_{t \in I}$.

**Proof.** Formula (21) follows from the definition (14) of the Conley–Zehnder index and the properties of the Leray index.

**Remark 11** The integer $\text{Inert } W_{xx}$ in appearing (21) is called “Morse’s index of concavity” [37] in the literature on periodic Hamiltonian orbits.
In terms on the explicit expression (A4) of the quadratic form $W$, that is
$$W(x, x') = \frac{1}{2} Px^2 - Lx \cdot x' + \frac{1}{2} Qx'^2$$
we have
$$W_{xx} = P - L - L^T + Q.$$ 
Thus, if the metaplectic isotopy $\hat{\Sigma}$ has endpoint $\hat{S}_{W, m}$ we have
$$\nu(\hat{\Sigma}) = m - \text{Inert}(P - L - L^T + Q).$$

3 Metaplectic Operators and their Weyl Symbols

The Weyl symbol of a metaplectic was introduced implicitly and without justification in the work [33] of Mehlig and Wilkinson, and was studied rigorously in [20] (also see [22, 21]).

3.1 Weyl operators

We will make use of the two following elementary but yet very useful unitary operators on $L^2(\mathbb{R}^n)$. For $z_0 = (x_0, p_0) \in L^2(\mathbb{R}^{2n})$ the displacement (or Weyl–Heisenberg) operator $\hat{T}(z_0)$ and the reflection (or Grossmann–Royer [28, 42, 21, 26]) operator $\hat{\Pi}(z_0)$ are defined by

$$\hat{T}(z_0) \psi(x) = e^{\frac{i}{\hbar} (p_0 x - \frac{1}{2} p_0 x_0)} \psi(x - x_0)$$

$$\hat{\Pi}(z_0) \psi(x) = e^{2i \frac{\hbar}{p_0} (x - x_0)} \psi(2x_0 - x).$$

One verifies that $\hat{\Pi}(z_0) = \hat{T}(z_0) \hat{\Pi}(0) \hat{T}(z_0)^*$ and that

$$\hat{T}(z_0 + z_1) = e^{-\frac{i}{2\hbar} \sigma(z_0, z_1)} \hat{T}(z_0) \hat{T}(z_1)$$

$$\hat{T}(z_0) \hat{T}(z_1) = e^{\frac{i}{\hbar} \sigma(z_0, z_1)} \hat{T}(z_1) \hat{T}(z_0).$$

For $a \in L^1(\mathbb{R}^{2n})$ the function $F_\sigma a$ defined by

$$F_\sigma a(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar} \sigma(z, z')} a(z) d^{2n} z'$$

is called the symplectic Fourier transform of $a$; we have $F_\sigma a(z) = F a(Jz)$ where $F$ is the usual Fourier transform on $\mathbb{R}^{2n}$. The operators $\hat{T}(z_0)$ and $\hat{\Pi}(z_0)$ are related by $F_\sigma$ in the following way:

$$\hat{\Pi}(z_0) \psi = 2^{-n} F_\sigma [\hat{T}(\cdot) \psi](z_0)$$
The displacement and reflection operators $\hat{\Pi}(z_0)$ allow to give very simple definitions of the cross-Wigner and cross-ambiguity functions [28, 42, 21]: for $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ they are defined [21, 26] by, respectively,

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \langle \hat{\Pi}(z_0)\psi | \phi \rangle \tag{28}$$

$$A(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \langle \psi | \hat{T}(z)\phi \rangle \tag{29}$$

A straightforward computation using the definitions (23) and (22) show that these equivalent to the usual expressions

$$W(\psi, \phi)(z) = \left(\frac{1}{2}\right)^n \frac{1}{2\pi\hbar} \int e^{-i\frac{\hbar}{2}p^m y} \psi(x + \frac{1}{2}y)\overline{\phi(x - \frac{1}{2}y)} d^n y \tag{30}$$

$$A(\psi, \phi)(z) = \left(\frac{1}{2}\right)^n \frac{1}{2\pi\hbar} \int e^{-i\frac{\hbar}{2}p^m y} \psi(y + \frac{1}{2}x)\overline{\phi(y - \frac{1}{2}x)} d^n y. \tag{31}$$

The functions $W(\psi, \phi)$ and $A(\psi, \phi)$ are related by the symplectic Fourier transform [21]

$$W(\psi, \phi) = F_{\sigma} A(\psi, \phi), \quad A(\psi, \phi) = F_{\sigma} W(\psi, \phi)$$

as immediately follows from formula (27).

Let $a \in \mathcal{S}(\mathbb{R}^{2n})$. The Weyl operator with symbol $a$ is defined by $\hat{A} = \text{Op}^W(a)$ where

$$\text{Op}^W(a)\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int \int e^{i\frac{\hbar}{2}(x-y)} a(y) \psi(y) d^n y. \tag{32}$$

Using the displacement operator this definition takes the more tractable form

$$\text{Op}^W(a) = \left(\frac{1}{2\pi\hbar}\right)^n \int a(z_0) \hat{\Pi}(z_0) d^{2n} z_0. \tag{33}$$

Setting $a_\sigma = F_{\sigma} a$ we also have, using the Plancherel theorem for the symplectic Fourier transform [21],

$$\text{Op}^W(a) = \left(\frac{1}{2\pi\hbar}\right)^n \int a_\sigma(z_0) \hat{T}(z_0) d^{2n} z_0. \tag{34}$$

The integrals above should be interpreted as vector-valued integrals (Bochner integrals). Using the generators $\hat{J}$, $\hat{V}_P$, and $\hat{M}_{L,m}$ defined by (A1), (A2), (A3) it is a simple exercise to show that the displacement and reflection operators satisfy the symplectic covariance relations

$$\hat{T}(S z_0) = S \hat{T}(z_0) S^{-1}, \quad \hat{\Pi}(S z_0) = S \hat{\Pi}(z_0) S^{-1} \tag{35}$$

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for every $\hat{S} \in \text{Mp}(n)$, $S = \pi^{\text{Mp}}(\hat{S})$, and using (34) it follows that Weyl operators satisfy the conjugation formula

$$\hat{S} \text{Op}^W(a) \hat{S}^{-1} = \text{Op}^W(a \circ S^{-1}). \quad (36)$$

We also have a conjugation formula for the operators $\hat{T}(z_0)$:

$$\hat{T}(z_0) \text{Op}^W(a) \hat{T}(z_0)^{-1} = \text{Op}^W(T(z_0)a) \quad (37)$$

where by definition $T(z_0)a(z) = a(z - z_0)$; this easily follows from formula (25).

Formula (30) implies that we have the following relation:

$$\langle \text{Op}^W(a)\psi, \phi \rangle = \langle \langle a, W(\psi, \phi) \rangle \rangle \quad (38)$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$; here $\langle \cdot, \cdot \rangle$ and $\langle \langle \cdot, \cdot \rangle \rangle$ are the distributional brackets on $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively. This formula allows to define $\hat{A} = \text{Op}^W(a)$ for arbitrary symbols $a \in \mathcal{S}'(\mathbb{R}^{2n})$ [21, 26].

**Proposition 12** Assume that the product $\hat{A}\hat{B} = \text{Op}^W(a)\text{Op}^W(b)$ is well defined. Then the twisted Weyl symbol of $\hat{C} = \hat{A}\hat{B}$ is given by the “twisted convolution” formula

$$c_\sigma(z) = \left(\frac{1}{2\pi \hbar}\right)^n \int e^{i\frac{\pi}{\hbar}(z,z')}a_\sigma(z - z')b_\sigma(z')d^{2n}z'. \quad (39)$$

This in particular applies when $a$ or $b$ is in $\mathcal{S}(\mathbb{R}^{2n})$.

**Proof.** See for instance [21], §6.3.2. ■

### 3.2 The Weyl symbol of a quadratic Fourier transform

Let us denote by $\text{Sp}_0(n)$ the subset of $\text{Sp}(n)$ consisting of all symplectic matrices having no eigenvalue equal to one:

$$\text{Sp}_0(n) = \{S \in \text{Sp}(n) : \det(S - I) \neq 0\}. \quad (40)$$

Let now $S \in \text{Sp}_0(n)$ and consider the family of operators $\hat{R}_\nu(S)$ defined, for $\nu \in \mathbb{R}$, by

$$\hat{R}_\nu(S) = \left(\frac{1}{2\pi \hbar}\right)^n \nu \sqrt{\det(S - I)} \int \hat{T}(Sz_0)\hat{T}(-z_0)d^{2n}z_0. \quad (41)$$
One verifies that for all $S \in \text{Sp}_0(n)$ and $\nu \in \mathbb{R}$ the operators $\hat{R}_\nu(S)$ satisfy the intertwining formula
\[ \hat{T}(Sz_0) = \hat{R}_\nu(S)\hat{T}(z_0)\hat{R}_\nu(S)^{-1}. \]

It follows, using the irreducibility of the Schrödinger representation of the Heisenberg group \cite{[17]}, that there exists a constant $c(S,\nu) \in \mathbb{C}$ such that $\hat{R}_\nu(S) = c(S,\nu)\hat{S}$ where $\pi^{\text{Mp}}(\hat{S}) = S$. It is moreover easy to check that the operators are $\hat{R}_\nu(S)$ unitary, hence $|c(S,\nu)| = 1$. The following result connects the integer $\nu$ in (41) to the Conley-Zehnder index when $\hat{R}_\nu(S)$ is a true metaplectic operator:

**Proposition 13** Let $\Sigma = (S_t)_{t \in I}$ be symplectic isotopy in $\text{Sp}(n)$ leading from the identity to $S \notin \text{Sp}_0(n)$. Let $\hat{\Sigma} = (\hat{S}_t)_{t \in I}$ be the metaplectic isotopy covering $\Sigma$ and $\hat{S} \in \text{Mp}(n)$ be its endpoint (thus $S = \pi^{\text{Mp}}(\hat{S})$). We have
\[ \hat{S} = \hat{R}_{\nu(\hat{\Sigma})}(S) \]
where $\nu(\hat{\Sigma}) = \nu(\Sigma) \mod 4$.

**Proof.** This results from the identity (20) (see \cite{[20]} and \cite{[22],[21]}). \hfill \blacksquare

The statement above has the following consequences when the endpoint of the symplectic isotopy $\Sigma$ is a free symplectic matrix $S_W$:

**Corollary 14** Let $\hat{S}_{W,m} \in \text{Mp}(n)$ be such that $S_W = \pi^{\text{Mp}}(\hat{S}_{W,m}) \notin \text{Sp}_0(n)$. We then have
\[ \hat{S}_{W,m} = \hat{R}_{m-\text{Inert } W_{xx}}(S) \] (42)
where $\text{Inert } W_{xx}$ is the index of inertia of the matrix $W_{xx}$ of second derivatives of the quadratic form $x \mapsto W(x,x)$ on $\mathbb{R}^n$.

This allows us to give a rigorous explicit formula for the twisted Weyl symbol of $\hat{S}_{W,m}$:

**Corollary 15** The twisted Weyl symbol of $\hat{S}_{W,m}$ with $S_W \notin \text{Sp}_0(n)$ is given by
\[ (s_W)_\sigma(z) = \frac{i^{m-\text{Inert } W_{xx}}}{\sqrt{|\det(S_W-I)|}} \exp \left( \frac{i}{2\hbar} M_W z \cdot z \right) \] (43)
where $M_W$ is the symplectic Cayley transform of $S_W$. 

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Proof. In view of (41) and (42) we have
\[ \widehat{S}_{W,m} = \left( \frac{1}{2\pi \hbar} \right)^n i^{m - \text{Inert}_{Wx}} \sqrt{\det(S_W - I)} \int \widehat{T}(S_W z_0) \widehat{T}(-z_0) d^{2n} z; \]
using formula (24) this can be rewritten
\[ \widehat{S}_{W,m} = \left( \frac{1}{2\pi \hbar} \right)^n i^{m - \text{Inert}_{Wx}} \sqrt{\det(S_W - I)} \int e^{-\frac{i}{\hbar} \sigma(S_W z_0, z_0)} \widehat{T}((S_W - I)z_0) d^{2n} z_0. \]
Making the change of variable \( z_0 \mapsto (S_W - I)^{-1} z_0 \) we get
\[ \widehat{S}_{W,m} = \left( \frac{1}{2\pi \hbar} \right)^n i^{m - \text{Inert}_{Wx}} \sqrt{\det(S_W - I)} \int e^{-\frac{i}{\hbar} \sigma((S_W - I)^{-1}z_0, (S_W - I)^{-1}z_0)} \widehat{T}(z_0) d^{2n} z_0 \]
hence the twisted Weyl symbol of \( \widehat{S}_{W,m} \) is
\[ (s_W)_\sigma(z) = i^{m - \text{Inert}_{Wx}} \sqrt{\det(S_W - I)} e^{-\frac{i}{\hbar} \sigma((S_W - I)^{-1}z_0, (S_W - I)^{-1}z_0)}. \]
A simple algebraic calculation shows that
\[ \sigma(S_W(S_W - I)^{-1}z_0, (S_W - I)^{-1}z_0) = \frac{1}{2} J(S_W + I)(S_W - I)^{-1} = M(S_W); \]
formula (44) follows. ■

Proposition 13 and formula (43) suggest that the Conley–Zehnder index is related to a choice of argument of the square root of the determinant of \( S - I \). This is indeed the case:

Proposition 16 Let \( \widehat{S}_{W,m} \in \text{Mp}(n) \) have projection \( S_W \notin \text{Sp}_0(n) \). We have
\[ \nu(\widehat{S}_{W,m}) = n + \frac{1}{\pi} \arg \det(S_W - I) \mod 2. \]
that is
\[ \nu(\widehat{S}_{W,m}) = \begin{cases} n \mod 2 & \text{if } S_W \in \text{Sp}_+(n) \\ n + 2 \mod 2 & \text{if } S_W \in \text{Sp}_-(n) \end{cases}. \]

Proof. The projection \( S_W = \pi_{\text{mp}}(\widehat{S}_{W,m}) \) is a free symplectic matrix, in block-matrix form
\[ S_W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \det B \neq 0. \]
A straightforward calculation yields the factorization

\[ S_W - I = \begin{pmatrix} 0 & B \\ I & D - I \end{pmatrix} \begin{pmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{pmatrix}. \]

Since \( S_W \in \text{Sp}(n) \) we have \( C - DB^{-1}A = -(B^T)^{-1} \) and hence

\[ C - (D - I)B^{-1}(A - I) = B^{-1}A + DB^{-1} - (B^T)^{-1} = W_{xx} \]

so that

\[ S_W - I = \begin{pmatrix} 0 & B \\ I & D - I \end{pmatrix} \begin{pmatrix} W_{xx} & 0 \\ B^{-1}(A - I) & I \end{pmatrix}. \]

It follows that

\[ \det(S_W - I) = (-1)^n \det B \det W_{xx} \]

and hence

\[ \arg \det(S_W - I) = n\pi + \arg \det B + \arg \det W_{xx} \mod 2\pi. \]

Noticing that \( \arg \det W_{xx} = \pi \text{ Inert } W_{xx} \) and that this is

\[ \arg \det(S_W - I) = n\pi + \arg \det B + \pi \text{ Inert } W_{xx} \mod 2\pi. \]

In view of formula (21) and (13) we have \( \arg \det(B) = m\pi \) (see Appendix A) and hence

\[ \arg \det(S_W - I) = (n + m - \text{ Inert } W_{xx})\pi \mod 2\pi \]

that is

\[ \arg \det(S_W - I) = (n + \nu(S_{W,m}))\pi \mod 2\pi \]

which yields (44).

### 3.3 Products of metaplectic operators

Each \( \hat{S} \in \text{Mp}(n) \) can be written as a product \( \hat{S}_{W,m}\hat{S}_{W',m'} \) (Appendix A, Proposition 25). It turns out that \( \hat{S}_{W,m} \) and \( \hat{S}_{W',m'} \) can be chosen so that their projections \( S_W \) and \( S_{W'} \) have no eigenvalue equal to one. This fact, together with the composition formula (39) leads to a complete characterization of the symbol of a metaplectic operator. When \( \hat{S} \) has projection \( S \not\in \text{Sp}_0(n) \) we have the following explicit result:
Proposition 17  Let $\hat{S} \in \text{Mp}(n)$ be such that $\pi^\text{Mp}(\hat{S}) \notin \text{Sp}_0(n)$.

(i) There exist $\hat{S}_{W,m}$ and $\hat{S}_{W',m'}$ such that $\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'}$; moreover these operators can be chosen so that $S_W = \pi^\text{Mp}(\hat{S}_{W,m}) \notin \text{Sp}_0(n)$ and $S_{W'} = \pi^\text{Mp}(\hat{S}_{W',m'}) \notin \text{Sp}_0(n)$.

(ii) We have

$$\hat{S} = \hat{R}_{\nu + \nu' + \frac{1}{2}\text{sign}(M)}(S) = \hat{R}_{\nu(S)}(S)$$

where $M = M_W + M_{W'}$ ($M_W$ and $M_{W'}$ the symplectic Cayley transforms of $S_W$ and $S_{W'}$), and

$$\nu = m - \text{Inert } W_{xx}, \quad \nu' = m' - \text{Inert } W'_{xx}$$

are the Conley–Zehnder indices of $\hat{S}_{W,m}$ and $\hat{S}_{W',m'}$;

(iii) The twisted Weyl symbol of $\hat{S}$ is given by

$$s_\sigma(z) = \frac{i^{\nu(S)}}{\sqrt{|\det(S - I)|}} \exp \left( \frac{i}{2\hbar} Mz \cdot z \right)$$

with

$$\nu(S) = \nu + \nu' + \frac{1}{2}\text{sign } M.$$  \hfill (49)

Proof.  See Proposition 10 in [20] or [21], §7.4 for detailed proofs. That $\hat{S}$ can always be factored as $\hat{S}_{W,m} \hat{S}_{W',m'}$ where $\hat{S}_{W,m}$ and $\hat{S}_{W',m'}$ have projections $S_W$ and $S_{W'}$ not in $\text{Sp}_0(n)$ was proven in [20]. For formula (48) the idea is to apply formula (39) to (43) and to use the Fresnel formula (4), which yields, after some calculations

$$c_\sigma(z) = \frac{i^{\nu + \nu' + \frac{1}{2}\text{sign}(M)}}{\sqrt{|\det((S_W - I)(S_{W'} - I))|}} e^{\frac{i}{\hbar} Mz \cdot z}.$$  \hfill (50)

A simple calculation taking into account the definition of the symplectic Cayley transforms shows that

$$(S_W - I)(S_{W'} - I)M = S - I$$

(M is invertible in view of Lemma [5]).

We have seen in Proposition 16 that the Conley–Zehnder index of a quadratic Fourier transform $\hat{S}_{W,m}$ is simply related to a choice of argument for $\text{det}(S_W - I)$. Using the result above, this observation can be generalized to the case of an arbitrary $\hat{S} \in \text{Mp}(n)$ with projection $S \notin \text{Sp}_0(n)$:
Corollary 18 Let $\hat{S} \in \text{Mp}(n)$ with $S = \pi^{\text{Mp}}(\hat{S}) \notin \text{Sp}_0(n)$. We have
\[
\nu(\hat{S}) = n + \frac{1}{\pi} \text{Arg det}(S - I) \mod 2. \tag{51}
\]

Proof. Writing $\hat{S} = \hat{S}_W, m \hat{S}_{W', m'}$ with $S_W$ and $S_{W'}$ not in $\text{Sp}_0(n)$ it follows from the identity (50) that
\[
\det [(S_W - I)(S_{W'} - I)M] = \det(S - I)
\]
with $M = M_W + M_{W'}$ and hence
\[
\text{arg det}(S - I) = \text{arg det}(S_W - I) + \text{arg det}(S_{W'} - I) + \text{arg det} M.
\]
Since $M = M_W + M_{W'}$ is invertible (Lemma 6) we have
\[
\text{arg det} M = \pi \text{Inert } M = -\pi \text{ Inert } M \mod 2\pi
\]
and hence, using formulas (44) and (19) together with the relation $\text{sign } M = 2(n - \text{Inert } M)$,
\[
\text{arg det}(S - I) = \nu(\hat{S}_W, m) + \nu(\hat{S}_{W', m}) - \pi(n - \frac{1}{2} \text{sign } M) \mod 2\pi
\]
\[
= \nu(\hat{S}_W, m) + \nu(\hat{S}_{W', m}) - n\pi + \frac{1}{2}\pi \text{sign } M \mod 2\pi
\]
\[
= \nu(\hat{S})\pi - n\pi \mod 2\pi
\]
proving formula (51). ■

4 Gaussian Density Operators

4.1 Generalities

A density operator on a complex Hilbert space $\mathcal{H}$ is a positive semidefinite (and hence selfadjoint) trace class operator $\hat{\rho}$ on $\mathcal{H}$ with unit trace. Every trace class operator is the product of two Hilbert–Schmidt operators, and is hence compact. The spectral theorem for compact operators implies that there exists a family $(\psi_j)$ of orthonormal vectors in $\mathcal{H}$ such that
\[
\hat{\rho} = \sum_j \lambda_j \hat{\Pi}_j \tag{52}
\]
where $\hat{\Pi}_j$ is the orthogonal projection on the vector $\psi_j$; the $\lambda_j$ are the eigenvalues corresponding to the eigenvectors $\psi_j$ and we have $\text{Tr}(\hat{\rho}) = \sum_j \lambda_j = 1$. 19
In what follows we will assume that $\mathcal{H} = L^2(\mathbb{R}^n)$. Let $a$ be the Weyl symbol of $\hat{\rho}$; by definition $\rho = (2\pi\hbar)^{-n}a$ is the Wigner distribution of $\hat{\rho}$. Taking (52) into account we have $\rho = \sum_j \lambda_j W\psi_j$ where $W\psi_j$ is the Wigner transform of $\psi_j$.

Let $\hat{A} = \text{Op}^W(a)$ and $\hat{B} = \text{Op}^W(b)$ be two trace class operators (or, more generally, Hilbert–Schmidt operators). Since Hilbert-Schmidt operators are precisely those with kernels in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ it follows that $a \in L^2(\mathbb{R}^{2n})$ and $b \in L^2(\mathbb{R}^{2n})$. The product $\hat{A}\hat{B}$ is of trace class and we have

$$\text{Tr}(\hat{A}\hat{B}) = \left(\frac{1}{2\pi\hbar}\right)^n \int a(z)b(z)d^{2n}z. \quad (53)$$

Let $a_\sigma = F_\sigma a$ be the symplectic Fourier transform of $a$ (see (26)). The formula

$$\text{Tr}(\hat{A}) = \left(\frac{1}{2\pi\hbar}\right)^n \int a(z)d^{2n}z = a_\sigma(0) \quad (54)$$

is often used in the literature; one should however be aware that it is only true if one assumes that in addition $a \in L^1(\mathbb{R}^{2n})$ (see [6, 15]). (It is instructive to read B. Simon’s analysis in [45] of trace formulas of this type; also see Shubin [44], §27).

### 4.2 Gaussian states

Let $\hat{\rho}$ be a density operators whose Wigner distribution is a Gaussian:

$$\rho(z) = (2\pi)^{-n}\sqrt{\det V}^{-1}e^{-\frac{1}{2}V^{-1}z \cdot z}. \quad (55)$$

The covariance matrix $V$ is a positive definite symmetric (real) $2n \times 2n$ matrix, and $z = (x,p)$ is in the phase space $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$. A necessary and sufficient condition for a function (55) to represent a quantum state is that the Hermitian matrix $V + (i\hbar/2)J$ ($J$ the standard symplectic matrix) has no negative eigenvalues; for short

$$V + \frac{i\hbar}{2}J \geq 0. \quad (56)$$

This condition ensures that the density operator $\hat{\rho}$ is indeed positive semidefinite and is equivalent in the Gaussian case to the Robertson–Schrödinger uncertainty principle: see de Gosson and Luef [27]). It will be convenient to set $V = \frac{1}{2\hbar}F^{-1}$; with that notation the Wigner distribution of the Gaussian state (55) can be written

$$\rho(z) = (\pi\hbar)^{-n}\sqrt{\det F}e^{-\frac{1}{2}Fz \cdot z} \quad (57)$$

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and the quantum condition (56) becomes $F^{-1} + iJ \geq 0$. The symplectic Fourier transform (26) of $\rho$ is given by

$$\rho_\sigma(z) = (2\pi\hbar)^{-n} e^{\frac{1}{4\hbar}JF^{-1}Jz \cdot z} = (2\pi\hbar)^{-n} e^{\frac{1}{4\hbar}F^{-1}Jz \cdot z}. \quad (58)$$

Notice that if $F = F^T \in \text{Sp}(n)$ then $JFJ = -F^{-1}$ hence, in this case

$$\rho_\sigma(z) = (2\pi\hbar)^{-n} e^{-\frac{1}{4\hbar}Fz \cdot z} \quad (59)$$

which is the symplectic Fourier transform of the Wigner transform of a generalized coherent state [32, 21] (see below).

The purity of $\hat{\rho}$ is by definition $\mu = \text{Tr}(\hat{\rho}^2)$, and we have in the Gaussian case

$$\mu = \left(\frac{\hbar}{2}\right)^n \det(V)^{-1/2} = \sqrt{\det F}. \quad (60)$$

Thus $\hat{\rho}$ is a pure state ($\mu = 1$) if and only if $\det(V) = (\hbar/2)^{2n}$, that is $\det F = 1$. One shows [32, 21] that this equivalent to the existence of $R \in \text{Sp}(n)$ such that $V = R^T R$. It follows that the only Gaussian pure states are those with Wigner distribution

$$\rho(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\hbar}S^t S \cdot z}. \quad (61)$$

Let $\phi_0$ be the standard coherent state: $\phi_0(x) = (\pi\hbar)^{-n/4} e^{-|x|^2/2\hbar}$. The action of the local metaplectic operators $\hat{M}_{L,m}$ and $\hat{V}_{-P}$ on $\phi_0$ is the $L^2$-normalized Gaussian $\psi_{X,Y} = \hat{V}_{-P} \hat{M}_{L,m} \phi_0$ given by

$$\psi_{X,Y}(x) = i^m (\pi\hbar)^{-n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar} (X+iY) x \cdot x}$$

where $X = L^T L$ and $Y = P$. The Wigner transform of $\psi_{X,Y}$ is ([32], [21] §8.5):

$$W_{\psi_{X,Y}}(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\hbar}G z \cdot z} \quad (62)$$

where $G$ is the positive definite symplectic and symmetric matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1}X^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}. \quad (63)$$

The ambiguity function $F_\sigma \rho_{X,Y}$ is easily calculated and one finds that

$$F_\sigma \rho_{X,Y}(z) = \left(\frac{1}{2\pi\hbar}\right)^n e^{-\frac{1}{4\hbar}G z \cdot z}. \quad (64)$$
5 Relative Phase Shifts

5.1 A general result

We will need the following generalization of formula (53):

**Lemma 19** Let $\hat{S} \in \text{Mp}(n)$ be such that $\pi^{\text{Mp}}(\hat{S}) \notin \text{Sp}_0(n)$ and $\rho$ the Gaussian distribution (57). The product $\hat{S}\rho$ is of trace class and we have

$$\text{Tr}(\hat{S}\rho) = \left(\frac{1}{2\pi}\right)^n \int s_\sigma(z) \rho_\sigma(z) d^{2n}z$$

(65)

where $s_\sigma$ is the twisted Weyl symbol of $\hat{S}$.

**Proof.** The product $\hat{C} = \hat{S}\rho$ is of trace class because trace class operators form a two-sided ideal in the algebra of bounded operators on $L^2(\mathbb{R}^n)$. We can however not apply directly formula (53) since $\hat{S}$ is not a Hilbert–Schmidt operator. Let us proceed as follows: in view of formula (39) in Proposition 12. The twisted Weyl symbol $c_\sigma$ of $\hat{C}$ is given by the absolutely convergent integral

$$c_\sigma(z) = \left(\frac{1}{2\pi}\right)^n \int e^{-\frac{i}{\hbar}\sigma(z,z')} s_\sigma(z') \rho_\sigma(z - z') d^{2n}z'$$

where the twisted symbol $s_\sigma$ of $\hat{S}$ is given by formula (48) and $\rho_\sigma$ is the Gaussian (58). Since $\rho_\sigma \in \mathcal{S}(\mathbb{R}^{2n})$ we have $c_\sigma \in \mathcal{S}(\mathbb{R}^{2n})$ and hence also $c \in \mathcal{S}(\mathbb{R}^{2n})$, and we may therefore apply the trace formula (54) which yields

$$\text{Tr}(\hat{S}\rho) = \text{Tr}(\hat{C}) = c_\sigma(0)$$

which is precisely formula (65). □

**Notation 20** To simplify the statements below it will be convenient to write $\nu(S_t)$ for the Conley–Zehnder index of the symplectic isotopy $t' \mapsto S_{t'}$ for $0 \leq t' \leq t$ and $\nu(\hat{S}_t)$ for the Conley–Zehnder index of the corresponding metaplectic isotopy.

Using this notation we have:

**Theorem 21** Let $\Sigma = (S_t)_{t \in I}$ be a symplectic isotopy with endpoint $S \in \mathcal{G}$. Let $\hat{\Sigma} = (\hat{S}_t)_{t \in I}$ be the associated metaplectic isotopy, and assume that $S_t \notin \text{Sp}_0(n)$. We have

$$\text{Tr}(\hat{S}_t\rho) = \frac{i^{\nu(\hat{S}_t)}}{\sqrt{|\det(S_t - I)|}} \det^{-1/2}(\frac{1}{2}F^{-1} + iM(S_T^T)).$$

(66)
The relative phase shift is thus given by the formula

$$\phi(t) = \frac{\pi}{2} \nu(\hat{S}_t) + \text{Arg det}^{-1/2}(\frac{1}{2} F^{-1} + iM(S_t^T)).$$  \hfill (67)$$

**Proof.** We have $\rho \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$ and $\hat{S}_t$ is bounded on $L^2(\mathbb{R}^n)$. Let $s_t$ be the Weyl symbol of $\hat{S}_t$; applying formula (65) in Lemma 19 we get

$$\text{Tr}(\hat{S}_t \hat{\rho}) = \int F_\sigma s_t(z) F_\sigma \rho(-z) d^{2n}z.$$ 

Since $F_\sigma s_t = (s_t)_\sigma$ is the twisted symbol of the metaplectic operator $\hat{S}_t$ (formula (43)) we get, taking the expression (58) into account

$$\text{Tr}(\hat{S}_t \hat{\rho}) = (2\pi \hbar)^{-n} i^{\nu(\hat{S}_t)} \sqrt{|\text{det}(S_t - I)|} \int e^{\frac{-1}{2\hbar}((\frac{1}{2} F^{-1} - iM_t) z^2)} d^{2n}z.$$ 

Taking $A = -\frac{1}{2} J F J - iM_t$ in the Fresnel formula (4), we get

$$\text{Tr}(\hat{S}_t \hat{\rho}) = i^{\nu(\hat{S}_t)} \sqrt{|\text{det}(S_t - I)|} \text{det}^{-1/2}(\frac{1}{2} F^{-1} + iM(S_t)J)$$

hence the result once $\nu(\hat{S}_t) = \nu_{CZ}(\Sigma)$ modulo four, since $JM(S_t)J = M(S_t^T)$ in view of the second formula (17).  \hfill \blacksquare

### 5.2 Application: harmonic oscillator and standard coherent state

Assume that the symplectic path $\Sigma$ consists of the rotations

$$S_t = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}.$$ 

Then $H = \frac{\omega}{2}(x^2 + p^2)$ and we have for $\omega t \notin \pi \mathbb{Z}$ and $\psi_0 \in \mathcal{S}(\mathbb{R})$

$$\hat{S}_t \psi_0(x) = i^{-[\omega t/\pi]} \sqrt{\frac{1}{2\pi \hbar |\sin \omega t|}} \int_{-\infty}^{\infty} e^{i W(x,x',t)} \psi_0(x') dx' \hfill (68)$$

where $m(\hat{S}_t) = -[\omega t/\pi]$ is the usual Maslov index ($[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$); the generating function is here

$$W(x, x', t) = \frac{1}{2 \sin \omega t} \left[ (x^2 + x'^2) \cos \omega t - 2xx' \right] \hfill (69)$$
(see e.g. [13], pp.196–198). One verifies by direct calculation that the function $\psi(\cdot, t) = S_t \psi_0$ satisfies

$$ih\frac{\partial \psi}{\partial t} = \frac{\omega}{2} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + x^2 \right) \psi, \quad \psi(\cdot, 0) = \psi_0.$$  

Choose $\rho(z) = (\pi \hbar)^{-1} e^{-|z|^2/\hbar}$ (it is the Wigner transform of the standard coherent state $\phi_0(x) = (\pi \hbar)^{-4} e^{-x^2/2\hbar}$). We have here $W_{xx} = -2 \tan(\omega t/2)$. Also,

$$M_t = \frac{1}{2} \begin{pmatrix} \cot(\omega t/2) & 0 \\ 0 & \cot(\omega t/2) \end{pmatrix}$$  

(70)

hence, since $F = I$ in this case,

$$\det \left( \frac{1}{2} I + iJ M_t J \right) = \frac{-e^{i\omega t}}{4 \sin^2(\omega t/2)}. \quad (71)$$

Using the prescriptions following Fresnel’s formula (4) we get, setting $A_t = \frac{1}{2} I + iJ M_t J$,

$$\det^{-1/2} A_t = \sqrt{-4 \sin^2(\omega t/2) e^{i\omega t}}$$

and hence, writing $\text{Arg} \det^{-1/2} A_t = \text{Arg}(t)$,

$$\text{Arg}(t) = \begin{cases} \frac{-\omega t + \pi}{2} & \text{for } 2k\pi < \omega t < (2k + 1)\pi \\ \frac{\omega t - \pi}{2} & \text{for } (2k + 1)\pi < \omega t < 2(k + 1)\pi. \end{cases}$$

On the other hand, using formula (66) in Theorem 21, the phase $\phi(t) = \text{Arg} \text{Tr}(\tilde{S}_t \tilde{\rho})$ is given by

$$\nu(\tilde{S}_t) = \left[ \frac{\omega t}{\pi} \right] - \text{Inert} \left( -\tan\left( \frac{\omega t}{2} \right) \right) \mod 4 \quad (72)$$

where $\text{Inert} \alpha = 0$ if $\alpha > 0$ and $\text{Inert} \alpha = 1$ if $\alpha < 0$; explicitly:

$$\nu(\tilde{S}_t) = -2(k + 1) \text{ for } 2k\pi < \omega t < 2(k + 1)\pi.$$  

Using formula (66) in Theorem 21 the phase $\varphi(t) = \text{Arg} \text{Tr}(\tilde{S}_t \tilde{\rho})$ is given by

$$\varphi(t) = -\frac{\pi}{2} \left( \left[ \frac{\omega t}{\pi} \right] + \text{Inert} \left( -\tan\left( \frac{\omega t}{2} \right) \right) \right) + \text{Arg}(t)$$

and hence, summarizing:

24
\[
\begin{array}{|c|c|c|c|}
\hline
\omega t & \nu(S_t) & \text{Arg}(t) & \varphi(t) \\
\hline
(2k\pi, (2k + 1)\pi) & -(2k + 1) & \frac{\omega t + \pi}{2} & 2k\pi - \frac{\omega t}{2} \\
(2k\pi, (2k + 1)\pi) & -(2k + 1) & \frac{\omega t - \pi}{2} & 2k\pi + \frac{\omega t}{2} \\
\hline
\end{array}
\]

which coincides with the results obtained by one of us in [40].

5.3 The generalized oscillator

We now consider symplectic isotopies associated with Hamiltonian functions of the type

\[
H(z) = \frac{1}{2}Kz \cdot z
\]

(73)

where \( K = K(t) \) is a positive definite symmetric real matrix. We recall Williamson’s symplectic diagonalization theorem (Folland [17], Ch.4 and [21], §8.3.1): there exists \( R \in \text{Sp}(n) \) such that

\[
K = R^T DR, \quad D = \begin{pmatrix}
\Omega & 0 \\
0 & \Omega
\end{pmatrix}
\]

(74)

where \( \Omega \) is a diagonal matrix whose diagonal entries \( \omega_j > 0 \) are such that the \( \pm i\omega_j \) are the eigenvalues of \( JK \). The numbers \( \omega_j \) are the symplectic eigenvalues of \( F \). We have

\[
H(R^{-1}z) = \frac{1}{2}Dz \cdot z = \sum_{j=1}^{n} \frac{\omega_j}{2}(x^2_j + p^2_j).
\]

(75)

Rearranging the phase space coordinates by replacing \( z = (x, p) \) with \( u = (x_1, p_1, ..., x_n, p_n) \) the symplectic flow \( S_t^{H \circ R^{-1}} \) is thus given by \( u(t) = S_t^{H \circ R^{-1}} u \) where \( (x_j(t), p_j(t)) = S_t^{(j)}(x_j, p_j) \) with

\[
S_t^{(j)} = \begin{pmatrix}
\cos \omega_j t & \sin \omega_j t \\
-\sin \omega_j t & \cos \omega_j t
\end{pmatrix}.
\]

The corresponding generating function will thus be \( W = \sum_{j=1}^{n} W_j \) where the \( W_j \) are given by formula (69):

\[
W_j(x_j, x'_j, t) = \frac{1}{2\sin \omega_j t} \left[(x^2_j + x'^2_j)\cos \omega_j t - 2x_j x'_j\right].
\]

(76)
Theorem 22 Let $\Sigma = (S^H)$ be the symplectic isotopy determined by (73). Let $\hat{\Sigma} = (\hat{S}^H)$ be the corresponding metaplectic isotopy. Let $\hat{\rho}$ a density matrix with Gaussian Wigner distribution (57). We have, for $\omega_j t \notin \pi \mathbb{Z}$,

$$\text{Tr}(\hat{S}^H_t \hat{\rho}) = \text{Tr}(\hat{S}^H_{t R^{-1}} \hat{\rho} R^{-1})$$

and hence

$$\varphi(t) = -\frac{\pi}{2} \sum_{j=1}^{n} \left( \frac{\omega_j t}{\pi} + \text{Inert} \left( -\tan\left(\frac{\omega_j t}{2}\right) \right) \right) + \arg \det^{-1/2}(\frac{-1}{\pi}(R^{-1})^T F R^T + i M(RS_t R^{-1})).$$

Proof. Let $R$ be as in the Williamson diagonalization (74); we have $S^H_t = R^{-1} S^H_{t R^{-1}} R$ and $\hat{S}^H_t = \hat{R}^{-1} \hat{S}^H_{t R^{-1}} R$ (Lemma 9) and thus

$$\text{Tr}(\hat{S}^H_t \hat{\rho}) = \text{Tr}(\hat{R}^{-1} \hat{S}^H_{t R^{-1}} \hat{R} \hat{\rho}) = \text{Tr}(\hat{S}^H_{t R^{-1}} R R^{-1} \hat{\rho} R^{-1})$$

hence (77) follows in view of the symplectic covariance result (36). Since $W = \sum_{j=1}^{n} W_j$ and $W_j$ being given by (76), it follows from formulas (68) and (76) that we have, for $\omega_j t \notin \pi \mathbb{Z}$,

$$\hat{S}^H_{t R^{-1}} \psi(x) = \left( \frac{1}{2\pi \hbar} \right)^{n/2} \Delta(W) \int \prod_{j=1}^{n} (e^{\frac{i}{\hbar} W_j(x, x', t)}) \psi(x') d^n x'$$

where

$$\Delta(W) = \left| m(t) \right| \sin \omega_1 t \cdots \sin \omega_n t$$

and $m(t) = -\sum_{j=1}^{n} \left( \frac{\omega_j t}{\pi} \right)$ is the Maslov index. We have, by definition (57) of $\rho$ and recalling that $\det R = 1$,

$$\rho \circ R^{-1}(z) = (\pi \hbar)^{-n} \sqrt{\text{det}(R^{-1})^T F R^{-1} e^{-i\frac{\hbar}{2}(R^{-1})^T F R^{-1} z \cdot z}}.$$  

On the other hand

$$\nu(\hat{S}^H_t) = \nu(\hat{R}^{-1} \hat{S}^H_{t R^{-1}} \hat{R}) = \nu(\hat{S}^H_{t R^{-1}})$$

(property 13 of the Conley–Zehnder index). We thus have (formula 72)

$$\nu(\hat{S}^H_{t R^{-1}}) = \nu(\hat{S}^{(1)}_t) + \nu(\hat{S}^{(2)}_t) + \cdots + \nu(\hat{S}^{(n)}_t)$$

where

$$\nu(\hat{S}^{(i)}_t) = -\left( \sum_{j=1}^{n} \left( \frac{\omega_j t}{\pi} \right) + \text{Inert} \left( -\tan\left(\frac{\omega_j t}{2}\right) \right) \right) \mod 4.$$  

(78)
On the other hand we have $M(R^{-1}S_t^{H_0 R^{-1}}R) = R^T M(S_t^{H_0 R^{-1}})R$ in view of
the second formula (17) and hence
\[ \frac{1}{2} F^{-1} + iJ M S_t^H J = \frac{1}{2} F^{-1} + iJ M(R^{-1}S_t^{H_0 R^{-1}})J = \frac{1}{2} F^{-1} + iJ R^T M(S_t^{H_0 R^{-1}})RJ = \frac{1}{2} F^{-1} + iR^{-1}J M(S_t^{H_0 R^{-1}})J(R^T)^{-1}. \]

We thus have
\[ \det(\frac{1}{2} F^{-1} + iJ M(S_t^H)J) = \det(\frac{1}{2} R F^{-1} R^T + iJ M(S_t^{H_0 R^{-1}})J) \]
where
\[ M^j_i = \frac{1}{2} \begin{pmatrix} \cot(\omega_j t/2) & 0 \\ 0 & \cot(\omega_j t/2) \end{pmatrix} \]
(79)
and thus
\[ \phi(t) = \frac{\pi}{2} \nu(\Sigma) + \text{Arg det}^{-1/2}(-\frac{1}{2} F + iM_t). \]

6 The Inhomogeneous Case

Our central result is placed in theorem 21 which now will be slightly generalized to a larger class of Hamiltonian dynamics, besides the quadratic term we will include affine transformations related to phase space displacements. Despite its simplicity, the new form of the Hamiltonian is widely used in the literature as an approximation for any Hamiltonian dynamical system, see for instance [9, 32, 39].

6.1 The groups HSp(n) and IMP(n)

The inhomogeneous symplectic group ISp(n) is the semi-direct product [7, 11]
\[ \text{ISp}(n) = \text{Sp}(n) \ltimes T(2n) \]
where $T(2n)$ is the group of phase space translations $T(z_0) : z \mapsto z + z_0$. Its elements are the affine symplectomorphisms $ST(z)$ (or $T(z)S$) where $S \in \text{Sp}(n)$ and $T(z) \in T(2n)$; note that
\[ ST(z) = T(Sz)S, \quad T(z)S = ST(S^{-1}z). \]
The group law of ISp(n) is given by
\[(S, z)(S', z') = (SS', S'^{-1}z + z')\]

More interesting is, in a sense, the group HSp(n) ([17, 7]; it is denoted by WSp(n) in [7]). It is defined as follows: Let H(2n) be the Heisenberg group, that is \(\mathbb{R}^{2n} \times S^1\) equipped with the group law
\[(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\sigma(z, z')).\]
The symplectic group acts on H(2n) by \(S(z, t) = (Sz, t)\) hence we can form the semidirect product Sp(n) ⋉ H(2n). By definition this group is HSp(n) the group law being given by
\[(S, (z, t))(S', (z', t')) = (SS', (S'^{-1}z, t'(z, t)), t')).\]

Let now \(\hat{T}: H(2n) \rightarrow U(L^2(\mathbb{R}^n))\) be the Schrödinger representation of H(2n) defined by \(\hat{T}(z, t) = e^{i\gamma t}\hat{T}(z)\).

We similarly denote by IMp(n) the group of unitary operators on \(L^2(\mathbb{R}^n)\) generated by the the operators \(\hat{S} \in Mp(n)\) and \(\hat{T}(z, t), z \in \mathbb{R}^{2n}\). It follows from the symplectic covariance relations
\[\hat{S}\hat{T}(z, t) = \hat{T}(Sz, t)\hat{S}, \quad \hat{T}(z, t)\hat{S} = \hat{S}\hat{T}(S^{-1}z, t)\]
that every element of IMp(n) can be written in the form \(\hat{S}\hat{T}(z)\) or \(\hat{T}(z)\hat{S}\).
This is often referred to as the extended metaplectic representation of HSp(n); the projection \(\pi_{\text{IMp}}: \text{IMp}(n) \rightarrow \text{HSp}(n)\) is given by
\[\pi_{\text{IMp}}(\hat{T}(z, t)\hat{S}) = (S, z, t).\]
Notice that if we restrict ourselves to the case \(t = 0\) this reduces to
\[\pi_{\text{IMp}}(\hat{T}(z)\hat{S}) = (S, z) \in \text{ISp}(n).\]

6.2 Symplectic paths in ISp(n)

Consider an affine metaplectic isotopy \((\tilde{U}_t)_{t \in \mathbb{R}}\) where \(\tilde{U}_t \in \text{IMp}(n)\) is of the type
\[\tilde{U}_t = \hat{T}(z_t, \gamma_t)\hat{S}_t = e^{i\gamma t}\tilde{T}(z_t)\tilde{S}_t; \quad (80)\]
here \(t \mapsto z_t = (x_t, p_t)\) is a \(C^1\) path in \(\mathbb{R}^{2n}\) and \((\tilde{S}_t)_{t \in \mathbb{R}}\) a metaplectic isotopy. The phase \(\gamma_t \in \mathbb{R}\) depends in a \(C^1\) fashion on \(t\). A straightforward
calculation taking into account the identity \( \hat{p} \hat{T}(z_t) - \hat{T}(z_t) \hat{p} = p_t \hat{T}(z_t) \) (cf. (37)) yields
\[
\frac{i \hbar}{dt} \hat{T}(z_t) = -\left( \frac{1}{2} \sigma(z_t, \dot{z}_t) + \sigma(\dot{z}_t, \hat{z}) \right) \hat{T}(z_t)
\]
where \( \sigma(\dot{z}_t, \hat{z}) \) is the operator \( J \dot{z}_t \cdot \hat{z} \) with \( \hat{z} \psi = (x \psi, -i \hbar \partial_x \psi) \); equivalently
\[
\frac{i \hbar}{dt} \hat{T}(z_t) = \left( \sigma(\hat{z} - z_t, \dot{z}_t) + \frac{1}{2} \sigma(z_t, \dot{z}_t) \right) \hat{T}(z_t).
\] (82)

On the other hand one easily verifies that
\[
\frac{i \hbar}{dt} \hat{S}_t = H(\hat{z}, t) \hat{S}_t
\]
where \( H(\hat{z}, t) \) is the Weyl quantization of the Hamiltonian function \( H(z, t) \) defined by (7), that is
\[
H(\hat{z}, t) = -\frac{1}{2} J \hat{S}_t \hat{S}_t^{-1} \hat{z} \cdot \hat{z}.
\]

Collecting these results we see that \( \hat{U}_t \) satisfies the Schrödinger equation
\[
\frac{i \hbar}{dt} \hat{U}_t = \left( -\dot{\gamma}_t - \frac{1}{2} \sigma(z_t, \dot{z}_t) - \sigma(\dot{z}_t, \hat{z}) + H(\hat{z} - z_t, t) \right) \hat{U}_t.
\]

We next observe that the operator
\[
\hat{H}_{z_t} = -\dot{\gamma}_t - \frac{1}{2} \sigma(z_t, \dot{z}_t) - \sigma(\dot{z}_t, \hat{z}) + H(\hat{z} - z_t, t)
\] (83)
occuring in this equation is the Weyl quantization of the inhomogeneous quadratic Hamilton function
\[
H_{z_t}(z, t) = -\dot{\gamma}_t - \frac{1}{2} \sigma(z_t, \dot{z}_t) - \sigma(\dot{z}_t, z) + H(z - z_t, t).
\]

The solutions of the associated Hamilton equations
\[
\dot{z} = J \partial_z H_{z_t}(z, t) = \dot{z}_t + J \partial_z H(z - z_t, t)
\]
are given by \( z = u + z_t \) where \( u \) is the solution of the Hamilton equations for \( H(z, t) \). Recalling that the flow determined by \( H(z, t) \) is the symplectic isotopy \( (S_t)_{t \in \mathbb{R}} \) we thus have
\[
z(t) = S_t(z(0) - z_0) + z_t.
\]
6.3 Application to relative phase shifts

We assume now that \( \rho \) is a Gaussian centered at a point \( \tau \in \mathbb{R}^{2n} \):

\[
\rho(z) = (2\pi)^{-n} \sqrt{\det V}^{-1} e^{-\frac{1}{2} V^{-1} (z-\tau) (z-\bar{\tau})} \tag{84}
\]

and that \( (\tilde{U}_t)_{t \in \mathbb{R}} \) is an affine metaplectic isotopy given by (80). We will use the following elementary result:

**Lemma 23** Let \( \tilde{A} = \text{Op}^W(a) \) and \( z_0 \in \mathbb{R}^{2n} \). We have \( \tilde{T}(z_0) \tilde{A} = \text{Op}^W(c) \) where

\[
c_\sigma(z) = a_\sigma(z-z_0) e^{-\frac{1}{2\pi} \delta(z,z_0)} \tag{85}
\]

**Proof.** The twisted Weyl symbol of \( \tilde{T}(z_0) \) is given by \( t_\sigma(z) = (2\pi\hbar)^n \delta(z-z_0) \); formula (85) follows using (39). \( \blacksquare \)

**Theorem 24** Let \( \tilde{\rho} \) be the density operator with Wigner distribution (84) and \( (\tilde{U}_t)_{t \in \mathbb{R}} \) the metaplectic isotopy defined by (80). We have

\[
\text{Tr}(\tilde{U}_t \tilde{\rho}) = \frac{i^{t_{2\pi}}(\Sigma) e^{\frac{1}{2} i J_z t \cdot z - \frac{i}{2} J_z t \cdot F^{-1} J_z t + t \Phi(z,t)}}{\sqrt{\det(S_t - I)}} \sqrt{\det(\frac{1}{2} F^{-1} + i M(S_t^T))},
\]

\[
\Phi(z,t) = \frac{1}{8\hbar} [(F^{-1} + iJ)J_z t - 2i\bar{z}] \cdot \left[ \frac{1}{2} F^{-1} + i M(S_t^T) \right]^{-1} [(F^{-1} + iJ)J_z t - 2i\bar{z}] \tag{86}
\]

**Proof.** In view of formula (85) in Lemma 23 the twisted Weyl symbol of \( \tilde{T}(z_t) \) is the function \( z \mapsto (s_t)_\sigma(z-z_t) e^{-\frac{1}{2\pi} \delta(z,z_t)} \). Proceeding as in the proof of Theorem 21 we have

\[
\text{Tr}(\tilde{T}(z_t) \tilde{\rho}) = \int (s_t)_\sigma(z-z_t) e^{-\frac{1}{2\pi} \delta(z,z_t)} \rho_\sigma(-z) d^{2n} z
\]

\[
= \int (s_t)_\sigma(z) e^{-\frac{1}{2\pi} \delta(z,z_t)} \rho_\sigma(z-t) d^{2n} z.
\]

Using the Fresnel formula (41) with \( (s_t)_\sigma(z) \) in (43) and \( \rho_\sigma \), the symplectic Fourier transform (26) of \( \rho \) in (84), eq. (86) follows. \( \blacksquare \)

The relative phase shift (2) for the Gaussian state in (84) subjected to the inhomogeneous dynamics in (80) will thus be (see Notation 20):

\[
\phi(t) = \frac{\pi}{2} \nu(\tilde{S}_t) + \frac{1}{\hbar} \gamma_t + \text{Arg} \Phi(z_t, \bar{z}) + \text{Arg} \det^{-1/2}(\frac{1}{2} F^{-1} + i M(S_t^T)) \tag{87}
\]

This formula reduces to the one in (67) when \( \tau = z_t = 0 \) and \( \gamma_t = 0 \), \( \forall t \in I \).
7 Discussion and Perspectives

The proof of the general result in Theorem 21 heavily relies on the fact that the integral giving the trace is easily calculable because the integrand is a Gaussian and can, as such, be explicitly determined by a Fresnel-type formula. This relative straightforwardness is due to the fact that the twisted Weyl symbol of the unitary evolution operator ($\hat{U}_t$) is here a family ($\hat{S}_t$) of metaplectic operators and is hence itself a (complex) Gaussian function, namely

$$s_\sigma(z) = \frac{\nu + \nu' + \frac{1}{2} \text{sign}(M)}{\sqrt{|\text{det}(S - I)|}} \exp \left( \frac{i}{2\hbar} M(S) z \cdot z \right)$$

when $\text{det}(S - I) \neq 0$. It would of course be interesting (and even essential) to extend Theorem 21 to more general situations. But even when $\hat{\rho}$ is still a Gaussian state we run into a major difficulty, which is the determination of the Weyl symbol (twisted, or not) of a general evolution operator ($\hat{U}_t$). Very little is actually known; to the best of our knowledge only sporadic attempts exist in the literature, and they usually consist in using non-rigorous Feynman-type path integral methods.

Fortunately, semiclassical propagation methods are very well developed in the context of the inhomogeneous dynamics presented in section ???. In this scenario, a generic analytic Hamiltonian function of $\hat{z}$ can be expanded up to second order, always giving rise to a quadratic structure (83), as proposed in [32]. The validity of this approximation is guaranteed for a time interval limited by the very known Ehrenfest time $\tau_E \sim \log(h^{-1})$ [9]. This scheme is very well suited for propagation of Gaussian states, since the operators (80) keep this set of states invariant. However, any quantum state can be expanded as a superposition of (Gaussian) coherent states, thus this method can be applied to the propagation of any initial state [9, 32]. An example for the propagation of states under a non-linear (and classically chaotic) Hamiltonian dynamics is given in [39].

On the other side, the study of the Pancharatnam phase for states outside of the Gaussian set is possible and can reveal new and interesting scenarios for the phase behavior. For instance, coherent and incoherent superpositions of Gaussian states [39] can be studied quite directly using the tools presented in this paper; the relation of the total phase for the interference fringes has not yet been explored in the literature. Even for more general states, the developed tools can also be applied using the Glauber–Sudarshan representation [34], which constitutes an expansion, in principle written for any quantum state, in terms of the standard coherent state basis. More
generally, the notion of Gabor (or Weyl–Heisenberg) frame could be useful in this context [16].

In the same spirit of the generalization of the total phase (1) defined in [38] to any density state (2), a possible generalization of the dynamical phase [38]

\[ \varphi_d = \text{Im} \int_0^t (\psi_t', \dot{\psi}_t') dt', \]
can be given, introducing the more general quantity

\[ \varphi'_d = \int_0^t \text{Tr} (\hat{H}(t')) dt'. \]

However, in [38] the geometric phase is defined to be \( \varphi - \varphi_d \) for \( \varphi \) in (1). The association of generalized geometric phase to \( \varphi - \varphi'_d \) should be carefully investigated, specially in what concerns its relation with the Conley–Zehnder index. This index can be viewed as a geometric quantity associated to paths on the space \( L^2(\mathbb{R}^n) \) connecting Gaussian states. This idea will be developed in forthcoming work.

**APPENDIX A: The Metaplectic Group**

For detailed studies of the symplectic group \( \text{Sp}(n) \) see [14, 21]; the properties of the metaplectic are studied in [17, 21].

**A.1 Definition**

The metaplectic group \( \text{Mp}(n) \) is a unitary representation on \( L^2(\mathbb{R}^n) \) of the double cover \( \text{Sp}_2(n) \) of the symplectic group \( \text{Sp}(n) \). The simplest (but not necessarily the most useful) way to describe \( \text{Mp}(n) \) is to use its elementary generators \( \hat{J}, \hat{V}_{-P}, \) and \( \hat{M}_{L,m} \) [17, 21]. Denoting by \( \pi_{\text{Mp}} \) the covering projection \( \text{Mp}(n) \rightarrow \text{Sp}(n) \) these operators and their projections are given by

\[ \hat{J} \psi(x) = e^{-in\pi/4} F \psi(x) , \quad \pi_{\text{Mp}}(\hat{J}) = J \]  
\[ \hat{V}_{-P} \psi(x) = e^{i/2P x^2} \psi(x) , \quad \pi_{\text{Mp}}(\hat{V}_{-P}) = V_{-P} \]  
\[ \hat{M}_{L,m} \psi(x) = i^m \sqrt{|\det L|} \psi(Lx) , \quad \pi_{\text{Mp}}(\hat{M}_{L,m}) = M_{L,m}. \]

Here \( F \) is the unitary \( h \)-Fourier transform

\[ F \psi(x) = \left( \frac{1}{2\pi h} \right)^n \int e^{-i\hat{x}x'} \psi(x') d^n x'. \]
and \( V_{-P} \) (\( P = P^T \)), \( M_{L,m} \) (\( \det L \neq 0 \)) are the symplectic matrices

\[
V_{-P} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}, \quad M_{L,m} = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}.
\]

The index \( m \) in \( \widehat{M}_{L,m} \) is an integer corresponding to a choice of \( \arg \det L \):

- \( m \) is even if \( \det L > 0 \)
- \( m \) is odd if \( \det L < 0 \).

It is called the \textit{Maslov index} of \( \widehat{M}_{L,m} \).

### A.2 Definition using quadratic Fourier transforms

Let \( P, Q \in \text{Sym}(n, \mathbb{R}) \) and \( L \in GL(n, \mathbb{R}) \), and let \( W \) be the real quadratic form on \( \mathbb{R}^n \times \mathbb{R}^n \) defined by

\[
W(x, x') = \frac{1}{2} P x^2 - L x \cdot x' + \frac{1}{2} Q x'^2.
\]

To \( W \) we associate \( \hat{S}_{W,m} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n) \) by the formula

\[
\hat{S}_{W,m} \psi(x) = e^{-ni\pi/4} \left( \frac{1}{2\pi\hbar} \right)^{n/2} i^m \sqrt{\det L} \int e^{\frac{i}{\hbar} W(x, x')} \psi(x') d^n x'
\]

where the integer \( m \) (which is only defined modulo 4) corresponds to a choice or \( \arg \det L \) as above. By definition, that integer is the Maslov index of \( \hat{S}_{W,m} \). One verifies by a simple calculation that we have

\[
\hat{S}_{W,m} = \hat{V}_{-P} \hat{M}_{L,m} \hat{V}^{-1}
\]

hence \( \hat{S}_{W,m} \in \text{Mp}(\mathbb{n}) \) is a unitary operator on \( L^2(\mathbb{R}^n) \). Using the formulas \( [A1] \rightarrow [A3] \) a simple calculation shows that \( S_W = \pi^{\text{Mp}}(\hat{S}_{W,m}) \) is given by

\[
S_W = \begin{pmatrix} L^{-1} Q & L^{-1} \\ PL^{-1} Q - L^T & L^{-1} P \end{pmatrix}.
\]

The operators \( \hat{S}_{W,m} \) are called quadratic Fourier transforms; one easily \( [31] [21] \) verifies that

\[
(\hat{S}_{W,m})^{-1} = \hat{S}_{W^*, m^*} \text{ with } W^*(x, x') = -W(x', x), \quad m^* = n - m.
\]

The quadratic Fourier transforms form a dense subset of \( \text{Mp}(\mathbb{n}) \). In fact they generate this group:
Proposition 25  Every $\hat{S} \in \text{Mp}(n)$ can be written as a the product of two quadratic Fourier transforms: $\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'}$ and $\pi^{\text{Mp}}(\hat{S}_{W,m}) = S_W$ where $S_W \in \text{Sp}(n)$ is generated by the quadratic form $W$, that is

$$(x, p) = S_W(x', p') \iff \begin{cases} p = \partial_x W(x, x') \\ p' = -\partial_{x'} W(x, x') \end{cases}.$$ 

Proof. See Leray [31], de Gosson [21]; for a detailed discussion of the notion of generating function see Arnol’d [4].

The factorization $\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'}$ of a metaplectic operator is by no means unique; for instance we can write the identity operator $I$ as $\hat{S}_{W,m} \hat{S}_{W}^{-1}$ for every quadratic Fourier transform $\hat{S}_{W,m}$. There is however an invariant attached to $\hat{S}$: the Maslov index. Denoting by Inert $R$ the index of inertia (= the number of negative eigenvalues) of the real symmetric matrix $R$ we have:

Proposition 26  Let $\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'} = \hat{S}_{W'',m''} \hat{S}_{W''',m'''}$. We have

$$m + m' - \text{Inert}(P' + Q) \equiv m'' + m''' - \text{Inert}(P'' + Q'') \mod 4. \quad (A8)$$

Proof. See Leray [31], de Gosson [18, 21].

It follows from formula (A8) that the class modulo 4 of the integer $m + m' - \text{Inert}(P' + Q)$ does not depend on the way we write $\hat{S} \in \text{Mp}(n)$ as a product $\hat{S}_{W,m} \hat{S}_{W',m'}$ of quadratic Fourier transforms; this class is denoted by $m(\hat{S})$ and called the Maslov index of $\hat{S}$. The mapping

$$m : \text{Mp}(n) \ni \hat{S} \longrightarrow m(\hat{S}) \in \mathbb{Z}_4$$

is called the Maslov index on $\text{Mp}(n)$. We have $m(\hat{S}_{W,m}) = m$, mod 4 ([31 [18]). The theory of the Maslov index has been further developed by Arnol’d, Leray, and by the author (see the review [8] by Cappell et al.).

8 APPENDIX B: Leray and Maslov Indices

B.1 The Leray index

Let Lag($n$) be the Lagrangian Grassmannian of the symplectic space $(\mathbb{R}^{2n}, \sigma)$. We have a natural action

$$\text{Sp}(n) \times \text{Lag}(n) \longrightarrow \text{Lag}(n).$$
Let \((\ell, \ell', \ell'') \in \text{Lag}^3(n)\); we denote by \(\tau(\ell, \ell', \ell'')\) the signature of the quadratic form
\[
Q(z, z', z'') = \sigma(z, z') + \sigma(z', z'') + \sigma(z'', z)
\]
on \(\ell \times \ell' \times \ell''\). It has the following properties:

- **Symplectic invariance:**
  \[
  \tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'') \quad \text{for all} \ S \in \text{Sp}(n);
  \]

- **Cocycle property:**
  \[
  \partial \tau(\ell, \ell', \ell'', \ell''') = 0 \quad (B1)
  \]
  where \(\partial\) is the usual coboundary operator;

- **Antisymmetry:**
  \[
  \tau(\pi(\ell, \ell', \ell'')) = (-1)^{\text{sign}(\pi)} \tau(\ell, \ell', \ell'')
  \]
  for every permutation \(\pi\) of \((\ell, \ell', \ell'')\).

We have
\[
\tau(\ell, \ell', \ell'') \equiv n + \partial \dim(\ell, \ell', \ell'') \mod 2 \quad (B2)
\]
where \(\dim(\ell, \ell') = \dim(\ell \cap \ell')\).

Let \(\pi_\infty : \text{Lag}_\infty(n) \to \text{Lag}(n)\) be the universal covering space of \(\text{Lag}(n)\) (“Maslov bundle”). We will write \(\ell = \pi^\text{Lag}_\infty(\ell_\infty)\). The Leray index is the only mapping
\[
\mu : \text{Lag}_\infty(n) \times \text{Lag}_\infty(n) \to \mathbb{Z}
\]
having the two following properties:

**LM1** It is locally constant on the set
\[
\{(\ell_\infty, \ell'_\infty) \in \text{Lag}_\infty(n) \times \text{Lag}_\infty(n) : \ell \cap \ell' = 0\};
\]

**LM2** Its coboundary descends to the signature: \(\partial \mu(\ell_\infty, \ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'')\), that is
\[
\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell''). \quad (B3)
\]

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Taking $\ell_\infty = \ell'_\infty$ in (B3) and using the antisymmetry of $\tau$ we get the relation
\[
\mu(\ell_\infty, \ell'_\infty) = -\mu(\ell'_\infty, \ell_\infty). \tag{B4}
\]
Identifying as usual the unitary group $U(n, \mathbb{C})$ with a subgroup $U(n)$ of $\text{Sp}(n)$ we have a transitive action
\[
U(n, \mathbb{C}) \times \text{Lag}(n) \rightarrow \text{Lag}(n).
\]
Let $\ell_P = 0 \times \mathbb{R}^n$ the mapping $\ell = u\ell_P \mapsto uu^T$ ($u \in U(n, \mathbb{C})$) induces a homeomorphism
\[
\text{Lag}(n) \rightarrow W(n, \mathbb{C}) = \{ w \in U(n, \mathbb{C}) : w = w^T \}
\]
and we have the identification with the set
\[
\text{Lag}_\infty(n) \equiv \{(w, \theta) : w \in W(n, \mathbb{C}), \det w = e^{i\theta} \};
\]
the projection $\pi_{\text{Lag}}^\infty$ is the mapping $(w, \theta) \mapsto w$. The Leray index can then be explicitly be defined in the transversal case $\ell \cap \ell' = 0$ by the Souriau [17] formula
\[
\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi}(\theta - \theta' + i \text{Tr} \text{Log}(-w(w')^{-1}) \tag{B5}
\]
when $\ell_\infty = (w, \theta)$ and $\ell'_\infty = (w', \theta')$. The condition $\ell \cap \ell' = 0$ is equivalent to $-w(w')^{-1}$ having no eigenvalue on the negative half-axis. In the non-transversal case one chooses $\ell''_\infty \in \text{Lag}_\infty(n)$ such that $\ell'' \cap \ell = \ell'' \cap \ell' = 0$ and one then defines
\[
\mu(\ell_\infty, \ell'_\infty) = \mu(\ell_\infty, \ell''_\infty) - \mu(\ell'_\infty, \ell''_\infty) + \tau(\ell, \ell', \ell''). \tag{B6}
\]
That the right-hand side in this formula is independent on the choice of $\ell''_\infty$ readily follows from the cocycle property (B3) of the signature $\tau$ [19, 21].

We have
\[
\mu(\ell_\infty, \ell'_\infty) \equiv n + \dim(\ell \cap \ell') \mod 2, \quad \mu(\ell_\infty, \ell'_\infty) = -\mu(\ell'_\infty, \ell_\infty) \tag{B7}
\]
(the first equality immediately follows from (B3) using (B2) and the second by taking $\ell''_\infty = \ell_\infty$ in (B3)). Let $\text{Sp}_\infty(n)$ be the universal covering group of $\text{Sp}(n)$. The natural group action
\[
\text{Sp}_\infty(n) \times \text{Lag}_\infty(n) \rightarrow \text{Lag}_\infty(n)
\]
such that
\[
(\alpha S_\infty)\ell_\infty = S_\infty(\beta^2 \ell_\infty) = \beta^2(S_\infty \ell_\infty) \tag{B8}
\]
where $\alpha$ and $\beta$ the generators of the cyclic groups $\pi_1[\text{Sp}(n)]$ and $\pi_1[\text{Lag}(n)]$, respectively \[31\]. We have
\[
\mu(\beta^r \ell_\infty, \beta^r' \ell_\infty') = \mu(\ell_\infty, \ell_\infty') + 2(r - r')
\] (B9)
for all $(r, r') \in \mathbb{Z}^2$.

The Leray index is invariant under the action of $\text{Sp}_\infty(n)$:
\[
\mu(S_\infty \ell_\infty, S_\infty' \ell_\infty') = \mu(\ell_\infty, \ell_\infty').
\] (B10)
This immediately follows from the fact that both functions $(\ell_\infty, \ell_\infty') \mapsto \mu(\ell_\infty, \ell_\infty')$ and $(\ell_\infty, \ell_\infty') \mapsto \mu(S_\infty \ell_\infty, S_\infty' \ell_\infty')$ satisfy the characteristic conditions (LM1) and (LM2) and that the signature is a symplectic invariant.

**B.2 Relative Maslov indices**

For $S_\infty \in \text{Sp}_\infty(n)$ and $\ell \in \text{Lag}(n)$ we define the Maslov index on $\text{Sp}_\infty(n)$ relative to $\ell$ by
\[
\mu_\ell(S_\infty) = \mu(S_\infty \ell_\infty, \ell_\infty).
\] (B11)

It follows from \[33\] that for every $\ell_\infty \in \text{Lag}_\infty(n)$ the function $\text{Sp}_\infty(n) \rightarrow \mathbb{Z}$ associating to $S_\infty$ the integer $\mu(S_\infty \ell_\infty, \ell_\infty)$ only depends on the projection $\ell = \pi_\infty(\ell_\infty)$, justifying the notation (B11).

Let $S_\infty, S'_\infty \in \text{Sp}_\infty(n)$ and $\ell \in \text{Lag}(n)$. We have the product formula
\[
\mu_\ell(S_\infty S'_\infty) = \mu_\ell(S_\infty) + \mu_\ell(S'_\infty) + \tau(\ell, S_\ell, S'_S_\ell) \] (B12)
(it readily follows from the coboundary property (B3) of the Leray index); taking $S'_\infty = S_\infty^{-1}$ in this formula it follows that
\[
\mu_\ell(S_\infty^{-1}) = -\mu_\ell(S_\infty). \] (B13)

The following identity describes the action of $\pi_1[\text{Sp}(n)]$ on the relative Maslov index: for every $r \in \mathbb{Z}$ we have from \[33\] that
\[
\mu_\ell(\alpha^r S_\infty) = \mu_\ell(S_\infty) + 4r. \] (B14)

It follows from the properties (LM1) and (LM2) of the Leray index that

**MA** The Maslov index relative to $\ell \in \text{Lag}(n)$ is the only mapping $\mu_\ell : \text{Sp}_\infty(n) \rightarrow \mathbb{Z}$ which is locally constant on the set $\{S_\infty : S_\ell \cap \ell = 0\}$ and satisfying the product formula (B12).

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