The renormalizability of Hořava–Lifshitz-type gravities

Domenico Orlando and Susanne Reffert

Institute for the Mathematics and Physics of the Universe, The University of Tokyo,
Kashiwa-no-Ha 5-1-5, Kashiwa-shi, 277-8568 Chiba, Japan

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Abstract
In this paper, we discuss the renormalizability of Hořava–Lifshitz-type gravity theories. Using the fact that Hořava–Lifshitz (HL) gravity is very closely related to the stochastic quantization of topologically massive gravity, we show that the renormalizability of HL gravity only depends on the renormalizability of topologically massive gravity. This is a consequence of the BRST and time-reversal symmetries pertinent to theories satisfying the detailed balance condition.

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1. Introduction
In [1], Hořava studied a non-Lorentz-invariant theory of gravity in (3 + 1) dimensions inspired by the Lifshitz model, which has appeared in the condensed matter context (see e.g. [2]). One of the main features of this theory is that unlike Einstein gravity, it is renormalizable by power-counting arguments. Hořava–Lifshitz (HL) gravity has sparked a lot of interest. A number of follow-up works have been published concerning its solutions (see e.g. [3, 4]) and cosmological implications (see e.g. [5–12]), but many fundamental questions have not yet been answered. In this paper, we investigate the renormalizability of this model in more detail.

Lifshitz-type models exhibit anisotropic scaling between space and time. The amount of this anisotropy is captured by the dynamical critical exponent $z$. In Lorentz-invariant theories, $z = 1$, while in the gravity theory proposed in [1], $z$ is set equal to 3 to make the theory renormalizable by power counting. Stochastic quantization (SQ) [13–15] proves to be a useful tool in the study of Lifshitz-type models. In [16], we have argued that the (scalar) quantum Lifshitz model can be understood to be resulting from the stochastic quantization of the free boson in one dimension less. We have further argued that adopting the manifestly supersymmetric formalism for stochastic quantization [17], which calls for the inclusion of...
fermionic terms into the action, allows for the consistent study of the quantum Lifshitz model and generalizations thereof.

In this paper, we follow the same philosophy. As already pointed out in [18], HL gravity can also be understood as the result of a stochastic quantization, namely of topologically massive gravity [19, 20]. Using arguments similar to those presented in [21], we will show that (super) Hořava–Lifshitz gravity is indeed renormalizable, at least if detailed balance is respected, and provided that its precursor theory, topologically massive gravity, is renormalizable, as discussed in [20]. Our main tools are the structure of the action which is implied by the detailed balance condition and the general properties exhibited by theories resulting from SQ. The main object in SQ is the Langevin equation, a stochastic differential equation which governs the time evolution of the field which is quantized. In the process of stochastic quantization, a $D$-dimensional Euclidean field theory is turned into a $(D + 1)$-dimensional quantum field theory, in which a new time direction has been added. This new time direction is necessarily on a different footing than the $D$ dimensions of the original theory, so such a theory is in general not Lorentz invariant. A nice property of the resulting $(D + 1)$-dimensional theory is that it automatically exhibits a supersymmetry in the new time dimension. Even though the resulting theory is in general not Lorentz invariant, its structure is thus very constrained and many of its properties depend largely on the original $D$-dimensional theory.

In fact, we will argue that the renormalizability of HL gravity rests on the renormalizability of the underlying topologically massive gravity. The additional structure of HL gravity can be understood in terms of the SQ process. We will show that this structure implied by detailed balance, and thus of the Langevin equation, is not altered by the renormalization group flow. To show this, we make use of the BRST symmetry of the theory and the time-reversal symmetry of the unrenormalized action. These two symmetries together constrain the form of the renormalized theory. To make use of the BRST symmetry, we adopt a formulation of SQ which calls for the introduction of fermionic fields. It is argued in [15] that the diagrammatic contributions of the fermions to equal-time correlators are cancelled by contributions from lines joining auxiliary fields to the field that is being quantized. As long as one remains on the same time slice, it is thus possible to completely drop all anti-commuting fields from the action, which in this case recovers precisely Hořava’s action without the fermionic fields we are working with.

As mentioned before, we heavily make use of the detailed balance condition. It was argued in [10] that this condition is phenomenologically undesirable. Our result does not extend to HL-type theories with additional terms which break detailed balance. For such cases, a different course must be pursued to find arguments for their renormalizability beyond power counting.

The plan of this paper is as follows. In section 2, we will briefly recall the essentials of HL gravity. In section 3, we will review some basic facts about stochastic quantization and apply them to the case at hand. In section 4, the renormalization properties of HL gravity are discussed. In section 5, we will summarize our results.

2. Preliminaries

We will be very brief in reviewing the action of HL gravity and only give the most necessary definitions without further explanation. For details, we refer the reader to [1].

Since HL theory has anisotropic scaling, the spacetime $\mathcal{M}$ has the structure of a codimension 1 foliation with topology $\mathbb{R} \times \Sigma$, and the theory is designed to be invariant under the foliation-preserving diffeomorphism group $\text{Diff}_\mathcal{F}(\mathcal{M})$. A Riemannian metric on
such a manifold can be decomposed à la ADM into the metric $g_{ij}$ induced along the leaves of the foliation, the shift variable $N_i$ and the lapse field $N$.

The basic objects that appear in the theory are the second fundamental form

$$K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i)$$  \hspace{1cm} (2.1)

and the DeWitt metric on the space of metrics

$$G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}.$$  \hspace{1cm} (2.2)

The parameter $\lambda$ is free; the value $\lambda = 1$ would be required for the full Diff($\mathcal{M}$) invariance to hold. The full action of HL gravity is given by

$$S = \int dt \int d^3x N \sqrt{g} \left\{ \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl} - \frac{\kappa^2}{2} C^{ij} - \frac{\mu}{2} \left( R_{ij} - \frac{1}{2} R g_{ij} + \Lambda_w g_{ij} \right) \right\}$$  \hspace{1cm} (2.3)

$$\times \left\{ \frac{1}{w^2} C^{kl} - \frac{\mu}{2} \left( R^{kl} - \frac{1}{2} R g^{kl} + \Lambda_w g^{kl} \right) \right\}.$$  \hspace{1cm} (2.3)

Here, $\kappa$, $\lambda$, and $w$ are dimensionless coupling constants. The coupling constant $\mu$ has dimension 1, $[\lambda_w] = 2$ and $C^{ij}$ is the Cotton tensor. The lack of Poincaré invariance of the theory is reflected by the fact that the indices $i, j, \ldots$ only refer to the coordinates on $\Sigma$ and the covariant derivative $\nabla_i$ is taken with respect to the metric $g_{ij}$.

The action is conveniently rewritten by introducing an auxiliary field $B^{ij}$ and observing that the quadratic term is the variation with respect to the metric of the action for topologically massive gravity:

$$\frac{1}{w^2} C^{ij} - \frac{\mu}{2} \left( R_{ij} - \frac{1}{2} R g_{ij} + \Lambda_w g_{ij} \right) = \frac{1}{\sqrt{g}} \frac{\delta S_{cl}(g)}{\delta g_{ij}},$$  \hspace{1cm} (2.4)

where

$$S_{cl} = \frac{1}{w^2} \int \omega_3(\Gamma) + \mu \int d^3x \sqrt{g}(R - 2\Lambda_w),$$  \hspace{1cm} (2.5)

with the gravitational Chern–Simons term being given by

$$\omega_3(\Gamma) = \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) = \varepsilon^{ijk} \left( \Gamma^m_{ij} \partial_j \Gamma^i_{km} + \frac{2}{3} \Gamma^m_{ij} \Gamma^i_{jm} \Gamma^m_{kn} \right) d^3x,$$  \hspace{1cm} (2.6)

where $\Gamma^i_{km}$ are the Christoffel symbols. Finally, one finds that

$$S = \int dt \int d^3x N \sqrt{g} \left\{ B^{ij} \left( K_{ij} + G_{ijkl} \frac{1}{\sqrt{g}} \frac{\delta S_{cl}}{\delta g_{kl}} \right) - B^{ij} G_{ijkl} B^{kl} \right\}.$$  \hspace{1cm} (2.7)

We will refer to actions of this form as satisfying the detailed balance condition.

3. HL gravity and stochastic quantization

As already hinted in [1], it is possible to understand HL gravity as the result of stochastically quantizing topologically massive gravity. In the following, we will make use of this connection to exploit the renormalization properties of stochastically quantized theories. We will follow the notation used in [14].

In the case of a scalar theory, stochastic quantization of a Euclidean field theory works as follows. We supplement the field $\phi(x)$ with an extra time dimension $t$ (which must not be confused with the Euclidean time $x_0$). Then we demand that the time evolution of $\phi(x, t)$ obeys a stochastic differential equation, the Langevin equation, which allows the relaxation to equilibrium:

$$\frac{d\phi(x, t)}{dt} = -\frac{\delta S_{cl}}{\delta \phi} + \eta(x, t),$$  \hspace{1cm} (3.1)
with $S_{cl}$ being the Euclidean action. A stochastic equation of this type, where the flow depends on the gradient of a function of the field, is said to satisfy the detailed balance condition. The correlations of $\eta$, which is a white Gaussian noise, are given by

$$\langle \eta(x,t) \rangle = 0, \quad \langle \eta(x_1,t_1) \eta(x_2,t_2) \rangle = 2\delta(t_1 - t_2)\delta^d(x_1 - x_2).$$

Equation (3.1) has to be solved given an initial condition at $t = t_0$ leading to an $\eta$-dependent solution $\phi_\eta(x,t)$. As a consequence, $\phi_\eta(x,t)$ is now also a stochastic variable. Its correlation functions are defined by

$$\langle \phi_\eta(x_1,t_1) \ldots \phi_\eta(x_k,t_k) \rangle = \int \mathcal{D}\eta \exp \left[ -\frac{1}{4} \int \! d^d x \int \! dt \, \eta^2(x,t) \right] \phi_\eta(x_1,t_1) \ldots \phi_\eta(x_k,t_k) \int \mathcal{D}\eta \exp \left[ -\frac{1}{4} \int \! d^d x \int \! dt \, \eta^2(x,t) \right].$$

One of the central points for stochastic quantization is that equilibrium is reached for $t \to \infty$ and that

$$\lim_{t \to \infty} \langle \phi_\eta(x_1,t) \ldots \phi_\eta(x_k,t) \rangle = \langle \phi(x_1) \ldots \phi(x_k) \rangle,$$

i.e. the equal-time correlators for $\phi_\eta$ tend to the corresponding quantum Green’s functions.

### 3.1. Stochastic quantization of gravity theories

In order to apply this formalism to HL gravity, we need to write a Diff$_F(M)$-invariant Langevin equation for the quantization of the classical action in equation (2.5). In particular, even though the field variable is $g_{ij}$, it was argued in [1] that instead of $\dot{g}_{ij}$, the second fundamental form $K_{ij}$ should appear on the lhs of the Langevin equation to maintain Diff$_F(M)$ invariance.

To reduce the number of possibly confusing indices, we introduce some notations as follows (see also [22]).

- In terms of $G$, the metric on 3D space is a contravariant vector:
  $$g_{ij} =: g^I;$$
  (3.5)

- Variations with respect to the metric $g_{ij}$ are indicated by $\partial_I$:
  $$\partial_I W(g) =: \frac{\delta W(g)}{\delta g_{ij}};$$
  (3.6)

- The metric on the space of metrics is expressed as a covariant metric tensor:
  $$G^{ijkl} = \sqrt{g} G^{ijkl} =: G_{IJ}.$$ 
  (3.7)

Note the presence of the $\sqrt{g}$ term, which did not appear in equation (2.2). Metric $G_{IJ}$ is a unique (up to the choice of parameter $\lambda$) metric with respect to which coordinate transformations of $g_{ij}$ are isometries (see [23]).

Indices of types $I$ and $J$ are raised and lowered using $G_{IJ}$ and its inverse $G^{IJ}$:

$$G_{IJ} G^{JK} = \delta^I_K, \quad G^{IJ} G_{KL} = \delta^I_K.$$ 
(3.8)

which in terms of space indices should be read as

$$G^{ijkl}_{\text{mink}} = \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^j_k \delta^i_l \right).$$
(3.9)

In the following, we will need the vielbein on the space of metrics,

$$E_A^I \, E_B^J G_{IJ} = \delta_{AB}.$$ 
(3.10)

Now we are ready to write down the Langevin equation for HL gravity:

$$K^I = -G^{IJ} \partial_J S_{cl} + \eta^I.$$ 
(3.11)
where $\eta$ is a noise. The measure for the noise $\eta^I$ depends on the 3D metric $g_{ij}$, which via the Langevin equation (3.11) in turn depends on $\eta^I$. This would introduce nonlinearities in the path integral that can be avoided if, using the vielbein, we introduce a new noise $\eta^A$ that is actually Gaussian. Its correlators are then defined independent of $g_{ij}$, in terms of $\delta_{AB}$.

More precisely,

$$\eta^A = E^A_I \eta^I,$$

and

$$\langle \eta^A(x, t) \eta^B(x', t') \rangle = \frac{2}{N(t)} \delta(x - x') \delta(t - t') \delta^{AB}.$$  (3.12)

(Note that indices $A, B$ in the non-coordinate basis are raised and lowered with $\delta_{AB}$.)

We can now express equation (3.11) in terms of $\eta^A$ (here expressed in the coordinates of the $D$-dimensional manifold):

$$K_{ij} = -G_{ijkl} \frac{\delta S_{cl}}{\sqrt{g}} \delta g_{kl} + \eta^\alpha E^I_{ij} \delta^\alpha_{AB}.$$  (3.14)

Because of the Langevin equation, the metric becomes a stochastic function. Its generating functional is given by

$$Z(J) = \langle e^{\int dt d^3x N J_I g_{II}} \rangle_{\eta} = \int D\eta \exp \left[ -\frac{1}{4} \int dt d^3x N (\eta^A \eta_A) + \int dt d^3x N J_I g_{II} \right].$$  (3.15)

Let $M^I_J$ be the variation

$$M^I_J(x, t) = \partial_J (K^I_I + G^{IK} \partial_K S_{cl} - \eta^I) = \tilde{M}^I_J - \eta^A \partial_I E^A_J.$$

Then the following identity holds:

$$1 = \int \mathcal{D}g \det(M) \delta(K^I_I + G^{IK} \partial_K S_{cl} - \eta^I).$$  (3.17)

The two factors in equation (3.17) can be expressed in terms of a bosonic auxiliary field $B$ and two fermionic auxiliary fields $\bar{\psi}, \psi$:

$$\det(M) = \int \mathcal{D}[\bar{\psi}] \mathcal{D}\psi \exp \left[ \frac{1}{2} \int dt d^3x N \bar{\psi}_I M^I_J \psi_J \right],$$

$$\delta(K^I_I + G^{IK} \partial_K S_{cl} - \eta^I) = \int \mathcal{D}B \exp \left[ \frac{1}{2} \int dt d^3x N B_I (K^I_I + G^{IK} \partial_K S_{cl} - \eta^I) \right].$$  (3.18)

Plugging identity (3.17) into the generating functional (3.15), one obtains

$$Z(J) = \int \mathcal{D}g \mathcal{D}[\bar{\psi}] \mathcal{D}\psi \mathcal{D}B \exp \left[ -S(g, \eta, \bar{\psi}, \psi, B) + \int dt d^3x N J_I g_{II} \right].$$  (3.20)

where

$$S(g, \eta, \bar{\psi}, \psi, B) = \frac{1}{2} \int dt d^3x N \left\{ \frac{1}{2} \eta^A \eta_A + B_I (K^I_I + G^{IK} \partial_K S_{cl} - \eta^I) \right. \left. - \bar{\psi}_I \tilde{M}^I_J \psi_J + \bar{\psi}_I \eta^A \partial_I E^A_J \psi_J \right\}.$$  (3.21)

Since the measure for $\eta^A$ is Gaussian, one can perform the path integral. This is equivalent to plugging the equation of motion,

$$\eta_A = E^A_I B_I - \partial_J E^A_J \bar{\psi}_I \psi_J,$$

into the action. With this,

$$Z(J) = \int \mathcal{D}g \mathcal{D}[\bar{\psi}] \mathcal{D}\psi \mathcal{D}B \exp \left[ -S(g, \bar{\psi}, \psi, B) + \int dt d^3x N J_I g_{II} \right].$$  (3.22)
where
\[
S(g, \bar{\psi}, \psi, B) = \frac{1}{2} \int dt d^3x N \left[ -\frac{1}{2} \left( E_A^I B_I - \partial_J E_A^I \bar{\psi}_J \psi^I \right) \delta^{AB} \right.
\]
\[
\times \left( E_B^K B_K - \partial_\ell E_B^K \bar{\psi}_K \psi^L \right)
\]
\[
+ B_I (K^I + G^{IK} \partial_K S_{\ell}) - \bar{\psi}_I \partial_J (K^I + G^{IK} \partial_K S_{\ell}) \psi^J \right],
\] (3.24)

Rearranging the quadratic term and using \(E_A^I E_B^H \delta^{AB} = G^{IK}\), the bosonic part of the action (in space coordinates) is given by
\[
S_B(g, B) = \frac{1}{2} \int dt d^3x \sqrt{g} \left[ -\frac{1}{2} B_{ij} G_{ijkl} B^{kl} + B_{ij} \left( K_{ij} + \frac{1}{\sqrt{g}} G_{ijkl} \delta \tilde{S}_{\ell} \delta g^{kl} \right) \right],
\] (3.25)

which recovers precisely the HL action in equation (2.7).

It is a common feature of stochastic quantization [14, 15] that if one considers only equal-time correlators for the metric (i.e. correlators on the same leaf of the foliation), the diagrammatic contributions of the fermions are exactly cancelled by the contributions from lines joining the \(B\) field to the metric. In this case, one can limit oneself to the bosonic part alone.

### 3.2. BRST invariance

The key point of our argument is that after adding the fermions, the action \(S(g, \bar{\psi}, \psi, B)\) is invariant under a BRST symmetry which is generated by
\[
\begin{align*}
\delta_\epsilon g^I &= \bar{\psi} I \\
\delta_\epsilon \psi^I &= 0 \\
\delta_\epsilon \bar{\psi}_I &= \bar{\psi} B_I \\
\delta_\epsilon B_I &= 0,
\end{align*}
\] (3.26)

where \(\bar{\psi}\) is a Grassmann variable. In fact, the variation of \(S\) is given by
\[
\delta_\epsilon S = -\delta_\epsilon \left( E_A^I B_I - \partial_J E_A^I \bar{\psi}_J \psi^I \right) \delta^{AB} \left( E_B^K B_K - \partial_\ell E_B^K \bar{\psi}_K \psi^L \right)
\]
\[
+ B_I \delta_\epsilon (K^I + G^{IK} \partial_K S_{\ell}) - \delta_\epsilon \bar{\psi}_I \tilde{M}_{j}^I \psi^J - \bar{\psi}_I \bar{M}_{j}^I \psi^J.
\] (3.27)

We find the following.

- The variation of the quadratic term gives
\[
\delta_\epsilon \left( E_A^I B_I - \partial_J E_A^I \bar{\psi}_J \psi^I \right) = B_I \partial_K E_A^I \bar{\psi} K - \partial_\ell \partial_J E_A^I \psi^J
\]
\[
- \bar{\psi} K \partial_\ell \partial_J E_A^I \bar{\psi}_J \psi^I = 0,
\] (3.28)
since the first two terms cancel each other and the last term vanishes because \(\partial_\ell \partial_J\) is symmetric under an exchange of the indices and \(\psi^K \psi^J\) is antisymmetric.

- The other variations vanish, since
\[
B_I \delta_\epsilon (K^I + G^{IK} \partial_K S_{\ell}) - \delta_\epsilon \bar{\psi}_I \tilde{M}_{j}^I \psi^J = B_I \bar{\psi} \psi^J \bar{M}_{j}^I - \bar{\psi} I \bar{M}_{j}^I \psi^J = 0,
\] (3.29)
and
\[
-\bar{\psi}_I \bar{\psi} K \partial_\ell \bar{M}_{j}^I \psi^J = -\bar{\psi}_I \bar{\psi} K \partial_\ell \bar{\psi} (K^I + G^{IK} \partial_K S_{\ell}) \psi^J = 0,
\] (3.30)
where we used the fact that \(\bar{M}\) is the variation of \(K^I + G^{IK} \partial_K S_{\ell}\).

Note that the BRST symmetry could also be reinterpreted as a supersymmetry in the time direction by introducing a superfield \(G^I = g^I + \partial \psi^I + \theta \psi^I + \theta \bar{\psi} B\). One should nevertheless be careful because of the presence of the \(\nabla_i N_j\) terms (that are analogous to gauge-fixing terms in the stochastic quantization of gauge theories). Moreover, since the ‘classical action’ in equation (2.5) is not bounded from below, supersymmetry would be spontaneously broken.
4. Renormalization properties of HL-type gravity

From the form of the action in equation (3.24), one can read off the dimensions of the auxiliary fields. In detail,

\[ [B^{ij}] = 3, \quad [\psi_{ij}] + [\bar{\psi}^{ij}] = 3, \quad [g_{ij}] = 0. \tag{4.1} \]

This means that after renormalization, the most general form of the action is

\[ S(R)(g, \bar{\psi}, \psi, B) = \int dt \, d^3x \, N \left\{ A_{IJ}(g) B_I B_J + C_{IJ}(g) B_I \bar{\psi}_J \psi^K + D^{IJ}_{KL}(g) \bar{\psi}_I \bar{\psi}_J \psi^K \psi^L 
+ B_I E^{(i)}(g) + \bar{\psi}_I H^{(i)}(g) \psi^J + I(g) \right\}, \tag{4.2} \]

where \( A, C, D \) are tensors of dimension 0 and depend only on the metric \( g_{ij} \), \( F^{(R)} \) and \( H \) have dimension less than or equal to 3 and are functions of the metric and its derivatives, and \( I(g) \) is a function of the metric and its derivatives of dimension less than 6.

4.1. Constraints from BRST

The renormalization group flow preserves the BRST symmetry generated by the relations in equation (3.26). It follows that the effective action \( S(R)(g, \bar{\psi}, \psi, B) \) satisfies the equation

\[ \psi^I \frac{\delta S(R)}{\delta g^I} + B_I \frac{\delta S(R)}{\delta \bar{\psi}_I} = 0. \tag{4.3} \]

The renormalized action thus takes the form

\[ S^{(R)}(g, \bar{\psi}, \psi, B) = \frac{1}{2} \int dt \, d^3x \, N \left\{ -\left( E^{(R)} \right)_{A}^{I} B_I + \partial_L E^{(R)}_{B} B_J \bar{\psi}_J \psi^L \right\} \delta^{AB} 
\times \left( E^{(R)}_{B} B_J + \partial_L E^{(R)}_{B} B_J \bar{\psi}_J \psi^L \right) + B_I E^{(i)}(g) \psi^J \psi^L \right\}, \tag{4.4} \]

Here we recognize the same structure of the stochastically quantized action as in equation (3.24). We can therefore reformulate the problem in terms of a Langevin equation:

\[ F^{(R)}_{ij}(g) = \eta^A E^{(R)}_{A}^{I}, \tag{4.5} \]

where \( \eta^A \) is again a white Gaussian noise.

4.2. Constraints from time reversal

In order to show that the above Langevin equation satisfies a detailed balance condition as the initial one in equation (3.11), we have to make use of another symmetry. Consider the generating functional in equation (3.15),

\[ Z(J) = \int \mathcal{D}_\eta \exp \left\{ -\frac{1}{4} \int dt \, d^3x \, N(\eta_I \eta^I) + \int dt \, d^3x \, N(J_I g^I) \right\}. \tag{4.6} \]

Substituting \( \eta \) in equation (3.15) by using the Langevin equation in equation (3.11), we find that the effective action is given by

\[ S(\eta) = \frac{1}{4} \int dt \, d^3x \, N(\eta_I \eta^I) = \frac{1}{4} \int dt \, d^3x \, N(K^I + G^{ij} \partial_j S_{\Delta})(K_I + \partial_j S_{\Delta}), \tag{4.7} \]

where the fields are thought of as functions of \( \eta \). Expanding the product, one can see that the cross term only gives a boundary contribution:

\[ 2 \int dt \, d^3x \, N K^I \partial_j S_{\Delta} = \int dt \, d^3x \left\{ \nabla_i (g_{ij} - N_j - N_j N_i) \frac{\delta S_{\Delta}}{\delta g_{ij}} \right\} 
= \int dt \, d^3x \left\{ \frac{d}{dt} S_{\Delta}(g) + \partial_i \left( N_j \frac{\delta S_{\Delta}}{\delta g_{ij}} \right) \right\}. \tag{4.8} \]
where we used the fact that $S_{cl}(g)$ preserves the diffeomorphisms of each leaf of the foliation. Expanding $K_{ij}$ and observing that the cross term in $K^I K_I$ is again a total derivative, we thus see that the unrenormalized action is *invariant under time reversal* and this property must be preserved under the RG flow.

Starting from the Langevin equation in equation (4.5), one can similarly write down the generating functional

$$Z(R)(J) = \int D\eta \exp \left[ -S(R)(\eta) + \int dt d^3x Ng^I J_I \right]$$

Knowing that in the initial theory $F^I = K^I + G^{IJ} \partial_J S_{cl}$, up to a rescaling of the noise, $F(R)^I (g)$ can be rewritten as

$$F(R)^I (g) = K^I + G^{IJ} \partial_J S_{cl}(g).$$

It follows that the effective action reads as

$$S(R)(\eta) = \frac{1}{4} \int dt d^3x N (K^I K_I + \Xi^I \Xi_I + 2K^I \Xi_I).$$

In order to preserve the time-reversal symmetry, $K^I \Xi_I$ must be a total derivative, which implies

$$\Xi_I = \partial_I S_{cl}(g),$$

where $S_{cl}(g)$ must preserve Lorentz invariance on the leaves. Summarizing, we find that after renormalization, the constraints of BRST symmetry and time reversal imply that the time dynamics is still described by a Langevin equation of the same form as in equation (3.11):

$$K_{ij} = -G_{ijkl} \frac{\delta S_{cl}(g)}{\delta g_{kl}} + \eta_{ab} F^{(R)^{ab}}_{ij}$$

or, equivalently, that the detailed balance structure of the action is preserved under the RG flow.

In other words, the renormalization properties of the theory in four dimensions are completely fixed by those of the ‘classical’ theory in three dimensions—in this case topologically massive gravity.

5. Conclusions

Hořava–Lifshitz gravity is a theory of gravity constructed to be renormalizable by power counting, even though at the price of sacrificing Lorentz invariance at short distances. Such a model is clearly relevant for anyone interested in the questions of quantum gravity and has thus generated a large echo. We believe that apart from phenomenological considerations which so far have been largely of a classical nature, the fundamental questions concerning the properties of the quantum theory need to be addressed in order to exclude issues of consistency.

In this paper, we have studied one such question, the one concerning the renormalization properties of HL gravity beyond power-counting arguments. In fact, our results confirm its renormalizability under certain conditions. We make use of the fact that (super) HL gravity can be taken to be the stochastic quantization of topologically massive gravity. Our argument relies on the renormalizability of the latter, which, even though not strictly proven, is thought to hold [20].
Some theories with additional terms breaking detailed balance were suggested to improve the phenomenology of HL gravity. Our derivation relies partly on detailed balance and cannot be applied directly to such models. We use the manifestly BRST invariant formalism of stochastic quantization, which is powerful enough to preserve the form of the action under the renormalization group flow.

The properties of HL gravity and its implications are still far from being completely understood and this field presents a large venue for investigation. Many fundamental questions remain to be answered.

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References

[1] Hořava P 2009 Quantum gravity at a Lifshitz point Phys. Rev. D 79 084008 (arXiv:0901.3775)
[2] Ardonne E, Fendley P and Fradkin E 2004 Topological order and conformal quantum critical points Ann. Phys. 310 493–551 (arXiv:cond-mat/0311466)
[3] H Lu, Mei J and Pope C N 2009 Solutions to Horava gravity arXiv:0904.1595
[4] Colgain E O and Yavartanoo H 2009 Dyonic solution of Horava–Lifshitz gravity arXiv:0904.4357
[5] Horava P 2009 Spectral dimension of the universe in quantum gravity at a Lifshitz point arXiv:0902.3657
[6] Calcagni G 2009 Cosmology of the Lifshitz universe arXiv:0904.0829
[7] Kiritis E and Kofinas G 2009 Horava–Lifshitz cosmology arXiv:0904.1334
[8] Mukohyama S 2009 Scale-invariant cosmological perturbations from Horava–Lifshitz gravity without inflation arXiv:0904.2190
[9] Brandenberger R 2009 Matter bounce in Horava–Lifshitz cosmology arXiv:0904.2835
[10] Nastase H 2009 On IR solutions in Horava gravity theories arXiv:0904.3604
[11] Cai R-G, Cao L-M and Ohta N 2009 Topological black holes in Horava–Lifshitz gravity arXiv:0904.3670
[12] Takahashi T and Soda J 2009 Chiral primordial gravitational waves from a Lifshitz point arXiv:0904.0554
[13] Parisi G and Y-S Wu 1981 Perturbation theory without gauge fixing Sci. Sin. 24 483
[14] Damgaard P H and Hüffel H 1987 Stochastic quantization Phys. Rep. 152 227–398
[15] Namiki M 1992 Stochastic Quantization (Lecture Notes in Physics) (Berlin: Springer)
[16] Dijkgraaf R, Orlando D and Reffert S 2009 Relating field theories via stochastic quantization arXiv:0903.0732
[17] Parisi G and Sourlas N 1982 Supersymmetric field-theories and stochastic differential-equations Nucl. Phys. B 206 321–32
[18] Horava P 2008 Quantum criticality and Yang–Mills gauge theory arXiv:0811.2217
[19] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories Ann. Phys. 140 372–411
[20] Deser S and Yang Z 1990 Is topologically massive gravity renormalizable? Class. Quantum Grav. 7 1603–12
[21] Zinn-Justin J 1986 Renormalization and stochastic quantization Nucl. Phys. B 275 135
[22] Rumpf H 1986 Stochastic quantization of Einstein gravity Phys. Rev. D 33 942–52
[23] DeWitt B S 1979 Quantum gravity: the new synthesis General Relativity ed S Hawking and W Israel (Cambridge: Cambridge University Press) pp 680–745