A primal-dual approach for a total variation Wasserstein flow

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Abstract

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A primal-dual approach for a TV-Wasserstein flow

GSI 2013, Paris, August 2013

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2 Primal-dual formulation of the problem
A relaxed optimality system of PDEs
The numerical approach
3 Numerical results
The 1-D case
The 2-D case with applications to denoising
4 Conclusions

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A highly nonlinear fourth-order PDE
For a regular domain \( \Omega \subset \mathbb{R}^d \), \( d = 1, 2 \) we consider: The problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \cdot (u \frac{\partial}{\partial x_1}), & q \in i|Du|, \\
\text{in } \Omega \times (0, T), & u(0, x) = u_0(x) \geq 0 \text{ in } \Omega \\
\text{where } \Omega u_0 dx &= 1 \text{ and the total variation of } u \text{ over } \Omega \text{ is defined as: } |Du| = \sup_{p \in C^0_0(\Omega; \mathbb{R}^d)} |u \cdot p dx|.
\end{align*}
\]

Subgradients of TV can be characterised such that: \( q \in i|Du| \Rightarrow q = - \cdot u | u| \text{ if } |u| = 0 \), which makes the problem above a nonlinear fourth-order PDE with severe restrictions.
and constraints for its numerical solution. . . Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 3 / 24

An L2-Wasserstein flow for density smoothing. An equivalent problem has been investigated by Burger, Franek, Schönlieb (2012). Therein, a smoothed version \( u \) of a given probability density \( u_0 \) was computed as a minimiser of: \( 1/2 W^2(u_0 L^d , u L^d) + \alpha E(u) \) smoothing term for different choices of \( E(u) \) (Dirichlet energy, Log-entropy, Fisher information, Total Variation...), e.g. \( u_0 \) could be a noisy MRI image or represent some real-world data (earthquakes or fires measurements). Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 4 / 24, An L2-Wasserstein flow for density smoothing. An equivalent problem has been investigated by Burger, Franek, Schönlieb (2012). Therein, a smoothed version \( u \) of a given probability density \( u_0 \) was computed as a minimiser of: \( 1/2 W^2(u_0 L^d , u L^d) + \alpha E(u) \) L2-Wasserstein distance for different choices of \( E(u) \) (Dirichlet energy, Log-entropy, Fisher information, Total Variation...), e.g. \( u_0 \) could be a noisy MRI image or represent some real-world data (earthquakes or fires measurements). Previous work in imaging by means of Wasserstein distance: S. Haker, L. Zhu and A. Tannenbaum (2004) for image registration; G. Peyr’è et al. (2013) for image color transfer; X. Bresson, T. Chan et al. (2009) for image segmentation; L. P. S. Demers et al. (2010) for particle image velocimetry. . . Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 4 / 24 The L2-Wasserstein metric \( (\Omega, d) \) is a metric space. The L2-Wasserstein distance between two probability measures \( \mu_1 , \mu_2 \in P^2(\Omega) \) (the space of all probability measures on \( \Omega \) with \( \mu \)-integrable second moment) is defined by \( W^2(\mu_1 , \mu_2)^2 := \min \Gamma(\mu_1 , \mu_2) \subset \Omega \times \Omega \) \( d(x, y)^2 \Gamma(\mu, \gamma) \). Here \( \Gamma(\mu_1 , \mu_2) \) denotes the space of pairings \( \gamma \in P(\Omega \times \Omega) \) such that: \( \mu_1 \) is the first marginal of \( \gamma \) and \( \mu_2 \) is the second marginal of \( \gamma \). The definition can be extended to \( (p, h) \)-Wasserstein distances. . . Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 5 / 24

Why TV-Wasserstein? Compared to smoother regularisers: Capability of preserving discontinuities and structures when regularising densities (Rudin, Osher, Fatemi ’92). Interest in Image Processing: discontinuities are the edges of the image = characteristic features in many imaging applications (bone density and brain images. . . ). Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 6 / 24

Previous results and our goal In their work Burger, Franek, Schönlieb have shown: Existence results (by standard technique in Calculus of Variations); Self-similarity properties of the solutions; Numerical results: augmented Lagrangian schemes solving the minimisation problem (for a fixed \( \alpha \), this means computing one timestep of the minimising movement scheme). Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 6 / 24

The minimisation problem and our PDE

\[ \text{The problem: } \frac{1}{2} W^2(u_0 L^d , u L^d) + \alpha E(u) \text{ has to be interpreted as a time discrete approximation of a solution of the gradient flow of } E \text{ with respect to the L2-Wasserstein metric: it represents one timestep of De Giorgi’s minimising movement scheme. Benning, Calatroni, D’urin, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 7 / 24} \]

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Outline

1 The problem
2 Primal-dual formulation of the problem
3 Numerical results

1 The problem

The relaxed problem

Merging the original equation

The relaxed equation we consider is: \( \frac{\partial u}{\partial t} = \nabla \cdot (u \nabla f) \), where \( f \) is a relaxed convex function. The relaxed function \( f \) is defined as:

\[
\tilde{f}(p) = \begin{cases} 
1 & \text{if } p < 1 \\
\frac{1}{1 - p} & \text{if } p > 1 \\
\frac{1}{2} \left| p \right| & \text{if } p = 1 
\end{cases}
\]

where \( \tilde{f}(p) \) is the relaxed version of the function \( f(p) \).

The relaxed problem is:

\[
\min_{u \in \mathbb{L}^2(\Omega)} \int_{\Omega} \left( \frac{1}{2} \left| \nabla u \right|^2 + f(u) \right) \, dx
\]

subject to:

\[
\int_{\Omega} u \, q \, dx = 0, \quad q \in \mathbb{L}^2(\Omega)
\]

with \( f(u) \) being a relaxed convex function.

1.1 The primal-dual approach

The primal-dual approach is a numerical method for solving the relaxed problem. It consists of:

- A primal-dual formulation of the problem
- A numerical approach

1.2 Numerical results

We present numerical results for the 1-D and 2-D cases, as well as applications to denoising.

1.3 Conclusions

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GSI 2013, Paris, August 2013 10 / 24 An alternative approach

The original equation we consider is: \( \frac{\partial u}{\partial t} = \nabla \cdot (u \nabla f) \) Benning, Calatroni, Düring, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow

GSI 2013, Paris, August 2013 11 / 24 The relaxed equation we consider is: \( \frac{\partial u}{\partial t} = \nabla \cdot (u \nabla f) \) Benning, Calatroni, Düring, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow

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GSI 2013, Paris, August 2013 13 / 24 A damped Newton method to solve the system

We discretise the differential operators and compute the numerical approximation of the solution \( u \) using the following scheme:

- Inner process (\( k \) superscripts) producing approximations of \( u \), \( v \) and \( \nabla f \)
- Outer iterations (\( n \) subscripts) for the time loop

1.4 Numerical results

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Some 1-D examples

We compare the TV-Wasserstein approach with the standard TV one:

- TV-Wasserstein comparison
- Initial condition TV result
- TV-Wasserstein result

(a) Gaussian in. cond. \(-0.5 \text{ to } 0.5\)

(b) \( \chi[a,b] \) in. cond. \(-0.5 \text{ to } 0.5\)

(c) Stair in. cond. Figure: Solutions for TV and TV-Wasserstein flows.
efficiently by using a nested damped Newton method that computes the numerical approximation of the solution in each time iteration; The results preserve smoothness of the higher-order TV-subgradients and relaxing via a penalty term leads to a system of nonlinear PDEs; The numerical solution is computed efficiently by using a nested damped Newton method that computes the numerical approximation of the solution in each time iteration; The results preserve the mass-conservation property and show good results in density smoothing (e.g. denoising in imaging), reducing artifacts compared to lower-order models; Q1 Rigorous analysis of the scheme? Barrier term? Stability properties? Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 18 / 24 Some 1-D examples We compare the TV-Wasserstein approach with the standard TV one: -0.5 0 0.5 1 1.5 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 x y TV−Wasserstein solution (c) Stair in. cond. Figure: Solutions for TV and TV-Wasserstein flows. \( \varepsilon = 10^{-5} \), \( t_0 = 1 \). Features: Similar with TV: Preservation of structure (i.e. discontinuities); Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 18 / 24 Some 1-D examples We compare the TV-Wasserstein approach with the standard TV one: -0.5 0 0.5 1 1.5 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 x y TV−Wasserstein solution (c) Stair in. cond. Figure: Solutions for TV and TV-Wasserstein flows. \( \varepsilon = 10^{-5} \), \( t_0 = 1 \). Features: Similar with TV: Preservation of structure (i.e. discontinuities); Different with TV: Decreasing of intensity -- enlarging of the support (because of the mass conservation); * Constant background = TV solutions: convergence to their mean. Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 18 / 24 Outline 1 The problem 2 Primal-dual formulation of the problem A relaxed optimality system of PDEs The numerical approach 3 Numerical results The 1-D case The 2-D case with applications to denoising 4 Conclusions Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 19 / 24 2-D results Solution of the TV-Wasserstein flow: (a) Initial condition. (b) TV result. Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 20 / 24 2-D results: applications to denoising Solution of the TV-Wasserstein flow: (a) Original pyramid. (b) Noisy pyramid. (c) TV-Wasserstein. Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 21 / 24 2-D results: applications to denoising (cont.) Solution of the TV-Wasserstein flow for real-world images: (a) Noisy LEGO. (b) TV result. (c) TV-Wasserstein result. Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 22 / 24 2-D results: applications to denoising (cont.) Solution of the TV-Wasserstein flow for real-world images: (a) Noisy LEGO. (b) TV result. (c) TV-Wasserstein result. Applications in MRI: the images of interest are densities restored from undersampled measurements and/or corrupted by noise or blur:. Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 23 / 24 Recap and future directions Tackling directly the non-smoothness of the higher-order TV-subgradients and relaxing via a penalty term leads to a system of nonlinear PDEs; The numerical solution is computed efficiently by using a nested damped Newton method that computes the numerical approximation of the solution in each time iteration; The results preserve the mass-conservation property and show good results in density smoothing (e.g. denoising in imaging), reducing artifacts compared to lower-order models; Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 24 / 24 Recap and future directions Tackling directly the non-smoothness of the higher-order TV-subgradients and relaxing via a penalty term leads to a system of nonlinear PDEs; The numerical solution is computed efficiently by using a nested damped Newton method that computes the numerical approximation of the solution in each time iteration; The results preserve the mass-conservation property and show good results in density smoothing (e.g. denoising in imaging), reducing artifacts compared to lower-order models; Q1 Rigorous analysis of the scheme? Barrier term? Stability properties? Q2 From the analysis of the 1-D case, more insights on the theory underlying the TV-Wasserstein gradient flow (joint work with M. Burger, D. Matthies). Benning, Calatroni, D‘uring, Sch‘onlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 24 / 24 Recap and future directions Tackling directly the non-smoothness of the higher-order TV-subgradients and relaxing via a penalty term leads to a system of nonlinear PDEs; The numerical solution is computed efficiently by using a nested damped Newton method that computes the numerical approximation of the solution in each time iteration; The results preserve the mass-conservation property and show good results in density smoothing (e.g. denoising in imaging), reducing artifacts compared to lower-order models; Q1 Rigorous analysis of the scheme? Barrier term? Stability properties?
the mass-conservation property and show good results in density smoothing (e.g. denoising in imaging), reducing artifacts compared to lower-order models; Q1 Rigorous analysis of the scheme? Barrier term? Stability properties? Q2 From the analysis of the 1-D case, more insights on the theory underlying the TV-Wasserstein gradient flow (joint work with M. Burger, D. Matthes). Thanks for listening! e-mail: l.calatroni@maths.cam.ac.uk Benning, Calatroni, D’uring, Schönlieb (CCA) A primal-dual approach for a TV-Wasserstein flow GSI 2013, Paris, August 2013 24 / 24