Anomalous Elasticity in Frictional Matter

Chandana Mondal, Michael Moshe, Itamar Procaccia*, Saikat Roy, Jin Shang, and Jie Zhang

1Dept. of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
2Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem, Israel 9190
3Dept. of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
4Department of Chemical Engineering, Indian Institute of Technology Ropar, Punjab 140001, India
5School of Physics and Astronomy, Shanghai Jiao Tong University, 200240 Shanghai, China
6Institute of Natural Sciences and School of Physics and Astronomy, Shanghai Jiao Tong University, 200240 Shanghai, China

In a recent paper [Phys. Rev. E 104, 024904] it was shown that mechanical strains in amorphous solids are screened via the formation of plastic events that are typically quadrupolar in nature. At low densities the screening effect is reminiscent of the role of dipoles in dielectrics, while the effect at high density has no immediate electrostatic analog, and is expected to change qualitatively the mechanical response, as seen for example in the displacement field. In this Letter we show that high-density screening results in undulating displacement field that strictly deviate from elasticity theory. We show that theoretical analysis, experimental measurements and numeric simulations of frictional granular amorphous assemblies are in agreement with each other and provide a strong support for the theory.

Introduction: In elastic materials the shear and bulk moduli serve to relate the mechanical response to external strains [1]. The displacement field associated with small strains can be computed using linear elasticity theory which is a well studied subject for many decades [2]. In perfect elastic materials one can also consider nonlinear responses, using nonlinear moduli to relate large deformations to higher order stress responses [3]. The applicability of this approach to amorphous solids had been however questioned. Firstly, it was shown that in amorphous solids plastic responses appear instantaneously for any amount of strain [4]. Secondly, it was found that nonlinear elastic constants are not well defined in amorphous solids, having unbounded sample to sample fluctuations in the thermodynamic limit [5]. Accordingly, it becomes necessary to re-examine the applicability of elasticity theory to amorphous solids.

Such a re-examination was presented in a recent publication [6]. It was found that plastic responses, which typically appear as quadrupolar (Eshelby-like) irreversible responses [4, 5], are very important in determining the mechanical responses of amorphous solids. When the density of these quadrupoles is low, they act only to renormalize the elastic moduli, but they do not change the form of the theory. Linear elasticity theory can still be used to predict, for example, displacement fields that result from small strains. This is analogous to the role of dipoles in dielectrics, where the dielectric constant is dressed, but the structure of electrostatic theory remains intact. On the other hand, when the density of quadrupoles is high, the gradients of their density cannot be ignored, and these are acting as dipoles. Dipole-dipole and dipole-displacement interaction become crucial, and these change the structure of the theory and the resulting mechanical responses. Details of this theory can be found in Ref. [4], below we will briefly review the results.

An entirely open question is what to expect in frictional amorphous granular matter. It had been recently demonstrated that there are anomalous effects in frictional matter, cf. Ref. [9]. In this Letter we focus on the screening effects due to plasticity in amorphous assemblies of frictional disks. We will show that plastic screening effects can be crucial, determining measured displacement fields that are not in accord with standard elasticity theory, but rather with the proposed theory of Ref. [6]. For concreteness we will study below assemblies of binary frictional disks of two different sizes (to achieve an amorphous material), bounded by a circular wall. In both experiments and simulations we will equilibrate the systems at a chosen target pressure, and then inflate one disk at the center of coordinates to study the ensuing displacement field. To this aim we present first the theoretical expectations for the displacement field in 2-dimensional radial geometry.

Brief review of the theory: In linear elasticity the equation satisfied by the displacement field reads

$$\Delta d + \lambda \nabla (\nabla \cdot d) = 0 , \quad \lambda \equiv \frac{1 + \nu}{1 - \nu} ,$$

(1)

with the appropriate boundary conditions, where $\nu$ is the 2-dimensional Poisson ratio. For our experiment and simulations we consider an annulus of radii $r_{in}$ and $r_{out}$, $r_{in} \ll r_{out}$, with an imposed displacement $d(r_{in}) = d_0 \hat{r}$ and $d(r_{out}) = 0$. The polar symmetry of the problem implies that $d(r) = d_r(r) \hat{r}$, in which case the equilibrium equation reduces to

$$\Delta d = 0 .$$

(2)

The solution to this differential equation that satisfies the boundary conditions is

$$d_r(r) = d_0 \frac{r^2 - r_{out}^2}{r_{in}^2 - r_{out}^2} \frac{r_{in}}{r} .$$

(3)
With the imposed displacement pushing outward, the solution \( \Delta d + \lambda \nabla (\nabla \cdot d) = -\mu d \) is always positive, and it decays like \( 1/r \), as is expected in standard elasticity theory. It was shown [6] that this form of the displacement field is expected to remain valid also when there exists a low density of plastic events, although the coefficients may get renormalized. We then refer to this situation as “quasi-elastic”.

On the other hand, when the density of plastic events becomes large and the gradients of their density cannot be neglected, Eq. (1) changes and assumes the form [6]:

\[
\Delta d + \lambda \nabla (\nabla \cdot d) = -\mu d
\]

The parameter \( \mu \) is related with the nucleation energy of a dipole and has the dimensions of length\(^{-2}\). The screening effect is negligible when \( \mu L^2 \ll 1 \) where \( L \) is the system size. Unlike the quadrupole screening, dipole screening leads to a qualitatively new behavior. In polar coordinates the equation for the displacement field assumes the form of Bessel equation

\[
d'' + \frac{1}{r}d' + (\kappa^2 - \frac{1}{r^2})d = 0
\]

with \( \kappa = \mu/(1+\lambda) \). A solution of this equation satisfying \( d_r(r_{in}) = d_0, \ d_r(r_{out}) = 0 \) reads

\[
d_r(r) = d_0 \frac{Y_1(\kappa r) J_1(\kappa r_{out}) - J_1(\kappa r) Y_1(\kappa r_{out})}{Y_1(\kappa r_{in}) J_1(\kappa r_{in}) - J_1(\kappa r_{in}) Y_1(\kappa r_{out})}.
\]

Here \( J_1 \) and \( Y_1 \) are the Bessel functions of the first and second kind respectively. It should be stressed that at this point we do not have a-priori theory for the numerical values of \( \kappa \), and in comparisons to experiments and simulations we need to fit this parameter. In the analysis of the experimental results below the values of the other parameters, namely \( r_{in} \) and \( r_{out} \) are not fitted; they are directly taken from the data of the experiments.

The set up of experiments and simulations: The experimental apparatus consists of a circular frame with a radius of 350 mm and height of 8 mm, that is stacked on a horizontal glass plate, which is shown as the blue circle in Fig. 1 panel (a). We have performed two sets of experiments using bi-disperse disks of two different stiffness constants, in order to produce two-dimensional amorphous packings at low and high dimensionless pressure values.

In the low dimensionless pressure experiment, we use bi-disperse acrylonitrile butadiene styrene (ABS) disks with a 1:1 number ratio, to prepare the amorphous packing. Firstly, we pad the outer circumference with a single layer of bi-disperse photo-elastic disks to act as pressure sensors, shown as the green layer in Fig. 1. Secondly, we fill up the rest of the circular frame with the ABS disk. The radii of the large and small disks are 7.0 mm and 5.5 mm, respectively, while the radii of the large and small photo-elastic disks are 7.0 mm and 5.0 mm, respectively. The area fraction of the whole system is 0.836 before the inflation at the center of the system.

For the high dimensionless pressure experiment, we fill the rest of the circular frame with the same bi-disperse photo-elastic disks as the ones at the circumference. The number ratio of small to large disks is 2:1, and the area fraction is 0.835 before the inflation. In the experiments, we apply photo-elastic techniques to measure the forces at the circumference and then use the normal force components of the contact force between boundary and the inside disks to obtain the values of pressure. When all the disks are photo-elastic, we can also measure pressure from the bulk since the pressure is the same in the bulk and at the boundary.

A conically shaped pusher (cf. Fig. 1 panel (b)) is placed in the center of the system to apply the inflation there, as shown by the black dot in Fig. 1 panel (a). The geometry and parameters are specified in Fig. 1 panel (b). Initially, this conically shaped pusher can contact the neighboring disks only by its skinny lower part. By pressing this pusher down smoothly, its broad cap starts to contact its surrounding disks, producing an effective inflation of the center disk.

We produce an initial amorphous configuration of a desired pressure by gradually adjusting the packing fraction, while at the same time we apply random tapping to eliminate any potential stress and local inhomogeneity of the amorphous assembly. The images of disks configurations are captured by a 2×2 array of high-resolution cameras above the system, whose four images can be stitched together through the calibration, using a checker board to achieve a spatial resolution of the disk position up to 10 pixel/mm. We can identify the position of the disks before and after the inflation at the center, to measure the displacement field. Specifically we report below how the radial component of the displacement field, averaged over a circle of radius \( r \), decays with the distance \( r \) from the inflating screw.

Since the ABS disks have a bulk modulus of \( B = 2.2 \) GPa and thickness of \( h_0 \sim 1 \) cm, it gives a microscopic pressure per disk of \( P_{mic} = Bh_0 = 2.2 \times 10^7 \) N/m. This gives, for a typical value of the desired pressure \( P \sim 10 \) N/m, a dimensionless pressure \( \tilde{P} = P/P_{mic} \sim 10^{-6} \) in the amorphous packings of ABS disks. Similarly, the photo-elastic disks have a bulk modulus of \( B = 4 \) MPa and thickness of \( h_0 = 6 \) mm, which gives a microscopic pressure per disk of \( P_{mic} = Bh_0 = 24 \times 10^3 \) N/m. This gives a dimensionless pressure \( \tilde{P} = P/P_{mic} \sim 10^{-3} \) in the amorphous packings of photo-elastic disks.

The simulations employ \( N = 16000 \) disks of two radii, half with a radius \( R_1 = 0.35 \) and the other half with a radius \( R_2 = 0.49 \). All the simulation units are quoted in SI units. The forces between the disk act only upon contact. The contact forces, which include both normal and tangential components due to friction, are modeled according to the discrete element method developed by Cundall and Strack [10], combining a Hertzian normal force and a tangential Mindlin component. The exact form of these forces is presented for example in the section “Materials and Methods” in Ref. [11]. The stiffness
constant in the Hertzian force law is \( k_n = 2 \times 10^6 \text{ N/m} \). Open source codes, LAMMPS [12] and LIGGGHTS [13] are used to perform the simulations. Initially the disks are placed in a circular box of fixed boundaries with an initial area fraction \( \phi = 0.45 \). Next, isotropic compression is implemented in a step-wise fashion by inflating each disk by a small factor (1.00004) followed by subsequent relaxation until the forces and torques on each disk are smaller than \( 10^{-7} \) in SI units. This process is continued until mechanically stable configurations are generated at a desired pressure \( \tilde{P} \). The dimensionless pressure \( \tilde{P} \) is obtained by dividing by the stiffness constant, \( \tilde{P} = P/k_n \).

After achieving a mechanically stable configuration at a target pressure, we choose the disk with larger diameter that is closest to the center of the simulation box and inflate it by a desired amount, and let the system evolve until equilibrium is reached. Once we reach the equilibrium state, we examine the displacement field that is induced by this inflation. Specifically we study how the radial component of the displacement field, averaged over a circle of radius \( r \), decays with the distance \( r \) from the inflated disk.

**Results:** As stated, at large pressures we observe a paucity of plastic responses, and accordingly the radial component of the displacement field is expected to follow normal elasticity theory up to a renormalization of the elastic moduli. This expectation is realized in both experiments and simulations as can be seen in Fig. 2. The observed radial displacement field is in close agreement with Eq. (3). In panels (a) and (b) we see typical simulation results, and in panels (c) and (d) we present experimental results. As expected in standard elasticity theory, the radial displacement is positive and monotonically decreasing over the whole range, and it decays at large distances like \( 1/r \).

On the other hand, at small pressures the density of
plastic responses increases sharply, and we expect that Eq. (6) will provide a good fit to the observed radial displacement field. In Fig. 3 we present simulational and experimental results including their fit to Eq. (6). Panels (a) and (b) refer to simulations at dimensionless pressure \( \tilde{P} = 2 \times 10^{-7} \) with an inflation of 80% of the central disk. The displacement field now becomes negative before it returns to zero at the outer boundary. There is only one fit parameter, i.e., \( \kappa \) in Eq. (6). In the simulations \( \kappa = 0.0525 \) and 0.0706 respectively in panels (a) and (b). The inner and outer radii \( r_{\text{in}} \) and \( r_{\text{out}} \) are not fitted, by rather used as provided in the simulations. Panels (c) and (d) refer to the experiment, with a very similar portion of negative radial displacement that cannot be assigned to the standard elastic solution. The values of \( r_{\text{in}} \) and \( r_{\text{out}} \) are again taken directly from the experimental data, and the values of \( \kappa \) are 1.4 \( \times 10^{-3} \) and 1.76 \( \times 10^{-3} \) in panels (c) and (d) respectively.

**Discussion:** It is quite obvious that the radial displacement fields sown in Fig. 3 cannot be possibly assigned to standard linear elasticity theory. The appearance of a range of distances in which the displacement is negative, is directly related to plastic responses that reduce the pressure in the bulk, resulting in disks displaced inwards, even though the inflation at the origin points outwards. The fit of the data to the theoretical prediction Eq. (6) is quite remarkable, especially since there is only one available fit parameter \( \kappa \). We note that the theoretical result of Eq. (6) predicts that for larger values of the screening parameter \( \kappa \), spatial oscillations will form in the displacement field. A hint for that is observed in the numerical simulations as shown in panels (a) and (b) of Fig. 3, where the displacement field has a short range of positive displacement close to the outer boundary, which decays to zero from above. It should be stated however that the fits in these figures are not perfect, and whether this is an indication of the failure of the continuum theory on scales of the disks, or whether this is due to the a spatial dependence of \( \kappa \), is still unknown, and further research is required. It is interesting to note that in the experimental results panels (c) and (d) this question does not arise, the displacement field is reaching the zero boundary condition from below, in a rather good agreement with the solution Eq. (6).

The change from quasi-elastic behavior at high pressure, cf. Fig. 2, to anomalous behavior at low pressure, cf. Fig. 3, is an indication of a possible phase transition at some intermediate value of the pressure, separating the two types of behavior. At present our systems, both in simulations and experiments, are too small to support a sharp phase transition. We find realizations at low pressure that show quasi-elastic behavior, and vice versa, realizations at high pressure that show displacement fields that cannot be fitted to the quasi-elastic solution. We thus defer the study of the possible phase transition to future work where much more data and other types of amorphous solids will be considered. It is noteworthy however that the type of transition alluded to, if it exists, is very reminiscent of the hexatic phase transition in 2-dimensional melting \([14][15]\), where a screening by dipoles replaces low-density quadrupoles. This analogy will be examined more closely in future publications.

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