We consider two-qubit undergoing local dissipation and subject to local driving. We then determine the optimal Markovian feedback action to preserve initial entanglement as well as to create stationary entanglement with the help of an $XY$ interaction Hamiltonian. Such feedback actions are worked out in a way not depending on the initial two-qubit state, whence called universal.

I. INTRODUCTION

Quantum entanglement has been recognized in recent decades as a resource for quantum information processing [1]. As such it should be controllable. Several efforts have been devoted to control entanglement [2]. Control can takes place with open-loop or and closed-loop strategies according to the principle of controllers design [3]. Quite generally closed loop control performs better than open loop control because it involves gathering information about the system state and then according to that actuate a corrective action on its dynamics, but results more difficult to implement [4]. A good compromise between these two tensions is probably represented by Markovian feedback [5], who brings the advantages of closed loop control but is not much difficult to realize. In fact it rests on an actuation based on the measurement result obtained immediately before, hence the name Markovian. Nevertheless it carries an inherent double optimization, over the measurement and over the actuation [6]. This makes designing optimal control a daunting task even for Markovian feedback, especially when dealing with composite systems and hence with entanglement control (we refer here to local control, i.e. measurement and actuation are both local operations). The best results (in terms of optimality) have been achieved in the context of Gaussian systems [7]. For qubit systems, due to their inherent nonlinearity, the situation is more complicate. With two qubit, on the one hand, a proof of principle of the effectiveness of Markovian feedback in stabilizing entanglement was given in [8], but it is not optimal. On the other hand, the effectiveness of Markovian feedback in protecting initial entangled states has shown in [9]. This action though optimal was derived in a way depending on the initial state.

Here we generalize these results by determining the optimal Markovian feedback action to preserve initial entanglement as well as to create stationary entanglement with the help of an $XY$ interaction Hamiltonian. Moreover, such feedback actions are worked out in a way not depending on the initial two-qubit state, whence referred to as universal.

The layout of the paper is as follows. We start by introducing the model feedback action in Sec.II. Then we address the issue of preserving initial entanglement in Sec. III and subsequently the issue of stabilizing entanglement in Sec. IV. Finally, Sec. V is for conclusion. Throughout the paper we will use $i$ to denote the imaginary unit.

II. THE MODEL

Consider a two-qubit system whose dynamics is governed by the following master equation

$$\dot{\rho} = -i [H, \rho] + \mathcal{D} [\sigma_1] \rho + \mathcal{D} [\sigma_2] \rho,$$  \hspace{1cm} (1)

where $H$ denotes the Hamiltonian and $\sigma_i, \sigma_i^\dagger$ ($i = 1, 2$) are the lowering, raising Pauli operator. Furthermore

$$\mathcal{D} [c] \rho \equiv c \rho c^\dagger - \frac{1}{2} c^\dagger c \rho - \frac{1}{2} \rho c^\dagger c,$$ \hspace{1cm} (2)

is the dissipative super operator and the way it appears in Eq. (1) shows qubit dissipation into local environments.
Following the reasoning of Ref. [5] and generalizing it, we may think at Eq. (1) as coming from averaging selective evolutions under local measurements with probability operator value elements

\[
\Omega_1(dt) = \sqrt{dt} \sigma_1, \\
\Omega_2(dt) = \sqrt{dt} \sigma_2, \\
\Omega_3(dt) = 1 - \left( iH + \frac{1}{2} \sigma_1^\dagger \sigma_1 + \sigma_2^\dagger \sigma_2 \right) dt,
\]

(3a) (3b) (3c)

describing detection (jump) on the first (resp. second) environment \( \Omega_1 \) (resp. \( \Omega_2 \)) and no detection \( \Omega_3 \). The measurement time is the infinitesimal \( dt \) as it is appropriate for continuous measurement. It is then easy to verify that the non selective evolution under this measurement

\[
\rho(t + dt) = \frac{3}{j=1} \Omega_j(dt) \rho(t) \Omega_j^\dagger(dt),
\]

(4)
is equivalent to the master equation (1).

The selective evolution allows us to incorporate the feedback action. This one, in order to be Markovian, must cause an immediate state change based only on the result of the measurement in the preceding infinitesimal time interval. Hence it must occur immediately after a detection and cause a finite amount of evolution. Let this finite evolution following a detection on qubit \( i \) at time \( t \) be as

\[
\tilde{\rho}_i(t + dt) = e^{K_i \sigma_i \rho(t) \sigma_i^\dagger dt},
\]

(5)

where \( K_i \)'s are Liouville super operators (tilde means that the density operator in unnormalized). Form Eq. (5) it is clear that the feedback action is local.

The nonselective evolution of the system is then given by

\[
\rho(t + dt) \propto \tilde{\rho}_1(t + dt) + \tilde{\rho}_2(t + dt) + \tilde{\rho}_3(t + dt),
\]

(6)

where \( \tilde{\rho}_3(t + dt) = \Omega_3(dt) \rho(t) \Omega_3^\dagger(dt) \). Since the latter is unchanged by feedback, we get for the normalized density operator

\[
\dot{\rho} = -i[H, \rho] + \sum_{i=1}^2 \left[ e^{K_i \sigma_i \rho} \sigma_i^\dagger - \frac{1}{2} \sigma_i^\dagger \sigma_i \rho - \frac{1}{2} \rho \sigma_i^\dagger \sigma_i \right].
\]

(7)

Assuming that \( K_i \)'s acting in a Hamiltonian way (so to avoid introducing further noise)

\[
K_i \rho = -i[F_i, \rho],
\]

(8)

we will further get

\[
\dot{\rho} = -i[H, \rho] + D \left[ e^{-iF_1 \sigma_1} \right] \rho + D \left[ e^{-iF_2 \sigma_2} \right] \rho,
\]

(9)

where \( F_1, F_2 \) are hermitian operators on \( \mathbb{C}^2 \) to be determined. They play the role of (local) feedback Hamiltonians and they concour to implement unitary local actuations \( e^{-iF_i} \). Hence these latter can be parameterized as follows:

\[
e^{-iF_i} = \begin{pmatrix}
  e^{-i(\alpha_i + \gamma_i)/2} \cos(\beta_i/2) & -e^{-i(\alpha_i - \gamma_i)/2} \sin(\beta_i/2) \\
  e^{i(\alpha_i - \gamma_i)/2} \sin(\beta_i/2) & e^{i(\alpha_i + \gamma_i)/2} \cos(\beta_i/2)
\end{pmatrix},
\]

(10)

with \( 0 \leq \alpha_i, \beta_i, \gamma_i \leq 2\pi \) (\( i = 1, 2 \)).

Furthermore it is

\[
e^{-iF_i \sigma_i} = \begin{pmatrix}
  -e^{-i(\alpha_i - \gamma_i)/2} \sin(\beta_i/2) & 0 \\
  e^{i(\alpha_i + \gamma_i)/2} \cos(\beta_i/2) & 0
\end{pmatrix}
\]

\[
= -e^{-i(\alpha_i - \gamma_i)/2} \sin(\beta_i/2) |1\rangle_i \langle 1| \\
+ e^{i(\alpha_i + \gamma_i)/2} \cos(\beta_i/2) |0\rangle_i \langle 1|
\]

(11)

having assumed \( |1\rangle_i \) as the excited state and \( |0\rangle_i \) as the ground state of the \( i \)th qubit.
Quite generally we can split the Hamiltonian $H$ into two contributions: a local driving term e.g.

$$
H_{\text{drive}} = \alpha \sigma_1^{(y)} + \alpha \sigma_2^{(y)},
$$

with driving amplitude $\alpha \in \mathbb{R}$ and an interaction term $H_{\text{int}}$ to be specified.

Then Eq. (9) explicitly becomes

$$
\dot{\rho} = -\frac{i}{\hbar} \left[ \alpha \sigma_1^{(y)} + \alpha \sigma_2^{(y)}, \rho \right] - \frac{i}{\hbar} \left[ H_{\text{int}}, \rho \right] + \sum_{i=1}^{2} \left( \frac{1}{2} (|A_i|^2 + |B_i|^2) (|1\rangle_i \langle 1| \rho + \rho |1\rangle_i \langle 1|) 
+ (A_i|1\rangle_i \langle 1| + B_i|0\rangle_i \langle 1|) \rho (A_i^*|1\rangle_i \langle 1| + B_i|1\rangle_i \langle 0|) \right),
$$

with

$$
A_i = e^{-i(\alpha_i - \gamma_i)/2} \sin(\beta_i/2),
$$
$$
B_i = e^{i(\alpha_i + \gamma_i)/2} \cos(\beta_i/2).
$$

This means that $|A_i|^2 + |B_i|^2 = 1$.

At this point the aim would be the optimization of a measure of entanglement over the parameters characterizing $F_1$ and $F_2$, or equivalently $e^{-iF_1}$ and $e^{-iF_2}$.

The figure of merit we shall employ for entanglement is the concurrence defined as

$$
C(\rho) := \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},
$$

where $\lambda_i$s are, in decreasing order, the non-negative square roots of the moduli of the eigenvalues of

$$
\rho (\sigma_1 - \sigma_1^+) \otimes (\sigma_2 - \sigma_2^+) \rho^* (\sigma_1 - \sigma_1^+) \otimes (\sigma_2 - \sigma_2^+).
$$

Furthermore, in the following we will distinguish two tasks: entanglement preservation and entanglement stabilization.

### III. PRESERVING ENTANGLEMENT

Suppose we want to preserve as much as possible an initial entangled state by using feedback and considering $H_{\text{int}} = 0$ in Eq. (13). Then, in the basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ such equation becomes

$$
\dot{\rho} = -\frac{i}{\hbar} \left( \begin{array}{cccc}
0 & -i\alpha & -i\alpha & 0 \\
-\alpha & 0 & -i\alpha & 0 \\
\alpha & 0 & 0 & -i\alpha \\
0 & \alpha & \alpha & 0
\end{array} \right) \rho + \frac{i}{\hbar} \left( \begin{array}{cccc}
0 & -i\alpha & -i\alpha & 0 \\
\alpha & 0 & 0 & -i\alpha \\
-\alpha & 0 & 0 & -i\alpha \\
0 & -\alpha & -\alpha & 0
\end{array} \right) \rho 
+ \left( \begin{array}{cccc}
A_1 & 0 & 0 & 0 \\
0 & A_1 & 0 & 0 \\
B_1 & 0 & 0 & 0 \\
0 & B_1 & 0 & 0
\end{array} \right) \rho + \left( \begin{array}{cccc}
A_1^* & 0 & B_1^* & 0 \\
0 & A_1^* & 0 & B_1^*
\end{array} \right) \rho 
+ \left( \begin{array}{cccc}
A_2 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
B_2 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0
\end{array} \right) \rho + \left( \begin{array}{cccc}
A_2^* & B_2^* & 0 & 0 \\
0 & A_2^* & B_2^* & 0
\end{array} \right) \rho 
- \frac{1}{2} \left( \begin{array}{cccc}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \rho - \frac{1}{2} \rho \left( \begin{array}{cccc}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array} \right).
$$

Writing

$$
\rho = \left( \begin{array}{cccc}
A & B_R + iB_I & C_R + iC_I & D_R + iD_I \\
B_R - iB_I & E & F_R + iF_I & G_R + iG_I \\
C_R - iC_I & F_R - iF_I & H & I_R + iI_I \\
D_R - iD_I & G_R - iG_I & I_R - iI_I & 1 - A - E - H
\end{array} \right),
$$
equation (17) can be put in the following form:

$$\dot{\mathbf{v}} = \mathbf{M}\mathbf{v} - \mathbf{w},$$  \hspace{1cm} (19) $$

where \( \mathbf{v} \) is the unknown vector

$$\mathbf{v} := (A, B_R, B_I, C_R, C_I, D_R, D_I, E, F_R, F_I, G_R, G_I, H, I_R, I_I),$$  \hspace{1cm} (20) $$

with entries depending on time \( t \), while

\[
\mathbf{M} := \left( \begin{array}{cccccccccc}
-|B_1|^2 - |B_2|^2 & -2\alpha & 0 & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
2\alpha & \chi_1 & 0 & 0 & 0 & -\alpha & 0 & -\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
i_2 & 0 & \chi_1 & 0 & 0 & 0 & -\alpha & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
r_1 + r_2 & 0 & 0 & 0 & -\alpha & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
i_1 & 0 & 0 & 0 & \chi_2 & 0 & -\alpha & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\_1 + r_2 + r_2 & 0 & 0 & 0 & 0 & 0 & -|B_1|^2 & 0 & 0 & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
0 & r_1 + r_2 + r_2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
0 & i_1 & r_1 + r_2 + r_2 & 0 & 0 & 0 & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\alpha & 0 & 0 & |B_2|^2 & 0 & \alpha & 0 & r_1 + 2\alpha & \alpha & 0 & -\frac{1}{2} & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0
\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
|B_1|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\alpha & |B_1|^2 & 0 & 0 & 0 & \alpha & 0 & \alpha & 0 & \alpha & 0 & 0 & 0 & 0 & r_2 + 2\alpha & -\frac{1}{2} & 0 & -\frac{1}{2}
\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0
\\
i_2 & 0 & 0 & 0 & 0 & 0 & 0 & i_2 & -\frac{1}{2} & -\frac{1}{2}
\end{array} \right)
\]

(21)

with

\[
r_i = \Re [A_i B_i^*],
\]

\[
i_i = \Im [A_i B_i^*],
\]

\[
\chi_i = -\left( \frac{3}{2} - |A_i|^2 \right),
\]

(22a, 22b, 22c)

and

$$\mathbf{w} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^\top.$$  \hspace{1cm} (23) $$

Notice that Eq. (19), thanks to (21), results independent of \( \gamma_i \). Let the initial condition be

$$\rho(0) = |\Psi\rangle \langle \Psi|,$$  \hspace{1cm} (24) $$

with a generic pure state \( |\Psi\rangle \) parametrized as

$$|\Psi\rangle = \cos \theta_3 |11\rangle + e^{i\phi_2} \cos \theta_1 \sin \theta_3 \sin \theta_2 |01\rangle + e^{i\phi_1} \cos \theta_2 \sin \theta_3 |10\rangle + e^{i\phi_3} \sin \theta_3 \sin \theta_2 \sin \theta_1 |00\rangle,$$  \hspace{1cm} (25) $$
being \( \theta_i \in [0, \pi/2] \) and \( \phi_i \in [0, 2\pi] \). In turn, this means

\[
A(0) = \cos^2 \theta_3, \\
B_R(0) = \cos \phi_1 \cos \theta_3 \cos \theta_2 \sin \theta_3, \\
B_I(0) = -\sin \phi_1 \cos \theta_3 \cos \theta_2 \sin \theta_3, \\
C_R(0) = \cos \phi_2 \cos \theta_3 \sin \theta_1 \sin \theta_3 \sin \theta_2, \\
C_I(0) = -\sin \phi_2 \cos \theta_3 \cos \theta_1 \sin \theta_3 \sin \theta_2, \\
D_R(0) = \cos \phi_3 \cos \theta_3 \sin \theta_2 \sin \theta_1, \\
D_I(0) = -\sin \phi_3 \cos \theta_3 \sin \theta_2 \sin \theta_1, \\
E(0) = \cos^2 \theta_2 \sin^2 \theta_3, \\
F_R(0) = \cos(\phi_1 - \phi_2) \cos \theta_2 \cos \theta_1 \sin^2 \theta_3 \sin \theta_2, \\
F_I(0) = \sin(\phi_1 - \phi_2) \cos \theta_2 \cos \theta_1 \sin^2 \theta_3 \sin \theta_2, \\
G_R(0) = \cos(\phi_1 - \phi_3) \cos \theta_2 \sin^2 \theta_3 \sin \theta_2 \sin \theta_1, \\
G_I(0) = \sin(\phi_1 - \phi_3) \cos \theta_2 \sin^2 \theta_3 \sin \theta_2 \sin \theta_1, \\
H(0) = \cos^2 \theta_1 \sin^2 \theta_3 \sin^2 \theta_2, \\
I_R(0) = \cos(\phi_2 - \phi_3) \cos \theta_1 \sin^2 \theta_3 \sin^2 \theta_2 \sin \theta_1, \\
I_I(0) = \sin(\phi_2 - \phi_3) \cos \theta_1 \sin^2 \theta_3 \sin^2 \theta_2 \sin \theta_1.
\]

(26)

Clearly depending on the values of parameters \( \theta_i \) and \( \phi_i \) the initial state can be entangled or factorable. However it is known that randomly picking these parameters entangled states are the most likely \([11]\). Hence we will consider the concurrence at a action, i.e. independent of the initial state. To this end initial states are chosen according to the following measure induced by \( \theta \) being

\[
\text{Eq.(19) is solved analytically (not reported here for the sake of simplicity) subjected to the initial condition (26). Then the average concurrence has been calculated numerically at each time } t \text{ using an ensemble of } 10^5 \text{ states. The procedure is repeated for values of } \alpha_i, \beta_i \text{ in the range } 0, 2\pi \text{ with step } \pi/12. \text{ Finally, the maximum value is taken as corresponding to optimal feedback and the value } \alpha_i = \beta_i = 0 \text{ is taken as corresponding to no feedback action.}
\]

The remarkable thing is that the average concurrence does not depend on the driving parameter \( \alpha \). Then the results comparing the average concurrence with optimal feedback and without feedback are reported in Fig.(1). We can see that feedback is advantageous at any time, although its benefit increases with time, has a maximum at \( t = 0.4 \), and then tends to decrease.

Actually it is useful to consider

\[
\theta_i := \arcsin \left( \xi_i^{1/(2i)} \right),
\]

(28)

so to have flat probability densities for \( \phi_i \in [0, 2\pi] \) and \( \xi_i \in [0, 1] \)

\[
Pr(\phi_i) = \frac{1}{2\pi}, \quad Pr(\xi_i) = 1.
\]

(29)

Eq.(12) is solved analytically (not reported here for the sake of simplicity) subjected to the initial condition (26). Then the average concurrence has been calculated numerically at each time \( t \) using an ensemble of \( 10^5 \) states. The procedure is repeated for values of \( \alpha_i, \beta_i \) in the range \( 0, 2\pi \) with step \( \pi/12 \). Finally, the maximum value is taken as corresponding to optimal feedback and the value \( \alpha_i = \beta_i = 0 \) is taken as corresponding to no feedback action.
Another remarkable result is that the optimal feedback is achieved by the same values of $\alpha_i, \beta_i$ at any time. These values are reported in Table I and show a clear asymmetry between the action on the two subsystems.

| $\alpha_1$ | $\beta_1$ | $\alpha_2$ | $\beta_2$ |
|-----------|-----------|-----------|-----------|
| $\pi/6$   | $5\pi/6$  | $5\pi/6$  | $\pi$     |

TABLE I: Values of parameters $\alpha_i, \beta_i$ realizing optimal feedback action.

IV. STABILIZING ENTANGLEMENT

Suppose now we want to stabilize entanglement, i.e. we want to achieve the maximum entanglement at stationary state. In this case we also need of an interaction Hamiltonian, e.g.

$$H_{int} = 2J\sigma_1^{(z)}\sigma_2^{(z)}.$$  \hspace{1cm} (30)

Then we have to solve (13) with null l.h.s. Subsequently maximize the concurrence (15) over $\alpha_i, \beta_i, \gamma_j$, as well as over $\alpha$ and $J$.

In the basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ the involved master equation reads

$$0 = -\iota \left( \begin{array}{cccc} 2J & -i\alpha & -i\alpha & 0 \\ i\alpha & -2J & 0 & -i\alpha \\ i\alpha & 0 & -2J & -i\alpha \\ 0 & i\alpha & i\alpha & 2J \end{array} \right) \rho + \iota \rho \left( \begin{array}{cccc} 2J & -i\alpha & -i\alpha & 0 \\ i\alpha & -2J & 0 & -i\alpha \\ i\alpha & 0 & -2J & -i\alpha \\ 0 & i\alpha & i\alpha & 2J \end{array} \right)$$

$$+ \left( \begin{array}{cccc} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \end{array} \right) \rho \left( \begin{array}{cccc} A_1^* & 0 & B_1^* & 0 \\ 0 & A_1^* & 0 & B_1^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$+ \left( \begin{array}{cccc} A_2 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \end{array} \right) \rho \left( \begin{array}{cccc} A_2^* & B_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_2^* & B_2^* \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$- \frac{1}{2} \left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \rho - \frac{1}{2} \rho \left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$  \hspace{1cm} (31)

Writing again $\rho$ as (18), equation (31) can be put in the same form of (19), where however now $M$ is defined as
Also in this case the dynamics results independent from the $\gamma_i$s. Furthermore we have no dependence from the initial state, hence the feedback action can again be considered universal.

The solution $\rho(\infty)$ is obtained analytically (not reported here for the sake of simplicity) and then optimization of concurrence has been pursued numerically by varying (for each value of $\alpha$ and $J$) $\alpha_\iota$ and $\beta_\iota$ in the range $0, 2\pi$ with step $\pi/12$. Finally, the maximum value is taken as corresponding to optimal feedback and the value $\alpha_\iota = \beta_\iota = 0$ is taken as corresponding to no feedback action. The concurrence $C(\rho(\infty))$ achieved with optimal feedback is plotted Fig. 2 vs $\alpha$ and $J$. The results have a mirror symmetry with respect to $\alpha = 0$, so only positive values of $\alpha$ are considered.

In Fig. 3 is reported the difference between the concurrence $C(\rho(\infty))$ achieved with optimal feedback and that without feedback action.

Furthermore, in order to show the supremacy of our optimized feedback, in Fig. 4 we plotted the difference between the concurrence $C(\rho(\infty))$ achieved with optimal feedback and that with suboptimal feedback of Ref. [8].

By referring to Fig. 2 we may notice that for each value of $J$ there is an optimal value of $\alpha$ giving the largest concurrence. This determines a curve $\alpha = \alpha(J)$ in the plane $J, \alpha$ along which we have maxima of concurrence. Then, in Figs. 5 and 6 we show the values of parameters $\alpha_\iota$ and $\beta_\iota$, respectively that allow to attain the maxima values of concurrence along $\alpha$. As we can see they oscillate and depend sensibly to the values of $\alpha$ and $J$.

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V. CONCLUSION

We have addressed two main problems when controlling entanglement in two-qubit dissipating into their own environments, namely protecting initial entanglement and stabilizing entanglement. We have determined for both tasks optimal Markovian feedback control by resorting to analytical solutions of the dynamics as well as to numerical optimization of concurrence. The feedback actions are worked out in a way not depending on the initial two-qubit state, hence resulting universal.

The present work fills the gap of Ref. [8] where a proof of principle of the effectiveness of Markovian feedback in stabilizing entanglement was given, but it was not optimal, as well as of Ref. [9] where the optimal Markovian feedback in protecting initial entangled states was derived in a way depending on the initial state and not continuous in time.

The found results could be helpful in designing experiments of entanglement control, particularly in settings such as cavity QED, trapped ions, solid-state based qubit [13].

The presented analysis can be extended rather easily to other interaction Hamiltonians, or even to more than two-qubit. More challenging seems the exploitation of Bayesian (state-estimation-based) feedback control of two-qubit entanglement, following
FIG. 5: Optimal values of $\alpha_1$ and $\alpha_2$ leading to maximum amount of entanglement along the curve $\alpha = a(J)$.

FIG. 6: Optimal values of $\beta_1$ and $\beta_2$ leading to maximum amount of entanglement along the curve $\alpha = a(J)$.

up single-qubit control performed in Ref. [14].

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