Local super anti-magic total face coloring on shackle graphs

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Abstract. We define graph $G$ as a nontrivial, finite, connected graph which contains vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. We also define $g$ as bijective function that mapping vertex, edge, and face labeling to natural number which starting from 1 until $|V(G)|$ for vertex label, from $|V(G)|+1$ until $|V(G)|+|E(G)|$ for edge label, and the last for face label from $|V(G)|+|E(G)|+1$ until $|V(G)|+|E(G)|+|F(G)|$. If there are different weights in any neighboring two faces $f_1$ and $f_2$ has $w(f_1) \neq w(f_2)$ for $f_1, f_2 \in F(G)$, so $g$ is considered a local super anti-magic total face labeling. A proper face coloring from local super anti-magic total face labeling caused by assigns the color of face weights to local super anti-magic total face coloring. The minimum number of colors needed for local super anti-magic total face coloring is called the chromatic number of the local super anti-magic total face coloring. Encryption keys can possibly be created from the result of local super anti-magic total face coloring that can be used to construct a modified Affine cipher and Cipher Feedback Mode. As a result, we have one theorem for the chromatic number of local super anti-magic total face coloring and two algorithms for establishing super anti-magic total face coloring on shackle graphs in Cipher Feedback Mode.

1. Introduction

In this work, we define graph $G$ as a nontrivial, finite, connected graph which contains vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions [7]. We also define $g$ as bijective function that mapping vertex, edge, and face labeling to natural number which starting from 1 until $|V(G)|$ for vertex label, from $|V(G)|+1$ until $|V(G)|+|E(G)|$ for edge label, and the last for face label from $|V(G)|+|E(G)|+1$ until $|V(G)|+|E(G)|+|F(G)|$. If there are different weights in any neighboring two faces $f_1$ and $f_2$ has $w(f_1) \neq w(f_2)$ for $f_1, f_2 \in F(G)$, so $g$ is considered a local super anti-magic total face labeling.

A proper face coloring from local super anti-magic total face labeling caused by assigns the color of face weights to local super anti-magic total face coloring. The minimum number of colors needed for local super anti-magic total face coloring is called the chromatic number of the local super anti-magic total face coloring. $\gamma_{latf}(G)$ can be denoted as the chromatic number of the local super anti-magic total face coloring.
For the first time local anti-magic vertex coloring of a graph was presented to Arumugam et al. [5]. In [5], they found the chromatic number of local antimagic vertex on some graphs such as path, cycle, complete, friendship, wheel, bipartite, and total bipartite graph. They also found the lower and upper bound of the local antimagic vertex chromatic number on the joint graph. Super edge-antimagic total labeling of $mK_n, n [6]$ is another analysis that relates to local super anti-magic total face labeling.

Another type of local anti-magic coloring exists. That is the local coloring of the anti-magic edge. Local anti-magic edge coloring has been studied by Agustin et al.[2]. Their results are the precise value of path, cycle, friendship, ladder, star, wheel, complete, prism, $C_n \odot mK_1$ and $G \odot mK_1$, as well as ($\gamma_{lea} \geq \Delta(G)$) lower bound of local edge anti-magic chromatic number. Other findings of the super local anti-magic total edge coloring with certain related wheel graphs [1] and of the comb graph [3] local edge anti-magic coloring of the comb product.

In this paper, we study the local super anti-magic total face coloring of the shackle of fan graph. Fan graph obtained by joining all vertices of $F_n$, $n \geq 2$ is a path $P_n$ to a further vertex, called the centre. We can see [8] for the definition of shackle graph operation. We apply the result of local super anti-magic total face coloring of the shackle of fan graph in developing Affine cipher and Cipher Feedback Mode key. We present observations for the lower bound of local super anti-magic total face chromatic number in this section to prove the main theorem.

**Observation 1** [10] For any $G$ graph, $\gamma_{latf}(G) \geq \chi_f(G)$, where $\chi_f(G)$ is chromatic number of face coloring in graph $G$.

2. The Results

The theorem below shows the existence of local super anti-magic total face chromatic number of shackle graphs.

**Theorem 1** Let $G$ be a Shack($F_2,v,n$) with $n$ is natural number and $n \geq 2$ then

$$\gamma_{latf}(G) = 2.$$  

**Proof.** Shackles of fan graphs ($F_2$) has $n$ copies of fan graphs ($F_2$). Shackles of fan graph ($F_2$) has vertex set as $V(G) = \{x_i; 1 \leq i \leq 2n\} \cup \{y_j; 1 \leq j \leq n + 1\}$, edge set as $E(G) = \{x_ix_{i+1}; 1 \leq i \leq 2n, i \text{ odd}\} \cup \{x_{2i-1}y_i; 1 \leq i \leq n\} \cup \{x_{2i}y_i; 1 \leq i \leq n\} \cup \{x_{2i-1}y_{i+1}; 1 \leq i \leq n\} \cup \{x_{2i}y_{i+1}; 1 \leq i \leq n\}$, and face set as $F(G) = \{f_i; 1 \leq i \leq 2n\}$. Shackles of fan graph ($F_2$) can be divided into two sets: a set of vertices, $V(G)$, and a set of edges, $E(G)$. The face set of $G$ can be represented as $\{(x_i, y_j); 1 \leq i \leq 2n, 1 \leq j \leq n + 1\}$. The chromatic number of face coloring in graph $G$ is at least 2, because the shackle of fan graph ($F_2$) has at least two adjacent faces. Hence, $\gamma_{latf}(G) = 2$. The lower bound of $\gamma_{latf}(G) \geq 2$. To get the upper bound of $\gamma_{latf}(G)$, we define the function of all the elements of $G$. The functions are as follows:
\[ g(x_i) = \begin{cases} 
\frac{i+1}{2}; & \text{for } i \text{ is odd} \\
2n + 1 - \frac{i}{2}; & \text{for } i \text{ is even} 
\end{cases} \]
\[ y_i = 2n + i \]
\[ x_{2i-1} y_i = 6n + 3 - 2i \]
\[ x_{2i} y_i = 6n + 2i \]
\[ x_{2i-1} y_{i+1} = 6n + 2 - 2i \]
\[ x_{2i} y_{i+1} = 6n + 1 + 2i \]
\[ x_i x_{i+1} = 3n + 1 + \frac{i+1}{2}, \text{ } i \text{ is odd} \]
\[ g(f_i) = \begin{cases} 
10n + 1 - i; & \text{for } i \text{ is odd} \\
10n + 3 - i; & \text{for } i \text{ is even} 
\end{cases} \]

From the function \( g \) above, we can get the face weights and it will show the differences of face weights. The face weights of local super anti-magic total face coloring on \( Shack(F_2, v, n) \) are as follows:

For \( w(f_i), \text{ } i \text{ is odd} \)
\[ w(f_i) = g(x_i) + g(x_{i+1}) + g(y_{\frac{i+1}{2}}) + g(x_{2\frac{i-1}{2}} y_{\frac{i+1}{2}}) + g(x_{2\frac{i}{2}} y_{\frac{i+1}{2}}) + g(x_i x_{i+1}) + g(f_i) \]
\[ = \frac{i+1}{2} + 2n + 1 - \frac{i}{2} + 2n + \frac{i+1}{2} + 6n + 3 - 2\frac{i}{2} + 6n + 2\frac{i+1}{2} + 3n + 1 + \frac{i+1}{2} + 10n + 1 - i \\
= 29n + 7 \]

For \( w(f_i), \text{ } i \text{ is even} \)
\[ w(f_i) = g(x_{i-1}) + g(x_i) + g(y_{\frac{i}{2}+1}) + g(x_{2\frac{i-1}{2}} y_{\frac{i}{2}+1}) + g(x_{2\frac{i}{2}} y_{\frac{i}{2}+1}) + g(x_i x_{i+1}) + g(f_i) \]
\[ = \frac{i}{2} + 2n + 1 - \frac{i}{2} + 2n + \frac{i}{2} + 1 + 6n + 2 - 2\frac{i}{2} + 6n + 1 + 2\frac{i}{2} + 3n + 1 + \frac{i}{2} + 10n + 3 - i \\
= 29n + 9 \]

From the face weights \( w(f_i) \) above, we can see the face weights contain only two types of weights. The face weights are \( 29n + 7 \) and \( 29n + 9 \). It concludes that we get \( \gamma_{latf}(G) \geq 2 \) and the local super anti-magic total face chromatic number of \( Shack(F_2, v, n) \) is 2.

Figure 1 shows the illustration of local anti-magic total face coloring of \( shackle(f_2, v, 4) \). From the illustration, we know the set of face weight contains only two elements. The face weights are 123 and 125.

![Figure 1. Local anti-magic total face coloring of shackle(f_2, v, 4).](image-url)
Ciphertext (C)

Figure 2. Block diagram of the encryption of the Affine cipher and the Cipher Feedback Mode
3. Establishing Cipher Feedback Mode

Cipher Feedback Mode allows data to be encrypted in units smaller than the block size [4]. We improved Affine cipher and Cipher Feedback Mode into one encryption process using local anti-magic total face labeling as the key. Our developed encryption process can be described in below.

- We divide the plaintext into some blocks as many as the chromatic number of local super anti-magic total face of graphs.
- The key system source is generated from the local anti-magic total face labeling of planar graphs.
- The length of the key is the length of all elements of graphs such as vertex, edge, and face.

3.1. Generating Key for Cipher Feedback Mode

To produce a key Cipher Feedback Mode, we change the local anti-magic total face algorithm to create the encryption key. The algorithm below generates the key from a set of labels of all elements of local super anti-magic total face which has the same face weight. We sort the set into the sequence and it will be the encryption key. The mechanism of the generated key system is encrypting each plaintext block which includes in the same face weight. We work on the 26 alphabets of English. Figure 2 shows how to encrypt plaintext by using Cipher Feedback Mode with key get from local anti-magic total face labeling. The following algorithm takes the key stream construction as follows.

Algorithm 1. Generating key system

1. To label graph elements, define $g$.
2. If the bijective function is $g$, do 3, otherwise return to 1.
3. Take a certain $b$ with the local anti-magic total face chromatic number.
4. Take $z_{ij}$ is the sequence of vertex, edge, and face labels in a face with the same face weight $i$ and the same face weight $1 \leq j \leq b$.
5. Put $z_{ij}$ and sort the sequence according to the smaller vertex, edge, and face label.
6. Take the $k =$ component of the $z_{ij}$ sequence.

3.2. Encryption and Decryption Algorithm

To create encryption in the Affine cipher and Cipher Feedback Mode, the key of the Cipher Feedback Mode produced by algorithm 1 is established. Using algorithm 2, the encryption step is performed out.

Algorithm 2. Encryption in Affine cipher and Cipher Feedback Mode

1. Define plaintext as $P = (p_l), 1 \leq l \leq h$
2. Divide $P$ into some blocks as many $b$ as the chromatic number of local super anti-magic total face of graphs.
3. Define $l = 1$ to $\lceil \frac{h}{b} \rceil$ and compute the plaintext blocks by using equation 1 and compute the ciphertext blocks using equation 2.

$$C_l = ((P_l + C_{l-1}) + K_l) \mod 26 \quad (1)$$

$$P_l = ((C_l - C_{l-1}) - K_l) \mod 26 \quad (2)$$

where $P_l$, $K_l$, and $C_l$ are the $b$-th block of plaintext, key sequence, and ciphertext, respectively. For $l = 1$, $C_{l-1}$ is a null vector.

Table 1 and Table 2 demonstrate how the key stream generated from algorithm 1 was being used to encrypt the plaintext “UNIVERSITASJEMBER” and generate the ciphertext “VQHEPQLZDBBUATFMJ”. It is able to do the decryption process in the reverse direction.
Table 1. Encryption Process

|          | P  | U  | N  | I  | V  | E  | R  | S  | I  | T  |
|----------|----|----|----|----|----|----|----|----|----|----|
| Block 1  |    |    |    |    |    |    |    |    |    |    |
|          | P  |    |    |    |    |    |    |    |    |    |
|          | Pl | 20 | 13 | 8  | 21 | 4  | 17 | 18 | 8  | 19 |
|          | Cl | 21 | 16 | 7  | 4  | 15 | 16 | 11 | 25 |
|          | Cl+1| 20| 34| 24| 28| 8| 32| 34| 19| 44|
|          | Kl | 1  | 8  | 9  | 2  | 7  | 10 | 3  | 6  | 11 |
|          | Cl | 21 | 42 | 33 | 30 | 15 | 42 | 37 | 25 | 55 |
|          | Cl mod 26 | 21 | 16 | 7  | 4  | 15 | 16 | 11 | 25 | 3  |
|          | C  | V  | Q  | H  | E  | P  | Q  | L  | Z  | D  |

|          | P  |    |    |    |    |    |    |    |    |    |
| Block 2  |    |    |    |    |    |    |    |    |    |    |
|          | P  |    |    |    |    |    |    |    |    |    |
|          | Pl | 0  | 18 | 9  | 4  | 12 | 1  | 4  | 17 |
|          | Cl | 0  | 1  | 1  | 20 | 0  | 19 | 5  | 12 |
|          | Cl+1| 0| 19| 10| 24| 12| 20| 9 | 29 |
|          | Kl | 1  | 8  | 10 | 2  | 7  | 11 | 3  | 6  |
|          | Cl | 27 | 20 | 26 | 19 | 31 | 12 | 35 |
|          | Cl mod 26 | 1 | 1 | 20 | 0 | 19 | 5 | 12 | 9 |
|          | C  | B  | B  | U  | A  | T  | F  | M  | J  |    |

Table 2. Decryption Process

|          | C  | V  | Q  | H  | E  | P  | Q  | L  | Z  | D  |
|----------|----|----|----|----|----|----|----|----|----|----|
| Block 1  |    |    |    |    |    |    |    |    |    |    |
|          | C  |    |    |    |    |    |    |    |    |    |
|          | Cl | 21 | 16 | 7  | 4  | 15 | 16 | 11 | 25 | 3  |
|          | Cl+1| 0| 21| 16| 7 | 4 | 15 | 16 | 11 | 25 |
|          | Cl+1| 21| -5| -9| -3| 11| 1| -5| 14| -22|
|          | Kl | 1  | 8  | 9  | 2  | 7  | 10 | 3  | 6  | 11 |
|          | P  | 20 | -13| -18| -5| 4| -9| -8| -8| -33|
|          | P  | 20 | 13 | 8  | 21 | 4  | 17 | 18 | 8  | 19 |
|          | P  | U  | N  | I  | V  | E  | R  | S  | I  | T  |

|          | C  |    |    |    |    |    |    |    |    |    |
| Block 2  |    |    |    |    |    |    |    |    |    |    |
|          | C  |    |    |    |    |    |    |    |    |    |
|          | Cl | 1  | 1  | 20 | 0  | 19 | 5  | 12 |
|          | Cl+1| 0| 1| 1| 20| 0| 19| 5| 12 |
|          | Cl+1| 1| 0| 19|-20| 19|-14| 7| -3 |
|          | Kl | 1  | 8  | 10 | 2  | 7  | 11 | 3  | 6  | 11 |
|          | P  | 0  | -8 | 9  | -22| 12| -25| 4  | 9  |
|          | P  | 0  | 18 | 9  | 4  | 12 | 1  | 4  | 17 |
|          | P  | A  | S  | J  | E  | M  | B  | E  | R  |    |

4. Concluding Remarks
In this research we have obtained the local super anti-magic total face coloring on shackle graphs. We recognize the local anti-magic total face chromatic number of shackle graphs is 2. We also have two algorithms for establishing local super anti-magic total face coloring on shackle graphs in Cipher Feedback Mode.

Open Problem 1 Find the local super anti-magic total face coloring on another operation graph.
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