Distance Bounds on Quantum Dynamics

Daniel A. Lidar
Department of Chemistry, Electrical Engineering, and Physics, Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, CA 90089

Paolo Zanardi
Department of Physics, Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, CA 90089

Kaveh Khodjasteh
Department of Physics, Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, CA 90089
Department of Physics and Astronomy, Dartmouth College, Hanover, NH 03755

We derive rigorous upper bounds on the distance between quantum states in an open system setting, in terms of the operator norm between the Hamiltonians describing their evolution. We illustrate our results with an example taken from protection against decoherence using dynamical decoupling.

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I. INTRODUCTION

When an open quantum system undergoes a dynamical evolution generated by a Hamiltonian, how far does its state evolve from itself as a function of the magnitude of the Hamiltonian? This is a fundamental question that is central to quantum information science [1] and quantum control [2, 3]. In order to make it more precise, suppose the system is “open”, it is coupled to a bath S, and its time-evolved state is given via the partial trace operation by $\rho_S(t) = \text{Tr}_B U(t) \rho(0) U(t)\dagger$ [4, 5]. In the context of quantum information, the question we have stated pertains to the problem of “quantum memory”, i.e., what is $\|\rho_S(t) - \rho_S(0)\|$ as a function of $\|\Omega(t)\|$ or $\|H(t)\|$? When the issue is quantum computation or general quantum control, one is interested in comparing two time-evolved states: the “ideal” state $\rho_S^\text{I}(t)$ that is error-free and is described by an “ideal” effective Hamiltonian $\Omega^\text{I}(t)$ (e.g., for a quantum algorithm), and the “actual” state $\rho_S(t)$, that underwent the full noisy dynamics described by the total effective Hamiltonian $\Omega(t)$. Then the question becomes: what is $\|\rho_S(t) - \rho_S^\text{I}(t)\|$ as a function of $\|\Omega(t) - \Omega^\text{I}(t)\|$ or $\|H(t) - H^\text{I}(t)\|$? The memory question can of course be seen as a special case of the computation question.

Here we prove bounds that answer these fundamental questions. Our bounds have immediate applications to problems in decoherence control [6] and fault-tolerant quantum computation [7], as they quantify the sense in which a distance between (effective) Hamiltonians describing the evolution should be made small, in order to guarantee a small distance between a desired and actual state.

To begin, we first recall the definition and key properties of so-called unitarily invariant norms, as we use such norms extensively (Section II). We mention the trace norm and operator norm, and introduce a new norm that mixes them. We then briefly review the accepted distance measures between states, so as to quantify the meaning of an expression such as $\|\rho_S(t) - \rho_S^0(t)\|$ (Section III). Next we discuss how to compute the effective Hamiltonian $\Omega(t)$ using the Magnus expansion or Thompson’s theorem, and introduce a generalized effective superoperator generator (Section IV). Since in many applications one is interested not in the distance between states generated by unitary, closed-system evolution, but instead in the distance between states of systems undergoing non-unitary, open-system dynamics, we prove an upper bound on the distance between such system-only states, in terms of the distance between the full “system plus bath” states (Section V). We then come to our main result: an upper bound on the distance between system states in terms of the distance between (effective) Hamiltonians describing the system+bath dynamics (Section VI). We present a discussion of our result in terms of an example borrowed from decoherence control using dynamical decoupling (Section VII). We conclude in Section VIII with some open questions.

II. UNITARILY INVARIANT NORMS

Unitarily invariant norms are norms satisfying, for all unitary $U, V$ [8]:
\[ \|UAV\|_{ui} = \|A\|_{ui}. \] (1)

We list some important examples.

(i) The trace norm:

\[ \|A\|_1 \equiv \text{Tr}|A| = \sum_i s_i(A), \] (2)

where \( |A| \equiv \sqrt{A^\dagger A} \), and \( s_i(A) \) are the singular values (eigenvalues of \( |A| \)).

(ii) The operator norm: Let \( \mathcal{V} \) an inner product space equipped with the Euclidean norm \( \|x\| \equiv \sqrt{\sum_i |x_i|^2(e_i,e_i)} \), where \( x = \sum_i x_i e_i \in \mathcal{V} \) and \( \mathcal{V} = \text{Sp}\{e_i\} \). Let \( \Lambda : \mathcal{V} \mapsto \mathcal{V} \). The operator norm is

\[ \|\Lambda\|_\infty \equiv \sup_{x \in \mathcal{V}} \frac{\|\Lambda x\|}{\|x\|} = \max_i s_i(\Lambda). \] (3)

Therefore \( \|Ax\| \leq \|A\|_\infty \|x\| \). In our applications \( \mathcal{V} = \mathcal{L}(\mathcal{H}) \) is the space of all linear operators on the Hilbert space \( \mathcal{H} \), \( A \) is a superoperator, and \( x = \rho \) is a normalized quantum state: \( \|\rho\|_1 = \text{Tr}\rho = 1 \).

(iii) The Frobenius, or Hilbert-Schmidt norm:

\[ \|A\|_2 \equiv \sqrt{\text{Tr} A^\dagger A} = \sqrt{\sum_i s_i(A)^2}. \] (4)

All unitarily invariant norms satisfy the important property of submultiplicativity \[\|AB\| \leq \|A\| \|B\| \].

The norms of interest to us are also multiplicative over tensor products and obey an ordering [9]:

\[ \|A \otimes B\|_i = \|A\|_i \|B\|_i \quad i = 1, 2, \infty, \]

\[ \|A\|_\infty \leq \|A\|_2 \leq \|A\|_1, \]

\[ \|AB\|_{ui} \leq \|A\|_\infty \|B\|_{ui}, \|B\|_\infty \|A\|_{ui}. \] (5)

There is an interesting duality between the trace and operator norm [9]:

\[ \|A\|_1 = \max\{ \text{Tr}(B^\dagger A) : \|B\|_1 \leq 1 \} \] (7)

\[ \|A\|_\infty = \max\{ \text{Tr}(B^\dagger A) : \|B\|_1 \leq 1, \|\text{Tr}(BA)\| \leq \|A\|_\infty \|B\|_1, \|B^\dagger\|_\infty \|A\|_1 \} \] (8)

In the last three inequalities \( A \) and \( B \) can map between spaces of different dimensions. If they map between spaces of the same dimension then

\[ \|A\|_1 = \max_{B^\dagger B = I} \|\text{Tr}(B^\dagger A)\|. \] (9)

We now define another norm, which we call the “operator-trace” norm (O-T norm):

\[ \|A\|_{\infty,1} = \sup_{\rho \in \mathcal{L}(\mathcal{H})} \frac{\|\Lambda\rho\|_1}{\|\rho\|_1} = \sup_{\|\rho\|_1 = 1} \|\Lambda\rho\|_1, \] (10)

where \( \Lambda : \mathcal{V} \mapsto \mathcal{V} \), and \( \mathcal{V} \) is a normed space equipped with the trace norm. Note that if \( \Lambda \rho \) is another normalized quantum state then \( \|\Lambda\|_{\infty,1} = 1 \). Also, by definition \( \|\Lambda\rho\|_1 \leq \|\Lambda\|_{\infty,1} \). Moreover, it follows immediately from the unitary invariance of the trace norm that the O-T norm is unitarily invariant. Indeed, let \( \Gamma = V \cdot V^\dagger \) be a unitary superoperator (i.e., \( V \) is unitary): then \( \|\Gamma A\|_{\infty,1} = \sup_{\rho} \|V_1 \Lambda V_1^\dagger\|_{\infty,1} = \sup_{\rho} \|\Lambda\rho\|_1 = \|\Lambda\|_{\infty,1} \). Therefore the O-T norm is also submultiplicative. However, note that unlike the case of the standard operator norm, there is no simple expression for \( \|\Lambda\|_{\infty,1} \) in terms of the singular values of \( \Lambda \).

### III. DISTANCE AND FIDELITY MEASURES

Various measures are known that quantify the notion of distance and fidelity between states. For example, the distance measure between quantum states \( \rho_1 \) and \( \rho_2 \) is the trace distance:

\[ D(\rho_1, \rho_2) \equiv \frac{1}{2} \|\rho_1 - \rho_2\|_1. \] (12)

The trace distance is the maximum probability of distinguishing \( \rho_1 \) from \( \rho_2 \). Namely, \( D(\rho_1, \rho_2) = \max_{\phi \in \mathcal{P} \subset I} \{ \langle P \rangle_1 - \langle P \rangle_2 \} \), where \( \langle P \rangle_1 = \text{Tr}(P)P \) and \( P \) is a projector, or more generally an element of a POVM (positive operator-valued measure) [9]. The fidelity between quantum states \( \rho_1 \) and \( \rho_2 \) is

\[ F(\rho_1, \rho_2) \equiv \|\sqrt{\rho_1} \sqrt{\rho_2}\|_1 = \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}, \] (13)

which reduces for pure states \( \rho_1 = |\psi\rangle \langle \psi| \) and \( \rho_2 = |\phi\rangle \langle \phi| \) to \( F(|\psi\rangle, |\phi\rangle) = |\langle \psi | \phi \rangle| \). The fidelity and distance bound each other from above and below [10]:

\[ 1 - D(\rho_1, \rho_2) \leq F(\rho_1, \rho_2) \leq \sqrt{1 - D(\rho_1, \rho_2)^2}, \] (14)

so that knowing one bounds the other. Many other measures exist and are useful in a variety of circumstances [9].

### IV. GENERATORS OF THE DYNAMICS

#### A. Effective superoperator generators

We shall describe the evolution in terms of an effective superoperator generator \( L(t) \), such that

\[ \rho(0) \rightarrow \rho(t) \equiv e^{L(t)}\rho(0). \] (15)
The advantage of this general formulation is that it incorporates non-unitary evolution as well. Nevertheless, here we focus primarily on the case of unitary evolution $\rho(t) = U(t)\rho(0)U(t)^\dagger$, with $\dot{U} = -\frac{i}{\hbar}HU$, for which we have

$$L(t) = -\frac{i}{\hbar}[\Omega(t), \cdot].$$

(16)

This follows immediately from the identity $e^{-iA}\rho e^{iA} = e^{-i[A, \cdot]}\rho \equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [A, \rho]$ is satisfied for any Hermitian operator $A$, where $[A, \rho]$ denotes a nested commutator, i.e., $[A, \rho] = [A, [A, \rho]]$, with $[0, \rho] = \rho$.

B. Magnus expansion

In perturbation theory the effective Hamiltonian can be evaluated most conveniently by using the Magnus expansion, which provides a unitary perturbation theory, in contrast to the Dyson series [11]. The Magnus expansion expresses $\Omega(t)$ as an infinite series: $\Omega(t) = \sum_{n=1}^{\infty} \Omega_n(t)$, where $\Omega_1(t) = \frac{1}{t} \int_0^t H(t_1)dt_1$, and the $n$th order term involves an integral over an $n$th level nested commutator of $H(t)$ with itself at different times. A sufficient (but not necessary) condition for absolute convergence of the Magnus series for $\Omega(t)$ in the interval $[0, t)$ is [12]:

$$\int_0^t \|H(s)\|_\infty ds < \pi.$$  

(17)

C. Relating the effective Hamiltonian to the “real” Hamiltonian

There is also a way to relate the effective Hamiltonian to the real Hamiltonian in a non-perturbative manner. To this end we make use of a recently proven theorem due to Thompson [13, 14].

Let us consider a general quantum evolution generated by a time-dependent Hamiltonian $H$, where $V$ is considered a perturbation to $H_0$ (in spite of this we will not treat $V$ perturbatively):

$$H(t) = H_0(t) + V(t).$$

(18)

The propagators satisfy:

$$\frac{dU(t, 0)}{dt} = -iH(t)U(t, 0),$$

(19)

$$\frac{dU_0(t, 0)}{dt} = -iH_0(t)U_0(t, 0).$$

(20)

We define the interaction picture propagator with respect to $H_0$, as usual, via:

$$\dot{U}(t, 0) = U_0(t, 0)^\dagger U(t, 0).$$

(21)

It satisfies the Schrödinger equation

$$\frac{d\tilde{U}(t, 0)}{dt} = -i\tilde{H}(t)\tilde{U}(t, 0),$$

(22)

with the interaction picture Hamiltonian

$$\tilde{H}(s) = U_0(t, 0)^\dagger V(t)U_0(t, 0).$$

(23)

See Appendix A for a proof.

The interaction picture propagator $\tilde{U}(t, 0)$ can be formally expressed as

$$\tilde{U}(t, 0) = T \left[ \exp \left( -i \int_0^t \tilde{H}(s) ds \right) \right] \equiv \exp[-it\tilde{\Omega}(t)],$$

(24)

where the second equality serves to define the effective interaction picture Hamiltonian $\tilde{\Omega}(t)$.

Lemma 1 There exist unitaries $\{W(s)\}$ such that

$$\tilde{\Omega}(t) \equiv \frac{1}{t} \int_0^t W(s)\tilde{H}(s)W(s)^\dagger ds.$$  

(26)

This is remarkable since it shows that the time-ordering problem can be converted into the problem of finding the (continuously parametrized) set of unitaries $\{W(s)\}$.

Proof. The formal solution [23] can be written as an infinite ordered product:

$$\tilde{U}(t, 0) = \lim_{N \to \infty} \prod_{j=0}^{N} \exp \left[ -i \left( \frac{t}{N} \right) \tilde{H} \left( \frac{jt}{N} \right) \right].$$

(27)

Thompson’s theorem [13, 14] states that for any pair of operators $A$ and $B$, there exist unitaries $V$ and $W$, such that $e^{A^t B} = e^{VAV^\dagger + WBW^\dagger}$. It follows immediately that if $\{A_j\}_{j=0}^{N}$ are Hermitian operators then it is always possible to find unitary operators $\{W_j\}_{j=1}^{N}$ such that

$$\prod_{j=0}^{N} \exp[ -iA_j ] = \exp[-i \sum_{j=0}^{N} W_j A_j W_j^\dagger].$$

(28)

The proof is non-constructive, i.e., the unitaries $\{W_j\}_{j=1}^{N}$ are not known. Applying this to Eq. [27] yields

$$\tilde{U}(t, 0) = \lim_{N \to \infty} \exp \left[ -i \left( \frac{t}{N} \right) \sum_{j=0}^{N} W_j \tilde{H} \left( \frac{jt}{N} \right) W_j^\dagger \right] \equiv \exp[-i \int_0^t W(s)\tilde{H}(s)W(s)^\dagger ds],$$

(29)

which is the claimed result with the effective Hamiltonian $\tilde{\Omega}(t)$ defined as in Eq. [20].

An immediate corollary of Lemma [11] is the following:
Corollary 1 The effective Hamiltonian \( \tilde{\Omega}(t) \) defined in Eq. (20) satisfies, for any unitarily invariant norm:

\[
\|\tilde{\Omega}(t)\|_{ui} \leq \frac{1}{t} \int_0^t ds \|V(s)\|_{ui} \equiv \|V\|_{ui} \quad (30)
\]

\[
\leq \sup_{0 < s < t} \|V(s)\|_{ui}. \quad (31)
\]

Proof. We have [Eq. (29)] \( \tilde{\Omega}(t) = U(t,0)\tilde{H}(s)U(t,0) \) and [Eq. (26)] \( \|\tilde{\Omega}(t)\|_{ui} = \|\frac{1}{t} \int_0^t W(s)\tilde{H}(s)W(s)ds\|_{ui} \). The result follows from the triangle inequality. \( \blacksquare \)

V. DISTANCE BEFORE AND AFTER PARTIAL TRACE

Since we are interested in the distances between states undergoing open system dynamics, we now prove the following:

Lemma 2 Let \( \mathcal{H}_S \) and \( \mathcal{H}_B \) be finite dimensional Hilbert spaces of dimensions \( d_S \) and \( d_B \), and let \( A \in \mathcal{H}_S \otimes \mathcal{H}_B \). Then for any unitarily invariant norm that is multiplicative over tensor products the partial trace satisfies the following norm inequality

\[
\|\text{tr}_B A\| \leq \frac{d_B}{\|I_B\|} \|A\|, \quad (33)
\]

where \( I \) is the identity operator.

This result was already known for the trace norm as a special case of the contractivity of trace-preserving quantum operations [1].

Proof. Consider a unitary irreducible representation \( \{U_G(g), g \in G\} \) of a compact group \( G \) on \( \mathcal{H}_B \). Then it follows from Schur’s lemma that the partial trace has the following representation [12]:

\[
\frac{1}{d_B} \text{tr}_B (X) \otimes I_B = \int_G [I_A \otimes U_B(g)]X[I_A \otimes U_B(g)]^\dagger d\mu(g), \quad (34)
\]

where \( d\mu(g) \) denotes the left-invariant Haar measure normalized as \( \int_G d\mu(g) = 1 \) and \( d_B \equiv \dim(\mathcal{H}_B) \). Then

\[
\|\text{tr}_B X\| = \frac{1}{\|I_B\|} \|\text{tr}_B (X) \otimes I_B\|
\]

\[
\leq \frac{\|I_B\|}{\|I_B\|} \int_G \|[I_A \otimes U_B(g)]X[I_A \otimes U_B(g)]^\dagger\| d\mu(g)
\]

\[
= \frac{d_B}{\|I_B\|} \int_G \|X\| d\mu(g) = \frac{d_B}{\|I_B\|} \|X\|. \quad (35)
\]

In particular, \( \|I_B\|_1 = d_B \), \( \|I_B\|_2 = \sqrt{d_B} \), and \( \|I_B\|_\infty = 1 \), and since the trace, Frobenius, and operator norms are all multiplicative over tensor products we have, specifically:

\[
\|\text{tr}_B X\|_1 \leq \|X\|_1, \quad (36)
\]

\[
\|\text{tr}_B X\|_2 \leq \sqrt{d_B} \|X\|_2, \quad (37)
\]

\[
\|\text{tr}_B X\|_\infty \leq d_B \|X\|_\infty. \quad (38)
\]

Note that not all unitarily invariant norms are multiplicative over tensor products. For instance, the Ky Fan \( k \)-norm \( \|\cdot\|_{(k)} \) is the sum of the \( k \) largest singular values, and is unitarily invariant [8], but it is not multiplicative in this way. For example, when \( d_A = d_B = d \geq k \geq 2 \) we have \( \|I_A\|_{(k)} = \|I_B\|_{(k)} = \|I_A \otimes I_B\|_{(k)} = k \). So, for \( X = I_A \otimes I_B \) we have \( \|\text{tr}_B X\|_{(k)} = \|I_A\|_{(k)} = dk \) but \( \|\text{tr}_B X\|_1 = (d/k)k = d \) which gives an inequality in the wrong direction.

VI. DISTANCE BOUND

We are now ready to prove our main theorem, that provides a bound on the distance between states in terms of the distance between effective superoperator generators. As an immediate corollary we obtain the bound in terms of the effective Hamiltonians.

How much does the deviation between \( L(t) \) and \( L^0(t) \) impact the deviation between \( \rho(t) \) and \( \rho^0(t) \)? We shall assume that the desired evolution is unitary, i.e., \( e^{iL^0(t)} = U^0(t) \cdot U^0(t)^\dagger \), where \( U^0(t) = e^{-i\tilde{\Omega}(t)} \).

Theorem 1 Consider two evolutions: the “desired” unitary evolution \( \rho(0) \mapsto \rho^0(t) \equiv e^{iL^0(t)}\rho(0) = e^{-i\tilde{\Omega}(t)}\rho(0) \), and the “actual” evolution \( \rho(t) \mapsto \rho(t) \equiv e^{iL(t)}\rho(0) \). Let \( \Delta L(t) \equiv L(t) - L^0(t) \).

\[
D[\rho(t),\rho^0(t)] \leq \min[1,\frac{1}{2}(e^{t\|\Delta L(t)\|_{\infty,1}} - 1)] \quad (39)
\]

If in addition \( t\|\Delta L(t)\|_{\infty,1} \leq 1 \) then

\[
D[\rho(t),\rho^0(t)] \leq t\Delta L(t)\|_{\infty,1}. \quad (40)
\]

Corollary 2 (less tight) For Hamiltonian evolution, where \( L(t) = -\frac{i}{\hbar}[\Omega(t),\cdot] \) and \( L^0(t) = -\frac{i}{\hbar}[\Omega^0(t),\cdot] \), and defining \( \Delta \Omega(t) \equiv \Omega(t) - \Omega^0(t) \):

\[
D[\rho(t),\rho^0(t)] \leq \min[1,\frac{1}{2}(e^{\frac{t}{\hbar}\|\Delta \Omega(t)\|_{\infty}} - 1)] \quad (41)
\]

\[
\leq \frac{2}{\hbar}t\|\Delta \Omega(t)\|_{\infty} \quad \text{if} \quad \frac{2}{\hbar}t\|\Delta \Omega(t)\|_{\infty} \leq 1. \quad (42)
\]

Note that in Corollary 2 we use the standard operator norm, while in Theorem 1 we use the O-T norm. There are two reasons that Corollary 2 is less tight. First, it involves an additional use of the triangle inequality,
which converts \( \|\Delta \Omega(t), \rho\|_1 \) to \( 2\|\Delta \Omega(t)\rho\|_1 \). Obviously, if \( \Delta \Omega(t) \) and \( \rho \) nearly commute then this results in a weak bound. Second, even though we do not have an interpretation of \( \|\Delta L(t)\|_{\infty,1} \) as a function of the singular values of \( \Delta L(t) \), it is convenient to imagine this to be the case; then, for Hamiltonian evolution, the effective superoperator generator \( \Delta L(t) \) has eigenvalues which are the eigenvalue (energy) differences of the corresponding effective Hamiltonian. Assuming the eigenvalues to be positive, their differences are always upper bounded by the largest eigenvalue, i.e., \( \|\Delta \Omega(t)\|_{\infty} \).

**Proof of Theorem 1.** Let us define Hermitian operators

\[
\mathcal{H}^0 \equiv itL^0(t) \quad \mathcal{H} \equiv itL(t),
\]

and consider (superoperator) unitaries generated by these operators as a function of a new time parameter \( s \) (we shall hold \( t \) constant):

\[
dU_0 / ds = -i\mathcal{H}^0 U_0, \quad dU / ds = -i\mathcal{H} U.
\]

Then

\[
U_0(s) = e^{-is\mathcal{H}^0} \quad U(s) = e^{-is\mathcal{H}},
\]

and we can define an interaction picture via

\[
U(s) = U_0(s) S(s),
\]

where the interaction picture “perturbation” is

\[
\tilde{V}(s) \equiv U_0^\dagger(s)(\mathcal{H} - \mathcal{H}^0) U_0(s) = itU_0^\dagger(s) \Delta L(t) U_0(s).
\]

Then, using (1) unitary invariance and (2) the definition of the \( \| \cdot \|_{\infty,1} \) norm [Eq. (11)]:

\[
D[\rho(t), \rho^0(t)] = \frac{1}{2} \| e^{itL^0(t)} (e^{-itL^0(t)} e^{itL(t)} - I) \rho(0) \|_1
\]

\[
\leq \frac{1}{2} \| (S(1) - I) \rho(0) \|_1 \quad (1)
\]

\[
\leq \frac{1}{2} \| S(1) - I \|_{\infty,1} \quad (2)
\]

which explains why we introduced \( S \). We can compute \( S \) using the Dyson series of time-dependent perturbation theory:

\[
dS / ds = -i\tilde{V} S,
\]

\[
S(s) = I + \sum_{m=1}^\infty S_m(s),
\]

\[
S_m(s) = \int_0^s ds_1 \int_0^{s_1} ds_2 \int_0^{s_1} \cdots ds_m \tilde{V}(s_1) \tilde{V}(s_2) \cdots \tilde{V}(s_m).
\]

Using submultiplicativity and the triangle inequality we can then show that (see Appendix [3] for the details):

\[
\| S(s) - I \|_{\infty,1} \leq e^{\| \Delta L(t) s \|_{\infty,1}} - 1,
\]

Thus, finally, we have from Eq. (48):

\[
D[\rho(t), \rho^0(t)] \leq \frac{1}{2} (e^{\| \Delta L(t) s \|_{\infty,1}} - 1).
\]

If additionally \( \| \Delta L \|_{\infty,1} \leq 1 \) then the inequality \( e^x - 1 \leq (e-1)x \) yields \( D[\rho(t), \rho^0(t)] \leq t\| \Delta L(t) \|_{\infty,1} \). On the other hand, note that \( D[\rho(t), \rho^0(t)] = \frac{1}{2} \| (S(1) - I) \rho(0) \|_1 \leq \frac{1}{2} \| S(1) \rho(0) \|_1 + \| \rho(0) \|_1 \) = 1. □

**Proof of Corollary 2.** For Hamiltonian evolution we have

\[
\hbar \| \Delta L(t) \|_{\infty,1} = \| [\Delta \Omega(t), \cdot] \|_{\infty,1}
\]

\[
= \sup_\rho \| [\Delta \Omega(t), \rho] \|_1 
\]

\[
\leq \sup_\rho 2\| \Delta \Omega(t) \rho \|_1
\]

\[
\leq \sup_\rho 2\| \Delta \Omega(t) \|_{\infty,1} \| \rho \|_1
\]

\[
= 2\| \Delta \Omega(t) \|_{\infty},
\]

where in the first inequality we used the triangle inequality, and in the second inequality we used Eq. (10). □

**VII. DISCUSSION**

A. The general bound

By putting together Eq. (33) and Corollaries 1 and 2 we can answer the question we posed in the Introduction:

**Theorem 2** Consider a quantum system \( S \) coupled to a bath \( B \), undergoing evolution under either the “actual” joint unitary propagator \( U(t) = e^{-\frac{i}{\hbar} \Omega(t)} \) or the “desired” joint unitary propagator \( U^0(t) = e^{-\frac{i}{\hbar} m \Omega^0(t)} \), generated respectively by \( H(t) \) and \( H^0(t) \). Then the trace distance between the actual time evolved system state \( \rho_S(t) = \text{tr}_B U(t) \rho(0) U^\dagger(t) \) and the desired one, \( \rho_S^0(t) = \text{tr}_B U^0(t) \rho(0) U^{0\dagger}(t) \), satisfies the bound

\[
D[\rho_S(t), \rho_S^0(t)] \leq \min [1, \frac{1}{2} (e^{\| H(t) \|_{\infty}} - (\| H(t) \|_{\infty}) - 1)]
\]

\[
\leq \min [1, \frac{1}{2} (e^{\| H \|_{\infty}} - (\| H_0 \|_{\infty}) - 1)].
\]

where \( \| X \|_{\infty} \equiv \frac{1}{\hbar} \int_0^t ds \| X(s) \|_\infty \).

This result shows that to minimize the distance between the actual and desired evolution it is sufficient to minimize the distance in the operator norm between the actual and desired effective Hamiltonians, or the difference of the time average of the operator norm of the real Hamiltonians. Techniques for doing so which explicitly use the Magnus expansion or operator norms include dynamical decoupling [10, 11] and quantum error correction for non-Markovian noise [18, 19, 20, 21, 22].
B. Illustration using concatenated dynamical decoupling

As an illustration, consider the scenario of a single qubit coupled to a bath via a general system-bath Hamiltonian $H_{SB} = H_S \otimes I_B + \sum_{\alpha=x,y,z} \sigma_\alpha \otimes B_\alpha$ ($H_S$ is the system-only Hamiltonian, $\sigma_\alpha$ are the Pauli matrices, and $B_\alpha \neq I_B$ are bath operators), with a bath Hamiltonian $H_B$, and controlled via a system Hamiltonian $H_C(t)$, so that the total Hamiltonian is $H(t) = H_C(t) \otimes I_B + H_{SB} + I_S \otimes H_B$. Suppose that one wishes to preserve the state of the qubit, i.e., we are interested in “quantum memory”, so that $\Omega(t) \equiv \Omega(0)$, and assuming zero-width pulses, the Magnus expansion yields the following upper bound:

$$T \| \Omega(T) - \Omega_0(T) \|_{ui}/\hbar \leq JT(\beta T/N^{1/2}) \log_4 N.$$

Here $\Omega_0(T) = I_S \otimes H_B$, $T = N\tau$ (the duration of a concatenated pulse sequence (23)), and assuming zero-width pulses, the Magnus expansion yields the following upper bound:

$$D[\rho_S(T),\rho_S^0(T)] \leq \frac{1}{2}(2^{J/T(\beta T/N^{1/2}) \log_4 N} - 1) \leq 2JT(\beta T/N^{1/2}) \log_4 N,$$

which shows that the distance between the actual and desired state is maintained arbitrarily well.

VIII. CONCLUSIONS

We have presented various bounds on the distance between states evolving quantum mechanically, either as closed or as open systems. These bounds are summarized in Theorem 24. We expect our bounds to be useful in a variety of quantum computing or control applications. An undesirable aspect of Eqs. (23) and (24) is that the operator norm can diverge if the bath spectrum is unbounded, as is the case, e.g., for an oscillator bath. A brute force solution is the introduction of a high-energy cutoff. However, a more elegant solution is to note 24 (lemma 8) that every system with energy constraints (such as a bound on the average energy), is essentially supported on a finite-dimensional Hilbert space. An even more satisfactory solution in the unbounded spectrum case is to find a distance bound involving correlation functions. This can be accomplished by performing a perturbative treatment in the system-bath coupling parameter, as is done in the standard derivation of quantum master equations [4, 5].

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APPENDIX A: INTERACTION PICTURE

We prove that $\hat{U}(t,0)$ defined in Eq. (24) satisfies the Schrödinger equation 22 with the interaction-picture Hamiltonian defined in Eq. (25). This requires proof since all the Hamiltonians considered are time-dependent, whereas usually one only considers the perturbation to be time-dependent. To see this we differentiate both sides of Eq. (24), while making use of Eqs. (22) and (20):

$$\frac{dU(t,0)}{dt} = \frac{d}{dt} \left[ U_0(t,0) \hat{U}(t,0) \right] = \frac{dU_0(t,0)}{dt} \hat{U}(t,0) + U_0(t,0) \frac{d\hat{U}(t,0)}{dt}.$$

which is the same differential equation as Eq. (19). The initial conditions of the equations are also the same $[U(0,0) = I]$, thus Eqs. (24) and (20) describe the propagator generated by $H(t)$.

APPENDIX B: DYSON EXPANSION BOUND

We prove Eq. (25). The proof makes use of (1) the triangle inequality, (2) submultiplicativity, and (3) unitary invariance.
\[ \|S(s) - I\|_{\infty, 1} = \left\| \sum_{m=1}^{\infty} S_m(s) \right\|_{\infty, 1} \]

\[ \leq \sum_{m=1}^{\infty} \|S_m(s)\|_{\infty, 1} \]

\[ = \sum_{m=1}^{\infty} \int_{0}^{s} ds_1 \cdots \int_{0}^{s_{m-1}} ds_m \prod_{i=1}^{m} \overline{V}(s_i) \]

\[ \leq \sum_{m=1}^{\infty} \int_{0}^{s} ds_1 \cdots \int_{0}^{s_{m-1}} ds_m \prod_{i=1}^{m} \|V(s_i)\|_{\infty, 1} \]

\[ \leq \sum_{m=1}^{\infty} \int_{0}^{s} ds_1 \cdots \int_{0}^{s_{m-1}} ds_m \|t \Delta L(t)\|^m_{\infty, 1} \]

\[ = \sum_{m=1}^{\infty} \|t \Delta L(t)\|^m_{\infty, 1} \frac{s^m}{m!} \]

\[ = e^{\|t \Delta L(t)s\|_{\infty, 1}} - 1 \]

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