 Article
Joint Universality of the Zeta-Functions of Cusp Forms

Renata Macaitienė

Regional Development Institute of Šiauliai Academy, Vilnius University, P. Višinskio Str. 25, LT-76351 Šiauliai, Lithuania; renata.macaitiene@sa.vu.lt

Abstract: Let $F$ be the normalized Hecke-eigen cusp form for the full modular group and $\zeta(s, F)$ be the corresponding zeta-function. In the paper, the joint universality theorem on the approximation of a collection of analytic functions by shifts $(\zeta(s + ih_1 \tau, F), \ldots, \zeta(s + ih_r \tau, F))$ is proved. Here, $h_1, \ldots, h_r$ are algebraic numbers linearly independent over the field of rational numbers.

Keywords: Hecke-eigen cusp form; joint universality; universality; zeta-function

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1. Introduction
The series of the types
\[ \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad s = \sigma + it, \]
where $\{\lambda_m\}$ is a nondecreasing sequence of real numbers and $\lim_{m \to \infty} \lambda_m = +\infty$ are called Dirichlet series. The majority of zeta-functions are meromorphic functions in some half-plane defined by Dirichlet series having a certain arithmetic sense. The most important of zeta-functions is the Riemann zeta-function
\[ \zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1. \]

In [1], Voronin discovered a very interesting and important property of $\zeta(s)$ to approximate a wide class of analytic functions by shifts $\zeta(s + it \tau)$, $\tau \in \mathbb{R}$, and called it universality. Later, it turned out that some other zeta-functions also are universal in the Voronin sense. This paper is devoted to the universality of zeta-functions of certain cusp forms.

Let $SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ be the full modular group. If the function $F(z)$ is holomorphic in the upper half-plane $\text{Im} z > 0$, and for all elements of $SL(2, \mathbb{Z})$ with some $\kappa \in 2\mathbb{N}$ satisfies the functional equation
\[ F\left( \frac{az + b}{cz + d} \right) = (cz + d)^\kappa F(z), \quad (1) \]
where $F(z)$ is called a modular form of weight $\kappa$ for the full modular group. Then, $F(z)$ has Fourier series expansion
\[ F(z) = \sum_{m=-\infty}^{\infty} c(m) e^{2\pi i m z}. \]
If \( c(m) = 0 \) for all \( m \leq 0 \), then \( F(z) \) is a cusp form of weight \( \kappa \). The corresponding zeta-function (or \( L \)-function) \( \zeta(s, F) \) is defined for \( \sigma > \frac{\kappa + 1}{2} \) by the Dirichlet series

\[
\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},
\]

and has the analytic continuation to an entire function. Additionally, we suppose that \( F(z) \) is a simultaneous eigenfunction of all Hecke operators \( T_m \)

\[
T_mF(z) = m^{\kappa-1} \sum_{a,d > 0 \atop ad = m} \frac{1}{d} \sum_{b \mod d} F(\frac{az + b}{d}), \quad m \in \mathbb{N}.
\]

In this case, \( c(1) \neq 0 \); therefore, the form \( F(z) \) can be normalized, and thus, we may suppose that \( c(1) = 1 \).

Now, we suppose that \( F(z) \) is a normalized Hecke-eigen cusp form of weight \( \kappa \) for the full modular group. Then, the zeta-function \( \zeta(s, F) \) can be written, for \( \sigma > \frac{\kappa + 1}{2} \), as a product over primes

\[
\zeta(s, F) = \prod_p \left( 1 - \frac{\alpha(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta(p)}{p^s} \right)^{-1},
\]

where \( \alpha(p) \) and \( \beta(p) \) are conjugate complex numbers satisfying the equality \( \alpha(p) + \beta(p) = c(p) \).

In the paper [2], the universality of the function \( \zeta(s, F) \) was proved. Let \( D_\kappa = \{ s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa + 1}{2} \}, \) \( K_F \) be the class of compact subsets of the strip \( D_\kappa \) with connected complements, and \( H_{0,F}(K), K \in K_F \) the class of continuous nonvanishing functions on \( K \) that are analytic in the interior of \( K \). Moreover, let \( \text{meas} A \) denote the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). Then, in [2], the following theorem was obtained.

**Theorem 1.** Suppose that \( K \in K_F \) and \( f(s) \in H_{0,F}(K) \). Then, for every \( \varepsilon > 0 \),

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon \right\} > 0.
\]

Theorem 1 shows that there are infinitely many shifts \( \zeta(s + i\tau, F) \) approximating a given function \( f(s) \in H_{0,F} \). In the shifts \( \zeta(s + i\tau, F) \) of Theorem 1, \( \tau \) takes arbitrary real values; therefore, the theorem is of continuous type. Further, discrete universality theorems for the function \( \zeta(s, F) \) are known. In [3,4], the discrete universality theorems with shifts \( \zeta(s + ikh, F), k \in \mathbb{N}, h > 0 \) being a fixed number, were proved. Denote by \( H(D_\kappa) \) the space of analytic on \( D_\kappa \) functions endowed with the topology of uniform convergence on compacta. The paper [5] is devoted to the universality for compositions \( \Phi(\zeta(s, F)) \) with certain operators \( \Phi : H(D_\kappa) \to H(D_\kappa) \). The results of the latter paper were applied in [6] for the functional independence of the compositions \( \Phi(\zeta(s, F)) \).

Let, for a fixed \( l \in \mathbb{N} \),

\[
\Gamma_0(l) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0(\text{mod } l) \right\}
\]

denote the Hecke subgroup of the group \( SL(2, \mathbb{Z}) \). If \( F(z) \) satisfies (1) for all elements of \( \Gamma_0(l) \), then \( F(z) \) is called a cusp form of weight \( \kappa \) and level \( l \). The form \( F(z) \) is called a new form if it is not a cusp form of level \( l_1 \mid l \). In [7], a universality theorem was obtained for zeta-functions of new forms.

The universality theorem of [2] was generalized in [8] for shifts \( \zeta(s + i\varphi(\tau), F) \) with differentiable function \( \varphi(\tau) \) satisfying the estimates \( (\varphi'(\tau))^{-1} = o(\tau) \) and \( \varphi(2\tau) \max_{\tau \leq t \leq 2\tau} (\varphi'(t))^{-1} = o(\tau) \).
\( \ll \) as \( \tau \to \infty \). The discrete version of results of [8] is given in [9]. In [10], the shifts \( \zeta(s + i\gamma_k, F) \), where \( \{\gamma_k : k \in \mathbb{N}\} \) is the sequence of nontrivial zeros of \( \zeta(s) \), are used.

The joint universality of zeta- and \( L \)-functions is a more complicated problem of analytic number theory. In this case, a collection of analytic functions are simultaneously approximated by a collection of shifts of zeta-functions. The first result in this direction also belongs to Voronin. He considered [11] the functional independence of Dirichlet \( L \)-functions \( L(s, \chi) \) with pairwise nonequivalent Dirichlet characters \( \chi \) and, for this, he obtained their joint universality. The paper [12] is devoted to the joint universality for zeta-functions of new forms twisted by Dirichlet characters, i.e., for the functions

\[
\sum_{m=1}^{\infty} \frac{c(m)\chi(m)}{m^s}, \quad \sigma > \frac{\kappa + 1}{2},
\]

with pairwise nonequivalent Dirichlet characters \( \chi_1, \ldots, \chi_r \).

Joint universality theorems with generalized shifts \( \zeta(s + i\varphi_j(k), F) \), \( j = 1, \ldots, r \), with some differentiable functions \( \varphi_j(\tau) \) can be found in [13]. Continuous and discrete joint universality theorems for more general zeta-functions are given in [14–16].

Our aim is to obtain a joint universality theorem for zeta-functions of normalized Hecke-eigen cusp forms by using different shifts. The first of the denseness results for shifts of a universal function were discussed in [17].

The main result of the paper is the following statement.

**Theorem 2.** Suppose that \( h_1, \ldots, h_r \) are real algebraic numbers linearly independent over the field of rational numbers \( \mathbb{Q} \). For \( j = 1, \ldots, r \), let \( K_j \subset \mathbb{R} \) and \( f_j(s) \in \mathcal{H}_{0,1}(K_j) \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{l \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + ih_j\tau, F) - f_j(s)| < \varepsilon \right\} > 0.
\]

Moreover "\( \liminf \)" can be replaced by "\( \lim \)" for all but at most countably many \( \varepsilon > 0 \).

For the proof of Theorem 2, we will apply the probabilistic approach based on a limit theorem in the space of analytic functions.

### 2. Mean Square Estimates

Recall the metric in the space \( H(D_k) \). Let \( \{K_l : l \in \mathbb{N}\} \subset D_k \) be a sequence of compact subsets such that

\[
D_k = \bigcup_{l=1}^{\infty} K_l.
\]

\( K_l \subset K_{l+1} \) for \( l \in \mathbb{N} \), and if \( K \subset D_k \) is a compact, then \( K \subset K_l \) for some \( l \). For example, we can take \( K_l \) closed rectangles. Then

\[
\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-1} \frac{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}{\sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D_k),
\]

is a metric in \( H(D_k) \) inducing the topology of uniform convergence on compacta. Let

\[
H'(D_k) = \left( H(D_k) \times \cdots \times H(D_k) \right) / r.
\]

For \( g = (g_1, \ldots, g_r) \in H'(D_k) \), \( j = 1, 2 \), define

\[
\rho(g_j, g_2) = \max_{1 \leq j \leq r} \rho(g_j, g_2).
\]
Then, $\rho$ is a metric in $H'(D_\kappa)$ inducing the product topology.
Let $\theta > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\left(-\left(\frac{m}{n}\right)^{\theta}\right), \quad m, n \in \mathbb{N}.$$ 

Then, the series

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^{s}},$$

in view of the estimate

$$c(m) \ll m^{\frac{\kappa-1}{2} + \epsilon},$$

is absolutely convergent in every fixed half plane $\sigma > \hat{\sigma}$. However, for our aim, this convergence is sufficient only for $\sigma > \frac{\kappa}{2}$.

For brevity, let $h = (h_1, \ldots, h_r)$,

$$\zeta(s + ih\tau, F) = (\zeta(s + ih_1\tau, F), \ldots, \zeta(s + ih_r\tau, F))$$

and

$$\zeta_n(s + ih\tau, F) = (\zeta_n(s + ih_1\tau, F), \ldots, \zeta_n(s + ih_r\tau, F)).$$

**Lemma 1.** For all $h$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho\left(\zeta(s + ih\tau, F), \zeta_n(s + ih\tau, F)\right) d\tau = 0.$$ 

**Proof.** By the definitions of the metrics $\rho$ and $\rho$, it suffices to show that, for every $h \in \mathbb{R}$ and compact set $K \subset D_\kappa$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K} |\zeta(s + ih\tau, F), \zeta_n(s + ih\tau, F)| d\tau = 0.$$ 

It is well known that for fixed $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$,

$$\int_{-T}^{T} |\zeta(\sigma + it, F)|^2 dt \ll_{\sigma} T,$$

where $\ll_{\sigma}$ means that the implied constant depends on $\sigma$. Therefore,

$$\int_{-T}^{T} |\zeta(\sigma + iht, F)|^2 dt \ll_{\sigma, h} T,$$

and, for $v \in \mathbb{R}$,

$$\frac{1}{T} \int_0^T |\zeta(\sigma + ih\tau + iv, F)|^2 dv \ll_{\sigma, h} 1 + |v|.$$ 

Let

$$l_n(s) = \frac{z^{\hat{\sigma}}(\frac{z}{\hat{\theta}})^n}{\hat{\theta}^n},$$
where \( \Gamma(z) \) denotes the Euler gamma-function and \( \theta \) is a number from the definition of \( v_n(m) \). Using the Mellin formula

\[
\frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(s) \alpha^s ds = e^{-\pi} \alpha, \quad \alpha, \beta > 0,
\]

we find that

\[
\exp \left\{ \left( \frac{m}{n} \right)^\theta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\theta - i\infty}^{\theta + i\infty} \Gamma \left( \frac{m}{n} \right) \frac{1}{\theta} - s \, ds.
\]

Therefore, in virtue of the definition of the function \( v_n(m) \), we obtain that, for \( \sigma > \frac{\kappa}{2} \),

\[
\zeta_n(s, F) = \frac{1}{\sqrt{2\pi i}} \sum_{m=1}^{\infty} \frac{c(m)}{m^s} \int_{\theta - i\infty}^{\theta + i\infty} \frac{\zeta(z)}{z} \frac{m^z}{z} \, dz
\]

or, using the Mellin formula

\[
\zeta_n(s, F) = \frac{1}{\sqrt{2\pi i}} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, F) l_n(z) \frac{dz}{z}. \tag{3}
\]

Let \( K \in D_\kappa \) be a fixed compact set. Then, there exists \( \epsilon > 0 \) such that, for all \( s = \sigma + it \in K \), the inequalities \( \frac{\kappa}{2} + 2\epsilon < \sigma < \frac{\kappa + 1}{2} - \epsilon \) are satisfied. We take, for such \( \sigma \),

\[
\theta_1 = \frac{\kappa}{2} + \epsilon - \sigma.
\]

Then, \( \theta_1 < 0 \). Therefore, by the residue theorem and (3),

\[
\zeta_n(s, F) - \zeta(s, F) = \frac{1}{\sqrt{2\pi i}} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} \zeta(s + z, F) l_n(z) \frac{dz}{z}.
\]

Hence, for all \( s \in K \),

\[
\zeta(s + ith, F) - \zeta_n(s + ith, F) = \frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} \zeta \left( \frac{\kappa}{2} + \epsilon + it + ith + iv, F \right) l_n \left( \frac{\kappa}{2} + \epsilon - s + iv \right) \frac{dv}{\frac{\kappa}{2} + \epsilon - s + iv}
\]

\[
= \frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} \zeta \left( \frac{\kappa}{2} + \epsilon + ith + iv, F \right) l_n \left( \frac{\kappa}{2} + \epsilon - s + iv \right) \frac{dv}{\frac{\kappa}{2} + \epsilon - s + iv}
\]

\[
\ll \frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} \left| \zeta \left( \frac{\kappa}{2} + \epsilon + ith + iv, F \right) \right| \sup_{s \in K} \left| l_n \left( \frac{\kappa}{2} + \epsilon - s + iv \right) \right| \frac{dv}{\frac{\kappa}{2} + \epsilon - s + iv}.
\]

Thus, in view of (2),

\[
\frac{1}{T} \int_{0}^{\infty} \left| \zeta(s + ith, F) - \zeta_n(s + ith, F) \right| \frac{d\tau}{T} \ll \left( \frac{1}{T} \int_{-\infty}^{\infty} \left| \zeta \left( \frac{\kappa}{2} + \epsilon + ith + iv, F \right) \right| ^2 \frac{d\tau}{T} \right)^{1/2} \sup_{s \in K} \left| l_n \left( \frac{\kappa}{2} + \epsilon - s + iv \right) \right| \frac{dv}{\frac{\kappa}{2} + \epsilon - s + iv}.
\]
\[ \ll_{\varepsilon, h, K} n^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |v|) \exp\{-c_1|v|\} dv \ll_{\varepsilon, h, K} n^{-\varepsilon} \]

Here, we used the estimate
\[ \Gamma \left( \frac{1}{\beta} \left( \frac{\kappa}{2} + \varepsilon - s + iv \right) \right) \ll \exp\left\{-\frac{c}{\beta} |v| - t \right\} \ll_\varepsilon \exp\{-c_1|v|\}, \quad c_1 > 0. \]

Estimate (4) proves the lemma. \( \Box \)

Let \( \mathbb{P} \) be the set of all prime numbers, and \( \gamma_p = \{ s \in \mathbb{C} : |s| = 1 \} \) for all \( p \in \mathbb{P} \). Define the set

\[ \Omega = \prod_{p \in \mathbb{P}} \gamma_p. \]

Then, the torus \( \Omega \) with product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on \( (\Omega, B(\Omega)) \) \( B(\mathcal{X}) \) is the Borel \( \sigma \)-field of the space \( \mathcal{X} \), the probability Haar measure \( m_H \) can be defined. Moreover, let

\[ \Omega = \Omega_1 \times \cdots \times \Omega_r, \]

where \( \Omega_j = \Omega \) for all \( j = 1, \ldots, r \). Once again, \( \Omega \) is a compact topological Abelian group. Therefore, on \( (\Omega, B(\Omega)) \) the probability Haar measure \( m_H \) exists. This gives the probability space \( (\Omega, B(\Omega), m_H) \). Denote by \( m_{jH} \) the Haar measure on \( (\Omega_j, B(\Omega_j)) \), \( j = 1, \ldots, r \). Then, \( m_{jH} \) is the product of the measures \( m_{1H}, \ldots, m_{rH} \). Now, denote by \( \omega = (\omega_1, \ldots, \omega_r) \) the elements of \( \Omega \), where \( \omega_j \in \Omega_j, j = 1, \ldots, r \). Let \( \omega_j(p) \) be the \( p \)-th component of an element \( \omega_j \in \Omega_j, j = 1, \ldots, r \), \( p \in \mathbb{P} \). Extend elements \( \omega_j(p) \) to the set \( \mathbb{N} \) by the formula

\[ \omega_j(m) = \prod_{p^{|m|} \mid m} \omega_j(p), \quad m \in \mathbb{N}, \]

and define \( H(D_k) \)-valued random element

\[ \zeta(s, \omega_j, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_j(m)}{m^s}, \quad j = 1, \ldots, r. \]

The later series is uniformly convergent on compact subsets of \( D_k \) for almost all \( \omega_j \). Moreover, for fixed \( \sigma \in \left( \frac{k}{2}, \frac{k+1}{2} \right) \)

\[ \int_{-T}^{T} |\zeta(s + it, \omega_j, F)|^2 dt \ll_\varepsilon T \]

for almost all \( \omega_j, j = 1, \ldots, r \) \[18\]. Define one more series

\[ \zeta_n(s, \omega_j, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_j(m)v_n(m)}{m^s}, \quad j = 1, \ldots, r, \]

which also, as \( \zeta_n(s, F) \), are absolutely convergent for \( \sigma > \frac{k}{2} \). Let

\[ \zeta(s + ih_1 \tau, \omega) = (\zeta(s + ih_1 \tau, \omega_1, F), \ldots, \zeta(s + ih_1 \tau, \omega_1, F)) \]

and

\[ \zeta_n(s + ih_1 \tau, \omega) = (\zeta_n(s + ih_1 \tau, \omega_1, F), \ldots, \zeta_n(s + ih_1 \tau, \omega_1, F)). \]
Then, repeating the proof of Lemma 1 and using estimate (5), we arrive to the following statement.

**Lemma 2.** For all $\mathbf{f}$ and almost all $\omega$,
\[
\lim_{T \to \infty} \limsup_{n \to \infty} \frac{1}{T} \int_{0}^{T} \rho\left(\xi(s+i\mathbf{f}, \omega), \xi(s+i\mathbf{f}, \omega)\right) d\mathbf{f} = 0.
\]

### 3. Limit Theorems

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define $H(D_\kappa)$-valued random element
\[
\xi(s, \omega, \mathbf{f}) = (\xi(s, \omega_1, \mathbf{f}), \ldots, \xi(s, \omega_1, F))
\]
and denote by $P_{T, \mathbf{f}}$ its distribution, i.e.,
\[
P_{T, \mathbf{f}}(A) = m_H\left\{ \omega \in \Omega : \xi(s, \omega, \mathbf{f}) \in A \right\}, \quad A \in \mathcal{B}(H'(D_\kappa)).
\]

**Theorem 3.** Suppose that $h_1, \ldots, h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$, and
\[
P_{T, \mathbf{f}}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \xi(s + i\mathbf{f}, \omega) \in A \right\}, \quad A \in \mathcal{B}(H'(D_\kappa)).
\]

Then, $P_{T, \mathbf{f}}$ converges weakly to $P_{T, \mathbf{f}}$ as $T \to \infty$.

We divide the proof of Theorem 3 into several lemmas.

**Lemma 3.** Suppose that $\lambda_1, \ldots, \lambda_r$ are algebraic numbers such that the system $\log \lambda_1, \ldots, \log \lambda_r$ is linearly independent over $\mathbb{Q}$. Then, for arbitrary algebraic numbers $\beta_0, \beta_1, \ldots, \beta_r$ that are not all zeros, the inequality
\[
|\beta_0 + \beta_1 \log \lambda_1 + \cdots + \beta_r \log \lambda_r| > h^{-c}
\]
holds. Here, $h$ denotes the height of the numbers $\beta_0, \beta_1, \ldots, \beta_r$, and $c$ is an effective constant depending on $r, \lambda_1, \ldots, \lambda_r$ and maximum of degrees of the numbers $\beta_0, \beta_1, \ldots, \beta_r$.

The lemma is a Baker result on linear forms of logarithm; see, for example, ref. [19].

For $A \in \mathcal{B}(\Omega)$, define
\[
Q_T(A) = \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \left( \left( p^{-i\mathbf{f}} : p \in \mathbb{P} \right), \ldots, \left( p^{-i\mathbf{f}} : p \in \mathbb{P} \right) \right) \in A \right\}.
\]

**Lemma 4.** Let $\lambda_1, \ldots, \lambda_r$ be the same as in Theorem 3. Then, $Q_T$ converges weakly to the Haar measure $\tilde{m}_H$ as $T \to \infty$.

**Proof.** We apply the Fourier transform method. Denote by $\mathcal{G}(k_1, \ldots, k_r), k_j = \{k_{pj} : k_{pj} \in \mathbb{Z}, p \in \mathbb{P}\}, j = 1, \ldots, r$ the Fourier transform of $Q_T$. By the definition of $Q_T$, we have
\[
\mathcal{G}(k_1, \ldots, k_r) = \prod_{j=1}^{r} \prod_{p \in \mathbb{P}}^{\ast} \omega_{pj}^{k_{pj}}(p) dQ_T
\]
\[
\frac{1}{T} \int_{0}^{T} \exp\left\{ -i\tau \sum_{j=1}^{r} \sum_{p \in \mathbb{P}}^{\ast} h_j k_{pj} \log p \right\} d\tau,
\]
where the star shows that only a finite number of integers $k_{pj}$ are not zero. Obviously,
\[
\mathcal{G}(0, \ldots, 0) = 1.
\]
Now, suppose that \((k_1, \ldots, k_r) \neq (0, \ldots, 0)\). Then, there exists a prime number \(p\) such that \(k_p \neq 0\) for some \(j\). Therefore,
\[
\beta_p \overset{\text{def}}{=} \sum_{j=1}^r h_j k_p \neq 0
\]
because the numbers \(h_1, \ldots, h_r\) are linearly independent over \(Q\). Thus, in view of Lemma 3,
\[
B_{k_1, \ldots, k_r} \overset{\text{def}}{=} \sum_{j=1}^k \sum_{p \in \mathbb{P}} h_j k_p \log p = \sum_{p \in \mathbb{P}}^s \beta_p \log p \neq 0.
\]
This and (6) imply
\[
\lim_{r \to \infty} g_T(k_1, \ldots, k_r)
\]
and this proves the lemma. \(\square\)

For \(A \in B(H^r(D_\kappa))\), define
\[
P_{T, n, F}(A) = \frac{1}{T} \meas \{ \tau \in [0, T] : \zeta_n(s + i\tau, F) \in A \}
\]
and
\[
P_{T, n, \Omega, F}(A) = \frac{1}{T} \meas \{ \tau \in [0, T] : \zeta_n(s + i\tau, \omega, F) \in A \}.
\]
Moreover, let the mapping \(u_n : \Omega \to H^r(D_\kappa)\) be given by
\[
u_n,F(\omega) = \zeta_n(s, \omega, F),
\]
and \(V_{n, F} = \overline{m}_H u_{n, F}^{-1}\), where
\[
V_{n, F}(A) = \overline{m}_H \left( u_{n, F}^{-1} A \right), \quad A \in B(H^r(D_\kappa)).
\]

Since the series for \(\zeta_n(s, \omega, j\tau)\) are absolutely convergent for \(\sigma > \frac{r}{2}\), the mapping \(u_{n, F}\) is continuous. Moreover, by the definitions of \(Q_F\) and \(P_{T, n, F}\), we have \(P_{T, n, F} = Q_F u_{n, F}^{-1}\). This equality, continuity of \(u_{n, F}\), Lemma 4, the well-known properties of weak convergence, and the invariance of the Haar measure \(\overline{m}_H\) lead to the following lemma.

**Lemma 5.** Let \(h_1, \ldots, h_r\) be the same as Theorem 3. Then, \(P_{T, n, F}\) and \(P_{T, n, \Omega, F}\) both converge weakly to the measure \(V_{n, F}\) as \(T \to \infty\).

Additionally to \(P_{T, F}\), define
\[
P_{T, \Omega, F}(A) = \frac{1}{T} \meas \{ \tau \in [0, T] : \zeta(s + i\tau, \omega, F) \in A \}, \quad A \in B(H^r(D_\kappa)).
\]

**Lemma 6.** Let \(h_1, \ldots, h_r\) be the same as Theorem 3. Then, on \((H^r(D_\kappa), B(H^r(D_\kappa)))\), there exists a probability measure \(P_F\) such that \(P_{T, F}\) and \(P_{T, \Omega, F}\) both converge weakly to \(P_F\) as \(T \to \infty\).
Proof. Since the series for $\zeta_n(s,F)$ is absolutely convergent, by a standard way it follows—see, for example [14,18]—that the sequence $\{V_{n,F} : n \in \mathbb{N}\}$ is tight, i.e., for every $\epsilon > 0$, there exists a compact set $K \subset H'(D_{\kappa})$ such that

$$V_{n,F}(K) > 1 - \epsilon$$

for all $n \in \mathbb{N}$. Hence, by the Prokhorov theorem, see [20], the sequence $\{V_{n,F}\}$ is relatively compact, i.e., each of its subsequences contains a subsequence $\{V_{n_k,F}\}$ such that $V_{n_k,F}$ converges weakly to a certain probability measure $P_F$ on $(H'(D_{\kappa}),\mathcal{B}(H'(D_{\kappa})))$ as $k \to \infty$.

Let $\xi_T$ be a random variable defined on a certain probability space with measure $\nu$ and uniformly distributed on $[0,T]$. Define the $H'(D_{\kappa})$-valued random element

$$X_{T,n,F} = X_{T,n,F}(s) = \zeta_n(s + ih\xi_T,F)$$

and denote by $X_{n,F} = X_{n,F}(s)$ the $H'(D_{\kappa})$-valued random element having the distribution $V_{n,F}$. Then, by Lemma 5, we have

$$X_{T,n,F} \overset{D}{\to} X_{n,F}$$

(8)

where $\overset{D}{\to}$ means the convergence in distribution. Moreover, since $V_{n_k,F}$ converges weakly to $P_F$, the relation

$$X_{n_k,F} \overset{D}{\to} P_F$$

(9)

is true. Let

$$X_{T,F} = X_{T,F}(s) = \zeta(s + ih\xi_T,F).$$

Then, using Lemma 1, we find that for every $\epsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \nu\left\{ \rho(X_{T,F},X_{T,n,F}) \geq \epsilon \right\} \leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\epsilon T} \int_0^T \rho\left(\zeta(s + ih\tau,F),\zeta_n(s + ih\tau,F)\right) d\tau = 0.$$

The later equality together with (8) and (9), and Theorem 4.2 of [20] lead to the relation

$$X_{T,F} \overset{D}{\to} P_F.$$  

(10)

This proves that $P_{T,F}$ converges weakly to $P_F$ as $T \to \infty$.

The relation (10) shows that the limit measure $P_F$ is independent of the subsequence $\{n_k\}$. Therefore, we have

$$X_{n_k,F} \overset{D}{\to} P_F.$$  

(11)

Define the $H'(D_{\kappa})$-valued random elements

$$X_{T,n,\xi,F} = X_{T,n,\xi,F}(s) = \zeta_n(s + ih\xi_T,\xi,F)$$

and

$$X_{T,\xi,F} = X_{T,\xi,F}(s) = \zeta(s + ih\xi_T,\xi,F).$$

Then, repeating the above arguments using Lemmas 2 and 5, and relation (11), we obtain that

$$X_{T,n,\xi,F} \overset{D}{\to} P_F,$$

and this is equivalent to weak convergence of $P_{T,\xi,F}$ to $P_F$ as $T \to \infty$. The lemma is proved. $\Box$
To prove Theorem 3, it remains to show that $P_F = P_{\xi_F}$. For this, we will apply some elements of the ergodic theory. For brevity, let
\[
h_\tau = \left( \left( p^{-ih_1\tau} : p \in \mathbb{P} \right), \ldots, \left( p^{-ih_r\tau} : p \in \mathbb{P} \right) \right), \quad \tau \in \mathbb{R}.
\]
Define the transformation of $\Omega$
\[
\varphi_\tau(\omega) = h_\tau \omega, \quad \omega \in \Omega.
\]
Since the Haar measure $m_H$ is invariant, the transformation $\varphi_\tau$ is measure-preserving and \{\varphi_\tau : \tau \in \mathbb{R}\} is a one-parameter group. A set $A \in B(\Omega)$ is called invariant with respect to the group \{\varphi_\tau\} if the sets $A$ and $\varphi_\tau(A)$, $\tau \in \mathbb{R}$, differ one from another at most by a set of $m_H$-measure zero.

**Lemma 7.** Let $h_1, \ldots, h_r$ be the same as Theorem 3. Then, the group \{\varphi_\tau\} is ergodic, i.e., the $\sigma$-field of invariant sets consists of sets having $m_H$-measure 1 or 0.

**Proof.** The characters $\chi$ of the group $\Omega$ are of the form
\[
\chi(\omega) = \prod_{j=1}^r \prod_{p \in \mathbb{P}} \alpha_{p^j}(p).
\]
This fact already was used in the proof of Lemma 4. Let $A$ be an arbitrary invariant set, $I_A$ its indicator function, and $\chi$ be a nontrivial character. Preserving the notation of the proof of Lemma 4, we have $(k_1, \ldots, k_r) \neq (0, \ldots, 0)$ and $B_{k_1, \ldots, k_r} \neq 0$. Therefore, there exists $\tau_0 \in \mathbb{R}$ such that
\[
\chi(h_\tau) = \exp \{ -i\tau_0 B_{k_1, \ldots, k_r} \} \neq 1. \quad (12)
\]
Moreover, in view of the invariance of $A$, we have
\[
I_A(h_\tau \omega) = I_A(\omega) \quad (13)
\]
for almost all $\omega \in \Omega$. Denote by $\hat{I}_A$ the Fourier transform of $I_A$. Then, by (13),
\[
\hat{I}_A(\chi) = \chi(h_m) \int_{\Omega} I_A(h_m \omega) \chi(\omega) d m_H = \chi(h_m) \hat{I}_A(\chi).
\]
This and (12) show that
\[
\hat{I}_A(\chi) = 0. \quad (14)
\]
Now, let $\chi_0$ denote the trivial character of $\Omega$, and suppose that $\hat{I}_A(\chi_0) = \alpha$. Then, in view of (14), we find that
\[
\hat{I}_A(\chi) = \alpha \int_{\Omega} \chi(\omega) d m_H = \alpha(\chi).
\]
Hence, $I_A(\omega) = \alpha$ for almost all $\omega \in \Omega$. Since $I_A$ is the indicator function, $I_A(\omega) = 1$ or $I_A(\omega) = 0$ for almost all $\omega$. Thus, $m_H(A) = 1$ or $m_H(A) = 0$, and the lemma is proved. \qed

**Proof of Theorem 3.** We have mentioned that it suffices to show that $P_F = P_{\xi_F}$. By Lemma 6 and the equivalent of weak convergence in terms of continuity sets, we have
\[
\lim_{T \to \infty} P_{T,\Omega} F(A) = P_F(A) \quad (15)
\]
for a continuity set $A$ of the measure $P_{\mathcal{T}}$, i.e., $P_{\mathcal{T}}(\partial A) = 0$, where $\partial A$ is the boundary of $A$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_{H})$, define the random variable

$$\xi(\omega) = \begin{cases} 1 & \text{if } \xi(s, \omega, F) \in A, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 7 implies the ergodicity of the random process $\xi(\varphi_{r}(\omega))$. Therefore, by the classical Birkhoff–Khintchine ergodic theorem, see, for example [21],

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \xi(\varphi_{r}(\omega)) \, d\tau = \mathbb{E}\xi = P_{\mathcal{T}}(A),$$

(16)

where $\mathbb{E}\xi$ is the expectation of $\xi$.

However, by the definitions of $\varphi_{r}$ and $\xi$,

$$\frac{1}{T} \int_{0}^{T} \xi(\varphi_{r}(\omega)) \, d\tau = \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \xi(s + i\tau, \omega, F) \in A \right\} = P_{\mathcal{T}}(A).$$

This and (16) show that

$$\lim_{T \to \infty} P_{\mathcal{T}}(A) = P_{\mathcal{T}}(A).$$

Therefore, by (15), we obtain that $P_{\mathcal{T}}(A) = P_{\mathcal{T}}(A)$ for all continuity sets $A$ of $P_{\mathcal{T}}(A)$. Hence, $P_{\mathcal{T}} = P_{\mathcal{T}}$ and the theorem is proved. $\square$

4. Proof of Theorem 2

Recall that the support of the measure $P_{\mathcal{T}}$ is a minimal closed set $S_{\mathcal{T}} \subset H'(D_{k})$ such that $P_{\mathcal{T}}(S_{\mathcal{T}}) = 1$.

Lemma 8. The support of the measure $P_{\mathcal{T}}$ is the set $\left( \{g \in H(D_{k}) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \} \right)^{r}$.

Proof. Since the space $H'(D_{k})$ is separable, we have [20],

$$\mathcal{B}(H'(D_{k})) = \left( \mathcal{B}(H(D_{k})) \times \cdots \times \mathcal{B}(H(D_{k})) \right)^{r}.$$

Therefore, it suffices to consider the measure $P_{\mathcal{T}}$ on the rectangular sets

$$A = A_{1} \times \cdots \times A_{r}, \quad A_{1}, \ldots, A_{r} \in H(D_{k}).$$

Let $\xi(s, \omega, F)$ be the $H(D_{k})$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_{H})$, where $m_{H}$ is the Haar measure. Then, it is known [10] that the support of $\xi(s, \omega, F)$ is the set $\{g \in H(D_{k}) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}$. Thus, the same set is the support of the distributions of $\xi(s, \omega_{j}, F)$, $j = 1, \ldots, r$. Since the measure $m_{H}$ is the product of the measures $m_{jH}$, $j = 1, \ldots, r$, we have

$$m_{H} \left\{ \omega \in \Omega : \xi(s, \omega, F) \in A \right\} = \prod_{j=1}^{r} m_{jH} \left\{ \omega_{j} \in \Omega_{j} : \xi(s, \omega_{j}, F) \in A_{j} \right\}.$$

This equality, the minimality of the support, and the support of the distributions of $\xi(s, \omega_{j}, F)$ prove the lemma. $\square$
Proof of Theorem 2. By the Mergelyan theorem on the approximation of analytic functions by polynomials [22], there exist polynomials \( p_1(s), \ldots, p_r(s) \) such that
\[
\sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\epsilon}{2}.
\]

Define the set
\[
G_\epsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D_\epsilon) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\epsilon}{2} \right\}.
\]

In view of Lemma 8, the set \( G_\epsilon \) is an open neighborhood of an element \( (e^{p_1(s)}, \ldots, e^{p_r(s)}) \) in support of the measure \( P_{\mathcal{L}, F} \). Hence,
\[
P_{\mathcal{L}, F}(G_\epsilon) > 0.
\]

This, Theorem 3 and the equivalent of weak convergence in terms of open sets, and the definitions of \( P_{T,F} \) and \( G_\epsilon \) prove the theorem with “lim inf”. Define one more set
\[
\hat{G}_\epsilon = \left\{ (g_1, \ldots, g_r) \in H^r(D_\epsilon) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \epsilon \right\},
\]

There \( \partial \hat{G}_{\epsilon_1} \cap \partial \hat{G}_{\epsilon_2} = \emptyset \) for \( \epsilon_1 \neq \epsilon_2 \). This shows that \( P_{\mathcal{L}, F}(\partial \hat{G}_\epsilon) = 0 \) for all but, for those countable, many \( \epsilon > 0 \). Moreover, (17) and (18) imply that \( P_{\mathcal{L}, F}(\hat{G}_\epsilon) > 0 \). This, Theorem 3 and the equivalent of weak convergence of probability measures in terms of continuity sets, and the definitions of \( P_{T,F} \) and \( \hat{G}_\epsilon \) prove the theorem with “lim”. \( \square \)

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References
1. Voronin, S.M. Theorem on the “universality” of the Riemann zeta-function. Math. USSR Izv. 1975, 9, 443–453. [CrossRef]
2. Laurinčikas, A.; Matsumoto, K. The universality of zeta-functions attached to certain cusp forms. Acta Arith. 2001, 98, 345–359. [CrossRef]
3. Laurinčikas, A.; Matsumoto, K.; Steuding, J. Discrete universality of \( L \)-functions for new forms. Math. Notes 2005, 78, 551–558. [CrossRef]
4. Laurinčikas, A.; Matsumoto, K.; Steuding, J. Discrete universality of \( L \)-functions for new forms. II. Litt. Math. J. 2016, 56, 207–218. [CrossRef]
5. Laurinčikas, A.; Matsumoto, K.; Steuding, J. Universality of some functions related to zeta-functions of certain cusp forms. Osaka J. Math. 2013, 50, 1021–1037.
6. Laurinčikas, A. On the functional independence of zeta-functions of certain cusp forms. Math. Notes 2020, 107, 609–617. [CrossRef]
7. Laurinčikas, A.; Matsumoto, K.; Steuding, J. The universality of \( L \)-functions associated with new forms. Izv. Math. 2003, 67, 77–90. [CrossRef]
8. Vaiginytė, A. Extention of the Laurinčikas-Matsumoto theorem. Chebyshevskii Sb. 2019, 20, 82–93.
9. Laurinčikas, A.; Šiaučiūnas, D.; Vaiginytė, A. Extension of the discrete universality theorem for zeta-functions of certain cusp forms. Nonlinear Anal. Model. Control 2018, 23, 961–973. [CrossRef]
10. Balčiunas, A.; Frančevič, V.; Garbaitauskiene, V.; Macaitiene, R.; Rimkevičienė, A. Universality of zeta-functions of cusp forms and non-trivial zeros of the Riemann zeta-function. Math. Model. Anal. 2021, 26, 82–93. [CrossRef]
11. Voronin, S.M. On the functional independence of Dirichlet \( L \)-functions. Acta Arith. 1975, 27, 493–503. (In Russian)
12. Laurinčikas, A.; Matsumoto, K. The joint universality of twisted automorphic \( L \)-functions. J. Math. Soc. Jpn. 2004, 56, 923–939. [CrossRef]
13. Laurinčikas, A.; Šiaučiūnas, D., Vaiginytė, A. On joint approximation of analytic functions by nonlinear shifts of zeta-functions of certain cusp forms. Nonlin. Anal. Model. Control 2020, 25, 108–125. [CrossRef]
14. Laurinčikas, A. Joint universality of zeta-functions with periodic coefficients. Izv. Math. 2010, 74, 515–539. [CrossRef]
15. Laurinčikas, A. Joint universality of zeta functions with periodic coefficients. II. Sib. Math. J. 2020, 61, 1064–1076. [CrossRef]
16. Laurinčikas, A. Joint discrete universality for periodic zeta-functions. III. Quaest. Math. 2020. [CrossRef]
17. Kaczorowski, J.; Laurinčikas, A.; Steuding, J. On the value distribution of shifts of universal Dirichlet series. *Monatsh. Math.* 2006, 147, 309–317. [CrossRef]

18. Kačenas, A.; Laurinčikas, A. On Dirichlet series related to certain cusp forms. *Lith. Math. J.* 1998, 38, 64–76. [CrossRef]

19. Baker, A. The theory of linear forms in logarithms. In *Transcendence Theory: Advances and Applications. Proceedings of a Conference Held in Cambridge in 1976*; Baker, A., Masser, D.W., Eds.; Academic Press: Boston, MA, USA, 1977; pp. 1–27.

20. Billingsley, P. *Convergence of Probability Measures*; Willey: New York, NY, USA, 1968.

21. Cramér, H.; Leadbetter, M.R. *Stationary and Related Stochastic Processes*; Willey: New York, NY, USA, 1967.

22. Mergelyan, S.N. Uniform approximations to functions of a complex variable. *Am. Math. Soc. Transl. Ser. 2* 1962, 3, 294–391.