Reduction of a bi-Hamiltonian hierarchy on $T^*U(n)$ to spin Ruijsenaars–Sutherland models

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Abstract

We first exhibit two compatible Poisson structures on the cotangent bundle of the unitary group $U(n)$ in such a way that the invariant functions of the $u(n)^*$-valued momenta generate a bi-Hamiltonian hierarchy. One of the Poisson structures is the canonical one and the other one arises from embedding the Heisenberg double of the Poisson-Lie group $U(n)$ into $T^*U(n)$, and subsequently extending the embedded Poisson structure to the full cotangent bundle. We then apply Poisson reduction to the bi-Hamiltonian hierarchy on $T^*U(n)$ using the conjugation action of $U(n)$, for which the ring of invariant functions is closed under both Poisson brackets. We demonstrate that the reduced hierarchy belongs to the overlap of well-known trigonometric spin Sutherland and spin Ruijsenaars–Schneider type integrable many-body models, which receive a bi-Hamiltonian interpretation via our treatment.

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1 Introduction

The many-body models of Calogero–Moser–Sutherland \[6, 28, 45\] and Ruijsenaars–Schneider \[37\] type are among the most interesting examples of finite-dimensional integrable systems both from the mathematical point of view and regarding their diverse physical applications. See, for example, the reviews \[29, 36, 46\] and references therein. The spin extensions of these models \[19, 47, 24\] are also important, and are currently subject to intense studies \[2, 8, 13, 14, 21, 33, 34, 38\].

The purpose of the present paper is to deepen the understanding of the Hamiltonian structure for a system of evolution equations that belongs to the above-mentioned family. The equations at issue have the form

\[ \dot{Q} = (iL^k)_0 Q, \quad \dot{L} = [R(Q)(iL^k), L], \]  

(1.1)

where \( Q \in T_n^{\text{reg}} \) is a diagonal unitary matrix with distinct eigenvalues, \( L \in \mathfrak{f}(n) \) is an \( n \times n \) Hermitian matrix, and the subscript 0 means diagonal part. The dynamical \( r \)-matrix \( R(Q) \) is the linear operator on \( \mathfrak{gl}(n, \mathbb{C}) \) that acts as zero on the diagonal matrices and acts on off-diagonal matrices according to

\[ R(Q) = \frac{1}{2}(\text{Ad}_Q + \text{id})(\text{Ad}_Q - \text{id})^{-1}. \]  

(1.2)

The inverse is well-defined on the off-diagonal subspace by virtue of the regularity of \( Q \); \( \text{Ad}_Q(X) = QXQ^{-1} \). The evolution equations associated with arbitrary \( k \in \mathbb{N} \) mutually commute if we restrict attention to ‘observables’ \( f(Q, L) \) that are invariant with respect to conjugations of \( L \) by diagonal unitary matrices. This means that the phase space of the system must be taken to be one of the two quotient spaces:

\[ T_n^{\text{reg}} \times (\mathfrak{f}(n)/T^n) \quad \text{or} \quad (T_n^{\text{reg}} \times \mathfrak{f}(n))/\mathcal{N}(n), \]  

(1.3)

where \( \mathcal{N}(n) \) is the normalizer of \( T^n < U(n) \) in \( U(n) \). The latter choice is actually more natural since it takes into account a hidden symmetry with respect to the permutation group \( S_n = \mathcal{N}(n)/T^n \). Accordingly, the physical observables are identified with the invariant real functions forming

\[ C^\infty(T_n^{\text{reg}} \times \mathfrak{f}(n))/\mathcal{N}(n). \]  

(1.4)

The system has a well-known Hamiltonian structure \[9, 17, 26, 32\], which arises via the parametrization

\[ L = p + (R(Q) + \frac{1}{2}\text{id})(\phi), \]  

(1.5)

where \( p \in \mathfrak{f}(n)_0 \) and \( \phi \in \mathfrak{f}(n)_\perp \), that is, they are Hermitian diagonal and off-diagonal matrices, respectively. The diagonal entries \( p_j \) of \( p \) and \( q_j \) in \( Q_j = e^{iq_j} \) represent canonically conjugate pairs, and are combined with the Poisson algebra carried by the quotient

\[ \mathfrak{f}(n)_\perp/T^n \equiv \mathfrak{u}(n) ^* /_0 \mathbb{T}^n. \]  

(1.6)

The quotient \( (1.6) \) embodies a Hamiltonian reduction \[31\] of the Lie-Poisson bracket of \( \mathfrak{u}(n) \) defined by utilizing the action of \( T^n < U(n) \) on \( \mathfrak{u}(n)^* \equiv \mathfrak{f}(n) \). In correspondence with \( (1.4) \), only the \( S_n \)-invariant elements of the full Poisson algebra are kept. The \( k = 1 \) member of the ‘hierarchy’ \( (1.1) \) is generated by the standard spin Sutherland Hamiltonian

\[ H_{\text{Suth}}(Q, p, \phi) = \frac{1}{2} \text{tr} \left(L(Q, p, \phi)-2\right) = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{8} \sum_{k \neq l} \frac{|\phi_{kl}|^2}{\sin^2 \frac{|\phi_{kl}|^2}{2}}, \]  

(1.7)
We stress that the Hamiltonian belongs to the space (1.4) and governs the time development of the physical observables.

In the ‘unparametrized form’ (1.1) the system can be viewed also as a degenerate limiting case of the spin Ruijsenaars–Schneider (RS) models introduced by Krichever and Zabrodin [24]. This interpretation was pointed out in the papers [3, 25], without noticing the coincidence with the spin Sutherland model. To be more exact, in these references the hyperbolic analogue of the system (1.1) was considered.

It is well-known that the restriction of the system (1.1) to a $2n$-dimensional symplectic leaf of the above-mentioned Poisson structure gives the spinless Sutherland model [20]. Interestingly, as explained below, another specialization gives the spinless trigonometric RS model. This latter specialization arises by restriction to a $2n$-dimensional symplectic leaf with respect to another Poisson structure, from which the same equations can be derived. To emphasize its double interpretation, the system (1.1) will be referred to as the trigonometric spin Ruijsenaars–Sutherland hierarchy.

The standard Poisson structure of the spin Sutherland model (1.7) results by applying a Poisson reduction [17, 20, 32] to the canonical Poisson structure of the cotangent bundle $T^* U(n)$. Our principal goal is to show that the cotangent bundle can be equipped with another Poisson structure, too, whose reduction induces another Poisson bracket on the space of observables (1.4). The two Poisson structures on $T^* U(n)$ as well as their reductions to (1.4) turn out to be compatible in the sense of bi-Hamiltonian geometry, and the evolution equations (1.1) as well as their unreduced avatars enjoy the bi-Hamiltonian property. This is the main result of the paper. (For background on bi-Hamiltonian systems, see e.g. [11, 41, 43].) As we shall see, the pertinent second Poisson structure is transferred to the cotangent bundle from the Heisenberg double [40] of the Poisson-Lie group $U(n)$.

The spinless trigonometric RS model was derived in [15, 16] by symplectic reduction of the free system on the Heisenberg double of $U(n)$ at a particular value of the corresponding moment map. It is true in general that the reduced phase spaces of symplectic reduction are symplectic leaves in the quotient of the original phase space defined by Poisson reduction. This explains how the spinless RS model appears on a symplectic leaf of the Ruijsenaars–Sutherland hierarchy with respect to its second Poisson structure. The reader may consult [13], too, where, we studied symplectic reductions of Heisenberg doubles at arbitrary moment map values; but without dealing with any bi-Hamiltonian aspect.

The bi-Hamiltonian structure of the hyperbolic analytic continuation of the trigonometric system (1.1) is described in [14]. However, in that case we do not have an explanation via a single Poisson reduction. Incidentally, a related problem is that no Hamiltonian reduction treatment of the real, repulsive hyperbolic spinless RS model is known.\footnote{This is so despite the fact that the holomorphic hyperbolic/trigonometric RS model is well-understood in more than one reduction approaches [1, 7, 18, 30]. Treating real forms of holomorphic integrable systems is a highly non-trivial task in general.}

Now we give an outline of the rest of the text. We start in Section 2 by presenting a tailor-made account of the Heisenberg double. In Section 3, we exhibit the bi-Hamiltonian structure on the cotangent bundle, and show that the free geodesic motion on $U(n)$ is encoded by a bi-Hamiltonian system. This is the content of Proposition 3.2 together with Lemma 3.3, which represent our first new result. Section 4 is the essential part of the paper, where we characterize the Poisson reduction of the bi-Hamiltonian manifold $\mathcal{M} := T^* U(n)$. Our main result is Theorem 4.5, which gives the compatible Poisson brackets on the space of functions (1.4). In addition, we show that the equations of motion (1.1) descend from the free system on $T^* U(n)$, and also display a large set of constants of motion. In Section 5, we give our conclusions and discuss how spin degrees of freedom can be introduced in relation to the second Poisson bracket.
The rudiments of the Heisenberg double

The material collected below is well known to experts (see e.g. [16, 27, 22, 23, 40]), except perhaps the presentation of the quasi-adjoint action that we shall give. We start by recalling that the Heisenberg double of the standard Poisson-Lie group $\text{U}(n)$ is the real Lie group $\text{GL}(n, \mathbb{C})$ equipped with a certain Poisson structure. This Poisson structure is actually symplectic, and it contains all information about the Poisson structure on $\text{U}(n)$ as well.

Before presenting the Poisson structure, we introduce two diffeomorphisms

$$m_1 : \text{GL}(n, \mathbb{C}) \to \text{U}(n) \times \text{B}(n), \quad m_2 : \text{U}(n) \times \text{B}(n) \to \text{U}(n) \times \mathfrak{P}(n),$$

(2.1)

where $\text{B}(n)$ is the subgroup of $\text{GL}(n, \mathbb{C})$ consisting of the upper triangular matrices with positive diagonal entries, and $\mathfrak{P}(n)$ contains the Hermitian, positive elements of $\text{GL}(n, \mathbb{C})$.

Every element $K \in \text{GL}(n, \mathbb{C})$ admits the unique decompositions

$$K = b_L g_L^{-1} = g_L b_R^{-1} \quad \text{with} \quad b_L, b_R \in \text{B}(n), \quad g_L, g_R \in \text{U}(n),$$

(2.2)

and $K$ can be recovered also from the pairs $(g_L, b_L)$ and $(g_R, b_R)$, by utilizing the decompositions

$$b_L^{-1} g_L = g_R^{-1} b_R.$$

(2.3)

It is easily seen from this that the map $m_1$ defined by

$$m_1(K) := (g_R, b_R)$$

(2.4)

is a diffeomorphism; and so is the map

$$m_2(g_R, b_R) := (g_R, b_R b_R^\dagger).$$

(2.5)

We shall use these maps to transfer the Poisson structure of $\text{GL}(n, \mathbb{C})$ to the model spaces $\text{U}(n) \times \text{B}(n)$ and $\text{U}(n) \times \mathfrak{P}(n)$.

Consider the real Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ and equip it with the non-degenerate, invariant bilinear form

$$\langle X, Y \rangle := \Im \text{tr}(XY), \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}).$$

(2.6)

Introduce the linear subspace of Hermitian matrices

$$\mathfrak{H}(n) := \text{i}\mathfrak{u}(n),$$

(2.7)

and the subalgebra

$$\mathfrak{b}(n) := \text{span}_\mathbb{R}\{E_{jj}, E_{kl}, iE_{kl} | 1 \leq j \leq n, \ 1 \leq k < l \leq n\},$$

(2.8)

where $E_{kl}$ is the elementary matrix of size $n$, having 1 at the $kl$ position. Both $\mathfrak{H}(n)$ and $\mathfrak{b}(n)$ are in duality with $\mathfrak{u}(n)$ with respect to the bilinear form (2.6). The real vector space decomposition

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{b}(n)$$

(2.9)

allows us to write every $X \in \mathfrak{gl}(n, \mathbb{C})$ in the form

$$X = X_{\mathfrak{u}(n)} + X_{\mathfrak{b}(n)}$$

(2.10)

with constituents in the respective subalgebras.
For any real function $f \in C^\infty(\text{GL}(n, \mathbb{C}))$, define the $\mathfrak{gl}(n, \mathbb{C})$-valued derivatives $\nabla f$ and $\nabla' f$ by

$$
\langle \nabla f(K), X \rangle := \left. \frac{d}{dt} \right|_{t=0} f(e^{tX}K), \quad \langle \nabla' f(K), X \rangle := \left. \frac{d}{dt} \right|_{t=0} f(Ke^{tX}), \ \forall X \in \mathfrak{gl}(n, \mathbb{C}).
$$

(2.11)

For any function $\phi \in C^\infty(\text{U}(n))$ introduce the $\mathfrak{b}(n)$-valued derivatives by

$$
\langle D\phi(g), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \phi(e^{tX}g), \quad \langle D'\phi(g), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \phi(ge^{tX}), \ \forall X \in \mathfrak{u}(n),
$$

(2.12)

and for any $\chi \in C^\infty(\text{B}(n))$ introduce the $\mathfrak{u}(n)$-valued derivatives by

$$
\langle D\chi(b), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \chi(e^{tX}b), \quad \langle D'\chi(b), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \chi(be^{tX}), \ \forall X \in \mathfrak{b}(n).
$$

(2.13)

Finally, for $\psi \in C^\infty(\mathfrak{H}(n))$, define the $\mathfrak{u}(n)$-valued derivative $d\psi$ by

$$
\langle d\psi(L), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \psi(L + tX), \ \forall X \in \mathfrak{H}(n).
$$

(2.14)

This definition makes sense since $(L + tX) \in \mathfrak{H}(n)$ for small $t$; remember that $\mathfrak{H}(n) := i\mathfrak{u}(n)$.

Following Semenov-Tian-Shansky [40], we introduce the (non-degenerate) Poisson bracket \{ , \}+ on $C^\infty(\text{GL}(n, \mathbb{C}))$ by

$$
\{ f_1, f_2 \}_+ = \langle \nabla f_1, R\nabla f_2 \rangle + \langle \nabla' f_1, R\nabla' f_2 \rangle,
$$

(2.15)

where $R := \frac{1}{2} (P_{\mathfrak{u}(n)} - P_{\mathfrak{b}(n)})$ is half the difference of the projection operators on $\mathfrak{gl}(n, \mathbb{C})$ associated with the decomposition (2.9).

We can express the Poisson structure of the Heisenberg double in terms of the variables $(g, b) \equiv (g_R, b_R)$ and $(g, L) \equiv (g_R, b_R b^R_R)$. In other words, the manifolds $\text{U}(n) \times \text{B}(n)$ and $\text{U}(n) \times \mathfrak{H}(n)$ carry unique Poisson structures $\{ , \}_+$ and $\{ , \}_2^+$ for which

$$
m_1 : (\text{GL}(n, \mathbb{C}), \{ , \}_+) \to (\text{U}(n) \times \text{B}(n), \{ , \}_1^+) \tag{2.16}
$$

and

$$
m_2 : (\text{U}(n) \times \text{B}(n), \{ , \}_1^+) \to (\text{U}(n) \times \mathfrak{H}(n), \{ , \}_2^+) \tag{2.17}
$$

are Poisson diffeomorphisms. Straightforward calculations lead to the following formulas.

**Proposition 2.1.** For $F \in C^\infty(\text{U}(n) \times \text{B}(n))$ denote $D_1 F$ and $D_2 F$ the derivatives with respect to the first and second arguments. The Poisson bracket of $F, \mathcal{H} \in C^\infty(\text{U}(n) \times \text{B}(n))$ can be written as follows:

$$
\{ F, \mathcal{H} \}_1^+(g, b) = \langle D_2^\mathcal{H}, b^{-1}(D_2 \mathcal{H})b \rangle - \langle D'_1 F, g^{-1}(D_1 \mathcal{H})g \rangle + \langle D_1 F, D_2 \mathcal{H} \rangle - \langle D_1 \mathcal{H}, D_2 F \rangle,
$$

(2.18)

where the derivatives on the right-hand side are taken at $(g, b) \in \text{U}(n) \times \text{B}(n)$.

**Proposition 2.2.** For $F \in C^\infty(\text{U}(n) \times \mathfrak{H}(n))$ denote $D_1 F$ and $d_2 F$ the derivatives with respect to the first and second arguments. We have the following formula:

$$
\{ F, H \}_2^+(g, L) = 4 \left. \langle Ld_2 F, (Ld_2 H)u_{(n)} \rangle \right|_{u_{(n)}} - \langle D'_1 F, g^{-1}(D_1 H)g \rangle + 2 \langle D_1 F, Ld_2 H \rangle - 2 \langle D_1 H, Ld_2 F \rangle,
$$

(2.19)

\[\text{If not specified otherwise, our spaces of } C^\infty\text{-functions always denote spaces of real functions.}\]
where the derivatives are taken at \((g, L) \in U(n) \times \mathfrak{p}(n)\), and \((2.11)\) is applied to \(X = (Ld_2H)\).

Referring to the decompositions \((2.2)\), let us now introduce the maps \(\Lambda_L, \Lambda_R\) from \(GL(n, \mathbb{C})\) to \(B(n)\), and the maps \(\Xi_L, \Xi_R\) from \(GL(n, \mathbb{C})\) to \(U(n)\) by

\[
\Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R, \quad \Xi_L(K) := g_L, \quad \Xi_R(K) := g_R.
\]  

(2.20)

The maps \(\Lambda_L\) and \(\Lambda_R\) are Poisson maps with respect to the standard multiplicative Poisson bracket on \(B(n)\), which is encoded by the first term of the formula \((2.18)\). Moreover, the map

\[
\Lambda(K) := b_L b_R,
\]

(2.21)

is also a Poisson map. Similarly, the maps \(\Xi_L\) and \(\Xi_R\) are Poisson maps, if \(U(n)\) is endowed with the Poisson structure that appears in the second term of \((2.18)\). It follows from general results that \(\Lambda\) is the moment map, in the sense of Lu \([27]\), for a certain Poisson action of \(U(n)\) on the Heisenberg double. This action was named ‘quasi-adjoint action’ by Klimčík \([22]\).

For any \(\eta \in U(n)\), let \(A_\eta\) denote the diffeomorphism of \(GL(n, \mathbb{C})\) associated with the quasi-adjoint action. It operates \([22]\) according to

\[
A_\eta(K) = \eta K \Xi_R(\eta \Lambda_L(K)).
\]  

(2.22)

The quasi-adjoint Poisson action

\[
A : U(n) \times GL(n, \mathbb{C}) \to GL(n, \mathbb{C}), \quad A(\eta, K) := A_\eta(K),
\]

(2.23)

gives rise to Poisson actions \(A^1\) and \(A^2\) on \(U(n) \times B(n)\) and on \(U(n) \times \mathfrak{p}(n)\), respectively, via the definitions

\[
A^1_\eta := m_1 \circ A_\eta \circ m_1^{-1} \quad \text{and} \quad A^2_\eta := m_2 \circ A_\eta \circ m_2^{-1}.
\]  

(2.24)

One can check that these actions obey the following formulas:

\[
A^1_\eta(g, b) = (\tilde{\eta} g \tilde{\eta}^{-1}, \Lambda_L(\tilde{\eta} b)) \quad \text{with} \quad \tilde{\eta} = \Xi_R(\eta \Lambda_L(m_1^{-1}(g, b)))^{-1}
\]  

(2.25)

and

\[
A^2_\eta(g, L) = (\tilde{\eta} g \tilde{\eta}^{-1}, \tilde{\eta} L \tilde{\eta}^{-1}) \quad \text{with} \quad \tilde{\eta} = \Xi_R(\eta \Lambda_L(m_1^{-1}(g, L)))^{-1}, \quad m := m_2 \circ m_1.
\]  

(2.26)

It is also not difficult to see that for any fixed \((g, b)\) the map \(\eta \mapsto \tilde{\eta}\) is a diffeomorphism of \(U(n)\). This leads to the following auxiliary statement.

**Lemma 2.3.** The actions \(\tilde{A}^1\) and \(\tilde{A}^2\) of \(U(n)\) on \(U(n) \times B(n)\) and on \(U(n) \times \mathfrak{p}(n)\) defined by the formulas

\[
\tilde{A}^1_\eta(g, b) = (\eta g \eta^{-1}, \Lambda_L(\eta b)), \quad \tilde{A}^2_\eta(g, L) = (\eta g \eta^{-1}, \eta L \eta^{-1}), \quad \forall \eta \in U(n),
\]

(2.27)

have the same orbits as the respective Poisson actions \(A^1\) and \(A^2\).

It is plain from Lemma 2.3 that the tilded and the corresponding untilded actions possess the same invariants. On the other hand, for any Poisson action, it is a standard fact that the Poisson bracket of any two invariant functions is again invariant. This leads to the next corollary.

**Corollary 2.4.** The ring of invariants \(C^\infty(U(n) \times \mathfrak{p}(n))^U(n)\), associated with the action \(\tilde{A}^2\), is a Poisson subalgebra of \(C^\infty(U(n) \times \mathfrak{p}(n))\) with respect to the Poisson bracket \(\{ , \}^\ast\).

An analogous result holds for the model \(U(n) \times B(n)\) of the Heisenberg double as well. We highlighted the statement of Corollary 2.4, since it will be used later. Incidentally, if a name is required at all, the action \(\tilde{A}^2\) of \(U(n)\) may be called *undressed quasi-adjoint action*.
3 Bi-Hamiltonian hierarchy on $T^*U(n)$

Let us consider the manifold

$$\mathcal{M} := U(n) \times \mathfrak{h}(n) := \{(g, L) \mid g \in U(n), L \in \mathfrak{h}(n)\}, \quad (3.1)$$

which (as explained below) serves as a model of the cotangent bundle $T^*U(n)$. Like in Section 2, for any function $F \in C^\infty(\mathcal{M})$, we have the derivatives

$$D_1F, D'_1F \in C^\infty(\mathcal{M}, \mathfrak{b}(n)) \quad \text{and} \quad d_2F \in C^\infty(\mathcal{M}, \mathfrak{u}(n)) \quad (3.2)$$

obeying the relation

$$\langle D_1F(g, L), X \rangle + \langle D'_1F(g, L), X' \rangle + \langle d_2F(g, L), Y \rangle = \frac{d}{dt} \bigg|_{t=0} F(e^{tX} ge^{tX'}, L + tY), \quad (3.3)$$

for every $X, X' \in \mathfrak{u}(n)$ and $Y \in \mathfrak{h}(n)$.

**Proposition 3.1.** The following formulas define two Poisson brackets on $C^\infty(\mathcal{M})$:

$$\{F, H\}_1(g, L) = \langle D_1F, d_2H \rangle - \langle D'_1H, d_2F \rangle + 2 \langle Ld_2F, d_2H \rangle, \quad (3.4)$$

and

$$\{F, H\}_2(g, L) = \langle D_1F, Ld_2H \rangle - \langle D'_1H, Ld_2F \rangle + 2 \langle Ld_2F, (Ld_2H)_{u(n)} \rangle - \frac{1}{2} \langle D'_1F, g^{-1}(D_1H)g \rangle, \quad (3.5)$$

where derivatives are taken at the point $(g, L)$ and we use the decomposition (2.10).

**Proof.** The first bracket is the canonical Poisson bracket of the cotangent bundle, expressed in terms of right-trivialization and taking $\mathfrak{h}(n) = \mathfrak{i}u(n)$ as the model of $u(n)^*$. To see this, note the identity

$$2 \langle Ld_2F, d_2H \rangle = \langle L, [d_2F, d_2H] \rangle. \quad (3.6)$$

The restriction of the second bracket to the open submanifold $U(n) \times \mathfrak{p}(n) \subset \mathcal{M}$ is a convenient multiple of the Heisenberg double Poisson bracket (2.19). Its algebraic nature guarantees that the Jacobi identity holds on the full manifold $\mathcal{M}$. For example, the Jacobi identity

$$\{\{L_a, L_b\}_2, L_c\}_2 \ + \ \text{c.p.} = 0, \quad (3.7)$$

for the linear functions $L_a(g, L) := \langle T_a, L \rangle$ defined by a basis $\{T_a\}$ of $u(n)$, requires the identity

$$\langle L[T_a, (LT_b)_{u(n)}] - L[T_b, (LT_a)_{u(n)}] + [LT_b, LT_a], (LT_c)_{u(n)} \rangle + \text{c.p.} = 0, \quad (3.8)$$

where c.p. means cyclic permutations of the indices $a, b, c$. Here, we used that

$$d_2L_a = T_a, \quad d_2\{L_a, L_b\}_2 = [T_a, (LT_b)_{u(n)}] - [T_b, (LT_a)_{u(n)}] + (T_bLT_a - T_aLT_b), \quad (3.9)$$

which is easily confirmed. We know that the expression (3.8) vanishes identically over the open subset $\mathfrak{p}(n) \subset \mathfrak{h}(n)$, because the Jacobi identity holds on the Heisenberg double. Thus it vanishes identically on the full $\mathfrak{h}(n)$, too, since it is given by a real analytic function of $L \in \mathfrak{h}(n)$. The same argument holds for any three functions chosen from the $L_a$ and real and imaginary parts of the matrix elements of $g$. This ensures the Jacobi identity for all smooth functions, since the $L_a$ and some matrix elements of $g$ can always be chosen locally as coordinate functions on $\mathcal{M}$. \qed
Define the Hamiltonians
\[ H_k(g, L) := \frac{1}{k} \tr(L^k), \quad \forall k \in \mathbb{N}. \] (3.10)

By using that \( d_2 H_k = iL^{k-1} \), one deduces the next statement.

**Proposition 3.2.** The Hamiltonians \( H_k \) pairwise Poisson commute with respect to both Poisson brackets of Proposition 3.1, and satisfy the relation
\[ \{ F, H_k \}_2 = \{ F, H_{k+1} \}_1, \quad \forall F \in C^\infty(\mathfrak{M}). \] (3.11)

The flows of the two Hamiltonian systems \((\mathfrak{M}, \{ \ , \}_2, H_k)\) and \((\mathfrak{M}, \{ \ , \}_1, H_{k+1})\) coincide, and are explicitly given by
\[ (g(t), L(t)) = \left( \exp(itL(0)^k)g(0), L(0) \right). \] (3.12)

The flow of \((\mathfrak{M}, \{ \ , \}_1, H_1)\), given by \((g(t), L(t)) = (e^{it}g(0), L(0))\), also commutes with the above family. We have a bi-Hamiltonian hierarchy, since the two Poisson brackets are compatible, i.e., their arbitrary linear combination is also a Poisson bracket. In order to show this, thanks to well-known results (see e.g. \[11, 43\]), it is sufficient to prove Lemma 3.3 below.

Introduce the vector field \( D \) on \( \mathfrak{M} \) that acts as the following derivation of the evaluation functions defined by the matrix elements of \( g \) and \( L \):
\[ D[g_{ij}] := 0, \quad D[L_{ij}] := \delta_{ij}. \] (3.13)

Using the unit matrix \( 1_n \), this is the vector field whose flow through \((g(0), L(0))\) reads
\[ (g(t), L(t)) = (g(0), L(0) + t1_n). \] (3.14)

**Lemma 3.3.** For \( F \in C^\infty(\mathfrak{M}) \), let \( D[F] \) denote the derivative along the vector field \( D \). The Poisson brackets of Proposition 3.1 enjoy the relation
\[ \{ F, H \}_1 = D[\{ F, H \}_2] - \{ D[F], H \}_2 - \{ F, D[H] \}_2, \] (3.15)
which means that the first bracket is the Lie derivative of the second one. In addition, we have
\[ D[\{ F, H \}_1] - \{ D[F], H \}_1 - \{ F, D[H] \}_1 = 0. \] (3.16)

**Proof.** It is enough to check the relation (3.15) for a set of coordinate functions on \( \mathfrak{M} \). Let \( L_a := \langle L, T_a \rangle \) be the component functions associated with a basis \( \{ T_a \} \) of \( u(n) \). The formula (3.15) certainly holds for coordinate functions on \( U(n) \) and the \( L_a \) if it holds for all elements of \( C^\infty(U(n)) \), which are regarded as \( L \)-independent elements of \( C^\infty(\mathfrak{M}) \), and all the functions \( L_a \). First, it is obvious that for \( F, H \in C^\infty(U(n)) \) both the left-hand side and the right-hand side of (3.15) give zero. Second, for \( F \in C^\infty(U(n)) \) and \( H = L_a \) we get
\[ D[\{ F, L_a \}_2] - \{ D[F], L_a \}_2 - \{ F, D[L_a] \}_2 = D[\{ F, L_a \}_2] = \langle D_1 F, T_a \rangle = \{ F, L_a \}_1. \] (3.17)

Finally for \( F = L_a \) and \( H = L_b \), we obtain
\[ D[\{ L_a, L_b \}_2] - \{ D[L_a], L_b \}_2 - \{ L_a, D[L_b] \}_2 = 2D[\{ LT_a, (LT_b)_{u(n)} \}] = 2\langle LT_a, T_b \rangle = \{ L_a, L_b \}_1, \] (3.18)
and thus the proof of (3.15) is complete. The equality (3.16) can be checked along similar lines. \( \square \)
According to standard terminology \[11, 41, 43\], \( M = T^*U(n) \) equipped with the two Poisson brackets subject to (3.15) and (3.16) is an example of an exact bi-Hamiltonian manifold. In conclusion, the identity (3.11) shows that the Hamiltonians \( H_k \) (3.10) generate a bi-Hamiltonian hierarchy on \( M \).

**Remark 3.4.** The fact that the free geodesic motion on the Poisson-Lie group \( U(n) \) corresponds to a Hamiltonian system on its Heisenberg double was pointed out in [48]. Our bi-Hamiltonian description of the free hierarchy is apparently new. It is customary to derive compatible Poisson brackets by linearization of quadratic Poisson structures, see, e.g., the paper [35]. Our construction is superficially similar, but we found a compatible pair on the whole of \( T^*U(n) \), whose existence is not implied by general linearization arguments.

### 4 Reduction under the conjugation action of \( U(n) \)

The essence of reduction with respect to a symmetry is that only those observables of the physical system are kept that are invariant under the action of the symmetry group. For the case at hand, this amounts to restriction to the ring of invariant functions \( C^\infty(M)^{U(n)} \), which is customarily identified as \( C^\infty(M/U(n)) \). Here, the invariance refers to the natural conjugation action of \( U(n) \) on the cotangent bundle. It can be viewed as an extension of the undressed quasi-adjoint action of Lemma 2.3, i.e., the action operates according to

\[
\tilde{A}_\eta^2(g, L) = (\eta g \eta^{-1}, \eta L \eta^{-1}), \quad \forall \eta \in U(n), (g, L) \in M.
\]

The reduction to invariant functions is often referred to as Poisson reduction. The following simple statement is important for us.

**Lemma 4.1.** The Poisson brackets \( \{ , \}_1 \) and \( \{ , \}_2 \) of Proposition 3.1 induce two compatible Poisson brackets on \( C^\infty(M)^{U(n)} \).

**Proof.** This follows from the compatibility of the two Poisson brackets on \( M \) and from the fact that the Poisson bracket of two smooth invariant functions is again invariant. The latter fact is obvious for the first bracket and it is a known property (Corollary 2.4) of the restriction of the second bracket to the Heisenberg double \( U(n) \times \mathfrak{p}(n) \). If \( F \) and \( H \) are \( U(n) \)-invariant real-analytic functions, then the validity of the invariance property,

\[
\{F, H\}_2 \circ \tilde{A}_\eta^2 = \{F, H\}_2,
\]

over the open submanifold \( U(n) \times \mathfrak{p}(n) \subset M \) implies that it holds over the full phase space \( M \). Indeed, both sides in the above equation represent real-analytic functions on \( M \). This ensures that the closure holds\(^3\) for \( C^\infty(M)^{U(n)} \), since every smooth invariant function on \( M \) can be expressed as a smooth function of a finite set of invariant polynomial functions in the matrix elements of \( g \) and \( L \).

We wish to study the reduced Poisson algebras given by the Lemma 4.1. In this paper, we make a technical assumption that simplifies the required analysis. Namely, we shall focus exclusively on the ‘regular part’ of the phase space \( M \). Indeed, both sides in the above equation represent real-analytic functions on \( M \). This ensures that the closure holds\(^3\) for \( C^\infty(M)^{U(n)} \), since every smooth invariant function on \( M \) can be expressed as a smooth function of a finite set of invariant polynomial functions in the matrix elements of \( g \) and \( L \).

Let \( \mathbb{T}^n \) denote the standard maximal torus of \( U(n) \). The dense open subset \( \mathbb{T}^n_{\text{reg}} \subset \mathbb{T}^n \) contains the elements

\[
Q = \text{diag}(Q_1, \ldots, Q_n) \in \mathbb{T}^n \quad \text{for which} \quad Q_i \neq Q_j, \quad \forall i \neq j.
\]

\(^3\)Incidentally, one can also work out a direct proof of the closure of \( C^\infty(M)^{U(n)} \) under \( \{ , \}_2 \) [35].
The dense open subset $U(n)_{\text{reg}} \subset U(n)$ is filled by the conjugacy classes passing through $T_n^{n}$. We define
\[ \mathcal{M}_{\text{reg}} := U(n)_{\text{reg}} \times \mathcal{H}(n). \]
(4.4)
Every $U(n)$ orbit in $\mathcal{M}_{\text{reg}}$ contains representatives in the submanifold
\[ T^{n}_{\text{reg}} \times \mathcal{H}(n) \subset \mathcal{M}_{\text{reg}}, \]
(4.5)
and this submanifold is preserved by the action of the normalizer, denoted $N(n)$, of $T^{n}$ in $U(n)$,
\[ N(n) \equiv \{ \eta \in U(n) \mid \eta Q \eta^{-1} \in T^{n}, \quad \forall Q \in T^{n} \}. \]
(4.6)
The ring of the $N(n)$-invariant functions on $T^{n}_{\text{reg}} \times \mathcal{H}(n)$ will serve as a model of $C^{\infty}(\mathcal{M}_{\text{reg}})^{U(n)}$.

**Lemma 4.2.** Let $\iota : T^{n}_{\text{reg}} \times \mathcal{H}(n) \to U(n)_{\text{reg}} \times \mathcal{H}(n)$ be the tautological embedding. For any $F \in C^{\infty}(\mathcal{M}_{\text{reg}})^{U(n)}$, $\iota^{*}F \equiv F \circ \iota$ belongs to $C^{\infty}(T^{n}_{\text{reg}} \times \mathcal{H}(n))^{N(n)}$, and
\[ \iota^{*} : C^{\infty}(\mathcal{M}_{\text{reg}})^{U(n)} \to C^{\infty}(T^{n}_{\text{reg}} \times \mathcal{H}(n))^{N(n)} \]
(4.7)
is an isomorphism of commutative algebras.

**Proof.** Any $F \in C^{\infty}(\mathcal{M}_{\text{reg}})^{U(n)}$ is uniquely determined by its restriction to $T^{n}_{\text{reg}} \times \mathcal{H}(n)$, and the restricted function belongs to $C^{\infty}(T^{n}_{\text{reg}} \times \mathcal{H}(n))^{N(n)}$. For any $g \in U(n)_{\text{reg}}$, we can choose $\sigma_{1}(g) \in T^{n}_{\text{reg}}$ and $\sigma_{2}(g) \in U(n)$ such that $\sigma_{1}(g) = \sigma_{2}(g)g\sigma_{2}(g)^{-1}$. Then, for any $f \in C^{\infty}(T^{n}_{\text{reg}} \times \mathcal{H}(n))^{N(n)}$, the formula
\[ F(g, L) := f(\sigma_{1}(g), \sigma_{2}(g) L \sigma_{2}(g)^{-1}) \]
(4.8)
gives a well-defined, $U(n)$-invariant function on $\mathcal{M}_{\text{reg}}$. To see that this function is smooth, we note that $U(n)_{\text{reg}}$ is the base of the (left) $N(n)$ principal fibre bundle with total space $T^{n}_{\text{reg}} \times U(n)$, $N(n)$ action given by $N(n) \ni \nu : (\tau, \eta) \mapsto (\nu \tau \nu^{-1}, \nu \eta)$, and bundle projection $U(n) \ni (\tau, \eta) \mapsto \eta^{-1} \tau \eta \in U(n)_{\text{reg}}$. Since this bundle is locally trivial, it admits smooth local sections,
\[ g \mapsto \sigma(g) = (\sigma_{1}(g), \sigma_{2}(g)). \]
(4.9)
Using such section $\sigma$ in [15,8] shows that $F$ is locally smooth. Because $F$ is a globally well-defined function on $\mathcal{M}_{\text{reg}}$, we see that it belongs to $C^{\infty}(\mathcal{M}_{\text{reg}})$. □

**Definition 4.3.** The reduced Poisson algebras $(C^{\infty}(T^{n}_{\text{reg}} \times \mathcal{H}(n))^{N(n)}, \{ \cdot, \cdot \}^{\text{red}}_{i})$ are defined by setting
\[ \{ F \circ \iota, H \circ \iota \}^{\text{red}}_{i} := \{ F, H \} \circ \iota \quad \text{for} \quad F, H \in C^{\infty}(\mathcal{M}_{\text{reg}})^{U(n)}, \quad i = 1, 2. \]
(4.10)

We shall establish an intrinsic description of the reduced Poisson brackets (4.10). In preparation, let us decompose $\mathfrak{g}l(n, \mathbb{C})$ as the direct sum of subalgebras
\[ \mathfrak{g}l(n, \mathbb{C}) = \mathfrak{g}l(n, \mathbb{C})_{+} + \mathfrak{g}l(n, \mathbb{C})_{0} + \mathfrak{g}l(n, \mathbb{C})_{-} \]
(4.11)
by means of the principal gradation, i.e., $\mathfrak{g}l(n, \mathbb{C})_{0}$ contains the diagonal matrices, and $\mathfrak{g}l(n, \mathbb{C})_{+}$ ($\mathfrak{g}l(n, \mathbb{C})_{-}$) contains the strictly upper (lower) triangular matrices. Correspondingly, any $X \in \mathfrak{g}l(n, \mathbb{C})$ can be written in the form $X = X_{+} + X_{0} + X_{-}$. We may also write $X = X_{0} + X_{\perp}$ with $X_{\perp} := X_{+} + X_{-}$. 

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For $Q \in \mathbb{T}_\text{reg}^n$, the linear operators $(\text{Ad}_Q - \text{id})|_{\mathfrak{gl}(n, \mathbb{C})}$ are invertible, and therefore one may introduce $\mathcal{R}(Q) \in \text{End}(\mathfrak{gl}(n, \mathbb{C}))$ by setting it equal to zero on $\mathfrak{gl}(n, \mathbb{C})_0$ and defining it otherwise as
\[
\mathcal{R}(Q)|_{\mathfrak{gl}(n, \mathbb{C})_+ + \mathfrak{gl}(n, \mathbb{C})_-} = \frac{1}{2}(\text{Ad}_Q + \text{id}) \circ \left((\text{Ad}_Q - \text{id})|_{\mathfrak{gl}(n, \mathbb{C})_+ + \mathfrak{gl}(n, \mathbb{C})_-}\right)^{-1},
\]
where $\text{Ad}_Q(X) = QXQ^{-1}$ for all $X \in \mathfrak{gl}(n, \mathbb{C})$. Incidentally, this is a well-known solution of the modified classical dynamical Yang-Baxter equation [10], which first appeared in [9]. Below, we apply the notation
\[
[X, Y]|_{\mathcal{R}(Q)} := [\mathcal{R}(Q)X, Y] + [X, \mathcal{R}(Q)Y], \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}).
\]
(4.13)

For any $f \in C^\infty(\mathbb{T}_\text{reg}^n \times \mathfrak{g}(n))$, the $\mathfrak{b}(n)_0 := \mathfrak{b}(n) \cap \mathfrak{gl}(n, \mathbb{C})_0$-valued derivative $D_1 f$ and the $\mathfrak{u}(n)$-valued derivative $d_2 f$ are defined naturally, in analogy with (3.3):
\[
\langle D_1 f(Q, L), X \rangle + \langle d_2 f(Q, L), Y \rangle = \frac{d}{dt}\bigg|_{t=0} f(e^{tX}Q, L + tY),
\]
(4.14)
for every $X \in \mathfrak{u}(n)_0 := \mathfrak{u}(n) \cap \mathfrak{gl}(n, \mathbb{C})$ and $Y \in \mathfrak{g}(n)$.

Lemma 4.4. Let $f := F \circ \circ$ for a function $F \in C^\infty(\mathfrak{M}_\text{reg}^n \times \mathfrak{g}(n))$. Then the following relations hold at any $(Q, L) \in \mathbb{T}_\text{reg}^n \times \mathfrak{g}(n)$:
\[
d_2 F(Q, L) = d_2 f(Q, L), \quad [L, d_2 f(Q, L)]_0 = 0,
\]
(4.15)
\[
D_1 F(Q, L) = D_1 f(Q, L) - [L, d_2 f(Q, L)]_+ - 2\mathcal{R}(Q)[L, d_2 f(Q, L)]_+,\n\]
(4.16)
where the subscripts 0 and + refer to the decomposition (4.11).

Proof. The first equality in (4.15) is trivial, and the second one follows from
\[
0 = \frac{d}{dt}\bigg|_{t=0} f(Q, e^{tX} Le^{-tX}) = \frac{d}{dt}\bigg|_{t=0} f(Q, L + t[X, L] + o(t))
\]
\[
= \langle d_2 f(Q, L), [X, L] \rangle = \langle [L, d_2 f(Q, L)]_0, X \rangle, \quad \forall X \in \mathfrak{u}(n)_0.
\]
(4.17)
In order to derive (4.16), let us take an arbitrary off-diagonal element $T \in \mathfrak{u}(n)$, and use the invariance of $F$ to write
\[
0 = \frac{d}{dt}\bigg|_{t=0} F(e^{tT}Qe^{-tT}, e^{tT}Le^{-tT}) = \langle T, D_1 f(Q, L) - D_1 f(Q, L) + [L, d_2 f(Q, L)] \rangle.
\]
(4.18)
On account of the relation $D_1 f(Q, L) = \text{Ad}_Q^{-1}(D_1 F(Q, L))$ and some obvious identities, equation (4.18) is equivalent to
\[
0 = \langle T, (\text{Ad}_Q - \text{id}) \circ \text{Ad}_Q^{-1}(D_1 F(Q, L))_+ + 2[L, d_2 F]_+ \rangle.
\]
(4.19)
To get this, we noticed that, with the decomposition $T = T_- + T_+$, we have
\[
\langle T, [L, d_2 f(Q, L)] \rangle = 2\langle T_-, [L, d_2 f(Q, L)] \rangle.
\]
(4.20)
As a result, we see that (4.19) is equivalent to
\[
(\text{Ad}_Q - \text{id}) \circ \text{Ad}_Q^{-1}(D_1 F(Q, L))_+ = -2[L, d_2 F]_+.
\]
(4.21)
Since $Q$ is regular, this can be solved for $(D_1 F(Q, L))_+$. To do this, using that the inverse is well-defined on $\mathfrak{gl}(n, \mathbb{C})_+$, we write
\[
(D_1 F(Q, L))_+ = -2\text{Ad}_Q \circ (\text{Ad}_Q - \text{id})^{-1}([L, d_2 f]_+) = -((\text{Ad}_Q - \text{id}) + (\text{Ad}_Q + \text{id})) \circ (\text{Ad}_Q - \text{id})^{-1}([L, d_2 f]_+) = -[L, d_2 f]_+ - 2\mathcal{R}(Q)[L, d_2 f]_+.
\]
(4.22)
Combining this with the equality $(D_1 F(Q, L))_0 = D_1 f(Q, L)$, which is a direct consequence of the definitions, one obtains the claimed formula (4.16).
The main result of this paper is the following description of the reduced Poisson brackets.

**Theorem 4.5.** For \( f, h \in C^\infty(\mathbb{T}_{\text{reg}}^n \times \mathfrak{g}(n))^{X(n)} \), the reduced Poisson brackets \( \{ f, h \} \) obey the explicit formulas

\[
\{ f, h \}_1^{\text{red}}(Q, L) = \langle D_1 f, d_2 h \rangle - \langle D_1 h, d_2 f \rangle + \langle L, [d_2 f, d_2 h]_{\mathcal{R}(Q)} \rangle, \tag{4.23}
\]

and

\[
\{ f, h \}_2^{\text{red}}(Q, L) = \langle D_1 f, L d_2 h \rangle - \langle D_1 h, L d_2 f \rangle + 2\langle L d_2 f, \mathcal{R}(Q)(L d_2 h) \rangle. \tag{4.24}
\]

The derivatives are evaluated at the point \((Q, L)\), and the notations \(4.12, 4.13\) are applied.

**Proof.** As detailed below, the claimed formulas result by substituting the formulas of Lemma 4.4 into the Poisson bracket formulas of Proposition 3.1, and performing some elementary algebraic manipulations.

To deal with the first Poisson bracket, note that at the point \((Q, L)\) we have

\[
-2\langle \mathcal{R}(Q)[L, d_2 f]_+, d_2 h \rangle = 2\langle [L, d_2 f]_+, \mathcal{R}(Q)(d_2 h) \rangle = \langle L, [d_2 f, \mathcal{R}(Q)(d_2 h)] \rangle. \tag{4.25}
\]

To get this, we used the anti-symmetric nature of \( \mathcal{R}(Q) \) together with the fact that it maps \( \mathfrak{u}(n) \) to \( \mathfrak{u}(n)_\perp \), and the obvious identity \( \langle X, Y \rangle = -\langle X^\dagger, Y^\dagger \rangle \). We also have

\[
-\langle [L, d_2 f]_+, d_2 h \rangle = -\frac{1}{2}\langle [L, d_2 f]_\perp, d_2 h \rangle = -\frac{1}{2}\langle L, [d_2 f, d_2 h] \rangle, \tag{4.26}
\]

where the last equality crucially depends on the property \([L, d_2 f]_0 = 0\). The formula \(4.23\) results by using these relations and Lemma 4.4 for the evaluation of

\[
\{ f, h \}_1^{\text{red}} = \langle (D_1 F)_0 + (D_1 F^\dagger)_+, d_2 H \rangle - \langle (D_1 H)_0 + (D_1 H^\dagger)_+, d_2 F \rangle + \langle L, [d_2 F, d_2 H] \rangle \tag{4.27}
\]
at the point \((Q, L)\).

Turning to the derivation of \(4.24\), at the point \((Q, L)\), we record the identities

\[
\langle D_1 F, L d_2 H \rangle = \langle D_1 f, L d_2 h \rangle - \langle (L d_2 h)_{\mathfrak{u}(n)}, \frac{1}{2}[L, d_2 f] + \mathcal{R}(Q)[L, d_2 f] \rangle = \langle D_1 f, L d_2 h \rangle + \langle 2\mathcal{R}(Q)(L d_2 h)_{\mathfrak{u}(n)} - (L d_2 h)_{\mathfrak{u}(n)}, L d_2 f \rangle, \tag{4.28}
\]

\[
2\langle L d_2 F, (L d_2 H)_{\mathfrak{u}(n)} \rangle = \langle L d_2 f, (L d_2 h)_{\mathfrak{u}(n)} \rangle - \langle L d_2 h, (L d_2 f)_{\mathfrak{u}(n)} \rangle. \tag{4.29}
\]

\[
\langle D_1 F, Q^{-1}(D_1 H) Q \rangle = 0. \tag{4.30}
\]

To verify \(4.29\), notice that \( \Im \text{tr}(L(d_2 F)L(d_2 H)) = 0 \), because \( \Im \text{tr}(X^\dagger) = -\Im \text{tr}(X) \) for all \( X \in \mathfrak{gl}(n, \mathbb{C}) \), then use the decomposition \(2.10\). With the aid of these identities, equation \(3.5\) gives

\[
\{ f, h \}_2^{\text{red}}(Q, L) = \langle D_1 f, L d_2 h \rangle - \langle D_1 h, L d_2 f \rangle + 2\langle L d_2 f, \mathcal{R}(Q)(L d_2 h)_{\mathfrak{u}(n)} \rangle - 2\langle \mathcal{R}(Q)(L d_2 h)_{\mathfrak{u}(n)}, L d_2 h \rangle. \tag{4.31}
\]

Finally, noting the identity

\[
-\langle \mathcal{R}(Q)(L d_2 f)_{\mathfrak{u}(n)}, L d_2 h \rangle = \langle (L d_2 f)_{\mathfrak{u}(n)}, \mathcal{R}(Q)(L d_2 h) \rangle = \langle L d_2 f, \mathcal{R}(Q)(L d_2 h)_{\mathfrak{u}(n)} \rangle, \tag{4.32}
\]
equation \(4.31\) is converted into \(4.24\). \(\square\)
Now we describe the reduction of the equations of motion of the bi-Hamiltonian hierarchy (3.11). Denote by $V_k$ the bi-Hamiltonian vector field on $\mathfrak{M}$ satisfying

$$V_k[F] = \{F, H_k\}_2 = \{F, H_{k+1}\}_1, \quad k \in \mathbb{N}. \quad (4.33)$$

This induces a derivation of $C^\infty(\mathfrak{M})^{U(n)}$, which in turn translates into a derivation of $C^\infty(\mathbb{T}_\text{reg}^n \times \mathfrak{H}(n))^{\mathcal{N}(n)}$. The latter derivation corresponds to a (non-unique) vector field $W_k$ on the manifold $\mathbb{T}_\text{reg}^n \times \mathfrak{H}(n)$, whose value at $(Q, L)$ takes the following form:

$$W_k(Q, L) = V_k(Q, L) + ([\zeta(Q, L), Q], [\zeta(Q, L), L]), \quad (4.34)$$

where $V_k(Q, L) = (iL^kQ, 0)$, according to (3.12), and $\zeta(Q, L) \in \mathfrak{u}(n)$ is subject to the condition

$$(iL^kQ + [(Q, L), Q])Q^{-1} \in \mathfrak{u}(n)_0. \quad (4.35)$$

In words, the ‘infinitesimal gauge transformation’ $\zeta(Q, L)$ ensures that $W_k$ is tangential to the manifold $\mathbb{T}_\text{reg}^n \times \mathfrak{H}(n)$. This holds since $\mathfrak{u}(n)_0 = \mathfrak{u}(n) \cap \mathfrak{gl}(n, \mathbb{C})_0$ is the Lie algebra of $\mathbb{T}^n$.

**Proposition 4.6.** The induced evolutional vector field $W_k$ of Eq. (4.34) is given by

$$W_k(Q, L) = (i(L^k)_0Q, [\mathcal{R}(Q)(iL^k), L]), \quad (4.36)$$

up to an arbitrary function $\delta\zeta_0(Q, L) \in \mathfrak{u}(n)_0$ that does not effect the induced derivatives of the elements of $C^\infty(\mathbb{T}_\text{reg}^n \times \mathfrak{H}(n))^{\mathcal{N}(n)}$. By choosing $\delta\zeta_0 = 0$, the evolution equation for $(Q(t), L(t)) \in T^{n}_\text{reg} \times \mathfrak{H}(n)$ associated with the vector field $W_k$ has the form (4.36).

**Proof.** One can check that

$$\zeta(Q, L) = \mathcal{R}(Q)(iL^k) - \frac{i}{2}L^k \quad (4.37)$$

is a particular solution of the condition (4.35). The general solution is obtained by adding $\delta\zeta_0$ to this one. Substitution of (4.37) into (4.34) gives (4.36). $\square$

To sum up, the message of Proposition 4.6 is that our Poisson reduction yields the trigonometric spin Ruijsenaars–Sutherland hierarchy as defined in Section 1. The reduction treatment equips this hierarchy with a bi-Hamiltonian structure. Indeed, our construction implies that the evolutional derivatives of the gauge invariant observables $f \in C^\infty(\mathbb{T}_\text{reg}^n \times \mathfrak{H}(n))^{\mathcal{N}(n)}$ satisfy

$$W_k[f] = \{f, h_k\}_2^{\text{red}} = \{f, h_{k+1}\}_1^{\text{red}}, \quad (4.38)$$

with the compatible Poisson brackets given by Theorem 4.5 and the reduced Hamiltonians $h_k$ obtained from $H_k$ (3.11). We next present a large set of constants of motion for this hierarchy.

**Proposition 4.7.** Let $\mathcal{P}(L, Q^{-1}LQ)$ be an arbitrary ‘non-commutative polynomial’, i.e., a linear combination of ordered products of powers of $L$ and $Q^{-1}LQ$. Then the $\mathcal{N}(n)$-invariant function $\text{tr}(\mathcal{P}(L, Q^{-1}LQ))$ is constant along the flow of the evolutional vector field $W_k$.

**Proof.** Denoting the derivative along $W_k$ by $\frac{d}{dt}$, we observe that

$$\frac{d}{dt}Q = iL^kQ + [\zeta(Q, L), Q] \quad \text{and} \quad \frac{d}{dt}L = [\zeta(Q, L), L] \quad (4.39)$$

imply

$$\frac{d}{dt}(Q^{-1}LQ) = [\zeta(Q, L), Q^{-1}LQ]. \quad (4.40)$$

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Proposition 4.7 provides constants of motion for the reduced bi-Hamiltonian dynamics on $\mathcal{M}_{\text{reg}}/U(n)$. These constants of motion are restrictions of well-defined functions on the whole of $\mathcal{M}/U(n)$. Indeed, the formula

$$
\text{tr} \left( \mathcal{P}(L, g^{-1} Lg) \right) 
$$

(4.42)

gives a $U(n)$-invariant, smooth function of $(g, L) \in \mathcal{M}$, which is a constant of motion for all the bi-Hamiltonian vector fields displayed in equation (3.11). It reproduces $\text{tr} \left( \mathcal{P}(L, Q^{-1} LQ) \right)$ upon restriction to $T_{\text{reg}}^n \times \mathfrak{h}(n)$. The Poisson brackets and the algebraic relations of these constants of motion will be studied in a future publication.

We end by remarking that arguments similar to those utilized by Reshetikhin [32, 33] can be applied to show the degenerate integrability of the reduced dynamics on generic symplectic leaves of any of the two reduced Poisson brackets. However, the details are rather complicated since $\mathcal{M}/U(n)$ is not a smooth manifold. This issue should be investigated further invoking the machinery of singular Hamiltonian reduction [31, 42]. One of the interesting open questions is whether the above exhibited polynomial constants of motion are sufficient for the degenerate integrability of the reduced system.

5 Discussion

The first new result of this paper is the bi-Hamiltonian description of the free motion on the group $U(n)$, developed in Section 3. We noticed that it is useful to present the Poisson structure [40] of the Heisenberg double in terms of the variables $(g_R, b_R b_R^\dagger) \in U(n) \times \mathfrak{u}(n)$, since in this way it admits extension to the cotangent bundle $\mathcal{M} = U(n) \times \mathfrak{u}(n)$. In Section 4, we demonstrated that Poisson reduction of the hierarchy of free motion leads to the trigonometric spin Ruijsenaars–Sutherland hierarchy governed by the evolution equations (1.1). This yields a bi-Hamiltonian interpretation for the dynamics of the gauge invariant functions of the variables $(Q, L)$, where the gauge group is given by the normalizer $\mathcal{N}(n)$ of the maximal torus $T_n$ inside $U(n)$.

The interpretation of the reduced system as a spin Sutherland model is supported by the the change of variables (1.5) that brings the Hamiltonian $\frac{1}{2} \text{tr}(L^2)$ into the form (1.7), and converts the reduced first Poisson bracket into the natural one carried by the phase space $(T^* T_{\text{reg}}^n \times (\mathfrak{u}(n)^{\ast}//_0 T^n)')/S_n$.

We now briefly discuss another change of variables, which is suited for the second reduced Poisson bracket upon restriction to the open submanifold arising from the Heisenberg double $U(n) \times \mathfrak{u}(n) \subset \mathcal{M}$. In this case we can write $L = bb^\dagger$, where $b = e^{p} b_+$ with a real diagonal matrix $p \in \mathfrak{b}(n)_0$ and an upper triangular matrix having unit diagonal, $b_+ \in B(n)_+$. By introducing $\lambda := b_+ Q^{-1} b_+ Q$, we obtain the invertible change of variables

$$
T_{\text{reg}}^n \times \mathfrak{u}(n) \ni (Q, L) \longmapsto (Q, p, \lambda) \in T_{\text{reg}}^n \times \mathfrak{b}(n)_0 \times B(n)_+;
$$

(5.1)
whereby every function $f(Q, L)$ is represented by a function $F(Q, p, \lambda)$. It can be shown (both by direct calculation or by applying Theorem 4.3 of [13]) that the reduced second Poisson bracket acquires the following decoupled form in terms of the new variables:

$$2\{F, H\}_2^{\text{red}}(Q, p, \lambda) = \langle D_Q F, d_p H \rangle - \langle D_Q H, d_p F \rangle + \langle D_\lambda' F, \lambda^{-1}(D_\lambda H) \lambda \rangle.$$  \hfill (5.2)

The derivatives on the right hand side are taken at $(Q, p, \lambda)$, $D_Q F \in b(n)_0$ and $d_p F = u(n)_0$ are defined in the obvious manner, and we take $D_\lambda F$ from the off-diagonal subspace of $u(n)$, according to the rule

$$\langle D_\lambda F(Q, p, \lambda), X_+ \rangle + \langle D_\lambda' F(Q, p, \lambda), Y_+ \rangle = \frac{d}{dt} \bigg|_{t=0} F(Q, p, e^{tX_+} \lambda e^{tY_+}), \quad (5.3)$$

$\forall X_+, Y_+ \in b(n)_+$. The subgroup $T^n$ of the gauge group acts by $(Q, p, \lambda) \mapsto (Q, p, \tau \lambda \tau^{-1})$, and the formula (5.2) defines a Poisson bracket on the $T^n$-invariant functions. Its last term can be recognized as the natural reduced Poisson bracket on $B(n)/T^n$, which is the Poisson-Lie analogue of $u(n)^*//_0 T^n$. The Hamiltonian $\text{tr}(L)$ has the ‘spin Ruijsenaars form’

$$\text{tr}(L) = \sum_{i=1}^n e^{2p_i} V_i(Q, \lambda) \quad \text{with} \quad V_i(Q, \lambda) = (b_+(Q, \lambda)b_+(Q, \lambda)^\dagger)_{ii}, \quad (5.4)$$

where $\lambda$ represents a ‘spin’ variable. An enlightening explicit formula of $V_i(Q, \lambda)$ is not available in general, but it is known that restriction to a particular symplectic leaf of $B(n)//_0 T^n$ gives the spinless trigonometric RS model [15]. An unpleasant feature of the new variables $(Q, p, \lambda)$ is that the action of the full gauge group $\lambda$ is the action of the full gauge group $N(n)$, and that of the permutation group $S_n = N(n)/T^n$, is not transparent in this setting (for the spinless case, see Section 4 of [16]).

There is a link between our results and the observation of Suris [44], who noticed that the spinless RS and Calogero–Moser hierarchies are governed by the same $R$-operators. In the trigonometric case, the pertinent $R$-operator is the sum of the one in (1.2) and a ‘correction term’. In this case the statement of [44] can be derived from our results by applying suitable restrictions and gauge fixings to the spin Ruijsenaars–Sutherland hierarchy.

An interesting open problem that stems from our work is that the global structure of the full reduced phase space should be explored in the future, dropping the restriction to $M_{\text{reg}} \subset M$. The issue of possible generalizations of the bi-Hamiltonian structure to the elliptic case and for other Lie groups should be also investigated. Finally, we wish to mention the question whether there is any relation between our results and the earlier studies [4, 12] of a bi-Hamiltonian structure for the rational, spinless Calogero model.

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\footnote{A complicated explicit formula for $b_+(Q, \lambda)$ can be obtained along the lines of Section 5.2 in [13].}
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