Composite Fermion states on the torus

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We extend the composite fermion construction to the torus geometry. We verify the validity of our
construction by computing the overlap of the composite fermion state to the exact diagonalization
ground state of both Coulomb interaction and Haldane-pseudopotential interaction $V_0$ ($V_1$) for
bosonic (fermionic) states.

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I. INTRODUCTION

A paramount impetus for the growing interest in
strongly correlated quantum matter is the discovery that
such systems can be topologically ordered. The first, and
most prominent, examples are the various incompressible
fractional quantum Hall (FQH) liquids. These are formed
at low temperatures, when very clean two-dimensional
electron gases are subjected to a strong perpendicular
magnetic field [1]. The most striking consequence of the
topological order in the FQH liquids is that the emergent
low-energy quasiparticles have fractional electric charge,
and are believed to obey fractional braiding statistics
[2]. Another hallmark of a topological phase of matter is
that the number of degenerate ground states depend on
the topology of the space on which the state is defined.
Again, the simplest example is the FQH liquids, where
numerical solutions of the microscopic Coulomb problem
can be compared to different theoretical predictions. In
this paper we develop techniques that make it possible
to make such comparisons for the Jain states, which are
an important set of actually observed FQH liquids.

The problem of many particles moving in a strong
magnetic field, and interacting via Coulomb forces, is
intractable, so one has to find an effective description.
While a natural approach is to construct effective low-
energy field theories [3, 4], a very fruitful alternative
route — following a seminal paper by Laughlin [5] —
has been to construct model wave functions with well defined
topological properties, and verify numerically that they
describe small systems accurately. The most common
method of verification is to compute the overlap with the
exact Coulomb ground state, but recently it was shown that
studying the entanglement spectrum [6] can give valuable
complementary insights.

Although most effort in FQH physics during the last
decade has been aimed at understanding different non-
Abelian states, we shall here concentrate on the family
of observed states in the lowest Landau level. There are
two successful theoretical approaches to describe these
states—the Haldane-Halperin hierarchy [7, 8] and the
composite fermion (CF) theory [9].

The Haldane-Halperin hierarchy describes a family of
incompressible FQH liquids formed via successive con-
densation of the low-energy excitations—quasiholes and
quasielectrons. It predicts that incompressible FQH liq-
uids may be found at filling fractions $\nu = p/q$, where $p$
and $q$ are relatively prime integers and $q$ is odd. It also
argues that the stability of the liquids decreases roughly as
$\sim 1/q$ with increasing denominator. The emergent quasi-
particles have fractional electric charge $1/q$ and obey
fractional Abelian braiding statistics. The simplest way
to obtain explicit wave functions for ground and quasi-
particle states, at all levels of the hierarchy, is by using
conformal field theory techniques [10, 11].

The CF theory can describe most of the observed FQH
liquids by mapping the problem of strongly interacting
fermions that fill a fraction $\nu$ of a Landau level, to that
of non-interacting (or at least weakly interacting) composite
fermions filling an integer number of effective Landau
levels in a reduced magnetic field. The latter describes
an incompressible state because of the finite gap between
the effective Landau levels. This approach also gives a
simple picture of the ground state and the low-energy
sector. A big initial success of the CF theory was the
very good agreement with the exact Coulomb eigenstates
obtained from finite-size numerical studies, both for the
ground state and the excited states. The two approaches
are not exclusive, but rather describe the same universal
features [12] and also give very similar predictions for
the relative stability of the various FQH liquids [8, 9].
Moreover, it was shown that for the important case of
the positive Jain series, the CF wave functions can be
obtained by a hierarchical construction, both in planar
and spherical geometry [13–16].

Both approaches to the Abelian FQH states in the low-
est Landau level—the hierarchy and the CF theory—are
well understood (and studied) in the disk and sphere geo-
metry; see e.g. [17] and references therein. The torus
geometry, however, has been studied far less and explicit
wave function were only known for certain model states
that are determined uniquely (up to center-of-mass trans-
lations) by their vanishing properties [18, 20]. A first
attempt to construct hierarchical wave functions on the
torus was done in Ref. [21] using conformal field theory
techniques. While it provided wave functions that are
very good approximations to the exact Coulomb ground
state in a certain parameter regime, the construction was
not satisfactory in that the wave functions did not trans-
form properly under modular transformations and were
uniquely defined only in the thermodynamic limit. It
has only recently been understood how to resolve these problems [22].

In this paper, we show how to generalize the CF construction to the torus geometry. The construction has no free parameters and gives unique model wave functions (up to center-of-mass translations) of the ground state and the excited states at filling fraction \( \nu = \frac{n}{np+1} \), \( n \) and \( p \) integers, corresponding to the positive Jain series.

There are several reasons why the torus geometry is interesting, even though it cannot be realized experimentally. We already mentioned the topological ground state degeneracy, but it is also important that numerical calculations are better defined on closed manifolds, such as the sphere and the torus, since they do not suffer from edge effects, which can be substantial for the system sizes one can reach numerically. While numerics on the sphere have proven very useful, there are still problems connected to finite size, most notably the so-called shift. It can happen that two states that are at the same (thermodynamic) filling fraction \( \nu \) appear at different magnetic fluxes \( N_\phi \) in the finite-size system. The most prominent examples are the Moore-Read state [23] and the CF Fermi liquid [24], which are both at filling 1/2 but have different shift. Numerical comparison of these two states on the sphere is therefore only indirect. On the torus, this issue does not arise, which allows for a direct comparison of these states [25].

Another advantage of the torus geometry is that one can change the shape of the torus—described by the modular parameter \( \tau \)—and thus get more information about a state without having to increase the system size. This was successfully used for entanglement entropy calculations, where one wants to extract a subleading constant term in the entropy. Ref. 26 showed that changing the aspect ratio of the torus and thus obtaining additional data, yielded much more accurate bounds on the topological constant than could be obtained from sphere calculations. Also, as shown by Avron et al. [27], by studying the response of QH liquid to an adiabatic change in \( \nu \), one can determine the odd part of the viscosity tensor. An explicit calculation in the case of the Laughlin states was made by Read [28], and the results in this paper could be used to perform similar calculations for the Jain states. Let us also note that the techniques introduced in this paper are not restricted to the positive Jain series, but can also be used to study more exotic states, such as the Bonderson-Slingerland states [29], the non-Abelian condensate states [30], as well as the closely related bipartite CF states [31].

Outline of the paper In Sec. II we first present the CF construction on the disk geometry and show how to generalize the approach to the torus geometry. We discuss single-particle states on the torus in Sec. II.B and give an expression for the product of two such states at different magnetic fluxes \( N_{\phi_1} \) and \( N_{\phi_2} \) in Sec. II.C. The derivation of this identity is given in the Appendix. In Sec. II.D we show how to evaluate CF states on the torus. Overlaps of some CF states with exact diagonalization results using both Coulomb and Haldane pseudo-potential interactions are calculated in Sec. III. These overlaps should be regarded solely as a proof of principle that the construction on the torus is sound. In Sec. IV we speculate on possible lowest Landau level projection schemes in real space, given that the torus places additional constraints on model wave functions.

II. GENERAL COMPOSITE FERMION CONSTRUCTION

In this Section we explain how to generalize the CF construction to the torus geometry. In Sec. II.A we first discuss the CF construction on the disk and sphere and point out some subtleties that become important on the torus. Sec. II.B contains a short review on the single-particle states on the torus. In Sec. II.C we derive formulas for the projection of a product of two single-particle states. In Sec. II.D we discuss properties of the CF states on the torus using the bosonic CF state at \( \nu = 2/3 \) as an explicit example.

A. Composite fermions on the disk geometry

There are already many good texts on the CF construction—see, for instance, [17] for an extensive and pedagogical review. Thus, we keep the discussion in this Section very brief and focus on properties that are relevant for the torus. In the following, we restrict ourselves to the positive Jain series at fillings \( \nu = \frac{n}{np+1} \), where \( n \geq 1 \) and \( p \) are integers. We expect the negative Jain series to work analogously, but we have not yet performed any explicit calculations. In the CF theory, one attaches an even (odd) number of vortices to strongly interacting fermions (bosons). The resulting fermionic particles are called composite fermions and one assumes that these composite particles are non-interacting or at least very weakly interacting. Due to the attachment of vortices, they feel a reduced magnetic field \( B^* = B - p\phi_0 \), where \( \phi_0 \) is the magnetic flux quantum and \( p \) is the two-dimensional density. For properly chosen \( p \) the reduction of magnetic flux is such that the CFs fill an integer number of effective Landau levels.

A trial wave function for the ground state of strongly interacting particles at filling \( \nu = \frac{n}{np+1} \) is then usually written as

\[
\Psi_\nu(\{z_j\}) = \mathcal{P}_{LLL} \left\{ \Phi_n(\{z_j, \bar{z}_j\}) \prod_{i<j} (z_i - z_j)^p \right\},
\]

where \( z = x + iy \) is a complex coordinate. \( \Phi_n(\{z_j, \bar{z}_j\}) \) is the many-body wave function (slater determinant) for the \( n \) lowest Landau levels filled and \( \mathcal{P}_{LLL} \) projects to the lowest Landau level. Equation (1) does not strictly speaking describe a proper lowest Landau level wave function on the disk, because it does not have the correct
Gaussian factor. Usually, one does not worry about this but just adds the correct factor by hand. However, this subtlety becomes important on the torus as explained in the next paragraph.

The naive guess of how to generalize Eq. (1) is to replace each part by the respective torus counterpart. In particular, this would amount to replacing the Jastrow factor with its periodized version [15]

$$\prod_{i<j}(z_i - z_j)^p \rightarrow \prod_{i<j} \theta_1(z_i - z_j | \tau)^p,$$

where $\theta_1$ is the odd Jacobi $\theta$ function (defined by setting $a = b = 1/2$ in Eq. (15)). We choose $\theta_1$ because it is the only $\theta$ function that has the correct short-distance behavior, i.e. is it the only antisymmetric $\theta$-function. However, this choice poses two obvious problems: first, the wave function does not obey the correct boundary conditions on the torus, see Eq. (7). [40] Second, there is no efficient way to project the many-body wave function to the lowest Landau level. To the best of our knowledge, no analog of the Girvin-Jach projection [32] is known on the torus. We will comment more on this in Sec. IV.

Instead of Eq. (1) we will consider the following expression:

$$\Psi_\mu(\{z_j\}) = \mathcal{P}_{LLL} \{ \Phi_n(\{z_j, \bar{z}_j\}) \Phi_1(\{z_j\})^p \}.$$  

The replacement of the Jastrow factor to $\Phi_1$ is of course trivial for both the disk and sphere—in the former case it only differs by a Gaussian factor. The point is, however, that (3) is a proper Landau level wave function on the disk, i.e. it has the correct Gaussian factor because the Gaussian factors of $\Phi_n$ and $\Phi_1^p$ combine to give the correct factor at the combined flux. In addition, using expression (3) solves both problems mentioned in the previous paragraph. It is straightforward to verify that $\Psi_\mu(\{z_j\})$ obeys the boundary conditions on the torus [7]. The projection onto the lowest Landau level can be implemented on the single-particle level, which is explained in Sec. II.C.

### B. Single-particle states on the torus

We consider a torus spanned by two, not necessarily orthogonal, translation vectors $\vec{L}_1$ and $\vec{L}_2$. A homogeneous external magnetic field—perpendicular to the surface of the torus—is described in terms of the vector potential $\vec{A} = -B \vec{r} \times \hat{\tau}$ using Landau gauge. The number of flux quanta piercing the torus is related to the area $A = |\vec{L}_1 \times \vec{L}_2|$ of the torus by $2\pi c \ell_B N_\phi = A = L_1 L_2 \sin(\theta)$ with magnetic length $\ell_B = \sqrt{\hbar c/(eB)}$ and $\theta$ being the angle between $\vec{L}_1$ and $\vec{L}_2$. The case of a rectangular torus corresponds to $\theta = \pi/2$. The shape of the torus is conveniently parametrized by the aspect ratio

$$\tau = \frac{L_2}{L_1} e^{i\theta}.$$  

In the presence of the magnetic field, any valid wave function on the torus must be invariant (up to an overall phase) under single-particle magnetic translations $t(\vec{L}_1)$ and $t(\vec{L}_2)$, where the magnetic translation operator is defined as

$$t(\vec{L}) = \exp \left[ \vec{L}(\vec{\nabla} - i \frac{e}{\hbar c} \vec{A} - i \frac{\vec{L} \times \vec{F}}{\tau^2} \right].$$

Let us define "small" magnetic translations

$$\hat{t}_1 \equiv i \left( \frac{\vec{L}_1}{N_\phi} \right) = \exp \left[ \frac{L_1}{N_\phi} \partial_x \right]$$

$$\hat{t}_2 \equiv i \left( \frac{\vec{L}_2}{N_\phi} \right) = \exp \left[ i \frac{L_2 \cos \theta}{L_1 N_\phi} + 2\pi i \frac{x}{L_1} \right]$$

$$\times \exp \left[ \frac{L_2 \sin \theta}{N_\phi} \partial_x + \frac{L_2 \sin \theta}{N_\phi} \partial_y \right].$$

The periodic boundary conditions of a wave function $\psi$ can, thus, be formulated as

$$\hat{t}_1^{N_\phi} \psi = e^{i\alpha_1 \tau} \psi$$

$$\hat{t}_2^{N_\phi} \psi = e^{i\alpha_2 \tau} \psi.$$  

In the remainder of the paper, we will set the solenoid fluxes $\alpha_1, \alpha_2 = 0$ without loss of generality.

As $\hat{t}_1$ and $\hat{t}_2$ do not commute with each other, we can choose the single-particle states to be eigenstates of only one of them. In the following, we will mostly use eigenfunctions of $\hat{t}_1$

$$\phi_{n,j}^{\hat{t}_1}(x, y) = N_n^{\ell_B} \sum_{k=-\infty}^{\infty} e^{-2\pi i(j+kN_\phi)z} e^{-y^2/(2\ell_B^2)}$$

$$\times \exp \left[ \frac{i\pi \tau}{N_\phi} (j+kN_\phi)^2 \right] H_n \left( \frac{2N_\phi^{\ell_B}}{L_1} (j+kN_\phi) - \frac{y}{\ell_B} \right),$$

where $z = (x + iy)/L_1$ is the dimensionless complex coordinate of the particles, $n = 0, 1, \ldots$ is the Landau level index and $j = 0, \ldots, (N_\phi - 1)$ the momentum index. $H_n$ denotes the $n$th Hermite polynomial. Note that the momentum is only defined modulo $N_\phi$, because any larger value can be absorbed into the sum over windings around the torus. The normalization constant is given by

$$N_n^{\ell_B} = \left( \frac{\sqrt{2N_\phi \mathcal{Z}(\tau)}}{(2^n n!)^2 \mathcal{A}} \right)^{1/2},$$

where $\mathcal{A} = 2\pi N_\phi^{\ell_B} N_\phi$ is the total area of the torus and $\mathcal{Z}(\tau) = (L_2/L_1) \sin(\theta)$ is the imaginary part of $\tau$. $\phi_{n,j}^{\hat{t}_1}$ is an eigenfunction of $\hat{t}_1$ with eigenvalue $\exp[-2\pi i j/N_\phi]$, while $\hat{t}_2$ shifts the momentum by 1: $\hat{t}_2 \phi_{n,j}^{\hat{t}_1} = \phi_{n,j-1}^{\hat{t}_1}$. As we will need to distinguish single-particle states at different flux later on, we keep the magnetic length $\ell_B$ as an explicit parameter in the single-particle state $\phi_{n,j}^{\hat{t}_1}$. 
C. Product of single-particle states on the torus

In complete analogy to the disk and sphere geometry, we can write a product of two single-particle states on the torus at magnetic flux $N_{\phi_1}$ and $N_{\phi_2}$ as

$$\phi_{n_1,j_1}^{f_1}(x,y)\phi_{n_2,j_2}^{f_2}(x,y) = \sum_{n=0}^{N_{\phi_1}} \sum_{j=0}^{N_{\phi_2}-1} C_{j_1,j_2}^{n_1,n_2;n} \phi_{n,j}^{f_1}(x,y), \quad (10)$$

where the magnetic lengths are related by $\ell^2 - \ell_1^2 = \ell_2^2$, which is equivalent to $N_{\phi} = N_{\phi_1} + N_{\phi_2}$. The constants $C_{j_1,j_2}^{n_1,n_2;n} = C_{j_1,j_2}^{n_1,n_2;n}(N_{\phi_1},N_{\phi_2},N_{\phi})$ depend on the fluxes $N_{\phi_1},N_{\phi_2}$ as well as the aspect ratio of the torus. They can be computed for arbitrary $n_1$ and $n_2$, but we have not been able to find a closed formula except for the two simplest cases $(n_1,n_2) = (0,0)$ and $(1,0)$. For the CF construction we need to know the coefficients for $n_2 = n = 0$, but arbitrary $n_1$. In the following, we restrict ourselves to these cases.

In order to simplify notation later on we define $Q$ as the greatest common divisor (gcd) of $N_{\phi_1}$ and $N_{\phi_2}$:

$$Q = \text{gcd}(N_{\phi_1},N_{\phi_2})$$

$$N_{\phi_1} = t_1 Q$$

$$N_{\phi_2} = t_2 Q$$

$$N_{\phi} = (t_1 + t_2)Q = tQ. \quad (11)$$

It follows that $Q = \text{gcd}(N_{\phi_1},N_{\phi}) = \text{gcd}(N_{\phi_2},N_{\phi})$. The different magnetic lengths are related to $t_1$ and $t_2$ by:

$$t_1 = \frac{L_1}{t}$$

$$t_2 = \frac{L_2}{t}. \quad (12)$$

For $n_1 = 0,1$ the coefficients in (10) become rather simple:

$$C_{j_1,j_2;j}^{0,0;0} = \sqrt{\frac{23(\tau)}{A(-i\tau)\sqrt{Q}t_1t_2}} \theta_3 \left( \frac{\pi(t_2j_1 - t_1j_2 + \beta t_1t_2Q)}{t_1t_2N_{\phi}} \right) \exp \left[ \frac{\pi N_{\phi}}{i\pi t_1t_2N_{\phi}} \right], \quad (13)$$

and $\theta_3(z|q) = \partial_2 \theta_3(z|q)$. For higher $n_1$, the coefficients can in principle still be represented with help of higher derivatives of the $\theta_3$-function, but they become increasingly cumbersome to evaluate. They can be written as:

$$C_{j_1,j_2;j}^{n_1,0;0} = \frac{N_{\phi}}{N_0} \left[ \frac{c_{j_1+j_2+\beta Qt_1}}{a_{\beta}} \right] \sum_{i=0}^{\lfloor n_1/2 \rfloor} \frac{n_1!}{2i!(2i)!} \left( -\frac{t_2}{t} \right)^i \left( -\frac{2i}{t} \right)^{n_2-i} \exp \left[ \frac{i\pi t_1t_2N_{\phi}}{s - \frac{t_2j_1 - t_1j_2 + \beta t_1t_2Q}{t_1t_2N_{\phi}}} \right]. \quad (16)$$

The derivation of Eq. (16) involves straightforward but tedious algebra, which is done in the Appendix. Equations (13) and (14) can be obtained from (16) by a Poisson resummation. Note that for $3(\tau)$ not too small, the...
summation over \( s \) converges rapidly. For numerical purposes, one needs to care only about the first few terms around zero. The problem of convergence for small \( \Im(\tau) \) can in principle be avoided by doing a Poisson resummation on the sum over \( s \)—similar to what was done in obtaining Eqs. \([13]\) and \([14]\).

D. Composite fermion wave functions on the torus

The formulas derived in the previous Section allow for evaluation of the Jain state at filling \( \nu = \frac{n}{p+1} \) given by

\[
\Psi_\nu(\{z_j\}) = \mathcal{P}_{\text{LLL}}[\Phi_n(\{z_j\})\Phi_1(\{z_j\})^p], \tag{17}
\]

where \( \Phi_j \) is the many-body wave function of the lowest \( j \) Landau levels completely filled. For \( p \) even (odd), this describes a fermionic (bosonic) state. As a sanity check, one may note that \( n = 1 \) reproduces the Laughlin states at filling \( \frac{1}{p+1} \). Expression \([17]\) can also be used to evaluate wave functions corresponding to quasihole and/or quasielectron excitations— in the same way as on the sphere or the disk. For the sake of simplicity, we focus on ground-state wave functions in the following discussion.

In principle, it is straightforward to evaluate \([17]\), by multiplying out the slater determinants and using the coefficients \([16]\) (repeatedly if \( p > 1 \)) to reduce the expression to a lowest Landau level wave function at the combined flux. For the simplest state, describing the bosonic Jain state at filling \( \nu = 2/3 \), the explicit expression becomes:

\[
\Psi_{2/3}(\{x_i, y_i\}) = \sum_{\sigma \in S_N} (-1)^\sigma \left( \prod_{a=0}^{N-1} \sum_{\beta_\alpha = 0}^2 \prod_{\alpha=0}^{N/2-1} \left( C^{10;0}_{\alpha, \sigma(\alpha); j_\alpha} C^{00;0}_{\alpha, \sigma(\alpha+N/2); j_\alpha+N/2} \right) \right) \sqrt{N!} \prod_{j=0}^{N-1} n_j! \, m_\mu(\{z_j\}), \tag{18}
\]

where \( j_\alpha = (\alpha \mod N_\phi + \sigma(\alpha) + \beta_\alpha Qt_\phi) \mod N_\phi \). The \( m_\mu(\{z_j\}) \)'s are the many-body basis states on the torus:

\[
m_\mu(\{z_j\}) = (z_1, \ldots, z_N|\mu), \tag{19}
\]

with \( \mu = \{n_0, \ldots, n_{N_\phi-1}\} \) and \( n_j = \sum_{a=0}^{N_\phi-1} \delta_{j, j_a} \) being the occupation number of the single-particle orbital with \( t_\phi \)-eigenvalue \( \exp[-2\pi ij/N_\phi] \). The state \( m_\mu(\{z_j\}) \) is as usual defined by the properly normalized slater determinant (for fermions) or permanent (for bosons) of all the occupied single-particle states \( \phi(x, y) \), see Eq. \([9]\).

The qualitative difference between Eq. \([18]\) to the corresponding disk and sphere expressions lies in the additional sums over \( \beta_1, \ldots, \beta_N \). In the disk and sphere geometry, the momentum of the product of two single-particle states on the torus is simply the sum of the two momenta. The projection, thus, involves evaluating \( N! \) terms in order to obtain the occupation numbers in the occupation number basis. On the torus, the momentum is only defined modulo the flux \( N_\phi \). This implies that the winding sums in the single-particle states at flux \( N_\phi \) and \( N_\phi \) yield different momenta at the final flux. For instance, in order to compute the CF state at filling fraction \( \nu = 2/3 \), one needs to evaluate \( N!3^N \) terms, which limits the system sizes one can reach. Note that this limitation becomes worse, if we increase \( p \) in Eq. \([17]\).

In addition, \([17]\) is in general not an eigenstate of the many-body translation operator \( \hat{T}_1 = \prod_{j=0}^{N-1} \hat{l}_1^{(j)} \), where \( \hat{l}_1^{(j)} \) translates the \( j \)th particle by \( \hat{L}_1/N_\phi \). An eigenstate can be obtained by either restricting to the correct momentum sector in the Fock basis or by applying the appropriate projection operator. As the system is translational invariant, we expect that we can write each momentum eigenstate as a product of a wave function \( \psi^{rel} \) that depends only on the relative coordinates and a wave function \( \psi^{com} \) that only depends on the center-of-mass coordinate \( Z = \sum_{j=1}^{N} z_j \) and incorporates the action of the many-body translation operators \( \hat{T}_a = \prod_{j=0}^{N-1} \hat{l}_a^{(j)} \), \( a = 1, 2 \). Thus, Eq. \([18]\) can be written as

\[
\Psi_{2/3}(\{x_i, y_i\}) = \sum_{j=0}^{2} c_j \psi^{rel}(\{z_j\}) \psi^{com}(Z), \tag{20}
\]

where \( j \) labels the many-body momentum and \( c_j \) are coefficients that may depend on \( \tau \) and \( N \). However, obtaining the explicit form of Eq. \([20]\) from the Fock decomposition is a very hard, unsolved problem, because of the infinite sums appearing in the \( \theta \) functions. For the Laughlin states—or more generally the Read-Rezayi series—one way to get around this problem is by guessing the correct form of \( \phi^{rel} \) and using the boundary conditions \([7]\) to derive \( \psi^{com} \). This is possible, because the Laughlin state is the unique ground state of a model Hamiltonian. Unfortunately, this is not true for general CF states, which is why the decomposition into \( \psi^{rel} \) and \( \psi^{com} \) is not known in these cases.

III. NUMERICAL ANALYSIS
In this Section, we present numerical checks on the wave functions obtained by Eq. (17). We computed the overlap between the exact diagonalization ground state and the bosonic CF state at filling $\nu = 2/3$ for system sizes up to $n = 10$ particles and in the fermionic case at filling $\nu = 2/5$ for system sizes up to $N = 6$ particles. The exact diagonalization was done both for Coulomb interaction and the smallest relevant Haldane pseudopotential $-V_0$ for bosons and $V_1$ for fermions. The shape of the torus was kept rectangular ($\theta = \frac{\pi}{2}$ in Eq. (4)), with aspect ratios $|\tau|$ varying from 0.1 to 10. Due to the invariance of the shape of the torus under the modular transformation $S : \tau \rightarrow -1/\tau$, we can restrict the analysis to aspect ratios $0.1 \leq |\tau| \leq 1$ without loss of generality, as a torus with $|\tau| > 1$ can be obtained by an $S$-transformation. The absolute value of the overlap $O = |\langle \Psi_{\text{CF}} | \Psi_{\text{ex}} \rangle|$ between the CF state and the exact diagonalization ground state is shown in Figs. 1 and 2 for varying aspect ratios.

When comparing overlaps of the torus and sphere geometry, we choose the most isotropic point, namely the square torus with $|\tau| = 1$. In the bosonic case, we find an overlap of $O = 0.996$ (Coulomb interaction) and $O = 0.973$ ($V_0$ interaction) for a square torus, which are slightly higher than the overlaps found in Ref. [33] for the spherical geometry. In the fermionic case, we find overlaps $O = 0.999$ (Coulomb interaction) and $O = 0.990$ ($V_1$ interaction), which are slightly lower, but still comparable to the overlaps found in Ref. [34].

The overlaps depend strongly on the shape of the torus, even though they remain quite high throughout the whole range of aspect ratios. On general grounds, we expect the overlap to approach unity in the limit of $|\tau| \rightarrow 0$ and $|\tau| \rightarrow \infty$. In Ref. [35] it was shown that the ground state of Hamiltonians with quite generic repulsive interactions becomes a product state for aspect ratios $|\tau| \rightarrow \infty$—the so-called thin torus limit. We checked numerically that the CF state has this property as well. The other limit $|\tau| \rightarrow 0$ can—for a rectangular torus—be mapped to the thin torus limit, when using the Landau gauge $\vec{A} = Bxy\hat{y}$ and eigenfunctions of the $t_2$ operator.

For all system sizes, we observe a dip (in the case of fermions several dips) in the overlap curve. The position of these dips depends on the system size— in the bosonic case, it seems to be shifted to lower values of $|\tau|$ for increasing system size. In the fermionic case, there is too little numerical data to make a statement. The origin of the dips is not clear at the moment, but one can note that the overlap of the fermionic Laughlin state at $\nu = 1/3$ with the Coulomb ground state has a qualitatively similar behavior as a function of $|\tau|$ to the one shown in Figs. 2.

IV. PROJECTION SCHEMES IN REAL SPACE

In principle, the method described in this paper allows one to compute any CF states on the torus. However, evaluating Eq. (17) becomes numerically hard for large $p$ when using Eq. (10). On the torus, one needs to evaluate $(N!)^{p+\bar{N}}$ terms to obtain the wave function in Fock space. Even on the disk and the sphere, where one does not have the additional complication of the $\beta_1, \ldots, \beta_N$ sums, the number of terms, which one needs to evaluate, is still $(N!)^p$. This restricts the system size to very small systems for large $p$.

A way around this, at least on the disk and the sphere, is to evaluate Eq. (17) in real space and use Monte Carlo techniques to study the resulting model wave functions. How to write the projection operator $P_{\beta_1 L \ldots L}$ in real space was shown by Girvin and Jach in Ref. [32]. It amounts to moving all anti-holomorphic components to the left and replacing them by derivative operators $\tilde{z} \rightarrow 2\tilde{z}$ with the assumption that the derivatives do not act on the
In this paper, we generalized the CF theory to the torus geometry. We showed the validity of our method by calculating the overlap between the CF states and the exact diagonalization ground state of Coulomb and the smallest relevant Haldane-pseudopotential interactions for filling fractions $\nu = 2/3$ and $\nu = 2/5$ and system sizes up to $N = 10(6)$ particles for the bosonic (fermionic) states. The overlaps on the square torus are comparable to the ones obtained in the disk and sphere geometry. It turns out that numerical evaluation of the wave function is harder on the torus than on the disk and sphere, because the winding sums mix different momentum sectors. We have also speculated on possible generalizations of the real space projection schemes to the torus geometry, though, unfortunately, we have not been able to find an explicit realization. Such schemes may allow one to reach larger system sizes than are possible with the method presented here.

Let us emphasize again that our method works for the whole low-energy sector of the CF states, even though we only treated the ground states explicitly in this paper. The techniques introduced here may be useful for systems that cannot be studied directly on the sphere, because they have different shifts, as e.g. the one studied in Ref. [39]. They can also be used to study generalizations of the Abelian Haldane-Halperin hierarchy, such as the Bonderson-Slingerland states [29], the non-Abelian condensate state [30], or the bipartite CF states [31]. Also it will clearly be interesting to see whether the exact agreement between the hierarchy and the CF wave functions that have been demonstrated on the plane and on the sphere, also holds true on the torus.

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### Appendix A: Derivation of product formula

In this Appendix, we derive formula (16) for an expansion of the product of two single-particle states at fluxes $N_{\phi_1}$ and $N_{\phi_2}$ in terms of the single-particle states at the combined flux $N_{\phi}$ with $N_{\phi_1} + N_{\phi_2} = N_{\phi}$. The product of two single-particle states is given by:
\[
\phi^\ell_1(x,y) \phi^\ell_2(x,y) = N_{n_1} N_{n_2} e^{-y^2/(2t^2)} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{-2\pi i (j_1 + k_1 N_{\phi_1} + j_2 + k_2 N_{\phi_2}) z} 
\times \exp \left[ i\pi \frac{(j_1 + k_1 N_{\phi_1})^2}{N_{\phi_1}} + \frac{(j_2 + k_2 N_{\phi_2})^2}{N_{\phi_2}} \right] H_{n_1} \left( \frac{2\pi \ell_1}{L_1} (j_1 + k_1 N_{\phi_1} - y \frac{\ell_1}{t_1}) \right). \tag{A1} \]

In the following, subscript 1 and 2 denote quantities of the two single-particle states respectively, while those without subscript denote those of the product. Note that \( N_{\phi} = N_{\phi_1} + N_{\phi_2} \) and \( \ell^{-2} = \ell_1^{-2} + \ell_2^{-2} \). Let us assume \( n_2 = 0 \) but \( n_1 \) may be arbitrary for the time being. We could also consider arbitrary \( n_2 \), but it will only complicate things unnecessarily.

Define \( Q \) as the greatest common divisor (gcd) of \( N_{\phi_1} \) and \( N_{\phi_2} \):

\[
Q = \gcd(N_{\phi_1}, N_{\phi_2}) \quad N_{\phi_1} = t_1 Q \quad N_{\phi_2} = t_2 Q \quad N_{\phi} = (t_1 + t_2)Q \equiv tQ. \tag{A2} \]

It follows that \( Q = \gcd(N_{\phi_1}, N_{\phi_2}) = \gcd(N_{\phi_2}, N_{\phi}) \). We use that the different magnetic lengths are related as:

\[
\frac{\ell}{\ell_1} = \sqrt{\frac{t_1}{t}}, \quad \frac{\ell}{\ell_2} = \sqrt{\frac{t_2}{t}}. \tag{A3} \]

We define the torus as in Sec. II. It is easy to check that the product of the two single-particle states obeys the correct boundary conditions for flux \( N_{\phi} \).

Let us first discuss how to rewrite the double sum over windings coming from both single-particle states, denoted by \( k_1 \) and \( k_2 \). We can choose integers \( k, s \in \mathbb{N} \), and \( \beta \in \{0, \ldots, t-1\} \) such that

\[
k_1 = \beta + k - t_2 s, \quad k_2 = k + t_1 s. \tag{A4} \]

which implies

\[
tk = t_1 k_1 + t_2 k_2 + \beta, \quad ts = \beta + k_2 - k_1. \tag{A5} \]

It is beneficial to introduce some more notation that will simplify expressions later on. We find that we can rewrite the phase factors of the single-particle states as

\[
j_1 + k_1 N_{\phi_1} = \frac{\ell}{\ell_1} A_k - t_1 t_2 Q Y_s \quad j_2 + k_2 N_{\phi_2} = \frac{\ell}{\ell_2} A_k + t_1 t_2 Q Y_s. \tag{A6} \]

where \( A_k \) depends only on \( k \) and \( \beta \), but not on \( s \), while \( Y_s \) depends only on \( s \) and \( \beta \), but not on \( k \):

\[
A_k = j_1 + j_2 + k N_{\phi} + \beta t_1 Q, \quad Y_s = s - \frac{t_2 j_1 - t_1 j_2 + \beta t_1 t_2 Q}{t_1 t_2 N_{\phi}}. \tag{A7} \]

Using these definitions we see that the \( z \)-dependent factor on the right-hand-side of Eq. (A1) does not depend on the summation index \( s \) and becomes rather simple:

\[
e^{-2\pi i (j_1 + k_1 N_{\phi_1} + j_2 + k_2 N_{\phi_2}) z} = e^{-2\pi i A_k z}. \tag{A8} \]

Let us now consider the factor that is exponential in the winding number. Using (A7) it can be rewritten as

\[
\exp \left[ i\pi \frac{(j_1 + k_1 N_{\phi_1})^2}{N_{\phi_1}} + \frac{(j_2 + k_2 N_{\phi_2})^2}{N_{\phi_2}} \right] = \exp \left[ i\pi \frac{A_k^2}{N_{\phi}} + t_1 t_2 N_{\phi} Y_s^2 \right] \quad \tag{A9} \]

i.e. it factorizes into two parts, each of which only depends on one of the summation indices.

Most of the complication lies in the Hermite polynomials, at least if \( n_1 \neq 0, 1 \). The Hermite polynomial in Eq. (A1) can be written as

\[
H_{n_1} \left( \frac{2\pi \ell_1}{L_x} (j_1 + k_1 N_{\phi_1}) - y \frac{\ell_1}{t_1} \right) = H_{n_1} \left( \frac{\ell}{\ell_1} \left[ \frac{2\pi \ell}{L_x} A_k - y \frac{\ell}{t} \right] - \left( \frac{2\pi \ell}{L_x} t_1 t_2 Q \right) Y_s \right). \tag{A10} \]

With the following identities

\[
H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_k(x)(2y)^{n-k}, \quad H_k(\gamma x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \gamma^{k-2i} (\gamma^2 - 1)^i \binom{k}{2i} \frac{(2i)!}{i!} H_{k-2i}(x) \tag{A11} \]
we find that
\[
H_{n_1} \left( \frac{2 \pi \ell}{\ell_1 L_1} A_k - \alpha_1 Y_s \right) = \sum_{l=0}^{n_1} \frac{l!}{l!} \left( -2 \alpha_1 Y_s \right)^{n_1-l} H_l \left( \frac{2 \pi \ell}{\ell_1 L_1} A_k \right) \\
= \sum_{l=0}^{n_1} \sum_{i=0}^{(l/2)} \frac{(2i)!}{i!} \left( \frac{2 \pi \ell}{\ell_1 L_1} \right)^{i-2i} \left( \frac{2 \pi \ell}{\ell_1 L_1} \right)^{i} \left( \frac{l}{l_1} \right)^{n_1} \times \left( -2 \alpha_1 Y_s \right)^{n_1-l} H_{l-2i} \left( \frac{2 \pi \ell}{L_1} A_k \right)
\]  

(A12)

We can now identify the coefficient \( C_{n_1}^{(i),n} \) to be
\[
C_{n_1, j_1, j_2, j_3}^{n_1, 0:n} = \frac{N_{n_1}^2 N_0^2}{n! N_n^2} \sqrt{\frac{\pi}{\ell_1}} \sum_{l=0}^{n_1} \frac{n_1!}{(n_1-n-2l)! l!} \left( -\frac{t_2}{t} \right)^i \left( -4 \pi t_1 t_2 \sqrt{\frac{3(\tau)Q}{2\pi t}} \right)^{n_1-n-2i} \times \sum_{s=-\infty}^{\infty} \left( s - \frac{t_2 j_1 - t_1 j_2 + \beta_1 t_2 Q}{t_1 t_2 N_0} \right)^{n_1-n-2i} \exp \left[ i\pi \tau t_1 t_2 N_0 \left( s - \frac{t_2 j_1 - t_1 j_2 + \beta_1 t_2 Q}{t_1 t_2 N_0} \right)^2 \right]
\]

(A13)

where \( j = (j_1 + j_2 + \beta t_1 Q) \mod N_0 \). The coefficients vanish for other values of \( j \). Equation (10) is obtained by setting \( n = 0 \). In order to find Eqs. (13) and (14) one must do a Poisson resummation on the sum over \( s \).

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