QUANTITATIVE UNIQUENESS PROPERTIES FOR $L^2$ FUNCTIONS WITH FAST DECAYING, OR SPARSELY SUPPORTED, FOURIER TRANSFORM

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Abstract. This paper builds upon two key principles behind the Bourgain-Dyatlov quantitative uniqueness theorem for functions with Fourier transform supported in an Ahlfors regular set. We first provide a characterization of when a quantitative uniqueness theorem holds for functions with very quickly decaying Fourier transform, thereby providing an extension of the classical Paneah-Logvinenko-Sereda theorem. Secondly, we derive a transference result which converts a quantitative uniqueness theorem for functions with fast decaying Fourier transform to one for functions with Fourier transform supported on a fractal set. As well as recovering the result of Bourgain-Dyatlov, we obtain analogous uniqueness results for denser fractals.

1. Introduction

The Fourier transform is the extension to $L^2(\mathbb{R}^d)$ of the operator which acts on $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \xi \cdot t} dm_d(t)$, where $m_d$ is the $d$-dimensional Lebesgue measure. This paper builds upon two principles underlying Bourgain and Dyatlov’s breakthrough uniqueness theorem for functions with Fourier transform supported in an Ahlfors regular set [3] (see Section 1.3 below):

1. Classical uniqueness theorems for functions with compactly supported Fourier transform extend to functions with sufficiently fast decaying Fourier transform, and
2. these results can be transferred to uniqueness theorems for functions with sparsely supported Fourier transform by appealing to the Beurling-Malliavin theorem.

In [3] these two principles are somewhat intertwined in the proof. Our goal here is to separate them and develop some theory for a general
weight function (in the spirit of Koosis’ books \cite{10, 11}). By doing so, we

(1) obtain a characterization of when a uniqueness theorem holds for functions with fast decaying Fourier transform (under a convexity assumption on the weight), see Theorem 1.3, and

(2) prove a general transference principle which converts a quantitative uniqueness theorem for functions with fast decaying Fourier transform to one for functions with sparsely supported Fourier transform (Theorem 1.6).

As well as recovering the uniqueness result in \cite{3}, this point of view enables one to obtain analogous results for functions whose Fourier transform is integrable with respect to the end-point weight given by

\[ \exp\left(\frac{|t|}{\log(e + |t|)}\right). \]

1.1. On the uniqueness (or strong annihilation) property for functions with fast decaying Fourier transform. Denote by \( m_d \) the Lebesgue measure on \( \mathbb{R}^d \), \( d \geq 1 \).

Definition 1.1. A Borel set \( E \subset \mathbb{R}^d \) is \((\gamma, \ell)\)-relatively dense if \( m_d(E \cap Q) \geq \gamma \) for any cube \( Q \subset \mathbb{R}^d \) of side-length \( \ell \).

The role of relatively dense sets in uniqueness theorems is exhibited by the classical Paneah-Logvinenko-Sereda theorem for band limited functions \((\cite{13, 19}, \text{see also} \cite{7, 12, 15, 18})\), one of the prototypical forms of the uncertainty principle, see Chapter 1 of \cite{20}.

The Paneah-Logvinenko-Sereda Theorem. Fix \( E \subset \mathbb{R}^d \). For every \( N > 0 \), there is a constant \( C > 0 \) such that

\begin{equation}
\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)} \text{ for every } f \text{ with } \text{supp}(\hat{f}) \subset B(0, N)
\end{equation}

if and only if \( E \) is \((\gamma, \ell)\)-relatively dense for some \( \gamma \in (0, 1) \) and \( \ell > 0 \).

In particular, the theorem says that a band limited function can be reconstructed uniquely by its values on a relatively dense set. The first result of this paper will be an extension of the Paneah-Logvinenko-Sereda theorem to functions which, instead of being band limited, have sufficiently fast decaying Fourier transform.

Definition 1.2. A weight \( W : [0, \infty) \to [0, \infty] \) has the Paneah-Logvinenko-Sereda (PLS) property if, for every \( d \in \mathbb{N}, \gamma \in (0, 1), \ell > 0, \) and \( C_W > 0 \), there exists a finite constant \( C = C(d, W, C_W, \gamma, \ell) > 0 \) such that if \( f \in L^2(\mathbb{R}^d) \) satisfies

\begin{equation}
\int_{\mathbb{R}^d} |\hat{f}(\xi)|W(|\xi|)d\xi \leq C_W^2\|f\|_{L^2(\mathbb{R}^d)}^2,
\end{equation}

and $E$ is a $(\gamma, \ell)$-relatively dense set, then

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)}.$$  

Notice that the ‘if’ direction of the Paneah-Logvinenko-Sereda theorem can be rephrased as the statement that for any $N > 0$, the weight

$$W(t) = \begin{cases} 1 & \text{for } |t| \leq N, \\ +\infty & \text{for } |t| > N \end{cases}$$

has the PLS property.

**Theorem 1.3.** Suppose that $W : [0, \infty) \to [0, \infty]$ satisfies

1. $W(0) = 1$, $W$ is non-decreasing, $W$ is lower-semicontinuous, and $\lim_{t \to \infty} W(t) = \infty$.
2. The mapping $\log r \mapsto \log W(r)$ is convex\(^1\) on $[1, \infty)$.  

Then $W$ satisfies the PLS property if and only if

$$\int_0^{\infty} \frac{\log W(t)}{1 + t^2} dm_1(t) = \infty.$$  

The constant $C > 0$ in (1.3) that is obtained in the proof takes quite an explicit form that can be calculated given a particular choice of $W$ satisfying (1.4), see Proposition 3.1 below.

Our motivation for formulating Theorem 1.3 came from the paper [3], where the following uniqueness theorem for functions with fast Fourier decay is presented: If $\delta \in (0, 1)$, and $\Theta(\xi) = |\xi| \log(|e^{+}\xi|)$, then

$$\|e^{\Theta(\xi)} \hat{f}(\xi)\|_{L^2(\mathbb{R})} \leq C_0 \|f\|_{L^2(\mathbb{R})}$$

implies $c \|f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(E)}$ where $E$ is an infinite union of well separated intervals of some fixed side-length, and $c$ depends on $\delta, C_0$, and the sidelength of the intervals (see (1.6) in [3]). Although stated in terms of intervals, it appears that one could adapt their proof to yield the stronger PLS property. Very recently, Han and Schlag [6] extended this estimate to several dimensions using Cartan set techniques.

As a consequence of Theorem 1.3, we observe that the end-point weight $W(\xi) = e^{\Theta(\xi)}$ with $\Theta(\xi) = \frac{|\xi|}{\log(e^{+}|\xi|)}$ has the PLS property. It is remarked in [3] (see Remark 2 after Lemma 3.1 in [3]) that this weight does not grow quickly enough for their proof to be applicable. As such, our approach is rather different to that taken in [3], or [6], relying on quasianalyticity rather than harmonic measure estimates.

\(^1\)This permits an interval $(t_0, \infty)$ on which $W(t) = +\infty$.  

1.2. From fast decay to sparse support: A transference principle. To develop a transference principle we shall lean on the scheme developed in [3]. In particular our considerations are based on use of the Beurling-Malliavin multiplier theorem (see, e.g. [8, 11]), which will restrict our discussion to uniqueness theorems in one dimension. Han and Schlag [6] adapted the techniques in [3, 9] to derive a multi-dimensional analogue of the Bourgain-Dyatlov fractal uncertainty principle for certain Ahlfors regular subsets of $\mathbb{R}^d$ with (possibly distorted) product structure, still making use of a multiplier theorem in one dimension. There is a natural analogue of Theorem 1.6 below in this product setting, but we do not explore such results here.

The condition of sparsity that arises is a modification of the short intervals condition (cf. the Beurling gap theorem [20]) taking into account that

- the result here is an $L^2$-theorem, so the condition of sparsity should be stable under translations in the Fourier domain, and,
- our conclusion is quantitative, so there should be some uniformity in the shortness condition.

With this in mind, we make the following definitions.

**Definition 1.4.** Fix a weight $W : [0, \infty) \to [0, \infty)$ with $W > 1$ on $[1, \infty)$.

- A collection $\{J_n\}_n$ is a $W$-short cover of a set $Q \subset \mathbb{R}$ if for every $n \in \mathbb{N}$, $J_n$ is comprised of intervals of length $\Omega_n = \log W(e^n)$ such that
  
  $\sum_{J \in J_n} \#J \geq Q \cap ([-e^{n+1}, -e^n] \cup [e^n, e^{n+1}])$, and

  $\|\{J_n\}_n\|_W := \sum_{n \in \mathbb{N}} \left( \frac{\Omega_n}{e^n} \right)^2 \text{card}(J_n) < \infty$.

- A set $Q$ is called $W$-sparse if, for every $t \in \mathbb{R}$, the set $Q - t$ has a $W$-short cover $\{J_n^{(t)}\}_n$, and moreover

  $\|Q\|_W = \sup_{t \in \mathbb{R}} \inf_{\{J_n^{(t)}\}_n \text{ a } W\text{-short cover of } Q - t} \|\{J_n^{(t)}\}_n\|_W < \infty$.

**Remark.** If $Q$ has a short $W$-cover and $\tilde{W} \leq W$, then $Q$ has a short $\tilde{W}$ cover. To see this cover each interval $J \in J_n$ of length $\log W(e^n)$ with no more than $\left\lfloor \frac{\log W(e^n)}{\log \tilde{W}(e^n)} \right\rfloor + 1$ intervals of length $\log \tilde{W}(e^n)$, and set $\tilde{J}_n$ to be the resulting collection of intervals of length $\log \tilde{W}(e^n)$.
Thus
\[ \sum_n \left( \frac{\log \tilde{W}(e^n)}{e^n} \right)^2 \text{card}(\tilde{J}_n) \leq 2 \sum_n \left( \frac{\log W(e^n) \log \tilde{W}(e^n)}{e^{2n}} \right) \text{card}(J_n), \]
and the right hand side is smaller than \( 2\|\{J_n\}_n\|_W \). As such, a slower growing weight \( W \) will have more \( W \)-sparse sets associated to it.

The transference principle that we prove will be for relatively dense sets with additional structure, namely that the sets contain intervals that are not too small.

**Definition 1.5.** Fix \( \gamma \in (0,1) \). A collection \( \{J_n\}_{n \in \mathbb{Z}} \) of intervals is called \( \gamma \)-dense if, for every \( n \in \mathbb{Z}, J_n \cap [n,n+1] \neq \emptyset \) and \( \ell(J_n) \geq \gamma \).

A set \( E \) is called a \( \gamma \)-dense collection of intervals if \( E = \bigcup_{n \in \mathbb{Z}} J_n \) for a \( \gamma \)-dense collection of intervals \( \{J_n\}_{n \in \mathbb{Z}} \).

Notice that, if \( E \) is a \( \gamma \)-dense collection of intervals, then \( E \) is a \((\gamma, 3)\)-relatively dense set.

**Theorem 1.6.** Fix an increasing weight \( W \geq 1 \) such that

1. for every \( \alpha > 0, W^\alpha \) has the PLS property\(^2\),
2. there is a constant \( C_{\text{doub}} \) such that \( \log W(et) \leq C_{\text{doub}} \log W(t) \), and \( \log W(t) \leq t/4 \), for every \( t > 1 \)
3. \( \|\log W\|_{\text{Lip}} = \Lambda_W < \infty \).

For every \( \Lambda > 0 \) and \( \gamma > 0 \), there is a constant \( C = C(C_{\text{doub}}, W, \Lambda_W, \Lambda, \gamma) \), such that for every \( W \)-sparse set \( Q \) with \( \|Q\|_W \leq \Lambda \), every \( \gamma \)-dense collection of intervals \( E \), and every \( f \in L^2(\mathbb{R}) \) with \( \text{supp}(\hat{f}) \subset Q \),

\[ \|f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(E)}. \]

**Remark.** One does not need the full strength of the PLS property in order to prove Theorem 1.6, it suffices for \( W \) to have an interval uniqueness property, where one only requires that \((1.2) \implies (1.3)\) in the case when \( E \) is a relatively dense collection of intervals. This appears to be a genuinely weaker property\(^3\).

The proof of Theorem 1.6 consists of a reorganization of the ideas presented in [3] combined with a localization trick. Very simple examples show that assuming a condition on the efficiency of covering \( Q \) alone (rather than all translations of \( Q \)) is not sufficient to reach

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\(^2\)Notice that conditions (1)–(2) of Theorem 1.3, and the validity of (1.4), are invariant under this transformation.

\(^3\)Koosis [10] describes this distinction in great detail in the context of qualitative uniqueness theorems for finite measures, for instance see the discussion around Kargaev’s theorem on p.236.
1.3. Application to regular sets. Combining Theorems 1.3 and 1.6 enables us to obtain uniqueness results for fractal sets denser than the class considered by Bourgain and Dyatlov in [3], and consequently to obtain an improvement of some of the main results in [3]. This is easiest to illustrate on the following class of regular sets.

Definition 1.7 ($\varphi$-regular sets). Fix an increasing continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$. A set $Q$ is $\varphi$-regular if, for every $N > 1$, $t \in \mathbb{R}$, and $1 \leq \ell \leq N$, there is a cover of the set $Q \cap [t-N, t+N]$ by $\varphi(N/\ell)$ intervals of length $\ell$.

For $\delta \in (0, 1)$, every $\delta$-regular set in the terminology of [3] is $\varphi$-regular with $\varphi(t) = Ct^\delta$ (see Lemma 2.8 of [3]).

Theorem 1.8. Suppose that $\varphi$ satisfies $\sum_{n \in \mathbb{N}} \frac{1}{n^2} \varphi(n) \leq C_0 < \infty$. For every $\gamma \in (0, 1)$, there is a constant $C = C(C_0, \gamma)$ such that for any $\gamma$-dense collection of intervals $E$, and $\varphi$-regular set $Q$,

$$\|f\|_{L^2(\mathbb{R})} \leq C \|f\|_{L^2(E)}$$

for every $f \in L^2(\mathbb{R})$ with $\text{supp}(\hat{f}) \subset Q$.

The uniqueness result in [3] corresponds to the case of a $\delta$-regular set with $\delta \in (0, 1)$.

Proof. Consider the weight $W(t) = \exp\left(\frac{t}{4 \log(\epsilon + t)}\right)$. Then $W$ satisfies the assumptions of Theorem 1.6 (for any $\alpha > 0$, $W^\alpha$ the PLS property from Theorem 1.3). Appealing to the $\varphi$-regularity of $Q$, we infer that for every $t \in \mathbb{R}$ and $n \geq 1$, the set $(Q-t) \cap [-e^{n+1}, -e^n] \cup [e^n, e^{n+1}]$ can be covered most $2\varphi((e-1)4(n+1)) \leq \varphi(8(n+1))$ intervals of length $\Omega_n = \log W(e^n)$ (notice that $\Omega_n \geq \frac{e^n}{4(n+1)}$, and $[e^n, e^{n+1}]$ has length $(e-1)e^n$). Therefore

$$\|Q\|_W \leq 2 \sum_{n \geq 1} \frac{1}{n^2} \varphi(8(n+1)) \leq 2 \sum_{n \geq 1} \frac{1}{n^2} \varphi(16n) \leq 2 \cdot 16^2 \cdot C_0,$$

and we conclude that (1.5) holds with $C = C(C_0, \gamma)$. \hfill $\Box$

Remark. In the setting of $\delta$-regular sets, Jin and Zhang [9] obtained a version of Corollary 1.8 with an effective bound in terms of the regularity parameter $\delta$. As $\delta$ tends to 1, the constant behaves in [9] as $\exp\left(\exp\left(\frac{C}{(1-\delta) \log(1-\delta)}\right)\right)$ (see Theorem 4.4 in [9]), where $\delta \in (0, 1)$ is the regularity parameter. The double exponential behaviour here is to be
expected (cf. Section 4.3 of [16]). If one incorporates the effective multiplier theorem\(^4\) given in [9] (see Theorem 3.2 in [9] or the appendix to [6]) into the arguments in this paper then one can also obtain effective bounds, but we do not explore this here.

2. Background material in quasianalytic functions required for Theorem 1.3

The main direction of the proof of Theorem 1.3 is the proof that (1.4) implies that the PLS property holds. The idea behind the proof is simple: The property on \(W\) yields that \(f\) belongs to a certain quasianalytic class. Using the localization principle behind the proof of the Paneah-Logvinenko-Sereda theorem [13] as presented in [12] or [15], we can reduce matters to a Remez-type inequality for quasianalytic functions, which is provided by an extension to several variables of a theorem of Nazarov-Sodin-Volberg [17]. This is carried out in Section 3. For readers who are not so concerned about the particular form of the constant \(C > 0\) in (1.3) that arises in the proof, we also provide a short proof of a more qualitative statement relying only on the Denjoy-Carleman theorem.

On the other hand, if (1.4) fails to hold, the Paley-Wiener multiplier theorem, see [10] p.97, yields the existence of functions supported on arbitrarily small balls for which 
\[
\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 W(|\xi|)^2 d\xi < \infty,
\]
thereby exhibiting that such \(W\) fail to satisfy the PLS property. For the benefit of the reader we sketch the argument in Section 4.

Until the conclusion of the proof of Theorem 1.3, assume that \(W\) is a weight satisfying hypotheses (1) and (2) of Theorem 1.3. There is no loss of generality by assuming that \(W \equiv 1\) on \([0, 1]\), and \(W\) grows faster than any power function at infinity,\(^5\) so we shall always do so. Set
\[
M_n = \sup_{\xi \in \mathbb{R}} \frac{|\xi|^n}{W(|\xi|)} = \sup_{t \geq 1} \frac{t^n}{W(t)}.
\]
Notice that \(M_n\) is an increasing log-convex sequence: \(M_n^2 \leq M_{n-1} M_{n+1}\). Therefore, setting \(M_0 = \max_{\xi \in \mathbb{R}} \frac{1}{W(|\xi|)} = 1\) we have that the sequence

\(^4\) [9] observed that one doesn’t need the full strength of the Beurling-Malliavin theorem to carry out the argument in [3], but rather a simpler result contained in [8]. See also [6].

\(^5\) The Fourier transform of a compactly supported bump function is in the Schwartz class, so power bounded weights certainly fail to satisfy the PLS property.
\[ \mu_n = \frac{M_{n-1}}{M_n} \] is non-increasing, and \( \mu_n \leq 1 \).

We begin by revisiting some very well-known elementary inequalities (e.g. [10]) in order to make our discussion self-contained. The property (2) of the weight \( W \) is used in the following lemma:

**Lemma 2.1.** For \( r > 1 \) with \( W(r) < \infty \), there exists an integer \( n \geq 0 \) with

\[ \log W(r) \leq (n + 1) \log r - \log M_n. \]

**Proof.** Fix \( r > 1 \) and choose some supporting line to the graph \( \{(\log t, \log W(t)) : t > 0\} \) with finite slope \( \nu \) at the point \( (\log r, \log W(r)) \) (\( \nu \geq 0 \) since \( W \) is increasing). With \( n \) equal the integer part of \( \nu \) we observe that \( M_n \leq (n + 1) \log r - \log W(r) \) (see [10] p.99-100), as required. \( \square \)

**Proposition 2.2.** The following inequalities hold:

\[ \sum_n \mu_n \leq \int_1^\infty \frac{\log W(t)}{t^2} \, dm(t) \leq \sum_n \mu_n + 1. \]

To prove this consider the Ostrowski function

\[ \rho(r) = \sup_{n \in \mathbb{N}} \frac{r^n}{M_n}. \]

Notice that, since the sequence \( \mu_n \) is decreasing, \( \rho(r) = \Pi_{\{n : r\mu_n > 1\}}(r\mu_n). \) The proposition is an immediate consequence of combining the following two lemmas.

**Lemma 2.3.** For \( r > 1 \),

\[ \log \rho(r) \leq \log W(r) \leq \log \rho(r) + \log r. \]

**Proof.** The left hand inequality is trivial. If \( r \) lies in the set \( I = \{W < \infty\} \), then we use Lemma 2.1 to fix \( n \geq 0 \) with \( \log W(r) \leq (n + 1) \log r - \log M_n \). But then \( \log W(r) \leq (n + 1) \log r - M_n \leq \rho(r) + \log r \), and so (2.1) holds in \( I \). In the case that \( I = [0, r_0) \) is a bounded interval, and \( W(r_0) = \infty \), then \( \lim_{r \to r_0^-} W(r) = \infty \) (\( W \) is increasing and lower semi-continuous), and since (2.1) holds in \( I \), we get that \( \rho(r_0) = +\infty \) and hence \( \rho(r) = \infty \) for all \( r > r_0 \) (\( \rho \) is non-decreasing). Finally, if \( I = [0, r_0) \) and \( W(r_0) < \infty \), \( M_n \leq r_0^n \) for every \( n \), and therefore \( \rho(r) \geq \sup_{n \in \mathbb{N}} \left( \frac{r}{r_0} \right)^n = \infty \) for \( r > r_0 \). \( \square \)

**Lemma 2.4.** The following identity holds:

\[ \int_1^\infty \frac{\log \rho(t)}{t^2} \, dm(t) = \sum_n \mu_n \]
Proof. The left hand side equals (using that $\mu_n \leq 1$)

$$
\int_1^{\infty} \sum_{n: \mu_n > 1} \frac{\log(t\mu_n)}{t^2} dm(t) = \sum_n \int_1^{\infty} \frac{\log(\mu_n t)}{t^2} dm(t).
$$

With a change of variable, we see that

$$
\int_1^{\infty} \frac{\log(\mu_n t)}{t^2} dm(t) = \mu_n \int_1^{\infty} \frac{\log(t)}{t^2} dm(t) = \mu_n,
$$

as required. $\square$

2.1. The Nazarov-Sodin-Volberg Theorem. Given any logarithmically convex sequence $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}}$ with $M_0 = 1$, we consider the class $C_\mathcal{M}([0,1])$ of smooth functions which satisfy $\|f^{(n)}\|_{L^\infty([0,1])} \leq M_n$ for every $n \geq 0$. A sequence $\mathcal{M}$ generates a quasi-analytic class if whenever $f \in C_\mathcal{M}([0,1])$ vanishes to infinite order at a point in $[0,1]$ ($f^{(k)}(x_0) = 0$ for every $k \geq 1$ for some $x_0 \in [0,1]$), then $f \equiv 0$ on $[0,1]$. The Denjoy-Carleman theorem (see e.g. [10]) ensures that $\mathcal{M}$ generates a quasi-analytic class if and only if

$$
\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty.
$$

With a slight abuse of notation, we call a logarithmically convex sequence $\mathcal{M}$ satisfying (2.2) quasi-analytic.

For $f \in C_\mathcal{M}([0,1])$, the Bang degree $n_f$ is defined by

$$
n_f = \sup \left\{ N : \sum_{\log \|f\|_{L^\infty([0,1])}^{-1} < n \leq N} \frac{M_{n-1}}{M_n} < e \right\}.
$$

A powerful theorem of Bang (see [1] or [17]) states that the Bang degree controls the number of zeroes of a function $f \in C_\mathcal{M}([0,1])$ counting multiplicities. It is therefore natural that it should depend on both the growth of the ratios of $M_{n-1}/M_n$ and a lower bound for $\|f\|_{L^\infty([0,1])}$. For our purposes we will want uniform bounds on the Bang degree of a function given the class $\mathcal{M}$. Therefore, we set, for $t \in (0,1]$,

$$
n_{\mathcal{M},t} = \sup \left\{ N : \sum_{-\log t < n \leq N} \frac{M_{n-1}}{M_n} < e \right\},
$$

so if $f \in C_\mathcal{M}([0,1])$ satisfies $\|f\|_{L^\infty([0,1])} \geq t$, then $n_f \leq n_{\mathcal{M},t}$. Following [17], we also define (compare with (1.7) in [17])

$$
\gamma_\mathcal{M}(n) = \sup_{1 \leq j \leq n} j \left[ \frac{M_{j+1}M_{j-1}}{M_j^2} - 1 \right], \text{ and } \Gamma_\mathcal{M}(n) = 4e^{4+4\gamma_\mathcal{M}(n)}.
$$
We are now in a position to state the Nazarov-Sodin-Volberg theorem, which builds upon the techniques developed by Bang [1].

**Theorem 2.5** (Theorem B from [17]). Suppose that $f \in C_{\mathcal{M}}([0, 1])$. Then for any interval $I \subset [0, 1]$ and measurable set $E \subset I$ with $m_1(E) > 0$, we have

$$\sup_I |f| \leq \left( \frac{\Gamma_{\mathcal{M}}(2n_f)|I|}{m(E)} \right)^{2n_f} \sup_E |f|.$$  

Again, the constant in this inequality must depend on the ratio of the value of $t = \|f\|_{L^\infty([0, 1])}$ to its apriori upper bound of $M_0 = 1$: the smaller the value of $t$, the more zeroes $f$ can have in the interval $[0, 1]$ while controlling the size of a fixed number of derivatives.

Since there has been interest in obtaining quantitative uniqueness bounds, see e.g. [9], we thought it worthwhile to present Theorem 2.5, where the constant is rather sharp. However, if the reader is not bothered by the particular form of the constant in Theorem 2.5, then the following qualitative result can be quickly derived from the Denjoy-Carleman theorem.

**Remark 2.6** (A quick qualitative bound). If $\gamma > 0$, $t > 0$, and $\mathcal{M}$ is a quasi-analytic sequence, then there is a finite constant $C = C(\gamma, t, \mathcal{M})$ such that whenever $f \in C_{\mathcal{M}}([0, 1])$ satisfies $\|f\|_{L^\infty([0, 1])} \geq t$ and $E \subset [0, 1]$ satisfies $m_1(E) \geq \gamma$, then

$$\|f\|_{L^\infty([0, 1])} \leq C(\gamma, t, \mathcal{M})\|f\|_{L^\infty(E)}.$$  

**Proof of Remark 2.6.** Suppose the result fails to hold, then for some $\gamma > 0$ and $t > 0$, there is a sequence $\{f_n\} \subset C_{\mathcal{M}}([0, 1])$ with $\|f_n\|_{L^\infty([0, 1])} \geq t$ and a set $E_n \subset [0, 1]$ with $m_1(E_n) \geq \gamma$ such that $\|f_n\|_{L^\infty(E)} \leq \frac{1}{n}\|f_n\|_{L^\infty([0, 1])} \leq \frac{1}{n}$. For any $k \geq 0$, the sequence $\{D^k f_n\} \subset C_{\mathcal{M}}([0, 1])$ is certainly equicontinuous, and so, with the aid of a diagonal argument and relabelling the sequence if necessary, we may assume that $f_n$ converges uniformly to a function $f \in C_{\mathcal{M}}([0, 1])$. But then $\|f\|_{L^\infty([0, 1])} \geq t$ (since $[0, 1]$ is compact), while $f \equiv 0$ on the set $E = \bigcap_n \bigcup_{m \geq m_n} E_m$ (if $x \in E$, then $x \in E_m$ for some subsequence $n_m \to \infty$, but then $|f(x)| = \lim_{m \to \infty} |f_{n_m}(x)| = 0$). Of course, $m_1(E) \geq \gamma$. However, a smooth function that vanishes on a set of positive measure has a zero of infinite order (for instance, at each Lebesgue point of the zero set), so $f \equiv 0$ on $[0, 1]$ since $\mathcal{M}$ generates a quasi-analytic class. This contradiction establishes (2.4). \hfill $\square$
Theorem 2.5 does not require the sequence $\mathcal{M}$ to be quasi-analytic, but we shall only use it in this case. We shall require an extension of Theorem 2.5 for quasi-analytic functions of several variables. To do this we shall appeal to an inductive argument of Fontes-Merz [5].

For $Q \subset \mathbb{R}^d$ a cube (whose sides are parallel to the coordinate axes), we say that $f : Q \to \mathbb{R}$ lies in $C_{\mathcal{M}}(Q)$ if for any multi-index $\alpha$ with order $|\alpha| := \alpha_1 + \cdots + \alpha_d = n$, it holds that $\|D^\alpha f\|_{L^\infty(Q)} \leq M_n$.

**Proposition 2.7.** For any $d \geq 1$, $t \in (0, 1]$, quasi-analytic class $\mathcal{M}$, and $s \in (0, 1]$, there is a finite constant $\Theta_{\mathcal{M}}(d, t, s)$ such that for any cube $Q \subset \mathbb{R}^d$ of sidelength 1 and whenever $f \in C_{\mathcal{M}}(Q)$ satisfies $\|f\|_{L^\infty(Q)} \geq t$, and $E \subset Q$ is a Borel measurable set with $m_d(E) \geq s$,

$$\|f\|_{L^\infty(Q)} \leq \Theta_{\mathcal{M}}(d, t, s) \|f\|_{L^\infty(E)}.$$ 

Moreover, we have the estimate

$$\Theta_{\mathcal{M}}(d, t, s) \leq \Theta_{\mathcal{M}}\left(1, t, \frac{s}{2}\right) \Theta_{\mathcal{M}}\left(d - 1, \frac{t}{\Theta_{\mathcal{M}}(1, t, \frac{s}{2})}, \frac{s}{2}\right).$$

Observe that Theorem 2.5 ensures that

$$\Theta_{\mathcal{M}}(1, t, s) \leq \left(\frac{\Gamma_{\mathcal{M}}(2n_{\mathcal{M}}, t)}{s}\right)^{2n_{\mathcal{M}, t}},$$

so one can calculate an effective bound on $\Theta_{\mathcal{M}}(d, \cdot, \cdot)$ for any dimension, albeit of a tower exponential form.

**Proof.** We follow the inductive scheme in [5]. Without loss of generality, assume $Q = [0, 1]^d$. The base case $d = 1$ is covered by the Nazarov-Sodin-Volberg theorem (or Remark 2.6). Suppose now that $d > 1$ and the proposition is proved for $d - 1$. Fix $f \in C_{\mathcal{M}}([0, 1]^{d - 1})$, $\|f\|_{L^\infty([0, 1]^{d - 1})} \geq t$ and $E \subset [0, 1]^d$ with $m_d(E) \geq s$. For $x \in \mathbb{R}^d$, put $x = (x', u)$ where $x' \in \mathbb{R}^{d - 1}$ and $u \in \mathbb{R}$. We set $E_u = \{x' \in \mathbb{R}^{d - 1} : (x', u) \in E\}$. Define the set

$$L = \{u \in [0, 1] : m_{d - 1}(E_u) \geq \frac{1}{2} m_d(E)\}.$$

Then

$$m_d(E) \leq \int_L m_{d - 1}(E_u) dm_1(u) + \int_{[0, 1] \setminus L} m_{d - 1}(E_u) dm_1(u).$$

We bound the first integral by $m_1(L)$ (since $E_u \subset [0, 1]^{d - 1}$), and the second integral by $\frac{1}{2} m_d(E)$ (for $u \in [0, 1] \setminus L$, $m_{d - 1}(E_u) < \frac{1}{2} m_d(E)$). Therefore, $m_1(L) \geq \frac{1}{2} m_d(E) \geq \frac{s}{2}$.

Suppose $(x', u) \in [0, 1]^d$ satisfies $|f(x', u)| = \sup_{x \in [0, 1]^d} |f(x)|$. Applying the $d = 1$ case to the function $f(x', \cdot) \in C_{\mathcal{M}}([0, 1])$ and the set
Let \( \varepsilon > 0 \) and fix \( u_0 \in L \) with \( |f(x', u_0)| + \varepsilon \geq \sup_{u \in L} |f(x', u)| \). Then by definition of \( L \), \( m_{d-1}(E_{u_0}) \geq m_d(E)/2 \geq s/2 \). Also, 
\[
\sup_{y' \in [0,1]^{d-1}} |f(y', u_0)| \geq |f(x', u_0)| \geq \frac{t}{\Theta_M(1, t, s/2)} - \varepsilon.
\]
Consequently, we may apply the inductive hypothesis that the proposition holds for \( d - 1 \) to the function \( f(\cdot, u_0) \) and the set \( E_{u_0} \) to obtain 
\[
\sup_{y' \in [0,1]^{d-1}} |f(y', u_0)| \leq \Theta_M \left( d - 1, \frac{t}{\Theta_M(1, t, s/2)} - \varepsilon, \frac{s}{2} \right) \sup_{y' \in E_{u_0}} |f(y', u_0)|.
\]
But if \( y' \in E_{u_0} \), then \( (y', u_0) \in E \) so \( \sup_{y' \in E_{u_0}} |f(y', u_0)| \leq \sup_{x \in E} |f(x)| \). Letting \( \varepsilon \to 0 \), we conclude that 
\[
\|f\|_{L^\infty([0,1]^d)} \leq \Theta_M(1, t, s/2) \Theta_M \left( d - 1, \frac{t}{\Theta_M(1, t, s/2)}, \frac{s}{2} \right) \sup_{x \in E} |f(x)|,
\]
as required. \( \square \)

We will require an \( L^2 \)-version of Proposition 2.7.

**Corollary 2.8.** Suppose that \( f \in \mathcal{C}_M([0,1]^d) \) satisfies \( \|f\|_{L^\infty([0,1]^d)} \geq t > 0 \). Then for any Borel measurable set \( E \subset [0,1]^d \) with positive measure, we have 
\[
\int_{[0,1]^d} |f|^2 dm_d \leq \frac{2\Theta_M(d, t, m_d(E)/2)^2}{m_d(E)} \int_E |f|^2 dm_d.
\]

**Proof.** Consider the set \( \tilde{E} = \{ x \in E : |f(x)|^2 \leq \frac{2}{m_d(E)} \int_E |f|^2 dm_d \} \).
Then \( m_d(\tilde{E}) \geq \frac{1}{2} m_d(E) \). Applying Proposition 2.7 with the set \( \tilde{E} \), it follows that 
\[
\sup_{[0,1]^d} |f| \leq \Theta_M(d, t, m_d(E)/2) \sup_{\tilde{E}} |f|.
\]
But \( \sup_{\tilde{E}} |f|^2 \leq \frac{2}{m_d(E)} \int_E |f|^2 dm_d \), and so 
\[
\int_{[0,1]^d} |f|^2 dm_d \leq \sup_{[0,1]^d} |f|^2 \leq \frac{2\Theta_M(d, t, m_d(E)/2)^2}{m_d(E)} \int_E |f|^2 dm_d,
\]
as required. \( \square \)
3. The sufficiency of (1.4) for the PLS property

Without loss of generality, we shall put $\ell = 1$ in the definition of relative density (for any $\ell > 0$, $W$ satisfies (1.4) if and only if $W(\ell \cdot)$ does). Suppose that
\[
\int_0^\infty \frac{\log W(t)}{1 + t^2} dm_1(t) = \infty.
\]
With $M_n = \max_{\xi \in \mathbb{R}} \frac{|\xi|^n}{W(|\xi|)}$ and $\mu_n = \frac{M_{n+1}}{M_n}$, we infer from Proposition 2.2 that $\sum \mu_n = \infty$, so $\mathcal{M} = \{M_n\}_{n \geq 0}$ is a quasi-analytic class with $M_0 = 1$.

A slightly modified quasi-analytic class will arise naturally in the proof, so we introduce it here. For $A > 1$, we define
\[
(3.1) \quad \mathcal{M}_A = \{\tilde{M}_n\}_{n \geq 0} \text{ with } \tilde{M}_0 = 1 \text{ and } \tilde{M}_n = A^n \frac{M_{n+d}}{M_d}.
\]
Observe that $\mathcal{M}_A$ is a log-convex sequence since $\mathcal{M}$ is log-convex.

Proposition 3.1. There exists $A = A(d) > 1$ such that for any $f \in L^2(\mathbb{R}^d)$ satisfying (1.2), $\gamma \in (0,1)$, and $(\gamma,1)$-relatively dense subset $E \subset \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} |f|^2 dm_d \leq 4 \gamma \Theta_{A,d} \left( d, \frac{1}{C_W A^{1+d} M_d} \right)^2 \gamma \int_E |f|^2 dm_d.
\]

Proof. Suppose $\|f\|_{L^2(\mathbb{R})} = 1$. Partition $\mathbb{R}^d$ into cubes of side-length 1. Fix $B > 2$. A cube $Q$ is said to be bad if there exists a multi-index $\alpha$ such that
\[
(3.2) \quad \int_Q |D^\alpha f|^2 dm_d > B^{2(|\alpha|+1)} M_{|\alpha|}^2 C_W \int_Q |f|^2 dm_d.
\]
If a cube isn’t bad, then it is called good. If $Q$ is a good cube, then we have good derivative control:
\[
(3.3) \quad \int_Q |D^\alpha f|^2 dm_d \leq B^{2(|\alpha|+1)} M_{|\alpha|}^2 C_W \int_Q |f|^2 dm_d \text{ for every } \alpha \in \mathbb{Z}_+^d.
\]
Notice that if $|\alpha| = n$, then by Plancherel’s identity, we have that $D^\alpha f \in L^2(\mathbb{R}^d)$, and moreover
\[
(3.4) \quad \int_{\mathbb{R}^d} |D^\alpha f|^2 dm_d = (2\pi)^{2n} \int_{\mathbb{R}^d} |\xi|^2 |\hat{f}(\xi)|^2 dm_d(\xi)
\leq (2\pi)^n \left[ \max_{\xi \in \mathbb{R}^d} \frac{|\xi|^n}{W(|\xi|)} \right]^2 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 W(|\xi|)^2 dm_d(\xi)
\leq (2\pi)^n M_n^2 C_W^2.
\]
Therefore, if \( B_n \) denotes the union of all cubes that are bad for derivatives of order \( n \) (i.e. the union of intervals for which (3.2) holds for some multi-index of order \( n \)), then
\[
\int_{B_n} |f|^2 dm \leq \frac{1}{B^{2(n+1)} M^2_n C^2_W} \sum_{\alpha : |\alpha| = n} \int_{\mathbb{R}^d} |D^\alpha f|^2 dm \leq \frac{(2\pi)^{2n} C(n)}{B^{2(n+1)}},
\]
where \( C(n) \) denotes the number of possible multi-indices of order \( n \). By induction one can readily see that \( C(n) \leq (n+1)^d \).

Consequently, if \( B \) denotes the union of all bad cubes, and \( B \) is large enough, then
\[
\int_B |f|^2 dm \leq \frac{1}{B^2} \sum_{n \geq 0} \frac{(2\pi)^{2n}(n+1)^d}{B^{2n}} \leq \frac{1}{2} = \frac{\|f\|^2_{L^2(\mathbb{R}^d)}}{2}.
\]
and so
\[(3.5) \quad \int_{\bigcup \{Q \text{ good}\}} |f|^2 dm \geq \frac{1}{2} \|f\|^2_{L^2(\mathbb{R}^d)}.\]

Now fix a good cube \( Q \) (which we recall has sidelength 1). Recall the elementary Sobolev inequality (see Chapter 1 of [14])
\[
\|g\|_{L^\infty(Q)} \leq C(d) \|g\|_{L^2(Q)} + C(d) \sum_{|\alpha| = d} \|\partial^\alpha g\|_{L^2(Q)}.
\]
From (3.3) we infer that for every \( n \geq 0 \) and \( |\alpha| = n \),
\[(3.6) \quad \|D^\alpha f\|_{L^\infty(Q)} \leq C(d) (B^{n+d+1} M_{n+d}) C_W \|f\|_{L^2(Q)} \]
\[\leq A^{n+d+1} M_{n+d} C_W \|f\|_{L^2(Q)},\]
for \( A = A(d) \).

Consider the function \( \tilde{f} = \frac{f}{A^{n+d+1} M_{n+d} C_W \|f\|_{L^2(Q)}} \). Then \( \tilde{f} \) belongs to the class \( C_{M_A}(Q) \) with the sequence \( M_A \) defined in (3.1). Also
\[
\|\tilde{f}\|_{L^\infty(Q)} \geq \frac{1}{C_W A^{1+d} M_d}.
\]
Therefore, applying Corollary 2.8 with the function \( \tilde{f} \) and the set \( E \cap Q \), which has measure at least \( \gamma \), results in
\[
\int_Q |\tilde{f}|^2 dm \leq \frac{2}{\gamma} \Theta_M \left( d, \frac{1}{C_W A^{1+d} M_d}, \frac{\gamma}{2} \right)^2 \int_{E \cap Q} |\tilde{f}|^2 dm.
\]
By homogeneity, this inequality also holds with \( f \) replacing \( \tilde{f} \).

Finally, summing over good cubes, we conclude from (3.5)
\[
\frac{1}{2} \int_{\mathbb{R}^d} |f|^2 dm \leq \frac{2}{\gamma} \Theta_M \left( d, \frac{1}{C_W A^{1+d} M_d}, \frac{\gamma}{2} \right)^2 \int_E |f|^2 dm.
\]
Proposition 3.1 is proved.

4. The necessity of (1.4) for the PLS property

We only consider \( d = 1 \). We shall assume

\[
\int_0^\infty \frac{\log W(t)}{1 + t^2} dt < \infty,
\]

and therefore (Proposition 2.2), \( \sum_n \mu_n < \infty \).

We shall sketch the Paley-Wiener construction (also the construction used in many presentations of the Denjoy-Carleman theorem, see e.g. [4, 10]) to show that there exist functions \( f \) supported on arbitrarily small intervals with \( \int_{\mathbb{R}} |\hat{f}(\xi)|^2 W(|\xi|)^2 d\xi < \infty \). Therefore \( W \) fails to have the PLS property.

Fix \( \varepsilon > 0 \). Choose \( n_0 \geq 10 \) such that \( \sum_{n \geq n_0} \mu_n < \varepsilon \). We set

\[
\hat{f}(\xi) = M_{n_0 - 1} \left( \frac{\sin((\varepsilon/n_0)\xi)}{(\varepsilon/n_0)\xi} \right)^{2n_0} \prod_{n \geq n_0} \frac{\sin(\mu_n\xi)}{\mu_n\xi}.
\]

As in (for example) Koosis, [10], p. 90-91, we infer that

- \( \hat{f} \) is the Fourier transform of a function that vanishes outside of an interval of width \( C\varepsilon \), for some absolute constant \( C > 0 \), and
- for \( n \geq 0 \), and \( |\xi| > 1 \),

\[
|\xi|^n |\hat{f}(\xi)| \leq \max(M_n, M_{n_0}) \left( \frac{n_0}{\varepsilon} \right)^{n_0} \left( \frac{|\sin((\varepsilon/n_0)\xi)|}{(\varepsilon/n_0)|\xi|} \right)^{n_0 + 1}
\]

(4.1)

\[
\leq C(n_0, \varepsilon) \frac{\max(M_n, M_{n_0})}{|\xi|^{n_0 + 1}}.
\]

From Lemma 2.1 we therefore infer that for \( |\xi| > 1 \) there exists \( n \) such that

\[
\log W(|\xi|) \leq (n + 1) \log |\xi| - \log M_n.
\]

But when combined with (4.1) this yields that

\[
W(|\xi|) \leq \frac{C(n_0, M_{n_0}, \varepsilon)}{|\hat{f}(\xi)||\xi|^{n_0}}.
\]

Therefore

\[
\int_{\mathbb{R}} |\hat{f}(\xi)|^2 W(|\xi|)^2 d\mu(\xi) < \infty.
\]
5. Tools for the proof of Theorem 1.6

Henceforth assume that $W$ is as in the statement of Theorem 1.6. The following form of the Beurling-Malliavin theorem [2] is obtained in [3], see Lemma 2.11.

**Theorem 5.1.** For every $\Lambda, \tilde{\gamma} > 0$, there exists $\beta = \beta_{\Lambda, \tilde{\gamma}} > 0$ such that

$$\int_0^\infty \frac{\log F(t)}{1 + t^2} \, dm(t) \leq \Lambda,$$

and $\|\log F\|_\infty \leq \Lambda$, then there exists a function $\psi$ satisfying $\text{supp}(\hat{\psi}) \subset [-\tilde{\gamma}, \tilde{\gamma}]$, $|\hat{\psi}| \leq F^{-\beta}$, and $\|\hat{\psi}\|_{L^2([-1,1])} \geq \beta$.

In what follows, we do not need the full strength of this theorem, and the reader can instead use Theorem 3.2 of [9] (see Theorem B.4 of [6] for a particularly clean formulation), since the logarithm of the weight $\tilde{W}$ constructed below in Section 5.1 has a Lipschitz continuous Hilbert transform. Following [8], the main difficulty in proving Theorem 5.1 in its full generality is constructing a majorant of $\log F$ with well behaved Hilbert transform. The aforementioned results in [6, 9] also have the advantage of providing more information about the form of $\beta$ in terms of $\tilde{\gamma}$ and $\Lambda$.

**Lemma 5.2.** Suppose that $\text{supp}(\hat{f}) \subset \bigcup_k I_k$, where $I_k = [t_k - 1, t_k + 1]$. Fix $\beta > 0$, and a sequence $\varphi_k \in L^2(\mathbb{R})$ with $\|\hat{\varphi}_k\|_{L^2([-1,1])} \geq \beta > 0$. Then

$$\|f\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{\beta} \int_{-2}^{2} \sum_k \|\hat{f}(\cdot - \tau - t_k)\hat{\varphi}_k\|_{L^2(\mathbb{R})}^2 \, d\tau.$$

**Proof.** Fix $k$ and observe that, with a change of variable,

$$\|\hat{f}\|_{L^2(I_k)}^2 = \|\hat{f}(\cdot - t_k)\|_{L^2([-1,1])} \leq \frac{1}{\beta} \int_{-1}^{1} \int_{-1}^{1} |\hat{f}(\xi - t_k)\hat{\varphi}_k(\xi)|^2 d\xi d\zeta \\
\leq \frac{1}{\beta} \int_{-2}^{2} \int_{-1}^{1} \int_{-1}^{1} |\hat{f}(\xi - t_k)\hat{\varphi}_k(\xi)|^2 d\zeta d\xi d\tau \\
\leq \frac{1}{\beta} \int_{-2}^{2} \|\hat{f}(\cdot - \tau - t_k)\hat{\varphi}_k\|_{L^2(\mathbb{R})}^2 \, d\tau.$$

The lemma follows by summation over $k$ (along with Plancherel’s identity).

5.1. An application of the Beurling-Malliavin theorem. In this section we shall apply the Beurling-Malliavin theorem (Theorem 5.1) to construct suitable functions $\varphi_\ell$. See [11] for (much) more information.
on the Beurling-Malliavin theorem and the instances when it can be applied.

Suppose that \( \{ \tilde{J}_n \} \) is a \( W \)-short cover of a set \( Q \) with \( \| \{ \tilde{J}_n \} \|_W \leq \Lambda \). We begin by regularizing the cover.

**Lemma 5.3.** Suppose that \( \{ \tilde{J}_n \} \) is a \( W \)-short cover of a set \( Q \). Then there is a \( W \)-short cover \( \{ J_n \} \) of \( Q \) satisfying that for every \( n \), \( \{ \frac{1}{2} J : J \in J_n \} \) are pairwise disjoint, and \( \| \{ J_n \} \|_W \leq 7 \| \{ \tilde{J}_n \} \|_W \).

**Proof.** Fix \( n \), and pick a maximal collection of \( \frac{\Omega_n}{2} \) separated points \( \{ t_m \} \) in \( Q \cap \left[ [ -e^{n+1} , -e^n ] \cup [ e^n , e^{n+1} ] \right] \). Consider the intervals \( J_m \) centred at \( t_m \) of sidelength \( \Omega_n \). Since \( \{ t_m \} \) are \( \frac{\Omega_n}{2} \) separated, \( \frac{1}{2} J_m \) are disjoint. On the other hand, by maximality, \( Q \cap \left[ [ -e^{n+1} , -e^n ] \cup [ e^n , e^{n+1} ] \right] \subset \bigcup_m J_m \). But at most 7 intervals \( J_m \) can intersect any interval \( J \in J_n \). Therefore, if \( J_n = \{ J_m \} \), then \( \text{card}(J_n) \leq 7 \cdot \text{card}(\tilde{J}_n) \).

Going back to our \( W \)-short cover \( \{ \tilde{J}_n \} \), we set \( \{ J_n \} \) as in the lemma, and so \( \| \{ J_n \} \|_W \leq 7 \Lambda \). Put \( \mathcal{J} = \bigcup_n J_n \).

**Claim 5.4.** For each \( J \in J_n \),

\[
3J \subset [ -e^{n+2} , -e^{n-1} ] \cap [ e^{n-1} , e^{n+2} ].
\]

**Proof.** Since \( \log W(e^n) \leq \frac{2^n}{4} \), the claim follows from the fact \( J \in J_n \) intersects \( [ e^n , e^{n+1} ] \cap [ -e^{n+1} , -e^n ] \).

**Claim 5.5.** There is a constant \( C > 0 \) depending on \( C_{\text{doub}} \) such that any interval \( 3J, J \in \mathcal{J} \), can intersect at most \( C \) of the intervals \( \{ 3J \}_{J \in \mathcal{J}} \).

**Proof.** Fix \( J \in \mathcal{J} \), so \( J \in J_n \) for some \( n \). From Claim 5.4, we infer that if \( I \in \mathcal{J} \) satisfies \( 3I \cap 3J \neq \emptyset \), then \( I \in J_m \) with \( |n - m| \leq 4 \). Fix such an \( m \) and consider all \( I \in J_m \) with \( 3I \cap 3J \neq \emptyset \). Since \( W \) satisfies the doubling condition, \( C_{\text{doub}}^{-4} \leq \left| \log W(e^n) / \log W(e^m) \right| = \left| \Omega_m / \Omega_n \right| \leq C_{\text{doub}}^4 \). Consequently, any such interval \( I \) is contained in the \( 15C_{\text{doub}}^4 \) dilation of \( J \), and has length at least \( C_{\text{doub}}^{-4} \ell(J) \). Finally, since the collection of intervals \( \{ \frac{1}{3} I : I \in J_m \} \) are pairwise disjoint, there can be at most \( \frac{15C_{\text{doub}}^4}{4C_{\text{doub}}^4} = 30C_{\text{doub}}^3 \) such intervals \( I \in J_m \). Since there are at most nine choices of \( m \), the lemma is proved.

Now, observe that since \( \log W \) is doubling, we obtain from Claim 5.4 that

\[
\log W(t) \leq \Omega_{n+2} \leq C_{\text{doub}}^2 \log W(e^n) \leq C_{\text{doub}}^3 \log W(t)
\]

for any \( t \in 3J \in J_n \).
\[ \int_{\bigcup_{3J : J \in \mathcal{J}}} \frac{\log W(t)}{1 + t^2} \, dm(t) \leq C \sum_n \Omega_n^2 \frac{\text{card}(\mathcal{J}_n)}{e^{2n}} \leq C \| \{ \mathcal{J}_n \} \|_W \leq C \Lambda. \]

For every \( J \in \mathcal{J}_n \), choose functions \( \eta_J \in \text{Lip}_0(3J) \) with \( \| \nabla \eta_J \|_\infty \leq 2/\Omega_n \) and \( \eta_J \equiv 1 \) on \( 3J \). We consider the weight \( \tilde{W} \),

\[ \log \tilde{W}(t) = \sqrt{\max(1, |t|)} + \sum_n \sum_{J \in \mathcal{J}_n} \Omega_n + 2 \eta_J. \]

(The precise form of the first term on the right hand side is not so important – any logarithmically Poisson summable function growing faster than the logarithm will suffice.) Observe from (5.1) that \( \log \tilde{W} \geq \log W \) on \( \bigcup_{J \in \mathcal{J}} 2J \) and

\[ \int_{\mathbb{R}} \frac{\log \tilde{W}(t)}{1 + t^2} \, dm(t) \leq C + C \int_{\bigcup_{3J : J \in \mathcal{J}}} \frac{\log W(t)}{1 + t^2} \, dm(t) \leq C + C \Lambda. \]

Since the intervals \( \{ 3J \}_{J \in \mathcal{J}} \) have bounded overlap,

\[ \| (\log \tilde{W})' \|_\infty \leq C + C \max_J \| (\eta_J \cdot \log W)' \|_\infty. \]

But, \( \| \eta_J (\log W)' \| \leq 2 \Lambda \), and \( \| \eta_J (\log W) \| \leq C_\text{doub}^2 \) (see (5.1)), so

\[ \| (\log \tilde{W})' \|_\infty \leq C(\Lambda_W + 1). \]

Therefore, for any choice \( \tilde{\gamma} > 0 \), we may apply Theorem 5.1 to get \( \beta > 0 \) depending on \( \tilde{\gamma} \), \( \Lambda_W \), \( C_\text{doub} \), and \( \Lambda \), and a function \( \varphi \) such that

1. \( |\hat{\varphi}| \leq W^{-\beta} \) on \( \bigcup_{J \in \mathcal{J}} 2J \)
2. \( |\hat{\varphi}| \leq e^{-\beta} \sqrt{\max(1, |t|)} \) on \( \mathbb{R} \)
3. \( \text{supp}(\varphi) \subset [-\tilde{\gamma}, \tilde{\gamma}] \), and
4. \( \| \hat{\varphi} \|_{L^2([-1,1])} \geq \beta. \)

Finally, we observe that since the weight \( W \) is increasing, there exists (a smallest) \( n_0 \) depending on \( W \), such that \( \ell(J) = \Omega_n \geq 4 \) whenever \( J \in \mathcal{J}_n \), \( n \geq n_0 \). Setting \( Q_2 \) to be the closed 2-neighbourhood of \( Q \), we therefore infer that

\[ \bigcup_{n \geq n_0} \bigcup_{J \in \mathcal{J}_n} 2J \supset Q_2 \cap [(-\infty, -e^{n_0}] \cup [e^{n_0}, \infty) \]

(recall that \( \bigcup_{n \geq n_0} \bigcup_{J \in \mathcal{J}_n} J \supset Q \cap [(-\infty, -e^{n_0}] \cap [e^{n_0}, \infty)] \)). But if \( t < e^{n_0} \), we have that \( W(t) \leq e^4 \). Thus, from (1) and (2) we get that

\[ |\hat{\varphi}| \leq CW^{-\beta} \) on \( Q_2, \]

for \( C = e^{4\beta} \leq e^4 \), for instance.
6. The proof of Theorem 1.6

Set $Q_2$ to be the closed 2-neighbourhood of $Q$.

Fix $\{t_\ell\}_\ell$ to be a maximal one-separated subset\(^7\) of $Q_2$, so $Q_2 \subset \bigcup \ell I_\ell$, where $I_\ell = [t_\ell - 1, t_\ell + 1]$.

Suppose that $Q$ and $f$ satisfy the hypotheses of Theorem 1.6 (so $\|Q\|_W \leq \Lambda$ and $\text{supp}(\hat{f}) \subset Q$). For every $\ell$, we can apply the construction of Section 5.1 with $\tilde{\gamma} = \gamma / 3$ to obtain a function $\varphi_\ell$ satisfying

1. $|\hat{\varphi}_\ell| \leq CW - \beta$ on $Q_2 - t_\ell$
2. $|\hat{\varphi}_\ell(t)| \leq e^{-\beta \sqrt{\max(1,|t|)}}$ on $\mathbb{R}$
3. $\text{supp}(\varphi_\ell) \subset [-\tilde{\gamma}, \tilde{\gamma}]$, and
4. $\|\hat{\varphi}_\ell\|_{L^2([-1,1])} \geq \beta$.

for some $\beta$ depending on $\Lambda$, $C_{\text{doub}}$, and $\Lambda_W$.

We will need the following simple auxiliary lemma.

**Lemma 6.1.** There is a constant $C = C(\beta) > 0$ such that for any $g \in L^2(\mathbb{R})$,

$$\sum_\ell \int_{-2}^2 \| \tilde{g}(\cdot - \tau - t_\ell) e^{-\frac{\beta}{2} \sqrt{\max(1,|\tau|)}} \|^2_{L^2(\mathbb{R})} d\tau \leq C \|g\|^2_{L^2(\mathbb{R})}$$

**Proof.** The left hand side of the inequality is bounded by

$$\int_{-2}^2 \sum_\ell \| \tilde{g} e^{-\frac{\beta}{2} \sqrt{|\cdot + \tau + t_\ell|}} \|^2_{L^2(\mathbb{R})} d\tau.$$ 

But, since the points $\{t_\ell\}_\ell$ are one-separated,

$$\sup_{\xi, \tau \in \mathbb{R}} \sum_\ell e^{-\beta \sqrt{|(\xi + \tau + t_\ell)|}} \leq C(\beta),$$

and the lemma follows. \[ \square \]

Set $\alpha = \frac{\beta}{2}$. For any $\tau \in [-2, 2]$, consider function

$$f_{\tau, \ell} = \mathcal{F}^{-1}(\hat{f}(\cdot - \tau - t_\ell) \hat{\varphi}_\ell) = (f \cdot e_{\tau + t_\ell}) * \varphi_\ell,$$

where $e_\ell(x) = e^{2\pi i \ell x}$. The function $f_{\tau, \ell}$ has its Fourier transform supported in the set $Q_2 - t_\ell$ and so satisfies that

$$|\hat{f}_{\tau, \ell}| \leq C |\hat{f}(\cdot - \tau - t_\ell)| \sqrt{|\hat{\varphi}_\ell| W^{-\alpha}} \text{ on } \mathbb{R}.$$ 

Consequently,

$$\|f_{\tau, \ell} W^\alpha\|_{L^2(\mathbb{R})} \leq C \|\hat{f}(\cdot - \tau - t_\ell) \sqrt{|\hat{\varphi}_\ell|}\|_{L^2(\mathbb{R})}$$

\(^7\)A maximal set satisfying $|t_\ell - t_{\ell'}| \geq 1$ if $\ell \neq \ell'$. 

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This passage discusses the proof of a theorem concerning Fourier transforms and the decay of functions. It introduces the necessary set $Q_2$ and describes the choice of a maximal one-separated subset $\{t_\ell\}_\ell$. The proof involves the construction of a function $\varphi_\ell$ satisfying certain decay conditions. An auxiliary lemma is also presented, which provides a key inequality used in the proof. The proof concludes with a final inequality that relates the Fourier transform of the function $f_{\tau, \ell}$ to the original function $\hat{f}$ and the decay of $\varphi_\ell$. The proof relies on properties of the Fourier transform and the decay of functions, ensuring that the conclusions follow logically from the hypotheses.
and so by combining property (2) of $\varphi_{\ell}$ and Lemma 6.1 we infer that

\begin{equation}
\int_{-2}^{2} \sum_{\ell} \left\| \hat{f}_{\tau,\ell} W^{\alpha} \right\|_{L^2(\mathbb{R})}^2 d\tau \leq C(\beta) \| f \|_{L^2(\mathbb{R})}^2.
\end{equation}

We apply a localization technique: Fix $A > 1$. We call a pair $(\tau, \ell)$ bad if

\[ \left\| \hat{f}_{\tau,\ell} W^{\alpha} \right\|_{L^2(\mathbb{R})}^2 > A \left\| \hat{f}_{\tau,\ell} \right\|_{L^2(\mathbb{R})}^2. \]

Otherwise $(\tau, \ell)$ is called good. If $(\tau, \ell)$ is good, then $f_{\tau,\ell}$ satisfies the condition to apply the PLS property with $C_{W} = A$. Notice first that,

\[ \int_{-2}^{2} \sum_{\ell \in \{\ell : (\tau, \ell) \text{ bad}\}} \left\| \hat{f}_{\tau,\ell} \right\|_{L^2(\mathbb{R})}^2 d\tau \leq \frac{1}{A} \int_{-2}^{2} \sum_{\ell \in \{\ell : (\tau, \ell) \text{ is good}\}} \left\| \hat{f}_{\tau,\ell} W^{\alpha} \right\|_{L^2(\mathbb{R})}^2 d\tau \]

\begin{equation}
\leq \frac{C(\beta)}{A} \| f \|_{L^2(\mathbb{R})}^2.
\end{equation}

Employing Lemma 5.2, we obtain

\begin{equation}
\left( \beta - \frac{C(\beta)}{A} \right) \| f \|_{L^2(\mathbb{R})}^2 \leq \int_{-2}^{2} \sum_{\ell \in \{\ell : (\tau, \ell) \text{ is good}\}} \left\| \hat{f}_{\tau,\ell} \right\|_{L^2(\mathbb{R})}^2 d\tau,
\end{equation}

and the left hand side of this inequality can be made at least $\frac{\beta}{2} \| f \|_{L^2(\mathbb{R})}^2$ by choosing $A = 2C(\beta)/\beta$.

We are now in a position to use the assumption of the PLS property for $W^{\alpha}$. Pick a $\gamma$-relatively dense collection of intervals $E = \bigcup_{n \in \mathbb{Z}} J_n$. Consider the collection $\tilde{E} = \bigcup_{n \in \mathbb{Z}} \frac{1}{2} J_n$ (so $\text{dist} (\tilde{E}, \mathbb{R} \setminus E) \geq \gamma/3$). Then $\tilde{E}$ is a $(\gamma/3, 3)$-relatively dense set. Since $W^{\alpha}$ has the PLS property, there is a constant $C = C(W, A, \gamma, \alpha)$ such that for every good pair $(\gamma, \ell)$ we have

\begin{equation}
\| f_{\tau,\ell} \|_{L^2(\mathbb{R})} \leq C \| f_{\tau,\ell} \|_{L^2(\tilde{E})}.
\end{equation}

Next, since $\text{supp}(\varphi_{\ell}) \subset [-\gamma, \gamma] = [-\gamma/3, \gamma/3]$ we infer that on $\tilde{E}$, \((f \cdot e_{t_\ell+\tau}) \ast \varphi_{\ell} = (f \chi_E e_{t_\ell+\tau}) \ast \varphi_{\ell}\), and hence by Plancherel’s identity

\[ \| f_{\tau,\ell} \|_{L^2(\tilde{E})} = \| (f \chi_E \cdot e_{t_\ell+\tau}) \ast \varphi_{\ell} \|_{L^2(\tilde{E})} \leq \| (f \chi_E \cdot e_{t_\ell+\tau}) \ast \varphi_{\ell} \|_{L^2(\mathbb{R})}. \]

Writing $\| (f \chi_E \cdot e_{t_\ell+\tau}) \ast \varphi_{\ell} \|_{L^2(\mathbb{R})} = \| \hat{f}_{\tau,\ell} \|_{L^2(\mathbb{R})}$, it follows by 6.3 that

\[ \int_{-2}^{2} \sum_{\ell \in \{\ell : (\tau, \ell) \text{ is good}\}} \left\| \hat{f}_{\tau,\ell} \right\|_{L^2(\mathbb{R})}^2 d\tau \leq C \int_{-2}^{2} \sum_{\ell} \left\| \hat{f}_{\tau,\ell} \right\|^2_{L^2(\mathbb{R})} d\tau \]

\[ \leq C \| \hat{f}_{\tau,\ell} \|_{L^2(\mathbb{R})}^2, \]

Lemma 6.1.
for some constant $C = C(W, A, \gamma, \alpha)$. Finally, bringing the estimates together yields
\[
\frac{2}{\pi} \| f \|^2_{L^2(R)} \leq C \| \hat{f} \chi_E \|^2_{L^2(R)} = C \| f \chi_E \|^2_{L^2(R)} \text{, i.e., } \| f \|^2_{L^2(R)} \leq C'' \| f \|^2_{L^2(E)} \text{ with a constant } C'' = C''(W, \gamma, \Lambda, C_{doub}, \Lambda_W), \text{ as required.}
\]

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