Well-posedness for the Navier-Stokes equations with data in homogeneous Sobolev-Lorentz spaces

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Abstract: In this paper, we study local well-posedness for the Navier-Stokes equations (NSE) with the arbitrary initial value in homogeneous Sobolev-Lorentz spaces \( \dot{H}^{s}_{q,r}(\mathbb{R}^d) := (-\Delta)^{-s/2}L^{q,r} \) for \( d \geq 2, q > 1, s \geq 0, 1 \leq r \leq \infty, \) and \( \frac{d}{q} - 1 \leq s < \frac{d}{q} \), this result improves the known results for \( q > d, r = q, s = 0 \) (see [4, 7]) and for \( q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2} \) (see [4, 9]). In the case of critical indexes \( (s = \frac{d}{q} - 1) \), we prove global well-posedness for NSE provided the norm of the initial value is small enough. The result that is a generalization of the result in [4] for \( q = r = d, s = 0 \).

§1. Introduction

We consider the Navier-Stokes equations in \( \mathbb{R}^d \):

\[
\begin{align*}
\partial_t u &= \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0,
\end{align*}
\]

which is a condensed writing for

\[
\begin{align*}
1 \leq k \leq d, \quad \partial_t u_k &= \Delta u_k - \sum_{l=1}^{d} \partial_l (u_l u_k) - \partial_k p, \\
\sum_{l=1}^{d} \partial_l u_l &= 0, \\
1 \leq k \leq d, \quad u_k(0, x) &= u_{0k}.
\end{align*}
\]

The unknown quantities are the velocity \( u(t, x) = (u_1(t, x), \ldots, u_d(t, x)) \) of the fluid element at time \( t \) and position \( x \) and the pressure \( p(t, x) \).

In the 1960s, mild solutions were first constructed by Kato and Fujita ([18], [19]) that are continuous in time and take values in the Sobolev spaces

\footnotesize
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$H^s(\mathbb{R}^d), (s \geq \frac{d}{2} - 1)$, say $u \in C([0, T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2} - 1)$ was given by Chemin [9]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2} - 1)$, see [4]. Results on the existence of mild solutions with value in $L^q(\mathbb{R}^d), (q > d)$ were established in the papers of Fabes, Jones and Rivi`ere [11] and of Giga [14]. Concerning the initial data in the space $L^\infty$, the existence of a mild solution was obtained by Cannone and Meyer in (4, 7). In 1994, Kato and Ponce [23] showed that the NSE are well-posed when the initial data belong to the homogeneous Sobolev spaces $\dot{H}^{\frac{d}{q} - 1}(\mathbb{R}^d), (d \leq q < \infty)$. Recently, the authors of this article have considered NSE in the mixed-norm Sobolev-Lorentz spaces, see [17]. In this paper, for $d \geq 2, q > 1, s \geq 0, 1 \leq r \leq \infty$, and $\frac{d}{q} - 1 \leq s < \frac{d}{q}$, we investigate mild solutions to NSE in the spaces $L^\infty([0, T]; \dot{H}^s_{L^r,r}(\mathbb{R}^d))$ when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}^s_{L^r,r}(\mathbb{R}^d)$, which are more general than the spaces $\dot{H}^s_q(\mathbb{R}^d), (\dot{H}^s_q(\mathbb{R}^d) = \dot{H}^s_{L^q,q}(\mathbb{R}^d))$. We obtain the existence of mild solutions with arbitrary initial value when $T$ is small enough, and existence of mild solutions for any $T > 0$ when the norm of the initial value in the Besov spaces $B^{s-d(\frac{1}{q} - \frac{1}{r}),\infty}_q(\mathbb{R}^d), (\frac{1}{2}(\frac{1}{q} + \frac{s}{d}) < \frac{1}{q} < \min\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\})$ is small enough.

In the particular case $(q > d, r = q, s = 0)$, we get the result which is more general than that of Cannone and Meyer (4, 7). Here we obtained a statement that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.

In the particular case $(q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2})$, we get the result which is more general than those of Chemin in [9] and Cannone in [4]. Here we obtained a statement that is stronger than those of Chemin in [9] and Cannone in [4] but under a much weaker condition on the initial data.

In the case of critical indexes $(1 < q \leq d, r \geq 1, s = \frac{d}{q} - 1)$, we get a result that is a generalization of a result of Cannone [5]. In particular, when $q = r = d, s = 0$, we get back the Cannone theorem (Theorem 1.1 in [5]).

The paper is organized as follows. In Section 2 we prove some inequalities for pointwise products in the Sobolev spaces and some auxiliary lemmas. In Section 3 we present the main results of the paper. In the sequence, for a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as $E$.

§2. Some auxiliary results

In this section, we recall the following results and notations.

Definition 1. (Lorentz spaces). (See [11].)
For $1 \leq p, r \leq \infty$, the Lorentz space $L^{p,r}(\mathbb{R}^d)$ is defined as follows: A measurable function $f \in L^{p,r}(\mathbb{R}^d)$ if and only if
\[
\|f\|_{L^{p,r}(\mathbb{R}^d)} := \left( \int_0^\infty (t^{\frac{1}{p}} f^*(t))^{\frac{r}{p}} \, dt \right)^{\frac{1}{r}} < \infty \text{ when } 1 \leq r < \infty,
\]
\[
\|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty \text{ when } r = \infty,
\]
where $f^*(t) = \inf \{\tau : M^d(\{x : |f(x)| > \tau\}) \leq t\}$, with $M^d$ being the Lebesgue measure in $\mathbb{R}^d$.

Before proceeding to the definition of Sobolev-Lorentz spaces, let us introduce several necessary notations. For real number $s$, the operator $\dot{\Lambda}^s$ is defined through Fourier translation by
\[
(\dot{\Lambda}^s f)^\wedge(\xi) = |\xi|^s \hat{f}(\xi).
\]
For $0 < s < d$, the operator $\dot{\Lambda}^s$ can be viewed as the inverse of the Riesz potential $I^s$ up to a positive constant
\[
I^s(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-s}} \, dy \text{ for } x \in \mathbb{R}^d.
\]
For $q > 1, r \geq 1$, and $0 \leq s < \frac{d}{q}$, the operator $I^s$ is continuous from $L^{q,r}$ to $L^{\tilde{q},\tilde{r}}$, where $\frac{1}{q} = -\frac{s}{d} - \frac{1}{\tilde{q}}$, see (26, Theorem 2.4 iii), p. 20).

**Definition 2.** (Sobolev-Lorentz spaces). (See [12].)
For $q > 1, r \geq 1$, and $0 \leq s < \frac{d}{q}$, the Sobolev-Lorentz space $\dot{H}^s_{L^{q,r}}(\mathbb{R}^d)$ is defined as the space $I^s(L^{q,r}(\mathbb{R}^d))$, equipped with the norm
\[
\|f\|_{\dot{H}^s_{L^{q,r}}} := \|\dot{\Lambda}^s f\|_{L^{q,r}}.
\]

**Lemma 1.** Let $q > 1, 1 \leq r \leq \tilde{r} \leq \infty$, and $0 \leq s < \frac{d}{q}$. Then we have the following imbedding maps
(a) \[
\dot{H}^s_{L^q} \hookrightarrow \dot{H}^s_{L^{q,r}} \hookrightarrow \dot{H}^s_{L^r} \hookrightarrow \dot{H}^s_{L^\infty}.
\]
(b) $\dot{H}^s_q = \dot{H}^s_{L^q}$ (equality of the norm).

**Proof.** It is easily deduced from the properties of the standard Lorentz spaces.

In the following lemmas, we estimate the pointwise product of two functions in $\dot{H}^s_q(\mathbb{R}^d), (d \geq 2)$ which is a generalization of the Holder inequality. In the case when $s = 0$ we get back the usual Holder inequality. Pointwise multiplication results for Sobolev spaces are also obtained in literature, see for example [10], [26], [22] and the references therein.
Lemma 2. Assume that

\[\frac{1}{p}, \frac{1}{q} < d, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{d}.\]

Then the following inequality holds

\[\|uv\|_{\dot{H}^{\frac{1}{r}}_1} \lesssim \|u\|_{\dot{H}^{\frac{1}{p}}_1} \|v\|_{\dot{H}^{\frac{1}{q}}_1}, \forall u \in \dot{H}^{\frac{1}{p}}_1, v \in \dot{H}^{\frac{1}{q}}_1,
\]

where \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{d}\).

**Proof.** By applying the Leibniz formula for the derivatives of a product of two functions, we have

\[\|uv\|_{\dot{H}^{\frac{1}{r}}_1} \simeq \sum_{|\alpha|=1} \|\partial^{\alpha}(uv)\|_{L^r} \leq \sum_{|\alpha|=1} \|\partial^{\alpha} u\|_{L^p} \|v\|_{L^q} + \sum_{|\alpha|=1} \|u\|_{L^p} \|\partial^{\alpha} v\|_{L^q}.
\]

By applying the Hölder and Sobolev inequalities we obtain

\[\sum_{|\alpha|=1} \|\partial^{\alpha} u\|_{L^r} \leq \sum_{|\alpha|=1} \|\partial^{\alpha} u\|_{L^p} \|v\|_{L^q} \lesssim \|u\|_{\dot{H}^{\frac{1}{p}}_1} \|v\|_{\dot{H}^{\frac{1}{q}}_1},
\]

where

\[\frac{1}{q_1} = \frac{1}{q} - \frac{1}{d}.
\]

Similar to the above reasoning, we have

\[\sum_{|\alpha|=1} \|u\|_{L^p} \|\partial^{\alpha} v\|_{L^q} \lesssim \|u\|_{\dot{H}^{\frac{1}{p}}_1} \|v\|_{\dot{H}^{\frac{1}{q}}_1}.
\]

This gives the desired result

\[\|uv\|_{\dot{H}^{\frac{1}{r}}_1} \lesssim \|u\|_{\dot{H}^{\frac{1}{p}}_1} \|v\|_{\dot{H}^{\frac{1}{q}}_1}.
\]

\[\Box\]

Lemma 3. Assume that

\[0 \leq s \leq 1, \frac{1}{p} > \frac{s}{d}, \frac{1}{q} > \frac{s}{d}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}.\]  \(1\)

Then the following inequality holds

\[\|uv\|_{\dot{H}^{\frac{1}{r}}_1} \lesssim \|u\|_{\dot{H}^{\frac{1}{p}}_1} \|v\|_{\dot{H}^{\frac{1}{q}}_1}, \forall u \in \dot{H}^{\frac{1}{p}}_1, v \in \dot{H}^{\frac{1}{q}}_1,
\]

where \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}.
\]
Proof. It is not difficult to show that if $p, q,$ and $s$ satisfy (1) then there exists numbers $p_1, p_2, q_1, q_2 \in (1, +\infty)$ (may be many of them) such that

$$\frac{1}{p} = \frac{1 - s}{p_1} + \frac{s}{p_2}, \quad \frac{1}{q} = \frac{1 - s}{q_1} + \frac{s}{q_2}, \quad \frac{1}{p} + \frac{1}{q} < 1,$$

$$p_2 < d, q_2 < d, \quad \text{and} \quad \frac{1}{p_2} + \frac{1}{q_2} < 1 + \frac{1}{d}.$$

Setting

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \quad \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{d},$$

we have

$$\frac{1}{r} = \frac{1 - s}{r_1} + \frac{s}{r_2}.$$ 

Therefore, applying Theorem 6.4.5 (page 152) of [1] (see also [25] for $\dot{H}_p^s$), we get

$$\dot{H}_p^s = [L^{p_1}, \dot{H}_q^1], \quad \dot{H}_q^s = [L^{q_1}, \dot{H}_p^1], \quad \dot{H}_r^s = [L^{r_1}, \dot{H}_r^1].$$

Applying the Holder inequality and Lemma 2 in order to obtain

$$\|uv\|_{L^{r_1}} \lesssim \|u\|_{L^{p_1}} \|v\|_{L^{q_1}}, \quad \forall \ u \in L^{p_1}, \ v \in L^{q_1},$$

$$\|uv\|_{\dot{H}_p^{r_2}} \lesssim \|u\|_{\dot{H}_p^{1}} \|v\|_{\dot{H}_q^{1}}, \quad \forall \ u \in \dot{H}_p^{1}, \ v \in \dot{H}_q^{1}.$$ 

From Theorem 4.4.1 (page 96) of [1] we get

$$\|uv\|_{\dot{H}_r^{s}} \lesssim \|u\|_{\dot{H}_p^{s}} \|v\|_{\dot{H}_q^{s}}.$$ 

\[ \square \]

Lemma 4. Assume that

$$q > 1, p > 1, 0 \leq \frac{s}{d} < \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \quad (2)$$

Then we have the inequality

$$\|uv\|_{\dot{H}_r^{s}} \lesssim \|u\|_{\dot{H}_p^{s}} \|v\|_{\dot{H}_q^{s}}, \quad \forall \ u \in \dot{H}_p^{s}, \ v \in \dot{H}_q^{s},$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$. 

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Proof. Denote by $[s]$ the integer part of $s$ and by $\{s\}$ the fraction part of the argument $s$. Using the formula for the derivatives of a product of two functions, we have

$$
\|uv\|_{\dot{H}^s} = \|\dot{A}^s(uv)\|_{L^r} = \|\dot{A}^s(\dot{H}^s)\|_{L^r} \simeq \\
\sum_{|\alpha|=|s|} \|\partial^\alpha \dot{A}^s(\dot{H}^s)\|_{L^r} = \sum_{|\alpha|=|s|} \|\dot{A}^s(\dot{H}^s)\|_{L^r} \\
= \sum_{|\alpha|=|s|} \|\partial^\alpha (uv)\|_{\dot{H}^s} \lesssim \sum_{|\gamma|+|\beta|=|s|} \|\partial^\gamma u \partial^\beta v\|_{\dot{H}^s}.
$$

Set

$$
\frac{1}{p} = \frac{1}{p} - \frac{s - |\gamma| - \{s\}}{d}, \quad \frac{1}{q} = \frac{1}{q} - \frac{s - |\beta| - \{s\}}{d}.
$$

Applying Lemma 3 and the Sobolev inequality in order to obtain

$$
\|\partial^\gamma u \partial^\beta v\|_{\dot{H}^s} \lesssim \|\partial^\gamma u\|_{\dot{H}^{s,p}_p} \|\partial^\beta v\|_{\dot{H}^{s,q}_q} \lesssim \|u\|_{\dot{H}^{s+1}_p} \|v\|_{\dot{H}^{s+1}_q} \lesssim \|u\|_{\dot{H}^{s}_p} \|v\|_{\dot{H}^{s}_q}.
$$

This gives the desired result

$$
\|uv\|_{\dot{H}^s} \lesssim \|u\|_{\dot{H}^{s}_p} \|v\|_{\dot{H}^{s}_q}.
$$

Lemma 5. Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.

(a) If $s < 1$ then the two quantities

$$
\left( \int_0^\infty (t^{-\frac{\alpha}{2}} \|e^{t^{\frac{\alpha}{2}}} \Lambda f\|_{L^q})^p \frac{dt}{t} \right)^{1/p}
$$

and $\|f\|_{\dot{B}^{s,p}_q}$ are equivalent.

(b) If $s < 0$ then the two quantities

$$
\left( \int_0^\infty (t^{-\frac{\alpha}{2}} \|e^{t^{\frac{\alpha}{2}}} f\|_{L^q})^p \frac{dt}{t} \right)^{1/p}
$$

and $\|f\|_{\dot{B}^{s,p}_q}$ are equivalent,

where $\dot{B}^{s,p}_q$ is the homogeneous Besov space.

Proof. See ([13], Proposition 1, p. 181 and Proposition 3, p. 182), or see ([26], Theorem 5.4, p. 45).

The following lemma is a generalization of the above lemma.

Lemma 6. Let $1 \leq p, q \leq \infty$, $\alpha \geq 0$, and $s < \alpha$. Then the two quantities

$$
\left( \int_0^\infty (t^{-\frac{\alpha}{2}} \|e^{t^{\frac{\alpha}{2}}} \Lambda f\|_{L^q})^p \frac{dt}{t} \right)^{\frac{1}{p}}
$$

and $\|f\|_{\dot{B}^{s,p}_q}$ are equivalent.
Proof. Note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^s_{q,p}$ to $\dot{B}^{s-s_0,p}$, see [3], then we can easily prove the lemma.

Lemma 7. Assume that $q > 1, 1 \leq r \leq \infty$, and $0 \leq s < \frac{d}{q}$. The following statement is true: If $u_0 \in \dot{H}^s_{L^q,r}$ then we can easily prove the lemma.

Proof. We have

$$\|e^{t\Delta}u_0\|_{\dot{H}^s_{L^q,r}} = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4t}} \|\dot{\Lambda}^s u_0(\cdot - \xi)\|_{L^q} d\xi$$

$$\leq \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4t}} \|\dot{\Lambda}^s u_0(\cdot - \xi)\|_{L^q} d\xi = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4t}} \|u_0\|_{\dot{H}^s_{L^q,r}} d\xi = \|u_0\|_{\dot{H}^s_{L^q,r}}.$$ 

Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [26], p. 227).

Theorem 1. Let $E$ be a Banach space, and $B : E \times E \to E$ be a continuous bilinear map such that there exists $\eta > 0$ so that

$$\|B(x, y)\| \leq \eta \|x\| \|y\|,$$

for all $x$ and $y$ in $E$. Then for any fixed $y \in E$ such that $\|y\| \leq \frac{1}{4\eta}$, the equation $x = y - B(x, x)$ has a unique solution $\bar{x} \in E$ satisfying $\|\bar{x}\| \leq \frac{1}{2\eta}$.

§3. Main results

Now, for $T > 0$, we say that $u$ is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial datum $u_0$ when $u$ solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes u(\tau, \cdot)) d\tau.$$

Above we have used the following notation: For a tensor $F = (F_{ij})$ we define the vector $\nabla \cdot F$ by $(\nabla \cdot F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors $u$ and $v$, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$\mathbb{P}f = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k, \quad (3)$$
where $R_j$ is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}}, \text{ i.e. } \hat{R}_j\hat{g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi)$$

with $\hat{}$ denoting the Fourier transform. The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|x|^2/4t} * u)(x).$$

If $X$ is a normed space and $u = (u_1, u_2, \ldots, u_d), u_i \in X, 1 \leq i \leq d$, then we write

$$u \in X, \|u\|_X = \left(\sum_{i=1}^{d} \|u_i\|_X^2\right)^{1/2}.$$ We define the auxiliary space $K_{s,\tilde{q},q,r,T}$ which is made up by the functions $u(t, x)$ such that

$$\|u\|_{K_{q,r,T}} = \sup_{0 \leq t \leq T} \|u(t, .)\|_{H_{Lr}^s} < \infty,$$

and

$$\lim_{t \to 0} \|u(t, .)\|_{H_{Lr}^s} = 0, \quad (4)$$

where $r, q, \tilde{q}, s$ being fixed constants satisfying

$$q, \tilde{q} \in (1, +\infty), r \geq 1, s \geq 0, \frac{s}{d} < \frac{1}{\tilde{q}} \leq \frac{1}{q} \leq \frac{s + 1}{d},$$

and

$$\alpha = \alpha(q, \tilde{q}) = d\left(\frac{1}{q} - \frac{1}{\tilde{q}}\right).$$

In the case $\tilde{q} = q$, it is also convenient to define the space $K_{q,r,T}$ as the natural space $L^\infty([0, T]; H_{Lr}^s(\mathbb{R}^d))$ with the additional condition that its elements $u(t, x)$ satisfy

$$\lim_{t \to 0} \|u(t, .)\|_{H_{Lr}^s} = 0. \quad (5)$$

**Remark 1.** The auxiliary space $K_{\tilde{q}} := K_{0,\tilde{q},r,T}$ ($\tilde{q} \geq d$) was introduced by Weissler and systematically used by Kato [20] and Cannone [5].

**Lemma 8.** Let $1 \leq r \leq \tilde{r} \leq \infty$. Then we have the following imbedding maps

$$K_{s,\tilde{q},r,T} \hookrightarrow K_{s,\tilde{q},\tilde{r},T} \hookrightarrow K_{s,\tilde{q},\infty,T} \hookrightarrow K_{s,\tilde{q},r,T}.$$  

**Proof.** It is easily deduced from Lemma 1 (a) and the definition of $K_{q,r,T}$.

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Lemma 9. If \( u_0 \in \dot{H}^{s,q,r}(\mathbb{R}^d) \) with \( q > 1, r \geq 1, s \geq 0, \) and \( \frac{s}{d} < \frac{1}{q} < \frac{s+1}{d} \) then for all \( \tilde{q} \) satisfying

\[
\frac{s}{d} < \frac{1}{\tilde{q}} < \frac{s}{d} + 1 \quad \text{and} \quad \frac{s}{d} < \frac{1}{q} < \frac{s+1}{d}
\]

we have

\[
te^{t\Delta} u_0 \in K^{s,\tilde{q},q}_{q,1,\infty},
\]

and the following imbedding map

\[
\dot{H}^{s,q,r}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}}),\infty}_{\tilde{q}}(\mathbb{R}^d).
\] (6)

Proof. Before proving this lemma, we need to prove the following lemma.

Lemma 10. Suppose that \( u_0 \in L^{q,r}(\mathbb{R}^d) \) with \( 1 \leq q \leq \infty \) and \( 1 \leq r < \infty \). Then \( \lim_{n \to \infty} \| \mathcal{X}_n u_0 \|_{L^{q,r}} = 0 \), where \( n \in \mathbb{N}, \mathcal{X}_n(x) = 0 \) for \( x \in \{ x : |x| < n \} \cap \{ x : |u_0(x)| < n \} \) and \( \mathcal{X}_n(x) = 1 \) otherwise.

Proof. With \( \delta > 0 \) being fixed, we have

\[
\{ x : |\mathcal{X}_n u_0(x)| > \delta \} \supseteq \{ x : |\mathcal{X}_{n+1} u_0(x)| > \delta \},
\] (7)

and

\[
\bigcap_{n=0}^{\infty} \{ x : |\mathcal{X}_n u_0(x)| > \delta \} = \{ x : |u_0(x)| = +\infty \}.
\] (8)

We prove that

\[
\mathcal{M}^d(\{ x : |u_0(x)| = +\infty \}) = 0,
\] (9)

with \( \mathcal{M}^d \) being the Lebesgue measure in \( \mathbb{R}^d \), assuming on the contrary

\[
\mathcal{M}^d(\{ x : |u_0(x)| = +\infty \}) > 0.
\]

We have \( u_0^*(t) := \inf \{ \tau : \mathcal{M}^d(\{ x : |u_0(x)| > \tau \}) \leq t \} = +\infty \) for all \( t \) such that \( 0 < t < \mathcal{M}^d(\{ x : |u_0(x)| = +\infty \}) \) and then \( \| u_0 \|_{L^{q,r}} = +\infty \), a contradiction.

Note that

\[
\mathcal{M}^d(\{ x : |\mathcal{X}_0 u_0(x)| > \delta \}) = \mathcal{M}^d(\{ x : |u_0(x)| > \delta \}).
\]

We prove that

\[
\mathcal{M}^d(\{ x : |u_0(x)| > \delta \}) < \infty,
\] (10)

assuming on the contrary

\[
\mathcal{M}^d(\{ x : |u_0(x)| > \delta \}) = \infty.
\]
We have $u_0^*(t) \geq \delta$ for all $t > 0$, from the definition of the Lorentz space, we get
\[ \|u_0\|_{L^q,r} = \left( \int_0^\infty \left( t^{\frac{q}{r}} u_0^*(t) \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} \geq \delta \left( \int_0^\infty t^{\frac{q}{r} - 1} \frac{dt}{t} \right)^{\frac{1}{r}} = \infty, \]
a contradiction.

From (7), (8), (9), and (10), we infer that
\[ \lim_{n \to \infty} M^d(\{x : |X_n u_0(x)| > \delta\}) = M^d(\{x : |u_0(x)| = +\infty\}) = 0. \quad (11) \]
Set
\[ u_n^*(t) = \inf \{\tau : M^d(\{x : |X_n u_0(x)| > \tau\}) \leq t\}. \]
We have
\[ u_n^*(t) \geq u_{n+1}^*(t). \quad (12) \]
Fixed $t > 0$. For any $\epsilon > 0$, from (11) it follows that there exists a number $n_0 = n_0(t, \epsilon)$ large enough such that
\[ M^d(\{x : |X_n u_0(x)| > \epsilon\}) \leq t, \forall n \geq n_0. \]
From this we deduce that
\[ u_n^*(t) \leq \epsilon, \forall n \geq n_0, \]
therefore
\[ \lim_{n \to \infty} u_n^*(t) = 0. \quad (13) \]
From (12) and (13), we apply Lebesgue’s monotone convergence theorem to get
\[ \lim_{n \to \infty} \|X_n u_0\|_{L^q,r} = \lim_{n \to \infty} \left( \int_0^\infty \left( t^{\frac{q}{r}} u_n^*(t) \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} = 0. \quad \square \]

Now we return to prove Lemma 9. We prove that
\[ \sup_{0 < \tau < \infty} t^{\frac{2}{h}} \|e^{\tau \Delta} u_0\|_{H^s_{L^{\tilde{q},1}}} \lesssim \|u_0\|_{H^s_{L^{q,r}}} \quad (14) \]
Set
\[ \frac{1}{\tilde{h}} = 1 + \frac{1}{q} - \frac{1}{\tilde{q}}. \]
Applying Proposition 2.4 (c) in ([26], pp. 20) for convolution in the Lorentz spaces, we have
\[ \|e^{\tau \Delta} u_0\|_{H^s_{L^{\tilde{q},1}}} = \|e^{\tau \hat{\Lambda}^s u_0}\|_{L^{\tilde{q},1}} = \frac{1}{(4\pi t)^{d/2}} \|e^{-\frac{|\cdot|^2}{4t}} \hat{\Lambda}^s u_0\|_{L^{\tilde{q},1}} \lesssim \frac{1}{t^{d/2}} \|e^{-\frac{|\cdot|^2}{4t}} \hat{\Lambda}^s u_0\|_{L^{\tilde{q},1}} = t^{-\frac{d}{2}} \|e^{-\frac{|\cdot|^2}{4t}} \hat{\Lambda}^s u_0\|_{L^{\tilde{q},1}} \lesssim t^{-\frac{d}{2}} \|u_0\|_{H^s_{L^{q,r}}}. \]
We claim now that
\[ \lim_{t \to 0} t^{\frac{d}{2}} \| e^{t\Delta} u_0 \|_{H^{s}_{L^{\frac{q}{p}}, 1}} = 0. \]

From Lemma 10 we have
\[ \lim_{n \to \infty} \left\| \mathcal{X}_{n,s} \hat{\Lambda}^s u_0 \right\|_{L^{q,r}} = 0, \] (15)

where \( \mathcal{X}_{n,s}(x) = 0 \) for \( x \in \{ x : |x| < n \} \cap \{ x : |\hat{\Lambda}^s u_0(x)| < n \} \) and \( \mathcal{X}_{n,s}(x) = 1 \) otherwise. We have
\[ t^{\frac{d}{2}} \| e^{t\Delta} u_0 \|_{H^{s}_{L^{\frac{q}{p}}, 1}} \leq \frac{t^{\frac{d}{2}} - \frac{d}{2}}{(4\pi)^{d/2}} \| e^{-\frac{t}{2}} \ast (\mathcal{X}_{n,s} \hat{\Lambda}^s u_0) \|_{L^{q,r}} + \]
\[ \frac{t^{\frac{d}{2}} - \frac{d}{2}}{(4\pi)^{d/2}} \| e^{-\frac{t}{2}} \ast ((1 - \mathcal{X}_{n,s}) \hat{\Lambda}^s u_0) \|_{L^{q,r}}. \] (16)

For any \( \epsilon > 0 \), applying Proposition 2.4 (c) in [26, pp. 20] and note that (15), we have
\[ \frac{t^{\frac{d}{2}} - \frac{d}{2}}{(4\pi)^{d/2}} \| e^{-\frac{t}{2}} \ast (\mathcal{X}_{n,s} \hat{\Lambda}^s u_0) \|_{L^{q,r}} \leq C_1 \| e^{-\frac{t}{2}} \|_{L^{q,r}} \left\| \mathcal{X}_{n,s} \hat{\Lambda}^s u_0 \right\|_{L^{q,r}} \leq C_2 \left\| \mathcal{X}_{n,s} \hat{\Lambda}^s u_0 \right\|_{L^{q,r}} < \frac{\epsilon}{2}, \] (17)

for large enough \( n \). Fixed one of such \( n \), applying Proposition 2.4 (a) in [26, pp. 20], we conclude that
\[ \frac{t^{\frac{d}{2}} - \frac{d}{2}}{(4\pi)^{d/2}} \| e^{-\frac{t}{2}} \ast ((1 - \mathcal{X}_{n,s}) \hat{\Lambda}^s u_0) \|_{L^{q,r}} \leq C_3 t^{\frac{d}{2}} \| e^{-\frac{t}{2}} \|_{L^{1}} \left\| (1 - \mathcal{X}_{n,s}) \hat{\Lambda}^s u_0 \right\|_{L^{q,r}} \]
\[ \leq C_4 t^{\frac{d}{2}} \| e^{-\frac{t}{2}} \|_{L^{1}} \| n(1 - \mathcal{X}_{n,s}) \|_{L^{q,r}} = C_5 n t^{\frac{d}{2}} \| (1 - \mathcal{X}_{n,s}) \|_{L^{q,r}} = C_6(n) t^{\frac{d}{2}} < \frac{\epsilon}{2}, \] (18)

for small enough \( t > 0 \). From the estimates (16), (17), and (18) it follows that
\[ t^{\frac{d}{2}} \| e^{t\Delta} u_0 \|_{H^{s}_{L^{\frac{q}{p}}, 1}} \leq C_2 \| \mathcal{X}_{n,s} \hat{\Lambda}^s u_0 \|_{L^{q,r}} + C_6(n) t^{\frac{d}{2}} < \epsilon. \]

Finally, the embedding (5) is derived from the inequality (14), Lemma 11 and Lemma 6.

**Remark 2.** In the case \( s = 0 \) and \( q = r = d \), Lemma 11 is a generalization of Lemma 9 in [8, p. 196].
In the following lemmas a particular attention will be devoted to study of the bilinear operator \( B(u, v)(t) \) defined by

\[
B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau) \otimes v(\tau)) \, d\tau.
\]

**Lemma 11.** Let \( s, q \in \mathbb{R} \) be such that

\[
s \geq 0, \quad q > 1, \quad \text{and} \quad s \frac{d}{q} < \frac{1}{2} + \frac{s + 1}{d}.
\]

Then for all \( \tilde{q} \) satisfying

\[
s \frac{d}{q} < \frac{1}{q} < \min \left\{ \frac{1}{2} + \frac{s + 1}{2d}, \frac{1}{q} \right\},
\]

the bilinear operator \( B(u, v)(t) \) is continuous from \( K_{s, \tilde{q}q, \tilde{q}, T} \) into \( K_{s, \tilde{q}q, 1, T} \) and the following inequality holds

\[
\|B(u, v)\|_{K_{s, \tilde{q}q, 1, T}} \leq C |T|^\frac{1}{2}(1+s-dq) \|u\|_{K_{s, \tilde{q}q, 1, T}} \|v\|_{K_{s, \tilde{q}q, 1, T}},
\]

where \( C \) is a positive constant independent of \( T \).

**Proof.** We have

\[
\|B(u, v)(t)\|_{H_{q, T}^s} \leq \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, .) \otimes v(\tau, .)) \right\|_{L_{q, T}^s} \, d\tau =
\int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \hat{A}^s (u(\tau, .) \otimes v(\tau, .)) \right\|_{L_{q, T}^s} \, d\tau.
\]

From the properties of the Fourier transform

\[
\left( e^{(t-\tau)\Delta} \mathbb{P} \nabla \hat{A}^s (u(\tau, .) \otimes v(\tau, .)) \right)^\wedge (\xi) =
e^{-t(t-\tau)} \xi^2 \sum_{l, k=1}^d \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_l) \left( \hat{A}^s (u_l(\tau, .)v_k(\tau, .)) \right)^\wedge (\xi),
\]

and then

\[
\left( e^{(t-\tau)\Delta} \mathbb{P} \nabla \hat{A}^s (u(\tau, .) \otimes v(\tau, .)) \right)_j\to^l =
\frac{1}{(t-\tau)^{\frac{d+2}{2}}} \sum_{l, k=1}^d K_{l, k, j} \left( \frac{.}{\sqrt{t-\tau}} \right)^* \left( \hat{A}^s (u_l(\tau, .)v_k(\tau, .)) \right),
\]

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where
\[
\hat{K}_{i,k,j}(\xi) = \frac{1}{(2\pi)^{d/2}} e^{-|\xi|^2} \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_i).
\]

Applying Proposition 11.1 ([26], p. 107) with \(|\alpha| = 1\) we see that the tensor \(\hat{K}(x) = \{\hat{K}_{i,k,j}(x)\}\) satisfies
\[
|\hat{K}(x)| \lesssim \frac{1}{(1 + |x|)^{d+1}}.
\] (24)

So, we can rewrite the equality (23) in the tensor form
\[
e^{(t-\tau)\Delta} \mathbb{P} \nabla \hat{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) =
\]
\[
\frac{1}{(t-\tau)^{d+1/2}} K \left( \frac{\cdot}{\sqrt{t-\tau}} \right) \ast \left( \hat{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right).
\] (25)

Set
\[
\frac{1}{r} = 2 \frac{s}{d} - \frac{1}{2}, \quad \frac{1}{h} = \frac{s}{d} - \frac{1}{q} + 1.
\] (26)

From the inequalities (19) and (20), we can check that the following conditions are satisfied
\[
1 < h, r < \infty \quad \text{and} \quad \frac{1}{q} + 1 = \frac{1}{h} + \frac{1}{r}.
\]

Applying Proposition 2.4 (c) in ([26], pp. 20) for convolution in the Lorentz spaces, we have
\[
\left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \hat{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{h,1}} \lesssim \frac{1}{(t-\tau)^{d+1/2}} \left\| K \left( \frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h,1}} \left\| \hat{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}}.
\] (27)

Applying Lemma 4 we obtain
\[
\left\| \hat{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}} \leq \left\| \hat{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^r} = \left\| u(\tau, \cdot) \otimes v(\tau, \cdot) \right\|_{H^q} \lesssim \left\| u(\tau, \cdot) \right\|_{H^q} \left\| v(\tau, \cdot) \right\|_{H^q}.
\] (28)

From the inequalities (24) and (26) we infer that
\[
\left\| K \left( \frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h,1}} = (t-\tau)^{\frac{d}{2h}} K \left. \right\|_{L^{h,1}} \simeq (t-\tau)^{\frac{d}{2h} - \frac{d}{2} + \frac{d}{2}}.
\] (29)
From the inequalities (27), (28), and (29) we deduce that
\[
\left\| e^{(t-\tau)\Delta} P \nabla \cdot \partial_t(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{q,1}} \lesssim (t-\tau)^{\frac{s}{d} - \frac{d}{2q} - \frac{1}{2}} \left\| u(\tau, \cdot) \right\|_{\dot{H}^s_q} \left\| v(\tau, \cdot) \right\|_{\dot{H}^s_q}.
\]
(30)
From the estimates (22) and (30), and note that from the inequalities (19) and (21), we can check that
\[
\left\| B\left( u, v \right) \right\|_{\dot{H}^s_q} \lesssim \int_0^t (t-\tau)^{\frac{s}{d} - \frac{d}{2q} - \frac{1}{2}} \left\| u(\tau, \cdot) \right\|_{\dot{H}^s_q} \left\| v(\tau, \cdot) \right\|_{\dot{H}^s_q} \ d\tau \lesssim \sup_{0<\eta<t} \eta^\frac{s}{2} \left\| u(\eta, \cdot) \right\|_{\dot{H}^s_q} \sup_{0<\eta<t} \eta^\frac{s}{2} \left\| v(\eta, \cdot) \right\|_{\dot{H}^s_q} \int_0^t (t-\tau)^{\frac{s}{d} - \frac{d}{2q} - \frac{1}{2}} \ d\tau \lesssim \sup_{0<\eta<t} \eta^\frac{s}{2} \left\| u(\eta, \cdot) \right\|_{\dot{H}^s_q} \sup_{0<\eta<t} \eta^\frac{s}{2} \left\| v(\eta, \cdot) \right\|_{\dot{H}^s_q} \int_0^t (t-\tau)^{\frac{s}{d} - \frac{d}{2q} - \frac{1}{2}} \ d\tau
\]
(31)
Let us now check the validity of the condition (4) for the bilinear term \( B(u, v)(t) \). Indeed, we have
\[
\lim_{t \to 0} \frac{\tau}{t} \left\| B(u, v)(t) \right\|_{\dot{H}^s_q} = 0,
\]
whenever
\[
\lim_{t \to 0} \frac{\tau}{t} \left\| u(t, \cdot) \right\|_{\dot{H}^s_q} = \lim_{t \to 0} \frac{\tau}{t} \left\| v(t, \cdot) \right\|_{\dot{H}^s_q} = 0.
\]
The estimate (21) is now deduced from the inequality (31).

**Remark 3.** In the case \( s = 0 \) and \( q = d \), Lemma 9 is a generalization of Lemma 10 in (8), p. 196.

**Lemma 12.** Let \( s, q \in \mathbb{R} \) be such that
\[
s \geq 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}.
\]
(32)
Then for all \( \tilde{q} \)
\[
\frac{1}{2} \left( \frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} \leq \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\},
\]
(33)
the bilinear operator \( B(u, v)(t) \) is continuous from \( K^{s,\tilde{q}}_{q,\tilde{q},T} \times K^{s,\tilde{q}}_{q,\tilde{q},T} \) into \( K^{s,\tilde{q}}_{q,\tilde{q},T} \) and the following inequality holds
\[
\left\| B(u, v) \right\|_{K^{s,\tilde{q}}_{q,\tilde{q},T}} \leq C(T) \left( \frac{1}{\tilde{q}} \left( 1 + s - \frac{s}{d} \right) \right) \left\| u \right\|_{K^{s,\tilde{q}}_{q,\tilde{q},T}} \left\| v \right\|_{K^{s,\tilde{q}}_{q,\tilde{q},T}},
\]
(34)
where \( C \) is a positive constant independent of \( T \).
Proof. Set
\[ \frac{1}{r} = \frac{2}{q} - \frac{s}{d}, \quad \frac{1}{h} = 1 + \frac{1}{q} - \frac{2}{r} + \frac{s}{d}. \] (35)
From the inequalities (32) and (33), we can check that \( h \) and \( r \) satisfy
\[ 1 < h, r < \infty \quad \text{and} \quad \frac{1}{q} + 1 = \frac{1}{h} + \frac{1}{r}. \]
From the equalities (24) and (35) it follows that
\[ \| e^{(t-\tau)A} \|_{L^r} \lesssim \frac{1}{(t-\tau)^{\frac{1}{r}}} \| K \|_{L^h} \| \Lambda^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \|_{L^r}. \] (36)
Applying Lemma 4 we have
\[ \| \Lambda^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \|_{L^r} \lesssim \| u(\tau, \cdot) \|_{H^s} \| v(\tau, \cdot) \|_{H^s}. \] (37)
From the inequalities (24) and (35) it follows that
\[ \| K \|_{L^h} = (t-\tau)^{\frac{1}{r}} \| K \|_{L^h} \lesssim (t-\tau)^{\frac{1}{r} + \frac{d}{q} - \frac{1}{2} + \frac{s}{q} + \frac{1}{2}}. \] (38)
From the estimates (36), (37), (38) we deduce that
\[ \| e^{(t-\tau)A} u(\tau, \cdot) \otimes v(\tau, \cdot) \|_{H^s} \lesssim (t-\tau)^{\frac{d}{q} - \frac{1}{2}} \| u(\tau, \cdot) \|_{H^s} \| v(\tau, \cdot) \|_{H^s} = (t-\tau)^{\alpha + \frac{d}{q} - \frac{1}{2}} \| u(\tau, \cdot) \|_{H^s} \| v(\tau, \cdot) \|_{H^s}. \]
From the inequalities (32) and (33), we can check that \( \alpha + \frac{d}{q} - \frac{1}{2} \) is negative and \( d \left( \frac{1}{q} - \frac{1}{2} \right) < 1 \), this gives the desired result
\[ \| B(u, v)(t) \|_{H^s} \lesssim \int_0^t (t-\tau)^{\alpha + \frac{d}{q} - \frac{1}{2} - \frac{1}{2}} \| u(\tau, \cdot) \|_{H^s} \| v(\tau, \cdot) \|_{H^s} d\tau \lesssim \sup_{0 < \eta < t} \eta \| u(\eta, \cdot) \|_{H^s} \| v(\eta, \cdot) \|_{H^s} \int_0^t (t-\tau)^{\alpha + \frac{d}{q} - \frac{1}{2}} \int_0^\eta (t-\tau)^{\alpha + \frac{d}{q} - \frac{1}{2}} \| u(\eta, \cdot) \|_{H^s} \| v(\eta, \cdot) \|_{H^s} d\tau d\eta \lesssim t^{\frac{1}{r} + \frac{1}{q} - \frac{1}{2}} \sup_{0 < \eta < t} \eta \| u(\eta, \cdot) \|_{H^s} \| v(\eta, \cdot) \|_{H^s}. \] (39)
Let us now check the validity of the condition (5) for the bilinear term $B(u, v)(t)$. Indeed, we have
\[
\lim_{t \to 0} \|B(u, v)(t)\|_{H^s_{L^q,1}} = 0
\]
whenever
\[
\lim_{t \to 0} \frac{1}{t} \|u(t, \cdot)\|_{H^s_q} = \lim_{t \to 0} \frac{1}{t} \|v(t, \cdot)\|_{H^s_q} = 0.
\]
The estimate (34) is now deduced from the inequality (39).

Combining Theorem 1 with Lemmas 7, 9, 11, 12, we obtain the following existence result.

**Theorem 2.** Let $s, q, \text{and } r \in \mathbb{R}$ be such that
\[
s \geq 0, \quad q > 1, \quad r \geq 1, \quad \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}.
\] (40)

(a) For all $\hat{q}$ satisfying
\[
\frac{1}{2} \left( \frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\hat{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{\hat{q}} \right\},
\] (41)
there exists a positive constant $\delta_{s,q,\hat{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_{L^q,r}(\mathbb{R}^d)$ with $\text{div}(u_0) = 0$ satisfying
\[
T^{\frac{1}{2} \left( \frac{1}{q} - \frac{s}{d} \right)} \sup_{0 < t < T} \|e^{t \Delta} u_0\|_{H^s_q} \leq \delta_{s,q,\hat{q},d},
\] (42)
NSE has a unique mild solution $u \in \mathcal{K}^s_{q,1,T} \cap L^\infty([0, T], \dot{H}^s_{L^q,r})$. In particular, for arbitrary $u_0 \in \dot{H}^s_{L^q,r}$ with $\text{div}(u_0) = 0$, there exists $T(u_0)$ small enough such that the inequality (42) holds.

(b) If $1 < q \leq d$, and $s = \frac{d}{q} - 1$ then for any $\hat{q}$ be such that
\[
\frac{1}{q} - \frac{1}{2d} < \frac{1}{\hat{q}} < \min \left\{ \frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{\hat{q}} \right\},
\] (41)
there exists a positive constant $\sigma_{q,\hat{q},d}$ such that if $\|u_0\|_{B^\frac{s}{\hat{q}}} \leq \sigma_{q,\hat{q},d}$ and $T = \infty$ then the inequality (42) holds.

**Proof.** From Lemmas 11 and 8, the bilinear operator $B(u, v)(t)$ is continuous from $K^{s,\hat{q}}_{q,\hat{q},T} \times K^{s,\hat{q}}_{q,\hat{q},T}$ into $K^{s,\hat{q}}_{q,\hat{q},T}$ and we have the inequality
\[
\|B(u, v)\|_{K^{s,\hat{q}}_{q,\hat{q},T}} \leq \|B(u, v)\|_{K^{s,\hat{q}}_{q,\hat{q},T}} \leq C_{s,q,\hat{q},d} T^{\frac{1}{2} \left( \frac{1}{q} - \frac{s}{d} \right)} \|u\|_{K^{s,\hat{q}}_{q,\hat{q},T}} \|v\|_{K^{s,\hat{q}}_{q,\hat{q},T}},
\] (41)
where $C_{s,q,d}$ is a positive constant independent of $T$. From Theorem 1 and the above inequality, we deduce following: for any $u_0 \in \dot{H}^s_{L^q,r}(\mathbb{R}^d)$ such that $\text{div}(u_0) = 0$, $T^\frac{1}{2}(1+s-\frac{d}{q}) \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \| \epsilon^t \Delta u_0 \|_{H^s_{L^q}} \leq \frac{1}{4C_{s,q,d}}$.

NSE has a mild solution $u$ on the interval $(0,T)$ so that
\begin{align*}
\| u \|_{L^\infty([0,T]; \dot{H}^s_{L^q,r})} \leq \sigma_{s,q,d}
\end{align*}
Lemma 12 and the relation (13) imply that $B(u,u) \in L^\infty([0,T]; \dot{H}^s_{L^q,r})$. On the other hand, from Lemma 7, we have $\epsilon^t \Delta u_0 \in L^\infty([0,T]; \dot{H}^s_{L^q,r})$. Therefore
\begin{align*}
u = \epsilon^t \Delta u_0 - B(u,u) \in L^\infty([0,T]; \dot{H}^s_{L^q,r}).
\end{align*}
From Lemma 9 and Lemma 11, we deduce that $u \in K_{q,T}$. From the definition of $K_{q,T}^s$ and Lemma 9, we deduce that the left-hand side of the inequality (12) converges to 0 when $T$ tends to 0. Therefore the inequality (12) holds for arbitrary $u_0 \in \dot{H}^s_{L^q,r}(\mathbb{R}^d)$ when $T(u_0)$ is small enough.

Remark 4. In the case when the initial data belong to the critical Sobolev-Lorentz spaces $\dot{H}^s_{L^q,r}(\mathbb{R}^d)$, $(1 < q \leq d, r \geq 1)$, from Theorem 2 (b), we get the existence of global mild solutions in the spaces $L^\infty([0,\infty); \dot{H}^s_{L^q,r}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{s}{q}-1}_{q,\infty}(\mathbb{R}^d)$ is small enough. Note that a function in $\dot{H}^s_{L^q,r}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^s_{L^q,r}(\mathbb{R}^d)$ norm but small in the $\dot{B}^{\frac{s}{q}-1}_{q,\infty}(\mathbb{R}^d)$ norm. This is deduced from the following imbedding maps (see Lemma 9)
\begin{align*}
\dot{H}^s_{L^q,r}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{s}{q}-1}_{q,\infty}(\mathbb{R}^d), \left( \frac{1}{q} - \frac{1}{d} < \frac{1}{q} < \frac{1}{q} \right).
\end{align*}
This result is stronger than that of Cannone. In particular, when $q = r = d, s = 0$, we get back the Cannone theorem (Theorem 1.1 in [5]).
Next, we consider the super-critical indexes $s > \frac{d}{q} - 1$.

**Theorem 3.** Let

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} < \frac{s + 1}{d}.$$  

Then for any $\tilde{q}$ be such that

$$\frac{1}{2} \left( \frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min\left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\},$$

there exists a positive constant $\delta_{s,q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_{L,q,r}(\mathbb{R}^d)$ with $\text{div}(u_0) = 0$ satisfying

$$T^\frac{1}{2}(1 + s - \frac{d}{q}) \left\| u_0 \right\|_{\dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}})}_{\tilde{q}}} \leq \delta_{s,q,\tilde{q},d},$$

NSE has a unique mild solution $u \in \mathcal{K}^{s,q,1}_{q,1,T} \cap L^\infty([0,T]; \dot{H}^s_{L,q,r})$.

**Proof.** Applying Lemma 6, the two quantities $\left\| u_0 \right\|_{\dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}})}_{\tilde{q}}}$ and

$$\sup_{0 < t < \infty} T^\frac{d}{2} \left( \frac{1}{q} - \frac{s}{d} \right) \left\| e^{t\Delta} u_0 \right\|_{\dot{H}^{s}_{\tilde{q}}}$$

are equivalent. Thus

$$\sup_{0 < t < \infty} T^\frac{d}{2} \left( \frac{1}{q} - \frac{s}{d} \right) \left\| e^{t\Delta} u_0 \right\|_{\dot{H}^{s}_{\tilde{q}}} \lesssim \left\| u_0 \right\|_{\dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}})}_{\tilde{q}}},$$

the theorem is proved by applying the above inequality and Theorem 2.

**Remark 5.** In the case when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}^s_{L,q,r}(\mathbb{R}^d), (q > 1, r \geq 1, s \geq 0, \text{ and } \frac{d}{q} - 1 < s < \frac{d}{q})$, we obtain the existence of mild solutions in the spaces $L^\infty([0,T]; \dot{H}^s_{L,q,r}(\mathbb{R}^d))$ for any $T > 0$ when the norm of the initial value in the Besov spaces $\dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}})}_{\tilde{q}}(\mathbb{R}^d)$ is small enough. Note that a function in $\dot{H}^s_{L,q,r}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^s_{L,q,r}(\mathbb{R}^d)$ norm but small in $\dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}})}_{\tilde{q}}(\mathbb{R}^d)$ norm. This is deduced from the following imbedding maps (see Lemma 9)

$$\dot{H}^s_{L,q,r}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-(\frac{d}{q} - \frac{d}{\tilde{q}})}_{\tilde{q}}(\mathbb{R}^d), \left( \frac{s}{d} < \frac{1}{q} < \frac{1}{\tilde{q}} \right).$$

Applying Theorem 3 for $q > d, r = q$ and $s = 0$, we get the following proposition which is stronger than the result of Cannone and Meyer ([4], [7]). In particular, we obtained a result that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.
Proposition 1. Let \( q > d \). Then for any \( \tilde{q} \) be such that
\[
q < \tilde{q} < 2q,
\]
there exists a positive constant \( \delta_{q, \tilde{q}, d} \) such that for all \( T > 0 \) and for all \( u_0 \in L^q(\mathbb{R}^d) \) with \( \text{div}(u_0) = 0 \) satisfying
\[
T^\frac{1}{2(1 - \frac{d}{q})} \| u_0 \|_{\dot{B}_{q}^{\frac{d}{q} - \frac{d}{q}, \infty}} \leq \delta_{q, \tilde{q}, d},
\]
NSE has a unique mild solution \( u \in K^{0, \tilde{q}, d, 1}_q \cap L^\infty([0, T]; L^q) \).

Remark 6. If in (44) we replace the \( \dot{B}_{q}^{\frac{d}{q} - \frac{d}{q}, \infty} \) norm by the \( L^q \) norm then we get the assumption made in ([4], [7]). We show that the condition (44) is weaker than the condition in ([4], [7]). In Remark 5 we have showed that
\[
L^q(\mathbb{R}^d) \hookrightarrow \dot{B}_{q}^{\frac{d}{q} - \frac{d}{q}, \infty}(\mathbb{R}^d), (\tilde{q} > q \geq d),
\]
but these two spaces are different. Indeed, we have \( |x|^{-\frac{d}{q}} \notin L^q(\mathbb{R}^d) \). On the other hand by using Lemma 6, we can easily prove that \( |x|^{-\frac{d}{q}} \in \dot{B}_{q}^{\frac{d}{q} - \frac{d}{q}, \infty}(\mathbb{R}^d) \) for all \( \tilde{q} > q \).

Applying Theorem 3 for \( q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2} \), we get the following proposition which is stronger than the results in Chemin in [9] and Cannone in [11]. In particular, we obtained the result that is stronger than that of Chemin and Cannone but under a much weaker condition on the initial data.

Proposition 2. Let \( \frac{d}{2} - 1 < s < \frac{d}{2} \). Then for any \( \tilde{q} \) be such that
\[
\frac{1}{2} \left( \frac{1}{2} + \frac{s}{d} \right) < \frac{1}{q} < \frac{1}{2},
\]
there exists a positive constant \( \delta_{s, \tilde{q}, d} \) such that for all \( T > 0 \) and for all \( u_0 \in \dot{H}^s(\mathbb{R}^d) \) with \( \text{div}(u_0) = 0 \) satisfying
\[
T^\frac{1}{2(1 + s - \frac{d}{2})} \| u_0 \|_{\dot{B}_{q}^{s - (\frac{d}{q} - \frac{s}{d}), \infty}} \leq \delta_{s, \tilde{q}, d},
\]
NSE has a unique mild solution \( u \in K^{s, \tilde{q}, d, 1}_q \cap L^\infty([0, T]; \dot{H}^s) \).

Remark 7. If in (45) we replace the \( \dot{B}_{q}^{s - (\frac{d}{q} - \frac{s}{d}), \infty} \) norm by the \( \dot{H}^s(\mathbb{R}^d) \) norm then we get the assumption made in ([9], [11]). We show that the condition (45) is weaker than the condition in ([2], [3]). In Remark 5 we showed that
\[
\dot{H}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{q}^{s - (\frac{d}{q} - \frac{s}{d}), \infty}, \quad \frac{1}{2} \left( \frac{1}{2} + \frac{s}{d} \right) < \frac{1}{q} < \frac{1}{2},
\]

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but that these two spaces are different. Indeed, we have $\dot{\Lambda}^{-s}|.|^{-\frac{d}{2}} \notin \dot{H}^{s}(\mathbb{R}^d)$, on the other hand by using Lemma 6 we easily prove that $\dot{\Lambda}^{-s}|.|^{-\frac{d}{2}} \in \dot{B}_{\dot{q}}^{s-(\frac{d}{2}-\frac{d}{q})_{\infty}}(\mathbb{R}^d)$ for all $\dot{q} > 2$.

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