Higher-order extension of Starobinsky inflation: Initial conditions, slow-roll regime, and reheating phase

G. Rodrigues-da-Silva∗
Departamento de Física, Universidade Federal do Rio Grande do Norte,
Campus Universitário, s/n - Lagoa Nova, CEP 59072-970, Natal, Rio Grande do Norte, Brazil

J. Bezerra-Sobrinho† and L. G. Medeiros‡
Escola de Ciências e Tecnologia, Universidade Federal do Rio Grande do Norte,
Campus Universitário, s/n - Lagoa Nova, CEP 59072-970, Natal, Rio Grande do Norte, Brazil
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The most current observational data corroborate the Starobinsky model as one of the strongest candidates in the description of an inflationary regime. Motivated by such success, extensions of the Starobinsky model have been increasingly recurrent in the literature. The theoretical justification for this is well grounded: higher-order gravities arise in high-energy physics in the search for the ultraviolet completeness of general relativity. In this paper, we propose to investigate the inflation due to the extension of the Starobinsky model characterized by the inclusion of the $R^3$ term. We make a complete analysis of the potential and phase space of the model, where we observe the existence of three regions with distinct dynamics for the scalar field. We can establish restrictive limits for the number of $e$-folds through a study of the reheating and by considering the usual couplings of the standard matter fields and gravity. Thereby, we duly confront our model with the observational data from Planck, BICEP3/Keck, and BAO. Finally, we discuss how the inclusion of the cubic term restricts the initial conditions necessary for the occurrence of a physical inflation.

I. INTRODUCTION

The idea of a quasi de Sitter type accelerated expansion regime through which the early universe experimented is, on the one hand, a necessity for the solution of a series of problems that arise in a decelerated Friedmann universe [1] and it is, on the other hand, a paradigm since it has the support of the most current observations from the Planck satellite [2]. Inflationary cosmology emerged with the aim of proposing a solution to problems such as the horizon, the flatness, and the magnetic monopoles [3]. However, over the years, the main motivation for building an inflationary model has become to provide a causal mechanism capable of describing the origin of the primordial inhomogeneities which will be responsible for producing the large-scale structures of the universe [4, 5]. We have lived in a golden moment in cosmology mainly because in the last two decades, it has become a branch of science capable of being duly confronted with observational data.

A consistent inflationary model must last long enough to deal with problems like the ones mentioned above and must also end up, through the graceful exit, giving way to Friedmann’s decelerated universe, in order to preserve the successful predictions of the standard model of cosmology. There are numerous models proposed in the literature suggesting the existence of a scalar field, or multiple ones, driving inflation [6–8]. Some of these models explore the idea of scalar fields, arising from some fundamental interaction, evolving in curved spacetimes described by general relativity (GR) [9–11], and some others are based on GR extensions, through the inclusion of higher-order curvature terms in the Einstein-Hilbert (EH) action [12–17].

Higher-order gravities are motivated by high-energy physics in the search for the ultraviolet (UV) completeness of GR. As it is well known, GR is a nonrenormalizable theory, and it is not possible to conventionally quantize it. However, the inclusion of such higher-order curvature terms — involving functions of the curvature scalar $R$, contractions of the Ricci tensor $R_{\mu\nu}$ and/or Riemann $R_{\kappa\lambda\mu\nu}$, as well as their derivatives — contribute to its renormalizability [18]. K. S. Stelle, for example, obtained a renormalizable system due to the inclusion of quadratic curvature correction terms in EH action [19]. In this sense, GR could be seen as an effective low-energy theory that requires higher-order curvature corrections as we increase the energy scale [20]. The inclusion of these terms is often accompanied by the problem of introducing ghost-type instabilities, which quantumly manifests itself with the loss of unitarity and, classically, with the absence of a lower limit for the Hamiltonian of the system [21]. However, this is not a problem that arises in $f(R)$ type extensions, known to be ghost-free [22].

The Starobinsky model [12], which is characterized by including the quadratic term $R^2$ in EH action, and consequently introducing only one additional parameter, is one of the strongest inflationary candidates, best fitting to the most current observational data [1]. By proposing an extension to GR, an inflationary regime can be obtained, for a certain range of the theory parameters, without the need of the extra fields used in part of the inflation mechanisms. This is due to the equivalence between higher-order gravities, whether $f(R)$ or $f(R, \Box R)$, and the scalar-tensor gravity theories [22–25].

* gesiel.neto.090@ufrn.edu.br
† jeremias.bs@gmail.com
‡ leo.medeiros@ufrn.br

1 Along with Higgs inflation [11], since they have the same potential.
In these cases, through a conformal transformation, we go from the original frame with equations for the metric of order \(2l + 2\) to the Jordan or Einstein frame with \(l\) scalar fields responsible for driving inflation [26]. With this scenario, researches considering extensions to the Starobinsky model has been increasingly recurrent in the literature [17, 27–34].

In this paper, we propose the generalization of Starobinsky inflation due to the inclusion of the cubic term \(R^3\) in the gravitational action. So we can write the action that describes this model as

\[
S (\bar{g}_{\mu\nu}, \chi) = \frac{M^2_{Pl}}{2} \int d^4x \sqrt{-\bar{g}} \left( R + \frac{1}{2\kappa_0} R^2 + \frac{\alpha_0}{3\kappa_0^2} R^3 \right),
\]

where \(M_{Pl}\) is the reduced Planck mass, so that \(M^2_{Pl} \equiv (8\pi G)^{-1}\), \(\kappa_0\) has squared mass units and the parameter \(\alpha_0\) is a dimensionless quantity. For consistency with the Starobinsky model, we assume \(\kappa_0 > 0\). We can view the action (1) as a classical theory of gravity containing higher-order energy corrections to GR.

The paper is structured as follows. In Sec. II, we present the action (1) in the context of a scalar-tensor theory in the Einstein frame and its associated potential. In Sec. III, we make a complete study of the potential and phase space of the model. We observe the existence of three regions with distinct dynamics for the scalar field, and in addition, we qualitatively discuss the influence that the cubic term has on the restriction of the initial conditions capable of leading the system to a consistent inflationary regime (physical inflation). In Sec. IV, we describe the entire dynamics of the scalar field, focusing (i) on the region where the slow-roll regime occurs and (ii) in the reheating region, where we develop a study that allows us to obtain expressions for the reheating number of \(e\)-folds and temperature valid for a wide range of models, and then we particularize for our case. In Sec. V, by using our results on the reheating period and considering the minimum hypothesis of the usual couplings of the standard matter fields with gravity, we can obtain restrictive limits for the inflation number of \(e\)-folds. In its turn, with the interval obtained for the number of \(e\)-folds, we confront our model with the observations of Planck, BICEP3/Keck and BAO [2, 35], by establishing the constraints on the theoretical values of scalar spectral index \(n_s\) and the tensor-to-scalar ratio \(r\). Finally, we discuss the fine-tuning problem of the initial conditions that the \(R^3\) term introduces for the occurrence of a physical inflation, as well as the conditions that lead to an eternal inflation regime. In Sec. VI, we make some final comments.

II. FIELD EQUATIONS

The action (1) can be conveniently rewritten as a scalar-tensor theory in the Einstein frame, in which we have a description through an auxiliary scalar field \(\chi\) minimally coupled with gravitation. By following the steps presented in Appendix A, we write

\[
S (\bar{g}_{\mu\nu}, \chi) = \frac{M^2_{Pl}}{2} \int d^4x \sqrt{-\bar{g}} \left[ R - 3 \left( \frac{1}{2} \bar{\partial}^\rho \bar{\partial}_\rho \chi + V (\chi) \right) \right],
\]

whose associated potential is

\[
V (\chi) = \frac{\kappa_0}{72\alpha_0^2} e^{-2\chi} \left( 1 - \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right) \times \left[ -1 + 8\alpha_0 (1 - e^\chi) + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right].
\]

The barred quantities are defined from the metric as \(\bar{g}_{\mu\nu} = e^\chi g_{\mu\nu}\) and the dimensionless field \(\chi\) is defined as

\[
e^\chi = 1 + \frac{R}{\kappa_0} + \alpha_0 \left( \frac{R}{\kappa_0} \right)^2.
\]

Addendum In order to recover the usual notation and dimensions of the scalar field and the potential, we must do

\[
\chi = \sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}} \quad \text{and} \quad \bar{V} (\phi) = 3\frac{M^2_{Pl}}{2} V (\chi).
\]

In this case, the action (2) is rewritten as

\[
S (\bar{g}_{\mu\nu}, \phi) = \int d^4x \sqrt{-\bar{g}} \left( \frac{M^2_{Pl}}{2} \bar{R} - \frac{1}{2} \bar{\partial}^\rho \bar{\partial}_\rho \phi - \bar{V} (\phi) \right).
\]

The action (2) gives us two field equations: one for the metric \(\bar{g}_{\mu\nu}\) and one for the scalar field \(\chi\). By taking the variation concerning the metric \(\bar{g}_{\mu\nu}\), we obtain

\[
\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \frac{1}{M^2_{Pl}} \bar{T}^{(eff)}_{\mu\nu},
\]

where

\[
\frac{1}{M^2_{Pl}} \bar{T}^{(eff)}_{\mu\nu} = 3 \left[ \bar{\partial}_\mu \chi \bar{\partial}_\nu \chi - \bar{g}_{\mu\nu} \left( \frac{1}{2} \bar{\partial}^\rho \chi \bar{\partial}_\rho \chi + V (\chi) \right) \right].
\]

In turn, the variation concerning the scalar field \(\chi\) results in

\[
\Box \chi - V' (\chi) = 0,
\]

where the box \(\Box \equiv \bar{\nabla}^\mu \bar{\nabla}_\mu\) represents the covariant d’Alembertian operator and the prime represents the derivative with respect to \(\chi\). By calculating \(V' (\chi)\) explicitly, we obtain

\[
V' (\chi) = \frac{\kappa_0}{36\alpha_0^2} e^{-2\chi} \left( -1 + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right) \times \left[ -1 + 2\alpha_0 (4 - e^\chi) + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right].
\]
III. POTENTIAL AND PHASE SPACE IN THE FRIEDMANN BACKGROUND

We begin this section by writing the field equations in the Friedmann background. Considering the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right),$$  \hspace{1cm} (10)

we obtain

$$H^2 = \frac{1}{2} \left(\frac{1}{2} \chi^2 + V(\chi)\right),$$  \hspace{1cm} (11)

$$\dot{H} = -\frac{3}{2} \left(\frac{1}{2} \chi^2\right),$$  \hspace{1cm} (12)

and

$$\ddot{\chi} + 3H \dot{\chi} + V' (\chi) = 0,$$  \hspace{1cm} (13)

where the dot represents time derivative and $H = \dot{a}/a$ is the Hubble parameter. The potential and its derivative are given by Eqs. (3) and (9), with $\chi = \chi (t)$.

The first steps to be analyzed are the characteristics of the potential $V (\chi)$. The structure of the potential depends on the parameter $\alpha_0$. For $V (\chi)$ to be well defined for all real numbers, it is necessary that

$$0 \leq \alpha_0 \leq \frac{1}{4}.$$  \hspace{1cm} (14)

The lower (upper) limit of $\alpha_0$ makes $V (\chi)$ a real function for any value of $\chi$ greater (less) than zero. Furthermore, $\alpha_0 \geq 0$ guarantees the stability of the gravitational model during the entire inflationary regime. Within the range given in Eq. (14), excepting $\alpha_0 = 0$, the potential has two critical points, namely,

$$\chi_0 = 0 \rightarrow \text{(minimum point)},$$  \hspace{1cm} (15)

$$\chi_c = \ln \left(4 + \sqrt{\frac{3}{\alpha_0}}\right) \rightarrow \text{(maximum point)}.$$  \hspace{1cm} (16)

The behavior of the potential for different values of $\alpha_0$ is shown in Fig. 1.

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2 Throughout this paper, we will assume that $\alpha_0$ is contained in this range.

3 The $f (R)$ models are stable whenever $f' (R) > 0$ and $f'' (R) \geq 0$ [22]. For our case

$$f (R) = R + \frac{1}{2\kappa_0} R^2 + \frac{\alpha_0}{3\kappa_0^2} R^3,$$

where

$$R = 6 \left(H + 2H^2\right) = -\frac{3}{2} \chi^2 + 6V (\chi).$$

During the inflationary regime, when $\chi^2 \ll V (\chi)$ and $\kappa_0 > 0$, the stability condition implies $\alpha_0 \geq 0$.

4 In the linearized description in the vicinity of the critical points of the system, when obtaining associated pure imaginary roots, as occurs in $(\chi, \dot{\chi})_0 = (0, 0)$, its nature is uncertain: it may be a center or a spiral point. Next, we show through numerical methods that the critical point $(\chi, \dot{\chi})_0$ is an attractor spiral point.  

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Figure 1. Potential $V (\chi)$ as a function of $\chi$, normalized to $\kappa_0 = 1$. The maximum values are located at $\chi_c \simeq 3.06, 5.18$ and $7.46$ for $\alpha_0 = 10^{-2}, 10^{-4}$ and $10^{-6}$, respectively.
potential plateau in Fig. 1, we have a slow-roll regime where \( \dot{\chi} \ll 1 \) and \( \chi \) is at least of the order of the unity. As expected, the slow-roll regime leads to an almost exponential expansion that characterizes inflation. That can be explicitly identified when comparing Eq. (11) with Eq. (12). In fact, in the plateau region we have \( \dot{\chi} \ll 1 \) and \( \chi \approx 3.06 \), respectively. In its turn, the red and black lines represent two trajectories that travel in opposite directions concerning the saddle point \((\chi_c, 0)\).

In this section, we describe the evolution of the scalar field \( \chi \) in the three regions of interest: the asymptotic region where \( V(\chi) \rightarrow 0 \) when \( \chi \gg 1 \); the plateau region, where the inflationary regime occurs; and the region of oscillation around the origin of the potential that characterizes the end of inflation.

A. Asymptotic region of the potential \( V(\chi) \)

We saw in Sec. III that depending on the initial conditions for the scalar field \( \chi \), it evolves to the right in the phase space (see Fig. 2). In this case, the scalar field \( \chi \) grows indefinitely and reaches an asymptotic region of the potential, namely, \( V(\chi) \rightarrow 0 \). The purpose of this subsection is to analyze the cosmic dynamics in this region.

In a sufficiently large region of \( \chi \), the potential (3) and its derivative (9) behave like

\[
V(\chi) \approx \frac{2\kappa_0}{9\sqrt{\alpha_0}} e^{-\frac{4}{9}\chi} \quad \text{and} \quad V'(\chi) \approx -\frac{\kappa_0}{9\sqrt{\alpha_0}} e^{-\frac{4}{9}\chi} \quad \text{for} \quad \chi \gg 1.
\]

In this case, the field equation (13) can be written as

\[
\ddot{\chi} + \frac{3}{\sqrt{2}} \left[ \frac{1}{2} \chi^2 + \frac{2\kappa_0}{9\sqrt{\alpha_0}} e^{-\frac{4}{9}\chi} \right] \dot{\chi} - \frac{\kappa_0}{9\sqrt{\alpha_0}} e^{-\frac{4}{9}\chi} \approx 0.
\]

A convenient ansatz for solutions of this equation is \( \ddot{\chi} = Ae^{-k\chi} \). Substituting this proposed solution in the previous equation, we obtain

\[
k = \frac{1}{4} \quad \text{and} \quad A = \frac{2}{3} \sqrt{\frac{\kappa_0}{35\sqrt{\alpha_0}}} \Rightarrow \chi \approx \frac{2}{3} \sqrt{\frac{\kappa_0}{35\sqrt{\alpha_0}}} e^{-\frac{4}{9}\chi}.
\]

By having \( \dot{\chi} \) and \( V(\chi) \), we can explicitly determine the pa-
rameter of the equation of state \(^5\)

\[
w = \frac{p_\chi}{\rho_\chi} = \frac{1}{2} \chi^2 - V(\chi) \approx -\frac{17}{18} \text{ for } \chi \gg 1.
\] (18)

The result obtained in Eq. (18) shows that when the scalar field rolls to the right side from the plateau of the potential, the parameter of the equation of state leaves a value of \(w \approx -1\) (plateau region) and moves asymptotically to \(w \approx -17/18\). In other words, the inflation dynamics migrates from an almost de Sitter regime to a power-law type inflation with \(a \approx t^{12}\). Therefore, in this region, the accelerated expansion never ends. This explicitly shows that the existence of the critical point \((\chi_c, 0)\) limits the initial conditions favorable to a physical inflation i.e. an inflation that connects to a radiation era type of a hot big-bang model. In Sec. V C, such a limitation on the initial conditions is further discussed.

B. Slow-roll leading-order inflationary regime

In this subsection, we investigate the slow-roll inflationary regime that occurs in the plateau region of Fig. 1. In leading-order, Eqs. (13) and (11) are approximated by

\[
3H\dot{\chi} \approx -V',
\] (19)

\[
H^2 \approx \frac{1}{2} V(\chi),
\] (20)

that when combined result in

\[
\dot{\chi}(\chi) \approx -\frac{\sqrt{2}V'}{3\sqrt{V}}.
\] (21)

The next step is obtaining \(V\) and \(V'\) in a leading order slow-roll regime. Defining a fundamental slow-roll parameter, namely, \(\delta \equiv e^{-\chi}\),\(^6\) we write the potential (3) as

\[
V(\chi) = -\frac{\kappa_0}{72\alpha_0^2} \left(-\delta + \delta \sqrt{1 - 4\alpha_0 \delta^{-1}} (\delta - 1)\right) \times
\]

\[
\left[-\delta + 8\alpha_0 (\delta - 1) + \delta \sqrt{1 - 4\alpha_0 \delta^{-1}} (\delta - 1)\right].
\]

Furthermore, we know that the maximum value of the potential consistent with a physical inflationary regime occurs for \(V(\chi_c)\), where

\[
e^{\chi_c} = 4 + \sqrt{\frac{3}{\alpha_0}} \Rightarrow \alpha_0 = \frac{3}{(e^{\chi_c} - 4)^2}.
\]

So, in a slow-roll leading-order regime, we have

\[
\alpha_0 \approx \frac{3}{e^{2\chi_c}} \equiv 3\delta_c^2.
\] (22)

In its turn, as the minimum value of \(\delta\) consistent with a physical inflation is \(\delta_c\), we have

\[
4\alpha_0 \delta^{-1} \approx 12\delta_c^2 \delta^{-1} \lesssim 12\delta \ll 1.
\] (23)

The relation (23) shows that the quantity \(4\alpha_0 \delta^{-1}\) contributes at most in the slow-roll leading-order regime. So, it is worth pointing out that \(\alpha_0\) is a second-order slow-roll term.

From the previous considerations, we can approximate the potential and its derivatives by

\[
V(\chi) \approx \frac{\kappa_0}{6} \left(1 - 2\delta - \frac{2}{3} \alpha_0 \delta^{-1}\right),
\] (24)

\[
V'(\chi) \approx \frac{\kappa_0}{3} \left(\delta - \frac{1}{3} \alpha_0 \delta^{-1}\right),
\] (25)

\[
V''(\chi) \approx -\frac{\kappa_0}{3} \left(\delta + \frac{1}{3} \alpha_0 \delta^{-1}\right).
\] (26)

Since we have determined approximate \(V(\chi)\) and \(V'(\chi)\) we can calculate \(\dot{\chi}\) in the slow-roll leading-order. Starting from (21) we have

\[
\dot{\chi} \approx -\frac{\sqrt{2}V'}{3\sqrt{V}} \Rightarrow \dot{\chi} \approx -\frac{2\sqrt{3\kappa_0}}{9} (\delta - \delta_c^2 \delta^{-1}).
\] (27)

That is, in the slow-roll leading-order, we have \(\dot{\chi} \sim \delta\) plus a correction proportional to \(\delta^2 \delta^{-1} \lesssim \delta\).

With the previous results, we can calculate the slow-roll parameters

\[
\epsilon \equiv -\frac{\dot{H}}{H^2} \text{ and } \eta \equiv -\frac{\dot{\epsilon}}{H \dot{\epsilon}}.
\] (28)

By using Eqs. (12), (20), (24) and (27), we get in the slow-roll leading-order, written in terms the fundamental parameter \(\delta\)

\[
\epsilon \approx \frac{4}{3} (\delta - \delta^{-1} \delta_c^2)^2,
\] (29)

\[
\eta \approx -\frac{8}{3} (\delta + \delta^{-1} \delta_c^2).
\] (30)

The expressions (29) and (30) make it clear that for \(\delta \ll 1\) we have \(\epsilon \ll 1\) and \(\eta \ll 1\), and therefore we are in an almost de Sitter expansion regime.

The next step is calculating the number of \(e\)-folds \(N\). By using Eqs. (24) and (25) we have

\[
N = \int \frac{3}{H} dt \approx \frac{3}{4} \int \frac{\delta}{\delta - \delta_c^2 \delta^{-1}} d\delta.
\] (31)

where \(\delta_c\) corresponds to \(\delta\) calculated at the end of inflation and it can be obtained by imposing \(\epsilon = 1\). The expression

\(^5\) Note \(\rho_\chi = \frac{1}{2} \chi^2 + V(\chi)\) has a squared mass units because by the adopted choice \(\chi\) is dimensionless.

\(^6\) A slow-roll regime, where the scalar field \(\chi\) lies in the vicinity of the plateau, occurs for \(\delta \ll 1\).
It is also worth noting that in a region where \( \delta_c \ll \delta \Rightarrow \delta_c N \ll 1 \), the slow-roll parameters (35) and (36) can be approximated by

\[
\epsilon \approx \frac{3}{4 N^2} \left[ 1 - \frac{1}{12} \left( \frac{16 \delta_c N}{3} \right)^2 \right],
\]

\[
\eta \approx \frac{2}{N} \left[ 1 + \frac{1}{12} \left( \frac{16 \delta_c N}{3} \right)^2 \right],
\]

where \( \delta_c N \) takes into account the first corrections for Starobinsky inflation.

### C. End of inflation and reheating

In this subsection, we investigate the end of inflation and the reheating period. In a relatively general way, this phase can be divided into two parts: preheating and thermalization. The preheating phase corresponds to the initial reheating phase, where a large number of matter particles are generated from a process known as parametric resonance. This process arises assuming that the transference of energy from the inflaton field to the matter fields occurs when the inflaton coherently oscillates around the minimum of the potential \([37, 38] \). Since preheating is an essentially nonthermal process, a thermalization stage is necessary for the universe to reach a radiation era domain with the matter in thermodynamic equilibrium. For details on the reheating phase, see Refs. [39, 40].

The detailed description of the entire reheating period is complex, as it involves nonperturbative nonequilibrium effects and depends on the interaction processes between the inflaton and the matter fields. However, from a phenomenological perspective, we can study the cosmic dynamics of this period through an equation of state\(^8\)

\[
p = w_{re} \rho(N) \rho,
\]

where the number of e-folds \( N \) takes into account the time dependence of \( w_{re} \). By construction, at the end of reheating, that is, the beginning of the radiation era, the parameter \( w_{re} \) of the equation of state must be \( 1/3 \). At the beginning of reheating, the parameter \( w_{re} \) depends on the behavior of the potential during the phase of coherent oscillations. Expanding the potential (3) around the minimum \( \chi = 0 \), we have

\[
V(\chi) \approx \frac{1}{2} V''(0) \chi^2 = \frac{k_0}{6} \chi^2.
\]

\(^8\) In this subsection, we will adopt the usual fourth mass dimension for \( \rho \). The relation of this quantity with \( \rho_\chi \) is

\[
\rho = \frac{3M_P^2}{2} \rho_\chi,
\]

where

\[
\rho_\chi = \frac{1}{2} \dot{\chi}^2 + V(\chi).
\]

See Eq. (5) for notation details.

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\( R^3 \): It is a unique feature of the model containing \( R^3 \). In fact, for the pure Starobinsky model, this does not happen, since \( \dot{\chi} \) is always nonzero in the attractor line.
Considering the mean behavior of the scalar field $\chi$ around this minimum and taking into account that $V \sim \chi^2$, we obtain $\langle w_{re} \rangle \approx 0$ [4, 41]. Thus, at the beginning of reheating, the universe behaves as in a matter domination era. Therefore, during the reheating phase $w_{re}(N)$ transits from 0 (matter domain) to $1/3$ (radiation era).

An important quantity in the characterization of reheating is the thermalization temperature $T_{re}$ reached at the beginning of the radiation era (end of reheating). This temperature is related to the energy density $\rho_{re}$ through the expression

$$\rho_{re} = \frac{\pi^2}{30} g_{re} T_{re}^4,$$  

(38)

where $g_{re}$ is the effective number of relativistic degrees of freedom at the end of reheating\(^9\). On the other hand, through Eq. (37) and the covariant conservation equation, we can relate $\rho_{re}$ with the energy density $\rho_e$ at the end of inflation

$$\dot{\rho} = -3H\rho[1 + w_{re}(N)] \Rightarrow \rho_{re} = \rho_e e^{-3N_{re}(1+w_e)},$$  

(39)

with the number of $e$-folds $N_{re}$ characterizing the duration of the reheating period and

$$w_e = \frac{1}{N_{re}} \int_{0}^{N_{re}} w_{re}(N) dN.$$  

(40)

Note the quantity $w_e$ represents the mean value of the parameter $w_{re}$ during reheating. Also, for $w_{re}$ monotonically increasing\(^10\), we have

$$0 \leq w_e \leq 1/3,$$  

(41)

so the closer to $1/3$, the more effective the rewarming is. By combining Eqs. (38) and (39) we obtain the reheating temperature in terms of $w_e$ and $N_{re}$

$$T_{re} = \left( \frac{30 \rho_e}{g_{re} \pi^2} \right)^{\frac{1}{4}} e^{-\frac{2}{3}(1+w_e)N_{re}}.$$  

(42)

The next step is by taking the logarithm of Eq. (43), resulting in

$$\ln \left( \frac{k}{a_0 H_k} \right) = -N_k - N_{re} + \ln \left( \frac{a_{re}}{a_{eq}} \right) + \ln \left( \frac{a_{eq}}{a_0} \right),$$  

(44)

with $N_{re} = \frac{\ln \left( \frac{a_{re}}{a_{eq}} \right)}{\ln \left( \frac{a_{eq}}{a_0} \right)}$ and $N_k = \frac{\ln \left( a_{eq}^3 T^3 \right)}{\ln \left( \frac{a_0}{a_k} \right)}$, where this last one is the number of $e$-folds of the inflationary period measured since the exit of the scale $k$ from the horizon. The last two terms of Eq. (44) can be rewritten to take into account the conservation of entropy $S = g_0 a^3 T^3$ in the radiation and matter eras [42, 48]. In this case, we can write

$$T_{eq} = \frac{a_0}{a_{eq}} T_0 \quad \text{and} \quad T_{eq} = \left( \frac{a_{re}}{a_{eq}} \right) \left( \frac{g_{re}}{g_0} \right)^{\frac{1}{4}} T_{re},$$  

(45)

where we consider that the relativistic degrees of freedom in the equivalence era and in the present day are identical, that is, $g_0 = g_{eq}$\(^12\). By substituting Eqs. (42) and (45) in Eq. (44), we get

$$N_{re} = \frac{4}{3} \left( w_e - \frac{1}{3} \right) \left\{ N_k + \ln \left( \frac{\rho_e^{1/4}}{H_k} \right) + \ln \left( \left( \frac{k}{a_0 T_0} \right) \left( \frac{30}{\pi^2} \right)^{\frac{1}{4}} \left( \frac{g_{re} T^4}{g_0^2} \right)^{\frac{1}{2}} \right) \right\}.$$  

(46)

Expressions (46) and (42) that determine the duration and temperature of the reheating are general and hold for a wide range of inflationary models. By particularizing these expressions for our model, it is necessary to explicitly determine the energy density at the end of inflation.

In single-field models, $\rho_e$ can be written in terms of the potential $V_e = (3M_{Pl}^2/2) V(\chi_e)$ given in Eq. (5). By imposing $\epsilon = 1$ (end of inflation) we obtain by Eqs. (11), (12) and (28) that $V(\chi_e) \approx \chi_e^2$. Thus,

$$\rho_e = \frac{3}{2} V_e = \frac{9M_{Pl}^4}{4} V(\chi_e).$$  

(47)

By Eq. (23), we see that $a_0$ is a typically small slow-roll second-order quantity. Thus, at the end of inflation, terms like $4a_0 e^{\chi_e}$ appearing in the potential are negligible and Eq. (3) can be approximated by the Starobinsky inflation potential

$$V(\chi_e) \approx \frac{K_0}{6} \left( 1 - e^{-\chi_e} \right)^2.$$  

(48)

\(^9\) By considering only particles from the standard model we have $g_{re} = 106.75$ [42].

\(^10\) Detailed numerical modeling usually indicates a monotonic behavior for the reheating equation of state [43–45].

\(^11\) Note that we are ignoring the dark energy era domain since it is only relevant very close to the present day.

\(^12\) By considering only particles from the standard model and disregarding the neutrinos masses, we have $g_0 = 3.94$ [42].
An estimate for $\chi_e$ is obtained by imposing $\epsilon = 1$ on the approximate slow-roll expression for the parameter $\epsilon$
\[
\epsilon \approx \frac{1}{3} \left( \frac{V'/(\chi_e)}{V(\chi_e)} \right)^2 = 1 \Rightarrow e^{-\chi_e} = 2\sqrt{3} - 3. \tag{49}
\]

Thus, substituting Eqs. (49) and (48) into Eq. (47), we obtain the energy density at the end of inflation
\[
\rho_e \approx \frac{3\zeta^4}{4} \left( 2 - \sqrt{3} \right)^2 M_{Pl}^4, \tag{50}
\]
where the dimensionless parameter $\zeta$ is defined as
\[
\zeta \equiv \left( \frac{\kappa_0}{M_{Pl}} \right)^{\frac{4}{7}}. \tag{51}
\]

Therefore, considering $H_k$ in the slow-roll leading-order ($H_k^2 \approx \kappa_0/12$) and substituting Eq. (50) in Eqs. (46) and (42) we get
\[
N_{re} = \frac{4}{3(\omega_a - \frac{1}{3})} \left[ N_k + 0.79 + \ln \left( \frac{k}{a_0 T_0^4} \right) + \frac{1}{12} \ln \left( \frac{g_{re}}{g_0} \right) \right], \tag{52}
\]
and
\[
T_{re} = \frac{0.64 \zeta}{g_{re}^{1/4}} e^{-\frac{4}{7}(1+\omega_a)N_{re}M_{Pl}}. \tag{53}
\]
We will see in the next section how constraints on $N_{re}$ and $T_{re}$ restrict the range of the number of $e$-folds $N_k$ of inflation.

V. OBSERVATIONAL CONSTRAINTS

Based on the developments of the previous sections, we now establish constraints to the proposed inflationary model. These constraints follow different approaches and are related to periods before, during, and after the inflationary regime.

A. Range in the number of $e$-folds $N_k$ of inflation

The description of the reheating period as presented in Sec. IV C with the assumption of later eras of radiation and matter dominance allows us to establish an interval for the number of $e$-folds $N_k$ of inflation. For this, it is necessary to provide some details about the minimum characteristics of the reheating period to be considered.

The cosmological period comprising energy scales above $10^4 \text{ GeV}$ until the end of inflation $10^{15} - 10^{16} \text{ GeV}$ is an uncertain period for particle physics. In fact, extensions of the standard model together with possible nonminimal couplings can provide extra processes of coupling the inflaton with the matter fields. However, even if these new processes are not present (or are not relevant for reheating) we can adopt as a minimum hypothesis the usual couplings of the standard matter fields with gravity. In this context, already in the Einstein frame, this means that the inflaton field $\phi$ decays predominantly in the Higgs doublet $h$, and in next-to-leading order, in a pair of gluons. By considering these two main decay channels, the inflaton decay rate is given by $\Gamma_\phi$ [49, 50]
\[
\Gamma_\phi \approx \frac{1}{24\pi} \left[ (1 - 6\xi)^2 + \frac{49\alpha_s^2}{4\pi^2} \right] M_{Pl}^4, \tag{54}
\]
where $\xi$ is the nonminimum coupling constant between the Higgs field and gravitation, that is $\xi |h|^2 R$, $\alpha_s$ is the coupling constant of QCD on the reheating energy scale [14] and $m_\phi$ is the effective mass of the inflaton, given by
\[
m_\phi^2 = V''(0) = \frac{\zeta^4}{3} M_{Pl}^2. \tag{55}
\]

The next step is determining $\zeta$ in terms of the model parameters and the scalar amplitude $A_s$ measured by the Planck satellite [53]. Considering the pivot scale $k = 0.002 \text{ Mpc}^{-1}$ as the scale of interest [13], we have in slow-roll-leading-order [54]
\[
A_s = \frac{3}{16\pi^2 M_{Pl}^2} \frac{V_k^3}{V_k'}, \tag{56}
\]
where the scalar amplitude $A_s^{0.002} = 2.3 \times 10^{-9}$ is obtained from the expression $A_s^k = A_s^k (k/k^*)^{n_s-1}$ taking into account that $A_s^k = 2.1 \times 10^{-3}$, $k^* = 0.05 \text{ Mpc}^{-1}$ and $n_s = 0.9665$. See Ref. [53] for details.

Thus, substituting Eqs. (24) and (25) into Eq. (56), we obtain
\[
\kappa_0 \approx 27\pi^2 M_{Pl}^2 A_s^{0.002} \left( \delta_e - \frac{1}{3} \delta_e^{-1} \alpha_0 \right)^2. \tag{57}
\]
Furthermore, we can rewrite the last equation in terms of the number of $e$-folds $N_k$ and the parameter $\zeta$ given in Eqs. (34) and (51), respectively
\[
\zeta^4 \approx 211\pi^2 A_s^{0.002} \left( \frac{\delta_e \exp \left( \frac{8}{3} \delta_e N_k \right)}{\exp \left( \frac{4}{3} \delta_e N_k \right) - 1} \right)^2, \tag{57}
\]
remembering that $\alpha_0 \approx 3\delta_e^2$. In the case of $\delta_e \to 0$, we recover the well-known relation of the Starobinsky model $\zeta^4 \approx 72\pi^2 A_s^{0.002} N_k^{-2}$.

By having determined the decay rate of the inflaton, we can estimate the reheating temperature $T_{re}$ as follows: the thermal equilibrium of the cosmic fluid is reached when $\Gamma_\phi \sim H$ [37]. Thus, the reheating temperature can be obtained from $3M_{Pl}^2 H_{re}^2 = \rho_{re} = \pi^2 g_{re} T_{re}^4/30$, which results in
\[
T_{re} \approx 0.5 \sqrt{\Gamma_\phi M_{Pl}} \approx 6 \times 10^{16} \zeta^3 \left( 1 - 6\xi \right)^2 + \frac{49\alpha_s^2}{4\pi^2} \text{ GeV}, \tag{58}
\]
13 The nonminimum coupling is necessary due to issues of the renormalizability of scalar fields in curved spaces [51].
14 By extrapolating the experimental results of the running coupling of $\alpha_s$ [52], we can estimate that during the reheating period $0.01 \lesssim \alpha_s \lesssim 0.03$.
15 We use this pivot scale in agreement with the observations presented in Refs. [2, 55] associated with observations $n_s \times r_{0.002}$. 
where we use \( g_{te} = 106.75 \) and \( M_{Pl} = 2.4 \times 10^{18} \) GeV. Note the value of \( T_{re} \) above depends on the inflationary parameters through \( \zeta \). Substituting Eq. (57) into Eq. (58), we obtain

\[
T_{re} \approx 3.4 \times 10^{13} \left\{ \frac{\delta_c \exp \left( \frac{8}{3} \delta_c N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right\}^\frac{3}{2} \times \sqrt{(1 - 6\xi)^2 + \frac{49\alpha^2}{4\pi^2}} \text{ GeV}.
\]  

(59)

Based on Eq. (59), we can estimate the order of magnitude of the reheating temperature. Considering \( N_k = 50 \) and \( \delta_c = 0 \) (Starobinsky), we have [49]

\[
T_{re} \sim \begin{cases} 
10^{10} & |1 - 6\xi| \text{ GeV if } \xi \neq 1/6 \\
10^8 & \text{ GeV if } \xi = 1/6 
\end{cases},
\]

where we adopt \( \alpha_s \approx 10^{-2} \). Although \( T_{re} \) depends on the value of \( \xi \), the expression above gives a minimum estimate for \( T_{re} \). In fact, even though the Higgs field has a conformal coupling with gravitation, that is \( \xi = 1/6 \), the inflaton decay channel in two gluons gives \( T_{re} \approx 10^8 \) GeV.

The previous construction allows establishing a minimum temperature \( T_{re}^{\text{(min)}} \) relatively independent of the details of the reheating phase. Even considering minimal hypotheses for reheating and a conformal coupling of the Higgs field with gravitation, we have

\[
T_{re}^{\text{(min)}} \approx 3.4 \times 10^{13} \left( \frac{7\alpha_s}{2\pi} \right) \left[ \frac{\delta_c \exp \left( \frac{8}{3} \delta_c N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right]^\frac{3}{2} \text{ GeV}.
\]

\[
\approx 3.8 \times 10^{11} \left[ \frac{\delta_c \exp \left( \frac{8}{3} \delta_c N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right]^\frac{3}{2} \text{ GeV}.
\]  

(60)

Since we characterized the reheating phase, let us see how the expressions of \( N_{re} \) and \( T_{re} \), obtained at the end of Sec. IV C provide restrictions for the number of e-folds \( N_k \).

By considering the scale of interest \( k = 0.002 \) Mpc\(^{-1}\), \( T_0 = 2.73 \) K, \( g_{te} = 106.75 \) and \( g_0 = 3.94 \), we can rewrite Eq. (52) as

\[
N_{re} = \frac{4}{1 - 3w_a} \left[ 64.36 - N_k + \ln \left( \zeta \right) \right]
= \frac{4}{1 - 3w_a} \left[ 61.87 - N_k + \frac{1}{2} \ln \left( \frac{\delta_c \exp \left( \frac{8}{3} \delta_c N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right) \right] (61)
\]

The upper limit for \( N_k \) is obtained taking into account that for physical consistency \( N_{re} \geq 0 \). So, due to Eq. (41), we have

\[
N_k \leq \frac{1}{2} \ln \left[ \frac{\delta_c \exp \left( \frac{8}{3} \delta_c N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right] \leq 61.87. \quad (62)
\]

Table I shows the upper limits of \( N_k \) for different values of \( \alpha_0 \).

| \( \alpha_0 \) | \( \delta_c \) | \( N_k \) |
| --- | --- | --- |
| 0 | 0 | 58.99 |
| 10^{-3} | 1.85 \times 10^{-4} | 58.99 |
| 10^{-4} | 5.8 \times 10^{-4} | 58.92 |
| 5 \times 10^{-3} | 1.43 \times 10^{-2} | 58.69 |
| 10^{-3} | 1.83 \times 10^{-2} | 58.45 |

In turn, the lower limit for the number of e-folds \( N_k \) is obtained taking into account that \( T_{re} \approx T_{re}^{\text{(min)}} \). Thus, by using Eqs. (53), (57), (60) and (61), we have

\[
\exp \left( \frac{16}{3} \delta_c N_k \right) \leq 1 \quad \exp \left( -3 \left( 1 + w_a \right) \bar{N}_{re} \right) \gtrsim \left( \alpha_0 \right)^{10^{-5}}, \quad (63)
\]

where

\[
\bar{N}_{re} = 61.87 - N_k + \frac{1}{2} \ln \left[ \frac{\delta_c \exp \left( \frac{8}{3} \delta_c N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right]. \quad (64)
\]

By expression (64), we see that as \( N_k \) decreases, \( \bar{N}_{re} \) increases and consequently the exponential that contains \( \bar{N}_{re} \) reduces the value of the left-hand side (lhs) of the expression (63). The decay speed of this exponential depends on the coefficient

\[
3 < \frac{3 \left( 1 + w_a \right)}{1 - 3w_a} \leq 65.079.
\]

where the lower and upper limits were established from Eq. (41). Thus, for the lower limit in \( N_k \) to be robust, i.e., relatively independent of the reheating period details, we must adopt \( w_a \to 0 \) in Eq. (63).\(^{18}\) In this case, the expression (63) is rewritten as

---

\(^{16}\) The debate over the range of allowable values for \( \xi \) is complex and is directly related to the stability of the Higgs field at high energies. This analysis depends on several factors such as the mass value of the top quark and the running coupling of \( \xi \) in a Friedmann background [55, 56].

\(^{17}\) Recovering the units, we have \( \log \left( \frac{k}{m_{Pl}^3} \right) = \ln \left( \frac{k}{m_{Pl}^3} \frac{\pi^2}{hB} \right) = -65.079. \)

\(^{18}\) Note that when \( w_a \to 0 \), the thermalization process proceeds as slowly as possible.
Table II shows lower limits of \( N_k \) for different values of \( \alpha_0 \).

Table II. Minimum values for \( N_k \) considering different values of \( \alpha_0 \).

| \( \alpha_0 \)    | \( \delta_c \) | \( N_k \)  |
|------------------|----------------|------------|
| \( 10^{-3} \)    | 0              | 53.32      |
| \( 10^{-4} \)    | 1.85 \times 10^{-3} | 53.31     |
| \( 5 \times 10^{-4} \) | 5.8 \times 10^{-3} | 53.23     |
| \( 10^{-3} \)    | 1.3 \times 10^{-2} | 52.91     |
|                  | 1.83 \times 10^{-2} | 52.57     |

Although it is not a very restrictive range, it was obtained from very general considerations. In this sense, the above result is quite robust and little dependent on the details of the reheating, radiation and matter eras that characterize the postinflationary universe. This range of \( N_k \) will be used in the next sections.

The previous modeling was developed using the thermal equilibrium temperature. However, in some inflationary models, the reheating temperature \( T_{re} \) is established long after the equilibrium between the decay rate \( \Gamma_c \) and the expansion \( H \) [57–59]. In this case, we can still obtain a result identical to Eq. (66) but using a different approach. We can treat the problem not through a reheating temperature \( T_{re} \), given in Eq. (58), but through a quasithermal energy density and decay rate of a universe in a quasithermal phase. In this case, the relation \( \rho_{qt} = 3M_{Pl}^2 \Gamma_c^2 \) is preserved, and similarly to what was developed previously, we can establish a minimum quasithermal energy density

\[
\rho_{qt}^{(min)} \approx 10^{48} \left[ \frac{\delta_c \exp \left( \frac{\delta_c}{3} N_k \right)}{\exp \left( \frac{16}{3} \delta_c N_k \right) - 1} \right]^6 \text{GeV}^4,
\]

where \( \delta_t \) is the slow-roll parameter.

In most cases, the relevant perturbations produced by inflationary models are the scalar and tensor ones. While the scalar perturbations are responsible for generating the primordial inhomogeneities, the tensor ones (gravitational waves) carry information directly from the inflationary period. In the simplest models involving only a canonical scalar field minimally coupled with general relativity\(^{19}\), the scalar and tensor perturbations are dominantly characterized by four parameters: scalar and tensor amplitudes \( A_s \) and \( A_t \), respectively; and scalar and tensor spectral indices, \( n_s \) and \( n_t \) respectively. Of these four parameters, only three are independent, since \( n_t \) can be written as a combination of \( A_s \) and \( A_t \).

From the point of view of current observations, the anisotropies in the CMB measure scalar quantities and establish an upper limit for tensor quantities \([2, 35]\). In this context, it is convenient to describe the upper constraint with the tensor-scalar ratio \( r \equiv A_t/A_s \).

In the slow-roll leading-order, we have \([54, 61]\)

\[
n_s = 1 + \eta - 2\epsilon \quad \text{and} \quad r = 16\epsilon,
\]

where \( \epsilon \) and \( \eta \) are the slow-roll parameters.\(^{20}\) For the proposed model, \( \epsilon \) and \( \eta \) are given in Eqs. (35) and (36) and therefore they depend on the number of \( e \)-folds \( N_k \) and the parameter \( \alpha_0 \) \( \approx 3.37 \times 10^{-11} \).

Figure 3 shows the parameter space \( n_s \times r_{0.002} \) containing the observational constraints (blue) obtained from Ref. \([35]\) and the theoretical evolution of the model (light green) obtained from Eq. (68). The analysis is done considering the interval in Eq. (66) for the number of \( e \)-folds.

The black dots in Fig. 3 represent the Starobinsky solution, where \( \alpha_0 = 0 \). As \( \alpha_0 \) increases, the region predicted by the model (light green) shifts to the left and slightly downwards. The 95\% C.L. is represented by the straight line joining the gray dots. This boundary establishes maximum values allowed for \( \alpha_0 \) being the largest of them \( 8.7 \times 10^{-5} \). Thus, we can consider that the CMB observations upper limit the value of \( \alpha_0 \) to \( 10^{-4} \).

Similar results were obtained in Refs. \([28, 32]\) by considering the \( \alpha_k^2 \) term as a small correction to Starobinsky inflation.

C. Initial conditions

As shown in Sec. III, the existence of a maximum point in the potential limits the initial conditions that drive inflation. In this sense, the purpose of this subsection is to estimate how severe this limitation is.

\(^{19}\) This is exactly our case when described in Einstein’s frame.

\(^{20}\) The scalar amplitude \( A_s \) is given by Eq. (56).
Substituting Eqs. (16) and (57) into Eq. (69), we obtain

\[
R_{\text{max}}(\alpha_0, N_k) \approx 1.5 \times 10^{-5} \alpha_0 \left[ \frac{\exp \left( \frac{4}{5} \sqrt{\frac{\pi}{N_k}} N_k \right)}{\exp \left( \frac{10}{3} \sqrt{\frac{\pi}{N_k}} N_k \right) - 1} \right]^2 \times \left( \frac{-1 + \sqrt{1 + 12\alpha_0 + 4\sqrt{5\alpha_0}}}{2\alpha_0} \right) M_{Pl}^2.
\]

As \(\alpha_0\) decreases, \(R_{\text{max}}\) increases and at the limit \(\alpha_0 \to 0\) (Starobinsky), we obtain \(R_{\text{max}} \to \infty\). In the following table, we show \(R_{\text{max}}\) for different values of \(\alpha_0\) considering \(N_k = 52\) and \(N_k = 59\):

| \(\alpha_0\) | \(R_{\text{max}}(M_{Pl})\) | \(\frac{R_{\text{max}}(M_{Pl})}{R_{\text{max}}(M_{Pl})}\) |
|-----------|-----------------|------------------|
| 0         | \(\infty\)      | \(\infty\)       |
| \(10^{-1}\) | 1.01            | 0.79             |
| \(10^{-2}\) | \(3 \times 10^{-7}\) | \(2 \times 10^{-7}\) |
| \(10^{-3}\) | \(8 \times 10^{-8}\) | \(6 \times 10^{-8}\) |

The probabilistic analysis that determines whether a generic \(R_{\text{ini}}\) produces physical inflation is not trivial. In fact, this analysis depends on the probability distribution of the curvature scalar in preinflationary models. However, if a uniform distribution is assumed for \(R_{\text{ini}}\), the results in Table III indicate severe limitations for the allowed initial conditions. For example, for \(\alpha_0 \sim 10^{-3}\), we must have \(R_{\text{ini}} \lesssim 10^{-7} M_{Pl}\), which corresponds to a probability of \(10^{-5}\)% that a generic \(R_{\text{ini}}\) leads to a physical inflationary regime. This probability increases as \(\alpha_0\) decreases. However, for very small \(\alpha_0\), the \(R^3\) term in the action becomes negligible and the model essentially behaves like the Starobinsky one. Note that this kind of limitation can be avoided if the pre-inflationary model has some dynamic mechanism that delays the start of inflation such that \(R_{\text{ini}}\) is always smaller than \(R_{\text{max}}\).

In addition to an upper limit, the initial condition \(R_{\text{ini}}\) also has a lower limit related to the need for a sufficiently long inflationary period \(N\). In fact, due to the constraint (66), the inflationary regime must satisfy the condition \(N > N_k\). By understanding the restrictions imposed by this condition, let us consider the region of the attractor line where physical inflation occurs. We know that the total "length" of this region is \(\chi_c - \chi_e\) where \(\chi_e\) is given by Eq. (49). On the other hand, we can select a subregion in which the number of \(e\)-folds is greater than or equal to a fixed \(N_k\). The length of this subregion is given by \(\chi_c - \chi_e\). So we define a probability function

\[
P(\alpha_0, N_k) = \frac{\chi_c - \chi_e}{\chi_e - \chi_c},
\]

which (approximately) measures how likely arbitrary initial conditions lead to an inflation with a number of \(e\)-folds \(N\) greater than or equal to \(N_k\). Note that in this analysis, we are excluding the entire set of initial conditions that lead \(\chi\) to the attractor line to the right side of the critical point \((\chi_e, 0)\). See Fig. 2 for details.

Substituting Eqs. (16), (34) and (49) into Eq. (70), we get

\[
P(\alpha_0, N_k) \approx \frac{\ln \left[ \frac{\exp \left( \frac{4N_k}{5} \sqrt{\frac{\pi}{N_k}} \right) + 1}{\exp \left( \frac{8N_k}{3} \sqrt{\frac{\pi}{N_k}} \right) - 1} \right]}{\ln \left( \frac{6 - 3\sqrt{5}}{\sqrt{\alpha_0}} \right)}.
\]

\[21\] For this estimate, we consider that \(R_{\text{ini}}/M_{Pl}^2\) has an equal probability of assuming a value between 0 and 1. The upper limit of 1 takes into account the ignorance of the theory of gravitation on the Planck scale.
In Table IV, we calculate $P(\alpha_0, N_\chi)$ for different values of $\alpha_0$ considering $N_\chi = 52$ and $N_\chi = 59$:

| $\alpha_0$ | $P(\alpha_0, 52)$ | $P(\alpha_0, 59)$ |
|------------|-------------------|-------------------|
| 0          | $\infty$         | $\infty$         |
| $10^{-10}$ | 0.83             | 0.82             |
| $10^{-9}$  | 0.37             | 0.35             |
| $10^{-8}$  | 0.22             | 0.19             |

The results in Table IV show that the probability of obtaining an inflationary period with at least $N_\chi$ e-folds increases as $\alpha_0$ decreases. Even so, for $\alpha_0 \sim 10^{-5}$ this probability does not reach 40%.

Still related to the initial conditions, a third point to be highlighted concerns the inflationary regime in the vicinity of the critical point $\chi_c$. In that neighborhood, quantum fluctuations dominate over the classical slow-roll regime, and the universe reaches an eternal inflation regime [62, 63].

According to Ref. [64] quantum fluctuations dominate whenever

$$\pi^2 M_p^2 \frac{\chi^2}{H^4} < 1. \quad (72)$$

In the slow-roll leading-order regime, we can rewrite the above condition as [65]

$$\frac{9}{8\pi^2 M_p^2} \frac{V^3}{V''} > 1. \quad (73)$$

Substituting Eqs. (24), (25) and (57) in Eq. (73), we get

$$\left| \delta - \frac{\alpha_0}{3} \right| < Q \sqrt{\frac{\alpha_0}{3}}, \quad (74)$$

where

$$Q \equiv \sqrt{24 A_k^{0.062} \left[ \exp \left( \frac{8}{3} \sqrt{\frac{\pi}{25}} N_\chi \right) \exp \left( \frac{16}{3} \sqrt{\frac{\pi}{25}} N_\chi \right) - 1 \right]}.$$ 

By analyzing the condition (74), we conclude that the eternal inflation regime is reached whenever $\delta = e^{-\chi}$ is within the interval

$$\delta^{(2)} < \delta < \delta^{(1)}, \quad (75)$$

where

$$\delta^{(1)} = \left( Q + \sqrt{Q^2 + 1} \right) \sqrt{\frac{\alpha_0}{3}},$$

$$\delta^{(2)} = \left( -Q + \sqrt{Q^2 + 1} \right) \sqrt{\frac{\alpha_0}{3}}.$$ 

Considering the intervals $52 \leq N_\chi \leq 59$ and $\alpha_0 \leq 10^{-4}$, a numerical analysis shows that the values of $\delta^{(1)}$ and $\delta^{(2)}$ depend only on the value of $\alpha_0$. In Table V, we calculate $\delta^{(1)}$ and $\delta^{(2)}$ for different values of $\alpha_0$.

Table V. Calculation of $\delta^{(1)}$ and $\delta^{(2)}$ for different values of $\alpha_0$ considering $N_\chi = 52$ and $\delta^2_c = \alpha_0/3$.

| $\alpha_0$ | $\delta^2_c$ | $\delta^{(2)}$ | $\delta^{(1)}$ |
|------------|-------------|----------------|----------------|
| $10^{-14}$ | $0.8 \times 10^{-14}$ | $2 \times 10^{-14}$ | $1.7 \times 10^{-14}$ |
| $10^{-10}$ | $0.8 \times 10^{-10}$ | $2 \times 10^{-10}$ | $1.7 \times 10^{-10}$ |
| $10^{-6}$  | $0.8 \times 10^{-6}$ | $2 \times 10^{-6}$ | $1.7 \times 10^{-6}$ |

The results in Table V show that for relevant values of $\alpha_0$ (typically $\alpha_0 \gtrsim 10^{-3}$), eternal inflation occurs only if $\chi$, when reaching the attractor line, is very close to $\chi_c$.

The discussions carried out previously allow us to conclude that the presence of an $R^3$ term as a correction to the Starobinsky model qualitatively changes the initial characteristics of inflation. If without the $R^3$ term (almost) any initial condition leads to an inflationary regime, with it, some kind of fine-tuning is necessary. The accuracy of this fine-tuning is highly uncertain and depends on the probabilistic structure of the pre-inflationary model. Even so, the attractive idea of chaotic inflation [4, 9] is lost with the inclusion of the $R^3$ term regardless of the value of $\alpha_0$.

Another possible scenario that must be taken into account is the one in which the inflaton field provides different initial conditions for different regions of space. In this scenario, the existence of an interval in phase space (however small it may be) that produces an eternal inflation regime becomes quite relevant. That occurs because in this regime an arbitrarily large volume is generated. Thus, different regions of space produce different causally disconnected "universes" achieving the multiverse idea [65]. In this context, it is reasonable to assume that we live in one of these universes even though the probability of having initial conditions that lead to eternal inflation is extremely low. Therefore, in such a scenario, the inclusion of the term $R^3$ in the Starobinsky model does not generate the need for a fine-tuning for physical inflation to take place.

VI. FINAL COMMENTS

In this paper, motivated by the success of the Starobinsky model in providing a consistent inflationary regime supported by the most current observational data, we propose an extension to it characterized by adding the higher-order term $R^3$ in the action.

We developed a complete study within the cosmological background. We analyzed the potential and the phase space of the model so that we observed the existence of three regions with different dynamics for the scalar field $\chi$. Such analysis enabled us to qualitatively verify that the introduction of the $R^3$ term in Starobinsky’s action restricts the initial conditions that lead the system to a physical inflation. We described the dynamics of the scalar field in each of the three regions, namely, the asymptotic region for $V(\chi) \to 0$ when
χ ≫ 1, the plateau region, where the slow-roll inflationary regime occurs, and the region where the reheating phase takes place. While the dynamics along the plateau enabled us to build the quantities with which we can compare the model with the observations, the reheating phase together with the minimal hypothesis of the usual couplings between the standard matter fields and gravity allowed us to obtain a restrictive minimal hypothesis of the usual couplings between the standard matter fields and gravity. Thus, we duly confronted the model with observational data from Planck, BICEP3/Keck, and BAO. We saw that as the parameter α₀ increases, the region in space $n_s \times r$ predicted by the model shifts to the left and slightly downwards. Furthermore, we can consider that observations from CMB anisotropies upper limit the value of α₀ to $10^{-4}$.

It is worth qualitatively commenting on the role that a higher-order $R^n$ term with $n > 3$ plays in Starobinsky $+ R^n$ models. Assuming that the higher-order $R^n$ term is a small correction to Starobinsky, the author of Ref. [28] finds that its associated potential is similar to the model discussed in our paper. In this case, we can say that in a classical context the existence of a maximum critical point with finite $\chi_c > 0$ in the potential may lead to a problem of fine-tuning in the initial conditions. On the other hand, the contribution of such higher-order $R^n$ terms becomes negligible for increasing values of $n$ as shown in Refs. [28, 66].

We concluded with the discussion about the limitation that the inclusion of an $R^3$ term imposes on the initial conditions: while in the Starobinsky model, practically any initial conditions lead the system to an inflationary regime, the inclusion of chaotic inflation is lost for a non-negligible value of α₀. On the other hand, we saw that in a scenario where the initial conditions are such that they produce an eternal inflation regime, the idea of a multiverse is achieved. In this case, the inclusion of the $R^3$ term in the Starobinsky model does not require a fine-tuning of the initial conditions for the occurrence of physical inflation.

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Appendix A: Einstein’s frame

In this appendix, we rewrite the action

$$S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left( R + \frac{1}{2\kappa_0} R^2 + \frac{\alpha_0}{3\kappa_0^2} R^3 \right), \quad (A1)$$

in Einstein’s frame.

We start by writing a second action in the form

$$\ddot{S} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-\bar{g}} \left[ \kappa_0 \left( \lambda + \frac{1}{2} \lambda^2 + \frac{\alpha_0}{3} \lambda^3 \right) + \mu \left( \frac{R}{\kappa_0} - \lambda \right) \right], \quad (A2)$$

where $\mu$ is a Lagrange multiplier. By taking the variation with respect to $\mu$, we obtain

$$\lambda = \frac{R}{\kappa_0},$$

which shows that $S = \ddot{S}$. Furthermore, taking the variation with respect to $\lambda$ we have

$$\mu = \kappa_0 \left( 1 + \lambda + \alpha_0 \lambda^2 \right). \quad (A3)$$

The next step is inverting Eq. (A3) and get $\lambda$ as a function of $\mu$. By rewriting (A3), we get the quadratic equation

$$\alpha_0 \lambda^2 + \lambda + 1 - \frac{\mu}{\kappa_0} = 0,$$

whose solution is

$$\lambda = \frac{-1 \pm \sqrt{1 - 4\alpha_0 \left( 1 - \frac{\mu}{\kappa_0} \right)}}{2\alpha_0}.$$  

For the limit $\alpha_0 \to 0$ to be well defined, we must choose the positive sign. Thus

$$\lambda = \frac{-1 + \sqrt{1 - 4\alpha_0 \left( 1 - \frac{\mu}{\kappa_0} \right)}}{2\alpha_0},$$

where we define $\bar{\mu} \equiv \mu/\kappa_0$. Then we must substitute $\lambda$ in Eq. (A2). So, using the quadratic equation itself, we obtain

$$\ddot{S} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-\bar{g}} \left\{ \bar{\mu} R - \kappa_0 \lambda \left[ \frac{2}{3} (\bar{\mu} - 1) - \frac{1}{6} \lambda \right] \right\}.$$  

By working only with the term that depends on $\lambda$, we can get

$$\dot{S} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left\{ \bar{\mu} R + \frac{\kappa_0}{24\alpha_0^2} \left[ -1 + \sqrt{1 - 4\alpha_0 (1 - \bar{\mu})} \right] \left[ -1 + 8\alpha_0 (1 - \bar{\mu}) + \sqrt{1 - 4\alpha_0 (1 - \bar{\mu})} \right] \right\}. $$

Therefore,

$$\dot{S} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left\{ \bar{\mu} R + \frac{\kappa_0}{24\alpha_0^2} \left[ -1 + \sqrt{1 - 4\alpha_0 (1 - \bar{\mu})} \right] \left[ -1 + 8\alpha_0 (1 - \bar{\mu}) + \sqrt{1 - 4\alpha_0 (1 - \bar{\mu})} \right] \right\}. $$
Then we must carry out the transformations
\[ \bar{\mu} = e^\chi \text{ and } \bar{g}_{\mu\nu} = e^{2\chi} g_{\mu\nu}, \]
in the action \( \bar{S} \). By using the results of Appendix D of Ref. [67], concerning the change in Ricci tensor and scalar curvature due to a conformal transformation, we get
\begin{align*}
\bar{S} & = \frac{M^2_{Pl}}{2} \int d^4x \sqrt{-\bar{g}} \left\{ \bar{R} - \frac{3}{2} e^{-\chi} \bar{g}^{\alpha\bar{\beta}} \partial_\alpha \chi \partial_{\bar{\beta}} \chi + 3e^{-2\chi} \Box e^\chi + \right. \\
& + \frac{\kappa_0 e^{-2\chi}}{24 \alpha_0^2} \left( -1 + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right) \left[ -1 + 8\alpha_0 (1 - e^\chi) + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right] \right\}.
\end{align*}
Furthermore,
\[ e^{-2\chi} \Box e^\chi = e^{-2\chi} \nabla_\rho \nabla^\rho e^\chi = \frac{1}{\sqrt{-\bar{g}}} \partial_\rho \left( g^{\rho\sigma} \sqrt{-\bar{g}} \partial_\sigma e^\chi \right), \]
is a surface term and it can be neglected. Thus,
\[ \bar{S} = \frac{M^2_{Pl}}{2} \int d^4x \sqrt{-\bar{g}} \left\{ \bar{R} - \frac{3}{2} \frac{\partial^2}{\partial^2} \chi \partial_\alpha \chi + \frac{\kappa_0 e^{-2\chi}}{24 \alpha_0^2} \left( -1 + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right) \left[ -1 + 8\alpha_0 (1 - e^\chi) + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right] \right\}. \]
We can write this expression as
\[ S(\bar{g}_{\mu\nu}, \chi) = \frac{M^2_{Pl}}{2} \int d^4x \sqrt{-\bar{g}} \left[ \bar{R} - 3 \left( \frac{1}{2} \bar{g}^{\rho\sigma} \partial_\rho \chi + V(\chi) \right) \right], \]
with
\[ V(\chi) = \frac{\kappa_0}{72 \alpha_0} e^{-2\chi} \left( 1 - \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right) \left[ -1 + 8\alpha_0 (1 - e^\chi) + \sqrt{1 - 4\alpha_0 (1 - e^\chi)} \right]. \]
It is interesting noting that the Starobinsky limit \( \alpha_0 \to 0 \) is well defined. It follows that
\[ \lim_{\alpha_0 \to 0} V(\chi) = \frac{\kappa_0}{6} (1 - e^{-\chi})^2, \]
as expected.

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