Tail Behaviour of Mexican Needlets

Claudio Durastanti

Department of Mathematics, University of Rome "Tor Vergata"

May 11, 2014

Abstract

In this paper we study the tail behaviour of Mexican needlets, a class of spherical wavelets introduced by [9]. In particular, we provide an explicit upper bound depending on the resolution level $j$ and a parameter $s$ governing the shape of the Mexican needlets.

Keywords and phrases: Wavelets, Mexican Needlets, Sphere, Localization.

AMS Classification: 42C40, 42C10, 42C15.

1 Introduction

A lot of interest has recently been focussed on various forms of spherical wavelets, see for instance [2], [3], [5], [11], [16], [21] and the references therein. This interest has also been fuelled by strong applied motivations, for instance in Astrophysics and Cosmology, cfr. for example [15]. Many theoretical and applied papers have been concerned, in particular, with so-called spherical needlets, which were introduced into the Functional Analysis literature by [16], [17]. Loosely speaking, the latter can be envisaged as a convolution of the spherical harmonics with a weight function which is smooth and compactly supported in the harmonic domain (more details will be given below). Localization properties in this framework were fully investigated by [16], [17].

*e-mail address: durastan@mat.uniroma2.it
†This research is supported by European Research Council Grant n. 277742 Pascal
Needlets have been recently generalized in various directions: for instance, spin needlets (see [6]) and mixed needlets (cfr. [7]) for spin fiber bundles, needlets were developed on the unitary ball in [18], while this framework has been also extended to allow for an unbounded support in the frequency domain by [9], see also [8], [10]; the latter construction is usually labelled Mexican needlets. It is to be noted that Mexican needlets can be considered asymptotically equivalent to the Spherical Mexican Hat Wavelets (SMHW), currently the most popular wavelet procedure in the applied Cosmological literature (see again [15]). Examples of applications, again related to the study of CMB radiation, can be found in [4], [12], [14], [19]. As described in details below, Mexican needlets enjoy tremendous localization properties in the real domain; in this paper, we investigate the relationship between the tail decay and the exact shape of the weight function. Indeed, the aim of this work is to provide analytic expressions to bound the tail behaviour in the real domain: we prove the tails are Gaussian up to a polynomial term, whose dependence on the choice of the kernel can be identified explicitly. In particular, we shall consider wavelet filters of the form

$$\Psi_{\varepsilon,s}(\vartheta) := \frac{1}{4\pi} \sum_{l=0}^{+\infty} (\varepsilon l)^{2s} \exp\left(- (\varepsilon l)^2\right) (2l+1) P_l(\cos \vartheta)$$

where \(P_l(\cdot)\) denotes the standard Legendre polynomial of degree \(l\). We shall be able to show that

$$|\Psi_{\varepsilon,s}(\vartheta)| \leq C_s \varepsilon^{-\frac{(\vartheta^2)}{\varepsilon^2}} \left(1 + \left|H_{2s} \left(\frac{\vartheta}{\varepsilon}\right)\right|\right),$$

where \(H_{2s}(\cdot)\) identifies the Hermite polynomial of degree \(2s\), cfr. Theorem 1 below.

It is important to remark that in [9] the authors obtained an analogous expression for the \(n\)-dimensional sphere, limiting their investigation to the case of the shape parameter \(s = 1\). In this paper we will extend this bound for any choice of \(s \in \mathbb{N}\); our argument exploits a technique similar to the one used by Narcowich, Petrushev and Ward in [16] (see also [13], [17]). In our proof, we will also exploit the analytic form of the weight function to compute exactly its Fourier transform in terms of Hermite polynomials; this will also allow us to investigate explicitly the roles of the resolution level \(j\) and of the shape parameter \(s\).
The plan of this paper is as follows. In Section 2 we recall the definition and some pivotal properties of Mexican needlets while in Section 3 we exploit our main theorem and some auxiliary results.

2 The construction of Mexican needlets

In this Section we shall review Mexican needlets, as developed by Geller and Mayeli, see [8], [9] and [10]. As mentioned, and similarly to standard needlets (cfr. [16], [17]), Mexican needlets can be viewed as a combination of Legendre polynomials weighted by a smooth function. Indeed, let us recall the well-known decomposition of the space of the square-integrable functions over the sphere, \( L^2(S^2) \), as

\[
L^2(S^2) = \bigoplus_{l \geq 0} H_l ,
\]

where \( H_l \) is the space of the homogeneous polynomials of degree \( l \), spanned by the spherical harmonics \( \{ Y_{lm}, l = 0, 1, 2, ..., m = -l, ..., l \} \). In [9], see also [8], [10], it was proven that, for any given resolution level \( j \in (-\infty, +\infty) \), a finite set of measurable regions \( \{ E_{jk} \}_{k=1}^{N_j} \) can be defined over the sphere, such that

\[
\bigcup_{k=1}^{N_j} E_{jk} = S^2 ,
\]

\[
E_{jk_1} \cap E_{jk_2} = \emptyset \text{ for any } k_1 \neq k_2 ,
\]

\[
diam(E_{jk_1}) \leq c_B B^{-j} ,
\]

where \( c_B > 0, B > 1 \); each of these regions can be indexed by a point \( \xi_{jk} \in E_{jk} \). Consider now the weight function

\[
f_s(x) := x^{2s} e^{-x^2} ,
\]

for \( s \in \mathbb{N} \), so that, for any \( l \geq 1 \), we have (cfr. [9])

\[
0 < \frac{m_B}{2 \log B} \leq \sum_{j=-\infty}^{+\infty} f_s^2 \left( \frac{x}{B^j} \right) \leq \frac{M_B}{2 \log B} < +\infty ;
\]
here

\[ m_B : = \eta_s \left( 1 - O \left( \left( \frac{B-1}{B} \right)^2 \log \left( \frac{B-1}{B} \right) \right) \right), \]

\[ M_B : = \eta_s \left( 1 + O \left( \left( \frac{B-1}{B} \right)^2 \log \left( \frac{B-1}{B} \right) \right) \right), \]

and

\[ \eta_s := \int_0^\infty f_s^2 (t) \frac{dt}{t} = \frac{\Gamma (2s)}{2^{2s}}. \quad (3) \]

Let

\[ K_{j:s} (x, y) := \sum_{l \geq 0} f_s \left( \frac{l}{B^j} \right) \frac{2l + 1}{4\pi} P_l (\langle x, y \rangle), \]

where \( P_l (\cdot) \) is the Legendre polynomial of degree \( l \), e.g.,

\[ P_l (x) := \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l, \]

see for instance [1].

Consider now the kernel operator:

\[ K_{j:s} F (x) = \int_{S^2} K_{j:s} (x, y) F (y) dy, \text{ for } F \in L^2 \left( S^2 \right). \]

It is shown in [9] that for \( \varepsilon > 0 \), for \( c_B \) sufficiently small, then

\[ (m_B - \varepsilon) \| F \|_{L^2 \left( S^2 \right)} \leq \sum_{j = -\infty}^{+\infty} \sum_{k=1}^{N_j} \lambda_{jk} |K_{j:s} F (\xi_{jk})|^2 \leq (M_B + \varepsilon) \| F \|_{L^2 \left( S^2 \right)}, \]

where \( \lambda_{jk} \) is the area of \( E_{jk} \); it can be shown that it is possible to choose the set \( E_{jk} \) so that, for any \( j, k \),

\[ \lambda_{jk} \geq c_B B^{-2j}. \]

**Remark 1** In [8], [9], [10] the argument \( l \) in (14) is replaced by \( \sqrt{-e_l}, \{e_l\} \) denoting the spectrum of the spherical Laplacian \( \Delta_{S^2} \). Recall that \( e_l = -l(l+1) \), whence

\[ (\Delta_{S^2} - e_l) Y_{lm} (x) = 0. \]
Hence we should write

\[ f_s \left( \frac{l}{B^j} \right) = \left( \frac{l(l+1)}{B^{2j}} \right)^s \exp \left( -\frac{l(l+1)}{B^{2j}} \right) ; \]

However we shall use instead (1) for the sake of simplicity; of course the difference is asymptotically negligible, considering that trivially

\[ \lim_{l \to \infty} \frac{\left( \frac{l(l+1)}{B^{2j}} \right)^s \exp \left( -\frac{l(l+1)}{B^{2j}} \right)}{\left( \frac{l^2}{B^{2j}} \right)^s \exp \left( -\frac{l^2}{B^{2j}} \right)} = 1. \]

Now define

\[ \psi_{jk;p}(x) := \sqrt{\lambda_{jk}} K_{j;s}(x, \xi_{jk}) \], \hspace{1cm} (4)

or more explicitly

\[ \psi_{jk;s}(x) = \sqrt{\lambda_{jk}} \sum_{l=0}^{+\infty} f_s \left( \frac{l}{B^j} \right) \sum_{m=-l}^{l} Y_{lm}(\xi_{jk}) Y_{lm}(x) ; \] \hspace{1cm} (5)

likewise, let us introduce the mexican needlet coefficients

\[ \beta_{jk;s} := \langle F, \psi_{jk;s} \rangle . \] \hspace{1cm} (6)

It is proven in [9] that there exists a constant \( C_0 = C_0(B, c_B, c'_B, f_s) \) such that

\[ (m_B - C_0) \| F \|_{L^2(S^2)} \leq \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{N_j} |\beta_{jk}|^2 \leq (M_B + C_0) \| F \|_{L^2(S^2)} . \]

Hence, if \( m_B - C_0 > 0 \), \( \{ \psi_{jk;s} \} \) describes a frame for \( L^2(S^2) \) with bounds \( (m_B - C_0) \) and \( (M_B + C_0) \) so that

\[ \frac{M_B + C_0}{m_B - C_0} \sim \frac{M_B}{m_B} = 1 + O \left( \left( \left( \frac{B - 1}{B} \right)^2 \log \left( \frac{B - 1}{B} \right) \right) \right) . \]

In particular, it was showed in ...that for \( s = 1 \)

\[ \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{N_j} |\beta_{jk;s}|^2 = \frac{\eta_s(1 + \delta)}{2 \log B} \| F \|_{L^2(S^2)} . \]
where \( \delta := \delta (B) = O \left( \left( \frac{B-1}{B} \right)^2 \log \left( \frac{B-1}{B} \right) \right) \) is such that

\[
\lim_{B \to 1} \delta (B) = 0 .
\]

With standard needlets it is possible to build a tight frame with tightness constant equal to 1, allowing for an exact reconstruction formula (cfr. \[16\], \[17\] and \[13\]). On the other hand, Mexican needlets have a non-compact support in the harmonic domain, and this makes perfect reconstruction unfeasible for the lack of an exact cubature formula. Despite these features, the Mexican needlets enjoy some remarkable advantages with respect to the standard ones: in particular, they have extremely good concentration properties in the real domain. Moreover, it is possible to choose the measurable disjoint sets \( E_{jk} \) with minimal conditions, and still ensure frame constants arbitrarily close to unity (and hence almost exact reconstruction). In this paper, we investigate the exact dependence of localization properties upon \( s \), an issue which is extremely relevant for applications (see for instance ScodeLLer). It may be noted that the choice of \( s \) represents a trade-off between localization in real and harmonic domain; the latter improves as \( s \) increases, while the reverse holds for the former.

Let us now introduce the geodesic distance (for \( \xi_{jk}, x \in S^2 \))

\[
\vartheta := \vartheta_{jk} (x) = d (x, \xi_{jk}) ,
\]

so that

\[
\Psi_{jk;s} (\vartheta) := \sqrt{\lambda_{jk}} \frac{1}{2\pi} \sum_{l=0}^{+\infty} f_s \left( \frac{(l + \frac{1}{2})}{B^l} \right) \left( l + \frac{1}{2} \right) P_l (\cos \vartheta) .
\] (7)

Before concluding this Section, by considering Remark 1, we can prove the following

**Lemma 1** For any \( s > 1 \),

\[
\psi_{jk;s} (x) := (-1)^s B^{-2js} (\Delta_{S^2})^s \psi_{jk;1} (x) .
\]

**Proof.** Easy calculations lead to

\[
-B^{-2j} \Delta_{S^2} \psi_{jk;s} (x)
\]
\[ \Delta s^2 \left( \sqrt{\lambda_{jk}} \sum_{l \geq 0} \left( \frac{l(l+1)}{B^{2j}} \right)^s \exp \left( -\frac{l(l+1)}{B^{2j}} \right) \sum_{m=-l}^{l} Y_{lm}(\xi_{jk}) Y_{lm}(x) \right) \]

\[ = \sqrt{\lambda_{jk}} \sum_{l \geq 0} \left( \frac{l(l+1)}{B^{2j}} \right)^{s+1} \exp \left( -\frac{l(l+1)}{B^{2j}} \right) \sum_{m=-l}^{l} Y_{lm}(\xi_{jk}) Y_{lm}(x) \]

\[ = \psi_{jk;s+1}(x) \]

Iterating the procedure, we obtain the statement. ■

3 The Localization property

The aim of this Section is to achieve an exhaustive proof of the so-called localization property, i.e. to establish an upper bound for the supremum of the modulus of the Mexican needlet defined as (7), remarking its dependence on the resolution level \( j \) and on the shape parameter \( s \), up to a multiplicative constant. This result is given in the Theorem [1]. We stress again that this achievement was pursued implicitly by Geller and Mayeli in [9], where the authors anyway found a similar result studying (7) for small and large angles, even if they limited their investigations to the case \( s = 1 \). Here, instead, we generalize this result for any value of the shape parameter \( s \) in (1) and for any generic value of \( \vartheta \) by a unique procedure, which resembles the one employed by Narcowich, Petrushev and Ward in [16] to exploit the localization property for standard needlets on the \( n \)-dimensional sphere \( S^n \) (see also [17], [13]). In this case, however, we will take advantage of the explicit formulation of the weight function (1), which allows us to compute exactly its Fourier transform in terms of Hermite polynomials and, through that, to exploit precisely the dependence on the resolution level \( j \) of the sup \( |\Psi_{jk,s}(\vartheta)| \). For the sake of simplicity, let

\[ \varepsilon = \varepsilon (B, j) := B^{-j} , \]

so that

\[ \Psi_{\varepsilon,s}(\vartheta) := \frac{1}{2\pi} \sum_{l=0}^{+\infty} f_s \left( \varepsilon \left( l + \frac{1}{2} \right) \right) \left( l + \frac{1}{2} \right) P_l(\cos \vartheta) . \]
Theorem 1 Let $\Psi_{\varepsilon, s}(\vartheta)$ be defined as in (8). Then, for any $s \in \mathbb{N}$, there exists $C_s > 0$ such that

$$|\Psi_{\varepsilon, s}(\vartheta)| \leq C_s \frac{e^{-\left(\frac{\vartheta}{2}\right)^2}}{\varepsilon^2} \left(1 + \left|H_{2s} \left(\frac{\vartheta}{\varepsilon}\right)\right|\right),$$

uniformly over $j$.

Proof. By using the Mehler-Dirichlet representation formula (see for instance [II]), the Legendre polynomial of degree $l$ can be written as

$$P_l(\vartheta) = \int_\vartheta^\pi \frac{\sin \left((l + \frac{1}{2}) \phi\right)}{\sqrt{\cos \vartheta - \cos \phi}} d\phi.$$

Hence we obtain

$$\Psi_{\varepsilon, s}(\vartheta) = \frac{1}{2\pi} \sum_{l=0}^{+\infty} f_s \left(\varepsilon \left(l + \frac{1}{2}\right)\right) \left(l + \frac{1}{2}\right) \int_\vartheta^\pi \frac{\sin \left((l + \frac{1}{2}) \phi\right)}{\sqrt{\cos \vartheta - \cos \phi}} d\phi$$

$$= \frac{1}{2\pi} \int_\vartheta^\pi \sum_{l=0}^{+\infty} f_s \left(\varepsilon \left(l + \frac{1}{2}\right)\right) \left(l + \frac{1}{2}\right) \frac{\sin \left((l + \frac{1}{2}) \phi\right)}{\sqrt{\cos \vartheta - \cos \phi}} d\phi$$

$$= \frac{1}{2\pi} \int_\vartheta^\pi K_{s}(\phi) \sqrt{\cos \vartheta - \cos \phi} d\phi,$$

where

$$K_{s}(\phi) := \sum_{l=0}^{\infty} f_s \left(\varepsilon \left(l + \frac{1}{2}\right)\right) \left(l + \frac{1}{2}\right) \sin \left((l + \frac{1}{2}) \phi\right)$$

$$= \sum_{l=0}^{\infty} g_{s, \varepsilon, \phi, s} \left(l + \frac{1}{2}\right)$$

$$= \frac{1}{2} \sum_{l=-\infty}^{+\infty} g_{s, \varepsilon, \phi, s} \left(l + \frac{1}{2}\right), \quad (9)$$

using, in the last equality, that

$$g_{s, \varepsilon, \phi, s}(u) := f_s \left(\varepsilon u\right) u \sin (u\phi)$$

is an even function.
From Lemma 2, we obtain
\[ |\Psi_{\varepsilon,s}(\vartheta)| \leq \tilde{C}_{2s+1}^{\varepsilon} \left| \int_{\theta}^{\pi} e^{-\left(\frac{\vartheta}{2}\right)^2} H_{2s+1} \left(\frac{\varphi}{2\varepsilon}\right) \frac{1}{\sqrt{\cos \vartheta - \cos \varphi}} d\varphi \right|. \quad (10) \]

We observe that
\[ \cos \vartheta - \cos \varphi = 2 \left(\phi^2 - \vartheta^2\right) \frac{\sin \left(\frac{\vartheta + \varphi}{2}\right)}{\frac{\vartheta}{2}} \frac{\sin \left(\frac{\varphi - \vartheta}{2}\right)}{\frac{\varphi}{2}}. \quad (11) \]

In order to estimate (10), we consider three different cases:

First of all, let \( \vartheta \in [\delta, \frac{\pi}{2}] \), where \( 0 < \delta < \varepsilon \), and observe that we have
\[ 0 < \frac{\vartheta + \varphi}{2} \leq \frac{3}{4} \pi, \]
\[ 0 \leq \frac{\phi - \vartheta}{2} \leq \frac{\pi}{2}. \]

Equation (11) becomes:
\[ \cos \vartheta - \cos \varphi \geq \frac{1}{2} \left(\phi^2 - \vartheta^2\right) \frac{\sqrt{2}}{3\pi} \frac{\sqrt{2}}{2} \quad = C \left(\phi^2 - \vartheta^2\right), \]

while the integral (10) can be rewritten as
\[ |\Psi_{\varepsilon,s}(\vartheta)| \leq \tilde{C}_{2s+1}^{\varepsilon} \left| \int_{\theta}^{\pi} e^{-\left(\frac{\vartheta}{2}\right)^2} H_{2s+1} \left(\frac{\varphi}{2\varepsilon}\right) \frac{1}{\sqrt{\phi^2 - \vartheta^2}} d\varphi \right|. \]

Recall (see for instance [1]) that, for \( n \) odd, the Hermite polynomials can be rewritten as
\[ H_n(x) = n! \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^r}{(2r+1)!} (2x)^{2r+1}, \]

so that
\[ |\Psi_{\varepsilon,s}(\vartheta)| \leq \tilde{C}_{2s+1}^{\varepsilon} \left(2s + 1\right)! \left| \sum_{r=0}^{s} \frac{(-1)^{s-r}}{(2r+1)(s-r)!} \left(\frac{2\phi}{2\varepsilon}\right)^{2r+1} \right| \]
\[ \quad \quad = \tilde{C}_{2s+1}^{\varepsilon} \left(2s + 1\right)! \left| \sum_{r=0}^{s} \frac{(-1)^{s-r}}{(2r+1)(s-r)!} \left(\frac{2\phi}{2\varepsilon}\right)^{2r+1} \right|. \]
Let us call
\[ Q_{\varepsilon, r}(\vartheta) := \int_{\theta}^{\pi} e^{-(\frac{\varphi}{\vartheta})^2} \left( \frac{\varphi}{\vartheta} \right)^{2r+1} \frac{d\varphi}{\sqrt{\varphi^2 - \vartheta^2}} ; \]
we obtain
\[ |\Psi_{\varepsilon,s}(\vartheta)| \leq \frac{\tilde{C}_{2r+1}}{\varepsilon^2} \left| n! \sum_{r=0}^{s} \frac{(-1)^{s-r} 2^{2r+1}}{(2r+1)!} Q_{\varepsilon,r}(\vartheta) \right| . \]
Observe that
\[ Q_{\varepsilon,r}(\vartheta) = \int_{\theta}^{\pi} e^{-(\frac{\varphi}{\vartheta})^2} \left( \frac{\varphi}{\vartheta} \right)^{2r+1} \left( \frac{\varphi}{\vartheta} \right)^{2s+1} \frac{1}{\sqrt{\varphi^2 - 1}} d\varphi ; \]
we use the substitution \( u = \varphi / \vartheta \), to obtain
\[ Q_{\varepsilon,r}(\vartheta) = e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} \int_{1}^{\frac{\vartheta}{\vartheta}} e^{-((\frac{\vartheta}{2\varepsilon})u)^2 + (\frac{\vartheta}{\vartheta})^2} u^{2r+1} du \]
\[ = e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} \int_{1}^{\frac{\vartheta}{\vartheta}} e^{-((\frac{\vartheta}{2\varepsilon})u)^2} (u^2 - 1) u^{2r} du \]
we hence apply the substitution \( t = (u^2 - 1)^{\frac{1}{2}} \), to obtain
\[ Q_{\varepsilon,r}(\vartheta) \]
\[ = e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} \int_{0}^{\infty} e^{-((\frac{\vartheta}{2\varepsilon})^2 - 1)^{\frac{1}{2}}} e^{-(\frac{\vartheta}{2\varepsilon})^2} (t^2 + 1)^r dt \]
\[ \leq e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} \int_{0}^{\infty} e^{-((\frac{\vartheta}{2\varepsilon})^2)^{\frac{1}{2}}} (t^2 + 1)^r dt \]
\[ = e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} \left( \int_{0}^{1} e^{-((\frac{\vartheta}{2\varepsilon})^2)^{\frac{1}{2}}} (t^2 + 1)^r dt + \int_{1}^{\infty} e^{-(\frac{\vartheta}{2\varepsilon})^2} (t^2 + 1)^r dt \right) . \]
Consider now that for \( t \in [0, 1] \), \( (t^2 + 1)^r \leq 2^r \), while for \( t \in (1, \infty) \), \( (t^2 + 1)^r \leq (2t)^{2r} \). Hence
\[ Q_{\varepsilon,r}(\vartheta) = e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} \left( \int_{0}^{1} e^{-(\frac{\vartheta}{2\varepsilon})^2} t^2 2^r dt + \int_{1}^{\infty} e^{-(\frac{\vartheta}{2\varepsilon})^2} (2t)^{2r} dt \right) \]
\[ \leq e^{-(\frac{\vartheta}{2\varepsilon})^2} \left( \frac{\vartheta}{2\varepsilon} \right)^{2r+1} 4^r \left( \int_{0}^{\infty} e^{-(\frac{\vartheta}{2\varepsilon})^2} t^2 2^r dt + \int_{0}^{\infty} e^{-(\frac{\vartheta}{2\varepsilon})^2} t^{2r} dt \right) . \]
Simple calculations lead to
\[
\int_0^\infty e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} t^2 dt = \frac{\sqrt{\pi}}{2} \left(\frac{\vartheta}{2\varepsilon}\right)^{-1},
\]
\[
\int_0^\infty e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} t^{2r} dt
\]
\[
= \frac{1}{2} \left(\frac{\vartheta}{2\varepsilon}\right)^{-(2r+1)} \int_0^\infty e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} \left(\frac{\vartheta}{2\varepsilon}\right)^2 \left(\frac{\vartheta}{2\varepsilon}\right)^2 2t dt
\]
\[
= \frac{1}{2} \left(\frac{\vartheta}{2\varepsilon}\right)^{-(2r+1)} \int_0^\infty e^{-z z^{-\frac{1}{2}}} dz
\]
\[
= \left(\frac{\vartheta}{2\varepsilon}\right)^{-(2r+1)} \frac{\Gamma \left(\frac{r}{2}\right)}{2},
\]
where we applied the substitution \( z = \left(\frac{\vartheta}{2\varepsilon}\right)^2 \). We obtain
\[
Q_{\varepsilon,r} (\vartheta) \leq C_r e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} \left(\frac{\vartheta}{2\varepsilon}\right)^{2r+1} \left(\frac{\sqrt{\pi}}{2} \left(\frac{\vartheta}{2\varepsilon}\right)^{-1} + \left(\frac{\vartheta}{2\varepsilon}\right)^{-(2r+1)} \frac{\Gamma \left(\frac{r}{2}\right)}{2}\right)
\]
\[
\leq C_r e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} \left(\frac{\vartheta}{2\varepsilon}\right)^{2r} \left(1 + \left(\frac{\vartheta}{2\varepsilon}\right)^{-2r}\right)
\]
\[
\leq C_r e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} \left(\frac{\vartheta}{2\varepsilon}\right)^{2r}.
\]
Thus
\[
|\Psi_{\varepsilon,s} (\vartheta)| \leq \frac{\tilde{C}_{s+1}^2}{\varepsilon^2} (2s + 1) \left(\sum_{r=0}^{s} \frac{(-1)^{s-r} 2^{2r+1}}{(2r+1)! (s-r)!} Q_{\varepsilon,r} (\vartheta)\right)
\]
\[
\leq \frac{\tilde{C}_{s+1}^2}{\varepsilon^2} e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} (2s)! \left(\sum_{r=0}^{s} \frac{C_r (-1)^{s-r} 2^{2r+1}}{(2r)! (s-r)!} \left(\frac{\vartheta}{2\varepsilon}\right)^{2r}\right)
\]
\[
= \frac{\tilde{C}_s^2}{\varepsilon^2} e^{-\left(\frac{\vartheta}{2\varepsilon}\right)^2} \left|H_{2s} \left(\frac{\vartheta}{2\varepsilon}\right)\right|.
\]
Consider now \( \vartheta \in \left[\frac{\pi}{2}, \pi\right] \) and define, as in [16], see also [13],
\[
\tilde{\vartheta} = \pi - \varphi, \quad \tilde{\vartheta} = \pi - \vartheta;
\]
we can easily observe that

\[
0 \leq \frac{\tilde{\vartheta} + \tilde{\phi}}{2} \leq \frac{\pi}{2}
\]

\[
0 \leq \frac{\tilde{\vartheta} - \tilde{\phi}}{2} \leq \frac{\pi}{2}
\]

so we have

\[
\cos \tilde{\phi} - \cos \tilde{\vartheta} \geq 2 \left( \tilde{\vartheta}^2 - \tilde{\phi}^2 \right) \frac{\sin \left( \frac{\tilde{\vartheta} - \tilde{\phi}}{2} \right) \sin \left( \frac{\tilde{\vartheta} + \tilde{\phi}}{2} \right)}{\left( \frac{\tilde{\vartheta} - \tilde{\phi}}{2} \right) \left( \frac{\tilde{\vartheta} + \tilde{\phi}}{2} \right)}
\]

\[
\geq 2 \left( \tilde{\vartheta}^2 - \tilde{\phi}^2 \right) \frac{2 \sqrt{2}}{\pi}
\]

\[
\geq C \left( \tilde{\vartheta}^2 - \tilde{\phi}^2 \right)
\]

By substituting in (10), we obtain, following the same procedure as above,

\[
|\Psi_{\varepsilon,s}(\tilde{\vartheta})| \leq \frac{\tilde{C}_{2s+1}'}{\varepsilon^2} \left| \int_0^{\tilde{\vartheta}} e^{-\left( \frac{\pi - \tilde{\phi}}{2\varepsilon} \right)^2} H_{2s+1} \left( \frac{\pi - \tilde{\phi}}{2\varepsilon} \right) \frac{\sqrt{\tilde{\vartheta}^2 - \tilde{\phi}^2}}{\sqrt{\tilde{\vartheta}^2 - \tilde{\phi}^2}} d\tilde{\phi} \right|
\]

\[
\leq \frac{C_{2s+1}'}{\varepsilon^2} \left| (2s + 1)! \sum_{r=0}^{s} \frac{(-1)^{s-r}}{(2r + 1)! (s - r)!} \tilde{Q}_{\varepsilon,r}(\tilde{\vartheta}) \right|
\]

where

\[
\tilde{Q}_{\varepsilon,r}(\tilde{\vartheta}) = \int_0^{\tilde{\vartheta}} e^{-\left( \frac{\pi}{2\varepsilon} \right)^2} \left( \frac{\pi - \tilde{\phi}}{2\varepsilon} \right)^{2r+1} \frac{\sqrt{\tilde{\vartheta}^2 - \tilde{\phi}^2}}{\sqrt{\tilde{\vartheta}^2 - \tilde{\phi}^2}} d\tilde{\phi}
\]

Now consider

\[
\exp \left[ - \left( \frac{\pi - \tilde{\phi}}{2\varepsilon} \right)^2 + \left( \frac{\tilde{\phi}}{2\varepsilon} \right)^2 \right] = \exp \left[ - \left( \frac{\pi}{2\varepsilon} \right)^2 + 2 \frac{\pi \tilde{\phi}}{2\varepsilon} \right]
\]

\[
\leq \exp \left[ - \frac{\pi^2}{2\varepsilon} \left( \frac{1}{2\varepsilon} - 1 \right) \right]
\]
It is easy to see that the function
\[ \gamma(x) = \exp \left[ -\frac{\pi^2}{2x} \left( \frac{1}{x} - 1 \right) \right] \]
has an absolute maximum for \( x = 1 \). Indeed,
\[ \gamma'(x) = \frac{\pi^2}{2x^2} \left( \frac{1}{x} - 1 \right) \exp \left[ -\frac{\pi^2}{2x} \left( \frac{1}{x} - 1 \right) \right] \]
so that
\[ \gamma'(x) = 0 \iff x = 1 , \]
and
\[ \gamma''(x) = \frac{3\pi^2}{2x^2} \left[ -\frac{1}{x^2} + \frac{1}{x} - \frac{1}{3} \right] \exp \left[ -\frac{\pi^2}{2x} \left( \frac{1}{x} - 1 \right) \right] , \]
so that
\[ \gamma''(1) = -\frac{\pi^2}{2} \exp \left( \frac{\pi^2}{4} \right) < 0 , \]
while
\[ \lim_{x \to \pm \infty} \gamma(x) = \lim_{x \to 0} \gamma(x) = 0 < \gamma(1) = \exp \left( \frac{\pi^2}{4} \right) . \]
Hence, we have
\[ \exp \left[ - \left( \frac{\pi - \tilde{\phi}}{2\varepsilon} \right)^2 + \left( \frac{\tilde{\phi}}{2\varepsilon} \right)^2 \right] \leq \exp \left[ \frac{\pi^2}{4} \right] . \]
On the other hand,
\[ \left( \frac{\pi - \tilde{\phi}}{2\varepsilon} \right)^{2r+1} = \left( \frac{\tilde{\phi}}{2\varepsilon} \right)^{2r+1} \left( \frac{\pi}{\tilde{\phi}} - 1 \right)^{2r+1} \leq \left( \frac{\tilde{\phi}}{2\varepsilon} \right)^{2r+1} , \]
so that
\[ \tilde{Q}_{\varepsilon,r}(\vartheta) \leq \tilde{C}_r \int_0^{\sqrt{2r+1}} e^{-\left( \frac{\tilde{\phi}}{2\varepsilon} \right)^2} \frac{\left( \frac{\tilde{\phi}}{2\varepsilon} \right)^{2r+1}}{\sqrt{\left( 1 - \left( \frac{\tilde{\phi}}{\vartheta} \right)^2 \right)}} d\tilde{\phi} \]
\[ = \tilde{C}_r \int_0^{\sqrt{2r+1}} e^{-\left( \frac{\tilde{\phi}}{2\varepsilon} \right)^2} \frac{\left( \frac{\tilde{\phi}}{2\varepsilon} \right)^{2r+1} \left( \frac{\tilde{\phi}}{\vartheta} \right)^{2r+1}}{\sqrt{\left( 1 - \left( \frac{\tilde{\phi}}{\vartheta} \right)^2 \right)}} 1 d\tilde{\phi} . \]
By substituting $u = \tilde{\varphi}/\tilde{\vartheta}$, we have

$$
\tilde{Q}_{\epsilon,r}(\vartheta) \leq \tilde{C}_r' \left( \frac{\tilde{\vartheta}}{2\varepsilon} \right)^{2r+1} e^{-\left( \frac{\tilde{\varphi}}{2\varepsilon} \right)^2} \int_0^1 \frac{e^{-\left( \frac{\tilde{\varphi}}{2\varepsilon} \right)^2(u^2-1)}}{\sqrt{1-u^2}} u^{2r+1} \, du
$$

and, choosing $t = \sqrt{1-u^2}$, we conclude that

$$
\tilde{Q}_{\epsilon,r}(\vartheta) \leq \tilde{C}_r'' \left( \frac{\tilde{\vartheta}}{2\varepsilon} \right)^{2r+1} e^{-\left( \frac{\tilde{\varphi}}{2\varepsilon} \right)^2} \int_0^1 \frac{e^{-\left( \frac{\tilde{\varphi}}{2\varepsilon} \right)^2(u^2-1)}}{\sqrt{1-u^2}} u^{2r+1} \, u \, dt
$$

We obtain therefore also in this case

$$
|\Psi_{\epsilon,s}(\vartheta)| \leq \frac{C_s''}{\varepsilon^2} e^{-\left( \frac{\varphi}{2\varepsilon} \right)^2} \left| H_{2s} \left( \frac{\vartheta}{2\varepsilon} \right) \right|.
$$

Finally, consider $\vartheta = 0$. In this case

$$
K_{\epsilon,s}(\phi) \leq \frac{1}{\varepsilon^2} \sum_{l \geq 0} f_s \left( \varepsilon \left( l + \frac{1}{2} \right) \right) \varepsilon^2 \left( l + \frac{1}{2} \right)
$$

$$
\leq \frac{C}{\varepsilon^2} \sum_{l \geq 0} (\varepsilon l)^{2s+1} e^{-l^2} \int e^{(l+1)} \, dx
$$

$$
\leq \frac{C}{\varepsilon^2} \int_0^\infty u^{2s+1} e^{-u^2} \, du
$$

$$
= \frac{C}{\varepsilon^2} \Gamma \left( s + \frac{3}{2} \right)
$$

$$
= \frac{C_s''}{\varepsilon^2}.
$$
Combining these results, we obtain, for \( \vartheta = [0, \pi] \),

\[
|\Psi_{\varepsilon} (\vartheta)| < C_s \frac{e^{-\left(\frac{\vartheta}{\varepsilon}\right)^2}}{\varepsilon^2} \left(1 + |H_{2s} \left(\frac{\vartheta}{\varepsilon}\right)|\right),
\]
as claimed.  

**Remark 2** Recall that

\[
\psi_{jk,s} (x) = \sqrt{\lambda_{jk}} \Psi_{j,s} (\vartheta (x)),
\]
where \( \vartheta (x) = d (x, \xi_{jk}) \), the geodesic distance over the sphere between \( x \) and \( \xi_{jk} \). Because \( \lambda_{jk} \leq cB^{-2j} \), we have

\[
|\psi_{jk,s} (x)| \leq C_s B^j e^{-B^2 \vartheta^2 (x)} \left(1 + |B \vartheta (x)|^{2s}\right).
\]

**Lemma 2** Let \( K_{\varepsilon,s} (\phi) \) be defined as in (9). Then there exists \( \tilde{C}_{2s+1} > 0 \) such that

\[
K_{\varepsilon,s} (\phi) \leq \frac{\tilde{C}_{2s+1}}{\varepsilon^2} e^{-\left(\frac{\phi}{\varepsilon}\right)^2} \left|H_{2s+1} \left(\frac{\phi}{2\varepsilon}\right)\right|.
\]

**Proof.** We introduce the following notation for the Fourier transform of \( f \in L^1 (\mathbb{R}) \):

\[
F [f] (\omega) := \int_{\mathbb{R}} f (x) e^{-i\omega x} dx =: \hat{f} (\omega).
\]
Recall also two standard properties for the Fourier transforms: under standard conditions, we have

\[
\frac{d^\alpha}{d\omega^\alpha} \hat{f} (\omega) = (-i)^\alpha F [x^\alpha f (x)] (\omega) ;
\]

\[
F \left[ \frac{d^\alpha}{dx^\alpha} f (x) \right] (\omega) = (-i)^\alpha \omega^\alpha F [f (x)] (\omega).
\]
Finally, recall the Poisson Summation Formula (PSF): if for \( \omega \in [0, 2\pi] \) and \( \alpha > 0 \)

\[
|f (x)| + \left|\hat{f} (\omega)\right| \leq \frac{C_\alpha}{1 + |x|^{\alpha+1}},
\]
then:

\[
\sum_{\tau = -\infty}^{+\infty} f (\tau) e^{-i\omega \tau} = \sum_{\nu = -\infty}^{+\infty} \hat{f} (\omega + 2\pi \nu).
\]
More details and discussions on Fourier transforms can be found, for instance, in the textbook [20].

Simple calculations lead us to

\[ F \left[ g_{\varepsilon, \phi; s} \left( x + \frac{1}{2} \right) \right] (\omega) = e^{i\frac{\omega}{2}} \hat{g}_{\varepsilon, \phi; s} (\omega) . \]

While, on one hand, we have

\[ F \left[ f_{s} (\varepsilon x) x \right] (\omega) = \int_{\mathbb{R}} f_{s} (\varepsilon x) x e^{-i\omega x} \]

\[ = \frac{1}{\varepsilon} \int_{\mathbb{R}} f_{s} (\varepsilon x) (\varepsilon x) e^{-i\omega x} dx \]

\[ = \frac{1}{\varepsilon \varepsilon} F \left[ f_{s} (x) x \right] (\frac{\omega}{\varepsilon}) , \]

on the other hand, by the definition (11) and by observing that

\[ F \left[ e^{-x^2} \right] (\omega) = \sqrt{\pi} e^{-\frac{\omega^2}{4}} \]

we can compute

\[ F \left[ f_{s} (x) x \right] (\omega) = \int_{\mathbb{R}} x^{2s+1} e^{-x^2 - i\omega x} dx \]

\[ = i^{2s+1} \frac{d^{2s+1}}{d\omega^{2s+1}} F \left[ e^{-x^2} \right] (\omega) \]

\[ = i^{2s+1} \sqrt{\pi} \frac{d^{2s+1}}{d\omega^{2s+1}} e^{-\frac{\omega^2}{4}} \]

\[ = (-1)^{s+\frac{1}{2}} \sqrt{\pi} \left( (-1)^{2s+1} H_{2s+1} \left( \frac{\omega}{2} \right) e^{-\frac{\omega^2}{4}} \right) \]

\[ = (-1)^{(s+\frac{1}{2})} \sqrt{\pi} H_{2s+1} \left( \frac{\omega}{2} \right) e^{-\frac{\omega^2}{4}} , \]

where \( H_{2s+1} (\cdot) \) is the Hermite polynomial of order \( 2s + 1 \). Recall that the polynomials composing \( H_n (x) \) are all even (odd) if \( n \) is even (odd), for more details, see for instance [1]. Collecting all these results, we have that

\[ F \left[ f_{s} (\varepsilon x) x \right] (\omega) = \frac{(-1)^{(s+\frac{1}{2})}}{\varepsilon^2} \sqrt{\pi} H_{2s+1} \left( \frac{\omega}{2\varepsilon} \right) e^{-\left( \frac{\pi}{\varepsilon^2} \right)^2} . \] (15)
We hence obtain:

\[
\hat{g}_{\varepsilon, \phi; s}(\omega) = F[sin(\phi x)](\omega) * F[f_s(\varepsilon x) x](\omega) = \frac{(-1)^s \pi^{\frac{3}{2}}}{\varepsilon^2} \left( \frac{\omega - \phi}{2\varepsilon} \right)^2 e^{-\left(\frac{\omega - \phi}{2\varepsilon}\right)^2} - H_{2s+1} \left( \frac{\omega + \phi}{2\varepsilon} \right) e^{-\left(\frac{\omega + \phi}{2\varepsilon}\right)^2}.
\]

Now, by using (14), we obtain

\[
\sum_{l=-\infty}^{+\infty} g_{\varepsilon, \phi; s}(l + \frac{1}{2}) = \sum_{\nu=-\infty}^{+\infty} e^{i\frac{2\pi\nu}{\varepsilon}} \hat{g}_{\varepsilon, \phi; s}(2\pi\nu)
\]

\[
= \sum_{\nu=-\infty}^{+\infty} e^{i\frac{2\pi\nu}{\varepsilon}} (-1)^s \pi^{\frac{3}{2}} \left( \frac{2\pi\nu - \phi}{2\varepsilon} \right) e^{-\left(\frac{2\pi\nu - \phi}{2\varepsilon}\right)^2} - H_{2s+1} \left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) e^{-\left(\frac{2\pi\nu + \phi}{2\varepsilon}\right)^2}
\]

\[
= 2 \sum_{\nu=-\infty}^{+\infty} e^{i\frac{2\pi\nu}{\varepsilon}} (-1)^{s+1} \pi^{\frac{3}{2}} \frac{2\pi\nu + \phi}{2\varepsilon} H_{2s+1} \left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) e^{-\left(\frac{2\pi\nu + \phi}{2\varepsilon}\right)^2},
\]

where the last equality takes in account that \(H_{2s+1}(\cdot)\) is odd. We have, therefore

\[
K_{\varepsilon, s}(\phi) = \frac{(-1)^{s+1} \pi^{\frac{3}{2}}}{\varepsilon^2} \sum_{\nu=-\infty}^{+\infty} e^{i\frac{2\pi\nu}{\varepsilon}} H_{2s+1} \left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) e^{-\left(\frac{2\pi\nu + \phi}{2\varepsilon}\right)^2}.
\]

We easily obtain that

\[
|K_{\varepsilon, s}(\phi)| = \frac{\pi^{\frac{3}{2}}}{\varepsilon^2} \sum_{\nu=-\infty}^{+\infty} H_{2s+1} \left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) e^{-\left(\frac{2\pi\nu + \phi}{2\varepsilon}\right)^2} \leq \frac{\pi^{\frac{3}{2}}}{\varepsilon^2} V_{\varepsilon, s}(\phi),
\]

where

\[
V_{\varepsilon, s}(\phi) = \sum_{\nu=-\infty}^{+\infty} \left| H_{2s+1} \left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) e^{-\left(\frac{2\pi\nu + \phi}{2\varepsilon}\right)^2} \right|.
\]
We observe, first of all, that
\[
V_{\varepsilon,s}(\phi) = \left| H_{2s+1}\left( \frac{\phi}{2\varepsilon} \right) \right| e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} + V_+ + V_- ,
\]
where
\[
V_+ = \sum_{\nu=1}^{+\infty} \left| H_{2s+1}\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) \right| e^{-\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right)^2} ,
\]
\[
V_- = \sum_{\nu=-\infty}^{-1} \left| H_{2s+1}\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) \right| e^{-\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right)^2} .
\]
Recall that, for \( n \) odd, we have
\[
H_n(x) = n! \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-k}}{(2k+1)!\left( \frac{n-1}{2} - k \right)!} (2x)^{2k+1} ,
\]
see for instance [1]. Hence we have for \( |x| > 1 \)
\[
\left| H_n(x) \right| \leq n! \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-k}}{(2k+1)!\left( \frac{n-1}{2} - k \right)!} (2x)^{2k+1} \\
\leq n! \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{(2k+1)!\left( \frac{n-1}{2} - k \right)!} (2x)^{2k+1} \\
\leq C_n' |x|^n .
\]
We therefore obtain
\[
\left| H_{2s+1}\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right) \right| e^{-\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right)^2} \\
\leq C_{2s+1} \left| \frac{2\pi\nu + \phi}{2\varepsilon} \right|^{2s+1} e^{-\left( \frac{2\pi\nu + \phi}{2\varepsilon} \right)^2} \\
= C_{2s+1} \left( \frac{2\pi\nu + \phi}{2\varepsilon} \right)^{2s+1} e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} e^{-\left( \frac{\pi\nu}{2\varepsilon} \right)^2} e^{-\frac{2\pi\nu\phi}{4\varepsilon^2}} \\
= e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} C_{2s+1} \left[ \left( \frac{\pi\nu}{\varepsilon} + \frac{\phi}{2\varepsilon} \right)^{2s+1} e^{-\left( \frac{\pi\nu}{2\varepsilon} \right)^2} e^{-\frac{2\pi\nu\phi}{4\varepsilon^2}} \right] \\
\leq e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} C_{2s+1} \left[ \left( \frac{\pi\nu}{\varepsilon} + \frac{\pi}{2\varepsilon} \right)^{2(s+1)} e^{-\left( \frac{\pi}{2\varepsilon} \right)^2} \right] .
\]
Because
\[ \frac{\pi x}{\varepsilon} + \frac{\pi}{2\varepsilon} < \frac{2\pi x}{\varepsilon}, \]
we have, considering that \( e^{-\nu \phi} < 1 \)

\[ V_+ \leq C_{2s+1} \sum_{\nu=1}^{+\infty} |H_{2(s+1)} \left( \frac{2\pi \nu + \phi}{2 \varepsilon} \right) e^{-\left( \frac{2\pi \nu + \phi}{2 \varepsilon} \right)^2} | \]

\[ \leq e^{-\left( \frac{\phi}{2 \varepsilon} \right)^2} C_{2s+1} \sum_{\nu=1}^{+\infty} \left( \frac{\pi \nu}{\varepsilon} + \frac{\pi}{2 \varepsilon} \right)^{2s+1} e^{-\left( \frac{\pi \nu}{2 \varepsilon} \right)^2} \]

\[ \leq e^{-\left( \frac{\phi}{2 \varepsilon} \right)^2} C_{2s+1} 2^{2s+1} \sum_{\nu=1}^{+\infty} \left( \frac{\pi \nu}{\varepsilon} \right)^{2s+1} e^{-\left( \frac{\pi \nu}{2 \varepsilon} \right)^2} \]

\[ \leq C'_{2s+1} e^{-\left( \frac{\phi}{2 \varepsilon} \right)^2}, \quad (18) \]

because the series \( \sum_{\nu=1}^{+\infty} \left( \frac{\pi \nu}{\varepsilon} \right)^{2s+1} e^{-\left( \frac{\pi \nu}{2 \varepsilon} \right)^2} \) is convergent, as easily proved by applying the D’Alembert’s criterion:

\[ \lim_{\nu \to \infty} \left( \frac{\pi (\nu+1)}{\varepsilon} \right)^{2s+1} e^{-\left( \frac{\pi (\nu+1)}{2 \varepsilon} \right)^2} \left( \frac{\pi \nu}{\varepsilon} \right)^{2s+1} e^{-\left( \frac{\pi \nu}{2 \varepsilon} \right)^2} \]

\[ = \lim_{\nu \to \infty} \left( 1 + \frac{1}{\nu} \right)^{2s+1} \exp \left( -\frac{\pi^2}{\varepsilon^2} (2\nu + 1) \right) = 0, \]
for all \( \nu > 1 \). On the other hand, if \( |x| \leq 1 \), we obtain

\[ |H_n(x)| \leq n! \left| \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-k}}{(2k+1)! \left( \frac{n-1}{2} - k \right)!} (2x)^{2k+1} \right| \]

\[ \leq n! \left| \sum_{k=0}^{\frac{n-1}{2}} \frac{1}{(2k+1)! \left( \frac{n-1}{2} - k \right)!} 2^{2k+1} \right| \]

\[ \leq C'_n. \quad (19) \]
Hence we obtain

\[ \left| H_{2s+1} \left( \frac{2\pi \nu + \phi}{2\varepsilon} \right) \right| e^{-\left( \frac{2\pi \nu + \phi}{2\varepsilon} \right)^2} \leq C_{2s+1}' e^{-\left( \frac{2\pi \nu + \phi}{2\varepsilon} \right)^2} \]

\[ = C_{2s+1}' e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} e^{-\left( \frac{\nu}{\varepsilon} \right)^2} e^{-\frac{2\pi \nu \phi}{4\varepsilon^2}} \]

\[ = e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} C_{2s+1}' \left[ e^{-\left( \frac{\nu}{\varepsilon} \right)^2} e^{-\frac{2\pi \nu \phi}{4\varepsilon^2}} \right] \]

\[ \leq e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} C_{2s+1}' e^{-\left( \frac{\nu}{\varepsilon} \right)^2}, \]

in order to have

\[ V_+ \leq C_{2s+1} \sum_{\nu=1}^{+\infty} H_{2s+1}(\nu) e^{-\left( \frac{2\pi \nu + \phi}{2\varepsilon} \right)^2} \]

\[ \leq e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} C_{2s+1} \sum_{\nu=1}^{+\infty} e^{-\left( \frac{\nu}{\varepsilon} \right)^2} \]

\[ \leq e^{-\left( \frac{\phi}{2\varepsilon} \right)^2} C_{2s+1} \sum_{\nu=1}^{+\infty} e^{-\left( \frac{\nu}{\varepsilon} \right)^2} \]

\[ \leq C_{2s+1}' e^{-\left( \frac{\phi}{2\varepsilon} \right)^2}, \quad (20) \]

because the series \( \sum_{\nu=1}^{+\infty} e^{-\left( \frac{\nu}{\varepsilon} \right)^2} \) is convergent, as easily proved by applying the D’Alembert’s criterion:

\[ \lim_{\nu \to \infty} \frac{e^{-\left( \frac{\nu}{\varepsilon} \right)^2}}{e^{-\left( \frac{(\nu+1)}{\varepsilon} \right)^2}} = \lim_{\nu \to \infty} \exp\left( -\frac{\pi^2}{\varepsilon^2} (2\nu + 1) \right) = 0, \]

for all \( \nu > 1. \)

Consider now the sum \( V_- \). Let us call \( \nu' = -\nu \), so that

\[ V_- = \sum_{\nu'=1}^{\infty} \left| H_{2s+1} \left( \frac{\phi - 2\pi \nu'}{2\varepsilon} \right) \right| e^{-\left( \frac{\phi - 2\pi \nu'}{2\varepsilon} \right)^2} \]

We have for Equation (17)

\[ \left| H_{2s+1} \left( \frac{\phi - 2\pi \nu'}{2\varepsilon} \right) \right| e^{-\left( \frac{\phi - 2\pi \nu'}{2\varepsilon} \right)^2} \]
\[ \leq C_{2s+1} \left| \frac{\phi - 2\pi \nu'}{2\varepsilon} \right|^{2s+1} e^{-\left( \frac{2\pi \nu'}{2\varepsilon} \right)^2} = e^{-\left( \frac{\phi}{2\pi} \right)^2} C_{2s+1} \left[ \frac{\phi - 2\pi \nu'}{2\varepsilon} \right|^{2s+1} e^{-\left( \frac{2\pi \nu'}{2\varepsilon} \right)^2} \right]. \]

Considering that \( \phi < \pi \), simple calculations lead to

\[ V_- \leq C_{2s+1} \sum_{\nu' = 1}^{+\infty} H_{2s+1} \left( \frac{\phi - 2\pi \nu'}{2\varepsilon} \right) e^{-\left( \frac{2\pi \nu'}{2\varepsilon} \right)^2} \]

\[ \leq e^{-\left( \frac{\phi}{2\pi} \right)^2} C_{2s+1} \sum_{\nu' = 1}^{+\infty} \left[ \frac{\phi - 2\pi \nu'}{2\varepsilon} \right|^{2s+1} e^{-\left( \frac{\nu'}{2\varepsilon} \right)^2} \right] \]

\[ \leq e^{-\left( \frac{\phi}{2\pi} \right)^2} C_{2s+1} \sum_{\nu' = 1}^{+\infty} \left[ \frac{\nu'}{\varepsilon} \right|^{2s+1} e^{-\left( \frac{\nu'}{2\varepsilon} \right)^2} \right] \]

\[ \leq C''_{2s+1} e^{-\left( \frac{\phi}{2\pi} \right)^2}, \quad (21) \]

because, again for the D'Alembert Criterion, we have

\[ \lim_{\nu' \to \infty} \left( \frac{\nu' + 1}{\nu'} \right)^{2s+1} \exp \left( -\frac{\nu'}{2\varepsilon} \right) = 0, \]

hence the series \( \sum_{\nu' = 1}^{+\infty} \left[ \frac{\nu'}{\varepsilon} \right|^{2s+1} e^{-\left( \frac{\nu'}{2\varepsilon} \right)^2} \right] \) is convergent.

Combining (18) and (21) in (16), we obtain that the term corresponding to \( \nu = 0 \) is dominant, hence we have

\[ V_{\varepsilon, s} (\phi) \leq e^{-\left( \frac{\phi}{2\pi} \right)^2} \left( \left| H_{2s+1} \left( \frac{\phi}{2\varepsilon} \right) \right| + C'_{2s+1} + C''_{2s+1} \right) \]

\[ \leq C_{2s+1} e^{-\left( \frac{\phi}{2\pi} \right)^2} H_{2s+1} \left( \frac{\phi}{2\varepsilon} \right). \]
Thus
\[ K_{\epsilon,s}(\phi) \leq \frac{\tilde{C}_{2s+1}}{\epsilon^2} e^{-\left(\frac{\phi}{2\epsilon}\right)^2} \left| H_{2s+1}\left(\frac{\phi}{2\epsilon}\right) \right|, \]
as claimed. \[ \blacksquare \]

**Corollary 1 (Boundedness on \(L^p(S^2)\)-norms)** For any \(p \in [1, +\infty)\), there exist \(c_p, C_p \in \mathbb{R}\) such that
\[ c_p B^{2j} \left(\frac{1}{2} - \frac{1}{p}\right) \leq \|\psi_{jk;s}\|_{L^p(S^2)} \leq C_p B^{2j} \left(\frac{1}{2} - \frac{1}{p}\right). \]
Furthermore, there exist \(c_\infty, C_\infty \in \mathbb{R}\) such that
\[ c_\infty B^j \leq \|\psi_{jk;s}\|_{L^\infty(S^2)} \leq C_\infty B^j. \]

The proof of this Corollary is very close to the one developed in the standard needlet framework in [17], the only remarkable difference concerning the estimate of the bounds for \(L^2(S^2)\) norms. In [17], this bound is proven as corollary of the tight-frame property. We establish a similar result for the Mexican needlet framework as follows. If \(dx\) denotes the uniform spherical measure, we have
\[ \|\psi_{jk;s}\|_{L^2(S^2)} = \int_{S^2} |\psi_{jk;s}(x)|^2 \, dx \]

\[ = \lambda_{jk} \int_{S^2} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} f_s \left( \frac{l}{B^j} \right) f_s \left( \frac{l'}{B^j} \right) \times \sum_{m=-l}^{l} \sum_{m'=-l}^{l} Y_{lm}(x) \overline{Y}_{l'm'}(x) \overline{Y}_{lm}(\xi_{jk}) Y_{l'm'}(\xi_{jk}) \, dx \]
\[ = \lambda_{jk} \sum_{l=0}^{\infty} f_s^2 \left( \frac{l}{B^j} \right) \sum_{m=-l}^{l} \overline{Y}_{lm}(\xi_{jk}) Y_{lm}(\xi_{jk}) \delta_{l}^{l'} \delta_{m}^{m'} \]
\[ = \lambda_{jk} \sum_{l=0}^{\infty} f_s^2 \left( \frac{l}{B^j} \right) \frac{2l + 1}{4\pi}. \]

Now, on one hand we have
\[ \lambda_{jk} \sum_{l=0}^{\infty} f_s^2 \left( \frac{l}{B^j} \right) \frac{2l + 1}{4\pi} \]
\[
\leq \frac{c}{2\pi} \frac{1}{B_j} \sum_{l=0}^{\infty} f^2_s \left( \frac{l}{B_j} \right) \frac{l + 1/2}{B_j} \\
= \frac{c}{2\pi} \sum_{l=0}^{\infty} \left( \frac{l}{B_j} \right)^{4s} \exp \left( -2 \left( \frac{l}{B_j} \right)^2 \right) \frac{l + 1/2}{B_j} \\
\leq \frac{c}{2\pi} \sum_{l=0}^{\infty} \left( \frac{l}{B_j} \right)^{4s} \exp \left( -2 \left( \frac{l}{B_j} \right)^2 \right) \frac{l}{B_j} \int_{\frac{l}{B_j}}^{\frac{l+1}{B_j}} dx \\
\leq \frac{c}{2\pi} \int_0^{\infty} x^{4s+1} \exp \left( -2x^2 \right) dx \leq C_2 ;
\]

on the other hand, we have similarly

\[
\lambda_{jk} \sum_{l=0}^{\infty} f^2_s \left( \frac{l}{B_j} \right) \frac{2l + 1}{4\pi} \geq c_2 .
\]

Now, following [17] (see also [13]), it is easy to observe that

\[
c_p B^{2j(\frac{1}{2} - \frac{1}{p})} \leq \|\psi_{jk}\|_{L^p(S^2)} \leq C_p B^{2j(\frac{1}{2} - \frac{1}{p})} .
\]

**Acknowledgement 1** The author wishes to thank Federico Cacciafesta for the helpful conversations and his enlightening suggestions and Domenico Marinucci for his precious hints and his fundamental corrections.

**References**

[1] Abramowitz, M.; Stegun, I. (1946), Handbook of Mathematical Functions, Dover, New York.

[2] Antoine, J.-P. and Vanderghynst, P. (2007), Wavelets on the Sphere and Other Conic Sections, *Journal of Fourier Analysis and its Applications*, 13, 369-386

[3] Dahlke, S., Steidtl, G., Teschke, G. (2007) Frames and Coorbit Theory on Homogeneous Spaces with a Special Guidance on the Sphere, *Journal of Fourier Analysis and its Applications*, 13, 387-404

[4] Durastanti, C., Lan, X., (2013), High-Frequency Tail Index Estimation by Nearly Tight Frames, *submitted*
[5] **Freeden, W., Schreiner, M. (1998)**, Orthogonal and nonorthogonal multiresolution analysis, scale discrete and exact fully discrete wavelet transform on the sphere. *Constr. Approx.* 14, 4, 493–515.

[6] **Geller, D. and Marinucci, D. (2010)**, Spin Wavelets on the Sphere, *Journal of Fourier Analysis and its Applications*, n.6, 840-884, arXiv: 0811.2835

[7] **Geller, D. and Marinucci, D. (2011)**, Mixed Needlets, *Journal of Mathematical Analysis and Applications*, n.375, 610-630.

[8] **Geller, D. and Mayeli, A. (2009)**, Continuous Wavelets on Manifolds, *Math. Z.*, Vol. 262, pp. 895-927, arXiv: math/0602201

[9] **Geller, D. and Mayeli, A. (2009)**, Nearly Tight Frames and Space-Frequency Analysis on Compact Manifolds, *Math. Z.*, Vol. 263 (2009), pp. 235-264, arXiv: 0706.3642

[10] **Geller, D. and Mayeli, A. (2009)**, Besov Spaces and Frames on Compact Manifolds, *Indiana Univ. Math. J.*, Vol. 58, pp. 2003-2042, arXiv:0709.2452.

[11] **Holschneider, M. and Iglewska-Nowak., I. (2007)**, Poisson Wavelets on the Sphere, *Journal of Fourier Analysis and its Applications*, 13, 405-420

[12] **Lan, X. and Marinucci, D. (2009)**, On the Dependence Structure of Wavelet Coefficients for Spherical Random Fields, *Stochastic Processes and their Applications*, 119, 3749-3766, arXiv:0805.4154

[13] **Marinucci, D. and Peccati, G. (2011)** *Random Fields on the Sphere. Representation, Limit Theorem and Cosmological Applications*, Cambridge University Press.

[14] **Mayeli, A. (2010)**, Asymptotic Uncorrelation for Mexican Needlets, *J. Math. Anal. Appl.* Vol. 363, Issue 1, pp. 336-344, arXiv: 0806.3009

[15] **McEwen, J.D., Vielva, P., Wiaux, Y., Barreiro, R.B., Cayon, L., Hobson, M.P., Lasenby, A.N., Martínez-Gonzalez, E., Sanz, J. (2007)**, Cosmological Applications of a Wavelet Analysis on the Sphere, *Journal of Fourier Analysis and its Applications*, 13, 495-510
[16] Narcowich, F.J., Petrushev, P. and Ward, J.D. (2006a) Localized Tight Frames on Spheres, *SIAM Journal of Mathematical Analysis* Vol. 38, pp. 574–594.

[17] Narcowich, F.J., Petrushev, P. and Ward, J.D. (2006b) Decomposition of Besov and Triebel-Lizorkin Spaces on the Sphere, *Journal of Functional Analysis*, Vol. 238, 2, 530–564.

[18] Petrushev, P., Xu, Y. (2008), Localized Polynomial Frames on the Ball, *Constr. Approx.* 27, 121-148

[19] Scodeller, S., Rudjord, O. Hansen, F.K., Marinucci, D., Geller, D. and Mayeli, A. (2011), Introducing Mexican needlets for CMB analysis: Issues for practical applications and comparison with standard needlets, *Astrophysical Journal*, 733, 121

[20] Stein, E.; Weiss, G. (1971), *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press.

[21] Wiaux, Y., McEwen, J.D., Vielva, P., (2007), Complex Data Processing: Fast Wavelet Analysis on the Sphere, *Journal of Fourier Analysis and its Applications*, 13, 477-494.