EVENT-TRIGGERED CONTROL OF INFINITE-DIMENSIONAL SYSTEMS

MASASHI WAKAIKI AND HIDEKI SANO

Abstract. This paper addresses the problem of event-triggered control for infinite-dimensional systems. We employ event-triggering mechanisms that compare the plant state and the error of the control input induced by the event-triggered implementation. Under the assumption that feedback operators are compact, a strictly positive lower bound on the inter-event times can be guaranteed. We show that if the threshold of the event-triggering mechanisms is sufficiently small, then the event-triggered control system with a bounded control operator and a compact feedback operator is exponentially stable. For infinite-dimensional systems with unbounded control operators, we employ two event-triggering mechanisms that are based on system decomposition and periodic event-triggering, respectively, and then analyze the exponential stability of the closed-loop system under each event-triggering mechanism.

Key words. event-triggered control, infinite-dimensional systems, stabilization

AMS subject classifications. 34G10, 93C25, 93C62, 93D15

1. Introduction. The aim of this paper is to develop resource-aware control schemes for infinite-dimensional systems. To this end, we employ event-triggering mechanisms and analyze exponential stability for infinite-dimensional event-triggered control systems. Event-triggering mechanisms invoke data transmissions if predefined conditions on the data are satisfied. As a result, network and energy resources are consumed only when the data is necessary for control. In addition to such networked-control applications, the analysis and synthesis of event-triggered control systems are interesting from a theoretical viewpoint, because the interaction of continuous-time and discrete-time dynamics in event-triggered control systems is different from that in periodic sampled-data systems. Most of the existing studies on event-triggered control have been developed for finite-dimensional systems, but some researchers have recently extended to infinite-dimensional systems, e.g., systems with output delays [15], first-order hyperbolic systems [8, 9, 21], and second-order parabolic systems [13, 14, 25]. These earlier studies deal with specific delay differential equations and partial differential equations. On the other hand, the infinite-dimensional system we study is described by abstract evolution equations as follows.

For Hilbert spaces $X, U$, we here consider the following system with state space $X$ and input space $U$:

\begin{equation}
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0; \quad x(0) = x^0 \in X,
\end{equation}

where $x(t) \in X$ is the state, $u(t) \in U$ is the input, $A$ is the generator of a strongly continuous semigroup $T(t)$ on $X$, and $B$ is a bounded linear operator from $U$ into the extrapolation space $X_{-1}$ associated with $T(t)$; see the notation and terminology section below for the definition of the extrapolation space $X_{-1}$. If the control operator $B$ is bounded from $U$ to $X$, $B$ is called bounded. Otherwise, $B$ is called unbounded. To illustrate the extrapolation space $X_{-1}$ and the unboundedness of the control operator $B$, consider the heat equation with Neumann boundary conditions, where $X =$

---

*Submitted to the editors DATE.

Funding: This work was supported by JSPS KAKENHI Grant Numbers JP17K14699.

†Graduate School of System Informatics, Kobe University, Nada, Kobe, Hyogo 657-8501, Japan (wakaiki@ruby.kobe-u.ac.jp, sano@crystal.kobe-u.ac.jp).
$L^2([0,1], \mathbb{C})$ and $Ax := x''$ with domain $D(A) = \{x \in W^{2,2}(0,1) : x'(0) = x'(1) = 0\}$. In this case, $X_{-1}$ can be regarded as the dual of $D(A)$ (with the graph norm of $A$) with respect to the pivot space $X$. Therefore, $X_{-1}$ contains Dirac delta functions, which implies that for the case where $B$ is unbounded, we can deal with point actuators. In contrast, if $B$ is bounded, then we consider only spatially-distributed actuators. We refer the reader to Section II.5 of [7] and Section 2.10 of [28] for more details on the extrapolation space $X_{-1}$.

To infinite-dimensional systems described by the abstract evolution equation (1.1), the results of periodic sampled-data control for finite-dimensional systems have been generalized in a number of papers; see [16–18,22–24]. Let an increasing sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) be the updating instants of the control input \( u \), and consider the feedback control
\[
    u(t) = Fx(t_k), \quad t_k \leq t < t_{k+1}, \ k \in \mathbb{N}_0,
\]
where \( F \) is a bounded linear operator from $X$ to $U$. In the standard sampled-data system, control updating is periodic, namely, \( t_{k+1} - t_k \) is constant for every \( k \in \mathbb{N}_0 \). In contrast, we employ the following event-triggering mechanisms in this paper:

\begin{align}
    t_0 &:= 0, \quad t_{k+1} := \inf \{ t > t_k : \|Fx(t) - Fx(t_k)\|_U > \varepsilon \|x(t_k)\|_X \} \ \forall k \in \mathbb{N}_0 \\
    t_0 &:= 0, \quad t_{k+1} := \inf \{ t > t_k : \|Fx(t) - Fx(t_k)\|_U > \varepsilon \|x(t)\|_X \} \ \forall k \in \mathbb{N}_0,
\end{align}

where \( \varepsilon > 0 \) is a threshold parameter. If the threshold \( \varepsilon \) is small, then the control input \( u(t) \) is frequently updated. Therefore, we would expect that the event-triggered control system is exponentially stable for all sufficiently small thresholds \( \varepsilon > 0 \) if \( A + BF \) generates an exponentially stable semigroup. One of the fundamental problems we consider is whether or not this intuition is correct.

In addition to stability, the minimum inter-event time \( \inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \) should be guaranteed to be bounded from below by a positive constant; otherwise, infinitely many events might occur in finite time. This phenomenon is called Zeno behavior [see (10)] and makes event-triggering mechanisms infeasible for practical implementation. The additional challenge of event-triggered control is to guarantee no occurrence of Zeno behavior. For finite-dimensional systems, the minimum inter-event time has been extensively investigated; see [1,5,27]. For example, in the finite-dimensional case, it has been shown in [27] that the event-triggering mechanism
\[
    t_0 := 0, \quad t_k := \inf \{ t > t_k : \|x(t) - x(t_k)\|_X > \varepsilon \|x(t_k)\|_X \} \ \forall k \in \mathbb{N}_0
\]
satisfies \( \inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \geq \theta \) for some \( \theta > 0 \). However, this is not true for the infinite-dimensional case, which is illustrated in Examples 3.1 and 3.2. This is the reason why we employ the event-triggering mechanisms (1.2) and (1.3), which compare the plant state and the error \( Fx(t) - Fx(t_k) \) of the control input induced by the event-triggering mechanism. We see in Section 3 that the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) defined by (1.2) satisfies \( \inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \geq \theta \) for some \( \theta > 0 \) if the feedback operator \( F \) is compact. The same result holds for the event-triggering mechanism (1.3) if a strictly positive lower bound on the decay of $T(t)$ is guaranteed.

In the analysis of minimum inter-event times, we exploit the assumption on the compactness of feedback operators, which may restrict applicability. In fact, as shown in Theorem 5.2.3 on p. 229 in [3], if the generator $A$ has residual or continuous spectra in the closed right half plane, then any compact feedback operator cannot guarantee the exponential stability of the semigroup generated by $A + BF$. However, if the unstable part of the system $(A,B)$ is finite-dimensional and controllable, then we
can design a compact feedback operator that guarantees the exponential stability of the semigroup generated by $A + BF$; see, e.g., Theorem 5.2.6 on p. 232 in [3]. One particular example of infinite-dimensional systems with compact feedback operators is partial differential equations (PDEs) in cascade with ordinary differential equations (ODEs). Stabilization of cascaded ODE-PDE systems have been recently studied, e.g., in [4, 26] and references therein. In the case where ODEs are located at the actuator side, the control input is applied to the finite-dimensional system whose dynamics is described by the ODEs, and the input space $U$ is finite-dimensional. Therefore, the feedback operator has finite rank and hence is compact.

After guaranteeing that the minimum inter-event time is positive in Section 3, we analyze the exponential stability of the event-triggered control system in Section 4 for the case where the control operator $B$ is bounded. First, we consider the event-triggering mechanism (1.2) with the time constraint $t_{k+1} - t_k \leq \tau_{\text{max}}$, $k \in \mathbb{N}_0$, where $\tau_{\text{max}} > 0$ can be chosen arbitrarily. Introducing a norm on the state space with respect to which the semigroup generated by $A + BF$ is a contraction, we provide a sufficient condition on the threshold $\varepsilon$ of the event-triggering mechanism for the exponential stability of the closed-loop system. Next we obtain a similar sufficient condition for the event-triggering mechanism (1.3). While we obtain the former result via a trajectory-based approach, a key element in the latter result is the application of the Lyapunov stability theorem.

In Section 5, we study the case where $B$ is unbounded. We first focus on a system with a finite-dimensional unstable part and a feedback operator that stabilizes the unstable part but does not act on the residual stable part. In this case, the feedback operator has a specific structure, but we can achieve the exponential stability of the closed-loop system by using less conservative event-triggering mechanisms for the finite-dimensional part. Second, we consider the case in which the semigroups $T(t)$ is analytic and the feedback operator $F$ has no specific structure but is compact. Moreover, the semigroup generated by $A_{BF}$, where $A_{BF}x = (A + BF)x$ with domain $D(A_{BF}) := \{x \in X : (A + BF)x \in X\}$, is assumed to be exponentially stable. Then we show that the closed-loop system is exponentially stable under periodic event-triggering mechanisms [11, 12] with sufficiently small sampling periods and threshold parameters.

In Section 6, we illustrate the obtained results with numerical examples. First, we study a cascaded system consisting of an ODE and a heat PDE as an example of an infinite-dimensional system with a bounded control operator. Second, we consider an Euler-Bernoulli beam for the case where $B$ is unbounded. From numerical simulations, we see that the event-triggered control systems achieve faster convergence with less control updates than the conventional periodic sampled-data control systems.

**Notation and terminology.** We denote by $\mathbb{Z}$ and $\mathbb{N}$ the set of integers and the set of positive integers, respectively. Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re} \, s > 0\}$, and $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. For $\alpha \in \mathbb{R}$, we define $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \text{Re} \, s > \alpha\}$. Let $X$ and $Y$ be Banach spaces. We denote the space of all bounded linear operators from $X$ to $Y$ by $\mathcal{B}(X,Y)$, and set $\mathcal{B}(X) := \mathcal{B}(X,X)$. We write $T^*$ for the adjoint operator of $T \in \mathcal{B}(X,Y)$. Let $A$ be a linear operator from $X$ to $Y$. The domain of $A$ is denoted by $D(A)$. For a subset $S \subset X$, let $A|_S$ denote the restriction of $A$ to $S$, namely,

$$A|_Sx = Ax \quad \forall x \in D(A) \cap S.$$  

The resolvent set and spectrum of a linear operator $A : D(A) \subset X \to X$ are denoted by $\rho(A)$ and $\sigma(A)$, respectively. Let $T(t)$ be a strongly continuous semi-
group on $X$. The exponential growth bound of $T(t)$ is denoted by $\omega(T)$, that is, $\omega(T) := \lim_{t \to \infty} \ln \|T(t)\|/t$.

We say that the strongly continuous semigroup $T(t)$ is exponentially stable if $\omega(T) < 0$. The space $X_{-1}$ denotes the extrapolation space associated with $T(t)$. More precisely, if $A$ is the generator of $T(t)$, then the space $X_{-1}$ is the completion of $X$ with respect to the norm $\|\cdot\|_{-1} := \|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)}$ for $\lambda \in \rho(A)$. Different choices of $\lambda \in \rho(A)$ lead to equivalent norms on $X_{-1}$. The semigroup $T(t)$ can be extended to a strongly continuous semigroup on $X_{-1}$, and its generator on $X_{-1}$ is an extension of $A$. We shall use the same symbols $T(t)$ and $A$ for the original ones and the associated extensions.

2. Infinite-dimensional system. Let an increasing sequence $\{t_k\}_{k \in \mathbb{N}_0}$ satisfy $t_0 = 0$ and $\inf_{k \in \mathbb{N}_0}(t_{k+1} - t_k) > 0$. We denote by $X$ and $U$ the state space and the input space, and both of them are Hilbert spaces. Let us denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the norm and the inner product of $X$, respectively. As in the periodic sampled-data case [17], where $t_{k+1} - t_k$ is constant for every $k \in \mathbb{N}_0$, consider the following infinite-dimensional system:

\begin{align}
(2.1a) \quad \dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0; \quad x(0) = x^0 \in X \\
(2.1b) \quad u(t) &= Fx(t_k), \quad t_k \leq t < t_{k+1}, \quad k \in \mathbb{N}_0,
\end{align}

where $x(t) \in X$ is the state and $u(t) \in U$ is the input for $t \geq 0$. We assume that $A$ is the generator of a strongly continuous semigroup $T(t)$ on $X$ and that the control operator $B$ and the feedback operator $F$ satisfy $B \in \mathcal{B}(U, X_{-1})$ and $F \in \mathcal{B}(X, U)$, respectively, where $X_{-1}$ is the extrapolation space associated with $T(t)$. We say that $B$ is bounded if $B \in \mathcal{B}(U, X)$; otherwise $B$ is unbounded. For example, if we control the temperature of a rod, then the control operator $B$ is bounded for spatially-distributed actuators but is unbounded for point actuators; see also Chapters 3 and 4 of [3].

The unique solution of the abstract evolution equation (2.1) is given by

\begin{align}
(2.2a) \quad x(0) &= x^0 \\
(2.2b) \quad x(t_k + \tau) &= T(\tau)x(t_k) + \int_0^\tau T(s)BFx(t_k)ds \quad \forall \tau \in (0, t_{k+1} - t_k], \quad \forall k \in \mathbb{N}_0.
\end{align}

In fact, considering $T(t)$ as a semigroup on $X_{-1}$, we find from the standard theory of strongly continuous semigroups that $x$ given by (2.2) satisfies

\begin{align}
(2.3) \quad x \in C(\mathbb{R}_+, X) \quad \text{and} \quad x|_{[t_k, t_{k+1}]} \in C^1([t_k, t_{k+1}], X_{-1}) \quad \forall k \in \mathbb{N}_0
\end{align}

and the following differential equation in $X_{-1}$:

\begin{align}
(2.4) \quad \dot{x}(t) &= Ax(t) + BFx(t_k) \quad \forall t \in (t_k, t_{k+1}), \quad \forall k \in \mathbb{N}_0.
\end{align}

Moreover, only $x$ defined by (2.2) satisfies the properties (2.3) and (2.4).

**Definition 2.1 (Exponential stability).** The system (2.1) is exponential stable if there exist $M \geq 1$ and $\gamma > 0$ such that $x$ given by (2.2) satisfies

\[ \|x(t)\| \leq Me^{-\gamma t}\|x^0\| \quad \forall x^0 \in X, \forall t \geq 0. \]

The supremum over all possible values of $\gamma$ is called the stability margin of the system (2.1).
3. Minimum inter-event time. We call $\inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k)$ the minimum inter-event time. If this value is zero, then the control input may be updated infinitely fast, which is not desirable for practical implementation. The objective of this section is to show that the minimum inter-event time of the following event-triggering mechanisms is bounded from below by a strictly positive constant:

$$
(3.1) \quad t_0 := 0, \quad t_{k+1} := \inf \left\{ t > t_k : \|F x(t) - Fx(t_k)\|_U > \varepsilon \|x(t_k)\| \right\} \quad \forall k \in \mathbb{N}_0
$$

$$
(3.2) \quad t_0 := 0, \quad t_{k+1} := \inf \left\{ t > t_k : \|F x(t) - Fx(t_k)\|_U > \varepsilon \|x(t)\| \right\} \quad \forall k \in \mathbb{N}_0.
$$

In this and the next section, we employ the event-triggering mechanisms (3.1) and (3.2), which compare the plant state and the error $Fx(t) - Fx(t_k)$ of the control input induced by the event-triggering implementation. On the other hand, for finite-dimensional systems, the event-triggering mechanism

$$
(3.3) \quad t_0 := 0, \quad t_{k+1} := \inf \left\{ t > t_k : \|x(t) - x(t_k)\| > \varepsilon \|x(t)\| \right\} \quad \forall k \in \mathbb{N}_0
$$

is commonly used; see, e.g., [27], in which it is proved that the minimum inter-event time of the event-triggering mechanism (3.3) is positive for finite-dimensional systems. However, in the infinite-dimensional case, there exists a triple of an infinite-dimensional systems, an initial state, and a feedback operator such that the inter-event time $t_{k+1} - t_k$ decreases to 0 in finite time.

**Example 3.1.** Let $X = L^2(0, \infty) := L^2([0, \infty), \mathbb{C})$ and consider the shift operator on $L^2(0, \infty)$:

$$
(T(t)x)(s) := x(t + s) \quad \forall x \in L^2(0, \infty), \forall s \geq 0.
$$

Then $T(t)$ is a strongly continuous semigroup on $L^2(0, \infty)$. As discussed in Remark 7 in [2], $T(t)$ is strongly stable but its adjoint $T(t)^*$ is not.

Define an initial state $x^0 \in L^2(0, \infty)$ by

$$
x^0(s) := \begin{cases} 1 & s \leq 1 \\ 0 & s > 1, \end{cases}
$$

and $x(t) = T(t)x^0$ for all $t \geq 0$, which means that the control operator $B$ is arbitrary but the feedback operator $F$ satisfies $F = 0$.

For $\varepsilon \in (0, 1)$, define a time sequence $\{t_k\}_{k \in \mathbb{N}_0}$ by

$$
(3.4) \quad t_0 := 0, \quad t_{k+1} := \inf \left\{ t > t_k : \|x(t) - x(t_k)\|_{L^2} > \varepsilon \|x(t_k)\|_{L^2} \right\} \quad \forall k \in \mathbb{N}_0.
$$

If $t_k \in [0, 1)$, then $\|x(t_k)\|_{L^2}^2 = 1 - t_k$ and

$$
\|T(\tau)x(t_k) - x(t_k)\|_{L^2}^2 = \int_{1-(t_k+\tau)}^{1-t_k} 1 \, ds = \tau \quad \forall \tau \in [0, 1 - t_k].
$$

It follows that

$$
t_{k+1} = t_k + \varepsilon^2 (1 - t_k) \quad \forall k \in \mathbb{N}_0.
$$

Similarly, if we define $\{t_k\}_{k \in \mathbb{N}_0}$ by the event-triggering mechanism (3.3), then

$$
t_{k+1} = t_k + \frac{\varepsilon^2}{1+\varepsilon^2} (1 - t_k) \quad \forall k \in \mathbb{N}_0.
$$

Both of the time sequences $\{t_k\}_{k \in \mathbb{N}_0}$ are monotonically increasing and converge to 1. Thus the minimum inter-event time $\inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k)$ is zero.
We next provide an example of infinite-dimensional event-triggered control systems in which $\inf_{x \in X} t_1 = 0$.

**Example 3.2.** Consider a metal rod of length 1 that is insulated at either end and can be heated along its length:

\begin{align}
\frac{\partial z}{\partial t}(\xi, t) &= \frac{\partial^2 z}{\partial \xi^2}(\xi, t) + v(\xi, t), \quad \xi \in [0, 1], \ t \geq 0 \\
\frac{\partial z}{\partial \xi}(0, t) &= 0, \quad \frac{\partial z}{\partial \xi}(1, t) = 0, \quad t \geq 0,
\end{align}

where $z(\xi, t)$ and $v(\xi, t)$ are the temperature of the rod and the addition of heat along the rod at position $\xi \in [0, 1]$ and time $t \geq 0$, respectively. We can reformulate the partial differential equation (3.5) as an abstract evolution equation (2.1a) with $X := L^2(0, 1) := L^2([0, 1], \mathbb{C})$, $U := L^2(0, 1)$, $x(t) := z(\cdot, t)$, and $u(t) := v(\cdot, t)$. As shown in Example 2.3.7 on p. 45 in [3], the generator $A$, the strongly continuous semigroup $T(t)$ generated by $A$, and the control operator $B$ are given by

$$Ax := -\sum_{n=0}^{\infty} n^2 \pi^2 \langle x, \phi_n \rangle_{L^2} \phi_n$$

with domain

$$D(A) := \left\{ x \in L^2(0, 1) : \sum_{n=0}^{\infty} n^4 \pi^4 \langle x, \phi_n \rangle_{L^2}^2 < \infty \right\}$$

and

$$T(t)x := \sum_{n=0}^{\infty} e^{-n^2 \pi^2 t} \langle x, \phi_n \rangle_{L^2} \phi_n \quad \forall x \in X, \ \forall t \geq 0; \quad B := I,$$

where $\phi_0(\xi) := 1$ and $\phi_n(\xi) := \sqrt{2} \cos(n\pi \xi)$, $n \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 1)$. Define the feedback operator $F \in \mathcal{B}(L^2(0, 1))$ by $Fx := -\langle x, \phi_0 \rangle_{L^2} \phi_0$. Although $T(t)$ is not exponentially stable, the strongly continuous semigroup $T_{BF}(t)$ generated by $A + BF$ is exponentially stable. For this system, we consider the event-triggering mechanism (3.3) and show $\inf_{x \in X} t_1 = 0$ for every threshold $\varepsilon > 0$.

Let the initial state $x^0$ be given by $x^0 := \phi_n$ with $n \in \mathbb{N}$. Then $Fx^0 = 0$ and hence $x(t) = e^{-n^2 \pi^2 t} \phi_n$ for every $t \in [0, t_1)$. Since

$$\|x(t) - x^0\| = 1 - e^{-n^2 \pi^2 t}, \quad \|x(t)\| = e^{-n^2 \pi^2 t} \quad \forall t \in [0, t_1)$$

it follows that $t_1 = \log(1 + \varepsilon)/(n^2 \pi^2) \to 0$ as $n \to \infty$. Thus, we obtain $\inf_{x \in X} t_1 = 0$ for every $\varepsilon > 0$. \[\square\]

In Example 3.1, we consider a situation where the state goes to zero in finite time with zero control input. In Example 3.2, we only show that the first inter-event time is close to zero if we choose a certain initial state. Therefore, we cannot say that the event-triggering mechanism (3.3) fails for practical control systems. However, these examples imply that there exists a triple of an infinite-dimensional system, an initial state, and a compact feedback operator for which the minimum inter-event time can be arbitrarily close to zero. This is the reason why we use the event-triggering mechanisms (3.1) and (3.2).
For $\tau \geq 0$, define the operator $S_{\tau} : U \to X$ by

$$S_{\tau}u := \int_0^\tau T(s)Bu.$$ 

The following lemma is useful when we evaluate the minimum inter-event time:

**Lemma 3.3 (Lemma 2.2 in [17]).** For any $\tau \geq 0$, $S_{\tau} \in B(U,X)$, and for any $\theta > 0$,

$$\sup_{0 \leq \tau \leq \theta} \|S_{\tau}\|_{B(U,X)} < \infty. \tag{3.6}$$

Moreover, if $F \in B(X,U)$ is compact, then

$$\lim_{\tau \to 0} \|S_{\tau}F\|_{B(X,U)} = 0. \tag{3.7}$$

For compact operators, the next lemma is also known:

**Lemma 3.4 (Lemma 2.1 in [17]).** Let $X, Y$, and $Z$ be Hilbert spaces and let $\Gamma : [0,1] \to B(X,Y)$ be given. If $\lim_{t \to 0} \Gamma(t)^*y = 0$ for all $y \in Y$ and if $\Lambda \in B(Y,Z)$ is compact, then

$$\lim_{t \to 0} \|\Lambda \Gamma(t)^*y\|_{B(X,Z)} = 0. \tag{3.8}$$

Using these lemmas, we obtain the following result:

**Lemma 3.5.** Assume that $T(t)$ is a strongly continuous semigroup on $X$, $B \in B(U,X-1)$, and $F \in B(X,U)$ is compact. Set

$$x(\tau) = (T(\tau) + S_{\tau}F)x^0 \quad \forall \tau > 0; \quad x^0 \in X. \tag{3.9}$$

For every $\varepsilon > 0$, there exists $\theta > 0$ such that for every $\tau \in [0,\theta)$,

$$\|Fx(\tau) - Fx^0\|_U \leq \varepsilon \|x^0\| \quad \forall x^0 \in X, \forall \tau \in [0,\theta).$$

**Proof.** Since (3.7) yields

$$x(\tau) - x^0 = (T(\tau) - I)x^0 + S_{\tau}Fx^0 \quad \forall \tau \geq 0,$$

it follows that

$$\|Fx(\tau) - Fx^0\|_U \leq \|F(T(\tau) - I) + FS_{\tau}F\|_{B(X,U)} \cdot \|x^0\| \quad \forall \tau \geq 0. \tag{3.10}$$

By Lemma 3.3,

$$\lim_{\tau \to 0} \|FS_{\tau}F\|_{B(X,U)} = 0. \tag{3.11}$$

Since $T(t)^*$ is strongly continuous (see, e.g., Theorem 2.2.6 on p. 37 in [3]), we obtain

$$\lim_{\tau \to 0} \|(T(\tau)^* - I)x\| = 0 \quad \forall x \in X,$$

and hence it follows from Lemma 3.4 that

$$\lim_{\tau \to 0} \|F(T(\tau) - I)\|_{B(X,U)} = 0. \tag{3.12}$$

Thus, for every $\varepsilon > 0$, there exists $\theta > 0$ such that for every $\tau \in [0,\theta)$,

$$\|F(T(\tau) - I) + FS_{\tau}F\|_{B(X,U)} \leq \|F(T(\tau) - I)\|_{B(X,U)} + \|FS_{\tau}F\|_{B(X,U)} < \varepsilon \tag{3.13}$$

Combining (3.10) and (3.13), we obtain the desired result. \qed
Therefore, it follows from (3.10) that there exists $M$ such that (3.11) holds for every $\varepsilon > 0$ and $\theta > 0$ such that

$$\inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \geq \theta$$

for every initial state $x^0 \in X$.

**Proof.** From (2.3), it follows that $x(t_k) \in X$ for every $k \in \mathbb{N}_0$. Therefore, we see from (2.2) and Lemma 3.5 that, for every $\varepsilon > 0$, there exists $\theta > 0$ such that

$$\|F(x(t_k + \tau) - F(x(t_k))\| \leq \varepsilon \|x(t_k)\| \quad \forall x^0 \in X, \forall \tau \in [0, \theta), \forall k \in \mathbb{N}_0.$$ 

Thus, the time sequence $\{t_k\}_{k \in \mathbb{N}_0}$ satisfies $\inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \geq \theta$.

We next investigate the minimum inter-event time of the event-triggering mechanism (3.2). To that purpose, we use the following estimate:

**Lemma 3.7.** Assume that $T(t)$ is a strongly continuous semigroup on $X$, $B \in \mathcal{B}(U, X_{-1})$, and $F \in \mathcal{B}(X, U)$ is compact. Define $x(t)$ as in (3.7). There exist $c_1 > 0$ and $s_1 > 0$ such that the semigroup $T(t)$ satisfies

$$(3.10) \quad \|T(s_1)x^0\| \geq c_1 \|x^0\| \quad \forall x^0 \in X$$

if and only if there exist $c_2 \geq 1$ and $\theta > 0$ such that

$$(3.11) \quad \|x^0\| \leq c_2 \|x(\tau)\| \quad \forall x^0 \in X, \forall \tau \in [0, \theta).$$

**Proof.** Suppose first that (3.10) holds for some $c_1 > 0$ and $s_1 > 0$. Since there exists $M \geq 1$ such that

$$\|T(\tau)\|_{\mathcal{B}(X)} \leq M \quad \forall \tau \in [0, s_1],$$

it follows from (3.10) that

$$c_1 \|x^0\| \leq \|T(s_1)x^0\| = \|T(s_1 - \tau)T(\tau)x^0\| \leq M \|T(\tau)x^0\| \quad \forall x^0 \in X, \forall \tau \in [0, s_1].$$

Therefore,

$$(3.12) \quad \|T(\tau)x^0\| \geq \frac{c_1}{M} \|x^0\| \quad \forall x^0 \in X, \forall \tau \in [0, s_1].$$

By (3.6), there exists $s_2 \in (0, s_1]$ such that

$$\|S_\tau Fx^0\| \leq \|S_\tau F\|_{\mathcal{B}(X)} \cdot \|x^0\| \leq \frac{c_1}{2M} \|x^0\| \quad \forall x^0 \in X, \forall \tau \in [0, s_2].$$

Combining (3.12) and (3.13), we obtain

$$(3.14) \quad \|x(\tau)\| \geq \|T(\tau)x^0\| - \|S_\tau Fx^0\| \geq \frac{c_1}{2M} \|x^0\| \quad \forall x^0 \in X, \forall \tau \in [0, s_2].$$

Therefore, (3.11) holds with $c_2 := 2M/c_1$ and $\theta := s_2$.

Conversely, if (3.11) holds for some $c_2 \geq 1$ and $\theta > 0$, then

$$\|x^0\| \leq c_2 \|T(\tau)x^0\| + c_2 \|S_\tau Fx^0\| \quad \forall x^0 \in X, \forall \tau \in [0, \theta).$$
Using (3.6) again, we find that there exists \( s_1 \in (0, \theta) \) such that
\[
\|S_\tau F\|_{B(X)} \leq \frac{1}{2c_2} \quad \forall \tau \in [0, s_1].
\]
Hence,
\[
\|T(\tau)x^0\| \geq \frac{1}{c_2} (\|x^0\| - c_2\|S_\tau Fx^0\|) \geq \frac{1}{2c_2}\|x^0\| \quad \forall x^0 \in X, \forall \tau \in [0, s_1].
\]
Thus the desired inequality (3.10) holds.

**Remark 3.8.** Suppose that \( B \) is bounded, i.e., \( B \in \mathcal{B}(U, X) \). Then
\[
\lim_{\tau \to 0} \|S_\tau \|_{\mathcal{B}(U, X)} = 0,
\]
and hence the compactness of \( F \) is not required in Lemma 3.7.

**Remark 3.9.** The condition (3.10) appears also in Theorem 2 of [19] for the applicability of the Lyapunov stability theorem, and Corollary 1 of [19] shows that \( T(t) \) satisfies (3.10) for some \( c_1 > 0 \) and \( s_1 > 0 \) and the range of \( T(t) \) is dense in \( X \) for some \( t \in (0, s_1) \) if and only if \( T(t) \) can be extended to a strongly continuous group on \( X \).

Using Lemmas 3.5 and 3.7, we show that the minimum inter-event time of the event-triggering mechanism (3.2) is bounded from below by a strictly positive constant if \( T(t) \) satisfies (3.10) for some \( c_1 > 0 \) and \( s_1 > 0 \).

**Theorem 3.10.** Assume that \( A \) generates a strongly continuous semigroup \( T(t) \) on \( X \), \( B \in \mathcal{B}(U, X_{-1}) \), and \( F \in \mathcal{B}(X, U) \) is compact. Assume further that the semigroup \( T(t) \) satisfies (3.10) for some \( c_1 > 0 \) and \( s_1 > 0 \). For the system (2.1), define the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) by the event-triggering mechanism (3.2). For every \( \varepsilon > 0 \), there exists \( \theta > 0 \) such that \( \inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \geq \theta \) for every initial state \( x^0 \in X \).

**Proof.** Combining Lemmas 3.5 and 3.7, we have that, for every \( \varepsilon > 0 \), there exists \( \theta > 0 \) such that
\[
\|Fx(t_k + \tau) - Fx(t_k)\|_U \leq \varepsilon \|x(t_k + \tau)\| \quad \forall x^0 \in X, \forall \tau \in [0, \theta), \forall k \in \mathbb{N}_0.
\]
Thus the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) satisfies \( \inf_{k \in \mathbb{N}_0} (t_{k+1} - t_k) \geq \theta \).

We conclude this section with a result on the continuous dependence of solutions of the evolution equation (2.1) under the event-triggering mechanism (3.2) on initial states. The technical difficulty is that the updating instants of control inputs are different depending on initial states in event-triggered control systems. The analysis of continuous dependence on initial states is straightforward but lengthy. The proof of the following theorem can be found in the Appendix A.

**Theorem 3.11.** Assume that \( A \) generates a strongly continuous semigroup \( T(t) \) on \( X \), \( B \in \mathcal{B}(U, X_{-1}) \), and \( F \in \mathcal{B}(X, U) \) is compact. Assume further that the semigroup \( T(t) \) satisfies (3.10) for some \( c_1 > 0 \) and \( s_1 > 0 \). Let \( x \) be the solution of the evolution equation (2.1) with the initial state \( x^0 \in X \) under the event-triggering mechanism (3.2) with an arbitrary threshold \( \varepsilon > 0 \). For every \( t_\varepsilon > 0 \) and every \( \delta > 0 \), there exists \( \delta_0 > 0 \) such that for every \( \zeta^0 \in X \) satisfying \( \|x^0 - \zeta^0\| < \delta_0 \),
\[
\|x(t) - \zeta(t)\| < \delta \quad \forall t \in [0, t_\varepsilon],
\]
where \( \zeta \) is the solution of the evolution equation (2.1) with the initial state \( \zeta^0 \) under the event-triggering mechanism (3.2).
4. Stability analysis under bounded control. In this section, we analyze closed-loop stability in the case where the control operator $B$ is bounded. The objective is to show that if the feedback operator $F$ is compact and if the semigroup generated by $A + BF$ is exponentially stable, then the event-triggered control system is exponentially stable, provided that the threshold $\varepsilon$ is sufficiently small.

Choose $\tau_{\text{max}} > 0$ arbitrarily. We first define a time sequence $\{t_k\}_{k \in \mathbb{N}_0}$ by

\begin{align}
(4.1a) \quad & t_0 := 0, \quad \psi_{k+1} := \inf \{ t > t_k : \| Fx(t) - Fx(t_k) \|_U > \varepsilon \| x(t_k) \| \} \\
(4.1b) \quad & t_{k+1} := \min \{ t_k + \tau_{\text{max}}, \; \psi_{k+1} \} \quad \forall k \in \mathbb{N}_0.
\end{align}

The above event-triggering mechanism is based on (3.1) and satisfies the time constraint $t_{k+1} - t_k \leq \tau_{\text{max}}$ for every $k \in \mathbb{N}_0$.

**THEOREM 4.1.** Assume that $A$ generates a strongly continuous semigroup $T(t)$ on $X$, $B \in \mathcal{B}(U, X)$, and $F \in \mathcal{B}(X, U)$ is compact. Assume that the semigroup $T_{BF}(t)$ generated by $A + BF$ is exponentially stable, i.e., there exists $M \geq 1$ and $\omega > 0$ such that

\begin{equation}
(4.2) \quad \| T_{BF}(t) \|_{\mathcal{B}(X)} \leq Me^{-\omega t} \quad \forall t \geq 0.
\end{equation}

If the threshold $\varepsilon > 0$ satisfies

\begin{equation}
(4.3) \quad \varepsilon < \frac{\omega}{M\|B\|_{\mathcal{B}(U, X)}},
\end{equation}

then for every $\tau_{\text{max}} > 0$, the system (2.1) with the event-triggering mechanism (4.1) is exponentially stable and its stability margin is at least $\gamma$ defined by

\begin{equation}
(4.4) \quad \gamma := -\frac{\log \left( (1-\varepsilon_0)e^{-\omega \tau_{\text{max}}} + \varepsilon_0 \right)}{\tau_{\text{max}}} = \frac{\varepsilon_0}{\omega},
\end{equation}

where $\varepsilon_0 := \frac{\varepsilon M\|B\|_{\mathcal{B}(U, X)}}{\omega}$.

**Proof.** As in the proof of Theorem 5.2 on p. 19 in [20] and Theorem 3.1 in [17], we introduce a new norm $\| \cdot \|$ on $X$, which is defined by

\begin{equation}
|x| := \sup_{t \geq 0} \| e^{\omega t} T_{BF}(t) x \|.
\end{equation}

This norm satisfies

\begin{equation}
(4.5) \quad \| x \| \leq |x| \leq M\|x\|, \quad \| T_{BF}(t) x \| \leq e^{-\omega t} |x| \quad \forall x \in X, \; \forall t \geq 0;
\end{equation}

see, e.g., the proof of Theorem 3.1 in [17].

Noting that

\begin{equation}
(4.6) \quad \dot{x}(t) = (A + BF)x(t) - B(Fx(t) - Fx(t_k)) \quad \forall t \in (t_k, t_{k+1}), \; \forall k \in \mathbb{N}_0,
\end{equation}

we see from a routine calculation (see, e.g., Exercise 3.3 in [3]) that $x(t_k + \tau)$ given in (2.2) can be written as

\begin{equation}
(4.7) \quad x(t_k + \tau) = T_{BF}(\tau)x(t_k) - \int_0^\tau T_{BF}(\tau - s)B(Fx(t_k + s) - Fx(t_k))ds
\end{equation}

for every $\tau \in (0, t_{k+1} - t_k]$. Since the event-triggering mechanism (4.1) guarantees

\begin{equation}
\sup_{0 \leq s < t_{k+1} - t_k} \| Fx(t_k + s) - Fx(t_k) \|_U \leq \varepsilon \| x(t_k) \| \quad \forall k \in \mathbb{N}_0,
\end{equation}

we have
the properties (4.5) yield

\[ |x(t_k + \tau)| \leq e^{-\omega \tau} |x(t_k)| + \int_0^\tau e^{-\omega (\tau - s)} |B(Fx(t_k + s) - Fx(t_k))| ds \]

\[ \leq e^{-\omega \tau} |x(t_k)| + \frac{1}{\omega} M\|B\|_{B(U, X)} \cdot \sup_{0 \leq s < \tau} \|Fx(t_k + s) - Fx(t_k)\|_U \]

\[ \leq ((1 - \delta_0) e^{-\omega \tau} + \delta_0) \cdot |x(t_k)| \quad \forall \tau \in (0, t_{k+1} - t_k], \forall k \in \mathbb{N}_0, \]

where \( \delta_0 \) is defined as in (4.1) and satisfies \( 0 < \delta_0 < 1 \) from (4.3).

The function \( f: (0, \infty) \rightarrow \mathbb{R} \) defined by

\[ f(\tau) := \frac{-\log ((1 - \delta_0)e^{-\omega \tau} + \delta_0)}{\tau} \]

is monotonically decreasing in \((0, \infty)\). Since

\[(1 - \delta_0)e^{-\omega \tau} + \delta_0 < (1 - \delta_0) + \delta_0 = 1 \quad \forall \tau > 0,\]

it follows that \( f(\tau) > 0 \) for every \( \tau > 0 \). Therefore, \( \gamma := f(\tau_{\text{max}}) \) satisfies \( \gamma > 0 \) and

\[ |x(t_k + \tau)| \leq e^{-\gamma \tau} |x(t_k)| \quad \forall \tau \in (0, t_{k+1} - t_k], \forall k \in \mathbb{N}_0. \]

Using (4.7) recursively, we obtain

\[ |x(t)| \leq e^{-\gamma t} |x^0| \quad \forall x^0 \in X, \forall t \geq 0. \]

Thus, (4.5) yields

\[ \|x(t)\| \leq |x(t)| \leq e^{-\gamma t} |x^0| \leq M e^{-\gamma t} \|x^0\| \quad \forall x^0 \in X, \forall t \geq 0. \]

This completes the proof.

Next we define the time sequence \( \{t_k\}_{k\in\mathbb{N}_0} \) by the event-triggering mechanism (3.2) and show the exponential stability of the closed-loop system. Instead of the trajectory-based approach in Theorem 4.1, we here apply the Lyapunov stability theorem.

**Theorem 4.2.** Assume that \( A \) generates a strongly continuous semigroup \( T(t) \) on \( X \), \( B \in B(U, X) \), and \( F \in B(X, U) \) is compact. Assume further that the semigroup \( T(t) \) satisfies (3.10) for some \( c_1 > 0 \) and \( s_1 > 0 \) and that the semigroup \( T_{BF}(t) \) generated by \( A + BF \) is exponentially stable. If the threshold \( \delta > 0 \) satisfies

\[ \delta < \frac{1}{2\|PB\|_{B(U, X)}} \quad \text{where} \quad PB := \int_0^\infty T_{BF}(t)^* T_{BF}(t) x dt \quad \forall x \in X, \]

then the system (2.1) with the event-triggering mechanism (3.2) is exponentially stable and its stability margin is at least \( \gamma \) defined by

\[ \gamma := \frac{1 - 2\delta\|PB\|_{B(U, X)}}{2 \int_0^\infty \|T_{BF}(t)\|^2_{B(X)} dt}. \]

**Proof.** There exists \( M \geq 1 \) such that

\[ \|T(\tau)\|_{B(X)} \leq M, \quad \|T_{BF}(\tau)\|_{B(X)} \leq M \quad \forall \tau \in [0, s_1]. \]
Using (3.10), we obtain
\[ c_1 \|x\| \leq \|T(s_1)x\| = \|T(s_1 - \tau)T(\tau)x\| \leq M\|T(\tau)x\| \quad \forall x \in X, \forall \tau \in [0, s_1]. \]

Hence
\[ (4.11) \quad \|T(\tau)x\| \geq \frac{c_1}{M}\|x\| \quad \forall x \in X, \forall \tau \in [0, s_1]. \]

Since \(T_{BF}(\tau)\) satisfies the variation of parameters formula (see Theorem 3.2.1 on p. 110 in [3]):
\[ (4.12) \quad T_{BF}(\tau)x = T(\tau)x + \int_0^\tau T_{BF}(\tau - s)BFT(s)x ds \quad \forall x \in X, \]

it follows from (4.10) and (4.11) that
\[ \|T_{BF}(\tau)x\| \geq \|T(\tau)x\| - \int_0^\tau \|T_{BF}(\tau - s)BFT(s)x\| ds \]
\[ \geq \left( \frac{c_1}{M} - \tau M^2\|BF\|_{B(X)} \right)\|x\| \quad \forall x \in X, \forall \tau \in [0, s_1]. \]

Therefore,
\[ (4.13) \quad \|T_{BF}(\tau)x\| \geq \frac{c_1}{2M}\|x\| \quad \forall x \in X, \forall \tau \in [0, s_3], \]

where
\[ s_3 := \min \left\{ s_1, \frac{c_1}{2M^3\|BF\|_{B(X)}} \right\} > 0. \]

Since \(T_{BF}(t)\) is exponentially stable, it follows from Theorem 5.1.3 on p. 217 in [3] that there exists a positive operator \(P \in B(X)\) such that the following Lyapunov equality holds:
\[ (4.14) \quad \langle (A + BF)x, Px \rangle + \langle Px, (A + BF)x \rangle = -\|x\|^2 \quad \forall x \in D(A), \]

and such an operator \(P\) is given as in (4.8). Using (4.13), we have from Theorem 2 in [19] that there exist \(\alpha, \beta > 0\) such that
\[ (4.15) \quad \alpha\|x\|^2 \leq \langle Px, x \rangle \leq \beta\|x\|^2 \quad \forall x \in X. \]

Since
\[ \langle Px, x \rangle = \int_0^\infty \|T_{BF}(t)x\|^2 dt \leq \int_0^\infty \|T_{BF}(t)\|^2 dt \cdot \|x\|^2, \]

we can choose \(\beta > 0\) in (4.15) so that
\[ \beta \leq \int_0^\infty \|T_{BF}(t)\|^2 dt. \]

Assume that \(x^0 \in D(A)\). Then the mild solution \(x\) of (2.2) is also a classical solution, i.e., \(x\) satisfies the differential equation (2.4) in \(X\); see e.g., Theorem 3.1.3 on p. 103 in [3]. Moreover, if we define \(e(t) := Fx(t) - F(t_k)\) for \(t_k \leq t < t_{k+1}\) and \(k \in \mathbb{N}_0\), which is the error induced by the event-triggered implementation, then
\[ \|e(t)\|_U \leq \varepsilon \|x(t)\| \quad \forall t \geq 0. \]
under the event-triggering mechanism (3.2). Therefore, using (4.6) and (4.14), we find that for every $t \in (t_k, t_{k+1})$ and every $k \in \mathbb{N}_0$, $V(t) := \langle P_x(t), x(t) \rangle$ satisfies

\[
\frac{dV}{dt}(t) \leq -\|x(t)\|^2 + 2\|PB\|_{B(U, X)} \cdot \|x(t)\| \cdot \|e(t)\| U \\
\leq -(1 - 2\varepsilon\|PB\|_{B(U, X)})\|x(t)\|^2 \\
\leq -2\gamma V(t),
\]

where $\gamma > 0$ is defined as in (4.9). We see from the positive definiteness (4.15) of $P$ that

\[
\alpha \|x(t)\|^2 \leq V(t) \leq e^{-2\gamma t} V(0) \leq \beta e^{-2\gamma t} \|x_0\|^2 \quad \forall t \geq 0,
\]

and hence

\[
(4.16) \quad \|x(t)\| \leq \sqrt{\frac{\beta}{\alpha}} e^{-\gamma t} \|x_0\| \quad \forall x_0 \in D(A), \forall t \geq 0.
\]

Finally, we show the exponential stability for all initial states in $X$. Fix $x_0 \in X$ and $t_e \geq 0$ arbitrarily, and let $x$ be the solution of the evolution equation (2.2) with the initial state $x_0$. Since $D(A)$ is dense in $X$, Theorem 3.11 shows that for every $\delta > 0$, there exists $\zeta_0 \in D(A)$ such that the solution $\zeta$ of the evolution equation (2.2) with the initial state $\zeta_0$ satisfies

\[
\|x(t) - \zeta(t)\| < \delta \quad \forall t \in [0, t_e].
\]

Therefore we have from (4.16) that

\[
\|x(t)\| \leq \|x(t) - \zeta(t)\| + \|\zeta(t)\| < \left(1 + \sqrt{\frac{\beta}{\alpha}}\right) \delta + \sqrt{\frac{\beta}{\alpha}} e^{-\gamma t} \|x_0\| \quad \forall t \in [0, t_e].
\]

Since $\delta > 0$ was arbitrary, we obtain

\[
\|x(t)\| \leq \sqrt{\frac{\beta}{\alpha}} e^{-\gamma t} \|x_0\| \quad \forall t \in [0, t_e]. \quad \Box
\]

Moreover, $t_e \geq 0$ was also arbitrary. Thus, the closed-loop system (2.1) is exponentially stable and its stability margin is at least $\gamma$.

**Remark 4.3.** If $M \geq 1$ and $\omega > 0$ satisfy $\|T_BF(t)\|_{B(X)} \leq Me^{-\omega t}$ for all $t \geq 0$, then we obtain $\|P\|_{B(X)} \leq M^2/(2\omega)$. Therefore, (4.8) and (4.9) can be rewritten as

\[
\varepsilon < \frac{\omega}{M^2\|B\|_{B(U, X)}}, \quad \gamma \geq \frac{\omega - \varepsilon M^2\|B\|_{B(U, X)}}{M^2}.
\]

5. **Stability analysis under unbounded control.** Throughout this section, we study the case $B \in B(U, X_{-1})$. We analyze the exponential stability of the closed-loop system under two event-triggering mechanisms. The first mechanism is based on system decomposition, and the second one employs a periodic event-triggering condition developed in [11,12].

5.1. **Event-triggered control based on system decomposition.**
5.1.1. System decomposition. In what follows, we shall place a number of assumptions on the infinite-dimensional system (2.1) and recall the decomposition of infinite-dimensional systems under unbounded control used in [16–18].

Assumption 5.1. There exists $\alpha < 0$ such that $\sigma(A) \cap \mathbb{C}_\alpha$ consists of finitely many eigenvalues of $A$ with finite algebraic multiplicities.

If Assumption 5.1 holds, then we can decompose $X$ by a standard technique (see, e.g., Lemma 2.5.7 on p. 71 in [3] or Proposition IV.1.16 on p. 245 in [7]) as follows. There exists a rectifiable, closed, simple curve $\Gamma$ in $\mathbb{C}$ enclosing an open set that contains $\sigma(A) \cap \mathbb{C}_\alpha$ in its interior and $\sigma(A) \cap (\mathbb{C} \setminus \mathbb{C}_\alpha)$ in its exterior. The operator $\Pi : X \to X$, defined by

$$\Pi := \frac{1}{2\pi i} \int_{\Gamma} (sI - A)^{-1} ds,$$

where $\Gamma$ is traversed once in the counterclockwise direction, is a projection operator, and we can decompose $X$ to be

$$X = X^+ \oplus X^-, \quad \text{where} \quad X^+ := \Pi X \quad \text{and} \quad X^- := (I - \Pi)X.$$

This decomposition satisfies $\dim X^+ < \infty$ and $X^+ \subset D(A)$. Moreover, $X^+$ and $X^-$ are $T(t)$-invariant for all $t \geq 0$. Define

$$A^+ := A|_{X^+}, \quad A^- := A|_{X^-}, \quad T^+(t) := T(t)|_{X^+}, \quad T^-(t) := T(t)|_{X^-}.$$

Then $\sigma(A^+) = \sigma(A) \cap \mathbb{C}_\alpha$ and $\sigma(A^-) = \sigma(A) \cap (\mathbb{C} \setminus \mathbb{C}_\alpha)$. Note that $A^+$ and $A^-$ generate the strongly continuous semigroups $T^+(t)$ on $X^+$ and $T^-(t)$ on $X^-$, respectively. Let $(X^-)_{-1}$ denote the extrapolation space associated with $T^-(t)$. The semigroup $T^-(t)$ can be extended to a strongly continuous semigroup on $(X^-)_{-1}$, and its generator on $(X^-)_{-1}$ is an extension of $A^-$ on $X^-$. The same symbols $T^-(t)$ and $A^-$ will be used to denote these extensions. Since the spectrum of the operator $A$ on $X$ is equal to the spectrum of the operator $A$ on $X_-$, the projection operator $\Pi$ on $X$ defined by (5.1) can be extended to a projection $\Pi_{-1}$ on $X_-$. If $\lambda \in \sigma(A)$ and if $\lambda I - A$ is considered as an operator in $\mathcal{B}(X, X_{-1})$, then $\Pi_{-1}$ is similar to $\Pi$, i.e.,

$$\Pi_{-1} = (\lambda I - A)\Pi(\lambda I - A)^{-1}$$

and satisfies $\Pi_{-1}X_{-1} = \Pi X = X^+$. Using the extended projection operator $\Pi_{-1}$, we can decompose the control operator $B \in \mathcal{B}(U, X_{-1})$:

$$B^+ := \Pi_{-1} B, \quad B^- := (I - \Pi_{-1})B.$$

Since $(X^-)_{-1}$ and $(X_{-1})^- := (I - \Pi_{-1})X_{-1}$ are both completions of $X^-$ endowed with the norm $\| \cdot \|_{-1}$, we can identify $(X^-)_{-1}$ and $(X_{-1})^-$ (see, e.g., the footnote 3 of p. 1213 in [16]). We also decompose the feedback operator $F \in \mathcal{B}(X, U)$:

$$F^+ := F|_{X^+}, \quad F^- := F|_{X^-}.$$

In addition to Assumption 5.1, we impose the following assumptions:

Assumption 5.2. The exponential growth bound $\omega(T^-)$ satisfies $\omega(T^-) < 0$.

Assumption 5.3. The pair $(A^+, B^+)$ is controllable.

Remark 5.4. It is shown in [17] that if $A$ generates an analytic semigroup and if there exists a compact operator $F \in \mathcal{B}(X, U)$ such that the semigroup generated by $A_{BF}$ is exponentially stable, then Assumptions 5.1–5.3 hold, where the operator $A_{BF} : D(A_{BF}) \subset X \to X$ by

$$A_{BF}x = (A + BF)x \quad \text{with domain} \quad D(A_{BF}) := \{ x \in X : (A + BF)x \in X \}.$$
5.1.2. Stability analysis. For every \( x \in X \), define \( x^+ := \Pi x \) and \( x^- := (I - \Pi)x \). For convenience, we use the notation \( x^+(t) \) and \( x^-(t) \) instead of \( x(t)^+ \) and \( x(t)^- \), respectively. Assume that the feedback operator \( F \in \mathcal{B}(X, U) \) satisfies \( F^- = F|_{X^-} = 0 \). Then the control input in (2.1b) is given by
\[
(5.5) \quad u(t) = Fx(t_k) = F^+ x^+(t_k), \quad t_k \leq t < t_{k+1}, \; k \in \mathbb{N}_0.
\]
In this subsection, we choose the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) so that the finite-dimensional system
\[
(5.6a) \quad x^+(0) = \Pi x^0 \in X^+ \\
(5.6b) \quad \dot{x}^+(t) = A^+ x^+(t) + B^+ F^+ x^+(t_k) \quad \forall t \in (t_k, t_{k+1}), \; \forall k \in \mathbb{N}_0.
\]
is exponentially stable and its stability margin is at least \( \gamma^+ > 0 \). For example, as in Theorem 4.2, we can define the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) by
\[
(5.7a) \quad t_0 := 0, \quad \psi_{k+1} := \inf \{ t > t_k : \|F^+ x^+(t) - F^+ x^+(t_k)\|_U > \varepsilon \|x^+(t)\| \} \\
(5.7b) \quad t_{k+1} := \min \{ t_k + \tau_{\max}, \psi_{k+1} \} \quad \forall k \in \mathbb{N}_0.
\]
Since \( X^+ \) is finite dimensional, we obtain less conservative conditions on the threshold \( \varepsilon \) for the closed-loop system (5.6) to be exponentially stable. In particular, we obtain the following result on the event-triggering mechanism (5.7):

**Proposition 5.5.** Consider the finite-dimensional system (5.6) and the event-triggering mechanism (5.7). Assume that the input space \( U \) is finite dimensional and choose \( \gamma^+ > 0 \). If there exist positive definite matrices \( P, Q \) and a positive scalar \( \kappa \) such that the following linear matrix inequalities are feasible:
\[
(5.8a) \quad \begin{bmatrix} Q - \varepsilon^2 \kappa I & -PB^+ \\ -(B^+)^*P & \kappa I \end{bmatrix} \succeq 0 \\
(5.8b) \quad (A^+ + B^+ F^+)^* P + P(A^+ + B^+ F^+) \preceq -(\gamma^+)^2 P - Q,
\]
them, for all \( \tau_{\max} > 0 \), the finite-dimensional system (5.6) is exponentially stable and its stability margin is at least \( \gamma^+ \).

**Proof.** We can prove Proposition 5.5 similarly to Theorem III.3 in [6]. Therefore, the proof is omitted.

**Theorem 5.6.** Suppose that Assumptions 5.1–5.3 hold. Let \( F^- = 0 \) and assume the feedback gain \( F^+ \) and the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) are chosen so that the finite-dimensional system (5.6) is exponentially stable, its stability margin is at least \( \gamma^+ > 0 \), and there exist \( \tau_{\max}, \tau_{\min} > 0 \) such that \( \tau_{\min} \leq t_{k+1} - t_k \leq \tau_{\max} \) for all \( x^+(0) \in X^+ \) and all \( k \in \mathbb{N}_0 \). Then the infinite-dimensional system (2.1) is exponentially stable and its stability margin is at least \( \min \{ \gamma^+, -\omega(T^-) \} \).

**Proof.** Setting \( S_+^\tau := \int_0^\tau T^+(s) B^+ ds \) and \( S^-_\tau := \int_0^\tau T^-(s) B^- ds \), we have from (2.2) that for every \( x \in (0, t_{k+1} - t_k] \) and every \( k \in \mathbb{N}_0 \),
\[
(5.9) \quad \begin{bmatrix} x^+(t_k + \tau) \\ x^-(t_k + \tau) \end{bmatrix} = \begin{bmatrix} T^+(\tau) & 0 \\ 0 & T^-(\tau) \end{bmatrix} \begin{bmatrix} x^+(t_k) \\ x^-(t_k) \end{bmatrix} + \begin{bmatrix} S^\tau_+ \\ S^-_\tau \end{bmatrix} \begin{bmatrix} F^+ \quad 0 \end{bmatrix} \begin{bmatrix} x^+(t_k) \\ x^-(t_k) \end{bmatrix}.
\]
Choose \( \gamma_0 \in (0, \min \{ \gamma^+, -\omega(T^-) \}) \) arbitrarily. The above \( x^+ \) is the unique solution of (5.6), and hence there exists \( \Gamma_1 \geq 1 \) such that
\[
(5.10) \quad \|x^+(t)\| \leq \Gamma_1 e^{-\gamma_0 t} \|x^+(0)\| \quad \forall t \geq 0.
\]
We will show that for every \( \gamma \in (0, \gamma_0) \), there exist \( \Gamma_2, \Gamma_3 > 0 \) such that for every \( \tau \in (0, t_{k+1} - t_k] \) and \( k \in \mathbb{N}_0 \),
\begin{equation}
\| x^-(t_k + \tau) \| \leq \Gamma_2 e^{-\gamma (t_k + \tau)} \| x^- (0) \| + \Gamma_3 e^{-\gamma (t_k + \tau)} \| x^+ (0) \|.
\end{equation}
It follows from (5.9) that for every \( \tau \in (0, t_{k+1} - t_k] \) and every \( k \in \mathbb{N}_0 \),
\begin{equation*}
x^-(t_k + \tau) = T^-(t_k + \tau) x^-(0) + S^- F^+ x^+(t_k) + \sum_{\ell=1}^k T^-(t_k + \tau - t_\ell) S^-_{t_{\ell+1}} F^+ x^+(t_{\ell+1}).
\end{equation*}
Since \( B^- \in \mathcal{B}(U, (X^-)_1^-) \) and since \( T^- (t) \) is a strongly continuous semigroup on \( (X^-)_1^- = (X^-)_1^- \), it follows from Lemma 3.3 that \( S^- \in \mathcal{B}(U, X^-) \) for every \( \tau \geq 0 \) and that there exists \( L \geq 0 \) such that
\begin{equation}
\sup_{0 \leq \tau \leq \tau_{\max}} \| S^- F^+ \|_{\mathcal{B}(X^-)} \leq L.
\end{equation}
Therefore, for all \( \tau \in (0, t_{k+1} - t_k] \) and all \( k \in \mathbb{N}_0 \),
\begin{equation}
\| x^-(t_k + \tau) \| \leq \| T^- (t_k + \tau) \|_{\mathcal{B}(X^-)} \cdot \| x^- (0) \| + L \| x^+ (t_k) \|
+ \sum_{\ell=1}^k \| T^- (t_k + \tau - t_\ell) \|_{\mathcal{B}(X^-)} \cdot L \| x^+ (t_{\ell+1}) \|.
\end{equation}
By the exponential stability of \( T^- (t) \), there exists \( M_1 \geq 1 \) such that
\begin{equation}
\| T^- (t) \|_{\mathcal{B}(X^-)} \leq M_1 e^{-\gamma_0 t} \quad \forall t \geq 0.
\end{equation}
From (5.10), (5.13), and (5.14), for all \( \tau \in (0, t_{k+1} - t_k] \) and all \( k \in \mathbb{N}_0 \),
\begin{align*}
\| x^-(t_k + \tau) \| &\leq M_1 e^{-\gamma_0 (t_k + \tau)} \| x^- (0) \| + L \Gamma_1 e^{-\gamma_0 t_k} \| x^+ (0) \|
+ \sum_{\ell=1}^k M_1 e^{-\gamma_0 (t_k + \tau - t_\ell)} \cdot \Gamma_1 e^{-\gamma_0 t_{\ell+1}} \| x^+ (0) \|
\leq M_1 e^{-\gamma_0 (t_k + \tau)} \| x^- (0) \| + (k + 1) L \Gamma_1 M_1 e^{-\gamma_0 (t_k + \tau - \tau_{\max})} \| x^+ (0) \|.
\end{align*}
Note that \( k \in \mathbb{N}_0 \) is the number of the control updates during \( [0, t_k + \tau] \) for all \( \tau \in (0, t_{k+1} - t_k] \) and hence satisfies
\begin{equation}
k \leq \frac{t_k + \tau}{\tau_{\min}} \quad \forall \tau \in (0, t_{k+1} - t_k].
\end{equation}
We also have that, for every \( \gamma \in (0, \gamma_0) \), there exists \( M_2 \geq 1 \) such that
\begin{equation*}
\left( \frac{t}{\tau_{\min}} + 1 \right) e^{-\gamma_0 t} \leq M_2 e^{-\gamma t} \quad \forall t \geq 0.
\end{equation*}
Then for every \( \tau \in (0, t_{k+1} - t_k] \) and \( k \in \mathbb{N}_0 \), (5.11) holds with \( \Gamma_2 := M_1 \) and \( \Gamma_3 := L \Gamma_1 M_1 M_2 e^{\gamma_0 \tau_{\max}} \).
Since
\begin{equation*}
\| x^+ (0) \| \leq \| \Pi \|_{\mathcal{B}(X)} \cdot \| x^0 \|, \quad \| x^- (0) \| \leq \left( 1 + \| \Pi \|_{\mathcal{B}(X)} \right) \cdot \| x^0 \| \quad \forall x^0 \in X,
\end{equation*}
it follows from (5.10) and (5.11) that
\[ \|x^+(t)\| \leq M^+ e^{-\gamma t} \|x^0\|, \quad \|x^-(t)\| \leq M^- e^{-\gamma t} \|x^0\| \quad \forall x^0 \in X, \forall t \geq 0, \]
where \( M^+ := \Gamma_1 \|\Pi\|_{\mathcal{B}(X)} \) and \( M^- := \Gamma_2 + (\Gamma_2 + \Gamma_3) \|\Pi\|_{\mathcal{B}(X)}. \) Thus we obtain
\[ \|x(t)\| = \|x^+(t) + x^-(t)\| \leq \|x^+(t)\| + \|x^-(t)\| \leq (M^+ + M^-) e^{-\gamma t} \|x^0\| \]
for all \( x^0 \in X \) and all \( t \geq 0. \) Thus, the infinite-dimensional system (2.1) is exponentially stable. Since the constants \( \gamma_0 \in (0, \min\{\gamma^+, -\omega(T^-)\}) \) and \( \gamma \in (0, \gamma_0) \) were arbitrary, the stability margin is at least \( \min\{\gamma^+, -\omega(T^-)\}. \)

5.2. Periodic event-triggered control. In Theorem 5.6, the feedback operator \( F \in \mathcal{B}(X, U) \) has a specific structure \( F^- = F|_{X^-} = 0. \) In contrast, we here assume that \( A \) generates an analytic semigroup, and use a periodic event-triggering condition proposed in [11,12]. Then we see that for every compact feedback operator \( F \in \mathcal{B}(U, X) \) for which the semigroup generated by \( A_{BF} \) in (5.4) is exponentially stable, there exists a periodic event-triggering condition such that the closed-loop system (2.1) is exponentially stable.

Fixing \( h > 0, \varepsilon > 0, \) and \( \ell_{\text{max}} \in \mathbb{N}, \) we define the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) by
\[
\begin{align*}
(5.16a) \quad t_0 &:= 0, \quad \psi_k := \min\{\ell h > t_k : \|Fx(\ell h) - Fx(t_k)\|_U > \varepsilon \|x(t_k)\|, \ell \in \mathbb{N}\} \\
(5.16b) \quad t_{k+1} &:= \min\{t_k + \ell_{\text{max}} h, \psi_k\} \quad \forall k \in \mathbb{N}_0,
\end{align*}
\]
which is a class of periodic event-triggering mechanisms [11,12]. Whereas the event-triggering mechanisms (3.1) and (3.2) require monitoring of the conditions continuously, the periodic event-triggering mechanism (5.16) verifies the conditions only periodically.

The result of this section is based on the following theorem on the exponential stability of periodic sampled-data systems:

**Theorem 5.7 (Theorem 4.8 in [17]).** Assume that \( A \) generates an analytic semigroup \( T(t) \) on \( X, B \in \mathcal{B}(U, X_{-1}), \) and \( F \in \mathcal{B}(X, U) \) is compact. If the semigroup generated by \( A_{BF} \) in (5.4) is exponentially stable, then there exists \( h^* > 0 \) such that for every \( h \in (0, h^*), \) the periodic sampled-data system (2.1) with \( t_{k+1} - t_k = h, k \in \mathbb{N}_0, \) is exponentially stable.

Theorem 5.8 below shows the existence of periodic event-triggering mechanisms achieving the exponential stability of the closed-loop system (2.1).

**Theorem 5.8.** Assume that \( A \) generates an analytic semigroup \( T(t) \) on \( X, B \in \mathcal{B}(U, X_{-1}), \) and \( F \in \mathcal{B}(X, U) \) is compact. Assume further that the semigroup generated by \( A_{BF} \) in (5.4) is exponentially stable. Choose \( h > 0 \) so that the periodic sampled-data system (2.1) with \( t_{k+1} - t_k = h, k \in \mathbb{N}_0, \) is exponentially stable. Then there exists \( \varepsilon^* > 0 \) such that the system (2.1) with the periodic event-triggering mechanism (5.16) is exponentially stable for every \( \varepsilon \in (0, \varepsilon^*) \) and every \( \ell_{\text{max}} \in \mathbb{N}. \)

**Proof.** Theorem 5.7 guarantees the existence of \( h > 0 \) such that the periodic sampled-data system (2.1) with \( t_{k+1} - t_k = h, k \in \mathbb{N}_0, \) is exponentially stable. By Lemma 2.3 in [17], this exponential stability is achieved if and only if the operator \( \Delta(h) := T(h) + S_h F \in \mathcal{B}(X) \) is power stable, that is, there exist \( M \geq 1 \) and \( \delta \in (0,1) \) such that
\[
\|\Delta(h)^k\|_{\mathcal{B}(X)} \leq M \delta^k \quad \forall k \in \mathbb{N}_0.
\]
For the time sequence \( \{t_k\}_{k \in \mathbb{N}_0} \) defined by (5.16), let \( t_k \in \mathbb{N}_0 \) satisfy \( t_k = \ell_k h \). We have from (2.2) that for all \( p \in \{0, \ldots, \ell_{k+1} - \ell_k\} \) and all \( k \in \mathbb{N}_0 \),

\[
(5.17) \quad x((\ell_k + p + 1)h) = T(h)x((\ell_k + p)h) + S_h F x(\ell_k h).
\]

Since \( S_h \in \mathcal{B}(U, X) \) by Lemma 3.3, we can regard the following proof of Theorem 5.8 as the discrete-time counterpart of Theorem 4.1.

For \( p \in \{0, \ldots, \ell_{k+1} - \ell_k - 1\} \) and \( k \in \mathbb{N}_0 \), define the error \( e \) by

\[
(5.18) \quad e((\ell_k + p)h) := F x((\ell_k + p)h) - F x(\ell_k h).
\]

By (5.17),

\[
x((\ell_k + p + 1)h) = \Delta(h)x((\ell_k + p)h) - S_h e((\ell_k + p)h).
\]

for every \( p \in \{0, \ldots, \ell_{k+1} - \ell_k - 1\} \) and every \( k \in \mathbb{N}_0 \). Applying induction the equation above, we obtain

\[
(5.19) \quad x(\ell_{k+1}h) = \Delta(h)^{\ell_{k+1} - \ell_k} x(\ell_k h) - \sum_{p=0}^{\ell_{k+1} - \ell_k - 1} \Delta(h)^{\ell_{k+1} - \ell_k - p - 1} S_h e((\ell_k + p)h)
\]

for all \( k \in \mathbb{N}_0 \).

We introduce a new norm \( |\cdot|_d \) on \( X \) defined by

\[
|x|_d := \sup_{\ell \in \mathbb{N}_0} |\delta^{-\ell} \Delta(h)^\ell x|.
\]

As in (4.5) for the continuous-time counterpart, this norm has the following properties:

\[
(5.20) \quad \|x\| \leq |x|_d \leq M\|x\|, \quad |\Delta(h)^k x|_d \leq \delta^k |x|_d \quad \forall x \in X, \forall k \in \mathbb{N}_0.
\]

Under the event-triggering mechanism (5.16), the error \( e \) given in (5.18) satisfies

\[
(5.21) \quad \|e((\ell_k + p)h)\| \leq \varepsilon \|x(\ell_k h)\| \quad \forall p \in \{0, \ldots, \ell_{k+1} - \ell_k - 1\}, \forall k \in \mathbb{N}_0.
\]

Combining (5.19)–(5.21), we obtain

\[
|x(\ell_{k+1}h)|_d \leq \delta^{\ell_{k+1} - \ell_k} |x(\ell_k h)|_d + \varepsilon M\|S_h\|_{\mathcal{B}(U, X)} \sum_{p=0}^{\ell_{k+1} - \ell_k - 1} \delta^{\ell_{k+1} - \ell_k - p - 1} |x(\ell_k h)|_d
\]

\[
= (\delta^{\ell_{k+1} - \ell_k} (1 - \varepsilon_0) + \varepsilon_0) \cdot |x(\ell_k h)|_d \quad \forall k \in \mathbb{N}_0,
\]

where

\[
\varepsilon_0 := \varepsilon \frac{M\|S_h\|_{\mathcal{B}(U, X)}}{1 - \delta}.
\]

Choose the threshold \( \varepsilon > 0 \) so that \( \delta (1 - \varepsilon_0) + \varepsilon_0 < 1 \), i.e.,

\[
\varepsilon < \frac{1 - \delta}{M\|S_h\|_{\mathcal{B}(U, X)}}.
\]

Define the function \( f(\ell) \) by

\[
f(\ell) := \frac{-\log(\delta^\ell (1 - \varepsilon_0) + \varepsilon_0)}{\ell h}.
\]
Then \( f(\ell) \) is positive and monotonically decreasing in \( \mathbb{N} \). Applying (5.20), we obtain
\[
\|x(\ell_{k+1}h)\| \leq |x(\ell_{k+1}h)| \leq e^{-\gamma(\ell_{k+1}-\ell_k)h}|x(\ell_kh)|d
\leq e^{-\gamma\ell_{k+1}h}|x_0|d \leq Me^{-\gamma\ell_{k+1}h}\|x_0\| \quad \forall k \in \mathbb{N}_0,
\]
where \( \gamma := f(\ell_{\max}) > 0 \).

From Lemma 3.3, there exists \( L \geq 1 \) such that \( \|\Delta(\tau)\|_{\mathcal{B}(X)} \leq L \) for every \( \tau \in [0, \ell_{\max}h] \). Therefore, we see from (2.2) and (5.22) that
\[
\|x(\ell_kh + \tau)\| \leq L\|x(\ell_kh)\|
\leq (e^{\gamma\ell_{\max}hLM} \cdot e^{-\gamma(\ell_kh+\tau)}\|x_0\|) \quad \forall \tau \in (0, (\ell_{k+1} - \ell_k)h], \forall k \in \mathbb{N}_0.
\]
Thus, the system (2.1) with the periodic event-triggering mechanism (5.16) is exponentially stable and its stability margin is at least \( \gamma > 0 \).

**Remark 5.9.** As easily seen from the proof of Theorem 5.8, one can obtain a similar result for the following event-triggering mechanism that uses the error of the state:
\[
t_0 := 0, \quad \psi_k := \min \{ \ell h > t_k : \|x(\ell h) - x(t_k)\| > \varepsilon\|x(t_k)\|, \ell \in \mathbb{N} \}
\]
\[
t_{k+1} := \min \{ t_k + \ell_{\max}h, \psi_k \} \quad \forall k \in \mathbb{N}_0.
\]

**Remark 5.10.** In Theorem 5.8, we assume that \( A \) generates an analytic semigroup and that \( F \) is compact. These assumptions are used for the existence of sampling periods with respect to which the periodic sampled-data system is exponentially stable. We can replace them with different assumptions such as those of Corollary 2.3 in [24].

**6. Numerical examples.** In this section, we provide numerical examples for both the case \( B \in \mathcal{B}(U,X) \) and \( B \notin \mathcal{B}(U,X) \). We consider a cascade ODE-PDE system for bounded \( B \) and an Euler-Bernoulli beam for unbounded \( B \).

**6.1. Bounded control.** We illustrate the event-triggering mechanism in Theorem 4.1 with a heat PDE in cascade with an ODE. Let \( b = [b_1 \cdots b_n] \in L^2([0,1],\mathbb{R})^{1 \times n}, G \in \mathbb{R}^{n \times n}, \) and \( H \in \mathbb{R}^{m \times m} \). For the space variable \( \xi \in [0,1] \) and the time variable \( t \geq 0 \), we consider the following ODE-PDE system:
\[
(6.1a) \quad \frac{\partial z_1}{\partial t}(\xi,t) = \frac{\partial^2 z_1}{\partial \xi^2}(\xi,t) + b(\xi)z_2(t), \quad \xi \in [0,1], \quad t \geq 0
\]
\[
(6.1b) \quad \frac{\partial z_1}{\partial \xi}(0,t) = 0, \quad \frac{\partial z_1}{\partial \xi}(1,t) = 0, \quad t \geq 0; \quad z_1(\xi,0) = z_1^0(\xi), \quad \xi \in [0,1]
\]
\[
(6.1c) \quad \dot{z}_2(t) = Gz_2(t) + Hu(t), \quad t \geq 0; \quad z_2(0) = z_2^0,
\]
where \( z_1(\xi,t) \) is the temperature at position \( \xi \in [0,1] \) and time \( t \geq 0 \), \( z_2(t) \) is the state of the ODE, and \( u(t) \) is the input.

**6.1.1. Exponential stability of continuous-time closed-loop system.** We can reformulate the cascaded ODE-PDE (6.1) as an abstract evolution equation (2.1a) in the following way. We write \( L^2(0,1) \) in place of \( L^2([0,1],\mathbb{C}) \). Define the state space \( X \) and the input space \( U \) by \( X := L^2(0,1) \times \mathbb{C}^n \) and \( U := \mathbb{C}^m \). Let \( z_1^0 \in L^2(0,1) \) and \( z_2^0 \in \mathbb{C}^n \). The state space \( X \) is a Hilbert space with the inner product
\[
\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle := \langle x_1, y_1 \rangle_{L^2} + \langle x_2, y_2 \rangle_{\mathbb{C}^n}.
\]
Set 
\[ x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ with } x_1(t) := z_1(\cdot, t) \text{ and } x_2(t) := z_2(t); \quad x^0 := \begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix} \in X. \]

Define \( \phi_0(\xi) := 1 \) and \( \phi_n(\xi) := \sqrt{2} \cos(n\pi \xi) \) for \( n \in \mathbb{N} \), which form an orthonormal basis for \( L^2(0, 1) \). Define \( A_1 : D(A_1) \subset L^2(0, 1) \rightarrow L^2(0, 1) \) by
\[ A_1 x_1 := -\sum_{n=0}^{\infty} n^2 \pi^2 \langle x_1, \phi_n \rangle_{L^2} \phi_n \]
with domain 
\[ D(A_1) := \left\{ x_1 \in L^2(0, 1) : \sum_{n=0}^{\infty} n^4 \pi^4 |\langle x_1, \phi_n \rangle_{L^2}|^2 < \infty \right\} \]
and \( B_1 : \mathbb{C}^n \rightarrow L^2(0, 1) \) by
\[ B_1 x_2 := bx_2 \quad \forall x_2 \in \mathbb{C}^n. \]

If we set 
\[ A := \begin{bmatrix} A_1 & B_1 \\ 0 & G \end{bmatrix} \text{ with } D(A) := D(A_1) \times \mathbb{C}^n, \quad B := \begin{bmatrix} 0 \\ H \end{bmatrix}, \]
then we can rewrite the cascaded ODE-PDE (6.1) as an abstract evolution equation in the form of (2.1a); see Example 2.3.7 on p. 45 in [3] for the expansion (6.2) of \( A_1 \).

Consider the situation where the state \( x(t) \) is observed at all \( t \geq 0 \). In the case of continuous-time control, we generate the input \( u \) as follows:
\[ u(t) = Fx(t), \quad t \geq 0, \]
where \( F := \begin{bmatrix} F_1 & F_2 \end{bmatrix} \) with \( F_1 \in \mathcal{B}(L^2(0, 1), \mathbb{C}^m) \) and \( F_2 \in \mathbb{C}^{m \times n} \). In this case, the dynamics of the closed-loop system is given by
\[ \dot{x}(t) = (A + BF)x(t), \quad t \geq 0; \quad x(0) = x^0 \in X. \]

The following proposition provides a sufficient condition for the strongly continuous semigroup \( T_{BF}(t) \) generated by \( A + BF \) to be exponentially stable.

**Proposition 6.1.** Consider the evolution equation (6.4) obtained from the ODE-PDE system (6.1) and the state-feedback controller (6.3) as above. Suppose that \( F_1 x_1 = f(x_1, \phi_0)_L \) for some \( f \in \mathbb{C}^m \). The strongly continuous semigroup \( T_{BF}(t) \) generated by \( A + BF \) is exponentially stable if the rational transfer functions \( \mathcal{G}_1(\lambda) := (\lambda - B^+(\lambda I - G - HF_2)^{-1}Hf)^{-1}, \quad \mathcal{G}_2(\lambda) := (\lambda I - G - HF_2)^{-1} \)
have poles only in \( \mathbb{C}_- \), where \( B^+ := [(b_1, \phi_0)_L \cdots (b_n, \phi_0)_L] \).

**Proof.** From the strong continuity of \( T_{BF}(t) \), it follows that \( x(t) := T_{BF}(t)x^0 \) is continuous for every \( t \geq 0 \). Moreover, there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|x(t)\| \leq Me^{\omega t} \) for every \( t \geq 0 \).

Let \( T_1(t) \) be the strongly continuous semigroup generated by \( A_1 \). Then
\[ T_1(t)x_1 = \sum_{n=0}^{\infty} e^{-n^2 \pi^2 t} \langle x_1, \phi_n \rangle_{L^2} \phi_n \quad \forall x_1 \in L^2(0, 1), \forall t \geq 0. \]
Since

\( x_1(t) = T_1(t)z_1^0 + \int_0^t T_1(t-s)B_1x_2(s)ds \quad \forall t \geq 0, \)

it follows that

\( x_1^+(t) = \langle z_1^0, \phi_0 \rangle_{L^2} + \int_0^t B^+x_2(s)ds \quad \forall t \geq 0, \)

where \( x_1^+(t) := \langle x_1(t), \phi_0 \rangle_{L^2} \) and \( B^+ := [\langle b_1, \phi_0 \rangle_{L^2} \cdots \langle b_n, \phi_0 \rangle_{L^2}] \).

On the other hand,

\( x_2(t) = e^{(G+HF_2)T}z_2^0 + \int_0^t e^{(G+HF_2)(t-s)}Hf x_1^+(s)ds \quad \forall t \geq 0. \)

Substituting this into (6.6), we obtain

\[
x_1^+(t) = \langle z_1^0, \phi_0 \rangle_{L^2} + \int_0^t \left( B^+ e^{(G+HF_2)s}z_2^0 + \int_s^t B^+ e^{(G+HF_2)(s-p)}Hf x_1^+(p)dp \right) ds
\]

for every \( t \geq 0. \) This integral equation can be solved by the Laplace transform. Denote by \( \mathcal{L}[x_1^+] \) the Laplace transform of \( x_1^+ \). For every sufficiently large \( \lambda > 0 \), we obtain

\[
\lambda \mathcal{L}[x_1^+](\lambda) = \langle z_1^0, \phi_0 \rangle_{L^2} + B^+ (\lambda I - G - HF_2)^{-1}z_2^0 + B^+ (\lambda I - G - HF_2)^{-1}Hf \mathcal{L}[x_1^+](\lambda),
\]

and hence

\( \mathcal{L}[x_1^+](\lambda) = G_1(\lambda)(\langle z_1^0, \phi_0 \rangle_{L^2} + B^+ G_2(\lambda)z_2^0), \)

where \( G_1(\lambda) := (\lambda - B^+ (\lambda I - G - HF_2)^{-1}Hf)^{-1} \) and \( G_2(\lambda) := (\lambda I - G - HF_2)^{-1}. \)

Suppose that \( G_1(\lambda) \) and \( G_2(\lambda) \) have poles only in \( \mathbb{C}_- \). By (6.8), there exist \( M_1 \geq 1 \) and \( \omega_1 > 0 \) such that \( |x_1^+(t)| \leq M_1e^{-\omega_1 t}\|x_0\| \) for every \( t \geq 0. \) This together with (6.7) shows that there exist \( M_2 \geq 1 \) and \( 0 < \omega_2 \leq \omega_1 \) such that \( \|x_2(t)\|_{L^2} \leq M_2 e^{-\omega_2 t}\|x_0\| \) for every \( t \geq 0. \)

Define the projection operator \( \Pi \) on \( L^2(0,1) \) by \( \Pi x_1 = \langle x_1, \phi_0 \rangle \phi_0 \) for \( x_1 \in L^2(0,1). \) Set \( x_1^- := (I - \Pi)x_1, \quad A_1^- := A_1|_{(I - \Pi)L^2(0,1)}, \) and \( B_1^- := (I - \Pi)B_1. \) Lemma 2.5.7 on p. 71 in [3] shows that \( T^-_1(t) := T_1(t)|_{(I - \Pi)L^2(0,1)} \) is the strongly continuous semigroup generated by \( A_1^- \) and is exponentially stable. Using (6.5), we obtain

\[
x_1^-(t) = T^-_1(t)(I - \Pi)z_1^0 + \int_0^t T^-_1(t-s)B^-_1x_2(s)ds \quad \forall t \geq 0.
\]

Therefore, there exist \( M_3 \geq 1 \) and \( 0 < \omega_3 \leq \omega_2 \) such that \( \|x_1^-(t)\| \leq M_3 e^{-\omega_3 t}\|x_0\| \) for every \( t \geq 0. \) Thus, \( T_{BF}(t) \) is exponentially stable. \( \square \)

6.1.2. Numerical simulation. Let \( n = m = 1, \) \( F_1 x_1 = f(x_1, \phi_0)_{L^2} \) for some \( f \in \mathbb{R}, \) and \( F_2 \in \mathbb{R}. \) Since \( G_1(\lambda) \) in Proposition 6.1 is given by

\[
G_1(\lambda) = \frac{\lambda - G - HF_2}{\lambda^2 - (G + HF_2)\lambda + B^+ Hf},
\]
it follows that \( T_{BF}(t) \) is exponentially stable if \( G + HF < 0 \) and \( B^+Hf < 0 \).

Set
\[
b = b_1 = 5\mathbb{1}_{[0.4, 0.6]}, \quad G = 0.5, \quad H = 1, \quad f = -1, \quad F_2 = -2.5,
\]
where \( \mathbb{1}_{[0.4, 0.6]} \) is the indicator function of the interval \([0.4, 0.6]\). We obtain \( B^+ = \langle b, \phi_0 \rangle_{L^2} = 1 \), and hence \( T_{BF}(t) \) is exponentially stable. Furthermore, as seen in the proof of Proposition 6.1, for every \( \omega \in (0, 1) \), there exists \( M \geq 1 \) such that
\[
\|T_{BF}(t)\|_{\mathcal{B}(X)} \leq Me^{-\omega t} \quad \forall t \geq 0.
\]
We here find a constant \( M = M(\omega) \) satisfying (6.9) by approximation as follows. For \( N \in \mathbb{N} \), define the projection operator \( \Pi_N \) on \( L^2(0, 1) \) by
\[
\Pi_N x_1 := \sum_{n=0}^N \langle x_1, \phi_n \rangle_{L^2} \phi_n \quad \forall x_1 \in L^2(0, 1)
\]
and the operators \( A_1(N), B_1(N) \), and \( F_1(N) \) on finite-dimensional spaces by
\[
A_1(N) := A_1|_{\Pi_N L^2(0, 1)}, \quad B_1(N) := \Pi_N B_1, \quad F_1(N) := F_1|_{\Pi_N L^2(0, 1)}.
\]
Set
\[
A(N) := \begin{bmatrix} A_1(N) & B_1(N) \\ 0_{1 \times (N+1)} & G \end{bmatrix}, \quad B(N) := \begin{bmatrix} 0_{(N+1) \times 1} \\ H \end{bmatrix}, \quad F(N) := \begin{bmatrix} F_1(N) & F_2 \end{bmatrix}.
\]
By the same argument in the proof of Proposition 6.1, we have that for every \( N \in \mathbb{N} \) and every \( \omega \in (0, 1) \), there exists \( M(N, \omega) \geq 1 \) such that
\[
\left\|e^{(A(N) + B(N)F(N))t}\right\|_{\mathcal{C}^{(N+2) \times (N+2)}} \leq M(N, \omega)e^{-\omega t} \quad \forall t \geq 0.
\]
For \( N \in \mathbb{N} \) and \( \omega \in (0, 1) \), set
\[
M_{\min}(N, \omega) := \sup_{t \geq 0} e^{\omega t} \left\|e^{(A(N) + B(N)F(N))t}\right\|_{\mathcal{C}^{(N+2) \times (N+2)}}^{\mathcal{C}^{(N+2) \times (N+2)}}
\]
which can be computed numerically. We choose a constant \( M = M(\omega) \) in (6.9) so that
\[
M \geq \lim_{N \to \infty} \sup_{N} M_{\min}(N, \omega).
\]
Set \( \omega = 0.5 \). As shown in Fig. 1, we numerically see that \( M_{\min}(N, 0.5) \) converges to the value less than 1.571 as \( N \to \infty \). Therefore we here assume that \( M = 1.571 \) satisfies (6.9) with \( \omega = 0.5 \). By Theorem 4.1, if the threshold \( \varepsilon \) of the event-triggering mechanism (4.1) satisfies \( \varepsilon \leq 0.31 \), then the system (2.1) with this event-triggering mechanism is exponentially stable for every \( \tau_{\max} > 0 \).

For the time responses, the initial states \( z_0^1 \) and \( z_0^2 \) are given by \( z_0^1(\xi) = 1 \) and \( z_0^2 = -1 \), respectively. In the simulations, we approximate \( L^2(0, 1) \) by the linear span of \( \{\phi_n : n \in \mathbb{N}_0, n \leq 20\} \). Fig. 2 depicts the state norm \( \|x(t)\| \) and the input \( u(t) \) by the event-triggering mechanism (4.1) with \( \varepsilon = 0.3 \) and \( \tau_{\max} = 1 \). As a comparison, we also show in Fig. 2 the case under periodic sampled-data control with \( t_{k+1} - t_k = 0.4 \). We see from Fig. 2a that the event-triggering mechanism achieves faster convergence of \( \|x(t)\| \) than the conventional periodic mechanism. Define \( T_s := \sup \{t \geq 0 : \|x(t)\| > 0.05\|x_0\|\} \). A small \( T_s \) means the fast convergence of the state.
We obtain $T_s = 3.838$ under the event-triggering mechanism and $T_s = 4.061$ under the conventional periodic mechanism. On the other hand, the numbers of the input updates on $(0, T_s)$ are 10 under both the mechanisms. This implies that the event-triggering mechanism achieves faster state convergence, by efficiently updating the control input. In fact, we observe from Fig 2b that the event-triggering mechanism updates the control input more frequently on the interval $[0, 1.5]$ when the (relative) change of $F \dot{x}(t)$ is large, but less frequently on the interval $[2, 5]$ when the change of $F \dot{x}(t)$ is small.

6.2. Unbounded control. As a numerical example in the case $B \notin B(U, X)$, we apply the event-triggering mechanism in Theorem 5.6 to an Euler-Bernoulli beam with structural damping [18]. Let $\xi \in [0, 1]$ and $t \geq 0$ denote the space and time variables. We assume that the Euler-Bernoulli beam is hinged at the one end of the beam $\xi = 0$ and has a freely sliding clamped end at the other end $\xi = 1$. Suppose that the shear force $u(t)$ is applied at $\xi = 1$. The dynamics of the Euler-Bernoulli beam is given by

\begin{equation}
\frac{\partial^2 z}{\partial t^2}(\xi, t) - 2\gamma \frac{\partial^3 z}{\partial \xi^2 \partial t}(\xi, t) + \frac{\partial^4 z}{\partial \xi^4}(\xi, t) = 0, \quad \xi \in [0, 1], \quad t \geq 0 \tag{6.12a}
\end{equation}

\begin{equation}
z(0, t) = 0, \quad \frac{\partial^2 z}{\partial \xi^2}(0, t) = 0, \quad \frac{\partial z}{\partial \xi}(1, t) = 0, \quad -\frac{\partial^3 z}{\partial \xi^3}(1, t) = u(t), \quad t \geq 0, \tag{6.12b}
\end{equation}

where $z(\xi, t)$ is the lateral deflection of the beam at location $\xi \in [0, 1]$ and time $t \geq 0$, $u(t)$ is the input, and $\gamma \in (0, 1)$ is the damping constant.
6.2.1. Abstract evolution equation of Euler-Bernoulli beams. We here recall the results developed in Section 5 of [18]: an abstract evolution equation of the form \( (2.1a) \) for the PDE (6.12).

We write \( L^2(0,1) \) and \( W^{4,2}(0,1) \) in place of \( L^2([0,1], \mathbb{C}) \) and \( W^{4,2}([0,1], \mathbb{C}) \), respectively. We introduce the operator \( A_0 : D(A_0) \subset L^2(0,1) \to L^2(0,1) \),

\[
A_0 \zeta := \frac{d^4 \zeta}{d \xi^4}
\]

with domain \( D(A_0) := \{ \zeta \in W^{4,2}(0,1) : \zeta(0) = 0, \ z''(0) = 0, \ z'(1) = 0, \ z''(1) = 0 \} \).

We consider the state space \( X := D(A_0^{1/2}) \times L^2(0,1) \), which is a Hilbert space with the inner product

\[
\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rangle := \langle A_0^{1/2} x_1, A_0^{1/2} y_1 \rangle_{L^2} + \langle x_2, y_2 \rangle_{L^2}.
\]

The input space \( U \) is given by \( U := \mathbb{C} \).

Define \( e_n \in L^2(0,1) \) and \( \lambda_n \in \mathbb{C} \) by

\[
e_n(\xi) := \sqrt{2} \sin \left( -\frac{\pi}{2} + n\pi \right) \xi \quad \forall \xi \in [0,1], \ \forall n \in \mathbb{N}
\]

\[
\lambda_{\pm n} := \left( -\gamma \pm i \sqrt{1 - \gamma^2} \right) \cdot \left( -\frac{\pi}{2} + n\pi \right)^2 \quad \forall n \in \mathbb{N}.
\]

The sets of functions \( \{ f_n \}_{n \in \mathbb{Z}^*} \) and \( \{ g_n \}_{n \in \mathbb{Z}^*} \) defined by

\[
f_{\pm n} := \frac{\sqrt{2}}{1 - \left( -\gamma \pm i \sqrt{1 - \gamma^2} \right)^2} \begin{bmatrix} e_n/\lambda_{\pm n} \\ e_n \end{bmatrix}, \quad g_{\pm n} := \frac{1}{\sqrt{2}} \begin{bmatrix} -e_n/\lambda_{\mp n} \\ e_n \end{bmatrix}
\]

are biorthogonal, that is,

\[
\langle f_n, g_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}
\]

Using \( \{ f_n \}_{n \in \mathbb{Z}^*} \) and \( \{ g_n \}_{n \in \mathbb{Z}^*} \), we define the operator \( A : D(A) \subset X \to X \) by

\[
Ax := \sum_{n \in \mathbb{Z}^*} \lambda_n \langle x, g_n \rangle f_n \quad \text{with} \quad D(A) := \left\{ x \in X : \sum_{n \in \mathbb{Z}^*} |\lambda_n|^2 \cdot |\langle x, g_n \rangle|^2 < \infty \right\},
\]

which is the generator of the strongly continuous semigroup \( T(t) \) given by

\[
T(t)x := \sum_{n \in \mathbb{Z}^*} e^{\lambda_n t} \langle x, g_n \rangle f_n \quad \forall x \in X, \ \forall t \geq 0.
\]

Then \( \sigma(A) = \{ \lambda_n : n \in \mathbb{Z}^* \} \). Let \( X_{-1} \) denote the extrapolation space associated with \( T(t) \), and define \( B \in \mathcal{B}(U, X_{-1}) \) by

\[
Bu := u \sum_{n \in \mathbb{Z}^*} (-1)^{|n|+1} f_n \quad \forall u \in \mathbb{C}.
\]

Introducing the state vector

\[
x(t) := \begin{bmatrix} z(\cdot, t) \\ \partial_z z(\cdot, t) \\ \partial_t z(\cdot, t) \end{bmatrix},
\]

we can rewrite the PDE (6.12) in the form \( \dot{x}(t) = Ax(t) + Bu(t) \), \( t \geq 0 \).
6.2.2. Numerical simulation. Since \( \sigma(A) = \{\lambda_n : n \in \mathbb{Z}^+\} \), we see that Assumption 5.1 is satisfied for every \( \alpha < 0 \). We here choose \( \alpha \in (-9\gamma \pi^2/4, -\gamma \pi^2/4) \) for the system decomposition of Section 5.1 and check whether or not Assumptions 5.2 and 5.3 hold. For such \( \alpha \), we obtain \( \sigma(A) \cap \mathbb{C}_\alpha = \{\lambda_{-1}, \lambda_1\} \) and \( \omega(T^-) = -9\gamma \pi^2/4 < 0 \). The subspace \( X^+ := \Pi X \) is spanned by \( \{f_{-1}, f_1\} \), and using this basis, we can rewrite \( A^+ := A|_{X^+} \) and \( B^+ := \Pi B \) as

\[
A^+ = \begin{bmatrix} \lambda_{-1} & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad B^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Clearly, \((A^+, B^+)\) is controllable. Thus Assumptions 5.2 and 5.3 are satisfied.

Set \( \gamma = 1/15 \) and define the feedback operator \( F \) by

\[
Fx := -\frac{13}{4} \gamma \pi^2 (\langle x, g_{-1} \rangle + \langle x, g_1 \rangle) \quad \forall x \in X.
\]

Similarly to \( A^+ \) and \( B^+ \), we can rewrite \( F^+ := F|_{X^+} \) with respect to the basis \( \{f_{-1}, f_1\} \) in the following way:

\[
F^+ = -\frac{13}{4} \gamma \pi^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

We see from Proposition 5.5 that if the threshold \( \varepsilon > 0 \) satisfies \( \varepsilon \leq 0.83 \), then this finite-dimensional system (5.6) with the event-triggering mechanism (5.7) is exponentially stable and its stability margin is at least \( 7\gamma \pi^2/4 \).

To compare event-triggered control and periodic sampled-data control, we compute the time responses of the Euler-Bernoulli beam with the following initial state:

\[
z(\xi, 0) = 1 - \cos(\pi \xi), \quad \frac{\partial z}{\partial \xi}(\xi, 0) = 0 \quad \forall \xi \in [0, 1].
\]

We apply the event-triggering mechanism (5.7) with \( \varepsilon = 0.70 \) and \( \tau_{\text{max}} = 1 \). For periodic sampled-data control, we set \( t_{k+1} - t_k \equiv 0.15 \). In the simulation, we approximate the state space \( X \) by the linear span of \( \{f_n : n \in \mathbb{Z}^+, |n| \leq 15\} \). Fig. 3a shows that the time responses of \( \|x(t)\| \) are close between event-triggered control and periodic sample-data control. However, we observe from Fig. 3b that the difference between the control input \( u(t) \) under the event-triggering mechanism and that under the periodic mechanism is large, particularly, during the time interval \([0, 1]\). Since the relative change of \( Fx(t) \) is small on \([0.3, 0.8]\), the event-triggering mechanism refrains from updating the input.

To check where or not the event-triggering mechanism can reduce the number of the input updates, we compute the time \( T_s := \sup\{t \geq 0 : \|x(t)\| > 0.05\|x^0\|\} \) and count how many times the control input is updated on \((0, T_s)\). Under the event-triggering mechanism (5.7), we obtain \( T_s = 1.882 \), and the control input is updated 10 times on \((0, T_s)\). On the other hand, under the periodic mechanism, the time \( T_s \) is \( T_s = 1.912 \) and the number of control updates is 12 on \((0, T_s)\). Hence, the event-triggering mechanism achieves faster convergence with less control updates than the periodic mechanism in this example.

7. Conclusion. We have investigated the minimum inter-event time and the exponential stability of infinite-dimensional event-triggered control systems. We have employed the event-triggering mechanisms that compare the plant state and the error of the control input induced by the event-triggered implementation. For these event-triggering mechanisms, we have shown that the minimum inter-event time is bounded.
from below by a strictly positive constant if the feedback operator is compact. Moreover, we have obtained sufficient conditions on the threshold of the event-triggering mechanisms for the exponential stability of the closed-loop system with a bounded control operator. For infinite-dimensional systems with unbounded control operators, we have also analyzed the exponential stability of the closed-loop system under two event-triggering mechanisms, which are based on system decomposition and periodic event-triggering, respectively. Future work involves extending the proposed analysis to semi-linear infinite-dimensional systems.

**Appendix A. Proof of Theorem 3.11.** For \( \tau \geq 0 \), define the operator \( \Delta(\tau) \in B(\mathcal{X}) \) by \( \Delta(\tau) := T(\tau) + S_{\tau} F \). By the argument in Section 2.2, if \( F \in B(\mathcal{X}, U) \) is compact and if (3.10) holds for some \( c_1 > 0 \) and \( s_1 > 0 \), then there exists \( \theta_m > 0 \) such that

\[
\| F \Delta(\tau)x^0 - Fx^0 \|_U \leq \varepsilon \| \Delta(\tau)x^0 \| \quad \forall x^0 \in \mathcal{X}, \forall \tau \in [0, \theta_m).
\]

The following lemma provides some properties of the operator \( \Delta \), which are useful to study the continuous dependence of solutions of the evolution equation (2.1) under the event-triggering mechanism (3.2) on initial states.

**Lemma A.1.** For any \( \theta > 0 \), the operator \( \Delta(\tau) \in B(\mathcal{X}) \) satisfies

\[
\sup_{0 \leq \tau \leq \theta} \| \Delta(\tau) \|_{B(\mathcal{X})} < \infty.
\]

Moreover, if \( F \in B(\mathcal{X}, U) \) is compact, then for every \( x^0 \in \mathcal{X} \) and every \( \tau \geq 0 \),

\[
\lim_{\kappa \to 0} \Delta(\tau + \kappa)x^0 = \Delta(\tau)x^0, \quad \lim_{\kappa \to 0} \Delta(\tau + \kappa)^*x^0 = \Delta(\tau)^*x^0.
\]

**Proof.** This immediately follows from Lemma 3.3 and

\[
\Delta(\tau + \tau_1)x - \Delta(\tau)x = T(\tau)(T(\tau_1) - I)x + \int_{\tau}^{\tau + \tau_1} T(s)BFxsds
\]

for every \( x \in \mathcal{X} \) and every \( \tau, \tau_1 \geq 0 \).}

The proof of Theorem 3.11 is based on the following lemma:
Lemma A.2. Assume that $B \in \mathcal{B}(U, X_{-1})$ and that $F \in \mathcal{B}(X, U)$ is compact. Assume further that the strongly continuous semigroup $T(t)$ on $X$ satisfies (3.10) for some $c_1 > 0$ and $s_1 > 0$. Let $x^0 \in X$ and $t_0 \geq 0$. Suppose that

\begin{equation}
 t_1 := \inf \{ t > t_0 : \| F \Delta(t-t_0)x^0 - Fx^0 \|_U > \varepsilon \| \Delta(t-t_0)x^0 \| \}
\end{equation}

satisfies $t_1 < \infty$. For every $\kappa_1 \in (0, \theta_m)$ and $\delta > 0$, there exist $\kappa_0 \in (0, \theta_m)$ and $\delta > 0$ such that for every $\eta_0 \geq 0$ and $\zeta^0 \in X$ satisfying

\begin{equation}
 |t_0 - \eta_0| < \kappa_0, \quad \| \eta_0 - \eta_0 \zeta^0 \| < \delta, \quad \| \zeta^0 \| < \delta
\end{equation}

we obtain

\begin{equation}
 |t_1 - \eta_1| < \kappa_1, \quad \| \Delta(t_1 - t_0)x^0 - \Delta(\eta_1 - \eta_0)\zeta^0 \| < \delta_1,
\end{equation}

where

\[ \eta_1 := \inf \{ t > \eta_0 : \| F \Delta(t - \eta_0)\zeta^0 - F\zeta^0 \|_U > \varepsilon \| \Delta(t - \eta_0)\zeta^0 \| \}. \]

Proof. Let $x^0 \in X$ and $t_0 \geq 0$ be given, and suppose that $t_1$ defined by (A.1) satisfies $t_1 < \infty$. Let us show the first inequality of (A.3). By the definition of $t_1$, for every $\kappa_1 \in (0, \theta_m)$, there exist $\kappa \in [0, \kappa_1]$ and $\varepsilon_0 > 0$ such that

\begin{equation}
 \| F \Delta(t_1 + \kappa - t_0)x^0 - Fx^0 \|_U = \varepsilon \| \Delta(t_1 + \kappa - t_0)x^0 \| + \varepsilon_0.
\end{equation}

Let $\eta_0 \geq 0$ and $\zeta^0 \in X$ satisfy (A.2) for some $\kappa_0 \in (0, \theta_m)$ and $\delta > 0$. Since $t_1 \geq t_0 + \theta_m$, it follows that $\eta_0 < \kappa_0 < t_1$. We obtain

\begin{align*}
 \| F \Delta(t_1 + \kappa - t_0)x^0 - Fx^0 \|_U & \leq \| F \Delta(t_1 + \kappa - \eta_0)\zeta^0 - F\zeta^0 \|_U + \| F(\Delta(t_1 + \kappa - t_0) - I)(x^0 - \zeta^0) \|_U \\
 & \quad + \| F\Delta(t_1 + \kappa - \eta_0)\zeta^0 - F\Delta(t_1 + \kappa - t_0)\zeta^0 \|_U \\
 \end{align*}

where

\[ \phi_1(\kappa_0, \delta_0) := \delta_0 \| F(\Delta(t_1 + \kappa - t_0) - I) \|_{\mathcal{B}(X, U)} \]

\[ + \left( \| x^0 \| + \delta_0 \right) \| F\Delta(t_1 + \kappa - \eta_0) - F\Delta(t_1 + \kappa - t_0) \|_{\mathcal{B}(X, U)}. \]

Lemmas 3.4 and A.1 show that

\[ \lim_{\eta_0 \to 0} \| F\Delta(t_1 + \kappa - \eta_0) - F\Delta(t_1 + \kappa - t_0) \|_{\mathcal{B}(X, U)} = 0. \]

Therefore, $\phi_1(\kappa_0, \delta_0)$ converges to zero as $(\kappa_0, \delta_0) \to (0, 0)$. Similarly,

\begin{equation}
 \| \Delta(t_1 + \kappa - \eta_0)\zeta^0 \| < \| \Delta(t_1 + \kappa - t_0)x^0 \| + \phi_2(\kappa_0, \delta_0),
\end{equation}

where

\[ \phi_2(\kappa_0, \delta_0) := \delta_0 \| \Delta(t_1 + \kappa - \eta_0) \|_{\mathcal{B}(X)} + \| \Delta(t_1 + \kappa - \eta_0)x^0 - \Delta(t_1 + \kappa - t_0)x^0 \|. \]

Using Lemma A.1 again, we find that $\phi_2(\kappa_0, \delta_0)$ converges to zero as $(\kappa_0, \delta_0) \to (0, 0)$. Combining (A.4)–(A.6), we obtain

\begin{align*}
 \| F\Delta(t_1 + \kappa - \eta_0)\zeta^0 - F\zeta^0 \|_U & > \varepsilon \| \Delta(t_1 + \kappa - \eta_0)\zeta^0 \| \\
 & \quad + (\varepsilon_0 - \phi_1(\kappa_0, \delta_0) - \varepsilon \phi_2(\kappa_0, \delta_0)).
\end{align*}
By the argument above, there exist $\kappa_0 \in (0, \theta_m)$ and $\delta_0 > 0$ such that for every $\eta_0 \geq 0$ and $\zeta^0 \in X$ satisfying (A.2),

$$\|F\Delta(t_1 + \kappa - \eta_0)x^0 - F\zeta^0\| > \varepsilon\|\Delta(t_1 + \kappa - \eta_0)x^0\|.$$

Hence $\eta_1 < t_1 + \kappa_1$ by definition. In the same way, we obtain $t_1 < \eta_1 + \kappa_1$. Thus the first inequality of (A.3) holds.

Next we prove the second inequality of (A.3). Let $\kappa_1 \in (0, \theta_m)$ and $\delta_1 > 0$ be given. We have shown that there exist $\kappa_0 \in (0, \theta_m)$ and $\delta_0 > 0$ such that $|t_1 - \eta_1| < \kappa_1$ for every $\eta_0 \geq 0$ and $\zeta^0 \in X$ satisfying (A.2). Since $\eta_1 > t_1 - \kappa_1 > t_0$, it follows that

$$\|\Delta(t_1 - t_0)x^0 - \Delta(\eta_1 - \eta_0)x^0\| < \|\Delta(t_1 - t_0)x^0 - \Delta(\eta_1 - \eta_0)x^0\| + \delta_0\|\Delta(\eta_1 - \eta_0)\|_{\mathcal{B}(X)}.$$

Lemma A.1 shows that

$$\lim_{\eta_1 \to t_1} \|\Delta(t_1 - t_0)x^0 - \Delta(\eta_1 - t_0)x^0\| = 0$$

and

$$\lim_{\eta_0 \to t_0} \|\Delta(\eta_1 - t_0)x^0 - \Delta(\eta_1 - \eta_0)x^0\| = 0.$$

Moreover,

$$\|\Delta(\eta_1 - \eta_0)\|_{\mathcal{B}(X)} \leq \sup_{0 \leq \tau \leq t_1 + 2\theta_m - t_0} \|\Delta(\tau)\|_{\mathcal{B}(X)} < \infty.$$

Thus we obtain the second inequality of (A.3) for all sufficiently small $\kappa_0 \in (0, \theta_m)$ and $\delta_0 > 0$.

We are now in a position to show that solutions of the evolution equation (2.1) continuously depend on initial states under the event-triggering mechanism (3.2).

**Proof of Theorem 3.11.** Let $x^0 \in X$ and $t_* > 0$ be given. Theorem 3.10 shows that there exist at most finitely many updating instants of the input $u$ induced by the event-triggering mechanism (3.2) on the interval $(0, t_*]$. We first assume that on the interval $(0, t_*)$, there exist no updating instants of the input $u$ derived from the initial state $x^0$. Then

$$\inf\{t > t_0 : \|F\Delta(t - t_0)x^0 - Fx^0\|_U > \varepsilon\|\Delta(t - t_0)x^0\|\} > t_*.$$

Note that $t_0 = \eta_0 = 0$. By Lemma A.2, there exists $\delta_0 > 0$ such that for every $\zeta^0 \in X$ satisfying $\|x^0 - \zeta^0\| < \delta_0$, the input $u$ derived from the initial state $\zeta^0$ has no updating instants on the interval $(0, t_*)$. Then

$$\|x(t) - \zeta(t)\| = \|\Delta(t)x^0 - \Delta(t)\zeta^0\| < \delta_0 \sup_{0 \leq \tau \leq t_*} \|\Delta(\tau)\|_{\mathcal{B}(X)} \quad \forall t \in [0, t_*],$$

and hence we obtain the desired conclusion by Lemma A.1.

Let $t_1, \ldots, t_p \in (0, t_*)$ with $t_1 < \cdots < t_p$ be the updating instants of the input $u$ derived from the initial state $x^0$. Without loss of generality, we may assume $t_p < t_*$, since otherwise we shift $t_p$ slightly so that $t_p < t_*$. Using Lemma A.2 iteratively, we find that for every $\kappa_1 \in (0, \theta_m)$ and $\delta_1 > 0$, there exists $\delta_0 > 0$ such that for every $\zeta^0 \in X$ satisfying $\|x^0 - \zeta^0\| < \delta_0$, the input $u$ derived from the initial state $\zeta^0$ has $p$ updating instants $\eta_1, \ldots, \eta_p \in (0, t_*)$ with $\eta_1 < \cdots < \eta_p$ and

$$|t_\ell - \eta_\ell| < \kappa_1, \quad \|x(t_\ell) - \zeta(\eta_\ell)\| < \delta_1 \quad \forall \ell \in \{1, \ldots, p\}.$$
Choose \( \ell \in \{1, \ldots, p\} \) arbitrarily. Assume first that \( \eta_\ell < t_\ell \). For every \( \tau \in [0, t_\ell - \eta_\ell) \),
\[
\|x(\eta_\ell + \tau) - \zeta(\eta_\ell + \tau)\| < \|x(t_\ell) - x(\eta_\ell + \tau)\| + \|\zeta(\eta_\ell + \tau) - \zeta(\eta_\ell)\| + \delta_1.
\]
We obtain
\[
\|x(t_\ell) - x(\eta_\ell + \tau)\| = \|\Delta(t_\ell - t_{\ell-1})x(t_{\ell-1}) - \Delta(\eta_\ell + \tau - t_{\ell-1})x(t_{\ell-1})\|
\]
and
\[
\|\zeta(\eta_\ell + \tau) - \zeta(\eta_\ell)\| < \delta_1\|\Delta(\tau) - I\|_{\mathcal{B}(X)} + \|\Delta(\tau)x(t_\ell) - x(t_\ell)\|.
\]
Since \( 0 \leq \tau < t_\ell - \eta_\ell \), it follows that \( |\tau| < \kappa_1 \) and
\[
|(t_\ell - t_{\ell-1}) - (\eta_\ell + \tau - t_{\ell-1})| = |t_\ell - \eta_\ell - \tau| < \kappa_1.
\]
Using Lemma A.1, we find that for every \( \delta > 0 \), there exist \( \kappa_1 \in (0, \theta_m) \) and \( \delta_1 > 0 \) such that the inequalities (A.7) imply
\[
\|x(\eta_\ell + \tau) - \zeta(\eta_\ell + \tau)\| < \delta \quad \forall \tau \in [0, t_\ell - \eta_\ell).
\]
We obtain a similar result for the case \( t_\ell \leq \eta_\ell \).

Define
\[
\ell = \max\{t_\ell, \eta_\ell\}, \quad \ell_{\ell+1} = \min\{t_{\ell+1}, \eta_{\ell+1}\} \quad \forall \ell \in \{0, \ldots, p\},
\]
where \( t_{p+1} := t_p \) and \( \eta_{p+1} := \eta_p \). It suffices to show that for every \( \delta > 0 \), there exist \( \kappa_1 \in (0, \theta_m) \) and \( \delta_1 > 0 \) such that the inequalities (A.7) imply
\[
\|x(\ell_{\ell_\ell} + \tau) - \zeta(\ell_{\ell_\ell} + \tau)\| < \delta \quad \forall \tau \in [0, z_{\ell_{\ell_\ell+1}} - z_{\ell_{\ell_\ell}}], \forall \ell \in \{0, \ldots, p\}.
\]
This follows from
\[
\|x(z_{\ell_{\ell_\ell}} + \tau) - \zeta(z_{\ell_{\ell_\ell}} + \tau)\| < \|\Delta(z_{\ell_{\ell_\ell}} + \tau - t_\ell)x(t_\ell) - \Delta(z_{\ell_{\ell_\ell}} + \tau - \eta_\ell)x(t_\ell)\|
\]
\[
+ \delta_1\|\Delta(z_{\ell_{\ell_\ell}} + \tau - \eta_\ell)\|_{\mathcal{B}(X)}.
\]
In fact,
\[
\|\Delta(z_{\ell_{\ell_\ell}} + \tau - t_\ell)x(t_\ell) - \Delta(z_{\ell_{\ell_\ell}} + \tau - \eta_\ell)x(t_\ell)\|
\]
\[
= \|T(\tau)(\Delta(|t_\ell - \eta_\ell|)x(t_\ell) - x(t_\ell))\|
\]
\[
\leq \sup_{0 \leq \tau \leq \tau_0} \|T(\tau)\|_{\mathcal{B}(X)} \cdot \|\Delta(|t_\ell - \eta_\ell|)x(t_\ell) - x(t_\ell)\|
\]
for every \( \tau \in [0, z_{\ell_{\ell_\ell+1}} - z_{\ell_{\ell_\ell}}] \) and every \( \ell \in \{0, \ldots, p\} \). Moreover, Lemma A.1 shows that
\[
\lim_{\eta_\ell \to t_\ell} \|\Delta(|t_\ell - \eta_\ell|)x(t_\ell) - x(t_\ell)\| = 0 \quad \forall \ell \in \{0, \ldots, p\}
\]
and that
\[
\|\Delta(z_{\ell_{\ell_\ell}} + \tau - \eta_\ell)\|_{\mathcal{B}(X)} \leq \sup_{0 \leq \tau \leq \tau_0} \|\Delta(t)\|_{\mathcal{B}(X)} < \infty.
\]
for every \( \tau \in [0, z_{\ell_{\ell_\ell+1}} - z_{\ell_{\ell_\ell}}] \) and every \( \ell \in \{0, \ldots, p\} \). This completes the proof. \( \Box \)
Acknowledgments. The authors are grateful to the associate editor and anonymous reviewers whose comments greatly improved the paper.

REFERENCES

[1] D. P. Borgers and W. P. M. H. Heemels, Event-separation properties of event-triggered control systems, IEEE Trans. Automat. Control, 59 (2014), pp. 2644–2656.
[2] R. F. Curtain and J. C. Oostveen, Necessary and sufficient conditions for strong stability of distributed parameter systems, Systems Control Lett., 37 (1999), pp. 11–18.
[3] R. F. Curtain and H. J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, New York: Springer, 1995.
[4] de Andrade, R. G. A., Vazquez, and D. J. Pagano, Backstepping stabilization of a linearized ODE-PDE Rijke tube model, Automatica, 96 (2018), pp. 98–109.
[5] V. S. Dolk, D. P. Borgers, and W. P. M. H. Heemels, Output-based and decentralized dynamic event-triggered control with guaranteed $L_p$-gain performance and Zeno-freeness, IEEE Trans. Automat. Control, 62 (2017), pp. 34–49.
[6] M. C. F. Donkers and W. P. M. H. Heemels, Output-based event-triggered control with guaranteed $L_\infty$-gain and improved and decentralized event-triggering, IEEE Trans. Automat. Control, 57 (2012), pp. 1362–1376.
[7] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, New York: Springer, 2000.
[8] N. Espitia, A. Girard, N. Marchand, and C. Prieur, Event-based control of linear hyperbolic systems of conservation laws, Automatica, 70 (2016), pp. 275–287.
[9] N. Espitia, A. Girard, N. Marchand, and C. Prieur, Event-based boundary control of a linear $2 \times 2$ hyperbolic system via backstepping approach, IEEE Trans. Automat. Control, 63 (2018), pp. 2686–2693.
[10] R. Goebel, R. Sanfelice, and A. Teel, Hybrid dynamical systems, IEEE Control Syst. Mag., 29 (2009), pp. 28–93.
[11] W. P. M. H. Heemels and M. C. F. Donkers, Model-based periodic event-triggered control for linear systems, Automatica, 49 (2013), pp. 698–711.
[12] W. P. M. H. Heemels, M. C. F. Donkers, and A. R. Teel, Periodic event-triggered control for linear systems, IEEE Trans. Automat. Control, 58 (2013), pp. 847–861.
[13] Z. Jiang, B. Cui, W. Wu, and B. Zhuang, Event-driven observer-based control for distributed parameter systems using mobile sensor and actuator, Comput. Math. Appl., 72 (2016), pp. 2854–2864.
[14] I. Karafyllis, M. Krstic, and K. Chrysafi, Adaptive boundary control of constant-parameter reaction-diffusion PDEs using regulation-triggered finite-time identification, Automatica, 103 (2019), pp. 166–179.
[15] D. Lehmann and J. Lunze, Event-based control with communication delays and packet losses, Int. J. Control, 85 (2012), pp. 563–577.
[16] H. Logemann, Stabilization of well-posed infinite-dimensional systems by dynamic sampled-data feedback, SIAM J. Control Optim., 51 (2013), pp. 1203–1231.
[17] H. Logemann, R. Rebarber, and S. Townley, Stability of infinite-dimensional sampled-data systems, Trans. Amer. Math. Soc., 355 (2003), pp. 3301–3328.
[18] H. Logemann, R. Rebarber, and S. Townley, Generalized sampled-data stabilization of well-posed infinite-dimensional systems, SIAM J. Control Optim., 44 (2005), pp. 1345–1369.
[19] A. Pazy, On the applicability of Lyapunov’s theorem in Hilbert space, SIAM J. Math. Anal., 3 (1972), pp. 291–294.
[20] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, New York: Springer, 1983.
[21] C. Prieur and A. Tanwani, Asymptotic stabilization of some finite and infinite dimensional systems by means of dynamic event-triggered output feedbacks, in Feedback Stabilization of Controlled Dynamical Systems, Cham: Springer, 2017, pp. 201–230.
[22] R. Rebarber and S. Townley, Generalized sampled data feedback control of distributed parameter systems, Systems & Control Letters, 34 (1998), pp. 229–240.
[23] R. Rebarber and S. Townley, Nonrobustness of closed-loop stability for infinite-dimensional systems under sample and hold, IEEE Trans. Automat. Control, 47 (2002), pp. 1381–1385.
[24] R. Rebarber and S. Townley, Robustness with respect to sampling for stabilization of Riesz spectral systems, IEEE Trans. Automat. Control, 51 (2006), pp. 1519–1522.
[25] A. Selivanov and E. Fridman, Distributed event-triggered control of diffusion semilinear
PDEs, Automatica, 68 (2016), pp. 344–351.

[26] G. Susto and M. Krstic, Control of PDE-ODE cascades with Neumann interconnections, J. Frankl. Inst., 347 (2010), pp. 284–314.

[27] P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, IEEE Trans. Automat. Control, 52 (2007), pp. 1680–1685.

[28] M. Tucsnak and G. Weiss, Well-posed systems–The LTI case and beyond, Automatica, 50 (2014), pp. 1757–1779.