VARIATIONAL EFFECT OF BOUNDARY MEAN CURVATURE ON ADM MASS IN GENERAL RELATIVITY

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Abstract. We extend the idea and techniques in [14] to study variational effect of the boundary geometry on the ADM mass of an asymptotically flat manifold. We show that, for a Lipschitz asymptotically flat metric extension of a bounded Riemannian domain with quasi-convex boundary, if the boundary mean curvature of the extension is dominated by but not identically equal to the one determined by the given domain, we can decrease its ADM mass while raising its boundary mean curvature. Thus our analysis implies that, for a domain with quasi-convex boundary, the geometric boundary condition holds in Bartnik’s minimal mass extension conjecture [4].

1. Introduction

Asymptotically flat manifolds are often used to model isolated systems in general relativity. A complete Riemannian manifold \((M^n, g)\) with dimension \(n \geq 3\) is called asymptotically flat if there is a compact set \(K \subset M\) and a diffeomorphism \(\Phi : M \setminus K \to \mathbb{R}^n \setminus \{|x| < 1\}\) such that, in the coordinate chart defined by \(\Phi\),

\[
|g_{ij}(x) - \delta_{ij}| + |x||g_{ij,k}(x)| + |x|^2|g_{ij,kl}(x)| = O(|x|^{-p})
\]

and

\[
|R(g)(x)| = O(|x|^{-q})
\]

for some \(p > \frac{n-2}{2}\) and some \(q > n\), where ";" denotes partial derivative in the coordinate chart and \(R(g)\) denotes the scalar curvature of \((M^n, g)\). The metric decay assumptions imply the existence of the limit

\[
m(g) = \frac{1}{4\omega_{n-1}} \lim_{r \to \infty} \int_{|x|=r} \sum_{i,j} (g_{ij,i} - g_{ii,j})\nu^j d\mu,
\]

where \(\omega_{n-1}\) is the volume of the standard unit sphere \(S^{n-1}\), \(d\mu\) is the Euclidean surface measure and \(\nu^j\) denotes the Euclidean unit normal. The quantity \(m(g)\) is called the total mass or ADM mass of \((M^n, g)\) [1]. It is a simple computation to show that if the metric \(g\) is conformally flat and scalar flat, then the total mass appears in the expansion of the conformal factor at infinity

\[
u = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{1-n})
\]

as \(m(g) = (n-1)A\).

A fundamental result relating the total mass of an asymptotically flat manifold and its local energy density (scalar curvature) is the Positive Mass Theorem (PMT),
first proved by R. Schoen and S.T. Yau [16] using minimal surface techniques and later by E. Witten [19] using spinors.

**Positive Mass Theorem**

Let \((M^n, g)\) be asymptotically flat with \(R(g) \geq 0\). If \(n \leq 7\) or \(M\) is spin, then the total mass of \((M^n, g)\) is non-negative, and is zero if and only if \((M^n, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, g_0)\).

Many other significant works have been made in the last two decades to understand the interplay between the total mass of \((M^n, g)\) and its geometry. Among them, one remarkable result is the following Riemannian Penrose Inequality proved by H. Bray [5] and G. Huisken and T. Ilmanen [11].

**Riemannian Penrose Inequality**

Let \((M^3, g)\) be asymptotically flat with \(R(g) \geq 0\). Let \(A\) be the area of the outermost minimal surface \(\Sigma\) in \((M^3, g)\). Then

\[
m(g) \geq \sqrt{\frac{A}{16\pi}},
\]

and the equality holds if and only if the part of \((M^3, g)\) outside \(\Sigma\) is isometric to the Schwarzschild manifold \((\mathbb{R}^3 \setminus B_{2\pi}(0), (1 + \frac{m^2}{r^2})4g_0)\) with \(m = m(g)\).

One natural question coming from the Penrose Inequality is, given an asymptotically flat \((M^n, g)\), what is the least contribution of a finite region \(\Omega \subset M\) to the total mass \(m(g)\)? Another way of asking the question is, between the notion of local energy density and the notion of the total mass, if there is a meaningful concept of the mass of a bounded region? There have been many attempts to define such a quasi-local mass function ([2], [7], [12] etc.), and one believes there should be an analog in Einstein’s gravity theory of the usual Newtonian measure of the mass of an extended body. In [2], R. Bartnik gave his quasi-local mass definition \(m_B(\Omega)\) from a variational point of view,

\[
m_B(\Omega) = \inf \{ m(\tilde{g}) \mid (\tilde{M}, \tilde{g}) \in PM \},
\]

where

\[
PM = \{ (\tilde{M}, \tilde{g}) \mid (\tilde{M}, \tilde{g}) \text{ is asymptotically flat with } R(\tilde{g}) \geq 0, \\
(\tilde{M}, \tilde{g}) \text{ contains } (\Omega, g) \text{ isometrically,} \\
\text{and no horizon lies outside } (\Omega, g) \}. \}
\]

It has been shown in [11] that \(m_B(\Omega) = 0\) if and only if \((\Omega, g)\) is locally Euclidean and \(\lim_{i \to \infty} m_B(\Omega_i) = m(g)\) if \(\{\Omega_i\}_{i=1}^\infty\) forms an exhaustion sequence of \((M, g)\).

There is a natural analogue between \(m_B(\Omega)\) and the usual definition of the electrostatic capacity of a conducting body,

\[
c(\Omega) = \inf \left\{ \int |\nabla u|^2 dx \mid u \in C^\infty_c(\mathbb{R}^3), \ u \equiv 1 \text{ on } \Omega \right\},
\]

where \(c(\Omega)\) is achieved by a harmonic function \(u\) on \(\mathbb{R}^3 \setminus \Omega\) that equals 1 on \(\partial \Omega\) and decays to 0 at infinity. It is interesting to know if similar things hold for \(m_B(\Omega)\), i.e. if there exists a metric \(\tilde{g}\) on \(M \setminus \Omega\) such that \(m(\tilde{g}) = m_B(\Omega)\), and if it exists, what kind of interior equation and boundary condition it satisfy?
Both of the research in [14] and in this paper are inspired by the above variational approach to the quasi-local mass problem. Motivated by the expectation that a metric achieving \( m_B(\Omega) \) might only be Lipschitz across \( \partial \Omega \), we established the positivity of the total mass of a class of piecewise smooth asymptotically flat manifolds containing \((\Omega, g)\) in [14]. In this paper, we focus on the variational effect of the boundary mean curvature on the total mass and relate it to the geometric boundary condition in Bartnik’s minimal mass extension conjecture [4].

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2. The Mass of Piecewise Smooth Manifolds

We first recall notations and results in [14]. Let \( M^n \) be a differentiable manifold which has the property that there exists a compact domain \( \Omega \) with smooth boundary such that \( M \setminus \Omega \) is diffeomorphic to \( \mathbb{R}^n \) minus a ball. Let \( n \geq 3 \) be a dimension for which the classical PMT [10] holds.

**Theorem 2.1.** [14] Let \( g_- \) and \( g_+ \) be smooth metrics defined on \( \overline{\Omega} \) and \( M \setminus \Omega \) so that \( g_-|_{\partial \Omega} = g_+|_{\partial \Omega} \) and \( g_+ \) is asymptotically flat. Suppose that both \( g_- \) and \( g_+ \) have non-negative scalar curvature and

\[
H(\partial \Omega, g_-) \geq H(\partial \Omega, g_+),
\]

where \( H(\partial \Omega, g_-) \) and \( H(\partial \Omega, g_+) \) represent the mean curvature of \( \partial \Omega \) in \((\overline{\Omega}, g_-)\) and \((M \setminus \Omega, g_+)\) with respect to unit normal vectors pointing to the unbounded region.

Then the mass of \( g_+ \) is non-negative. If \( H(\partial \Omega, g_-) > H(\partial \Omega, g_+) \) at some point on \( \partial \Omega \), then \( g_+ \) has a strict positive mass. If \( n = 3 \) and the mass of \( g_+ \) is zero, then \((\overline{\Omega}, g_-)\) can be isometrically embedded in \( \mathbb{R}^3 \) and \((M \setminus \Omega, g_-)\) is isometric to its complement.

**Remark.** Our sign convention for the mean curvature is that \( H(S^{n-1}, g_0) = n - 1 \), where \( S^{n-1} \) is the unit sphere in the Euclidean space \((\mathbb{R}^n, g_0)\).

The proof of this theorem in [14] was based on Schoen-Yau’s original proof of the classical PMT and a metric mollification proposition which interprets the difference of the mean curvature as scalar curvature concentration along the boundary. To state that proposition precisely, we let \( U_{\epsilon}^- \) and \( U_{\epsilon}^+ \) be \( 2\epsilon \)-tubular neighborhoods of \( \Sigma \) in \((\overline{\Omega}, g_-)\) and \((M \setminus \Omega, g_+)\) such that \( U_{\epsilon}^- \) and \( U_{\epsilon}^+ \) are diffeomorphic to \( \Sigma \times (-2\epsilon, 0) \) and \( \Sigma \times [0, 2\epsilon) \), and \( g_-|_{U_{\epsilon}^-} \) and \( g_+|_{U_{\epsilon}^+} \) have the form

\[
\begin{align*}
(1) & \quad g_- = g_{-ij}(x, t)dx^i dx^j + dt^2 \\
(2) & \quad g_+ = g_{+ij}(x, t)dx^i dx^j + dt^2
\end{align*}
\]

where \( t \) is the standard coordinate for \((-2\epsilon, 0)\) and \([0, 2\epsilon)\), and \((x^1, \ldots, x^n)\) are local coordinates for \( \Sigma \). Identifying \( U = U_{\epsilon}^+ \cup U_{\epsilon}^- \) with \( \Sigma \times (-2\epsilon, 2\epsilon) \), we define \( \tilde{M} \) to be a possibly new differentiable manifold with the background topological space \( M \) and the differential structure determined by the open covering \( \{\Omega, M \setminus \overline{\Omega}, U\} \), where \( U \) carries the differential structure induced from \( \Sigma \times (-2\epsilon, 2\epsilon) \). It follows from the fact \( g_-|_U = g_+|_U \) that \( g_- \) and \( g_+ \) determine a continuous metric \( g \) on \( \tilde{M} \) such that \( g|_U \) has the form

\[
g = g_{ij}(x, t)dx^i dx^j + dt^2,
\]

where \( g_{ij}(x, t) = g_{-ij}(x, t) \) when \( t \leq 0 \) and \( g_{ij}(x, t) = g_{+ij}(x, t) \) when \( t \geq 0 \). For such a metric \( g \), we have the following proposition.
Proposition 2.1. [14] There exists a family of $C^2$ metrics $\{g_\delta\}_{0<\delta<\epsilon}$ on $\bar{M}$ such that $g_\delta$ agrees with $g$ outside $\Sigma \times (-\frac{\delta}{2}, \frac{\delta}{2})$, $g_\delta$ is uniformly close to $g$ in $C^0$ topology and the scalar curvature of $g_\delta$ satisfies

\begin{align}
R_\delta(x,t) &= O(1), \text{ for } (x,t) \in \Sigma \times \{\frac{\delta^2}{100} < |t| \leq \frac{\delta}{2}\} \\
R_\delta(x,t) &= O(1) + \{H(\Sigma, g_-(x)) - H(\Sigma, g_+(x))\} \left\{100\phi \left(\frac{100t}{\delta^2}\right)\right\},
\end{align}

for $(x,t) \in \Sigma \times [-\frac{\delta^2}{100}, \frac{\delta^2}{100}]$, where $O(1)$ represents bounded quantity depending only on $g$, but not on $\delta$.

3. Variational Effect of Boundary Geometry

The main goal of this paper is to investigate the boundary mean curvature equality $H(\Sigma, g_-) \equiv H(\Sigma, g_+)$ from a variational point of view. We will briefly discuss its implication to the quasi-local mass question in the end.

Let $(M^n, g)$ be a smooth asymptotically flat manifold with $R(g) \geq 0$. Let $\Omega \subset M$ be a compact domain with smooth boundary $\Sigma$. We define

$$
\mathcal{M}_\infty = \{(M_\infty, g_+) \mid (M_\infty, g_+) \text{ is a smooth asymptotically flat manifold} \}
$$

with boundary $\Sigma$ such that $R(g_+) \geq 0$, $g|_{\Sigma} = g_+|_{\Sigma}$ and $H(\Sigma, g) \geq H(\Sigma, g_+)$.}

Our next theorem states that, if $\partial \Omega$ is quasi-convex in the sense of [18] and if $(M_\infty, g_+) \in \mathcal{M}_\infty$ and $H(\Sigma, g_+)$ does not agree with $H(\Sigma, g)$ identically, we can decrease $m(g_+)$ while raising $H(\Sigma, g_+)$ to be almost $H(\Sigma, g)$.

Theorem 3.1. Assume that $\Sigma$ has positive scalar curvature with respect to the induced metric and $H(\Sigma, g) > 0$. For any $(M_\infty, g_+) \in \mathcal{M}_\infty$, if $H(\Sigma, g_+) > 0$ and $H(\Sigma, g_+)$ does not agree with $H(\Sigma, g)$ identically, then for any $\epsilon > 0$, there exists a $(M_\infty, \tilde{g}_+) \in \mathcal{M}_\infty$ such that $m(\tilde{g}_+) < m(g_+)$ and $H(\Sigma, \tilde{g}_+) \geq H(\Sigma, g) - \epsilon$.

Proposition 2.1 indicates that strict jump of the mean curvature at some point on $\Sigma$ suggests that there is positive singular scalar curvature of the piecewise smooth manifold $(g, g_+)$ at $\Sigma$. Thus we expect to level down the singular scalar curvature to reduce $m(g_+)$. Unlike the proof in [14], we must keep the interior geometry of $(\Omega, g)$ fixed. For that purpose, we first push the singular scalar curvature at $\Sigma$ into the interior of $M_\infty$, then we apply conformal deformation similar to that in [14] outside $\bar{\Omega}$ to decrease $m(g_+)$. For notation consistency, we let $(\tilde{\Omega}, g_-)$ denote $(\Omega, g)$.

3.1. A Metric “Bridge” near the Boundary. We establish the existence of a metric “bridge” that connects $g_-$ and $g_+$ near $\Sigma$ in a way that the singular scalar curvature is propagated into the interior of $M_\infty$.

Proposition 3.1. Assume that $\Sigma$ has positive scalar curvature with respect to the induced metric and $H(\Sigma, g_-) > 0$. For any $(M_\infty, g_+) \in \mathcal{M}_\infty$, if $H(\Sigma, g_+) > 0$ and $H(\Sigma, g_-) \not\equiv H(\Sigma, g_+)$, then there exists a tubular neighborhood $N_\sigma = \Sigma \times [0, \sigma]$ of $\Sigma$ in $(M_\infty, g_+)$ and a scalar flat metric $g_c$ on $N_\sigma$ such that

\begin{align}
\begin{cases}
g_c|_{\Sigma} = g_-|_{\Sigma}, & H(\Sigma, g_c) = H(\Sigma, g_-) \\
g_c|_{\Sigma_\sigma} = g_+|_{\Sigma_\sigma}, & H(\Sigma_\sigma, g_c) \geq H(\Sigma_\sigma, g_+),
\end{cases}
\end{align}

where $\Sigma_\sigma = \Sigma \times \{\sigma\}$ and “$f \not\equiv h$” means that “$f \geq h$ but $f$ is not identically $h$.”
To prove Proposition 3.1, we adopt the following quasi-spherical metric type construction, which was first developed by R. Bartnik in [3] and recently has been used by B. Smith and G. Weinstein in [18] and Y. Shi and L. Tam in [17].

Let \( \Sigma \) be a smooth compact manifold without boundary with dimension \( n - 1 \). Let \( N = \Sigma \times [0, \infty) \) be the product manifold equipped with a smooth background metric \( g \), which has the form

\[
g(x, t) = g_t(x)dx^i dx^j + dt^2,
\]

where \( t \) is the coordinate on \([0, \infty)\) and \((x^1, x^2, \ldots, x^{n-1})\) are coordinates on \( \Sigma \). Given a function \( \tilde{R} \), we want to find a function \( u > 0 \) such that the metric \( \tilde{g} \) defined by

\[
\tilde{g}(x, t) = g_t(x)dx^i dx^j + u^2(x, t)dt^2
\]

has the prescribed scalar curvature \( \tilde{R} \). One basic motivation to such a construction is that \( \tilde{R} \) and \( H \) represent the mean curvature of \( \Sigma_t = \Sigma \times \{ t \} \) in \((N, \tilde{g})\) and \((N, g)\) with respect to the vector \( \frac{\partial}{\partial t} \). The following equation on \( u \) was derived in several literature (for example, see [3], [17]).

**Lemma 3.1.** \( \tilde{g} \) has the scalar curvature \( \tilde{R} \) if and only \( u \) satisfies

\[
H \frac{\partial u}{\partial t} = u^2 \Delta_{g_t} u + \frac{1}{2} (u - u^3)R(g_t) - \frac{1}{2} u R(g) + \frac{1}{2} u^3 \tilde{R}.
\]

Here \( \Delta_{g_t}(\cdot) \) denotes the Laplacian operator of \((\Sigma_t, g_t)\), \( R(g_t) \) is the scalar curvature of \((\Sigma, g_t)\) and \( R(g) \) is the scalar curvature of \((N, g)\).

The following short time existence of solutions follows directly from the fact that (10) is a non-linear parabolic PDE of \( u \) if \( H \) is positive and an implicit function theorem type argument (See [3]).

**Lemma 3.2.** For any positive \( u_0 \) on \( \Sigma \), there exists a small constant \( \sigma > 0 \) and a positive \( u = u(x, t) \) on \( \Sigma \times [0, \sigma] \) so that \( u \) solves

\[
\begin{align*}
H \frac{\partial u}{\partial t} &= u^2 \Delta_{g_t} u + \frac{1}{2} (u - u^3)R(g_t) - \frac{1}{2} u R(g) + \frac{1}{2} u^3 \tilde{R} \\
\frac{\partial u}{\partial t}\big|_{t=0} &= u_0
\end{align*}
\]

on \( N_\sigma = \Sigma \times [0, \sigma] \).

For our interest in decreasing the mass of \( g \) in case \( g \) is asymptotically flat, we start with \( R(g) \geq 0 \) and choose \( \tilde{R} = 0 \). Then (11) is reduced to

\[
\begin{align*}
H \frac{\partial u}{\partial t} &= u^2 \Delta_{g_t} u + \frac{1}{2} (u - u^3)R(g_t) - \frac{1}{2} u R(g) \\
\frac{\partial u}{\partial t}\big|_{t=0} &= u_0.
\end{align*}
\]

One nice thing about such a choice is that we have a maximum principle on the solution to (12), whose proof is exactly the same as the proof of the standard maximum principle for second order linear parabolic equations.

**Lemma 3.3.** Assume that the foliation \( \{ (\Sigma_t, g_t) \}_{t>0} \) has positive scalar curvature and positive mean curvature. If \( u \) is a positive solution to (12) on \( N_T = \Sigma \times [0, T] \) and \( u_0 \leq 1 \), then

\[
\max_{N_T} u \leq 1.
\]
Proof of Proposition 3.1. We choose
\[ 0 < u_0 = \frac{H(\Sigma, g_+)}{H(\Sigma, g_-)} \leq 1 \]
and let \( u \) be a solution to (12) with \( g \) replaced by \( g_+ \) on a Gaussian tubular neighborhood \( N_\sigma = \Sigma \times [0, \sigma] \) of \( \Sigma \) in \((\mathcal{M}_\infty, g_+)\). It follows from Lemma 3.3 that \( u \leq 1 \) on \( N_\sigma \). Since \( u_0 \neq 1 \), by continuity we may shrink \( \sigma \) so that \( u(x, \sigma) \neq 1 \). On \( N_\sigma \) we define
\[ g_c = g_+(x)dx^i dx^j + u^2 dt^2, \]
(6) follows directly from (9) and Lemma 3.1.

\[ \square \]

3.2. Mass Decrease due to Boundary Effect. We are now in a position to prove Theorem 3.1. The main idea is to first apply Proposition 3.1 to propagate the singular scalar curvature at \( \Sigma \) a fixed distance into the interior of \( \mathcal{M}_\infty \), then to apply Proposition 2.1 and argument similar to that in \[14\] to decrease \( m(g_+) \). We divide the proof into several steps.

Step 1. Tilt down the mean curvature to allow a strict gap:
For technical reasons, we first approximate \((\mathcal{M}_\infty, g_+)\) by \(\{ (\mathcal{M}_\infty, g_{s+}) \}_{s>0}\) where \( H(\Sigma, g_{s+}) < H(\Sigma, g_+) \). Let \( \psi \) be a solution to
\[ \begin{cases} 
\triangle g_+ \psi = 0 \text{ on } \mathcal{M}_\infty \\
\psi = 0 \text{ on } \Sigma \\
\psi \to 1 \text{ at } \infty.
\end{cases} \]
(15)
For each \( s \in (0,1) \), we define
\[ g_{s+} = (1 - s\psi)^\frac{1}{n-2} g_+. \]
We have that
\[ \lim_{s \to 0} m(g_{s+}) = m(g_+), \quad g_{s+}|_\Sigma = g_+|_\Sigma \]
and
\[ H(\Sigma, g_{s+}) = H(\Sigma, g_+) - \left( \frac{2s}{n-2} \frac{\partial \psi}{\partial \vec{n}} \right), \]
where \( \frac{\partial \psi}{\partial \vec{n}} > 0 \) by the strong maximum principle. Since \( H(\Sigma, g_+) > 0 \), we may assume that \( H(\Sigma, g_{s+}) > \frac{1}{2} H(\Sigma, g_+) > 0 \) for sufficiently small \( s \).

Step 2. Propagate the singular scalar curvature to the interior of \( \mathcal{M}_\infty \):
We apply Proposition 3.1 in a slightly different way in order to keep the strict mean curvature gap at \( \Sigma \). For each small \( s > 0 \), we let
\[ u_0^s = \frac{H(\Sigma, g_{s+})}{H(\Sigma, g_-) - \frac{2s}{n-2} \frac{\partial \phi}{\partial \vec{n}}} = \frac{H(\Sigma, g_+) - \frac{2s}{n-2} \frac{\partial \phi}{\partial \vec{n}}}{H(\Sigma, g_-) - \frac{2s}{n-2} \frac{\partial \phi}{\partial \vec{n}}}, \]
(19)
and let \( u_s \) be a short time solution to (12) with \( g \) replaced by \( g_{s+} \). Since \( g_{s+} \) and \( u_0^s \) have smooth dependence on \( s \), there exist constants \( \sigma_0 > 0 \) and \( s_0 > 0 \) so that \( u_s \) exists on \( N_{\sigma_0} = \Sigma \times [0, \sigma_0] \) for any \( s \in [0, s_0] \) and \( u_s \) depends smoothly on \( s \). On \( N_{\sigma_0} \) we define
\[ g_c^s = g_{s+}(x,t)dx^i dx^j + u_s^2 dt^2 \]
(20)
where
\begin{equation}
\sum = g_{s+}(x,t)dx^i dx^j + dt^2.
\end{equation}

It follows from the fact $H(\Sigma, g_-) \neq H(\Sigma, g_+)$ and the proof of Proposition 3.1 that $g_s$ is a scalar flat metric and
\begin{equation}
\begin{cases}
g_c^s|_{\Sigma} = g_-|_{\Sigma}, & H(\Sigma, g_c^s) = H(\Sigma, g_-) - \frac{2s}{n-2} \frac{\partial \phi}{\partial n} \\
g_c^s|_{\Sigma_{s_0}} = g_{s+}|_{\Sigma_{s_0}}, & H(\Sigma_{s_0}, g_c^s) \geq (1 + f)H(\Sigma_{s_0}, g_{s+}),
\end{cases}
\end{equation}
where $0 \leq f \leq 1$ is a function on $\Sigma_{s_0}$ that is not identically zero and depends only on $\frac{\partial H(\Sigma, g_-)}{\partial (\Sigma, g_-)}$. We note that, by choosing $s_0$ sufficiently small, we may also assume that $H(\Sigma_{s_0}, g_{s+}) > H(\Sigma_{s_0}, g_+) > \frac{1}{2}H(\Sigma, g_+) > 0$.

\underline{Step 3.} Smooth $(g_c^s, g_{s+})$ at $\Sigma_{s_0}$.

For each $s \in [0, s_0]$, we let $\tilde{M}^s_\infty$ be the modified differentiable manifold on which $(g_c^s, g_{s+})$ determines a continuous metric $g_s$ as in Section 2. It follows from the proof of Proposition 3.1 in [14] that there exists a family of smooth metrics $\{g_c^s\}_{s_0 > \delta > 0}$ on $\tilde{M}^s_\infty$, where $\delta_0$ is independent on $s$, such that $g_s = g_c$ outside $\Sigma_{s_0} \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$, $g_s^c$ approaches to $g_c$ in $C^0$ topology uniformly with respect to $s$, and $R^c_s$, the scalar curvature of $g_s^c$, satisfies
\begin{equation}
R^c_s(x,t) = O(1), \text{ for } (x,t) \in \Sigma_{s_0} \times \left\{\frac{\delta^2}{100} < |t| \leq \frac{\delta}{2}\right\}
\end{equation}
\begin{equation}
R^c_s(x,t) = O(1) + \left\{H(\Sigma_{s_0}, g_s^c)(x) - H(\Sigma_{s_0}, g_{s+})(x)\right\} \left\{\frac{100}{\delta^2} - \frac{100}{\delta^2}\right\},
\end{equation}
where $O(1)$ represents quantity that is bounded by constants depending only on $g_0^c$ and $g_{s+}$, but not on $\delta$ and $s$.

\underline{Step 4.} Annihilate the negative scalar curvature:

For each fixed $s$, we consider the solution to the following equation for small $\delta$
\begin{equation}
\begin{cases}
\Delta g^s u^s + c_n R^s u_s \quad = \quad 0 \quad \text{on } \tilde{M}^s_\infty \\
u^s \quad = \quad 1 \quad \text{on } \Sigma \\
u^s \quad \to \quad 1 \quad \text{at } \infty.
\end{cases}
\end{equation}

Similar to Proposition 4.1 in [14], we have that
\begin{equation}
\lim_{\delta \to 0} \sup_{\tilde{M}^s_\infty} \{u_s^s - 1\} = 0 \quad \text{and} \quad \|u_s^s\|_{C^{2,\alpha}(K)} \leq C_K
\end{equation}
for any compact $K \subset \tilde{M}^s_\infty \setminus \Sigma_{s_0}$. Hence, passing to a subsequence, $u_s^s$ converges to 1 on $\Sigma \times [0, \frac{\delta}{2}]$ in $C^2$ topology, which implies that
\begin{equation}
\lim_{\delta \to 0} \sup_{\Sigma} \frac{\partial u_s^s}{\partial n} = 0,
\end{equation}
where $\tilde{n}$ denotes the outward unit normal vector to $\Sigma$ determined by $g^s_c$. We define
\begin{equation}
\tilde{g}^s = (u_s^s)^{\frac{2}{n-2}} g^s_c,
\end{equation}
similar to Lemma 4.2 in [14], we have that
Lemma 3.4.  

\[ \lim_{\delta \to 0} m(\tilde{g}_\delta^s) = m(g_{s+}) \quad \text{and} \quad \lim_{\delta \to 0} H(\Sigma, \tilde{g}_\delta^s) = H(\Sigma, g_{s+}^s). \]

Proof: The second limit follows directly from (27) and

\[ H(\Sigma, \tilde{g}_\delta^s) = H(\Sigma, g_{s+}^s) + \frac{2}{n-2} \frac{\partial u_s^*}{\partial n}. \]

To see the first limit, we recall that

\[ m(\tilde{g}_\delta^s) = m(g_{s+}^s) + (n-1)A_s^s, \]

where \( A_s^s \) is given by the expansion \( u_s^* = 1 + A_s^s |x|^{2-n} + O(|x|^{1-n}). \) Applying integration by parts to (25) multiplied by \( u_s^* \), we have that

\[ \int_{M^\infty} \left[ |\nabla_{g_{s+}} v_s^*|^2 - c_n R_{s+}^s u_s^* \right] dg_{s+} + \frac{\partial u_s^*}{\partial n} d\mu, \]

where \( \mu \) is the induced surface measure by \( g_{s+} \) on \( \Sigma \). (31) and (32) imply that

\[ m(g_{s+}^s) = m(g_{s+}^s) + \frac{n-1}{n-2} \omega_{n-1} \left\{ \int_{M^\infty} \left[ |\nabla_{g_{s+}} v_s^*|^2 - c_n R_{s+}^s u_s^* \right] dg_{s+} + \frac{\partial u_s^*}{\partial n} d\mu \right\}. \]

It follows from (24) and the proof of (25) that the integral term above goes to zero. Hence, we have that \( \lim_{\delta \to 0} m(\tilde{g}_\delta^s) = \lim_{\delta \to 0} m(g_{s+}^s) = m(g_{s+}). \)

Step 5. Level down the positive scalar curvature:

To make use of the scalar curvature concentration near \( \Sigma_{\sigma_0} \) as \( \delta \to 0 \), we let \( v_s^\delta \) be a positive solution to

\[ \begin{cases} 
\triangle v_s^\delta - c_n \tilde{R}_s^\delta v_s^\delta = 0 & \text{on } M^\infty \\
v_s^\delta = 1 & \text{on } \Sigma \\
v_s^\delta \to 1 \text{ at } \infty,
\end{cases} \]

and define

\[ \tilde{g}_\delta^s = (v_s^\delta)^{-\frac{4}{n-2}} \tilde{g}_\delta^s. \]

Like (31) and (32), we have that

\[ m(\tilde{g}_\delta^s) = m(g_{s+}^s) + (n-1)A_s^s, \]

where \( A_s^s \) is given by the expansion \( v_s^s(x) = 1 + A_s^s |x|^{2-n} + O(|x|^{1-n}) \) and can be written explicitly as

\[ (2-n)\omega_{n-1}A_s^s = \int_{M^\infty} \left[ |\nabla_{\tilde{g}_\delta^s} v_s^\delta|^2 + c_n \tilde{R}_s^\delta (v_s^\delta)^2 \right] d\tilde{g}_\delta^s + \frac{\partial (v_s^\delta)^2}{\partial n} d\mu. \]

Proposition 3.2.

\[ \lim inf_{s \to 0} \left\{ \lim inf_{\delta \to 0} \left\{ \int_{M^\infty} \left[ |\nabla_{\tilde{g}_\delta^s} v_s^\delta|^2 + c_n \tilde{R}_s^\delta (v_s^\delta)^2 \right] d\tilde{g}_\delta^s + \frac{\partial (v_s^\delta)^2}{\partial n} d\mu \right\} \right\} > 0 \]

Proof: Assume that (38) is not true, then there exist sequences \( \{s_k\} \) and \( \{\delta_k\} \) so that

\[ \lim_{k \to \infty} s_k = 0, \quad \lim_{k \to \infty} \delta_k = 0. \]
and

\[
\lim_{k \to \infty} \left\{ \int_{\tilde{M}^{s_k}_{\infty} \setminus N^{s_k}_{\infty}} \left[ |\nabla g^{s_k}_{\delta_k} v^{s_k}_{\delta_k}|^2 + c_n R^{s_k}_{\delta_k} (v^{s_k}_{\delta_k})^2 \right] d\tilde{g}^{s_k}_{\delta_k} + \int_{\Sigma} \frac{\partial v^{s_k}_{\delta_k}}{\partial n} d\mu \right\} = 0.
\]

(41) implies that, passing to a subsequence, \( \{v^{s_k}_{\delta_k}\} \) converges to a \( g^{\delta}_c \)-harmonic function \( v \) on the compact set \( \Sigma \times [0, \frac{N}{8}] \) in \( C^2 \) topology, where \( 0 \leq v \leq 1 \) by the maximum principle. Hence,

\[
\lim_{\delta \to 0^+} \{\sup_{\Sigma} \left( \frac{\partial v^{s_k}_{\delta_k}}{\partial n} - \frac{\partial v}{\partial n} \right) \} = 0.
\]

We claim that \( v \equiv 1 \). If not, the strong maximum principle implies that

\[
\sup_{\Sigma} \frac{\partial v}{\partial n} < 0 \quad \text{and} \quad 0 < v(x) < 1 \quad \text{for} \quad x \in \Sigma \times (0, \frac{\sigma_0}{4}).
\]

We let \( \theta \in (0, 1) \) denote the supremum of \( v \) on \( \Sigma \times \{ \frac{2\sigma}{3} \} \) and let \( w_k \) be the solution to

\[
\begin{align*}
\Delta_{\tilde{g}_{\delta_k}} w_k &= 0 \quad \text{on} \quad \tilde{M}^{s_k}_{\infty} \setminus N^{s_k}_{\infty} \\
w_k &= \theta \quad \text{on} \quad \Sigma^{s_k}_{\infty} \\
w_k(x) &\to 1 \quad \text{at} \quad \infty,
\end{align*}
\]

where \( N^{s_k}_{\infty} = \Sigma \times [0, \frac{2\sigma}{3}] \). It follows from (44), (45) and the maximum principle that \( w_k \geq v^{s_k}_{\delta_k} \), which implies that

\[
A_{\delta_k} \leq B_k,
\]

where \( B_k \) is given by the expansion \( w_k = 1 + B_k |x|^{2-n} + O(|x|^{1-n}) \) and can be written explicitly as

\[
(2-n)\omega_{n-1} B_k = \int_{\tilde{M}^{s_k}_{\infty} \setminus N^{s_k}_{\infty}} |\nabla g^{s_k}_{\delta_k} w_k|^2 d\tilde{g}^{s_k}_{\delta_k} + \int_{\Sigma^{s_k}_{\infty}} \frac{\partial w_k}{\partial n} d(\tilde{g}^{s_k}_{\delta_k} |\Sigma^{s_k}_{\infty}).
\]

By the maximum principle, we have that \( \frac{\partial w_k}{\partial n} \geq 0 \). Hence, (46) shows that

\[
(2-n)\omega_{n-1} B_k \geq \int_{\tilde{M}^{s_k}_{\infty} \setminus N^{s_k}_{\infty}} |\nabla g^{s_k}_{\delta_k} w_k|^2 d\tilde{g}^{s_k}_{\delta_k}.
\]

Since \( \theta \in (0, 1) \), (46) implies that

\[
\lim_{k \to \infty} \int_{\tilde{M}^{s_k}_{\infty} \setminus N^{s_k}_{\infty}} |\nabla g^{s_k}_{\delta_k} w_k|^2 d\tilde{g}^{s_k}_{\delta_k} \geq \frac{1}{2} E(g, \theta) > 0,
\]

where \( E(g, \theta) = \inf \{ \int_{\tilde{M}^{s_k}_{\infty} \setminus N^{s_k}_{\infty}} |\nabla g^{s_k}_{\delta_k} \psi|^2 d\tilde{g}^{s_k}_{\delta_k} | \psi = \theta \text{ on } \Sigma^{s_k}_{\infty}, \psi \to 1 \text{ at } \infty \} \). Thus it follows from (44), (46) and (47) that

\[
\lim_{k \to \infty} (2-n)\omega_{n-1} A_{\delta_k} \geq \frac{1}{2} E(g, \theta).
\]

By (46) we have a contradiction to (40) and \( v \equiv 1 \) on \( \Sigma \times [0, \frac{2\sigma}{3}] \). Now (41) imply that

\[
\lim_{k \to \infty} \{\sup_{\Sigma} \left( \frac{\partial v^{s_k}_{\delta_k}}{\partial n} \right) \} = 0.
\]
Now it follows from (17), (22), Lemma 3.4 and Proposition 3.2 that the re exist

The mean curvature relation

and the fact that, passing to a subsequence, \( \lim_{k \to \infty} \int_{M^+_{\delta_k}} \left[ |\nabla g_{\delta_k}^\epsilon|^2 + c_n R^\epsilon_{\delta_k}(v^\epsilon_{\delta_k})^2 \right] d\tilde{g}_{\delta_k}^\epsilon = 0. \)

Now we are in a situation that is as same as in Proposition 4.2 in [14]. The proof in [14] shows that (50) can not hold.

To complete the proof of Theorem 3.1, we fix a constant \( \epsilon > 0 \) and define

\[
g_{\delta t}^\epsilon = (1 + t(v^\epsilon_s - 1)) \frac{\delta}{\epsilon} \tilde{g}_{\delta}.
\]

The mean curvature relation

\[
H(\Sigma, g_{\delta t}^\epsilon) = H(\Sigma, \tilde{g}_{\delta}) + t \frac{\partial v^\epsilon_s}{\partial \tilde{n}} < H(\Sigma, \tilde{g}_{\delta})
\]

and the fact that, passing to a subsequence, \( \{v^\epsilon_s\} \) converges to a \( g^0_{\delta} \)-harmonic function \( v \) on \( \Sigma \times [0, \frac{\delta}{\epsilon}] \) in \( C^2 \) topology as \( \delta, s \to 0 \) imply that there exists a constant \( t_0 > 0 \) depending only on \( \epsilon, v \) but not on \( \delta, s \) such that

\[
H(\Sigma, g_{\delta t}^\epsilon) - \frac{1}{2} \epsilon < H(\Sigma, g_{\delta t_0}^\epsilon) < H(\Sigma, g_{\delta}^s).
\]

Now it follows from [14], [22], Lemma 3.4 and Proposition 3.2 that there exist \( 0 < s < s_0 \) and \( 0 < \delta < \delta_0 \) such that

\[
\left\{ \begin{array}{c}
m(g_{s+}) < m(g_s) + \frac{b_k A}{4 n - 2} \frac{\partial v^\epsilon_s}{\partial \tilde{n}} \\
H(\Sigma, g^s_{\delta}) = H(\Sigma, g^s_{\delta}) - \frac{2s}{n - 2} \frac{\partial v^\epsilon_s}{\partial \tilde{n}} \\
m(g^s_{\delta}) < m(g_{s+}) + \frac{b_k A}{4 n - 2} \\
H(\Sigma, g^s_{\delta}) < H(\Sigma, g^s_{\delta}) + \frac{2s}{n - 2} \frac{\partial v^\epsilon_s}{\partial \tilde{n}} \\
m(g^s_{\delta}) > m(g_{s+}) - \frac{2s}{n - 2} \frac{\partial v^\epsilon_s}{\partial \tilde{n}} \\
m(g^s_{\delta}) < m(g^s_{\delta}) - \frac{2s}{n - 2} A
\end{array} \right.
\]

where \( \left| \frac{s}{n - 2} \frac{\partial v^\epsilon_s}{\partial \tilde{n}} \right| < \frac{\delta}{\epsilon} \) and

\[
A = \liminf_{s \to 0} \left\{ \liminf_{\delta \to 0} \left\{ \int_{M^s_{\delta}} \left[ |\nabla \tilde{g}_{\delta}^\epsilon|^2 + c_n R^\epsilon_{\delta}(v^\epsilon_{\delta})^2 \right] d\tilde{g}_{\delta}^\epsilon + \int_{\Sigma} \frac{\partial v^\epsilon_s}{\partial \tilde{n}} d\tilde{g}_{s+} \right\} \right\} > 0.
\]

Hence, we have \( (\tilde{M}^s_{\delta t_0}, g^s_{\delta t_0}) \in M^s_{\infty} \) such that

\[
\left\{ \begin{array}{c}
m(g^s_{\delta t_0}) < m(g_{s+}) \\
H(\Sigma, g^s_{\delta t_0}) > H(\Sigma, g_{s+}) - \epsilon.
\end{array} \right.
\]

\[
3.3. \text{A Note on Quasi-local Mass.} \text{ Theorem 3.1 suggests that, when considering the quasi-local mass question, we may first focus on a domain} (\Omega, g) \subset (M^s, g) \text{ which has quasi-convex boundary, i.e.} \partial \Omega \text{ has positive scalar curvature and positive mean curvature. For such a domain} (\Omega, g), \text{ we modify Bartnik’s definition slightly to define the following metric extension class}
\]

\[
\overline{F}M = \{ (M^s_{\infty}, g_{s+}) | (M^s_{\infty}, g_{s+}) \text{ is a smooth asymptotically flat manifold with boundary} \Sigma \text{ such that} R(g_{s+}) \geq 0, g|_{\Sigma} = g_{s+}|_{\Sigma}, \\
H(\Sigma, g) \geq H(\Sigma, g_{s+}) > 0, \text{ and there is no} S^2 \text{ outside} \Sigma \text{ with area less than or equal to} \Sigma. \}.
\]
Like the no horizon assumption in Bartnik’s definition, the last restriction on \( \overline{\mathcal{M}} \) is imposed to prevent the infimum of the total mass functional over \( \overline{\mathcal{M}} \) from being trivially zero. A careful examination of our construction in Section 3.2 reveals that, if we start with a \((M_\infty, g_+)\) in \( \overline{\mathcal{M}} \), the resulting comparison manifold \((M_\infty, \tilde{g}_{\delta_{60}})\) can also be chosen in \( \overline{\mathcal{M}} \). Hence, we have the following characterization of \( \overline{\mathcal{M}} \).

**Proposition 3.3.** For any given \( \epsilon > 0 \),
\[
\inf\{m(g_+) | g_+ \in \overline{\mathcal{M}}\} = \inf\{m(g_+) | g_+ \in \overline{\mathcal{M}}, H(\Sigma, g_+) \geq H(\Sigma, g) - \epsilon\}.
\]

Proposition 3.3 implies that, for any mass minimizing sequence \( \{(M_\infty, g_+)\}_i \) in \( \overline{\mathcal{M}} \), it can always be replaced by a new mass minimizing sequence \( \{(M_\infty, \tilde{g}_{+i})\}_i \) such that \( \{H(\Sigma, \tilde{g}_{+i})\}_i \) monotonically increases to \( H(\Sigma, g_-) \). In particular, we may assume that the Hawking mass of \( \Sigma \) in \((M_\infty, \tilde{g}_{+i})\)
\[
\text{Area}(\Sigma)^{\frac{2}{n}} \left\{ 16\pi - \int_{\Sigma} H(\Sigma, \tilde{g}_{+i})^2 \right\}
\]
monotonically decrease to the Hawking mass of \( \Sigma \) in \((\bar{\Omega}, g_-)\).

We call a manifold \((M_\infty, g_{min}) \in \overline{\mathcal{M}}\) a **minimal mass extension** of \((\bar{\Omega}, g)\) if
\[
m(g_{min}) = \inf\{m(g_+) | (M_\infty, g_+) \in \overline{\mathcal{M}}\}.
\]
So far, it is an open question if there exists a minimal mass extension of \((\bar{\Omega}, g)\) in \( \overline{\mathcal{M}} \). However, the following proposition shows that if such an extension exists, it must be a static metric \([8]\) with zero scalar curvature and satisfy Bartnik’s geometric boundary condition \([4]\). For notation consistency, we again let \( g_- \) denote \( g \) on \( \bar{\Omega} \).

**Proposition 3.4.** If \((M_\infty, g_+)\) is a minimal mass extension of \((\bar{\Omega}, g_-)\), then, in the interior of \( M_\infty \), \( g_+ \) is a scalar flat and static metric and, at the boundary \( \Sigma \), \( g_+ \) satisfies
\[
\begin{cases}
g_-|_{\Sigma} = g_+|_{\Sigma} \\
H(\Sigma, g_-) = H(\Sigma, g_+).
\end{cases}
\]

**Remark.** For a great introduction to static metrics, readers are referred to \([8]\).

**Proof:** The boundary condition \((57)\) follows directly from Theorem 3.1. To derive the interior equation, we first show that \( \bar{R}(g_+) \) is identically zero. Assume not, then there exists a positive solution \( u \) to
\[
\begin{cases}
\triangle_{g_+} u - c_n R(g_+) u &= 0 & \text{on } M_\infty \\
u &= 1 & \text{on } \Sigma \\
u &\to 1 & \text{at } \infty,
\end{cases}
\]
and \( u \) has an asymptotic expansion at \( \infty \),
\[
u(x) = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{1-n}).
\]
It follows from the strong maximum principle that \( A < 0 \). We consider a path of metrics \( \{g_{+t}\}_{0 \leq t \leq 1} \) defined by \( g_{+t} = \frac{v_t}{1 + t(u - 1)} g_+ \), where
\[
v_t = (1 - t) + tu = 1 + t(u - 1).
\]
It follows from \((58)\) that
\[
\triangle_{g_+} v_t - c_n R(g_+) v_t = c_n(t - 1) R(g_+) \leq 0.
\]
Hence, \((M_\infty, g_{t_+})\) is asymptotically flat with non-negative scalar curvature. At \(\Sigma\), we have that

\[
\begin{align*}
\left. g_{t_+} \right|_\Sigma &= g_+|_\Sigma \\
H(\Sigma, g_{t_+}) &= H(\Sigma, g_+) + \frac{2}{n-2} \frac{\partial u}{\partial n},
\end{align*}
\]

where \(\vec{n}\) is the outward unit normal vector field along \(\Sigma\) determined by \(g_+\). By the strong maximum principle, \(\frac{\partial u}{\partial \vec{n}} < 0\) at every point on \(\Sigma\). Hence, it follows from (60) and (62) that, for sufficiently small \(t_0\), \(\{g_{t_+}\}_{0 \leq t \leq t_0} \subset \overline{\mathcal{PM}}\). On the other hand, straightforward calculation reveals that

\[
m(g_{t_+}) = m(g_+) + (n-1)tA.
\]

The fact \(A < 0\) implies that \(m(g_{t_+}) < m(g_+)\) for small positive \(t\), which contradicts the fact \((M_\infty, g_+)\) minimizes the total mass. Hence, \(g_+\) must have vanishing scalar curvature on \(M_\infty\).

Now assume that \(g_+\) is not static on \(M_\infty\), the scalar curvature deformation result of J. Corvino [8] then implies that there exists a manifold \((M_\infty, \tilde{g}) \in \overline{\mathcal{PM}}\) such that \(\tilde{g}\) agrees with \(g\) outside a compact set \(K \subset (M_\infty \setminus \Sigma)\) and \(\tilde{R}(\tilde{g}) > 0\) on \(K\). Thus \((M_\infty, \tilde{g})\) is a minimal mass extension with non-zero scalar curvature, which is a contradiction to what we have just proved. Hence, \(g_+\) is a static metric. \(\square\)

Proposition 3.4 suggests one interesting metric extension question known as the static metric extension conjecture proposed by R. Bartnik in [4]. For a partial answer to the corresponding small data solution, readers are referred to [13].

\section*{References}

[1] R. Arnowitt, S. Deser, and C. W. Misner. Coordinate invariance and energy expressions in general relativity. \textit{Phys. Rev.} (2), 122:997–1006, 1961.

[2] Robert Bartnik. New definition of quasilocal mass. \textit{Phys. Rev. Lett.}, 62(20):2346–2348, 1989.

[3] Robert Bartnik. Quasi-spherical metrics and prescribed scalar curvature. \textit{J. Differential Geom.}, 37(1):31–71, 1993.

[4] Robert Bartnik. Energy in general relativity. In \textit{Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991)}, pages 5–27. Internat. Press, Cambridge, MA, 1997.

[5] Hubert Bray. Proof of the riemannian penrose conjecture using the positive mass theorem. \textit{J. Differential Geom.}, 59(2):177–267, 2001.

[6] Hubert Bray and Felix Finster. Curvature estimates and the positive mass theorem. \textit{Comm. Anal. Geom.}, 10(2):291–306, 2002.

[7] J. David Brown and James W. York, Jr. Quasilocal energy and conserved charges derived from the gravitational action. \textit{Phys. Rev. D} (3), 47(4):1407–1419, 1993.

[8] Justin Corvino. Scalar curvature deformation and a gluing construction for the Einstein constraint equations. \textit{Comm. Math. Phys.}, 214(1):137–189, 2000.

[9] Gilbarg David and Trudinger Neil S. \textit{Elliptic Partial Differential Equations of Second Order}. Berlin: Springer-Verlag, 1983.

[10] Heinz Hopf. \textit{Differential geometry in the large}, volume 1000 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, second edition, 1989. Notes taken by Peter Lax and John W. Gray, With a preface by S. S. Chern, With a preface by K. Voss.

[11] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. \textit{J. Differential Geom.}, 59(3):353–437, 2001.

[12] Chiu-Chu Melissa Liu and Shing-Tung Yau. New definition of quasilocal mass and its positivity. \textit{http://arXiv.org/abs/gr-qc/0303019}, 2003.

[13] Pengzi Miao. On existence of static metric extensions in general relativity. \textit{Comm. Math. Phys.}, to appear.

[14] Pengzi Miao. Positive mass theorem on manifolds admitting corners along a hypersurface. \textit{Adv. Theor. Math. Phys.}, 6(6), 2002.
[15] Richard Schoen. Variational theory for the total scalar curvature functional for riemannian metrics and related topics. In Topics in the Calculus of Variations. Lecture Notes in Math. 1365, pages 120–154. Berlin: Springer-Verlag, 1987.

[16] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. Comm. Math. Phys., 65(1):45–76, 1979.

[17] Yuguang Shi and Luen-Fai Tam. Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. http://arXiv.org/abs/math.DG/0301047, 2003.

[18] Brian Smith and Gilbert Weinstein. On the connectedness of the space of initial data for the Einstein equations. Electron. Res. Announc. Amer. Math. Soc., 6:52–63 (electronic), 2000.

[19] Edward Witten. A new proof of the positive energy theorem. Comm. Math. Phys., 80(3):381–402, 1981.

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