A State Dependent Thermo-Hyperelastic Model for Geomaterials

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Abstract. A hyperelastic model for the state dependent and thermo-mechanical coupled nonlinear elastic behavior of geomaterials is developed in this paper. In the model the stress-density state dependent elastic moduli and the elastic instability are predicted by the high order terms of elastic strain invariants in the elastic potential function. The effects of true cohesion on the elastic potential of bonded geomaterials such as bonded sands, natural structured clays and rocks are also taken into account. As a result, unified relations between the density, the degradable cohesion, the confining stress and the elastic moduli can be derived for different geomaterials. Meanwhile, such an approach theoretically results in a state boundary and thus a strength criterion for geomaterials from the stability of elasticity. Based on such a hyperelastic approach, the nonlinear thermo-elastic coupled behavior is further considered by defining an equivalent elastic volumetric strain and taking into account a volumetric elastic thermal expansion coefficient dependent on both the volume fraction and thermal property of bound water absorbed on mineral surfaces.

Keywords. Nonlinear thermo-hyperelastic coupling, thermodynamics, cohesion degradation.

1. Introduction
In the geotechnical engineering, it is important to study the nonlinear elastic behavior of geomaterials such as granular soils, clays and rocks, which largely depends on the states of stress, density and temperature. Experimental results in small strain tests and bend element tests show that the elastic moduli of rocks and soils can be defined as a power function of the confining stress with an index number ranging from 0.4 to 0.7 [1-4]. For soils and some soft rocks, the elastic moduli also remarkably depend on their stress histories that result in different densities and over-consolidation ratios. The prediction of such nonlinear and state-dependent elastic behavior is therefore one of the crucial issues for the modeling of geomaterials. The linear elastic model extensively used in geomechanics, however, may result in a large deviation of deformation from the real value. In fact, even under extremely small strain, the elastic moduli of geomaterials cannot be simply referred to as constants [5-7].

The temperature of natural geomaterials usually changes with the atmosphere temperature and the depths below ground surface, which further leads to the variations in their elastic properties [8]. Generally speaking, the elastic behavior of clay can be more sensitive to the temperature than that of other geomaterials. The resulted thermo-elastic coupled behavior of geomaterials is especially important in some geotechnical engineering areas such as geothermal engineering and nuclear waste
storage [9]. In these fields, the temperature-induced variations in the elastic moduli and deformation of geomaterials should be concerned for accurate modeling.

The importance of nonlinear elasticity also lies in its intrinsic link with the strength criterion (or failure criterion) of geomaterials, since that the strength concept to a large content refers to the limit of elastic resistance to deformation under specific stress and temperature conditions. This fact had been ignored by most existing models in which certain strength criteria defined were completely independent of the elastic behavior. Instead, it had been noted that the strength criteria of solids can be naturally determined by their stabilities of nonlinear elasticity without additional definitions and parameters [10-11]. The elastic model with such a feature can be developed using the hyperelastic method in which the definition of elastic potential is needed.

For different types of geomaterials, the stress-density-temperature dependent elasticity and the strength criterion can differ significantly due to the differences in their microstructures, absorbed bound water contents and bonding effects. For rocks, natural structured clays and artificially bonded soils, the bonding effects will not only increase the elastic modulus and the strength but also lead to a nonlinear strength criterion with certain tensile strength. The main purpose of this paper is thus to develop a nonlinear model covering the above-mentioned elastic behavior unified for different geomaterials. The nonlinear and thermal elastic stress-strain relationship, instead of being defined directly like in Cauchy elasticity (e.g. the Duncan-Chang model [12]), is proposed based on the hyperelastic approach [11, 13] with a definition of elastic potential dependent on density, degradable bonding and temperature. The elastic moduli, thermo-elastic deformation and strength criteria of different geomaterials can thus be predicted within a unified theoretical framework.

2. Theory Development

2.1. General Background

In order to establish a nonlinear elastic model described above, an elastic potential density function generalized for different geomaterials should be defined in terms of elastic strain, density and temperature. The following three invariants of the elastic strain tensor $\varepsilon^e_{ij}$ are usually used for this purpose:

$$\varepsilon^e_{kk}, \varepsilon^e_{ij} = \sqrt{\varepsilon^e_{ij} \varepsilon^e_{ij}}, \quad \varepsilon^e_t = \sqrt{\varepsilon^e_{ij} \varepsilon^e_{jk} \varepsilon^e_{ki}}$$

(1)

where $\varepsilon^e_{kk}$ is just the elastic volumetric strain and $\varepsilon^e_{ij} = \varepsilon^e_{ij} - \delta_{ij}/3$ is the deviatoric elastic strain.

The effective stress of geomaterials, denoted as $\sigma^e_{ij}$, can thus be:

$$\sigma^e_{ij} = \frac{\partial \omega_e(\varepsilon^e_{ij}, T, \rho)}{\partial \varepsilon^e_{ij}}$$

(2)

where $T$ and $\rho$ are the temperature and dry density of the specific geomaterial. In the simplest case, the elastic potential function $\omega_e$ for linear elastic material under isothermal conditions can be defined as:

$$\omega_e = \frac{1}{2} K_e (\varepsilon^e_{kk})^2 + G_e (\varepsilon^e_t)^2$$

(3)

where $K_e$ and $G_e$ are the elastic bulk and shear moduli, respectively.

For the nonlinear hyperelasticity, $\omega_e$ could be defined by extending equation (3) to a higher-order function of elastic strain invariants, e.g. in the model by Houlsby, Amorosi and Rojas model [14] and the model by Houlsby [15]. Both of them well capture the confining stress dependency of elastic moduli, and the latter also gives a constant maximum stress ratio for soil. The Granular Elasticity is another hyperelastic model for granular solids extended from the theory of nonlinear contact stiffness between sphere particles [11, 16]. The elastic potential in Granular Elasticity is originally defined by
where $B$ is a material parameter with stress dimension, $\alpha$ is the ratio of $G_\text{e}$ to $K_\text{e}$ under isotropic compression. This model gives well predictions of the elastic behavior for sands, while is not applicable for clays and bonded geomaterials. Therefore, based on similar ideas, a more generalized hyperelastic model unified for different types of saturated geomaterials will be developed in this section considering the bonding and thermal effects. In this paper, all stresses and strains are taken as positive under compression.

2.2. Hyperelasticity for Saturated Geomaterials

Saturated geomaterials in this paper is classified into three categories including granular soils, unbounded clayey soils and bonded geomaterials. Only the bonded geomaterials, e.g. artificially bonded sands, naturally structured clays and rocks, are considered to be with true cohesion [17]. It should be noted that there is no true cohesion in most saturated clays, and thus the shear and tensile strengths of them should completely vanish under a zero confining stress [18]. However, as shown in figure 1, the state boundary line (also the failure line) of such saturated unbounded clays may be nonlinear in effective stress space, leading to a kind of fictitious cohesion when the Mohr-Column strength criterion is applied. The stress boundary of granular soils like sands, also without any bonding effects, is a straight line through the origin in effective stress space. On the contrary, a negative effective stress region is allowed for bonded geomaterials with a curved stress boundary line, as shown in figure 1. These differences between geomaterials should be considered in the thermo-hyperelastic model of geomaterials.

Figure 1. Typical state boundary lines (failure lines) of geomaterials in effective stress space.

Figure 2. Evolutions of elastic strain invariants in drained shearing ($A = 30$ GPa; $B = 100$ GPa; $m = n$ elastic shear strain and the dash lines are ones for the elastic volumetric strain).

Under isothermal conditions, such a hyperelastic model can be derived by defining the elastic moduli as a function of the elastic volumetric deformation and the cohesion; e.g., $K_\text{e} \propto (\varepsilon_\text{v} + c)^m$ and $G_\text{e} \propto (\varepsilon_\text{v} + c)^n$, where $K_\text{e}$ and $G_\text{e}$ are the elastic bulk and shear moduli, respectively, $c$ is a cohesion index representing the degree of bonding within geomaterials and $m$ and $n$ are two state-dependency indexes. Once the elastic potential function is determined, the effective mean stress $p = \partial \omega_\text{e} / \partial \varepsilon_\text{v}$ and shear stress invariant $q = \sqrt{2s_{ij}s_{ij}} = \partial \omega_\text{e} / \partial \varepsilon_\text{v}$, where $s_{ij} = \sigma_{ij} - p\delta_{ij}$ is the deviatoric stress tensor. Thus, further condition the effect of density on the elastic stiffness, the following formulation of $\omega_\text{e}$ can be derived for geomaterials:
where $\omega$ and $\omega'$ are two stress-dimension parameters, and $f(\rho)$ is an exponential function of dry density, i.e.

$$f(\rho) = e^{k_1 \rho}$$

where $k_1$ is a material constant and $\rho$ is the dry density. The relations above can be reduced to the cases for unbounded geomaterials just by taking $c = 0$. The two nonlinear indexes, $m$ and $n$, control the stress dependency of elastic moduli of geomaterials. They also determine the distinction between the state boundary lines of granular soils and unbounded clays as shown in Figure 1. According to the material types, the values of $c$, $m$ and $n$ should meet the following conditions:

$$\begin{cases} 
  c = 0, \quad m = n & \text{Granular soils} \\
  c = 0, \quad m \neq n & \text{Unbounded clays} \\
  c \neq 0 & \text{Bonded Geomaterials}
\end{cases} \quad (7)$$

Substituting equation (7) into equation (2), the effective stress of saturated geomaterials can be obtained in terms of $\varepsilon_{\sigma}$, $\varepsilon_{\varepsilon}$ and $\rho$:

$$\sigma_{ij} = f(\rho)\{2A(\varepsilon_{\sigma} + c)^m \varepsilon_{ij} + [B(\varepsilon_{\varepsilon} + c)m\varepsilon_{\sigma} + An(\varepsilon_{\varepsilon} + c)n^{-2} \varepsilon_{\sigma}^2]\delta_{ij}\} \quad (8)$$

Equation (8) leads to the following elastic moduli:

$$K_\varepsilon = f(\rho)[B(\varepsilon_{\varepsilon} + c)m + An(\varepsilon_{\varepsilon} + c)n^{-2} \varepsilon_{\sigma}^2] \quad (9a)$$

$$G_\varepsilon = 2f(\rho)A(\varepsilon_{\sigma} + c)m \quad (9b)$$

from which the relations between elastic moduli, effective confining pressure and cohesion can be derived as follows:

$$\frac{K_\varepsilon}{B_0 e^{k_1 \rho}} \frac{1}{m} - \frac{\rho'}{K_\varepsilon} = c \quad (10a)$$

$$\frac{G_\varepsilon}{A_0 e^{k_1 \rho}} \frac{1}{m} - \frac{\rho'}{G_\varepsilon} = c \quad (10b)$$

In the derivation of equation (10), $m = \eta$ is assumed for simplification. Using such a kind of hyperelasticity, the nonlinear elastic stress-strain relations can be simulated for different geomaterials. Figure 2 shows a simulation example of the elastic strain evolutions under a given stress path with constant incremental stress ratios. It is obvious that shear-induced elastic volumetric deformation can be observed.

### 2.3. Thermo-elastic Coupled Hyperelastic Relations

The elastic potential can also be stored or released due to the elastic thermal expansion or contraction of geomaterials, which may further lead to changes in elastic moduli and strengths. Denoting the secant elastic volumetric thermal expansion coefficient as $\beta$, an equivalent elastic volumetric strain can be defined as:

$$E_\beta = \varepsilon_{\beta} + \beta \Delta T \quad (11)$$

where $\Delta T = T - T_0$ is the temperature variation with respect to the reference temperature $T_0$ and $\beta \Delta T$ is the corresponding maximum expansive thermal volumetric strain. The elastic potential density and the effective stress expression with thermal effects can thus be obtained by replacing the variable $\varepsilon_{\sigma}$ in equations (7 and 8) with $E_\beta$, i.e.,
For saturated geomaterials without clay minerals (e.g., granular soils), \( \beta \) mainly depends on the thermal properties of the mineral compositions and can be taken as a constant. However, for saturated clays or other geomaterials with clay minerals, \( \beta \) could be sensitive to the thermal properties of bound water absorbed on the clay mineral surfaces. The bound water usually possesses certain shear stiffness and can be considered to deform like a solid together with the mineral solid. Thus, both the thermal expansions/contractions of mineral solid and bound water contribute to the thermally induced volumetric deformation of geomaterials. The thermal property of free water, however, does not affect the elastic thermal deformation of the skeleton of saturated geomaterials.

Therefore, the value of \( \beta \) can be determined by considering the reversible thermal volume change of a skeleton element with a total volume of \( V \), a mineral solid volume of \( V_s \), a bound water volume of \( V_b \), and a free water volume of \( V_f \) (\( V = V_s + V_b + V_f \)). The total volume change of mineral solid and bound water induced by a temperature variation \( \Delta T \) is \( \Delta V_1 = (\beta_s V_s + \beta_b V_b) \Delta T \), where \( \beta_s \) and \( \beta_b \) are the intrinsic volumetric thermal expansion coefficients of mineral solid and bound water, respectively. Under thermo-elastic conditions, it can be assumed that the void volume occupied by free water changes in accordance with the same thermal expansion coefficient as the total volume of mineral solid and bound water, indicating a void volume change \( \Delta V_2 = \Delta V_1 / (V_s + V_b) \cdot V_f \). Thus, the reversible volume change of the skeleton element just equals \( \Delta V_1 + \Delta V_2 \). \( \beta \) can be then defined as follows:

\[
\beta = \frac{1}{V} \frac{\Delta V_1 + \Delta V_2}{\Delta T} = \frac{\beta_s V_s + \beta_b V_b}{\phi_s + \phi_b}
\]  
(13)

where \( \phi_s = V_s/V \) and \( \phi_b = V/V_b \) are the volume fractions of mineral solid and bound water, respectively.

In general, temperature variations will lead to the changes in \( \phi_s \), \( \phi_b \), and \( \beta_b \), resulting in the nonlinear thermo-elastic coupling with a non-constant \( \beta \). The evolutions of \( \phi_s \) and \( \phi_b \) can be determined according to the following mass conservation equations:

\[
\frac{d(\rho_s \phi_s)}{dt} = \rho_s \phi_s \dot{\varepsilon}_v
\]  
(14a)

\[
\frac{d(\rho_b \phi_b)}{dt} = \rho_b \phi_b \dot{\varepsilon}_v - \rho_b \phi_b \alpha_{b,f} \dot{T}
\]  
(14b)

where \( \varepsilon_v \) is the volumetric strain (\( \varepsilon_v = \varepsilon_v^e \) for pure elastic processes); \( \rho_s \) and \( \rho_b \) are the intrinsic densities of mineral solid and bound water, respectively; \( \alpha_{b,f} \) is the free water mass converted from unit mass of bound water induced by a unit temperature rise within unit time. The thermally induced conversion from bound water to free water is considered here as a basic physical property of bound water and as a key mechanism governing the thermo-elastic behavior of saturated geomaterials with bound water absorbed by clay minerals. For geomaterials in which the content of bound water is ignorable (e.g., granular soils), it is obviously from equation (13) that \( \beta \) is just the thermal expansion coefficient of the mineral solid. Equations (13 and 14) therefore provide a unified description of the thermal elastic behavior of different saturated geomaterials.

The intrinsic densities of mineral solid and bound water, \( \rho_s \) and \( \rho_b \), can be sensitive to the temperature variation. It can be defined that \( \rho_s = \rho_{s0} (1 - \beta_s \Delta T) \) and \( \rho_b = \rho_{b0} (1 - \beta_b \Delta T) \), where \( \rho_{s0} \) and \( \rho_{b0} \) are respectively the values of \( \rho_s \) and \( \rho_b \) at \( T = T_0 \). From equation (14), we have

\[
\rho_s = \rho_{s0} \phi_s = \rho_{s0} \exp(\Delta \varepsilon_v)
\]  
(15a)

\[
\phi_b = \frac{\rho_{b0}}{1 - \beta_b \Delta T} \exp(\Delta \varepsilon_v - \alpha_{b,f} \Delta T)
\]  
(15b)
where $\Delta \varepsilon_v$ is the increment of volumetric strain; $\rho_0$ and $\phi_{b0}$ are the initial values of $\rho_v$ and $\phi_{b'}$ respectively. Substituting equation (15) into equation (13) and then substituting equation (13) into equations (11-12), the nonlinear elastic thermal deformation coupled with certain stress conditions can be predicted. Detailed discussions will be made in section 3.

3. Stability of Isothermal Elastic Behavior

In this section, the elastic stability concept is employed to derive the strength criteria of various geomaterials. As shown in figure 3, when subjected to a pure elastic load path with constant incremental stress ratio, elastic instability will be triggered when the peak stress state is reached. In the effective stress space, such instable states can also be defined as the stress state boundary of geomaterials. On the stress state boundary, both the density and the stress should reach their extreme value states which requires $d\rho = 0$ and $d\phi' = dq = 0$. We can therefore have

$$\frac{\partial^2 \omega_e}{\partial \varepsilon_v^2} \frac{\partial^2 \omega_e}{\partial \rho^2} - \left( \frac{\partial^2 \omega_e}{\partial \varepsilon_v^2 \partial \rho} \right)^2 = 0 \quad (16)$$

which is just the stress state boundary equation. The strength criterion in terms of elastic strains unified for bonded and unbonded geomaterials can be then derived by substituting equation (5) into equation (16). Combining with the hyperelastic relation equation (8), the corresponding strength criterion in stress space is also available. The state boundary concept above also implicates that, for triaxially sheared geomaterials with strain hardening under drained conditions, the critical state at which the density and the stress stop changing should be reached once the stress path reaches the stress state boundary. In contrast, for those with strain softening, the volumetric deformation will be transited from contraction to dilation on the stress boundary, triggering the occurrence of strain softening.

Equation (16) in fact can also be derived from the stability of thermodynamic equilibrium state which requires the entropy of a thermodynamic system reach a maximum (i.e. $\delta S = 0$ and $\delta^2 S < 0$ where $S$ is the entropy). Considering a solid element at a transient equilibrium state of $\delta S = 0$ with zero energy dissipation rate, the corresponding states of the element such as stress and density can exist stably only when the equilibrium state is able to remain stable. Otherwise, any external disturbances could drive the system deviating from the equilibrium state to another state with different stress and density. From the perspective of mechanics, the stability of thermodynamic equilibrium state is equivalent to the elastic stability of a material that determines the strength criterion. For a material element at a certain equilibrium state with a volume of $V$, the stability of equilibrium state can be expressed by $\delta^2 S = -T^{-1} \int_V \delta^2 \omega_e < 0$. The elastic stability therefore requires the second order variation of elastic potential be positive, i.e. $\delta^2 \omega_e > 0$.

Accordingly, the stability of elastic state requires

$$\delta^2 \omega_e = \begin{pmatrix} \delta \varepsilon_v & \delta \varepsilon_v^T \\ \delta \varepsilon_v & \delta \rho \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \omega_e}{\partial \varepsilon_v^2} & \frac{\partial^2 \omega_e}{\partial \rho \partial \varepsilon_v} \\ \frac{\partial^2 \omega_e}{\partial \rho \partial \varepsilon_v} & \frac{\partial^2 \omega_e}{\partial \rho^2} \end{pmatrix} \begin{pmatrix} \delta \varepsilon_v \\ \delta \rho \end{pmatrix} > 0 \quad (17)$$

the matrix in the right side of equation (17) is just the Hessian matrix of $\omega_e$ with respect to $\varepsilon_v$, $\varepsilon_v^T$ and $\rho$. equation (17) explicitly indicates that this matrix should be positive definiteness at a stable elastic state. It can be used as the generalized theoretical description of the strength criterion of a material.

With a density dependency of $\omega_e$ defined by equation (6), the positive definiteness of the matrix is just determined by $(\partial^2 \omega_e/\partial \varepsilon_v^2)(\partial^2 \omega_e/\partial \varepsilon_v^2) - (\partial^2 \omega_e/\partial \varepsilon_v^2 \partial \varepsilon_v^T)^2 > 0$. Thus, the strength criterion defined in equation (16) just represents the critical condition for the destabilization of elasticity. Under
an instable elastic state, materials should deviate from the thermodynamic equilibrium state and hence show plastic deformation. Thus, the state boundary of geomaerials in effective stress space can be determined by substituting the states of elastic strain into the hyperelastic relation in equation (8). For bonded geomaerials, the shear-induced cohesion degradation occurs may change the state boundary in stress space. Therefore, for a more generalized purpose, the cohesion index \( c \) should be a variable degrading with the shear process. From the thermodynamics, the evolution law of \( c \) can be derived as follow:

\[
\dot{c} = -k_c \frac{\partial \omega_\varepsilon}{\partial c} \tag{18}
\]

where \( k_c \) is a parameter controlling the degradation rate of cohesion. From equation (25), the cohesion index \( c = c_0 \exp\left(-\int_{t_0}^{t} k_c \frac{\partial \omega_\varepsilon}{\partial c} dt\right) \), where \( c_0 \) is the initial cohesion index and the integral term is stress-path dependent and may be coupled with the plastic deformation. It is also obvious that a higher level of shear stress will lead to a larger degradation of \( c \). Here, as a simplified application of the \( c \) expression derived above, the \( c \) value on failure line can be simply considered to be an exponentially decreasing function of \( q \). Figure 4 shows the simulated state boundary of cement-bonded sand samples with different initial bonding levels, in which \( A = 100 \text{ MPa}, B = 25 \text{ MPa}, k = 0.0075 \text{ kg/m}^3 \text{ and } m = n = 1 \) are used. The cohesion degradation is calculated by \( c = c_0 \exp\left(-2.05 \times 10^{-4} q\right) \) (unit of \( q \): Pa). With such a cohesion degradation defined, the failure lines of bonded sands with different cementation contents are well predicted by the model, as shown in figure 3.

**Figure 3.** Elastic instability concept.  
**Figure 4.** Predicted failure lines of cement bonded sand with cohesion degradation (scatters are measured data [19]).

### 4. Nonlinear Thermo-hyperelastic Coupling

Under non-isothermal conditions, it had been known that thermally induced deformation and pore pressure variation can be observed in saturated geomaerials [20]. In this paper the nonlinear thermo-elastic coupling behavior is discussed based on the hyperelastic model proposed.

Comparing equation (12) with equation (8), it can be found that the temperature does not affect both the \( G_s(K_p) = p' - \rho - c \) relation and the strength criterion in \( p' - q - \rho \) space. In other words, the temperature effects on the elastic moduli and strength of geomaerials should be determined by their density changes in response to thermal loads. In some cases for saturated clays heated, there is even an irreversible thermal volume contraction that results in the thermal hardening with an increased density. More fundamentally, in other cases where the thermo-elasticity is dominant (e.g. the heating problems in highly over-consolidated clays [21]), nonlinear evolutions of elastic thermal volumetric strain can also be observed. As far as what is concerned in the thermo-hyperelasticity proposed here, it is also essential to model the irreversible part of thermal volumetric strain for geomaerials.
For a geomaterial element under isotropic stress state, the effective mean stress is

\[ p' = B_\alpha e^{k_\alpha} (E'_v + c)^m E'_v \]  

(19)

where the equivalent volumetric strain \( E'_v \) is defined in equation (11). Keeping \( p' \) constant and then heating the element from an initial temperature \( T_0 \) to a higher temperature \( T \) under drained conditions, the density and the elastic volumetric strain will be changed in response to the thermal loading. Only considering pure thermo-elasticity, the following evolution law for \( E'_v \) can be derived from conditions \( \dot{p}' = 0 \) and \( \dot{\rho} = \rho \dot{e}_v = \rho \dot{e}'_v \).

\[ \dot{e}'_v = -\beta'_e \dot{T} \]  

(20a)

\[ \beta'_e = \frac{k_T}{k_T + k'_p \rho} \beta'_p \]  

(20b)

\[ K'_e = B_\alpha e^{k'_\alpha} [m(E'_v + c)^{m-1}E'_v + (E'_v + c)^m] \]  

(20c)

\[ \beta'_p = \frac{\beta p}{\beta T} \Delta T + \beta \]  

(20d)

where \( K'_e \) is the tangent elastic bulk modulus and \( \beta'_e \) is the tangent elastic volumetric thermal expansion coefficient. In most cases, \( K'_e/(K_e + k'_p \rho) \approx 1 \) and thus \( \beta'_e \approx \beta'_p \). As discussed in Section 2.2, the secant elastic volumetric thermal expansion coefficient \( \beta \) is a function of intrinsic thermal coefficients of solid and bound water (Equation (13)). The intrinsic thermal coefficients of bound water, \( \beta_b \), also depends on temperature according to \( [21] \)

\[ \beta_b = \frac{10^{-4}}{r-20} [11941.497 - 0.274 T + 0.0651 T^2 \]  

\[ -0.0091237 T^3 - 11591.52 (T^{273.15} - 1)^{0.025} ] \]  

(21)

The relation between \( \beta'_e \) and \( T \) can be obtained by substituting equations (21 and 14b) into equation (13), as shown in figure 5. It is shown that, due to the heating-induced conversion of bound water to free water, \( \beta'_e \) gradually decreases with temperature from positive to negative and a higher conversion rate \( (\alpha_{\beta_p}) \) leads to a faster decrease in \( \beta'_e \). As a result, there is a critical temperature at which the thermal volumetric deformation is transited from expansion to contraction, as shown in figure 6. It is consistent with the laboratory observation for highly over-consolidated clays [20]. Such nonlinearity is resulted from both the temperature dependency of \( \beta'_b \) and the thermally induced conversion between bound water and free water. Furthermore, the thermal volumetric strain is significantly sensitive to the bound water content absorbed in the specific geomaterial. The higher the bound water content, the larger the thermal volumetric deformation will develop. In this sense, the thermo-elastic coupled behavior of geomaterials is to a large extent controlled by their capabilities of absorbing bound water on mineral surfaces. If without any bound water \( (\phi_b = 0) \), the tangent thermal expansion coefficient can be reduced to \( \beta'_e = \beta_b \) which leads to a linear relation between thermal volumetric strain and temperature. That is just the case for sands or other dry geomaterials.
5. Conclusion

A thermo-hyperelastic model is developed in this paper for both bonded and unbonded saturated geomaterials. It is shown that the state dependent elastic behavior of geomaterials can be described by the hyperelasticity based on the concept of elastic potential function. The following conclusions can be made according to the analyses using this model. Analytical relations among the elastic bulk/shear modulus, the effective confining stress, the cohesion and the density are derived. Moreover, the hyperelastic model proposed well predicts not only the state dependency of elastic behavior but also the strength criterion dependent on the density and the cohesion as well as the cohesion degradation.

The thermo-elastic coupling of saturated geomaterials is described using the concept of equivalent elastic volumetric strain defined by an elastic volumetric thermal expansion coefficient dependent on the thermal properties of free water and bound water within pores in geomaterials. The effects of temperature variation on elastic stiffness and strength criterion are considered to mainly depend on the density changes induced by thermal loads. The nonlinear development of thermal elastic volumetric strain in response to pure thermal loads is well predicted by the model and is attributed to both the temperature dependency of the thermal expansion coefficient of bound water and the thermally induced conversion between free water and bound water.

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