CONDITIONAL EXPECTATIONS THROUGH BOOLEAN CUMULANTS AND SUBORDINATION - TOWARDS A BETTER UNDERSTANDING OF THE LUKACS PROPERTY IN FREE PROBABILITY

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Abstract. We use here a recent idea of studying functions of free random variables using Boolean cumulants. We develop idea of explicit calculations of conditional expectation using Boolean cumulants. We demonstrate Boolean cumulants approach allows to calculate explicitly some conditional expectations of functions in free random variables. We present how Boolean cumulants together with subordination simplify proofs of some results which are known in research literature.

1. Introduction

In this paper we study characterizations of probability measures in terms of free random variables. Similar type of problems were studied for long time in classical probability. Among many examples the most prominent one is Bernstein’s theorem which says that for independent random variables $X, Y$, vector $(U, V) = (X + Y, X - Y)$ has independent components if and only if $X$ and $Y$ have Gaussian distribution with the same variance. Another example is Lukacs theorem which states that for independent $X, Y$ random variables vector $(X + Y, X/(X + Y))$ has independent components if and only if $X, Y$ have Gamma distributions with the same scale parameter (cf. [11]).

It was observed that in the framework of free probability many classical characterizations of probability measures have their counterparts. The analogue of Bernstein’s theorem was studied by Nica (see [13]) and says that for free $X, Y$, random variables $U = X + Y$ and $V = X - Y$ are free if and only if $X, Y$ have Wigner semicircle distributions with the same variance. The free analogue of Lukacs theorem was studied in [19].

It was also observed (see [9, 3, 22]) that the strong assumption that both vectors $(X, Y)$ and $(U, V)$ consist of independent (respectively free) random variables can be weakened and it is enough to assume only constancy of some conditional moments of $U$ given $V$. This phenomenon was observed both in context of commutative, independent random variables, as well as for non-commutative, free random variables (c.f. [4, 20]). On the other hand calculation of conditional moments of functions of non-commutative random variables is highly non-trivial. In [6] the powerful method of subordination of free convolutions was applied quite naturally to some characterization problems and simplified the proofs. However the method developed in there does not cover all known examples of characterizations. For example characterizations studied in [18] could not be treated directly by this method. It turns out that for one of the cases considered in [18] in order to apply subordination method, one needs to calculate some non-trivial conditional expectation.

Recently discovered connections between Boolean cumulants [7, 10] and free probability allows to overcome that difficulty. In particular it was observed that Boolean cumulants appear quite naturally in calculations of conditional expectations of some functions of free random variables. And we apply the results form [10] to the conditional expectation we are interested in.

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More precisely we present a new approach to the result considered in [18], i.e. we work in dual scheme to Lukacs theorem (observe that in the framework of Lukacs theorem one has $X = UV$ and $Y = V(1 - U)$), we assume that some conditional moments of $V^{1/2}(1 - U)V^{1/2}$ given $U^{1/2}VU^{1/2}$ are constant, and conclude that $U$ has a free Binomial distribution and $V$ has a free Poisson distribution. As we mentioned above, it turns out, that the characterizations considered in [18] are not a straightforward application of the subordination technique from [6]

The main technical result of the present paper is an explicit formula for conditional expectation

$$\mathbb{E}_V \left( (1 - U)^{-1} U^{1/2} z U^{1/2} V U^{1/2} (1 - z U^{1/2} V U^{1/2})^{-1} U^{1/2} (1 - U)^{-1} \right),$$

where $\mathbb{E}_V$ denotes the conditional expectation on the algebra generated by $V$ and random variables $U, V$ are free. Next we present easier proofs of main results of [18].

We also present an easy proof of free Lukacs property for free Poisson distributed random variables. That is we show that for free random variables $X$ and $Y$ both having free Poisson (Marchenko-Pastur) distribution, random variables $X + Y$ and $(X + Y)^{-1/2} X (X + Y)^{-1/2}$ are free. In [19] we presented a "hands on", direct combinatorial proof, our strategy was to show vanishing of mixed free cumulants of $X + Y$ and $(X + Y)^{-1/2} X (X + Y)^{-1/2}$. In particular we calculated explicitly joint free cumulants of $X, X^{-1}$ for invertible, free Poisson distributed random variable $X$. Here we show that free Lukacs property follows from results proved in [5].

Except from Introduction this paper has 4 more sections. In Section 2 we set up the framework and recall notions used in this paper. Section 3 is devoted to explicit calculation of the conditional expectation (1). In Section 4 we present simplified proofs of results from [18]. Section 5 contains the simple proof of Lukacs property based on results from [5].

2. Notation and Background

In this section we introduce notions and facts from non-commutative probability. We restrict the background to essential facts which are necessary in the subsequent sections. Readers, who are not familiar with non-commutative (in particular, free) probability, to get a wider perspective may want to consult one of the books [12, 14]. We assume that $\mathcal{A}$ is a unital $*$-algebra and $\varphi : \mathcal{A} \mapsto \mathbb{C}$ is a linear functional which is normalized (that is, $\varphi (1_\mathcal{A}) = 1$, where $1_\mathcal{A}$ is a unit of $\mathcal{A}$), positive, tracial and faithful. We will refer to the pair $(\mathcal{A}, \varphi)$ as a non-commutative probability space.

2.1. Freeness, free and Boolean cumulants. The concept of freeness was introduced by Voiculescu in [21] and among several existing notions of non-commutative "independence" is the most prominent one. Here we recall the definition.

**Definition 2.1.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. We say that subalgebras $(\mathcal{A}_i)_{1 \leq i \leq n}$ of algebra $\mathcal{A}$ are free if for any $X_k \in \mathcal{A}_{i_k}$ which are centred, i.e. $\varphi (X_k) = 0$, $k = 1, \ldots, n$,

$$\varphi (X_1 \cdots X_n) = 0$$

whenever neighbouring random variables come from different subalgebras, that is when $i_k \neq i_{k+1}$ for all $k = 1, \ldots, n - 1$.

It turns out that freeness has a nice combinatorial description which uses the lattice of non-crossing partitions.

**Definition 2.2.**

(1) By a partition of a finite totally ordered set $S$ we understand a family of subsets of $B_1, \ldots, B_k \subseteq S$ such that $\bigcup_{j=1}^k B_j = S$ where $B_j$ are non-empty and pairwise disjoint. Set of all partition of a set $S$ is denoted by $\mathcal{P}(S)$. In the special case $S = [n] := \{1, \ldots, n\}$ we denote it by $\mathcal{P}(n)$. For $\pi \in \mathcal{P}(n)$ suppose that $\pi = \{B_1, \ldots, B_k\}$, then the
sets $B_j$ for $j = 1, \ldots, k$ are called blocks of $\pi$, the number of blocks is denoted by $|\pi|$, i.e. we have $|\pi| = k$.

(2) We say that $\pi \in \mathcal{P}(n)$ is a non-crossing partition if for any $B_1, B_2 \in \pi$ and $0 \leq i_1 < j_1 < i_2 < j_2 \leq n$,

$$(i_1, i_2 \in B_1 \text{ and } j_1, j_2 \in B_2) \Rightarrow B_1 = B_2.$$  

The set of all non-crossing partitions of $[n]$ is denoted by $NC(n)$.

(3) We say that $\pi \in \mathcal{P}(n)$ is an interval partition if for any $B_1, B_2 \in \pi$ and $0 \leq i_1 < j_1 < i_2 \leq n$

$$(i_1, i_2 \in B_1 \text{ and } j_1 \in B_2) \Rightarrow B_1 = B_2.$$  

The set of all interval partitions of $[n]$ is denoted by $Int(n)$.

Next we will recall of a partial order on $NC(n)$ called of them is reversed refinement order and is denoted denoted by $\leq$.

**Definition 2.3.** For $\pi, \sigma \in \mathcal{P}(n)$ we say that that $\pi \leq \sigma$ if for any block $B \in \pi$ there exists a block $C \in \sigma$ such that $B \subseteq C$. Partial order $\leq$ also makes sense on the sets $NC(n)$ and $Int(n)$. By $1_n$ we denote the maximal partition with respect to $\leq$, i.e. the partition with one block equal to $\{1, \ldots, n\}$.

It turns out that $(NC(n), \leq)$ and $(Int(n), \leq)$ have a lattice structure, for details we refer to [14] Lectures 9 and 10.

Next we will define cumulant functionals relative to non-crossing and interval partitions, called free and Bollean cumulants, respectively. Free cumulants are important tools in free probability introduced in [16], while Boolean cumulants are related to the so called Boolean independence defined in [17].

**Definition 2.4.** For every $n \geq 1$ and any $X_1, \ldots, X_n \in \mathcal{A}$ one defines free cumulants $\kappa_n(X_1, \ldots, X_n)$, $i_1, \ldots, i_r \in [n]$, $r \geq 1$, recursively via equations

$$\varphi(X_1, \ldots, X_n) = \sum_{\pi \in NC(n)} \kappa_\pi(X_1, \ldots, X_n)$$

where for $\pi = \{B_1, \ldots, B_k\} \in NC(n)$

$$\kappa_\pi(X_1, \ldots, X_n) = \prod_{j=1}^{k} \kappa_{|B_j|}(X_i; i \in B_j).$$

Similarly for every $n \geq 1$ and any $X_1, \ldots, X_n \in \mathcal{A}$ one defines Boolean cumulants $\beta_r(X_{i_1}, \ldots, X_{i_r})$, $i_1, \ldots, i_r \in [n]$, $r \geq 1$, recursively via equation

$$\varphi(X_1, \ldots, X_n) = \sum_{\pi \in Int(n)} \beta_\pi(X_1, \ldots, X_n),$$

where for $\pi = \{B_1, \ldots, B_k\} \in Int(n)$

$$\beta_\pi(X_1, \ldots, X_n) = \prod_{j=1}^{k} \beta_{|B_j|}(X_i; i \in B_j).$$

It turns out that freeness can be described in terms of free cumulants, more precisely random variables $X_1, \ldots, X_n$ are free if and only if for any $r \geq 2$ we have $\kappa_r(X_{i_1}, \ldots, X_{i_r}) = 0$ for any non-constant choice of $i_1, \ldots, i_r \in \{1, \ldots, n\}$.

**Remark 2.1.** In the sequel we will need the formula for Boolean cumulants with products as entries (see e.g. [17]) Fix two integers $m, n$ such that $0 < m + 1 < n$ and numbers $1 \leq i_1 <
2.2. Conditional expectation. Assume that \((\mathcal{A}, \varphi)\) is a \(W^*\)-probability spaces, that is \(\mathcal{A}\) is a finite von Neumann algebra and \(\varphi\) a faithful normal tracial state. Then for any von Neumann subalgebra \(\mathcal{B} \subset \mathcal{A}\) there exists a faithful, normal projection \(E_B : A \to B\) such that \(\varphi \circ E_B = \varphi\). This projection is the conditional expectation onto the subalgebra \(\mathcal{B}\) with respect to \(\varphi\). If \(X \in \mathcal{A}\) is self-adjoint then \(E_B(X)\) defines a unique self-adjoint element in \(\mathcal{B}\). For \(X \in \mathcal{A}\) by \(E_X\) we denote conditional expectation given the von Neumann subalgebra generated by \(X\) and \(1_A\). In order to prove that \(E_B(Y) = Z\) it is enough to show that for any \(X \in \mathcal{B}\) one has \(\varphi(YX) = \varphi(ZX)\).

2.3. Free Poisson and free Binomial distributions. In this subsection we recall definitions and some basic facts about distributions which we study in this paper.

First we discuss so called free Poisson distribution.

Remark 2.2.

1. The Marchenko–Pastur (or free Poisson) distribution \(\mu = \mu(\alpha, \lambda)\) is defined by
\[
\mu = \max\{0, 1 - \lambda\} \delta_0 + \hat{\mu},
\]
where \(\alpha, \lambda > 0\) and the measure \(\hat{\mu}\), supported on the interval \((\alpha(1 - \sqrt{\lambda})^2, \alpha(1 + \sqrt{\lambda})^2)\), has the density (with respect to the Lebesgue measure)
\[
\hat{\mu}(dx) = \frac{1}{2\pi \alpha x} \sqrt{4(1 + \lambda)^2 - (x - \alpha(1 + \lambda))^2} \, dx.
\]

2. The \(R\)-transform of the free Poisson distribution \(\mu(\alpha, \lambda)\) is of the form
\[
r_{\mu(\alpha, \lambda)}(z) = \frac{\alpha \lambda}{1 - \alpha z}.
\]

3. For free Poisson distribution \(\mu(\alpha, \lambda)\) the \(S\)-transform is of the form
\[
S_{\mu(\alpha, \lambda)}(z) = \frac{1}{\alpha \lambda + \alpha z}.
\]

Free binomial distribution will also play an important role in this paper.

Remark 2.3. Free-binomial distribution \(\nu = \nu(\sigma, \theta)\) is defined by
\[
\nu = (1 - \sigma) \mathbb{1}_{0 < \sigma < 1} \delta_0 + \tilde{\nu} + (1 - \theta) \mathbb{1}_{0 < \theta < 1} \delta_1,
\]
where \(\tilde{\nu}\) is supported on the interval \((x_-, x_+),\)

\[
x_\pm = \left(\sqrt{\frac{\sigma}{\sigma + \theta}} \left(1 - \frac{1}{\sigma + \theta}\right) \pm \sqrt{\frac{1}{\sigma + \theta} \left(1 - \frac{\sigma}{\sigma + \theta}\right)}\right)^2,
\]
and has the density
\[
\tilde{\nu}(dx) = (\sigma + \theta) \frac{\sqrt{(x - x_-)(x_+ - x)}}{2\pi x (1 - x)} \, dx.
\]
where \((\sigma, \theta) \in \{(\sigma, \theta) : \frac{\sigma + \theta}{\sigma + \theta - 1} > 0, \frac{\sigma \theta}{\sigma + \theta - 1} > 0\}\). The n-th free convolution power of distribution
\[
p \delta_0 + (1 - p) \delta_{1/n}
\]
is free-binomial distribution with parameters \(\sigma = n(1 - p)\) and \(\theta = np\), which justifies the name of the distribution (see [15]).
Its Cauchy transform is of the form (see e.g. the proof of Cor. 7.2 in [3])

\[ G_{\sigma,\theta}(z) = \frac{(\sigma + \theta - 2)z + 1 - \sigma - \sqrt{[(\sigma + \theta - 2)z + 1 - \sigma]^2 - 4(1 - \sigma - \theta)z(z - 1)}}{2z(1 - z)}. \]

For free Binomial distribution \( \nu(\sigma, \theta) \) the \( S \)-transform is of the form

\[ S_{\nu(\sigma, \theta)}(z) = 1 + \frac{\theta}{\sigma + z}. \]

3. Calculation of conditional expectation

This section is devoted to calculation of the conditional expectation announced in the introduction. We are able to calculate it using Boolean cumulants and results from [10, 7], surprisingly calculations which use Boolean cumulants seems to be simpler than the one which uses free cumulants. First we will recall relevant facts about subordination functions, for details see [2].

For \( \Psi_X(z) = zX(1 - zX)^{-1} \) and for all \( z \in \mathbb{C}^+ \) the following formulas for conditional expectations with respect to \( V \) and \( U \), respectively, hold

\[ \mathbb{E}_V \Psi_{V^{1/2}UV^{1/2}}(z) = \Psi_V(\omega_1(z)) \]
\[ \mathbb{E}_U \Psi_{UV^{1/2}V^{1/2}}(z) = \Psi_U(\omega_2(z)). \]

Note that since we assume that \( \varphi \) is tracial and both \( U \) and \( V \) are positive we have that moments of \( UV, U^{1/2}VU^{1/2} \) and \( V^{1/2}UV^{1/2} \) are the same, so \( M_{UV} = M_{U^{1/2}VU^{1/2}} = M_{V^{1/2}UV^{1/2}}. \) Since \( M_X(z) = \varphi(\Psi_X(z)) \) identities (6) and (7) imply

\[ M_{UV}(z) = M_U(\omega_1(z)) = M_U(\omega_2(z)). \]

For future use we will also denote \( \Psi_X := \Psi_X(1) \). In the sequel, we will also use the symbol \( \psi \) for a formal power series \( \psi(x) = \sum_{k \geq 1} x^k \), where \( x \) is from some (unspecified) algebra over a real vector space. We will also need the following two formulas involving Boolean cumulants. The first is taken from [10] and the second is a simple consequence of one of the main results from [7]; it is also closely related to characterization of freeness from [8].

**Proposition 3.1.** Let \( \{X_1, \ldots, X_{n+1}\} \) and \( \{Y_1, \ldots, Y_n\} \) be free, \( n \geq 1 \). Then

\[ \varphi(X_1Y_1 \ldots X_nY_n) \]
\[ = \sum_{k=0}^{n-1} \sum_{0=j_0<j_1<\ldots<j_{k+1}=n} \varphi(Y_{j_1} \ldots Y_{j_{k+1}}) \prod_{\ell=0}^{k} \beta_{2(\ell t+1-j_t)-1}(X_{j_{t+1}}, Y_{j_{t+1}}, \ldots, Y_{j_{t+1}}) \]

and

\[ \beta_{2n+1}(X_1, Y_1, \ldots, X_n, Y_n, X_{n+1}) \]
\[ = \sum_{k=2}^{n+1} \sum_{1=j_1<\ldots<j_k=n+1} \beta_k(X_{j_1}, \ldots, X_{j_k}) \prod_{\ell=1}^{k-1} \beta_{2(\ell t+1-j_t)-1}(Y_{j_{t+1}}, X_{j_{t+1}}, \ldots, X_{j_{t+1}}). \]

We note for future use a simple consequence of the first formula above together with its reformulation

**Remark 3.2.** (1) We have that in some neighbourhood of zero

\[ \omega_1(z) = \sum_{k=1}^{\infty} \beta_{2k-1}(Y, X, Y, \ldots, X, Y)z^k. \]

Similarly the second subordination function \( \omega_2 \), satisfies

\[ \mathbb{E}_\varphi[\Psi_{X^{1/2}Y^{1/2}}(z)|B_Y] = \Psi_X(\omega_2(z)). \]
and we have
\[ \omega_2(z) = \sum_{k=1}^{\infty} \beta_{2k-1}(X, Y, X, \ldots, Y, X) z^k. \]

(2) Equation (10) can be reformulated in less transparent but still useful way, where now \( i_k \) records how many \( Y \)'s there is between \( k \)-th and \( k+1 \)-st \( Y \) which was picked to the same block with \( Y_n \).

\[ \varphi(X_1Y_1 \ldots X_nY_n) = \sum_{k=1}^{n} \sum_{i_1 + \ldots + i_k = n-k} \varphi(Y_{n-i_1-\ldots-i_k-1-\ldots-(k-1)} Y_{n-i_1-\ldots-i_k-1-\ldots-(k-1)}) \prod_{j=1}^{k} \beta_{2i_j+1} \left( (X_{n-i_1-\ldots-i_j-(j-1)}, Y_{n-i_1-\ldots-i_j-(j-1)}), \ldots, X_{n-i_1-\ldots-i_{j-1}-(j-1)} \right) \]

where we set \( i_0 = 0 \).

The remaining part of this section is devoted to calculation of conditional expectation (11). Actually, we will calculate a more general conditional expectation
\[ \mathbb{E}_V f(U)U^{-1/2} \Psi_{U^{1/2}VU^{1/2}}(z)U^{-1/2} g(U), \]
for suitable functions \( f \) and \( g \). For \( f = g = \psi \) we obtain (11). To this end below we introduce a mapping \( \varphi^f(U, f) \) as follows.

Below we present the main technical result of this paper. We calculate the conditional expectation (12). It is worth to point out that our result says, that this conditional expectation can be written quite naturally using moments and Boolean cumulants, while free cumulants are absent. This confirms findings from [10] that Boolean cumulants serve as a useful tool to calculate conditional moments of functions of free random variables.

**Proposition 3.3.** Assume that \( U, V \) are free, \( 0 \leq U < 1 \) and \( V \) is bounded. Let \( f \) and \( g \) be such that \( f(U) \) and \( g(U) \) are bounded. Then for \( z \) in some neighbourhood of \( 0 \) and \( \omega_1, \omega_2 \) satisfying (6) and (7)
\[ \mathbb{E}_V f(U)U^{-1/2} \Psi_{U^{1/2}VU^{1/2}}(z)U^{-1/2} g(U) \]
\[ = \omega_2(z) \eta^f_U(\omega_2(z)) + z \eta^f_U(\omega_2(z)) \eta^g_U(\omega_2(z)) V (1 + \Psi_V(\omega_1(z))), \]
where
\[ \eta^f(z) = \sum_{\ell=0}^{\infty} \beta_{\ell+2}(f(U), U, \ldots, U, g(U)) z^\ell, \]
\[ \eta^g(z) = \sum_{\ell=0}^{\infty} \beta_{\ell+1}(f(U), U, \ldots, U) z^\ell. \]

**Proof.** We will calculate the conditional expectation using Remark 3.2.

For \( z \) sufficiently small we can write \( \Psi_{U^{1/2}VU^{1/2}}(z) = \sum_{n=1}^{\infty} z^n U^{1/2}(V(UV)^{n-1}) U^{1/2} \). It suffices to calculate
\[ \sum_{n=1}^{\infty} z^n \varphi(f(U)V(UV)^{n-1} g(U) H), \]
for any \( H \) in the von Neumann algebra generated by \( \{1, V\} \). Thus we reduce the problem to the calculation of
\[ \varphi(f(U)V(UV)^{n-1} g(U) H). \]
Next we use formula (9) to express the above moment in terms of moments of $H$ and $V$ and Boolean cumulants of $f(U), g(U), U$ and $V$. The two free families in (9) are $\{f(U), U, \ldots, U, g(U)\}_{n+1}$ and $\{V, \ldots, V, H\}$. Denoting $i_\ell = j_{\ell+1} - j_\ell$ in (9) we get

$$\varphi \left( f(U) V(UV)^{n-1} g(U) H \right) = \varphi(H) \beta_{2n+1} \left( f(U), V, U, \ldots, U, V, g(U) \right) + \sum_{k=1}^{n} \varphi \left( V^k H \right) \sum_{i_1 + \ldots + i_{k+1} = n-k} \beta_{2i_1+1} \left( f(U), V, \ldots, V, U \right) \cdots \beta_{2i_{k+1}+1} \left( U, V, \ldots, U, V, g(U) \right),$$

where the Boolean cumulants which are hidden under $\cdots$ in the formula above are of the form $\beta_{2k+1}(U, V, \ldots, U, V, U)$.

Thus the conditional expectation assumes the form

$$\mathbb{E}_V \left( 1 - U \right)^{-1} U^{1/2} \Psi_{U^{1/2} V U^{1/2}}(z) U^{1/2} (1 - U)^{-1} = \sum_{n=1}^{\infty} z^n \left( \beta_{2n+1} \left( f(U), V, U, \ldots, U, V, g(U) \right) + \sum_{k=1}^{n} V^k \sum_{i_1 + \ldots + i_{k+1} = n-k} \beta_{2i_1+1} \left( f(U), V, \ldots, V, U \right) \cdots \beta_{2i_{k+1}+1} \left( U, V, \ldots, U, V, g(U) \right) \right).$$

Let us denote

$$A_1(z) = \sum_{n=0}^{\infty} \beta_{2n+1} \left( f(U), V, U, \ldots, U, V, U \right) z^n, \quad A_2(z) = \sum_{n=0}^{\infty} \beta_{2n+1} \left( U, V, U, \ldots, U, V, g(U) \right) z^n,$$

$$B(z) = \sum_{n=1}^{\infty} \beta_{2n+1} \left( f(U), V, U, \ldots, U, V, g(U) \right) z^n,$$

$$C_1(z) = \sum_{n=0}^{\infty} \beta_{2n+1} \left( U, V, U, \ldots, U, V, U \right) z^n, \quad C_2(z) = \sum_{n=0}^{\infty} \beta_{2n+1} \left( V, U, \ldots, V, U, V \right) z^n.$$

Observe that the change of order of summation gives

$$\sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} V^k \sum_{i_1 + \ldots + i_{k+1} = n-k} \beta_{2i_1+1} \left( f(U), V, \ldots, V, U \right) \cdots \beta_{2i_{k+1}+1} \left( U, V, \ldots, U, V, g(U) \right)$$

$$= zV \sum_{n=1}^{\infty} \sum_{k=1}^{n} V^{k-1} z^{k-1} \sum_{i_1 + \ldots + i_{k+1} = n-k} \beta_{2i_1+1} \left( f(U), V, \ldots, V, U \right) z^{i_1} \cdots \beta_{2i_{k+1}+1} \left( U, V, \ldots, U, V, g(U) \right) z^{i_{k+1}}$$

$$= zA_1(z)A_2(z)V \sum_{k=1}^{\infty} \left[ zC_1(z) V \right]^{k-1} = zA_1(z)A_2(z)V \left( 1 + \Psi_{V}(zC_1(z)) \right).$$

Remark 3.2 implies that $zC_1(z)$ is exactly the subordination function $\omega_1(z)$.

Thus, we finally get

$$\mathbb{E}_V \left( f(U) U^{-1/2} \Psi_{U^{1/2} V U^{1/2}}(z) U^{-1/2} g(U) \right) = B(z) + zA_1(z)A_2(z)V \left( 1 + \Psi_{V}(\omega_1(z)) \right).$$
We will use below (10) for free collections \( \{ f(U), U, \ldots, U \} \) and \( \{ V, \ldots, V \} \) for any \( i \geq 0 \) (with indices changed in the way we did when using (9) earlier in this proof)

\[
A_1(z) = \sum_{i=0}^{\infty} \sum_{l=0}^{i} \beta_{i+1}(f(U), U, \ldots, U) z^l \beta_{i+1}(V, U, \ldots, U, V) z^i \ldots \beta_{i+1}(V, U, \ldots, U, V) z^{i-l} = \sum_{l=0}^{\infty} \beta_{l+1}(f(U), U, \ldots, U) z^l C_2^l(z)
\]

where \( zC_2(z) = \omega_2(z) \). Consequently, \( A_1(z) = \eta_{U}^f(\omega_2(z)) \) - see (14).

Boolean cumulants are invariant under reflection

\[
\beta_{2i+1}(U, V, \ldots, U, V, g(U)) = \beta_{2i+1}(g(U), V, U, \ldots, U, V)
\]

and thus, by the same calculation as for \( A_1 \), we get \( A_2(z) = \eta_{U}^g(\omega_2(z)) \).

To find \( B \) we begin by repeating (with obvious modifications) the first part of the calculation which has been done above for \( A_1 \). It gives the representation

\[
B(z) = \sum_{l=0}^{\infty} \beta_{l+2}(f(U), U, \ldots, U, g(U)) z^{l+1} C_2^{l+1}(z)
\]

Consequently, \( B(z) = \omega_2(z) \eta_{U}^g(\omega_2(z)) \) - see (13).

\[\square\]

It appears that both the terms \( \eta_{U}^f, g \) and \( \eta_{U}^f \) can be conveniently expressed in terms of an operation \( \varphi_D^T \) which we are going to define now.

Let \( H \) be a formal power series \( H(T) = \sum_{k \geq 0} h_k T^k \) where \( T \) is a variable from some algebra. Let \( D \) denote the zero derivative i.e. linear operator which is defined on a power series in \( T \in A \) through its action on monomials: \( D T^k = T^{k-1} \) for \( k \geq 1 \) and \( D T^k = 0 \) for \( k = 0 \). Similarly by \( D \) we denote the zero derivative acting on complex power series.

For a power series \( f : A \rightarrow A \) and \( T \in A \) we define a new operator \( \varphi_D^T(H, f) \) (acting on complex functions)

\[
(16) \quad \varphi_D^T(H, f) := \sum_{k \geq 0} h_k \varphi(D^k f(T)) D^k.
\]

Note that \( \varphi_D^T(H, f) \) is a formal series of weighted zero derivatives of increasing orders (with the weight \( h_k \varphi(D^k f(T)) \) for the derivative of order \( k, k \geq 0 \)). The result of its application to an analytic function in general is a formal series (which in some cases may converge). Note that for integer \( r \geq 0 \)

\[
\varphi_D^T(H, T^r) = \sum_{k=0}^{r} h_k \varphi(T^{r-k}) D^r,
\]

and, in particular,

\[
\varphi_D^T(H, 1) = h_0 \quad \text{and} \quad \varphi_D^T(H, T) = h_0 \varphi(T) + h_1 D.
\]

Therefore, e.g. for \( H := \psi \) we have

\[
(17) \quad \varphi_D^T(\psi, 1) = 0 \quad \text{and} \quad \varphi_D^T(\psi, T) = D.
\]

On the other hand since for \( \psi(z) = z(1-z)^{-1}, \ |z| < 1 \), we have \( D^k \psi = 1 + \psi \) we see that

\[
\varphi_D^T(H, \psi) = \varphi(1 + \Psi_T) H(D).
\]
In particular, 

$$\varphi_D^T(\psi, \psi) = \varphi(1 + \Psi_T)\psi(D).$$  \hfill (18)

Now we are ready to give explicit formulas for $\eta_U^{f,g}$ and $\eta_U^L$ defined in (14) and (15), respectively.

**Proposition 3.4.** Assume that $f$ and $g$ are analytic functions on the unit disc. Let $U$ be a non-commutative variable such that $0 \leq U < 1$ with the Boolean transform $\eta_U$.

Then

$$\eta_U^{f,g} = \left[ \varphi_D^U(\psi, f) \circ \varphi_D^U(\psi, g) \right] \eta_U,$$

and

$$\eta_U^L(z) = z \varphi_D^U(\psi, f)(D\eta_U)(z) + \varphi_D^U(\psi, f)(\eta_U)(0).$$  \hfill (19)  \hfill (20)

**Proof.** First, we consider $\eta_U^{f,g}$.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. Expanding $f$ and $g$ we get

$$\eta_U^{f,g}(z) = \sum_{\ell=0}^{\infty} z^\ell \sum_{i,j \geq 0} a_i b_j \beta_{l+2}(U^i, U, \ldots, U, U^j).$$

Since, $\beta_{l+2}(1, U, \ldots, U, U^j) = \beta_{l+2}(U^i, U, \ldots, U, 1) = 0$ the inner double sum starts with $i = j = 1$.

Using the definition of Boolean cumulants and the formula for Boolean cumulants with products as entries (2) one can easily obtain the following formula

$$\beta_{r+1} \left(G, U, \ldots, U, U^i\right) = \beta_{r+1} \left(U^i, U, \ldots, U, G\right) = \sum_{m=1}^{i} \beta_{r+m} \left(G, U, \ldots, U, \underbrace{U}_{r+m-1}\right) \varphi \left(U^{i-m}\right).$$  \hfill (21)

Applying (21) we obtain

$$\eta_U^{f,g}(z) = \sum_{\ell=0}^{\infty} z^\ell \sum_{i,j \geq 1} a_i b_j \sum_{k=1}^{i} \sum_{m=1}^{j} \beta_{\ell+k+m}(U) \varphi(U^{i-k}) \varphi(U^{j-m}).$$

Changing several times the order of inner summations and the variables we get

$$\eta_U^{f,g}(z) = \sum_{\ell=0}^{\infty} z^\ell \sum_{k,m \geq 1} \beta_{\ell+k+m}(U) \varphi \left(\sum_{i \geq k} a_i U^{i-k}\right) \varphi \left(\sum_{j \geq m} b_j U^{j-m}\right)$$

$$= \sum_{\ell=0}^{\infty} z^\ell \sum_{k,m \geq 1} \beta_{\ell+k+m}(U) \varphi \left(D^k f(U)\right) \varphi \left(D^m g(U)\right)$$

$$= \sum_{\ell=0}^{\infty} z^\ell \sum_{r=2}^{\infty} \beta_{\ell+r}(U) \sum_{k=1}^{r-1} \varphi \left(D^k f(U)\right) \varphi \left(D^{r-k} g(U)\right)$$

$$= \sum_{r=2}^{\infty} \left( \sum_{k=1}^{r-1} \varphi \left(D^k f(U)\right) \varphi \left(D^{r-k} g(U)\right) \right) \sum_{\ell=0}^{\infty} \beta_{\ell+r}(U) z^\ell$$

$$= \sum_{r=2}^{\infty} \left( \sum_{k=1}^{r-1} \varphi \left(D^k f(U)\right) \varphi \left(D^{r-k} g(U)\right) \right) D^r \eta_U(z).$$
Consequently,
\[
\eta_U^{fg} = \sum_{r=2}^{\infty} \left( \sum_{k=1}^{r-1} \left[ \varphi(D^k f(U))D^k \right] \left[ \varphi(D^{r-k} g(U))D^{r-k} \right] \right) \eta_U.
\]

Thus (21) yields
\[
\beta_1(f(U)) = \sum_{i \geq 1} a_i \beta_1(U^i).
\]

3.5 Remark

As follows

Similarly (20) means

Finally, we calculate the conditional expectation from (1).

3.6 Remark

For \( f(z) = \sum_{i \geq 1} a_i z^i \) we have

\[
\beta_1(f(U)) = \sum_{i \geq 1} a_i \beta_1(U^i).
\]

Thus (21) yields

\[
\beta_1(f(U)) = \sum_{i \geq 1} a_i \sum_{k=1}^{i} \beta_k \varphi(U^{i-k}) = \sum_{k \geq 1} \beta_k \varphi(D^k f(U)).
\]

Since \( \beta_k = (D^k \eta_U)(0) \) we see that

\[
\beta_1(f(U)) = \sum_{k \geq 1} \varphi(D^k f(U))(D^k \eta)(0) = \varphi_D^U(\psi, f, U)(\eta_U)(0).
\]

By definition \( \eta_U^f(z) = \sum_{i \geq 1} \beta_1 (f(U)) \). For the first term we use (19) with \( g = \text{id} \). Therefore the second identity of (17) yields

\[
\eta_U^f(z) = \sum_{i \geq 1} \beta_1 (f(U)) \varphi(\psi U, f) D\eta(z) + \beta_1 (f(U))
\]

and thus (20) follows.

Remark 3.5. Note that due to the second formula in (17) it follows from (19) that

\[
\eta_U^{\text{id}, \text{id}} = D^2 \eta_U
\]

and (20) yields

\[
\eta_U^{\text{id}} = D \eta_U.
\]

The last identity extends to any function \( f_r \) defined as \( f_r(T) = T^r \), where \( r \geq 1 \) is an integer, as follows

\[
\eta_U^{f_r} = \sum_{j=1}^{r} \varphi(U^{r-j}) D^j \eta_U.
\]

Remark 3.6. One can rewrite the equation (19) in a more straightforward form as

\[
\eta_U^{fg}(z) = \sum_{r=2}^{\infty} \left( \sum_{k=1}^{r-1} \varphi \left( \sum_{i \geq k} a_i U^{i-k} \right) \right) \left( \sum_{j \geq r-k} b_j U^{j-(r-k)} \right) \sum_{\ell=0}^{\infty} \beta_{\ell+r}(U) z^\ell.
\]

Similarly (20) means

\[
\eta_U^f(z) = \sum_{r=1}^{\infty} \varphi \left( \sum_{i \geq r} a_i U^{i-r} \right) \sum_{\ell=0}^{\infty} \beta_{\ell+r}(U) z^\ell.
\]

Finally, we calculate the conditional expectation from (1).

Proposition 3.7. Assume that \( U, V \) are free, \( 0 \leq U < 1 \) and \( V \) is bounded. Then for \( z \) in some neighbourhood of 0

\[
\mathbb{E}_V (1 - U)^{-1/2} \Psi_{U^{1/2} V U^{1/2}}(z) U^{1/2} (1 - U)^{-1} = B(z) + z A^2(z)V (1 + \Psi_V(\omega_1(z))),
\]

(22)
Thus, since
\[ A(z) = \frac{\eta_U(\omega_2(z)) - \eta_U(1)}{\omega_2(z) - 1} \varphi((1 - U)^{-1}), \]
\[ B(z) = \frac{\omega_2(z)(\eta_U(\omega_2(z)) - \eta_U(1))}{(\omega_2(z) - 1)^2} \varphi^2((1 - U)^{-1}) \]
and \( \omega_1, \omega_2 \) satisfy (5).

Proof. From (13) with \( f = g = \psi \) we see that the conditional expectation at the left hand side of (22) is of the form
\[ \omega_2(z)\eta_U^\psi(\omega_2(z)) + z \left( \eta_U^\psi(\omega_2(z)) \right)^2 (1 + \Psi_V(\omega_1(z))) V, \]
i.e. we need only to show that \( \eta_U^\psi(\omega_2(z)) = A(z) \), where \( A \) is defined in (23) and \( \omega_2(z)\eta_U^\psi(\omega_2(z)) = B(z) \), where \( B \) is defined in (24).

We first compute \( \eta_U^\psi \). To this end we rely on (20). Observe that (18) yields
\[ \varphi^U_{D}(\psi, \psi)(D\eta_U) = \varphi(1 + \Psi_U) \sum_{k \geq 1} D^{k+1} \eta_U = \varphi(1 + \Psi_U)(D\psi(D))\eta_U. \]
Note that for a power series \( h(z) = \sum_{j \geq 0} h_j z^j \) we have
\[ \psi(D)(h)(z) := \sum_{k \geq 1} D^k h(z) = \sum_{k \geq 1} \sum_{j \geq k} h_j z^{j-k} = \sum_{j \geq 1} h_j \sum_{k=1}^j z^{j-k} = \frac{h(z)-h(1)}{z-1}. \]
Therefore, using (25) we get
\[ D\psi(D)(h)(z) = D \left( \frac{h(z)-h(1)}{z-1} \right) = \frac{Dh(z)-h(1)}{z-1}. \]
Thus, since \( zD\eta_U(z) = \eta_U(z) \), we get
\[ z\varphi^U_{D}(\psi, \psi)(D\eta_U)(z) = \varphi(1 + \Psi_U) \frac{\eta_U(z)-\eta_U(1)}{z-1}. \]
Note also that due (24) and \( \eta_U(0) = 0 \) we obtain
\[ \varphi^U_{D}(\psi, \psi)(\eta_U)(0) = \varphi(1 + \Psi_U)(\psi(D)\eta_U)(0) = \varphi(1 + \Psi_U)\eta_U(1) \]
Thus it follows from (20) that \( A(z) := \eta_U^\psi(\omega_2(z)) \).

Now we calculate \( \eta_U^\psi \). Using first (19) and then (25) once, we get
\[ \eta_U^\psi = \varphi^2(1 + \Psi_U)\psi^2(D)V\eta_U = \varphi^2(1 + \Psi_U)\psi(D)K, \]
where the function \( K \) is defined by
\[ K(w) = \frac{\eta_U(w)-\eta_U(1)}{w-1}. \]
Since \( K(1) = \eta_U(1) \), upon using again (25), we get
\[ \eta_U^\psi(w) = \frac{\eta_U(w)-\eta_U(1)}{w-1} - \eta_U(1) \]
and the formula for \( B \) follows. \( \square \)

4. Dual Lukacs regressions of negative orders

In this section we present simplified proofs of results from [18].

Theorem 4.1. Let \( U > 0 \) and \( V > 0 \) be free and such that \( \varphi(V) \), \( \varphi(V^{-1}) \), \( \varphi(U) \) and \( \varphi((1 - U)^{-1}) \) are finite. Assume that for some \( b, c \in \mathbb{R} \)
\[ E_{V^{1/2}U^{1/2}} V^{1/2}(1 - U)V^{1/2} = bI, \]
\[ E_{V^{1/2}U^{1/2}} [V^{1/2}(1 - U)V^{1/2}]^{-1} = cI. \]
Then \( bc > 1 \) and, with \( \alpha = \varphi(\Psi_U) > 0 \),
It is easy to see through purely algebraic manipulations that for non-commutative \( W > 0 \) and \( T > 0 \)

\[
L := \varphi \left( (1 - U)^{-1}V^{-1/2}\Psi_{V^1/2UV^1/2}(z)V^{-1/2} \right) = cM_{V^1/2UV^1/2}(z).
\]

It is easy to see through purely algebraic manipulations that for non-commutative \( W > 0 \) and \( T > 0 \)

\[
W^{-1/2}\Psi_{V^1/2UV^1/2}(z)TW^{-1/2} = zT^{1/2} (\Psi_{T^1/2WT^1/2}(z) + 1) T^{1/2}.
\]

Therefore applying (29) with \((W, T) = (V, U)\) to (28) we get

\[
L = \varphi \left( z(1 - U)^{-1}U^{1/2}(\Psi_{U^1/2VU^1/2}(z) + 1)U^{1/2} \right)
\]

\[
= \varphi \left( z(1 - U)^{-1}U^{1/2}(E_U \Psi_{U^1/2VU^1/2}(z) + 1)U^{1/2} \right)
\]

\[
= z\varphi \left( (1 - U)^{-1}U(\Psi_U(\omega_2(z)) + 1) \right).
\]

By simple algebra

\[
U(1 - U)^{-1}(\Psi_U(t) + 1) = \frac{1}{t - 1} (\Psi_U(t) - \Psi_U(1))
\]

and thus (we write below \( \omega_2 = \omega_2(z) \))

\[
L = \frac{z M_U(\omega_2) - \alpha}{\omega_2 - 1}.
\]

Consequently, (8) and (28) yield

\[
z(M_U(\omega_2) - \alpha) = c(\omega_2 - 1)M_U(\omega_2).
\]

Similarly, we multiply both sides of (26) by \( z\Psi_{V^1/2UV^1/2}(z) \) and obtain

\[
M := z\varphi \left( ((1 - U)V^{1/2}\Psi_{V^1/2UV^1/2}(z)V^{1/2}) \right) = zbM_{V^1/2UV^1/2}(z).
\]

Applying (29) with \((W, T) = (U, V)\) we see that

\[
M = \phi((1 - U)U^{-1/2}\Psi_{U^1/2VU^1/2}(z)U^{-1/2}) - z\phi((1 - U)V).
\]

Thus traciality and subordination (7) yield

\[
M = \phi(\Psi_U^{-1} \Psi_U(\omega_2(z))) - zb
\]

since (26) implies \( \phi((1 - U)V) = b \). We rewrite (30) as

\[
\Psi_U^{-1}\Psi_U(t) = t + (t - 1)\Psi_U(t)
\]

and plug it into \( M \). Taking additionally into account (8) at the RHS of (32) we finally get

\[
\omega_2(z) + (\omega_2 - 1)M_U(\omega_2) = bz(M_U(\omega_2) + 1).
\]

Identity \( M_{UV}(z) = M_U(\omega_2(z)) \) can be written in terms of inverse functions as \( \omega_2(M_U^{-1}(s)) = M_U^{-1}(s) \) (for the discussion about the existence of an inverse see [1]). Thus rewriting (33) and (31) in terms of \( M_U^{-1}(s) \) and \( M_U^{-1}(s) \) we obtain the following system of linear equations

\[
\begin{cases}
  b(1 + s)M_U^{(1)}(s) = (1 + s)M_U^{(1)}(s) - s, \\
  (s - \alpha)M_U^{(1)}(s) = c \left( M_U^{(1)}(s) - 1 \right).
\end{cases}
\]

We solve this system in terms of \( M_U^{-1} \) and \( M_U^{-1} \) and thus obtain \( S \)-transforms using the standard formula \( S_X(s) = \frac{1 + s}{s} M_U^{-1}(s) \) (see Lecture 18 [14]) we get

\[
S_U(s) = 1 + \frac{bc}{\alpha + (bc - 1)s}
\]

\[
S_{UV}(s) = \frac{c}{\alpha + s(bc - 1)}.
\]
By freeness of $U$ and $V$ we know that $S_{UV} = S_U S_V$, which allows to compute the $S$-transform of $V$,

$$S_V(z) = \frac{c}{bc + \alpha + (bc - 1)s}.$$  

Since the $S$-transform determines the distribution uniquely, the result follows. \hfill \Box

**Theorem 4.2.** Let $0 < U < 1$ and $V > 0$ be free, $V$ bounded. Assume that for some for $c, d \in \mathbb{R}$ condition (27) holds and

$$\mathbb{E}_{V^{1/2}UV^{1/2}} [V^{1/2}(1 - U)V^{1/2}]^{-2} = dI.$$  

Then $d > c^2$ and with $\alpha = \varphi(\Psi_U)$

- $V$ has free Poisson distribution $\mu \left( \frac{d-c^2}{c^2}, \frac{c^2\alpha + d}{d-c^2} \right)$,
- $U$ has free binomial distribution $\nu \left( \frac{c^2\alpha}{d-c^2}, \frac{d-c^2}{d-c^2} \right)$.

**Proof.** As in the previous proof we conclude that (27) implies (31). Then we consider (35). We multiply its both sides by $\Psi_U(\cdot) = (1) = (2)$ and with

$$\mathbb{E}_{V^{1/2}UV^{1/2}} [V^{1/2}(1 - U)V^{1/2}]^{-2} = dM_{V^{1/2}UV^{1/2}}$$

Using traciality of $\varphi$ and identity (29) with $(W, T) = (V, U)$ we obtain

$$N = z\varphi \left( (1 - U)^{-1}V^{-1}(1 - U)^{-1}U^{1/2} (\Psi_{U^{1/2}UV^{1/2}}(z) + 1) U^{1/2} \right)$$

$$= z\varphi \left( (1 - U)^{-1}U^{1/2}\Psi_{U^{1/2}UV^{1/2}}(z)U^{1/2}(1 - U)^{-1} + z\varphi \left( U^{-1}(1 - U)^{-2}U \right) \right)$$

$$= z\varphi \left( (1 - U)^{-1}U^{1/2}\Psi_{U^{1/2}UV^{1/2}}(z)U^{1/2}(1 - U)^{-1} + z\varphi \left( U^{-1}(1 - U)^{-2}U \right) \right).$$

By Proposition 3.7 we get

$$N = zB(z)\phi(V^{-1}) + z^2A^2(z)(1 + M_V(\omega_1(z))) + z\varphi \left( U^{-1}V^{-1}U^{-2} \right).$$

Since $\eta_U = \frac{M_U}{1 + M_U}$ it follows that $\eta_U = \frac{M_U}{1 + M_U}$. Also $M_U(1) = \phi \left( \sum_{k=1}^{\infty} kU^{k-1} \right) = \varphi(\Psi_U)$ and $M_U(1) = \phi \left( U \sum_{k=1}^{\infty} kU^{k-1} \right) = \phi(U(1 - U)^{-2})$. Moreover, (27) implies

$$\varphi(V^{-1})\varphi((1 - U)^{-1}) = c \quad \text{and} \quad \phi(\Psi_U) = c \varphi(U)\phi(V)$$

and (35) yields

$$\varphi(U(1 - U)^{-2})\varphi(V^{-1}) = d\varphi(U)\varphi(V).$$

Additionally, we easily see that

$$\varphi(V^{-1}) = \frac{c}{\alpha + 1}, \quad \eta_U(1) = \frac{\alpha}{1 + \alpha},$$

$$M_U'(1) = \frac{\alpha(1 + \alpha)d}{c^2}, \quad \eta_U'(1) = \frac{\alpha d}{(1 + \alpha)c^2}.$$

Moreover, $M_V(\omega_1(z)) = M_U(\omega_2(z))$. Summing up, (below $\omega_2 = \omega_2(z)$)

$$N = \frac{c\omega_2}{(\omega_2-1)^2} \left[ \frac{M_U(\omega_2) - \alpha}{M_U(\omega_2) + 1} \right] \left( \frac{z}{\omega_2 - 1} \right)^2 \left[ \frac{M_U(\omega_2) - \alpha}{M_U(\omega_2) + 1} \right] + \frac{d}{c(\omega_2 - 1)}.$$

Since $M_{V^{1/2}UV^{1/2}}(z) = M_U(\omega_2)$ at the RHS of (36) this equation upon multiplication both sides by $z$ can be written as

$$\left( \frac{z}{c(\omega_2 - 1)} \right)^2 \left[ \frac{M_U(\omega_2) - \alpha}{M_U(\omega_2) + 1} \right] c^2 \left[ \omega_2 - z \left[ M_U(\omega_2) - \alpha \right] \right] = d z \left( \frac{z}{c(\omega_2 - 1)} \right) \alpha + M_U(\omega_2).$$

Using now (31) in the form $\frac{c}{c(\omega_2 - 1)} = \frac{M_U(\omega_2)}{M_U(\omega_2) + 1}$ we get

$$\frac{c^2}{M_U(\omega_2) + 1} \left( \omega_2 - z \left[ M_U(\omega_2) - \alpha \right] \right) = dz.$$
Now, plug in $z [M_U(\omega_2) - \alpha] = c(\omega_2 - 1)M_U(\omega_2)$, which is another reformulation of (31), to conclude that

$$\omega_2 + (\omega_2 - 1)M_U(\omega_2) = \frac{d}{cz}(M_U(\omega_2) + 1).$$

Note that (37) upon substitution $b = d/c^2$ is the same as (33). Therefore (31) and (37) is the same system of equations as in the previous proof and thus the result follows. \hfill $\square$

5. Free Lukacs property for Marchenko-Pastur law

Here we point out a simple proof of Lukacs property of free Poisson distribution. The proof presented below uses a result from [20]. The original proof of the Lukacs property for free Poisson distribution from [19] relies heavily on combinatorics of free probability. In particular we calculated there joint free cumulants of $X$ and $X^{-1}$ for free Poisson distributed $X$.

**Theorem 5.1.** Let $X, Y$ be free both free Poisson distributed $\mu(\lambda, \alpha)$, $\mu(\kappa, \alpha)$, where $\lambda + \kappa > 1$, then random variables

$$U = (X + Y)^{-1/2}X(X + Y)^{-1/2} \quad \text{and} \quad V = X + Y$$

are free.

**Proof.** It suffices to show that

$$\varphi \left( \prod_j U_{m_j} V_{n_j} \right) = \phi \left( \prod_j U_{1}^{m_j} V_{1}^{n_j} \right)$$

for $U_1$ and $V_1$ free with distributions free binomial $(\lambda, \kappa)$ and free Poisson $(\lambda + \kappa, \alpha)$ respectively.

Define $X_1 = V_1^{1/2}U_1 V_1^{1/2}$, $Y_1 = V_1 - X_1$. From [20, Th. 3.2] we get that $X_1$ and $Y_1$ are free with free Poisson distributions $\nu(\lambda, \alpha)$ and $\nu(\kappa, \lambda)$ respectively.

But $\prod_j U_{m_j} V_{n_j} = g(X, Y)$ for some function $g$. Since $\lambda + \kappa > 1$ the spectrum of $V$ is bounded away from zero and thus $U$ is in $C^*$-subalgebra generated by $X$ and $Y$. Consequently, it can be approximated by polynomials in free variables $X$ and $Y$. Therefore, the distribution of $g(X, Y)$ by distributions of $X$ and $Y$, see e.g. [14].

Of course, $\prod_j U_{1}^{m_j} V_{1}^{n_j} = g(X_1, Y_1)$ for the same function $g$.

Since the quantity $\varphi(g(X, Y))$ for $X$ and $Y$ free depends only on distributions of $X$ and $Y$ so $\varphi(g(X, Y)) = \phi(g(X_1, Y_1))$ and the proof is completed. \hfill $\square$

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