CONVOLUTION-TYPE DERIVATIVES, HITTING-TIMES OF SUBORDINATORS AND TIME-CHANGED $C_0$-SEMIGROUPS

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Abstract. This paper takes under consideration subordinators and their inverse processes (hitting-times). The governing equations of such processes are presented by means of convolution-type integro-differential operators similar to the fractional derivatives. Furthermore the concept of time-changed $C_0$-semigroup is discussed in case the time-change is performed by means of the hitting-time of a subordinator. Such time-change gives rise to bounded linear operators governed by integro-differential time-operators. Because these operators are non-local the presence of long-range dependence is investigated.

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1. Introduction

The study of subordinators and their hitting-times has attracted the attention of many researchers since the 1940’s. In particular a great effort has been dedicated to the study of the relationships between Bochner subordination and Cauchy problems (Bochner [7, 8]). See Feller [11]; Jacob [14]; Schilling et al. [35] and the references therein for more information on Bochner subordination. A subordinator $\sigma(t)$, $t \geq 0$, is a stochastic process with non-decreasing paths for which $E e^{-\lambda \sigma(t)} = e^{-t f(\lambda)}$
where $f$ is a Bernstein function (see Bertoin [5, 6] for more details on subordinators). Its inverse process is defined as

$$f^L(t) = \inf \{ s \geq 0 : f^\sigma(s) > t \}$$

(1.1)

and is the hitting-time of $f^\sigma$. When the function $f$ is $f(\lambda) = \lambda^\alpha$, $\alpha \in (0, 1)$, the related subordinator is called the $\alpha$-stable subordinator and the inverse process $L^\alpha(t) = \inf \{ s > 0 : \sigma^\alpha(s) > t \}$ is called the inverse stable subordinator (see Meerschaert and Sikorskii [24]; Meerschaert and Straka [25]; Samorodnitsky and Taqqu [33] for more information on the stable subordinator and its inverse process). The relationships between such processes and partial differential equations have been object of intense study in the past three decades and have gained considerable popularity together with the study of fractional calculus (for fractional calculus the reader can consult Kilbas et al. [15]). As pointed out in Orsingher and Beghin [27, 28], fractional PDEs are indeed related to time-changed processes while the relationships between time-fractional Cauchy problems and the inverse of the stable subordinator was explored for the first time by Baeumer and Meerschaert [2]; Meerschaert et al. [21]; Saichev and Zaslavsky [31]; Zaslavsky [38]. Equations of fractional order appear in a lot of physical phenomena (Meerschaert and Sikorskii [24]) and in particular for modeling anomalous diffusions (see for example Benson et al. [3]; D’Ovidio [9]).

In the present paper we deal with the inverse processes $f^L(t)$, $t \geq 0$, of subordinators $f^\sigma(t)$, $t \geq 0$, with Laplace exponent the Bernstein function $f$ having the following representation

$$f(x) = a + bx + \int_0^\infty (1 - e^{-sx}) \bar{\nu}(ds)$$

(1.2)

for a non-negative measure $\bar{\nu}$ on $(0, \infty)$ (Bernstein [4]; Schilling et al. [35]). We consider the case in which the tail $s \rightarrow \bar{\nu}(s) = a + \bar{\nu}(s, \infty)$ is absolutely continuous on $(0, \infty)$ and we define integro-differential operators similar to the fractional derivatives. In particular we show how the operator

$$f^D_t u(t) = b \frac{d}{dt} u(t) + \int_0^t \frac{d}{dt} u(t - s) \nu(s) ds$$

(1.3)

allows us to write the governing equations of

$$T_t u = \int_0^\infty T_s u l_t(ds), \quad u \in \mathcal{B},$$

(1.4)

where $l_t(B) = \Pr \{ f^L(t) \in B \}$ is the distribution of $f^L$ and $T_s$ is a $C_0$-semigroup on the Banach space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$. We call the operator $T_t$ a time-changed $C_0$-semigroup. In fact the main result of the present paper shows that $T_t u$, $u \in \mathcal{B}$, is a bounded strongly continuous linear operator on $\mathcal{B}$ and solves the problem

$$\begin{cases}
  f^D_t q(t) = A q(t), & 0 < t < \infty, \\
  q(0) = u \in \text{Dom}(A),
\end{cases}$$

(1.5)

where $A$ is the infinitesimal generator of the $C_0$-semigroup $T_t u$, $u \in \mathcal{B}$.

A central role in our analysis is played by the tail $\nu(s)$ of the Lévy measure $\bar{\nu}$ since it emerges through all the results of the paper. It appears in the definitions of convolution-type derivatives of the form (1.3) we will discuss in Section 2.
Furthermore we prove the following convergence in distribution
\[
\lim_{\gamma \to 0} \left( bt + \sum_{j=1}^{N(t \nu(\gamma))} Y_j \right) \xrightarrow{\text{law}} f(\sigma(t), t \geq 0),
\]
where \(Y_j\) are i.i.d. random variables with distribution
\[
\Pr\{Y_j \in dy\} = \frac{1}{\nu(\gamma)} (\tilde{\nu}(dy) + a\delta_0) \mathbb{1}_{y > \gamma}, \quad \gamma > 0, \forall j = 1, \ldots, n,
\]
and \(N(t), t \geq 0,\) is a homogeneous Poisson process independent from the r.v.'s \(Y_j.\)

The symbol \(\delta_0\) stands for the Dirac point mass at infinity.

**List of symbols.** Here is a list of the most important notations adopted in the paper.

- With \(\mathcal{L} [u(\cdot)](\lambda) = \tilde{u}(\lambda)\) we denote the Laplace transform of the function \(u.\)
- \(\mathcal{F} [u(\cdot)](\xi) = \tilde{u}(\xi)\) indicates the Fourier transform of the function \(u.\)
- With \(f(\sigma(t), t \geq 0,\) we denote the subordinator with Laplace exponent \(f.\)
- \(\mu_t(B) = \Pr\{f(\sigma(t)) \in B\}\) indicates the convolution semigroup (transition probabilities) associated with the subordinator \(f(\sigma(t), t \geq 0).\) When the measure \(\mu_t\) has a density we adopt the abuse of notation \(\mu_t(ds) = \mu_t(s)ds\) where \(\mu_t(s)\) indicates the density of \(\mu_t.\)
- \(f(L(t), t \geq 0,\) indicates the inverse of the subordinator \(f(\sigma(t), t \geq 0.\)
- The symbol \(l_t(B) = \Pr\{f(L(t)) \in B\}\) indicates the distribution of \(f(L(t), t \geq 0.\) With abuse of notation we denote by \(l_t(s)\) the density of \(l_t(ds).\)
- With \(A\) we denote the infinitesimal generator of the semigroup \(T_t u\) for \(u \in \mathcal{B}\) (\(\mathcal{B}\) is a Banach space).

**2. Convolution-type derivatives**

In this section we define convolution-type operators similar to the fractional derivatives. The logic of our definitions starts from the observation of the fractional derivative of order \(\alpha \in (0, 1)\) (in the Riemann-Liouville sense) can be considered the first-order derivative of the Laplace convolution \(u(t) \ast t^{-\alpha}/\Gamma(1 - \alpha)\) (see Kilbas et al. [15])
\[
\frac{d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds.
\] (2.1)

For fractional calculus and applications the reader can also consult Mainardi [20] and for different form of fractional derivatives Achar et al. [1]; Kochubei [16]; Lorenzo and Hartley [19]; Meerschaert and Scheffer [23]; Meerschaert and Sikorskii [24]. Formula (2.1) can be formally viewed as \(\frac{d}{dt} \tilde{u}(\xi)\) for \(\alpha \in (0, 1).\) Here we generalize this idea to a Bernstein function (Bernstein [4]). A Bernstein function is a function \(f(x) : (0, \infty) \to \mathbb{R}\) of class \(C^\infty, f(x) \geq 0, \forall x > 0\) for which
\[
(-1)^k f^{(k)}(x) \leq 0, \quad \forall x > 0 \text{ and } k \in \mathbb{N}.
\] (2.2)

A function \(f\) is a Bernstein function if, and only if, admits the representation
\[
f(x) = a + bx + \int_0^\infty (1 - e^{-sx}) \tilde{\nu}(ds), \quad x > 0,
\] (2.3)
where \(a, b \geq 0\) and \(\bar{\nu}(ds)\) is a non-negative measure on \((0, \infty)\) satisfying the integrability condition
\[
\int_0^\infty (z \wedge 1) \bar{\nu}(dz) < \infty.
\] (2.4)

According to the literature we refer to the measure \(\bar{\nu}\) and to the triplet \((a, b, \bar{\nu})\) as the Lévy measure and the Lévy triplet of the Bernstein function \(f\). The representation (2.3) is called the Lévy-Khintchine representation of \(f\).

The Bernstein functions are closely related to the so-called completely monotone functions (see more on Bernstein function in Jacob [14]; Schilling et al. [35]). The function \(g(x) : (0, \infty) \to \mathbb{R}\) is completely monotone if has derivatives of all orders satisfying
\[
(-1)^k g^{(k)}(x) \geq 0, \quad \forall x > 0 \text{ and } k \in \{0\} \cup \mathbb{N}.
\] (2.5)

By Bernstein Theorem (see [4]) the function \(g\) is completely monotone if and only if
\[
g(x) = \int_0^\infty e^{-sx} m(ds), \quad x > 0,
\] (2.6)
when the above integral converges \(\forall x > 0\) and where \(m(ds)\) is a non-negative measure on \([0, \infty)\). Here and all throughout the paper the following symbology and definitions will be the same. We use \(f(\cdot)\) to denote the Bernstein function with representation (2.3) and we consider the completely monotone function
\[
g(x) = \frac{f(x)}{x}, \quad x > 0,
\] (2.7)
with representation
\[
g(x) = b + \int_0^\infty e^{-sx} \nu(s) ds,
\] (2.8)
where \(\nu(s)\) is the tail of the Lévy measure appearing in (2.3)
\[
\nu(s) ds = (a + \bar{\nu}(s, \infty)) ds.
\] (2.9)

The representations (2.7) and (2.8) define a completely monotone function and are valid for every Bernstein function \(f\) (see for example Schilling et al. [35] Corollary 3.7 (iv)). We observe that \(\nu(s)\) is in general a right-continuous and non-increasing function for which
\[
\int_0^1 (a + \bar{\nu}(s, \infty)) ds = \int_0^1 \nu(s) ds < \infty.
\] (2.10)

Furthermore we note that
\[
\bar{\nu}(s, \infty) < \infty, \text{ for all } s > 0.
\] (2.11)

In order to justify (2.11) we recall the inequality
\[
(1 - e^{-1}) (t \wedge 1) \leq 1 - e^{-t}, \quad t \geq 0,
\] (2.12)
which can be extended as
\[
(1 - e^{-t}) (t \wedge \epsilon) \leq (1 - e^{-t}), \quad \text{for all } 0 < \epsilon \leq 1, t \geq 0.
\] (2.13)

By taking into account (2.13) we can rewrite for all \(0 < \epsilon \leq 1\) the integrability condition (2.4) as
\[
\int_0^\infty (t \wedge \epsilon) \bar{\nu}(dt) < \infty, \quad \text{for all } 0 < \epsilon \leq 1,
\] (2.14)
since 

\[ \int_0^{\infty} (t \wedge \epsilon) \bar{\nu}(dt) \leq \frac{e^\epsilon}{e^\epsilon - 1} \int_0^{\infty} (1 - e^{-t}) \bar{\nu}(dt) = \frac{e^\epsilon}{e^\epsilon - 1} f(1) < \infty \]  

(2.15) 

and this implies (2.11). When the Lévy measure has finite mass, that is 

\[ \bar{\nu}(0, \infty) < \infty, \]

(2.16) 

and if \( b = 0 \), the corresponding Bernstein function \( f \) is bounded.

2.1. Convolution-type derivatives on the positive half-axis. In this section we define a generalization, with respect to a Bernstein function \( f \), of the classical Riemann-Liouville fractional derivative and we discuss some of its fundamental properties. Here is the first definition.

**Definition 2.1.** Let \( 0 < c \leq d < \infty \) and \( u \in AC([c, d]) \) that is the space of absolutely continuous function on \([c, d]\). Let \( f \) be a Bernstein function with representation (2.3) and let \( \bar{\nu} \) be the corresponding Lévy measure with tail \( \nu(s) = a + \bar{\nu}(s, \infty) \). Assume that \( s \to \nu(s) \) is absolutely continuous on \((0, \infty)\). We define the generalized Riemann-Liouville derivative according to the Bernstein function \( f \) as 

\[ fD_t^{(c,d)} u(t) := \frac{d}{dt} \left[ bu(t) + \int_0^{t-c} u(t-s) \nu(s) ds \right], \quad t \in [c, d]. \]  

(2.17) 

The representation (2.17) can be extended to define the derivative on the half-axis \( \mathbb{R}^+ \) as it is done for the classical Riemann-Liouville fractional derivative (see Kilbas et al. [15] page 79). Hence we write 

\[ fD_t^{(0,\infty)} u(t) := \frac{d}{dt} \left[ bu(t) + \int_0^t u(t-s) \nu(s) ds \right]. \]  

(2.18) 

**Lemma 2.2.** Let \( fD_t^{(c,\infty)} u(t), \ t \geq c \geq 0, \) be as in Definition 2.1 and let \( |u(t)| \leq Me^{\lambda t} \) for some \( \lambda_0, \ M > 0 \). We have the following result 

\[ \mathcal{L} \left[ fD_t^{(c,\infty)} u(t) \right] (\lambda) = f(\lambda) \bar{u}(\lambda) - be^{-\lambda c} u(c), \quad \Re \lambda > \lambda_0. \]  

(2.19) 

**Proof.** The Laplace transform can be evaluated explicitly as follows 

\[ \mathcal{L} \left[ fD_t^{(c,\infty)} u(t) \right] (\lambda) = b\lambda \bar{u}(\lambda) - be^{-\lambda c} u(c) + \mathcal{L} \left[ \frac{d}{dt} \int_0^{t-c} u(t-s) \nu(s) ds \right] (\lambda) \]

\[ = b\lambda \bar{u}(\lambda) - be^{-\lambda c} u(c) + \lambda \int_0^{t-c} \mathcal{L} \left[ u(t-s) \nu(s) ds \right] (\lambda) \]

\[ = b\lambda \bar{u}(\lambda) - be^{-\lambda c} u(c) + \lambda \int_0^{\infty} \int_{s+c}^{\infty} e^{-\lambda t} u(t-s) \nu(s) dt ds \]

\[ = \lambda g(\lambda) \bar{u}(\lambda) - be^{-\lambda c} u(c) = f(\lambda) \bar{u}(\lambda) - be^{-\lambda c} u(c). \]  

(2.20) 

In the last steps we used (2.7) and (2.8). 

In view of the previous Lemma we note that our definition is consistent and generalizes the Riemann-Liouville fractional derivatives of order \( \alpha \in (0, 1) \) in a reasonable way.
Remark 2.3. Let the function \( f \) of Definition 2.1 be \( f(x) = x^\alpha, \ x > 0, \ \alpha \in (0, 1) \), for which (2.3) becomes

\[
x^\alpha = \int_0^\infty (1 - e^{-sx}) \frac{\alpha s^{-\alpha-1}}{\Gamma(1 - \alpha)} ds,
\]

that is to say \( a = 0 \) and \( b = 0 \) and

\[
\tilde{\nu}(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1 - \alpha)} ds
\]

and therefore

\[
\nu(s)ds = ds \int_s^\infty \frac{\alpha z^{-\alpha-1}}{\Gamma(1 - \alpha)} dz = \frac{s^{-\alpha} ds}{\Gamma(1 - \alpha)}.
\]

By performing these substitutions in Definition 2.1 it is easy to show that

\[
f D_t^{(0, +\infty)} u(t) = \frac{R d^\alpha}{dt^\alpha} u(t)
\]

where

\[
\frac{R d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds
\]

is the Riemann-Liouville fractional derivative.

By following the logic inspiring the fractional Dzerbayshan-Caputo derivative (see [15]) defined, for an absolutely continuous function \( u(t), \ t > 0 \), as

\[
C d^\alpha dt^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds,
\]

we can give the following alternative definition of generalized derivative with respect to a Bernstein function.

**Definition 2.4.** Let \( 0 < c \leq d < \infty \) and \( u \in AC([c,d]) \). Let \( f \) and \( \nu \) be as in Definition 2.1 and \( u(t) \in AC([c,d]) \). We define the generalized Dzerbayshan-Caputo derivative according to the Bernstein function \( f \) as

\[
f D_t^{(c,d)} u(t) := b \frac{d}{dt} u(t) + \int_c^d \frac{\partial}{\partial t} u(t-s) \nu(s) ds, \quad t \in [c,d].
\]

As already done for the classical Dzerbayshan-Caputo derivative we can extend (2.27) to the half-axis \( \mathbb{R}^+ \) (see for example [15] page 97) by

\[
f D_t^{(0, \infty)} u(t) := b \frac{d}{dt} u(t) + \int_0^t \frac{\partial}{\partial t} u(t-s) \nu(s) ds.
\]

Throughout the paper we will write for the sake of simplicity \( f D_t \) instead of \( f D_t^{(0, \infty)} \).

**Lemma 2.5.** Let \( f D_t \) be as in (2.28) and let \( |u(t)| \leq Me^{\lambda_0 t}, \) for some \( \lambda_0, \ M > 0 \). We obtain

\[
\mathcal{L} \left[ f D_t u(t) \right] (\lambda) = f(\lambda) \bar{u}(\lambda) - \frac{f(\lambda)}{\lambda} u(0), \quad \Re \lambda > \lambda_0.
\]
Proof. By evaluating explicitly the Laplace transform we obtain
\[
\mathcal{L} \left[ f D_t u(t) \right] (\lambda) = b\lambda \tilde{u}(\lambda) - bu(0) + \int_0^\infty e^{-\lambda t} \int_0^t \frac{d}{dt} u(t-s) \nu(s) ds \, dt
\]
\[
= b\lambda \tilde{u}(\lambda) - bu(0) + \int_0^\infty e^{-\lambda s} \int_0^s \frac{d}{ds} u(s) \nu(t-s) ds \, ds
\]
\[
= b\lambda \tilde{u}(\lambda) - bu(0) + \int_0^\infty \nu(t) (\lambda \tilde{u}(\lambda) - u(0))
\]
\[
= f(\lambda) \tilde{u}(\lambda) - \frac{f(\lambda)}{\lambda} u(0)
\]
(2.30)
where we used the relationships (2.7) and (2.8).

Remark 2.6. By performing the same substitutions of Remark 2.3 it is easy to show that
\[
f D_t u(t) = C d^\alpha dt^\alpha u(t)
\]
(2.31)
where \(C d^\alpha dt^\alpha\) is the Dzerbayshan-Caputo derivative defined in (2.26).

It is well known that the Riemann-Liouville fractional derivative of a function \(u \in AC([c,d])\) exist almost everywhere in \([c,d]\) and can be written as (see Kilbas et al. [15] page 73)
\[
R d^\alpha dt^\alpha u(t) = C d^\alpha dt^\alpha u(t) + (t-c)^{-\alpha} \Gamma(1-\alpha) u(c).
\]
(2.32)
Here is a more general result.

Proposition 2.7. Let \(f D_t^{(c,d)}\) and \(f D_t^{(c,d)}\) be respectively as in Definitions 2.1 and 2.4. We have that \(f D_t^{(c,d)} u(t)\) exists almost everywhere in \([c,d]\) and can be written as
\[
f D_t^{(c,d)} u(t) = f D_t^{(c,d)} u(t) + \nu(t-c) u(c).
\]
(2.33)
Proof. Let
\[
V(s) = \int_0^s \nu(w) dw
\]
(2.34)
which is convergent in view of (2.10). Since \(u \in AC([c,d])\) we have for \(c < s < d\)
\[
u(t) (\lambda \tilde{u}(\lambda) - u(0))
\]
(2.30)
where we used the relationships (2.7) and (2.8).

Remark 2.6. By performing the same substitutions of Remark 2.3 it is easy to show that
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Here is a more general result.

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\[
f D_t^{(c,d)} u(t) = f D_t^{(c,d)} u(t) + \nu(t-c) u(c).
\]
(2.33)
Proof. Let
\[
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\]
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which is convergent in view of (2.10). Since \(u \in AC([c,d])\) we have for \(c < s < d\)
\[
u(t) (\lambda \tilde{u}(\lambda) - u(0))
\]
(2.30)
where we used the relationships (2.7) and (2.8).
In the fourth step we performed an integration by parts. □

2.2. Convolution-type derivatives on the whole real axis. In this section we develop a generalized space-derivative with respect to a Bernstein function \( f \) with domain on the whole real axis \( \mathbb{R} \), by following the logic inspiring the Weyl derivatives.

**Definition 2.8.** Let \( f \) and \( \nu(s) \) be as in Definition 2.1. We define the generalized Weyl derivative, according to the Bernstein function \( f \), on the whole real axis as

\[
\mathcal{D}_x^\nu u(x) := \left[ b \frac{d}{dx} u(x) + \int_0^\infty \frac{\partial}{\partial x} u(x - s) \nu(s) ds \right], \quad x \in \mathbb{R},
\]

and

\[
\mathcal{D}_x^- u(x) := -\left[ b \frac{d}{dx} u(x) + \int_0^\infty \frac{\partial}{\partial x} u(x + s) \nu(s) ds \right], \quad x \in \mathbb{R}.
\]

Some remarks on the domain of definition of (2.37) and (2.38) are stated in Section 5.1.

**Lemma 2.9.** Let \( \mathcal{D}_x^\pm \) be as in Definition 2.8. We have that

\[
\mathcal{F} \left[ \mathcal{D}_x^\nu u(x) \right] (\xi) = f(-i\xi) \tilde{u}(\xi)
\]

and

\[
\mathcal{F} \left[ \mathcal{D}_x^- u(x) \right] (\xi) = f(i\xi) \tilde{u}(\xi).
\]

**Proof.** By evaluating the first Fourier transform explicitly, we obtain

\[
\mathcal{F} \left[ \mathcal{D}_x^\nu u(x) \right] (\xi) = -bi\xi \tilde{u}(\xi) - i\xi \mathcal{F} \left[ \int_0^\infty u(x - s) \nu(s) ds \right] (\xi)
\]

\[
= -bi\xi \tilde{u}(\xi) - i\xi \int_0^\infty \int_{\mathbb{R}} e^{i\xi z + i\xi s} u(z) dz \nu(s) ds
\]

\[
= -bi\xi \tilde{u}(\xi) - i\xi \int_0^\infty ds e^{i\xi s} \left( a + \int_s^\infty \tilde{\nu}(ds) \right) \tilde{u}(\xi)
\]

and by integrating by parts we get that

\[
\mathcal{F} \left[ \mathcal{D}_x^\nu u(x) \right] (\xi) = a \tilde{u}(\xi) - bi\xi \tilde{u}(\xi) + \int_0^\infty (1 - e^{i\xi s}) \tilde{\nu}(ds) \tilde{u}(\xi)
\]

\[
= f(-i\xi) \tilde{u}(\xi).
\]

By repeating the same calculation one can easily prove (2.40). □
Remark 2.10. Definitions (2.37) and (2.38) are consistent with the Weyl definition of fractional derivatives on the whole real axis which are, for $\alpha \in (0,1)$ and $x \in \mathbb{R}$, (see [15])

$$\frac{+d^\alpha}{dx^\alpha}u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^x \frac{u(s)}{(x-s)\alpha} ds, \quad \text{right derivative,}$$

(2.43)

and

$$\frac{-d^\alpha}{dx^\alpha}u(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{u(s)}{(s-x)\alpha} ds, \quad \text{left derivative.}$$

(2.44)

We have

$$fD^\pm_x u(x) = \frac{\pm d^\alpha}{dx^\alpha} u(x), \quad x \in \mathbb{R}.$$  

(2.45)

We resort to the fact that (see [15] page 90)

$$\mathcal{F} \left[ \frac{\pm \partial^\alpha}{\partial x^\alpha} u(x) \right] (\xi) = (\mp i\xi)^\alpha \hat{u}(\xi)$$

(2.46)

and thus by combining (2.46) with Lemma 2.9 the proof of (2.45) is complete. The reader can also check the result by performing the substitution $b = 0$ and

$$\nu(s)ds = \frac{s^{-\alpha}}{\Gamma(1-\alpha)} ds \quad (2.47)$$

in (2.37) and (2.38) which yields (2.43) and (2.44) with a change of variable.

3. Subordinators, hitting-times and continuous time random walks

A subordinator $f\sigma(t), t \geq 0$, is a stochastic process in continuous time with non-decreasing paths (see more on subordinators in Bertoin [5, 6]) and values in $[0, \infty]$ where $\infty$ is an absorbing state (cemetery). The process $\sigma^f(t), t \geq 0$, is a subordinator if it has independent and homogeneous increments on $[0, \zeta)$ where

$$\zeta = \inf \{ t \geq 0 : \sigma^f(t) = \infty \}.$$  

(3.1)

If $\zeta \equiv \infty$ the process $\sigma^f$ is said to be a strict subordinator since it has stationary and independent increments in the ordinary sense. In this case the Laplace exponent of $f\sigma$ is

$$f(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda s}) \bar{\nu}(ds)$$

(3.2)

that is $a = 0$.

The transition probabilities of subordinators $\mu_t(B) = \Pr \{ f\sigma(t) \in B \}, B \subset [0, \infty)$ Borel, $t > 0$, are convolution semigroups of sub-probability measure with the following property concerning the Laplace transform

$$\mathcal{L}[\mu_t](\lambda) = e^{-tf(\lambda)}$$

(3.3)

where $f$ is a Bernstein function having representation (2.3). A family $\mu_t, t > 0,$ of sub-probability measures on $\mathbb{R}^n$ is called a convolution semigroup on $\mathbb{R}^n$ if it satisfies the conditions

- $\mu_t([0, \infty)) \leq 1, \forall t \geq 0$;
- $\mu_s * \mu_t = \mu_{t+s}, \forall s, t \geq 0,$ and $\mu_0 = \delta_0$;
- $\mu_t \rightharpoonup \delta_0$, vaguely as $t \to 0$. 


where we denoted by $\delta_0$ the Dirac point mass at zero. The fact that the tail function $s \to \nu(s)$ of the Lévy measure $\tilde{\nu}$ is absolutely continuous on $(0,\infty)$ and that $\tilde{\nu}(0,\infty) = \infty$ is a sufficient condition for saying the transition probabilities of the corresponding subordinator are absolutely continuous (see Sato [34], Theorem 27.7).

It has been shown that any subordinator has a Laplace exponent as in (3.3) and that any Bernstein function with representation (2.3) is the Laplace exponent of a subordinator (see for example [6]). A subordinator is a step process if its associated Bernstein function $f$ is bounded. Looking at the representation (2.3) we see that a Bernstein function is bounded if $\tilde{\nu}(0,\infty) < \infty$ and $b = 0$. If these conditions are not fulfilled (and thus $b > 0$ and $\tilde{\nu}(0,\infty) = \infty$) the subordinator is a strictly increasing process.

The inverse process of a subordinator is defined as

$$\mathcal{I}_s(t) = \inf \{ s > 0 : \mathcal{I}_s(s) > t \}, \quad s, t > 0,$$

and thus $\mathcal{I}_s$ is the hitting-time of $\mathcal{I}_s$ since $\mathcal{I}_s$ has non-decreasing paths (see Bertoin [5, 6]). With this in hand we note that $\mathcal{I}_s$ is again a non-decreasing process but in general it has non-stationary and non-independent increments. In what follows we develop some properties of the distribution of $\mathcal{I}_s(t)$, $t \geq 0$, denoted by $l_t(B) = \Pr \{ \mathcal{I}_s(t) \in B \}$.

**Lemma 3.1.** Let $\mathcal{I}_s(t)$, $t \geq 0$, and $\mathcal{I}_s(B)$, $t \geq 0$, be respectively a subordinator and its inverse. Let $\nu$ be the Laplace exponent of $\mathcal{I}_s$ represented as in (2.3) for $a, b \geq 0$. Let $\nu(s)$ be the tail of the Lévy measure $\tilde{\nu}$ and $l_t(B)$ the distribution of $\mathcal{I}_s$. Suppose that $s \to \nu(s)$ is absolutely continuous and that $\tilde{\nu}(0,\infty) = \infty$. We have that

$$\mathcal{L}[l_t(s,\infty)](\lambda) = \frac{1}{\lambda} e^{-s\mathcal{L}^-(\lambda)}.$$  

**Proof.** We resort to the fact that $\mathcal{I}_s$ has non-decreasing paths and thus, in view of the construction (3.4) of $\mathcal{I}_s$ we have

$$\Pr \{ \mathcal{I}_s(t) > s \} = \Pr \{ \mathcal{I}_s(s) < t \}.\quad (3.6)$$

In view of (3.6) we observe that

$$\int_0^\infty e^{-\lambda t} l_t(s,\infty) dt = \int_0^\infty e^{-\lambda t} \mu_0(0,t) dt \quad (3.7)$$

and thus

$$\int_0^\infty e^{-\lambda t} l_t(s,\infty) dt = \int_0^\infty e^{-\lambda t} \int_0^t \mu_s(dz) dt = \frac{1}{\lambda} e^{-s\mathcal{L}^-(\lambda)}.\quad (3.8)$$

**Proposition 3.2.** Let $\mathcal{I}_s(t)$, $t \geq 0$, be the subordinator with Laplace exponent $f$ represented by (2.3) for $a \geq 0$, $b \geq 0$. Let $\nu$ be the tail of the Lévy measure $\tilde{\nu}$. Assume that $\tilde{\nu}(0,\infty) = \infty$ and that $s \to \nu(s) = a + \tilde{\nu}(s,\infty)$ is absolutely continuous on $(0,\infty)$. Let $\mathcal{I}_s(t)$, $t \geq 0$, be the inverse of $\mathcal{I}_s$, in the sense of (3.4), with distribution $l_t(B) = \Pr \{ \mathcal{I}_s(t) \in B \}$. We have the following results.

1. The distribution $l_t$ have a density such that $l_t(ds) = l_t(s)ds$ and $l_t(s) = b\mu_s(t) + (\nu(t) + \mu_s(t))$ where with abuse of notation we denoted with $l_t(s)$ and $\mu_s(t)$ respectively the density of $l_t(ds)$ and $\mu_s(dt)$ and the symbol $\ast$ stands
for the Laplace convolution \( \int_0^t \mu_s(t-z)\nu(z)dz \). Furthermore \( \mathcal{L}[l_s](\lambda) = \frac{f(\lambda)}{\lambda} e^{-sf(\lambda)} \).

(2) \( \lim_{h \to 0} l_{t+h} = l_t \) \( \forall t \geq 0 \) and \( \lim_{t \to 0} l_t[0,\infty) = \delta_0[0,\infty) \).

(3) \( l_t(0) = \nu(t) \) \( \forall t > 0 \).

(4) \( l_t[0,\infty) = 1 \), \( \forall a, b \geq 0 \).

Proof. (1) Since we assume \( \bar{\nu}(0,\infty) = \infty \) and \( s \to \nu(s) \) is absolutely continuous on \( (0,\infty) \), we have that from Theorem 27.7 in [34] the transition probabilities \( \mu_t(dx) \) are absolutely continuous and therefore have a density \( \mu_t(x) \). Thus we write

\[
\mathcal{L}[b\mu_s(\cdot) + (\mu_s(\cdot) * \nu(\cdot))](\lambda) = be^{-sf(\lambda)} + \int_0^\infty e^{-\lambda t} \int_0^t \mu_s(t-z)\nu(z)dz \, dt
\]

\[
= be^{-sf(\lambda)} + \int_0^\infty dz \int_z^\infty e^{-\lambda t} \mu_s(t-z)\nu(z)
\]

\[
= \frac{f(\lambda)}{\lambda} e^{-sf(\lambda)}, \tag{3.9}
\]

where we used (2.8). From (3.9) we get

\[
\int_s^\infty \mathcal{L}[b\mu_w(\cdot) + (\mu_w(\cdot) * \nu(\cdot))](\lambda) \, dw = \int_s^\infty \frac{f(\lambda)}{\lambda} e^{-w f(\lambda)} \, dw = \frac{1}{\lambda} e^{-sf(\lambda)}. \tag{3.10}
\]

Since (3.10) coincides with (3.8) we can write

\[
\int_s^\infty (b\mu_w(t) + (\mu_w(t) * \nu(t))) \, dw = l_t(s, \infty) \tag{3.11}
\]

which completes the proof.

(2) We have

\[
\lim_{h \to 0} l_{t+h}[s,\infty) = \lim_{h \to 0} \int_s^\infty \left( b\mu_s(t+h) + \int_0^{t+h} \mu_s(t+z)\nu(z)dz \right) \, ds
\]

\[
= l_t[s,\infty) \tag{3.12}
\]

since \( \mu_s(t) \) is a density. Furthermore

\[
\lim_{t \to 0} l_t[0,\infty) = \lim_{t \to 0} \int_0^\infty \left( b\mu_s(t) + \int_0^t \mu_s(t+z)\nu(z)dz \right) \, ds = \delta_0[0,\infty). \tag{3.13}
\]

(3) This is obvious since for \( t > 0 \), \( l_t(0) = b\mu_0(t) + \nu(t) \ast \mu_0(t) = \nu(t) \).

(4) The proof of this can be carried out by observing that

\[
\int_0^\infty e^{-\lambda t} l_t[0,\infty) \, dt = \int_0^\infty e^{-\lambda t} l_t[s,\infty) \, dt \bigg|_{s=0}^{s=\infty} = \frac{1}{\lambda} e^{-sf(\lambda)} \bigg|_{s=0}^{s=\infty} = \frac{1}{\lambda}. \tag{3.14}
\]

Subordinators are related to Continuous Time Random Walks (CTRWs). The CTRWs (introduced in Montroll and Weiss [26]) are processes in continuous time in which the number of jumps performed in a certain amount of time \( t \) is a random variable, as well as the jump’s length. For example, the stable subordinator can be viewed (in distribution) as the limit of a CTRW performing a Poissonian number of power-law jumps (see for example Meerschaert and Sikorskii [24]). In Meerschaert and Scheffer [22], among other things, the authors pointed out that the limit process of a CTRW with infinite-mean waiting times converge to a Lévy
motion time-changed by means of the hitting-time $L^\alpha(t)$, $t \geq 0$, of the stable subordinator $\sigma^\alpha(t)$, $t \geq 0$. Since subordinators are also Lévy processes they can be decomposed according to the Lévy-Itô decomposition (Itô [13]). By following the logic of the Lévy-Itô decomposition we derive a CTRW converging (in distribution) to a subordinator with laplace exponent $f$ and having a hitting-time converging to its inverse. Our CTRW is therefore the sum of a pure drift and a compound Poisson. The distribution of the jumps’ length need some attention. In particular we define i.i.d. random variables $Y_j$ representing the random length of the jump, with law

$$p_{Y_j}(dy) = \frac{1}{\nu(\gamma)} (\nu(dy) + a \delta_\infty) \mathbf{1}_{y>\gamma}, \quad \gamma > 0, \forall j = 1, \ldots, n,$$  

(3.15)

where $\delta_\infty$ indicates the Dirac point mass at $\infty$ and $a \geq 0$. In (3.15) $\nu$ and $\nu$ are respectively the Lévy measure and its tail as defined in equations from (2.3) to (2.9) and upon which the definitions of convolution-type derivatives of previous section are based. The parameter $a \geq 0$ is that in (2.3) and it is known in literature as the killing rate of the subordinator. The distribution (3.15) can be taken as follows. The probability of a jump of length $y > \gamma > 0$ is given by the normalized Lévy measure when $a = 0$. When $a > 0$ the probability of a jump of infinite length increases since $\nu(y) \xrightarrow{y \to \infty} 0$ and thus $\Pr\{Y \in dy\} / dy \xrightarrow{y \to \infty} a / \nu(\gamma)$. When constructing a CTRW with Poisson waiting times and jump length’s distribution (3.15) by choosing $a > 0$ we obtain a limit process (for $\gamma \to 0$) assuming value $+\infty$ from a certain time $\zeta < \infty$ on. Usually $\zeta$ is called the lifetime of the process (see [6]). The case $a > 0$ in (3.15) therefore gives rise to the so-called killed subordinators. A killed subordinator $f(\sigma)_t$, is defined as

$$f(\sigma)_t = \begin{cases} f(\sigma)_t, & t < \zeta, \\ +\infty, & t \geq \zeta, \end{cases}$$  

(3.16)

where $\zeta$ is the lifetime defined in (3.1). Obviously $a = 0$ implies $\zeta = \infty$. For simplicity we will use the notation $f(\sigma)_t$ both for killed and non-killed subordinators when no confusion arises. We are ready to prove the following convergences in distribution inspired by the Lévy-Ito decomposition and useful in order to understand the role of the Lévy measure $\nu$ and its tail $\nu(s)$.

**Proposition 3.3.** Let $N(t), t \geq 0$, be a homogeneous Poisson process with parameter $\theta = 1$ independent from the i.i.d. random variables $Y_j$ with distribution (3.15). Let $f$ be the Bernstein function with representation (2.3) Laplace exponent of the subordinator $f(\sigma(t), t \geq 0$, and let $f(L(t), t \geq 0$ be the inverse of $f(\sigma$ as in (3.4). Let $\nu(s)$ be the tail of the Lévy measure $\nu$. The following convergences in distribution are true.

1. $$\left( bt + \sum_{j=0}^{N(t \nu(\gamma))} Y_j \right) \xrightarrow{\text{law}} f(\sigma(t) \quad \text{as} \quad \gamma \to 0, \quad (3.17)$$

2. $$\inf \left\{ s > 0 : bs + \sum_{j=0}^{N(s \nu(\gamma))} Y_j > t \right\} \xrightarrow{\text{law}} f(L(t) \quad \text{as} \quad \gamma \to 0. \quad (3.18)$$
Proof. In order to prove (1) we consider the following Laplace transform

$$
\mathbb{E} \exp \left\{ -\lambda bt - \lambda \sum_{j=0}^{N(t\nu(\gamma))} Y_j \right\}
$$

$$
= e^{-\lambda bt} \mathbb{E} \left[ \left( e^{-\lambda Y} \right)^{N(t\nu(\gamma))} \right]
$$

$$
= \exp \left\{ -\lambda bt e^{t\nu(\gamma)(1-e^{-\lambda Y})} \right\}
$$

$$
= \exp \left\{ -t \left( b\lambda + \nu(\gamma) \int_{\gamma}^{\infty} (1-e^{-\lambda y}) \bar{\nu}(dy) \right) \right\}, \quad (3.19)
$$

where $p_{Y_j}(dy)$ is the one in (3.15). In the previous steps we used the independence of the random variables $Y_j$ and the fact that

$$
\mathbb{E} e^{-\lambda N(t\nu(\gamma))} = e^{-t\nu(\gamma)(1-e^{-\lambda})}, \quad (3.20)
$$

By performing the limit for $\gamma \to 0$ in (3.19) we obtain

$$
\lim_{\gamma \to 0} \mathbb{E} \exp \left\{ -\lambda bt - \lambda \sum_{j=0}^{N(t\nu(\gamma))} Y_j \right\}
$$

$$
= \exp \left\{ -t \left( a + b\lambda + \int_{0}^{\infty} (1-e^{-\lambda y}) \bar{\nu}(dy) \right) \right\}
$$

$$
= e^{-tf(\lambda)}, \quad (3.21)
$$

and this proves (1).

Now we prove (2). Let $Z(t) = \inf \left\{ s > 0 : bs + \sum_{j=0}^{N(s\nu(\gamma))} Y_j > t \right\}$. By definition we have that

$$
\Pr \{ Z(t) > s \} = \Pr \left\{ bs + \sum_{j=0}^{N(s\nu(\gamma))} Y_j < t \right\} \quad (3.22)
$$

and thus

$$
\mathcal{L} [\Pr \{ Z(\cdot) > s \}] (\lambda) = \mathcal{L} \left[ \Pr \left\{ bs + \sum_{j=0}^{N(s\nu(\gamma))} Y_j < \cdot \right\} \right] (\lambda). \quad (3.23)
$$

By taking profit of calculation (3.19) we obtain

$$
\mathcal{L} [\Pr \{ Z(\cdot) > s \}] (\lambda) = \frac{1}{\lambda} \exp \left\{ -s \left( a + b\lambda + \int_{0}^{\infty} (1-e^{-\lambda y}) \bar{\nu}(dy) \right) \right\} \quad (3.24)
$$

and by performing the limit for $\gamma \to 0$ we arrive at

$$
\lim_{\gamma \to 0} \mathcal{L} [\Pr \{ Z(\cdot) > s \}] (\lambda) = \frac{1}{\lambda} e^{-sf(\lambda)}. \quad (3.25)
$$

Since (3.25) coincides with (3.8) the proof is complete. □

**Remark 3.4.** For $f(x) = x^\alpha$, $\alpha \in (0, 1)$ result (3.17) becomes

$$
\lim_{\gamma \to 0} \sum_{j=0}^{N(t\nu(\gamma))} Y_j \xrightarrow{law} \sigma^\alpha(t), \quad (3.26)
$$
where $\sigma^\alpha(t), t \geq 0$, is the stable subordinator of order $\alpha \in (0,1)$ and the i.i.d. random variables $Y_j$ have power-law distribution
\[
\Pr \{ Y \in dy \} / dy = \alpha \gamma^\alpha y^{-\alpha-1} \mathbf{1}_{y > \gamma}, \quad \gamma > 0,
\]
which can be obtained from (3.15) by performing the substitutions
\[
\nu(y) = \frac{\alpha y^{-\alpha-1}}{\Gamma(1-\alpha)} dy, \quad \text{and} \quad \nu(\gamma) = \frac{\gamma^{-\alpha}}{\Gamma(1-\alpha)},
\]
due to the fact that $f(x) = x^\alpha = (2.21)$ ($a = 0, b = 0$). The result (3.26) is well-known (see, for example, Meerschaert and Sikorskii [24]) and represents the convergence in distribution of a CTRW with power-law distributed jumps to the stable subordinator.

4. Densities and related governing equations

In this section we present in a unifying framework the governing equations of the densities of subordinators and their inverses, by making use of the operators defined in Section 2.

**Theorem 4.1.** Let $\sigma(t), t \geq 0$, and $\ell(t), t \geq 0$, be respectively a subordinator and its inverse. Let $\bar{\nu}$ be the Lévy measure such that $\bar{\nu}(0, \infty) = \infty$ and let $\nu(s) = a + \bar{\nu}(s, \infty)$. Assume $s \rightarrow \nu(s)$ is absolutely continuous on $(0, \infty)$.

(1) The probability density $\mu_t(x)$ of the subordinator $\sigma$ is the solution to the problem
\[
\begin{aligned}
\frac{\partial}{\partial t} \mu_t(x) &= -fD_{\bar{\nu}}^{(bt, +\infty)} \mu_t(x), \quad x > bt, 0 < t < \infty, b \geq 0, \\
\mu_t(bt) &= 0, \quad 0 < t < \infty, \\
\mu_0(x) &= \delta(x),
\end{aligned}
\]

(2) The probability density $l_t(x)$ of $\ell(t)$, $t \geq 0$, is the solution to the equation
\[
\begin{aligned}
fD_{\nu}^{(0, \infty)} l_t(x) &= -\frac{\partial}{\partial x} l_t(x), \quad t > 0, \quad \text{and} \quad \begin{cases} 
0 < x < \frac{1}{b} < \infty, & \text{if } b > 0, \\
0 < x < \infty, & \text{if } b = 0,
\end{cases}
\end{aligned}
\]
subject to
\[
\begin{aligned}
l_t(t/b) &= 0, \\
l_t(0) &= \nu(t), \\
l_0(x) &= \delta(x).
\end{aligned}
\]
The operator $fD_{\nu}^{(bt, +\infty)}$ is the one of Definition 2.1.

**Proof.** As already pointed out the conditions assumed on $\bar{\nu}$ and $\nu(s)$ ensure that $\mu_t(B)$ and $l_t(B)$ are absolutely continuous and therefore have densities we denote again by $\mu_t(x)$ and $l_t(x)$.

(1) First we note that $\mu_t(x) = 0$ for $x \leq bt$, $b \geq 0$, indeed from Proposition 3.3
\[
\Pr \{ f\sigma(t) > bt \} = \lim_{\gamma \to 0} \Pr \left\{ bt + \sum_{j=0}^{N(\nu(\gamma))} Y_j > bt \right\} = 1.
\]
The Laplace transform of \( \mu_t(x) \) is \( \mathcal{L}[\mu_t(x)](\phi) = e^{-tf(\phi)} \) and therefore \( \mathcal{L}[\tilde{\mu}_t(x)](\lambda) = 1/(f(\lambda) + \phi) \). In view of Lemma 2.2 the Laplace transform of (4.1) with respect to \( x \) is
\[
\frac{\partial}{\partial t} \tilde{\mu}_t(\phi) = -f(\phi)\tilde{\mu}_t(\phi) + be^{-bt}\mu_t(bt)
\] (4.5)
and therefore by performing the Laplace transform with respect to \( t \) we obtain
\[
\tilde{\mu}_\lambda(\phi) = \frac{1}{f(\lambda) + \phi}
\] (4.6)
where we used the facts that \( \tilde{\mu}_0(\phi) = 1 \) and \( \mu_t(bt) = 0 \). This completes the proof of (1).

(2) First we show that \( \ell_t(x) = 0 \) for \( x \geq \frac{t}{b} \) when \( b > 0 \). By considering Proposition 3.3 we have
\[
Pr\left\{ fL(t) < \frac{t}{b} \right\} = Pr\left\{ f\sigma \left( \frac{t}{b} \right) > t \right\} = \lim_{\gamma \to 0} Pr\left\{ t + \sum_{j=0}^{N(\frac{t}{b},\gamma)} Y_j > t \right\} = 1. \tag{4.7}
\]
The function \( t \to \ell_t(x) \) is differentiable since from Proposition 3.2 we have that \( \ell_t(x) = b\mu_t(t) + \int_0^t \mu_x(t-z)\nu(z)dz \) and \( t \to \mu_x(t) \) is differentiable in view of Theorem 28.1 of [34].

The double Laplace transform of \( \ell_t(x) \) reads
\[
\mathcal{L} \left[ \mathcal{L} \left[ \ell_t(x) \right] \right](\phi)(\lambda) = \frac{f(\lambda)/\lambda}{\phi + f(\lambda)}, \tag{4.8}
\]
where we used Proposition 3.2. From this point we temporary assume that \( b > 0 \). We consider the Laplace transform with respect to \( x \) of (4.2) and we obtain
\[
f\mathcal{D}_{(0,\infty)}^\tilde{\ell}_t(\phi) = -\phi\tilde{\ell}_t(\phi) + \ell_t(0) - e^{-\phi(t/b)}\ell_t(t/b). \tag{4.9}
\]
Considering the Laplace transform with respect to \( t \) of (4.9) and by taking into account (4.3) we get
\[
f(\lambda)\tilde{\ell}_\lambda(\phi) - b\tilde{l}_0(\phi) = -\phi\tilde{l}_\lambda(\phi) + \frac{f(\lambda)}{\lambda} - b \tag{4.10}
\]
where we used the fact that
\[
\int_0^\infty e^{-\lambda t}\nu(t)dt = \frac{f(\lambda)}{\lambda} - b \tag{4.11}
\]
and Lemma 2.2. The conditions (4.3) imply \( \tilde{l}_0(\phi) = 1 \) and thus
\[
\tilde{\ell}_\lambda(\phi) = \frac{f(\lambda)/\lambda}{\phi + f(\lambda)}. \tag{4.12}
\]
The proof for \( b = 0 \) can be carried out equivalently.
4.1. Some remarks on the long-range correlation. The operators $\mathcal{D}_t$ and
$\mathcal{D}_t^{c,\infty}$ are non-local and govern processes with different memory properties. The
presence of long-range correlation can be detected in several ways (see for example
Samorodnitsky [32]). Here we will explore the rate by which the correlation of
the inverses of subordinators decays (a similar approach can be found in Leonenko
et al. [17] applied to a fractional Pearson diffusion). In Veillette and Taqqu [37]
the authors derive an explicit formula for the moments of the inverse processes of
subordinators. Define
\[ E \left[ f_{L_t}^{m_1} \cdots f_{L_t}^{m_n} \right] = U(t_1, \ldots, t_n; m_1, \ldots, m_n). \]  
(4.13)
Formula (4.13) obeys the recursion formula
\[ U(t_1, \ldots, t_n; m_1, \ldots, m_n) = \int_0^{t_{\min}} \sum_{i=1}^n m_i U(t_1 - \tau, \ldots, t_n - \tau, m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n) U(d\tau) \]  
(4.14)
where $t_{\min} = \min(t_1, \ldots, t_n)$. If $n = 1$ and $m_1 = 1$ the function (4.13) reduces to
\[ U(x) = E \left[ f_{L_t}^x \right] = E \left[ \int_0^{\infty} 1_{\{\sigma(t) \leq x\}} dt \right] \]  
(4.15)
and is known as the renewal function since it is the distribution function of the
renewal measure $U(dx)$. The renewal measure is the potential measure of a subor-
dinator and it is given by
\[ U(B) = E \int_0^{\infty} 1_{\{\sigma(t) \in B\}} dt = \int_0^{\infty} \mu_t(B) dt, \quad \text{for } B \subseteq [0, \infty), \]  
(4.16)
the reader can consults Song and Vondraček [36] for further information. We recall
the renewal function is subadditive that is
\[ U(x + y) \leq U(x) + U(y), \quad \forall x, y \geq 0 \]  
(4.17)
and that
\[ \int_0^{\infty} e^{-\lambda x} U(dx) = \frac{1}{f(\lambda)} \int_0^{\infty} e^{-\lambda x} U(x) dx = \frac{1}{\lambda f(\lambda)}. \]  
(4.18)
Furthermore it is well-known (see, for example, [6], Proposition 1.4) that there exist
positive constants $c$ and $c'$ such that
\[ c U(x) \leq \frac{1}{f(\frac{x}{2})} \leq c' U(x). \]  
(4.19)
By applying (4.14) we write
\[ E \left[ f_{L_t}^s f_{L_t}^{t} \right] = \int_0^{s \land t} (U(s - \tau) + U(t - \tau)) U(d\tau) \]  
(4.20)
which can be interpreted as a long-range dependency property. We can write for
$w > 0$,
\[ E \left( f_{L_t}^w f_{L(t + s)} \right) = \int_0^{t \land (t + s)} (U(t - \tau) + U(t + s - \tau)) U(d\tau) \]
\[ \geq \int_0^t U(s + 2t - 2\tau) U(d\tau) \]
\[ \geq \int_0^t c' f \left( \frac{1}{s + 2t - 2\tau} \right) U(d\tau) \]  

(4.21)

where we applied (4.17) and (4.19). We recall that \(1/f\) is monotone and thus we can write

\[ \lim_{s \to \infty} \int_0^t c' f \left( \frac{1}{s + 2t - 2\tau} \right) U(d\tau) = \int_0^t \lim_{s \to \infty} c' f \left( \frac{1}{s + 2t - 2\tau} \right) U(d\tau) > 0 \]  

(4.22)

since \(\lim_{z \to 0} f(z) \geq 0\). Fix \(w, t > 0\) and use formula (4.22), we have

\[ \int_w^{\infty} E^{fL(t)} fL(t + s) ds = +\infty, \quad \forall w, t > 0. \]  

(4.23)

5. On the governing equations of time-changed \(C_0\)-semigroups

In this section we discuss the concept of time-changed \(C_0\)-semigroups on a Banach space \((\mathcal{B}, \|\cdot\|_B)\) (see more on semigroup theory in Engel and Nagel [10]; Jacob [14]) which we define as the Bochner integral

\[ T_t u = \int_0^{\infty} T_s u l_t(ds) \]  

(5.1)

where \(T_t\) is a \(C_0\)-semigroup and \(l_t\) is the distribution of the inverse \(fL(t), t \geq 0\) of \(f\sigma(t), t \geq 0\). We recall that a \(C_0\)-semigroup of operators on \(\mathcal{B}\) is a family of linear operators \(T_t\) (bounded and linear) which maps \(\mathcal{B}\) into itself and is strongly continuous that is

\[ \lim_{t \to 0} \|T_t u - u\|_B = 0, \quad \forall u \in \mathcal{B}. \]  

(5.2)

In other words a bounded linear operator \(T_t\) acting on a function \(u \in \mathcal{B}\) is said to be a \(C_0\)-semigroup if, \(\forall u \in \mathcal{B},\)

- \(T_0 u = u\) (is the identity operator),
- \(T_t T_s u = T_{t+s} u, \forall s, t \geq 0,\)
- \(\lim_{t \to 0} \|T_t u - u\|_B = 0.\)

The infinitesimal generator of a \(C_0\)-semigroup is the operator

\[ A u := \lim_{t \to 0} \frac{T_t u - u}{t}, \]  

(5.3)

for which

\[ \text{Dom}(A) := \left\{ u \in \mathcal{B} : \lim_{t \to 0} \frac{T_t u - u}{t} \right. \text{ exists as strong limit} \}. \]  

(5.4)

The aim of this section is to write the initial value problem associated with \(T_t\) by making use of the convolution-type time-derivatives of Definition 2.4.

**Theorem 5.1.** Let \(fL(t), t \geq 0,\) be the inverse process of a subordinator with Laplace exponent \(f\) and let \(l_t\) be the distribution of \(fL.\) Let \(\hat{\nu}(0, \infty) = \infty\) and \(s \to \nu(s) = a + \hat{\nu}(s, \infty)\) be absolutely continuous on \((0, \infty)\). Let \(T_t u, u \in \mathcal{B},\) be a (strongly continuous) \(C_0\)-semigroup on the Banach space \((\mathcal{B}, \|\cdot\|_B)\) such that \(\|T_t u\|_B \leq \|u\|_B.\) Let \((A, \text{Dom}(A))\) be the generator of \(T_t u.\) The operator defined by the Bochner integral

\[ T_t u = \int_0^{\infty} T_s u l_t(ds) \]  

(5.5)

acting on a function \(u \in \mathcal{B}\) is such that
(1) $\mathcal{T}_t u$ is a uniformly bounded linear operator on $\mathcal{B}$,

(2) $\mathcal{T}_t u$ is strongly continuous $\forall u \in \mathcal{B}$,

(3) $\mathcal{T}_t u$ solves the problem

$$\begin{cases}
{\mathcal{D}} q(t) = A q(t), & 0 < t < \infty, \\
q(0) = u \in \text{Dom}(A)
\end{cases} \tag{5.6}$$

where the time-operator $^{f} \mathcal{D}_t$ is the one appearing in Definition 2.4.

**Proof.** Now we prove the Theorem for $b > 0$ which is the case requiring some additional attention. The proof for $b = 0$ can be carried out equivalently and therefore is a particular case.

(1) At first we show that the operator $\mathcal{T}_t u$ is uniformly bounded on $(\mathcal{B}, \| \cdot \|_\mathcal{B})$.

From the hypothesys we have

$$\| \mathcal{T}_t \| \leq 1, \quad t \geq 0, \tag{5.7}$$

In view of (5.7) we can write

$$\| \mathcal{T}_t u \|_\mathcal{B} = \left\| \int_0^\infty T_s u l_t(\text{d}s) \right\|_\mathcal{B}$$

$$\leq \int_0^\infty \| T_s u \|_\mathcal{B} l_t(\text{d}s) \leq \| u \|_\mathcal{B}, \tag{5.8}$$

since $l_t[0, \infty) = 1$, $\forall t \geq 0$, as showed in Proposition 3.2.

(2) The strong continuity follows from the fact that

$$\lim_{h \to 0} \| \mathcal{T}_h u - u \|_\mathcal{B} = \left\| \int_0^\infty T_s u l_h(\text{d}s) - u \right\|_\mathcal{B}$$

$$\leq \int_0^\infty \| T_s u - u \|_\mathcal{B} l_h(\text{d}s) \xrightarrow{h \to 0} 0, \tag{5.9}$$

since $l_h \to \delta_0$ as $h \to 0$ and $\mathcal{T}_s$ is strongly continuous.

(3) Since $\mathcal{T}_t$ is a $C_0$-semigroup generated by $(A, \text{Dom}(A))$ we have

$$\frac{d}{dt} \mathcal{T}_t u = A \mathcal{T}_t u = T_t A u, \quad \forall u \in \text{Dom}(A). \tag{5.10}$$

Now let

$$A_s = \frac{T_s u - u}{s}. \tag{5.11}$$

We note that

$$A_s \mathcal{T}_t u = A_s \int_0^\infty T_z u l_t(\text{d}z)$$

$$= \int_0^\infty \frac{T_{z+s} u - T_z u}{s} l_t(\text{d}z)$$

$$= \int_0^\infty T_z \left( \frac{T_s u - u}{s} \right) l_t(\text{d}z) \tag{5.12}$$

and since for $u \in \text{Dom}(A)$ the limit for $s \to 0$ on the right-hand side exists we have that $\mathcal{T}_t$ maps $\text{Dom}(A)$ into itself.
By using Lemma 2.5 we note that the Laplace transform of (5.6) becomes

\[
\begin{aligned}
\begin{cases}
 f(\lambda)\tilde{q}(\lambda) - \frac{f(\lambda)}{\lambda} q(0) = A\tilde{q}(\lambda) \\
 q(0) = u.
\end{cases}
\end{aligned}
\]  
(5.13)

Now define the operator

\[
 f_{R,\lambda,A} := \int_0^\infty e^{-\lambda t} T_t dt = \frac{f(\lambda)}{\lambda} R_{f(\lambda),A}
\]  
(5.14)

where

\[
 R_{f(\lambda),A} = \int_0^\infty e^{-tf(\lambda)} T_t dt.
\]  
(5.15)

We recall that since we assume \((A, \text{Dom}(A))\) generate a \(C_0\)-semigroup for which \(\|T_t u\|_{\mathcal{B}} \leq \|u\|_{\mathcal{B}}\), we necessarily have that \(A\) is closed and densely defined. Furthermore for all \(\lambda \in \mathbb{C}\) with \(\Re \lambda > 0\) we must have that \(\lambda \in \rho(A)\) and

\[
 R_{\lambda,A} = \int_0^\infty e^{-\lambda t} T_t dt
\]  
(5.16)

is the resolvent operator and \(\rho(A)\) is the resolvent set of \(A\). The integral (5.15) is justified since every Bernstein function has an extension onto the right complex half-plane \(\mathbb{H} = \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \}\) which satisfies (see [35], Proposition 3.5)

\[
 \Re f(\lambda) = a + b\Re \lambda + \int_0^\infty (1 - e^{-s\Re \lambda} \cos \Im \lambda) \check{\nu}(ds) > 0.
\]  
(5.17)

By computing we can evaluate the following Laplace transform

\[
\int_0^\infty e^{-\lambda t} f_{\lambda} T_t u dt =
\]

\[
\begin{aligned}
&= \left[ b \int_0^\infty e^{-\lambda t} \lim_{h \to 0} \frac{T_{t+h} - T_t}{h} u dt + \int_0^\infty e^{-\lambda t} \lim_{h \to 0} \frac{T_{t+h} - T_t}{h} u dt \right] \\
&= \left[ \lim_{h \to 0} b \frac{e^{\lambda h}}{h} \int_0^\infty e^{-\lambda t} T_t u dt - b \lim_{h \to 0} \frac{1}{h} \int_0^\infty e^{-\lambda t} T_t u dt \right] \\
&\quad + \int_0^\infty ds \nu(s) \int_s^\infty e^{-\lambda t} \lim_{h \to 0} \frac{T_{t+h} - T_t}{h} u dt \\
&= \left[ b \lim_{h \to 0} \frac{e^{\lambda h} - 1}{h} f_{R\lambda} u - b \lim_{h \to 0} \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T_t u dt \right] \\
&\quad + \left( \frac{f(\lambda)}{\lambda} - b \right) \left[ \lim_{h \to 0} \frac{e^{\lambda h} - 1}{h} f_{R\lambda} u - \lim_{h \to 0} \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T_t u dt \right] \\
&= \left[ \frac{f(\lambda)}{\lambda} \left( \lim_{h \to 0} \frac{e^{\lambda h} - 1}{h} f_{R\lambda} u - \lim_{h \to 0} \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} T_t u dt \right) \right] \\
&= f(\lambda) f_{R\lambda} u - \frac{f(\lambda)}{\lambda} u,
\end{aligned}
\]  
(5.18)

where in the third step we used (2.8).
With this in hand we note that $fR_{\lambda,A}$ satisfies
\[ \| fR_{\lambda} \| \leq \int_0^\infty \| e^{-\lambda t} \|_{T_{A}} dt = \frac{1}{\Re \lambda}, \tag{5.19} \]
where we used (5.8). Note that we can formally write
\[ \int_0^\infty e^{-\lambda t} dt = \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} T_s ds = \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-s(f(\lambda) - A)} ds = \frac{f(\lambda)}{\lambda} \frac{1}{f(\lambda) - A}, \tag{5.20} \]
where we used Proposition 3.2 to state that $L[\xi(s)](\lambda) = \frac{f(\lambda)}{\lambda} e^{-sf(\lambda)}$ and $\xi(s)$ represents by abuse of notation the density of $l_t(ds)$. In (5.20) we used the exponential representation $T_t = e^{tA}$. Since we do not assume that $A$ is bounded the symbol $e^{tA}$ should be intended as $e^{tA}u = \text{strong-lim}_{\lambda \to \infty} e^{tA\lambda}u$ (Yosida approximation) where $A_{\lambda} := \lambda AR_{\lambda}$.

Now we have to prove that $\forall u \in \text{Dom}(A)$ we must have $fR_{\lambda}u \in \text{Dom}(A)$ and
\[ (f(\lambda) - A) fR_{\lambda}u = fR_{\lambda} (f(\lambda) - A) u = \frac{f(\lambda)}{\lambda} u. \tag{5.21} \]
Now by the definition
\[ A_h = \frac{1}{h} (T_h u - u) \tag{5.22} \]
for which $\lim_{h \to 0} A_h = A$, we find
\[
\begin{align*}
A_h fR_{\lambda}u &= \frac{T_h - I}{h} \int_0^\infty e^{-\lambda t} \int_0^\infty T_s u l_t(ds) dt \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty T_{s+h} - T_s u l_t(ds) dt \\
&= \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} T_{s+h} - T_s u ds \\
&= \frac{e^{hf(\lambda)}}{h} \frac{f(\lambda)}{\lambda} \int_0^h e^{-sf(\lambda)} T_s u ds - \frac{1}{h} \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} T_s u ds \\
&= \frac{f(\lambda)}{\lambda} \frac{e^{hf(\lambda)}}{h} - \frac{1}{h} \int_0^\infty e^{-zf(\lambda)} T_z u dz - \frac{f(\lambda)}{\lambda} \int_0^h e^{-sf(\lambda)} T_s u ds \\
& \xrightarrow{h \to 0} f(\lambda) fR_{\lambda} u - \frac{f(\lambda)}{\lambda} u. \tag{5.23}
\end{align*}
\]
This proves that $fR_{\lambda}u \in \text{Dom}(A)$ and that $(f(\lambda) - A) fR_{\lambda}u = \frac{f(\lambda)}{\lambda} u$. Furthermore we find
\[
\begin{align*}
fR_{\lambda} Au &= \int_0^\infty e^{-\lambda t} T_s A u dt = \int_0^\infty e^{-\lambda t} \int_0^\infty T_s A u l_t(ds) dt \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{d}{ds} T_s A u l_t(ds) ds \\
&= \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} \frac{d}{ds} T_s u ds \\
&= -\frac{f(\lambda)}{\lambda} u + f(\lambda) fR_{\lambda} u, \tag{5.24}
\end{align*}
\]
which completes the proof.

5.1. Convolution-type space-derivatives and Phillips’ formula. Let \( T_t \) be a \( C_0 \)-semigroup acting on functions \( u \in \mathcal{B} \), where \((\mathcal{B}, \| \cdot \|_{\mathcal{B}})\) is Banach space. Let \( \mu_t \) be a convolution semigroup of sub-probability measures on \([0, \infty)\) such that \( L[\mu_t] = e^{-tf} \) where \( f \) is a Bernstein function. The operator defined by the Bochner integral

\[
\int_0^\infty T_s u \mu_t(ds), \quad u \in \mathcal{B},
\]

is called a subordinate semigroup in the sense of Bochner. A classical result due to Phillips [29] state that the infinitesimal generator \((\int A, \text{Dom}(\int A))\) of the subordinate semigroup \(\int T_t\) on \(u \in \mathcal{B}\) is written as

\[
\int A u = -f(-A) u = -au + bAu + \int_0^\infty (T_s u - u) \nu(ds),
\]

with \(\text{Dom}(A) \subseteq \text{Dom}(\int A)\).

In Definition 2.8 we developed the convolution-type space-derivatives \(\int \mathcal{D}_x^\pm\) defined on the whole real axis. We have shown that they becomes, for \(f(x) = x^\alpha\), \(\alpha \in (0,1)\), the Weyl space-fractional derivatives defined in (2.43) and (2.44). In this section we show that \(-\int \mathcal{D}_x^\pm\) can be viewed as the infinitesimal generator of the subordinate semigroup in the sense of Bochner

\[
Q_t u(x) = \int_0^\infty T_s u(x) \mu_t(ds)
\]

where \(T_t' u(x) = u(x + t), u \in L^p(\mathbb{R})\), is the left translation semigroup.

Remark 5.2. We recall that the left translation operator \( T_t' u = u(x + t), t \geq 0, u \in L^p(\mathbb{R})\), defines a strongly continuous \(C_0\)-semigroup on \(L^p(\mathbb{R})\) (see for example [10] page 66) and has infinitesimal generator \(A = \frac{d}{dx}\) with \(\text{Dom}(A) = W^{1,p}, 1 \leq p < \infty\), where

\[
W^{1,p}(\mathbb{R}) = \{ u \in L^p(\mathbb{R}) : u \text{ absolutely continuous and } u' \in L^p(\mathbb{R}) \}.
\]

This implies that \(-\int \mathcal{D}_x^\pm\) have to coincide with Phillips’ representation (5.26) with \(A = \frac{d}{dx}\).

Proposition 5.3. Let \(\int \sigma(t)\) be a subordinator with Laplace exponent \(f\) and transition probabilities \(\mu_t\). Let \(\zeta = \inf \{ t \geq 0 : \int \sigma(t) = +\infty \}\). The solution to the initial value problem

\[
\begin{cases}
\frac{\partial}{\partial t} q(x, t) = -\int \mathcal{D}_x^- q(x, t), & x \in \mathbb{R}, 0 < t < \infty, \\
q(x, 0) = u(x) \in W^{1,p}(\mathbb{R}),
\end{cases}
\]

is given by the contractive strongly continuous semigroup of operators on \(L^p(\mathbb{R})\)

\[
Q_t u(x) = \int_0^\infty u(x + y) \mu_t(dy), \quad t < \infty
\]

which is the subordinate translation semigroup \(T_t' u(x) = u(x + t)\), in the sense of Bochner. The operator \(-\int \mathcal{D}_x^-\) is that of Definition 2.8 and \(W^{1,p}\) is defined in (5.28).
Proof. Since $Q_t u$ is a subordinate semigroup in the sense of Bochner, it defines again a $C_0$-semigroup on $L^p(\mathbb{R})$. By applying Phillips’ result ([29]) we know that the infinitesimal generator of $Q_t u$ is written as
\begin{equation}
-f \left( -\frac{\partial}{\partial x} \right) u(x) = -au(x) + b \frac{\partial}{\partial x} u(x) + \int_0^\infty (T_s^t u(x) - u(x)) \bar{\nu}(ds). \tag{5.31}
\end{equation}

Since
\begin{equation}
\left\| -f \left( -\frac{\partial}{\partial x} \right) u(x) \right\|_p \leq a \left\| u(x) \right\|_p + b \left\| \frac{\partial}{\partial x} u(x) \right\|_p + \int_0^\infty \left\| T_s^t u(x) - u(x) \right\|_p \bar{\nu}(ds)
\end{equation}

by applying the well-known inequality (see for example Jacob [14])
\begin{equation}
\| T_t u(x) - u(x) \| \leq (t \| Au(x) \| \wedge 2 \| u(x) \|), \quad u \in \Dom(A) \tag{5.33}
\end{equation}

which is valid in general for a strongly continuous semigroup $T_t u(x)$ on a Banach space $(\mathcal{B}, \| \cdot \|)$ and infinitesimal generator $(A, \Dom(A))$, we can write
\begin{equation}
\left\| -f \left( -\frac{\partial}{\partial x} \right) \right\|_p \leq a \left\| u(x) \right\|_p + b \left\| \frac{\partial}{\partial x} u(x) \right\|_p + \int_0^\infty z\bar{\nu}(dz) \left\| \frac{\partial}{\partial x} u(x) \right\|_p
\end{equation}

\begin{equation}
+ 2 \int_0^\infty \bar{\nu}(dz) \left\| u(x) \right\|_p. \tag{5.34}
\end{equation}

This shows that (5.31) is defined on
\begin{equation}
\begin{cases}
W^{1,p}(\mathbb{R}), & \text{if } b > 0, \\
W^{1,p}(\mathbb{R}), & \text{if } b = 0 \text{ and } \bar{\nu}(0, \infty) = \infty, \\
L^p(\mathbb{R}), & \text{if } b = 0 \text{ and } \bar{\nu}(0, \infty) < \infty.
\end{cases} \tag{5.35}
\end{equation}

since for $\bar{\nu}(0, \infty) < \infty$ we can choose $\epsilon = 0$ in (5.34).

The operator $f D_x^-$
\begin{equation}
-f D_x^- u(x) = b \frac{\partial}{\partial x} u(x) + \int_0^\infty \frac{\partial}{\partial x} u(x + s) \nu(s) ds \tag{5.36}
\end{equation}

for $u \in W^{1,p}(\mathbb{R})$ can be rewritten as
\begin{equation}
-f D_x^- u(x) = b \frac{\partial}{\partial x} u(x) + \int_0^\infty \frac{\partial}{\partial s} u(x + s) (a + \bar{\nu}(s, \infty)) ds
\end{equation}

\begin{equation}
= -au(x) + b \frac{\partial}{\partial x} u(x) + \int_0^\infty \int_0^\infty \frac{\partial}{\partial s} T_s^t u(x) ds \bar{\nu}(dz)
\end{equation}

\begin{equation}
= -au(x) + b \frac{\partial}{\partial x} u(x) + \int_0^\infty (T_s^t u(x) - u(x)) \bar{\nu}(dz) \tag{5.37}
\end{equation}

which coincides with (5.31). This completes the proof. \qed

6. Example: The Tempered Stable Subordinator

By setting the Bernstein function considered in previous sections to be $f(x) = x^\alpha$, $\alpha \in (0, 1)$, we retrieve the stable subordinator $\sigma^\alpha(t)$, $t \geq 0$, for which $E e^{-\lambda \sigma^\alpha(t)} = e^{-t \lambda^\alpha}$, and its inverse process $L^\alpha(t)$, $t \geq 0$. Therefore by performing the substitution $f(x) = x^\alpha$ all throughout the paper we retrieve the results related to fractional calculus. In this section we take as example the Bernstein function
\begin{equation}
f(x) = (x + \vartheta)^\alpha - \vartheta^\alpha = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-xs}) e^{-\vartheta s} s^{-1 - \alpha} ds, \tag{6.1}
\end{equation}
where \( \vartheta > 0, \alpha \in (0, 1) \). The Bernstein function (6.1) is the Laplace exponent of the subordinator \( \vartheta \sigma^\alpha(t) \) such that

\[
\mathbb{E} e^{-\lambda \vartheta \sigma^\alpha(t)} = e^{-t((\lambda + \vartheta)^\alpha - \vartheta^\alpha)},
\]

(6.2)

The process \( \vartheta \sigma^\alpha(t), t \geq 0 \), is known in literature as the relativistic stable subordinator since it appears in the study of the stability of the relativistic matter (Lieb [18]) but it is also known as the tempered stable subordinator (see for example Meerschaert and Sikorskii [24] page 207, Rosiński [30] or Zolotarev [39], Lemma 2.2.1).

From (6.1) we know that the Lévy measure has the explicit representation

\[
\nu(ds) = \frac{\alpha e^{-\vartheta^\alpha s - \alpha - 1}}{\Gamma(1-\alpha)} ds,
\]

(6.3)

and has infinite mass (\( f(x) \) is not bounded). Furthermore its tail becomes

\[
\nu(s) = \left( \frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right),
\]

(6.4)

where

\[
\Gamma(-\alpha, s) = \int_s^\infty e^{-z} z^{-\alpha - 1} dz
\]

(6.5)

is the incomplete Gamma function. It is well-known that the governing equation of \( \vartheta \sigma^\alpha(t), t \geq 0 \), is written by using the so-called tempered fractional derivative

\[
\partial_x \vartheta \sigma^\alpha u(x) = e^{-\vartheta \sigma^\alpha x} R^{\vartheta \sigma^\alpha} \left[ e^{\vartheta \sigma^\alpha x} u(x) \right] - \vartheta \sigma^\alpha u(x), \quad \alpha \in (0, 1),
\]

(6.6)

as

\[
\frac{\partial}{\partial t} \mu_t^{\vartheta \sigma^\alpha}(x) = -\partial_x \vartheta \sigma^\alpha \mu_t^{\vartheta \sigma^\alpha}(x), \quad x > 0, t > 0,
\]

(6.7)

see [24] page 209 and the references therein. According to Theorem 4.1 we must have

\[
\frac{\partial}{\partial t} \mu_t^{\vartheta \sigma^\alpha}(x) = -f \mathcal{D}_x^{(0, \infty)} \mu_t^{\vartheta \sigma^\alpha}(x), \quad x > 0, t > 0,
\]

(6.8)

and indeed it is easy to show that if \( f(\lambda) = (\lambda + \vartheta)^\alpha - \vartheta^\alpha \)

\[
f \mathcal{D}_x^{(0, \infty)} u(x) = \frac{d}{dx} \int_0^x u(x-s) \left( \frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right) ds = \partial_x^{\vartheta \sigma^\alpha} u(x).
\]

(6.9)

This can be done for example by observing that

\[
\mathcal{L} \left[ \frac{d}{dx} \int_0^x u(x-s) \left( \frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right) ds \right](\lambda) = \mathcal{L} \left[ \partial_x^{\vartheta \sigma^\alpha} u(x) \right](\lambda).
\]

(6.10)

The time operator \( f \mathcal{D}_t \) governing the density of

\[
\vartheta \sigma^\alpha(t) = \inf \{ s > 0 : \vartheta \sigma^\alpha(s) > t \},
\]

(6.11)

becomes in this case

\[
f \mathcal{D}_t^{(0, \infty)} t_t^{\vartheta \sigma^\alpha}(x) = \frac{\partial}{\partial t} \int_0^t t_t^{\vartheta \sigma^\alpha}(x) \left( \frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right) ds,
\]

(6.12)

and therefore \( t_t^{\vartheta \sigma^\alpha}(x), t > 0, \) is the solution to

\[
\begin{cases}
\frac{\partial}{\partial t} \int_0^t t_t^{\vartheta \sigma^\alpha}(x) \left( \frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right) ds = -\frac{\partial}{\partial x} t_t^{\vartheta \sigma^\alpha}(x), & t > 0, x > 0, \\
\int_0^t t_t^{\vartheta \sigma^\alpha}(0) = \frac{\alpha \vartheta^\alpha \Gamma(-\alpha, t)}{\Gamma(1-\alpha)}, & t > 0, \\
t_0^{\vartheta \sigma^\alpha}(x) = \delta(x),
\end{cases}
\]

(6.13)
Finally, in view of Proposition 3.3, we are able to write the CTRW converging in distribution to \( \theta^\sigma(t), t \geq 0 \). We have

\[
\lim_{\gamma \to 0} N \left( \frac{\gamma \alpha}{\Gamma(\gamma, \alpha)} \right) \sum_{j=0}^\infty Y_j \xrightarrow{\text{law}} \theta^\sigma(t) \quad (6.14)
\]

where \( Y_j \) are i.i.d. random variables with distribution

\[
\Pr\{Y_j \in dy\} = \frac{e^{-\psi y^{-\alpha-1}}}{\psi \Gamma(-\alpha, \gamma)} \mathbb{1}_{[y>\gamma]}, \quad \gamma > 0, \forall j = 1, \ldots, n, \quad (6.15)
\]

and \( N(t), t \geq 0, \) is an independent homogeneous Poisson process with parameter \( \theta = 1 \).

7. Acknowledgements

Thanks are due to the referee whose remarks and suggestions have certainly improved a previous draft of the paper.

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