The sixth Painlevé equation arising from $D_4^{(1)}$ hierarchy

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Abstract

The sixth Painlevé equation arises from a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ by similarity reduction.

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Introduction

The Drinfeld-Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy [DS]. It is known that their similarity reductions imply several Painlevé equations [AS, KK1, NY1]. For the sixth Painlevé equation ($P_{VI}$), the relation with the $A_2^{(1)}$-type hierarchy is investigated [KK2]. On the other hand, $P_{VI}$ admits a group of symmetries which is isomorphic to the affine Weyl group of type $D_4^{(1)}$ [O]. Also it is known that $P_{VI}$ is derived from the Lax pair associated with the algebra $\hat{so}(8)$ [NY3]. However, the relation between $D_4^{(1)}$-type hierarchies and $P_{VI}$ has not been clarified. In this paper, we show that the sixth Painlevé equation is derived from a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ by similarity reduction.

Consider a Fuchsian differential equation on $\mathbb{P}^1(\mathbb{C})$

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0, \quad (0.1)$$

with the Riemann scheme

$$\left\{ \begin{array}{cccccccc}
x = t_0 & x = t_1 & x = t_3 & x = t_4 & x = \lambda & x = \infty \\
0 & 0 & 0 & 0 & 0 & \rho \\
\theta_0 & \theta_1 & \theta_3 & \theta_4 & 2 & \rho + 1 
\end{array} \right\},$$
satisfying the relation
\[ \theta_0 + \theta_1 + \theta_3 + \theta_4 + 2\rho = 1. \]

We also let \( \mu = \text{Res}_{x=\lambda} p_2(x) \, dx \). Then the monodromy preserving deformation of the equation (0.1) is described as a system of partial differential equations for \( \lambda \) and \( \mu \). This system can be regarded as the symmetric representation of \( P_{VI} \) [Kaw]. We discuss a derivation of the symmetric representation in the case
\[
\begin{align*}
\theta_0 &= \alpha_0, & \theta_1 &= \alpha_1 - 1, & \theta_3 &= \alpha_3 - 1, & \theta_4 &= \alpha_4 - 1, & \rho &= \alpha_2.
\end{align*}
\]

Note that
\[ \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 4. \]

With the notation
\[
F_0 = \lambda + t, \quad F_1 = \lambda + \frac{t + 1}{t - 1}, \quad F_2 = \mu, \quad F_3 = \lambda - \frac{t - 1}{t + 1}, \quad F_4 = \lambda - \frac{1}{t},
\]
the dependence of \( \lambda \) and \( \mu \) on \( t \) is given by
\[
\vartheta(F_j) = 2F_0F_1F_2F_3F_4 - (\alpha_0 - 1)F_1F_3F_4 - (\alpha_1 - 1)F_0F_1F_4 - (\alpha_4 - 1)F_0F_1F_3 + \Theta_j,
\]
for \( j = 0, 1, 3, 4 \) and
\[
\vartheta(F_2) = -F_2^2(F_0F_1F_3 + F_0F_1F_4 + F_0F_3F_4 + F_1F_3F_4)
+ F_2\{(\alpha_3 + \alpha_4 - 2)F_0F_1 + (\alpha_1 + \alpha_4 - 2)F_0F_3 + (\alpha_1 + \alpha_3 - 2)F_0F_4
+ (\alpha_0 + \alpha_4 - 2)F_1F_3 + (\alpha_0 + \alpha_3 - 2)F_1F_4 + (\alpha_0 + \alpha_1 - 2)F_3F_4\}
- \alpha_2\{(\alpha_0 + \alpha_2 - 1)F_0 + (\alpha_1 + \alpha_2 - 1)F_1 + (\alpha_3 + \alpha_2 - 1)F_3
+ (\alpha_4 + \alpha_2 - 1)F_4\};
\]
where
\[ \vartheta = \Theta_0 \frac{d}{dt}, \quad \Theta_i = \prod_{j=0,1,3,4; j \neq i} (F_i - F_j). \]

Note that the system (0.2), (0.3) is equivalent to the Hamiltonian system:
\[
\begin{align*}
\frac{d\lambda}{dt} &= \frac{\partial H'}{\partial \mu}, & \frac{d\mu}{dt} &= -\frac{\partial H'}{\partial \lambda},
\end{align*}
\]
where the Hamiltonian $H' = H'(\lambda, \mu, t)$ is given by

\[
\Theta_0 H' = F_0 F_1 F_2^2 F_3 F_4 - (\alpha_0 - 1) F_1 F_2 F_3 F_4 - (\alpha_1 - 1) F_0 F_2 F_3 F_4 \\
- (\alpha_3 - 1) F_0 F_1 F_2 F_4 - (\alpha_4 - 1) F_0 F_1 F_2 F_3 + \alpha_2 F_0 \{(\alpha_0 - 1) F_0 \\
+ (\alpha_1 + \alpha_2 - 1) F_1 + (\alpha_3 + \alpha_2 - 1) F_3 + (\alpha_4 + \alpha_2 - 1) F_4 \}.
\]

We also remark that the system (0.4) is transformed into the Hamiltonian system for $P_{V_1}$ as in [IKSY]

\[
\frac{dq}{ds} = \frac{\partial H}{\partial p}, \quad \frac{dp}{ds} = -\frac{\partial H}{\partial q},
\]

with the Hamiltonian

\[
s(s - 1) H = q(q - 1)(q - s)p^2 - \frac{1}{4}\{((\alpha_1 - 4)q(q - 1) \\
+ \alpha_3 q(q - s) + \alpha_4 (q - 1)(q - s)\}p + \frac{1}{16}\alpha_2 (\alpha_0 + \alpha_2)q,
\]

by the canonical transformation $(\lambda, \mu, t, H') \to (q, p, s, H)$ defined as

\[
q = \frac{(t + \frac{t-1}{t+1})F_4}{(\frac{t-1}{t+1} - \frac{1}{t})F_0}, \quad p = \frac{(\frac{t-1}{t+1} - \frac{1}{t})F_0 (F_0 F_2 + \alpha_2)}{4(t + \frac{t-1}{t+1})(t + \frac{1}{t})},
\]

and

\[
s = -\frac{(t + \frac{t-1}{t+1})(\frac{t+1}{t} + \frac{1}{t})}{(t - \frac{t+1}{t+1})(\frac{t-1}{t+1} - \frac{1}{t})}.
\]

This paper is organized as follows. In Section 1, we recall the definition of the affine Lie algebra $g = g(D_4^{(1)})$. In Section 2, a Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$ is formulated. In Sections 3 and 4, we show that its similarity reduction implies the symmetric representation of $P_{V_1}$.

## 1 Affine Lie algebra

In the notation of [Kac], the affine Lie algebra $g = g(D_4^{(1)})$ is the Lie algebra generated by the Chevalley generators $e_i, f_i, \alpha_i^\vee (i = 0, \ldots, 4)$ and the scaling element $d$ with the fundamental relations

\[
(ad e_i)^{1-a_{ij}}(e_j) = 0, \quad (ad f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j), \]

\[
[a_i^\vee, a_j^\vee] = 0, \quad [a_i^\vee, e_j] = a_{ij} e_j, \quad [a_i^\vee, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{i,j} \alpha_i^\vee, \]

\[
[d, a_i^\vee] = 0, \quad [d, e_i] = \delta_{i,0} e_0, \quad [d, f_i] = -\delta_{i,0} f_0,
\]

}\]
for $i, j = 0, \ldots, 4$, where $A = (a_{ij})_{i,j=0}^4$ is the generalized Cartan matrix of type $D_4^{(1)}$ defined by

$$
A = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 \\
-1 & -1 & 2 & -1 & -1 \\
0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 2
\end{pmatrix}.
$$

We denote the Cartan subalgebra of $\mathfrak{g}$ by

$$
\mathfrak{h} = \bigoplus_{j=0}^4 \mathbb{C} \alpha_j^\vee \oplus \mathbb{C} d.
$$

The canonical central element of $\mathfrak{g}$ is given by

$$
K = \alpha^\vee_0 + \alpha^\vee_1 + 2\alpha^\vee_2 + \alpha^\vee_3 + \alpha^\vee_4.
$$

The normalized invariant form $(| \, |) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is determined by the conditions

$$
(a^\vee_i | a^\vee_j) = a_{ij}, \quad (e_i | f_j) = \delta_{i,j}, \quad (a^\vee_i | e_j) = (a^\vee_i | f_j) = 0,
$$

$$
(d | d) = 0, \quad (d | a^\vee_j) = \delta_{0,j}, \quad (d | e_j) = (d | f_j) = 0,
$$

for $i, j = 0, \ldots, 4$.

We consider the $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k(s)$ of type $s = (1, 1, 0, 1, 1)$ by setting

\begin{align*}
\deg \mathfrak{h} &= \deg e_2 = \deg f_2 = 0, \\
\deg e_i &= 1, \quad \deg f_i = -1 \quad (i = 0, 1, 3, 4).
\end{align*}

If we take an element $d_s \in \mathfrak{h}$ such that

$$
(d_s | \alpha^\vee_2) = 0, \quad (d_s | \alpha^\vee_j) = 1 \quad (j = 0, 1, 3, 4),
$$

this gradation is defined by

$$
\mathfrak{g}_k(s) = \{ x \in \mathfrak{g} \mid [d_s, x] = kx \} \quad (k \in \mathbb{Z}).
$$

In the following, we choose

$$
d_s = 4d + 2\alpha^\vee_1 + 3\alpha^\vee_2 + 2\alpha^\vee_3 + 2\alpha^\vee_4.
$$

We set

$$
\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k(s), \quad \mathfrak{g}_{\geq0} = \bigoplus_{k\geq0} \mathfrak{g}_k(s).
$$
We choose the graded Heisenberg subalgebra $s = \bigoplus_{k \in \mathbb{Z}} s_k(s)$ of $g$ of type $s = (1, 1, 0, 1, 1)$ with
\[ s_1(s) = \mathbb{C}\Lambda_{1,1} \oplus \mathbb{C}\Lambda_{1,2}, \]
where
\[
\Lambda_{1,1} = -e_0 + e_1 + e_3 - e_{21} + e_{23} + e_{24}, \quad \\
\Lambda_{1,2} = e_1 - e_3 + e_4 + e_{20} + e_{21} + e_{23}.
\]
Here we denote
\[ e_{2j} = [e_2, e_j], \quad f_{2j} = [f_2, f_j] \quad (j = 0, 1, 3, 4). \]
We remark that
\[ s = \{ x \in g \mid [\Lambda_{1,1}, x] \in \mathbb{C}K \}. \]
and
\[ s_0(s) = \mathbb{C}K, \quad s_{2k}(s) = 0 \quad (k \neq 0). \]
Each $s_{2k-1}(s)$ is expressed in the form
\[ s_{2k-1}(s) = \mathbb{C}\Lambda_{2k-1,1} \oplus \mathbb{C}\Lambda_{2k-1,2}, \]
with certain elements $\Lambda_{2k-1,i}$ $(i = 1, 2)$ satisfying
\[
[\Lambda_{2k-1,i}, \Lambda_{2l-1,j}] = (2k - 1)\delta_{i,j}\delta_{k+l,1}K \quad (i, j = 1, 2; k, l \in \mathbb{Z}).
\]
For $k = 0$, we have
\[
\Lambda_{-1,1} = \frac{1}{2}(-2f_0 + f_1 + f_3 + f_{21} - f_{23} - 2f_{24}), \quad \\
\Lambda_{-1,2} = \frac{1}{2}(f_1 - f_3 + 2f_4 - 2f_{20} - f_{21} - f_{23}).
\]

Remark 1.1. In the notation of \cite{C}, the Heisenberg subalgebra $s$ corresponds to the conjugacy class $D_4(a_1)$ of the Weyl group $W(D_4)$; see \cite{DF}.

2 Drinfeld-Sokolov hierarchy

In the following, we use the notation of infinite dimensional groups
\[ G_{<0} = \exp(\widehat{g}_{<0}), \quad G_{\geq 0} = \exp(\widehat{g}_{\geq 0}), \]
where $\widehat{g}_{<0}$ and $\widehat{g}_{\geq 0}$ are completions of $g_{<0}$ and $g_{\geq 0}$ respectively.
Introducing the time variables $t_{k,i}$ ($i = 1, 2; k = 1, 3, 5, \ldots$), we consider the Sato equation for a $G_{<0}$-valued function $W = W(t_{1,1}, t_{1,2}, \ldots)$

$$\partial_{k,i}(W) = B_{k,i}W - W\Lambda_{k,j} \quad (i = 1, 2; k = 1, 3, 5, \ldots), \quad (2.1)$$

where $\partial_{k,i} = \partial/\partial t_{k,i}$ and $B_{k,i}$ stands for the $g_{\geq 0}$-component of $W\Lambda_{k,i}W^{-1} \in \hat{g}_{<0} \oplus g_{\geq 0}$. We understand the Sato equation (2.1) as a conventional form of the differential equation

$$\partial_{k,i} - B_{k,i} = W(\partial_{k,i} - \Lambda_{k,i})W^{-1} \quad (i = 1, 2; k = 1, 3, 5, \ldots), \quad (2.2)$$

defined through the adjoint action of $G_{<0}$ on $\hat{g}_{<0} \oplus g_{\geq 0}$. The Zakharov-Shabat equation

$$[\partial_{k,i} - B_{k,i}, \partial_{j,l} - B_{j,l}] = 0 \quad (i, j = 1, 2; k, l = 1, 3, 5, \ldots), \quad (2.3)$$

follows from the Sato equation (2.2).

The $g_{\geq 0}$-valued functions $B_{1,i}$ ($i = 1, 2$) are expressed in the form

$$B_{1,i} = \Lambda_{1,i} + U_i, \quad U_i = \sum_{j=0}^{4} u_{j,i} \alpha_j^\vee + x_i e_2 + y_i f_2. \quad (2.4)$$

The Zakharov-Shabat equation (2.3) for $k = 1$ is equivalent to

$$\partial_{1,i}(U_j) - \partial_{1,j}(U_i) + [U_j, U_i] = 0, \quad [\Lambda_{1,i}, U_j] - [\Lambda_{1,j}, U_i] = 0, \quad (2.5)$$

for $i, j = 1, 2$. Then we have

**Lemma 2.1.** Under the Sato equation (2.2), the following equations are satisfied:

$$(d_s|\partial_{1,i}(U_j)) + \frac{1}{2}(U_i|U_j) = 0 \quad (i, j = 1, 2). \quad (2.6)$$

**Proof.** The system (2.2) for $k = 1$ is equivalent to

$$\partial_{1,i} - \Lambda_{1,i} - U_i = W(\partial_{1,i} - \Lambda_{1,i})W^{-1} \quad (i = 1, 2). \quad (2.7)$$

Set

$$W = \exp(w), \quad w = \sum_{k=1}^{\infty} w_{-k}, \quad w_{-k} \in g_{-k}(s).$$

Then the system (2.7) implies

$$U_i = \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}(w)^{k-1}\partial_{1,i}(w) + \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}(w)^k(\Lambda_{1,i}) \quad (i = 1, 2). \quad (2.8)$$
Comparing the component of degree \(-k\) in (2.8), we obtain

\[ U_i = \text{ad}(w_{-1})(\Lambda_{1,i}) \quad (i = 1, 2), \]

for \(k = 0\);

\[ \text{ad}(w_{-2})(\Lambda_{1,i}) + \frac{1}{2}\text{ad}(w_{-1})^2(\Lambda_{1,i}) + \partial_{1,i}(w_{-1}) = 0 \quad (i = 1, 2), \quad (2.9) \]

for \(k = 1\);

\[
\sum_{i_1 + \ldots + i_\ell = k+1} \frac{1}{\ell!}\text{ad}(w_{-1})\ldots\text{ad}(w_{-i_\ell})(\Lambda_{1,i}) \\
+ \sum_{i_1 + \ldots + i_\ell = k} \frac{1}{\ell!}\text{ad}(w_{-i_1})\ldots\text{ad}(w_{-i_\ell})\partial_{1,i}(w_{-i_\ell}) = 0 \quad (i = 1, 2),
\]

for \(k \geq 2\). On the other hand, we have

\[ (\Lambda_{1,i}|\text{ad}(\Lambda_{1,j})(x)) = 0 \quad (i, j = 1, 2; x \in g_{-2}(s)), \]

and

\[ (\Lambda_{1,i}|x) = (d_s|\text{ad}(\Lambda_{1,i})(x)) \quad (i = 1, 2; x \in g_{-1}(s)). \]

Hence it follows that

\[
(\Lambda_{1,j}|\text{LHS of (2.9)}) = \frac{1}{2}(\Lambda_{1,j}|\text{ad}(w_{-1})^2(\Lambda_{1,i})) + (\Lambda_{1,j}|\partial_{1,i}(w_{-1})) \\
= -\frac{1}{2}(U_i|U_j) - (d_s|\partial_{1,i}(U_j)).
\]

\[ \square \]

**Remark 2.2.** Let \(X(0) \in G_{<0}G_{\geq 0}\) and define

\[ X = X(t_{1,1}, t_{1,2}, \ldots) = \exp(\xi)X(0), \quad \xi = \sum_{i=1,2} \sum_{k=1,3} t_{k,i}\Lambda_{k,i}. \]

Then a solution \(W \in G_{<0}\) of the system (2.1) is given formally via the decomposition

\[ X = W^{-1}Z, \quad Z \in G_{\geq 0}. \]
3 Similarity reduction

Under the Sato equation (2.2), we consider the operator
\[ \mathcal{M} = W \exp(\xi) d_s \exp(-\xi) W^{-1}, \quad \xi = \sum_{i=1,2} \sum_{k=1,3,...} t_{k,i} \Lambda_{k,i}. \]

Then the operator \( \mathcal{M} \) satisfies
\[ \partial_{k,i}(\mathcal{M}) = [B_{k,i}, \mathcal{M}] \quad (i = 1, 2; k = 1, 3, 5, \ldots). \]

Note that
\[ \mathcal{M} = d_s - \sum_{i=1,2} \sum_{k=1,3,...} kt_{k,i} W \Lambda_{k,i} W^{-1} - d_s(W) W^{-1}. \]

Assuming that \( t_{k,1} = t_{k,2} = 0 \) for \( k \geq 3 \), we require that the similarity condition \( \mathcal{M} \in g_{\geq 0} \) is satisfied. Then we have
\[ \partial_{1,i}(\mathcal{M}) = [B_{1,i}, \mathcal{M}] \quad (i = 1, 2). \]

where \( \mathcal{M} = d_s - t_{1,1} B_{1,1} - t_{1,2} B_{1,2} \), or equivalently
\[ [d_s - M, \partial_{1,i} - B_{1,i}] = 0 \quad (i = 1, 2), \quad (3.1) \]

where \( M = t_{1,1} B_{1,1} + t_{1,2} B_{1,2} \). Under the Zakharov-Shabat equation
\[ [\partial_{1,1} - B_{1,1}, \partial_{1,2} - B_{1,2}] = 0, \]

the system (3.1) is equivalent to
\[ \sum_{j=1,2} t_{1,j} \partial_{1,j}(B_{1,i}) = [d_s, B_{1,i}] - B_{1,i} \quad (i = 1, 2). \]

In terms of the operators \( U_i \), this similarity condition can be expressed as
\[ \sum_{j=1,2} t_{1,j} \partial_{1,j}(U_i) + U_i = 0 \quad (i = 1, 2). \quad (3.2) \]

We regard the systems (2.5), (2.6) and (3.2) as a similarity reduction of the Drinfeld-Sokolov hierarchy of type \( D_{1}^{(1)} \).

In the notation (2.4), these systems are expressed in terms of the variables \( u_{j,i}, x_i, y_i \) as follows:
\[ \partial_{1,1}(x_2) - \partial_{1,2}(x_1) - (u_{1,1} - u_{3,1} - u_{0,2} + u_{4,2})x_1 + (u_{0,1} - u_{4,1} + u_{1,2} - u_{3,2})x_2 = 0, \]
\[ \partial_{1,1}(y_2) - \partial_{1,2}(y_1) + (u_{1,1} - u_{3,1} - u_{0,2} + u_{4,2})y_1 - (u_{0,1} - u_{4,1} + u_{1,2} - u_{3,2})y_2 = 0, \]
\[ \partial_{1,1}(w_{2,2}) - \partial_{1,2}(w_{2,1}) - x_1 y_2 + x_2 y_1 = 0, \]
\[ \partial_{1,1}(u_{j,2}) - \partial_{1,2}(u_{j,1}) = 0 \quad (j = 0, 1, 3, 4), \]
and

\begin{align*}
  u_{1,1} - 2u_{2,1} + u_{3,1} + 2u_{4,1} - u_{1,2} + u_{3,2} &= 0, \\
  u_{1,1} - u_{3,1} - 2u_{0,2} - u_{1,2} + 2u_{2,2} - u_{3,2} &= 0, \\
  u_{1,1} - u_{3,1} + u_{1,2} + u_{3,2} - 2u_{4,2} + 2x_1 &= 0, \\
  2u_{0,1} - u_{1,1} - u_{3,1} - u_{1,2} + u_{3,2} + 2x_2 &= 0, \\
  u_{1,1} - u_{3,1} + 2u_{0,2} - u_{1,2} - u_{3,2} + 2y_1 &= 0, \\
  u_{1,1} + u_{3,1} - 2u_{4,1} - u_{1,2} + u_{3,2} + 2y_2 &= 0,
\end{align*}

(3.3)

for the system (2.5):

\[
\begin{align*}
  \sum_{l=0,1,3,4} 4\partial_{l,i}(u_{l,j}) \\
  + \sum_{l=0,1,3,4} (2u_{l,i} - u_{2,i})(2u_{l,j} - u_{2,j}) + 2(x_iy_j + y_ix_j) &= 0 \quad (i, j = 1, 2),
\end{align*}
\]

for the system (2.6):

\[
\begin{align*}
  t_{1,1}\partial_{1,1}(x_i) + t_{1,2}\partial_{1,2}(x_i) + x_i &= 0, \\
  t_{1,1}\partial_{1,1}(y_i) + t_{1,2}\partial_{1,2}(y_i) + y_i &= 0, \\
  t_{1,1}\partial_{1,1}(u_{j,i}) + t_{1,2}\partial_{1,2}(u_{j,i}) + u_{j,i} &= 0, \quad (i = 1, 2; j = 0, \ldots, 4),
\end{align*}
\]

for the system (3.2). In the next section, we show that they imply the sixth Painlevé equation.

Under the similarity condition (3.2), the system (2.6) implies

\[
2(d_s|U_i) - t_{1,1}(U_i|U_1) - t_{1,2}(U_i|U_2) = 0 \quad (i = 1, 2).
\]

It is expressed in terms of the variables \(u_{j,i}, x_i, y_i\) as follows:

\[
\begin{align*}
  &\sum_{l=0,1,3,4} 4u_{l,i} - \sum_{l=0,1,3,4} t_{1,1}(2u_{l,i} - u_{2,i})(2u_{l,1} - u_{2,1}) - 2t_{1,1}(x_iy_1 + y_ix_1) \\
  &- \sum_{l=0,1,3,4} t_{1,2}(2u_{l,i} - u_{2,i})(2u_{l,2} - u_{2,2}) - 2t_{1,2}(x_2y_2 + y_2x_2) = 0 \quad (i = 1, 2).
\end{align*}
\]

(3.4)

**Remark 3.1.** The systems (2.5) and (3.2) can be regarded as the compatibility condition of the Lax form

\[
d_s(\Psi) = M\Psi, \quad \partial_{1,i}(\Psi) = B_{1,i}\Psi \quad (i = 1, 2),
\]

(3.5)

where \(\Psi = W \exp(\xi)\).
4 The sixth Painlevé equation

In the previous section, we have derived the system of the equations

\[ \partial_{i,j}(U_j) - \partial_{j,i}(U_i) + [U_j, U_i] = 0, \quad [\Lambda_{1,i}, U_j] - [\Lambda_{1,j}, U_i] = 0, \]
\[ (d_s \partial_{i,j}(U_j)) - \frac{1}{2}(U_j | U_j) = 0, \quad \sum_{i=1,2} t_{1,i} \partial_{i,j}(U_i) + U_i = 0 \quad (i, j = 1, 2), \]  

(4.1)

for the \( g_0 \)-valued functions \( U_i = U_i(t_{1,1}, t_{1,2}) \) (\( i = 1, 2 \)), as a similarity reduction of the \( D_4^{(1)} \) hierarchy of type \( s = (1, 1, 0, 1, 1) \). In terms of the operators \( B_{1,i} = \Lambda_{1,i} + U_i \) and \( M = t_{1,1} B_{1,1} + t_{1,2} B_{1,2} \), the system (4.1) is expressed as

\[ [\partial_{1,1} - B_{1,1}, \partial_{1,2} - B_{1,2}] = 0, \quad [d_s - M, \partial_{1,i} - B_{1,i}] = 0 \quad (i = 1, 2), \]

with the equations for normalization (2.6). In this section, we show that the sixth Painlevé equation is derived from them.

The operator \( M \) is expressed in the form

\[ M = \sum_{i=1,2} t_{1,i} \Lambda_{1,i} + \sum_{j=0,1,3,4} \kappa_j \alpha_j^\vee + \eta \alpha_2^\vee + \varphi \psi_2 + \psi f_2, \]

so that

\[ \kappa_j = t_{1,1} u_{j,1} + t_{1,2} u_{j,2} \quad (j = 0, 1, 3, 4), \quad \eta = t_{1,1} u_{2,1} + t_{1,2} u_{2,2}, \]
\[ \varphi = t_{1,1} x_1 + t_{1,2} x_2, \quad \psi = t_{1,1} y_1 + t_{1,2} y_2. \]  

(4.2)

The system (3.1) implies that the variables \( \kappa_j \) (\( j = 0, 1, 3, 4 \)) are independent of \( t_{1,i} \) (\( i = 1, 2 \)). Then the following lemma is obtained from (3.3), (3.1) and (1.2).

**Lemma 4.1.** The variables \( u_{j,i}, x_i, y_i \) (\( i = 1, 2; j = 0, \ldots, 4 \)) are determined uniquely as polynomials in \( \eta, \varphi \) and \( \psi \) with coefficients in \( \mathbb{C}(t_{1,i})[\kappa_j] \). Furthermore, the following relation is satisfied:

\[ \eta^2 - (\kappa_0 + \kappa_1 + \kappa_3 + \kappa_4)(\eta + 1) + \kappa_0^2 + \kappa_1^2 + \kappa_3^2 + \kappa_4^2 + \varphi \psi = 0. \]

Thanks to this lemma, the system (4.1) can be rewritten into a system of first order differential equations for \( \eta \) and \( \varphi \); we do not give the explicit formulas here.

We denote by \( n_+ \) the subalgebra of \( g \) generated by \( e_j \) (\( j = 0, \ldots, 4 \)), and by \( b_+ \) the borel subalgebra of \( g \) defined by \( b_+ = h \oplus n_+ \). We look for a dependent variable \( \lambda \) such that

\[ \tilde{M} = \exp(-\lambda f_2)M \exp(\lambda f_2) - \exp(-\lambda f_2)d_s(\exp(\lambda f_2)) \in b_+, \]
\[ \tilde{B}_{1,i} = \exp(-\lambda f_2)B_{1,i} \exp(\lambda f_2) - \exp(-\lambda f_2)\partial_{1,i}(\exp(\lambda f_2)) \in b_+ \quad (i = 1, 2), \]

(4.3)
namely
\[ \varphi \lambda^2 + (2\eta - \kappa_0 - \kappa_1 - \kappa_3 - \kappa_4)\lambda - \psi = 0, \]
\[ \partial_{1,i}(\lambda) + x_i\lambda^2 - (u_{0,i} + u_{1,i} - 2u_{2,i} + u_{3,i} + u_{4,i})\lambda - y_i = 0 \quad (i = 1, 2). \quad (4.3) \]

Note that the definition of \( \tilde{M} \) and \( \tilde{B}_{1,i} \) arises from the gauge transformation \( \Psi \to \Phi \) defined by \( \Phi = \exp(-\lambda f_2)\Psi \) on the Lax form (3.5). By Lemma 4.1 together with the system (4.1), we can show that
\[ \lambda = -\frac{1}{8\varphi} (8\eta - \alpha_0^2 - \alpha_1^2 - \alpha_3^2 - \alpha_4^2 + 4), \]

satisfies the equation (4.3), where \( \alpha_j \) \((j = 0, 1, 3, 4)\) are constants defined by
\[ \kappa_j = -\frac{1}{16} (8\alpha_j - \alpha_0^2 - \alpha_1^2 - \alpha_3^2 - \alpha_4^2 - 4). \]

We also let \( \mu \) by a dependent variable defined by \( \mu = \varphi \) so that
\[ \eta = -\lambda \mu + \frac{1}{8} (\alpha_0^2 + \alpha_1^2 + \alpha_3^2 + \alpha_4^2 - 4), \quad \varphi = \mu. \]

Then the system (4.1) can be regarded as a system of differential equations for variables \( \lambda \) and \( \mu \) with parameters \( \alpha_j \) \((j = 0, 1, 3, 4)\).

We now regard the system (4.1) as a system of ordinary differential equations with respect to the independent variable \( t = t_{1,1} \) by setting \( t_{1,2} = 1 \).

Then the operator \( \tilde{M} \) is written in the form
\[ \tilde{M} = \frac{1}{16} (\alpha_0^2 + \alpha_1^2 + \alpha_3^2 + \alpha_4^2 - 4)K - \sum_{j=0,1,3,4} \frac{1}{2} (\alpha_j - 1)\alpha_j^\gamma \]
\[ + F_2 e_2 - F_0 e_0 + (t - 1)F_1 e_1 - (t + 1)F_3 e_3 - tF_4 e_4 \]
\[ + e_{20} - (t - 1)e_{21} + (t + 1)e_{23} + te_{24}, \]
where
\[ F_0 = \lambda + t, \quad F_1 = \lambda + \frac{t + 1}{t - 1}, \quad F_2 = \mu, \quad F_3 = \lambda - \frac{t - 1}{t + 1}, \quad F_4 = \lambda - \frac{1}{t}. \]

The operator \( \tilde{B} = \tilde{B}_{1,1} \) is written in the form
\[ \tilde{B} = \tilde{u}_2 K + \sum_{j=0,1,3,4} \tilde{u}_j \alpha_j^\gamma + \tilde{x} e_2 \]
\[ - e_0 + (\lambda + 1)e_1 - (\lambda - 1)e_3 - \lambda e_4 - e_{21} + e_{23} + e_{24}, \]
where $\tilde{u}_2$ is a polynomial in $\lambda$, $\mu$ and the other coefficients are given by

$$
\Theta_0 \tilde{u}_j = F_0 F_1 F_2 F_3 F_4 F_j^{-1} - \sum_{i=0,1,3,4; \ i \neq j} \frac{1}{2} (\alpha_i + \alpha_j - 2) F_0 F_1 F_3 F_4 F_i^{-1} F_j^{-1} - \frac{1}{2} (\alpha_j - 1) F_0 (F_0 - F_1 - F_3 - F_4) \quad (j = 0, 1, 3, 4),
$$

$$
\Theta_0 \tilde{x} = F_0 F_2 (F_0 - F_1 - F_3 - F_4) + (\alpha_0 + \alpha_2 - 1) F_0 + (\alpha_1 + \alpha_2 - 1) F_1 + (\alpha_3 + \alpha_2 - 1) F_3 + (\alpha_4 + \alpha_2 - 1) F_4,
$$

with

$$
\Theta_0 = (F_0 - F_1)(F_0 - F_3)(F_0 - F_4), \quad \alpha_2 = -\frac{1}{2} (\alpha_0 + \alpha_1 + \alpha_3 + \alpha_4 - 1).
$$

Since $\tilde{M}$ and $\tilde{B}$ is obtained from $M$ and $B_{1,1}$ by the gauge transformation, they satisfy

$$
\left[ ds - \tilde{M}, \frac{d}{dt} - \tilde{B} \right] = 0.
$$

By rewriting this compatibility condition into differential equations for $F_j$ ($j = 0, \ldots, 4$), we obtain the same system as (0.2), (0.3).

**Theorem 4.2.** Under the specialization $t_{1,1} = t$ and $t_{1,2} = 1$, the system (4.1) is equivalent to the sixth Painlevé equation (0.2), (0.3).

**Remark 4.3.** The system (0.2), (0.3) can be regarded as the compatibility condition of the Lax pair

$$
ds(\Phi) = \tilde{M} \Phi, \quad \frac{d\Phi}{dt} = \tilde{B} \Phi,
$$

where $\Phi = \exp(-\lambda f_2) W \exp(\xi)$. Let

$$
\Omega = \exp(\omega_1 \alpha_1 + \omega_2 \alpha_2 + \omega_3 \alpha_3 + \omega_4 \alpha_4) \exp(F_0^{-1} e_2) \Phi,
$$

where

$$
\omega_1 = \frac{1}{2} \log(t^2 + 2t - 1)(t^2 + 1), \quad \omega_2 = \log F_0,
$$

$$
\omega_3 = \frac{1}{2} \log(1 + 2t - t^2)(t^2 + 1), \quad \omega_4 = \frac{1}{2} \log(1 + 2t - t^2)(t^2 + 2t - 1).
$$

Then the system (4.1) is transformed into the Lax pair of the type of [NY3] by the gauge transformation $\Phi \to \Omega$. 

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Finally, we define the group of symmetries for $P_{VI}$ following [NY2]. Consider the transformations

$$r_i(X) = X \exp(-e_i) \exp(f_i) \exp(-e_i) \quad (i = 0, \ldots, 4),$$

where

$$X = \exp(\xi)X(0) = W^{-1}Z, \quad \xi = \sum_{i=1,2} \sum_{k=1,3} t_{k,i} \Lambda_{k,i}.$$  

Under the similarity condition $M \in \mathfrak{g}_{\geq 0}$, their action on $W$ is given by

$$r_i(W) = \exp(\lambda f_2) \exp\left( \frac{(\alpha_i^\vee|d_s - \tilde{M})}{(f_i|d_s - \tilde{M})} f_i \right) \exp(-\lambda f_2)W \quad (i = 0, 1, 3, 4),$$

$$r_2(W) = W.$$  

We also define

$$r_i(\alpha_j) = \alpha_j - \alpha_i a_{ij} \quad (i, j = 0, \ldots, 4).$$

Then the action of them on the variables $\lambda, \mu$ is described as

$$r_i(F_j) = F_j - \frac{\alpha_i}{F_i} u_{ij} \quad (i, j = 0, \ldots, 4),$$

where $U = (u_{ij})_{i,j=0}^4$ is the orientation matrix of the Dynkin diagram defined by

$$U = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-1 & -1 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$  

Note that the transformations $r_i$ ($i = 0, \ldots, 4$) satisfy the fundamental relations for the generators of the affine Weyl group $W(D^{(1)}_4)$.

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