Casimir effect of a Lorentz-violating scalar in magnetic field

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In this paper I study the Casimir effect caused by a charged and massive scalar field that breaks Lorentz invariance in a CPT-even, aether-like manner. The scalar field satisfies Dirichlet boundary conditions on a pair of very large plane parallel plates. I use the \( \zeta \)-function regularization technique to study the effect of a constant magnetic field, orthogonal to the plates, on the Casimir energy and pressure. I investigate the cases of a timelike Lorentz asymmetry, a spacelike Lorentz asymmetry in the direction perpendicular to the plates, and a spacelike asymmetry in the plane of the plates and, in all those cases, derive simple analytic expressions for the zeta function, Casimir energy and pressure in the limits of small plate distance, strong magnetic field and large scalar field mass. I discover that the Casimir energy and pressure, and their magnetic corrections, all strongly depend on the direction of the unit vector that implements the breaking of the Lorentz symmetry.

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I. INTRODUCTION

The first theoretical prediction of an attractive force of a purely quantum nature between two uncharged and conducting parallel plates in vacuum is due to Hendrik Casimir [1]. His prediction was confirmed by sophisticated experiments ten years later [2], with many and more accurate experimental verifications following them [3, 4]. The Casimir effect strongly depends on the boundary conditions at the plates of the quantum field under consideration. Dirichlet or Neumann boundary conditions cause an attraction between the plates, mixed (Dirichlet-Neumann) boundary conditions cause a repulsive force [5].

While in standard quantum field theory Lorentz invariance is strictly obeyed, other theories propose models where a violation of Lorentz symmetry leads to a space-time anisotropy [6, 7]. Mechanisms for breaking Lorentz symmetry have been proposed in modified quantum gravity [8, 9], in theories that propose variation of some coupling constants [10–12], and in string theory [13], where non-vanishing expectation values of some vector and tensor fields components lead to a spontaneous violation of Lorentz invariance at the Planck energy scale. A comprehensive list of papers that study various consequences of Lorentz symmetry breaking is available in the work by Cruz et al. [14]. Implications of a Lorentz asymmetry in the Casimir effect have been studied in the case of Lorentz-breaking extensions of QED [15–17], and in the case of a real scalar field in vacuum [14] and in a medium at finite temperature [18]. Both of these papers investigate a modified Klein-Gordon model that breaks Lorentz symmetry in a CPT-even, aether-like manner.

While several authors have studied magnetic corrections to the Casimir effect in a symmetric spacetime with unbroken Lorentz symmetry [19, 21], there has not been a study of magnetic corrections to the Casimir effect of a charged scalar field that breaks the Lorentz symmetry. This work intends to fill that gap and provide theoretical predictions of the magnetic field effects on the quantum vacuum of the modified Klein-Gordon model introduced in Ref. [14]. In this paper I will investigate the effect of a uniform magnetic field \( \vec{B} \) on the Casimir energy and pressure due to a Lorentz-violating scalar field, by studying a model similar to the one first presented in Ref. [14]: a charged scalar field that breaks Lorentz symmetry and satisfies Dirichlet boundary conditions on a pair of very large parallel plates at distance \( a \) from each other.

All Casimir effect calculations require a method to regularize the energy of the vacuum between the plates, because a naive calculation of the vacuum energy, without regularization, will produce an infinite result. While Ref. [14] uses the Abel-Plana method to regularize the Casimir energy, in this study I will use the \( \zeta \) function regularization technique [22, 23], a modern method also used extensively within the framework of finite temperature field theory [24, 26].

In Sec. [I] of this paper I briefly describe the theoretical model of a charged scalar field that breaks Lorentz symmetry in an aether-like and CPT-even manner, and calculate its \( \zeta \) function. In Sec. [II] I examine the case of a timelike Lorentz asymmetry and calculate the Casimir energy, obtaining simple analytic expressions for the energy in the short plate distance limit, large magnetic field limit, and large mass limit. In Sec. [IV] I investigate the case of a spacelike
Lorentz anisotropy in the same plane as the plates, calculate the Casimir energy, and obtain simple expressions for it in the three limits listed above. In Sec. VII I focus on a spacelike Lorentz anisotropy perpendicular to the plates, calculate the Casimir energy, and obtain simple expressions for it in the aforementioned three limits. In Sec. VII I calculate the Casimir pressure for all the cases described above. My conclusions, along with a discussion of my results are presented in Sec. VII.

II. THE MODEL AND ITS ZETA FUNCTION

I will investigate the Casimir effect due to a charged scalar field of mass \( m \) that breaks the Lorentz symmetry, using the theoretical model of a scalar field that produces an aether-like and CPT-even Lorentz symmetry breaking presented in Ref. [14]. In this model, the Lorentz symmetry violation is caused by a coupling of the derivative of the scalar field to a fixed unit four-vector \( u^\mu \). The Klein Gordon equation for this field is

\[
\Box + \lambda (u \cdot \partial)^2 + m^2 \phi = 0,
\]

where the dimensionless parameter \( \lambda \ll 1 \), and the unit four-vector \( u^\mu \) indicates the space-time direction in which the Lorentz symmetry is broken. My goal is to study how the space-time anisotropy and the presence of a magnetic field modify the Casimir effect. I am considering two square plates of side \( L \) perpendicular to the \( z \) axis, and a constant magnetic field \( \vec{B} \) pointing in the \( z \) direction. The two plates are located at \( z = 0 \) and \( z = a \). I will use the generalized zeta function technique to study this problem, and impose Dirichlet boundary conditions at the plates. I will study the cases when the unit four-vector \( u^\mu \) is timelike, spacelike and perpendicular to \( \vec{B} \), and spacelike and parallel to \( \vec{B} \).

I use Euclidean time \( \tau \) and begin by writing the eigenvalues of the Klein Gordon operator \( D_E \) of Eq. (1)

\[
D_E = \frac{\partial^2}{\partial \tau^2} - \nabla^2 + \lambda (u \cdot \partial)^2 + m^2,
\]

notice that at this initial stage the magnetic field is not present and I will introduce it later. When \( u^\mu \) is timelike in the \( x-y \) plane, I take it as \( u^\mu = \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \) and the eigenvalues of \( D_E \) are:

\[
\left\{ (1 - \lambda)k_0^2 + k_x^2 + k_y^2 + \left( \frac{n\pi}{a} \right)^2 + m^2 \right\},
\]

where \( k_0, k_x, k_y \in \mathbb{R} \) and \( n = 0, 1, 2, \ldots \). When \( u^\mu \) is spacelike and in the \( x-y \) plane, I take it as \( u^\mu = \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \) and the eigenvalues of \( D_E \) are:

\[
\left\{ k_0^2 + \left( 1 - \frac{\lambda}{2} \right) (k_x^2 + k_y^2) + \left( \frac{n\pi}{a} \right)^2 + m^2 \right\}.
\]

When \( u^\mu \) is in the \( z \) direction, I find the following eigenvalues of \( D_E \):

\[
\left\{ k_0^2 + k_x^2 + k_y^2 + (1 - \lambda) \left( \frac{n\pi}{a} \right)^2 + m^2 \right\}.
\]

Now I introduce the magnetic field, and its presence modifies the eigenvalues of Eqs. [3] [4] [5] by changing \( k_x^2 + k_y^2 \) into \( 2eB(\ell + \frac{1}{2}) \), where \( e \) is the charge of the scalar field and \( \ell = 0, 1, 2, \ldots \), labels the Landau levels. At this point, I can write the generalized zeta function, with the magnetic field contribution, for the three aforementioned cases of \( u^\mu \). When \( u^\mu \) is timelike,

\[
\zeta(s) = \mu^{2s} \frac{L^2 eB}{2\pi} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left( (1 - \lambda)k_0^2 + (2\ell + 1)eB + \left( \frac{n\pi}{a} \right)^2 + m^2 \right)^{-s},
\]

where the parameter \( \mu \) with dimension of mass keeps \( \zeta(s) \) dimensionless for all values of \( s \) and \( \frac{L^2 eB}{2\pi} \) takes into account the degeneracy of the Landau levels. When \( u^\mu \) is spacelike and perpendicular to \( \vec{B} \),

\[
\zeta(s) = \mu^{2s} \frac{L^2 eB}{2\pi} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left( k_0^2 + (1 - \frac{\lambda}{2}) (2\ell + 1)eB + \left( \frac{n\pi}{a} \right)^2 + m^2 \right)^{-s}.
\]
When \( u^\mu \) is spacelike and parallel to \( \vec{B} \),
\[
\zeta(s) = \mu^{2s} \frac{L^2 eB}{2\pi} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \left[ k_0^2 + (2\ell + 1)eB + (1 - \lambda) \left( \frac{n\pi}{a} \right)^2 + m^2 \right]^{-s}. \tag{8}
\]

In the next three sections I will evaluate the zeta function and obtain the Casimir energy for the three cases of timelike Lorentz anisotropy, spacelike anisotropy in the \( x-y \) plane, and spacelike anisotropy in the \( z \) direction.

### III. TIMELIKE ANISOTROPY

I change the variable of integration in the \( k_0 \)-integral of Eq. (9), and use the identity
\[
z^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-zt}dt,
\]
where \( \Gamma(s) \) is the Euler gamma function, to obtain
\[
\zeta(s) = \frac{\mu^{2s} L^2}{\Gamma(s)} \frac{eB}{4\pi^2} \sqrt{1 - \lambda} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int_{0}^{\infty} t^{s-1}e^{-[k_0^2 + (2\ell + 1)eB + (1 - \lambda) \left( \frac{n\pi}{a} \right)^2 + m^2]t}dt.
\tag{10}
\]

I do the \( k_0 \) integration and use
\[
\sum_{\ell=0}^{\infty} e^{-(2\ell + 1)ct} = \frac{1}{\sinh(\epsilon Bt)}, \tag{11}
\]
to find
\[
\zeta(s) = \frac{\mu^{2s} L^2}{\Gamma(s)} \frac{eB}{8\pi^2} \sqrt{1 - \lambda} \sum_{n=0}^{\infty} \int_{0}^{\infty} t^{s-3/2}e^{-[\frac{n\pi}{a}]^2 + m^2]t} \csc(\epsilon Bt)dt.
\tag{12}
\]

This zeta function cannot be evaluated in a closed form in the general case, but can be reduced to a very simple form in some limiting cases. Notice that the term with \( n = 0 \) is independent of \( a \), does not contribute to the measurable Casimir pressure and therefore can be neglected.

In the short plate distance limit, \( a^{-1} \gg \sqrt{\epsilon B} \) and \( a^{-1} \gg m \), I take \( e^{-m^2t} \approx 1 - m^2t \) and
\[
eB \csc(\epsilon Bt) \approx \frac{1}{t} - \frac{e^2B^2}{6t}, \tag{13}
\]
to obtain
\[
\zeta(s) = \frac{\mu^{2s} L^2}{\Gamma(s)} \frac{1}{8\pi^2} \frac{\sqrt{1 - \lambda}}{\sqrt{1 - \lambda}} \sum_{n=1}^{\infty} \int_{0}^{\infty} t^{s-5/2}e^{-\frac{n^2\pi^2}{a^2}t} \left( 1 - \frac{e^2B^2}{6t^2} \right)(1 - m^2t)dt. \tag{14}
\]

After I do the integration and neglect a smaller term proportional to \( e^2B^2m^2 \), I find
\[
\zeta(s) = \frac{\mu^{2s} L^2}{\Gamma(s)} \frac{1}{8\pi^2} \left( \frac{\pi}{a} \right)^{3-2s} \left[ \Gamma(s - \frac{3}{2})\zeta_R(2s - 3) - \frac{m^2 a^2}{\pi^2} \Gamma(s - \frac{1}{2})\zeta_R(2s - 1) - \frac{e^2B^2a^4}{64\pi^4} \Gamma(s + \frac{1}{2})\zeta_R(2s + 1) \right], \tag{15}
\]
where \( \zeta_R(s) \) is the Riemann zeta function. In the case of a complex scalar field the Casimir energy, \( E_C \), is related to the derivative of \( \zeta(s) \) by
\[
E_C = -\zeta'(0). \tag{16}
\]

Using the following power series expansions, valid for \( s \ll 1 \),
\[
\left( \frac{\mu a}{\pi} \right)^{2s} \frac{\Gamma(s - \frac{3}{2})}{\Gamma(s)} \zeta_R(2s - 3) \approx \frac{\sqrt{\pi}}{90}s + O(s^2), \tag{17}
\]
\[
\left(\frac{\mu a}{\pi}\right)^{2s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta_R(2s - 1) \simeq \frac{\sqrt{\pi}}{6} s + \mathcal{O}(s^2),
\]
(18)

\[
\left(\frac{\mu a}{\pi}\right)^{2s} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)} \zeta_R(2s + 1) \simeq \frac{\sqrt{\pi}}{2} + \sqrt{\pi} \left[ \gamma_E + \ln \left(\frac{\mu a}{2\pi}\right)\right] s + \mathcal{O}(s^2),
\]
(19)

where \(\gamma_E = 0.57721\ldots\) is the Euler-Mascheroni constant, I obtain the Casimir energy in the short plate distance limit

\[
E_C = -\frac{\pi^2 L^2}{8} \frac{a}{\alpha^3} (1 - \lambda)^{-\frac{1}{2}} \left(\frac{1}{90} - \frac{m^2 \alpha^2}{6\pi^2} - \frac{e^2 B^2 \alpha^4}{6\pi^4} \left[ \gamma_E + \ln \left(\frac{\mu a}{2\pi}\right)\right]\right),
\]
(20)

where I need to fix the parameter \(\mu\) and will make the obvious choice \(\mu = \max\{m, \sqrt{eB}\}\). The leading order term of Eq. (20) agrees with the leading term calculated by Cruz et al. [14] in the short plate distance limit, since \((1 - \lambda)^{-\frac{1}{2}} \simeq (1 + \lambda)^{\frac{1}{2}}\), when \(\lambda \ll 1\). Notice that my result, when compared to that of Ref. [14], carries an extra factor of two because I am considering a complex scalar field, not a real one. Finally, Eq. (20) shows that the timelike anisotropy causes a slight increase of the magnetic field correction to \(E_C\), when compared to the magnetic correction to \(E_C\) in isotropic spacetime.

In the strong magnetic field limit, \(\sqrt{eB} \gg a^{-1}, m\). Therefore, in Eq. (12), I can approximate \(\text{csch}(eBt) \simeq 2e^{-eBt}\) and do a Poisson resummation of the series to find

\[
\zeta(s) = \frac{\mu^{2s}}{\Gamma(s)} \frac{eBL^2}{4\pi^2} \frac{a}{\sqrt{1 - \lambda}} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-2} e^{-\frac{m^2 a^2}{t}} e^{-(eB+m^2) t} dt,
\]
(21)

where, again, I neglected the \(n = 0\) term which is easily evaluated to be

\[
\zeta_0(s) = \frac{L^2 a (eB + m^2) eB}{8\pi^2 \sqrt{1 - \lambda} \left(\frac{a}{\sqrt{eB + m^2}}\right)^s} \frac{\Gamma(s - 1)}{\Gamma(s)},
\]
(22)

and produces a uniform energy density term that does not contribute to the Casimir pressure as long as the magnetic field is present inside and outside the plates [27]. I change integration variable in Eq. (21) and find

\[
\zeta(s) = \frac{\mu^{2s}}{\Gamma(s)} \frac{eBL^2}{4\pi^2} \frac{a}{\sqrt{1 - \lambda}} \left(\frac{a}{\sqrt{eB + m^2}}\right)^{s-1} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-2} e^{-na\sqrt{eB+m^2} (t+1/t)} dt.
\]
(23)

In the strong magnetic field limit \(a\sqrt{eB} \gg 1\), and all terms with \(n > 1\) in the sum of Eq. (23) are negligibly small and can be left out. I evaluate the remaining integral using the saddle point method and find

\[
\zeta(s) = \left(\frac{\mu^{2s}}{\sqrt{eB + m^2}}\right)^s \frac{L^2}{4\pi^{3/2}} \frac{eB}{\sqrt{1 - \lambda}} \frac{(eB + m^2)^{\frac{1}{2}}}{\sqrt{a}} e^{-2a\sqrt{eB+m^2}} \frac{e^{-2eBt} t^s}{\Gamma(s)}
\]
(24)

The derivative of \(\zeta(s)\) is easily calculated using

\[
\frac{A^s}{\Gamma(s)} \simeq s + \mathcal{O}(s^2),
\]
(25)

to obtain the strong magnetic field Casimir energy

\[
E_C = -\frac{L^2}{4\pi^{3/2}} \frac{eB}{\sqrt{1 - \lambda}} \frac{(eB + m^2)^{\frac{1}{2}}}{\sqrt{a}} e^{-2a\sqrt{eB+m^2}}.
\]
(26)

Notice that the dominant term is the exponential suppression term.

Finally I will examine the large mass limit, \(m \gg \sqrt{eB}, a^{-1}\). I do a Poisson resummation of the zeta function of Eq. (12) and find

\[
\zeta(s) = \frac{\mu^{2s}}{\Gamma(s)} \frac{L^2}{8\pi^2} \frac{eB \alpha}{\sqrt{1 - \lambda}} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-2} e^{-\left(m^2 t + \frac{a^2}{t}\right)} \text{csch}(eBt) dt,
\]
(27)
after neglecting the \( n = 0 \) term in the sum because it is a uniform energy density term that will not contribute to the Casimir energy. Next, I change the integration variable to find

\[
\zeta(s) = \left( \frac{\mu^2 a}{m} \right)^s \frac{e^{BL^2}}{8\pi^2 \Gamma(s)} \frac{m}{\sqrt{1 - \lambda}} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-2} e^{-n\lambda m (t+\frac{1}{2})} \text{csch} (neB a/m) dt,
\]

where all the terms with \( n > 1 \) are negligible because \( am \gg 1 \). I do the integral using the saddle point method, use (29) to take the derivative of \( \zeta(s) \), and find the large mass limit of the Casimir energy

\[
E_C = -\frac{L^2}{8} \left( \frac{m}{\pi a} \right)^{\frac{3}{2}} e^{-2am} \sqrt{1 - \lambda} F(z),
\]

where \( z = eB a/m \) is dimensionless and

\[
F(z) = z \text{csch}(z)
\]

is the magnetic field correction to \( E_C \). Notice that \( F(z) \to 1 \) when \( z \to 0 \), and thus my result agrees with the large mass limit of Ref. [14] when \( B \to 0 \).

IV. SPACELIKE ANISOTROPY IN THE X-Y PLANE

When the Lorentz anisotropy is spacelike and perpendicular to \( \vec{B} \), the zeta function (7) can be written as

\[
\zeta(s) = \frac{\mu^{2s} L^2 eB}{4\pi^2} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} dt \Gamma(s) \left[ k_n^2 + (1 - \frac{1}{2}) (2\ell + 1) eB + (\frac{m}{a})^2 + m^2 \right] t.
\]

Comparing (31) with (10), it is evident that (31) can be obtained by taking (10) and replacing \( eB \) with \( (1 - \frac{1}{2}) eB \) in it. Therefore \( E_C \) for anisotropy in the \( x - y \) plane is obtained, in each of the three asymptotic limits, by taking Eqs. (20), (23), (29) and making the same replacement.

In the short plate distance limit, I find

\[
E_C = -\frac{\pi^2 L^2}{8 a^3} (1 - \lambda)^{-\frac{3}{2}} \left[ \frac{1}{90} - \frac{m^2 a^2}{6\pi^2} - \frac{(1 - \lambda)}{6\pi^4} \left( \gamma_E + \ln \left( \frac{\mu a}{2\pi} \right) \right) \right],
\]

showing a slight decrease of the magnetic field contribution to \( E_C \) when compared to the cases of timelike anisotropy and isotropic spacetime. As I did in Eq. (20), I choose \( \mu = \max\{m, \sqrt{eB}\} \). When \( B \to 0 \), the leading order term of this result agrees with the leading order term obtained in Ref. [14] under the same conditions.

In the strong magnetic field limit, the Casimir energy is given by

\[
E_C = -\frac{L^2 eB}{4\pi^{3/2}} \left( \frac{(1 - \frac{1}{2}) eB + m^2}{\sqrt{a}} \right)^{\frac{3}{2}} e^{-2\omega \sqrt{1 - \frac{1}{2}} eB + m}.
\]

Finally, in the large mass limit, I obtain

\[
E_C = -\frac{L^2}{8} \left( \frac{m}{\pi a} \right)^{\frac{3}{2}} e^{-2am} \sqrt{1 - \lambda} F(z \sqrt{1 - \lambda}),
\]

in agreement with the large mass limit of Ref. [14] when \( B \to 0 \). Notice that the magnetic correction obtained here is smaller than the one obtained in the case of timelike anisotropy.

V. SPACELIKE ANISOTROPY IN THE Z DIRECTION

When the spacelike Lorentz anisotropy is parallel to \( \vec{B} \), the zeta function can be written as

\[
\zeta(s) = \frac{\mu^{2s} eBL^2}{4\pi^2} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} dt \Gamma(s) \left[ k_n^2 + (2\ell + 1) eB + (\frac{m}{a})^2 + m^2 \right] t.
\]
This zeta function can be obtained immediately by multiplying the zeta function of Eq. (10) by $\sqrt{1 - \lambda}$ and then replacing $\lambda$ with $\frac{m^2}{a^2}$ in it. Therefore $E_C$ for anisotropy in the $z$ direction is obtained by making the same modifications to the three asymptotic forms of $E_C$ of Eqs. (20), (26), and (29).

In the limit of short plate distance, I obtain

$$E_C = -\frac{\pi^2 L^2}{8a^3}(1 - \lambda)^{\frac{3}{2}} \left[ \frac{1}{90} - \frac{m^2 a^2}{6\pi^2(1 - \lambda)} - \frac{e^2 B^2 a^4}{6\pi^2(1 - \lambda)^2} \left( \gamma_E + \ln \left( \frac{\mu a}{2\pi} \right) + \frac{\lambda}{2} \right) \right],$$

(36)

where I used $\ln[(1 - \lambda)^{-\frac{1}{2}}] \simeq \lambda/2$ and $\mu = \max\{m, \sqrt{eB}\}$. When $B \to 0$, the leading order term agrees with the leading order term of the short plate distance limit of Ref. [14] for anisotropy in the $z$ direction. Notice that the magnetic field correction, apart from the smaller term proportional to $\lambda/2$, is the same in this limit as it is for a timelike anisotropy when the plate distance is short, as shown in Eq. (20).

In the strong magnetic field limit I find the following Casimir energy

$$E_C = -\frac{L^2 eB}{4\pi^3} \frac{(eB + m^2)^{\frac{1}{4}}}{\sqrt{a}}(1 - \lambda)^{\frac{3}{4}} e^{-2a\sqrt{\frac{eB + m^2}{a}}} F\left(\frac{z}{\sqrt{1 - \lambda}}\right).$$

(37)

Last, in the large mass limit, I obtain

$$E_C = -\frac{L^2}{8} \left( \frac{m\sqrt{1 - \lambda}}{\pi a} \right)^{\frac{3}{4}} e^{-2\sqrt{\frac{a^2 eB}{m^2}}} F\left(\frac{z}{\sqrt{1 - \lambda}}\right),$$

(38)

a result that, when $B \to 0$, agrees with the result obtained in Ref. [14].

VI. CASIMIR PRESSURE

The Casimir pressure is given by

$$P_C = -\frac{1}{L^2} \frac{\partial E_C}{\partial a}.$$ 

(39)

Keeping the assumption that the magnetic field is present inside and outside the plates, all uniform energy density terms neglected previously will not contribute to the Casimir pressure. Furthermore, once the Casimir pressure for a timelike anisotropy is presented, the pressure in the cases of spacelike anisotropy in the $z$ direction or $x - y$ plane is immediately obtained with a simple substitution. In the presence of a timelike anisotropy, the short plate distance Casimir pressure is

$$P_C = -\frac{\pi^2 L^2}{8a^3}(1 - \lambda)^{\frac{3}{2}} \left[ \frac{1}{30} - \frac{m^2 a^2}{6\pi^2} - \frac{e^2 B^2 a^4}{6\pi^2} \left[ 1 + \gamma_E + \ln \left( \frac{\mu a}{2\pi} \right) \right] \right],$$

(40)

where the parameter $\mu$ is $\mu = \max\{m, eB\}$. The presence of a magnetic field contributes an almost constant attractive term, aside from a weak logarithmic dependence on $a$. The short plate distance pressure in the presence of a spacelike anisotropy in the $x - y$ direction is obtained from Eq. (40) by replacing $eB$ with $(1 - \frac{\lambda}{2})eB$, and the short plate distance pressure for anisotropy in the $z$ direction is obtained by replacing $a$ with $\frac{\sqrt{a}}{\sqrt{1 - \lambda}}$ in Eq. (40).

The strong magnetic field limit of the Casimir pressure, in the presence of a timelike anisotropy, is

$$P_C = -\frac{eB}{2\pi^{3/2}} \frac{(eB + m^2)^{\frac{1}{4}}}{\sqrt{a}}(1 - \lambda)^{\frac{3}{4}} e^{-2a\sqrt{eB + m^2}} \left[ 1 + \frac{1}{4a\sqrt{eB + m^2}} \right],$$

(41)

where the dominant term is the exponential suppression, since $a\sqrt{eB} \gg 1$. In the case of a spacelike anisotropy in the $x - y$ direction, the strong field limit of the pressure is obtained from (41) by replacing $eB$ with $(1 - \frac{\lambda}{2})eB$, which produces a slightly weaker exponential suppression of $P_C$, when compared to timelike anisotropy or isotropic spacetime. On the other hand, the pressure in the strong field limit for anisotropy in the $z$ direction has a slightly increased exponential suppression when compared to a timelike anisotropic or isotropic spacetime. This expression of $P_C$ is obtained from (41) by replacing $a$ with $\frac{\sqrt{a}}{\sqrt{1 - \lambda}}$.

Last, the Casimir pressure in the large mass limit in the presence of a timelike anisotropy is

$$P_C = \frac{m}{4} \left( \frac{m}{\pi a} \right)^{3/2} \frac{e^{-2am}}{\sqrt{1 - \lambda}} F(z) \left( 1 + \frac{1}{4am} + \frac{eB}{2m^2} \coth z \right),$$

(42)

where $z = \frac{\sqrt{a}}{m}$ and $F(z)$ is defined in (20). Notice that, also here, the exponential suppression is the dominant term, since $am \gg 1$. The large mass limit of the pressure in the case of spacelike anisotropy, either in the $z$ or in the $x - y$ direction, is obtained from (12) by making the substitutions outlined in the previous paragraph.
VII. DISCUSSION AND CONCLUSIONS

In this work I used the generalized zeta function technique to study the Casimir effect of a Lorentz-violating scalar field in the presence of a magnetic field. This massive and charged complex scalar field satisfies a modified Klein-Gordon equation that breaks Lorentz symmetry in a CPT-even aether-like manner by the direct coupling of the field derivative to a constant unit four-vector \( u^\mu \). I investigated the case of this field having Dirichlet boundary conditions on two plane parallel plates, perpendicular to the magnetic field. My results show a strong dependence of \( E_C \) and \( P_C \) on the dimensionless quantity \( \lambda \) that parametrizes the breaking of the Lorentz symmetry. This strong dependence on \( \lambda \) was also observed in Ref. [14]. When the magnetic field is switched off my results agree, to leading order, with the results of Ref. [14], and also agree with the well known results for isotropic spacetime when \( \lambda \to 0 \). In the short plate distance limit, I find that the magnetic correction decreases the Casimir pressure in all three cases of a timelike and spacelike \( u^\mu \), with the biggest correction happening when \( u^\mu \) is spacelike and orthogonal to \( \vec{B} \). In Sec. VI I obtained simple analytic expressions of the Casimir pressure in the case of a strong magnetic field I discover that \( P_C \) is exponentially suppressed in all three cases, and find that the pressure is largest when \( u^\mu \) is spacelike and orthogonal to \( \vec{B} \). Finally, I find that \( P_C \) is exponentially suppressed also in the large mass limit, and the largest magnetic correction happens when \( u^\mu \) is spacelike and parallel to \( \vec{B} \).

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