Molecular superfluid phase in one-dimensional multicomponent fermionic cold atoms

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(Dated: February 11, 2008)

PACS numbers: 03.75.Mn, 71.10.Fd, 03.75.Ss, 71.10.Pm

We study a simple model of \( N \)-component fermions with contact interactions which describes fermionic atoms with \( N = 2F + 1 \) hyperfine states loaded into a one-dimensional optical lattice. We show by means of analytical and numerical approaches that, for attractive interaction, a quasi-long-range molecular superfluid phase emerges at low density. In such a phase, the pairing instability is strongly suppressed and the leading instability is formed from bound-states made of \( N \) fermions. At small density, the molecular superfluid phase is generic and exists for a wide range of attractive contact interactions without an SU(\( N \)) symmetry between the hyperfine states.

Because of rapid progress in recent years, cold atom systems have become a major field of research for investigating the physics of strong correlations in a widely tunable range and in unprecedentedly clean systems [1]. Ultracold atomic systems also offer direct access to the study of spin degeneracy since the hyperfine spin \( F \) can be larger than 1/2, resulting in \( 2F + 1 \) hyperfine states. In non-magnetic traps, such as optical traps, this high degeneracy might give rise to novel exotic quantum phases. The superfluid state of optically trapped alkali fermions with hyperfine spin \( F > 1/2 \) has been studied with an emphasis on the general structure of the large-spin Cooper pairs [2]. The spin-degeneracy in fermionic atoms is also expected to give rise to more complex superfluid phases. In particular, a molecular superfluid (MS) phase might be stabilized where more than two fermions form a bound state. Such a non-trivial superfluid behavior has already been found in different contexts. In nuclear physics, a four-particle condensate—the \( \alpha \) particle—is favored over deuteron condensation [3] and it may have implications for light nuclei and asymmetric matter in nuclear stars [4]. This quartet condensation can also occur in semiconductors with the formation of biexcitons [5]. A quartetting phase, which stems from the pairing of Cooper pairs, has also been found in a model of one-dimensional (1D) Josephson junctions [6]. A similar phase has also been reported in exact-diagonalization calculations of the two-dimensional t-J model at low doping [7]. More recently, the emergence of quartets and triplets (three-fermion bound states) has been proposed to occur in the context of ultracold fermionic atoms [8, 9, 10, 11].

In view of this increasing interest in the formation of complex superfluid condensates, it would be highly desirable to have at one’s disposal a simple paradigmatic \( N \)-component fermionic model which displays this exotic physics. It will be shown in this letter that such a model is provided by the 1D \( N \)-component fermionic Hubbard model with attractive contact interaction:

\[
\mathcal{H} = -t \sum_{i,\alpha} [c_{\alpha,i}^\dagger c_{\alpha,i+1} + \text{H.c.}] + \frac{U}{2} \sum_{i} n_i^2, \tag{1}
\]

where \( c_{\alpha,i}^\dagger \) is the fermion creation operator corresponding to the \( \alpha \)-hyperfine state \( \alpha = 1, \ldots, N \) and \( n_i = \sum_{\alpha} c_{\alpha,i}^\dagger c_{\alpha,i} \) is the density at site \( i \). Model (1) displays an extended U(\( N \))=U(1)×SU(\( N \)) symmetry and it has been recently introduced in the context of ultracold fermionic atoms [12]. A possible experimental realization of this model (1) for \( N = 3 \) would be a system of \( ^6\text{Li} \) atoms loaded into a 1D optical lattice with a carefully tuned combination of external magnetic and optical fields to make three internal states exhibit SU(3) symmetry [13]. The \( N = 4 \) case might also be relevant to the optical trap of four hyperfine states of \(^{40}\text{K} \) (\( F = 9/2 \) atoms) [14]. The SU(\( N \)) symmetry of Eq. (1) has an important consequence since, when \( N > 2 \), even for \( U < 0 \) there can be no pairing between fermions: there is no way to form a SU(\( N \)) singlet with only two fermions. The only superfluid instability that can be stabilized is a molecular one where \( N \) fermions form a SU(\( N \)) singlet: \( M^j = c_{1,i}^\dagger c_{2,i}^\dagger \cdots c_{N,i}^\dagger \). In this letter, we shall show, by means of a combination of analytical and numerical results obtained by the density-matrix renormalization group (DMRG) technique [15], that this MS phase emerges in the phase diagram of model (1) for \( U < 0 \) and at small enough density \( \eta \). The latter phase is not an artifact of the extended SU(\( N \)) symmetry of model (1)—it is robust to symmetry breaking terms toward more realistic situations. In this respect, we believe that the Hubbard model (1) captures the main generic features responsible for the formation of the MS phase.

Low-energy approach. The low-energy effective field theory corresponding to the SU(\( N \)) Hubbard chain (1) can be derived, as usual, from the linearization at the two Fermi points (\( \pm k_F \)) of the dispersion relation of free \( N \)-component
fermions\textsuperscript{[16, 17]}. The derivation of the low-energy Hamiltonian is straightforward (see for instance Ref.\textsuperscript{[18]} for details) and, away from half-filling, it separates into two commuting charge and spin pieces: $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s$. This is the famous spin-charge separation which is the hallmark of 1D electronic systems\textsuperscript{[16, 17]}. Within this low-energy description, the $U(1)$ charge excitations are described by a free massless bosonic field $\Phi$ with Hamiltonian density:

$$\mathcal{H}_c = \frac{v}{2} \left[ \frac{1}{K} (\partial_x \Phi)^2 + K (\partial_x \Theta)^2 \right], \quad (2)$$

where $v$ and $K$ are respectively the charge velocity and the Luttinger parameter. A perturbative estimate of the equal-time MS correlations associated with a charge-density wave (CDW) and the equal-time density correlation for

$$\langle N(x) \rangle = \langle M(x) \rangle \sim x^{-1/2}$$

is straightforward (see for instance Ref.\textsuperscript{[18]} for details)

> ![FIG. 1: (Color online) SU(3) model: triplet and density correlations vs distance obtained by DMRG with $L = 153, n = 1/3$ and $U/t = -4$. (a) dominant triplet over CDW correlations can both be fitted with $K = 2.7$. We also see the $k_F$ (respectively $2k_F$) oscillations of $|N(x)|$ (respectively $|M(x)|$). (b) 1-particle Green function $G(x)$ and pairing correlations $P(x)$ vs distance. Both are short-range and with the same correlation length $\xi = 0.68$.](https://example.com/fig1.png)

For $U < 0$ a spectral gap $m$ opens. The low-energy spectrum in the hyperfine spin sector consists of $N - 1$ branches with masses $m_n = m \sin(\pi x / N)$ ($r = 1, \ldots, N - 1$). The dominant instability which governs the physics of this phase is the one with the slowest decaying correlations at zero temperature. Both the 1-particle Green function $G(x) = \langle c_{\alpha,i} c_{\alpha,i+x} \rangle$ and (onsite) pairing correlations $P(x) = \langle c_{\alpha,i}^+ c_{\alpha,i+x}^+ c_{\beta,i} c_{\beta,i+x} \rangle$ are short-range. On the contrary, the equal-time density correlation $N(x) = \langle n_i n_{i+x} \rangle$ associated with a charge-density wave (CDW) and the equal-time MS correlations $M(x) = \langle M_i M_{i+x}^\dagger \rangle$ have the following power-law decay at long distance\textsuperscript{[24]}:

$$N(x) \sim \cos(2k_F x)x^{-2K/N}$$

$$M(x) \sim x^{-N/(2K)} \quad \text{for } N \text{ even} \quad (3)$$

$$M(x) \sim x^{-N/(2K)} \quad \text{for } N \text{ odd} \quad (5)$$

We thus see that CDW and MS instabilities compete and the key point of the analysis is the one which dominates. At issue is the value of the Luttinger parameter $K$. In particular a dominant MS instability requires $K > N/2$ ($K > N/\sqrt{3}$) for $N$ even (odd, respectively) and thus a fairly large value of $K$ which, with only short range interaction, is not guaranteed. However a simple argument suggests that this may be realized at sufficiently small density at large negative $U$. Indeed, when $n \ll 1$ and $|U|/t \gg 1$, a dilute gas of strongly bound $N$-fermion objects forms and\textsuperscript{17} behaves as essentially free hardcore bosons ($N$ even) or free fermions ($N$ odd) with an effective hopping $t^N / |U|^{N-1}$. One can therefore estimate $M(x)$ in this limit as the free bosonic Green function,

$$M(x) \sim x^{-1/2}$$

when $N$ is even and, as the free fermion Green function,

$$M(x) \sim \sin(k_F x) / x$$

when $N$ is odd. By comparing with Eqs.\textsuperscript{24, 25}, we deduce an upper bound for $K$ which is $K_{\text{max}} = N$\textsuperscript{[22]}. From the perturbative estimate we see that $K > 1$ and $K$ increases with $|U|$, so that there is room to stabilize an MS phase for sufficiently strong attractive interaction and small density. In addition, at zero density, the $N$-component Fermi gas with an SU$(N)$ symmetry is known to be exactly-solvable by means of the Bethe-ansatz approach and bound states of $N$ fermions are formed for attractive interaction\textsuperscript{[23]}. Outside these cases of infinite attractive interaction or vanishing density, the existence and stability of this MS phase stem from the full non-perturbative behavior of the Luttinger parameter $K$ as a function of the density $n$ and the interaction $U$. We shall now evaluate numerically this parameter in the simplest odd and even cases $N = 3, 4$ by computing dominant correlations with the DMRG technique to conclude on the extension of the MS phase.

**Numerical Results.** We have performed extensive DMRG calculations for both the $N = 3$ and $N = 4$ cases and for a wide range of densities $n$ and interactions $U$\textsuperscript{[24, 25]}. We show in Fig.\textsuperscript{1}a-b and Fig.\textsuperscript{2}a-b our data for $N = 3$ and $N = 4$ respectively at typical values of $n$ and $U = -4t$. In both cases, and in agreement with the low-energy approach, we find that a gap opens in the hyperfine spin sector and that the one-particle and pairing correlations are always short ranged. From Fig.\textsuperscript{1}b and Fig.\textsuperscript{2}b, one can compute the one- and two-particle correlation lengths $\xi$ that is expected to vary as the inverse gap : $\xi \sim 1 / m$. We find that the ratio $R = m_1 / m_2$ is close to 1 and $1/\sqrt{2}$ respectively for $N = 3$ and $N = 4$ as expected from the low-energy approach. In contrast, we see in Fig.\textsuperscript{1}a and Fig.\textsuperscript{2}a that the density and MS correlations $N(x)$ and $M(x)$ display power-law behavior. Clearly, triplet and quartet correlations dominate over CDW at these densities ($n = 1/3$ for $N = 3$ and $n = 0.5$ for $N = 4$). The phase diagrams for both SU(3) and SU(4) models are presented in Fig.\textsuperscript{3} and Fig.\textsuperscript{4} which give a map of $K$ vs interaction and density. The values of $K$ were ob-
At filling $\nu = 0.25$, the same $K \approx 2.7$ is used to give a rough estimate of the exponent. (b) 1- and 2-particle correlations vs distance. Both are short-range and the ratio of the two correlation lengths is $\xi_2/\xi_1 = 0.68 \approx 1/\sqrt{2}$.

We have shown that a quasi-long-range perturbation limit [25]. For example in the SU(4) case, the $K(\nu)$ function is almost $U$-independent for $|U|/t > 2$ so that lines of equal $K$ are parallel to the $U$ axis.

Effect of symmetry breaking perturbations. At this point, the natural question is whether the molecular superfluid phases survives to the breaking of the SU($N$) symmetry. This is an important question since in most of the realistic situations, the actual symmetry is expected to be much smaller. Part of the answer is given in 1D systems by the accepted view that, at sufficiently low energies and for generic interactions, the dynamical symmetry is most likely to be enlarged [26]; though the SU($N$) symmetry is not an exact symmetry, it is physically meaningful as an effective low-energy theory. As an example, we consider the SU(4) case relevant for spin-3/2 cold atoms and add to the Hubbard Hamiltonian [1] a singlet-pairing coupling $V \sum_i P_{00,i}^\dagger P_{00,i}$ where $P_{00,i}^\dagger = c_{1/2,i}^0 c_{-1/2,i}^0 - c_{1/2,i}^0 c_{-1/2,i}^0$. As shown in Ref. [27], the pairing term reduces the SU(4) symmetry down to SO(5).

We show typical data for $U/t = -4$ and $V/t = -2$ at the density $n = 1/2$ in Fig. [5]. We clearly see that the equal-time pairing correlation function $P(x) = \langle P_{00,i}^\dagger P_{00,i+x} \rangle$ admits an exponential decay i.e. there is no BCS instability. In contrast, quartet correlations are (quasi) long ranged and dominate over CDW ones. Remarkably, we observe from Fig. [5] that the gap ratio $R$ is very similar to the one for the full SU(4) symmetric model when $V = 0$. This means that the SU(4) model [1] is a very good starting point to explore the main features of the quartet phase. Of course, for large negative $V$, a BCS phase does appear [28] but the main point here is to show that the quartet molecular phase is not an artifact of the SU(4) symmetry and does exist in more realistic models [29]. A more detailed study will be presented elsewhere.

Concluding remarks. We have shown that a quasi-long-range general MS phase can emerge in 1D for attractive interactions at low density. This 1D phase, characterized by
Both are short-range and the ratio of the two correlation lengths is \( \frac{\xi_2}{\xi_1} \approx 0.72 \pm 1/\sqrt{3} \).

a bound-state made of \( N \) fermions, can be viewed as a nematic Luttinger liquid and a simple paradigmatic model to describe its main physical properties is the attractive SU(\( N \)) Hubbard chain \( [1] \). The triplet and quartet phases in the simplest \( N = 3, 4 \) cases might be explored experimentally in the context of spinor ultracold fermionic atoms. As a first step, we have assumed here a homogeneous optical lattice and neglect in the first approximation the parabolic confining potential of the atomic trap. We expect that this potential will not affect the properties of this molecular phase at low density. Such an effect could be investigated by DMRG calculations for quantitative comparisons \( [30] \). In the context of cold atoms experiments, the triplet and quartet phases can be probed by radio-frequency spectroscopy to measure the excitation gaps of the successive triplet-quartet dissociation process. We hope that future experiments in ultracold fermionic atoms will reveal the existence of these triplet and quartet phases.

We would like to thank F. H. L. Essler, P. Schuck, G. V. Shlyapnikov, and A. M. Tsvelik for useful discussions and their interest. SC and GR thank IDRIS (Orsay, France) and CALMIP (Toulouse, France) for use of supercomputers. SRW acknowledges the support of the NSF under grant DMR-0605444.