DOMAINS OF HOLOMORPHY FOR IRREDUCIBLE UNITARY REPRESENTATIONS OF SIMPLE LIE GROUPS

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1. Introduction

Let us consider a unitary irreducible representation \((\pi, \mathcal{H})\) of a simple, non-compact and connected algebraic Lie group \(G\). Let us denote by \(K\) a maximal compact subgroup of \(G\). According to Harish-Chandra, the Lie algebra submodule \(\mathcal{H}_K\) of \(K\)-finite vectors of \(\pi\) consists of analytic vectors for the representation, i.e. for all \(v \in \mathcal{H}_K\) the orbit map

\[ f_v : G \rightarrow \mathcal{H}, \ g \mapsto \pi(g)v \]

is real analytic. For these functions \(f_v\) we determine, and in full generality, their natural domain of definition as holomorphic functions (see Theorem 5.1 below):

**Theorem 1.1.** Let \((\pi, \mathcal{H})\) be a unitary irreducible representation of \(G\). Let \(v \in \mathcal{H}\) be a non-zero \(K\)-finite vector and \(f_v\) be the corresponding orbit map. Then there exists a unique maximal \(G \times K_C\)-invariant domain \(D_\pi \subseteq G_C\), independent of \(v\), to which \(f_v\) extends holomorphically. Explicitly:

(i) \(D_\pi = G_C\) if \(\pi\) is the trivial representation.
(ii) \(D_\pi = \Xi^+ K_C\) if \(G\) is Hermitian and \(\pi\) is a non-trivial highest weight representation.
(iii) \(D_\pi = \Xi^- K_C\) if \(G\) is Hermitian and \(\pi\) is a non-trivial lowest weight representation.
(iv) \(D_\pi = \Xi K_C\) in all other cases.

In the theorem above \(\Xi, \Xi^+, \Xi^-\) are certain \(G\)-domains in \(X_C = G_C/K_C\) over \(X = G/K\) with proper \(G\)-action. These domains are studied in this paper because of their relevance for the theorem above (see [5]). Let us mention that \(\Xi\) is the familiar crown domain and that the inclusion \(\Xi K_C \subset D_\pi\) traces back to our joint work with Robert Stanton ([6], [7]).

**Acknowledgment:** I am happy to point out that this paper is related to joint work with Eric M. Opdam [5]. Also I would like to thank Joseph Bernstein who, over the years, helped me with his comments to understand the material much better.

Finally I appreciate the work of a very good referee who made many useful remarks on style and organization of the paper.
2. Notation

Throughout this paper $G$ shall denote a connected simple non-compact Lie group. We denote by $G_C$ the universal complexification of $G$ and suppose:
- $G \subseteq G_C$;
- $G_C$ is simply connected.

We fix a maximal compact subgroup $K < G$ and form $X = G/K$, the associated Riemannian symmetric space of the non-compact type. The universal complexification $K_C$ of $K$ will be realized as a subgroup of $G_C$. We set

$$X_C = G_C/K_C$$

and call $X_C$ the affine complexification of $X$. Note that

$$X \hookrightarrow X_C, \quad gK \mapsto gK_C$$
defines a $G$-equivariant embedding which realizes $X$ as a totally real form of the Stein symmetric space $X_C$. We write $x_0 = K_C \in X_C$ for the standard base point in $X_C$.

However, the natural complexification of $X$ is not $X_C$, but the crown domain $\Xi \subset X_C$ whose definition we recall now. We shall provide the standard definition of $\Xi$, see [1].

Lie algebras of subgroups $L < G$ will be denoted by the corresponding lower case German letter, i.e. $l < g$; complexifications of Lie algebras are marked with a $C$-subscript, i.e. $l_C$ is the complexification of $l$.

Let us denote by $p$ the orthogonal complement to $\mathfrak{f}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form. We set

$$\hat{\Omega} = \{ Y \in p \mid \text{spec(ad} Y) \subset (-\pi/2, \pi/2) \}.$$ 

Then

$$\Xi = G \exp(i\hat{\Omega}) \cdot x_0 \subset X_C$$
is a $G$-invariant neighborhood of $X$ in $X_C$, commonly referred to as crown domain. Sometimes it is useful to have an alternative, although less invariant picture of the crown domain: if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace and $\Omega := \hat{\Omega} \cap \mathfrak{p}$, then

$$\Xi = G \exp(i\Omega) \cdot x_0.$$
The set $\Omega$ is nicely described through the restricted root system $\Sigma = \Sigma(g, a)$:

$$\Omega = \{ Y \in a \mid \alpha(Y) < \pi/2 \ \forall \alpha \in \Sigma \}.$$ 

If $W$ is the Weyl group of $\Sigma$, then we note that $\Omega$ is $W$-invariant.

Sometimes we will employ the root space decomposition $g = a \oplus m \oplus \bigoplus_{\alpha \in \Sigma} g^\alpha$ with $m = \mathfrak{z}(a)$ as usual. We choose a positive system $\Sigma^+ \subset \Sigma$ and form the nilpotent subalgebra $n = \bigoplus_{\alpha \in \Sigma^+} g^\alpha$.

2.1. The example of $G = \text{Sl}(2, \mathbb{R})$

For illustration and later use we will exemplify the above notions at the basic case of $G = \text{Sl}(2, \mathbb{R})$.

We let $K = \text{SO}(2, \mathbb{R})$ be our choice for the maximal compact subgroup and identify $X = G/K$ with the upper half plane $D^+ := \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$. We recall that

$$X_C = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \text{diag}[\mathbb{P}^1(\mathbb{C})]$$

with $G_C$ acting diagonally by fractional linear transformations. The $G$-embedding of $X = D^+$ into $X_C$ is given by

$$z \mapsto (z, \overline{z}) \in X_C.$$ 

If $D^-$ denotes the lower half plane, then the crown domain is given by

$$\Xi = D^+ \times D^- \subseteq X_C.$$ 

In addition we record two $G$-domains in $X_C$ which sit above $\Xi$, namely:

$$\Xi^+ = D^+ \times \mathbb{P}^1(\mathbb{C}) \setminus \text{diag}[\mathbb{P}^1(\mathbb{C})],$$

$$\Xi^- = \mathbb{P}^1(\mathbb{C}) \times D^- \setminus \text{diag}[\mathbb{P}^1(\mathbb{C})].$$

Observe that $\Xi = \Xi^+ \cap \Xi^-.$

3. Remarks on $G$-invariant domains in $X_C$ with proper action

One defines elliptic elements in $X_C$ by

$$X_{C,\text{ell}} = G \exp(i\mathfrak{p}) \cdot x_0 = G \exp(i\mathfrak{a}) \cdot x_0.$$ 

The main result of [1] was to show that $\Xi$ is a maximal domain in $X_{C,\text{ell}}$ with $G$-action proper. In particular, $G$ acts properly on $\Xi$.

It was found in [5] that $\Xi$ in general is not a maximal domain in $X_C$ for proper $G$-action: the domains $\Xi^+$ and $\Xi^-$ from (2.2)-(2.3) yield
counterexamples. To know all maximal domains is important for the theory of representations [5], Sect. 4.

That $\Xi$ in general is not maximal for proper action is related to the unipotent model for the crown which was described in [5]. To be more precise, we showed that there exists a domain $\hat{\Lambda} \subseteq \mathfrak{n}$ containing 0 such that

\begin{equation}
\Xi = G \exp(i\hat{\Lambda}) \cdot x_0.
\end{equation}

Now there is a big difference between the unipotent parametrization (3.1) and the elliptic parametrization (2.1): If we enlarge $\Omega$ the result is no longer open; in particular, $X_{\mathbb{C},\text{ell}}$ is not a domain. On the other hand, if we enlarge the open set $\hat{\Lambda}$ the resulting set is still open; in particular $X_{\mathbb{C},\text{u}} := G \exp(i\mathfrak{n}) \cdot x_0$ is a domain. Thus, if there were a bigger domain than $\Xi$ with proper action, then it is likely by enlargement of $\hat{\Lambda}$.

We need some facts on the boundary of $\Xi$.

### 3.1 Boundary of $\Xi$

Let us denote by $\partial \Xi$ the topological boundary of $\Xi$ in $X_{\mathbb{C}}$. One shows that

$\partial_{\text{ell}} \Xi := G \exp(i\partial \Omega) \cdot x_0 \subseteq \partial \Xi$

(cf. [7]) and calls $\partial_{\text{ell}} \Xi$ the elliptic part of $\partial \Xi$. We define the unipotent part $\partial_{\text{u}} \Xi$ of $\partial \Xi$ to be the complement to the elliptic part:

\[ \partial_{\text{u}} \Xi = \partial \Xi \setminus \partial_{\text{ell}} \Xi. \]

The relevance of $\partial_{\text{u}} \Xi$ is as follows. Let $X \subset D \subseteq X_{\mathbb{C}}$ denote a $G$-domain with proper $G$-action. Then $D \cap \partial_{\text{ell}} \Xi = \emptyset$ by the above cited result of [1]. Thus if $D \not\subset \Xi$, then one has

$D \cap \partial_{\text{u}} \Xi \neq \emptyset$.

Let us describe $\partial_{\text{u}} \Xi$ in more detail. For $Y \in \mathfrak{a}$ we define a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$ by

$\mathfrak{g}_{\mathbb{C}}[Y] = \{ Z \in \mathfrak{g}_{\mathbb{C}} \mid e^{-2i\text{ad}(Y)} \circ \sigma(Z) = Z \}$

with $\sigma$ the Cartan involution on $\mathfrak{g}_{\mathbb{C}}$ which fixes $\mathfrak{k} + i\mathfrak{p}$. Then there is a partial result on $\partial_{\text{u}} \Xi$, for instance stated in [2]:

\begin{equation}
\partial_{\text{u}} \Xi \subseteq \{ G \exp(e) \exp(iY) \cdot x_0 \mid Y \in \partial \Omega, \}
\end{equation}

\begin{equation}
0 \neq e \in \mathfrak{g}_{\mathbb{C}}[Y] \cap i\mathfrak{g} \text{ nilpotent}. 
\end{equation}

If $Y$ is such that only one root, say $\alpha$, attains the value $\pi/2$, then we call $Y$ and as well the elements in the boundary orbit $G \exp(e) \exp(iY)$.
Accordingly we define the regular unipotent boundary $\partial_{u,\text{reg}} \Xi = \{ z \in \partial \Xi | z \text{ regular} \}$. Note that $g_C[Y]$ is of especially simple form for regular $Y$, namely

$$g_C[Y] = i a \oplus m \oplus g[\alpha]^{-\theta} \oplus i g[\alpha]^{\theta}$$

where $g[\alpha] = g^\alpha \oplus g^{-\alpha}$. Hence, in the regular situation, one can choose $e$ above to be in $i g[\alpha]^{\theta} + i a$. We summarize our discussion:

**Proposition 3.1.** Let $X \subset D \subset X_C$ be a $G$-invariant domain with proper $G$-action which is not contained in $\Xi$. Then $D \cap \partial_{u,\text{reg}} \Xi \neq \emptyset$. More precisely, there exists $Y \in \partial \Omega$ regular (with $\alpha \in \Sigma$ the unique root attaining $\pi/2$ on $Y$) and a non-zero nilpotent element $e \in i g[\alpha]^{\theta} + i a$ such that

$$\exp(e) \exp(i Y) \cdot x_0 \in \partial_{u,\text{reg}} \Xi \cap D.$$

4. Maximal domains for proper action

The aim of this section is to classify all maximal $G$-domains in $X_C$ which contain $X$ and maintain proper action. The answer will depend whether $G$ is of Hermitian type or not.

4.1. Non-Hermitian groups.

The objective is to prove the following theorem:

**Theorem 4.1.** Suppose that $G$ is not of Hermitian type. If $X \subset D \subset X_C$ is a $G$-invariant domain with proper $G$-action, then $D \subset \Xi$.

Before we can give the proof of the theorem some preparation is needed. The proof relies partly on a structural fact characterizing non-Hermitian groups (see Lemma 4.4 below) and on a precise knowledge of the basic case of $G = \text{Sl}(2, \mathbb{R})$.

Let us begin with the relevant facts for $G = \text{Sl}(2, \mathbb{R})$. With $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ our choices for $a$ and $n$ are

$$a = \mathbb{R} \cdot T \quad \text{and} \quad n = \mathbb{R} \cdot E.$$

Note that $\Omega = (-\pi/4, \pi/4)T$.

The a slight modification of results in [5], Sect. 3 and 4 yield:

**Lemma 4.2.** Let $G = \text{Sl}(2, \mathbb{R})$ and $J \subset \mathbb{R}$ be an open subset. Then

$$\Xi_J := G \exp(i J \cdot E) \cdot x_0$$

is a $G$-invariant open subset of $X_C$ and the following holds:
(i) $G$ does not act properly if $\{-1, 1\} \subset J$.
(ii) $\Xi = \Xi_{(-1,1)}$.
(iii) $\Xi^{+} = \Xi_{(-1,\infty)}$.
(iv) $\Xi^{-} = \Xi_{(-\infty,1)}$.

We also need that $\partial \Xi$ is a fiber bundle over the affine symmetric space $G/H$ where $H = SO_{e}(1,1)$. Notice that $H$ is the stabilizer of the boundary point

$$z_{H} := \exp(-i\pi T/4) \cdot x_{0} = (1, -1) \in \partial_{\text{aff}} \Xi.$$ 

Write $\tau$ for the involution on $G$, resp. $g$, fixing $H$, resp. $h$, and denote by $g = h + q$ the corresponding eigenspace decomposition. The $h$-module $q$ breaks into two eigenspaces $q = q^{+} \oplus q^{-}$ with

$$q^{\pm} = \mathbb{R} \cdot e^{\pm} \quad \text{where} \quad e^{\pm} = \begin{pmatrix} 1 & \mp 1 \\ \pm 1 & -1 \end{pmatrix}.$$ 

Finally write

$$C = \mathbb{R}_{\geq 0} \cdot e^{+} \cup \mathbb{R}_{\geq 0} \cdot e^{-}$$

and $C^{\times} = C \setminus \{0\}$. Note that both $C$ and $C^{\times}$ are $H$-stable. We cite [5], Th. 3.1:

Lemma 4.3. Let $G = \text{Sl}(2,\mathbb{R})$. Then the map

$$G \times_{H} C \rightarrow \partial \Xi, \quad [g, e] \mapsto g \exp(i e) \cdot z_{H}$$

is a $G$-equivariant homeomorphism. Moreover,

(i) $\partial_{\text{aff}} \Xi = G \cdot z_{H} \simeq G/H$,
(ii) $\partial_{u} \Xi = G \exp(i C^{\times}) \cdot z_{H} \simeq G \times_{H} C^{\times}$,
(iii) $\partial_{u} \Xi = G \exp(i E) \cdot x_{0} \Pi G \exp(-i E) \cdot x_{0}$.

As a last piece of information we need a structural fact which is only valid for non-Hermitian groups.

Lemma 4.4. Suppose that $G$ is not of Hermitian type. Then for all $\alpha \in \Sigma$ and $E \in g^{\alpha}$ there exists an $m \in M = Z_{K}(a)$ such that

$$\text{Ad}(m)E = -E.$$ 

Proof. Let us remark first that we may assume that $G$ is of adjoint type. If $G$ is complex, then the assertion is clear as $T := \exp(i a) \subset M$ provides us with the elements we are looking for. More generally for $\dim g^{\alpha} > 1$ one knows (Kostant) that $M_{0} = \exp(m)$ acts transitively on the unit sphere in $g^{\alpha}$ (cf. [3]).

In the sequel we use the terminology and tables of the classification of real simple Lie algebras as found in the monograph [3], App. C. As $G$ is not Hermitian, Kostant’s result leaves us with the following cases.
for \( \mathfrak{g} \): \( \mathfrak{sl}(n, \mathbb{R}) \) for \( n \geq 3 \), \( \mathfrak{so}(p, q) \) for \( 0, 2 \neq p, q \) and \( p + q > 2 \), \( E \), \( E I, E V, E VI, E VII, E IX, F I \) and \( G \).

Now we make the following observation. The lemma is true for \( G = \text{Sl}(3, \mathbb{R}) \) as a simple matrix computation shows. Suppose that \( \alpha \) is such that it can be put into an \( A_2 \)-subsystem of \( \Sigma \). As \( \dim \mathfrak{g}^\alpha \) is one-dimensional (by our reduction) this means that we can put \( E \in \mathfrak{g}^\alpha \) in a subalgebra isomorphic to \( \mathfrak{sl}(3, \mathbb{R}) \). Now it is important to recall the nature of the component group of \( M \), see \cite{3}, Th. 7.55. It follows that the \( M \)-group of \( \text{Sl}(3, \mathbb{R}) \) (isomorphic to \((\mathbb{Z}/2\mathbb{Z})^2\)) embeds into the \( M \)-group of \( G \).

The \( A_2 \)-reduction described above deletes most of the cases in our list. We remain with the orthogonal cases \( \text{so}(p, q) \) for \( 0, 2 \neq p, q \) and \( p \neq q \). A simple matrix computation, which we leave to the reader, finishes the proof. \( \square \)

**Proof.** (of Theorem 4.1) Suppose that \( G \) is not of Hermitian type. Let \( X \subset D \subset \Xi \) be a \( G \)-invariant domain with proper \( G \)-action which is not contained in \( \Xi \). We shall show that \( D \) does not exist.

According to Proposition 3.1 we find a regular \( Y \in \partial \Omega \) and a non-zero nilpotent \( e \in \mathfrak{g}_C[Y] \cap i\mathfrak{g} \) such that

\[
\exp(e) \exp(iY) \cdot x_0 \in \partial_{u, \text{reg}} \Xi \cap D.
\]

Let \( \alpha \in \Sigma \) be the root corresponding to \( Y \). Write \( Y = Y^\alpha + Y' \) with \( Y^\alpha, Y' \in \mathfrak{a} \) such that \( \alpha(Y') = 0 \). It is known that \( Y^\alpha \in \partial \Omega \) and \( Y' \in \Omega \). Hence we may use \( \mathfrak{sl}(2) \)-reduction which in conjunction with Lemma 4.3 implies the existence of \( E^\alpha \in \mathfrak{g}^\alpha \) such that:

- \( \{E^\alpha, \theta(E^\alpha), [E^\alpha, \theta(E^\alpha)]\} \) is an \( \mathfrak{sl}(2) \)-triple,
- \( \exp(iE^\alpha) \exp(iY') \cdot x_0 \in \partial_{u, \text{reg}} \Xi \cap D \),

Now, as \( G \) is not of Hermitian type, Lemma [\?] implies that there exists an element \( m \in M \) such that \( \text{Ad}(m)E^\alpha = -E^\alpha \). Hence

\[
\exp(-iE^\alpha) \exp(iY') \cdot x_0 \in \partial_{u, \text{reg}} \Xi
\]

as well. But this contradicts Lemma 4.2(i). \( \square \)

### 4.2. Hermitian groups

Let now \( G \) be of Hermitian type and \( G \subseteq P^- K^- P^+ \) be a Harish-Chandra decomposition of \( G \) in \( G_C \). We define flag varieties

\[
F^+ = G_C/K^- P^+ \quad \text{and} \quad F^- = G_C/K^- P^-
\]

and inside of them we declare the flag domains

\[
D^+ = G K^- P^+ / K^- P^+ \quad \text{and} \quad D^- = G K^- P^- / K^- P^- .
\]
Then
\[(4.1) \quad X^C \hookrightarrow F^+ \times F^-, \quad gK_C \mapsto (gK_CP^+, gK_CP^-)\]
identifies \(X^C\) as a Zariski open affine piece of \(F^+ \times F^-\). In more detail: As \(G\) is of Hermitian type, there exist \(w_0 \in N_{G_C}(K_C)\) such that \(w_0P^\pm w_0^{-1} = P^\mp\). In turn, this element induces a \(G_C\)-equivariant biholomorphic map:
\[\phi : F^+ \to F^-, \quad gK_CP^+ \mapsto gw_0K_CP^- .\]
With that the embedding (4.1) gives the following identification of \(X^C\):
\[(4.2) \quad X^C = \{(z, w) \in F^+ \times F^- | \phi(z) \, \mathbf{T} w\},\]
where \(\mathbf{T}\) stands for the transversality notion in the flag variety \(F^-\). We recall what it means to be transversal. First note that the notion is \(G_C\)-invariant, i.e. for \(z, w \in F^-\) and \(g \in G_C\) one has \(z \, \mathbf{T} w\) if and only if \(gz \, \mathbf{T} gw\). Now for the base point \(z^- = K_CP^- \in F^-\) one has \(z^- \, \mathbf{T} w\) if and only if \(w \in P^- w_0z^-\).

We keep the realization of \(X^C\) in \(F^+ \times F^-\) (cf. (4.1) in mind and recall the description of \(\Xi\):
\[\Xi = D^+ \times D^-\]
(see [7]).

For subsets \(X^\pm \subset F^\pm\) we write \(X^+ \times_\mathbf{T} X^-\) for those elements \((x^+, x^-) \in X^+ \times X^-\) which are transversal, i.e. \(\phi(x^+) \, \mathbf{T} x^-\). With this terminology in mind we finally define
\[\Xi^+ = D^+ \times_\mathbf{T} F^- , \quad \Xi^- = F^+ \times_\mathbf{T} D^- .\]

4.2.1. Basic structure theory of \(\Xi^+\) and \(\Xi^-\). It is obvious that both \(\Xi^+\) and \(\Xi^-\) are open and \(G\)-invariant. However, as was pointed out by the referee, it is a priori not clear that they are connected. In order to see this let \(p_+ : \Xi^+ \to D^+\) be the projection onto the first factor. Likewise we define \(p_- : \Xi^- \to D^-\).

**Proposition 4.5.** Let \(\epsilon \in \{-, +\}\). The map \(p_\epsilon : \Xi^\epsilon \to D^\epsilon\) induces the structure of a holomorphic fiber bundle with fiber isomorphic to \(P^\epsilon\).

**Proof.** We confine ourselves with the case \(\epsilon = +\).

As \(p_+\) is \(G\)-equivariant and \(D^+\) is \(G\)-homogeneous, it is sufficient to determine the fiber \(p_+^{-1}(z^+)\). Recall that \(z^+ = K_CP^+ \in F^+\) is the base point. Now
\[p_+^{-1}(z^+) = \{(z^+, w) \in F^+ \times F^- | \phi(z^+) \, \mathbf{T} w\} .\]
Observe that $\phi(z^+) = w_0z^-$ and that $w_0z^- \tau w$ is equivalent to $z^- \tau w_0^{-1}w$. By the definition of transversality this means that $w_0^{-1}w \in P^-w_0z^-$ or $w \in w_0P^-w_0z^-$. It is no loss of generality to assume that $w_0 = w_0^{-1}$. So we arrive at $w \in P^+z^-$ and this concludes the proof of the proposition.

\textbf{Corollary 4.6.} Both $\Xi^+$ and $\Xi^-$ are contractible.

It was observed by the referee that Proposition 4.5 allows the following interesting reformulation.

\textbf{Corollary 4.7.} The map $G \times_K P^+ \to \Xi^+$, $[g,p] \mapsto (gz^+, gpz^-)$ is a $G$-equivariant diffeomorphism. In particular $\Xi^+$ is $G$-biholomorphic to $T^{0,1}D^+$, the antiholomorphic tangent bundle of $D^+$. Likewise, $\Xi^-$ is $G$-biholomorphic to $T^{0,1}D^-$. 

Corollary 4.7 combined with the Harish-Chandra decomposition implies that $\Xi^\epsilon \simeq D^\epsilon \times P^\epsilon$ as complex manifolds. In particular $\Xi^\epsilon$ is Stein.

The fact that $K_C$ normalizes $P^\epsilon$ allows us to speak of $G \times P^\epsilon$-invariant domains in $X_C$. It follows from [1.1] and Corollary 4.7 that $\Xi^\epsilon$ is $G \times P^\epsilon$-invariant.

\textbf{Proposition 4.8.} Let $\epsilon \in \{-, +\}$. The real group $G$ acts properly on $\Xi^\epsilon$. Moreover $\Xi^\epsilon$ is a maximal $G \times P^\epsilon$-invariant domain in $X_C$ for proper $G$-action.

\textbf{Proof.} As the $G$-action is proper on $D^\epsilon$, it follows that $G$ acts properly on $\Xi^\epsilon$. In the sequel we deal with $\epsilon = +$ only. It remains to show that $\Xi^+$ is a maximal $G \times P^+$-invariant domain in $X_C$ for proper $G$-action.

We argue by contradiction and suppose that $D \supset \Xi^+$ is a $G \times P^+$-domain in $X_C$ with proper $G$-action. Then $D = (D_0 \times F^-) \cap X_C$ with $D_0 \supset D^+$ a $G$-domain with proper action. Now recall the following facts:

- There are only finitely many $G$-orbits in $F^+$.
- There are precisely two orbits with proper $G$-action: $D^+$ and $\phi^{-1}(D^-)$.

The assertion follows.

\textbf{Remark 4.9.} Suppose that $G$ is of Hermitian type. Then it can be shown that if $X \subseteq D \subseteq X_C$ is a $G$-invariant domain with proper $G$-action, then $D \subseteq \Xi^+$ or $D \subseteq \Xi^-$. 

As we will not need this fact, we refrain from a proof.
If \( D \subseteq X_C \) is a subset, then we write \( DK_C \) for its preimage in \( G_C \) under the canonical projection \( G_C \to X_C \).

**Proposition 4.10.** The following assertions hold:

(i) \( \Xi^+ K_C = GK_C P^+ \),

(ii) \( \Xi^- K_C = GK_C P^- \).

**Proof.** It suffices to prove (i). Recall the embedding (4.1), and the definition of transversality condition. We deduce that \( P^+ \subset \Xi^+ K_C \). As \( \Xi^+ K_C \) is \( G \times K_C \)-invariant, it follows that \( GP^+ K_C = GK_C P^+ \subset \Xi^+ K_C \).

Conversely, Corollary [4.7] implies that \( GP^+ \) maps onto \( \Xi^+ \) and thus \( \Xi^+ \subset GP^+ K_C \). \(\square\)

We conclude this subsection with some easy facts on the structure of \( \Xi^+ \) and \( \Xi^- \) which will be used later on.

### 4.2.2. Unipotent model for \( \Xi^+ \) and \( \Xi^- \)

We begin with the unipotent parameterization of \( \Xi^+ \) and \( \Xi^- \). Some terminology is needed.

According to C. Moore, \( \Sigma \) is of type \( C_n \) or \( BC_n \). Hence we find a subset \( \{ \gamma_1, \ldots, \gamma_n \} \) of long strongly orthogonal restricted roots. We fix \( E_j \in \mathfrak{g}^{\gamma_j} \) such that \( \{ E_j, \theta(E_j), [E_j, \theta E_j] \} \) becomes an \( \mathfrak{sl}(2) \)-triple. Set \( T_j := 1/2[E_j, \theta E_j] \) and note that

\[
\Omega = \bigoplus_{j=1}^n (-\pi/2, \pi/2) T_j.
\]

We set \( V = \bigoplus_{j=1}^n \mathbb{R} \cdot E_j \) and take a cube inside \( V \) by

\[
\Lambda = \bigoplus_{j=1}^n (-1, 1) E_j.
\]

In [5], Sect. 8, we have shown that

\[
\Xi = G \exp(i\Lambda) \cdot x_0.
\]

In this parametrization of \( \Xi \) the unipotent boundary piece has a simple description:

\[
(4.3) \quad \partial_u \Xi = G \exp(i\partial\Lambda) \cdot x_0.
\]

The strategy now is to enlarge \( \Xi \) by enlarging \( \Lambda \) while maintaining that the object stays a domain on which \( G \) acts properly. But now we have to be a little bit careful with our choice of \( E_j \). Replacing \( E_j \) by \( -E_j \) has no effect for the matters cited above, but for the sequel.
Our choice is such that $\gamma_1, \ldots, \gamma_n$ are positive roots (this determines the non-compact roots in $\Sigma^+$ uniquely). We set

$$\Lambda^+ = \bigoplus_{j=1}^n (-1, \infty) E_j \quad \text{and} \quad \Lambda^- = \bigoplus_{j=1}^n (-\infty, 1) E_j.$$ 

Then, a direct generalization of Lemma 4.2(iii),(iv) yields:

**Proposition 4.11.** The following assertions hold:

(i) $\Xi^+ = G \exp(i\Lambda^+) \cdot x_0$,

(ii) $\Xi^- = G \exp(i\Lambda^-) \cdot x_0$.

**Remark 4.12.** If we define subcones of the nilcone $N \subseteq \mathfrak{g}$ by

$$N^+ = \text{Ad}(K) \left[ \bigoplus_{j=1}^n [0, \infty) E_j \right] \quad \text{and} \quad N^- = -N^+,$$

then one can show that the maps

$$G \times_K N^\pm \to \Xi^\pm, \quad [g, Y] \mapsto g \exp(iY) \cdot x_0$$

are homeomorphic.

5. **Representation theory**

Let $(\pi, \mathcal{H})$ be a unitary representation of $G$ and $\mathcal{H}_K$ the underlying Harish-Chandra module of $K$-finite vectors. Notice that $\mathcal{H}_K$ is naturally a module for $K_C$.

We say that $(\pi, \mathcal{H})$ is a highest, resp. lowest, weight representation if $G$ is of Hermitian type and $\mathfrak{p}^+ = \text{Lie}(P^+)$, resp. $\mathfrak{p}^-$, acts on $\mathcal{H}_K$ in a finite manner.

We turn to the main result of this paper.

**Theorem 5.1.** Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of $G$. Let $v \in \mathcal{H}$ be a non-zero $K$-finite vector and

$$f_v : G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

the corresponding orbit map. Then there exists a unique maximal $G \times K_C$-invariant domain $D_\pi \subseteq G_C$, independent of $v$, to which $f_v$ extends holomorphically. Explicitly:

(i) $D_\pi = G_C$ if $\pi$ is the trivial representation.

(ii) $D_\pi = \Xi^+ K_C$ if $G$ is Hermitian and $\pi$ is a non-trivial highest weight representation.

(iii) $D_\pi = \Xi^- K_C$ if $G$ is Hermitian and $\pi$ is a non-trivial lowest weight representation.
(iv) $D_x = \Xi K_C$ in all other cases.

Proof. If $\pi$ is trivial, then the assertion is clear. So let us assume that $\pi$ is non-trivial in the sequel. Fix a nonzero $K$-finite vector $v$ and consider the orbit map $f_v : G \to H$. We recall the following two facts:

- $f_v$ extends to a holomorphic $G$-equivariant map $f_v : \Xi K_C \to H$ (see [7], Th. 1.1).
- If $D_v \subseteq G C$ is a $G \times K_C$-invariant domain to which $f_v$ extends holomorphically, then $G$ acts properly on $D_v/K_C$ (see [5], Th. 4.3).

We begin with the case where $G$ is not of Hermitian type. Here the assertion follows from the bulleted items above in conjunction with Theorem 4.1. So we may assume for the remainder that $G$ is of Hermitian type. If $\pi$ is a highest weight representation, then it is clear that $f_v$ extends to a holomorphic map $G K_C P^+ \to H$. Thus, in this case $\Xi^+ K_C = G K_C P^+$ (cf. Proposition 4.10) is a maximal domain of definition for $f_v$ by Proposition 4.8 and the second bulleted item from above. Likewise, if $(\pi, H)$ is a lowest weight representation, then $\Xi^- K_C$ is a maximal domain of definition of $f_v$. As both $\Xi^+$ and $\Xi^-$ are simply connected with sufficiently regular boundary, it follows that these maximal domains are in fact unique.

It remains to show:

- If $f_v$ extends holomorphically on a domain $D \supset \Xi$ such that $D \cap [\Xi^+ \setminus \Xi] \neq \emptyset$, then $(\pi, H)$ is a highest weight representation.
- If $f_v$ extends holomorphically on a domain $D \supset \Xi$ such that $D \cap [\Xi^- \setminus \Xi] \neq \emptyset$, then $(\pi, H)$ is a lowest weight representation.

It is sufficient to deal with the first case. So suppose that $f_v$ extends to a bigger domain $D$ such that $D \cap [\Xi^+ \setminus \Xi] \neq \emptyset$. Taking derivatives and applying the fact that $d\pi(U(g_C))v = H_K$, we see that $f_u$ extends to $D$ for all $u \in H_K$. By Proposition 3.1 (4.3) and our assumption we find $1 \leq j \leq n$ be such that $\exp(iE_j) \exp(iY) \cdot x_0 \in D$ for some $Y \in \Omega$ with $\gamma_j(Y) = 0$. Let $G_j < G$ be the analytic subgroup corresponding to the $\mathfrak{sl}(2)$-triple $\{E_j, \theta(E_j), [E_j, \theta(E_j)]\}$. Basic representation theory of type I-groups in conjunction with [5], Th. 4.7, yields that $\pi |_{G_j}$ breaks into a direct sum of highest weight representations. Applying $N_K(a)$ (which in particular permutes the $G_k$ and preserves $H_K$) we see that above matters hold for any other $G_k$ as well (note that $Y$ might change but this does not matter as $\Omega$ is $N_K(a)$-invariant). It follows that $\pi$ is a highest weight representation and completes the proof of the theorem. □
Remark 5.2. The domains $\Xi$, $\Xi^+$ and $\Xi^-$ are independent of the choice of the connected group $G$. Accordingly, the above theorem holds for all simple connected non-compact Lie groups $G$, i.e. we can drop the assumption that $G \subseteq G_C$ and $G_C$ simply connected.

Problem 5.3. The above theorem should hold true for all irreducible admissible Banach representations of $G$ under the reservation that (i) gets modified to : $D_\pi = G_C$ if $\pi$ is finite dimensional.

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