DECAY RATE AND RADIAL SYMMETRY OF THE EXPONENTIAL ELLIPTIC EQUATION

KIN MING HUI AND SUNGHOON KIM

Abstract. Let $n \geq 3$, $\alpha, \beta \in \mathbb{R}$, and let $v$ be a solution $\Delta v + \alpha e^v + \beta x \cdot \nabla e^v = 0$ in $\mathbb{R}^n$, which satisfies the conditions $\lim_{R \to \infty} \frac{1}{\log R} \int_1^R \rho^{1-a} \left( \int_{B_\rho} e^v \, dx \right) \, d\rho \in (0, \infty)$ and $|x|^2 e^v(x) \leq A_1$ in $\mathbb{R}^n$. We prove that $\frac{v(x)}{\log |x|} \to -2$ as $|x| \to \infty$ and $\alpha > 2\beta$. As a consequence under a mild condition on $v$ we prove that the solution is radially symmetric about the origin.

1. Introduction

In this paper we will study various properties of the solution $v$ of the nonlinear elliptic equation

$$\Delta v + \alpha e^v + \beta x \cdot \nabla e^v = 0 \quad \text{in } \mathbb{R}^n \quad \text{(1.1)}$$

for any $n \geq 3$ where $\alpha, \beta \in \mathbb{R}$, are some constants. Let $v = \log u$. Then $u$ satisfies

$$\Delta \log u + \alpha u + \beta x \cdot \nabla u = 0, \quad u > 0, \quad \text{in } \mathbb{R}^n \quad \text{(1.2)}$$

As observed by S.Y. Hsu [H3], the radial symmetric solution of (1.2) is the singular limit of the radial symmetric solutions of the nonlinear elliptic equation,

$$\Delta (u^m/m) + \alpha u + \beta x \cdot \nabla u = 0, \quad u > 0, \quad \text{in } \mathbb{R}^n \quad \text{(1.3)}$$

as $m \searrow 0$. On the other hand as observed by P. Daskalopoulos and N. Sesum [DS], K.M. Hui and S.H. Kim [HK1], [HK2], [L2] also arises in the study of the extinction behaviour and global behaviour of the solutions of the logarithmic diffusion equation,

$$u_t = \Delta \log u, \quad u > 0, \quad \text{in } \mathbb{R}^n \quad \text{(1.4)}$$

(1.2) also arises in the study of self-similar solutions of (1.4) (DS, HK1, HK2, V1, V2). Hence in order to understand the behaviour of the solutions of (1.3) and (1.4) it is important to understand the properties of solutions of (1.1).

In [HS2] S.Y. Hsu proved that there exists a radially symmetric solution of (1.1) (or equivalently (1.2)) if and only if either $\alpha \geq 0$ or $\beta > 0$. She also proved that when $n \geq 3$ and $\alpha > \max(2\beta, 0)$, then any radially symmetric solution $v$ of (1.1) satisfies

$$\lim_{|x| \to \infty} |x|^2 e^v(x) = \frac{2(n-2)}{\alpha - 2\beta}. \quad \text{(1.5)}$$

By (1.5) and a direct computation the radially symmetric solution $v$ of (1.1) satisfies

$$\lim_{|x| \to \infty} \frac{v(x)}{\log |x|} = -2, \quad \text{(1.6)}$$

Date: Sept 3, 2012.

2010 Mathematics Subject Classification. Primary 35B40 Secondary 35B06, 35J60.

Key words and phrases. decay rate, exponential elliptic equation, radial symmetry.
(1.7) \[ A_0 := \lim_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{|x|<\rho} e^v \, dx \right) \, d\rho \in (0, \infty) \]
and
(1.8) \[ |x|^2 e^{v(x)} \leq A_1 \quad \forall x \in \mathbb{R}^n \]
for some constant \( A_1 > 0 \). A natural question is if \( v \) is a solution of (1.1) which satisfies (1.7) and (1.8) for some constant \( A_1 > 0 \), will \( v \) satisfy (1.6) and is \( v \) radially symmetric about the origin? We answer the first question in the affirmative in this paper. For the second question we prove that under some conditions on the solution \( v \) of (1.1), \( v \) is radially symmetric about the origin.

For any solution \( v \) of (1.1) we define the rotation operator \( \Phi_{ij} \) by
\[ \Phi_{ij}(x) = x_i v_j(x) - x_j v_i(x), \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, i \neq j, i, j = 1, \ldots, n. \]
Note that if we write \( x_1 = \rho \cos \theta \) and \( x_2 = \rho \sin \theta \) where \( \rho = \sqrt{x_1^2 + x_2^2} \), then \( \Phi_{12}(x) = \frac{\partial v}{\partial \theta}(x) \). We are now ready to state the main results of this paper.

**Theorem 1.1.** Let \( n \geq 3 \) and \( \alpha, \beta \in \mathbb{R} \). Suppose \( v \) is a solution of (1.1) which satisfies (1.7) and (1.8) for some constant \( A_1 > 0 \). Then \( v \) satisfies (1.6) and \( \alpha > 2\beta \).

**Corollary 1.2.** Let \( n \geq 3 \). Suppose \( \alpha \leq 2\beta \). Then (1.1) does not have any solution that satisfies both (1.7) and (1.8) for some constant \( A_1 > 0 \).

**Theorem 1.3.** Let \( n \geq 3 \) and \( 2\beta < \alpha < n\beta \). Suppose \( v \) is a solution of (1.1) which satisfies (1.7), (1.8),
\[ \|x \cdot \nabla v\|_{L^\infty(\mathbb{R}^n)} \leq C < \infty. \]
and
(1.10) \[ \lim_{|x| \to \infty} |x|^{n-2} |\Phi_{ij}(x)| = 0, \quad \forall i \neq j, i, j = 1, \ldots, n. \]
Then there exists a constant \( R_0 > 0 \) such that if \( v \) is radially symmetric in \( B_{R_0} \), then \( v \) is radially symmetric in \( \mathbb{R}^n \).

Note that although there are many research done on the radial symmetry of elliptic equations without first order term by B. Gidas, W.M. Ni, and L. Nirenberg [GNN], L. Caffaralli, B. Gidas and J. Spruck [CGS], W. Chen and C. Li [CL], S.D. Taliaferro [T] and others, very little is known about the radial symmetry of elliptic equations with non-zero first order term. The reason is that one cannot use the moving plane technique to prove the radial symmetry of the solution for elliptic equations with non-zero first order term. The recent paper [KM] by E. Kamalinejad and A. Moradifam is one of the few papers that studies the radial symmetry of elliptic equations with non-zero first order term. Hence our result on radial symmetry is new.

The plan of the paper is as follows. In section two we will prove Theorem 1.1 and Corollary 1.2. In section three we will prove Theorem 1.3. For any \( r > 0 \), \( x_0 \in \mathbb{R}^n \), let \( B_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < r \} \) and \( B_r = B_r(0) \). Let \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \). We will let \( n \geq 3, \alpha, \beta \in \mathbb{R} \), and let \( v \) be a solution of (1.1) which satisfies both (1.7) and (1.8) for some constant \( A_1 > 0 \) for the rest of the paper. We will also let \( A_0 \) be given by (1.6) for the rest of the paper.

2. DECAY RATE OF THE SOLUTION OF (1.1)

In this section we will use a modification of the technique of [Hs1] to prove the decay rate (1.6) for \( v \). We first start with a lemma.
Lemma 2.1. For any $x_0 \in \mathbb{R}^n$, we have

\begin{equation}
(2.1) \lim_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho = A_0.
\end{equation}

Proof. Let $x_0 \in \mathbb{R}^n$ and $\rho > |x_0|$. Since $B_{\rho-|x_0|} \subset B_\rho(x_0)$, for any $R > R_0 > |x_0| + 1$,

\begin{align*}
&\frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho \\
&\quad \geq \frac{1}{\log R} \int_{R_0}^R \frac{1}{\rho^{n-1}} \left( \int_{B_{\rho-|x_0|}} e^v \, dx \right) \, d\rho \\
&\quad \geq \frac{1}{\log R} \left( \frac{R_0 - |x_0|}{R_0} \right)^{n-1} \int_{R_0-|x_0|}^{R-|x_0|} \frac{1}{\rho^{n-1}} \left( \int_{B_\rho} e^v \, dx \right) \, d\rho \\
&\quad = -\frac{1}{\log R} \left( \frac{R_0 - |x_0|}{R_0} \right)^{n-1} \int_{1}^{R_0-|x_0|} \frac{1}{\rho^{n-1}} \left( \int_{B_\rho} e^v \, dx \right) \, d\rho \\
&\quad + \log(R - |x_0|) \left( \frac{R_0 - |x_0|}{R_0} \right)^{n-1} \left[ \frac{1}{\log(R - |x_0|)} \int_1^{R-|x_0|} \frac{1}{\rho^{n-1}} \left( \int_{B_\rho} e^v \, dx \right) \, d\rho \right].
\end{align*}

Letting $R \to \infty$,

\begin{equation}
(2.2) \liminf_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho \geq \left( \frac{R_0 - |x_0|}{R_0} \right)^{n-1} A_0
\end{equation}

\Rightarrow \liminf_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho \geq A_0 \quad \text{as } R_0 \to \infty

Similarly,

\begin{equation}
(2.3) \limsup_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho \leq A_0.
\end{equation}

By (2.2) and (2.3), we get (2.1) and the lemma follows. \hfill \Box

Lemma 2.2. For any $x_0 \in \mathbb{R}^n$, we have

\begin{equation}
(2.4) \lim_{\rho \to \infty} \rho^2 \int_{|\sigma| = 1} e^{v(x_0 + \rho \sigma)} \, d\sigma = (n-2)A_0
\end{equation}

and

\begin{equation}
(2.5) \lim_{\rho \to \infty} \left[ \frac{1}{\log \rho} \int_{|\sigma| = 1} v(x_0 + \rho \sigma) \, d\sigma \right] = -(\alpha - 2\beta)A_0.
\end{equation}

Proof. Let $x_0 \in \mathbb{R}^n$ and $\rho > 0$. We first observe that (1.1) can be rewritten as

\begin{equation}
(2.6) \Delta v + (\alpha - n\beta)e^v + \beta \text{div}(xe^v) = 0 \quad \text{in } \mathbb{R}^n.
\end{equation}
Integrating (2.6) over $B_\rho(x_0)$,
\[
0 = \rho^{-1} \int_{|\sigma| = 1} \frac{\partial v}{\partial \rho}(x_0 + \rho \sigma) \, d\sigma + (\alpha - n\beta) \int_{B_\rho(x_0)} e^v \, dx + \beta \rho^n \int_{|\sigma| = 1} e^{(x_0 + \rho \sigma)} \, d\sigma
\]
\[
+ \beta \rho^{n-1} \int_{|\sigma| = 1} e^{(x_0 + \rho \sigma)}(x_0 \cdot \sigma) \, d\sigma. \tag{2.7}
\]

Let $R > R_0 \geq 1$. Dividing (2.7) by $\rho^n$ and integrating over $\rho \in (1, R)$,
\[
\int_{|\sigma| = 1} v(x_0 + R \sigma) \, d\sigma - \int_{|\sigma| = 1} v(x_0 + \sigma) \, d\sigma
\]
\[
= -(\alpha - n\beta) \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho - \beta \int_1^R \rho \left( \int_{|\sigma| = 1} e^{(x_0 + \rho \sigma)} \, d\sigma \right) \, d\rho
\]
\[
- \beta \int_1^R \left( \int_{|\sigma| = 1} e^{(x_0 + \rho \sigma)}(x_0 \cdot \sigma) \, d\sigma \right) \, d\rho. \tag{2.8}
\]

Let $\{\rho_i\}_{i=1}^\infty$ be a sequence such that $\rho_i > 2|x_0| + 1$ for all $i \in \mathbb{Z}^+$ and $\rho_i \to \infty$ as $i \to \infty$. Then by (1.3) there exists a constant $C_1 > 0$ such that
\[
\rho_i^2 \int_{|\sigma| = 1} e^{v(x_0 + \rho_i \sigma)} \, d\sigma \leq C_1 \quad \forall i \in \mathbb{Z}^+.
\]

Hence the sequence $\{\rho_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself such that
\[
\rho_i^2 \int_{|\sigma| = 1} e^{v(x_0 + \rho_i \sigma)} \, d\sigma
\]
converges to some non-negative number as $i \to \infty$. On the other hand by (2.1), there exists a constant $R_1 > 1$ such that
\[
\frac{A_0}{2} \leq \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho(x_0)} e^v \, dx \right) \, d\rho \leq \frac{1}{(n-2)\log R} \int_{B_R(x_0)} e^v \, dx \quad \forall R \geq R_1 \tag{2.9}
\]
\[
\Rightarrow \int_{B_{\rho_i}(x_0)} e^v \, dx \geq \frac{(n-2)}{2} A_0 \log \rho \to \infty \quad \text{as } \rho \to \infty.
\]

Then by (1.7), (2.1), (2.9), and the l'Hospital rule,
\[
A_0 = \lim_{\rho \to \infty} \frac{\rho_i^{-1} \int_{B_{\rho_i}(x_0)} e^v \, dx}{\rho_i^{-1} \int_{|\sigma| = 1} e^{v(x_0 + \rho_i \sigma)} \, d\sigma} = \lim_{i \to \infty} \rho_i^{n-2} \int_{B_{\rho_i}(x_0)} e^v \, dx
\]
\[
= \lim_{i \to \infty} \rho_i^{n-1} \int_{|\sigma| = 1} e^{v(x_0 + \rho_i \sigma)} \, d\sigma = \frac{1}{(n-2)} \lim_{i \to \infty} \rho_i^2 \int_{|\sigma| = 1} e^{v(x_0 + \rho_i \sigma)} \, d\sigma
\]

Since the sequence $\{\rho_i\}_{i=1}^\infty$ is arbitrary, (2.4) follows. Then by (2.4) for any $0 < \varepsilon < 1$ there exists a constant $R_0 > 1$ such that
\[
\rho_i^2 \int_{|\sigma| = 1} e^{v(x_0 + \rho_i \sigma)} \, d\sigma - (n-2)A_0 < (n-2)A_0 \varepsilon \quad \forall \rho \geq R_0. \tag{2.10}
\]
By (2.10),
\[
\int_{1}^{R_0} \rho \left( \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} \, d\sigma \right) \, d\rho + (1-\varepsilon)(n-2)A_0 \log(R/R_0)
\leq \int_{1}^{R} \rho \left( \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} \, d\sigma \right) \, d\rho
\leq \int_{1}^{R_0} \rho \left( \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} \, d\sigma \right) \, d\rho + (1+\varepsilon)(n-2)A_0 \log(R/R_0) \quad \forall R > R_0
\]
and
\[
\left| \int_{1}^{R} \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} (x_0 \cdot \sigma) \, d\sigma \, d\rho \right|
\leq |x_0| \int_{1}^{R_0} \left( \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} \, d\sigma \right) \, d\rho + 2(n-2)A_0|x_0| \left( \frac{1}{R_0} - \frac{1}{R} \right)
\]
for any $R > R_0$. Dividing (2.11) and (2.12) by $\log R$ and letting first $R \to \infty$ and then $\varepsilon \to 0$, we get
\[
\lim_{R \to \infty} \frac{1}{\log R} \int_{1}^{R} \rho \left( \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} \, d\sigma \right) \, d\rho = (n-2)A_0
\]
and
\[
\lim_{R \to \infty} \frac{1}{\log R} \int_{1}^{R} \int_{|\sigma|=1} e^{v(x_0+\rho \sigma)} (x_0 \cdot \sigma) \, d\sigma \, d\rho = 0.
\]
Dividing (2.8) by $\log R$ and letting $R \to \infty$, by (1.7), (2.13) and (2.14), we get (2.5) and the lemma follows.

We now let
\[
\begin{align*}
\left\{ \begin{array}{ll}
w_1(x) = \frac{1}{n(2-n)\omega_n} \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \, e^v(y) \, dy & \forall x \in \mathbb{R}^n \\
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
\quad \left\{ \begin{array}{ll}
w_{2,R}(x) = \frac{1}{n(2-n)\omega_n} \int_{|y| \leq R} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \, \text{div} \left( e^v(y) \right) \, dy & \forall x \in B_R, \ R > 0 \\
\end{array} \right.
\end{align*}
\]
where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Then $w_1$ and $w_{2,R}$ are well-defined in $\mathbb{R}^n$ and $B_R$ respectively with $w_1 \in C(\mathbb{R}^n)$ and $w_{2,R} \in C(B_R)$. Moreover
\[
\begin{align*}
\begin{array}{ll}
\quad \left\{ \begin{array}{ll}
\Delta w_1 = e^v & \text{in } \mathbb{R}^n \\
\Delta w_{2,R} = \text{div} \left( e^v(y) \right) & \text{in } B_R \quad \forall R > 0.
\end{array} \right.
\end{align*}
\]

Lemma 2.3. As $R \to \infty$, $w_{2,R}$ will converge uniformly on every compact subset of $\mathbb{R}^n$ to
\[
\begin{align*}
w_2(x) &= \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} e^v(y) \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \cdot y \, dy \\
&= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \left( \frac{(x-y) \cdot y}{|x-y|^n} + \frac{1}{|y|^{n-2}} \right) e^v(y) \, dy.
\end{align*}
\]
Proof. Let $R_0 > 1$, $R > 2R_0$ and $|x| \leq R_0$. Now

$$w_{2,R}(x) = \frac{1}{n(2-n)\omega_n} \int_{|y|=R} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) e^{v(y)} y \cdot v(y) d\sigma_R(y)$$

(2.18)

$$+ \frac{1}{n(n-2)\omega_n} \int_{|y|<R} e^{v(y)} \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \cdot y dy$$

$$= I_{1,R}(x) + I_{2,R}(x)$$

where $v(y) = \frac{y}{|y|}$. Since there exists a constant $C_1 > 0$ such that

$$\left| \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right| \leq C_1 \frac{|x|}{|y|^{n-1}} \quad \forall |y| \geq 2|x|,$$

by (2.11), there exists a constant $C_2 > 0$ such that

$$\sup_{|x|<R_0} |I_{1,R}(x)| \leq \frac{C_2 R_0}{R} \to 0 \quad \text{as } R \to \infty.$$  

By the Taylor expansion (P.231 of [GNN]),

$$\frac{1}{|x-y|^n} = \frac{1}{|y|^n} \left( 1 + \frac{n}{|y|^2} \sum_{j=1}^{n} x_j y_j + O \left( \frac{1}{|y|^2} \right) \right) \quad \forall |y| \geq 2|x|, \ x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n).$$

Hence there exists a constant $C_3 > 0$ such that

$$\left| \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \cdot y \right| = (n-2) \left| \frac{(x-y) \cdot y}{|x-y|^n} + \frac{1}{|y|^{n-2}} \right|$$

$$= (n-2) \left| \frac{x \cdot y}{|y|^n} \left( 1 + \frac{n}{|y|^2} \sum_{j=1}^{n} x_j y_j + \cdots \right) - \frac{1}{|y|^{n-2}} \left( \frac{n}{|y|^2} \sum_{j=1}^{n} x_j y_j + \cdots \right) \right|$$

(2.21)

$$\leq C_3 \frac{|x|}{|y|^{n-1}}, \quad \forall |y| \geq 2|x|, \ x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n).$$

By (2.10) and (2.21),

$$\int_{2R_0 < |y| < R} e^{v(y)} \left| \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \cdot y \right| dy$$

(2.22)

$$< \int_{2R_0}^{R} \frac{C_3 R_0}{\rho^2} \int_{|\sigma|=1} e^{v(\rho \sigma)} d\sigma d\rho \leq C'' R_0 \int_{2R_0}^{R} \frac{1}{\rho^2} d\rho \leq C'' < \infty$$

holds for any $|x| \leq R_0$ and $R > 2R_0$. On the other hand since $v$ is continuous and $v$ satisfies (1.8), $e^{v} \in L^\infty(\mathbb{R}^n)$. Hence by (2.21),

$$\int_{|y|<2R_0} e^{v(y)} \left| \nabla_y \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) \cdot y \right| dy$$

(2.23)

$$\leq C_3 R_0 \int_{|y|<3R_0} \frac{1}{|y|^{n-1}} dy = 3n\omega_n C_3 R_0^2 < \infty.$$
By (2.22) and (2.23) and the Lebesgue Dominated Convergence Theorem,

\begin{equation}
I_{2,R}(x) \to \frac{1}{n \omega_n} \int_{\mathbb{R}^n} \left( \frac{(x-y) \cdot y}{|x-y|^n} + \frac{1}{|y|^{n-2}} \right) e^{v(y)} \, dy
\end{equation}

uniformly on \( \{|x| \leq R_0\} \) as \( R \to \infty \). By (2.18), (2.20) and (2.24), the lemma follows. \( \square \)

**Lemma 2.4.** There exists a constant \( C > 0 \) such that

\begin{equation}
|w_1(x)| \leq C \log |x| \quad \forall |x| \geq 2.
\end{equation}

**Proof.** Let \( |x| \geq 2 \). We first split \( w_1 \) into two parts as follows.

\[
w_1(x) = \frac{1}{n(2-n)\omega_n} \int_{|y| > 2|x|} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) e^{v(y)} \, dy \\
+ \frac{1}{n(2-n)\omega_n} \int_{|y| \leq 2|x|} e^{v(y)} \, dy + \frac{1}{n(n-2)\omega_n} \int_{|y| \leq 2|x|} \frac{e^{v(y)}}{|y|^{n-2}} \, dy
\]

\begin{equation}
:= I_1 + I_2 + I_3.
\end{equation}

By (2.10),

\begin{equation}
\left| \int_{|\sigma|=1} e^{v(\rho \sigma)} \, d\sigma \right| \leq \frac{C}{\rho^2}, \quad \forall \rho > 0
\end{equation}

for some constant \( C > 0 \). Hence by (2.19) and (2.27),

\begin{equation}
|I_1| \leq C \int_{2|x|}^{\infty} \frac{|x|}{\rho^{n-1}} \cdot \frac{1}{\rho^2} \cdot \rho^{n-1} \, d\rho = C' < \infty \quad \forall x \in \mathbb{R}^n.
\end{equation}

for some constant \( C' > 0 \). By (2.27),

\[
I_3 = \frac{1}{n(n-2)\omega_n} \int_{0 < |y| \leq 1} \frac{e^{v(y)}}{|y|^{n-2}} \, dy + \frac{1}{n(n-2)\omega_n} \int_{1 \leq |y| \leq 2|x|} \frac{e^{v(y)}}{|y|^{n-2}} \, dy
\]

\begin{equation}
\leq C \int_{0}^{1} \rho^{n-2} \cdot \rho^{n-1} \, d\rho + C \int_{1}^{2|x|} \frac{1}{\rho^{n-2}} \cdot \frac{1}{\rho^2} \cdot \rho^{n-1} \, d\rho
\]

\[= C (1 + \log(2|x|))\]

\[\leq C' \log |x| \quad \forall |x| \geq 2.
\]

On the other hand by (1.8),

\begin{equation}
|I_2| \leq \int_{D_1(x)} A_1 \frac{1}{|x-y|^{n-2} |y|^2} \, dy + \int_{D_2(x)} A_1 \frac{1}{|x-y|^{n-2} |y|^2} \, dy = I_{2,1} + I_{2,2}
\end{equation}

where

\begin{equation}
\begin{aligned}
D_1(x) &= \{ y \in \mathbb{R}^n : |y| \leq 2|x| \text{ and } |x - y| \leq \frac{|x|}{2} \} \\
D_2(x) &= \{ y \in \mathbb{R}^n : |y| \leq 2|x| \text{ and } |x - y| \geq \frac{|x|}{2} \}.
\end{aligned}
\end{equation}

Since

\[|x - y| \leq \frac{|x|}{2} \Rightarrow |y| \geq \frac{|x|}{2},\]

we have

\begin{equation}
I_{2,1} \leq \frac{C}{|x|^2} \int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|^{n-2}} \, dy = C' < \infty \quad \forall x \neq 0
\end{equation}
for some constant $C' > 0$. Finally on $D_2(x)$,

$$(2.33) \quad I_{2,2} \leq \frac{C}{|x|^{n-2}} \int_{|y| \leq 2|x|} \frac{1}{|y|^2} \, dy = C'' < \infty \quad \forall x \neq 0$$

for some constant $C'' > 0$. By (2.26), (2.28), (2.29), (2.30), (2.32) and (2.33), we get (2.25) and lemma follows.

**Lemma 2.5.** The following holds.

$$v(x) + (\alpha - n\beta)w_1(x) + \beta w_2(x) = v(0) \quad \forall \mathbb{R}^n.$$  

**Proof.** Let $q = v + (\alpha - n\beta)w_1 + \beta w_2$. Then by (2.16) and Lemma 2.3

$$\Delta q = 0 \quad \text{in} \quad \mathbb{R}^n \quad \text{in the distribution sense.}$$

Let $x_0, x_1 \in \mathbb{R}^n$. By (1.8) there exists a constant $R_0 > 0$ such that

$$(2.34) \quad v(x) < 0 \quad \forall |x| \geq R_0.$$  

By the mean value theorem for harmonic functions,

$$(2.35) \quad q(x_0) - q(x_1) = \frac{1}{|B_R|} \left( \int_{B_R(x_0)} q \, dx - \int_{B_R(x_1)} q \, dx \right) \quad \forall R > 0.$$  

Let $a = |x_0 - x_1|$ and $R > R_0 + 2 + 2a + |x_0| + |x_1|$. Since $B_{R_0} \subset B_{R-2a}(x_1) \subset B_{R-a}(x_0) \subset B_{R}(x_1)$,

$$(2.36) \quad \mathbb{R}^n \setminus B_{R-a}(x_0) \subset \mathbb{R}^n \setminus B_{R-2a}(x_1) \subset \mathbb{R}^n \setminus B_{R_0}.$$  

Then by (2.34), (2.36) and Lemma 2.2

$$\frac{1}{|B_R|} \left( \int_{B_R(x_0)} v \, dx - \int_{B_R(x_1)} v \, dx \right)$$

$$= \frac{1}{|B_R|} \left( \int_{B_R(x_0) \setminus B_{R-a}(x_0)} v \, dx - \int_{B_R(x_1) \setminus B_{R-a}(x_0)} v \, dx \right)$$

$$\leq -\frac{1}{|B_R|} \int_{B_R(x_1) \setminus B_{R-a}(x_0)} v \, dx$$

$$\leq -\frac{1}{|B_R|} \int_{B_R(x_1) \setminus B_{R-2a}(x_1)} v \, dx$$

$$\leq -\frac{1}{|B_R|} \left( \inf_{R-2a \leq \rho \leq R} \left[ \frac{1}{\log \rho} \int_{|\sigma| = 1} v(x_1 + \rho \sigma) \, d\sigma \right] \right) \int_{R-2a}^R \rho^{n-1} \log \rho \, d\rho$$

$$\leq -\frac{2aR^{n-1} \log R}{|B_R|} \left( \inf_{R-2a \leq \rho \leq R} \left[ \frac{1}{\log \rho} \int_{|\sigma| = 1} v(x_1 + \rho \sigma) \, d\sigma \right] \right)$$

$$\to 0 \quad \text{as} \ R \to \infty.$$  

On the other hand, since $(B_R(x_0) \setminus B_R(x_1)) \cup (B_R(x_1) \setminus B_R(x_0)) \subset (B_{R+a}(x_0) \setminus B_{R-a}(x_0))$,

$$\frac{1}{|B_R|} \left( \int_{B_R(x_0)} ((\alpha - n\beta)w_1 + \beta w_2) \, dx - \int_{B_R(x_1)} ((\alpha - n\beta)w_1 + \beta w_2) \, dx \right)$$

$$\leq \frac{1}{|B_R|} \int_{B_{R+a}(x_0) \setminus B_{R-a}(x_0)} \|w_1\| \, dx + \frac{\beta}{|B_R|} \int_{B_{R+a}(x_0) \setminus B_{R-a}(x_0)} |w_2| \, dx$$

$$:= \|\alpha - n\beta\| J_1 + \|\beta\| J_2.$$  

By Lemma 2.4
\( J_1 \leq \frac{C_1}{|B_R|} \int_{R-a \leq |x-x_0| \leq R+a} \log |x| \, dx \leq \frac{2C_1 a \log(|x_0| + R + a) |\partial B_{R+a}|}{|B_R|} \to 0 \quad \text{as } R \to \infty \)
for some constant \( C_1 > 0 \). We next observe that by (2.17),
\[
J_2 \leq \frac{1}{n \omega_n |B_R|} \int_{B_{R+a}(x_0) \setminus B_{R-a}(x_0)} \left| \int_{|y| > 2|x|} \left( \frac{(x-y) \cdot y}{|y|^n} + \frac{1}{|y|^{n-2}} \right) e^{\nu(y)} \, dy \right| \, dx \\
+ \frac{1}{n \omega_n |B_R|} \int_{B_{R+a}(x_0) \setminus B_{R-a}(x_0)} \left[ \int_{|y| \leq 2|x|} \frac{|y|}{|x-y|^{n-2}} e^{\nu(y)} \, dy \right]
\]
\( (2.40) \quad := J_{2,1} + J_{2,2} \).
By (2.21) and (2.27),
\[
J_{2,1} \leq \frac{C}{|B_R|} \int_{B_{R+a}(x_0) \setminus B_{R-a}(x_0)} |x| \left( \int_{2|x|}^{\infty} \frac{1}{\rho^{n-1}} \frac{1}{\rho^2} \rho^{n-1} \, d\rho \right) \, dx \\
= \frac{C}{2 |B_R|} |B_{R+a}(x_0) \setminus B_{R-a}(x_0)| \leq C'' (R + a)^{n-1} \quad \text{as } R \to \infty.
\]

Let
\[
\tilde{I}_1 = \int_{|y| \leq 2|x|} \frac{|y|}{|x-y|^{n-1}} e^{\nu(y)} \, dy \quad \text{and} \quad \tilde{I}_2 = \int_{|y| \leq 2|x|} \frac{e^{\nu(y)}}{|y|^{n-2}} \, dy.
\]

Then by the proof of Lemma 2.4
\( \tilde{I}_2 \leq C \log |x| \quad \forall |x| \geq 2 \).

On the other hand by (1.8),
\[
\tilde{I}_1 \leq \int_{|y| \leq 2|x|} \frac{A_1}{|x-y|^{n-1}|y|} \, dy = \int_{D_1(x)} \frac{A_1}{|x-y|^{n-1}|y|} \, dy + \int_{D_2(x)} \frac{A_1}{|x-y|^{n-1}|y|} \, dy
\]
\( (2.43) \quad = \tilde{I}_{1,1} + \tilde{I}_{1,2} \)
where \( D_1(x), D_2(x) \), are as given by (2.31). Since \( |y| \geq |x|/2 \) for all \( y \in D_1(x) \),
\[
\tilde{I}_{1,1} \leq \frac{C}{|x|} \int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|^{n-1}} \, dy = C' < \infty
\]
and
\[
\tilde{I}_{1,2} \leq \frac{2^{n-1} A_1}{|x|^{n-1}} \int_{|y| \leq 2|x|} \frac{1}{|y|} \, dy = C'' < \infty
\]
for some constants \( C' > 0, C'' > 0 \). By (2.42), (2.43), (2.44) and (2.45),
\[
J_{2,2} \leq \frac{C}{|B_R|} \int_{B_{R+a}(x_0) \setminus B_{R-a}(x_0)} (1 + \log |x|) \, dx
\]
\( (2.46) \quad \leq C(1 + \log(|x_0| + R + a)) \frac{|B_{R+a}(x_0) \setminus B_{R-a}(x_0)|}{|B_R|} \leq C'(R + a)^{n-1} \log(a + |x_0| + R) \to 0 \quad \text{as } R \to \infty.\)
Letting \( R \to 0 \) in (2.35), by (2.37), (2.38), (2.39), (2.40), (2.41) and (2.46), we get
\[
q(x_0) - q(x_1) \leq 0 \quad \forall x_0, x_1 \in \mathbb{R}^n.
\]
Since \( x_0, x_1 \) are arbitrary, by interchanging the roles of \( x_0 \) and \( x_1 \) in the above argument we get
\[
q(x_1) - q(x_0) \leq 0 \quad \forall x_0, x_1 \in \mathbb{R}^n.
\]
Hence
\[
q(x_0) - q(x_1) = 0 \quad \forall x_0, x_1 \in \mathbb{R}^n.
\]
Thus \( q \) is a constant. Hence \( q(x) = q(0) = v(0) \) for any \( x \in \mathbb{R}^n \) and the lemma follows. \( \square \)

**Lemma 2.6.**
\[
\lim_{|x| \to \infty} \frac{w_1(x)}{\log |x|} = \frac{A_0}{n\omega_n} \quad \text{as} \quad |x| \to \infty.
\]

**Proof.** Let \( I_1, I_2, \) and \( I_3 \) be the same as the proof of Lemma 2.4. Then \( I_1 \) satisfies (2.28). By the proof of Lemma 2.4 there exists a constant \( C_1 > 0 \) such that
\[
I_3 = \frac{1}{n(n - 2)\omega_n} \int_{|y| \leq |x|} e^{v(y)} |y|^{-n-2} dy + \frac{1}{n(n - 2)\omega_n} \int_{|y| \leq |x| \leq 2|x|} e^{v(y)} |y|^{-n-2} dy.
\]
By Lemma 2.2 there exists a constant \( R_0 > 1 \) such that (2.10) holds with \( \epsilon = 1/3 \). Now
\[
I_{3,1} = \frac{1}{n(n - 2)\omega_n} \int_{|y| \leq |x|} e^{v(y)} |y|^{-n-2} dy \geq C \int_{R_0}^{|x|} \frac{d\rho}{\rho} = C \log \left( \frac{|x|}{R_0} \right) \to \infty \quad \text{as} \quad |x| \to \infty,
\]
by (2.4) and l'Hospital rule,
\[
\lim_{|x| \to \infty} \frac{I_{3,1}}{\log |x|} = \frac{\lim_{|x| \to \infty} |x|^2 \int_{|\sigma| = 1} e^{v(|x|\sigma)} d\sigma}{n(n - 2)\omega_n} = \frac{A_0}{n\omega_n}.
\]
By (2.10),
\[
I_{3,2} = \frac{1}{n(n - 2)\omega_n} \int_{|x|}^{2|x|} \rho \left( \int_{|\sigma| = 1} e^{v(\rho \sigma)} d\sigma \right) d\rho \leq C_2 \int_{|x|}^{2|x|} \frac{d\rho}{\rho} = C_2 \log 2 < \infty \quad \forall |x| > R_0,
\]
for some constant \( C_2 > 0 \). By (2.48), (2.50) and (2.51),
\[
\lim_{|x| \to \infty} \frac{I_3}{\log |x|} = \frac{A_0}{n\omega_n}.
\]
Hence by (2.28), (2.47) and (2.52), we get
\[
\lim_{|x| \to \infty} \frac{w_1(x)}{\log |x|} = \frac{A_0}{n\omega_n}
\]
and lemma the follows. \( \square \)

**Lemma 2.7.**
\[
\lim_{|x| \to \infty} \frac{w_2(x)}{\log |x|} = \frac{(n - 2)A_0}{n\omega_n} \quad \text{as} \quad |x| \to \infty.
\]
Proof. By (2.17),
\[ w_2(x) = \frac{1}{n\omega_n} \int_{|y| \leq |x|} \frac{(x-y) \cdot y}{|x-y|^n} e^{v(y)} \, dy + \frac{1}{n\omega_n} \int_{|y| > |x|} \left( \frac{(x-y) \cdot y}{|x-y|^n} + \frac{1}{|y|^{n-2}} \right) e^{v(y)} \, dy \]
(2.54)
\[ + \frac{1}{n\omega_n} \int_{|y| \leq |x|} \frac{e^{v(y)}}{|y|} \, dy \]
\[ = I_1 + I_2 + (n-2)I_3. \]

Since
\[ |I_1| \leq \frac{1}{n\omega_n} \int_{|y| \leq |x|} \frac{|y|}{|x-y|^{n-1}} e^{v(y)} \, dy, \]
by the proof of Lemma 2.5 there exists a constant \( C > 0 \) such that
(2.55)
\[ |I_1| \leq C \quad \forall x \in \mathbb{R}^n. \]
By (2.21) and (2.27),
(2.56)
\[ |I_2| \leq C|x| \int_{|y| \geq |x|} \frac{e^{v(y)}}{|y|^{n-1}} \, dy \leq C'|x| \int_{2|x|}^{\infty} \frac{d\rho}{\rho^{n-1}} = C'' < \infty \]
for some constants \( C' > 0, C'' > 0 \). By the proof of Lemma 2.6, \( I_3 \) satisfies (2.52). By (2.52), (2.54), (2.55) and (2.56), we get (2.53) and the lemma follows. \( \square \)

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.5, Lemma 2.6 and Lemma 2.7,
\[ \lim_{|x| \to \infty} \frac{v(x)}{\log |x|} = -L \]
where
\[ L = \frac{(\alpha - 2\beta)A_0}{n\omega_n}. \]
Suppose that \( L > 2 \). Then there exist constants \( a \in (2, n) \) and \( R_1 > 1 \) such that
\[ e^{v(x)} \leq |x|^{-a} \quad \forall |x| \geq R_1. \]
Hence
\[ 0 \leq \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_{\rho}} e^v \, dy \right) \, d\rho \leq \frac{C}{\log R}, \quad \forall R > R_1 \]
\[ \Rightarrow \lim_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_{\rho}} e^v \, dy \right) \, d\rho = 0 \quad \text{as } R \to \infty \]
which contradicts (1.7). Hence \( L \leq 2 \). Suppose that \( L < 2 \). Then there exist constants \( b \in (0, 2) \) and \( R_2 > 1 \) such that
(2.57)
\[ e^{v(x)} \geq |x|^{-b} \quad \forall |x| \geq R_2. \]
By (2.57) and a direct computation,
\[
\frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho} e^v \, dy \right) \, d\rho \geq \frac{C_1 R^{2-b}}{\log R} - \frac{C_2}{\log R} \quad \forall R > R_2
\]
\[
\Rightarrow \lim_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left( \int_{B_\rho} e^v \, dy \right) \, d\rho = \infty \quad \text{as } R \to \infty
\]
which again contradicts (1.7). Hence \( L = 2 \) and the theorem follows. \( \Box \)

**Corollary 2.8.** If \( v \) is a solution of (1.1) which satisfies (1.7) and (1.8), then \( \alpha > 2 \beta \) and \( A_0 = \frac{2n\omega_n}{\alpha - 2\beta} \).

### 3. Radial Symmetry of the Solution

In this section we will prove that under some condition on the solution \( v \) of (1.1), \( v \) is radially symmetric about the origin. We first start with a proposition.

**Proposition 3.1.** Let \( n \geq 3, \alpha > 2 \beta, a_0 \in \mathbb{R}, \) and let \( v \) be the unique radially symmetric solution of (1.1) with \( v(0) = a_0 \) constructed in [Hs2]. Then (1.5) holds and
\[
\lim_{r \to \infty} rv'(r) = -2.
\]

**Proof.** We first observe that (1.5) is proved in [Hs2]. We next observe that by putting \( x_0 = 0 \) in (2.7) and simplifying,
\[
rv'(r) = \frac{n\beta - \alpha}{r^{n-2}} \int_0^r \rho^{n-1} e^{v(\rho)} \, d\rho - \beta r^2 e^{v(r)}.
\]
Hence by (1.5),
\[
\lim_{r \to \infty} rv'(r) = \lim_{r \to \infty} \frac{n\beta - \alpha}{r^{n-2}} \int_0^r \rho^{n-1} e^{v(\rho)} \, d\rho - \beta \lim_{r \to \infty} r^2 e^{v(r)}
\]
\[
= (\frac{n\beta - \alpha}{n-2}) \lim_{r \to \infty} \frac{r^{n-1} e^{v(r)}}{(n-2)r^{n-3}} - \beta \lim_{r \to \infty} r^2 e^{v(r)}
\]
\[
= \left[ \frac{n\beta - \alpha}{n-2} - \beta \right] \lim_{r \to \infty} r^2 e^{v(r)}
\]
\[
= -2.
\]

\( \Box \)

**Proposition 3.2.** Let \( n \geq 3, \alpha > 2 \beta, \) and let \( v \) be a solution of (1.1) which satisfies (1.7), (1.8) and (1.9). Then
\[
x \cdot \nabla v \to -2 \quad \text{uniformly as } |x| \to \infty.
\]

**Proof.** Let
\[
V(x) = x \cdot \nabla v(x).
\]
By (1.1) and a direct computation \( V \) satisfies
(3.1)
\[
\Delta V = F(x) \quad \text{in } \mathbb{R}^n
\]
where
(3.2)
\[
F(x) \equiv -(2\alpha + (\alpha + 2\beta)V + \beta V^2) e^v - \beta (x \cdot \nabla V) e^v.
\]
Let \( 0 < \mu < 1 \) and \( x \in \mathbb{R}^n \) be such that \( |x| > 1 \). For any \( y \in B_{|x|/2}(x) \), let
\[
d_y = \text{dist} \left( y, \partial B_{|x|/2}(x) \right)
\]
and
\[
R = \frac{1}{3} d_y.
\]
Then by Green’s representation formula,
\[
V(z) = H(z) + N(z) \quad \forall z \in B_{2\mu R}(y)
\]
for some harmonic function \( H \) in \( B_{2\mu R}(y) \) where
\[
N(z) = \frac{1}{n(2-n)\omega_n} \int_{B_{2\mu R}(y)} |z-w|^{2-n} F(w) \, dw
\]
is the Newtonian potential of \( F \) in \( B_{2\mu R}(y) \). Since \( d_z \geq d_y/3 = R \) for any \( z \in B_{2\mu R}(y) \), by (3.3), (3.4), and Theorem 2.1 and Lemma 4.1 of [GT],
\[
d_y|\nabla N(y)| = \frac{d_y}{n(n-2)\omega_n} \left| \int_{B_{2\mu R}(y)} \nabla_y |y-w|^{2-n} F(w) \, dw \right|
\]
\[
\leq C_1 \mu d_y^2 \sup_{B_{2\mu R}(y)} |F|
\]
\[
\leq C_1 \mu \|d_z^2 F(z)\|_{L^\infty(B_{|x|}/2(x))}
\]
and
\[
d_y|\nabla H(y)| \leq \frac{C_2 d_y}{\mu R} \sup_{B_{2\mu R}(y)} |H| \leq \frac{3C_2}{\mu} \left( \sup_{B_{2\mu R}(y)} |V| + \mu^2 R^2 \sup_{B_{2\mu R}(y)} |F| \right)
\]
\[
\leq \frac{3C_2}{\mu} \left( \|V\|_{L^\infty(B_{|x|}/2(x))} + \mu^2 \|d_z^2 F(z)\|_{L^\infty(B_{|x|}/2(x))} \right)
\]
for some constants \( C_1 > 0, \ C_2 > 0 \). Hence
\[
\|d_y \nabla V(y)\|_{L^\infty(B_{|x|}/2(x))} \leq \frac{C_3}{\mu} \left( \|V\|_{L^\infty(B_{|x|}/2(x))} + \mu^2 \|d_z^2 F(z)\|_{L^\infty(B_{|x|}/2(x))} \right)
\]
for some constant \( C_3 > 0 \). Since \( d_z \leq |x|/2 \leq |z| \) for any \( z \in B_{|x|/2}(x) \),
\[
d_z e^{\nu(z)} \leq A_1 \quad \forall z \in B_{|x|/2}(x).
\]
Then by (1.3), (3.2) and (3.6),
\[
|d_z^2 F(z)| \leq C_4 (1 + \|V\|_{L^\infty(B_{|x|/2}(x))} + \|V\|^2_{L^\infty(B_{|x|/2}(x))} + \|d_w \nabla V(w)\|_{L^\infty(B_{|x|/2}(x))})
\]
for some constant \( C_4 > 0 \) and any \( z \in B_{|x|/2}(x) \). Hence by (3.5) and (3.7),
\[
\|d_y \nabla V(y)\|_{L^\infty(B_{|x|}/2(x))} \]
\[
\leq \frac{C_5}{\mu} \left( 1 + \|V\|_{L^\infty(B_{|x|}/2(x))} + \|V\|^2_{L^\infty(B_{|x|}/2(x))} \right)
\]
for some constant \( C_5 > 0 \). We now choose \( \mu = 1/(2C_5) \). Then by (3.8),
\[
\frac{|x|}{4} \|\nabla V\|_{L^\infty(B_{|x|/4}(x))} \leq \|d_y \nabla V(y)\|_{L^\infty(B_{|x|/2}(x))} \leq 2C_5 \left( 1 + \|V\|_{L^\infty(\mathbb{R}^n)} + \|V\|^2_{L^\infty(\mathbb{R}^n)} \right).
\]
Hence
\[
\|\nabla V\|_{L^\infty(B_{|x|}/2(x))} \leq \frac{C_6}{|x|} \left( 1 + \|V\|_{L^\infty(\mathbb{R}^n)} + \|V\|^2_{L^\infty(\mathbb{R}^n)} \right).
\]
where $C_6 = 8C_5$. Let $r > 1$. Taking supremum over all $|x| \geq r$ in (3.9), we get
\begin{equation}
\|\nabla V\|_{L^\infty(\mathbb{R}^n \setminus B_r)} \leq \frac{C_6}{r^2} \left( 1 + \|V\|_{L^\infty(\mathbb{R}^n)} + \|V\|^2_{L^\infty(\mathbb{R}^n)} \right).
\end{equation}
We next choose $\varepsilon_0 > 0$ such that for any $\sigma, \sigma' \in S^{n-1}$, the line segment $l_{\sigma, \sigma'}$ joining $\sigma$ and $\sigma'$ is outside $B_{\frac{1}{2}}$, i.e., $|\xi| \geq \frac{1}{2}$ for all $\xi \in l_{\sigma, \sigma'}$. Then by (3.10), \( \forall x = r\sigma, x' = r\sigma', |\sigma - \sigma'| < \varepsilon_0, \)
\[ |x \cdot \nabla v(x) - x' \cdot \nabla v(x')| = |V(x) - V(x')| \]
\[ = \left| \int_0^1 \frac{\partial}{\partial t} V(tx + (1-t)x') \, dt \right| \]
\[ \leq |x - x'| \cdot \sup_{0 \leq t \leq 1} |\nabla V(tx + (1-t)x')| \]
\[ \leq |\sigma - \sigma'| \cdot \sup_{0 \leq t \leq 1} C \frac{t\sigma + (1-t)\sigma'}{t\sigma + (1-t)\sigma'} \]
\[ \leq C|\sigma - \sigma'| \]
for some constant $C > 0$. Hence the family \( \{rv_r(r\sigma)\}_{r>1} \) is equi-Hölder continuous on $S^{n-1}$. Let \( \{r_i\}_{i=1}^\infty \) be a sequence of positive numbers such that $r_i \to \infty$ as $i \to \infty$. Then \( \{r_i\}_{i=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly on $S^{n-1}$ as $i \to \infty$. Then by the l'Hospital rule and Theorem 1.3,
\[ \lim_{i \to \infty} r_i v_{r_i}(r_i\sigma) = \lim_{i \to \infty} \frac{v(r_i\sigma)}{\log r_i} = -2 \quad \text{uniformly on } S^{n-1} \text{ as } i \to \infty. \]
Since the sequence \( \{r_i\}_{i=1}^\infty \) is arbitrary, \( rv_r(r\sigma) \to -2 \) uniformly on $S^{n-1}$ as $r \to \infty$ and the proposition follows. \hfill \square

We will now prove Theorem 1.3.

**Proof of Theorem 1.3** Since any rotation in $\mathbb{R}^n$ can be decomposed into a finite number of rotations in 2-dimensional planes, it suffices to prove that \( \Phi_{12} \equiv 0 \) in $\mathbb{R}^n$. By direct computation $\Phi_{12}$ satisfies
\[ \triangle q + \alpha e^v q + \beta (x \cdot \nabla v) e^v q + \beta (x \cdot \nabla q) e^v = 0. \]
Let $w(x) = \Phi_{12}(x)/g(x)$ where $g(x) = |x|^{2-n}$. Then $w$ satisfies
\[ \triangle w + \left( \frac{2\nabla g}{g} + \beta e^v x \right) \cdot \nabla w + \left( \alpha + \beta (x \cdot \nabla v) + \frac{\beta (x \cdot \nabla q)}{g} \right) e^v w = 0 \quad \text{in } \mathbb{R}^n. \]
If $\beta \leq 0$, then $\alpha < n\beta \leq 2\beta$. Then by Corollary 1.2 (1.1) has no solution and contradiction arises. Hence $\beta > 0$. We now choose $\varepsilon > 0$ such that $\alpha - n\beta + \varepsilon \beta < 0$. By Proposition 3.2 there exists a constant $R_0 > 0$ such that
\[ x \cdot \nabla v < -2 + \varepsilon \quad \forall |x| \geq R_0. \]
Then
\[ \alpha + \beta (x \cdot \nabla v) + \frac{\beta (x \cdot \nabla q)}{g} \leq \alpha - n\beta + \varepsilon \beta < 0 \quad \forall |x| \geq R_0 \]
Suppose $v$ is radially symmetric in $B_{R_0}$. Then $\Phi_{12} \equiv 0$ in $\overline{B_{R_0}}$. Hence
\[ w \equiv 0 \quad \text{in } \overline{B_{R_0}}. \]
We next observe that by (1.10),

$$|w| \leq |x|^{n-2} \Phi_{12}(x) \to 0 \quad \text{as } |x| \to \infty.$$ 

Then by the maximum principle for $w$ in $\mathbb{R}^n \setminus \overline{B_{R_0}}$,

$$w \equiv 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B_{R_0}}.$$ 

Hence

$$\Phi_{12} \equiv 0 \quad \text{in } \mathbb{R}^n.$$ 

Thus $v$ is radially symmetric in $\mathbb{R}^n$. □

Acknowledgement Sunghoon Kim supported by Priority Research Centers Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012047640).

REFERENCES

[CGS] L. Caffaralli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), 271–297.

[CL] W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. 63 (1991), no. 3, 615–622.

[DS] P. Daskalopoulos and N. Sesum, *On the extinction profile of solutions to fast diffusion*, J. Reine Angew Math. 622 (2008), 95–119.

[GN] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry and Related Properties via the Maximum Principle*, Commun. Math. Phys. 68 (1979), 209-243.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren der Math. Wiss., vol. 224, Springer-Verlag, Heidelberg, 1983.

[Hs1] S.Y. Hsu, *Non-existence and behaviour at infinity of solutions of some elliptic equations*, Discrete Contin. Dyn. Syst. 10 (2004), no. 3, 769-786.

[Hs2] S.Y. Hsu, *Classification of radially symmetric self-similar solutions of $u_t = \Delta \log u$ in higher dimensions*, Differential Integral Equations 18 (2005), no. 10, 1175-1192.

[Hs3] S.Y. Hsu, *Singular limit and exact decay rate of a nonlinear elliptic equation*, Nonlinear Analysis TMA 75 (2012), 3443-3455.

[HK1] K.M. Hui and S.H. Kim, *Large time behaviour of higher dimensional logarithmic diffusion equation*, Proceedings of the Royal Society of Edinburgh, Section: A Mathematics (to appear).

[HK2] K.M. Hui and S.H. Kim, *Extinction profile of the logarithmic diffusion equation*, [http://arxiv.org/abs/1012.1915](http://arxiv.org/abs/1012.1915).

[KM] E. Kamalinejad and A. Moradifam, *Radial Symmetry of Large Solutions of Semilinear Elliptic Equations with convection*, [http://arxiv.org/abs/1207.3105](http://arxiv.org/abs/1207.3105).

[T] S.D. Taliaferro, *Radial symmetry of large solutions of nonlinear elliptic equations*, Proc. Amer. Math. Soc. 124 (1996), no. 2, 447–455.

[V1] J.L. Vazquez, *Smoothing and decay estimates for nonlinear diffusion equations*, Oxford Lecture Series in Mathematics and its Applications 33, Oxford University Press, Oxford, 2006.

[V2] J.L. Vazquez, *The porous medium equation-Mathematical Theory*, Oxford Mathematical Monographs, Oxford University Press, 2007.

Kin Ming Hui: Institute of Mathematics, Academia Sinica., Taipei, Taiwan, R.O.C.

E-mail address: kmhui@gate.sinica.edu.tw

Sunghoon Kim: Department of Mathematics and PMI (Pohang Mathematics Institute), Pohang University of Science and Technology (POSTECH)!, Hyoja-Dong San 31, Nam-gu, Pohang 790-784, South Korea

E-mail address: math.s.kim@postech.ac.kr