The BIC of a singular foliation defined by an abelian group of isometries

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Abstract

We study the cohomology properties of the singular foliation $\mathcal{F}$ determined by an action $\Phi: G \times M \to M$ where the abelian Lie group $G$ preserves a riemannian metric on the compact manifold $M$. More precisely, we prove that the basic intersection cohomology $I H^p_p(M/\mathcal{F})$ is finite dimensional and verifies the Poincaré Duality. This duality includes two well-known situations:

- Poincaré Duality for basic cohomology (the action $\Phi$ is almost free).
- Poincaré Duality for intersection cohomology (the group $G$ is compact and connected).

This paper deals with an action $\Phi: G \times M \to M$ of an abelian Lie group on a compact manifold $M$ preserving a riemannian metric on it. The orbits of this action define a singular foliation $\mathcal{F}$ on $M$. Putting together the orbits of the same dimension we get a stratification $\mathcal{S}_\mathcal{F}$ of $M$. This structure is still very regular. The foliation $\mathcal{F}$ is in fact a conical foliation and we can define the basic intersection cohomology $H^p_p(M/\mathcal{F})$ (cf. [13]). This invariant becomes the basic cohomology $H^p_p(M/\mathcal{F})$ when the action $\Phi$ is almost free, and the intersection cohomology $H^p_p(M/G)$ when the Lie group $G$ is compact and connected (cf. [13]).

The aim of this work is to prove that this cohomology $H^p_p(M/\mathcal{F})$ is finite dimensional (cf. Theorem 4.2.2) and verifies the Poincaré Duality (cf. Theorem 4.3.6). This result generalizes [6], for the almost free case. When $G$ is compact and connected we find these results in [8].

The paper is organized as follows. In Section 1 we present the conical foliations. Section 2 is devoted to the study of isometric actions and the induced foliations. For this kind of foliations we can define the basic intersection cohomology. This is done in the Section 3. The main results of the work are proved in the Section 4.

In the sequel $M$ is a connected, second countable, Haussdorff, without boundary and smooth (of class $C^\infty$) manifold of dimension $m$. All the maps are considered smooth unless something else is indicated.

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1 Conical foliations

A singular foliation whose associated stratification is conical in the sense of Goresky-MacPherson (cf. [8]) is just a conical foliation.

1.1 Singular foliations. A regular foliation on a manifold $M$ is a partition $\mathcal{F}$ of $M$ by connected immersed submanifolds, called leaves, in such a way that each $x \in M$ possesses the following local model:

$$(\mathbb{R}^m, \mathcal{H})$$

where leaves are defined by $\{dx_1 = \cdots = dx_p = 0\}$. We shall say that $(\mathbb{R}^m, \mathcal{H})$ is a simple foliation. Notice that the leaves have the same dimension.

A singular foliation on a manifold $M$ is a partition $\mathcal{F}$ of $M$ by connected subsets, called leaves, in such a way that each $x \in M$ possesses the following local model:

$$(\mathbb{R}^{m-n} \times \mathbb{R}^n, \mathcal{H} \times \mathcal{K})$$

where $(\mathbb{R}^{m-n}, \mathcal{H})$ is a simple foliation and $(\mathbb{R}^n, \mathcal{K})$ is a singular foliation having the origin as an unique 0-dimensional leaf. When $n = 0$ we just have a regular foliation. The above condition is an empty one when $x$ is an isolated 0-dimensional leaf. Notice that the leaves are connected immersed submanifolds whose dimensions may vary. This local model is exactly the local model of a foliation of Sussman [15] and Stefan [14]; so, these foliations are singular in our sense.

Classifying the points of $M$ following the dimension of the leaves one gets a stratification $\mathcal{S}_F$ of $M$ whose elements are called strata. The foliation is regular when this stratification has just one stratum $\{M\}$.

A smooth map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ between singular foliated manifolds is foliated if it sends the leaves of $\mathcal{F}$ into the leaves of $\mathcal{F}'$. When $f$ is an open foliated embedding then it preserves the dimension of the leaves and therefore it sends the strata of $M$ into the strata of $M'$. Examples, more properties and the singular version of the Frobenius theorem the reader can find in [13, 1, 14, 15, 16].

1.1.1 Examples.

(a) In any open subset $U \subset M$ we have the singular foliation $\mathcal{F}_U = \{\text{connected components of } L \cap U / L \in \mathcal{F}\}$. The associated stratification is $\mathcal{S}_{\mathcal{F}_U} = \{\text{connected components of } S \cap U / S \in \mathcal{S}_F\}$.

(b) Consider $(N, \mathcal{K})$ a connected regular foliated manifold. In the product $N \times M$ we have the singular foliation $\mathcal{K} \times \mathcal{F} = \{L_1 \times L_2 / L_1 \in \mathcal{K}, L_2 \in \mathcal{F}\}$. The associated stratification is $\mathcal{S}_{\mathcal{K} \times \mathcal{F}} = \{N \times S / S \in \mathcal{S}_F\}$.

(c) Consider $S^{n-1}$ a sphere with positive dimension endowed with a singular foliation $\mathcal{G}$ without 0-dimensional leaves. Identify the disk $\mathbb{D}^n$ with the cone $cS^{n-1} = S^{n-1} \times [0, 1] / S^{n-1} \times \{0\}$ by the map $x \mapsto [x/||x||, ||x||]$ where $[u, t]$ is a generic element of the cone. We shall consider on $\mathbb{D}^n$ the singular foliation

$$c\mathcal{G} = \{F \times \{t\} / F \in \mathcal{G}, t \in [0, 1]\} \cup \{\vartheta\},$$

where $\vartheta$ is the vertex $[u, 0]$ of the cone. The induced stratification is

$$\mathcal{S}_{c\mathcal{G}} = \{S \times ]0, 1[ / S \in \mathcal{S}_{\mathcal{G}}\} \cup \{\vartheta\},$$
since $\mathcal{G}$ does not possesses 0-dimensional leaves. For technical reasons we allow $n$ to take the value 0; in this case $S^n = \emptyset$ and $cS^{n-1} = \{\varnothing\}$.

Same considerations apply to the $\infty$-cone $c_\infty S^{n-1} = S^{n-1} \times [0, \infty[ \setminus S^{n-1} \times \{0\}$. In particular, $\mathbb{R}^n = c_\infty S^{n-1}$. Notice that the map $f : (cS^{n-1}, c\mathcal{G}) \rightarrow (c_\infty S^{n-1}, c_\infty \mathcal{G})$, defined by $f[\theta, t] = [\theta, \tan(t\pi/2)]$, is a foliated diffeomorphism.

The strata of a singular foliation are not necessarily manifolds and their relative position can be very wild. Consider $(\mathbb{R}, \mathcal{F})$ where $\mathcal{F}$ is given by a vector field $\frac{\partial}{\partial t}$; there are two kind of strata. The connected components of $f^{-1}(\mathbb{R} - \{0\})$ and these of $f^{-1}(0)$. In other words, any connected closed subset of $\mathbb{R}$ can be a stratum. In order to support an intersection cohomology, the stratification $\mathcal{S}_\mathcal{F}$ asks for a certain amount of regularity and conicalicity (see [8] for the case of stratified pseudomanifolds). This leads us to introduce the following definition.

### 1.2 Conical foliations.

This definition is made by induction on the dimension of $M$. A singular foliation $(M, \mathcal{F})$ is said to be a conical foliation if for any point $x \in M$ we can find a foliated diffeomorphism

$$\varphi : (\mathbb{R}^{m-n} \times cS^{n-1}, H \times c\mathcal{G}) \rightarrow (U, \mathcal{F}_U),$$

where

- $U \subset M$ is an open neighborhood of $x$,
- $(S^{n-1}, \mathcal{G})$ is a conical foliation without 0-dimensional leaves, called link of $x$, and
- $\varphi(0, \varnothing) = x$.

We shall say that $(U, \varphi)$ is a conical chart of $x$. Notice that, if $S$ is the stratum containing $x$ then $\varphi^{-1}(S \cap U) = \mathbb{R}^{m-n} \times \{\varnothing\}$. A regular foliation can be considered as a conical foliation for which the link is empty. If $m = 1$, a conical foliation has to be regular. If $m = 2$ a conical foliation is either regular or it has leaves of dimension 0 and 1. The link of a singular leaf is $S^1$ with the one leaf foliation. The typical example is given by the standard action of $S^1$ on $\mathbb{R}^1$.

We also say that $(M \times [0, 1]^p, \mathcal{F} \times \mathcal{I})$, where $\mathcal{I}$ is the pointwise foliation of $[0, 1]^p$, is a conical foliated manifold.

Notice that each stratum is an embedded submanifold of $M$. Although a point $x$ may have several charts the integer $n$ is an invariant: it is the codimension of the stratum containing $x$. This integer cannot to be 1 since the conical foliation $(S^{n-1}, \mathcal{G})$ has not 0-dimensional leaves.

The above local description implies some important facts about the stratification $\mathcal{S}_\mathcal{F}$. Notice for example that the family of strata is finite in the compact case and locally finite in the general case. The closure $\overline{S}$ of a stratum $S \in \mathcal{S}_\mathcal{F}$ is a union of strata. Put $S_1 \preceq S_2$ if $S_1, S_2 \in \mathcal{S}_\mathcal{F}$ and $S_1 \subseteq \overline{S}_2$. This relation is an order relation and therefore $(\mathcal{S}_\mathcal{F}, \preceq)$ is a poset.

The depth of $\mathcal{S}_\mathcal{F}$, written depth $\mathcal{S}_\mathcal{F}$, is defined to be the largest $i$ for which there exists a chain of strata $S_0 \prec S_1 \prec \cdots \prec S_i$. So, depth $\mathcal{S}_\mathcal{F} = 0$ if and only if the foliation $\mathcal{F}$ is regular. It is always finite because of the locally finiteness of $\mathcal{S}_\mathcal{F}$. We also have depth $\mathcal{S}_{\mathcal{F}_U} \leq$ depth $\mathcal{S}_\mathcal{F}$ for any open subset $U \subset M$ and depth $\mathcal{G} = \text{depth } \mathcal{S}_{\mathcal{H} \times \mathcal{G}} < \text{depth } \mathcal{S}_{\mathcal{H} \times \mathcal{G}}$ (cf. 1.1.1).

The minimal strata are exactly the closed strata. An inductive argument shows that the maximal strata are the strata of dimension $m$. They are called regular strata and the others singular strata. Since the codimension of singular strata is at least 2, then only one regular stratum $R_{\mathcal{F}}$ appears, which is an open dense subset. The union of singular strata shall be denoted by $\Sigma_{\mathcal{F}}$. 
The restriction of \( \mathcal{F} \) to a stratum \( S \) defines a regular foliation \( \mathcal{F}_S \). Moreover, if \( S_1 \prec S_2 \) then \( \dim \mathcal{F}_{S_1} < \dim \mathcal{F}_{S_2} \). So, the biggest leaves of \( \mathcal{F} \) live in \( R_F \). The dimension \( \dim \mathcal{F} \) of the foliation \( \mathcal{F} \) is the dimension of these leaves, that is \( \dim \mathcal{F} = \dim \mathcal{F}_{R_F} \). Notice that for any singular stratum \( S \) we have the equality

\[
\dim \mathcal{F} - \dim \mathcal{F}_S = \dim \mathcal{G},
\]

where \( (\mathbb{S}^{n-1}, \mathcal{G}) \) is the link of a point of \( S \) (see (5)).

1.2.1 Examples.

(a) Any regular foliation is a conical foliation.

(b) In the examples of section 1.1.1 we obtain conical foliations if we replace singular foliation by conical foliation.

(c) Foliated bundles. Consider \((N, \mathcal{N})\) a conical foliation, \( B \) a manifold, \( \tilde{B} \) its universal covering and \( h: \pi_1(B) \to \text{Diff}(N, \mathcal{N}) \) a representation into the group of diffeomorphisms of \( N \) preserving \( \mathcal{N} \). The foliated bundle is the crossed product \( M = \tilde{B} \times_{\pi_1(B)} N \) endowed with the foliation \( \mathcal{F} \) whose leaves are obtained from \( \tilde{B} \times L, L \in \mathcal{N} \). This foliated bundle is conical. The dynamic of the leaves, strata, ... can be very complicated.

1. Put \( B = S^1 \), \( N = c_\infty \mathbb{S}^n \), \( n > 0 \), \( \mathcal{N} = c\mathcal{G} \), \( \mathcal{G} \) the one-leaf foliation and \( h(1) \) defined by \( h(1)(u) = u/2 \). The minimal stratum \( S = S^1 \) does not possesses a small saturated tubular neighborhood.

2. Put \( B = S^1 \), \( N = c\mathbb{S}^{2n+1} \), \( \mathcal{G} \) the Hopf foliation and \( h(1) \) defined by

\[
h(1)[(z_1, \ldots, z_n), t] = [(e^{2\pi it} \cdot z_1, \ldots, e^{2\pi it} \cdot z_n), t]
\]

and \( \mathcal{F} = c\mathcal{G} \). Here the leaves are diffeomorphic to cylinders (in irrational \( t \)-levels) or to tori (in non-zero rational \( t \)-levels) or to a circle (for \( t = 0 \)) and they are completely mixed up. The minimal stratum \( S = S^1 \) possesses small saturated tubular neighborhoods, but these neighborhoods do not retract to \( S \) through a foliated retraction.

(d) Singular Riemannian foliations. A singular Riemannian foliation, SRF for short, is a singular foliated manifold \((M, \mathcal{F})\) endowed with a Riemannian metric such that every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets (cf. [10]). The restriction of \( \mathcal{F} \) to each stratum \( S \) of \( \mathcal{F}_S \) is a regular Riemannian foliation. The local structure of \( \mathcal{F} \) can be described as follows (cf. [10] and [2]). Given \( x \in M \) we can find a chart modeled on \((\mathbb{R}^{n-n} \times \mathbb{R}^n, \mathcal{H} \times \mathcal{K})\) where \((\mathbb{R}^{n-n}, \mathcal{H})\) is a simple foliation, \((\mathbb{R}^n, \mathcal{K})\) is a SRF having the origin as unique 0-dimensional leaf and being invariant by homotheties.

The sphere \( \mathbb{S}^{n-1} \) inherits a singular Riemannian foliation \( \mathcal{G} \) with \( \mathcal{K} = c\mathcal{G} \) since \( \mathcal{K} \) is invariant by homotheties [10]. We conclude that a SRF is a conical foliation.

(e) Isometric actions. The foliation determined by an isometric action induces a singular Riemannian foliation and therefore a conical foliation. This example will be treated more extensively in the second Section.

(f) The product. Given two conical foliated manifolds \((M_1, \mathcal{F}_1)\) and \((M_2, \mathcal{F}_2)\) their product \((M_1 \times M_2, \mathcal{F}_1 \times \mathcal{F}_2)\) is also conical.
The BIC of a linear foliation.

Proof. Let us see that. We proceed by induction on depth $S_F + 1 + \text{depth } S_{F'}$. When this number is 0 the product foliation is conical since is regular. The proof of the general case is just a local matter. Taking into account (b) it suffices to consider the product $(cS^{n_1}, cG_1) \times (cS^{n_2}, cG_2)$, where $(S^{n_1}, G_1)$ and $(S^{n_2}, G_2)$ are conical. By induction hypothesis it suffices to check the local conicalicity near $(\vartheta_1, \vartheta_2)$. We are going to construct a conical foliated manifold $(S^{n_1+n_2+1}, G)$ and a foliated embedding $\zeta: (cS^{n_1+n_2+1}, cG) \to (cS^{n_1}, cG_1) \times (cS^{n_2}, cG_2)$ sending $\vartheta$ to $(\vartheta_1, \vartheta_2)$.

Since the foliation on the cone $cS^{n_1}$ is tangent to each sphere $S^{n_1} \times \{t\}$ then the submanifold $N = \{(\vartheta_1, t_1), (\vartheta_2, t_2)\} \in S^{n_1} \times S^{n_2} / t_1^2 + t_2^2 = 1/2$ is a foliated submanifold. Moreover, the map

$$\psi: (N \times [0,1], (cG_1 \times cG_2) \times \mathcal{I}) \to (cS^{n_1} \times cS^{n_2}, cG_1 \times cG_2),$$

defined by $\psi([\vartheta_1, t_1], [\vartheta_2, t_2]) = ([\vartheta_1, t \cdot t_1], [\vartheta_2, t \cdot t_2])$ is a foliated embedding. We claim that $(N, (cG_1 \times cG_2) \times \mathcal{I})$ is conical. Again it is a local question, so it suffices to prove that $(N - \{t_1 = 0\}, (cG_1 \times cG_2)_{N - \{t_1 = 0\}})$ is conical (similarly for $N - \{t_2 = 0\}$). Since it is foliated diffeomorphic to $(S^{n_1} \times [0,1] \times cS^{n_2}, G_1 \times \mathcal{I} \times cG_2)$, where $\mathcal{I}$ is the 0-dimensional foliation of $[0,1]$, then it suffices to apply the induction hypothesis and we get the claim.

Notice that $N$ is diffeomorphic to the sphere $S^{n_1+n_2+1}$, by $(\vartheta_1, t_1, \vartheta_2, t_2) \mapsto (\sqrt{t_1} \vartheta_1, \sqrt{t_2} \vartheta_2, t_2)$. The induced foliation $(S^{n_1+n_2+1}, \mathcal{G})$ is conical. Under this diffeomorphism, the foliated embedding $\psi$ becomes the foliated embedding

$$\zeta: (S^{n_1+n_2+1} \times [0,1], \mathcal{G} \times \mathcal{I}) \to (cS^{n_1} \times cS^{n_2}, cG_1 \times cG_2),$$

defined by $\zeta((u, v) = (x_0, \ldots, x_{n_1}, y_0, \ldots, y_{n_2}), t) = (\left[\frac{u}{\|u\|}, \frac{t\|u\|}{\sqrt{2}}\right], \left[\frac{v}{\|v\|}, \frac{t\|v\|}{\sqrt{2}}\right])$. But $\zeta$ extends to the foliated embedding $\zeta: (cS^{n_1+n_2+1}, c\mathcal{G}) \to (cS^{n_1} \times cS^{n_2}, cG_1 \times cG_2)$ by putting $\zeta(\vartheta) = (\vartheta_1, \vartheta_2)$. This ends the proof.

The following result will be useful in the sequel.

**Lemma 1.2.2** Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two conical foliated manifolds with same dimension and let $f: M \times [0,1] \to M' \times [0,1]$ be an embedding. If the restriction

(2) $$f: (M \times [0,1], \mathcal{F} \times \mathcal{I}) \to (M' \times [0,1], \mathcal{F}' \times \mathcal{I})$$

is foliated then

(3) $$f: (M \times [0,1], \mathcal{F} \times \mathcal{I}) \to (M' \times [0,1], \mathcal{F}' \times \mathcal{I})$$

is also foliated.

**Proof.** Notice first that, when the two foliations are regular, the result comes directly from the local description of $f$.

Consider on the other hand $S \in S_M$ a minimal stratum. From (2) there exists $S' \in S_{M'}$ with $f(S \times [0,1]) \subset S' \times [0,1]$ and therefore $f(S \times [0,1]) \subset \overline{S'} \times [0,1]$. We claim that

$$f(S \times [0,1]) \subset S' \times [0,1].$$

For that purpose, let us suppose that there exists $S'_0 \in S_{M'}$ with $S'_0 \prec S'$ and $f(S \times \{0\}) \cap (S'_0 \times \{0\}) \neq \emptyset$. Since $f(M \times [0,1])$ is an open subset of $M' \times [0,1]$ then $f(M \times [0,1]) \cap (S'_0 \times [0,1]) \neq \emptyset$. But this is not possible since the map $f: (M \times [0,1], \mathcal{F} \times \mathcal{I}) \to (f(M \times [0,1]), \mathcal{F}' \times \mathcal{I})$ is a foliated diffeomorphism and $S \times [0,1]$ a minimal stratum of $S_{M \times \mathcal{I}}$.\hfill\clubsuit
We proceed now by induction on depth $S_F$. When this depth is 0 then $F$ is a regular foliation and the above considerations give $f((M - S_{\text{min}}) \times [0, 1], F \times I) \rightarrow (M' \times [0, 1], F' \times I)$ is a foliated map. Consider now the general case. Denote $S_{\text{min}}$ the union of closed strata of $F$. By induction hypothesis the restriction

$$f: ((M - S_{\text{min}}) \times [0, 1], F \times I) \rightarrow (M' \times [0, 1], F' \times I)$$

is a foliated map. Consider now $S \in S_F$ a singular stratum. We have seen that there exists $S' \in S_{F'}$ with $f(S \times [0, 1]) \subset S' \times [0, 1]$. It remains to prove that

$$f: (S \times [0, 1], F \times I) \rightarrow (S' \times [0, 1], F' \times I)$$

is a foliated map. We get the result since the two foliations are regular.

1.3 Local blow up. Consider a conical chart $(U, \varphi)$ of a point $x$ of a singular stratum $S$ of $M$. The composition map

$$P_{\varphi}: (\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1], \mathcal{H} \times cG \times I) \rightarrow (U, F_U),$$

defined by $P_{\varphi}(u, \theta, t) = \varphi(u, [\theta, t])$, is said to be a local blow up. It is a foliated smooth map verifying

- The restriction $P_{\varphi}: \mathbb{R}^{m-n} \times S^{n-1} \times ]0, 1[ \rightarrow U - S$ is a diffeomorphism.
- The restriction $P_{\varphi}: \mathbb{R}^{m-n} \times S^{n-1} \times \{0\} \rightarrow U \cap S$ is a fiber bundle whose fiber is just $S^{n-1}$.

In other words, the local blow up replaces each point of the minimal stratum by a link.

The following result shows that the local blow up is essentially unique and that the link of $x$ is also unique.

**Proposition 1.3.1** Let $(U_1, \varphi_1)$, $(U_2, \varphi_2)$ be two charts of a point $x$ of $M$ with $U_1 \subset U_2$. There exists an unique embedding

$$\Phi_{1,2}: \mathbb{R}^{m-n} \times S^{n-1} \times [0, 1] \rightarrow \mathbb{R}^{m-n} \times S^{n-1} \times [0, 1]$$

making the following diagram commutative

$$\begin{array}{ccc}
\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1] & \xrightarrow{\Phi_{1,2}} & \mathbb{R}^{m-n} \times S^{n-1} \times [0, 1] \\
P & \downarrow & P \\
\mathbb{R}^{m-n} \times cS^{n-1} & \xrightarrow{\varphi_2^{-1} \circ \varphi_1} & \mathbb{R}^{m-n} \times cS^{n-1}.
\end{array}$$

where $P(u, \theta, t) = (u, [\theta, t])$.

Moreover, if the two charts are conical charts, modeled respectively on $(\mathbb{R}^{m-n} \times cS^{n-1}, \mathcal{H}_1 \times cG_1)$ and $(\mathbb{R}^{m-n} \times cS^{n-1}, \mathcal{H}_2 \times cG_2)$, then

$$\Phi_{1,2}: (\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1], \mathcal{H}_1 \times G_1 \times I) \rightarrow (\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1], \mathcal{H}_2 \times G_2 \times I)$$

is a foliated embedding.
Proof. Notice that $\varphi_2^{-1} \circ \varphi_1 : \mathbb{R}^{m-n} \times cS^{n-1} \to \mathbb{R}^{m-n} \times cS^{n-1}$ is an embedding.

Uniqueness. It comes from density of $\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1]$ and from the fact that $P : \mathbb{R}^{m-n} \times S^{n-1} \times \mathbb{R}$ is a diffeomorphism.

Existence. Denote $(f, g) : \mathbb{R}^{m-n} \times \mathbb{R}^n \to \mathbb{R}^{m-n} \times \mathbb{R}^n$ the embedding $\varphi_2^{-1} \circ \varphi_1$. The components $f$ and $g$ are smooth with $g(0) = 0$. So, the map $h : \mathbb{R}^{m-n} \times S^{n-1} \times [0, 1] \to \mathbb{R}^n$ defined by $h(u, \theta, t) = g(u, t \cdot \theta)/t$ is smooth and without zeroes. Finally, we define

$$
\Phi_{1,2}(u, \theta, t) = \left( f(u, t \cdot \theta), \frac{h(u, \theta, t)}{||h(u, \theta, t)||}, t \cdot ||h(u, \theta, t)|| \right).
$$

Embedding. Notice first the following. Since $(f, g)$ is an embedding with $g(0) = 0$ then each $g(u, -) : \mathbb{R}^n \to \mathbb{R}^n$ is an embedding. Put $G_u$ its derivative at 0, which is an isomorphism. By construction we have $h(u, \theta, 0) = G_u(\theta)/||G_u(\theta)||$. So, each restriction $\Phi_{1,2} : \{u\} \times S^{n-1} \times \{0\} \to \{f(u, 0)\} \times S^{n-1} \times \{0\}$ is a diffeomorphism.

Now, consider two points (resp. two vectors) on $\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1]$ sent by $\Phi_{1,2}$ to the same point (resp. vector). Since $\varphi_2^{-1} \circ \varphi_1$ is an embedding then they live on a fiber $\{u\} \times S^{n-1} \times \{0\}$. Since $G_u$ is an isomorphism, we get the claim.

Foliated. The embedding $\Phi_{1,2}$ extends the foliated map $\varphi_2^{-1} \circ \varphi_1$. We apply Lemma 1.2.2. ✷

An important consequence of this Proposition is that the links $(S^{n-1}, G)$ of two points of the same stratum $S$ are foliated diffeomorphic. We shall write $G_S = G$.

2 Foliations determined by an abelian isometric action.

We deal in this paper with an abelian isometric action $\Phi : G \times M \to M$ defined on a compact smooth manifold. As we study the induced foliation $F$, we may assume that the group $G$ is connected; it suffices to replace $G$ by the connected component $G_0$ containing the unity element.

For technical reasons, we also need to work with non compact manifolds. The tame actions are introduced for this purpose.

2.1 Tame actions. A smooth action $\Phi : G \times M \to M$ of a abelian Lie group on a smooth manifold $M$ is tame if it extends to a smooth action $\Phi : K \times M \to M$ where $K$ is an abelian compact Lie group containing $G$. We say that $K$ is a tamer group of $G$. When $M$ is compact, this notion is equivalent to that of isometric action (cf. [9]).

When necessary, we can suppose that the group $G$ is dense on $K$.

The restriction of the action of $G$ to a $K$-invariant submanifold of $M$ is again a tame action. For a subgroup $H \subset G$ the restriction $\Phi : H \times M \to M$ is also a tame action.

The connected components of the orbits of the tame action $\Phi$ determine a foliation $F$ on $M$. Since the action $\Phi : G \times M \to M$ is isometric then $F$ is a conical foliation (cf. 1.2.1 (e)).

2.1.1 Three particular actions.

In this work we shall use the particular actions we describe now.

(a) The action $\Xi : G \times K \to K$, defined by $\Xi(g, k) = g \cdot k$.

The action is tame since it extends to the action $\Xi : K \times K \to K$ defined by $\Xi(g, k) = g \cdot k$. We write $F_K$ the induced foliation, we have $\dim F_K = \dim G$. For each $u \in g$, Lie algebra of $G$, we shall write $X_u \in \mathcal{X}(K)$ the associated fundamental vector field.
(b) The action $\Psi : G \times K/H \to K/H$, defined by $\Psi(g,kH) = (g \cdot k)H$, where $H \subset K$ a closed subgroup.

The action is tame since it extends to the action $\Psi : K \times K/H \to K/H$ defined by $\Psi(g,kH) = (g \cdot k)H$. We write $\mathcal{F}_{K/H}$ the induced foliation, we have $\dim \mathcal{F}_{K/H} = \dim GH/H = \dim G/(G \cap H)$.

For each $u \in \mathfrak{g}$ we shall write $Y_u \in \mathcal{X}(K/H)$ the associated fundamental vector field. If we write $\pi : K \to K/H$ the canonical projection then we have $\pi_*X_u = Y_u$.

(c) The action $\Gamma : G \cap H \times H \to H$, defined by $\Gamma(g,h) = g \cdot h$.

The action is tame since it extends to the action $\Gamma : H \times H \to H$ defined by $\Gamma(g,h) = g \cdot h$. We write $\mathcal{F}_H$ the induced foliation, we have $\dim \mathcal{F}_H = \dim G \cap H$. For each $u \in \mathfrak{g} \cap \mathfrak{h}$ we shall write $Z_u \in \mathcal{X}(H)$ the associated fundamental vector field. Here $\mathfrak{h}$ is the Lie algebra of $H$.

2.2 Twisted product. This is the first geometrical tool we use for the study of $\mathcal{F}$.

Take $K$ a connected compact abelian Lie group and $G$ a connected subgroup. We shall suppose that $G$ is dense on $K$. Consider an orthogonal action $\Theta : H \times \mathbb{R}^n \to \mathbb{R}^n$ where $H$ is a compact subgroup of $K$. The twisted product $K \times_H \mathbb{R}^n$ is the quotient of $K \times \mathbb{R}^n$ by the equivalence relation $(k \cdot h^{-1}, \Theta(h,z)) \sim (k,z)$. The element of the twisted product corresponding to $(k,z)$ is denoted by $<k,z>$. It is a manifold endowed with the action

$$\Phi : G \times (K \times_H \mathbb{R}^n) \longrightarrow (K \times_H \mathbb{R}^n),$$

defined by $\Phi(g, <k,z>) = <g \cdot k,z>$. It is clearly a tame action and we denoted by $\mathcal{F}_{tw}$ the induced conical foliation.

The restriction $\Theta : G \cap H \times \mathbb{R}^n \to \mathbb{R}^n$ is also a tame action. The induced conical foliation is denoted by $\mathcal{F}_{tw};$ a tamer group is given by $H$. The couple $(\mathbb{R}^n, \mathcal{F}_{tw})$ is a slice of the twisted product.

The product $K \times \mathbb{R}^n$ is endowed with the conical foliation $\mathcal{F}_K \times \mathcal{F}_{\mathbb{R}^n}$. The natural projection $R : K \times \mathbb{R}^n \to K \times_H \mathbb{R}^n$ gives the relations $\mathcal{F}_{tw} = R_* (\mathcal{F}_K \times \mathcal{F}_{\mathbb{R}^n})$ and

$$S_{\mathcal{F}_{tw}} = \{R(K \times S) / S \in S_{\mathcal{F}_{\mathbb{R}^n}}\} = R(\{K\} \times S_{\mathcal{F}_{\mathbb{R}^n}}).$$

2.3 Tubular neighborhood. This is the second geometrical tool we use for the study of $\mathcal{F}$.

Consider a singular stratum $S$ of $S_{\mathcal{F}}$. Since $\dim G(k \cdot x) = \dim G(x)$ for each $k \in K$ and each $x \in S$ then $S$ is a $K$-invariant proper submanifold of $M$. It possesses a $K$-invariant tubular neighborhood $(T, \tau, S, \mathbb{R}^n)$ whose structural group is $O(n)$. We mean that $\tau$ is $K$-invariant and that there exists an atlas $\mathcal{B}$ such that for two charts $(U, \varphi)$, $(V, \psi) \in \mathcal{B}$ and for a $k \in K$ the composition

$$\psi \circ k \circ \varphi^{-1} : (U \cap (k^{-1}V)) \times \mathbb{R}^n \longrightarrow ((kU) \cap V) \times \mathbb{R}^n$$

is of the form $(x,u) \mapsto (k \cdot x, A_{x,k}(u))$ with $A_{x,k} \in O(n)$.

Recall that there are the following smooth maps associated with this neighborhood:

+ The radius map $\rho : T \to [0,1[$ defined fiberwise from the assignation $[x,t] \mapsto t$. Each $t \neq 0$ is a regular value of the $\rho$. The pre-image $\rho^{-1}(0)$ is $S$. This map is $K$-invariant, that is, $\rho(k \cdot z) = \rho(z)$.

+ The contraction $H : T \times [0,1] \to T$ defined fiberwisely from $([x,t], r) \mapsto [x, rt]$. The restriction $H_t : T \to T$ is an embedding for each $t \neq 0$ and $H_0 \equiv \tau$. We shall write $H(z,t) = t \cdot z$. This map is $K$-invariant, that is, $t \cdot (k \cdot z) = k \cdot (t \cdot z)$. 
These two maps are related by \( \rho(t \cdot z) = t\rho(z) \).

The hypersurface \( D = \rho^{-1}(1/2) \) is the tube of the tubular neighborhood. It is a \( K \)-invariant submanifold of \( T \). Notice that the map
\[
\nabla : D \times [0, 1] \rightarrow T,
\]
defined by \( \nabla(z, t) = (2t) \cdot z \) is a \( K \)-equivariant smooth map, where \( K \) acts trivially on the \([0, 1]-\)factor. Its restriction \( \nabla : D \times [0, 1] \rightarrow T - S \) is a \( K \)-equivariant diffeomorphism and its restriction \( \nabla : D \times \{0\} \equiv D \rightarrow S \) is \( \tau \).

The foliation \( \mathcal{F} \) induces on the fibers of \( \tau \) a tame foliation. Let us see that. Write \( G_S = G_{x_0} \), the isotropy subgroup of a point (and therefore, any point) \( x_0 \in S \). This group acts effectively on \( \tau^{-1}(x_0) \). The trace on \( \tau^{-1}(x_0) \) of \( \mathcal{F} \) is given by the action of \( G_S \). Using a chart of the atlas \( \mathcal{B} \) we identify \( \tau^{-1}(x_0) \) with \( \mathbb{R}^n \). The induced foliated manifold \((\mathbb{R}^n, \mathcal{F}_R^n)\) is the slice of the tube.

The action of \( G_S \) induces the orthogonal action \( \Theta : G_S \times S^{n-1} \rightarrow S^{n-1} \). This action is an effective tame action. We shall write \( G_S \) the induced conical foliation. In fact, \((S^{n-1}, G_S)\) is the link of \( S \). The formula (1) becomes
\[
(5) \quad \dim \mathcal{F} = \dim \mathcal{F}_S + \dim G_S.
\]

The tubular neighborhood gives rise to a local blow up:

**Proposition 2.3.1** Given a conical chart \((U, \varphi) \in \mathcal{B}\) there exists a commutative diagram
\[
\begin{array}{ccc}
\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1] & \xrightarrow{\varphi'} & D \times [0, 1] \\
\downarrow P & & \downarrow \nabla \\
\mathbb{R}^{m-n} \times cS^{n-1} & \xrightarrow{\varphi} & T,
\end{array}
\]
where \( \varphi' : (\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1], \mathcal{H} \times \mathcal{G} \times \mathcal{I}) \rightarrow (D \times [0, 1], \mathcal{F} \times \mathcal{I}) \) is a foliated embedding.

**Proof.** The existence of the embedding \( \varphi' \) and the commutativity \( \tau \circ \varphi' = \varphi \circ P \) is guaranteed by Lemma 1.3.1. We also know that the restriction \( \varphi' = \nabla^{-1} \circ \varphi \circ P : (\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1], \mathcal{H} \times \mathcal{G} \times \mathcal{I}) \rightarrow (D \times [0, 1], \mathcal{F} \times \mathcal{I}) \) is a foliated embedding. Applying Lemma 1.2.2 we get that \( \varphi' : (\mathbb{R}^{m-n} \times S^{n-1} \times [0, 1], \mathcal{H} \times \mathcal{G} \times \mathcal{I}) \rightarrow (D \times [0, 1], \mathcal{F} \times \mathcal{I}) \) is a foliated embedding. \( \blacklozenge \)

### 2.4 Molino’s blow up.

The Molino’ blow up \([10]\) of the foliation \( \mathcal{F} \) produces a new foliation \( \hat{\mathcal{F}} \) of the same kind but of smaller depth. The main idea is to replace each point of closed strata by the fiber of a convenient tubular neighborhood. But in order to avoid the boundary that appears in this procedure, we take the double.

We suppose depth \( S_F > 0 \). Denote \( \min_S \) the union of closed (minimal) strata and choose \( T_{\min} \) a disjoint family of \( K \)-invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by \( D_{\min} \). Notice that the induced map \( \nabla_{\min} : D_{\min} \times [0, 1] \rightarrow T_{\min} - S_{\min} \) is a \( K \)-equivariant diffeomorphism. The blow up of \( M \) is the manifold
\[
\hat{M} = \left\{ \left( D_{\min} \times [1, 1] \right) \coprod \left( (M - S_{\min}) \times \{ -1, 1 \} \right) \right\} / \sim,
\]
where \((z, t) \sim (\nabla_{\min}(z, |t|), t/|t|)\), and the map \(\mathcal{L}: \hat{M} \rightarrow M\) defined by

\[
\mathcal{L}(v) = \begin{cases} 
\nabla_{\min}(z, |t|) & \text{if } v = (z, t) \in D_{\min} \times ]-1, 1[ \\
z & \text{if } v = (z, j) \in (M - S_{\min}) \times \{-1, 1\}.
\end{cases}
\]

Notice that \(\mathcal{L}\) is a continuous map whose restriction \(\mathcal{L}: \hat{M} - \mathcal{L}^{-1}(S_{\min}) \rightarrow (M - S_{\min})\) is a \(K\)-equivariant smooth trivial 2-covering.

Since the map \(\nabla_{\min}\) is \(K\)-equivariant then \(\Phi\) induces the action \(\tilde{\Phi}: K \times \hat{M} \rightarrow \hat{M}\) by saying that the blow-up \(\mathcal{L}\) is \(K\)-equivariant. The open submanifolds \(\mathcal{L}^{-1}(T_{\min})\) and \(\mathcal{L}^{-1}(T_{\min} - S_{\min})\) are clearly \(K\)-diffeomorphic to \(D_{\min} \times [-1, 1[\) and \(D_{\min} \times [-1, 0[\cup]0, 1[\) respectively. Notice that the depth of the restrictions of \(\mathcal{F}\) to \(D_{\min}, T_{\min} - S_{\min}\) and \(M_{\min} - S_{\min}\) is strictly smaller than depth \(S_\mathcal{F}\).

The restriction \(\tilde{\Phi}: G \times \hat{M} \rightarrow \hat{M}\) is a tame action, whose tamer is just \(\tilde{\Phi}: K \times \hat{M} \rightarrow \hat{M}\). The induced foliation is \(\tilde{\mathcal{F}}\). The associated foliations \(\mathcal{F}\) and \(\tilde{\mathcal{F}}\) are related by \(\mathcal{L}\) which is a foliated map. Moreover, if \(S\) is a not minimal stratum of \(S_\mathcal{F}\) then there exists an unique stratum \(S'\) of \(S_{\tilde{\mathcal{F}}}\) such that \(\mathcal{L}^{-1}(S) \subset S'\). The family \(\{S' / S \in S_\mathcal{F}\}\) covers \(\hat{M}\) and verifies the relationship: \(S_1 < S_2 \Leftrightarrow S_1' < S_2'\). We conclude the important property

\[
\text{depth } S_{\tilde{\mathcal{F}}} < \text{depth } S_\mathcal{F}.
\]

For any perversity \(\mathcal{p}\) on \(M\) we define the perversity \(\mathcal{p}\) on \(\hat{M}\) by \(\mathcal{p}(S') = \mathcal{p}(S)\).

### 2.5 Orbit type stratification.

Following the action \(\Phi: K \times M \rightarrow M\), the points of \(M\) are classified by this equivalence relation:

\[
x \sim y \iff K_x = K_y.
\]

The induced partition \(S_\Phi\) is the orbit type stratification of \(M\) (see for example [3]). The elements of this stratification are connected \(K\)-invariants submanifolds, called ot-strata. This stratification is finer than the stratification \(S_\mathcal{F}\) defined by the action \(\Phi: K \times M \rightarrow M\), but it verifies similar properties, in particular, \((S_\Phi, <)\) is a poset with finite depth.

Since an ot-stratum is a \(K\)-invariant submanifolds then it possesses an invariant neighborhood. A blow up can be constructed as in the previous framework: the ot-blow-up. We write \(\mathcal{N}: \hat{M} \rightarrow M\) the ot-blow-up, which is a \(K\)-equivariant continuous map relatively to an action \(\tilde{\Phi}: K \times \hat{M} \rightarrow \hat{M}\). We have

\[
\text{depth } S_{\tilde{\Phi}} < \text{depth } S_\Phi.
\]

### 3 Basic Intersection cohomology

The basic cohomology of a foliated space is the right cohomological tool to study the transverse structure foliation. A conical foliation has a transverse structure which reminds the stratified pseudomanifolds of [8], and for these kind of singular spaces the most adapted cohomological tool is the intersection cohomology. In this section we mix up the two ingredients and we introduce the basic intersection cohomology.

For the rest of this section, we fix \((M, \mathcal{F})\) a conical foliated manifold.

A differential form defined on the regular stratum may have a wild behavior relatively to the singular strata. But there are some of which a good contact with the singular part. These are the perversive forms, and from them we are going to construct the basic intersection cohomology.
There are several ways to define perverse forms: using a system of tubular neighborhoods (cf. [5]), using a global blow up (cf. [12]) . . . ; in this work we introduce a more intrinsical way, using the local blow ups we already have seen.

We are going to deal with differential forms on products (manifold) \( \times [0,1]^p \); these forms are restrictions of differential forms defined on (manifold) \( \times ]-1,1]^p \).

### 3.1 Perverse forms.

A differential form defined on the regular stratum may have a wild behavior relatively to singular strata. But there are some of them with a good contact with the singular part. These are the perverse forms, and from them we are going to construct the basic intersection cohomology. Roughly speaking, perverse forms are differential forms defined on the regular stratum \( R_\mathcal{F} \) which are extendable through local blow ups.

The differential complex \( \Pi^*_\mathcal{F}(M \times [0,1]^p) \) of perverse forms of \( M \times [0,1]^p \) is introduced by induction on depth \( S_\mathcal{F} \). When this depth is 0 then

\[
\Pi^*_\mathcal{F}(M \times [0,1]^p) = \Omega^*(M \times [0,1]^p).
\]

Consider now the generic case. A perverse form of \( M \times [0,1]^p \) is first of all a differential form \( \omega \) defined on \( R_\mathcal{F} \times [0,1]^p \subset M \times [0,1]^p \) such that, for a conical chart \( (U, \phi) \), there exists a perverse form \( \omega_\phi \) of \( \mathbb{R}^{m-n} \times S^{n-1} \times [0,1]^{p+1} \) with

\[
\omega_\phi = (P_\phi \times \text{identity}_{[0,1]^p})^* \omega \quad \text{on} \quad \mathbb{R}^{m-n} \times R_G \times [0,1] \times [0,1]^p.
\]

Notice that this condition makes sense since the restriction of the local blow up

\[
P_\phi: \mathbb{R}^{m-n} \times R_G \times [0,1] \quad \rightarrow \quad U \cap R_\mathcal{F}
\]

is a diffeomorphism.

We notice that

\[
P_\phi^*: \Pi^*_\mathcal{F}(U) \rightarrow \Pi^*_n(G \times I)\left(\mathbb{R}^{m-n} \times S^{n-1} \times [0,1]\right)
\]

is a differential graded commutative algebra (dgca in short) isomorphism.

The complex \( \Pi^*_\mathcal{F}(M) \) is a dgca since \( (\omega + \eta)_\phi = \omega_\phi + \eta_\phi, (\omega \wedge \eta)_\phi = \omega_\phi \wedge \eta_\phi \) and \( (d\omega)_\phi = d\omega_\phi \).

#### 3.1.1 Remarks.

(a) The notion of perverse form depends on the foliation \( \mathcal{F} \) through the stratification \( S_\mathcal{F} \).

(b) A local blow up produces a factor \([0,1]^p\). Further desingularisation would produce extra \([0,1]^p\) factors. This is the reason for introducing directly \([0,1]^p\).

(c) The perversity condition does not depend on the choice of the conical chart. In fact, if the condition (8) is satisfied for a conical chart then it is also satisfied for another conical chart (cf. Proposition 1.3.1).

(d) The local blow up of the cone \( cS^{n-1} \) is essentially the product \( S^{n-1} \times [0,1] \). So, we have the natural identification \( \Pi^*_g(S^{n-1} \times [0,1]) = \Pi^*_g(cS^{n-1}) \) given by the assignation \( \omega \mapsto \omega|_{R_G \times [0,1]} \).

(e) Given a tubular neighborhood \( \tau: T \rightarrow S \) of a singular stratum (cf.2.3), the map \( \nabla: D \times [0,1] \rightarrow T \) gives an isomorphism between \( \Pi^*_\mathcal{F}(T) \) and \( \Pi^*_{\mathcal{F} \times I}(D \times [0,1]) \) (cf. Proposition 2.3.1 and Proposition 1.3.1).

(f) Let us illustrate this notion with an example. Consider the isometric action \( \Phi: \mathbb{R} \times S^7 \rightarrow S^7 \) defined by \( \Phi(t, (z_0,z_1,z_2,z_3)) = (e^{a\pi i}z_0,e^{b\pi i}z_1,e^{c\pi i}z_2,z_3) \) with \( (a,b,c) \neq (0,0,0) \). Recall that the
induced foliation $\mathcal{F}$ is conical (cf. 1.2.1 (e)). We can see $S^7$ as the join $S^5 \ast S^1$ where $\mathbb{R}$ acts freely on the first factor, inducing the foliation $\mathcal{G}$, and trivially on the second factor. There is just one singular stratum, namely $S^1$, whose link is $(S^5, \mathcal{G})$. This stratum has a “global conical chart” $(\mathcal{S}^1 \times cS^5, \mathcal{I} \times c\mathcal{G})$ then

$$\Pi^\ast_r(S^7) = \Omega^\ast(S^5 \times D^2).$$

(g) There are differential forms on $R_{\mathcal{F}}$ which are not perverse. Any differential form $\omega$ of $M$ is perverse, that is, $\Omega^\ast(M) \subset \Pi^\ast_r(M)$. In fact we have $\omega_\varphi = P^\ast_{\varphi, \omega}$ on $\mathbb{R}^{n-m} \times S^{n-1} \times [0,1]$ for any conical chart $(U, \varphi)$.

Proof. For the first part it is sufficient to consider a smooth function on $R_{\mathcal{F}}$ going to infinity when approaching to the singular part. For the second part we proceed by induction on $\dim S_\mathcal{F}$. Consider $\omega \in \Pi^\ast_r(M \times [0,1]^p)$. For a conical chart $(U, \varphi)$, we have that $\omega_\varphi = P^\ast_{\varphi, \omega} \in \Omega^\ast(\mathbb{R}^{m-n} \times S^{n-1} \times [0,1]^p)$. Here, we apply the induction hypothesis ♣

(h) An open foliated embedding $f: (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ between two foliated conical manifolds induces the dgea operator $\varphi^* : \Pi^\ast_r(M_2) \to \Pi^\ast_r(M_1)$.

Proof. The embedding $f$ preserves conical charts and therefore perverse forms. ♣

3.2 Perverse degree. The amount of transversallity of a perverse form $\omega \in \Pi^\ast_r(M)$ with respect to a singular stratum $S$ is measured by the perverse degree $||\omega||_S$. We define first the local perverse degree $||\omega||_U$ for a conical chart $(U, \varphi)$ of a point of $S$.

Notice that a local blow up replaces $U \cap S$ by $\mathbb{R}^{m-n} \times R_G \times \{0\}$ and that the restriction

$$P^\ast_{\varphi} : \mathbb{R}^{m-n} \times R_G \times \{0\} \to U \cap S$$

is (isomorphic to) a trivial bundle. Since the differential form $\omega$ is a perverse form, then the differential form $\varphi^* \omega$ (living on $\mathbb{R}^{m-n} \times R_G \times \{0\}$) extends to the differential form $\omega_\varphi$ (living on $\mathbb{R}^{m-n} \times R_G \times [0,1]$). Roughly speaking, the perverse degree $||\omega||_U$ is the vertical degree of $\omega_\varphi$ relatively to the added part, that is, the fibration (10).

More precisely, when $\omega_\varphi = 0$ on $\mathbb{R}^{m-n} \times R_G \times \{0\}$ we put $||\omega||_U = -\infty$ and in the other cases

$$||\omega||_U = \min \left\{ k \in \mathbb{N} \bigg/ \begin{array}{l} \omega_\varphi(u_0, \ldots, u_k, \ldots) \equiv 0 \\
\text{for each family } \{u_0, \ldots, u_k\} \\
\text{of tangent vectors to the fibers of } P^\ast_{\varphi} : \mathbb{R}^{m-n} \times R_G \times \{0\} \to U \cap S \end{array} \right\}. $$

This number does not depend on the choice of the conical chart.

Proof. Take $(U, \varphi_1)$ and $(U, \varphi_2)$ two foliated charts. From Lemma 1.3.1 we have $\omega_\varphi_1 = \Phi^\ast_{1,2} \omega_\varphi_2$ and that the restriction $\Phi^\ast_{1,2} : \mathbb{R}^{m-n} \times R_G \times \{0\} \to \mathbb{R}^{m-n} \times R_G \times \{0\}$ is a diffeomorphism. This implies that $||\omega||^\varphi_1_U = ||\omega||^\varphi_2_U$. ♣

Finally, we define the perverse degree $||\omega||_S$ by

$$||\omega||_S = \sup \{||\omega||_U / (U, \varphi) \text{ is a conical chart of a point of } S\}.$$ (11)

For two perverse forms $\omega$ and $\eta$ and a singular stratum $S$ we have:

$$||\omega + \eta||_S \leq \max \{||\omega||_S, ||\eta||_S\}, \quad ||\omega \cdot \eta||_S \leq ||\omega||_S \cdot ||\eta||_S.$$ (12)

For a perverse form $\omega$ the perverse degree is smaller of the usual degree and verifies

$$||\omega||_S \leq \dim R_G S = \codim_M S - 1.$$ (13)
3.2.1 Remarks.

(a) The notion of perverse degree depends on the foliation $\mathcal{F}$ through the stratification $S_{\mathcal{F}}$.
(b) The perverse degree of a differential form of $M$ is $-\infty$ or 0 (cf. 3.1.1 (g)).
(c) Consider $S^{n-1}$ endowed with a conical foliation without 0-dimensional leaves and the disk $cS^{n-1}$ endowed with the induced conical foliation. Then the perverse degree of a form $\omega \in \Pi^* (cS^{n-1}) = \Pi^*_0 (S^{n-1} \times [0, 1])$ is just the degree of the restriction $\omega|_{S^{n-1} \times \{0\}}$, where the degree of 0 is $-\infty$.
(d) Given a tubular neighborhood $\tau: T \to S$ (cf. 2.3) we know that we can identify $\Pi^*_T (T)$ with $\Pi^*_{T \times S} (D \times [0, 1])$ through $\nabla$ (cf. 3.1.1 (e)). Since $\nabla: D \times [0, 1] \to T - S$ is a $K$-diffeomorphism we have that the family of strata of $T$ is in fact the manifold $\Pi$.

(e) Let us illustrate this notion with the example of 3.1.1 (f). The perverse forms are just the differential forms of $S^5 \times D^2$. The perverse degree $||\cdot||_{S^1}$ is measured relatively to the trivial fibration $S^5 \times S^1 \to S^1$. So, if we have a volume form of $S^5$, $\theta_1$ a volume form of $S^1$ and $(x, y)$ the coordinates of $D^2$, we have $||\theta_1||_{S^1} = 0$ ; $||\theta_2||_{S^1} = 5$ ; $||\theta_3 \wedge (x dx + y dy)||_{S^1} = -\infty$.

(f) A perverse form with $||\omega||_S \leq 0$ and $||d\omega||_S \leq 0$ induces a differential form $\omega_S$ on $S$. When this happens for each stratum $S$ we conclude that $\omega \equiv \{\omega_S\}$ is a Verona’s controlled form (cf. [17]).

Proof. Consider $(U_2, \varphi_2)$ a conical chart of a point of a singular stratum $S$. The conditions $||\omega||_S \leq 0$ and $||d\omega||_S \leq 0$ give the existence of a form $\eta \varphi_2 \in \Omega^*(U_2 \cap S)$ with $\omega \varphi_2 = P^*_\varphi_2 \eta \varphi_2$ on $\mathbb{R}^{m-n} \times R_G \times \{0\}$. Let $(U_1, \varphi_1)$ be another conical chart of a point of $S$ with $(U_1, \varphi_1) \subset (U_2, \varphi_2)$. From the Proposition 1.3.1 we have

$$P^*_\varphi_1 \eta \varphi_1 = \omega \varphi_1 = \Phi^*_{1, 2} \omega \varphi_2 = \Phi^*_{1, 2} P^*_\varphi_2 \eta \varphi_2 = P^*_\varphi_2 \eta \varphi_2$$

and therefore $\eta \varphi_1 = \eta \varphi_2$ on $U_1 \cap S$. This implies that the forms $\{\eta \varphi_i\}$ overlap and define a differential form $\omega_S \in \Omega^*(S)$.

(g) An open foliated embedding $f: (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ between two foliated conical manifolds induces the dgca operator $f^*: \Pi^*_\mathcal{F}_2 (M_2) \to \Pi^*_\mathcal{F}_1 (M_1)$ which preserves the perverse degree (cf. Proposition 1.3.1).

3.3 Basic cohomology.

The basic cohomology of a foliation $\mathcal{F}$ is an important tool in the study its transversal structure and plays the rôle of the cohomology of the leaf space $M/\mathcal{F}$, which can be a wild topological space.

Consider $(M, \mathcal{F})$ a foliated manifold. A differential form $\omega \in \Omega^*(M)$ is basic if

$$i_X \omega = i_X d\omega = 0,$$

for each vector field $X$ on $M$ tangent to the foliation $\mathcal{F}$. For example, a function $f$ is basic iff $f$ is constant on the leaves. We shall denoted by $\Omega^*(M/\mathcal{F})$ the complex of basic forms. Since the sum and the product of basic forms are still basic forms, then the complex of basic forms is a dgca. Its cohomology $H^*(M/\mathcal{F})$ is the basic cohomology of $(M, \mathcal{F})$. We also use the relative basic cohomology $H^*((M, \mathcal{F})/F)$, that is, the cohomology computed from the complex of basic forms vanishing on the saturated set $F$.

When the foliation comes from a fibration $f: M \to B$ with connected fibers, then the leaf space $M/\mathcal{F}$ is in fact the manifold $B$ and the basic cohomology $H^*(M/\mathcal{F})$ is the cohomology of $B$. 
3.3.1 Remarks.

(a) The basic cohomology does not use the stratification $S_F$.

(b) Let us illustrate this notion with an example. Consider the isometric action $\Psi: \mathbb{R} \times S^5 \to S^5$ defined by $\Psi(t, (z_0, z_1, z_2)) = (e^{\alpha_it} z_0, e^{\beta_it} z_1, e^{\gamma_it} z_2)$ with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. The induced foliation $G$ is a regular one and we have that $H^i(S^5/G)$ is

\[
\begin{array}{cccccc}
i & 0 & 1 & 2 & 3 & 4 \\
\mathbb{R} & 0 & \mathbb{R} \cdot |e| & 0 & \mathbb{R} \cdot |e^2|
\end{array}
\]

Here, the cycle $e \in \Omega^2(S^5/G)$ is an Euler form. Notice that this cohomology is finite dimensional and verifies the Poincaré Duality. These facts are always true for any regular isometric flow on a compact manifold (see [6] and [7]).

Consider now the singular isometric flow defined in 3.1.1 (f). Here we have that $\Omega^i(S^7/F)$ is

\[
\begin{array}{ccccccc}
i & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\mathbb{R} & 0 & 0 & 0 & \mathbb{R} \cdot \beta \cdot e & 0 & \mathbb{R} \cdot \beta \cdot e^2
\end{array}
\]

The cycle $\beta \in \Omega^2(S^7/F)$ is the form induced by 1 from the double suspension $S^7 \equiv \Sigma \Sigma S^5$.

Notice that this cohomology is finite dimensional. This is true for any isometric flow on a compact manifold (see [18], [11]). On the other hand, the Poincaré Duality is lost. We introduce in this work the basic intersection cohomology in order to recover this property.

(c) A smooth foliated map $f: (M_1, F_1) \to (M_2, F_2)$ induces the dgca operator $f^*: \Omega^*(M_2/F_2) \to \Omega^*(M_1/F_1)$.

3.4 Basic intersection cohomology. The stratification $S_F$ induced by a conical foliation $F$ is rich enough to support an intersection cohomology theory.

Consider $(M, F)$ a conical foliated manifold. A perversity is a map $\mathfrak{p}: S^*_F \to \mathbb{Z}$, where $S^*_F$ is the family of singular strata. There are two particular perversities we are going to use:

- the constant perversity $\mathfrak{p}$ defined by $\mathfrak{p}(S) = i$, where $i \in \mathbb{Z}$, and

- the (basic) top perversity $\mathfrak{p}$ defined by $\mathfrak{p}(S) = \text{codim}_M F - \text{codim}_S F_S - 2 = \text{codim}_M S - \dim S_S - 2$.

Any two perversities can be added.

Associated to an open foliated embedding $f: (M', F') \to (M, F)$ there exists a perversity on $(M', F')$, still written $\mathfrak{p}$, defined by $\mathfrak{p}(S') = \mathfrak{p}(S)$ where $S' \in S^*_F$ and $S \in S^*_F$ with $f(S') \subset S$.

The basic intersection cohomology appears when one considers basic forms whose perverse degree is controlled by a given perversity. We shall write

\[
\Omega^i_{\mathfrak{p}}(M/F) = \left\{ \omega \in \Omega^i(R_F/F_{R_F}) \cap \Pi^i_F(M) \mid \text{max}(||\omega||_S, ||d\omega||_S) \leq \mathfrak{p}(S) \quad \forall S \in S^*_F \right\}
\]

the complex of basic perverse forms whose perverse degree (and that of their differential) is bounded by the perversity $\mathfrak{p}$. It is a differential complex, but it is not an algebra, in fact the wedge product acts in the following way:

\[
\wedge: \Omega^i_{\mathfrak{p}}(M/F) \times \Omega^j_{\mathfrak{p}}(M/F) \to \Omega^{i+j}_{\mathfrak{p}+\mathfrak{p}}(M/F)
\]

(see (12)). The cohomology $H^i_{\mathfrak{p}}(M/F)$ of this complex is the basic intersection cohomology of $(M, F)$, or BIC for short, relatively to the perversity $\mathfrak{p}$ (cf. [13]).
3.4.1 Remarks.

(a) The basic intersection cohomology coincides with the basic cohomology when the foliation \( F \) is regular, that is when depth \( S_F = 0 \). But it also generalizes the intersection cohomology of Goresky-MacPherson (cf. \cite{goresky-macpherson}) when the leaf space \( B \) lies in the right category, that of stratified pseudomanifolds (cf. \cite{macpherson}).

(b) The perverse degree of a perverse form verifies (13). But when this form is also a basic one we have

\[
||\omega||_S \leq \dim R_G - \dim F R_G = \text{codim}_M F - \text{codim}_S F_S - 1 = (\bar{t} + 1)(S)
\]

(cf. (1)).

(c) If \( \omega \in \Omega^\ell_T (M/F) \) et \( \varphi : (\mathbb{R}^{m-n} \times cS^{n-1}, \mathcal{H} \times cG) \to (U, F) \) is a conical chart then

\[
\omega \varphi \equiv 0 \quad \text{on} \quad \mathbb{R}^{m-n} \times R_G \times \{0\},
\]

where \( \ell = \text{codim}_M F \).

\[\text{Proof.}\] Put \( S \) the stratum of \( S_F \) containing \( \varphi(\mathbb{R}^{m-n} \times \{0\}) \). Notice that we have \( \bar{t}(S) = \ell - ((m - n) - \dim \mathcal{H}) - 2 = n - \dim G - 2 \). If the above assertion is not true then there exists \((x, z) \in \mathbb{R}^{m-n} \times R_G, \{v_1, \ldots, v_{m-n} \} \in T_x \mathbb{R}^{m-n} \text{ and } \{w_1, \ldots, w_{n-1} \} \in T_z S^{n-1}\) with \( \omega(x, z, 0) (v_1, \ldots, v_{m-n}) \neq 0 \). This is not possible since the vectors \( \{w_1, \ldots, w_{n-1}\} \) are tangent to the fibers of \( P_\varphi \) and \( n - 1 - \dim G > \bar{t}(S) \). \( \diamond \)

(d) Consider a tubular neighborhood \( \tau : T \to S \) (cf. 2.3) and \( \overline{p} \) a perversity on \( T \). From 3.2.1 (d) we get that \( \nabla^* \) establishes an isomorphism between \( \Omega^\ell_T (T/F) \) and

\[
\left\{ \omega \in \Omega^\ell_T \left( D \times [0,1]/F \times I \right) / \left| \omega \right|_{(D \cap R_F)} \leq \overline{p}(S) \text{ and } \left| d\omega \right|_{(D \cap R_F)} \leq \overline{p}(S) \right\}.
\]

Here, the perversity \( \overline{p} \) on \( D \times [0,1] \) defined by \( \overline{p}(S' \times [0,1]) = \overline{p}(\nabla(S' \times [0,1])) \).

(e) Let us illustrate this notion with the example of 3.1.1 (f) (see also 3.3.1 (b)). A direct calculation gives that \( H^\overline{p}_*(S^7/F) \) is

\[
\begin{array}{cccccccc}
  & i = 0 & i = 1 & i = 2 & i = 3 & i = 4 & i = 5 & i = 6 \\
  \overline{p} \leq -1 & 0 & 0 & \mathbb{R} \cdot [\beta] & 0 & \mathbb{R} \cdot [\beta \wedge e] & 0 & \mathbb{R} \cdot [\beta \wedge e^2] \\
  \overline{p} = 0 & \mathbb{R} & 0 & 0 & 0 & \mathbb{R} \cdot [\beta \wedge e] & 0 & \mathbb{R} \cdot [\beta \wedge e^2] \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \overline{p} = \bar{t} & \mathbb{R} & 0 & \mathbb{R} \cdot [e] & 0 & \mathbb{R} \cdot [\beta \wedge e] & 0 & 0 \\
  \overline{p} \geq \bar{t} + 1 & \mathbb{R} & 0 & \mathbb{R} \cdot [e] & 0 & \mathbb{R} \cdot [e^2] & 0 & 0 \\
\end{array}
\]

The first line is the relative basic cohomology \( H^\overline{p}_*(S^7/(S^1/F)) \), the second line is the basic cohomology \( H^\overline{p}_*(S^7/F) \) and the last line is the basic cohomology \( H^\overline{p}_*((S^7 - S^1)/F) \). These cohomologies are finite dimensional and verify the following Poincaré Duality:

\[
H^\overline{p}_*(M/F) \cong H^{\overline{p} - i}_*(M/F),
\]

for \( i + j = 6 \) and \( \overline{p} + \overline{p} = \bar{t} \).

We shall prove in this work that all these facts are always true for any abelian isometric flow on a compact manifold.

(f) Given an open foliated embedding \( f : (M_1, F_1) \to (M_2, F_2) \) we have the induced differential operator \( f^* : \Omega^p_*(M_2/F_2) \to \Omega^\overline{p}_*(M_1/F_1) \) (cf. 3.1.1 (h) and 3.3.1 (c)). If \( f \) is a foliated diffeomorphism then \( f^* \) is a differential isomorphism.

We present now some of the technical tools used in this work. We fix \( (M, F) \) a conical foliated manifold and \( \overline{p} \) a perversity.
3.5 Local calculations

The intersection basic cohomology, as the basic cohomology, is not easily computable. It becomes computable for singular foliations defined by an abelian isometric group as will be observed in the next section. Nevertheless, the typical calculations for the BIC are the classical ones.

**Proposition 3.5.1** Let \((\mathbb{R}^k, \mathcal{H})\) be a simple foliation. Put \(\overline{\mathcal{P}}\) the perversity defined on the conical foliated manifold \((\mathbb{R}^k \times M, \mathcal{H} \times \mathcal{F})\) by \(\overline{\mathcal{P}}(\mathbb{R}^k \times S) = \overline{\mathcal{P}}(S)\). The canonical projection \(pr: \mathbb{R}^k \times M \to M\) induces the isomorphism

\[
\overline{\mathcal{H}}^i(\mathbb{R}^k \times M \times \mathcal{H} \times \mathcal{F}) \cong \overline{\mathcal{H}}^i(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F}).
\]

**Proof.** Fix \(a \in \mathbb{R}^k\) a basis point and put \(\iota: M \to \mathbb{R}^k \times M\) the inclusion defined by \(\iota(x) = (a, x)\). Notice that a conical chart \((U, \varphi)\) on \(M\) induces the conical chart \((\mathbb{R}^k \times U, \text{Identity}_{\mathbb{R}^k} \times \varphi)\) on \(\mathbb{R}^k \times M\). Under these charts the projection \(pr\) becomes the canonical projection \(\mathbb{R}^k \times \mathbb{R}^{m-n} \times c\mathbb{S}^{n-1} \to \mathbb{R}^{m-n} \times c\mathbb{S}^{n-1}\) defined by \((u, c, \zeta) \mapsto (v, \zeta)\). The inclusion \(\iota\) becomes the inclusion \(\mathbb{R}^{m-n} \times c\mathbb{S}^{n-1} \to \mathbb{R}^k \times \mathbb{R}^{m-n} \times c\mathbb{S}^{n-1}\) defined by \((v, \zeta) \mapsto (a, v, \zeta)\). An inductive argument on the depth gives that the operators \(pr^*: \overline{\mathcal{H}}^0(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F}) \to \overline{\mathcal{H}}^0(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F})\) and \(\iota^*: \overline{\mathcal{H}}^0(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F}) \to \overline{\mathcal{H}}^0(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F})\), are well-defined differential operators.

Since the composition \(\iota^* \circ pr^*\) is the identity then it suffices to prove that \(pr^* \circ \iota^*\) is homotopic to the identity.

The foliated homotopy \(k_1: \mathbb{R}^k \times [0, 1] \to \mathbb{R}^k\) defined by \(k_0(u, t) = tu\) induces the homotopy \(k_1: \mathbb{R}^k \times M \times [0, 1] \to \mathbb{R}^k \times M\) defined by \(k_1(u, x, t) = (k_0(u, t), x)\). This homotopy does not involve the \(M\)-factor, so it induces the morphism \(k_1^*: \Pi^*_\mathcal{H} \times \mathcal{F}(\mathbb{R}^k \times M) \to \Pi^*_\mathcal{H} \times \mathcal{F}(\mathbb{R}^k \times M \times [0, 1])\) which preserves the perverse degree. We also have \(d \circ k_1^* = k_1^* \circ d\) and therefore the differential morphism

\[
k_1^*: \overline{\mathcal{H}}^0(\mathbb{R}^k \times M) \to \overline{\mathcal{H}}^0(\mathbb{R}^k \times M \times [0, 1]).
\]

The integration along the \([0, 1]\)-factor does not involves \(M\). So, the operator \(K: \Pi^*_\mathcal{H} \times \mathcal{F}(\mathbb{R}^k \times M) \to \Pi^{-1}_\mathcal{H} \times \mathcal{F}(\mathbb{R}^k \times M)\), given by \(K\omega = \int_0^1 k_1^* \omega\), is well-defined and preserves the perverse degree. On the other hand, it verifies the homotopy equality

\[
d \circ K + K \circ d = pr^* \circ \iota^* - \text{Identity}.
\]

This implies that \(dK\) also preserves the perverse degree. We conclude that

\[
K: \overline{\mathcal{H}}^0(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F}) \to \overline{\mathcal{H}}^{-1}(\mathbb{R}^k \times M / \mathcal{H} \times \mathcal{F})
\]

is well-defined and that is a homotopy operator between \(pr^* \circ \iota^*\) and the identity. \(\text{\&}\)

For the cone \((c\mathbb{S}^{n-1}, c\mathcal{G})\) (cf. 1.1.1 (c)) we have:

**Proposition 3.5.2** Let \(\mathcal{G}\) be a conical foliation without 0-dimensional leaves on the sphere \(\mathbb{S}^{n-1}\). A perversity \(\overline{\mathcal{P}}\) on \(c\mathbb{S}^{n-1}\) gives the perversity \(\overline{\mathcal{P}}\) on \(\mathbb{S}^{n-1}\) defined by \(\overline{\mathcal{P}}(S) = \overline{\mathcal{P}}(S \times [0, 1])\). The canonical projection \(\chi: \mathbb{S}^{n-1} \times [0, 1] \to \mathbb{S}^{n-1}\) induces the isomorphism

\[
\overline{\mathcal{H}}^i(c\mathbb{S}^{n-1}/c\mathcal{G}) = \begin{cases} 
\overline{\mathcal{H}}^i(\mathbb{S}^{n-1}/\mathcal{G}) & \text{if } i \leq \overline{\mathcal{P}}(\{\emptyset\}) \\
0 & \text{if } i > \overline{\mathcal{P}}(\{\emptyset\}).
\end{cases}
\]

**Proof.** The statement about perversities is clear. From 3.2.1 (c) we have

\[
\overline{\mathcal{H}}^i(c\mathbb{S}^{n-1}/c\mathcal{G}) = \begin{cases} 
\overline{\mathcal{H}}^i(\mathbb{S}^{n-1} \times [0, 1]/\mathcal{G} \times I) & \text{if } j < \overline{\mathcal{P}}(\{\emptyset\}) \\
\overline{\mathcal{H}}^i(\mathbb{S}^{n-1} \times [0, 1]/\mathcal{G} \times I) \cap d^{-1} \ker \iota^* & \text{if } j = \overline{\mathcal{P}}(\{\emptyset\}) \\
\overline{\mathcal{H}}^i(\mathbb{S}^{n-1} \times [0, 1]/\mathcal{G} \times I) \cap \ker \iota^* & \text{if } j > \overline{\mathcal{P}}(\{\emptyset\})
\end{cases}
\]

Here \(\iota: \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \times [0, 1]\) is the inclusion defined by \(\iota(\theta) = (\theta, 0)\). Proceeding as in the above Proposition we get the isomorphism

\[
\overline{\mathcal{H}}^i(c\mathbb{S}^{n-1}/c\mathcal{G}) = \begin{cases} 
\overline{\mathcal{H}}^i(\mathbb{S}^{n-1}/\mathcal{G}) & \text{if } i \leq \overline{\mathcal{P}}(\{\emptyset\}) \\
0 & \text{if } i > \overline{\mathcal{P}}(\{\emptyset\}).
\end{cases}
\]

\(\text{\&}\)
3.6 Mayer-Vietoris.

An open covering \{U, V\} of \(M\) by saturated open subsets is a basic covering when there exists a subordinated partition of the unity made up of controlled basic functions. They may or may not exist. For a such covering we have the Mayer-Vietoris short sequence

\[
0 \to \Omega^*_\mathcal{F}(M/F) \to \Omega^*_\mathcal{F}(U/F) \oplus \Omega^*_\mathcal{F}(V/F) \to \Omega^*_\mathcal{F}((U \cap V)/F) \to 0,
\]

where the map are defined by \(\omega \mapsto (\omega, \omega)\) and \((\alpha, \beta) \mapsto \alpha - \beta\). The third map is onto since the elements of the partition of the unity are controlled basic functions. Thus, the sequence is exact. This result is not longer true for more general coverings.

The Mayer-Vietoris sequence allows to make computations when the conical foliated manifold is covered by finite suitable covering. The passage from the finite case to the general case may be done using an adapted version of the Bredon’s trick of [4, page 289]:

**Bredon’s Trick 3.6.1** Let \(X\) be a paracompact topological space (resp. compact topological space) and let \(\{U_\alpha\}\) be an open covering, closed for finite intersection. Suppose that \(Q(U)\) is a statement about open subsets of \(X\), satisfying the following three properties:

**BT1** \(Q(U_\alpha)\) is true for each \(\alpha\);

**BT2** \(Q(U), Q(V)\) and \(Q(U \cap V) \implies Q(U \cup V)\), where \(U\) and \(V\) are open subsets of \(X\);

**BT3** \(Q(U_i) \implies Q\left( \bigcup U_i \right)\), where \(\{U_i\}\) is a disjoint family (resp. finite disjoint family) of open subsets of \(X\).

Then \(Q(X)\) is true.

3.7 Compact supports.

For the study of the Poincaré Duality we shall need the notion of cohomology with compact supports. We define the support of a perverse form \(\omega \in \Pi^*_\mathcal{F}(M)\) as the closure (in \(M\!\!\!)\)

\[
supp \omega = \{ x \in M - \Sigma \mathcal{F} / \omega(x) \neq 0 \}.
\]

We have the relations \(\supp(\omega + \omega') \subset \supp \omega \cup \supp \omega\)', \(\supp(\omega \cap \omega') \subset \supp \omega \cap \supp \omega'\) and \(\supp d\omega \subset \supp \omega\). We denote

\[
\Omega^*_\mathcal{F}(M/F) = \left\{ \omega \in \Omega^*_\mathcal{F}(M/F) / \supp \omega \text{ is compact} \right\}
\]

the complex of basic differential forms with compact support. It is a differential complex, but it is not an algebra, in fact the wedge product acts in this way:

\[
\wedge: \Omega^j\mathcal{F}(M/F) \times \Omega^k\mathcal{F}(M/F) \to \Omega^{j+k}\mathcal{F}(M/F)
\]

(see (12)). The cohomology \(\mathcal{H}^*_\mathcal{F}(M/F)\) of this complex is the basic intersection cohomology with compact support of \((M, F)\), relatively to the perversity \(\mathcal{F}\). Of course, when \(M\) is compact we have \(\mathcal{H}^*_\mathcal{F}(M/F) = \mathcal{H}^*_\mathcal{F}(M/F)\).

For \(\Sigma \mathcal{F} = \emptyset\), this notion generalizes the basic cohomology with compact supports.

Given a basic covering \{\(U, V\)\} of \(M\) we have the Mayer-Vietoris short sequence

\[
0 \to \Omega^*_\mathcal{F}(U \cap V)/F) \to \Omega^*_\mathcal{F}(U/FU) \oplus \Omega^*_\mathcal{F}(V/FV) \to \Omega^*_\mathcal{F}(M/F) \to 0,
\]

where the map are defined by \(\omega \mapsto (\omega, \omega)\) and \((\alpha, \beta) \mapsto \alpha - \beta\). The third map is onto since the elements of the partition of the unity are controlled basic functions. Thus, the sequence is exact.

We give now some local calculations. Given a simple foliation \((\mathbb{R}^k, \mathcal{H}) \equiv (\mathbb{R}^a \times \mathbb{R}^b, \mathcal{J} \times \mathcal{I})\), where \(\mathcal{J}\) is the one-leaf foliation of \(\mathbb{R}^a\) and \(\mathcal{I}\) the pointwise foliation of \(\mathbb{R}^b\), we have \(\mathcal{H}_c^*(\mathbb{R}^k/\mathcal{H}) = \mathbb{R}\) generated by \([f \, dx_1 \wedge \cdots \wedge dx_b]\) where \(f\) is a bump function: \(f \in C^\infty(\mathbb{R}^b)\) with \(\int_{\mathbb{R}^b} f = 1\) and compact support.
Proposition 3.7.1 Let $(\mathbb{R}^k, \mathcal{H})$ be a simple foliation. We have the isomorphism

$$\mathbb{H}_{\mathcal{F},c}^i(M/\mathcal{F}) \cong \mathbb{H}_{\mathcal{F},c}^{i+1}(\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F})$$

given by $[\beta] \mapsto \int f \, dx_1 \wedge \cdots \wedge dx_b \wedge \beta$, where $f \in C^\infty(\mathbb{R}^k)$ is bump function.

Proof. Notice first $\Omega^*_\mathcal{F,c}(\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F}) = \Omega^*_{\mathcal{F},c}(\mathbb{R}^b \times M/\mathcal{H} \times \mathcal{F})$. It suffices to prove the case $b = 1$.

Before executing the calculation let us introduce some notation. Let $\Delta$ be a differential form on $\Omega^*(R_\mathcal{F} \times M)$ which does not include the $dt$ factor. By $\int_0^c s \beta(s) \wedge ds$ and $\int_0^c s \beta(s) \wedge ds$ we denote the forms on $\Omega^*(R_\mathcal{F} \times M)$ obtained from $\beta$ by integration with respect to $s$: $\int_0^c \beta(s) \wedge ds(x,t)(\vec{v}_1,\ldots,\vec{v}_i) = \int_0^c \beta(x,s)(\vec{v}_1,\ldots,\vec{v}_i) \, ds$ and $\int_0^t \beta(x,s)(\vec{v}_1,\ldots,\vec{v}_i) = \int_0^c \beta(x,s)(\vec{v}_1,\ldots,\vec{v}_i) \, ds$ where $c \in ]a, b[\, (x,t) \in R_\mathcal{F} \times ]a, b[$ and $(\vec{v}_1,\ldots,\vec{v}_i) \in T_{(x,t)}(R_\mathcal{F} \times ]a, b[)$.

A generic differential form $\omega$ of $\Omega^*_\mathcal{F,c}(\mathbb{R} \times M/\mathcal{H} \times \mathcal{F})$ is of the form

$$\omega = \alpha + \beta \wedge dt,$$

where $\alpha, \beta \in \Omega^*_\mathcal{F,c}(\mathbb{R} \times M/\mathcal{H} \times \mathcal{F})$ do not contain $dt$. Consider the differential operators

$$\Delta : \Omega^*_\mathcal{F,c}(M/\mathcal{F}) \rightarrow \Omega^{-1}_\mathcal{F,c}(\mathbb{R} \times M/\mathcal{H} \times \mathcal{F}) \quad \text{and} \quad \nabla : \Omega^*_\mathcal{F,c}(\mathbb{R} \times M/\mathcal{H} \times \mathcal{F}) \rightarrow \Omega^{*+1}_\mathcal{F,c}(M/\mathcal{F}),$$

defined by

$$\Delta(\alpha + \beta \wedge dt) = \int_{-\infty}^\infty \beta(s) \wedge ds \quad \text{and} \quad \nabla(\beta) = f \, dt \wedge \beta.$$

Notice that $\Delta \circ \nabla = \text{Identity}$ (up to a sign) which gives $\nabla^* \circ \Delta^* = \text{Identity}$ (up to a sign). We prove now $\Delta^* \circ \nabla^* = \text{Identity}$. Consider $[\omega = \alpha + \beta \wedge dt] \in \mathbb{H}^i_{\mathcal{F},c}(\mathbb{R} \times M/\mathcal{H} \times \mathcal{F})$. Define $\eta \in \Omega^{-1}_{\mathcal{F,c}}(\mathbb{R} \times M/\mathcal{H} \times \mathcal{F})$ by

$$\eta = \left( \int_{-\infty}^t f(s) \, ds \right) \left( \int_{-\infty}^\infty \beta(s) \wedge ds \right) - \left( \int_{-\infty}^t f(s) \, ds \right) \left( \int_{-\infty}^\infty \beta(s) \wedge ds \right).$$

A straightforward calculation gives

$$d\eta = f \, dt \wedge \int_{-\infty}^\infty \beta(s) \wedge ds - \beta \wedge dt - \alpha = \Delta^* \circ \nabla^*(\omega) - \omega.$$

This ends the proof.

For the computation of $\mathbb{H}^*_{\mathcal{F}}(c\mathbb{S}^{n-1}/c\mathcal{G})$ we consider $g \in C^\infty([0,1])$ with $g \equiv 1$ on $[0,1/4]$, $g \equiv 0$ on $[3/4,1]$ and $\int_0^1 g = 1$.

Proposition 3.7.2 Let $\mathcal{G}$ be a conical foliation without 0-dimensional leaves on the sphere $\mathbb{S}^{n-1}$. We have the isomorphism

$$\mathbb{H}^*_{\mathcal{F},c}(c\mathbb{S}^{n-1}/c\mathcal{G}) = \left\{ \begin{array}{ll} 0 & \text{if } i \leq \overline{p}(\{\vartheta\}) + 1 \smallskip \mathbb{H}^{i+1}_{\mathcal{F}}(\mathbb{S}^{n-1}/\mathcal{G}) & \text{if } i \geq \overline{p}(\{\vartheta\}) + 2 \end{array} \right.$$
Proof. From 3.2.1 (c) we have

\[
\Omega^1_{\pi,c}(cS^{n-1}/cG) = \begin{cases} 
\Omega_{\pi,c}^i(S^{n-1} \times [0,1]/G \times I) & \text{if } i < \overline{p}(\emptyset) \\
\Omega_{\pi,c}^i(S^{n-1} \times [0,1]/G \times I) \cap d^{-1} \ker \iota^* & \text{if } i = \overline{p}(\emptyset) \\
\Omega_{\pi,c}^i(S^{n-1} \times [0,1]/G \times I) \cap \ker \iota^* & \text{if } i > \overline{p}(\emptyset)
\end{cases}
\]

Here \( \iota : S^{n-1} \to S^{n-1} \times [0,1] \) is the inclusion defined by \( \iota(\theta) = (\theta, 0) \). Consider a cycle \( \omega = \alpha + dt \wedge \beta \in \Omega^i_{\pi,c}(cS^{n-1}/cG) \). Notice the equality

\[
\omega = d\left( \int_0^1 \beta(s) \wedge ds \right),
\]

with \( \left( \int_0^1 \beta(s) \wedge ds \right) \in \Omega^{i-1}_{\pi,c}(S^{n-1} \times [0,1]/G \times I) \). This gives \( H^i_{\overline{p}}(cS^{n-1}/cG) = 0 \) if \( i \leq \overline{p}(\emptyset) + 1 \).

Now, it suffices to prove that the assignment \([\omega] \mapsto [g \ dt \wedge \omega] \) establishes an isomorphism between \( H^{i-1}_{\overline{p}}(S^{n-1}/G) \) and \( H^i\left( \Omega^*_{\pi,c}(S^{n-1} \times [0,1]/G \times I) \cap \ker \iota^* \right) = H^i_{\overline{p}}(c(S^{n-1} \times [0,1],S^{n-1} \times \{0\}))/G \times I) \). The proof is exactly the same of the previous Proposition, replacing \(-\infty\) by 0.

\[\blacklozenge\]

3.8 Twisted product.

We show in this section how to compute the BIC of the twisted product \( K \times_\mathcal{H} \mathbb{R}^n \) (cf. 2.2) in terms of the group \( K \) and the slice \( \mathbb{R}^n \).

Notice first that a perversity \( \overline{p} \) on \((K \times \mathbb{R}^n, \mathcal{F}_K \times \mathcal{F}_{\mathbb{R}^n})\) (resp. \((K \times_\mathcal{H} \mathbb{R}^n, \mathcal{F}_{tw})\)) is determined by a perversity \( \overline{p} \) on \((\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n})\) by putting \( \overline{p}(K \times S) = \overline{p}(S) \) (resp. \( \overline{p}(R(K \times S)) = \overline{p}(S) \), see (4)).

Lemma 3.8.1 The natural projection \( R : K \times \mathbb{R}^n \to K \times_\mathcal{H} \mathbb{R}^n \) induces the differential monomorphism

\[
R^* : \Omega^*_{\overline{p}}(K \times_\mathcal{H} \mathbb{R}^n/\mathcal{F}_{tw}) \to \Omega^*_{\overline{p}}(K \times \mathbb{R}^n/\mathcal{F}_K \times \mathcal{F}_{\mathbb{R}^n}),
\]

for any perversity \( \overline{p} \).

Proof. We proceed in several steps.

(a) A foliated atlas for \( \pi : K \to K/\mathcal{H} \).

Since \( \pi \) is a \( \mathcal{H} \)-principal bundle then it possesses an atlas \( \mathcal{A}_{\#} = \{ \varphi : \pi^{-1}(U) \to U \times \mathcal{H} \} \) made up with \( \mathcal{H} \)-equivariant foliated charts. The \( \mathcal{H} \)-equivariance means \( \varphi(h \cdot k) = (\pi(k), h \cdot h_0) \) if \( \varphi(k) = (\pi(k), h_0) \).

We study the foliation \( \varphi_* \mathcal{F}_K \). This equivariance property gives \( \varphi_* X_u = (0, Z_u) \) for each \( u \in g \cap h \). Thus, the trace of the foliation \( \varphi_* \mathcal{F}_K \) on the fibers of the canonical projection \( pr : U \times \mathcal{H} \to U \) is \( \mathcal{F}_H \). On the other hand, since the map \( \pi \) is a \( \mathcal{G} \)-equivariant submersion then \( \pi_* \mathcal{F}_K = \mathcal{F}_{K/\mathcal{H}} \), which gives \( \varphi_* \mathcal{F}_K = \mathcal{F}_{K/\mathcal{H}} \). We conclude that \( \varphi_* \mathcal{F}_K \subset \mathcal{F}_{K/\mathcal{H}} \times \mathcal{F}_H \). By dimension reasons we get \( \varphi_* \mathcal{F}_K = \mathcal{F}_{K/\mathcal{H}} \times \mathcal{F}_H \). The atlas \( \mathcal{A}_{\#} \) is an \( \mathcal{H} \)-equivariant foliated atlas of \( \pi \).

(b) A foliated atlas of \( R : K \times \mathbb{R}^n \to K \times_\mathcal{H} \mathbb{R}^n \).

We claim that \( \mathcal{A} = \{ \overline{\varphi} : \pi^{-1}(U) \times \mathbb{R}^n \to U \times \mathbb{R}^n / (U, \varphi) \in \mathcal{A}_{\#} \} \) is a foliated atlas of \( R \) where the map \( \overline{\varphi} \) is defined by \( \overline{\varphi}(<k,z>) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k),0))^{-1} \cdot k, z))) \). This map is a diffeomorphism whose inverse is \( \overline{\varphi}^{-1}(u,z) = <\varphi^{-1}(u,0), z> \). It verifies

\[
\overline{\varphi}_* R_* (\mathcal{F}_K \times \mathcal{F}_{\mathbb{R}^n}) = \overline{\varphi}_* R_* (\varphi^{-1} \times \text{Identity}_{\mathbb{R}^n})_* (\mathcal{F}_{K/\mathcal{H}} \times \mathcal{F}_H \times \mathcal{F}_{\mathbb{R}^n}).
\]

A straightforward calculation shows \( \overline{\varphi}_0 R_0 (\varphi^{-1} \times \text{Identity}_{\mathbb{R}^n}) = (\text{Identity}_{U} \times \Theta) \). Since \( \mathcal{F}_H \) is defined by the action \( \Gamma : G \cap H \times H \to H \) then \( \Theta_* (\mathcal{F}_H \times \mathcal{F}_{\mathbb{R}^n}) = \mathcal{F}_{\mathbb{R}^n} \). Finally we obtain \( \overline{\varphi}_* \mathcal{F}_{tw} = \mathcal{F}_{K/\mathcal{H}} \times \mathcal{F}_{\mathbb{R}^n} \).

(c) Last step.
Given \((U, \varphi) \in \mathcal{A}_\#\), we have the commutative diagram

\[
\begin{array}{ccc}
U \times H \times \mathbb{R}^n & \overset{\varphi}{\longrightarrow} & K \times \mathbb{R}^n \\
P \downarrow & & \downarrow R \\
U \times \mathbb{R}^n & \overset{\overline{\varphi}}{\longrightarrow} & K \times_H \mathbb{R}^n,
\end{array}
\]

where \(P(u, h, z) = (u, \Theta(h, z))\) and \(R^{-1}(\text{Im } \overline{\varphi}) = \text{Im } \varphi\). We claim that

\[
R^*: \Omega^*_p(K \times_H \mathbb{R}^n / \mathcal{F}_H) \longrightarrow \Omega^*_p(K \times \mathbb{R}^n / \mathcal{F}_K \times \mathcal{F}_H)
\]

is a well-defined morphism. Since it is a local question and we have \(R^{-1}(\text{Im } \overline{\varphi}) = \text{Im } \varphi\), then it suffices to prove that the induced map \(P^*: \Omega^*_p(U \times \mathbb{R}^n / \mathcal{F}_K \times \mathcal{F}_H) \longrightarrow \Omega^*_p(U \times H \times \mathbb{R}^n / \mathcal{F}_K \times \mathcal{F}_H \times \mathcal{F}_R)\) is well-defined. This comes from the fact that the map

\[
\nabla: (U \times H \times \mathbb{R}^n, \mathcal{F}_K \times \mathcal{F}_H \times \mathcal{F}_R) \longrightarrow (U \times H \times \mathbb{R}^n, \mathcal{F}_K \times \mathcal{F}_H \times \mathcal{F}_R),
\]

defined by \(\nabla(u, h, z) = (u, h, \Theta(h, z))\) is a foliated diffeomorphism. Since \(\text{pr}_0 \circ \nabla = R\), with \(\text{pr}_0: U \times H \times \mathbb{R}^n \rightarrow U \times \mathbb{R}^n\) canonical projection then it suffices to apply 3.4.1 (f) and Proposition 3.5.1. The injectivity of \(R^*\) comes from the fact that \(R\) is a surjection.

\[
\text{Lemma 3.8.2} \quad \text{Let } \Phi: G \times M \rightarrow M \text{ be a tame action. Put } K \text{ a connected tame group of } G. \text{ We write } V_u \in \mathcal{A}(K) \text{ the fundamental vector field associated to an element } u \text{ of the Lie algebra of } K. \text{ For any perversity } \overline{p} \text{ the interior operator }
\]

\[
i_{V_u}: \left(\Omega^*_p(M / \mathcal{F})\right)^K \longrightarrow \left(\Omega^{-1}_p(M / \mathcal{F})\right)^K.
\]

is well-defined.

\[
\text{Proof.} \text{ Since } K \text{ is connected then the vector field } V_u \text{ is } K\text{-invariant. So, the contraction operator } i_{V_u} \text{ preserves the } K\text{-invariants differential forms. Moreover, if } \omega \in \left(\Omega^*_p(M / \mathcal{F})\right)^K \text{ and } X \text{ is a vector field on } M \text{ tangent to the foliation } \mathcal{F} \text{ then we have: }
\]

\[
i_X i_{V_u} \omega = -i_{V_u} i_X \omega = 0 \text{ and } i_X d i_{V_u} \omega = -i_X i_{V_u} d \omega = i_{V_u} i_X d \omega = 0.
\]

We end the proof if we show that \(\omega \in \left(\Omega^*_p(M \times [0, 1])\right)^K\) implies \(i_{V_u} \omega \in \Omega^*_p(M \times [0, 1])\). We proceed by induction on the depth of \(S_\mathcal{F}\).

1. \textbf{First step:} depth \(S_\mathcal{F} = 0\). The result is clear.

2. \textbf{Induction step.} Since the question is a local one we can consider \(M = T\) a \(K\)-invariant tubular neighborhood of a singular stratum \(S\). Write \(\hat{V}_u \in \mathcal{A}(D \times [0, 1])\) the fundamental vector field associated to \(u\). This vector field is tangent to the boundary of \(D \times [0, 1]\) write \(U\) its restriction. On the other hand, since \(\nabla: D \times [0, 1] \rightarrow T\) is a \(K\)-equivariant map, then \(\nabla_* V_u = V_u\).

We have, for each \(\alpha \in \Omega^*(D)\), the relationship: \(i_U \alpha(v_1, \ldots, v_j) \neq 0 \Rightarrow \alpha(v_1, \ldots, v_j) \neq 0\). This gives the inequality \(||i_U \alpha||_\tau \leq ||\alpha||_\tau\).

Denote \(I = \text{Identity}_{[0, 1]^p}\). We have \((\nabla \times I)^* \omega \in \Omega^*_p(D \times [0, 1]^{[p+1]})\) with \(||(\nabla \times I)^* \omega\|_{D \times [0, 1]^{[p+1]}} \leq \overline{p}(S)\) and \(||(\nabla \times I)^* d \omega\|_{D \times [0, 1]^{[p+1]}} \leq \overline{p}(S)\) (cf. 3.4.1 (d)). By induction hypothesis we have

\[
(\nabla \times I)^*(i_{V_u} \omega) = i_{\hat{V}_u}(\nabla \times I)^* \omega \in \Omega^*_p(D \times [0, 1]^{[p+1]})
\]

The result comes now from: 
\[
\left| \left| \left( \nabla \times I \right)^* (iV_u \omega) \right| \right|_{D \times \{0\} \times [0,1]^p} = \left| \left| \left( i\bar{V}_u (\nabla \times I)^* \omega \right) \right| \right|_{D \times \{0\} \times [0,1]^p} = \left| \left| iU \left( (\nabla \times I)^* \omega \right) \right| \right|_{D \times \{0\} \times [0,1]^p} 
\]
\[
\leq \left| \left| (\nabla \times I)^* \omega \right| \right|_{D \times \{0\} \times [0,1]^p} \leq \overline{p}(S),
\]
and
\[
\left| \left| \left( (\nabla \times I)^* (d(iV_u \omega)) \right) \right| \right|_{D \times \{0\} \times [0,1]^p} = \left| \left| - (\nabla \times I)^* (iV_u d\omega) \right| \right|_{D \times \{0\} \times [0,1]^p} \leq \overline{p}(S)
\]
since \( \omega \) is \( K \)-invariant and 3.4.1 (d).

3.8.3 Fixing some notations about Lie algebras.

Write \( \mathfrak{f}, \mathfrak{g} \) and \( \mathfrak{h} \) the Lie algebras of \( K, G \) and \( H \) respectively. Choose \( \kappa \) an invariant riemannian metric on \( K \), which exists by compactness. Consider

\[
\{u_1, \ldots, u_a, u_{a+1}, \ldots, u_b, u_{b+1}, \ldots, u_c, u_{c+1}, \ldots, u_f\}
\]
an orthonormal basis of \( \mathfrak{f} \) with \( \{u_1, \ldots, u_b\} \) basis of \( \mathfrak{g} \) and \( \{u_{a+1}, \ldots, u_c\} \) basis of \( \mathfrak{h} \).

For each index \( i \) we write \( X_i \in \mathcal{X}(K) \) the associated invariant vector field to \( u_i \) (cf. 2.1.1 (a)). Let \( \gamma_i \in \Omega^1(K) \) be the dual form of \( X_i \), that is \( \gamma_i = i_{X_i} \kappa \). It is a cycle and it is invariant by \( K \), that is, \( k^* \gamma_i = \gamma_i \) for each \( k \in K \). Since \( K/H \) is an abelian Lie group then \( H^* (K/H) = \bigwedge^* (\gamma_1, \ldots, \gamma_a, \gamma_{c+1}, \ldots, \gamma_f) \). The \( \mathcal{F}_{K/H} \)-basic differential forms on \( \bigwedge (\gamma_1, \ldots, \gamma_a, \gamma_{c+1}, \ldots, \gamma_f) \) are exactly \( \bigwedge^* (\gamma_{c+1}, \ldots, \gamma_f) \). This gives,

\[
H^* (K/H/\mathcal{F}_{K/H}) = \bigwedge^* (\gamma_{c+1}, \ldots, \gamma_f).
\]

Proposition 3.8.4 Let \( K \times_H \mathbb{R}^n \) be a twisted product. Let us suppose that the group \( G \) is connected and dense in the group \( K \). Then

\[
H^*_p(K \times_H \mathbb{R}^n/\mathcal{F}_{tw}) \cong H^* \left( K/H/\mathcal{F}_{K/H} \right) \otimes \left( H^*_p(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H,
\]

for any perversity \( \overline{p} \).

Proof. Since the operator \( R^*: \Omega^*_p(K \times \mathbb{R}^n/\mathcal{F}_{tw}) \rightarrow \Omega^*_p(K \times \mathbb{R}^n/\mathcal{F}_{K} \times \mathcal{F}_{\mathbb{R}^n}) \) is a monomorphism (cf. Lemma 3.8.1) the it suffices to compute the cohomology of \( \text{Im} R^* \). We describe this complex in several steps.

\(< i >\) Description of \( \Omega^*_p( K \times R_{\mathcal{F}_{\mathbb{R}^n}} ) \).

A differential form of \( \Omega^*_p( K \times R_{\mathcal{F}_{\mathbb{R}^n}} ) \) is of the form

\[
\eta + \sum_{1 \leq i_1 < \cdots < i_f \leq \ell} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_f} \wedge \eta_{i_1, \ldots, i_f},
\]

where the forms \( \eta, \eta_{i_1, \ldots, i_f} \in \Omega^*( K \times R_{\mathcal{F}_{\mathbb{R}^n}} ) \) verify \( i_{X_i} \eta = i_{X_i} \eta_{i_1, \ldots, i_f} = 0 \) for all indices.

\(< ii >\) Description of \( \Pi^*_p( K \times R_{\mathcal{F}_{\mathbb{R}^n}} ) \).

Since the foliation \( \mathcal{F}_K \) is regular then we always can choose a conical chart of the form \( (U_1 \times U_2, \varphi_1 \times \varphi_2) \) where \( (U_1, \varphi_1) \) is a foliated chart of \( (K, \mathcal{F}_K) \) and \( (U_2, \varphi_2) \) is a conical chart of \( (\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}) \). The local blow up of the chart \( (U_1 \times U_2, \varphi_1 \times \varphi_2) \) is constructed from the second factor without modifying the first one. So, the differential forms \( \gamma_i \) are always perverses forms and a differential form \( \omega \in \Pi^*_p( K \times R_{\mathcal{F}_{\mathbb{R}^n}} ) \) is of the form \( (15) \) where \( \eta, \eta_{i_1, \ldots, i_f} \in \Pi^*_p( K \times R_{\mathcal{F}_{\mathbb{R}^n}} ) \) verify \( i_{X_i} \eta = i_{X_i} \eta_{i_1, \ldots, i_f} = 0 \) for all indices.
Consider the operator $\Delta: \left(\Omega^*_{\mathfrak{p}}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n})\right)^H \rightarrow \bigwedge^* (\gamma_{b+1}, \ldots, \gamma_f) \otimes \left(\Omega^*_{\mathfrak{p}}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n})\right)^H$, defined by

$$\Delta(\beta) = \beta + \sum_{b+1 \leq i_1 < \cdots < i_c \leq c} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_c} \wedge (i_{W_{i_1}} \cdots i_{W_{i_2}} \beta).$$
A straightforward computation gives that the operator $\Delta$ is a differential operator and that the restriction

$$\Delta: \left( \Omega^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n) \right)^H \rightarrow A^*,$$

is an isomorphism. The inverse operator is

$$\Delta^{-1} \left( \xi_0 + \sum_{b+1 \leq i_1 < \cdots < i_c} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_c} \wedge \xi_{i_1, \ldots, i_c} \right) = \xi_0$$

(cf. Lemma 3.8.2). We conclude that the differential complex $\text{Im } R^*$ is isomorphic to

$$\bigwedge^\ast (\gamma_{c+1}, \ldots, \gamma_f) \otimes \left( \Omega^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n) \right)^H$$

$< vi >$ Last step.

Consider now the the operator

$$H^* \left( \left( \Omega^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n) \right)^H \right) \hookrightarrow \left( \Pi^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n) \right)^H$$

induced by the inclusion $\iota: \left( \Omega^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n) \right)^H \hookrightarrow \Omega^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n)$. The usual arguments show that this operator is an isomorphism:

- Monomorphism: $\omega = d\eta \Rightarrow \omega = d\int_H \Upsilon^*_h \eta \, dh$

- Epimorphism: $\omega - \Upsilon^*_h \omega = d\eta_h$ for each $h \in H \implies \omega - \int_H \Upsilon^*_h \omega \, dh = d\int_H \eta_h \, dh$

We get

$$\Pi^*_p(\mathbb{R}^n / \mathcal{F}_{\text{tw}}) \cong H^* \left( K / H \bigg/ \mathcal{F}_{K/H} \right) \otimes \left( \Pi^*_p(\mathbb{R}^n / \mathcal{F}_\mathbb{R}^n) \right)^H,$$

(cf. (14)).

3.8.5 Remark.

The same procedure gives that the differential operator $\nabla = (\text{Identity} \wedge^\ast (\gamma_{1}, \ldots, \gamma_{a}, \gamma_{c+1}, \ldots, \gamma_{f}) \otimes \Delta^{-1}) \circ R^*$

gives the isomorphism

$$\left( \Pi^*_p(\mathbb{R}) \right)^K \cong \bigwedge^\ast (\gamma_{1}, \ldots, \gamma_{a}, \gamma_{c+1}, \ldots, \gamma_{f}) \otimes \left( \Pi^*_p(\mathbb{R}^n) \right)^H = H^* \left( K / H \right) \otimes \left( \Pi^*_p(\mathbb{R}^n) \right)^H.$$

4 Cohomological properties of the BIC

We prove in this section that the BIC of a compact foliated manifold $(M, \mathcal{F})$, determined by an abelian isometric action $\Phi: G \times M \rightarrow M$, is finite dimensional and verifies the Poincaré Duality.

When the orbits of this action have the same dimension, that is when depth $S_\mathcal{F} = 0$, then the foliation $\mathcal{F}$ is a (regular) riemannian foliation (cf. [10]) and the BIC becomes the usual basic cohomology $H^* (M / \mathcal{F})$. We already know from [7] that $H^* (M / \mathcal{F})$ is finite dimensional and verifies the Poincaré Duality. For the generic case we are going to proceed by induction on the depth of $S_\mathcal{F}$.

But first of all we show how the BIC generalizes the usual basic cohomology. The same situation appears for the intersection homology of a stratified pseudomanifold (cf. [8]).

**Proposition 4.1** Let $(M, \mathcal{F})$ be a foliated manifold determined by a tame action. Then
i) \( \mathcal{H}^*_\overline{\eta}(M/F) \cong H^*(R_F/F) \) if \( \overline{\eta} > 7. \)

ii) \( \mathcal{H}^*_0(M/F) \cong H^*(M/F) \).

iii) \( \mathcal{H}^*_p(M/F) \cong H^*((M, \Sigma_F)/F) \) if \( \overline{p} < 0. \)

**Proof.** The map \( \omega \mapsto \omega \) gives the differential operator \( I_M: \Omega^*_\overline{\eta}(M/F) \to \Omega^*(R_F/F) \). The restriction map \( \omega \mapsto \omega_{R_F} \) defines the differential operators \( J_M: \Omega^*(M/F) \to \Omega^*_0(M/F) \) and \( K_M: \Omega^*((M, \Sigma_F)/F) \to \Omega^*_0(M/F) \) (cf. 3.2.1 (b)). We prove by induction on the depth of \( S_F \) the following assertions.

\[ \mathfrak{A}_1(M, F) = "I_M \text{ is a quasi-isomorphism}". \]

\[ \mathfrak{A}_2(M, F) = "J_M \text{ is a quasi-isomorphism}". \]

\[ \mathfrak{A}_3(M, F) = "K_M \text{ is a quasi-isomorphism}". \]

1. **First step:** \( \text{depth } S_F = 0. \) The singular part \( \Sigma_F \) is empty and therefore \( \mathcal{H}^*_\overline{\eta}(M/F) = \mathcal{H}^*_0(M/F) = \mathcal{H}^*_p(M/F) = H^*(M/F) \) with \( I_M = J_M = K_M = \text{Identity} \).

2. **Induction step:** The family \( \{M - S_{\min}, T_{\min}\} \) is a \( K \)-invariant open covering of \( M \) (cf. 2.4). Choose \( \alpha: [0, 1] \to \mathbb{R} \) a smooth map with \( \alpha \equiv 1 \) on \([0, 1/4]\) and \( \alpha \equiv 0 \) on \([3/4, 1]\). Write \( f = \alpha \circ \rho_{\min}: M \to \mathbb{R} \), which is a \( K \)-invariant map and therefore \( F \)-basic. Since \( \text{supp } f \subset T_{\min} \) and \( \text{supp } (1 - f) \subset M - S_{\min} \), we conclude that the covering is a basic one. From 3.6 we have a Mayer-Vietoris sequence and we get

\[ \mathfrak{A}_i(T_{\min} - S_{\min}, F), \mathfrak{A}_i(M - S_{\min}, F) \text{ and } \mathfrak{A}_i(T_{\min}, F) \implies \mathfrak{A}_i(M, F), \]

for \( i = 1, 2, 3. \) The induction hypothesis gives \( \mathfrak{A}_i(T_{\min} - S_{\min}, F) \) and \( \mathfrak{A}_i(M - S_{\min}, F) \), it remains to prove \( \mathfrak{A}_i(T_{\min}, F) \).

From 3.2.1 (d) we know that we can identify the perverse forms of \( T_{\min} \) with the perverse forms of \( D_{\min} \times [0, 1] \). This identification sends basic forms to basic forms and preserves the perverse degrees relatively to any stratum different from those of \( S_{\min} \). The perverse degree of a perverse form \( \omega \) of \( D_{\min} \times [0, 1] \) relatively to \( S_{\min} \) becomes the vertical degree of the restriction \( \omega \) relatively to \( \nabla_{\min} \equiv \tau_{\min}: D_{\min} \times \{0\} \equiv D_{\min} \to S_{\min} \) (cf. 3.2.1 (d)). That is,

- \( \Omega^*_\overline{\eta}(T_{\min}/F) \) becomes \( \Omega^*_\overline{\eta}(D_{\min} \times [0, 1]/F \times I) \) (cf. 3.4.1 (b))
- \( \Omega^*_\overline{\eta}(T_{\min}/F) \) becomes \( \{ \omega \in \Omega^*_\overline{\eta}(D_{\min} \times [0, 1]/F \times I) / \omega|_{D_{\min}} = \tau^* \min \eta \text{ with } \eta \in \Omega^*(S_{\min}/F) \} \)
- \( \Omega^*_\overline{\eta}(T_{\min}/F) \) becomes \( \{ \omega \in \Omega^*_\overline{\eta}(D_{\min} \times [0, 1]/F \times I) / \omega|_{D_{\min}} = 0 \}. \)

Proceeding as in Proposition 3.5.1 we prove that

\[ \mathcal{H}^*_\overline{\eta}(T_{\min}/F) \cong \mathcal{H}^*_\overline{\eta}(D_{\min}/F) \]

\[ \mathcal{H}^*_\overline{\eta}(T_{\min}/F) \cong H^*(S_{\min}/F) \]

\[ \mathcal{H}^*_\overline{\eta}(T_{\min}/F) \cong 0 \]

Notice that \( I_{T_{\min}} \) is induced by \( \text{pr} \circ \nabla^{-1}: T_{\min} - \Sigma_F \to D_{\min} \times [0, 1] \to D_{\min} \) and \( J_T \) becomes \( \tau^* \).

On the other hand, since \( \nabla_{\min}: (D_{\min} \times [0, 1], F \times I) \to (T_{\min} - \Sigma_F, F) \) is a foliated diffeomorphism, we get

\[ H^*((T_{\min} - \Sigma_F)/F) \cong H^*(D_{\min} \times [0, 1]/F \times I) \cong H^*(D_{\min}/F), \]

where the isomorphism is induced by \( \text{pr} \circ \nabla^{-1} \). The induction hypothesis gives that \( I_{T_{\min}} \) is a quasi-isomorphism.
The inclusion $t_{\min}: S_{\min} \to T_{\min}$ and the projection $\tau_{\min}: T_{\min} \to S_{\min}$ are foliated maps with $\tau_{\min} \circ t_{\min} = \text{Identity}$, so they induce the operators $t^*: \Omega^*(S_{\min}/F) \to \Omega^*(T_{\min}/F)$ and $\tau^*: \Omega^*(T_{\min}/F) \to \Omega^*(S_{\min}/F)$ verifying $t_{\min}^* \circ \tau_{\min}^* = \text{Identity}$. The composition $t_{\min}^* \circ \tau_{\min}^*$ is homotopic to the identity by a foliated homotopy. This homotopy is just $H: T_{\min} \times [0, 1] \to T_{\min}$ (cf. 2.4). So, the operator $\tau_{\min}^*$ induces the isomorphism

$$H^* (T_{\min}/F) \cong H^* (S_{\min}/F).$$

This proves that $J_{T_{\min}}$ is a quasi-isomorphism.

Let $\omega \in \Omega^*((T_{\min}, \Sigma_F)/F)$ be a cycle. The above homotopy operator gives the relation $\omega = d \int_0^1 H^* \omega$. Since the homotopy $H$ preserves the foliation $F$ then $\int_0^1 H^* \omega \in \Omega^*((T, \Sigma_F)/F)$ and therefore

$$H^* ((T_{\min}, \Sigma_F)/F) = 0.$$

This proves that $K_{T_{\min}}$ is a quasi-isomorphism. \hfill \(\blacklozenge\)

### 4.2 Finiteness.

We prove that the BIC of the conical foliation $F$ induced by an abelian isometric action on a compact manifold is finite dimensional. We proceed by induction on the depth of $S_F$. In order to decrease the depth we use the blow up of Molino. This will lead us to the twisted product through the invariant tubular neighborhood.

**Proposition 4.2.1** Let $(M, F)$ be a conical foliated manifold determined by a tame action. Consider $(T, \tau, S, \mathbb{R}^n)$ a $K$-invariant tubular neighborhood of a compact singular stratum $S$. If the BIC of the slice $(\mathbb{R}^n, F_{\mathbb{R}^n})$ is finite dimensional then the BIC of the tube $(T, F)$ is also finite dimensional.

**Proof.** We can suppose that $G$ is connected and we fix a connected tamper group $K$ where $G$ is dense.

We consider the orbit type stratification induced by the action $\Phi: K \times S \to S$ (cf. 2.5). We prove by induction on depth $S_{\Phi}$ the following statement.

$$\mathfrak{A}(T, F) = \text{“The BIC } HH^*_p(T/F) \text{ is finite dimensional for each perversity } p.”$$

Fix $p$ a perversity. Recall that any $K$-invariant submanifold of $M$ inherits naturally a perversity (cf. 3.4), written also $p$. We proceed in two steps.\hfill

1. **First step:** depth $S_{\Phi} = 0$. The isotropy subgroup of any point of $S$ is a compact subgroup $H \subset K$. The orbit space $S/K$ is a manifold and the natural projection $\pi: S \to S/K$ is a locally trivial bundle with fiber $K/H$. Fix $\{U_\alpha\}$ a good open covering of $S/K$ (cf. [4]), which is closed for finite intersections. For an open subset $U \subset S/K$ we consider the statement

$$Q(U) = \text{“The BIC } HH^*_p(\tau^{-1}\pi(U)/F) \text{ is finite dimensional”}.$$\hfill

Notice that $Q(S) = \mathfrak{A}(T, F)$. We get the result if we verify the three conditions of the Bredon’s trick (cf. Proposition 3.6.1).

(BT1) Since $U_\alpha$ is contractible we can identify $\pi^{-1}(U_\alpha)$ with $U_\alpha \times K/H$. The group $K$ acts by $k_0 \cdot (x, k_1 H) = (x, k_0 k_1 H)$. Fix $\{x_0\}$ a basis point of $U_\alpha$ and identify $\{x_0\} \times K/H$ with $K/H$. The contractibility of the open subset $U_\alpha$ gives a $K$-invariant $O(n)$-isomorphism between $(T, \tau, S, \mathbb{R}^n)$ and $(U_\alpha \times \tau^{-1}(K/H), \text{Identity}_{U_\alpha} \times \tau, U_\alpha \times K/H, \mathbb{R}^n)$. Notice that, identifying $\tau^{-1}(eH)$ with $\mathbb{R}^n$, the map $< k, u > \mapsto k \cdot u$ realizes a $K$ diffeomorphism between $\tau^{-1}(K/H)$ and the twisted product $K \times_H \mathbb{R}^n$.

The contractibility of $U_\alpha$ gives:

$$HH^*_p(\tau^{-1}\pi(U)/F) \cong HH^*_p(U_\alpha \times \tau^{-1}(K/H)/T \times T) \cong HH^*_p(\tau^{-1}(K/H)/F) \cong HH^*_p(K \times_H \mathbb{R}^n/F),$$

which is finite dimensional following Proposition 3.8.4.
The lifting of a partition of the unity subordinated to the covering \{U, V\} of S/K gives a controlled and basic partition of the unity subordinated to the covering \{τ⁻¹π¹(U), τ⁻¹π¹(V)\} of T (cf. 3.2.1 (b)). This covering is a basic one. Now we apply 3.6.

(BT3) Clear since \( \Pi^*_F \left( \tau^{-1} \pi^1 \left( \bigcup_{i=0}^m U_i \right) \right) / \mathcal{F} = \bigoplus_{i=0}^m \Pi^*_F \left( \tau^{-1} \pi^1 (U_i) / \mathcal{F} \right) \).

Notice that the compactness asked in the statement of the Proposition is used here by considering a finite covering \{U_i / 1 \leq i \leq m\}. For a non finite covering the property (BT3) could be false.

2. Induction step. Denote \( S^\text{min}_\Phi \) the union of closed (minimal) strata of \( S_\Phi \) and choose \( T^\text{min}_\Phi \) a disjoint family of K-invariant tubular neighborhoods of the closed strata. The projection map is written \( \tau^\text{min}_\Phi : T^\text{min}_\Phi \to S^\text{min}_\Phi \). This set is not empty since depth \( S_\Phi > 0 \). The union of associated tubes is denoted by \( D^\text{min}_\Phi \). It is a compact K-invariant submanifold verifying

\[
\text{depth } S_\Phi : K \times D^\text{min}_\Phi \to D^\text{min}_\Phi < \text{depth } S_\Phi.
\]

The induced map \( \nabla^\text{min}_\Phi : D^\text{min}_\Phi \times [0, 1] \to T^\text{min}_\Phi - S^\text{min}_\Phi \) is a K-equivariant diffeomorphism, trivial action on the \([0, 1]\)-factor. The radius map \( \rho^\text{min}_\Phi : T^\text{min}_\Phi \to [0, 1] \) is a K-invariant map.

The family \( \{ \tau^{-1}((S - S^\text{min}_\Phi), \tau^{-1}(T^\text{min}_\Phi) \} \) is a K-invariant open covering of T. Choose \( \alpha : [0, 1] \to \mathbb{R} \) a smooth map with \( \alpha \equiv 1 \) on \([0, 1/4]\) and \( \alpha \equiv 0 \) on \([3/4, 1]\). Write \( f = \alpha \circ \rho^\text{min}_\Phi \circ \tau : T \to \mathbb{R} \), which is a K-invariant map and therefore \( \mathcal{F} \)-basic. Since \( \text{supp } f \subset \tau^{-1}(T^\text{min}_\Phi) \) and \( \text{supp } (1 - f) \subset \tau^{-1}(S - S^\text{min}_\Phi) \) we conclude that the covering is a basic one. From 3.6 we get an exact Mayer-Vietoris sequence

\[
0 \to \Omega^*_F (\tau^{-1}((S - S^\text{min}_\Phi) / \mathcal{F}) \to \Omega^*_F (\tau^{-1}(S - S^\text{min}_\Phi) / \mathcal{F}) \oplus \Omega^*_F (\tau^{-1}(T^\text{min}_\Phi) / \mathcal{F}) \to \Omega^*_F (T / \mathcal{F}) \to 0.
\]

The Five Lemma gives

\[ \mathfrak{A}(\tau^{-1}((S - S^\text{min}_\Phi) / \mathcal{F}), \mathfrak{A}(\tau^{-1}(S - S^\text{min}_\Phi) / \mathcal{F}) \) and \( \mathfrak{A}(\tau^{-1}(S - S^\text{min}_\Phi) / \mathcal{F}), \mathfrak{A}(T / \mathcal{F}) \), \( \mathfrak{A}(T / \mathcal{F}) \).

We check now these three conditions.

(a) \( \mathfrak{A}(\tau^{-1}((S - S^\text{min}_\Phi) / \mathcal{F}) \). The K-equivariant diffeomorphism \( \nabla^\text{min}_\Phi \) produces by pull-back the commutative diagram

\[
\begin{array}{ccc}
\tau^{-1}(D^\text{min}_\Phi) \times [0, 1] & \xrightarrow{\nabla^\text{min}_\Phi} & \tau^{-1}(T^\text{min}_\Phi - S^\text{min}_\Phi) \\
\tau \times \text{Identity} [0, 1] & \downarrow & \tau \\
D^\text{min}_\Phi \times [0, 1] & \xrightarrow{\Sigma^\text{min}_\Phi} & T^\text{min}_\Phi - S^\text{min}_\Phi
\end{array}
\]

where \( \nabla^\text{min}_\Phi \) is a K-diffeomorphism. So, we get

\[
\Pi^*_F (\tau^{-1}((S - S^\text{min}_\Phi) / \mathcal{F}) \cong \Pi^*_F (\tau^{-1}(D^\text{min}_\Phi) \times [0, 1] / \mathcal{F} \times \mathcal{I}) \cong \Pi^*_F (\tau^{-1}(D^\text{min}_\Phi) / \mathcal{F}),
\]

which is finite from the induction hypothesis. (see (16)).

(b) \( \mathfrak{A}(\tau^{-1}(T^\text{min}_\Phi) / \mathcal{F}) \). The idea is the following. We prove that the inclusion \( \tau^{-1}(S^\text{min}_\Phi) \hookrightarrow \tau^{-1}(T^\text{min}_\Phi) \) induces an isomorphism

\[
\Pi^*_F (\tau^{-1}(T^\text{min}_\Phi) / \mathcal{F}) \cong \Pi^*_F (\tau^{-1}(S^\text{min}_\Phi) / \mathcal{F})
\]

Now, since the depth of \( S_\Phi : K \times S^\text{min}_\Phi \to S^\text{min}_\Phi \) is 0, it suffices to apply the first step.
The BIC of a linear foliation.

The contraction \( H^{\min} : T^{\min} \times [0, 1] \to T^{\min} \) is a \( K \) invariant map with \( H^{\min}_0 = \iota \circ \tau^{\min} \) and \( H^{\min}_1 = \text{Identity} \circ \tau^{-1}(T^{\min}) \), where \( \iota : S^{\min} \hookrightarrow T^{\min} \) is the natural inclusion. Notice that \( \tau^{\min}_1 \circ \iota = \text{Identity} \circ \tau^{-1}(S^{\min}) \).

The map \( H^{\min}_1 \) is locally the map \( H^{\min} : U \times \mathbb{R}^m \times [0, 1] \to U \times \mathbb{R}^m \) defined by \( H^{\min}_1(x, v, t) = (x, tv) \).

Consider the induced commutative diagram by pull-back

\[
\begin{array}{ccc}
\tau^{-1}(T^{\min}) \times [0, 1] & \xrightarrow{H^{\min}} & \tau^{-1}(T^{\min}) \\
\tau \times \text{Identity} \downarrow & & \downarrow \\
T^{\min} \times [0, 1] & \xrightarrow{H^{\min}_1} & T^{\min}.
\end{array}
\]

Put \( \tau^{\min}_1 \) and \( \iota \) the pull backs of \( \tau^{\min} \) and \( \iota \) respectively. We have \( H^{\min}_1 = \iota \circ \tau^{\min}_1 \) and \( \text{Identity} = H^{\min}_1 \circ \tau^{\min}_1 \).

The operator \( H^{\min}_1 \) is \( K \)-invariant and therefore is a foliated morphism: \( H^{\min}_1 : F = F \times I \). It is locally of the form \( H^{\min} : U \times \mathbb{R}^m \times \mathbb{R}^n \times [0, 1] \to U \times \mathbb{R}^m \times \mathbb{R}^n \) with \( H^{\min}(x, v, w, t) = (x, tv, w) \). Since the stratification induced by \( S_F \) is \( \{ U \times \mathbb{R}^m \times S' \times [0, 1] / S' \in S_F \} \) then the perverse condition and the perverse degree are read on the \( \mathbb{R}^n \)-factor. So, the induced operator

\[
\left. H^{\min}_1 \right| : \Omega^*_{\mathbb{F}}(\tau^{-1}(T^{\min})/F) \to \Omega^*_{\mathbb{F}}(\tau^{-1}(T^{\min} \times [0, 1]/F \times I))
\]

is well-defined. The integration along the \([0, 1]\)-factor does not involve \( \mathbb{R}^n \). So, the integration operator

\[
K : \Omega^*_{\mathbb{F}}(\tau^{-1}(T^{\min})/F) \to \Omega^*_{\mathbb{F}}(\tau^{-1}(T^{\min})/F),
\]

given by \( K(\omega) = \int_0^1 H^{\min}_1 \omega \), is well-defined. On the other hand, it verifies the homotopy equality:

\[
d \circ K + K \circ d = \left( H^{\min}_1 \right)^* - \left( H^{\min}_0 \right)^* = \left( H^{\min}_1 \right)^* - \text{Identity}.
\]

This gives \( \tau^{\min}_1 \circ \iota^\ast = \text{Identity} \). Since \( \iota \circ \tau^{\min}_1 = \text{Identity} \) we get (17).

**c)** \( \mathbb{F}(\tau^{-1}(S - S^{\min}), F) \). The idea is to construct a \( K \)-invariant tubular neighborhood \((E, \nu, \widetilde{S}, \mathbb{R}^n)\) and a \( K \)-equivariant commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{M} & T \\
\nu \downarrow & & \tau \downarrow \\
\widetilde{S} & \xrightarrow{N} & S
\end{array}
\]

verifying

i) The map \( N : \widetilde{S} \to S \) is the ot-blow-up of \( S \) relatively to the action \( \Phi : K \times S \to S \) and depth \( \Phi \cdot S \) is \( \text{depth} \ S_\Phi \), where \( \Phi \) is the induced action of \( K \) on \( \widetilde{S} \).

ii) The restrictions \( M : M^{-1}\tau^{-1}(S - S^{\min}) \to \tau^{-1}(S - S^{\min}) \) and \( M : M^{-1}\tau^{-1}(T^{\min} - S^{\min}) \to \tau^{-1}(T^{\min} - S^{\min}) \) are two trivial 2-coverings.

iii) The invariant tubular neighborhood \((M^{-1}\tau^{-1}(T^{\min}), \nu, N^{-1}(T^{\min}), \mathbb{R}^n)\) is \( K \)-equivariantly diffeomorphic to \((\tau^{-1}(D^{\min}), \tau \times \text{Identity} , (D^{\min}) \times \mathbb{R}^n)\).
The conditions ii) and iii) give

\[ \mathfrak{A}(\tau^{-1}(S - S^{\min}), \mathcal{F}) \iff \mathfrak{A}(\mathcal{M}^{-1}\tau^{-1}(S - S^{\min}), \mathcal{E}) \]
\[ (19) \]
\[ \mathfrak{A}(\tau^{-1}(T^{\min} - S^{\min}), \mathcal{F}) \iff \mathfrak{A}(\mathcal{M}^{-1}\tau^{-1}(T^{\min} - S^{\min}), \mathcal{E}) \]
\[ \mathfrak{A}(\tau^{-1}(D^{\min}), \mathcal{F}) \iff \mathfrak{A}(\mathcal{M}^{-1}\tau^{-1}(T^{\min}), \mathcal{E}), \]

where \( \mathcal{E} \) is the foliation induced by the action of \( K \) on \( E \).

The family \( \{ \mathcal{M}^{-1}\tau^{-1}(S - S^{\min}), \mathcal{M}^{-1}\tau^{-1}(T^{\min}) \} \) is a \( K \)-invariant open covering of \( E \). Choose \( \alpha: [0, 1] \rightarrow \mathbb{R} \) a smooth map with \( \alpha \equiv 1 \) on \( [0, 1/4] \) and \( \alpha \equiv 0 \) on \( [3/4, 1] \). Denote \( f = \alpha \circ \rho^{\min} \circ \mathcal{N} \circ \nu: E \rightarrow \mathbb{R} \), which is a \( K \)-invariant map and therefore \( \mathcal{E} \)-basic, where \( \mathcal{E} \) is the foliation induced by the action of \( K \) on \( E \). Since \( \text{supp } f \subset \mathcal{M}^{-1}\tau^{-1}(T^{\min}) \) and \( \text{supp } (1 - f) \subset \mathcal{M}^{-1}\tau^{-1}(S - S^{\min}) \) we conclude that the covering is a basic one. From 3.6 we get an exact Mayer-Vietoris sequence

\[ 0 \rightarrow \Omega^*_\mathcal{F}(\mathcal{M}^{-1}\tau^{-1}(T^{\min} - S^{\min})/\mathcal{E}) \rightarrow \Omega^*_\mathcal{F}(\mathcal{M}^{-1}\tau^{-1}(S - S^{\min})/\mathcal{E}) \oplus \Omega^*_\mathcal{F}(\mathcal{M}^{-1}\tau^{-1}(T^{\min})/\mathcal{E}) \rightarrow \]
\[ \rightarrow \Omega^*_\mathcal{F}(E/\mathcal{E}) \rightarrow 0. \]

We check now these three conditions. The Five Lemma and (19) give

\[ \mathfrak{A}(\tau^{-1}(T^{\min} - S^{\min}), \mathcal{F}), \mathfrak{A}(\tau^{-1}(D^{\min}), \mathcal{F}) \text{ and } \mathfrak{A}(E, \mathcal{E}) \Rightarrow \mathfrak{A}(\tau^{-1}(S - S^{\min}), \mathcal{F}). \]

We check now these three conditions.

(c1) \( \mathfrak{A}(\tau^{-1}(T^{\min} - S^{\min}), \mathcal{F}) \). It is the condition (a).

(c2) \( \mathfrak{A}(\tau^{-1}(D^{\min}), \mathcal{F}) \). By induction hypothesis since we have (16).

(c3) \( \mathfrak{A}(E, \mathcal{E}) \). By induction hypothesis since we have (7) by i).

It remains to construct (18). Consider the manifold

\[ \tilde{S} = \left\{ \left( D^{\min} \times \right] -1, 1 \left[ \right) \coprod \left( S - S^{\min} \times \{ -1, 1 \} \right) \right\} / \sim, \]

where \( (z, t) \sim (\nabla^{\min}(z, |t|), t/|t|) \), and the map \( \mathcal{N}: \tilde{S} \rightarrow S \) defined by

\[ \nabla^{\min}(z, |t|) \quad \text{if } v = (z, t) \in D^{\min} \times \right] -1, 1 \left[ \right] \]
\[ z \quad \text{if } v = (z, j) \in (S - S^{\min}) \times \{ -1, 1 \}. \]

This is the ot-blow-up \( \mathcal{N}: \tilde{S} \rightarrow S \) induced by the action \( \Phi: K \times S \rightarrow S \).

Consider the manifold

\[ E = \left\{ \left( \tau^{-1}(D^{\min}) \times \right] -1, 1 \left[ \right) \coprod \left( \tau^{-1}(S - S^{\min}) \times \{ -1, 1 \} \right) \right\} / \sim, \]

where \( (z, t) \sim (\nabla^{\min}(z, |t|), t/|t|) \) the map \( \mathcal{M}: E \rightarrow T \) defined by

\[ \nabla^{\min}(z, |t|) \quad \text{if } v = (z, t) \in \tau^{-1}(D^{\min}) \times \right] -1, 1 \left[ \right] \]
\[ z \quad \text{if } v = (z, j) \in \tau^{-1}(S - S^{\min}) \times \{ -1, 1 \} \]

and the map \( \nu: E \rightarrow \tilde{S} \) defined by

\[ \nu(v) = \left\{ \begin{array}{ll}
(\tau(z), t) & \text{if } v = (z, t) \in \tau^{-1}(D^{\min}) \times \right] -1, 1 \left[ \right] \\
(\tau(z), j) & \text{if } v = (z, j) \in \tau^{-1}(S - S^{\min}) \times \{ -1, 1 \}
\end{array} \right\} \]
Since $\nabla^{-1}_{\min}$ and $\tilde{\nabla}^{-1}_{\min}$ are $K$-equivariant embeddings then $\tilde{S}$ and $E$ are $K$-manifolds. The maps $N$ and $M$ are $K$-equivariant continuous maps. Since the map $\tau$ is $K$-equivariant then the map $\nu$ is a $K$-equivariant map. The diagram (18) is clearly commutative.

We have that $(E,\nu,S,\mathbb{R}^n)$ is a tubular neighborhood since $(\tau^{-1}(S-S^{-1}_{\min}),\tau,S^{-1}_{\min},\mathbb{R}^n)$ and $(\tau^{-1}(D^{-1}_{\min})\times]-1,1[,\tau \times \text{Identity}|-1,1[,D^{-1}_{\min}\times]-1,1[,\mathbb{R}^n)$ are compatible tubular neighborhoods. It remains to verify the properties i)-iii).

i) By construction.

ii) By construction $\mathcal{M}^{-1}\tau^{-1}(S-S^{-1}_{\min}) = \tau^{-1}(S-S^{-1}_{\min}) \times \{-1,1\}$ and $\mathcal{M}$ is the projection on the first factor.

iii) By construction $\mathcal{N}^{-1}(T^{-1}_{\min}) = D^{-1}_{\min}\times]-1,1[,\mathcal{M}^{-1}\tau^{-1}(T^{-1}_{\min}) = \tau^{-1}(D^{-1}_{\min})\times]-1,1[\nu becomes $\tau \times \text{Identity}|-1,1|$. This ends the proof.

The first main result of this section is the following

**Theorem 4.2.2** The BIC of the foliation determined by an isometric action of an abelian Lie group on a compact manifold is finite dimensional.

**Proof.** Given a conical foliated manifold $(N,\mathcal{N})$ we consider the statement

$$\mathfrak{A}(N,\mathcal{N}) = \text{“The BIC } H^\bullet_{\mathfrak{p}}(N/\mathcal{N}) \text{ is finite dimensional, for any perversity } \mathfrak{p}.”$$

Consider $\Phi: G \times M \to M$ an isometric action of an abelian Lie group $G$ on a compact manifold. This equivalent to say that the action is tame. We denote by $\mathcal{F}$ the induced conical foliation. Let us suppose that $G$ is connected and dense in the (connected) tamper group $K$. We prove $\mathfrak{A}(M,\mathcal{F})$ by induction on depth $S_{\mathcal{F}}$.

1. **First step:** depth $S_{\mathcal{F}} = 0$. The foliation $\mathcal{F}$ is a (regular) riemannian foliation (cf. [10]) and the BIC is just the basic cohomology (cf. 3.4.1 (a)) The result comes directly from [7].

2. **Induction step:** The family $\{M-S^{-1}_{\min},T^{-1}_{\min}\}$ is a $K$-invariant basic open covering of $M$ (cf. proof of Proposition 4.1). From 3.6 we get an exact Mayer-Vietoris sequence

$$0 \to \Omega^\mathfrak{p}_M(T^{-1}_{\min}-S^{-1}_{\min}/\mathcal{F}) \to \Omega^\mathfrak{p}_M(M-S^{-1}_{\min}/\mathcal{F}) \oplus \Omega^\mathfrak{p}_M(T^{-1}_{\min}/\mathcal{F}) \to \Omega^\mathfrak{p}_M(M/\mathcal{F}) \to 0.$$ 

The Five Lemma gives

$$\mathfrak{A}(T^{-1}_{\min}-S^{-1}_{\min},\mathcal{F}), \mathfrak{A}(T^{-1}_{\min},\mathcal{F}) \text{ and } \mathfrak{A}(M-S^{-1}_{\min},\mathcal{F}), \implies \mathfrak{A}(M,\mathcal{F}).$$

We check now these three conditions.

(a) $\mathfrak{A}(T^{-1}_{\min}-S^{-1}_{\min},\mathcal{F})$. Since $\nabla^{-1}_{\min}$ is a $K$-equivariant diffeomorphism we have

$$\mathfrak{A}(T^{-1}_{\min}-S^{-1}_{\min},\mathcal{F}) \iff \mathfrak{A}(D^{-1}_{\min}\times]-1,1[,\mathcal{F},\times I) \iff \mathfrak{A}(D^{-1}_{\min},\mathcal{F}),$$

which is true since the depth of $(D^{-1}_{\min},\mathcal{F})$ is strictly smaller than depth $S_{\mathcal{F}}$ (cf. 2.5).

(b) $\mathfrak{A}(T^{-1}_{\min},\mathcal{F})$. If we prove $\mathfrak{A}(\mathbb{R}^n,\mathcal{F}_{\mathbb{R}^n})$, for the slice of a tubular neighborhood $(T,S,\tau,\mathbb{R}^n)$, then it suffices to apply the Proposition 4.2.1. Recall that $\mathcal{F}_{\mathbb{R}^n}$ is defined by an orthogonal action $\Theta: G \cap H \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\Theta: G \cap H \times S^{n-1} \to S^{n-1}$ is a tame action without fixed points defining $\mathcal{G}$ and verifying $(\mathbb{R}^n,\mathcal{F}_{\mathbb{R}^n}) = (e\mathbb{S}^{n-1},G_{\mathbb{S}^{n-1}})$. We have $H^\mathfrak{p}_\mathcal{F}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \overset{3.5.2}{=} H^\mathfrak{p}_{\mathbb{S}^{n-1}}(\mathbb{S}^{n-1}/G_{\mathbb{S}^{n-1}})$, the tronqued cohomology, which is finite dimensional since depth $S_{G_{\mathbb{S}^{n-1}}}$ < depth $S_{\mathcal{F}}$ implies $\mathfrak{A}(\mathbb{S}^{n-1},G_{\mathbb{S}^{n-1}})$. 

29 Octobre 2018.
(c) \( \mathfrak{A}(M - S_{\text{min}}, F) \). The family \( \{ \mathcal{L}^{-1}(M - S_{\text{min}}, \mathcal{L}^{-1}(T_{\text{min}}) \} \) is a \( K \)-invariant open covering of \( M \). Choose \( \alpha: [0,1] \to \mathbb{R} \) a smooth map with \( \alpha \equiv 1 \) on \([0,1/4]\) and \( \alpha \equiv 0 \) on \([3/4,1]\). Write \( f = \alpha \circ \rho_{\text{min}} \circ \mathcal{L}: \hat{M} \to \mathbb{R} \), which is a \( K \)-invariant map and therefore \( \hat{\mathcal{F}} \)-basic. Since \( \text{supp} \ f \subset \mathcal{L}^{-1}(T_{\text{min}}) \) and \( \text{supp} \ (1 - f) \subset \mathcal{L}^{-1}(M - S_{\text{min}}) \) we conclude that the covering is a basic one. From 3.6 we get an exact Mayer-Vietoris sequence

\[
0 \to \Omega^*(\mathcal{L}^{-1}(T_{\text{min}} - S_{\text{min}})/\hat{\mathcal{F}}) \to \Omega^*(\mathcal{L}^{-1}(M - S_{\text{min}})/\hat{\mathcal{F}}) \oplus \Omega^*(\mathcal{L}^{-1}(T_{\text{min}})/\hat{\mathcal{F}}) \to \Omega^*(\hat{M}/\hat{\mathcal{F}}) \to 0.
\]

Recall that, following 2.5, we have that \( \mathcal{A}(\mathcal{L}^{-1}(M - S_{\text{min}}), \hat{\mathcal{F}}) \equiv \mathcal{A}(M - S_{\text{min}}, \mathcal{F}) \). Now, the Five Lemma gives

\[
\mathcal{A}(\mathcal{L}^{-1}(T_{\text{min}} - S_{\text{min}}), \hat{\mathcal{F}}), \mathcal{A}(\mathcal{L}^{-1}(T_{\text{min}}), \hat{\mathcal{F}}) \text{ and } \mathcal{A}(\hat{M}, \hat{\mathcal{F}}), \implies \mathcal{A}(M - S_{\text{min}}, \mathcal{F}).
\]

We check now these three conditions.

(c1) \( \mathcal{A}(\mathcal{L}^{-1}(T_{\text{min}} - S_{\text{min}}), \hat{\mathcal{F}}) \). Since \( \mathcal{L}^{-1}(T_{\text{min}} - S_{\text{min}}) \) is \( K \)-diffeomorphic to two copies of \( T_{\text{min}} - S_{\text{min}} \) (cf. 2.4) then we have

\[
\mathcal{A}(\mathcal{L}^{-1}(T_{\text{min}} - S_{\text{min}}), \hat{\mathcal{F}}) \iff \mathcal{A}(T_{\text{min}} - S_{\text{min}}, \mathcal{F}).
\]

Now we apply (a).

(c2) \( \mathcal{A}(\mathcal{L}^{-1}(T_{\text{min}}), \hat{\mathcal{F}}) \). From 2.4 we know that \( \mathcal{L}^{-1}(T_{\text{min}}) \) is \( K \)-diffeomorphic to \( D_{\text{min}} \times ]1,1[ \). Now we proceed as in (a).

(c3) \( \mathcal{A}(\hat{M}, \hat{\mathcal{F}}) \). Because depth \( S_{\hat{\mathcal{F}}} < S_{\mathcal{F}} \) (cf. 2.4).

\[\blacklozenge\]

### 4.3 Poincaré Duality

We prove in this section that the BIC of a conical foliation \( \mathcal{F} \) defined on an oriented manifold \( M \) and determined by a tame action, verifies the Poincaré Duality:

\[
\mathcal{H}_\mathcal{E}(M/\mathcal{F}) \cong \mathcal{H}_\mathcal{E}^{-}(M/\mathcal{F}).
\]

Here \( \ell \) (or \( \ell_M \)) is the codimension of the foliation \( \mathcal{F} \). The two perversities \( \overline{\mathcal{P}} \) and \( \overline{\mathcal{Q}} \) are complementary, that is, \( \overline{\mathcal{P}} + \overline{\mathcal{Q}} = \overline{\mathcal{I}} \).

The proof follows the path of 4.2, but firstly we define the morphism \( P_M \) giving (20), it depends on the notion of a tangent volume form.

#### 4.3.1 Tangent volume form

For the definition of the pairing \( P_M \) we need a volume form tangent to the leaves of \( \mathcal{F} \).

Consider \( \Phi: G \times M \to M \) a tame action defining \( \mathcal{F} \). We can choose the group \( G \) connected and the action \( \Phi \) effective; so, \( b = \text{dim } G = \text{dim } \mathcal{F} \). We also fix \( \{ u_1, \ldots, u_b \} \) a basis of the Lie algebra \( \mathfrak{g} \) of \( G \). The associated fundamental vector fields on \( M \) are denoted by \( \{ V_1, \ldots, V_b \} \).

A tangent volume form of \( (M, \mathcal{F}) \) is a \( G \)-invariant differential form \( \eta \in \Pi_\mathcal{F}^b(M) \) verifying

\[
\eta(V_1, \ldots, V_b) = 1.
\]

Notice that \( d\eta(V_1, \ldots, V_b, -) = 0 \).

**Lemma 4.3.2** Consider \( K \) a tamer group of \( G \). There exists a \( K \)-invariant tangent volume form \( \eta \) of \((M, \mathcal{F})\) verifying the following properties:

(a) For each \( \omega \in \Omega^b(M/\mathcal{F}) \) the product \( \omega \wedge \eta \) does not depend on the tangent volume form \( \eta \).
(b) For each $\omega \in \Omega^{\ell-1}_F(M/F)$ the product $\omega \wedge d\eta$ is 0.

(c) For each $\omega \in \Omega^{\ell}_F(M/F)$ the integral $\int_{R_F} \omega \wedge \eta$ is finite.

(d) For each $\omega \in \Omega^{\ell-1}_F(M/F)$ the integral $\int_{R_F} d(\omega \wedge \eta)$ is 0.

Proof. We proceed in several steps.

First step. Existence.

We prove the following statement by induction on depth $S_F$:

"There exists a $K$-invariant differential form $\eta \in \Pi^b_{F \times I}(M \times [0,1]^p)$ verifying:

\[ \eta((V_1,0),\ldots,(V_b,0)) = 1, \]

for each $p \in \mathbb{N}$".

The existence is proven by taking $p = 0$.

When depth $S_F = 0$ then we define $\eta_0$ on the orbits of $\Phi$ by (22) and we extend to a differential form of $\Omega^b(M \times [0,1]^p)$. The differential form $\eta = \int_K k^*\eta_0$ is $K$-invariant, lives in $\Pi^{b}_{F \times I}(M \times [0,1]^p)$ and verifies (22) since each $k$ is a $K$-equivariant diffeomorphism.

Consider now the case depth $S_F > 0$. By induction hypothesis there exists a $K$-invariant differential form $\eta_0 \in \Pi^b_{F \times I}(M \times [0,1]^p)$ verifying (22). Associated to the Molino’s blow up we have the $K$-equivariant imbedding $S_\sigma : (M - S_{\min}) \rightarrow L^\sigma^{-1}(M - S_{\min})$, defined by $\sigma(z) = (z,1)$. The differential form $\eta = (\sigma \times \text{identity}_{[0,1]^p})^*\eta_0$ belongs to $\Omega^b(R_F \times [0,1]^p)$. It is $K$-invariant and verifies (22) since $\sigma$ is a $K$-equivariant imbedding. It remains to prove that $\eta \in \Pi^b_{F \times I}(M \times [0,1]^p)$, which is a local property. So, we can consider that $M$ is a tubular neighborhood $T$ of a singular stratum of $S_F$ and prove $(\nabla \times \text{identity}_{[0,1]^p})^*\eta \in \Pi^b_{F \times I}(D \times [0,1])$ (cf. 3.1.1 (e)). This is the case since $\sigma \circ \nabla: D \times [0,1] \rightarrow D \times [0,1]$ is just the inclusion and $\eta_0 \in \Pi^b_{F \times I}(D \times [0,1])$.

Second step. The condition (a).

Let $\eta'$ be another tangent volume form associated to $F$ through $\Phi$ and $\{u_1, \ldots, u_b\}$. By degree reasons it suffices to prove the equality $i_{V_1} \cdots i_{V_b} (\omega \wedge \eta) = i_{V_1} \cdots i_{V_b} (\omega \wedge \eta')$. Since $\omega$ is a basic form, we have $i_{V_1} \cdots i_{V_b} (\omega \wedge \eta) = (-1)^{\ell_b}\omega \wedge (i_{V_1} \cdots i_{V_b}\eta) = (-1)^{\ell_b}\omega \wedge (i_{V_1} \cdots i_{V_b}\eta') = i_{V_1} \cdots i_{V_b} (\omega \wedge \eta')$.

Third step. The condition (b).

For degree reasons it suffices to prove that $i_{V_1} \cdots i_{V_b} (\omega \wedge d\eta) = 0$. Since $\omega$ is a basic form, we can write $i_{V_1} \cdots i_{V_b} (\omega \wedge d\eta) = (-1)^{\ell_b}\omega \wedge i_{V_1} \cdots i_{V_b} d\eta = 0$.

Third step. The condition (c).

It suffices to prove that $\int_{R_F \times [0,1]^p} \gamma < \infty$, where $\gamma \in \Pi^{\ell + p}_{F \times I}(M \times [0,1]^p)$ with compact support. We proceed by induction on the depth of $S_F$. When the foliation is regular then the result is clear. In the general case we know that the result is true for $M - S_{\min} \times [0,1]^p$ and $(T_{\min} - S_{\min}) \times [0,1]^p$. It remains to consider $T_{\min} \times [0,1]^p$. From 3.4.1 (d) we know that we can identify the perverse forms of $T_{\min} \times [0,1]^p$ with the perverse forms of $D_{\min} \times [0,1]^{p+1}$ through the map

$\nabla_{\min} \times \text{Identity}_{[0,1]^p} : D_{\min} \times [0,1] \times [0,1]^p \equiv D_{\min} \times [0,1]^{p+1} \rightarrow T_{\min} \times [0,1]^p$.

Since this map is a diffeomorphism between $D_{\min} \times [0,1] \times [0,1]^p$ and $(T_{\min} - S_{\min}) \times [0,1]^p$, then we have

$$\int_{R_{D_{\min} \times [0,1]^{p+1}}} \gamma = \int_{R_{D_{\min} \times [0,1]^{p+1}}} \gamma = \int_{R_{D_{\min} \times [0,1]^{p+1}}} \gamma = \int_{R_{D_{\min} \times [0,1]^{p+1}}} \gamma.$$
The induction hypothesis gives that this integral is finite.

**Fourth step. The condition (d).**

Since $\text{supp } \omega$ is compact then it suffices to prove $\int_{U \cap R_F} d(\omega \wedge \eta) = 0$ where $(U, \varphi)$ is a conical chart of $F$ and $\omega \in \Omega^\ell_{\tau,c}(U/F)$ with $\text{supp } \omega \subset U$. We have:

$$\int_{U \cap R_F} d(\omega \wedge \eta) = \int_{R^{m-n-1} \times R_G \times [0,1]} d(P^\varphi_\omega \wedge P^\varphi_\eta) = \int_{R^{m-n-1} \times R_G \times [0,1]} d(\omega \varphi \wedge \eta \varphi) = 0.$$

This ends the proof.

4.3.3 The pairing. Let $\eta$ be a tangent volume form. Consider the differential operator given by

$$P_M: \Omega^\ast_p(M/F) \times \Omega^{\ell\ast}_{\tau,c}(M/F) \longrightarrow \mathbb{R}$$

where $P_M(\alpha, \beta) = \int_{R_F} \alpha \wedge \beta \wedge \eta$. Notice that the manifold $R_F \subset M$ is an oriented manifold. From Proposition 4.3.2 (c) we have that this integration is well-defined. This operator depends on the action $\Phi: G \times M \rightarrow M$ and on the choice of the basis $\{u_1, \ldots, u_b\}$ of $\mathfrak{g}$ (cf. Proposition 4.3.2 (a)).

The pairing is the induced operator

$$P_M: \mathcal{H}^\ast_p(M/F) \times \mathcal{H}^{\ell\ast}_{\tau,c}(M/F) \longrightarrow \mathbb{R}$$

defined by

$$P_M([\alpha], [\beta]) = \int_{R_F} \alpha \wedge \beta \wedge \eta.$$

This operator is well-defined (cf. Proposition 4.3.2 (b) and (d)). The Poincaré Duality stands that $P_M$ is a non degenerate pairing, that is, the operator

$$P_M: \mathcal{H}^\ast_p(M/F) \longrightarrow \text{Hom} \left( \mathcal{H}^{\ell\ast}_{\tau,c}(M/F), \mathbb{R} \right)$$

defined by

$$P_M([\alpha])([\beta]) = \int_{R_F} \alpha \wedge \beta \wedge \eta$$

is an isomorphism.

The first step to get the Poincaré Duality is the following.

**Proposition 4.3.4** Consider a twisted product $K \times_H \mathbb{R}^n$ as in 2.2. Suppose that the action $\Phi: G \times (K \times_H \mathbb{R}^n) \rightarrow (K \times_H \mathbb{R}^n)$ is effective. We fix a basis of $\mathfrak{t}$ as in 3.8.3. The BIC of $(K \times_H \mathbb{R}^n, F_{tw})$ verifies the Poincaré Duality when the BIC of its slice $(\mathbb{R}^n, F_{R^n})$ verifies the Poincaré Duality.

**Proof.** Since the action $\Phi: G \times (K \times_H \mathbb{R}^n) \rightarrow (K \times_H \mathbb{R}^n)$ defining $F_{tw}$ is effective then the action $\Theta: (G \cap H) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defining $F_{R^n}$ is also effective. The Proposition makes sense. We proceed now in two steps.

(a) **Construction of the tangent volume form.**

Consider a tangent volume form $\eta_0 \in \left( \Pi^{\ell-a}_{\mathcal{F}_{R^n}}(\mathbb{R}^n) \right)^H$ of the slice $(\mathbb{R}^n, F_{R^n})$. We construct the tangent volume form $\eta$ of the twisted product in terms of $\eta_0$. 

Put $\eta = \nabla^{-1}(\gamma_1 \wedge \cdots \wedge \gamma_a \wedge \eta_0) = R^{-\ast}(\gamma_1 \wedge \cdots \wedge \gamma_a \wedge \Delta'(\eta_0))$. It belongs to $(R^\ast_x(K \times_H \mathbb{R}^n))^K$ (cf. 3.8.5). We prove now (21). Since $R_x X_i = V_i$ for $i \in \{1, \ldots, f\}$ then,

$$
\eta(V_1, \ldots, V_b) = (\gamma_1 \wedge \cdots \wedge \gamma_a \wedge \Delta'(\eta_0))(X_1, \ldots, X_b) = \Delta'(\eta_0)(X_{a+1}, \ldots, X_b) =
$$

$$
= \left( \eta_0 + \sum_{a+1 \leq i_1 < \cdots < i_c \leq e} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_c} \wedge \left( i_{W_{i_1}} \cdots i_{W_{i_c}} \eta_0 \right) \right)(X_{a+1}, \ldots, X_b) =
$$

$$
= i_{W_b} \cdots i_{W_{a+1}} \eta_0 = 1
$$

from (21) for $\eta_0$. We conclude that $\eta$ is a tangent volume form of the twisted product. Recall that the pairing $P_M$ can be defined using this form $\eta$ (cf. Proposition 4.3.2 (a)).

(a) Poincaré Duality.

Consider now two complementary perversities $\overline{p}$ and $\overline{q}$ on $\dim K \times_H \mathbb{R}^n$. The induced perversities $\overline{p}$ and $\overline{q}$ on $\mathbb{R}^n$ are also two complementary perversities. Let us see that, for each stratum $S \in S_{\mathcal{F}_n}$, we have:

$$
\overline{p}(S) + \overline{q}(S) = \overline{p}(K \times_H S) + \overline{q}(K \times_H S) = \overline{t}(K \times_H S) =
$$

$$
= \dim K \times_H \mathbb{R}^n - \dim K \times_H S - \dim G_{K \times_H S} - 2 =
$$

$$
= \dim \mathbb{R}^n - \dim S - \dim (G \cap H)_S - 2 = \overline{t}(S).
$$

By hypothesis the pairing $P_{\mathbb{R}^n} : H^*_p(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \times H^{\ell_{\mathbb{R}^n}}_{\mathbb{R}^n}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \to \mathbb{R}$ is non degenerate. Since $\eta_0$ is $H$-invariant then the pairing $P_{\mathbb{R}^n} : \left( H^*_p(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H \times \left( H^{\ell_{\mathbb{R}^n}}_{\mathbb{R}^n}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H \to \mathbb{R}$ is also non degenerate.

On the other hand, it is clear that the pairing $P : \wedge^\ast(\gamma_{c+1}, \ldots, \gamma_f) \times \wedge^{f-c-\ast}(\gamma_{c+1}, \ldots, \gamma_f) \to \mathbb{R}$, defined by

$$
P(\gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u}, \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}}) = \int_{K/H} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u} \wedge \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}}
$$

is non degenerate. Notice the equality $\ell_{K \times_H \mathbb{R}^n} = f + a + n - c - b = \ell_{\mathbb{R}^n} + f - c$.

We prove that the pairing $P_{K \times_H \mathbb{R}^n} : H^*_p(K \times_H \mathbb{R}^n) \times H^{\ell_{K \times_H \mathbb{R}^n}}_{\mathbb{R}^n}(K \times_H \mathbb{R}^n) \to \mathbb{R}$ is not degenerate. We know, from Proposition 3.8.4, that this is the case if the following diagram commutes (up to a sign):

$$
\begin{array}{ccc}
\left( \wedge^\ast(\gamma_{c+1}, \ldots, \gamma_f) \otimes \left( H^*_p(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H \right) \times \left( \wedge^{f-c-\ast}(\gamma_{c+1}, \ldots, \gamma_f) \otimes \left( H^{\ell_{\mathbb{R}^n}}_{\mathbb{R}^n}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H \right) & \xrightarrow{P \otimes P_{\mathbb{R}^n}} & \mathbb{R} \\
\nabla_-^* \times \nabla_-^* & \downarrow \text{Identity} & \\
H^*_p(K \times_H \mathbb{R}^n) \times H^{\ell_{K \times_H \mathbb{R}^n}}_{\mathbb{R}^n} & \xrightarrow{P_{K \times_H \mathbb{R}^n}} & \mathbb{R}
\end{array}
$$

Let us see that. For each $[\alpha] \in \left( H^*_p(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H$ and each $[\beta] \in \left( H^{\ell_{\mathbb{R}^n}}_{\mathbb{R}^n}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \right)^H$, we have, for degree reasons,

$$
P_{K \times_H \mathbb{R}^n}(\nabla_-^* \times \nabla_-^*)(\gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u} \otimes [\alpha], \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}} \otimes [\beta]) =
$$

$$
\int_{K \times_H \mathbb{R}^n} R^{-\ast}(\gamma_1 \wedge \cdots \wedge \gamma_a \wedge \alpha \wedge \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}} \wedge \beta \wedge \gamma_1 \wedge \cdots \wedge \gamma_a \wedge \eta_0) =
$$

$$
\int_{K \times \mathbb{R}^n} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u} \wedge \alpha \wedge \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}} \wedge \beta \wedge \gamma_1 \wedge \cdots \wedge \gamma_a \wedge \eta_0 \wedge \gamma_{a+1} \wedge \cdots \wedge \gamma_c =
$$

$$
\int_{K \times \mathbb{R}^n} \gamma_1 \wedge \cdots \wedge \gamma_a \wedge \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u} \wedge \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}} \wedge \gamma_{a+1} \wedge \cdots \wedge \gamma_c =
$$

$$
P(\gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u}, \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}}) \cdot P_{\mathbb{R}^n}(\alpha, [\beta]) =
$$

$$
(\alpha \wedge [\alpha], \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_u} \otimes [\alpha], \gamma_{j_1} \wedge \cdots \wedge \gamma_{j_{f-c-u}} \otimes [\beta]).
$$
up to a sign.

We study now the Poincaré Duality of the tubular neighborhoods of the strata of $S_{\mathcal{F}}$ (cf. 2.3). Consider $(T, \tau, S, \mathbb{R}^n)$ a $K$-invariant tubular neighborhood of a singular stratum $S$, where $K$ is a tamer group of $G$ with $K = \overline{S}$. Put $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n})$ the slice of the tubular neighborhood. The foliation $\mathcal{F}_{\mathbb{R}^n}$ is defined by an effective tame action $\Theta: (G_S)_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. We fix $\{u_1, \ldots, u_b\}$ a basis of $g$ and suppose that $\{u_{a+1}, \ldots, u_b\}$ is a basis of the Lie algebra of $G_S$.

**Proposition 4.3.5** Under the above conditions, if the BIC of the slice $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n})$ verifies the Poincaré Duality then the BIC of the tube $(T, \mathcal{F})$ also verifies the Poincaré Duality.

**Proof.** The proof of the Proposition is the same of that of the Proposition 4.2.1 by changing

$$\mathfrak{A}(T, \mathcal{F}) = \text{“The BIC } \mathcal{H}_p^\ell(T/\mathcal{F}) \text{ is finite dimensional for each perversity } \overline{p}.”$$

by

$$\mathfrak{A}(T, \mathcal{F}) = \text{“The pairing } P_T: \mathcal{H}_p^\ell(T/\mathcal{F}) \rightarrow \mathcal{H}_{\overline{p}}^{\ell-\ast}(T/\mathcal{F}) \text{ is non degenerate, for any two complementary perversities } \overline{p} \text{ and } \overline{q}.”$$

We consider the orbit type stratification of $S$ induced by the action $\Phi_S: K \times S \rightarrow S$ of $\Phi$. We proceed by induction on the depth of this stratification. By using the Mayer-Vietoris technics of 3.6 and 3.7 we can suppose that $\Phi_S$ defines a fiber bundle $\pi: S \rightarrow S/K$ whose fiber is $K/H$. Considering a good covering of $S/K$, the Mayer-Vietoris procedure leads us to the case $S/K = \text{point}$, that is, to the case where $T$ is the twisted product $K \times_H \mathbb{R}^n$. Here, we apply the Proposition 4.3.4.

The second main result of this section is the following

**Theorem 4.3.6** The BIC of the foliation determined by an isometric action of an abelian Lie group on an oriented compact manifold verifies the Poincaré Duality.

**Proof.** In fact, we prove the result for a foliation $\mathcal{F}$ determined by a tame action $\Phi: G \times M \rightarrow M$ on an oriented manifold not necessarily compact. We can suppose that $G$ is connected and that the action $\Phi$ is an effective one. We fix $K$ a tamer group where $G$ is dense.

The proof of the Proposition is the same of that of the Theorem 4.2.2 by changing

$$\mathfrak{A}(M, \mathcal{F}) = \text{“The BIC } \mathcal{H}_p^\ell(M/\mathcal{F}) \text{ is finite dimensional for each perversity } \overline{p}.”$$

by

$$\mathfrak{A}(M, \mathcal{F}) = \text{“The pairing } P_T: \mathcal{H}_p^\ell(M/\mathcal{F}) \rightarrow \mathcal{H}_{\overline{p}}^{\ell-\ast}(M/\mathcal{F}) \text{ is non degenerate, for any two complementary perversities } \overline{p} \text{ and } \overline{q}.”$$

By using the Mayer-Vietors technics of 3.6, 3.7 and the Proposition 4.3.5 we reduce the problem to prove $\mathfrak{A}(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n})$, where $(\mathbb{R}^n, \mathcal{F}_{\mathbb{R}^n}) \equiv (cS^{n-1}, cG_S)$ is a slice of a tubular neighborhood of a singular stratum $S$ of $S_{\mathcal{F}}$. In other words, we need to prove that the pairing

$$P_{\mathbb{R}^n}: \mathcal{H}_p^\ell(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \times \mathcal{H}_{\overline{p}}^{\ell-\ast}(\mathbb{R}^n/\mathcal{F}_{\mathbb{R}^n}) \rightarrow \mathbb{R},$$

is non degenerate.

From Propositions 3.5.1, 3.5.2, 3.7.1 and 3.7.2 we have

$$P_{\mathbb{R}^n} = \begin{cases} \mathcal{H}_p^\ell(S^{n-1}/G_S) & \text{if } i \leq \overline{p}(\vartheta) \\ 0 & \text{if } i \geq \overline{p}(\vartheta) + 1 \end{cases}$$

(23)
and
\[ II_{\pi,e}^{F_{n-1}}(\mathbb{R}^n/F_{\mathbb{R}^n}) = \begin{cases} 0 & \text{if } i \geq \ell_{\mathbb{R}^n} - \overline{\pi}(\{\vartheta\}) - 1 \\ II_{\pi}^{F_{n-1}}(S^{n-1}/G_S) & \text{if } i \leq \ell_{\mathbb{R}^n} - \overline{\pi}(\{\vartheta\}) - 2. \end{cases} \]

Since \( \overline{\pi} \) and \( \overline{\eta} \) are complementary perversities on \( \mathbb{R}^n \) then we have:
\[ \overline{\pi}(\{\vartheta\}) + \overline{\eta}(\{\vartheta\}) = \overline{t}(\{\vartheta\}) = n - \dim F_{\mathbb{R}^n} - 2 = n - \dim G - 2 = \ell_{\mathbb{G}^{n-1}} - 1 = \ell_{\mathbb{R}^n} - 2. \]

These formulas give:
\[ (24) \quad II_{\pi,e}^{F_{n-1}}(\mathbb{R}^n/F_{\mathbb{R}^n}) = \begin{cases} 0 & \text{if } i \geq \overline{\pi}(\{\vartheta\}) + 1 \\ II_{\pi}^{F_{n-1}}(S^{n-1}/G_S) & \text{if } i \leq \overline{\pi}(\{\vartheta\}). \end{cases} \]

Now, an argument on the depth of \( S_F \) gives the property \( \mathfrak{A}(\mathbb{R}^n, F_{\mathbb{R}^n}) \) from these three facts
(i) \( \mathfrak{A}(S^{n-1}, G_S) \).

(ii) The pairing \( P_{\mathbb{R}^n} \) becomes the pairing \( P_{\mathbb{G}^{n-1}} \) through the isomorphism induced by (23) and (24).

The operator \( \mathcal{N}_23: \mathcal{I}_{\pi}^{\mathbb{R}}(S^{n-1}/G_S) \rightarrow \mathcal{I}_{\pi}^{\mathbb{R}}(cS^{n-1}/cG_S) \) defining (23) is \( \mathcal{N}_23([\alpha]) = [\alpha] \); the operator \( \mathcal{N}_24: \mathcal{I}_{\pi}^{\mathbb{R}}(S^{n-1}/G_S) \rightarrow \mathcal{I}_{\pi}^{\mathbb{R}}(cS^{n-1}/G_S) \) defining (24) is \( \mathcal{N}_24([\beta]) = [g dt \wedge \beta] \).

Since the action of \( \Theta \) lies on \( S^{n-1} \) then we can take a common tangent volume form \( \eta \) for \( F_{\mathbb{R}^n} \) and \( G_S \) (cf. proof of Proposition 4.3.2). Now, for \( [\alpha] \in \mathcal{I}_{\pi}^{\mathbb{R}}(S^{n-1}/G_S) \) and \( [\beta] \in \mathcal{I}_{\pi,e}^{F_{n-1}}(S^{n-1}/G_S) \) we have
\[ P_{\mathbb{R}^n}(\mathcal{N}_23[\alpha], \mathcal{N}_24[\beta]) = \int_{\mathbb{R}^n \times R_{F_{\mathbb{R}^n}} \times [0,1]} \alpha \wedge g \wedge dt \wedge \beta \wedge \eta = \left( \int_{R_{F_{\mathbb{R}^n}}} \alpha \wedge \beta \wedge \eta \right) \left( \int_0^1 g dt \right) = P_{\mathbb{G}^{n-1}}([\alpha], [\beta]). \]

(iii) The perversities \( \overline{\pi} \) and \( \overline{\eta} \) are complementary on \( S^{n-1} \).

We have, for any stratum \( S \in S_G \) the equalities \( \overline{\pi}(S) + \overline{\eta}(S) = \overline{\pi}(S \times [0,1]) + \overline{\eta}(S \times [0,1]) = \overline{t}(S \times [0,1]) = \text{codim } \mathbb{R}^n F_{\mathbb{R}^n} - \text{codim } S \times [0,1](F_S \times \mathcal{I}) - 2 = \text{codim } \mathbb{G}^{n-1} G_S - \text{codim } S F_S - 2 = \overline{t}(S) \).

Hau amaia da.

\[ \star \]

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