Torus actions of complexity one and their local properties

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Abstract. We consider an effective action of a compact \((n - 1)\)-torus on a smooth 2\(n\)-manifold with isolated fixed points. We prove that under certain conditions the orbit space is a closed topological manifold. In particular, this holds for certain torus actions with disconnected stabilizers. There is a filtration of the orbit manifold by orbit dimensions. The subset of orbits of dimensions less than \(n - 1\) has a specific topology which we axiomatize in the notion of a sponge. In many cases the original manifold can be recovered from its orbit manifold, the sponge, and the weights of tangent representations at fixed points.

1. Introduction

An action of a compact torus \(G\) on a topological space \(X\) is a classical object of study [4]. For a point \(x \in X\) let \(G_x \subset G\) denote the stabilizer subgroup and \(Gx\) the orbit of \(x\). Let \(p: X \to X/G\) be the projection to the orbit space. Let \(S(G)\) denote the set of all closed subgroups of \(G\) endowed with the lower interval topology. There is continuous map

\[ \tilde{\lambda}: X/G \to S(G) \]

which maps an orbit \(x \in X/G\) to the stabilizer subgroup \(G_x\), see [5].

The classical idea in the study of torus actions is the following. It is assumed that the projection map \(p: X \to X/G\) admits a section. Then, given the orbit space \(Q = X/G\), and the continuous map \(\tilde{\lambda}: Q \to S(G)\) one builds a topological model

\[ X_{(Q,\tilde{\lambda})} = (Q \times G)/\sim \]

which is equivariantly homeomorphic to the original space \(X\). The method of constructing model spaces was used by Davis and Januszkiewicz [11] for the classification of manifolds which are now called quasitoric [9]. This idea traces back to the works of Vinberg [19].

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The method can be naturally extended to the locally standard actions of $G \cong T^n$ on $2n$-manifolds \[20\]. In this case the projection may not admit the global section, however, it always admits a section locally, and there exists a topological model space of such action.

Buchstaber–Terzić \[6, 7, 8\] introduced a theory of $(2n, k)$-manifolds in order to study the orbit spaces of more general torus actions and to obtain topological models for such actions. Grassmann manifolds and flag manifolds are important families of $(2n, k)$-manifolds. In this theory a manifold is subdivided into strata $X_\sigma$, so that the action has the same stabilizer $T_\sigma$ for all points of a stratum. It is essential in the definition of $(2n, k)$-manifold that there is a convex polytope $P^k$ and a $T^k$-equivariant generalized moment map $X^{2n} \to P^k$. Every stratum $X_\sigma$ is then represented as a principal $T/T_\sigma$-bundle over the product $P_\sigma \times M_\sigma$, where $P_\sigma$ is a certain subpolytope of $P$ and $M_\sigma$ is an auxiliary space of dimension $2(n-k)$ called the space of parameters. Therefore the orbit space $X^{2n}/T^k$ is represented as the union $\bigsqcup P_\sigma \times M_\sigma$. The theory of $(2n, k)$-manifolds provided the specific methods to describe the topology of this union.

Whenever a compact $k$-torus acts effectively on a $2n$-manifold we call the number $n-k$ the \textit{complexity of the action}. While actions of complexity zero are well studied, the actions of positive complexity constitute a harder problem. It is generally assumed that the actions of complexity $\geq 2$ are extremely complicated in general. The actions of complexity one take an intermediate position: they were studied from several different viewpoints. Algebraical theory of complexity one actions was developed in the works of many authors, in particular, the classification of such actions even in nonabelean case was given by Timashëv \[17, 18\]. Hamiltonian complexity one actions on symplectic manifolds are also well studied: see e.g. the work of Karshon–Tolman \[14\] and references therein. Circle action on a 4-manifold is a classical subject, see e.g. \[10, 13, 16\].

In this paper we study complexity one actions from the topological viewpoint. Our approach is different from the one used in \[5\] and \[7\]. Instead of trying to stratify the manifold so that the action on each stratum admits a section, we partition the manifold by orbit types. Under two restrictions we prove that the orbit space $Q = X/T$ is a topological manifold, see Theorem \[2.10\] for the precise statement. Note that for this result it is not required that the stabilizers of the action are connected. Such restriction was imposed in the theory of $(2n, k)$-manifolds, however there exist natural examples of the actions which have finite stabilizers but still the orbit space is a manifold.

We make a remark on the main difference from situations considered in toric topology: the typical action of complexity one does not admit a section, even locally.

Natural examples of complexity one actions which we keep in mind are the following.

1. The $T^3$ action on the complex Grassmann manifold $G_{4,2}$.
2. The $T^2$ action on the manifold $F_3$ of full complex flags in $\mathbb{C}^3$.
3. Quasitoric manifolds $X^{2n}_{(P,\Lambda)}$ with the induced action of a generic subtorus $T \subset G$, $\dim T = n - 1$.
4. The space of isospectral periodic tridiagonal Hermitian matrices of size $n \geq 3$.

Using the theory of $(2n, k)$-manifolds, Buchstaber–Terzić proved that the orbit space of the Grassmann manifold $G_{4,2}$ is $S^5$, and the orbit space of the flag manifold $F_3$ is $S^4$. These
two examples motivated our study. In Theorem [5.1] we prove that the orbit space of a quasitoric manifold by the action of $T$ is also homeomorphic to a sphere $S^{n+1}$.

A space of isospectral tridiagonal $n \times n$-matrices is a more interesting object. This space will be studied in details in the subsequent paper [2]. This space depends on the spectrum and for some degenerate spectra it is not smooth. However, if it is a smooth manifold, we will prove that its orbit space is $S^4 \times T^{n-3}$. In [2] we describe non-free part of the torus action using the regular permutohedral tiling of the space. This allows to understand the topology of the whole space, not just its orbit space.

The study of the space of periodic tridiagonal matrices raised several questions about actions of complexity one. One of the questions is the topological classification of such actions. In this paper we prove that under certain restrictions the space $X$ with complexity one action is determined by the orbit manifold $Q = X/T$, the set of non-free orbits $Z \subset Q$, and the weights of tangent representations at fixed points. See Theorem [5.5] and Proposition [5.7] The set of non-free orbits have a specific topology which we axiomatized in the notion of sponge. Sponges seem to be the objects of independent interest.

2. Appropriate actions of complexity one

In the following, $T$ usually denotes the compact torus of dimension $n-1$ and $G$ denotes compact tori of other dimensions. We refer to the classical monograph of Bredon [4] for general information of group actions on manifolds.

Let us specify the type of actions to be considered in the paper. For a smooth action of $G$ on a smooth manifold $X$ define the fine partition on $X$ by orbit types

$$X = \bigsqcup_{H \in S(G)} X^H.$$ 

Here $H$ runs over all closed subgroups of $G$ and $X^H = \tilde{\lambda}^{-1}(H) = \{x \in X \mid G_x = H\}$.

**Definition 2.1.** An effective action of $G$ on a compact smooth manifold $X$ is called *appropriate* if

- the fixed points set $X^G$ is finite;
- (adjoining condition) the closure of every connected component of a partition element $X^H$, $H \neq G$, contains a point $x'$ with $\dim G_{x'} > \dim H$.

If, moreover, the stabilizer subgroup of every point is a torus, we call the action *strictly appropriate*.

**Remark 2.2.** The adjoining condition implies that whenever a subset $X^H$ is closed in the topology of $X$, then it is the fixed point set $X^G$.

**Remark 2.3.** A subgroup $H$ of a torus has the form $H_t \times H_f$, where $H_t$ is a torus and $H_f$ is a finite abelian group. For strictly appropriate actions the finite components $H_f$ of all stabilizers vanish. In other words, a strictly appropriate action is an action with all stabilizers being connected.
Example 2.4. Let an algebraical torus \((\mathbb{C}^\times)^k\) act algebraically on a smooth variety \(X\) with finitely many fixed points. Then the induced action of a compact subtorus \(T^k \subset (\mathbb{C}^\times)^k\) on \(X\) is appropriate, as follows from Bialynicki-Birula method [3]. Indeed, for a given point \(x \in X \setminus X^T\) consider the 1-dimensional algebraical torus \(\mathbb{C}^\times \subset (\mathbb{C}^\times)^k\) which acts on \(x\) nontrivially. Consider the point \(x' = \lim_{t \to 0^+} tx\), where \(0 < t \leq 1\), \(t \in \mathbb{C}^\times\). The point \(x'\) is connected with \(x\) and has stabilizer of bigger dimension (since \(x\) is stabilized by \((\mathbb{C}^\times)^k\) as well as by \(\mathbb{C}^\times\)). Iterating this procedure, we arrive at some fixed point.

In particular, the action of a compact torus on a complex GKM-manifold (see [12]) is appropriate.

Example 2.5. The effective action of \(T^{n-1}\) on \(F_n\), the manifold of complete complex flags in \(\mathbb{C}^n\) is strictly appropriate. The effective action of \(T^{n-1}\) on a Grassmann manifold \(G_{n,k}\) of complex \(k\)-planes in \(\mathbb{C}^n\) is also strictly appropriate.

Example 2.6. Let the action of \(G \cong T^n\) on a smooth manifold \(X^{2n}\) be locally standard (see definition in Section 5). The orbit space \(P = X^{2n}/G\) is a manifold with corners. This action is appropriate whenever every face of \(P\) contains a vertex. If it is appropriate, then it is strictly appropriate. In particular, quasitoric manifolds provide examples of strictly appropriate torus actions.

Example 2.7. Let the action of \(G\) on \(X\) be appropriate, and the induced action of a subtorus \(T \subset G\) on \(X\) has the same fixed points set. Then the action of \(T\) is also appropriate. Indeed, the partition element \((X')^K\) of the \(T\)-action for \(K \subset T\) have the form

\[
(X')^K = \bigcup_{H \subset G : H \cap T = K} X^H.
\]

Therefore the adjoining condition for \(G\)-action implies the adjoining condition for the induced \(T\)-action.

Now we restrict to actions of complexity one, that is to the case \(\dim T = n - 1\), \(\dim X = 2n\). Let \(x \in X^T\) be a fixed point, and \(\alpha_1, \ldots, \alpha_n \in N = \text{Hom}(T, S^1) \cong \mathbb{Z}^{n-1}\) be the weights of the tangent representation at \(x\). This means,

\[
T_xX \cong V(\alpha_1) \oplus \cdots \oplus V(\alpha_n),
\]

where \(V(\alpha)\) is the standard 1-dimensional complex representation given by

\[
 tz = \alpha(t) \cdot z, \quad z \in \mathbb{C}.
\]

If there is no complex structure on \(X\), then we have an ambiguity in choice of signs of \(\alpha_i\). These signs do not affect the following definitions.

Definition 2.8. A representation of \(T^{n-1}\) on \(\mathbb{C}^n\) is called in general position if every \(n - 1\) of its \(n\) weights are linearly independent. An action of \(T = T^{n-1}\) on \(X = X^{2n}\) is called an action in general position if its tangent representation at any fixed point is in general position.
Remark 2.9. For a given $n$-tuple of weights $\alpha_1, \ldots, \alpha_n$ there is a relation $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ in $N \cong \mathbb{Z}^{n-1}$. The action is in general position if $c_i \neq 0$ for $i = 1, \ldots, n$.

Theorem 2.10. Consider an appropriate action of $T = T^{n-1}$ on $X = X^{2n}$ and assume it is in general position. Then the orbit space $Q = X/T$ is a topological manifold.

Proof. First we prove the local statement near fixed points.

Lemma 2.11. For a representation of $T = T^{n-1}$ on $\mathbb{C}^n$ in general position we have $\mathbb{C}^n/T \cong \mathbb{R}^{n+1}$.

Proof. Consider the standard action of $G = T^n$ on $\mathbb{C}^n$ which rotates the coordinates. The weights of the standard action $e_1, \ldots, e_n$ is the standard basis of the character lattice $\text{Hom}(G, S^1) \cong \mathbb{Z}^n$. Consider the lattice homomorphism $\phi : \mathbb{Z}^n \to N$ given by $\phi(e_i) = \alpha_i$, $i = 1, \ldots, n$. This homomorphism is induced by some homomorphism $\phi^* : T \to G$ of tori. The given action of $T$ is the composition of $\phi^*$ with the standard action.

So far we may assume that there is an action of a subtorus $T' = f(T) \subset G$ where $G$ acts in a standard way. The torus $T'$ is given by $\{t_1^{c_1} \cdots t_n^{c_n} = 1\}$, where $(c_1, \ldots, c_n)$ is a linear relation on the weights $\alpha_i$ and $\gcd\{c_i\} = 1$. The condition of general position implies that all $c_i \neq 0$. Hence the intersection of $T'$ with each coordinate circle in $G$ is a finite subgroup.

Let us denote the space $\mathbb{C}^n/T = \mathbb{C}^n/T'$ by $Q$. We have the map $g : Q \to \mathbb{C}^n/G \cong \mathbb{R}^n_{\geq 0}$, which sends $T$-orbit to its $G$-orbit. For every $p \in \mathbb{R}^n_{\geq 0}$ the preimage $g^{-1}(p)$ is a circle $G/T'$. For every $p \in \partial \mathbb{R}^n_{\geq 0}$, the preimage $g^{-1}(p)$ is a single point, since the product of $T'$ with any nontrivial coordinate subtorus generate the whole torus $G$. Therefore we have $Q = \mathbb{R}^n_{\geq 0} \times S^1/\sim$, where $\sim$ collapses circles over $\partial \mathbb{R}^n_{\geq 0}$. We have

$$(\mathbb{R}^n_{\geq 0} \times S^1/\sim) \cong (\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} \times S^1)/\sim \cong \mathbb{R}^{n-1} \times \mathbb{C},$$

which proves the lemma.

We now prove the theorem by induction on the dimension of stabilizer subgroup. If $\dim H = n - 1$, that is $H = T$, Lemma 2.11 shows that $X/T$ is a manifold near the fixed point set $X^T/T$. Now let $[x] \in X/T$ be an orbit such that $T_x = H$, that is $x \in X^H$. Due to the adjoining condition, there exists a point $x'$ such that the local representations at $x$ and $x'$ coincide and $x'$ is close to a partition element $X^{H'}$ with $\dim H' > \dim H$. Here by the local representation we mean a representation of $T_x$ on the normal space $T_x X/T_x T(x)$ to the orbit.

By induction, the space $X/T$ is a manifold near $X^{H'}/T$ therefore there exists a neighborhood of $[x']$ homeomorphic to $\mathbb{R}^{n+1}$. Therefore there exists a neighborhood of $[x]$ homeomorphic to $\mathbb{R}^{n+1}$. Indeed, both neighborhoods are homeomorphic to the orbit space of the local representation according to slice theorem.

Remark 2.12. Let $v_1, \ldots, v_{n-1} \in \mathbb{R}^{n-1}$ be the basis of a vector space and $v_n = -v_1 - \cdots - v_{n-1}$. Consider the subset $C$ of $\mathbb{R}^{n-1}$ given by

$$C = \bigcup_{I \subseteq [n], |I| = n-2} \text{Cone}(v_i \mid i \in I).$$
This subset is homeomorphic to the \((n-2)\)-skeleton of the standard nonnegative cone \(\mathbb{R}^n_{\geq 0}\).

The subset \(C\) is the \((n-2)\)-skeleton of the simplicial fan \(\Delta_{n-1}\) of type \(A_{n-1}\); it comes equipped with the natural filtration

\[ C_0 \subset \cdots \subset C_{n-2} = C \]

where \(C_k\) is the union of cones of dimension \(k\) of \(\Delta_{n-1}\). This filtration can be defined topologically: we say that \(x \in \mathbb{R}^{n-1}\) has type \(k\) if \(C\) cuts a small disc \(B_x\) around \(x\) into \(n-k\) chambers. Then \(C_k\) consists of all points of type \(\leq k\).

Next we introduce a notion of the subspace in a topological manifold which is locally modeled by the subset \(C \subset \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}\). Assume we are given a topological manifold \(Q\) of dimension \(n+1\) and a subset \(Z \subset Q\).

**Definition 2.13.** A subset \(Z \subset Q\) is called a *sponge* if, for any point \(x \in Z\), there is a neighborhood \(U_x \subset Q\) such that \((U_x, U_x \cap Z)\) is homeomorphic to \((V \times \mathbb{R}^2, (V \cap C) \times \mathbb{R}^2)\), where \(V\) is an open subset of the space \(\mathbb{R}^{n-1} \subset C\).

Every sponge is filtered in a natural way compatible with the filtration of \(C\). We say that a point \(x \in Z \subset Q\) has type \(k\) if \(H^2(U_x \setminus Z; \mathbb{Z}) \cong \mathbb{Z}^{n-k-1}\) for a small disc neighborhood \(x \in U_x \subset Q\). Then \(Z_k\) consists of all points of type at most \(k\). Note that \(\dim Z_k = k\). Informally speaking, the sponge set is a collection of \((n-2)\)-manifolds with corners, and the corners are stacked together like maximal cones in \(C\). The case \(n = 4\) is shown on Fig.1

![Figure 1. Local structure of the sponge for \(n = 4\).](image)

**Construction 2.14.** For a general action of a torus \(G\), \(\dim G = m\) on \(X\) we can consider the *coarse filtration*:

\[ X_0 \subset X_1 \subset \cdots \subset X_m = X \]

where \(X_i = \bigcup_{\dim H \geq m-i} X^H\) is the union of all orbits of dimension at most \(i\). In particular, the set \(X \setminus X_{m-1}\) is the locus of almost free action (“almost free” action means “have only finite stabilizers”). There is an induced coarse filtration on \(Q = X/G\):

\[ Q_0 \subset Q_1 \subset \cdots \subset Q_m \]
Remark 2.15. The terms “fine partition” and “coarse filtration” refer to the following fact. The fine partition distinguishes different subgroups of the torus. However, coarse filtration distinguishes only the dimensions of the subgroups.

Proposition 2.16. For an appropriate action in general position of $T^{n-1}$ on $X^{2n}$ we get a topological manifold $Q = X/T$. The coarse filtration on $Q$ has the form

$Z_0 \subset Z_1 \subset \cdots \subset Z_{n-2} = Z \subset Q$

where $Z \subset Q$ is a sponge. The filtration by orbit dimensions coincides with the sponge filtration defined topologically.

Proof. The local statement near fixed points is proved in Lemma 2.11. The global case follows by the adjoining condition similar to the proof of Theorem 2.10. □

3. Characteristic data

Assume there is an appropriate action of $T = T^{n-1}$ in general position on $X = X^{2n}$. We allow $X$ to have a boundary, however in this case we require that the action is free on $\partial X$. The same arguments as before show that $Q = X/T$ is a topological manifold with boundary and its boundary $\partial Q$ is $\partial X/T$.

In this section we assume that the actions are strictly appropriate. This means that there are no finite components in stabilizer subgroups and therefore, the face partition of a coarse filtration coincides with the fine partition on $Q$. With a strictly appropriate action in general position we assign the characteristic data $(Q, Z, \mu, e)$ consisting of the following information

- $Q = X/T$, the orbit space.
- $Z \subset Q^*$, the sponge subset determined by the action:

$Z_0 \subset Z_1 \subset \cdots \subset Z_{n-2} = Z$,

where $Z_i \subset Q$ is the set of orbits of dimension at most $i$. The closure of a connected component of $Z_i \setminus Z_{i-1}$ is called a face of $Z$ of dimension $i$. Faces of dimension $n-2$ are called facets. Every face of dimension $i$ is contained in exactly $\binom{n-1}{i}$ different facets. The stabilizer remains the same for all points in the interior of any given face $F$, since no finite components are allowed in the stabilizers. This stabilizer will be denoted $T_F$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be the set of facets of $Z$.
- $\mu$ is a characteristic map

$\mu: \mathcal{F} \to \{1\text{-dimensional subgroups of } T^{n-1}\} = \text{Hom}(S^1, T^{n-1}) \cong \mathbb{Z}^{n-1}$

It sends a facet $F_k$ to the oriented stabilizer subgroup $T_{F_k}$ of any of its interior points (we introduce orientation arbitrarily, see details in Section 4). For any face $F$ of dimension $i$ we have $F = \bigcap_{i \in I} F_i$ for certain subset $I \subset [m], |I| = \binom{n-1}{2}$. The stabilizer $T_F$ is the product of $\mu(F_i) = T_{F_i}$ inside the torus $T^{n-1}$. Note that this product is generally not free, since it has dimension $n-1-i$. However, it can be seen that characteristic map $\mu$ determines the stabilizers of all points.
• $e \in H^2(Q \setminus Z; H^2(BT))$ is the Euler class of the free part of the action. In details, every orbit in $Q \setminus Z$ is full-dimensional and there are no finite stabilizer subgroups, therefore the free part of the action is a principal $T$-bundle $p: X^{\text{free}} \to Q \setminus Z$. This bundle is classified by the homotopy class of a map

$$Q \setminus Z \to BT \cong (\mathbb{C}P^\infty)^{n-1} \cong K(\mathbb{Z}^{n-1}; 2).$$

Such maps also classify the second cohomology group of $Q \setminus Z$. Therefore, we have the classifying element

$$e \in H^2(Q \setminus Z; \mathbb{Z}^{n-1}), \quad \text{where } \mathbb{Z}^{n-1} \cong H_2(BT; \mathbb{Z}) \cong H_1(T; \mathbb{Z}).$$

**Remark 3.1.** Note that unlike the half-dimensional torus actions the characteristic data of complexity one actions can not be arbitrary. It will be shown in this and the next section that the Euler class $e$ and the characteristic function $\mu$ determine each other to much extent. Moreover, the Euler class of complexity one actions is always nontrivial.

Let $x \in Z \subset Q$ be a point of type $k \leq n - 2$. Let $U_x$ be a small neighborhood of $x$ in $Q$, homeomorphic to an open disc. Let $i_x: U_x \to Q$ be the inclusion map. We have an induced homomorphism

$$H^2(Q \setminus Z; H_1(T)) \to H^2(U_x \setminus Z; H_1(T))$$

The image of $e \in H^2(Q \setminus Z; H_1(T))$ by this homomorphism is denoted

$$e_x \in H^2(U_x \setminus Z; H_1(T)) \cong \mathbb{Z}^{n-k-1} \otimes H_1(T)$$

and called the local Euler class at $x$. Recall that the type of the point is defined by the rank of the second cohomology of $U_x \setminus Z$, see section 2.

In particular, if $x$ has type $n - 2$ (i.e. $x$ lies in the interior of a facet), the neighborhood $U$ can be chosen in a way that $U_x \cap Z \cong \mathbb{R}^{n-2}$. In this case we have $U_x \setminus Z \cong \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-2}$ and

$$H^2(U_x \setminus Z; H_1(T)) \cong H^2(\mathbb{R}^{n+1} \setminus \mathbb{R}^{n-2}; H_1(T)) \cong H^1(T).$$

The last isomorphism is canonical provided $Q$ (hence $U_x$) is oriented.

**Definition 3.2.** The Euler class $e$ and characteristic function $\mu$ are called compatible if the following condition holds: for any $x \in Z$, the map $H_1(T) \to H_1(T/T_x)$ induced by the quotient map $T \to T/T_x$ sends $e_x \in H^1(T)$ to zero.

**Proposition 3.3.** Assume there is an appropriate action in general position of $T = T^{n-1}$ on a manifold $X = X^{2n}$. Then its characteristic data $e$ and $\mu$ are compatible.

**Proof.** As before, let $Q = X/T$ be the orbit space, $Z \subset Q$ the set of orbits of dimensions $\leq n - 2$, and $p: X \to Q$ the projection map. Take any point $x \in Z \subset Q$. We can choose a small neighborhood $U_x \ni x$, $U_x \subset Q$ such that stabilizers of any point $y \in U_x$ are contained in $T_x$ and $U_x \cong \mathbb{R}^{n+1}$. Consider the map

$$f: p^{-1}(U_x)/T_x \stackrel{T/T_x}{\longrightarrow} U_x$$

taking the remaining quotient. Since all stabilizers of points in $U_x$ are contained in $T_x$, the map $f$ is a principal $T/T_x$-bundle. It is a trivial bundle since $U_x$ is contractible,
therefore the induced $T/T_x$-bundle over $U_x \setminus Z$ is also trivial. Hence its Euler class vanishes. However, this Euler class is the image of $e_x \in H^2(U_x \setminus Z; H_1(T))$ under the induced map $H_1(T) \to H_1(T/T_x)$, which proves the statement. 

\textbf{Remark 3.4.} For a point $x$ in the interior of a facet $F_i$ the stabilizer $T_x$ is one-dimensional. In this case the compatibility condition tells that $e_x$ is proportional to the fundamental class of $T_x = \mu(F_i)$:

$$e_x = k_i \mu(F_i) \in H_1(T; \mathbb{Z}) \cong \text{Hom}(S^1, T).$$

The constants $k_i \in \mathbb{Z}$ can be determined from the weights of the tangent representation at any fixed point adjacent to $F_i$. It will be shown in Section 4 that all these constants are actually $\pm 1$ for strictly appropriate actions.

\textbf{Construction 3.5.} Let us construct a topological model space given abstract compatible characteristic data. Assume a topological $(n+1)$-manifold $Q$ is given, and let $Z \subset Q$ be a sponge with facets $F_1, \ldots, F_m$. Let $\mu$ be a map assigning a 1-dimensional subgroup of $T = T^{n-1}$ to any facet $F_i$ with the following property: if a $k$-dimensional face $F$ of a sponge lies in facets $F_i$ with labels $i \in I$, $|I| = \binom{n-k}{2}$, then

$$\dim \prod_{i \in I} \mu(F_i) = k.$$

The subgroup $\prod_{i \in I} \mu(F_i)$ will be denoted $T_x$ if $x$ lies in interior of $F$. If $x \in Q \setminus Z$ we set $T_x = \{1\} \subset T$. Finally, fix a class $e \in H^2(Q \setminus Z; H_1(T))$ compatible with $\mu$. With all this information fixed, introduce a space $Y = Y(Q, Z, \mu, e)$. As a set,

$$Y = \bigsqcup_{x \in Q} T/T_x.$$

The topology is introduced in two steps.

1. The topology on a subset

$$Y^{\text{free}} = \bigsqcup_{x \in Q \setminus Z} T/T_x = \bigsqcup_{x \in Q \setminus Z} T \subset Y$$

is introduced in a way such that the natural projection $Y^{\text{free}} \to Q \setminus Z$ is the principal $T$-bundle classified by $e \in H^2(Q \setminus Z; H_1(T))$.

2. For a point $y$ in $\bigsqcup_{x \in Z} T/T_x$ we specify the basis of topology. Let $x \in Z$ and $t_x \in T/T_x \subset Y$. To define the base of topology near $t_x$, we fix a small open neighborhood $U_x \subset Q$ of $x$ and for each $x' \in U_x$ take a projection of tori $p_{x'}: T/T_x \to T/T_{x'}$. This is well defined since $U_x$ is assumed small enough so that $T_x$ contains any other stabilizer $T_{x'}$. Let $V$ be a neighborhood of $t_x$ in $T/T_x$. The subsets of the form

$$\bigsqcup_{x' \in U_x} p_{x'}^{-1}(V)$$

form the base of topology around $t_x$. Note that since $e$ and $\mu$ are compatible, we have a trivial principal $T/T_x$-bundle over $A \to U_x \setminus Z$ therefore the topology defined in (2) is compatible with the one defined in (1) on a subset $U_x \setminus Z$. 

Finally, define the $T$-action on each fiber $T/T_x$ as given by the projection $T \to T_x$. It can be seen that $Y$ is a compact Hausdorff topological space carrying the continuous action of $T$. Its orbit space is homeomorphic to $Q$.

The constructed space $Y = Y(Q,Z,\mu,e)$ is not necessarily a manifold.

**Example 3.6.** Assume $e_x = 0$ for some point $x$ lying in interior of a facet $F_j$. Then $Y$ is not a manifold over $x$. Indeed, by construction, a neighborhood of $x$ in $Y$ is homeomorphic to $U_x \times T/\sim$, where $(x',t') \sim (x'',t'')$ whenever $x' = x'' \in Z$ and $t'(t'')^{-1} \in \mu(F_j)$. This subset is not a manifold, which can be shown by computing its local homology groups for points lying over $Z$.

**Proposition 3.7.** Let $X = X^{2n}$ be a manifold with strictly appropriate action of $T = T^{n-1}$ in general position. Let $(Q,Z,\mu,e)$ be its characteristic data. Let $Y$ be the model space constructed from the data $(Q,Z,\mu,e)$. Then there is a $T$-equivariant homeomorphism $h: Y \to X$ which induces the identity homeomorphism on the orbit space $Q$:

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow p_Y & & \downarrow p_X \\
Q & \longrightarrow & Q
\end{array}
\]

**Proof.** The equivariant homeomorphism over $Q\setminus Z$ follows immediately, since both $p_X^{-1}(Q\setminus Z)$ and $p_Y^{-1}(Q\setminus Z)$ are the principal $T$-bundles classified by $e$. For a point $x \in Z \subset Q$, the equivariant homeomorphism $h: p_Y^{-1}(U_x\setminus Z) \to p_X^{-1}(U_x\setminus Z)$ is extended uniquely to the equivariant homeomorphism $h: p_Y^{-1}(U_x) \to p_X^{-1}(U_x)$. Indeed, there is a unique equivariant homeomorphism $\hat{h}: p_Y^{-1}(U_x)/T_x \to p_X^{-1}(U_x)/T_x$ which extends the homeomorphism $h/T_x: p_Y^{-1}(U_x\setminus Z)/T_x \to p_X^{-1}(U_x\setminus Z)/T_x$, since both spaces are trivial $(T/T_x)$-bundles over $U_x$ (according to compatibility condition) and $U_x\setminus Z$ is dense in $U_x$. For a point $t_x \in T/T_x \subset p_Y^{-1}(U_x)$ over $x$ there is a unique point $\alpha \in p_X^{-1}(U_x)$ such that $\hat{h}([t_x]) = [\alpha]$, since the projection map $p_X^{-1}(U_x) \to p_X^{-1}(U_x)/T_x$ is a bijection over $x$. Hence we can extend $h$ by putting $h(t_x) = \alpha$.

This procedure defines an equivariant continuous bijection between compact spaces $Y$ and $X$. Since $X$ is compact and $Y$ is Hausdorff it is an equivariant homeomorphism. \qed

4. Orientation issues and details

Consider a representation of $T = T^{n-1}$ on $\mathbb{C}^n$ in general position. The weights $\alpha_1, \ldots, \alpha_n \in \text{Hom}(T,S^1)$ are defined up to sign.

**Definition 4.1.** An omniorientation is a choice of the orientation of $T$ (hence the orientation of the lattice $N = \text{Hom}(T,S^1)$) and the choice of signs of all vectors $\alpha_i$.

**Construction 4.2.** Assume there is a fixed basis in the lattice $N$, so that $\alpha_j$ is written in coordinates $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n-1})$. For each $i = 1, \ldots, n$ consider the determinant of the matrix formed by $\alpha_j$ with $j \neq i$:

\[
\bar{c}_i = (-1)^{i} \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \wedge \cdots \wedge \alpha_n \in \Lambda^{n-1}N \cong \mathbb{Z}
\]
Since \( \alpha_i \) are in general position we have \( \bar{c}_i \neq 0 \) for all \( i = 1, \ldots, n \). Cramer’s rule implies

\[
\bar{c}_1 \alpha_1 + \cdots + \bar{c}_n \alpha_n = 0.
\]

Let \( c_{\gcd} = \gcd(\bar{c}_1, \ldots, \bar{c}_n) \) and \( c_i = \bar{c}_i / c_{\gcd} \). Let \( G = T^n \) act on \( \mathbb{C}^n \) in a standard way

\[
(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)
\]

and let \( T \) be a subtorus

\[
T' = \{ t_1^{c_1} \cdots t_n^{c_n} = 1 \} \subset G
\]

The proof of Lemma 2.11 implies that the orbit space of the representation of \( T \) on \( \mathbb{C}^n \) coincides with the orbit space of the induced action of \( T' \) on \( \mathbb{C}^n \), therefore we may not distinguish these two cases.

**Lemma 4.3.** The representation action of \( T = T' \) on \( \mathbb{C}^n \) in general position is strictly appropriate if and only if \( c_i = \pm 1 \), that is all parameters \( \bar{c}_i \) coincide up to sign.

**Proof.** The point \((0, \ldots, 0, 1, 0, \ldots, 0) \) with unit at \( j \)-th position has the stabilizer \( T' \cap G_j \), where \( T' \) is given by \( c_i \) and \( G_j \) is the \( j \)-th coordinate circle of \( G \cong T^n \). This stabilizer is isomorphic to the cyclic group \( \mathbb{Z}_{c_i} \). If the action is strictly appropriate, then there are no finite components in stabilizer subgroups, so far \( c_i \) is necessarily \( \pm 1 \).

The converse statement is similar. The stabilizers of \( T' \)-action on \( \mathbb{C}^n \) have the form \( T' \cap G_I \) for all possible coordinate subtori \( G_I \in G, I \subseteq [n] \). This group has finite component of order \( \gcd(c_i | i \in I) \). Hence, if all \( c_i \) are \( \pm 1 \), these finite components vanish. \( \square \)

Recall that \( C \subset \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1} \) denotes the \((n - 2)\)-skeleton of the fan of type \( A_{n-1} \). This space is the sponge of an appropriate representation action of \( T \) on \( \mathbb{C}^n \).

In the following we only consider strictly appropriate actions. The facets \( \{ F_{i,j} \mid 1 \leq i < j \leq n \} \) of \( Z \) are labeled in a way that \( F_{i,j} \) is “spanned” by all weights except \( \alpha_i \) and \( \alpha_j \). Let us fix an orientation on 1-dimensional stabilizers of the action (this corresponds to the choice of the signs of the characteristic values \( \mu(F_{i,j}) \in \text{Hom}(S^1, T)) \). These orientations determine the orientation of the orbit \( Tx \cong T/\mu(F_{i,j}) \) for \( x \in F_{i,j}^* \). The preimage of \( F_{i,j}^* \) under the projection map has the form \( \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j = 0, z_k \neq 0 \text{ for } k \neq i, j \} \), this space has a canonical orientation determined by the complex structure on \( \mathbb{C}^n \). Therefore the orientations of the stabilizer circles determine the orientations of facets \( F_{i,j} \).

Finally, since the orientation on \( \mathbb{C}^n/T \cong \mathbb{R}^{n+1} \) is fixed, the orientation of \( F_{i,j} \) determines the orientation of a small 2-sphere \( S^2_{i,j} \) around \( F_{i,j} \). Let us describe the Euler class of the free part of action.

**Proposition 4.4.** The Euler class \( e \in H^2(Q \setminus Z; H_1(T)) \) of a strictly appropriate representation action of \( T \) on \( \mathbb{C}^n \) is given by the condition

\[
\langle e, [S^2_{i,j}] \rangle = \frac{c_i}{c_j} \mu(F_{i,j}) \in H_1(T) \cong \text{Hom}(S^1, T),
\]

for a small 2-sphere around facet \( F_{i,j} \), \( 1 \leq i < j \leq n \).
The constants $c_i$ were defined earlier in this section. Lemma 4.3 shows that for strictly appropriate actions $c_i = \pm 1$. Note that $\frac{c_i}{c_j} = \frac{\tilde{c}_i}{\tilde{c}_j}$, and parameters $\tilde{c}_i, \tilde{c}_j$ can be computed from the weight vectors.

**Proof.** Assume $i = 1$, $j = 2$ for simplicity. The preimage of a sphere $S^2_{1,2}$ in the space $\mathbb{C}^n$ has the form

$$M = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + |z_2|^2 = \varepsilon, |z_k| = \varepsilon, k > 2\}$$

for small $\varepsilon > 0$.

The subtorus $T = \{t_1^{c_1} \cdots t_n^{c_n} = 1\} \subset G$ acts freely on $M$. The stabilizer $T_x = \mu(F_{1,2})$ for $x \in F^\circ_{1,2}$ has the form

$$T_x = \{t_1^{c_1} t_2^{c_2} = 1, t_k = 1, k > 2\}.$$  

The induced action of $T/T_x$ on $M/T_x$ is a trivial principal bundle, therefore Euler class of $T$-action on $M$ coincides with the image of the Euler class of $T_x$-action on $M$ under the inclusion map $i_x: T_x \rightarrow T$. The $T_x$-action on $M$ is the Hopf bundle if $c_1, c_2$ have the same sign, and “anti-Hopf” bundle if $c_1, c_2$ have different signs. Its Euler class is $\mu(F_{1,2}) \in H^2(S^2_{1,2}; H_1(T))$ in the first case and $-\mu(F_{1,2})$ in the second case.

$$\square$$

**Figure 2.** Orienting three facets with a common face

**Remark 4.5.** Note that there exist relations on the cycles $[S_{i,j}] \in H_2(Q \backslash C; \mathbb{Z}) \cong \mathbb{Z}^{n-1}$. Every triple of indices $i, j, k$ determines the $(n - 3)$-face $F_{i,j,k} \subset C$ which lies in facets $F_{i,j}, F_{j,k}, F_{i,k}$. If we choose a small circle around $F_{i,j,k} \subset \mathbb{R}^{n-1}$ and orient the facets $F_{i,j}, F_{j,k}, F_{i,k}$ consistently (see schematic Fig.2), we get a relation in $H_2(Q \backslash C; \mathbb{Z})$:

$$[S_{i,j}] + [S_{j,k}] + [S_{i,k}] = 0.$$  

It implies the cocycle relation for stabilizers:

$$\frac{c_i}{c_j} \mu(F_{i,j}) + \frac{c_j}{c_k} \mu(F_{j,k}) + \frac{c_i}{c_k} \mu(F_{i,k}) = 0.$$  

This relation is not surprising. Indeed, the product of the circle subgroups $\mu(F_{i,j}), \mu(F_{j,k}), \mu(F_{i,k})$ inside the torus $T$ have dimension 2, therefore there should be exactly one linear relation on their fundamental classes.

Proposition 4.4 implies that for strictly appropriate torus actions we have $e_x = \pm [\mu(F)]$ for $x \in F^\circ$, since this holds in the local chart around fixed point.
5. Reductions of locally standard actions

A smooth manifold $X = X^{2n}$ with the action of $G = T^n$ is called \textit{locally standard} if the action is locally modeled by the standard representation of $G = T^n$ on $\mathbb{C}^n$. Since $\mathbb{C}^n/T^n \cong \mathbb{R}^n_{\geq 0}$, the orbit space $P = X/G$ gets a natural structure of a manifold with corners. Manifolds with locally standard actions are classified up to equivariant homeomorphism (see [20]) by the following characteristic data

1. The manifold with corners $P$, dim $P = n$. There is a requirement that every $k$-dimensional face of $Q$ is contained in precisely $n-k$ facets. Such manifolds with corners are called \textit{nice} in [15] or just \textit{manifolds with faces}.

2. The characteristic function $\lambda$ which maps a facet $F$ of $P$ into a circle subgroup of $G$: the stabilizer of any interior point of $F$. Characteristic function satisfies the celebrated (*)-condition: whenever facets $F_1, \ldots, F_k$ intersect in $P$, the subgroups $\lambda(F_1), \ldots, \lambda(F_k)$ form a direct product inside $G$. Since every circle subgroup of $G$ determines a primitive integral vector in $\text{Hom}(S^1, G) \cong \mathbb{Z}^n$ up to sign, it will be convenient to assume that $\lambda$ takes values in the lattice.

3. The Euler class $e \in H^2(Q; H_1(G)) \cong H^2(P^c; H_1(G))$, which classifies the principal $G$-bundle $X^\text{free} \to X^\text{free}/G = P^c$, where $X^\text{free}$ is the free part of the $G$-action.

In the following we assume that every face of $P$ contains a vertex so that the action is \textit{appropriate}.

A manifold $X$ with a locally standard action of $G$ is called a \textit{quasitoric manifold} if $P = X/G$ is isomorphic to a simple polytope as a manifold with corners. The free part of action is a trivial $G$-bundle, since $P$ is contractible. So far, the Euler class vanishes for quasitoric manifolds.

A fixed point $v$ of a locally standard action of $G$ on $X$ corresponds to a vertex $v$ of $P$ (we denote them by the same letter). We have $v = F_1 \cap \cdots \cap F_n$ for some facets $F_i \subset Q$. Then the weights $\alpha_1, \ldots, \alpha_n \in \text{Hom}(G, S^1) = N$ of the tangent representation at $v$ is the dual basis to $\lambda(F_1), \ldots, \lambda(F_n) \in \text{Hom}(S^1, G) = N^\ast$.

Let $\{\alpha_{v,i}\}$ be a collection of all weights at all fixed points. We can choose a generic homomorphism of the lattices

$$\phi: \text{Hom}(G, S^1) \cong \mathbb{Z}^n \to \mathbb{Z}^{n-1}$$

such that the images $\phi(\alpha_{v,1}), \ldots, \phi(\alpha_{v,n}) \in \mathbb{Z}^{n-1}$ are in general position for any fixed point $v$. The homomorphism $\phi$ is determined by some homomorphism of tori $\phi^\ast: T^{n-1} \to G$. Therefore the action of the subtorus $T = \phi^\ast(T^{n-1}) \subset G$ on $X$ is in general position.

**Theorem 5.1.** Let $X = X^{2n}$ be a quasitoric manifold with the action of $G \cong T^n$. Let $T \subset G$ be a subtorus of dimension $n-1$ such that the induced action of $T$ on $X$ is an action in general position. Then $X/T \cong S^{n+1}$.

**Proof.** Denote the orbit space $X/T$ by $Q$ and the orbit space $X/G$ by $P$. By the definition of a quasitoric manifold, $P$ is a simple polytope, dim $P = n$. We have a map $g: Q \to P$, which sends a $T$-orbit to the $G$-orbit in which it lies. For any point $x$ in the interior of $P$ we have $g^{-1}(x) \cong S^1$. Since the action is in general position, the preimage of
a point $x \in \partial P$ is a single point (this fact was actually proved in Lemma 2.11 for a local chart). Since $P$ is contractible, the map $g: Q \to P$ admits a section over $P$. Therefore we have

$$Q \cong P \times S^1 / \sim$$

where $\sim$ collapses circles over $\partial P$. Since $P$ is homeomorphic to the $n$-disc $D^n$, we have

$$Q \cong D^n \times S^1 / \sim \cong \partial(D^n \times D^2) \cong S^{n+1},$$

which proves the statement.

We further investigate the characteristic data of the induced action of $T \cong T^{n-1}$ on a quasitoric manifold. The arguments in the proof of Theorem 5.1 imply the following statement.

**Proposition 5.2.** The sponge of the $T$-action on a quasitoric manifold $X$ has the form

$$Z \subset S^{n-1} \subset \Sigma^2 S^{n-1} \cong Q,$$

where $S^{n-1}$ is identified with the boundary of the polytope $P$ and $Z$ is its $(n-2)$-skeleton. The facets of $Z$ are exactly the faces of $P$ of codimension 2.

Note that characteristic function $\lambda$ of the $G$-action determines the characteristic function $\mu$ of the $T$-action. Let $F$ be a codimension-2 face of $P$ (hence a facet of $Z$). Then $F = F_1 \cap F_2$, where $F_1, F_2$ are the facets of $P$. We have

$$\mu(F) = \lambda(F_1) \times \lambda(F_2) \cap T$$

Here $\lambda(F_1) \times \lambda(F_2)$ is a 2-torus in $G$, and since $T$ is a codimension-1 subtorus of $G$ in general position, the intersection $\lambda(F_1) \times \lambda(F_2) \cap T$ is a 1-dimensional subgroup, which is the stabilizer of the $T$-action on interior of $F$. If we want this subgroup to be a circle (recall that the definition of strictly appropriate action requires that stabilizers don’t have finite components), then the subgroup $T \subset G$ is subject to some additional restrictions. Namely, the subgroup $T \subset G$ determines the character $\alpha_T \in \text{Hom}(G, S^1)$, $\alpha_T: G \to G/T$. The next lemma easily follows from Lemma 4.3.

**Lemma 5.3.** The induced action of $T$ on a locally standard $G$-manifold $X$ is strictly appropriate if and only if $\langle \alpha, \lambda(F_i) \rangle = \pm 1$ for all facets $F_i$.

**Example 5.4.** Let $c: \{F_1, \ldots, F_m\} \to [n]$ be a proper coloring of facets of a simple polytope $P$. This means, whenever $F_i$ and $F_j$ are adjacent, their colors $c(F_i), c(F_j)$ are different. Given such coloring we can construct a special characteristic function $\lambda_c: \{F_1, \ldots, F_m\} \to \mathbb{Z}^n$ which associates to $F_i$ the basis vector $\lambda(F_i) = e_{c(F_i)} \in \mathbb{Z}^n$. Such characteristic functions and corresponding quasitoric manifolds $X_{P,\lambda_c}$ were called pullbacks from linear model in [11]. It can be seen that the induced action of the subtorus

$$T = \{t_1^{c_1} t_2^{c_2} \cdots t_n^{c_n} = 1\} \subset G$$

on $X_{P,\lambda_c}$ is strictly appropriate.

Note that there exist other examples of strictly appropriate induced actions which do not come from colored characteristic functions.
The Euler class \( e \) of the induced action of \( T \) on a quasitoric manifold \( X \) determines the action.

**Theorem 5.5.** Let \( X' \) and \( X'' \) be two manifolds with strictly appropriate actions in general position. Let \((Q', S^{n+1}, Z', \mu', e')\) and \((Q'', S^{n+1}, \mu'', e'')\) be their characteristic data. Suppose there is a homeomorphism of pairs \((Q', Z') \cong (Q'', Z'')\) and \( e'_x = e''_x \) for any point \( x \) in a sponge. Then \( X' \) and \( X'' \) are equivariantly homeomorphic.

**Proof.** Taking \( x \) in the interior of a facet \( F \) of a sponge \( Z' \cong Z'' \), we see that \( \mu'(F) = \mu''(F) \) since \( e'_x \) is the fundamental class of \( \mu'(F) \) and \( e''_x \) is the fundamental class of \( \mu''(F) \). Hence \( \mu' = \mu'' \).

Let \((Q, Z)\) be either \((Q', Z')\) or \((Q'', Z'')\) and let \( U = \bigcup_{x \in Z} U_x \) be a neighborhood of \( Z \) in \( Q \). As before, \( U_x \) denotes a small neighborhood of \( x \in Z \) homeomorphic to an open ball. The local classes \( e_x \) determine the classes \( e'_x \in H^3(U_x, U_x \setminus Z; \mathbb{Z}^{n-1}) \) due to the exact sequence

\[
\begin{array}{c}
H^2(U_x; \mathbb{Z}^{n-1}) \\ \downarrow \\
0 \\
\end{array} \longrightarrow \begin{array}{c}
H^2(U_x \setminus Z; \mathbb{Z}^{n-1}) \\ \downarrow e_x \\
e'_x \\
\end{array} \longrightarrow \begin{array}{c}
H^3(U_x, U_x \setminus Z; \mathbb{Z}^{n-1}) \\ \downarrow \\
H^3(U_x; \mathbb{Z}^{n-1}) \\
\end{array}
\]

The classes \( \{e'_x \mid x \in Z\} \) determine a unique element \( e' \in H^3(U, U \setminus Z; \mathbb{Z}^{n-1}) \) such that \( i^*_x(e') = e'_x \) for an inclusion \( i_x : U_x \hookrightarrow U \) according to Mayer–Vietoris argument. By excision, we can view \( e' \) as an element in \( H^3(Q, Q \setminus Z; \mathbb{Z}^{n-1}) \cong H^3(U, U \setminus Z; \mathbb{Z}^{n-1}) \). Recall that \( Q \cong S^{n+1} \). From the exact sequence

\[
\begin{array}{c}
H^2(Q; \mathbb{Z}^{n-1}) \\ \downarrow \\
0 \\
\end{array} \longrightarrow \begin{array}{c}
H^2(Q \setminus Z; \mathbb{Z}^{n-1}) \\ \downarrow e \\
e' \\
\end{array} \longrightarrow \begin{array}{c}
H^3(Q, Q \setminus Z; \mathbb{Z}^{n-1}) \\ \downarrow \\
H^3(Q; \mathbb{Z}^{n-1}) \\
\end{array}
\]

we extract a unique element \( e \in H^2(Q, Z; \mathbb{Z}^{n-1}) \) which projects to \( e_x \) for any point \( x \in Z \).

Characteristic data \((Q' \cong S^{n+1}, Z', \mu', e')\) and \((Q'' \cong S^{n+1}, Z'', \mu'', e'')\) coincide, hence the spaces \( X' \) and \( X'' \) are equivariantly homeomorphic to the model space according to Proposition 3.7. Thus they are homeomorphic to each other. \( \Box \)

**Remark 5.6.** Instead of the equality \( e'_x = e''_x \) one can require the equality of characteristic functions \( \mu' = \mu'' \), and for a small 2-sphere around each facet \( F \) specify the type of its preimage (whether it is Hopf or anti-Hopf bundle, see Proposition 4.4). If the types agree for \( X \) and \( X' \) then equality \( \mu' = \mu'' \) would imply equality of local classes \( e'_x = e''_x \).

In order to study certain examples, we need a modification of Theorem 5.5. Let \( M \) be a closed manifold of dimension \( n - 1 \). Assume there is a regular simple cell subdivision on \( M \) which means there is a given regular cell structure in which every \( k \)-dimensional cell is contained in exactly \( n - k \) maximal cells. Its \((n - 2)\)-skeleton \( Z_M = M_{n-2} \) is a sponge. Consider the manifold with boundary \( Q_M = M \times D^2 \). We consider \( M \) as a subset \( M \times 0 \subset Q_M \).
PROPOSITION 5.7. Let \((X, \partial X)\) be a 2n-dimensional manifold with boundary, and assume there is an appropriate action of \(T = T^{n-1}\) on \(X\) with characteristic data \((Q_M, Z_M, \mu_M, e_M)\). We also assume that the action is free on the boundary \(\partial X\) and the principal \(T\)-bundle \(\partial X \to \partial X/T = \partial Q_M \cong M \times \partial D^2\) is trivial. Then the class \(e_M \in H^2(Q_M \setminus Z_M; \mathbb{Z}^{n-1})\) is uniquely determined by the local classes \(e_x, x \in Z_M\).

PROOF. There is an exact sequence of the pair \((Q_M \setminus Z_M, \partial Q_M)\):
\[
H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1}) \longrightarrow H^2(Q_M \setminus Z_M; \mathbb{Z}^{n-1}) \longrightarrow H^2(\partial Q_M; \mathbb{Z}^{n-1})
\]
The class \(e \in H^2(Q_M \setminus Z_M; \mathbb{Z}^{n-1})\) maps to zero since the free part of action is a trivial \(T\)-bundle over \(\partial Q\). Hence there exists \(\tilde{e} \in H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1})\) which maps to \(e\), and \(\tilde{e}\) is uniquely determined by the class \(\tilde{e}\). We have
\[(Q_M \setminus Z_M)/\partial Q_M \cong \Sigma^2(M \setminus Z_M)\]
hence \(H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1}) \cong \tilde{H}^0(M \setminus Z_M)\). The space \(M \setminus Z_M\) is the disjoint union of open top-dimensional cells of \(M\). It can be seen that cohomology classes of \(H^2(Q_M \setminus Z_M, \partial Q_M; \mathbb{Z}^{n-1})\) are localized near \(Z_M\) thus are completely determined by the local classes. \(\square\)

COROLLARY 5.8. Under the assumptions of Proposition 5.7, the equivariant homeomorphism type of \(X\) is determined by \((Q_M, Z_M)\) and the weights of all tangent representations at all fixed points.

CONSTRUCTION 5.9. The examples of the actions above can be constructed in the following way. We consider a manifold \(P \cong M \times [0, 1]\) with boundary \(\partial P = \partial_0 P \sqcup \partial_1 P\), \(\partial_0 P = M \times \{0\}\), and endow it with the structure of a nice manifold with corners. Namely, we subdivide the boundary component \(\partial_0 P\) according to the subdivision of \(M\) and do nothing with \(\partial_1 P\) (this boundary component is considered a single face of dimension \(n-1\)). Now we may take an abstract characteristic function satisfying \((*)\)-condition:
\[
\lambda: \{\text{facets of } \partial_0 P\} \to \text{Hom}(S^1, G) \cong \mathbb{Z}^n,
\]
and construct a topological manifold
\[
X = (P \times G)/ \sim
\]
with boundary \(\partial X = \partial_1 P \times G\). Here \(G \cong T^n\) and \(\sim\) collapses tori over \(\partial_0 P\) according to characteristic function (refer to [11, 20, 9] for details). These particular manifolds with boundary were studied in [11].

We take a generic \((n-1)\)-dimensional subtorus \(T \subset G\) so that the induced action of \(T\) on \(X\) is strictly appropriate and in general position. It can be seen that the orbit space \(Q = X/T\) is homeomorphic to
\[
Q \cong P \times S^1/\sim = (M \times [0, 1]) \times S^1/\sim
\]
where the circles collapse over \(\partial_0 P = M \times \{0\}\). Therefore \(Q \cong M \times D^2\). The sponge of the \(T\)-action is the \((n-2)\)-skeleton of \(M = M \times \{0\} \subset M \times D^2\). Finally, the free \(T\)-action over \(M \times \partial D^2\) is a trivial bundle, since the \(G\)-action is trivial over \(\partial_1 P\).
6. Grassmann and flag manifolds

Next we review two classical examples motivating our study.

**Example 6.1.** The standard action of a compact torus $T^4$ on $\mathbb{C}^4$ induces the action of $T^4$ on a Grassmann manifold $G_{4,2}$ of complex 2-planes in $\mathbb{C}^4$. This action has non-effective kernel $\Delta(T^1) \subset T^4$, hence we have an effective action of $T = T^4/\Delta(T^1) \cong T^3$ on $G_{4,2}$, $\dim \mathbb{R} G_{4,2} = 8$. There are 6 fixed points, and it is not difficult to find the weights of their tangent representations. The easiest way to do this is to look at the image of the moment map, which coincides with a regular octahedron $\Delta_{4,2}$. Its vertices correspond to the fixed points, and the primitive lattice vectors along the edges of octahedron correspond to the weights of the tangent representation. For example, the edges from the top vertex $(0, 0, 1)$ of octahedron are

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (0, 1, -1), \quad \alpha_3 = (-1, 0, -1), \quad \alpha_4 = (0, -1, -1).$$

Every 3 of them are linearly independent, hence the action is in general position. The action is strictly appropriate.

It was proved in [7], that the orbit space $G_{4,2}/T$ is homeomorphic to $S^5$. The sponge $Z$ of the action is obtained by taking the boundary of octahedron $\partial \Delta_{4,2}$, and attaching three squares along the equatorial cycles as shown on Fig.3.

**Figure 3.** The sponge of $G_{4,2}$ consists of the boundary of octahedron with 3 squares attached along the equators

**Example 6.2.** The standard action of $T^3$ on $\mathbb{C}^3$ induces the effective action of $T = T^3/\Delta(T^1)$ on the manifold $F_3$ complete complex flags in $\mathbb{C}^3$. We have $\dim T = 2$, $\dim F_3 = 6$. There are 6 fixed points and the tangent representation at each point is in general position. The action has no finite components in stabilizers.

Using the technique of [7] (see [2] for alternative proof) it can be shown that the orbit space $F_3/T$ is homeomorphic to $S^4$. The sponge of the action has dimension 1. This is simply the GKM-graph of the action, which is well known. This graph is shown on Fig.4. As an abstract graph it is a complete bipartite graph $K_{3,3}$. The figure on the right shows how to realize this graph as a 1-skeleton of simple cell structure on a 2-torus $\mathcal{T}$. Actually, $\mathcal{T}$ can be embedded in $S^4 = F_3/T$ in a canonical way and the preimage of its small neighborhood $U_\mathcal{T}$ under the projection map is described by Construction 5.9. This subject will be covered in detail in a subsequent paper [2].
Note the geometrical difference of these two examples from the induced $T$-action on a quasitoric manifold. In case of $T$-action on a quasitoric manifold, the sponge, which is an $(n - 2)$-dimensional complex, can be embedded in $S^{n-1}$ (since it is the $(n - 2)$-skeleton of a polytope). However the sponges of $G_{4,2}$ and $F_3$ do not embed in a sphere as codimension one complexes. In case of $F_3$ the graph $K_{3,3}$ is well-known to be non-planar. The sponge of $G_{4,2}$, which is the octahedron with 3 squares attached, cannot be embedded in $\mathbb{R}^3$.

**Remark 6.3.** Whenever the orbit space $Q = X/T$ is a sphere $S^{n+1}$, Alexander duality implies $H^2(Q \setminus Z; R) \cong H_{n-2}(Z; R)$ for a sponge $Z \subset Q$. The homology class corresponding to $e \in H^2(Q \setminus Z; H_1(T))$ is represented by the chain

$$\sigma = \sum_{F: \text{facet of } Z} e_x \cdot [F] \in C_{n-2}(Z; H_1(T)),$$

Here $[F]$ is the fundamental class of a facet $F$ and $e_x \in H_2(U_x; H_1(T)) \cong H_1(T)$ is the local Euler class in an interior point $x \in F^o$. The chain $\sigma$ is a cycle according to relation (4.2).

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