Disentangling topological degeneracy in the entanglement spectrum of one-dimensional symmetry protected topological phases

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One-dimensional valence bond solid (VBS) states represent the simplest symmetry protected topological phases. We show that their ground state entanglement spectrum contains both topological and non-topological structures. For the SO(3) symmetric VBS states with odd-integer spins, the two-fold topological degeneracy is associated with an underlying $Z_2 \times Z_2$ symmetry that protects the corresponding topological phase. In general, for the SO($2S+1$) symmetric VBS states with integer spins $S$, the corresponding protecting symmetry is identified as the $(Z_2 \times Z_2)^2$ symmetry, yielding the $2S$-fold topological degeneracy. The topological degeneracy and associated protecting symmetry can be identified by a non-local unitary transformation, which changes the topological order of the VBS states into conventional ferromagnetic order.

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INTRODUCTION

Topological properties of low-dimensional quantum many-body systems have been attracting considerable interest in both quantum information sciences and condensed matter physics. Remarkably, it is understood that important information of the topological phase is encoded in the von Neumann entanglement entropy of its ground state.\textsuperscript{[1,3]} In a seminal paper, Li and Haldane\textsuperscript{[4]} proposed that the entanglement spectrum (ES) labeled by the quantum numbers of symmetries preserved by the Schmidt singular-value decomposition contains more information than the entanglement entropy. In particular, they found the largest eigenvalues of the ground state reduced density matrix mimics the physics edge spectrum of a gapped quantum system, thus revealing the bulk topological order. A large number of works have followed, suggesting that the “low-energy” structure of ES reveals the bulk topological order.\textsuperscript{[2,5]}

Recently, a critique on this body of work was given by Chandran, Khemani, and Sondhi\textsuperscript{[6]}, who posed a question, “How universal is the entanglement spectrum?” In this paper we will address this important question by studying the ES of a family of one-dimensional symmetry protected topological (SPT) phases.\textsuperscript{[10–12]} The SPT phases have robust gapless edge excitations and can not be continuously connected to trivial product state without breaking the protecting symmetry or closing the energy gap. Moreover, the one-dimensional SPT phases have been generally classified by group cohomologies completely.\textsuperscript{[11,11,13]} ES of these phases are sufficiently simple and can be calculated analytically for model wave functions,\textsuperscript{[14]} thus providing an ideal setting for addressing this issue. Our results demonstrate that ES of topological phases include both topological/universal and non-topological parts. In certain cases the universal part can be isolated, and is extremely useful in revealing the bulk topological order.

The simplest example of SPT phases is the Haldane gapped phase of the antiferromagnetic spin-1 chain,\textsuperscript{[12]} which is protected by any one of the following discrete symmetries: time reversal symmetry, link inversion symmetry, or the $D_2 \cong Z_2 \times Z_2$ symmetry comprising π rotations about two orthogonal axes.\textsuperscript{[16–18]} The fixed point wave functions for these phases are the valence bond solid (VBS) states.\textsuperscript{[19]} We show their ES contains a topological part, which is universal, and a non-topological or non-universal part. The topological part can be isolated by a topological disentangler, a non-local unitary transformation. For the familiar SO(3) symmetric VBS states with odd-integer spins, their ES is given by a single energy level with $(S+1)$-fold degeneracy. However, the bulk topological order only gives rise to (or protects) a two-fold degeneracy, which is associated with the $Z_2 \times Z_2$ symmetry. The remaining $(S+1)/2$-fold degeneracy is not protected by the bulk topological order.

Furthermore, for VBS phases with higher symmetry, we can find corresponding high topological degeneracy and associated protecting symmetry via a generalized non-local unitary transformation. Specifically, we find in the SO(5) symmetric VBS state with spin-2,\textsuperscript{[20,21]} the topological degeneracy are four-fold and the protecting symmetry can be identified as $(Z_2 \times Z_2)^2$. In general, for the SO($2S+1$) symmetric VBS states with integer spin-$S$,\textsuperscript{[20,24]} the corresponding protecting symmetry is given by $(Z_2 \times Z_2)^S$, and the topological degeneracy is $2^S$-fold.
VBS STATES AND ENTANGLEMENT SPECTRUM

The SO(3) symmetric VBS states with arbitrary integer spin for a periodic spin chain is given by \[ |\text{VBS}\rangle = \prod_{i=1}^{N} \left( a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger \right)^S |\text{vac}\rangle, \] (1)

where the Schwinger spin-boson representation with a local constraint \( a_i^\dagger a_i + b_i^\dagger b_i = 2S \) has been used as \[ s_i^+ = a_i^\dagger b_i, s_i^- = b_i^\dagger a_i, s_i^z = \frac{1}{2} \left( a_i^\dagger a_i - b_i^\dagger b_i \right). \]

One can construct an SU(2) spin invariant model Hamiltonians, whose ground states are exactly given by these VBS states. In the thermodynamic limit, the spin correlation function decays exponentially with a correlation length \( \xi = 1/\ln(1 + 2/S) \), implying that the VBS state is a gapped disordered state.

For a finite length chain with open boundary condition, the ground state wave function is expressed as

\[
|\text{VBS} (\tau_i^z = \alpha, \tau_{N}^z = \beta)\rangle = \left( a_1^\dagger \right)^{\frac{\alpha}{2} + \alpha} \left( b_1^\dagger \right)^{\frac{\alpha}{2} - \alpha} \times \prod_{i=1}^{N-1} \left( a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger \right)^S \left( a_i^\dagger \right)^{\frac{\beta}{2} + \beta} \left( b_i^\dagger \right)^{\frac{\beta}{2} - \beta} |\text{vac}\rangle,
\]

where \( \alpha \) and \( \beta \) can be chosen from \(-S/2, -S/2 + 1, \ldots, S/2\), representing the eigenvalues of the edge spins of \( \tau_1^z \) and \( \tau_N^z \). There are total \((S + 1)^2\) degenerate edge states.

As model wave functions of a family of one-dimensional SPT phases, the VBS states exhibit the simplest ES. To calculate the ES for the wave functions \(|\text{VBS}(\alpha, \beta)\rangle\), we cut the spin chain into two parts A and B (see Fig.1a), and the VBS wave function can be expressed as

\[
|\text{VBS} (\alpha, \beta)\rangle = \sum_{n=0}^{S} (-1)^{S-n} C_n^S |\text{VBS} (\tau_i^z = \alpha, \tau_{N}^z = n - S/2)\rangle_A \times |\text{VBS} (\tau_{N+1}^z = S/2 - n, \tau_N^z = \beta)\rangle_B,
\]

where \( C_n^S = S! / n!(S - n)! \). After the normalization of all wave functions are carefully considered, the resulting wave function is written as

\[
|\text{VBS} (\alpha, \beta)\rangle = \sum_{n=0}^{S} (-1)^{S-n} e^{-\zeta S^2 / 2} \times |\text{VBS} (\alpha, n - S/2)\rangle_A |\text{VBS} (S/2 - n, \beta)\rangle_B,
\]

where a ratio of the length dependent factors has been used: \( \zeta = C(N) / C(N+S) \), and the entanglement energy level is just given by a single level \( \zeta = \ln(S + 1) \) with a degeneracy \((S + 1)\), which exactly corresponds to the degeneracy of the VBS wave function with one edge spin. So the degeneracy of each edge spin and the degeneracy of the entanglement level has one-to-one correspondence. However, a natural question to ask is whether all the degeneracies of the entanglement energy level are topologically protected or not.

NON-LOCAL UNITARY TRANSFORMATION

In order to characterize the topological features of these VBS states, non-local string order parameters (SOP) were first proposed for the spin-1 case \([20]\), and then generalized to arbitrary integer spins \([27, 28]\):

\[
O_\mu^{(j,k)} = \lim_{|k-j| \to \infty} \langle \text{VBS}| s_j^\mu \exp \left( i \pi \sum_{i=j}^{k-1} s_i^x \right) s_k^\mu |\text{VBS}\rangle
\]

where \( \mu = z, x \) or \( y \). In the thermodynamic limit, the SOP is found to be

\[
O_\mu^{(j,k)} = \left( \frac{S + 1}{S + 2} \right)^2 \delta_{S, \text{odd}},
\]

which reveals the fundamental difference between VBS states with odd- and even-integer spins. Only VBS states with odd-integer spins represent topologically non-trivial phases. An alternative and more useful characterization was revealed through a non-local unitary transformation \([27, 29]\):

\[
U = \prod_{k=2}^{N} \prod_{j=1}^{k-1} e^{i \pi s_j^x s_k^x},
\]

from which the local spin operators are transformed to

\[
\tilde{s}_j^z = U^{-1} s_j^z U = s_j^z \exp \left( i \pi \sum_{i=j}^{j-1} s_i^x \right),
\]

\[
\tilde{s}_j^x = U^{-1} s_j^x U = s_j^x \exp \left( i \pi \sum_{i=j+1}^{N} s_i^x \right),
\]

where a string operator in \( \tilde{s}_j^x \) involves all the sites from left edge site up to \( j - 1 \), while the string operator in \( \tilde{s}_j^x \)
involves all the sites from the site \( j+1 \) up to the right edge site. These properties will become crucial in studying the transformed VBS wave function. Meanwhile, the nonlocal SOP becomes to the local spin correlation via this transformation:

\[
U^{-1}s_j^\mu s_k^\mu U = s_j^\mu \exp \left( i\pi \sum_{i=3}^{k-1} s_i^\mu \right) s_k^\mu, \quad \mu = z, x. \tag{9}
\]

TRANSFORMED VBS STATES AND ITS ES

For the convenience of performing the non-local unitary transformation, the VBS wave functions are expressed in terms of the eigenvalue basis of the left edge spin \( \tau_1^z \) and the right edge spin in the \( \tau_N^z \):

\[
|\text{VBS}(\tau_1^z = \alpha, \tau_N^z = \beta)\rangle = (a_1^\dagger)^{+\alpha} (b_1^\dagger)^{-\alpha} \times \prod_{k=1}^{N-1} (a_k^b a_{k+1}^b - b_k^a b_{k+1}^a)^S \left( c_{k}^N \right)^{+\beta} \left( d_{k}^N \right)^{-\beta} |\text{vac}\rangle,
\]

where \( c_j^a = (a_j^a + b_j^a) / \sqrt{2} \) and \( d_j^a = (a_j^a - b_j^a) / \sqrt{2} \). The transformed VBS states are ferromagnetic long-range ordered states with two finite magnetizations \( \vec{m}_J^z \) and \( \vec{m}_N^z \) as local order parameters of the transformed states with two finite magnetizations \( \vec{m}_J^z \) and \( \vec{m}_N^z \) as local order parameters.\]

\[
\langle \text{VBS}(\alpha, \beta) | s_j^z | \text{VBS}(\alpha, \beta) \rangle = (-1)^{\alpha+S/2} \frac{(S+1)}{(S+2)} \delta_{S, \omega, \text{odd}}.
\]

\[
\langle \text{VBS}(\alpha, \beta) | s_j^z | \text{VBS}(\alpha, \beta) \rangle = (-1)^{\beta+S/2} \frac{(S+1)}{(S+2)} \delta_{S, \omega, \text{odd}}.
\]

Therefore the non-local transformation with the discrete symmetry \( Z_2 \times Z_2 \) plays the role of a topological disentangler, because it turns topological order of the original VBS state into conventional ferromagnetic order. The eigenvalues \( \alpha \) and \( \beta \) of the edge spins are used to classify the transformed degenerate ground states \( |\text{VBS}(\alpha, \beta)\rangle\).

There exist four different classes of long-range ordered ferromagnetic states with the degeneracy \((S+1)/2\) each. These degeneracies of each ferromagnetic ordered state only appear for \( S > 1 \) odd-integer spins and results from the partial polarization of the spins. The detailed ferromagnetic configurations are summarized in Table I.

To consider the ES of the transformed VBS state, we cut the spin chain into two parts A and B (see Fig.1b). When the edge spin \( \tau_{L+1}^z \) is transformed into the \( \tau_{L+1}^z \) basis, the VBS wave function is written as

\[
|\text{VBS}(\alpha, \beta)\rangle = \sum_{n=0}^{S} \sum_{m=0}^{S} (-1)^{S-n} C^n_S F(S, n, m) \times |\text{VBS}(\tau_1^z = \alpha, \tau_L^z = n - S/2)\rangle_{A} \times |\text{VBS}(\tau_{L+1}^z = m - S/2, \tau_N^z = \beta)\rangle_{B}, \tag{10}
\]

where

\[
F(S, n, m) = \sum_{p=0}^{S-n} \sum_{q=0}^{n} (-1)^{n-q} C^n_S C^p_{S-n} \delta_{m, p+q},
\]

and we have expressed the edge spins in the wave function for part A in terms of \( \tau_1^z \) and \( \tau_L^z \) basis, and the edge spins in the wave function of part B in terms of \( \tau_{L+1}^z \) and \( \tau_N^z \) basis.

When the non-local unitary transformation is applied to the VBS wave function, we separate the transformation into three parts \( U = U_A U_B U_{AB} \), where

\[
U_A = \prod_{j<k\in A} e^{ix\tau_j^z \tau_k^z}, U_B = \prod_{j<k\in B} e^{ix\tau_j^z \tau_k^z}, U_{AB} = e^{ix\tau_A^z \tau_B^z}.
\]

\[
U_{AB} \text{ plays the essential role in determining the new ES.}
\]

\[
\begin{array}{c|c|c|c|c}
\text{left edge } & \text{right edge } & m_1^z & m_2^z & m_3^z \\
\hline
\frac{S}{2}, \frac{S}{2}, \ldots, -\frac{S}{2} & \frac{S}{2}, \frac{S}{2}, \ldots, -\frac{S}{2} & \frac{S}{2} & \frac{S}{2} & \frac{S}{2} \\
-\frac{S}{2}, -\frac{S}{2}, \ldots, \frac{S}{2} & -\frac{S}{2}, -\frac{S}{2}, \ldots, \frac{S}{2} & -\frac{S}{2} & -\frac{S}{2} & -\frac{S}{2} \\
-\frac{S}{2}, -\frac{S}{2}, \ldots, \frac{S}{2} & -\frac{S}{2}, -\frac{S}{2}, \ldots, \frac{S}{2} & -\frac{S}{2} & -\frac{S}{2} & -\frac{S}{2} \\
-\frac{S}{2}, -\frac{S}{2}, \ldots, \frac{S}{2} & -\frac{S}{2}, -\frac{S}{2}, \ldots, \frac{S}{2} & -\frac{S}{2} & -\frac{S}{2} & -\frac{S}{2} \\
\end{array}
\]

TABLE I: The transformed SO(3) symmetric VBS states with odd-integer spins. They are ferromagnetic states classified into four groups by the eigenvalues of the edge spins \( \tau_1^z \) and \( \tau_N^z \), each with \((S + 1)/2\)-fold degeneracy.

and the total density matrix of \( |\text{VBS}(\alpha, \beta)\rangle \) is thus obtained. After the normalization factors are properly taken into account, we can trace out the degrees of freedom of the wave function of the part A, and the reduced density matrix is derived as

\[
[p_B(\alpha, \beta)]_{m\cdot m'} = \sum_{n=0}^{S} \sum_{m=0}^{m'} \sum_{m'=0}^{S} \frac{1}{C^n_S C^p_{S-n}} \times F(S, n, m) F(S, n, m') e^{ix(n-\frac{S}{2})(m-m')} \tag{12}
\]

To complete the summations, some further algebra is performed to rewrite the function \( F(S, n, m) F(S, n, m') \) in terms of the Schwinger bosons. When \((m - m')\) is even, the phase factor disappears \( e^{ix(n-\frac{S}{2})(m-m')} = 1 \), and the diagonal elements of the reduced density matrix is obtained

\[
[p_B(\alpha, \beta)]_{m\cdot m'} = \delta_{m, m'} \tag{13}
\]

However, when \((m - m')\) is odd, there is a phase factor \((-1)^{(n-\frac{S}{2})}\) in the summations, and non-diagonal elements of the reduced density matrix is given by

\[
[p_B(\alpha, \beta)]_{m\cdot m'} = (-1)^{(n-\frac{S}{2})} \frac{1}{S+1} \delta_{m, m', S} \tag{14}
\]
For the even-integer spins \( S \), all the non-diagonal matrix elements are zero because \( m + m' \) can not be even, and then \( \rho_E \) is a diagonal matrix with the same element 1/(\( S + 1 \)). This implies that the ES of the VBS states with even-integer spins remains the same after the non-local unitary transformation. Thus the degeneracy in ES of VBS states with even-integer spins is not of topological nature, consistent with the absence of the non-local SOP found previously and triviality of this phase.\(^{18}\)

However, for the odd-integer spins \( S \), the above results show that \( \rho_E \) has diagonal elements 1/(\( S + 1 \)) and skew diagonal elements \((-1)^{\alpha-S/2}/(S+1)\), so it is a X-form matrix. For any edge configuration, \( \rho_E \) has one eigenvalue 1/(\( S + 1 \)), and the entanglement energy level is dramatically changed and given by \( \zeta = \ln(S + 1) - \ln 2 \) with \((S + 1)/2\) fold degeneracy. In the spin-1 case, the transformed VBS wave function is a ferromagnetic product state, and the entanglement energy level is non-degenerate.\(^{31}\) For the \( S > 1 \) odd-integer spins, however, the transformed VBS wave functions are still entangled states, which are spin partially polarized ferromagnetic states with a degeneracy \((S + 1)/2\) for each class. The ES degeneracy of the transformed VBS states depends on the precise form of the resultant ferromagnetic states, and is thus non-topological. The crucial point, however, is that the non-local unitary transformation with the discrete symmetry \( Z_2 \times Z_2 \) has lifted the two-fold topological degeneracy, which is shared by all of these odd-integer spin VBS states belonging to the same topological phase.\(^{18}\) Meanwhile, the topological degeneracy and the topologically protecting symmetry can be read off from the non-local unitary transformation.

**SO(5) SYMMETRIC VBS STATE**

In order to put the above results in a larger context, we consider VBS states with larger symmetry groups which represent SPT phases with higher symmetry. We start with a topologically nontrivial VBS state formed in a spin-2 chain. Viewing each spin-2 as formed by two spin-3/2’s, we can construct an \( SO(5) \) symmetric VBS state.\(^{20,21}\) As pointed out in Ref.\(^{32,33}\), the \( \pm 3/2, \pm 1/2 \) states of a spin-3/2 can be regarded as the four states of the spinor representation of \( SO(3) \). Similarly one can also represent the \( \pm 2, \pm 1, 0 \) states of spin-2 as the five-dimensional vector irreducible representation (IR) of \( SO(5) \). Then we can view the vector IR as the symmetric component of the tensor product of two virtual spinor IR’s, i.e.,

\[
4 \otimes 4 = 1 \oplus 5 \oplus 10. \tag{15}
\]

The numerals are the dimensions of the \( SO(5) \) IR’s. The tensor product of two \( \frac{5}{2} \)'s on adjacent sites decomposes into

\[
5 \otimes 5 = 1 \oplus 10 \oplus 14. \tag{16}
\]

Comparing Eqs.\(^{15}\) and\(^{16}\), we can regard Eq.\(^{15}\) as the tensor product of two neighboring virtual spins after their respective partners have form \( SO(5) \) singlet with other virtual spins. Then one can find that \( SO(5) \) singlet 1 and the antisymmetric 10 appear in the decomposition but the symmetric 14 is absent. Therefore, if \( H = \sum_i P_{44}(i,i + 1) \), the \( SO(5) \) symmetric VBS state where neighboring virtual 4’s pair into \( SO(5) \) singlet will be the ground state. The operator \( P_{44}(i,i + 1) \) can be expressed in terms of the \( SO(5) \) generators

\[
P_{44}(i,j) = \frac{1}{2} \sum_{1 \leq a < b \leq 5} L_{ij}^{ab} L_{ij}^{ab} + \frac{1}{10} \left( \sum_{1 \leq a < b \leq 5} L_{ij}^{ab} L_{ij}^{ab} \right)^2 + \frac{1}{5}. \tag{17}
\]

Because the physical spin \( SU(2) \) is a subgroup of \( SO(5) \), each IR of \( SO(5) \) must decompose into an integral number of \( SU(2) \) multiplets. Thus the 14 discussed above must be expressible as the direct sum of \( SU(2) \) IR obtained by decomposing the direct product of two \( S = 2 \) multiplets. Since the 14-dimensional IR is symmetric upon the exchange of site indices, it must only contain even-spin \( SU(2) \) multiplets, i.e. \( 14 \rightarrow S_t = 2 \oplus S_t = 4 \).

For an open chain, there exist a free spin-3/2 at each end of the chain, leading to 4-fold degeneracy which is topologically protected by a larger symmetry. The corresponding topological ES degeneracy should also be 4-fold. For comparison, in the \( SO(3) \) symmetric VBS states, topology only protects a half edge spin and 2-fold ES degeneracy for odd-integer spins, while there is no protection at all for even-integer spins.

**SO(2S + 1) SYMMETRIC VBS STATES WITH INTEGER SPINS**

It is straightforward to promote the symmetry of the VBS states and demand that the spin-S states on each site transform under the \((2S + 1)\)-dimensional vector representation of \( SO(2S + 1) \), which can be formed by tensor decomposition of two virtual \( 2S \)-dimensional spinors.\(^{20,22}\) The main issue here is to identify the topological degeneracy. Since \( SO(2S + 1) \) is a rank-S algebra, there are \( S \) mutually commuting Cartan generators: \( \{ L^{12}, L^{34}, \ldots, L^{2S−1,2S} \} \). At each site, the quantum states are classified by the eigenvalues of these Cartan generators as

\[
L^{2a−1,2a}|m_\alpha \rangle = m_\alpha |m_\alpha \rangle, \quad (m_\alpha = 0, \pm 1). \tag{18}
\]

Thus the single-site states are associated with \( S \) quantum numbers \( \{ m_1, \cdots, m_S \} \), and they are subjected to the constraint

\[
m_\alpha m_\beta = 0, \quad (\alpha \neq \beta). \tag{19}
\]

The topological feature of these VBS states can be characterized by the following generalized non-local SOP

\[
\sigma^{ab} = \lim_{|j−i|→∞} \langle L_{ij}^{ab} \prod_{r=i}^{j−1} \exp(i\pi L_{rr}^{\alpha\beta}) L_{ij}^{ab} \rangle. \tag{20}
\]

Since the ground state is \( SO(2S + 1) \) rotational invariant, the above non-local order parameters should all be equal.
to each other. To determine the value of these parameters, only $\mathcal{O}^{12}$ needs to be evaluated. In the $L^{12}$ channel, the role of the phase factor in Eq. (20) is to correlate the finite spin polarized states in the $m_1$ channel at the two ends of the string. If nonzero $m_1$ takes the same value at the two ends, then the phase factor is equal to 1. On the other hand, if nonzero $m_1$ takes two different values at the two ends, then the phase factor is equal to $-1$. Thus the value of $\mathcal{O}^{12}$ is determined purely by the probability of $m_1 = \pm 1$ appearing at the two ends of the string. It is straightforward to show that the probability of the states $m_1 = \pm 1$ appearing at one lattice site is $2/(2S+1)$ and thus $\mathcal{O}^{12} = 4/(2S+1)^2$.

In the $SO(2S+1)$ Lie algebra, $(L^{2α−1, 2α}, L^{2α−1, 2S+1}, L^{2α, 2S+1})$ span an $SO(3)$ sub-algebra in which $\exp(i\pi L^{2α, 2S+1})$ plays the role of flipping the quantum number $m_α$. This exponential operator can flip the quantum numbers of $m_α$ without disturbing the quantum states in all other channels. Thus the following non-local unitary transformation with $Z_2 \times Z_2$ discrete symmetry

$$U_α = \prod_{j<i} \exp \left( i\pi L_{ij}^{2α−1, 2α} L_{ij}^{2α, 2S+1} \right), \quad (21)$$

can change the spin configuration in the $m_α$ channel into a ferromagnetically ordered one. Furthermore, by performing this non-local transformation successively in all the channels $U = \prod_{α=1}^S U_α$, all the configurations of the ground state will become ferromagnetically ordered. Applying $U$ to the Cartan generators, it can be shown that

$$UL_i^{αβ}U^{-1} = L_i^{αβ} \exp(i\pi \sum_{j=1}^{i-1} L_j^{αβ}). \quad (22)$$

Substituting this formula to Eq. (20), we find that

$$\mathcal{O}^{αβ} = \lim_{|i−j|→∞} \langle L_i^{αβ} L_j^{αβ} \rangle_U. \quad (23)$$

Thus the non-local SOP $\mathcal{O}^{αβ}$ of the Cartan generators become the ordinary two-point correlation functions of local operators. Under this general non-local unitary transformation with the $(Z_2 \times Z_2)^S$ symmetry, the topological order of the original VBS states is transformed into conventional ferromagnetic order, just like in the $SO(3)$ case.

When an open chain system is considered, there appear $2^S$ degenerate edge states at each end of the chain, which are also topologically protected by the discrete symmetry $(Z_2 \times Z_2)^S$. Accordingly the topological degeneracy in the ES can be read off as $2^S$-fold. We thus find that a higher symmetry in the SPT phase protects a higher ES degeneracy.

### CONCLUSION

We have shown through explicit examples that entanglement spectra of topological ground states contain both universal and non-universal structures. The topological degeneracy of the lowest entanglement energy level may be lifted or isolated by a generalized non-local unitary transformation or topological disentangler, which is determined by the minimal symmetry protecting the topological phases. Our results shed significant light on the issue of using entanglement spectrum to identify topological order.

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