Operator Regularization and
Noncommutative Chern Simons Theory

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1 Abstract

We examine noncommutative Chern Simons theory using operator regularization. Both the \( \zeta \)-function and the \( \eta \)-function are needed to determine one loop effects. The contributions to these functions coming from the two point function is evaluated. The \( U(N) \) noncommutative model smoothly reduces to the \( SU(N) \) commutative model as the noncommutative parameter \( \theta_{\mu\nu} \) vanishes.
2 Introduction

Normally, Chern Simons theory is a purely topological theory as it is metric independent [1-3]. However, if we consider Chern Simons theory in a noncommutative space [4,5] in which

\[ [x_\mu, x_\nu] = -i\theta_{\mu\nu} \]  

metric dependence inevitably arise. The consequences of the presence of \( \theta_{\mu\nu} \) in noncommutative Chern Simons theory have been considered in [6]. In particular, the one loop contribution to the two point function has been computed in [7].

In this paper, we would like to examine this one loop, two point function using operator regularization [8], a generalization of \( \zeta \)-function regularization [9]. The modulus and phase of the functional determinant associated with the effective action at one loop order are associated with the \( \zeta \)-function [8,9] and \( \eta \)-function [1, 10-12] respectively. This approach has been used in conventional Chern Simons theory [11-15].

3 The \( \zeta \)- and \( \eta \)-functions

The usual Chern Simons action is given by

\[ S = \int d^3x \, Tr \epsilon_{ijk} \left( A_i \partial_j A_k - \frac{2i}{3} A_i A_j A_k \right) \]  

where \( A_i = A_i^a T^a \) with \( T^a \) being the generator of a Lie group \( G \). If \( G \) is a unitary group \( U(N) \), then (2) can be generalized by converting the products occurring in (2) to Moyal
product in which [4,5]

\[ A_i(x) \ast A_j(x) = e^{-\frac{i}{2} \partial^x \times \partial^y} A_i(x)A_j(y) \bigg|_{x=y} \]  

(3)

where \( a \times b = \theta_{ij}a_i b_j \).

In order to use operator regularization to compute radiative effects, it is necessary to employ background field quantization [16-18]. We begin by splitting \( A_i \) into the sum of background and quantum fields

\[ A_i \to A_i + Q_i. \]  

(4)

A gauge fixing Lagrangian

\[ L_{gf} = Tr (ND_i(A)Q_i) \]  

(5)

is chosen so as to leave the symmetry

\[ A_i \to A_i + D_i(A)\Omega \]  

(6a)

\[ Q_i \to Q_i - i (Q_i \ast \Omega - \Omega \ast Q_i) \]  

(6b)

\[ \equiv Q_i - i [Q_i, \Omega] \]

present in (2) unbroken. In (5), \( N \) is a Nakanishi-Lautraup field and \( D_i(A) \) is the covariant derivative

\[ D_i(A)f = \partial_i f - i [A_i, f]. \]  

(7)

The ghost Lagrangian associated with the gauge fixing of eq. (5) and the gauge transformation

\[ A_i \to A_i + D_i(A + Q)\Lambda \]  

(8a)
\[ Q_i \rightarrow Q_i \] 

is

\[ L_{\text{ghost}} = \overline{C} D_i(A) D_i(A + Q) C. \] 

From (2), (5) and (9) it is evident that the terms in the effective Lagrangian \( L + L_{gf} + L_{\text{ghost}} \) that are bilinear in the quantum fields are

\[ L^{(2)} = (Q^a_1, N^a) \left( \begin{array}{cc} \epsilon_{ijp} D^{ab}_p(A) & -D^{ab}_i(A) \\ D^{ab}_j(A) & 0 \end{array} \right) \left( \begin{array}{c} Q^b_j \\ N^b \end{array} \right) \]

\[ + \overline{C}^a \left( D^{ab}_i(A) D^{bc}_i(A) \right) C^c, \] 

so that the one loop generating functional is given by

\[ W^{(1)}(A) = \left( \ln \det^{-1/2} H_I \right) + (\ln \det H_{II}) \] 

where \( H_I \) and \( H_{II} \) are the two operators in (10). As \( H_I \) is linear in derivatives, it is necessary to make the replacement

\[ \ln \det^{-1/2} H_I \rightarrow \ln \det^{-1/4} H_I^2. \] 

(A possible loss of phase in this replacement will be considered below.) Since

\[ \epsilon_{ijk} D_j(A) D_k(A) f = -\frac{i}{2} \epsilon_{ijk} [F_{jk}, f] \] 

where

\[ F_{jk} = \partial_i A_j - \partial_j A_i - i (A_i * A_j - A_j * A_i), \]

it is evident that

\[ H^2_I = \left( \begin{array}{cc} -D^2 \delta_{ij} & -\epsilon_{imn} F_{mn} \\ \epsilon_{mnj} F_{mn} & -D^2 \end{array} \right). \]
We note that
\[ [M, N] = M^a T^a \ast N^b T^b - N^b T^b \ast M^a T^a \]
\[ = \frac{1}{2} ([T^a, T^b] \{ M^a, N^b \} + \{ T^a, T^b \} [M^a, N^b]) \]
\[ \equiv (i f^{abc} \{ M^a, N^b \} + d^{abc} [M^a, N^b]) T^c \]

and this becomes, on account of the Moyal product defined in (3)
\[ = 2i \left[ f^{abc} \cos \left( \frac{p \times q}{2} \right) + d^{abc} \sin \left( \frac{p \times q}{2} \right) \right] (M^a N^b T^c) \]

(17)

where \( p \) and \( q \) are the momenta of \( M^a \) and \( N^b \) respectively. This is to be utilized when employing a perturbative expansion of \( W^{(1)}(A) \) in powers of \( A \). (Conventions for \( f^{abc} \) and \( d^{abc} \) are those of [19].)

In operator regularization, one first uses
\[ \ln det H = tr \ln H \]
\[ = \lim_{s \to 0} -d \frac{d}{ds} (tr H^{-s}) \]
\[ = -d \frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-Ht} \right] \]
\[ \equiv -\zeta'(0) \]

(18)

and then employs an expansion due to Schwinger [20]
\[ tr e^{-(H_0 + H_1) t} = tr \left[ e^{-H_0 t} + \left( \frac{t}{1} \right) e^{-H_0 t} H_1 \right. \]
\[ + \frac{(-t)^2}{2} \int_0^1 du e^{-(1-u)H_0 t} H_1 e^{-uH_0 t} H_1 + \ldots \]  

(19)
to effect an expansion in powers of the background field. (The dependence of $H$ on the
background field resides entirely in $H_1$.)

It is now possible to apply (18) and (19) to compute the contribution of the two point
function to the $\zeta$ function associated with $\ln \det^{-1/4} H_I^2$ and $\ln \det H_{II}$. We find that

$$W^{(1)}(A) = -\frac{d}{ds} \bigg|_0 \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{tr} \left\{ -\frac{1}{4} \exp -t \begin{pmatrix} -D^2 \delta_{ij} & -\epsilon_{imn} F_{mn} \\ \epsilon_{mij} F_{mn} & -D^2 \end{pmatrix} + \exp -t(-D^2) \right\}.$$  

(20)

Employing the expansion of eq. (19) and computing functional traces in momentum space
with [20]

$$< p|f|q> = \frac{1}{(2\pi)^{3/2}} f(p-q)$$  

(21)

we find that the contribution to the two point function coming from (20) is

$$W^{(1)}_2(A) = -\frac{d}{ds} \bigg|_0 \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \int \frac{dp \, dq}{(2\pi)^3} \left\{ -\frac{1}{4}(-t)^2 \int_0^1 du \left[ e^{-(1-u)p^2 + uq^2} \right] \left( f^{apb} f^{p}_{mn} (p-q) \cos \frac{p \times q}{2} + d^{pab} f_{mn} (p-q) \sin \frac{p \times q}{2} \right) \right\}$$  

(22)

Upon making the usual shift in momentum variables, this becomes

$$= \frac{N}{2} \frac{d}{ds} \bigg|_0 \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s+1} \int \frac{dp \, dq}{(2\pi)^3} \int_0^1 du \ e^{-[q^2 + u(1-u)p^2]t} \left( f^{ap}_{mn} (p) f^{b}_{mn} (-p) \left( \delta^{ab} - \delta^{a0} \delta^{b0} \cos(p \times q) \right) \right).$$  

(23)

The standard integrals [21]

$$\int \frac{d^n k}{(2\pi)^n} e^{-k^2 t} = \frac{1}{(4\pi t)^{n/2}}$$  

(24a)
\[
\int_0^\infty dt \: t^{\nu-1}e^{-At} = \Gamma(\nu)A^{-\nu} \tag{24b}
\]

\[
\int_0^\infty dt \: t^{\nu-1}e^{-\gamma t - \beta/t} = 2 \left(\frac{\beta}{\gamma}\right)^{\nu/2} K_{\nu}(2\sqrt{\beta\gamma}) \tag{24c}
\]

can be used to reduce (23) to

\[
W_2^{(1)}(A) = \frac{N}{2} \frac{d}{ds} \left|_0^1 \frac{1}{\Gamma(s)} \int \frac{d^3p}{(4\pi)^{3/2}} \int_0^1 du f^{a}_{mn}(p)f^{b}_{mn}(-p) \right.
\]

\[
\left. \left[ \Gamma \left( s + \frac{1}{2} \right) \left( u(1-u)p^2 \right)^{-s-1/2} \delta^{ab} \right. \right.
\]

\[
-2 \left( \frac{\tilde{p}^2}{4u(1-u)p^2} \right)^{s+1/2} K_{s+1/2} \left( 2\sqrt{\frac{u(1-u)p^2\tilde{p}^2}{4}} \right) \delta^{a0}\delta^{b0} \right]\]

where \( p \times q \equiv \tilde{p} \cdot q \). It is now possible to evaluate \( \frac{d}{ds} \left|_0^1 \right. \) in (25) explicitly, leaving us with

\[
W_2^{(1)}(A) = \frac{N}{2} \int \frac{d^3p}{(4\pi)^{3/2}} \int_0^1 du f^{a}_{mn}(p)f^{b}_{mn}(-p) \left[ \Gamma(s) \Gamma(s+1) \Gamma(s+1/2) \Gamma(s+1/2) \right]
\]

\[
\left. \left[ \Gamma(s) \Gamma(s+1) \Gamma(s+1/2) \Gamma(s+1/2) \right] \left( u(1-u)p^2 \right)^{-s-1/2} \delta^{ab} \right.
\]

\[
-2 \left( \frac{\tilde{p}^2}{4u(1-u)p^2} \right) K_{s+1/2} \left( 2\sqrt{\frac{u(1-u)p^2\tilde{p}^2}{4}} \right) \delta^{a0}\delta^{b0} \right].
\]

Since \( K_{s+1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z} \), it is possible to compute both integrals over \( u \) in (26) using [21]

\[
\int_0^1 du u^{a-1}(1-u)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{27a}
\]

\[
\int_0^1 du \frac{1}{\sqrt{u(1-u)}}e^{-A\sqrt{u(1-u)}} = 2 \int_0^{\pi/2} d\theta e^{-\frac{A}{2}\cos\theta} = \pi I_0(A/2) \tag{27b}
\]

leaving us with

\[
W_2^{(1)}(A) = \frac{N}{16} \int \frac{d^3p}{\sqrt{p^2}} \left[ f^{a}_{mn}(p)f^{b}_{mn}(-p) \right] \left( \delta^{ab} - I_0 \left( \frac{\sqrt{p^2\tilde{p}^2}}{2} \right) \delta^{a0}\delta^{b0} \right). \tag{28}
\]

In the limit \( \theta^{\mu\nu} \to 0 \) (so that \( \tilde{p}^2 \to 0 \)), \( I_0 \left( \frac{\sqrt{p^2\tilde{p}^2}}{2} \right) \to 1 \) leaving only the \( SU(N) \) contribution to (28).
We can now consider the loss of phase associated with the replacement of eq. (12). This phase has been discussed extensively in [1, 10-12]; it is associated with so-called \( \eta \)-function,

\[
\eta(s) = \frac{1}{\Gamma \left( \frac{s+1}{2} \right)} \int_0^\infty dt \ t^{\frac{s+1}{2}} \text{Tr} \left( H_I e^{-H_I^2 t} \right).
\]

(29)

If a parameter \( \lambda \) is inserted into \( H_I \) so that \( H_I(\lambda = 1) = H_I \), then from (29) it is easily seen that

\[
\frac{d\eta(s)}{d\lambda} = -s \frac{1}{\Gamma \left( \frac{s+1}{2} \right)} \int_0^\infty dt \ t^{\frac{s-1}{2}} \text{Tr} \left( \frac{dH_I(\lambda)}{d\lambda} e^{-H_I^2(\lambda)t} \right).
\]

(30)

As only \( \eta(0) \) is required to determine the phase we are interested in, it is sufficient to compute the poles arising in the integral over \( t \) in (30) on account of the explicit factor of \( s \) arising in front of the integral.

If now

\[
H_I(\lambda) \equiv \begin{pmatrix} \epsilon_{ipj} (\partial_p - i\lambda[A_p]) - (\partial_i - i\lambda[A_i]) \\ (\partial_j - i\lambda[A_j]) & 0 \end{pmatrix}
\]

(31)

then it is possible to expand the right side of (30) in powers of the background field using a second expansion due to Schwinger [20]

\[
e^{-(H_0 + H_1)t} = \left[ e^{-H_0t} + (-t) \int_0^1 du \ e^{-(1-u)H_0t} H_1 e^{-uH_0t} \\
+ (-t)^2 \int_0^1 du \int_0^1 dv \ e^{-(1-u)H_0t} H_1 e^{-u(1-v)H_0t} \\
+ H_1 e^{-uvH_0t} + \ldots \right].
\]

(32)

From (15), (31) and (32), it is evident that the contribution to (30) that is bilinear in the background field \( A_\mu \) is (upon following the approach used above with the \( \zeta \)-function)

\[
\frac{d\eta^{(2)}(s)}{d\lambda} = \frac{N\lambda s}{\Gamma \left( \frac{s+1}{2} \right)} \int_0^\infty dt \ t^{\frac{s+1}{2}} \int_0^1 du \int \frac{dp \ dq}{(2\pi)^3} e^{-\left( q^2 + u(1-u)p^2 \right)t}
\]

(33)
\[
\left( \epsilon_{ijk} A^q_i(p) f^{b}_{jk}(-p) \right) \left( \delta^{ab} - \delta^{a0} \delta^{b0} \cos(p \times q) \right).
\]

The integrals appearing in (33) are identical in form to those in (23); just as we obtain (28) we find that

\[
\frac{d\eta^{(2)}_\lambda(s)}{d\lambda} = \frac{N\lambda s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^1 du \int \frac{dp}{(4\pi)^{3/2}} \epsilon_{ijk} A^q_i(p) f^{b}_{jk}(-p) \\
\left[ \delta^{ab} \Gamma\left(\frac{s}{2}\right) (u(1-u)p^2)^{-s/2} \right] \\
-2\delta^{a0} \delta^{b0} \left( \frac{\bar{p}^2}{4u(1-u)p^2} \right)^{s/4} \frac{K_{s/2}}{K_{s/2}} \left( \sqrt{u(1-u)p^2} \bar{p} \right) \right].
\]  

(34)

In the limit \( x \to 0, K_\nu(x) \to 2^{\nu-1} \Gamma(\nu)x^{-\nu} \) and thus when \( \theta_{\mu\nu} \to 0 (\bar{p}^2 \to 0) \), (34) reduces to

\[
\approx \frac{N\lambda s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^1 du \int \frac{dp}{(4\pi)^{3/2}} \epsilon_{ijk} A^q_i(p) f^{b}_{jk}(-p) \\
\left[ \Gamma\left(\frac{s}{2}\right) (u(1-u)p^2)^{-s/2} \right] \\
\left[ \delta^{ab} - \delta^{a0} \delta^{b0} \right].
\]  

(35)

As in the case of the \( \zeta \)-function, only the \( SU(N) \) contribution to \( \frac{d}{dx}\eta_\lambda(0) \) survives in the commutative limit.

4 Discussion

Chern Simons theory is difficult to regulate on account of the presence of the tensor \( \epsilon_{ijk} \). Operator regularization appears however to be a suitable way of dealing with this problem as it does not involve altering the original Lagrangian. This permits one to deal with both the usual commutative \( SU(N) \) Chern Simons model and as well noncommutative \( U(N) \) Chern Simons theory. An analysis of the one loop two point function done here has shown
that in the commutative limit, the non-commutative $U(N)$ model smoothly reduces to the
commutative $SU(N)$ model.

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