ON SOME LIOUVILLE TYPE THEOREMS FOR THE STATIONARY
MHD AND HALL-MHD EQUATIONS

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Abstract. We prove several Liouville type results for the stationary MHD and Hall-MHD
equations. In particular, we show that the velocity and magnetic field, belonging to some
Lorentz spaces or satisfying a priori decay assumption, must be zero.

Keywords: Liouville type theorem; MHD equations; Lorentz spaces
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1. Introduction

In this paper, we investigate the following three-dimensional steady-state incompressible
MHD equations
\[
\begin{cases}
u \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{H} \cdot \nabla \mathbf{H}, \\
u \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{u} = \Delta \mathbf{H}, \\
\text{div} \mathbf{u} = 0, \\
\text{div} \mathbf{H} = 0.
\end{cases}
\]
and Hall-MHD equations
\[
\begin{cases}
u \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H}, \\
\nabla \times ((\nabla \times \mathbf{H}) \times \mathbf{H}) - \nabla \times (\mathbf{u} \times \mathbf{H}) = \Delta \mathbf{H}, \\
\text{div} \mathbf{u} = 0, \\
\text{div} \mathbf{H} = 0.
\end{cases}
\]

Here \(\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))\), \(p(x, t)\) denote the velocity and pressure of the
fluid respectively, and \(\mathbf{H}(x, t) = (H_1(x, t), H_2(x, t), H_3(x, t))\) is the magnetic field vector.
The Hall term \(\nabla \times ((\nabla \times \mathbf{H}) \times \mathbf{H})\) in (1.2) is included due to the Ohm’s law, which is believed
to be a key issue for understanding magnetic reconnection in geo-dynamo [1], neutron stars
[6] and star formation [20].

Recently, there are many works has been devoted to the well-posedness theory for the
classical Hall-MHD and MHD equations. We refer the reader to interesting papers [3] [10] [21]
and references therein.

Liouville type theorem for partial differential equations has drawn much attention. Actually,
Liouville type theorem naturally arises when considering the regularity of solutions to
partial differential equations, like Navier-Stokes equations and (Hall) MHD equations. However,
the development of the Liouville type theorem for the stationary (Hall) MHD equations
is slow. Admittedly, there are many works on the Liouville type theorem for the steady-state
incompressible Navier-Stokes and MHD equations. The related works can be found readily
in [4] [7] [11] [12] [14] [17].

In particular, for MHD equations, Chae [3] generalized Galdi’s work which is well-known
for the Navier-Stokes equations, to Hall-MHD equations under the assumption
\[
(\mathbf{u}, \mathbf{H}) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \quad \text{and} \quad (\nabla \mathbf{u}, \nabla \mathbf{H}) \in L^2(\mathbb{R}^3).
\]
Further, Gala demonstrated that a solution \((u, H)\) to the 3D stationary MHD equations is zero by adding the assumption
\[
(u, H) \in L^2(\mathbb{R}^3) \quad \text{and} \quad \nabla H \in L^2(\mathbb{R}^3)
\]
in [8]. On the other hand, Chae and Weng [4] showed that the smooth solution \((u, H) \equiv 0\) to the Hall-MHD equations (1.1) provided
\[
(u, H) \in L^3(\mathbb{R}^3) \quad \text{and} \quad (\nabla u, \nabla H) \in L^2(\mathbb{R}^3).
\]
Another interesting result ref. [13] proved that the condition
\[
(u, H) \in L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)
\]
implies \((u, H) \equiv 0\) as well. It’s worth noting that Schulz’s work is the first result without the requirement
\[
(\nabla u, \nabla H) \in L^2(\mathbb{R}^3).
\]

However, the classical Liouville type problem to the steady-state Navier-Stokes and (Hall) MHD equations is still an open problem. Very recently, Seregin and Wang [16] showed that when the velocity field belongs to some Lorentz spaces, Navier-Stokes equations satisfy Liouville type theorems. In [15], Seregin proved that \(u \equiv 0\) under some decay assumption conditions.

Motivated by [14, 15, 16], just those \((u, H)\) which are in Lorentz spaces or satisfy specific a priori decay assumption, shall be considered in our paper. Compared with the result in [13], we relax the restriction that \((u, H) \in L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)\). Another point is that we do not demand further integrability, i.e., \(\nabla u \in L^2(\mathbb{R}^3)\) which is different from the result in [3]. We also show that the condition \(|(u(x), H(x))| \leq \frac{C}{(1 + |x|)^\mu}\) with \(x' = (x_1, x_2)\) and \(\mu \approx 0.67\) implies \((u, H) \equiv 0\).

Define \(M_{\gamma,q,\ell}(u,H), R) := R^{\gamma - \frac{2}{q}}\|((u,H))\|_{L^q,\ell(B(R) \setminus B(\frac{R}{2}))}\), then our main results can be stated as follows.

**Theorem 1.1.** Let \((u, H)\) be a smooth solution to MHD (1.1).

(i) For \(q > 3, 3 \leq \ell \leq \infty\), \((\text{or } q = \ell = 3)\), \(\gamma > \frac{2}{3}\), assume that
\[
\liminf_{R \to \infty} M_{\gamma,q,\ell}(u,H), R) < \infty.
\]
Then \((u, H) \equiv 0\).

In particular, if \(\gamma = \frac{2}{3}\), for some \(0 < \delta < \frac{1}{C(q, \ell)}\)
\[
\liminf_{R \to \infty} M_{\gamma,q,\ell}(u,H), R) \leq \delta \int_{\mathbb{R}^3} |\nabla (u - H)|^2 \, dx
\]
need to be required additionally.

(ii) For \(\frac{12}{5} < q < 3, 1 \leq \ell \leq \infty, \gamma > \frac{1}{3} + \frac{1}{q}\), suppose that
\[
\liminf_{R \to \infty} M_{\gamma,q,\ell}(u,H), R) < \infty.
\]
Then \((u, H) \equiv 0\).

**Remark 1.1.** If we let \(q = \ell = 3\) and assume \((u, H) \in L^3(\mathbb{R}^3)\), we see that \(M_{\gamma,3,3} \to 0\) as \(R \to \infty\) for every \(\gamma \in (\frac{2}{3},1]\). Hence, Chae and Weng’s result in [4] follows from Theorem 1.1.

On the other hand, we can establish the following conciser result, i.e.,

**Theorem 1.2.** Let \((u, H)\) be a smooth solution to (1.1) and satisfy
\[
|(u(x), H(x))| \leq \frac{C}{(1 + |x'|)^\mu}
\]
for any $x = (x', x_3)$ and $\mu > \frac{2}{3}$. Then $(u, H) \equiv 0$.

**Remark 1.2.** Noted that there are some similar results for Navier–Stokes equations, see \cite{15, 19}.

Fortunately, MHD equations satisfy Galilean Invariance. Motivated by \cite{14}, we apply a similar technique to show Theorem 1.3.

**Theorem 1.3.** Let $(u, H)$ be a smooth solution to (1.1), $\frac{3}{2} < q \leq 3$, and $2 \leq s \leq 6$. Suppose that

\begin{equation}
M((u, H), R) := \sup_{R > 0} R^{1 - \frac{3}{q}} \| (u, H) \|_{L^q (B(R))} < \infty,
\end{equation}

and

\begin{equation}
N((u, H), R) := \sup_{R > 0} R^{\frac{3}{2} - \frac{2}{s}} \| (u, H) \|_{L^s (B(R))} < \infty.
\end{equation}

Then $(u, H) \equiv 0$.

For Hall-MHD equations, the Liouville type theorems can be stated as follow.

**Theorem 1.4.** Let $(u, H)$ be a smooth solution to Hall-MHD (1.2) satisfying $\nabla H \in L^2 (\mathbb{R}^3)$ and $q > 3$, $3 \leq \ell \leq \infty$, (or $q = \ell = 3$), $\gamma > \frac{2}{3}$, assume that

\begin{equation}
\lim_{R \to \infty} M_{\gamma, q, \ell}((u, H), R) < \infty.
\end{equation}

Then $(u, H) \equiv 0$.

In particular, if $\gamma = \frac{2}{3}$, for some $0 < \delta < \frac{1}{C(q, \ell)}$

\begin{equation}
\lim_{R \to \infty} M_{\frac{3}{2}, q, \ell}((u, H), R) \leq \delta \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla H|^2) \, dx
\end{equation}

need to be required additionally.

**Remark 1.3.** If we let $q = \ell = \frac{9}{2}$ and assume $(u, H) \in L^\frac{9}{2} (\mathbb{R}^3)$, we observe that $M_{\frac{9}{2}, \frac{9}{2}, \frac{9}{2}} \to 0$ as $R \to \infty$. Therefore, Gala’s result in \cite{8} follows from Theorem 1.4.

The rest of this paper is organized as follows. In Section 2, we provide some elementary facts. In Section 3, we obtain the Caccioppoli type inequality, which is the key of our proof. Finally, we will show the complicate proof of Theorem 1.1–1.4, respectively in Section 4–7.

**Notations.** Throughout this paper, $L^{p, \infty}(\Omega)$ stands for a weak Lebesgue space, which is a particular Lorentz space $L^{p, q}(\Omega)$ and $L^{p, p}(\Omega) = L^p(\Omega)$ is a usual Lebesgue space. $\| \cdot \|_{L^{p, q}(\Omega)}$ denotes the semi-norm of the Lorentz space $L^{p, q}(\Omega)$. $B(R)$ is a ball of radius $R$ centered at the origin, i.e., $B(R) = \{ x \in \mathbb{R}^3 \mid |x| < R \}$.

2. Preliminaries

In this section, we will give some elementary facts and useful lemmas which will be used in the next section.

**Lemma 2.1** (Calderón-Zygmund Inequality, See \cite{9}). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $f$ a function in $L^p(\Omega)$, $1 < p < \infty$, and $w$ the Newtonian potential of $f$. Then $w \in W^{2, p}(\Omega)$, $\Delta w = f$ a.e. $\Omega$, and

$$
\int_\Omega |\nabla^2 w|^p \, dx \leq C \int_\Omega |f|^p \, dx,
$$

where constant $C > 0$ only depends on $n$ and $p$. 
Lemma 2.2 (Galilean Invariance). Let \((u, H, p, \lambda)\) be a solution to the MHD system \((1.1)\) and \(\lambda \in \mathbb{R}\). Then

\[
\begin{align*}
\mathbf{u}_\lambda(x, t) &= \lambda \mathbf{u}(\lambda x, \lambda^2 t), \\
p_{\lambda}(x, t) &= \lambda^2 p(\lambda x, \lambda^2 t), \\
\mathbf{H}_\lambda(x, t) &= \lambda \mathbf{H}(\lambda x, \lambda^2 t),
\end{align*}
\]

also solves the MHD equations \((1.1)\).

**Proof.** It is easily to find that

\[
\begin{align*}
(u_{\lambda} \cdot \nabla u_{\lambda})(x, t) &= \lambda^3(u \cdot \nabla u)(\lambda x, \lambda^2 t), \\
(H_{\lambda} \cdot \nabla H_{\lambda})(x, t) &= \lambda^3(H \cdot \nabla H)(\lambda x, \lambda^2 t), \\
(H_{\lambda} \cdot \nabla u_{\lambda})(x, t) &= \lambda^3(H \cdot \nabla u)(\lambda x, \lambda^2 t), \\
(u_{\lambda} \cdot \nabla H_{\lambda})(x, t) &= \lambda^3(u \cdot \nabla H)(\lambda x, \lambda^2 t), \\
(p_{\lambda})(x, t) &= \lambda^2(p)(\lambda x, \lambda^2 t), \\
(\Delta u_{\lambda})(x, t) &= \lambda^3(\Delta u)(\lambda x, \lambda^2 t), \\
(\Delta H_{\lambda})(x, t) &= \lambda^3(\Delta H)(\lambda x, \lambda^2 t).
\end{align*}
\]

Hence, \((u_{\lambda}, p_{\lambda}, H_{\lambda})\) satisfies the MHD equations \((1.1)\). \(\square\)

Lemma 2.3 (Sobolev embedding, [2]). If \(s\) is in \([0, \frac{d}{2}]\), then the space \(\dot{H}^s(\mathbb{R}^d)\) is continuously embedded in \(L^{\frac{2d}{d-s}}(\mathbb{R}^d)\).

3. **Caccioppoli Type Inequalities**

We begin with an auxiliary lemma about Caccioppoli type inequality, which is the key of our proof. We state this inequality below.

**Proposition 3.1.** Let \((u, H)\) be a smooth solution to \((1.1)\) and \(v = u + H, T = u - H\). Then the following Caccioppoli type inequalities hold:

(i) For \(q > 3, 3 \leq \ell \leq \infty\),

\[
\int_{B(\frac{R}{2})} |\nabla v|^2 \, dx \leq CR^{-2} \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx + D_1,
\]

where \(D_1 := C(q, \ell) R^{2-\frac{2}{q}} \|T\|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))}^2\).

(ii) For \(0 < \delta < 1, 6 \frac{3-\delta}{2-\delta} < q < 3\),

\[
\int_{B(\frac{R}{2})} |\nabla v|^2 \, dx \leq CR^{-2} \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx + D_2,
\]

where \(D_2 := C(\delta) \left(R^{2-\frac{9-3\delta}{q} - \frac{\delta}{2}} \|T\|_{L^{9,\infty}(B(R) \setminus B(\frac{R}{2}))} \|v\|_{L^{\frac{9}{\delta},\infty}(B(R) \setminus B(\frac{R}{2}))}^2 \right)^{\frac{2}{2-\delta}}\).

**Proof.** Given \(R > 0\), fix numbers \(q\) and \(r\) so that \(\frac{3R}{4} \leq q < r \leq R\). Now, choose a cut-off function \(\varphi \in C_0^\infty(B(R))\) satisfying the following conditions:

\[
\varphi(x) = \begin{cases} 
1, & \text{if } x \in B(q) \\
0, & \text{if } x \in B(r)^c 
\end{cases}
\]

\(0 \leq \varphi \leq 1\) and \(|\nabla \varphi(x)| \leq \frac{c}{(r - q)}\).

Considerate the following Dirichlet problem

\[
\begin{align*}
\Delta \psi &= \text{div}(\varphi v) & \text{in } B(R) \setminus B(\frac{R}{2}), \\
\psi &= 0 & \text{on } \partial B(R) \cup B(\frac{2R}{3}).
\end{align*}
\]
From the standard elliptic equations theory, there is a unique \( \psi \in W^{1,s}_0(B(R) \setminus B(\frac{2R}{3})) \cap W^{2,s}(B(R) \setminus B(\frac{2R}{3})) \) solving this Dirichlet problem. Therefore \( w = \nabla \psi \in W^{1,s}_0(B(R) \setminus B(\frac{2R}{3})) \) such that \( \text{div } w = \text{div}(\varphi v) = \nabla \varphi \cdot v \). Applying Lemma 2.1, we can deduce the following inequality.

\[
\int_{B(R) \setminus B(\frac{2R}{3})} |\nabla w|^s \, dx \leq C \int_{B(R) \setminus B(\frac{2R}{3})} |\nabla^2 \psi|^s \, dx
\]

(3.3)

where \( C \) is independent of \( R \) and only depends on \( s \) (\( 1 < s < \infty \)).

According to the general Marcinkiewicz interpolation theorem, we find

\[
\|\nabla w\|_{L^{s,t}(B(r) \setminus B(\frac{2r}{3}))} \leq C(q) \|\nabla \varphi \cdot v\|_{L^{s,t}(B(r) \setminus B(\frac{2r}{3}))}.
\]

(3.4)

Adding the equation (1.1) and (1.2), multiplying (3.5) \( \varphi \), and only depends on \( C \), we get:

\[
\int_{B(r)} \varphi |\nabla v|^2 \, dx = -\int_{B(r)} \nabla v : (\nabla \varphi \otimes v) \, dx + \int_{B(r)} \nabla w : \nabla v \, dx
\]

(3.6)

\[
= \sum_{i=1}^{4} I_i.
\]

Notice that \( R > r > \frac{3R}{4} > \frac{2r}{3} > \frac{R}{2} \), Hölder’s inequality gives:

\[
|I_1| = \left| \int_{B(r)} \nabla v : (\nabla \varphi \otimes v) \, dx \right|
\]

(3.7)

\[
\leq C \left( \int_{B(r)} |\nabla v|^2 \, dx \right)^\frac{1}{2} \left( \int_{B(r) \setminus B(\frac{2r}{3})} |\nabla \varphi \otimes v|^2 \, dx \right)^\frac{1}{2}
\]

\[
\leq \frac{C}{r - \theta} \left( \int_{B(r)} |\nabla v|^2 \, dx \right)^\frac{1}{2} \left( \int_{B(R) \setminus B(\frac{2R}{3})} |v|^2 \, dx \right)^\frac{1}{2}.
\]

By (3.3), we deduce

\[
|I_2| = \left| \int_{B(r)} \nabla w : \nabla v \, dx \right|
\]

(3.8)

\[
\leq C \left( \int_{B(r)} |\nabla v|^2 \, dx \right)^\frac{1}{2} \left( \int_{B(r) \setminus B(\frac{2r}{3})} |\nabla w|^2 \, dx \right)^\frac{1}{2}
\]

\[
\leq \frac{C}{r - \theta} \left( \int_{B(r)} |\nabla v|^2 \, dx \right)^\frac{1}{2} \left( \int_{B(R) \setminus B(\frac{2R}{3})} |v|^2 \, dx \right)^\frac{1}{2}.
\]
Proof of inequality (3.1). To estimate $I_3$ and $I_4$, we are going to use the fact that $\text{div } T = \text{div } u - \text{div } H = 0$. Integration by parts gives

$$I_3 = - \int_{B(r)} (T \cdot \nabla v) \cdot \varphi v \, dx = -\frac{1}{2} \int_{B(r)} \varphi \text{div}(T|v|^2) \, dx = \frac{1}{2} \int_{B(r)} (T|v|^2) \cdot \nabla \varphi \, dx.$$

Using the Hölder inequality in Lorentz spaces, assuming that $q > 3$ and $\ell \geq 3$, we have

$$|I_3| = \frac{1}{2} \int_{B(r)} |T||v|^2 \cdot \nabla \varphi \, dx$$

$$\leq \frac{C(r-\varrho)}{(r-\varrho)} \int_{B(r) \setminus B(\varrho)} |T||v|^2 \, dx$$

$$\leq \frac{C}{(r-\varrho)} \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}^2 \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}$$

$$\leq \frac{C(q, \ell)}{(r-\varrho)} R^{3-\frac{2}{q}} \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}^2 \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}.$$

Thanks to (3.4), $I_4$ can be evaluated as follow:

$$|I_4| = \left| \int_{B(r)} (T \cdot \nabla v) \cdot \varphi w \, dx \right| = \left| \int_{B(r) \setminus B(\varrho)} (T \otimes v) : \nabla w \, dx \right|$$

$$\leq C \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|\nabla w\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}$$

$$\times \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}$$

\begin{equation}
\leq \frac{C}{(r-\varrho)} \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}^2 \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}.
\end{equation}

Remark 3.1. When $q = \ell = 3$, $\|1\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} = 1$.

Thus, inserting (3.4) into (3.6) leads to

$$\int_{B(\varrho)} |\nabla v|^2 \, dx \leq \int_{B(r)} \varphi |\nabla v|^2 \, dx$$

$$\leq \frac{C}{r-\varrho} \left( \int_{B(r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(r) \setminus B(\varrho)} |v|^2 \, dx \right)^{\frac{1}{2}}$$

$$+ \frac{C(q, \ell)}{(r-\varrho)} R^{3-\frac{2}{q}} \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}^2 \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}$$

$$\leq \frac{1}{2} \int_{B(r)} |\nabla v|^2 \, dx + \frac{C}{(r-\varrho)^2} \int_{B(r) \setminus B(\varrho)} |v|^2 \, dx$$

$$+ \frac{C(q, \ell)}{(r-\varrho)} R^{3-\frac{2}{q}} \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}^2 \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}.$$

Repeating the same arguments in two ball $B(\varrho)$ and $B(\varrho R)$ with $\frac{R}{2} < \frac{3R}{4} < \varrho$, we find

$$\int_{B(\varrho R)} |\nabla v|^2 \, dx \leq \frac{1}{2} \int_{B(\varrho)} |\nabla v|^2 \, dx + CR^{-2} \int_{B(R) \setminus B(\varrho)} |v|^2 \, dx$$

$$+ \frac{C(q, \ell)}{(3R/4 - \varrho)} R^{3-\frac{2}{q}} \|T\|_{L^{q,\ell}(B(R) \setminus B(\varrho))} \|v\|_{L^{q,\ell}(B(R) \setminus B(\varrho))}^2 \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))} \frac{1}{L^{q,\ell}(B(R) \setminus B(\varrho))}.$$

As a result

$$\int_{B(\frac{\varrho R}{2})} |\nabla v|^2 \, dx \leq CR^{-2} \int_{B(R) \setminus B(\varrho)} |v|^2 \, dx + D_1,$$
which yields the inequality (3.1).

Proof of inequality (3.2). We only need to re-estimate \( I_3 \) and \( I_4 \). To the end, we introduce \( \bar{v} = \chi - [\chi]_{B(r) \setminus B(\frac{3r}{4})} \), where \([\chi]_{\Omega}\) is the mean value of \( \chi \) over a domain \( \Omega \). Thanks to the integration by parts, we find

\[
I_3 = -\frac{1}{2} \int_{B(r)} (T \cdot \nabla |\chi|^2) \, \varphi \, dx = -\frac{1}{2} \int_{B(r)} (T \cdot \nabla (|\chi|^2 - [\chi]_{B(r) \setminus B(\frac{3r}{4})}^2)) \, \varphi \, dx
\]

(3.11)

\[
= \frac{1}{2} \int_{B(r) \setminus B(\rho)} (T \cdot \nabla \varphi) (|\chi|^2 - [\chi]_{B(r) \setminus B(\frac{3r}{4})}^2) \, dx,
\]

and, since \( \frac{R}{2} < \frac{2r}{3} < \frac{3R}{4} \leq \rho \),

\[
|I_3| \leq \frac{C}{r - \rho} \int_{B(r) \setminus B(\frac{3r}{4})} |\nabla \varphi| |\chi| |\chi| \, dx.
\]

By the assumptions \( 0 < \delta \leq 1 \), \( \frac{6(3 - \delta)}{6 - \delta} < q < 3 \), we get

\[
0 < \beta := 1 - \frac{3 - \delta}{q} - \frac{\delta}{6} < 1.
\]

Thanks to the Hölder inequality for Lorentz spaces, we show

\[
|I_3| \leq \frac{C}{r - \rho} \int_{B(r) \setminus B(\frac{3r}{4})} \left|\nabla \varphi\right|^{1-\delta} \left|\chi\right|^\delta \left|\chi\right| \left|\chi\right| \, dx
\]

\[
\leq \frac{C}{r - \rho} \left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

\[
\times \left(\left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

\[
\leq \frac{C}{r - \rho} R^{3\beta} \left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

\[
\times \left(\left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

Using the inequality

\[
\left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \leq C \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))},
\]

and by Gagliardo-Nireberg-Sobolev inequality give

\[
|I_3| \leq \frac{C}{r - \rho} R^{3\beta} \left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

\[
\leq \frac{1}{8} \int_{B(r) \setminus B(\frac{3r}{4})} \left|\nabla \varphi\right|^2 \, dx + C(\delta) \left(\frac{R^{3\beta}}{r - \rho} \left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

\[
|I_4| = \left|\int_{B(r) \setminus B(\frac{3r}{4})} (T \cdot \nabla \nabla \cdot \varphi) \, dx \right| = \left| - \int_{B(r) \setminus B(\frac{3r}{4})} (T \otimes \nabla \varphi) : \nabla \varphi \, dx \right|
\]

\[
\leq \frac{C}{r - \rho} \left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]

\[
\leq \frac{1}{8} \int_{B(r) \setminus B(\frac{3r}{4})} \left|\nabla \varphi\right|^2 \, dx + C(\delta) \left(\frac{R^{3\beta}}{r - \rho} \left|\nabla \varphi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))} \left|\chi\right|_{L^{q,\infty}(B(r) \setminus B(\frac{3r}{4}))}
\]
Therefore,
\[
\int_{B(\varrho)} |\nabla v|^2 \, dx \leq 2 \frac{C}{r - \varrho} \left( \int_{B(r)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx \right)^{\frac{1}{2}} + \frac{1}{4} \int_{B(r)} |\nabla v|^2 \, dx \\
+ C(\delta) \left( \frac{R^{3\beta}}{r - \varrho} \|v\|_{L^\infty(B(R) \setminus B(\frac{R}{2}))} \|T\|_{L^\infty(B(R) \setminus B(\frac{R}{2}))} \right)^{\frac{2}{\beta}}
\]
\[
\leq \frac{1}{2} \int_{B(r)} |\nabla v|^2 \, dx + C \frac{(r - \varrho)^2}{(r - \varrho)^2} \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx \\
+ C(\delta) \left( \frac{R^{3\beta}}{r - \varrho} \|v\|_{L^\infty(B(R) \setminus B(\frac{R}{2}))} \|T\|_{L^\infty(B(R) \setminus B(\frac{R}{2}))} \right)^{\frac{2}{\beta}},
\]
for any \(3R^4 \leq \varrho < r \leq R\). The following Caccioppoli type inequality can be obtained by the standard iterative arguments.
\[
\int_{B(\frac{R}{2})} |\nabla v|^2 \, dx \leq \frac{C}{R^2} \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx + D_2.
\]
We complete the proof of Proposition 3.1.

**Proposition 3.2.** Let \((u, H)\) be a smooth solution to (1.2) satisfying \(\nabla H \in L^2(\mathbb{R}^3)\) and \(q > 3, 3 \leq \ell \leq \infty\). Then the following Caccioppoli type inequalities hold:
\[
\int_{B(\frac{R}{2})} (|\nabla u|^2 + |\nabla H|^2) \, dx \leq \frac{C}{R^2} \| (u, H) \|^2_{L^2(B(R) \setminus B(\frac{R}{2}))} + \frac{C}{R^2} \| u \|^2_{L^2(B(R) \setminus B(\frac{R}{2}))} \\
+ \frac{C}{R} \| u \|^2_{L^2(B(R) \setminus B(\frac{R}{2}))} + D_3,
\]
where \(D_3 := C(q, \ell) R^{2 - \frac{q}{2}} \| u \|^3_{L^q(B(R) \setminus B(\frac{R}{2}))}\).

**Proof.** In order to prove our result, we firstly recall the following two fundamental identities in vector analysis
\[
f \times (\nabla \times f) = \frac{1}{2} \nabla |f|^2 - (f \cdot \nabla) f, \\
\nabla \times (f \times g) = (g \cdot \nabla) f - (f \cdot \nabla) g + f \text{ div } g - g \text{ div } f.
\]
By using above vector identity, system (1.2) leads to
\[
\begin{align*}
\begin{cases}
u \cdot \nabla u - \Delta u + \nabla p &= -\frac{1}{2} \nabla |H|^2 + H \cdot \nabla H, \\
\nabla \times ((\nabla \times H) \times H) &= \nabla \cdot \nabla H - H \cdot \nabla u = \Delta H, \\
\text{div } u &= 0, \\
\text{div } H &= 0.
\end{cases}
\end{align*}
\]

Given \(R > 0\), fix numbers \(\varrho\) and \(r\) so that \(3R^4 \leq \varrho < r \leq R\). Now, choose a cut–off function \(\varphi \in C_0^\infty(B(R))\) satisfying the following conditions:
\[
\varphi(x) = \begin{cases} 1, & \text{if } x \in B(\varrho) \\
0, & \text{if } x \in B(r)^c
\end{cases}
\]
\(0 \leq \varphi \leq 1\) and \(|\nabla \varphi(x)| \leq \frac{C}{(r - \varrho)}\).
Consider the following Dirichlet problem
\[
\begin{cases}
\Delta \psi = \text{div}(\varphi u) & \text{in } B(R) \setminus B(\frac{2R}{3}), \\
\psi = 0 & \text{on } \partial B(R) \cup \partial B(\frac{2R}{3}).
\end{cases}
\]

From the standard elliptic equations theory, there is a unique \( \psi \in W^{1,s}_0(B(R) \setminus B(\frac{2R}{3})) \cap W^{2,s}_0(B(R) \setminus B(\frac{2R}{3})) \) solving this Dirichlet problem. Therefore \( w = \nabla \psi \in W^{1,s}_0(B(R) \setminus B(\frac{2R}{3})) \) such that \( \text{div } w = \text{div}(\varphi u) = \nabla \varphi \cdot u \). Applying Lemma 2.1, we can deduce the following inequality.

\[
\int_{B(R) \setminus B(\frac{2R}{3})} |\nabla w|^s \, dx \leq \frac{C}{(r - \rho)^s} \int_{B(R) \setminus B(\frac{2R}{3})} |u|^s \, dx
\]

where \( C \) is independent of \( R \) and only depends on \( s \) \((1 < s < \infty)\).

According to the general Marcinkiewicz interpolation theorem, we find

\[
\| \nabla w \|_{L^{s,\ell}(B(R) \setminus B(\frac{2R}{3}))} \leq C(\rho) \| \nabla \varphi \cdot u \|_{L^{s,\ell}(B(R) \setminus B(\frac{2R}{3}))}.
\]

Multiplying (3.13) by \((\varphi u - w)\) and (3.14) by \(\varphi H\), respectively, integrating by parts over \(B(r)\), adding the result together, we get:

\[
\int_{B(r)} \varphi |\nabla u|^2 \, dx + \int_{B(r)} \varphi |\nabla H|^2 \, dx
\]

\[
= -\int_{B(r)} \nabla u : (u \otimes \nabla \varphi) \, dx - \int_{B(r)} \nabla H : (H \otimes \nabla \varphi) \, dx + \int_{B(r)} \nabla u : \nabla w \, dx
\]

\[
- \int_{B(r)} (u \cdot \nabla u) \cdot (\varphi u - w) \, dx - \int_{B(r)} \nabla (p + \frac{1}{2}|H|^2) \cdot (\varphi u - w) \, dx
\]

\[
+ \int_{B(r)} (H \cdot \nabla H) \cdot (\varphi u - w) + (H \cdot \nabla u) \cdot \varphi H \, dx
\]

\[
- \int_{B(r)} \nabla \times ((\nabla \times H) \times H) \cdot \varphi H \, dx - \int_{B(r)} (u \cdot \nabla H) \cdot \varphi H \, dx
\]

\[
= \sum_{i=1}^{8} II_i.
\]

Similar to the treatment of (3.7), \(II_1\) and \(II_2\) can be estimated as

\[
|II_1| = \left| \int_{B(r)} \nabla u : (u \otimes \nabla \varphi) \, dx \right|
\]

\[
\leq C \left( \int_{B(r)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(r) \setminus B(\rho)} |\nabla \varphi \otimes u|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C}{r - \rho} \| \nabla u \|_{L^2(B(r))} \| u \|_{L^2(B(R) \setminus B(\frac{2R}{3}))}
\]

and

\[
|II_2| \leq \frac{C}{r - \rho} \| \nabla H \|_{L^2(B(r))} \| H \|_{L^2(B(R) \setminus B(\frac{2R}{3}))}.
\]

By (3.15), we deduce

\[
|II_3| = \left| \int_{B(r)} \nabla w : \nabla u \, dx \right|
\]

\[
\leq C \left( \int_{B(r)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(r) \setminus B(\frac{2R}{3})} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C}{r - \rho} \| \nabla u \|_{L^2(B(r))} \| u \|_{L^2(B(R) \setminus B(\frac{2R}{3}))}.
\]
For $II_4$, we have

$$II_4 = \int_{B(r)} -u \cdot \nabla u \cdot \varphi u \, dx + \int_{B(r)} u \cdot \nabla u \cdot w \, dx = II_{41} + II_{42}.$$ 

Using the Hölder inequality in Lorentz spaces for $II_{41}$, assuming that $q > 3$ and $\ell \geq 3$, we have

$$|II_{41}| = \left| \frac{1}{2} \int_{B(r)} u|u|^2 \cdot \nabla \varphi \, dx \right|
\leq \frac{C}{(r - \varrho)} \int_{B(r) \setminus B(\varrho)} |u|^3 \, dx
\leq \frac{C}{(r - \varrho)} \|u\|_{L^q,\ell(B(r) \setminus B(\varrho))}^3 \|\nabla \varphi\|_{L^{q,\ell}(B(r) \setminus B(\varrho))} \leq \frac{C(q, \ell)}{(r - \varrho)} R^{3 - \frac{2}{q}} \|u\|_{L^q,\ell(B(R) \setminus B(\frac{R}{2}))}^3. $$

Thanks to (3.16), $II_{42}$ can be evaluated as follow:

$$|I_{42}| = \left| \int_{B(r)} (u \cdot \nabla u) \cdot w \, dx \right| = \left| \int_{B(r) \setminus B(\frac{R}{2})} (u \otimes u) : \nabla w \, dx \right|
\leq C \|u\|_{L^q,\ell(B(r) \setminus B(\frac{R}{2}))}^2 \|\nabla w\|_{L^{q,\ell}(B(r) \setminus B(\frac{R}{2}))} \times \|1\|_{L^{q,\ell}(B(r) \setminus B(\frac{R}{2}))}
\leq \frac{C}{(r - \varrho)} \|u\|_{L^q,\ell(B(R) \setminus B(\frac{R}{2}))}^3 \|\nabla w\|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))}
\leq \frac{C(q, \ell)}{(r - \varrho)} R^{3 - \frac{2}{q}} \|u\|_{L^q,\ell(B(R) \setminus B(\frac{R}{2}))}^3. $$

For $II_5$, since $\varphi u - w$ is divergence-free, we see that

$$II_5 = -\int_{B(r)} \nabla (p + \frac{1}{2}|H|^2) \cdot (\varphi u - w) \, dx = \int_{B(r)} (p + \frac{1}{2}|H|^2) \text{div}(\varphi u - w) \, dx = 0.$$

The Hölder inequality and Lemma 2.3 imply

$$|II_6| = \left| \int_{B(r)} ((H \cdot \nabla H) \cdot (\varphi u) + (H \cdot \nabla u) \cdot (\varphi H) - (H \cdot \nabla H) \cdot w) \, dx \right|
= \left| \int_{B(r)} (\varphi H \cdot \nabla (H \cdot u) - (H \cdot \nabla H) \cdot w) \, dx \right|
\leq \int_{B(r)} |H|^2 |u| |\nabla \varphi| + |H|^2 |\nabla w| \, dx
\leq \frac{C}{(r - \varrho)} \|H\|_{L^{q,\ell}(B(r) \setminus B(\frac{R}{2}))}^2 \|u\|_{L^q(B(r) \setminus B(\frac{R}{2}))} + |H|^2 \|\nabla w\|_{L^2(B(r) \setminus B(\frac{R}{2}))}
\leq \frac{C}{(r - \varrho)} \|\nabla H\|_{L^2(B(r) \setminus B(\frac{R}{2}))}^2 \|u\|_{L^q(B(r) \setminus B(\frac{R}{2}))} + \frac{C}{(r - \varrho)^2} \|\nabla H\|_{L^2(B(r))}^2 \|u\|_{L^q(B(r))}^2.$$
Similarly, we obtain
\[
|II_7| = \left| \int_{B(r)} \nabla \times ((\nabla \times \mathbf{H}) \times \mathbf{H}) \cdot \varphi \mathbf{H} \, dx \right|
\]
\[
= \left| \int_{B(r)} ((\nabla \times \mathbf{H}) \times \mathbf{H}) \cdot (\mathbf{H} \times \nabla \varphi) \, dx \right|
\]
\[
(3.24)
\]
\[
\leq \int_{B(r)} |\nabla \mathbf{H}| |\mathbf{H}|^2 |\nabla \varphi| \, dx
\]
\[
\leq \frac{C}{(r-q)} |\nabla \mathbf{H}|_{L^2(B(r) \setminus B(q))}^2 |\mathbf{H}|_{L^2(B(R) \setminus B(q))}^2
\]
\[
\leq \frac{C}{(r-q)} |\nabla \mathbf{H}|_{L^2(\mathbb{R}^3)}^3
\]

and
\[
|II_8| = \left| \int_{B(r)} u_i \partial_i H_j H_j \varphi \, dx \right| = \left| \int_{B(r)} u_i \frac{1}{2} \partial_i |H_j|^2 \varphi \, dx \right| = \left| \int_{B(r)} \frac{1}{2} |\mathbf{H}|^2 u \cdot \nabla \varphi \, dx \right|
\]
\[
(3.25)
\]
\[
\leq \frac{C}{(r-q)} |\mathbf{H}|_{L^2(B(R) \setminus B(q))}^2 |u|_{L^2(B(R) \setminus B(q))}^2
\]
\[
\leq \frac{C}{(r-q)} |\nabla \mathbf{H}|_{L^2(\mathbb{R}^3)}^2 |u|_{L^2(B(R) \setminus B(q))}^3
\]

Thus, inserting (3.18)–(3.25) into (3.17), noting \( \nabla \mathbf{H} \in L^2(\mathbb{R}^3) \) leads to
\[
\int_{B(q)} (|\nabla u|^2 + |\nabla H|^2) \, dx \leq \int_{B(r)} \varphi (|\nabla u|^2 + |\nabla H|^2) \, dx
\]
\[
\leq \frac{C}{r-q} \left( \int_{B(r)} \left| \nabla u \right|^2 \, dx \right)^\frac{3}{2} \left( \int_{B(r) \setminus B(q)} \left| u \right|^2 \, dx \right)^\frac{1}{2}
\]
\[
+ \frac{C}{r-q} \left( \int_{B(r)} |\nabla H|^2 \, dx \right)^\frac{3}{4} \left( \int_{B(r) \setminus B(q)} |H|^2 \, dx \right)^\frac{1}{4} + \frac{C}{r-q} \left( \int_{B(r) \setminus B(q)} |u|^2 \, dx \right)^\frac{3}{4}
\]
\[
+ \frac{C}{r-q} \left( \int_{B(r) \setminus B(q)} |u|^2 \, dx \right)^\frac{1}{4} + \frac{C}{r-q} \left( \int_{B(r) \setminus B(q)} |H|^2 \, dx \right)^\frac{3}{4}
\]
\[
\leq \frac{1}{2} \int_{B(r)} |\nabla u|^2 \, dx + \frac{C}{r-q} \int_{B(r) \setminus B(q)} |u|^2 \, dx + \frac{1}{2} \int_{B(r)} |\nabla H|^2 \, dx
\]
\[
+ \frac{C}{r-q} \int_{B(r) \setminus B(q)} |H|^2 \, dx + \frac{C}{r-q} \left( \int_{B(r) \setminus B(q)} |u|^2 \, dx \right)^\frac{3}{4}
\]
\[
+ \frac{C}{r-q} \left( \int_{B(r) \setminus B(q)} |u|^2 \, dx \right)^\frac{1}{4} + \frac{C}{r-q} \left( \int_{B(r) \setminus B(q)} |H|^2 \, dx \right)^\frac{3}{4}
\]

Repeating the same arguments in two ball \( B(q) \) and \( B(\frac{3R}{4}) \) with \( \frac{R}{2} < \frac{3R}{4} < q \), we find
\[
\int_{B(\frac{3R}{4})} (|\nabla u|^2 + |\nabla H|^2) \, dx
\]
\[
\leq \frac{1}{2} \int_{B(q)} (|\nabla u|^2 + |\nabla H|^2) \, dx + \frac{C}{R^2} \int_{B(R) \setminus B(q)} (|u|^2 + |H|^2) \, dx
\]
\[
+ \frac{C}{R} \left( \int_{B(R) \setminus B(q)} |u|^2 \, dx \right)^\frac{3}{4} + \frac{C}{R^2} \left( \int_{B(R) \setminus B(q)} |u|^2 \, dx \right)^\frac{1}{4}
\]
\[
+ \frac{C}{\left( \frac{3R}{4} - q \right)} R^{3-\frac{2}{q}} \left| u \right|_{L^{\frac{3q}{q-2}}(B(R) \setminus B(q))}^3 + \frac{C}{R}.
\]
As a result
\[ \int_{B(\frac{R}{2})} (|\nabla u|^2 + |\nabla H|^2) \, dx \leq \frac{C}{R^2} \| (u, H) \|_{L^2(B(R) \setminus B(\frac{R}{2}))}^2 + \frac{C}{R^2} \| u \|_{L^2(B(R) \setminus B(\frac{R}{2}))}^2 
\]
\[ + C \| u \|_{L^2(B(R) \setminus B(\frac{R}{2}))} + D_3, \]
which yields the inequality (3.13). \qed

4. Proof of Theorem 1.1

With Proposition 3.1 in hand, we are now ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1 (i).** It is easy to check that, for \( q > 2 \), the following estimate is valid:
\[ R^{-2} \left( \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx \right) \leq R^{-2} \| \frac{2q}{2q-2} \|_{L^2(B(R) \setminus B(\frac{R}{2}))}^2 \| v \|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))}^2 
\]
\[ \leq C(q, \ell) R^{1-\frac{6}{q}} \| v \|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))}^2 
\]
\[ = C(q, \ell) R^{1-2\gamma} M^2_{\gamma,q,\ell}((u, H), R). \]

Note that
\[ D_1 \leq C(q, \ell) R^{2-3\gamma} M^3_{\gamma,q,\ell}((u, H), R). \]

By the condition (1.3) and (1.4), we find
\[ \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \leq \begin{cases} 
C(q, \ell) \liminf_{R \to \infty} \left( R^{2-3\gamma} M^3_{\gamma,q,\ell}((u, H), R) \right) & \text{if } \gamma > \frac{2}{3}, \\
C(q, \ell) \liminf_{R \to \infty} M^3_{\gamma,q,\ell}((u, H), R) < \int_{\mathbb{R}^3} |\nabla v|^2 \, dx & \text{if } \gamma = \frac{2}{3}. 
\end{cases} \]

Then \( v \equiv 0 \). Hence, \( u \equiv -H \).

Substituting this relation into (1.1) and (1.3), we know that
\[ \begin{cases} 
\Delta u = 0, \\
\text{div } u = 0. 
\end{cases} \]

As before, we can also find a \( w \in W^{1,\infty}_0(B(r) \setminus B(\frac{2r}{3})) \) such that \( \text{div } w = \text{div}(\varphi u) \). Here \( \varphi \) is the same cut-off function that we used in the proof of the Proposition 3.1. Testing (1.1) with \( \varphi u - w \), and the integration by parts find the following identity
\[ \int_{B(r)} \varphi \nabla u \cdot \nabla w \, dx = - \int_{B(r)} \nabla u : (\nabla \varphi \otimes u) \, dx + \int_{B(r)} \nabla u : \nabla w \, dx. \]

Thus, once again we obtain
\[ \int_{B(\frac{R}{2})} |\nabla u|^2 \, dx \leq CR^{-2} \int_{B(R) \setminus B(\frac{R}{2})} |u|^2 \, dx \]
\[ \leq CR^{-2} \| v \|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))}^2 \]
\[ \leq C(q, \ell) R^{1-\frac{6}{q}} \| u \|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))}^2. \]

As a result, let \( R \to \infty \), we recover \( u \equiv 0 \). The proof of Theorem 1.1 (i) is completed.

**Proof of Theorem 1.1 (ii)** Based on Proposition 3.1 (ii), using Hölder’s inequality, we have
\[ R^{-2} \int_{B(R) \setminus B(\frac{R}{2})} |v|^2 \, dx \leq R^{-2} \| v \|_{L^{q,\infty}(B(R) \setminus B(\frac{R}{2}))}^2 \| 1 \|_{L^{\frac{2q}{2q-2}}(B(R) \setminus B(\frac{R}{2}))}^2 \]
\[ \leq R^{\frac{1-2}{q}} \| v \|_{L^{q,\infty}(B(R) \setminus B(\frac{R}{2}))}^2 \]
\[ \leq R^{\frac{1-2}{q}} (R^{\frac{1}{q}+rac{1}{2}-\gamma} \| (u, H) \|_{L^{q,\infty}(B(R) \setminus B(\frac{R}{2}))}^2)^2 \]
\[ = R^{\frac{1-2}{q}} (R^{\frac{1}{q}+rac{1}{2}-\gamma} M^2_{\gamma,q,\infty}((u, H), R))^2 \]
and
\[ D_2 \leq C(\delta) \left( M_{\gamma, q, \infty}^2(v, R) M_{\gamma, q, \infty}(T, R) R^{3\beta - 1+ (\gamma - \frac{2}{3})(3-\delta)} \right)^{\frac{2}{z - \delta}} \]
\[ \leq C(\delta) \left( M_{\gamma, q, \infty}((u, H), R) M_{\gamma, q, \infty}((u, H), R) R^{2 - \frac{2}{3} - \gamma (3 - \delta)} \right)^{\frac{2}{z - \delta}} \]
\[ \leq C(\delta) \left( M_{\gamma, q, \infty}^3 R^{2 - \frac{2}{3} - \gamma (3 - \delta)} \right)^{\frac{2}{z - \delta}}. \]
Hence
\[ \int_{B(\frac{R}{2})} |\nabla v|^2 \, dx \leq CR^{1 - 2\gamma} M_{\gamma, q, \infty}^2 ((u, H), R) + C(\delta) \left( M_{\gamma, q, \infty}^3 R^{2 - \frac{2}{3} - \gamma (3 - \delta)} \right)^{\frac{2}{z - \delta}}. \]

Now, for any given \( q \in \left( \frac{12}{5}, 3 \right) \), we can find \( q_1 \) satisfying the following relationship
\[ q > q_1 > \frac{12}{5}, \quad \gamma > \frac{1}{3} + \frac{1}{q_1} > \frac{1}{3} + \frac{1}{q}. \]
Given \( q_1 \), there is a real number \( \delta \in (0, 1) \) such that
\[ q_1 = \frac{6(3 - \delta)}{6 - \delta} < q. \]
Noticing that
\[ 2 - \frac{\delta}{2} - \gamma (3 - \delta) = 2 - \frac{3(3 - \delta)}{6 - q_1} = \gamma (3 - \frac{6(3 - q_1)}{6 - q_1}) = \frac{3 + q_1 - 3q_1 \gamma}{6 - q_1} = \frac{3q_1 (\frac{1}{3} + \frac{1}{q} - \gamma)}{6 - q_1} < 0, \]
we have \( v \equiv 0 \) via letting \( R \to \infty \). Hence, once again we deduce \( u \equiv -H \).

Now, our goal is to prove \( u \equiv 0 \). Using the relation \( u = -H \) as we did in the proof Theorem 1.1 (i) we can find \( \|f\| \) and the Caccioppoli inequality
\[ \int_{B(\frac{R}{2})} |\nabla u|^2 \, dx \leq \frac{C}{R^2} \int_{B(R) \setminus B(\frac{R}{2})} |u|^2 \, dx. \]
Therefore
\[ \int_{B(\frac{R}{2})} |\nabla u|^2 \, dx \leq CR^{1 - \frac{q}{3}} \|u\|^2_{L^{q, \infty}(B(R) \setminus B(\frac{R}{2}))}. \]
Considering \( B(\frac{R}{2}) \), \( u \equiv 0 \) can be yielded by passing \( R \to \infty \). We complete the proof of Theorem 1.1 \( \square \)

5. PROOF OF THEOREM 1.2

Proof of Theorem 1.2 Let \( C(R) := \{ x = (x', x_3) \in \mathbb{R}^3 \mid |x'| \leq R, |x_3| \leq R \} \), which is the cylindrical region.

Thank to the Caccioppoli type inequality (3.2), we have
\[ \int_{C(\frac{\sqrt{2}R}{2})} |\nabla v|^2 \, dx \leq CR^{-2} \int_{C(R) \setminus C(\frac{\sqrt{2}R}{2})} |v|^2 \, dx \]
\[ + C(\delta) \left( R^{2 - \frac{q - \frac{3\delta}{q}}{q - \frac{3\delta}{q}} - \frac{4}{q}} \|T\|^2_{L^{q, \infty}(C(R) \setminus C(\frac{\sqrt{2}R}{2}))} \|v\|^2_{L^{q, \infty}(C(R) \setminus C(\frac{\sqrt{2}R}{2}))} \right)^{\frac{2}{z - \delta}} \]
\[ \leq C\|v\|^2_{L^{q}(C(R))} R^{1 - \frac{q}{3}} + C(\delta, q) (R^{2 - \frac{q - \frac{3\delta}{q}}{q} - \frac{4}{q}} \|T\|^2_{L^{q}(C(R))} \|v\|^{2 - \frac{q}{3}}_{L^{q}(C(R))})^{\frac{2}{z - \delta}} \]
for \( \frac{12}{5} < q < 3 \), where we used the fact that \( B(R) \subset C(R) \subset B(\sqrt{2}R) \) and the property
\[ \|v\|_{L^{q, \infty}(\Omega)} \leq C(q, \ell)\|v\|_{L^{q, \ell}(\Omega)}. \]
Introducing the polar coordinates, the decay assumption (1.6) yields that
\[
\|v\|_{L^q(C(R))} = \left( \int_{-R}^{R} \int_{|x'|<R} |v(x)|^q \, dx' \, dx_3 \right)^{\frac{1}{q}} \\
\leq C \left( \int_{-R}^{R} \int_{|x'|<R} \frac{1}{(1+|x'|)^\mu q} \, dx' \, dx_3 \right)^{\frac{1}{q}} \\
= C(2R)^{\frac{1}{q}} \left( \int_0^{2\pi} \int_0^R \frac{1}{(1+\rho)^\mu q} \, \rho \, d\rho \, d\theta \right)^{\frac{1}{q}}.
\]
For \(\mu q > 2\), we have
\[
\|v\|_{L^q(C(R))} \leq C(4\pi R)^{\frac{1}{q}} \left( \int_0^R (1+\rho)^{1-\mu q} \, d\rho \right)^{\frac{1}{q}} \leq C(\mu, q) R^{\frac{2}{q}}.
\]
Similarly,
\[
\|T\|_{L^q(C(R))} \leq C(\mu, q) R^{\frac{2}{q}}.
\]
Combining the above estimates together, we obtain
\[
(5.1) \quad \int_{C(\frac{\sqrt{2}}{4}R)} |\nabla v|^2 \, dx \leq C(\mu, q) R^{1-\frac{2}{q}} + C(\delta, \mu, q) R^{\left(2-\frac{6-2\delta}{2} - \frac{6-2\delta}{q}\right) \frac{\delta}{2-\delta}}.
\]
For fixed \(\mu > \frac{2}{3}\), there is a constant \(\delta \in (0, 1)\) such that
\[
\frac{2}{\mu} < 4 \frac{3-\delta}{4-\delta}
\]
Since \(\delta > 0\), we know \(6 (4-\delta) < 4 (6-\delta)\). Then we let
\[
q := \frac{1}{2} \left( \max \left\{ \frac{6-\delta}{6-\delta}, \frac{2}{\mu} \right\} + 4 \frac{3-\delta}{4-\delta} \right)
\]
It is easily to find
\[
\frac{12}{5} < \max \left\{ \frac{6-\delta}{6-\delta}, \frac{2}{\mu} \right\} < q < 4 \frac{3-\delta}{4-\delta} < 3 \quad \text{and} \quad \mu q > 2.
\]
Then
\[
2 - \frac{\delta}{2} - \frac{6-2\delta}{q} < 0.
\]
Passing \(R \to \infty\), it follows from (5.1) that
\[
\int_{\mathbb{R}^3} |\nabla v| \, dx = 0,
\]
which implies \(v \equiv 0\). Hence \(u \equiv -H\).

Similar to the proof of Theorem 1.1, repeating above arguments yield \(u \equiv H \equiv 0\). \(\square\)

6. PROOF OF THEOREM 1.3

Before the proof of Theorem 1.3 we first notice that MHD equations satisfy the Galilean invariance (Lemma 2.2) like Navier-Stokes equations. By the method in [17], the following property with respect to Caccioppoli type inequality may be obtained (with a slight modification).

Lemma 6.1 (See [17]). Assume that smooth functions \((v, T)\) satisfy (3.5). Then exists a positive number \(C\) depending only on \(M\) and \(q\) such that
\[
\int_{B(R)} |\nabla v|^2 \, dx \leq C(M, q) R^{-2} \int_{B(2R)} |v - [v]_{B(2R)}|^2 \, dx.
\]
We are now in a position to proof Theorem 1.3.
Proof of Theorem 1.4. Hölder’s inequality gives

\[ \frac{1}{R^2} \int_{B(2R)} |\nabla v|^2 \, dx \leq C \left( \frac{1}{R^2} \int_{B(2R)} |v|^2 \, dx \right)^{\frac{1}{2}} \left( \frac{1}{R^2} \int_{B(2R)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \]

for any \( R > 0 \), where \( C \) is independent on \( R \). Let \( R \to \infty \), we can see

\[ \int_{\mathbb{R}^3} |\nabla v|^2 \, dx < \infty \]

Recall the following Poincaré’s inequality on balls

\[ \left( \int_{B(2R)} |v - [v]_{B(2R)}|^3 \, dx \right)^{\frac{1}{3}} \leq C \left( \int_{B(2R)} |\nabla v|^3 \, dx \right)^{\frac{1}{3}} \]

with a universal positive constant \( C \). Combining (6.2) and Lemma 6.1 yields

\[ \frac{1}{|B(R)|} \int_{B(R)} |\nabla v|^2 \, dx \leq C \left( \frac{1}{|B(2R)|} \int_{B(2R)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}}, \]

where \( C \) is independent of \( x_0 \) and \( R \).

Define the function \( h := |\nabla v|^\frac{2}{3} \). By (6.1), we know \( h \in L^\frac{2}{3} (\mathbb{R}^3) \). And let

\[ M_h(x_0) := \sup_{R>0} \left( \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} h(x) \, dx \right), \]

be the Hardy-Littlewood maximal function of \( h \).

Now, we can rewrite (6.3) as

\[ M_{h^\frac{2}{3}} \leq CM_h^{\frac{2}{3}} \quad \text{in} \quad \mathbb{R}^3. \]

From the properties of the maximal function in \( L^p (\mathbb{R}^3) \) \((p > 1)\), we know that there is a universal constant \( C > 0 \) such that

\[ \int_{\mathbb{R}^3} M_{h^\frac{2}{3}}(x) \, dx \leq C \int_{\mathbb{R}^3} h^\frac{2}{3}(x) \, dx = C \int_{\mathbb{R}^3} |\nabla v|^2 \, dx < +\infty. \]

Hence we have already demonstrated that both \( h^\frac{2}{3} \) and its maximal function \( M_{h^\frac{2}{3}} \) are all of the class of \( L^1 (\mathbb{R}^3) \), which means \( h \equiv 0 \) (ref. [18]). Therefore once again we arrive at \( u \equiv -H \).

Then \( u = H \equiv 0 \) via repeating above proof.

Remark 6.1. With Lemma 6.1 in hand, compared with the result in [13], our result does not require \((u, H) \in BMO^{-1}(\mathbb{R}^3)\).

7. Proof of Theorem 1.4

Along with Proposition 3.1, we are now ready to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Analogously to the proof of Theorem 1.1, it is easy to check that, for \( q > 2 \), the following estimate is also valid:

\[ \frac{1}{R^2} \|(u, H)\|_{L^2(B(R) \setminus B(\frac{R}{2}))}^2 \leq \frac{1}{R^2} \|(u, H)\|_{L^2(\nabla(B(R) \setminus B(\frac{R}{2}))}^2 \leq CR^{2(\frac{1}{2} - \frac{2}{q})} \|(u, H)\|_{L^q, \ell(B(R) \setminus B(\frac{R}{2}))}^2 \]

\[ \leq C(q, \ell) R^{1 - \frac{2}{q}} \|(u, H)\|_{L^q, \ell(B(R) \setminus B(\frac{R}{2}))}^2 \]

\[ = C(q, \ell) R^{1 - 2\gamma} M_{\gamma, q, \ell}((u, H), R). \]

Notice that \( 1 - 2\gamma < 0 \) if \( \gamma > \frac{2}{3} \), by (1.9), we have

\[ \liminf_{R \to \infty} \frac{1}{R^2} \|(u, H)\|_{L^2(B(R) \setminus B(\frac{R}{2}))}^2 = 0, \]

\[ \liminf_{R \to \infty} \frac{1}{R} \|u\|_{L^2(B(R) \setminus B(\frac{R}{2}))}^2 = 0. \]
and
\[
\liminf_{R \to \infty} \frac{1}{R^2} \|u\|_{L^2(B(R) \setminus B(\frac{R}{2}))} = 0.
\]

Note that
\[
D_3 \leq C(q, \ell) R^{2-3\gamma} M^{3}_{\gamma, q, \ell}((u, H), R).
\]

By the condition (1.9) and (1.10), we find
\[
\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla H|^2) \, dx \leq \begin{cases} 
C \liminf_{R \to \infty} (R^{2-3\gamma} M^{3}_{\gamma, q, \ell}((u, H), R)) & \gamma > \frac{2}{3}, \\
C \liminf_{R \to \infty} M^{3}_{\gamma, q, \ell}((u, H), R) & \gamma = \frac{2}{3}, \\
\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla H|^2) \, dx & \gamma = \frac{2}{3},
\end{cases}
\]
where \(C = C(q, \ell)\). Hence \(u = H \equiv 0\).

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