Abstract

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions. Define $X^+(t)$ and $Y^-(t)$ as the trajectories of the processes $X(t)$ and $Y(t)$ pushed upwards and, respectively, downwards by $B(t)$, according to Skorohod-reflection. In the recent paper [8], Jon Warren proves inter alia that $Z(t) := X^+(t) - Y^-(t)$ is a three-dimensional Bessel-process. In this note, we present an alternative, elementary proof of this fact.

1 Introduction

The study of 1d Brownian trajectories pushed up or down by Skorohod-reflection on some other Brownian trajectories (running backwards in time) was initiated in [5] and motivated in [7] by the construction of the object what is today called the Brownian Web, see [3]. It turns out that these Brownian paths, reflected on one another, have very interesting, sometimes surprising properties. For further studies of Skorohod-reflection of Brownian paths on one another see also [6], [1], [8] etc. In particular, in [8] Warren considers two interlaced families of Brownian paths with paths belonging to the second family reflected off paths belonging to the first (in Skorohod’s sense) and derives a determinantal formula for the distribution of coalescing Brownian motions.

A particular case of Warren’s formula is the following: fix a Brownian path and let two other Brownian paths be pushed upwards and respectively downwards by Skorohod-reflection on the trajectory of the first one. The difference of the last two will be a
three-dimensional Bessel-process. In the present note, we give an alternative, elementary proof of this fact.

1.1 Skorohod-reflection

Let $T \in (0, \infty)$ and $b, x : [0, T) \to \mathbb{R}$ be continuous functions. Assume $x(0) \geq b(0)$. The construction of the following proposition is due to Skorohod. Its proof can be found either in [4] (see Lemma 2.1 in Chapter VI) or in [5] (see Lemma 2 in Section 2.1).

**Proposition 1.** (1) There exists a unique continuous function $x_{b_1} : [0, T) \to \mathbb{R}$ with the following properties

- The function $x_{b_1} - b$ is non-negative.
- The function $x_{b_1} - x$ is non-decreasing.
- The function $x_{b_1} - x$ increases only when $x_{b_1} = b$. That is

$$
\int_0^T \mathbb{1}\{x_{b_1}(t) \neq b(t)\} \, d(x_{b_1}(t) - x(t)) = 0.
$$

(2) The function $t \mapsto x_{b_1}(t)$ is given by the construction

$$
x_{b_1}(t) = x(t) + \sup_{0 \leq s \leq t} (x(s) - b(s))_-.\n$$

(3) The map $C([0, T)) \times C([0, T)) \ni (b(\cdot), x(\cdot)) \mapsto (b(\cdot), x_{b_1}(\cdot)) \in C([0, T)) \times C([0, T))$ is continuous in supremum distance.

We call the function $t \mapsto x_{b_1}(t)$ the upwards Skorohod-reflection of $x(\cdot)$ on $b(\cdot)$. As it is remarked in [5], the term Skorohod-pushup of $x(\cdot)$ by $b(\cdot)$ would be more adequate. Skorohod-reflection on paths $b(t) = \text{const.}$ plays a fundamental role in the proper formulation and proof of Tanaka’s formula, see Chapter VI of [4].

The downwards Skorohod-reflection or Skorohod-pushdown is defined for continuous functions $b, y : [0, T) \mapsto \mathbb{R}$ with $y(0) \leq b(0)$ by

$$
y_{b_1} := -((-y)_{(-b)_1}), \quad y_{b_1}(t) = y(t) - \sup_{0 \leq s \leq t} (y(s) - b(s))_+.
$$

Given three continuous trajectories $b, x, y : [0, T) \to \mathbb{R}$ with $y(0) \leq b(0) \leq x(0)$, the map $C([0, T)) \times C([0, T)) \times C([0, T)) \ni (b(\cdot), x(\cdot), y(\cdot)) \mapsto (b(\cdot), x_{b_1}(\cdot), y_{b_1}(\cdot)) \in C([0, T)) \times C([0, T)) \times C([0, T))$ is clearly continuous in supremum distance.

1.2 The result

Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions starting from 0 and define
\begin{align}
X^+(t) &:= X_{B_1(t)}, \quad \hat{X}(t) := X^+ - B(t), \\
Y^-(t) &:= Y_{B_1(t)}, \quad \hat{Y}(t) := -Y^-(t) + B(t). 
\end{align}

We are interested in the difference process
\[ Z(t) := X^+(t) - Y^-(t) = \hat{X}(t) + \hat{Y}(t). \]

It is straightforward that \(2^{-1/2}\hat{X}(t)\) and \(2^{-1/2}\hat{Y}(t)\) are both standard reflected Brownian motions. They are, of course, strongly dependent.

The following fact is a particular consequence of the main results in [8]:

**Theorem.** The process \(2^{-1/2}Z(t)\) is BES\(^3\), that is standard 3d Bessel-process.

\[ dZ(t) = 2 \frac{1}{Z(t)} \, dt + \sqrt{2} \, dW(t), \quad Z(0) = 0. \]

In the next section, we present an elementary proof of this fact.

## 2 Proof

### 2.1 Discrete Skorohod-reflection

Define the following square lattices inbedded in \(\mathbb{R} \times \mathbb{R}\):

\[ \mathcal{L} := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is even}\}, \quad \mathcal{L}^* := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is odd}\}. \]

In both of the lattices, the points \((t_1, x_1)\) and \((t_2, x_2)\) are connected with an edge if and only if \(|t_1 - t_2| = |x_1 - x_2| = 1\). Note that \(\mathcal{L}\) and \(\mathcal{L}^*\) are Whitney-duals of each other.

We define the discrete analogue of the Skorohod-reflection in \(\mathcal{L}\) and \(\mathcal{L}^*\). Later on, we say that the function \(y : [0, T] \cap \mathbb{Z} \to \mathbb{Z}\) is a walk in the lattice \(\mathcal{L}\) or \(\mathcal{L}^*\) if the consecutive elements of the sequence \((0, y(0)), (1, y(1)), \ldots, (T, y(T))\) are edges in \(\mathcal{L}\) or \(\mathcal{L}^*\).

Let \(b : [0, T] \cap \mathbb{Z} \to \mathbb{Z}\) and \(x : [0, T] \cap \mathbb{Z} \to \mathbb{Z}\) be two walks in the lattices \(\mathcal{L}\) and \(\mathcal{L}^*\), respectively. Assume that \(x(0) \geq b(0)\). An analogue of Proposition 1 holds in this case, but the proof is even easier.

**Proposition 2.** (1) There is a unique walk \(x_{b1} : [0, T] \cap \mathbb{Z} \to \mathbb{Z}\) in \(\mathcal{L}^*\) with the following properties:

- The function \(x_{b1} - b\) is non-negative.
- The function \(x_{b1} - x\) is non-decreasing.
- The function \(x_{b1} - x\) increases only when \(x_{b1} = b + 1\), i.e.

\[ \sum_{t=1}^{T} \mathbb{1}\{x_{b1}(t) = b(t) > 1\} \left[ (x_{b1}(t) - x(t)) - (x_{b1}(t - 1) - x(t - 1)) \right] = 0. \]
(2) The function \( t \mapsto x_{b}^{↑}(t) \) can be expressed as
\[
x_{b}^{↑}(t) = x(t) + \sup_{s \in [0,t] \cap \mathbb{Z}} (x(s) - b(s)) - 1.
\]

We call the function \( t \mapsto x_{b}^{↑}(t) \) the discrete upwards Skorohod-reflection of \( x(\cdot) \) on \( b(\cdot) \). The discrete downwards Skorohod-reflection is defined similarly. If \( y : [0,T] \cap \mathbb{Z} \to \mathbb{Z} \) is a walk in \( \mathcal{L} \) and \( b : [0,T] \cap \mathbb{Z} \to \mathbb{Z} \) is a walk in \( \mathcal{L}^* \) with \( y(0) \leq b(0) \), then
\[
y_{b}^{↓} := -\left(-y_{(b)}\right), \quad y_{b}^{↓}(t) = y(t) - \sup_{s \in [0,t] \cap \mathbb{Z}} (y(s) - b(s)) - 1.
\]

In this paper, we use the same notation for the discrete Skorohod-reflection and the continuous one (defined as Skorohod-reflection), but it will be always clear from the context which is the adequate one.

2.2 Approximation of reflected Brownian motions

Let \( M(t) \) be a random walk on the lattice \( \mathcal{L} \) with jumps from \((t, x)\) to \((t + 1, x + 1)\) or \((t + 1, x - 1)\) with probability \( 1/2 - 1/2 \) and \( M(0) = 0 \). We define the random walks \( U(t) \) and \( L(t) \) on \( \mathcal{L}^* \) with the same transition probabilities, which are independent of each other and of \( M(t) \). The initial values are \( U(0) = 1 \) and \( L(0) = -1 \). We extend our walks for non-integral values of \( t \) linearly, so the trajectories are continuous.

Since all these three random walks have steps with mean 0 and variance 1, it follows that
\[
\left( \frac{M(nt)}{\sqrt{n}}, \frac{U(nt)}{\sqrt{n}}, \frac{L(nt)}{\sqrt{n}} \right) \xrightarrow{d} (B(t), X(t), Y(t)) \quad (n \to \infty). \tag{5}
\]

We established earlier that the map \( (b(\cdot), x(\cdot), y(\cdot)) \mapsto (b(\cdot), x_{b}^{↑}(\cdot), y_{b}^{↓}(\cdot)) \) is continuous in supremum distance. From Donsker’s invariance principle (see e.g. Section 7.6 of [2]), we conclude that
\[
\left( \frac{M(nt)}{\sqrt{n}}, \frac{U_{M(n)}^{↑}(nt)}{\sqrt{n}}, \frac{L_{M(n)}^{↓}(nt)}{\sqrt{n}} \right) \xrightarrow{d} (B(t), X^{+}(t), Y^{-}(t)) \tag{6}
\]
in distribution as \( n \to \infty \). Note that we can use the discrete Skorohod-reflection to transform \( U \) and \( L \), because the difference is only the addition of 1, which vanishes in the limit. At this point, it suffices to show that
\[
2^{-1/2} \frac{U_{M(n)}^{↑}(nt) - L_{M(n)}^{↓}(nt)}{\sqrt{n}}
\]
converges to a BES\(^{3}\)-process.

For \( x, y \in \mathbb{Z}^+ \), we define the stochastic matrix
\[
P_{xy} = \begin{cases} \frac{1}{2} & \text{if } y = x \\ \frac{1}{4} & \text{if } |y - x| = 1 \\ 0 & \text{otherwise} \end{cases}
\]
It is well known that if $X_n$ is a homogeneous Markov-chain with transition probabilities $(P_{xy})_{x,y \in \mathbb{Z}^+}$, then its diffusive limit is BES$^3$, i.e. for every $T > 0$ the process $\sqrt{2(n^{-1/2}X_n)}_{0 \leq t \leq T}$ converges to a 3d Bessel-process in the Skorohod-topology as $n \to \infty$. So the proof of our theorem relies on the following

**Lemma 1.** $\frac{1}{\sqrt{2}}(U_{M_1}(t) - L_{M_1}(t))$ is a Markov-chain and its transition matrix is $(P_{xy})_{x,y \in \mathbb{Z}^+}$, where $U_{M_1}$ and $L_{M_1}$ are discrete Skorohod-reflections.

### 2.3 Markov-property of the distance of the two reflected walks

We introduce a different notation for the triple $(M, U_{M_1}, L_{M_1})$, which is just a linear transformation. Let $K_n := L_{M_1}(n)$ be the position of the lower reflected walk. With the definition $D_n := \frac{1}{2}(U_{M_1}(n) - L_{M_1}(n))$, the distance of the two reflected walks is $2D_n$. $P_n := \frac{1}{2}(M(n) - L_{M_1}(n) - 1)$, which means that the position of $M$ related to the lower walk is $2P_n + 1$. The vector $(K_n, D_n, P_n)$ is clearly a Markov-chain.

We are only interested in the coordinate $D_n$, which turns out to be also Markov and to have transition matrix $(P_{xy})_{x,y \in \mathbb{Z}^+}$. To show this, we have to determine the conditional distribution of $P_n$, because in certain cases it modifies the transition rules of $D_n$.

**Lemma 2.** The following identities hold

$$\mathbb{P}(P_n = x \mid D_n^0) = \frac{1}{D_n} \mathbb{1}(x \in \{0, 1, \ldots, D_n - 1\}),$$

(7)

$$\mathbb{P}(D_{n+1} = y \mid D_n^0) = P_{D_{n}y}$$

(8)

where $D_n^0$ means the sequence of variables $D_0, \ldots, D_n$.

**Proof.** The two identities (7), respectively, (8) of the lemma are proved by a common induction on $n$. Since $D_0 = 1$ and $P_0 = 0$, the case $n = 0$ is trivial.

For the induction step, we have to enumerate the possible transitions of the Markov-chain $(K_n, D_n, P_n)$. For the sake of simplicity, we only prove for $D_n = D_{n-1} - 1$, the other cases are similar. It is easy to check that the transition $(k, d, p) \to (k + 1, d - 1, p)$ has probability $\frac{1}{8} \mathbb{1}(p \in \{0, 1, \ldots, d - 2\})$, this will be called type A events. Type B events are the transitions $(k, d, p) \to (k + 1, d - 1, p - 1)$, which happen with probability $\frac{1}{8} \mathbb{1}(p \in \{1, 2, \ldots, d - 1\})$. No other cases give $d \to d - 1$. 

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Proof of (7): Let \( x, y \in \mathbb{Z}^+ \). We suppose that \( y = D_{n-1} - 1 \).

\[
\mathbb{P} \left( P_n = x \mid D_n = y, D_0^n \right) = \sum_{z \in \mathbb{Z}} \mathbb{P} \left( P_n = x \mid P_{n-1} = z, D_n = y, D_0^{n-1} \right) \mathbb{P} \left( P_{n-1} = z \mid D_n = y, D_0^{n-1} \right)
\]

\[
= \sum_{z \in \mathbb{Z}} \frac{\mathbb{P} \left( P_n = x, D_n = y \mid P_{n-1} = z, D_0^{n-1} \right)}{\mathbb{P} \left( D_n = y \mid P_{n-1} = z, D_0^{n-1} \right)} \mathbb{P} \left( P_{n-1} = z \mid D_n = y, D_0^{n-1} \right)
\]

\[
= \sum_{z=x}^{x+1} \mathbb{P} \left( P_n = x, D_n = y \mid P_{n-1} = z, D_0^{n-1} \right) \frac{\mathbb{P} \left( P_{n-1} = z \mid D_0^{n-1} \right)}{\mathbb{P} \left( D_n = y \mid D_0^{n-1} \right)}
\]

\[
= \mathbb{P} \left( P_n = x, D_n = y \mid P_{n-1} = x, D_0^{n-1} \right) \frac{\mathbb{P} \left( P_{n-1} = x \mid D_0^{n-1} \right)}{\mathbb{P} \left( D_n = y \mid D_0^{n-1} \right)}
\]

\[
+ \mathbb{P} \left( P_n = x, D_n = y \mid P_{n-1} = x + 1, D_0^{n-1} \right) \frac{\mathbb{P} \left( P_{n-1} = x + 1 \mid D_0^{n-1} \right)}{\mathbb{P} \left( D_n = y \mid D_0^{n-1} \right)}
\]

\[
= \frac{1}{8} \mathbb{I} \left( x \in \{0, 1, \ldots, D_{n-1} - 2\} \right) \frac{1}{D_{n-1}} \mathbb{I} \left( x \in \{0, 1, \ldots, D_{n-1} - 1\} \right)
\]

\[
+ \frac{1}{8} \mathbb{I} \left( x \in \{0, 1, \ldots, D_{n-1} - 2\} \right) \frac{1}{D_{n-1}} \mathbb{I} \left( x \in \{-1, 0, \ldots, D_{n-1} - 2\} \right)
\]

\[
= \frac{1}{D_{n-1} - 1} \mathbb{I} \left( x \in \{0, \ldots, D_{n-1} - 2\} \right) = \frac{1}{y} \mathbb{I} \left( x \in \{0, 1, \ldots, y - 1\} \right).
\]

First, we used the law of total probability and the definition of conditional probability and the identity \( \mathbb{P}(E|F)/\mathbb{P}(F|E) = \mathbb{P}(E)/\mathbb{P}(F) \) on a conditional probability space. As remarked at the beginning of this proof, there are only two cases to reduce the value of \( D \), so the sum has only two terms. Then, we used both inductional hypotheses to evaluate the conditional probabilities. The remaining steps are obvious.

Proof of (8): We spell out the proof for \( D_{n+1} = D_n - 1 \), the cases \( D_{n+1} = D_n \) and \( D_{n+1} = D_n + 1 \) are similar.

\[
\mathbb{P} \left( D_{n+1} = D_n - 1 \mid D_0^n \right) = \sum_{x=0}^{D_{n-1}} \mathbb{P} \left( D_{n+1} = D_n - 1 \mid P_n = x, D_0^n \right) \mathbb{P} \left( P_n = x \mid D_0^n \right)
\]

\[
= \sum_{x=0}^{D_{n-1}} \left( \frac{1}{8} \mathbb{I} \left( x \in \{0, 1, \ldots, D_n - 2\} \right) + \frac{1}{8} \mathbb{I} \left( x \in \{1, 2, \ldots, D_n - 1\} \right) \right) \frac{1}{D_n}
\]

\[
= \frac{1}{4} \frac{D_n - 1}{D_n} = \mathbb{P}_{D_n(D_n-1)}.
\]
In the second step, only type \( A \) and \( B \) events can cause the transition \( D_{n+1} = D_n - 1 \). We applied the first part of this lemma to evaluate the second conditional probability factor.

As a consequence, we see that the distribution of \( D_{n+1} \) conditioned on \( D_n \) depends only on \( D_n \), which means that \( D_n \) is a Markov-chain with transition matrix \((P_{xy})_{xy}\). From this, the assertion of the theorem follows.

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