Abstract. We study the correspondence theory of intuitionistic modal logic in modal Fairtlough–Mendler semantics (modal FM semantics) (Fairtlough and Mendler in Inf Comput 137(1):1–33, 1997), which is the intuitionistic modal version of possibility semantics (Holliday in UC Berkeley working paper in logic and the methodology of science, 2022. http://escholarship.org/uc/item/881757qn). We identify the fragment of inductive formulas (Goranko and Vakarelov in Ann Pure Appl Logic 141(1–2):180–217, 2006) in this language and give the algorithm ALBA (Conradie and Palmigiano in Ann Pure Appl Logic 163(3):338–376, 2012) in this semantic setting. There are two major features in the paper: one is that in the expanded modal language, the nominal variables, which are interpreted as atoms in perfect Boolean algebras, complete join-prime elements in perfect distributive lattices and complete join-irreducible elements in perfect lattices, are interpreted as the refined regular open closures of singletons in the present setting, similar to the possibility semantics for classical normal modal logic (Zhao in J Logic Comput 31(2):523–572, 2021); the other feature is that we do not use conominals or diamond, which restricts the fragment of inductive formulas significantly. We prove the soundness of ALBA with respect to modal FM-frames and show that ALBA succeeds on inductive formulas, similar to existing settings like (Conradie and Palmigiano in Ann Pure Appl Logic 163(3):338–376, 2012; Zhao 2021, in: Cia-battoni, Pimentel, Queiroz (eds) Logic, language, information, and computation, Springer International Publishing, Cham, 2022).

Keywords: Possibility semantics, Fairtlough–Mendler semantics, Correspondence theory, Nucleus, Complete Heyting algebra, Intuitionistic modal logic.

1. Introduction

1.1. Possibility Semantics

Possibility semantics, which is proposed by Humberstone [22], is a generalization of possible world semantics for modal logic, based on partial possibilities instead of complete possible worlds like the ones in standard possible
world semantics. In recent years there have been a lot of studies in possibility semantics [5,11,17–19,35], to name but a few. For a comprehensive study of possibility semantics, see [20,21].

1.2. Intuitionistic Study of Possibility Semantics

In [3], Bezhanishvili and Holliday use the tools of nuclei to study the equivalence between Fairtlough–Mendler semantics (FM semantics for short) [14], Dragalin semantics [12, 13] and nuclear semantics of intuitionistic logic, which can be regarded as different ways to realize possibility semantics of intuitionistic logic. In [4], Bezhanishvili and Holliday study the different semantics of intuitionistic logic, which form a hierarchy. Among these semantics, Dragalin semantics is more like neighbourhood semantics of modal logic [30,31,33] in the sense of using a function $X \to P(P(X))$ and Beth semantics of intuitionistic logic [2] in the sense of using paths or developments, while nuclear semantics is more like algebraic semantics in the sense of using a nucleus on the Heyting algebra of upsets, and FM semantics is more like relational semantics in the sense of using two relations. In [26], Massas provides choice-free representation theorems for distributive lattice, Heyting algebras and co-Heyting algebras. In [28], Massas studies the B-frame duality, where B-frames can be seen as a generalization of posets, which play an important role in the representation theory of Heyting algebras.

1.3. Correspondence Theory for Possibility Semantics

In [36], Yamamoto studies Sahlqvist correspondence theory for full possibility frames. In [21, Theorem 7.20], Holliday shows that all inductive formulas are filter-canonical, therefore every normal modal logic axiomatized by inductive formulas is sound and complete with respect to its canonical full possibility frame. In [37], Zhao shows that inductive formulas have first-order correspondents in full possibility frames as well as in filter-descriptive possibility frames, using algebraic and algorithmic correspondence theory methods [7,8]. In this setting, the algebraic structure of regular open subsets of a given full possibility frame is a complete Boolean algebra with a complete operator, which is not necessarily an atomic Boolean algebra, therefore it is not necessarily perfect.

1.4. Intuitionistic Modal Logic

Intuitionistic modal logic studies modal logic over an intuitionistic propositional basis. In the literature, there are many different approaches to the study of intuitionistic modal logic [1,29,34]. From a relational semantic point
of view, the semantic structure used to interpret intuitionistic modal formulas are typically with two relations, one for the modality and one for the partial order of intuitionistic propositional logic. In our present work, we use a different semantics from most existing works, where we have three relations, one for the modality and two for interpreting intuitionistic propositional logic. For an intuitive interpretation of this semantics, see [4, Remark 4.29].

1.5. Our Methodology

Our aim is to see if Sahlqvist-type correspondence theory could work for semantics whose algebraic counterpart is based on locales, i.e., complete Heyting algebras, which are not necessarily perfect (where every element is join-generated by complete join-primes).

We study the correspondence theory of intuitionistic modal logic in the modal version of Fairtlough–Mendler semantics, using algorithmic correspondence theory methods¹, as explained in [7,8]. We define the class of inductive formulas for this semantics as well as the Ackermann Lemma Based Algorithm ALBA which computes the first-order correspondents of inductive formulas.

Following the methodology of [37], our semantic analysis of the modal FM-frames is based on their dual algebraic structures, which are complete Heyting algebras with complete operators (not necessarily perfect), where complete join-primes are not always available, in contrast to settings like [8]. We use the representation of complete Heyting algebras as the refined regular open subsets in Fairtlough–Mendler frames, and we identify one Heyting algebra with an operator (HAO) and one Boolean algebra with an operator (BAO) as the dual algebraic structures of a given modal FM-frame \( F \): namely, the HAO \( \mathbb{H}_{\text{RO12}} \) of refined regular open subsets of \( F \) and the BAO \( \mathbb{B}_K \) of arbitrary subsets of \( F \). Therefore, we can define a natural order-embedding \( e : \mathbb{H}_{\text{RO12}} \rightarrow \mathbb{B}_K \). It is completely meet-preserving, therefore it has a left adjoint \( c : \mathbb{B}_K \rightarrow \mathbb{H}_{\text{RO12}} \) sending a subset of the domain of \( F \) to the smallest refined regular open subset containing it. We will use \( c \) substantially in the interpretations of the expanded modal language \( L^+ \) of the algorithm ALBA, which will play an important role in the relational semantics and the refined regular open translation of the expanded modal language.

To summarize, the principles for choosing the interpretations of the expanded modal language are the following:

¹Strictly speaking, our semantics is not the original Fairtlough–Mendler semantics, but an equivalent form defined in [3]. For more discussions, see Section 2.1.
• The set of possible interpretations of nominals is join-dense in the complete algebra of possible values of propositional variables;
• The set of possible interpretations of conominals is meet-dense in the complete algebra of possible values of propositional variables (only in settings when this is possible);
• The interpretation of black connectives should be the adjoints of modal-ities in the basic propositional language;
• These interpretations should be expressible in a first-order way.

1.5.1. Structure of the Paper The paper is organized as follows: Section 2 introduces some relevant structures and notions that will be used in later sections. Section 3 presents preliminaries on modal Fairtlough–Mendler semantics. Section 4 analyzes the semantic environment for the interpretation of the expanded modal language. Section 5 introduces the expanded modal language formally as well as the refined regular open translation. Section 6 gives the syntactic definition of inductive formulas. Section 7 introduces the algorithm ALBA with an example. Section 8 gives its success proof on inductive formulas and Section 9 gives its soundness proof. Section 10 gives the conclusions.

2. Preliminaries

In this section, we introduce some topological and relational structures and the notion of nucleus, which will be useful in later sections. For more details, see [3,26,27].

2.1. Topological and Relational Structures

Definition 2.1. (Refined Bitopological Space, Definition 5.1.1 in [26]) A refined bitopological space is a triple \((X, τ_1, τ_2)\) where \(τ_1\) and \(τ_2\) are topologies on \(X\) and \(τ_1 ⊆ τ_2\).

Definition 2.2. (Refined Birelational Frame) A refined birelational frame is a tuple \((X, ≤_1, ≤_2)\) where \(≤_2 ⊆ ≤_1\) are both partial orders on \(X\).

For any poset \((X, ≤)\), define the topology \(τ_{≤}\) taking all the upsets of \((X, ≤)\) as open sets. Therefore we can identify a partial order \((X, ≤)\) with its corresponding Alexandroff topology \((X, τ)\). Since \(≤_2 ⊆ ≤_1\) iff \(τ_{≤_1} ⊆ τ_{≤_2}\), we can identify refined birelational frames with their corresponding refined bi-Alexandroff spaces.
In what follows, we will call refined birelational frames also Fairtlough-Mendler frames (FM-frames for short) [3,14]. Strictly speaking, what we call Fairtlough-Mendler frames are not exactly the same as the original definition in [14]. The definition in [14] (originally called Kripke constraint model, here we only consider the frame part) is as follows:

**Definition 2.3.** (Definition 3.1 in [14]) A Fairtlough-Mendler frame (FM-frame for short) is a tuple \( F = (X, \leq_1, \leq_2, F) \), where \( \leq_1 \) and \( \leq_2 \) are preorders (i.e., reflexive and transitive binary relations) on the set \( X \), \( \leq_2 \subseteq \leq_1 \), and \( F \subseteq X \) is an \( \leq_1 \)-upset in the sense that for any \( w, v \in X \), if \( w \in F \) and \( w \leq_1 v \) then \( v \in F \).

FM-frames are used to interpret propositional lax logic, while here we use them to interpret intuitionistic modal logic.

In [3], Bezhanishvili and Holliday further define the following constraints on FM-frames:

- An FM-frame \( F = (X, \leq_1, \leq_2, F) \) is normal if \( F = \emptyset \);
- An FM-frame \( F = (X, \leq_1, \leq_2, F) \) is partially ordered if \( \leq_1 \) and \( \leq_2 \) are partial orders.

In [3, Corollary 4.10], Bezhanishvili and Holliday prove that a poset \( P \) is a locale iff there is a partially ordered normal FM-frame \( F \) such that \( P \) is isomorphic to the algebra of fixpoints in the nuclear algebra of \( F \). Together with [3, Proposition 4.2] which says that from an FM-frame we can get an equivalent nuclear algebra whose fixpoints form a locale, we have that every FM-frame can be transformed into an equivalent partially ordered normal FM-frame \( F \), which is essentially the same as the FM-frame we define here.

### 2.2. Nucleus

In this subsection we introduce the notion of nucleus on Heyting algebras. For more details, see [23–25].

**Definition 2.4.** (Nucleus) A nucleus on a Heyting algebra \( H \) is a map \( j : H \to H \) satisfying the following conditions:

- \( j(a \land b) = j(a) \land j(b) \);
- \( a \leq j(a) \);
- \( j(j(a)) = j(a) \).

It is clear that a nucleus on \( H \) is also a closure operator, and the following properties follow from the definition:
• \(j(\top) = \top\);
• if \(a \leq b\), then \(j(a) \leq j(b)\).

We call a nucleus dense if \(j(\bot) = \bot\).

**Definition 2.5.** (Nuclear algebra) A nuclear algebra is a pair \((\mathbb{H}, j)\) where \(\mathbb{H}\) is a Heyting algebra and \(j\) is a nucleus on \(\mathbb{H}\). It is a localic nuclear algebra if \(\mathbb{H}\) is a locale, i.e., a complete Heyting algebra.

The following result is well-known, see e.g., [13, page 71].

**Theorem 2.6.** If \(\mathbb{H} = (H, \bot, \top, \land, \lor, \rightarrow, j)\) is a nuclear algebra, then \(\mathbb{H}_j = (H_j, \bot, \top, \land_j, \lor_j, \rightarrow_j)\) is a Heyting algebra where \(H_j = \{a \in H \mid a = j(a)\}\) and for \(a, b \in H_j\):

• \(\bot_j = j(\bot)\);
• \(a \land_j b = a \land b\);
• \(a \lor_j b = j(a \lor b)\);
• \(a \rightarrow_j b = a \rightarrow b\).

If \(H\) is a localic nuclear algebra, then \(H_j\) is a locale, where for \(Y \subseteq H_j\):

• \(\land_j Y = \land Y\);
• \(\lor_j Y = j(\lor Y)\).

In a given refined bitopological space \((X, \tau_1, \tau_2)\), we use \(I_i\) to denote the interior map in \(\tau_i\) and \(C_i\) to denote the closure map in \(\tau_i\). We use \(\text{RO}_{12}(X)\) to denote \(\{Y \subseteq X \mid Y = I_1 C_2(Y)\}\).

The following proposition will be useful in our semantic setting:

**Proposition 2.7.** Let \(\mathcal{F} = (X, \leq_1, \leq_2)\) be an FM-frame and \((X, \tau_1, \tau_2)\) be its corresponding refined bitopological space. Let the \(\text{RO}_{12}\) be the algebra \((\text{RO}_{12}(X), \emptyset, H, \cap, \lor_{\text{RO}_{12}}, \rightarrow_{\text{RO}_{12}})\) where \(Y \lor_{\text{RO}_{12}} Z := l_1 C_2(Y \cup Z)\), \(Y \rightarrow_{\text{RO}_{12}} Z := l_1((X - Y) \cup Z)\). Then the operator \(l_1 C_2\) is a nucleus on the complete Heyting algebra of opens in \(\tau_1\). Therefore \(\text{RO}_{12}\) is a complete Heyting algebra.

**Proof.** See [26, Theorem 5.1.2 and Corollary 5.1.3]. Here we give the proofs for the readers’ convenience.

We first show that \(l_1 C_2\) is a nucleus on the complete Heyting algebra of opens in \(\tau_1\):

• First of all, since both the \(\tau_1\)-interior map \(l_1\) and the \(\tau_2\)-closure map \(C_2\) are monotone, we have that their composition \(l_1 C_2\) is monotone, i.e. if \(U \subseteq V\) for \(U, V \in \tau_1\), then \(l_1 C_2(U) \subseteq l_1 C_2(V)\).
• $U \subseteq l_1C_2(U)$ for any $U \in \tau_1$: since $U \in \tau_1$, we have that $U \subseteq l_1(U)$. Since $U \subseteq C_2(U)$, by monotonicity of $l_1$, we have $U \subseteq l_1(U) \subseteq l_1C_2(U)$.

• $l_1C_2(U) = l_1C_2l_1C_2(U)$ for any $U \in \tau_1$: first of all, for any $U \subseteq X$, since $\tau_1 \subseteq \tau_2$, we have that $l_1(U) \subseteq l_2(U)$ and $C_2(U) \subseteq C_1(U)$. In addition, for any $U \in \tau_2$, we have $C_2l_2C_2(U) = C_2(U)$. Since $\tau_1 \subseteq \tau_2$, we have that $C_2l_2C_2(U) = C_2(U)$ holds for any $U \in \tau_1$. Therefore, by the monotonicity of $l_1$, we have that for any $U \in \tau_1$, $l_1C_2l_1C_2(U) \subseteq l_1C_2l_2C_2(U) = l_1C_2(U)$. The other direction follows from the monotonicity of $l_1C_2$.

• Then we prove that $l_1C_2(U \cap V) = l_1C_2(U) \cap l_1C_2(V)$ for any $U, V \in \tau_1$. The inclusion $l_1C_2(U \cap V) \subseteq l_1C_2(U) \cap l_1C_2(V)$ follows from the monotonicity of $l_1C_2$. For the other inclusion, since $U \in \tau_1 \subseteq \tau_2$, we have that $U \cap C_2(V) = l_2(U) \cap C_2(V) \subseteq C_2(U \cap V)$. By monotonicity of $l_1$, we have $l_1(U \cap C_2(V)) \subseteq l_1C_2(U \cap V)$. Since $l_1$ distributes over intersections and $U \in \tau_1$, we have $U \cap l_1C_2(V) = l_1(U) \cap l_1C_2(V) \subseteq l_1C_2(U \cap V)$. By applying the same method twice, we get $l_1C_2(U \cap l_1C_2(V)) \subseteq l_1C_2(l_1C_2(U \cap V)) \subseteq l_1C_2l_1C_2(U \cap V) = l_1C_2(U \cap V)$.

The definition of $\vee_{RO_{12}}$ and $\rightarrow_{RO_{12}}$ follows from Theorem 2.6.

The fact that $\mathbb{H}_{RO_{12}}$ is a complete Heyting algebra follows from that $l_1C_2$ is a nucleus on the complete Heyting algebra of opens in $\tau_1$.

3. Preliminaries on Modal Fairtlough–Mendler Semantics

In the present section we collect the preliminaries on modal Fairtlough–Mendler semantics. For more details, see e.g., [14,26,27]. The style of presentation follows [8,37].

3.1. Language

Let Prop be the set of propositional variables. The basic modal language $\mathcal{L}$ is defined as follows:

\[
\varphi:: = p \mid \bot \mid T \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid \square \varphi,
\]

where $p \in \text{Prop}$.

In the algorithm, we will use inequalities $\varphi \leq \psi$, whose truth in a model is equivalent to the global truth of $\varphi \rightarrow \psi$ in the model. We will also define quasi-inequalities $\varphi_1 \leq \psi_1 \& \ldots \& \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$, where $\&$ is the meta-level conjunction and $\Rightarrow$ is the meta-level implication. We say that a formula is pure if it does not contain occurrences of propositional variables.
3.2. Semantics

For any \( R \subseteq W \times W \), we denote \( R[X] = \{ w \in W \mid (\exists x \in X) Rxw \} \), \( R^{-1}[X] = \{ w \in W \mid (\exists x \in X) Rwx \} \), \( R[w] := R[\{ w \}] \) and \( R^{-1}[\{ w \}] := R^{-1}[\{ w \}] \), respectively.

**Definition 3.1.** (Modal FM-frames and models)

A modal FM-frame is a tuple \( \mathbb{F} = (X, \leq_1, \leq_2, R) \), where \((X, \leq_1, \leq_2)\) is an FM-frame, \( R \subseteq W \times W \) is such that \( \Box_{RO_{12}}(X) := \{ w \in W \mid R[w] \subseteq X \} \in RO_{12}(X) \) for any \( X \in RO_{12}(X) \). A modal FM-model is a pair \( \mathbb{M} = (\mathbb{F}, V) \) where \( V : \text{Prop} \rightarrow RO_{12}(X) \) is a valuation on \( \mathbb{F} \).

Given any modal FM-model \( \mathbb{M} = (X, \leq_1, \leq_2, R, V) \) and any \( w \in X \), the satisfaction relation is defined as follows:

\[
\begin{align*}
\mathbb{M}, w &\models p \quad \text{iff} \quad w \in V(p) \\
\mathbb{M}, w &\not\models \bot \quad \text{never} \\
\mathbb{M}, w &\not\models \top \quad \text{always} \\
\mathbb{M}, w &\models \varphi \land \psi \quad \text{iff} \quad \mathbb{M}, w \models \varphi \land \mathbb{M}, w \models \psi \\
\mathbb{M}, w &\models \varphi \lor \psi \quad \text{iff} \quad (\forall v \geq_1 w)(\exists u \geq_2 v)(\mathbb{M}, u \models \varphi \lor \mathbb{M}, u \models \psi) \\
\mathbb{M}, w &\models \varphi \rightarrow \psi \quad \text{iff} \quad (\forall v \geq_1 w)(\mathbb{M}, v \models \varphi \Rightarrow \mathbb{M}, v \models \psi) \\
\mathbb{M}, w &\models \Box \varphi \quad \text{iff} \quad \forall v (Rwv \Rightarrow \mathbb{M}, v \models \varphi).
\end{align*}
\]

We use \( \llbracket \varphi \rrbracket^\mathbb{M} = \{ w \in X \mid \mathbb{M}, w \models \varphi \} \) to denote the truth set of \( \varphi \) in \( \mathbb{M} \).

We say that \( \varphi \) is globally true on \( \mathbb{M} \) (notation: \( \mathbb{M} \models \varphi \)) if \( \mathbb{M}, w \models \varphi \) for all \( w \in X \). We say that \( \varphi \) is valid on \( \mathbb{F} \) (notation: \( \mathbb{F} \models \varphi \)) if \( \varphi \) is globally true on \( (\mathbb{F}, V) \) for all \( V \) on \( X \). The semantics for inequalities and quasi-inequalities is as follows:

\[
\begin{align*}
\mathbb{M} &\models \varphi \leq \psi \quad \text{iff} \quad (\text{for all } w \in X) (\mathbb{M}, w \models \varphi \Rightarrow \mathbb{M}, w \models \psi) \\
\mathbb{M} &\models \wedge_{i=1}^n (\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi \quad \text{iff} \quad (\mathbb{M} \models \varphi_i \leq \psi_i \text{ for all } i) \Rightarrow (\mathbb{M} \models \varphi \leq \psi).
\end{align*}
\]

An inequality (resp. quasi-inequality) is valid on \( \mathbb{F} \) if it is globally true on \( (\mathbb{F}, V) \) for all \( V \).

The next proposition is obvious, which justifies our usage of \( \leq \) in place of \( \rightarrow \) on the global truth and validity level:

**Proposition 3.2.** For any modal FM-model \( \mathbb{M} = (X, \leq_1, \leq_2, R, V) \) and any \( w \in X \),

\[
\begin{align*}
\mathbb{M} &\models \varphi \rightarrow \psi \quad \text{iff} \quad \llbracket \varphi \rrbracket^\mathbb{M} \subseteq \llbracket \psi \rrbracket^\mathbb{M} \quad \text{iff} \quad \mathbb{M} \models \varphi \leq \psi \\
\mathbb{F} &\models \varphi \rightarrow \psi \quad \text{iff} \quad \mathbb{F} \models \varphi \leq \psi.
\end{align*}
\]
4. Semantic Environment of our Setting

In this section, we use some algebraic structures to give the semantic definition of how the expanded modal language used in the algorithm will be defined. The style of presentation follows [37].

4.1. The Heyting Algebra with Operator $\mathbb{H}_{\text{RO}_{12}}$

**Definition 4.1.** (Heyting algebra with operator) A Heyting algebra with operator (HAO) is a tuple $\mathbb{H} = (H, \bot, \top, \land, \lor, \rightarrow, \Box)$, where the $\Box$-free part is a Heyting algebra and $\Box$ is a unary operation such that $\Box \top = \top$ and $\Box(a \land b) = \Box a \land \Box b$ for any $a, b \in H$. An HAO is complete if its Heyting algebra part is complete. An HAO is completely multiplicative if $\Box$ preserves all existing meets.

**Definition 4.2.** For any modal FM-frame $F = (X, \leq_1, \leq_2, R)$, let the $\mathbb{H}_{\text{RO}_{12}} = (\text{RO}_{12}(X), \varnothing, H, \cap, \lor_{\text{RO}_{12}}, \rightarrow_{\text{RO}_{12}}, \Box_{\text{RO}_{12}})$ where $Y \lor_{\text{RO}_{12}} Z := I_1 C_2(Y \cup Z)$, $Y \rightarrow_{\text{RO}_{12}} Z := I_1((X \setminus Y) \cup Z)$.

**Proposition 4.3.** (cf. [21, Theorem 5.6(2)]) For any modal FM-frame $F$, $\mathbb{H}_{\text{RO}_{12}}$ is a complete and completely multiplicative HAO.

**Proof.** The complete HAO part follows from Proposition 2.7. The complete multiplicativity follows from the definition of $\Box_{\text{RO}_{12}}$ that $\Box_{\text{RO}_{12}}$ preserves arbitrary intersections.

The essential difference between the correspondence for Kripke semantics for intuitionistic modal logic and the current setting is that the algebra $\mathbb{H}_{\text{RO}_{12}}$ is not perfect in general, i.e., it is not join-generated by completely join-primes.

4.2. The Auxiliary BAO $\mathbb{B}_K$

Consider any modal FM-frame $F = (X, \leq_1, \leq_2, R)$, there is another way of viewing it, namely taking it as a trirelational frame $F_3 = (X, \leq_1, \leq_2, R)$, the complex algebra of which is a Boolean algebra with three operators $\mathbb{B}_K$.

The formal definition of the BAO $\mathbb{B}_K$ is given as follows:

**Definition 4.4.** For any modal FM-frame $F = (X, \leq_1, \leq_2, R)$, define $\mathbb{B}_K = (P(X), \varnothing, W, \cap, \cup, -, \Box_K, \Box_{\leq_1}, \Box_{\leq_2})$, where $\cap, \cup, -$ are set-theoretic intersection, union and complementation respectively, $\Box_K(a) = \{w \in W \mid R[w] \subseteq a\}$, and $\Box_{\leq_i}(a) = \{w \in W \mid (\forall v \geq_i w)(v \in a)\}$.

**Lemma 4.5.** $e : \mathbb{H}_{\text{RO}_{12}} \hookrightarrow \mathbb{B}_K$ is a completely meet-preserving order-embedding such that $e \circ \Box_{\text{RO}_{12}} = \Box_K \circ e$. 
Proof. Since \( \mathcal{H}_{\text{RO}}(X) \subseteq P(X) \), we have that \( e : \mathcal{H}_{\text{RO}} \hookrightarrow \mathcal{B}_K \) is an order-embedding according to the order of the two algebras. Since in \( \mathcal{H}_{\text{RO}} \), arbitrary intersections of refined regular open subsets are again refined regular open, \( e \) is completely meet-preserving. Notice also that \( \Box_{\text{RO}} \) is the restriction of \( \Box_K \) to \( \mathcal{H}_{\text{RO}} \).

However, since \( \mathcal{H}_{\text{RO}} \) and \( \mathcal{B}_K \) have different definitions of join, \( \mathcal{H}_{\text{RO}} \) is not a sublattice of \( \mathcal{B}_K \).

We have the following corollary:

**Corollary 4.6.** \( e : \mathcal{H}_{\text{RO}} \hookrightarrow \mathcal{B}_K \) has a left adjoint \( c : \mathcal{B}_K \twoheadrightarrow \mathcal{H}_{\text{RO}} \) defined, for every \( a \in \mathcal{B}_K \), by

\[
c(a) = \bigwedge_{\mathcal{H}_{\text{RO}}} \{ b \in \mathcal{H}_{\text{RO}} \mid a \leq e(b) \}.
\]

Proof. By [10, Proposition 7.34], if a map \( e : \mathcal{H}_{\text{RO}} \hookrightarrow \mathcal{B}_K \) is completely meet-preserving between two partial orders, then it has a left-adjoint \( c : \mathcal{B}_K \twoheadrightarrow \mathcal{H}_{\text{RO}} \), and \( c(a) = \bigwedge_{\mathcal{H}_{\text{RO}}} \{ b \in \mathcal{H}_{\text{RO}} \mid a \leq e(b) \} \) for every \( a \in \mathcal{B}_K \).

Clearly, \( c(Y) = l_1 C_2(Y) \) for any \( \leq_1 \)-upset \( Y \subseteq X \), and \( c(Y) = l_1 C_2(\uparrow_1 Y) \) for any \( Y \subseteq W \) where \( \uparrow_1 Y \) is the least \( \leq_1 \)-upset containing \( Y \). Indeed, by definition \( c(X) = \bigwedge_{\mathcal{H}_{\text{RO}}} \{ Y \in \text{RO}(X) \mid X \leq e(Y) \} = \bigcap \{ Y \in \text{RO}(X) \mid X \subseteq Y \} \), which is the smallest refined regular open set containing \( X \). The closure operator \( c \) will be called the refined regular open closure map and \( c(a) \) the refined regular open closure of \( a \).

### 4.3. Interpreting the Additional Symbols

#### 4.3.1. Nominals

Nominals are originally introduced in hybrid logic (see e.g., [6, Chapter 14]). They are used as special propositional variables and are interpreted as singletons to refer to a world of the domain. In correspondence theory, nominals are used for computing the minimal valuation of propositional variables so as to eliminate them.

In the literature, nominals are interpreted as atoms (in complete atomic Boolean algebra based settings), completely join-prime elements (in perfect distributive lattice based settings), completely join-irreducible elements (in perfect (non-distributive) lattice based settings), regular open closures of singletons (in the setting of possibility semantics of classical normal modal logic). The common feature is that the selected class of elements join-generates the relevant complete lattices.

Therefore, in our setting, we need to find a subset of \( \mathcal{H}_{\text{RO}} \) which join-generates the whole \( \mathcal{H}_{\text{RO}} \). We take \( \text{Nom}(\mathcal{H}_{\text{RO}}) := \{ c(\{x\}) \mid x \in X \} \) to be the set of refined regular open closures of singletons in \( \mathcal{B}_K \), which will be shown to be the join-dense set of \( \mathcal{H}_{\text{RO}} \).
Correspondence Theory for Modal...

Propositional base

- perfect Boolean algebras
- perfect distributive lattices
- perfect general lattices
- possibility semantics

Nominals/join-generators

- atoms
- complete join-primes
- complete join-irreducibles
- regular open closures of singletons

**Proposition 4.7.** For any \( Y \in \mathbb{H}_{RO_{12}} \),

\[
Y = \bigvee_{RO_{12}} \{c(\{x\}) \mid x \in Y \} = \bigvee_{RO_{12}} \{Z \in \text{Nom}(\mathbb{H}_{RO_{12}}) \mid Z \subseteq Y \}.
\]

**Proof.** For the first equality, it is easy to see that for any \( x \in Y \), \( c(\{x\}) \subseteq c(Y) = Y \), so \( \bigvee_{RO_{12}} \{c(\{x\}) \mid x \in Y \} \subseteq Y \). For the other direction, for any \( x \in Y \), \( x \in c(\{x\}) \subseteq \bigvee_{RO_{12}} \{c(\{x\}) \mid x \in Y \} \). The second equality is easy.

4.3.2. **Black Diamond** The black diamond \( \Diamond \) comes from tense logic, which is the backward looking diamond “there was a moment where...”. When \( \Box \) is interpreted on the relation \( R \), \( \Diamond \) is interpreted on the relation \( R^{-1} \). These two connectives have the following property: \( \varphi \rightarrow \Box \psi \) is globally true in a model iff \( \Diamond \varphi \rightarrow \psi \) is globally true in that model. This property is called the adjunction property algebraically. In correspondence theory, we use the black diamond to compute the minimal valuation of a propositional variable, and the adjunction property is exactly what we need. Therefore, our task here is to find the left adjoint of \( \Box_{RO_{12}} \).

We know from lattice theory that in complete lattices, a map has a left adjoint iff it is completely meet-preserving. Since \( \mathbb{H}_{RO_{12}} \) and \( \mathbb{B}_K \) are both complete, and \( \Box_{RO_{12}} : \mathbb{H}_{RO_{12}} \rightarrow \mathbb{H}_{RO_{12}} \) and \( \Box_{K} : \mathbb{B}_K \rightarrow \mathbb{B}_K \) are both completely meet-preserving, they both have left adjoints, which are denoted by \( \Diamond_{RO_{12}} \) and \( \Diamond_{K} \), respectively.

For \( \Diamond_{K} \), the following lemma holds:

**Lemma 4.8.** \( \Diamond_{K}(X) = R[X] \) for any \( X \subseteq W \).

**Proof.** It suffices to show that \( R[X] \subseteq Y \) iff \( X \subseteq \Box_{K}(Y) \).

\( \Rightarrow \): Suppose \( R[X] \subseteq Y \). Then for any \( w \in X \), \( R[w] \subseteq R[X] \subseteq Y \), so \( w \in \Box_{K}(Y) \). Therefore \( X \subseteq \Box_{K}(Y) \).

\( \Leftarrow \): Suppose \( X \subseteq \Box_{K}(Y) \). Then for any \( v \in R[X] \), there is a \( w \in X \) such that \( Rwv \). By \( X \subseteq \Box_{K}(Y) \), we have that \( w \in \Box_{K}(Y) \), so \( R[w] \subseteq Y \). since \( v \in R[w] \), we have \( v \in Y \). Therefore \( R[X] \subseteq Y \).

For \( \Diamond_{RO_{12}} \), we have the following lemma:

**Lemma 4.9.** \( \Diamond_{RO_{12}}(Y) = (c \circ \Diamond_{K} \circ e)(Y) \).
**Proof.** We have the following chain of equalities:

\[
\diamond_{RO_{12}}(Y) = \bigwedge_{RO_{12}} \{ Z \in \mathbb{H}_{RO_{12}} | Y \leq \Box_{RO_{12}}(Z) \} \\
= \bigwedge_{RO_{12}} \{ Z \in \mathbb{H}_{RO_{12}} | e(Y) \leq (\diamond_{K} \circ e)(Z) \} \quad \text{(Lemma 4.5)} \\
= \bigwedge_{RO_{12}} \{ Z \in \mathbb{H}_{RO_{12}} | (\diamond_{K} \circ e)(Y) \leq e(Z) \} \\
= (c \circ \diamond_{K} \circ e)(Y) \\
= c(R[Y]) \quad \text{(Lemma 4.8)} \\
= l_1 C_2(\uparrow R[Y]) \quad \text{(Corollary 4.6)}
\]

5. Preliminaries on Algorithmic Correspondence

In the present section, we collect preliminaries on algorithmic correspondence for modal FM semantics. We will define the expanded modal language \( L^+ \) which will be used in the algorithm ALBA, the first-order correspondence language \( L^1 \) and the refined regular open translation of \( L^+ \) into \( L^1 \) which will be used as the language for the first-order correspondents. Our treatment is similar to [8,37].

5.1. The Expanded Modal Language \( L^+ \)

The expanded modal language \( L^+ \) contains the basic modal language \( L \) and a set \( \text{Nom} \) of nominals, to be interpreted as elements in \( \text{Nom}(\mathbb{H}_{RO_{12}}) \), and the black connective \( \diamond \), to be interpreted as the left adjoint of \( \Box \). Its definition is given as follows:

\[
\varphi ::= p \mid \bot \mid T \mid i \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi \mid \Box \varphi \mid \diamond \varphi,
\]

where \( p \in \text{Prop} \) and \( i \in \text{Nom} \).

We extend the valuation \( V \) to nominals such that \( V(i) \in \text{Nom}(\mathbb{H}_{RO_{12}}) \) and use \( i \) to denote the element that \( V(i) = l_1 C_2(\uparrow 1 \{i\}) \).

The satisfaction relation for the additional symbols is given as follows:

**Definition 5.1.** In any modal FM-model \( \mathcal{M} = (X, \leq_1, \leq_2, R, V) \),

\[
\mathcal{M}, w \models i \iff (\forall v \geq_1 w)(\exists u \geq_2 v)(i \leq_1 u).
\]

\[
\mathcal{M}, w \models \diamond \varphi \iff (\forall v \geq_1 w)(\exists u \geq_2 v)(\exists t \leq_1 u)\exists s(Rst \text{ and } \mathcal{M}, s \models \varphi).
\]

Truth set and validity are defined similarly to the basic modal language.
5.2. The Correspondence Languages $\mathcal{L}^1$

In order to express the first-order correspondents of modal formulas, we need to define the first-order correspondence language $\mathcal{L}^1$.

**Syntax.** The non-logical symbols of the first-order correspondence language $\mathcal{L}^1$ consists of the following:

- A set of unary predicate symbols $P_n$, each of which corresponds to a propositional variable $p_n$ and is going to be interpreted as a refined regular open subset of the domain;
- Three binary relation symbols $\leq_1, \leq_2$ and $R$ corresponding to the relations denoted with the same symbol;
- A set of individual symbols $i_n$, each of which corresponds to a nominal $i_n$. Notice that we allow the individual symbols $i_n$ to be quantified by quantifiers $\forall i_n, \exists i_n$.

5.3. The Refined Regular Open Translation

In the present section, we will define the refined regular open translation of $\mathcal{L}^+$ into $\mathcal{L}^1$.

**Definition 5.2.** (Syntactic Refined Regular Open Closure) Given a first-order formula $\alpha(x)$ with at most $x$ free, the syntactic regular open closure $\text{RO}^{12}_x(\alpha(x))$ is defined as $(\forall y \geq 1 x)(\exists z \geq 2 y)(\exists z' \leq 1 z)\alpha(z')$.

**Definition 5.3.** (Refined Regular Open Translation) The refined regular open translation of a formula in $\mathcal{L}^+$ into $\mathcal{L}^1$ is given as follows:

$$\begin{align*}
\text{RROT}_x(p_i) & := P_i(x); \\
\text{RROT}_x(\bot) & := x \neq x; \\
\text{RROT}_x(\top) & := x = x; \\
\text{RROT}_x(i) & := \text{RO}^{12}_x(i = x); \\
\text{RROT}_x(\varphi_1 \land \varphi_2) & := \text{RROT}_x(\varphi_1) \land \text{RROT}_x(\varphi_2); \\
\text{RROT}_x(\varphi_1 \lor \varphi_2) & := (\forall y \geq 1 x)(\exists z \geq 2 y)(\text{RROT}_y(\varphi_1) \lor \text{RROT}_y(\varphi_2)); \\
\text{RROT}_x(\varphi_1 \rightarrow \varphi_2) & := (\forall y \geq 1 x)(\text{RROT}_y(\varphi_1) \rightarrow \text{RROT}_y(\varphi_2)); \\
\text{RROT}_x(\Box \varphi) & := \forall y(Rxy \rightarrow \text{RROT}_y(\varphi)); \\
\text{RROT}_x(\Diamond \varphi) & := \text{RO}^{12}_x(\exists y(Ryx \land \text{RROT}_y(\varphi))).
\end{align*}$$

The refined regular open translations of inequalities $\varphi \leq \psi$ and quasi-inequalities are given as follows (notice that they are interpreted on the level of global truth of models):

$$\begin{align*}
\text{RROT}(\varphi \leq \psi) & := \forall x(\text{RROT}_x(\varphi) \rightarrow \text{RROT}_x(\psi)); \\
\text{RROT}(\&_{j=1}^n(\varphi_j \leq \psi_j) \Rightarrow \varphi \leq \psi) & := \bigwedge_{j=1}^n \forall x(\text{RROT}_x(\varphi_j) \rightarrow \text{RROT}_x(\psi_j)) \rightarrow \forall x(\text{RROT}_x(\varphi) \rightarrow \text{RROT}_x(\psi)).
\end{align*}$$
The following proposition justifies the translation defined above:

**Proposition 5.4.** For any modal FM-model $\mathcal{M} = (X, \leq_1, \leq_2, R, V)$, any $w \in X$, any $\mathcal{L}^+$-formula $\varphi$, any $\mathcal{L}^+$-inequality $\text{Ineq}$ and any $\mathcal{L}^+$-quasi-inequality $\text{Quasi}$,

- $\mathcal{M}, w \vDash \varphi$ iff $\mathcal{M} |\vDash RROT_x(\varphi)[w]$;
- $\mathcal{M} \vDash \varphi$ iff $\mathcal{M} |\vDash \forall x RROT_x(\varphi)$;
- $\mathcal{M} \vDash \text{Ineq}$ iff $\mathcal{M} |\vDash RROT(\text{Ineq})$;
- $\mathcal{M} \vDash \text{Quasi}$ iff $\mathcal{M} |\vDash RROT(\text{Quasi})$.

where $\vDash$ is the first-order satisfaction relation.

**Proof.** By the definition of refined regular open translation and the semantics of $\mathcal{L}^+$, it suffices to prove by induction on the structure of the formulas in $\mathcal{L}^+$. ■

6. Inductive Formulas

In this section, we define inductive formulas for our setting, which will be shown to have first-order correspondents. The definition is similar to [38].

We define *positive formulas* with variables in $A \subseteq \text{Prop}$ as follows:

$$\text{POS}_A ::= p | \perp | \top | \Box \text{POS}_A | \text{POS}_A \land \text{POS}_A | \text{POS}_A \lor \text{POS}_A$$

where $p \in A$.

We define the *dependence order* on propositional variables as any irreflexive and transitive binary relation $<_{\Omega}$ on them.

We define the *PIA formulas*\(^2\) with main variable $p$ as follows:

$$\text{PIA}_p ::= p | \perp | \top | \Box \text{PIA}_p | \text{POS}_{A_p} \rightarrow \text{PIA}_p$$

where $A_p = \{q \in \text{Prop} | q <_{\Omega} p\}$.

We define the *inductive antecedent* as follows:

$$\text{Ant} ::= \text{PIA}_p | \text{Ant} \land \text{Ant} | \text{Ant} \lor \text{Ant}$$

where $p \in \text{Prop}$.

We define the *inductive succedent* as follows:

$$\text{Suc} ::= \text{POS}_{\text{Prop}} | \text{PIA}_q \rightarrow \text{Suc} | \Box \text{Suc} | \text{Suc} \land \text{Suc}$$

where $q \in \text{Prop}$.

\(^2\)For the name, see e.g., [32, Remark 3.24].
Finally, an \( \Omega \)-inductive formula is a formula of the form \( \text{Ant} \rightarrow \text{Suc} \). An inductive formula is an \( \Omega \)-inductive formula for some \( <_{\Omega} \).

**Example 6.1.** Formulas like \( \Box p \rightarrow p \), \( \Box p \rightarrow \Box \Box p \), \( \Box (p \rightarrow q) \land \Box \Box p \rightarrow \Box q \) are examples of inductive formulas.

**Remark 6.2.** Since in our settings, we have not yet found a meet-dense set of \( \mathbb{H}_{\text{RO}_{12}} \), and we will only compute minimal valuations rather than maximal valuations, we do not use the order-type and signed generation tree style definitions like in [8]. If a meet-dense set of \( \mathbb{H}_{\text{RO}_{12}} \) can be found, then we can compute maximal valuations of propositional variables, and then we can compute the first-order correspondents of the KC formula \( \neg p \lor \neg \neg p \) and of the LC formula \( (p \rightarrow q) \lor (q \rightarrow p) \). We cannot compute many interesting non-modal intermediate logic axioms in our setting for this reason.

- Since we do not have diamond in the basic language, the fragment that we have is much smaller than typical definitions of inductive formulas in some existing settings like [8]. In particular, we do not have standard Sahlqvist formulas with \( \Diamond \) like \( \Box p \rightarrow \Diamond p \), \( \Diamond \Diamond p \rightarrow \Diamond p \), \( p \rightarrow \Box \Diamond p \) or \( \Diamond \Diamond p \rightarrow \Box \Diamond p \). Therefore, incorporating \( \Diamond \) into this framework would be an interesting future direction.

7. **The Algorithm ALBA**

In the present section, we define the algorithm ALBA which computes the first-order correspondence of the input formula, in the style of [8,38].

The algorithm ALBA executes in three stages.

1. **Preprocessing and First approximation:** ALBA receives a formula \( \text{Ant} \rightarrow \text{Suc} \) as input and transforms it into the inequality \( \text{Ant} \leq \text{Suc} \). Then we apply the following rules:

(a) We apply the following **distribution rules** exhaustively:

- In \( \text{Ant} \), rewrite every subformula of the former form into the latter form:
  - \( (\beta \lor \gamma) \land \alpha, (\beta \land \alpha) \lor (\gamma \land \alpha) \)
  - \( \alpha \land (\beta \lor \gamma), (\alpha \land \beta) \lor (\alpha \land \gamma) \)
- In \( \text{Suc} \), rewrite every subformula of the former form into the latter form:
  - \( (\beta \land \gamma) \lor \alpha, (\beta \lor \alpha) \land (\gamma \lor \alpha) \)
  - \( \alpha \lor (\beta \land \gamma), (\alpha \lor \beta) \land (\alpha \lor \gamma) \)
  - \( \alpha \rightarrow \beta \land \gamma, (\alpha \rightarrow \beta) \land (\alpha \rightarrow \gamma) \)
- □(α ∧ β), □α ∧ □β

(b) Apply the splitting rules:

\[
\begin{align*}
α ≤ β ∧ γ & \quad α ≥ β \quad α ≤ γ \\
α ≤ γ & \quad β ≤ γ
\end{align*}
\]

Now for each obtained inequality φ_i ≤ ψ_i, we apply the first-approximation rule:

\[
φ_i ≤ ψ_i \quad \xrightarrow{i_0 ≤ φ_i} \quad i_0 ≤ ψ_i
\]

Now we call each quasi-inequality i_0 ≤ φ_i ⇒ i_0 ≤ ψ_i a system, and use S to denote a meta-conjunction of inequalities. When S is empty, we denote it as ∅.

2. The reduction-elimination cycle: In this stage, for each system i_0 ≤ φ_i ⇒ i_0 ≤ ψ_i, we apply the following rules to eliminate all the propositional variables:

(a) Splitting rules:

\[
\begin{align*}
S ⇒ α ≤ β ∧ γ \\
S ⇒ α ≤ β \quad S ⇒ α ≤ γ \\
S & α ≤ β ∧ γ ⇒ φ ≤ ψ \\
S & α ≤ β & α ≤ γ ⇒ φ ≤ ψ \\
S ⇒ α ∨ β ≤ γ \\
S ⇒ α ≤ γ \quad S ⇒ β ≤ γ \\
S & α ∨ β ≤ γ ⇒ φ ≤ ψ
\end{align*}
\]

(b) Residuation rules:

\[
\begin{align*}
S ⇒ α ≤ □β \\
S ⇒ •α ≤ β \\
S & α ≤ □β ⇒ φ ≤ ψ \\
S & •α ≤ β ⇒ φ ≤ ψ \\
S ⇒ α ≤ β ⇒ γ \\
S ⇒ α ∧ β ≤ γ \\
S & α ≤ β ⇒ γ ⇒ φ ≤ ψ \\
S & α ∧ β ≤ γ ⇒ φ ≤ ψ
\end{align*}
\]

(c) Approximation rule:

\[
S ⇒ φ ≤ ψ \quad \xrightarrow{S & i ≤ φ} \quad i ≤ ψ
\]
The nominal introduced by the approximation rule must not occur in the system before applying the rule.

(d) Deleting rules:

\[
\begin{align*}
& S \& \alpha \leq \top \Rightarrow \varphi \leq \psi \\
& S \Rightarrow \varphi \leq \psi \\
& S \Rightarrow \alpha \leq \top \\
& \emptyset \Rightarrow \alpha \leq \top
\end{align*}
\]

(e) Right-handed Ackermann rule. This rule eliminates propositional variables and is the core of the algorithm:

\[
\begin{align*}
& \land_{i=1}^n \theta_i \leq p \land \land_{j=1}^m \eta_j \leq \iota_j \Rightarrow \varphi \leq \psi \\
& \land_{j=1}^m \eta_j(\theta/p) \leq \iota_j(\theta/p) \Rightarrow \varphi(\theta/p) \leq \psi(\theta/p)
\end{align*}
\]

where:

i. \( p \) does not occur in \( \theta_1, \ldots, \theta_n \);
ii. Each \( \eta_i, \psi \) is positive, and each \( \iota_i, \varphi \) negative in \( p \), for \( 1 \leq i \leq m \);
iii. \( \theta := \land \theta_1 \lor \ldots \lor \theta_n \).

3. Output: If in Stage 2, the algorithm gets stuck for some systems, i.e., some propositional variables cannot be eliminated, then the algorithm stops and outputs “failure”. Otherwise, each initial system after the first approximation rule has been reduced to a set of pure quasi-inequalities \( \text{Reduce}(i_0 \leq \varphi_i \Rightarrow i_0 \leq \psi_i) \), and then the output is a set of pure quasi-inequalities \( \bigcup_{i \in J} \text{Reduce}(i_0 \leq \varphi_i \Rightarrow i_0 \leq \psi_i) \). Then we can use the conjunction of the refined regular open translations of the quasi-inequalities to obtain the first-order correspondent (notice that in the refined regular open translation of each quasi-inequality, we need to universally quantify over all the individual variables).

Example 7.1. Here we give a simple example. For the sake of clarity we add propositional quantifiers and nominal quantifiers before the quasi-inequality.

\[
\begin{align*}
& \forall p(\square p \rightarrow p) \\
& \forall p(\square p \leq p) \\
& \forall p \forall i(i \leq \square p \Rightarrow i \leq p) \\
& \forall p \forall i(\Diamond i \leq p \Rightarrow i \leq p) \\
& \forall i(i \leq \Diamond i) \\
& \forall i \forall x(RROT_x(i) \rightarrow RROT_x(\Diamond i)) \\
& \forall i \forall x(RO_x^{12}(i = x) \rightarrow RO_x^{12}(\exists y(Ryx \land RO_x^{12}(i = y))))
\end{align*}
\]
Remark 7.2. • There is an implementation of a similar correspondence algorithm SQEMA for the basic modal logic case, see [15]. For the moment there is no implementation for ALBA for non-classical logics other than basic modal logic and basic modal logic with universal modality.

• As we can see from the example, even for simple formulas like $\Box p \rightarrow p$, its first-order correspondent is already very complex. Therefore, there are some natural questions to ask:
  - Whether the first-order correspondent of $\Box p \rightarrow p$ can be simplified manually;
  - Whether there are systematic tricks for simplifying the outputs of ALBA;
  - Whether there are intuitionistic modal formulas with simple first-order correspondents;
  - What is the exact complexity of the correspondence algorithm ALBA.

8. Success

In the present section, we show the success of ALBA on any inductive formula $\phi \rightarrow \psi$, i.e., for any input inductive formula, the algorithm will terminate and output a set of pure quasi-inequalities and a first-order formula which will be shown to be the first-order correspondent of the input inductive formula in Section 9.

**Theorem 8.1.** ALBA succeeds on any inductive formula $\phi \rightarrow \psi$, i.e., it terminates on any input inductive formula $\phi \rightarrow \psi$ and outputs a set of pure quasi-inequalities and a first-order formula.

**Proof.** We check the shape of the inequalities or systems in each stage, for the input formula $\text{Ant} \rightarrow \text{Suc}$:

**Stage 1.**

After applying the distribution rules, it is easy to see that $\text{Ant}$ becomes of the form $\bigvee \bigwedge \text{PIA}_p$, and $\text{Suc}$ becomes of the form $\bigwedge \text{Suc}'$, where

$$\text{Suc}' := \text{POS}_{\text{Prop}} | \text{PIA}_q \rightarrow \text{Suc}' | \Box \text{Suc}'.$$  

Then by applying the splitting rules, we get a set of inequalities of the form $\bigwedge \text{PIA}_p \leq \text{Suc}'$.

After the first approximation rule, we get systems of the form $i_0 \leq \bigwedge \text{PIA}_p \Rightarrow i_0 \leq \text{Suc}'$.

**Stage 2.** In this stage, we deal with each system $i_0 \leq \bigwedge \text{PIA}_p \Rightarrow i_0 \leq \text{Suc}'$.  

For the inequality $i_0 \leq \bigwedge PIA_p$, by first applying the splitting rule for $\land$ and then exhaustively applying the residuation rules for $\Box$ and $\rightarrow$, we get inequalities of the form $\text{MinVal}_p \leq p$ or $\text{MinVal}_p \leq \top$ or $\text{MinVal}_p \leq \bot$, where

$$\text{MinVal}_p ::= i_0 \mid \Diamond \text{MinVal}_p \mid \text{MinVal}_p \land \text{POS}_{A_p},$$

where $A_p = \{q \in \text{Prop} \mid q < \omega p\}$.

Now we deal with the $i_0 \leq \text{Suc}'$ part.

- If the system is of the form $S \& \& (\text{MinVal}_p \leq p) \Rightarrow i_0 \leq PIA_q \rightarrow \text{Suc}'$, then we have the following execution of ALBA:

  $$S \& \& (\text{MinVal}_p \leq p) \Rightarrow i_0 \land PIA_q \leq \text{Suc}'$$
  $$S \& \& (\text{MinVal}_p \leq p) \& \& j \leq i_0 \land PIA_q \Rightarrow j \leq \text{Suc}'$$
  $$S \& \& (\text{MinVal}_p \leq p) \& \& j \leq i_0 \Rightarrow j \leq \text{Suc}'.

- If the system is of the form $\& (\text{MinVal}_p \leq p) \Rightarrow i_0 \leq \Box \text{Suc}'$, then we have the following execution of ALBA:

  $$\& (\text{MinVal}_p \leq p) \Rightarrow \Diamond i_0 \leq \text{Suc}'$$
  $$\& (\text{MinVal}_p \leq p) \& \& j \leq \Diamond i_0 \Rightarrow j \leq \text{Suc}'.

Therefore, by the reduction strategies above, we get a quasi-inequality of the form (here Pure is a meta-conjunction of inequalities without propositional variables):

$$\& (\text{MinVal}_p \leq p) \& \& \text{Pure} \Rightarrow j \leq \text{POS}_{\text{Prop}}.$$  

Now we can apply the right-handed Ackermann rule to an $\Omega$-minimal variable $q$ to eliminate it. Since there are only finitely many propositional variables, we can always find another $\Omega$-minimal variable to eliminate. Finally we eliminate all propositional variables and get pure quasi-inequalities and the refined regular open translation. It is clear that the algorithm terminates.

9. Soundness

In this section, we prove the soundness of ALBA with respect to modal FM-frames, i.e., we show that the first-order formula computed from the algorithm is really the first-order correspondent of the input inductive formula. The basic proof structure is similar to [8,9,38].
Theorem 9.1. (Soundness) If ALBA runs according to the success proof in the previous section on an input inductive formula $\varphi \rightarrow \psi$ and outputs a first-order formula $\text{FO}(\varphi \rightarrow \psi)$, then for any modal FM-frame $F = (X, \leq_1, \leq_2, R)$,

$$F \models \varphi \rightarrow \psi \text{ iff } F \models \text{FO}(\varphi \rightarrow \psi).$$

Proof. The proof goes similarly to [8, Theorem 8.1]. Let $\varphi \rightarrow \psi$ denote the input formula, $\{i_0 \leq \varphi_i \Rightarrow i_0 \leq \psi_i\}_{i \in I}$ denote the set of systems after the first-approximation rule, let $\{\text{Reduce}(i_0 \leq \varphi_i \Rightarrow i_0 \leq \psi_i)\}_{i \in I}$ denote the sets of quasi-inequalities after Stage 2, let $\text{FO}(\varphi \rightarrow \psi)$ denote the refined regular open translation of the quasi-inequalities in Stage 3 into first-order formulas. Then it suffices to show the equivalence from (1) to (4) given below:

1. $F \models \varphi \rightarrow \psi$  
2. $F \models i_0 \leq \varphi_i \Rightarrow i_0 \leq \psi_i$, for all $i \in I$  
3. $F \models \text{Reduce}(i_0 \leq \varphi_i \Rightarrow i_0 \leq \psi_i)$, for all $i \in I$  
4. $F \models \text{FO}(\varphi \rightarrow \psi)$

The equivalence between (1) and (2) follows from Proposition 9.2;  
The equivalence between (2) and (3) follows from Propositions 9.3, 9.4;  
The equivalence between (3) and (4) follows from Proposition 5.4. $
$
In what follows, we will prove the soundness of each rule in each stage.

Proposition 9.2. (Soundness of the rules in Stage 1) The distribution rules and the splitting rules are sound in both directions in $F$.

Proof. For the soundness of the distribution rules, it follows from the validity of the following equivalences in $F$:

1. $(\alpha \lor \beta) \land \gamma \leftrightarrow (\alpha \land \gamma) \lor (\beta \land \gamma)$;
2. $\alpha \land (\beta \lor \gamma) \leftrightarrow (\alpha \land \beta) \lor (\alpha \land \gamma)$;
3. $\Box (\alpha \land \beta) \leftrightarrow \Box \alpha \land \Box \beta$;
4. $(\alpha \land \beta) \lor \gamma \leftrightarrow (\alpha \lor \gamma) \land (\beta \lor \gamma)$;
5. $\alpha \lor (\beta \land \gamma) \leftrightarrow (\alpha \lor \beta) \land (\alpha \lor \gamma)$;
6. $(\alpha \rightarrow \beta \land \gamma) \leftrightarrow (\alpha \rightarrow \beta) \land (\alpha \rightarrow \gamma)$. 

In what follows, we will prove the soundness of each rule in each stage.
For the soundness of the splitting rules, it follows from the following fact:

\[
\begin{align*}
F \models \alpha \leq \beta \land \gamma & \iff (F \models \alpha \leq \beta \text{ and } F \models \alpha \leq \gamma); \\
F \models \alpha \lor \beta \leq \gamma & \iff (F \models \alpha \leq \gamma \text{ and } F \models \beta \leq \gamma).
\end{align*}
\]

For the soundness of the first-approximation rule, by Proposition 4.7,

\[
V(\varphi_i) = \bigvee_{\text{RO12}} \{Z \in \text{Nom}(H_{\text{RO12}}) \mid Z \subseteq V(\varphi_i)\}.
\]

Now we have the following chain of equivalences:

\[
\begin{align*}
F \models \varphi_i \leq \psi_i & \\
& \iff \text{for any } V, \bigvee_{\text{RO12}} \{Z \in \text{Nom}(H_{\text{RO12}}) \mid Z \subseteq V(\varphi_i)\} \subseteq V(\psi_i) \\
& \iff \text{for any } V, (\forall Z \in \text{Nom}(H_{\text{RO12}}) \text{ s.t. } Z \subseteq V(\varphi_i))(Z \subseteq V(\psi_i)) \\
& \iff \text{for any } V, (\forall Z \in \text{Nom}(H_{\text{RO12}})(Z \subseteq V(\varphi_i) \Rightarrow Z \subseteq V(\psi_i)) \\
& \iff \text{for any } V', V'(i) \subseteq V'(\varphi_i) \Rightarrow V'(i) \subseteq V'(\psi_i) \\
& \iff F \models i \leq \varphi_i \Rightarrow i \leq \psi_i.
\end{align*}
\]

For the soundness of each rule in Stage 2, we introduce the following notations: for each rule, before its application we have a system \( S \Rightarrow \text{Ineq} \), after its application we get a system \( S' \Rightarrow \text{Ineq}' \) (indeed, for the splitting rules we might have two systems after the application, but their soundness is trivial), so the soundness of Stage 2 is the equivalence of the following:

- \( F \models S \text{ } \Rightarrow \text{Ineq} \);
- \( F \models S' \Rightarrow \text{Ineq}' \).

It suffices to show the following property for the splitting rules, residuation rules and the deleting rules:

For any \( F \), any \( V \),

\[
F, V \models S \Rightarrow \text{Ineq} \text{ iff } F, V \models S' \Rightarrow \text{Ineq}'.
\]

For the first-approximation rule, we prove it directly.

For the right-handed Ackermann rule, we also prove it directly.

**Proposition 9.3.** The splitting rules, the approximation rule, the residuation rules and the deleting rules in Stage 2 are sound in both directions in \( F \).

**Proof.** The soundness proofs for the splitting rules and the residuation rules are similar to the soundness of the same rules in [8].
• For the splitting rules, it follows from the following equivalence: for any modal FM-frame $F$, any valuation $V$ on $F$,

$F, V \vDash \alpha \leq \beta \land \gamma \iff F, V \vDash \alpha \leq \beta$ and $F, V \vDash \alpha \leq \gamma$;

$F, V \vDash \alpha \lor \beta \leq \gamma \iff F, V \vDash \alpha \leq \gamma$ and $F, V \vDash \beta \leq \gamma$.

• For the residuation rules, it follows from the following equivalence: for any modal FM-frame $F$, any valuation $V$ on $F$,

$- F, V \vDash \alpha \leq \Box \beta \iff F, V \vDash \Diamond \alpha \leq \beta$;

$- F, V \vDash \alpha \leq \beta \rightarrow \gamma \iff F, V \vDash \alpha \land \beta \leq \gamma$.

The equivalences above follow from the fact that the interpretations of $\Diamond$ and $\Box$ form an adjunction pair, and the interpretations of $\land$ and $\rightarrow$ form a residuation pair.

• For the approximation rule, the soundness proof is similar to the first-approximation rule: for any modal FM-frame $F$, any valuation $V$,

\[
 F, V \vDash \varphi_i \leq \psi_i
\]

iff

$V(\varphi_i) \subseteq V(\psi_i)$

iff

\[
\bigvee_{\text{RO}_{12}} \{ Z \in \text{Nom}(H_{\text{RO}_{12}}) \mid Z \subseteq V(\varphi_i) \} \subseteq V(\psi_i)
\]

iff

$(\forall Z \in \text{Nom}(H_{\text{RO}_{12}}) \text{ s.t. } Z \subseteq V(\varphi_i))(Z \subseteq V(\psi_i))$

iff

$(\forall Z \in \text{Nom}(H_{\text{RO}_{12}})(Z \subseteq V(\varphi_i) \Rightarrow Z \subseteq V(\psi_i))).$

Therefore, if $F \vDash S \Rightarrow \varphi_i \leq \psi_i$, then for any $V$, if $F, V \vDash S$ and $V(i) \subseteq V(\varphi_i)$, then $V(\varphi_i) \subseteq V(\psi_i)$, so for $V(i) \in \text{Nom}(H_{\text{RO}_{12}})$, from the above equivalences we have $V(i) \subseteq V(\psi_i)$, i.e., $F, V \vDash i \leq \psi_i$. Thus $F \vDash S \& i \leq \varphi_i \Rightarrow i \leq \psi_i$.

If $F \vDash S \& i \leq \varphi_i \Rightarrow i \leq \psi_i$, then for any valuation $V$, if $F, V \vDash S$, then for any $Z \in \text{Nom}(H_{\text{RO}_{12}})$, if $Z \subseteq V(\varphi_i)$, take $V' := V_Z^i$, then since $i$ does not occur in $S$, we have $F, V' \vDash S$. We also have $V'(i) = Z \subseteq V(\varphi_i)$, so from $F \vDash S \& i \leq \varphi_i \Rightarrow i \leq \psi_i$ we get $Z = V'(i) \subseteq V'(\psi_i) = V(\psi_i)$, therefore $F, V \vDash \varphi_i \leq \psi_i$. Therefore we get $F \vDash S \Rightarrow \varphi_i \leq \psi_i$.

• The soundness of the deleting rule is trivial, since $\alpha \leq \top$ always holds in any modal FM-model.

PROPOSITION 9.4. The right-handed Ackermann rule is sound in $F$.

PROOF. Without loss of generality we assume that $n = 1$ and $m = 1$. Then it suffices to show the following equivalence:
\[\mathbb{F} \models \theta \leq p \& \eta \leq \iota \Rightarrow \varphi \leq \psi;\]
\[\mathbb{F} \models \eta(\theta/p) \leq \iota(\theta/p) \Rightarrow \varphi(\theta/p) \leq \psi(\theta/p).\]

\[\downarrow: \text{Assume that } \mathbb{F} \models \theta \leq p \& \eta \leq \iota \Rightarrow \varphi \leq \psi. \text{ Then for any valuation } V \text{ on } \mathbb{F}, \text{ if } \mathbb{F}, V \models \eta(\theta/p) \leq \iota(\theta/p), \text{ then take } V' = V^p_V(\theta), \text{ then since } p \text{ does not occur in } \theta, \text{ we have } V'(\theta) = V(\theta) = V'(p), \text{ therefore } V(\eta(\theta/p)) = V'(\eta(\theta/p)) = V'(\eta), \text{ similarly } V(\iota(\theta/p)) = V'(\iota), \text{ so from } \mathbb{F}, V \models \eta(\theta/p) \leq \iota(\theta/p) \text{ we get } V'(\eta) \subseteq V'(\iota). \text{ Therefore } \mathbb{F}, V' \models \varphi \leq \psi. \text{ Therefore } V'(\varphi) \subseteq V'(\psi). \text{ Similar to } \eta \text{ and } \iota \text{ we get } V'(\varphi) = V(\varphi(\theta/p)) \text{ and } V'(\psi) = V(\psi(\theta/p)), \text{ so } \mathbb{F}, V \models \varphi(\theta/p) \leq \psi(\theta/p). \text{ By the arbitrariness of } V \text{ we get } \mathbb{F} \models \eta(\theta/p) \leq \iota(\theta/p) \Rightarrow \varphi(\theta/p) \leq \psi(\theta/p).\]

\[\uparrow: \text{Assume } \mathbb{F} \models \eta(\theta/p) \leq \iota(\theta/p) \Rightarrow \varphi(\theta/p) \leq \psi(\theta/p). \text{ Then for any valuation } V \text{ on } \mathbb{F}, \text{ if } \mathbb{F}, V \models \theta \leq p \& \eta \leq \iota, \text{ then } V(\theta) \subseteq V(p), V(\eta) \subseteq V(\iota). \text{ Therefore by the polarity of } p \text{ in } \eta(\theta/p) \text{ and } \iota(\theta/p) \text{ we have that } V(\eta(\theta/p)) \subseteq V(\eta) \subseteq V(\iota) \subseteq V(\iota(\theta/p)). \text{ So from } \mathbb{F} \models \eta(\theta/p) \leq \iota(\theta/p) \Rightarrow \varphi(\theta/p) \leq \psi(\theta/p) \text{ we get } V(\varphi) \subseteq V(\varphi(\theta/p)) \subseteq V(\psi(\theta/p)) \subseteq V(\psi), \text{ so } \mathbb{F}, V \models \varphi \leq \psi. \text{ Therefore by the arbitrariness of } V \text{ we get } \mathbb{F} \models \theta \leq p \& \eta \leq \iota \Rightarrow \varphi \leq \psi.\]

10. Conclusions

In this paper, we studied the correspondence theory of intuitionistic modal logic in modal Fairtlough-Mendler semantics, which is the intuitionistic modal counterpart of possibility semantics. Our study could be regarded as the study of correspondence theory for complete Heyting algebras with complete operators which are not necessarily perfect.

We proposed the general principles to choose the interpretations of the expanded modal language used in the algorithm ALBA, and applied it in the current setting. Therefore, we interpreted the nominals in our setting as the refined regular open closures of singletons, which are join-dense in \(\mathbb{H}_{\text{RO}_{12}}\) and can be expressed in a first-order way.

For future directions, we mention the following:

- In the present paper, we use a join-dense set of \(\mathbb{H}_{\text{RO}_{12}}\) to interpret the nominals. Whether a meet-dense set of \(\mathbb{H}_{\text{RO}_{12}}\) which can be expressed in a first-order way can be found is a question related to whether we can expand the fragment of inductive formulas here, which is also related to finding correspondents of the KC formula \(\neg p \lor \neg \neg p\) and of the LC formula \((p \to q) \lor (q \to p)\) in the present semantic setting.
• In the present paper, we only have □ as the modality, since it interacts well with arbitrary intersection. A future direction is to find a way to incorporate ◊ into the picture.

• In the present paper, we consider complete Heyting algebra expansions, therefore it is natural to also consider the correspondence theory of complete lattice expansions which are not necessarily perfect.

• In the present paper, we do not consider the canonicity and completeness concepts related to the intuitionistic modal version of possibility semantics, which is a future direction.

• It was proved in [21] that there is a modal logic that is not the logic of any class of Kripke frames but is the logic of a class of possibility frames. It is natural to ask whether there is a natural intuitionistic modal logic that is not the logic of any class of modal Kripke frames but is the logic of a class of modal FM-frames.

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Z. ZHAO
Taishan University
Taian
China
zhaozhiguang23@gmail.com