Condensation and Clustering in the Driven Pair Exclusion Process

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We investigate particle condensation in a driven pair exclusion process on one- and two-dimensional lattices under the periodic boundary condition. The model describes a biased hopping of particles subject to a pair exclusion constraint that each particle cannot stay at the same site with its pre-assigned partner. The pair exclusion causes a mesoscopic condensation characterized by the scaling of the condensate size $m_{\text{con}} \sim N^a$ and the number of condensates $N_{\text{con}} \sim N^b$ with the total number of sites $N$. Those condensates are distributed randomly without hopping bias. We find that the hopping bias generates a spatial correlation among condensates so that a cluster of condensates appears. Especially, the cluster has an anisotropic shape in the two-dimensional system. The mesoscopic condensation and the clustering are studied by means of numerical simulations.

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I. INTRODUCTION

In a driven diffusive system, hopping bias in dynamics drives a system out of equilibrium and can lead to a variety of effects. The hopping bias is irrelevant for non-interacting or single-particle systems. Biased diffusion of a particle, for example, is equivalent to unbiased diffusion in a co-moving frame. Particle interaction can make a difference. Consider, as an example, the asymmetric simple exclusion process (ASEP) \cite{1} in one dimension (1D), in which particles hop under the constraint that each site can be occupied by at most a single particle [exclusion interaction]. When particles hop symmetrically, the system belongs to the Edwards-Wilkinson universality class in which the dynamic correlation length $\xi$ grows in time $t$ as $\xi \sim t^{\frac{3}{2}}$ \cite{2}. With a hopping bias, however, the system exhibits a scaling $\xi \sim t^a$ of the Kardar-Parisi-Zhang universality class \cite{3}.

In general, the hopping bias can also change the nature of a stationary state probability distribution. We focus on systems displaying a condensation transition. Condensation occurs in various forms such as Bose-Einstein condensation, traffic jams, hub formation in evolving networks, and so on \cite{2}. Condensation can be studied in the context of driven diffusive systems \cite{3,4}, and the effect of hopping bias has been studied in those systems \cite{5,6,7,8}.

The zero range process (ZRP) is useful in studying particle condensation \cite{4,10,11}. Consider, for simplicity, a one-dimensional ring of $N$ sites $i = 1, 2, \cdots, N$ under a periodic boundary condition. There are $M$ particles, and the occupation number at site $i$ is denoted as $n_i = 0, 1, 2, \cdots$. A particle at site $i$ selected randomly decreases by one ($n_i \to n_i - 1$) at a rate $u(n_i)$ and then hops to a neighboring site $j = i + 1$ with probability $p$ or $j = i - 1$ with probability $q = 1 - p$. The jumping rate depends on the occupation number at a source site, whose functional form reflects the nature of on-site interactions among particles. For example, if particles hop independently, the jumping rate should be linearly proportional to the occupation number, $u(m) \propto m$. A jumping rate function $u(m)$ growing sublinearly or decreasing in $m$ corresponds to an attractive interaction.

The stationary state of the ZRP is known exactly. The probability $P_{st}(m)$ to find the system in a configuration $m = (m_1, \cdots, m_L)$ in the stationary state is given by a product form

$$P(m) = \frac{\delta(M - \sum_{i=1}^{N} m_i)}{Z(N, M)} \prod_{i=1}^{N} \frac{1}{[\prod_{l=1}^{m_i} u(l)]}$$

with a normalization factor $Z(N, M)$. The exact solution allows one to understand the condition for condensation. Suppose that the jumping rate is given by $u(m) = 1 + b/m$. When $b < 2$, particles are distributed uniformly at all values of particle density $\rho = M/N$. On the other hand, when $b > 2$, there emerges a single macroscopic condensate of size $m_{\text{con}} = (\rho - \rho_c)N$ for $\rho > \rho_c$ with $\rho_c = 1/(b - 2)$. The ZRP has the same stationary state at all values of $p$ and $q$ irrespective of the hopping bias.

An interesting generalization of the ZRP was considered in Ref. \cite{6}, where the particle jumping rate is constant, but a jump is accepted with probability $v(m)$ depending on the occupation number $m$ at target sites. Such a model is called the target process (TP). Without a hopping bias ($p = q$), the stationary state of the TP is mapped to that of the ZRP with jumping rate $u(m + 1) = 1/v(m)$. Hence, the unbiased TP displays macroscopic condensation under an appropriate condition, e.g., $v(m) = 1/(1 + b/(m + 1))$ with $b > 2$ and $\rho > 1/(b - 2)$. However, when a hopping bias with $p \neq q$ exists, the mapping breaks down, and the stationary state of the TP is not given by the product form. Moreover, the hopping bias destroys the condensation \cite{6}.

The conserved mass aggregation (CMA) model also
exhibits a condensation phenomenon \[5\]. In this model, all particles at a site \(i\) hop \((m_i \rightarrow 0)\) to a neighboring site at the unit rate or a single particle is chipped off \((m_i \rightarrow m_i - 1)\) to a neighboring site at a constant rate \(\omega\). Particles at different sites aggregate through the former process while they are scattered out by the latter. Competition between them results in condensation when \(\rho \geq \rho_c(\omega) = \sqrt{\omega + 1} - 1\) in all dimensions when the hopping and the chipping are symmetric \[2\]. However, a bias in the hopping and the chipping was shown to inhibit condensation in one-dimension \[3\].

In this work, we study a particle condensation phenomenon in a driven pair exclusion process (PEP). The PEP was first introduced in Ref. \[12\] as a model for hub formation in evolving networks \[13\]. In the PEP, particles hop under the so-called pair exclusion constraint, which will be explained later. With symmetric hopping, the system in the stationary state exhibits an intriguing condensation state characterized by multiple mesoscopic condensates of size \(n_{\text{con}} \sim N^{1/2}\) and of number \(N_{\text{con}} \sim N^{1/2}\), where \(N\) is the total number of particles with a logarithmic correction \[12\]. Those condensates are distributed randomly without any spatial correlation. We will investigate the effect of the hopping bias on the nature of mesoscopic condensation.

The paper is organized as follows: In Sec. \[II\] we introduce the PEP with and without a hopping bias. The condensation phase transition of the driven PEP has been investigated via numerical simulations, results of which are presented in Sec. \[III\]. We summarize the paper in Sec. \[IV\].

**II. DRIVEN PAIR EXCLUSION PROCESS**

There are \(M\) particles on a \(d\)-dimensional hypercubic lattice of \(N = L^d\) sites under a periodic boundary condition. The occupation number at site \(i\) is denoted as \(m_i\). Let us assume that there are \(M/2\) distinct particle species and that each species has two elements. Pair exclusion means that no pair of particles of the same species is not allowed to stay at a same site. Such an interaction appears naturally in evolving networks \[13\].

The particle hopping dynamics is given as follows: At each time step, (i) we select a source site \(i\) at random from among \(N\) sites. (ii) When \(m_i > 0\), we select one particle from among \(m_i\) particles and attempt a particle hopping with the probability

\[
u(m_i) = \left(1 + \frac{b}{m_i}\right)/\left(1 + b\right).
\]

(iii) There are \(2d\) possible target sites \(\{i \pm e_k | k = 1, \cdots, d\}\), where \(e_k\) denotes the unit vector in the \(k\)th direction. The target site is chosen from among forward sites \(i + e_k | k = 1, \cdots, d\) with probability \(p\) or from among backward sites \(i - e_k | k = 1, \cdots, d\) with probability \(q = 1 - p\). (iv) The hopping attempt is accepted only if it does not violate pair exclusion. The hopping is symmetric when \(p = q = 1/2\) while it is biased to the forward (backward) direction when \(p > q (p < q)\). The particle dynamics is illustrated in Fig. \[1\].

Due to pair exclusion, a hopping attempt is accepted or rejected depending on a particle’s species distribution. In Ref. \[12\], the accepting probability was shown to be approximately given by

\[
u(m) \approx 1 - \frac{m}{M},
\]

where \(m \ll M\) is the occupation number at a target site. Using the approximation, we can map the PEP to a driven diffusive system whose particle hopping probability from site \(i\) to \(j\) is given by

\[
W_{ji} = \frac{\nu(m_j)u(m_i)}{\nu(m_i)u(m_j)}
\]

with the hopping probability \(p_h = p\ (q)\) to the forward (backward) direction.

When the hopping is symmetric \((p = q)\), the model is solvable, and its stationary state probability distribution is given by a product form as in Eq. \[1\]. In fact, the model has the same stationary state as the ZRP with the hopping rate function given by

\[
u_ZRP(m) = \nu(m)/\nu(m - 1) \propto 1 + \frac{b}{m} + \frac{m}{M}.
\]

Note that the additional factor \([\frac{b}{m}]\) accounts for pair exclusion. Such a factor is irrelevant when \(m = O(1)\), but it plays a crucial role in the condensation phase. It suppresses a macroscopic condensate, and multiple mesoscopic condensates of size \(m_{\text{con}} \sim N^{1/2}\) and of number \(N_{\text{con}} \sim N^{1/2}\) appear \[12\]. Those mesoscopic condensates are spatially uncorrelated because the probability distribution has a factorized product form.

In this work, we investigate the driven PEP with \(p = 1\) and \(q = 0\). The hopping bias invalidates the mapping of the PEP to the ZRP; hence, the stationary state probability distribution is not given by a product form. One obvious question is whether macroscopic or mesoscopic condensation occurs or not. Another interesting question is about a spatial correlation. Since the stationary state is not given by a factorized form, a spatial correlation in the particle distribution exists. Numerical simulation results on these issues are presented in the following section.

![FIG. 1: The driven pair exclusion process in one (a) and two dimensions (b). Particle species are distinguished with filled patterns in (a). Pair exclusion forbids the particle represented by an empty circle to hop to the right.](image-url)
III. NUMERICAL RESULTS

We have performed extensive Monte Carlo simulations for the driven PEP in $d = 1$ and $d = 2$ dimensions. Particles are allowed to hop only in the preferred forward direction ($p = 1$ and $q = 0$) (see Fig. 1). We adopt the jumping probability in Eq. (2) with $b = 4$. This particular value of $b$ is chosen because it allows mesoscopic condensation for the symmetric PEP \cite{12}. We start with $M$ particles distributed randomly on $N = L^d$ sites, and data are measured in the stationary state over a time interval $T \geq 10^8$. The system sizes are up to $L = 16000$ in 1D and $L = 100$ in 2D. The condensation transition is examined with the occupation number distribution

$$P(m) = \frac{1}{L^d} \left( \sum_{i=1}^{L^d} \delta(m_i, m) \right),$$

which is the probability of a site having $m$ particles.

![FIG. 2: The occupation number distribution $P(m)$ is shown at various system sizes $N = L^d$, for 1D in (a) and 2D in (b). Parameter values are $\rho = 4$ and $b = 4$. The inset shows $P(m)$ for systems of size $N = 16000$ in 1D and $N = 100^2$ in 2D when the particle density $\rho$ is 1/4, 1/2, and 4, which are below, equal to, and above the critical density $\rho_c = 1/2$, respectively. The symbols represent the critical distribution of the corresponding ZRP given in Eq. (7) with $b = 4$.](image)

When the particle density is low, the distribution function decays exponentially (see the insets in Fig. 2) in $m$. Without pair exclusion, the PEP reduces to the ZRP, which does not show condensation at low particle density. Since pair exclusion suppresses condensation, naturally, condensation does not occur.

When the particle density is so low that there is no condensate in the system, the effect of pair exclusion represented by Eq. (3) can be negligible in the infinite size limit. Namely, the PEP becomes equivalent to the ZRP in the normal phase without condensates. We expect that the equivalence persists up to the critical density $\rho_c = 1/(b-2)$ at which the ZRP undergoes a condensation transition \cite{4}. In the insets of Fig. 2, we compare the occupation number distribution at $\rho = \rho_c$ with the critical occupation number distribution of the ZRP \cite{4}

$$P_c(m) \propto \frac{\Gamma(m+1)}{\Gamma(m+b+1)}$$

with $\Gamma(x) = (x-1)!$, which scales as $P_c(m) \sim m^{-\beta}$ for large $m$. The numerical data are in good agreement with the critical distribution with $b = 4$. This comparison shows that the driven PEP undergoes a condensation transition at the same threshold $\rho_c = 1/(b-2)$.

When the density is high, a broad peak in $P(m)$ appears. The peak position moves to the right as $L$ increases, as shown in Fig. 2. The peak represents condensates. The peak is broad, so we quantify the size of a typical condensate, $m_{\text{con}}$, with the highest peak position. The dependence of the condensate size on the number of lattice sites is plotted in Fig. 3. It follows a power-law scaling as

$$N_{\text{con}}(N) \sim N^{\beta}$$

with $\beta \simeq 0.56$ in 1D and $\beta \simeq 0.62$ in 2D. The condensate is not macroscopic but mesoscopic with $0 < \beta < 1$.

We note that the spectral weight of the condensate peak is much larger than $1/N$, which implies that there are multiple condensates. We quantify the number of condensates, $N_{\text{con}}$, from the total spectral weight of the peak. There is a local minimum in $P(m)$ separating the occupation number into two regions. We estimated $N_{\text{con}}$ as the total spectral weight beyond the local minimum multiplied by $N$. Numerical data in Fig. 4 show that it also follows a power-law scaling as

$$N_{\text{con}} \sim N^{\alpha}$$

with $\alpha \simeq 0.51$ in 1D and $\alpha \simeq 0.34$ in 2D. The total number of particles belonging to the condensates is proportional to $N$. 

![FIG. 3: Power-law scaling of the size of a typical condensate $m_{\text{con}}$ (□), the number of condensates $N_{\text{con}}$ (○), and the total number of particles belonging to the condensates $m_{\text{total}}$ (●) in 1D (filled symbols) and 2D (empty symbols). Parameter values are $b = 4$ and $\rho = 4$. Solid line are guides for the eye.](image)
We have shown that the driven PEP exhibits mesoscopic condensation characterized by the scaling \( m_{\text{con}} \sim N^\beta \) and \( N_{\text{con}} \sim N^\alpha \). It contrasts with the CMA and the TP in which condensation does not occur in the presence of a hopping bias. Although the driven PEP exhibits a similar type of mesoscopic condensation as the symmetric PEP, significant differences exist. The symmetric PEP can be approximated as a driven diffusive system with the hopping probability given in Eq. (4), which can be mapped to the ZRP. Since the stationary state of the ZRP is given by a factorized product form, no spatial correlation exists. Consequently, the ZRP in all dimensions has the same property \([12]\). The variation of \( \alpha \) and \( \beta \) with respect to \( d \) in the driven PEP suggests that a spatial correlation does matter.

The spatial correlation is clearly seen from the spatial distribution of the occupation number. When condensation occurs, one can locate the site \( i_M \) at which the occupation number is maximum. Then, one can measure the mean occupation number \( C(r) \) at site \( j = i_M + r \) displaced from \( i_M \) by \( r \). The distribution function is plotted for 1D at \( b = 4 \) and \( \rho = 4 \) in Fig. 4 (a). The occupation number is high near \( r = 0 \) and then decays to a constant value at large \( r \). This shows that mesoscopic condensates are bound to each other to form a cluster. We speculate that the clustering may originate from the jamming of driven condensates due to pair exclusion.

The occupation number distribution turns out to follow a scaling form

\[
C(r) = N^\beta F(r/N^\delta)
\]

with \( \beta = 0.56 \) and \( \delta = 0.30 \) (see Fig. 4 (b)). The scaling function decays exponentially. There are \( N_{\text{con}} \sim N^\alpha \) condensates, so one may expect that \( \delta = \alpha \). However, the numerical analysis in Fig. 4 (b) yields a value of \( \delta = 0.30 \), which is smaller than \( \alpha = 0.51 \). The discrepancy may be explained by the assumption that there are \( N_{\text{cl}} \sim N^{\alpha - \delta} \) such condensate clusters. In fact, the inset of Fig. 4(a) shows that there are a few condensate clusters. With small values of \( (\alpha - \delta) \simeq 0.2 \), the expected number of clusters is small, and we could not verify the power-law scaling numerically.

We also study the clustering of mesoscopic condensates in 2D. Measuring the occupation number distribution \( C(r) \) at all lattice sites \( j = i_M + r \) with \( r = x\hat{e}_1 + y\hat{e}_2 \), we found that all condensates form a single cluster. Interestingly, the condensate cluster has an elongated shape. Presented in Fig. 5 are numerical data \( C_{\parallel}(x) \), the occupation number at sites \( j = i_M + x(\hat{e}_1 + \hat{e}_2) \) along the direction parallel to the bias, and \( C_{\perp}(x) \), the occupation number at sites \( j = i_M + x(\hat{e}_1 - \hat{e}_2) \) along the direction perpendicular to the bias. They are fitted well to the scaling form

\[
C_{\parallel, \perp}(x) = N^\beta \mathcal{C}_{\parallel, \perp}(x/N^{\delta_{\parallel, \perp}}).
\]

The exponent \( \beta = 0.62 \) is the same as the one for the condensate size. On the other hand, the two exponents describing the characteristic width of the cluster have different values of \( \delta_{\parallel} \simeq 0.25 \) and \( \delta_{\perp} \simeq 0.1 \). The total number of mesoscopic condensates inside the cluster scales as \( N^{\delta_{\parallel} + \delta_{\perp}} \sim N^{0.35} \), which is close to the previous estimate \( N^\alpha \) with \( \alpha = 0.34 \).

We add a few remarks on the shape of the occupation number distribution function. We observe that the distribution is symmetric as \( C(r) = C(-r) \) in 1D. In 2D, the distribution in the transverse direction is trivially symmetric as \( C_{\perp}(x) = C_{\perp}(-x) \). However, it is asymmetric in the longitudinal direction as \( C_{\parallel}(x) \neq C_{\parallel}(-x) \). The origin of the asymmetry is not understood yet.

From the spatial distribution of the occupation number, we conclude that there is a clustering of condensates in the driven PEP. Condensate clustering was observed in interacting particle systems \([19, 20]\), where the clustering was caused by a particle attraction. Clustering in the driven PEP has a different origin. Clustering does
not occur in the unbiased PEP where mesoscopic condensates are distributed randomly. When there is a hopping bias, there is congestion due to the pair exclusion. The jamming leads to clustering.

IV. SUMMARY

We have investigated particle condensation in the driven PEP in 1D and 2D. The PEP with symmetric hopping displays mesoscopic condensation, which is characterized by uncorrelated condensates of number $N_{\text{con}} \sim N^\alpha$ and of size $m_{\text{con}} \sim N^\beta$ with $\alpha = \beta = 1/2$ in all dimensions [12]. The driven PEP also exhibits mesoscopic condensation. However, a fundamental difference exists. The driven PEP is characterized by a spatial correlation that is manifested in the $d$-dependent exponents and the clustering of condensates. The clustering is a consequence of jamming caused by the combined effect of the hopping bias and the pair exclusion. The shape of the condensate cluster follows the scaling form of Eqs. (10) and (11). It is interesting to note that the cluster is anisotropic in 2D. Our results show that the spatial correlation leads to rich behaviors of condensation phenomena, which need to be investigated thoroughly in the future.

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