Entanglement Witnesses for Indistinguishable Particles

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We study the problem of witnessing entanglement among indistinguishable particles. For this purpose, we derive a set of equations which results in necessary and sufficient conditions for probing multipartite entanglement between arbitrary systems of Bosons or Fermions. The solution of these equations yields the construction of optimal entanglement witnesses for partial and full entanglement. Our approach unifies the verification of entanglement for distinguishable and indistinguishable particles. We provide general solutions for certain observables to study quantum entanglement in systems with different quantum statistics in noisy environments.

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I. INTRODUCTION

Nonlocal correlations among many particles or quantized fields are one key element of the quantum nature of physics [1–4]. Applications in metrology use this quantum feature to beat classical limitations [5–8]. Quantum entanglement has been also studied as a resource for quantum information technologies [9]. For a fundamental characterization, a lot of attention has been devoted to entanglement between distinguishable particles (DP) or multiple degrees of freedom [10, 11].

Indistinguishable particles (IP), on the other hand, are indispensable for understanding the properties of many-particle quantum systems. For IP systems having a certain spin statistics [12], however, even the notion of entanglement itself has no generally accepted definition [13]. For example, the two-Fermion state,

\[ |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle \equiv |\uparrow\rangle \wedge |\downarrow\rangle, \tag{1} \]

is a Bell-like entangled state using the tensor product \(\otimes\), and, at the same time, it is a product state in the notion of the antisymmetric product \(\wedge\). This ambiguity originates from the fact that the (anti)symmetrization requirement of (Fermion)Boson systems has formally the same structure as a nonlocal superposition. The left-hand side of Eq. (1) is closely related to the well-established theory of entanglement between distinguishable subsystems [14]. Hence, we will focus on the right-hand side [15, 16]. That is, for Fermions, the state (1) is separable in the exterior algebra. Similarly, we will refer to product states of the symmetric algebra, with the associated symmetric product \(\vee\), as separable Boson states. In general, we will focus on the question: "How to certify quantum entanglement, which does not rely on (anti)symmetrization?"

For bipartite pure states, the relation between entanglement for DP and the tensor product is represented by the Schmidt decomposition [17]. Whenever a single tensor product state is sufficient to expand a pure state, it is separable. In analogy, entanglement between IP is characterized by the Slater decomposition [15, 16]. A pure Fermion or Boson state is separable, if it is a single anti-symmetric or symmetric product state, given in terms of Slater determinants or permanents, respectively. A classical mixture of product states extends the corresponding definitions to mixed quantum states. If such a representation is impossible, the state under study is entangled.

Irregardless of the product for constructing compound Hilbert spaces, \(\otimes, \wedge, \vee\), we can detect quantum correlation via so-called entanglement witnesses [15, 18]. A witness is an observable which is non-negative for all separable states, and may be negative for entangled states. Such criteria have been successfully applied to experimentally probe quantum correlations [19–24]. In the same dimensions, the characterization and application of entanglement in systems of IP gained an increasing importance during the last years; see, e.g., [25–30]. Recently, a method for the construction of optimized multipartite entanglement witnesses for DP has been proposed [31] and applied to perform a full entanglement analysis of experimentally generated multimode states [32].

In the present contribution, we derive equations which allow the construction of optimized, necessary and sufficient entanglement probes for Boson and Fermion systems. The formalism is applicable to arbitrary numbers of particles, partially and fully entangled states, and discrete and continuous variable quantum systems. Our method unifies the detection of entanglement for DP and IP. Furthermore, we explicitly construct witnesses to demonstrate the strength of this technique to certify entanglement for different spin statistics in noisy environments.

II. \(K\)-SEPARABLE FERMION AND BOSON STATES

The formulation of multipartite entanglement for DP is based on the tensor product structure of compound Hilbert spaces \(\mathcal{H}^\otimes N\). A \(N\)-partite quantum state \(\hat{\sigma}\) is fully separable, if it can be written as a convex combina-
tion of product states of the subsystems [14],
\[ \hat{\sigma} = \int dP(a_1, \ldots, a_N) \frac{|a_1, \ldots, a_N \rangle \langle a_1, \ldots, a_N|}{|a_1, \ldots, a_N|_1, \ldots, a_N|_{a_1, \ldots, a_N}_N} \]  
(2)

Here, \(|a_1, \ldots, a_N| = |a_1| \otimes \cdots \otimes |a_N|\) are, in general, unnormalized \(N\)-partite product vectors, and \(P\) is a classical probability distribution.

A fundamental postulate of quantum mechanics for Bosons or Fermions is that the quantum states are symmetric or antisymmetric upon exchange of the subsystems, respectively. This restricts the physical states to the (anti)symmetric subspace of the \(N\)-fold tensor product Hilbert space, \(\mathcal{H}^{\otimes N}, \mathcal{H}^{\vee N} \subset \mathcal{H}^\otimes N\). A projection from the tensor product space to these subspaces is given by the permutation operator,
\[ \hat{\Pi}^\pm |a_1, \ldots, a_N| = \sum_{\sigma \in S_N} \frac{(\pm 1)^{|\sigma|}}{N!} |a_{\sigma(1)}, \ldots, a_{\sigma(N)}|, \]  
(3)

where \(|\sigma|\) denotes the parity of the permutation \(\sigma \in S_N\). Other projections might be similarly studied, which allow a generalization to other parastatistics. Now, the (anti)symmetric product states can be identified as [39]:

\[ |a_1 \land \cdots \land |a_N| \equiv \hat{\Pi}^- |a_1, \ldots, a_N|, \]  
(4)

and

\[ |a_1 \lor \cdots \lor |a_N| \equiv \hat{\Pi}^+ |a_1, \ldots, a_N|. \]  
(5)

Equations (4) and (5) define fully or \(N\)-separable Fermions and Bosons, respectively. More involved is the notion of \(K\)-separable states, see [10, 11] for introductions. In systems of DP a \(K\)-separable vector \(|\psi_K| \in \mathcal{H}_K\) is defined as a product vector
\[ |\psi_K| = |b_1| \otimes \cdots \otimes |b_K| = |b_1, \ldots, b_K|, \]  
(6)

for positive integers \((N_1, \ldots, N_K)|, |b_k| \in \mathcal{H}^\otimes N_k\), and \(\sum_{k=1}^K N_k = N\). Consequently, a \(K\)-separable (anti)symmetric vector is defined as

\[ |\psi_K^\pm| = \hat{\Pi}^\pm |\psi_K| = \hat{\Pi}^\pm |b_1, \ldots, b_K|. \]  
(7)

Based on the (anti)symmetric product, one gets a general definition of \(N\)-separability for IP [34, 39]. Additionally, a \(N\)-Fermion or \(N\)-Boson quantum state \(\hat{\sigma}\) is \(K\)-separable, \(1 \leq K \leq N\), if it can be written as a convex combination of (anti)symmetric product states [40]
\[ \hat{\sigma} = \int dP(b_1, \ldots, b_K) \frac{|b_1, \ldots, b_K \rangle \langle b_1, \ldots, b_K|}{|b_1, \ldots, b_K|_1 \cdots |b_1, \ldots, b_K|_K} \]  
(8)

Exchanging the tensor product \(\otimes\) by either the symmetric product \(\lor\) or the antisymmetric product \(\land\), cf. Eqs. (4) and (5), keeps the separability definition (2) of DP structurally preserved. If a state cannot be written according to definition (3) for \(K = N\), then entanglement between IP is certified beyond any correlation that can arise from the (anti)symmetrization requirement of the quantum statistics itself.

III. CONSTRUCTION OF ENTANGLEMENT WITNESSES

Since Eq. (8) defines a closed, convex set of states, the Hahn-Banach separation theorem is applicable. This means that for any \(K\)-entangled state of IP, \(\hat{\sigma} = \hat{\Pi}^\pm \hat{\sigma} \hat{\Pi}^\pm\), exists a bounded Hermitian operator \(\hat{L}\), such that:
\[ \text{tr}(\hat{L} \hat{\sigma}) > \sup \{ \text{tr}(\hat{L} \hat{\sigma}) : \forall \hat{\sigma} \text{ in Eq. (8)} \}. \]  
(9)

In this form we have a necessary and sufficient condition in terms of observables \(\hat{L}\) probing multipartite entanglement between IP in finite and infinite dimensional spaces.

It is sufficient to take the least upper bound on the right-hand-side of inequality (9) over all normalized product vectors, being the extremal points of the given convex set of \(K\)-separable states. Moreover, the operator \(\hat{\Pi}^\pm\) plays the role of the identity in the (anti)symmetric subspace. Combining these facts allows us to write condition (9) in terms of the expectation value of the entanglement witness operator:
\[ \hat{W}_\pm = G \hat{\Pi}^\pm - \hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm, \]  
(10)

\[ G = \sup \left\{ \frac{\langle b_1, \ldots, b_K| \hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm |b_1, \ldots, b_K\rangle}{\langle b_1, \ldots, b_K| \hat{\Pi}^\pm |b_1, \ldots, b_K\rangle} \right\}, \]  
(11)

where the least upper bound is taken over all product vectors \(|b\rangle\), see also [15, 16]. Interestingly, the simple modification \(\hat{\Pi}^\pm \to \hat{1}\) in Eqs. (10) and (11) yields the corresponding construction of witnesses for DP [32, 34].

The least upper bound in Eq. (11) defines an optimization of the Rayleigh quotient,
\[ g = \frac{\langle b_1, \ldots, b_K| \hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm |b_1, \ldots, b_K\rangle}{\langle b_1, \ldots, b_K| \hat{\Pi}^\pm |b_1, \ldots, b_K\rangle} \to G. \]  
(12)

The optimization is carried out as a derivative of the Rayleigh quotient:
\[ \frac{\partial g}{\partial |b_j|} = 0 \]  
for \(j = 1, \ldots, K. \)  
(13)

We define for \(\hat{X} \in \{ \hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm \hat{\Pi}^\pm \}\) the operators \(\hat{X}_{|b_j|} = \langle b_1, \ldots, b_j-1, b_{j+1}, \ldots, b_K| \hat{X}|b_1, \ldots, b_j-1, b_{j+1}, \ldots, b_K\rangle\).
Thus, the derivatives can be written as
\[ \frac{\partial g}{\partial |b_j|} = \frac{\langle \hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm |b_j|_{\hat{\Pi}^\pm} b_j \rangle}{\langle |b_j| |\hat{\Pi}^\pm |b_j\rangle} - \frac{g(\hat{\Pi}^\pm |b_j\rangle_{\hat{\Pi}^\pm} |b_j\rangle)}{\langle b_j| \langle b_j|_{\hat{\Pi}^\pm} b_j \rangle}. \]
(14)

The resulting set of algebraic equations,
\[ (\hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm)_{|b_j|} |b_j\rangle = g(\hat{\Pi}^\pm |b_j\rangle_{\hat{\Pi}^\pm} |b_j\rangle) \]  
for \(j = 1, \ldots, K, \) define the separability eigenvalue (SEValue) equations for IP. The common eigenvalue \(g\) is denoted as the SEValue, and the (anti)symmetric product vector \(\hat{\Pi}^\pm |b_1, \ldots, b_K\rangle\) is the corresponding separability eigenvector (SEvector) for IP.
The SEvalue equations for IP represent a system of $K$ coupled eigenvalue equations. Remarkably, it turns out that they have the same structure as the corresponding equations for DP \cite{37}. For DP, we use the $N$-fold identity $\hat{1}$ that replaces the projector $\hat{1}^\pm$ in \cite{15}. More properties of the SEvalue equations for IP are derived in Appendix B, including a second form of these equations as a single perturbed eigenvalue problem and the invariance under local unitary operations.

We found that the SEvalue $g$ corresponds to an optimal expectation value of $\hat{L}$ for $K$-separable Boson or Fermion states. Therefore, we get the bound in the entanglement criterion \cite{9} as

$$\sup \{\text{tr}(\hat{L}\hat{\sigma}) : \text{for all } \hat{\sigma} \text{ in Eq. } (9)\} = \sup \{g\},$$

(16)
i.e., the initial convex optimization problem is solved by the largest SEvalue of the equations \cite{15}. Now, the entanglement condition for the state $\hat{\gamma}$ may be written as

$$\langle \hat{L} \rangle = \text{tr}(\hat{\gamma}\hat{L}) > \sup \{g\}.$$  

(17)

Alternatively, this condition can be written in terms of witnesses constructed from Eqs. \cite{10} and \cite{11}:

$$\hat{W}_\pm = G\hat{\Pi}^\pm - \hat{\Pi}^\pm \hat{L}\hat{\Pi}^\pm, \text{ with } G = \sup \{g\},$$

(18)

which reads as $\text{tr}(\hat{\gamma}\hat{W}_\pm) < 0$. Hence, by solving the algebraic problem \cite{15} of observables $\hat{L}$, we are able to construct, in principle, any optimal entanglement witnesses for multiple correlated Bosons or Fermions. Moreover, the SEvalue equations for IP might be also used for a numerical optimization if an analytical solution is not available. Since the criterion in \cite{17} and the witnessing approach are equivalent, we study from now on solely the former one.

**IV. BIPARTITE EXAMPLE**

First, we aim to witness bipartite entanglement. As our observable we may choose the rank one operator:

$$\hat{L} = |\psi\rangle \langle \psi|.$$  

(19)

For DP, we have the well-known Schmidt decomposition \cite{9} to represent the state,

$$|\psi\rangle = \sum_{i,j=1}^{d} \psi_{i,j}|i, j\rangle = \sum_{n=1}^{d} \lambda_n |u_n, v_n\rangle,$$

(20)
in terms of orthonormal sets $\{|u_n\rangle\}$ and $\{|v_n\rangle\}$ ($\lambda_n \geq 0$). For a Fermion or Boson state we get the Slater decomposition – as studied, for example, in Refs. \cite{16} \cite{39} \cite{42} – as:

$$|f\rangle = \hat{\Pi}^- |\psi\rangle = \sum_{i,j=1}^{d} f_{i,j} |i, j\rangle$$

$$= \sum_{n=1}^{(d/2)} \kappa_n (|w_{2n-1}, w_{2n}\rangle - |w_{2n}, w_{2n-1}\rangle),$$

(21)

$$|b\rangle = \hat{\Pi}^+ |\psi\rangle = \sum_{i,j=1}^{d} b_{i,j} |i, j\rangle = \sum_{n=1}^{d} \kappa'_n |w'_n, w'_n\rangle,$$

(22)

for $f_{i,j} = -f_{j,i}$, orthonormal $|w_n\rangle$, $|x\rangle$ denoting the largest integer less or equal to $x$, and $\kappa_n \geq 0$; and for $b_{i,j} = b_{j,i}$ with orthonormal $|w'_n\rangle$ and $\kappa'_n \geq 0$.

The solution of the SEvalue equation for IP and the observable in \cite{19} is computed in Appendix C. It is worth pointing out that the nontrivial solutions, i.e. $g \neq 0$, have forms which are directly related to the decompositions \cite{21} and \cite{22}. Namely, we get for Fermions the SEvalues $g_n = 2\kappa_n^2$ for the SEvectors $|w_{2n-1}\rangle \wedge |w_{2n}\rangle$, and for Bosons the SEvalues $g_n = \kappa_n^2$ and $g_{k,l} = \kappa_k^2 + \kappa_l^2$ for the SEvectors $|w'_n\rangle \vee |w_n\rangle$ and $|w'_k\rangle \vee |w'_l\rangle$ (with $|w'_k\rangle = \sqrt{\kappa_k^2} |w'_k\rangle + i \sqrt{\kappa_l^2} |w'_l\rangle$), respectively. Now, the entanglement condition \cite{17} can be written in terms of the fidelities:

$$\langle b | \hat{\gamma} | b \rangle > \max_{1 \leq k < l \leq d} \{ \kappa_k^2 + \kappa_l^2 \},$$

(23)

or $\langle f | \hat{\gamma} | f \rangle > \max_{1 \leq n \leq (d/2)} \{ 2\kappa_n^2 \},$  

(24)

for bipartite, mixed or pure entangled states of Bosons, $\hat{\gamma}_B$, or Fermions, $\hat{\gamma}_F$. Note that, for the case of DP, we get the separable bound \cite{43}: $\max_{1 \leq n \leq d} \{ \lambda_n^2 \}$; cf. Eq. \cite{20}.

Let us apply the method to the pure state $|\psi\rangle$ which is mixed with white noise,

$$\hat{\rho} = \hat{p} |\psi\rangle \langle \psi| \hat{1} + (1 - \hat{p}) \frac{\hat{1}}{\text{tr} \hat{1}},$$

(25)

with $p \in [0, 1]$ being a noise parameter, $\hat{1} \in \{1, \hat{\Pi}^+, \hat{\Pi}^-, \}$, and the second term being separable. Replacing $\hat{1}$ with other projections we could additionally study entanglement for other parastatistics, e.g., for anyons \cite{44} \cite{45}.

In Fig. \cite{1} we compare different quantum statistics regarding their entanglement properties for the mixed state \cite{25} in dependence on the dimensionality of the single particle’s Hilbert space, $d = \dim \mathcal{H}$, applying the test operator in \cite{19}. As long as $\langle \hat{L} \rangle > \sup \{g\}$ (gray area in Fig. \cite{1}), we have identified entanglement for the mixing parameter $p$ for DP (plot: SR$>1$), Fermions or Bosons. Since the structure of (anti)symmetric product states is related to Bell-like states, cf. Eq. \cite{11}, we also consider Schmidt rank (SR) two states. The calculation of the corresponding bounds is done in \cite{40} and applied in \cite{47}. For any $p$ in gray area of the plot SR$>2$, we can conclude that more than two tensor-product states have to be superimposed to describe the state \cite{25}. Thus, our approach allows the detection of different forms of entanglement based on a single observable.
where $p$ is a classical probability distribution. The expectation value of the observable (26) together with the bound (27), yields the entanglement condition

$$\langle \hat{L} \rangle = \int_{|q|<1} d^2q p(q)(1-|q|^2)(q + q^*) > \frac{1}{2^{K-1}}, \quad (30)$$

for details see Appendix D.

The identification of multipartite entangled Bosons and Fermions as well as DP is shown in Fig. 2 for a dephasing channel, i.e.: the amplitude $|q|$ is fixed and phase $\arg q$ is randomized. As long as the expectation value is above the dashed lines, we certified that the state cannot be a $K$-separable one. This example demonstrates the general possibility to construct spin statistics independent entanglement tests with our approach.

FIG. 2: The expectation value in Eq. (30) is plotted for $N = 5$ (solid curve), an amplitude $|q| = 1/\sqrt{3}$, and a uniformly distributed phase in the interval $\arg q \in [-\delta, \delta]$. If $\langle \hat{L} \rangle$ is above the dashed line $K$, then the state cannot be a $K$-separable one – independently of the quantum statistics.

VI. CONCLUSIONS

We derived a method which allows the construction of entanglement probes in systems of Bosons and Fermions. These necessary and sufficient conditions are capable of determining full and partial entanglement for any number of particles. The optimization of these criteria is based on a set of generalized eigenvalue equations. These equations yield a structural unification of entanglement for distinguishable particles, Fermions and Bosons, and it can be generalized to other parastatistics. Examples for determining entanglement of bipartite and multipartite as well as discrete and continuous variable quantum states demonstrate the wide range of applications of our technique, even in the presence of noise. Additionally, the construction of spin-statistics independent entanglement probes has been established. Since entanglement witnesses are formulated in terms of observables, we believe that our approach will provide a versatile tool to

FIG. 1: The mixing in terms of $p$ for the state (25) depending on $d = \dim \mathcal{H}$ is plotted. As long as $p$ is in the gray shaded area, we successfully detected entanglement. The coefficients in Eqs. (20) and (22) for DP and Bosons, respectively, are chosen to be equal $\lambda_k = \kappa_k = d^{-1/2} (k = 1, \ldots, d)$. In the case of Fermions, we chose $\kappa_k = (2[d/2])^{-1/2} (k = 1, \ldots, [d/2])$, see Eq. (21), yielding a different behavior for even and odd dimensions $d$.

V. MULTIPARTITE EXAMPLE

In the following we will study an entanglement test, which is even independent of the spin statistics of a multipartite system under study. We further assume $\dim \mathcal{H} = \infty$. The observable is

$$\hat{L} = |1, \ldots, N\rangle\langle N+1, \ldots, 2N| + |N+1, \ldots, 2N\rangle\langle 1, \ldots, N|, \quad (26)$$

In Appendix D, we solve the SEvalue equations for DP and IP. The maximal bounds for $K$-separable states is

$$\sup\{|g\rangle\} = (1/2)^{K-1}, \quad (27)$$

independently of the quantum statistics. The observable (26) may be applied to a GHZ-type of state [48]

$$|q\rangle = \sqrt{\nu(\hat{1})} \sum_{n=0}^{\infty} \sqrt{1-|q|^2} q^n |nN+1, \ldots, (n+1)N\rangle, \quad (28)$$

with $\hat{1} \in \{\hat{1}, \hat{\Pi}^+, \hat{\Pi}^-\}$, $\nu(\hat{\Pi}^+) = N!$, and $\nu(\hat{1}) = 1$. This state is of a GHZ-type structure, because for each mode $j$ holds that the individual vectors $|nN+j\rangle$ are orthonormal for different $n$ [49]. The pure state might be perturbed due to a randomly distributed $q (|q| < 1)$:

$$\hat{\rho} = \int_{|q|<1} d^2q p(q)|q\rangle\langle q|, \quad (29)$$

where $p$ is a classical probability distribution. The expectation value of the observable (26) together with the bound (27), yields the entanglement condition

$$\langle \hat{L} \rangle = \int_{|q|<1} d^2q p(q)(1-|q|^2)(q + q^*) > \frac{1}{2^{K-1}}, \quad (30)$$

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characterize entanglement in future experiments, with applications to Bose-Einstein condensates or ultra-cold Fermi systems.

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Appendix A: Symmetrizing operators and $K$-partitions

Let $\mathcal{H}^{\otimes N}$ be an $N$-fold Hilbert space. The symmetric and anti-symmetric projection operators $\hat{\Pi}^+$ and $\hat{\Pi}^-$, respectively, are defined as

$$\hat{\Pi}^\pm = \sum_{\sigma \in S_N} \frac{(\pm 1)^{|\sigma|}}{N!} \hat{P}_{\sigma}, \quad (A1)$$

with $\hat{P}_{\sigma}|a_1, \ldots, a_N\rangle = |a_{\sigma(1)}, \ldots, a_{\sigma(N)}\rangle$ for any permutation $\sigma \in S_N$. It holds $(\hat{\Pi}^\pm)^\dagger = \hat{\Pi}^\pm$. We may study Hermitian operators in a product basis operator expansion, given by terms of the form

$$\hat{X} = \hat{Y}_1 \otimes \cdots \otimes \hat{Y}_N, \quad (A2)$$

with $\hat{Y}_j = \hat{Y}_j$ for $j = 1, \ldots, N$, together with its symmetric form,

$$\hat{X}^{(\text{sym})} = \sum_{\sigma \in S_N} \frac{1}{N!} \hat{Y}_{\sigma(1)} \otimes \cdots \otimes \hat{Y}_{\sigma(N)}. \quad (A3)$$

We claim

$$\hat{\Pi}^\pm \hat{X} \hat{\Pi}^\pm = \hat{X}^{(\text{sym})} \hat{\Pi}^\pm = \hat{\Pi}^\pm \hat{X}^{(\text{sym})}. \quad (A4)$$

The first equality can be directly computed, since for all $|a_1, \ldots, a_N\rangle$ holds:

$$\hat{\Pi}^\pm \hat{X} \hat{\Pi}^\pm |a_j\rangle = \hat{\Pi}^\pm \sum_{\sigma \in S_N} \frac{(\pm 1)^{|\sigma|}}{N!} \hat{Y}_{\sigma(j)} |a_{\sigma(j)}\rangle$$

$$= \sum_{\sigma, \tau \in S_N} \frac{(\pm 1)^{|\sigma|+|\tau|}}{N! \mu!} \hat{Y}_{\tau(j)} \hat{\Pi} |a_{\sigma(j)}\rangle$$

$$= \sum_{\tau \in S_N} \frac{1}{N!} \hat{Y}_{\tau(j)} \sum_{\mu \in S_N} \frac{(\pm 1)^{|\mu|}}{N!} \hat{P}_{\mu} |a_{\mu(j)}\rangle$$

$$= \hat{X}^{(\text{sym})} \hat{\Pi}^\pm |a_j\rangle, \quad (A5)$$

where we used a substitution $\mu = \tau \circ \sigma$ and $(\pm 1)^{|\tau|+|\sigma|} = (\pm 1)^{|\tau \circ \sigma|}$. The second equality in $(A4)$ follows from the fact that $\hat{X}$ and $\hat{\Pi}^\pm$ are Hermitian operators,

$$\hat{\Pi}^\pm \hat{X}^{(\text{sym})} = \hat{\Pi}^\pm \hat{X} \hat{\Pi}^\pm = (\hat{\Pi}^\pm \hat{X} \hat{\Pi}^\pm)^\dagger$$

$$= (\hat{\Pi}^\pm \hat{X}^{(\text{sym})})^\dagger = \hat{X}^{(\text{sym})} \hat{\Pi}^\pm. \quad (A6)$$

For $\hat{X} = \hat{1} \otimes \cdots \otimes \hat{1}$, we get from $(A4)$ that $\hat{\Pi}^\pm$ is idempotent. An observable $\hat{L}$ which solely acts on the corresponding subspaces ($\mathcal{H}_{K,N}^N$ and $\mathcal{H}^{\otimes N}$) should fulfill the commutation relation $[\hat{L}, \hat{\Pi}^\pm] = 0$. From $(A4)$ follows that this is fulfilled for every $\hat{L} = \hat{L}^{(\text{sym})}$.

For every $|\psi\rangle \in \mathcal{H}^{\otimes N}$, we get the (anti)symmetric vector in the projected subspace as

$$|\psi^\pm\rangle = \hat{\Pi}^\pm |\psi\rangle. \quad (A7)$$

In systems of DP a $K$-separable vector $|\psi_K\rangle \in \mathcal{H}_N$ is defined as a product state for some positive integers $(N_1, \ldots, N_K)$

$$|\psi_K\rangle = |b_1\rangle \otimes \cdots \otimes |b_K\rangle, \quad (A8)$$

with $|b_k\rangle \in \mathcal{H}^{\otimes N_k}$ and $\sum_{k=1}^K N_k = N$.

The tuple $(N_1, \ldots, N_K)$ defines a partitioning of the $N$-fold Hilbert space. Further note that $|b_k\rangle$ is not necessarily a product state in $\mathcal{H}^{\otimes N_k}$, and it could even required that $|\psi_K\rangle$ is not $(K+1)$-separable. A $K$-separable (anti)symmetric vector is defined as

$$|\psi_K^{\pm}\rangle = \hat{\Pi}^\pm |\psi_K\rangle. \quad (A9)$$

Since the permutation is applied, the initial ordering of the Hilbert spaces does not play a role, i.e. the element ordering in $(A9)$ is, without loss of generality, given by the tensor product sequence $\mathcal{H}^{\otimes N_1} \otimes \cdots \otimes \mathcal{H}^{\otimes N_K}$. However, let us point out that $(N_1, \ldots, N_K)$ and $(N_1', \ldots, N_K')$ define, in general, different partitions if these tuples which are not identical – up to a permutation of indices. For example, the three partition $(2, 3, 1)$ of a six mode system describes the same partitioning as $(1, 2, 3)$, but it differs from $(2, 2, 2)$.

Appendix B: Separability eigenvalue equations for indistinguishable particles

In this section we study some properties of the SEvalue equations for IP. Even though we find a strong relation to the case of DP, cf. [37], we want to point out that neither the distinguishable case includes the indistinguishable one, nor vice versa. This is due to the properties of the non-invertible operator $\hat{\Pi}^\pm$.

1. Details on the optimization

Let us reconsider the optimization which has been outlined in the main body. The Rayleigh quotient $q = \rho(b_1, \ldots, b_K)$ is defined for the partition $(N_1, \ldots, N_K)$ as

$$\rho(b_1, \ldots, b_K) = \frac{\langle b_1, \ldots, b_K | \hat{\Pi}^\pm \hat{L} \hat{\Pi}^\pm | b_1, \ldots, b_K \rangle}{\langle b_1, \ldots, b_K | \hat{\Pi}^\pm \hat{\Pi}^\pm | b_1, \ldots, b_K \rangle},$$

$$= \frac{\langle b_1, \ldots, b_K | \hat{L}^{(\text{sym})} \hat{\Pi}^\pm | b_1, \ldots, b_K \rangle}{\langle b_1, \ldots, b_K | \hat{\Pi}^\pm | b_1, \ldots, b_K \rangle}, \quad (B1)$$
whose infimum(supremum) yields the minimal(maximal) expectation value of $\hat{L}$ for all $K$-separable and normalized states. Hence, we get the optimization problem

$$\rho(b_1,\ldots,b_K) \to \text{optimum},$$  
(B2)

under the constraint that the denominator exists, i.e. $\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle \neq 0$. Using the abbreviation $\hat{X}_{\pi}$ for the operator which is defined by the relation

$$\forall|x\rangle,|y\rangle \in \mathcal{H}^{\otimes N_k} :$$
$$\langle x|\hat{X}_{\pi}|y\rangle = \langle b_1,\ldots,b_{k-1},x,b_{k+1},\ldots,b_K|\hat{X}\rangle \times |b_1,\ldots,b_{k-1},y,b_{k+1},\ldots,b_K\rangle,$$

we get the derivative as

$$0 = \frac{\partial \rho(b_1,\ldots,b_K)}{\partial b_k} = \frac{\partial}{\partial b_k} \frac{\langle b_k|\hat{\Pi}^{\pm}\hat{L}\hat{\Pi}^{\pm}\rangle_{\text{sym}}|b_k\rangle}{\langle b_k|\hat{\Pi}^{\pm}\rangle_{\text{sym}}|b_k\rangle}$$

(B4)

$$= \frac{\langle b_k|\hat{\Pi}^{\pm}\hat{L}\hat{\Pi}^{\pm}\rangle_{\text{sym}}|b_k\rangle}{\langle b_k|\hat{\Pi}^{\pm}\rangle_{\text{sym}}|b_k\rangle} - \frac{\langle b_k|\hat{\Pi}^{\pm}\hat{L}\hat{\Pi}^{\pm}\rangle_{\text{sym}}|b_k\rangle}{\langle b_k|\hat{\Pi}^{\pm}\rangle_{\text{sym}}|b_k\rangle^2}.$$

As discussed in the main body, this expression can be reformulated in the form of an eigenvalue problem

$$(\hat{L}^{\text{sym}})\hat{\Pi}^{\pm}|b_k\rangle = g(\hat{\Pi}^{\pm}|b_k\rangle), \text{ for } k = 1,\ldots,K,$$

(B5)

with $g = \rho(b_1,\ldots,b_K)$. Finally, we get the upper bound

$$G = \sup_{\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle \neq 0} \rho(b_1,\ldots,b_K) = \sup \{g\}. \quad \text{(B6)}$$

Note that, similarly to the construction in the main body, a witness can be constructed using the lower bound of the Rayleigh quotient $\rho(b_1,\ldots,b_K)$ as

$$\hat{W}_{\pm} = \hat{\Pi}^{\pm}\left[\hat{L} - \inf \{g\}\right]^{\hat{\Pi}^{\pm}}.$$  
(B7)

2. Second form

Equivalent to the form of the SEvalue equations for IP in (B5), one might formulate a second form. The solutions $g$ and $\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle$ of the Hermitian operator $\hat{L}$ can be found by solving

$$\hat{L}^{\text{sym}}\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle = g\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle + |\chi\rangle,$$  
(B8)

with the perturbation term $|\chi\rangle$, which has to fulfill for all $j = 1,\ldots,K$ and for all $|x\rangle \in \mathcal{H}^{\otimes N_j}$ an orthogonality relation of the form

$$\langle b_1,\ldots,b_{j-1},x,b_{j+1},\ldots,b_K|\chi\rangle = 0$$  
(B9)

to be equivalent with the first form in (B5). Note that the perturbation $|\chi\rangle$ is an element of the (anti)symmetric subspace, since $\hat{\Pi}^{\pm}[\hat{L}^{\text{sym}} - g|\chi\rangle] = |\chi\rangle$. The coupled set of equations of the first form (B5) is transformed into a single, but perturbed, eigenvalue equation of the second form (B8).

3. Transformation properties

Another important property of the SEvalue equations for IP is the behavior under certain transformations of the observable. Local unitaries $\hat{U}$ and shifting of $\hat{L}$, leading to a transformed observable

$$\hat{L}' = \left[\hat{U}^{\otimes N}\right]^{\dagger} \left[\lambda_1\hat{L} + \lambda_2\hat{\Pi}^{\pm}\right] \left[\hat{U}^{\otimes N}\right],$$  
(B10)

with $\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\}$, can be directly passed onto the solutions. If the SEvalue $g$ together with the SEvector $\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle$ is a solutions of the SEvalue equations for IP of $\hat{L}$, then the operator $\hat{L}'$ has the corresponding solutions

$$g' = \lambda_1 g + \lambda_2,$$

(B11)

$$\hat{\Pi}^{\pm}|b'_1,\ldots,b'_K\rangle = \hat{U}^{\otimes N}\hat{\Pi}^{\pm}|b_1,\ldots,b_K\rangle$$

$$\quad = \hat{\Pi}^{\pm} \left[\hat{U}^{\otimes N}\right] |b_1,\ldots,b_K\rangle.$$  
(B12)

Hence, by solving the SEvalue equations for IP for a given $\hat{L}$ we gain the solutions for a whole class of observables. Additionally, we get for $\lambda_1 = 1$ and $\lambda_2 = 0$, that the SEvalues are invariant under local transformations.

Appendix C: Bipartite Example

As a first example, we consider witnesses for $N = 2$ based on the projection

$$\hat{L} = |\psi\rangle\langle\psi|,$$  
(C1)

with $|\psi\rangle \in \mathcal{H}^{\otimes 2}$. In case of Fermions, we have an antisymmetric state $|f\rangle = \hat{\Pi}^{-}|\psi\rangle \in \mathcal{H}^{\otimes 2}$, and for Bosons we have a symmetric state $|b\rangle = \hat{\Pi}^{+}|\psi\rangle \in \mathcal{H}^{\otimes 2}$. For simplicity, we will assume in the following $\hat{L} = |f\rangle\langle f| (\hat{L} = |b\rangle\langle b|)$ for the two-Fermion (two-Boson) system which yields $\hat{L} = \hat{\Pi}^{+}\hat{\Pi}^{-} (\hat{L} = \hat{\Pi}^{+}\hat{\Pi}^{-})$.

1. Fermions

First, we give a unitary state representation for arbitrary $d$-dimensional ($d \in \mathbb{N} \cup \{\infty\}$) pure states of two Fermions. We start with a Fermion state,

$$|f\rangle = \sum_{i,j=1}^{d} f_{i,j}|i,j\rangle, \text{ with } f_{i,j} = -f_{j,i},$$  
(C2)

and introduce a skew-symmetric coefficient matrix

$$M_f = (f_{i,j})_{i,j} = -M_f^T.$$  
(C3)
Using the Autonne-Takagi factorization in Ref. [42], we find the Slater decomposition of this coefficient matrix,
\[ \hat{M}_f = \hat{V} \hat{D} \hat{V}^T, \]  
with \( \hat{D} \) being a block diagonal matrix containing anti-diagonal \( 2 \times 2 \) blocks and \( \lambda_j \geq 0 \). Thus, we get
\[ |f\rangle = \hat{V} \otimes \hat{V} \sum_{n=1}^{[d/2]} \lambda_j (|2n - 1, 2n\rangle - |2n, 2n - 1\rangle), \]  
with \( \lfloor \cdot \rfloor \) being the floor function. Let us note that the SEvalues are invariant under unitary separable operations \( \hat{V} \otimes \hat{V} \), which solely rotate the SEvectors. Hence, without loss of generality, we assume \( \hat{V} = 1 \).

Now, we consider operators having an expansion as
\[ \hat{L} = \sum_{m,n=1}^{[d/2]} L_{m,n}(|2m - 1, 2m\rangle - |2m, 2m - 1\rangle) \times (|2n - 1, 2n\rangle - |2n, 2n - 1\rangle), \]  
which is in the special case \( L_{m,n} = \lambda_m \lambda_n \) the desired projection operator \((C1)\). Since \( \hat{L} = \hat{\Pi}^\dagger \hat{\Pi} \), the SEvalue equations for Fermions in the second form read as
\[ \hat{L}|a_1, a_2\rangle = \frac{1}{2} (|a_1, a_2\rangle - |a_2, a_1\rangle) + |\chi\rangle, \]  
Using \( \gamma_n = (|2n - 1, 2n\rangle - |2n, 2n - 1\rangle)|a_1, a_2\rangle \), we get
\[ \hat{L}|a_1, a_2\rangle = \sum_{m=1}^{[d/2]} \left[ \sum_{n=1}^{[d/2]} L_{m,n} \gamma_n \right] (|2m - 1, 2m\rangle - |2m, 2m - 1\rangle). \]  
We find that \( \hat{L}|a_1, a_2\rangle \) is already diagonalized in the form \((C5)\). Hence, the orthogonality of \( |a_1, a_2\rangle \) to the perturbation \( |\chi\rangle \) is fulfilled if
\[ \hat{\Pi}^\dagger |a_1, a_2\rangle = \frac{1}{2} (|2n - 1, 2n\rangle - |2n, 2n - 1\rangle) \]
\[ \cong |2n - 1\rangle \wedge |2n\rangle, \]  
\[ g = 2L_{n,n}, \]  
\[ |\chi\rangle = \sum_{m \neq n} L_{m,n} (|2m - 1, 2m\rangle - |2m, 2m - 1\rangle), \]  
for all \( n = 1, \ldots, [d/2] \). For the special case \( L_{m,n} = \lambda_m \lambda_n \), we get the maximal SEvalue as
\[ G = \max_n \{2\lambda_n^2\}. \]  

2. Bosons

Again, we first give the state representation according to Ref. [42] for arbitrary \( d \)-dimensional pure state of two-Bosons. This means that the symmetric state
\[ |b\rangle = \sum_{i,j=1}^{d} b_{i,j} |i, j\rangle, \]  
with \( |i, j\rangle \) being the floor function. Let us note that the SEvalues are invariant under unitary separable operations \( \hat{V} \otimes \hat{V} \), which solely rotate the SEvectors. Hence, without loss of generality, we assume \( \hat{V} = 1 \).

As in the previous example for Fermions, let us consider the more general operator
\[ \hat{L} = \sum_{m,n=1}^{d} L_{m,n} |m, m\rangle \langle n, n| \]  
The relation \( \hat{L} = \hat{\Pi}^\dagger \hat{\Pi} \) simplifies the SEvalue equations for Bosons in the second form to
\[ \hat{L}|a_1, a_2\rangle = \frac{1}{2} (|a_1, a_2\rangle + |a_2, a_1\rangle) + |\chi\rangle. \]  
Using \( \gamma_n = \langle n, n|a_1, a_2\rangle \), we can now write
\[ \hat{L}|a_1, a_2\rangle = \sum_{m=1}^{d} \left[ \sum_{n=1}^{d} L_{m,n} \gamma_n \right] |m, m\rangle. \]  
Hence one class of solutions is given by
\[ \hat{\Pi}^\dagger |a_1, a_2\rangle = |n, n\rangle \cong |n\rangle \lor |n\rangle, \]  
\[ g = L_{n,n}, \]  
\[ |\chi\rangle = \sum_{m \neq n} L_{m,n} |m, m\rangle. \]  
Unlike in the Fermion case, we have to take a brief look on the decomposition of product states of Bosons. Namely the state \( \hat{\Pi}^\dagger |a_1, a_2\rangle \) for any \( |a_1\rangle \neq |a_2\rangle \) has a decompositions, cf. Eq. \((C16)\), as
\[ \hat{\Pi}^\dagger |a_1, a_2\rangle = \hat{V} \otimes \hat{V} (\lambda_1 |1, 1\rangle + \lambda_2 |2, 2\rangle), \]  
for \( |a_1\rangle = \hat{V} \left( \sqrt{\lambda_1} |1\rangle + i\sqrt{\lambda_2} |2\rangle \right) \) and \( |a_2\rangle = \hat{V} \left( \sqrt{\lambda_1} |1\rangle - i\sqrt{\lambda_2} |2\rangle \right) \).
which includes the scenario \( X_1', X_2' \neq 0 \). Hence, we get a more involved set of solutions of the form (for \( k \neq l \)):

\[
\hat{\Pi}^+|a_1, a_2 \rangle = \lambda_k^2|k, k \rangle + \lambda_l^2|l, l \rangle, \quad (C24)
\]

\[
|\chi \rangle = \sum_{m \neq k, l} (L_{m,k} \lambda_k^2 + L_{m,l} \lambda_l^2)|m, m \rangle, \quad (C25)
\]

where the coefficients \( \lambda_k^2 \) and \( \lambda_l^2 \) are determined from the SEValue equation (C18). We insert (C24) and (C25) into

\[
\hat{L} \hat{\Pi}^+|a_1, a_2 \rangle - |\chi \rangle = \hat{g} \hat{\Pi}^+|a_1, a_2 \rangle, \quad (C26)
\]

and find that the remaining terms to be computed are

\[
(L_{k,k} \lambda_k^2 + L_{k,l} \lambda_l^2)|k, k \rangle + (L_{l,k} \lambda_k^2 + L_{l,l} \lambda_l^2)|l, l \rangle = g(\lambda_k^2|k, k \rangle + \lambda_l^2|l, l \rangle). \quad (C27)
\]

This is a standard eigenvalue problem in \( C^2 \), which has the solutions

\[
g^\pm = \frac{L_{k,k} + L_{l,l} \pm \Delta}{2}, \quad \Delta = \sqrt{(L_{k,k} - L_{l,l})^2 + 4|L_{k,l}|^2},
\]

\[
\lambda_k^2 = 2L_{k,k} \lambda_l^2 \quad \text{and} \quad \lambda_l^2 = L_{l,l} - L_{k,k} \pm \Delta, \quad (C28)
\]

with the Hermiticity condition \( L_{i,k} = L_{k,i}^\dagger \).

Again, in the particular case \( L_{m,n} = \lambda_m \lambda_n \), we get the simplified solutions

\[
g = \lambda_n^2, \quad g^- = 0, \quad \text{and} \quad g^+ = \lambda_k^2 + \lambda_l^2. \quad (C29)
\]

Please also note that in this case the SEvector to the SEValue \( g^+ \) is (up to a scaling) \( \lambda_k|k, k \rangle + \lambda_l|l, l \rangle \); see also (C23). Since \( \lambda_k^2 + \lambda_l^2 \geq \lambda_i^2 \), we get the maximal SEValue as

\[
G = \max_{k \neq l}\{\lambda_k^2 + \lambda_l^2\}. \quad (C30)
\]

**Appendix D: Multipartite Example**

In the main body, we additionally considered an interference operator

\[
\hat{L} = |N+1, \ldots, 2N\rangle \langle N+1, \ldots, 2N| + |N+1, \ldots, 2N\rangle \langle 1, \ldots, N|, \quad (D1)
\]

whose expectation value is the real part of an off-diagonal element of the density operator \( \hat{\rho} \). \( \langle \hat{L} \rangle = 2 \text{Re}(\hat{\rho}(1, \ldots, N), (N+1, \ldots, 2N)) \). Local unitary operations allow the generalization to other off-diagonal elements or phase shifts, \( \text{Re}(\exp(i\tilde{\phi})\hat{\rho}(1, \ldots, N), (N+1, \ldots, 2N)) \).

The injective transformations \( \hat{T}_j \) of the basis \( \{|n\rangle\}_{n \in \mathbb{N}} \)

\[
\hat{T}_j|n\rangle = |nN+j\rangle \quad \text{for} \quad j = 1, \ldots, N, \quad (D2)
\]

are constructed such that one can directly see that the for all \( j, j' = 1, \ldots, N \) and \( n, n' \in \mathbb{N} \) an orthogonality is given,

\[
|n + j|n'N + j'\rangle = \delta_{n,n'}\delta_{j,j'}. \quad (D3)
\]

Therefore, we get for \( \hat{I} \in \{\hat{1}, \hat{I}^+, \hat{I}^-\} \) orthogonality

\[
|1, \ldots, N\rangle \langle N+1, \ldots, 2N| = 0 \quad (D4)
\]

as well as the normalizations

\[
|N+1, \ldots, 2N\rangle \langle N+1, \ldots, 2N| \quad (D5)
\]

\[
= |1, \ldots, N\rangle \langle 1, \ldots, N| = 1/\nu(\hat{1}),
\]

with \( \nu(\hat{\Pi}^\pm) = N! \) and \( \nu(\hat{1}) = 1 \). Due to this fact, we may define the \( K \)-separable vectors

\[
|v_1, \ldots, v_K\rangle = |1, \ldots, N\rangle, \quad (D6)
\]

\[
|w_1, \ldots, w_K\rangle = |N+1, \ldots, 2N\rangle, \quad (D7)
\]

which are orthogonal for Fermions, Bosons, and DP and any partition \( (N_1, \ldots, N_K) \).

As a last fact before we solve the SEValue equations for this operator, let us recall an example of the standard eigenvalue problem:

\[
\hat{M} = m|m\rangle \langle m| + m^*|m\rangle \langle m|, \quad (D8)
\]

\[
|\mu_\pm\rangle = \left( |m\rangle \pm \frac{m^*}{|m|} |m\rangle \right) / \sqrt{2}, \quad (D9)
\]

\[
\mu_\pm = \pm |m|, \quad (D10)
\]

with complex \( m \neq 0 \), orthonormal \( \{|m\rangle, |m\rangle\} \), and \(|\mu_\pm\rangle \) being the eigenvectors of \( \hat{M} \) to the eigenvalues \( \mu_\pm \).

Now, let us use the first form of the SEValue equation for IP and DP of the operator (D1). Since the spanned subspace of \( \hat{L} \) is span\( \{v_1, \ldots, v_K\}, \{w_1, \ldots, w_K\} \}, let us expand

\[
|b_j\rangle = \beta_{v,j}|v_j\rangle + \beta_{w,j}|w_j\rangle. \quad (D11)
\]

We get for the \( j \)th SEValue equation the two components

\[
\langle v_j\rangle \langle \hat{L} \hat{\Pi}^+ \rangle_{b_j} |b_j\rangle = g\langle v_j\rangle \langle \hat{1}\rangle |b_j\rangle, \quad (D12)
\]

\[
\langle w_j\rangle \langle \hat{L} \hat{\Pi}^+ \rangle_{b_j} |b_j\rangle = g\langle w_j\rangle \langle \hat{1}\rangle |b_j\rangle. \quad (D13)
\]

Equivalently, we get by a rescaling with \( \nu(\hat{1}) \)

\[
\prod_{i \neq j} (\beta_{v,i} \beta_{w,i}) \left| \beta_{w,j} \right|^2 = g \prod_{i \neq j} (|\beta_{v,i}|^2 + |\beta_{w,i}|^2) \left| \beta_{v,j} \right|^2, \quad (D14)
\]

\[
\prod_{i \neq j} (\beta_{v,i}^* \beta_{v,i}) \left| \beta_{v,j} \right|^2 = g \prod_{i \neq j} (|\beta_{v,i}|^2 + |\beta_{w,i}|^2) \left| \beta_{w,j} \right|^2. \quad (D15)
\]

which has the structure of the eigenvalue problem in (D8) with the solution in Eqs. (D9) and (D10). Hence for each \( j \) we get the solution for components with \( |\beta_{v,j}| = |\beta_{w,j}| = 1/\sqrt{2} \), yielding \( |\beta_{v,i}|^2 + |\beta_{w,i}|^2 = 1 \) and the eigenvalues

\[
g = \pm \prod_{i \neq j} \left| \beta_{v,i} \beta_{v,i} \right| = \pm \left( \frac{1}{2} \right)^{K-1}. \quad (D16)
\]
Note for $\prod_{j=1}^N (\beta_{v_j}^*, \beta_{w_j}) = 0$, we get the trivial SE value $g = 0$ and, for example, the SE vector $\hat{\|} b_1, \ldots, b_K \rangle = \hat{\|} v_1, \ldots, v_K \rangle$. Finally, the maximal SE value is

$$G = \sup \{ g \} = \left( \frac{1}{2} \right)^{K-1}. \quad (D17)$$

Note that this result is independent of the particular $K$-partition $(N_1, \ldots, N_K)$ and, due to especially chosen orthonormality in $(D3)$, the result is also independent of the spin statistics.

The particularly considered state for the plot is given by $q = r \exp[i\varphi]$ (with the optimal choice $r = 1/\sqrt{3}$),

$$|q\rangle = \sqrt{\nu(\hat{\|})} \sum_{n=0}^{\infty} (1 - r^2)^{1/2} r^n \exp[i\varphi n]$$

$$\times |nN + 1, \ldots, (n + 1)N\rangle, \quad (D18)$$

with $\langle q|q \rangle = 1$. A uniform dephasing in the interval $\varphi \in [-\delta, +\delta]$ results in the density matrix

$$\hat{\rho} = \int_{-\delta}^{+\delta} \frac{d\varphi}{2\delta} [r \exp[i\varphi] \langle r \exp[i\varphi] |$$

$$= \sum_{n,n'=0}^{\infty} (1 - r^2)^{n+n'} \sin[\delta(n-n')] \nu(\hat{\|})$$

$$\times \hat{\|} |nN + 1, \ldots, (n + 1)N\rangle \langle n'N + 1, \ldots, (n' + 1)N| \hat{\|}, \quad (D19)$$

Note that for a full dephasing, $\delta = \pi$, this state is diagonal in product states and, therefore, separable. For no dephasing, $\delta = 0$, we have a pure GHZ-type entangled state.
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