Well Extend Partial Well Orderings

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Abstract
In this paper, we prove that any partially well-ordered structure \((A, R)\) can be extended to a well-ordered structure. This result also applies to a well-founded structure because such a well-founded relation can be easily extended to a partial well ordering. The idea is to first decompose elements of \(A\) by their relative ranks under \(R\), afterwards linearly extend them with different \(R\)-ranks in ascending order, and finally well extend those with the same \(R\)-rank. Then, we discuss the problem that whether every linear extension of \((A, R)\) could be a well-ordered structure.

1 INTRODUCTION
Given a structure \((A, R)\) where \(R\) is a relation on \(A\), we define the following notations:

\begin{definition}
\textit{t} \in A \text{ is said to be an } R\text{-minimal element of } A \text{ iff there is no } x \in A \text{ for which } x R t.
\end{definition}

\begin{definition}
\(R\) is said to be well founded iff every nonempty subset of \(A\) has an \(R\)-minimal element.
\end{definition}

\begin{definition}
\(R\) is called a partial well ordering if it is a transitive well-founded relation.
\end{definition}

Clearly if \(B \not\subseteq \text{fld } R\), then any \(t\) in \(B - \text{fld } R\) is an \(R\)-minimal element. A partial well ordering is also a strict partial ordering because any well-founded relation by definition 1.2 is irreflexive otherwise if \(x R x\) then the set \(\{x\}\) has no \(R\)-minimal element.

By Order-Extension Principle [1], any partial ordering can be extended to a linear ordering. Similarly, E. S. Wolk proved that a non-strict partial ordering \(R\) defined on \(A\) is a non-strict partial well ordering iff every linear extension of \(R\) is a well ordering of \(A\) [4]. However, this result does not apply to strict partial well orderings by definition 1.3 where irreflexivity is mandatory. Take \((\mathbb{Z}, \emptyset)\) as an example in which \(\mathbb{Z}\) is the set of integers. \((\mathbb{Z}, \emptyset)\) is a partially well-ordered structure, however the normal ordering of \(\mathbb{Z}\) is obviously a linear extension of \(\emptyset\) but not a well ordering. The reason is that \(\emptyset\) is not a legal partial well ordering by the definition in [1]. In this paper, we prove that in spite of strict partialness an arbitrary partial well-ordered structure can still be extended to a well-ordered structure. Later, we discuss the problem that whether every linear extension of \((A, R)\) could be a well-ordered structure.

2 WELL EXTENSION
In this section, we prove that:

\begin{theorem}
Any partially well-ordered structure \((A, R)\) can be extended to a well-ordered structure \((A, W)\) in which \(R \subseteq W\).
\end{theorem}

Actually theorem 2.1 also applies to a well-founded structure because such a well-founded relation can be first extended to a partial well ordering:

\begin{lemma}
If \((A, R)\) is a well-founded structure, then \(R\) can be extended to a partial well ordering on \(A\).
\end{lemma}

\begin{proof}
\(R\)'s transitive extension \(R'\) is a partial well ordering. Please refer to [2] for details of this well-known result.
\end{proof}

Clearly if either \(A = \emptyset\) or \(R = \emptyset\), the extension is trivial by Well-Ordering Theorem. In the sequel, we assume that both \(A\) and \(R\) are not empty. The idea is to first decompose elements of \(A\) by their relative ranks under \(R\), afterwards linearly extend them with different \(R\)-ranks in ascending order, and finally well extend those with the same \(R\)-rank. To be more precise, let \(R\)-rank be denoted as \(R K\), then \(R K\) is a function for which \(R K(t) = \{R K(x) \mid x R t\}\). We will prove that ran\(R K\) and each \(R K(t)\) are ordinals. Next, we construct \(W\) as following:
1. if $\text{RK}(x) \in \text{RK}(y)$, add $(x, y)$ to $W$.

2. if $\text{RK}(x) \ni \text{RK}(y)$, add $(y, x)$ to $W$.

3. if $\text{RK}(x) = \text{RK}(y)$ and $x \neq y$, then $x$ and $y$ have no relation at all in $R$. By Well-Ordering Theorem, there exists a well ordering $\prec$ on the set $\{t \in A \mid \text{RK}(t) = \text{RK}(x)\}$. If $x < y$, add $(x, y)$ to $W$; otherwise add $(y, x)$ to $W$.

$\text{RK}$ is defined by the transfinite recursion theorem schema on well-founded structures. Take $\gamma_1(f, t, z)$ to be the formula $z = \text{ran } f$, then there exists a unique function $\text{RK}$ on $A$ for which

$$
\text{RK}(t) = \text{ran}(\text{RK} \upharpoonright \{x \in A \mid x Rt\}) = \text{RK} \{x \mid x Rt\} = \{\text{RK}(x) \mid x Rt\}
$$

$\text{RK}$ is similar to the "$\epsilon$-image" of well-ordered structures, and has the following properties:

**Lemma 2.3.**

(a) $\text{RK}(t) \notin \text{RK}(t)$ for any $t \in A$.

(b) For any $s$ and $t$ in $A$,

$$
s Rt \Rightarrow \text{RK}(s) \in \text{RK}(t)
$$

$$
\text{RK}(s) \in \text{RK}(t) \Rightarrow \exists s' \in A \text{ with } \text{RK}(s') = \text{RK}(s) \text{ and } s'Rt
$$

(c) $\text{RK}(t)$ is an ordinal for any $t \in A$.

(d) $\text{ran}\text{RK}$ is an ordinal.

**Proof.**

(a) Let $S$ be the set of counterexamples:

$$
S = \{t \in A \mid \text{RK}(t) \notin \text{RK}(t)\}
$$

If $S$ is nonempty, then there exists a minimal $\hat{t} \in S$. Since $\text{RK}(\hat{t}) \in \text{RK}(\hat{t})$, there is some $s R \hat{t}$ with $\text{RK}(s) = \text{RK}(\hat{t})$ by definition of $\text{RK}$. But then $\text{RK}(s) \in \text{RK}(s)$, contradicting the fact that $\hat{t}$ is minimal in $S$.

(b) By definition.

(c) Let

$$
B = \{t \in A \mid \text{RK}(t) \text{ is an ordinal}\}
$$

We use Transfinite Induction Principle to prove that $B = A$. For a minimal element $\hat{t} \in A$, $\text{RK}(\hat{t}) = \emptyset$ which is an ordinal. So $\hat{t} \in B$, and $B$ is not empty. Assume $\operatorname{seg} t = \{x \in A \mid x Rt\} \subset B$, then $\text{RK}(t) = \{\text{RK}(x) \mid x Rt\}$ is a set of ordinals. If $u \in v \in \text{RK}(t)$, there exist $x, y$ in $A$ with $u = \text{RK}(x), v = \text{RK}(y), x R y$ and $y Rt$. Because $R$ is a transitive relation, then $x Rt$ and $u \in \text{RK}(t)$. $\text{RK}(t)$ is a transitive set of ordinals, which implies that it is an ordinal and $t \in B$.

(d) If $u \in \text{RK}(t) \in \text{ran}\text{RK}$, then there is some $x Rt$ with $u = \text{RK}(x)$; consequently $u \in \text{ran}\text{RK}$.

Then $\text{ran}\text{RK}$ is a transitive set of ordinals, therefore itself is an ordinal too.

In the sequel, $\text{ran}\text{RK}$ will be denoted as $\lambda$. To be noted, $\text{RK}$ is not a homomorphism of $A$ onto $\lambda$.

We define

$$
\text{RVRK} = \{(\alpha, B) \mid (\alpha \in \lambda) \land (B \subseteq A) \land (t \in B \iff \text{RK}(t) = \alpha)\}
$$

$\text{RVRK}$ is a function from $\lambda$ into $\mathcal{P}(A)$, because it is a subset of $\lambda \times \mathcal{P}(A)$ and is single rooted. In addition, $\text{RVRK}$ is one-to-one. The purpose of $\text{RVRK}$ is to decompose $A$. 

\[\square\]
We then define

\[ T = \{(B, \prec) \mid (B \subseteq A) \land (\prec \text{ is a well ordering on } B)\} \]

\( T \) is a set, because if \((B, \prec) \in T\), then \( (B, \prec) \in \mathcal{P}(A) \times \mathcal{P}(A \times A) \). By Axiom of Choice, there exists a function \( GW \subseteq T \) with \( \text{dom } GW = \text{dom } T = \mathcal{P}(A) \). That is, \( GW(B) \) is a well ordering on \( B \subseteq A \). \( GW \) is one-to-one too.

Finally we enumerate elements of \( A \) to construct the desired well ordering. Let \( \gamma_2(f, y) \) be the formula:

(i) If \( f \) is a function with domain an ordinal \( \alpha \in \lambda \), \( y = (GW \circ \text{RVRK}(\alpha)) \cup ((\bigcup \text{RVRK}[\alpha]) \times \text{RVRK}(\alpha)) \).

(ii) Otherwise, \( y = \emptyset \).

Then transfinite recursion theorem schema on well-ordered structures gives us a unique function \( F \) with domain \( \lambda \) such that \( \gamma_2(F \upharpoonright \text{seg}(\alpha, F(\alpha))) \) for all \( \alpha \in \lambda \). Because \( \text{seg} \alpha = \alpha \), we get \( \gamma_2(F \upharpoonright \alpha, F(\alpha)) \).

We claim that:

**Lemma 2.4.** \( W = \bigcup \text{ran } F \) is a well ordering extended from \( R \).

**Proof.** Suppose \( x, y, z \in A \), and \( \alpha, \beta, \theta \in \lambda \) are their \( R \)-ranks respectively.

1. \( (x, y) \in R \Rightarrow \alpha \in \beta \)

\( \Rightarrow (x, y) \in (\bigcup \text{RVRK}[\beta]) \times \text{RVRK}(\beta) \)

\( \Rightarrow (x, y) \in F(\beta) \)

\( \Rightarrow (x, y) \in W \)

Therefore \( R \subseteq W \).

2. There are three possible relations between \( \alpha \) and \( \beta \):

(i) \( \alpha \in \beta \), then \( x \not\in y \) and \( x W y \) according to the construction of \( W \).

(ii) \( \alpha \ni \beta \), then \( x \not\in y \) and \( y W x \).

(iii) \( \alpha = \beta \). Let \( \prec = GW \circ \text{RVRK}(\alpha) \), then \( x = y \), \( x < y \), or \( y < x \). This implies that \( x = y \), \( x W y \), or \( y W x \).

Furthermore suppose \( x W y \) and \( y W z \), then \( \alpha \not\in \beta \not= \theta \). If \( \alpha \in \theta \), then \( x W z \). Otherwise, \( \alpha = \beta = \theta \).

Let \( \prec = GW \circ \text{RVRK}(\alpha) \), then \( x < y \) and \( y < z \). Because \( \prec \) is a well ordering, \( x \prec z \) and then \( x W z \).

From the above, \( W \) satisfies trichotomy on \( A \) and is a transitive relation, therefore \( W \) is a linear ordering.

3. Suppose \( B \) is a nonempty subset of \( A \), then \( \text{RK}[B] \) is a nonempty set of ordinals by Axiom of Replacement. Such a set has a least element \( \sigma \). Let \( C = B \cap \text{RVRK}(\sigma) \) and \( \prec = GW \circ \text{RVRK}(\sigma) \). \( C \) is a nonempty subset of \( \text{RVRK}(\sigma) \), so it has a least element \( \check{t} \) under \( \prec \). For any \( x \) in \( B \) other than \( \check{t} \), either \( \sigma \in \alpha \) or \( \sigma = \alpha \). In both cases, \( \check{t} W x \) and \( \check{t} \) is indeed the least element of \( B \).

\[ \square \]

Here we conclude that an arbitrary well-founded relation or partial well ordering can be extended to a well ordering.

3 DISCUSSION

Going back to the claim by E. S. Wolk, we consider a similar problem: under what circumstances will every linear extension of \((A, R)\) be a well-ordered structure when talking about strict-partialness? Here are some facts.

**Lemma 3.1.** If every linear extension of a partially well-ordered structure \((A, R)\) is a well-ordered structure, then \( \text{RVRK}(\alpha) \) is finite for all \( \alpha \in \lambda \).
Proof. Suppose that there exists \( \alpha \in \lambda \) in which RVRK(\( \alpha \)) is infinite. Then it has a countably infinite subset \( D \), and let \( f \) be the one-to-one function from \( D \) onto the set of integers \( \mathbb{Z} \). We induce a linear ordering \( S \) on \( D \) by:

\[
x Sy \iff f(x) < f(y) \quad \text{where} \quad < \text{is the normal ordering of} \mathbb{Z}
\]

Clearly \( S \) is also a partial ordering on RVRK(\( \alpha \)). During the construction of \( W \) in theorem 2.3, we take an arbitrary linear extension of \( S \) on RVRK(\( \alpha \)) instead of \( GW \circ RVRK(\alpha) \). Then \( W \) is not a well ordering, otherwise \( S \) is also a well ordering on \( D \) which is obviously false. \( \square \)

Lemma 3.2. There exists a partially well-ordered structure \((A, R)\) in which RVRK(\( \alpha \)) is finite for all \( \alpha \in \lambda \) and one of its linear extension is not a well-ordered structure.

Proof. The idea is to take a countably infinite binary tree, and linearly extend such a tree by letting the left subtree of each node greater than its right subtree.

To be more precise, let \( < \) be the normal ordering on the set of natural numbers \( \omega \), and \( R_1 \) be the ordering on \( \omega \) in which \( n R_1 (2 \times n + 1) \land n R_1 (2 \times n + 2) \). \((\omega, R_1)\) is a well-founded structure because \( R_1 \subseteq < \). Let \( R \) be the transitive extension of \( R_1 \), then \((\omega, R)\) is a partially well-ordered structure with the following properties:

(a) \( x R y \Rightarrow \exists t_1, t_2, \ldots, t_n \in \omega \land x R t_1 R t_2 R \cdots R t_n R y \)
(b) \( \lambda = \text{ran} \text{RK} = \omega \)
(c) \( \text{RVRK}(\( n \)) = \{(2^n - 1), 2^n, \ldots, (2^{n+1} - 2)\} \) for all \( n \in \omega \), therefore \( \text{card} \text{RVRK}(\( n \)) = 2^n \in \omega \).

We then define the following function for each element to get the descendants:

\[
\text{GD} = \{(x, B) \mid (x \in \omega) \land (B \subseteq \omega) \land (t \in B \iff x R t)\}
\]

GD is a function from \( \omega \) into \( \mathcal{P}(\omega) \), because it is a subset of \( \omega \times \mathcal{P}(\omega) \) and is single rooted.

Let \( \gamma_3(f, y) \) be the formula:

(i) \( f \) is a function with domain a natural number \( n \in \omega \). Let \( \text{RVRK}(\( n \)) = \{x_1, x_2, \ldots, x_{2^n}\} \) for which \( x_1 < x_2 < \cdots < x_{2^n} \). Then \( y = \bigcup_{1 \leq i \leq 2^n} (\text{GD}(x_i) \times \text{GD}(x_i)) \)

(ii) otherwise, \( y = \emptyset \).

Transfinite recursion theorem schema gives us a unique function \( G \) with domain \( \omega \) such that \( \gamma_3(G \upharpoonright \text{seg} n, G(n)) \) for all \( n \in \omega \). That is, \( \gamma_3(G \upharpoonright n, G(n)) \).

Then \( L = (\bigcup \text{ran} G) \cup R \) is a linear extension of \( R \). The proof is straightforward, and we omit the details. Let \( g : \omega \rightarrow \omega \) be the function for which \( g(n) = 2^{n+2} - 3 \). It is easy to verify that \( g(n^+) L g(n) \) for all \( n \in \omega \). Therefore \( g \) is a descending chain and \( L \) is not a well ordering. \( \square \)

References

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