Regularity of envelopes in Kähler classes

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We establish the $C^{1,1}$ regularity of quasi-psh envelopes in a Kähler class, confirming a conjecture of Berman.

1. Introduction

Let $(X^n, \omega)$ be a compact Kähler manifold, and $\theta = \omega + \sqrt{-1} \partial \bar{\partial} v$ a closed real $(1,1)$-form cohomologous to $\omega$, where $v \in C^\infty(X, \mathbb{R})$. The envelope (or extremal function) $u_\theta$ is defined by

$$u_\theta(x) = \sup \{ u(x) \mid u \in PSH(X, \theta), u \leq 0 \}$$

$$= -v + \sup \{ u(x) \mid u \in PSH(X, \omega), u \leq v \},$$

is a $\theta$-psh function with minimal singularities in the class $[\omega]$, and has received much attention recently (see for example [6, 7, 14] and references therein). By Berman-Demailly [8], we know that the complex Hessian (or equivalently the Laplacian) of $u_\theta$ belongs to $L^\infty(X)$, and so in particular $u_\theta$ is $C^{1,\alpha}(X)$ for all $0 < \alpha < 1$. A direct PDE proof was given by Berman in [4].

Here we establish the optimal regularity result for the envelope, which was previously only known when $[\omega] \in H^2(X, \mathbb{Q})$ by [3] (see also [5]). This resolves affirmatively a conjecture of Berman [3, Conjecture 1.10]:

**Theorem 1.1.** The envelope $u_\theta$ is in $C^{1,1}(X)$.

This is in general optimal, see e.g. [3] Example 5.2] for examples on toric manifolds.

In fact, combining our result with the arguments in [12, Proof of Theorem 2.5], we obtain the same $C^{1,1}$ regularity result for the “rooftop envelopes”

$$P(v_1, \ldots, v_k)(x) = \sup \left\{ u(x) \mid u \in PSH(X, \omega), u \leq \min_{j=1,\ldots,k} v_j \right\},$$

where the $v_j$’s are $C^{1,1}$ functions, see Theorem [3.1] below.
Also, using Theorem 1.1 together with the arguments in [3, Theorem 3.4], we obtain a slightly shorter proof of the identity

\[(\theta + \sqrt{-1}\partial\bar{\partial}u_\beta)^n = \chi_{\{u_\beta = 0\}}\theta^n,\]

which clearly implies

\[(1.2) \quad \int_{\{u_\beta = 0\}} \theta^n = \int_X \omega^n,\]

and which was proved (in more generality) in [8, Corollary 2.5]. Indeed it is classical that the Monge-Ampère operator \((\theta + \sqrt{-1}\partial\bar{\partial}u_\beta)^n\) vanishes outside the contact set \(\{u_\beta = 0\}\) (see e.g. [3, Proposition 3.1] or [6, Proposition 2.10]), and by Theorem 1.1 we know that \(\nabla_i u_\beta\) is Lipschitz (for any \(1 \leq i \leq n\), working in a local coordinate chart) and so \(\nabla_i u_\beta = 0\) a.e. on the set \(\{\nabla_i u_\theta = 0\}\) (see e.g. [1, Theorem 3.2.6]), which contains the contact set. Therefore a.e. on the contact set we have \(\nabla^2 u_\theta = 0\) and so \(\theta + \sqrt{-1}\partial\bar{\partial}u_\theta = \theta\), which proves (1.1).

The proof of the Theorem 1.1 which is given in section 2 is obtained by using Berman’s result [4] that the envelope \(u_\theta\) is in fact the limit of solutions of a 1-parameter family of complex Monge-Ampère equations, together with the technique recently introduced by Chu, Weinkove and the author [9, 10] to obtain uniform \(C^{1,1}\) estimates for such equations. A generalization of this result to “rooftop envelopes” (in the sense of [12]) is proved in section 3.

After the first version of this paper was posted on the arXiv, we were informed that J. Chu and B. Zhou independently proved Theorem 1.1 in [11].

2. \(C^{1,1}\) regularity of envelopes

In this section we give the proof of Theorem 1.1. Following the approach of [4], we consider the family of complex Monge-Ampère equations

\[(2.1) \quad (\theta + \sqrt{-1}\partial\bar{\partial}u_\beta)^n = e^{\beta u_\beta} \omega^n,\]

where \(\beta \in \mathbb{R}_{\geq 0}\), the function \(u_\beta\) is smooth and \(\theta + \sqrt{-1}\partial\bar{\partial}u_\beta\) is a Kähler metric on \(X\). This is solvable thanks to the work of Aubin [2] and Yau [15]. Recall also that we write \(\theta = \omega + \sqrt{-1}\partial\bar{\partial}v\) for a smooth function \(v\).
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Berman shows in [4] that

\begin{align}
\|u_\beta\| & \leq C, \quad \|\Delta_g u_\beta\| \leq C, \quad \|u_\beta - u_\theta\| \leq C \frac{\log \beta}{\beta}, \\
\end{align}

for a uniform constant $C$ independent of $\beta$ (and which depends only on the $C^{1,1}$ norm of $v$), from which it follows that $u_\beta$ converges to $u_\theta$ in $C^{1,\alpha}(X)$ for any $0 < \alpha < 1$, as $\beta \to \infty$.

Our main result is that for all $\beta \in \mathbb{R}_{\geq 0}$ we have

\begin{align}
\|\nabla^2 u_\beta\|_g & \leq C,
\end{align}

for a uniform $C$, which immediately implies Theorem 1.1. As will be apparent from the proof, the constant $C$ depends only on the $C^{1,1}$ norm of $v$. Let $\varphi = u_\beta + v$ and rewrite (2.1) as

\begin{align}
(\omega + \sqrt{-1} \partial\bar{\partial} \varphi)^n = e^{\beta (\varphi - v)} \omega^n,
\end{align}

where $\tilde{\omega} := \omega + \sqrt{-1} \partial\bar{\partial} \varphi$ is a Kähler metric, and the idea is to follow very closely the method introduced by Chu, Weinkove and the author in [9, 10]. We thus let $\lambda_1(\nabla^2 \varphi)$ be the largest eigenvalue of $\nabla^2 \varphi$ with respect to $g$, and the goal is to prove that $\lambda_1(\nabla^2 \varphi) \leq C$ for a uniform constant $C$. Indeed, once we prove this, since the trace of $\nabla^2 \varphi$ is $\Delta_g \varphi$ which is bounded below by $-n$, we will conclude that

\begin{align}
\|\nabla^2 \varphi\|_g & \leq C,
\end{align}

which implies (2.3). To this end, we apply the maximum principle to

\begin{align}
Q = \log \lambda_1(\nabla^2 \varphi) + h(\|\partial \varphi\|^2_g) - A \varphi,
\end{align}

(defined on the set where $\lambda_1(\nabla^2 \varphi) > 0$, which we may assume is nonempty) where $A > 0$ is a uniform constant to be determined and

\begin{align}
h(s) = -\frac{\lambda}{2} \log \left(1 + \sup_M |\partial \varphi|^2_g - s\right),
\end{align}

where $\lambda = (1 + 2 \sup_X |\partial v|^2_g)^{-1} \leq 1$, is a small uniform constant. The only difference between this quantity and the corresponding one in [10] is that
there we just took $\lambda = 1$. We have

$$\frac{\lambda}{2} \geq h' \geq \frac{\lambda}{2 + 2 \sup_M |\partial \varphi|^2_g} > 0, \quad \text{and} \quad h'' = \frac{2}{\lambda} (h')^2 \geq 2(h')^2,$$

where we are evaluating $h$ and its derivatives at $|\partial \varphi|^2_g$. The bounds (2.2) show that the last two terms in $Q$ are uniformly bounded.

We work at a point $x_0$ where the maximum is achieved, and as in [10] we choose local normal coordinates for $g$ near $x_0$, so that $(\tilde{g}_{\gamma \nu})_{x_0}$ is diagonal, as well as constant vector fields $\{V_\alpha\}$ near $x_0$ which at that point form an orthonormal basis of eigenvectors of $\nabla^2 \varphi$, with $\nabla^2 \varphi(V_1, V_1)(x_0) = \lambda_1$.

We also apply the same perturbation argument as in [10], so that $Q$ gets replaced by the local quantity $\hat{Q}$ defined near $x_0$ as in [10] by

$$\hat{Q} = \log \lambda_1(\Phi) + h(|\partial \varphi|_g^2) - A \varphi,$$

where $\Phi$ is the endomorphism of $TX$ given by

$$\Phi^\nu_{\nu} = g^\mu\nu (\nabla^2 \varphi_{\gamma \nu} - \delta_{\gamma \nu} + V_1^\gamma V_1^\nu),$$

where $(V_1^\nu)$ are the components of $V_1$. The largest eigenvalue $\lambda_1(\Phi)$ now varies smoothly near $x_0$ and $\hat{Q}$ achieves a local maximum at that point. Writing $\lambda_\alpha = \lambda_\alpha(\Phi)$, the goal is to show that $\lambda_1(x_0) \leq C$, for a uniform constant $C$. We claim that at $x_0$ we have

$$0 \geq \Delta \hat{Q} \geq 2 \sum_{\alpha > 1} \frac{\tilde{g}^{ij} |\partial_i (\varphi V_\alpha V_i)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^{pq} \tilde{g}^{qr} |V_1(\tilde{g}_{pq})|^2}{\lambda_1} - \frac{\tilde{g}^{ij} |\partial_i (\varphi V_1 V_i)|^2}{\lambda_1^2}$$

$$+ h' \sum_k \tilde{g}^{ij} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + \beta h' |\partial \varphi|_g^2 + h'' \tilde{g}^{ij} |\partial_i |\partial \varphi|_g^2|^2$$

$$+ (A - C) \sum_i \tilde{g}^{ii} - An + \frac{\beta}{4}$$

$$\geq 2 \sum_{\alpha > 1} \frac{\tilde{g}^{ij} |\partial_i (\varphi V_\alpha V_i)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^{pq} \tilde{g}^{qr} |V_1(\tilde{g}_{pq})|^2}{\lambda_1} - \frac{\tilde{g}^{ij} |\partial_i (\varphi V_1 V_i)|^2}{\lambda_1^2}$$

$$+ h' \sum_k \tilde{g}^{ij} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + h'' \tilde{g}^{ij} |\partial_i |\partial \varphi|_g^2|^2$$

$$+ (A - C) \sum_i \tilde{g}^{ii} - An.$$
Indeed, as in [10, (2.7)] we have

\begin{equation}
\Delta \hat{g} \hat{\phi} = \frac{\Delta_g (\lambda_1)}{\lambda_1} - \frac{\tilde{g}^\tilde{\tau} |\partial_i (\varphi V_{\lambda_1} V_1)|^2}{\lambda_1^2} + h' \Delta_g (|\partial \varphi|_g^2) + h'' \tilde{g}^\tilde{\tau} |\partial_i |\partial \varphi|_g^2|^2 + A \sum_i \tilde{g}^\tilde{\tau} - A_n,
\end{equation}

and as in [10, (2.8)]

\begin{equation}
\Delta_g (\lambda_1) \geq 2 \sum_{\alpha \geq 1} \frac{\tilde{g}^\tilde{\tau} |\partial_i (\varphi V_{\lambda_1} V_1)|^2}{\lambda_1 - \lambda_\alpha} + \tilde{g}^\tilde{\tau} V_1 (\tilde{g}^\tilde{\tau} V_1) - C \lambda_1 \sum_i \tilde{g}^\tilde{\tau}.
\end{equation}

The Monge-Ampère equation (2.4) in local coordinates reads

\begin{equation}
\log \det \tilde{g} = \log \det g + \beta \varphi - \beta v,
\end{equation}

and so applying $V_1 V_1$ to this and evaluating at $x_0$ we obtain

\begin{equation}
\tilde{g}^\tilde{\tau} V_1 (\tilde{g}^\tilde{\tau} V_1) = \tilde{g}^{\tilde{\rho} \tilde{\tau}} \tilde{g}^{\tilde{\rho} \tilde{\tau}} |V_1 (\tilde{g}^{\tilde{\rho} \tilde{\tau}})|^2 + V_1 V_1 (\log \det g)
+ \beta V_1 V_1 (\varphi) - \beta V_1 V_1 (v)
\geq \tilde{g}^{\tilde{\rho} \tilde{\tau}} \tilde{g}^{\tilde{\rho} \tilde{\tau}} |V_1 (\tilde{g}^{\tilde{\rho} \tilde{\tau}})|^2 + V_1 V_1 (\log \det g) + \beta (\lambda_1 - C)
\geq \tilde{g}^{\tilde{\rho} \tilde{\tau}} \tilde{g}^{\tilde{\rho} \tilde{\tau}} |V_1 (\tilde{g}^{\tilde{\rho} \tilde{\tau}})|^2 + V_1 V_1 (\log \det g) + \frac{\beta}{2} \lambda_1,
\end{equation}

since we may assume that at $x_0$ the largest eigenvalue $\lambda_1$ is large. This gives

\begin{equation}
\Delta_g (\lambda_1) \geq 2 \sum_{\alpha \geq 1} \frac{\tilde{g}^\tilde{\tau} |\partial_i (\varphi V_{\lambda_1} V_1)|^2}{\lambda_1 - \lambda_\alpha} + \tilde{g}^{\tilde{\rho} \tilde{\tau}} \tilde{g}^{\tilde{\rho} \tilde{\tau}} |V_1 (\tilde{g}^{\tilde{\rho} \tilde{\tau}})|^2
- C \lambda_1 \sum_i \tilde{g}^\tilde{\tau} + \frac{\beta}{2} \lambda_1.
\end{equation}
Next, at $x_0$,

$$
\Delta_g (|\partial \varphi|^2_g) = \sum_k g^{i\bar{j}} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + 2\beta \text{Re} \left( \sum_k \varphi_k (\varphi - v) \right)
$$

$$
+ g^{i\bar{j}} \partial_i \partial_{\bar{j}} (g^{k\ell}) \varphi_k \varphi_{\ell}
\geq \sum_k g^{i\bar{j}} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C \sum_i g^{i\bar{j}} + 2\beta |\partial \varphi|^2_g
$$

$$
- 2\beta \text{Re} \left( \sum_k \varphi_k v \right)
\geq \sum_k g^{i\bar{j}} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C \sum_i g^{i\bar{j}} + \beta |\partial \varphi|^2_g - \beta |\partial v|^2_{v_g},
$$

where to derive the first line we have applied $\partial_i$ to (2.10). But then

$$\beta h' \text{Re} |\partial \varphi|^2_g \leq \frac{\beta \lambda}{2} \text{Re} |\partial \varphi|^2_g \leq \frac{\beta \sup_X |\partial \varphi|^2_g}{2 + 4 \sup_X |\partial \varphi|^2_g} \leq \frac{\beta}{4},$$

and so combining this with (2.8), (2.12) and (2.13), we see that (2.7) holds.

Now the rest of the proof proceeds exactly as in [10], since (2.7) is the same as [10, (2.6)], and the specific form of the PDE (2.4) is not used anymore in [10] after that point. At a couple of places we used that $h'' = 2(h')^2$, but in fact the inequality $h'' \geq 2(h')^2$ is enough, and this holds in our case. The constant $A$ is chosen at the end of the argument of [10] Proof of Theorem 1.2, and it equals $A = C + 3$, where $C$ is the uniform constant in (2.7). This completes the proof of Theorem 1.1.

3. Rooftop envelopes

In this section we consider a generalization of Theorem 1.1 as follows.

Suppose we are now given $C^{1,1}$ functions $v_j, j = 1, \ldots, k$ on a compact Kähler manifold $(X, \omega)$, and we consider the “rooftop envelope”

$$P(v_1, \ldots, v_k)(x) = \sup \{ u(x) \mid u \in PSH(X, \omega), u \leq \min_{j=1,\ldots,k} v_j \}.$$ 

When $k = 1$ this is essentially the same as the envelope we considered in Theorem 1.1 but with a weaker regularity assumption. Darvas-Rubinstein proved in [12] that $P(v_1, \ldots, v_k)$ has bounded Laplacian on $X$, in particular it is in $C^{1,\alpha}(X)$ for all $0 < \alpha < 1$, and that if $[\omega] \in H^2(X, \mathbb{Q})$ then $P(v_1, \ldots, v_k)$ is in $C^{1,1}(X)$. This last point used the regularity results of
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Berman and Demailly [3, 8], and another proof was also given by Berman [5]. Using Theorem 1.1, we can prove the $C^{1,1}$ regularity of $P(v_1, \ldots, v_k)$ in general Kähler classes:

**Theorem 3.1.** The rooftop envelope $P(v_1, \ldots, v_k)$ is in $C^{1,1}(X)$.

**Proof.** The argument in [12, Proof of Theorem 2.5] reduces this result to proving the case when $k = 1$. So we have a function $v \in C^{1,1}(X)$, and consider the envelope

$$P(v)(x) = \sup\{u(x) \mid u \in PSH(X, \omega), u \leq v\},$$

and the goal is to show that $P(v)$ is also in $C^{1,1}(X)$.

By using convolution in local charts and gluing them with a partition of unity (see e.g. the appendix in [13]) can choose a sequence $v_j$ of smooth functions which converge to $v$ in $C^{1,\alpha}(X)$ for some fixed $0 < \alpha < 1$, and such that $\|v_j\|_{C^{1,1}(X, g)} \leq C$ for all $j$. For each $j$ and $\beta \geq 0$ solve

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\beta(\varphi - v_j)}\omega^n,$$

where $\varphi = \varphi_{j,\beta}$ and $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$. As mentioned earlier, Berman [4] proved that

$$(3.2) \quad |\varphi| \leq C, \quad |\Delta_g \varphi| \leq C, \quad |\varphi - P(v_j)| \leq C \frac{\log \beta}{\beta},$$

for a uniform constant $C$ independent of $j, \beta$, from which it follows that for any $j$ fixed $\varphi$ converges to $P(v_j)$ in $C^{1,\alpha}(X)$ for any $0 < \alpha < 1$, as $\beta \to \infty$. From Theorem 1.1 and its proof, we also have that

$$|\nabla^2 \varphi|_g \leq C,$$

independent of $j, \beta$. Therefore $\|P(v_j)\|_{C^{1,1}(X, g)} \leq C$ for all $j$. On the other hand we have that $P(v_j) \to P(v)$ uniformly as $j \to \infty$, which follows easily from the definition, and so we conclude that $P(v) \in C^{1,1}(X)$ as well. □

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