From Subgroups of Direct Sums to Virtuality

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Abstract

We start by an original investigation on subgroups of (even infinite) direct sums in the first 4 sections, that largely generalizes Remak’s known theorem; inspired by that general picture we have elsewhere extended this elementary "virtual" diagrammatic situation (in diagrammatic length 2 meaning set-theoretic fixation of vertices) by generalizing to the notion of "virtuality" in module extensions and diagrams in modular representation theory.

Our first approach starts with an appropriately defined equivalence relation, which is precisely what allows for treating the confusing case of multiple factors, thus giving a deeper insight into the structure of such subgroups.

Several applications and new techniques arising from that approach are examined, even ones concerning basic properties of homomorphisms, extending well-known elementary ones.

1 Introduction

Although the first part of this article concerns some basic group theory, that is justified not only in view of the numerous applications that follow but also by the fact that there are, astonishingly, still many obscurities about the subgroups of a direct product of \( n \) groups, for \( n > 2 \), already in its general outset. There is, on the other hand, an increasing tendency to look at subdirect products in more specific contexts and instances, as one may for example see in [1].

I got my first motivation to consider this kind of questions while working in modular representation theory and trying to understand the subtle inner structure of modules in a "virtual" framework, by which I was then lead to analogue but in some sense more general group-theoretic considerations. The analogue lines can only be drawn by depicting results through "fan-like" diagrams, which is also a fundamental kind of problem one encounters in the effort to attach diagrams to modules in an optimal way, so that that they are somehow analogue to those diagrammatic depictions of subdirect products regarding that particular feature of the latter, that their vertices are also set-theoretically fixed. This
last feature being impossible for diagrammatic length (=Loewy length, speaking now of modules) greater than 2, we may achieve the best possible analogue in the frame of the "Virtual Category" (see \cite{4} and \cite{3}). This bridge to more general diagrammatic methods (in Representation Theory) is stressed here with remarks \cite{18} and \cite{13} and finally in the last small section 6.

In particular, by combining the approach and our results here with "subdirect presentations through homomorphisms" we are lead by an original way to both known and unknown facts about homomorphisms in general (section 5). It should also be pointed out that our results are generalizable to operator groups (/operator subgroups) by properly extending/specializing the proofs.

We are giving an outline of our approach:

The introduction of an equivalence relation in a subgroup \( U \leq A = A_1 \times A_2 \times \ldots \times A_n \) is critical for our insight into its structure, although we can \textit{a posteriori} also determine a normal subgroup \( I \) of \( U \), of which the cosets are actually the classes ("adhesive fibres") of our relation; this is the key to our approach, leading in particular to our theorems 17 and 31 which generalize Remak’s theorem about the structure (through some "structural" isomorphisms, intrinsic to the subgroup inclusion of \( U \) in \( A \)) of subgroups of the direct product of two groups to the case of any arbitrary (even infinite) number of factors. Namely, in theorem 17 we show an analogue of that, for any choice of a subset of the set of direct factors, while in theorem 31 we determine a necessary and sufficient condition for \( U \) to have an analogue \( n \)-fold structure, i.e. at all places, as for the case \( n = 2 \). Finally, the optimal generalization of theorem 31 is done with theorem 41. In all these cases we proceed by means of the specifically important structure of that normal subgroup \( I \), called the \textbf{core} of the particular inclusion of \( U \) as a subgroup of \( A_1 \times A_2 \times \ldots \times A_n \); we are lead to that subgroup and its relevant to the subdirect product analysis by the key role of its generating subgroups \( E_i \) to our equivalence relation, as elucidated in the proofs of proposition 5 and lemma 6. Very critical for our most general case, treated in theorem 41 is the notion of cohesive components of the core and its related ("cohesion") decomposition as their product (propositions 27, 71).

It is also important to stress that our results may also be applied to the case of a direct sum of any countable sequence of groups, see remarks 4, 12 and 20.

There are two reasons for us to begin with the case \( n = 2 \), although at least the final results are well known in this case: (a) It has been precisely this method, that has lead our intuition to the generalization for any \( n \), (b) There is also another, methodological reason: Our proof on the one hand does not depend on the fundamental theorem for group homomorphisms, on the other this proof, along with the overall point of view of our method, leads us to the aforementioned results on homomorphisms, which also represent hitherto unknown generalizations of the fundamental homomorphism theorem, a theorem which is then obtained as a very special case (as corollary 62).

In the following subsection 4.2 we investigate the action of a kind of "projectivization" of \((\text{Aut}R)^n\) on subdirect products over \( R \), which, apart from defining some orbits in a family of subdirect products, also gives us a whole orbit once we have one of them. In subsection 4.3 we investigate conditions for a group \( G \).
to be expressible as a subdirect product of groups belonging to any class $\mathcal{E}$ of groups, while we also specialize in some particular classes of special interest.

Another approach proceeds in section 5, by presenting a subdirect product through "diagonalizing" homomorphisms emanating from a single group; by that we gain an independent, quite different view of them - but then also some basic general results about homomorphisms, by applying the conclusions of the previous sections on such presentations of subdirect products, results considerably extending classical/elementary ones (see proposition [61] and its corollaries). The possibilities of the techniques arising from this approach are not exhausted here, they are just opened.

The achievements and insights in this work, with the intense focus on the level of subsets and their elements, somehow pave the way toward a virtual approach to module extension and diagram theory, the fundamentals of which are laid in [4] and in [3].

Key words: Presentations of subdirect products, decomposition as a subdirect product, adhesive fibres, subdirect AutR-classes, Subdirect (in)decomposability, Subdirect presentations.

About notation:
We shall automatically consider elements of subgroups of any subproduct of the original one as elements of the latter too - and vice versa, according to the context. The phrygic hat $\hat{\ }$ above an index designates omitting, as usual. Whenever we have some "basic" set $\Omega$ and a subset $M \subseteq \Omega$, $\hat{M}$ shall denote its complement in $\Omega$, so as to have a partition $\Omega = M \cup \hat{M}$. $1$ may both denote the identity element or the trivial (sub)group. $\pi$ with the proper indices shall designate projections from direct products. The symbol " $\times$ " may denote not only outer, but also inner direct product, which in all cases should be clear from the context. Direct products are denoted either as $Dr \prod_{i=1}^{m} W_i$ (like in [12], f.ex.) or just $\prod_{i=1}^{m} W_i$, while in case of infinite factors we are only using direct sums here.

2 The case of two direct factors
The first theorem here is a well known one; the reason to repeat it here is, as already mentioned, that we are getting to it through a totally different, original approach, that is then also applicable to the general case of more than two factors, yielding results and insights which, to my knowledge, are new. Take notice of the fact that no use of the fundamental theorem for group homomorphisms is made in its proof; it is thus meaningful to get another proof of the latter based on it, in fact as a special case of something much more general (section 5).
Theorem 1 (10; can be found for example in 13), also in (9,11,13) Given the direct product \( A \times B \) of two groups and a subgroup \( U \leq A \times B \), there exists a unique isomorphism \( \sigma : \pi_1(U) / U \cap A \to \pi_2(U) / U \cap B \), which is thus "structural", as it determines the discrete "pair cosets" of which \( U \) consists; it is also natural, in the sense that, for a given subdirect product \( U \) of \( A \times B \) and a homomorphism \((f_A, f_B)\) from it to \( A \times B \), sending \( U \) to \( U \) and inducing \( \sigma \)' from \( \sigma \), this \( \sigma \)' is precisely the "structural isomorphism" of \( U \), as above. Furthermore, the isomorphism \( \sigma \) can be (naturally) continued to \( R := U / (U \cap A) \times (U \cap B) \). Conversely, given a normal subgroup \( K \) of a subgroup \( H_A \) of \( A \), respectively, a normal subgroup \( L \) of a subgroup \( H_B \) of \( B \) (i.e. \( K \leq H_A \leq A, L \leq H_B \leq B \)) and an isomorphism \( \sigma : H_A / K \to H_B / L \), a subgroup \( U \leq A \times B \) is uniquely determined as consisting of the \( \sigma \)-determined pair-cosets. This amounts, of course, to realizing \( U \) as a fiber product of \( \pi_A(U) \) and \( \pi_B(U) \), over a fixed isomorphic copy \( T \) of the two sides of \( \sigma \), with respect to \( \pi_X(U) \)'s \((X = A, B)\) epimorphisms on it, that must be so coordinated, as to induce precisely that isomorphism \( \sigma \) (as above), that determines \( U \)'s pair-fibres correctly.

Proof. We define a relation "\( \sim \)" on \( U \leq \pi_1(U) \times \pi_2(U) \) by stipulating first that "adjacent" pairs are related, i.e. \((a, b) \sim (a', b), (a, b) \sim (a, b')\) for any \((a, b), (a', b), (a, b') \in U\) and then taking as "\( \sim \)" the transitive hull of this first stipulation. Reflectivity and symmetricity being apparent, it is clear that "\( \sim \)" is an equivalence relation on \( U \). A crucial property of this relation is that, (m) if \((a_1, b_1) \sim (a'_1, b'_1)\) and \((a_2, b_2) \sim (a'_2, b'_2)\), then \((a_1a_2, b_1b_2) \sim (a'_1a'_2, b'_1b'_2)\); the relationship between two pairs means the existence of a finite sequence of pairs starting with the first and ending with the second of those two, such that any two subsequent pairs have the same first or second coordinate (we will call such a sequence an adjacency sequence). To see our claim, consider such sequences for the two given relationships and make them of equal length by repeating the last term of the shortest one as many times as necessary; we shall have to produce a sequence with these two as terminal members, consisting of subsequent "adjacent" terms, i.e. ones sharing the same coordinate. To achieve this, take the products of the terms of same order in those two sequences of equal length, to get first a sequence of the same length, whose first and last terms are, respectively, \((a_1a_2, b_1b_2)\) and \((a'_1a'_2, b'_1b'_2)\). Those subsequent terms in this new sequence that come from the multiplication of subsequent terms in the original sequences which in both share the same-order coordinate, do probably also share same order terms as well; there is a slight problem whenever they come from multiplication of subsequent terms sharing the first term in the one, the second in the other, hence like this:

\[(\alpha, \beta), (\alpha, \delta) \text{ in the one (sharing the first coordinate) and } (a, b), (c, b) \text{ in the other (sharing the second coordinate), thus yielding the subsequent terms } (\alpha a, \beta b), (\alpha c, \delta b) \text{ of the new sequence (of products), which do not share any term; but then it will suffice to insert the new term } (\alpha c, \beta b) = (\alpha, \beta)(c, b) \in U \text{ between them.} \]
It is, on the other hand, immediate to see that \((a,b) \sim (a_1,b_1)\) implies \((a^{-1},b^{-1}) \sim (a_1^{-1},b_1^{-1})\), just by taking the inverses of all terms in the finite sequence.

We need to notice the obvious fact that \((\alpha, \beta) \in U\) and \((\alpha, 1) \in U\) imply \((1, \beta) = (\alpha, \beta)(\alpha, 1)^{-1} \in U\); by symmetry this yields that, for \((\alpha, \beta) \in U\), \((\alpha, 1) \in U \iff (1, \beta) \in U\) (s).

We shall now show how these properties also imply that, whenever \((\alpha, 1)\) (respectively, \((1, b)\)) is an element of \(U\), the first coordinate in its equivalence class \([\{(\alpha, 1)]\) (resp., the second in \([\{(1, b)]\) runs over a normal subgroup of \(\pi_A(U)\); in order to see this, we shall show that, whenever \((\alpha, \beta) \sim (\gamma, 1)\), it turns out that \((\alpha, 1)\) (and indeed \((1, \beta)\) as well) is also an element of \(U\) (and, of course, in the same equivalence class) (S).

Indeed, let \(x_0 = (\gamma, 1), x_1, ..., x_{n+1} = (\alpha, \beta)\) be an adjacency sequence for the relationship \((\alpha, \beta) \sim (\gamma, 1)\); it is then immediate to see that \(x_0^{-1}x_1x_2^{-1}...x_{n+1}^{-1} = (\alpha, 1)\) or \((1, \beta)\) or \((\alpha, 1)^{-1} = (\alpha^{-1}, 1)\) or \((1, \beta)^{-1}\), which then according to our observation (s) above all of these 4 elements shall belong to \(U\), in particular \((\alpha, 1) \in U\), as stated.

Hence given any \((\alpha, \beta) \in U\), which belongs to the equivalence class of \((1, 1)\), this can be inside that class decomposed as \((\alpha, \beta) = (\alpha, 1)(1, \beta)\). In particular, this implies that any element of the equivalence class of \((1, 1) \in U\) is generated by elements adjacent to \((1, 1)\), i.e. having 1 in the one coordinate. If we now also observe that being adjacent for any two elements of \(U\) is the same as getting any of them by multiplying the other by such a generating element (i.e. adjacent to \((1, 1)\)), this yields that the equivalence class of \((1, 1)\) is in fact a subgroup of \(I\) of \(U\), which we shall call its "subdirect core". It is thus also clear in our case (i.e., \(n=2\)) that this subdirect core equals \((U \cap A) \times (U \cap B)\). (Remark: This decomposition of the subdirect core does not hold in general for \(n > 2\).

The observation (S) above means also that the first coordinate of the class \([\{(\alpha, 1)] = I\) runs over the same set as its elements with second coordinate 1; on the other hand, every element of this form in \(U\) apparently belonging to the same class, this special equivalence class is a subgroup of \(U\), which is generated by the elements of the form \((\alpha, 1)\) or \((1, \beta)\), hence it must also be normal both in \(U\) and in \(\pi_A(U)\), as conjugation by elements of \(U\) yields elements of the same form, hence in the same class-subgroup and in \(A\) as well, apart from being in advance clear that the set of elements of this form in \(U\) is the group \(U \cap \pi_A(U)\). (This gives also an alternative way to see the normality of \(U \cap A = U \cap \pi_A(U)\) in \(U\) and in \(\pi_A(U)\), otherwise clear from the fact that \(\pi_A(U) \times \pi_B(U)\) normalizes \(\pi_A(U)\), hence \(U \leq \pi_A(U) \times \pi_B(U)\) must normalize \(U \cap \pi_A(U)\) - and then it follows immediately that also \(\pi_A(U)\) has to normalize \(U \cap \pi_A(U)\); on the other hand, one sees that the kernel of the restriction to \(U\) of \(\pi_A(U) \times \pi_B(U)\) 's projection onto its second factor is precisely \(U \cap \pi_A(U) = U \cap A\.) When we transfer the same remarks to the second factor, we see immediately that the equivalence class of the elements of \(U\) of the form \((\alpha, 1)\), also containing \((1, 1)\) and the elements \((1, b)\) in \(U\), equals the group \((U \cap A) \times (U \cap B)\).

We wish next to see another interpretation of our equivalence relation on \(U\)
through its special properties which we have seen:

Assume, so, that \((a, b) \sim (c, d)\); invoking the multiplication property, we may multiply this with the trivial relationship \((a^{-1}, b^{-1}) \sim (a^{-1}, b^{-1})\) to get \((a^{-1}c, b^{-1}d) \sim (1, 1) \iff (a^{-1}c, b^{-1}d) \in [(1, 1)] = (U \cap A) \times (U \cap B) \iff c \in a(U \cap A) \land d \in b(U \cap B) \iff (c, d) \in a(U \cap A) \times b(U \cap B)\), which shows that the class of \((a, b) \in U\) is the Cartesian product of the left (and right) cosets \(a(U \cap A)\) and \(b(U \cap B)\) of \(U \cap A\), resp. of \(U \cap B\), in \(\pi_A(U)\), resp. in \(\pi_B(U)\). Now, \(a(U \cap A) \times b(U \cap B)\) being an equivalence class, in case there were also a class \(a(U \cap A) \times d(U \cap B)\), while from the definition of our relation is \((a, b) \sim (a, d)\), this “new” class has to be identical with \(a(U \cap A) \times b(U \cap B)\), i.e. \(d(U \cap B) = b(U \cap B)\). This crucial remark establishes a bijection \(\sigma : \pi_A(U) / U \cap A \rightarrow \pi_B(U) / U \cap B\), which is bound to be a group homomorphism (hence an isomorphism), considering the special property (1) of our relation. This means that \(U\) is partitioned into classes which may be described as “pair fibres” of the form \(a(U \cap A) \times (a(U \cap A))^{\pi}\), which at the same time determines a unique coset \((a, b)(U \cap A) \times (U \cap B)\), where \(b\) may be any element of the coset \(a(U \cap A)^{\pi}\) in \(\pi_B(U)\). This observation “prolongs” the bijection \(\sigma : \pi_A(U) / U \cap A \leftrightarrow \pi_B(U) / U \cap B\) to \(\pi_A(U) / U \cap A \leftrightarrow \pi_B(U) / U \cap B \leftrightarrow U / (U \cap A) \times (U \cap B)\), still in a homomorphic manner, as property (m) suggests.

As for the converse assertion, we can easily prove that the given isomorphism \(\sigma\) defines a unique subgroup \(U \leq A \times B\) - first as a subset, while the group structure follows from that of the direct product, where it is embedded; then \(U\), according to the previous, determines a unique such isomorphism, which hence must be \(\sigma\).

Concerning the interpretation as fiber products, that is quite clear - see for example [5].

Another way to realize \(U\) could be to view it as a total space of a bundle \((U, p, R, (U \cap A) \times (U \cap B))\), \(R\) being the base group, with “typical fibre” \((U \cap A) \times (U \cap B)\), where \(p : U \rightarrow R = U / (U \cap A) \times (U \cap B)\) is the canonical projection, and for \(t \in R\) the fibre (”\(\sim\”-equivalence class or ”adhesive fibre”) is \(A_t \times B_t\) with \(A_t, B_t\) its \((\sigma-)\) corresponding \((U \cap A)\)-, resp. \((U \cap B)\)-, cosets in \(\pi_A(U)\), resp. \(\pi_B(U)\), so that \(p^{-1}(t) = A_t \times B_t\). This point of view can also be adapted to our theorems [77, 81] and 31 below.

For later use in section 6 we finally prove (independently of theorem 1) the following

**Lemma 2** \(\pi_A(U) \cong U / U \cap B\), \(\pi_B(U) \cong U / U \cap A\).

**Proof.** Observe that \(\ker(\pi_A|_U) = U \cap B\), \(\ker(\pi_B|_U) = U \cap A\).

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6
3 The generic case

Let \( U \leq A = A_1 \times A_2 \times \ldots \times A_n \); we assume further that \( \pi_i(U) \cap A_i \) is not trivial for any \( i \in \{1, \ldots, n\} \) and then introduce the following subgroups of this product:

\[
E_s = \pi_{1 \ldots \hat{i} \ldots n}(U) \cap U, \text{ which is clearly equal to } Ker\pi_s \mid U = Ker\pi_s \cap U = \left( \bigcap_{i \notin s} A_i \right) \cap U,
\]

consisting of the elements of \( U \), having 1 in the \( s \)-coordinate. More generally, for any set \( \Lambda \) of indices \( i_1 \{i_2 | \ldots | i_s \} \) from \( \{1, \ldots, n\} \), define \( E_{i_1 i_2 \ldots i_s} := \pi_{1 \ldots \hat{i_1} \ldots \hat{i_2} \ldots \hat{i_s} \ldots n}(U) \cap U \), consisting of the elements of \( U \), having 1 in the \( i_1, i_2, \ldots, i_s \)-coordinates.

It is obvious that

\[
E_{1 \ldots i_2 \ldots i_s} = \bigcap_{t \in \{i_1, \ldots, i_s\}} E_t = \bigcap_{t \in \{i_1, \ldots, i_s\}} \bigcap_{t \in \{i_1, \ldots, i_s\}} \left( E_t \cap E_t' \right) = \bigcap_{t \in \{i_1, \ldots, i_s\}} \bigcap_{t \in \{i_1, \ldots, i_s\}} E_t = E_t \cap E_t' \cap \ldots \cap E_t = Ker\pi_{\Lambda \mid U},
\]

\( E_{\{1, \ldots, n\}} \) is the trivial subgroup of \( I \).

**Definition 3** We define, as in the case \( n=2 \), a relation \( \sim \) on \( U \), in two steps: first, we stipulate that \( \pi = (a_1, \ldots, a_n) \sim \pi' = (b_1, \ldots, b_n) \) whenever \( a_\lambda = b_\lambda \), for some \( \lambda \in \{1, \ldots, n\} \) (we will then say that \( \pi \) and \( \pi' \) touch one another or are adjacent at the \( \lambda \)-coordinate), then take the minimal transitive extension of this first germ relation, to get an equivalence relation on \( U \); we shall designate equivalence classes of \( \sim \) by using square brackets \([\cdot]\), which shall also be called “adhesive fibres” of the subgroup \( U \). In particular, the equivalence class \( I = [[1, \ldots, 1]] \) of \( U \)’s identity element shall be referred to as the **(subdirect) core** of the subgroup \( U \) of \( A_1 \times A_2 \times \ldots \times A_n \).

**Remark 4** Our results from this section onward are easily seen to be extendable to the case of a subgroup \( U \) of an infinite direct sum \( \bigsqcup_{j \in J} A_j \), where the index set \( J \) is a totally ordered and countably infinite one (f.e. \( \mathbb{N} \)), in a similar manner (and along the ordering of \( J \), upwards). As for the definition of the above relation in this case, notice that also in the case of infinitely many direct summands only finite “connecting” sequences of adjacent elements shall be entailed for any assertion of equivalence between two elements of \( U \).

Next we shall show \( I \) to be a normal subgroup of \( U \) and the equivalence classes of \( \sim \) indeed the same as \( I \)’s cosets in \( U \).

The following subgroups of \( U \), which are readily seen to be normal subgroups in \( U \), shall play a crucial role in our investigation:

Define for any subsequence \( \Lambda \) of indices \( i_1 \{i_2 | \ldots | i_s \} \) from \( J = \{1, \ldots, n\} \), \( L_\Lambda = L_{i_1 \ldots i_2 \ldots i_s} \) to be \( (A_{i_1} \times A_{i_2} \times \ldots \times A_{i_s}) \cap I \), set also \( L_{\emptyset} \) to be the trivial subgroup of \( U \). Let us call them “subcores” of \( U \). Notice that \( E_{1 \ldots \hat{i} \ldots n} = L_s \), as they both consist of the elements of \( U \), having all but their \( s \)-coordinate equal to 1; also, \( L_{1 \ldots \hat{i} \ldots n} = E_s \). More generally, \( L_\Lambda = L_{i_1 i_2 \ldots i_s} = E_{1 \ldots \hat{i} \ldots \hat{i_2} \ldots \hat{i_s} \ldots n} = Ker\pi_\Lambda \), hence a normal subgroup of \( U \) for any proper subset \( \Lambda \) of \( J = \{1, \ldots, n\} \).
However this cannot be likewise concluded when $\Lambda = \{1, ..., n\}$, then yielding $I$ as $L_{\Lambda}$; instead, we are proving that in the following proposition.

Notice that, for $M, N \subset \{1, ..., n\}$, with $M \cap N = \emptyset$, it follows that $L_M \cap L_N = 1$, therefore $L_M L_N = L_N L_M$: in fact, they even commute elementwise (from the original direct product). Set also $L_\emptyset = 1$, $E_\emptyset = I$.

For any $M \subset \{1, ..., n\}$, $\pi_M$ denotes the corresponding projection from $A_1 \times A_2 \times \cdots \times A_n$ to $Dr \prod_{i \in M} A_i$.

We want to recall here the infinite symmetric group $\Sigma_\infty$, definable f.e.x. on the set of positive integers and consisting of the permutations of it, that fix all but a finite subset of it. We shall also use the fact that the symmetric group $S_n$ is generated by its involutions, i.e. transpositions, which is also the case for $\Sigma_\infty$. The latter equals the injective limit of all the symmetric groups $S_n$, $n \in \mathbb{N}^*$. Actually we may just choose the $n-1$ transpositions $(i, i+1)$, $i = 1, ... n-1$, as generators for $S_n$ - hence for $\Sigma_\infty$ too.

**Proposition 5** The equivalence class $I = [(1, ..., 1)]$ of $U$ 's identity element is a normal subgroup of $U$, henceforth to be called the subdirect core of the subgroup $U$ of $A_1 \times A_2 \times \cdots \times A_n$, generated by its subgroups $E_i$, $i = 1, ..., n$; any $E_i$ is normal both in $U$ and in $\pi_{1,...,n}(U)$. Furthermore, $I = \prod_{i=1}^n E_i = \prod_{i=1}^n E_{\tau(i)}$, where $\tau$ is any permutation in $S_n$, a result that also holds for the subgroup $I_M$ of $I$ generated by any non-empty subset of $\{E_1, ..., E_n\}$, corresponding to a subset $M$ of $\{1, ..., n\}$. In the case that we have a subgroup $U$ of an infinite direct sum $\bigsqcup_{j \in J} A_j$, where the index set $J$ is countably infinite, we define similarly the core $I$ as the subgroup $I = \bigsqcup_{j \in J} E_j$ of $U$, wherein it is now again possible to permute summands by any element of the infinite symmetric group $\Sigma_\infty$.

**Proof.** $E_s \trianglelefteq U$, $s = 1, 2, ..., n$, because $E_s$ consists precisely of the elements of $U$, that have 1 in the $s$-coordinate - a property maintained through conjugation in $U$. Alternatively, we might just use that $E_A = Ker\pi_A|U$.

Observe that $\overline{\sigma} = (a_1, ..., a_n) \in I \iff \overline{\sigma} \sim (1, ..., 1) \iff \exists$ a finite sequence $\overline{a^0} = \overline{\sigma}, \overline{a^1}, ..., \overline{a^{\mu+1}} = (1, ..., 1)$, set $\overline{a^k} = (a_{i_{\kappa}}^0, ..., a_{i_{\kappa}}^\mu)$ for $\kappa = 0, 1, ..., \mu + 1$, such that any two neighbouring terms $a_{i_{\kappa}}^{\mu-1}, a_{i_{\kappa}}^\mu$ share, say, their $i_{\kappa}$-coordinate. (By assuming this sequence to be of minimal length, we get that $a_{i_{\kappa}}^\mu \neq 1 \forall i \in \{1, ..., n\}$ whenever $\kappa(\mu)$. Then, by taking the sequence of the inverses we get such an ”adjacency sequence” yielding the relationship $\overline{a^{\mu+1}} \sim (1, ..., 1)$, proving that $\overline{\sigma}^{-1} \in I$ as well.

Before proceeding to prove that, given another $\overline{\tau} = (b_1, ..., b_n) \in I$, $\overline{\sigma\tau}$ shall belong to $I$ too, we must make a crucial remark: in the above ”adjacency sequence” for $\overline{\sigma} \sim (1, ..., 1)$, $\overline{\sigma}^{-1} a_{i_{\kappa}}^{-1} \in U$ with 1 in the $i_{\kappa}$-coordinate, hence $\overline{a_{i_{\kappa}}}^{-1} a_{i_{\kappa}}^{-1} \in E_{i_{\kappa}}$, allowing us to replace the condition for the existence of an ”adjacency sequence” for $\overline{\tau} \sim (1, ..., 1)$ with the possibility to write $\overline{\sigma}$ as a product of elements belonging to the several $E_i$’s: indeed,
\[
\bar{\alpha} = \bar{\alpha}^0 = \bar{\alpha}^\mu \left( \bar{\alpha}^{\mu-1} \bar{\alpha}^{\mu-2} \right) \left( \bar{a}^{\mu-1} \bar{a}^{\mu-2} \right) \left( \bar{a}^{\mu-1} \bar{a}^{\mu-2} \right) \left( \bar{a}^{\mu-1} \bar{a}^{\mu-2} \right) = \]

\[
= \bar{\alpha}^{\mu+1} \bar{\alpha}^{\mu} ... \bar{\alpha} + 1, \text{ any } \bar{\alpha} \in E_{\kappa}, \text{ where it is obvious what we have substituted the greek } \bar{\alpha}' \text{ for; conversely, given such an expression of } \bar{\alpha} = \bar{\alpha}^\mu \text{ as } \bar{\alpha}^{\mu+1} \bar{\alpha}^{\mu} ... \bar{\alpha} + 1, \text{ with any } \bar{\alpha} \in E_{\kappa}, \text{ we get the adjacency sequence } \bar{\alpha} = \bar{\alpha}^\mu = \bar{\alpha}^{\mu+1} \bar{\alpha}^{\mu} ... \bar{\alpha} + 1, \text{ i.e. } \bar{\alpha} = \bar{\alpha}^{\mu+1} \bar{\alpha}^{\mu} ... \bar{\alpha} + 1, \text{ with } \bar{\alpha} = \bar{\alpha}^{\mu+1}, \bar{\alpha}^\mu + 1 = (1, ..., 1). \text{ Hence we may also write } \mathbb{b} \in I \text{ as a product of elements of the several } E_i \text{'s, say } \mathbb{b} = \beta^\mu \beta^{\mu-1} ... \beta^2 \beta^1, \text{ therefore it becomes obvious through this new equivalent condition for an element of } U \text{ to belong to } I \text{ that also the product } \mathbb{a}^\mu \beta = \mathbb{a}^{\mu+1} \mathbb{a}^{\mu} ... \mathbb{a}^{\mu+1} \beta^\mu, \text{ i.e. } \mathbb{a}^\mu \beta \in \mathbb{a}^{\mu+1}, \text{ hence it becomes obvious that } U \text{ is generated by its subgroups } E_i, i = 1, ..., n - \text{ i.e., } I = \left\{ \bigcap_{i=1}^n E_i \right\}. \text{ This, combined with the normality of the } E_i \text{'s in } U, \text{ assures that } I \text{ is normal in } U.
\]

As for the last claim, it will suffice to prove that, for any } a \in A, \text{ it is possible to write it as a product } \bar{\alpha}^{\mu+1} \bar{\alpha}^{\mu} ... \bar{\alpha} + 1, \text{ where } \bar{\alpha} \in E_{\kappa}, \text{ in such a way, that } i_{\mu+1}(i_{\mu}(...(i_1 - \text{ or even in a way such that this ordering will first be valid after application of the (random) permutation } \tau_0); \text{ to see this, it will obviously be enough to prove that, given a product } \mathbb{e}_i \mathbb{e}_j \text{ with } \mathbb{e}_i \in E_i, \mathbb{e}_j \in E_j, \text{ it is always possible to write it in the form } \mathbb{e}_i' \mathbb{e}_j', \text{ where } \mathbb{e}_i' \in E_i, \mathbb{e}_j' \in E_j; \text{ in particular, we need that just for } j = i + 1, \text{ as we can then generate any permutation } \tau. \text{ For notational convenience we prove it for } i = 1, j = 2; \text{ so, let } \mathbb{e}_1 \in E_1, \mathbb{e}_2 \in E_2. \text{ By taking their commutator } [\mathbb{e}_1, \mathbb{e}_2], \text{ one sees directly that it belongs to } E_1 \cap E_2 = E_{12}, \text{ hence } \mathbb{e}_1 \mathbb{e}_2 = \mathbb{e}_2 \mathbb{e}_1 [\mathbb{e}_1, \mathbb{e}_2], \text{ which is a product of } \mathbb{e}_2 \in E_2 \text{ and } \mathbb{e}_1 [\mathbb{e}_1, \mathbb{e}_2] \in E_1. \text{ (Alternatively, it is enough to remember that the } E_i \text{'s, as well as any (finite) products of them, are all normal subgroups of } U).\]

We emphasize here that this does not in general mean that the elements of } E_i \text{ commute with those of } E_j \text{ (with } i \neq j), \text{ unless } n = 2, \text{ in which case the commutator above becomes the identity element of } U, \text{ as } E_{12} \text{ is then the trivial subgroup. ■}

It cannot be overstressed that the core } I \text{ does NOT, in general, pertain to the group } U, \text{ but to its particular given inclusion } U \leq A_1 \times A_2 \times \ldots \times A_n \text{ as a subgroup of } I \text{'s direct product.}

Lemma 6 For } \bar{\alpha}, \mathbb{b} \in U, \text{ it holds that } \bar{\alpha} \sim \mathbb{b} \text{ iff } \mathbb{b} \in I \text{ ; hence equivalence classes of } "\sim" \text{ is the same thing as } I \text{'s cosets in } U.

Proof. } \bar{\alpha} \sim \mathbb{b} \iff \exists \text{ a finite sequence } \bar{\alpha} = \bar{\alpha}^\mu, \bar{\alpha}^\mu - 1, ..., \bar{\alpha}^0 = \mathbb{b}, \text{ such that any two neighbouring terms } \bar{\alpha}^{\mu-1} \bar{\alpha} \text{ share, say, their } i_{\kappa} \text{-coordinate for } \kappa = 1, ..., \mu + 1, \text{ meaning that } \bar{\alpha}^{\mu-1} \bar{\alpha} = \bar{\alpha}^{\mu-1} \bar{\alpha} \text{.\"'}
Remark 7 Despite the "simplifying" assertion of the preceding lemma it is however important for our understanding and analysis also to continue identifying the cosets of I as "adhesive fibres", according to our original definition of the equivalence relation. It is meanwhile precisely that understanding, which has lead us to this new approach and insight into the subdirect structure - while, besides being the starting point of view for the bulk of our general analysis here, it also proves crucial later for the proper understanding of the following subsections.

The first four points of the following lemma are a direct consequence of the definitions:

Lemma 8 Let $\emptyset \not= M, M', N \subseteq \{1, \ldots, n\}$. Then, (i) $E_M \cap E_N = E_{M \cap N}$, (ii) For $M' \subseteq M$, $L_{M'} \subseteq L_M$, (iii) Hence $L_{M\cap N} \supseteq L_M L_N$, $L_N L_M$, which are both equal to $L_M \times L_N$ in case $M \cap N = \emptyset$, (iv) $L_{M\cap N} = L_M \cap L_N$. Furthermore, for any subset $\Lambda$ of the set $J$ of indices, $E_\Lambda \subseteq \pi_\Lambda(U)$.

Proof. We need only to prove the last assertion.

We shall show that $E_s \subseteq \pi_{1,\ldots,n}(U)$. Just for notational convenience, we will show this for $s=1$, i.e. that $E_1 \subseteq \pi_{2,\ldots,n}(U)$. Let, so, $\pi = (a_2, \ldots, a_n) \in E_1$ and $\overrightarrow{\pi} = (b_2, \ldots, b_n) = (1,b_2, \ldots, b_n) \in \pi_{2,\ldots,n}(U)$, which means that there exists some $b_1 \in A_1$, such that $\overrightarrow{\pi} = (b_1, \ldots, b_n) \in U$; then $\pi^\sigma = \overrightarrow{\pi} \in U$, therefore $\pi^\sigma \in \pi_{2,\ldots,n}(U) \cap U = E_1$, completing the argument. The argument in the general case of any subset $\Lambda$ of the set $J$ of indices is completely similar. ■

The next lemma is a very practical one, although quite obvious:

Lemma 9 For $M \cap N = \emptyset$ above, non-equality in $L_{M\cap N} \supseteq L_M \times L_N$ means the existence of some element $\pi = (a_1; a_M; 1, \ldots, 1) \in I$, such that $\pi_M(\overrightarrow{\pi}) = (a_M) \notin I$ ($\Rightarrow \pi_N(\overrightarrow{\pi}) \notin I$). By an equivalent formulation, for $M \cap N = \emptyset$, $L_{M\cap N} = L_M \times L_N$ iff for any $\pi \in L_{M\cap N}$, $\pi_M(\overrightarrow{\pi}) \in L_{M\cap N}$ ($\Leftrightarrow \pi_N(\overrightarrow{\pi}) \in L_{M\cap N}$).

Definition 10 Let $\Lambda$ be any set of indices $i_1,i_2,\ldots,i_s$ from $\{1,\ldots,n\}$, $L_{\Lambda} = L_{i_1,i_2,\ldots,i_s}$ the corresponding "subcore" $(A_{i_1} \times A_{i_2} \times \ldots \times A_{i_s}) \cap I$ of $U$. We shall call such a non-trivial subcore $L_{\Lambda} = L_{i_1,i_2,\ldots,i_s}$ cohesive in $U \leq A_1 \times A_2 \times \ldots \times A_n$ if there is no non-trivial partition $\Lambda = \{i_1, i_2, \ldots, i_s\} = M \cup N$ of $\Lambda$ ($M \cap N = \emptyset$), with $L_M$, $L_N$ non-trivial and $L_{\Lambda} = L_M \times L_N$, i.e., so that $L_{\Lambda}$ split over $L_M$ ($\Leftrightarrow$over $L_N$).

A subcore $L_{\Lambda}$ shall be called reducible if there is a proper subset $M \subset \Lambda$, such that $L_M = L_{\Lambda}$; otherwise we shall designate it as a non-reducible subcore.

The last lemma [9] is crucial up to the proof of the following one:
Lemma 11 Assume $\emptyset \neq M, N \subseteq \{1, \ldots, n\}$ with $M \cap N \neq \emptyset$, such that the subcores $L_M, L_N$ be cohesive and non-reducible. Then $L_{M \cup N}$ is cohesive too. Therefore there are uniquely determined maximal cohesive subcores $L_N$'s, $i = 1, 2, \ldots, n$, which we shall call the cohesive components of $I$, and we have correspondingly the finest possible decomposition $I = L_{N_1} \times L_{N_2} \times \ldots \times L_{N_n}$ of the core.

Proof. Assume, to the contrary, that there exists a non-trivial partition $S \sqcup S'$ of $M \cup N$, so that (1) $L_{M \cup N} = L_S \times L_{S'}$. Set now $S \cap M = S_1, S \cap N = S_2, S' \cap M = S'_1, S' \cap N = S'_2$. We shall show that $L_M = L_{S_1} \times L_{S'_1}$ and $L_N = L_{S_2} \times L_{S'_2}$; by elementary set-theoretic arguments on the index-sets it is immediate to see that, in any case, at least one of the above direct decompositions is non-trivial, which yields a contradiction to the cohesiveness. The proof being similar in both cases, we are restricting ourselves to showing the first one.

Take any arbitrary $\mathfrak{t} \in L_M \subseteq L_{M \cup N}$, hence by (1) and lemma 9, $\pi_S(\mathfrak{t}) \in I$, implying both, $\pi_S(\mathfrak{t}) \in L_S$ and $\pi_S(\mathfrak{t}) \in L_M$, therefore by lemma 8(iv) $\pi_S(\mathfrak{t}) \in L_{S \cap M} = L_{S_1}$ and, a fortiori, $\pi_{S_1}(\mathfrak{t}) \in L_{S_1}$; similarly, $\pi_{S'_1}(\mathfrak{t}) \in L_{S'_1}$. Then, again by lemma 9, we get $L_M = L_{S_1} \times L_{S'_1}$, as wished.

This property guarantees that any cohesive subcore is contained in a maximal one; then, clearly by virtue of maximality, the (direct) product of all the maximal cohesive subcores gives the whole of $I$. ■

Remark 12 In case we had an infinite sum (see remark 7) instead of the finite case we have considered in our proofs, the first part of the last lemma would ensure that the set of cohesive subcores is inductively ordered, hence Zorns lemma applies, to give that any cohesive subcore is contained in a maximal one also in this case.

Definition 13 In case the cohesive components of $I$ are precisely the $L_i$'s, $i = 1, 2, \ldots, n$, or, equivalently, $I = L_1 \times L_2 \times \ldots \times L_n$, we shall call the subgroup $U \leq A_1 \times A_2 \times \ldots \times A_n$ a (cohesively) smashed one. If for a non-empty, proper subset $M$ of $\{1, \ldots, n\}$, $I = L_M \times L_{\tilde{M}}$, then we shall say that $I$ splits over $L_M$ (or over $L_{\tilde{M}}$) - or even, by a simplifying controlled abuse of language, over $M$ (or $\tilde{M}$).

At the other extreme of such a non-trivial decomposition of the core, we want to make another distinction, that of a deltoid subcore $L_{\Lambda}$ of $U$, meaning that for any proper subset of indices $M$ of $\Lambda$, $L_M$ is trivial. A non-interesting special case of that occurs whenever $|\Lambda| = 1$, such subcores shall be referred to as trivial ones. $U$ itself shall be called deltoid if all $E_i$'s are trivial; in that case it may also be viewed as a (non-proper) deltoid subcore of itself.

The following two lemmata are quite immediate to see:

Lemma 14 Let a $M$ be a non-empty, proper subset of $\{1, \ldots, n\}$; then $L_M \times L_{\tilde{M}} \subseteq I$, with "=" holding iff $I$ splits over $M$. 

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Lemma 15 If, for a non-empty, proper subset \( M \) of \( \{1, \ldots, n\} \), \( L_M \) is a maximal deltoid subcore of \( U \) (even trivially, with \(|M| = 1\)), then \( L_M \) is a direct factor of \( I \).

Lemma 16 Let \( U \leq A_1 \times A_2 \times \ldots \times A_n \) and assume also that all \( \pi_i(U) \)'s are non-trivial (otherwise we should have considered the maximal subproduct satisfying this condition); then, the following are true:

\textbf{a.} For \( \overline{\pi}, \overline{\sigma} \in U \), with \( \pi \) fixed, \( \sigma \) variable, so that \( \pi_s(\overline{\sigma}) = \pi_s(\overline{\pi}) \), the variation domain of \( \overline{\sigma} \) is the coset \( \pi E_s \) in \( U \) while the variation domain of \( \pi_{1,\ldots,n}(\overline{\sigma}) = \pi_{1,\ldots,n}(\overline{\pi}) \) in \( U \); conversely, by varying only the \( s \)-coordinate, the variation space of \( \overline{\sigma} \) is the coset \( \pi E_{1,\ldots,n} \) in \( U \).

More generally, for any variable \( \overline{\pi} = (a_1, \ldots, a_n) \in U \) and any sequence of indices \( i_1 \langle i_2 \langle \ldots \langle i_s \rangle \), by fixing the \( \{i_1, i_2, \ldots, i_s\} \)-coordinates \( \{a_1, \ldots, a_s\} \) and varying the others, so that \( \overline{\pi} \) remain in \( U \), the variation domain of \( \overline{\pi} \) is the coset \( \pi E_{i_1,\ldots,i_s} \); conversely, by varying the complementary set of coordinates, the variation domain of \( \overline{\pi} \) becomes \( \pi E_{1,\ldots,\hat{i_1},\ldots,\hat{i_s},\ldots,n}(\overline{\pi}) \). Corresponding to that, the variation domain of \( \pi_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n}(\overline{\pi}) \) is, in the first case, the coset \( \pi_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n}(\overline{\pi}) E_{i_1,\ldots,i_s} \pi_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n}(U) \); accordingly, by varying the \( \{i_1, i_2, \ldots, i_s\} \)-coordinates \( \{a_1, \ldots, a_s\} \) and fixing the others, so that \( \overline{\pi} \) remain in \( U \), the variation space of \( \pi_{i_1,\ldots,i_s}(\overline{\pi}) \) (while varying inside \( \pi_{i_1,\ldots,i_s}(U) \)) is the coset \( \pi_{i_1,\ldots,i_s}(\overline{\pi}) E_{1,\ldots,\hat{i_1},\ldots,\hat{i_s},\ldots,n} \), for any particular (“original”) value of \( \overline{\pi} \).

\textbf{b.} For any non-empty proper subset \( M = \{i_1, i_2, \ldots, i_s\} \) of \( \{1, \ldots, n\} \), say \( i_1 \langle i_2 \langle \ldots \langle i_s \rangle \), there is a unique isomorphism \( \sigma : \pi_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n}(U) / E_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n} \mapsto \pi_{i_1,\ldots,i_s}(U) / E_{i_1,\ldots,i_s} \), with the “structural” property that, for any \( \overline{\pi} = (a_1, \ldots, a_n) \in U \), \( \sigma \) sends the coset \( \pi_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n}(\overline{\pi}) E_{i_1,\ldots,i_s} \) to \( \pi_{i_1,\ldots,i_s}(\overline{\pi}) E_{i_1,\ldots,i_s} \). In other words \( U \) may be realized as a fiber product of \( \pi_{1,\ldots,i_1,\ldots,\hat{i_2},\ldots,i_n}(U) \) and \( \pi_{i_1,\ldots,i_s}(U) \), over a fixed isomorphic copy of the two sides of \( \sigma \) with respect to their apparent epimorphisms on it.

As for the converse, a subgroup \( U \) is now determined by the following data:

A normal subgroup \( I \) of \( U \), with the property that it only contains one “~” equivalence class, a partition \( \{1, \ldots, n\} = M \cup \hat{M} \), subgroups \( W_M \leq Dr \prod_{i \in M} A_i \) and \( W_{\hat{M}} \leq Dr \prod_{i \in \hat{M}} A_i \), respectively containing \( L_M \) and \( L_{\hat{M}} \), together with a "structural isomorphism" \( \sigma : W_M / L_M \mapsto W_{\hat{M}} / L_{\hat{M}} \), where \( L_M = \pi_M(I) \), \( L_{\hat{M}} = \pi_{\hat{M}}(I) \). \( U \) thus is determined set-theoretically as a subset of the direct product, which in turn fully determines its group structure.

\textbf{Proof.} (We are treating the general case of (a), the first one just being a special case of that.)

We shall throughout keep on the convention of viewing the image of any projection from the original direct product as contained (embedded) in that product too, in the obvious way.
a. So, let \( \overrightarrow{a} \in U \) have the same \( \{i_1, i_2, \ldots, i_s\} \)-coordinates \( (\{a_{i_1}, \ldots, a_{i_s}\}) \) as \( \overline{a} \), i.e. \( \pi_{i_1i_2\ldots i_s}(\overrightarrow{a}) = \pi_{i_1i_2\ldots i_s}(\overline{a}) \); then obviously \( \overline{a} \cdot \overrightarrow{a} \in E_{i_1i_2\ldots i_s} = E_M \) \( \iff \overrightarrow{a} = \overline{a} \) and \( \overrightarrow{a} \in \pi E_M = \pi L_{\overline{M}} \), meaning that \( \pi E_M = \pi L_{\overline{M}} \) is the
variation domain of the conditionally (i.e., lying inside \( U \)) variable \( \overline{a} \) - which, while the \( \{i_1, i_2, \ldots, i_s\} \)-coordinates remain constant and only the rest defines the variation, is equivalent to that \( \pi_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n}(\overrightarrow{a}) = \\
= \pi_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n}(\overline{a}) = \\
= \pi_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n}(\overline{a}) \), i.e., while \( \overrightarrow{a} \) varies in the prescribed way, the variation domain of the really changing part, \( \pi_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n}(\overline{a}) \), is \( \pi_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n}(\overrightarrow{a}) E_{i_1\ldots i_2\ldots i_s} \), or, by another notation, \( \pi_{\hat{M}}(\overline{a}) L_{\overline{M}} \).

Conversely, by holding the (complementary) set of \( \hat{M} \)-coordinates of \( \overline{a} \) fast, the variation domain of \( \overline{a} \) becomes, similarly, \( \pi \Pi E_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n} = \pi L_M = \pi \overline{M} \) - while that of its really variable part \( \pi_{i_1i_2\ldots i_s}(\overline{a}) \) shall be \( \pi_{i_1i_2\ldots i_s}(\overrightarrow{a}) E_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n} = \pi_M(\overline{a}) L_M \).

In this way, any element \( \overline{a} \in U \) determines in this way an assignment of the coset \( \pi_{\hat{M}}(\overline{a}) L_{\overline{M}} \) from \( \pi_{\hat{M}}(\overline{a}) L_{\overline{M}} \) to the coset \( \pi M(\overline{a}) L_M \) from \( \pi M(\overline{a}) L_M \) - and vice versa. We are going to show now that these assignments indeed define an isomorphism.

It is at first clear that, while \( \overline{a} \) varies over \( U \), the union of the cosets
\( \pi_{i_1i_2\ldots i_s}(\overline{a}) E_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n} = \pi M(\overline{a}) L_M \) gives the whole of \( \pi M(U) \) and, likewise, the union of the cosets \( \pi_{1\ldots i_1\ldots i_2\ldots i_s\ldots i_n}(\overrightarrow{a}) E_{i_1i_2\ldots i_s} \) \( \pi_{\hat{M}}(\overrightarrow{a}) L_{\overline{M}} \) gives all of \( \pi_{\hat{M}}(U) \).

b. Let us now deal with the bijectivity.

To that end we are also here introducing a special (w.r.t. \( M \)) relation on \( U \leq \pi M(U) \times \pi_{\hat{M}}(U) \), where in this last product we shall denote the typical element \( \overline{a} \) as \( \overline{a} = (\overline{a}_M; \overline{a}_{\overline{M}}) \) with \( \overline{a}_M = \pi_M(\overline{a}), \overline{a}_{\overline{M}} = \pi_{\overline{M}}(\overline{a}) \); we shall still allow ourselves to view \( \overline{a}_M; \overline{a}_{\overline{M}} \) as elements of the original direct product as well, without notification. Notice that as such they (as well as their resp. inverses) commute, because of the direct product.

Let us then say for two such elements \( \overline{a}, \overline{b} = (\overline{a}_M; \overline{a}_{\overline{M}}) \) to be "related" if either \( \overline{a}_M = \overline{b}_M \) or \( \overline{a}_{\overline{M}} = \overline{b}_{\overline{M}} \). This rule of course corresponds to our two "variations" above, while keeping some coordinates fixed, respectively the one way or the other, and amounts to either \( \overline{a}^{-1}\overline{b} \in L_M \) or \( \overline{a}^{-1}\overline{b} \in L_{\overline{M}} \). We then let "r" be the least transitive relation generated by that rule, which we may immediately see to be an equivalence relation, in fact one quite reminiscent of that defined in our proof of theorem 1.

It is then immediate to see that for any given \( \overline{a} = (\overline{a}_M; \overline{a}_{\overline{M}}) \in U \), the r-equivalence class \( [\overline{a}] \) to which \( \overline{a} \) belongs is \( [\overline{a}] = \{ \overline{a}_M L_M; \overline{a}_{\overline{M}} L_{\overline{M}} \} = \{ \overline{a}_M L_M; \overline{a}_{\overline{M}} L_{\overline{M}} \} \). By comparing this to the assignment determined by any element, call it now \( \overline{a} \), the way we did in part A of the proof, and by virtue of the guaranteed distinctiveness of the equivalent classes, it becomes clear that the assignments in A altogether amount to a bijective function \( \sigma : \pi M(U) / L_M \longrightarrow \pi_{\hat{M}}(U) / L_{\overline{M}} \), with the "structural" property of assigning \( \overline{a}_M L_M \longrightarrow \overline{a}_{\overline{M}} L_{\overline{M}} \), for any \( \overline{a} = (\overline{a}_M; \overline{a}_{\overline{M}}) \in U \). This assignment through \( \overline{a} = (\overline{a}_M; \overline{a}_{\overline{M}}) \in U \) is thus also bound
to \((\mathfrak{f}_M \mathcal{L}_M, \mathfrak{f}_M \mathcal{L}_M) = (\mathfrak{f}_M, \mathfrak{f}_M) \mathcal{L}_M \mathcal{L}_M = \mathfrak{f}_M \mathcal{L}_M \mathcal{L}_M = \mathfrak{f}_M (L_M \times \mathcal{L}_M)\), hence bijection \(\sigma\) is extended, \(\sigma : \pi_M (U) / \mathcal{L}_M = \pi_{\mathcal{L}_M} (U) / \mathcal{L}_M \rightarrow U / \mathcal{L}_M \times \mathcal{L}_M\) by the totality of the (elementwise "structural") assignments \(\mathfrak{f}_M \mathcal{L}_M \mapsto \mathfrak{f}_{\mathcal{L}_M} (\mathcal{L}_M)\) for all \(\mathfrak{f} \in U\).

\(\mathcal{L}_M\). It remains now only to prove the homomorphic property of the (extended) \(\sigma\).

However this follows from its "structural property" and the componentwise multiplication in the direct product, i.e. the fact that \((\mathfrak{f}\mathcal{G})_M = \mathfrak{f}_M \mathcal{G}_M, (\mathfrak{f}\mathcal{G})_{\mathcal{L}_M} = \mathfrak{f}_{\mathcal{L}_M} \mathcal{G}_{\mathcal{L}_M}\) - and so on. The situation is very similar to that in our proof of theorem 1 (case \(n=2\)).

As for the converse, we construct the \((\mathcal{L}_M, \mathcal{L}_M)\)-type "pair fibres" of the suitable \(U\) thanks to the "structural property" of \(\sigma\), so that \(\pi_M (U) = \mathcal{W}_M\) and \(\pi_{\mathcal{L}_M} (U) = \mathcal{W}_M\).

\textbf{Theorem 17} For any non-empty proper subset \(M = \{i_1, i_2, \ldots, i_s\}\) of \(\{1, \ldots, n\}\), there is a unique isomorphism \(\sigma : \pi_M (U) / \mathcal{L}_M = \pi_{\mathcal{L}_M} (U) / \mathcal{L}_M \rightarrow U / \mathcal{L}_M \times \mathcal{L}_M\), which further extends to an isomorphism to \(\mathcal{R} = U / \mathcal{I}\) if and only if \(I\) splits over \(L_M\); if that is not the case, then \(\mathcal{R}\) is isomorphic to a quotient of \(U / \mathcal{L}_M \times \mathcal{L}_M\).

Furthermore the above isomorphism \(\sigma\) has the "structural" property that, for any \(\mathfrak{f} = (a_1, \ldots, a_n) \in U\), \(\sigma\) sends the coset \(\pi_M (\mathfrak{f}) L_M\) over to \(\pi_{\mathcal{L}_M} (\mathfrak{f}) \mathcal{L}_M\), which implies the partition of \(U\) into distinct "pair fibres".

That implies that also the converse of the statement is true, meaning that we may restore \(U\) from the following data: A partition \(\{1, \ldots, n\} = M \cup \mathcal{M}\), subgroups \(W_M \leq \mathcal{D} \prod_{i \in M} A_i\) and \(\mathcal{W}_M \leq \mathcal{D} \prod_{i \in \mathcal{M}} A_i\), a normal subgroup \(I\) of \(W_M \times \mathcal{W}_M\), which is "adhesive" as a subset of the original product, i.e. it only contains one ~"~-equivalence class, together with a "structural isomorphism" \(\sigma : W_M / \mathcal{L}_M = \mathcal{W}_M / \mathcal{L}_M\), where \(L_M = \pi_M (I), \mathcal{L}_M = \pi_{\mathcal{L}_M} (I)\).

\textbf{Proof.} Directly from lemmata [10] & [14].

\textbf{Remark 18} This theorem may be considered as a generalization of the case \(n=2\).

In both cases we may depict the subgroup \(U\) diagrammatically as \(\wedge\) - with 2 edges. The converse statement in both theorems shows that the subgroup \(U\) is fully determined once the following data is given: (i) The 2 edge-groups, call them \(W_M, \mathcal{W}_M\), amounting to the subgroups \(\pi_M (U), \pi_{\mathcal{L}_M} (U)\) in the last theorem, (ii) A normal subgroup \(I\) of \(W_M \times \mathcal{W}_M\), such that, by defining \(L_M = \pi_M (I), \mathcal{L}_M = \pi_{\mathcal{L}_M} (I)\) (which shall correspond to the bottom vertices,
call them socles, of the two edges), the factor groups $W_M/L_M$, $W_{\tilde{M}}/L_{\tilde{M}}$ are isomorphic, both corresponding to the "head" vertex of the two edges, (iii) An actual such isomorphism $\sigma : W_M/L_M \rightarrow W_{\tilde{M}}/L_{\tilde{M}}$, which shall serve as the "structural isomorphism" for determining the "pair fibres" of $U$. The condition for $U$ to be a subgroup in the original direct product $A_1 \times A_2 \times \ldots \times A_n$ is that $I$ shall be "cohesive" as a subset of that, where of course $W_M \leq \prod_{i \in M} A_i$ and $W_{\tilde{M}} \leq \prod_{i \in \tilde{M}} A_i$. Notice that, in case $I$ splits over $L_M$, the cohesive property amounts to just the cohesiveness of $L_M$ and $L_{\tilde{M}}$.

This picture does actually suggest that a whole "class" of such subgroups may be defined by just varying the structural isomorphism $\sigma$: This is actually the subject of subsection 4.2, where we however only actualize the case that $I$ splits over $L_M$, but in the generalized context of theorem 41.

Similarly to the lemma 2 we prove also here the following:

**Lemma 19** For any non-empty proper subset $M$ of $\{1, \ldots, n\}$, $\pi_M(U) \cong U/L_{\tilde{M}}$, $\pi_{\tilde{M}}(U) \cong U/L_M$.

**Proof.** Observe again that $\ker \pi_M|_U = L_{\tilde{M}}$, $\ker \pi_{\tilde{M}}|_U = L_M$. 

---

**Remark 20** In continuation of remarks 4 and 12, by going through the arguments of our proofs in this section, it is easy to see that they are generalizable to the case of an infinite direct sum. Notice that corresponding to theorem 17, we shall then have a partition $J = M \sqcup \tilde{M}$ (where $\tilde{M} = J \setminus M$, as usually) of the indexing set $J$.

The same generalizability of our results to the infinite case remains true throughout the following section, however and for space economy we are not going to point it out again and again in what follows.

### 4 The general structure

#### 4.1 The general theorems

Let now $U \leq A_1 \times A_2 \times \ldots \times A_n$ ($n > 1$) be a subdirect product - i.e., $\pi_i(U) = A_i$ for all $i$'s.
Lemma 21 Assume that $U$ above is deltoid, i.e. all $E_i$'s are trivial; then all $A_i$'s are isomorphic to $U$, and there is a system of ("structural") isomorphisms between any two of them, such that $U$ consist of $n$-tuples of through those isomorphisms corresponding elements. The converse holds (trivially) too. In particular, the same holds for any deltoid subcore $L_A$ of $U$, by considering it as a subdirect product of $\prod_{i \in A} \pi_i(U)$.

Proof. By assuming that we might have two elements $\overline{a}, \overline{b} \in U$ with one, say the $i$'th, coordinate in common and with at least another coordinate not in common, that would give the contradiction that $1 \neq \overline{b}^{-1} \overline{a} \in E_i$. This shows that $U$ entirely consists of mutually disjoint $n$-tuples $\overline{a} = (a_1, ..., a_n)$; this, combined with the assumption $\pi_i(U) = A_i$ for all $i$'s, establishes a system of bijective maps between any two of the direct factors $A_i$. That these are group homomorphisms, simply amounts to the group structure and the coordinatewise multiplication in $U$.

Remark 22 As $L_i \leq E_j$ for any $j \neq i$, the hypothesis of the lemma yields that also all $L_i$'s are trivial.

For a non-empty, proper subset $M$ of $\{1, ..., n\}$, such that $L_M$ is a maximal deltoid subcore of $U$, let $\kappa$ be any arbitrary coordinate contained in $M$, and let $M_\kappa$ denote the set $M - \{\kappa\}$. Then we shall call $L_{M_\kappa}$ a submaximal deltoid subcore of $U$ subject to the (apparently uniquely determined) maximal $M$.

Proposition 23 With the above notation, for a submaximal core $L_{M_\kappa}$ let us simplify the notation by setting $V = \pi_{M_\kappa}(U)$; we may consider $V$ as a subgroup of $\prod_{i \in M_\kappa} A_i$ via its isomorphism to $A_\kappa < \prod_{i \in M_\kappa} A_i$ (see lemma 21). Then, with the obvious meaning of notation, we have the following:

(i) $V$ is a subdirect product of $\prod_{i \in M_\kappa} A_i$.

(ii) $L_\kappa(V) \cong L_M(U) \cong L_M$, $L_\kappa(V) = L_M(U) \cong L_M$.

(iii) $I(V) = L_M(V) \times L_\kappa(V) \cong I(U) = L_M \times L_M$, $V/I(V) \cong U/I$ and $V \cong U$.

It is thus possible, by continuing just as with the substitution of $V$ for $U$ here, to substitute a subdirect product $U$ by another (subdirect in a subproduct of the original $\prod_{i=1}^n A_i$) which is isomorphic to it and has a similar subdirect structure but with non-trivial deltoid subcores.

Proof. (i) is an immediate consequence of the definition of $V$ and the properties of projections.

(ii) Notice that $\widehat{M}$ here means the complement of $\{\kappa\}$ in $\widehat{M} \cup \{\kappa\}$, i.e. $L_\kappa(V) = Ker\pi_{M_\kappa}(U)$, which is isomorphic to $L_M(U)$ because of lemma 21. The second one follows because any element $x$ of $U$, with $\pi_\kappa(x) = 1$, belongs to $I(U)$, hence by lemma 21 also $\pi_M(x) = 1$, therefore $x \in L_M(U)$.
(iii) The equalities follow from lemmata 15 and 14 then apply (ii). Now \( \pi_\kappa (V) = \pi_\kappa (U) = A_\kappa \) and lemma 24 again shows that \( \pi_\kappa (U) \cong \pi_M (U) \), therefore \( \pi_\kappa (V) \cong \pi_M (U) \), while in this last isomorphism \( L_\kappa (V) \) corresponds to \( L_M (U) \) (as in (ii)), therefore also \( \pi_\kappa (V) / L_\kappa (V) \cong \pi_M (U) / L_M (U) \) (1).

On the other hand by theorem 17 applied twice, \( V / I (V) \cong \pi_\kappa (V) / L_\kappa (V) \cong \pi_M (U) / L_M (U) \) (see (1)) \( \cong U / I \), as \( I = I (U) \) splits over \( L_M = L_M (U) \). But, also theorem 17 this last quotient is also isomorphic to \( \pi_M (U) / L_M (U) \), while the first in the above sequence of isomorphisms \( V / I (V) \cong \pi_M (V) / L_M (V) \), therefore also \( \pi_M (U) / L_M (U) \cong \pi_M (V) / L_M (V) \). If we now observe that \( \pi_M (U) = \pi_M (\pi_\kappa (U)) = \pi_M (V) \) and \( L_M (U) \subseteq L_M (V) \), we deduce from this last isomorphism that \( L_M (U) = L_M (V) \).

Now, the converse in theorem 17 as explained in remark 18 makes it clear how to define an isomorphism \( V \cong U \).

**Condition 24** By this proposition we may from now on assume that our subgroup \( U \) of \( \prod_{i=1}^n A_i \) under consideration contains no non-trivial deltoid subcores.

Let now be given a partition of a subset of the index-set of the original direct sum, i.e. \( \Lambda = M \sqcup N \).

As \( L_\Lambda = \left( \prod_{i \in \Lambda} A_i \right) \cap U \), our theorem 17 is applicable to the subgroup \( L_\Lambda \) of \( \prod_{i \in \Lambda} A_i \), while the definition of \( L_M \), \( L_N \) as subgroups of \( L_\Lambda \leq \prod_{i \in \Lambda} A_i \) still remains unchanged inside \( U \subseteq A_1 \times A_2 \times \ldots \times A_n \) (since they were already subgroups of \( \prod_{i \in \Lambda} A_i \) inside \( \prod_{i=1}^n A_i \)); hence, by that theorem, at any rate is \( L_M \times L_N \leq L_\Lambda \), while equality here would mean \( \pi_M (L_\Lambda) = L_M \) and, equivalently, \( \pi_N (L_\Lambda) = L_N \), since in this case the groups of the isomorphism \( \sigma \) in theorem 17 are trivial. In this connection it is important to notice that, considering a subgroup \( U \) of \( \prod_{i=1}^n A_i \) in the case that \( U \)'s projection on some of the direct factors \( A_i \) is trivial makes our analysis too blurry and useless, by short-circuiting it at effect at a trivial level; consequently one should have to exclude at least those direct factors \( A_i \), on which \( U \)'s projection is trivial, by taking the subkernel that corresponds to the direct factors, on which the projection of \( U \) is non-trivial. Further, analyzing all subkernels of \( U \), could take us closer to a diagrammatic representation of \( U \)'s structure - which at any rate is limited by the (complexity of the) structure of the direct factors \( A_i \) themselves.

- In elementary terms, the condition \( L_M \times L_N = L_\Lambda \) means (by theorem 17 applied on \( L_\Lambda \)) that, for any \( x \in L_\Lambda \), the element \( \pi_M (x) \) of \( \prod_{i \in M} A_i \) also

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Lemma 26 The cohesive components of $I$ forces $\pi_I(x) \in L_N$. By now viewing $L_\Lambda$ as a subgroup of $Dr \prod_{i \in \Lambda} A_i$, while forgetting for a moment about the original $U \leq A_1 \times A_2 \times \ldots \times A_n$, we get the following

**Lemma 25** For $U \leq A_1 \times A_2 \times \ldots \times A_n$, $\emptyset \neq M \subset \{1, \ldots, n\}$, $M \neq \{1, \ldots, n\}$, the condition $\pi_M(U) \leq U$ implies the following (by theorem 17 equivalent) facts:

$\pi_M(U) = L_M$, $\pi_M(U) = L_{\Lambda M}$, $U = L_M \times L_{\Lambda M}$

**Proof.** Enhance the preceding discussion with the remark, following from the definition of the core $I$ of $U$, that $M$ being a proper subset of $\{1, \ldots, n\}$ immediately means that the relation $\pi_M(U) \leq U$ implies $\pi_M(U) \leq I(U)$, which then forces $\pi_M(U) = L_M$. ■

- We are now pointing out a relevant implication of part (a) of theorem 17 in order to prove, on the contrary, that $L_M \times L_N \neq L_\Lambda$, it is enough just to find one $x \in L_\Lambda$, with the property that $\pi_M(x) \notin L_\Lambda$ (or, equivalently, $\notin L_M$).

**Lemma 26** The cohesive components of $I$ intersect each other trivially.

**Proof.** At first, notice that $M \cap N = \emptyset \Rightarrow L_M \cap L_N = 1$ and, therefore,

$L_M \cap L_N \neq 1 \Rightarrow M \cap N \neq \emptyset$. In view of this, combined with the maximality of the cohesive components from their definition, it will suffice to prove the following:

"If $M$, $N$, $P$ are mutually disjoint non empty index sets, such that $L_{M \cup P}$, $L_{N \cup P}$ be cohesive, then $L_{M \cup N \cup P}$ is cohesive too."

Assume to the contrary, that there is a non-trivial decomposition $L_{M \cup N \cup P} = L_R \times L_S$ (1) ($R$, $S$ non-empty, $R \cap S = \emptyset$, $R \cap S = M \cup N \cup P$).

On account of lemma 25 cohesiveness of $L_{M \cup P}$, $L_{N \cup P}$ implies that neither $R$ nor $S$ may be contained in either $M \cup P$ or $N \cup P$, which in turn means that $R$ as well as $S$ have non-trivial intersections with $M$ and $N$. So, by means of of lemma 25 we get through (1) a non-trivial decomposition of the subgroups $L_{M \cup P}$, $L_{N \cup P}$ of $L_{M \cup N \cup P}$, contrary to their cohesiveness. ■

**Proposition 27** There is always a unique (up to ordering of factors) decomposition $I = L_{N_1} \times \ldots \times L_{N_n}$ of the core $I$ as the product of its cohesive components; this will be referred to as the (total) cohesion decomposition of the core $I$.

**Proof.** If $I = L_{1 \ldots n}$ is cohesive, then we are done with $m=1$; otherwise, we continue examining its factors, until they cannot be any further decomposed, meaning that they are cohesive. Uniqueness is a consequence of the previous lemma. ■

**Remark 28** Given that one might have the situation $N \subset M$ ($N \neq M$) and still $L_N = L_M$, to a cohesion decomposition is, to begin with, not necessarily attached a unique partition of $\{1, \ldots, n\}$. To remedy that, we agree from now on (unless otherwise specified) to take the maximal such subsets of $\{1, \ldots, n\}$. 

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Definition 29 We shall call a subgroup $U \subseteq A_1 \times A_2 \times \ldots \times A_n$, for $n > s + 2$, an $r$-weakly smashed one if there exists a partition of $\{1, \ldots, n\}$ into subsets of cardinality at least $r$, such that for every subset $N = \{i_1, i_2, \ldots, i_s\}$ in the partition ($r \leq s$), $E_{i_1i_2\ldots i_s}$ is contained in (the direct product) $\prod_{\kappa \notin N} L_\kappa$.

(Trivially, for $t < s$, $t$-weak smashedness also implies $s$-weak smashedness.)

For $n > 2$ the condition that all $E_{ij}$, $i \neq j$, be trivial, of course also implies that all $L_i$'s are trivial.

Lemma 30 "1-weakly smashed" means for $U$ the same as "(cohesively) smashed".

Proof. "⇒": As the core $I$ is generated by the $E_i$'s, $E_i \subseteq L_1 \times \ldots \times L_n \Rightarrow I \subseteq L_1 \times \ldots \times L_n$, while the converse inclusion is trivial.

"⇐": Trivial. ■

The following theorem on smashed subdirect products is a crucial step toward reaching to the theorem about the general case:

Theorem 31 Let $U \subseteq A_1 \times A_2 \times \ldots \times A_n \ (n > 2)$ be a subdirect product (i.e., $\pi_i(U) = A_i$ for all i's) which is smashed. Then there is a (uniquely determined) "structural" system of isomorphisms of the $A_i/L_i$'s, all those being isomorphic to $U/I = U/(L_1 \times L_2 \times \ldots \times L_n)$, in a perfect generalization of the case $n=2$.

This, again, amounts to realizing $U$ as a fibre product of the $A_i$'s over $R := A_1/L_1$, with respect to each $A_i$'s epimorphism on it, gotten by composing the canonical $A_i \rightarrow A_i/L_i$, with $A_i/L_i \rightarrow A_1/L_1$ from the mentioned "structural" system of isomorphisms. The converse is again true. Also, for any subset of indices $i_1(i_2\ldots i_s$ from $\{1, \ldots, n\}$, we have uniquely determined structural isomorphisms $R \simeq \pi_{i_1i_2\ldots i_s}(U)/(L_{i_1} \times \ldots \times L_{i_s})$.

Proof. Set, now, $A'_i = A_i/L_i$ and use the previous proposition for the projection $U'$ of $U$, as a subgroup of $A'_1 \times A'_2 \times \ldots \times A'_n \cong (A_1 \times A_2 \times \ldots \times A_n)/(L_1 \times L_2 \times \ldots \times L_n)$ = $(A_1 \times A_2 \times \ldots \times A_n)/I$; by the component-dependent definition of the core it is clear that the core of $U'$ is trivial, hence also its generating subgroups $E'_i$; therefore we may apply lemma 24 on $U'$, then we lift back to $U$.

As for the converse, we set $I = L_1 \times L_2 \times \ldots \times L_n$, $A'_i = A_i/L_i$, consider $(A_1 \times A_2 \times \ldots \times A_n)/I \simeq A'_1 \times A'_2 \times \ldots \times A'_n$, apply the converse of the previous proposition and lift back.

Alternatively, we could again use the method of determining the "fibres of $n$-tuples" (turning out to be cosets of $I$ in $U$) as equivalence classes in $U$, as we did in the case $n = 2$.

As for the last assertion, it suffices to apply theorem 17 since $E_{i_1i_2\ldots i_s}(U) \cap \pi_{i_1i_2\ldots i_s}(U) = (U \cap I) = \pi_{i_1i_2\ldots i_s}(U) \cap (L_1 \times \ldots \times L_n) = L_{i_1} \times \ldots \times L_{i_s}$.

■
Remark 32  This theorem may also be viewed as a generalization, in another 
direction, of the case \( n = 2 \).

Corollary 33  A smashed subdirect product of \( A_1 \times A_2 \times \ldots \times A_n \) may always 
be taken as a pull-back of \( n \) epimorphisms.

Corollary 34  If \( U \leq A_1 \times A_2 \times \ldots \times A_n \) such that, for all \( i \in N = \{ i_1, i_2, \ldots, i_s \} \), 
\( E_i \) is contained in \( (L_{i_1} \times \ldots \hat{L}_i \times \ldots \times L_{i_s}) \prod_{\kappa \notin N} E_\kappa \) (equivalently, just in 
\( (L_{i_1} \times \ldots \times L_{i_s}) \prod_{\kappa \notin N} E_\kappa \)), then the preceding theorem is applicable for 
\( \pi_{i_1 \ldots i_s} \) (\( U \)) as a subgroup of the direct product of its projections on \( A_{i_1}, \ldots, A_{i_s} \).

Now we prove the following analogue to lemmata 2 and 19.

Lemma 35  Let \( U \leq A_1 \times A_2 \times \ldots \times A_n \) \( (n)2 \) be a smashed subdirect product.
Then for any sequence of indices \( i_1 \hat{\ldots} i_s \) \{ from \( \{ 1, \ldots, n \} \}, \quad \pi_{i_1 \ldots i_s} \) \( U \) \( \cong \) 
\( U/\text{Dr} \prod_{i \notin \{ i_1, i_2, \ldots, i_s \} } L_i \). In particular \( A_i \cong U/\text{Dr} \prod_{i \notin \{ i_1, i_2, \ldots, i_s \} } L_i \).

Proof.  Combine lemma 19 with smashedness, which implies that \( E_{i_1 \ldots i_s} \) \( L_{i_1} \times \ldots \times L_{i_s} \), \( (E_{i_1 \ldots i_s} =) L_{i_1} \times \ldots \times L_{i_s} \) = \( \text{Dr} \prod_{i \notin \{ i_1, i_2, \ldots, i_s \} } L_i \). \( \blacksquare \)

Example 36  Assume that we have a normal subgroup \( B = B_1 \times \ldots \times B_n \) of 
a group \( G \), \( n \geq 2 \), hence every single direct factor \( B_i \) is normal in \( G \). Let 
\( G/B \cong R \).

Set \( K_i = \text{Dr} \prod_{j \neq i} B_j \), \( \sigma_i : G \to G/K_i \), \( i = 1, \ldots, n \), the natural epimorphisms 
and, finally, \( A_i = G/K_i \), \( A = \text{Dr} \prod_{i=1}^n A_i \).

Then we get a faithful representation \( \sigma \) of \( G \) as a subdirect product of the 
direct product \( A \), as follows:
\[ \sigma : G \ni g \mapsto (\sigma_1 (g), \ldots, \sigma_n (g)) \in A \]

It is obvious that this is a monomorphism (we are in the just following showing its injectivity) - and we set \( U = \sigma (G) \leq A = \text{Dr} \prod_{i=1}^n A_i \). We will be using the 
terminology that we have established above; it is immediate to see the following facts: \( \pi_i (U) = \sigma_i (G) = A_i \), 
\( L_i = \sigma (B_i) \cong \text{(realizable as "\( = \)" inside \( A \)) \( \sigma_i (B_i) \) \( = \) \( \text{(inside \( A_i \)) \( B_i/K_i \)} \cong B_i, \text{therefore also } \sigma_i^{-1} (1) = \sigma_i^{-1} (L_1 \cap L_2) = \sigma_i^{-1} (L_1) \cap \sigma_i^{-1} (L_2) = B_1 \cap B_2 = 1 \) } \), proving the injectivity of \( \sigma \); on the other hand, \( \sigma (B) = \text{Dr} \prod_{i=1}^n \sigma_i (B_i) = 
\text{Dr} \prod_{i=1}^n L_i \) and, \( \sigma \) being an isomorphism between \( G \) and \( U \), \( U/\text{Dr} \prod_{i=1}^n L_i = 
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σ(G)/σ(B) ≃ G/B ≃ R. For any set of indices \(i_1, i_2, \ldots, i_s\) from \(\{1, \ldots, n\}\),
\[
E_{i_1 \ldots i_2 \ldots i_s} = \sigma(K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_s}) = \sigma\left(\prod_{j \notin \{i_1, \ldots, i_s\}} B_j\right) = Dr \prod_{j \notin \{i_1, \ldots, i_s\}} \sigma(B_j) = Dr \prod_{j \notin \{i_1, \ldots, i_s\}} L_j;
\]
in particular,
\[
E_i = \sigma(K_i) = Dr \prod_{j \notin i} \sigma(B_j) = Dr \prod_{j \notin i} \sigma(B_j) = Dr \prod_{j \notin i} L_j,
\]
showing that \(U\) is a smashed subgroup of the direct product \(A\), hence our last theorem 19 applies; therefore, its core is just being \(I = L_1 \times L_2 \times \ldots \times L_n\), we have \(U/I = U/Dr \prod_{i=1}^n L_i \simeq R\) and then, according to theorem 19, also \(\pi_i(U)/L_i \simeq R\) - a fact at which we also can arrive directly, as \(\pi_i(U)/L_i = (G/K_i)/B_iK_i/K_i \simeq G/B \simeq R\).

Also from the same theorem, more generally \(R \simeq \pi_{i_1, i_2, \ldots, i_s}(U)/L_{i_1} \times \ldots \times L_{i_s}\), for any proper subset \(\{i_1, i_2, \ldots, i_s\} \subset \{1, \ldots, n\}\) (\(i_1, i_2, \ldots, i_s\)); if we assumed that \(\pi_{i_1, i_2, \ldots, i_s}(U) \leq U\), which would immediately also imply, due to the core \(I\)'s definition, that \(\pi_{i_1, i_2, \ldots, i_s}(U) = L_{i_1} \times \ldots \times L_{i_s}\), then, according to theorem 5, all three isomorphic factor groups given by it should be trivial, hence \(U = L_{i_1} \times \ldots \times L_{i_s}\), which in our case equals \(L_1 \times \ldots \times L_n\) - which, through the isomorphism \(\sigma\), would then yield that \(R \simeq G/B = 1\) and \(G = B = B_1 \times \ldots \times B_n\) (compare as well with lemma 12).

**Conclusion 37** Given a subgroup \(B = B_1 \times \ldots \times B_n\) of a group \(G\), \(n \geq 2\), so that every single direct factor \(B_i\) is normal in \(G\), we get a faithful representation \(\sigma\) of \(G\) as a smashed subdirect product \(U\) of the direct product \(A = Dr \prod_{i=1}^n G/K_i\),
where \(K_i = Dr \prod_{j \neq i} B_j\).

**Example 38** As a case of particular interest for our (quite general) example, we may for example use as \(B\) one of the normal subgroups (for example, the maximal of them, by starting off with all minimal normal subgroups, at least for \(G\) finite) of \(G\) given by the following theorem of R. Remak (as well):

**Theorem 39** (R. Remak) \[9\] Let \(B_1, \ldots, B_m (m > 0)\) be minimal normal subgroups of a group \(G\) and set \(B = \prod_{i=1}^m B_i\). Then there exists a subset \(\{i_1, i_2, \ldots, i_n\} \subset \{1, \ldots, m\}\), such that \(B = B_{i_1} \times \ldots \times B_{i_n}\).

**Problem 40** Conclusion \[37\] may also prompt us to the more general "inverse" problem, of investigating the ways to (faithfully) represent a given group as a subdirect product; of particular interest would be to get to non-smashed representations. We are looking at this problem in our last subsection 4.3 here.

Some orientation on this kind of problems in general Group Theory can be found in \[5\]; it has already been addressed to since 1930 in \[10\].
Theorem 41 Given $U \leq A = A_1 \times A_2 \times \ldots \times A_n$, let $I = L_{N_1} \times \ldots \times L_{N_m}$ be the (total) cohesive decomposition of its core $I$; denote, also, by $\pi^i, i = 1, \ldots, m$, the projection from the product $A$ to its subproduct attributed to the subset $N_i$ of $\{1, \ldots, n\}$. Set $R = U/I$; then all quotients $\pi^i(U)/L_{N_i}$, for $i = 1, \ldots, m$, are isomorphic to each other and to $R$ in a "structural" way, as in our previous theorems (see theorem 37). $U$ may be realized as a fiber product (pull-back) of the $\pi^i(U)$'s ("structurally coordinated") epimorphisms onto $R$; in other words, $U$ may be realized as a smashed subdirect product.

Also, for any sequence of indices $i_1i_2\ldots i_s$ from $\{1, \ldots, m\}$, we have structural isomorphisms $R \cong \pi^{i_1i_2\ldots i_s}(U) / (L_{N_{i_1}} \times \ldots \times L_{N_{i_s}})$, where $\pi^{i_1i_2\ldots i_s}$ denotes the projection from the product $A$ to its subproduct attributed to the subset $N_{i_1} \cup \ldots \cup N_{i_s}$ (a disjoint union) of $\{1, \ldots, n\}$. Thanks to the "structural" property of the above isomorphisms to $R$, the statement is also here true, meaning that we may restore $U$ from the following data: A partition $\{1, \ldots, n\} = \bigcup_{i=1}^m N_i$, subgroups $W_i \leq \bigcap_{j \in N_i} A_j$, a normal adhesive (inside $\bigcap_{j \in N_i} A_j$) subgroup $L_{N_i}$ of $W_i$, such that all of $W_i/L_{N_i}$ be isomorphic to $R := W_m/L_{N_m}$, together with a sequence $(\sigma_1, \ldots, \sigma_{m-1})$ of "structural isomorphisms" $\sigma_i : W_i/L_{N_i} \rightarrow R$, which shall determine the "adhesive fibres" of a subgroup $U$, having core $I = L_{N_1} \times \ldots \times L_{N_m}$ and such that $\pi^i(U) = W_i$.

Proof. Thanks to the commutativity amongst the factors $A_1, A_2, \ldots, A_n$ of the direct product $A$, we may rearrange them in an order that fits into the sequence $N_1, \ldots, N_m$ of our partition of $\{1, \ldots, n\}$ and renumber; then, by setting $B_\kappa = \bigcap_{j \in N_\kappa} A_j, \kappa = 1, \ldots, m$, we get $A = B_1 \times B_2 \times \ldots \times B_m$, indeed a smashed product, whereupon we now may apply our theorem 11 and get exactly what we are looking for. ■

This last theorem 41 is a generalization of theorem 31; this may also be fruitfully combined with theorem 37.

Notice that the cohesive decomposition of the core involved in the theorem may also be chosen to be an arbitrary one, i.e. not necessarily the total but a coarser one.

Now we may also give the full generalization of lemmata 2, 19 and 35 which again follows from the totally smashed case of this last lemma 35.

Lemma 42 Same situation as in theorem 41; then for any sequence of indices $i_1i_2\ldots i_s$ from $\{1, \ldots, m\}$, $\pi^{i_1i_2\ldots i_s}(U) \cong U/\bigcap_{i \in \{i_1, i_2, \ldots, i_s\}} L_{N_i}$. In particular $\pi^i(U) \cong U/L_1 \times \ldots \times L_1 \times L_2 \times \ldots \times L_n$. 22
Remark 43  Theorem 44 allows us to make diagrammatic depictions here, analogous to that of remark 15. In this case we may depict the subgroup \( U \) diagrammatically as \( U^m = W_i \) with \( m \) edges \( W_i, i = 1, ..., m \), with "socles" \( L_{N_i} \) and heads isomorphic to \( R \), where the "structural isomorphisms" of the heads to \( R \) shall be needed to restore the "adhesive fibres" in \( Dr \prod_{i=1}^m W_i \), which define \( U \) set-theoretically. Its group structure is then obtained from that of \( Dr \prod_{i=1}^m W_i \).

4.2 Subdirect \( AutR \)-classes

In what follows in this subsection we start off with our last theorem 44; however we could equally well apply this theory in the situation theorem 17, as already mentioned in our remark 19; on the other hand theorem 1 is a special case of the a cohesive decomposition as in theorem 44.

Let us so just adopt the notation of theorem 44.

We shall denote by \( \mathfrak{P}^m (AutR) \) the set of equivalence classes in \( (AutR)^m \) under the following relation: \( (\sigma_1, ..., \sigma_m) \sim (\tau_1, ..., \tau_m) \) iff there exists \( \rho \in AutR: (\sigma_1 \rho, ..., \sigma_m \rho) = (\tau_1, ..., \tau_m) \). We may also define multiplication in \( (AutR)^m \) in the obvious way, and it is then equally apparent that left multiplication by another element preserves equivalence of two elements.

We need also to identify all \( m \) factor groups \( \pi^i (U) / L_{N_i} \), with \( R \) through some fixed isomorphisms: we allow ourselves further a notational convention, that we may w.r.t. the canonical epimorphisms \( \xi_i: \pi^i (U) \twoheadrightarrow \pi^i (U) / L_{N_i} \cong R \) identify the preimage \( \xi_i^{-1}(r) \) of some \( r \in R \) as \( r L_{N_i} \), which may not cause any confusion, inasmuch as these coset representatives "\( r \)" in different \( \pi^i (U) \) shall never interact with one another. We shall then remember that, whenever considering the fixed representatives (transversals) "\( r \)" in different \( \pi^i (U) \), they do only make a group when considered modulo \( L_{N_i} \). This identification may be viewed as a step toward the idea of "virtuality"/"virtual category".

It is immediate to check through the universal property of a fiber product that the fiber products \( \{ (\xi_i , i = 1, ..., m) : R \} \) and \( \{ (\rho \circ \xi_i , i = 1, ..., m) : R \} \) for any \( \rho \in AutR \) are identical; on the other hand we may also look directly into these fiber products as subsets (and subgroups) of \( \prod_{i=1}^m \pi^i (U) \leq A = A_1 \times A_2 \times ... \times A_n \), by looking at their adhesive fibres:

We can immediately see that composing all the \( m \) canonical epimorphisms \( \xi_i : \pi^i (U) \twoheadrightarrow \pi^i (U) / L_{N_i} \) (and similarly any other set of epimorphisms \( \sigma_i \circ \xi_i : \pi^i (U) \twoheadrightarrow \pi^i (U) / L_{N_i} \)) for some isomorphisms \( \sigma_i \) of \( \pi^i (U) / L_{N_i} \cong R \) with a \( \rho \in AutR \) from the left and then taking the corresponding fiber-product for \( \{ (\rho \circ \xi_i , i = 1, ..., m) : R \} \) results in precisely the same subgroup, as the one gotten as the original pull-back \( U \) of \( \{ (\xi_i , i = 1, ..., m) : R \} \) over \( R \), inasmuch as it gives precisely the same adhesive fibres: The connective bundles that constitute the original fiber product \( U \) are precisely \( \{ (\xi_1^{-1}(r), ..., \xi_m^{-1}(r)) \}, r \in R \).
which by the notational convention mentioned above may also be described
as \( \{(rL_{N_1}, ..., rL_{N_m}) \mid r \in R\} \), wherein we are also recalling our standard
notation around \( U \) (see theorem [11]). By letting \( \mathbf{\pi}(\rho, ..., \rho) \in (\text{Aut}R)^m \) act on
the \( m \) canonical epimorphisms \( \xi_i : \pi^i(U) \to \pi^i(U) / L_{N_i} \) through composition
as above, we now get the new fibre product as a subgroup of \( \prod_{i=1}^m \pi^i(U) \), consisting
of the adhesive fibres
\[
\{(\rho \circ \xi_i)^{-1}(r), ..., (\rho \circ \xi_m)^{-1}(r) \mid r \in R\} = \\
= \{(\xi_i^{-1}(r), ..., \xi_m^{-1}(r)) \mid r \in R\} = \\
= \{(\xi_1^{-1}(r^\pi_1), ..., \xi_m^{-1}(r^\pi_m)) \mid r \in R\}
\]
which by substituting \( r \to r^\pi \) is rewritten as
\[
\{(\xi_1^{-1}(r), ..., \xi_m^{-1}(r)) \mid r \in R\} = \{(rL_{N_1}, ..., rL_{N_m}) \mid r \in R\}
\]
We may also more generally let any \( \mathbf{\pi} = (\sigma_i, i = 1, ..., m) \in (\text{Aut}R)^m \) act on
\( \prod_{i=1}^m \pi^i(U) / L_{N_i} \), thus yielding a new subdirect product in \( \prod_{i=1}^m \pi^i(U) \), out of the
original one \( U \), obtained as the fiber product \( U^\mathbf{\pi} : \{(\sigma_i \circ \xi_i, i = 1, ..., m) \mid R\} \) over
\( \text{R} \). The resulting \( U^\mathbf{\pi} \) consists of the adhesive fibres
\[
\{(\sigma_1 \circ \xi_1)^{-1}(r), ..., (\sigma_m \circ \xi_m)^{-1}(r) \mid r \in R\} = \\
= \{(\xi_1^{-1}(r^\pi_1), ..., \xi_m^{-1}(r^\pi_m)) \mid r \in R\} = \left\{ (r^\pi_1 L_{N_1}, ..., r^\pi_m L_{N_m}) \mid r \in R \right\};
\]
had we acted with its \"-equivalent \)-tuple \( \mathbf{\pi} = (\tau_1, ..., \tau_m) = (\sigma_1 \rho, ..., \sigma_m \rho) \) of
\( R \)-automorphisms above, we would again get the exactly same set of adhesive
fibres, now described as
\[
\{(r^\pi_1 L_{N_1}, ..., r^\pi_m L_{N_m}) \mid r \in R\} = \\
= \left\{ (r^\pi_1 \sigma_1^{-1} L_{N_1}, ..., r^\pi_m \sigma_m^{-1} L_{N_m}) \mid r \in R \right\}
\]
which, by substituting \( r \to r^\rho \), is rewritten as
\[
\{(r^\pi_1 L_{N_1}, ..., r^\pi_m L_{N_m}) \mid r \in R\};
\]
thus we have obtained an equality between the subgroups \( U^\mathbf{\pi} \) and \( U^\mathbf{\rho} \). Conversely, if we had \( U^\mathbf{\pi} = U^\mathbf{\rho} \) for
some \( \mathbf{\pi} = (\sigma_i, i = 1, ..., m), \mathbf{\rho} = (\tau_1, ..., \tau_m) \in (\text{Aut}R)^m \), then their corresponding
sets of adhesive fibres must be identical, i.e.
\[
\{(r^\pi_1 L_{N_1}, ..., r^\pi_m L_{N_m}) \mid r \in R\} = \left\{ (r^\pi_1 L_{N_1}, ..., r^\pi_m L_{N_m}) \mid r \in R \right\}.
\]
from which we conclude the existence of a unique bijection \( \rho : R \to R \) such that
\[
\{(r^\rho_1 L_{N_1}, ..., r^\rho_m L_{N_m}) \mid r \in R\} = \left\{ (r^\rho_1 \sigma_1^{-1} L_{N_1}, ..., r^\rho_m \sigma_m^{-1} L_{N_m}) \mid r \in R \right\},
\]
where this last expression is the fibre product of \( \{(\sigma_i \circ \rho \circ \xi_i, i = 1, ..., m) \mid R\} \)
over \( \text{R} \), while the first is the one of \( \{(\tau_i \circ \xi_i, i = 1, ..., m) \mid R\} \) over \( \text{R} \), forcing
\( \sigma_i \circ \rho = \tau_i \). That \( \rho \) is homomorphic modulo \( I \) follows from the multiplication
rules of the adhesive fibres.

These findings amount to the following

**Proposition 44** In the above described way, two \( m \)-tuples \( \mathbf{\pi}, \mathbf{\sigma} \in (\text{Aut}R)^m \)
give rise to the same subgroup if and only if the \( m \)-tuples \( \mathbf{\pi}, \mathbf{\sigma} \) of automor-
phisms of \( R \) are equivalent under the introduced "right projective" equivalence
relation \( \sim \). The so determined action of such a \( \sigma \) on \( \prod_{i=1}^{m} \pi^i(U) / L_{N_i} \) may be realized by changing the coordinated structural isomorphisms of theorem 44 all the way through, a change effectuated by "twisting" every \( \pi^i(U) / L_{N_i} \) by \( \sigma_i^{-1} \).

By denoting the \((\text{Aut}R)^m\)-orbit of \( U \) as \( \mathcal{P}(U) \), we do so finally get an induced faithful and transitive action of \( \mathcal{P}^{m-1}(\text{Aut}R) \) on \( \mathcal{P}(U) \), where by \( \mathcal{P}^{m-1}(\text{Aut}R) \) we mean \( (\text{Aut}R)^m / \sim \).

We point out that the twistings above establish the new "alignments", that determine the new adhesive fibres that constitute \( U^\sigma \) out of those of the original \( U \).

**Proposition 45** Let \( \sigma = (\sigma_1, \ldots, \sigma_m) \in (\text{Aut}R) \), where \( \sigma_1 = \text{id}_R \), and let \( \Sigma = \langle \sigma_2, \ldots, \sigma_m \rangle \); then \( U^\sigma \cap U \) consists of the \( R^{\Sigma} \)-adhesive fibres in \( U \), where \( R^{\Sigma} \) is the subgroup of \( R \) consisting of the \( \Sigma \)-fixed points on it - that is \( U^\sigma \cap U = \{ (rL_{N_1}, \ldots, rL_{N_m}) : r \in R^{\Sigma} \} \), by using our notation explained above. In particular, all groups in the \( \mathcal{P}^{m-1}(\text{Aut}R) \)-orbit \( \mathcal{P}(U) \) of \( U \) contain the core \( I = L_{N_1} \times \ldots \times L_{N_m} \) of the original \( U \).

**Proof.** It follows from our discussion above, by comparing the adhesive fibres of \( U^\sigma \) and \( U \).

It is clear that every equivalence class in \((\text{Aut}R)^m\) contains a representative of the form of \( \sigma \) in the proposition above, i.e. with the first component equal to \( \text{id}_R \). We shall call it its \((1-)\text{canonical representative}\); further we shall call the subgroup \( \Sigma \) of \( \text{Aut}R \) generated by all components of the canonical representative its breadth group. We could have defined corresponding breadth groups by demanding the \( i \)-th automorphism to be trivial instead; it is nonetheless immediate to check that any of these choices results in the same breadth group.

We want next to examine, whether we can determine **conditions to ensure the existence of a homomorphism** \( \alpha \) of \( U \), coinduced by \( \sigma = (\sigma_1, \ldots, \sigma_m) \); i.e. which acts trivially on the core \( I \) of \( U \) and induces \( \sigma_i \) on \( \pi^i(U) / L_{N_i} \); such one would probably establish a very convenient isomorphism between \( U \) and \( U^\sigma \).

Due to the original direct product, this issue boils down to the corresponding question for every \( \pi^i(U) \) (except that we are now looking for automorphisms \( \alpha : \pi^i(U) \to \pi^i(U) \)).

This would in general seem too good to be true: In order to come closer to some sufficient conditions for such a cute set-up, let us further **assume that \( U \) splits over \( I \), i.e. that \( U \cong R \ltimes I \), which again is equivalent to \( \pi^i(U) = R_i \ltimes L_{N_i} \), for \( i = 1, \ldots, m \), where \( R_i \cong R \).**

Let us therefore define such a map \( \alpha \) on \( U \), by determining its \( N_i \)-coordinates through \( \pi^i(\alpha(r)) = \sigma_i(r) \) or, with exponential notation, \( r^{\sigma_i} \), for \( r \in R_i \), \( \pi^i(\alpha(l)) = l \) for \( l \in L_{N_i} \). This is clearly a bijective map; we are now going to find conditions for it to be homomorphic:
Let $x, x' \in U$, with $\pi^i (x) = r_i l_i$, $\pi^i (x') = r' \ell'_i$, where $l_i, \ell'_i \in L_{N_i}, r_i, r'_i \in R_i$. On the one side we have that $\pi^i (\alpha (x x')) = \alpha (\pi^i (x x')) = \alpha (r_i l_i r'_i \ell'_i) = \alpha (r_i l_i r'_i \ell'_i) = \alpha (r_i r'_i \ell'_i l'_i) = \alpha (r_i r'_i u_i l'_i) = \alpha (r_i r'_i u_i l'_i)$ (1), on the other is $\pi^i (\alpha (x) \alpha (x')) = \pi^i (\alpha (x)) \pi^i (\alpha (x')) = \alpha (r_i l_i) \alpha (r'_i \ell'_i) = \alpha (r_i r'_i u_i l'_i) = \alpha (r_i r'_i u_i l'_i)$ (2). Then for $\alpha$ to be a homomorphism one should have $\alpha (x x') = \alpha (x) \alpha (x')$, or equivalently that $\pi^i (\alpha (x x')) = \pi^i (\alpha (x)) \pi^i (\alpha (x'))$, which by (1) & (2) means $\ell'_i = \ell'_i$, that is, $r_i^{-1} r'_i$ centralizes $l_i \forall l_i \in L_{N_i}, \forall r_i \in R_i$ - i.e. that every element of the form $r_i^{-1} r'_i$ in each $R_i$ centralizes $L_{N_i}, i = 1, ..., m$. Assume now further that $\varphi \in (Aut R)^m$ is the 1-canonical representative of its class in $\mathcal{P}^{m-1} (Aut R)$. Notice that, with such an 1-canonical $\varphi \in (Aut R)^m$, the condition we have found is automatically trivially satisfied by $R_1$.

**Proposition 46 a.** If $U$ splits over its core $I$, then the necessary and sufficient condition for the existence of an isomorphism $\alpha$ from $U$ to $U^\varphi$ (with $\varphi$ 1-canonical) acting trivially on the core $I$ of $U$ and inducing $\sigma_i$ on $R_i$, $i = 1, ..., m$ is that every element of the form $r_i^{-1} r'_i$ in each $R_i$ centralizes $L_{N_i}, i = 2, ..., m$.

b. Assuming further that, for every $i = 2, ..., m$, $\sigma_i$ is a fixed-point free automorphism of $R_i$ and that every $R_i$ is either finite or abelian and Artinian (as a $\mathbb{Z}$-module), the condition above is equivalent to the statement that $R_i$ is contained in the center $\mathcal{Z} (\pi^i (U))$ of $\pi^i (U)$, hence that $\pi^i (U) = L_{N_i} \times R_i, i = 2, ..., m$.

In particular, that latter is the case if already $R$ itself centralizes $I$, i.e. if $U \cong R \times I$, which is equivalent to $\pi^1 (U) = L_{N_i} \times R_i, i = 1, ..., m$.

**Proof.** It is now sufficient to prove part (b).

Assuming that the automorphism $\sigma_i$ is fixed-point free, the map (not a homomorphism, in general) $\phi_i$ sending $x \in R_i$ to $x^{-1} x^\sigma_i \in R_i$ is injective: for $x^{-1} x^\sigma_i = y^{-1} y^\sigma_i \Rightarrow y x^{-1} = (y x^{-1})^\sigma_i$, whence the assumption on $\sigma_i$ gives $y = x$.

If $R_i$ is finite, then $\phi_i$ is clearly bijective.

Let us now suppose that $R_i$ is abelian and Artinian (as a $\mathbb{Z}$-module).

Then $\phi_i$ is suddenly homomorphic, actually a monomorphism. Then the Artinian property (DCC) forces the monomorphism $\phi_i$ to be surjective, hence an automorphism: Suppose that $\phi_i$ is not surjective. So, there exists some $0 \neq y \in R_i$ that does not belong to $\text{Im} \phi_i$, therefore $\phi_i (y) \notin \text{Im} \phi_i^2$. On the other hand $\phi_i (y)$, obviously belonging to $\text{Im} \phi_i$, cannot be $0$, due to $\phi_i$'s injectivity; that proves that the obvious inclusion $\text{Im} \phi_i \supset \text{Im} \phi_i^2$ is strict; by a similar argument the strictness of inclusion continues inductively in the infinite tower $\text{Im} \phi_i \supset \text{Im} \phi_i^2 \supset \text{Im} \phi_i^3 \supset \text{Im} \phi_i^4 \supset ...$, which contradicts the Artinian DCC. Therefore is $\phi_i$ an automorphism.

That means that in both cases of (b) every element of $R_i$ may be written in the form $x^{-1} x^\sigma_i$, for some $x \in R_i$; therefore, the condition for the existence of a $\mathcal{P}$-coinduced homomorphism $\alpha$ of $U$ becomes that every element of $R_i$ centralize $L_{N_i}$, according to (a), i.e. that $R_i$ centralize $L_{N_i}$. 


Notice that, in case the condition of (b) on \( R_i, i = 1, ..., m, \) is satisfied for all but for \( i = \kappa, \) then one should prefer the \( \kappa \)-canonical representative of the class of \( \varphi \) in \( \mathbb{P}^m - 1 \) (Aut\( R \)), afterwards examine if all \( \sigma_i \) on \( R_i \) for \( i \neq \kappa \) are fixed-point free.

**Remark 47** Here is a situation, where the above proposition is applicable, possibly (depending on \( \varphi \)) its part (b) too: Let \( U \) above be of finite order \( \kappa \lambda, \) \( (\kappa, \lambda) = 1, |R| = \kappa, |I| = \lambda, \) with \( I \) abelian; in that case it is known (for example. [2, IV 3.13, Remark]) that the sequence \( I \hookrightarrow U \twoheadrightarrow R \) splits - and, consequently (or just by the same arguments, as \( |L_N| \) is a divisor of \( |I| \)), \( \pi^i(U) \) splits over \( L_N, \) (i.e., \( L_N \hookrightarrow \pi^i(U) \twoheadrightarrow R_i \) splits).

**Lemma 48** Define \( \phi_i : R_i \rightarrow R_i \) as the map sending \( x \) to \( x^{-1}x^{\sigma_i}; \) by assuming \( R_i \) to be abelian, \( \phi_i \) becomes a group homomorphism. Then the restriction of \( \sigma_i \) to the image \( \phi_i(R_i) = \text{Im} \phi_i \) is fixed-point free.

**Proof.** Observe that \( \ker \phi_i = R_i^{(\sigma_i)} \), where \( R_i^{(\sigma_i)} \) is the subgroup of \( \sigma_i \)-fixed points, hence we get the natural isomorphism \( \text{Im} \phi_i \cong R_i/R_i^{(\sigma_i)} \) so that the \( \sigma_i \)-action on \( \text{Im} \phi_i \) be equivalent to the one induced on \( R_i/R_i^{(\sigma_i)} \).

**Example 49** Let us just take a simple example, just to assist visualization:

Consider the epimorphisms \( \xi_1 : \mathbb{Z}_{15} \ni x \mapsto x \mod 3 \in \mathbb{Z}_3 \) and \( \xi_2 : \mathbb{Z}_{21} \ni y \mapsto y \mod 3 \in \mathbb{Z}_3 \); their fiber product over \( \mathbb{Z}_3 \) is then the subgroup \( U = \{ (x, y) \in \mathbb{Z}_{15} \times \mathbb{Z}_{21} : x \mod 3 = y \mod 3 \} \), clearly a subdirect product of \( \mathbb{Z}_{15} \times \mathbb{Z}_{21} \). Let \( \text{Aut}\mathbb{Z}_3 = \{ \sigma \}, \sigma^2 = 1; \) let us determine the subgroup \( U^\sigma, \) with \( \sigma = (1, \sigma) \sim (\sigma^{-1}, 1) \) which, according to proposition 44, is effectuated through ”twisting” by \( (\sigma, 1) \): That means, \( U^\sigma = \{ (x, y) \in \mathbb{Z}_{15} \times \mathbb{Z}_{21} : (x \mod 3)^\sigma = y \mod 3 \}. \)

Notice that both \( U \) and \( U^\sigma \) convey similar diagrammatic depictions as \( \mathbb{Z}_5 \twoheadrightarrow \mathbb{Z}_3 \triangleleft \mathbb{Z}_7 \).

### 4.3 Subdirect \( \mathfrak{C} \)-(in)decomposability

In the spirit of remark 20 we shall rather be speaking of subdirect sums than products; of course in the case of a finite number of summands (in which we mostly use the term ”product” here) meaning of the two terms is identical.

One might ask about the decomposability of any arbitrary group as a subdirect product and what does a particular kind of decomposition mean in terms of the structure of the group. To meet this kind of questions, also by getting inspiration from our example above, we come to the propositions below.

Before proceeding to see them, we wish to generalize the notion of a subdirect group product, by allowing its definition up to isomorphism and without the restriction about the finite number of direct factors:
**Definition 50** U shall be called a (generalized) subdirect sum of the family \( \{ A_j : j \in J \} \) if there exists a monomorphism \( \mu : U \to \prod_{j \in J} A_j \), such that all composites \( \pi_j \circ \mu \) be epimorphisms, where \( \pi_j : \prod_{j \in J} A_j \to A_j \) are the canonical projections.

**Proposition 51** U is a subdirect sum of the family \( \{ A_j : j \in J \} \) if and only if there exists a family \( \{ E_j : j \in J \} \) of normal subgroups of U, so that \( U/E_j \cong A_j \), and \( \bigcap_{j \in J} E_j = 1 \).

**Proof.** If the family of normal subgroups is given, in order to define \( \mu : U \to \prod_{j \in J} U/E_j \) it suffices to define all \( \pi_j \circ \mu : U \to U/E_j \); we simply define them as the canonical maps. It is clear that \( \ker \mu = \bigcap_{j \in J} E_j = 1 \), therefore \( \mu \) is monomorphic.

Conversely, given a subdirect sum \( U \) as in the definition, let \( E_i \ (i \in J) \) be defined the way we have done it earlier, i.e. \( E_i = \ker (\pi_i \circ \mu) \); but since \( \pi_i \circ \mu : U \to A_i \) has been assumed to be epi-, we get readily \( U/E_i \cong A_i \), as wished. On the other hand the kernel of the monomorphism \( \mu \), being trivial, is also equal to \( \bigcap_{j \in J} \ker (\pi_j \circ \mu) = \bigcap_{j \in J} E_j \), and we are done. 

**Corollary 52** A group cannot be (isomorphically and non-trivially) written as a subdirect sum if and only if the intersection of all its non-trivial normal subgroups is non-trivial. Such groups may be called subdirectly indecomposable.

There is much more that can in a similar manner be derived from the last proposition; to state them in generality, let \( \mathfrak{E} \) be a property referring to factor groups by normal subgroups of a given group \( G \). We might also refer to \( \mathfrak{E} \) as a class of groups, and then consider the normal subgroups of \( G \), such that the corresponding factor group belong to the class \( \mathfrak{E} \). We shall call the intersection of all such normal subgroups the \( \mathfrak{E} \)-residue of \( G \). The factor group corresponding to the \( \mathfrak{E} \)-residue the shall then be called the \( \mathfrak{E} \)-residual of \( G \). It becomes then immediate to see the following

**Proposition 53** The necessary and sufficient condition for a group \( G \) to be expressible as a subdirect sum of groups belonging to the class \( \mathfrak{E} \) is that the \( \mathfrak{E} \)-residue of \( G \) is trivial - while there are at least two non-trivial, proper normal subgroups of \( G \), such that the corresponding factor group belongs to \( \mathfrak{E} \).

We mention some examples in the next
Corollary 54 If $G$ is a torsion group, $\varpi_1, \varpi_2, \ldots, \varpi_s$ pairwise distinct sets of primes, then $G$ can be expressed as a subdirect product of respectively $\varpi_1$, $\varpi_2, \ldots, \varpi_s$-groups if and only if $\bigcap_{i=1}^{s} \mathcal{D}^{\varpi_i} = 1$, where $G/\mathcal{D}^{\varpi}$ is the $\varpi$-residual of $G$ (for definition see for example, [12, 3.44]), $\mathcal{D}^{\varpi}$ the "$\varpi$-residue".

Notice that, unless $G$ is a $\varpi'_\kappa$-group for all $\kappa \in \{1, \ldots, s\}$, the condition $\bigcap_{i=1}^{s} \mathcal{D}^{\varpi'_i} = 1$ also forces that at least one of all those $\varpi'_\kappa$-residues $\mathcal{D}^{\varpi'_\kappa}$ is less than $G$. The result may also be extended to the case of an infinite collection of $\varpi'_\kappa$'s:

Corollary 55 If $G$ is a torsion group and $\{\varpi_n/n \in \mathbb{N}\}$ a collection of pairwise distinct sets of primes, then $G$ can be expressed as a subdirect sum of $\bigcap_{n \in \mathbb{N}} G/\mathcal{D}^{\varpi_n}$. (To secure non-triviality at all places we may demand that $\mathcal{D}^{\varpi_n} < G$ for all $n \in \mathbb{N}$).

Another interesting special case is given by the complete reducible residue $CR(G)$ of a group $G$, defined as the intersection of all normal subgroups of $G$, such that the corresponding factor groups be simple. Then simplicity yields the following:

Corollary 56 $CR(G)$ is the unique normal subgroup of $G$, such that the factor group is completely reducible (i.e. isomorphic to the direct sum of simple groups).

We remind that a group $G$ is called quasisimple if it is perfect ($[G, G] = G$) and $G/Z(G)$ is simple. The quasisimple residual $QCR(G)$ of a group $G$ is defined as the intersection of all kernels of epimorphisms of $G$ onto quasisimple groups (see f.ex. [5, intr.], where also the last corollary is stated as well).

Corollary 57 A group $G$ is expressible as a subdirect sum of quasisimple groups if and only if its quasisimple residual $QCR(G)$ is trivial. In that case it follows that it actually becomes a direct or central product.

Corollary 58 The quasisimple residue $QCR(G)$ of a group $G$ is its unique smallest normal subgroup such that the corresponding factor group is expressible as a subdirect sum of quasisimple groups.

Proposition 59 Every group $G$ may be written as a subdirect product of groups that are either simple or subdirectly indecomposable.
**Proof.** Let us assign to each $x \neq 1$ in $G$ a normal subgroup $K_x$, maximal among the normal subgroups not containing $x$; obviously $\bigcap_{x \neq 1} K_x = 1$. By invoking the elementary fact of the 1–1 correspondence between the lattices of normal subgroups of $G$ containing $K_x$, and of normal subgroups of $G/K_x$ (see for example [12 3.29]), we see immediately that any normal subgroup of $G/K_x$ has to contain the (non-trivial) canonical image of $x$ in $G/K_x$, otherwise the 1–1 correspondence would yield a normal subgroup of $G$, properly containing $K_x$, contradicting the maximality of $K_x$. In case there is no normal subgroup of $G/K_x$ either, containing the canonical image of $x$ in $G/K_x$, this group is obviously simple, while otherwise is $G/K_x$ subdirectly indecomposable (corollary 52). Proposition 51 now gives the result. ■

**Remark 60** Of course this proposition does not tell us anything about how interesting the guaranteed subdirect decomposition might be. For example in the case of a simple $G$ the proof of the proposition actually gives us a subdirect representation of $G$ as the diagonal in the product $\prod_{x \in G} G_x$, where each $G_x = G$.

That is, in the case of a simple $G$ (and only in this!), the procedure in the proof of proposition 59 yields a "deltoid subdirect product" (compare our definition 10), in the sense that its whole core $I$ is trivial. That is of course uninteresting, it makes therefore sense to substitute $H$ for any diagonal $\Delta H$ like in the case of a simple $G$ in the outcome of the procedure, in the spirit of condition 24, still to get a subdirect decomposition of the form guaranteed by proposition 59 (except only for the case that the given group was simple) but a more interesting one. We notice also that such a decomposition is not in general uniquely determined, as the choice of a maximal $K_x$ is not so either.

I believe that this new view/realization of our subject may provide a key to progress on other questions, even in better understanding some already known results and thereby also enhancing their deepening or implementation; as a possible such reviewing might be thought the issue of the lattice of such subgroups, also of the normal subgroups; on this subject it should anyway be expedient to revisit, among many others of course, [8], [14], [15].

5 Subdirect presentations & applications to homomorphisms

It is in itself interesting to look at subdirect products from another point of view (and compare), but we may furthermore gain important new insight and basic results about homomorphisms/endomorphisms on the way, considerably extending classical/elementary ones; results which hitherto have (amazingly)
remained hidden, while we come across them very naturally and unrestrained by the present approach. Also our approach here, for which we already have been predisposed by example 24, remains general - but it would be interesting and very fruitful, I believe, to apply our results and techniques in more specific contexts or in concrete situations.

5.1 The case of two factors

Let \( f_i : A \twoheadrightarrow G_i, \ i = 1, 2, \) be non-trivial group epimorphisms (we may always get to epimorphisms, by substituting the target group with the image of the homomorphism). Let also \( \Delta : A \ni a \mapsto (a, a) \in A \times A \) be the diagonal monomorphism, \( F : A \times A \ni (a_1, a_2) \mapsto (f_1(a_1), f_2(a_2)) \in G_1 \times G_2 \) and define \( u := F \circ \Delta, U := u(A) \leq G_1 \times G_2. \)

We shall subsequently be using all our previous terminology and symbols about \( U \leq G_1 \times G_2. \) Apparently, \( \ker(F) = \ker(f_1) \times \ker(f_2) \leq A \times A, \) therefore is \( \ker(u) = \ker(f_1) \cap \ker(f_2) \) (1). Of course, the assumed surjectivity of the \( f_i \) ‘s means that the chosen \( U \) indeed is a subdirect product. Due to (1), the condition \( \bigcap_{i=1}^2 \ker(f_i) = 1 \) is equivalent to \( u \) being injective; in this case, we may describe \( U \) more explicitly (set-theoretically), as \( U = \{(f_1(a), f_2(a)) / a \in A\}. \)

As \( L_1 = E_2 = U \cap G_1 = f_1(\ker(f_2)) \) and \( L_2 = E_1 = U \cap G_2 = f_2(\ker(f_1)). \)

By taking \( G_2 = A \) and \( f_2 = id_A, \) we get

**Corollary 62** Given an epimorphism \( f_1 : A \twoheadrightarrow G_1 \) of groups, we have \( G_1 \simeq A/\ker(f_1) \)

which, of course, is the elementary first homomorphism theorem: at this point, it is essential to notice that our proof of theorem \[\text{[4]}\] on which proposition \[\text{[6]}\] depends, does not apply this homomorphism theorem!

In this view, the first isomorphism in Prop. \[\text{[6]}\] above is seen to be a generalization of the first homomorphism theorem; as such one, we reformulate it here:

**Corollary 63** Given group homomorphisms \( f_i : A \rightarrow G_i, \ i = 1, 2, \) we have \( \text{Im}(f_1)/f_1(\ker(f_2)) \simeq \text{Im}(f_2)/f_2(\ker(f_1)). \)

Of particular interest is to specialize to the case, in which we have endomorphisms instead of homomorphisms, when we shall drop surjectivity, so as to have the same target group \( A \) in all cases; it is often convenient to take isomorphic copies \( A_i \) of \( A, \) through fixed isomorphisms: we may subsequently do it at convenience, even without special notification.
Corollary 64 Given $\rho_i \in \text{End}(A)$, $i = 1, 2$, it holds that $\text{Im}(\rho_1) / \rho_1(\ker(\rho_2)) \simeq \text{Im}(\rho_2) / \rho_2(\ker(\rho_1)) \simeq U / \rho_1(\ker(\rho_2)) \times \rho_2(\ker(\rho_1))$, where $U = u(A)$, as above.

5.2 The case of $n$ ($\geq 2$) factors

The general outset is just a generalization of the case $n = 2$; thus, let $f_i : A \rightarrow G_i, \ i = 1, \ldots, n$, be non-trivial group epimorphisms (we can always reduce to the case of epimorphisms, by taking the images as domains of the $f_i$'s). Let also $\Delta : A \ni a \mapsto (a, \ldots, a) \in A^n$ be the diagonal monomorphism, $F : A^n \ni (a_1, \ldots, a_n) \mapsto (f_1(a_1), \ldots, f_n(a_n)) \in \text{Dr} \prod_{i=1}^n G_i$ and define $u := F \circ \Delta$, $U := u(A) \leq \text{Dr} \prod_{i=1}^n G_i$. As before, $\ker(F) = \text{Dr} \prod_{i=1}^n \ker(f_i) \leq A^n$, whence $\ker(u) = \bigcap_{i=1}^n \ker(f_i)$, so that the assumption of its triviality, i.e. that $\bigcap_{i=1}^n \ker(f_i) = 1$, amount to $u$'s injectivity, in which case we may describe $U$ set-theoretically as, $U = \{(f_1(a), \ldots, f_n(a)) / a \in A\}$.

Our previous terminology shall apply to our $U$ here too.

$u$ may of course be viewed as a representation of the group $A$ as subdirect product; by taking our outview from a subdirect product, however, one might be looking for a suitable $u$, i.e. an $A$ with the right homomorphisms, to get to such a "presentation" of a given $U$.

Definition 65 For a subdirect product $U$ of $\text{Dr} \prod_{i=1}^n G_i$, we shall be calling a homomorphism $u = F \circ \Delta : A \rightarrow \text{Dr} \prod_{i=1}^n G_i$ as above, such that $u(A) = U$, a "presentation of $U$ by homomorphisms"; we shall also denote this subdirect product $U$ by $[A; \{f_1, \ldots, f_n\}]$. It shall be called "terse", if it is injective, i.e. if $\bigcap_{i=1}^n \ker(f_i) = 1$.

Let $i_1, i_2, \ldots, i_s$ sequence of indices from $\{1, \ldots, n\}$; set $\Lambda = \{i_1, i_2, \ldots, i_s\}$ and write $\{1, \ldots, n\} = \Lambda \cup \hat{\Lambda}$, a disjoint union.

Set $K_i = \ker(f_i)$ and, for any subset $\Lambda$ of indices as above, $K_\Lambda = K_{i_1i_2\ldots i_s} = K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_s}$. Set, furthermore, $\xi_\Lambda = \pi_\Lambda \circ u$. Clearly, $\xi_\Lambda(A) = \pi_\Lambda(U)$.

We see immediately that

Lemma 66 (a) $\ker(\xi_\Lambda) = K_\Lambda$, (b) $L_\Lambda(U) = u(K_\Lambda) = E_\Lambda(U)$, and (c) The core $I$ of $U$ is, $I = \langle E_i / i = 1, \ldots, n \rangle = \langle u(K_i) / i = 1, \ldots, n \rangle = u(\langle K_i / i = 1, \ldots, n \rangle = u\left(\prod_{i=1}^n K_i\right)$.
Lemma 67 Given a presentation by homomorphisms \( u \) of the subdirect product \( U \) as above, we can always get to a terse presentation of \( U \) as a subdirect product of the same direct product.

Proof. Since \( K_{12...n} = \bigcap_{i=1}^{n} \ker (f_i) \) is contained in the kernel of any \( f_i \), all \( f_i \)'s factor through \( \overline{A} = A/K_{12...n} \), giving rise to \( \overline{f_i} : \overline{A} \to G_i, i = 1, ..., n \) and, thus, a terse presentation. ■

It is immediate to verify the following remarkable lemma:

Lemma 68 Given a “tersely presented by homomorphisms” smashed subdirect product \( U = [A; (f_1, ..., f_n)] \), we can readily get a usual definition of \( U \) as a pull-back out of it. Conversely, given a definition of a subdirect product \( U \) as a pull-back, we get to a terse presentation of it as \( U = [U; (p_1, ..., p_n)] \), where \( p_i \) is the \( i \)'th projection from the direct product, which may be considered as trivial in the sense that the homomorphism \( u = (p_1, ..., p_n) \circ \Delta \) is the identity map on \( U \).

Proof. We restrict ourselves to show it here for \( n = 2 \) (in which case \( U \) is always smashed), as the technic is the same for any \( n \); the generalization for an arbitrary \( n > 2 \) is obtained by use of theorem 31.

For the first part, we set \( G_1/f_1(\ker (f_2)) \simeq G_2/f_2(\ker (f_1)) \simeq R; \) for the pull-back, we set off from the epimorphisms \( \tau_i : G_i \to R \), that have to be chosen so that together they induce the structural \( \sigma : G_1/f_1(\ker (f_2)) \to G_2/f_2(\ker (f_1)) \) (the structural correspondence of pair-fibres), for example by letting \( G_2/f_2(\ker (f_1)) := R \) and, denoting with \( \pi_i : G_i \to G_i/f_i(\ker (f_j)) \), \( i \neq j \), the canonical epimorphisms, take \( \tau_1 = \sigma \circ \pi_1 \) and \( \tau_2 = \pi_2; \)

\[
\begin{array}{ccc}
U & \rightarrow & G_1 \\
\downarrow & & \downarrow \tau_1 \\
G_2 & \tau_2 & R
\end{array}
\]

For the other direction, let \( U \) be given as the pull-back of the epimorphisms \( \tau_i : G_i \to R \), i.e. \( U = \{(g_1, g_2) \in G_1 \times G_2 / \tau_1 (g_1) = \tau_2 (g_2)\}; \) take \( U; (p_1, p_2) \), set \( K_i = \ker (f_i) \) (\( i=1,2 \)) and observe that \( \ker (p_1) = 1 \times L_2, \ker (p_2) = L_1 \times 1, p_i (\ker (p_j)) = L_i (i \neq j). \)

So, we see that homomorphic presentation of a subgroup of a direct product is not necessarily bound to its smashedness, as its definition as a pull-back does.

We may apply theorem 17 to our “homomorphically” presented” subdirect product \( U \), even without demanding our \( u \) to be injective (“terse”). Taking into account lemma 66(b), we get:

Proposition 69 With the notation above, \( \xi_A (A) / u (K_\Lambda) \cong \xi_A (A) / u (K_\Lambda) \cong u (A) / (u (K_\Lambda) \times u (K_\Lambda)) \).
For \( n=2 \), this amounts to proposition \[ 61 \] let us see what we get for \( n=3 \):

**Corollary 70** Given the non-trivial group epimorphisms \( f_i : A \to G_i, i = 1, 2, 3 \), we have the double isomorphism:
\[
G_1 / f_1(K_{23}) \cong \xi_{23}(A) / \xi_{23}(K_1) \cong u(A) / (f_1(K_{23}) \times \xi_{23}(K_1))
\] - and two more, symmetrically.

We recall now a previous note, from just above lemma \[ 25 \]
- In elementary terms, this condition means that \( L_M \times L_N = L_\Lambda \) iff, for any \( x \in L_\Lambda \), the element \( \pi_M(x) \) of \( Dr \ \prod_{i \in M} A_i \) also belongs to \( L_M \) - or, equivalently, for any \( x \in L_\Lambda \), \( \pi_N(x) \in L_N \).

By using this criterion, we get readily to the following

**Proposition 71** Assume that \( K_{12...n} = 1 \), i.e. that \( u \) is injective (: a terse presentation of \( U \)) and let \( \Lambda = \{i_1, i_2, ..., i_s\} = M \cup N \) \((M \cap N = \emptyset)\) be a partition of the subset \( \Lambda \) of \( \{1, ..., n\} \). The condition \( L_\Lambda = L_M \times L_N \) is in our case of such a "homomorphically" presented subdirect product \( U \) equivalent to \( \hat{K}_\Lambda = \hat{K}_M \times \hat{K}_N \) (an "internal" direct product), where \( \hat{M}, \hat{N} \) are the complements of \( M, N \) (respectively) inside the index set \( \{1, ..., n\} \).

**Proof.** By implementing the above mentioned criterion, if \( L_\Lambda = L_M \times L_N \) then, for any \( x \in K_{\hat{\Lambda}} \) (\( \Leftrightarrow u(x) \in L_\Lambda \)), \( \pi_M(u(x)) = \xi_M(x) \in L_M \), meaning that there is some \( x_M \in A \), with \( u(x_M) \in L_M \) (\( \Leftrightarrow f_i(x_M) = 1 \) for any \( i \in \hat{M} \equiv x_M \in K_{\hat{M}} \)), such that \( \pi_M(u(x_M)) = \pi_M(u(x)) \); correspondingly, we have that \( \pi_N(u(x)) = \xi_N(x) \in L_N \), meaning that there is some \( x_N \in K_{\hat{N}} \), such that \( \pi_N(u(x_N)) = \pi_N(u(x)) \).

We claim now that \( u(x_Mx_N) = u(x) \); since both parts do apparently belong to \( L_\Lambda \) and \( \Lambda = M \cup N \), it suffices to prove that both projections \( \pi_M \) and \( \pi_N \), when applied to them, give the same result.

However, \( \pi_M(u(x_Mx_N)) = \pi_M(u(x_M)) \pi_M(u(x_N)) = \pi_M(u(x_M)) = \pi_M(u(x)) \), as \( x_M \in K_{\hat{M}} \subset K_M = ker(\pi_M \circ u) \) and, similarly, \( \pi_N(u(x_Mx_N)) = \pi_N(u(x)) \); hence, as noticed, \( u(x_Mx_N) = u(x) \), which, by invoking to the injectivity of \( u \), implies that \( x = x_Mx_N \).

We have thus proven that \( K_{\hat{\Lambda}} \subset K_{\hat{M}}K_{\hat{N}} \); the other inclusion being apparent, this implies \( K_{\hat{\Lambda}} = K_{\hat{M}}K_{\hat{N}} \). But, as \( M \cap N = \emptyset \) implies \( \hat{M} \cup \hat{N} = \{1, ..., n\} \), we get that \( K_{\hat{M}} \cap K_{\hat{N}} = K_{1...n} = 1 \), as \( U \)'s presentation is "terse", implying that the product of those two normal subgroups of \( K_{\hat{\Lambda}} \) is, indeed, direct.

The converse implication becoming now quite apparent, our proposition has been proven.

**Corollary 72** For the tersely presented \( U \) above, the condition for it to be smashed (see def. 11) is, that the subgroup of \( A \), generated by the normal subgroups \( K_i, i = 1, ..., n \), is the (necessarily direct, due to terseness) product of all \( K_i \)'s - i.e., \( \prod_{i=1}^{n} K_i = \prod_{i=1}^{n} K_i \).
We may also specialize to endomorphisms instead of the above homomorphisms $f_i$ - in which case, of course, we shall have to abandon surjectivity for the defining homomorphisms, in the general case.

**Example 73** Let $U = [A; (f_1, \ldots, f_n)], f_i : A \to G_i, i = 1, \ldots, n \ (n \geq 2)$, where $A$ has a normal subgroup $B = B_1 \times \ldots \times B_n$, hence every single direct factor $B_i$ is normal in $A$. Let $G/B \simeq R$.

Set $K_i = Dr \prod_{j \neq i} B_j, \ker (f_i) = K_i, G = Dr \prod_{i=1}^n G_i$.

In the notation that we have introduced in this last section, $K_i = \bigcap_{j \neq i} K_j = B_i$ and, by the corollary above, $U$ is smashed.

Of course, this here is a re-visiting (and notational updating) of example 36.

**Example 74** We construct an example of a non-smashed subdirect product:

Let $U = [A; (f_1, \ldots, f_5)], f_i : A \to G_i, i = 1, \ldots, 5$, where $A$ has a normal subgroup $B = B_{12} \times B_3 \times B_4 \times B_5$, and let $C_1, C_2$ be normal subgroups of $A$ contained in $B_{12}$, with trivial intersection (or otherwise the presentation wouldn’t be terse) but so that $C_1 C_2 \neq B_{12}$, with $K_1 = \ker (f_1) = C_2 B_3 B_4 B_5$, $K_2 = \ker (f_2) = C_1 B_3 B_4 B_5$, $K_3 = \ker (f_3) = B_{12} B_3 B_5$, $K_4 = \ker (f_4) = B_{12} B_3 B_5$, $K_5 = \ker (f_5) = B_{12} B_3 B_4$. The presentation is terse, as $K_{12345} = 1$.

By lemma 35(b), we have: $L_1 = u(K_{2345}) = u(C_1), L_2 = u(K_{1345}) = u(C_2), L_{12} = u(K_{145}) = u(B_{12});$ by applying proposition 40, we see that $L_{12}$ is cohesive, which means that $U$ here is not smashed. The cohesive components are readily seen to be $L_{12}, L_3 = u(K_{1245}) = u(B_3), L_4 = u(K_{1235}) = u(B_4), L_5 = u(K_{1234}) = u(B_5)$. We may now apply theorem 29, to deduce that:

$[A; (f_1, f_2)]/u(B_{12}) \simeq G_3 / f_3(B_3) \simeq G_4 / f_4(B_4) \simeq G_5 / f_5(B_5) \simeq \ldots$

It is not difficult to make a generalization of this example.

**Remark 75** If we let $f_i : A \to G_i, i = 1, \ldots, n$, be just group homomorphisms, not necessarily surjective, we get obvious actions of $\text{Aut} (A)$ and $\text{Dr} \prod_i \text{Aut} (G_i)$, further also of $\text{End} (A)$ and $\text{Dr} \prod_i \text{End} (G_i)$, on the set (/semigroup) of subgroups of $\text{Dr} \prod_i G_i$. Of particular interest is the case, when all $G_i$’s are isomorphic - in particular, to $A$.

6 Through a virtual dualization to diagrams

As we have seen, subgroups of direct products may be viewed as pull-backs, i.e., fiber products. Their structure conveys "naturally" diagrammatic depictions of
the form \[ \/\|\ \] - with \( m \) edges in the general case of theorem \( \text{[11]} \) we remind especially the important remarks \( \text{[18]} \) and \( \text{[13]} \).

Although we have not given a proper general definition for diagrams, its suggested use in this case just corresponds to the structure theorem \( \text{[11]} \) and does certainly have the special restriction, that it refers to a particular representation of a group \( U \) as a subdirect product. It has, nevertheless, the basic characteristic that we might expect of any diagram: Consisting of just two layers (levels), the vertices of the lower correspond to (a direct product of) subobjects (subgroups), the vertex on top is a factor group, namely one that is a factor group in many different ways according to our general theorem \( \text{[11]} \). Let us, for our ease, allow ourselves to call the vertex on top the head of our depiction of \( U \), the direct product of the lower level its socle; notice that this refers only to the particular embedding of \( U \) as a subdirect product.

If we now, conversely, use the investigated structure of such subgroups in order to deduce properties for such a basic diagram, then our theorem \( \text{[11]} \), our analysis of the subdirect product structure and, in particular, lemmata \( \text{[2]}, \text{[19]}, \text{[35]} \) and \( \text{[42]} \) make it clear that:

**Lemma 76.**

\( a. \) Any subdiagram of the suggested subdirect product representation diagram of such a subgroup \( U \) comprising a single edge (or any proper subset of the set of edges) corresponds to a certain factor group - but never to a subgroup of \( U \). Consequently, there is no proper subgroup of \( U \) that corresponds to any subdiagram containing the top vertex.

\( b. \) The diagrammatic properties of any subdiagram as in (a), comprising any number of edges, corresponding to a factor group of \( U \), are the same as of the whole diagram of \( U \) - i.e., property (a) "repeats itself".

Notice that whenever we speak of subdiagrams here, we shall mean that they are connected (unless otherwise stated) and that they include any edge of the given one if and only if they also include both its ends.

But there is another major feature to justify taking this kind of simple diagrams as a major cornerstone for a diagrammatic theory: Namely, what makes such a diagrammatic depiction especially interesting and worth studying for us is its virtuality, in the sense that the multiple direct factors of the "socle" are also determined set-theoretically (of well defined sections of \( U \), in some extended set-theoretical sense) and not just up to isomorphism - meaning that: Their vertices correspond to well-defined subsets of well-defined subsections. Notice that this virtuality could not possibly be claimed just by reference to pull-backs, as these are only defined up to isomorphism; this is why our first approach has been through "subgroups of direct products".

It would next be natural to think of considering the dual case, i.e. the virtual counterpart of push-outs.

This, however, becomes cumbersome in the category of groups: In it coproducts (push-outs) are namely realizable by free products (by amalgamated free products). In particular, in the case of extensions of an arbitrary group \( G \) by an
abelian group $A$, the push-out is the extension that is (functorially) induced by $G$-module homomorphisms $A \rightarrow A'$, while the ones induced by homomorphisms $G' \rightarrow G$ are as usually obtained as pull-backs. That push-out is a quotient not of the direct, but of the semidirect (see [2, IV 3, exercise 1(b); see also ex. 2]).

On the other hand, by moving into the category $\mathfrak{Ab}$ of abelian groups, duality works fine: then the push-out of a family of morphisms $S \rightarrow A_i, i \in I$, is realized as a certain factor group of their coproduct (direct sum); one may compare this with our example.

Our main focus with diagrammatic methods shall therefore from now on however shift from groups to modules - and to representation theory. We intend to get a new kind of diagrams there, ones having "virtual properties" in a sense that generalizes the basic "virtuality" described above. The original motivation toward the main subject of this article has actually been this: to begin understanding and substantializing "virtuality" in modules as well as possible. Then I chose to generalize by considering the more difficult category of groups, instead of those of abelian groups or modules, while also viewing it as very interesting for its own sake.

This shift of area (category) shall also allow us to dualize, so as to get the virtual counterpart of push-outs.

This and much more is done in [4] and its natural continuation in [3].

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