Nonlinear self-adjointness and conservation laws

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Abstract

The general concept of nonlinear self-adjointness of differential equations is introduced. It includes the linear self-adjointness as a particular case. Moreover, it embraces the strict self-adjointness (definition 1) and quasi-self-adjointness introduced earlier by the author. It is shown that the equations possessing nonlinear self-adjointness can be written equivalently in a strictly self-adjoint form by using appropriate multipliers. All linear equations possess the property of nonlinear self-adjointness, and hence can be rewritten in a nonlinear strictly self-adjoint form. For example, the heat equation \( u_t - \Delta u = 0 \) becomes strictly self-adjoint after multiplying by \( u^{-1} \). Conservation laws associated with symmetries are given in an explicit form for all nonlinearly self-adjoint partial differential equations and systems.

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1. Preliminaries

The concept of self-adjointness of nonlinear equations was introduced [1] (it is called the strict self-adjointness in what follows, see definition 1) for constructing conservation laws associated with symmetries of differential equations. To extend the possibilities of the new method for constructing conservation laws [2], the notion of quasi-self-adjointness was used in [3]. I suggest here the general concept of nonlinear self-adjointness. It embraces the strict self-adjointness and quasi-self-adjointness introduced earlier as well as the usual linear self-adjointness. Moreover, it will be shown that all linear equations possess the property of nonlinear self-adjointness.

It is demonstrated that the equations possessing nonlinear self-adjointness can be written equivalently in a strictly self-adjoint form by using appropriate multipliers. Consequently, any linear equation can be rewritten in an equivalent nonlinear form which is strictly self-adjoint.
For example, the heat equation $u_t - \Delta u = 0$ becomes strictly self-adjoint if we rewrite it in the form $u^{-1}(u_t - \Delta u) = 0$.

The construction of conservation laws demonstrates a practical significance of the nonlinear self-adjointness. Namely, conservation laws can be associated with symmetries for all linear and nonlinear self-adjoint differential equations.

1.1. Notation

Let $x = (x^1, \ldots, x^n)$ be independent variables. We consider two sets of dependent variables $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^m)$ with respective partial derivatives

$$u_i = \{u_i^1\}, \quad u_{ij} = \{u_i^j\}, \ldots, \quad u_{i_1\cdots i_k} = \{u_{i_1\cdots i_k}^1\}$$

and

$$v_i = \{v_i^1\}, \quad v_{ij} = \{v_i^j\}, \ldots, \quad v_{i_1\cdots i_k} = \{v_{i_1\cdots i_k}^1\},$$

where

$$u_i^s = D_i(u^s), \quad u_{ij}^s = D_iD_j(u^s), \quad u_{i_1\cdots i_k}^s = D_{i_1} \cdots D_{i_k}(u^s).$$

Here and in what follows, $D_i$ denotes the operator of total differentiation:

$$D_i = \frac{\partial}{\partial x^i} + u_i^s \frac{\partial}{\partial u^s} + v_i^s \frac{\partial}{\partial v^s} + u_{ij}^s \frac{\partial}{\partial u^s} + v_{ij}^s \frac{\partial}{\partial v^s} + u_{i_1\cdots i_k}^s \frac{\partial}{\partial u_{i_1\cdots i_k}^s} + \cdots.$$

1.2. Adjoint equations

We will consider systems of $m$ differential equations (linear or nonlinear)

$$F^\alpha(x, u, u_1, \ldots, u_m) = 0, \quad \alpha = 1, \ldots, m,$$

with $m$ dependent variables. The adjoint equations to equations (1) are written as [1]

$$F^\alpha_*(x, u, v, u_1, \ldots, u_m, v_1, \ldots, v_m) = 0, \quad \alpha = 1, \ldots, m,$$

with the adjoint operator $F^\alpha_*$ defined by

$$F^\alpha_*(x, u, v, u_1, \ldots, u_m, v_1, \ldots, v_m) = \frac{\delta L}{\delta u^\alpha},$$

where $L$ is the formal Lagrangian for equations (1) given by

$$L = \sum_{\beta=1}^m \delta \beta F^\beta(x, u, u_1, \ldots, u_m)$$

and $\delta/\delta u^\alpha$ is the variational derivative

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^\infty (-1)^s D_{i_1} \cdots D_{i_k} \frac{\partial}{\partial u_{i_1\cdots i_k}^\alpha}.$$

For a linear equation $L[u] = 0$, the adjoint operator defined by (3) is identical with the classical adjoint operator $L^*[v]$ determined by the equation $vL[u] - uL^*[v] = D_i(p^i)$.

The adjointness of linear operators $L$ is a symmetric relation, namely $(L^*)^* = L$. Nonlinear equations do not possess this property so that, in general, $(F^\alpha)^* \neq F$. 

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1.3. Self-adjointness

A linear operator $L$ is said to be self-adjoint if $L^* = L$. Then, we also say that the equation $L[u] = 0$ is self-adjoint. Thus, the self-adjointness of a linear equation $L[u] = 0$ means that the adjoint equation $L^*[v] = 0$ coincides with $L[u] = 0$ upon the substitution $v = u$.

This property has been extended to nonlinear equations in [1]. It will be called here the strict self-adjointness and defined as follows.

**Definition 1.** Equation (1) is said to be strictly self-adjoint if the adjoint equation (2) becomes equivalent to the original equation (1) upon the substitution (5). For example, the Korteweg–de Vries (KdV) equation $u_t = u_{xxx} + uu_x$ is self-adjoint. Indeed, its adjoint equation (2) has the form $v_t = v_{xxx} + u v_x$ and coincides with the KdV equation upon setting $v = u$.

The concept of quasi-self-adjointness introduced in [3] generalizes definition 1 by replacing (5) with the substitution of the form $v = \phi(u)$, $\phi'(u) \neq 0$.

Thus, equation (1) is quasi-self-adjoint if the adjoint equation (2) becomes equivalent to equation (1) upon the substitution (6). Let us consider as an example the equation $u_t - u^2 u_{xx} = 0$ (7) describing the nonlinear heat conduction in solid hydrogen

> It becomes equivalent to equation (7) not after the substitution (5) but after the following substitution of the form (6):

$v = u^{-2}$.

1.4. Theorem on conservation laws

We will use the following statement proved in [2].

**Theorem 1.** Any infinitesimal symmetry (Lie point, Lie–Bäcklund, nonlocal)

\[ X = \xi^i(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^a(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u^a} \]

of equations (1) leads to a conservation law $D_i (C^i) = 0$ constructed by the formula

\[ C^i = \xi^i \mathcal{L} + W^a \left[ \frac{\partial \mathcal{L}}{\partial u^a} - D_j \left( \frac{\partial \mathcal{L}}{\partial u^a_{ij}} \right) + D_jD_k \left( \frac{\partial \mathcal{L}}{\partial u^a_{ijk}} \right) - \cdots \right] + D_j \left( W^a \right) \left[ \frac{\partial \mathcal{L}}{\partial u^a_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u^a_{ijk}} \right) + \cdots \right] + D_jD_k \left( W^a \right) \left[ \frac{\partial \mathcal{L}}{\partial u^a_{ijk}} - \cdots \right], \]  

where $W^a = \eta^a - \xi^i u^a_i$ and $\mathcal{L}$ is the formal Lagrangian (4). In applying formula (9), the formal Lagrangian $\mathcal{L}$ should be written in the symmetric form with respect to all mixed derivatives $u^a_{ij}$, $u^a_{ijk}$, $\cdots$.

1 Recall that (7) is related to the classical (1+1)-dimensional heat equation by a differential substitution [4] or a reciprocal transformation [5]. This connection, together with its extensions, allows the analytic solution of certain moving boundary problems in nonlinear heat conduction [6].
2. Definition and main properties of nonlinear self-adjointness

2.1. Heuristic discussion

Definition (4) of the formal Lagrangian $L$ shows that the vector (9) involves the ‘non-physical’ variable $v$. Therefore, the validity of the conservation equation $D_t (C^i) = 0$ requires that we should take into account not only equations (1) but also the adjoint equations (2). But if the system (1) is quasi-self-adjoint (in particular self-adjoint), one can eliminate $v$ via substitution (6) and obtain a conservation law for equations (1).

However, the quasi-self-adjointness is not the only case when the variables $v$ can be eliminated from the conserved vector (9). Let us note first of all that we can relax the condition $\varphi' (u) \neq 0$ in (6) since it is used only to guarantee the equivalence of equation (2) to equation (1) after eliminating $v$ by setting $v = \varphi(u)$. In constructing conservation laws, it is only important that $v$ does not vanish identically, because otherwise $L = 0$ and (9) gives the trivial vector $C^i = 0$. Therefore, we can replace the condition $\varphi' (u) \neq 0$ in (6) with the weaker condition $\varphi (u) \neq 0$. Secondly, substitution (6) can be replaced with a more general substitution where $\varphi$ involves not only the variable $u$ but also its derivatives as well as the independent variables $x$.

This will be a differential substitution

$$v^a = \varphi^a (x, u, u_{(1)}, \ldots), \quad \alpha = 1, \ldots, m.$$ (10)

The only requirement is that not all $\varphi^a$ vanish. Moreover, $\varphi$ may involve nonlocal variables, e.g. $D^{-1}_{i} (u^a)$. Then, it is more convenient to determine $v$ implicitly by

$$\Phi_{\alpha} (x, u, u_{(1)}, \ldots; v, v_{(1)}, \ldots) = 0, \quad \alpha = 1, \ldots, m.$$ (11)

2.2. Definition

In this communication, we will consider the substitutions (10) that do not involve the derivatives and use the following definition of nonlinear self-adjointness.

Definition 2. The system (1) is said to be nonlinearly self-adjoint if the adjoint system (2) is satisfied for all solutions $u$ of equations (1) upon a substitution

$$v^a = \varphi^a (x, u), \quad \alpha = 1, \ldots, m,$$ (12)

such that

$$\varphi (x, u) \neq 0.$$ (13)

In other words, the following equations hold:

$$F^*_a (x, u, \varphi (x, u), u_{(1)}, \varphi (\sigma)) = \lambda^a_{\beta} F_\beta (x, u, u_{(1)}, \varphi (\sigma)), \quad \alpha = 1, \ldots, m,$$ (14)

where $\lambda^a_{\beta}$ are undetermined coefficients. Here, $\varphi$ is the $m$-dimensional vector $\varphi = (\varphi^1, \ldots, \varphi^m)$ and $\varphi (\sigma)$ are its derivatives:

$$\varphi (\sigma) = [D_{u_1} \cdots D_{u_\sigma} (\varphi^a (x, u))], \quad \sigma = 1, \ldots, s.$$

Equation (13) means that not all components $\varphi^a (x, u)$ of the vector $\varphi$ vanish simultaneously.
2.3. Properties

**Proposition 1.** The system (1) is nonlinearly self-adjoint if and only if there exist functions $v^a$ given by (12) and satisfying the condition (13) that solve the adjoint system (2) for all solutions $u(x)$ of equations (1).

**Proposition 2.** Any linear equation is nonlinearly self-adjoint.

**Proof.** This is a consequence of the fact that the adjoint equation $L^*[v] = 0$ to the linear equation $L[u] = 0$ does not involve the variable $u$. Therefore, any non-vanishing solution $v = \psi(x)$ of the adjoint equation gives a vector function (12) which is independent of $u$ and hence satisfies the requirement of definition 2. □

In the case of one dependent variable, i.e. $m = 1$, we can easily prove the following.

**Proposition 3.** Equation (1) is nonlinearly self-adjoint if and only if it becomes strictly self-adjoint (see definition 1) upon rewriting it in the equivalent form

$$\mu(x, u)F(x, u, u(1), \ldots, u(s)) = 0, \quad \mu(x, u) \neq 0,$$

(15)

with an appropriate multiplier $\mu(x, u)$. In particular, any linear equation can be made self-adjoint in the sense of definition 1.

**Proof.** The computation reveals the following relation between the multiplier (15) and the substitution (12):

$$\psi(x, u) = u\mu(x, u).$$

(16)

Namely, if equation (1) is nonlinearly self-adjoint with the substitution (12), then equation (15) whose multiplier $\mu$ is determined by (16) is strictly self-adjoint, and vice versa. □

2.4. Example: the Kompaneets equation

Let us write the Kompaneets equation [7] in the form

$$u_t = \frac{1}{x^2} D_x [x^4 (u_t + u + u^2)].$$

(17)

The computation shows that the adjoint equation to equation (17),

$$v_t + x^2 v_{xx} - x^2 (1 + 2u) v_t + 2(x + 2u - 1) v = 0,$$

does not have a solution of the form (6) but it has the solution of the form (12), namely

$$v = x^2.$$

Hence, equation (17) is not quasi-self-adjoint, but it is nonlinearly self-adjoint. Equation (16) provides the multiplier $\mu = x^2/u$. Hence, the Kompaneets equation becomes strictly self-adjoint if we write it in the form

$$\frac{x^2}{u} u_t = \frac{1}{u} D_x [x^4 (u_t + u + u^2)].$$

(18)
3. Time-dependent conservation laws for the Kadomtsev–Petviashvili (KP) equation

We will use the KP equation \[ \frac{u_t}{u} - u u_x - u_{xxx} = u_{yy} \] \[ (19) \]
written as the system (see, e.g. [9], p 241, and the references therein)
\[ u_t - uu_x - u_{xxx} - \omega_y = 0, \quad \omega_x - u_y = 0. \] \[ (20) \]

3.1. Nonlinear self-adjointness

The formal Lagrangian (4) for equations (20) is written as
\[ L = v(u_t - uu_x - u_{xxx} - \omega_y) + z(\omega_x - u_y) \] \[ (21) \]
and equation (3) yields the following adjoint system to the system (20):
\[ v_t - uu_x - v_{xxx} - zy = 0, \quad zx - vy = 0. \] \[ (22) \]
Equations (22) become identical with the KP equations (20) upon the substitution
\[ v = u, \quad z = \omega. \] \[ (23) \]
It means that the system (20) is nonlinearly self-adjoint.

3.2. Symmetries

The system (20) admits the infinite-dimensional Lie algebra spanned by the operators
\[ X_f = 3f \frac{\partial}{\partial t} + \left( f'x + \frac{1}{2} f''y^2 \right) \frac{\partial}{\partial x} + 2f'y \frac{\partial}{\partial y} - \left[ 2f' + f''x + \frac{1}{2} f'''y^2 \right] \frac{\partial}{\partial u} - \frac{1}{6} f^{(4)}y^3 \frac{\partial}{\partial \omega}, \] \[ (24) \]
\[ X_g = 2g \frac{\partial}{\partial y} + g' y \frac{\partial}{\partial x} - g''y \frac{\partial}{\partial u} - \left[ g' + g''x + \frac{1}{2} g'''y^2 \right] \frac{\partial}{\partial \omega}, \] \[ (25) \]
\[ X_h = h \frac{\partial}{\partial x} - h' \frac{\partial}{\partial u} - h''y \frac{\partial}{\partial \omega}, \] \[ (26) \]
where \( f, g, h \) are three arbitrary functions of \( t \). We will ignore the obvious symmetry
\[ X_\alpha = \alpha(t) \frac{\partial}{\partial \omega} \]
describing the addition to \( \omega \) an arbitrary function of \( t \).

Note that the operators (24)–(26) considered without the term \( \frac{\partial}{\partial \omega} \) span the infinite-dimensional Lie algebra of symmetries of the KP equation (19). They coincide (up to normalizing coefficients) with the symmetries of the KP equation that were first obtained by Schwarz in 1982 (see also [10, 11] and the references therein).
3.3. Conservation laws

Noether’s theorem is not applicable to the system (20). But theorem 1 from section 1.4 is applicable. Applying formula (9) to the formal Lagrangian (21) and to the symmetry (24), and then eliminating \( \nu, z \) by the substitution (23), we obtain the conservation law

\[
[D_t(C^1) + D_x(C^2) + D_y(C^3)]_{20} = 0
\]

(27)

with the following components of the conserved vector \( C = (C^1, C^2, C^3) \):

\[
\begin{align*}
C^1 &= -\frac{1}{2} f u^2 - (x f'' + \frac{1}{2} y^2 f'') u, \\
C^2 &= (u_{xx} + \frac{1}{2} u^3 - \frac{1}{2} u_x^2 - \frac{1}{2} \omega^2) f' + (x u_{xx} + \frac{1}{2} x u^3 - u_x) f'' + \frac{1}{2} (y^2 u^3 + 2y^2 u_x u - 4xy \omega) f''' - \frac{1}{2} y^3 \omega f^{(4)}, \\
C^3 &= u x f' + x \omega f'' + (xy + \frac{1}{2} y^2 \omega) f''' + \frac{1}{6} y^3 u f^{(4)}.
\end{align*}
\]

The conservation equation (27) for the vector (28) has the form

\[
D_t(C^1) + D_x(C^2) + D_y(C^3)
\]

\[
= \left( uf' + x f'' + \frac{1}{2} y^2 f''' \right) (u_{xxx} + uu_x + \omega_x - u_x)
\]

\[
+ \left( \omega f' + xy f'' + \frac{1}{6} y^3 f^{(4)} \right) (u_y - \omega_y).
\]

(29)

Since \( f = f(t) \) is an arbitrary function, (28) provides an infinite set of conserved vectors. Note that the subscript \( (20) \) in equation (27) refers to the restriction on the solution manifold found, e.g., in [12–14]. In particular, the Lagrangian approach and the Noether theorem are used in [14] which contains an interesting discussion of the infinite set of conservation laws and can be easily computed by formula (9).

Different approaches to the construction of conservation laws for the KP equation can be found, e.g., in [12–14]. In particular, the Lagrangian approach and the Noether theorem are used in [14] which contains an interesting discussion of the infinite set of conservation laws. Note that the second equation of the system (20) guarantees that the vector field \((u, \omega)\) has the potential defined by \( u = \phi_x, \omega = \phi_y \). Then, the system (20) is replaced by the potential KP equation \( \phi_{xt} - \phi_x \phi_{xx} - \phi_{xxx} - \phi_{yy} = 0 \) which, unlike equation (19) or the system (20), has a Lagrangian, namely

\[
L = -\frac{1}{2} \phi_x \phi_t + \frac{1}{6} \phi_x^3 + \frac{1}{2} \phi_y^2 - \frac{1}{2} \phi_{xx}^2.
\]

Now the Noether theorem can be used upon rewriting the symmetries of the KP equation in terms of the potential \( \phi \).

4. Conservation laws for linear equations

One can obtain the conserved vector by formula (9) for any linear equation because linear equations are nonlinearly self-adjoint by proposition 2. Consider, e.g., the heat equation

\[
u_t - \Delta u = 0
\]

(30)

with any number of spatial variables \( x = (x^1, \ldots, x^n) \). Applying formula (9) to the dilation generator \( X = u \frac{\partial}{\partial u} \) we obtain the conservation law \([D_t(\tau) + \nabla \cdot \chi]_{(30)} = 0\) with

\[
\tau = \nu \phi(t, x), \quad \chi = u \nabla \psi(t, x) - \psi(t, x) \nabla u,
\]

where \( \nu = \psi(t, x) \) is an arbitrary solution of the adjoint equation \( \nu_t + \Delta \nu = 0 \) to equation (30). The conserved vector (31) embraces the conserved vectors associated with all other symmetries of equation (30). In particular, the projective symmetry

\[
X = \tau \frac{\partial}{\partial \tau} + x \frac{\partial}{\partial x} - \frac{|x|^2 + 2nt}{4} \frac{\partial}{\partial u}
\]
of equation (30) gives the conserved vector
\[ \tau = \frac{|x|^2 - 2nt}{4} u, \quad \chi^i = \frac{x^i}{2} u - \frac{|x|^2 - 2nt}{4} u_i. \]

It corresponds to (31) with the particular solution \( v = (|x|^2 - 2nt)/4 \) of the adjoint equation. In the one-dimensional case \( (n = 1) \), it is shown in [15] (see also [16]) by direct calculation that all conserved vectors for the heat equation \( u_t = u_{xx} \) have the form (31). It is shown in [17], theorem 3, that the result similar to (31) is true for any even-order homogeneous linear evolution equation with one spatial variable.

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