A MIXED PROBLEM FOR A BOUSSINESQ HYPERBOLIC EQUATION WITH INTEGRAL CONDITION

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Abstract. A hyperbolic problem which combines a classical (Dirichlet) and a non-local constraint is considered. The existence and uniqueness of strong solutions are proved, we use a functional analysis method based on a priori estimate and on the density of the range of the operator generated by the considered problem.

1. INTRODUCTION

The first study of evolution problems with a nonlocal condition - the so called energy specification - goes back to Cannon[5], 1963 Using an integral condition, we proved the existence and uniqueness of the solution of a mixed problem which combine a classical (Dirichlet) and an integral condition for the equation. Problems involving local and integral condition for hyperbolic equations are investigated by the energy inequalities method in [1], [6], [7], [8], [9], [10], [11], [12]. In this paper, we prove the existence and uniqueness of the solution for the mixed problem (1) – (5). Our proof is based on a priori estimate and on the fact that the range of the operator generated by the considered problem is dense.

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2. Formulation of the problem

In the region $Q = (0, l) \times (0, T)$, with $l < \infty$ and $T < \infty$, we shall consider the problem

\begin{align*}
(1) \quad & L u = u_{tt} - (b(x,t)u_x)_x - \beta \frac{\partial^4 u}{\partial t^2 \partial x^2} = f(x,t), \forall (x,t) \in Q \\
(2) \quad & l_1 u = u(x,0) = \varphi_1(x), \quad x \in (0,l) \\
(3) \quad & l_2 u = u_t(x,0) = \varphi_2(x), \quad x \in (0,l) \\
(4) \quad & u(0,t) = 0, \quad t \in (0,T) \\
(5) \quad & \int_0^l xu(x,t) \, dx = 0, \quad t \in (0,T)
\end{align*}

where $\beta \in \mathbb{IR}^*_+$ and $b(x,t)$ and its derivatives satisfy the conditions:

$C_1 : b_0 \leq b(x,t) \leq b_1$, $b_t(x,t) \leq b_2$, $b_x(x,t) \leq b_3$, for any $(x,t) \in \overline{Q}$,

$C_2 : b_{tt}(x,t) \leq b_4$, $b_{xt}(x,t) \leq b_5$, for any $(x,t) \in \overline{Q}$.

The functions $f$, $\varphi_1$ and $\varphi_2$ are known functions which satisfy the compatibility conditions:

$\varphi_1(0) = \varphi_2(0) = \int_0^l x \varphi_1(x) \, dx = \int_0^l x \varphi_2(x) \, dx = 0.$
3. functional Spaces

The problem (1)-(5) can be put in the following operator form: \( Lu = \mathcal{F} \), \( u \in D(L) \), where:

\[
Lu = (\mathcal{L}u, l_1u, l_2u) \quad \text{and} \quad \mathcal{F} = (f, \varphi_1, \varphi_2).
\]

The operator \( L \) is considered from \( B \) to \( H \), where \( B \) is the Banach space consisting of functions \( u \in L^2(Q) \), satisfying conditions (4) and (5) with the finite norm:

\[
\|u\|_B^2 = \sup_{0 \leq \tau \leq T} \left[ \|u(\cdot, \tau)\|_{L^2(0, l)}^2 + \|u_t(\cdot, \tau)\|_{L^2(0, l)}^2 \right]
\]

and \( F \) is the Hilbert space \( L^2(Q) \times L^2(0, l) \times L^2(0, l) \) equipped with the norm:

\[
\|F\|_H^2 = \|f\|_{L^2(Q^T)}^2 + \|\varphi_1\|_{L^2(0, l)}^2 + \|\varphi_2\|_{L^2(0, l)}^2.
\]

Let \( D(L) \) denote the domain of \( L \), which is the set of all functions \( u \in L^2(Q) \) for which \( u_t, u_x, u_{tx}, u_{tt}, u_{ttt} \in L^2(Q) \) and satisfying conditions (4) and (5).

4. A priori estimate and its consequences

**Theorem 1**: For any function \( u \in D(L) \) satisfies conditions \( C_1-C_2 \) there exists a positive constant \( c \), such that

\[
\|u\|_B \leq c \|Lu\|_H,
\]

**Proof**: We consider the scalar product in \( L^2(Q^T) \) of the operator \( \mathcal{L}u \) and \( Mu \), where \( Mu = x^* \mathcal{S}_x^* u_t - \mathcal{S}_x^* (\rho u_t) \), with \( Q^T = (0, l) \times (0, \tau), \ 0 \leq \tau \leq T \), and \( \mathcal{S}_x^* v = \int_x v(\xi, t) d\xi \), we obtain
we establish the equalities:

\[(8)\]
\[
(\mathcal{L} u, M u)_{L^2(Q^\tau)} = (u_{tt}, x \Psi^*_x u_t)_{L^2(Q^\tau)} - ((b (x, t) u_x)_x, x \Psi^*_x u_t)_{L^2(Q^\tau)}
- \beta (u_{ttt}, x \Psi^*_x u_t)_{L^2(Q^\tau)} - (u_{tt}, \Psi^*_x (\rho u_t))_{L^2(Q^\tau)}
+ ((b (x, t) u_x)_x, \Psi^*_x (\rho u_t))_{L^2(Q^\tau)} + \beta (u_{ttt}, \Psi^*_x (\rho u_t))_{L^2(Q^\tau)}.
\]

Making use of conditions (2)-(5) and integrating by parts we establish the equalities:

\[(9)\]
\[
(u_{tt}, x \Psi^*_x u_t)_{L^2(Q^\tau)} = \frac{1}{2} \| \Psi^*_x u_t(., \tau) \|_{L^2(0, l)}^2
- \frac{1}{2} \| \Psi^*_x \varphi_2 \|_{L^2(0, l)}^2 - (\Psi^*_x u_{tt}, u_t)_{L^2_\beta(Q^\tau)},
\]

\[(10)\]
\[
- ((b (x, t) u_x)_x, x \Psi^*_x (u_t))_{L^2(Q^\tau)}
= \frac{1}{2} \| \sqrt{b (., \tau)} u(., \tau) \|_{L^2(0, l)}^2
- \frac{1}{2} \| \sqrt{b (., 0)} \varphi_1 \|_{L^2(0, l)}^2
- \frac{1}{2} \| \sqrt{b (., t)} u \|_{L^2(Q^\tau)}^2
- (b (x, t) u_x, u_t)_{L^2_\beta(Q^\tau)}
- (b (x, t) u_x, u_t)_{L^2_\beta(Q^\tau)},
\]

\[(11)\]
\[
- \beta (u_{ttt}, x \Psi^*_x (u_t))_{L^2(Q^\tau)} = \frac{\beta}{2} \| u_t(., \tau) \|_{L^2(0, l)}^2
- \frac{\beta}{2} \| \varphi_2 \|_{L^2(0, l)}^2 - \beta (u_{ttt}, u_t)_{L^2_\beta(Q^\tau)}.
\]

\[(12)\]
\[
- (u_{tt}, \Psi^*_x (\rho u_t))_{L^2(Q^\tau)} = (\Psi^*_x u_{tt}, u_t)_{L^2_\beta(Q^\tau)},
\]

\[(13)\]
\[
((b (x, t) u_x)_x, \Psi^*_x (\rho u_t))_{L^2(Q^\tau)} = (b (x, t) u_x, u_t)_{L^2_\beta(Q^\tau)} ,
\]

\[(14)\]
\[
\beta (u_{ttt}, \Psi^*_x (\rho u_t))_{L^2(Q^\tau)} = \beta (u_{ttt}, u_t)_{L^2_\beta(Q^\tau)}.
\]
Combining equalities (9)-(14) and (8) we obtain:

\[
\frac{1}{2} \left\| \mathcal{S}_x^* u_t (., \tau) \right\|^2_{L^2(0,l)} + \frac{1}{2} \left\| \sqrt{b (., t)} u (., \tau) \right\|^2_{L^2(0,l)} + \frac{\beta}{2} \left\| u_t (., \tau) \right\|^2_{L^2(0,l)}
\]

\[
= \frac{1}{2} \left\| \mathcal{S}_x^* \varphi_2 \right\|^2_{L^2(0,l)} + \frac{1}{2} \left\| \sqrt{b (., t)} \varphi_1 \right\|^2_{L^2(0,l)} + \frac{\beta}{2} \left\| \varphi_2 \right\|^2_{L^2(0,l)}
\]

\[
+ \frac{1}{2} \left\| \varphi_2 \right\|^2_{L^2(0,l)} + \frac{1}{2} \left\| \sqrt{b_t u} \right\|^2_{L^2(Q^r)} + (b_x (x, t) u, \mathcal{S}_x^* u_t)_{L^2(Q^r)}
\]

\[
+ (\mathcal{L} u, x \mathcal{S}_x^* u_t)_{L^2(Q^r)} - (\mathcal{L} u, \mathcal{S}_x^* (\rho u_t))_{L^2(Q^r)}.
\]

By applying the Cauchy inequality to the last three terms on the right-hand side of the inequality (15) and making use conditions \(C_1\), combining with (15), we obtain

\[
\frac{1}{2} \left\| u (., \tau) \right\|^2_{L^2(0,l)} + \frac{1}{2} \left\| u_t (., \tau) \right\|^2_{L^2(0,l)} + \frac{1}{2} \left\| \mathcal{S}_x^* u_t (., \tau) \right\|^2_{L^2(0,l)}
\]

\[
\leq k \left[ \left\| f \right\|^2_{L^2(Q^r)} + \left\| \varphi_1 \right\|^2_{L^2(0,l)} + \left\| \varphi_2 \right\|^2_{L^2(0,l)}
\]

\[
+ \left\| u \right\|^2_{L^2(Q^r)} + \left\| u_t \right\|^2_{L^2(Q^r)} + \left\| \mathcal{S}_x^* u_t \right\|^2_{L^2(Q^r)} \right].
\]

where \(k = \frac{\max(2, b_1, \beta + l^2, b_2, l^4)}{\min(1, b_0, \beta)}\).

Applying the Gronwall lemma to (16), and eliminating the term \(\left\| \mathcal{S}_x^* u_t (., \tau) \right\|^2_{L^2(0,l)}\) of the left-hand side of the inequality we obtain

\[
\frac{1}{2} \left\| u (., \tau) \right\|^2_{L^2(0,l)} + \frac{1}{2} \left\| u_t (., \tau) \right\|^2_{L^2(0,l)}
\]

\[
\leq k \exp(k T) \left( \left\| f \right\|^2_{L^2(Q^r)} + \left\| \varphi_1 \right\|^2_{L^2(0,l)} + \left\| \varphi_2 \right\|^2_{L^2(0,l)} \right).
\]
Since the left-hand side of (17) does not depend on \( \tau \), we take the supremum with \( \tau \) from 0 to \( T \), then the estimate (7) follows with \( c = \sqrt{k} \exp(k\frac{T}{2}) \).

5. Solvability of the problem

**Proposition 1.** The operator \( L \) acting from \( B \) to \( H \) have a closure.

**Proof.** (see [3])

Let be \( \bar{L} \) the closure of \( L \), \( D(\bar{L}) \) its domain.

**Definition.** The solution of \( \bar{L}u = F \) for any \( u \in D(\bar{L}) \) is strong solution of problem (1)-(5). we take the limit in the inequality (7), we obtain \( \|u\|_B \leq c \|\bar{L}u\|_H, \forall u \in D(\bar{L}) \).

From the inequality we have:

**Corollary 1:** The strong solution of problem (1)-(5) when it exists, it’s unique, and depends continuously of data \( f, \varphi_1, \varphi_2 \).

**Corollary 2:** The set of values \( R(\bar{L}) \) of the operator \( \bar{L} \) is equal to the closure \( \overline{R(\bar{L})} \) of \( R(L) \).

**Theorem 2:** If the conditions C1-C2 are satisfying, then for any \( \mathcal{F} = (f, \varphi_1, \varphi_2) \in H \), there exists a strong unique solution \( u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}\mathcal{F}} \) of the problem (1)-(5) where the estimate \( \|u\|_B \leq c \|\mathcal{F}\|_H \) is satisfying, where \( c \) is a positive constant does not depend of \( u \).

**Proof:** From (22) we conclude that the operator \( \bar{L} \) acting from \( D(\bar{L}) \) in \( R(\bar{L}) \) have an inverse \( \bar{L}^{-1} \), and from corollary 2, we conclude that the range \( R(\bar{L}) \) of the operator \( \bar{L} \) is closed. Then we will be proved the density of the set \( R(L) \) in the space \( H \) (i.e) \( \overline{R(L)} = H \).

For this we need the following proposition:
**Proposition 2:** If, for all functions \( u \in D_0(L) \), where

\[
D_0(L) = \{ u/u \in D(L) : l_1 u = l_2 u = 0 \}
\]

and for some function \( \omega \in L^2(Q) \), we have

(18) \[ (\mathcal{L}u, \omega)_{L^2(Q)} = 0, \]

then \( \omega \) vanishes almost everywhere in \( Q \).

**Proof of the proposition 2:** The relation (18) is given for all \( u \in D_0(L) \); we can express it in a particular form. Let \( u_{tt} \) be a solution of:

(19) \[ b(\sigma, t) [x \mathcal{S}_x^* u_{tt} - \mathcal{S}_x^*(\rho u_{tt})] = h(x, t), \]

where \( \sigma \) is a constant in \((0, l)\) and \( h(x, t) = \int_t^T \omega(x, \tau) d\tau \).

And let \( u \) be the function defined by:

(20) \[ u = \begin{cases} 0, & \text{si } 0 \leq t \leq s, \\ \int_s^t (t - \tau) u_{\tau\tau} d\tau, & \text{si } s \leq t \leq T. \end{cases} \]

(19) and (20) follows \( u \) is in \( D_0(L) \) and:

(21) \[ \omega(x, t) = \mathcal{S}_x^{-1} h = -[b(\sigma, t) (x \mathcal{S}_x^* u_{tt} - \mathcal{S}_x^*(\rho u_{tt}))]_t = [b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}]_t. \]

To continue the proof we need the following lemma:

**Lemma 2.** The function \( \omega \) defined by (21), belongs to the space \( L^2(Q) \).

**Proof of lemma 2:** We start with the proof of this inequality\( \| \mathcal{S}_x^*(\rho - x) u_{tt} \|_{L^2(0, l)}^2 \leq \frac{t^4}{12} \| u_{tt} \|_{L^2(0, 1)}^2. \)

From this inequality and since the conditions \( C_1 \) are satisfied we conclude that \( b_t(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt} \) belongs to \( L^2(Q) \).

Because \( \omega(x, t) = [b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}]_t = b_t(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt} + b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{ttt}, \) then we will
be prooved that: $b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{ttt} \in L^2(Q)$.

For this we introduce the $t$-averaging operators $\rho_\varepsilon$ of the form

$$(\rho_\varepsilon f)(x, t) = \frac{1}{\varepsilon} \int_0^T \omega\left(\frac{t-s}{\varepsilon}\right) f(x, s) ds,$$

where $\omega \in C_0^\infty(0, T), \omega \geq 0,
\int_{-\infty}^{+\infty} \omega(s) ds = 1, \omega \equiv 0$ for $t \leq 0$ and $t \geq T$.

Applying the operators $\rho_\varepsilon$ and $\frac{\partial}{\partial t}$ to the equation

$$-b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt} = h(x, t),$$

we obtain

$$\frac{\partial}{\partial t}\left(-b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}\right) = \frac{\partial}{\partial t} \left[-b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt} + \rho_\varepsilon (b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt})\right] - \frac{\partial}{\partial t}\rho_\varepsilon h.$$

Then

$$\|b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}\|_{L^2(Q)}^2 \leq 2 \left\| \frac{\partial}{\partial t} \left[b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt} - \rho_\varepsilon (b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt})\right]\right\|_{L^2(Q)}^2 + 2 \left\| \frac{\partial}{\partial t}\rho_\varepsilon h\right\|_{L^2(Q)}^2.$$

Since $\rho_\varepsilon f \xrightarrow{\varepsilon \to 0} f$, and $\frac{\partial}{\partial t} (b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt})$ is bounded in $L^2(Q)$, then $\omega \in L^2(Q)$.

Now we return to the 2nd proposition, we remplace $\omega$ in (18) by its representation given by (21) we have:

(22) $$(u_{tt}, [b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}]_t)_{L^2(Q)} = ((b(x, t) u_x)_x, [b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}]_t)_{L^2(Q)} + \beta (u_{ttxx}, [b(\sigma, t) \mathcal{S}_x^*(\rho - x) u_{tt}]_t)_{L^2(Q)}.$$
Making use conditions (3)-(5), and from the particular form of $u$ given by (19) and (20), the equality (22) can be simplified. For this integrating by parts each term of the equality on the sup-domain $Q_s = (0, l) \times (s, T)$ where $0 \leq s \leq T$

\begin{equation}
(23) \quad (u_{tt} \left[ b(\sigma, t) \mathcal{S}_x^*(\rho - x)u_{tt}\right]_t)_{L^2(Q)}
= \frac{1}{2} \left\| \sqrt{b(\sigma, s)} \mathcal{S}_x^*u_{tt}(., s) \right\|_{L^2(0, l)}^2 - \frac{1}{2} \left\| \sqrt{b_t(\sigma, .)} \mathcal{S}_x^*u_{tt} \right\|_{L^2(Q_s)}^2,
\end{equation}

\begin{equation}
(24) \quad (b(x, t)u_x)_x \left[ b(\sigma, t) \mathcal{S}_x^*(\rho - x)u_{tt}\right]_t)_{L^2(Q)}
= -\frac{1}{2} \left\| \sqrt{b(., T)} b(\sigma, T)u_t(., T) \right\|_{L^2(0, l)}^2
+ \frac{1}{2} \int_{Q_s} \left[ 3b_t(x, t) b(\sigma, t) + b(x, t) b_t(\sigma, t) \right] (u_t)^2 dxdt
- \int_0^l b_t(x, T) b(\sigma, T) u(x, T) u_t(x, T) dx
+ \int_{Q_s} \left[ b_{tt}(x, t) b(\sigma, t) + b_t(x, t) b_t(\sigma, t) \right] uu_t dxdt
+ \int_{Q_s} \left[ b_x(x, t) u_t + b_{xt}(x, t) u \right] b(\sigma, t) \mathcal{S}_x^*u_{tt} dxdt,
\end{equation}

\begin{equation}
(25) \quad \beta \left[ u_{ttxx} \left[ b(\sigma, t) \mathcal{S}_x^*(\rho - x)u_{tt}\right]_t\right]_{L^2(Q)}
= \frac{\beta}{2} \left\| \sqrt{b_t(\sigma, .)} u_{tt} \right\|_{L^2(Q_s)}^2 - \frac{\beta}{2} \left\| \sqrt{b(\sigma, s)} u_{tt}(., s) \right\|_{L^2(0, l)}^2.
\end{equation}
Substitution of (23)-(25) into (22) gives

\begin{align*}
(26) & \quad \frac{1}{2} \left\| \sqrt{b(\sigma, s)} \mathbb{I}^* x u_{tt}(\cdot, s) \right\|_{L^2(0, l)}^2 \\
& + \frac{1}{2} \left\| \sqrt{b(\cdot, T) b(\sigma, T)} u_t(\cdot, T) \right\|_{L^2(0, l)}^2 \\
& + \frac{\beta}{2} \left\| \sqrt{b(\sigma, s)} u_{tt} \right\|_{L^2(0, l)}^2 \\
& = \frac{1}{2} \left\| \sqrt{b_t(\sigma, s)} \mathbb{I}_x^* u_{tt} \right\|_{L^2(Q_s)}^2 + \frac{\beta}{2} \left\| \sqrt{b_t(\sigma, \cdot)} u_{tt} \right\|_{L^2(Q_s)}^2 \\
& + \frac{1}{2} \int_{Q_s} \left[ 3b_t(x, t) b(\sigma, t) + b(x, t) b_t(\sigma, t) \right] (u_t)^2 \, dx \, dt \\
& - \int_0^l b_t(x, T) b(\sigma, T) u(x, T) u_t(x, T) \, dx \\
& + \int_{Q_s} \left[ b_{tt}(x, t) b(\sigma, t) + b_t(x, t) b_t(\sigma, t) \right] uu_t \, dx \, dt \\
& + \int_{Q_s} \left[ b_x(x, t) u_t + b_{xt}(x, t) u \right] b(\sigma, t) \mathbb{I}_x^* u_{tt} \, dx \, dt.
\end{align*}

By applying the Cauchy inequality and Cauchy inequality with \( \varepsilon \) to estimate the last three terms on the right-hand side of the inequality (26) and making use conditions \( C_1, C_2 \), combining the estimates and (26) taking into account
that $\varepsilon = \frac{b_0^2}{2b_1}$ we obtain

\begin{equation}
\frac{b_0}{2} \left[ \|\mathcal{S}_x u_{tt}(.,s)\|^2_{L^2(0,l)} + \frac{b_0}{2} \|u_t(.,T)\|^2_{L^2(0,l)} + \beta \|u_{tt}(.,s)\|^2_{L^2(0,l)} \right] \leq \left( b_1^2 + \frac{b_2}{2} \right) \|\mathcal{S}_x u_{tt}\|^2_{L^2(Q_s)} + \frac{\beta b_2}{2} \|u_{tt}\|^2_{L^2(Q_s)} + \frac{b_2^2 + b_1 + 4b_1 b_2 + b_3^2}{2} \|u_t\|^2_{L^2(Q_s)} + \frac{b_2^2 + b_4 + b_5^2}{2} \|u\|^2_{L^2(Q_s)} + \frac{b_2^2 b_2^2}{b_0^2} \|u(.,T)\|^2_{L^2(0,l)}
\end{equation}

By virtue of the elementary inequality

\begin{equation}
\frac{b_1^2 b_2^2}{b_0^2} \|u(.,T)\|^2_{L^2(0,l)} \leq \frac{b_1^2 b_2^2}{b_0^2} \|u\|^2_{L^2(Q_s)} + \frac{b_1^2 b_2^2}{b_0^2} \|u_t\|^2_{L^2(Q_s)},
\end{equation}

we estimate the last term of the right-hand side of the inequality (27), we obtain

\begin{equation}
\|\mathcal{S}_x u_{tt}(.,s)\|^2_{L^2(0,l)} + \frac{b_0}{2} \|u_t(.,T)\|^2_{L^2(0,l)} + \beta \|u_{tt}(.,s)\|^2_{L^2(0,l)} \leq \left( \frac{2b_1^2 + b_2}{b_0} \right) \|\mathcal{S}_x u_{tt}\|^2_{L^2(Q_s)} + \frac{\beta b_2}{b_0} \|u_{tt}\|^2_{L^2(Q_s)} + \frac{2b_1^2 b_3^2}{b_0^2} + 2b_1 b_2 + (b_2 + b_1)^2 + b_3^2 \|u_t\|^2_{L^2(Q_s)} + \frac{b_2^2 \left(b_2^2 + b_4^2 + b_5^2\right) + 2b_1^2 b_2^2}{b_0^2} \|u\|^2_{L^2(Q_s)}
\end{equation}

For estimate the last term of the right-hand side of the inequality (29), we will be prove the inequality $\|u\|^2_{L^2(Q_s)} \leq 24T^2 \|u_t\|^2_{L^2(Q_s)}$, combining the last inequality and (29) we get
The inequality (30) it be
\[ \| \mathcal{S}_x^* u_{tt}(\cdot, s) \|^2_{L^2(0,1)} + \| u_{tt}(\cdot, s) \|^2_{L^2(0,1)} + \| u_t(\cdot, T) \|^2_{L^2(0,1)} \leq k \left[ \| \mathcal{S}_x^* u_{tt} \|^2_{L^2(Q_s)} + \| u_{tt} \|^2_{L^2(Q_s)} + \| u_t \|^2_{L^2(Q_s)} \right], \]
where \( k = \frac{\max(\beta b_2, (2b_2^2 + b_2) b_0 k(b_i, T))}{b_0 \min(1, \beta, \frac{b_0}{2})} \).

To continue, we introduce the new function \( v(x, t) = \int_t^T u_{\tau \tau} d\tau \), then \( u_t(x, t) = v(x, s) - v(x, t) \), and \( u_t(x, T) = v(x, s) \).

The inequality (30) it be
\[ \| \mathcal{S}_x^* u_{tt}(\cdot, s) \|^2_{L^2(0,1)} + \| u_{tt}(\cdot, s) \|^2_{L^2(0,1)} + (1 - 2k(T - s)) \| v(\cdot, s) \|^2_{L^2(0,1)} \leq 2k \left( \| \mathcal{S}_x^* u_{tt} \|^2_{L^2(Q_s)} + \| u_{tt} \|^2_{L^2(Q_s)} + \| v \|^2_{L^2(Q_s)} \right). \]

If \( s_0 > 0 \) satisfies \( (1 - 2k(T - s_0)) = \frac{1}{2} \), then the inequality (31) implies
\[ \| \mathcal{S}_x^* u_{tt}(\cdot, s) \|^2_{L^2(0,1)} + \| u_{tt}(\cdot, s) \|^2_{L^2(0,1)} + \| v(\cdot, s) \|^2_{L^2(0,1)} \leq 4k \left( \| \mathcal{S}_x^* u_{tt} \|^2_{L^2(Q_s)} + \| u_{tt} \|^2_{L^2(Q_s)} + \| v \|^2_{L^2(Q_s)} \right), \]
for all \( s \in [T - s_0, T] \). We denote
\[ Y(s) = \| \mathcal{S}_x^* u_{tt} \|^2_{L^2(Q_s)} + \| u_{tt} \|^2_{L^2(Q_s)} + \| v \|^2_{L^2(Q_s)}. \]
We get:
\[ Y'(s) = -\| \mathcal{S}_x^* u_{tt}(\cdot, s) \|^2_{L^2(0,1)} - \| u_{tt}(\cdot, s) \|^2_{L^2(0,1)} - \| v(\cdot, s) \|^2_{L^2(0,1)}. \]
Then and from (32) we obtain $-Y'(s) \leq 4kY(s)$. Then $-\frac{\partial}{\partial s} (Y(s) \exp(4ks)) \leq 0$.

Integrating this inequality on $(s, T)$ and taking into account that $Y(T) = 0$, we obtain $Y(s) \exp(4ks) \leq 0$. Then $Y(s) = 0$ for all $s \in [T - s_0, T]$. Then $\omega = 0$ almost everywhere in $Q_{T-s_0}$, proceeding in this way step by step, we prove that $\omega = 0$ almost everywhere in $Q$.

This achieves the proof of proposition. Now we return to prove the théorème. We will be prove that $R(L) = H$. Since $H$ is a Hilbert space, the equality $R(L) = H$ is true, if from

$$
(Lu, W)_H = (Lu, \omega)_{L^2(Q)} + (l_1u, \omega_1)_{L^2(0, l)} + (l_2u, \omega_2)_{L^2(0, l)} = 0,
$$

where $W = (\omega, \omega_1, \omega_2) \in R(L)^\perp$, we get $\omega \equiv 0, \omega_1 \equiv 0$ and $\omega_2 \equiv 0$ in $Q$, for any element of $D_0(L)$.

From (33) we obtain $\forall u \in D_0(L), (Lu, \omega)_{L^2(Q)} = 0$. Then by virtue of the 2nd proposition, we conclude that $\omega \equiv 0$. Then for (33), we obtain $(l_1u, \omega_1)_{L^2(0, l)} + (l_2u, \omega_2)_{L^2(0, l)} = 0$.

Since the quantities $l_1u$ and $l_2u$ can vanish independently and the ranges of the trace operators $l_1$ and $l_2$ are dense in the Hilbert space $L^2(0, l)$, then $\omega_1 = \omega_2 = 0$. Thus to conclude that $W = 0$. 
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