THE SUBSPACE PROBLEM
FOR WEIGHTED INDUCTIVE LIMITS
OF SPACES OF HOLOMORPHIC FUNCTIONS

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The aim of the present article is to solve in the negative a well-known open problem raised by Bierstedt, Meise and Summers in [BMS1] (see also [BM1]). We construct a countable inductive limit of weighted Banach spaces of holomorphic functions, which is not a topological subspace of the corresponding weighted inductive limit of spaces of continuous functions. As a consequence the topology of the weighted inductive limit of spaces of holomorphic functions cannot be described by the weighted sup-seminorms given by the maximal system of weights associated with the sequence of weights defining the inductive limit. The main step of our construction shows that a certain sequence space is isomorphic to a complemented subspace of a weighted space of holomorphic functions. To do this we make use of a special sequence of outer holomorphic functions and of the existence of radial limits of holomorphic bounded functions in the disc.

Weighted spaces and weighted inductive limits of spaces of holomorphic functions on open subsets of \(\mathbb{C}^N\) \((N \in \mathbb{N})\) arise in many fields like linear partial differential operators, convolution equations, complex and Fourier Analysis and distribution theory. Since the structure of general locally convex inductive limits is rather complicated and many pathologies can occur, the applications of weighted inductive limits have been restricted. The reason was that it did not seem possible to describe the inductive limit, its topology, and in particular a fundamental system of seminorms in a way that permits direct estimates and computations. In the theory of Ehrenpreis [E] of “analytically uniform spaces”, he needed that the topology of certain weighted inductive limits of spaces of entire functions, which are the Fourier-Laplace transforms of spaces of test functions or ultradistributions, has a fundamental system of weighted sup-seminorms. Berenstein and Dostal [BD] reformulated the problem in a more general setting and used the term ”complex representation”. This corresponds exactly with the term ”projective description” used by Bierstedt, Meise and Summers [BMS1] which is the one we will also utilize in this paper. In [BMS1] it was proved that countable weighted inductive limits of Banach spaces of holomorphic functions on arbitrary open subsets \(G\) of \(\mathbb{C}^N\) admit such a canonical projective description by weighted sup-seminorms whenever the linking maps between the generating Banach spaces are compact. This theorem extended previous work by B. A. Taylor [T] with a more functional analytic approach and was very satisfactory from the point of view of applications. It remained open whether the projective description theorem continued to

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hold for weighted inductive limits of spaces of holomorphic functions without any restriction on the linking maps. This problem is solved here.

1. Notation and preliminaries.

All the vector spaces are defined over the complex scalar field \( \mathbb{C} \). We denote by \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_+^0 \)) the space of strictly positive reals (resp. \( \mathbb{R}_+ \cup \{0\} \)).

Let \( V = (v_k)_{k=1}^{\infty} \) be a decreasing sequence of continuous strictly positive weight functions defined on an open subset \( G \) of \( \mathbb{C}^N, \; N \in \mathbb{N} \). We denote by \( V_C(G) \) (resp. \( V_H(G) \)) the inductive limits \( \text{ind}_{k} C v_k(G) \) and \( \text{ind}_{k} H v_k(G) \), where \( C v_k(G) \) (respectively, \( H v_k(G) \)) denotes the Banach space

\[
\{ f : G \to \mathbb{C} \text{ continuous (resp. holomorphic)} \mid \quad p_{v_k}(f) := \sup_{z \in G} v_k(z) |f(z)| < \infty \}.
\]

The canonical embedding \( V_H(G) \hookrightarrow V_C(G) \) is continuous, and it is a well known open problem, if the topologies of \( V_H(G) \) and \( V_C(G) \) coincide on \( V_H(G) \). (See [BM1], Section 1 or [BiBo4], Section 4, Problem 5.) This is a particular case of the so-called subspace problem for locally convex inductive limits.

In order to describe the topology of the weighted inductive limits \( V_C(G) \) and \( V_H(G) \), Bierstedt, Meise and Summers [BMS1] introduced the system of weights \( \mathcal{V} \), associated with the sequence \( V \),

\[
\mathcal{V} = \{ \sigma : G \to \mathbb{R}_+ \text{ continuous} \mid \forall k \in \mathbb{N} \exists C_k > 0 \text{ such that } \sigma \leq C_k v_k \}.
\]

The projective hulls \( C\mathcal{V}(G) \) (resp. \( H\mathcal{V}(G) \)) of \( V_C(G) \) (resp. \( V_H(G) \)) is the locally convex space

\[
\{ f : G \to \mathbb{C} \text{ continuous (resp. holomorphic)} \mid \quad p_{\sigma}(f) := \sup_{z \in G} \sigma(z) |f(z)| < \infty \text{ for all } \sigma \in \mathcal{V} \}.
\]

endowed with the locally convex topology defined by the seminorms \( p_{\sigma} \) as \( \sigma \) varies in \( \mathcal{V} \). Clearly the inclusions \( V_C(G) \hookrightarrow C\mathcal{V}(G) \) and \( V_H(G) \hookrightarrow H\mathcal{V}(G) \) are continuous. In [BMS1] it was proved that \( V_C(G) = C\mathcal{V}(G) \) and \( V_H(G) = H\mathcal{V}(G) \) hold algebraically and that the two spaces in each equality have the same bounded sets. Moreover one of the main results in [BMS1] shows that if \( V \) satisfies condition \( (S) \)

\[
(S) \quad \text{for all } k \text{ there is } l \text{ such that } v_l/v_k \text{ vanishes at infinity on } G
\]

then \( V_H(G) = H\mathcal{V}(G) \) holds topologically and \( V_H(G) \) is a topological subspace of \( V_C(G) \). In [BM2], [Ba] and [BiBo3] the topological identity \( V_C(G) = C\mathcal{V}(G) \) was characterized in terms of a condition \( (D) \) on the sequence \( V \). We present here an example showing that if condition \( (S) \) does not hold the space \( V_H(G) \) need not be a topological subspace of \( V_C(G) \) and \( V_H(G) = H\mathcal{V}(G) \) need not hold topologically.

In the construction of our example we need weighted inductive limits of spaces of sequences on \( \mathbb{N} \). We recall the notations from [BMS2]. We will denote here by
\(\Lambda = (\lambda_k)_{k=1}^{\infty}\) a decreasing sequence of strictly positive weights on \(\mathbb{N}\), \(\lambda_{nk} := \lambda_k(n)\) for \(k, n \in \mathbb{N}\). The corresponding weighted inductive limit is denoted by \(k_\infty = \text{ind}_k l_\infty(\lambda_k)\). The system of weights associated with \(\Lambda\) is denoted by \(\lambda\) and \(\lambda \in \bar{\lambda}\) if and only if \(\bar{\lambda}(n) > 0\) for every \(n \in \mathbb{N}\) and for every \(k \in \mathbb{N}\) there is \(C_k > 0\) with \(\lambda \leq C_k \lambda_k\) on \(\mathbb{N}\). The projective hull of the inductive limit \(k_\infty\) is denoted by \(K_\infty\), and it is the space

\[
\{ x = (x_n) | p_{\lambda}(x) := \sup_{n \in \mathbb{N}} \lambda(n)|x_n| < \infty \text{ for all } \lambda \in \bar{\lambda} \}.
\]

The spaces \(K_\infty\) and \(k_\infty\) always coincide algebraically and they have the same bounded sets, but there are examples of sequences \(\Lambda\) such that \(K_\infty\) and \(k_\infty\) do not coincide topologically, \(K_\infty\) has bounded sets which are not metrizable and it is not bornological. See [BMS2], [BiBo1], [K] and [V]. We refer to [BiBo4] for a survey article on spaces of type \(V\).

### 2. Main construction.

In this section we construct a sequence of weights \(W = (w_k)_{k=1}^{\infty}\) on an open bounded set \(G_1\) of \(\mathbb{C}\) such that the projective hull \(H\bar{W}(G_1)\) contains a complemented subspace isomorphic to a space of sequences \(K_\infty\) which is not bornological. Consequently, the space \(H\bar{W}(G_1)\) is not bornological and, hence, it does not coincide topologically with \(V\).

We first select a decreasing sequence \(\Lambda = (\lambda_k)_{k=1}^{\infty}\) of strictly positive functions \(\lambda_k(n) = \lambda_{nk}, n, k \in \mathbb{N}\), on \(\mathbb{N}\) such that \(1/n^2 < \lambda_{nk} \leq 1\) for all \(n, k\), the corresponding space \(K_\infty\) is not bornological and it contains bounded sets which are not metrizable. For example combine [K], Section 31.7 with [BM1], Theorem 9 and [BiBo1]. In this case the system of weights \(\lambda\) associated with \(\Lambda\) satisfies that for each \(p \in \bar{\lambda}\) there are \(\lambda \in \bar{\lambda}\) and \(C > 0\) with \(p \leq C \lambda\) and \(1/n^2 < \lambda(n) \leq 1\) for all \(n \in \mathbb{N}\). Indeed, given \(p \in \bar{\lambda}\), we select \(c_k > 0\) such that \(p \leq \inf_k c_k \lambda_k\). We put \(d_k = \max(c_k, 1)\) for all \(k\) and we set \(\lambda = \min(\inf_k d_k \lambda_k, \lambda_1) \in \bar{\lambda}\). Accordingly \(K_\infty\) has a fundamental system of seminorms \(P\) given by multiples of elements \(\lambda \in \bar{\lambda}\) satisfying \(1/n^2 < \lambda(n) \leq 1\) for all \(n \in \mathbb{N}\).

We denote \(G_1 = \{ z \in \mathbb{C} \mid 1/2 < |z| < 1, 0 < \arg z < \pi \}\) and we define the system \(W = (w_k)_{k=1}^{\infty}\) of weight functions on \(G_1\) by

\[
w_k(re^{i\theta}) = \hat{w}_k(\theta),
\]

where \(\hat{w}_k : [0, \pi] \to \mathbb{R}^+\) satisfies

\[
\hat{w}_k(\theta) = \lambda_{nk}
\]

for \(\theta \in I_n := [\theta_n - 1/(2^5 n^2), \theta_n + 1/(2^5 n^2)], \theta_n := 1/(2n)\), for all \(n \in \mathbb{N}\),

\[
\hat{w}_k(\theta_n + 1/(2^4 n^2)) = \hat{w}_k(0) = \hat{w}_k(\pi) = 1,
\]

for all \(n\), and \(\hat{w}_k\) is extended affinely for other \(\theta\).
Now we define a sequence of elements of $H(G_1)$ which will be essential in our construction. For all $n \in \mathbb{N}$, let $\varepsilon_n > 0$ be a number satisfying
\[ \varepsilon_n < 2^{-n-16}n^{-6}; \]
because of the choice of $(\lambda_{nk})$ we have, in particular,
\[ \varepsilon_n < 2^{-n-16}n^{-4}\lambda_{nk}. \]
For all $n$, let $e_n$ be an analytic function on the disc defined by
\[
e_n(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \varphi_n(\theta) d\theta \right),
\]
where $\varphi_n : [0, 2\pi] \to \mathbb{R}^+$ is the measurable function
\[
\varphi_n(\theta) := \begin{cases} 
1 & \text{for } \theta \in J_n := [\theta_n - \varepsilon_n, \theta_n + \varepsilon_n] \\
\varepsilon_n 2^{-m-4} & \text{for } \theta \in J_m, m \neq n \\
\varepsilon_n & \text{for other } \theta.
\end{cases}
\]
In fact, by $[R]$, Theorem 17.16, $e_n \in H^\infty$ and
\[
|e_n^*(e^{i\theta})| = \varphi_n(\theta),
\]
where $e_n^*(e^{i\theta}) := \lim_{r \to 1} e_n(re^{i\theta})$, holds for a.e. $\theta \in [0, 2\pi]$. We also denote by $e_n$ the restrictions of $e_n$ to $G_1$.

For $n \in \mathbb{N}$ we denote
\[ D_n := \{ z \in G_1 \mid |z - e^{i\theta}| < 1/(50n^2) \}, \quad C_n := G_1 \setminus D_n. \]
We have $D_n \subset \{ z \in G_1 \mid z = re^{i\theta}, \theta \in I_n \}$ and, moreover, $|e^{i\theta} - z| > 1/(2^n n^2)$ for $\theta \in J_n$ and $z \in C_n$. Since
\[
\int_0^{2\pi} (1 - |z|^2)/|e^{i\theta} - z|^2 d\theta = 2\pi,
\]
we can apply the Jensen inequality ($[R]$, Theorem 3.3; take exp for the convex function and $(1 - |z|^2)/(2\pi|e^{i\theta} - z|^2) d\theta$ for the probability measure) to get for $z \in C_n$
\[
|e_n(z)| \leq \exp \left( \frac{1}{2\pi} \Re \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \varphi_n(\theta) d\theta \right)
\]
\[
= \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \log \varphi_n(\theta) d\theta \right)
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \varphi_n(\theta) d\theta
\]
\[
\leq \frac{1}{2\pi} \int_{J_n} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta + \frac{\varepsilon_n}{2\pi} \int_{[0,2\pi] \setminus J_n} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta
\]
\[ \leq \pi^{-1}\varepsilon_n 2^{12}n^4 + \varepsilon_n \leq 2^{-4-n}. \]

Analogously, one can show \(|e_n(z)| \leq 1\) for all \(z \in G_1\).

In the proof of our main result in this section we need the following technical lemma. It shows that given an arbitrary weight function \(\overline{w}' \in \overline{W}\) we can choose a dominating weight function on \(G_1\) which has some specific properties.

**Lemma 1.** Given a weight function \(\overline{w}' \in \overline{W}\) we can find a weight function \(\overline{w} \in \overline{W}\) with the following properties:

1°. There exists \(C > 0\) such that \(C\overline{w}' \leq \overline{w} \leq 1\).

2°. If \(\overline{w}'' \in \overline{W}\) is defined as \(w_k\) except that \(\lambda_{nk}\) is replaced by \(1/n^2\), we have \(\overline{w}'' \leq \overline{w}\).

3°. The weight \(\overline{w}\) is constant on every \(D_n\) so that \(\overline{\lambda}(n) : N \rightarrow \mathbb{R}^+, \overline{\lambda}(n) := \overline{w}(z), z \in D_n\), satisfies \(1/n^2 \leq \overline{\lambda}(n)\) for all \(k\) and \(n\) and \(\overline{\lambda} \in \overline{\Lambda}\).

**Proof.** Let \(\varrho(n) := \max\{1/n^2, \sup\{\overline{w}'(z) | \arg(z) \in I_n\}\}\) and define \(\overline{w}^{(1)}\) as \(w_k\) but replace \(\lambda_{nk}\) by \(\varrho(n)\). Now it is easy to see that the weight \(\overline{w}\), defined by

\[ \overline{w}(z) = \min\{w_1(z), \max\{\overline{w}^{(1)}, \overline{w}'(z)\}\} \]

for \(z \in G_1\), has all the desired properties; the property 3° follows from the facts that \(\overline{w}^{(1)}\) and \(w_1\) are constants on \(D_n\) and \(\overline{w}^{(1)} \geq \overline{w}'\) on \(D_n\). \( \square \)

Our next lemma is essentially known.

**Lemma 2.** Let \(E\) and \(F\) be complete locally convex spaces. Let \(\psi : E \rightarrow F\) and \(\phi : F \rightarrow E\) be continuous linear maps such that \(\phi \psi : E \rightarrow E\) satisfies the following condition: there is a fundamental system of seminorms \(P\) on \(E\) and there is \(0 < \delta < 1\) such that

\[ p((\phi \psi - \text{id}_E) x) \leq \delta p(x) \quad \forall x \in E \forall p \in P. \]

Then \(E\) is isomorphic to a complemented subspace of \(F\).

**Proof.** We put \(B := \phi \psi - \text{id}_E\) and we define \(A : E \rightarrow E\) by

\[ Ax := \sum_{n=0}^{\infty} (-1)^n B^n x, \ x \in E. \]

Then \(A\) is a well defined continuous linear operator on \(E\). Indeed, for \(x \in E\), the series \(\sum_{n=0}^{\infty} (-1)^n B^n x\) is absolutely summable in \(E\) and, for \(p \in P, x \in E\), we have

\[ p(\sum_{n=0}^{\infty} (-1)^n B^n x) \leq \sum_{n=0}^{\infty} p(B^n x) \leq \sum_{n=0}^{\infty} \delta^n p(x) \leq C p(x). \]
Moreover \( \phi(\psi A) = \text{id}_E \). Indeed

\[
\phi(\psi A) = (\phi \psi) A = (\text{id}_E + B) \sum_{n=0}^{\infty} (-1)^n B^n = \text{id}_E.
\]

This implies that \( \psi A \phi \) is a projection on \( F \) whose image is isomorphic to \( E \) (see e.g. [H], pp. 122-123).

**Theorem 3.** The space \( \overline{HW}(G_1) \) contains a complemented subspace isomorphic to the non-bornological space \( K_\infty \). In particular \( \overline{HW}(G_1) \) does not coincide topologically with the weighted inductive limit \( WH(G_1) \).

**Proof.** We construct continuous linear maps \( \psi : K_\infty \to \overline{HW}(G_1) \) and \( \phi : \overline{HW}(G_1) \to K_\infty \) satisfying the assumptions of lemma 2.

First define \( \psi : K_\infty \to \overline{HW}(G_1) \) by \( \psi(a) := \sum_{n=1}^{\infty} a_n e_n \) for \( a = (a_n)_{n=1}^{\infty} \in K_\infty \). To see that \( \psi \) is well defined and continuous, we fix \( w' \in W \), and we select \( w \in W \) and \( \lambda \in \Lambda \) as in Lemma 1. If \( (a_n)_{n=1}^{\infty} \) is a sequence of scalars such that \( \sup_n \lambda(n) |a_n| = 1 \), we have \( |a_n| \leq n^2 \) for all \( n \). Every \( z_0 \in G_1 \) has a neighbourhood \( U \) which intersects at most one of the sets \( D_n \). It follows from the estimates of \( |e_n(z)| \) established after the definition of \( e_n \) that \( \sum a_n e_n(z) \) converges uniformly for \( z \in U \) and thus defines a holomorphic function of \( G_1 \). Moreover, denoting \( D := \bigcap_{n=1}^{\infty} C_n \), by the choice of \( (a_n) \), we have

\[
\sup_{z \in G_1} \overline{w}(z) \sum_{n=1}^{\infty} \lambda(n)^{-1} |e_n(z)| \
= \sup_{m \in \mathbb{N}} \sup_{z \in D_m} (\overline{w}(z) \sum_{n \in \mathbb{N}, n \neq m} \lambda(n)^{-1} |e_n(z)| + \overline{w}(z) \lambda(m)^{-1} |e_m(z)|) \
+ \sup_{z \in D} \overline{w}(z) \sum_{n=1}^{\infty} \lambda(n)^{-1} |e_n(z)| \leq 
\sup_{m \in \mathbb{N}} \left( \sum_{n=1}^{\infty} \lambda(n)^{-1} 2^{-4n} + \lambda(m) \lambda(m)^{-1} \right) 
+ \sum_{n=1}^{\infty} \lambda(n)^{-1} 2^{-4n} \leq 3 = 3 \sup_n \lambda(n) |a_n|.
\]

This shows that the map \( \psi \) is continuous.

To define \( \phi : \overline{HW}(G_1) \to K_\infty \) we need radial values of elements of \( \overline{HW}(G_1) \). We fix \( f \in \overline{HW}(G_1) \). There is \( k \in \mathbb{N} \) such that \( f \in Hw_k(G_1) \). Given \( n \in \mathbb{N} \),
since the weight \( w_k \) is constant in \( \{re^{i\theta} : \theta \in \mathcal{I}_n, 1/2 < r < 1 \} \), it follows that \( f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) exists a.e. for \( \theta \in \mathcal{I}_n \). See \( [HF] \), pp. 34 and ff. or \( [D] \), pp. 170. Accordingly we define \( f^*(e^{i\theta}) \) a.e. \( \theta \in \mathcal{I}_n \), which is an element of \( L^\infty(\mathcal{I}_n) \). Observe that the radial limits of \( e_n \) in each \( \mathcal{I}_n \) are the restriction of the ones of \( e_n \) in the disc, and that, if \( (a_n) \in K^\infty \), it follows from the inequalities established in the first part of this proof, that \( (\sum a_ne_n^*(e^{i\theta}))^n = \sum a_ne_n^*(e^{i\theta}) \). We set \( \chi_n(\theta) := e^{-i\arg e_n^*(e^{i\theta})} \) and we define \( \phi : HW(G_1) \to K^\infty \) by

\[
\phi(f) := ((2\varepsilon_n)^{-1}\int_{\mathcal{I}_n} f^*(e^{i\theta})\chi_n(\theta)d\theta)_{n \in \mathbb{N}}
\]

We first check that \( \phi \) is well defined and continuous. Given \( \bar{\lambda} \in \Lambda \) with \( 1/n^2 \leq \bar{\lambda}(n) \leq 1 \) for all \( n \), we define the weight \( \bar{w} \in W \) as \( w_k \), but replacing \( \lambda_{nk} \) by \( \lambda(n) \). If \( f \in HW(G_1) \) and \( n \in \mathbb{N} \) we have

\[
\bar{\lambda}(n)(2\varepsilon_n)^{-1}\int_{\mathcal{I}_n} f^*(e^{i\theta})\chi_n(\theta)d\theta | \leq \bar{\lambda}(n) \sup_{\mathcal{I}_n} |f^*(e^{i\theta})| \leq \sup_{z \in G_1} \bar{w}(z)|f(z)|.
\]

This shows \( \phi(f) \in K^\infty \) and the continuity of \( \phi \).

It remains to show that \( \phi \psi - id_{K^\infty} \) satisfies the condition in lemma 2. First observe that for all \( n \in \mathbb{N} \)

\[
(2\varepsilon_n)^{-1}\int_{\mathcal{I}_n} e_n^*(e^{i\theta})\chi_n(\theta)d\theta = 1.
\]

On the other hand, for \( n \in \mathbb{N} \) fixed, \( |e_n^*(e^{i\theta})| = \varepsilon_n 2^{-n-4} \) for all \( j \in \mathbb{N}, j \neq n \), a.e. \( \theta \in \mathcal{I}_n \) and the series \( \sum a_j\varepsilon_j \) converges absolutely for \( (a_n) \in K^\infty \). Indeed, select \( \bar{\lambda} \in \Lambda \) with \( 1/n^2 \leq \bar{\lambda}(n) \leq C \) for all \( n \in \mathbb{N} \). We have \( S := \sup_j \bar{\lambda}(j)|a_j| < \infty \) and \( \varepsilon_j < 2^{-j-16}j^{-6} < 2^{-j-16}j^{-4}\bar{\lambda}(j) \), then \( \sum |a_j|\varepsilon_j < S \sum 2^{-j-16} \). In particular

\[
\sum |a_j|\varepsilon_j \leq (1/8)\sup_n \bar{\lambda}(n)|a_n| \ \forall \bar{\lambda} \in \Lambda.
\]

Moreover

\[
(\phi \psi - id_{K^\infty})((a_n)) = ((2\varepsilon_n)^{-1}\sum_{j \neq n} a_j \int_{\mathcal{I}_n} e_n^*(e^{i\theta})\chi_n(\theta)d\theta)_{n \in \mathbb{N}}.
\]

If \( \bar{\lambda} \in \Lambda \) satisfies \( 1/n^2 \leq \bar{\lambda}(n) \leq 1 \) for all \( n \in \mathbb{N} \) and \( a \in K^\infty \) is such that \( p_{\bar{\lambda}}(a) = \sup_n \bar{\lambda}(n)|a_n| = 1 \) we have, for \( n \in \mathbb{N} \),

\[
|\bar{\lambda}(n)(2\varepsilon_n)^{-1}\sum_{j \neq n} a_j \int_{\mathcal{I}_n} e_n^*(e^{i\theta})\chi_n(\theta)d\theta | \leq \bar{\lambda}(n)(2\varepsilon_n)^{-1}\sum_{j \neq n} |a_j| \int_{\mathcal{I}_n} |e_n^*(e^{i\theta})\chi_n(\theta)|d\theta \leq
\]

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$$\bar{\lambda}(n) \sum_{j \neq n} |a_j| \sup_{\theta \in J_n} |e^*_j(e^{i\theta})| \leq 2^{-n-4}\bar{\lambda}(n) \sum_{j \neq n} |a_j| \varepsilon_j \leq 1/128.$$  

Therefore  

$$p_\bar{\lambda}(\phi \psi - id_{K_\infty})(a) \leq (1/128)p_\bar{\lambda}(a).$$

Since the multiples of the weights \(\bar{\lambda} \in \bar{\lambda}\) with \(1/n^2 \leq \bar{\lambda}(n) \leq 1\) for all \(n \in \mathbb{N}\) form a fundamental system of seminorms of the space \(K_\infty\), the conclusion follows from lemma 2.  \(\Box\)

3. The subspace problem.

In this section we denote \(G = G_1 \times \mathbb{C} \subset \mathbb{C}^2\) and we construct a decreasing sequence \(V = (v_k)_{k=1}^\infty\) of weight functions on \(G\) such that \(\mathcal{V}C(G) = C\mathcal{V}(G)\) holds topologically, but \(\mathcal{V}H(G)\) is not a topological subspace of \(\mathcal{V}C(G)\). Moreover the projective hull \(H\mathcal{V}(G)\) is not even a (DF)-space.

If \(z_1 \in G_1\), we write \(d(z_1)\) to denote the distance of \(z_1\) to the complement of \(G_1\).

We define the system \(V = (v_k)_{k=1}^\infty\) of weight functions on \(G\) by 

\[v_k(z_1, z_2) = w_k(z_1)u_k(z_1, |z_2|),\]

where \(u_k : G_1 \times \mathbb{R}^+ \to \mathbb{R}^+\) is defined by 

\[u_k(z_1, t) = \begin{cases} (1 + t)^{-\frac{k-1}{2k}}, & t \geq k + 1 \\ (1 + \frac{1}{d(z_1)} + t)^{-\frac{k-1}{2k}}, & t \leq k \end{cases}\]

and, for each fixed \(z_1\), \(u_k(z_1, t)\) is extended affinely for \(k < t < k + 1\). It is easy to see that the functions \(v_k\) are continuous on \(G\).

Bierstedt and Meise [BM2] introduced the following condition \((M)\) on the sequence \(V = (v_k)_{k=1}^\infty\) : for each \(k \in \mathbb{N}\) and each subset \(Y\) of \(G\) which is not relatively compact, there exists \(k' = k'(k, Y) > k\) with \(\inf_{y \in Y} v_{k'}(y)/v_k(y) = 0\).

They proved that this condition is equivalent to the fact that \(C\mathcal{V}(G)\) induces the compact open topology on each bounded subset and that condition \((M)\) implies the topological identity \(\mathcal{V}C(G) = C\mathcal{V}(G)\). Moreover, if \(V\) satisfies \((M)\), then \(H\mathcal{V}(G)\) is a Montel space. It was an open problem (see also [Bi]) whether \(\mathcal{V}H(G)\) is a Montel space when \(V\) satisfies \((M)\). In fact, if \(\mathcal{V}H(G)\) is a Montel space, then \(\mathcal{V}H(G) = H\mathcal{V}(G)\) holds topologically by a direct application of the Baernstien open mapping lemma as in [BMS1].

**Proposition 4.** The sequence \(V = (v_k)_{k=1}^\infty\) satisfies condition \((M)\). Consequently, \(\mathcal{V}C(G) = C\mathcal{V}(G)\) holds algebraically and topologically, and \(H\mathcal{V}(G)\) is a Montel space with metrisable bounded sets.

Proof. Let \(k \in \mathbb{N}\) be given and let \(Y\) be a subset of \(G\) which is not relatively compact. We have two possibilities: either 

\[(i) \exists (z^{(m)}) = ((z_1^{(m)}, z_2^{(m)})) \subset Y : \sup_m |z_2^{(m)}| = \infty, \text{ or} \]

\[(ii) \exists (z^{(m)}) = ((z_1^{(m)}, z_2^{(m)})) \subset Y : |z_2^{(m)}| \leq M \text{ and } \inf_m d(z_1^{(m)}) = 0.\]
In the first case it follows easily from the definition of $u_k$ that, taking $k' = k + 1,$

$$
\sup_{m \in \mathbb{N}} \frac{v_k(z^{(m)})}{u_{k+1}(z^{(m)})} \geq \sup_{m \in \mathbb{N}} \frac{u_k(z_1^{(m)}, |z_2^{(m)}|)}{u_{k+1}(z_1^{(m)}, |z_2^{(m)}|)} \\
\geq \sup_{m \in \mathbb{N}, |z_2^{(m)}| > k+2} |z_2^{(m)}| - \frac{k+1}{2k+1} + \frac{k}{z^{(k+1)}} = \infty.
$$

In case (ii), we choose $k' > M$ and $k' > k + 1$ to get

$$
\sup_{m \in \mathbb{N}} \frac{v_k(z^{(m)})}{u_{k'}(z^{(m)})} \geq \sup_{m \in \mathbb{N}} \left( \frac{1}{d(z_1^{(m)})} + 1 + M \right)^{-\frac{k+1}{2k+1}} \cdot \left( \frac{1}{d(z_1^{(m)})} \right)^{\frac{k}{2k+1}} = \infty. \quad \square
$$

It is very easy to see that every $f \in H\overline{\nabla}(G)$ is constant with respect to the second variable. Indeed, if $f \in H\overline{\nabla}(G)$, there are $k \in \mathbb{N}$ and $C > 0$ such that $p_{v_k}(f) \leq C$, so that for every fixed $z_1 \in G_1$

$$
C \geq \sup_{z \in \mathbb{C}} w_k(z_1)u_k(z_1, |z_2|)|f(z_1, z_2)|
$$

$$
\geq w_k(z_1) \sup_{z \in \mathbb{C}} (1 + \frac{1}{d(z_1)} + |z_2|)^{-1/2}|f(z_1, z_2)|.
$$

Now it is an elementary fact of complex analysis that a holomorphic $g : \mathbb{C} \to \mathbb{C}$ satisfying $\sup\{|z| + c_0|^{1/2}g(z)| \mid z \in \mathbb{C}\} < \infty$ for some constant $c_0$, must be constant. Accordingly $f$ must be constant with respect to $z_2$. Now we define $A : H\overline{\nabla}(G) \to H\overline{\nabla}(G_1)$ by $Af(z_1) = f(z_1, 0)$. To show that $A$ is well defined, we observe that

$$
p_{w_k}(Af) := \sup_{z \in G_1} w_k(z)|f(z, 0)| \leq \sup_{z \in G_1} w_k(z)C_ku_k(z, k+1)|f(z, 0)|
$$

$$
\leq C_kp_{v_k}(f)
$$

for all $f \in H\nu_k(G)$ and $C_k := (k + 2)^{(k-1)/(2k)}$.

Given $g \in H\overline{\nabla}(G_1)$, we define $\overline{g} : G \to \mathbb{C}$ by $\overline{g}(z_1, z_2) = g(z_1)$ for all $(z_1, z_2) \in G$. To show that $\overline{g} \in H\overline{\nabla}(G)$, we fix $k \in \mathbb{N}$ with $g \in H\nu_k(G_1)$, then we have the estimate

$$
\sup_{(z_1, z_2) \in G} v_k(z_1, z_2)|\overline{g}(z_1, z_2)| \leq \sup_{z_1 \in G_1} w_k(z_1)|g(z_1)|,
$$

since $0 \leq u_k \leq 1$. This shows that $A$ is bijective and that $A^{-1} : H\nu_k(G_1) \to H\nu_k(G)$ is continuous for every $k \in \mathbb{N}$. By the closed graph theorem for (LB)-spaces, this also yields that $A : V\overline{H}(G) \to WH(G_1)$ is a topological isomorphism. Moreover, $A^{-1} : H\overline{\nabla}(G_1) \to H\overline{\nabla}(G)$ is continuous. Indeed, if $\overline{g} \in \overline{V}$ is given, we define $\overline{w}(z_1) = \sup_{z_2 \in \mathbb{C}} \overline{g}(z_1, z_2)$ for $z_1 \in G_1$. We have for all $k$

$$
\overline{w}(z_1) = \sup_{z_2 \in \mathbb{C}} \overline{g}(z_1, z_2) \leq \sup_{z_2 \in \mathbb{C}} C_kv_k(z_1, z_2) \leq C_kw_k(z_1),
$$

...
hence \( \pi \in \mathbb{W} \). Moreover,

\[
p_\pi(f) = \sup_{(z_1, z_2) \in G} \pi(z_1, z_2) |f(z_1, 0)| \leq \sup_{z_1 \in G_1} \{ |f(z_1, 0)| \sup_{z_2 \in \mathbb{C}} \{ \pi(z_1, z_2) \} \} = p_\pi(Af).
\]

On the other hand, \( A : H\mathcal{V}(G) \to H\mathcal{W}(G_1) \) is not continuous. In fact, \( H\mathcal{W}(G_1) \) and \( H\mathcal{V}(G) \) cannot be isomorphic, since the first one contains a complemented subspace isomorphic to \( K_\infty \) by theorem 3; hence it contains bounded sets which are not metrizable; while every bounded subset of \( H\mathcal{V}(G) \) is metrizable by proposition 4.

**Theorem 5.** The space \( H\mathcal{V}(G) \) is not bornological, \( \mathcal{V}H(G) = H\mathcal{V}(G) \) does not hold topologically, \( \mathcal{V}H(G) \) is not a topological subspace of \( \mathcal{V}C(G) \) and it is not a Montel space. Moreover, \( H\mathcal{V}(G) \) is not a (DF)-space.

Proof. If \( H\mathcal{V}(G) \) is bornological (or equivalently if \( \mathcal{V}H(G) = H\mathcal{V}(G) \) holds topologically), then the linear map \( A^{-1} : H\mathcal{W}(G_1) \to \mathcal{V}H(G) \) is continuous. Consequently the identity \( \text{id} = AA^{-1} : H\mathcal{W}(G_1) \to \mathcal{W}H(G_1) \) is continuous. This implies \( H\mathcal{W}(G_1) \) is bornological; which contradicts theorem 3.

By proposition 4, \( \mathcal{V}C(G) = C\mathcal{V}(G) \) holds topologically. Since \( H\mathcal{V}(G) \) is clearly a topological subspace of \( C\mathcal{V}(G) \), we conclude that \( \mathcal{V}H(G) \) is not a topological subspace of \( \mathcal{V}C(G) \). If \( \mathcal{V}H(G) \) were Montel, we could apply directly Banach open mapping lemma (see e.g. [BPC]; 8.6.8(5)) to conclude that \( \mathcal{V}H(G) \) would be a topological subspace of \( \mathcal{V}C(G) \); a contradiction.

Finally assume \( H\mathcal{V}(G) \) is a (DF)-space. Since it is a complete (DF)-space which is Montel, \( H\mathcal{V}(G) \) is bornological (cf. [BPC]; 8.3.48). This is a contradiction. \( \Box \)

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