Questions About Extreme Points

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Abstract. We discuss the geometry of the unit ball—specifically, the structure of its extreme points (if any)—in subspaces of $L^1$ and $L^\infty$ on the circle that are formed by functions with prescribed spectral gaps. A similar issue is considered for kernels of Toeplitz operators in $H^\infty$.

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1. Introduction

Given a Banach space $X = (X, \| \cdot \|)$, we write

$$\text{ball}(X) := \{ x \in X : \| x \| \leq 1 \}.$$

An element $x$ of ball$(X)$ is said to be an extreme point thereof if it is not expressible as $x = \frac{1}{2}(u + v)$ with two distinct points $u, v \in$ ball$(X)$. Clearly, every extreme point $x$ of ball$(X)$ satisfies $\| x \| = 1$.

In what follows, the role of $X$ is played by certain function spaces on the circle $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$ which are defined in spectral terms. First of all, letting $m$ stand for the normalized arc length measure on $\mathbb{T}$, we introduce the (Lebesgue) spaces $L^p = L^p(\mathbb{T}, m)$ in the usual way, and we denote the standard $L^p$ norm by $\| \cdot \|_p$. Further, we recall that the Fourier coefficients of a function $f \in L^1$ are given by

$$\hat{f}(k) := \int_\mathbb{T} \overline{\zeta}^k f(\zeta) \, dm(\zeta), \quad k \in \mathbb{Z},$$

and the set

$$\text{spec } f := \{ k \in \mathbb{Z} : \hat{f}(k) \neq 0 \}$$

is called the spectrum of $f$.

For $1 \leq p \leq \infty$, the Hardy space $H^p$ is then defined by

$$H^p := \{ f \in L^p : \text{spec } f \subset \mathbb{Z}_+ \},$$

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where $\mathbb{Z}_+ := \{0, 1, 2, \ldots \}$. (We also introduce the notation $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{Z}_+$ for future reference.) As usual, we may view elements of $H^p$ as holomorphic functions on the open unit disk when convenient; see [15, Chapter II] for the underlying theory and basic properties of $H^p$ spaces.

More generally, given a nonempty set $\Lambda \subset \mathbb{Z}_+$, we consider the \textit{lacunary} (or \textit{punctured}) Hardy spaces

$$H^p(\Lambda) := \{ f \in L^p : \text{spec } f \subset \Lambda \}, \quad 1 \leq p \leq \infty,$$

normed by $\| \cdot \|_p$ as before. We are concerned with the extreme points of ball($H^p(\Lambda)$), so only the endpoint exponents $p = 1$ and $p = \infty$ are of interest. Indeed, for $1 < p < \infty$, the uniform convexity of $L^p$ implies that every unit-norm function is extreme.

In the classical setting, it is well known that the extreme points of ball($H^1$) are precisely the outer functions $F \in H^1$ with $\|F\|_1 = 1$, whereas the extreme points of ball($H^\infty$) are the functions $f \in H^\infty$ satisfying $\|f\|_\infty = 1$ and

$$\int_T \log(1 - |f|) \, dm = -\infty. \quad (1)$$

Both results can be found in [4]; alternatively, see [15, Chapter IV] or [17, Chapter 9].

Recently, the author was able to establish the corresponding extreme point criteria in $H^p(\Lambda)$, with $p = 1, \infty$, under the hypothesis that the underlying set $\Lambda$ is either small or large. Precisely speaking, it was assumed that either

$$\# \Lambda < \infty \quad (2)$$

or

$$\#(\mathbb{Z}_+ \setminus \Lambda) < \infty. \quad (3)$$

In the case of $H^1(\Lambda)$, the extreme points of the unit ball were described in [9] under condition (2), and in [10,11] under condition (3); the case of $H^\infty(\Lambda)$ was treated in [12] for both types of $\Lambda$'s.

Little seems to be known about the extreme points in $H^1(\Lambda)$ and $H^\infty(\Lambda)$ when neither (2) nor (3) holds. The questions we ask below are largely motivated by our curiosity in this regard. Sometimes, however, we find it natural to adopt a more general viewpoint. Namely, letting $\Lambda$ be a subset of $\mathbb{Z}$ (not necessarily of $\mathbb{Z}_+$), we extend our attention to the \textit{lacunary} $L^p$ spaces

$$L^p_\Lambda := \{ f \in L^p : \text{spec } f \subset \Lambda \}, \quad p = 1, \infty,$$

with norm $\| \cdot \|_p$.

2. Questions, Problems, and a Bit of Discussion

Here are some of the questions that puzzle us.
**Question 1.** Given a set $\Lambda \subset \mathbb{Z}$, which unit-norm functions from $L^1_{\Lambda}$ (if any) are extreme points for $\text{ball}(L^1_{\Lambda})$? Also, what are the extreme points of $\text{ball}(L^\infty_{\Lambda})$?

Clearly, of concern are the cases that do not reduce to the existing results on $H^p(\Lambda)$ as described above. Furthermore, it may well happen for a suitable $\Lambda$ that $\text{ball}(L^1_{\Lambda})$ has no extreme points at all. (A classical example is provided by taking $\Lambda = \mathbb{Z}$, in which case $L^1_{\Lambda}$ becomes the “full” $L^1$.) In fact, the mere existence of extreme points seems to present a nontrivial problem, which we now state and discuss in some detail.

**Question 2.** For which sets $\Lambda \subset \mathbb{Z}$ does $\text{ball}(L^1_{\Lambda})$ possess an extreme point? In particular, for which sets $\Lambda$ of the form

$$\Lambda = E \cup \mathbb{Z}_+, \quad \text{with } E \subset \mathbb{Z}_-, \quad (4)$$

does this happen?

Our interest in this last class of sets reflects an attempt to interpolate, so to speak, between $H^1$ and $L^1$ (i.e., between the cases $E = \emptyset$ and $E = \mathbb{Z}_-$), where two different things occur. Namely, the unit ball has plenty of extreme points in the former case, and none at all in the latter.

Now, let us say that a set $\Lambda \subset \mathbb{Z}$ is *periodic* if there is a positive integer $n$ such that

$$\Lambda + n = \Lambda \quad (5)$$

(as usual, $\Lambda + n$ stands for $\{k + n : k \in \Lambda\}$). For instance, any arithmetic progression in $\mathbb{Z}$ is obviously periodic.

To introduce another type of sets that we need here, we first recall the notation $M(\mathbb{T})$ for the space of all finite Borel complex measures on $\mathbb{T}$. Also, for $\mu \in M(\mathbb{T})$, we let $\text{spec } \mu$ denote the set of those indices $k \in \mathbb{Z}$ for which $\hat{\mu}(k) := \int_{\mathbb{T}} z^k d\mu$ is nonzero. Finally, a subset $\Lambda$ of $\mathbb{Z}$ is said to be a *Riesz set*, written as $\Lambda \in \mathcal{R}$, if every measure $\mu \in M(\mathbb{T})$ with $\text{spec } \mu \subset \Lambda$ is absolutely continuous with respect to $m$.

The classical F. and M. Riesz theorem (see, e.g., [15, Chapter II]) tells us that $\mathbb{Z}_+ \in \mathcal{R}$. A deeper study and further examples of Riesz sets can be found in [16, Part One, Chapter 1]. Among these examples are the $\Lambda$’s given by (4), where $E$ is one of the following sets:

$$\{-2^k : k \in \mathbb{N}\}, \quad \{-k^2 : k \in \mathbb{N}\}, \quad \{-p : p \text{ prime}\}.$$

The next result provides a bit of information on Question 2 (but is a far cry from answering it completely).

**Theorem 2.1.** Let $\Lambda \subset \mathbb{Z}$. If either $\Lambda$ is periodic or $\#(\mathbb{Z} \setminus \Lambda) < \infty$, then $\text{ball}(L^1_{\Lambda})$ has no extreme points. On the other hand, if $\Lambda \in \mathcal{R}$ then $\text{ball}(L^1_{\Lambda})$ does possess extreme points.

**Proof.** Suppose that $\Lambda$ is periodic, so that (5) holds for some $n \in \mathbb{N}$. Now let $f \in L^1_{\Lambda}$ be an arbitrary function with $\|f\|_1 = 1$. To show that $f$ is not an extreme point of $\text{ball}(L^1_{\Lambda})$, it suffices to find a real-valued function $h \in L^\infty$ such that $fh \in L^1_{\Lambda}$ and $h$ is nonconstant on the set $\{\zeta \in \mathbb{T} : f(\zeta) \neq 0\}$.

(The existence of such an $h$ is actually equivalent to the statement that $f$ is
nonextreme for ball($L^1_\Lambda$). We refer to [14, Chapter V, Section 9] or [11, Lemma 2.1], where the equivalence is proved in the context of $H^1$ and its subspaces; the case of a general subspace in $L^1$ is similar.) One possible choice is

$$h(z) = \text{Re}(z^n) = \frac{1}{2} (z^n + \overline{z}^n), \quad z \in \mathbb{T}.$$ 

Indeed, the assumption that $\text{spec } f \subset \Lambda$ implies, in conjunction with (5), that

$$\text{spec } (z^n f) \subset \Lambda \quad \text{and} \quad \text{spec } (\overline{z}^n f) \subset \Lambda.$$ 

Hence $\text{spec } (fh) \subset \Lambda$, so that $fh \in L^1_\Lambda$.

Now suppose that $\mathbb{Z} \setminus \Lambda$ is a finite set, say, of cardinality $N$. Thus,

$$\mathbb{Z} \setminus \Lambda = \{k_1, \ldots, k_N\}, \quad (6)$$

where the $k_j$’s are pairwise distinct integers. Once again, given an arbitrary unit-norm function $f$ in $L^1_\Lambda$, we prove that $f$ is a nonextreme point of ball($L^1_\Lambda$) by constructing a real-valued function $h \in L^\infty$ that satisfies $fh \in L^1_\Lambda$ and is nonconstant on the support of $f$. In fact, we claim that for a suitable nonzero vector

$$\alpha = (\alpha_1, \ldots, \alpha_{2N+1}) \in \mathbb{R}^{2N+1}, \quad (7)$$

the function

$$h_\alpha(z) := \text{Re} \left( \sum_{j=1}^{2N+1} \alpha_j z^j \right), \quad z \in \mathbb{T},$$

does the job. To check this, we associate with each vector (7) the numbers

$$\gamma_\nu(\alpha) := (fh_\alpha)(k_\nu), \quad \nu = 1, \ldots, N, \quad (8)$$

and consider the linear map $S: \mathbb{R}^{2N+1} \to \mathbb{R}^N$ defined by

$$S\alpha = (\text{Re } \gamma_1(\alpha), \text{Im } \gamma_1(\alpha), \ldots, \text{Re } \gamma_N(\alpha), \text{Im } \gamma_N(\alpha)).$$

The rank of $S$ is of course bounded by $2N$, and we deduce from the rank-nullity theorem (see, e.g., [2, p.63]) that the kernel of $S$ has dimension at least 1; in particular, the kernel is nontrivial. Now, if $\alpha \in \mathbb{R}^{2N+1}$ is a nonzero vector with $S\alpha = 0$, then the numbers (8) are all null, whence $fh_\alpha \in L^1_\Lambda$.

Also, the function $h_\alpha$ (which is obviously real-valued and bounded) is then nonconstant on any set $\mathcal{E} \subset \mathbb{T}$ with $m(\mathcal{E}) > 0$. Our claim is thereby verified.

Finally, suppose that $\Lambda \in \mathcal{R}$. Consider the space $C := C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$, and put

$$C^\Lambda := \{f \in C : \text{spec } f \subset \tilde{\Lambda}\},$$

where

$$\tilde{\Lambda} := \{-k : k \in \mathbb{Z} \setminus \Lambda\}.$$ 

As usual, we identify the dual of $C$ with $M := M(\mathbb{T})$, the functional induced by a measure $\mu \in M$ being $g \mapsto \int_{\mathbb{T}} g d\mu$. The dual of the quotient space $C/C^\Lambda$ is then $C^\Lambda^\perp$, the annihilator of $C^\Lambda$ in $M$. On the other hand,

$$(C^\Lambda)^\perp = \{\mu \in M : \text{spec } \mu \subset \Lambda\}.$$
This last set of measures embeds in $L^1$ (the $\mu$’s involved are absolutely continuous with respect to $m$ because $\Lambda \in \mathcal{R}$), so it coincides with $L^1_\Lambda$. Consequently, we have

$$(C/C^\Lambda)^* = (C^\Lambda)^\perp = L^1_\Lambda.$$  

The existence of extreme points in $\text{ball}(L^1_\Lambda)$ is now guaranteed by the Krein–Milman theorem; see, e.g., [17, Chapter 9].

Our next question is motivated by the conjecture—or perhaps a vague feeling—that if $\Lambda \subset \mathbb{Z}_+$ and if $\Lambda$ contains “most” of $\mathbb{Z}_+$, then the extreme points of $\text{ball}(H^1(\Lambda))$ are “not too far” from being outer functions. Indeed, when $\Lambda$ is all of $\mathbb{Z}_+$, our space is just $H^1$ and its extreme points are precisely the outer functions of norm 1; see [4]. Furthermore, it was shown in [11] (see also [10]) that if $\mathbb{Z}_+ \setminus \Lambda$ is a finite set, say with $\#(\mathbb{Z}_+ \setminus \Lambda) = N$, and if $f$ is an extreme point of $\text{ball}(H^1(\Lambda))$, then the inner factor of $f$ is necessarily a finite Blaschke product with at most $N$ zeros. In light of these facts, it seems tempting to conjecture that when $\mathbb{Z}_+ \setminus \Lambda$ is appropriately “thin” (or “sparse”) in $\mathbb{Z}_+$, the inner factors corresponding to the extreme points of $\text{ball}(H^1(\Lambda))$ are still fairly “tame,” in some sense or another. It would be nice to have a rigorous result to that effect.

**Question 3.** Suppose that $F$ is a suitably sparse (infinite) subset of $\mathbb{Z}_+$, and let $\Lambda = \mathbb{Z}_+ \setminus F$. What can we say about the inner factors of functions that arise as extreme points of $\text{ball}(H^1(\Lambda))$? To be more specific, what happens when $F$ is $\{2^k : k \in \mathbb{Z}_+\}$ or $\{2^{2k} : k \in \mathbb{Z}_+\}$?

On the other hand, the case of $H^1(\Lambda)$ where $\Lambda$ (rather than $\mathbb{Z}_+ \setminus \Lambda$) is a sparse—say, Hadamard lacunary—subset of $\mathbb{Z}_+$ is also worth studying; that would provide a natural extension to what was done in [9].

Turning to the $L^\infty$ part of Question 1, we now make a few observations pertaining to that setting. First we show that if $\Lambda$ is obtained from $\mathbb{Z}$ by removing a finite number of elements, then the extreme points in $L^\infty_\Lambda$ are precisely the unimodular functions, just as it happens for $L^\infty(=L^\infty_\mathbb{Z})$.

**Proposition 2.2.** Suppose that $\Lambda \subset \mathbb{Z}$ and $\#(\mathbb{Z} \setminus \Lambda) < \infty$. In order that a function $f \in L^\infty_\Lambda$ with $\|f\|_\infty = 1$ be an extreme point of $\text{ball}(L^\infty_\Lambda)$, it is necessary and sufficient that $|f| = 1$ a.e. on $\mathbb{T}$.

**Proof.** The sufficiency is obvious, since $L^\infty_\Lambda \subset L^\infty$ and every unimodular function is an extreme point of $\text{ball}(L^\infty)$. To prove the necessity, let (6) be an enumeration of $\mathbb{Z} \setminus \Lambda$. Now suppose $f$ is a unit-norm function in $L^\infty_\Lambda$ that satisfies $|f| < 1$ on a set of positive measure on $\mathbb{T}$. We then define $g := 1 - |f|$, so that $g$ is a non-null function in $L^\infty$; clearly, we also have $g \geq 0$ a.e. on $\mathbb{T}$. Further, with each vector

$$\beta = (\beta_0, \beta_1, \ldots, \beta_N) \in \mathbb{C}^{N+1}$$

we associate the polynomial

$$p_\beta(z) := \sum_{j=0}^N \beta_j z^j, \quad z \in \mathbb{T},$$

and consider the linear map $T: \mathbb{C}^{N+1} \to \mathbb{C}^N$ that acts by the rule
\[
T\beta = \left( \left( \hat{(g\beta)}(k_1), \ldots, \hat{(g\beta)}(k_N) \right) \right).
\]
The rank of $T$ being obviously bounded by $N$, we invoke the rank-nullity theorem to conclude that the kernel of $T$ is nontrivial.

Now, if $\beta \in \mathbb{C}^{N+1}$ is a nonzero vector with $T\beta = 0$, then the corresponding polynomial $p = p_\beta$ is non-null and satisfies $gp \in L^\infty_\Lambda \setminus \{0\}$. We may assume in addition that \( \|p\|_\infty = 1 \), which yields
\[
|f \pm gp| \leq |f| + |g| \leq |f| + g = 1
\]
almost everywhere on $T$. Consequently, $f + gp$ and $f - gp$ are two distinct points of $\text{ball}(L^\infty_\Lambda)$, and the identity
\[
f = \frac{1}{2}(f + gp) + \frac{1}{2}(f - gp)
\]
shows that $f$ fails to be extreme for the ball. \hfill \Box

At the same time, it is not hard to produce a set $\Lambda \subset \mathbb{Z}$ with $\sup \Lambda = \infty$ and $\inf \Lambda = -\infty$ for which $\text{ball}(L^\infty_\Lambda)$ has a much richer supply of extreme points. To this end, we first introduce a bit of terminology. Following [16], we say that a set $\Lambda(\subset \mathbb{Z})$ is a $\mathcal{D}$-set if it has the following property: whenever $\mu \in M(\mathbb{T})$ is a measure with $\text{spec} \mu \subset \Lambda$ whose total variation $|\mu|$ assigns zero mass to a set of positive $m$-measure (length) on $\mathbb{T}$, we have $\mu = 0$.

As a classical example of a $\mathcal{D}$-set, we mention $\mathbb{Z}_+$; indeed, an $H^1$ function that vanishes on a set $E \subset \mathbb{T}$ with $m(E) > 0$ must be null. For more sophisticated examples, we refer the reader to [16, Part One, Chapter 1]. In particular, it is shown there that if $E = \{-n^k : k \in \mathbb{N}\}$ with an integer $n \geq 2$, then $E \cup \mathbb{Z}_+$ is a $\mathcal{D}$-set.

**Proposition 2.3.** Let $\Lambda$ be a $\mathcal{D}$-set. Suppose further that $f \in L^\infty_\Lambda$ is a function with $\|f\|_\infty = 1$ for which
\[
m\left(\{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}\right) > 0.
\] Then $f$ is an extreme point of $\text{ball}(L^\infty_\Lambda)$.

**Proof.** We want to check that the only function $g \in L^\infty_\Lambda$ satisfying
\[
\|f + g\|_\infty \leq 1 \quad \text{and} \quad \|f - g\|_\infty \leq 1
\]
is $g \equiv 0$. Since
\[
|f|^2 + |g|^2 = \frac{1}{2} \left( |f + g|^2 + |f - g|^2 \right),
\]
it follows from (10) that $|g|^2 \leq 1 - |f|^2$ a.e. on $\mathbb{T}$. Consequently, $g = 0$ a.e. on
\[
E_f := \{\zeta \in \mathbb{T} : |f(\zeta)| = 1\},
\]
while (9) tells us that $m(E_f) > 0$. The desired conclusion that $g \equiv 0$ is now ensured by the hypothesis that $\Lambda$ is a $\mathcal{D}$-set. (To see why, identify $g$ with the measure $\mu_g \in M(\mathbb{T})$ given by $d\mu_g = g \, dm$. Note also that
\[
\text{spec} \mu_g = \text{spec} g \subset \Lambda
\]
and use the identity $|\mu_g|(\mathcal{E}_f) = \int_{\mathcal{E}_f} |g| \, dm = 0$ to deduce that $\mu_g$, and hence $g$, is null.) We are done. 

We mention in passing that, by a theorem of Amar and Lederer (see [1]), the unit-norm $H^\infty$ functions that obey (9) are precisely the exposed points of ball($H^\infty$). (Recall that, for a Banach space $X$, a point $x$ in ball($X$) is said to be exposed for the ball if there exists a functional $\phi \in X^*$ of norm 1 such that the set $\{y \in$ ball($X$) : $\phi(y) = 1\}$ equals $\{x\}$. It is well known, and easily shown, that every exposed point is extreme.) The following question might be of interest in this connection.

**Question 4.** Does there exist a set $\Lambda \subset \mathbb{Z}$ such that the extreme points of ball($L^\infty_\Lambda$) are characterized, among the unit-norm functions $f \in L^\infty_\Lambda$, by condition (9)?

From (9), we now turn to the weaker condition (1) which characterizes the extreme points of ball($H^\infty$). This time, we ask whether the criterion remains unchanged for suitably perturbed $H^\infty$-spaces of the form $L^\infty_\Lambda$, provided that $\Lambda$ is “not too different” from $\mathbb{Z}_+$. 

**Question 5.** For which sets $F \subset \mathbb{Z}_+$ is it true that (1) characterizes the extreme points $f$ of ball($H^\infty(\mathbb{Z}_+ \setminus F)$)? Also, for which sets $E \subset \mathbb{Z}_-$ does (1) characterize the extreme points $f$ of ball($L^\infty_\Lambda$), where $\Lambda = E \cup \mathbb{Z}_+$?

The function $f$ to be tested is, of course, always assumed to be a unit-norm element of the space in question. Now, if $\#F < \infty$, then the corresponding extreme point criterion is indeed given by (1) (see [12, Theorem 2.1]), and a similar fact is true if $\#E < \infty$. The same criterion should apply when $F$ (resp., $E$) is appropriately sparse in $\mathbb{Z}_+$ (resp., $\mathbb{Z}_-$), and we would like to see a reasonably sharp sparseness condition that ensures this.

We note, however, that taking $F$ to be the set of odd positive integers, we get $\mathbb{Z}_+ \setminus F = 2\mathbb{Z}_+$ and the extreme points $f$ of ball($H^\infty(2\mathbb{Z}_+)$) are again described by (1) (see [12] for a more detailed discussion of this example). Thus, $F$ need not be any thinner than $\mathbb{Z}_+ \setminus F$ in this situation.

Going back to our description of the extreme points of ball($H^1(\Lambda)$) and ball($H^\infty(\Lambda)$), as obtained previously in the cases (2) and (3), we now want to extend these results in yet another direction.

**Question 6.** What happens to the results just mentioned, as well as to their $L^p_\Lambda$ versions, in higher dimensions (say, on $\mathbb{T}^d$ in place of $\mathbb{T}$)? Also, what happens when passing from $\mathbb{T}$ to $\mathbb{R}$ (or $\mathbb{R}^d$)?

Of course, the lacunary Hardy spaces $H^p(\Lambda)$ (resp., the $L^p_\Lambda$ spaces) on the torus $\mathbb{T}^d$ should be defined appropriately in terms of a given set of multi-indices $\Lambda \subset \mathbb{Z}^d_+$ (resp., $\Lambda \subset \mathbb{Z}^d$). In particular, the analogue of (3) should now read $\#(\mathbb{Z}^d_+ \setminus \Lambda) < \infty$.

Moving to the real line, we fix a closed set $\Lambda \subset \mathbb{R}$ and define $L^p_\Lambda = L^p_\Lambda(\mathbb{R})$ with $p = 1, \infty$ as the space of all functions $f \in L^p(\mathbb{R})$ whose Fourier transform $\hat{f}$ vanishes on $\mathbb{R} \setminus \Lambda$ (when $p = \infty$, we interpret $\hat{f}$ in the sense of distributions). Now, as a natural counterpart of (2), we may impose the condition that $\Lambda$ be a compact set of positive length; the corresponding Paley–Wiener type
spaces $L^p_\Lambda$ are actually of special interest. In the simplest case where $\Lambda$ is an interval, the extreme (and exposed) points of ball($L^1_\Lambda(\mathbb{R})$) were characterized in [6]. A similar study of the “second simplest” case, where $\Lambda$ is made up of two disjoint intervals, was recently carried out in [18] (also in the $L^1$ setting), and little—if anything—is known for more general Paley–Wiener spaces of the $L^1_\Lambda$ type.

When $\Lambda$ is contained in $\mathbb{R}_+ := [0, \infty)$, we call $L^p_\Lambda(\mathbb{R})$ a lacunary Hardy space on $\mathbb{R}$ and we denote it by $H^p_\mathbb{R}(\Lambda)$. The usual Hardy spaces on $\mathbb{R}$ are thus $H^p_\mathbb{R}(\mathbb{R}) := H^p_\mathbb{R}(\mathbb{R}_+)$. The elements of $H^p_\mathbb{R}(\Lambda)$ are precisely the functions in $L^p_\Lambda(\mathbb{R})$ whose Poisson integral extension to the upper half-plane is holomorphic there.

We now mention a simple situation where the extreme points of ball($H^\infty_\mathbb{R}(\Lambda)$) are easy to describe. Namely, this happens when $\mathbb{R}_+ \setminus \Lambda$ is a bounded set, a condition that can be viewed as an analogue of (3). The next result provides a counterpart to [12, Theorem 2.1], where the disk version was treated.

**Proposition 2.4.** Suppose $\Lambda$ is a closed subset of $\mathbb{R}_+$ such that $\mathbb{R}_+ \setminus \Lambda$ is bounded. Assume also that $f \in H^\infty_\mathbb{R}(\Lambda)$ and $\|f\|_{\infty} = 1$. Then $f$ is an extreme point of ball($H^\infty_\mathbb{R}(\Lambda)$) if and only if

$$\int_{\mathbb{R}} \frac{\log(1 - |f(t)|)}{1 + t^2} \, dt = -\infty. \quad (11)$$

**Proof.** The “if” part follows from the inclusion $H^\infty_\mathbb{R}(\Lambda) \subset H^\infty_\mathbb{R}$, coupled with the fact that the extreme points of ball($H^\infty_\mathbb{R}$) are characterized by (11).

To prove the “only if” part, assume that (11) fails and let $G(\in H^\infty)$ be the outer function with modulus $1 - |f|$ on $\mathbb{R}$. Observe further that, for a suitably large number $A > 0$, the function

$$g(x) = g_A(x) := e^{iAx} G(x), \quad x \in \mathbb{R},$$

will be in $H^\infty_\mathbb{R}(\Lambda)$; indeed, the spectrum of $g$ is contained in $[A, \infty)$. Since $|f \pm g| \leq |f| + |g| = |f| + |G| = 1$ a.e. on $\mathbb{R}$, the identity

$$f = \frac{1}{2}(f + g) + \frac{1}{2}(f - g)$$

shows that $f$ is not an extreme point of ball($H^\infty_\mathbb{R}(\Lambda)$). \qed

Our last question deals with a different type of subspaces in $H^\infty$ (we are back to $\mathbb{T}$ now), where the structure of extreme points seems to be unclear. Given a function $\varphi$ in $L^\infty = L^\infty(\mathbb{T})$, we put

$$K_p(\varphi) := \{ f \in H^p : z\varphi f \in H^p \}, \quad 1 \leq p \leq \infty,$$

so that $K_p(\varphi)$ is the kernel in $H^p$ of the Toeplitz operator with symbol $\varphi$.

**Question 7.** Let $\varphi \in L^\infty$ and assume that $K_\infty(\varphi) \neq \{0\}$. What are the extreme points of ball($K_\infty(\varphi)$)?

When $\varphi = \overline{\theta}$ for an inner function $\theta$, $K_\infty(\varphi)$ becomes the model subspace $H^\infty \cap \theta zH^\infty$, and the problem of determining its extreme points was posed earlier in [8]. Furthermore, if $\varphi(z) = z^{N+1}$ for some $N \in \mathbb{Z}_+$, then $K_\infty(\varphi)$ coincides with $H^\infty(\Lambda_N)$, where $\Lambda_N := \{0, 1, \ldots, N\}$, and is formed by the
polynomials of degree at most $N$. In this last case, the extreme points are known (see [7] or [12]). On the other hand, the extreme points of $\operatorname{ball}(K_1(\varphi))$ admit a neat description for a general $\varphi \in L^\infty$; this can be found in [5].

We remark, in conclusion, that there are related geometric concepts—such as exposed or strongly extreme points of the unit ball—which are also worth studying in the context of lacunary $H^p$ or $L^p$ spaces, as well as in $K_p(\varphi)$, with $p = 1, \infty$. In fact, even for the usual (nonlacunary) $H^1$, the structure of its exposed points is far from being understood; the case of $H^1(\Lambda)$ is touched upon in [9, 11] for the sets $\Lambda$ that obey (2) or (3). As regards strongly extreme points, we refer to [3] for the definition and a characterization of these in the classical $H^p$ setting; see also [13] for further results involving subspaces of $H^\infty$.

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