SOLVING THE YAMABE PROBLEM BY AN ITERATIVE METHOD ON A SMALL RIEMANNIAN DOMAIN

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Abstract. We introduce an iterative scheme to solve the Yamabe equation $-a \Delta_g u + Su = \lambda u^{p-1}$ on small domains $(\Omega, g) \subset \mathbb{R}^n$ equipped with a Riemannian metric $g$. Thus $g$ admits a conformal change to a constant scalar curvature metric. The proof does not use the traditional functional minimization. Applications to the Yamabe problem on closed manifolds, manifolds with boundary, and noncompact manifolds are given in forthcoming papers.

1. Introduction

In this paper, we solve the Yamabe equation on small domains $(\Omega, g)$ in $\mathbb{R}^n$ equipped with a Riemannian metric $g$. We introduce an iterative method developed for hyperbolic operators [9], [10] and elliptic operators [18], with a long history in PDE theory dating back to [15, 16]. This local solution will be applied in forthcoming papers by the second author to global settings including closed manifolds, manifolds with boundary, and certain noncompact manifolds.

For a brief history, in 1960 Yamabe proposed the following generalization of the classical uniformization theorem for surfaces:

**The Yamabe Conjecture.** Given a compact Riemannian manifold $(M, g)$ of dimension $n \geq 3$, there exists a metric conformal to $g$ with constant scalar curvature.

Let $S = S_g$ be the scalar curvature of $g$, and let $\tilde{S}$ be the scalar curvature of the conformal metric $\tilde{g} = e^{2f}g$. Set $e^{2f} = u^{p-2}$, where $p = \frac{2n}{n-2}$ and $u > 0$. Then

$$\tilde{S} = u^{1-p} \left(-4 \cdot \frac{n-1}{n-2} \Delta_g u + Su\right),$$

where the Laplacian $\Delta_g = -d^*d$ is negative definite. Setting $a = 4 \cdot \frac{n-1}{n-2} > 0$, we have that $\tilde{g} = u^{p-2}g$ has constant scalar curvature $\lambda$ if and only if $u$ satisfies the Yamabe equation

$$-a \Delta_g u + Su = \lambda u^{p-1},$$

The solution of the Yamabe conjecture for closed manifolds involved three major steps (see [11] for a thorough treatment):

1. Yamabe, Trudinger and Aubin proved that if the minimum of the Yamabe functional $Y(g) = \int_M S \, dVol_g/(Vol(M, g))^{(n-2)/n}$ on a conformal class of metrics on a closed manifold $(M, g)$ is smaller than the minimum on the conformal class of the standard metric on $S^n$, then (2) has a solution;
2. Aubin then used Step 1 to prove that if $\dim M \geq 6$ and $(M, g)$ is not locally conformally flat, then (2) has a solution;
3. Finally, Schoen used the positive mass theorem to prove that (2) has a solution if $M$ has dimension 3, 4, 5 or is locally conformally flat, and $M$ is not conformal to the standard sphere.

There are also results for manifolds with boundary [4, 5, 6, 14] and open manifolds [3, 8, 19] with certain restrictions.
In contrast, our methods treat small domains in all dimensions greater than two. (To be honest, there is one place in the proof of Theorem 2.8 where an easy estimate depends on the dimension.) The main result is:

**Theorem.** Let \((\Omega, g)\) be a connected domain with smooth boundary in \(\mathbb{R}^n\), and let \(g\) be a Riemannian metric on \(\Omega\). If the \(g\)-volume and the Euclidean diameter of \(\Omega\) are sufficiently small, then there is a conformal change \(\tilde{g} = u^{p-2}g\) of \(g\) to a constant scalar curvature metric. On \(\partial\Omega\), we can arrange that \(\tilde{g} = g\).

The proof that \(u\) exists, is smooth, and is positive is contained in Theorems 2.3, 2.5, 3.1. The last statement of the Theorem is Remark 3.1.

The proof has technical advantages over previous proofs: (i) Yamabe obtained the Yamabe equation (2) as the Euler-Lagrange equation of \(Y(g)\), while we solve (2) directly, without discussing whether a minimum of \(Y(g)\) exists; (ii) In contrast to Yamabe and Trudinger’s arguments, which treated the subcritical case \(s < p\) of (2) before passing to the limit \(s = p\), we work directly with (2); (iii) Because our argument is local and does not involve minimizing a functional, it has applications to closed manifolds, manifolds with boundary, and noncompact manifolds. The main disadvantage is that we work with (2) directly, we cannot assume that \(u\) is positive as in previous approaches; the proof of positivity requires a separate argument.

The paper is organized as follows. In §2, we apply the iterative method to solve (2) on a small bounded domain \(\Omega \subset \mathbb{R}^n, n \not\equiv 2 \pmod{8}\), with constant Dirichlet boundary conditions (Theorem 2.3). The size of \(\Omega\) is determined in the proof. The dimensional restriction ensures that \(u^{p-1}\) is well defined, even if \(u\) is possibly negative. The main technical difficulty is that the nonlinearity in (2) involves the function \(x^{p-1}\), which is not globally Lipschitz on \(\mathbb{R}^+\); the easier case of an elliptic equation with globally Lipschitz nonlinearity is treated in [18]. The added difficulty is handled by familiar techniques: the Gagliardo-Nirenberg inequality, the Poincaré inequality, Li-Yau estimates for the first eigenvalue of \(\Delta_g\), and elliptic estimates. The solution obtained is a weak solution in the Sobolev space \(H^1(\Omega, g)\). In Theorem 2.8, we prove that the solution is in fact smooth, using arguments adapted from Yamabe and Trudinger’s work in the subcritical case.

In §3, we remove the dimension restriction by proving that the iterative method leads to a positive solution to the Yamabe equation (Theorem 3.1). Although we could have proven positivity in §2, the argument is fairly technical, so we have given it its own section.

Appendix A proves a technical result from §3, and Appendix B gives a table of the constants used in the article.

### 2. The Yamabe Problem on a Riemannian Domain

In this section, we start with an open, bounded subset \(\Omega \subset \mathbb{R}^n, n \not\equiv 2 \pmod{8}\), where we assume that \(\Omega\) is a smooth manifold with boundary. We apply an iterative method to solve the Yamabe equation (2) on \(\Omega\) with constant Dirichlet boundary conditions, where \(\Omega\) is equipped with a Riemannian metric \(g\) which extends smoothly to \(\Omega\). There are two steps: in Theorem 2.3 we prove that a weak solution exists, and in Theorem 2.8 we prove that the solution is in fact smooth.

We call \((\Omega, g)\) a Riemannian domain.

Thus we consider the boundary value problem:

\[-a\Delta_g u + Su = \lambda u^{p-1}\text{ in }\Omega; \quad u = c > 0\text{ on }\partial\Omega.\]  

(3)

Here \(a = \frac{4(n-1)}{n-2}, p = \frac{2n}{n-2}, S\) is the scalar curvature of \(g\), and \(c\) is a fixed positive constant. \(\lambda\) is an unspecified nonzero constant. When \(n \not\equiv 2 \pmod{8}\), \(\frac{n+2}{n-2}\) in lowest terms has odd denominator, so for real valued functions \(u\), \(u^{p-1} = u^{\frac{n+2}{n-2}}\) is well-defined. For the rest of this section, we assume this condition on \(n\).
On $\Omega$, we have $g = g_{ij} dx^i \otimes dx^j$ in the standard coordinates on $\mathbb{R}^n$, with volume form $d\text{Vol}_g = \sqrt{\text{det}(g_{ij})} dx^1 \ldots dx_n$. $(v, w)_g$ and $|v|_g = (v, v)^{1/2}_g$ denote the inner product and norm with respect to $g$.

We define two equivalent versions of the $L^p$ norms and two equivalent versions of the Sobolev norms on $(\Omega, g)$.

**Definition 2.1.** Let $(\Omega, g)$ be a Riemannian domain. For real valued functions $u$, we set:

1. For $1 \leq p < \infty$,
   
   $L^p(\Omega)$ is the completion of $\left\{ u \in C_c^\infty(\Omega) : \|u\|_p^p := \int_\Omega |u|^p dx < \infty \right\}$,

   $L^p(\Omega, g)$ is the completion of $\left\{ u \in C_c^\infty(\Omega) : \|u\|_{p, g}^p := \int_\Omega |u|^p d\text{Vol}_g < \infty \right\}$.

2. For $\nabla$ the Levi-Civita connection of $g$, and for $u \in C^\infty(\Omega)$,
   
   $|\nabla^k u|^2_g := (\nabla^{\alpha_1} \ldots \nabla^{\alpha_k} u)(\nabla_{\alpha_1} \ldots \nabla_{\alpha_k} u)$.

In particular, $|\nabla^0 u|^2_g = |u|^2_g$ and $|\nabla^1 u|^2_g = |\nabla u|^2_g$.

3. For $s \in \mathbb{N}, 1 \leq p < \infty$,
   
   $W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \|u\|_{W^{s,p}(\Omega)}^p := \int_\Omega \sum_{j=0}^s |D_j u|^p dx < \infty \right\}$,

   $W^{s,p}(\Omega, g) = \left\{ u \in L^p(\Omega, g) : \|u\|_{W^{s,p}(\Omega, g)}^p = \sum_{j=0}^s \int_\Omega |D_j u|^p d\text{Vol}_g < \infty \right\}$.

Here $|D_j u|^p := \sum_{|\alpha| = j} |\partial^\alpha u|^p$ in the weak sense. Similarly, $W^{s,p}_0(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the $W^{s,p}$-norm. In particular, $H^s := W^{s,2}(\Omega)$ and $H^s(\Omega, g) := W^{s,p}(\Omega, g)$ are the usual Sobolev spaces, and we similarly define $H^s_0(\Omega), H^s_0(\Omega, g)$.

**Remark 2.1.** It is clear that the two $L^p$ norms are equivalent, the two $H^s$ norms are equivalent, and the two $W^{s,p}$ norms are equivalent on $\Omega$. Thus there are constants $C_2 > C_1 > 0$ such that

\begin{align*}
    C_1 \|u\|_{H^s(\Omega)} &\leq \|u\|_{H^s(\Omega, g)} \leq C_2 \|u\|_{H^s(\Omega)} \\
    C_1 \|u\|_{W^{s,p}(\Omega)} &\leq \|u\|_{W^{s,p}(\Omega, g)} \leq C_2 \|u\|_{W^{s,p}(\Omega)} \\
    C_1 \|u\|_{L^p(\Omega)} &\leq \|u\|_{L^p(\Omega, g)} \leq C_2 \|u\|_{L^p(\Omega)}.
\end{align*}

In Riemannian normal coordinates centered at $p \in \Omega$, $g$ agrees with the Euclidean metric up to terms of order $O(r^2)$, where $r$ is the distance to $p$. Thus there exists a neighborhood $U_p$ of $p$ on which we may assume $C_1 \geq 1/2, C_2 \leq 2$ in (3) for $u \in C_c^\infty(U_p)$. Since we will eventually assume that the diameter of $\Omega$ is sufficiently small, and since $C_2/C_1$ for $\Omega'$ is smaller than $C_2/C_1$ for $\Omega$ when $\Omega' \subset \Omega$, we can assume that $C_2/C_1 \in [1, 4]$.

The main tools used to solve (3) are (i) the version of the Gagliardo-Nirenberg (GN) 1 warning interpolation inequality for the zero trace case; (ii) a version of the extension theorem; (iii) the Poincaré inequality with respect to Laplace-Beltrami operator.

**Proposition 2.1. (GN trace zero case)** [2] Thm. 3.70 Let $q, r, l$ be real numbers with $1 \leq q, r, l \leq \infty$, and let $j, m$ be integers with $0 \leq j < m$. Define $\alpha$ by solving

\[ \frac{1}{l} = \frac{j}{n} + \alpha \left( \frac{1}{r} - \frac{m}{n} \right) + \frac{1 - \alpha}{q}, \]
as long as \( l > 0 \). If \( \alpha \in \left[ \frac{j}{m}, 1 \right] \), then there exists a constant \( C_{m,j,q,r} \), depending only on \( n, m, j, q, r, \alpha \) such that for all \( u \in C_c^\infty (\mathbb{R}^n) \),
\[
\| \nabla^j u \|_{L^\infty (\mathbb{R}^n)} \leq C_{m,j,q,r,\alpha} \| \nabla^m u \|_{L^\infty (\mathbb{R}^n)}^\alpha \| u \|_{L^\infty (\mathbb{R}^n)}^{1-\alpha}.
\] (8)

(If \( r = \frac{n}{m-j} \neq 1 \), then (8) is not valid for \( \alpha = 1 \).)

**Remark 2.2.** For fixed \( n, m, j, q, r, \alpha \), we can leave \( C_{m,j,q,r,\alpha} \) unchanged in (8) if we restrict the support of \( u \) to a domain.

**Proposition 2.2.** (Extension Operator) [11 Thm. 5.22] Let \( \Omega \) be a bounded, open, connected subset of \( \mathbb{R}^n \) with smooth boundary. Then there exists a bounded linear operator \( E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n) \), the extension operator, such that \( Eu \) has compact support, \( Eu = u \) a.e. on \( \Omega \), and
\[
\| Eu \|_{W^{k,p}(\mathbb{R}^n)} \leq K(k,p,\Omega) \| u \|_{W^{k,p}(\Omega)}.
\] (9)

If \( \Omega \) is fixed, we write \( K(k,p,\Omega) = K(k,p) \). Note that \( K(k,p) \geq 1 \).

**Proposition 2.3.** (Poincaré inequality) Let \((\tilde{M}, g)\) be a compact manifold with smooth boundary and with interior \( M \). Let \( \lambda_1 \) be the first non-zero eigenvalue of \(-\Delta_g\) on \( u \in H^1_0(M,g) \). We have
\[
\| u \|_{L^2(M,g)} \leq \lambda_1^{-1/2} \| \nabla g u \|_{L^2(M,g)}.
\] (10)

Moreover, \( \lambda_1^{-1/2} \) is the optimal constant for (10) holds.

To control \( \lambda_1 \) here, we need the following theorem of Li and Yau.

**Theorem 2.1.** [13 Thm. 7] Let \((\tilde{M}, g)\) be a compact manifold with smooth boundary, let \( r_{inj} \) be the injectivity radius of \( M \), and let \( h_g \) be the minimal of the mean curvature of \( \partial M \). Choose \( K \geq 0 \) such that \( \text{Ric}_g \geq -(n-1)K \). For \( \lambda_1 \) as in Proposition 2.3,
\[
\lambda_1 \geq \frac{1}{\gamma} \left( \frac{1}{4(n-1)r_{inj}^2} (\log \gamma)^2 - (n-1)K \right),
\] (11)
where
\[
\gamma = \max \left\{ \exp[1 + (1 - 4(n-1)K^2)^{1/2}], \exp[-2(n-1)K r_{inj}] \right\}.
\] (12)

**Remark 2.3.** (i) We will apply Proposition 2.3 and Theorem 2.1 only in the case \( \tilde{M} = \Omega \).

(ii) As in Remark 2.1 in Riemannian normal coordinates centered at \( p \in \Omega \), \( g \) agrees with the Euclidean metric up to terms of order \( O(r^2) \). Thus if \( \Omega \) is a \( g \)-geodesic ball of radius \( r \), the mean curvature of \( \partial \Omega \) is close \( (n-1)/r \), the mean curvature of a Euclidean \( r \)-ball in \( \mathbb{R}^n \). In (12), as \( r \to 0 \), \( K \) can be taken to be unchanged (since \( g \) is independent of \( r \)), \( r_{inj} \to 0 \), and \( h \cdot r_{inj} \to n-1 \). Thus \( \gamma \to e^2 \), the right hand side of (11) goes to infinity as \( r \to 0 \), and \( \lambda_1 \to \infty \).

If \( \Omega \) is a general Riemannian domain with a small enough injectivity radius, then \( \Omega \) sits inside a \( g \)-geodesic ball \( \Omega'' \) of small radius. By the Rayleigh quotient characterization of \( \lambda_1 \), we have \( \lambda_1^{\Omega''} \leq \lambda_1^{\Omega} \). Thus for all Riemannian domains \((\Omega, g)\), \( \lambda_1^{-1} \to 0 \) as the radius of \( \Omega \) goes to zero.

We recall the basic elliptic estimate for the Dirichlet problem.

**Theorem 2.4.** [17 Ch. 5, Thm. 1.3] Let \((\Omega, g)\) be a Riemannian domain, and let \( L \) be a second order elliptic operator of the form \( Lu = -\Delta_g u + Xu \) where \( X \) is a first order differential operator with smooth coefficients on \( \Omega \). For \( f \in L^2(\Omega, g) \), a solution \( u \in H^1_0(\Omega, g) \) to \( Lu = f \) in \( \Omega \) with \( u \equiv 0 \) on \( \partial \Omega \) belongs to \( H^2(\Omega, g) \), and
\[
\| u \|_{H^2(\Omega, g)} \leq C^* \left( \| f \|_{L^2(\Omega, g)} + \| u \|_{H^1(\Omega, g)} \right).
\] (13)

\( C^* = C^*(L, \Omega, g) \) depends on \( L \) and \((\Omega, g)\).
Remark 2.4. If $u, f$ have support in $\Omega' \subset \Omega$, we can set $C^*(L, \Omega', g|_{\Omega'}) = C^*(L, \Omega, g)$ in (13), since for $u \in H^2(\Omega', g)$, we have $\|u\|_{H^2(\Omega', g)} = \|u\|_{H^2(\Omega, g)}$, etc.

We are now ready to prove the main theorem of this section by an iteration scheme.

Theorem 2.3. Let $(\Omega, g)$ be Riemannian domain in $\mathbb{R}^n$, $n \neq 2 \pmod{8}$, with $C^\infty$ boundary, and with $\text{Vol}_g(\Omega)$ and the Euclidean diameter of $\Omega$ sufficiently small. Then the Yamabe equation (3) has a real solution $u \in H^1(\Omega, g)$ in the weak sense.

To be more precise, we start with $(\Omega, g)$ and as necessary pass to sub-Riemannian domains $(\Omega', g|_{\Omega'}) \subset (\Omega, g)$, with $C^\infty$ boundary, such that $\text{Vol}_g(\Omega')$ and $r_{inj}(\Omega')$ are sufficiently small. This “smallness” is discussed after the proof in Remark 2.5. Throughout the proof, we discuss the weak form of linear elliptic PDE, i.e., we discuss the form $B[u, v] = (f, v)_g, \forall v \in H^1_0(\Omega)$ where $B[u, v] = (-a\Delta gu, v)_g$ and $(h, k)_g$ is the $L^2(\Omega, g)$ inner product.

Proof. We first consider the linear elliptic PDE with constant boundary condition:

$$au_0 - a\Delta gu_0 = f \quad \text{in } \Omega; \; u_0 = c \quad \text{on } \partial \Omega.$$  

(14)

By taking $\bar{u}_0 = u_0 - c$, (14) is equivalent to

$$a\bar{u}_0 - a\Delta g\bar{u}_0 = f - ac \quad \text{in } \Omega; \; \bar{u}_0 \equiv 0 \quad \text{on } \partial \Omega.$$  

(15)

For any $f \in L^2(\Omega, g)$, the Lax-Milgram Theorem implies that (15) has a unique solution $\bar{u}_0 \in H^1_0(\Omega, g)$. Since $C^\infty(\Omega)$ is dense in $H^1_0(\Omega, g)$, we can assume either that $\bar{u}_0 \in C^\infty(\Omega)$ or $\bar{u}_0 \in H^1_0(\Omega, g)$.

By the Poincaré inequality and (13), we observe that

$$\langle a\bar{u}_0 - a\Delta g\bar{u}_0, \bar{u}_0 \rangle_g = (f - ac, \bar{u}_0)_g \Rightarrow \|\bar{u}_0\|^2_{H^1(\Omega, g)} \leq \frac{1}{a} \|f - ac\|_{L^2(\Omega, g)} \|\bar{u}_0\|_{L^2(\Omega, g)}$$

$$\Rightarrow \|\bar{u}_0\|^2_{H^1(\Omega, g)} \leq \frac{1}{a} \|f - ac\|^2_{L^2(\Omega, g)} \lambda_1^{-\frac{1}{2}} \|
abla \bar{u}_0\|_{L^2(\Omega, g)} \leq \frac{1}{a} \|f - ac\|^2_{L^2(\Omega, g)} \lambda_1^{-\frac{1}{2}} \|\bar{u}_0\|_{H^1(\Omega, g)}$$

$$\Rightarrow \|\bar{u}_0\|_{H^1(\Omega, g)} \leq \frac{1}{a} \lambda_1^{-\frac{1}{2}} \|f - ac\|_{L^2(\Omega, g)}.$$  

(16)

(The first implication uses $\|\bar{u}_0\|^2_{H^1(\Omega, g)} = (\bar{u}_0, \bar{u}_0)_g + (\nabla \bar{u}_0, \nabla \bar{u}_0)_g = (\bar{u}_0, \bar{u}_0)_g + (\Delta g \bar{u}_0, \bar{u}_0)_g$.)

Applying Theorem 2.2 to (15), we have

$$\|u_0\|_{H^2(\Omega, g)} \leq C^*(\|f - ac\|_{L^2(\Omega, g)} + \|\bar{u}_0\|_{H^1(\Omega, g)}) \leq C^* \left(1 + \frac{1}{a^2} \lambda_1^{-\frac{1}{2}}\right) \|f - ac\|_{L^2(\Omega, g)}$$

$$\Rightarrow \|u_0\|_{H^2(\Omega, g)} \leq \|f - ac\|_{L^2(\Omega, g)} + \|c\|_{H^2(\Omega, g)} := C\|f - ac\|_{L^2(\Omega, g)} + c \cdot \bar{C}^{2}.$$  

(17)

It follows that $\bar{u}_0 \in H^1_0(\Omega, g) \cap H^2(\Omega, g)$. In particular,

$$\bar{C} := \text{Vol}_g(\Omega)^{\frac{1}{2}}$$

decreases as $\text{Vol}_g(\Omega)$ shrinks. Furthermore, $C = C(-\Delta g, \Omega, g)$ is nonincreasing as $\Omega$ shrinks. Indeed, as $\Omega$ shrinks, $C = C^*(1 + a^{-1} \lambda_1^{-1/2})$ is bounded above by Remarks 2.3(ii) and 2.4. For fixed $c$, we can take $\Omega$ of small enough $g$-volume and choose $f$ so that $C\|f - ac\|_{L^2(\Omega, g)} + c \cdot \bar{C}^{2} \leq 1$, so by (16)

$$\|u_0\|_{H^2(\Omega, g)} \leq 1, \|\bar{u}_0\|_{H^2(\Omega, g)} < 1.$$  

(18)

We apply the iteration scheme by defining $u_k$ to be the solution of

$$au_k - a\Delta gu_k = au_{k-1} - S u_{k-1} + \lambda u_{k-1}^{-1} \quad \text{in } (\Omega, g), \; u_k = c \quad \text{on } \partial \Omega, \; k = 1, 2, \ldots.$$  

(19)

The first main step is to prove the boundedness of $u_k$ in $H^2(\Omega, g)$ (see (31)). For

$$\tilde{u}_k = u_k - c,$$  

(20)
\[ u_k - \Delta_g \bar{u}_k = au_{k-1} - Su_{k-1} + \lambda u_{k-1}^p - ac \text{ in } (\Omega, g), \bar{u}_k = 0 \text{ on } \partial \Omega, k = 1, 2, \ldots \] \hfill \tag{21}\]

By Lax-Milgram, for \( k = 1 \), Proposition 2.1 has a unique solution \( \bar{u}_1 \). As with \( \bar{u}_0 \), for the same \( C \) as in (16), we obtain
\[ \| \bar{u}_1 \|_{H^2(\Omega, g)} \leq C \| au_0 - Su_0 + \lambda u_0^p - ac \|_{L^2(\Omega, g)} \]
\[ \leq aC \| u_0 \|_{L^2(\Omega, g)} + C \sup |S| \| u_0 \|_{L^2(\Omega, g)} + C |\lambda| \| u_0^{p-1} \|_{L^2(\Omega, g)} + acC \tilde{C}_\frac{1}{2}. \]
\hfill \tag{22}

We now apply Proposition 2.1 to bound \( \| u_0^{p-1} \|_{L^2(\Omega, g)} \) in (22) by \( \| u_0 \|_{H^2(\Omega, g)} \). Since \( C^\infty(\Omega) \) is dense in \( H^2(\Omega, g) \), we may assume that \( u_0 \in C^\infty(\Omega) \cap H^2(\Omega, g) \).

We start with
\[ \| u_0^{p-1} \|^2_{L^2(\Omega, g)} = \int_\Omega \| u_0^{p-1} \|^2 \, dVol_g = \left( \int_\Omega |u_0|^{2p-2} \, dVol_g \right)^{\frac{2p-2}{p-2}} = \| u_0 \|^{2p-2}_{L^{2p-2}(\Omega)}. \]

For \( l = 2p - 2, q = r = 2, j = 0, m = 2 \) in (7), \( \alpha = \frac{n}{n+2} \in [0, 1) \), so we can apply (8) and (9) to the compactly supported \( \text{Eu} \) of \( u \) and obtain
\[ \| u_0 \|_{L^{2p-2}(\Omega, g)} \leq C_2 \| u_0 \|_{L^{2p-2}(\Omega)} \leq C_2 \| \text{Eu}_0 \|_{L^{2p-2}(\mathbb{R}^n)} \leq C_2 C_0 \| \nabla^2 \text{Eu}_0 \|_{L^{\frac{n}{n+2}}(\mathbb{R}^n)} \| \text{Eu}_0 \|_{L^{\frac{2p}{p-2}}(\mathbb{R}^n)} \]
\[ \leq C_2 C_0 \| \text{Eu}_0 \|_{H^2(\mathbb{R}^n)} \leq C_2 C_0 K(2, 2) \| u_0 \|_{H^2(\Omega)} \leq C_0 K(2, 2) \frac{C_2}{C_1} \| u_0 \|_{H^2(\Omega, g)}. \]
\hfill \tag{23}

Here we can take \( C_0 = C_{2,0,2,2, \frac{n}{n+2}} \) as in (8), but for later purposes we set
\[ C_0 := \max \left\{ C_{2,0,2,2, \frac{n}{n+2}}, C_{1,0,2,2, \frac{n}{n+2}} \right\}. \]
\hfill \tag{24}

Hence
\[ \| u_0^{p-1} \|^2_{L^2(\Omega, g)} = \| u_0 \|^{p-1}_{L^{2p-2}(\Omega, g)} \leq C_0^{p-1} K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \| u_0 \|^{p-1}_{H^2(\Omega, g)}. \]
\hfill \tag{25}

We cannot directly apply the Poincaré inequality to the first two terms on the right hand side of (22), since \( u_0 \) does not have zero trace. This is not a serious problem, since
\[ aC \| u_0 \|_{L^2(\Omega, g)} \leq aC \| \bar{u}_0 \|_{L^2(\Omega, g)} + acC \text{Vol}^{\frac{1}{2}}_g \leq aC \lambda_1^{-\frac{1}{2}} \| \nabla \bar{u}_0 \|_{L^2(\Omega, g)} + acC \tilde{C}_\frac{1}{2} \]
\[ = aC \lambda_1^{-\frac{1}{2}} \| \nabla u_0 \|_{L^2(\Omega, g)} + acC \tilde{C}_\frac{1}{2}. \]
\hfill \tag{26}

Plugging (25) and (26) into (22), and using (13), (18), we get
\[ \| u_1 \|_{H^2(\Omega, g)} \leq \| \bar{u}_1 \|_{H^2(\Omega, g)} + c\tilde{C}_\frac{1}{2} \]
\[ \leq aC \| u_0 \|_{L^2(\Omega, g)} + C \sup |S| \| u_0 \|_{L^2(\Omega, g)} + C |\lambda| \| u_0^{p-1} \|_{L^2(\Omega, g)} + (aC + 1)c\tilde{C}_\frac{1}{2} \]
\[ \leq aC \lambda_1^{-\frac{1}{2}} \| \nabla u_0 \|_{L^2(\Omega, g)} + C \lambda_1^{-\frac{1}{2}} \text{sup} |S| \| \nabla u_0 \|_{L^2(\Omega, g)} \]
\[ + C |\lambda| C_0^{p-1} K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \| u_0 \|^{p-1}_{H^2(\Omega, g)} + (BC + 1)c\tilde{C}_\frac{1}{2} \]
\[ \leq \left( aC \lambda_1^{-\frac{1}{2}} + C \text{sup} |S| \lambda_1^{-\frac{1}{2}} + |\lambda| C_0^{p-1} K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \right) \| u_0 \|_{H^2(\Omega, g)} \]
\[ + (BC + 1)c\tilde{C}_\frac{1}{2} \]
\[ \leq \left( C \lambda_1^{-\frac{1}{2}} (a + \text{sup} |S|) + |\lambda| C_0^{p-1} K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \right) + (BC + 1)c\tilde{C}_\frac{1}{2}, \]
for \( B = 2a + \max |S| \).
We can choose $\Omega$ of small enough diameter and volume so that
\[
C\lambda_1^{-\frac{1}{2}} (a + \sup |S|) + (BC + 1)c\tilde{C}^{\frac{1}{2}} \leq \frac{1}{2}, \quad \frac{2}{ac}(p-1)\lambda_1^{-\frac{1}{p+2}} C_0 < 1, \quad \lambda_1^{-\frac{1}{2}} \frac{1}{ac} < 1.
\] (28)

We will use the last two inequalities later.) Indeed, as $\Omega$ shrinks, we know $C = C^* (1 + a^{-1}\lambda_1^{-1/2})$ is bounded above, $C_0$ in (24) is nonincreasing by Remark 2.2, $\tilde{C} \to 0$ in (17) as the volume of $\Omega$ shrinks, and $C_2/C_1$ is bounded by Remark 2.1.

Once $\Omega$ is chosen so that (28) holds, the constant $K(2, 2) = K(2, 2, \Omega)$ in (9) is fixed. Since the choice of the constant scaling $\lambda$ by a positive constant does not affect the solvability of (19), we can choose $\lambda$ such that
\[
\left( C\lambda_1^{-\frac{1}{2}} (a + \sup |S|) + |\lambda|CC_0^{p-1}K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \right) + (BC + 1)c\tilde{C}^{\frac{1}{2}} \leq 1.
\] (29)

It follows from (27), (28), (29) that
\[
\|u_1\|_{H^2(\Omega, g)} \leq \left( C\lambda_1^{-\frac{1}{2}} (a + \sup |S|) + |\lambda|CC_0^{p-1}K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \right) \|u_{k-1}\|_{H^2(\Omega, g)} + (BC + 1)c\tilde{C}^{\frac{1}{2}} \leq 1.
\] (30)

For any positive integer $k$, we repeat the argument starting with (19), and conclude that
\[
\|u_k\|_{H^2(\Omega, g)} \leq \left( C\lambda_1^{-\frac{1}{2}} (a + \sup |S|) + |\lambda|CC_0^{p-1}K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{p-1} \right) \|u_{k-1}\|_{H^2(\Omega, g)} + (BC + 1)c\tilde{C}^{\frac{1}{2}},
\]

since by induction $\|u_{k-1}\|_{H^2(\Omega, g)} \leq 1$. Note that the constants and hence the choice of $\lambda$ are independent of $k$. Therefore,
\[
\|u_k\|_{H^2(\Omega, g)} \leq 1, \forall k \in \mathbb{Z}_{\geq 0}.
\] (31)

We thus have a bounded sequence $\{u_k\}$ in $H^2(\Omega, g)$ of solutions to (19); equivalently, $\{\tilde{u}_k\}$ is a bounded sequence of solutions in $H^1_0(\Omega, g) \cap H^2(\Omega, g)$.

The second main step is to prove that $\{\tilde{u}_k\}$ (and not just a subsequence) converges to some $\tilde{u} \in H^1_0(\Omega, g)$, and hence $\{u_k\}$ converges to $u$ in $H^1(\Omega, g)$.

Since $C^\infty(\Omega)$ is dense in $H^1_0(\Omega, g)$ in the $H^1$-norm, we may assume again that $\{\tilde{u}_k\} \subset C^\infty(\Omega)$. Then $u_k = \tilde{u}_k + c \in C^\infty(\Omega) \cap H^1_0(\Omega, g) \subset C^\infty(\Omega) \cap H^2(\Omega, g)$. To prove the convergence, take (19) for $k$ and $k + 1$:
\[
au_k - a\Delta_g u_k = au_{k-1} - S\tilde{u}_{k-1} + \lambda u_{k-1},
\]
\[
au_{k+1} - a\Delta_g u_{k+1} = au_k - S\tilde{u}_k + \lambda u_k.
\] (32)

Subtract the first equation in (32) from the second, and pair both sides with $\tilde{u}_{k+1} - \tilde{u}_k$. Noting that $\tilde{u}_{k+1} - \tilde{u}_k = u_{k+1} - u_k$, we obtain
\[
\langle a\|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega, g)}^2 = \langle a(u_{k+1} - u_k) + (-a\Delta_g)(\tilde{u}_{k+1} - \tilde{u}_k), \tilde{u}_{k+1} - \tilde{u}_k \rangle_g
\]
\[\quad = \langle a(\tilde{u}_k - \tilde{u}_{k-1}), \tilde{u}_{k+1} - \tilde{u}_k \rangle_g + (-S(\tilde{u}_k - \tilde{u}_{k-1}), \tilde{u}_{k+1} - \tilde{u}_k \rangle_g
\]
\[\quad + \left( \lambda \left( \langle u_{k+1} - u_k \rangle_g, \tilde{u}_{k+1} - \tilde{u}_k \rangle_g, \right) ,
\]

where we recall that $\langle , \rangle_g$ is the $L^2(\Omega, g)$ inner product.
For the first two terms on the last line of (33), we apply the Poincaré inequality (10):

\[ (-S(\tilde{u}_k - \tilde{u}_{k-1}), \tilde{u}_{k+1} - \tilde{u}_k)_g \]
\[ \leq \sup\|S\| \|\tilde{u}_k - \tilde{u}_{k-1}\|_g \|\tilde{u}_{k+1} - \tilde{u}_k\|_g \]
\[ \leq \sup\|S\lambda_1^{-1}\|_g \|\nabla(\tilde{u}_k - \tilde{u}_{k-1})\|_g \|\nabla(\tilde{u}_{k+1} - \tilde{u}_k)\|_g \]
\[ \leq \sup\|S\lambda_1^{-1}\|_g \|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega,g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega,g)} , \]

and similarly,

\[ (a(\tilde{u}_k - \tilde{u}_{k-1}), \tilde{u}_{k+1} - \tilde{u}_k)_g \leq a\lambda_1^{-1}\|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega,g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega,g)} . \]

To treat the last term on the last line of (33), we apply the mean value theorem in the form

\[ |f(y) - f(x)| \leq |y - x| \sup_{0 \leq t \leq 1} |f'(x + t(y - x))| \]

for \( f(z) = z^{p-1} \) and \( x, y \) replaced by \( u_{k-1}(x), u_k(x) \), resp.: \( |u_k^{p-1}(x) - u_{k-1}^{p-1}(x)| \leq (p - 1)\|u_k(x) - u_{k-1}(x)\|_g \sup_{0 \leq t \leq 1} |t_k(x)u_k(x) + (1 - t_k(x))u_{k-1}(x)|^{p-2} . \)

Write \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \), where

\[ \Omega_1 = \{ x \in \Omega : u_k(x) > u_{k-1}(x) \} ; \]
\[ \Omega_2 = \{ x \in \Omega : u_k(x) < u_{k-1}(x) \} ; \]
\[ \Omega_3 = \{ x \in \Omega : u_k(x) = u_{k-1}(x) \} ; \]

It is clear that \( t_k(x) = 1 \) on \( \Omega_1 \), and \( t_k(x) = 0 \) on \( \Omega_2 \); on \( \Omega_3 \), both sides of (33) vanish. Thus

\[ |u_k^{p-1}(x) - u_{k-1}^{p-1}(x)| \leq (p - 1)\|u_k(x) - u_{k-1}(x)\|_{H^1(\Omega,g)}^{p-2} \quad \text{on} \ \Omega_1 , \]
\[ |u_k^{p-1}(x) - u_{k-1}^{p-1}(x)| \leq (p - 1)\|u_k(x) - u_{k-1}(x)\|_{H^1(\Omega,g)}^{p-2} \quad \text{on} \ \Omega_2 . \]

Since \( u_k - u_{k-1} = \tilde{u}_k - \tilde{u}_{k-1} \), we get

\[ \left( \lambda \left( u_k^{p-1} - u_{k-1}^{p-1} \right), \tilde{u}_{k+1} - \tilde{u}_k \right)_g \]
\[ \leq |\lambda| \int_{\Omega_1} \left| u_k^{p-1} - u_{k-1}^{p-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ = |\lambda| \int_{\Omega_1} \left| u_k^{p-1} - u_{k-1}^{p-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g + |\lambda| \int_{\Omega_2} \left| u_k^{p-1} - u_{k-1}^{p-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ \leq |\lambda| \int_{\Omega_1} (p - 1) \left| u_k \right|^{p-2} \left| u_k - u_{k-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ + |\lambda| \int_{\Omega_2} (p - 1) \left| u_k \right|^{p-2} \left| u_k - u_{k-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ \leq |\lambda| \int_{\Omega} (p - 1) \left| u_k \right|^{p-2} \left| u_k - u_{k-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ + |\lambda| \int_{\Omega} (p - 1) \left| u_k \right|^{p-2} \left| u_k - u_{k-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ = |\lambda| \int_{\Omega} (p - 1) \left| u_k \right|^{p-2} \left| \tilde{u}_k - \tilde{u}_{k-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g \]
\[ + |\lambda| \int_{\Omega} (p - 1) \left| u_k \right|^{p-2} \left| \tilde{u}_k - \tilde{u}_{k-1} \right| \left| \tilde{u}_{k+1} - \tilde{u}_k \right| \, d\text{Vol}_g . \]
Applying Hölder’s inequality to $p_1, p_2, p_3$ with $p_1 = \frac{n+2}{n-2}$, $p_2 = p_3 = \frac{2(n+2)}{n}$ (so $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$), and recalling that $p - 2 = \frac{4}{n-2}$, we obtain

\[
\left(\lambda \left( u_{k-1}^{p-1} - u_{k-1}^{p-1} \right), \tilde{u}_{k+1} - \tilde{u}_k \right)_g \\
\leq (p - 1)|\lambda| \left( \int_{\Omega} |u_k|^{\frac{4p_1}{n-2}} dV \right)^{\frac{1}{p_1}} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega,g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{L^{p_2}(\Omega,g)} \\
+ (p - 1)|\lambda| \left( \int_{\Omega} |u_{k-1}|^{\frac{4p_1}{n-2}} dV \right)^{\frac{1}{p_1}} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega,g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{L^{p_2}(\Omega,g)} \\
= (p - 1)|\lambda| \left\| u_k \right\|_{L^{\frac{4p_1}{n-2}}(\Omega,g)} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega,g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{L^{p_2}(\Omega,g)} \\
+ (p - 1)|\lambda| \left\| u_{k-1} \right\|_{L^{\frac{4p_1}{n-2}}(\Omega,g)} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega,g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{L^{p_2}(\Omega,g)} .
\]

Note that

\[
P_1 = \frac{n + 2}{2} \Rightarrow \frac{4p_1}{n - 2} = \frac{2(n + 2)}{n - 2} = 2p - 2.
\]

For the terms $\|u_k\|_{L^{\frac{4p_1}{n-2}}(\Omega,g)}$, $\|u_{k-1}\|_{L^{\frac{4p_1}{n-2}}(\Omega,g)}$ in the last two lines of (37), we apply (23) and (31) to get

\[
\|u_k\|_{L^{\frac{4p_1}{n-2}}(\Omega,g)} \leq C_0 K(2, 2) \frac{C_2}{C_1} \|u_k\|_{H^2(\Omega,g)} \\
\Rightarrow \|u_k\|_{L^{\frac{4p_1}{n-2}}(\Omega,g)} \leq \left( C_0 K(2, 2) \frac{C_2}{C_1} \right)^{-\frac{n-2}{4}} \|u_k\|_{H^2(\Omega,g)} \leq \left( C_0 K(2, 2) \frac{C_2}{C_1} \right)^{-\frac{n-2}{4}},
\]

(38)

We next consider terms like $\|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega,g)} = \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{\frac{2(n+2)}{n}}(\Omega,g)}$ in the last two lines of (37). Since we may assume $\{\tilde{u}_k\} \subset C^\infty_c(\Omega)$, we can use Proposition 2.1 with $l = \frac{2(n+2)}{n}$, $q = r = 2$, $j = 0$, $m = 1$ in (7), and obtain

\[
\frac{n}{2(n+2)} = \alpha \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{1 - \alpha}{2} \Rightarrow \alpha = \frac{n}{n + 2}.
\]

Thus $\alpha \in (0, 1)$, and it follows from (3) and (24) that

\[
\|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega,g)} \leq C_1 \lambda_{\ell, 0, 2, 2, \frac{1}{n+2}} \|\nabla (\tilde{u}_k - \tilde{u}_{k-1})\|_{L^\frac{n}{\alpha \frac{n+2}{n-2}}(\Omega,g)} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^\frac{2}{\alpha \frac{n+2}{n-2}}(\Omega,g)} \\
\leq C_0 \lambda_1 \|\nabla (\tilde{u}_k - \tilde{u}_{k-1})\|_{L^\frac{n}{\alpha \frac{n+2}{n-2}}(\Omega,g)} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega,g)} ,
\]

(39)

\[
\|\tilde{u}_{k+1} - \tilde{u}_k\|_{L^{p_2}(\Omega,g)} \leq C_0 \lambda_1 \|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega,g)} .
\]
Plugging (38) and (39) into (37), we conclude that the last term of (33) satisfies
\[
\left( \lambda \left( u_k^{p-1} - u_{k-1}^{p-1} \right), \tilde{u}_{k+1} - \tilde{u}_k \right)_{L^2(\Omega, g)} 
\leq 2(p - 1)|\lambda| \left( C_0 K(2, 2) \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{L^{p_2}(\Omega, g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{L^{p_2}(\Omega, g)} 
\leq 2(p - 1)|\lambda| \left( C_0 K(2, 2) \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} C_0^2 \lambda_1^{-\frac{2}{p-1}} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega, g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega, g)} 
\leq 2(p - 1)|\lambda| \left( K(2, 2) \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} C_0^p \lambda_1^{-\frac{2}{p-1}} \|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega, g)} \|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega, g)}. 
\]

It follows from (33), (34), (35), and (40) that
\[
\|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega, g)} 
\leq \left( \lambda_1^{-1} \left( 1 + a^{-1} \sup |S| \right) + 2a^{-1}(p - 1)|\lambda| \left( K(2, 2) \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} C_0^p \lambda_1^{-\frac{2}{p-1}} \right) \cdot \|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega, g)},
\]
where we have cancelled \(\|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega, g)}\) from both sides of (11). By (28), we have
\[
\lambda_1^{-1} \left( 1 + \frac{1}{a} \sup |S| \right) = C \lambda_1^{-\frac{1}{2}}(a + \sup |S|) \cdot \lambda_1^{-\frac{1}{2}} \frac{1}{a C} < C \lambda_1^{-\frac{1}{2}}(a + \sup |S|),
\]
\[
2a^{-1}(p - 1)|\lambda| \left( K(2, 2) \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} C_0^p \lambda_1^{-\frac{2}{p-1}} = \left| \lambda |CC_0^{p-1} \left( \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} K(2, 2)^{p-1} \right| \cdot \left( \frac{2}{a C} (p - 1) \lambda_1^{-\frac{2}{p-1}} C_0 \right) \cdot \left( K(2, 2) \frac{C_2}{C_1} \right)^{-1} < |\lambda|CC_0^{p-1} \left( \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} K(2, 2)^{p-1},
\]
where we use \(K(2, 2) \geq 1, C_2/C_1 \geq 1\). Combining these two estimates and applying (29), we observe that
\[
\lambda_1^{-1} \left( 1 + \frac{1}{a} \sup |S| \right) + 2a^{-1}(p - 1)|\lambda| \left( K(2, 2) \frac{C_2}{C_1} \right)^{\frac{1}{p-1}} C_0^p \lambda_1^{-\frac{2}{p-1}} < C \lambda_1^{-\frac{1}{2}}(a + \sup |S|) + |\lambda|CC_0^{p-1} K(2, 2)^{p-1} \left( \frac{C_2}{C_1} \right)^{-1} \geq 1 - (BC + 1)c C_1^{\frac{1}{2}}. 
\]
By (29),
\[
A := 1 - (BC + 1)c C_1^{\frac{1}{2}} \in (0, 1).
\]
Thus (11) becomes
\[
\|\tilde{u}_{k+1} - \tilde{u}_k\|_{H^1(\Omega, g)} < A \|\tilde{u}_k - \tilde{u}_{k-1}\|_{H^1(\Omega, g)},
\]
which implies that \(\{\tilde{u}_k\}\) is a Cauchy sequence in \(H^1(\Omega, g)\). By (20), \(u_k\) converges to some \(u \in H^1(\Omega, g)\). Taking the limit on both sides of (19), it follows that
\[
-\alpha \Delta_g u + Su = \lambda u^{p-1} \text{ in } \Omega.
\]
in the weak sense. Since \(\tilde{u} = \lim \tilde{u}_k\) has zero trace, \(u = c\) on \(\partial \Omega\). Thus \(u\) solves (3). \(\square\)
Remark 2.5. (i) We discuss where in the proof we may have to shrink $\Omega$ and decrease the choice of $\lambda$ in (19).

1. To obtain $C_2/C_1 \in [1,4]$ in Remark 2.1 we may have to decrease $\text{diam} E(\Omega)$.
2. For (18), we may have to decrease $\text{Vol}_g(\Omega)$.
3. For (28), we may have to decrease both $\text{Vol}_g(\Omega)$ and $\text{diam} E(\Omega)$.

In particular, (28) and (29) depend on $\max |S|$ on $\Omega$.

(ii) In the case $\lambda = 0$, if the conformal Laplacian $-a\Delta_g + S_g$ has zero as first eigenvalue, then by the Fredholm alternative $-a\Delta_g u + S_g u = 0$, $u \equiv c > 0$ on $\partial \Omega$ cannot have a solution, which would contradict Theorem 2.4 However, it is easy to check that for $\Omega$ small enough, the conformal Laplacian has positive first eigenvalue.

(iii) In fact, $u \in C(\Omega)$. For the three cases prove that $u \in W^{2,p}(\Omega)$ for $p > 0$. By the Extension Proposition, $Eu \in W^{2,p}(\mathbb{R}^n)$ and thus $Eu$ is continuous by the Sobolev Embedding Theorem. Since $Eu = u$ a.e. on $\Omega$, we can extend $u$ continuously to $\partial \Omega$ by $Eu$.

We now prove (Theorem 2.8) that the solution $u$ of (3) obtained in Theorem 2.3 is smooth. In §3, we show that $u > 0$ pointwise in all dimensions $n \geq 3$.

We need familiar analytic tools stated below: a weak maximum principle for elliptic operators, various Sobolev embedding theorems, interior elliptic regularity, and Schauder estimates. We assume familiarity with the H"older spaces $C^{0,\alpha}(\Omega)$ and the Schauder spaces $C^{s,\alpha}(\Omega)$.

**Theorem 2.4.** (i) [7, Cor. 3.2] (Weak Maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^2$ boundary. Let $L$ be a second order elliptic operator of the form

$$Lu = -\sum_{|\alpha|=2} -a_{\alpha}(x)\partial^{\alpha}u + \sum_{|\beta|=1} -b_{\beta}(x)\partial^{\beta}u + c(x)u$$

where $a_{\alpha}, b_{\beta}, c \in C^\infty(\Omega)$ are smooth and bounded real-valued functions on $\Omega$. Let $u \in C^2(\overline{\Omega})$. Suppose that in $\Omega$, we have $Lu \geq 0, c(x) \geq 0$. Then for $u^- := \min(u,0)$,

$$\inf_{\Omega} u = \inf_{\partial \Omega} u^-.$$

(ii) [7, Thm. 3.5] (Strong Maximum Principle) Assume that $\partial \Omega$ is smooth. Let $L$ be a second order uniformly elliptic operator as above. If $Lu \geq 0, c(x) \geq 0$, and if $u(x) = 0$ at an interior point $x \in \Omega$, then $u \equiv 0$ on $\Omega$.

**Theorem 2.5.** [1, Ch. 4] (Sobolev Embeddings) Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary $\partial \Omega$.

(i) For $s \in \mathbb{N}$ and $1 \leq p \leq p' < \infty$ such that

$$\frac{1}{p} - \frac{s}{n} < \frac{1}{p'},$$

$W^{s,p}(\Omega)$ continuously embeds into $L^{p'}(\Omega)$: for some $K = K(s,p,p',\Omega,g) > 0$,

$$\|u\|_{L^{p'}(\Omega,g)} \leq K\|u\|_{W^{s,p}(\Omega,g)}.$$ (46)

(ii) For $s \in \mathbb{N}$, $1 \leq p < \infty$ and $0 < \alpha < 1$ such that

$$\frac{1}{p} - \frac{s}{n} \leq -\frac{\alpha}{n},$$

Then $W^{s,p}(\Omega)$ continuously embeds in the Hölder space $C^{0,\alpha}(\Omega)$: for some $K' = K'(s,p,p',\Omega,g) > 0$,

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq K'\|u\|_{W^{s,p}(\Omega,g)}.$$ (48)

**Theorem 2.6.** [7, Thm 7.22] (Kondarachov-Rellich Compactness Theorem) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial \Omega$. Then $W^{1,p}(\Omega)$ compactly embeds in $L^q(\Omega)$ for $q < \frac{np}{n-p}$, provided $p < n$. 


Lemma 2.1. Let $u$ be a solution of $-\Delta_g u = f$.

(i) (Interior Regularity) If $f \in W^{s,p}(\Omega,g)$ and $\partial \Omega$ is $C^\infty$, then $u \in W^{s+2,p}(\Omega,g)$.

(ii) (Schauder Estimates) If $f \in C^{s,\alpha}(\Omega)$ and $\partial \Omega \in C^{s,\alpha}$, then $u \in C^{s+2,\alpha}(\Omega)$. Also, if $u \in C^{0,\alpha}(\Omega)$, then

$$\|u\|_{C^{s+2,\alpha}(\Omega)} \leq D_2(\|-\Delta_g u\|_{C^{s,\alpha}(\Omega)} + \|u\|_{C^{0,\alpha}(\Omega)}),$$

for some $D_2 = D_2(s,p,-\Delta_g,\Omega,\partial \Omega) > 0$.

Remark 2.6. Before we prove the smoothness weak solutions $u \in H^1(\Omega,g)$ of the Yamabe equation (3), we prove that $u$ is actually in $H^2(\Omega,g)$.

Theorem 2.7. [11] Thm 2.4 Let $(\Omega,g)$ be a Riemannian domain in $\mathbb{R}^n$, and let $u \in H^1_0(\Omega,g)$ be a weak solution of $-\Delta_g u = f$.

(i) (Interior Regularity) If $f \in W^{s,p}(\Omega,g)$ and $\partial \Omega$ is $C^\infty$, then $u \in W^{s+2,p}(\Omega,g)$.

Theorem 2.8. Let $(\Omega,g)$ be a Riemannian domain in $(\mathbb{R}^n, g)$, $n \not\equiv 2 \pmod{8}$, as in Theorem 2.3. The solution $u$ of (3) obtained in Theorem 2.3 lies in $H^2(\Omega,g)$.

Proof. By the equivalence of norms in Remark 2.1, it suffices to show $u \in H^2(\Omega)$. For $u = \bar{u} + c$ as above, we only need to show $\bar{u} \in H^2(\Omega) \cap H^1_0(\Omega)$, where $\bar{u} = \lim \bar{u}_k$ in $H^1_0(\Omega)$.

By (2), (31), $\|\bar{u}_k\|_{H^2(\Omega)} \leq C_1^{-1}$ for all $k$, so there exists a subsequence, also denoted $\{\bar{u}_k\}$, such that $\bar{u}_k \to w$ weakly in $H^2(\Omega)$, i.e.,

$$f(\bar{u}_k) \to f(w), \forall f \in H^{-2}(\Omega).$$

Since $i : H^2(\Omega) \hookrightarrow H^1(\Omega)$ is a compact inclusion, there exists a subsequence, again denoted $\{\bar{u}_k\}$, such that $i(\bar{u}_k) \to w'$ strongly and hence also weakly in $H^1_0(\Omega)$. Thus for all $g \in H^{-1}(\Omega)$, $g(i(\bar{u}_k)) \to g(w')$. The pullback $i^* : H^{-1}(\Omega) \to H^{-2}(\Omega)$ is continuous, so $g \circ i = i^* g \in H^{-2}(\Omega)$ for $g \in H^{-1}(\Omega)$. It follows from (30) that

$$g(i(w)) = g(w'), \forall g \in H^{-1}(\Omega).$$

Hence $i(w) = w'$ in $H^1_0(\Omega)$. By the proof of Theorem 2.3, the (original) sequence $\{\bar{u}_k\}$ converges to $\bar{u}$ strongly in $H^1_0(\Omega)$, so it follows that $\bar{u} = w \in H^2(\Omega)$.

Note that we do not claim that $\bar{u}_k \to \bar{u}$ in $H^2(\Omega)$.

Remark 2.6. Since we now know that $u \in H^2(\Omega,g)$, it follows that $u$ solves the Yamabe equation $-\Delta_g u = -Su + \lambda u^p - 1$ in the $L^2(\Omega)$-sense with $u \equiv c$ on $\partial \Omega$ in the trace sense.

Theorem 2.8. Let $(\Omega,g)$ be a Riemannian domain in $(\mathbb{R}^n, g)$, $n \not\equiv 2 \pmod{8}$, as in Theorem 2.3. The weak real solution $u \in H^1(\Omega,g)$ of the Yamabe equation (3) in Theorem 2.3 is a smooth solution.

The proof is similar to Yamabe and Trudinger’s original arguments as well as the approach in [11], but avoids working with subcritical exponents.

Proof. The first step is to show that $u \in C^{2,\alpha}(\Omega)$. By Lemma 2.1, $u \in H^2(\Omega,g)$. By the GN inequality Proposition 2.1, $\bar{u}$ and therefore $u = \bar{u} + c$ lie in $L^r(\Omega,g)$, where $r$ satisfies (17), i.e.,

$$\frac{1}{r} = \beta \left(\frac{1}{2} - \frac{2}{n}\right) + \frac{1-\beta}{2}, 0 \leq \beta < 1 \Rightarrow \frac{1}{r} = \frac{n-4\beta}{2n}, 0 \leq \beta < 1.$$  

(51)

There are three cases, depending on $n = \dim(M)$.

Case I. $n = 3$ or 4. For $n = 3, 4$ and an arbitrary $r \geq 2$, there exists $\beta \in [0,1)$ such that (51) holds. Since $w^{p-1} \in L^{\frac{1}{r+1}}(\Omega,g) \subset L^r(\Omega,g)$,

$$-\Delta_g u = -Su + \lambda u^{p-1} \in L^r(\Omega,g),$$
for \( r \geq 2 \). By Theorem \( 2.7(i) \), \( u \in W^{2,r}(\Omega,g) \).

For \( r \gg 0 \), \( (7) \) holds for some \( \alpha \in (0,1) \), and applying Theorem \( 2.5(ii) \) to \( u \), we obtain \( u \in C^{0,\alpha}(\Omega) \). By the Schauder estimates in Theorem \( 2.7(ii) \), we conclude that \( u \in C^{2,\alpha}(\Omega) \).

**Case II.** \( n = 5 \) or \( 6 \). When \( n \geq 5 \), \( (51) \) gives

\[
\frac{2n}{n-4} = 0 \leq \beta < 1 \Rightarrow r = \frac{2n}{n-4} - \epsilon,
\]

where \( \epsilon > 0 \) can be arbitrarily small by choosing \( \beta \) close to 1. In particular, for \( \epsilon \) small enough, \( r > p = \frac{2n}{n-2} \).

As in the previous case, we have \( \tilde{u} \in L^r \subseteq L^{\frac{r}{p-1}} \) and \( -\Delta_g u \in L^{\frac{r}{p-1}} \), so elliptic regularity (Theorem \( 2.7(i) \)) implies \( u \in W^{2,\frac{r}{p-1}}(\Omega,g) \). The Sobolev embedding condition \( (45) \) implies

\[
u \in L^{r'}(\Omega,g), \quad \text{for} \quad \frac{p-1}{r} - \frac{2}{n} \leq \frac{1}{r'}.
\]

When \( n = 5 \), \( (53) \) holds for any \( r' \geq 1 \); when \( n = 6 \), \( (53) \) holds for \( r' \gg 0 \). We again conclude that \( u \in C^{2,\alpha}(\Omega) \).

**Case III.** \( n \geq 7 \). The case of equality in \( (53) \) is

\[
r' = \frac{nr}{np - n - 2r}.
\]

Plugging in \( r \) from \( (52) \) and using \( p = \frac{2n}{n-2} \), we get

\[
r' - r = \frac{n^2(n-6)}{(n-2)(n-4)} + 2\epsilon > 0,
\]

for \( n \geq 7 \). As above, \( u \in L^{r'} \) implies \( u \in W^{2,\frac{r'}{p-1}}(\Omega) \). Then solving

\[
\frac{p-1}{r'} - \frac{2}{n} = \frac{1}{r'},
\]

we obtain \( r'' > r' > r > p \) and \( u \in W^{2,\frac{r''}{p-1}}(\Omega,g) \). Plugging \( (53) \) for \( 1/r' \) into \( (54) \), we get \( \frac{1}{r'} = \frac{(p-1)^2}{r} - (1+(p-1))\frac{2}{n} \). Iterating this process, after \( M \) steps we find that \( u \in L^r(\Omega,g) \) where

\[
\frac{1}{r} \geq \left( \frac{p-1}{r} \right)^M - \left( \sum_{m=0}^{M-1} \frac{2}{n} \right) \cdot \frac{2}{n} = \left( \frac{p-1}{r} \right)^M - \left( \frac{(p-1)^M - 1}{p} \right) \cdot \frac{2}{n} \]

\[
= \frac{(p-1)^M}{r} - \frac{(p-1)^M - 1}{p} \quad \text{since} \quad (p-2)\frac{n}{2} = p
\]

Since \( 1/r - 1/p < 0 \), the last line is negative for \( M \gg 0 \). We conclude that \( u \in W^{2,q}(\Omega,g) \) for \( q \gg 1 \). It follows from Theorem \( 2.5(ii) \) that \( u \in C^{0,\alpha}(\Omega) \) for some \( \alpha \in (0,1) \). As above, we obtain \( u \in C^{2,\alpha}(\Omega) \).

Thus in all cases, we have \( u \in C^{2,\alpha}(\Omega) \). Using the Schauder estimates in Theorem \( 2.7(ii) \) and the limiting arguments involving \( \tilde{u} \) and \( \{u_n\} \) above, we bootstrap to get \( u \in C^\infty(\Omega) \). \( \square \)

**Remark 2.7.** In the classical approach, one proves \( u_\varepsilon > 0 \) for solutions to the Yamabe problem at subcritical exponents \( \varepsilon \); the main problem is to show that the weak limit \( u \) of the \( u_\varepsilon \) is not
identify 0 at the critical exponent. In our case, since \( u \equiv c > 0 \) on \( \partial \Omega \), we immediately see that \( u \) is nontrivial.

3. Positivity of the Solution of the Yamabe Problem

In this section, we prove that there exists a positive solution to the Yamabe equation in all dimensions \( n \geq 3 \), except in one rare case.

In Theorem 3.1, we treat two cases, depending on \( \text{sgn}(\lambda) \). In the proof, it is convenient to assume

\[
S_g \in \left( -\frac{a}{2}, \frac{a}{2} \right),
\]

which can always be achieved by scaling \( g \). We prove that the solution \( u \) is positive by showing that each \( u_k \geq 0 \) for all \( k \) in the iteration steps (19). It follows that \( u_k^{p-1} \geq 0 \) is well-defined, which allows us to remove the restriction \( n \neq 2 \) (mod 8) in the previous section.

**Theorem 3.1.** Let \((\Omega, g)\) be Riemannian domain in \((\mathbb{R}^n, g), n \geq 3\), with \( \text{Vol}_g(\Omega) \) and the Euclidean diameter of \( \Omega \) sufficiently small. Then (5) has a real, smooth, positive solution \( u \).

**Proof.** We analyze the positivity in two cases: (i) \( \lambda \geq 0 \); (ii) \( \lambda < 0 \).

**Case I.** \( \lambda \geq 0 \). From the first iteration step (14), if we choose \( f_0 > 0 \) and \( f_0 \in C^\infty(\bar{\Omega}) \), then \( u_0 \in C^\infty(\Omega) \cap C^0(\Omega) \) and \( au_0 - a\Delta_g u_0 > 0 \). By the weak maximum principle Theorem 2.4(i), \( u_0 > 0 \) since \( \inf_{\partial \Omega} \min(u_0, 0) = 0 \). By the strong maximum principle Theorem 2.4(ii), if \( u = 0 \) at some point in \( \Omega \) then \( u \equiv 0 \). This contradicts \( u = c > 0 \) on \( \partial \Omega \), so \( u_0 > 0 \). Inductively, assume \( u_{k-1} > 0 \) on \( \Omega \). By (19),

\[
a u_k - a \Delta_g u_k = au_{k-1} - S_g u_{k-1} + \lambda u_{k-1}^{p-1} > 0,
\]

(55)
since \( a - S_g \geq \frac{a}{2} > 0 \) and \( \lambda > 0 \). As above, we conclude that \( u_k > 0 \) on \( \Omega \), and by Theorem 2.8 \( u_k \in C^\infty(\Omega) \cap C^0(\Omega) \). Since each \( u_k > 0 \), it follows that \( u \geq 0 \). Since \( u \equiv c > 0 \) on \( \partial \Omega \) and \( u \in C^\infty(\Omega) \cap C^0(\Omega) \) by bootstrapping, we conclude that \( u > 0 \) by the strong maximum principle.

**Case II.** \( \lambda < 0 \). Set

\[
L = \frac{3a}{2}.
\]

As in (14), (15), we consider the initial step

\[
a u_0 - a \Delta_g u_0 = f_0 \quad \text{in} \quad \Omega, u_0 \equiv c \quad \text{on} \quad \partial \Omega;
\]

(57)

\[
a \tilde{u}_0 - a \Delta_g \tilde{u}_0 = f_0 - ac \quad \text{in} \quad \Omega, \tilde{u}_0 \equiv 0 \quad \text{on} \quad \partial \Omega.
\]

Assuming \( f_0 \in C^\infty(\Omega) \cap (\cap_{p \geq 1} L^p(\Omega, g)) \), elliptic regularity implies \( u_0 \in C^\infty(\Omega) \). By Theorem 2.5(i),

\[
u_0, \tilde{u}_0 \in H^2(\Omega, g) \Rightarrow u_0, \tilde{u}_0 \in L^{r_1}(\Omega, g) \quad \text{for} \quad \frac{1}{r_1} \geq \frac{n - 4}{2n}.
\]

By the interior regularity in Theorem 2.7(i), \( \tilde{u}_0 \in W^{2,r_1}(\Omega, g) \). Applying Theorem 2.7(i) again, we conclude that

\[
\tilde{u}_0, u_0 \in W^{2, r_1}(\Omega, g) \Rightarrow \tilde{u}_0, u_0 \in L^{r_2}(\Omega, g) \quad \text{for} \quad \frac{1}{r_2} \geq \frac{1}{r_1} - \frac{2}{n} \geq \frac{n - 8}{2n}.
\]

Continuing, we have the following bootstrapping for \( u_0 \) and \( \tilde{u}_0 \):

\[
u_0, \tilde{u}_0 \in W^{2,2} \Rightarrow u_0, \tilde{u}_0 \in L^{r_1} \Rightarrow u_0, \tilde{u}_0 \in W^{2, r_1} \Rightarrow u_0, \tilde{u}_0 \in L^{r_2} \Rightarrow \ldots
\]

\[
\Rightarrow u_0, \tilde{u}_0 \in W^{2, r_j} \Rightarrow u_0, \tilde{u}_0 \in L^{r_{j+1}} \Rightarrow \ldots,
\]

where the \( r_j \) are increasing, and each \( r_j \) satisfies

\[
\frac{1}{r_j} \geq \frac{n - 4j}{2n}.
\]
The right hand side of (58) is nonpositive for
\[ j \geq \left\lceil \frac{n+3}{4} \right\rceil := M_n, \] (59)
so once \( j \geq M_n, \tilde{u}_0, u_0 \in L^r(M, g) \) and \( \tilde{u}_0, u_0 \in W^{2,r}(\Omega, g) \) for all \( r \geq 1 \). Note that \( (n-4j)/2n > 0 \) for \( j = 1, \ldots, M_n - 1 \). Set
\[ C' = \max \{ aD_1 + 1, aD_1 + D_1 \}, \quad B_1' = B_1 = \sum_{l=0}^{M_n} (C'K)^l + c\text{Vol}_g(\Omega)^{\frac{1}{r_1}}. \] (60)

For \( 1 \leq j \leq M_n \), applying (49) to \( \tilde{u}_0 \), we have
\[ \|\tilde{u}_0\|_{W^{2,j}(\Omega, g)} \leq D_1 \left( \|f_0 - ac - a\tilde{u}_0\|_{L^{j}(\Omega, g)} + \|\tilde{u}_0\|_{L^{j}(\Omega, g)} \right) \leq D_1 \|f_0\|_{L^{j}(\Omega, g)} + (aD_1 + 1)\|\tilde{u}_0\|_{L^{j}(\Omega, g)} + B_1' \] (61)

For \( 1 \leq j \leq M_n - 1 \), the Sobolev embedding theorem again gives
\[ \|\tilde{u}_0\|_{L^{j+1}(\Omega, g)} \leq K\|\tilde{u}_0\|_{W^{2,j}(\Omega, g)}, \|\tilde{u}_0\|_{L^{j+1}(\Omega, g)} \leq K\|\tilde{u}_0\|_{H^2(\Omega, g)} < K; \] (62)
\[ \|u_0\|_{L^{j+1}(\Omega, g)} \leq \|\tilde{u}_0\|_{L^{j+1}(\Omega, g)} + c\text{Vol}_g(\Omega)^{\frac{1}{r_1}}, \|u_0\|_{L^{j+1}(\Omega, g)} \leq \|\tilde{u}_0\|_{L^{j+1}(\Omega, g)} + c\text{Vol}_g(\Omega)^{\frac{1}{r_1}}. \]

We can choose \( f_0 > 0 \) small enough so that
\[ \|\tilde{u}_0\|_{W^{2,j}(\Omega, g)} \leq 1, \|\tilde{u}_0\|_{L^{j+1}(\Omega, g)} \leq K, \|u_0\|_{L^{j+1}(\Omega, g)} \leq K + B_1; \]
\[ \|\tilde{u}_0\|_{W^{2,j}(\Omega, g)} \leq (C'K)^j (L + 1)^j + L \cdot B_1 \sum_{l=0}^{j-1} K^l(C')^{l+1} (L + 1)^l \] (63)
\[ + B_1' \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n; \]
\[ \|u_0\|_{L^{j}(\Omega, g)} \leq K^j(C')^{j-1}(L + 1)^{j-1} + L \cdot B_1 \sum_{l=1}^{j-1} (C'K)^l (L + 1)^{l-1} + B_1, j = 2, \ldots, M_n, \]
since this involves only a finite number of choices for \( f_0 \). The justification for the complicated terms in (63) is given by the Claim below. Furthermore, \( f_0 > 0 \) implies \( u_0 > 0 \), as in Case I.

Consider the first iteration
\[ au_1 - a\Delta g u_1 = au_0 - S_\gamma u_0 + \lambda u_0^{p-1} \text{ in } \Omega, u_1 \equiv c \text{ on } \partial \Omega; \] (64)
\[ a\tilde{u}_1 - a\Delta g \tilde{u}_1 = au_0 - S_\gamma u_0 + \lambda u_0^{p-1} - ac \text{ in } \Omega, \tilde{u}_1 \equiv 0 \text{ on } \partial \Omega. \]
\( u_0 \in C^\infty(\Omega) \) implies \( u_1 \in C^\infty(\Omega) \) by elliptic regularity. Since \( \lambda < 0 \) and \( u_0 > 0 \), if
\[ au_0 - S_\gamma u_0 + \lambda u_0^{p-1} \geq 0, \] (65)
then \( u_1 \geq 0 \). (65) holds if we choose \( \lambda \) such that
\[ |\lambda| \leq \frac{a}{\sup u_0^{p-2} - \frac{a - S_\gamma}{\sup u_0^{p-2}}}. \] (66)
We eventually want to bound $|\lambda|$ independent of the $u_k$. To begin, by (64) and Sobolev embedding in Theorem 2.5(ii) we conclude that
\[
|u_0| \leq |\tilde{u}_0| + c \leq \|\tilde{u}_0\|_{C^{k,0}(\Omega)} + c \leq D_2\|\tilde{u}_0\|_{W^{2,r,M_n}(\Omega,g)} + c
\]
\[
\leq K\cdot \left( (C'K)^{M_n} (L+1)^{M_n} + L \cdot B_1 \sum_{l=0}^{M_n-1} K^l (C')^{l+1} (L+1)^l + B_1' \sum_{l=0}^{M_n-1} K^l (C')^{l+1} \right) + c
\]
\[
:= C_{M_n}.
\]
We note that to apply Theorem 2.5(ii), we need $1/r_{M_n} - 2/n \leq -\alpha/n$, which holds if $r_{M_n} \geq n$. This can be arranged, since by (58) and (59), $r_{M_n}$ can be arbitrarily large. Hence by (65)-(67), $u_1 \geq 0$ if
\[
|\lambda| \leq \frac{\alpha}{C_{M_n}^{n/2}}.
\]
In fact, $u_1 > 0$ by the maximum principle. By (30), we still have $\|u_1\|_{H^2(\Omega,g)} \leq 1$, after possibly shrinking $|\lambda|$ in (68). Since (65) now holds, we have
\[
a \tilde{u}_1 - a \Delta_g \tilde{u}_1 = au_0 - S_g u_0 + \lambda u_0^{n-1} - ac
\]
\[
\Rightarrow |a \tilde{u}_1 - a \Delta_g \tilde{u}_1| \leq |au_0 - S_g u_0 + \lambda u_0^{n-1}| + ac \leq au_0 - S_g u_0 + \lambda u_0^{n-1} + ac
\]
\[
\leq Lu_0 + ac;
\]
\[
\Rightarrow \|a \Delta_g \tilde{u}_1\|_{L^c_j(\Omega,g)} \leq \|a \tilde{u}_1 - a \Delta_g \tilde{u}_1\|_{L^c_j(\Omega,g)} + a\|\tilde{u}_1\|_{L^c_j(\Omega,g)}
\]
\[
\leq L\|u_0\|_{L^c_j(\Omega,g)} + a\|u_1\|_{L^c_j(\Omega,g)} + B_1', j = 1, \ldots, M_n.
\]
It follows from (49) and (69) that for $j = 1, \ldots, M_n$,
\[
\|\tilde{u}_1\|_{W^{2,r_j}(\Omega,g)} \leq D_1 \left( \|a \Delta_g \tilde{u}_1\|_{L^c_j(\Omega,g)} + \|\tilde{u}_1\|_{L^c_j(\Omega,g)} \right)
\]
\[
\leq C' \left( L\|u_0\|_{L^c_j(\Omega,g)} + \|\tilde{u}_1\|_{L^c_j(\Omega,g)} + B_1' \right).
\]
Recalling that Theorem 2.5(i) implies $\|u\|_{L^c_j(\Omega,g)} \leq K\|u\|_{W^{2,r_j-1}(\Omega,g)}$ by the construction of the $r_j$, and using (70) repeatedly, we have
\[
\|\tilde{u}_1\|_{W^{2,r_j}(\Omega,g)} \leq C' L\|u_0\|_{L^c_j(\Omega,g)} + C'\|\tilde{u}_1\|_{L^c_j(\Omega,g)} + C'B_1'
\]
\[
\leq C' L\|u_0\|_{L^c_j(\Omega,g)} + C'K\|\tilde{u}_1\|_{W^{2,r_j-1}(\Omega,g)} + C'B_1'
\]
\[
\leq C' L\|u_0\|_{L^c_j(\Omega,g)} + (C')^2 K \left( L\|\tilde{u}_0\|_{L^c_j-1(\Omega,g)} + \|\tilde{u}_1\|_{L^c_{j-1}(\Omega,g)} + B_1' \right) + C'B_1'
\]
\[
\leq C' L\|u_0\|_{L^c_j(\Omega,g)} + (C')^2 K L\|u_0\|_{L^c_{j-1}(\Omega,g)} + (C')^2 K \|\tilde{u}_1\|_{L^c_{j-1}(\Omega,g)}
\]
\[
+ (C')^2 KB_1' + C'B_1'
\]
\[
\leq \ldots
\]
for \( j = 1, \ldots, M_n \). Continuing until the right hand side contains \( \|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \) and recalling that and \( \|u\|_{L^r(\Omega, g)} \leq K \|u\|_{W^{2,2}(\Omega, g)} \), we obtain

\[
\|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \leq L \sum_{l=0}^{j-1} K^l(C')^{l+1} \|u_0\|_{L^{r_{j-1}}(\Omega, g)} + (C' K)^j \|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \\
+ B_1^j \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n;
\]

\[
\|\tilde{u}_1\|_{\mathcal{L}^r(\Omega, g)} \leq L \sum_{l=1}^{j-1} (C' K)^l \|u_0\|_{\mathcal{L}^{r_{j-1}}(\Omega, g)} + K^j (C')^{j-1} \|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \\
+ B_1^j \sum_{l=0}^{j-1} (C' K)^l, j = 2, \ldots, M_n.
\]

We now obtain stronger estimates on \( \tilde{u}_1 \) and \( u_1 \).

**Claim:** We have

\[
\|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \leq L, \|\tilde{u}_1\|_{\mathcal{L}^{r_1}(\Omega, g)} \leq K, \|u_1\|_{\mathcal{L}^{r_1}(\Omega, g)} \leq K + B_1
\]

\[
\|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \leq (C' K)^j (L + 1)^j + L \cdot B_1 \sum_{l=1}^{j-1} K^l(C')^{l+1} (L + 1)^l \\
+ B_1^j \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n;
\]

\[
\|u_1\|_{\mathcal{L}^{r_j}(\Omega)} \leq K^j (C')^{j-1}(L + 1)^{j-1} + L \cdot B_1 \sum_{l=1}^{j-1} (C' K)^l (L + 1)^{l-1} + B_1, j = 2, \ldots, M_n.
\]

This is proved in Appendix A.

The estimates in the Claim for \( \tilde{u}_1, u_1 \) are exactly the same as for \( \tilde{u}_0, u_0 \) in (63). Therefore, we can repeat the estimates in (67) and get

\[
u_1 = |u_1| \leq C_{M_n}.
\]

For \( \lambda \) as in (68), (76) implies that

\[
au_1 - S_g u_1 + \lambda u_1^{p-1} \geq 0.
\]

It follows (as above for \( u_1 \)) that \( u_2 > 0 \) is a smooth solution for the second line in (32) with \( k = 1 \). Inductively, we consider the general iteration step

\[
a u_k - a \Delta_g u_k = a u_{k-1} - S_g u_{k-1} + \lambda u_{k-1}^{p-1} \text{ in } \Omega, u_k \equiv c \text{ on } \partial \Omega;
\]

\[
a \tilde{u}_k - a \Delta_g \tilde{u}_k = a \tilde{u}_{k-1} - S_g \tilde{u}_{k-1} + \lambda \tilde{u}_{k-1}^{p-1} - ac \text{ in } \Omega, \tilde{u}_k \equiv 0 \text{ on } \partial \Omega.
\]
with the following estimates:
\[
\|\bar{u}_{k-1}\|_{W^{2,2}(\Omega)} \leq 1, \|\bar{u}_{k-1}\|_{L^1(\Omega)} \leq K, \|u_{k-1}\|_{L^1(\Omega)} \leq K + B_1;
\]
\[
\|\bar{u}_{k-1}\|_{W^{2,r_j}(\Omega)} \leq (C'K)^j (L + 1)^j + L \cdot B_1 \sum_{l=0}^{j-1} K^l(C')^l (L + 1)^l
\]
\[
+ B_1' \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n; \tag{78}
\]
\[
\|u_{k-1}\|_{L^{r_j}(\Omega)} \leq K^j(C')^{j-1}(L + 1)^{j-1} + L \cdot B_1 \sum_{l=1}^{j-1} (C'K)^l (L + 1)^{l-1} + B_1, j = 2, \ldots, M_n;
\]
\[
u_{k-1} = |u_{k-1}| \leq C_{M_n}.
\]
From (78), we conclude that for \( \lambda \) in (67), we have
\[
a u_{k-1} - S_g u_{k-1} + \lambda u_{k-1}^{p-1} \geq 0,
\]
and so \( u_k > 0 \) is a smooth solution of (77). By induction, (71) and (72) are replaced by
\[
\|\tilde{u}_k\|_{W^{2,r_j}(\Omega,g)} \leq L \sum_{l=0}^{j-1} K^l(C')^l ||u_{k-1}\|_{L^{r_j-l}(\Omega,g)} + (C'K)^j ||\tilde{u}_k||_{W^{2,2}(\Omega,g)}
\]
\[
+ B_1' \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n;
\]
\[
\|\tilde{u}_k\|_{L^{r_j}(\Omega,g)} \leq \sum_{l=1}^{j-1} (C'K)^l ||u_{k-1}\|_{L^{r_j-l}(\Omega,g)} + K^j(C')^{j-1} ||\tilde{u}_k||_{W^{2,2}(\Omega,g)}
\]
\[
+ B_1' \sum_{l=0}^{j-1} (C'K)^l, j = 2, \ldots, M_n.
\]
Using the estimates in (78) and arguing as in Appendix A, we conclude that (78) holds with the index shift \( k-1 \rightarrow k \):
\[
\|\tilde{u}_k\|_{W^{2,2}(\Omega,g)} \leq 1, \|\tilde{u}_k\|_{L^1(\Omega,g)} \leq K, \|u_k\|_{L^1(\Omega,g)} \leq K + B_1;
\]
\[
\|\tilde{u}_k\|_{W^{2,r_j}(\Omega,g)} \leq (C'K)^j (L + 1)^j + L \cdot B_1 \sum_{l=0}^{j-1} K^l(C')^l (L + 1)^l
\]
\[
+ B_1' \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n; \tag{79}
\]
\[
\|u_k\|_{L^{r_j}(\Omega,g)} \leq K^j(C')^{j-1}(L + 1)^{j-1} + L \cdot B_1 \sum_{l=1}^{j-1} (C'K)^l (L + 1)^{l-1} + B_1, j = 2, \ldots, M_n;
\]
\[
u_k = |u_k| \leq C_{M_n}.
\]
In summary, the upper bounds in (79) hold for all \( k \in \mathbb{Z}_{\geq 0} \) for fixed \( \lambda \) satisfying (68). By the argument starting at (29), where \( \lambda \) must be independent of \( k \), we conclude that \( \{u_k\} \) is a Cauchy sequence in \( H^1(\Omega,g) \). Thus \( \lim_{k \to \infty} u_k = u \) exists in \( H^1(\Omega,g) \) (with \( u \in H^2(\Omega,g) \) by Lemma 2.1) solves the Yamabe equation (3), and satisfies \( u \geq 0 \). By the usual maximum principle argument, \( u > 0 \).

This finishes Case II and the theorem. \( \square \)
Remark 3.1. We can always change the boundary condition to \( c = 1 \) by scaling \( u \) to \( c^{-1}u \), which scales \( \lambda \) to \( e^{p-2}\lambda \). This may force us to shrink \( \Omega \) due to (29). The advantage is that the new constant scalar curvature metric \( \tilde{g} \) associated to \( c^{-1}u \) equals \( g \) at \( \partial \Omega \). Thus we solve the Yamabe problem while keeping the scalar curvature of \((\partial \Omega, g_{\partial \Omega})\) unchanged.

Appendix A. Proof of the claim

Claim: We have

\[
\|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \leq 1, \|\tilde{u}_1\|_{L^1(\Omega, g)} \leq K, \|u_1\|_{\mathcal{L}^1(\Omega, g)} \leq K + B_1 \tag{80}
\]

\[
\|\tilde{u}_1\|_{W^{2,r_j}(\Omega, g)} \leq (C'K)^j (L + 1)^j + L \cdot B_1 \sum_{l=0}^{j-1} K^l(C')^{l+1} (L + 1)^l + B_1, j = 1, \ldots, M_n. \tag{81}
\]

\[
\|u_1\|_{\mathcal{L}^j(\Omega)} \leq K^j(C')^{j-1}(L + 1)^{j-1} + L \cdot B_1 \sum_{l=1}^{j-1} (C'K)^l (L + 1)^{l-1} + B_1, j = 2, \ldots, M_n. \tag{82}
\]

Proof. The three parts of (80) follow from (i) applying (31); (ii) the first line of (62) with \( \tilde{g} \) replaced with \( \tilde{u}_1 \); (iii)

\[
\|u_1\|_{L^2(\Omega)} \leq \|\tilde{u}_1\|_{L^2(\Omega)} + \|c\|_{L^1(\Omega, g)} \leq K + c \text{Vol}_g(\Omega)^{1/r_1} \leq K + B_1.
\]

For (81), we recall (71):

\[
\|\tilde{u}_1\|_{W^{2,r_j}(\Omega, g)} \leq L \sum_{l=0}^{j-1} K^l(C')^{l+1} \|u_0\|_{\mathcal{L}^{r_j-1}(\Omega, g)} + (C'K)^j \|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} + B_1 \sum_{l=0}^{j-1} K^l(C')^{l+1}, j = 1, \ldots, M_n.
\]

The last terms in (71) and (81) are equal. Insert the estimate for \( \|\tilde{u}_0\|_{L^{r_j-1}(\Omega, g)} \) in (63) into the first term on the right hand side of (71). Since \( \|\tilde{u}_1\|_{W^{2,2}(\Omega, g)} \leq 1 \), we get

\[
\|\tilde{u}_1\|_{W^{2,r_j}(\Omega, g)} \leq L \sum_{l=0}^{j-2} K^l(C')^{l+1} \left[ K^{j-l}(C')^{j-l-1}(L + 1)^{j-l-1} + L \cdot B_1 \sum_{s=1}^{j-l-1} (C'K)^s (L + 1)^{s-1} + B_1 \right] + LK^{j-1}(C')^j (K + B_1) + (C'K)^j + B_1 \sum_{l=0}^{j-1} K^l(C')^{l+1}
\]

\[
= L(KC')^j(L + 1)^j \sum_{l=0}^{j-1} (L + 1)^{-l} + L^2 B_1 \sum_{l=0}^{j-2} K^l(C')^{l+1} \sum_{s=1}^{j-l-1} (C'K)^s (L + 1)^{s-1} + LB_1 \sum_{l=0}^{j-1} K^l(C')^{l+1} + (C'K)^j + B_1 \sum_{l=0}^{j-1} K^l(C')^{l+1}.
\]

The fourth term on the RHS of (83) involves the estimate for \( \|\tilde{u}_0\|_{L^1(\Omega, g)} \) in (63), as the estimate on the last line of (63) is valid for \( j \geq 2 \). To pass from (83) to (81), we use (i) the first term on the RHS of (81) combines the first term on the RHS of (83) with the subterm \( LK^{j-1}(C')^jK \) in the fourth term on the RHS of (83); (ii) the third term on the RHS of (81) combines the term
Plugging \( (86) \) into \((85)\) gives

\[
L(KC')^j(L + 1)^{j - 1} \sum_{l=0}^{j-1} (L + 1)^{-l} = L(C'K)^j \frac{(L + 1)^j - 1}{L + 1 - 1} = (C'K)^j(L + 1)^j - (C'K)^j.
\]

Thus \((84)\) becomes

\[
\|\hat{u}_1\|_{W^{2,r_j}((\Omega, \delta_g)} \leq (C'K)^j(L + 1)^j + L^2B_1 \sum_{l=0}^{j-2} K^l(C')^{l+1} \sum_{s=1}^{j-l-1} (K'C)^s(L + 1)^{s-1} \]

\[
+ LB_1 \sum_{l=0}^{j-1} K^l(C')^{l+1} + B'_1 \sum_{l=0}^{j-1} K^l(C')^{l+1}.
\]

We simplify the second and the third terms on the RHS of \((85)\) by expanding out the second term in powers of \(l\) and then collecting powers of \(KC'\):

\[
L^2B_1 \sum_{l=0}^{j-2} K^l(C')^{l+1} \sum_{s=1}^{j-l-1} (K'C)^s(L + 1)^{s-1} + LB_1 \sum_{l=0}^{j-1} K^l(C')^{l+1}
\]

\[
= C'L^2B_1 \left[ \sum_{s=1}^{j-1} (K'C)^s(L + 1)^{s-1} + K'C' \sum_{s=1}^{j-2} (K'C)^s(L + 1)^{s-1} \right]
\]

\[
+ (K'C')^2 \sum_{s=1}^{j-3} (K'C)^s(L + 1)^{s-1} + \ldots + (K'C)^j \sum_{s=1}^{j-2} (K'C)^s(L + 1)^{s-1}
\]

\[
+ C'LB_1 \left( \frac{(K'C)^j - 1}{K'C' - 1} \right)
\]

\[
= C'L^2B_1 \left[ C'K + (C'K)^2((L + 1) + 1) \right.
\]

\[
+ \ldots + (C'K)^j - 1 ((L + 1)^{j-2} + (L + 1)^{j-2} + \ldots + 1) \bigg] + C'LB_1 \left( \frac{(K'C)^j - 1}{K'C' - 1} \right)
\]

\[
= C'L^2B_1 \sum_{l=1}^{j-1} (K'C)^l \left( \frac{(L + 1)^l - 1}{L + 1 - 1} \right) + C'LB_1 \left( \frac{(K'C)^j - 1}{K'C' - 1} \right)
\]

\[
= C'LB_1 \sum_{l=1}^{j-1} (K'C)^l (L + 1)^l - \sum_{l=1}^{j-1} (K'C)^l + C'LB_1 \left( \frac{(K'C)^j - 1}{K'C' - 1} \right)
\]

\[
= C'LB_1 \left[ \sum_{l=1}^{j-1} (K'C)^l (L + 1)^l \right].
\]

Plugging \((86)\) into \((85)\) gives

\[
\|\hat{u}_1\|_{W^{2,r_j}((\Omega, \delta_g)} \leq (C'K)^j(L + 1)^j + L \cdot B_1 \sum_{l=0}^{j-1} K^l(C')^{l+1} (L + 1)^l + B'_1 \sum_{l=0}^{j-1} K^l(C')^{l+1},
\]

which is \((81)\).
The proof of (52) is similar. We plug the last line of (63) into (72) and proceed as above. □

Appendix B. Table of constants

| Constant   | First appearance |
|------------|------------------|
| $a, p, c, \lambda$ | Below (3) |
| $C_1, C_2$ | (8) |
| $C_{m,j,q,r,\alpha}$ | (5) |
| $K(k, p)$ | (9) |
| $\lambda_1$ | (10) |
| $C^*$ | (13) |
| $C'$ | (16) |
| $\tilde{C}'$ | (17) |
| $C_0$ | (24) |

| Constant   | First appearance |
|------------|------------------|
| $B$ | (27) |
| $A$ | (13) |
| $K$ | (16) |
| $K'$ | (48) |
| $D_1, D_2$ | Theorem 2.7 |
| $L$ | (56) |
| $M_n$ | (59) |
| $C', B'_1, B_1$ | (60) |
| $C_{M_n}$ | (61) |

References

[1] R. Adams and J. Fournier. *Sobolev Spaces*. Pure and Applied Mathematics. Academic Press, Cambridge, MA, 2nd edition, 2003.
[2] T. Aubin. *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*. Grundlehren der mathematischen Wissenschaften. Springer, Berlin, Heidelberg, New York, 1982.
[3] P. Aviles and R. McOwen. Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds. *J. Differential Geom.*, 27(2):225–239, 1988.
[4] S. Brendle and S.-Z. Chen. An existence theorem for the Yamabe problem on manifolds with boundary. *J. Eur. Math. Soc. (JEMS)*, 16(5):991–1016, 2014.
[5] J. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.
[6] J. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.
[7] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer, Berlin, Heidelberg, New York, 2001.
[8] N. Grosse. The Yamabe equation on manifolds of bounded geometry. *Comm. Anal. Geom.*, 21(5):957–978, 2013.
[9] P. Hintz. Global analysis of quasilinear wave equations on asymptotically de Sitter spaces. *Annales de l’Institut Fourier*, 66(4):1285–1408, 2016.
[10] P. Hintz and A. Vasy. Global analysis of quasilinear wave equations on asymptotically Kerr-de Sitter spaces. *International Mathematics Research Notices*, 2016(17):5355–5426, 2016.
[11] J. Lee and T. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.
[12] P. Li. Poincaré inequalities on Riemannian manifolds. *Annals of Mathematics Studies*, 102:73–83, 1982.
[13] P. Li and S.T. Yau. Estimates of eigenvalues of a compact Riemannian manifold. *Proceedings of Symposia in Pure Mathematics*, 36, January 1980.
[14] F. Marques. Existence results for the Yamabe problem on manifolds with boundary. *Indiana Univ. Math. J.*, 54(6):1599–1620, 2005.
[15] J. Moser. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Nat. Acad. Sci. U.S.A.*, 47:1824–1831, 1961.
[16] J. Moser. A rapidly convergent iteration method and nonlinear differential equations. II. *Ann. Scuola Norm. Sup. Pisa*, 20(3):499–535, 1966.
[17] M. Taylor. *Partial Differential Equations I*. Springer-Verlag, New York, New York, 2011.
[18] J. Xu. Iterative methods for globally Lipschitz nonlinear Laplace equations, arXiv:1911.10192. Submitted.
[19] J. Zhiren. A counterexample to the Yamabe problem for complete noncompact manifolds. In S.-S. Chern, editor, *Partial Differential Equations Proceedings of a Symposium held in Tianjin, June 23 – July 5, 1986*, number 1308 in Springer Lecture Notes in Mathematics, pages 93–101, 1988.
