The Family BlowUp Formula of the Family Seiberg-Witten Invariants

Ai-Ko Liu*

October 29, 2018

In this paper one studies the family blowup formula of the family Seiberg-Witten invariants [LL]. The paper is one among a series of papers aiming at studying the family Seiberg-Witten invariants. The family blowup formula has several interesting applications in symplectic geometry and in enumerative geometry. The applications of the formula will be presented in the other papers. Recently, the author has applied the technique of the family blowup formula to resolve some conjecture [Liu1] by Götsche [Got] and Götsche-Yau-Zaslow [YZ], [Got]. Furthermore, a proof of the Harvey-Moore conjecture [Liu2] is given along the line of [Liu1], in which the existence of family blowup formula has played an essential role. The current paper contains the material which provides the foundation of these applications. It is also interesting to compare the results in this paper with the one in [LL1] about the wall crossing formula of the family Seiberg-Witten invariants.

The derivation of blowup formula has a long history in Donaldson theory. After being conjectured by the various experts about its existence, it was calculated first by R. Stern and R. Fintushal [FS2] the universal formula. Soon after the Seiberg-Witten theory had been developed in [W], the much more simplified blowup formula were derived by the various experts immediately. Despite of its simplicity, it has played a very crucial role in understanding the four-manifold topology. It was in the two papers [LL1], [LL2], T.J. Li and the author found out the link between blowup formula and certain discrepancy of Taubes’ “SW=Gr” in the $b_2^+ = 1$ category. Later it was D. Mcduff who modified the definition of Gromov-Taubes invariants in this special case and proposed a modified definition of

---

*email address: akliu@math.berkeley.edu, Current Address: Mathematics Department of U.C. Berkeley
the Gromov-Taubes invariant that was believed to be identifiable to the Seiberg-Witten invariants.

It was in [LL1] that the family Seiberg-Witten invariants were defined and studied by the current author and T.J. Li. Soon after our study, it was found that the family invariants shared the same discrepancy as the ordinary Seiberg-Witten invariant of $b_2^+ = 1$ four-manifold. It was proposed by the author in a discussion with T.J. Li to use the blowup formula in studying this phenomena. It turns out that the formula has some rather interesting applications in enumerative geometry, too. The details will be presented elsewhere [Liu1].

The author wants to thank Prof. C. H. Taubes and Prof. S.T. Yau for their encouragement. The author also likes to thank T.J. Li with whom the theory of family Seiberg-Witten invariants were jointly developed [LL1].

The organization of the current paper is as following. In the first section, we set up the family blowup construction and set up the notations that will be frequently used in the following sections. In section 2, one derives the family blowup formula in the $C^\infty$ category using the language of $\text{spin}^c$ spinors and connections. The readers with an algebraic background can skip over the derivation in section 2 and jump to section 3.

After deriving the blowup formula, we outline a few applications of the blowup formula in the various sub-sections of section 2. In sub-section 2.2, the application to enumeration of singular curves with prescribed singular multiplicities [Liu1] is addressed in certain detail.

In section 3, one develops a version of algebraic Seiberg-Witten invariants $\mathcal{ASW}$ for algebraic surfaces. The definitions are separated into cases and are discussed in different sub-sections. In sub-section 3.3.1 we discuss the relationship between $\mathcal{ASW}$ and the usual Seiberg-Witten invariants. In section 5 we prove the family blowup formula for the algebraic Seiberg-Witten invariants. Finally, in the subsection 5.1, we construct the universal obstruction bundles for the universal families.

As a preliminary, let us start by stating the basic facts about family Seiberg-Witten invariants and $\text{spin}^c$ structures.

A $\text{spin}^c$ structure on a four-manifold $M$ determines a $U(2)$ bundle over $M$. Following the usual convention, we use its determinant line bundle $\mathcal{L}$ to parametrize the $\text{spin}^c$ structures on $M$. Thus, $\text{spin}^c$ structures on $M$ can be identified non-canonically with (up to torsions) $H^2(M, \mathbb{Z})$.

The Seiberg-Witten invariant on $M$ are defined using the Seiberg-
Witten moduli spaces with the expected dimension formula 
\[ d_{SW}(\mathcal{L}) = \frac{c_1(\mathcal{L})^2 - 2\chi(M) - 3\sigma(M)}{4} \] 
onumber
on the moduli spaces. For \( b_2^+ > 1 \) manifolds, the invariants defined are diffeomorphism invariants of the four-manifolds. For \( b_2^+ = 1 \) manifolds, the invariants defined depends on additional chamber structures of the SW equations.

The family Seiberg-Witten invariants are a natural generalization of Seiberg-Witten invariants to fiber bundles \( \mathcal{X} \hookrightarrow B \) of smooth four-manifolds.

Given a monodromy invariant fiberwise \( spin^c \) structure and a fiberwise homotopic class of section \([B, \mathcal{X}]_{fiber}\), one may define \( FSW \) (see [LL1]).

The expected dimension formula of a relative \( spin^c \) structure \( \mathcal{L} \) is
\[
\dim_{\mathbf{R}} B + \frac{c_1(\mathcal{L})^2 - 2\chi(\mathcal{X}/B) - 3\sigma(\mathcal{X}/B)}{4}.
\]

If \( \dim_{\mathbf{R}} B < b_2^+ - 1 \), the family invariant defined are independent to the relative smooth Riemannian metrics and the choices of families of relative self-dual two forms used to define the family SW equations.

If \( \dim_{\mathbf{R}} B \geq b_2^+ - 1 \), the family invariant defined may depend on the additional chamber structures.

Let us state the main theorems in this paper. The detailed discussion on the notations will be discussed in section 1.

Let \( \mathcal{X} \hookrightarrow B \) be a smooth fiber bundle over a smooth oriented even dimensional base \( B \) of oriented four-manifolds with \( b_2^+ \geq 1 \). Let \( s : B \hookrightarrow \mathcal{X} \) be a smooth cross section such that the normal bundle \( N_{s(B)}\mathcal{X} \) is identified with a complex rank two bundle \( N_s \).

Through tubular neighborhood theorem it induces fiberwise almost complex structures in a neighborhood of \( s(B) \subset \mathcal{X} \).

Let \( \mathcal{X}' \) be the relative almost complex blowing up of \( \mathcal{X} \) along \( s(B) \), let \( E \) denote the exceptional line bundle associated to the exceptional locus \( \cong P_B(N_s) \).

Let \( \mathcal{L} \) denote a relative \( spin^c \) structure of \( \mathcal{X} \hookrightarrow B \) and let \( \mathcal{L}_0 \) denote the pull-back of \( \mathcal{L} \) by \( s : B \hookrightarrow \mathcal{X} \). In the additive notation, \( \mathcal{L} + mE, m \) odd, represents the \( spin^c \) structure associated with the tensor product \( \mathcal{L} \otimes E^\otimes m \).

Fix a fiberwise homotopic class of \( C^\infty \) sections \([B, \mathcal{X}']\), it induces a fiberwise homotopic class of sections \([B, \mathcal{X}]\) through the blowing down map \( \mathcal{X}' \hookrightarrow \mathcal{X} \). The pure and mixed family invariants of \( \mathcal{L} \), \( \mathcal{L} + mE \) are defined as in [LL1].

3
Main Theorem 0.1 (Blowup formula for pure Family invariants)

Let \( m \) be an odd integer, then we have the following family blowup formulae relating the pure invariant of \( \mathcal{L} + mE \) with the mixed invariants of \( \mathcal{L} \). Suppose that \( m \geq 3 \) and both of the spin\(^c\) structures have non-negative family Seiberg-Witten dimensions, i.e.

\[
\dim_{\mathbb{R}} B + \frac{c_1(\mathcal{L})^2 - 2\chi(\mathcal{X}/B) - 3\sigma(\mathcal{X}/B)}{4} - \frac{m^2 - 1}{4} \geq 0,
\]

then

\[
FSW_B(1, \mathcal{L}+mE) = \sum_{i \geq 0} FSW_B(c_i(\sqrt{L_0} \otimes \det(N_s)^{-1} \otimes S^{m-3}(N_s \oplus C_B)), \mathcal{L}).
\]

If \( m \) is a negative odd integer, then

\[
FSW_B(1, \mathcal{L}+mE) = \sum_{i \geq 0} FSW_B(c_i(\sqrt{L_0} \otimes \det(N_s)^{-1} \otimes S^{m-3}(N_s^* \oplus C_B)), \mathcal{L}).
\]

Let \( m = 1 \), then \( FSW_B(1, \mathcal{L} + mE) = FSW_B(1, \mathcal{L}) \).

The similar blowup formula of the mixed invariants will be stated and proved in the section 2.

In section 3, we define a version of algebraic Seiberg-Witten invariants on algebraic surfaces \( M \) for cohomology classes \( C = c_1(\mathcal{L} \otimes K_M) \in H^{1,1}(M, \mathbb{C}) \cap H^{2}(M, \mathbb{Z}) \).

Main Theorem 0.2 Let \( M \) be an algebraic surface and let \( C \) be an integral \((1,1)\) cohomology class, then there exists an \( ASW(C) \in \mathbb{Z} \) defined in terms of the moduli space of algebraic curves dual to \( C \).

For \( p_g = 0 \) surfaces, the \( ASW(C) \) can be identified (up to signs) with the usual \( SW \) invariant of the class \( 2C - c_1(K_M) \in H^2(M, \mathbb{Z}) \) in the chamber deformed by large multiples of Kahler forms.

The major distinction of algebraic Seiberg-Witten invariants from the usual \( SW \) on symplectic four-manifolds is that \( ASW \) are not of simple type, i.e. \( ASW(C) \) may be non-zero for classes with \( d_{GT}(C) = \frac{C^2 - C \cdot c_1(K_M)}{2} > 0 \).

The construction for algebraic Seiberg-Witten invariants (based on Kuranishi models) can be extended to algebraic families \( \mathcal{X} \to B \) or its relative blowing up \( \mathcal{X}' \to B \). Then we have the corresponding family blowing up formulae relating the \( AFSW \) of different classes,
Main Theorem 0.3 (Blowup Formula for Algebraic Family Seiberg-Witten Invariants).

Let $c$ be a class of algebraic cycle in $B$. Then the algebraic mixed invariants of $C + mE$ and $C$ are related by

$$\mathcal{AFSW}_{X' \mapsto B}(c, C + mE) = \sum_{i \geq 0} \mathcal{ASW}_{X' \mapsto B}(c \cap c_i (E \otimes (S^{m-2}(C_B \oplus N_{s(B)}^n X))), C)$$

for $m \geq 2$ and

$$\mathcal{AFSW}_{X' \mapsto B}(c, C + mE) = \sum_{i \geq 0} \mathcal{ASW}_{X' \mapsto B}(c \cap c_i (E \otimes (S^{-m-1}(C_B \oplus N_{s(B)}^n X))), C)$$

for $m \leq -1$.

For $m = 0, 1$, we have

$$\mathcal{AFSW}_{X' \mapsto B}(c, C + mE) = \mathcal{AFSW}_{X \mapsto B}(c, C).$$

For the assumptions imposed on $X', X$ and the detailed assumptions of theorem 0.3 please consult section 5 and theorem 5.1.

1 The family blowing up construction

Suppose that one is given a fiber bundle $X$ over a compact oriented manifold $B$ whose typical fibers are diffeomorphic to an oriented four-manifold $M$ with $b_2^+ > 0$. Let $s : B \mapsto X$ be a smooth cross section and a tubular neighborhood in $X$ is denoted by $\mathcal{N}$. The tubular neighborhood theorem allows us to identify $\mathcal{N}$ with a real four dimensional vector (ball) bundle over $s(B)$. One imposes the extra condition on the section $s$ requiring that the real four-plane bundle carries complex structures and fixes one complex structure in our discussion. Therefore, the normal bundle is viewed as a rank two complex vector bundle, denoted by $N_s$.

Let $C_B$ be the trivial complex line bundle over $B$ and let $P = \overline{P_B(N_s \oplus C_B)}$ be the projectification of the bundle $N_s \oplus C_B$, with the fiber-wise orientation reversed. The trivial factor $C_B$ in $N_s \oplus C_B$ defines a smooth section to $P$ whose tubular neighborhood is diffeomorphic to $N_s$, the total space of the bundle $N_s$, with bundle orientation reversed. With the preceding convention understood, we can perform the fiberwise connected sum by deleting the two
tubular neighborhoods of \( X \) and \( \bar{P} \) and gluing their complements via a fiber-wise orientation-reversing diffeomorphism. The new fiber bundle, denoted by \( X_s' \), \( X_s' \cong X_s' \bar{P} \), is called the family blowing up of \( X \) along \( s \). Unlike the case when \( B \) is a point, the existence of \( s \) is no longer a completely trivial matter. However, if the fiber bundle \( X \hookrightarrow B \) has a fiber-wise almost complex structure, then any section \( s \) can be blown up topologically. Another new feature is that different choices of fiberwise homotopy classes of these cross sections may result in non-diffeomorphic fiber bundles \( X_s' \).

When the cross section \( s \) has been fixed, we may drop the subscript \( s \) in \( X_s' \) and denote the resulting fiber bundle by \( X' \).

**Remark 1.1** If the fiber bundle carries a family of fiberwise symplectic forms (parameterized by the base), then it induces a fiber-wise almost complex structure on the fibers and we can blow up any smooth cross section. Moreover, one can mimic the symplectic blowing up construction of Guillemin-Sternberg-Mcduff [Mc] to construct a new family of fiberwise symplectic forms on the “blown up” fiber bundle. Even if we have considered the family blowing ups in the symplectic category, nevertheless we do not require the additional condition that the total space of the fiber bundle to be symplectic. Sometimes the additional condition is met and the family blowing up construction is really the blowing-up of a real codimension four symplectic cross section in a symplectic total space. We definitely want to relax the condition here as some natural families (e.g. twistor families of \( K3 \) or \( T^4 \) or its induced families on the universal spaces [Liu1]) simply do not satisfy this additional condition. However fiberwise blowing ups along the cross sections of these non-symplectic fiber bundles is crucial in counting holomorphic curves in the twistor families[Liu1].

It is a well known fact that the cohomology of a \( \mathbb{CP}^2 \) bundle is, as a module of the cohomology over the base \( H^*(B; \mathbb{Z}) \), generated by the various powers of “hyper-plane class”. The choices of the hyper-plane classes are not unique. Different choice of the hyper-plane class give different generators of the same module. The same assertion applies to \( \bar{P} \) as well once we flip the orientation. However, if we view \( \bar{P} \) as the projectification of \( N_s \oplus C_B \) with a reversed orientation, it does give us a canonical choice of exceptional class \( E \). From now on let us fix this choice implicitly. Then \( E_2^2 \cdot \pi_{\bar{P}}[B] = -1 \) and \( E^3 \) lies in \( \oplus_{i=0}^2 H^*(B)E^i \).
2 The Blowup Formula and the Proof of the Family Blowup Formula

Recall that the usual blowup formula of the Seiberg-Witten invariants relates the Seiberg-Witten invariant of a given Spin$^c$ structure on $M$ with the invariant of a corresponding Spin$^c$ structure on $M\sharp\mathbb{C}P^2$. Before we give a proof of the family blowup formula, let us review the idea behind the proof of the original formula. The proof of the original formula is based on the usage of the long neck metrics. Suppose there is a smooth metric on $M\sharp\mathbb{C}P^2$ such that the metric is isometric to the product metric on the cylindrical collar $S^3 \times [-L, L]$ between $M - B_{p_1}(\epsilon_1)$ and $\mathbb{C}P^2 - B_{p_2}(\epsilon_2)$. We let $L$ go to $\infty$ and discuss the structure of the moduli spaces. As $S^3$ has positive scalar curvature metrics, the spinor part of the Seiberg-Witten solutions tend to vanish on the long neck. This implies that any solution can be decomposed into one over $M - B_{p_1}(\epsilon_1)$ and the other one is over $\mathbb{C}P^2 - B_{p_2}(\epsilon_2)$. Conversely, we can glue solutions on $M - B_{p_1}(\epsilon_1)$ and $\mathbb{C}P^2 - B_{p_2}(\epsilon_2)$ back to solutions on the connected sum $M\sharp\mathbb{C}P^2$. As $\mathbb{C}P^2$ is negative definite, we can choose the metric such that the solutions on $\mathbb{C}P^2$ are reducible. Namely, $\psi \equiv 0$ and the connection is anti-self-dual. The reader can consult [FS], page 226, theorem 8.5, where the authors considered long necks of lens spaces $L(p^2, 1-p)$ instead of $S^3$.

The spin$^c$ structure reduces to some odd multiple of $E$ on $\mathbb{C}P^2$. We denote the spin$^c$ determinant line bundle on $M$ by $\mathcal{L}$. The $\mathcal{L}$ induces a spin$^c$ determinant line bundle on $M\sharp\mathbb{C}P^2$, also denoted by the same symbol.

If the absolute value of the multiplicity is equal to one, then $\dim(\mathcal{M}_L) = \dim(\mathcal{M}_{L \pm E})$, and $SW(\mathcal{L}) = SW(\mathcal{L} \pm E)$, where the left hand side is a Seiberg-Witten invariant of $M$ and the right hand side is that of $M\sharp\mathbb{C}P^2$.

On the other hand if the multiplicity is bigger than one in absolute value, the glued moduli space is smooth $\mathcal{M}_{L+mE} \cong \mathcal{M}_L$ but it is not of the expected dimension. Thus the obstruction bundle must be inserted in the calculation of the Seiberg-Witten invariants.

As usual let $e$ denote the Euler class of the $S^1$ bundle $e$ over $\mathcal{M}_{L+mE}$ constructed from the quotient of global $U(1)$ gauge transformations. Let $\text{Obs}_m$ denote the obstruction bundle on $\mathcal{M}_{L+mE}$.

Then the Seiberg-Witten invariant $SW(\mathcal{L} + mE)$ is calculated by
the following expression

$$
\int_{\mathcal{M}_\mathcal{L}} e^{\dim \mathcal{M}_{\mathcal{L}+mE}/2} \cup e(\text{Obs}_m)
$$

$$
= \int_{\mathcal{M}_\mathcal{L}} e^{\dim \mathcal{M}_{\mathcal{L}+mE}/2} \cup c_{\text{top}}(\text{Obs}_m)
$$

where $e(\text{Obs}_m)$ denotes the Euler class of the vector bundle $\text{Obs}_m$.

To prove that $SW(\mathcal{L}) = SW(\mathcal{L} + mE)$, it suffices to prove that $e(\text{Obs}_m)$ is a $\frac{m^2-1}{8}$ power of $e$.

We sketch a proof of the following simple lemma which is well known to the experts.

Lemma 2.1 Let $\tilde{e}$ be the complex line bundle over $\mathcal{M}_{\mathcal{L}+mE}$ induced by the principal $S^1$ bundle $e$. The obstruction bundle $\text{Obs}_m$ over $\mathcal{M}_{\mathcal{L}+mE}$ is isomorphic to the bundle $\tilde{e} \otimes \mathbb{C}^{\frac{m^2-1}{8}}$.

Proof of lemma 2.1

In fact the formal dimension of the moduli space $\mathcal{M}_{mE}$ on $\overline{\mathbb{C}P^2}$ is $\frac{-m^2-3}{4}$. On the other hand the expression $2\chi + 3\sigma$ decreases by one after a single blowing up. Thus the real formal dimensions of the moduli spaces $\dim \mathcal{M}_{\mathcal{L}}$ and $\dim \mathcal{M}_{\mathcal{L}+mE}$ differ by $\frac{-m^2+1}{4}$.

Because the unique solution of the $\text{spin}^c$ structure $mE$ on $\overline{\mathbb{C}P^2}$ is reducible, the solution is fixed by the $S^1$ action. Thus the obstruction bundle over $\mathcal{M}_{mE}$ is nothing but a $\frac{m^2-1}{8}$ dimensional complex vector space over a single point (the complex structure of the bundle is inherited from the complex spinors). Under the gluing construction the reducible solution on $\overline{\mathbb{C}P^2}$ is glued to a solution on $\mathcal{M}$. As a result, the final $S^1$ action on the glued based moduli space comes from the diagonal embedding of the $S^1$ actions on both sides of configuration spaces. Under this action, the obstruction bundle is identified with $\tilde{e} \otimes \mathbb{C}^{\frac{m^2-1}{8}}$. $\square$

After reviewing the idea to derive the blowup formula, we may generalize it to the family invariants. Recall that in defining the family invariant, the tautological class $e$ is not canonically defined. It depends on the choices of a homotopic classes of cross sections of the fiber bundles $\mathcal{X} \mapsto B$, $\mathcal{X}' \mapsto B$. The choice was made implicitly in the definition of the family invariants.

Given the data as in section 1 we relate the family Seiberg-Witten invariants of $\mathcal{L} + mE$ on $\mathcal{X}'_{\mathbb{R}_f} \overline{\mathbb{P}}$ and $\mathcal{L}$ on $\mathcal{X}$. As before the
family dimensions of the moduli spaces of these two spin$^c$ structures differ by $-\frac{m^2+1}{4}$. Using the same type of analysis as before, we would like to stretch the long neck such that the metric becomes the product metric along the long neck.

Lemma 2.2 The unit sphere bundle of $N_s$ carries a fiberwise positive scalar curvature metric.

Proof: Given the hermitian metric on $N_s$, it induces Euclidean metrics on the fibers of $N_s$. Then the unit spheres in the fibers are given the induced Riemannian metric over which $SO(4)$ acts transitively. Then the unit spheres $\cong S^3$ carries the round metric, which is well known to be of positive scalar curvature. $\square$

Because the $S^3$ bundle carries fiberwise positive scalar curvature metric, one can mimic the previous discussion on long neck metric (see also [FS]) to decompose or glue the family moduli spaces. By using the special kind of fiberwise metric, the family moduli spaces for $L + mE$ is the fiber product of the corresponding family moduli spaces for $L$ over $X \mapsto B$ and for $M_{\pi^*PL_0 + mE}$ over $\bar{P} \mapsto B$, where $M_{\pi^*PL_0 + mE} \cong M_{mE}$ is diffeomorphic to $B$ under the natural projection map. Here the line bundle $\mathcal{L}_0$ over $B$ denotes the pullback of $\mathcal{L}$ by $s : B \mapsto X$. Because $M_{mE}$ is not of the expected family dimension, it carries an obstruction bundle of real rank $\frac{m^2+3}{4}$.

If $m = \pm 1$, we may still conclude that these two family Seiberg-Witten invariants coincide. $FSW_B(L \pm E) = FSW_B(L)$.

When $m \neq \pm 1$, the analogue of lemma 2.1 is

Proposition 2.1 Let $\mathcal{X}' = \mathcal{X}_{sf} \bar{P}$ be the fiberwise connected sum of $\mathcal{X}$ with $\bar{P}$ with the long neck metric. Then there is a real $\frac{m^2-1}{4}$ dimensional obstruction bundle $\text{Obs}_m$ over $M_{L + mE} \cong M_{L}$.

Let $A_0$ denote the unique fiberwise anti-self-dual connections over $\bar{P} \mapsto B$ for the spin$^c$ line bundle $\pi^*_P\mathcal{L}_0 + mE$ with respect to the positive scalar curvature fiberwise metrics on $\bar{P}$. The obstruction bundle $\text{Obs}_m$ over $M_{L + mE}$ can be identified with $\bar{e} \otimes_{C} \pi^*_M\mathcal{W}_m$, where $\mathcal{W}_m$ is the $\frac{m^2-1}{8}$ dimensional complex vector bundle over $B$, the cokernel bundle of $D_{A_0}$.

The derivation of the proposition is parallel to lemma 2.1 except that a vector space $C^{\frac{m^2-1}{2}}$ should be replaced by a vector bundle of the same rank.
Proof of prop 2.1 The line bundle $\pi^*_\bar{P}L_0$ are pull-back from the base $B$ and it has a trivial first Chern class on the fibers. Its presence does not affect the dimension count.

Given the unique reducible solution $(A_0, 0)$ on $\bar{P}/B$ of $\pi^*_\bar{P}L_0 + mE$, consider the deformation complex of the family Seiberg-Witten equations at $(A_0, 0)$,

$$d^* \oplus d^+ \oplus D_{A_0} : \Omega^1_{\bar{P}/B} \oplus \Gamma(\bar{P}/B, S^+_{\pi^*_\bar{P}L_0 + mE}) \rightarrow \Omega^0_{\bar{P}/B} \oplus \Omega^2_{\bar{P}/B} \oplus \Gamma(\bar{P}/B, S^-_{\pi^*_\bar{P}L_0 + mE}).$$

Because $\bar{P}$ is simply connected with the positive scalar curvature metric, the kernel of the above deformation complex is trivial. Because $\bar{P}$ is negative definite, $\text{Coker}(d^+) = 0$. Thus, the cokernel of the deformation complex is equal to $R_B \oplus \text{Coker}(D_{A_0})$, where $R_B$ denotes the trivial real rank one line bundle of constant functions on $B$. This direct factor can be identified with the lie algebra of the global $S^1$ gauge action which fixes $(A_0, 0)$. By index calculation, $\text{Coker}(D_{A_0}) \subset \Gamma(\bar{P}/B, S^-_{\pi^*_\bar{P}L_0 + mE})$ is of complex rank $\frac{m^2-1}{8}$.

After we graft the reducible solution on $\bar{P}$ to $\mathcal{X}'$, the non-reducible solution on $\mathcal{X}'$ is not fixed by the $S^1$ action. Thus, only the $W_m = \text{Coker}(D_{A_0})$ factor of the obstruction bundle is grafted to an obstruction bundle on $\mathcal{M}_{L+mE}$.

By the same argument as in lemma 2.1, the complex line bundle $\tilde{e}$ (which depends on the choice of a cross section $B \mapsto \mathcal{X}'$) is tensored with $\pi^*_\mathcal{M}_\mathcal{L} W_m$.

Thus, we have

$$\text{Obs}_m \cong \tilde{e} \otimes \pi^*_\mathcal{M}_\mathcal{L} W_m, W_m = \text{Coker}(D_{A_0}).$$

Proposition 2.2 Given a positive and odd $m$, the vector bundle $W_m$ can be identified with

$$\sqrt{L_0} \otimes \text{det}(N_s)^{-1} \otimes S^{\frac{m+3}{2}}(N_s \oplus C_B),$$

in the rational K group $K(B) \otimes \mathbb{Q}$.

For $m$ odd and negative, the vector bundle $W_m$ can be identified with

$$\sqrt{L_0} \otimes \text{det}(N_s)^{-1} \otimes S^{\frac{m+3}{2}}(N_s \oplus C_B),$$

in the rational K group $K(B) \otimes \mathbb{Q}$. 

10
The proposition is the key to prove the family blowup formula. Proof of proposition 2.2: It suffices to calculate the Chern character of $W_m$ and show that it is equal to the Chern character of the right hand side bundle. In the proof of the previous proposition prop. 2.1, we know that $[W_m] = [-IND(D_{A_0})]$ in $K(B)$, the negative of family index of $D_{A_0}$.

In general the Chern character of the index bundle is calculated by the family index theorem [BGV]. As the calculation is purely topological, it does not depend on the explicit choice of the connection $A_0$. From now on we ignore the dependence of the family index virtual bundle on $A_0$. The Chern character $ch(IND(D_{A_0}))$ is calculated by the following expression,

$$
\int_{P \rightarrow B} \hat{A}_{P/B}ch(\sqrt{L_0} \otimes E^\otimes m).
$$

The symbol $\int_{P \rightarrow B}$ denotes the push-forward map from $H^*(\tilde{P})$ to $H^*(B)$.

We notice that $P = P(N_s \oplus C_B)$ and $\tilde{P}$ have the opposite orientations. If we flip the orientation of $\tilde{P}$ back to $P$ and switch the positive and negative spinors as well,

$$
ch(-IND(D_{A_0})) = - \int_{P \rightarrow B} \hat{A}_{P/B}ch(\sqrt{L_0} \otimes E^\otimes m)
$$

$$
= \int_{P \rightarrow B} \hat{A}_{P/B}ch(\sqrt{L_0} \otimes H^\otimes -m).
$$

In the above formula, the line bundle $E$ has been replaced by $H^*$, the tautological line bundle of the projective bundle $P(N_s \oplus C_B)$.

To continue the calculation, we apply the following simple lemma,

**Lemma 2.3** Let $V$ be a complex rank $n$ vector bundle over a smooth manifold $B$. Let us denote $P(V)$ to be the projective space bundle over $B$ formed by projectifying $V \mapsto B$. Then the first Chern class of the relative tangent bundle along the fibers, $c_1(T_{P(V)/B})$ is given by $nH + \pi^*_P(V)c_1(V)$.

**Proof of lemma 2.3**

Recall the following well known short exact sequence of $T_{\mathbb{CP}^{n-1}},$

$$
0 \rightarrow C \rightarrow H \otimes \pi^*_P \mathbb{C} \rightarrow T_{\mathbb{CP}^{n-1}} \rightarrow 0.
$$

The relative version of the sequence on $V$ gives
\[ 0 \to C_{P(V)} \to H \otimes \pi_{P(V)}^* V \to T_{P(V)/B} \to 0. \]

Then

\[ c_1(T_{P(V)/B}) = c_1(H \otimes \pi_{P(V)}^* V) = nH + \pi_{P(V)}^* c_1(V). \]

\[ \square \]

In our discussion, we take \( n = 3 \) and \( V = N_s \oplus C_B \).

Having identified the relative first Chern class of \( P \), we are ready to continue the calculation. Rewrite this family index push-forward as

\[
\int_{P/B} \hat{A}_{P/B} ch(\sqrt{H^3 \otimes \det(N_s)}) ch(\sqrt{\mathcal{L}_0 \otimes H^{\otimes(-m-3)} \otimes \det(N_s)^{-1}}).
\]

The relative \( \hat{A}_{P/B} \) and \( ch(\sqrt{H^3 \otimes \det(N_s)}) \) combine into the relative Todd class. And the calculation is reduced to a Grothendieck-Riemann-Roch type calculation on the projective space bundle \( P \hookrightarrow B \),

\[
\int_{P/B} \text{Todd}_{P/B} ch(\sqrt{\mathcal{L}_0 \otimes H^{\otimes(-m-3)} \otimes \det(N_s)^{-1}}) = ch(IND(\bar{\partial} + \bar{\partial}^*)) \cdot ch(\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1}}).
\]

\[
\bar{\partial} + \bar{\partial}^* : \Omega_{P/B}^{0,0} \otimes H^{\frac{m-3}{2}} \oplus \Omega_{P/B}^{0,2} \otimes H^{\frac{m-3}{2}} \to \Omega_{P/B}^{0,1} \otimes H^{\frac{m-3}{2}}
\]

If \( m \leq -3, \frac{-m-3}{2} \in \mathbb{N} \cup \{0\} \), then the above Family Riemann-Roch formula can be re-interpreted as (up to tensoring with \( \sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1}} \)) the Chern character of the push-forward of \( H^{\frac{m-3}{2}} \) along the projective bundle \( P \hookrightarrow B \).

Let us cite the following well known fact on projective spaces,

**Lemma 2.4** Let \( P(V) \) be the projective space formed by projectifying a complex vector space \( V \) and \( H \) denote the holomorphic hyperplane bundle on \( P(V) \) with the standard \( \bar{\partial} \) operator, then for \( p \geq 0 \), \( H^p_\partial(P(V), H^p) \) is naturally isomorphic to \( S^p(V^*) \), the \( p \)-th-symmetric power of linear functionals on \( V \).
We adopt the convention that $S^0(V) = \mathbb{C}$.

By applying the family version of lemma 2.4, the Chern character of $W_m$ is equivalent to the Chern character of the following bundle,

$$\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1} \otimes S^\frac{m-3}{2} (N_s^* \oplus C_B)}.$$

For $m \geq 3$ we apply relative Serre duality. Notice that the usual surface Riemann-Roch theorem [GH] for $\mathbb{C}P^2$ has the following structure,

$$h^0(\mathbb{C}P^2, H^p) - h^1(\mathbb{C}P^2, H^p) + h^2(\mathbb{C}P^2, H^p) = 1 + \frac{(p^2 + 3p)}{2} = \frac{(p + 1)(p + 2)}{2}.$$

It is easy to see that Serre duality $H^p \mapsto H^{-p-3}$ induces a symmetry on the formula. If $m - 3 \geq 0$, we replace $\frac{m-3}{2}$ by $-\frac{(m-3)}{2} = \frac{(m-3)}{2}$. Notice that

$$H^2(\mathbb{C}P^2, H^{-\frac{(m+3)}{2}}) \cong H^0(\mathbb{C}P^2, H^{\frac{m-3}{2}})^*.$$

And in this case the chern character is equivalent to that of

$$\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1} \otimes S^\frac{m-3}{2} (N_s \oplus C_B)}.$$

This ends the proof of proposition 2.2. □

We are ready to prove the following blowup formula of the family invariants.

**Theorem 2.1** (Blowup formula for the Pure Invariants) Let $\mathcal{X}, \mathcal{X}'$, $N_s$ be as defined in section 1 and let $\mathcal{L}, \mathcal{L}_0$ be the spin$^c$ determinant line bundle over $\mathcal{X} \mapsto B$ and its pull-back by $s : B \mapsto \mathcal{X}$.

Let $m > 1$ be an odd integer bigger such that the spin$^c$ structure $\mathcal{L} + mE$ has non-negative family Seiberg-Witten dimension, then we have the following blowup formula relating the pure invariant of $\mathcal{L} + mE$ over $\mathcal{X}' \mapsto B$ with the mixed invariants of $\mathcal{L}$ over $\mathcal{X} \mapsto B$,

$$FSW_B(1, \mathcal{L} + mE) = \sum_{i \geq 0} FSW_B(c_i(\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1} \otimes S^\frac{m-3}{2} (N_s \oplus C_B)}), \mathcal{L}).$$

Let $m < -1$ be an odd integer, then we have

$$FSW_B(1, \mathcal{L} + mE) = \sum_{i \geq 0} FSW_B(c_i(\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1} \otimes S^\frac{m-3}{2} (N_s^* \oplus C_B)}), \mathcal{L}).$$
Let $m = \pm 1$, then we have
\[ FSW_B(\mathcal{L} + mE) = FSW_B(\mathcal{L}). \]

Proof: The theorem is a consequence of proposition 2.2.

Fix a fiberwise homotopic class of fiberwise metrics and self-dual two forms pair on $\mathcal{X}$. Also fix a fiberwise homotopic class of cross sections $B \mapsto \mathcal{X}$. Following [LL1], one may define the family Seiberg-Witten invariant of $\mathcal{L} + mE$ in a specific chamber.

For $m = \pm 1$,
\[
FSW_B(1, \mathcal{L} \pm mE) = \int_{\mathcal{M}_\mathcal{L}} e^{\frac{\text{dim} B}{2} + \frac{c_1(\mathcal{L})^2 - 2\chi(M) - 3\sigma(M)}{8}} = FSW_B(1, \mathcal{L}).
\]

For $m \neq 1$, the pure invariant is equal to
\[
FSW_B(1, \mathcal{L} + mE) = \int_{\mathcal{M}_\mathcal{L}} e^{\frac{\text{dim} B}{2} + \frac{c_1(\mathcal{L})^2 - 2\chi(M) - 3\sigma(M) - m^2 + 1}{8}} c_{\text{top}}(\tilde{e} \otimes W_m).
\]

By proposition 2.2 we can replace $W_m$ by either
\[
\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1} \otimes S^{\frac{m-3}{2}}(N_s \oplus C_B)}
\]

or
\[
\sqrt{\mathcal{L}_0 \otimes \det(N_s)^{-1} \otimes S^{\frac{m-3}{2}}(N_s^* \oplus C_B)}
\]

depending on $m \geq 3$ or $m \leq -3$. We get the blowup formula by expanding the top Chern class in terms of the powers of $c_1(\tilde{e}) = e$.

Similarly, by inserting an even dimensional cohomology class $\eta \in H^*(B, \mathbb{Z})$ into the definition of the family invariant, we can get the similar formula relating the mixed invariants before and after the blowup process.

**Theorem 2.2 (Blowup formula for the Family Mixed Invariants)**

Suppose that $m \geq 3$, then it follows that
\[
FSW_B(\eta, \mathcal{L} + mE) = \sum_{i \geq 0} FSW_B(\eta \cup c_i(\sqrt{\mathcal{L}_0} \otimes \det(N_s)^{-1} \otimes S^{\frac{m-3}{2}}(N_s \oplus C_B)), \mathcal{L}).
\]

Suppose that $m \leq -3$, then
\[
FSW_B(1, \mathcal{L} + mE) = \sum_{i \geq 0} FSW_B(c_i(\sqrt{\mathcal{L}_0} \otimes \det(N_s)^{-1} \otimes S^{\frac{m-3}{2}}(N_s^* \oplus C_B)), \mathcal{L}).
\]
The proof is almost identical to theorem 2.1. We omit it. □

The family blowup formula has a very interesting dependence on $\mathcal{L}$ through $\mathcal{L}_0$, unlike the usual blowup formula of the Seiberg-Witten theory. Only in the special cases where the $\mathcal{L}_0$ become trivial, e.g. we have a trivial constant cross section in the trivial product family, does the dependence go away. The formula also depends on the complex rank two normal bundle $N_s$ explicitly. The gluing technique localizes the effect of the blowing up to the family invariant into a vicinity of the cross section $s : B \mapsto X$. In a sense the blowup formula should be viewed as a localization theorem of the Family Seiberg-Witten invariants.

We will spend some time in the next subsection 2.1 to find out the geometric meaning of the dependence. The formula also have a nontrivial dependence on the odd integer $m$, while the original blowup formula has no explicit dependence in $m$ at all. The reader may notice that there is a duality symmetry between $m \mapsto -m$. In the proof we see that the $\mathbb{Z}_2$ symmetry roots at the Serre duality of $\mathbb{CP}^2$.

2.1 Applications to Counting Singular Curves

In this subsection, we discuss the relation between the family blowup formula and the counting singular curves with prescribed multiplicities. The purpose is to link up the cohomological information of the family blowup formula derived in the previous section with the algebraic-geometric data which also appears in the discussion upon ideal sheaves of points.

We would like to achieve a few goals in this subsection:

I. Understand the algebraic structure appearing in the family blowup formula. Relate it with a pseudo-holomorphic or algebraic geometric question of counting curves with singularities of prescribed multiplicities.

II. Motivate the definition of algebraic Seiberg-Witten invariant (see section 3) and an algebraic proof of family blowup formula. (see section 5)

III. In proposition 2.3 use an example to illustrate why the $SW$ simple type condition for $b^+_2 > 1$ symplectic four-manifolds implies the vanishing of all the family invariants used to count singular curves. Again, motivate to define algebraic Seiberg-Witten invariant without the simple type property.
The blowing up construction does not require the fiber bundle \( \mathcal{X} \rightarrow B \) to carry fiber-wise almost complex structures. If it does, the tubular neighborhood of \( s : B \rightarrow \mathcal{X} \) inherits the almost complex structures from the ambient space. As there are abundant examples of fiber bundles with fiberwise almost complex structures, there are countless examples for which the family blowup formula can be applied to.

We also recall that when a four-manifold \( M \) carries an almost complex structure, any \( \text{spin}^c \) determinant line bundle \( L \) can be rewritten as \( 2C + K_M^{-1} \) in the additive notation. The class \( C \) is the cohomology class appearing in the Gromov-Taubes theory such that \( SW(L) = Gr(C) \). Considers a family which carries fiber-wise almost complex structures, then \( \det(N_s) \) is isomorphic to \( K_M^{-1} \). Then the expression \( \sqrt{L_0 \otimes \det(N_s)^{-1}} \) is nothing but the restriction of \( \sqrt{L \otimes K_M} \) to the section \( s : B \rightarrow \mathcal{X} \). This is the first hint that the pure topological derivation of the family blowup formula may have something to do with the Gromov-Taubes theory.

The dependence on \( N_s \) is manifestly present at the term \( S_{\leq \frac{m-3}{2}}^m(N_s \oplus C_B) = \oplus_{i \leq \frac{m-3}{2}} S^i(N_s) \). Given the complex normal bundle \( N_s \), it is exactly the bundle of polynomial algebra in \( N_s \) of degree less than or equal to \( \frac{m-3}{2} \).

To understand the structure, let us consider a very special case of the family blowup formula.

**Example 2.1** Let \( M \) be a complex surface. Consider \( \mathcal{X} = M \times M \rightarrow M \) to be the product fiber bundle. Instead of using the trivial constant cross section, we consider the diagonal cross section \( \Delta : M \rightarrow M \times M \). It is well known that the normal bundle of \( \Delta(M) \subset M \times M \) is isomorphic to the tangent bundle \( T_M \) itself. Blowing up \( \Delta(M) \subset M \times M \) in the complex category produces \( \mathcal{X}' = \text{Blowup}_{\Delta(M)}M \times M \). \( \mathcal{X}' \rightarrow M \) is the blown up fibration from \( M \times M \rightarrow M \).

Let \( E_C \) be a complex line bundle on \( M \) with \( c_1(E) = C \in H^2(M, \mathbb{Z}) \). The pull-back of \( E_C \) to \( \mathcal{X} = M \times M \) induces a line bundle on the fiber bundle \( M \times M \rightarrow M \). The restriction of any complex line bundle \( E_C \) to the diagonal section \( \Delta : M \rightarrow M \times M \), is isomorphic to \( E_C \). In the example take \( L = E_C^2 \otimes K_M^{-1} \). As usual let \( E \) denote the
exceptional class of the blowing up.

According to family blowup formula for $m = -2p - 1 < -1$,

$$FSW_M(1, \mathcal{L} - (2p+1)E) = \sum_{i} FSW_B(c_i(E_C \otimes (\oplus_{0 \leq k \leq p-1} S^k(T^*_M))), \mathcal{L}).$$

The obstruction bundle whose Chern classes are inserted in the blowup formula is given by

$$E_C \otimes (\oplus_{k \leq p-1} S^i(T^*_M)).$$

Taking a closer look at the bundle, one realizes that it is nothing but the $p - 1$jets bundle of the line bundle $E_C$. Take an arbitrary point $x \in M$. Consider an arbitrary germ of smooth section $s$ of $E_C$. Recall that given a connection on a smooth line bundle $E_C$, the covariant derivative defines a map

$$\nabla : \Gamma(U, E_C) \otimes T_M \mapsto \Gamma(U, E_C),$$

for open sets $U$ containing $x$.

We consider $(s(x), (\nabla s)(x), S^2(\nabla \nabla) s(x), \ldots, S^{p-1}(\nabla \cdots \nabla)s(x))$, where $S^k(\nabla \cdots \nabla)$ denotes the symmetrized $k$–th order covariant derivative operator. When we let the point $x$ move along the base manifold $M$, we find that the datum exactly have the right dependence as a section of $E_C \otimes (\oplus_{i \leq p-1} S^i(T^*_M))$.

Now we are ready to perform the calculation. From the family blowup formula (see page 13), the pure family invariant on the blown up fiber bundle is expressed as the sum three terms of mixed family invariants over $M \times M \mapsto M$. The first term is the pure invariant. The second term is the mixed invariant with $c_1(E_C \otimes S^{p-1}(T^*_M \oplus C_M))$ inserted. The third term is the mixed family invariant with $c_2(E_C \otimes S^{p-1}(T^*_M \oplus C_M))$ inserted, calculated with respect to the $spin^c$ structure $\mathcal{L} = E^2_C \otimes K^{-1}_M$ on $M \times M \mapsto M$.

Because the fiber bundle $\mathcal{X} = M \times M \mapsto M = B$ is a trivial product bundle, it is easy to see that the first two terms always vanish. It is because the family moduli space of $\mathcal{L} = E^2_C \otimes K^{-1}_M$ over $B = M$ will be a trivial product. It implies the vanishing of the pure and mixed invariants unless the base class cohomological insertion has a degree equal to the base dimension. Thus the pure invariant of the blown up fiber bundle $FSW_M(1, \mathcal{L} - (2p + 1)E)$ is equal to the product of usual $SW$ invariant of $\mathcal{L}$ (over $B =$ a point) and $\int_M c_2(E_C \otimes S^{p-1}(T^*_M \oplus C_M))$. 

17
Suppose we choose \( p > 0 \), it is not hard to see that

\[
\frac{c_1(\mathcal{L} + mE)^2 - 2\chi(M) - 3\sigma(M) + 1}{4} + 4 < \frac{c_1(\mathcal{L})^2 - 2\chi - 3\sigma}{4}.
\]

In order that \( FSW_M(1, \mathcal{L} - (2p + 1)E) \) to be nonzero, the moduli space dimension of \( \mathcal{L} \) over \( B = pt \), must be strictly positive.

Suppose that \( M \) is a symplectic manifold with \( b^+_2 > 1 \), then according to Taubes’ result [T2], the manifold is of Seiberg-Witten simple type. That is to say that all the moduli spaces of basic classes (whose \( SW \neq 0 \)) are of expected dimension zero (see [FM] for a derivation for Kahler surfaces). This observation gives us the following vanishing result,

**Proposition 2.3** Let \( m \) be an odd integer whose absolute value is bigger than one. Let \( \mathcal{L} \) be an arbitrary spin\(^c\) structure on a Kahler surface with \( p_g > 0 \) (or more generally any symplectic four manifold with \( b^+_2 > 1 \)), then the pure Seiberg-Witten invariants of \( \mathcal{L} + mE \), \( FSW_B(1, \mathcal{L} + mE) \) on \( M_2 = \text{Blowup}_{\Delta(M)}(M \times M) \hookrightarrow M \) vanish.

The space \( M_2 \) is the \( l = 2 \) version of the universal space \( M_l \) (see [Liu1]).

**Proof of the proposition:** From

\[
FSW_B(1, \mathcal{L} - (2p + 1)E) = \int_M c_2(E_C \otimes S^{p-1}(T_M^* \oplus C_M)) \cdot SW(\mathcal{L}),
\]

in order that \( FSW_B(1, \mathcal{L} - (2p + 1)E) \neq 0 \), \( SW(\mathcal{L}) \) must be non-zero. But this implies that \( \mathcal{L} \) is a basic class. If the surface has \( p_g > 0 \), \( c_1(\mathcal{L})^2 - 2\chi - 3\sigma = 0 \). This violates the bound on the expected dimension we get above. The argument for \( m > 1 \) case is almost identical to the \( m = -(2p + 1) < -1 \) case. If \( M \) is a \( b^+_2 > 1 \) symplectic four-manifold, one may replace complex blowing up by almost complex blowing up and the same argument works, too. \( \square \)

Let us explain the relevance of this piece of calculation with symplectic geometry. If we consider \( \mathcal{L} = E_C^2 \otimes K^{-1}_M \), then the Seiberg-Witten invariant of \( \mathcal{L} \) is equal to the enumeration of pseudo holomorphic curves Poincare dual to the cohomology class \( C = c_1(E_C) \).

When we consider the singular curves dual to \( C \) with a singularity of the prescribed multiplicity \( p \), they can be resolved into smooth curves on the blown up manifold dual to \( C - pE \). Or in term of spin\(^c\) structure, \( 2(E - pE) + K^{-1}_M - E = \mathcal{L} - (2p + 1)E \). When the
singular point is allowed to move on the whole $B = M$, the calculation of $FSW_M(1, L - (2p + 1)E)$ should be directly related to the counting of curves dual to $C$ with a multiplicity $p$ singularity in $M$. Thus the vanishing result prop. 2.3 on $FSW_M(1, L + mE)$ indicates that such counting of the pseudo-holomorphic singular curves always gives the answer 0 when the corresponding symplectic four-manifold is of Seiberg-Witten simple type.

On the other hand, if the symplectic four-manifold $M$ is of $b_2^+ = 1$, then it is not of non-simple type in the Taubes chamber (determined by large deformations of its symplectic forms) and the usual SW invariants of $L$ for $c^2(L) - 2X - 3\sigma > 0$ is calculated by the so-called wall crossing formula [KM], [LL2] and is ±1 for $b_1 = 0$ manifolds. Then the family blowup formula predicts that the pure invariants on $M$ can be calculated by the so-called wall crossing formula [KM], [LL2] and is ±1 for $b_1 = 0$ manifolds.

In the case when $b_2^+ = 3$ and the manifolds carry hyper-winding families of symplectic forms, one thickens the base by multiplying with a copy of $S^{b_2^+ - 1}$.

2.2 The Family Blowup Formula and the Universal Family

One important application of the family blowup formula is in the long paper [Liu1], concerning the enumeration of singular curves with nodal or other singularities. Let us give a slightly slow paced discussion about its relationship with family blowup formula. This subsection is an extension of the example 2.1.

Let $M$ be a symplectic four manifold with a compatible almost complex structure $J : TM \mapsto TM$. We fix such an almost complex structure and view $M$ as an almost complex manifold. Recall that the sequence of universal spaces $M_k, k \in \mathbb{N}$ and $f_k : M_{k+1} \mapsto M_k$ can be constructed inductively by the recipe in [Liu1]. Set $M_0 = pt$ and $M_1 = M$. Then define $f_0 : M_1 \mapsto M_0$ to be the constant map. Suppose that $M_l, l \leq k, l \in \mathbb{N}$ and $f_{l-1} : M_l \mapsto M_{l-1}$ have been defined for all $l \leq k$ such that $f_l$ are smooth pseudo-holomorphic submersions. We define $M_{k+1}$ and $f_k : M_{k+1} \mapsto M_k$ by the following recipe.

Take the fiber product of $M_k \times_{M_{k-1}} M_k$ through $f_{k-1} : M_k \mapsto M_{k-1}$ and $f_k : M_k \mapsto M_k$. Then $f_{k-1} : M_k \mapsto M_{k-1}$ maps relatively into the diagonal of $M_k \times_{M_{k-1}} M_k$ as an almost complex manifold. Consider the almost complex blowing up of $\Delta_{M_{k-1}} : M_k \mapsto M_k \times_{M_{k-1}} M_k$ as a complex codimension two sub-manifold.
Define \( M_{k+1} \) to be the blown up manifold and it maps naturally to either copy of \( M_k \) surjectively.

By almost complex blowing up, we mean the following: The normal bundle of the relative diagonal is well known to be isomorphic to the relative tangent bundle of \( M_k \mapsto M_{k-1} \). Because \( f_{k-1} \) is pseudo-holomorphic, the relative tangent bundle \( T_{M_k/M_{k-1}} \), as the kernel of \( df_{k-1} \) is stable under the almost complex structure of \( M_k \) and becomes a complex rank two vector bundle over \( M_k \). Through the isomorphism, the relative tangent bundle \( T_{\Delta(M_k)} \) of \( \Delta(M_k) \) inherits the structure of a complex vector bundle. To construct the almost complex blowing up of \( M_k \), replace \( \Delta(M_k) \) by \( \mathbb{P}(T_{M_k/M_{k-1}}) \). The almost complex structure outside of the blown-up locus \( M_k \times M_{k-1} \) is unchanged. The almost complex structure on \( T_{M_{k+1}} \) is induced from the natural almost complex structure from \( \mathbb{P}(T_{M_k/M_{k-1}}) \) and \( N_{\Delta(M_k)} M_k \times M_k \) to \( \mathbb{P}(T_{M_{k+1}}) \).

It is easy to check that the almost complex structure induced in this way is \( C^\infty \) on the whole \( M_{k+1} \). Moreover the natural map \( f_k : M_{k+1} \mapsto M_k \), defined as the composition of \( M_{k+1} \mapsto M_k \times M_{k-1} \) and \( M_k \times M_{k-1} \mapsto M_k \) (either copy) is a composition of pseudo-holomorphic maps and is therefore pseudo-holomorphic.

**Lemma 2.5** The fiber bundle \( f_k : X_{k+1} = M_{k+1} \mapsto M_k = B \) can be constructed from the product fiber bundle \( X_0 = M \times M_k \mapsto M_k \) by \( k \) consecutive blowing ups \( X_i, 1 \leq i \leq k \).

(i). \( X_i \mapsto M_k = B \) is constructed from \( X_{i-1} \mapsto M_k = B \) by blowing up a cross section \( M_k \mapsto X_{i-1} \).

(ii). Consider \( f_{k-1,i} : M_k \mapsto M_i, i \leq k-1 \) to be \( f_{k-1,i} = f_i \circ f_{i+1} \circ \cdots \circ f_{k-1} \) and \( f_i : M_{i+1} \mapsto M_i \), then \( X_i \) can be identified with the fiber product \( M_k \times M_i, M_{i+1} \) of \( f_{k-1,i} \) with \( f_i \), the pull-back of \( f_i : M_{i+1} \mapsto M_i \) by \( f_{k-1,i} : M_k \mapsto M_i \).

Proof of the lemma: The proof is essentially the same as the proof of lemma 3.1. in [Liu]. \( \square \)

Knowing that fiber bundle \( X_k \) and the original fiber bundle \( X_0 = M \times M_k \) are related by \( k \) different blowing ups, the cohomology class \( C - \sum_{i \leq k} m_i E_i \) on \( X_k \) is related to \( C \) on \( X_0 \) through \( C - m_1 E_1, C - m_1 E_1 - m_2 E_2, \cdots \). To simplify the notation, we have identified \( C \) on \( X_0 \) with its pull-back on the blown-up manifolds \( X_i, i \leq k \) and denote them by the same symbol \( C \).
Proposition 2.4 For $C^2 - C \cdot c_1(K_M) - \sum_{i<k}(m_i^2 - m_i) \geq 0$, the pure family invariant on $X_k \mapsto M_k$, $FSW_{M_k}(1,E_{2C-\sum_{i<k}(2m_i+1)}E_i \otimes K_M^{-1})$ is equal to

$$SW(\mathbb{E}_{2C} \otimes K_M^{-1}): \int_{M_k} c_{2k}(\mathbb{E}_C \otimes \mathbb{S}^{m_i-1}(f_{k-1,i}^* T_{M/k} \oplus \mathcal{C}_{M_k}) \otimes \mathbb{E}_C - m_i E_i \otimes \mathbb{S}^{m_j-1}(f_{k-1,j}^* T_{M/k} \oplus \mathcal{C}_{M_k}))$$

$$\cdots \mathbb{E}_C - \sum_{i \leq k-1} m_i E_i \otimes \mathbb{S}^{m_k-1}(f_{k-1,k}^* T_{M/k} \oplus \mathcal{C}_{M_k})).$$

Proof of the proposition: By using family blowup formula consecutively, one may relate the pure family invariant $FSW_{M_k}(1,E_{2C-\sum_{i<k}(2m_i+1)}E_i \otimes K_M^{-1})$ on $X_k \mapsto M_k$ to a combination of the mixed family invariants $FSW_{M_k}(\eta,E_{2C} \otimes K_M^{-1})$ over $M \times M_k \mapsto M_k$, $\eta \in H^*(M_k,Z)$. Because the map $f_{k-1,i}$ factors through $f_i$, then $X_i$ can be viewed as the pull-back of $M_{i+1} \times_M M_i$ by $f_{k-1,i+1}: M_k \mapsto M_{i+1}$.

Thus, the relative diagonal of $M_{i+1} \times_M M_i$ pulls back to a cross section $s_i$ of $X_i \mapsto M_k$. On the other hand, pull-back by $M_k \mapsto M_{i+1}$ of the blowing up of the $\Delta_M \times M_i$ is nothing but $X_{i+1}$. Thus, $X_{i+1}$ can be thought as constructed from $X_i$ by the pullback of the relative diagonal section $\Delta_M \times M_i$.

Therefore, one may identify the complex rank two normal bundle $N_{s_i(M_k)}X_i$ to be $f_{k-1,i+1}^* T_{M_{i+1}/f_i^*M_i}$.

The class $\eta$ should be a combination of cup products of various Chern classes of the bundles $E_{C-\sum_{j \leq i-1} m_j E_j \otimes \mathbb{S}^{m_j-1}(f_{k-1,i}^* T_{M/k} \oplus \mathcal{C}_{M_k})}$.

By using the product rule of the total Chern classes under bundle addition, $\eta$ can be identified with

$$c_{total}(\oplus_{i \leq k} E_{C-\sum_{j \leq i-1} m_j E_j \otimes \mathbb{S}^{m_j-1}(f_{k-1,i}^* T_{M/k} \oplus \mathcal{C}_{M_k}))).$$

However, because $X_0 = M \times M_k \mapsto M_k = B$ is a product fiber bundle, only the grade $4k$ component of $\eta$ contributes to the mixed invariant. The pure family invariant is equal to

$$\int_{M_k} c_{2k}(\oplus_{i \leq k} E_{C-\sum_{j \leq i-1} m_j E_j \otimes \mathbb{S}^{m_j-1}(f_{k-1,i}^* T_{M/k} \oplus \mathcal{C}_{M_k})),FSW_{M_k}([M_k],E_{2C} \otimes K_M^{-1}))$$

$$= \int_{M_k} c_{2k}(\oplus_{i \leq k} E_{C-\sum_{j \leq i-1} m_j E_j \otimes \mathbb{S}^{m_j-1}(f_{k-1,i}^* T_{M/k} \oplus \mathcal{C}_{M_k})),SW(E_{2C} \otimes K_M^{-1}).$$

This ends the proof of the proposition. $\Box$
Remark 2.1 One may calculate the integral of the $2k$ – th Chern class $c_{2k}$ over $M_k$ by pushing forward along $f_i: M_{i+1} \mapsto M_i$ consecutively. By using the blowup formula of the Chern classes of tangent bundles, the final answer depends on $C = c_1(E_C), c_1(T_M), c_2(T_M)$ and all the multiplicities $m_1, m_2, \cdots, m_k$. It has to be a universal (manifold independent) polynomial of $C^2[M], C \cdot c_1(M)[M], c_1^2(M)[M], c_2(M)[M], P_{m_1,m_2,\cdots,m_k}(C^2[M], C \cdot c_1(M)[M], c_1^2(M)[M], c_2(M)[M])$.

2.3 The Existence of Pseudo-Holomorphic Singular Curves with Prescribed Singular Multiplicities

Given a symplectic four manifold $M$ and a cohomology class $C$ whose Gromov moduli space has non-negative dimension. If $FSW = FGT$ is known to hold for those general families, then the number of pseudo-holomorphic curves singular (with restriction on the orders of the singularities) at a finite number of points are symplectic invariants and the invariants are determined by the Gromov invariants of the class and the Chern classes information of the tangent bundles, etc. If $M$ is a symplectic four manifold with $b_2^+ > 1$, Taubes [T1,T2, T3] proves that $M$ has simple type. It means that once the moduli space dimension is positive, its invariant vanishes. On the other hand the dimension of the moduli spaces of singular curves are strictly smaller than that of the original moduli spaces. Hence it reasonable to speculate that the singular curves invariants actually all vanish.

At first the speculation may look incompatible with the intuition we have. On every algebraic surface $X$ we are able to count e.g. nodal curves on $X$. The enumerative problem is well known in algebraic geometry. It does gives nontrivial answers in general. However our theorem tells us that the counting problem in algebraic geometric way does not give us the symplectic invariants when $b_2^+ > 1$. Only when $b_2^+ = 1$, the algebraic calculation and the symplectic calculation coincide completely. On the other hand, it does not mean that the symplectic counting problem does not make sense for $b_2^+ > 1$ symplectic four manifolds. If fact, if we can construct some family of symplectic four manifolds which become non-simple type, then we can still make sense of the symplectic singular curves counting and get a nonzero answer. A good example is K3. As $K3$ is of $b_2^+ = 3$, it falls into the category discussed in the corollary. If we count the singular invariants in the usual way, we always get zero. As $K3$ has
hyperkahler structures, a generic K3 has no integral (1, 1) class, not mentioning any effective holomorphic curves. However we can still consider the $S^2$-family and the corresponding family invariants. In the final section, we make a systematic study upon the algebraic family Seiberg-Witten “invariants”. It will be indicated that the algebraic “invariants” and the smooth invariants coincide only when $b^+_2 = 1$. Otherwise the algebraic “invariants” correspond to a “local” $p_g$ dimensional smooth family invariant. The existence of the local family invariants was previously speculated independently by T.J. Li and the author. This also gives a philosophical explanation why the algebraic geometers can count the curves when the symplectic geometers claim the triviality of the invariants.

From the previous discussion we learn that in the Taubes’ chamber, the invariants are all $\pm 1$ if $C^2 \geq -2$. Using the previous formula we learn that if we count the singular curves in the $S^2$ family, we get nontrivial results. To demonstrate the power of the blowup formula, let us discuss some other explicit examples.

Let us discuss another interesting applications of the family blowup formulas to the special families considered previously. Let $M$ be a symplectic four-manifold with $b^+_2 = 1$. Take $B$ to be $M_k$, the $k$-th universal space (see [Liu1]). Take $X_0 = M \times M_k \mapsto M_k = B$ and $X_k = M_{k+1} \mapsto M_k$ which can be constructed from $X_0$ by $k$ consecutive blowing ups of cross sections. Given a class $C \in H^2(M, \mathbb{Z})$, it determines a smooth line bundle $E_C$ on $M$. Consider the $spin^c$ determinant line bundle $2E^2 - \sum_{i \leq k}(2m_i + 1)E_i - K_M$ (in additive notation).

Suppose one chooses the class $\eta = [M_k] \in H^{top}(M_k, \mathbb{Z})$ and considers the corresponding blowup formula of mixed family invariants expressing $FSW_{M_k}(\eta, 2E^2_C - \sum_{i \leq k}(2m_i + 1)E_i - K_M)$ in terms of the mixed invariants from $M \times M_k \mapsto M_k$.

One concludes the following interesting corollary,

**Corollary 2.1** Let $M$ be a symplectic four-manifold with $b^+_2 = 1$. Let $C$ be a cohomology class $\in H^2(M, \mathbb{Z})$ whose Gromov-Taubes invariant $Gr(C)$ is nonzero (see [T1], [T2]). Given any tuple of positive integers $m_i, 1 \leq i \leq k$ with $
abla \frac{C^2 - C \cdot c_1(K_M)}{2} - \sum_{i \leq k}(m_i^2 - m_i) \geq 0$ and $k$ distinct points on $M$, then there exists a pseudo-holomorphic curve Poincare dual to $C$ passing through these $k$ distinct points of $M$ with multiplicities at the $i$-th point not less than $m_i$.

Proof: Continue the discussion before the statement of the corollary. As $\eta$ has exhausted the dimension of the base $M_k$, the extra Chern
classes of the obstruction bundle will not be able to contribute to the invariants.

Thus, we conclude $FSW_B(\eta, 2E_C^2 - \sum_{i\leq k}(2m_i + 1)E_i - K_M)$ is equal to $FSW_B(\eta, 2E_C^2 - K_M)$ on $M \times M_k \mapsto M_k$. On the other hand, the particular mixed invariant of $M \times M_k \mapsto M_k$ is nothing but the usual $SW(2E_C - K_M)$, which according to Taubes theorem ‘$SW=Gr$’, is equal to $Gr(C)$.

Thus one concludes that $FSW_B(\eta, 2E_C^2 - \sum_{i\leq k}(2m_i + 1)E_i - K_M)$ is always nonzero if

1. The family Seiberg-Witten dimension of $2E_C^2 - \sum_{i\leq k}(2m_i + 1)E_i - K_M$ is non-negative, which is reduced to the condition $\frac{C^2 - C.c_1(K_M)}{2} - \sum_i (m_i^2 - m_i) \geq 0$

2. The ordinary Seiberg-Witten invariant of the class $2E_C - K_M$ in Taubes’ chamber $= Gr(C)$ is nonzero.

For any $k$ distinct points $p_1, p_2, \ldots, p_k$ in $M$, it determines a point in the top (open) stratum of $M_k$.

As the particular choice of $\eta = [M_k]$ is poincare dual to the zero cycle on $M_k$, the family invariant $FSW_{M_k}([M_k], 2E_C^2 - \sum_{i\leq k}(2m_i + 1)E_i - K_M)$ can be re-interpreted as the a counting of Seiberg-Witten solutions on the fiber above the point in $M_k$. The fiber above the given point in $M_k$ is a symplectic blowing up $\tilde{M}$ of $M$ at these $k$ distinct points. Then the family blowup formula asserts that if we deform the family Seiberg-Witten equations by a large deformation of a given family of symplectic forms, there is always some solutions of the Seiberg-Witten equations above the given point in $M_k$. Otherwise, the invariant count is zero for an empty moduli space. In particular, the same conclusion still holds if one takes the $r \mapsto \infty$ limit in the family Seiberg-Witten equations.

On the other hand, by Taubes’ analysis the solution of Seiberg-Witten equation of large symplectic form perturbation will converges (in the $r \mapsto \infty$ limit) to a $(1,1)$ current which can be regularized to be a pseudo-holomorphic curve dual to $C - \sum m_i E_i$. The readers can consult [T1, T2, T3] for details.

Finally, a pseudo-holomorphic curve dual to $C - \sum m_i E_i$ in the blown up manifold $\tilde{M}$ gives rise to a pseudo-holomorphic curve dual to $C$ in $M$. The curve has to develop a multiplicity $m_i$ singularity at $p_i$, $i \leq k$ as the intersection number of its ‘proper transformation’ in $\tilde{M}$ has intersection multiplicity $m_i$ with the exceptional curve $E_i$ above $p_i$. □

The corollary tells us that one can construct a singular divisor(curve) on $\tilde{M}$ with prescribed singular multiplicities and singular
Let $M$ be an algebraic surface with $p_g = 0, q(M) = 0$ and let $I_Z$ be the ideal sheaf of $Z = \sum_{i \leq k} m_i p_i, p_i \in M, m_i \in \mathbb{N}$, then one has the following sheaf short exact sequence

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_Z \rightarrow 0.$$  

For a locally free $\mathcal{E}$ with $c_1(\mathcal{E}) = C$,

$$0 \rightarrow I_Z \otimes \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Z \otimes \mathcal{E} \rightarrow 0$$

induces the long exact sequence

$$0 \rightarrow H^0(M, I_Z \otimes \mathcal{E}) \rightarrow H^0(M, \mathcal{E}) \rightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{E}) \rightarrow H^1(M, I_Z \otimes \mathcal{E}) \cdots.$$  

If $\mathcal{E}$ is sufficiently very ample to imply the vanishing of $h^1(M, I_Z \otimes \mathcal{E})$, the long exact sequence truncates to a short exact sequence. A curve in the linear system $\mathbb{P}(H^0(M, \mathcal{E}))$ has the prescribed singular multiplicities on $p_i, i \leq k$ if and only if the restriction of the corresponding defining section to the non-reduced $Z$ vanishes.

Given that $h^0(Z, \mathcal{O}_Z \otimes \mathcal{E}) = \sum_i \frac{m_i(m_i-1)}{2}$ and $h^0(M, \mathcal{E}) \geq \chi(M, \mathcal{E}) = 1 + \frac{C^2 - c_1(K_M) \cdot C}{2}$ (assuming $h^2(M, \mathcal{E}) = 0$), $h^0(M, I_Z \otimes \mathcal{E}) \geq \chi(M, \mathcal{E}) - \sum_{i < k} \frac{m_i^2 - m_i}{2} \geq 1$ if $\frac{C^2 - c_1(K_M) \cdot C}{2} - \sum_i \frac{m_i(m_i-1)}{2} \geq 0$.

By applying the family blowup formula and Taubes’ technique in symplectic geometry, corollary 2.1 can be viewed as the symplectic version of the theorem in $b^+_3 = 1$ category. As the corresponding technique in algebraic geometry has play an essential role in the study of linear system, one believes that the present symplectic version should also plays a similar role. The parallelism between the symplectic and the algebraic argument also suggests that the family blowup formula should have its algebraic geometric origin. This also motivates the definition of algebraic $SW$ invariant in section 3 and the algebraic proof of family blowup formula in section 5.

**Example 2.2** Let $M$ be $\mathbb{C}P^2$. Let the cohomology class $C$ denote $dH, d \geq 1$. Then the expected dimension of $SW$ (or Gromov-Taubes) moduli spaces of $(2d + 3)H (dH)$ are of $\frac{d^2 + 3d}{2}$ dimension. Let us consider the following enumeration problem. We are interested in counting the curves which have one nodal (ordinary double) point in $\mathbb{C}P^2$. It is well known that a nodal condition decreases
the moduli spaces by real two dimension. Therefore the nodal moduli spaces (which consist of nodal curves or their compactifications) are real \(d^2 + 3d - 2\) dimensional. As \(d^2 + 3d - 2 = (d + 1)(d + 2) - 4\) is always even, we can consider \((d + 1)(d + 2)/2 - 2\) generic points on \(\mathbb{CP}^2\) and require the nodal curves to pass through these generic points. After imposing these point passing conditions, generically (with respect to the almost complex structures and the points) there are finite number of nodal curves passing through these points.

To calculate the number, we notice that by the family blowup formula we find the answer to be \(\int_{\mathbb{CP}^2} c_2(H^d \otimes (T^*_{\mathbb{CP}^2} \oplus C_{\mathbb{CP}^2})) \cdot SW(H^{2d+3})\). As we know that \(SW(H^{2d+3}) = 1\) by the wall crossing formula, we conclude that the invariants are calculated by \(c_2(H^d \otimes (T^*_{\mathbb{CP}^2} \oplus C_{\mathbb{CP}^2}))|_{\mathbb{CP}^2}\). After some simple calculation one gets \(3(d-1)^2\). This answer is well known to algebraic geometers and is a very special case of Severi degrees.

Next let us discuss another simple but interesting example.

**Example 2.3** Let \(M\) be a minimal symplectic four manifold with \(b_2^+ = 3\). Suppose that there exists an \(S^2\) family of symplectic forms \(\omega_x, x \in S^2\) on \(M\) such that the projection to \(H^2_+(M, \mathbb{R}) - \{0\} \cong S^2 \times \mathbb{R}^*\) is of mapping degree 1. The family of symplectic forms is called a hyper-twisting family of symplectic forms on \(M\).

Such manifolds must have \(c_1(M) = 0\). \(K3\) and \(T^4\) are the only known examples in the Kahler category which have hyperwinding families of symplectic forms. These \(S^2\) families of forms can be constructed from hyperkahler families of Kahler forms.

Let us consider the \(S^2\) hyperwinding family of symplectic forms of \(M\). Consider a primitive cohomology class \(C \in H^2(M, \mathbb{Z})\) with square zero. There are an infinite number of these classes on such \(M\). Firstly, consider the fiber bundle \(M \times S^2 \hookrightarrow S^2\) with a hyperwinding family of symplectic forms on the fibers. The family wall crossing formula implies that the family invariant \(FSW_{S^2}(1, 2C)\) over \(M \times S^2 \hookrightarrow S^2\) is equal to \(\pm 1\) in the first winding chamber \([LL1]\).

Consider an \(S^2\) family of almost complex structures compatible with an \(S^2\) family of fiberwise Riemannian metrics. Consider the blown up fiber bundle \(X' = S^2 \times M_2 \hookrightarrow S^2 \times M_1 = B\) using the \(S^2\) family of almost complex structures. It is easy to see that one can use the symplectic blowup construction to construct an \(S^2\) family of symplectic forms on \(S^2 \times M_2 \hookrightarrow S^2 \times M_1 = B\). By a similar
calculation as in example 2.1, the family invariant $\text{FSW}_{S^2}(1, 2C - \sum 5E)$, evaluated in the winding chamber, is equal to

$$c_2(E_C \otimes (T^*M \oplus C_M))[M] \cdot \text{FSW}_{S^2}(1, 2C) = \pm c_2(E_C \otimes (T^*M \oplus C_M))$$

$$= c_2(M)[M] = \chi(M).$$

We have used $C^2 = 0, c_1(T^*M \oplus C_M) = 0$ and $\text{FSW}_{S^2}(1, 2C) = \pm 1$ in the winding chamber over $S^2 \times M \mapsto S^2 = B$.

On the other hand, the Riemann-Roch formula for almost complex manifolds implies that on $M$ we have

$$2 - \frac{b_1(M)}{2} = 1 - \frac{b_1(M) + b_2^+(M)}{2} = \text{ind}(\bar{\partial}) = \frac{\int_M (c_1^2 + c_2)}{24} = \frac{\chi(M)}{24}.$$

Thus we find that the family invariant $\text{FSW}_{S^2}(1, 2C - 5E)$ over $S^2 \times M \mapsto S^2 \times M$ is $\pm 24$ for $b_1(M) = 0$ manifold $M$.

The answer 24 does not come out accidentally. If we consider a generic elliptic K3 surface elliptic fibered over $\mathbb{C}P^1$, then it is well known there are 24 singular nodal fibers.

The tool of the family blowup formula implies that on a simply connected hyperwinding family of symplectic four-manifold $M$, one can recover the number of single node nodal rational curves within the $S^2$ family as the family invariant $\text{FSW}_{S^2 \times M_1}(1, E_2C - 3E)$, and is identical to the number of singular nodal fibers of an elliptic K3 surface.

This above picture supports the following conjecture,

**Conjecture 2.1** Let $M$ be a simply connected symplectic four-manifold with an $S^2$ family of hyper-winding symplectic forms, then $M$ is diffeomorphic to the underlying smooth manifold of the K3 surface and the hyperwinding family of symplectic forms is homotopic to the $S^2$ families of hyperkähler structures of the K3 surfaces through $S^2$ families of symplectic forms.

One can formulate a similar conjecture for $T^4$ or other primary Kodaira surfaces (with $b_2^+ = 2$). The uniqueness of the $S^2$ or $S^1$ families up to homotopies plays a crucial role in understanding the symplectic structures of these $c_1 = 0$ symplectic four-manifolds.
3 The Definition of Algebraic Seiberg-Witten Invariants

In this section we discuss the definition of algebraic Seiberg-Witten invariant on algebraic surfaces. The proof of main theorem 0.2 occupies the whole section. Because of the algebraic nature of the discussion, we will make use of algebraic (holomorphic) vector bundles and locally free sheaves frequently. We adopt the convention that if we use the bold character to denote an algebraic vector bundle, e.g. \( \mathbf{E} \), the calligraphic character, \( \mathcal{E} \), will denote the locally free sheaf of sections of \( \mathbf{E} \) and vice versa.

Given an algebraic surface \( M \), there are two important holomorphic invariants associated with \( M \), \( q(M) \), the irregularity of \( M \) and \( p_g \), the geometric genus of \( M \). They are related to the homological invariant \( b_1(M), b_2^+(M) \) by the relationship

\[
b_1(M) = q(M), \quad b_2^+(M) = 1 + 2p_g.
\]

Both of the invariants are essential in defining the algebraic version of (family) Seiberg-Witten invariants. Let us discuss briefly before the formal mathematical treatment. If the geometric genus of the surface is 0, then the algebraic family invariant coincides with the topological family Seiberg-Witten invariant defined in [LL1]. On the other hand, the algebraic Seiberg-Witten invariant differ from the usual topological Seiberg-Witten invariant in that it 'formally' corresponds to topological family Seiberg-Witten invariant of a germ of a high dimensional family.

Given an algebraic surface \( M \), the set of spin\(^c\) structures on the underlying smooth four-manifold of \( M \) is isomorphic (up to torsions) to \( H^2(M, \mathbb{Z}) \), which is isomorphic to the set of isomorphism classes of \( \mathcal{C}^\infty \) line bundles on \( M \). Thus, the algebraic Seiberg-Witten invariant of \( M \) can be viewed as a map

\[
\mathcal{A}SW : H^*(M, \mathbb{Z}) \mapsto \mathbb{Z}.
\]

For generic classes, the formal base dimensions are \( p_g \). But for some non-generic classes (defined slightly later), the formal base dimensions of the infinitesimal germs are in-between 0 and \( p_g \).

Let \( M \) be an algebraic surface with \( p_g = 0 \), then algebraic and topological Seiberg-Witten invariants coincide. If \( q(M) = 0 \), then a \( \mathcal{C}^\infty \) topological line bundle can be given a unique holomorphic structure. On the other hand, the holomorphic structures of a fixed \( \mathcal{C}^\infty \)
line bundle on a $q(M) > 0$ surface have non-trivial moduli and the algebraic Seiberg-Witten invariant counts holomorphic curves from all the different holomorphic structures of a fixed topological line bundle. If one would like to enumerate the holomorphic curves from a particular holomorphic structure of the $C^\infty$ line bundle, additional cohomological insertion has to be made on the family invariant.

**Remark 3.1** The readers with an algebraic geometric background may notice that the enumeration of holomorphic curves from the zero sections of different holomorphic structures of a fixed $C^\infty$ line bundle corresponds to curve counting in a non-linear system, while enumerations of curves from a fixed holomorphic structure corresponds to curve counting in a linear system.

There have been different versions of Symplectic or Algebraic Gromov type invariants aiming at curve enumerations ([Be], [LiT1], [LiT2], [R], [RT1], [RT2], [S], [T3]). It may be desirable to clarify the difference of $\mathcal{ASW}$ from the usual Gromov-Witten invariants.

To summarize,

I. the algebraic (family) Seiberg-Witten invariant is an algebraic device used to enumerate curves as the divisors on a given algebraic surface than holomorphic maps from domain curves to the target $M$.

II. Because usual Seiberg-Witten theory has compact moduli spaces, the algebraic (family) Seiberg-Witten invariants are defined using compact moduli spaces as well.

III. As its definition does not involve the domain curves, the (compactification of) Deligne-Mumford moduli spaces $\mathcal{M}_{g,n}$ do not come into our picture. Thus, the algebraic Seiberg-Witten invariants do not have nice combinatorial structures from the domain curves as the usual Gromov-Witten invariants do.

IV. On the other hand, the algebraic Seiberg-Witten invariants are related to Surface Riemann-Roch formula closely. It can be seen directly from the dimension formula of $\mathcal{ASW}$ (see the discussion in the following section).

V. Unlike the usual Seiberg-Witten invariants (usually defined by perturbation argument using $C^\infty$ topology), the $\mathcal{ASW}$ are defined as the intersection numbers of various Chern classes on a neighborhood of the algebraic Seiberg-Witten moduli spaces based on the construction of algebraic Kuranishi models. Unless the reduced
algebraic Seiberg-Witten moduli space (cut off by a finite number of codimension one cycles, determined by its formal dimension formula) happens to be of zero dimensional, the invariant usually does not correspond to actual counting of the number of curves dual to a class $C$. Thus, one should view $\mathcal{ASW}$ as a formal enumeration (in the sense of intersection theory).

VI. In most of the cases, the data of algebraic Kuranishi models involves algebraic vector bundles and algebraic bundle maps between these vector bundles. In some minor cases (discussed later), it involves algebraic vector bundles and non-algebraic bundle maps between them. We still call the corresponding defined invariant “algebraic” as the algebraic Seiberg-Witten moduli space itself and the Chern classes of the algebraic vector bundles involved in defining $\mathcal{ASW}$ are algebraic objects. VII. Finally, by its definition the $\mathcal{ASW}$ is “NOT” an invariant in the traditional sense. Namely, it is not transparent that it is independent of the complex structure of $M$. But we still call it an “invariant” because

(i). it is independent to the choices of the algebraic Kuranishi models chosen to define the invariant.

(ii). In many situations, it is related to the topological Seiberg-Witten invariant.

(iii). As will be proved in section 5, it shares the same functorial properties under blowing ups as the usual (family) Seiberg-Witten invariants.

(iv). After explicit calculation (e.g. by wall crossing formula, etc), usually one can compute $\mathcal{ASW}$ explicitly and find it to be independent of the analytic information (like the complex structure) of $M$.

4 The Definition of Algebraic Seiberg-Witten Invariant for Algebraic Surfaces with zero Geometric Genera

Recall that the original Seiberg-Witten invariant $SW$ is defined for all $spin^c$ structures on a given smooth four-manifold with $b_2^+ \geq 1$ (with dependence on chamber structures when $b_2^+ = 1$. On the other hand, the usage of $spin^c$ structure is not particularly convenient for
our discussion of algebraic Seiberg-Witten invariants. Therefore, we will adopt a slightly different notation in the holomorphic or algebraic category.

To begin our discussion, we start by recalling (see e.g [FM]) how is the Seiberg-Witten invariant defined in terms of the holomorphic data.

Let \((M, \omega_M)\) be a Kahler surface and let \(\omega_M\) be the Kahler form. Then \(\omega_M\) splits the \(\text{spin}^c\) spinor vector bundle into

\[ S^+_L \cong E \oplus E \otimes K^{-1}_M. \]

Following the usual convention, one can denote \(\alpha \in \Gamma(E)\) and \(\beta \in \Gamma(E \otimes K^{-1}_M)\) the smooth sections of the \(C^\infty\) line bundles and the corresponding Dirac equation and Seiberg-Witten equation are reduced to scalar valued equation and read as

\[
\overline{\partial_a} \alpha + \overline{\partial_a}^\ast \beta = 0, \\
F_a^{0,2} = \overline{\partial_a}^2 = \beta \cdot \overline{\alpha}, \\
F_a^{1,1} \wedge \omega_M = i\frac{|\alpha|^2 - |\beta|^2 - 1}{2} \omega_M^2.
\]

A standard argument implies that \(\alpha \equiv 0\) or \(\beta \equiv 0\) identically (which depends on whether \(c_1(E) \cdot \omega_M > 0\) or \(c_1(E) \cdot \omega_M < 0\). If \(c_1(E) \cdot \omega_M > 0\), the smooth section \(\beta\) is identically zero and

\[
\overline{\partial_a} : \Gamma(E) \longrightarrow \Gamma(E \otimes \omega_M^{0,1})
\]

satisfies \(\overline{\partial_a}^2 = 0\) and thus defines a holomorphic structure on the \(C^\infty\) line bundle \(E\). The Dirac equation \(\overline{\partial_a} \alpha + \overline{\partial_a}^\ast \beta = 0\) is then reduced to the d-bar equation

\[
\overline{\partial_a} \alpha = 0
\]

and \(\alpha\) is a non-zero holomorphic section of the holomorphic structure induced on the line bundle \(E\). The \((1,1)\)-projection of the Kahler-Seiberg-Witten equation can be reduced to the Kazdan-Warner equation on \(M\).

**Definition 4.1** Given a cohomology class \(C \in H^2(M, \mathbb{Z})\), the Gromov-Taubes dimension of the class is defined to be

\[ d_{GT}(C) = \frac{C \cdot C - c_1(K_M) \cdot C}{2}. \]
The formula is nothing but the dimension formula of Seiberg-Witten invariant, formulated in terms of $C$ instead of $L = 2C - c_1(K_M)$ (in additive notation). It was first discovered by C. Taubes [T].

4.1 The $p_g = 0$ and $q(M) = 0$ Case

In the following, we discuss the algebraic Seiberg-Witten invariant in the easiest case. Later we will extend our discussion to the $q(M) > 0$ case and then to the $p_g > 0$ case as well.

Let us fix a $c_1(E) = C \in H^2(M, \mathbb{Z})$ and define the algebraic Seiberg-Witten invariant of $C$ on the algebraic surface $M$. Because $h^{2,0} = p_g = 0$, the class $C$ is automatically a $(1, 1)$ class.

Because our major interest is to study the holomorphic curves poincare dual to $C$, we require $C \cdot \omega_M > 0$. Otherwise, $C$ cannot be represented by holomorphic curves and we simply define $\mathcal{ASW}(C) = 0$. The major tool for the definition of the algebraic Seiberg-Witten invariant is the existence of the algebraic Kuranishi model, defined in definition 4.2 and definition 4.4, etc.

If $q(M) = 0$, there exists a unique holomorphic structure on any given $C^\infty$ line bundle $E$. By abusing the notation, we use the same symbol $E$ to denote the holomorphic line bundle and the underlying $C^\infty$ line bundle.

We start by defining $\mathcal{ASW}(C) = \mathcal{ASW}_{pt}(1, C)$ in this special case. Heuristically speaking, the number $\mathcal{ASW}(C)$ should enumerate the number of holomorphic curves dual to $C = c_1(E)$, passing through the $d_{GT}(C) = \frac{C^2 - c_1(K_M)}{2}$ number of generic points on $M$.

Let $D_C$ be an effective Weil divisor defined by the zero locus of a non-zero holomorphic section of $E$. Then one has the following short exact sheaves sequence,

$$0 \to \mathcal{O}_M \to \mathcal{O}_M(D_C) \to \mathcal{O}_{D_C}(D_C) \to 0.$$

We have the following vanishing lemma for $h^2(M, E)$.

**Lemma 4.1** Let $M$ be an algebraic surface with $p_g(M) = q(M) = 0$. Suppose that $h^0(M, E) > 0$, then $h^2(M, E) = 0$.

Proof: Suppose that $h^0(M, E) > 0$, there is a non-zero holomorphic section of the holomorphic line bundle $E$. Pick one non-trivial section and denote the corresponding zero locus by $D_C$. 

32
Then it is well known by spectral sequence argument that the sheaf cohomology group $H^i(M, \mathcal{O}_M(D_C))$ is isomorphic to d-bar cohomology group $H^i_{\partial}(M, \mathcal{E})$.

To argue the vanishing of $H^2_{\partial}(M, \mathcal{E})$, it suffices to show that $H^2(M, \mathcal{O}_M(D_C)) = 0$.

Let us take the right derived long exact sequence of the above short exact sequence and consider the last three terms,

$$H^2(M, \mathcal{O}_M) \hookrightarrow H^2(M, \mathcal{O}_M(D_C)) \hookrightarrow H^2(D_C, \mathcal{O}_{D_C}(D_C)) \mapsto 0.$$  

One observes that the first term and the third term are both trivial. The $H^2(D_C, \mathcal{O}_{D_C}(D_C))$ is trivial because $D_C$ is complex one dimensional. On the other hand, $\dim CH^2(M, \mathcal{O}_M) = p_g(M)$ is assumed to be zero, then the middle term $H^2(M, \mathcal{O}_M(D_C))$ must be trivial. □

Knowing the triviality of the second $d-bar$ cohomology, we have the following dimension count

$$h^0_{\partial}(M, \mathcal{E}) - h^1_{\partial}(M, \mathcal{E}) = h^0(M, \mathcal{O}_M(D_C)) - h^1(M, \mathcal{O}_M(D_C))$$

$$= \chi(\mathcal{O}_M(D_C)) = \frac{D_C \cdot D_C - K_M \cdot D_C}{2} + 1 = \frac{C \cdot C - c_1(K_M) \cdot C}{2} + 1.$$  

We have abused the notation $\cdot$ a bit. The pairing $D_C \cdot D_C$ denotes the intersection pairing on divisors, while the pairing $C \cdot C$ means the cohomology pairing on $H^2(M, \mathbb{Z})$.

If $h^1(M, \mathcal{O}_M(D_C)) = 0$, then $h^0(M, \mathcal{O}(D_C)) = \frac{C^2 - c_1(K_M) \cdot C}{2} + 1$ and the complete linear system $|D_C|$ is a projective space of the expected Gromov-Taubes dimension $\frac{C^2 - c_1(K_M) \cdot C}{2}$. The counting indicates that when $h^1(M, \mathcal{O}_M(D_C)) \neq 0$, the vector space dimension $h^0(M, \mathcal{O}_M(D_C))$ may be different from the “expected dimension” $\frac{C^2 - c_1(K_M) \cdot C}{2} + 1$ and $H^1(M, \mathcal{O}_M(D_C)) \cong H^1(D_C, \mathcal{O}_{D_C}(D_C))$ represents the obstruction space.

We define the algebraic moduli space of curves dual to $C$ to be the projective space formed by the vector space $H^0(M, \mathcal{O}_M(D_C)) \cong H^0_{\partial}(M, \mathcal{E})$, which is of $h^0(M, \mathcal{O}_M(D_C)) - 1$ dimension.

We adopt the following formulation which will be extended to $p_g > 0$ or $q(M) > 0$ cases. As the algebraic family moduli space is not of the expected dimension, an insertion of the top Chern class of the obstruction bundle is necessary.
Definition 4.2 Consider the triple \((V, W, \Phi_{VW})\) with \(V, W\) be finite dimensional vector spaces and let \(\Phi_{VW}\) be a linear map from \(V\) to \(W\),

\[ \Phi_{VW} : V \rightarrow W. \]

It is said to be an algebraic Kuranishi model of the class \(C\) over \(M\) if \(\text{Ker}(\Phi_{VW}) \cong H^0(M, \mathcal{O}_M(D_C))\) and \(\text{Cokernel}(\Phi_{VW}) \cong H^1(M, \mathcal{O}_M(D_C))\).

Given an algebraic Kuranishi model of \(C\), one defines the algebraic family Seiberg-Witten invariant by the following recipe.

Consider the projective space \(\pi_{\mathbb{P}(V)} : \mathbb{P}(V) \rightarrow \text{pt}\) and the corresponding tautological line bundle \(\mathcal{H}^*\), the dual of hyperplane line bundle. The vector space morphism \(\Phi_{VW}\) induces a bundle map \(\mathcal{H}^* \mapsto \pi_{\mathbb{P}(V)}^* \mathcal{H}^*\) and therefore a canonical section of the obstruction bundle \(\mathcal{H} \otimes \pi_{\mathbb{P}(V)}^* \mathcal{H}^*\).

The algebraic cycle of the moduli space of curves poincare dual to \(C\) is represented by the top Chern class \(c_{\text{top}}(\mathcal{H} \otimes \pi_{\mathbb{P}(V)}^* \mathcal{H}^*)\).

Requiring the curve to pass through a generic point imposes an additional \(c_1(\mathcal{H})\) insertion.

One defines \(\text{ASW}(C)\) to be

\[
\text{ASW}(C) = \int_{\mathbb{P}(V)} c_1^{\dim V - \dim W - 1}(\mathcal{H}) c_{\text{top}}(\mathcal{H} \otimes \mathcal{C} W).
\]

Remark 4.1 It is easy to see that the definition of \(\text{ASW}_{pt}\) is independent of the choices of \((V, W, \Phi_{VW})\) and is equal to 1, which is up to a sign equal to the wall crossing number \([KM]\) of a \(b_2^+ = 1, b_1(M) = 0\) four-manifold.

In the following, we extend the definition of algebraic Seiberg-Witten invariants to \(q(M) \neq 0\) case.

4.2 The Definition of ASW for \(q(M) \neq 0, p_g = 0\) Algebraic Surfaces

Let \(M\) be an \(p_g = 0\) algebraic surface with \(q(M) \neq 0\) and let \(C\) be an element in \(H^2(M, \mathbb{Z})\). As \(H^{2,0}(M, \mathbb{C}) = H^{0,2}(M, \mathbb{C}) = 0\), \(C\) is
automatically a $(1,1)$ class, which becomes the first Chern class of a holomorphic line bundle $E$. However, the holomorphic structures on $E$ are not unique and form a continuous moduli known to be $\text{Pic}^0(M)$, the connected component of the Picard group $\text{Pic}(M)$ containing identity.

Consider the Hodge decomposition

$$H^1(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^{1,0}(M, \mathbb{C}) \oplus H^{0,1}(M, \mathbb{C}),$$

which induces an real vector space isomorphism

$$i : H^1(M, \mathbb{R}) \cong H^{0,1}(M, \mathbb{C}) \cong H^1(M, \mathcal{O}_M).$$

The following is well known to algebraic geometers,

**Proposition 4.1** Let $M$ be a Kahler surface with $q(M) \neq 0$, then the connected component of the Picard variety $\text{Pic}(M)$ containing identity, $\text{Pic}^0(M)$, can be identified as a complex variety to be the following complex torus

$$T(M) = H^1(M, \mathcal{O}_M)/i(H^1(M, \mathbb{Z})) \cong H^{0,1}(M, \mathbb{C})/i(H^1(M, \mathbb{Z})).$$

For a discussion on $\text{Pic}^0(M) = T(M)$, please consult [BPV], page 36, section 13. Given a $C^\infty$ topological line bundle, there is a universal holomorphic line bundle $E$ over $M \times T(M)$ such that the restriction, $E|_{M \times \{t\}}$, $t \in T(M)$, is the holomorphic line bundle over $M$, $c_1(E|_{M \times \{t\}}) = C$, with the specific holomorphic structure parametrized by $t \in T(M)$. One way to construct this is to consider the Poincare line bundle over $M \times T(M)$ and tensor it with any holomorphic line bundle over $M$ with first Chern class $= C$.

Let us consider a class $C \in H^2(M, \mathbb{Z})$ which satisfies $(i). C \cdot C - c_1(K_M) \cdot C \geq 0$, $(ii). C \cdot \omega_M > 0$.

Then the surface Riemann-Roch formula for $E$, $c_1(E) = C$, gives

$$\chi(E) = \frac{c_1(E) \cdot c_1(E) - c_1(K_M) \cdot c_1(E)}{2} + 1 - q(M) + p_g$$

$$= \frac{C \cdot C - c_1(K_M) \cdot C}{2} - q(M) + 1 = d_{GT}(C) - q(M) + 1,$$

which differs from the Gromov-Taubes dimension formula by $1 - q(M)$. Consider the projection $\pi : M \times T(M) \mapsto T(M)$ and push for-
ward the sheaf of holomorphic sections of $E$, $\mathcal{E}$, along $M \times T(M) \mapsto T(M)$.

There are three right derived image sheaves $R^i\pi_*(\mathcal{E})$, $0 \leq i \leq 2$, to consider.

Case 1. $(c_1(K_M) - C) \cdot \omega_M < 0$. For most of the classes $C$ with high enough degree (energy) $C \cdot \omega_M \gg 0$, the pairing $c_1(K_M) \cdot \omega_M - C \cdot \omega_M$ is negative.

Under this additional assumption on $C$, it follows from relative Serre duality that the sheaf $R^2\pi_*(\mathcal{E})$ over $T(M)$ vanishes. Otherwise $K_M \otimes \mathcal{E}^*$ has non-zero sections, which implies that the degree (energy) $c_1(K_M) \otimes \omega_M = c_1(K_M) \cdot \omega_M - C \cdot \omega_M > 0$.

The sheaves $R^0\pi_*(\mathcal{E})$ and $R^1\pi_*(\mathcal{E})$ may not be locally free. When $t \in T(M)$ moves, the dimension of $H^0(M, \mathcal{E}|_{M \times t})$ may vary and can even be zero at the generic points of $T(M)$.

Given an effective divisor $D_C$, which is the zero locus of some holomorphic section of $E_{M \times t}$, $t \in T(M)$, the short exact sequence

$$0 \mapsto \mathcal{O}_M \mapsto O_{M}(D_C) \mapsto O_{D_C}(D_C) \mapsto 0$$

has a corresponding long exact sequence for $p_g = 0$ surfaces,

$$0 \mapsto C \mapsto H^0(M, \mathcal{O}_M(D_C)) \mapsto H^0(D_C, \mathcal{O}_{D_C}(D_C)) \mapsto H^1(M, \mathcal{O}_M)$$

$$\mapsto H^1(M, \mathcal{O}_M(D_C)) \mapsto H^1(D_C, \mathcal{O}_{D_C}(D_C)) \mapsto 0.$$
indicates that the tangent direction in \( T_t(T(M)) \) which gives rise
to non-trivial infinitesimal curve deformations is in the kernel of the
tangent obstruction map

\[
T_t(T(M)) \cong H^1(M, \mathcal{O}_M) \mapsto H^1(M, \mathcal{O}_M(D_C)).
\]

Following the same idea as before, we define the algebraic family
Kuranishi model of the class \( C \),

**Definition 4.4** Let \( V, W \) be two algebraic vector bundles over the
\( q(M) \) dimensional torus \( T(M) \) and let \( \Phi_{VW} : V \mapsto W \) be an
algebraic bundle map from \( V \) to \( W \). We denote the corresponding
sheaves of sections and the sheaf morphism by \( V, W, \Phi_{VW} \), respec-
tively.

The data \((V, W, \Phi_{VW})\) is said to be an algebraic Kuranishi model
of \( C \) over \( M \times T(M) \) if there exist sheaf isomorphisms \( \text{Ker}(\Phi_{VW}) \cong \mathcal{R}^0\pi_*(\mathcal{E}) \) and \( \text{Coker}(\Phi_{VW}) \cong \mathcal{R}^1\pi_*(\mathcal{E}) \).

In the following, we prove the existence of the algebraic Kuranishi
model of \( C \).

**Proposition 4.2** Suppose that \( C \in H^2(M, \mathbb{Z}) \) on an algebraic sur-
face \( M \) with \( p_g(M) = 0, q(M) > 0 \) satisfies
(i). \( C \cdot C - c_1(K_M) \cdot C \geq 0 \),
(ii). \( (c_1(K_M) - C) \cdot \omega_M < 0 \).

Then there exists an algebraic Kuranishi model of \( C \) over \( M \times T(M) \).

Proof of the proposition: Because \( M \) is algebraic, we consider an
ample effective divisor \( D \) on \( M \). Then \( D = D \times T(M) \) is relative
ample along \( \pi : M \times T(M) \mapsto T(M) \).

We consider the short exact sequence

\[
0 \mapsto \mathcal{O}_{M \times T(M)} \mapsto \mathcal{O}_{M \times T(M)}(nD) \mapsto \mathcal{O}_{nD}(nD) \mapsto 0,
\]

for a large \( n \), whose value is yet to be determined.

By tensoring with \( \mathcal{E} \) and by taking the right derived long exact
sequence,

\[
0 \mapsto \mathcal{R}^0\pi_*(\mathcal{E}) \mapsto \mathcal{R}^0\pi_*(\mathcal{E}(nD)) \mapsto \\
\mathcal{R}^1\pi_*(\mathcal{O}_{nD} \otimes \mathcal{E}(nD)) \mapsto \mathcal{R}^1\pi_*(\mathcal{E})
\]
\[ \mapsto R^1\pi_*(\mathcal{E}(nD)) \mapsto R^1\pi_*(\mathcal{O}_{nD} \otimes \mathcal{E}(nD)) \mapsto 0, \]

where we have used the vanishing of \( R^2\pi_*(\mathcal{E}) \) over \( T(M) \).

To show that \( R^0\pi_*(\mathcal{E}(nD)) \) and \( R^0\pi_*\pi_*(\mathcal{O}_{nD} \otimes \mathcal{E}(nD)) \) are locally free and the sequence is truncated to a four-term long exact sequence, it suffices to prove that the sheaf \( R^1\pi_*(\mathcal{E}(nD)) \) vanishes on \( T(M) \). Once this has been done, the sheaf exact sequence truncates to a four-term exact sequence and \( R^1\pi_*(\mathcal{O}_{nD} \otimes \mathcal{E}(nD)) \) vanishes as well.

Then \( R^0\pi_*(\mathcal{O}_{nD} \otimes \mathcal{E}(nD)) \) is locally free as \( H^0(nD, \mathcal{O}_{nD}(nD) \otimes \mathcal{E}|_{M \times t}) \) is of constant rank independent of \( t \in T(M) \) (see [Ha] theorem 21.11).

To show that \( R^1\pi_*(\mathcal{E}(nD)) \) vanishes on \( T(M) \), it suffices to check that \( h^1(M, \mathcal{O}_M(nD) \otimes \mathcal{E}|_{M \times t}) \) is zero for all \( t \in T(M) \).

By Nakai criterion, one can make \( \mathcal{O}(nD) \otimes K^{-1}_M \otimes \mathcal{E}|_{M \times \{t\}} \) ample if \( n \) is chosen to be large enough. Then the vanishing of the \( h^1(M, \mathcal{O}_M(nD) \otimes \mathcal{E}|_{M \times t}) \) is a simple consequence of Kodaira vanishing theorem. \( \square \)

We define the algebraic Seiberg-Witten invariant of \( C \) to be

\[ \text{ASW}(C) = \int_{P(V)} c_1^{d(C)}(\mathcal{H}) c_{\text{top}}(\mathcal{H} \otimes \pi_{P(V)}^* \mathcal{W}) \]

\[ = c_1^{d(C)}(\mathcal{H}) \cap c_{\text{top}}(\mathcal{H} \otimes \pi_{P(V)}^* \mathcal{W}) \in A_0(P(V)), \]

where the bold face \( c_i \) denotes the Chern classes as algebraic cycle classes in \( A_*(P(V)) \). The \( \cap \) is the intersection product on the cycles.

The definition is independent to the choices of the algebraic Kuranishi models of \( C \) as we notice that \( \text{ASW} \) can be computed by using Grothendieck-Riemann-Roch formula on \( [V - W] \in K(T(M)) \) and is identical to the wall crossing calculation of \( 2C - K_M \) performed in [LL2].

**Remark 4.2** The algebraic Seiberg-Witten invariant defined in this way corresponds to counting curves from all the possible holomorphic structures on the line bundle with first Chern class \( C \). To count curves from a fixed holomorphic structure (linear system) we have to modify the definition of \( \text{ASW} \) by the following recipe

\[ \int_{P(V)} c_1^{d(C) - q(M)}(\mathcal{H}) \pi_{P(V)}^* [T(M)] c_{\text{top}}(\mathcal{H} \otimes \pi_{P(V)}^* \mathcal{W}), \]

38
by inserting the pull back of the fundamental class \([T(M)]\) and dropping the power of \(c_1(H)\).

In general, one can consider a finite number of classes in \(H_1(M, \mathbb{Z})\) which generate a finite number of elements \(\eta_i\) in \(H^1(T(M), \mathbb{Z})\). One may insert the cup product of these elements \(\wedge \eta_i\) into the integral (intersection pairing) and the corresponding algebraic Seiberg-Witten invariant counts the algebraic curves from the holomorphic structures in a locus of \(T(M)\) poincare dual to \(\wedge \eta_i \in H^*(T(M), \mathbb{Z})\).

The corresponding wall crossing number has been calculated in [L][LL1], and we omit the details here.

Case 2: \((c_1(K_M) - C) \cdot \omega_M > 0\). If both pairings \(C \cdot \omega_M\) and \((c_1(K_M) - C) \cdot \omega_M\) are positive, \(\mathcal{R}^2\pi_*(\mathcal{E})\) may not be trivial and the previous argument is not applicable.

Nevertheless, we still have the following proposition,

**Proposition 4.3** The supports of the coherent sheaves \(\mathcal{R}^0\pi_*(\mathcal{E})\) and \(\mathcal{R}^2\pi_*(\mathcal{E})\) do not intersect on \(T(M)\).

Proof: The proposition is in fact a generalization of lemma 4.1. Firstly, we mark that by [Ha] chapter 2, section 5, exercise 5.6(c), the support of a coherent sheaf is closed. If the intersection of these two supports is non-empty, then by base change there exists at least a \(t \in T(M)\) such that \(H^0(M, \mathcal{E}|_{M \times t})\) and \(H^2(M, \mathcal{E}|_{M \times t})\) are both non-trivial. Then the chosen \(t \in T(M)\) defines a holomorphic line bundle structure on the \(\mathcal{C}^\infty\) line bundle with first Chern class \(C\). Then one applies lemma 4.1 to this situation and finds the contradiction! Thus, the supports of the two coherent sheaves can never intersect.

\(\Box\)

The positivity of both \((c_1(K_M) - C) \cdot \omega_M\) and \(C \cdot \omega_M\) implies that \(c_1(K_M) \cdot \omega_M > 0\).

Given the class \(c_1(K_M)\) with positive pairing with \(\omega_M\), we separate into two cases depending on whether \(c_1(K_M)\) is poincare dual to a holomorphic curve in \(M\) or not.

(i). \(c_1(K_M)\) is poincare dual to a holomorphic curve in \(M\).

Because \(p_g = \dim \mathbb{C}H^0(M, K_M) = 0\), the holomorphic curve is an effective divisor of some holomorphic line bundle which has an underlying \(\mathcal{C}^\infty\) line bundle as \(K_M\).
According to surface classification result, such algebraic surface is either a general type surface or an elliptic surface. For both types of surfaces, $K_{M_{\text{min}}}$ is numerically effective. Namely, $c_1(K_{M_{\text{min}}})$ has non-negative pairings with effective classes.

By repeating blowing down $M$, one may get a unique minimal model of $M$, denote by $M_{\text{min}}$, and the adjunction equality states that

$$c_1(K_M) = c_1(K_{M_{\text{min}}}) + \sum E_i,$$

where $E_i \in H^2(M, \mathbb{Z})$ are the exceptional $-1$ class of the blowing down map $M \rightarrow M_{\text{min}}$.

We may write $C$ as $C_{\text{min}} + \sum m_i E_i$, with $C_{\text{min}} \in H^2(M_{\text{min}}, \mathbb{Z}) \subseteq H^2(M, \mathbb{Z})$. A direct calculation shows that

$$C_{\text{min}}^2 - c_1(K_{\text{min}}) \cdot C_{\text{min}} = C^2 - c_1(K_M) \cdot C + \sum (m_i^2 - m_i) \geq C^2 - c_1(K_M) \cdot C \geq 0,$$

independent of the signs of these $m_i$.

If $C_{\text{min}} = 0 \in H^2(M_{\text{min}}, \mathbb{Z})$, then $C = \sum m_i E_i$ is a multiple of exceptional classes. The condition $C^2 - C \cdot K_M = -\sum (m_i^2 - m_i) \geq 0$ forces $m_i(m_i - 1) = 0$ for all $i$ and therefore either $m_i = 1$ or $m_i = 0$.

Define $J = \{i | m_i = 1, 1 \leq i \leq n\}$. Thus, $C = \sum_{i \in J} E_i$ can and only can be represented by the sum of holomorphic $-1$ curves. In this case, we define $\text{ASW}(\sum E_i)$ to be 1. If $C_{\text{min}} \neq 0$, then the image of the holomorphic curve dual to $C$ under the blowing down map $M \rightarrow M_{\text{min}}$ is a holomorphic curve dual to $C_{\text{min}}$.

Denote $L = C_{\text{min}} - c_1(K_{M_{\text{min}}}) \in H^2(M_{\text{min}}, \mathbb{Z})$. If $L = 0$, $C$ has to be equal to $c_1(K_{M_{\text{min}}}) + \sum_{i \in J} E_i$ for some sub-collection $J$ of $-1$ classes. If $-L$ is not poincare dual to any holomorphic curve in $M_{\text{min}}$, then $c_1(K_M) - C = -L + \sum_{1 \leq i \leq n} (1 - m_i) E_i$ is not representable by holomorphic curves in $M$, either. In this case, $R^2 \pi_*(\mathcal{E})$ vanishes. The definition of $\text{ASW}(C)$ is identical to the $(c_1(K_M) - C) \cdot \omega_M < 0$ case.

So we may suppose that $-L$ is representable by holomorphic curves in $M_{\text{min}}$ and therefore $-L \cdot \omega_{M_{\text{min}}} > 0$.

Because $K_{M_{\text{min}}}$ is numerically effective and both $C_{\text{min}}$ and $c_1(K_{M_{\text{min}}}) - C_{\text{min}}$ are represented by holomorphic curves in $M_{\text{min}}$, it follows that $c_1(K_{M_{\text{min}}}) \cdot C_{\text{min}}$ and $c_1(K_{M_{\text{min}}}) \cdot (c_1(K_{M_{\text{min}}}) - C_{\text{min}})$ are non-negative.

Therefore

$$C_{\text{min}}^2 = (C_{\text{min}}^2 - c_1(K_{M_{\text{min}}}) \cdot C_{\text{min}}) + c_1(K_{M_{\text{min}}}) \cdot C_{\text{min}} \geq 0 + 0 = 0,$$
(c_1(K_{\text{min}}) - C_{\text{min}})^2 = ((c_1(K_{\text{min}}) - C_{\text{min}})^2 - c_1(K_{\text{min}}) \cdot (c_1(K_{\text{min}}) - C_{\text{min}}))
\quad + c_1(K_{\text{min}}) \cdot (c_1(K_{\text{min}}) - C_{\text{min}})
\quad = (C_{\text{min}}^2 - c_1(K_{\text{min}}) \cdot C_{\text{min}}) + c_1(K_{\text{min}}) \cdot (c_1(K_{\text{min}}) - C_{\text{min}}) \geq 0 + 0 = 0.

Because both the classes $c_1(K_{\text{min}}) - C_{\text{min}}$ and $C_{\text{min}}$ are in the forward light cone, then by the line cone lemma [LL3], their intersection pairing

$$(c_1(K_{\text{min}}) - C_{\text{min}}) \cdot C_{\text{min}} = -(C_{\text{min}}^2 - C_{\text{min}} \cdot c_1(K_{\text{min}})) \geq 0.$$ 

Thus $(c_1(K_{\text{min}}) - C_{\text{min}}) \cdot C_{\text{min}} = 0$, which can only occur either if $c_1(K_{\text{min}}) - C_{\text{min}}$ and $C_{\text{min}}$ are parallel to each other (up to torsions) in $H^2(M, \mathbb{Z})$ and both $c_1(K_{\text{min}}) - C_{\text{min}}$ and $C_{\text{min}}$ lie on the boundary of the light cone, i.e. $M_{\text{min}}$ is a minimal elliptic surface with $c_1^2(K_{\text{min}}) = 0$ and $C_{\text{min}} = r c_1(K_{\text{min}})$ (up to torsions) for some $r \in \mathbb{Q}, |r| \leq 1$.

The definition of the algebraic Seiberg-Witten invariant $\text{ASW}(C)$ in this situation deserves some additional discussion. Because of the presence of the sheaf $R^2 \pi_*(\mathcal{E})$, the algebraic Kuranishi model for $C$ is not defined over the whole $T(M)$ as in the $(c_1(K_M) - C) \cdot \omega_M < 0$ case.

Instead one considers the support of $R^0 \pi_*(\mathcal{E})$ and $R^2 \pi_*(\mathcal{E})$ and denote them by $Z_0 \subset T(M)$ and $Z_2 \subset T(M)$, respectively. According to [Ha] section 2.5, exercise 5.6., $Z_i, i = 0, 2$ are compact sub-varieties of $T(M)$.

As before consider the derived long exact sequence of the short exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}(nD) \rightarrow \mathcal{O}_nD \otimes \mathcal{E}(nD) \rightarrow 0,$$

for an ample divisor $D \subset M, D = D \times T(M) \subset M \times T(M)$, and a sufficiently large $n$.

As in the $(c_1(K_M) - C) \cdot \omega_M < 0$ case, the sheaf $R^2 \pi_*(\mathcal{E})$ is trivial over Zariski open $Z^*_2 = T(M) - Z_2$. Thus, one still can prove the existence of

$$\Phi_{\text{ASW}} : \mathcal{V} \rightarrow \mathcal{W}$$

over $Z^*_2$, where $\mathcal{V}$ and $\mathcal{W}$ are locally free sheaves over $Z^*_2$ such that $\text{Ker}(\Phi_{\text{ASW}}) \cong R^0 \pi_*(\mathcal{E}|_{M \times Z^*_2})$ and $\text{Coker}(\Phi_{\text{ASW}}) \cong R^1 \pi_*(\mathcal{E}|_{M \times Z^*_2}).$
Let \( V, W \) be the corresponding algebraic vector bundles over \( Z_c^2 \), then we still have \( \text{rank}_C(V - W) = \frac{C^2 - c_1(K_M)C - q(M)}{2} + 1 \).

Consider the projective space bundle \( \pi_P : P_{Z_c^2}(V) \to Z_c^2 \) as the ambient space. Then \( c_{\text{top}}(H \otimes \pi_P^* W) \) determines an algebraic cycle class \([M_C] \subset P_{Z_c^2}(V)\) of dimension \( q(M) + (\text{rank}_C V - 1) - \text{rank}_C W = 0 \) representing \( c_{\text{top}}(H \otimes \pi_P^* W) \). The zero dimensional cycle class is an integral multiple \( m_C \) of the generator \([pt] \subset A_0(P_{Z_c^2}(V))\).

We define \( \mathcal{ASW}(C) \) to be \( m_C \).

**Remark 4.3** One may construct an algebraic Kuranishi model for both \( R_i \pi^*(E), i = 0, 2 \) over \( T(M) \), but the \( c_{\text{top}}(H \otimes W) \) over the projective space bundle \( P_{T(M)}(V) \) represents the wall crossing number of the associated spin\(^c\) structure and is calculable by the universal wall crossing formula \([LL2]\). It is equal to zero for \( C = \alpha c_1(K_M) + \sum E_i \) on \( p_q = 0 \) elliptic surfaces, which indicates that besides the algebraic cycle class associated with \( R^0 \pi_* (E) \), the algebraic cycle class associated with \( R^2 \pi_* (E) \) gives an opposite contribution.

This indicates that the algebraic Seiberg-Witten invariant, which is supposed to be equal to the Seiberg-Witten invariant under large deformation on the Seiberg-Witten equations by the symplectic form, is identical to the Seiberg-Witten invariant in the metric chamber (because the wall crossing number is 0).

In fact, for an elliptic surface \( M \to \Sigma \) over a higher genus curve \( \Sigma, T(M) \) can be identified with \( J(\Sigma) \), the Jacobian variety of \( \Sigma \) and the Seiberg-Witten invariant of the spin\(^c\) class \( 2C - K_M \) (in additive notation) was determined in \([FM2]\) and was closed related to the intersection theory on the symmetric product \( S^d \Sigma \) for some \( d \in \mathbb{N} \). The space \( S^d \Sigma \) can be identified with \( \coprod_{t \in Z_0} P(H^0(M, E_M \times \{t\})) \) in our picture and the support \( Z_0 \) can be identified with the image of \( S^d \Sigma \to J(\Sigma) = T(M) \).

For the details of the enumeration of \( \mathcal{ASW}(C) \) based on curve theory, please consult \([FM2]\).

(ii). \( c_1(K_M) \) is not poincare dual to any holomorphic curve in \( M \).

If \( R^0 \pi_* (E) \) is trivial, namely, for all \( t \in T(M) \) the global sections \( \Gamma(M \times \{t\}, E_{M \times \{t\}}) = 0 \), then \( C \) is not poincare dual to any holomorphic curve in \( M \). In this case, we simply define \( \mathcal{ASW}(C) \) to be 0.

On the other hand, suppose \( R^0 \pi_* (E) \) is not trivial, then \( C \) is poincare dual to some holomorphic curve in \( M \). This implies that
$c_1(K_M) - C$ cannot be poincare dual to any holomorphic curve in $M$. Otherwise, the sum of the homology classes of the holomorphic curves dual to $C$ and $c_1(K_M) - C$ is poincare dual to $c_1(K)$, violating the assumption that $c_1(K_M)$ is not represented by holomorphic curves in $M$.

Thus, the sheaf $R^2\pi_*(\mathcal{E})$ must vanish. Then we define $\mathcal{A}SW(C)$ by the same recipe as in $(c_1(K_M) - C) \cdot \omega_M < 0$ case on page 36.

4.3 The Algebraic Family Seiberg-Witten Invariants for $p_g > 0$ Algebraic Surfaces

In the previous subsection, we defined the algebraic Seiberg-Witten invariants for $p_g = 0$ surfaces, based on discussions on the class $C$. The invariants defined as intersection numbers of (subspaces) of projective space bundles are equal to the topological Seiberg-Witten invariants of these surfaces in the specific chambers corresponding to large deformations of symplectic (Kahler) forms.

In general, their values are directly related to the wall crossing formula calculated in [KM], [LL2].

However, the ordinary Seiberg-Witten invariant (which are equal to Gromov-Taubes invariant through Taubes' theorem 'SW=Gr') for $p_g > 0$ algebraic surface has the simple type property. Namely, the invariant vanishes if the expected dimension $d_{GT}(C)$ of the $spin^c$ structure is positive.

On the other hand, we will define a version of algebraic Seiberg-Witten invariant of $C \in H^2(M, \mathbb{Z})$, which is non-zero for most classes $C$ with $C \cdot \omega_M > 0$, $C^2 - c_1(K_M) \cdot C = 2d_{GT}(C) \geq 0$. The readers should be aware that $\mathcal{A}SW$ defined in this section is not directly related to the usual Seiberg-Witten invariant $SW(2C - K_M)$ of the $spin^c$ class $2C - K_M$.

It turns out that for most of the $C$ specified above, we can construct algebraic Kuranishi models by algebraic vector bundles and algebraic bundle morphisms. But for a few minor cases, we have to move out of the algebraic category and consider non-algebraic bundle maps, despite that the moduli space of curves dual to $C$ is still an algebraic object.

Firstly, We begin by addressing the definition of the invariant $\mathcal{A}SW$ and then at the end we give a brief discussion about its relationship with the family Seiberg-Witten invariant.
The key difference from the usual Seiberg-Witten invariant is the dimension formula of $\mathcal{ASW}$.

For a class $C \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{C})$ with $C \cdot \omega_M > 0$, its expected family moduli space dimension is $C^2 - C \cdot c_1(K_M) + fbd(C, M)$, where $fbd(C, M) \in \mathbb{N} \cup \{0\}$, the abbreviation of the formal base dimension, is a correction term depending on $C$.

As usual, let $\mathcal{E} \mapsto M \times T(M)$ denote the locally free sheaf associated to $E$, with the holomorphic structures parametrized by $T(M)$.

**Definition 4.5** If $R^0\pi_*(\mathcal{E})$ is the zero sheaf over $T(M)$, then define $fbd(C, M) = 0$. Otherwise, $C$ is Poincaré dual to the fundamental class of a holomorphic curve, which is the zero locus of some $s_t \in H^0(M, \mathcal{E}_M \times \{t\})$.

Given the pair $(s_t, t)$, $t \in T(M)$ and $s_t \in H^0(M, \mathcal{E}_M \times \{t\}) - \{0\}$, the tensor product with $s_t$ induces the sheaf morphism

$$\otimes s_t : \mathcal{O}_M \rightarrow \mathcal{E}_M \times \{t\}$$

and then the induced morphism of cohomologies

$$(\otimes s_t)_* : H^2(M, \mathcal{O}_M) \mapsto H^2(M, \mathcal{E}_M \times \{t\})$$

Then the number $fbd(C, M)$ is defined to be the dimension of $\cap_{(s_t, t)} \ker ((\otimes s_t)_*)$, with $t \in T(M)$ and $s_t \in H^0(M, \mathcal{E}_M \times \{t\}) - \{0\}$.

It follows from the definition that $0 \leq fbd(C, M) \leq \dim H^2(M, \mathcal{O}_M) = p_g$. In the following, we assume that $C \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{C})$ satisfies $C \cdot \omega_M > 0$ and $\frac{C^2 - C \cdot c_1(K_M) C}{2} + fbd(C, M) \geq 0$.

As in the $p_g(M) = 0$ case, we separate into different cases.

**Case 1:** $(c_1(K_M) - C) \cdot \omega_M < 0$: This condition implies the vanishing of the second derived image sheaf $R^2\pi_*(\mathcal{E})$. It follows from the definition of $fbd$ that $fbd(C, M) = p_g$.

As in the $p_g = 0$ case, we can construct algebraic Kuranishi model $(V, W, \Phi_{VW})$ over $T(M)$, with the understanding that $T(M) = \text{Pic}^0(M)$ reduces to a point when $q(M) = 0$. Then one defines $\mathcal{ASW}(C)$ to be

$$\int_{\mathcal{P}_T(M)(V)} \frac{c_1^2 - C \cdot c_1(K_M)}{2} + p_g(H) \cup c_{\text{top}}(H \otimes \pi_*(\mathcal{V} \otimes W))$$

44
Lemma 4.2 Let $M$ be an algebraic surface with $p_g(M) > 0$ and $M_{min}$ denotes the unique minimal model of $M$. Let $E_1, E_2, \cdots E_n$ denote the $-1$ classes $\in H^2(M, \mathbb{Z}) = H^2(M_{min}, \mathbb{Z})$ of the blowing down map $p : M \mapsto M_{min}$. Then for $C = \sum_{i \in I} E_i, I \subset \{1, 2, \cdots , n\}$, the number $fbd(C, M)$ is zero.

Proof of the lemma: By adjunction formula $c_1(K_M) = c_1(K_{M_{min}}) + \sum_{i=1}^{n} E_i$. Let $K_M(K_{M_{min}})$ denote the canonical sheaf associated to $K_M(K_{M_{min}})$, respectively. By abusing the notation, we use the same symbol $E_i$ to denote the $-1$ cohomology class and the unique effective $-1$ divisor it associates with. As usual $\mathcal{E}$ is the locally free sheaf over $M \times T(M)$ with $c_1(\mathcal{E}) = \sum_{i \in I} E_i$.

Then

$$H^0(M, K_M \otimes \mathcal{O}(- \sum_{i \in I} E_i)) \longrightarrow H^0(M, K_M), I \subset \{1, 2 \cdots , n\}$$

is an isomorphism. Denote $p : M \mapsto M_{min}$ to be the blowing down map. The isomorphism follows from rewriting $K_M \otimes \mathcal{O}(- \sum_{i \in I} E_i)$ and $K_M$ as $p^* K_{M_{min}} \otimes \mathcal{O}(\sum_{i \in I} E_i)$ and $p^* K_{M_{min}} \otimes \mathcal{O}(\sum_{i=1}^{n} E_i)$ and there is a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
H^0(M_{min}, K_{M_{min}}) & \longrightarrow & H^0(M, p^* K_{M_{min}}) \\
\downarrow & & \downarrow \\
H^0(M, p^* K_{M_{min}} \otimes \mathcal{O}(\sum_{i \in I} E_i)) & \longrightarrow & H^0(M, K_M)
\end{array}
$$
By Serre duality, this implies that for some $t \in T(M)$, $H^2(M, \mathcal{O}_M) \mapsto H^2(M, \mathcal{E}_{M \times \{t\}})$ is an isomorphism. Thus, the formal base dimension $fbd(C, M) = 0$ for such classes. □

This implies that the expected dimension of the classes $\sum_{i \in I} E_i$ is 0, the same as $p_g(M) = 0$ case. The moduli space of curves is a single regular point and $\text{ASW}(\sum_{i \in I} E_i)$ is defined to be 1.

(i). If $C = \alpha c_1(K_{M_{\text{min}}}) + \sum_{i \in I} E_i, \quad 0 \neq |\alpha| \leq 1, \quad I \subset \{1, 2, \ldots, n\}$, on an elliptic surface $M$ with the minimal model $M_{\text{min}}$, then $f : M_{\text{min}} \mapsto \Sigma$ is an elliptic fibration without multiple fibers. $K_{M_{\text{min}}} = f^* K$, where $K$ is a line bundle over $\Sigma$, $\deg_{\Sigma} K = g(\Sigma) - 1 + p_g(M)$.

In this case we know that,

Proposition 4.4 Let $C = \alpha c_1(K_{M_{\text{min}}}) + \sum_{i \in I} E_i, \quad 0 \neq |\alpha| \leq 1$ on an elliptic surface $M$, then $fbd(C, M) = 0$.

Proof of the proposition: We first show that $fbd(C, M) = fbd(\alpha c_1(K_{M_{\text{min}}}), M_{\text{min}})$ for such classes.

Let $\mathcal{E}_{\text{min}}$ be the locally free sheaf over $M_{\text{min}} \times T(M_{\text{min}})$ with $c_1(\mathcal{E}_{\text{min}}) = C_{\text{min}} = \alpha c_1(K_{M_{\text{min}}}) \in H^2(M_{\text{min}}, \mathbb{Z})$. Then for the blowing down map $p : M \mapsto M_{\text{min}}, \quad \mathcal{E} = p^* \mathcal{E}_{\text{min}} \otimes \mathcal{O}_M(\sum_{i \in I} E_i)$.

Pick an arbitrary $t \in T(M_{\text{min}}) \cong T(M)$ and consider the following commutative diagram

\[
\begin{array}{ccc}
H^2(M_{\text{min}}, \mathcal{O}_{M_{\text{min}}}) & \mapsto & H^2(M_{\text{min}}, \mathcal{E}_{\text{min}}|_{M_{\text{min}} \times \{t\}}) \\
\downarrow & & \downarrow \\
H^2(M, \mathcal{O}_M) & \mapsto & H^2(M, \mathcal{E}|_{M \times \{t\}})
\end{array}
\]

It is not hard to show that both the vertical arrows are isomorphisms. The equality of the formal base dimensions follows.

From now on we may assume that $M = M_{\text{min}}$ is a minimal elliptic surface and $\mathcal{E} = \mathcal{E}_{\text{min}}$. We would like to show that for any given array in $H^2(M, \mathcal{O}_M)$, there exists a pair $(t, s), t \in T(M), s \in H^0(M, \mathcal{E}|_{M \times \{t\}}) - \{0\}$ such that this array is mapped injectively under $\otimes s : H^2(M, \mathcal{O}_M) \mapsto H^2(M, \mathcal{E}_{M \times \{t\}})$.

By Serre duality, it suffices to show that for any given ray in $H^0(M, \mathcal{K}_M)$, there exists a pair $(t, s)$ as above such that

\[ (\otimes s)^* : H^0(M, \mathcal{K}_M \otimes \mathcal{E}^*|_{M \times \{t\}}) \mapsto H^0(M, \mathcal{K}_M) \]
maps onto this ray. Because both $C$ and $c_1(K_M)$ are pulled back from the base $\Sigma$ of the elliptic fibration $f : M \mapsto \Sigma$, the above statement can be translated into showing that: For all non-zero sections in $H^0(\Sigma, K)$, $K_M = f^*K$, $deg_\Sigma K = g(\Sigma) - 1 + p_g(M)$, there is an invertible sheaf $D$ over $\Sigma$, $deg_\Sigma D = (1 - \alpha)deg_\Sigma K$ and a section $s$ in $H^0(\Sigma, K \otimes D^*)$ such that the image of $s \otimes H^0(\Sigma, D) \mapsto H^0(\Sigma, K)$ contains these sections in $H^0(\Sigma, K) = H^0(\Sigma, O_\Sigma(\sum l_m x_l))$.

Let $\sum l_m x_l, x_l \in \Sigma$ be an effective divisor in the linear system of $|K|$, we know that $\sum l_m = deg_\Sigma K$. Choose a tuple $n_l \leq m_l, \forall l$ such that $\sum l_n = (1 - \alpha)deg_\Sigma K$. Then we can take $D = O_\Sigma(\sum l_n x_l)$ and $\sum (m_l - n_l)x_l$ defines a ray (up to $C^*$ action) of sections $s \in H^0(\Sigma, O_\Sigma(\sum (m_l - n_l)x_l)) \cong H^0(\Sigma, K \otimes D^*)$. It is apparent that the map induced by tensoring with $s$ maps onto the ray of sections defining $\sum l_m x_l$. □

(ii). $-2fbd(C, M) \leq C^2 - c_1(K_M) \cdot C < 0$. We do not classify $C$ in this situation. Any such $C$ does not correspond to basic classes for ordinary Seiberg-Witten invariants because of the negativity of its Gromov-Taubes dimension. These classes are candidates of the exceptional classes.

In the following, we discuss the construction of algebraic Kuranishi model of a $C \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{C})$ satisfying (i)' or (ii) above. In these cases the bundle map of the Kuranishi model will not be algebraic.

**Proposition 4.5** Let $M$ be an algebraic surface with $p_g(M) > 0$ and let $C \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{C})$ be an integral $(1, 1)$ class satisfying (i)' and (ii) on page 46. Then there exist algebraic vector bundles $V, W$ and an algebraic bundle map $\Phi_{VW} : V \mapsto W$ such that

(i). There exist another pair of algebraic bundles $\tilde{W}, \tilde{V}$ and the bundle map $\Psi_{\tilde{VW}} : \tilde{W} \mapsto \tilde{V}$ such that

$$Coker(\Psi_{\tilde{VW}}) \cong (R^2 \pi_*(\mathcal{E})) \cong R^0 \pi_*(\mathcal{E}^* \otimes K_M)^*,$$

by relative Serre duality.

(ii). The locally free sheaf $\mathcal{W}$ associated with the vector bundle $W$ is a sub-sheaf of the locally free sheaf $\mathcal{W} \oplus O_{T(M)}^{p_g-fbd(C,M)}$. And we have

$$Ker(\Phi_{VW}) \cong R^0 \pi_*(\mathcal{E}),$$

47
and \( \text{Coker}(\Phi_{VW}) \) contains a sub-sheaf isomorphic to \( R^1\pi_*(\mathcal{E}) \).

(iii). \( \text{rank}_C V - \text{rank}_C W = \frac{C^2 - c_1(K_M) - C}{2} - q(M) + p_g(M) + \text{fbd}(C, M) + 1 \).

Proof of the proposition: We prove (i), (ii), and (iii) step by step.

Step one: As usual, we choose an effective ample divisor \( D \) on \( M \) and consider the following short exact sequence for a large enough \( n \). By choosing \( n \) large enough, we may assume \( \Sigma \in |nD| \) to be a smooth curve in the complete linear system \( |nD| \). In the following, we assume that \( \Sigma = nD \) is a smooth curve.

\[
0 \rightarrow \mathcal{O}_M(\mathcal{E}) \rightarrow \mathcal{O}(\Sigma) \otimes \mathcal{E} \rightarrow \mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E} \rightarrow 0.
\]

We have the following long exact sequences and sheaf isomorphisms,

\[
0 \rightarrow R^0\pi_*(\mathcal{E}) \rightarrow R^0\pi_*(\mathcal{O}(\Sigma) \otimes \mathcal{E}) \rightarrow R^0\pi_*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E})
\]

\[
\rightarrow R^1\pi_*(\mathcal{E}) \rightarrow 0,
\]

\[
R^1\pi_*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}) \cong R^2\pi_*(\mathcal{E}).
\]

The difference from the previous situations is that \( R^0\pi_*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}) \) and \( R^1\pi_*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}) \) may not be locally free over \( T(M) \).

To remedy this, consider a sufficiently very ample divisor \( \Delta \) on the smooth curve \( \Sigma \in |nD| \) and take the derived long exact sequence of the following short exact sequence,

\[
0 \rightarrow \mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}|_{\Sigma \times T(M)} \rightarrow \mathcal{O}_\Sigma(\Sigma + \Delta) \otimes \mathcal{E}|_{\Sigma \times T(M)} \rightarrow \mathcal{O}_\Delta(\Sigma + \Delta) \otimes \mathcal{E}|_{\Sigma \times T(M)} \rightarrow 0.
\]

Then we have the following four-terms long exact sequence on the derived sheaves when \( \Delta \) is sufficiently very ample on \( \Sigma \),

\[
0 \rightarrow R^0\pi_*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}|_{\Sigma \times T(M)}) \rightarrow R^0\pi_*(\mathcal{O}_\Sigma(\Sigma + \Delta) \otimes \mathcal{E}|_{\Sigma \times T(M)}) \rightarrow
\]

\[
R^0\pi_*(\mathcal{O}_\Delta(\Sigma + \Delta) \otimes \mathcal{E}|_{\Delta \times T(M)}) \rightarrow R^1\pi_*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}|_{\Sigma \times T(M)}) \rightarrow 0.
\]

Define \( \mathcal{V} \), \( \tilde{\mathcal{W}} \) and \( \tilde{\mathcal{V}} \) to be the locally free sheaves.
\[ R^0 \pi_*(O(\Sigma) \otimes \mathcal{E}), \ R^0 \pi_*(O_\Sigma(\Sigma+\Delta) \otimes \mathcal{E}|_{\Sigma \times T(M)}) \text{ and } R^0 \pi_*(O_\Delta(\Sigma+\Delta) \otimes \mathcal{E}|_{\Delta \times T(M)}) \text{ over } T(M), \text{ respectively. As usual let } V, W \text{ and } \tilde{V} \text{ be the corresponding algebraic vector bundles.} \]

Then there are bundle maps \( \Phi'_{VW} : V \mapsto \tilde{W} \), and \( \Psi_{WV} : \tilde{W} \mapsto \tilde{V} \).

The map \( \Phi'_{VW} \) is induced by the sheaf morphism composition

\[
R^0 \pi_*(O(\Sigma) \otimes \mathcal{E}) \mapsto R^0 \pi_*(O_\Sigma(\Sigma) \otimes E|_{\Sigma \times T(M)}),
\]

while \( \Psi_{WV} \) is induced by the above four term long exact sequence.

We have \( \text{Ker}(\Phi'_{VW}) \cong R^0 \pi_*(\mathcal{E}) \).

By construction,

\[
\text{Coker}(\Psi_{WV}) \cong R^1 \pi_*(O_\Sigma(\Sigma) \otimes E|_{\Sigma \times T(M)}) \cong R^2 \pi_*(\mathcal{E} \otimes \mathcal{K}_M). \]

Thus, statement one has been proved.

In the following, we will implicitly identify \( \text{Coker}(\Psi_{WV}) \) with \( R^2 \pi_*(\mathcal{E}) \).

Step two: The definition of the number \( fbd(C, M) \) implies that there exists a surjective sheaf morphism

\[
\mathcal{O}^p_{T(M)} \subset \mathcal{O}^p_{T(M)} \cong R^2 \pi_*(\mathcal{O}_M \times T(M)) \mapsto \mathcal{R}^2 \pi_*(\mathcal{E}) \mapsto 0.
\]

We define the sheaf \( \mathcal{W} \) to be the direct sum

\[
\mathcal{W} = \text{Ker}(\mathcal{W} \mapsto \tilde{V}) \oplus \text{Ker}(\mathcal{O}^p_{T(M)} \mapsto \mathcal{R}^2 \pi_*(\mathcal{E})).
\]

Thus, \( \mathcal{W} \) is naturally a subsheaf of \( \tilde{\mathcal{W}} \oplus \mathcal{O}^p_{T(M)} \).

**Lemma 4.3** The sheaf \( \mathcal{W} \) is locally free.

Proof of the lemma: To show that \( \mathcal{W} \) is locally free, it suffices to show that ([Ha], page 174, chapter II. 8.9.) for all \( x \in T(M) \), the \( k(x) \) vector space \( \mathcal{W} \otimes k(x) \) is of constant rank independent of \( x \).

But this follows from the equality

\[
\text{rank}_{k(x)}(\mathcal{W} \otimes k(x)) = \text{rank}_{k(x)}\text{Ker}(\tilde{\mathcal{W}} \mapsto \tilde{V}) \otimes k(x)
\]

49
\[ \text{rank}_k(x) \mathcal{Ker}(\mathcal{O}_T^{p_g-fbd(C,M)} \mapsto \mathcal{R}^2\pi_*(\mathcal{E})) \otimes k(x) \]

\[ = \text{rank}_k(x) \mathcal{W} \otimes k(x) - \text{rank}_k(x) \tilde{\mathcal{W}} \otimes k(x) + \text{rank}_k(x) \mathcal{R}^2\pi_*(\mathcal{E})) \otimes k(x) \]

\[ + \text{rank}_k(x) \mathcal{O}_T^{p_g-fbd(C,M)} \otimes k(x) - \text{rank}_k(x) \mathcal{R}^2\pi_*(\mathcal{E})) \otimes k(x) \]

\[ = \text{rank}_k(x) \mathcal{W} - \text{rank}_k(x) \mathcal{\tilde{W}} + p_g - fbd(C,M), \]

and is independent of \( x \in T(M) \) because \( \tilde{\mathcal{V}}, \tilde{\mathcal{W}} \) are locally free.

The lemma has been proved. \( \Box \)

Once we realize \( \mathcal{W} \) to be a locally free sheaf, we are ready to construct \( \Phi_{\mathcal{VW}} : \mathcal{V} \mapsto \mathcal{W} \).

Because the image \( \Phi'_{\mathcal{VW}}(\mathcal{V}) \) lies in the sub-sheaf \( \mathcal{R}_0^0 \mathcal{\pi}^*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E}) \), the kernel of \( \Psi_{\mathcal{VW}} : \text{Im}(\Phi'_{\mathcal{VW}}) \) actually lies inside \( \mathcal{W} \). Thus, the map \( \Phi'_{\mathcal{VW}} \) factors through a sheaf morphism \( \Phi_{\mathcal{VW}} : \mathcal{V} \mapsto \mathcal{W} \) and

\[ \text{Ker}(\Phi_{\mathcal{VW}}) = \text{Ker}(\text{Phi}'_{\mathcal{VW}}) \cong \mathcal{R}_0^0 \mathcal{\pi}^*(\mathcal{E}). \]

On the other hand,

\[ \text{Coker}(\Phi_{\mathcal{VW}}) \supset \text{Ker}(\text{Phi}'_{\mathcal{VW}})/\text{Im}(\text{Phi}'_{\mathcal{VW}}) \]

\[ = \mathcal{R}_0^0 \mathcal{\pi}^*(\mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{E})/\text{Im}(\mathcal{R}_0^0 \mathcal{\pi}^* \text{bigl}(\mathcal{O}(\Sigma) \otimes \mathcal{E}) \cong \mathcal{R}_1^1 \mathcal{\pi}^*(\mathcal{E}). \]

This ends the proof of the second statement.

Step Three: We have

\[ \text{rank}_C \mathcal{V} - \text{rank}_C \mathcal{W} = \text{rank}_C \mathcal{V} - (\text{rank}_C \tilde{\mathcal{W}} + (p_g - fbd(C,M)) - \text{rank}_C \tilde{\mathcal{W}}) \]

\[ = \chi(\mathcal{O}(\Sigma) \otimes \mathcal{E}) - \chi(\mathcal{O}(\Sigma + \Delta) \otimes \mathcal{E}) - \text{deg}(\Delta) - p_g + fbd(C,M) \]

\[ = \chi(\mathcal{E}) - p_g + fbd(C,M) = \frac{C^2 - c_1(K_M) \cdot C}{2} - q(M) + fbd(C,M) + 1, \]

by surface Riemann-Roch theorem. This finishes the proof of step three and therefore the proposition. \( \Box \)

By the algebraic Kuranishi model \( \Phi_{\mathcal{VW}} : \mathcal{V} \mapsto \mathcal{W} \) constructed in proposition 4.5, the algebraic moduli space of curves dual \( C \) can be realized as a projective cone determined by the coherent sheaf \( \mathcal{R}_0^0 \mathcal{\pi}^*(\mathcal{E}) \) and is embedded inside \( \mathcal{P}_T(M)(\mathcal{V}) \) as the zero locus of a canonical section \( s_{\Phi_{\mathcal{VW}}} : \Gamma(\mathcal{P}_T(M)(\mathcal{V}), \mathcal{H}_{\mathcal{P}_T(M)} \otimes \mathcal{W}). \)

Thus, one defines \( \mathcal{A}_W(C) \) to be the intersection product of the algebraic cycle classes
At the end of this subsection, we explain why the above definitions of $\mathcal{ASW}$ in the various cases are independent to the choices of the algebraic Kuranishi models.

**Proposition 4.6** The algebraic Seiberg-Witten invariants defined in this section are independent to the choices of algebraic Kuranishi models $(V, W, \Phi_{VW})$ used to define them.

Proof of the proposition: As has been pointed out earlier, in a few cases ($p_g = 0$) the calculation of $\mathcal{ASW}$ can be identified with the wall crossing formula of Seiberg-Witten invariants [LL2] or some known calculation of Seiberg-Witten invariants [FM2]. Thus the answers are known to be independent to extra data like the choices of Kuranishi models. Nevertheless, we offer an algebraic proof for all the different cases.

The algebraic Seiberg-Witten invariants for $q(M) = 0$ algebraic surface is $\pm 1$ and the proof of the proposition is trivial in these cases. In the following, we assume $q(M) > 0$ and separate into two different situations.

(i). The sheaf $R_2^\pi_*(E) = 0$ over $T(M)$. In this case, $\mathcal{ASW}(C)$ is equal to

$$
\frac{c_1(H) \cap C^2 - c_1(K_M) \cdot C}{2} + fbd(C, M) \cap c_{\text{top}}(H \otimes W) + p_g(H) \cdot c_{\text{top}}(H \otimes \pi^*_P V W),
$$

for some algebraic Kuranishi model $(V, W, \Phi_{VW})$.

One may evaluate $\mathcal{ASW}(C)$ directly and find the answer to be

$$
\int_{\pi(P(V))} \pi^*_P(V) c_1(H) + p_g(H) \cdot c_{\text{top}}(H \otimes \pi^*_P V W),
$$

Then the independence to the algebraic Kuranishi models is due to the equality in the $K$ group of coherent sheaves on $T(M)$,

$$
W - V = R_1^\pi(\mathcal{E}) - R_0^\pi(\mathcal{E}).
$$

An alternative way without evaluating $\mathcal{ASW}$ is by stabilization and lemma 5.3 on page 68. Suppose that $(V_a, W_a, \Phi_{V_a W_a})$, $(V_b, W_b, \Phi_{V_b W_b})$ are two different algebraic Kuranishi models of the class $C$. 

51
Then by lemma 5.3 it is not hard to see that both the original integral expressions can be stabilized to

$$\int_{\mathbf{P}(V_a \oplus V_b)} c_1^{C_2 - c_1(KM) C_2} \cdot p_g(H) \cdot c_{top}(H \otimes \pi^*_{V_a \oplus V_b}(W_b \oplus V_a))$$

$$= \int_{\mathbf{P}(V_a \oplus V_b)} c_1^{C_2 - c_1(KM) C_2} \cdot p_g(H) \cdot c_{top}(H \otimes \pi^*_{V_a \oplus V_b}(W_b \oplus V_a)).$$

by using $[W_b - V_b] = [W_a - V_a]$ in the $K$ group of vector bundles.

(ii). Suppose that $\mathcal{R}^2 \pi^* \mathcal{E} \neq 0$ on $T(M)$, we show the independence to the Kuranishi models by a stabilization argument.

Case I: $p_g(M) = 0$. Denote the support of the coherent $\mathcal{R}^2 \pi^* \mathcal{E}$ to be $Z_2$. Suppose there are two algebraic Kuranishi models $(V_a, W_a, \Phi_{V_a W_a})$, $(V_b, W_b, \Phi_{V_b W_a})$ over $Z_2$ for $\mathcal{M}_C$, the identical stabilization argument as in (i). works except the Kuranishi models are defined over $Z_2$ instead of the whole $T(M)$.

Case II: $p_g(M) > 0$. In this case the $\mathcal{ASW}(C)$ is defined to be an intersection number on the total space of an algebraic vector bundle (over a projective space bundle).

Suppose that we are given two different algebraic Kuranishi models of $\mathcal{M}_C$, with the corresponding algebraic vector bundles given by $V_1, W_1, \tilde{W}_1$ and $V_2, W_2, \tilde{W}_2$, respectively.

Then we have to show that

$$c_1(H) \cap \{c_1^{C_2 - c_1(KM) C_2} + fbd(C,M)\} \cap c_{top}(H \otimes W_1) \in \mathcal{A}_0(P_{T(M)}(V_1)) \cong \mathbb{Z}$$

and

$$c_1(H) \cap \{c_1^{C_2 - c_1(KM) C_2} + fbd(C,M)\} \cap c_{top}(H \otimes W_2) \in \mathcal{A}_0(P_{T(M)}(V_2)) \cong \mathbb{Z}$$

coincide.

By lemma 5.3, we may replace the intersection numbers of algebraic cycles by

$$c_1(H) \cap \{c_1^{C_2 - c_1(KM) C_2} + fbd(C,M)\} \cap c_{top}(H \otimes (W_1 \oplus V_2)) \in \mathcal{A}_0(P_{T(M)}(V_1 \oplus V_2))$$
and
\[ \mathbf{c}_1(H) \cap \{ \frac{c_2 - c_1(K_M)}{2} + fbd(C, M) \} \cap \mathbf{c}_{top}(H \otimes (W_2 \oplus V_1)) \in \mathcal{A}_0(P_{T(M)}(V_1 \oplus V_2)). \]

As before, it suffices to show the following equality in the $K$ group of coherent sheaves $K_0(T(M))$,

\[ [\mathcal{V}_2] + [\mathcal{W}_1] = [\mathcal{V}_1] + [\mathcal{W}_2] \]

and it is equivalent to

\[ [\mathcal{V}_2] + [\tilde{\mathcal{W}}_1] + [\mathcal{O}_{T(M)}^{-fbd(C, M)}] - [\tilde{\mathcal{V}}_1] = [\mathcal{V}_1] + [\tilde{\mathcal{W}}_2] + [\mathcal{O}_{T(M)}^{-fbd(C, M)}] - [\tilde{\mathcal{V}}_2] \]

and is then equivalent to

\[ [\tilde{\mathcal{W}}_1] - [\mathcal{V}_1] - [\tilde{\mathcal{V}}_1] = [\tilde{\mathcal{W}}_2] - [\mathcal{V}_2] - [\tilde{\mathcal{V}}_2]. \]

By using the exact sequence in the proof of the proposition, both sides of the yet to be proved equality in $K_0(T(M))$ can be identified with

\[ -\mathcal{R}^0_\pi_*(\mathcal{E}) + \mathcal{R}^1_\pi_*(\mathcal{E}) - \mathcal{R}^2_\pi_*(\mathcal{E}), \]

therefore, both sides must be equal. \qed

### 4.3.1 The Relationship of $\mathcal{ASW}$ with Family Seiberg-Witten Invariants

In this subsection, we outline the relationship between $\mathcal{ASW}$ and the family Seiberg-Witten invariant.

For $p_g(M) = 0(b_2^+ = 1)$ algebraic surfaces, all the classes $C \in H^2(M, \mathbb{Z})$ automatically become $(1, 1)$ classes under the Hodge decomposition. And the algebraic Seiberg-Witten invariants of $C$ are equal to the Seiberg-Witten invariants of the $\text{spin}^c$ class $2C - K_M$ (in additive notation) in the chambers of deformations by large Kahler forms. On the other hand, for algebraic surfaces with positive geometric genera, the algebraic Seiberg-Witten invariant of $C$ is defined only for $C \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{C})$ and its dimension formula depends on the holomorphic invariant $0 \leq fbd(C, M) \leq p_g$. 
Suppose that $D_C$ is an effective curve on $M$ representing the cohomology class $C$ and $s_{D_C}$ is a defining global section of $D_C$. Then we analyze the long exact sequence associated to

\[ 0 \rightarrow \mathcal{O}_M \xrightarrow{\otimes s_{D_C}} \mathcal{O}_M(D_C) \rightarrow \mathcal{O}_{D_C}(D_C) \rightarrow 0, \]

\[ 0 \rightarrow H^0(M, \mathcal{O}_M) \rightarrow H^0(M, \mathcal{O}_M(D_C)) \rightarrow H^0(M, \mathcal{O}_{D_C}(D_C)) \]

\[ \rightarrow H^1(M, \mathcal{O}_M) \rightarrow H^1(M, \mathcal{O}_M(D_C)) \rightarrow H^1(M, \mathcal{O}_{D_C}(D_C)) \]

\[ \rightarrow H^2(M, \mathcal{O}_M) \rightarrow H^2(M, \mathcal{O}_M(D_C)) \rightarrow 0. \]

Let $\mathcal{T}_M$ and $\mathcal{T}_M^*$ denote the tangent and cotangent sheaves of $M$. As $C \in H^{1,1}(M, \mathcal{C}) \cong H^1(M, \mathcal{T}_M^*) = H^1(M, \Omega^1_M)$, the restriction of the cup product pairing

\[ \cup : H^1(M, \mathcal{T}_M) \otimes H^1(M, \Omega^1_M) \longrightarrow H^2(M, \mathcal{O}_M) \]

gives rises to the linear map

\[ H^1(M, \mathcal{T}_M) \xrightarrow{\cup[C]} H^2(M, \mathcal{O}_M), \]

sending the infinitesimal complex deformations of $M$ to the infinitesimal deformation of Hodge structures. In the fixed complex vector space $H^2(M, \mathcal{C})$, the deformation of the decomposition

\[ H^2(M, \mathcal{C}) = H^{2,0}(M, \mathcal{C}) \oplus H^{1,1}(M, \mathcal{C}) \oplus H^{0,2}(M, \mathcal{C}) \]

is a so-called deformation of the Hodge structures. The tangent space of infinitesimal deformations of $H^{1,1}(M, \mathcal{C}) \oplus H^{2,0}(M, \mathcal{C}) \subset H^2(M, \mathcal{C})$ can be identified with

\[ \text{Hom}_{\mathcal{C}}(H^{1,1}(M, \mathcal{C}) \oplus H^{2,0}(M, \mathcal{C}), H^2(M, \mathcal{C})/H^{1,1}(M, \mathcal{C}) \oplus H^{2,0}(M, \mathcal{C})) \]

\[ \cong \text{Hom}_{\mathcal{C}}(H^{1,1}(M, \mathcal{C}) \oplus H^{2,0}(M, \mathcal{C}), H^{0,2}(M, \mathcal{C})). \]

Then one may interpret the map $H^1(M, \mathcal{T}_M) \xrightarrow{\cup[C]} H^2(M, \mathcal{O}_M)$ as the composition of the infinitesimal period map

\[ H^1(M, \mathcal{T}_M) \mapsto \text{Hom}_{\mathcal{C}}(H^{1,1}(M, \mathcal{C}) \oplus H^{2,0}(M, \mathcal{C}), H^{0,2}(M, \mathcal{C})) \]

and
$$\text{Hom}_C(H^{1,1}(M, C) \oplus H^{2,0}(M, C), H^{0,2}(M, C)) \longrightarrow \text{Hom}_C(C, H^{0,2}(M, C)) \cong H^2(M, \mathcal{O}_M)$$

through the embedding $C \subset H^{1,1}(M, C)$.

In general, the linear map $\cup[C] : H^1(M, \mathcal{T}_M) \to H^2(M, \mathcal{O}_M)$ may not be surjective. For example, when $C = \alpha \cdot c_1(K_M)$ and when $H^1(M, \mathcal{T}_M)$ is un-obstructed, the map $\cup[C]$ is always trivial as the canonical class $c_1(K_M)$ persists to be of type $(1,1)$ under complex deformations of $M$.

Given an invertible sheaf $\mathcal{L}$, a connection on $\mathcal{L}$ is a compatible family of 1st order differential operators $\nabla$, such that for $U$ open, $X \in \Gamma(U, \mathcal{T}_M)$ and $s \in \Gamma(U, \mathcal{L})$, $\nabla_X(s) = X(f) \cdot s + f \cdot \nabla_X(s)$ for $f \in \Gamma(U, \mathcal{O}_M)$.

Suppose $D_C$ is an effective divisor and $s_{D_C}$ is a defining global section of $D_C$ in $\Gamma(M, \mathcal{O}(D_C))$. Then $\nabla(s)|_{D_C}$ establishes a morphism from $H^1(M, \mathcal{T}_M)$ to $H^1(D_C, \mathcal{O}_{D_C}(D_C))$.

Then we have the following commutative diagram

$$
\begin{array}{ccc}
H^1(M, \mathcal{T}_M) & \longrightarrow & H^1(D_C, \mathcal{O}_{D_C}(D_C)) \\
\downarrow & & \downarrow \delta \\
H^2(M, \mathcal{O}_M) & = & H^2(M, \mathcal{O}_M)
\end{array}
$$

where $\delta$ is the connection homomorphism in the long exact sequence and the left vertical arrow is the infinitesimal period map.

We can make the following conclusion on the comparison between $\text{ASW}_{\text{pt}}(1, C)$ and the family Seiberg-Witten theory.

(i). Suppose that $fbd(C, M) = p_g(M)$, then the expected dimension of the moduli space is $C^2 - c_1(K_M) \cdot C + p_g$. Let $B^{p_g}(\epsilon)$ denote the radius $\epsilon$ ball in the complex space $\mathbb{C}^{p_g}$.

Suppose that there exists a germ of deformation of complex structures of $M$, $\pi : \mathcal{X} \to B^{p_g}(\epsilon) \subset \mathbb{C}^{p_g}$ such that

(a). $\pi^{-1}(0)$ is bi-holomorphic to $M$.

and

(b). the infinitesimal composite period map

$$
\mathcal{T}_0B^{p_g} \to H^1(M, \mathcal{T}_M) \xrightarrow{\cup[C]} H^2(M, \mathcal{O}_M)
$$

is an isomorphism.

Then $C$ fails to be a $(1,1)$ class in the nearby fibers.

In the given local family of complex manifolds, the family moduli space of curves in $C$ localizes to be above $0 \in B^{p_g}(\epsilon)$. In such
situations, the dimension formula of ordinary \( SW \) theory and the expected virtual dimension differ by \( p_g \). But the latter matches up with the family \( SW \) theory dimension formula of a \( p_g \) family. Then \( \text{ASW}(C) \) is actually equal to the (local) family Seiberg-Witten invariant of the spin\(^c\) class \( 2C - K_M \) (in additive notation) with the family Seiberg-Witten equation deformed by the large fiberwise Kahler forms.

**Remark 4.4** The local condition (a) and (b) hold when there are \( p_g \) dimensional un-obstructed infinitesimal complex deformation of \( M \) map injectively into the infinitesimal deformation of Hodge structures.

The global version asks for the existence of \( \pi : \mathcal{X} \mapsto B \), where \( B \) is a compact smooth \( p_g \) dimensional variety and the total space \( \mathcal{X} \) is Kahler.

(a). \( \pi^{-1}(b) \) is bi-holomorphic to \( M \) for some \( b \in B \).

(a)’. \( C \in H^2(\mathcal{X,Z}) \) and \( i_b^*C \in H^{1,1}(\pi^{-1}(b), \mathbf{C}) \) for the inclusion map \( i_b : \pi^{-1}(b) \mapsto \mathcal{X} \).

(b). If \( C \in H^{1,1}(\pi^{-1}(b), \mathbf{C}) \) for any \( b \in B \), then the infinitesimal composite period map

\[
\mathbf{T}_bB \mapsto H^1(\pi^{-1}(b), T\pi^{-1}(b)) \mapsto H^2(\pi^{-1}(b), \mathcal{O}_{\pi^{-1}(b)})
\]

is an isomorphism.

It turns out that the algebraic surfaces with trivial canonical bundles \( K_M \cong \mathcal{O}_M \), i.e. \( K3 \) surfaces or abelian surfaces, carry the so-called twistor families of complex structures which have the desired global properties (a), (a)’, (b). In these situations, the \( \text{ASW} \) can be interpreted as the family Seiberg-Witten invariants in the chambers deformed by large \( S^2 \) family of hyperkahler forms [LL1].

Besides hyperkahler \( S^2 \) families of complex structures, it is probably too strong to find the germs of smooth families \( \mathcal{X} \mapsto B \) with smooth fibers which satisfy all (a), (a)’ and (b).

Suppose that \( c_1(M) = 0 \), then all the \( C \in H^2(M, \mathbf{Z}) \), \( C \cdot \omega_M > 0 \) satisfy \( \text{fbd}(C, M) = p_g \). On the other hand let \( E \) be sufficiently relative very ample on \( M \times T(M) \mapsto T(M) \), Kodaira vanishing theorem also implies that \( \text{fbd}(C, M) = p_g \) for such classes \( c_1(E) = C \). In particular, the algebraic Seiberg-Witten invariant of \( C \) for such \( E \) can be thought formally as the family invariant on a germ of \( p_g \) dimensional infinitesimal family.
(ii). Suppose that \( f_{bd}(C, M) = 0 \), e.g. \( C = \sum E_i \), then the dimension formula \( \frac{C^2 - C \cdot \eta(K_M)}{2} \) is identical to the \( p_g = 0 \) case.

By combining with the family blowing up formula, the independence of dimension formula on \( p_g \) explains why type I exceptional curves in the universal family [L1] of \( p_g > 0 \) algebraic surfaces obey the same dimension formula in the \( p_g = 0 \) algebraic surfaces.

(iii). Suppose that \( 0 < f_{bd}(C, M) < p_g \), then the dimension formula of \( C \) is shifted to \( \frac{C^2 - C \cdot \eta(K_M)}{2} + f_{bd}(C, M) \). We discuss (ii) and (iii) together.

By the definition of the number \( f_{bd}(C, M) \), there exists a subspace \( F \) of \( H^2(M, \mathcal{O}_M) \) of dimension \( f_{bd}(C, M) \) which is the intersection of all the kernels of \( H^2(M, \mathcal{O}_M) \otimes \mathbb{C} \to H^2(M, \mathcal{E}_M \times \{ t \}) \). Then the subspace \( F \) defines a \( f_{bd}(C, M) \) dimensional subspace of infinitesimal deformation of Hodge structures deforming \( C \) away from being a \( (1, 1) \) class. For positive \( f_{bd}(C, M) \), the \( F \) defines a trivial factor on the obstruction bundle of the Kahler Seiberg-Witten theory, which causes the usual SW invariant to vanish. By removing the trivial factor \( F \), the intersection number defining \( ASW(C) \) are generically nonzero. Only when certain conditions like (a), (a)', (b) (on page 56) hold, the operation of removing a trivial factor in the obstruction bundle can be interpreted as extending \( M \) into a local family.

5 The Algebraic Proof of the Family Blowup Formula

Having defined the algebraic Seiberg-Witten invariants in section 3, we offer an algebraic proof of the family blowup formula which includes the important special cases of the universal families \( M_{n+1} \mapsto M_n, n \in \mathbb{N} \) and its sub-families \( M_{n+1} \times M_n Y(\Gamma) \mapsto Y(\Gamma) \) on the closures of admissible strata See [Liu1]).

Suppose that \( \pi : \mathcal{X} \mapsto B \) is an algebraic family of smooth algebraic surfaces over a smooth algebraic base manifold \( B \) and let \( s : B \mapsto \mathcal{X} \) denote an algebraic cross section of the fibration. Let \( C \) be an element in \( H^{1,1}(\mathcal{X}, \mathbb{Z}) \) which restricts to a monodromy invariant class on the fibers. The inclusion \( s(B) \subset \mathcal{X} \) is a codimension two smooth sub-variety of \( \mathcal{X} \) with normal bundle \( N_{s(B)}\mathcal{X} \). Blowing
up $s(B) \subset X$ produces the blown up variety $X'$ with an exceptional divisor $E_B \cong \mathbb{P}_B(\mathbb{N})$. We denote the fiberwise exceptional curve (and the corresponding cohomology class) of $X' \to B$ by the notation $E$. Then the blowup formula is expected to relate the algebraic family invariants of $C$ over $X \to B$ and $C' = C + mE, m \in \mathbb{Z}$ over $X' \to B$.

Given the family $\pi : X \to B$, the family of the relative Picard group $\text{Pic}^0$ associated to the fiber algebraic surfaces as in proposition 4.1 forms a fiber bundle of complex tori over $B$, denoted by $\mathcal{T}_B(X)$. The fibration $\mathcal{T}_B(X) \to B$ can be identified with the quotient $\mathcal{R}^1\pi_*\mathcal{O}_X/\mathcal{R}^1\pi_*(\mathbb{Z})$.

As in the $B = pt$ case, there is a holomorphic line bundle $E$ over $\mathcal{T}_B(X)$ with first Chern class (the pull-back of) $C$. When $X \to B$ is the universal family $M_{n+1} \to M_n$ or its sub-families, it is easy to use the following proposition, prop. 5.1 repeatedly to show that $\mathcal{T}_B(X)$ is isomorphic to the trivial product $T(M) \times B$.

**Proposition 5.1** Let $X' \to B$ denote the blown up fibration from $X \to B$ along $s : B \to X$. Then $\mathcal{T}_B(X') = \mathcal{T}_B(X)$. Namely, the torus fibration is invariant under the blowing up along a cross section.

Proof: The fibers of $\mathcal{T}_B(X)$ are constructed from the Hodge group $H^{0,1}$ of the fibers of $X \to B$. The $(0,1)$ component of the Hodge decomposition of an algebraic surface is invariant under blowing ups. By applying the observation to a family that is why $\mathcal{T}_B(X)$ is invariant under blowing ups along cross sections.

We introduce some technical conditions on the fibration which guarantees the existence of algebraic Kuranishi models.

**Definition 5.1** An algebraic fibration $X \to B$ of relative dimension two is said to be relatively good if there exists an effective very ample divisor $D$ in $X$ which is of relative dimension one under $X \to B$. The algebraic fibration $X \to B$ is said to be two-relatively good if there exist two effective very ample divisors $D_1, D_2$ such that

(i). $D_1$ and $D_2$ are both smooth and are of relative dimension one over $B$.

(ii). $D_1 \cap D_2 \subset D_i, i = 1, 2$ is a smooth divisor in $D_1$ and $D_2$ and is of relative dimension zero over $B$. 

58
We introduce the following definition of formal excess base dimension $\text{febd}(C, \mathcal{X}/B)$, extending definition 4.5 for $B = \text{pt}$.

**Definition 5.2** Let $\pi : \mathcal{X} \rightarrow B$ be an algebraic fiber bundle of algebraic surfaces and let $C \in H^{1,1}(\mathcal{X}, \mathbb{C}) \cap H^2(\mathcal{X}, \mathbb{Z})$ be a monodromy invariant fiberwise cohomology class. Let $\mathcal{E}$ be the invertible sheaf over $T_B(\mathcal{X})$ and consider the following natural pairing

$$\mathcal{R}^0_{\pi_*}(\mathcal{E}) \otimes \mathcal{R}^2_{\pi_*}(\mathcal{O}_\mathcal{X}) \rightarrow \mathcal{R}^2_{\pi_*}(\mathcal{E}),$$

Define $\text{Ann}(\mathcal{R}^0_{\pi_*}(\mathcal{E})) \subset \mathcal{R}^2_{\pi_*}(\mathcal{O}_\mathcal{X})$ to be the annihilator of $\mathcal{R}^0_{\pi_*}(\mathcal{E})$ under the pairing.

If $\mathcal{R}^2_{\pi_*}(\mathcal{E}) = 0$, define $\text{febd}(C, \mathcal{X}/B) = p_g$ in this trivial case.

If $\mathcal{R}^2_{\pi_*}(\mathcal{E}) \neq 0$, then the formal excess base dimension $\text{febd}(C, \mathcal{X}/B)$ is defined to be the rank of the maximal trivial locally free subsheaves of $\text{Ann}(\mathcal{R}^0_{\pi_*}(\mathcal{E}))$.

To simplify our notations, in the following theorem and its proof we do not write explicitly the pull-back maps on line bundles and the cohomologies $H^*(\mathcal{X}, \mathbb{Z}) \rightarrow H^*(\mathcal{X}', \mathbb{Z})$ induced by the blowing down map $\mathcal{X}' \rightarrow \mathcal{X}$. Thus, we use the same symbol $C$ to denote the fiberwise class on $\mathcal{X}$ and its pull-back to $\mathcal{X}'$. The readers should be able to determine from the context whether we refer to the class $C$ on $\mathcal{X}$ or its pull-back to $\mathcal{X}'$.

Let $\mathcal{E}$ be an invertible sheaf on $\mathcal{X}' \times_B T_B(\mathcal{X}')$, then $c_1(\mathcal{E})$ pulls back to a class in $H^2(\mathcal{X}', \mathbb{Z})$. We abuse the notation and denote it by the same symbol $c_1(\mathcal{E})$. In the following, we assume that the pull-back first Chern class $c_1(\mathcal{E}) = C$ or $C + mE$ in $H^2(\mathcal{X}', \mathbb{Z})$.

If the geometric genus $p_g$ of the fiber algebraic surfaces is zero or if $c_1(K_{X/B}) - C$ is of non-positive degree (with respect to an ample polarization), we assume that fiber bundle $\pi' : \mathcal{X}' \rightarrow B$ is relatively good. If $p_g$ of the fiber surfaces is greater than zero and the relative degree of the class $c_1(K_{X/B}) - C$ is positive, then we assume that $\pi' : \mathcal{X}' \rightarrow B$ is two-relatively good.

Once we make such assumptions on the fiber bundle $\mathcal{X}' \rightarrow B$, we can mimic the construction in section 3 and construct the algebraic family Kuranishi models of the family moduli spaces of curves dual to $C$ and $C + mE$. Because the current construction is basically the family extension of the construction for $B = \text{pt}$, we omit much of the details. Nevertheless, we address the analogue of proposition 4.5 which constructs the algebraic family Kuranishi model $(\mathcal{V}, \mathcal{W}, \Phi_{\mathcal{VW}})$ of the case $p_g > 0$ and $\mathcal{R}^2_{\pi_*}(\mathcal{E}) \neq 0$. 

59
By definition 5.2 of \( \text{febd}(c_1(\mathcal{E}), \mathcal{X}'/B) \), there exists an inclusion 
\( \mathcal{O}_{\mathcal{T}_B(\mathcal{X}')}^{\text{febd}(c_1(\mathcal{E}), \mathcal{X}'/B)} \subset \mathcal{R}^2\pi'_*(\mathcal{O}_{\mathcal{X}'}) \). Define the quotient locally free
sheaf \( \mathcal{R}^2\pi'_*(\mathcal{O}_{\mathcal{X}'})/\mathcal{O}_{\mathcal{T}_B(\mathcal{X}')}^{\text{febd}(c_1(\mathcal{E}), \mathcal{X}'/B)} \) to be \( \mathcal{F} \). Denote \( i_{\mathcal{F}} \) to be the
surjective morphism \( \mathcal{F} \hookrightarrow \mathcal{R}^2\pi'_*(\mathcal{E}) \).

**Proposition 5.2** Suppose that the algebraic fiber bundle \( \pi : \mathcal{X}' \hookrightarrow B \) is two-relatively good, then for the given invertible \( \mathcal{E} \) over \( \mathcal{X}' \times_B \mathcal{T}_B(\mathcal{X}') \),

(i). There exists a pair of locally free sheaves \( \tilde{\mathcal{W}}, \tilde{\mathcal{V}} \) and a sheaf
morphism \( \Psi_{\tilde{\mathcal{W}}\tilde{\mathcal{V}}} : \tilde{\mathcal{W}} \hookrightarrow \tilde{\mathcal{V}} \) such that
\( \text{Coker}(\Psi_{\tilde{\mathcal{W}}\tilde{\mathcal{V}}}) \cong \mathcal{R}^2\pi'_*(\mathcal{E}) \).

(ii). The locally free sheaf \( \mathcal{W} \) associated to \( \mathcal{W} \) is taken to be
\( \text{Ker}(\Psi_{\tilde{\mathcal{W}}\tilde{\mathcal{V}}}) \oplus \text{Ker}(i_{\mathcal{F}}) \) and there exists the algebraic family Kuranishi morphism
\( \Phi_{\mathcal{V}\mathcal{W}} : \mathcal{V} \hookrightarrow \mathcal{W} \) such that
\( \text{Ker}(\Phi_{\mathcal{V}\mathcal{W}}) \cong \mathcal{R}^0\pi'_*(\mathcal{E}) \) and \( \text{Coker}(\Phi_{\mathcal{V}\mathcal{W}}) \) contains
\( \mathcal{R}^0\pi'_*(\mathcal{E}) \) as a sub-sheaf.

(iii). There is the following formula on the virtual rank,

\[
\text{rank}_C \mathcal{V} - \text{rank}_C \mathcal{W} = \frac{c_1^2(\mathcal{E}) - c_1(\mathcal{E}) \cdot c_1(K_{\mathcal{X}'}/B)}{2} - q + \text{febd}(c_1(\mathcal{E}), \mathcal{X}'/B) + 1.
\]

Proof of the proposition:
Step One: By using the condition of \( \pi' : \mathcal{X}' \hookrightarrow B \) being “2-relatively
good”, the proof of the statement (i) is parallel to the \( B = \text{pt} \) case.

As before, we use bold character like \( \mathcal{W} \) to denote the algebraic
vector bundle and use the calligraphic character like \( \mathcal{W} \) to denote
the corresponding sheaf of sections.

We set \( D_1 = D \) and \( lD_2 = \Delta \) and take \( n, l \) to be sufficiently large
integers. We take \( \mathcal{V}, \mathcal{V}, \mathcal{W} \) to be \( \mathcal{R}^0\pi'_*(\mathcal{O}_{\mathcal{X}'}(nD)\otimes \mathcal{E}), \mathcal{R}^0\pi'_*(\mathcal{O}_{nD+\Delta}(nD+\Delta)\otimes \mathcal{E}) \) and \( \mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD+\Delta)\otimes \mathcal{E}) \), respectively.

Then we have for sufficiently large \( n \) and \( l \),

\[
0 \hookrightarrow \mathcal{R}^0\pi'_*(\mathcal{E}) \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD)\otimes \mathcal{E}) \hookrightarrow \mathcal{R}^1\pi'_*(\mathcal{E}) \hookrightarrow 0,
\]

\[
0 \hookrightarrow \mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD)\otimes \mathcal{E}) \hookrightarrow \tilde{\mathcal{V}} \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{R}^1\pi'_*(\mathcal{O}_{nD}(nD)\otimes \mathcal{E}) \hookrightarrow 0,
\]

and

\[
\mathcal{R}^1\pi'_*(\mathcal{O}_{nD}(nD)\otimes \mathcal{E}) \cong \mathcal{R}^2\pi'_*(\mathcal{E}),
\]

similar to the \( B = \text{pt} \) case.
Step Two: For statement (ii), we focus on the locally free-ness of $W = \text{Ker}(\Psi_\tilde{W}) \oplus \text{Ker}(i_F)$. According to [Ha] page 174, chapter II lemma 8.9, it suffices to check that for all $y \in T_B(X')$, $\dim_{k(y)} W \otimes k(y)$ is a constant independent of $y$.

We demonstrate this by showing (using the exact sequences listed in step one)

$$\dim_{k(y)} W \otimes k(y) = \dim_{k(y)} \text{Ker}(\Psi_\tilde{W}) \otimes k(y) + \dim_{k(y)} \text{Ker}(i_F) \otimes k(y)$$

$$= (\dim_{k(y)} \tilde{W} \otimes k(y) - \dim_{k(y)} \tilde{V} \otimes k(y) + \dim_{k(y)} \mathcal{R}^2 \pi_*(\mathcal{E}) \otimes k(y)) +$$

$$(\dim_{k(y)} \mathcal{F} \otimes k(y) - \dim_{k(y)} \mathcal{R}^2 \pi_*(\mathcal{E}) \otimes k(y)) = \text{rank}_C \tilde{W} - \text{rank}_C \tilde{V} + \text{rank}_C \mathcal{F}.$$

Thus, the sheaf $W$ is locally free. The remaining conclusions about $\text{Ker}$ or $\text{Coker}$ are identical to the original argument in prop. 4.5 and we omit it here.

Step Three: For statement (iii) in the proposition, we notice that the construction of $\tilde{V}, \tilde{W}$ and $V$ are the family extension of the construction in prop 4.3 and their ranks are independent to this extension, we still have as before

$$\text{rank}_C \tilde{V} + \text{rank}_C \tilde{W} - \text{rank}_C \tilde{W} = \frac{c_1(\mathcal{E})^2 - c_1(\mathcal{E}) \cdot c_1(K_{X'/B})}{2} - q + p_g + 1$$

by surface Riemann-Roch formula.

Then the formula in (iii). can be derived by using the calculation in the step two and the formula $\text{rank}_C \mathcal{F} = p_g - \text{febd}(c_1(\mathcal{E}), X'/B)$. □

**Example 5.1** The rank calculation in the step two of the proof can be strengthened to imply that

$$[W] = [\tilde{W}] - [\tilde{V}] + [\mathcal{F}]$$

in the $K$ group of coherent sheaves on $T_B(X')$, $K_0(T_B(X'))$. This identity will be used implicitly in the derivation of the algebraic family blowup formula.
By using the data \((V, W, \Phi_{VW})\) of an algebraic family Kuranishi model, we define the algebraic family Seiberg-Witten invariants similar to the \(B = pt\) case. The corresponding mixed invariants for \(C + mE = c_1(\mathcal{E})\) will be defined to be

\[
\mathcal{A}FSW_{\mathcal{X}' \to B}(c, C + mE) = c_1 \left( \frac{c_1^2(\mathcal{E}) - c_1(\mathcal{E}) - c_1(\mathcal{K}_{\mathcal{X}'/B})}{2} + f_{ebd}(c_1(\mathcal{E}), \mathcal{X}'/B) \right)(H) \cap c_{top}(H \otimes W) \in A_0(\mathcal{P}_{TB}(\mathcal{X}'')(V)),
\]

for \(c \in A_*(B)\).

**Theorem 5.1** Let \(\mathcal{X}' \to B\) and \(\mathcal{X} \to B\) and \(C\) be as described above with the appropriated relatively good conditions, and let \(m\) be an integer such that family dimension \(\int_{\mathcal{X}/B} (C^2 - c_1(\mathcal{K}_{\mathcal{X}/B}))C + f_{ebd}(C, \mathcal{X}/B) - \frac{m^2 - m}{2} + \dim_C B \geq 0\). Then for any \(c \in H^*(B, \mathbb{Z})\) (or any class \(c\) of algebraic cycles in \(B\)) the mixed algebraic family Seiberg-Witten invariant of the class \(C + mE\) over the blown up fibration \(\mathcal{X}' \to B\) is related to the algebraic family Seiberg-Witten invariant of \(C\) by the following formula,

\[
\mathcal{A}FSW_{\mathcal{X}' \to B}(c, C + mE) = \sum_{i \geq 0} \mathcal{ASW}_{\mathcal{X} \to B}(c \cap c_i(E \otimes (S^{m-2}(C_B \oplus N_{s(B)}\mathcal{X}))), C)
\]

for \(m \geq 1\) and

\[
\mathcal{A}FSW_{\mathcal{X}' \to B}(c, C + mE) = \sum_{i \geq 0} \mathcal{ASW}_{\mathcal{X} \to B}(c \cap c_i(E \otimes (S^{-m-1}(C_B \oplus N^*_{s(B)}\mathcal{X}))), C)
\]

for \(m \leq 0\).

**Remark 5.1** One may assume additionally that \(\mathcal{X}' \to B\) carries cross sections. Then the image of each cross section to \(\mathcal{X}'\) under the blowing down map \(\mathcal{X}' \to \mathcal{X}\) induces a cross section on \(\mathcal{X} \to B\). By using the cross sections, one may interpret formally the algebraic family Seiberg-Witten invariant as a virtual count of holomorphic curves through generic cross sections of \(\mathcal{X}' \to B\), \(\mathcal{X} \to B\) and the family blowup formula relates these invariants.

Proof of the theorem: Suppose that abstractly we are given an algebraic Kuranishi model of the fiberwise class \(C\) over \(\mathcal{X} \to B\). Namely,
there is a bundle map $\Phi_{VW} : V \to W$ over $T_B(\mathcal{X})$ and the algebraic family invariant is defined either as an intersection number involving $c_{top}(H \otimes W)$ over the whole projective space bundle $P_{T_B(\mathcal{X})}$ or over an open subset of it. As usual, we use $V$ and $W$ to denote the corresponding locally free sheaves associated to $V, W$.

Let $\mathcal{E}$ and $\mathcal{E}_m$ denote the invertible sheaves associated with $C$ and $C + mE$ on $\mathcal{X}'$, respectively. By proposition 5.1 we can identify their base spaces to be $T_B(\mathcal{X}') = T_B(\mathcal{X})$.

Case One: We assume that $m > 0$, then there is a short exact sequence relating $E_m$ and $E$,

$$
0 \to \mathcal{E} \to \mathcal{E}_m \to O_{mE_B}(mE_B) \otimes \mathcal{E} \to 0,
$$

where $E_B = P_B(N_s(B)\mathcal{X})$ denotes the exceptional divisor of the blowing down map $\mathcal{X}' \to \mathcal{X}$.

Firstly, we establish the following lemma,

Lemma 5.1 Let $E_B = P_B(N_s(B)\mathcal{X})$ denote the exceptional divisor of $\mathcal{X}' \to \mathcal{X}$. Then for all $m > 0$, the sheaf $\mathcal{R}^0\pi'_*(O_{mE_B}(mE_B) \otimes \mathcal{E})$ over $T_B(\mathcal{X})$ is the zero sheaf and the sheaf $\mathcal{R}^0\pi'_*(O_{mE_B}(mE_B) \otimes \mathcal{E})$ is locally free.

Proof: By induction we assume that for $k = m - 1$ the sheaf $\mathcal{R}^0\pi'_*(O_{kE_B}(kE_B))$ over has been proved to be the zero sheaf. We prove that for $k + 1 = m$, $\mathcal{R}^0\pi'_*(O_{(k+1)B}((k+1)E_B))$ is also trivial.

Recall that for two effective divisors $A$ and $B$ on an algebraic manifold, we have the following short exact sheaf sequence (see for example [Fr], pp 15, equation (1.9)),

$$
0 \to O_{A}(A) \to O_{A+B}(A+B) \to O_B(A+B) \to 0.
$$

In our case we take $A = kE_B$ and $B = E_B$, then

$$
0 \to O_{kE_B}(kE_B) \to O_{(k+1)E_B}((k+1)E_B) \to O_{E_B}((k+1)E_B) \to 0.
$$

Then we have the following (portion of) derived long exact sequence by pushing forward along $\pi' : \mathcal{X}' \to B$,

$$
0 \to \mathcal{R}^0\pi'_*(O_{kE_B}(kE_B)) \to \mathcal{R}^0\pi'_*(O_{(k+1)E_B}((k+1)E_B)) \to \mathcal{R}^0\pi'_*(O_{E_B}((k+1)E_B)).
$$
It suffices to argue that $\mathcal{R}^0\pi'_*(\mathcal{O}_{EB}((k+1)E_B))$ vanishes. By base change to all closed points $b \in B$, it suffices to check that $h^0(\mathcal{P}^1, \mathcal{O}_{\mathcal{P}^1}(-k-1)) = 0$ for $k \geq 0$. It follows from the negativity of the degree $\text{deg}\mathcal{O}_{\mathcal{P}^1}(-k-1)$ on $\mathcal{P}^1$. Then by curve-Riemann-Roch and Grauert criterion (see [Ha] page 288 cor. 12.9) the sheaf $\mathcal{R}^1\pi'_*(\mathcal{O}_{(k+1)E_B}((k+1)E_B))$ is locally free.

By tensoring with $\mathcal{E}$ from the base $\mathcal{T}_B(\mathcal{X})$ and by moving $\mathcal{E}$ into the right derived image functor we find that $\mathcal{R}^1\pi'_*(\mathcal{O}_{(k+1)E_B}((k+1)E_B) \otimes \mathcal{E})$ is locally free and $\mathcal{R}^0\pi'_*(\mathcal{O}_{(k+1)E_B}((k+1)E_B) \otimes \mathcal{E})$ is the zero sheaf.

Moreover, we have the following short exact sequence of right derived sheaves, which will be used in identifying the relative obstruction sheaves appearing in the statement of the theorem,

$$0 \to \mathcal{R}^1\pi'_*(\mathcal{O}_{(m-1)E_B}((m-1)E_B)) \to \mathcal{R}^1\pi'_*(\mathcal{O}_{mE_B}(mE_B)) \to \mathcal{R}^1\pi'_*(\mathcal{O}_{EB}(mE_B)) \to 0.$$

By using the fact $\mathcal{R}^0\pi'_*(\mathcal{O}_{\mathcal{P}_B(N_s(B))(x)}(-E_B)) = N^*_s(B), \mathcal{X}$, the conormal sheaf of $s(B) \subset \mathcal{X}$ and the relative Serre duality along $\mathcal{P}_B(N_s(B)\mathcal{X}) \to B$, one can identify $\mathcal{R}^1\pi'_*(\mathcal{O}_{EB}(mE_B))$ with the $m - 2$ symmetric power $\mathcal{S}^{m-2}(N^*_s(B)\mathcal{X})$.

Thus, there is an identity among total Chern classes of these derived sheaves,

$$c_{\text{total}}(\mathcal{R}^1\pi'_*(\mathcal{O}_{EB}(mE_B))) = c_{\text{total}}(\mathcal{R}^1\pi'_*(\mathcal{O}_{(m-1)E_B}((m-1)E_B)) \otimes \mathcal{E}) \cdot c_{\text{total}}(\mathcal{S}^{m-2}(N^*_s(B)\mathcal{X}) \otimes \mathcal{E}).$$

This identity will be used in the proof of the theorem 5.1. This finishes the proof of lemma 5.1. □

Firstly, we deal with the case when $\mathcal{R}^2\pi'_*(\mathcal{E}), \mathcal{R}^2\pi'_*(\mathcal{E}_m)$ vanish. It happens when the fiber surfaces have a trivial geometric genus or when $c_1(K_{(X/B - C)}$ is of negative relative degree.

By using lemma 5.1 we have

$$\mathcal{R}^0\pi'_*(\mathcal{E}) \cong \mathcal{R}^0\pi'_*(\mathcal{E}_m)$$

and

$$0 \to \mathcal{R}^1\pi'_*(\mathcal{E}) \to \mathcal{R}^1\pi'_*(\mathcal{E}_m) \to \mathcal{R}^1\pi'_*(\mathcal{O}_{mEB}(mE_B) \otimes \mathcal{E}) \to 0.$$

This suggests that one can build up algebraic Kuranishi models of $\mathcal{C}, \mathcal{C} + m\mathcal{E}$ by $\Phi_{\mathcal{V}' \mathcal{W}} : \mathcal{V} \to \mathcal{W}, \Phi_{\mathcal{V}' \mathcal{W}} : \mathcal{V}' \to \mathcal{W}', \mathcal{V}' = \mathcal{V},$ with an short exact sequence relating the obstruction sheaves,
0 \mapsto W \mapsto W' \mapsto R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}) \mapsto 0.

To achieve this, we choose a large \( n \) and take \( R^0\pi'_*(\mathcal{O}(nD) \otimes \mathcal{E}_m) = \mathcal{V}' \) and \( R^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) = \mathcal{W}' \), similar to subsection 4.2.

Let \( R^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) \mapsto R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}) \) be the composition of two surjective maps

\[ R^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) \mapsto R^1\pi'_*(\mathcal{E}_m) \]

and

\[ R^1\pi'_*(\mathcal{E}_m) \mapsto R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}). \]

**Proposition 5.3** Define \( W \) to be the kernel of

\[ R^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) \mapsto R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}) \mapsto 0. \]

Then \( W \) is locally free and there is a four term exact sheaf sequence

\[ 0 \mapsto R^0\pi'_*(\mathcal{E}) \mapsto R^0\pi'_*(\mathcal{O}(nD) \otimes \mathcal{E}_m) \mapsto W \mapsto R^1\pi'_*(\mathcal{E}) \mapsto 0. \]

Because of this exact sequence, one may take \( \mathcal{V} = \mathcal{V}', \mathcal{W} \to define \) a Kuranishi model of \( C \).

**Proof:**

The sheaf \( W \) is locally free if and only if \( \phi(x) = \dim_{k(x)} W_x \otimes k(x) \) is a constant, which follows from the locally freeness of both \( R^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) \) and \( R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}) \) (see lemma 5.1). Because \( R^0\pi'_*(\mathcal{E}) \cong R^0\pi'_*(\mathcal{E}_m) \), the injectivity of \( R^0\pi'_*(\mathcal{E}) \mapsto R^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) \) follows.

On the other hand, the exactness of

\[ 0 \mapsto R^0\pi'_*(\mathcal{E}_m) \mapsto \mathcal{V}' \mapsto \mathcal{W}' \mapsto R^1\pi'_*(\mathcal{E}_m) \mapsto 0 \]

implies the composition \( \mathcal{V}' \mapsto R^1\pi'_*(\mathcal{E}_m) \) is trivial. Thus \( \mathcal{V}' \mapsto R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}) \) is also trivial and then \( \mathcal{V}' \mapsto \mathcal{W}' \) must factor through \( \mathcal{W} \mapsto \mathcal{W}' \), the kernel of \( \mathcal{W}' \mapsto R^1\pi'_*(\mathcal{O}_{mE_B}(mE_B) \otimes \mathcal{E}) \).

This implies that there is a four term exact sequence

\[ 0 \mapsto R^0\pi'_*(\mathcal{E}) \mapsto \mathcal{V}' \xrightarrow{\Phi_{\mathcal{V}'\mathcal{W}}} \mathcal{W} \mapsto \text{Cokernel}(\Phi_{\mathcal{V}'\mathcal{W}}) \mapsto 0. \]
It suffices to show that $Coker(\Phi_{V'W'}) \cong \mathcal{R}^1\pi'_*(\mathcal{E})$ is a natural isomorphism. This follows from the following commutative diagram,

$$
\begin{align*}
Coker(\Phi_{V'W'}) & \mapsto \Phi_{V'W'} & \mapsto \mathcal{R}^1\pi'_*(O_{mE_B}(mE_B) \otimes \mathcal{E}) \\
\mathcal{R}^1\pi'_*(\mathcal{E}) & \mapsto \mathcal{R}^1\pi'_*(\mathcal{E}_m) & \mapsto \mathcal{R}^1\pi'_*(O_{mE_B}(mE_B) \otimes \mathcal{E})
\end{align*}
$$

Both rows are short exact sequences and the vertical arrows are natural isomorphisms. Then $\Phi_{V'W'} \cong \mathcal{R}^1\pi'_*(\mathcal{E}_m)$ induces the isomorphism $\Phi_{V'W'} \cong \mathcal{R}^1\pi'_*(\mathcal{E})$.

When one pulls back $W', W$ to $P_{T_{B}(\mathcal{X})}(\mathcal{V})$, one has the following identity on the top Chern class

$$
c_{top}(\mathcal{H} \otimes W') = c_{top}(\mathcal{H} \otimes W) \cdot c_{top}(\mathcal{H} \otimes (\mathcal{R}^1\pi'_*(O_{mE_B}(mE_B) \otimes \mathcal{E}))) = \\
c_{top}(\mathcal{H} \otimes \mathcal{W}) \cdot c_{top}(\mathcal{H} \otimes (\oplus_{0 \leq j \leq m-2} S^j(N_{s(B)}\mathcal{X})) \otimes \mathcal{E}).
$$

We have inductively used the Chern class identity at the end of the proof of lemma 5.1.

After finishing the vanishing $\mathcal{R}^2\pi'_*(\mathcal{E})$ case, we have to deal with $\mathcal{R}^2\pi'_*(\mathcal{E})$ non-vanishing case.

Because of the relative Serre duality

$$
\mathcal{E}, \mathcal{E}_m = \mathcal{E} \otimes O(mE_B) \Longrightarrow \mathcal{K}_{\mathcal{X}'/B} \otimes \mathcal{E}^*, \mathcal{K}_{\mathcal{X}'/B} \otimes (\mathcal{E}_m)^* = \mathcal{K}_{\mathcal{X}'/B} \otimes \mathcal{E}^* \otimes O(-mE_B),
$$

the case can be reduced to a parallel discussion with $m < 0$. We will address it at the end of the subsection on page 72.

Case Two: We assume that $m \leq 0$. The $m = 0$ case is trivial and we consider negative $m = -k, k \in \mathbb{N}$. We consider the following short exact sheaf sequence,

$$
0 \hookrightarrow \mathcal{E}_m \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{O}_{kE_B} \otimes \mathcal{E} \twoheadrightarrow 0.
$$

Its derived long exact sequence relates the derived images of $\mathcal{E}_m$ and $\mathcal{E}$,

$$
0 \hookrightarrow \mathcal{R}^0\pi'_*(\mathcal{E}_m) \hookrightarrow \mathcal{R}^0\pi'_*(\mathcal{E}) \twoheadrightarrow \mathcal{R}^0\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E}) \twoheadrightarrow \\
\mathcal{R}^1\pi'_*(\mathcal{E}_m) \hookrightarrow \mathcal{R}^1\pi'_*(\mathcal{E}) \twoheadrightarrow \mathcal{R}^1\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E}).
$$
Similar to lemma 5.1, we have the following lemma regarding
\[ R^i \pi'_*(O_{kE_B} \otimes \mathcal{E}) = R^i \pi'_*(O_{kE_B}) \otimes \mathcal{E}, \ i = 0, 1. \]

**Lemma 5.2** For \( k \in \mathbb{N} \), the first derived image sheaf \( R^1 \pi'_*(O_{kE_B}) \) is the zero sheaf and \( R^0 \pi'_*(O_{kE_B}) \) is locally free. Moreover, \( c_{\text{total}}(R^0 \pi'_*(O_{kE_B})) \) is equal to \( c_{\text{total}}(S^{k-1}(O_{B} \oplus N_s(B)^* \mathcal{X})) \).

Proof: The argument of this lemma is very similar to lemma 5.1. We consider the following short exact sequence

\[ 0 \rightarrow O_{E_B}(- (k-1)E_B) \rightarrow O_{kE_B} \rightarrow O_{(k-1)E_B} \rightarrow 0, \]

for two divisors \( A = (k-1)E_B \) and \( B = E_B \) and their sum \( A + B = kE_B \). Then we have to take the derived sequence and make an induction on \( k \).

The key to identify the total Chern classes is to show that for \( E_B = P_B(N_{s(B)}^* \mathcal{X}) \), \( R^0 \pi'_*(O_{E_B}(-(r-1)E_B)) = S^{r-1}(N_{s(B)}^* \mathcal{X}) \), \( r \in \mathbb{N} \), which follows from the projection formula (see [Ha] page 253 exercise 8.4.) of the \( \mathbb{P}^1 \) bundle. We leave the remaining detail as an exercise to the reader, based on the argument of lemma 5.1.

If one works in the \( C^\infty \) category, the derived long exact sequence of \( \mathcal{E}_m, \mathcal{E} \) and lemma 5.2 is enough for us to derive the desired family blowup formula stated in the theorem 5.1. To give an algebraic proof, we have to go through a detour.

To simplify the discussion, firstly we assume (i) \( p_g \) of the fiber surface to be 0 or (ii) \( c_1(K_{\mathcal{X}/B}) - C \) to be of non-constant degree with respect to an ample relative polarization. Under these assumptions, the second derived sheave \( R^2 \pi'_*(\mathcal{E}) \) vanishes.

By assumption, the fibration \( \mathcal{X}' \rightarrow B \) is relatively good. Consider an effective ample divisor \( D \) on the total space \( \mathcal{X}' \), relative dimension one over \( B \), and a large multiple \( nD \) for \( n \gg 0 \). The same argument as in subsection 4.2 allows us to construct explicit algebraic Kuranishi models for \( C - kE \) and \( C \) over \( T_B(\mathcal{X}') \). The key idea is to relate the algebraic Kuranishi models of \( C - kE \) and \( C \).

Consider the short exact sequence

\[ 0 \rightarrow \mathcal{E}_m \otimes O(nD) \rightarrow \mathcal{E} \otimes O(nD) \rightarrow O_{kE_B}(nD) \otimes \mathcal{E} \rightarrow 0. \]

By choosing \( n \) large enough one can make the first derived image sheaves \( R^1 \pi'_*(O(nD) \otimes \mathcal{E}_m), R^1 \pi'_*(O(nD) \otimes \mathcal{E}) \) vanish.

Thus, by lemma 5.2 one gets a short exact sequence of locally free sheaves,
On the other hand, one may get a short exact sequence on the obstruction sheaves of the algebraic Kuranishi models,

\[ 0 \mapsto R^0_\pi'_*(E \otimes \mathcal{O}(nD)) \mapsto R^0_\pi'_*(O_{kE_B}(nD)) \otimes \mathcal{E} \mapsto 0. \]

The following short exact sequence on the locally free sheaves is vital to the remaining discussion.

\[ 0 \mapsto R^0_\pi'_*(O_{nD}(nD) \otimes \mathcal{E}_m) \mapsto R^0_\pi'_*(O_{nD}(nD) \otimes \mathcal{E}) \mapsto R^0_\pi'_*(t_{kE_B \cap nD}(nD)) \otimes \mathcal{E} \mapsto 0, \]

due to the vanishing of higher derived image sheaves \( R^i_\pi'_*(O_{nD}(nD) \otimes \mathcal{E}_m), R^i_\pi'_*(O_{nD}(nD) \otimes \mathcal{E}) \), \( i \geq 1 \).

The following short exact sequence on the locally free sheaves is

\[ 0 \mapsto R^0_\pi'_*(O_{kE_B}) \otimes \mathcal{E} \mapsto R^0_\pi'_*(O_{kE_B}(nD)) \otimes \mathcal{E} \mapsto R^0_\pi'_*(t_{kE_B \cap nD}(nD)) \otimes \mathcal{E} \mapsto 0. \]

Consider the following lemma,

**Lemma 5.3** Let \( V \) be an algebraic vector bundle over \( T_B(X) \) and \( 0 \mapsto V \mapsto V' \mapsto U \mapsto 0 \) be a bundle extension of \( V \) by \( U \). Denote \( H^* \) to be the tautological line bundle over \( P_{T_B(X)}(V') \). Then the projective space bundle \( P_{T_B(X)}(V) \) can be identified with the subspace of \( P_{T_B(X)}(V') \) defined by the zero locus of a canonical section of \( H \otimes U \).

**Proof:** Consider the bundle surjection \( V' \mapsto U \), it induces a map from the tautological line bundle \( H^* \) to \( U \) and therefore a canonical section of \( H \otimes U \) over \( P_{T_B(X)}(V') \). On the other hand, the fibers of \( H^* \) can be identified with the rays of \( V' \), which map trivially to \( U \) if and only if the rays are from the sub-bundle \( V \). A direct investigation shows that the \( P_{T_B(X)}(V) \) is the transversal zero locus of the canonical section of \( H \otimes U \). \( \square \)

This lemma implies that one may thicken the projective space bundle by adding \( H \otimes U \) to the obstruction bundle.

Now one compares the algebraic Kuranishi models of \( C - kE \) and \( C \). Let \( V, V' \) and \( U \) be the algebraic vector bundles associated with the locally free sheaves \( R^0_\pi'_*(E \otimes \mathcal{O}(nD)), R^0_\pi'_*(E \otimes \mathcal{O}(nD)) \) and \( R^0_\pi'_*(O_{kE_B}(nD)) \otimes \mathcal{E} \), respectively.

Then there is a bundle short exact sequence \( 0 \mapsto V \mapsto V' \mapsto U \mapsto 0 \). The original algebraic Kuranishi model of \( C - kE \) and
are realized as intersection numbers in (open sets of) \( P_{\mathcal{T}(\mathcal{X})}(V) \) and \( P_{\mathcal{T}(\mathcal{X})}(V') \). By lemma 5.3, one may embed \( P_{\mathcal{T}(\mathcal{X})}(V) \) in \( P_{\mathcal{T}(\mathcal{X})}(V') \) as the transversal zero locus of \( \mathbf{H} \otimes \mathbf{U} \).

Thus, to compare the algebraic family invariants of \( C - kE \) and \( C \), it suffices to compare the obstruction bundles \( W' \) and \( W \) and their canonical sections. Let \( W, W' \) and \( \tilde{U} \) be the algebraic vector bundle over \( \mathcal{T}_B(\mathcal{X}) \) associated with \( \mathcal{R}^{0}\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m), \mathcal{R}^{0}\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}) \) and \( \mathcal{R}^{0}\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}) \). Then there is also a bundle short exact sequence

\[
0 \to W \to W' \to U \to 0
\]

and the following diagram is commutative

\[
\begin{array}{ccc}
\Phi_{VW} : & V & \to & W \\
\downarrow & & \downarrow & \\
\Phi_{V'W'} : & V' & \to & W'
\end{array}
\]

While the family moduli space of curves in \( C - kE \) and in \( C \) are both viewed as subspaces in \( P_{\mathcal{T}(\mathcal{X})}(V') \), the obstruction bundles of the class \( C - kE \) and \( C \) are \( \mathbf{H} \otimes (W \oplus U_V), \mathbf{H} \otimes W', \) respectively. One see easily that

\[
[W] + [U] = [W'] + [U_V] - [U_W]
\]

in the \( K \) group of \( \mathcal{T}_B(\mathcal{X}) \). By using

\[
0 \to \mathbf{E} \otimes (S^{k-1}(C_B \oplus N^*_s(B)) \mathcal{X})) \to U \to \tilde{U} \to 0,
\]

one may conclude

\[
[W] + [U_V] = [W'] + [E \otimes (S^{k-1}(C_B \oplus N^*_s(B)) \mathcal{X}))]
\]

\[
\text{rank}_C W + \text{rank}_C U = \text{rank}_C W' + \text{rank}_C S^{k-1}(C_B \oplus N^*_s(B)) \mathcal{X}.
\]

and the following identity on top Chern classes

\[
c_{top}(\mathbf{H} \otimes (W \oplus U_V)) = c_{top}(\mathbf{H} \otimes W') \cdot c_{top}(\mathbf{H} \otimes \mathbf{E} \otimes (S^{k-1}(C_B \oplus N^*_s(B)) \mathcal{X})).
\]

The identity on \( \mathcal{AFSW} \) of \( C \) and \( C - kE \) follows from the equality of top Chern classes.
Finally let us derive the family blowup formula for $p_g > 0$ fiber surfaces with a fiberwise invariant class $C$ with positive relative degree on $c_1(K_{X/B}) - C$. The derivation is much more evolved than the previous case as several derived image sheaves fail to be locally free in this case.

By the assumption of the theorem, the fibration $X' \hookrightarrow B$ is assumed to be two-relatively good. Namely, there are two ample divisors $D_1, D_2$ which are (1). of relative dimension one over $B$. (2). The intersection $D_1 \cap D_2$ is of relative dimension zero over $B$.

Following subsection 4.3, we adopt the asymmetric notation $D = D_1$ and $\Delta = lD_2$.

**Remark 5.2** Basically we have to extend the construction in subsection 4.3 to a relative version. The essential difference is that in the $B = pt$ case, we may choose a single $\Sigma \in |nD|$ to be smooth. Then the vanishing of certain derived image sheaves on $\Sigma \times T(M) \hookrightarrow T(M)$ follows from Kodaira vanishing theorem on $\Sigma$. When $X' \hookrightarrow B$ is a non-trivial fiber bundle, it may be hard to find an effective divisor in $|nD|$ which is relatively smooth of dimension one over $B$ unless one makes some additional assumption on $X' \hookrightarrow B$. But the vanishing result on the first derived image sheaves of sufficiently very ample invertible sheaves on a non-smooth (or even non-reduced) relative dimension one divisor $nD$ can still be derived by using the exact sequence $0 \hookrightarrow \mathcal{O} \hookrightarrow \mathcal{O}(nD) \hookrightarrow \mathcal{O}_{nD}(nD) \hookrightarrow 0$ suitably.

We have the following commutative diagram of coherent sheaves for a sufficiently large $n$, and $l$.

\[
\begin{array}{ccc}
\mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) & \rightarrow & \mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}) \\
\downarrow & & \downarrow
\\
\mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD + \Delta) \otimes \mathcal{E}_m) & \rightarrow & \mathcal{R}^0\pi'_*(\mathcal{O}_{nD}(nD + \Delta) \otimes \mathcal{E}) \\
\downarrow & & \downarrow
\\
\mathcal{R}^0\pi'_*(\mathcal{O}_{nD\cap\Delta}(nD + \Delta) \otimes \mathcal{E}_m) & \rightarrow & \mathcal{R}^0\pi'_*(\mathcal{O}_{nD\cap\Delta}(nD + \Delta) \otimes \mathcal{E}) \\
\downarrow & & \downarrow
\\
\mathcal{R}^1\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}_m) & \rightarrow & \mathcal{R}^1\pi'_*(\mathcal{O}_{nD}(nD) \otimes \mathcal{E}) \\
\downarrow & & \downarrow
\\
\mathcal{R}^1\pi'_*(\mathcal{O}_{nD\cap\Delta}(nD) \otimes \mathcal{E}) & \rightarrow & \mathcal{R}^1\pi'_*(\mathcal{O}_{nD\cap\Delta}(nD) \otimes \mathcal{E})
\end{array}
\]

In the commutative diagram there are twelve derived image sheaves and form four rows and three columns. The first row and the fourth row are connected by the connecting homomorphism, which is not included in the diagram. Thus, the first row is left exact but generally not right exact. Because $D \hookrightarrow B$ is of relative dimension one,
by base change one sees that the fourth row is right exact. On the other hand, the three columns are parts of derived long exact sequences from sheaves short exact sequences. If we choose $l$ and $n$ to be large enough, the three columns are four terms exact sequences. This also implies that the second row is a short exact sequence. Because $nD \cap \Delta$ is relative dimension one (non-reduced) over the base, the third row is also short exact.

Similar to the arguments of lemma 5.1 and lemma 5.2, the derived image sheaves in the second and the third rows of the diagram are locally free sheaves. The $(2, 1)$th, $(3, 1)$th, $(2, 2)$th, and $(3, 2)$th entries are used in constructing the algebraic Kuranishi models of $C - kE$ and $C$.

Proof of the Case with non-zero $R^2\pi'_*(E)$:

Denote the vector bundle associated with the $(2, 1)$th, $(2, 2)$th, $(3, 1)$, $(3, 2)$th and $(2, 3)$th entries of the sheaves commutative diagram on page 70 by $\tilde{W}, \tilde{W}', \tilde{V}, \tilde{V}'$ and $R$, respectively. Then one may construct $W$ and $W'$ by using the recipe of proposition 4.5. Denote the algebraic vector bundles associated with the locally free $R^0\pi'_*(O(nD) \otimes E_m), R^0\pi'_*(O(nD) \otimes \mathcal{E})$ by $V, V'$. Then a parallel discussion similar to proposition 4.5 implies that the algebraic family Kuranishi models of $C - kE$ and $C$ can be built from the algebraic bundle maps

$$\Phi_{VW} : V \to W,$$

and

$$\Phi_{V'W'} : V' \to W'.$$

We concentrate on how does the relative obstruction bundle of the family blowup formula appear in the current picture.

Firstly, notice that there are short exact sequences

$$0 \to V \to V' \to U_V \to 0,$$

$$0 \to \tilde{V} \to \tilde{V}' \to \tilde{U}_V \to 0,$$

$$0 \to \tilde{W} \to \tilde{W}' \to \tilde{U}_W \to 0,$$

with $U_V, \tilde{U}_V$ being the algebraic vector bundle associated with the locally free sheaves $R^0\pi'_*(O_{kE_B}(nD) \otimes \mathcal{E})$ and $R^0\pi'_*(O_{nD \cap kE_B \cap \Delta}(nD + \Delta) \otimes \mathcal{E})$. 71
Recall that \( [\mathbf{W}] = [\mathbf{W}] - [\mathbf{V}] + [\mathbf{F}] \) and \( [\mathbf{W}'] = [\mathbf{W}'] - [\mathbf{V}'] + [\mathbf{F}'] \) in the \( K \) group of algebraic vector bundles on \( T_B(\mathcal{X}') \), by remark 3.1.

It is easy to see that for the formal excess base dimensions, we have \( \text{febd}(C, \mathcal{X}'/B) = \text{febd}(C - mE, \mathcal{X}'/B) \) by using the isomorphism \( R^2\pi'_*(\mathcal{E}_m) \cong R^2\pi'_*(\mathcal{E}) \) and then \( F \cong F' \). (read the paragraph before prop. 5.2 for the definition of the corresponding locally free sheaf \( \mathcal{F} \)).

Again by lemma 5.3 and the same argument in the previous case, it suffices to show the following identity on the virtual bundles

\[
[\mathbf{U}_V \oplus \mathbf{U}_V - \mathbf{U}_W] = [E \otimes C \left(S^{k-1}(C_B \oplus N^{*}_{s(B)}\mathcal{X})\right)]
\]

in the \( K \) group of algebraic vector bundles on \( T_B(\mathcal{X}) \) and the corresponding equality on their virtual ranks.

The follows from the following calculation in the \( K \) group of coherent sheaves \( K_0(T_B(\mathcal{X})) \) and the short exact sequences,

\[
-R^0\pi'_*(\mathcal{O}_{nD \cap kE_B}(nD + \Delta) \otimes \mathcal{E}) + R^0\pi'_*(\mathcal{O}_{kE_B}(nD) \otimes \mathcal{E}) + R^0\pi'_*(\mathcal{O}_{nD \cap kE_B \cap \Delta}(nD + \Delta) \otimes \mathcal{E})
\]

\[
= -(R^0\pi'_*(\mathcal{O}_{nD \cap kE_B}(nD + \Delta) \otimes \mathcal{E}) - R^0\pi'_*(\mathcal{O}_{nD \cap kE_B \cap \Delta}(nD + \Delta) \otimes \mathcal{E})) + R^0\pi'_*(\mathcal{O}_{kE_B}(nD) \otimes \mathcal{E})
\]

\[
= R^0\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E}) - R^1\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E})
\]

\[
= R^0\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E}) - R^0\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E}) = R^0\pi'_*(\mathcal{O}_{kE_B} \otimes \mathcal{E}).
\]

On page 67, lemma 5.2, we have already identified the total Chern classes of the two coherent sheaves \( R^0\pi'_*(\mathcal{O}_{kE_B}) \) and the symmetric power \( S^{k-1}(\mathcal{O}_B \oplus N^{*}_{s(B)}\mathcal{X}) \), so the proof of \( m < 0, R^2\pi'_*(\mathcal{E}) \neq 0 \) case is done.

At the end of the proof, let us address the \( m > 0, R^2\pi'_*(\mathcal{E}) \neq 0 \) case leftover on page 68. As in the above discussion, \( \mathcal{X}' \mapsto B \) is assumed to be two relatively good. We have chosen \( nD \) and \( \Delta = lD_{2|nD} \) to construct the algebraic family Kuranishi models.

By interchanging the roles of \( \mathcal{E} \) and \( \mathcal{E}_m \) symbolically and by replacing \( k \) by \( m \), there is a corresponding twelve-term commutative exact diagram similar to the one on page 70. Following the convention on page 71, we take \( V, \mathbf{W}, \mathbf{V} \) and \( \mathbf{V}', \mathbf{W}', \mathbf{V}' \) to be the algebraic bundles associated to the algebraic family Kuranishi models of \( C \) and \( C + mE \), respectively.

Then as before there are exact sequences
\[0 \mapsto \mathbf{V} \mapsto \mathbf{V}' \mapsto \mathbf{U}_V \mapsto 0,\]
\[0 \mapsto \tilde{\mathbf{V}} \mapsto \tilde{\mathbf{V}}' \mapsto \tilde{\mathbf{U}}_V \mapsto 0,\]

and
\[0 \mapsto \mathbf{W} \mapsto \mathbf{W}' \mapsto \mathbf{U}_W \mapsto 0.\]

In this case, \( \mathbf{F} \cong \mathbf{F}' \) may not hold, but we still have \([\mathbf{F}] = [\mathbf{F}']\) in the reduced \(K\) group as both of their associated locally free sheaves \(\mathcal{F}\) and \(\mathcal{F}'\) are constructed from \(\mathcal{R}^2 \pi'_*(\mathcal{O}_\mathcal{X})\) by quotienting out some trivial sub-factors \(\mathcal{O}_{\mathcal{E}}(\mathcal{C},\mathcal{X}')/B\) and \(\mathcal{O}_{\mathcal{E}}(\mathcal{C}+mE,\mathcal{X}')/B\), respectively.

Our goal is to prove that,

(i). 
\[-[\tilde{\mathbf{U}}_W - \tilde{\mathbf{U}}_V - \mathbf{U}_V] = -[E \otimes S^{m-2}(C_B \oplus N_{s(B)}\mathcal{X})] \text{ in the } K \text{ group.}\]

(ii). The virtual ranks of the virtual vector bundles in (i). match.

We can go through the same calculation above on page 72, except that the invertible sheaf \(E \otimes S^{m-2}(C_B \oplus N_{s(B)}\mathcal{X})\) which replaces \(E\) cannot be pulled out of the derived images of \(\pi'\).

The final answer is
\[\mathcal{R}^0 \pi'_*(\mathcal{O}_{mE}(mE_B) \otimes \mathcal{E}) - \mathcal{R}^1 \pi'_*(\mathcal{O}_{mE}(mE_B) \otimes \mathcal{E})\]

by lemma 5.1

\[= -\mathcal{R}^1 \pi'_*(\mathcal{O}_{mE}(mE_B) \otimes \mathcal{E}) - \mathcal{R}^1 \pi'_*(\mathcal{O}_{mE}(mE_B)) \otimes \mathcal{E}.\]

this ends the proof of the theorem in the \(\mathcal{R}^2 \pi'_*(\mathcal{E}) \neq 0, m < 0\) case and therefore the proof of the whole theorem. \(\square\)

5.1 The Family Blowup Formula of the Universal Family

In the previous section, section 5, we have derived the family blowup formula of the algebraic family invariant. When we prove the formula, we have to construct algebraic family Kuranishi models based on relatively very ample divisors on these families. The readers should be aware that the non-uniqueness of the choices of the very ample divisors and therefore the choices of algebraic family Kuranishi models. In this sub-section, we focus on the universal family.
$M_{n+1} \mapsto M_n$ and make a brief remark regarding the 'canonical obstruction bundles' for these special families. Following the notation in [Liu1], we would like to explain how does the blowup formula relate to the canonical obstruction bundles of these families.

Let $E$ be the invertible sheaf over $M \times T(M)$ whose first Chern class over $M \times \{t\}, t \in T(M)$ is $C \in H^{1,1}(M, \mathbb{Z})$. One may apply the blowing down map $T(M) \times M_{n+1} \mapsto T(M) \times M \times M_n = T_{M_n}(M_{n+1})$ to pull $E$ back, which defines an invertible sheaf over $T(M) \times M_{n+1}$. We abuse the notation and denoting them by the same symbol. Let $E_1, E_2, \cdots, E_n$ denote the exceptional divisors of the blowing down map $M_{n+1} \mapsto M \times M_n$. Then we have the following short exact sequence

$$0 \mapsto \mathcal{E}(-\sum m_i E_i) \mapsto \mathcal{E} \mapsto \mathcal{O} \sum m_i E_i \otimes \mathcal{E} \mapsto 0.$$  

Taking the right derived images of this sequence along $T(M) \times M_{n+1} \mapsto T(M) \times M_n$, then we get a long exact sequence of coherent sheaves on $T(M) \times M_n$. In the long paper [Liu1], our main concern is study the case when $E$ is sufficiently very ample. So let us make a simplifying assumption on $E$,

**Assumption 1** The invertible sheaf $\mathcal{E} \otimes K^{-1}_M$ is ample.

By Nakai criterion, this assumption is reduced to an assumption of the positivity of $C - c_1(K_M)$ on the curve cone of $M$. If $E$ is a high power of an ample invertible sheaf, the assumption always holds. Under this assumption, the higher sheaf cohomologies of $\mathcal{E}$ vanish. Then the sheaf short exact sequence induces a derived long exact sequence

$$0 \mapsto R^0(f_n)_*(\mathcal{E}(-\sum m_i E_i)) \mapsto R^0(f_n)_*(\mathcal{E}) \mapsto R^0(f_n)_*(\mathcal{O} \sum_{i \leq n} m_i E_i \otimes \mathcal{E})$$

$$\mapsto R^1(f_n)_*(\mathcal{E}(-\sum_{i \leq n} m_i E_i)) \mapsto 0.$$  

Because $\mathcal{E}$ is pulled back from $M \times T(M)$ to the universal family, $R^0(f_n)_*(\mathcal{E})$ is pulled back from $T(M)$ and is constant along the $M_n$ factor. By the similar argument as in lemma 5.2 we can prove the following,
Proposition 5.4 The coherent sheaf $R^0(f_n)_*(O_{\sum_{i\leq n} m_i E_i} \otimes \mathcal{E})$ is locally free of rank $\sum_{i\leq n} \frac{(m_i+1)m_i}{2}$.

Proof: Because the sheaf is the zero-th derived image of a relatively one dimensional fibration, it suffices to prove inductively that the first derived image sheaf vanishes.

This follows from the following short exact sequence

$$0 \to O_{m_n E_n} \left(\sum_{i\leq n-1} m_i E_i\right) \otimes \mathcal{E} \to O_{\sum_{i\leq n} m_i E_i} \otimes \mathcal{E} \to O_{\sum_{i\leq n-1} m_i E_i} \otimes \mathcal{E} \to 0.$$ 

The remaining of the proof is very similar to the lemmas 5.1 and 5.2, we omit the details. □

As in [Liu1], the $n+1$th universal space $M_{n+1}$ can be constructed by blowing up the relative diagonal $M_n \mapsto M_n \times_{M_{n-1}} M_n$ from the fiber product

$$\begin{array}{ccc}
M_n \times_{M_{n-1}} M_n & \longrightarrow & M_n \\
\downarrow & & \downarrow f_n \\
M_n & \mapsto & M_{n-1}
\end{array}$$

The space $M_n \times_{M_{n-1}} M_n$ fibers over $M_n$ and the map $M_n \times_{M_{n-1}} M_n \mapsto M_n$ is smooth of relative dimension two. One may view the relative diagonal as a cross section from the base $M_n$ to $M_n \times_{M_{n-1}} M_n$ and the normal bundle of the cross section is isomorphic to the relative tangent bundle of $M_{n-1} \mapsto M_{n-1}$, $T_{M_{n/\mathcal{M}_{n-1}}} \cong T_{M_n}/f_n^* TM_{n-1}$.

Knowing that $R^0(f_n)_*(O_{\sum_{i\leq n} m_i E_i} \otimes \mathcal{E})$ is locally free, its associated vector bundle can be used to build the canonical obstruction bundle of the family Seiberg-Witten invariant (under the additional assumption □).

Definition 5.3 Under the assumption \[Liu\] on $\mathcal{E}$, let $V_{\text{canon}}$ and $W_{\text{canon}}$ denote the algebraic vector bundles associated to the locally free $R^0(f_n)_*(\mathcal{E})$ and $R^0(f_n)_*(O_{\sum_{i\leq n} m_i E_i} \otimes \mathcal{E})$.

Then for a given tuple $(m_1, m_2, m_3, \cdots, m_n)$ of singular multiplicities, the tuple

$$(V_{\text{canon}}, W_{\text{canon}}, \Phi_{V_{\text{canon}}W_{\text{canon}}})$$

with the bundle map
\[ \Phi_{V_{\text{canon}}} W_{\text{canon}} : V_{\text{canon}} \mapsto W_{\text{canon}} \]

induced from

\[ R^0(f_n)_*(\mathcal{E}) \mapsto R^0(f_n)_*(\mathcal{O}_{\sum_{i \leq n} m_i E_i} \otimes \mathcal{E}) \]

is called the canonical algebraic family Kuranishi model of \( C - \sum_i m_i E_i \).

The vector bundle \( \text{Obs}_{\text{canonical}; C - \sum_{i \leq n} m_i E_i} = H \otimes \pi^*_{P(V_{\text{canon}})} W_{\text{canon}} \)

over the projective space bundle \( P(V_{\text{canon}}) \) is defined to be the canonical obstruction bundle of the class \( C - \sum_i m_i E_i \).

Moreover, one see explicitly that the first term of the following short exact sequence relating the canonical obstruction bundles of \( C - \sum_{i \leq n} m_i E_i \) and \( C - \sum_{i \leq n-1} m_i E_i \),

\[ 0 \mapsto H \otimes \pi^*_{P(V_{\text{canon}})}(E \otimes S^{m_n - 1}(\mathcal{O}_B \oplus T^{*}_{\mathbb{M}_{n}/\mathbb{M}_{n-1}})) \mapsto \text{Obs}_{\text{canonical}; C - \sum_{i \leq n} m_i E_i} \mapsto \text{Obs}_{\text{canonical}; C - \sum_{i \leq n-1} m_i E_i} \mapsto 0. \]

This gives a transparent explanation why the mixed invariants \( \mathcal{AFSW}_{\mathbb{M}_{n+1} \mapsto \mathbb{M}_n}(c, C - \sum_{i \leq n} m_i E_i) \) is equal to the combination of mixed invariants \( \sum_{p \leq \frac{m_n(m_n+1)}{2}} \mathcal{AFSW}_{\mathbb{M}_{n+1} \mapsto \mathbb{M}_n}(c \cup c_p(S^{m_n - 1}(C_B \oplus T^{*}_{\mathbb{M}_{n}/\mathbb{M}_{n-1}})), C - \sum_{i \leq n-1} m_i E_i) \).

References

[Be] K. Behrend, Gromov-Witten Invariants in Algebraic Geometry, Inventiones Math., 127 pp601-617 (1997)

[BGV] N. Berline, E Getzler, M. Vergne Heat Kernels and Dirac Operators Grundlehren der mathematischen Wissenschaften 298 New York: Springer-Verlag (1991)

[BPV] W. Barth, C. Peters, A. Van De Ven, Compact Complex Surfaces Ergebnisse der Math. 4 New York: Springer-Verlag (1984)

[BT] R. Bott and L. Tu, Differential Forms in Algebraic Topology Graduate texts in mathematics; 82

[F] W. Fulton. Intersection Theory. A series of Modern Surveys in Mathematics, Springer-Verlag, (1984).
[Fr] Robert Friedman. *Algebraic Surfaces and Holomorphic Vector Bundles* Universitext, Springer-Verlag, (1998).

[FM] R. Friedman and J. Morgan *Algebraic Surfaces and Seiberg-Witten Invariants*. Journal of Algebraic Geometry, 6, no.6, pp445-479, (1997).

[FM2] R. Friedman and J. Morgan *Obstruction Bundles, Semiregularity, and Seiberg-Witten Invariants*. Communications of Analysis and Geometry, 7, no.3, pp451-495, (1999).

[FS] R. Fintushel, R. Stern *Rational Blowdowns of Smooth 4-Manifolds*. Journal of Differential Geometry, 46, pp.181-235, (1997).

[FS2] R. Fintushel, R. Stern *Blowup Formula For Donaldson Invariants*. Annals of Mathematics (2), 143, pp.529-546, (1996).

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. New York: John Wiley and Sons, (1978).

[Got] L. Göttscbe. *A Conjectural Generating Function for Numbers of Curves on Surfaces*. Preprint. [alg-geom/9711012](1997).

[Ha] R. Hartshorne, *Algebraic Geometry*. Graduate texts in Mathematics, 52

[HM1] J. Harvey and G. Moore, *Algebras, BPS states, and Strings*. [hep-th/9510182](1995).

[K] M. Karoubi *K-Theory, an introduction*. Grundlehren der mathematischen Wissenschaften 226, New York: Springer-Verlag (1970).

[KM] P. Kronheimer, and T. Mrowka. *The Genus of Imbedded Surfaces in the Projective Spaces*. Math. Research Letters. 1, pp. 797-808 (1994).

[LiT1] J. Li, G. Tian *Virtual Moduli Spaces and Gromov-Witten Invariants of Algebraic Varieties*. Journal of Amer. Math. Soc. 11, pp119-174 (1998)

[LiT2] J. Li, G. Tian *Virtual Moduli Spaces and Gromov-Witten Invariants of General Symplectic Manifolds*. Preprint [alg-geom/9608032](1996).

[Liu] A. K. Liu, *Ph.D thesis* Harvard University, (1996).
[Liu1] A. K. Liu, Family Blowup Formula, Admissible Graphs and the Enumeration of Singular Curves (I). Journal of Differential Geometry 56 pp381-579 (2001)

[Liu2] A. K. Liu, Cosmic String and Family Seiberg-Witten Theory in preparation, (2003)

[Liu3] A. K. Liu, Family Switching Formula and the $-n$ Exceptional Rational Curves in preparation, (2003)

[LL1] T. J. Li and A. K. Liu, Family Seiberg-Witten Invariant and Wall Crossing Formula. to appear in Communications of Analysis and Geometry.

[LL2] T. J. Li and A. K. Liu. General Wall Crossing Formula. Mathematical Research Letters. 2 pp. 97-118, (1995).

[LL3] T. J. Li and A. K. Liu. The Symplectic Structures of Rational and Ruled Surfaces and the generalized Adjunction inequality. 2 pp. 453-471, (1995).

[Mc] D. Mcduff. Remark on the Uniqueness of Symplectic Blowing Ups, London Math. Soc. Lecture Note Ser.192 Symplectic Geometry, pp157-167, (1993)

[R] Y. Ruan. Virtual Neighborhood and Pseudo-Holomorphic Curves, [alg-geom/9611021] preprint, (1996).

[RT1] Y. Ruan and G. Tian. The Mathematical Theory of Quantum Cohomology, Journal of Differential Geometry. 42 no. 2. Sep. pp 259-367, (1995).

[RT2] Y. Ruan and G. Tian. Higher Genus Symplectic Invariants and Sigma Models coupled with Gravity, Inventiones Mathematicae. 130 no. 3. pp. 455-516, (1997).

[S] B. Siebert Gromov-Witten Invariants for General Symplectic Manifolds, [dg-ga/9608005] preprint, (1996)

[T1] C.H. Taubes. SW $\rightarrow$ Gr, from the Seiberg-Witten Equations to Pseudo-holomorphic Curves. Journal of American Mathematical Society. 9 no. 3. (1996).

[T2] C.H. Taubes. Gr $\rightarrow$ SW, from the Pseudo-holomorphic Curves to the Solutions to Seiberg-Witten Equations, Preprint. (1995).

[T3] C.H. Taubes, The Seiberg Witten Invariants and Gromov Invariant, Mathematical Research Letters. (1995).
[V] Israel Vainsencher. *Enumeration of n-fold tangent hypersurfaces to a surface*, Journal of Algebraic Geometry. 4 pp503-526. (1995).

[YZ] S. T. Yau and E. Zaslow. *BPS states, string duality, and Nodal Curves on K3*. Nuclear Physics B. 471 no. 3. pp503-512, (1996).