A method for calculation of the reliability indices of the restorable technical systems with independent elements and arbitrary reliability structure

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Abstract. This scientific paper deals with analysis of the reliability indices of the technical systems, which consist of a set of different and independent restorable elements with the given failure rates, repair rates and reliability structure. An offered by the authors combinatorial method for calculation of the reliability indices of restorable technical systems with arbitrary reliability structure is discussed. The obtained by the authors formulas for calculation of the stationary availability factor, mean time to failure and mean time to repair are also presented. Finally, an example of calculation of the reliability indices for the technical system with the bridge reliability structure is also given.

1. Introduction

Complex technical systems are an essential part of the modern world and they are widely used in business and manufacturing processes of enterprises [1, 2]. To provide high reliability and fault-tolerance of technical systems the different reliability methods, such as structural, functional and informational redundancy, are used. Reliability and fault-tolerance of technical systems directly affect the efficiency, stability and safety of the processes, provided by technical systems. Therefore, development of the reliability models and methods for analysis of the reliability indices of technical systems is quite urgent scientific task.

Modern reliability theory [3-6] offers the mathematical modeling based on Markov chains [7, 8] for analysis of the reliability indices of the restorable technical systems.

In simple cases, when the technical system consists of $n$ identical elements and the reliability structure is simple (sequential, parallel structure or $k$-modular redundancy), the Markov birth-death chain with $n+1$ states can be used for the reliability analysis.

In more complex cases, when elements of the system are different or the reliability structure is complex (bridge structure), the Markov chain with $2^n$ states is required for the reliability analysis, if the elements are independent on failures and repairs. If the elements of the system have complex dependencies on failures or repairs, then number of states is even greater.

Obviously, the number of states of the Markov chain rapidly grows with increase of the number of elements of system. For example for a restorable technical system, which consist of the $n = 10$
independent and different elements, the number of states of the Markov chain will be $2^{10} = 1024$, and obtaining of the formulas for calculation of the stationary availability factor, mean time to failure and mean time to repair for the given system is a quite difficult mathematical task.

Within the research work in the field of reliability models of technical systems [9, 10] the authors obtained the generalized formulas and offered combinatorial method for calculation of the stationary availability factor, mean time to failure and mean time to repair of the technical systems, which consist of a set of different and independent restorable elements with the given failure rates, repair rates and reliability structure.

2. Combinatorial method for calculation of the reliability indices

Let us discuss a generalized reliability model of the restorable technical system with arbitrary reliability structure, which consists of $n$ different restorable elements with the given failure rates $\lambda_i$ and repair rates $\mu_i$. We will use the Markov chains as the mathematical base of the reliability model.

For simplification of the analysis, we will assume that the elements are independent on failures and repairs and have exponential distribution law for failure and repair times. Moreover, we will limit our analysis to derivation of the calculation formulas for three key reliability indices: stationary availability factor, mean time to failure and mean time to repair.

Taking into account aforesaid the generalized reliability model of the system with $n$ different and independent restorable elements could be represented by the following Markov chain, which contains $2^n$ states (figure 1).

![Markov chain](image)

**Figure 1.** Markov chain for generalized reliability model of the technical systems with $n$ different and independent restorable elements.

Each of the state of Markov chain represents the specific combination (subset) of failed elements of the technical system.

Let us divide the set of states of Markov chain to the «columns» $q = 0 \ldots n$, where $q$ is the exact number of failed elements.

Let us designate as $\Omega(n, q)$ the set of subsets, which represent the combinations of $q$ failed elements among $n$ elements of technical system.

Also let us designate as $\Psi$ the specific subset, which represents the specific combination of $q$ failed elements among $n$ elements of technical system.

As an example, for $n = 3$ we have the following set of the subsets:

- For $q = 0$ we have the set $\Omega(3, 0) = \{\{\}\}$, which contains only one empty subset: $\Psi = \{\}$. 

• For \( q = 1 \) we have the set \( \Omega(3, 1) = \{ \{1\}, \{2\}, \{3\} \} \), which contains 3 subsets with one element of the system in each subset: \( \Psi = \{1\}, \Psi = \{2\} \) and \( \Psi = \{3\} \).

• For \( q = 2 \) we have the set \( \Omega(3, 2) = \{ \{1, 2\}, \{1, 3\}, \{2, 3\} \} \), which contains 3 subsets with two elements of the system in each subset: \( \Psi = \{1, 2\}, \Psi = \{1, 3\} \) and \( \Psi = \{2, 3\} \).

• Finally, for \( q = 3 \) we have the set \( \Omega(3, 3) = \{ \{1, 2, 3\} \} \), which contains only one subset with all three elements of the system: \( \Psi = \{1, 2, 3\} \).

Now, taking into account the independency of elements of the system, the probability of the specific state \( \Psi \), which represents specific combination of \( q \) failed elements and remaining \( n - q \) operable elements, can be calculated as the product of probabilities of the failed state of the elements, which are in the subset \( \Psi \), multiplied by the product of probabilities of the operable state of the remaining elements, which are not in the subset \( \Psi \):

\[
P_{\Psi} = \prod_{j \in \Psi} q_j \prod_{j \notin \Psi} p_j.
\]  

(1)

In turn, taking into account that the elements are independent on failures and repairs and have the given constant failure rates \( \lambda_j \) and repair rates \( \mu_j \), the stationary probabilities of the operable and failed states for the \( j \)-th restorable element can be calculated by the following simple formulas \([5, 6]\):

\[
p_j = \frac{\mu_j}{\lambda_j + \mu_j}; \quad q_j = \frac{\lambda_j}{\lambda_j + \mu_j}.
\]

(2)

Finally, taking into account aforesaid, we obtain the following formula for calculation of the probability of the specific state \( \Psi \) for the technical system with \( n \) independent restorable elements:

\[
P_{\Psi} = \frac{n}{\prod_{j=1}^{n}(\lambda_j + \mu_j)} \frac{\prod_{j \in \Psi} \left( \frac{\lambda_j}{\mu_j} \right)}{\prod_{j=1}^{n} \left(1 + \frac{\lambda_j}{\mu_j}\right)}.
\]

(3)

Now, let us divide each of the set \( \Omega(n, q) \), \( q = 0...n \), to the following two subsets:

1. \( \Omega'(n, q) \) – subset of combinations of \( q \) failed elements among \( n \) elements, at which the system is operable.

2. \( \Omega''(n, q) \) – subset of combinations of \( q \) failed elements among \( n \) elements, at which the system is not operable.

Obviously, for each \( q = 0...n \) the subsets \( \Omega'(n, q) \) of combinations of the failed elements, at which system is operable, are completely determined by the reliability structure of the technical system, and they can be obtained by enumeration of the combinations in the set \( \Omega(n, q) \) and highlighting the combinations, at which system is operable.

Accordingly, the stationary availability factor of the technical system with independent restorable elements and given reliability structure can be calculated as the sum for all \( q = 0...n \) of sums of the probabilities of all states \( \Psi \) belonging to \( \Omega'(n, q) \), at which the system is operable:

\[
K_S = \sum_{q=0}^{n} \sum_{\Psi \in \Omega'(n, q)} P_{\Psi} = \frac{\sum_{q=0}^{n} \left( \sum_{\Psi \in \Omega'(n, q)} \left( \prod_{j \in \Psi} \left( \frac{\lambda_j}{\mu_j} \right) \right) \right)}{\prod_{j=1}^{n} \left(1 + \frac{\lambda_j}{\mu_j}\right)}.
\]

(4)
Next, to calculate the mean time to failure we need to take into account not only the states \( \Psi \in \Omega^*(n, q) \), \( q = 0 \ldots n \), at which the system is operable, but also the transitions from these states to the other states, at which the system is not operable.

It should be noted that in the discussed Markov chain any of transition with the rate \( \lambda_i \) from the operable state \( \Psi \in \Omega^*(n, q) \) has meaning of failure of the subsequent element \( i \notin \Psi \) (one of the remaining operable elements) and it always transfers the system to the state \( \Psi^* = \Psi \cup \{i\} \), which includes the element \( i \) and belongs to the subset \( \Omega(n, q+1) \). The subset \( \Omega(n, q+1) \) is located in the right neighbor column \( q + 1 \) in Markov chain. Accordingly, for calculation of the mean time to repair we should take into account only those transitions, which transfer the system to the non-operable states \( \Psi^* \in \Omega^*(n, q+1) \). Moreover, we should also take into account that for \( q = n \), when the system is in the state \( \{1, \ldots, n\} \), there are no more elements, which can fail.

Now, taking into account aforesaid, we obtain the following formula for calculation of the mean time to failure of the technical system with independent restorable elements and given reliability structure, using the topological method for the reliability models based on the Markov chains [5, 6]:

\[
T_F = \frac{\sum_{q=0}^{n-1} \sum_{\Psi \in \Omega^*(n, q)} P_{\Psi} \left( \sum_{i \in \Psi \setminus \Psi^*} \lambda_i \right)}{\sum_{q=0}^{n-1} \sum_{\Psi \in \Omega^*(n, q)} P_{\Psi}} = \frac{\sum_{q=0}^{n-1} \sum_{\Psi \in \Omega^*(n, q)} \left( \prod_{j \notin \Psi} \left( \frac{\lambda_j}{\mu_j} \right) \right)}{\sum_{q=0}^{n-1} \sum_{\Psi \in \Omega^*(n, q)} \left( \prod_{j \notin \Psi} \left( \frac{\lambda_j}{\mu_j} \right) \right)}, \tag{5}
\]

Finally, for calculation of the mean time to repair, we can use the well-known identity \( K_S = T_F / (T_F + T_R) \) for the reliability models based on the Markov chains [5, 6], which allows us to calculate the \( T_R \) using the stationary availability factor \( K_S \) and mean time to failure \( T_F \).

Accordingly, we obtain the following formula for calculation of the mean time to repair of the technical system with independent restorable elements and given reliability structure:

\[
T_R = T_F \left( 1 - \frac{K_S}{K_S} \right) = \frac{\sum_{q=0}^{n-1} \sum_{\Psi \in \Omega^*(n, q)} \left( \prod_{j \notin \Psi} \left( \frac{\lambda_j}{\mu_j} \right) \right)}{\sum_{q=0}^{n-1} \sum_{\Psi \in \Omega^*(n, q)} \left( \prod_{j \notin \Psi} \left( \frac{\lambda_j}{\mu_j} \right) \right)}, \tag{6}
\]

3. Example of calculation of the reliability indices

Let us overview an example of the system, which consists of the \( n = 5 \) independent restorable elements with the given failure rates \( \lambda_j \), repair rates \( \mu_j \) and bridge reliability structure (figure 2).

![Figure 2. System with \( n = 5 \) independent restorable elements and bridge reliability structure.](Image)

At first, let us provide the analysis of the system reliability structure and formation of the subsets \( \Omega^*(5, q) \) of combinations of \( q \) failed elements among \( n = 5 \) elements, at which system is operable, and the subsets \( \Omega (5, q) \) of combinations of the failed elements, at which system is not operable.
• For $q = 0$ the subset $\Omega'(5, 0)$ contains only one empty combination of the failed elements $\emptyset$, at which the system is operable. In turn, the subset $\Omega(5, 0)$ is empty and does not contain any of the combinations of the failed elements.

• For $q = 1$ the subset $\Omega'(5, 1)$ contains 5 combinations of 1 failed element: $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, at which the system is operable. In turn, the subset $\Omega(5, 1)$ is empty and does not contain any of the combinations of failed elements.

• For $q = 2$ the subset $\Omega'(5, 2)$ contains 8 combinations of 2 failed elements: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$, at which system is operable. In turn, the subset $\Omega(5, 2)$ contains 2 combinations: $\{1,2\}$, $\{4,5\}$, at which system is not operable.

• For $q = 3$ the subset $\Omega'(5, 3)$ contains 2 combinations of 3 failed elements: $\{1,3,4\}$, $\{2,3,5\}$, at which system is operable. In turn, the subset $\Omega(5, 3)$ contains 8 combinations: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{3,4,5\}$, at which system is not operable.

• For $q = 4$ the subset $\Omega'(5, 4)$ is empty and does not contain any of combinations of failed elements, at which the system is operable. In turn, the subset $\Omega(5, 4)$ contains 5 combinations of 4 failed elements: $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$, $\{1,3,4,5\}$, $\{2,3,4,5\}$, at which system is not operable.

• Finally, for $q = 5$ the subset $\Omega'(5, 5)$ is empty and does not contain any of the combinations of failed elements, at which the system is operable. In turn, the subset $\Omega(5, 5)$ contains only one combination of 5 failed elements: $\{1,2,3,4,5\}$, at which system is not operable.

Now, using the formed above subsets of operable and non-operable combinations of the failed elements and formula (4), we obtain the following formula for calculation of the stationary availability factor of the system with 5 independent restorable elements and bridge reliability structure:

$$K_s = \frac{1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_3}{\mu_3} + \frac{\lambda_4}{\mu_4} + \frac{\lambda_5}{\mu_5} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2} + \frac{\lambda_1\lambda_3}{\mu_1\mu_3} + \frac{\lambda_1\lambda_4}{\mu_1\mu_4} + \frac{\lambda_1\lambda_5}{\mu_1\mu_5} + \frac{\lambda_2\lambda_3}{\mu_2\mu_3} + \frac{\lambda_2\lambda_4}{\mu_2\mu_4} + \frac{\lambda_2\lambda_5}{\mu_2\mu_5} + \frac{\lambda_3\lambda_4}{\mu_3\mu_4} + \frac{\lambda_3\lambda_5}{\mu_3\mu_5} + \frac{\lambda_4\lambda_5}{\mu_4\mu_5}}{1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_3}{\mu_3} + \frac{\lambda_4}{\mu_4} + \frac{\lambda_5}{\mu_5} + \frac{\lambda_2\lambda_3}{\mu_2\mu_3} + \frac{\lambda_2\lambda_4}{\mu_2\mu_4} + \frac{\lambda_2\lambda_5}{\mu_2\mu_5} + \frac{\lambda_3\lambda_4}{\mu_3\mu_4} + \frac{\lambda_3\lambda_5}{\mu_3\mu_5} + \frac{\lambda_4\lambda_5}{\mu_4\mu_5}} \cdot (7)$$

Next, taking into account the transition rates from the operable states of the system, which belongs to the subsets $\Omega'(5, q)$, to the non-operable states, which belongs to the subsets $\Omega(5, q+1)$, for all $q = 0 \ldots 4$, and using the formula (5), we obtain the following formula for calculation of the mean time to failure of the system:

$$T_f = \frac{\lambda_1\lambda_2}{\mu_1\mu_2} + \frac{\lambda_2\lambda_3}{\mu_2\mu_3} + \frac{\lambda_2\lambda_4}{\mu_2\mu_4} + \frac{\lambda_2\lambda_5}{\mu_2\mu_5} + \frac{\lambda_3\lambda_4}{\mu_3\mu_4} + \frac{\lambda_3\lambda_5}{\mu_3\mu_5} + \frac{\lambda_4\lambda_5}{\mu_4\mu_5} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2} + \frac{\lambda_2\lambda_3}{\mu_2\mu_3} + \frac{\lambda_2\lambda_4}{\mu_2\mu_4} + \frac{\lambda_2\lambda_5}{\mu_2\mu_5} + \frac{\lambda_3\lambda_4}{\mu_3\mu_4} + \frac{\lambda_3\lambda_5}{\mu_3\mu_5} + \frac{\lambda_4\lambda_5}{\mu_4\mu_5} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2} + \frac{\lambda_2\lambda_3}{\mu_2\mu_3} + \frac{\lambda_2\lambda_4}{\mu_2\mu_4} + \frac{\lambda_2\lambda_5}{\mu_2\mu_5} + \frac{\lambda_3\lambda_4}{\mu_3\mu_4} + \frac{\lambda_3\lambda_5}{\mu_3\mu_5} + \frac{\lambda_4\lambda_5}{\mu_4\mu_5}}{} \cdot (8)$$
Finally, using the formula (6), we obtain the following formula for calculation of the mean time to repair of the system:

$$T_R = \frac{\lambda_1 \lambda_2 + \lambda_3 \lambda_5 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_2 \lambda_5 + \lambda_1 \lambda_3 \lambda_4 + \lambda_1 \lambda_3 \lambda_5}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} + \frac{\lambda_1 \lambda_4 \lambda_5 + \lambda_2 \lambda_4 \lambda_5 + \lambda_1 \lambda_4 \lambda_5 + \lambda_2 \lambda_4 \lambda_5 + \lambda_1 \lambda_2 \lambda_4 + \lambda_2 \lambda_3 \lambda_5}{\mu_1 \mu_3 \mu_4 \mu_5} + \frac{\lambda_1 \lambda_2 \lambda_4 \lambda_5 + \lambda_1 \lambda_2 \lambda_4 \lambda_5 + \lambda_1 \lambda_3 \lambda_5 + \lambda_1 \lambda_2 \lambda_4 \lambda_5 + \lambda_1 \lambda_3 \lambda_4 \lambda_5 + \lambda_1 \lambda_3 \lambda_5}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} + \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5}$$ (9)

For the particular case, when the system consists of identical elements with the given failure rate $\lambda$ and repair rate $\mu$, the calculation formulas can be significantly simplified.

Accordingly, the formula for calculation of the stationary availability factor of the system with $n = 5$ independent and identical restorable elements and bridge reliability structure is as follows:

$$K_S = \frac{1 + 5 \frac{\lambda}{\mu} + 8 \frac{\lambda^2}{\mu^2} + 2 \frac{\lambda^3}{\mu^3}}{1 + \frac{\lambda}{\mu}^3}.$$ (10)

Next, the formula for calculation of the mean time to failure is as follows:

$$T_F = \frac{1 + 5 \frac{\lambda}{\mu} + 8 \frac{\lambda^2}{\mu^2} + 2 \frac{\lambda^3}{\mu^3}}{\lambda \left(4 \frac{\lambda}{\mu} + 18 \frac{\lambda^2}{\mu^2} + 4 \frac{\lambda^3}{\mu^3}\right)}.$$ (11)

Finally, the formula for calculation of the mean time to repair is as follows:

$$T_R = \frac{2 \frac{\lambda^2}{\mu^2} + 8 \frac{\lambda^3}{\mu^3} + 5 \frac{\lambda^4}{\mu^4} + \frac{\lambda^5}{\mu^5}}{\lambda \left(4 \frac{\lambda}{\mu} + 18 \frac{\lambda^2}{\mu^2} + 4 \frac{\lambda^3}{\mu^3}\right)} = \frac{1 + 5 \frac{\mu}{\lambda} + 8 \frac{\mu^2}{\lambda^2} + 2 \frac{\mu^3}{\lambda^3}}{\left(4 \frac{\mu}{\lambda} + 18 \frac{\mu^2}{\lambda^2} + 4 \frac{\mu^3}{\lambda^3}\right)}.$$ (12)

4. Conclusion
Thus, within the scope of this scientific paper the authors offer the generalized formulas and an combinatorial method for calculation of the stationary availability factor, mean time to failure and mean time to repair of the technical systems, which consist of a set of different and independent restorable elements with the given failure, repair rates and reliability structure.
The advantage of the method is that it does not require mathematical modeling based on Markov chains and offers the «direct» formulas for calculation of the reliability indices, which are formed by using the combinatorial analysis of the operable and non-operable states of the technical system for any given reliability structure.

However, it should be noted, that the obtained formulas are derived on condition that the elements of the system are independent on failures and repairs. In particular, it is assumed that after reaching of any failed state, the system will not be shutdown, and the remaining operable elements may fail up to failure of all elements of the system. It is also assumed that there are unlimited capabilities for simultaneous repair of any number of the failed elements.

In a number of cases, the technical systems may have different kind of dependencies and limitations on failure and repair of the elements. Therefore, in such cases the offered by the authors method cannot be applied and appropriate reliability models should be developed for analysis of the reliability indices.

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