Poisson Hopf algebras associated to quantized enveloping algebras

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Abstract

We study certain Poisson structures related to quantized enveloping algebras. In particular, we give a description of the Poisson structure of a certain manifold associated to the ring of differential operators.

0 Introduction

Let $U$ be the quantized enveloping algebra corresponding to a finite dimensional complex simple Lie algebra $\mathfrak{g}$. It is a Hopf algebra over $\mathbb{Q}(q, q^{-1})$. In [4] De Concini-Procesi introduced a certain $\mathbb{Q}[q, q^{-1}]$-form $U_{\mathbb{Q}[q, q^{-1}]}$, and studied its properties (see also [1]). For $z \in \mathbb{C}^\times$ we denote by $U_z$ the specialization of $U_{\mathbb{Q}[q, q^{-1}]}$ at $q = z$. Let $\ell$ be a positive odd integer, and let $\zeta \in \mathbb{C}^\times$ be a primitive $\ell$-th root of 1 (we assume that $\ell$ is prime to 3 for type $G_2$). Then $U_\zeta$ is canonically isomorphic to the specialization of the De Concini-Kac form at $q = \zeta$. Denote by $Z_\zeta$ the central Hopf subalgebra of $U_\zeta$ generated by the $\ell$-th powers of typical generators. It is called the Frobenius center of $U_\zeta$. Then one of the main results of [4] (and [1]) is the following isomorphisms of Hopf algebras:

\[(0.1) \quad Z_\zeta \cong U_1 \cong \mathbb{C}[M],\]

where $M$ is a certain algebraic group associated to $\mathfrak{g}$. The three Hopf algebras appearing in (0.1) are endowed with natural Poisson Hopf algebra structures, and the isomorphisms in (0.1) is in fact that of Poisson Hopf algebras. In
De Concini-Procesi constructed an isomorphism $U_1 \cong \mathbb{C}[M]$ by giving
a correspondence between generators and verifying the necessary relations
among generators by a lengthy calculation. Later a more natural construction
of the isomorphism in terms of the Drinfeld paring was found by Gavarini [5].

In this note we present a slightly different proof of (0.1) which is still based
on Gavarini’s construction of the isomorphism. In fact in [5] the statement
about the Poisson structure is deduced from its dual statement, but our
argument is more direct (we do not claim that our proof is simpler than the
one in [5]). We also give a description of the Poisson algebra associated to
the ring of differential operators.

In this paper we shall use the following notation for a Hopf algebra $H$
over a field $\mathbb{k}$. The comultiplication, the counit, and the antipode of $H$ are
denoted by

\begin{align*}
\Delta_H : H &\to H \otimes \mathbb{k} H, \\
\varepsilon_H : H &\to \mathbb{k}, \\
S_H : H &\to H
\end{align*}

respectively. The subscript $H$ will often be omitted. For $n \in \mathbb{Z}_{>0}$ we denote by

$\Delta_n : H \to H^\otimes n+1$

the algebra homomorphism given by

$\Delta_1 = \Delta, \quad \Delta_n = (\Delta \otimes id_{H^\otimes n-1}) \circ \Delta_{n-1},$

and write

$\Delta(h) = \sum_{(h)} h(0) \otimes h(1), \quad \Delta_n(h) = \sum_{(h)_n} h(0) \otimes \cdots \otimes h(n) \quad (n \geq 2).$

I would like to express my appreciation to F. Gavarini for pointing out
the reference [5] and informing me of several valuable comments on the first
version of this manuscript.

1 Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be its Cartan
subalgebra. We denote by $\Delta \subset \mathfrak{h}^*, Q \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h}^*)$ the set of roots,
the root lattice $\sum_{\alpha \in \Delta} \mathbb{Z}\alpha$, and the Weyl group respectively. We fix a set of
simple roots \( \{ \alpha_i \}_{i \in I} \), and denote the corresponding set of positive roots and simple reflections by \( \Delta^+ \subset \mathfrak{h}^* \) and \( \{ s_i \}_{i \in I} \subset W \) respectively. Set

\[ Q^+ = \sum_{\alpha \in \Delta^+} \mathbb{Z}_{\geq 0} \alpha = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^*. \]

We denote the longest element of \( W \) by \( w_0 \). Let

(1.1) \((\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}\)

be the \( W \)-invariant symmetric bilinear form satisfying \((\beta, \beta)/2 = 1\) for short roots \( \beta \in \Delta \). We define subalgebras \( n^+, n^- \) of \( \mathfrak{g} \) by

\[ n^\pm = \bigoplus_{\beta \in \Delta^\pm} \mathfrak{g}_{\pm \beta}, \]

where

\[ \mathfrak{g}_{\pm \beta} = \{ x \in \mathfrak{g} \mid [h, x] = \pm \beta(h)x \ (h \in \mathfrak{h}) \}. \]

Then we have

\[ \mathfrak{g} = n^+ \oplus \mathfrak{h} \oplus n^- \]

For each \( i \in I \) we take \( e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}, h_i \in \mathfrak{h} \) such that \([e_i, f_i] = h_i\) and \( \alpha_i(h_i) = 2\).

We define a subalgebra \( \mathfrak{m} \) of \( \mathfrak{g} \) by

\[ \mathfrak{m} = \{(h + x, -h + y) \mid h \in \mathfrak{h}, x \in n^+, y \in n^- \}. \]

Set

\[ \mathfrak{m}^0 = \{(h, -h) \mid h \in \mathfrak{h}\}, \quad \mathfrak{m}^+ = \{(x, 0) \mid x \in n^+\}, \quad \mathfrak{m}^- = \{(0, y) \mid y \in n^-\}. \]

They are subalgebras of \( \mathfrak{m} \), and we have \( \mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^0 \oplus \mathfrak{m}^- \). Moreover, we have isomorphisms

(1.2) \( \mathfrak{h} \simeq \mathfrak{m}^0 \quad (h \leftrightarrow (h, -h)) \),

(1.3) \( n^+ \simeq \mathfrak{m}^+ \quad (x \leftrightarrow (x, 0)) \),

(1.4) \( n^- \simeq \mathfrak{m}^- \quad (y \leftrightarrow (0, y)) \)

of Lie algebras. We denote by

(1.5) \( \iota : \mathfrak{h} \to \mathfrak{m}^0 \)

the isomorphism (1.2). For \( i \in I \) set

\[ x_i = (e_i, 0) \in \mathfrak{m}^+, \quad y_i = (0, f_i) \in \mathfrak{m}^-, \quad t_i = (h_i, -h_i) \in \mathfrak{m}^0. \]

Let \( G \) be the adjoint group of \( \mathfrak{g} \). We denote by \( M \) the connected closed subgroup of \( G \times G \) with Lie algebra \( \mathfrak{m} \). Let \( M^0, M^\pm \) be the connected closed subgroups of \( M \) with Lie algebras \( \mathfrak{m}^0, \mathfrak{m}^\pm \) respectively.
2 Quantized enveloping algebras

For \( n \in \mathbb{Z}_{\geq 0} \) we set
\[
[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \quad [n]_t! = [n]_t[n-1]_t \cdots [2]_t[1]_t \in \mathbb{Z}[t, t^{-1}].
\]

For \( n \in \mathbb{Z} \) and \( m \in \mathbb{Z}_{\geq 0} \) we set
\[
\left[ \frac{n}{m} \right]_t = [n]_t[n-1]_t \cdots [n-m+1]_t/[m]_t!.
\]

The quantized enveloping algebra \( U = U_q(\mathfrak{g}) \) of \( \mathfrak{g} \) is an associative algebra over \( \mathbb{Q}(q) = \mathbb{Q}(q) \) with identity element 1 generated by the elements \( K_\lambda (\lambda \in Q), E_i, F_i (i \in I) \) satisfying the following defining relations:

\begin{align*}
(2.1) & \quad K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu} \quad (\lambda, \mu \in Q), \\
(2.2) & \quad K_\lambda E_i K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_i, \quad (\lambda \in Q, i \in I), \\
(2.3) & \quad K_\lambda F_i K_\lambda^{-1} = q^{-\lambda, \alpha_i} F_i \quad (\lambda \in Q, i \in I), \\
(2.4) & \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I), \\
(2.5) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n E_i^{(1-a_{ij})-n} E_i E_i^{(n)} = 0 \quad (i, j \in I, i \neq j), \\
(2.6) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n F_i^{(1-a_{ij})-n} F_i F_i^{(n)} = 0 \quad (i, j \in I, i \neq j),
\end{align*}

where \( q_i = q^{(\alpha_i, \alpha_i)}/2 \), \( K_i = K_{\alpha_i} \), \( a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \) for \( i, j \in I \), and
\[
E_i^{(n)} = E_i^n/[n]_q!, \quad F_i^{(n)} = F_i^n/[n]_q!.
\]

For \( i \in I \) and \( n \in \mathbb{Z}_{\geq 0} \). Algebra homomorphisms \( \Delta : U \to U \otimes U, \varepsilon : U \to \mathbb{Q}(q) \) and an algebra anti-automorphism \( S : U \to U \) are defined by:

\begin{align*}
(2.7) & \quad \Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \\
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, & \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \\
(2.8) & \quad \varepsilon(K_\lambda) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \\
(2.9) & \quad S(K_\lambda) = K_\lambda^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i,
\end{align*}

and \( U \) is endowed with a Hopf algebra structure with the comultiplication \( \Delta \), the counit \( \varepsilon \) and the antipode \( S \).
We define subalgebras $U^0, U^{\gtrless 0}, U^{\lesssim 0}, U^+, U^-$ of $U$ by

\begin{align}
U^0 &= \langle K_\lambda \mid \lambda \in Q \rangle, \\
U^{\gtrless 0} &= \langle K_\lambda, E_i \mid \lambda \in Q, i \in I \rangle, \\
U^{\lesssim 0} &= \langle K_\lambda, F_i \mid \lambda \in Q, i \in I \rangle, \\
U^+ &= \langle E_i \mid i \in I \rangle, \\
U^- &= \langle F_i \mid i \in I \rangle.
\end{align}

Note that $U^0, U^{\gtrless 0}, U^{\lesssim 0}$ are Hopf subalgebras of $U$, while $U^+$ and $U^-$ are not Hopf subalgebras.

The following result is standard.

**Proposition 2.1.**

(i) $\{K_\lambda \mid \lambda \in Q\}$ is a $\mathbb{Q}(q)$-basis of $U^0$.

(ii) $U^+$ (resp. $U^-$) is isomorphic to the $\mathbb{Q}(q)$-algebra generated by $\{E_i \mid i \in I\}$ (resp. $\{F_i \mid i \in I\}$) with defining relation (2.5) (resp. (2.6)).

(iii) $U^{\gtrless 0}$ (resp. $U^{\lesssim 0}$) is isomorphic to the $\mathbb{Q}(q)$-algebra generated by $\{E_i, K_\lambda \mid i \in I, \lambda \in Q\}$ (resp. $\{F_i, K_\lambda \mid i \in I, \lambda \in Q\}$) with defining relations (2.1), (2.2), (2.5) (resp. (2.1), (2.3), (2.6)).

(iv) The linear maps

$$U^- \otimes U^0 \otimes U^+ \rightarrow U \leftarrow U^+ \otimes U^0 \otimes U^-, \quad U^+ \otimes U^0 \rightarrow U^{\gtrless 0} \leftarrow U^{\lesssim 0} \otimes U^0 \otimes U^-$$

induced by the multiplication are all isomorphisms.

For $\gamma \in Q$ we set

$$U^\pm_\gamma = \{x \in U^\pm \mid K_\lambda x K_\lambda^{-1} = q^{(\lambda, \gamma)} x \ (\lambda \in Q)\}.$$  

We have $U^\pm_{\pm \gamma} = \{0\}$ unless $\gamma \in Q^+$, and

$$U^\pm = \bigoplus_{\gamma \in Q^+} U^\pm_{\pm \gamma}.$$  

For $i \in I$ we can define an algebra automorphism $T_i$ of $U$ by

\begin{align}
T_i(K_\mu) &= K_{s_i \mu} \quad (\mu \in Q), \\
T_i(E_j) &= \begin{cases} 
\sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (j \in I, j \neq i), \\
- F_i K_i & (j = i),
\end{cases} \\
T_i(F_j) &= \begin{cases} 
\sum_{k=0}^{-a_{ij}} (-1)^k q_i^k F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (j \in I, j \neq i), \\
- K_i^{-1} E_i & (j = i).
\end{cases}
\end{align}
For \( w \in W \) we define an algebra automorphism \( T_w \) of \( U \) by \( T_w = T_{i_1} \cdots T_{i_n} \) where \( w = s_{i_1} \cdots s_{i_n} \) is a reduced expression. The automorphism \( T_w \) does not depend on the choice of a reduced expression (see Lusztig [9]).

We fix a reduced expression

\[
w_0 = s_{i_1} \cdots s_{i_N}
\]

of \( w_0 \), and set

\[
\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leq k \leq N).
\]

Then we have \( \Delta^+ = \{ \beta_k \mid 1 \leq k \leq N \} \). For \( 1 \leq k \leq N \) set

\[
(2.15) \quad E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}), \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}).
\]

Then \( \{E_{\beta_k}^{m_1} \cdots E_{\beta_k}^{m_N} \mid m_1, \ldots, m_N \geq 0\} \) (resp. \( \{F_{\beta_k}^{m_1} \cdots F_{\beta_k}^{m_N} \mid m_1, \ldots, m_N \geq 0\} \)) is a \( \mathbb{Q}(q) \)-basis of \( U^+ \) (resp. \( U^- \)), called the PBW-basis (see Lusztig [8]). We have \( E_{\alpha_i} = E_i \) and \( F_{\alpha_i} = F_i \) for any \( i \in I \). For \( 1 \leq k \leq N, m \geq 0 \) we also set

\[
(2.16) \quad E_{\beta_k}^{m} = E_{\beta_k}^{m}/[m]_{q_{\beta_k}!}, \quad F_{\beta_k}^{m} = F_{\beta_k}^{m}/[m]_{q_{\beta_k}!},
\]

where \( q_{\beta} = q^{(\beta,\beta)/2} \) for \( \beta \in \Delta^+ \).

There exists a unique bilinear form

\[
(2.17) \quad \tau : U^{\geq 0} \times U^{\leq 0} \to \mathbb{Q}(q),
\]

called the Drinfeld paring, which is characterized by

\[
(2.18) \quad \tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in U^{\geq 0}, y_1, y_2 \in U^{\leq 0}),
\]

\[
(2.19) \quad \tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U^{\geq 0}, y \in U^{\leq 0}),
\]

\[
(2.20) \quad \tau(K_\lambda, K_\mu) = q^{-\langle \lambda, \mu \rangle} \quad (\lambda, \mu \in Q),
\]

\[
(2.21) \quad \tau(K_\lambda, F_i) = \tau(E_i, K_\lambda) = 0 \quad (\lambda \in Q, i \in I),
\]

\[
(2.22) \quad \tau(E_i, F_j) = \delta_{ij}/(q_{i}^{-1} - q_{i}) \quad (i, j \in I).
\]

It satisfies the following (see Tanisaki [11], Lusztig [9]).

**Lemma 2.2.** (i) \( \tau(S(x), S(y)) = \tau(x, y) \) for \( x \in U^{\geq 0}, y \in U^{\leq 0} \).

(ii) For \( x \in U^{\geq 0}, y \in U^{\leq 0} \) we have

\[
y x = \sum_{(x)_{2}, (y)_{2}} \tau(x_{(0)}, S(y_{(0)}))\tau(x_{(2)}, y_{(2)})x_{(1)}y_{(1)},
\]

\[
x y = \sum_{(x)_{2}, (y)_{2}} \tau(x_{(0)}, y_{(0)})\tau(x_{(2)}, S(y_{(2)}))y_{(1)}x_{(1)}.
\]
(iii) \( \tau(xK_\lambda, yK_\mu) = q^{-(\lambda, \mu)}\tau(x, y) \) for \( \lambda, \mu \in Q, x \in U^+, y \in U^- \).

(iv) \( \tau(U^+_\beta, U^-_\gamma) = \{0\} \) for \( \beta, \gamma \in Q^+ \) with \( \beta \neq \gamma \).

(v) For any \( \beta \in Q^+ \) the restriction of (2.17) to \( U^+_\beta \times U^-_{-\beta} \) is non-degenerate.

We have the following explicit computation of (2.17) in terms of PBW-bases ([6], [7], [10]).

**Proposition 2.3.** We have

\[
\tau(E_{\beta_1}^{m_1} \cdots E_{\beta_N}^{m_N}, F_{\beta_1}^{n_1} \cdots F_{\beta_N}^{n_N}) = \prod_{k=1}^N \delta_{m_k, n_k} (-1)^m_{\lambda_k}[q_{\beta_k}] q_{\beta_k}^{m_k(m_k-1)/2} (q_{\beta_k} - q_{\beta_k}^{-1})^{-m_k}.
\]

The quantized enveloping algebra \( V = U_q(m) \) of \( m \) is an associative algebra over \( \mathbb{Q}(q) \) with identity element 1 generated by the elements \( Z_\lambda (\lambda \in Q), X_i, Y_i (i \in I) \) satisfying the following defining relations:

\begin{align*}
(2.23) & \quad Z_0 = 1, \quad Z_\lambda Z_\mu = Z_{\lambda + \mu} \quad (\lambda, \mu \in Q), \\
(2.24) & \quad Z_\lambda X_i Z_\lambda^{-1} = q^{(\lambda, \alpha_i)} X_i \quad (\lambda \in Q, i \in I), \\
(2.25) & \quad Z_\lambda Y_i Z_\lambda^{-1} = q^{(\lambda, \alpha_i)} Y_i \quad (\lambda \in Q, i \in I), \\
(2.26) & \quad X_i Y_j - Y_j X_i = 0 \quad (i,j \in I), \\
(2.27) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n X_i^{(1-a_{ij})-n} X_j X_i^{(n)} = 0 \quad (i,j \in I, i \neq j), \\
(2.28) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n Y_i^{(1-a_{ij})-n} Y_j Y_i^{(n)} = 0 \quad (i,j \in I, i \neq j),
\end{align*}

where

\[
X_i^{(n)} = X_i^n/[n]_q!, \quad Y_i^{(n)} = Y_i^n/[n]_q!.
\]

\( V \) is endowed with a structure of Hopf algebra by

\begin{align*}
(2.29) & \quad \Delta(Z_\lambda) = Z_\lambda \otimes Z_\lambda, \\
& \quad \Delta(X_i) = X_i \otimes 1 + Z_i \otimes X_i, \quad \Delta(Y_i) = Y_i \otimes Z_i + 1 \otimes Y_i, \\
(2.30) & \quad \varepsilon(Z_\lambda) = 1, \quad \varepsilon(X_i) = \varepsilon(Y_i) = 0, \\
(2.31) & \quad S(Z_\lambda) = Z_\lambda^{-1}, \quad S(X_i) = -Z_i^{-1} X_i, \quad S(Y_i) = -Y_i Z_i^{-1},
\end{align*}

where \( Z_i = Z_{\alpha_i} \) for \( i \in I \).
We define subalgebras $V^0, V_{\geq 0}, V_{\leq 0}, V^+, V^-$ of $V$ by

\begin{align*}
V^0 &= \langle Z_\lambda \mid \lambda \in Q \rangle, \\
V_{\geq 0} &= \langle Z_\lambda, X_i \mid \lambda \in Q, i \in I \rangle, \\
V_{\leq 0} &= \langle Z_\lambda, Y_i \mid \lambda \in Q, i \in I \rangle, \\
V^+ &= \langle X_i \mid i \in I \rangle, \\
V^- &= \langle Y_i \mid i \in I \rangle.
\end{align*}

Then $V^0, V_{\geq 0}, V_{\leq 0}$ are Hopf subalgebras of $V$.

Similarly to Proposition 2.1 we have the following.

**Proposition 2.4.**

(i) $\{Z_\lambda \mid \lambda \in Q\}$ is a $\mathbb{Q}(q)$-basis of $V^0$.

(ii) $V^+$ (resp. $V^-$) is isomorphic to the $\mathbb{Q}(q)$-algebra generated by $\{X_i \mid i \in I\}$ (resp. $\{Y_i \mid i \in I\}$) with defining relation (2.27) (resp. (2.28)).

(iii) $V_{\geq 0}$ (resp. $V_{\leq 0}$) is isomorphic to the $\mathbb{Q}(q)$-algebra generated by $\{X_i, Z_\lambda \mid i \in I, \lambda \in Q\}$ (resp. $\{Y_i, Z_\lambda \mid i \in I, \lambda \in Q\}$) with defining relations (2.23), (2.24), (2.27) (resp. (2.23), (2.25), (2.28)).

(iv) The linear maps

\[ V^- \otimes V^0 \otimes V^+ \to V \leftarrow V^+ \otimes V^0 \otimes V^-, \]

\[ V^+ \otimes V^0 \to V_{\geq 0} \leftarrow V^0 \otimes V^+, \quad V^- \otimes V^0 \to V_{\leq 0} \leftarrow V^0 \otimes V^- \]

induced by the multiplication are all isomorphisms.

For $\gamma \in Q$ we set

\[ V^\pm_\gamma = \{ x \in V^\pm \mid Z_\lambda x Z_\lambda^{-1} = q^{\pm(\lambda, \gamma)}x \ (\lambda \in Q) \}. \]

We have $V^\pm_\gamma = \{0\}$ unless $\gamma \in Q^+$, and

\[ V^\pm = \bigoplus_{\gamma \in Q^+} V^\pm_\gamma. \]

By Proposition 2.1 and Proposition 2.3 we have isomorphisms

\[ \eta^\leq_0 : V_{\leq 0} \to U_{\leq 0} \quad (Y_i \mapsto F_i, Z_\lambda \mapsto K_{-\lambda}), \]

\[ \eta^\geq_0 : V_{\geq 0} \to U_{\geq 0} \quad (X_i \mapsto E_i, Z_\lambda \mapsto K_{\lambda}) \]

of Hopf algebras.
We define a bilinear form

\[(2.37) \quad \sigma : U \times V \to \mathbb{Q}(q)\]

by

\[
\sigma(u_+ u_0 S(u_-), v_- v_+ v_0) = \tau(u_+, \eta \leq 0(v_-)) \tau(u_0, \eta \leq 0(v_0)) \tau(\eta \geq 0(v_+), u_-)
\]

\[
(u_\pm \in U^\pm, u_0 \in U^0, v_\pm \in V^\pm, v_0 \in V^0).
\]

Note that

\[
\sigma(u_+ u_0 \leq 0, v_\geq 0) = \tau((S \circ \eta \geq 0)(v_\geq 0), u_0) \varepsilon(u_+)
\]

\[
(v_\geq 0 \in V^\geq 0, u_+ \in U^+, u_0 \in U^0),
\]

\[
\sigma(u_0 S(u_-), v_\leq 0) = \tau(u_0, \eta \leq 0(v_\leq 0)) \varepsilon(u_-)
\]

\[
(v_\leq 0 \in V^\leq 0, u_- \in U^-, u_0 \in U^0).
\]

The following result is a consequence of Gavarini [5, Theorem 6.2].

**Proposition 2.5.** We have

\[
\sigma(u, v v') = (\sigma \otimes \sigma)(\Delta(u), v \otimes v') \quad (u \in U, v, v' \in V).
\]

### 3 A-forms

We fix a subring \(A\) of \(\mathbb{Q}(q)\) containing \(\mathbb{Q}[q, q^{-1}]\). We denote by \(U^L_A\) the Lusztig \(A\)-form of \(U\), i.e., \(U^L_A\) is the \(A\)-subalgebra of \(U\) generated by the elements

\[
E_i^{(m)} , F_i^{(m)} , K_\lambda \quad (i \in I, m \geq 0, \lambda \in \mathbb{Q}).
\]

Set

\[
U^L_{\geq 0} = U^L_A \cap U^\geq , \quad U^L_{\leq 0} = U^L_A \cap U^\leq ,
\]

\[
U^L_A = U^L_{\geq 0} \cup U^L_{\leq 0}.
\]

Then \(U^L_{\geq 0}, U^L_{\leq 0}, U^L_\geq, U^L_\leq\) are endowed with structures of Hopf algebras over \(A\) via the Hopf algebra structure on \(U\), and the multiplication of \(U^L_A\) induces isomorphisms

\[
U^L_A \simeq U^L_{\geq} \otimes U^L_{\leq} \otimes U^L_{\geq} \simeq U^L_{\geq} \otimes U^L_{\leq} \otimes U^L_{\leq},
\]

\[
U^L_{\geq} \simeq U^L_{\geq} \otimes U^L_{\leq} \simeq U^L_{\geq} \otimes U^L_{\leq},
\]

\[
U^L_{\leq} \simeq U^L_{\leq} \otimes U^L_{\leq} \simeq U^L_{\leq} \otimes U^L_{\leq}.
\]
of \( A \)-modules. Moreover, \( U^{L,+}_A, U^{L,-}_A, U^{L,0}_A \) are free \( A \)-modules with bases

\[
\{ E^{(m_N)}_{\beta N} \ldots E^{(m_1)}_{\beta 1} \mid m_1, \ldots, m_N \geq 0 \},
\{ F^{(m_N)}_{\beta N} \ldots F^{(m_1)}_{\beta 1} \mid m_1, \ldots, m_N \geq 0 \},
\left\{ \prod_{i \in I} \left( K^{\varepsilon_i}_i \left[ K_i \right] / n_i \right) \mid n_i \geq 0, \varepsilon_i = 0 \text{ or } 1 \right\}
\]

respectively, where

\[
\left[ K_i \right] / m = \prod_{s=0}^{m-1} \frac{q_i^{-s} K_i - q_i^s K_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (m \geq 0).
\]

We denote by \( V_A \) the \( A \)-subalgebra of \( V \) generated by the elements

\[
X_i^{(m)}, Y_i^{(m)}, Z_{i \pm}^{\pm}, \left[ Z_i \right] / m \quad (i \in I, m \geq 0),
\]

where

\[
\left[ Z_i \right] / m = \prod_{s=0}^{m-1} \frac{q_i^{-s} Z_i - q_i^s Z_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (m \geq 0).
\]

Set

\[
V^+_A = V_A \cap V^+, \quad V^0_A = V_A \cap V^0,
V^\geq 0_A = V_A \cap V^{\geq 0}, \quad V^{\leq 0}_A = V_A \cap V^{\leq 0},
V_{\pm \gamma}^\pm_A = V_A \cap V_{\pm \gamma}^\pm \quad (\gamma \in Q^+).
\]

Then we have

\[
V^{\pm}_A = \langle X_i^{(m)} \mid i \in I, m \geq 0 \rangle, \quad V^0_A = \langle Y_i^{(m)} \mid i \in I, m \geq 0 \rangle
\]

as \( A \)-algebras, and \( V^0_A \) is a free \( A \)-module with basis consisting of the elements

\[
\prod_{i \in I} \left( Z_i^{\varepsilon_i} \left[ Z_i \right] / m_i \right) \quad (m_i \geq 0, \varepsilon_i = 0 \text{ or } 1).
\]

Moreover, the multiplication of \( V_A \) induces isomorphisms

\[
V_A \cong V^+_A \otimes V^0_A \otimes V^+_A \cong V^+_A \otimes V^0_A \otimes V^-_A,
V^{\geq 0}_A \cong V^0_A \otimes V^+_A \cong V^+_A \otimes V^0_A,
V^{\leq 0}_A \cong V^0_A \otimes V^-_A \cong V^-_A \otimes V^0_A.
\]
of $A$-modules. Note that
\[ \eta^\geq_0(V^<_A) = U^L_A \cap U^0_A, \quad \eta^\geq_0(V^+_A) = U^L_A \cap U^0_A, \quad \eta^\geq_0(V^0_A) = U^L_A, \]
\[ \eta^\leq_0(V^<_A) = U^L_A \cap U^0_A, \quad \eta^\leq_0(V^-_A) = U^L_A, \quad \eta^\leq_0(V^0_A) = U^L_A. \]

We define root vectors $X_{\beta_k} \in V^+_\beta_k$, $Y_{\beta_k} \in V^-_{\beta_k}$ $(1 \leq k \leq N)$ by
\[ \eta^\geq_0(X_{\beta_k}) = E_{\beta_k}, \quad \eta^\leq_0(Y_{\beta_k}) = F_{\beta_k}, \]
and set
\[ X_{\beta_k}^{(m)} = \frac{X_{\beta_k}^m}{[m]_{q_{\beta_k}}}, \quad Y_{\beta_k}^{(m)} = \frac{Y_{\beta_k}^m}{[m]_{q_{\beta_k}}} \quad (1 \leq k \leq N, \ m \geq 0). \]

Then we have free bases \{ $X_{\beta_N}^{(m)} \cdots X_{\beta_1}^{(m)}$ | $m_1, \ldots, m_N \geq 0$ \} and \{ $Y_{\beta_N}^{(m_N)} \cdots Y_{\beta_1}^{(m_1)}$ | $m_1, \ldots, m_N \geq 0$ \} of $V^+_A$ and $V^-_A$ respectively.

We define root vectors $X_{\beta_k} \in V^+_\beta_k$, $Y_{\beta_k} \in V^-_{\beta_k}$ $(1 \leq k \leq N)$ by
\[ \eta^\geq_0(X_{\beta_k}) = E_{\beta_k}, \quad \eta^\leq_0(Y_{\beta_k}) = F_{\beta_k}, \]
and set
\[ X_{\beta_k}^{(m)} = \frac{X_{\beta_k}^m}{[m]_{q_{\beta_k}}}, \quad Y_{\beta_k}^{(m)} = \frac{Y_{\beta_k}^m}{[m]_{q_{\beta_k}}} \quad (1 \leq k \leq N, \ m \geq 0). \]

Then we have free bases \{ $X_{\beta_N}^{(m_N)} \cdots X_{\beta_1}^{(m_1)}$ | $m_1, \ldots, m_N \geq 0$ \} and \{ $Y_{\beta_N}^{(m_N)} \cdots Y_{\beta_1}^{(m_1)}$ | $m_1, \ldots, m_N \geq 0$ \} of $V^+_A$ and $V^-_A$ respectively.

Set
\[ U_A = \{ u \in U \mid (u, U^0_A) \subset A \}, \]
\[ U^+_A = U^+ \cap U_A, \quad U^0_A = U^0 \cap U_A, \]
\[ U^0_A \cap U_A, \quad U^0_A = U^0 \cap U_A, \]
\[ U^0_A \cap U_A \}
(\beta \in Q^+). \]

We can easily check that
\[ U^+_A = \{ x \in U^+ \mid (x, U^L_A) \in A \}, \]
\[ U^-_A = \{ y \in U^- \mid (U^L_A, y) \in A \} \]
\[ U^0_A = \sum_{\lambda \in P} A K \lambda. \]

Moreover, the multiplication of $U$ induces the isomorphism
\[ U_A \cong U^+_A \otimes U^0_A \otimes U^-_A, \]
\[ U^0_A \cap U^+_A \cong U^+_A \otimes U^0_A, \]
\[ U^0_A \cong U^0_A \otimes U^-_A \cong U^-_A \otimes U^0_A \]
of $A$-modules.

For $i \in I$ we set
\[ A_i = (q_i - q_i^{-1})E_i, \quad B_i = (q_i - q_i^{-1})F_i. \]

For $1 \leq k \leq N$ we also set
\[ A_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})E_{\beta_k}, \quad B_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})F_{\beta_k}. \]

Then we have the following results (see Gavarini [5]).
Lemma 3.1. \( \{ A_{\beta_1}^{m_1} \cdots A_{\beta_i}^{m_i} \mid m_1, \ldots, m_N \geq 0 \} \) (resp. \( \{ B_{\beta_1}^{m_1} \cdots B_{\beta_i}^{m_i} \mid m_1, \ldots, m_N \geq 0 \} \)) is an \( \mathbb{A} \)-basis of \( U^+_\mathbb{A} \) (resp. \( U^-_\mathbb{A} \)). In particular, we have \( U^\pm_\mathbb{A} \subset U^{L, \pm}_\mathbb{A} \).

Proposition 3.2. (i) \( U^0_\mathbb{A}, U^+_\mathbb{A}, U^-_\mathbb{A}, U^\geq_\mathbb{A}, U^\leq_\mathbb{A} \) are \( \mathbb{A} \)-subalgebras of \( U \).

(ii) \( U^0_\mathbb{A}, U^\geq_\mathbb{A}, U^\leq_\mathbb{A}, U^+ \) are Hopf algebras over \( \mathbb{A} \).

Remark 3.3. It follows from Lemma 3.1 that the \( \mathbb{A} \)-form \( U^+_\mathbb{A} \) of \( U \) is the same as the one considered in De Concini-Procesi [4].

By (3.7), (3.8) and Lemma 3.1 the bilinear form \( \tau : U^{\geq} \times U^{\leq} \rightarrow \mathbb{Q}(q) \) induces

\[
\tau^0 : U^0_\mathbb{A} \times U^0_\mathbb{A} \rightarrow \mathbb{A}, \\
\tau^L : U^0_\mathbb{A} \times U^\leq_\mathbb{A} \rightarrow \mathbb{A}, \\
\tau^0 : U^0_\mathbb{A} \times U^\geq_\mathbb{A} \rightarrow \mathbb{A}
\]

and \( \sigma : U \times V \rightarrow \mathbb{Q}(q) \) induces a bilinear form

\[
\sigma^0 : U^+_\mathbb{A} \times V_\mathbb{A} \rightarrow \mathbb{A}.
\]

4 Specialization

Fix \( z \in \mathbb{C}^x \) and set

\[ \mathbb{A}_z = \{ f/g \mid f, g \in \mathbb{Q}[q, q^{-1}], g(z) \neq 0 \} \subset \mathbb{Q}(q) . \]

We set

\[
U^L_z = \mathbb{C} \otimes \mathbb{A}_z U^L_{\mathbb{A}_z}, \quad V_z = \mathbb{C} \otimes \mathbb{A}_z V_{\mathbb{A}_z}, \quad U_z = \mathbb{C} \otimes \mathbb{A}_z U_{\mathbb{A}_z},
\]

where the specialization \( \mathbb{A}_z \rightarrow \mathbb{C} \) is given by \( q \mapsto z \). We also define \( U^L_z, U^L_0, U^L_{\geq}, U^L_{\leq}, U^L_{\pm, \beta} \) (\( \beta \in \mathbb{Q}^+ \)), \( V^L_z, V^0_z, V^\geq_z, V^\leq_z, V^\pm_z, V^\pm_{z, \pm, \beta} \) (\( \beta \in \mathbb{Q}^+ \)) similarly. We denote by

\[
p^L_z : U^L_{\mathbb{A}_z} \rightarrow U^L_z, \quad p_z : V_{\mathbb{A}_z} \rightarrow V_z, \quad \pi_z : U_{\mathbb{A}_z} \rightarrow U_z
\]

the natural homomorphisms. The bilinear forms (3.12), (3.13) for \( R = \mathbb{A}_z \) induce bilinear forms

\[
\tau^0_z : U^0_z \otimes U^L_z \rightarrow \mathbb{C}, \quad \tau^L_z : U^L_z \otimes U^\leq_z \rightarrow \mathbb{C}, \\
\tau^0_z : U^0_z \otimes U^\geq_z \rightarrow \mathbb{C},
\]

the natural homomorphisms. The bilinear forms (3.12), (3.13) for \( R = \mathbb{A}_z \) induce bilinear forms
and (3.14) for $R = A_z$ induces a bilinear form

\[ \sigma_z : U_z \times V_z \to \mathbb{C}. \]  

Set

\[ J_z = \{ v \in V_z \mid \sigma_z(U_z, v) = \{0\} \}. \]

**Lemma 4.1.** $J_z$ is a Hopf ideal of $V_z$, and we have $J_z = V_z^{-} - V_z^+ (J_z \cap V_z^0)$.

**Proof.** It easily follows from Proposition 2.5 that $J_z$ is a two-sided ideal.

Set $J^0_z = J_z \cap V_z^0$ and $V_z' = V_z / V_z^{-} - V_z^+ J^0_z$. Since the multiplication of $V_z$ induces an isomorphism $V_z \simeq V_z^{-} \otimes V_z^+ \otimes V^0$, we have

\[ V_z' \simeq (V_z^{-} \otimes V_z^+ \otimes V^0) / (V_z^{-} \otimes V_z^+ \otimes J^0_z) \simeq V_z^{-} \otimes V_z^+ \otimes (V_z^0 / J^0_z). \]

Let $\sigma'_z : U_z \times V_z' \to \mathbb{C}$ be the bilinear form induced by $\sigma_z$. Then we see easily that $\{ v \in V_z' \mid \sigma'_z(U_z, v) = \{0\} \} = \{0\}$. Hence $J_z = V_z^{-} - V_z^+ J^0_z$. It remains to show $\Delta(J_z) \subset J_z \otimes V_z + V_z \otimes J_z$. By the above argument we are reduced to showing $\Delta(J^0_z) \subset J^0_z \otimes V_z^0 + V_z^0 \otimes J^0_z$. We can easily check this from the definition of $\sigma$. \qed

We define a Hopf algebra $\overline{V}_z$ by

\[ \overline{V}_z = V_z / J_z. \]

We denote by

\[ \overline{\sigma}_z : V_{A_z} \to \overline{V}_z \]

the canonical homomorphism. Let

\[ \overline{\sigma}_z : U_z \times \overline{V}_z \to \mathbb{C} \]

be the bilinear form induced by (4.8). Denote the images of $V_z^0$, $V_z^\pm$, $V_z^{\geq 0}$, $V_z^{\leq 0}$ under $V_z \to \overline{V}_z$ by $\overline{V}_z^0$, $\overline{V}_z^\pm$, $\overline{V}_z^{\geq 0}$, $\overline{V}_z^{\leq 0}$ respectively. Then the multiplication of $\overline{V}_z$ induces isomorphisms

\[ \overline{V}_z \simeq \overline{V}_z^0 \otimes \overline{V}_z^+, \overline{V}_z^\pm \otimes \overline{V}_z^0, \overline{V}_z^{\geq 0} \simeq \overline{V}_z^+ \otimes \overline{V}_z^0, \overline{V}_z^{\leq 0} \simeq \overline{V}_z^- \otimes \overline{V}_z^0. \]

**Lemma 4.2.** The bilinear form (4.8) is perfect in the sense that

\[ u \in U_z, \quad \overline{\sigma}_z(u, \overline{V}_z) = \{0\} \implies u = 0, \]

\[ v \in \overline{V}_z, \quad \overline{\sigma}_z(U_z, v) = \{0\} \implies v = 0. \]
Proof. (4.10) is clear from the definition. The proof of (4.9) is reduced to showing

\[ u \in U_z^0, \quad \sigma_z(u, V_z^0) = \{0\} \implies u = 0. \]

This can be shown by a direct computation. Details are omitted. □

Set

\[
I_z^0 = \eta^0(J_z \cap V_z^0) \subset U_z^{L,0},
I_z^{\geq 0} = U_z^{L,+}I_z^0 \subset U_z^{L,\geq 0}, \quad I_z^{\leq 0} = U_z^{L,-}I_z^0 \subset U_z^{L,\leq 0},
I_z = U_z^{L,-}U_z^{L,+}I_z^0 \subset U_z^L.
\]

Lemma 4.3. \( I_z^0, I_z^{\geq 0}, I_z^{\leq 0}, I_z \) are Hopf ideals of \( U_z^{L,0}, U_z^{L,\geq 0}, U_z^{L,\leq 0}, U_z^L \) respectively.

Proof. By Lemma 4.1 we see easily that \( J_z, V_z^+, J_z^0, V_z^- J_z^0 \) are Hopf ideals of \( V_z^0, V_z^{\geq 0}, V_z^{\leq 0} \) respectively. Since \( \eta^{\geq 0}|_{V_z^0} : V_z^0 \to U_z^{L,0}, \eta^{\geq 0} : V_z^{\geq 0} \to U_z^{L,\geq 0}, \eta^{\leq 0} : V_z^{\leq 0} \to U_z^{L,\leq 0} \) are isomorphisms of Hopf algebras, \( I_z^0, I_z^{\geq 0}, I_z^{\leq 0} \) are Hopf ideals of \( U_z^{L,0}, U_z^{L,\geq 0}, U_z^{L,\leq 0} \) respectively. Then the assertion for \( I_z \) follows from those for \( U_z^{L,0}, U_z^{L,\geq 0}, U_z^{L,\leq 0} \). □

We define a Hopf algebra \( \overline{U}_z^L \) by

\[ (4.11) \quad \overline{U}_z^L = U_z^L / I_z. \]

We denote by

\[ (4.12) \quad p_z : U_z^L \to \overline{U}_z^L \]

the canonical homomorphism. Denote the images of \( U_z^{L,0}, U_z^{L,\pm}, U_z^{L,\geq 0}, U_z^{L,\leq 0} \) under \( U_z^L \to \overline{U}_z^L \) by \( \overline{U}_z^{L,0}, \overline{U}_z^{L,\pm}, \overline{U}_z^{L,\geq 0}, \overline{U}_z^{L,\leq 0} \) respectively. We also denote by

\[ (4.13) \quad \overline{\eta}_z^{\geq 0} : \overline{V}_z^{\geq 0} \to \overline{U}_z^{L,\geq 0}, \quad \overline{\eta}_z^{\leq 0} : \overline{V}_z^{\leq 0} \to \overline{U}_z^{L,\leq 0} \]

the Hopf algebra isomorphisms induced by \( \eta^{\geq 0} \) and \( \eta^{\leq 0} \). The bilinear forms \( (4.3) \) induce

\[ (4.14) \quad \overline{\tau}_z^{0,L} : \overline{U}_z^{\geq 0} \times \overline{U}_z^{L,\leq 0} \to \mathbb{C}, \quad \overline{\tau}_z^{L,0} : \overline{U}_z^{L,\geq 0} \times \overline{U}_z^{\leq 0} \to \mathbb{C}. \]
5 Specialization to 1

For an algebraic groups $S$ over $\mathbb{C}$ with Lie algebra $\mathfrak{s}$ we will identify the coordinate algebra $\mathbb{C}[S]$ of $S$ with a subspace of the dual space $U(\mathfrak{s})^*$ of the enveloping algebra $U(\mathfrak{s})$ by the canonical Hopf paring

$$\mathbb{C}[S] \otimes U(\mathfrak{s}) \to \mathbb{C}.$$ 

We see easily that $J_1$ is generated by the elements $p_1(Z_\lambda) \in V_1$ for $\lambda \in Q$. From this we see easily the following.

**Lemma 5.1.** (i) We have an isomorphism $\overline{V}_1 \cong U(\mathfrak{m})$ of Hopf algebras satisfying

$$\overline{p}_1(X_i) \leftrightarrow x_i, \quad \overline{p}_1(Y_i) \leftrightarrow y_i,$$

$$\overline{p}_1 \left( \left[ Z_i \right] \right) \leftrightarrow \left[ \frac{t^m}{m!} \right] := t(t - 1) \cdots (t - m + 1)/m!.$$ 

(ii) We have an isomorphism $\overline{U}_1 \cong U(\mathfrak{g})$ of Hopf algebras satisfying

$$\overline{p}_1^L(E_i) \leftrightarrow e_i, \quad \overline{p}_1^L(F_i) \leftrightarrow f_i,$$

$$\overline{p}_1^L \left( \left[ K_i \right] \right) \leftrightarrow \left[ \frac{h^m}{m!} \right] := h(h - 1) \cdots (h - m + 1)/m!.$$ 

In the rest of this paper we will occasionally identify $\overline{V}_1$ and $\overline{U}_1^L$ with $U(\mathfrak{m})$ and $U(\mathfrak{g})$ respectively.

In [4] De Concini-Procesi proved an isomorphism

$$U_1 \cong \mathbb{C}[M]$$

of Poisson Hopf algebras. They established (5.1) by giving a correspondence between generators of both sides and proving the compatibility after a lengthy calculation. Later Gavarini [5] gave a more natural approach to the isomorphism (5.1) using the Drinfeld paring. Namely we have the following.

**Theorem 5.2** (Gavarini [5]). The bilinear form

$$\sigma_1 : U_1 \times \overline{V}_1 \to \mathbb{C}$$

induces an an isomorphism

$$\Upsilon : U_1 \to \mathbb{C}[M] (\subset U(\mathfrak{m})^* \simeq \overline{V}_1^*)$$

of Hopf algebras.
Gavarini \([5]\) also proved that \((5.2)\) is an isomorphism of Poisson algebras. This point will be discussed later in Section \(7\) below.

For convenience of readers we give a more concrete description of \(\Upsilon\). Let us first give a description of \(\mathbb{C}[M]\) as a subspace of \(U(\mathfrak{m})^*\). The enveloping algebra \(U(\mathfrak{m}^\pm)\) has the direct sum decomposition

\[
U(\mathfrak{m}^\pm) = \bigoplus_{\beta \in Q^+} U(\mathfrak{m}^\pm)^*_{\pm \beta},
\]

where \(U(\mathfrak{m}^\pm)^*_{\pm \beta} = \{ x \in U(\mathfrak{m}^\pm) \mid [\iota(h), x] = \beta(h)x \ (h \in \mathfrak{h}) \}\) for \(\beta \in Q^+\). Then we have

\[
\mathbb{C}[M^\pm] = \bigoplus_{\beta \in Q^+} (U(\mathfrak{m}^\pm)^*_{\pm \beta})^* \subset U(\mathfrak{m}^\pm)^*.
\]

Moreover, we have

\[
\mathbb{C}[M^0] = \bigoplus_{\lambda \in Q} \mathbb{C} \chi_{\lambda} \subset U(\mathfrak{h})^*
\]

where \(\chi_{\lambda} : U(\mathfrak{m}^0) \to \mathbb{C}\) is the algebra homomorphism given by \(\chi_{\lambda}(\iota(h)) = \lambda(h) \ (h \in \mathfrak{h})\). The isomorphism

\[
M^- \times M^+ \times M^0 \simeq M^- \times M^+ \times M^0 \simeq M \quad ((g_-, g_+, g_0) \mapsto g_- g_+ g_0)
\]

of algebraic varieties induced by the product of the group \(M\) gives an identification

\[
(5.3) \quad \mathbb{C}[M^+] \otimes \mathbb{C}[M^-] \otimes \mathbb{C}[M^0] \simeq \mathbb{C}[M]
\]

of vector spaces. The multiplication of the algebra \(U(\mathfrak{m})\) induces an identification

\[
U(\mathfrak{m}^+) \otimes U(\mathfrak{m}^-) \otimes U(\mathfrak{m}^0) \simeq U(\mathfrak{m}).
\]

Then the canonical embedding \(\mathbb{C}[M] \subset U(\mathfrak{m})^*\) is given by

\[
\mathbb{C}[M] \simeq \mathbb{C}[M^+] \otimes \mathbb{C}[M^-] \otimes \mathbb{C}[M^0] \subset U(\mathfrak{m}^+)^* \otimes U(\mathfrak{m}^-)^* \otimes U(\mathfrak{m}^0)^*
\]

\[
\subset (U(\mathfrak{m}^+) \otimes U(\mathfrak{m}^-) \otimes U(\mathfrak{m}^0))^* = U(\mathfrak{m})^*.
\]

Let \(\tilde{\Upsilon} : U_1 \to U(\mathfrak{m})^* (\simeq \bar{\mathcal{V}}_1^*)\) be the linear map induced by \(\bar{\sigma}_1\). Then we have

\[
\tilde{\Upsilon}(u_+ u_0 S(u_-))(v_- v_+ v_0) = \Upsilon^+(u_+)(v_-) \cdot \Upsilon^0(u_0)(v_0) \cdot \Upsilon^-(u_-)(v_+)
\]

\[
(u_\pm \in U_1^\pm, u_0 \in U_1^0, v_\pm \in U(\mathfrak{m})^\pm, v_0 \in U(\mathfrak{m})^0).
\]
where $\Upsilon^\pm : U_1^\pm \to U(m^\mp)^*$ and $\Upsilon^0 : U_1^0 \to U(m^0)^*$ are given by
\[
\Upsilon^+(u_+(v)) = \pi^0_1(u_+ + \eta^0_1(u_+)), \\
\Upsilon^-(u_-(v)) = \pi^0_1(u_- + \eta^0_2(v + u_+)), \\
\Upsilon^0(K_\lambda) = \chi_\lambda.
\]

For $i \in I$ we define $a_i \in \mathbb{C}[M^-] \subset U(m^-)^*$, $b_i \in \mathbb{C}[M^+] \subset U(m^+)^*$ by
\[
\langle a_i, U(m^-)_\beta \rangle = 0 \quad (\beta \neq \alpha_i), \\
\langle b_i, U(m^+)_\beta \rangle = 0 \quad (\beta \neq \alpha_i), \\
\langle a_i, y_i \rangle = -1, \\
\langle b_i, x_i \rangle = 1.
\]

We identify $\mathbb{C}[M^\pm], \mathbb{C}[M^0]$ with subalgebras of $\mathbb{C}[M]$ via $\Upsilon^\pm$, and regard $a_i, b_i, \chi_\lambda (i \in I, \lambda \in Q)$ as elements of $\mathbb{C}[M]$. We see easily the following.

**Lemma 5.3.** Under the identification $\Upsilon^\pm$ of Theorem 5.2 we have
\[
\pi_1(A_i) \leftrightarrow a_i, \quad \pi_1(B_i) \leftrightarrow b_i\chi_{-\alpha_i}, \quad \pi_1(K_\lambda) \leftrightarrow \chi_\lambda \quad (i \in I, \lambda \in Q).
\]

### 6 Specialization to roots of 1

We fix a positive odd integer $\ell$. We assume that $\ell$ is prime to 3 if $g$ is of type $G_2$. We denote by $\zeta \in \mathbb{C}^\times$ a fixed primitive $\ell$-th root of 1.

**Remark 6.1.** Denote by $U^\mathcal{DK}_{\mathbb{Q}[q,q^{-1}]}$ the De Concini-Kac $\mathbb{Q}[q,q^{-1}]$-form of $U$ (see [2]). Namely, $U^\mathcal{DK}_{\mathbb{Q}[q,q^{-1}]}$ is the $\mathbb{Q}[q,q^{-1}]$-subalgebra of $U$ generated by $\{K_i, E_i, F_i \mid i \in I\}$. Then we have $U_\zeta \simeq \mathbb{C} \otimes_{\mathbb{Q}[q,q^{-1}]} U^\mathcal{DK}_{\mathbb{Q}[q,q^{-1}]}$, where the specialization $\mathbb{Q}[q,q^{-1}] \to \mathbb{C}$ is given by $q \mapsto \zeta$.

We denote by $\xi^\ell : U^L_\zeta \to U^L_1$ Lusztig's Frobenius morphism (see [8]). Namely, $\xi^\ell$ is an algebra homomorphism given by
\[
\xi^\ell(p^L_\zeta(E_i^{(n)})) = \begin{cases} p^L_1(E_i^{(n/\ell)}) & (\ell \mid n) \\ 0 & (\ell \nmid n) \end{cases},
\]
\[
\xi^\ell(p^L_\zeta(F_i^{(n)})) = \begin{cases} p^L_1(F_i^{(n/\ell)}) & (\ell \mid n) \\ 0 & (\ell \nmid n) \end{cases},
\]
\[
\xi^\ell(p^L_\zeta(K_i^{m/\ell})) = \begin{cases} p^L_1(K_i^{m/\ell}) & (\ell \mid m) \\ 0 & (\ell \nmid m) \end{cases},
\]
\[
\xi^\ell(p^L_\zeta(K_\lambda)) = p^L_1(K_\lambda) \quad (\lambda \in Q).
\]

It is a Hopf algebra homomorphism.
Lemma 6.2. We have $\tilde{\xi}_L(I_\zeta) \subset I_1$.

Proof. It is sufficient to show $\tilde{\xi}(I_\zeta^0) \subset I_1^0$. For $z \in \mathbb{C}^*$, $m = (m_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$, and $\varepsilon = (\varepsilon_i)_{i \in I} \in \{0, 1\}^I$ set

$$K_{m,\varepsilon}(z) = p_z^L\left(\prod_{i \in I} \left(K_i^{\varepsilon_i} \left[K_i^m\right]\right)\right) \in U_z^{L,0}.$$ 

Then any element $u$ of $U_z^{L,0}$ is uniquely written as a finite sum

$$u = \sum_{m,\varepsilon} c_{m,\varepsilon} K_{m,\varepsilon}(z) \quad (c_{m,\varepsilon} \in \mathbb{C}).$$

Then we have $u \in I^0$ if and only if

$$\sum_{m,\varepsilon} c_{m,\varepsilon} q_i^{(\lambda,\alpha_i^\vee)} \left(\lambda, \alpha_i^\vee\right) m_i_{q_i} \bigg|_{q=z} = 0 \quad (\forall \lambda \in Q).$$

Hence it is sufficient to show that

$$(6.5) \quad \sum_{m,\varepsilon} c_{m,\varepsilon} q_i^{(\lambda,\alpha_i^\vee)} \left(\lambda, \alpha_i^\vee\right) m_i_{q_i} \bigg|_{q=\zeta} = 0 \quad (\forall \lambda \in Q)$$

implies

$$(6.6) \quad \sum_{m,\varepsilon} c_{m,\varepsilon} \left(\mu, \alpha_i^\vee\right) m_i_{\ell m_i} = 0 \quad (\forall \mu \in Q).$$

Indeed $[6.6]$ follows by setting $\lambda = \ell \mu$ in $(6.5)$. \qed

We denote by

$$\xi^L : \mathcal{U}_\zeta^L \to \mathcal{U}_1^L$$

the Hopf algebra homomorphism induced by $\tilde{\xi}^L$.

Lemma 6.3. There exists a Hopf algebra homomorphism

$$\xi : \mathcal{V}_\zeta \to \mathcal{V}_1$$

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satisfying

\[
\xi(\bar{\pi}_\zeta(X^{(n)}_i)) = \begin{cases} 
\bar{\pi}_1(X^{(n/\ell)}_i) & (\ell | n) \\
0 & (\ell \not| n),
\end{cases}
\]

\[
\xi(\bar{\pi}_\zeta(Y^{(n)}_i)) = \begin{cases} 
\bar{\pi}_1(Y^{(n/\ell)}_i) & (\ell | n) \\
0 & (\ell \not| n),
\end{cases}
\]

\[
\xi\left(\bar{\pi}_\zeta\left(\left[\frac{Z_i}{m}\right]\right)\right) = \begin{cases} 
\bar{\pi}_1\left(\left[\frac{Z_i}{m/\ell}\right]\right) & (\ell | m) \\
0 & (\ell \not| m),
\end{cases}
\]

\[
\xi((\bar{\pi}_\zeta(Z)) = 1 \quad (\lambda \in Q).
\]

**Proof.** By the isomorphisms \(\tilde{V}_z^{\geq 0} \cong \tilde{U}_z^{\geq 0}, \tilde{V}_z^{\leq 0} \cong \tilde{U}_z^{\leq 0}\) for \(z \in \mathbb{C}^*\) induced by \(\eta^{\geq 0}, \eta^{\leq 0}\) we obtain Hopf algebra homomorphisms \(\xi^{\geq 0} : \tilde{V}_z^{\geq 0} \rightarrow \tilde{V}_1^{\geq 0}\) and \(\xi^{\leq 0} : \tilde{V}_z^{\leq 0} \rightarrow \tilde{V}_1^{\leq 0}\) corresponding to \(\xi^L|_{\tilde{U}_z^{\geq 0}}\) and \(\xi^L|_{\tilde{U}_z^{\leq 0}}\) respectively. By \([\tilde{V}_z^+, \tilde{V}_z^-] = 0\) we obtain the desired Hopf algebra homomorphism \(\xi : \tilde{V}_z \rightarrow \tilde{V}_1\) satisfying \(\xi|_{\tilde{V}_z^{\geq 0}} = \xi^{\geq 0}\) and \(\xi|_{\tilde{V}_z^{\leq 0}} = \xi^{\leq 0}\). \(\square\)

By \(^8\) and the construction of \(\xi\) we have the following.

**Lemma 6.4.** We have

\[
\xi(\bar{\pi}_\zeta(X^{(n)}_{\beta_k})) = \begin{cases} 
\bar{\pi}_1(X^{(n/\ell)}_{\beta_k}) & (\ell | n) \\
0 & (\ell \not| n),
\end{cases}
\]

\[
\xi(\bar{\pi}_\zeta(Y^{(n)}_{\beta_k})) = \begin{cases} 
\bar{\pi}_1(Y^{(n/\ell)}_{\beta_k}) & (\ell | n) \\
0 & (\ell \not| n),
\end{cases}
\]

**Proposition 6.5.** There exists a unique linear map

\[
t^\xi : U_1 \rightarrow U_\zeta
\]

satisfying

\[
\sigma_\zeta(t^\xi(u), v) = \sigma_1(u, \xi(v)) \quad (u \in U_1, \ v \in \tilde{V}_\zeta).
\]

It is an injective Hopf algebra homomorphism whose image is contained in the center of \(U_\zeta\).
Proof. By a direct computation the linear map \( t\xi : U_1 \to U_\zeta \) defined by

\[
(6.11) \quad t\xi(\pi_1(A^{\ell_1}_{\beta_1} \cdots A^{\ell_{r_1}}_{\beta_1} K_\Lambda S(B^{s_1}_{\beta_1} \cdots B^{s_{r_1}}_{\beta_1}))) = \pi_2(A^{\ell_2}_{\beta_2} \cdots A^{\ell_{r_2}}_{\beta_2} K_\Lambda S(B^{s_2}_{\beta_2} \cdots B^{s_{r_2}}_{\beta_2}))
\]

satisfies \((6.10)\). The uniqueness and the injectivity of \( t\xi \) follow from Lemma 4.12. \( t\xi \) is a homomorphism of coalgebras by Proposition 2.2.

Let us show that \( t\xi(u) \) is a central element for any \( u \in U_1 \). We may assume that \( u \in U_1^0 \) or \( u \in U_1^+ \) or \( u \in S(U_1^-) \). If \( u \in U_1^0 \), then \( t\xi(u) \) is a central element since it is a linear combination of the elements of the form \( K_\Lambda(\lambda \in Q) \). Assume \( u \in U_1^+ \). Let us show

\[
(6.12) \quad t\xi(u)x = x t\xi(u) \quad (x \in U_\zeta^+).
\]

It is sufficient to show \( \nu_\zeta t\xi(u)x, y = \nu_\zeta x t\xi(u), y \) for any \( y \in V_\zeta^- \). By

\[
\nu_\zeta (t\xi(u)x, y) = (\nu_\zeta \otimes \nu_\zeta)(x \otimes t\xi(u), \Delta(y)) = (\nu_\zeta \otimes \nu_\zeta)(x \otimes u, (1 \otimes \xi)(\Delta(y))),
\]

\( \nu_\zeta (x t\xi(u), y) = (\nu_\zeta \otimes \nu_\zeta)(x \otimes t\xi(u), \Delta'(y)) = (\nu_\zeta \otimes \nu_\zeta)(x \otimes u, (1 \otimes \xi)(\Delta'(y))) \),

it is sufficient to show \( (1 \otimes \xi)(\Delta(y)) = (1 \otimes \xi)(\Delta'(y)) \) for any \( y \in V_\zeta^- \). Here \( \Delta' \) is the opposite comultiplication. We may assume \( y = \bar{\nu}_\zeta Y_i^{(a_i)} \). Then we have

\[
\Delta(\bar{\nu}_\zeta Y_i^{(a_i)}) = \sum_{r=0}^n \zeta_i^{r(n-r)} \bar{\nu}_\zeta Y_i^{(r)} \otimes \bar{\nu}_\zeta Y_i^{(n-r)} Z_i^r,
\]

where \( \zeta_i = \zeta^{(a_i, a_i)/2} \), and hence

\[
(1 \otimes \xi)(\Delta(\bar{\nu}_\zeta Y_i^{(a_i)}))) = \sum_{r \geq 0, \ell \leq n} \zeta_i^{r \ell(n-r)} \bar{\nu}_\zeta Y_i^{(r)} \otimes \bar{\nu}_\zeta Y_i^{(n-r)} (1 \otimes \xi)(\Delta'(Y_i^{(a_i)})))
\]

(note that \( \bar{\nu}_\zeta Z_i^\ell = 1 \)). \((6.12)\) is proved. By Proposition 2.3 we have

\[
(6.13) \quad \tau_\zeta^{0,0} t\xi(x, y) = \varepsilon(x) \varepsilon(y) \quad (x \in U_1^{\geq 0}, y \in U_\zeta^{\leq 0}).
\]

Hence by Lemma 2.2 we obtain

\[
t\xi(u)y = \sum_{(u)_{x,y}} \tau_\zeta^{0,0}(t\xi(u_{(0)}), y_{(0)}) \tau_\zeta^{0,0}(t\xi(u_{(2)}), S(y_{(2)})) y_{(1)} t\xi(u_{(1)})
\]

\[
= \sum_{(u)_{x,y}} \varepsilon(u_{(0)}) \varepsilon(y_{(0)}) \varepsilon(u_{(2)}) \varepsilon(y_{(2)}) y_{(1)} t\xi(u_{(1)}) = y t\xi(u)
\]

for any \( y \in U_\zeta^{\leq 0} \). Therefore, \( t\xi(u) \) is a central element for any \( u \in U_1^+ \). Similarly, we can show that \( t\xi(u) \) is a central element for any \( u \in S(U_1^-) \). We have shown that the image of \( t\xi \) is contained in the center of \( U_\zeta \). It also follows from this and \((6.11)\) that \( t\xi \) is an algebra homomorphism.

\( \square \)
Remark 6.6. Some of the arguments in our proof of Proposition 6.5 is similar to those for the dual statement in Gavarini [5, Theorem 7.9].

We set

\[(6.14) \quad Z_\zeta = \text{Im}(^{t}\xi).\]

By Proposition 6.5 it is a Hopf subalgebra of \(U_\zeta\) contained in the center.

7 Poisson structures

By Theorem 5.2 and Proposition 6.5 we have isomorphisms

\[(7.1) \quad Z_\zeta \simeq U_1 \simeq \mathbb{C}[M]\]

due to Hopf algebras. They are in fact isomorphisms of Poisson Hopf algebras with respect to certain canonical Poisson structures (De Concini-Procesi [4]). In this section we will give an account of those Poisson structures.

We first recall standard facts on Poisson structures (see e.g. [4]). A commutative associative algebra \(R\) over a field \(K\) equipped with a bilinear map

\[\{ , \} : R \times R \to R\]

is called a Poisson algebra if it satisfies

(a) \(\{a, a\} = 0 \quad (a \in R),\)

(b) \(\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad (a, b, c \in R),\)

(c) \(\{a, bc\} = b\{a, c\} + \{a, b\}c \quad (a, b, c \in R).\)

A map \(F : R \to R'\) between Poisson algebras \(R, R'\) is called a homomorphism of Poisson algebras if it is a homomorphism of associative algebras and satisfies \(F(\{a_1, a_2\}) = \{F(a_1), F(a_2)\}\) for any \(a_1, a_2 \in R\). The tensor product \(R \otimes_K R'\) of two Poisson algebras \(R, R'\) over \(K\) is equipped with a canonical Poisson algebra structure given by

\[\begin{align*}
(a_1 \otimes b_1)(a_2 \otimes b_2) &= a_1a_2 \otimes b_1b_2, \\
\{a_1 \otimes b_1, a_2 \otimes b_2\} &= \{a_1, a_2\} \otimes b_1b_2 + a_1a_2 \otimes \{b_1, b_2\}
\end{align*}\]

for \(a_1, a_2 \in R, b_2, b_2 \in R'\). A commutative Hopf algebra \(R\) over a field \(K\) equipped with a bilinear map

\[\{ , \} : R \times R \to R\]

is called a Poisson algebra if it satisfies

(a) \(\{a, a\} = 0 \quad (a \in R),\)

(b) \(\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad (a, b, c \in R),\)

(c) \(\{a, bc\} = b\{a, c\} + \{a, b\}c \quad (a, b, c \in R).\)
is called a Poisson Hopf algebra if it is a Poisson algebra and the comultiplication $\Delta : R \to R \otimes_K R$ is a homomorphism of Poisson algebras (in this case the counit $\varepsilon : R \to K$ and the antipode $S : R \to R$ become automatically a homomorphism and an anti-homomorphism of Poisson algebras respectively).

A smooth affine algebraic variety $X$ over $\mathbb{C}$ is called a Poisson variety if we are given a bilinear map

$$\{ , \} : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}[X]$$

so that $\mathbb{C}[X]$ is a Poisson algebra. In this case $\{f, g\}(x)$ for $f, g \in \mathbb{C}[X]$ and $x \in X$ depends only on $df_x, dg_x$, and hence we have $\delta \in \Gamma(X, \wedge^2 \Theta_X)$ such that

$$\{f, g\}(x) = \delta_x(df_x, dg_x),$$

where $\Theta_X$ denotes the sheaf of vector fields. We call $\delta$ the Poisson tensor of the Poisson variety $X$.

A linear algebraic group $S$ over $\mathbb{C}$ is called a Poisson algebraic group if we are given a bilinear map

$$\{ , \} : \mathbb{C}[S] \times \mathbb{C}[S] \to \mathbb{C}[S]$$

so that $\mathbb{C}[S]$ is a Poisson Hopf algebra. Let $\delta$ be the Poisson tensor of $S$ as a Poisson variety, and define $\varepsilon : S \to \wedge^2 \mathfrak{s}$ by $(d\ell_g)(\varepsilon(g)) = \delta_g$ for $g \in S$. Here, $\mathfrak{s}$ is the Lie algebra of $S$ which is identified with the tangent space $T_eS$ at the identity element $e \in S$, and $\ell_g : S \to S$ is given by $x \mapsto gx$. By differentiating $\varepsilon$ at $e$ we obtain a linear map $\mathfrak{s} \to \wedge^2 \mathfrak{s}$. It induces an alternating bilinear map $[ , ] : \mathfrak{s}^* \times \mathfrak{s}^* \to \mathfrak{s}^*$. Then this $[,]$ gives a Lie algebra structure on $\mathfrak{s}^*$. Moreover, the following bracket product gives a Lie algebra structure on $\mathfrak{s} \oplus \mathfrak{s}^*$:

$$[(a, \varphi), (b, \psi)] = ([a, b] + \varphi b - \psi a, a\psi - b\varphi + [\varphi, \psi]).$$

Here, $\mathfrak{s} \times \mathfrak{s}^* \ni (a, \varphi) \to a\varphi \in \mathfrak{s}^*$ and $\mathfrak{s}^* \times \mathfrak{s} \ni (\varphi, a) \to \varphi a \in \mathfrak{s}$ are the coadjoint actions of $\mathfrak{s}$ and $\mathfrak{s}^*$ on $\mathfrak{s}^*$ and $\mathfrak{s}$ respectively. In other words $(\mathfrak{s} \oplus \mathfrak{s}^*, \mathfrak{s}^*)$ is a Manin triple with respect to the symmetric bilinear form on $\mathfrak{s} \oplus \mathfrak{s}^*$ given by $((a, \varphi), (b, \psi)) = \varphi(b) + \psi(a)$. We say that $(a, b, c)$ is a Manin triple with respect to a symmetric bilinear form $(, )$ on $a$ if

(a) $a$ is a Lie algebra,

(b) $(, )$ is $a$-invariant and non-degenerate,

(c) $b$ and $c$ are subalgebras of $a$ such that $a = b \oplus c$ as a vector spaces,
(d) \((b, b) = (c, c) = \{0\}\).

Conversely, if \((a, b, c)\) is a Manin triple and \(B\) is a linear algebraic group with Lie algebra \(b\), then we can associate a natural Poisson Hopf algebra structure on \(\mathbb{C}[B]\) by reversing the above process.

Now let us return to our original setting. Note that \(m\) is a subalgebra of \(g \oplus g\). Set
\[(7.2) \quad \mathfrak{k} = \{ (x, x) \mid x \in g \} \subset g \oplus g.\]

We have a natural isomorphism
\[(7.3) \quad \theta : g \to \mathfrak{k} \quad (\theta(x) = (x, x))\]
of Lie algebras. It is easily seen that \((g \oplus g, m, \mathfrak{k})\) is a Manin triple with respect to the symmetric bilinear form \(\tilde{\kappa}\) on \(g \oplus g\) given by
\[(7.4) \quad \tilde{\kappa}((x_1, y_1), (x_2, y_2)) = \kappa(x_1, x_2) - \kappa(y_1, y_2) \quad (x_1, x_2, y_1, y_2 \in g),\]
where
\[(7.5) \quad \kappa : g \times g \to \mathbb{C}\]
is the \(g\)-invariant symmetric bilinear form which induces the symmetric bilinear form \((1.1)\) on \(h^*\). It follows that \(\mathbb{C}[M]\) is endowed with a natural Poisson Hopf algebra structure.

**Lemma 7.1.** \(\mathbb{C}[M]\) is generated by \(\{a_i, b_i, \chi_\lambda \mid i \in I, \lambda \in Q\}\) as a Poisson algebra.

**Proof.** See De Concini-Procesi [4, Section 14.5].

On the other hand we have a natural Poisson Hopf algebra structure on \(U_1\) given by
\[(7.6) \quad \{\pi_1(a), \pi_1(b)\} = \pi_1 \left( \frac{[a, b]}{(q - q^{-1})} \right) \quad (a, b \in U_{\mathcal{A}})\]
(see De Concini-Procesi [4]).

The definition of the Poisson structure on \(Z_\zeta\) is more subtle. Let \(C_\zeta\) be the center of \(U_\zeta\). We have a Poisson algebra structure on \(C_\zeta\) given by
\[(7.7) \quad \{\pi_\zeta(a), \pi_\zeta(b)\} = \pi_\zeta \left( \frac{[a, b]}{\ell(q^\ell - q^{-\ell})} \right) \quad (a, b \in U_{\mathcal{A}_\zeta}, \pi_\zeta(a), \pi_\zeta(b) \in C_\zeta).\]
If \(Z_\zeta\) is closed under the Poisson bracket \((7.7)\), then this gives a Poisson Hopf algebra structure on \(Z_\zeta\) (see De Concini-Procesi [4]).
Theorem 7.2 (De Concini-Procesi [4]). $Z_{\zeta}$ is closed under the Poisson bracket (7.7). Moreover, the isomorphisms in (7.1) preserve Poisson structures.

Gavarini [5] also gave a natural proof of the fact that the isomorphism $U_1 \simeq \mathbb{C}[M]$ in (7.1) preserves the Poisson structures using his definition of the isomorphism in terms of the Drinfeld paring. In fact, he gave a proof of the statement dual to it concerning Poisson coalgebra structure of the dual objects, and deduced the above statement from it. In the rest of this section we give a direct proof of this statement.

Lemma 7.3. Let $i \in I$, $\gamma \in \mathbb{Q}^+$ and $b \in U_{1,-\gamma}$. Write

$$\Delta(b) = \sum_{\gamma_1, \gamma_2 \in \mathbb{Q}^+, \gamma_1 + \gamma_2 = \gamma} b_{\gamma_1, \gamma_2} (1 \otimes \pi_1(K_{-\gamma_1})) (b_{\gamma_1, \gamma_2} \in U_{1,-\gamma_1} \otimes U_{1,-\gamma_2}),$$

and define $b', b'' \in U_{1,-(\gamma-\alpha_i)}$ by

$$\begin{align*}
&b_{\alpha_i, \gamma-\alpha_i} = \pi_1(B_i) \otimes b', \\
&b_{\gamma-\alpha_i, \alpha_i} = b'' \otimes \pi_1(B_i).
\end{align*}$$

Then we have

$$\begin{align*}
\{\pi_1(A_i), b\} &= \frac{\langle \alpha_i, \alpha_i \rangle}{2} (b'' \pi_1(K_i) - b' \pi_1(K_i^{-1})), \\
\{\pi_{\zeta}(A_i^\ell), t \xi(b)\} &= \frac{\langle \alpha_i, \alpha_i \rangle}{2} (t \xi(b'') \pi_{\zeta}(K_i^\ell) - t \xi(b') \pi_{\zeta}(K_i^{-\ell})).
\end{align*}$$

Proof. Note that (7.9) can be regarded as a special case of (7.10) when $\ell = 1$. Hence we will only prove (7.10). We can write

$$\Delta_2(A_i^\ell) = A_i^\ell \otimes 1 \otimes 1 + K_i^\ell \otimes A_i^\ell \otimes 1 + K_i^\ell \otimes K_i^\ell \otimes A_i^\ell + (q^\ell - q^{-\ell}) \sum_j X_j \otimes X_j' \otimes X_j''$$

for some $X_j, X_j', X_j'' \in U_{k_{\zeta}}^\geq 0$. By $(\varepsilon \otimes 1 \otimes \varepsilon) \Delta_2(A_i^\ell) = A_i^\ell$ we have

$$\sum_j \varepsilon(X_j) \varepsilon(X_j'') X_j' = 0.$$
Take $B \in U_{h_{-\ell_{\gamma}}}^-$ such that $\pi_\zeta(B) = ^t\xi(b)$. Then we have
\[
A^\ell_i B
= \sum_{(B)_2} \tau(A^\ell_i, B(0))\tau(1, SB(2))B(1) + \sum_{(B)_2} \tau(K^\ell_i, B(0))\tau(1, SB(2))B(1)A^\ell_i
\]
\[+ \sum_{(B)_2} \tau(K^\ell_i, B(0))\tau(A^\ell_i, SB(2))B(1)K^\ell_i
\]
\[+ (q^\ell - q^{-\ell}) \sum_{(B)_2,j} \tau(X_j, B(0))\tau(X''_j, SB(2))B(1)X'_j,
\]
and hence
\[
\ell\{\pi_\zeta(A^\ell_i), ^t\xi(b)\}
= \pi_\zeta\left( \left( \sum_{(B)} \tau(A^\ell_i, B(0))B(1) + \sum_{(B)} \tau(A^\ell_i, SB(1))B(0)K^\ell_i \right)/ (q^\ell - q^{-\ell}) \right)
\]
\[+ \pi_\zeta\left( \sum_{(B)_2,j} \tau(X_j, B(0))\tau(X''_j, SB(2))B(1)X'_j \right).\]

Note
\[
\sum_{(B)_2, j} \pi_\zeta(B(0)) \otimes \pi_\zeta(B(1)) \otimes \pi_\zeta(B(2)) = (\pi_\zeta \otimes \pi_\zeta \otimes \pi_\zeta)(\Delta_2(B))
\]
\[= \Delta_2(\pi_\zeta(B)) = \Delta_2(^t\xi(b)) = (^t\xi \otimes ^t\xi \otimes ^t\xi)(\Delta_2(b)).\]

Hence by (6.13) and (7.11) we have
\[
\pi_\zeta\left( \sum_{(B)_2,j} \tau(X_j, B(0))\tau(X''_j, SB(2))B(1)X'_j \right)
= \sum_{(B)_2,j} \tau_{\pi_\zeta(X_j), ^t\xi(b(0))} \tau_{\pi_\zeta(X_j'), ^t\xi(b(1))} \tau_{\pi_\zeta(X''_j), ^t\xi(b(2))} \pi_\zeta(B(1)) \pi_\zeta(X'_j)
\]
\[= \sum_{(B)_2, j} \tau_{\pi_\zeta(X_j), ^t\xi(b(0))} \tau_{\pi_\zeta(X''_j), ^t\xi(b(1))} \pi_\zeta(X'_j)
\]
\[= ^t\xi(b)\pi_\zeta\left( \sum_j \varepsilon(X_j)\varepsilon(X''_j)X'_j \right) = 0.
\]

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Write
\[ \Delta(B) = \sum_{\gamma_1, \gamma_2 \in Q^+, \gamma_1 + \gamma_2 = \ell \gamma} B_{\gamma_1, \gamma_2}(1 \otimes K_{-\gamma_1}) \quad (B_{\gamma_1, \gamma_2} \in U_{A_{-\gamma_1}} \otimes U_{A_{-\gamma_2}}), \]
and define \( B', B'' \in U_{A_{-\ell(\gamma - \alpha_i)}} \) by
\[ B_{\ell \alpha_i, \ell(\gamma - \alpha_i)} = B'_i \otimes B', \quad B_{\ell(\gamma - \alpha_i), \ell \alpha_i} = B'' \otimes B'_i. \]
Then by
\[ (\ell \xi \otimes \ell \xi)(\Delta(b)) = \Delta(\ell \xi(b)) = \Delta(\pi \xi(B)) = (\pi \xi \otimes \pi \xi)(\Delta(B)) \]
we have \( \pi \xi(B') = \ell \xi(b') \) and \( \pi \xi(B'') = \ell \xi(b''). \) Hence we obtain
\[
\begin{align*}
\ell \{ \pi \xi(A_{i}^{(l)}), y \xi(b) \} & = \pi \xi \left( \left( \tau(A_{i}^{(l)}, B_{i}^{(l)}) B_{i}^{(l)} K_{-\ell}^{(l)} + \tau(A_{i}^{(l)}, S(B_{i}^{(l)} K_{-\ell(\gamma - \alpha_i)})) B''_{i} K_{i}^{(l)} \right) / (q^{\ell} - q^{-\ell}) \right) \\
& = \ell(\alpha_i, \alpha_i)(\pi \xi(B'') \pi \xi(K_{i}^{(l)}) - \pi \xi(B') \pi \xi(K_{-\ell}^{(l)}))/2 \\
& = \ell(\alpha_i, \alpha_i)(\ell \xi(b'') \pi \xi(K_{i}^{(l)}) - \ell \xi(b') \pi \xi(K_{-\ell}^{(l)})). \\
\end{align*}
\]
\[ \boxdot \]

For \( i \in I \) we set
\[ \zeta_{i} = \zeta^{(\alpha_i, \alpha_i)}/2. \]

For \( F \in U_{A_{\zeta_i} - \gamma} \) \((\gamma \in Q^+)\) define \( \varphi^{i}_{r,s}(F) \in U_{A_{\zeta_i} - (\gamma - (r + s) \alpha_i)} \) \((r, s \geq 0, i \in I)\) by
\[ \Delta_2(F) = \sum_{r, s} F_{i}^{(r)} \otimes \varphi^{i}_{r,s}(F) K_{i}^{(r)} \otimes F_{i}^{(s)} K_{-\gamma + s \alpha_i} \]
\[ + \bigoplus_{(\gamma_1, \gamma_2, \gamma_3) \in \Xi} U_{A_{\zeta_i} - \gamma_1} \otimes U_{A_{\zeta_i} - \gamma_2} K_{-\gamma_1} \otimes U_{A_{\zeta_i} - \gamma_3} K_{-\gamma_2} \]
where \( \Xi \) consists of \((\gamma_1, \gamma_2, \gamma_3) \in (Q^+)^3\) such that \( \gamma_1 + \gamma_2 + \gamma_3 = \gamma \) and \((\gamma_1, \gamma_3) \notin \mathbb{Z}_{\geq 0} \alpha_i \times \mathbb{Z}_{\geq 0} \alpha_i \).

**Lemma 7.4.** Let \( i \in I \). For \( F \in U_{A_{\zeta_i} - \gamma} \) \((\gamma \in Q^+)\) we have
\[ \zeta^{L}(\mathcal{P}_{\zeta}^{L}(\varphi_{i}^{i}(F))) = \zeta^{rs}_{i} \zeta^{L}(\mathcal{P}_{\zeta}^{L}(\varphi_{i + s,0}^{i}(F))). \]
Proof. For \( X \in U_{\rho}^{+} \) we have

\[
\tau(A_{i}^{\ast}XA_{i}^{\ast}, F) = (\tau \otimes \tau \otimes \tau)(A_{i}^{\ast} \otimes X \otimes A_{i}^{\ast}, \Delta_{2}(F)) = \tau(A_{i}^{\ast}, F_{i}^{(r)}) \tau(X, \varphi_{r,s}(F)) \tau(A_{i}^{\ast}, F_{i}^{(s)}) = (-1)^{r+s} \left( \frac{r(r-1)+s(s-1)}{2} \right) \tau(X, \varphi_{r,s}(F)),
\]

and hence

\[
\tau_{\xi}^{\ast}(x, \pi_{\xi}^{\ast}((\varphi_{r,s}(F))) = (-1)^{r+s} \zeta \left( \frac{r(r-1)+s(s-1)}{2} \right) \tau_{\xi}(\pi_{\xi}(A_{i}))^{x} \pi_{\xi}(A_{i})^{r}, \pi_{\xi}^{\ast}(F))
\]

for any \( x \in U_{\rho}^{+} \). It follows that

\[
\tau_{\xi}^{\ast}(x', \pi_{\xi}^{\ast}((\varphi_{r,s}(F))) = \tau_{\xi}^{\ast}(\pi_{\xi}(A_{i})^{x'} \pi_{\xi}(A_{i})^{r}, \pi_{\xi}^{\ast}(F))
\]

for any \( x' \in U_{\rho}^{+} \). Here we have used the fact that the image of \( t \xi \) is contained in the center. \( \square \)

**Lemma 7.5.** Let \( i \in I \), \( \gamma \in Q^{+} \) and \( b \in U_{1,\gamma}^{+} \).

(i) Let \( f \in U_{1,-(\gamma+\alpha_{i})}^{L,-} \). Write

\[
\Delta(f) = \sum_{\gamma_{1}, \gamma_{2} \in Q^{+}, \gamma_{1}+\gamma_{2}=\gamma+\alpha_{i}} f_{\gamma_{1}, \gamma_{2}} \quad (f_{\gamma_{1}, \gamma_{2}} \in U_{1,-\gamma_{1}}^{L,-} \otimes U_{1,-\gamma_{2}}^{L,-}),
\]

and define \( f' \in U_{1,-\gamma}^{L,-} \) by \( f_{\gamma_{1}, \gamma_{2}} = f' \otimes f_{i} \). Then we have

\[
(7.14) \quad \tau_{\xi}^{\ast}(\pi_{\gamma}(A_{i}), b, f) = \tau_{\xi}^{\ast}(b, \frac{\alpha_{i}, \alpha_{i}}{2} [f, e_{i}] - \frac{\alpha_{i}^{\gamma}, \gamma}{2} f').
\]

(ii) Let \( f \in U_{1,-(\gamma+\alpha_{i})}^{L,-} \). Write

\[
\Delta(f) = \sum_{\gamma_{1}, \gamma_{2} \in Q^{+}, \gamma_{1}+\gamma_{2}=\gamma+\alpha_{i}} f_{\gamma_{1}, \gamma_{2}} (1 \otimes \pi_{\xi}^{\ast}(K-\gamma_{1}))
\]

and define \( f' \in U_{1,-\gamma}^{L,-} \) by \( f_{\gamma_{1}, \gamma_{2}} = f' \otimes \pi_{\xi}^{\ast}(F_{i}^{(\ell)}) \). Then we have

\[
(7.15) \quad \tau_{\xi}^{\ast}(\pi_{\gamma}(A_{i}), t \xi(b), f) = \tau_{\xi}^{\ast}(b, \frac{\alpha_{i}, \alpha_{i}}{2} [\xi^{\ell}(f), e_{i}] - \frac{\alpha_{i}^{\gamma}, \gamma}{2} \xi^{\ell}(f')).
\]

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Hence we will only prove (ii). Take $F \in U^L_{\kappa,\gamma}$ such that $\mathcal{P}_F = f$. By

\[
\Delta_2(E_i^{(t)}) = \sum_{r+s+t=\ell} q_i^{r+s+t} E_i^{(r)} K_i^{s+t} \otimes E_i^{(t)}
\]

and Lemma 2.2 we have

\[
E_i^{(t)} F = \sum_{r+s+t=\ell} \sum_{(F)_2} q_i^{r+s+t} \tau(E_i^{(r)}, F_0) \tau(E_i^{(t)}, SF_2) \tau(F_1) E_i^{(s)} K_i^{t}.
\]

By $E_i^{(t)} F \in U^L_{\kappa}$ we have $X_s \in U^L_{\kappa}$ for $0 \leq s \leq \ell$. Note that $X_\ell = F$ and

\[
X_0 = \sum_{r+t=\ell} X_{0,r,t},
\]

where

\[
X_{0,r,t} = (-1)^r q_i^{(1-\ell)(\alpha_i',\gamma)} \tau(E_i^{(r)}, F_i^{(r)}) \tau(E_i^{(t)}, F_i^{(t)}) \varphi_{r,t}(F) K_i^{r+t}.
\]

Note also that

\[
\tau(E_i^{(m)}, F_i^{(m)}) = \frac{(-1)^m q_i^m m(m-1) / [m] q_i^{-1}}{(m)!(q_i - q_i^{-1})^m}.
\]

Hence $X_{0,r,t} \in U^L_{\kappa}$ for $r + t = \ell$, $r \neq 0$, $t \neq 0$. By $X_0 \in U^L_{\kappa}$ we also have $X_{0,0,t} \in U^L_{\kappa}$. From this we obtain

\[
\varphi_{\ell,0}(F) K_i^{-\ell} - q_i^{(1-\ell)(\alpha_i',\gamma)} \varphi_{0,\ell}(F) K_i^{\ell} \in (q^\ell - q^{-\ell}) U^L_{\kappa},
\]

or equivalently,

\[
\varphi_{\ell,0}(F) K_i^{-\ell} - q_i^{\ell} \varphi_{0,\ell}(F) K_i^{\ell} \in (q^\ell - q^{-\ell}) U^L_{\kappa}.
\]
Let us show

\[ (7.16) \quad \xi^L(\overline{\mathcal{L}}^L_p(\sum_{r+t=\ell, r>0, t>0} X_{0,r,t})) = \frac{\ell - 1}{2\ell} \xi^L(\overline{\mathcal{L}}^L_\zeta(\varphi^i_{\ell,0}(F))) \]

By Lemma 7.4 we have

\[ \xi^L(\overline{\mathcal{L}}^L_p(\sum_{r+t=\ell, r>0, t>0} X_{0,r,t})) = \sum_{t=1}^{\ell-1} (-1)^t \zeta^t_i \frac{1}{[\ell-t]_{\zeta_i} ![t]_{\zeta_i}} \xi^L(\overline{\mathcal{L}}^L_\zeta(\varphi^i_{\ell,0}(F))) \]

and hence it is sufficient to show

\[ \sum_{t=1}^{\ell-1} \frac{(-1)^t \zeta^t_i}{[\ell-t]_{\zeta_i} ![t]_{\zeta_i}} = (\zeta_i - \zeta_i^{-1})^{1 - 2\ell} \frac{1}{2\ell}. \]

Indeed we have

\[ 2 \sum_{t=1}^{\ell-1} \frac{(-1)^t \zeta^t_i}{[\ell-t]_{\zeta_i} ![t]_{\zeta_i}} = \sum_{t=1}^{\ell-1} \frac{(-1)^t \zeta^t_i}{[\ell-t]_{\zeta_i} ![t]_{\zeta_i}} + \sum_{t=1}^{\ell-1} \frac{(-1)^t \zeta^{t-t}_i}{[\ell-t]_{\zeta_i} ![t]_{\zeta_i}} = \sum_{t=1}^{\ell-1} \frac{(-1)^t (\zeta_i - \zeta_i^{-1})^{t-1} \zeta^{t-1}_i}{[\ell-1]_{\zeta_i} ![t-1]_{\zeta_i}} \]

\[ = (\zeta_i - \zeta_i^{-1}) \frac{1}{\ell} \left( 1 - \sum_{s=0}^{\ell-1} (-1)^s \left[ \begin{array}{c} \ell - 1 \\ s \end{array} \right]_{\zeta_i} \right) = (\zeta_i - \zeta_i^{-1}) \frac{1}{\ell} \left( 1 - \prod_{j=0}^{\ell-2} (\zeta_i^{\ell-2-2j} - 1) \right) \]

\[ = (\zeta_i - \zeta_i^{-1}) \frac{1}{\ell} \left( 1 - \prod_{j=1}^{\ell-1} (1 - \zeta_i^{-2j}) \right) = (\zeta_i - \zeta_i^{-1})^{1 - \ell}. \]

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(1.16) is proved. On the other hand we have

\[ \xi^L(\mathcal{P}_\zeta^L(X_{0,\ell,0} + X_{0,0,\ell})) \]
\[ = \xi^L \left( \mathcal{P}_\zeta^L \left( -q_i^{(\ell-1)/2} \frac{\varphi_{i,0}^i(F) K_i^{-\ell} - q_i^{\ell(1-\ell - (\alpha_i^\gamma, \gamma))} \varphi_{0,\ell}^i(F) K_i^\ell}{[\ell]_q! (q_i - q_i^{-1})^{\ell}} \right) \right) \]
\[ = - \xi^L \left( \mathcal{P}_\zeta^L \left( \frac{\varphi_{i,0}^i(F) K_i^{-\ell} - \varphi_{0,\ell}^i(F) K_i^\ell}{[\ell]_q! (q_i - q_i^{-1})^{\ell}} \right) \right) \]
\[ - \xi^L \left( \mathcal{P}_\zeta^L \left( \frac{(1 - q_i^{\ell(1-\ell - (\alpha_i^\gamma, \gamma))}) \varphi_{0,\ell}^i(F) K_i^\ell}{[\ell]_q! (q_i - q_i^{-1})^{\ell}} \right) \right) \]
\[ = - \xi^L \left( \mathcal{P}_\zeta^L \left( \frac{\varphi_{i,0}^i(F) K_i^{-\ell} - \varphi_{0,\ell}^i(F) K_i^\ell}{\ell(q_i^\ell - q_i^{-\ell})} \right) \right) \]
\[ + \frac{1 - \ell - (\alpha_i^\gamma, \gamma)}{2\ell} \xi^L \left( \mathcal{P}_\zeta^L (\varphi_{0,\ell}^i(F)) \right) . \]

and hence

\[ [\xi^L(f), e_i] = \xi^L(\mathcal{P}_\zeta^L([F, E_i^{(\ell)}])) = - \sum_{t=0}^\ell \xi^L(\mathcal{P}_\zeta^L(X_{0,\ell-t,\ell})) \]
\[ = \xi^L \left( \mathcal{P}_\zeta^L \left( \frac{\varphi_{i,0}^i(F) K_i^{-\ell} - \varphi_{0,\ell}^i(F) K_i^\ell}{\ell(q_i^\ell - q_i^{-\ell})} \right) \right) + \frac{(\alpha_i^\gamma, \gamma)}{2} \xi^L \left( \mathcal{P}_\zeta^L (\varphi_{0,\ell}^i(F)) \right) . \]

Write

\[ \Delta(F) = \sum_{\gamma_1, \gamma_2 \in Q^+, \gamma_1 + \gamma_2 = \ell(\gamma + \alpha_i)} F_{\gamma_1, \gamma_2} (1 \otimes K_{-\gamma_1}) \]
\[ (F_{\gamma_1, \gamma_2} \in U_{\alpha_i - \gamma_1}^{L_+} \otimes U_{\alpha_i - \gamma_2}^{L_+}) . \]

Then we have

\[ F_{\ell_0, \ell} = F_i^{(\ell)} \otimes \varphi_{i,0}^i(F), \quad F_{\ell, \ell_0} = \varphi_{0,\ell}^i(F) \otimes F_i^{(\ell)} . \]
Take $B \in U_{\kappa,\ell}^*$ such that $\pi_\zeta(B) = \xi(b)$. Then we have

$$\tau^\theta_L(\{\pi_\zeta(A_i^\ell, \xi(b)), f\})$$

$$= (\tau(A_i^\ell B - BA_i^\ell, F)/\ell(q^\ell - q^{-\ell}))|_{q = \zeta}$$

$$= ((\tau \otimes \tau)(B \otimes A_i^\ell - A_i^\ell \otimes B, \Delta(F))/\ell(q^\ell - q^{-\ell}))|_{q = \zeta}$$

$$= (-q_i^{(\ell-1)/2}(\phi_{\ell,0}(F) - \phi_{\ell,0}(F))/\ell(q^\ell - q^{-\ell}))|_{q = \zeta}$$

$$= (\tau(B, \phi_{\ell,0}(F) - \phi_{\ell,0}(F))/\ell(q^\ell - q^{-\ell}))|_{q = \zeta}$$

$$= (\tau(B, \phi_{\ell,0}(F)K_i^{-\ell} - \phi_{\ell,0}(F)K_i^{\ell})/\ell(q^\ell - q^{-\ell}))|_{q = \zeta}$$

$$= \tau^\theta_L \left(\xi(b), \phi_L \left(\frac{\phi_{\ell,0}(F)K_i^{-\ell} - \phi_{\ell,0}(F)K_i^{\ell}}{\ell(q^\ell - q^{-\ell})}\right)\right)$$

$$= \frac{\alpha_i, \alpha_i}{2} \tau^\theta_L \left(\xi^L \left(\frac{\phi_{\ell,0}(F)K_i^{-\ell} - \phi_{\ell,0}(F)K_i^{\ell}}{\ell(q^\ell - q^{-\ell})}\right)\right)$$

$$= \frac{\alpha_i, \alpha_i}{2} \tau^\theta_L \left(\xi^L(\phi_{\ell,0}(F))\right)$$

$$= \frac{\alpha_i, \alpha_i}{2} \tau^\theta_L \left(\xi^L(\phi_{\ell,0}(F))\right)$$

We are done.

Now let us finish the proof of Theorem 7.2. Regarding $\mathbb{C}[M]$ as a subspace of $U(m)^*$ the Poisson bracket of $\mathbb{C}[M]$ is uniquely determined by the following properties.

(7.17) $\langle\{\phi, \psi\}, 1 \rangle = 0 \quad (\phi, \psi \in \mathbb{C}[M])$,

(7.18) $\langle\{\phi, \psi\}, a \rangle = \tilde{\kappa}[(d\phi)_e, (d\psi)_e], a \quad (\phi, \psi \in \mathbb{C}[M], a \in m)$,

(7.19) $\langle\{\phi, \psi\}, w \rangle$

$$= \sum_{(\phi, \psi)} \langle\phi(0)\psi(0) \otimes \{\phi(1), \psi(1)\} + \{\phi(0), \psi(0)\} \otimes \phi(1)\psi(1), u \otimes v \rangle$$

$$(\phi, \psi \in \mathbb{C}[M], u, v \in U(m)).$$

In (7.18) we have identified the cotangent space $m^*$ of $M$ at the identity element $e$ with the Lie algebra $\mathfrak{k}$ via $\tilde{\kappa}$. Define $\text{res} : \mathbb{C}[M] \rightarrow \mathfrak{k}$ as the composite of

$$\mathbb{C}[M] \hookrightarrow U(m)^* \rightarrow m^* \simeq \mathfrak{k},$$

where $U(m)^* \rightarrow m^*$ is induced by the canonical embedding $m \hookrightarrow U(m)$ and $m^* \simeq \mathfrak{k}$ is given by $\tilde{\kappa}$. Then we have $(d\phi)_e = \text{res}(\phi)$ for any $\phi \in m$. 

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\( \mathbb{C}[M] \). Therefore, the Poisson bracket of \( \mathbb{C}[M] \) is uniquely characterized as the bilinear map

\[
\{ , \} : \mathbb{C}[M] \times \mathbb{C}[M] \to \mathbb{C}[M]
\]

satisfying

(a) \( \mathbb{C}[M] \) becomes a Poisson Hopf algebra,
(b) \( \text{res}(\{ \varphi, \psi \}) = [\text{res}(\varphi), \text{res}(\psi)] \quad (\varphi, \psi \in \mathbb{C}[M]). \)

Assume that we are given a bilinear map (7.20) satisfying the condition (a). Set

\[ D = \{ a_i, b_i, \chi_\lambda \mid i \in I, \lambda \in Q \} \subset \mathbb{C}[M]. \]

By Lemma 7.1 and the general formula

\[
\text{res}(\varphi \psi) = \varepsilon(\varphi) \text{res}(\psi) + \varepsilon(\psi) \text{res}(\varphi) \quad (\varphi, \psi \in \mathbb{C}[M])
\]

we see that the condition (b) is satisfied if and only if

(c) \( \text{res}(\{ \varphi, \psi \}) = [\text{res}(\varphi), \text{res}(\psi)] \quad (\varphi \in D, \psi \in \mathbb{C}[M]). \)

Note that \( \mathbb{C}[M] \) is generated as an algebra by \( \mathbb{C}[M^+] \), \( \mathbb{C}[M^-] \) and \( \mathbb{C}[M^0] \), where \( \mathbb{C}[M^0], \mathbb{C}[M^\pm] \) are regarded as subalgebras of \( \mathbb{C}[M] \) by (5.3). Hence by (7.21) we see that the condition (c) is satisfied if and only if

(d) \( \text{res}(\{ \varphi, \psi \}) = [\text{res}(\varphi), \text{res}(\psi)] \quad (\varphi \in D, \psi \in \mathbb{C}[M^+] \cup \mathbb{C}[M^-] \cup \mathbb{C}[M^0]). \)

Let us show that the isomorphism \( \Upsilon : U_1 \to \mathbb{C}[M] \) preserves the Poisson structure. Set \( r = \text{res} \circ \Upsilon : U_1 \to \mathfrak{k} \), and

\[
D' = \{ \pi_1(A_i), \pi_1(B_i), \pi_1(K_\lambda) \mid i \in I, \lambda \in Q \} \subset U_1.
\]

By the above argument it is sufficient to show

\[
r(\{ a, b \}) = [r(a), r(b)] \quad (a \in D', b \in U_1^+ \cup U_1^- \cup U_1^0),
\]

where the Poisson bracket \( \{ , \} \) of \( U_1 \) is given by (7.4). We have

\[
r(\pi_1(A_i)) = \frac{(\alpha_i, \alpha_i)}{2} \theta(e_i), \quad r(\pi_1(B_i)) = -\frac{(\alpha_i, \alpha_i)}{2} \theta(f_i),
\]

\[
r(\pi_1(K_\lambda)) = \frac{1}{2} \theta(h_\lambda),
\]

where \( h_\lambda \in \mathfrak{h} \) is such that \( \kappa(h_\lambda, h) = \lambda(h) \) for any \( h \in \mathfrak{h} \).
Let us show (7.23) for \( a = \pi_1(K_\lambda) \). If \( b = \pi_1(B) \) for \( B \in U_\lambda^0 \), then we have \( \{a, b\} = 0 \) by \([U_0^0, U_0^\perp]\) = 0, and \([r(a), r(b)] = 0 \) by \( r(a), r(b) \in \theta(h) \). Assume \( b = \pi_1(B) \) for \( B \in U_{\lambda, \pm, \gamma}^\perp \). Then we have

\[
\{a, b\} = \pi_1((K_\lambda B - BK_\lambda)/(q - q^{-1})) = \pi_1((q^{\pm(\lambda, \gamma)} - 1)BK_\lambda/(q - q^{-1})) = \pm (\lambda, \gamma)b/2.
\]

On the other hand by \( r(b) \in \theta g_{\pm, \gamma} \) we have \([r(a), r(b)] = \pm (\lambda, \gamma)r(b)/2 \). Hence (7.23) is proved for \( a = \pi_1(K_\lambda) \). Note that the above argument also give the proof for the case \( b \in \pi_1(B_i) \), and \( b \in U_{1^-}^\perp \).

Let us consider the case \( a = \pi_1(A_i) \) and \( b \in U_{1^-}^\perp \) for \( \gamma \in Q^+ \). If \( \gamma - \alpha_i \notin \Delta \cup \{0\} \), then both sides of (7.23) are zero by \([r(a), r(b)] = (\lambda, \gamma)r(b)/2 \) \( \in \theta(g_{\gamma - \alpha_i}) \). In the case \( \gamma = 0 \) we can easily check that both sides of (7.23) coincide with zero. In the case \( \gamma = \alpha_i \), we can also easily check that the both sides of (7.23) coincide with \((\alpha_i, \alpha_i)\theta(h_{\alpha_i})/2 \). Therefore, we may assume that \( \gamma - \alpha_i \in \Delta^+ \). In this case it is sufficient to show

\[
\kappa(r(\{a, b\}, (x, 0))) = \kappa([r(a), r(b)], (x, 0))
\]

for any \( x \in g_{\gamma - \alpha_i} \). By Lemma 7.3 we have

\[
\{a, b\} = \frac{\sigma_i}{2}(b''\pi_1(K^i) - b'\pi_1(K^{-1}_i)),
\]

where \( b', b'' \) are as in (7.8), and hence

\[
\kappa(r(\{a, b\}, (x, 0))) = \frac{\sigma_i}{2}\kappa(r(b''\pi_1(K^i) - b'\pi_1(K^{-1}_i)), (x, 0)) = \frac{\sigma_i}{2}(b''\pi_1(K^i) - b'\pi_1(K^{-1}_i), (x, 0)) = \frac{\sigma_i}{2}(b'' - b', (x, 0)).
\]

On the other hand we have

\[
\kappa([r(a), r(b)], (x, 0)) = -\kappa(r(b), [r(a), (x, 0)]) = -\frac{1}{2}\sigma_i(b, [x_i, (x, 0)]/2 = -\frac{1}{2}\sigma_i(b', (x, 0)) - x_i)/2
\]

\[
= -\frac{1}{2}\sigma_i((\pi_i(B^i), x_i)\sigma_i(b''\pi_1(K^{-1}_i), (x, 0))) - \sigma_i(b'', (x, 0))\sigma_i(\pi_1(B_iK_{-(\gamma - \alpha_i)}), x_i)/2
\]

\[
= -\alpha_i(\alpha_i)\sigma_i(b'' - b', (x, 0))/2.
\]
is proved in the case $a = \pi_i(A_i)$ and $b \in U_i^-$. 

Let us next consider the case $a = \pi_1(A_i)$ and $b \in U_i^\gamma$. We have $r(a) \in \theta(g_{\alpha_i})$, $r(b) \in \theta(g_\gamma)$, $r\{a, b\} \in \theta(g_{\gamma + \alpha_i})$, and hence we may assume that $\gamma + \alpha_i \in \Delta^+$. If $\gamma = 0$, then the both sides of \eqref{eq:7.23} is zero. Hence we may also assume that $\gamma \in \Delta^+$. Then it is sufficient to show

$$\tilde{\kappa}(r\{a, b\}, (0, y)) = \tilde{\kappa}([r(a), r(b)], (0, y))$$

for $y \in g_{-(\gamma + \alpha_i)}$. By Lemma \ref{lem:7.3} we have

$$\tilde{\kappa}(r\{a, b\}, (0, y)) = \bar{\sigma}_1(a, b, (0, y)) = \pi^{0, L}_{1}([a, b], y) = \frac{(\alpha_i, \alpha_i)}{2}\pi^{0, L}_{1}(b, [y, e_i]).$$

On the other hand we have

$$\tilde{\kappa}([r(a), r(b)], (0, y)) = -\tilde{\kappa}(r(b), [r(a), (0, y)])$$

$$= - (\alpha_i, \alpha_i)\tilde{\kappa}(r(b), (0, [e_i, y]))/2 = (\alpha_i, \alpha_i)\bar{\sigma}_1(b, [y, e_i]) /2$$

$$= (\alpha_i, \alpha_i)\pi^{0, L}_{1}(b, [y, e_i]) /2.$$

\eqref{eq:7.23} is proved in the case $a = \pi_i(A_i)$ and $b \in U_i^+$. The remaining case $a = \pi_1(B_i)$ is proved similarly to the case $a = \pi_1(A_i)$. We omit the details. Now we have proved that the isomorphism $\Upsilon : U_1 \to \mathbb{C}[M]$ preserves the Poisson structure.

It remains to show that $Z_\zeta$ is closed under the Poisson bracket and that the isomorphism $^t\xi : U_1 \simeq Z_\zeta$ preserves the Poisson structure. By the above argument and by Lemma \ref{lem:7.1} we see that the set $D'$ (see \eqref{eq:7.22}) generates the Poisson algebra. Therefore, it is sufficient to show

\begin{equation}
^t\xi\{a, b\} = \{^t\xi(a), ^t\xi(b)\} \quad (a \in D', b \in U_i^+ \cup U_i^0 \cup U_i^-).
\end{equation}

The case $a = \pi_1(K_\lambda)$ is easy. Hence it is sufficient to show \eqref{eq:7.24} in the cases $a = \pi_1(A_i)$ or $a = \pi_1(B_i)$, and $b \in U_i^+ \cup U_i^-$. Assume $a = \pi_1(A_i)$. If $b \in U_i^-$, the assertion follows from Lemma \ref{lem:7.3}. If $b \in U_i^+$, then we see easily by Lemma \ref{lem:7.5} that

$$\bar{\sigma}_\zeta(\{^t\xi(a), ^t\xi(b)\}, y) = \bar{\sigma}_\zeta(\{^t\xi(a), ^t\xi(b)\}, y)$$

for any $y \in V_\zeta$. Since $\{^t\xi(a), ^t\xi(b)\}, ^t\xi\{a, b\} \in U_\zeta^+$, \eqref{eq:7.24} holds in this case. The proof for the case $a = \pi_1(B_i)$ is similar to that for $a = \pi_1(A_i)$. Details are omitted. The proof of Theorem \ref{thm:7.2} is now complete.
8 Poisson manifold associated to rings of differential operators

We denote by $F$ the subspace of $U^*$ spanned by the matrix coefficients of finite dimensional $U$-modules $E$ such that

$$E = \bigoplus_{\lambda \in \mathbb{Q}} E_\lambda \quad \text{with} \quad E_\lambda = \{ v \in E \mid K_\mu v = q^{(\lambda,\mu)} v \ (\forall \mu \in \mathbb{Q}) \}.$$ 

It is endowed with a structure of Hopf algebra via

$$\langle \phi \psi, u \rangle = \langle \phi \otimes \psi, \Delta(u) \rangle \quad (\phi, \psi \in F, \ u \in U),$$

$$\langle 1, u \rangle = \varepsilon(u) \quad (u \in U),$$

$$\langle \Delta(\phi), u \otimes u' \rangle = \langle \phi, uu' \rangle \quad (\phi \in F, \ u, u' \in U),$$

$$\varepsilon(\phi) = \langle \phi, 1 \rangle \quad (\phi \in F),$$

$$\langle S(\phi), u \rangle = \langle \phi, S(u) \rangle \quad (\phi \in F, \ u \in U),$$

where $\langle \ , \ \rangle : F \times U \to \mathbb{Q}(q)$ is the canonical paring. $F$ is also endowed with a structure of $U$-bimodule by

$$\langle u' \phi u'', u \rangle = \langle \phi, uu''u' \rangle \quad (\phi \in F, u, u', u'' \in U).$$

For a subring $A$ of $\mathbb{Q}(q)$ containing $\mathbb{Q}[q, q^{-1}]$ we set

$$F_A = \{ \phi \in F \mid \langle \phi, U^L_A \rangle \subset A \}.$$ 

It is a Hopf algebra over $A$ and a $U^L_A$-bimodule. For $z \in \mathbb{C}^\times$ we set

$$F_z = \mathbb{C} \otimes_{A_z} F_{A_z},$$

where $A_z \to \mathbb{C}$ is given by $q \mapsto z$. Then $F_z$ is a Hopf algebra over $\mathbb{C}$ and a $U^L_z$-bimodule. In the following we will only be concerned with $F_1$, which is canonically isomorphic to the coordinate algebra $\mathbb{C}[G]$ of the adjoint group of $g$.

Denote by $\phi \mapsto \overline{\phi}$ the canonical homomorphism $F_{A_1} \to F_1 = \mathbb{C}[G]$. We have a natural Poisson Hopf algebra structure on $\mathbb{C}[G]$ given by

$$\{ \overline{\phi}, \overline{\psi} \} = [\overline{\phi}, \overline{\psi}]/(q - q^{-1}) \quad (\phi, \psi \in F_{A_1}).$$

It is known that this Poisson Hopf algebra structure of $\mathbb{C}[G]$ coincides with the one coming from the Manin triple $(g \oplus g, m, t)$ by identifying $t$ with $g$ (see De Concini-Lyubashenko) [3].

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We define a $\mathbb{Q}(q)$-algebra structure on

$$D = F \otimes_{\mathbb{Q}(q)} U$$

by

$$(\varphi \otimes u)(\varphi' \otimes u') = \sum_{(u)} \varphi(u_{(0)}\varphi') \otimes u_{(1)}u' \quad (\varphi, \varphi' \in F, u, u' \in U).$$

The algebra $D$ is an analogue of the ring of differential operators on $G$. We will identify $U$ and $F$ with subalgebras of $D$ by the embeddings $U \ni u \mapsto 1 \otimes u \in D$ and $F \ni \varphi \mapsto \varphi \otimes 1 \in D$ respectively.

Let $A$ be a subring of $\mathbb{Q}(q)$ containing $\mathbb{Q}[q, q^{-1}]$. We have a natural $A$-form

$$D'_A = F_A \otimes_A U_A$$

of $D$ whose specialization

$$D'_1 = C \otimes_A D'_A = F_1 \otimes U_1$$

at $q = 1$ is almost isomorphic to the ring $\mathbb{C}[G] \otimes_C U(g)$ of differential operators on $G$. However, in the following we will be concerned with a different $A$-form

$$D_A = F_A \otimes_A U_A.$$

For $z \in \mathbb{C}^\times$ we set

$$D_z = C \otimes_{A_z} D_{A_z} = F_z \otimes U_z$$

where $A_z \to \mathbb{C}$ is given by $q \mapsto z$.

**Lemma 8.1.** $D_1$ is a commutative algebra. In particular, it is identified as an algebra with the coordinate algebra $\mathbb{C}[G] \otimes \mathbb{C}[M]$ of $G \times M$.

**Proof.** By the definition of the multiplication of $D$ it is sufficient to show $u\varphi = \varepsilon(u)\varphi$ for $u \in U_1$, $\varphi \in F_1$. Let $\iota : U_1 \to \overline{U'_1}$ be the algebra homomorphism induced by $U_h \subset U'_h$. Then we have

$$\langle u\varphi, u' \rangle = \langle \varphi, u'\iota(u) \rangle \quad (u' \in \overline{U'_1}),$$

and hence it is sufficient to show $\iota(u) = \varepsilon(u)1$ for any $u \in U_1$. We may assume that $u$ is one of $K_\lambda$ ($\lambda \in Q$), $A_{\beta_k}$, $B_{\beta_k}$ ($1 \leq k \leq N$). In these cases the assertion follows from $\iota(K_\lambda) = 1$, $\iota(A_{\beta_k}) = \iota(B_{\beta_k}) = 0$.

**Remark 8.2.** We can show that $D_1$ is isomorphic to a central subalgebra of $D_\zeta$, where $\zeta$ is as in Section 6.
By Lemma 8.1 we have a natural Poisson algebra structure of \( D_1 = \mathbb{C}[G] \otimes \mathbb{C}[M] \) given by
\[
\{ \Phi, \Phi' \} = [\Phi, \Phi']/(q - q^{-1}) \quad (\Phi, \Phi' \in D_\lambda),
\]
where \( D_\lambda \ni \Phi \mapsto \Phi \in D_1 = \mathbb{C}[G] \otimes \mathbb{C}[M] \) is the natural homomorphism. Let us describe this Poisson bracket more explicitly.

By definition the canonical inclusions \( \mathbb{C}[G] \ni \phi \mapsto \phi \otimes 1 \in D_1 \) and \( \mathbb{C}[M] \ni \psi \mapsto 1 \otimes \psi \in D_1 \) are homomorphisms of Poisson algebras. Since the Poisson structures of \( \mathbb{C}[G] \) and \( \mathbb{C}[M] \) are already described explicitly, we have only to give a description of \( \{ \phi, \psi \} \) for \( \phi \in \mathbb{C}[G], \psi \in \mathbb{C}[M] \).

In general, for an algebraic group \( S \) with Lie algebra \( \mathfrak{s} \) we denote by \( \langle , \rangle : \mathbb{C}[S] \times U(\mathfrak{s}) \rightarrow \mathbb{C} \) the canonical Hopf paring. We have a \( U(\mathfrak{s}) \)-bimodule structure of \( \mathbb{C}[S] \) given by
\[
\langle u'\phi u'', u \rangle = \langle \phi, uu' \rangle.
\]
For \( a \in \mathfrak{s}, \phi \in \mathbb{C}[S], s \in S \) we have
\[
(a\phi)(s) = \frac{d}{dt} \phi(s \exp(ta))|_{t=0}, \quad (\phi a)(s) = \frac{d}{dt} \phi(\exp(ta)s)|_{t=0}.
\]
For \( a \in \mathfrak{s} \) we denote by \( L_a \) (resp. \( R_a \)) the left (resp. right) invariant vector field on \( S \) given by \( L_a(\phi) = a\phi \) (resp. \( R_a(\phi) = \phi a \)). For \( b \in \mathfrak{s}^* \) we denote by \( L_b^*(\text{resp. } R_b^*) \) the left (resp. right) invariant 1-form on \( S \) given by \( \langle L_a, L_b^* \rangle = \langle a, b \rangle \) (resp. \( \langle R_a, R_b^* \rangle = \langle a, b \rangle \)).

**Proposition 8.3.** For \( \phi \in \mathbb{C}[G], \psi \in \mathbb{C}[M] \) we have
\[
\{ \phi, \psi \} = - \sum_{r=1}^{\dim \mathfrak{g}} \langle L_{\xi_r}(\phi) \rangle(\psi),
\]
where \( \{ \xi_r \}_{r=1}^{\dim \mathfrak{g}} \) and \( \{ \eta_r \}_{r=1}^{\dim \mathfrak{g}} \) are bases of \( \mathfrak{g} \) and \( \mathfrak{m} \) respectively such that \( \kappa(\theta(\xi_r), \eta_s) = \delta_{rs} \).

**Proof.** Our assertion is equivalent to the identity
\[
\sum_{(x)} (x(0)f) \otimes x(1) - f \otimes x)/\(q - q^{-1}\) = \sum_r \xi_r \overline{f} \otimes \overline{x} \eta_r \quad (f \in F_{\mathfrak{h}_1}, x \in U_{\mathfrak{h}_1})
\]

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in $D_1 = F_1 \otimes U_1 \cong \mathbb{C}[G] \otimes \mathbb{C}[M]$. For $u \in U_{A_1}^L$ we have
\[
\sum_x \langle x_{(0)} f, u \rangle x_{(1)} - \langle f, u \rangle x = \sum_x \langle f, ux_{(0)} \rangle x_{(1)} - \langle f, u \rangle x,
\]
\[
\sum_r \langle \xi_r f, u \rangle \overline{\xi_r} = \sum_r \langle f, \overline{\xi_r} \rangle \overline{\xi_r},
\]
and hence our assertion is further equivalent to the identity
\[
\frac{\left( \sum_x u x_{(0)} \otimes x_{(1)} - u \otimes x \right)}{(q - q^{-1})} = \sum_r \xi_r \otimes \overline{\xi_r} \quad (u \in U_{A_1}^L, x \in U_{A_1})
\]
in $U_1^L \otimes U_1 \cong U(\mathfrak{g}) \otimes \mathbb{C}[M]$. This statement follows from its special case $u = 1$:
\[
\frac{\left( \sum_x x_{(0)} \otimes x_{(1)} - 1 \otimes x \right)}{(q - q^{-1})} = \sum_r \xi_r \otimes \overline{\xi_r} \quad (x \in U_{A_1}).
\]
Let $\{x_j\}_j$ be a free basis of $U_{A_1}$, and define $v_j \in U_{A_1}^* = \text{Hom}_{A_1}(U_{A_1}, A_1)$ by $\langle v_j, x_k \rangle = \delta_{jk}$. Then for $v \in U_{A_1}^*$ we have
\[
\sum_{(x)} \langle v, x_{(1)} \rangle x_{(0)} - \langle v, x \rangle 1 = \sum_{(x), j} \langle v, x_{(1)} \rangle \langle v_j, x_{(0)} \rangle x_j - \langle v, x \rangle 1
\]
\[
= \sum_j \langle v_j v, x \rangle x_j - \langle v, x \rangle 1,
\]
\[
\sum_r \langle \overline{v}, \overline{x} \rangle \xi_r = \sum_r \langle \overline{\xi_r}, \overline{\xi_r} \rangle \xi_r.
\]
Here, the multiplication of $U_{A_1}^*$ is induced by the comultiplication of $U_{A_1}$. Therefore, we have only to show the identity
\[
\frac{\left( \sum_j x_j \otimes v_j v - 1 \otimes v \right)}{(q - q^{-1})} = \sum_r \xi_r \otimes \overline{\eta_r} \quad (v \in U_{A_1}^*)
\]
in the completion $\text{Hom}_C(\mathbb{C}[M], U(\mathfrak{g}))$ of $U(\mathfrak{g}) \otimes \mathbb{C}[M]^*$. This statement follows from its special case $v = 1_{U_{A_1}^*} = \varepsilon$:
\[
\frac{\left( \sum_j x_j \otimes v_j - 1 \otimes 1 \right)}{(q - q^{-1})} = \sum_r \xi_r \otimes \eta_r.
\]
This follows from Lemma 8.4 below.
Lemma 8.4. Let $\Xi : \mathbb{C}[M] \to U(\mathfrak{g})$ be the map induced by
$$U_{\kappa_1} \ni x \mapsto (x - \varepsilon(x)1)/(q - q^{-1}) \in U_{\kappa_1}^L.$$Then we have $\text{Im}(\Xi) \subset \mathfrak{g}$ and
$$\tilde{\kappa}(\vartheta(\Xi(\varphi)), \eta) = \langle \varphi, \eta \rangle \quad (\varphi \in \mathbb{C}[M], \eta \in \mathfrak{m}).$$This can be shown by a direct computation in terms of root vectors. Details are omitted.

Remark 8.5. In terms of the Poisson tensor $\delta$ of the Poisson manifold $G \times M$ Proposition 8.3 can be reformulated as follows. Under the identification $\mathfrak{g} \cong \mathfrak{m}^*$, $\mathfrak{m}^* \cong \mathfrak{g}$ via $\tilde{\kappa}$ we have
$$\delta_{(g,m)}((L^*_\eta)_g, (R^*_\xi)_m) = -\tilde{\kappa}(\xi, \eta) \quad ((g, m) \in G \times M, \eta \in \mathfrak{m}, \xi \in \mathfrak{g}).$$

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