Symmetry and Its Importance in the Oscillation of Solutions of Differential Equations

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Abstract: Oscillation and symmetry play an important role in many applications such as engineering, physics, medicine, and vibration in flight. The purpose of this article is to explore the oscillation of fourth-order differential equations with delay arguments. New Kamenev-type oscillatory properties are established, which are based on a suitable Riccati method to reduce the main equation into a first-order inequality. Our new results extend and simplify existing results in the previous studies. Examples are presented in order to clarify the main results.

Keywords: delay differential equations; fourth-order; oscillation

1. Introduction

Fourth-order differential equations have enormous potential for applications in engineering, medicine, aviation, physics, etc. In past years, significant attention has been devoted to the oscillation theory of various classes of equations, see [1–9].

In this work, we are concerned with the fourth-order delay differential equation:

\[
\left( j_3(y)\left(j_2(y)(j_1(y)\xi'(y))'\right)' + \sum_{r=1}^{m} \pi_r(y)f(z_r(y)) \right) = 0, \quad y \geq y_0. \tag{1}
\]

Throughout this article, we suppose that

(H_1) \pi_r \in C([y_0, \infty), \mathbb{R}) \text{ is non-negative, } r = 1, 2, \ldots, m, \text{ } z_r \in C^1([y_0, \infty), \mathbb{R}), \text{ } z_r(y) \leq y \text{ and } \lim_{y \to \infty} z_r(y) = \infty, \text{ } f \in C(\mathbb{R}, \mathbb{R}), \text{ and there exists a constant } k > 0 \text{ such that } f(u)/u \geq k, \text{ for } u \neq 0.

(H_2) j_i \in C^3([y_0, \infty), \mathbb{R}), \text{ } i = 1, 2, 3 \text{ are positive and }

\[
\int_{y_0}^{\infty} \frac{1}{j_i(s)} \, ds = \infty. \tag{2}
\]

Delay differential equations can also be used in engineering and the modeling of dynamical networks of interacting free-bodies. Finally, the properties of delay differential equations are used in the study of singular differential equations of fractions, see [10–14].

It is clear that the form of problem Equation (1) is more general than all the problems considered in [12,14], where the authors in [12,14] discussed the oscillatory properties of differential equations of the neutral type with a canonical operator, and they used the comparison method and integral averaging technique to obtain these properties. Their approach...
is based on using these mentioned methods to reduce Equation (1) into a second-order equation, while in our article we discuss the oscillatory properties of differential equations with a middle term and with a non-canonical operator of the delay-type, and we employ a different approach based on using the Riccati technique to reduce the main equation into a first-order inequality to obtain more effective Kamenev-type oscillatory properties.

The aim of this article is to establish the oscillatory properties of Equation (1).

Several studies have had very interesting results related to the oscillatory properties of solutions of differential equations.

Dzurina et al. [15] obtained sufficient conditions for oscillation for equation

\[
\left( f_1(y) \left( f_2(y) (j_1(y) \xi(y))^\prime \right)^\prime \right) + p(y) \xi(y) + \pi(y) \xi(\tau(y)) = 0. \tag{3}
\]

They also used the technique of comparison.

In Grace et al. [16], some comparison criteria have been studied when \( \tau(y) \leq y \), and some oscillation criteria for Equation (1) are given when Equation (2) holds.

In addition, the results obtained in [17] are presented for Equation (1) when

\[
\lim_{y \to \infty} \inf_y \frac{1}{f_1(y)} \left( \int_{f_1(y)}^{\infty} \frac{1}{f_2(v)} \int_{f_2(v)}^{\infty} A_1(v) dv du \right) ds > \frac{1}{4} \tag{4}
\]

and

\[
\lim_{y \to \infty} \inf_y \left( \int_{f_1(y)}^{\infty} \frac{1}{f_2(v)} \int_{f_2(v)}^{\infty} \frac{1}{f_3(y)} ds du \right) \int_{f_1(y)}^{\infty} A_2(v) dv > \frac{1}{4} \text{ for } y \geq y_1, \tag{5}
\]

where there are positive functions \( A_1, A_2 \in C([y_0, \infty), \mathbb{R}^+) \).

The purpose of this article is to explore the oscillation of Equation (1). New oscillation theorems are established, which are based on a suitable Riccati-type method.

This article is organized as follows. In Section 2, we introduce some auxiliary lemmas and some notations. In Section 3, we present new oscillation results for Equation (1) by Riccati transformation. Finally, two examples with specific values of parameters are offered to illustrate our main theorems.

2. Some Lemmas

We start with the following important Lemmas.

**Lemma 1.** [18] Let \( \alpha \geq 1, C > 0 \) and \( D \) be constant. Then

\[
D \xi - C \xi^{(a+1)/a} \leq \frac{\alpha^n}{(a + 1)^{a+1}} \frac{D^{a+1}}{C^a}, \tag{6}
\]

for all positive \( \xi \).

**Lemma 2.** [19] If the function \( \xi \) satisfies \( \xi^{(m)}(y) > 0, m = 0, 1, \ldots, n, \) and \( \xi^{(n+1)}(y) < 0 \) for \( y \geq y_0 \), then

\[
\frac{\xi(y)}{y^n/n!} \geq \frac{\xi^{(n)}(y)}{y^{n-1}/(n - 1)!}. \tag{7}
\]

**Lemma 3.** [17] Let \( \xi \) be an eventually positive solution of Equation (1).

Then, we find the following cases:

\((N_1)\) \( \xi'(y) > 0, (j_1 \xi')'(y) > 0, (j_2 (j_1 \xi')')'(y) > 0, (j_3 (j_2 (j_1 \xi')')')'(y) < 0. \tag{8}\)

\((N_2)\) \( \xi'(y) > 0, (j_1 \xi')'(y) < 0, (j_2 (j_1 \xi')')'(y) > 0, (j_3 (j_2 (j_1 \xi')')')'(y) < 0. \tag{9}\)
Let \( \zeta \) be an eventually positive solution of Equation (1) hold. If \((N_1)\) holds and there exists a function \( \theta_1 \in C^1([y_0, \infty), \mathbb{R}_+^+) \) such that

\[
\zeta(y) := \theta_1(y) \left( \frac{j_3 \left( j_2 (j_1 \zeta')' \right) (y)}{\left( j_2 (j_1 \zeta')' \right) (y)} \right) > 0, \tag{14}
\]

then

\[
\zeta'(y) \leq -k \theta_1(y) \sum_{r=1}^{m} \pi_r(y) A(y) + \frac{j_3(y) (\theta'_1(y))^2}{2 \theta_1(y)}, \tag{15}
\]

In addition, if \((N_2)\) holds and there exists a function \( \theta_2 \in C^1([y_0, \infty), \mathbb{R}_+^+) \) such that

\[
w(y) := \theta_2(y) \left( \frac{(j_1 \zeta')'(y)}{\zeta(y)} \right) > 0, \tag{16}
\]

then

\[
w'(y) \leq -B(y) + \frac{j_3(y) (\theta_2(y))^2}{4 \theta_2(y)}, \tag{17}
\]

where \( \zeta(y) \) and \( w(y) \) are called Riccati transformations.

**Proof.** Let \( \zeta \) be an eventually positive solution of Equation (1) hold. From Lemma 3 there exist two possible cases \((N_1)\) and \((N_2)\).

Let case \((N_1)\) hold. From \((j_1 \zeta')'(y) > 0\) and \( \left( j_3 \left( j_2 (j_1 \zeta')' \right)' \right)'(y) < 0 \) for \( y \geq y_1 \), we obtain

\[
\left( j_2 (j_1 \zeta')' \right)'(y) = \left( j_2 (j_1 \zeta')' \right)'(y_1) + \int_{y_1}^{y} \frac{ds}{j_3(s)} \left( j_2 (j_1 \zeta')' \right)'(s) ds \tag{18}
\]

\[
\geq j_3(y) \left( \int_{y_1}^{y} \frac{ds}{j_3(s)} \right) \left( j_2 (j_1 \zeta')' \right)'(y). 
\]
Thus for \( y \geq y_1 \), we have

\[
\left( \frac{j_2(j_1\xi')'}{\delta(y)} \right)' = \left( \frac{j_2(j_1\xi')'}{\delta(y)} \right)'(y) \delta(y) - \left( \frac{j_2(j_1\xi')'}{\delta^2(y)} \right) \delta'(y) 
\]

(19)

\[
= \left( \frac{j_2(j_1\xi')'}{\delta(y)} \right)'(y) \delta(y) - \left( \frac{j_2(j_1\xi')'}{\delta^2(y)} \right) \delta'(y) 
\]

\[
\leq \left( \frac{j_2(j_1\xi')'}{\delta^2(y)} \right) \left[ \frac{\delta(y)}{j_3(y)} \int_{y_1}^{y} \frac{ds}{j_2(s)} - \delta'(y) \right] 
\]

\[
\leq 0. 
\]

Therefore, \( (j_2(j_1\xi')')(y)/\delta(y) \) is a non-increasing function for \( y_2 \geq y_1, y \geq y_2 \). Then, we get

\[
(j_1\xi')(y) = (j_1\xi')(y_2) + \int_{y_2}^{y} \frac{j_2(s)(j_1\xi')'(s)\delta(s)}{j_2(s)\delta(s)} ds 
\]

(20)

\[
\geq \frac{j_2(y)}{\delta(y)} \left( \int_{y_2}^{y} \delta(s) ds \right) (j_1\xi')'(y). 
\]

Thus, from Equation (20), we obtain

\[
\left( \frac{(j_1\xi')(y)}{\sigma(y)} \right)' = \left( \frac{(j_1\xi')(y)}{\sigma(y)} \right)'(y) \sigma'(y) - \left( \frac{(j_1\xi')(y)}{\sigma^2(y)} \right) \sigma'(y) 
\]

(21)

\[
= \left( \frac{(j_1\xi')(y)}{\sigma(y)} \right)' \left[ \frac{\sigma(y)}{j_2(y)} \int_{y_2}^{y} \frac{\delta(s) ds}{j_2(s)\sigma(y)} - \sigma'(y) \right] 
\]

\[
\leq 0. 
\]

Therefore, \( ((j_1\xi')(y))/\sigma(y) \) is a non-increasing function for \( y_3 \geq y_2, y \geq y_3 \). So we get

\[
\xi(y) = \xi(y_3) + \int_{y_3}^{y} \frac{(j_1\xi')'(s)\sigma(s)}{j_1(s)\sigma(s)} ds 
\]

(22)

\[
\geq \frac{j_1(y)}{\sigma(y)} \left( \int_{y_3}^{y} \sigma(s) ds \right) \xi'(y). 
\]

From Equation (1), we have

\[
\left( j_3(y) \left( j_2(y)(j_1(y)\xi'(y))' \right) \right)' \leq - \sum_{r=1}^{m} \pi_r(y) f(\xi(z_r(y))). 
\]

(23)

By using condition \( f(u)/u \geq k \), we see that

\[
\left( j_3(y) \left( j_2(y)(j_1(y)\xi'(y))' \right) \right)' \leq - k \sum_{r=1}^{m} \pi_r(y) \xi(z_r(y)). 
\]

(24)

Since \( (j_2(j_1\xi')')(y)/\delta(y) \) is non-increasing, we get

\[
\frac{(j_2(j_1\xi')')(z_r(y))}{\delta(z_r(y))} \geq \frac{(j_2(j_1\xi')')(y)}{\delta(y)}, \quad z_r(y) \leq y, 
\]

(25)
i.e.,
\[
\frac{(j_3 \xi')(z_r(y))}{(j_1 \xi')(y)} \geq \frac{j_2(y)\delta(z_r(y))}{j_1(z_r(y))\delta(y)}
\]  
(26)

Thus, from Equations (20), (22) and (26), we have
\[
\frac{\xi(z_r(y))}{(j_1 \xi')(y)} \geq \frac{1}{j_2(y) \sigma(z_r(y))} \left( \int_{y_3}^{z_r(y)} \sigma(s)ds \right) \left( \frac{j_2}{j_1} \right)
\]

\[
\frac{\xi(z_r(y))}{(j_1 \xi')(y)} \geq \frac{1}{\delta(y)\sigma(z_r(y))} \left( \int_{y_3}^{z_r(y)} \sigma(s)ds \right) \left( \frac{j_2}{j_1} \right).
\]  
(27)

From ξ(y), we obtain
\[
\xi'(y) = \theta_1(y) \left[ \frac{j_3}{j_2(j_1 \xi')(y)} \right] + \theta_1(y) \left( \frac{j_3}{j_2(j_1 \xi')(y)} \right) - \theta_1(y) \frac{j_3}{j_2(j_1 \xi')(y)} \left( \frac{j_2}{j_1} \right)^2.
\]

Using Equations (14), (24) and (27), we obtain
\[
\xi'(y) \leq \frac{\theta_1(y)}{\theta_1(y)^2} \xi(y) - k\theta_1(y) \sum_{r=1}^{m} \pi_r(y) \frac{\xi(z_r(y))}{(j_1 \xi')(y)} - \frac{1}{j_3(y)\theta_1(y)} \xi^2(y),
\]  
(28)

which yields
\[
\xi'(y) \leq -k\theta_1(y) \sum_{r=1}^{m} \pi_r(y)A(y) + \frac{\theta_1(y)}{\theta_1(y)^2} \xi(y) - \frac{1}{j_3(y)\theta_1(y)} \xi^2(y).
\]  
(29)

Using Lemma 1 with \( C = 1/(j_3(y)\theta_1(y)) \), \( D = \theta_1(y)/\theta_1(y) \) and \( \xi = \xi(y) \), we get
\[
\frac{\theta_1(y)}{\theta_1(y)} \xi(y) - \frac{1}{j_3(y)\theta_1(y)} \xi^2(y) \leq \frac{j_3(y)\left( \theta_1(y)^2 \right)^2}{2\theta_1(y)}.
\]  
(30)

From Equations (29) and (30), we get
\[
\xi'(y) \leq -k\theta_1(y) \sum_{r=1}^{m} \pi_r(y)A(y) + \frac{j_3(y)\left( \theta_1(y)^2 \right)^2}{2\theta_1(y)}.
\]  
(31)

Thus, Equation (15) holds.

Let case (N_2) hold. From \( w(y) \), we find that
\[
w'(y) = \theta_2(y) \frac{(j_1 \xi')(y)}{\xi(y)} + \theta_2(y) \frac{(j_1 \xi')(y)}{\xi(y)} - \theta_2(y) \frac{(j_1 \xi')^2(y)}{\xi^2(y)}.
\]  
(32)
Hence by Equation (16), we get
\[ w'(y) = \frac{\theta_2'(y)}{\theta_2(y)} w(y) + \theta_2(y) \frac{(j_1 \xi')(y)}{\xi(y)} - \frac{w^2(y)}{j_1(y) \theta_2(y)}. \] (33)

From Lemma 2, we find
\[ \xi(y) \geq \frac{y}{3} \xi'(y), \quad \text{where } n = 3. \] (34)

Integrating Equation (34) from \( z_r(y) \) to \( y \), we obtain
\[ \frac{\xi(z_r(y))}{\xi(y)} \geq \frac{z^3_r(y)}{y^3}. \] (35)

Integrating Equation (1) from \( y \) to \( u \), we obtain
\[ \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(u) - \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(y) + \int_y^u \sum_{r=1}^m \pi_r(s) f(\xi(z_r(s))) ds \leq 0. \] (36)

Easily we find that
\[ \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(u) - \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(y) + \int_y^u \sum_{r=1}^m \pi_r(s) \xi(z_r(s)) ds \leq 0. \] (37)

From \( \xi'(y) > 0 \) and Equation (35), we have
\[ \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(u) - \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(y) + \int_y^u \sum_{r=1}^m \pi_r(s) \left( \frac{z^3_r(s)}{s^3} \right) \xi(z_r(s)) ds \leq 0. \] (38)

Letting \( u \to \infty \), we arrive at the inequality
\[ - \left( j_3 \left( j_2(j_1 \xi')' \right)' \right)(y) + \xi(y) \int_y^\infty \sum_{r=1}^m \pi_r(s) \left( \frac{z^3_r(s)}{s^3} \right) ds \leq 0. \] (39)

Thus,
\[ \left( j_2(j_1 \xi')' \right)'(y) \geq \xi(y) \left[ \frac{1}{j_3(s)} \int_y^\infty \sum_{r=1}^m \pi_r(s) \left( \frac{z^3_r(s)}{s^3} \right) ds \right]. \] (40)

Integrating Equation (40) from \( y \) to \( \infty \) we obtain
\[ \left( j_1 \xi' \right)'(y) + \xi(y) \frac{1}{j_2(y)} \int_y^\infty \left[ \frac{1}{j_3(s)} \int_s^\infty \sum_{r=1}^m \pi_r(v) \left( \frac{z^3_r(v)}{v^3} \right) dv \right] ds \leq 0. \] (41)

Hence, by Equation (41) in Equation (33), we find
\[ w'(y) \leq -B(y) + \frac{\theta_2'(y)}{\theta_2(y)} w(y) - \frac{w^2(y)}{j_1(y) \theta_2(y)}. \] (42)

Thus, we have
\[ w'(y) \leq -B(y) + \frac{j_1(y) (\theta_2'(y))^2}{4 \theta_2(y)}. \] (43)

Thus, Equation (17) holds. This completes the proof. \( \square \)

In the next theorem, we establish new Kamenev-type oscillatory properties for Equation (1).
Theorem 1. Let Equation (2) hold. Assume that there exist positive functions \( \theta_1, \theta_2, \delta, \sigma \in C^1([y_0, \infty)) \) and an integer \( n \in \mathbb{N} \). If
\[
\limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y-s)^n \left( k \theta_1(s) \sum_{r=1}^{m} \pi_r(s) A(s) - \frac{j_3(s) (\theta_1'(s))^2}{2 \theta_1(s)} \right) ds = \infty
\] (44)
and
\[
\limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y-s)^n \left( B(s) - \frac{j_1(s) (\theta_2(s))^2}{4 \theta_2(s)} \right) ds = \infty,
\] (45)
then Equation (1) is oscillatory.

Proof. Let \( \xi \) be a non-oscillatory solution of Equation (1). Without loss of generality, we can assume that \( \xi(y) \) is eventually positive. For case (N), from Lemma 4, we get that Equation (15) holds. Thus, we have
\[
- \int_{y_0}^{y} (y-s)^n \xi'(s) ds \geq \int_{y_0}^{y} (y-s)^n \left( k \theta_1(s) \sum_{r=1}^{m} \pi_r(s) A(s) - \frac{j_3(s) (\theta_1'(s))^2}{2 \theta_1(s)} \right) ds.
\] (46)
Since
\[
\int_{y_0}^{y} (y-s)^n \xi'(s) ds = n \int_{y_0}^{y} (y-s)^{n-1} \xi(s) ds - (y-y_0)^n \xi(y_0),
\] (47)
Thus, we get
\[
\left( \frac{y-y_0}{y} \right)^n \xi(y_0) - \frac{n}{y^n} \int_{y_0}^{y} (y-s)^{n-1} \xi(s) ds \geq \frac{1}{y^n} \int_{y_0}^{y} (y-s)^n \left( k \theta_1(s) \sum_{r=1}^{m} \pi_r(s) A(s) - \frac{j_3(s) (\theta_1'(s))^2}{2 \theta_1(s)} \right) ds.
\]
Hence,
\[
\frac{1}{y^n} \int_{y_0}^{y} (y-s)^n \left( k \theta_1(s) \sum_{r=1}^{m} \pi_r(s) A(s) - \frac{j_3(s) (\theta_1'(s))^2}{2 \theta_1(s)} \right) ds \leq \left( \frac{y-y_0}{y} \right)^n \xi(y_0),
\] (48)
and so
\[
\limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y-s)^n \left( k \theta_1(s) \sum_{r=1}^{m} \pi_r(s) A(s) - \frac{j_3(s) (\theta_1'(s))^2}{2 \theta_1(s)} \right) ds \leq \xi(y_0),
\] (49)
which contradicts Equation (44).

For case (N2), from Lemma 4, we find Equation (17) holds. Thus, we see
\[
- \int_{y_0}^{y} (y-s)^n w'(s) ds \geq \int_{y_0}^{y} (y-s)^n \left( B(y) - \frac{j_1(y) (\theta_2'(y))^2}{4 \theta_2(y)} \right) ds.
\] (50)
From Equations (47) and (50), we get
\[
\left( \frac{y-y_0}{y} \right)^n w(y_0) - \frac{n}{y^n} \int_{y_0}^{y} (y-s)^{n-1} w(s) ds \geq \frac{1}{y^n} \int_{y_0}^{y} (y-s)^n \left( B(s) - \frac{j_1(s) (\theta_2(s))^2}{4 \theta_2(s)} \right) ds,
\]
which yields
\[
\limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y - s)^n \left( B(s) - \frac{j_1'(s) \theta_1(s)^2}{4 \theta_2(s)} \right) ds \leq w(y_0),
\]
which contradicts Equation \((45)\).

Theorem 1 has been proved. \(\Box\)

Now, we give some interesting examples to demonstrate the applicability of the obtained criteria in the main results.

**Example 1.** Consider a differential equation
\[
\left( y \left( y \xi(y) \right)' \right)' + y \zeta(ay) = 0, \quad y \geq 1,
\]
where \(a \in (0, 1)\) is a constant. Let \(j_1 = j_2 = j_3 = y\), \(\pi(y) = ay\), \(z(y) = y\). Moreover, we have
\[
\int_{y_0}^{\infty} \frac{ds}{s} = \infty.
\]

If we now set \(\theta_1(y) = \theta_2(y) = k = 1\), we can easily find that the conditions of Theorem 1 are satisfied. So, Equation \((52)\) is oscillatory. As a matter of fact, one such solution is \(\xi(y) = \sin(y)\).

**Example 2.** Consider the equation
\[
\zeta^{(4)}(y) + \frac{q_0}{y^4} \zeta(y) = 0, \quad y \geq 1, \quad q_0 > 0.
\]

Let \(j_1 = j_2 = j_3 = 1\), \(n = 4\), \(z(y) = y\) and \(\pi(y) = q_0/y^4\). Hence, it is easy to see that
\[
\delta(y) = y, \quad \sigma(y) = \frac{1}{2} y^2
\]
and
\[
A(y) = \frac{1}{\delta(y) \sigma(z_3(y))} \left( \int_{y_3}^{\infty} \frac{\sigma(s) ds}{j_1(s)} \right) \left( \int_{y_2}^{\infty} \frac{\delta(s) ds}{j_2(s)} \right)
\]
\[
= \frac{2}{y^2} \left( \frac{y^2}{6} \right) \left( \frac{y^2}{2} \right) = \frac{y^2}{6}.
\]

Now, if we set \(\theta_1(y) = \theta_2(y) = y\) and \(k = 1\), then we see
\[
\limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y - s)^n \left( k \theta_1(y) \sum_{r=1}^{m} \pi_r(y) A(y) - \frac{j_3(y) \left( \theta_1'(y) \right)^2}{2 \theta_1(y)} \right) ds
\]
\[
= \limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y - s)^2 \left( \frac{q_0}{6s} - \frac{1}{2s} \right) ds
\]
\[
= \limsup_{y \to \infty} \left( \frac{q_0}{6} - \frac{1}{2} \right) \frac{1}{y^2} \int_{y_0}^{y} (y - s)^2 \frac{1}{s} ds
\]
and
\[
\limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y - s)^n \left( B(y) - \frac{j_1(y) (\theta_2'(y))^2}{4 \theta_2(y)} \right) ds
\]
\[= \limsup_{y \to \infty} \frac{1}{y^n} \int_{y_0}^{y} (y - s)^2 \left( \frac{\pi_0}{6s} - \frac{1}{4s} \right) ds.
\]

So, the conditions become
\[q_0 > 3\]
and
\[q_0 > 1.5.\] (55)

Thus, by Theorem 1, Equation (53) is oscillatory if \(q_0 > 3\).

4. Conclusions

It’s clear that the form of problem Equation (1) is more general than all the problems considered in [12,14]. In this paper, using the suitable Riccati-type transformation, we have offered some new sufficient conditions that ensure that any solution of Equation (1) oscillates under assumption \(\int_{y_0}^{\infty} \frac{1}{y} ds = \infty\).

In addition, it would be useful to extend our results to fourth-order differential equations of the form
\[
\left( \left( j_3(y) \left( j_2(y) \left( j_1(y) \xi'(y) \right) \right) \right) \right) + \sum_{r=1}^{m} \pi_r(y) f(z_r(y)) = 0, \quad (56)
\]
under condition \(\int_{y_0}^{\infty} \frac{1}{j_1(s)} ds < \infty\).

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