A Counterexample and Fix to a Minimum Distance Duality Theorem

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Abstract

We consider dual optimization problems to the fundamental problem of finding the minimum distance from a point to a subspace. We provide a counterexample to a theorem which has appeared in the literature, relating the minimum distance problem to a maximization problem in the predual space. The theorem was stated in a series of papers by Zames and Owen in the early 1990s in conjunction with a non-standard definition, which together are true, but the theorem is false when assuming standard definitions, as it would later appear. Reasons for the failure of this theorem are discussed; in particular, the fact that the Hahn-Banach Theorem cannot be guaranteed to provide an extension which is an element of the predual space. The condition needed to restore the theorem is derived; namely, that the annihilator of the pre-annihilator return the original subspace of interest. This condition is consistent with the non-standard definition initially used, and it is further shown to be necessary in a sense.

1 Introduction

We consider minimum distance problems in real normed vector spaces. A well-known duality result for these problems is a generalization of the projection theorem in the dual space. We consider the validity of similar duality results in the predual space. This could be of use for problems arising in spaces for which the predual is much easier to characterize than the dual, including functions of total bounded variation BV(Ω) [11], trace-class operators / nuclear operators, and the Hardy space $H_\infty$ [16, 10].

After introducing some terminology in Section 2, we review the main duality results for minimum distance problems in Section 3, and state (as a conjecture) a previously stated theorem providing a duality result in the predual space. We provide counterexamples to this conjecture in Section 4, discuss reasons why certain attempted proofs would fail in Section 4.1, and then show how the result can be restored in Section 5.

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2 Preliminaries

We define the main terms and spaces that will be needed.

Definition 1. Given a normed space \( X \), its \textit{(normed) dual} is the space of all bounded linear functionals on \( X \), and is denoted by \( X^* \).

Definition 2. Given a normed space \( X \), its \textit{predual} is the normed space denoted by \( \overset{*}{X} \) such that \((\overset{*}{X})^* \simeq X\).

Dual pairings between an element in a normed space and in its dual space are then denoted by \( \langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R} \).

Definition 3. Given a subset of a normed space \( S \subset X \), its \textit{annihilator}, denoted by \( S^\perp \subset X^* \), is given by:

\[
S^\perp = \{ \lambda \in X^* \mid \langle x, \lambda \rangle = 0 \ \forall \ x \in S \},
\]
and its \textit{pre-annihilator}, denoted by \( \overset{\perp}{S} \subset \overset{*}{X} \), is given by:

\[
\overset{\perp}{S} = \{ \nu \in \overset{*}{X} \mid \langle \nu, x \rangle = 0 \ \forall \ x \in S \}.
\]

Remark 4. This is the standard definition of the pre-annihilator, as given in [12, 13, 1] and many other texts. In general, we then have \( S \subset \overset{\perp}{(S^\perp)} \subset (\overset{\perp}{S})^\perp \) [13]. The issues addressed in this paper can be viewed as having all stemmed from the false assumption that these sets are equivalent when \( S \) is a subspace.

Let \( \mathbb{Z} \) represent the set of integers, \( \mathbb{Z}_+ \) the set of nonnegative integers, and \( \mathbb{Z}_{++} \) the set of positive integers.

For \( 1 \leq p < \infty \), let

\[
\ell_p = \left\{ x : \mathbb{Z}_+ \to \mathbb{R} \mid \sum_{k=0}^{\infty} |x_k|^p < \infty \right\}
\]

with norm \( \|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p} \). Further let

\[
\ell_\infty = \left\{ x : \mathbb{Z}_+ \to \mathbb{R} \mid \sup_{k \in \mathbb{Z}_+} |x_k| < \infty \right\}
\]

with norm \( \|x\|_\infty = \sup_{k \in \mathbb{Z}_+} |x_k| \). We also define \( c_0 \subset \ell_\infty \) as

\[
c_0 = \{ x \in \ell_\infty \mid \lim_{k \to \infty} x_k = 0 \}.
\]

The pairings between any sequences that we will consider will be given by:

\[
\langle x, \lambda \rangle = \sum_{k=0}^{\infty} \lambda_k x_k.
\]
3 Minimum Distance Results

We review the minimum distance problem and its main duality results.

We first state a theorem which relates finding the minimum distance from a point to a subspace with a maximization problem in the dual space. This can be thought of as a generalization of the projection theorem in Hilbert spaces, and is the first of the two main theorems in [12, Ch. 5]. The use of max indicates that there is a variable for which the sup is realized.

**Theorem 5.** Let $X$ be a real normed linear space, let $S \subset X$ be a subspace, and let $y \in X$. Then:

$$\inf_{x \in S} \|y - x\| = \max_{\lambda \in S^\perp} \langle y, \lambda \rangle.$$  

We now state the other main theorem of [12, Ch. 5], which instead relates a minimum distance problem in the dual space with a maximization problem in the original/primal space. The use of min similarly indicates that there is a variable for which the inf is realized.

**Theorem 6.** Let $X$ be a real normed linear space, let $S \subset X$ be a subspace, and let $\zeta \in X^*$. Then:

$$\min_{\lambda \in S^\perp} \|\zeta - \lambda\| = \sup_{x \in S} \langle x, \zeta \rangle.$$  

This theorem was restated in [14, 15, 16], in a manner which was then repeated in several papers including [10, 4, 5, 7, 6, 9, 2, 3, 8]. The basic idea was that given this relation between a problem in the dual space and one in the primal space, it could be rewritten as a relation between a problem in the primal space and one in the predual space. We now take the theorem from those papers and state it as a conjecture.

**Conjecture 7.** Let $X$ be a real normed linear space, let $S \subset X$ be a subspace, and let $y \in X$. Then:

$$\min_{x \in S} \|y - x\| = \sup_{\nu \in S^\perp} \langle \nu, y \rangle.$$  

Looking at Theorem 5 along with Conjecture 7, we see that given a problem of finding the minimum distance from a point to a subspace, we could choose to solve an equivalent maximization problem in either the dual or the predual space, over either the annihilator or the pre-annihilator, with the only difference being that the maximum is only guaranteed to be realized in the dual space.

Note that in some of these papers, including [14, 15, 16], the conjecture was stated along with a non-standard definition of the pre-annihilator. This indeed alters its veracity, and is discussed further in Remark 13.
4 Counterexample

We now present a counterexample to the conjecture.

Example 8. Let \( X = \ell_1 \). It is then well-established \([12, 13]\) that \( X^* \simeq \ell_\infty \) and that \( ^*X \simeq c_0 \).

Now consider the subspace \( S \subset X \), motivated by an exercise in \([13]\), given by:

\[
S = \left\{ x \in \ell_1 \mid \sum_{k=0}^{\infty} x_k = 0 \right\}.
\]

Then, considering \( x \in S \) with \( x_0 = -1 \), \( x_i = 1 \) for some \( i \in \mathbb{Z}_{++} \), and \( x_j = 0 \ \forall \ j \notin \{0, i\} \),

\[
\lambda \in S^\perp \implies \langle x, \lambda \rangle = \sum_{k=0}^{\infty} \lambda_k x_k = \lambda_i - \lambda_0 = 0
\]

\[
\implies \lambda_i = \lambda_0.
\]

Since \( i \in \mathbb{Z}_{++} \) was arbitrary, we have \( \lambda \) constant as a necessary condition for \( \lambda \in S^\perp \). Then, taking any \( x \in S \), and any \( \alpha \in \mathbb{R} \):

\[
\lambda_k = \alpha \ \forall \ k \in \mathbb{Z}_+ \implies \langle x, \lambda \rangle = \sum_{k=0}^{\infty} \lambda_k x_k = \alpha \sum_{k=0}^{\infty} x_k = 0
\]

\[
\implies \lambda \in S^\perp.
\]

So this is a sufficient condition as well, and we have the annihilator as:

\[
S^\perp = \{ \lambda \in \ell_\infty \mid \lambda_k = \alpha \ \forall \ k \in \mathbb{Z}_+ \text{ for some } \alpha \in \mathbb{R} \}.
\]

In trying to work out the pre-annihilator, we would follow the same steps (just with the order of the pairing reversed), and draw the same conclusion, but we further need \( ^\perp S \subset c_0 \). There is only one element which is constant, and for which the sequence converges to zero, and so we have:

\[
^\perp S = \{ 0 \}.
\]

Now, consider \( y \in \ell_1 \) such that \( y_0 = 1 \) and \( y_k = 0 \) for all \( k \in \mathbb{Z}_{++} \); i.e., \( y = \{1, 0, 0, 0, \ldots\} \)

(actually, any \( 0 \neq y \in \ell_1 \) will do).

If \( x \in S \), then,

\[
\sum_{k=0}^{\infty} x_k = 0 \implies \left| \sum_{k=1}^{\infty} x_k \right| = |x_0| \implies \sum_{k=1}^{\infty} |x_k| \geq |x_0|,
\]
and then:

\[ \|y - x\|_{\ell_1} = |1 - x_0| + \sum_{k=1}^{\infty} |x_k| \]

\[ \geq |1 - x_0| + \sum_{k=1}^{\infty} x_0 \]

\[ \geq 1. \]

This bound is clearly achievable by choosing \( x = 0 \), and so we have:

\[ \min_{x \in S} \|y - x\| = 1. \]

But,

\[ \sup_{\nu \in \perp S \atop \|\nu\| \leq 1} \langle \nu, y \rangle = 0, \]

and so Conjecture 7 is false.

This may raise the question of whether a counterexample can be found for which the pre-annihilator is not trivial, especially since the supremum would typically be approached as the norm of the predual variable approached one. We thus also provide the following adjusted example.

**Example 9.** Again let \( X = \ell_1 \), with \( X^* \simeq \ell_\infty \) and \( *X \simeq c_0 \).

Now consider the subspace \( S \subset X \) given by:

\[ S = \left\{ x \in \ell_1 \mid x_0 = 0, \sum_{k=1}^{\infty} x_k = 0 \right\}. \]

By similar arguments, we find that the annihilator is:

\[ S^\perp = \left\{ \lambda \in \ell_\infty \mid \lambda_k = \alpha \ \forall k \in \mathbb{Z}_{++} \text{ for some } \alpha \in \mathbb{R} \right\}; \]

that is, the first element may be any real number, and it must be some constant thereafter, and that the pre-annihilator is:

\[ \perp S = \left\{ \lambda \in \ell_\infty \mid \lambda_k = 0 \ \forall k \in \mathbb{Z}_{++} \right\}; \]

that is, the first element may be any real number, and it must be zero thereafter.
We then consider $y \in \ell_1$ such that $y_0 = 1$, $y_1 = 1$ and $y_k = 0$ for all $2 \leq k \in \mathbb{Z}_+$; i.e., $y = \{1, 1, 0, 0, \ldots\}$. Then, if $x \in S$:

$$
\|y - x\|_{\ell_1} = |1 - x_0| + |1 - x_1| + \sum_{k=2}^{\infty} |x_k|
$$

$$
= 1 + |1 - x_1| + \sum_{k=2}^{\infty} |x_k|
$$

$$
\geq 1 + |1 - x_1| + |x_1|
$$

$$
\geq 2,
$$

and so, since this bound can also be achieved with $x = 0$,

$$
\min_{x \in S} \|y - x\| = 2.
$$

But,

$$
\sup_{\nu \in \perp S, \|\nu\| \leq 1} \langle \nu, y \rangle = \sup_{\|\nu\| \leq 1} \nu_0 = 1.
$$

4.1 Discussion

We briefly discuss where one would fail if attempting to prove the conjecture by following steps similar to those in the proofs of Theorem 5 or Theorem 6.

In either case, the same steps can be used to establish that the minimization will result in a value greater than or equal to that of the maximization, and the problem arises when trying to establish that a particular variable exists for one which will achieve the value of the other, which would then establish the equality.

In the proof of Theorem 5, a linear functional of unit norm which vanishes on $S$ is defined on the subspace $[y + S] = \{\alpha y + x \mid \alpha \in \mathbb{R}, x \in S\}$, which is then extended to the entire space $X$ via the Hahn-Banach theorem. This provides a $\lambda \in S^\perp$ of unit norm such that $\langle y, \lambda \rangle = \inf_{x \in S} \|y - x\|$, establishing the equality. The unrecoverable problem when trying to apply the framework of this proof to the conjecture, is that the Hahn-Banach theorem provides an extension to a functional on the primal space, and thus to an element of the dual space, but cannot be guaranteed to provide an extension to an element of the predual space.

Suppose we take the value of the supremum to be $m$, and instead try to adapt the framework of the proof of Theorem 6 to prove the conjecture. We can define $\tilde{y} : \perp S \to \mathbb{R}$ by restriction $\tilde{y} = y|_{\perp S}$, such that $\|\tilde{y}\| = m$. We then can indeed use the Hahn-Banach theorem to extend this to $\tilde{y} : X \to \mathbb{R}$, i.e., to $\tilde{y} \in X$, with $\|\tilde{y}\| = m$, and with $\langle \nu, \tilde{y} \rangle = \langle \nu, y \rangle$ if $\nu \in \perp S$. If we then let $\hat{x} = y - \tilde{y}$, we get $\|y - \hat{x}\| = \|\tilde{y}\| = m$ as desired. The problem then arises when trying to verify that $\hat{x} \in S$. Given an arbitrary $\nu \in \perp S$, this implies that $\langle \nu, \hat{x} \rangle = \langle \nu, y - \tilde{y} \rangle = 0$, but this only establishes that $\hat{x} \in (\perp S)^\perp$, which in general is a superset of $S$. 

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5 Fix

We now state a few versions of the fixed duality relation.

**Theorem 10.** Let $X$ be a real normed linear space, let $S \subset X$ be a subspace, and let $y \in X$. Then:

\[
\inf_{x \in S} \|y - x\| \geq \min_{x \in (S^\perp)^\perp} \|y - x\| = \sup_{\nu \in (S^\perp)^\perp, \|\nu\| \leq 1} \langle \nu, y \rangle.
\]

**Proof.** The inequality follows immediately from the fact that $S \subset (S^\perp)^\perp$. The equality follows immediately from Theorem 6.

This has the following obvious and more useful corollary.

**Corollary 11.** Let $X$ be a real normed linear space, let $S \subset X$ be a subspace such that $(S^\perp)^\perp = S$, and let $y \in X$. Then:

\[
\min_{x \in S} \|y - x\| = \sup_{\nu \in S, \|\nu\| \leq 1} \langle \nu, y \rangle.
\]

**Remark 12.** This condition is later verified to hold for the subspace of interest in many of the papers which used the conjecture, including [14, 15, 16], so the main results therein are unaffected. The condition should obviously be checked for other papers which used the conjecture as well.

**Remark 13.** Occasionally, the pre-annihilator has been defined as the set $S^\perp$ for which $(S^\perp)^\perp \simeq S$. This includes the initial series of papers with the conjecture [14, 15, 16], as well as [4, 5, 6, 2, 3]. While this does avoid the problem with the conjecture, and ultimately makes it equivalent to Corollary 11, it is not the standard definition, and as we can see from the counterexamples, it is not always commensurate with the standard definition. In Example 8, for instance, $(S^\perp)^\perp \simeq X$.

**Remark 14.** Note that closure is not enough to guarantee that this condition will hold, as is already evident from the counterexamples, in which $S$ was indeed a closed subspace. It follows from $S$ being a closed subspace that $S = ^\perp(S^\perp)$ [12, p. 118], but in general we still have $S \subset (S^\perp)^\perp$.

Assuming closure does allow our condition to become necessary as well as sufficient for this strong duality to hold for any point, as stated in our final theorem.

**Theorem 15.** Let $X$ be a real normed linear space, and let $S \subset X$ be a closed subspace. Then:

\[
\min_{x \in S} \|y - x\| = \sup_{\nu \in S, \|\nu\| \leq 1} \langle \nu, y \rangle
\]

for all $y \in X$, if and only if $(S^\perp)^\perp = S$. 

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Proof. Sufficiency follows from Corollary 11.

Now assume that \( S \subset (\perp S)^+ \) is a proper subset, and choose \( y \in (\perp S)^+ \setminus S \). It follows from Theorem 10 that
\[
\sup_{\nu \in \perp S} \langle \nu, y \rangle = \min_{x \in (\perp S)^+} \|y - x\| = 0.
\]

If we had \( \inf_{x \in S} \|y - x\| = 0 \), then for any \( \varepsilon > 0 \), there would exist \( x \in S \) such that \( \|y - x\| \leq \varepsilon \), making \( y \) a limit point of \( S \), and since \( S \) is closed this gives \( y \in S \), a contradiction.

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