GENERALIZED STRICHARTZ ESTIMATES AND SCATTERING FOR 3D ZAKHAROV SYSTEM

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ABSTRACT. We obtain scattering for the 3D Zakharov system with non-radial small data in the energy space with angular regularity of degree one. The main ingredient is a generalized Strichartz estimate for the Schrödinger equation in the space of $L^2$ angular integrability.

1. Introduction

The aim of this paper is twofold. We firstly obtain generalized Strichartz estimates for radial dispersive equations. Secondly, making use of these estimates we prove scattering for the 3D Zakharov system with non-radial initial data.

Strichartz estimates. To begin with, let us consider the following Schrödinger-type dispersive equations

$$i\partial_t u + D^a u = 0, \quad u(0, x) = f(x) \quad (1.1)$$

where $u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $D = \sqrt{-\Delta}$, $a > 0$. Two typical examples are the wave equation ($a = 1$) and the Schrödinger equation ($a = 2$). The space time estimates which are called Strichartz estimates address the estimates

$$\| e^{itD^a} P_0 f \|_{L^q_t L^p_x} \lesssim \| f \|_{L^2} \quad (1.2)$$

where $\hat{P_0} f \approx 1_{|\xi| \sim 1} \hat{f}$ (See the end of this section for the precise definition). Strichartz [40] proved (1.2) for the case $q = p$ by drawing connection between the estimate (1.2) and Fourier restriction estimate which is known as Tomas-Stein theorem [44] (also see [38] p.364-369 and references therein). Since then, the estimates were substantially extended by various authors, e.g. [10, 23] for $a = 1$, and [9, 46] for $a = 2$. It is now well-known (see [19]) that (1.2) holds whenever the following admissible condition is satisfied:

$$\text{AP}(a) : \begin{cases} 2 \leq q, p \leq \infty, \frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{p}), (q, p, d) \neq (2, \infty, 3); \quad a = 1, \\ 2 \leq q, p \leq \infty, \frac{1}{q} \leq \frac{d}{2}(\frac{1}{2} - \frac{1}{p}), (q, p, d) \neq (2, \infty, 2); \quad a \neq 1. \end{cases} \quad (1.3)$$

If the function $f$ is assumed to be radially symmetric, as expected naturally (1.2) holds for a wider range of $(q, p)$ than AP(a) (for example, see [21, 36, 7, 33]). The optimal range for (1.2) under radial symmetry assumption is known except one endpoint case ($d \geq 2$). Namely, if $f$ is radial, then (1.2) holds if the following condition holds:

$$\text{RAP}(a) : \begin{cases} 2 \leq q, p \leq \infty, \frac{1}{q} < (d - 1)(\frac{1}{2} - \frac{1}{p}) \text{ or } (q, p) = (\infty, 2); \quad a = 1, \\ 2 \leq q, p \leq \infty, \frac{1}{q} < (d - \frac{1}{2})(\frac{1}{2} - \frac{1}{p}), (q, p) \neq (2, \frac{4d-3}{2d-3}); \quad a \neq 1. \end{cases} \quad (1.4)$$
The sharp range was obtained in [15] except some endpoints when \( a > 1 \) and the remaining endpoint estimates were later obtained in [6, 18] independently. See [6] for the estimates for dispersive equations defined by more general pseudo-differential operators.

Inspired by the result for radial functions, one may try to find a weaker variant of Strichartz estimate which is valid for non-radial functions and has the wider admissible range. There are two notable approaches in this direction. The first is to consider the estimate with additional angular regularity

\[
\| e^{itD^a} P_0 f \|_{L_t^q L_x^p} \lesssim \| f \|_{H^{0,s}_\omega} \quad (1.5)
\]

in which some angular regularity is traded off by the extension of admissible range. See the end of this section for the definition of \( H^{0,s}_\omega \). When \( a = 1 \), the estimate with almost sharp regularity was obtained in [39] (also see [8], [17] and [6]). When \( a \neq 1 \), in [6] the authors obtained (1.5) for some \( s > 0 \) and \( (q, p) \) satisfying RAP(a). However the problem of obtaining (1.5) with the optimal regularity is still open.

The latter is to consider the estimate with weaker angular integrability, namely

\[
\| e^{itD^a} P_0 f \|_{L_t^q L_x^p L^s_\omega} \lesssim \| f \|_{L^2_x} \quad (1.6)
\]

for \( s < p \). Here the norm \( L_t^q L_x^p L^s_\omega \) for function \( u(t, x) \) on \( \mathbb{R} \times \mathbb{R}^d (d \geq 2) \) is defined as follows

\[
\| u \|_{L_t^q L_x^p L^s_\omega} = \left( \int_{\mathbb{R}} \left[ \int_0^\infty \int_{S^{d-1}} |u(t, \rho x')|^s d\omega(x') \right]^{\frac{q}{s}} \rho^{d-1} d\rho \right)^{\frac{1}{q}} dt .
\]

Since \( f \) is assumed to be in \( L^2 \) spaces, in view of orthogonality of spherical harmonics the \( L^2 \) estimate

\[
\| e^{itD^a} P_0 f \|_{L_t^q L_x^p L^s_\omega} \lesssim \| f \|_{L^2_x} \quad (1.6)
\]

is most convenient to work with. These type of norm was used in [42] to obtain the endpoint case of Strichartz estimate for 2D Schrödinger (see [24] for 3D wave equation) which is not allowed in the usual mixed norm spaces. For the wave equation \( (a = 1) \), it was known that (1.6) also holds for RAP(1) pairs (see [35], [17] and [6]). However, when \( a \neq 1 \), as far as the authors know, it seems that (1.6) is known only for some RAP(1) pair with additional condition \( q \geq p \) (see Theorem 1.7 in [17]).

The first purpose of this paper is to consider the generalized estimates of the type (1.6) for the case \( a \neq 1 \). The other motivation is from the recent study of Zakharov system. From the viewpoint of application, the estimate (1.6) works better than (1.5), because there is no loss of angular regularity. The first result of this paper is the following.

**Theorem 1.1.** Let \( a > 1, d \geq 3 \) and \( p(d) = \frac{6d-7+\sqrt{4d^2+16d-7}}{4d-7} \). Then the estimate (1.6) holds if

\[
2 \leq q, p \leq \infty, \quad \frac{1}{q} < \frac{p(d)}{p(d) - 2} \left( \frac{1}{2} - \frac{1}{p} \right) \text{ or } (q, p) = (\infty, 2).
\]  

For the Schrödinger case \( a = 2 \), our results are new. The range is strictly contained in RAP(2), but wider than RAP(1) which is crucial for the application to Zakharov system. For example, for \( d = 3 \), \( p(3) \approx 3.48 \), then \( (2, 4) \) satisfies the condition.
Now we sketch the ideas in proving Theorem 1.1. We first use spherical harmonic expansion of $L^2(S^{d-1})$ to reduce the estimates (1.6) to the uniform boundedness of one-dimensional oscillatory integral operators associated with Bessel functions of different orders. To show (1.6) we need to obtain the estimates which are uniform along the orders of Bessel functions. For the wave case $a = 1$, uniform decay estimates of $J_\mu$ are sufficient. However, to get (1.6) on the range wider than RAP(1) for the case $a > 1$, we need to exploit the oscillatory effect due to the non-vanishing second derivative ($a > 1$). For this purpose we split Bessel function $J_\mu$ of order $\mu$ into two parts so that $J_\mu = J_\mu^M + J_\mu^E$ (see (2.6)). For the error term $J_\mu^E$, one has better uniform decay estimates, and for the main term $J_\mu^M$ we basically rely on $TT^*$ method and need to obtain uniform kernel estimates for which we carry out rather delicate analysis based on the stationary phase method.

Scattering for 3D Zakharov system. We now consider the scattering problem for the 3D Zakharov system which was introduced by Zakharov [47] as a mathematical model for the Langmuir turbulence in unmagnetized ionized plasma:

\[
\begin{align*}
  i \dot{u} - \Delta u &= nu, \\
  \frac{\dot{n}}{\alpha^2} - \Delta n &= -\Delta |u|^2,
\end{align*}
\]

(1.8)

with the initial data

\[
  u(0, x) = u_0, \quad n(0, x) = n_0, \quad \dot{n}(0, x) = n_1,
\]

(1.9)

where $(u, n)(t, x) : \mathbb{R}^{1+3} \to \mathbb{C} \times \mathbb{R}$, and $\alpha > 0$ denotes the ion sound speed. It preserves $\|u(t)\|_{L^2_x}$ and the energy

\[
E = \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|D^{-1}\dot{n}|^2/\alpha^2 + |n|^2}{2} - n|u|^2 dx.
\]

(1.10)

The natural energy space for initial data is

\[
(u_0, n_0, n_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times \dot{H}^{-1}(\mathbb{R}^3).
\]

(1.11)

Local wellposedness (without symmetry) was well understood. The well-posedness in the energy space was proved in [5] for $d = 2, 3$ and in [12] for $d = 1$, and in weighted Sobolev space in [20]. Unconditional uniqueness for 3D in energy space was shown in [26]. Improvement to the critical regularity was obtained in [12] for $d = 1, 2$, and to the full subcritical regularity in [12] for $d \geq 3$. Global well-posedness with small norm in energy space was proved in [5]. In [11] the well-posedness for the system on the torus was studied. See [31] results on the subsonic limit to NLS (as $\alpha \to \infty$). Concerning the long time and blow-up behavior, Merle [27] obtained blow-up in finite or infinite time for negative energy, and the scattering was studied in [33] [11] [29].

Under radial symmetry assumption the scattering for 3D Zakharov system with small energy was shown in [13] and global dynamics below ground state was obtained in [14]. Very recently, the scattering for non radial case was obtained by Hani-Pusateri-Shatah [16] under the assumption that the initial data are small enough and have sufficient regularity and decay.

Here we take a different direction by considering initial data with additional angular regularity. Compared with the condition imposed for the scattering result in
our condition on the initial data is much weaker in view of the simple embedding relation between angular regularity and weighted Sobolev space. Moreover, our result contains the existence of the wave operator, and hence the scattering operator is constructed. The following is our second result.

**Theorem 1.2.** Assume \( \|(u_0, n_0, n_1)\|_{H^1_ω \times H^{0,1}_ω \times \dot{H}^{-1,1}_ω} = \varepsilon \) for \( \varepsilon > 0 \) sufficiently small. Then the global solution \((u, n)\) to (1.8) belongs to \( C^0_ωH^{1,1}_ω \times C^0_ωH^{0,1}_ω \cap \dot{C}^1_ω\dot{H}^{-1,1}_ω \), and scatters in this space: there exists \( (u_{0,±}, n_{0,±}, n_{1,±}) \in H^{1,1}_ω \times H^{0,1}_ω \times \dot{H}^{-1,1}_ω \) such that

\[
\lim_{t \to \pm \infty} \|(u, n, \dot{n}) - (u_{0,±}, n_{0,±}, n_{1,±})\|_{H^{1,1}_ω \times H^{0,1}_ω \times \dot{H}^{-1,1}_ω} = 0 \tag{1.12}
\]

where \((u_{0,±}, n_{0,±}, n_{1,±})\) is the solution to the linear Zakharov system with the initial datum \((u_{0,±}, n_{0,±}, n_{1,±})\). Moreover, for any \((u_{0,±}, n_{0,±}, n_{1,±}) \in H^{1,1}_ω \times H^{0,1}_ω \times \dot{H}^{-1,1}_ω \) with sufficiently small norm, there exists a unique global solution \((u, n)\) to (1.8) such that (1.12) holds.

We now give some words for the proof of Theorem 1.2. The main difficulty lies in derivative loss and slow dispersion of the wave equation together with the quadratic nonlinearity. We basically follows the idea in [13] by making use of the generalized Strichartz estimates (1.16). In fact, we combine (1.16) and the normal form technique (for example, see [32] and [30]) to capture some nonlinear oscillations. In order to get around the weak angular integrability, we add some angular regularity so that the resulting space in angular variable becomes a Banach algebra.

**Notations.** Finally we close this section by listing the notation.

- \( \mathcal{F}(f) \) and \( \hat{f} \) denote the Fourier transform of \( f \). For \( a \geq 1 \), \( S_a(t) = e^{itD^a} = \mathcal{F}^{-1} e^{it|\xi|^a} \mathcal{F} \), and \( S(t) = S_2(t) \).
- \( \eta : \mathbb{R} \to [0,1] \) is an even, non-negative smooth function which is supported in \( \{ \xi : |\xi| \leq 8/5 \} \) and \( \eta \equiv 1 \) for \( |\xi| \leq 5/4 \).
- For \( k \in \mathbb{Z} \), \( \chi_k(\xi) = \eta(\xi/2^k) - \eta(\xi/2^{k-1}) \) and \( \chi_{\leq k}(\xi) = \eta(\xi/2^k) \).
- \( P_k, P_{\leq k} \) are defined on \( L^2(\mathbb{R}^d) \) by \( P_k u(\xi) = \chi_k(|\xi|) \hat{u}(\xi), P_{\leq k} u(\xi) = \chi_{\leq k}(|\xi|) \hat{u}(\xi) \).
- \( \Delta_ω \) denotes the Laplace-Beltrami operator on the unite sphere \( S^{d-1} \) endowed with the standard metric \( g \) measure \( d\omega \) and \( \Lambda_ω = \sqrt{1-\Delta_ω} \).
- For \( 1 \leq i, j \leq n, X_{ij} = x_i \partial_j - x_j \partial_i \). It is well-known that for \( f \in C^2(\mathbb{R}^n) \)

\[
\Delta_ω(f)(x) = \sum_{1 \leq i < j \leq n} X_{ij}^2(f)(x).
\]

Denote \( L^p_ω = L^p(\mathbb{R}^{d-1} : d\omega), H^s_ω = H^s(\mathbb{R}^{d-1}) = \Lambda_ω^{-s} L^p_ω \).

- \( L^p(\mathbb{R}^d) \) denotes the usual Lebesgue space, and \( L^p(\mathbb{R}^+ : r^{d-1} dr) \).
- \( L^p_t L^q_ω \) and \( L^p_t H^s_ω \) are Banach spaces defined by the following norms

\[
\|f\|_{L^p_t L^q_ω} = \|\|f(r\omega)\|_{L^q_ω}\|_{L^p_t}, \|f\|_{L^p_t H^s_ω} = \|\|f(r\omega)\|_{H^s_ω}\|_{L^p_t}.
\]

- \( H^s_ω, \dot{H}^s_ω \) are the usual Sobolev (Besov) spaces on \( \mathbb{R}^d \).
- \( \mathcal{B}^s_{(p,q),r} \) denotes the Besov-type space given by the norm

\[
\|f\|_{\mathcal{B}^s_{(p,q),r}} = \left( \sum_{k \in \mathbb{Z}} 2^{ksr} \|P_k f\|_{L^p_t L^q_ω}^r \right)^{1/r}.
\]
• $H^{s,\alpha}_{p,\omega}$ is the space with the norm \( \|f\|_{H^{s,\alpha}_{p,\omega}} = \|\Lambda^\alpha f\|_{L^p}, \) and the spaces $ \dot{H}^{s,\alpha}_{p,\omega}$, $B^{s,\alpha}_{p,\omega}$, $\dot{B}^{s,\alpha}_{p,q,\omega}$, and $\dot{B}^{s,\alpha}_{(p,q),\omega}$ are defined similarly.

• For simplicity, we denote $H^{s,\alpha}_\omega = H^{s,\alpha}_{2,\omega}$, $\dot{H}^{s,\alpha}_\omega = \dot{H}^{s,\alpha}_{2,\omega}$, $B^{s,\alpha}_\omega = B^{s,\alpha}_{2,\omega}$, $\dot{B}^{s,\alpha}_{(p,q),\omega} = \dot{B}^{s,\alpha}_{(p,2),\omega}$.

• Let $X$ be a Banach space on $\mathbb{R}^d$. $L^q_X$ denotes the space-time space on $\mathbb{R} \times \mathbb{R}^d$ with the norm $\|u\|_{L^q_X} = \|\|u(t, \cdot)\|_X\|_{L^q_t}$.

2. GENERALIZED STRICHTZ Estimates

In this section, we prove Theorem 1.1. First, we make some reductions. To prove (1.6), it is equivalent to show

$$
\|T_a f\|_{L^2_t L^2_x} \lesssim \|f\|_{L^2}.
$$

where

$$
T_a f(t, x) = \int_{\mathbb{R}^d} e^{it|\xi|^2} |\lambda(t)|^{\alpha} f(\xi) d\xi.
$$

Now we expand $f$ by the orthonormal basis $\{Y^l_k\}$, $k \geq 0, 1 \leq l \leq d(k)$ of spherical harmonics with $d(k) = C^k_{n+k-1} - C^{k-2}_{n+k-3}$, such that

$$
f(\xi) = f(\rho \sigma) = \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} a^l_k(\rho) Y^l_k(\sigma).
$$

Using the identities (see [37])

$$
\hat{Y}^l_k(\rho \sigma) = c_{d,k} \rho^{-d/2} J_{\nu} (\rho) Y^l_k(\sigma)
$$

where $c_{d,k} = (2\pi)^{d/2} k^{-k}$, \( \nu = \nu(k) = \frac{d-2+2k}{2} \), then we get

$$
T_a f(t, x) = \sum_{k,l} c_{d,k} a^l_k(T^\nu_{a_l})(t, |x|) Y^l_k(x/|x|),
$$

where

$$
T^\nu_{a_l}(h)(t, r) = r^{-d/2} \int e^{-it\rho^\alpha} J_{\nu} (\rho) \rho^{d/2} \lambda(\rho) h(\rho) d\rho.
$$

Here $J_{\nu}(r)$ is the Bessel function

$$
J_{\nu}(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2) \pi^{1/2}} \int_1^\infty e^{it(1 - t^2)^{\nu-1/2}} dt, \quad \nu > -1/2.
$$

Thus (2.1) becomes

$$
\|T^\nu_{a_l}(a^l_k)\|_{L^2_t L^2_x} \lesssim \|a^l_k(\rho)\|_{L^2_{\rho, \nu}}.
$$

To prove (2.2), it is equivalent to show

$$
\|T^\nu_{a_l}(h)\|_{L^2_t L^2_x} \lesssim \|h\|_{L^2},
$$

with a bound independent of $\nu$, since $q, p \geq 2$.

By the classical Strichartz estimates (see the endpoint estimates in [19] [24]), we can get $\|1_{\nu \leq T_{a_l}^\nu}(h)\|_{L^2_t L^2_x} \lesssim \|h\|_{L^2}$. Thus it remains to show

$$
\|1_{\nu > T_{a_l}^\nu}(h)\|_{L^2_t L^2_x} \lesssim \|h\|_{L^2},
$$

(2.4)
with a bound independent of \( \nu \). For any \( R \gg 1 \), define
\[
S_R^{\nu,a}(t,r) = \chi_0(t \frac{r}{R}) \int e^{-it\theta^a} J_\nu(r) \chi_0(\rho) h(\rho) d\rho.
\]
Then
\[
1_{r \geq 1} T^\nu_a \| S_{\nu,a} \|_{L^q \to L^p} \lesssim \sum_{j \geq 5} 2^j \left( \frac{d-1}{q} - \frac{d-2}{p} \right) \| S_{2j}^{\nu,a} \|_{L^2 \to L^q L^p}.
\]
Then to prove (2.4), it suffices to show for some \( \delta > 0 \)
\[
\| S_{\nu,a}^{\nu,\delta}(h) \|_{L^q L^p} \leq C R \frac{d-2}{\nu} \frac{d-1}{p} - \delta \| h \|_{L^2},
\] (2.5)
where \( C \) is independent of \( \nu \). By interpolation, we only need to show (2.5) for \((q, p) = (2, p)\).

In the radial case, the estimates (2.3) can be reduced to (2.5) with \( \nu = d-2 \), see [15]. The same argument in [15] also works for fixed \( \nu \), but with \( \nu \)-dependent bound, see also [6]. The difficulty in (2.5) is to obtain a uniform bound as \( \nu \to \infty \), thus the proof in [15] [6] does not work. We need to exploit the uniform properties of the Bessel function with respect to \( \nu \). In order to do so, we use the Schl"afli’s integral representation (see p. 176, [45]):
\[
J_\nu(r) = \frac{1}{\pi} \Re \int_0^\pi e^{i(r \sin \theta - \nu \theta)} d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu \tau - r \sinh \tau} d\tau
\]
\[
:= J^M_\nu(r) - J^E_\nu(r).
\] (2.6)
Obviously, the main term in (2.6) is \( J^M_\nu(r) \). First we recall the Van der Corput Lemma (see p. 334, [38]):

**Lemma 2.1** (Van der Corput). Suppose \( \phi \) is real-valued and smooth in \( (a, b) \), and that \( |\phi^{(k)}(x)| \geq 1 \) for all \( x \in (a, b) \). Then
\[
\left| \int_a^b e^{i\lambda \phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left[ |\psi(b)| + \int_a^b |\psi'(x)| dx \right]
\]
holds when (i) \( k \geq 2 \), or (ii) \( k = 1 \) and \( \phi'(x) \) is monotonic. Here \( c_k \) is a constant depending only on \( k \).

Next, we recall the uniform decay estimates for the Bessel functions, which can be found in [15] p. 229 (7). For completeness, we present a proof here because we need the estimate for the error term.

**Lemma 2.2** (Uniform decay estimates). Assume \( r, \nu > 10 \). Then we have
\[
\left| \int_0^\pi e^{i(r \sin \theta - \nu \theta)} d\theta \right| \leq C(1 + |r^2 - \nu^2|)^{-1/4},
\] (2.7)
As a consequence,
\[
|J_\nu(r)| + |J'_\nu(r)| \leq C(1 + |r^2 - \nu^2|)^{-1/4},
\] (2.8)
and for any \( R \geq 1 \),
\[
\int_{r \sim R} |J_\nu(r)|^2 + |J'_\nu(r)|^2 dr \lesssim 1.
\] (2.9)
Proof. We need only to show (2.7), since (2.8) follows immediately from (2.7), the fact that $2J_{\nu}'(r) = J_{\nu-1}(r) - J_{\nu+1}(r)$ (see p. 45 (2) in [45]) and the following observation

$$|J_{\nu}^E(r)| + |(J_{\nu}^E)'(r)| \lesssim (r + \nu)^{-1}. \quad (2.10)$$

We have

$$\left| \int_0^\pi e^{i(r \sin \theta - \nu \theta)} d\theta \right| \lesssim \int_0^{\pi/2} e^{i(r \sin \theta - \nu \theta)} d\theta + \int_{\pi/2}^\pi e^{i(r \sin \theta - \nu \theta)} d\theta := I + II.$$

Using Lemma 2.1 and integration by part, we easily get

$$II \lesssim \int_{\pi/2}^\pi e^{i(r \sin \theta - \nu \theta)} d\theta + \int_{\pi/2}^{\pi} e^{i(r \sin \theta - \nu \theta)} d\theta \lesssim \min(\nu^{-1/2}, (1 + r + \nu)^{-1})$$

which suffices to give the bound as desired. It remains to control the term $I$. We have the trivial bound $I \lesssim 1$. Then we show $I \lesssim |r^2 - \nu^2|^{-1/4}$ case by case.

Case 1: $\nu > r$. If $\nu \geq 2r$, then we have $I \lesssim \nu^{-1}$, which suffices to give the bound. Now we assume $r < \nu < 2r$. Fixing a $\varepsilon > 0$, then we get from the stationary phase and integration by part that

$$I \lesssim \int_0^\varepsilon e^{i(r \sin \theta - \nu \theta)} d\theta + \int_\varepsilon^\pi e^{i(r \sin \theta - \nu \theta)} d\theta \lesssim \min(\varepsilon, |\nu - r|^{-1}) + (r\varepsilon)^{-1/2} \lesssim r^{-1/2} + (r\varepsilon)^{-1/2}.$$

Thus setting $\varepsilon = r^{-1/2}|r - \nu|^{1/2}$, we obtain the bound, as desired.

Case 2: $\nu \leq r$. We may assume further $\nu < r$. Let $\theta_0 = \arccos \frac{\nu}{r}$, then we have $\theta_0 \sim \sqrt{1 - \frac{\nu^2}{r^2}}$. Denote $E = \{|\theta - \theta_0| \geq \frac{\theta_0}{r^2} \cap [0, \pi/2]\}$ and $A = r^{-1/2}(r^2 - \nu^2)^{1/8}$. Note that $|r \cos \theta - \nu| \gtrsim r\theta_0^2$ on $E \cap [0, \pi/2]$. If $\theta_0 \gtrsim A$, then

$$I \lesssim \int_{\{|\theta - \theta_0| < \frac{\theta_0}{r^2} \cap [0, \pi/2]\}}^\pi e^{i(r \sin \theta - \nu \theta)} d\theta + \int_{E \cap [0, \pi/2]} e^{i(r \sin \theta - \nu \theta)} d\theta \lesssim (r\theta_0)^{-1/2} + (r\theta_0^2)^{-1} \lesssim |r^2 - \nu^2|^{-1/4},$$

else if $\theta_0 \ll A$, namely $r^{-1/2}(r^2 - \nu^2)^{3/8} \ll 1$, then

$$I \lesssim \int_{E \cap [0, \pi/2]} e^{i(r \sin \theta - \nu \theta)} d\theta + \int_{E \cap [0, A]} e^{i(r \sin \theta - \nu \theta)} d\theta + \int_{E \cap [A, \pi/2]} e^{i(r \sin \theta - \nu \theta)} d\theta \lesssim (r\theta_0)^{-1/2} + A + (rA)^{-1} \lesssim |r^2 - \nu^2|^{-1/4}.$$

Thus the proof of (2.7) is completed.

We will also need to exploit the following asymptotical property of the Bessel functions. The following lemma was obtained in [12], see e.g. Lemma 3 in [2].

Lemma 2.3 (Asymptotical property). Let $\nu > 10$ and $r > \nu + \nu^{1/3}$, we have

$$J_{\nu}(r) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\theta(r)} + e^{-i\theta(r)}}{(r^2 - \nu^2)^{1/4}} + h(\nu, r),$$

where

$$\lim_{r \to \infty} h(\nu, r) = 0.$$
where
\[ \theta(r) = (r^2 - \nu^2)^{1/2} - \nu \arccos \frac{\nu}{r} - \frac{\pi}{4} \]
and
\[ |h(\nu, r)| \lesssim \left( \frac{\nu^2}{(r^2 - \nu^2)^{7/4}} + \frac{1}{r} \right) 1_{[\nu + \nu^{1/3}, 2\nu]}(r) + r^{-1} 1_{[2\nu, \infty)}(r). \]

With these uniform properties of Bessel function and the \( L^2_t L^2_r \) estimate, which is nothing but the local smoothing effect, we are already able to obtain some new estimates.

**Lemma 2.4.** Assume \( a > 0 \). For \( \nu > 10 \), \( R \geq 1 \), \( 2 \leq p \leq \infty \)
\[
\| S^\nu,\lambda_R(h) \|_{L^2_t L^p_r} \lesssim \| h \|_{L^2}, \quad \| S^\nu,\lambda_R(h) \|_{L^\infty_t L^2_r} \lesssim \| h \|_{L^2}. \tag{2.12}
\]

**Proof.** For the first inequality, we only need to prove the estimate for \( p = 2, \infty \). For \( p = 2 \), using Plancherel’s equality in \( t \) and Lemma 2.3 we get
\[
\| \chi_0 \left( \frac{r}{R} \right) \int e^{-it\nu} J_\nu(r \rho) \chi_0(\rho) h(\rho) d\rho \|_{L^2_t L^2_r} \sim \| \chi_0 \left( \frac{r}{R} \right) J_\nu(r \rho) \chi_0(\rho) h(\rho) \|_{L^2_t L^2_r} \lesssim \| h \|_2.
\]
Similarly, for \( p = \infty \), we have
\[
\| S^\nu,\lambda_R(h) \|_{L^\infty_t L^2_r} \lesssim \| S^\nu,\lambda_R(h) \|_{L^2_t L^2_r} + \| \partial_r(S^\nu,\lambda_R(h)) \|_{L^\infty_t L^2_r} \lesssim \| h \|_2.
\]
The second inequality follows immediately from Minkowski’s inequality. Thus, the proof is completed. \( \square \)

By the lemma above and interpolation, we see that (2.5) holds for RAP(1) pairs. In the rest of this section, we will refine the estimate for \( S^\nu,\lambda_R \), assuming \( a = 2 \), by making use of the stationary phase method. The proof for the other case \( 1 < a < 2 \) is identical. For simplicity, we write \( S^\nu_R = S^\nu_{\lambda_R} \).

Fixing \( \lambda \geq 100 R^{1/3} \), we decompose \( S^\nu_R(h) = \sum_{j=1}^3 S^\nu_{R,j}(h), \) where
\[
S^\nu_{R,j}(h) = \chi_0 \left( \frac{r}{R} \right) \int e^{-it\nu} J_\nu(r \rho) \gamma_j \left( \frac{r \rho - \nu}{\lambda} \right) \chi_0(\rho) h(\rho) d\rho,
\]
with \( \gamma_1(x) = \eta(x) \), \( \gamma_2(x) = (1 - \eta(x)) 1_{x<0} \), and \( \gamma_3(x) = (1 - \eta(x)) 1_{x>0} \). Then as \( S^\nu_R \) we have for \( j = 1, 2, 3, 2 \leq p \leq \infty \)
\[
\| S^\nu_{R,j}(h) \|_{L^2_t L^p_r} \lesssim \| h \|_{L^2}. \tag{2.13}
\]
We will refine the estimates for \( S^\nu_{R,j}, j = 1, 2, 3, \) respectively.

**Lemma 2.5.** Assume \( R \geq 1 \), \( \lambda \geq 100 R^{1/3} \), \( 2 \leq p \leq \infty \). Then
\[
\| S^\nu_{R,1}(h) \|_{L^2_t L^p_r} \lesssim \lambda^{1/4} R^{-1/4} \| h \|_{L^2}, \tag{2.14}
\]
\[
\| S^\nu_{R,2}(h) \|_{L^2_t L^p_r} \lesssim \left( \lambda^{-1} R^{1/4} \right)^{1/2} \| h \|_{L^2}, \tag{2.15}
\]
\[
\| S^\nu_{R,3}(h) \|_{L^2_t L^p_r} \lesssim \left( \lambda^{4/3} + \left( \lambda^{-1} R^{1/4} \right)^{2/p} + R^{-1/p} \right) \| h \|_{L^2}. \tag{2.16}
\]

**Proof.** By interpolation, we only need to show the estimates for \( p = 2, \infty \).

**Step 1:** estimate of \( S^\nu_{R,1} \).
As in the proof of Lemma 2.4, by Lemma 2.3 we have
\[
\| S^\nu_{R,1}(h) \|_{L^2_t L^p_r} \lesssim \| J_\nu(r) 1_{|r - \nu| \leq \lambda} \|_2 \| h \|_2 \lesssim \lambda^{1/4} R^{-1/4} \| h \|_2.
\]
Similarly, we have
\[ \| S_{R,1}^{\nu}(h) \|_{L_t^2 L_r^\infty} \lesssim \lambda^{1/4} R^{-1/4} \| h \|_2. \]

**Step 2: estimate of \( S_{R,2}^{\nu} \).**

By the support of \( \gamma_2 \), we have \( \nu > r\rho + \lambda \) in the support of \( \gamma_2 \). Thus we use the formula (2.6). Without loss of generality, we assume \( J_\nu^M = \frac{1}{i} \int_0^\pi e^{i(r\rho \sin \theta - \nu \theta)} d\theta \) (its conjugate part can be handled in the same way), and decompose
\[ S_{R,2}^{\nu}(h) := M_{R,2}^{\nu}(h) + E_{R,2}^{\nu}(h) \]
where
\[
M_{R,2}^{\nu}(h) = \chi_0(\frac{r}{R}) \int e^{-it\rho^2} \left( \int_0^{\pi/2} e^{i(r\rho \sin \theta - \nu \theta)} d\theta \right) \gamma_2(\frac{r\rho - \nu}{\lambda}) \chi_0(\rho) h(\rho) d\rho,
\]
\[
E_{R,2}^{\nu}(h) = \chi_0(\frac{r}{R}) \int e^{-it\rho^2} \left( J_\nu^M + \int_{\pi/2}^{\pi} e^{i(r\rho \sin \theta - \nu \theta)} d\theta \right) \gamma_2(\frac{r\rho - \nu}{\lambda}) \chi_0(\rho) h(\rho) d\rho.
\]

By (2.10) and (2.11), we can easily get for \( 2 \leq p \leq \infty \)
\[ \| E_{R,2}^{\nu}(h) \|_{L_t^2 L_r^p} \lesssim R^{-1/2} \| h \|_2. \]

It remains to bound \( M_{R,2}^{\nu} \). Denote \( \phi(r, \rho, \theta) = r\rho \sin \theta - \nu \theta \). Integrating by part, we can decompose further
\[
M_{R,2}^{\nu}(h)(t, r) = \chi_0(\frac{r}{R}) \int e^{i\phi(r, \rho, \theta)} \left| \frac{\partial}{\partial \theta} \right|_{\theta=\pi/2} - \frac{\partial}{\partial \theta} \left|_{\theta=0} \right. e^{i\phi(r, \rho, \theta)} \right. \bigg|_{\theta=\pi/2} - \bigg|_{\theta=0} \bigg. \left. \int_0^{\pi/2} e^{i(r\rho \sin \theta - \nu \theta)} \rho \sin \theta (r\rho \cos \theta - \nu)^2 d\theta \right) \gamma_2(\frac{r\rho - \nu}{\lambda}) e^{-it\rho^2} \chi_0(\rho) h(\rho) d\rho.
\]
\[
:= M_{R,2,1}^{\nu}(h) - M_{R,2,2}^{\nu}(h) - M_{R,2,3}^{\nu}(h).
\]

It suffices to prove: for \( R \geq 1, j = 1, 2, 3 \)
\[ \| M_{R,2,j}^{\nu}(h) \|_{L_t^2 L_r^\infty} \lesssim \lambda^{-1} R^{1/4} \| h \|_{L^2}. \quad (2.17) \]

Indeed, we have
\[
M_{R,2,1}^{\nu}(h) = \chi_0(\frac{r}{R}) \int e^{i\phi(r, \rho, \theta)} \left. \frac{\partial}{\partial \theta} \right|_{\theta=\pi/2} - \frac{\partial}{\partial \theta} \bigg|_{\theta=0} \left. \frac{\partial}{\partial \theta} \right|_{\theta=\pi/2} - \bigg|_{\theta=0} \left. \frac{\partial}{\partial \theta} \right|_{\theta=\pi/2} \gamma_2(\frac{r\rho - \nu}{\lambda}) e^{-it\rho^2} \chi_0(\rho) h(\rho) d\rho,
\]
\[
M_{R,2,2}^{\nu}(h) = \chi_0(\frac{r}{R}) \int \frac{1}{i(r\rho - \nu)} \gamma_2(\frac{r\rho - \nu}{\lambda}) e^{-it\rho^2} \chi_0(\rho) h(\rho) d\rho.
\]

First we consider \( j = 2 \). By TT* argument, it is equivalent to show
\[ \| M_{R,2,2}^{\nu}(M_{R,2,2}^{\nu})^* f \|_{L_t^2 L_r^\infty} \lesssim \lambda^{-2} R^{1/2} \| f \|_{L_t^2 L_r^1} \]
where
\[ M_{R,2,2}^{\nu}(M_{R,2,2}^{\nu})^* f = \int K_2(t - t', r, r') f(t', r') dt' dr' \]
with the kernel
\[ K_2(t, r, r') = \int e^{-it\rho^2} \frac{\chi_0(\frac{r'}{R})}{r' - \nu} \gamma_2(\frac{r\rho - \nu}{\lambda}) \chi_0(\frac{r'}{R}) \gamma_2(\frac{r'\rho - \nu}{\lambda}) \lambda_0^2(\rho) d\rho. \]
It suffices to prove
\[ \|K_2\|_{L^1_t L^\infty_{r,r'}} \lesssim \lambda^{-2} R^{1/2}. \]  
(2.18)

Denote \( F_2 = \frac{\chi_0(r)}{i(r\rho - \nu)} \gamma_2(\frac{r\rho - \nu}{\lambda}) \frac{\chi_0(r')}{i(r'\rho - \nu)} \gamma_2(\frac{r'\rho - \nu}{\lambda}) \). First we have the trivial bound \( |K_2| \lesssim \lambda^{-2} \). Then by Lemma 2.1 we get
\[ |K_2(t, r, r')| \lesssim |t|^{-1/2} \int |\partial_r F_2| \, dr \lesssim \lambda^{-2} |t|^{-1/2}. \]

On the other hand, if \(|t| \gg R\), using integration by part twice, get
\[ |K_2| \lesssim \int |t|^{-2} \left| \partial_r \left[ \rho^{-1} \partial_r (\rho^{-1} F_2) \right] \right| \, dr \lesssim |t|^{-2} R \lambda^{-3}. \]

Then eventually we have
\[ |K_2| \lesssim \lambda^{-2} |t|^{-1/2} \text{if } |t| \lesssim R + |t|^{-2} R \lambda^{-3} \text{if } |t| \gg R \]
which implies the bound (2.18) as desired.

For \( j = 1 \), the proof follows in an similar and easier way as \( j = 2 \) since \( \nu \gtrsim R \). Actually, we have
\[ \|M^\nu_{R,2,1}(h)\|_{L^2_t L^\infty_x} \lesssim R^{-3/4} \|h\|_{L^2}. \]  
(2.19)

Now we consider \( j = 3 \). Similarly, the kernel of \( M^\nu_{R,2,3}(M^\nu_{R,2,3})^* \) is
\[
K_3(t - t', r, r') = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} e^{-i(t - t')\rho^2} e^{i(r\rho \sin \theta - \nu \theta)} \frac{\chi_0(r)}{i(r\rho \cos \theta - \nu)} \gamma_2(\frac{r\rho - \nu}{\lambda}) \times e^{-i(r'\rho \sin \theta' - \nu \theta')} \frac{\chi_0(r')}{i(r'\rho \cos \theta' - \nu)} \gamma_2(\frac{r'\rho - \nu}{\lambda}) \chi_0(\rho) \, d\rho \, d\theta \, d\theta'.
\]

It suffices to prove
\[ \|K_3\|_{L^1_t L^\infty_{r,r'}} \lesssim \lambda^{-2} R^{1/2}. \]

Denote \( F_3(\rho) = \frac{\chi_0(r)}{i(r\rho \cos \theta - \nu)} \gamma_2(\frac{r\rho - \nu}{\lambda}) \frac{\chi_0(r')}{i(r'\rho \cos \theta' - \nu)} \gamma_2(\frac{r'\rho - \nu}{\lambda}) \). First we have the trivial bound by integrating on \( \theta, \theta' \)
\[ |K_3| \lesssim \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{r \rho \sin \theta \gamma_2(\frac{r\rho - \nu}{\lambda})}{(r \rho \cos \theta - \nu)^2} \cdot \frac{r' \rho \sin \theta' \gamma_2(\frac{r'\rho - \nu}{\lambda})}{(r' \rho \cos \theta' - \nu)^2} \chi_0(\rho) \, d\rho \, d\theta \, d\theta'. \]

Then by Lemma 2.1 let
\[
R_\theta := r \rho \cos \theta - \nu, \quad R'_\theta := r' \rho \cos \theta' - \nu,
\]  
(2.20)
we have
\[ |K_3| \lesssim \int_0^{\pi/2} \int_0^{\pi/2} |t|^{-1/2} \int \left| \partial_\rho F_3 \right| dpd\theta d\theta' \]
\[ \lesssim |t|^{-1/2} \int_0^{\pi/2} \int_0^{\pi/2} \int \left| \partial_\rho \partial_\theta R^{-1}_\theta \right| \left| \partial_\rho (R'_\theta)^{-1} \right| \left| \gamma_2(R_\theta/\lambda) \gamma_2(R'_\theta/\lambda) \chi_0^2(\rho) \right| dpd\theta d\theta' \]
\[ + |t|^{-1/2} \int_0^{\pi/2} \int_0^{\pi/2} \int \left| \partial_\theta R^{-1}_\theta \right| \left| \partial_\rho \partial_\rho (R'_\theta)^{-1} \right| \left| \gamma_2(R_\theta/\lambda) \gamma_2(R'_\theta/\lambda) \chi_0^2(\rho) \right| dpd\theta d\theta' \]
\[ + |t|^{-1/2} \int_0^{\pi/2} \int_0^{\pi/2} \int \left| \partial_\rho R^{-1}_\theta \right| \left| \partial_\rho (R'_\theta)^{-1} \right| \left| \partial_\rho \left[ \gamma_2(R_\theta/\lambda) \gamma_2(R'_\theta/\lambda) \chi_0^2(\rho) \right] \right| dpd\theta d\theta' \]
\[ \lesssim |t|^{-1/2} \lambda^{-2}, \]

where we used the fact that \( \partial_\rho \partial_\theta R^{-1}_\theta, \partial_\theta R^{-1}_\theta \) do not change the sign. On the other hand, for \( |t| \gg R \), denoting \( \phi_1 = -2t\rho + r \sin \theta - r' \sin \theta' \), we get
\[ |K_3| \lesssim \int_0^{\pi/2} \int_0^{\pi/2} \int \left| \partial_\rho (\phi_1^{-1} \partial_\rho [\phi_1^{-1} F]) \right| dpd\theta d\theta' \lesssim |t|^{-2} \lambda^{-3} R \ln(R). \]

Then we get
\[ |K_3| \leq \lambda^{-2} (1 + |t|)^{-1/2} 1_{|t| \leq R} + |t|^{-2} \ln(R) \lambda^{-3} R 1_{|t| \gg R} \]

which implies that \( \|K_3\|_{L^1_t L^\infty_r} \lesssim \lambda^{-2} R^{1/2} \) as desired.

**Step 3: estimate of** \( S^{\nu}_{R,3} \).

By the support of \( \gamma_3 \), we have \( r\rho > \nu + \lambda > \nu + \nu^{1/3} \) in the support of \( \gamma_3(\frac{r\rho - \nu}{\lambda}) \). Thus we use the Lemma 2.3 and decompose
\[ S^{\nu}_{R,3}(h) := M^{\nu}_{R,3}(h) + E^{\nu}_{R,3}(h) \]

where
\[ M^{\nu}_{R,3}(h) = \chi_0 \left( \frac{r}{R} \right) \int e^{-ir\rho^2} \frac{e^{i\theta(r\rho)} + e^{-i\theta(r\rho)}}{2\sqrt{2\pi} (r^2 \rho^2 - \nu^2)^{1/4}} \gamma_3(\frac{r\rho - \nu}{\lambda}) \chi_0(\rho) h(\rho) d\rho, \]
\[ E^{\nu}_{R,3}(h) = \chi_0 \left( \frac{r}{R} \right) \int e^{-ir\rho^2} h(\nu, \rho) \gamma_3(\frac{r\rho - \nu}{\lambda}) \chi_0(\rho) h(\rho) d\rho, \]

with \( \theta(r), h(\nu, r) \) given in Lemma 2.3.

First, we consider \( M^{\nu}_{R,3} \). We only estimate
\[ \tilde{M}^{\nu}_{R,3}(h) = \chi_0 \left( \frac{r}{R} \right) \int e^{-ir\rho^2} \frac{e^{i\theta(rp)}}{(r^2 \rho^2 - \nu^2)^{1/4}} \gamma_3(\frac{r\rho - \nu}{\lambda}) \chi_0(\rho) h(\rho) d\rho, \]

since the other term is similar. It is easy to see \( \|\tilde{M}^{\nu}_{R,3}(h)\|_{L^2_r L^\infty_\nu} \lesssim \|h\|_2 \), as in the proof of Lemma 2.4. We will prove
\[ \|\tilde{M}^{\nu}_{R,3}(h)\|_{L^2_r L^\infty_\nu} \lesssim \lambda^{-1/4} \|h\|_{L^2}. \]

Similarly as in Step 2, the kernel for \( \tilde{M}^\nu_{R,3}(\tilde{M}^\nu_{R,3})^* \) is
\[ K(t - t', r, r') = \int e^{-i(t-t')r^2 - \theta(rp) + \theta'(r'p)} \frac{\chi_0 \left( \frac{r}{R} \right) \gamma_3 \left( \frac{r\rho - \nu}{\lambda} \right) \chi_0 \left( \frac{r'}{R} \right) \gamma_3 \left( \frac{r'\rho - \nu}{\lambda} \right) \chi_0^2(\rho) d\rho}{(r^2 \rho^2 - \nu^2)^{1/4} (r'^2 \rho^2 - \nu^2)^{1/4} \lambda_0^2(\rho)}. \]
It suffices to show
\[ \|K\|_{L^1_t L^\infty_x} \lesssim \lambda^{-1/2}. \]

Obviously, we have a trivial bound
\[ |K| \lesssim \lambda^{-1/2} R^{-1/2}. \]

Recall \( \theta(r) = (r^2 - \nu^2)^{1/2} - \nu \arccos \frac{\nu}{r} - \frac{\pi}{4} \), then direct computation shows
\[
\begin{align*}
\theta'(r) &= (r^2 - \nu^2)^{1/2} r^{-1}, \\
\theta''(r) &= (r^2 - \nu^2)^{-1/2} - (r^2 - \nu^2)^{1/2} r^{-2} = (r^2 - \nu^2)^{-1/2} \nu^2 r^{-2}, \\
\theta'''(r) &= (r^2 - \nu^2)^{-3/2} \nu^2 (-3 + \frac{2\nu^2}{r^2}).
\end{align*}
\]

Denoting \( G = \frac{\chi_0(r)}{(r^2 - \nu^2)^{3/4}}\), \( \phi_2 = t \rho^2 - \theta(r \rho) + \theta(r' \rho) \). Then
\[
\begin{align*}
\partial_\rho(\phi_2) &= 2t \rho - \theta'(r \rho) r + \theta'(r' \rho) r' = 2t \rho - \frac{\rho(r^2 - \nu^2)}{\sqrt{r^2 \rho^2 - 2 + \sqrt{r^2 \rho^2 - \nu^2}}} \\
\partial^2_\rho(\phi_2) &= 2t - \theta''(r \rho)r^2 + \theta''(r' \rho)r'^2 \\
&= 2t + \frac{(r^2 - \nu^2)}{\sqrt{r^2 \rho^2 - 2 + \sqrt{r^2 \rho^2 - \nu^2}}}, \\
\partial^3_\rho(\phi_2) &= -\theta'''(r \rho)r^3 + \theta'''(r' \rho)r'^3.
\end{align*}
\]

We observe that if \( |\partial_\rho(\phi_2)| \ll |t| \), then \( |\partial^2_\rho(\phi_2)| \gg |t| \). This observation is also true for the case \( a > 1 \), but not true if \( a < 1 \).

If \( |t| \lesssim R \), we divide \( K \)
\[
K = \int e^{-i\phi_2} G t_0(\frac{100 \partial_\rho(\phi_2)}{t}) d\rho + \int e^{-i\phi_2} G [1 - t_0(\frac{100 \partial_\rho(\phi_2)}{t})] d\rho := I_1 + I_2.
\]

By Lemma 2.1, we obtain
\[
|I_1| \lesssim |t|^{-1/2} \left( \int |\partial_\rho G t_0(\frac{100 \partial_\rho(\phi_2)}{t})| d\rho + \int |G t_0(\frac{100 \partial_\rho(\phi_2)}{t}) \frac{100 \partial^2_\rho(\phi_2)}{t}| d\rho \right)
\lesssim |t|^{-1/2} \lambda^{-1/2} R^{-1/2},
\]

where for the first term, we estimate it as \( K_3 \), while for the second term, we only need to observe that \( t_0(\frac{100 \partial_\rho(\phi_2)}{t}) \frac{100 \partial^2_\rho(\phi_2)}{t} \) has fixed sign depending only on \( t \).
For $I_2$, without loss of generality, we assume $r^2 - r'^2 > 0$. Then integrating by part, we get
\[
|I_2| \lesssim \int \left| \partial_\rho \left( (\partial_\rho \phi_2)^{-1} - \frac{\partial_\rho \phi_2}{t} \right) G[1 - \eta_0 \left( \frac{100 \partial_\rho \phi_2}{t} \right)] \right| d\rho
\]
\[
\lesssim \int \left| \frac{\partial^2 \phi_2}{(\partial_\rho \phi_2)^2} G[1 - \eta_0 \left( \frac{100 \partial_\rho \phi_2}{t} \right)] \right| d\rho
\]
\[
+ |t|^{-1} \left( \int |\partial_\rho G| d\rho + \int |G(\eta_0 \left( \frac{100 \partial_\rho \phi_2}{t} \right) \frac{100 \partial^2 \phi_2}{t}| d\rho \right)
\]
\[
\lesssim \int \frac{\partial^2 \phi_2}{(\partial_\rho \phi_2)^2} G[1 - \eta_0 \left( \frac{100 \partial_\rho \phi_2}{t} \right)] d\rho + |t|^{-1} \lambda^{-1/2} R^{-1/2}
\]
\[
\lesssim |t|^{-1} \lambda^{-1/2} R^{-1/2},
\]
where we used the fact that $\partial^2 \phi_2$ changes the sign at most once.

If $|t| \gg R$, we have $|\partial_\rho (\phi_2)| \sim |t|$. Thus integrating by part, we get
\[
|K| \lesssim \int \left| \partial_\rho \left( (\partial_\rho \phi_2)^{-1} \partial_\rho ((\partial_\rho \phi_2)^{-1} G) \right) \right| d\rho
\]
\[
\lesssim \int \left| (\partial_\rho \phi_2)^{-3} \partial^3 \phi_2 G \right| d\rho + \int \left| (\partial_\rho \phi_2)^{-2} \partial^2 \phi_2 G \right| d\rho
\]
\[
+ \int \left| (\partial_\rho \phi_2)^{-3} \partial^2 \phi_2 \partial_\rho G \right| d\rho + \int \left| (\partial_\rho \phi_2)^{-2} \partial^2 \phi_2 \partial_\rho G \right| d\rho
\]
\[
:= II_1 + II_2 + II_3 + II_4.
\]

As for $I_2$, we can obtain
\[
II_2 + II_3 + II_4 \lesssim |t|^{-2} \lambda^{-1/2} R^{-1/2} R^2 \lambda^{-2} \lesssim |t|^{-2} \lambda^{-5/2} R^{3/2}.
\]

For $II_1$, we have
\[
II_1 \lesssim |t|^{-3} \lambda^{-1/2} R^{-1/2} \int (-\theta'''(r) \rho^3 - \theta''(r') \rho'^3 \gamma_3 \frac{r\rho - \nu}{\lambda}) \gamma_3 \frac{r' \rho - \nu}{\lambda} d\rho
\]
\[
\lesssim |t|^{-3} \lambda^{-1/2} R^{-1/2} \sup_{\rho, r \rho \rho \rho > \nu + \lambda} \theta''(r) \rho^2 \lesssim |t|^{-3} \lambda^{-1} R.
\]

Thus, eventually we get
\[
|K| \lesssim |t|^{-1/2} \lambda^{-1/2} R^{-1/2} \left( |t|^{-2} \lambda^{-5/2} R^{3/2} + |t|^{-3} \lambda^{-1} R \right) 1_{|t| \leq R}
\]
which implies $\|K\|_{L^1_t L^\infty} \lesssim \lambda^{-1/2}$ as desired, if $\lambda \gtrsim R^{1/3}$.

It remains to bound $E'_{R,3}$. First, we have for any $f \in L^2$
\[
\|E'_{R,3}(f)\|_{L^2_t L^\infty} \lesssim \|S'_{R,3}(f)\|_{L^2_t L^\infty} + \|M'_{R,3}(f)\|_{L^2_t L^\infty} \lesssim \|f\|_{L^2}.
\]

On the other hand, using the decay estimate of $h(\nu, r), we get
\[
\|E'_{R,3}(f)\|_{L^2_t L^2} \lesssim (\lambda^{-5/4} R^{1/4} + R^{-1/2}) \|f\|_{L^2}.
\]

Therefore, we complete the proof. \(\square\)

Now we are ready to prove Theorem 14. At first, we notice that we need to assume $d \geq 3$, due to the $R^{-1/p}$ bound which comes from the estimates of $E'_{R,3}$ (see
Next, we optimize the choice of $\lambda$. The main bounds in Lemma 2.5 are $\lambda^{1/4} R^{-1/4}, \lambda^{-1} (1-\frac{2}{p})$. We can make them equal by choosing
\[
\lambda = R \frac{p}{8(p-1)} \geq R^{1/3}.
\]
Thus (1.6) holds, if $q = 2$ and $p \geq 2$ satisfies
\[
\frac{d-1}{p} - \frac{d-2}{2} + \frac{p}{8(p-1)} - \frac{1}{4} < 0,
\]
which is equivalent to
\[
p > p(d) = \frac{6d - 7 + \sqrt{4d^2 + 4d - 7}}{4d - 7}.
\]
Then by interpolation, we can obtain Theorem 1.1.

In order to apply these new Strichartz estimates, we use Christ-Kiselev lemma to derive the inhomogeneous linear estimates.

**Corollary 2.6.** Assume $(q,p), (\tilde{q}, \tilde{p})$ both satisfies (1.7), $(q,p) \neq (\infty,2)$, and $q > \tilde{q}'$. Then
\[
\left\| \int_0^t S_a(t-s) P_0 f(s) ds \right\|_{L^q_t L^p_x} \lesssim \| f \|_{L^\tilde{q}_t L^\tilde{p}_x}.
\]

**Proof.** By Theorem 1.1 and interpolation with (1.2) for AP(a), we get slightly stronger estimates: for $(q,p)$ satisfy (1.7), and $(q,p) \neq (\infty,2)$, then
\[
\| S_a(t) P_0 f \|_{L^q_t L^p_x} \lesssim \| f \|_{L^2_t}.
\]
Thus this corollary following immediately from Christ-Kiselev lemma.

## 3. Scattering for Zakharov system

This section is devoted to proving Theorem 1.2. The main ingredients are the normal form reduction and the generalized Strichartz estimates in Theorem 1.1.

### 3.1. Normal form transform.

We recall the normal form transform that was used in [13]. It is convenient first to change the system into first order as usual. Let
\[
N := n - iD^{-1} \hat{n}/\alpha,
\]
then $n = \text{Re} N = (N + \tilde{N})/2$ and the equations for $(u, N)$ are
\[
\begin{cases}
(i\partial_t - \Delta) u = Nu/2 + \tilde{N}u/2, \\
(i\partial_t + \alpha D) N = \alpha D|u|^2.
\end{cases}
\]

In our proof, the term $\tilde{N}u$ makes no essential difference than $Nu$, and hence for simplicity, we assume the nonlinear term in first equation of (3.2) is $Nu$.

We use $S(t), W_\alpha(t)$ to denote the Schrödinger, wave propagators:
\[
S(t) \phi = \mathcal{F}^{-1} e^{it|\xi|^2} \hat{\phi}, \quad W_\alpha(t) \phi = \mathcal{F}^{-1} e^{i\alpha t|\xi|^2} \hat{\phi}, \quad \hat{\phi} = \mathcal{F} \phi.
\]
For a quadratic term $uv$, we use $(uv)_{LH}, (uv)_{HH}, (uv)_{HL}$ to denote the three different interactions
\[
(uv)_{LH} = \sum_{k \in \mathbb{Z}} P_{\leq k-5} u P_k v, (uv)_{HL} = \sum_{k \in \mathbb{Z}} P_k u P_{\leq k-5} v, (uv)_{HH} = \sum_{|k_1 - k_2| \leq 4, k_1, k_2 \in \mathbb{Z}} P_{k_1} u P_{k_2} v.
\]
To distinguish the resonant interaction, we also use

\[(uv)_{\alpha L} = \sum_{|k - \log_2 \alpha| \leq 1, \ k \in \mathbb{Z}} P_k u P_{\leq k-5} v, (uv)_{XL} = \sum_{|k - \log_2 \alpha| > 1, \ k \in \mathbb{Z}} P_k u P_{\leq k-5} v, \tag{3.3}\]

and similarly \((uv)_{La}, (uv)_{LX}\). It is obvious that we have

\[uv = (uv)_{HH} + (uv)_{LH} + (uv)_{HL} = (uv)_{HH} + (uv)_{La} + (uv)_{LX} + (uv)_{\alpha L} + (uv)_{XL}. \tag{3.4}\]

Moreover, for any such index \(* = HH, HL, \alpha L, \text{etc.}, \) we denote the bilinear symbol (multiplier) by

\[\mathcal{F}(uv)_* = \int \mathcal{P}_* \hat{u}(\xi - \eta)\hat{v}(\eta) d\eta, \tag{3.5}\]

and finite sum of those bilinear operators are denoted by the sum of indices:

\[(uv)_{*1,*2,+…} = (uv)_{*1} + (uv)_{*2} + ... \tag{3.6}\]

By normal norm reduction (see [13]), we obtain that (3.2) is equivalent to the following integral equation

\[u = S(t)u_0 + S(t)\Omega(N, u)(0) - \Omega(N, u)(t) - i\alpha \int_0^t S(t-s)\Omega(D|u|^2, u)(s) ds - \frac{i}{2} \int_0^t S(t-s)\Omega(N, Nu)(s) ds - i \int_0^t S(t-s)(Nu)_{LH+HH+\alpha L} ds, \tag{3.7}\]

\[N = W_\alpha(t)N_0 + W_\alpha(t)D\tilde{\Omega}(u, u)(0) - D\tilde{\Omega}(u, u)(t) - \int_0^t W_\alpha(t-s)D(u\bar{u})_{HH+\alpha L+La} ds - \int_0^t W_\alpha(t-s)(D\tilde{\Omega}(Nu, u) + D\tilde{\Omega}(u, Nu))(s) ds, \tag{3.8}\]

where \(\Omega, \tilde{\Omega}\) is a bilinear Fourier multiplier in the form

\[\Omega(f, g) = \mathcal{F}^{-1} \int \mathcal{P}_{XL} \omega^{-1} \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta, \]

\[\tilde{\Omega}(f, g) = \mathcal{F}^{-1} \int \mathcal{P}_{XL+LX} \frac{\hat{f}(\xi - \eta)\hat{g}(\eta)}{|\xi - \eta|^2 - |\eta|^2 - \alpha|\xi|} d\eta. \]

The equations after normal form reduction look "roughly"

\[(i\partial_t + D^2)(u - \Omega(N, u)) = (Nu)_{LH+HH+\alpha L} + \Omega(D|u|^2, u) + \Omega(N, Nu), \]

\[(i\partial_t + \alpha D)(N - D\tilde{\Omega}(u, u)) = D|u|^2_{HH+\alpha L+La} + D\tilde{\Omega}(Nu, u) + D\tilde{\Omega}(u, Nu). \tag{3.9}\]

3.2. The angular Strichartz space and main estimates. Inspired by [13], we introduce the Strichartz norm we need, and present the main nonlinear estimates. For \(u\) and \(N\), we use the Strichartz norms with angular regularity:

\[u \in X = \langle D \rangle^{-1} (L^\infty_t H^0_{\omega, 1} \cap L^2_t \tilde{B}^{1/4+\varepsilon, 1}_{(\varepsilon, 2+), \omega} \cap L^2_t \tilde{B}^{0, 1}_{6-\omega}) \tag{3.10}, \]

\[N \in Y = L^\infty_t H^0_{\omega, 1} \cap L^2_t \tilde{B}^{-1/4-\varepsilon, 1}_{(\varepsilon, 2+), \omega} \tag{3.11}, \]
for fixed $0 < \varepsilon \ll 1$, where $q(\cdot)$ is defined by
\[ \frac{1}{q(\varepsilon)} = \frac{1}{4} + \frac{\varepsilon}{3}. \] (3.12)
The condition $0 < \varepsilon \ll 1$ ensures that
\[ \frac{7}{2} < q(\varepsilon) < 4 < q(-\varepsilon) < \infty, \] (3.13)
such that the norms in (3.10)-(3.11) satisfy the condition in Theorem 1.1.

We intend to apply the generalized Strichartz estimates to the integral equations.

To do the nonlinear estimates, we use some representation theory of $SO(3)$. Let $\mu$ be the Haar measure of $SO(3)$. In the sequel, we denote $L^q_A = L^q(SO(3), \mu)$.

**Lemma 3.1.** (a) For any $1 \leq p, q \leq \infty$, we have
\[ \|f\|_{L^p_t L^q_x} \sim \|f(Ax)\|_{L^p_t L^q_x}. \]
(b) For any $1 < q < \infty$, we have
\[ \|f\|_{L^p_t H^s_x} \sim \|f\|_{L^p_t L^q_x} + \sum_{i,j} \|X_{i,j}f\|_{L^p_t L^q_x}, \]
where $X_{i,j} = x_i \partial_j - x_j \partial_i$.

**Proof.** Part (a) is direct. For the proof of part (b), see [43]. \qed

**Lemma 3.2.** Assume $1 \leq p_1, p_2, q_1, q_2, p, q \leq \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then
\[ \|f \ast g\|_{L^p_t L^q_x} \leq C \|f\|_{L^{p_1} L^{q_1}_x} \|g\|_{L^{p_2} L^{q_2}_x}. \]

**Proof.** By Lemma 3.1, we have
\[
\|f \ast g\|_{L^p_t L^q_x} \sim \left\| \int f(Ax - y)g(y)dy \right\|_{L^p_t L^q_x} \\
\sim \left\| \int f(Ax - Ay)g(Ay)dy \right\|_{L^p_t L^q_x} \leq \|f\|_{L^{p_1} L^{q_1}_x} \|g\|_{L^{p_2} L^{q_2}_x},
\]
where in the last inequality we used Minkowski's and Hölder's inequalities in $A$, then Young's inequality in $x$. \qed

For a symbol $m$ on $\mathbb{R}^6$, define $T_m$ to be the bilinear operator on $\mathbb{R}^3$:
\[ T_m(f, g)(x) = \int_{\mathbb{R}^6} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix(\xi + \eta)} d\xi d\eta. \]
The following lemma plays a crucial role.

**Lemma 3.3.** Let $1 \leq p, p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$. Assume $m(\xi, \eta) = h(|\xi|, |\eta|)$ for some function $h$, $m$ is bounded and satisfies for all $\alpha, \beta$
\[ |\partial^\alpha_x \partial^\beta_\eta m(\xi, \eta)| \leq C_{\alpha, \beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|}, \quad \xi, \eta \neq 0. \]

Then for $q > 2$,
\[ \|T_m(P_{k_1} f, P_{k_2} g)\|_{L^p_t H^s_x} \leq C \|f\|_{L^{p_1} L^{q_1}_x} \|g\|_{L^{p_2} L^{q_2}_x} \]
holds for any $k_1, k_2 \in \mathbb{Z}$, with an uniform constant $C$. 

Proof. We can write
\[ T_m(P_k f, P_k g)(x) = \int K(x - y, x - y') f(y)g(y')dydy' \]
where the kernel is given by
\[ K(x, y) = \int m(\xi, \eta) \chi_k(\xi) \chi_k(\eta) e^{i\xi x + i\eta y} d\xi d\eta. \]
From the assumption on \( m \), and integration by parts, we get a pointwise bound of the kernel:
\[ |K(x, y)| \lesssim 2^{3k_1} (1 + |2^{k_1} x|)^{-4} 2^{3k_2} (1 + |2^{k_2} y|)^{-4}. \]
(3.14)
Since \( m \) is both radial in \( \xi, \eta \), then \( K \) is both radial in \( x, y \). Then by Lemma 3.1 (b) we get
\[ \|T_m(P_k f, P_k g)\|_{L^p_t L^q_x} \lesssim \|T_m(P_k f, P_k g)\|_{L^p_t L^q_x} + \sum_{i,j} \|X_{i,j} T_m(P_k f, P_k g)\|_{L^p_t L^q_x} \]
\[ := I + II. \]
For the term \( I \), by Minkowski’s inequality and (3.14), we have
\[ I \lesssim \left\| \int K(Ax - y, Ax - y') f(y)g(y')dydy' \right\|_{L^p_t L^q_x} \]
\[ \lesssim \left\| \int K(x - y, x - y') f(Ay)g(Ay')dydy' \right\|_{L^p_t L^q_x} \]
\[ \lesssim \|f(Ax)\|_{L^p_1 L^\infty_x} \cdot \|g(Ax)\|_{L^p_2 L^\infty_x} \lesssim \|f\|_{L^p_1 L^\infty_x} \|g\|_{L^p_2 L^\infty_x}, \]
then by the Sobolev embedding \( H^1_\omega(S^2) \hookrightarrow L^\infty_\omega \), this suffices to give the desired bound. Similarly, for the term \( II \), we have
\[ II \lesssim \sum_{i,j} \|T_m(P_k X_{i,j} f, P_k g)\|_{L^p_t L^q_x} + \sum_{i,j} \|T_m(P_k f, P_k X_{i,j} g)\|_{L^p_t L^q_x} \]
\[ \lesssim \sum_{i,j} \|X_{i,j} f\|_{L^p_1 L^\infty_x} \|g\|_{L^p_2 L^\infty_x} + \sum_{i,j} \|X_{i,j} g\|_{L^p_1 L^\infty_x} \|f\|_{L^p_2 L^\infty_x}, \]
which gives the desired bound, by Sobolev embedding. \( \square \)

With these lemmas above, we can follow the proof with slight modifications in [13] to prove the main estimates. The following two lemmas can be proved similarly as Lemma 3.2-3.3 in [13]. The main difference is that we use Lemma 3.3 for every bilinear dyadic piece.

Lemma 3.4 (Bilinear terms I). (1) For any \( N \) and \( u \), we have
\[ \|(Nu)_{LH}\|_{L^1_t H^1_x} \lesssim \|N\|_{L^2_t B^{1/4-\varepsilon, 1}_q(\xi(-\varepsilon, 2\omega))} \|\langle D\rangle u\|_{L^2_t B^{1/4+\varepsilon, 1}_q(\xi(\varepsilon, 2\omega))}, \]
\[ \|(Nu)_{HH}\|_{L^1_t H^1_x} \lesssim \|N\|_{L^2_t B^{1/4-\varepsilon, 1}_q(\xi(-\varepsilon, 2\omega))} \|\langle D\rangle u\|_{L^2_t B^{1/4+\varepsilon, 1}_q(\xi(\varepsilon, 2\omega))}. \]
(2) If \( 0 \leq \theta \leq 1 \), \( \frac{1}{q} = \frac{1}{2} - \frac{\theta}{2}, \frac{1}{r} = \frac{1}{4} + \frac{\varepsilon}{3} + \frac{\theta}{3} \), then for any \( N \) and \( u \)
\[ \|(Nu)_{\alpha L}\|_{\langle D\rangle^{-1} L^{r'}_t B^{\frac{1}{2}-\frac{\theta}{2} - \frac{\varepsilon}{3}, 1}_r} \lesssim \|N\|_{L^2_t B^{1/4-\varepsilon, 1}_q(\xi(-\varepsilon, 2\omega))} \|u\|_{L^\infty_t H^0_{\omega} \cap L^2_t B^{1/4+\varepsilon, 1}_q(\xi(\varepsilon, 2\omega))}. \]
Remark 1. In application, we will use Lemma 3.4 (2) and Lemma 3.6 (2) by fixing a $0 < \theta_0 \ll 1$ such that by this choice $(\tilde{q}, \tilde{r})$ is admissible to apply Corollary 2.6.

Note the fact that $X_{i,j}(fg) = X_{i,j}fg + fX_{i,j}g$, and $X_{i,j}$ commute with the radial Fourier multiplier operator. Then the following two lemmas are just the bilinear and trilinear estimates, Lemma 3.5 and Lemma 3.7 obtained in [13] after applying $X_{i,j}$, and not by Lemma 3.3.

Lemma 3.6 (Boundary terms). (a) For any $N_0$ and $u_0$, we have

$$\|\Omega(N_0, u_0)\|_{H^{1,1}_0} \lesssim \|N_0\|_{H^{0,1}_0} \|u_0\|_{H^{1,1}_0},$$

$$\|\tilde{D}\Omega(u_0)\|_{H^{0,1}_0} \lesssim \|u_0\|_{H^{1,1}_0} \|u_0\|_{H^{1,1}_0}.$$  

As a consequence, for any $N$ and $u$

$$\|\Omega(N, u)\|_{L^{1,1}_r H^{1,1}_0} \lesssim \|N\|_{L^{1,1}_r H^{0,1}_0} \|u\|_{L^{1,1}_r H^{1,1}_0},$$

$$\|\tilde{D}\Omega(u, u)\|_{L^{1,1}_r H^{0,1}_0} \lesssim \|u\|_{L^{1,1}_r H^{1,1}_0} \|u\|_{L^{1,1}_r H^{1,1}_0}.$$  

(b) For any $N$ and $u$ we have

$$\|\langle D\rangle\Omega(N, u)\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{(1,1),\omega}} \lesssim \|N\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}} \|u\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}},$$

$$\|\tilde{D}\Omega(u, u)\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{(1,1),\omega}} \lesssim \|u\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}} \|u\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}}.$$  

Lemma 3.7 (Cubic terms). For any $N$ and $u$ we have

$$\|\Omega(D|u|^2, u)\|_{H^{1,1}_0} \lesssim \|u\|^2_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}} \|u\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}},$$

$$\|\langle D\rangle\Omega(N, Nu)\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{(1,1),\omega}} \lesssim \|u\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}} \|N\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}}^2,$$

$$\|\tilde{D}\Omega(Nu, u)\|_{H^{1,1}_0} \lesssim \|u\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}} \|N\|_{L^{1/2}_r B^{1/4+\varepsilon,1}_{\infty,\omega}}.$$  

With these estimates, we can prove the scattering part of Theorem 1.1, see [13]. For the wave operator, see the appendix of [14]. We omit the details.

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