$L_0$ regularized estimation for nonlinear models that have sparse underlying linear structures

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Abstract

We study the estimation of $\beta$ for the nonlinear model $y = f(X^T \beta) + \epsilon$ when $f$ is a nonlinear transformation that is known, $\beta$ has sparse nonzero coordinates, and the number of observations can be much smaller than that of parameters ($n \ll p$). We show that in order to bound the $L_2$ error of the $L_0$ regularized estimator $\hat{\beta}$, i.e., $\|\hat{\beta} - \beta\|_2$, it is sufficient to establish two conditions. Based on this, we obtain bounds of the $L_2$ error for (1) $L_0$ regularized maximum likelihood estimation (MLE) for exponential linear models and (2) $L_0$ regularized least square (LS) regression for the more general case where $f$ is analytic. For the analytic case, we rely on power series expansion of $f$, which requires taking into account the singularities of $f$.

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1 Introduction

Regularized estimation for sparse models that have a large number of parameters comparing to that of observations has become an important topic in statistics, machine learning, and a few other areas (Bunea et al. 2007, Candès & Tao 2007, Donoho et al. 2006, Efron et al. 2004, Field 1994, Natarajan 1995, Zhao & Yu 2006). The research in these areas has been focused on regularized least square (LS) regression for sparse linear models $y = X\beta + \epsilon$, where $y \in \mathbb{R}^n$ is the response vector, $X \in \mathbb{R}^{n \times p}$ the design matrix, $\beta \in \mathbb{R}^p$ the vector of parameters, and $\epsilon \in \mathbb{R}^n$ the random error vector that has mean 0 given $X$. By sparse we mean the number of nonzero coordinates of $\beta$ is much smaller than $p$ (Wasserman & Roeder 2009).

On the other hand, nonlinear models such as logistic models that have underlying linear structures are widely used. The general form of such models is

$$y = f(X^T \beta) + \epsilon,$$

(1.1)
where \( f : \mathbb{R} \to \mathbb{R} \) is a nonlinear function that may or may not be known. Here and henceforth, for \( x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n \), we denote
\[
f(x) = (f(x_1), \ldots, f(x_n))^\top.
\]

The need for nonlinear models with sparse underlying linear structure is clearly laid out in several recent works in neuroscience (Sharpee et al. 2008, 2004) and some algorithms based on information criteria have been proposed to estimate not only \( \beta \) but also \( f \). However, at this point, it seems very hard to evaluate the estimation precision of those algorithms.

In this article we are content to establish the \( L_2 \) precision of \( L_0 \) regularized estimator of \( \beta \) for sparse models, when the design matrix \( X \) is fixed and \( f \) is known. We shall allow \( n \ll p \). Despite its limitation from a computational point of view, the \( L_0 \) regularization is an important and conceptually simple instrument for parameter estimation and model selection (Akaike 1974, Huang et al. 2008, Schwarz 1978). Besides, since many improvements over the \( L_0 \) regularization are achieved by taking advantage of properties of linear models that may fail to be had by nonlinear models (Zhao & Yu 2006), it is reasonable to take \( L_0 \) regularization as a prototype for further study on nonlinear models. With this in mind, our concern is whether good estimation precision could be achieved instead of how fast to achieve it.

In Section 2, we establish a basic result. We show that provided two conditions are satisfied, the \( L_2 \) error of the \( L_0 \) regularized estimator satisfies a quadratic inequality which yields the estimation precision. Consequently, establishing the estimation precision is reduced to establishing the two conditions. As a minor benefit of the result, independence of the coordinates of \( \epsilon \) in general need not be assumed.

We will also set up notation and collect other preliminary results in Section 2. After that, we shall establish the alluded conditions for exponential linear models and for analytic models, i.e., models with analytic \( f \). Although a special case of analytic models, exponential linear models are much simpler to handle due to its explicit expression of the conditional density of \( y \) given \( X \). For these models, we consider the maximum likelihood estimator (MLE). The discussion is in Section 3. For analytic models, we will consider the LS regression. Sections 4 and 5 establish the two conditions, respectively. In Section 5, the approach is to use infinite power series expansion of \( f \). The main complexity of the approach arises when \( f \) has singularities on \( \mathbb{C} \). To illustrate, we will use as working examples the logistic regression model in Section 3 and a noise corrupted version of it in Section 5. Most of the proofs are collected in Section 6.

## 2 Preliminaries

### 2.1 Notation
Denote by \( X_1^\top, \ldots, X_n^\top \) the row vectors of \( X \), with \( X_i \in \mathbb{R}^n \). Denote by \( V_1, \ldots, V_p \) the column vectors of \( X \). We shall always assume that \( X \) is fixed and impose the
condition that \( V_j \neq 0 \). In fact, if a column vector of \( X \) is 0, then it has no effect on \( y \) and should be removed. In the subsequent discussion, the column vectors of \( X \) should be understood as unnormalized. It is therefore helpful to think of \( X \) as a collection of covariate vectors registered exactly as they are observed.

For \( S = \{i_1, \ldots, i_k\} \), with \( 1 \leq i_1 < \ldots < i_k \leq p \), denote \( X_S = (V_{i_1}, \ldots, V_{i_k}) \), and for \( u \in \mathbb{R}^p \), denote \( u_S = (u_{i_1}, \ldots, u_{i_k})^\top \). The support of \( u \) is \[ \text{spt}(u) = \{i : u_i \neq 0\}. \]

Denote by \( \|u\|_p \) the \( L_p \) norm of \( u \). If \( A \) is a set, denote by \( |A| \) its cardinality. The \( L_0 \) norm of \( u \) refers to \( |\text{spt}(u)| \) and is often denoted by \( \|u\|_0 \). We choose the notation \( |\text{spt}(u)| \) since it seems more intuitive.

For \( \varphi = (\varphi_1, \ldots, \varphi_n) \) and \( x \in \mathbb{R}^n \), where each \( \varphi_i : \mathbb{R} \to \mathbb{R} \), denote \[ \varphi(x) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))^\top. \]

### 2.2 General form of estimator and line of argument

The general form of an \( L_0 \) regularized estimator is

\[ \hat{\beta} = \arg \min_{u \in D} [\ell(y, Xu) + c_r|\text{spt}(u)|], \]

where \( D \) is a pre-selected search domain in \( \mathbb{R}^p \), \( \ell(y, Xu) \) is certain loss function, and \( c_r > 0 \) is a tuning parameter. For the MLE, \( \ell(y, Xu) \) is the minus log likelihood, while for the LS regression, it is \( \|y - Xu\|^2_2 \). For linear regression, \( D \) is typically set equal to \( \mathbb{R}^p \). However, for nonlinear regression, our position is that some constraint on \( D \) is needed in order to control the potentially large variation of the functional property of \( f \) at different possible values of \( X\beta \).

For both the MLE and LS regression, the argument to establish the precision of \( \hat{\beta} \) proceeds as follows. First, it is easy to show that \( \hat{\beta} \) satisfies an inequality of the following form,

\[ G(\psi(X\hat{\beta}) - \psi(X\beta)) \leq 2|\epsilon, \varphi(X\hat{\beta}) - \varphi(X\beta)| - c_r(|\text{spt}(\hat{\beta})| - |\text{spt}(\beta)|), \]

where \( G \) is a function \( \mathbb{R}^n \to \mathbb{R} \), \( \psi = (\psi_1, \ldots, \psi_n) \) and \( \varphi = (\varphi_1, \ldots, \varphi_n) \), with \( \psi_i \) and \( \varphi_i \) being functions \( \mathbb{R} \to \mathbb{R} \). Then the following two conditions will be established.

**Condition H1** Given \( q \in (0, 1) \), there is \( c_1 = c_1(X, \beta, \varphi, q) > 0 \), such that

\[ \Pr \left\{ \left| \epsilon, \varphi(Xu) - \varphi(X\beta) \right| \leq c_1 \sqrt{n} \|u - \beta\|_1, \text{ all } u \in D \right\} \geq 1 - 2q. \]

The coefficient 2 in \( 1 - 2q \) is nonessential. It is for ease of notation in the statements of main results.

**Condition H2** There is \( c_2 = c_2(X, \beta, \psi) > 0 \), such that for all \( u \in D \),

\[ G(\psi(Xu) - \psi(X\beta)) \geq c_2 n \|u - \beta\|^2_2. \]
The constants $c_1$ and $c_2$ will be explicitly constructed. In general, both depend on $X$. Since we only consider fixed design, they are nonrandom.

We will check the conditions respectively for the MLE and LS regression. Once this is done, using the next result, we then obtain a bound on $\|\hat{\beta} - \beta\|$. Note that the result is stated in a little more general form as it does not require that $\hat{\beta}$ be the one defined by (2.1).

**Proposition 2.1** Suppose Conditions H1 and H2 are satisfied. If $\hat{\beta} \in D$ is a random variable that always satisfies the inequality (2.2) with $c_r = 3c_1^2/c_2$, then, letting $\kappa_r = 3c_1/c_2$,

$$\Pr \left\{ \|\hat{\beta} - \beta\| \leq \frac{\kappa_r \sqrt{|\text{spt}(\beta)|}}{\sqrt{n}} \right\} \geq 1 - 2q.$$ 

In order for the bounds to be meaningful, we need to make sure $\kappa_r$ is not too large, at least comparing to $\sqrt{n}$. This will be the main consideration when we try to establish Conditions H1 and H2.

Because Proposition 2.1 plays a fundamental role in our study, we give its proof below. This is the only result whose proof appears in the main text.

**Proof of Proposition 2.1.** Denote $T = \text{spt}(\beta)$ and $S = \text{spt}(\hat{\beta})$. Under Conditions H1 and H2, with probability at least $1 - 2q$,

$$c_2n\|\hat{\beta} - \beta\|^2 \leq 2c_1\sqrt{n}\|\hat{\beta} - \beta\|_1 - c_r(|S| - |T|) \leq 2c_1\sqrt{n}\|S \cup T\|\|\hat{\beta} - \beta\|_2 - c_r(|S| - |T|),$$

where the second inequality is due to $\text{spt}(\beta - \hat{\beta}) \subset S \cup T$ and Cauchy-Schwartz inequality. Let $t = \|\hat{\beta} - \beta\|_2$ and $b = c_1/c_2$. Then

$$t^2 - \frac{2b\sqrt{|S \cup T|}}{\sqrt{n}}t + 3b^2(|S| - |T|) \leq 0.$$

The left hand side is a quadratic function in $t$. In order for the inequality to hold, there have to be $|S \cup T| \geq 3(|S| - |T|)$ and

$$0 \leq t \leq \frac{b}{\sqrt{n}} \left[ \sqrt{|S \cup T|} + \sqrt{|S \cup T| + 3(|T| - |S|)} \right].$$

Let $T_1 = T \setminus S$ and $S_1 = S \setminus T$. By $|S \cup T| = |S_1| + |T|$ and $|T| - |S| = |T_1| - |S_1|$, then

$$0 \leq t \leq \frac{b}{\sqrt{n}} \left( \sqrt{|T| + |S_1|} + \sqrt{|T| + 3|T_1| - 2|S_1|} \right).$$

It is easy to see that due to $|T_1| \leq |T|$, the right hand side is a decreasing function in $|S_1|$ on $[0,(|T| + 3|T_1|)/2]$, and hence is no greater than its value at 0, which is $(b/\sqrt{n})(\sqrt{|T| + 3|T_1|}) \leq 3b\sqrt{|T|}/\sqrt{n}$. □

To establish Conditions H1 and H2, certain assumptions are needed. We next discuss the major assumptions used by both the MLE and LS regression.
2.3 Tail assumption on errors

To establish Condition H1, we will need the following assumption on \( \epsilon \).

**Tail assumption.** There is \( \sigma > 0 \), such that for any \( t, a_1, \ldots, a_n \in \mathbb{R} \),

\[
\Pr \left\{ \left( \sum_{i=1}^n a_i \epsilon_i \right)^2 \geq t^2 \sum_{i=1}^n a_i^2 \right\} \leq 2 \exp \left\{ -\frac{t^2}{2\sigma^2} \right\}. \tag{2.3}
\]

The tail assumption (2.3) rather mild. If \( \epsilon \sim N(0, \sigma^2 \Sigma) \) and the spectral radius of \( \Sigma \) is no greater than 1, then (2.3) holds. In this case, \( \epsilon_1, \ldots, \epsilon_n \) need not be independent. Moreover, if \( \epsilon_i \) are independent, such that \( \mathbb{E}(\epsilon_i) = 0 \) and \( |\epsilon_i| \leq \sigma \) for all \( i \), then by Hoeffding’s inequality (Pollard 1984), (2.3) holds.

2.4 Coherence and restricted domains

In order to identify \( \beta \), some conditions on the correlations between the column vectors of \( X \) are needed. The maximum correlation between columns of \( X \) is

\[
\mu(X) = \sup_{1 \leq i < j \leq p} \frac{|V_i^\top V_j|}{\|V_i\|_2 \|V_j\|_2}.
\]

Conditions on \( \mu(X) \) are often referred to as coherence property (Bunea et al. 2007, Candès & Plan 2009). The following function

\[
n(\nu) = (1 - \nu)[1 + 1/\mu(X)] \tag{2.4}
\]

will be regularly used in our discussion.

**Proposition 2.2** Fix \( \nu \in [0, 1] \). (1) For \( u \in \mathbb{R}^p \), if \( |\text{spt}(u)| \leq n(\nu) \), then

\[
\|Xu\|_2^2 \geq \nu[1 + \mu(X)] \sum_{j=1}^p |u_j|^2 \|V_j\|_2^2.
\]

(2) For \( u, v \in \mathbb{R}^p \), if \( |\text{spt}(u) \cup \text{spt}(v)| \leq n(\nu) \), then

\[
\|X(u - v)\|_2^2 \geq \nu[1 + \mu(X)] \sum_{j=1}^p |u_j - v_j|^2 \|V_j\|_2^2.
\]

In particular, the inequality holds if \( |\text{spt}(u)| \lor |\text{spt}(v)| \leq n(\nu)/2 \).

As mentioned earlier, for the estimator (2.1), we need to impose some constraints on the search domain \( D \). For this purpose, we define several sets. For \( I \subset \mathbb{R} \), let

\[
\mathcal{D}(I) = \{ u \in \mathbb{R}^p : X_i^\top u \in I, \ 1 \leq i \leq n \}, \tag{2.5}
\]
and for $h \geq 1$, let
\[
\mathcal{D}(I, h) = \mathcal{D}(I) \cap \{u \in \mathbb{R}^p : |\text{spt}(u)| \leq h\}.
\] (2.6)

Apparently, denoting by $T$ the mapping $u \to Xu$, $\mathcal{D}(I) = T^{-1}(I^n)$.

One constraint that will be regularly imposed is $D \subset \mathcal{D}(I, n(\nu)/2)$ for some $\nu \in (0, 1)$. The implied constraint that $X_i^\top u \in I$ for every $i$ is to make sure that the functions involved in the estimator (2.1), i.e., $G$, $\psi_i$ and $\phi_i$, have good enough properties for all candidate values of $\beta$, especially properties determined by derivatives. This constraint on the functional properties is needed when we establish both Conditions H1 and H2. For linear regression, roughly speaking, this is not a concern and one can simply choose $I = \mathbb{R}$, simply because the derivative of a linear function is constant, and so the pertinent functional properties are uniform.

The constraint $D \subset \mathcal{D}(I, n(\nu)/2)$ also imposes a constraint on $|\text{spt}(\hat{\beta})|$. As Proposition 2.2 indicates, one consequence of the constraint is that any two candidate estimates of $\beta$ can be well separated by their corresponding values of $Xu$, so that a large portion of $\beta$ can be correctly identified. For this reason, the constraint will be needed when we establish Condition H2. Clearly, the smaller $\mu(X)$ is, the milder the constraint. Under mild conditions, $\mu(X)$ can be as small as $O(\sqrt{n^{-1}\ln p})$; see Candès & Plan (2009) and also the comments at the end of Section 3.3. This results in a constraint of the form $|\text{spt}(\hat{\beta})| \leq C\sqrt{n/\ln p}$, which is quite mild even when $p$ is much larger than $n$, for example, $p = n^a$ for some $a > 1$.

We shall need the following properties of $\mathcal{D}(I, h)$.

**Proposition 2.3** (1) If $I$ is closed, then $\mathcal{D}(I, 1) \subset \mathcal{D}(I, 2) \subset \cdots$ are closed and (2) if $I$ is compact and $h < n(0) = 1 + \mu(X)^{-1}$, then $\mathcal{D}(I, h)$ is compact.

### 3 Exponential linear models

#### 3.1 Setup and main result

Let $\mu$ be a Borel measure on $\mathbb{R}$ with $\mu(\mathbb{R}) > 0$. Suppose $I \subset \mathbb{R}$ is an nonempty open interval and $\{P_t : t \in I\}$ is a family of probability distributions on $\mathbb{R}$, such that with respect to $\mu$ each $P_t$ has a density
\[
p_t(y) = \exp\{ty - \Lambda(t)\}, \text{ with } \Lambda(t) = \ln \left[\int e^{ty} \mu(dy)\right].
\] (3.1)

As is well known, $\Lambda \in C^\infty(I)$ and for $t \in I$,
\[
E(\xi) = \Lambda'(t), \quad \text{Var}(\xi) = \Lambda''(t) > 0, \quad \text{if } \xi \sim P_t.
\] (3.2)

For example, if $\mu = N(0, \sigma^2)$, then $\Lambda(t) = \sigma^2t^2/2$ and $P_t = N(\sigma^2t, \sigma^2)$. If $\mu$ is the counting measure on $\{0, 1\}$, then $\Lambda(t) = \ln(1 + e^t)$ and $P_t$ is the Bernoulli distribution with parameter $e^t/(1 + e^t)$. We notice that given $y$, $g(t) := p_t(y)$ can
be analytically extended to the domain \( \{ z \in \mathbb{C} : \text{Re}(z) \in I \} \). This fact is not needed in the rest of the section.

Assume that given \( X, y_1, \ldots, y_n \) are independent, such that each \( y_i \sim P_{t_i} \) with \( t_i = X_i^\top \beta \). The joint likelihood of \( y_1, \ldots, y_n \) is then

\[
\prod_{i=1}^n \exp \left\{ y_i X_i^\top \beta - \Lambda(X_i^\top \beta) \right\} = \exp \left\{ y^\top X \beta - \sum_{i=1}^n \Lambda(X_i^\top \beta) \right\}.
\]

From the expression, the \( L_0 \) regularized MLE for \( \beta \) is

\[
\hat{\beta} = \arg \max_{u \in D} \left[ y^\top X u - \sum_{i=1}^n \Lambda(X_i^\top u) - c_r |\text{spt}(u)| \right]. \tag{3.3}
\]

If \( \beta \in D \), then

\[
y^\top X \beta - \sum_{i=1}^n \Lambda(X_i^\top \beta) - c_r |\text{spt}(\beta)| \leq y^\top X \hat{\beta} - \sum_{i=1}^n \Lambda(X_i^\top \hat{\beta}) - c_r |\text{spt}(\hat{\beta})|,
\]

and hence

\[
\sum_{i=1}^n \left[ \Lambda(X_i^\top \hat{\beta}) - \Lambda(X_i^\top \beta) - \Lambda'(X_i^\top \beta) X_i^\top (\hat{\beta} - \beta) \right] \\
\leq \langle \epsilon, X \hat{\beta} - X \beta \rangle - c_r (|\text{spt}(\hat{\beta})| - |\text{spt}(\beta)|),
\]

where \( \epsilon_i = y_i - \mathbb{E}(y_i) = y_i - \Lambda'(X_i^\top \beta) \) has mean 0 for each \( i \). It is seen that the inequality gives rise to (2.2) once we define

\[
G(x) = \sum_{i=1}^n x_i, \quad \psi_i(z) = \Lambda(z) - \Lambda'(X_i^\top \beta) z, \quad \varphi_i(z) = z/2, \tag{3.4}
\]

for \( x \in \mathbb{R}^n, z \in \mathbb{R} \) and \( 1 \leq i \leq n \).

**Theorem 3.1** Suppose \( \epsilon_1, \ldots, \epsilon_n \) satisfy (2.3) for some \( \sigma > 0 \). Fix \( \nu \in (0, 1) \). Let \( D = \mathbb{D}(I, n(\nu)/2) \) in (3.3), where \( n(\nu) \) is defined in (2.4). Suppose

\[
\delta := \inf_{t \in I} \Lambda''(t) > 0. \tag{3.5}
\]

Fix \( q \in (0, 1/2) \). Let

\[
c_r = \frac{3\sigma^2 \ln(p/q)}{\nu \delta [1 + \mu(X)]} \max_j \|V_j\|_2^2
\]

in (3.3). Then, provided \( \beta \in D \),

\[
\Pr \left\{ \|\hat{\beta} - \beta\|_2 \leq \frac{\kappa_r \sqrt{|\text{spt}(\beta)|}}{\sqrt{n}} \right\} \geq 1 - 2q, \tag{3.6}
\]

where \( \kappa_r = \frac{3\sigma \sqrt{2 \ln(p/q)}}{\nu \delta [1 + \mu(X)]} \times \frac{\sqrt{n} \max_j \|V_j\|_2}{\min_j \|V_j\|_2^2} \).
3.2 Comments

Some comments on Theorem 3.1 are in order, many of them also apply to the results we shall establish later. First, on the constraint \( \beta \in D(I, n(\nu)/2) \). As noted in Section 2.4, under mild conditions, for \( p \) with \( \ln p = o(n), n(\nu) \approx \sqrt{n/\ln p} \). In many cases, since it is reasonable to assume that \( |spt(\beta)| = O(1) \) (Wasserman & Roeder 2009), the constraint then is very mild.

Second, on \( \|\hat{\beta} - \beta\|_2 \), which is determined by \( \kappa_r \sqrt{|spt(\beta)|/\sqrt{n}} \) in (3.6). By (3.6), \( \kappa_r = O(R \sqrt{\ln p}) \), where

\[
R = \frac{\sqrt{n} \max_j \|V_j\|_2}{\min_j \|V_j\|_2^2} = \frac{\max_j \|V_j\|_2}{\min_j \|V_j\|_2^2/\sqrt{n}}.
\]

Under mild conditions, \( R \) grows very slowly with \( n \). For example, \( R = 1 \) if \( X \) is such that \( \|V_j\|_2 = \sqrt{n} \) (recall all \( V_j \in \mathbb{R}^n \)). We shall see such an example related to the logistic regression. As another example, suppose all the \( np \) entries of \( X \) are i.i.d. \( \sim \mathcal{Z} \). If \( Z \) is bounded, then clearly \( \max_j \|V_j\|_2/\sqrt{n} = O(1) \). If \( Z \sim \mathcal{N}(0,1) \), then for any \( 0 < \eta < 1/2 \),

\[
\Pr\left\{ \max_{1 \leq j \leq p} \|V_j\|_\infty \leq \sqrt{2 \ln(np/\eta)} \right\} \geq 1 - 2\eta.
\]

Since \( \max_j \|V_j\|_2 \leq \sqrt{n} \max_j \|V_j\|_\infty \), then with high probability, \( \max_j \|V_j\|_2/\sqrt{n} = O(\sqrt{\ln(np)}) \). At the same time, given \( 0 < c < \mathbb{E}(Z^2) \),

\[
\Pr\left\{ \frac{1}{n} \min_{1 \leq j \leq p} \|V_j\|_2^2 \leq c \right\} \leq p \Pr \{ Z_1^2 + \cdots + Z_n^2 \leq nc \} \leq p \psi(c)^n,
\]

where \( \psi(c) = \inf_{t > 0} E[e^{tc - tz^2}] < 1 \). Therefore, for large \( n \) and \( p \), with high probability, we have \( \max_j \|V_j\|_2/\sqrt{n} = O(\sqrt{\ln(np)}) \) or even \( O(1) \) on the one hand, and \( \min_j \|V_j\|_2^2/\sqrt{n} \geq c \) on the other, provided \( \ln p = o(n) \). In particular, suppose \( p = O(n^a) \) for some \( a > 0 \). Then it is seen that \( R = O(\sqrt{\ln n}) \) or even \( O(1) \), and hence, by (3.6), with high probability, \( \|\hat{\beta} - \beta\|_2 = O(\ln n/\sqrt{n}) \) or \( O(\sqrt{\ln p/\sqrt{n}}) \).

Finally, the precision also depends on \( \delta = \inf_{t \in I} \Lambda''(t) \). To see why \( \delta \) matters, consider the case where \( \Lambda'(t) \) is uniformly small in an interval \( I \) that contains all of \( X_i^\top \beta \). This implies that \( \Lambda'(t) \) has little change on \( I \), so by (3.2), \( \mathbb{E}(y_1), \ldots, \mathbb{E}(y_n) \) are close to each other, and at the same time each \( y_i \) has little variation. This gives rise to a nearly “flat” plot of \( y_i \) vs \( X_i^\top \beta \), which makes the identification of \( \beta \) difficult. That is to say the precision of the estimate cannot be high. Certainly, if \( \Lambda''(t) \) has a wide range on \( I \), then using \( \inf_{t \in I} \Lambda''(t) \) to set \( c_\delta \) can be quite conservative. However, as \( X_i^\top \beta \) are unknown, it is the only way to account for all the possible values of \( X_i^\top \beta \), including the least ideal one.

3.3 Logistic regression

Suppose \( y_1, \ldots, y_n \) are independent Bernoulli random variables, such that

\[
\Pr\{y_i = 1\} = e^{x_i^\top \beta} / (1 + e^{x_i^\top \beta}), \quad i = 1, \ldots, n.
\]
The corresponding parametric family of densities is $p_t(y) = \exp\{ty - \Lambda(t)\}$ with respect to the counting measure on $\{0, 1\}$, with $\Lambda(t) = \ln(1 + e^t)$.

For $i = 1, \ldots, n$, $\varepsilon_i = y_i - \Pr\{y_i = 1\} \in (-1, 1)$. Therefore, by Hoeffding’s inequality (Pollard 1984), (2.3) holds with $\sigma = 1$. Given $I \subset \mathbb{R}$, by direct calculation,

$$\inf_{t \in I} \Lambda''(t) = \left(2 \cosh \frac{M_I}{2}\right)^{-2}, \quad \text{with} \quad M_I = \sup_{t \in I} |t|.$$ 

Given $q \in (0, 1)$, let

$$c_r = \frac{12 \ln(p/q)}{\nu[1 + \mu(X)]} \times \frac{\max_j \|V_j\|_2^2}{\min_j \|V_j\|_2^2} \times \cosh^2 \frac{M_I}{2},$$

and

$$\kappa_r = \frac{12 \sqrt{2 \ln(p/q)}}{\nu[1 + \mu(X)]} \times \sqrt{n} \frac{\max_j \|V_j\|_2}{\min_j \|V_j\|_2} \times \cosh^2 \frac{M_I}{2}.$$ 

By Theorem 3.1, if $\beta \in \mathcal{D}(I, n(\nu)/2)$, then, with probability at least $q$, (3.6) holds for the estimator

$$\hat{\beta} = \arg \max \left\{ y^\top X u - \sum_{i=1}^n \ln(1 + e^{X_i^\top u}) - c_r|\text{spt}(u)| : u \in \mathcal{D}(I, n(\nu)/2) \right\}.$$ 

If $X$ is binary, i.e., $X_{ij} = 0$ or $1$, the result can be somewhat simplified. Let $\tilde{X} \in \mathbb{R}^{n \times (p+1)}$ such that $\tilde{X}_{ij} = 2X_{ij} - 1$, for $j \leq p$ and $\tilde{X}_{i,p+1} = 1$. Also let $\tilde{\beta} \in \mathbb{R}^{p+1}$ such that $\tilde{\beta}_j = \beta_j/2$ for $j \leq p$ and $\tilde{\beta}_{p+1} = \sum_{j=1}^p \beta_j/2$. Then $X_i^\top \hat{\beta} = \tilde{X}_i^\top \tilde{\beta}$. Let $\tilde{V}_1, \ldots, \tilde{V}_{p+1}$ be the column vectors of $\tilde{X}$. Then $\|V_j\|_2 = \sqrt{n}$. If we regress $y$ on $\tilde{X}$ to estimate $\tilde{\beta}$, then

$$c_r = \frac{12 \ln((p+1)/q)}{\nu[1 + \mu(X)]} \times \cosh^2 \frac{M_I}{2}, \quad \kappa_r = \frac{12 \sqrt{2 \ln(p/q)}}{\nu[1 + \mu(X)]} \times \cosh^2 \frac{M_I}{2}.$$ 

In the example, $\mu(\tilde{X})$ can be very small. If $X_{ij}$ are i.i.d. with $\Pr\{X_{ij} = 0\} = \Pr\{X_{ij} = 1\} = 1/2$, then for any $1 \leq j < k \leq p + 1$, $\tilde{V}_j^\top \tilde{V}_k \sim \sum_{i=1}^n \eta_i$, where $\eta_i$ are i.i.d. with $\Pr\{\eta_i = 1\} = \Pr\{\eta_i = -1\} = 1/2$. By Hoeffding’s inequality, given $t > 0$,

$$\Pr\left\{ \frac{|\tilde{V}_j^\top \tilde{V}_k|}{\sqrt{\sum_{i=1}^n \eta_i}} \geq t\sqrt{n} \right\} = \Pr\left\{ \sum_{i=1}^n \eta_i \geq t\sqrt{n} \right\} \leq 2e^{-t^2/2}.$$ 

It follows that given $\delta \in (0, 1)$,

$$\Pr\left\{ \mu(\tilde{X}) \geq \sqrt{\frac{2n \ln \left(\frac{p+1}{\delta}\right)}{\delta}} \right\} \leq \frac{p(p+1)}{2} \Pr\left\{ \frac{|\tilde{V}_1^\top \tilde{V}_2|}{\sqrt{\|V_1\|_2^2 \|V_2\|_2}} \geq \sqrt{\frac{2n \ln \left(\frac{p+1}{\delta}\right)}{\delta}} \right\} \leq \delta.$$ 

Therefore, with high probability, $\mu(\tilde{X}) = O(\sqrt{\ln p/n})$, which is very small for reasonably large $p$ and $n$. 

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4 Least square regression: preliminaries

4.1 Reformulation and Condition H2

Suppose that, with $X$ fixed,

$$y_i = f(X_i^T \beta) + \epsilon_i, \quad 1 \leq i \leq n,$$

where $\epsilon_i$ are independent with mean 0. The $L_0$ regularized LS estimator for $\beta$ is

$$\hat{\beta} = \arg \min_{u \in D} \left[ \| y - f(Xu) \|_2^2 + c_r|\text{spt}(u)| \right], \quad (4.1)$$

where, as in $(3.3)$, $D$ is a suitable search domain in $\mathbb{R}^p$ and $c_r$ is a regularization parameter. If $\beta \in D$, then

$$\| y - f(X\hat{\beta})\|_2^2 + c_r|\text{spt}(\hat{\beta})| \leq \| y - f(X\beta)\|_2^2 + c_r|\text{spt}(\beta)|,$$

and hence

$$\| f(X\hat{\beta}) - f(X\beta)\|_2^2 \leq 2(\epsilon, f(X\hat{\beta}) - f(X\beta)) - c_r(|\text{spt}(\hat{\beta})| - |\text{spt}(\beta)|),$$

which implies (2.2) once we define

$$G(x) = \|x\|_2^2, \quad \psi_i(z) = \varphi_i(z) = f(z), \quad (4.2)$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $1 \leq i \leq n$. By Proposition 2.1, all we need to do then is to find suitable constants $c_1$ and $c_2$ so that Conditions H1 and H2 are satisfied.

For $I \subset \mathbb{R}$ that contains at least two points, denote

$$d(f,I) = \inf \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x \in I, y \in I, x \neq y \right\}.$$ 

We start with the easier task of establishing Condition H2.

**Proposition 4.1** Let $I \subset \mathbb{R}$ be an interval with positive length. Suppose $f$ is defined on $I$ with $d(f,I) > 0$. Fix $\nu \in (0,1)$. Let $D$ in (4.1) be a subset of $\mathcal{D}(I,n(\nu)/2)$. If $\beta \in D$, then for $G$ and $\psi$ defined as in (4.2), Condition H2 is satisfied with

$$c_2 = \frac{d(f,I)^2 \nu [1 + \mu(X)]}{n} \min_{1 \leq j \leq p} \|V_j\|_2^2.$$

As noted in Section 3.2, under mild conditions, for large $n$ and reasonably large $p$, $c_2 \asymp 1$. Therefore, by Proposition 2.1, in order for the estimate $\hat{\beta}$ to have some reasonable precision, the coefficient $c_1$ in Condition H1 has to be of order $o(\sqrt{n})$. To this end, depending on how well the nonlinear function $f$ behaves, some extra constraints need to be imposed on the domain $D$. Section 5 is devoted to establishing Condition H1 for the LS regression. Below we outline the steps to be taken.
4.2 Observations that point to Condition H1

Recall that Condition H1 stipulates an upper bound on $|\langle \epsilon, f(Xu) - f(X\beta) \rangle|$ that has to hold simultaneously for all $u$. If $f(x) = x$, such a bound is easy to find due to the conjugate relation $\langle \epsilon, f(Xu) - f(X\beta) \rangle = \langle X^\top \epsilon, u - \beta \rangle$, as it then suffices to find a bound for $\|X^\top \epsilon\|_\infty$, which can be derived from the tail assumption on $\epsilon$ (Candès & Plan 2009, Zhang 2009). For nonlinear $f$, in general, there are no similar applicable relations. However, like $e^x/(1 + e^x)$, in many cases, $f$ is analytic and so we may exploit its power series expansions around different points. By working with, say $f(x) = x^2$, one could imagine a kind of power series expansion

$$f(Xu) = \sum M_\alpha h_\alpha(u),$$

such that each $M_\alpha$ is some type of (row-wise) monomial transformation of $X$, and $h_\alpha(u)$ a vector resulting from a similar transformation of $u$. This makes it possible to rewrite $\langle \epsilon, f(Xu) - f(X\beta) \rangle$ as an infinite sum of $\langle M_\alpha^\top \epsilon, h_\alpha(u) - h_\alpha(\beta) \rangle$, which could lead to a desirable bound.

The method works if $f$ is analytic on the entire $\mathbb{C}$, or, more generally, when all the coordinates of $Xu$ and $X\beta$ fall into the disc of convergence of the power series expansion of $f$ at 0. On the other hand, when $f$ has poles as $e^x/(1 + e^x)$ does, the coordinates of $Xu$ and $X\beta$ may fall into different discs of convergence of power series expansion. Roughly, to deal with this problem, our approach is to cover the line segment connecting $Xu$ and $X\beta$ with different discs of convergence of power series, apply the result obtained for the case of single analytic disc, and patch together the resulting bounds. This turns out to account for most of the complexity in our treatment of the analytic case.

One question is whether we can just use a finite Taylor expansion to derive bounds for $\langle \epsilon, f(Xu) - f(X\beta) \rangle$, thus dispensing with the assumption of analyticity. The answer seems to be no in general. Unless $f$ is a polynomial, a finite Taylor expansion of $f(Xu) - f(X\beta)$ has a remainder term of the form $R_\alpha(u)[h_\alpha(u) - h_\alpha(\beta)]$, where $R_\alpha(u)$ is a matrix that in general depends on $u$. As a result, although for each individual $u$, we can get a bound for $\langle \epsilon^\top R_\alpha(u), h_\alpha(u) - h_\alpha(\beta) \rangle$ that holds with high probability, there is no guarantee to get that with high probability, the bounds hold simultaneously for all $u$, which is needed for establishing the precision of $\hat{\beta}$.

5 Least square regression: continued

5.1 Setup

Let $I \subset \mathbb{R}$ be a closed interval with positive length. In this section, we assume that $f : I \rightarrow \mathbb{R}$ is analytic in a neighborhood of $I$, i.e., $f$ has a (unique) analytic extension onto an open set in $\mathbb{C}$ containing $I$. This is equivalent to saying that
$f \in C^\infty(I)$ and for each $t \in I$, there is $r > 0$, such that

$$\sum_{k=0}^{\infty} |a_k| r^k < \infty,$$
where $a_k = \frac{f^{(k)}(t)}{k!} \in \mathbb{R}$,

and $f(z + t) = \sum_{k=0}^{\infty} a_k z^k$, for all $z \in (-r, r)$ with $z + t \in I$. \hfill (5.1)

The radius of convergence of the power series (5.1), henceforth denoted by $\rho(f, t)$, can be determined by (Rudin 1987)

$$\rho(f, t) = \left(\lim_{k \to \infty} |a_k|^{1/k}\right)^{-1}$$

If $|z| < \rho(f, t)$, then we say $f(z + t)$ has a convergent power series expansion at $t$.

We will regularly use the following weighted $L_1$ norm

$$\|u\|_{1,s} = \sum_{j=1}^{p} |u_j| \|V_j\|_s, \quad u \in \mathbb{R}^p, \ s \geq 1.$$ \hfill (5.2)

Recall that it is assumed from the beginning that $V_j \neq 0$ for all $j$. Therefore, $\|u\|_{1,s}$ is indeed a norm. Finally, if $(\mathcal{E}, \|\cdot\|)$ is a normed linear space, then denote by

$$B(u, a; \|\cdot\|) = \{v \in \mathcal{E} : \|v - u\| < a\}$$

the sphere centered at $u \in \mathcal{E}$ with radius $a > 0$ under the norm $\|\cdot\|$, and by

$$\delta(E; \|\cdot\|) = \inf\{a : E \subset B(u, a; \|\cdot\|) \text{ for some } u\}.$$ 

the infimum of the radii of spheres under the norm $\|\cdot\|$ that contain $E \subset \mathcal{E}$.

### 5.2 Single analytic disc

We first consider the case where all $f(X_1^T u), \ldots, f(X_n^T u)$ have convergent power series expansions at 0. The main result of this section is as follows.

**Theorem 5.1** Suppose $0 \in I$ and $d(f, I) > 0$. Fix $\nu \in (0,1)$ and $\theta \in (0,1)$. Suppose

$$D = \mathcal{D}(I, n(\nu)/2) \cap \{u \in \mathbb{R}^p : \|u\|_{1,\infty} \leq \theta \rho(f, 0)/2\}$$

in (4.1) and $\epsilon$ satisfies (2.3) for $\sigma > 0$. Given $q \in (0,1)$, let $\lambda_p = \ln[p(1 + q^{-1})]$. If $\beta \in D$, then the conclusion of Proposition 2.1 holds with

$$c_1 = \sigma \sqrt{2\lambda_p} \sum_{k=1}^{\infty} \left[ \frac{\sqrt{K} f^{(k)}(0)}{(k - 1)!} \left[\theta \rho(f, 0)\right]^{k-1} \times n^{-\frac{1}{2}} \max_{1 \leq j \leq p} \|V_j\|_{2k} \right],$$

and $c_2$ as in Proposition 4.1.
If \( f \) is linear, then the expression of \( c_1 \) is simplified into
\[
c_1 = \sigma \sqrt{2\lambda_p |f'(0)|} \max_{1 \leq j \leq p} \|V_j\|_2 / \sqrt{n}.
\]
In the general case, as \( n^{-1/2k} \max_j \|V_j\|_k \leq \max_j \|V_j\|_\infty \),
\[
c_1 \leq \sigma \sqrt{2\lambda_p K} \max_j \|V_j\|_\infty, \quad \text{with} \quad K = \sum_{k=1}^{\infty} \frac{\sqrt{2}|f^{(k)}(0)|}{(k-1)!} |\theta g(f,0)|^{k-1}.
\]

Since \( g(f,0) = (\lim_k |f^{(k)}(0)/k!|^{1/k})^{-1} \), it is easy to see that \( c_1 < \infty \). As noted in Section 3.2, under mild conditions, \( \max_j \|V_j\|_\infty = O(\sqrt{\ln(np)}) \). Since \( \lambda_p = O(\ln p) \) and \( K \) is a constant, \( c_1 = O(\sqrt{\ln(np)\ln p}) \). Therefore, for reasonably large \( p \), such as \( p = n^a \), \( c_1 = O(\sqrt{\ln n}) \). Moreover, as seen previously, under mild conditions, it is possible that \( c_1 = O(\ln n) \). Combining the comment after Proposition 4.1, it is seen that the regression estimator (4.1) can have good precision.

### 5.3 Multiple analytic discs

We first need some preparation. Let \( N \subset \mathbb{C} \) be an open set containing \( I \) such that \( f \) has an analytic extension on \( N \). Let \( J = N \cap \mathbb{R} \). For \( u \in D(J) \), \( i = 1, \ldots, n \), and \( k \in \mathbb{N} \), define functions,
\[
a_{ik}(u) = \frac{f^{(k)}(X_i^\top u)}{k!}, \quad A_k(u) = \max_{1 \leq i \leq n} |a_{ik}(u)|, \quad r(u) = \min_{1 \leq i \leq n} g(f, X_i^\top u). \quad (5.3)
\]

It is easy to see that \( r(u) > 0 \). Given any function \( b(u) \) on \( D(J) \) satisfying
\[
0 < b(u) < r(u)
\]
and given any set \( E \subset D(J) \), denote
\[
b(E) = \inf_{u \in E} b(u), \quad r(E) = \inf_{u \in E} r(u), \quad A_k(E) = \sup_{u \in E} A_k(u). \quad (5.5)
\]

If \( E \) is finite, then it is easy to see that \( r(E) > b(E), \) and, by \( \lim_k |a_{ik}(u)|^{1/k} = 1/g(f, X_i^\top u) \) for \( u \in D(J) \) and \( i = 1, \ldots, n, \)
\[
\lim_{k \to \infty} A_k(E)^{1/k} = \max_{u \in E} \lim_{k \to \infty} |a_{ik}(u)|^{1/k} = \frac{1}{r(E)}. \quad (5.6)
\]

Let \( G \) be a subset of \( D(J) \). If
\[
E \subset \bigcup_{u \in G} \mathcal{O}_u, \quad \text{with} \quad \mathcal{O}_u = B(u, b(u)/2; \| \cdot \|_{1,\infty}), \quad (5.7)
\]
then \( G \) will be referred to as a “\( b/2 \)-covering grid”, or simply “covering grid” for \( E \). By this definition, for each point \( u \) in a covering grid and \( i = 1, \ldots, n \), \( f \) is analytic.
at $X_i^T u$ with $g(f, X_i^T u) > b(u)$. Note that a covering grid of $E$ need not be its subset. If $E$ is compact, it always has a finite covering grid.

Finally, for $E \subset \mathbb{R}^p$, denote

$$C(E) = \{(1-s)u + sv : s \in [0,1], u, v \in E\},$$

i.e., the union of all the line segments connecting pairs of points in $E$. If $E$ is bounded (resp. compact), then $C(E)$ is bounded (resp. compact). If $|\text{spt}(u)| \leq a$ for every $u \in E$, then $|\text{spt}(v)| \leq 2a$ for every $v \in C(E)$. However, $C(E)$ may not be convex, and for unbounded closed $E$, $C(E)$ may not be even closed.

After all the preparation, the main result can be stated as follows.

**Theorem 5.2** Suppose $I$ is compact and $d(I, f) > 0$. Fix $\nu \in (0,1)$. In the regression (4.1), let $D$ be a closed subset of $\mathcal{D}(I, n(\nu)/2)$. Fix $b(u)$ satisfying (5.4). Let $G$ be a finite $b/2$-covering grid of $C(D)$. Given $q \in (0,1)$, let $\lambda_p = \ln p(1+q^{-1})$. If $\beta \in D$, then the conclusion of Proposition 2.1 holds with

$$c_1 = \sqrt{2} \sigma \sum_{k=1}^{n} \left[ k \sqrt{\ln |G| + k \lambda_p A_k(G)b(G)^{k-1}} \times n^{-\frac{3}{4}} \max_{1 \leq k \leq p} \|V_j\|_{2k}\right]$$

and $c_2$ as in Proposition 4.1.

To get $c_1$, it is enough to assume $D$ is a compact subset of $\mathcal{D}(J)$. The stronger assumption that $D \subset \mathcal{D}(I, n(\nu)/2)$ is needed in order to get both $c_1$ and $c_2$. By Proposition 2.3, $\mathcal{D}(I, n(\nu)/2)$ is compact. Therefore, if $D \subset \mathcal{D}(I, n(\nu)/2)$ is closed, it is compact as well.

Unlike in Theorem 5.1, here $c_1$ depends on $|G|$. In order for the regression estimator (4.1) to have good precision, $|G|$ has to be controlled. The smaller $|G|$ is, the higher the precision we can claim for $\hat{\beta}$. To see what might be an acceptable level of $|G|$, observe that

$$c_1 \leq \sqrt{2} \sigma K \sqrt{\ln |G| + \lambda_p \max_{1 \leq j \leq p} \|V_j\|_{\infty}} = O \left( \sqrt{\ln(p|G|)} \max_{j} \|V_j\|_{\infty} \right),$$

where $K = \sum_k k^{3/2} A_k(G)b(G)^{k-1}$ is finite by (5.6). From the comment after Proposition 4.1, it is seen that $\hat{\beta}$ has good precision if $\sqrt{\ln(p|G|)} \max_j \|V_j\|_{\infty} = o(\sqrt{n})$.

Provided $\max_j \|V_j\|_{\infty} = O(\sqrt{\ln(np)})$ and $p = n^a$, this implies there should be $\ln |G| = o(n/\ln n)$. Certainly, $|G|$ depends on the choice of the search domain $D$ in (4.1) and the property of $f$. We next get some upper bounds of $|G|$.

### 5.4 Upper bounds on the cardinality of covering grid

We follow the notation in Section 5.3. Recall that $f$ is analytic on some open domain $N \subset \mathbb{C}$ containing $I = [a, b]$ and $J = N \cap \mathbb{R}$. The next result says that $|G|$ can be as small as 1 in Theorem 5.2. It follows directly from the definition of covering grid.
**Proposition 5.3** Let $D \subset B(w, d/2; \| \cdot \|_{1, \infty})$ for some $w \in \mathcal{D}(J)$ and $0 < d < r(w)$. Then for any $b$ satisfying (5.4) and $d < b(w)$, $\{ w \}$ is a $b/2$-covering grid for $C(D)$.

As an example, if $f$ is analytic in a neighborhood of 0 and $\| u \|_{1, \infty} \leq \theta g(f, 0)/2$ for all $u \in D$, where $0 < \theta < 1/2$, then, since $r(0) = g(f, 0)$, $\{ 0 \}$ is a $b/2$-covering grid of $C(D)$ for any $b$ satisfying (5.4) with $b(0) > \theta g(f, 0)$.

We next consider more general cases. For ease of notation, for $E \subset \mathbb{R}^p$ and $S \subset \{ 1, \ldots, p \}$, denote $\delta(E) = \delta(E; \| \cdot \|_{1, \infty})$ and $E_S = \{ u \in E : \text{spt}(u) \subset S \}$.

**Proposition 5.4** Fix $b(u)$ satisfying (5.4) and $h \in \mathbb{N}$. Let $D \subset \mathcal{D}(I, h/2)$ be compact and $K = C(D)$.

1. If $J = \mathbb{R}$ and $\bar{d}_b := \inf_{u \in \mathcal{D}(J)} b(u) > 0$, then $K$ has a $b/2$-covering grid with cardinality no greater than

$$\sum_{|S| = h: \ K_S \neq \emptyset} [2\delta(K_S)/\bar{d}_b + 1]^h \leq \left( \frac{p}{h} \right)^h [2\delta(D)/\bar{d}_b + 1]^h.$$ 

2. In general, if $d_b := \inf_{u \in \mathcal{D}(I,h)} b(u) > 0$, then $K$ has a $b/2$-covering grid with cardinality no greater than

$$\sum_{|S| = h: \ K_S \neq \emptyset} [4\delta(K_S)/d_b + 1]^h \leq \left( \frac{p}{h} \right)^h [4\delta(D)/d_b + 1]^h.$$ 

Note that, since $I$ is compact, $\inf_{u \in \mathcal{D}(I,h)} r(u) \geq \inf_{x \in I} g(f, x) > 0$, so there are always functions $b(u)$ satisfying (5.4) and $d_b > 0$. For example, $b(u) = r(u)/2$.

Finally, in Theorem 5.2, $c_1$ depends on the choice of $G$, so it may not be easy to use. Using the above bounds on $|G|$, we have some more convenient choices for $c_1$, although they are larger than the one in (5.8).

**Proposition 5.5** Let $D$ be a compact subset of $\mathcal{D}(I, h/2)$ in regression (4.1).

1. Let $\bar{d}_k = \sup_{x \in I} | f(k)(x) | /k!$ and $\bar{g}_0 = \inf_{x \in I} g(f, x)$. Suppose $J = \mathbb{R}$, $\bar{g}_0 > 0$, and for any $\bar{q}_1 \in (0, \bar{g}_0)$, $\sup_{|z| \leq \bar{q}_1} \| f'(z) \| < \infty$. Then the radius of convergence of $\sum_{k \geq 1} \bar{d}_k \bar{g}_0^k$ is $\bar{q}_0$ and given $\bar{q}_1 \in (0, \bar{q}_0)$, $c_1$ in (5.8) can be set equal to

$$c_1 = \sqrt{2\sigma} \sum_{k \geq 1} \left[ k \sqrt{h \ln(pQ)} + k \lambda_p d_k \bar{q}_1^{k-1} \times n^{-\frac{1}{2k}} \max_{1 \leq j \leq p} \| V_j \|_{2k} \right],$$

where $Q = 2\delta(D)/\bar{q}_1 + 1$.

2. Let $d_k = \sup_{x \in I} | f(k)(x) | /k!$ and $\bar{g}_0 = \inf_{x \in I} g(f, x)$. Then $\bar{g}_0 > 0$ is equal to the radius of convergence of $\sum_{k \geq 1} d_k \bar{g}_0^k$, and given $\bar{q}_1 \in (0, \bar{g}_0)$, $c_1$ in (5.8) can be set equal to

$$c_1 = \sqrt{2\sigma} \sum_{k \geq 1} \left[ k \sqrt{h \ln(pQ)} + k \lambda_p d_k \bar{q}_1^{k-1} \times n^{-\frac{1}{2k}} \max_{1 \leq j \leq p} \| V_j \|_{2k} \right],$$

where $Q = 4\delta(D)/\bar{q}_1 + 1$. 

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In (5.9), because the radius of convergence of $\sum_{k \geq 1} \bar{d}_k z^k$ is $\bar{\delta}_0$, $c_1 < \infty$. As $\lambda_p = \ln[p(1+q^{-1})]$, $c_1 = O(\sqrt{n} \ln p \max_j \|V_j\|_{\infty})$. Therefore, under mild conditions, for large $n$, as long as $h$ is not too large, the regression (4.1) still has good precision.

### 5.5 Logistic regression with binary noise

Let $y_1, \ldots, y_n$ be the same random variables as in Section 3.3. However, we only see their randomly “flipped” versions $z_1, \ldots, z_n \in \{0, 1\}$, such that

$$\Pr\{z_1, \ldots, z_n \mid y_1, \ldots, y_n\} = \prod_{i=1}^{n} p_{y_i, z_i},$$

where $p_{ab} \geq 0$ and $p_{a0} + p_{a1} = 1$ for $a = 0, 1$. Suppose all $p_{ab}$ are known. The regression model now is $E(z_i) = f(X_i \top \beta)$ with

$$f(t) = \frac{p_{01} + p_{11} e^t}{1 + e^t}.$$

If $p_{01} = p_{11}$, then $z_i$ is independent of $y_i$ with $\Pr\{z_i = 1\} = p_{11}$, making inference impossible. Therefore, we will assume $\Delta_p = |p_{11} - p_{01}| > 0$.

Since $f$ is analytic on $\mathbb{C} \setminus \{t_k, \ k \in \mathbb{Z}\}$, where $t_k = (2k + 1)\pi i$, we shall apply Proposition 5.3(1). First, since $\epsilon_i = z_i - E(z_i)$ are independent and $|\epsilon_i| \leq 1$, they satisfy the tail assumption (2.3) with $\sigma = 1$. Since $\varrho(f, x)$ is the distance from $z$ to the closest pole, for any $x \in \mathbb{R}$, $\varrho_0 = |0 - t_1| = \pi$. Simple calculation gives $f'(t) = (p_{11} - p_{01})[2 \cosh(t/2)]^{-2}$. By $2|\cosh(a + bi)| \geq e^{\|a\|} - e^{-\|a\|}$ for $a, b \in \mathbb{R}$, it is easy to see that for $y \in (0, \pi)$,

$$M(y) := \sup_{|\ln z| \leq y} |2 \cosh(z/2)|^{-2} < \infty,$$

Fix $\bar{\varrho}_1 \in (0, \pi)$, $r > 0$ and $\nu \in (0, 1)$. Let $I = [-r, r]$ and

$$D = \{ u \in \mathbb{R}^p : \|u\|_{1,\infty} \leq r, \ |\spt(u)| \leq n(\nu)/2 \}.$$

Apparently, $D \subset D(I, n(\nu)/2)$ and $\delta(D) \leq r$, where, as in Proposition 5.3, $\delta(D) = \delta(D; \|\cdot\|_{1,\infty})$.

Let $\theta \in (\bar{\varrho}_1/\pi, 1)$. For any $x \in \mathbb{R}$ and $k \geq 1$, by Cauchy’s contour integral,

$$\frac{|f^{(k)}(x)|}{k!} \leq \frac{1}{2k\pi} \oint_{|z-x| = \bar{\varrho}_1/\theta} |f'(z)| dz \leq \frac{\Delta_p M(\bar{\varrho}_1/\theta)}{k(\bar{\varrho}_1/\theta)^{k-1}},$$

giving $d_k \leq \Delta_p M(\bar{\varrho}_1/\theta)/[k(\bar{\varrho}_1/\theta)^{k-1}]$. Therefore, by Proposition 5.3(1),

$$c_1 \leq \sqrt{2} \Delta_p M(\bar{\varrho}_1/\theta) \sum_{k=1}^{\infty} \left( R + k\lambda_p g^{k-1} \times n^{-\frac{k}{k+1}} \max_{1 \leq j \leq p} \|V_j\|_{2k} \right),$$

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Since

\[ m(r) := \inf_{x \in [-r, r]} |2 \cosh(x/2)|^{-2} > 0. \]

Therefore, by Proposition 4.1,

\[ c_2 \geq \frac{\Delta^2_p m(r)^2 \nu [1 + \mu(X)]}{n} \min_{1 \leq j \leq p} \| V_j \|^2. \]

Similar to Section 3.3, if all the entries of \( X \) are \( \pm 1 \), then the results can be simplified so that \( D = \{ u \in \mathbb{R}^p : \sum_i |u_i| \leq r, |\text{spt}(u)| \leq n(\nu)/2 \} \), and

\[ c_1 \leq \sqrt{2} \Delta_p M(\tilde{\gamma}_1/\theta) \sum_{k=1}^{\infty} \sqrt{R + k \lambda_p \theta^{k-1}}, \quad c_2 \geq \Delta^2_p m(r)^2 \nu [1 + \mu(X)]. \]

### 6 Technical details

#### 6.1 Preliminary results

**Proof of Proposition 2.2.** (1) Let \( S = \text{spt}(u) \). If \( |S| = 0 \), then \( u = 0 \) and the inequality trivially holds. Suppose \( |S| \geq 1 \). Since \( Xu = \sum_{j \in S} u_j V_j \),

\[
\| Xu \|_2^2 = \sum_{j \in S} |u_j|^2 \| V_j \|_2^2 + \sum_{i,j \in S, i \neq j} u_i u_j V^T_i V_j \\
\geq \sum_{j \in S} |u_j|^2 \| V_j \|_2^2 - \mu(X) \sum_{i,j \in S, i \neq j} |u_i||u_j|\|V_i\|_2\|V_j\|_2 \\
= [1 + \mu(X)] \sum_{j \in S} |u_j|^2 \| V_j \|_2^2 - \mu(X) \left( \sum_{j \in S} |u_j|\|V_j\|_2 \right)^2.
\]

By Cauchy-Schwartz inequality,

\[
\| Xu \|_2^2 \geq (1 + \mu(X) - \mu(X)|S|) \sum_{j \in S} |u_j|^2 \| V_j \|_2^2.
\]

Since \( |S| \leq n(\nu) = (1 - \nu)[1 + 1/\mu(X)] \), then \( 1 + \mu(X) - \mu(X)|S| \geq \nu[1 + \mu(X)] \), which implies the desired inequality.

(2) By \( \text{spt}(u - v) \subset \text{spt}(u) \cup \text{spt}(v) \) and the assumption, \( |\text{spt}(u - v)| \leq n(\nu) \). The inequality then follows from (1). \( \square \)

**Proof of Proposition 2.3.** (1) Because \( I \) is closed and the mapping \( T : u \rightarrow Xu \) is continuous, \( \mathcal{D}(I) = T^{-1}(I^n) \) is closed. Also, \( \mathcal{V}_h := \{ u \in \mathbb{R}^p : |\text{spt}(u)| \leq h \} \) is closed. Thus \( \mathcal{D}(I, h) = \mathcal{D}(I) \cap \mathcal{V}_h \) is closed. It is easy to see that \( \mathcal{D}(I, h) \subset \mathcal{D}(I, h') \) when \( h < h' \).
Because of (1), to show that \( \mathcal{D}(I, h) \) is compact for \( h < n(0) \), it suffices to show the set is bounded. Since \( h < n(0) \), there is \( \nu \in (0, 1) \) such that \( h \leq n(\nu) \). Let \( u \in \mathcal{D}(I, h) \). Then \( |\text{spt}(u)| \leq n(\nu) \), so by Proposition 2.2,

\[
\|u\|_2^2 \leq \frac{\|Xu\|_2^2}{\nu(1 + \mu(X)) \min_{1 \leq j \leq p} \|V_j\|_2^2}.
\]

Since \( X_i^\top u \in I \) for each \( i \), then \( \|Xu\|_2^2 \leq n \max_i |X_i^\top u|^2 \leq n \sup_{x \in I} |x|^2 \). Because \( I \) is bounded, it is seen \( \|u\|_2^2 \) is bounded for \( u \in \mathcal{D}(I, h) \). \( \square \)

### 6.2 Exponential linear models

In this section, we prove the next two lemma.

**Lemma 6.1** Condition \( H_1 \) is by satisfied for \( \varphi = (\varphi_1, \ldots, \varphi_n) \) with

\[
c_1 = \sigma \sqrt{\frac{\ln(p/q)}{2n}} \max_{1 \leq j \leq p} \|V_j\|_2. \tag{6.1}
\]

**Lemma 6.2** Condition \( H_2 \) is satisfied by \( G \) and \( \psi = (\psi_1, \ldots, \psi_n) \) with

\[
c_2 = \frac{\nu \delta [1 + \mu(X)]}{2n} \min_{1 \leq j \leq p} \|V_j\|_2^2. \tag{6.2}
\]

By Proposition 2.1, if \( c_r = 3c_1^2/c_2 \) in (3.3), then (3.6) holds with \( \kappa_r = 3c_1/c_2 \). Therefore, once the lemmas are proved, we get the expressions of \( c_r \) and \( \kappa_r \) as in Theorem 3.1.

As in (3.4), let \( G(x) = x_1 + \cdots + x_n \) for \( x \in \mathbb{R}^n \), and \( \varphi_i(z) = z/2 \), \( \psi_i(z) = \Lambda(z) - \Lambda'(X_i^\top \beta)z \) for \( 1 \leq i \leq n \) and \( z \in \mathbb{R} \).

**Proof of Lemma 6.1.** By (2.3) and \( \epsilon^\top V_j = \sum_{i=1}^n X_{ij} \epsilon_i \),

\[
\Pr \left\{ \left| \epsilon^\top V_j \right| \leq \sqrt{2 \ln(p/q)} \sigma \|V_j\|_2, \text{ all } j = 1, \ldots, p \right\} \\
\geq 1 - \sum_{j=1}^p \Pr \left\{ \left| \epsilon^\top V_j \right|^2 > 2 \ln(p/q) \sigma^2 \|V_j\|_2^2 \right\} \geq 1 - 2q.
\]

Consequently, with probability at least \( 1 - 2q \),

\[
\|X^\top \epsilon\|_\infty = \max_{1 \leq j \leq p} \left| \epsilon^\top V_j \right| \leq \sqrt{2 \ln(p/q)} \sigma \max_{1 \leq j \leq p} \|V_j\|_2 = 2c_1 \sqrt{n},
\]

which implies condition \( H_1 \) due to the fact that for all \( u \in \mathbb{R}^p \),

\[
|\langle \epsilon, \varphi(Xu) - \varphi(X\beta) \rangle| = \frac{1}{2} |\langle \epsilon, Xu - X\beta \rangle| = \frac{1}{2} |(X^\top \epsilon)^\top (u - \beta)| \leq \frac{1}{2} \|X^\top \epsilon\|_\infty \|u - \beta\|_1. \quad \square
\]
Proof of Lemma 6.2. Given \( u \in \mathcal{D}(I, n(\nu)/2) \), for \( t \in [0, 1] \), let

\[
h(t) = \sum_{i=1}^{n} \psi_i((1-t)X_i^\top \beta + tX_i^\top u),
\]

which is well-defined as \((1-t)X_i^\top \beta + tX_i^\top u \in I\). Let \( \Delta = G(\psi(Xu) - \psi(X\beta)) \). Then

\[
\Delta = \sum_{i=1}^{n} \left[ \psi_i(X_i^\top u) - \psi_i(X_i^\top \beta) \right] = h(1) - h(0).
\]

Observe that \( \psi'_i(X_i^\top \beta) = 0 \). Then \( h'(0) = \sum_i X_i^\top (u - \beta) \psi'_i(X_i^\top \beta) = 0 \), so by Taylor expansion, \( \Delta = h''(\tau)/2 \) for some \( \tau \in (0, 1) \). By \( \psi''_i(z) = \Lambda''(z) \) and \( \inf_{t \in I} \Lambda''(t) = \delta > 0 \),

\[
\Delta = \frac{1}{2} \sum_{i=1}^{n} [X_i^\top (u - \beta)]^2 \psi''_i((1-t)X_i^\top \beta + tX_i^\top u) \\
\geq \frac{\delta}{2} \sum_{i=1}^{n} [X_i^\top (u - \beta)]^2 = \frac{\delta\|X(u - \beta)\|_2^2}{2}.
\]

By \( |\text{spt}(u - \beta)| \leq |\text{spt}(u) \cup \text{spt}(\beta)| \leq n(\nu) \) and Proposition 2.2,

\[
\Delta \geq \frac{\delta\nu[1 + \mu(X)]}{2} \min_{1 \leq j \leq n} \|V_j\|_2^2 \cdot \|u - \beta\|_2^2,
\]

and so Condition H2 is satisfied with \( c_2 \) set as in (6.2). \( \square \)

6.3 Proofs for LS regression: the case of single analytic disc

First, we establish Condition H2.

Proof of Proposition 4.1. For \( i = 1, \ldots, n \) and \( u \in D \), since \( X_i^\top \beta \in I \) and \( X_i^\top u \in I \),

\[
\|f(Xu) - f(X\beta)\|_2^2 = \sum_{i=1}^{n} |f(X_i^\top u) - f(X_i^\top \beta)|^2 \\
\geq \sum_{i=1}^{n} d(f, I)^2 |X_i^\top u - X_i^\top \beta|^2 = d(f, I)^2 \|X(u - \beta)\|_2^2.
\]

Since \( |\text{spt}(u - \beta)| \leq |\text{spt}(u)| + |\text{spt}(\beta)| \leq n(\nu) \), then by Proposition 2.2,

\[
\|f(Xu) - f(X\beta)\|_2^2 \geq d(f, I)^2 \nu[1 + \mu(X)] \min_{1 \leq j \leq p} \|V_j\|_2^2 \times \|u - \beta\|_2^2.
\]

Because the right hand side is \( c_2 n \|u - \beta\|_2^2 \), the proof is complete. \( \square \)

The main result in this section is Proposition 6.5, which together with Proposition 4.1 immediately leads to Theorem 5.1. For brevity, in the rest of this section, we shall denote \( \Pi = \{1, \ldots, p\} \).
6.3.1 Power series expansion and tail assumption

To facilitate subsequent discussions, we first consider
\[ \varphi(x) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))^\top, \quad x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n, \]
where \( \varphi_1, \ldots, \varphi_n \) are real-valued functions that may be different from each other.

Suppose each \( \varphi_i \) can be analytically extended to a neighborhood of 0 in \( \mathbb{C} \). Let
\[ a_{ik} = \frac{\varphi_i^{(k)}(0)}{k!} \in \mathbb{R}, \quad (6.3) \]
Then \( g(\varphi_i, 0) = \left( \lim_{k \to \infty} |a_{ik}|^{1/k} \right)^{-1} \). Since we are interested in \( \varphi(Xu) - \varphi(Xv) \) instead of \( \varphi(Xu) \) itself, without loss of generality, let \( \varphi_i(0) = 0 \).

For vector \( v = (v_1, \ldots, v_p)^\top \) and \( k \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \Pi^k \), denote by \( v_\alpha \) the product of \( v_{\alpha_1}, \ldots, v_{\alpha_k} \). For example, if \( p = 3 \) and \( k = 4 \), then \( v_{(1,3,1,2)} = v_1v_3v_1v_2 = v_1^2v_2v_3 \). With this notation, for \( i = 1, \ldots, n \), \( Xu = X_{i\alpha} \) \( X_{i\alpha_1} \cdots X_{i\alpha_k} \). For each \( j = 1, \ldots, p \), let \( n_j(\alpha) = |\{i : \alpha_i = j\}| \). Clearly, \( n_1(\alpha) + \cdots + n_p(\alpha) = k \).

By (6.3), for \( i = 1, \ldots, n \), provided \( |X_i^\top u| < g(\varphi_i, 0) \),
\[ \varphi_i(X_i^\top u) = \sum_{k=1}^{\infty} a_{ik}(X_i^\top u)^k = \sum_{k=1}^{\infty} a_{ik} \left( \sum_{\alpha \in \Pi^k} X_{i\alpha} u_\alpha \right). \]
Therefore, if \( |X_i^\top u| < g(\varphi_i, 0) \) for all \( i \), then
\[ \langle \epsilon, \varphi(Xu) \rangle = \sum_{i=1}^{n} \epsilon_i \varphi_i(X_i^\top u) = \sum_{k=1}^{\infty} \sum_{\alpha \in \Pi^k} u_\alpha \sum_{i=1}^{n} \epsilon_i a_{ik} X_{i\alpha}. \quad (6.4) \]

**Lemma 6.3** Suppose \( \epsilon \) satisfy (2.3). Let \( q_1, q_2, \ldots \geq 0 \) with \( q := \sum_k q_k < 1/2 \). Given real numbers \( \theta_{ik} \), \( 1 \leq i \leq n \), \( 1 \leq k \leq p \), consider the condition
\[ \left| \sum_{i=1}^{n} \epsilon_i \theta_{ik} X_{i\alpha} \right| \leq \sigma \sqrt{2 \ln(p^k/q_k) \sum_{i=1}^{n} \theta_{ik}^2 X_{i\alpha}^2}, \quad (6.5) \]
where \( \sigma \) is the constant in (2.3) and \( \ln 0 \) is defined to be \(-\infty\). Then
\[ \Pr \left\{ (6.5) \text{ holds for all } k \geq 1 \text{ and } \alpha \in \Pi^k \right\} \geq 1 - 2q. \quad (6.6) \]

**Proof.** The left hand side of (6.6) is at least
\[ 1 - \sum_{k=1}^{\infty} \sum_{\alpha \in \Pi^k} \Pr \{ (6.5) \text{ does not hold for } k \text{ and } \alpha \} \]
Since \( |\Pi^k| = p^k \), it suffices to show that for each \( k \) and \( \alpha = (\alpha_1, \ldots, \alpha_k) \),
\[ \Pr \left\{ \left| \sum_{i=1}^{n} \epsilon_i \theta_{ik} X_{i\alpha} \right|^2 > 2\sigma^2 \ln(p^k/q_k) \sum_{i=1}^{n} \theta_{ik}^2 X_{i\alpha}^2 \right\} \leq 2p^{-k} q_k, \quad (6.7) \]
which directly follows from (2.3). \( \Box \)
6.3.2 Establishing Condition H1

Recall the following multinomial formula: for any \( j = 1, \ldots, p \),

\[
\sum_{\alpha \in \Pi^k} n_j(\alpha) x_j^{n_j(\alpha)-1} \prod_{s \neq j} x_s^{n_s(\alpha)} = k(x_1 + \cdots + x_p)^{k-1},
\]

(6.8)
as the left hand side is equal to

\[
\sum_{k_1 + \cdots + k_p = k} \binom{k}{k_1 \cdots k_p} x_j^{k_j-1} \prod_{s \neq j} x_s^{k_s} = \frac{\partial}{\partial x_j} \left[ \sum_{k_1 + \cdots + k_p = k} \binom{k}{k_1 \cdots k_p} x_j^{k_j} \prod_{s \neq j} x_s^{k_s} \right] = \frac{\partial \left[ \sum_i x_i^k \right]}{\partial x_j}.
\]

For each \( j = 1, \ldots, p \), let

\[
\omega_{jk} = a_{1k}^2 x_{1j}^{2k} + \cdots + a_{nk}^2 x_{nj}^{2k}.
\]

(6.9)

Lemma 6.4 Suppose that, with \( \theta_{ik} = a_{ik} \), (6.5) holds for all \( k \geq 1 \) and \( \alpha \in \Pi^k \).

Given \( u \) and \( v \), let \( d_j = |u_j - v_j| \) and \( m_j = |u_j| \vee |v_j| \) for \( j = 1, \ldots, p \). If

\[
\sum_{j=1}^{p} m_j \max_{1 \leq i \leq n} \frac{|X_{ij}|}{\rho(\varphi_i, 0)} < 1,
\]

(6.10)
then, letting \( \xi = \langle \epsilon, \varphi(X_u) - \varphi(X_v) \rangle \),

\[
|\xi| \leq \sigma \sqrt{2} \sum_{k=1}^{\infty} \left[ k \sqrt{\ln(p^k/q_k)} \left( \sum_{j=1}^{p} m_j \omega_{jk}^{2k} \right)^{k-1} \sum_{j=1}^{p} d_j \omega_{jk}^{2k} \right].
\]

(6.11)

Proof. By (6.10), for any \( i \),

\[
|X_i^\top u| \leq \sum_{j=1}^{p} |u_j X_{ij}| \leq \rho(\varphi_i, 0) \sum_{j=1}^{p} m_j \frac{|X_{ij}|}{\rho(\varphi_i, 0)} < \rho(\varphi_i, 0),
\]

and likewise \( |X_i^\top v| < \rho(\varphi_i, 0) \). Therefore, by (6.4),

\[
\xi = \sum_{k=1}^{\infty} \sum_{\alpha \in \Pi^k} \left[ (u_\alpha - v_\alpha) \sum_{i=1}^{n} \epsilon_i a_{ik} X_{i\alpha} \right].
\]
By the assumption, (6.5) holds with $\theta_{ik} = a_{ik}$ for all $k \geq 1$ and $\alpha \in \Pi^k$. Thus

$$|\xi| \leq \sum_{k=1}^{\infty} \sum_{\alpha \in \Pi^k} \left[ |u_{\alpha} - v_{\alpha}| \sum_{i=1}^{n} \epsilon_i a_{ik} X_{i\alpha} \right]$$

$$\leq \sum_{k=1}^{\infty} \sigma \sqrt{2 \ln(p^k/g_k)} \left[ \sum_{\alpha \in \Pi^k} |u_{\alpha} - v_{\alpha}| \sqrt{M_\alpha} \right]$$

(6.12)

where $M_\alpha = \sum_{i=1}^{n} a_{ik}^2 X_{i\alpha}^2$. Given $k \geq 1$, for each $\alpha \in \Pi^k$, by $n_1(\alpha) + \cdots + n_p(\alpha) = k$ and Cauchy-Schwartz inequality,

$$M_\alpha = \sum_{i=1}^{n} a_{ik}^2 \prod_{j=1}^{p} X_{ij}^{2n_j(\alpha)} \leq \prod_{j=1}^{p} \left( \sum_{i=1}^{n} a_{ik}^2 X_{ij}^{2k} \right)^{n_j(\alpha)/k} \leq \prod_{j=1}^{p} \omega_j n_j(\alpha)/k,$$

where the last inequality is due to the notation in (6.9). On the other hand,

$$|u_{\alpha} - v_{\alpha}| = \left| \prod_{j=1}^{p} u_j^{n_j(\alpha)} - \prod_{j=1}^{p} v_j^{n_j(\alpha)} \right|$$

$$\leq \sum_{j=1}^{p} \left| u_j^{n_j(\alpha)} - v_j^{n_j(\alpha)} \right| \prod_{s=1}^{j-1} |v_s|^{n_s(\alpha)} \prod_{s=j+1}^{n} |u_s|^{n_s(\alpha)}$$

$$\leq \sum_{j=1}^{p} n_j(\alpha) d_j m_j^{n_j(\alpha) - 1} \prod_{s \neq j} m_s^{n_s(\alpha)}.$$

Therefore,

$$\sum_{\alpha \in \Pi^k} |u_{\alpha} - v_{\alpha}| \sqrt{M_\alpha} \leq \sum_{\alpha \in \Pi^k} \left\{ \left[ \prod_{j=1}^{p} n_j(\alpha) d_j m_j^{n_j(\alpha) - 1} \prod_{s \neq j} m_s^{n_s(\alpha)} \right] \prod_{j=1}^{p} \omega_j n_j(\alpha)/(2k) \right\}$$

$$= \sum_{j=1}^{p} d_j \omega^{\frac{1}{2k}} \sum_{\alpha \in \Pi^k} n_j(\alpha) \left[ m_j \omega^{\frac{1}{2k}} \right]^{n_j(\alpha) - 1} \prod_{s \neq j} \left[ m_s \omega^{\frac{1}{2k}} \right]^{n_s(\alpha)}$$

$$= k \left( \sum_{j=1}^{p} m_j \omega^{\frac{1}{2k}} \right)^{k-1} \sum_{j=1}^{p} d_j \omega^{\frac{1}{2k}},$$

where the last equality is due to the multinomial formula (6.8). Now by (6.12), the inequality in (6.11) is proved.

\[ \square \]

**Proposition 6.5** Fix $\theta \in (0,1)$. Let $D = \{ u \in \mathbb{R}^p : \|u\|_{1,\infty} \leq \theta \varrho(f,0)/2 \}$ in Condition H1 and $\epsilon$ satisfy (2.3) for $\sigma > 0$. If $\beta \in D$, then Condition H1 is satisfied by setting $c_1$ as in Theorem 5.1.

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Proof. We have \( \varphi_i = f \) and \( g(\varphi_i, 0) = g(f, 0) \). For \( u \in D \), let \( d = u - \beta \) and \( m = (m_1, \ldots, m_n)^T \), with \( m_j = |u_j| \vee |\beta_j| \). Then
\[
\|m\|_1 \leq \|u\|_1 + \|\beta\|_1 \leq \theta\|g(f, 0)\|.
\] (6.13)

As a result
\[
\sum_{j=1}^p m_j \max_{1 \leq i \leq n} |X_{ij}| = \|m\|_1 \leq \theta
\]
and (6.10) is satisfied. Let \( q_k = \left( \frac{q}{1+q} \right)^k \). Then \( \sum_k q_k = q \), so by Lemmas 6.3 and 6.4, with probability at least 1 - \( 2q \), (6.11) holds. For each \( k \geq 1 \), by the notation in (6.9), \( \omega_{jk} = \left( |f^{(k)}(0)|/k! \right)^2 ||V_j||_2^2 \). Recall that in Theorem 5.1, \( \lambda_p \) is defined to be \( \ln[p(1+q^{-1})] \). Since \( \sqrt{\ln(p^k/q_k)} = k\lambda_p \),
\[
k \sqrt{\ln(p^k/q_k)} \left( \sum_{j=1}^p m_j \omega_{jk}^{1/2} \right)^{k-1} \sum_{j=1}^p d_j \omega_{jk}^{1/2} = \frac{\sqrt{k\lambda_p} f^{(k)}(0)}{(k-1)!} \times \|m\|_{1,2k}^{-1} \times \|d\|_{1,2k},
\] (6.14)
where the weighted \( L_1 \) norm \( \| \cdot \|_{1,s} \) is defined in (5.2) and satisfies
\[
\|u\|_{1,s} = \begin{cases} n^{1/s} \|u\|_1, & s \geq 1, \\ \max_{1 \leq j \leq p} \|V_j\|_s \times \|u\|_1, & s \geq 1. \end{cases}
\]

Then by (6.13),
\[
\|m\|_{1,2k}^{-1} \times \|d\|_{1,2k} \leq \left( n^{1/2} \|m\|_1 \right)^{-1} \times \max_{1 \leq j \leq p} \|V_j\|_{2k} \times \|d\|_1
\]
\[
\leq \sqrt{n \theta g(f, 0)}^{-1} \times n^{-1/2} \max_{1 \leq j \leq p} \|V_j\|_{2k} \times \|d\|_1.
\]

Together with (6.11) and (6.14), this yields the proof. \( \square \)

6.4 LS regression: multiple analytic disc case

6.4.1 Proof of Theorem 5.2

We first restate Lemma 6.3 as follows.

Lemma 6.6 Let \( \epsilon \) satisfy (2.3). Let \( E \subset D(J) \) be finite and for \( k \geq 1 \) and \( u \in E \), let \( q_{k,u} \geq 0 \), such that \( q := \sum_k \sum_{u \in E} q_{k,u} < 1/2 \). Consider the condition
\[
\left| \sum_{i=1}^n \epsilon_i a_{ik}(u) X_{ia} \right| \leq \sigma \sqrt{2 \ln(p^k/q_{k,u}) \left( \sum_{i=1}^n a_{ik}(u)^2 X_{ia}^2 \right)},
\] (6.15)
where \( \sigma > 0 \) is the constant in (2.3). Then

\[
\Pr \left\{ (6.15) \text{ holds for all } k \geq 1, \alpha \in \Pi^k, \text{ and } u \in E \right\} \geq 1 - 2\eta.
\]

The next result provides a bound on \(|\langle \epsilon, f(Xu) - f(Xv) \rangle|\) for suitable \( u \) and \( v \). The method of its proof is described at the end of Section 4.

**Lemma 6.7** Given \( b(u) \) satisfying (5.4), let \( G \) be a finite \( b/2 \)-covering grid of a set \( K \subset D(J) \). Fix \( q_k \geq 0 \) such that \( q := \sum_k q_k < 1/2 \) and \( \ln q_k = O(k) \) over \( \mathcal{J} = \{ k \in \mathbb{N} : A_k(G) > 0 \} \). Suppose that, with \( E = G \) and \( q_k, u = q_k / |G| \), (6.15) holds for all \( k \geq 1, \alpha \in \Pi^k, \) and \( u \in G \). If \( u, v \in K \) and the entire line segment connecting them is in \( K \), then, letting \( \xi = \langle \epsilon, f(Xu) - f(Xv) \rangle \) and \( d = v - u \),

\[
|\xi| \leq \sigma \sqrt{2n} H(b(G), d)
\]

where \( H(b(G), d) < \infty \), with

\[
H(z, d) = \sum_{k=1}^{\infty} k \sqrt{\ln |G| + \ln(p^k/q_k)} A_k(G) \times n^{-\frac{1}{2}} \|d\|_{1,2k} \times z^{k-1}.
\]

**Proof.** Since \( G \) is finite, \( b(G) < r(G) \). Given \( \eta \in (0, r(G)/b(G) - 1) \), let \( T = \left[ \frac{2\|d\|_{1,\infty}}{\eta b(G)} \right] \).

By the assumption, \( u + \theta d \in K \) for \( \theta \in [0, 1] \). For \( t = 0, \ldots, T \), let \( u(t) = u + t d / T \). Then \( u(0) = u, u(T) = v \), and \( u(t) \in K \). Fix \( t = 1, \ldots, T \). Then

\[
\|u(t) - u(t-1)\|_{1,\infty} = \|d\|_{1,\infty} / T \leq \eta b(G) / 2.
\]

By the definition of \( G \), we can find some \( w \in G \), such that \( \|u(t) - w\|_{1,\infty} \leq b(G) / 2 \). Then \( \|u(t-1) - w\|_{1,\infty} \leq (1 + \eta)b(G)/2 \). Let \( \varphi(x) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))^\top \), with

\[
\varphi_i(z) = f(z + X_i^\top w) - f(X_i^\top w), \quad 1 \leq i \leq n.
\]

Let \( \tilde{u} = u(t) - w, \tilde{v} = u(t-1) - w \). Then

\[
\varphi(X\tilde{u}) = f(Xu(t)) - f(Xw), \quad \varphi(X\tilde{v}) = f(Xu(t-1)) - f(Xw),
\]

and, as shown just now,

\[
\|\tilde{u}\|_{1,\infty} \leq (1 + \eta)b(G)/2, \quad \|\tilde{v}\|_{1,\infty} \leq (1 + \eta)b(G)/2.
\]

Let \( m = (m_1, \ldots, m_p)^\top \) with \( m_j = |\tilde{u}_j| \vee |\tilde{v}_j| \). From the above equalities we get

\[
\|m\|_{1,\infty} \leq \|\tilde{u}\|_{1,\infty} + \|\tilde{v}\|_{1,\infty} \leq (1 + \eta)b(G).
\]

(6.17)
and hence, by \( g(\varphi_i, 0) = g(f, X_i^T w) \geq r(w), \)
\[
\sum_{j=1}^{p} m_j \max_{1 \leq i \leq n} |X_{ij}| \leq \frac{\|m\|_{1,\infty}}{r(w)} \leq \frac{(1 + \eta)b(G)}{r(G)} < 1.
\]

Now Lemma (6.4) can be applied to \( \varphi \), with \( u, v \), and \( q_k \) therein replaced with \( \tilde{u}, \tilde{v}, \) and \( q_k/|G| \), respectively. Then
\[
\left| \langle \epsilon, f(Xu^{(t)}) - f(Xu^{(t-1)}) \rangle \right| \leq \sigma \sqrt{2 \sum_{k=1}^{\infty} k}\sqrt{\ln(|G|^p/kq_k)} M_k^{(t)},
\]
where
\[
M_k^{(t)} = \left( \sum_{j=1}^{p} m_j \omega_{jk}^2 \right)^{k-1} \sum_{j=1}^{p} |u_j^{(t)} - u_j^{(t-1)}| \omega_{jk}^2
\]
with \( \omega_{jk} = \sum_{i=1}^{n} a_{ik}^2(u)|X_{ij}|^{2k} \leq A_k^2(G)||V_j||^{2k} \leq nA_k^2(G)||V_j||^{2k}.
\]
Since \( u^{(t)} - u^{(t-1)} = d/T \), it follows that
\[
M_k^{(t)} \leq \left( \sum_{j=1}^{p} m_j n \frac{\|A_k^\frac{1}{2}(G)||V_j||_\infty}{\|d\|_{1,2k}} \right)^{k-1} \times \frac{\|A_k^\frac{1}{2}(G)||V_j||_\infty}{T} \sum_{j=1}^{p} d_j ||V_j||_{2k}
\]
\[
= \frac{\sqrt{n}A_k(G)}{T} \|m\|_{1,\infty}^{k-1} \times n^{-\frac{k}{2}} \|d\|_{1,2k}
\]
\[
\leq \frac{\sqrt{n}A_k(G)}{T} [(1 + \eta)b(G)]^{k-1} \times n^{-\frac{k}{2}} \|d\|_{1,2k},
\]
where the last inequality is due to (6.17). Consequently,
\[
|\xi| \leq \sum_{t=1}^{T} \left| \langle \epsilon, f(Xu^{(t)}) - f(Xu^{(t-1)}) \rangle \right| = H((1 + \eta)b(G), d).
\]

By (5.6) and \( \ln q_k = O(k) \) over \( J \), the radius of convergence of the power series defining \( g(z) = H(z, d) \) is \( r(G) > b(G) \). As \( (1 + \eta)b(G) < r(G) \), we can let \( \eta \to 0 \) and apply dominated convergence. The proof is then complete. \( \square \)

**Proposition 6.8** In Condition H1, let \( D \) be a compact subset of \( \mathcal{D}(J) \). Suppose \( \epsilon \) satisfies (2.3) for some \( \sigma > 0 \). Let \( G \) be a finite \( b/2 \)-covering grid of \( C(D) \). If \( \beta \in D \), then Condition H1 is satisfied by setting \( c_1 \) as in Theorem 5.2.

**Proof.** Since \( C(D) \) is compact, it indeed has a finite \( b/2 \)-covering grid, justifying the assumption on \( G \). As in the proof of Proposition 6.5, let \( q_k = (\frac{\sigma}{1+q})^k \). Then by Lemmas 6.6 and 6.7, with probability at least \( 1 - 2q_1 \), (6.16) holds. The rest of the proof follows that for Proposition 6.5 and hence is omitted for brevity. \( \square \)
Proof of Theorem 5.2. First, by $D \subset \mathcal{D}(I, n(\nu)/2)$ and $d(I, f) > 0$, Proposition 4.1 can be applied to yield $c_2$. Second, $C(D)$ is compact and since $I$ is an interval, $C(D) \subset \mathcal{D}(I)$. Then $C(D) \subset \mathcal{D}(J)$. Proposition 6.8 can be applied to $K = C(D)$ to get $c_1$. □

6.4.2 Other technical results

Proof of Proposition 5.4. Because $D \subset \mathcal{D}(I, h/2)$ and is compact, $K = C(D) \subset \mathcal{D}(I, h)$ and is compact.

First, fix $S$ with $|S| = h$ and $K_S \neq \emptyset$. Let $\psi_S : \mathbb{R}^p \rightarrow \mathbb{R}^S$ be the natural projection and $\iota_S : \mathbb{R}^S \rightarrow \mathbb{R}^p$ the immersion, such that $\iota_S(y) = z \in \mathbb{R}^p$, with $z_j = y_j$ for $j \in S$ and $z_j = 0$ for $j \notin S$. Define the weighted $L_1$ norm $\| \cdot \|_S$ on $\mathbb{R}^S$ such that $\|u\|_S = \sum_{j \in S} |\iota_S(u_j)||V_j|_\infty$. For ease of notation, denote $B_S(w, a) = B(w, a; \| \cdot \|_S)$ and $\delta_S(E) = \delta(E; \| \cdot \|_S)$. Likewise, denote $B(w, a) = B(w, a; \| \cdot \|_{1,\infty})$ and $\delta(E) = \delta(E; \| \cdot \|_{1,\infty})$.

Fix $d > 0$. Later we will set $d$ to specific values. Let $E = \psi_S(K_S)$. It is easy to verify that $\delta_S(E) = \delta(K_S)$. By simple geometric argument, it is seen that $E$ can be covered by no more than $[\delta(K_S)/d + 1]^h$ spheres $B_S(\tilde{u}_k, d)$, with each one intersecting with $E$. Let $u_k = \iota_S(\tilde{u}_k)$.

In case (1), let $d = d_b/2$. By $J = \mathbb{R}$, $f$ is analytic at every $X_i^T u_k$. Then, by

$$K_S = \iota_S(E) \subset \bigcup_k \iota_S(B(\tilde{u}_k, d)) \subset \bigcup_k B(u_k, d) \subset \bigcup_k B(u_k, b(u_k)/2),$$

$u_1, \ldots, u_m$ is a $b/2$-covering grid of $K_S$.

In case (2), Let $d = d_b/4$. Since $f$ may not be analytic at every $X_i^T u_k$, we cannot directly take $u_1, \ldots, u_m$ as a covering grid. For each $i = 1, \ldots, m$, choose an arbitrary $\tilde{w}_k \in B_S(\tilde{u}_k, d) \cap E$ and let $w_k = \iota_S(\tilde{w}_k)$. As $w_k \subset K_S$, $f$ is analytic at every $X_i^T w_k$. It is easy to check that $B_S(\tilde{w}_k, 2d)$ contains $B_S(\tilde{u}_k, d)$. Therefore,

$$K_S = \iota_S(E) \subset \bigcup_k \iota_S(B(\tilde{w}_k, 2d)) \subset \bigcup_k B(w_k, 2d) \subset \bigcup_k B(w_k, b(u_k)/2),$$

so $w_1, \ldots, w_m$ is a $b/2$-covering grid of $K_S$.

Denote by $G_S$ the covering grid as above in either case. As $K = \bigcup_{|S|=h} K_S$, $G = \bigcup_{|S|=h: K_S \neq \emptyset} G_S$ is a $b/2$-covering grid of $K$ and

$$|G| \leq \sum_{|S|=h: K_S \neq \emptyset} |G_S|$$

We already know $|G_S| \leq [\delta(K_S)/d + 1]^h$. By $\delta(K_S) \leq \delta(K) = \delta(D)$,

$$|G_S| \leq [\delta(D)/d + 1]^h.$$

Finally, there are at most $\binom{p}{h}$ subsets $S$ with $|S| = h$ and $K_S \neq \emptyset$. The proof for the bounds on $|G|$ is thus complete. □
Proof of Proposition 5.5. (1) If $c > \tilde{g}_0$, then there is $t \in J$ such that $g(f, t) < c$. Since $\lim_k |f^{(k)}(t)/k|^{1/k} = g(f, t)^{-1}$,

$$\lim_{k \to \infty} \tilde{d}_k c^k \geq \lim_{k \to \infty} \frac{|f^{(k)}(t)c^k|}{k!} = \infty.$$ 

Therefore, the radius of convergence of $\sum_{k \geq 1} \tilde{d}_k z^k$ is at most $\tilde{g}_0$. To show that the radius of convergence is $\tilde{g}_0$, it suffices to show that $\tilde{d}_k c^k$ is bounded for any $c \in (0, \tilde{g}_0)$. By assumption $M := \sup_{|\Im(z)| \leq c} |f'(z)| < \infty$. Fix $x \in \mathbb{R}$. For any $z$ with $|z - x| = c$, $|\Im(z)| \leq c$. Therefore, by Cauchy’s contour integral,

$$\frac{|f^{(k)}(x)|}{k!} = \left| \frac{1}{2k\pi} \oint_{|z-x|=c} \frac{f'(z)dz}{(z-x)^k} \right| \leq \frac{1}{2k\pi} \oint_{|z-x|=c} \frac{|f(z)|dz}{|z-x|^k} \leq \frac{M}{kc^{k-1}}.$$

Take supremum over $x \in \mathbb{R}$. Then we get $\tilde{d}_k c^k \leq M/c < \infty$ for all $k \geq 1$.

From the definitions in (5.3), it is clear that $A_k(u) \leq \tilde{d}_k$ and $r(u) \geq \tilde{g}_0$ for $u \in D$. Given any $\tilde{g}_1 \in (0, \tilde{g}_0)$, let $b(u) \equiv \tilde{g}_1$. By Proposition 5.4 (1), there is a $b/2$-covering grid $G$ for $C(D)$ with $|G| \leq p^h(2\delta(D)/\tilde{g}_0 + 1)^h$. Therefore, $c_1$ can be set as in (5.9).

(2) For each $x \in I$, $g(f, x) > 0$. Since $I$ is compact, it is covered by a finite number of intervals $(x_i - g(f, x_i)/2, x_i + g(f, x_i)/2)$. Let $c = \min_i g(f, x_i)/2$. Then $c > 0$. For any $x \in I$, there is $x_i$ such that $|z - x| < c$. Let $c_i = \{z \in \mathbb{C} : |z - x| < g(f, x_i)\}$. Hence $f$ is analytic at $z$. As a result, $f$ is analytic in the disc centered at $x$ with radius $c$, and so $g(f, x) \geq c$. This leads to $g_0 = \inf_{x \in I} g(f, x) \geq c$. For $c \in (0, g_0)$, since $I_c = \{z \in \mathbb{C} : |z - x| \leq c\}$ for some $x \in I$ is compact, $M := \sup_{z \in I_c} |f'(z)| < \infty$. Using Cauchy’s contour integral as in (1), it can be shown that $g_0$ is the radius of convergence of $\sum_{k \geq 1} \tilde{d}_k z^k$. The rest of (2) can be proved following the argument for (1). \hfill \Box

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