Fractional D$p$-Branes with Transverse and Tangential Dynamics in the Presence of Background Fluxes

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Abstract

We shall construct two boundary states which are corresponding to a dynamical fractional D$p$-brane in the presence of the fluxes of the Kalb-Ramond field and a $U(1)$ gauge potential, in the partially orbifold spacetime $\mathbb{R}^{1,5} \times \mathbb{C}^2/\mathbb{Z}_2$. These states accurately describe the D$p$-brane in the twisted and untwisted sectors under the orbifold projection. We use them to compute the interaction of two parallel fractional D$p$-branes with the transverse velocities, tangential rotations and tangential linear motions. Various properties of the interaction, such as its long-range force, will be discussed.

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1 Introduction

D-branes represent important key roles in understanding superstring theory and M-theory [1]-[4]. They can be studied through a powerful and adequate formalism, known as the “boundary state” formalism. A boundary state is a physical state of closed string that prominently encodes all properties of a D-brane such as tension, internal fields and dynamical variables. This state elaborates that a D-brane is a source for emitting all closed string states. Therefore, overlap of two boundary states via the closed string propagator specifies the interaction amplitude of the corresponding D-branes. Hence, this strong procedure for various configurations of the D-branes has been vastly used [5]-[21].

Among the various branes configurations the fractional branes reveal profitable behaviors [22]-[32]. For example, by a special system of the fractional branes one can demonstrate the gauge/gravity correspondence with help of open/closed string duality [27, 32]. In addition, the fractional branes drastically appear in the various subjects of the string and M-theories. For instance, the fractional branes give some clues for defining the Matrix theory [33]-[35].

On one hand, we have the dynamical branes which have a widespread application in string theory. On the other hand, the dressed branes, with background and internal fields, exhibit various interesting properties. For example, interactions of the branes are accurately controlled by these fields. Simultaneous consideration of the dynamics, background fields, internal fields and fractionality of the branes, in the framework of superstring theory, motivated and stimulated us to calculate the boundary states and interaction of two parallel fractional Dp-branes with the foregoing properties. Thus, we shall consider the background field $B_{\mu\nu}$ and two $U(1)$ internal potentials $A_{\alpha}^{(1,2)}$ on the worldvolumes of the branes. In this setup each brane has tangential rotation, tangential and transverse linear motions. The transverse dynamics of the branes are along a non-orbifold perpendicular direction. In the twisted superstring theory, via our orbifold, the background spacetime partially is non-compact orbifold with the topological structure $\mathbb{R}^{1,5} \times \mathbb{C}^2/\mathbb{Z}_2$. Finally, we shall separate the contribution of the massless states of closed superstring from the total interaction. Our procedure will be the boundary state formalism.

This paper is organized as in the following. In Sec. 2, we shall compute the boundary states corresponding to a dressed-fractional Dp-brane with tangential and transverse dynamics, both for the untwisted and twisted sectors. In Sec. 3, the interaction amplitude
of two parallel fractional Dp-branes with the above properties will be calculated. In Sec. 4, we shall investigate the behavior of the interaction for large distances of the branes to obtain the long-range force of our setup. Section 5 will be devoted to the conclusions.

2 The boundary states associated with our Dp-brane

We consider a dressed-dynamical fractional Dp-brane which lives in the d-dimensional spacetime and completely is transverse to the non-compact orbifold $\mathbb{C}^2/\mathbb{Z}_2$. The complex coordinates of $\mathbb{C}^2$ are constructed from $\{x^a|a = d-4, d-3, d-2, d-1\}$, so that the $\mathbb{Z}_2$-group acts on them. This group has the structure $\{e, h| h^2 = e\}$, in which under the action of the element $h$ we have $x^a \rightarrow -x^a$. The Dp-brane has stuck at the orbifold fixed-points, which define a $(d-4)$-dimensional hyperplane at the location $x^a = 0$. In the d-dimensional orbifolded spacetime the brane can possess the dimension $p \leq d - 5$.

In order to construct the boundary state of a dynamical fractional Dp-brane with background fields, we begin with the following sigma-model action for closed string

$$S = - \frac{1}{4\pi\alpha'} \int d^2\sigma \left( \sqrt{-g} g^{rs} G_{\mu\nu} \partial_r X^\mu \partial_s X^\nu + \epsilon^{rs} B_{\mu\nu} \partial_r X^\mu \partial_s X^\nu \right)$$

$$+ \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} d\sigma \left( A_\alpha \partial_\tau X^\alpha + \omega_{\alpha\beta} J^{\alpha\beta}_\tau \right),$$

(2.1)

where $B_{\mu\nu}$ is the Kalb-Ramond field, and $A_\alpha(X)$ is a $U(1)$ gauge potential. $g_{rs}$ and $G_{\mu\nu}$ are the metrics of the string worldsheet and spacetime, respectively. The area $\Sigma$ indicates the worldsheet of the emitted closed string, and $\partial\Sigma$ is its boundary. The set $\{x^\alpha|\alpha = 0, 1, \ldots, p\}$ specifies the directions along the Dp-brane worldvolume.

Here we assume the background fields $G_{\mu\nu}$ and $B_{\mu\nu}$ to be constant, with $G_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. We use the conventional gauge $A_\alpha = -\frac{1}{2} F_{\alpha\beta} X^\beta$ with a constant field strength $F_{\alpha\beta}$. The brane’s rotation-motion term comprises a constant antisymmetric angular velocity $\omega_{\alpha\beta}$. The matrix elements $\omega_{0\alpha}$ and $\omega_{\bar{\alpha}\bar{\beta}}$, with $\bar{\alpha}, \bar{\beta} \in \{1, 2, \ldots, p\}$, represent the linear and angular velocities of the brane, respectively. The variable $J^{\alpha\beta}_\tau = X^\alpha \partial_\tau X^\beta - X^\beta \partial_\tau X^\alpha$ shows the angular momentum density. Therefore, at present the dynamics of the brane is inside its volume. We shall afterward add a transverse motion too. We should note that presence of the background field $B_{\mu\nu}$ and the internal field $A_\alpha$ gives rise to some preferred alignments inside the brane worldvolume. Thus, the Lorentz invariance in the worldvolume has been explicitly broken. This clarifies that the dynamics
of the brane inside its volume is meaningful.

2.1 The bosonic branch of the boundary state

Vanishing the variation of the action yields the following boundary state equations for the twisted (T) and the untwisted (U) sectors, via the orbifold projection,

\[
\left[(\eta_{\alpha\beta} + 4\omega_{\alpha\beta})\partial_{\tau}X^\beta + \mathcal{F}_{\alpha\beta}\partial_{\sigma}X^\beta\right]_{\tau=0}|B_x\rangle^{U\cap T} = 0, \\
(X^I - y^I)_{\tau=0}|B_x\rangle^{U\cap T} = 0, \\
(X^a - y^a)_{\tau=0}|B_x\rangle^{U\cap T} = 0,
\]

where \( \mathcal{F}_{\alpha\beta} = B_{\alpha\beta} - F_{\alpha\beta} \) is the total field strength. The directions \( \{x^I| I = p+1, \ldots, d-5\} \) will be used for both sectors. In the twisted sector they refer to the non-orbifold directions which are perpendicular to the brane worldvolume. For both sectors the parameters \( \{y^I| I = p+1, \ldots, d-5\} \) specify the position of the brane, and the other position parameters, due to the presence of the orbifold, for both sectors are zero, i.e. \( \{y^a = 0| a = d-4, \ldots, d-1\} \). However, as we see there are mixed boundary conditions along the brane worldvolume.

Now we introduce a transverse velocity to the brane. Since the brane has stuck at the orbifold fixed-points it cannot move along the orbifold directions. Let the boost direction be a member of the set \( \{x^{p+1}, \ldots, x^{d-5}\} \), which we call it \( x^{i_0} \). Hence, Eqs. (2.2) under the boost find the features

\[
\left[\partial_{\tau}(X^0 - v^{i_0}X^{i_0}) + 4\omega^0_\beta\partial_{\tau}X^{\beta} + \mathcal{F}^0_\beta\partial_{\sigma}X^{\beta}\right]_{\tau=0}|B_x\rangle^{U\cap T} = 0, \\
\left[\partial_{\tau}X^{\alpha} + 4\gamma^2\omega^{\alpha}_0\partial_{\tau}(X^0 - v^{i_0}X^{i_0}) + 4\omega^{\alpha}_\beta\partial_{\tau}X^{\beta}
+\gamma^2\mathcal{F}^\alpha_0\partial_{\sigma}(X^0 - v^{i_0}X^{i_0}) + \mathcal{F}^\alpha_\beta\partial_{\sigma}X^{\beta}\right]_{\tau=0}|B_x\rangle^{U\cap T} = 0, \\
(X^{i_0} - v^{i_0}X^0 - y^{i_0})_{\tau=0}|B_x\rangle^{U\cap T} = 0, \\
(X^{i_0} - y^{i_0})_{\tau=0}|B_x\rangle^{U\cap T} = 0, \\
(X^a)_{\tau=0}|B_x\rangle^{U\cap T} = 0,
\]

where \( \gamma = 1/\sqrt{1-(v^{i_0})^2} \), and \( i \in \{p+1, \ldots, i_0, \ldots, d-5\} \), i.e. \( i \neq i_0 \). In order to have a transverse motion the dimension of the brane is restricted by \( p \leq d-6 \).

For each sector, by applying the mode expansion of the closed string coordinates into Eqs. (2.3), the boundary state equations will be written in terms of the string oscillators.
For the twisted sector the mode expansions of the closed string coordinates $X^\alpha$, $X^{i_0}$ and $X^i$ possess the common form

$$X^\rho(\sigma, \tau) = x^\rho + 2\alpha' p^\rho \tau + \frac{i}{2} \sqrt{2\alpha'} \sum_{m \neq 0} \frac{1}{m} \left( \alpha_m e^{-2im(\tau - \sigma)} + \tilde{\alpha}_m e^{-2im(\tau + \sigma)} \right), \quad \rho \in \{\alpha, i_0, i\},$$

(2.4)

and the string coordinates along the orbifold have the feature

$$X^\alpha(\sigma, \tau) = \frac{i}{2} \sqrt{2\alpha'} \sum_{r \in \mathbb{Z} + 1/2} \frac{1}{r} \left( \alpha_r e^{-2ir(\tau - \sigma)} + \tilde{\alpha}_r e^{-2ir(\tau + \sigma)} \right).$$

(2.5)

For the untwisted sector the mode expansion of the string coordinates are as the conventional cases.

On the basis of the mode expansions we obtain

$$\left[ \left( \gamma(\delta^0 \lambda - v^{i_0} \delta_0 \lambda) + \gamma(4\omega^0 \partial - \mathcal{F}^0 \partial) \delta^0 \lambda \right) \alpha^\lambda_m + \right. \left. \right] |B_x^{(osc)}\rangle^\text{osc} \langle \text{ osc}^{(osc)}| = 0, \nonumber$$

$$\left[ \left( \delta^0 \lambda + \gamma^2(4\omega^0 \partial - \mathcal{F}^0 \partial)(\delta^0 \lambda - v^{i_0} \delta_0 \lambda) + (4\omega^0 \partial + \mathcal{F}^0 \partial) \delta^0 \lambda \right) \alpha^\lambda_m + \right. \left. \right] |B_x^{(osc)}\rangle^\text{osc} \langle \text{ osc}^{(osc)}| = 0, \nonumber$$

$$\left[ \left( \delta^0 \lambda - v^{i_0} \delta_0 \lambda \right) \alpha^\lambda_m - \left( \delta_0 \lambda - v^{i_0} \delta^0 \lambda \right) \tilde{\alpha}^\lambda_m \right] |B_x^{(osc)}\rangle^\text{osc} \langle \text{ osc}^{(osc)}| = 0, \nonumber$$

(2.6)

where $\lambda \in \{\alpha, i_0\}$. For the zero-mode parts of both sectors Eqs. (2.3) yield

$$\left[ p^0 - v^{i_0} p^{i_0} + 4\omega^0 \partial \mathcal{P} \partial \right] |B_x^{(0)}\rangle^\text{osc} \langle \text{ osc}^{(0)}| = 0, \nonumber$$

$$\left[ p^\partial + 4\gamma^2 \omega^0 \partial (p^0 - v^{i_0} p^{i_0}) + 4\omega^0 \partial \mathcal{P} \partial \right] |B_x^{(0)}\rangle^\text{osc} \langle \text{ osc}^{(0)}| = 0, \nonumber$$

$$\left( x^{i_0} - v^{i_0} x^{i_0} - y^{i_0} \right) |B_x^{(0)}\rangle^\text{osc} \langle \text{ osc}^{(0)}| = 0, \nonumber$$

$$\left( x^{i_0} - y^{i_0} \right) |B_x^{(0)}\rangle^\text{osc} \langle \text{ osc}^{(0)}| = 0, \nonumber$$

$$\left( x^a \right) |B_x^{(0)}\rangle^\text{osc} \langle \text{ osc}^{(0)}| = 0, \nonumber$$

(2.7)

where we exerted the decomposition $|B_x\rangle^\text{osc} \langle \text{ osc}| = |B_x^{(0)}\rangle^\text{osc} \langle \text{ osc}^{(0)}| \otimes |B_x^{(osc)}\rangle^\text{osc} \langle \text{ osc}^{(osc)}|$. 

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According to the first two equations of Eqs. (2.7) we receive the following relations among the eigenvalues of the momentum components

\[
p^0 - v^{i_0} p^{i_0} = -4 \omega^0_{\beta} p^{\beta},
\]

\[
\Omega^{\alpha}_{\beta} p^{\beta} = 0,
\]

\[
\Omega^{\alpha}_{\beta} \equiv \delta^{\alpha}_{\beta} - 16 \gamma^2 \omega^0_{\alpha} \omega^0_{\beta} + 4 \omega^{\alpha}_{\beta}.
\]  

(2.8)

We observe that the tangential dynamics of the brane connects the momentum components of the emitted closed string. For the third relation we have two choices: if the \( p \times p \) matrix \( \Omega^{\alpha}_{\beta} \) is invertible, i.e. \( \det \Omega^{\alpha}_{\beta} \neq 0 \), then all \( p^{\alpha} \)'s must vanish, and hence \( p^0 \) and \( p^{i_0} \) identically become zero. The second choice is \( \det \Omega^{\alpha}_{\beta} = 0 \), which is a constraint between the \( 1 + p(p + 1)/2 \) parameters \( \{ \omega_{\alpha\beta}, v^{i_0} \} \). In this case, \( p^{\alpha} \)'s can be nonzero, so \( p^0 \) and \( p^{i_0} \) can also be nonzero. This non-vanishing momentum extremely is different from the usual case in which the closed strings are emitted perpendicular to the brane. The nonzero momentum implies that the brane dynamics effectively induces a peculiar potential on the emitted closed strings. For example, the second choice for the D2-brane eventuates to the following equation between the tangential, normal and angular velocities of the brane

\[
(1 + 16 \omega_1^2)(1 - v_\perp^2) - 16 v_\parallel^2 = 0,
\]  

(2.9)

where \( v_\perp = v^{i_0} \) and \( v_\parallel^2 = \omega_{01}^2 + \omega_{02}^2 \). For simplification of the calculations we shall apply the first choice.

The coherent state method elaborates the following solutions for the oscillating parts of the boundary state

\[
|B^{(osc)}_x\rangle^T = \sqrt{\det Q} \exp \left[ -\sum_{m=1}^{\infty} \left( \frac{1}{m} \alpha_{-m}^\rho S_{\rho\rho'} \tilde{\alpha}_{m}^{\rho'} \right) \right] \times \exp \left[ \sum_{r=1/2}^{\infty} \left( \frac{1}{r} \alpha_{-r} \tilde{\alpha}_{r} \right) \right] |0\rangle_{\alpha} \otimes |0\rangle_{\tilde{\alpha}},
\]

\[
|B^{(osc)}_x\rangle^U = \sqrt{\det Q} \exp \left[ -\sum_{m=1}^{\infty} \left( \frac{1}{m} \alpha_{-m}^\mu \tilde{S}_{\mu\nu} \tilde{\alpha}_{m}^{\nu} \right) \right] |0\rangle_{\alpha} \otimes |0\rangle_{\tilde{\alpha}},
\]  

(2.10)
where $\rho, \rho' \in \{\alpha, i, i_0\}$ and $\mu, \nu \in \{\alpha, a, i, i_0\}$. The matrices $S$ and $\tilde{S}$ are defined by

$$
S_{\rho \rho'} = ((Q^{-1}N)_{\lambda \lambda'}, -\delta_{ij}),
$$
$$
\tilde{S}_{\mu \nu} = ((Q^{-1}N)_{\lambda \lambda'}, -\delta_{ij}, -\delta_{ab}),
$$

$$
Q^0_{\lambda} = \gamma (\delta^0_{\lambda} - v^{i_0} \delta^{i_0}_{\lambda}) + \gamma (4 \omega^0_{\bar{\alpha}} - \mathcal{F}^0_{\bar{\alpha}}) \delta_{\bar{\alpha}}_{\lambda},
$$
$$
Q^{\bar{\alpha}}_{\lambda} = \delta^{\bar{\alpha}}_{\lambda} + \gamma^2 (4 \omega^{0}_{\bar{\alpha}} - \mathcal{F}^{0}_{\bar{\alpha}}) (\delta^0_{\lambda} - v^{i_0} \delta^{i_0}_{\lambda}) + (4 \omega^{\bar{\alpha}}_{\bar{\beta}} - \mathcal{F}^{\bar{\alpha}}_{\bar{\beta}}) \delta_{\bar{\beta}}_{\lambda},
$$
$$
Q^{i_0}_{\lambda} = \delta^{i_0}_{\lambda} - v^{i_0} \delta^0_{\lambda},
$$

$$
N^0_{\lambda} = \gamma (\delta^0_{\lambda} - v^{i_0} \delta^{i_0}_{\lambda}) + \gamma (4 \omega^0_{\bar{\alpha}} + \mathcal{F}^0_{\bar{\alpha}}) \delta_{\bar{\alpha}}_{\lambda},
$$
$$
N^{\bar{\alpha}}_{\lambda} = \delta^{\bar{\alpha}}_{\lambda} + \gamma^2 (4 \omega^{0}_{\bar{\alpha}} + \mathcal{F}^{0}_{\bar{\alpha}}) (\delta^0_{\lambda} - v^{i_0} \delta^{i_0}_{\lambda}) + (4 \omega^{\bar{\alpha}}_{\bar{\beta}} + \mathcal{F}^{\bar{\alpha}}_{\bar{\beta}}) \delta_{\bar{\beta}}_{\lambda},
$$
$$
N^{i_0}_{\lambda} = -\delta^{i_0}_{\lambda} + v^{i_0} \delta^0_{\lambda}.
$$

The normalization factors in Eqs. (2.10) can be deduced from the disk partition function [36]-[38]. Precisely, the quadratic form of the tangential dynamics term, accompanied by the gauge $A_{\alpha} = -\frac{1}{2} F_{\alpha \beta} X^\beta$, induces a quadratic form for the boundary portion of the action (2.1). Therefore, path integration on this Gaussian action manifestly introduces the prefactor $\prod_{n=1}^{\infty} (\det Q)^{-1}$ to Eqs. (2.10). Using the regularization $\prod_{n=1}^{\infty} a \to 1/\sqrt{a}$ the prefactors find the above square root feature.

Note that the coherent state method gives the boundary states (2.10) under the conditions $S S^T = 1$ and $\tilde{S} \tilde{S}^T = 1$. These equations eventuate to the following relations among the variables $\{\omega_{\alpha \beta}, F_{\alpha \beta}, B_{\alpha \beta}, v^{i_0}\}$,

$$
\omega^0_{\bar{\alpha}} F^0_{\bar{\alpha}} = 0,
$$
$$
F^{\bar{\alpha}}_{\bar{\beta}} \omega^{0 \beta} + \omega^{\bar{\alpha}}_{\bar{\beta}} F^{0 \beta} = 0,
$$
$$
F^{\bar{\alpha}}_{\bar{\beta}} \omega^{\bar{\beta}}_{\bar{\kappa}} + F^{\bar{\beta}}_{\bar{\kappa}} \omega^{\bar{\alpha}}_{\bar{\kappa}} - \gamma^2 \left( \omega^{\alpha}_0 F^{\bar{\beta}}_0 + \omega^{\bar{\beta}}_0 F^{\bar{\alpha}}_0 \right) = 0.
$$

Thus, from the total $d-p-5+3p(p+1)/2$ parameters of each brane, i.e. $\{\omega_{\alpha \beta}, F_{\alpha \beta}, B_{\alpha \beta}, v^{i_0}, y^i\}$, only $p^2 + d - p - 6$ of them remain independent.

Making use of the commutation relation $[x^\mu, p^\nu] = i \eta^{\mu \nu}$ the zero-mode parts of the
boundary states find the solutions

\[ |B_x^{(0)}T^{T} = \frac{T_p}{2} \delta \left( x^io - v^io x^0 - y^io \right) |p^io = 0 \rangle \prod_i [\delta \left( x^i - y^i \right) |p^i = 0 \rangle] \prod_\alpha |p^\alpha = 0 \rangle, \]

\[ |B_x^{(0)}U^T = \frac{T_p}{2} \delta \left( x^io - v^io x^0 - y^io \right) |p^io = 0 \rangle \prod_i [\delta \left( x^i - y^i \right) |p^i = 0 \rangle] \times \prod_a [\delta \left( x^a \right) |p^a = 0 \rangle] \prod_\alpha |p^\alpha = 0 \rangle. \]

The constant factor \( T_p \) is the tension of the Dp-brane.

In the bosonic string theory the total boundary states, corresponding to the two sectors, are given by

\[ |B_x \rangle^{U\cap T} = |B_x^{(osc)}\rangle^{U\cap T} \otimes |B_x^{(0)}\rangle^{U\cap T} \otimes |B_{gh} \rangle, \]

where \( |B_{gh} \rangle \) is the known boundary state, associated with the conformal ghosts, and for both sectors obviously is the same.

2.2 The fermionic branch of the boundary state

The worldsheet supersymmetry implies that we can exert the following replacements on the bosonic boundary state equations (2.3) to extract their fermionic counterparts

\[ \partial_+ X^\mu(\sigma, \tau) \rightarrow -i\eta \psi_+^\mu(\tau + \sigma), \]

\[ \partial_- X^\mu(\sigma, \tau) \rightarrow -\psi_-^\mu(\tau - \sigma), \]

(2.14)

where \( \partial_\pm = (\partial_\tau \pm \partial_\sigma)/2 \), and \( \eta = \pm 1 \) is saved for the GSO projection on the boundary states.

Similar to the bosonic part of the boundary state the fermionic part also includes the twisted and untwisted sectors. For constructing the boundary state equations in terms of the fermionic oscillators we use the fermionic mode expansions of each sector. For the untwisted sector the worldsheet fields \( \psi_\pm^\mu \) have the well-known mode expansions, and for the twisted sector they have the following expansions

\[ \psi_+^\mu = \sum_t \tilde{\psi}_t^\mu e^{-2it(\tau+\sigma)}, \]

\[ \psi_-^\mu = \sum_t \tilde{\psi}_t^\mu e^{-2it(\tau-\sigma)}, \]

(2.15)
where in the twisted NS-NS sector there are
\[ \psi_t^\rho \text{ and } \bar{\psi}_t^\rho, \quad t \in \mathbb{Z} + 1/2, \]
\[ \psi_r^a \text{ and } \bar{\psi}_r^a, \quad r \in \mathbb{Z}, \]
and in the twisted R-R sector we have
\[ \psi_t^\rho \text{ and } \bar{\psi}_t^\rho, \quad t \in \mathbb{Z}, \]
\[ \psi_r^a \text{ and } \bar{\psi}_r^a, \quad r \in \mathbb{Z} + 1/2. \]

The indices “\( \rho \)” and “\( a \)” indicate the non-orbifold and orbifold directions, respectively. Since for the superstring theory the critical dimension is \( d = 10 \), we select the sets \( \{a \mid a = 6, 7, 8, 9\} \) and \( \{\rho \mid \rho = 0, 1, 2, 3, 4, 5\} \) for the orbifold and non-orbifold directions, respectively.

Introducing the replacements (2.14) into Eqs. (2.3) and using the above mode expansions, we obtain

\[
\left[ (\gamma(\delta^0_\lambda - v^{i0}\delta^0_\lambda) + \gamma(4\omega^0_\alpha - \mathcal{F}^0_\alpha)\delta^\alpha_\lambda) \psi^\lambda_t \right. \\
- i\eta \left( \gamma(\delta^0_\lambda - v^{i0}\delta^0_\lambda) + \gamma(4\omega^0_\alpha + \mathcal{F}^0_\alpha)\delta^\alpha_\lambda \right) \bar{\psi}^\lambda_{-t} \left| B_{\psi}, \eta \right\rangle_{\text{U}T} = 0 ,
\]
\[
\left[ (\delta^\alpha_\lambda + \gamma^2(4\omega^\alpha_0 - \mathcal{F}^\alpha_0)(\delta^0_\lambda - v^{i0}\delta^0_\lambda) + (4\omega^\alpha_\beta - \mathcal{F}^\alpha_\beta)\delta^\beta_\lambda) \psi^\lambda_t \right. \\
- i\eta \left( \delta^\alpha_\lambda + \gamma^2(4\omega^\alpha_0 + \mathcal{F}^\alpha_0)(\delta^0_\lambda - v^{i0}\delta^0_\lambda) + (4\omega^\alpha_\beta + \mathcal{F}^\alpha_\beta)\delta^\beta_\lambda \right) \bar{\psi}^\lambda_{-t} \left| B_{\psi}, \eta \right\rangle_{\text{U}T} = 0 ,
\]
\[
\left[ (\delta^0_\lambda - v^{i0}\delta^0_\lambda)\psi^\lambda_t + i\eta(\delta^0_\lambda - v^{i0}\delta^0_\lambda)\bar{\psi}^\lambda_{-t} \right] \left| B_{\psi}, \eta \right\rangle_{\text{U}T} = 0 ,
\]
\[
(\psi_t^i + i\eta\bar{\psi}_{-t}^i)\left| B_{\psi}, \eta \right\rangle_{\text{U}T} = 0 ,
\]
\[
(\psi_r^a + i\eta\bar{\psi}_{-r}^a)\left| B_{\psi}, \eta \right\rangle_{\text{U}T} = 0 .
\]

Now let decompose \( \left| B_{\psi}, \eta \right\rangle_{\text{U}T} = \left| B_{\psi}^{(\text{osc})}, \eta \right\rangle_{\text{U}T} \otimes \left| B_{\psi}^{(0)}, \eta \right\rangle_{\text{U}T} \). Thus, the oscillating part of these equations can be rewritten in the compact forms

\[
\left( \psi_t^\rho - i\eta S^\rho \bar{\psi}_{-t}^\rho \right) \left| B_{\psi}^{(\text{osc})}, \eta \right\rangle_{\text{T}} = 0 ,
\]
\[
(\psi_r^a + i\eta \bar{\psi}_{-r}^a) \left| B_{\psi}^{(\text{osc})}, \eta \right\rangle_{\text{T}} = 0 ,
\]
\[
(\psi_t^\mu - i\eta \tilde{S}^\mu \bar{\psi}_{-t}^\mu) \left| B_{\psi}^{(\text{osc})}, \eta \right\rangle_{\text{U}} = 0 ,
\]

(2.17)
where for the first and third equations $t \in \mathbb{Z} - \{0\}$ ($t \in \mathbb{Z} + 1/2$) is related to the R-R (NS-NS) sector, and in the second equation there is $r \in \mathbb{Z} + 1/2$ ($r \in \mathbb{Z} - \{0\}$) for the R-R (NS-NS) sector.

The zero-mode parts of Eqs. (2.16), for the NS-NS and R-R sectors, take the features

\[
\begin{align*}
\left( \psi_0^\rho - i\eta S_0^\rho \tilde{\psi}_0^\rho \right) |B_\psi^{(0)}; \eta\rangle_R^T &= 0 , \\
\left( \psi_0^a + i\eta \tilde{\psi}_0^a \right) |B_\psi^{(0)}; \eta\rangle_{NS}^T &= 0 , \\
\left( \psi_0^\mu - i\eta \tilde{S}_\mu^\nu \tilde{\psi}_0^\nu \right) |B_\psi^{(0)}; \eta\rangle_R^U &= 0 .
\end{align*}
\]

(2.18)

Note that, unlike the untwisted sector, the NS-NS portion of the twisted sector also possesses a zero-mode part which originates from the orbifold directions.

2.2.1 The boundary states of the NS-NS sectors

On the basis of the coherent state method the oscillating parts of the NS-NS boundary states for both sectors are given by

\[
|B_\psi; \eta\rangle_{NS}^T = \exp \left[ i\eta \sum_{t=1/2}^{\infty} \psi_0^\rho S^\rho_{\rho'} \tilde{\psi}_0^\rho' \right] \exp \left[ i\eta \sum_{r=1}^{\infty} \psi_0^a \tilde{\psi}_r^a \right] |B_\psi^{(0)}; \eta\rangle_{NS}^T ,
\]

\[
|B_\psi; \eta\rangle_{NS}^U = \exp \left[ i\eta \sum_{t=1/2}^{\infty} \psi_0^\mu \tilde{S}^\mu_{\nu} \tilde{\psi}^\nu_0 \right] |0\rangle_{NS}^U .
\]

(2.19)

The second equation of Eqs. (2.18) elucidates that the state $|B_\psi^{(0)}; \eta\rangle_{NS}^T$ is independent of the background fields and dynamics of the brane. It has the solution \[26], \[29],

\[
|B_\psi^{(0)}; \eta\rangle_{NS}^T = \left( \tilde{C} \frac{1 + i\eta \tilde{\Gamma}}{1 + i\eta} \right)_{LM} |L\rangle \otimes |\tilde{M}\rangle ,
\]

(2.20)

where $\tilde{C}$ is the charge conjugation matrix of the group $SO(4)$, $\tilde{\Gamma} = \hat{\Gamma}^6 \hat{\Gamma}^7 \hat{\Gamma}^8 \hat{\Gamma}^9$, and $|L\rangle$ and $|\tilde{M}\rangle$ are spinor states of $SO(4)$. 

10
2.2.2 The boundary states of the R-R sectors

By solving Eqs. (2.17) we acquire the following solutions for the oscillating parts of the R-R boundary states

\[
|B_{\psi}; \eta\rangle^T_R = \frac{1}{\sqrt{\det Q}} \exp \left[ i\eta \sum_{t=1}^{\infty} \psi^\rho_t S_{\rho\rho'} \bar{\psi}^\rho'_{-t} \right] \times \exp \left[ i\eta \sum_{r=1/2}^{\infty} \psi^a_r \bar{\psi}^a_{-r} \right] |B_{\psi}^{(0)}; \eta\rangle^T_R ,
\]

\[
|B_{\psi}; \eta\rangle^U_R = \frac{1}{\sqrt{\det Q}} \exp \left[ i\eta \sum_{t=1}^{\infty} \psi^\mu_t \bar{S}_{\mu\nu} \bar{\psi}^\nu_{-t} \right] |B_{\psi}^{(0)}; \eta\rangle^U_R . \tag{2.21}
\]

The reversed determinants, in contrast with the bosonic part, i.e. Eqs. (2.10), is due to the Grassmannian nature of the fermionic variables.

The zero-mode boundary states \(|B_{\psi}^{(0)}; \eta\rangle^T_R\) and \(|B_{\psi}^{(0)}; \eta\rangle^U_R\) are the solutions of the first and third equations of (2.18). The explicit forms of them are given by

\[
|B_{\psi}^{(0)}; \eta\rangle^T_R = \gamma \left( \tilde{C}(\Gamma^0 + v^{i_0} \Gamma^{i_0}) \Gamma^1 \ldots \Gamma^p \frac{1 + i\eta \tilde{\Gamma}^G}{1 + i\eta} G' \right)_{A'B'} |A\rangle \otimes |\tilde{B}\rangle ,
\]

\[
|B_{\psi}^{(0)}; \eta\rangle^U_R = \gamma \left( C(\Gamma^0 + v^{i_0} \Gamma^{i_0}) \Gamma^1 \ldots \Gamma^p \frac{1 + i\eta \Gamma^{i_1}}{1 + i\eta} G \right)_{AB} |A\rangle \otimes |\tilde{B}\rangle . \tag{2.22}
\]

In the twisted sector \(\tilde{C}\) is the charge conjugate matrix of the group \(SO(1,5)\), the \(\Gamma'\)-matrices satisfy the Clifford algebra in the six dimensions and \(\tilde{\Gamma} = \Gamma^0 \Gamma^1 \ldots \Gamma^5\), then \(|A\rangle\) and \(|\tilde{B}\rangle\) are spinors of \(SO(1,5)\). In the untwisted sector \(C\) is the charge conjugate matrix of \(SO(1,9)\), and \(|A\rangle\) and \(|\tilde{B}\rangle\) are spinors of this group. The \(32 \times 32\) and \(8 \times 8\) matrices \(G\) and \(G'\) satisfy the following equations

\[
\Gamma^{\lambda\lambda'} G' - M^\lambda_{\chi'} G' \Gamma^{\chi\lambda'} - v^{i_0} \Gamma^{i_0} \Gamma^{\lambda\lambda'} G' - v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} M^\lambda_{\chi'} G' \Gamma^{\chi'0} = 0,
\]

\[
\Gamma^{\lambda\lambda'} G - M^\lambda_{\chi'} G \Gamma^{\chi\lambda'} - v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} G - v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} M^\lambda_{\chi'} G \Gamma^{\chi'0} = 0, \tag{2.23}
\]

where \(M^\lambda_{\chi'} = (Q^{-1} N)^\lambda_{\chi'}\). Using the algebra of the Dirac matrices these equations can be rewritten in the suitable forms

\[
\Gamma^{\lambda}(G' + v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} G') - M^\lambda_{\chi'} (G' + v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} G') \Gamma^{\chi'\lambda'} = 2v^{i_0} \eta^{i_0} \Gamma^{\lambda0} G' ,
\]

\[
\Gamma^{\lambda}(G + v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} G) - M^\lambda_{\chi'} (G + v^{i_0} \Gamma^{i_0} \Gamma^{\lambda0} G) \Gamma^{\chi'\lambda'} = 2v^{i_0} \eta^{i_0} \Gamma^{\lambda0} G . \tag{2.24}
\]
Therefore, $G'$ and $G$ explicitly find the solutions

\[
G' = \left[ (1 + v^i \Gamma^i \Gamma^0) - 2v^i \Gamma^i \Gamma^0 (1 + R^i \Gamma^i \Gamma^0) \right]^{-1} \\
\times \exp \left( \frac{1}{2} \Phi \Gamma \Gamma' \right) : \ ,
\]

\[
G = \left[ (1 + v^i \Gamma^i \Gamma^0) - 2v^i \Gamma^i \Gamma^0 (1 + R^i \Gamma^i \Gamma^0) \right]^{-1} \\
\times \exp \left( \frac{1}{2} \Phi \Gamma \Gamma' \right) : \ ,
\]

\[
\bar{\Phi} = (\Phi - \Phi^T)/2 \ ,
\]

\[
\Phi_{\lambda\lambda'} \equiv ((PM + 1)^{-1}(PM - 1))_{\lambda\lambda'} \ ,
\]

where $R = PM$, and the matrix $P$ is defined by $P^\lambda_{\lambda'} = (\delta^\alpha_{\beta}, -\delta^i_{i_0})$ with $P^\alpha_{i_0} = 0$. The conventional notation $:\,$ implies that we must expand the exponentials with the convention that all Dirac matrices anticommute, hence only a finite number of terms remain. In the absence of the dynamical variables we have $\bar{\Phi}_{\alpha\beta} = \mathcal{F}_{\alpha\beta}$ which is in accordance with the conventional results of the literature.

For instance, the antisymmetric matrix $\bar{\Phi}_{\lambda\lambda'}$ corresponding to a dressed fractional D2-brane, which is parallel to the $x^1 x^2$-plane and its tangential velocity is along the $x^3$-direction, has the following structure

\[
\bar{\Phi}_{\lambda\lambda} = 0, \\
\bar{\Phi}_{01} = \left[ 2(1-v^2) (\mathcal{F}_{02}\omega_{12} + \mathcal{F}_{12}(4\omega_{01}\omega_{12} - \omega_{02})) - 8\mathcal{F}_{02}\omega_{01}\omega_{02} \\
-\mathcal{F}_{01} ((1-v^2)(1+8\omega_{12}^2) - 8\omega_{02}^2) \right]/W, \\
\bar{\Phi}_{02} = \left[ -2(1-v^2)(\mathcal{F}_{01}\omega_{12} - \mathcal{F}_{12}(4\omega_{02}\omega_{12} + \omega_{01})) - 8\mathcal{F}_{01}\omega_{01}\omega_{02} \\
-\mathcal{F}_{02} ((1-v^2) (1+8\omega_{12}^2) - 8\omega_{01}^2) \right]/W, \\
\bar{\Phi}_{03} = \nu, \\
\bar{\Phi}_{12} = \left[ -2 (\mathcal{F}_{01}\omega_{02} - \mathcal{F}_{02}\omega_{01}) - 8 (\mathcal{F}_{01}\omega_{01}\omega_{12} + \mathcal{F}_{02}\omega_{02}\omega_{12}) \\
-\mathcal{F}_{12} (1-v^2 - 8(\omega_{01}^2 + \omega_{02}^2)) \right]/W, \\
\bar{\Phi}_{13} = 0, \\
\bar{\Phi}_{23} = 0, \\
W \equiv (1-v^2) (16\omega_{12}^2 - 1) + 16 (\omega_{01}^2 + \omega_{02}^2) .
\]
2.3 The total boundary states

For eliminating the closed string tachyon and preserving the supersymmetry we should apply the GSO projection. The total GSO-projected boundary state is a linear combination of two states with $\eta = \pm 1$. Thus, the total physical boundary states in the twisted and untwisted sectors are given by

$$|B\rangle^{T}_{\text{NS,R}} = \frac{1}{2} (|B, +\rangle^{T}_{\text{NS,R}} + |B, -\rangle^{T}_{\text{NS,R}}),$$
$$|B\rangle^{U}_{\text{NS}} = \frac{1}{2} (|B, +\rangle^{U}_{\text{NS}} - |B, -\rangle^{U}_{\text{NS}}),$$
$$|B\rangle^{U}_{\text{R}} = \frac{1}{2} (|B, +\rangle^{U}_{\text{R}} + |B, -\rangle^{U}_{\text{R}}),$$

(2.27)

where the states $|B; \eta\rangle^{\text{U,T}}_{\text{NS,R}}$ are defined by the partial states

$$|B; \eta\rangle^{\text{U,T}}_{\text{NS,R}} = |B^{x}\rangle^{\text{U,T}} \otimes |B^{\psi}; \eta\rangle^{\text{U,T}}_{\text{NS,R}} \otimes |B^{gh}\rangle \otimes |B^{sgh}; \eta\rangle_{\text{NS,R}}.$$

As we see the GSO-projection selects different combinations in the NS-NS and R-R portions of the untwisted sector while for the twisted sector, because of the orbifold projection, it chooses similar structure for both NS-NS and R-R parts.

Note that the ghost and superghost boundary states $|B^{gh}\rangle$ and $|B^{sgh}; \eta\rangle_{\text{NS,R}}$ are not influenced by the dynamics of the brane, the orbifold projection and the background fields. For the next purposes we bring in the boundary state of the R-R super-conformal ghosts

$$|B^{sgh}; \eta\rangle_{\text{R}} = \exp \left[ i\eta \sum_{n=1}^{\infty} (\gamma_{-n}\bar{\beta}_{-n} - \beta_{-n}\bar{\gamma}_{-n}) \right]$$

$$+ i\eta\gamma_{0}\tilde{\beta}_{0}] |P = -1/2, \tilde{P} = -3/2\rangle,$$

(2.28)

where the vacuum of the superghosts is in the picture $(-1/2, -3/2)$, and it is annihilated by $\tilde{\gamma}_{0}$ and $\beta_{0}$.

3 The D-branes interaction

The D-branes interactions have extensively appeared in the main subjects of physics. For example, the following phenomena have satisfactory descriptions via the D-branes interactions: the gauge/gravity correspondence [27, 32], presence of the dark matter [39], extra gravity inside our universe [40, 41], origin of the inflation [42, 43], creation of the
Big-Bang by the D-branes collision \[44\], and may other physical phenomena, e.g. see \[45, 46, 47\].

The boundary state formalism allows us to directly compute the cylinder amplitude in the closed string channel. The ends of the cylinder lie on the D-branes and represent the boundaries of the closed superstring worldsheet. This superstring is emitted by one of the branes and then is absorbed by the other one. Since each D-brane couples to the all closed superstring states via its corresponding boundary state, it obviously is a source for procreating any closed superstring states. The interaction amplitude is calculated as the tree-level diagram between the two boundary states, associated with the D-branes. Therefore, two D-branes prominently interact through the exchange of closed superstrings.

The orbifold projection imposes two parts for the interaction: one part due to the untwisted sector and another portion from the twisted sector

\[
A^{\text{Total}} = A^U + A^T,
\]

\[
A^{U\Gamma}_{\text{NS}} = U^\Gamma_{\text{NS}} \langle B_1 | D^{U\Gamma}_{\text{NS}} | B_2 \rangle_{\text{NS}} + U^\Gamma_{\text{R}} \langle B_1 | D^{U\Gamma}_{\text{R}} | B_2 \rangle_{\text{R}},
\]

\[
D^{U\Gamma}_{\text{NS, R}} = 2\alpha' \int_0^\infty dt \ e^{-tH^{U\Gamma}_{\text{NS, R}}},
\]

where \(D^{U\Gamma}\) and \(H^{U\Gamma}\) are the closed superstring propagators and Hamiltonians, respectively. One should apply the GSO-projected boundary states from Eqs. (2.27). The Eqs. (3.1) clarify that in the interaction all possible forces between two D-branes, which are exchanged by the R-R and NS-NS states of closed superstring, have been taken into account. However, the total Hamiltonians are given by

\[
H^{T\Gamma}_{\text{NS}} = H^T_{x} + H^T_{\psi, \text{NS}} + H_{\text{gh}} + H_{\text{sgh, NS}},
\]

\[
H^{T\Gamma}_{\text{R}} = H^T_{x} + H^T_{\psi, \text{R}} + H_{\text{gh}} + H_{\text{sgh, R}}.
\]

For the twisted sector we have

\[
H^T_x = \alpha' \rho \rho_p + 2 \sum_{n=1}^{\infty} (\alpha^\rho_n \alpha_{n\rho} + \bar{\alpha}^\rho_n \bar{\alpha}_{n\rho}) + 2 \sum_{r=1/2}^{\infty} (\alpha^a_r \alpha_{ra} + \bar{\alpha}^a_r \bar{\alpha}_{ra}) - \frac{2}{3},
\]

\[
H^T_{\psi, \text{NS}} = 2 \sum_{t=1/2}^{\infty} (t \psi^\rho_t \psi_{tp} + t \bar{\psi}^\rho_t \bar{\psi}_{tp}) + 2 \sum_{r=1}^{\infty} (r \psi^a_{-r} \psi_{ra} + r \bar{\psi}^a_{-r} \bar{\psi}_{ra}) + \frac{1}{6},
\]

\[
H^T_{\psi, \text{R}} = 2 \sum_{t=1}^{\infty} (t \psi^\rho_t \psi_{tp} + t \bar{\psi}^\rho_t \bar{\psi}_{tp}) + 2 \sum_{r=1/2}^{\infty} (r \psi^a_{-r} \psi_{ra} + r \bar{\psi}^a_{-r} \bar{\psi}_{ra}) + \frac{2}{3}.
\]
Note that, by enumerating the ghosts and superghosts contributions, the zero-point energies of the total Hamiltonians $H^T_{\text{NS}}$ and $H^T_{\text{R}}$ vanish. The untwisted sector comprises the following total Hamiltonians

$$H^U_{\text{NS}} = H_{\text{gh}} + H_{\text{sgNh}} + \alpha' p^\mu p_\mu + 2 \sum_{n=1}^\infty (\alpha^-_n \alpha_{n\mu} + \alpha^-_n \bar{\alpha}_{n\mu})$$

$$H^U_{\text{R}} = H_{\text{gh}} + H_{\text{sgRh}} + \alpha' p^\mu p_\mu + 2 \sum_{n=1}^\infty (\alpha^-_n \alpha_{n\mu} + \alpha^-_n \bar{\alpha}_{n\mu})$$

$$+ 2 \sum_{t=1}^\infty (t \psi^-_t \psi^t_\mu + t \bar{\psi}^-_t \bar{\psi}^t_\mu) - \frac{5}{2}.$$

(3.3)

The total Hamiltonian $H^U_{\text{NS}}$ possesses nonzero vacuum energy, while the zero-point energy of the total Hamiltonian $H^U_{\text{R}}$ vanishes.

### 3.1 Partial amplitudes

Now we study the various parts of the total interaction amplitude for two parallel dynamical fractional D$p$-brane with background fields. The branes completely sit at the orbifold fixed-points with the transverse velocities $v^i_1$ and $v^i_2$ along the non-orbifold perpendicular direction $x^i$, the tangential dynamics $\omega^{(1)}_{\alpha\beta}$ and $\omega^{(2)}_{\alpha\beta}$, the internal field strength $F^{(1)}_{\alpha\beta}$ and $F^{(2)}_{\alpha\beta}$, and the background field $B_{\mu\nu}$.

#### 3.1.1 The amplitude in the untwisted sector

Since the untwisted sector is not affected by the orbifold projection here we merely give the final result for its amplitude

$$A^U = \frac{T_p^2 \alpha' V_p}{8(2\pi)^{8-p}} \frac{1}{V} \int_0^\infty dt \left( \sqrt{\frac{\pi}{\alpha' t}} \right)^{8-p} \exp \left( -\frac{1}{4\alpha' t} \sum_i (y^i_1 - y^i_2)^2 \right)$$

$$\times \left\{ \frac{1}{q} \sqrt{\det(Q^1_1 Q^2_2)} \left[ \prod_{n=1}^\infty \frac{\det (1 + M^T_1 M_2 q^{2n-1})}{\det (1 - M^T_1 M_2 q^{2n})} \frac{1 - q^{2n}}{1 + q^{2n-1}} \right]^{p-6} - \prod_{n=1}^\infty \frac{\det (1 - M^T_1 M_2 q^{2n-1})}{\det (1 - M^T_1 M_2 q^{2n})} \frac{1 - q^{2n}}{1 + q^{2n-1}} \right\} + \xi' \right\},$$

(3.4)
where $q = e^{-2t}$, $V = |v_1^{i_0} - v_2^{i_0}|$ is the relative velocity of the branes, $V_p$ is the volume of each brane, and $\xi$ and $\xi'$ are defined by

$$\xi \equiv \frac{1}{2} \text{Tr} \left( C G_2 C^{-1} G_1^T \left[ -1 + v_1^{i_0} v_2^{i_0} + (v_1^{i_0} - v_2^{i_0})(\Gamma^{0})^T (\Gamma^{i_0})^T \right] \right),$$

$$\xi' \equiv \text{Tr} \left( C G_2 C^{-1} G_1^T \left[ -1 + v_1^{i_0} v_2^{i_0} + (v_1^{i_0} - v_2^{i_0})(\Gamma^{0})^T (\Gamma^{i_0})^T \Gamma^{11} \right] \right).$$

(3.5)

The four terms in Eq. (3.4) originate from the NS-NS, NS-NS$(-1)^F$, R-R and R-R$(-1)^F$. In fact, the factor $\xi'$ for the usual configurations of the branes, e.g. stationary and bare of the internal and background fields, vanishes. In other words, some special setups, such as our setup, receive nonzero values for $\xi'$. In Eq. (3.4) the three possible signs of $(\xi, \xi')$ reveal the interactions of the brane-brane, antibrane-antibrane and brane-antibrane systems. The exponential factor shows the damping nature of the interaction with respect to the square distance of the branes. This damping factor will also appear in the twisted sector.

### 3.1.2 The amplitude in the twisted NS-NS sector

Applying the total GSO-projected boundary state of the twisted NS-NS sector we acquire the following partial amplitude

$$A_{\text{NS-NS}}^T(\eta_1, \eta_2) = \frac{T_p^2 \alpha' V_p}{4(2\pi)^{4-p}} \frac{1}{V} \sqrt{\text{det}(Q_1 Q_2)} \int_0^\infty dt \left\{ \left( \frac{\pi}{\alpha' t} \right)^{4-p} \right. $$

$$\times \delta_{\eta_1 \eta_2, 1} \exp \left( -\frac{1}{4\alpha' t} \sum_i (y_i^1 - y_i^2)^2 \right)$$

$$\times \prod_{n=1}^\infty \left[ \frac{\text{det} \left( 1 + M_1^T M_2 q^{2n-1} \right)}{\text{det} \left( 1 - M_1^T M_2 q^{2n} \right)} \left( \frac{1 - q^{2n}}{1 + q^{2n-1}} \right)^{p-2} \left( \frac{1 + q^{2n}}{1 - q^{2n-1}} \right) \right] \left\} \right. $$

(3.6)

The six factors in the infinite product come from the oscillators, and have the following origins. The determinants of the numerator and denominator respectively are the effects of the fermions and bosons, along the directions of the worldvolumes and transverse velocities. The exponent of the factor $\prod_{n=1}^\infty (1 - q^{2n})^{p-2}$ is $2+(p-4)$, where $+2$ is the ghosts contribution and $p-4$ is for the bosons along the non-orbifold perpendicular directions except the transverse velocity direction. The exponent of the product $\prod_{n=1}^\infty (1 + q^{2n-1})^{-2-p}$ is $-2 + (4 - p)$, where -2 is the superghosts contribution and $4 - p$ turns up from the fermions along the non-orbifold perpendicular directions except the transverse velocity
direction. The factors \(\prod_{n=1}^{\infty} (1 + q^{2n})^4\) and \(\prod_{n=1}^{\infty} (1 - q^{2n-1})^{-4}\) arise from the fermions and bosons in the orbifold directions, respectively.

We observed that the zero-mode part of the boundary state in the twisted NS-NS sector, i.e. Eq. (2.20), has a non-trivial structure. Hence, the spin structure NS-NS\((-1)^F\) does not contribute to the interaction.

### 3.1.3 The amplitude in the twisted R-R sector

In the calculation of the interaction amplitude of the twisted R-R sector we receive a contribution from the zero-modes of the super-conformal ghosts which is divergent, i.e., according to the Eq. (2.28) we obtain

\[
R\langle B^{(0)}_{\text{sgh}}; \eta_1 | B^{(0)}_{\text{sgh}}; \eta_2 \rangle_R = \sum_{n=0}^{\infty} (-\eta_1 \eta_2)^n,
\]

where for \(\eta_1 \eta_2 = +1\) is an alternating series, and for \(\eta_1 \eta_2 = -1\) becomes divergent. Hence, it requires a suitable regularization. Similar to the Refs.\[7, 48\] we insert the regulator \(\mathcal{R}(x) = x^{2G_0}\) as follows

\[
\lim_{x \to 1} T_R \langle B^{(0)}_1; \eta_1 | B^{(0)}_2; \eta_2 \rangle_R^T \equiv \lim_{x \to 1} \frac{T_R \langle B^{(0)}_1; \eta_1 | \mathcal{R}(x) | B^{(0)}_2; \eta_2 \rangle_R^T}{T_R \langle B^{(0)}_1; \eta_1 | B^{(0)}_2; \eta_2 \rangle_R^T}
\]

\[
= \lim_{x \to 1} \left[ R\langle B^{(0)}_{\text{sgh}}; \eta_1 | x^{2G_0} | B^{(0)}_{\text{sgh}}; \eta_2 \rangle_R \times \frac{T_R \langle B^{(0)}_1; \eta_1 | B^{(0)}_2; \eta_2 \rangle_R^T}{R\langle B^{(0)}_1; \eta_1 | B^{(0)}_2; \eta_2 \rangle_R} \right],
\]

where \(G_0 = -\gamma_0 \beta_0\). The superghost factor eventuates to the result \(1/(1 + \eta_1 \eta_2 x^2)\). Thus, by the following insertion of \(\beta_0, \gamma_0, \tilde{\beta}_0\) and \(\tilde{\gamma}_0\) in the superghost part of Eq. (3.7) for \(\eta_1 = -\eta_2 \equiv \eta\) we acquire

\[
\lim_{x \to 1} R\langle B^{(0)}_{\text{sgh}}; \eta | x^{2G_0} \delta \left(\beta_0 - \frac{1}{4\pi} \gamma_0\right) \delta \left(\tilde{\beta}_0 + \frac{1}{4\pi} \tilde{\gamma}_0\right) | B^{(0)}_{\text{sgh}}; -\eta \rangle_R = 1.
\]

For \(\eta_1 = -\eta_2\) this defines a regular amplitude in the twisted R-R sector. Therefore, we receive

\[
\lim_{x \to 1} T_R \langle B^{(0)}_1; \eta_1 | \mathcal{R}(x) | B^{(0)}_2; \eta_2 \rangle_R^T = \text{Tr} \left\{ \tilde{C} \left( \Gamma^{i\alpha} + v_2^{i\alpha} \Gamma^{i\alpha} \right) \Gamma^{i\alpha} \ldots \Gamma^{i\rho} \frac{1 + i\eta_2 \tilde{\Gamma}}{1 + i\eta_2} G_s^\prime \right\} \tilde{C}^{-1}
\]

\[
\times \left[ \tilde{C} \left( \Gamma^{i\alpha} + v_1^{i\alpha} \Gamma^{i\alpha} \right) \Gamma^{i\alpha} \ldots \Gamma^{i\rho} \frac{1 + i\eta_1 \tilde{\Gamma}}{1 - i\eta_1} \right]^\top \tilde{C}^{-1}
\]

\[
\equiv \bar{\xi} \delta_{\eta_1,1} + \bar{\xi}' \delta_{\eta_1,-1}
\]

(3.9)
where $\tilde{\xi}$ and $\tilde{\xi}'$ have the definitions

$$
\tilde{\xi} \equiv \frac{1}{2} \text{Tr} \left( \tilde{C} G_2' \tilde{C}^{-1}(G_1')^T \left[ -1 + v_1^{10} v_2^{10} + (v_1^{io} - v_2^{io})(\Gamma'^0)^T (\Gamma'^0) \right] \right),
$$

$$
\tilde{\xi}' \equiv \text{Tr} \left( \tilde{C} G_2' \tilde{C}^{-1}(G_1')^T \left[ -1 + v_1^{10} v_2^{10} + (v_1^{io} - v_2^{io})(\Gamma'^0)^T (\Gamma'^0) \right] \tilde{\Gamma} \right). \tag{3.10}
$$

In fact, this regularization has been also used in the untwisted R-R sector with the $\Gamma$-matrices of the group $SO(1,9)$. Since the superghost boundary state is independent of the orbifold projection we applied the result (3.8) to the untwisted R-R sector.

Adding all these together the partial amplitude of the twisted R-R sector is accomplished as

$$
\mathcal{A}_{R-R}^T(\eta_1, \eta_2) = \frac{T^2 p'V_p}{16(2\pi)^{4-p}} \frac{1}{V} \int_0^\infty dt \left( \sqrt{\frac{\pi}{\alpha' t}} \right)^{4-p} \exp \left( -\frac{1}{4\alpha' t} \sum_i (y_1^i - y_2^i)^2 \right) \times \left[ \delta_{\eta_{1,2}} \prod_{n=1}^\infty \frac{\det (1 + M_1^T M_2 q^{2n})}{\det (1 - M_1^T M_2 q^{2n})} \left( \frac{1 - q^{2n}}{1 + q^{2n}} \right)^{p-2} \left( \frac{1 + q^{2n-1}}{1 - q^{2n-1}} \right)^4 + \delta_{\eta_{1,2}, -1} \tilde{\xi}' \right]. \tag{3.11}
$$

These two terms are corresponding to the R-R and R-R($-1)^F$ spin structures, respectively. The R-R($-1)^F$ portion merely receives a non-vanishing contribution from the fermionic zero-modes. The origins of the six factors in the infinite product in Eq. (3.11) are analogous to the description after Eq. (3.6), in which here the fermions and superconformal ghosts live in the R-R sector. Similar to the untwisted R-R sector, in Eq. (3.11) the normalizing factors of the fermions and bosons exactly cancel each other.

### 3.2 The total interaction amplitude

As we said the total interaction amplitude possesses two main parts $\mathcal{A}^{\text{Total}} = \mathcal{A}^U + \mathcal{A}^T$. The untwisted part was exhibited by Eq. (3.4). The twisted part is specified by the following summation

$$
\mathcal{A}^T = \mathcal{A}^T(+, +) + \mathcal{A}^T(+, -) + \mathcal{A}^T(-, +) + \mathcal{A}^T(-, -),
$$

$$
\mathcal{A}^T(\eta_1, \eta_2) = \mathcal{A}^T_{\text{NS-NS}}(\eta_1, \eta_2) + \mathcal{A}^T_{R-R}(\eta_1, \eta_2), \tag{3.12}
$$
where $\eta_1, \eta_2 \in \{+1, -1\}$. The signs of the quantities $\tilde{\xi}$ and $\tilde{\xi}'$ in the amplitude (3.12) indicate the interactions of the brane-brane, antibrane-antibrane and brane-antibrane systems.

By comparing the amplitudes of the untwisted sector $A^U$ and twisted sector $A^T$ we note that presence of the orbifold directions drastically induces significant effects on the interaction. However, presence of the various parameters in the setup, i.e., the matrix elements of the $U(1)$ field strengths and the Kalb-Ramond tensor, the tangential and transverse and angular velocities of the branes, the dimension of the branes, the coordinates of the branes locations, and the orbifold effects inspires a general feature to the total interaction amplitude $A^{\text{Total}}$. The strength of the interaction is accurately controlled by these parameters.

In fact, the relative transverse velocity of the branes generally breaks the supersymmetry. Therefore, our setup does not preserve enough value of the supersymmetry, and hence it does not satisfy the BPS no-force condition. This can be manifestly seen by the fact that, for the $Dp$-branes with the same angular velocity $\omega_{(1)\alpha\beta} = \omega_{(2)\alpha\beta}$ and identical internal fields, the attraction force of the NS-NS states is not compensated by the repulsive force of the R-R states.

Note that since the ghost and superghost parts of the boundary states completely are independent of the background fields, the branes dynamics and orbifold projection, their contributions have been introduced by manipulation into the partial amplitudes (3.4), (3.6) and (3.11).

4 Interaction of the branes with large distance

In each interaction theory behavior of the associated amplitude after an enough long time represents a reliable long-range force of the theory. Thus, for the interacting distant branes the massless closed superstring states extremely possess a dominant contribution to the interaction, while the contribution of all massive states, except the tachyon state, vanish. The long-range amplitude $A^{(0)}$ is obtained by taking the limit $t \rightarrow \infty$ of the oscillating portions of the total amplitude $A^{\text{Total}}$. Note that the superstring states are merely defined by the oscillators. Hence, since the other time dependent factors come from the bosonic zero-modes we shall not take the limit of them.
The total interaction amplitude of the distant branes is

\[ \mathcal{A}^{(0)\text{Total}} = \mathcal{A}^{(0)\text{U}} + \mathcal{A}^{(0)\text{T}}_{\text{NS-NS}} + \mathcal{A}^{(0)\text{T}}_{\text{R-R}}, \]

where the untwisted part of this amplitude is given by

\[ \mathcal{A}^{(0)\text{U}} = \frac{T_p^2 \alpha' V_p}{8(2\pi)^{8-p}} \frac{1}{V} \int_0^\infty dt \left\{ \left( \sqrt{\frac{\pi}{\alpha' t}} \right)^{8-p} \exp \left( -\frac{1}{4\alpha' t} \sum_i (y_i^1 - y_i^2)^2 \right) \right\} \times \left( \sqrt{\det(Q_1^\dagger Q_2)} \left[ 12 - 2p + 2\text{Tr}(M_1^T M_2) \right] + \xi + \xi' \right). \]

According to Eq. (3.6), for computing the contribution of the twisted NS-NS sector, we must apply the limit

\[ \lim_{t \to \infty} \prod_{n=1}^{\infty} \left[ \frac{\det \left( 1 + M_1^T M_2 q^{2n-1} \right)}{\det \left( 1 - M_1^T M_2 q^{2n} \right)} \right] \left( \frac{1 - q^{2n}}{1 + q^{2n-1}} \right)^{p-2} \left( \frac{1 + q^{2n}}{1 - q^{2n-1}} \right)^4 \rightarrow (1 + [6 - p + \text{Tr}(M_1^T M_2)] e^{-2t}). \]

Therefore, we acquire the partial amplitude

\[ \mathcal{A}^{(0)\text{T}}_{\text{NS-NS}} = \frac{T_p^2 \alpha' V_p}{2(2\pi)^{4-p}} \frac{1}{V} \sqrt{\det(Q_1^\dagger Q_2)} \int_0^\infty dt \left\{ \left( \sqrt{\frac{\pi}{\alpha' t}} \right)^{4-p} \right\} \times \exp \left( -\frac{1}{4\alpha' t} \sum_i (y_i^1 - y_i^2)^2 \right) \times \lim_{t \to \infty} (1 + [6 - p + \text{Tr}(M_1^T M_2)] e^{-2t}). \]

The brackets in the last lines of Eqs. (4.2) and (4.4) reveal the valuable contribution of the graviton, dilaton and Kalb-Ramond states to the long-range force. As we see the contribution of these states in the twisted sector is exponentially damped. This is because of the fact that under the orbifold projection these states become massive. This effect was done by deforming the zero-point energy of the corresponding Hamiltonian. In fact, due to this modified zero-point energy, the ground state of closed superstring is changed to a massless state. Hence, the long-range force (4.4) completely originates from the exchange of this massless state.

Making use of Eq. (3.11), for the twisted R-R sector we should exert the limit

\[ \lim_{t \to \infty} \prod_{n=1}^{\infty} \left[ \frac{\det \left( 1 + M_1^T M_2 q^{2n} \right)}{\det \left( 1 - M_1^T M_2 q^{2n} \right)} \right] \left( \frac{1 - q^{2n}}{1 + q^{2n}} \right)^{p-2} \left( \frac{1 + q^{2n-1}}{1 - q^{2n-1}} \right)^4 \rightarrow (1 + 8e^{-2t} + 2 [(10 - p) + \text{Tr}(M_1^T M_2)] e^{-4t}). \]
Thus, the contribution of the massless states of this sector to the long-range force is given by

\[ A^{(0)}_{R-R} = \frac{T^2 \alpha' V}{8(2\pi)^{4-p}} \left( \xi + \xi' \right) \int_0^\infty dt \left\{ \left( \frac{\pi}{\alpha' t} \right)^{4-p} \exp \left( -\frac{1}{4\alpha' t} \sum_i (y_i^1 - y_i^2)^2 \right) \right\}. \] (4.6)

As we saw the zero-point energy of the total Hamiltonian of the twisted R-R sector is zero. Thus, the ground state of closed superstring in this sector is massless. This elucidates that the long-range force (4.6) purely comes from the exchange of this ground state.

## 5 Conclusions and summary

We constructed two boundary states in the untwisted and twisted sectors of superstring theory, corresponding to a fractional Dp-brane. The brane lives at the fixed-points of the orbifold \( \mathbb{C}^2/\mathbb{Z}_2 \), and was dressed by the Kalb-Ramond field and a \( U(1) \) internal gauge potential. Besides, transverse and tangential linear motions and rotation were imposed to it. We saw that the orbifold directions, background fields and dynamics of the brane prominently affected the boundary states. For example, the orbifold projection induced a zero-mode part to the twisted NS-NS boundary state.

We observed that the momentum of the emitted closed string possesses components along the brane worldvolume and along the direction of the transverse motion. This noticeable result is unlike the conventional case, and manifestly originates from the brane dynamics. Thus, the emitted closed string receives an effective potential via the brane dynamics. In fact, having this effect strictly puts a restriction on the matrix elements of \( \omega_{\alpha\beta} \) and transverse velocity.

We obtained the total interaction amplitude for two parallel dynamical-fractional Dp-branes (or a brane and an anti-brane), in the foregoing setup. The amplitude of the NS-NS part of the twisted sector does not receive any contribution from the GSO-projected parts of the boundary states with different spin structures. Presence of the various parameters in the setup, accompanied by the orbifold projection, gave a generalized form to the total amplitude. The interaction strength can be accurately adjusted to any desirable value by these parameters. However, because of the effects of the parameters and the orbifold directions the configuration does not satisfy the BPS no-force condition.

We separated a special part of the interaction which merely occurs by the exchange of
the massless states of closed superstring. The untwisted part of this interaction elaborates exchange of the graviton, dilaton, Kalb-Ramond and the usual R-R massless states. The twisted part of the long-range force specifies exchange of the NS-NS and R-R ground states which, in the projected spectrum, are massless. Note that the spectrum of the projected superstring theory, by the orbifold, does not comprise the usual NS-NS and R-R massless states.

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