Braided anti-flexible bialgebras

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Abstract

We introduce the concept of braided anti-flexible bialgebra and construct cocycle bicrossproduct anti-flexible bialgebras. As an application, we solve the extending problem for anti-flexible bialgebras by using some non-abelian cohomology theory.

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1 Introduction

As a special type of Lie-admissible algebras, anti-flexible algebras have been studied by Anderson, Outcalt, Kosier and Rodabaugh in [7, 16, 22]. Very recently, the concept of anti-flexible bialgebras is introduced by Dassoundo, Bai and Hounkonnou in [11]. The theory of Manin triples and anti-flexible Yang-Baxter equation for anti-flexible algebras are developed in the same paper. Pre-anti-flexible algebras and pre-anti-flexible bialgebras are studied by Dassoundo in [12, 13].

On the other hand, the theory of extending structure for many types of algebras were well developed by A. L. Agore and G. Militaru in [1, 2, 3, 4, 5, 6]. Let $A$ be an algebra and $E$ a vector space containing $A$ as a subspace. The extending problem is to describe and classify all algebra structures on $E$ such that $A$ is a subalgebra of $E$. They show that associated to any extending structure of $A$ by a complement space $V$, there is an unified product on the direct sum space $E \cong A \oplus V$. Recently, extending structures for Lie bialgebras, 3-Lie algebras, infinitesimal bialgebras and Lie conformal superalgebras were studied in [15, 27, 28, 29, 30].

Since the extending structure for anti-flexible algebras and the cohomology theory of anti-flexible bialgebras have not been developed in the literature. The aim of this paper is to fill in these gaps. The motivation is from quantum group theory. In [21, 14, 9, 10], the concept of braided Hopf algebras was provided and the construction of cross product bialgebras was studied in detail. See also [17, 18, 20, 23]. In [23, 19, 26], the notion of braided Lie bialgebras was introduced and the construction of cocycle bicrossproducts Lie bialgebras was developed. It is a natural question whether there are similar constructions in theory of anti-flexible bialgebras.

In this paper, we provided the concept of braided anti-flexible bialgebras. It is showed that this new concept will play a key role in considering extending problem for anti-flexible
bialgebras. Secondly, the theory of unified product for anti-flexible bialgebras is also developed
and the construction of cocycle bicrossproduct anti-flexible bialgebras is given. Finally, we
solve the extending problem for anti-flexible bialgebras by using some non-abelian cohomology
theory.

This paper is organized as follows. In Section 2, we recalled some definitions and fixed some
notations about anti-flexible algebras. In Section 3, we introduced the concept of braided anti-
flexible bialgebras and proved the bosonisation theorem associating braided anti-flexible bial-
gebras to ordinary anti-flexible bialgebras. In Section 4, we defined the notion of matched pairs
of braided anti-flexible bialgebras. Besides, we constructed cocycle bicrossproduct anti-flexible
bialgebras through two generalized braided anti-flexible bialgebras. In Section 5, we studied
the extending problems for anti-flexible bialgebras and proved that they can be classified by
some non-abelian cohomology theory.

Throughout the following of this paper, all vector spaces will be over a fixed field of character
zero. An algebra or a coalgebra is denoted by \((A, \cdot)\) or \((A, \Delta)\). The identity map of a vector
space \(V\) is denoted by \(\text{id}_V : V \to V\) or simply \(\text{id} : V \to V\). The flip map \(\tau : V \otimes V \to V \otimes V\)
is defined by \(\tau(u \otimes v) = v \otimes u\) for all \(u, v \in V\).

2 Preliminaries

**Definition 2.1.** Let \(A\) be a vector space equipped with a multiplication \(\cdot : A \otimes A \to A\). Then
\(A\) is called an anti-flexible algebra if the following anti-flexible identity is satisfied:
\[
(a, b, c) = (c, b, a),
\]
or equivalently,
\[
(a \cdot b) \cdot c - a \cdot (b \cdot c) = (c \cdot b) \cdot a - c \cdot (b \cdot a),
\]
where \(a, b, c \in A\) and the associator is denoted by \((a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c)\). In the
following, we always omit “ \(\cdot\) ” and write the multiplication by \(ab\) for simplicity.

**Definition 2.2.** An anti-flexible coalgebra \(A\) is a vector space equipped with a comultiplication \(\Delta : A \to A \otimes A\) such that the following anti-flexible condition is satisfied,
\[
(\Delta \otimes \text{id})\Delta(a) - (\text{id} \otimes \Delta)\Delta(a) = \tau_{13}((\Delta \otimes \text{id})\Delta(a) - (\text{id} \otimes \Delta)\Delta(a)),
\]
where \(\tau_{13}(a \otimes b \otimes c) = c \otimes b \otimes a\). We denote an anti-flexible coalgebra by \((A, \Delta)\).

**Definition 2.3.** An anti-flexible bialgebra \(A\) is a vector space equipped simultaneously
with an anti-flexible algebra structure \((A, \cdot)\) and an anti-flexible coalgebra structure \((A, \Delta)\)
such that the following compatibility conditions are satisfied,
\[
\Delta(ab) + \tau \Delta(ba) = \sum a_1 b \otimes a_2 + ba_2 \otimes a_1 + b_1 \otimes ab_2 + b_2 \otimes b_1 a, \tag{4}
\]
\[
(\text{id} - \tau)(a_1 \otimes a_2 b + ab_1 \otimes b_2 - ba_1 \otimes a_2 - b_1 \otimes b_2 a) = 0, \tag{5}
\]
where we use the sigma notation $\Delta(a) := \sum a_1 \otimes a_2$. We denote an anti-flexible bialgebra by $(A, \cdot, \Delta)$.

**Remark 2.4.** The anti-flexible bialgebras in the above Definition 2.3 is same as in [11, Definition 3.4], but we use different notations as [11], in which (4) and (5) were written as

\[
\Delta(ab) + \tau\Delta(ba) = (\tau(\text{id} \otimes L(b)) + R(b) \otimes \text{id})\Delta(a) + (\tau(R(a) \otimes \text{id}) + \text{id} \otimes L(a))\Delta(b),
\]

\[
(\tau(\text{id} \otimes R(b)) - \text{id} \otimes R(b) \otimes L(b) \otimes \text{id})\Delta(a) = (\tau(\text{id} \otimes R(a)) - \text{id} \otimes R(a) - \tau(L(a) \otimes \text{id}) + L(a) \otimes \text{id})\Delta(b),
\]

where $L(a)$ and $R(a)$ denote the left and right multiplication operators respectively. It is easy to see that (6) and (7) are equivalent to (4) and (5) using the sigma notation. For simplicity, we also would like to denote

\[
\Delta(a) \cdot b := \sum a_1 \otimes a_2 b = (\text{id} \otimes R(b))\Delta(a),
\]

\[
a \cdot \Delta(b) := \sum ab_1 \otimes b_2 = (L(a) \otimes \text{id})\Delta(b),
\]

\[
\Delta(a) \bullet b := \sum a_1 b \otimes a_2 = (R(b) \otimes \text{id})\Delta(a),
\]

\[
a \bullet \Delta(b) := \sum b_1 \otimes ab_2 = (\text{id} \otimes L(a))\Delta(b).
\]

Thus we also write the compatibility conditions (6) and (7) as

\[
\Delta(ab) + \tau\Delta(ba) = \sum \Delta(a) \bullet b + b \cdot \tau\Delta(a) + a \cdot \Delta(b) + \tau\Delta(b) \cdot a,
\]

\[
(\text{id} - \tau)\left(\Delta(a) \cdot b + a \cdot \Delta(b) - b \cdot \Delta(a) - \Delta(b) \cdot a\right) = 0.
\]

**Definition 2.5.** Let $A$ be an anti-flexible algebra and $V$ be a vector space. Then $V$ is called an $A$-bimodule if there is a pair of linear maps $\triangleright : A \otimes V \rightarrow V, (a, v) \mapsto a \triangleright v$ and $\triangleleft : V \otimes A \rightarrow V, (v, a) \mapsto v \triangleleft a$ such that the following conditions hold:

\[
(ab) \triangleright v - a \triangleright (b \triangleright v) = (v \triangleleft b) \triangleleft a - v \triangleleft (ba),
\]

\[
(a \triangleright v) \triangleleft b - a \triangleright (v \triangleleft b) = (b \triangleright v) \triangleleft a - b \triangleright (v \triangleleft a),
\]

for all $a, b \in A$ and $v \in V$.

The category of bimodules over $A$ is denoted by $\mathcal{AM}_A$.

**Definition 2.6.** Let $A$ be an anti-flexible coalgebra, $V$ a vector space. Then $V$ is called an $A$-bicomodule if there is a pair of linear maps $\phi : V \rightarrow A \otimes V$ and $\psi : V \rightarrow V \otimes A$ such that the following conditions hold:

\[
(\Delta_A \otimes \text{id}_V)\phi(v) - (\text{id}_A \otimes \phi)\phi(v) = \tau_{13}\left((\psi \otimes \text{id}_A)\psi(v) - (\text{id}_V \otimes \Delta_A)\psi(v)\right),
\]

\[
(\phi \otimes \text{id}_A)\psi(v) - (\text{id}_A \otimes \psi)\phi(v) = \tau_{13}\left((\phi \otimes \text{id}_A)\psi(v) - (\text{id}_A \otimes \psi)\phi(v)\right).
\]
If we denote by \( \phi(v) = v_{(-1)} \otimes v_{(0)} \) and \( \psi(v) = v_{(0)} \otimes v_{(1)} \), then the above equations can be written as
\[
\Delta_A(v_{(-1)}) \otimes v_{(0)} - v_{(-1)} \otimes \phi(v_{(0)}) = \tau_{13}\left((\Delta_A \otimes \text{id}_H)\psi(a) - (\text{id}_A \otimes \psi)\Delta_A(a)\right),
\]
and
\[
\phi(v_{(0)}) \otimes v_{(1)} - v_{(-1)} \otimes \psi(v_{(0)}) = \tau_{13}\left(\phi(v_{(0)}) \otimes v_{(1)} - v_{(-1)} \otimes \psi(v_{(0)})\right).
\]

The category of bicomodules over \( A \) is denoted by \( AM^A \).

**Definition 2.7.** Let \( H \) and \( A \) be anti-flexible algebras. An action of \( H \) on \( A \) is a pair of linear maps \( \triangleright : H \otimes A \rightarrow A, (x, a) \rightarrow x \triangleright a \) and \( \triangleleft : A \otimes H \rightarrow A, (a, x) \rightarrow a \triangleleft x \) such that \( A \) is an \( H \)-bimodule and the following conditions hold:
\[
(x \triangleright a)b - x \triangleright (ab) = (ba) \triangleleft x - b(a \triangleleft x),
\]
\[
a(x \triangleright b) - (a \triangleleft x)b = b(x \triangleright a) - (b \triangleleft x)a,
\]
for all \( x \in H \) and \( a, b \in A \). In this case, we call \((A, \triangleright, \triangleleft)\) to be an \( H \)-bimodule algebra.

**Definition 2.8.** Let \( H \) and \( A \) be anti-flexible coalgebras. An coaction of \( H \) on \( A \) is a pair of linear maps \( \phi : A \rightarrow H \otimes A \) and \( \psi : A \rightarrow A \otimes H \) such that \( A \) is an \( H \)-bicomodule and the following conditions hold:
\[
(\phi \otimes \text{id}_H)\Delta_A(a) - (\text{id}_H \otimes \Delta_A)\phi(a) = \tau_{13}\left((\Delta_A \otimes \text{id}_H)\psi(a) - (\text{id}_A \otimes \psi)\Delta_A(a)\right),
\]
and
\[
(\psi \otimes \text{id}_A)\Delta_A(a) - (\text{id}_A \otimes \phi)\Delta_A(a) = \tau_{13}\left((\psi \otimes \text{id}_A)\Delta_A(a) - (\text{id}_A \otimes \phi)\Delta_A(a)\right).
\]
If we denote by \( \phi(a) = a_{(-1)} \otimes a_{(0)} \) and \( \psi(a) = a_{(0)} \otimes a_{(1)} \), then the above equations can be written as
\[
\phi(a_1) \otimes a_2 - a_{(-1)} \otimes \Delta_A(a_{(0)}) = \tau_{13}\left(\Delta_A(a_{(0)}) \otimes a_{(1)} - a_1 \otimes \psi(a_2)\right),
\]
\[
\psi(a_1) \otimes a_2 - a_1 \otimes \phi(a_2) = \tau_{13}(\psi(a_1) \otimes a_2 - a_1 \otimes \phi(a_2)).
\]
for all \( a \in A \). In this case, we call \((A, \phi, \psi)\) to be an \( H \)-bicomodule coalgebra.

**Definition 2.9.** Let \((A, \cdot)\) be a given anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra), \( E \) a vector space. An extending system of \( A \) through \( V \) is an anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) on \( E \) such that \( V \) is a complement subspace of \( A \) in \( E \), the canonical injection map \( i : A \rightarrow E, a \mapsto (a, 0) \) or the canonical projection map \( p : E \rightarrow A, (a, x) \mapsto a \) is an anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) homomorphism. The extending problem is to describe and classify up to an isomorphism the set of all anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) structures that can be defined on \( E \).

We remark that our definition of extending system of \( A \) through \( V \) contains not only extending structure in [1, 2, 3] but also the global extension structure in [4]. In fact, the canonical injection map \( i : A \rightarrow E \) is an anti-flexible (co)algebra homomorphism if and only if \( A \) is an anti-flexible sub(co)algebra of \( E \).
Definition 2.10. Let $A$ be an anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra), $E$ be an anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) such that $A$ is a subspace of $E$ and $V$ a complement of $A$ in $E$. For a linear map $\varphi : E \to E$ we consider the diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A & \overset{i}{\rightarrow} & E & \overset{\pi}{\rightarrow} & V & \rightarrow & 0 \\
\downarrow{id_A} & & \downarrow{\varphi} & & \downarrow{id_V} & & \downarrow{\pi'} & & \downarrow{0} \\
0 & \rightarrow & A & \overset{i'}{\rightarrow} & E & \overset{\pi'}{\rightarrow} & V & \rightarrow & 0
\end{array}
\]

where $\pi, \pi' : E \to V$ are the projection maps and $i, i' : A \to E$ are the inclusion maps. We say that $\varphi : E \to E$ stabilizes $A$ if the left square of the diagram (22) is commutative.

Let $(E, \cdot)$ and $(E', \cdot')$ be two anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) structures on $E$. $(E, \cdot)$ and $(E', \cdot')$ are called equivalent, and we denote this by $(E, \cdot) \equiv (E', \cdot')$, if there exists an anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) isomorphism $\varphi : (E, \cdot) \to (E', \cdot')$ which stabilizes $A$. Denote by $\text{Ext}_H(E, A)$ the set of equivalent classes of anti-flexible algebra (anti-flexible coalgebra, anti-flexible bialgebra) structures on $E$.

3 Braided anti-flexible bialgebras

In this section, we introduce the concepts of anti-flexible Hopf bimodule and braided anti-flexible bialgebra which are key concepts in the following sections.

3.1 Anti-flexible Hopf bimodules and braided anti-flexible bialgebras

Definition 3.1. Let $H$ be an anti-flexible bialgebra. An anti-flexible Hopf bimodule over $H$ is a space $V$ endowed with maps

\[\triangleright : H \otimes V \to V, \quad \triangleleft : V \otimes H \to V, \quad \phi : V \to H \otimes V, \quad \psi : V \to V \otimes H,\]

such that $V$ is simultaneously an $H$-bimodule, an $H$-bicomodule and the following compatibility conditions hold:

(HM1) $\phi(x \triangleright v) + \tau \psi(v \triangleleft x) = v(-1) \otimes (x \triangleright v(0)) + v(1) \otimes (v(0) \triangleleft x),$

(HM2) $\psi(x \triangleright v) + \tau \phi(v \triangleleft x) = (x_1 \triangleright v) \otimes x_2 + v(0) \otimes xv(1) + v(0) \otimes v(-1)x + (v \triangleleft x_2) \otimes x_1,$

(HM3) $(x \triangleright v(0)) \otimes v(1) - (v \triangleleft x_1) \otimes x_2 - v(0) \otimes v(1)x$

\[= \tau(xv(-1) \otimes v(0) + x_1 \otimes (x_2 \triangleright v) - v(-1) \otimes (v(0) \triangleleft x)).\]

We denote the category of anti-flexible Hopf bimodules over $H$ by $H_H^H \mathcal{M}^H_H$.

Definition 3.2. Let $H$ be an anti-flexible bialgebra. Let $A$ be an anti-flexible algebra and an anti-flexible coalgebra in $H_H^H \mathcal{M}^H_H$, we call $A$ a braided anti-flexible bialgebra, if the following braided compatibility conditions are satisfied:
(BB1) \( \Delta_A(ab) + \tau \Delta_A(ba) \)
\[
= a_1b \otimes a_2 + ba_2 \otimes a_1 + b_1 \otimes ab_2 + b_2 \otimes b_1a \\
+ (a_{(-1)} \triangleright b) \otimes a_{(0)} + (b \triangleleft a_{(1)}) \otimes a_{(0)} + b_{(0)} \otimes (a \triangleleft b_{(1)}) + b_{(0)} \otimes (b_{(-1)} \triangleright a).
\]

(BB2) \( (id - \tau) \left( a_1 \otimes a_2 b - ba_1 \otimes a_2 - b_1 \otimes b_2 a + ab_1 \otimes b_2 \right) \)
\[
+ (id - \tau) \left( a_{(0)} \otimes (a_{(1)} \triangleright b) - (b \triangleleft a_{(-1)}) \otimes a_{(0)} - b_{(0)} \otimes (b_{(1)} \triangleright a) + (a \triangleleft b_{(-1)}) \otimes b_{(0)} \right) = 0.
\]

Here \( A \) is an anti-flexible algebra and an anti-flexible coalgebra in \( H_H^* \), which means that \( A \) is simultaneously an \( H \)-bimodule anti-flexible algebra (anti-flexible coalgebra) and \( H \)-bicomodule anti-flexible algebra (anti-flexible coalgebra).

Now we construct anti-flexible bialgebras from braided anti-flexible bialgebras. Let \( H \) be an anti-flexible bialgebra, \( A \) be an anti-flexible algebra and an anti-flexible coalgebra in \( H_H^* \).

We define multiplication and comultiplication on the direct sum vector space \( E := \mathbb{A} \oplus H \) by
\[
(a, x)(b, y) := (ab + x \triangleright b + a \triangleleft y, xy), \\
\Delta_E(a, x) := \Delta_A(a) + \phi(a) + \psi(a) + \Delta_H(x).
\]

This is called biproduct of \( A \) and \( H \) which will be denoted by \( A \bowtie H \).

**Theorem 3.3.** Let \( H \) be an anti-flexible bialgebra. Then the biproduct \( A \bowtie H \) forms an anti-flexible bialgebra if and only if \( A \) is a braided anti-flexible bialgebra in \( H_H^* \).

**Proof.** It is easy to prove the multiplication and the comultiplication are anti-flexible. Next, we show the first compatibility condition:
\[
\Delta_E((a, x)(b, y)) + \tau \Delta_E((b, y)(a, x))
\]
\[
= \Delta_E((a, x)) \bullet (b, y) + (b, y) \cdot \tau \Delta_E((a, x)) + (a, x) \bullet \Delta_E((b, y)) + \tau \Delta_E((b, y)) \cdot (a, x).
\]

By direct computations, the left hand side is equal to
\[
\Delta_E((a, x)(b, y)) + \tau \Delta_E((b, y)(a, x))
\]
\[
= \Delta_E(ab + x \triangleright b + a \triangleleft y, xy) + \tau \Delta_E(ba + y \triangleright a + b \triangleleft x, yx)
\]
\[
= \Delta_A(ab) + \phi(ab) + \psi(ab) + \Delta_A(x \triangleright b) + \phi(x \triangleright b) + \psi(x \triangleright b)
\]
\[
+ \Delta_A(a \triangleleft y) + \phi(a \triangleleft y) + \psi(a \triangleleft y) + \Delta_H(xy)
\]
\[
+ \tau \Delta_A(ba) + \tau \phi(ba) + \tau \psi(ba) + \tau \Delta_A(y \triangleright a) + \tau \phi(y \triangleright a) + \tau \psi(y \triangleright a)
\]
\[
+ \tau \Delta_A(b \triangleleft x) + \tau \phi(b \triangleleft x) + \tau \psi(b \triangleleft x) + \tau \Delta_H(yx),
\]

and the right hand side is equal to
\[
\Delta_E((a, x)) \bullet (b, y) + (b, y) \cdot \tau \Delta_E((a, x)) + (a, x) \bullet \Delta_E((b, y)) + \tau \Delta_E((b, y)) \cdot (a, x)
\]
\[
= (a_1 \otimes a_2 + a_{(-1)} \otimes a_{(0)} + a_{(0)} \otimes a_{(1)} + x_1 \otimes x_2) \bullet (b, y)
\]
\[
+ (b, y) \cdot (a_2 \otimes a_1 + a_{(0)} \otimes a_{(-1)} + a_{(1)} \otimes a_{(0)} + x_2 \otimes x_1)
\]
\[
+ (a, x) \bullet (b_1 \otimes b_2 + b_{(-1)} \otimes b_{(0)} + b_{(0)} \otimes b_{(1)} + y_1 \otimes y_2)
\]
Thus the second compatibility condition holds if and only if
\(\phi - \Delta_A(b_1) = 0\). Finally, we show the second compatibility condition:

\(\Delta_A(ab) + \tau\Delta_A(ba) = a_1b \otimes a_2 + ba_2 \otimes a_1 + b_1 \otimes ab_2 + b_2 \otimes b_1a + (a_{(-1)} \triangleright b) \otimes a_1 + (b \triangleleft a_{(1)}) \otimes a_0 + (a \triangleleft b_{1(1)}) + b_{0(1)} \otimes (b_{(-1)} \triangleright a)\).

Then the two sides are equal to each other if and only if

\((1)\Delta_A(ab) + \tau\Delta_A(ba) = a_2 b \otimes a_2 + ba_2 \otimes a_1 + b_1 \otimes ab_2 + b_2 \otimes b_1a + (a_{(-1)} \triangleright b) \otimes a_1 + (b \triangleleft a_{(1)}) \otimes a_0 + (a \triangleleft b_{1(1)}) + b_{0(1)} \otimes (b_{(-1)} \triangleright a)\).

By direct computation, we have

\[
\Delta_E((a, x)) \cdot (b, y) - (b, y) \cdot \Delta_E((a, x)) - \Delta_E((b, y)) \cdot (a, x) + (a, x) \cdot \Delta_E((b, y))
\]

Thus the second compatibility condition holds if and only if

\(6)\ (id - \tau)\left(a_1 \otimes a_2 b - ba_1 \otimes a_2 - b_1 \otimes b_2 a + ab_1 \otimes b_2\right) = 0.\)
\begin{align*}
+ (\text{id} - \tau) \left( a_{(0)} \otimes (a_{(1)} \triangleright b) - (b \triangleleft a_{(-1)}) \otimes a_{(0)} - b_{(0)} \otimes (b_{(1)} \triangleright a) + (a \triangleleft b_{(-1)}) \otimes b_{(0)} \right) = 0,
\end{align*}
(7) \quad a_{(-1)} \otimes a_{(0)} b - b_{(-1)} \otimes b_{(0)} a = \tau(ab_{(0)} \otimes b_{(1)}) - ba_{(0)} \otimes a_{(1)}),
\begin{align*}
(8) (\text{id} - \tau)((x \triangleright b_1) \otimes b_2) - b_1 \otimes (b_2 \triangleleft x) = 0,
\end{align*}
(9) \quad (x \triangleright b_{(0)}) \otimes b_{(1)} - (b \triangleleft x_1) \otimes x_2 - b_{(0)} \otimes b_{(1)} x
\begin{align*}
= \tau(xb_{(-1)} \otimes b_{(0)} + x_1 \otimes (x_2 \triangleright b) - b_{(-1)} \otimes (b_{(0)} \triangleleft x)).
\end{align*}

Combine with all the conditions above, we have found that (2)–(3) and (9) are the conditions for \( A \) to be an anti-flexible Hopf bimodule; and (4)–(5) with (7)–(8) are the conditions for \( A \) to be an anti-flexible algebra and anti-flexible coalgebra in \( \mathcal{H}_H^M \); finally, (1) and (6) are the conditions for \( A \) to be a braided anti-flexible bialgebra.

The proof is completed. \qed

3.2 From quasitriangular anti-flexible bialgebra to braided anti-flexible bialgebra

Let \((A, \cdot)\) be an anti-flexible algebra and \( r = \sum_i u_i \otimes v_i \in A \otimes A \). Set
\begin{align*}
& r_{12} = \sum_i u_i \otimes v_i \otimes 1, \quad r_{13} = \sum_i u_i \otimes 1 \otimes v_i, \quad r_{23} = \sum_i 1 \otimes u_i \otimes v_i, \quad (23)
\end{align*}

In this section, we consider a special class of anti-flexible bialgebras, that is, the anti-flexible bialgebra \((A, \Delta_r)\) on an anti-flexible algebra \((A, \cdot)\), with the map \( \Delta_r \) defined by
\begin{align*}
\Delta_r(a) = \sum_i u_i \otimes av_i + v_i a \otimes u_i. \quad (24)
\end{align*}

Lemma 3.4. ([11]) Let \((A, \cdot)\) be an anti-flexible algebra and \( r \in A \otimes A \). Let \( \Delta_r : A \to A \otimes A \) be a map defined by \( (24) \). If in addition, \( r \) is skew-symmetric and \( r \) satisfies
\begin{align*}
r_{12}r_{13} - r_{23}r_{12} + r_{13}r_{23} = 0, \quad (25)
\end{align*}

which is called the anti-flexible Yang-Baxter equation (AFYB). Then \((A, \Delta_r)\) is an anti-flexible bialgebra.

This kind of anti-flexible bialgebra is called a quasitriangular anti-flexible bialgebra.

Theorem 3.5. Let \((A, \cdot, r)\) be a quasitriangular anti-flexible bialgebra and \( M \) an \( A \)-bimodule. Then \( M \) becomes an anti-flexible Hopf bimodule over \( A \) with maps \( \phi : M \to A \otimes M \) and \( \psi : M \to M \otimes A \) given by
\begin{align*}
\phi(m) := \sum_i u_i \otimes m \triangleleft v_i, \quad \psi(m) := \sum_i v_i \triangleright m \otimes u_i \quad (26)
\end{align*}

Proof. We first prove that \( M \) is an \( A \)-bicomodule:
\begin{align*}
(\Delta_r \otimes \text{id}) \phi(m) - (\text{id} \otimes \phi) \phi(m) = \tau_{13} \left( (\psi \otimes \text{id}) \psi(m) - (\text{id} \otimes \Delta_r) \psi(m) \right).
\end{align*}
In fact, the left hand side is equal to
\[
(\Delta_r \otimes \text{id}) \phi(m) - (\text{id} \otimes \phi) \phi(m) = \Delta_r(u_i) \otimes m \triangleleft v_i - u_i \otimes \phi(m \triangleleft v_i) = u_j \otimes u_i v_j \otimes m \triangleleft v_i + v_j u_i \otimes u_j \otimes m \triangleleft v_i - u_i \otimes u_j \otimes (m \triangleleft v_i) \triangleleft v_j = u_i \otimes u_j \otimes (v_i v_j) - u_i \otimes u_j \otimes (m \triangleleft v_i) \triangleleft v_j,
\]
where the last equal holds by the (AFYB) and the antisymmetry of \( r \), then right hand side is equal to
\[
\tau_{13} \left( (\psi \otimes \text{id}) \psi(m) - (\text{id} \otimes \Delta_r) \psi(m) \right) = \tau_{13} \left( \psi(v_i \triangleright m) \otimes u_i - (v_i \triangleright m) \otimes \Delta_r(u_i) \right) = \tau_{13} \left( (v_j \triangleright (v_i \triangleright m)) \otimes u_j \otimes u_i - (v_i \triangleright m) \otimes u_j \otimes u_i v_j - (v_i \triangleright m) \otimes v_j u_i \otimes u_j \right) = \tau_{13} \left( (v_j \triangleright (v_i \triangleright m)) \otimes u_j \otimes u_i + (u_i \otimes v_j \otimes v_i v_j - (u_i \triangleright m) \otimes u_j v_i \otimes v_j \right) = \tau_{13} \left( (v_j \triangleright (v_i \triangleright m)) \otimes u_j \otimes u_i - ((u_i u_j) \triangleright m) \otimes v_i \otimes v_j \right) = u_i \otimes u_j \otimes (v_j \triangleright (v_i \triangleright m)) - v_j \otimes v_i \otimes ((u_i u_j) \triangleright m) = u_i \otimes u_j \otimes (v_j \triangleright (v_i \triangleright m)) - u_j \otimes u_i \otimes ((v_i v_j) \triangleright m) = u_i \otimes u_j \otimes ((v_i v_j) \triangleright m).
\]
Thus the two sides are equal to each other if and only if
\[
m \triangleleft (v_i v_j) - (m \triangleleft v_i) \triangleleft v_j = v_j \triangleright (v_i \triangleright m) - (v_j v_i) \triangleright m,
\]
which holds obviously by \((ab) \triangleright v - a \triangleright (b \triangleright v) = (v \triangleleft b) \triangleleft a - v \triangleleft (ba)\).

Then we need to prove
\[
(id - \tau_{13}) \left( (\phi \otimes \text{id}) \psi(m) - (\text{id} \otimes \psi) \phi(m) \right) = 0.
\]
We have
\[
(\phi \otimes \text{id}) \psi(m) - (\text{id} \otimes \psi) \phi(m) = (\phi \otimes \text{id})((v_i \triangleright m) \otimes u_i) + (\text{id} \otimes \psi)(u_i \otimes (m \triangleleft v_i)) = u_j \otimes ((v_j \triangleright m) \triangleleft v_j) \otimes u_i - u_i \otimes (v_j \triangleright (m \triangleleft v_i)) \otimes u_j = u_i \otimes ((v_j \triangleright m) \triangleleft v_i) \otimes u_j - u_i \otimes (v_j \triangleright (m \triangleleft v_i)) \otimes u_j,
\]
then
\[
\tau_{13} \left( (\phi \otimes \text{id}) \psi(m) - (\text{id} \otimes \psi) \phi(m) \right) = u_i \otimes ((v_i \triangleright m) \triangleleft v_j) \otimes u_j - u_j \otimes (v_j \triangleright (m \triangleleft v_i)) \otimes u_i = u_i \otimes ((v_i \triangleright m) \triangleleft v_j) \otimes u_j - u_i \otimes (v_i \triangleright (m \triangleleft v_j)) \otimes u_j.
\]
Since \( M \) is an \( A \)-bimodule, by
\[
(a \triangleright m) \triangleleft b - a \triangleright (m \triangleleft b) = (b \triangleright m) \triangleleft a - b \triangleright (m \triangleleft a),
\]
we have \((\text{id} - \tau_{13})((\phi \otimes \text{id})\psi(m) - (\text{id} \otimes \psi)\phi(m)) = 0\) holds. Thus \(M\) is an \(A\)-bicomodule.

Secondly, we check that \(M\) is an anti-flexible Hopf bimodule. For (HM1), we have

\[
\phi(x \triangleright m) + \tau \psi(m \triangleleft x) = u_i \otimes ((x \triangleright m) \triangleleft v_i) + u_i \otimes (v_i \triangleright (m \triangleleft x)) = u_i \otimes (x \triangleright (m \triangleleft v_i)) + u_i \otimes ((v_i \triangleright m) \triangleleft x) = m_{(-1)} \otimes (x \triangleright m_{(0)}) + m_{(1)} \otimes (m_{(0)} \triangleleft x).
\]

For (HM2), we have

\[
\psi(x \triangleright m) + \tau \phi(m \triangleleft x) = (v_i \triangleright (x \triangleright m)) \otimes u_i + ((m \triangleleft x) \triangleleft v_i) \otimes u_i = ((v_i \triangleright x) \triangleright m) \otimes u_i + (u_i \triangleright m) \otimes xv_i - (m \triangleleft u_i) \otimes v_i x + (m \triangleleft (v_i x)) \otimes u_i = (u_i \triangleright m) \otimes xv_i + ((v_i \triangleright x) \otimes m \otimes u_i + (v_i \triangleright m) \otimes xu_i + (m \triangleleft u_i) \otimes v_i x + (m \triangleleft (v_i x)) \otimes u_i = (x_1 \triangleright m) \otimes x_2 + m_{(0)} \otimes xm_{(1)} + m_{(0)} \otimes m_{(-1)} x + (m \triangleleft x_2) \otimes x_1.
\]

For (HM3), we have

\[
(x \triangleright m_{(0)}) \otimes m_{(1)} - (m \triangleleft x_1) \otimes x_2 - m_{(0)} \otimes m_{(1)} x = (x \triangleright (v_i \triangleright m)) \otimes u_i - (m \triangleleft u_i) \otimes xv_i - (m \triangleleft (u_i x)) \otimes u_i - (v_i \triangleright m) \otimes u_i x = (x \triangleright (v_i \triangleright m)) \otimes u_i + (m \triangleleft u_i) \otimes xv_i - (m \triangleleft (v_i x)) \otimes u_i + (u_i \triangleright m) \otimes v_i x = -(m \triangleleft v_i) \otimes u_i + (m \triangleleft v_i) \otimes xu_i + ((xv_i) \triangleright m) \otimes u_i + (u_i \triangleright m) \otimes v_i x = \tau(-u_i \otimes ((m \triangleleft v_i) \otimes x) + xu_i \otimes (m \triangleleft v_i) + u_i \otimes ((xv_i) \triangleright m) + v_i x \otimes (u_i \triangleright m)) = \tau(xm_{(-1)} \otimes m_{(0)} + x_1 \otimes (x_2 \triangleright m) - m_{(-1)} \otimes (m_{(0)} \triangleleft x)).
\]

This completed the proof. \(\Box\)

**Theorem 3.6.** Let \((A, \cdot, r)\) be a quasitriangular anti-flexible bialgebra. Then \(A\) becomes a braided anti-flexible bialgebra over itself with \(M = A\) and \(\phi : M \to A \otimes M\) and \(\psi : M \to M \otimes A\) are given by

\[
\phi(a) := \sum_i u_i \otimes av_i, \quad \psi(a) := \sum_i v_i a \otimes u_i, \quad (27)
\]

**Proof.** All we need to do is to verify the braided compatibility conditions (BB1) and (BB2).

For (BB1), we have the right hand side is equal to the left hand side by

\[
a_1 b \otimes a_2 + ba_2 \otimes a_1 + b_1 \otimes ab_2 + b_2 \otimes b_1 a + a_{(-1)} b \otimes a_{(0)} + ba_{(1)} \otimes a_{(0)} + b_{(0)} \otimes ab_{(1)} + b_{(0)} \otimes b_{(-1)} a = u_i b \otimes av_i + (v_i a) b \otimes u_i + b(av_i) \otimes u_i + bu_i \otimes v_i a + u_i \otimes a(bv_i) + v_i b \otimes au_i.
\]
Thus (BB2) holds. This completed the proof.

where we use the conditions that \( r \) is skew-symmetric and the anti-flexible Yang-Baxter equation (AFYB). Thus (BB1) holds.

For (BB2), by similar calculations, we obtain

\[
\begin{align*}
+a_1 \otimes a_2 b - b a_1 \otimes a_2 - b_1 \otimes b_2 a + a b_1 \otimes b_2 \\
+ a_{(0)} \otimes a_{(1)} b - b a_{(1)} \otimes a_{(0)} - b_{(0)} \otimes b_{(1)} a + a b_{(1)} \otimes b_{(0)} \\
= u_i \otimes (u_i b) + v_i a \otimes u_i b - b u_i \otimes a v_i - b (v_i a) \otimes u_i - u_i \otimes (u_i b) a - v_i b \otimes u_i a + a u_i \otimes b v_i \\
+ a (v_i b) \otimes u_i + v_i a \otimes u_i b - b u_i \otimes a v_i - v_i b \otimes u_i a + a u_i \otimes b v_i \\
= u_i \otimes a (v_i b) + v_i a \otimes u_i b - b u_i \otimes a v_i - (v_i a) a \otimes u_i - u_i \otimes (u_i b) a - v_i b \otimes u_i a + a u_i \otimes b v_i \\
+ a (v_i b) \otimes u_i + v_i a \otimes u_i b - b u_i \otimes a v_i - v_i b \otimes u_i a + a u_i \otimes b v_i \\
= v_i a \otimes (v_i b) + a (v_i b) \otimes u_i - v_i b \otimes u_i a + a u_i \otimes b v_i - u_i \otimes b (v_i a) - (v_i a) a \otimes u_i + v_i a \otimes u_i b - b u_i \otimes a v_i \\
+ a (v_i b) \otimes u_i + v_i a \otimes u_i b - b u_i \otimes a v_i - v_i b \otimes u_i a + a u_i \otimes b v_i \\
= \tau \left( u_i \otimes (v_i b) + v_i a \otimes u_i b - b u_i \otimes a v_i - (v_i a) a \otimes u_i - u_i \otimes (u_i b) a - v_i b \otimes u_i a + a u_i \otimes b v_i \\
+ a (v_i b) \otimes u_i + v_i a \otimes u_i b - b u_i \otimes a v_i - v_i b \otimes u_i a + a u_i \otimes b v_i \right) \\
= \tau \left( a_1 \otimes a_2 b - b a_1 \otimes a_2 - b_1 \otimes b_2 a + a b_1 \otimes b_2 \\
+ a_{(0)} \otimes a_{(1)} b - b a_{(1)} \otimes a_{(0)} - b_{(0)} \otimes b_{(1)} a + a b_{(1)} \otimes b_{(0)} \right).
\end{align*}
\]

Thus (BB2) holds. This completed the proof. \(\square\)

**Remark 3.7.** In fact, we find that the braided terms \((a_{(-1)} \triangleright b) \otimes a_{(0)} + (b \triangleleft a_{(1)}) \otimes a_{(0)} + b_{(0)} \otimes (a \triangleleft b_{(1)}) + b_{(0)} \otimes (b_{(-1)} \triangleright a)\) and \((\text{id} - \tau)(a_{(0)} \otimes (a_{(1)} \triangleright b) - (b \triangleleft a_{(1)}) \otimes a_{(0)} - b_{(0)} \otimes (b_{(1)} \triangleright a) + (a \triangleleft b_{(1)}) \otimes b_{(0)})\) are equal to zero in the above proof. Thus what we have obtained is a braided anti-flexible bialgebra with zero braided terms. It is an open question for us whether there exists a braided anti-flexible bialgebra with nonzero braided terms coming from a quasitriangular anti-flexible bialgebra. For this direction, one should give or classify all anti-flexible Hopf bimodule structures which are different from \(\phi\) and \(\psi\) given by us in Theorem 3.5. Unfortunately, we have not found other natural anti-flexible Hopf bimodule beyond above until now.
4 Cocycle bicrossproducts of anti-flexible bialgebras

4.1 Matched pair of braided anti-flexible bialgebras

In this section, we construct anti-flexible bialgebra from the double cross biproduct of a matched pair of braided anti-flexible bialgebras.

Let $A, H$ be both anti-flexible algebras and anti-flexible coalgebras. For any $a, b \in A, x, y \in H$, we denote maps

\[
\vdash: H \otimes A \to A, \quad \lhd: A \otimes H \to A,
\]

\[
\triangleright : A \otimes H \to H, \quad \langle : H \otimes A \to H,
\]

\[
\phi : A \to H \otimes A, \quad \psi : A \to A \otimes H,
\]

\[
\rho : H \to A \otimes H, \quad \gamma : H \to H \otimes A,
\]

by

\[
\vdash (x \otimes a) = x \rightarrow a, \quad \lhd (a \otimes x) = a \leftarrow x,
\]

\[
\triangleright (a \otimes x) = a \triangleright x, \quad \langle (x \otimes a) = x \langle a,
\]

\[
\phi(a) = \sum a_{(-1)} \otimes a_{(0)}, \quad \psi(a) = \sum a_{(0)} \otimes a_{(1)},
\]

\[
\rho(x) = \sum x_{[-1]} \otimes x_{[0]}, \quad \gamma(x) = \sum x_{[0]} \otimes x_{[1]}.
\]

**Definition 4.1.** (II) A matched pair of anti-flexible algebras is a system $(A, H, \triangleright, \lhd, \langle, \rightarrow)$ consisting of two anti-flexible algebras $A$ and $H$ and four bilinear maps $\triangleright: H \otimes A \to H,$ $\lhd: A \otimes H \to A, \rightarrow: H \otimes A \to A$ such that $(H, \triangleright, \langle)$ is an $A$-bimodule, $(A, \rightarrow, \lhd)$ is an $H$-bimodule and satisfying the following compatibilities:

\[
\text{(AM1)} \quad x \rightarrow (ab) + (ba) \leftarrow x = (x \rightarrow a)b + b(a \leftarrow x) + (x \langle a) \rightarrow b + b \leftarrow (a \triangleright x),
\]

\[
\text{(AM2)} \quad (a \leftarrow x)b + (a \triangleright x) \rightarrow b - a(x \rightarrow b) - a \leftarrow (x \langle a)
\]

\[
= (b \triangleright x) \rightarrow a + (b \leftarrow x)a - b(x \rightarrow a) - b \leftarrow (x \langle a),
\]

\[
\text{(AM3)} \quad (xy) \lhd a + a \triangleright (yx) = x(y \lhd a) + (a \triangleright y)x + x \triangleright (y \rightarrow a) + (a \leftarrow y) \triangleright x,
\]

\[
\text{(AM4)} \quad (x \triangleright a)y + (x \rightarrow a) \triangleright y - x(a \triangleright y) - x \lhd (a \leftarrow y)
\]

\[
= (y \triangleright a)x + (y \rightarrow a) \triangleright x - y(a \triangleright x) - y \lhd (a \leftarrow x).
\]

**Lemma 4.2.** (II) Let $(A, H, \langle, \triangleright, \lhd, \rightarrow)$ be a matched pair of anti-flexible algebras. Then $A \bowtie H := A \otimes H,$ as a vector space, with the multiplication defined for all $a, b \in A$ and $x, y \in H$ by

\[
(a, x)(b, y) := (ab + a \leftarrow y + x \rightarrow b, \ a \triangleright y + x \lhd b + xy)
\]

is an anti-flexible algebra called the bicrossed product associated to the matched pair of anti-flexible algebras $A$ and $H.$
Now we introduce the notion of matched pairs of anti-flexible coalgebras, which is the dual version of matched pairs of anti-flexible algebras.

**Definition 4.3.** A matched pair of anti-flexible coalgebras is a system \((A, H, \phi, \psi, \rho, \gamma)\) consisting of two anti-flexible coalgebras \(A\) and \(H\) and four bilinear maps \(\phi : A \to H \otimes A, \psi : A \to A \otimes H, \rho : H \to A \otimes H, \gamma : H \to H \otimes A\) such that \((H, \rho, \gamma)\) is an \(A\)-bicomodule, \((A, \phi, \psi)\) is an \(H\)-bicomodule and satisfying the following compatibility conditions:

\[
\begin{align*}
(MC1) & \quad \phi(a_1) \otimes a_2 + \gamma(a_{(-1)}) \otimes a_{(0)} - a_{(-1)} \otimes \Delta_A(a_{(0)}) \\
& = \tau_{13}((\Delta_A(a_{(0)}) \otimes a_{(1)}) - a_1 \otimes \psi(a_2) - a_{(0)} \otimes \rho(a_{(1)})), \\
(MC2) & \quad \psi(a_1) \otimes a_2 + \rho(a_{(-1)}) \otimes a_{(0)} - a_1 \otimes \phi(a_2) - a_{(0)} \otimes \gamma(a_{(1)}) \\
& = \tau_{13}(\psi(a_1) \otimes a_2 + \rho(a_{(-1)}) \otimes a_{(0)} - a_1 \otimes \phi(a_2) - a_{(0)} \otimes \gamma(a_{(1)})), \\
(MC3) & \quad \rho(x_1 \otimes x_2 + \psi(x_{[1]}) \otimes x_{[0]} - x_{[1]} \otimes \Delta_H(x_{[0]}) \\
& = \tau_{13}((\Delta_H(x_{[0]}) \otimes x_{[1]} - x_{[0]} \otimes \phi(x_{[1]}) - x_1 \otimes \gamma(x_2)), \\
(MC4) & \quad \gamma(x_1 \otimes x_2 + \phi(x_{[1]}) \otimes x_{[0]} - x_{[0]} \otimes \psi(x_{[1]}) - x_1 \otimes \rho(x_2)) \\
& = \tau_{13}(\gamma(x_1) \otimes x_2 + \phi(x_{[1]}) \otimes x_{[0]} - x_{[0]} \otimes \psi(x_{[1]}) - x_1 \otimes \rho(x_2)).
\end{align*}
\]

**Lemma 4.4.** Let \((A, H, \phi, \psi, \rho, \gamma)\) be a matched pair of anti-flexible coalgebras. We define \(E = A \bowtie H\) as the vector space \(A \oplus H\) with comultiplication

\[
\Delta_E(a) = (\Delta_A + \phi + \psi)(a), \quad \Delta_E(x) = (\Delta_H + \rho + \gamma)(x),
\]

that is

\[
\Delta_E(a) = \sum a_1 \otimes a_2 + \sum a_{(-1)} \otimes a_{(0)} + \sum a_{(0)} \otimes a_{(1)}, \\
\Delta_E(x) = \sum x_1 \otimes x_2 + \sum x_{[1]} \otimes x_{[0]} + \sum x_{[0]} \otimes x_{[1]}.
\]

Then \(A \bowtie H\) is an anti-flexible coalgebra which is called the bicrossed coproduct associated to the matched pair of anti-flexible coalgebras \(A\) and \(H\).

**Proof.** We need to prove that

\[
(\Delta_E \otimes \text{id})\Delta_E(a, x) - (\text{id} \otimes \Delta_E)\Delta_E(a, x) = \tau_{13}((\Delta_E \otimes \text{id})\Delta_E(a, x) - (\text{id} \otimes \Delta_E)\Delta_E(a, x)).
\]

The left hand side is equal to

\[
\begin{align*}
(\Delta_E \otimes \text{id})\Delta_E(a, x) - (\text{id} \otimes \Delta_E)\Delta_E(a, x) &= (\Delta_E \otimes \text{id})\left((a_1 \otimes a_2 + a_{(-1)} \otimes a_{(0)} + a_{(0)} \otimes a_{(1)}) + x_1 \otimes x_2 + x_{[-1]} \otimes x_{[0]} + x_{[0]} \otimes x_{[1]}\right) \\
&\quad - (\text{id} \otimes \Delta_E)\left((a_1 \otimes a_2 + a_{(-1)} \otimes a_{(0)} + a_{(0)} \otimes a_{(1)}) + x_1 \otimes x_2 + x_{[-1]} \otimes x_{[0]} + x_{[0]} \otimes x_{[1]}\right) \\
&= \Delta_A(a_1) \otimes a_2 + \phi(a_1) \otimes a_2 + \psi(a_1) \otimes a_2 \\
&\quad + \Delta_H(a_{(-1)}) \otimes a_{(0)} + \rho(a_{(-1)}) \otimes a_{(0)} + \gamma(a_{(-1)}) \otimes a_{(0)} \\
&\quad + \Delta_A(a_{(0)}) \otimes a_{(1)} + \phi(a_{(0)}) \otimes a_{(1)} + \psi(a_{(0)}) \otimes a_{(1)}
\end{align*}
\]
\[+\Delta_H(p_1) \otimes x_2 + \rho(x_1) \otimes x_2 + \gamma(x_1) \otimes x_2\]
\[+\Delta_A(x_{[-1]}) \otimes x_0 + \phi(x_{[-1]}) \otimes x_0 + \psi(x_{[-1]}) \otimes x_0\]
\[+\Delta_H(x_0) \otimes x_1 + \rho(x_0) \otimes x_1 + \gamma(x_0) \otimes x_1\]
\[-a_1 \otimes \Delta_A(a_2) - a_1 \otimes \phi(a_2) - a_1 \otimes \psi(a_2)\]
\[-a_{(-1)} \otimes \Delta_A(a_{(0)}) - a_{(-1)} \otimes \phi(a_{(0)}) - a_{(-1)} \otimes \psi(a_{(0)})\]
\[-a_{(0)} \otimes \Delta_H(a_{(1)}) - a_{(0)} \otimes \rho(a_{(1)}) - a_{(0)} \otimes \gamma(a_{(1)})\]
\[-x_1 \otimes \Delta_H(x_2) - x_1 \otimes \rho(x_2) - x_1 \otimes \gamma(x_2)\]
\[-x_{[-1]} \otimes \Delta_H(x_{[0]} - x_{[-1]} \otimes \rho(x_{[0]} - x_{[-1]} \otimes \gamma(x_{[0]}))\]
\[-x_0 \otimes \Delta_A(x_{[1]} - x_0 \otimes \phi(x_{[1]}) - x_0 \otimes \psi(x_{[1]})\).

The right hand side can be computed similarly. Thus the two side are equal to each other if and only if \((A, H, \phi, \psi, \rho, \gamma)\) is a matched pair of anti-flexible coalgebras.

In the following of this section, we construct anti-flexible bialgebra from the double cross biproduct of a matched pair of braided anti-flexible bialgebras. First we generalize the concept of anti-flexible Hopf bimodule to the case of \(A\) is not necessarily an anti-flexible bialgebra but it is both an anti-flexible algebra and an anti-flexible coalgebra. But by abuse of notation, we also call it anti-flexible Hopf bimodule.

**Definition 4.5.** Let \(A\) be an anti-flexible algebra and an anti-flexible coalgebra. An anti-flexible Hopf bimodule over \(A\) is a space \(V\) endowed with maps

\[\triangleright: A \otimes V \to V, \quad \triangleleft: V \otimes A \to V,\]
\[\phi: V \to A \otimes V, \quad \psi: V \to V \otimes A,\]

such that \(V\) is simultaneously a bimodule, a bicomodule over \(A\) and satisfying the following compatibility conditions

(HM4) \[\phi(a \triangleright v) + \tau\psi(v \triangleleft a) = v_{[-1]} \otimes (a \triangleright v_{(0)}) + v_{(1)} \otimes (v_{(0)} \triangleleft a),\]

(HM5) \[\psi(a \triangleright v) + \tau\phi(v \triangleleft a) = (a_1 \triangleright v) \otimes a_2 + v_{(0)} \otimes av_{(1)} + v_{(0)} \otimes v_{[-1]}a + (v \triangleleft a_2) \otimes a_1,\]

(HM6) \[(a \triangleright v_{(0)}) \otimes (v_{(1)} - (v \triangleleft a_1) \otimes a_2 - v_{(0)} \otimes v_{(1)}a)
= \tau(au_{[-1]} \otimes v_{(0)} + a_1 \otimes (a_2 \triangleright v) - v_{[-1]} \otimes (v_{(0)} \triangleleft a)).\]

then \(V\) is called an anti-flexible Hopf bimodule over \(A\).

We denote the category of anti-flexible Hopf bimodules over \(A\) by \(\mathcal{A}_A \mathcal{M}_A\).

**Definition 4.6.** If \(A\) be an anti-flexible algebra and anti-flexible coalgebra and \(H\) is an anti-flexible Hopf bimodule over \(A\). If \(H\) is an anti-flexible algebra and an anti-flexible coalgebra in \(\mathcal{A}_A \mathcal{M}_A\), then we call \(H\) a braided anti-flexible bialgebra over \(A\), if the following conditions are satisfied:
\[(BB3) \quad \Delta_H(xy) + \tau \Delta_H(yx) \\
= x_1 y \otimes x_2 + y_2 x_1 + y_1 \otimes y_2 + y_2 \otimes y_1 \\
+ (x_{(-1)} \triangleright y) \otimes x(0) + (y \triangleright (x_{(1)} \otimes x(0)) + y(0) \otimes (x \triangleright y(1)) + y(0) \otimes (y_{(-1)} \triangleright x),
\]

\[(BB4) \quad (\text{id} - \tau)\left( x_1 \otimes x_2 y - y x_1 \otimes x_2 - y_1 \otimes y_2 x + xy_1 \otimes y_2 \right) \\
+ (\text{id} - \tau)\left( x(0) \otimes (x_{(1)} \triangleright y) - (y \triangleright x_{(-1)}) \otimes x(0) - y(0) \otimes (y_{(1)} \triangleright x) + (x \triangleright y_{(-1)}) \otimes y(0) \right) = 0.
\]

**Definition 4.7.** Let \( A, H \) be both anti-flexible algebras and anti-flexible coalgebras. If the following conditions hold:

\[(DM1) \quad \phi(ab) + \tau \psi(ba) \\
= (b \triangleright a(1)) \otimes a(0) + b(1) \otimes b(0) a + (a_{(-1)} \triangleleft b) \otimes a(0) + b_{(-1)} \otimes ab(0),
\]

\[(DM2) \quad \rho(xy) + \tau \gamma(yx) \\
= (y \triangleright x_1[0] \otimes x_0[1] + y_{[0]} \otimes y_0[1]) x + (x_{(-1)} \triangleright y) \otimes x_0[1] + y_{(-1)} \otimes xy_0[1],
\]

\[(DM3) \quad \Delta_A(x \rightarrow b) + \tau \Delta_A(b \leftarrow x) \\
= (b \rightarrow x_{[0]} \otimes x_{[-1]} + b_2 \otimes (b_1 \leftarrow x) + (x_{[0]} \rightarrow b) \otimes x_{[1]} + b_1 \otimes (x \rightarrow b_2),
\]

\[(DM4) \quad \Delta_H(a \rhd x) + \tau \Delta_H(x \triangleleft a) \\
= (x \triangleleft a(0)) \otimes a_{(-1)} + x_2 \otimes (x_1 \triangleleft a) + (a(0) \rhd x) \otimes a(1) + x_1 \otimes (a \rhd x_2)
\]

\[(DM5) \quad \phi(x \rightarrow b) + \tau \psi(b \leftarrow x) + \gamma(x \triangleleft b) + \tau \rho(b \rhd x) \\
= (b \triangleright x_{[1]} \otimes x_{[-1]} + b_1 \otimes b(0) \leftarrow x) + (x_{[0]} \triangleleft b) \otimes x_{[1]} + b_{(-1)} \otimes (x \rightarrow b(0)),
\]

\[(DM6) \quad \psi(x \rightarrow b) + \tau \phi(b \leftarrow x) + \rho(x \triangleleft b) + \tau \gamma(b \rhd x) \\
= (b \leftarrow x_{[1]} \rhd x + bx_{[1]} \otimes x_{[0]} + b_2 \otimes (b_1 \rhd x) + b(0) \otimes b_{(-1)} x
\]

\[(DM7) \quad a(0) \otimes (a(1) \triangleleft b) + ab(0) \otimes b(1) - b(0) \otimes (b_1 \triangleleft a) - ba(0) \otimes a(1)
\]

\[(DM8) \quad (\text{id} - \tau)(x_1 \otimes (x_2 \triangleleft b) + (x \triangleleft b(0)) \otimes b(1) - b_{(-1)} \otimes (b_0 \triangleright x)) - (b \triangleright x_1) \otimes x_2 = 0,
\]

\[(DM9) \quad x_{[0]} \otimes (x_{[1]} \triangleleft y) + xy_{[0]} \otimes y_{[1]} - y_{[0]} \otimes (y_{[1]} \triangleleft x) - yx_{[0]} \otimes x_{[1]}
\]

\[(DM10) \quad (\text{id} - \tau)(x_{[-1]} \otimes (x_0 \rightarrow b) + (x \rightarrow b_1) \otimes b_2 - b_1 \otimes (b_2 \leftarrow x) - (b \leftarrow x_{[0]}) \otimes x_{[1]} = 0,
\]

\[(DM11) \quad b_{(-1)} \otimes (b(0) \triangleright x) + (b \triangleright x_{[0]}) \otimes x_{[1]} - x_1 \otimes (x_2 \rightarrow b) - x_{[0]} \otimes (b_1 \triangleright b_2 - b_2 \triangleright b_{(-1)} \otimes b(0)
\]

then \((A, H)\) is called a **double matched pair.**
Theorem 4.8. Let \((A, H)\) be matched pair of anti-flexible algebras and anti-flexible coalgebras, \(A\) is a braided anti-flexible bialgebra in \(H \mathcal{A}_H\), \(H\) is a braided anti-flexible bialgebra in \(A \mathcal{A}_A\). If we define the double cross biproduct of \(A\) and \(H\), denoted by \(A \bowtie H\), \(A \bowtie H = A \otimes H\) as anti-flexible algebra, \(A \bowtie H \bowtie H\) as anti-flexible algebra, \(A \bowtie H = A \bowtie H\) as anti-flexible coalgebra, then \(A \bowtie H\) becomes an anti-flexible bialgebra if and only if \((A, H)\) forms a double matched pair.

Theorem 1.8 is a special case of Theorem 4.1 in next section, and the proof is by direct computations, so we omit the details.

4.2 Cocycle bicrossproduct anti-flexible bialgebras

In this section, we construct cocycle bicrossproduct anti-flexible bialgebras, which is a generalization of double cross biproduct.

Let \(A, H\) be both anti-flexible algebras and anti-flexible coalgebras. For \(a, b \in A, x, y \in H\), we denote maps

\[
\sigma : H \otimes H \to A, \quad \theta : A \otimes A \to H,
\]

\[
P : A \to H \otimes H, \quad Q : H \to A \otimes A,
\]

by

\[
\sigma(x, y) \in H, \quad \theta(a, b) \in A,
\]

\[
P(a) = \sum a_{<1>} \otimes a_{<2>}, \quad Q(x) = \sum x_{[1]} \otimes x_{[2]}.
\]

A bilinear map \(\sigma : H \otimes H \to A\) is called a cocycle on \(H\) if

(CC1) \(\sigma(xy, z) - \sigma(x, yz) + \sigma(x, y) = z - x \to \sigma(y, z) = \sigma(zy, x) - \sigma(y, x) + \sigma(z, y) = x - z \to \sigma(y, x)\).

A bilinear map \(\theta : A \otimes A \to H\) is called a cocycle on \(A\) if

(CC2) \(\theta(ab, c) - \theta(a, bc) + \theta(a, b) \triangleleft c - a \triangleright \theta(b, c) = \theta(cb, a) - \theta(c, ba) + \theta(c, b) \triangleleft a - c \triangleright \theta(b, a)\).

A bilinear map \(P : A \to H \otimes H\) is called a cycle on \(A\) if

(CC3) \(\Delta_H(a_{<1>}) \otimes a_{<2>} - a_{<1>} \otimes \Delta_H(a_{<2>}) + P(a_{(0)}) \otimes a_{(1)} - a_{(-1)} \otimes P(a_{(0)})\)

\[= \tau_{13} (\Delta_H(a_{<1>}) \otimes a_{<2>} - a_{<1>} \otimes \Delta_H(a_{<2>}) + P(a_{(0)}) \otimes a_{(1)} - a_{(-1)} \otimes P(a_{(0)})\).

A bilinear map \(Q : H \to A \otimes A\) is called a cycle on \(H\) if

(CC4) \(\Delta_A(x_{[1]}) \otimes x_{[2]} - x_{[1]} \otimes \Delta_A(x_{[2]}) + Q(x_{[0]}) \otimes x_{[1]} - x_{[-1]} \otimes Q(x_{[0]})\)

\[= \tau_{13} (\Delta_A(x_{[1]}) \otimes x_{[2]} - x_{[1]} \otimes \Delta_A(x_{[2]}) + Q(x_{[0]}) \otimes x_{[1]} - x_{[-1]} \otimes Q(x_{[0]}))\).

In the following definitions, we introduced the concept of cocycle anti-flexible algebras and cycle anti-flexible coalgebras, which are in fact not really ordinary anti-flexible algebras and anti-flexible coalgebras, but generalized ones.
Definition 4.9. (i): Let $\sigma$ be a cocycle on a vector space $H$ equipped with multiplication $H \otimes H \to H$, satisfying the following cocycle identity:

(CC5) \[(xy)z - x(yz) + \sigma(x, y) \triangleright z - x \triangleleft \sigma(y, z) = (zy)x - z(yx) + \sigma(z, y) \triangleright x - z \triangleleft \sigma(y, x).\]

Then $H$ is called a $\sigma$-anti-flexible algebra which is denoted by $(H, \sigma)$.

(ii): Let $\theta$ be a cocycle on a vector space $A$ equipped with a multiplication $A \otimes A \to A$, satisfying the following cocycle identity:

(CC6) \[(ab)c - a(bc) + \theta(a, b) \to c - a \leftarrow \theta(b, c) = (cb)a - c(ba) + \theta(c, b) \to a - c \leftarrow \theta(b, a).\]

Then $A$ is called a $\theta$-anti-flexible algebra which is denoted by $(A, \theta)$.

(iii) Let $P$ be a cycle on a vector space $H$ equipped with a comultiplication $\Delta : H \to H \otimes H$, satisfying the following cycle identity:

(CC7) \[\Delta_H(x_1) \otimes x_2 - x_1 \otimes \Delta_H(x_2) + P(x_{[-1]}) \otimes x_{[0]} - x_{[0]} \otimes P(x_{[1]}) = \tau_{13} \left( \Delta_H(x_1) \otimes x_2 - x_1 \otimes \Delta_H(x_2) + P(x_{[-1]}) \otimes x_{[0]} - x_{[0]} \otimes P(x_{[1]}) \right).\]

Then $H$ is called a $P$-anti-flexible coalgebra which is denoted by $(H, P)$.

(iv) Let $Q$ be a cycle on a vector space $A$ equipped with a comultiplication $\Delta : A \to A \otimes A$, satisfying the following cycle identity:

(CC8) \[\Delta_A(a_1) \otimes a_2 - a_1 \otimes \Delta_A(a_2) + Q(a_{[-1]}) \otimes a_{[0]} - a_{[0]} \otimes Q(a_{[1]}) = \tau_{13} \left( \Delta_A(a_1) \otimes a_2 - a_1 \otimes \Delta_A(a_2) + Q(a_{[-1]}) \otimes a_{[0]} - a_{[0]} \otimes Q(a_{[1]}) \right).\]

Then $A$ is called a $Q$-anti-flexible coalgebra which is denoted by $(A, Q)$.

Definition 4.10. A cocycle cross product system is a pair of $\theta$-anti-flexible algebra $A$ and $\sigma$-anti-flexible algebra $H$, where $\sigma : H \otimes H \to A$ is a cocycle on $H$, $\theta : A \otimes A \to H$ is a cocycle on $A$ and the following conditions are satisfied:

(CP1) \[x \to (ab) + (ba) \leftarrow x + \sigma(x, \theta(a, b)) + \sigma(\theta(b, a), x) = (x \to a)b + b(a \leftarrow x) + (x \triangleleft a) \to b + b \leftarrow (a \triangleright x),\]

(CP2) \[(a \leftarrow x)b + (a \triangleright x) \to b - a(x \leftarrow b) - a \leftarrow (x \triangleleft b) = (b \triangleright x) \to a + (b \leftarrow x)a - b(x \to a) - b \leftarrow (x \triangleleft a),\]

(CP3) \[(xy) \leftarrow a + \sigma(x, y)a - x \leftarrow (y \to a) - \sigma(x, y \triangleleft a) = (a \leftarrow y) \leftarrow x - a \leftarrow (yx) + \sigma(a \triangleright y, x) - a\sigma(y, x),\]

(CP4) \[x \leftarrow a \leftarrow y + \sigma(x \triangleleft a, y) - x \leftarrow (a \leftarrow y) - \sigma(x, a \triangleright y) = (y \to a) \leftarrow x + \sigma(y \triangleleft a, x) - y \to (a \leftarrow x) - \sigma(y, a \triangleright x),\]

(CP5) \[(xy) \triangleleft a + a \triangleright (xy) + \theta(\sigma(x, y), a) + \theta(a, \sigma(y, x)) = x(y \triangleleft a) + (a \triangleright y)x + x \triangleleft (y \to a) + (a \leftarrow y) \triangleright x,\]

(CP6) \[x \triangleleft a)y + (x \leftarrow a) \triangleright y - x(a \triangleright y) - x \triangleleft (a \leftarrow y) = (y \triangleleft a)x + (y \leftarrow a) \triangleright x - y(a \triangleright x) - y \triangleleft (a \leftarrow x),\]
(CP7) \((ab) \triangleright x + \theta(a, b)x - a \triangleright (b \triangleright x) - \theta(a, b) \leftarrow x\)
\[= (x \triangleright b) \triangleleft a + \theta(x \rightarrow b, a) - x\theta(b, a) - x \triangleright (ba),\]

(CP8) \((a \triangleright x) \triangleleft b + \theta(a \leftarrow x, b) - a \triangleright (x \triangleright b) - \theta(a, x \rightarrow b)\)
\[= (b \triangleright x) \triangleleft a + \theta(b \leftarrow x, a) - b \triangleright (x \triangleright a) - \theta(b, x \rightarrow a).\]

**Lemma 4.11.** Let \((A, H)\) be a cocycle cross product system. If we define \(E = A_\sigma \# \theta H\) as the vector space \(A \otimes H\) with the multiplication
\[(a, x)(b, y) = (ab + x \rightarrow b + a \leftarrow y + \sigma(x, y), xy + x \triangleright b + a \triangleright y + \theta(a, b)),\] (29)
then \(E = A_\sigma \# \theta H\) forms an anti-flexible algebra which is called the cocycle cross product anti-flexible algebra.

**Proof.** We have to check that
\[((a, x)(b, y))(c, z) - (a, x)((b, y)(c, z)) = ((c, z)(b, y))(a, x) - (c, z)((b, y)(a, x)).\]

By direct computations, the left hand side is equal to
\[
\begin{align*}
((a, x)(b, y))(c, z) - (a, x)((b, y)(c, z)) &= \left(\left(ab \triangleright c + (x \triangleright b)c + (a \triangleright y)c + \sigma(x, y)c + (xy) - c + (x \triangleright b) - c + (a \triangleright y) - c + \theta(a, b) - c + (ab) - z + (x \triangleright b) - z + (a \leftarrow y) - z + \sigma(x, y) - zight)
+ \sigma(xy, z) + \sigma(x \triangleright b, z) + \sigma(a \triangleright y, z) + \sigma(\theta(a, b), z), \quad (xy)z + (x \triangleright b)z + (a \triangleright y)z
+ \theta(a, b)z + (xy)z + (x \triangleright b)z + (a \triangleright y)z + \theta(a, b)z + (xy)z + (x \triangleright b)z + (a \triangleright y)z
+ (a \leftarrow y)z + \sigma(x, y)z + \theta(ab, c) + \theta(x \rightarrow b, c) + \theta(a \leftarrow y, c) + \theta(\sigma(x, y, c), c)
- \left(ab + (a \triangleright y - c) + (a \leftarrow z) + a(\sigma(y, z)) + x \rightarrow (bc) + x \rightarrow (y \leftarrow c)
+ z \rightarrow (b \leftarrow z) + x \rightarrow \sigma(y, z) + a \triangleright (yz) + a \rightarrow (y \triangleright c) + a \rightarrow (b \triangleright z) + a \leftarrow \theta(b, c)
+ \sigma(x, yz) + \sigma(x, y \triangleright c) + \sigma(x, b \triangleright z) + \sigma(x, \theta(b, c)), \quad x(yz) + x(y \triangleright c) + x(b \triangleright z)
+ x\theta(b, c) + x \leftrightarrow (bc) + x \leftrightarrow (y \rightarrow c) + x \leftrightarrow (b \rightarrow z) + x \leftrightarrow \sigma(y, z) + a \triangleright (yz) + a \triangleright (y \leftarrow c)
+ a \triangleright (b \triangleright z) + a \triangleright \theta(b, c) + \theta(a, bc) + \theta(a, y \rightarrow c) + \theta(a, b \leftarrow z) + \theta(a, \sigma(y, z))\right).\n\end{align*}
\]

We can compute the right hand side in the same way. Thus the two sides are equal to each other if and only if (CP1)–(CP8) hold. \(\blacksquare\)

**Definition 4.12.** A cycle cross coproduct system is a pair of \(P\)-anti-flexible coalgebra \(A\) and \(Q\)-anti-flexible coalgebra \(H\), where \(P : A \rightarrow H \otimes H\) is a cycle on \(A\), \(Q : H \rightarrow A \otimes A\) is a cycle on \(H\) such that following conditions are satisfied:

(CCP1) \(\phi(a_1) \otimes a_2 + \gamma(a_{(-1)}) \otimes a_{(0)} - a_{(-1)} \otimes \Delta_A(a_{(0)}) - a_{<1>} \otimes Q(a_{<2>})\)
\[= \tau_{13}(\Delta_A(a_{(0)}) \otimes a_{(1)}) + Q(a_{<1>}) \otimes a_{<2>} - a_1 \otimes \psi(a_2) - a_{(0)} \otimes \rho(a_{(1)}),\]

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Lemma 4.13. Let \((A,H)\) be a cycle cross coproduct system. If we define \(E = A^P \# QH\) as the vector space \(A \oplus H\) with the comultiplication

\[
\Delta_E(a) = (\Delta_A + \phi + \psi + P)(a), \quad \Delta_E(x) = (\Delta_H + \rho + \gamma + Q)(x),
\]

that is

\[
\Delta_E(a) = a_1 \otimes a_2 + a_{(-1)} \otimes a_{(0)} + a_{(0)} \otimes a_{(1)} + a_{<1>} \otimes a_{<2>},
\]

\[
\Delta_E(x) = x_1 \otimes x_2 + x_{[-1]} \otimes x_{[0]} + x_{[0]} \otimes x_{[1]} + x_{[1]} \otimes x_{[2]},
\]

then \(A^P \# QH\) forms an anti-flexible coalgebra which we will call it the cycle cross coproduct anti-flexible coalgebra.

Proof. We have to check

\[
(\Delta_E \otimes \text{id}) \Delta_E(a, x) - (\text{id} \otimes \Delta_E \otimes \text{id}) \Delta_E(a, x) = \tau_{13} \left( (\Delta_E \otimes \text{id}) \Delta_E(a, x) - (\text{id} \otimes \Delta_E \otimes \text{id}) \Delta_E(a, x) \right).
\]

By direct computations, the left hand side is equal to

\[
(\Delta_E \otimes \text{id}) \Delta_E(a, x) - (\text{id} \otimes \Delta_E \otimes \text{id}) \Delta_E(a, x)
\]
and the right hand side is calculated in the same way. Thus the two sides are equal to each other if and only if (CCP1)–(CCP8) hold.

**Definition 4.14.** Let A, H be both anti-flexible algebras and anti-flexible coalgebras. If the following conditions hold:

(CDM1) $\phi(ab) + \tau \psi(ba) + \gamma(\theta(a, b)) + \tau \rho(\theta(b, a))$

$$= \theta(b, a_2) \otimes a_1 + b_\triangleright a_1, a_2 \otimes a_0 + b_\triangleright a_2 \otimes (b_\triangleright a_1 \rightarrow a) + b_1 \otimes b_0, a$$

$$+ \theta(a_1, b) \otimes a_2 + (a_1 < b) \otimes a_0 + b_\triangleright a_1 \otimes (a \leftarrow b_\triangleright a_2),$$

(CDM2) $\rho(xy) + \tau \gamma(yx) + \psi(\sigma(x, y)) + \tau \phi(\sigma(y, x))$

$$= \sigma(y, x) \otimes x_1 + (y \rightarrow x_1) \otimes x_0 + y_1 \otimes y_0 \otimes x + y_2 \otimes (y_1 \leftarrow x)$$

$$+ \sigma(x_1, y) \otimes x_2 + (x_1 \leftarrow y) \otimes x_0 + y_1 \otimes x_0 \otimes y_1 \otimes (x \leftarrow y_2),$$

(CDM3) $\Delta_A(x \rightarrow b) + \tau \Delta_A(b \leftrightarrow x) + Q(x \leftarrow b) + \tau Q(b \triangleright x)$

$$= (b \leftarrow x_0) \otimes x_1 \leftarrow x + b_1 \otimes (b \leftarrow x) + b_0 \otimes \sigma(b_1, x)$$

$$+ (x_0 \rightarrow b) \otimes x_1 \leftarrow x_0 + b_1 \otimes (x \rightarrow b) + b_0 \otimes \sigma(x, b_0),$$

(CDM4) $\Delta_H(a \triangleright y) + \tau \Delta_H(y \triangleright a) + P(a \leftarrow y) + \tau P(y \rightarrow a)$

$$= (y \triangleright a_0) \otimes a_1 + y_1 \otimes a_0 \rightarrow y_2 \otimes (y_1 \rightarrow a) + y_0 \otimes \theta(y_1 \rightarrow a)$$

$$+ (a_0 \triangleright y) \otimes a_1 + a_1 \rightarrow a \otimes y_1 \rightarrow y_2 \otimes (y_0 \triangleright \theta(a, y_1)).$$

(CDM5) $\Delta_H(\theta(a, b)) + \tau \Delta_H(\theta(b, a)) + P(ab) + \tau P(ba)$

$$= \theta(b, a_0) \otimes a_1 + (b \triangleright a_0) \otimes a_1 \leftarrow a_2 + b_1 \otimes \theta(b_0, a) + b_2 \otimes (b_\triangleright a_1 \leftarrow a_2)$$

$$+ \theta(a_0, b) \otimes a_1 + (a_1 \leftarrow b) \otimes a_2 \leftarrow a_1 \leftarrow a_2 \otimes \theta(a, b_0) \otimes b_0 \otimes \sigma(a, b_0),$$

(CDM6) $\Delta_A(\sigma(x, y)) + \tau \Delta_A(\sigma(y, x)) + Q(xy) + \tau Q(yx)$

$$= (y \rightarrow x_2) \otimes x_1 \leftarrow x_0 \leftarrow x_0 \otimes \sigma(y_0 \triangleright x) + y_1 \otimes \sigma(y_1 \rightarrow x)$$

$$+ \sigma(x_0, y) \otimes x_1 \leftarrow y_1 \otimes x_2 \leftarrow y_2 \otimes (y_1 \rightarrow x) + \sigma(x_0 \rightarrow y) \otimes x_1 \rightarrow y_0 \otimes \sigma(y, y_0) \otimes y_1 \otimes (x \rightarrow y_2),$$
(CDM7) $\phi(x \rightarrow b) + \tau \psi(b \leftarrow x) + \gamma(x \triangleleft b) + \tau \rho(b \triangleright x)$
\[
= (b \triangleright x_0) \otimes x_{-1} + \theta(b, x_{[1]}) \otimes x_{[1]} + b_{(1)} \otimes (b_0 \leftarrow x) + b_{<2>} \otimes \sigma(b_{<1>}, x) \\
+ (x_0 \leftarrow b) \otimes x_{[1]} + \theta(x_{[1]}, b) \otimes x_{[2]} + b_{(1)} \otimes (x \rightarrow b_0) + b_{<1>} \otimes \sigma(x, b_{<2>}),
\]

(CDM8) $\psi(x \rightarrow b) + \tau \phi(b \leftarrow x) + \rho(x \triangleleft b) + \gamma(b \triangleright x)$
\[
= (b \leftarrow x_2) \otimes x_1 + b x_{[1]} \otimes x_0 + b_2 \otimes (b_1 \triangleright x) + b_0 \otimes b_{(-1)} x \\
+ (x_1 \rightarrow b) \otimes x_2 + x_{[-1]} b \otimes x_0 + b_1 \otimes (x \triangleleft b_2) + b_0 \otimes x b_1),
\]

(CDM9) $a_1 \otimes \theta(a_2, b) + a_0 \otimes (a_{<1>} \triangleleft b) + b_{(0)} \otimes b_{(1)} + (a \leftarrow b_{<1>}) \otimes b_{<2>}$
\[
- b_1 \otimes \theta(b_2, a) - b_{(0)} \otimes (b_{<1>} \triangleleft a) - b_{a_{(0)}} \otimes a_{(1)} - (b \leftarrow a_{<1>}) \otimes a_{<2>}
= \tau (a_{(-1)} \otimes a_{(0)} b + a_{<1>} \otimes (a_{<2>} \leftarrow b) + \theta(a, b_1) \otimes b_2 + (a \triangleright b_{(-1)}) \otimes b_{(0)} \\
- b_{(-1)} \otimes b_{(0)} a - b_{<1>} \otimes (b_{<2>} \leftarrow a) - \theta(b, a_{1}) \otimes a_{2} - (b \triangleright a_{(-1)}) \otimes a_{(0)}),
\]

(CDM10) $(id - \tau)(a_{(-1)} \otimes \theta(a_0, b) + a_{<1>} \otimes (a_{<2>} \triangleleft b) + \theta(a, b_{(0)}) \otimes b_{(1)} + (a \triangleright b_{<1>}) \otimes b_{<2>})$
\[
= (id - \tau)(b_{(-1)} \otimes \theta(b_0, a) + b_{<1>} \otimes (b_{<2>} \triangleleft a) + \theta(b, a_{(0)}) \otimes a_{(1)} + (b \triangleright a_{<1>}) \otimes a_{<2>}),
\]

(CDM11) $(id - \tau)(x_1 \otimes (x_2 \triangleleft b + x_0 \otimes \theta(x_{[1]}, b) + (x \triangleleft b_{(0)}) \otimes b_{(1)} + x_0 \triangleright x_{<2>})$
\[
= (id - \tau)(b_{(-1)} \otimes (b_0 \triangleright x) + b_{<1>} \otimes b_{<2>} x + (b \triangleright x_1) \otimes x_2 + \theta(b, x_{[-1]}) \otimes x_0),
\]

(CDM12) $(id - \tau)(x_{[-1]} \otimes (x_0 \rightarrow b) + x_{[1]} \otimes x_{[2]} b + (x \rightarrow b_1) \otimes b_2 + \sigma(x, b_{(-1)}) \otimes b_{(0)})$
\[
= (id - \tau)(b_1 \otimes (b_2 \rightarrow x) + b_0 \otimes \sigma(b_{(1)}, x) + (b \leftarrow x_{[0]}) \otimes x_{[1]} + b x_{[1]} \otimes x_{[2]}),
\]

(CDM13) $x_1 \otimes (x_2 \rightarrow b) + x_0 \otimes x_{[1]} b + (x \triangleleft b_1) \otimes b_2 + x b_{(-1)} \otimes b_{(0)}$
\[
- b_{(-1)} \otimes (b_0 \leftarrow x) - b_{<1>} \otimes \sigma(b_{<2>}, x) - (b \triangleright x_{[0]}) \otimes x_{[1]} - \theta(b, x_{[1]}) \otimes x_{[2]}$
\[
= \tau (x_{[-1]} \otimes (x_0 \leftarrow b) + x_{[1]} \otimes \theta(x_{[2]}, b) + (x \rightarrow b_{(0)}) \otimes b_{(1)} + \sigma(x, b_{<1>}) \otimes b_{<2>}
- b_1 \otimes (b \triangleright x) - b_0 \otimes b_{(1)} x - (b \leftarrow x_{1}) \otimes (x_2 - b_{-} x_{[-1]} \otimes x_{[0]}),
\]

(CDM14) $x_1 \otimes \sigma(x_{2}, y) + x_0 \otimes (x_{[1]} \rightarrow y) + y x_{[0]} \otimes y_{[1]} + (x \triangleleft y_{[1]}) \otimes y_{[2]}$
\[
- y_1 \otimes \sigma(y_{2}, x) - x_{[0]} \otimes (y_{[1]} \leftarrow x) - y x_{[0]} \otimes x_{[1]} - (y \triangleleft x_{[1]}) \otimes x_{[2]}$
\[
= \tau (x_{[-1]} \otimes x_{[0]} y + x_{[1]} \otimes (x_{[2]} \triangleright y) + \sigma(x, y_{1}) \otimes y_2 + (x \rightarrow y_{[-1]}) \otimes y_{[0]}
- y_{(-1)} \otimes y_{[0]} x - y_{[1]} \otimes (y_{[2]} \triangleright x) - \sigma(y, x_{1}) \otimes x_2 + (y \rightarrow x_{[-1]}) \otimes x_{[0]}),
\]

(CDM15) $(id - \tau)(y_{[-1]} \otimes \sigma(y_{[0]}, x) + y_{[1]} \otimes (y_{[2]} \rightarrow x) + \sigma(x, y_{[0]}) \otimes y_{[1]} + (x \rightarrow y_{[1]}) \otimes y_{[2]})$
\[
= (id - \tau)(y_{[-1]} \otimes \sigma(y_{[0]}, x) + y_{[1]} \otimes (y_{[2]} \leftarrow x) + \sigma(y, y_{[0]}) \otimes x_{[1]} + (y \rightarrow x_{[1]}) \otimes x_{[2]}),
\]

then $(A,H)$ is called a cocycle double matched pair.

**Definition 4.15.** (i) A cocycle braided anti-flexible bialgebra $A$ is simultaneously a cocycle anti-flexible algebra $(A, \theta)$ and a cycle anti-flexible coalgebra $(A, Q)$ satisfying the conditions

(CBB1) $\Delta_A(ab) + \tau \Delta_A(ba) + Q\theta(a, b) + \tau Q\theta(b, a)$
\[
= ba_2 \otimes a_1 + b_2 \otimes b_{1a} + a_1b \otimes a_2 + b_1 \otimes ab_2 \\
+ (b \leftarrow a_{(1)}) \otimes a_{(0)} + b_{(0)} \otimes (b_{(-1)} \rightarrow a) + (a_{(-1)} \rightarrow b) \otimes a_{(0)} + b_{(0)} \otimes (a \leftarrow b_{(1)}),
\]

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(CBB2) \((\text{id} - \tau) (a_1 \otimes a_2 b + ab_1 \otimes b_2 - b_1 \otimes b_2 a - ba_1 \otimes a_2)\)
\[ + (\text{id} - \tau) (a(0) \otimes (a(1) \rightarrow b) + (a \leftarrow b(-1)) \otimes (b(0) \rightarrow a) - (b \leftarrow a(-1)) \otimes a(0)) = 0.\]

(ii) A 

A cocycle braided anti-flexible bialgebra \(H\) is simultaneously a cocycle anti-flexible algebra \((H, \sigma)\) and a cycle anti-flexible coalgebra \((H, P)\) satisfying the conditions

(CBB3) \(\Delta_H(xy) + \tau \Delta_H(yx) + P\sigma(x, y) + \tau P\sigma(y, x)\)
\[ = yx_2 \otimes x_1 + y_2 \otimes y_1 x + x_1 y \otimes x_2 + y_1 \otimes xy_2\]
\[ + (y \triangleleft x[-1]) \otimes x[0] + y[0] \otimes (y[-1] \triangleright x) + (x[-1] \triangleright y) \otimes x[0] + y[0] \otimes (x \triangleleft y[1]),\]

(CBB4) \((\text{id} - \tau)(x_1 \otimes x_2 y + xy_1 \otimes y_2 - y_1 \otimes y_2 x - yx_1 \otimes x_2)\)
\[ + (\text{id} - \tau)(x[0] \otimes (x[1] \triangleright y) + (x \triangleleft y[-1]) \otimes y[0] - y[0] \otimes (y[1] \triangleright x) - (y \triangleleft x[-1]) \otimes x[0]) = 0.\]

The next theorem says that we can obtain an ordinary anti-flexible bialgebra from two cocycle braided anti-flexible bialgebras.

**Theorem 4.16.** Let \(A, H\) be cocycle braided anti-flexible bialgebras, \((A, H)\) be a cocycle cross product system and a cycle cross coproduct system. Then the cocycle cross product algebra and cycle cross coproduct coalgebra fit together to become an ordinary anti-flexible bialgebra if and only if \((A, H)\) forms a cocycle double matched pair. We will call it the cocycle bicrossproduct anti-flexible bialgebra and denote it by \(A^P \#_\theta H\).

The proof is by direct computations, so we omit the details.

5 Extending structures for anti-flexible bialgebras

In this section, we will study the extending problem for anti-flexible bialgebras. We will find some special cases when the braided anti-flexible bialgebra is reduced into an ordinary anti-flexible bialgebra. It is proved that the extending problem can be solved by using of the non-abelian cohomology theory based on cocycle bicrossed product for braided anti-flexible bialgebras in last section.

5.1 Extending structures for anti-flexible algebras (coalgebras)

First we are going to study extending problem for anti-flexible algebras (coalgebras).

There are two cases for \(A\) to be an anti-flexible algebra in the cocycle cross product system defined in last section, see condition (CC6). The first case is when we let \(\triangleright, \triangleleft\) to be trivial and \(\theta \neq 0\), then from condition (CP1) we get \(\sigma(x, \theta(a, b)) + \sigma(\theta(a, b), x) = 0\). Since \(\theta \neq 0\) we assume \(\sigma = 0\) for simplicity, thus we obtain the following type \((a1)\) unified product for anti-flexible algebras.

**Lemma 5.1.** Let \(A\) be an anti-flexible algebra and \(V\) a vector space. An extending datum of \(A\) by \(V\) of type \((a1)\) is \(\Omega^{(1)}(A, V) = (\triangleright, \triangleleft, \theta, \cdot)\) consisting of bilinear maps
\[ \triangleright : A \otimes V \rightarrow V, \quad \triangleleft : V \otimes A \rightarrow V, \quad \theta : A \otimes A \rightarrow V, \quad \cdot : V \otimes V \rightarrow V.\]
Denote by $A\#\theta V$ the vector space $E = A \oplus V$ together with the multiplication given by

$$(a,x)(b,y) = (ab, xy + x \triangleleft b + a \triangleright y + \theta(a,b)).$$(30)

Then $A\#\theta V$ is an anti-flexible algebra if and only if the following compatibility conditions hold for all $a, b \in A, x, y, z \in V$:

(A1) $(xy)\triangleleft a + a \triangleright (yx) = x(y\triangleleft a) + (a \triangleright y)x$,

(A2) $(x \triangleleft a)y - x(a \triangleright y) = (y \triangleleft a)x - y(a \triangleright x)$,

(A3) $(ab)\triangleright x + \theta(a,b)x - a \triangleright (b\triangleright x) = (x \triangleleft b)\triangleleft a - x\theta(b,a) - x \triangleleft (ba)$,

(A4) $(a \triangleright x)\triangleleft b - a \triangleright (x \triangleleft b) = (b \triangleright x)\triangleleft a - b \triangleright (x \triangleleft a)$,

(A5) $\theta(ab,c) - \theta(a,bc) + \theta(a,b)\triangleleft c - a \triangleright \theta(b,c) = \theta(cb,a) - \theta(c,ba) + \theta(c,b)\triangleleft a - c \triangleright \theta(b,a)$,

(A6) $(xy)z - x(yz) = (zy)x - z(xy)$.

Note that (A1)–(A4) are deduced from (CP5)–(CP8) and by (A6) we obtain that $V$ is an anti-flexible algebra. Furthermore, $V$ is in fact an anti-flexible subalgebra of $A\#\theta V$ but $A$ is not although $A$ is itself an anti-flexible algebra.

Denote also the set of all algebraic extending datum of $A$ by $V$ of type (a1) by $\Omega^{(1)}(A,V)$ by abuse of notations.

In the following, we always assume that $A$ is a subspace of a vector space $E$, there exists a projection map $p : E \to A$ such that $p(a) = a$, for all $a \in A$. Then the kernel space $V := \ker(p)$ is also a subspace of $E$ and a complement of $A$ in $E$.

**Lemma 5.2.** Let $A$ be an anti-flexible algebra and $E$ a vector space containing $A$ as a subspace. Suppose that there is an anti-flexible algebra structure on $E$ such that $V$ is an anti-flexible subalgebra of $E$ and the canonical projection map $p : E \to A$ is an anti-flexible algebra homomorphism. Then there exists an algebraic extending datum $\Omega^{(1)}(A,V)$ of $A$ by $V$ such that $E \cong A\#\theta V$.

**Proof.** Since $V$ is a subalgebra of $E$, we have $x \cdot_E y \in V$ for all $x, y \in V$. We define the extending datum of $A$ through $V$ by the following formulas:

$$\triangleleft : V \otimes A \to V, \quad x \triangleleft a := x \cdot_E a,$$

$$\theta : A \otimes A \to V, \quad \theta(a,b) := a \cdot_E b - p(a \cdot_E b),$$

$$\cdot_V : V \otimes V \to V, \quad x \cdot_V y := x \cdot_E y.$$

for all $a, b \in A$ and $x, y \in V$. It is easy to see that the above maps are well defined and $\Omega^{(1)}(A,V)$ is an extending system of $A$ through $V$ and

$$\varphi : A\#\theta V \to E, \quad \varphi(a,x) := a + x$$

is an isomorphism of anti-flexible algebras. □
Lemma 5.3. Let $\Omega^{(1)}(A, V) = (\triangleright, \triangleleft, \cdot, \cdot')$ and $\Omega^{(1)}(A, V) = (\triangleright', \triangleleft', \cdot', \cdot')$ be two extending datums of $A$ by $V$ of type $(a1)$ and $A\#_{a} V$, $A\#_{a'} V$ be the corresponding unified products. Then there exists a bijection between the set of all homomorphisms of anti-flexible algebras $\varphi : A\#_{a, \triangleright} V \to A\#_{a', \triangleright'} V$ whose restriction on $A$ is the identity map and the set of pairs $(r, s)$, where $r : V \to A$ and $s : V \to V$ are two linear maps satisfying

\begin{align*}
    r(x \triangleleft a) &= r(x) \cdot' a, \quad (31) \\
    r(a \triangleright x) &= a \cdot' r(x), \quad (32) \\
    a \cdot' b &= ab + r\theta(a, b), \quad (33) \\
    r(xy) &= r(x) \cdot' r(y), \quad (34) \\
    s(x) \triangleleft a + \theta'(r(x), a) &= s(x \triangleleft a), \quad (35) \\
    a \triangleright' s(y) + \theta'(a, r(y)) &= s(a \triangleright y), \quad (36) \\
    \theta'(a, b) &= s\theta(a, b), \quad (37) \\
    s(xy) &= s(x) \cdot' s(y) + s(x \triangleleft r(y) + r(x) \triangleright' s(y) + \theta'(r(x), r(y)). \quad (38)
\end{align*}

for all $a \in A$ and $x, y \in V$.

Under the above bijection the homomorphism of anti-flexible algebras $\varphi = \varphi_{r, s} : A\#_{a} V \to A\#_{a'} V$ to $(r, s)$ is given by $\varphi(a, x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. Moreover, $\varphi = \varphi_{r, s}$ is an isomorphism if and only if $s : V \to V$ is a linear isomorphism.

Proof. Let $\varphi : A\#_{a} V \to A\#_{a'} V$ be a homomorphism whose restriction on $A$ is the identity map. Then $\varphi$ is determined by two linear maps $r : V \to A$ and $s : V \to V$ such that $\varphi(a, x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. In fact, we have to show

$$
\varphi((a, x)(b, y)) = \varphi(a, x) \cdot' \varphi(b, y).
$$

The left hand side is equal to

\[
\varphi((a, x)(b, y)) = \varphi(ab, x \triangleleft b + y \triangleleft a + xy + \theta(a, b)) = (ab + r(x \triangleleft b) + r(y \triangleleft a) + r(xy) + r\theta(a, b), s(x \triangleleft b) + s(y \triangleleft a) + s(xy) + s\theta(a, b),)
\]

and the right hand side is equal to

\[
\varphi(a, x) \cdot' \varphi(b, y) = (a + r(x), s(x)) \cdot' (b + r(y), s(y)) = ((a + r(x)) \cdot' (b + r(y)), s(x) \triangleleft (b + r(y)) + (a + r(x)) \triangleright' s(y) + s(x) \cdot' s(y) + \theta'(a + r(x), b + r(y))).
\]

Thus $\varphi$ is a homomorphism of anti-flexible algebras if and only if the above conditions hold. $\blacksquare$

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The second case is when $\theta = 0$, we obtain the following type (a2) unified product for anti-flexible algebras.

**Theorem 5.4.** Let $A$ be an anti-flexible algebra and $V$ a vector space. An extending datum of $A$ through $V$ of type (a2) is a system $\Omega^{(2)}(A, V) = (\triangleright, \triangleright\triangleright, \leftrightarrow, \sigma, \cdot)$ consisting of linear maps

$$\triangleright : V \otimes A \rightarrow A, \quad \leftrightarrow : A \otimes V \rightarrow A, \quad \triangleright\triangleright : A \otimes V \rightarrow V$$

$$\sigma : V \otimes V \rightarrow A, \quad \cdot : V \otimes V \rightarrow V$$

Denote by $A_\sigma\#H$ the vector space $E = A \oplus V$ together with the multiplication

$$(a, x)(b, y) = (ab + x \mapsto b + a \mapsto y + \sigma(x, y), xy + x \triangleleft b + a \triangleright y). \quad (39)$$

Then $A_\sigma\#H$ is an anti-flexible algebra if and only if the following compatibility conditions hold for all $a, b, c \in A$, $x, y, z \in V$:

- **(B1)** $x \mapsto (ab) + (ba) \mapsto x = (x \mapsto a)b + b(a \mapsto x) + (x \triangleleft a) \mapsto b + b \mapsto (a \triangleright x)$,

- **(B2)** $(a \mapsto x)b + (a \triangleright x) \mapsto b - a(x \mapsto b) - a \mapsto (x \triangleleft b) = (b \triangleright x) \mapsto a + (b \mapsto x)a - b(x \mapsto a) - b \mapsto (x \triangleleft a)$,

- **(B3)** $(xy) \mapsto a + \sigma(x, y)a - x \mapsto (y \mapsto a) - \sigma(x, y \triangleleft a) = (a - y) \mapsto x - a \mapsto (yx) + \sigma(a \triangleright y, x) - a\sigma(y, x)$,

- **(B4)** $(x \mapsto a) \mapsto y + \sigma(x \triangleleft a, y) - x \mapsto (a \mapsto y) - \sigma(x, a \triangleright y) = (y \mapsto a) \mapsto x + \sigma(y \triangleleft a, x) - y \mapsto (a \mapsto x) - \sigma(y, a \triangleright x)$,

- **(B5)** $(xy) \triangleleft a + a \triangleright (yx) = x(y \triangleleft a) + (a \triangleright y)x + x \triangleleft (y \mapsto a) + (a \mapsto y)\triangleright x$,

- **(B6)** $(x \triangleleft a)y + (x \mapsto a)\triangleright y - x(a \triangleright y) - x \triangleleft (a \mapsto y) = (y \triangleleft a)x + (y \mapsto a)\triangleright x - y(a \triangleright x) - y \triangleleft (a \mapsto x)$,

- **(B7)** $(ab)\triangleright x - a \triangleright (b \triangleright x) = (x \triangleleft b) \triangleleft a - x \triangleleft (ba)$,

- **(B8)** $(a \triangleright x)\triangleleft b - a \triangleright (x \triangleleft b) = (b \triangleright x)\triangleleft a - b \triangleright (x \triangleleft a)$,

- **(B9)** $\sigma(xy, z) - \sigma(x, yz) + \sigma(x, y) \mapsto z - x \mapsto \sigma(y, z)$

$$= \sigma(zy, x) - \sigma(z, yx) + \sigma(z, y) \mapsto x - z \mapsto \sigma(y, x),$$

- **(B10)** $(xy)z - x(yz) + \sigma(x, y)\triangleright z - x \triangleleft \sigma(y, z) = (zy)x - z(yx) + \sigma(z, y)\triangleright x - z \triangleleft \sigma(y, x)$.

Similar to Theorem 5.2, we obtain.

**Theorem 5.5.** Let $A$ be an anti-flexible algebra, $E$ a vector space containing $A$ as a subspace. If there is an anti-flexible algebra structure on $E$ such that $A$ is an anti-flexible subalgebra of $E$. Then there exists an algebraic extending structure $\Omega^{(2)}(A, V) = (\triangleleft, \triangleright, \leftrightarrow, \sigma, \cdot)$ of $A$ through $V$ such that there is an isomorphism of anti-flexible algebras $E \cong A_\sigma\#H$.  

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Lemma 5.6. Let $\Omega^{(2)}(A,V) = (\sigma', \psi', \sigma, \psi)$ and $\Omega^{(2)}(A,V) = (\sigma', \psi', \sigma, \psi')$ be two algebraic extending structures of $A$ through $V$ and $A_\sigma \# V$, $A_\sigma' \# V$ the associated unified products. Then there exists a bijection between the set of all homomorphisms of anti-flexible algebras $\psi : A_\sigma \# V \rightarrow A_\sigma' \# V$ which stabilize $A$ and the set of pairs $(r,s)$, where $r : V \rightarrow A$, $s : V \rightarrow V$ are linear maps satisfying the following compatibility conditions for all $x \in A$, $u, v \in V$:

$(M1) \ r(x \cdot y) = r(x) \cdot r(y) + \sigma'(s(x), s(y)) - \sigma(x, y) + r(x) \leftarrow' s(y) + s(x) \rightarrow' r(y),$

$(M2) \ s(x \cdot y) = r(x) \triangleright s(y) + s(x) \triangleleft r(y) + s(x) \leftarrow' s(y),$

$(M3) \ r(x \cdot a) = r(x) \cdot a - x \rightarrow a + s(x) \rightarrow' a,$

$(M4) \ r(a \cdot x) = a \cdot r(x) - a \leftarrow x + a \rightarrow' s(x),$

$(M5) \ r(a \cdot x) = a \cdot r(x) - a \leftarrow x + a \rightarrow' a,$

Under the above bijection the homomorphism of anti-flexible algebras $\varphi = \varphi_{(r,s)} : A_\sigma \# H \rightarrow A_\sigma' \# H$ corresponding to $(r,s)$ is given for all $a \in A$ and $x \in V$ by:

$$\varphi(a, x) = (a + r(x), s(x)).$$

Moreover, $\varphi = \varphi_{(r,s)}$ is an isomorphism if and only if $s : V \rightarrow V$ is an isomorphism linear map.

Let $A$ be an anti-flexible algebra and $V$ a vector space. Two algebraic extending systems $\Omega^{(i)}(A,V)$ and $\Omega^{(i)}(A,V)$ are called equivalent if $\varphi_{(r,s)}$ is an isomorphism. We denote it by $\Omega^{(i)}(A,V) \equiv \Omega^{(i)}(A,V)$. From the above lemmas, we obtain the following result.

Theorem 5.7. Let $A$ be an anti-flexible algebra, $E$ a vector space containing $A$ as a subspace and $V$ be a complement of $A$ in $E$. Denote $\mathcal{H}A(V,A) := \Omega^{(1)}(A,V) \sqcup \Omega^{(2)}(A,V)/\equiv$. Then the map

$$\Psi : \mathcal{H}A(V,A) \rightarrow Extd(E,A),$$

$$\Omega^{(1)}(A,V) \mapsto A \#_0 V, \quad \Omega^{(2)}(A,V) \mapsto A_\sigma \# V$$

(40)

is bijective, where $\overline{\Omega^{(i)}(A,V)}$ is the equivalence class of $\Omega^{(i)}(A,V)$ under $\equiv$.

Next we consider the anti-flexible coalgebra structures on $E = A^p \#_Q V$.

There are two cases for $(A, \Delta_A)$ to be an anti-flexible coalgebra. The first case is when $Q = 0$, then we obtain the following type (c1) unified coproduct for anti-flexible coalgebras.

Lemma 5.8. Let $(A, \Delta_A)$ be an anti-flexible coalgebra and $V$ a vector space. An extending datum of $A$ by $V$ of type (c1) is $\Omega^{(3)}(A,V) = (\phi, \psi, \rho, \gamma, P, \Delta_V)$ with linear maps

$$\phi : A \rightarrow V \otimes A, \quad \psi : A \rightarrow A \otimes V,$$
\[\rho : V \to A \otimes V, \quad \gamma : V \to V \otimes A,\]
\[P : A \to V \otimes V, \quad \Delta_V : V \to V \otimes V.\]

Denote by \(A^P\#V\) the vector space \(E = A \oplus V\) with the linear map \(\Delta_E : E \to E \otimes E\) given by
\[\Delta_E(a) = (\Delta_A + \phi + \psi + P)(a), \quad \Delta_E(x) = (\Delta_V + \rho + \gamma)(x),\]
that is,
\[\Delta_E(a) = a_1 \otimes a_2 + a_{(-1)} \otimes a_{(0)} + a_{(0)} \otimes a_{(1)} + a_{<1>} \otimes a_{<2>},\]
\[\Delta_E(x) = x_1 \otimes x_2 + x_{[-1]} \otimes x_{[0]} + x_{[0]} \otimes x_{[1]},\]

Then \(A^P\#V\) is an anti-flexible coalgebra with the comultiplication given above if and only if the following compatibility conditions hold:

(C1) \[\phi(a_{(-1)}) \otimes a_{(0)} - a_{(-1)} \otimes \Delta_A(a_{(0)}) = \tau_{13}(\Delta_A(a_{(0)}) \otimes a_{(1)} - a_{(-1)} \otimes \psi(a_{(2)}) - a_{(0)} \otimes \rho(a_{(1)})),\]

(C2) \[\psi(a_{(-1)}) \otimes a_{(0)} + a_{(-1)} \otimes \phi(a_{(2)}) - a_{(0)} \otimes \gamma(a_{(1)}) = \tau_{13}(\psi(a_{(1)}) \otimes a_{(2)} + \rho(a_{<1>}) \otimes a_{<2>}) - a_{(0)} \otimes \Delta_V(a_{(1)}) - a_{(-1)} \otimes P(a_{(2)}),\]

(C3) \[a_{(-1)} \otimes \psi(a_{(0)}) + a_{<1>} \otimes \rho(a_{<2>}) - a_{(-1)} \otimes \phi(a_{(0)}) - a_{(-1)} \otimes a_{(1)} - \gamma(a_{<1>} \otimes a_{<2>} = \tau_{13}(a_{(-1)} \otimes \psi(a_{(1)}) + a_{<1>} \otimes \rho(a_{<2>}) - a_{(-1)} \otimes a_{(1)} - \gamma(a_{<1>} \otimes a_{<2}>),\]

(C5) \[\rho(x_{[0]}) \otimes x_{[1]} - x_{[-1]} \otimes \Delta_A(x_{[1]}),\]

(C6) \[\gamma(x_{[0]}) \otimes x_{[1]} = \tau_{13}(\gamma(x_{[0]}) \otimes x_{[1]} - x_{[-1]} \otimes \Delta_A(x_{[1]})),\]

(C8) \[\rho(x_{[0]}) \otimes x_{[1]} - x_{[-1]} \otimes \gamma(x_{[0]}),\]

(C9) \[\Delta_V(a_{<1>} \otimes a_{<2>}) = \tau_{13}(\Delta_V(a_{<1>}) \otimes a_{<2>} - a_{<1>} \otimes \Delta_V(a_{<2>}) = \tau_{13}(\Delta_V(a_{<1>}) \otimes a_{<2>} - a_{<1>} \otimes \Delta_V(a_{<2>}) = \tau_{13}(\Delta_V(a_{<1>}) \otimes a_{<2>} - a_{<1>} \otimes \Delta_V(a_{<2>}) = \tau_{13}(\Delta_V(a_{<1>}) \otimes a_{<2>} - a_{<1>} \otimes \Delta_V(a_{<2>}) = \tau_{13}(\Delta_V(a_{<1>}) \otimes a_{<2>} - a_{<1>}

Denote also the set of all coalgebraic extending datum of \(A\) by \(V\) of type (c1) by \(\Omega^{(3)}(A, V)\) by abuse of notations.
Lemma 5.9. Let \((A, \Delta_A)\) be an anti-flexible coalgebra and \(E\) a vector space containing \(A\) as a subspace. Suppose that there is an anti-flexible coalgebra structure \((E, \Delta_E)\) on \(E\) such that \(p : E \to A\) is an anti-flexible coalgebra homomorphism. Then there exists a coalgebraic extending system \(\Omega^{(3)}(A, V)\) of \((A, \Delta_A)\) by \(V\) such that \((E, \Delta_E) \cong A^P \# V\).

Proof. Let \(p : E \to A\) and \(\pi : E \to V\) be the projection map and \(V = \ker(p)\). Then the extending datum of \((A, \Delta_A)\) by \(V\) is defined as follows:

\[
\begin{align*}
\phi : A &\to V \otimes A, \quad \phi(a) = (\pi \otimes p)\Delta_E(a), \\
\psi : A &\to A \otimes V, \quad \psi(a) = (p \otimes \pi)\Delta_E(a), \\
\rho : V &\to A \otimes V, \quad \rho(x) = (p \otimes \pi)\Delta_E(x), \\
\gamma : V &\to V \otimes A, \quad \gamma(x) = (\pi \otimes p)\Delta_E(x), \\
\Delta_V : V &\to V \otimes V, \quad \Delta_V(x) = (\pi \otimes \pi)\Delta_E(x), \\
Q : V &\to A \otimes A, \quad Q(x) = (p \otimes p)\Delta_E(x) \\
P : A &\to V \otimes V, \quad P(a) = (\pi \otimes \pi)\Delta_E(a).
\end{align*}
\]

One check that \(\varphi : A^P \# V \to E\) given by \(\varphi(a, x) = a + x\) for all \(a \in A, x \in V\) is an anti-flexible coalgebra isomorphism. \(\square\)

Lemma 5.10. Let \(\Omega^{(3)}(A, V) = (\phi, \psi, \rho, \gamma, P, \Delta_V)\) and \(\Omega'^{(3)}(A, V) = (\phi', \psi', \rho', \gamma', P', \Delta'_V)\) be two coalgebraic extending datums of \((A, \Delta_A)\) by \(V\). Then there exists a bijection between the set of anti-flexible coalgebra homomorphisms \(\varphi : A^P \# V \to A'^P \# V\) whose restriction on \(A\) is the identity map and the set of pairs \((r, s)\), where \(r : V \to A\) and \(s : V \to V\) are two linear maps satisfying

\[
\begin{align*}
P'(a) &= s(a_{<1>}) \otimes s(a_{<2>}), \\
\phi'(a) &= s(a_{<-1>}) \otimes a(0) + s(a_{<1>}) \otimes r(a_{<2>}), \\
\psi'(a) &= a(0) \otimes s(a_{<1>}) + r(a_{<1>}) \otimes s(a_{<2>}), \\
\Delta'_A(a) &= \Delta_A(a) + r(a_{<-1>}) \otimes a(0) + a(0) \otimes r(a_{<1>}) + r(a_{<1>}) \otimes r(a_{<2>}), \\
\Delta'_V(s(x)) + P'(r(x)) &= (s \otimes s)\Delta_V(x), \\
\rho'(s(x)) + \psi'(r(x)) &= r(x_1) \otimes s(x_2) + x_{[1]} \otimes s(x_{[0]}), \\
\gamma'(s(x)) + \phi'(r(x)) &= s(x_1) \otimes r(x_2) + s(x_{[0]}) \otimes x_{[1]}, \\
\Delta'_A(r(x)) &= r(x_1) \otimes r(x_2) + x_{[1]} \otimes r(x_{[0]}) + r(x_{[0]}) \otimes x_{[1]}.
\end{align*}
\]

Under the above bijection the anti-flexible coalgebra homomorphism \(\varphi = \varphi_{r, s} : A^P \# V \to A'^P \# V\) to \((r, s)\) is given by \(\varphi(a, x) = (a + r(x), s(x))\) for all \(a \in A\) and \(x \in V\). Moreover, \(\varphi = \varphi_{r, s}\) is an isomorphism if and only if \(s : V \to V\) is a linear isomorphism.

Proof. Let \(\varphi : A^P \# V \to A'^P \# V\) be a coalgebra homomorphism whose restriction on \(A\) is the identity map. Then \(\varphi\) is determined by two linear maps \(r : V \to A\) and \(s : V \to V\) such that \(\varphi(a, x) = (a + r(x), s(x))\) for all \(a \in A\) and \(x \in V\). We will prove that \(\varphi\) is a homomorphism

\(\square\)
of anti-flexible coalgebras if and only if the above conditions hold. First it is easy to see that
\( \Delta_E' \varphi(a) = (\varphi \otimes \varphi) \Delta_E(a) \) for all \( a \in A. \)

\[
\Delta_E' \varphi(a) = \Delta_E'(a) = \Delta_A'(a) + \phi'(a) + \psi'(a) + P'(a),
\]
and

\[
(\varphi \otimes \varphi) \Delta_E(a) = \Delta_E(r(x), s(x)) = \Delta_E(r(x)) + \Delta_E(s(x)) = \Delta_A'(r(x)) + \phi'(r(x)) + \psi'(r(x)) + P'(r(x)) + \Delta'_V(s(x)) + \rho'(s(x)) + \gamma'(s(x)),
\]
and

\[
(\varphi \otimes \varphi) \Delta_E(x) = (\varphi \otimes \varphi)(\Delta_V(x) + \rho(x) + \gamma(x))
\]

Thus we obtain that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) if and only if the conditions (45), (46), (47) and (48) hold. Then we consider that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) for all \( x \in V. \)

Thus we obtain that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) if and only if the conditions (41), (42), (43) and (44) hold. Then we consider that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) for all \( x \in V. \)

Thus we obtain that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) if and only if the conditions (41), (42), (43) and (44) hold. Then we consider that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) for all \( x \in V. \)

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Thus we obtain that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) if and only if the conditions (41), (42), (43) and (44) hold. Then we consider that \( \Delta_E' \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) for all \( x \in V. \)

The second case is \( \phi = 0 \) and \( \psi = 0 \), and from (CCP3), we get \( P = 0 \) when \( Q \neq 0 \). Then we obtain the following type (c2) unified coproduct for anti-flexible coalgebras.

**Lemma 5.11.** Let \((A, \Delta_A)\) be an anti-flexible coalgebra and \(V\) a vector space. An extending datum of \((A, \Delta_A)\) by \(V\) of type (c2) is \(\Omega^{(4)}(A, V) = (\rho, \gamma, Q, \Delta_V)\) with linear maps

\[
\rho : V \to A \otimes V, \quad \gamma : V \to V \otimes A, \quad \Delta_V : V \to V \otimes V, \quad Q : V \to A \otimes A.
\]

Denote by \( A^\#QV \) the vector space \(E = A \oplus V\) with the comultiplication \(\Delta_E : E \to E \otimes E\) given by

\[
\Delta_E(a) = \Delta_A(a), \quad \Delta_E(x) = (\Delta_V + \rho + \gamma + Q)(x),
\]

\[
\Delta_E(a_1 \otimes a_2) = a_1 \otimes a_2, \quad \Delta_E(x_1 \otimes x_2 + x_{[-1]} \otimes x_{[0]} + x_{[0]} \otimes x_{[1]} + x_{[1]} \otimes x_{[2]}).
\]

Then \( A^\#QV \) is an anti-flexible coalgebra with the comultiplication given above if and only if the following compatibility conditions hold:
Let \( \rho(x_1) \otimes x_2 - x_{[-1]} \otimes \Delta_V(x_0) = \tau_{13}(\Delta_V(x_0) \otimes x_{[1]} - x_1 \otimes \gamma(x_2)) \),

\( \gamma(x_1) \otimes x_2 - x_{[-1]} \otimes \rho(x_2) = \tau_{13}(\gamma(x_1) \otimes x_2 - x_1 \otimes \rho(x_2)) \),

\( \Delta_A(x_{[-1]}) \otimes x_0 + Q(x_1) \otimes x_2 - x_{[-1]} \otimes \rho(x_0) = \tau_{13}(\gamma(x_0) \otimes x_{[1]} - x_0 \otimes \Delta_A(x_{[1]}) - x_1 \otimes Q(x_2)) \),

\( \rho(x_0) \otimes x_{[1]} - x_{[-1]} \otimes \gamma(x_0) = \tau_{13}(\rho(x_0) \otimes x_{[1]} - x_{[-1]} \otimes \gamma(x_0)) \).

\( \Delta_A(x_{[1]}) \otimes x_{[2]} - x_{[1]} \otimes \Delta_A(x_{[2]}) + Q(x_0) \otimes x_{[1]} - x_{[-1]} \otimes Q(x_0) = \tau_{13}(\Delta_A(x_{[1]}) \otimes x_{[2]} - x_{[1]} \otimes \Delta_A(x_{[2]}) + Q(x_0) \otimes x_{[1]} - x_{[-1]} \otimes Q(x_0)) \).

\( \Delta_V(x_1) \otimes x_2 - x_1 \otimes \Delta_V(x_2) = \tau_{13}(\Delta_V(x_1) \otimes x_2 - x_1 \otimes \Delta_V(x_2)) \).

Note that in this case \((V, \Delta_V)\) is an anti-flexible coalgebra.

Denote the set of all coalgebraic extending datum of \( A \) by \( V \) of type \((c_2)\) by \( \Omega^{(4)}(A, V) \). Similar to the anti-flexible algebra case, one show that any anti-flexible coalgebra structure on \( E \) containing \( A \) as an anti-flexible subcoalgebra is isomorphic to such an unified coproduct.

**Lemma 5.12.** Let \((A, \Delta_A)\) be an anti-flexible coalgebra and \( E \) a vector space containing \( A \) as a subspace. Suppose that there is an anti-flexible coalgebra structure \((E, \Delta_E)\) on \( E \) such that \((A, \Delta_A)\) is an anti-flexible subcoalgebra of \( E \). Then there exists a coalgebraic extending system \( \Omega^{(4)}(A, V) \) of \((A, \Delta_A)\) by \( V \) such that \((E, \Delta_E) \cong A \#^Q V \).

**Proof.** Let \( p : E \to A \) and \( \pi : E \to V \) be the projection map and \( V = \ker(p) \). Then the extending datum of \((A, \Delta_A)\) by \( V \) is defined as follows:

\[
\rho : V \to A \otimes V, \quad \phi(x) = (p \otimes \pi)\Delta_E(x),
\]

\[
\gamma : V \to V \otimes A, \quad \phi(x) = (\pi \otimes p)\Delta_E(x),
\]

\[
\Delta_V : V \to V \otimes V, \quad \Delta_V(x) = (\pi \otimes \pi)\Delta_E(x),
\]

\[
Q : V \to A \otimes A, \quad Q(x) = (p \otimes p)\Delta_E(x).
\]

One check that \( \varphi : A \#^Q V \to E \) given by \( \varphi(a, x) = a + x \) for all \( a \in A, x \in V \) is an anti-flexible coalgebra isomorphism. \( \square \)

**Lemma 5.13.** Let \( \Omega^{(4)}(A, V) = (\rho, \gamma, Q, \Delta_V) \) and \( \Omega'^{(4)}(A, V) = (\rho', \gamma', Q', \Delta'_V) \) be two coalgebraic extending datums of \((A, \Delta_A)\) by \( V \). Then there exists a bijection between the set of coalgebra homomorphisms \( \varphi : A \#^Q V \to A \#^Q V \) whose restriction on \( A \) is the identity map and the set of pairs \((r, s)\), where \( r : V \to A \) and \( s : V \to V \) are two linear maps satisfying

\[
\rho'(s(x)) = r(x_1) \otimes s(x_2) + x_{[-1]} \otimes s(x_0),
\]

\[
\gamma'(s(x)) = s(x_1) \otimes r(x_2) + s(x_0) \otimes x_{[1]},
\]

\[
\Delta'_V(s(x)) = (s \otimes s)\Delta_V(x)
\]

\[
\Delta'_A(r(x)) + Q'(s(x)) = r(x_1) \otimes r(x_2) + x_{[-1]} \otimes r(x_0) + r(x_0) \otimes x_{[1]} + Q(x).
\]
Under the above bijection the coalgebra homomorphism \( \varphi = \varphi_{r,s} : A \#^Q V \to A \#^Q V \) to \((r,s)\) is given by \( \varphi(a,x) = (a + r(x), s(x)) \) for all \( a \in A \) and \( x \in V \). Moreover, \( \varphi = \varphi_{r,s} \) is an isomorphism if and only if \( s : V \to V \) is a linear isomorphism.

**Proof.** The proof is similar to the proof of Lemma 5.10. Let \( \varphi : A \#^Q V \to A \#^Q V \) be a coalgebra homomorphism whose restriction on \( A \) is the identity map. First it is easy to see that \( \Delta'_E \varphi = (\varphi \otimes \varphi) \Delta_E(a) \) for all \( a \in A \). Then we consider that \( \Delta'_E \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) for all \( x \in V \).

\[
\Delta'_E \varphi(x) = \Delta'_E(r(x), s(x)) = \Delta'_E(r(x)) + \Delta'_E(s(x)) = \Delta'_A(r(x)) + \Delta'_V(s(x)) + \rho(s(x)) + \gamma'(s(x)) + Q'(s(x)),
\]
and

\[
(\varphi \otimes \varphi) \Delta_E(x) = (\varphi \otimes \varphi)(\Delta_V(x) + \rho(x) + \gamma(x) + Q(x)) = (\varphi \otimes \varphi)(x_1 \otimes x_2 + x_{-1} \otimes x_{0} + x_{0} \otimes x_{1} + Q(x)) = r(x_1) \otimes r(x_2) + r(x_1) \otimes s(x_2) + s(x_1) \otimes r(x_2) + s(x_1) \otimes s(x_2) + x_{-1} \otimes r(x_{0}) + x_{-1} \otimes s(x_{0}) + r(x_{0}) \otimes x_{1} + s(x_{0}) \otimes x_{1} + Q(x).
\]

Thus we obtain that \( \Delta'_E \varphi(x) = (\varphi \otimes \varphi) \Delta_E(x) \) if and only if the conditions (51), (52), (53) and (54) hold. By definition, we obtain that \( \varphi = \varphi_{r,s} \) is an isomorphism if and only if \( s : V \to V \) is a linear isomorphism.

Let \((A, \Delta_A)\) be an anti-flexible coalgebra and \( V \) a vector space. Two coalgebraic extending systems \( \Omega^{(i)}(A,V) \) and \( \Omega^{(i)}(A,V) \) are called equivalent if \( \varphi_{r,s} \) is an isomorphism. We denote it by \( \Omega^{(i)}(A,V) \equiv \Omega^{(i)}(A,V) \). From the above lemmas, we obtain the following result.

**Theorem 5.14.** Let \((A, \Delta_A)\) be an anti-flexible coalgebra, \( E \) a vector space containing \( A \) as a subspace and \( V \) be a \( A \)-complement in \( E \). Denote \( \mathcal{HC}(V,A) := \Omega^{(3)}(A,V) \uplus \Omega^{(4)}(A,V) \). Then the map

\[
\Psi : \mathcal{HC}(V,A) \to C\text{Extd}(E,A), \quad \Omega^{(3)}(A,V) \to A \#^P V, \quad \Omega^{(4)}(A,V) \to A \#^Q V
\]

is bijective, where \( \Omega^{(i)}(A,V) \) is the equivalence class of \( \Omega^{(i)}(A,V) \) under \( \equiv \).

### 5.2 Extending structures for anti-flexible bialgebras

Let \((A, \cdot, \Delta_A)\) be an anti-flexible bialgebra. From (CBB1) we have the following two cases.

The first case is that we assume \( Q = 0 \) and \( - \to \) to be trivial. Then by the above Theorem 4.16 we obtain the following result.
Theorem 5.15. Let \((A, \cdot, \Delta_A)\) be an anti-flexible bialgebra and \(V\) a vector space. An extending datum of \(A\) by \(V\) of type (I) is \(\Omega^{(I)}(A, V) = (\triangleright, \triangleleft, \theta, \phi, \psi, P, \rho, \gamma, \Delta_V)\) consisting of linear maps

\[
\triangleright: A \otimes V \to V, \quad \triangleleft: V \otimes A \to V, \quad \theta: A \otimes A \to V, \quad P: A \to V \otimes V, \quad \gamma: V \otimes V \to V, \\
\phi: A \to V \otimes A, \quad \psi: A \otimes A \to V, \quad \rho: V \to A \otimes V, \quad \Delta_V: V \otimes V \to V.
\]

Then the unified biproduct \(A^P \#_\theta V\) with multiplication

\[(a, x)(b, y) := (ab, xy + a \triangleright y + x \triangleleft b + \theta(a, b)) \quad (55)\]

and comultiplication

\[
\Delta_E(a) = \Delta_A(a) + \phi(a) + \psi(a) + P(a), \quad \Delta_E(x) = \Delta_V(x) + \rho(x) + \gamma(x) \quad (56)
\]

forms an anti-flexible bialgebra if and only if \(A \#_\theta V\) forms an anti-flexible algebra, \(A^P \# V\) forms an anti-flexible coalgebra and the following conditions are satisfied:

\[\begin{align*}
(E1) & \quad \phi(ab) + \tau \psi(ba) + \gamma(\theta(a, b)) + \tau \rho(\theta(b, a)) \\
& = \theta(b, a_2) \otimes a_1 + (b \triangleright a_{(1)}) \otimes a_{(0)} + b_{(1)} \otimes b_{(0)} a \\
& + \theta(a_1, b) \otimes a_2 + (a_{(-1)} \triangleleft b) \otimes a_{(0)} + b_{(-1)} \otimes ab_{(0)}, \\
(E2) & \quad \rho(xy) + \tau \gamma(yx) = y_{[1]} \otimes y_{[0]} x + y_{[-1]} \otimes xy_{[0]}, \\
(E3) & \quad \Delta_V(a \triangleright y) + \tau \Delta_V(y \triangleleft a) \\
& = (y \triangleleft a_{(0)}) \otimes a_{(-1)} + ya_{<2>} \otimes a_{<1>} + y_2 \otimes (y_1 \triangleleft a) + y_{[0]} \otimes \theta(y_{[-1]}, a) \\
& + (a_{(0)} \triangleright y) \otimes a_{(1)} + a_{<1>} y \otimes a_{<2>} + y_1 \otimes (a \triangleright y_2) + y_{[0]} \otimes \theta(a, y_{[1]}), \\
(E4) & \quad \Delta_V(\theta(a, b)) + \tau \Delta_V(\theta(b, a)) + P(ab) + \tau P(ba) \\
& = \theta(b, a_{(0)}) \otimes a_{(-1)} + (b \triangleright a_{<2>}) \otimes a_{<1>} + b_{(1)} \otimes \theta(b_{(0)}, a) + b_{<2>} \otimes (b_{<1>} \triangleright a) \\
& + \theta(a_{(0)}, b) \otimes a_{(1)} + (a_{<1>} \triangleleft b) \otimes a_{<2>} + b_{(-1)} \otimes \theta(a, b_{(0)}) + b_{<1>} \otimes (a \triangleright b_{<2>}), \\
(E5) & \quad \gamma(x \triangleleft b) + \tau \rho(b \triangleright x) = (b \triangleright x_{[0]}) \otimes x_{[-1]} + (x_{[0]} \triangleleft b) \otimes x_{[1]}, \\
(E6) & \quad \rho(x \triangleleft b) + \tau \gamma(b \triangleright x) \\
& = bx_{[1]} \otimes x_{[0]} + b_2 \otimes (b_1 \triangleright x) + b_{(0)} \otimes b_{(-1)} x \\
& + x_{[-1]} b \otimes x_{[0]} + b_1 \otimes (x \triangleleft b_2) + b_{(0)} \otimes xb_{(1)}, \\
(E7) & \quad a_1 \otimes \theta(a_2, b) + a_{(0)} \otimes (a_{(1)} \triangleleft b) + ab_{(0)} \otimes b_{(1)} \\
& - b_{(1)} \otimes \theta(b_2, a) - b_{(0)} \otimes (b_{(1)} \triangleleft a) - ba_{(0)} \otimes a_{(1)} \\
& = \tau\left(a_{(-1)} a_{(0)} b + \theta(a_{(1)}, b_1) \otimes b_2 + (a \triangleright b_{(-1)}) \otimes b_{(0)} \\
& - b_{(-1)} \otimes b_{(0)} a - \theta(b_{(1)} a_{(0)} \otimes a_{(2)} - (b \triangleright a_{(1)}) \otimes a_{(0)})\right), \\
(E8) & \quad (\text{id} - \tau)(a_{(-1)} \otimes \theta(a_{(0)}, b) + a_{<1>} \otimes (a_{<2>} \triangleleft b) + \theta(a, b_{(0)}) \otimes b_{(1)} + (a \triangleright b_{<1>}) \otimes b_{<2>}) \\
& = (\text{id} - \tau)(b_{(-1)} \otimes \theta(b_{(0)}, a) + b_{<1>} \otimes (b_{<2>} \triangleleft a) + \theta(b, a_{(0)}) \otimes a_{(1)} + (b \triangleright a_{<1>}) \otimes a_{<2>}),
\end{align*}\]
\[(E9)\] \[x_0 \otimes x_1 + (a \triangleleft b_1) \otimes b_2 + x_1 \delta \otimes b_0 - (b \triangleright x_0) \otimes x_1 \]
\[= \tau \left( x_0 \otimes x_1 \right) \otimes b - b_1 \otimes (b_2 \triangleright x) - b_0 \otimes b_1 x - bx_0 \otimes x_0 \],

\[(E10)\] \[(\text{id} - \tau)(x_0 \otimes (x_0 \otimes \theta(x_1), b) + (x \otimes b_0) \otimes b_1) + xb_{<1>} \otimes x_{<2>}) = (\text{id} - \tau)(b_1 \otimes (b_0 \triangleright x) + b_{<1>} \otimes b_{<2>} x) + (b_1 \otimes x_1) \otimes x_2 + \theta(b, x_{[-1]} \otimes x_0),
\]

\[(E12)\] \[xy_0 \otimes y_1 - yx_0 \otimes x_1 = \tau \left( x_1 \otimes x_0 y - y_1 \otimes y_0 x \right),
\]

\[(E13)\] \[\Delta_V(xy) + \tau \Delta_V(yx) = yx_2 \otimes x_1 + y_2 \otimes y_1 x + x_1 y \otimes x_2 + y_1 \otimes y_2 \]
\[+ (y \otimes x_1) \otimes x_0 + y_0 \otimes (y_{[-1]} \triangleright x) + (x_{[-1]} \triangleright y) \otimes x_0 + y_0 \otimes (x \otimes y_{[1]}),
\]

\[(E14)\] \[(\text{id} - \tau)(x_0 \otimes x_2 y + xy_1 \otimes y_2 - y_1 \otimes y_2 x - yx_1 \otimes x_2) + (\text{id} - \tau)(x_0 \otimes (x_{[1]} \triangleright y) + (x \otimes y_{[-1]} \otimes y_0 \otimes (y_{[1]} \triangleright x) - (y \otimes x_{[-1]} \otimes x_0) = 0.
\]

Conversely, any anti-flexible bialgebra structure on $E$ with the canonical projection map $p : E \to A$ both an algebra homomorphism and a coalgebra homomorphism is of this form.

Note that in this case, $(V, \cdot, \Delta_V)$ is a braided anti-flexible bialgebra. Although $(A, \cdot, \Delta_A)$ is not a sub-bialgebra of $E = AP\#_\theta V$, but it is indeed an anti-flexible bialgebra and a subspace $E$. Denote the set of all anti-flexible bialgebraic extending datum of type (I) by $\Omega^{(I)}(A, V)$.

The second case is that we assume $P = 0, \theta = 0$ and $\phi, \psi$ to be trivial. Then by the above Theorem 4.16, we obtain the following result.

**Theorem 5.16.** Let $A$ be a anti-flexible bialgebra and $V$ a vector space. An extending datum of $A$ by $V$ of type (II) is $\Omega^{(II)}(A, V) = (\to, \leftarrow, \cdot, \cdot, \sigma, \rho, \gamma, Q, \gamma, \Delta_V)\) consisting of linear maps

$\to : V \otimes A \to A, \leftarrow : A \otimes V \to A, \cdot : V \otimes A \to V, \cdot : A \otimes V \to V, \cdot : V \otimes V \to V, \Delta_V : V \otimes V \to A$.

Then the unified biproduct $A_\sigma \#^Q V$ with multiplication

\[(a, x)(b, y) = (ab + x - a < b + a \triangleright y, xy + x \triangleleft b + a \triangleright y), \quad \quad (57)\]

and comultiplication

\[\Delta_E(a) = \Delta_A(a), \quad \Delta_E(x) = \Delta_V(x) + \rho(x) + \gamma(x) + Q(x) \quad (58)\]

forms an anti-flexible bialgebra if and only if $A_\sigma \# V$ forms an anti-flexible algebra, $A \#^Q V$ forms an anti-flexible coalgebra and the following conditions are satisfied:

\[(F1) \quad \rho(xy) + \gamma(yx) = \sigma(y, x_2) \otimes x_1 + (y \otimes x_1) \otimes x_0 + y_{[1]} \otimes y_0 x + y_{[2]} \otimes (y_{[1]} \triangleright x) + \sigma(x_1, y) \otimes x_2 + (x_{[1]} \leftarrow y) \otimes x_0 + y_{[-1]} \otimes xy_0 + y_{[1]} \otimes (x \triangleleft y_{[2]}), \quad \quad (59)\]

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\[(F2)\quad \Delta_A(x \leftarrow b) + \tau \Delta_A(b \leftarrow x) + Q(x \triangleleft b) + \tau Q(b \triangleright x)
= (b \leftarrow x_0) \otimes x_{[-1]} + bx_{[2]} \otimes x_{[1]} + b_2 \otimes (b_1 \leftarrow x)
+ (x_{[0]} \rightarrow b) \otimes x_{[1]} + x_{[1]}b \otimes x_{[2]} + b_1 \otimes (x \rightarrow b_2),
\]

\[(F3)\quad \Delta_V(a \triangleright y) + \tau \Delta_V(y \triangleleft a) = y_2 \otimes (y_1 \triangleleft a) + y_1 \otimes (a \triangleright y_2),
\]

\[(F4)\quad \Delta_A(\sigma(x, y)) + \tau \Delta_A(\sigma(y, x)) + Q(xy) + \tau Q(yx)
= (y \rightarrow x_{[2]} \otimes x_{[1]} + \sigma(y, x_{[0]} \otimes x_{[-1]} + y_{[1]} \otimes \sigma(y_{[0]}, x) + y_{[2]} \otimes (y_{[1]} \leftarrow x)
+ \sigma(x_{[0]}, y) \otimes x_{[1]} + (x_{[1]} \leftarrow y) \otimes x_{[2]} + y_{[-1]} \otimes \sigma(x, y_{[0]} + y_{[1]} \otimes (x \rightarrow y_{[2]})),
\]

\[(F5)\quad \gamma(x \triangleleft b) + \tau \rho(b \triangleright x) = (b \triangleright x_{[0]}) \otimes x_{[-1]} + (x_{[0]} \triangleleft b) \otimes x_{[1]},
\]

\[(F6)\quad \rho(x \triangleleft b) + \tau \gamma(b \triangleright x)
= (b \leftarrow x_{[2]} \otimes x_{[1]} + bx_{[1]} \otimes x_{[0]} + b_2 \otimes (b_1 \triangleright x)
+ (x_1 \rightarrow b) \otimes x_2 + x_{[-1]}b \otimes x_{[0]} + b_1 \otimes (x \triangleleft b_2),
\]

\[(F7)\quad x_1 \otimes (x_2 \rightarrow b) + x_{[0]} \otimes x_{[1]} \leftarrow b_{[-1]} \otimes x_{[0]} \otimes x_{[1]}
= \tau(\sigma(x_{[-1]} \otimes (x_{[0]} \triangleleft b) - b_1 \otimes (b_2 \triangleright x) - (b \leftarrow x_1) \otimes x_2 - bx_{[-1]} \otimes x_{[0]}),
\]

\[(F8)\quad (\text{id} - \tau)(x_1 \otimes (x_2 \triangleleft b) - (b \triangleright x_1) \otimes x_2) = 0,
\]

\[(F9)\quad x_1 \otimes \sigma(x_2, y) + x_{[0]} \otimes (x_{[1]} \leftarrow y) + xy_{[0]} \otimes y_{[1]} + (x \triangleleft y_{[1]}) \otimes y_{[2]}
= (y_1 \otimes \sigma(y_2, x) - y_1y_{[0]} \otimes (y_{[1]} \leftarrow x) - yx_{[0]} \otimes x_{[1]} - (y \triangleleft x_{[1]}) \otimes x_{[2]})
= \tau(x_{[-1]} \otimes x_{[0]} y + x_{[1]} \otimes (x_2 \triangleright y) + \sigma(x, y_1) \otimes y_2 + (x \rightarrow y_{[-1]}) \otimes y_{[0]}
- y_{[-1]} \otimes y_{[0]} x - y_{[1]} \otimes (y_{[2]} \triangleright x) - \sigma(y, x_1) \otimes x_2 - (y \rightarrow x_{[-1]}) \otimes x_{[0]},
\]

\[(F10)\quad (\text{id} - \tau)(x_{[-1]} \otimes (x_{[0]} \rightarrow b) + x_{[1]} \otimes x_{[2]} b + (x \rightarrow b_1) \otimes b_2
= (\text{id} - \tau)(b_1 \otimes (b_2 \leftarrow x) + (b \leftarrow x_{[0]}) \otimes x_{[1]} + bx_{[1]} \otimes x_{[2]}),
\]

\[(F11)\quad (\text{id} - \tau)(x_{[-1]} \otimes \sigma(x_{[0]}, y) + x_{[1]} \otimes (x_{[2]} \leftarrow y) + \sigma(x, y_{[0]}) \otimes y_{[1]} + (x \rightarrow y_{[1]})) \otimes y_{[2]}
= (\text{id} - \tau)(y_{[-1]} \otimes \sigma(y_{[0]}, x) + y_{[1]} \otimes (y_{[2]} \leftarrow x) + \sigma(y, x_{[0]}) \otimes x_{[1]} + (y \rightarrow x_{[-1]})) \otimes x_{[2]},
\]

\[(F12)\quad \Delta_V(xy) + \tau \Delta_V(yx)
= yx_{[2]} \otimes x_{[1]} + y_2 \otimes y_1 x + x_{[1]} y \otimes x_{[2]} + y_1 \otimes yx_2
+ (y \triangleleft x_{[1]}) \otimes x_{[0]} + y_{[0]} \otimes (y_{[-1]} \triangleright x) + (x_{[-1]} \triangleright y) \otimes x_{[0]} + y_0 \otimes (x \triangleleft y_{[1]}),
\]

\[(F13)\quad (\text{id} - \tau)(x_1 \otimes x_2 y + xy_{[1]} \otimes y_{[2]} - y_1 \otimes y_2 x - yx_1 \otimes x_2)
+ (\text{id} - \tau)(x_{[0]} \otimes (x_{[1]} \triangleright y) + (x \triangleleft y_{[-1]})) \otimes y_{[0]} - y_0 \otimes (y_{[1]} \triangleright x) - (y \triangleleft x_{[-1]}) \otimes x_{[0]} = 0.
\]

Conversely, any anti-flexible bialgebra structure on \(E\) with the canonical injection map \(i : A \rightarrow E\) both an anti-flexible algebra homomorphism and an anti-flexible coalgebra homomorphism is of this form.

Note that in this case, \((A, \cdot, \Delta_A)\) is an anti-flexible sub-bialgebra of \(E = A_\sigma \#^Q V\) and \((V, \cdot, \Delta_V)\) is a braided anti-flexible bialgebra. Denote the set of all anti-flexible bialgebraic extending datum of type (II) by \(\Omega^{(II)}(A, V)\).
In the above two cases, we find that the braided anti-flexible bialgebra $V$ play a special role in the extending problem of anti-flexible bialgebra $A$. Note that $A^P \#_\theta V$ and $A_\sigma \#^Q V$ are all anti-flexible bialgebra structures on $E$. Conversely, any anti-flexible bialgebra extending system $E$ of $A$ through $V$ is isomorphic to such two types. Now from Theorem 5.15 and Theorem 5.16 we obtain the main result of in this section, which solve the extending problem for anti-flexible bialgebra. 

**Theorem 5.17.** Let $(A, \cdot, \Delta_A)$ be an anti-flexible bialgebra, $E$ a vector space containing $A$ as a subspace and $V$ be a complement of $A$ in $E$. Denote by

$$\mathcal{HLB}(A,V) := \Omega^{(I)}(A,V) \sqcup \Omega^{(II)}(A,V) / \equiv.$$ 

Then the map

$$\Upsilon : \mathcal{HLB}(A,V) \rightarrow BExtd(E,A),$$

$$\overline{\Omega^{(I)}(A,V)} \mapsto A^P \#_\theta V, \quad \overline{\Omega^{(II)}(A,V)} \mapsto A_\sigma \#^Q V$$

is bijective, where $\overline{\Omega^{(I)}(A,V)}$ and $\overline{\Omega^{(II)}(A,V)}$ are the equivalence classes of $\Omega^{(I)}(A,V)$ and $\Omega^{(II)}(A,V)$ under $\equiv$ respectively.

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