TRIVIALITY RESULTS FOR COMPACT $k$-YAMABE SOLITONS

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Abstract. In this paper, we show that any compact gradient $k$-Yamabe soliton must have constant $\sigma_k$-curvature. Moreover, we provide a certain condition for a compact $k$-Yamabe soliton to be gradient.

1. Introduction and main results

The concept of gradient $k$-Yamabe soliton, introduced in the celebrated work [3], corresponds to a natural generalization of gradient Yamabe solitons. We recall that a Riemannian manifold $(M^n, g)$ is a $k$-Yamabe soliton if it admits a constant $\lambda \in \mathbb{R}$ and a vector field $X \in \mathfrak{X}(M)$ satisfying the equation

$$\frac{1}{2} \mathcal{L}_X g = 2(n-1)(\sigma_k - \lambda)g,$$

where $\mathcal{L}_X g$ and $\sigma_k$ stand, respectively, for the Lie derivative of $g$ in the direction of $X$ and the $\sigma_k$-curvature of $g$. Recall that, if we denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of the symmetric endomorphism $g^{-1}A_g$, where $A_g$ is the Schouten tensor defined by

$$A_g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{\text{scal}_g}{2(n-1)} g \right),$$

then the $\sigma_k$-curvature of $g$ is defined as the $k$-th symmetric elementary function of $\lambda_1, \ldots, \lambda_n$, namely

$$\sigma_k = \sigma_k(g^{-1}A_g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \text{for} \quad 1 \leq k \leq n.$$

Since $\sigma_1$ is the trace of $g^{-1}A_g$, the 1-Yamabe solitons simply correspond to gradient Yamabe solitons [1, 2, 3, 4, 5, 6, 7, 8, 9]. For simplicity, the soliton will be denoted by $(M^n, g, X, \lambda)$. It may happen that $X = \nabla f$ is the gradient field of a smooth real function $f$ on $M$, in which case the soliton $(M^n, g, \nabla f, \lambda)$ is referred to as a gradient $k$-Yamabe soliton. Equation (1) then becomes

$$\nabla^2 f = 2(n-1)(\sigma_k - \lambda)g,$$

where $\nabla^2 f$ is the Hessian of $f$. Moreover, when either $f$ is a constant function or $X$ is a Killing vector field, the soliton is called trivial and, in this case, the metric $g$ is of constant $k$-curvature $\sigma_k = \lambda$.

In recent years, much efforts have been devoted to study the geometry of $k$-Yamabe solitons. For instance, Hsu in [9] shown that any compact gradient 1-Yamabe soliton is trivial. For $k > 1$, the extension of the previous result was investigated by Catino et al. [3], and Bo et al. [2]. In [3], the authors proved that any compact, gradient $k$-Yamabe soliton with nonnegative Ricci tensor is trivial. On the other hand, the authors in [2] showed that any compact, gradient $k$-Yamabe soliton with constant negative scalar curvature must be trivial.

In this paper, we extend the above results as follows.

Theorem 1.1. Any compact gradient $k$-Yamabe soliton $(M^n, g, \nabla f, \lambda)$ is trivial, i.e., has constant $\sigma_k$-curvature $\sigma_k = \lambda$.

In the scope of $k$-Yamabe solitons, we provide the following extension of Theorem 1.3 in [2].

Theorem 1.2. The compact $k$-Yamabe soliton $(M^n, g, X, \lambda)$ is trivial if one of the following conditions holds:

(a) $k = 1$.

(b) $k \geq 2$ and $(M^n, g)$ is locally conformally flat.
The Hodge-de Rham decomposition theorem (see [1, 15]), shows that any vector field \( X \) on a compact oriented Riemannian manifold \( M \) can be decompose as follows:

\[
X = \nabla h + Y,
\]

where \( h \) is a smooth function on \( M \) and \( Y \in \mathfrak{X}(M) \) is a free divergence vector field. Indeed, just consider the 1-form \( X^\flat \). Hence applying the Hodge-de Rham theorem, we decompose \( X^\flat \) as follows:

\[
X^\flat = d\alpha + \delta\beta + \gamma.
\]

Taking \( Y = (\delta\beta + \gamma)^\flat \) and \( (d\alpha)^\flat = \nabla h \) we arrive at the desired result.

Now we notice that the same result obtained in [12] for compact almost Yamabe solitons also works for compact \( k \)-Yamabe solitons. More precisely, we have the following theorem.

**Theorem 1.3.** The compact \( k \)-Yamabe soliton \( (M^n, g, X, \lambda) \) is gradient if, and only if,

\[
\int_{M^n} \operatorname{Ric}(\nabla h, Y) dv_g \leq 0,
\]

where \( h \) and \( Y \) are the Hodge-de Rham decomposition components of \( X \).

As a consequence of Theorem 1.1 and Theorem 1.3, we derive the following triviality result.

**Corollary 1.4.** Let \( (M^n, g, X, \lambda) \) be a compact \( k \)-Yamabe soliton \( (k \geq 2) \) and \( X = \nabla h + Y \) the Hodge-de Rham decomposition of \( X \). If

\[
\int_{M^n} \operatorname{Ric}(\nabla h, Y) dv_g \leq 0,
\]

then \( (M^n, g) \) is a trivial \( k \)-Yamabe soliton.

An immediate consequence of the above corollary is the next result.

**Corollary 1.5.** Any compact \( k \)-Yamabe soliton \( (M^n, g, X, \lambda) \) with \( k \geq 2 \) and nonpositive Ricci curvature is trivial.

Finally, taking into account the \( L^2(M) \) orthogonality of the Hodge-de Rham decomposition, we obtain.

**Theorem 1.6.** Let \( (M^n, g, X, \lambda) \) be a compact \( k \)-Yamabe soliton \( (k \geq 2) \) and \( X = \nabla h + Y \) the Hodge-de Rham decomposition of \( X \). If

\[
\int_{M^n} g(\nabla h, X) dv_g \leq 0,
\]

then \( (M^n, g) \) is a trivial \( k \)-Yamabe soliton.

2. Proofs

**Proof of Theorem 1.2:** If \( k = 1 \), then \( (M^n, g) \) is a Yamabe soliton and the result is well known from [6]. Now, consider \( k \geq 2 \) and suppose \( (M^n, g) \) locally conformally flat. It was proved in [8, 14] that, on a compact, locally conformally flat, Riemannian manifold, one has

\[
\int_{M^n} g(\nabla \sigma_k, X) dv_g = 0,
\]

for every conformal Killing vector field \( X \) on \( (M^n, g) \). From the structure equation [1], we know that \( X \) is a conformal Killing vector field; hence, it follows that

\[
0 = \int_{M^n} g(\nabla \sigma_k, X) dv_g = -\int_{M^n} \sigma_k(\text{div} X) dv_g = -2n(n - 1) \int_{M^n} \sigma_k(\sigma_k - \lambda) dv_g,
\]

where in the second equality we have used the divergence theorem. On the other hand, again from the divergence theorem, we obtain

\[
0 = \int_{M^n} \text{div} X dv_g = 2n(n - 1) \int_{M^n} (\sigma_k - \lambda) dv_g.
\]

Jointly equations (4) and (5), we conclude that

\[
2n(n - 1) \int_{M^n} (\sigma_k - \lambda)^2 dv_g = 0,
\]

which implies that \( \sigma_k = \lambda \) and \( \mathcal{L}_X g = 0 \). Hence \( (M^n, g) \) is trivial.
Proof of Theorem 1.1: If \( k = 1 \), then \((M^n, g)\) is a gradient Yamabe soliton and the result is well known from [9]. Now, consider \( k \geq 2 \) and suppose by contradiction that \( f \) is nonconstant. From Theorem 1.1 of [9], we obtain that \((M^n, g)\) is rotationally symmetric and \( M^n \setminus \{N,S\}\) is locally conformally flat. Here \( N,S \) corresponds to the extremal points of \( f \) in \( M \). From the structure equation [2], we know that \( \nabla f \) is a conformal Killing vector field; hence, we can apply Theorem 5.2 of [14] to deduce

\[
\sigma(0) = \int_{M^n} g(\nabla \sigma_k, \nabla f) dv_g = -2n(n-1) \int_{M^n} \sigma_k(\sigma_k - \lambda) dv_g,
\]

where in the last equality we have used the divergence theorem. On the other hand, again from the divergence theorem, we get

\[
0 = \int_{M^n} \Delta f dv_g = 2n(n-1) \int_{M^n} (\sigma_k - \lambda) dv_g.
\]

Jointly equations (6) and (7), we conclude that

\[
2n(n-1) \int_{M^n} (\sigma_k - \lambda)^2 dv_g = 0,
\]

which implies that \( \sigma_k = \lambda \) and \( f \) is harmonic. Since \( M^n \) is compact, \( f \) is a constant, which leads to a contradiction. This proves that \( f \) is constant.

\[\square\]

Proof of Theorem 1.3: From the Hodge-de Rham decomposition [4], we deduce that

\[
\frac{1}{2} \mathcal{L}_Y g = \frac{1}{2} \mathcal{L}_X g - \frac{1}{2} \mathcal{L}_{\nabla h} g = 2(n-1)(\sigma_k - \lambda) g - \nabla^2 h.
\]

Therefore, to prove that \((M^n, g)\) admits a gradient \( k \)-Yamabe soliton structure, it is necessary and sufficient to show that \( \mathcal{L}_Y g = 0 \). From [5], we arrive at

\[
\frac{1}{4} \int_{M^n} |\mathcal{L}_Y g|^2 dv_g = \int_{M^n} \left[ 4n(n-1)^2(\sigma_k - \lambda)^2 - 4(n-1)g(\nabla^2 h, (\sigma_k - \lambda)g) + |\nabla^2 h|^2 \right] dv_g
\]

\[= \int_{M^n} \left[ |\nabla^2 h|^2 - 4n(n-1)^2(\sigma_k - \lambda)^2 \right] dv_g.
\]

We are going to compute the right-hand side of (9) using the following identity

\[
\int_{M^n} 2\text{Ric}(\nabla h, Y) dv_g = \int_{M^n} \left[ \text{Ric}(X, X) - \text{Ric}(\nabla h, \nabla h) - \text{Ric}(Y, Y) \right] dv_g.
\]

Taking the divergence of (8), we get

\[
\frac{1}{2} \text{div}(\mathcal{L}_Y g)(Y) = \frac{1}{2} \text{div}(\mathcal{L}_X g)(Y) - \frac{1}{2} \text{div}(\mathcal{L}_{\nabla h} g)(Y)
\]

\[= 2(n-1)\text{div}(\sigma_k - \lambda)(Y) - \frac{1}{2} \text{div}(\mathcal{L}_{\nabla h} g)(Y)
\]

\[= 2(n-1)g(\nabla \sigma_k, Y) - \frac{1}{2} \text{div}(\mathcal{L}_{\nabla h} g)(Y).
\]

Hence, from the Bochner formula (see Lemma 2.1 of [11]), we can express (11) as follows

\[
\frac{1}{2} |\nabla h|^2 - |\nabla Y|^2 + \text{Ric}(Y, Y) = 4(n-1)g(\nabla \sigma_k, Y) - 2\text{Ric}(\nabla h, Y) - 2g(\nabla \Delta h, Y),
\]

and using the compactness of \( M^n \), we arrive at equation

\[
\int_{M^n} 2\text{Ric}(\nabla h, Y) dv_g = \int_{M^n} \left[ |\nabla Y|^2 - \text{Ric}(Y, Y) \right] dv_g.
\]

On the other hand, the same argument as above shows that

\[
\frac{1}{2} |\nabla X|^2 - |\nabla X|^2 + \text{Ric}(X, X) = -2(n-1)(n-2)g(\nabla \sigma_k, X).
\]

Since

\[
\int_{M^n} |\nabla X|^2 dv_g = \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 + 2g(\nabla \nabla h, Y) \right] dv_g
\]

\[= \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 + 2g(\nabla \Delta h + \text{Ric}(\nabla h), Y) \right] dv_g
\]

\[= \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 + 2\text{Ric}(\nabla h, Y) \right] dv_g,
\]

\[\square\]
we may integrate (14) over $M^n$ to deduce

$$
\int_{M^n} \text{Ric}(X,X) dv_g = \int_{M^n} [||\nabla X||^2 - 2(n-1)(n-2)g(\nabla \sigma_k, X)] dv_g
$$
\[(15)\]

$$
= \int_{M^n} [||\nabla^2 h||^2 + ||\nabla Y||^2 - 2\text{Ric}(\nabla h, Y) + 4n(n-1)^2 \times \\
\times (n-2)(\sigma_k - \lambda)^2] dv_g.
$$

Again, the same argument based on Lemma 2.1 of [11], allow us to deduce that

$$
\int_{M^n} \text{Ric}(\nabla h, \nabla h) dv_g = \int_{M^n} [4n^2(n-1)^2(\sigma_k - \lambda) - ||\nabla^2 h||^2] dv_g.
$$
\[(16)\]

Now, replacing back (15), (15) and (16) into (10), we get

$$
\int_{M^n} [||\nabla^2 h||^2 - 4n(n-1)^2(\sigma_k - \lambda)^2] dv_g = \int_{M^n} \text{Ric}(\nabla h, Y) dv_g,
$$
which combining with (10) produce the desired result. 

**Proof of Theorem 1.6**: Since the Hodge-de Rham decomposition is orthogonal on $L^2(M)$, we get

$$
\int_{M^n} g(\nabla h, X) dv_g = \int_{M^n} g(\nabla h, \nabla h + Y) dv_g = \int_{M^n} |\nabla h|^2 dv_g.
$$

Therefore, if

$$
\int_{M^n} g(\nabla h, X) dv_g \leq 0,
$$

we obtain that $\nabla h = 0$ and, consequently, $X = Y$. Now, since $Y$ is a free divergence vector field, we deduce

$$
0 = \text{div} Y = \text{div} X = 2n(n-1)(\sigma_k - \lambda),
$$

which implies that $\sigma_k = \lambda$ and $\mathcal{L}_X g = 0$, hence, trivial.

\[\square\]

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