Complex response analysis of a non-smooth oscillator under harmonic and random excitations

Shichao MA¹,², Xin NING¹,², Liang WANG³,⁴, Wantao JIA³, Wei XU³

1. School of Astronautics, Northwestern Polytechnical University, Xi’an 710072, China;
2. National Key Laboratory of Aerospace Flight Dynamics, Xi’an 710072, China;
3. School of Mathematics and Statistics, Northwestern Polytechnical University, Xi’an 710129, China

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Abstract It is well-known that practical vibro-impact systems are often influenced by random perturbations and external excitation forces, making it challenging to carry out the research of this category of complex systems with non-smooth characteristics. To address this problem, by adequately utilizing the stochastic response analysis approach and performing the stochastic response for the considered non-smooth system with the external excitation force and white noise excitation, a modified conducting process has proposed. Taking the multiple nonlinear parameters, the non-smooth parameters, and the external excitation frequency into consideration, the steady-state stochastic P-bifurcation phenomena of an elastic impact oscillator are discussed. It can be found that the system parameters can make the system stability topology change. The effectiveness of the proposed method is verified and demonstrated by the Monte Carlo (MC) simulation. Consequently, the conclusions show that the process can be applied to stochastic non-autonomous and non-smooth systems.

Key words non-autonomous system, non-smooth system, random excitation

1 Introduction

Non-smooth systems are a major area of interest within the field of non-linear dynamics, which can be applied to the modeling of many nonlinear dynamic phenomena in the fields of aerospace, structural engineering, and mechanical engineering[1–5]. The impact and friction described by these systems are often encountered in the components of the systems or between

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† Corresponding author, E-mail: liangwang1129@nwpu.edu.cn
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the component and the constraint surface\cite{6}. At the same time, random perturbations are unavoidable in non-smooth systems. Therefore, it is of significant importance to make it clear whether mechanical systems can work normally under the influence of these random factors\cite{7}. Stochastic P-bifurcations of non-smooth systems refer to the sudden change in the topology of the non-smooth systems with variable parameters\cite{8–10}. Using the limiting averaging principle, Xia et al.\cite{11} studied the abrupt phenomenon of stationary probability density functions (PDFs) of a three-degree-of-freedom nonlinear stochastic system. Wang et al.\cite{12} studied the stochastic P-bifurcation of a smooth and discontinuous smooth and discontinuous (SD) oscillator via the numerical method. Obviously, carrying out the research of the stochastic responses of such systems under random perturbation excitations is urgent and necessary.

With the development of nonlinear disciplines and the upgrading of computers, there are many effective methods to study the responses of such systems, including analytical methods and numerical algorithms\cite{13–16}. Since the non-smooth properties of stochastic non-smooth systems are difficult to be handled, more and more scholars are currently working in the stochastic responses of such non-smooth systems\cite{17–20}. However, due to the existence of non-smooth factors, an approximate transformation is used for many methods to convert the original non-smooth system to a smooth system, and then the system responses are obtained. The drawback is obvious, which may cause the non-smooth characteristics that are unique to this type of system to be destroyed. Unless performing the approximate transformation, there are seldom rigorous analytical methods used to solve non-smooth systems directly. For strong nonlinear systems, the analytical method is difficult to solve and the derivation process is complicated. Relatively speaking, numerical methods possess better calculating performances. The Monte Carlo (MC) simulation method is the most common numerical method that can guarantee the correctness of the results. Unfortunately, this method puts forward higher requirements to the amount of samples and computation time to get the exact responses. Many thanks to Hsu, a generalized cell mapping (GCM) method has been proposed, and can be utilized to solve these problems\cite{21–22}. A large number of references have verified the rationality of the GCM algorithm\cite{23–25}. As a further development, Wang et al.\cite{26} proposed an improved cell mapping algorithm for solving original non-smooth impact systems. However, we find that the algorithm proposed in Ref. [26] is limited to autonomous systems, and cannot solve the external excitation problem of the considered non-autonomous systems. Motivated by the aforementioned observations, the main aim of this paper is to investigate a more optimizing process which is suitable for general non-autonomous impact systems. The main innovations can be summarized as follows. Based on Ref. [22], in order to retain the non-smooth characteristics of the system, no non-smooth transformation is imposed on the vibro-impact system. A special and fast cell mapping method is used to calculate the PDFs of non-autonomous vibro-impact system for the first time. With the MC simulation, the accuracy of the method is illustrated by random steady state bifurcation analysis.

This paper is arranged as follows. Section 2 gives a stochastic elastic impact with the external excitation force as a research object and the corresponding stochastic GCM (SGCM) method. Section 3 analyzes the stochastic P-bifurcation phenomena by variable parameters, including the nonlinear parameters, the non-smooth parameters, and the external excitation frequency, using the optimization cell mapping method. Conclusions are given in Section 4.

2 Model and corresponding method

We study a stochastic non-smooth system with external excitations as shown in Fig. 1. A mass block can be reciprocated by the external excitation and random force. Once the displacement reaches a certain distance, the mass block will collide with the constraint surface. For practical purposes, it is assumed that the mass block and constraint surface are both regarded as elastomers\cite{6}. Thus, the elastic impact needs to satisfy the Hertzian contact law\cite{27–28}. It is a non-autonomous periodic system and can be described by
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\[ \ddot{x} + a_0 \dot{x} + b_0 x + c_0 x^3 + g x^2 \dot{x} + f_e(x, \dot{x}) = F \cos(\omega t) + \xi(t), \quad (1) \]

\[ f_e(x, \dot{x}) = \begin{cases} k_1 (x - q) + c_1 \dot{x}, & x \geq q, \\ 0, & x < q. \end{cases} \quad (2) \]

Equation (2) means the elastic impact process with the damping constant \( c_1 \) and the spring stiffness \( k_1 \). When \( x \geq q \), the elastic impact occurs. The parameters \( a_0, b_0, \) and \( c_0 \) are fixed, and then this system is perturbed by the Gaussian white noise \( \xi(t) \) which satisfies

\[ E(\xi(t)) = 0, \quad E(\xi(t)\xi(t + \tau)) = 2\sigma \delta(\tau). \quad (3) \]

For the stochastic elastic impact system with the external excitation, we examine its dynamic behavior by studying its stochastic responses. Now, the corresponding SGCM method is described to gain the stochastic responses. Under the external excitation of period force, the non-autonomous system can be expressed as

\[ \dot{x} = f(x, t, \xi(t)) = f(x, t + T, \xi(t)). \quad (4) \]

For this system, suppose that time is the explicit state variable, and rewrite the system as

\[ \dot{x} = y, \quad \dot{y} = F \cos(\omega \cdot \theta) - a_0 y - b_0 x - c_0 x^3 - g x^2 y - f_e(x, y) + \xi(t), \quad \dot{\theta} = 1, \quad (5) \]

\[ f_e(x, \dot{x}) = \begin{cases} k_1 (x - q) + c_1 \dot{x}, & x \geq q, \\ 0, & x < q. \end{cases} \quad (6) \]

For the systems (5)–(6), we first need to subdivide the state space into smaller intervals. Then, each interval produces a number of random sample trajectories. The one-step transition probability matrix can be derived by calculating the sample probability numbers falling into each interval. Thus, on the basis of the Chapman-Kolmogorov (C-K) equation

\[ p(x, nT) = \int p(x, T|x_0, 0)p(x_0, (n-1)T)dx_0, \quad (7) \]

we choose \( T \) as the length of one-step iteration to satisfy one complete impact process. Therefore, the probability density distribution after iterations can be calculated easily by

\[ p(n) = P \cdot p(n-1) = P \cdot p(n-2) = \cdots = P^n \cdot p(0), \quad n \in \mathbb{N}. \quad (8) \]

3 Stochastic P-bifurcation analysis

The stochastic P-bifurcation studies the shape of the invariant measure (steady-state response probability density) of the stochastic dynamic system with the change of the parameters, including the significant changes in the number and position of peaks. The PDF of the system can be obtained by using the above process which can study the steady-state stochastic P-bifurcation phenomenon of the non-smooth system under the variable parameters. At the same time, the MC simulation is used to illustrate the validity of the process.
For the systems (5)–(6), the parameters are fixed at $a_0 = -0.1$, $b_0 = -0.5$, $c_0 = 0.5$, $k_1 = 0.8$, $q = 0$, and $F = 0.23$, where the noise intensity is 0.02. The domain $D = (-3 \leq x \leq 3, -3 \leq y \leq 3)$ is divided into 2500 intervals, and each interval produces 200 random sample trajectories. First, the effects of damping constants are considered. Choosing $g = 0.1$ and $\omega = 0.5$, Figs. 2–3 provide the steady-state marginal PDFs with $c_1 = 0.02, 0.06, 0.10, 0.16$, respectively. As shown in Fig. 2, the PDFs of $x$ present a relatively uniform distribution with $c_1 = 0.02$. But as $c_1$ increases, some parts of the PDFs start to bulge. It indicates that the impact probability of the elastic surface would be increased with the increasing stiffness coefficient of the impact surface. Different from the PDFs of $x$, the topological structure of marginal PDFs of velocity $y$ is bimodal-shaped when $c_1 = 0.02$ in Fig. 3. But the shape tends to have one peak as the stiffness coefficient increases to $c_1 = 0.16$. Figure 4 demonstrates the joint PDFs with changing damping constants. Among them, the bright areas represent the locations where the
PDF is larger. When $c_1 = 0.02$, the bright areas of joint PDFs exhibit a circle. It is apparent that the bright areas of joint PDFs become a pie distribution with $c_1 = 0.06$. As $c_1$ continues to increase, the bright areas begin to shrink, and the color becomes darker. Certainly, numerical results are identical with the MC simulation, which verifies the feasibility of the process.

Then, how the adjustable parameter $g$ on a small scale induces the generation of stochastic P-bifurcation comes into consideration under fixed $c_1 = 0.04$ and $\omega = 0.5$. Figure 5 shows the steady-state marginal PDFs of $x$ with $g = 0.07, 0.10, 0.15, 0.20$. When $g = 0.07$, the topological figure of the marginal PDF is bimodal. The distribution of PDFs is relatively uniform with $g = 0.10$. As $g$ continues to increase, the shape of PDFs is a bimodal structure again, but both peaks are getting taller and leaner, which indicates a more concentrated PDFs distribution. In Fig. 6, the marginal PDFs of $y$ have obviously transformed into a single peak with the increase in $g$. Similarly, the MC simulation results are in good agreement with the method. Figure 7
shows the results of the correlational analysis of joint PDFs with $g = 0.07, 0.10, 0.15, 0.20$. Closer inspection of this figure shows that the bright areas of joint PDFs also change from ring to pie and finally to sheet as $g$ increases. The results in this section indicate that the nonlinear parameter $g$ could induce the instability of this non-smooth system on a small scale, and it can also be said that the stochastic P-bifurcation occurs at this time.

Finally, consider the impact of the external excitation frequency $\omega$. Select the parameters $g = 0.1, c_1 = 0.04$, and the range of stochastic P-bifurcation parameter $\omega = 0.25, 0.75, 1.0$. As shown in Figs. 8–10, the marginal and joint PDFs exhibit the change of the topology. In Figs. 8–9, the marginal PDFs of $x$ and $y$ are both evolved from double peaks to a single peak. It is more like that one of the peaks is falling, and the other peak is rising. For joint PDFs, the bright areas of PDFs are ring-shaped with $\omega = 0.25$, while when $\omega = 0.75$, the bright areas of PDF change into a small area. With the continuous increase in $\omega$, there is a clear trend of the
overall PDF becoming a half-moon structure in Fig. 10. These results indicate that not only the nonlinear and non-smooth parameters of the system but also the external excitation frequency will affect the stability characteristics of the system, the latter of which also make qualitative changes to the system.

4 Conclusions

The improved cell mapping method is used to derive the PDF responses of non-autonomous and non-smooth systems under the external excitation force and noise perturbation. At the same time, it is proven that the method is well coincident with the MC simulation method. Overall, this study strengthens the idea that there is no need to change the one-step transition probability matrix once it is obtained, and thus, we just need to calculate the matrix iteration multiplication to obtain the PDFs. Then, the response calculation time will be greatly reduced compared with the MC simulation method obviously, especially for steady-state responses. In addition, the stochastic P-bifurcation phenomena are also discussed in this paper. By the proposed process, the results reveal that the nonlinear and non-smooth parameters of the system can conduct the stochastic P-bifurcation phenomena, and the change of the external excitation frequency can also cause the topology of PDFs to change. Without any approximate transformation of the original system, this approach will prove useful in expanding the studies for the stochastic responses of non-smooth systems in practical engineering.

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