ARC NUMBERS FROM GAUSS DIAGRAMS

TOBIAS HAGGE

Abstract. We characterize planar diagrams which may be divided into \( n \) arc embeddings in terms of their chord diagrams, generalizing a result of Taniyama for the case \( n = 2 \). Two algorithms are provided, one which finds a minimal arc embedding (in quadratic time in the number of crossings), and one which constructs a minimal subdiagram having same arc number as \( D \).

1. Introduction

An undecorated chord diagram \( D = (C, H, X) \) consists of an oriented circle \( C \), a set \( H \subset C \) half-crossings having finite, even cardinality, and a partition \( X \) of \( H \) into two element sets, called (full) crossings. Given another undecorated chord diagram \( D' = (C', H', X') \), \( D \) and \( D' \) are equivalent as undecorated chord diagrams (write \( D \sim D' \)) if there is an orientation-preserving homeomorphism \( \phi : C \rightarrow C' \) such that \( \{ \phi(x) | x \in X \} = X' \). We say \( D' \) is a subdiagram of \( D \) if \( X' \subset X \) and \( H' = \cup X' \).

In an undecorated chord diagram \( D = (C, H, X) \), a regular point \( x \) is an element \( x \in C \setminus H \). A regular arc \( r \) is an open arc \( r \subset C \) such that \( \partial r \subset C \setminus H \). Then \( r \) is properly embedded if it contains no full crossings. An embedded partition of \( D \) is a finite set of regular points, called cut points, such that the components of the complement are properly embedded arcs. Two embedded partitions of \( D \) are equivalent if they differ by an isotopy of \( C \) which fixes half-crossings. An arc embedding of \( D \) is minimal if \( D \) has no embedded partition with fewer properly embedded arcs. The arc number of \( D \) is the number of arcs in a minimal embedded partition, or, equivalently, the number of cut points.

Arc number first appeared in [2], where it was shown that every knot has a diagram with arc number two. The result was used to provide a characterization of knot groups in terms of direct products of free groups. The same result was rediscovered in [3] as part of the construction of a knot invariant. It was used in [1] to show that every knot contains a diagram containing at most four odd-sided polygons. In [4], Taniyama gave a classification of planar diagrams of arc number two in terms of their Gauss diagrams. In [5] he conjectured the correct classification for arc number three. The present work completes this classification for arbitrary arc number and provides an efficient method for finding minimal embedded partitions for a given diagram.

Two regular arcs \( r, r' \) are equivalent as regular arcs if there is a regular arc \( r'' \) containing both \( r \) and \( r' \) such that each of \( r, r', r'' \), and \( r \cap r' \) contain the same set of half-crossings. For example, in a diagram with one crossing there are six equivalence classes of nonempty arcs. Up to arc equivalence, each regular arc is specified by an ordered pair of half crossings \( (b(r), f(r)) \). The front boundary \( f(r) \) is the hindmost half-crossing in \( C \setminus r \), using the preferred orientation; similarly, the back boundary \( b(r) \) is the foremost half-crossing in \( C \setminus r \).
A regular arc \( r' \) is a front extension of a regular arc \( r \) if \( b(r) = b(r') \) and \( r \subset r' \). The extension is proper if \( r \sim r' \). A properly embedded regular arc \( r \) is an \( f \) arc if no proper front extension of \( r \) is properly embedded. Given two inequivalent \( f \) arcs \( r \) and \( r' \), one may be properly contained in another. In this case \( f(r) = f(r') \).

In the sequel, regular arcs and embedded partitions are considered only up to equivalence. For computational purposes each open arc \( (h_i, h_{i+1}) \) between adjacent half-crossings is identified with a single point, and computations are performed in the quotient topology. For the exposition, however, the language of chord diagrams is retained.

By convention, when points in a finite set \( X \subset C \) are expressed with numbered subscripts, the subscripts are defined modulo \(|X|\). The cyclic ordering of the subscripts indicates the ordering of the points on \( C \), respecting the orientation.

2. Finding the Arc Number of an Undecorated Chord Diagram

**Lemma 2.1.** Every undecorated chord diagram \( D \) has a minimal embedded partition in which at most one component is not an \( f \)-arc.

**Proof.** Let \( (p_0, \ldots, p_{n-1}) \) be a minimal embedded partition. Let \( q_0 = p_0 \). Continuing around the circle for \( i \in [1, n-2] \), each \( q_i \) is given by sliding \( p_i \) in the direction of the preferred orientation so that the arc \( (q_{i-1}, q_i) \) is front maximal. It is not possible to slide \( p_i \) past \( p_{i+1} \), as this would contradict minimality. The result is a minimal embedded partition in which all components are \( f \)-arcs, except perhaps \( (q_{n-1}, q_0) \).

**Theorem 2.2.** The arc number of an undecorated chord diagram \( D = (C, \{h_1, \ldots, h_{2n}\}, X) \) may be computed in \( O(n^2) \) time.

**Proof.** For each regular arc \( r = (h_i, h_{i+1}) \) in \( D \), construct an embedded partition \( P = \{q_0, \ldots, q_{n-1}\} \) by placing the first cut point \( p_0 \) on \( r \) and then constructing a sequence of \( f \)-arcs, stopping when we reach \( p_0 \) again. This takes \( O(n) \) time. By construction, at least one such \( P \) is equivalent to a partition constructed in Lemma 2.1 and is therefore minimal.

A star \( S_{t,a} \) for \( a \in \mathbb{N}, t \in \mathbb{N}^+ \) is an undecorated chord diagram with \( 1 + (a + 1)t \) crossings, these being of the form \( c_j = \{h_j, h_{j+2(t-1)}\} \), where \( j \) ranges over the values in \([0, 1 + 2(a + 1)t]\) having a fixed parity. Up to equivalence \( S_{t,a} \) depends only on \( t \) and \( a \). Some examples of stars are given in Figure 1.

A star ordering of \( S_{t,a} \) is the ordering \( (c_j, c_{j+2t}, \ldots, c_{j+2(a+1)t}) \) starting with some \( j \). Given such, for \( n < 1 + (a + 1)t \) let \( S^n_{t,a} \) be the proper subdiagram consisting of the first \( n \) crossings in \( S_{t,a} \). If \( t \leq t' \) then it is easy to see that \( S^n_{t,a} \cong S^{n}_{t',a} \).

Thus one may ignore \( t \) and write \( S^n_{a} \) without ambiguity. Note that \( S^n_{a} \) is not a star unless \( n \leq a + 1 \), in which case \( S^n_{a} = S_{1,n-1} \).

**Lemma 2.3.** The star \( S_{t,a} \) has arc number \( a + 2 \).

**Proof.** The star \( S_{t,a} \) is symmetric under a rotation that sends \( c_j \) to \( c_{j+2t} \). Thus one need only construct two embedded partitions to find a minimal one, starting either just before or just after the back half of a crossing. Consider the constructed partitions beginning just before \( b(c_0) \) and just after \( b(c_{a+1}) \). In both cases the first \( a + 1 \) obstructing crossing are the same, forming \( S^{a+1}_{a} \cong S_{1,a-1} \). A single additional arc completes each partition, and both are minimal.
For every f-arc \( m \), there is a unique obstructing crossing \( c(m) \) that prevents further front extension. Since \( m \) has a preferred orientation, \( c_m \) inherits front (obstructing) and back half-crossings \( f(c(m)) \) and \( b(c(m)) \) respectively from \( m \), as well as a preferred properly embedded regular arc \( r(c(m)) \sim (b(c(m)), f(c(m))) \) contained in \( m \).

If two disjoint f-arcs have the same obstructing crossing, then the diagram has arc number two. In any case, f-arc components \( m_1 \neq m_2 \) of an embedded partition have disjoint \( r(c(m_1)) \) and \( r(c(m_2)) \). More generally, given a pair of disjoint f-arcs \( m \) and \( m' \), \( c(m) \) is never properly contained in \( r(c(m')) \).

**Theorem 2.4.** For \( a > 0 \), an undecorated chord diagram \( D \) has arc number \( \geq a + 2 \) iff it contains \( S_{t,a} \) as a subdiagram for some \( t \in \mathbb{N}^+ \).

**Proof.** Lemma 2.3 gives one direction. For the other, the case \( a = 0 \) is obvious. Suppose \( a > 0 \) and \( D \) has arc number \( a + 2 \). Since \( S_a^{a+1} = S_{1,a-1}, S_{t,a} \) contains \( S_{1,b} \) for \( 1 \leq b < a \). Thus it suffices to show that \( D \) contains \( S_{t,a} \) for some \( t \).

Construct a minimal embedded partition \( P = (p_0, \ldots, p_{a+1}) \) as in Theorem 2.2. Since for \( i \in [0, a] \) the arc \( (p_i, p_{i+1}) \) is front maximal, each \( p_i \) with \( i > 1 \) has an associated obstructing crossing \( c_i \). Since obstructing crossings given by an embedded partition bound disjoint f-arcs, these \( c_i \) form a copy of \( S_{1,a-1} = S_a^{a+1} \).
Now, starting at \( p_{a+2} \), continue around the circle, extending embedded arcs to \( f \)-arcs by sliding the \( p_k \) as was done in Lemma 2.1\(^2\). We claim that at some point the generated obstructing crossings give the desired star.

Inductively assume that at the beginning of each stage \( k > a+1 \), the constructed obstructing crossings \( (c_1, \ldots, c_{k-1}) \) form a star ordered copy of \( S_{k-1}^a \). Thus each \( f(c_j) \) lies in \( r(c_j-1-x(a+1)) \) for each \( x > 0 \) such that \( j-1-x(a+1) \geq 0 \), and similarly each \( b(c_j) \) lies in \( r(c_j-2-x(a+1)) \) for each \( x > 0 \) such that \( j-2-x(a+1) \geq 0 \).

Construct \( c_k \) by sliding \( p_k \) so that \( (p_{k-1}, p_k) \) is maximal, and let \( c_k \) be the associated obstructing crossing. We will show \( c_k \neq c_l \) for \( l < k \). First, minimality of the resulting partition and front maximality of \( p_{k-a}, p_{k-a+1}, \ldots, p_k \) imply that \( c_{k-a}, c_{k-a+1}, \ldots, c_k \) form a star ordered copy of \( S_{1,a-1} \), the crossings of which bound disjoint arcs. Second, for each \( i < k-a-1 \), \( f(c_i) \) and \( b(c_i) \) each lie in one of \( r(c_{k-a-1}), \ldots, r(c_{k-1}) \), and so at least one of these lies in one of \( r(c_{k-a}), \ldots, r(c_{k-1}) \). These two statements imply that \( c_k \neq c_l \) unless possibly \( c_k = c_{k-a-1} \). In this case, however, \( p_k \) may be slid all the way to \( p_{k-a-1} \), contradicting minimality of the partition. Thus for \( l < k \), \( c_{k_l} \neq c_{j_l} \).

Consider the locations of \( b(c_k) \) and \( f(c_k) \) relative to the other \( c \) at stage \( k \). There can be no crossing \( c_l \) such that \( b(c_l) \) lies in \( (f(c_{k-1}), b(c_k)) \) since otherwise, to avoid proper containment, \( f(c_l) \) would lie behind \( f(c_k) \) and \( c_l \) would be the obstructing crossing at the \( k \)th stage, not \( c_k \). Similarly, there can be no \( f(c_l) \in (f(c_{k-1}), b(c_k)) \), as otherwise either \( b(c_{l+1}) \in (b(c_k), f(c_{k-1})) \), in which case \( c_{l+1} \) is the obstructing crossing at stage \( k \), not \( c_j \), or else \( b(c_k) \in (f(c_l), b(c_{l+1})) \), in which case \( c_k \) would have been the obstructing crossing at stage \( l+1 \), rather than \( c_{l+1} \). Thus no \( c_l \) for \( l < k \) has half-crossings in \( (f(c_{k-1}), b(c_k)) \).

Next, either \( f(c_k) \) lies in \( r(c_{k-x(a+1)}) \) for all \( x > 0 \) such that \( k-x(a+1) \geq 0 \), or there is at least one \( x > 0 \) such that \( k-x(a+1) \geq 0 \) and \( f(c_k) \notin r(c_{k-x(a+1)}) \). In the first case, the resulting diagram is \( S_{a}^k \). In the second case, since \( c_k \) does not contain \( c_{k-x(a+1)} \), \( f(c_k) \) lies between \( b(c_{k-x(a+1)}) \) and \( b(c_k) \). In this case the crossings \( c_{j_k-x(a+1)}, c_{j_k-x(a+1)+1}, \ldots, c_{j_k} \) form \( S_{a,x} \). Since the number of crossings is finite and the \( c_i \) are all distinct, the second case must eventually occur. □

References

[1] C. Adams, R. Shinjo and K. Tanaka, Complementary regions of knot and link diagrams, [arXiv:0812.2558](http://arxiv.org/abs/0812.2558) (2008).
[2] G. Hotz, Arkadenfadendarstellung von Knoten und eine neue Darstellung der Knotengruppe (German), *Abh. Math. Sem. Univ. Hamburg*, 24 (1960), 132-148.
[3] M. Ozawa, Edge number of knots and links, [arXiv:0705.4348](http://arxiv.org/abs/0705.4348) (2007).
[4] K. Taniyama, Circle Immersions that can be divided into two arc embeddings, *Proc. Amer. Math. Soc.*, 138 (2010), 753-751.
[5] K. Taniyama, Circle Immersions that can be divided into two arc embeddings, Knots in Washington XXVIII, Washington D.C., February 28th, 2009.