Differentiator-Based Output-Feedback Controller for Uncertain Nonautonomous Nonlinear Systems With Unknown Relative Degree

JANG-HYUN PARK, SEONG-HWAN KIM, (Associate Member, IEEE), AND TAE-SIK PARK, (Member, IEEE)
Department of Electrical and Control Engineering, Mokpo National University, Chonnam 58554, South Korea
Corresponding author: Tae-Sik Park (tspark@mokpo.ac.kr)

This work was supported by the Korea Electric Power Corporation under Grant R18XA04.

ABSTRACT A novel output-feedback controller for uncertain nonautonomous nonlinear systems with unknown relative-degree is proposed in this study. With the assumption that the upper bound of the relative degree is known, the proposed control scheme is designed based on the input filter and higher-order switching differentiator (HOSD) with over dimension. Using the overestimated time-derivatives of tracking error by HOSD, the proposed controller compensates for the effects of uncertainties including an unknown relative degree in the controlled system. Assuming that the internal zero dynamics, if they exist, are stable, the asymptotic stability of the closed-loop system is guaranteed.

INDEX TERMS Differentiator-based control, output-feedback, uncertain nonlinear system, unknown relative degree.

I. INTRODUCTION
Designing controllers for nonlinear systems that contain uncertainties has been a challenging issue for decades because uncertainties are widely intrinsic in real physical systems [1]–[17]. There are two conventional approaches used to deal with unstructured or unmodeled uncertainties in the controlled nonlinear system in particular: First, adaptive control schemes with universal approximators (refer to [1]–[9] and references therein) are widely researched. In universal approximator-based adaptive controllers, neural networks or fuzzy logic systems are used to capture and compensate for the unknown dynamic structure. Second, sliding-mode control (SMC) algorithms for continuous and discrete nonlinear systems [10]–[16] are also being actively researched and adopted. In the SMC, some discontinuous switching control action forces the system dynamics to the sliding surface regardless of the unmodelled uncertainties in the controlled system. More recently, some approximation-free control laws for uncertain nonlinear systems have been proposed [18]–[22]. The advantages of these control algorithms are that the complexities and dynamic orders of the controllers are relatively decreased, and that the prescribed output tracking performance is guaranteed regardless of the system uncertainties. Specifically, differentiator-based controllers [19]–[21] that combine a time-derivative estimator with a control input filter for generating control laws have good stabilizing property irrespective of the unmodelled uncertainties and nonautonomous property in the nonaffine nonlinear system.

Although substantial efforts have been made in this research area, most of the previous controllers have been designed under the assumption that the time-invariant relative degree is known in advance. The actual physical system must have a finite system order, and the relative order must therefore also be bounded. Taking this fact together with the difficulty of designing a controller without knowing at least the bounding constant of the relative degree, most of the previous research papers assume that the maximum value of the relative degree is known. Regarding linear systems, some existing literature [23]–[25] deals with designing controllers without relative degree information. For instance, in [23], an active disturbance rejection controller that requires no relative degree for the SISO LTI plant has been proposed. In [24], [25], some results on the linear system with an upper bound for the relative degree have been derived via filters with over dimension. Very few studies have examined the controller design for uncertain nonlinear systems with
unknown relative degrees. One example is [11], in which a homogeneous higher-order SMC with a known upper bound for the variable relative degree is proposed. In [26], an automatic identification algorithm for a relative degree with a known upper limit is proposed. In [27], a controller based on two switching algorithms is proposed for global output-feedback stabilization for nonlinear systems with an unknown relative degree. However, [11, 26] consider affine-in-the-control nonlinear systems; in [26], there is no rigorous stability proof of the closed-loop system including a switching scheme between controllers that have different orders. The system considered in [27] is more restrictive than that of [11] and [26], since it is a linear-in-the-control autonomous nonlinear system whose nonlinearities are the functions of the system output only. That is, none of the current results can be applied to more general nonaffine and nonautonomous nonlinear systems.

Therefore, in this paper, a novel differentiator-based output-feedback controller for more general uncertain nonaffine and nonautonomous nonlinear systems with the unknown relative degree is proposed. The proposed control scheme is based on the input filter and higher-order switching differentiator (HOSD) [19] with over dimension under the assumption that the relative degree’s maximum value is available. The main contributions of the proposed control algorithm can be summarized as follows:

1) Compared to the previous controllers in [11], [26], [27], the nonlinear system considered in this paper is a relatively broader class of nonaffine-in-the-control nonautonomous nonlinear systems. To the authors’ knowledge, there has been no research examining controller for the considered general nonlinear systems with the unknown relative degree.

2) The proposed scheme combines the HOSD [28] and the differentiator-based controller [20] to deal with uncertainties containing an unknown relative degree in the controlled system. Due to the properties of the HOSD, the proposed controller shows no peaking or chattering, and it guarantees asymptotic output tracking performance.

3) The proposed control scheme structure is relatively simple, and there is no lengthy or complicated control formula involving switching functions. There are only two design constants in the control formula, which demonstrates the compactness of the proposed controller.

This paper is organized as follows. In Section II, the dynamics of the considered nonlinear system with the control objective are described and the HOSD is introduced. Section III describes the structure of the proposed controller with the main theorem. A stability analysis and the prerequisite of an input signal to HOSD are also presented and discussed in Section III. In Section IV, simulations using example systems with various relative degrees are then conducted to show the proposed controller’s performance and compactness. Finally, Section V presents concluding remarks.
Assumption 5: The function \( f \) and \( h \) in (1) are smooth functions. The partial derivatives \( \frac{\partial f}{\partial t}(x, u, t) \) for \( i = 1, 2, \cdots, n+1 \) and \( \frac{\partial h}{\partial t}(x, t) \) for \( j = 1, 2, \cdots, n+2 \) are all bounded on \( \mathcal{D} = \Psi \times [0, \infty) \).

The proposed differentiator-based controller adopts HOSD to estimate time-derivatives of the tracking error. Before introducing HOSD dynamics, it is necessary to present some definitions. Let \( \Phi \) be a set of all strictly increasing infinite time sequences such that

\[
\Phi \overset{\Delta}{=} \{(t_i)_{i=0}^{\infty} | t_0 = 0, t_i < t_{i+1} \forall i \in \mathbb{N}_0\} \tag{4}
\]

where \( \mathbb{N}_0 = \{0, 1, 2, \cdots\} \). For a sequence \( T = (t_i) \in \Phi, \Omega_T \) denotes a set of functions that discontinuous at some or all \( t_i \).

Definition 1: \([28]\) For \( T = (t_i) \in \Phi \), define the set of functions as follows:

\[
\Omega_T^L \overset{\Delta}{=} \left\{ f(t) \mid f(t) \in \Omega_T, \sup_{t_i \leq t < t_{i+1}} |f(t)| \leq L < \infty \right\} \tag{5}
\]

where \( L > 0 \) is a constant. The functions in \( \Omega_T^L \) are bounded in the piecewise sense (BPWS) below \( L \).

Lemma 1: \([28]\) Suppose the time-derivatives of a time-varying signal \( a(t) \) are BPWS such that \( a^{(j+1)} \in \Omega_T^L \) for \( j = 1, 2, \cdots, n \) where \( L_j \)'s are positive constants and \( T \in \Phi \). \( a^{(n+2)} \) is also assumed to be BPWS. Consider the following HOSD dynamics

\[
\dot{\sigma}_j = k_j e_{aj} + \sigma_j \geq 0, \quad j = 1, 2, \cdots, n \tag{6}
\]

where \( e_{aj} = \sigma_{j-1} - \sigma_j \) with \( \sigma_0 = a \). If the design constants are chosen such that \( k_j > 0 \) and \( L_j > L_j^* \) for all \( j = 1, 2, \cdots, i \), then:

\[
\sigma_j(t) \rightarrow a^{(i)}, \quad j = 1, 2, \cdots, n + 1 \tag{7}
\]

holds.

The detailed proof of Lemma 1 is shown in \([28]\).

The number of design constants in (6) is \( 2n \); for the design convenience of HOSD, it is necessary to reduce the number of constants to be determined. Let us define a constant \( L^* \) as \( L^* \overset{\Delta}{=} \max\{L_1^*, L_2^*, \cdots, L_n^*\} \) where \( L_j^* \)'s are bounding constants satisfying \( a^{(j+1)} \in \Omega_T^L \) as described in Lemma 1. Then, choose \( L_j \)'s such that \( L_1 = L_2 = \cdots = L_n = L \) with \( L > L^* \). Along with the additional determination of \( k_j \overset{\Delta}{=} \beta_j L \) where \( \beta_j \)'s are pre-defined constants, the HOSD (6) can be reexpressed as follows.

\[
\dot{\sigma}_j = \beta_j e_{aj} + \sigma_j \geq 0, \quad j = 1, 2, \cdots, n \tag{8}
\]

Through numerous simulations, the constants \( \beta_j \)'s up to \( j = 6 \) have been determined as

\[
\beta_1 = 10, \quad \beta_2 = 7, \quad \beta_3 = 5.5, \quad \beta_4 = 4.8, \quad \beta_5 = 4.4, \quad \beta_6 = 4.2 \tag{9}
\]

That is, substituting \( k_j \) for \( \beta_j L \) with predetermined \( \beta_j \)'s in (9) eliminates the need to determine the design constant \( k_j \) separately if the only \( L \) is determined. For example, the specific HOSD dynamics for \( n = 3 \) are described as shown in (43). The only design constant \( L \) must be increased to improve the estimation performance of the HOSD.

### III. CONTROLLER DESIGN

As assumed previously, the relative degree \( r \) is assumed to be unknown, but its upper bound \( n \) is known, i.e., \( r \leq n \). The original system order \( m \) has no direct relation with \( n \). However, if \( m \) is known, then \( n \) can be chosen as \( m \), since the relative degree cannot exceed the system dynamic order. If the zero dynamics exist, it is assumed to be stable according to Assumption 2. The proposed control scheme is based on the control input filter and HOSD with over dimension.

### A. CONTROL INPUT FILTERING

The following simple linear time-invariant (LTI) filter is introduced to constitute a signal that is fed into the HOSD (8)

\[
\dot{w}_i = -cw_i + w_{i+1}, \quad i = 1, \cdots, n - 1 \tag{10}
\]

\[
w_n = -cw_n + u
\]

where \( c > 0 \) is a design constant. The signal \( a(t) \) that is fed into the HOSD is generated as

\[
a = e - w_1. \tag{11}
\]

Then, from Lemma 1, the following equalities hold for \( \sigma_j \) \( (j = 1, 2, \cdots, n) \):

\[
\sigma_1 = \dot{a} + d_1(t) = \dot{e} - p_1(w) - w_2 + d_1(t) \tag{12}
\]

\[
\sigma_2 = \dot{d}_1 + d_2(t) = \dot{e} - p_2(w) - w_3 + d_2(t) \tag{13}
\]

\[\vdots\]

\[
\sigma_{n-1} = \dot{d}_{n-1} + d_{n-1}(t) = e^{(n-1)} - p_{n-1}(w) - w_n + d_{n-1}(t) \tag{14}
\]

\[
\sigma_n = \dot{d}_n + d_n(t) = e^{(n)} - p_n(w) - u + d_n(t) \tag{15}
\]

where \( d_i(t) \)'s are estimation errors that vanish asymptotically and \( p_i(w) \)'s are the polynomials of \( w \) satisfying \( w_i^{(l)}(t) = p_j + w_{j+1} \) for \( j = 1, \cdots, n \) with \( w_{n+1} = u \). The \( p_j \)'s are easily calculated for \( i = 1, \cdots, 6 \) as follows:

\[
p_1(w) = -cw_1 \tag{16}
\]

\[
p_2(w) = c^2 w_1 - 2cw_2 \tag{17}
\]

\[
p_3(w) = -c^3 w_1 + 3c^2 w_2 - 3cw_3 \tag{18}
\]

\[
p_4(w) = c^4 w_1 - 4c^3 w_2 + 6c^2 w_3 - 4cw_4 \tag{19}
\]

\[
p_5(w) = -c^5 w_1 + 5c^4 w_2 - 10c^3 w_3 + 10c^2 w_4 - 5cw_5 \tag{20}
\]

\[
p_6(w) = c^6 w_1 - 6c^5 w_2 + 15c^4 w_3 - 20c^3 w_4 + 15c^2 w_5 - 6cw_6 \tag{21}
\]
All the states of the filter (10), i.e., $w_i$, are all bounded as described in the following lemma.

**Lemma 2 ([29]):** Under Assumption 2, the following inequalities hold

$$|w_i| < \frac{r_i}{\kappa^{n-i+1}} \quad (22)$$

for $i = 1, \ldots, n$.

Note that the scheme of using a control input filter to make the signal into the differentiator is originally proposed in [19], [21].

**B. CONTROL LAW AND STABILITY ANALYSIS**

Let the tracking error vector be $e = [e, \dot{e}, \ldots, e^{(n-1)}]^T \in \mathbb{R}^n$; then, its estimate can be obtained using (12)-(14) as

$$\dot{\hat{e}} = \begin{bmatrix} e \\ \sigma_1 + p_1 + w_2 \\ \vdots \\ \sigma_n + p_n + w_n \end{bmatrix} \in \mathbb{R}^n \quad (23)$$

which becomes $e$ asymptotically by Lemma 1. The control law is determined as

$$u = -\sigma_n - p_n - k^T \dot{\hat{e}} \quad (24)$$

where the constant vector $k = [k_1, k_2, \ldots, k_n]^T$ is chosen such that

$$(s + \kappa)^n = s^n + k_n s^{n-1} + \cdots + k_2 s + k_1 \quad (25)$$

with $\kappa > 0$ is a design constant.

The main result of the proposed control scheme is described in the following theorem.

**Theorem 1:** Consider system (1) under Assumptions 1 through 5. The control input (24) using the HOSD (8) and input filter (10) makes the tracking error vector $e$ asymptotically stable.

**Proof:** The following equality is easily induced from (24) and (15).

$$u = -\sigma_n - p_n - k^T \dot{e}$$

$$= -e^{(n)} - p_n - u + d_n(t) - p_n - k^T e - k^T d(t)$$

$$= -e^{(n)} + u - k^T e - d_n(t) - k^T d(t). \quad (26)$$

where $d(t) = [0, d_1(t), \ldots, d_{n-1}(t)]^T$. The following resultant equation is easily derived from the last equality of (26).

$$e^{(n)} = -k^T e - d_n(t) - k^T d(t) \quad (27)$$

This can be described more concisely as a vector form equation

$$\dot{\hat{e}} = \mathbf{A} e + \mathbf{b} d(t) \quad (28)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & -k_2 & -k_3 & \cdots & -k_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (29)$$

and $d(t) = -d_n(t) - k^T d(t)$. There exist positive definite matrices $P$ and $Q$ such that $A^T P + PA + Q = 0$. For Lyapunov function $V = e^T P e$, its time derivative is induced as

$$\dot{V} = -e^T Q e + 2e^T P d(t)$$

\[ \leq -\lambda_{\text{min}}(Q)\|e\|^2 + 2\|e\|\lambda_{\text{max}}(P)\|d(t)\| \quad (30) \]

From the last inequality, it is concluded that if $|e| > \lambda|d(t)|$ where $\lambda = \frac{2\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(Q)}$, then $\dot{V} < 0$. This means that, since $d(t)$ converges to zero asymptotically, $|e|$ is also asymptotically stable. \qed

According to Lemma 1, the condition on the input signal $a(t)$ into HOSD is that $a^{(j)}$s for $j = 1, 2, \ldots, n + 2$ are all BPWS.

**Corollary 1:** The time-derivatives $e^{(j)}$s for $j = 1, 2, \ldots, n + 2$ are all BPWS under Assumptions 4 and 5.

**Proof:** Because a function that adds two BPWS functions is also BPWS, if $e^{(j)}$ and $w^{(j)}$ ($j = 1, 2, \ldots, n + 2$) are each BPWS, then $e^{(j)}(= e^{(j)} - w^{(j)})$ can be said to be BPWS. Also note that a bounded continuous function is trivially BPWS. From Assumption 4, since $y_{\min}$'s for $j = 1, 2, \ldots, n + 2$ are bounded, they are BPWS. Therefore, showing that the time-derivatives of system output $y^{(j)}$s for $j = 1, 2, \ldots, n + 2$ are BPWS is sufficient to prove that $e^{(j)}$s are BPWS. The first time-derivative of $y$ is derived as follows.

$$\dot{y} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t} \quad \triangleq \quad \zeta_1(x, t) \quad (31)$$

By Assumption 5, $\zeta_1(x, t)$ is smooth and bounded on $\mathcal{D}$. It is also clear that $\frac{\partial \zeta_1}{\partial t}(x, t)$ is bounded on $\mathcal{D}$ for $i = 1, \ldots, n - 1$. The next time-derivative is induced as follows.

$$\ddot{y} = \frac{\partial \zeta_1}{\partial x} f + \frac{\partial \zeta_1}{\partial t} \quad \triangleq \quad \zeta_2(x, t) \quad (32)$$

Again by Assumption 5, $\zeta_2(x, t)$ is also smooth and bounded on $\mathcal{D}$. It is also true that $\frac{\partial \zeta_2}{\partial t}(x, t)$ is bounded on $\mathcal{D}$ for $i = 1, \ldots, n$. In this sense, for $j = 1, \ldots, r - 1$, the following equations are induced

$$y^{(j)} = \frac{\partial \zeta_{j-1}}{\partial x} f + \frac{\partial \zeta_{j-1}}{\partial t} \quad \triangleq \quad \zeta_j(x, t, u) \quad (33)$$

with $\zeta_0 = h$. By Assumption 5, $\zeta_j$'s for $j = 1, \ldots, r - 1$ are smooth and bounded on $\mathcal{D}$ and $\frac{\partial \zeta_j}{\partial u}$ is also bounded on $\mathcal{D}$ for $i = 1, 2, \ldots, n + 2 - j$.

For $j = r$, the control input $u$ appears in $y^{(r)}$ as

$$y^{(r)} = \frac{\partial \zeta_{r-1}}{\partial x} f + \frac{\partial \zeta_{r-1}}{\partial t} \quad \triangleq \quad \zeta_r(x, t, u) \quad (34)$$

and since $\zeta_r$ is still a smooth function, $y^{(r)}$ is bounded on $\mathcal{D}$. For $j = r + 1$, the following differential equation is derived

$$\frac{\partial y^{(r+1)}}{\partial x} f + \frac{\partial y^{(r+1)}}{\partial t} + \frac{\partial \zeta_r}{\partial u} \frac{\partial u}{\partial t} \quad \triangleq \quad \zeta_{r+1}(x, t, u, \dot{u}) \quad (35)$$
The time-derivative of the control input is easily induced from (24) as

\[
\ddot{u} = -\sigma_n - \dot{p}_n - k^T \dot{e}
\]

\[
= \mu_1(\text{sgn}(e_{a1}), \ldots, \text{sgn}(e_{an}), w_1, \ldots, w_n, u, \dot{e})
\]

(36)

where \(\mu_1(\cdot)\) is a linear function of its variables. Because the \(\text{sgn}(\cdot)\) function and \(w_i\)s are all bounded, and since \(\dot{e}\) has been shown to be bounded in the first step of this proof, it is certain that \(\ddot{u}\) is bounded and, thus, BPWS. This leads to the boundedness of \(y^{(r+1)}\). Before the next step, it should be noted that \(\frac{d^i}{dt^i} \text{sgn}(\cdot)\)s for \(i = 1, 2, \ldots, n\) are all in \(\mathbb{R}_T^0\) with some interval \(T_i \in \Phi\) since \(\text{sgn}(\cdot)\) is constant within the subintervals.

The next time-derivative of \(y\) is described as

\[
y^{(r+2)} = \frac{\partial \xi_{r+1}}{\partial x} f + \frac{\partial \xi_{r+1}}{\partial t} \dot{f} + \sum_{j=0}^{i-1} \frac{\partial \xi_{r+1}}{\partial u} \dot{u}^{(j+1)}
\]

\[
= \xi_{r+1}(x, t, u, \dot{u}, \ddot{u}, \ldots, u^{(i)})
\]

(37)

The \(u\) is derived from (36) as

\[
\ddot{u} = \mu_2 \left( \frac{d}{dt} \text{sgn}(e_{a1}), \ldots, \frac{d}{dt} \text{sgn}(e_{an}), w_1, \ldots, w_n, \dot{u}, \ddot{u}, \dot{e} \right)
\]

(38)

where \(\mu_2\) is another linear function of its variables. Since \(\ddot{u}\) and \(\dddot{e}\) have been proven to be bounded in the previous steps, and since the time-derivatives of the \(\text{sgn}(\cdot)\) functions are all BPWS, \(u\) is clearly BPWS. Thus, \(y^{(r+2)}\) is also BPWS. In general, for \(i = 1, \ldots, n + 2 - r\), \(y^{(r+i)}\) and \(u^{(i)}\) can be respectively described as follows.

\[
y^{(r+i)} = \frac{\partial \xi_{r+i-1}}{\partial x} f + \frac{\partial \xi_{r+i-1}}{\partial t} \dot{f} + \sum_{j=0}^{i-1} \frac{\partial \xi_{r+i-1}}{\partial u} \dot{u}^{(j+1)}
\]

\[
= \xi_{r+i}(x, t, u, \dot{u}, \ddot{u}, \ldots, u^{(i)})
\]

(39)

\[
\dot{u}^{(i)} = \mu_i \left( \frac{d^i}{dt^i} \text{sgn}(e_{a1}), \ldots, \frac{d^i}{dt^i} \text{sgn}(e_{an}), w_1, \ldots, w_n, u^{(i-1)}, e^{(0)} \right)
\]

(40)

Based on the facts that \(u^{(i-1)}\) and \(e^{(0)}\) have been shown to be BPWS in the former steps and \(\mu_i\) is a linear-in-the-parameters function, \(\dot{u}^{(i)}\) is also BPWS. This leads to the conclusion that \(y^{(r+i)}\) is BPWS. In this sense, the boundednesses in the piecewise sense of \(y^{(r+i)}\) from \(i = 1\) to \(i = n + 2 - r\) can be shown recursively.

For the time-derivatives of \(w_1\), the following holds for \(j = 1, 2, \ldots, n - 1\)

\[
w_1^{(j)} = p_1(w) + w_{j+1}
\]

(41)

further, since \(w_i\)s \((i=1, \ldots, n)\) are all bounded from Lemma 2 and \(p_1(w)\)s are all linear functions of them, \(w_i^{(j)}\)s \((j = 1, \ldots, n - 1)\) are all bounded and, thus, BPWS. The higher-order derivatives are described as

\[
w_1^{(n)} = p_n(w) + u
\]

\[
w_1^{(n+1)} = p_{n+1}(w) + \dot{u}
\]

\[
w_1^{(n+2)} = p_{n+2}(w) + \ddot{u}
\]

(42)

where \(p_{n+1}(w)\) and \(p_{n+2}(w)\) are some linear functions of \(w_i\)s \((i = 1, \ldots, n)\). From the fact that \(u\) (Assumption 1), \(\dot{u}\), and \(\ddot{u}\) are all BPWS, as shown in the previous steps, it is clear that \(w_1^{(n)}, w_1^{(n+1)}, w_1^{(n+2)}\) are also BPWS.

Taken together, these findings indicate that \(u^{(i)}(= e^{(0)} - w_1^{(j)})\) for \(j = 1, \ldots, n + 2\) are all BPWS.

Remark 1: The proposed control scheme relies heavily on the performance of the differentiator. In the nonlinear system control, a high-gain observer (HGO) [30] and a higher-order sliding-mode differentiator (HOSMD) [31] are widely adopted to estimate the time-derivatives of tracking error or system output. However, HOSD proposed in [19] shows better transient and steady-state responses than HGO or HOSMD while guaranteeing asymptotical tracking performance. Moreover, HOSD has additional performance merits since its estimations show no peaking or chattering. For these reasons, the HOSD is adopted in this paper.

Remark 2: In the proposed control law, there are only three design constants \(L, \epsilon, \) and \(\kappa\). Moreover, it is confirmed through various simulations that the value of \(\kappa\) has no significant effect on controller performance. It is sufficient to choose \(\epsilon = 1\), and, in this sense, the number of design constants is only two. It is also worth noting that no time derivatives of \(y_{d}(t)\) are required in the controller design. This is beneficial because, in real physical systems, they are often tough to measure or calculate.

The overall design steps for the controller are summarized as follows.

1. For the \(n\)th-order system (1), construct the HOSD (8) with appropriately determined constant \(L\).
2. Construct the LTI filter (10) with constant \(c\). By observing numerous simulations, as mentioned previously, choosing \(c = 1\) is suitable for most cases.
3. Formula (24) with a properly chosen \(\kappa\) and calculated vector \(k\) using (25) is used to generate the control input.

These steps will be applied to an example controller in the next section.

IV. SIMULATIONS

In this section, simulations are performed using three example nonlinear systems that have various relative degrees. Although the relative degrees of those example systems are different, the structures of the controllers applied are all the same if the systems have the same maximum values of the relative degrees.

A. CONTROLLER DESIGN FOR EXAMPLE SYSTEMS

The maximum values of the relative degrees of the example systems are all assumed to be 3 (i.e., \(r \leq 3\)). The design procedure of the controller is as follows. First, the dynamics of HOSD that estimate \(\dot{a}, \ddot{a}, \) and \(\dddot{a}\), where \(a = e - w_1\), are as follows:

\[
\dot{a}_1 = 10Le_{a1} + \sigma_1 \text{ with } e_{a1} = a(t) - \alpha_1
\]

\[
\dot{\sigma}_1 = L \text{sgn}(e_{a1})
\]

\[
\dot{\sigma}_2 = 7Le_{a2} + \sigma_2 \text{ with } e_{a2} = \sigma_1 - \alpha_2
\]
\[ \dot{\sigma}_2 = L \text{sgn}(e_{\alpha 2}) \]
\[ \dot{\sigma}_3 = 5.5 Le_{\alpha 3} + \sigma_3 \text{ with } e_{\alpha 3} = \sigma_2 - \alpha_3 \]
\[ \dot{\sigma}_3 = L \text{sgn}(e_{\alpha 3}) \] (43)

From Lemma 1, it is guaranteed that \( \sigma_1 \to \dot{a} \), \( \sigma_2 \to \ddot{a} \), and \( \sigma_3 \to a \). Note that the only design constant of (43) is \( L \).

To generate filtered signals \( w_1, w_2, \) and \( w_3 \), the following 3rd-order LTI filter using control input is constructed.

\[ \dot{w}_1 = -cw_1 + w_2 \]
\[ \dot{w}_2 = -cw_2 + w_3 \]
\[ \dot{w}_3 = -cw_3 + u \] (44)

The controller (24) in this case (i.e., \( n = 3 \)) is

\[ u = -\sigma_3 - p_3 - k^T \hat{e} \] (45)

where

\[ \hat{e} = \begin{bmatrix} e \\ \sigma_1 + p_1 + w_2 \\ \sigma_2 + p_2 + w_3 \end{bmatrix} \] (46)

and \( p_j \)s for \( j = 1, 2, 3 \) are defined in (16), (17), and (18), respectively. The constant \( c \) in (44) is set as 1, and the initial values for the control input filter (44) and HOSD (43) in all the following simulations are all zeros.

### B. 1st EXAMPLE

The first example is the following 1st-order nonlinear system with \( r = m = 1 \).

\[ \dot{x} = \cos x - x^3 + u \]
\[ y = x \] (47)

It is assumed that the structure and the relative degree are not known to the controller except \( r \leq n = 3 \). The HOSD parameter is chosen as \( L = 10 \). The controller (45) is applied with the calculated vector \( k = [125, 75, 15]^T \) using \( \kappa = 5 \) and (25). The initial state and desired output are \( x(0) = -0.5 \) and \( y_d(t) = \sin t \), respectively. The simulation results are shown in Fig. 1 and Fig. 2; as shown in the figures, system output \( y \) tracks \( y_d \) well, and all the time-varying signals are bounded.

### C. 2nd EXAMPLE

The second example is the 2nd-order system (\( r = m = 2 \)) with the following dynamics.

\[ \dot{x}_1 = x_1 + x_2 + 0.2 x_2^3 \]
\[ \dot{x}_2 = x_1 x_2 + u + \frac{u^3}{7} \]
\[ y = x_1 \] (48)

As in the previous case, it is assumed that the controller does not know the structure and relative degree of the system except \( r \leq n = 3 \). Thus, the same controller (45) applied. The control objective is to track the desired output \( y_d(t) = \sin t \). The design constants are \( L = 11 \) and \( \kappa = 1.2 \). The calculated \( k \) vector using \( \kappa = 1.2 \) and (25) is \( k = [1.728, 4.32, 3.6]^T \). The initial state is \( x(0) = [-0.1, 0]^T \), and the simulation results are shown in Fig. 3 and Fig. 4; as shown in the
D. 3rd EXAMPLE

The third example is the 3rd-order system \( m = 3 \) with internal dynamics:

\[
\begin{align*}
\dot{x}_1 &= 2x_2 \\
\dot{x}_2 &= 2x_3x_2 + \sin x_2 + 0.5u \\
\dot{x}_3 &= -x_3 + \exp(2x_2) \\
y &= x_1
\end{align*}
\]

The controller does not know the actual relative degree \( r = 2 \) but has the information \( r \leq 3 \). Thus, the same controller (45) is also applied here with the desired output \( y_{d}(t) = -0.5 \exp(-2t) \). The design constant of HOSD (43) is \( L = 15 \). The calculated \( k \) vector using selected \( \kappa = 6 \) and (25) is \( k = [216, 108, 18]^T \). The initial state vector of the system is \( x(0) = [0, 0, 0]^T \). As depicted in Fig. 5 and Fig. 6, system output \( y \) tracks \( y_{d} \) well, and all the signals are bounded.

Remark 3: As shown by the simulation results, the controller (45) with (43) and (44) can be used as a universal controller for systems whose dynamic orders are less than or equal to 3 (i.e., \( m \leq 3 \)). The controller (45) is also applicable for systems with a varying relative degree as long as the time-varying relative degree \( r(t) \) satisfies \( r(t) \leq 3 \) for all \( t > 0 \).

V. CONCLUSION

A novel output-feedback controller for a fairly general class of uncertain nonlinear systems with an unknown relative degree is proposed in this paper. With the assumption that the upper bound of the relative degree is known, the proposed control scheme combines the LTI input filter and HOSD with over dimension. To the best of the authors’ knowledge, no study has attempted to design a controller for the general nonlinear system considered in this paper with an unknown relative degree. The proposed differentiator-based controller depends substantially on the estimation performance of HOSD. The effects of unstructured uncertainties containing unknown relative degree, which are intrinsic in the controlled system, are captured by time-derivative estimations of the output tracking error using HOSD. The proposed control law is very compact and has only two design constants. It is proved that asymptotic tracking performance is guaranteed and that the condition on the input signal into HOSD is matched. This paper has demonstrated the performance and compactness of the proposed controller through simulations of various example systems.

REFERENCES

[1] J.-H. Park, S.-H. Kim, and T.-S. Park, “Output-feedback adaptive neural controller for uncertain pure-feedback nonlinear systems using a high-order sliding mode observer,” IEEE Trans. Neural Netw. Learn. Syst., vol. 30, no. 5, pp. 1596–1601, May 2019.

[2] J. Wan, T. Hayat, and F. E. Alsaadi, “Adaptive neural globally asymptotic tracking control for a class of uncertain nonlinear systems,” IEEE Access, vol. 7, pp. 19054–19062, 2019.

[3] J.-H. Park, S.-H. Kim, and C.-J. Moon, “Adaptive neural control for strict-feedback nonlinear systems without backstepping,” IEEE Trans. Neural Netw., vol. 20, no. 7, pp. 1204–1209, Jul. 2009.
I. S. Dimanidis, C. P. Bechlioulis, and G. A. Rovithakis, “Output feedback control for the SISO system with the unknown order and the unknown relative degree,” *Automatica*, vol. 51, no. 1, pp. 54–63, Apr. 2015.

J.-H. Park, T.-S. Park, and S.-H. Kim, “Asymptotically convergent higher-order switching differentiator,” *Mathematics*, vol. 8, no. 2, p. 185, Feb. 2020.

J.-E. Slotine and W. Li, *Applied Nonlinear Control*. Upper Saddle River, NJ, USA: Prentice-Hall, 1991.

H. K. Khalil, “High-gain observers in feedback control: Application to permanent magnet synchronous motors,” *IEEE Control Syst. Mag.*, vol. 37, no. 3, pp. 25–41, Jun. 2017.

A. Levant, “Higher-order sliding modes, differentiation and output-feedback control,” *Int. J. Control*, vol. 76, nos. 9–10, pp. 924–944, Jan. 2003.

JANG-HYUN PARK received the B.S., M.S., and Ph.D. degrees in electrical engineering from Korea University, Seoul, South Korea, in 1995, 1997, and 2002, respectively. He is currently a Professor with the Department of Electrical and Control System Engineering, Mokpo National University, Mokpo, South Korea. His main research interests include robust control, fuzzy control, and their implementations to real plants.

SEONG-HWAN KIM (Associate Member, IEEE) received the B.S., M.S., and Ph.D. degrees in electrical engineering from Korea University, Seoul, South Korea, in 1991, 1995, and 1998, respectively. He is currently a Professor with the Department of Electrical and Control System Engineering, Mokpo National University, Mokpo, South Korea. His research interests include neuro-control, fuzzy control, adaptive nonlinear control, and power electronics.

TAE-SIK PARK (Member, IEEE) received the B.S., M.S., and Ph.D. degrees in electrical engineering from Korea University, Seoul, South Korea, in 1994, 1996, and 2000, respectively. He is currently an Associate Professor with the Department of Electrical and Control System Engineering, Mokpo National University, Mokpo, South Korea. His main research interest includes DSP-based real-time control of electrical drives.