Dissipative or just Nonextensive hydrodynamics?  
- Nonextensive/Dissipative correspondence -

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We argue that there is correspondence between the perfect nonextensive hydrodynamics and the usual dissipative hydrodynamics, which we call nonextensive/dissipative correspondence (NexDC). It leads to simple expression for dissipative entropy current and allows for predictions for the ratio of bulk and shear viscosities to entropy density, \( \zeta/s \) and \( \eta/s \).

Recently there is renewed interest in dissipative hydrodynamical models [1, 2, 3, 4, 5] prompted by the success of perfect hydrodynamics in describing RHIC data [6] and by recent calculations of transport coefficients of strongly interacting quark-gluon system using the AdS/CFT correspondence [7]. The question is whether dissipative hydrodynamics (d-hydrodynamics) is really needed and how it should be implemented regarding problems with its formulation like ambiguities in the form equations used [3], unphysical instability of the equilibrium state in the first order theory [4] or loss of causality in the first order equation approach [5]. We would like to propose a novel view on the dissipative hydrodynamics based on nonextensive formulation of the ideal hydrodynamical model, which we call the q-hydrodynamics [8]. When applied to ideal q-fluid it can be solved exactly, in way similar to the usual ideal hydrodynamics. However, it contains additional terms which can be interpreted as due to dissipative effects expressed by the nonextensivity parameter \( q \) - a single parameter here. Therefore, the following nonextensive/dissipative correspondence (NexDC) emerges: ideal q-fluid is apparently equivalent to some viscous fluid with its transport coefficients being (implicit) functions of parameter \( q \). This parameter combines information about all possible intrinsic fluctuations and correlations existing in the collision process (in particular in the QGP being formed). Referring to [8] for more information on q-statistics it is enough to say here that it is based on (indexed by \( q \) and nonextensive, see left panel of Fig. 1) Tsallis rather than Boltzmann-Gibbs (BG) entropy to which it converges for \( q \to 1 \). Characteristic feature here is appearance of q-exponentials, \( \exp_q(-X) = [1 - (1 - q)X]^{1/(1-q)} \to \exp(-X) \) for \( q \to 1 \). Among other things \( q - 1 \) measures scaled variance of the corresponding intensive quantities like, for example, temperature \( T \), or the amount of nonvanishing in the hydrodynamical limit correlations (see left panel of Fig. 1 [8]). Although q-hydrodynamics does not fully solve the problems of d-hydrodynamics, nevertheless it allows us to extend the usual perfect fluid approach (using only one new parameter \( q \)) well behind its usual limits toward the regions reserved for dissipative approach only.

In this note we can only explain main points of our proposition leaving interested reader to [8] and references there for details. Our idea is visualized in Fig. 1. Left panel shows that in the usual hydrodynamics there is some spacial scale \( L_{\text{hyd}} \), such that volume \( L_{\text{hyd}}^3 \) contains enough particles composing our fluid. However, in case when there are some fluctuations and/or correlations in the system characterized by some typical correlation length \( l \) and when of \( l > L_{\text{hyd}} \), taking the usual limit \( L_{\text{hyd}} \to 0 \) removes the explicit dependence on the scale \( L_{\text{hyd}} \) but the correlation length \( l \) leaves its imprint as parameter \( q \) and one has to use nonextensive entropy (one can argue that in this case...
\( q \sim l/L_{\text{hyd}} \geq 1 \). The situation encountered is shown on right panel of Fig. 1. Locally conserved BG entropy current \( s^q(x) \) and expensive entropy \( S \) is replaced by locally conserved entropy \( S_q \) local equilibrium is replaced by a kind of stationary state (or \( q \)-equilibrium containing dynamic leading to the assumed intrinsic fluctuations/correlations and summarily characterized by the parameter \( q \)).

**Non-extensivity for system with correlations**

\[ L_{\text{hyd}} \leq l/L \leq 1 \]

\[ 0 \leq L_{\text{hyd}} < 1 \]

\[ q \leq 1 \]

\[ q > 1 \]

- extensive
- \( q \)-modified
- stationary

**Standard hydrodynamics vs. \( q \)-hydrodynamics**

\[ s^q(x) = -k_B \int \frac{d^3p}{(2\pi \hbar)^3} p^\mu \left( f_q^q \ln f_q - f_q \right), \]

where \( \ln_q f_q \equiv \left[ f_q^{1-q} - 1 \right]/(1-q) \) (1)

where \( f_q = f_q(x,p) \) is nonextensive version of phase space distribution function in space-time position \( x \) and momentum \( p \). One finds that \( \partial_\mu \sigma^\mu_q \geq 0 \) at any space-time point, i.e., relativistic local \( H \)-theorem is valid [8, 9]. Demanding now that \( \partial_\mu \sigma^\mu_q \equiv 0 \) one gets

\[ f_q(x,p) = \left[ 1 - (1-q)\frac{p_\mu u^\mu_q(x)}{k_B T_q(x)} \right]^{1/(1-q)} \] (2)

where \( T_q(x) \) is the \( q \)-temperature [8] and \( u^\mu_q(x) \) is the \( q \)-hydrodynamical flow four-vector. The state characterized by \( f_q(x,p) \) is the local \( q \)-equilibrium state, i.e., a kind of stationary state which includes already some interactions between particles composing our fluid (see right panel of Fig. 1). The symmetry of collision term in the nonextensive Boltzmann equation [8, 9] and energy-momentum conservation in two particle collisions result in the nonextensive version of local energy-momentum conservation,

\[ T^\mu_{\nu;q} = 0, \quad \text{where} \quad T^\mu_{\nu;q}(x) = \frac{1}{2(2\pi \hbar)^3} \int \frac{d^3p}{p^0} p^\mu p^\nu f_q^q(x,p). \] (3)

Assuming now that this \( q \)-energy-momentum tensor can be decomposed in the usual way in terms of the \( q \)-modified energy density \( \varepsilon_q \) and \( q \)-pressure \( P_q \) by using the \( q \)-modified flow \( u^\mu_q \) (such that \( u^\mu_q = (1,0,0,0) \) in the rest frame) one obtains the perfect \( q \)-hydrodynamical equation \( \Delta^\mu_{\nu,q} = g^\mu_{\nu} - u^\mu_q u^\nu_q \) (covariant derivative notation is used here, see [8]):

\[ T^\mu_{\nu;q;\mu} = \left[ \varepsilon_q(T_q) u^\mu_q u^\nu_q - P_q(T_q) \Delta^\mu_{\nu,q} \right]_{;\mu} = 0. \] (4)
One the other hand one can also decompose $T_{q}^{\mu\nu}$, using the usual 4-velocity fluid field $u^\mu$ and obtain equation

$$
\left[ \tilde{\varepsilon}(T_q) u^\mu u^\nu - \tilde{P}(T_q) \Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu} \right]_{;\mu} = 0,
$$

(5)

where $(\delta u^\mu_q \equiv u^\mu_q - u^\mu)$ and $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu} u^{\nu})$ whereas

$$
\tilde{\varepsilon} = \varepsilon_q + 3P; \quad \tilde{P} = P_q + \Pi; \quad W^\mu = w_q[1 + x] \Delta^\mu \delta u^\lambda_q; \quad \pi^{\mu\nu} = \frac{W^\nu W^\mu}{w_q[1 + x]^2} + \Pi \Delta^{\mu\nu} = w_q \delta u^<\mu \delta u^>_{\nu}; \quad \Pi = \frac{1}{3} w_q[x^2 + 2x]
$$

(6)

are, respectively, energy density ($\tilde{\varepsilon}$), pressure ($\tilde{P}$), energy or heat flow vector ($W^\mu$), shear (symmetric and traceless) pressure tensor ($\pi^{\mu\nu}$) and bulk pressure ($\Pi$). Notation used is: $A^{(\mu} B^{\nu)} = \frac{1}{2}(A^{\mu} B^{\nu} + A^{\nu} B^{\mu})$; $w_q \equiv \varepsilon_q + P_q$; $x \equiv u_\mu \delta u^\mu_q = -\frac{1}{2} \delta u_{\mu q} \delta u^\mu_q$; $a^{<\mu} b^{>\nu} \equiv \left[ \frac{1}{2} (\Delta^\mu \Delta^\nu + \Delta^\nu \Delta^\mu) - \frac{1}{2} \Delta^{\mu\nu} \Delta_{\lambda\sigma} \right] a^\lambda b^\sigma$. Notice that whereas the time evolution of $\Pi$ is controlled by $q$-hydrodynamics (via the respective time dependencies of $\varepsilon_q$, $P_q$ and $x$) its form is determined by the assumed constraints which must assure that the local entropy production in the standard 2nd order theory $\Pi$ is never negative.

The crucial point of our work is assumption that there exists some temperature $T$ and velocity $\delta u^\mu_q$ satisfying the following NexDC relations:

$$
P(T) = P_q(T_q), \quad \varepsilon(T) = \varepsilon_q(T_q) + 3\Pi
$$

(7)

($\varepsilon$ and $P$ are energy density and pressure defined in the usual Boltzmann-Gibbs statistics, i.e., for $q = 1$). In this case one can transform equation (5) into the following usual d-hydrodynamical equation (8):

$$
\left[ \varepsilon(T) u^\mu u^\nu - (P(T) + \Pi) \Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu} \right]_{;\mu} = 0.
$$

(8)

This completes demonstration of the equivalence of perfect $q$-hydrodynamics represented by Eq. (4) and its $d$-hydrodynamics counterpart represented by Eq. (8). We propose to call this equivalence NexDC: nonextensive/dissipative correspondence. In [8] we have successfully applied $q$-hydrodynamics to description of RHIC data on particle production (in a limited fashion, however; to apply it to flow effects and correlation phenomena, like HBT effect, one must go out the one-dimensional approximation used here - such work is now in progress).

The most important point in in NexDC is the fact that although in ideal $q$-hydrodynamics the $q$-entropy is conserved, i.e., $[\varepsilon_q u^\mu]_{;\mu} = 0$, we can rewrite it in the form corresponding to dissipative fluid with entropy production: $[\varepsilon^\mu u^\mu]_{;\mu} = -\frac{1}{P_T} \delta T^\mu_{;\mu}$. When applied to description of multiparticle production processes this fact is seen in the prediction of $q$-dependent increase of multiplicity of produced particles [8]. The most general expression for the full order dissipative entropy current in the NexDC approach:

$$
\sigma^\mu_{\text{full}} \equiv su^\mu + \frac{W^\mu}{T} - \frac{2T}{T_q} \left[ 1 - \sqrt{1 - \frac{3\Pi}{w}} \right] su^\mu + \frac{2(T - T_q)}{T_q} \frac{W^\mu}{T}.
$$

(9)

One can now calculate bulk and shear viscosities emerging from the NexDC. Here we shall present only our main result, namely the sum rule connecting ratios of bulk and shear viscosities over the entropy density $s$:

$$
\frac{1}{\zeta/s} + \frac{3}{\eta/s} = \frac{w\sigma^\mu_{\text{full};\mu}}{\Pi^2}.
$$

(10)
To disentangle it some additional input is needed. Results presented in Fig. 2 are obtained assuming that total entropy is generated by action of the shear viscosity only. This can be confronted with AdS/CFT conjecture \([7]\) that \(\eta/s \geq 1/4\pi\).

To summarize: we claim that nonextensive approach to hydrodynamics can be regarded as a new phenomenological way to deal with viscous fluids in which many different dynamical features (already known or yet to be discovered) are summarily represented by a single parameter \(q\) describing a kind of \(q\)-ideal fluid by means of ideal \(q\)-hydrodynamics, which is apparently much simpler to handle than the usual dissipative hydrodynamics.

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