Analytical continuation and resummed perturbation theory

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Abstract

A pattern of partial resummation of perturbation theory series inspired by analytical continuation is discussed for some physical observables.

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1 INTRODUCTION

At low energy $\mu \sim m_\tau$ an expansion parameter of perturbation theory the strong coupling constant $\alpha_s = \alpha_s(m_\tau) \simeq 0.35$ is rather large (e.g. [1]) and, therefore, for a generic QCD observable $\sigma$ (without Born term)

$$\sigma \sim \alpha_s(1 + \sigma_1 \alpha_s + \sigma_2 \alpha_s^2 + \ldots)$$  \hspace{1cm} (1)

the series of perturbation theory approximation converges badly the last term being about 10% of the leading one.

To improve predictions that is required by experimental data for some observables at present, higher order corrections have to be included. This, however, is technically difficult due to the necessity of computing many-loop integrals representing Feynman diagrams in high orders of perturbation theory. There is a little hope to obtain next terms in (1) (beyond three-four loops) for many processes in realistic models.

It should be stressed that phenomenologically (leaving apart general considerations of possible power corrections that are necessary for two point correlators and lead to resonances) there is no much intrinsic reason to go beyond perturbation theory: wild asymptotic behavior is not seen yet. Also the freedom of the choice of the renormalization schemes allows one to render the series convergent for a given observable (or some set of observables) at least in an heuristic sense that further known terms decrease [2]. There is no strict indication that perturbation theory is broken though the accuracy it can provide in a number of cases is not sufficient for confronting predictions with experimental data. Thus, at the level of phenomenology just needs of precision require an improvement of perturbation theory predictions and because further terms are not available going beyond perturbation theory in different ways is now widely discussed [3].

Before adding genuine nonperturbative terms which is not obvious in cases when Wilson operator product expansion is not directly applicable one tries to go beyond the finite order perturbation theory by a resummation of a particular subset of terms that can be explicitly generated. The simplest one is due to running of the coupling constant.

This resummation is ambiguous to a great extent in particular it can change the analytic properties that exist in any finite order of perturbation theory and are established on a general ground of quantum field theory and even can make them wrong [4], i.e. resummed quantities can not satisfy some general requirements that leads to necessity of interpretation of the results. The ambiguities that are produced by the resummation and the change of analytic properties are analyzed in some details.
2 AMBIGUITY FOR THE $\tau$ LEPTON WIDTH IN MS SCHEME

The $\tau$ lepton width became a real laboratory for investigation of properties and numerical validity of low energy perturbation theory \[5\].

The spectral density $R(s)$ for a two point correlator

$$\Pi(x) = \langle 0 | T j(x) j(0) | 0 \rangle$$

where $j(x)$ is a weak charged current of light quarks generates Adler’s function

$$D(Q^2) = Q^2 \int_0^\infty \frac{R(s) ds}{(s + Q^2)^2}, \quad Q^2 = -q^2$$

that can be calculated in Euclidean domain in terms of perturbation theory series in the coupling constant $\alpha_s(\mu)$

$$D(Q^2) = \alpha(\mu) + \alpha(\mu)^2 (\beta_0 \ln \frac{\mu^2}{Q^2} + c) + \ldots$$

From the last expression the spectral density can be found in any finite order of perturbation theory in the form

$$R(s) = \alpha(\mu) + \alpha(\mu)^2 (\beta_0 \ln \frac{\mu^2}{s} + c) + \ldots$$

(3)

The $\tau$ lepton decay width are given by

$$r_\tau = \int_0^{m_\tau^2} R(s) W(s/m_\tau^2) ds/m_\tau^2,$$

(4)

where

$$W(x) = (1 - x)^2 (1 + 2x).$$

In a finite order of perturbation theory the expression for the $\tau$ lepton width has the form

$$r_\tau = \alpha(\mu) + \alpha(\mu)^2 (\beta_0 \ln \frac{\mu^2}{m_\tau^2} + \tilde{c}) + \ldots$$

Above formulas are given for normalization only and $\beta_0$ is the first coefficient of the $\beta$ function in QCD. Within the formal perturbation theory in the strong coupling constant $\alpha_s$ every term ($\ln^n \frac{\mu^2}{Q^2}$) has correct analytic properties in $q^2$: a cut along the positive semiaxes. Using formulas (2) and (3) with known renormalization group properties of $R(s)$ and $r_\tau$ ($\mu$-independence) one can go beyond the predictions of finite order perturbation theory and perform a partial resummation in many different ways. First it can be done directly on the cut \[3\]. Because the spectral density itself is nonintegrable at low energy after using the renormalization group improvement on the cut ($\mu^2 = s$)

$$R(s) \sim \alpha_s(s) = \frac{1}{\beta_0 \ln(s/\Lambda^2)}$$
one can take the discontinuity after RG summation in Euclidean domain to obtain (in the leading order)

\[ R(s) = \frac{1}{\pi} \arctan \pi \alpha(s) = \alpha(s) - \frac{\pi^2}{3} \alpha(s)^3 + \ldots \]  

(5)

Now the integral in (4) can be done that provides an improvement of PT prediction through resummation on the cut. This technique works up to third order of PT that is available now, i.e., commutation of RG summation and taking the discontinuity makes the integral in (4) regular.

At third order of perturbation theory, however, there appears another way of definition of the quantity in question (4) that is connected with the change of renormalization scheme that reduces to a redefinition of the coupling constant. In the effective charge scheme [7] an effective \( \beta_\tau(\alpha_\tau) \) has a zero that leads to an IR fixed point [8] and integrals (4) can be explicitly done in this scheme [8]. This possibility however depends crucially on the order of PT and is absent in the second order. It is unknown whether it persists in the fourth order. So for this technique the natural requirement that the method of resummation is stable in every order of PT is not fulfilled.

There are also other possibilities of resummation, for instance, different kinds of optimization [9, 10].

Another recipe (more perturbative because it is formulated in the complex plane and not on the physical cut) is to use the analytic properties given by (2) and define

\[ \int R(s)ds = \frac{1}{2\pi i} \int_C \Pi(z)dz. \]

Cauchy theorem requires no singularities inside the contour so if

\[ D(z) \sim D(z)^{PT} \sim \alpha(z) = \frac{1}{\beta_0 \ln(-z/\Lambda^2)} \]

then \( D(z)^{PT} \) has wrong analytic properties and there is a difference (nonperturbative) which is proportional to \( \Lambda = m_\tau \exp(-2\pi/9\alpha_s(m_\tau)) \) with the result of direct summation on the cut.

One should stress that the modified minimal subtraction scheme is always used for the definition of the charge and the change of an observable within PT is formally of higher order in the coupling constant. The real problem is that this difference is large enough to be caught by experiment. Then the theoretical predictions can differ by the amount that depends on the procedure used in computation and is not negligible. It is unclear how to single out the best numerical value. There is here even more ambiguity than the simple freedom in the choice of the renormalization scheme.

Thus, \( \alpha_{MS}(m_\tau) \) depends rather strongly on the resummation procedure that should be explicitly explained when precise comparison of different predictions is made.

Having this ambiguity in mind and noting that \( \alpha_{\overline{MS}}(m_\tau) \) by itself cannot be measured because it is unphysical quantity we next consider more strict test of pQCD that involves only observables and therefore is free of the renormalization scheme ambiguity. Still an ambiguity due to resummation contrary to the finite order analysis [12] is present.
3 A TEST OF pQCD FOR DIRECTLY MEASURED QUANTITIES

For the analysis the moments of $e^+e^-$ annihilation and $r_\tau$ are chosen because

- these observables are generated by the same Green’s function in pQCD ($m_q = 0$) that reduces unknown possible nonperturbative effects
- the integration scale for the moments can be adjusted in such a way to avoid the renormalization group evolution that would require the use of the $\beta$ function and introduce further uncertainties
- both moments and $r_\tau$ can be directly measured with high precision that allows to pin down the theoretical difference that is parametrically of the next (fifth) order in the coupling constant and is fairy small.

Notations for further analysis are as follows. The whole spectral density

$$R(s) = 2(1 + \frac{4}{9} r(s))$$

is defined through the reduced one

$$r(s) = \frac{9}{4\pi} \alpha + \ldots$$

that determines the reduced Adler’s function

$$\tilde{d}(Q^2) = Q^2 \int_0^\infty \frac{r(s)ds}{(s + Q^2)^2}.$$ 

$$= \frac{\alpha}{\pi} + k_1 \left(\frac{\alpha}{\pi}\right)^2 + k_2 \left(\frac{\alpha}{\pi}\right)^3 + k_3 \left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

The moments of $e^+e^-$ annihilation rate are defined by

$$r_n = (n + 1) \int_0^{m^2_\tau} \frac{ds}{m^2_\tau} \left(\frac{s}{m^2_\tau}\right)^n r(s)$$

$$r_\tau = 2r_0 - 2r_2 + r_3.$$ 

Coefficients $k_i$ summarize all information from perturbation theory and are the only ingredient for testing the theory. We factor out all known nonperturbative corrections due to condensates. The technique of resummation based on integration along the contour in the complex plane is adopted. Because $r_\tau$ and $r_n$ are renormalization group invariant it is convenient to use renormalization group invariant approach from the very beginning.

Introduce

$$d_\tau = d(m^2_\tau) = m^2_\tau \int_0^\infty \frac{r(s)ds}{(s + m^2_\tau)^2}$$
for which the RG equation is

\[ z \frac{d}{dz} d = -d^2 (1 + \rho_1 d + \rho_2 d^2 + \rho_3 d^3 + \ldots) \]

where \( \rho_i \) are renormalization scheme invariants, \( \rho_1 = 0.79, \rho_2 = 1.035, \rho_3 = 2k_3 - 2.97953 \) (\( \overline{\text{MS}} \) parameterization) with recently computed coefficient of the \( \beta \) function \([13]\).

Moments are defined as integrals along the contour in the complex plane and therefore unphysical singularity is included. If

\[ p(z) : -z \frac{d}{dz} p(z) = d(z) \]

then

\[ r_n = \frac{n + 1}{2\pi i} \int_{|x|=1} x^n p(m_x^2 x) dx. \]

The machinery of computing consists now in finding functions \( f_n(.) \) and \( g(.) \) and then inverting the function \( g(.) \) to represent moments through the only input parameter \( r_\tau \)

\[ r_n = f_n(d_\tau), \quad r_\tau = g(d_\tau), \quad d_\tau = g^{-1}(r_\tau), \]

\[ r_n = f_n[g^{-1}(r_\tau)] = (f_n \otimes g^{-1})(r_\tau). \]

Analytic properties of the function \( f_n \otimes g^{-1}(.) \) in \( r_\tau \) determine a radius of convergence for our series \([12]\). We could not determine these properties completely. However the simpler question – analytic properties of \( f_n(x) \) with respect to \( x \) – can be answered completely (in \( \overline{\text{MS}} \) scheme without \( k_3 \)). To the leading order the explicit formula is

\[ f_0(x) = \frac{1}{2\pi x} \int_{-\pi}^{\pi} \frac{e^{i\phi} d\phi}{1 + ix\phi} = 1 + 2x + \ldots \]

that gives \( x < 1/\pi \) or \( \alpha < \frac{4}{\pi^2} \) \([13]\).

Generalization to higher orders is straightforward, one has to find a singularity of the solution of the renormalization group equation \([4]\)

\[ z \frac{d}{dz} d(z) = \beta(d(z)), \quad d(m_\tau^2) = d_\tau. \]

The result is \( \alpha^{(1)} < \frac{4}{9}0.744, \alpha^{(2)} < \frac{4}{9}0.697, \alpha^{(3)} < \frac{4}{9}0.674 \) \([4]\).

The actual value of \( \alpha^{(3)} = 0.3540 \) obtained from \( r_\tau^{\text{exp}} = 0.487 \pm 0.011 \) \([16]\) lies outside convergence regions. The behavior of the decay rate in higher orders of perturbation theory is presented in Table 1 \([4]\). If the resummation does catch a dominant behavior then next several orders of perturbation theory do not improve much and do not show wild asymptotic behavior either. It forms a kind of dead zone where no qualitatively new behavior starts.

Results for the moments are presented in Table 2. The label “PT” stands for finite order perturbation theory results \([12]\), while “\( \rho_2 \)” and “\( \rho_3 \)” columns contain resummed results in third and fourth order approximation for the \( \beta \) function \([13]\). “Exp” gives
Table 1: Higher order perturbative predictions for the reduced part of the semileptonic $\tau$ decay width using $a^{(3)}_\tau = 0.2535$

| $n$ | $r_\tau$ | $n$ | $r_\tau$ | $n$ | $r_\tau$ |
|-----|----------|-----|----------|-----|----------|
| 1   | 0.2535   | 5   | 0.4993   | 9   | 0.7400   |
| 2   | 0.4021   | 6   | 0.4622   | 10  | 1.5577   |
| 3   | 0.4870   | 7   | 0.4402   | 11  | 4.059    |
| 4   | 0.5153   | 8   | 0.4894   | 12  | 11.905   |

Table 2: Predictions for moments of $e^+e^-$ annihilation rate through $\tau$ lepton width

|   | PT         | $\rho_2$ | $\rho_3$ | exp |
|---|------------|----------|----------|-----|
| $R_0$ | 2.28 ± 0.05 | 2.334    | 2.338    | 2.15 |
| $R_1$ | 2.14 ± 0.08 | 2.219    | 2.216    | 2.06 |
| $R_2$ | 2.12 ± 0.12 | 2.228    | 2.230    | 2.00 |
| error | trunca-    | ~ 0.07 | ~ 0.07 | ~ 10% |
| tion | input      | input   | sys |

The only quantity that is given at the finite order of perturbation theory in our approach is the $\beta$ function and one has to check its validity at least heuristically in terms of decreasing with order. The worst pattern of convergence for the $\beta$ function in the course of the analysis is given by (numerically)

$$\beta(\alpha) \sim 1 + 0.284 + 0.134 + 0.0467\rho_3 + \ldots$$
at the point \( \alpha = 0.36 \). In all other points on the contour the convergence is better. One can see that the numerical value of \( \rho_3 \) is rather important for application of discussed technique even though the change of results is small.

4 CONCLUSIONS

As conclusions to my talk I summarize results of investigation done in [4, 12, 13]:

- the coupling constant extracted from different processes even in the same renormalization scheme (for instance, \( \overline{\text{MS}} \)) requires for precise comparison an explicit mentioning of the resummation procedure
- the recipe based on analytic continuation is rather stable against inclusion of higher order corrections to the \( \beta \) function
- moments of \( e^+e^- \) annihilation at \( \mu \sim m_\tau \) are estimated to be smaller than required by pQCD with resummation of effects of running of the coupling constant.

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