Towards Understanding the Predictability of Stock Markets from the Perspective of Computational Complexity*

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Abstract

This paper initiates a study into the century-old issue of market predictability from the perspective of computational complexity. We develop a simple agent-based model for a stock market where the agents are traders equipped with simple trading strategies, and their trades together determine the stock prices. Computer simulations show that a basic case of this model is already capable of generating price graphs which are visually similar to the recent price movements of high tech stocks. In the general model, we prove that if there are a large number of traders but they employ a relatively small number of strategies, then there is a polynomial-time algorithm for predicting future price movements with high accuracy. On the other hand, if the number of strategies is large, market prediction becomes complete in two new computational complexity classes CPP and BCPP, where $P^{NP[O(\log n)]} \subseteq BCPP \subseteq CPP = PP$. These computational completeness results open up a novel possibility that the price graph of an actual stock could be sufficiently deterministic for various prediction goals but appear random to all polynomial-time prediction algorithms.

1 Introduction

The issue of market predictability has been debated for more than a century (see [1] for earlier papers and [3, 12, 15, 17] for more recent viewpoints). In 1900, the pioneering work “Theory of Speculation” of Louis Bachelier used Brownian motion to analyze the stochastic properties of security prices [7]. Since then, Brownian motion and its variants have become textbook tools for modeling financial assets. Relatively recently, the radically different methodology of Mandelbrot used fractals to approximate price graphs deterministically [18]. In this paper, we initiate a study into this long-running issue from the perspective of computational complexity.

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We develop a simple agent-based model for a stock market [8, 16]. The agents are traders equipped with simple trading strategies, and their trades together determine the stock prices. We first consider a basic case of this model where there are only two strategies, namely, momentum and contrarian strategies. The choice of this base model and thus our general model is justified at two levels: (1) Experimental and empirical studies in the finance literature [1, 4, 6, 9–11, 14] show that a large number of traders primarily follow these two strategies. (2) Our own simulation results show that despite its simplicity, the base model is capable of generating price graphs which are visually similar to the recent price movements of high tech stocks (Figures 1 and 2).

With these justifications, we then consider the issue of market predictability in the general model. We prove that if there are a large number of traders but they employ a relatively small number of strategies, then there is a polynomial-time algorithm to predict future price movements with high accuracy (Theorem 3). On the other hand, if there are also a large number of strategies, then the problem of predicting future prices becomes computationally very hard. To describe this hardness, we define two new computational complexity classes called CPP and BCPP (Definitions 8 and 9). We show that some market prediction problems are complete for these two classes (Theorems 14 and 17) and that P^{NP[O(\log n)]} \subseteq \text{BCPP} \subseteq \text{CPP} = \text{PP}.

These computational completeness results open up the possibility that the price graph of an actual stock could be sufficiently deterministic for various prediction purposes but appear random to all polynomial-time prediction algorithms. This is in contrast to the most popular academic belief that the future price of a stock cannot be predicted from its historical prices because the latter are statistically random and contain no information. This new viewpoint also differs from the fractal-based methodology in that the price graph of a stock could be a fractal but the fractal might not be computable in polynomial time. The findings in this paper can by no means settle the debate on market predictability. Our goal is only that this alternative approach could provide new insights to the predictability issue in a systematic manner. In particular, it could provide a general framework to investigate the many documented technical trading rules [20] and to generate novel and significant interdisciplinary research problems for computer science and finance.

The rest of the paper is organized as follows. Section 2 discusses the basic market model. Section 3 formulates the general model. Section 4 proves the complexity results for market prediction in the general model. We conclude the paper with some directions for future research in Section 5.

2 A Basic Market Model

In this section, we present a very simple market model, called the deterministic-switching MC (DSMC) model. The letter M stands for a momentum strategy, and the letter C for a contrarian strategy. These two strategies and the model itself are defined in Section 2.1. Some computer simulations for this model are reported in Section 2.2.

Intuitively, these strategies are heuristics (“rules of thumb”) used by traders in the absence of reliable asset valuation models. As discussed in [11], a momentum trader may observe a sequence of “up” trades (price increments) and execute a buy trade in the anticipation that she will not be one of the last buyers, knowing very well that the asset is overpriced. Similarly, she may see some “down” trades (price decrements) and then make a sell trade in the hope that there will be more sellers after her. In contrast, after detecting a number of “up” (respectively, down) trades, a contrarian trader may submit a sell (respectively, buy) trade, anticipating a price reversal.

Both experimental and empirical studies have shown that traders look at past price dynamics to form their expectations of future prices, and a large number of them primarily follow momentum or contrarian strategies [1, 4, 6, 10]. In addition, the traders may switch between these two diamet-
rically opposite strategies. Momentum and contrarian strategies are dominant in the behavior of professional market timers as well. The use of momentum and contrarian strategies sometimes signifies gambling tendencies among traders. In fact, a market model with momentum and contrarian traders can also be interpreted as a market with noise traders and rational traders, where the noise traders essentially follow a momentum strategy while the rational traders attempt to exploit the noise traders by following a contrarian strategy.

2.1 Defining the DSMC Model

In the DSMC model, there is only one stock traded in the market. The model is completely specified by three integer parameters $m, L, k > 0$, and a real parameter $\alpha > 0$ as follows.

There are $m$ traders in the market, and each trader’s strategy set consists of momentum ($M$) and contrarian ($C$) strategies. At the beginning of day 1 of the investment period, each trader randomly chooses her initial strategy from $\{M, C\}$ and an integer $\ell_i \in [2, L]$ with equal probability, where $L$ is the maximum strategy switching period. This is the only source of randomness in the DSMC model; from this point onwards, there is no random choice.

Rule 1 (Deterministic Strategy Switching Rule) For days $1, \ldots, k+1$, there is no trading. Each trader starts trading from day $k+2$ using her initial strategy. Trader $i$ uses the same strategy for $\ell_i$ days and switches it at the beginning of every $\ell_i$ days.

The next rule defines the two strategies with respect to a given memory size $k$, which is the same for all traders.

Rule 2 (Trading Rule) At the beginning of day $t$, observe the stock prices $P_f$ of days $f \in [t-(k+1), t-1]$. For $g \in [t-k, t-1]$, count the number $k_u$ of days $g$ when $P_g > P_{g-1}$; and the number $k_d$ of days when $P_g < P_{g-1}$. The $k$-day trend is defined as $\text{Tr}(k, t) = k_u - k_d$. Then, if $\text{Tr}(k, t) \geq 0$ (respectively, $< 0$), the momentum strategy $M$ buys (respectively, sells) one share of the stock at the market price determined by Rule 3 below. In contrast, the contrarian strategy $C$ sells (respectively, buys) one share of the stock.

For instance, suppose that $k = 2$, and investor $i$ picks her initial strategy $M$ and $\ell_i = 2$ at the beginning of day 1. She then observes the prices of days 1, 2, 3, which are, say, $80, 82, 90$. At the beginning of day 4, she issues a market order to buy one share of the stock. The orders issued by the traders on day 4 together determine the price of day 4 as specified by Rule 3 below. Suppose that the price of day 4 is $91$, then investor $i$ issues another market buy order at the beginning of day 5. Since her $\ell_i$ is 2, at the beginning of day 6, she switches her strategy from $M$ to $C$.

Rule 3 (Price Adjustment Rule) The prices for days $1, \ldots, k+1$ are given. On day $t \geq k+2$, let $m_b$ and $m_s$ be the total numbers of buys and sells, respectively. Then, the price $P_t$ on day $t$ is determined by the following equation:

$$P_t - P_{t-1} = \alpha \cdot (m_b - m_s),$$

where $\alpha$ is the unit of price change.
Figure 1: A one-year price sequence generated using the DSMC model. Parameters: number of traders $m = 20$, memory size $k = 2$, maximum strategy switching period $L = 8$, unit of price change $\alpha = 0.25$, number of trading days $= 250$. The price graph appears strikingly similar to the recent price movements of high tech stocks.

Figure 2: A one-year price sequence generated using the DSMC model. The parameters are the same as those for Figure 1.
2.2 Computer Simulation on the DSMC Model

We have conducted some computer simulations of the DSMC model to test whether it can generate realistic price graphs. Because we had to examine the graphs visually, our time constraints limited the number of these simulations to only about six hundred. For a large fraction of them, we set \( m = 20 \), \( L = 8 \), and the initial \( k \) prices in the range of $70 to $90. We then focused on testing the effect of memory size \( k \). Two main findings are as follows:

- For \( k = 1 \), the price graphs were not visually real.
- For \( k = 2 \), about one out of four graphs were strikingly similar to those of recent high tech stocks, which was a major positive surprise to us. Two representatives of such graphs are shown in Figures 1 and 2.

These two statements are based on our subjective impressions and limited simulations. To further understand the DSMC model, it would be useful to automate statistical analysis on the price graphs generated by this model and compare them with real stock prices.

3 A General Market Model

In this section, we define a market model, called the AS model, where the word AS stands for arbitrary strategies. It can be verified in a straightforward manner that the DSMC model is a special case of the AS model.

In the AS model, there is only one stock traded in the market. The model is completely specified as follows with five parameters: (1) the number \( m \) of traders, (2) a unit \( \alpha > 0 \) of price change, (3) a set \( \Pi = \{S^1, \ldots, S^h\} \) of strategies, (4) a price adjustment rule (Equation 1 or 2 below), and (5) a joint distribution of the population variables \( X_1, \ldots, X_h \).

Rule 4 (Market Initialization) There are \( m \) traders in the market. At the beginning of day 1 of the investment period, each trader randomly chooses her initial strategy from \( \Pi \). Let \( X_i \) be the number of traders who choose \( S^i \). Then, each \( X_i \) is a random variable, which is the only source of randomness in the model. (Unlike the DSMC model, because the allowable generality of \( \Pi \), the AS model does not need strategy switching.)

Different joint distributions of the variables \( X_i \) lead to different specific models and prediction problems. In Section 1.2, we consider joint distributions that tend to Gaussian in the limit as the number \( m \) of traders becomes large. In Section 1.3, we consider the case where the variables \( X_i \) are independent, and each is 0 or 1 with equal probability.

Rule 5 (Trading Strategies) There is no trading on day 0. At the beginning of day \( t \geq 1 \), a trader observes the historical prices \( P_0, \ldots, P_{t-1} \) and reacts by issuing a market order to buy one share of the stock, hold (i.e., do nothing), or sell one share according her strategy. Formally, a strategy is a collection of functions \( S = \{S_1, S_2, \ldots, S_t, \ldots\} \), where each \( S_t \) maps \( P_0, \ldots, P_{t-1} \) to \( +1 \) (buy), \( 0 \) (hold), or \( -1 \) (sell).

The price \( P_t \) of day \( t \) is determined at the end of the day by the day’s \( m \) market orders using Rule 4. Since the traders choose their strategies randomly, the sequence \( P_0, P_1, \ldots, P_t, \ldots \) is a stochastic process. We write \( F_t \) for the probability space induced by all possible sequences \( \langle P_0, \ldots, P_t \rangle \). Then, we think of each function \( S_t \) as a random variable on \( F_{t-1} \).

We distinguish between strategies that react to price movements and those that ignore them.
• $\mathcal{S}$ is an active strategy if the functions $S_t$ may or may not be constant functions. An active trader is one with an active strategy.

• $\mathcal{S}$ is a passive strategy if the functions $S_t$ all are constant functions. A passive trader is one with a passive strategy.

Rule 6 (Price Adjustment) The price $P_0$ is given. At the end of day $t \geq 1$, the price $P_t$ is determined by the day’s market orders to buy or sell from the traders. We consider two simple rules:

With the proportional increment (PI) rule,

$$P_t = P_{t-1} + \alpha \cdot \sum_{i=1}^{h} X_i \cdot S^i_t,$$

where $\alpha$ is the unit of price change. Thus we can observe directly the net difference between the number of buyers and sellers on day $t$.

With the fixed increment (FI) rule,

$$P_t = P_{t-1} + \alpha \cdot \text{sign} \left( \sum_{i=1}^{h} X_i \cdot S^i_t \right).$$

In this case, the market moves up or down depending on whether the majority of traders are buying or selling, but the amount by which it moves is fixed at $\alpha$.

For notational brevity, an $\mathcal{AS}+$FI model refers to an AS model with the fixed increment rule, and an $\mathcal{AS}+$PI model refers to an AS model with the proportional increment rule.

In reality, the price tends to move up if there are more buy orders than sell orders; similarly, the price tends to move down if there are more sell orders than buy orders. The FI rule is meant to model the sign but not the magnitude of the slope of this correlation, while the PI rule attempts to model both. Clearly, there can be many other increment rules, which this paper leaves for future research.

4 Predicting the Market

Informally, the market prediction problem at the beginning of day $t$ is defined as follows:

• The data consists of (1) the five parameters of an AS-model, i.e., $m$, $\alpha$, $\Pi$, $X_i$, and a price adjustment rule, and (2) a price history $P_0, \ldots, P_{t-1}$.

• The goal is to predict the price $P_t$ by estimating the conditional probabilities $\Pr[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}]$, $\Pr[P_t < P_{t-1} \mid P_0, \ldots, P_{t-1}]$, and $\Pr[P_t = P_{t-1} \mid P_0, \ldots, P_{t-1}]$.

Note that $\Pr[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}]$ is symmetric to $\Pr[P_t < P_{t-1} \mid P_0, \ldots, P_{t-1}]$ and $\Pr[P_t = P_{t-1} \mid P_0, \ldots, P_{t-1}] = 1 - \Pr[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}] - \Pr[P_t < P_{t-1} \mid P_0, \ldots, P_{t-1}]$. Thus, from this point onwards, our discussion focuses on estimating $\Pr[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}]$.

From an algorithmic perspective, we sometimes assume that the price adjustment rule and the joint distribution of the variables $X_i$ are fixed, and that the input to the algorithm is $m$, $\alpha$, a description of $\Pi$, and the price history. This allows different algorithms for different model families as well as side-steps the issue of how to represent the possibly very complicated joint distribution of the variables $X_i$ as part of the input. As for the description of $\Pi$, we only need $S^1_t, \ldots, S^\ell_t$ for each $S^i_t \in \Pi$ instead of the whole $\Pi$, and the description of these functions can simplified by restricting their domains to consist of the price sequences consistent with the given price history.
4.1 Markets as Systems of Linear Constraints

In the AS+FI model with parameters $m$ and $\alpha$, a price sequence $P_0, \ldots, P_t$ and $\Pi$ can yield a set of linear inequalities in the population variables $X_i$ as follows. If the price changes on day $t$, we have

$$\text{sign}(P_t - P_{t-1}) \sum_{i=1}^{h} S_i^t X_i > 0.$$  

(3)

If the price does not change, we have instead the equation

$$\sum_{i=1}^{h} S_i^t X_i = 0.$$  

(4)

Furthermore, any assignment of the variables $X_i$ that satisfies either inequality is feasible with respect to the corresponding price movement on day $t$. In both cases, $S_i^t$ is computable from the price sequence $P_0, \ldots, P_{t-1}$. The same statements hold for days $1, \ldots, t - 1$. Therefore, given $m$ and $\alpha$, we can extract from $\Pi$ and $P_0, \ldots, P_t$ a set of linear constraints on the variables $X_i$. The converse holds similarly. We formalize these two observations in Lemmas 1 and 2 below.

**Lemma 1** In the AS+FI model with parameters $m$ and $\alpha$, given $\Pi$ and a price sequence $P_0, \ldots, P_\beta$, there are matrices $A$ and $B$ with coefficients in $\{-1, 0, +1\}$, $h$ columns each, and $\beta$ rows in total. The rows of $A$ (respectively, $B$) correspond to the days when $P_j \neq P_{j-1}$ (respectively, $P_j = P_{j-1}$). Furthermore, the column vectors $x = (X_1, \ldots, X_h)^\top$ consistent with $\Pi$ and $P_0, \ldots, P_\beta$ are exactly those that satisfy $Ax > 0$ and $Bx = 0$. The matrices $A$ and $B$ can be computed in time $O(h\beta T)$, where $T$ is an upper bound on the time to compute a single $S_i^j$ from $P_0, \ldots, P_\beta$ over all $j \in [1, \beta]$ and $S_i^t$.

**Proof:** Follows immediately from Equations 3 and 4. ■

**Lemma 2** In the AS+FI model with parameters $m$ and $\alpha$, given a system of linear inequalities $Ax > 0, Bx = 0$, where $A$ and $B$ have coefficients in $\{-1, 0, +1\}$ with $h$ columns each, and $\beta$ rows in total, there exist (1) a set $\Pi$ of $h$ strategies corresponding to the $h$ columns of $A$ and $B$, and (2) a $(\beta + 1)$-day price sequence $P_0, \ldots, P_\beta$ with the latter $\beta$ days corresponding to the $\beta$ rows of $A$ and $B$. Furthermore, the values of the population variables $X_1, \ldots, X_n$ are feasible with respect to the price movement on day $j$ if and only if column vector $x = (X_1, \ldots, X_n)^\top$ satisfies the $j$-th constraint in $A$ and $B$. Also, $P_0, \ldots, P_\beta$ and a description of $\Pi$ can be computed in $O(h\beta)$ time.

**Proof:** Follows immediately from Equations 3 and 4. ■

In the AS+PI model we obtain only equations, of the form:

$$\sum_{i=1}^{h} S_i^t X_i = \frac{1}{\alpha} (P_t - P_{t-1}).$$  

(5)

In this case there is a direct correspondence between market data and systems of linear equations. We formalize this correspondence in Lemmas 3 and 4 below.

**Lemma 3** In the AS+PI model with parameters $m$ and $\alpha$, given $\Pi$ and a price sequence $P_0, \ldots, P_\beta$, there is a matrix $B$ with coefficients in $\{-1, 0, +1\}$, $h$ columns, and $\beta$ rows, and a column vector $b$
of length $h$, such that the column vectors $x = (X_1, \ldots, X_h)^\top$ consistent with $\Pi$ and $P_0, \ldots, P_\beta$ are exactly those that satisfy $Bx = b$. The coefficients of $B$ and $b$ can be computed in time $O(h\beta T)$, where $T$ is an upper bound on the time to compute a single $S^i_j$ from $P_0, \ldots, P_\beta$ over all $j \in [1, \beta]$ and $S^i$.

**Proof:** Follows immediately from Equation 5. ■

**Lemma 4** In the AS+PI model with parameters $m$ and $\alpha$, given a system of linear equations $Bx = b$, where $B$ is a $\beta \times h$ matrix with coefficients in $\{-1, 0, +1\}$, there exist (1) a set $\Pi$ of $h$ strategies corresponding to the $h$ columns of $B$, and (2) a $(\beta + 1)$-day price sequence $P_0, \ldots, P_\beta$ with the last $\beta$ days corresponding to the $\beta$ rows of $B$. Furthermore, the values of the population variables $X_1, \ldots, X_n$ are feasible with respect to the price movement on day $j$ if and only if column vector $x = (X_1, \ldots, X_n)^\top$ satisfies the $j$-th constraint in $B$. Also, $P_0, \ldots, P_\beta$ and a description of $\Pi$ can be computed in $O(h\beta)$ time.

**Proof:** Follows immediately from Equation 5. ■

### 4.2 An Easy Case for Market Prediction: Many Traders but Few Strategies

In Section 4.2.1, we show that if an AS+FI market has far more traders than strategies, then it takes polynomial time to estimate the probability that the next day’s price will rise. In Section 4.2.2, we discuss why the same analysis technique does not work for an AS+PI market.

#### 4.2.1 Predicting an AS+FI Market

For the sake of emphasizing the dependence on $m$, let $\Pr_m[E]$ be the probability that event $E$ occurs when there are $m$ traders in the market.

This section makes the following assumptions:

E1 The input to the market prediction problem is simply a price history $P_0, \ldots, P_{t-1}$. The output is $\lim_{m \to \infty} \Pr_m[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}]$.

E2 The market follows the AS+FI model.

E3 $\Pi$ is fixed. The values $S^j_i$ over all $i \in [1, h]$ are computable from the input in total time polynomial in $j$.

E4 Each of the $m$ traders independently chooses a random strategy $S^i$ from $\Pi$ with fixed probability $p_i > 0$, where $p_1 + \cdots + p_h = 1$.

The parameter $\alpha$ is irrelevant.

Notice that the column vector $X = (X_1, \ldots, X_h)^\top$ is the sum of $m$ independent identically-distributed vector-valued random variables with a center at $p = m \cdot (p_1, \ldots, p_h)^\top$. We recenter and rescale $X$ to $Y = (X - m \cdot (p_1, \ldots, p_h)^\top)/\sqrt{m}$. Then, by the Central Limit Theorem (see, e.g., [3, Theorem 29.5]), as $m \to +\infty$, $Y$ converges weakly to a normal distribution centered at the $h$-dimensional vector $(0, \ldots, 0)^\top$. In Theorem 3 below, we rely on this fact to estimate the probability that the market rises for price histories that occur with nonzero probability.
Theorem 5 Assume that \( \lim_{m \to \infty} \Pr_m[P_0, \ldots, P_{t-1}] > 0 \). Then there is a fully polynomial-time approximation scheme for estimating \( \lim_{m \to \infty} \Pr_m[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}] \) from \( P_0, \ldots, P_{t-1} \). The time complexity of the scheme is polynomial in (1) the length \( t \) of the price history, (2) the inverse of the relative error bound \( \epsilon \), and (3) the inverse of the failure probability \( \eta \).

Remark. We omit the explicit dependency of the running time in \( h \) and \( p_1, \ldots, p_h \) in order to concentrate on the main point that market prediction is easy with this section’s four assumptions. The parameters \( h \) and \( p_1, \ldots, p_h \) are constant under the assumptions.

Proof: We use Lemma 4 to convert the price history \( P_0, \ldots, P_{t-1} \) into a system of linear constraints \( AX > 0 \) and \( BX = 0 \), with the next day’s price change \( P_t - P_{t-1} \) determined by \( \text{sign}(c \cdot X') \) for some \( c \). Since the values \( S_i^j \) are computable in time polynomial in \( j \), this conversion takes time polynomial in \( t \).

Then, \( \Pr_m[P_0, \ldots, P_{t-1}] = \Pr_m[AX > 0 \land BX = 0] \). Since \( \lim_{m \to \infty} \Pr_m[AX > 0 \land BX = 0] > 0 \), the constraints in \( B \) must be vacuous; in other words, for each \( P_i = 0 \) with \( i \in [0, t-1] \), the corresponding constraint in \( B \) is \( 0 \cdot X_i + \cdots + 0 \cdot X_{h-1} = 0 \). Therefore, \( \Pr_m[P_0, \ldots, P_{t-1}] = \Pr_m[AX > 0] \). Furthermore, since both \( A \) and \( c \) are constant with respect to \( m \),

\[
\lim_{m \to \infty} \Pr_m[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}] = \frac{\lim_{m \to \infty} \Pr_m[AX > 0 \land c \cdot X > 0]}{\lim_{m \to \infty} \Pr_m[AX > 0]}. \tag{6}
\]

So to compute the desired \( \lim_{m \to \infty} \Pr_m[P_t > P_{t-1} \mid P_0, \ldots, P_{t-1}] \), we compute \( \lim_{m \to \infty} \Pr_m[AX > 0 \land c \cdot X > 0] \) and \( \lim_{m \to \infty} \Pr_m[AX > 0] \) as follows.

To avoid the degeneracy caused by \( \sum_{i=1}^h X_i = m \), we work with \( X' = (X_1, \ldots, X_{h-1})^\top \) instead of \( X \) by replacing \( X_h \) with \( m - \sum_{i=1}^{h-1} X_i \) and making related changes. Let \( p' = (p_1, \ldots, p_{h-1})^\top \), which is the center of \( X' \). As is true for \( Y \), as \( m \to +\infty \), the vector \( Y' = (X' - m \cdot p')/\sqrt{m} \) converges weakly to a normal distribution centered at the \((h-1)\)-dimensional point \((0, \ldots, 0)^\top \).

Under the assumption that each \( p_i \) is nonzero, the distribution of \( Y' \) is full-dimensional (within its restricted \((h-1)\)-dimensional space), as in the limit the variance of each coordinate \( Y_i' \) is nonzero conditioned on the values of the other coordinates, which implies that the smallest subspace containing the distribution must contain all \( h-1 \) axes. We can calculate the covariance matrix of \( Y' \) directly from the \( p_i \), as it is equal to the covariance matrix for a single trader: on the diagonal, \( C_{ii} = p_i - p_i^2 \); and for off-diagonal elements, \( C_{ij} = -p_i p_j \). Given \( C \), \( Y' \) has density \( \rho(x) = ae^{x^\top C x} \) for some constant \( a \), and we can evaluate this density in \( O(h^2) \) time given \( x \), which is \( O(1) \) time under our assumption that \( \Pi \) is fixed.

Let \( A_i \) be the \( i \)-th constraint of \( A \), i.e., \( A_{i,1} X_1 + \cdots + A_{i,h} X_h > 0 \). Let \( A_i' \) denote the constraint \( (A_{i,1} - 1) X_1 + \cdots + (A_{i,h} - 1) X_h > 0 \). Let \( c' = (c_1 - c_h, \ldots, c_{h-1} - c_h) \).

We next convert the constraints of \( A \) on \( X \) into constraints on \( Y' \). First of all, notice that \( A_i X = \sqrt{m} (A_i' Y') + m A_i p \). So \( A_i X > 0 \) if and only if \( A_i' Y' > -\sqrt{m} A_i p \). The term \( -\sqrt{m} A_i p \) may not be constant. In such a case, as \( m \to \infty \), the hyper plane bounding the half space \( A_i' Y' > -\sqrt{m} A_i p \) keeps moving away from the origin, which presents some technical complication. To remove this problem, we analyze the term in three cases. If \( A_i p < 0 \), then since \( m \cdot p \) is the center of \( X \), as \( m \to \infty \), \( \Pr_m[A_i X < 0] \) converges to 1. In other words, \( A_i \) is infeasible with probability 1 in the limit. Then, since \( \lim_{m \to \infty} \Pr_m[P_0, \ldots, P_{t-1}] > 0 \), such \( A_i \) cannot exist in \( A \). Similarly, if \( A_i p > 0 \), then \( \lim_{m \to \infty} \Pr_m[A_i X > 0] = 1 \) and \( A_i \) is vacuous. The interesting constraints are those for which \( A_i p = 0 \); in this case, by algebra, \( A_i X > 0 \) if and only if \( A_i' Y' > 0 \). Thus, let \( D \) be the matrix formed by these constraints; \( D \) can be computed in \( O(h t) \) time. Then, since \( D \) is constant with respect to \( m \), \( \lim_{m \to \infty} \Pr_m[AX > 0] = \lim_{m \to \infty} \Pr_m[D Y' > 0] \). Similarly, \( \Pr_m[AX > 0 \land c \cdot X > 0] \) converges to (1) \( \Pr_m[D Y' > 0] \), or (2) \( \Pr_m[D Y' > 0 \land c' \cdot Y' > 0] \) for case (1) \( c \cdot p < 0 \), case (2) \( c \cdot p > 0 \), or case (3) \( c \cdot p = 0 \), respectively.
Therefore, by Equation \[\lim_{m \to \infty} \Pr_m[P_t > P_{t-1} | P_0, \ldots, P_{t-1}]\] equals 0 for case (1) and equals 1 for case (2). Case (3) requires further computation.

\[
\lim_{m \to \infty} \Pr_m[P_t > P_{t-1} | P_0, \ldots, P_{t-1}] = \frac{\lim_{m \to \infty} \Pr_m[DY' > 0 \land c \cdot Y' > 0]}{\lim_{m \to \infty} \Pr_m[DY' > 0]}. \tag{7}
\]

The numerator and denominator of the ratio in Equation 7 are both integrals of the distribution of \(Y'\) in the limit over the bodies of possibly infinite convex polytopes. To deal with the possible infiniteness of the convex bodies \(DY' > 0 \land c \cdot Y' > 0\) and \(DY' > 0\), notice that the density drops exponentially. So we can truncate the regions of integration to some finite radius around the \((h - 1)\)-dimensional origin \((0, \ldots, 0)^\top\) with only exponentially small loss of precision. Finally, since the distribution of \(Y'\) in the limit is normal, by applying the Applegate-Kaman integration algorithm for log-concave distributions \[\text{Algorithm for log-concave distributions}\] to the numerator and denominator separately, we can approximate \(\lim_{m \to \infty} \Pr_m[P_t > P_{t-1} | P_0, \ldots, P_{t-1}]\) within the desired time complexity.

### 4.2.2 Remarks on Predicting an AS+PI Market

The probability estimation technique based on taking \(m\) to \(\infty\) does not appear to be applicable to the AS+PI model for the following reasons.

First of all, by Lemma \[\text{Lemma}\] the input price history induces a system of linear equations \(BX = b\). If any equation in \(BX = b\) is not equivalent to \(X_1 + \cdots + X_h = m\) or \(0 \cdot X_1 + \cdots + 0 \cdot X_h = 0\), then \(\lim_{m \to \infty} \Pr_m[P_0, \ldots, P_{t-1}] = 0\).

A natural attempt to overcome this seemingly technical difficulty would be to (1) solve \(BX = b\) to choose a maximal set \(U\) of independent variables \(X_j\) and (2) evaluate \(\Pr_m[P_0, \ldots, P_{t-1}]\) in the probability space induced by this set. Still, a single constraint such as \(B_{1,1} \cdot X_1 + \cdots + B_{1,h} \cdot X_h = \alpha \cdot m_0\) with \(B_{i,j} \geq 0\) for all \(j \in [1, h]\) and \(B_{i,j'} > 0\) for some \(X_j' \in U\) forces \(\lim_{m \to \infty} \Pr_m[P_0, \ldots, P_{t-1}] = 0\) in the new probability space. This is due to the fact that \(m_0\) is constant with respect to \(m\).

A further attempt would be to evaluate \(\lim_{m \to \infty} \Pr_m[P_t > P_{t-1} | P_0, \ldots, P_{t-1}]\) by directly working with the probability space induced by \(P_0, \ldots, P_{t-1}\). This also does not work because we show below that the market prediction problem can be reduced to the case where taking a limit in \(m\) has no effect on the distribution of the strategy counts. Suppose that we are given a market which follows the assumptions \[\text{Assumptions}\] of Section \[\text{Section}\] except that this market uses the PI rule and has \(m_0\) traders. We construct a new market with any \(m \geq m_0\) traders with the following modifications:

1. The price history \(P_0, \ldots, P_{t-1}\) is extended with an extra day into \(P_0', \ldots, P_{t-1}', P_t'\), where \(P_j' = P_j\) for \(0 \leq j \leq t - 1\). Each strategy \(S_i\) is extended into a new strategy \(S_i'\) where (1) on day \(j \in [1, t - 1]\), \(S_i'(P_0, \ldots, P_{j-1}) = S_i(P_0, \ldots, P_{j-1})\), (2) on day \(t\), \(S_i'\) always buys, and (3) on day \(t + 1\), \(S_i'(P_0', \ldots, P_t') = S_i(P_0, \ldots, P_{t-1})\). Thus, \(P_t' = P_{t-1}' + \alpha \cdot m_0\).

2. Add a passive strategy \(S_{h+1}'\) that always holds.

3. Let \(p_i' = \frac{1}{2} p_i\) for \(1 \leq i \leq h\) and \(p_{h+1}' = \frac{1}{2}\).

Note that since \(P_t' - P_{t-1}' = \alpha \cdot m_0\), \(m - m_0\) traders choose the passive strategy \(S_{h+1}\). Also, the new market and the new price history can accommodate any \(m \geq m_0\) traders. Note that because of the constraint \(P_t' - P_{t-1}' = \alpha \cdot m_0\), the probability distribution of \((X_1, \cdots, X_h)^\top\) conditioned on \(P_0', \ldots, P_t'\) in the new market for each \(m \geq m_0\) is identical to the probability distribution of \((X_1, \cdots, X_h)^\top\) conditioned on \(P_0, \ldots, P_{t-1}\) in the original market with \(m = m_0\). Furthermore,
\[
\Pr_m[P_{t+1} > P_t', P_t', \ldots, P_t'] = \Pr_m[P_t > P_{t-1}, P_0, \ldots, P_{t-1}].
\]
So we have obtained the desired reduction.

Consequently, we are left with a situation where the number of active strategies may be comparable to the number of traders. Such a market turns out to be very hard to predict, as shown next in Section 4.3.

### 4.3 A Hard Case for Market Prediction: Many Strategies

Section 4.2 shows that predicting an AS+FI market is easy (i.e., takes polynomial time) when the number \(m\) of traders vastly exceeds the number \(h\) of strategies. In this section, we consider the case where every trader may have a distinct strategy, and show that predicting an AS+FI or AS+PI market becomes very hard indeed.

We now define two decision-problem versions of market prediction. Both versions make the following assumption:

- Each \(X_i\) is independently either 0 or 1 with equal probability.

The **bounded** market prediction problem is:

- Input: a set of \(n\) passive strategies and a price history spanning \(n\) days such that the probability that the market rises on day \(n + 1\) conditioned on the price history is either (1) greater than \(2/3\) or (2) less than \(1/3\).

- Question: Which case is it, case (1) or case (2)?

The **unbounded** market prediction problem is:

- Input: a set of \(n\) passive strategies and a price history spanning \(n\) days.

- Question: Is the probability that the market rises on day \(n + 1\) conditioned on the price history greater than \(1/2\) (without the usual \(\epsilon\) term)?

The unbounded market prediction problem has less financial payoff than the bounded one due to different probability thresholds. For each of these two problems, there are in effect two versions, depending on which price increment rule is used; however, both versions turn out to be equally hard. These two problems can be analyzed by similar techniques, and our discussion below focuses on the bounded market prediction problem with a hardness theorem for the unbounded market prediction problem in Section 4.3.4.

We show in Section 4.3.1 how to construct passive strategies and price histories such that solving bounded market prediction is equivalent to estimating the probability that a Boolean circuit outputs 1 on a random input conditioned on a second circuit outputting 1. In Section 4.3.2, we show that this problem is hard for \(P^{NP[O(\log n)]}\) and complete for a class that lies between \(P^{NP[O(\log n)]}\) and PP. Thus bounded market prediction is not merely NP-hard, but cannot be solved in the polynomial-time hierarchy at all unless the hierarchy collapses to a finite level.

#### 4.3.1 Reductions from Circuits to Markets

Lemma \(\text{[lemma1]}\) converts a circuit into a system of linear inequalities, while Lemma \(\text{[lemma2]}\) converts a system of linear inequalities into a system of linear equations. These systems can then be converted into AS+FI and AS+PI market models using Lemmas \(\text{[lemma3]}\) and \(\text{[lemma4]}\), respectively.
Note that the restriction in Lemma 6 to circuits consisting of 2-input NOR gates is not an obstacle to representing arbitrary combinatorial circuits (with constant blow-up), as 2-input NOR gates are universal.

**Lemma 6** For any $n$-input Boolean circuit $C$ consisting of $m$ 2-input NOR gates, there exists a system $Ax > 0$ of $3m + 2$ linear constraints in $n + m + 2$ unknowns and a length $n + m + 2$ column vector $c$ with the following properties:

1. Both $A$ and $c$ have coefficients in $\{-1, 0, +1\}$ that can be computed in time $O((n + m)^2)$.
2. Any 0-1 vector $(x_1, \ldots, x_n)$ has a unique 0-1 extension $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m+2})$ satisfying $Ax > 0$.
3. If $Ax > 0$, then $cx > 0$ if and only if $C(x_1, x_2, \ldots, x_n) = 1$.

**Proof:** Let $x_{n+k}$ represent the output of the $k$-th NOR gate, where $1 \leq k \leq m$. Without loss of generality we assume that gate $m$ is the output gate.

The variables $x_{n+m+1}$ and $x_{n+m+2}$ are dummies to allow for a zero right-hand-side in $Ax > 0$; our first two constraints are $x_{n+m+1} > 0$ and $x_{n+m+2} > 0$.

Suppose gate $k$ has inputs $x_i$ and $x_j$. The NOR operation is implemented by the following three linear inequalities:

$$
\begin{align*}
x_i + x_{n+k} &< 2; \\
x_j + x_{n+k} &< 2; \\
x_i + x_j + x_{n+k} &> 0.
\end{align*}
$$

The first two constraints ensure that the output is never 1 if an input is 1, while the last requires that the output is 1 if both inputs are 0; the constraints are thus satisfied if and only if $x_{n+k} = \overline{x_i \lor x_j}$.

Using the dummy variables, the first two constraints are written as

$$
\begin{align*}
-x_i - x_{n+k} + x_{n+m+1} + x_{n+m+2} &> 0; \\
-x_j - x_{n+k} + x_{n+m+1} + x_{n+m+2} &> 0.
\end{align*}
$$

Let $Ax > 0$ be the system obtained by combining all of these inequalities. Then for each $(x_1, \ldots, x_n)$, $Ax > 0$ determines $x_{n+k}$ for all $k \geq 1$. The vector $c$ is chosen so that $cx = x_{n+m}$.

One might suspect that the fixed increment rule’s ability to hide the exact values of the left-hand side of each constraint is critical to disguise the inner workings of the circuit. However, by adding slack variables we can translate the inequalities into equations, allowing the use of a proportional increment rule without revealing extra information.

**Lemma 7** Let $Ax > 0$ be a system of $m$ linear inequalities in $n$ variables where $A$ has coefficients in $\{-1, 0, +1\}$. Then there is a system $By = 1$ of $mn - m + 1$ linear equations in $2mn - 3m + n + 1$ variables with the following properties:

1. $B$ has coefficients in $\{-1, 0, +1\}$ that can be computed in time $O((mn)^2)$.
2. There is a bijection $f : x \mapsto y$ between the 0-1 solutions $x$ to $Ax > 0$ and the 0-1 solutions $y$ to $By = 1$, such that $x_j = y_j$ for $1 \leq j \leq n$ whenever $y = f(x)$.
**Proof:** For each \( 1 \leq i \leq m \), let \( A_i \) be the constraint \( \sum_j A_{ij}x_j > 0 \). To turn these inequalities into equations, we add slack variables to soak up any excess over 1, with some additional care taken to ensure that there is a unique assignment to the slack variables for each setting of the variables \( x_j \).

We will use the following 0-1 variables, which we think of as alternate names for \( y_1 \) through \( y_{2mn-3m+n+1} \):

| Variables | Purpose | Indices | Count |
|-----------|---------|---------|-------|
| \( x_j \) | original variables | \( 1 \leq j \leq n \) | \( n \) |
| \( u \) | constant 1 | none | 1 |
| \( s_{ij} \) | slack variables for \( A_i \) \( 1 \leq i \leq m, 1 \leq j \leq n-1 \) | \( m(n-1) \) |
| \( t_{ij} \) | slack variables for \( s_{ij} \) \( 1 \leq i \leq m, 1 \leq j \leq n-2 \) | \( m(n-2) \) |

Observe that for each \( i \), \( \sum_j s_{ij} \) can take on any integer value \( \sigma_i \) between 0 and \( n-1 \), and that for any fixed value of \( \sigma_i \), the \( S_{ij} \) constraints uniquely determine the values of \( s_{ij} \) and \( t_{ij} \) for all \( j \). So each constraint \( B_i \) permits \( \chi_i = \sum_j A_{ij}x_j \) to take on precisely the same values 1 to \( n \) that \( A_i \) does, and each \( \chi_i \) uniquely determines \( \sigma_i \) and thus the assignment of all \( s_{ij} \) and \( t_{ij} \). □

### 4.3.2 Conditional Probability Complexity Classes

Suppose that we take a polynomial-time probabilistic Turing machine, fix its inputs, and use the usual Cook’s Theorem construction to turn it into a circuit whose inputs are the random bits used during its computation. Then, we can feed the resulting circuit to Lemmas 6 and 2 to obtain an AS+FI market model in which there is exactly one assignment of population variables for each set of random bits, and the price rises on the last day if and only if the output of the Turing machine is 1. By applying Lemma 7 to the intermediate system of linear inequalities, we can similarly convert a circuit to an AS+PI model. It follows that bounded market prediction is BPP-hard for either model. But with some cleverness, we can exploit the conditioning on past history to show that bounded market prediction is in fact much harder than this. We do so in Section 4.3.3 after a brief detour through computational complexity in this section.

We proceed to define some new counting classes based on conditional probabilities. One of these, BCPP, has the useful feature that bounded market prediction is BCPP-complete. We will use this fact to relate the complexity of bounded market prediction to more traditional complexity classes.

The usual counting classes of complexity theory (PP, BPP, R, ZPP, \( C_m \), etc.) are defined in terms of counting the relative numbers of accepting and rejecting states of a nondeterministic Turing machine. We will define a new family of counting classes by adding a third decision state that does not count for the purposes of determining acceptance or rejection.

A noncommittal Turing machine is a nondeterministic Turing machine with three decision states: accept, reject, and abstain. We represent a noncommittal Turing machine as a deterministic Turing machine which takes a polynomial number of random bits in addition to its input; each
assignment of the random bits gives a distinct computation path. A computation path is accept-
ing/rejecting/abstaining if it ends in an accept/reject/abstain state, respectively. We often write 1, 0, or ⊥ as shorthand for the output of an accepting, rejecting, or abstaining path.

Conditional versions of the usual counting classes are obtained by carrying over their definitions from standard nondeterministic Turing machines to noncommittal Turing machines, with some care in handling the case of no accepting or rejecting paths. We can still think of these modified classes as corresponding to probabilistic machines, but now the probabilities we are interested in are conditioned on not abstaining.

**Definition 8** The conditional probabilistic polynomial-time class (CPP) consists of those languages $L$ for which there exists a polynomial-time noncommittal Turing machine $M$ such that $x \in L$ if and only if the number of accepting paths when $M$ is run with input $x$ exceeds the number of rejecting paths.

**Definition 9** The bounded conditional probabilistic polynomial-time class (BCPP) consists of those languages $L$ for which there exists a constant $\epsilon > 0$ and a polynomial-time noncommittal Turing machine $M$ such that (1) $x \in L$ implies that a fraction of at least $\frac{1}{2} + \epsilon$ of the total number of accepting and rejecting paths are accepting and (2) $x \notin L$ implies that a fraction of at least $\frac{1}{2} + \epsilon$ of the total number of accepting and rejecting paths are rejecting.

**Definition 10** The conditional randomized polynomial-time class (CR) consists of those languages $L$ for which there exists a constant $\epsilon > 0$ and a polynomial-time noncommittal Turing machine $M$ such that (1) $x \in L$ implies that a fraction of at least $\epsilon$ of the total number of accepting and rejecting paths are accepting, and (2) $x \notin L$ implies that there are no accepting paths.

As we show in Theorems 11 and 12, CPP and CR turn out to be the same as the unconditional classes PP and NP, respectively.

**Theorem 11** CPP = PP.

**Proof:** First of all, PP $\subseteq$ CPP because a PP machine is a CPP machine that happens not to have any abstaining paths. For the inverse direction, represent each abstaining path of a CPP machine by a pair consisting of one accepting and one rejecting path, and each accepting or rejecting path by two accepting or rejecting paths. Then the resulting PP machine accepts if and only if the CPP machine does.

**Theorem 12** CR = NP.

**Proof:** To show NP $\subseteq$ CR, replace each rejecting path of an NP machine with an abstaining path in a CR machine. For the inverse direction, replace each abstaining path of the CR machine with a rejecting path in the NP machine.

BCPP appears to be a more interesting class. Since it is clearly a subset of CPP, we have:

**Corollary 13** BCPP $\subseteq$ PP.
Proof: Immediate from Theorem 1 and the definition of BCPP and CPP. □

On the other hand, BCPP appears to be much stronger than the analogous non-conditional class BPP. For example, it is straightforward to show that NP ⊆ BPP. Use the representation of an NP-machine as a deterministic machine $M$ that takes some polynomial number of “hint” bits in addition to its input, and replace these $N$ hint bits with $N$ random bits $r$. In addition, supply another $2N$ random bits $r'$, which will be used to amplify the conditional probability of accepting paths. Now let $M'(x,r,r')$ accept if $M(x,r)$ accepts; reject if $M(x,r)$ rejects and $r' = \overline{0}$; and abstain if $M(x,r)$ rejects and $r' \neq \overline{0}$. Then if $M$ has any accepting path on input $x$, $M'$ has at least $2^{2N}$ accepting paths and at most $2^N - 1$ rejecting paths, for an exponentially large probability of accepting — since we have amplified the small number of accepting paths so that they overwhelm the few rejectors. Alternatively, if $M(x,r)$ never accepts, neither does $M'$.

By repeating this sort of amplification of “good” paths, we can in fact simulate $O(\log n)$ queries of an NP-oracle, as stated in the following theorem.

**Theorem 14** $\text{P}^{\text{NP}[O(\log n)]} \subseteq \text{BCPP}$.

**Proof:** Let $M(x,h_1,h_2,\ldots,h_k)$ be a deterministic implementation a $\text{P}^{\text{NP}[O(\log n)]}$ machine, where $k = O(\log n)$ and each $h_i$ supplies a witness for the $i$-th oracle query. We will show that the language $L(M)$ accepted by $M$ is in BCPP.

To simplify the presentation, we assume that each oracle query is a Boolean formula with a fixed number $m$ of variables, where $m$ is polynomial in $n = |x|$, and that $h_i$ is an assignment for those variables. We assume that $M$ consists of a sequence of functions $M_1, M_2, \ldots, M_k$ for generating oracle queries, a set of $k$ verifiers $V_1, \ldots, V_k$ for verifying the witnesses $h_i$, and a combining function $M_s$ that produces the output from the input and the outputs of the $V_i$. Each function $M_i$ takes as input the input $|x|$ and the outputs of $V_1$ through $V_{i-1}$. $V_i$ sees the output of $M_i$ and the input $h_i$. $M_s$ sees the input $|x|$ and the outputs of $V_1$ through $V_k$. The output of the combined machine $M$ is the output of any computation path where $h_i$ is chosen so that $V_i(M_i,h_i) = 1$ whenever such an $h_i$ exists. In other words, we demand that $h_i$ be a satisfying assignment when possible, and ignore those paths where satisfiable queries are issued but satisfying assignments are not supplied.

We will represent $M$ with a noncommittal machine $M'(x,r_1,\ldots,r_k,r')$, where each $r_i$ is a random bit-vector replacing the corresponding $h_i$, and $r'$ is an extra supply of random bits used to amplify the good computation paths to overwhelm the bad ones. This amplification process is a little complicated, because it is not enough to amplify paths that find good witnesses to particular queries; it may be that a bad witness for an earlier query causes some $M_i$ to issue a different query from the correct one. So we must amplify a path that finds a witness to query $i$ enough to overwhelm not only the exponentially many invalid witnesses to query $i$, but also the exponentially many valid witnesses that might be returned to instances queries $i+1$ through $k$ based on an incorrect answer to query $i$.

Let $v = (v_1,v_2,\ldots,v_k)$ be the vector of outputs of $V_i$. For each $v$, we define an amplification exponent $A(v)$ as follows:

$$A(v) = \sum_{i=1}^{k} v_i 2^{k-i}(mk + 1).$$

We will write $A_i$ for the coefficient $2^{(k-i)}(mk + 1)$; these coefficients $A_i$ are chosen to make Equation 8 work below.

Now let $M'(x,r_1,\ldots,r_k,r') = M(x,r_1,\ldots,r_k)$ whenever $r'_i = 0$ for all $i > A(v)$, where $v$ is the output of $V_1$ through $V_k$ in the computation of $M$. If $r'_i \neq 0$ for some $i > A(v)$,
$M'(x, r_1, \ldots, r_k, r') = \bot$. The effect of the $r'$ bits is to set the weight of each non-abstaining path to $2^A(v)$.

Clearly $M'$ can be computed in polynomial time as long as $|r'|$ is polynomial. The number of $r'$ bits needed is the maximum value of $A(v)$, which is $(mk + 1)(2^{k+1} - 1) = O(n^c k 2^h)$. For this to be polynomial, we need $k = O(\log n)$.

A good path $p$ is precisely one for which each $v_i$ is the correct output of the NP-oracle. A bad path $p'$ is one in which one or more of the $v'_i$ values is incorrect. We will match bad paths with good paths, and show that the weight of each good path is much larger than the total weight of all bad paths mapped to it.

Identify each path with the sequence $r_1, \ldots, r_k$ that generates it. Let $b = b_1, \ldots, b_k$ generate some bad path. Let $i$ be the first point at which $b_i$ is an invalid witness to a satisfiable query. Then there is a good path $a = a_1, \ldots, a_k$ such that $a_j = b_j$ for $j < i$. Furthermore, if $v^a$ and $v^b$ are the vectors of verifier outputs for $a$ and $b$, then not only is $v^b_j = v^a_j$ for $j < i$, but also $v^a_i = 1$ while $v^b_i = 0$ since the only false verifier outputs are false negatives.

The maximum value for $A(v^b)$ is obtained if $v^b_j = 1$ for $j > i$; so we have

$$A(v^a) - A(v^b) \geq \left( \sum_{j=1}^{i-1} A_j v^a_j + A_i \right) - \left( \sum_{j=1}^{i-1} A_j v^b_j + \sum_{j=i+1}^{k} A_j \right) = A_i - \sum_{j=i+1}^{k} A_j. \tag{8}$$

Now

$$A_i = 2^{k-i}(mk + 1) = (mk + 1) + \sum_{j=i+1}^{k} 2^{k-j}(mk + 1) = (mk + 1) + \sum_{j=i+1}^{k} A_j,$$

so $A_i - \sum_{j=i+1}^{k} A_j = mk + 1$. But the ratio between the weight of $b$ and its corresponding good path $a$ is then at most $2^{A(v^b) - A(v^a)} = 2^{-mk-1}$. Since there are only $2^{mk}$ paths altogether, there are fewer than $2^{mk}$ bad paths; thus, the ratio between the total weight of all bad paths mapped to $a$ and the weight of $a$ is less than $2^{mk}2^{-mk-1} = \frac{1}{2}$. Summing over all good paths shows that the total weight of all bad paths is less than half the total weight of all good paths, so at least $\frac{2}{3}$ of all non-abstaining paths are good. It follows that, conditioned on not abstaining, $M'$ accepts with probability greater than $\frac{2}{3}$ if $M$ accepts, and accepts with probability less than $\frac{1}{3}$ if $M$ rejects. Hence $L(M) \in \text{BCPP}$. 

An interesting open question is where exactly BCPP lies between $\text{P}^{\text{NP}[O(\log n)]}$ and PP. It is conceivable that by cleverly exploiting the power of conditioning to amplify low-probability events one could show $\text{BCPP} = \text{PP}$. However, we will content ourselves with the much easier result of showing that the usual amplification technique for BPP also applies to BCPP.

**Theorem 15** If $L \in \text{BCPP}$, then there exists a noncommittal Turing machine $M$ such that the probability that $M$ accepts conditioned on not abstaining is at least $1 - f(n)$ if $x \in L$ and at most $f(n)$ if $x \notin L$, where $n = |x|$ and $f(n)$ is any function of the form $2^{-O(n^c)}$ for some constant $c > 0$.

**Proof:** Let membership in $L$ be computed by $M'$. Assume $x \in L$ (the case $x \notin L$ is symmetric). Consider $k$ independent executions of $M'$ with input $x$; call the random variables representing their outputs $Y_1, Y_2, \ldots, Y_k$. Because the executions are independent, for any 0-1 vector of values $y_i \quad \Pr[\forall i Y_i = y_i \mid \forall i Y_i \neq \bot] = \prod_i \Pr[Y_i = y_i \mid Y_i \neq \bot]$. So conditioning on no abstentions, $\sum_i Y_i$ has a binomial distribution with $p = \frac{1}{2} + \epsilon$, and Chernoff bounds imply $\Pr[\sum_i Y_i < \frac{k}{2} \mid \forall i Y_i \neq \bot]$ is exponentially small in $ke$. Since $\epsilon$ is constant and we can make $k$ polynomially large in $n$, the result follows. 

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4.3.3 Bounded Market Prediction Is BCPP-complete

In Section 4.3.2, we have defined the complexity class BCPP and have shown that it contains the powerful class \( \mathsf{P}^{\mathsf{NP}[O(\log n)]} \). In this section, we show that bounded market prediction is complete for BCPP. In a sense, this result says that market prediction is a universal prediction problem: if we can predict a market, we can predict any event conditioned on past history as long as we can sample from an underlying discrete probability space whose size is at most exponential.

It also says that bounded market prediction is very hard. That is, using Theorems 15 and 14, even if the next day’s price is determined with all but an exponentially small probability, it cannot be solved in the polynomial-time hierarchy unless the hierarchy collapses to a finite level.

**Theorem 16** The bounded market prediction problem is complete for BCPP, in either the AS+FI or the AS+PI model.

**Proof:** First we show that bounded market prediction is a member of BCPP. Given a market, construct a noncommittal Turing machine \( M \) whose input is the price history and strategies, and whose random inputs supply the settings for the population variables \( X_i \). Let \( M \) abstain if the price history is inconsistent with the input and population variables; depending on the model, this is either a matter of checking the linear inequalities produced by Lemma 1 or the equations produced by Lemma 3. Otherwise, \( M \) accepts if the market rises and rejects if the market falls on the next day. The probability that \( M \) accepts thus equals the probability that the market rises: either more than \( \frac{2}{3} \) or less than \( \frac{1}{3} \). Since the problem is to distinguish between these two cases, \( M \) solves the problem within the definition of a BCPP-machine.

In the other direction, we reduce from any BCPP-language \( L \). Suppose \( L \) is accepted by some BCPP-machine \( M \). We will translate \( M \) and its input \( x \) into a bounded market prediction problem. First use Theorem 15 to amplify the conditional probability that \( M \) accepts to either more than \( \frac{2}{3} \) or less than \( \frac{1}{3} \) as bounded market prediction demands. Then convert \( M \) into two polynomial-size circuits, one computing

\[
C_L(r) = \begin{cases} 
0 & \text{if } M(x, r) = \perp; \\
1 & \text{if } M(x, r) \neq \perp,
\end{cases}
\]

and the other computing

\[
C_1(r) = \begin{cases} 
0 & \text{if } M(x, r) \neq 1; \\
1 & \text{if } M(x, r) = 1.
\end{cases}
\]

Without loss of generality we may assume that \( C_L \) and \( C_1 \) are built from NOR gates. Applying Lemma 3 to each yields two sets of constraints \( A_L y > 0 \) and \( A_1 y > 0 \) and column vectors \( c_L \) and \( c_1 \) such that \( c_L y > 0 \) if and only if \( C_L y = 1 \) and \( c_1 x > 0 \) if and only if \( C_1(x) = 1 \), where \( y \) satisfies the previous linear constraints and \( x \) is the initial prefix of \( y \) consisting of variables not introduced by the construction of Lemma 3. We also have from Lemma 3 that there is a one-to-one correspondence between assignments of \( x \) and assignments of \( y \) satisfying the \( A \) constraints, so probabilities are not affected by this transformation.

Now use Lemma 2 to construct a market model in which \( A_L y > 0 \), \( A_1 y > 0 \), and \( c_L y > 0 \) are enforced by the strategies and price history, and \( \text{sign}(c_1 y) \) determines the price change on the next day of trading. Thus the consistent settings of the variables \( X_i \) are precisely those corresponding to settings of \( r \) for which \( C_L(r) = 1 \), or, in other words, those yielding computation paths that do not abstain. The market rises when \( C_1(r) = 1 \), or when \( M \) accepts. So if we can predict whether the market rises or falls with conditional probability at least \( \frac{2}{3} \), we can predict the likely output of \( M \). It follows that bounded market prediction for the AS+FI model is BCPP-hard.
To show the similar result for the AS+PI model, use Lemma 7 to convert the constraints $A_\perp y > 0$, $A_1 y > 0$ into a system of linear equations $Bz = 1$, and then proceed as before, using Lemma 4 to convert this system to a price history and letting $c_1 z$ determine the price change (and thus the sign of the price change) on the next day of trading.

4.3.4 Unbounded Market Prediction is CPP-complete

The unbounded market prediction problem seems harder because the probability threshold in question is $\frac{1}{2}$ with no $\epsilon$ bound in contrast to the thresholds $\frac{2}{3}$ and $\frac{1}{3}$ for the bounded market prediction problem. The following theorem reflects this intuition. However, since we do not know whether BCPP is distinct from PP, we do not know whether unbounded prediction is strictly harder.

**Theorem 17** The unbounded market prediction problem is complete for CPP = PP, in either the AS+FI or the AS+PI model.

**Proof:** Similar to the proof of Theorem 16.

5 Future Research Directions

There are many problems left open in this paper. Below we briefly discuss some general directions for further research.

We have reported a number of simulation and theoretical results for the AS model. As for empirical analysis, it would be of interest to fit actual market data to the model. We can then use the estimated parameters to (1) test whether the model has any predicative power and (2) test the effectiveness of new or known trading algorithms. This direction may require carefully choosing “realistic” strategies for $\Pi$. Besides the momentum and contrarian strategies, there are some popular ones which are worth considering, such as those based on support levels. Investment newsletters could be a useful source of such strategies.

The AS model is an idealized one. We have chosen such simplicity as a matter of research methodology. It is relatively easy to design highly complicated models which can generate very complex market behavior. A more challenging and interesting task is to design the simplest possible model which can generate the desired market characteristics. For instance, a significant research direction would be to find the simplest model in which market prediction is computationally hard. On the other hand, it would be of great interest to find the most general models in which market prediction takes only polynomial time. For this goal, we can consider injecting more realism into the model by introducing resource-bounded learning (the generality of $\Pi$ is equivalent to unbounded learning), variable memory size, transaction costs, buying power, limit orders, short sell, options, etc.

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