Recovery of signals under the condition on RIC and ROC via prior support information

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Abstract In this paper, the sufficient condition in terms of the RIC and ROC for the stable and robust recovery of signals in both noiseless and noisy settings was established via weighted $l_1$ minimization when there is partial prior information on support of signals. An improved performance guarantee has been derived. We can obtain a less restricted sufficient condition for signal reconstruction and a tighter recovery error bound under some conditions via weighted $l_1$ minimization. When prior support estimate is at least 50\% accurate, the sufficient condition is weaker than the analogous condition by standard $l_1$ minimization method, meanwhile the reconstruction error upper bound is provably to be smaller under additional conditions. Furthermore, the sufficient condition is also proved sharp.

Keywords Compressed sensing, Restricted isometry property, Restricted orthogonality constant, Weighted $l_1$ minimization, Sparse signal recovery

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1 Introduction

Compressed sensing shows that it is highly possible to reconstruct sparse signals from what was previously believed to be incomplete information\textsuperscript{11}\textsuperscript{13}. The fundamental goal in compressed sensing is to recover a high dimensional sparse signal based on a small number of linear measurements, possibly corrupted by noise. This can be compactly described via

$$y = Ax + z,$$  \hfill (1.1)
where $A$ is a given $n \times N$ sensing matrix with $n \ll N$, i.e., using very few measurements, $y \in \mathbb{R}^n$ is a vector of measurements, and $z \in \mathbb{R}^n$ is the measurement error ($z = 0$ means no noise). One needs to reconstruct the unknown signal $x \in \mathbb{R}^N$ based on $A$ and $y$. In general, the solutions to the underdetermined systems of linear equations (1.1) are not unique. In order to recover $x$ uniquely, additional assumptions on $A$ such as restricted isometry property and $x$ such as sparsity are needed.

A vector $x \in \mathbb{R}^N$ is $k$-sparse if $\|x\|_0 = |\text{supp}(x)| \leq k$, where $\text{supp}(x) = \{i : x_i \neq 0\}$ is the support of $x$. Then the most natural approach for solving this problem is to find the sparsest solution in the feasible set of possible solutions. In the noiseless case, it can be cast as the $l_0$ minimization problem as below [10, 13, 20, 26]:

$$\minimize_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{subject to} \quad Ax = y. \quad (1.2)$$

It was proved that when measurements $n > 2k$ and $A$ is in general position (any collection of $n$ columns of $A$ is linearly independent), then any $k$-sparse signals can be exactly recovered [14]. However, $l_0$ minimization problem is a combinatorial problem which becomes intractable in the high dimensional settings. Hence, solving it directly is NP-hard.

Candès and Tao [12] then proposed the following constrained $l_1$ minimization method:

$$\minimize_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad \|y - Ax\|_2 \leq \epsilon. \quad (1.3)$$

It can be viewed as a convex relaxation of $l_0$ minimization. To recover sparse signals via constrained $l_1$ minimization, Candès and Tao [12] also introduced the notion of Restricted Isometry Property (RIP), which is one of the most commonly used frameworks for compressive sensing. The definition of RIP is as follows.

**Definition 1.1.** Let $A \in \mathbb{R}^{n \times N}$ be a matrix and $1 \leq k \leq N$ is an integer. The restricted isometry constant (RIC) $\delta_k$ of order $k$ is defined as the smallest nonnegative constant that satisfies

$$(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2,$$

for all $k$-sparse vectors $x \in \mathbb{R}^N$. Note that for $k_1 \leq k_2$, $\delta_{k_1} \leq \delta_{k_2}$.

Thus, $l_1$ minimization has been proved an effective way to recover sparse signals in many settings [2, 3, 5–9, 12, 23]. Candès, Romberg and Tao first gained the sufficient condition for stable recovery by $l_1$ minimization method [9]. In [9], Cai and Zhang applied the following $l_1$ minimization

$$\minimize_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad \|y - Ax\|_2 \in \mathcal{B}, \quad (1.4)$$

where $\mathcal{B}$ is a bounded set determined by the noise structure. In particular, $\mathcal{B}$ is taken to be $\{0\}$ in the noiseless case. Here they considered the following $l_2$ bounded noise and Dantzing Selector noise settings

$$\mathcal{B}^{|z|_2}(\epsilon) = \{z : \|z\|_2 \leq \epsilon\} \quad (1.5)$$
and

\[ B^{DS}(\varepsilon) = \{ z : \| A^T z \|_\infty \leq \varepsilon \}. \quad (1.6) \]

Cai and Zhang [6] provided a sharp sufficient condition \( \delta_{tk} < \sqrt{\frac{t}{t-1}} \) with \( t \geq 4/3 \) which can guarantee the exact recovery of all \( k \)-sparse signals in the noiseless case and stable recovery of approximately sparse signals in the noise case by \( l_1 \) minimization method with (1.4) and (1.6).

In addition, the restricted orthogonality constant is also important in compressed sensing [2, 3, 7].

**Definition 1.2.** Let \( A \in \mathbb{R}^{n \times N} \) be a matrix and \( 1 \leq k_1, k_2 \leq N \) be integers with \( k_1 + k_2 \leq N \), the restricted orthogonality constant (ROC) \( \theta_{k_1,k_2} \) of order \((k_1, k_2)\) is defined as the smallest nonnegative constant that satisfies

\[ |\langle Au, Av \rangle| \leq \theta_{k_1,k_2} \| u \|_2 \| v \|_2, \]

for all \( k_1 \)-sparse vectors \( u \in \mathbb{R}^N \) and \( k_2 \)-sparse vectors \( v \in \mathbb{R}^N \) with disjoint supports. Note that for \( k_1 \leq k_2 \) and \( k_1' \leq k_2' \), \( \theta_{k_1,k_2} \leq \theta_{k_1',k_2'} \).

It also has been shown that \( l_1 \) minimization can recover a sparse signal under various conditions on \( \delta_k \) and \( \theta_{k_1,k_2} \) [2, 3, 7, 11, 12, 15, 17]. For example, \( \delta_k + \theta_{k,k} + \theta_{k,2k} < 1 \) [12], \( \delta_{1.5k} + \theta_{k,1.5k} < 1 \) [15], and \( \delta_{1.25k} + \theta_{k,1.25k} < 1 \) [12]. Cai and Zhang [7] also established a sharp sufficient condition in terms of RIC and ROC to achieve the stable and robust recovery of signals in both noiseless and noisy cases via \( l_1 \) minimization method. In fact, Cai and Zhang [7] proved that \( \delta_a + C_{a,b,k} \theta_{a,b} < 1 \) can ensure stable and robust recovery of signals via \( l_1 \) minimization method with (1.4) and (1.6). Moreover, for any \( \varepsilon > 0 \), \( \delta_a + C_{a,b,k} \theta_{a,b} < 1 + \varepsilon \) is not sufficient to guarantee the exact and stable recovery of all \( k \)-sparse signals via any methods.

It is worthy of noting that compressed sensing is a nonadaptive data acquisition technique since \( A \) is independent of \( x \), the signal being measured. The \( l_1 \) minimization method is also itself nonadaptive as a result of no prior information on the signal \( x \) being used in (1.4). In practical examples, however, the estimate of the support of the signal or of its largest coefficients may be possible to be drawn. Incorporating prior information is very useful for recovering signals from compressive measurements. Thus, the following weighted \( l_1 \) minimization method which incorporates partial support information of the signals has been introduced to replace standard \( l_1 \) minimization

\[
\text{minimize} \quad \| x \|_{1,w} \quad \text{subject to} \quad \| y - Ax \|_2 \leq \varepsilon, \quad (1.7)
\]

where \( w \in [0,1]^N \) and \( \| x \|_{1,w} = \sum w_i |x_i| \). Reconstructing compressively sampled signals with partially known support has been previously studied in the literature; see [1, 16, 18, 19, 21, 22, 24].
Borries, Miosso and Potes in [1], Khajehnejad et al. in [19], and Vaswani and Lu in [24] introduced the problem of signal recovery with partially known support independently. The works by Borries et al. in [1], Vaswani and Lu in [21, 24, 25] and Jacques in [18] incorporated known support information using weighted $l_1$ minimization approach with zero weights on the known support, namely, given a support estimate $\tilde{T} \subset \{1, 2, \ldots, N\}$ of unknown signal $x$, setting $w_i = 0$ whenever $i \in \tilde{T}$ and $w_i = 1$ otherwise, and derived sufficient recovery conditions. Friedlander et al. in [16] extended weighted $l_1$ minimization approach to nonzero weights. They allow the weights $w_i = \omega \in [0, 1]$ if $i \in \tilde{T}$. Since Friedlander et al. incorporated the prior support information and consider the accuracy of the support estimate, they derived the stable and robust recovery guarantees for weighted $l_1$ minimization which generalize the results of Candès, Romberg and Tao in [9]. They actually improved the recovery guarantees of $l_1$ minimization problem (1.3) by using weighted $l_1$ minimization problem (1.7). Friedlander et al. [16] pointed out that once at least 50% of the support information is accurate, a less conservative sufficient condition for guaranteeing stably and robustly signal reconstruction as well as a tighter reconstruction error bound can be obtained. Furthermore, they also pointed out sufficient conditions are weaker than those of [24] when $\omega = 0$.

In this paper, we consider the following weighted $l_1$ minimization method:

$$\min_{x \in \mathbb{R}^N} \|x\|_{1,w} \quad \text{subject to} \quad y - Ax \in \mathcal{B}$$

$$\text{with} \quad w_i = \begin{cases} 1, & i \in \tilde{T}^c \\ \omega, & i \in \tilde{T}. \end{cases} \quad (1.8)$$

where $0 \leq \omega \leq 1$ and $\tilde{T} \subset \{1, 2, \ldots, N\}$ is a given support estimate of unknown signal $x$. $\mathcal{B}$ is also a bounded set determined by the noise settings (1.5) and (1.6). Our goal is to generalize the results of Cai and Zhang [7] via the weighted $l_1$ minimization method (1.8). We establish the sufficient condition on RIC and ROC for the stable and robust recovery of signals with partially known support information from (1.1). We also show that the recovery by weighted $l_1$ minimization method (1.8) is stable and robust under weaker sufficient conditions compared to the standard $l_1$ minimization method (1.4) when we have the partial support information with accuracy better than 50%. Meanwhile, we obtain the smaller upper bounds on the reconstruction error under additional conditions. By means of weighted $l_1$ minimization method (1.8), that is to say, the requirement on the RIC and ROC of the sensing matrix for guaranteeing stable and robust signal recovery can be further relaxed if at least 50% of the support estimate is accurate; in addition, the reconstruction error upper bound is provably to be smaller under additional conditions. Our result implies that the achievable performance of signal recovery via weighted $l_1$ minimization method (1.8) is actually better than the works by Cai and Zhang [7] under some conditions.

The rest of the paper is organized as follows. In Section 2, we will introduce some notations and some basic lemmas that will be used. The main results are given in Section 3 and the proofs
of our main results are presented in Section 4.

2 Preliminaries

Let us begin with basic notations. For arbitrary $x \in \mathbb{R}^N$, $x_{\max(k)}$ is defined as $x$ with all but the largest $k$ entries in absolute value set to zero, i.e. $x_{\max(k)}$ is the best $k$-term approximation of $x$, and $x_{-\max(k)} = x - x_{\max(k)}$. Let $T_0$ be the support of $x_{\max(k)}$, with $T_0 \subseteq \{1, \ldots, N\}$ and $|T_0| \leq k$. Let $\tilde{T} \subseteq \{1, \ldots, N\}$ be the support estimate of $x$ with $|\tilde{T}| = \rho k$, where $\rho \geq 0$ represents the ratio of the size of the estimated support to the size of the actual support of $x_{\max(k)}$ (or the support of $x$ if $x$ is $k$-sparse). Denote $\tilde{T}_0 = T_0 \cap \tilde{T}$ and $\tilde{T}_c = T_0^c \cap \tilde{T}$ with $|\tilde{T}_0| = \alpha |\tilde{T}| = \alpha \rho k$ and $|\tilde{T}_c| = \beta |\tilde{T}| = \beta \rho k$, where $\alpha$ denotes the ratio of the number of indices in $T_0$ that were accurately estimated in $\tilde{T}$ to the size of $\tilde{T}$ and $\alpha + \beta = 1$. For arbitrary nonnegative number $\zeta$, we denote by $\lfloor \zeta \rfloor$ an integer satisfying $\zeta \leq \lfloor \zeta \rfloor < \zeta + 1$.

Moreover, for given set $T \subseteq \{1, \ldots, N\}$, we denote by $x_T$ the vector which equals to $x$ on $T$ and 0 on the component $T^c$.

We first state three key technical tools used in the proof of the main result. Lemma 2.1 was introduced by Cai and Zhang ([7], Lemma 5.1) which provides a way to estimate the inner product by the ROC when only one component is sparse. Lemma 2.2 introduced by Cai and Zhang ([8], Lemma 5.3) provides an inequality between the sum of the $\alpha$th power of two sequences of nonnegative numbers based on the inequality of their sums. Cai, Wang and Xu ([2], Lemma 1) supplied Lemma 2.3 that reveals the relationship between ROC’s of different orders.

**Lemma 2.1 ([7], Lemma 5.1).** Let $k_1, k_2 \leq N$ and $\lambda \geq 0$. Assume $u, v \in \mathbb{R}^N$ have disjoint supports and $u$ is $k_1$-sparse. If $\|v\|_1 \leq \lambda k_2$ and $\|v\|_\infty \leq \lambda$, then

$$|\langle Au, Av \rangle| \leq \theta_{k_1, k_2} \|u\|_2 \cdot \lambda \sqrt{k_2}.$$ 

**Lemma 2.2 ([8], Lemma 5.3).** Assume $m \geq k$, $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$, $\sum_{i=1}^{k} a_i \geq \sum_{i=k+1}^{m} a_i$, then for all $\alpha \geq 1$,

$$\sum_{j=k+1}^{m} a_j^\alpha \leq \sum_{i=1}^{k} a_i^\alpha.$$ 

More generally, assume $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$, $\lambda \geq 0$ and $\sum_{i=1}^{k} a_i + \lambda \geq \sum_{i=k+1}^{m} a_i$, then for all $\alpha \geq 1$,

$$\sum_{j=k+1}^{m} a_j^\alpha \leq k^{1/\alpha} \left( \sum_{i=1}^{k} \frac{a_i^\alpha}{k} + \frac{\lambda}{k} \right)^{\alpha}.$$ 

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Lemma 2.3 ([2], Lemma 1). For any \( \tau \geq 1 \) and positive integers \( k, k' \) such that \( \tau k' \) is an integer, then
\[
\theta_{k,\tau k'} \leq \sqrt{\tau} \theta_{k,k'}.
\]

As we mentioned in the introduction, Cai and Zhang [7] provided the sharp sufficient condition for ensuring exact and stable sparse signals reconstruction via \( l_1 \) minimization (1.4). Their main result can be stated as below.

Theorem 2.1 ([7], Theorem 2.6). Let \( y = Ax + z \) with \( \|z\|_2 \leq \varepsilon \) and \( \tilde{x}^{l_2} \) is the minimizer of (1.4) with \( B = B^{l_2}(\eta) = \{z : \|z\|_2 \leq \eta\} \) for some \( \eta \geq \varepsilon \). If
\[
\delta_a + C_{a,b,k} \theta_{a,b} < 1 \tag{2.1}
\]
for some positive integers \( a \) and \( b \) with \( 1 \leq a \leq k \), where
\[
C_{a,b,k} = \max \left\{ \frac{2k-a}{\sqrt{ab}}, \sqrt{\frac{2k-a}{a}} \right\}, \tag{2.2}
\]
then
\[
\|\tilde{x}^{l_2} - x\|_2 \leq C_0(\varepsilon + \eta) + C_1 \cdot 2 \|x - \max (k)\|_1, \tag{2.3}
\]
where
\[
C_0 = \frac{\sqrt{2}(1+\delta_a)k/a}{1 - \delta_a - C_{a,b,k} \theta_{a,b}}, \quad C_1 = \frac{\sqrt{2k}C_{a,b,k} \theta_{a,b}}{(1 - \delta_a - C_{a,b,k} \theta_{a,b})(2k-a)} + \frac{1}{\sqrt{k}}. \tag{2.4}
\]

Theorem 2.2 ([7], Theorem 2.7). Let \( y = Ax + z \) with \( \|A^Tz\|_\infty \leq \varepsilon \) and \( \tilde{x}^{DS} \) is the minimizer of (1.4) with \( B = B^{DS}(\eta) = \{z : \|A^Tz\|_\infty \leq \eta\} \) for some \( \eta \geq \varepsilon \). If \( \delta_a + C_{a,b,k} \theta_{a,b} < 1 \) for some positive integers \( a \) and \( b \) with \( 1 \leq a \leq k \), where \( C_{a,b,k} = \max \left\{ \frac{2k-a}{\sqrt{ab}}, \sqrt{\frac{2k-a}{a}} \right\} \), then
\[
\|\tilde{x}^{DS} - x\|_2 \leq C'_0(\varepsilon + \eta) + C'_1 \cdot 2 \|x - \max (k)\|_1, \tag{2.5}
\]
where
\[
C'_0 = \frac{\sqrt{2k}}{1 - \delta_a - C_{a,b,k} \theta_{a,b}}, \quad C'_1 = C_1. \tag{2.6}
\]

Cai and Zhang pointed out that the sufficient condition (2.1) is sharp in Theorem 2.8 (see [7]). Namely, if \( \delta_a + C_{a,b,k} \theta_{a,b} = 1 \), there does not exist any method that can exactly recover all \( k \)-sparse signals in noiseless case. Also, in noisy case, for any \( \varepsilon > 0 \), \( \delta_a + C_{a,b,k} \theta_{a,b} < 1 + \varepsilon \) can not guarantee the stable recovery of all \( k \)-sparse signals.
3 Main results

**Theorem 3.1.** Let \( x \in \mathbb{R}^N \) be an arbitrary signal and its best \( k \)-term approximation support on \( T_0 \subseteq \{1, \ldots, N\} \) with \( |T_0| \leq k \). Let \( \tilde{T} \subseteq \{1, \ldots, N\} \) be an arbitrary set and denote \( \rho \geq 0 \) and \( 0 \leq \alpha \leq 1 \) such that \( |	ilde{T}| = \rho k \) and \( |	ilde{T} \cap T_0| = \alpha \rho k \). Let \( y = Ax + z \) with \( \|z\|_2 \leq \varepsilon \) and \( \tilde{x}^l_2 \) is the minimizer of (1.8) with (1.5). If

\[
\delta_a + C_{a,b,k}^{\alpha,\omega} \theta_{a,b} < 1
\]

(3.1)

for some positive integers \( a \) and \( b \) with \( 1 \leq a \leq k \), where

\[
C_{a,b,k}^{\alpha,\omega} = \max \left\{ \frac{s}{\sqrt{ab}}, \frac{s}{\sqrt{a}} \right\}
\]

(3.2)

with

\[
s = \left[ \left( k - a + \omega k + (1 - \omega)\sqrt{(1 + \rho - 2\alpha \rho)k} \cdot \max\left\{ \sqrt{(1 + \rho - 2\alpha \rho)k}, \sqrt{a}\right\} \right) \right].
\]

(3.3)

Then

\[
\|\tilde{x}^l_2 - x\|_2 \leq D_0(2\varepsilon) + D_1 \cdot 2 \left( \omega \|x_{T_0}\|_1 + (1 - \omega)\|x_{\tilde{T} \cap T_0}\|_1 \right),
\]

(3.4)

where

\[
D_0 = \frac{\sqrt{2(1 + \delta_a)d/a}}{1 - \delta_a - C_{a,b,k}^{\alpha,\omega} \theta_{a,b}},
\]

\[
D_1 = \frac{\sqrt{2dC_{a,b,k}^{\alpha,\omega} \theta_{a,b}}}{(1 - \delta_a - C_{a,b,k}^{\alpha,\omega} \theta_{a,b})s + \frac{1}{\sqrt{d}}},
\]

(3.5)

Let \( y = Ax + z \) with \( \|A^Tz\|_\infty \leq \varepsilon \). Assume that \( \tilde{x}^{DS} \) is the minimizer of (1.8) with (1.6) and (3.1) holds. If

\[
\delta_a + C_{a,b,k}^{\alpha,\omega} \theta_{a,b} < 1
\]

for some positive integers \( a \) and \( b \) with \( 1 \leq a \leq k \), where

\[
C_{a,b,k}^{\alpha,\omega} = \max \left\{ \frac{s}{\sqrt{ab}}, \frac{s}{\sqrt{a}} \right\},
\]

(3.6)

where \( s \) is given in (3.3). Then

\[
\|\tilde{x}^{DS} - x\|_2 \leq D'_0(2\varepsilon) + D'_1 \cdot 2 \left( \omega \|x_{T_0}\|_1 + (1 - \omega)\|x_{\tilde{T} \cap T_0}\|_1 \right),
\]

(3.7)

where

\[
D'_0 = \frac{\sqrt{2d}}{1 - \delta_a - C_{a,b,k}^{\alpha,\omega} \theta_{a,b}}, \quad D'_1 = D_1.
\]

(3.8)
Here
\[
d = \begin{cases} 
k, & \omega = 1, \\
\max\{k, (1 + \rho - 2\alpha\rho)k\}, & 0 < \omega < 1. \end{cases}
\] (3.8)

**Remark 3.1.** In Theorem 2.1, we observed that every signal \( x \in \mathbb{R}^N \) can be stably and robustly recovered. And if \( B = \{0\} \) and \( x \) is a \( k \)-sparse signal, then Theorem 3.1 ensures exact recovery of the signal \( x \).

When the the measurement model (1.1) is with Gaussian noise, the above results on the bounded noise case can be directly applicable to the case where the noise is Gaussian by using the same argument as in [2, 3]. This is due to the fact Gaussian noise is essentially bounded. The concrete content is stated as follows.

**Remark 3.2.** Let \( x \in \mathbb{R}^N \) be an arbitrary signal and its best \( k \)-term approximation support on \( T_0 \subseteq \{1, \ldots, N\} \) with \( |T_0| \leq k \). Let \( \tilde{T} \subseteq \{1, \ldots, N\} \) be an arbitrary set and define \( \rho \geq 0 \) and \( 0 \leq \alpha \leq 1 \) such that \( |\tilde{T}| = \rho k \) and \( |\tilde{T} \cap \tilde{T}_0| = \alpha \rho k \). Assume that \( z \sim N_n(0, \sigma^2 I) \) in (1.1) and \( \delta_a + C_{\alpha,\omega}^a \theta_{a,b} \leq 1 \) for some positive integers \( a \) and \( b \) with \( 1 \leq a \leq k \), where \( C_{\alpha,\omega}^a = \max \left\{ \frac{s}{\sqrt{ab}}, \sqrt{\frac{a}{\alpha}} \right\} \) with \( s = \left\lceil \left( k - a + \omega k + (1 - \omega)\sqrt{(1 + \rho - 2\alpha\rho)k} \right) \max\{\sqrt{(1 + \rho - 2\alpha\rho)k}, a\} \right\rceil \). Let \( B^{l_2} = \{ z : \|z\|_2 \leq \sigma\sqrt{n + 2\sqrt{n \log n}} \} \) and \( B^{DS} = \{ z : \|A^Tz\|_\infty \leq \sigma\sqrt{2\log N} \} \). \( \hat{x}^{l_2} \) and \( \hat{x}^{DS} \) is the minimizer of (1.8) with \( B^{l_2} \) and \( B^{DS} \), respectively. Then, with probability at least \( 1 - 1/n \),
\[
\|\hat{x}^{l_2} - x\|_2 \leq D_0(2\sigma\sqrt{n + 2\sqrt{n \log n}}) + D_1 \cdot 2 \left( \omega \|x_{T_0^c}\|_1 + (1 - \omega)\|x_{\tilde{T} \cap T_0^c}\|_1 \right),
\]
and
\[
\|\hat{x}^{DS} - x\|_2 \leq D'_0(2\sigma\sqrt{2\log N}) + D'_1 \cdot 2 \left( \omega \|x_{T_0^c}\|_1 + (1 - \omega)\|x_{\tilde{T} \cap T_0^c}\|_1 \right),
\]
with probability at least \( 1 - 1/\sqrt{n \log N} \).

**Theorem 3.2.** Let \( 1 \leq a \leq s \leq k \), \( a + s \leq N \) and \( b \geq 1 \), where \( s \) is defined as (3.3). Then there exists a sensing matrix \( A \in \mathbb{R}^{n \times N} \) satisfying \( \delta_a + C_{\alpha,\omega}^a \theta_{a,b} = 1 \) where \( C_{\alpha,\omega}^a = \max \left\{ \frac{s}{\sqrt{ab}}, \sqrt{\frac{a}{\alpha}} \right\} \) and some \( k \)-sparse vector \( \eta \in \mathbb{R}^N \) such that the weighted \( l_1 \) minimization method (1.8) fails to exactly recover the \( k \)-sparse vector \( \eta \) in the noiseless case and stably recover the \( k \)-sparse vector \( \eta \) in the noise case.

**Remark 3.3.** Theorem 3.2 implies that for arbitrarily \( \varepsilon > 0 \), \( \delta_a + C_{\alpha,\omega}^a \theta_{a,b} < 1 + \varepsilon \) is not sufficient to guarantee the exact recovery of all \( k \)-sparse vectors in noiseless case and the stable recovery of all \( k \)-sparse vectors in noise case.

**Proposition 3.1.** Let \( s \) be defined as (3.3) and \( d \) be defined as (3.8).
(1) If $\omega = 1$, then $s = 2k - a, d = k$. The sufficient condition (3.4) of Theorem 3.1 is identical to that of Theorem 2.1 and Theorem 2.2 with (2.1), and $D_0 = C_0, D_1 = C_1, D'_0 = C'_0, D'_1 = C'_1$. Moreover, the condition is sharp.

(2) If $\alpha = \frac{1}{2}$, then $s = 2k-a$ and $d = k$. The sufficient condition (3.4) of Theorem 3.1 is identical to that of Theorem 2.1 and Theorem 2.2 with (2.1), and $D_0 = C_0, D_1 = C_1, D'_0 = C'_0, D'_1 = C'_1$. Moreover, the condition is sharp.

(3) Assume $0 \leq \omega < 1$. If $\alpha > \frac{1}{2}$, then $s < 2k - a$ and $d = k$. The sufficient condition (3.4) in Theorem 3.1 is weaker than that of Theorem 2.1 and Theorem 2.2 with (2.1), and $D_0 < C_0, D'_0 < C'_0$.

(4) Suppose $0 \leq \omega < 1$. If $\alpha > \frac{1}{2}$ and $b \leq s$, then $D_1 < C_1$.

(5) Suppose $0 \leq \omega < 1$. If $\alpha > \frac{1}{2}$ and $s < b \leq 2k-a$, then $D_1 < C_1$ if and only if $1 - \delta_a - C_{a,b}^\omega \theta_{a,b} < \sqrt{\frac{2k - a - b}{\sqrt{a - b}}} \theta_{a,b}$.

(6) Suppose $0 \leq \omega < 1$. If $\alpha > \frac{1}{2}$ and $b > 2k - a$, then $D_1 < C_1$ if and only if $1 - \delta_a - C_{a,b}^\omega \theta_{a,b} < \sqrt{\frac{2k - a}{a}} \theta_{a,b}$.

4 Proofs

Proof of Theorem 3.1. Firstly, we show the estimate (3.4). Let $h = \tilde{x}^{l_2} - x$, where $x$ is the original signal and $\tilde{x}^{l_2}$ is the minimizer of (1.8) with (1.5). We can express $h$ as $h = \sum_{i=1}^{N} c_i u_i$, where $\{c_i\}_{i=1}^{N}$ are nonnegative and decreasing, i.e. $c_1 \geq c_2 \geq \cdots \geq c_N \geq 0$. Let $\{u_i\}_{i=1}^{N}$ are different unit vectors with one entry of $\pm 1$ and other entries of zeros. From the following inequality proved by Friedlander et al. (see (21) in [16])

$$
\|h_T\|_1 \leq \omega \|h_T\|_1 + (1 - \omega) \|h_{T_0 \cup \tilde{T} \setminus T_0}\|_1 + 2(\omega \|x_T\|_1 + (1 - \omega) \|x_{\tilde{T} \cap T_0}\|_1),
$$

(4.1)

we have

$$
\sum_{i=k+1}^{N} c_i = \|h_{-\max(k)}\|_1 \leq \omega \|h_T\|_1 + (1 - \omega) \|h_{T_0 \cup \tilde{T} \setminus T_0}\|_1 + 2(\omega \|x_T\|_1 + (1 - \omega) \|x_{\tilde{T} \cap T_0}\|_1).
$$

Noting that $|T_0 \cup \tilde{T} \setminus T_0| = (1 + 2\alpha \rho)k$, thus

$$
\|h_{-\max(a)}\|_\infty = c_{a+1} = \frac{\sum_{i=1}^{a} c_i}{a} \leq \frac{\|h_{\max(a)}\|_1}{a} \leq \frac{\|h_{\max(a)}\|_2}{\sqrt{a}},
$$

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\[
\|h_{-\max(a)}\|_1 = \sum_{i=0+1}^{k} c_i + \sum_{i=k+1}^{N} c_i \\
\leq \frac{k-a}{k} \sum_{i=1}^{k} c_i + \omega \|h_{T_0}\|_1 + (1-\omega)\|h_{T_0 \cup \tilde{T}_0\setminus T_0}\|_1 + 2 \left(\omega \|x_{T_0}\|_1 + (1-\omega)\|x_{\tilde{T}_0\cap T_0}\|_1\right) \\
\leq \frac{k-a}{a} \|h_{\max(a)}\|_1 + \omega \sqrt{k} \|h_{T_0}\|_2 + (1-\omega)\sqrt{(1+\rho-2\alpha\rho)k} \|h_{T_0 \cup \tilde{T}_0\setminus T_0}\|_2 \\
+ 2 \left(\omega \|x_{T_0}\|_1 + (1-\omega)\|x_{\tilde{T}_0\cap T_0}\|_1\right) \\
= \left(k-a + \omega k + (1-\omega)\sqrt{(1+\rho-2\alpha\rho)k} \cdot \max\{\sqrt{1+\rho-2\alpha\rho}, \sqrt{a}\}\right) \|h_{\max(a)}\|_2 \sqrt{a} \\
+ 2 \left(\omega \|x_{T_0}\|_1 + (1-\omega)\|x_{\tilde{T}_0\cap T_0}\|_1\right) \\
\leq s \|h_{\max(a)}\|_2 \sqrt{a} + 2 \left(\omega \|x_{T_0}\|_1 + (1-\omega)\|x_{\tilde{T}_0\cap T_0}\|_1\right),
\]
where \(s = \left[\left(k-a + \omega k + (1-\omega)\sqrt{(1+\rho-2\alpha\rho)k} \cdot \max\{\sqrt{1+\rho-2\alpha\rho}, \sqrt{a}\}\right)\right].\) Taking \(k_1 = a, k_2 = s, \lambda = \frac{\|h_{\max(a)}\|_2}{\sqrt{a}} + \frac{2\left(\omega \|x_{T_0}\|_1 + (1-\omega)\|x_{\tilde{T}_0\cap T_0}\|_1\right)}{s},\) from above inequalities and Lemma 2.1, we obtain

\[
|\langle Ah_{\max(a)}, Ah_{-\max(a)}\rangle| \leq \theta_{\alpha, s} \|h_{\max(a)}\|_2 2\sqrt{s} \cdot \left(\frac{\|h_{\max(a)}\|_2}{\sqrt{a}} + \frac{2\left(\omega \|x_{T_0}\|_1 + (1-\omega)\|x_{\tilde{T}_0\cap T_0}\|_1\right)}{s}\right).
\]

Combining the definition of \(\delta_k\) and the fact that

\[
\|Ah\|_2 = \|A\tilde{x}_0 - Ax\|_2 \leq \|y - A\tilde{x}_0\|_2 + \|Ax - y\|_2 \leq 2\varepsilon,
\]
we have

\[
|\langle Ah_{\max(a)}, Ah\rangle| \leq \|Ah_{\max(a)}\|_2 \|Ah\|_2 \\
\leq \sqrt{1 + \delta_a} \|h_{\max(a)}\|_2 \cdot (2\varepsilon).
\]
Hence,

\[(2\varepsilon)\sqrt{1 + \delta_a} \|h_{\text{max}(a)}\|_2 \geq |\langle Ah_{\text{max}(a)}, Ah \rangle| \]

\[\geq A h_{\text{max}(a)}\|_2^2 - |\langle Ah_{\text{max}(a)}, Ah_{-\text{max}(a)} \rangle| \]

\[\geq (1 - \delta_a)\|h_{\text{max}(a)}\|_2^2 - \theta_{a,s}\|h_{\text{max}(a)}\|_22\sqrt{s} \cdot \left(\frac{\|h_{\text{max}(a)}\|_2}{\sqrt{a}} + \frac{2(\omega\|x_{T_0^n}\|_1 + (1 - \omega)\|x_{\bar{T}_c \cap T_0^n}\|_1)}{s}\right) \]

\[= \left(1 - \delta_a - \sqrt{\frac{\theta_{a,s}}{a}}\right)\|h_{\text{max}(a)}\|_2^2 - \theta_{a,s}\|h_{\text{max}(a)}\|_2^2 \frac{2(\omega\|x_{T_0^n}\|_1 + (1 - \omega)\|x_{\bar{T}_c \cap T_0^n}\|_1)}{\sqrt{s}}. \]

It follows from the above inequality that

\[\|h_{\text{max}(a)}\|_2 \leq \frac{\sqrt{1 + \delta_a} (2\varepsilon)}{1 - \delta_a - \sqrt{\frac{\theta_{a,s}}{a}}} + \frac{\theta_{a,s}}{1 - \delta_a - \sqrt{\frac{\theta_{a,s}}{a}}} \frac{2(\omega\|x_{T_0^n}\|_1 + (1 - \omega)\|x_{\bar{T}_c \cap T_0^n}\|_1)}{\sqrt{s}}. \]

Define

\[d = \begin{cases} 
  k, & \text{for } \omega = 1, \\
  \max\{k, (1 + \rho - 2\alpha \rho)k\}, & 0 \leq \omega < 1.
\end{cases} \]

With (4.1), it is clear that

\[\|h_{-\text{max}(d)}\|_1 \leq \|h_{\text{max}(d)}\|_1 + 2(\omega\|x_{T_0^n}\|_1 + (1 - \omega)\|x_{\bar{T}_c \cap T_0^n}\|_1). \]

From Lemma 2.2, we have

\[\|h_{-\text{max}(d)}\|_2 \leq \|h_{\text{max}(d)}\|_2 + \frac{2(\omega\|x_{T_0^n}\|_1 + (1 - \omega)\|x_{\bar{T}_c \cap T_0^n}\|_1)}{\sqrt{d}}. \]
Therefore,
\[
\|h\|_2 = \sqrt{\|h_{\max}(d)\|_2^2 + \|h_{\max}(d)\|_2^2} \\
\leq \sqrt{\|h_{\max}(d)\|_2^2 + \left(\|h_{\max}(d)\|_2 + \frac{2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}{\sqrt{d}}\right)^2} \\
\leq \sqrt{2\|h_{\max}(d)\|_2^2 + \frac{2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}{\sqrt{d}}} \\
= \sqrt{2 \sum_{i=1}^{d} c_i^2 + \frac{2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}{\sqrt{d}}} \\
\leq \sqrt{2 \sum_{i=1}^{d} c_i^2 + \frac{2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}{\sqrt{d}}} \\
= \sqrt{\frac{2d}{a}\|h_{\max}(a)\|_2 + \frac{2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}{\sqrt{d}}} \\
\leq \sqrt{\frac{2(1 + \delta_a)d/a}{1 - \delta_a - \sqrt{s}\theta_{a,s}}(2\varepsilon) + \left(\frac{\sqrt{2d/a}\theta_{a,s}}{1 - \delta_a - \sqrt{s}\theta_{a,s}} + \frac{1}{\sqrt{d}}\right) \cdot 2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}.
\]

Since
\[
\theta_{a,s} = \theta_{a,\min\{b,s\}} \leq \sqrt{\frac{s}{\min\{b,s\}}} \theta_{a,\min\{b,s\}} \leq \max\{\sqrt{\frac{s}{b}}, 1\} \theta_{a,b} = \sqrt{\frac{s}{b}} C_{a,b,k} \theta_{a,b},
\]
where \( C_{a,b,k} = \max\{\frac{s}{\sqrt{a}}, \sqrt{\frac{s}{a}}\} \), and the first inequality follows from Lemma 2.3. Consequently,
\[
\|h\|_2 \leq \sqrt{\frac{2(1 + \delta_a)d/a}{1 - \delta_a - C_{a,b,k} \theta_{a,b}}(2\varepsilon) + \left(\frac{\sqrt{2d} C_{a,b,k} \theta_{a,b}}{1 - \delta_a - C_{a,b,k} \theta_{a,b}} + \frac{1}{\sqrt{d}}\right) \cdot 2\left(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1\right)}.
\]

So, (3.4) is obtained.

Next, we can prove (3.6) going along similar lines to that of (3.4). To prove (3.6), we only need to use the following (4.4) and (4.5) instead of (1.2) and (1.3), respectively.

\[
\|A^T A\|_{\infty} = \|A^T A(\widehat{x}_{DS} - x)\|_{\infty} \\
\leq \|A^T (A\widehat{x}_{DS} - y)\|_{\infty} + \|A^T (y - Ax)\|_{\infty} \\
\leq 2\varepsilon, \tag{4.4}
\]

\[
|\langle Ah_{\max(a)}, Ah \rangle| = |\langle h_{\max(a)}, A^T Ah \rangle| \\
\leq \|h_{\max(a)}\|_1 \|A^T Ah\|_{\infty} \\
\leq \sqrt{a}\|h_{\max(a)}\|_2 \cdot (2\varepsilon). \tag{4.5}
\]
This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Firstly, let $L = a + s$, and

$$\xi_1 = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \xi(i) = \xi(a, \rho) \in \mathbb{R}^N,$$  

if $L - k > \rho k$,

or

$$\xi_1 = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \xi(i) = \xi(b, \rho) \in \mathbb{R}^N,$$  

if $L - k \leq \rho k$.

Due to $\|\xi_1\|_2 = 1$, we extend $\xi_1$ into an orthonormal basis $\{\xi_1, \ldots, \xi_N\}$ of $\mathbb{R}^N$. Next, we define the linear map $A : \mathbb{R}^N \to \mathbb{R}^N$ such that for all $x = \sum_{i=1}^{N} c_i \xi_i \in \mathbb{R}^N$,

$$Ax = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \xi(i) = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \xi(i) = \xi(a, \rho) \in \mathbb{R}^N.$$

Then for any $a$-sparse signal $x$, we can easily gain

$$\|Ax\|_2^2 = \left(1 + \frac{L - s}{L + s}\right) \left(\|x\|_2^2 - |\langle \xi_1, x \rangle|^2\right),$$

and

$$|\langle \xi_1, x \rangle|^2 \leq \|x\|_2^2 \cdot \sum_{i \in \text{supp}(x)} |\xi_1(i)|^2 \leq \|x\|_2^2 \cdot \|\xi_{1, \text{max}(a)}\|_2^2 \leq \frac{a}{L} \|x\|_2^2 = \frac{L - s}{L} \|x\|_2^2.$$

Hence,

$$\left(1 + \frac{L - s}{L + s}\right) \|x\|_2^2 \geq \|Ax\|_2^2 \geq \left(1 + \frac{L - s}{L + s}\right) (1 - \frac{L - s}{L}) \|x\|_2^2 = \left(1 - \frac{L - s}{L + s}\right) \|x\|_2^2,$$

which deduces

$$\delta_a \leq \frac{L - s}{L + s}.$$

Finally, we estimate $\theta_{a,b}$. For arbitrary $a$-sparse vector $u \in \mathbb{R}^N$ and $b$-sparse vector $v \in \mathbb{R}^N$ with disjoint supports, we define $u = \sum_{i=1}^{N} l_i \xi_i$ and $v = \sum_{i=1}^{N} d_i \xi_i$. It follows immediately that

(1) When $b \leq s$, through a simple calculation, it can be concluded that

$$|l_1| = \|\langle \xi_1, u \rangle\| \leq \|u\|_2 \cdot \left(\sum_{i \in \text{supp}(u)} |\xi_1(i)|^2\right)^{1/2} \leq \|u\|_2 \cdot \|\xi_{1, \text{max}(a)}\|_2 \leq \sqrt{\frac{a}{L}} \|u\|_2,$$

and

$$|d_1| = \|\langle \xi_1, v \rangle\| \leq \|v\|_2 \cdot \left(\sum_{i \in \text{supp}(v)} |\xi_1(i)|^2\right)^{1/2} \leq \|v\|_2 \cdot \|\xi_{1, \text{max}(b)}\|_2 \leq \sqrt{\frac{b}{L}} \|v\|_2.$$
It then follows that

\[
\frac{1}{1 + \frac{L-s}{L+s}}|\langle Au, Av \rangle| = \left| \sum_{i=2}^{N} l_i d_i \right| = \left| -l_1 d_1 \right| \leq \frac{\sqrt{ab}}{L} \|u\|_2 \|v\|_2.
\]

Accordingly,

\[
\theta_{a,b} \leq (1 + \frac{L-s}{L+s}) \frac{\sqrt{ab}}{L}.
\]

Therefore,

\[
\delta_a + C_{a,b,k}^{\alpha,\omega} \theta_{a,b} \leq \frac{L-s}{L+s} + \max \left\{ \frac{s}{\sqrt{ab}}, \frac{s}{a} \right\} \cdot (1 + \frac{L-s}{L+s}) \frac{\sqrt{ab}}{L}
\]

\[
= \frac{L-s}{L+s} + \frac{s}{\sqrt{ab}} \left( 1 + \frac{L-s}{L+s} \right) \frac{\sqrt{ab}}{L}
\]

\[
= 1.
\]

(ii) When \( b > s \), without loss of generality, we can suppose that \( u \) and \( v \) are nonzero. If \( u = 0 \) or \( v = 0 \), clearly \( \langle Au, Av \rangle = 0 \leq C \|u\|_2 \|v\|_2 \) holds for all \( C > 0 \). We normalize \( u \) and \( v \) such that \( \|u\|_2 = \|v\|_2 = 1 \). Because \( u \) is \( a \)--sparse and \( v \) is \( b \)--sparse, and \( u, v \) have disjoint supports, we conclude

\[
|l_1| = |\langle \xi_1, u \rangle| \leq \sqrt{\frac{a}{L}} \|u\|_2 = \sqrt{\frac{a}{L}} = \sqrt{\frac{a}{s+a}},
\]

and

\[
\left| d_1 \pm \sqrt{\frac{a}{s}} l_1 \right| = \left| \langle \xi_1, v \pm \sqrt{\frac{a}{s}} u \rangle \right| \leq \| v \pm \sqrt{\frac{a}{s}} u \|_2
\]

\[
= \sqrt{\|v\|_2^2 + \frac{a}{s} \|u\|_2^2} = \sqrt{\frac{s+a}{s}}.
\]
In view of $|l_1| \leq \sqrt{\frac{a}{a+s}}$ and $1 \leq a \leq s,$

$$
\frac{1}{1 + \frac{L-s}{L+s}}|(Au, Av)| = \left| \sum_{i=2}^{N} l_i d_i \right| = | - l_1 d_1 |
$$

$$
= \left( \max \left\{ \left| d_1 + \sqrt{\frac{a}{s}} l_1 \right|, \left| d_1 - \sqrt{\frac{a}{s}} l_1 \right| \right\} - \sqrt{\frac{a}{s}} l_1 \right) \cdot |l_1|
$$

$$
\leq |l_1| \left( \sqrt{\frac{s+a}{s}} - \sqrt{\frac{a}{s}} |l_1| \right)
$$

$$
= -\sqrt{\frac{a}{s}} \left( |l_1|^2 - \frac{1}{2} \sqrt{s + a} \frac{a}{s} |l_1| \right) + \frac{s + a}{4\sqrt{as}}
$$

$$
\leq -\sqrt{\frac{a}{s}} \left( \sqrt{\frac{s}{s + a} - \frac{1}{2} \sqrt{\frac{s + a}{a}} \right} + \frac{s + a}{4\sqrt{as}}
$$

$$
= \frac{\sqrt{s+a}}{s+a} = \frac{\sqrt{as}}{L},
$$

which implies

$$
\theta_{a,b} \leq (1 + \frac{L-s}{L+s}) \frac{\sqrt{as}}{L}.
$$

Hence,

$$
\delta_a + C_{a,b,k}^{\alpha,\omega} \theta_{a,b} \leq \frac{L-s}{L+s} + \max \left\{ \frac{s}{\sqrt{ab}}, \sqrt{\frac{s}{a}} \right\} \cdot (1 + \frac{L-s}{L+s}) \frac{\sqrt{as}}{L}
$$

$$
= \frac{L-s}{L+s} + \sqrt{s} \left( 1 + \frac{L-s}{L+s} \frac{\sqrt{as}}{L} \right)
$$

$$
= 1.
$$

In a word, $\delta_a + C_{a,b,k}^{\alpha,\omega} \theta_{a,b} \leq 1$ has been proved.

Next, we define

$$
\eta = (1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^N,
$$

$$
\gamma = (0, \ldots, 0, -1, \ldots, -1, 0, \ldots, 0, -1, \ldots, -1, 0, \ldots, 0) \in \mathbb{R}^N, \quad \text{if } L-k > \rho k,
$$

$$
\text{or} \quad \gamma = (0, \ldots, 0, -1, \ldots, -1, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0) \in \mathbb{R}^N, \quad \text{if } L-k \leq \rho k.
$$

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From $1 \leq a \leq s \leq k$ and $L = a + s$, we have $L - k \leq k$. Hence $\gamma$ and $\gamma$ are $k$-sparse. Moreover, $||\eta||_1, w = k$, $||\gamma||_1, w \leq L - k \leq k$. Note that $||\gamma||_1, w \leq ||\eta||_1, w$ and $\xi_1 = \frac{1}{\sqrt{L}}(\eta - \gamma)$. Since $A\xi_1 = 0$, we obtain $A\eta = A\gamma$.

(i) $||\gamma||_1, w < ||\eta||_1, w$.

In the noiseless case $y = A\eta$, if weighted $l_1$ minimization method (1.8) can exactly recover $\eta$, namely, $\hat{\eta} = \eta$. Clearly, $||\hat{\eta}||_1, w = ||\eta||_1, w$. It contradicts that $||\gamma||_1, w < ||\eta||_1, w$.

In the noise case $y = A\eta + z$, suppose weighted $l_1$ minimization method (1.8) can stable recover $\eta$ with constraint $B$, i.e., $\lim_{z \to 0} \hat{\eta} = \eta$. Due to $y - A(\hat{\eta} - \eta + \gamma) = y - A\hat{\eta} \in B$ and the definition of $\hat{\eta}$, it follows immediately that $||\hat{\eta}||_1, w \leq ||\hat{\eta} - \eta + \gamma||_1, w$. Thus, we have $||\eta||_1, w \leq ||\gamma||_1, w$ as $z \to 0$. It contradicts that $||\gamma||_1, w < ||\eta||_1, w$.

(ii) $||\gamma||_1, w = ||\eta||_1, w$. The weighted $l_1$ method (1.8) does not distinguish $k$-sparse signals $\eta$ and $\gamma$ based on $y$ and $A$.

Hence the weighted $l_1$ method (1.8) does not exactly and stably recover the $k$-sparse signal $\eta$ based on $A$ and $y$. Combining Theorem 3.1, we have $\delta_a + C_{a, b, k}^{\alpha, \omega} \theta_{a, b} = 1$. This completes the proof of the theorem.

Proof of Proposition 3.1. For (1) and (2), when $\omega = 1$ or $\alpha = \frac{1}{2}$, by simple calculation, we have $s = 2k - a, d = k$. Then, it is easy to imply (1) and (2) by comparing Theorem 3.1 with Theorem 2.1 and Theorem 2.2.

(3) Let $0 \leq \omega < 1$. If $\alpha > \frac{1}{2}$, by means of the definition of $s$ in (3.3) and $d$ in (3.8), it follows immediately that $s < 2k - a, d = k$.

When $b \leq s$, $C_{a, b, k}^{\alpha, \omega} = \frac{s}{\sqrt{ab}} < \frac{2k - a}{\sqrt{ab}} = C_{a, b, k}$.

When $s < b \leq 2k - a$, $C_{a, b, k}^{\alpha, \omega} = \sqrt{\frac{s}{a}} < \frac{2k - a}{\sqrt{ab}} = C_{a, b, k}$.

When $b \geq 2k - a$, $C_{a, b, k}^{\alpha, \omega} = \sqrt{\frac{s}{a}} < \frac{2k - a}{\sqrt{ab}} = C_{a, b, k}$.

For any positive integers $a$ and $b$ with $1 \leq a \leq k$, in short, we obtain $C_{a, b, k}^{\alpha, \omega} < C_{a, b, k}$, which implies $\delta_a + C_{a, b, k}^{\alpha, \omega} \theta_{a, b} < \delta_a + C_{a, b, k} \theta_{a, b}$. Thus the condition $\delta_a + C_{a, b, k}^{\alpha, \omega} \theta_{a, b} < 1$ in (3.1) is weaker than $\delta_a + C_{a, b, k} \theta_{a, b} < 1$ in (3.21) and $D_0 = \frac{\sqrt{2(1 + \delta_a)\frac{k}{a}}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} < \frac{\sqrt{2(1 + \delta_a)\frac{k}{a}}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} = C_0$, $D_0' = \frac{\sqrt{2k}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} = C_0'$, which implies (3).

(4) Assume $0 \leq \omega < 1$. If $\alpha > \frac{1}{2}$ and $b \leq s$, we have $d = k$ and $\delta_a + C_{a, b, k}^{\alpha, \omega} \theta_{a, b} < \delta_a + C_{a, b, k} \theta_{a, b}$.

Combining the definition of $C_1$ and $D_1$, obviously, $D_1 = \frac{\sqrt{2k}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} + \frac{1}{\sqrt{k}} < \frac{\sqrt{2k}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} + \frac{1}{\sqrt{k}} = C_1$.

(5) Since $\alpha > \frac{1}{2}$ and $s < b \leq 2k - a$, $C_{a, b, k}^{\alpha, \omega} = \sqrt{\frac{s}{a}}$, $C_{a, b, k} = \frac{2k - a}{\sqrt{ab}}$. Thus, to prove $D_1 < C_1$, we just need to prove $\frac{\sqrt{2k}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} + \frac{1}{\sqrt{k}} < \frac{\sqrt{2k}}{1 - \delta_a - C_{a, b, k} \theta_{a, b}} + \frac{1}{\sqrt{k}}$. It is equal to prove that $1 - \delta_a < \frac{2k - a - s}{\sqrt{a(\sqrt{b} - \sqrt{s})}} \theta_{a, b}$, namely, $1 - \delta_a - C_{a, b, k} \theta_{a, b} < \frac{2k - a - s}{\sqrt{a(\sqrt{b} - \sqrt{s})}} \theta_{a, b}$. 

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(6) Due to $\alpha > \frac{1}{2}$ and $b > 2k - a$, we have $C_{a,b,k}^{\alpha,\omega} = \sqrt{\frac{a}{a}}$ and $C_{a,b,k} = \sqrt{\frac{2k-a}{a}}$. To show $D_1 < C_1$ is equal to prove that

$$\frac{\sqrt{2k} \sqrt{\frac{a}{a}} \theta_{a,b}}{(1-\delta_a - \sqrt{\frac{a}{a}} \theta_{a,b}) \sqrt{a}} + \frac{1}{\sqrt{k}} < \frac{\sqrt{2k} \sqrt{\frac{2k-a}{a}} \theta_{a,b}}{(1-\delta_a - \sqrt{\frac{2k-a}{a}} \theta_{a,b}) \sqrt{2k-a}} + \frac{1}{\sqrt{k}}.$$ 

It suffices to prove $1 - \delta_a < \frac{\sqrt{2k-a} + \sqrt{a}}{\sqrt{a}} \theta_{a,b}$, i.e., $1 - \delta_a - C_{a,b,k}^{\alpha,\omega} \theta_{a,b} < \sqrt{\frac{2k-a}{a}} \theta_{a,b}$.  

\[\square\]

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