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Topological quantization of energy transport in micro- and nano-mechanical lattices

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Topological effects typically discussed in the context of quantum physics are emerging as one of the central paradigms of physics. Here, we demonstrate the role of topology in energy transport through dimerized micro- and nano-mechanical lattices in the classical regime, i.e., essentially “masses and springs”. We show that the thermal conductance factorizes into topological and non-topological components. The former takes on three discrete values and arises due to the appearance of edge modes that prevent good contact between the heat reservoirs and the bulk, giving a length-independent reduction of the conductance. In essence, energy input at the boundary mostly stays there, an effect robust against disorder and nonlinearity. These results bridge two seemingly disconnected disciplines of physics, namely topology and thermal transport, and suggest ways to engineer thermal contacts, opening a direction to explore the ramifications of topological properties on nanoscale technology.

Topological effects typically discussed in the context of quantum physics are emerging as one of the central paradigms of physics. Here, we demonstrate the role of topology in energy transport through dimerized micro- and nano-mechanical lattices in the classical regime, i.e., essentially “masses and springs”. We show that the thermal conductance factorizes into topological and non-topological components. The former takes on three discrete values and arises due to the appearance of edge modes that prevent good contact between the heat reservoirs and the bulk, giving a length-independent reduction of the conductance. In essence, energy input at the boundary mostly stays there, an effect robust against disorder and nonlinearity. These results bridge two seemingly disconnected disciplines of physics, namely topology and thermal transport, and suggest ways to engineer thermal contacts, opening a direction to explore the ramifications of topological properties on nanoscale technology.

The mechanical lattice we examine has vibrational energy transport

The topological nature of the lattice can be seen by considering \( \bar{H}_q = R_x \sigma_x + R_y \sigma_y \), where \( \sigma_x, \sigma_y \) are the Pauli matrices. The curve \( (R_x = K_1 + K_2 \cos(q), R_y = -K_2 \sin(q)) \) may or may not wrap around the origin in the complex plane as \( q \) goes from 0 to \( 2\pi \). Counting how many times the curve encircles the origin gives the

winding number:

\[
W = \begin{cases} 
1, & K_1 < K_2 \\
0, & K_1 > K_2
\end{cases}
\]

This number is an important topological property of a 1D system’s band structure. The Zak phase is the 1D Berry phase and is \( 2\pi \) times the winding number. When the winding number is nonzero, the lattice is topologically non-trivial and edge modes appear, decaying exponentially from the edges with a decay length \( \xi = -\log(K_1/K_2) \). Without loss of generality, we use the convention that if only one edge mode is present, it is on the left. The number of left (\( N_L \)) and right (\( N_R \)) edge modes is thus

\[
N_L = W, \quad N_R = 1 - e^{i\pi(N+\bar{W})}/2.
\]
This is the maximum rate at which a harmonic lattice can transport heat between two equilibrium reservoirs at different temperatures. Since it depends only on the bulk band structure, $\Omega$, it is independent of winding number, i.e., swapping the order of $K_1$ and $K_2$—or changing the parity of the lattice—will not affect it. Reaching this conductance in practice, however, requires that all phonon modes are sufficiently in contact with the reservoirs so that they are supplied ample thermal energy. In the presence of topological edge modes this limit is never reached, and the thermal conductance is always lower than $\kappa_0$, regardless of the system length. This is rather surprising, considering the fact that there are at most two edge modes, whereas the number of modes grows linearly with the system size.

We note that the ability of a specific mode $q$ to conduct heat will depend on its contact with the external reservoirs and its intrinsic conductance (determined by its group velocity). In the setup of Fig. 1(b), the strength of the contact of a specific mode $q$ with the reservoirs is given by $\gamma u_q^2$ and $\gamma u_q^2$ for the left and right, respectively. The coupling (i.e., damping rate) $\gamma$ is the strength of contact of the reservoirs to the cantilever beam at the
Many of the results below hold over a large range of the damping parameter $\gamma$ for arbitrary $\kappa$. Perturbing the boundaries – the conductance for mode $\ell$ – increases the spatial extent of the edge states grows as the temperature $T$ increases. (a) When $K_1 < K_2$ in an even site lattice, there are two edge modes (purple and wine lines) that reside in the gap between the two phonon bands (outlined with blue and red for the upper and lower bands, respectively). As $K_1$ increases the spatial extent of the edge states grows until they merge with the bulk states at $K_1 = K_2$. This process is shown for alternating couplings only (i.e., $m_1/m_2 = 1$; $\omega_1/\omega_2 = 1$) and for the masses alternating (i.e., $m_1/m_2 = 2$; $\omega_1/\omega_2 = 1$). (b) The edge modes also exist in nonlinear lattices and persist even as the nonlinearity increases with temperature $T$ (at high enough temperatures, the nonlinearity merges the edge states with the bulk). The persistence of the edge modes is relevant to MEMS/NEMS, which are often operated in nonlinear regimes. Prior studies also show that classical nonlinear systems can exhibit topological excitations.

The polarization vector of the mode on the boundaries, $u^m_{n\eta}$ with $n = 1$ or $N$, attenuates the coupling of the mode $q$ to the reservoirs. When the lattice weakly contacts the reservoirs – in order to minimally perturb the boundaries – the conductance for mode $q$, $\kappa_q$, is due to two contributions in series (see the SM)

$$\frac{k_B}{\kappa_q} = \frac{1}{\gamma u^m_{q1}} + \frac{1}{\gamma u^m_{qN}},$$

where the first term is from the left interface and the second from the right interface. To describe behavior for arbitrary $\gamma$, the bulk contribution $-N/v_g$, the intrinsic ability of the mode to transfer heat – and an overdamping contribution proportional to $\gamma$ would need to be included in Eq. (6). Many of the results below hold up to moderate values of $\gamma$, as explained in the SM.

The edge modes have an exponentially vanishing amplitude, $u^2_{q\eta} \approx 0$, for either $n = 1$ or $N$, which yields $\kappa_q \approx 0$ for $q \in \mathcal{E}$, where $\mathcal{E}$ is the set of edge modes. The total conductance will then be

$$\kappa = \frac{1}{2\pi} \int dq \lim_{N \to \infty} N\kappa_q,$$

where the integral is over only the phonon bands and thus the edge state contribution – which would be a separate sum – is absent. This equation has a similar form to Eq. (5) but $\kappa_q$ contains the non-ideal contact to the external heat source and sink.

We proceed by giving a heuristic derivation of the effect of topology, and a rigorous derivation is in the SM. Considering all normal modes of a lattice, one has simple “sum rules” for the boundary amplitudes, $\sum q u^2_{q1}$ and $\sum q u^2_{qN} = 1/m$ for the case when $m_1 = m_N = m$, that reflect the (mass) scaling and orthogonal transformations that yield the normal modes. In the absence of edge modes, the bulk modes have a contact strength $\kappa_n \propto \gamma/(mN)$ for $n = 1$ and $N$. Using this value for $u^2_{q\eta}$, the non-topological interfacial conductance for an even length lattice is

$$\tilde{\kappa} = \frac{k_B \gamma}{2m},$$

which is limited by the coupling of the external reservoirs to the lattice, i.e., the heat injected is the bottleneck to current flow (a similar situation occurs in electronic transport).

In the presence of edge modes – states localized at the boundaries – the total coupling of the bulk to the reservoirs is reduced: $\sum_{q \neq 0\in E} u^2_{q\eta} = 1/m - \sum_{q \in \mathcal{E}} u^2_{q\eta}$. The bulk modes therefore have a contact strength $\gamma \left(1 - m\sum_{q \in \mathcal{E}} u^2_{q\eta}\right)/(mN)$. The amplitude squared of an edge mode on a boundary of its origin is $(1-e^{-2\xi})/m$, which follows from the normalization of an exponentially decaying state (see the SM). The bulk modes therefore have contact $\gamma u^2_{m} \propto \gamma \exp(-2\xi)/(mN)$ for $n = 1$ and $N$. The conductance in the presence of edge modes is then

$$\kappa = \frac{k_B \gamma e^{-2\xi}}{2m} = e^{-2\xi}\tilde{\kappa}.$$
topological levels

$$\Xi = \frac{2}{e^{2N\xi} + e^{2N\bar{\xi}}}.$$  \tag{11}$$

The quantity $\Xi$ is thus a function of winding number, as $N\xi(K)$ depend on it through Eq. (4). Out of the different configurations (using the parameters that give the same bulk properties but ordering them in different ways), Eq. (11) will give only three possible values, corresponding to the presence of 0, 1 or 2 edge modes. All trivial mass effects (at the boundaries) are in $\bar{\xi}$.

The effect of topological edge modes on thermal conductance is demonstrated in Figure 3, which discusses a uniform lattice with alternating $K_1$ and $K_2$ only (solid lines), a lattice with $m_1 = 2m_2$ and $\omega_1 = \omega_2$ (dashed lines) and a lattice with $m_1/m_2 = K_1/K_2$, $\omega_1 = \omega_2$ (dash-dotted lines). For each bulk lattice, we show the case with zero edge modes (green), two edge modes (purple), and one edge mode (red and blue for a left and right edge mode, respectively).

Figure 3(a) plots the thermal conductance $\kappa$ versus the ratio $K_1/K_2$. The conductance can take on essentially any value (by changing masses and on-site frequencies, one can fill in the whole plot). However, when taking simple ratios, $\kappa/\bar{\kappa}$, of the thermal conductance, a simple quantization emerges, as shown in Fig. 3(b). These ratios take on just three values given by $\Xi$. Generally, the introduction of edge modes suppresses the conductance, as it reduces the contact between the energy sources/sinks and the bulk states. Non-topological effects (i.e., the changing bulk state structure as $K_1/K_2$ increases) can also significantly influence the conductance for certain sets of parameters.

Since the suppression of the thermal conductance is a topological effect, it will not depend on the specific details of the system and reservoirs and is anticipated to be robust against various modifications to the lattice (so long as the topology is maintained). We demonstrate this robustness in Fig. 3(c) where we plot the normalized conductance versus $K_1/K_2$ for three additional lattices: the FPU-\(\beta\) lattice of Fig. 2(c) and two disordered, dimerized lattices. For all these cases, the conductance follows Eq. (10), showing the universality of the topology-induced reduction of thermal conductance. If the Langevin reservoirs are replaced by uniform harmonic lattices with constant coupling $0 < K < \min(K_1,K_2)$ representing trivial topology, the edge modes and their influence on thermal conductance should still survive at the boundary due to a change of topology.

We further note that Eq. (6) is a general result. It entails, therefore, that even non-topological localized modes (e.g., due to a light mass at the boundary) can suppress the thermal conductance. However, non-topological modes will not display the quantized conductance of Eq. (10) and shown in Fig. 3b,c. As well, there are many channels for heat/energy transport. In the setup envisioned in Fig. 1, heat will also be carried by vibrations of the underlying crystal lattice. Therefore, it is necessary to use a low thermal conductivity material so that vibrations of the cantilevers are the dominant channel for energy transport.

Just as thermal transport can serve as a probe of non-linear structural transitions, these results show that signatures of nontrivial topologies appear in classical (or quantum) energy transport in conventional physical systems, such as MEMS/NEMS or at crystal-polymer interfaces. In particular, a combination of laser-induced heating and optical/electronic readout will allow for the topological characterization of energy transport in micro- or nano-mechanical lattices and control of heat flow. The emergence of edge states may help design, e.g., thermoelectric devices, where the lattice thermal conductance needs to be suppressed independently of the electronic conductance. Moreover, energy flow and thermal properties are critical to the operation of nanotechnologies, where they can limit and even define the functionality of devices. The results presented here thus generate exciting prospects for observing topological properties in conventional physical systems and utilizing them to design micro- and nano-scale devices.

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Figure 3. Topological quantization of the conductance. (a) The conductance versus $K_1/K_2$ of infinite length dimerized lattices (in the setup shown in Fig. 1(b)). We show only a subset of the curves for clarity, see the SM\(^\text{29}\) for details. (b) When taking the ratio of the conductance to the non-topological component, $\bar{\kappa}$, all the curves in (a) collapse onto three distinct levels. Green, purple, and red-blue lines show the cases with zero, two, and one edge mode(s) (left or right), respectively, generically suppressing the conductance. (c) The topological conductance is robust to nonlinearity and disorder. The yellow circles and wine squares show the normalized conductance of two disordered lattices (with both an odd and even number of sites), and the magenta diamonds show the normalized conductances of the FPU lattice. Empty (full) symbols are for one (two) edge modes. Since the nonlinearity also introduces an overall shift in the parameters, the data points are plotted versus an effective $\bar{\kappa}$, all the curves in (a) collapse onto three distinct levels.

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