Binary Nonlinearization of Lax pairs of Kaup-Newell Soliton Hierarchy

Wen-Xiu Ma†‡, Qing Ding†, Wei-Guo Zhang* and Bao-Qun Lu†

†Institute of Mathematics, Fudan University, Shanghai 200433, P. R. of China
‡FB Mathematik und Informatik, Universität Paderborn, D-33098 Paderborn, Germany
*Dept. of Math. & Mech., Changsha Railway University, Changsha 410075, P. R. of China

Summary. — Kaup-Newell soliton hierarchy is derived from a kind of Lax pairs different from the original ones. Binary nonlinearization procedure corresponding to the Bargmann symmetry constraint is carried out for those Lax pairs. The proposed Lax pairs together with adjoint Lax pairs are constrained as a hierarchy of commutative, finite dimensional integrable Hamiltonian systems in the Liouville sense, which also provides us with new examples of finite dimensional integrable Hamiltonian systems. A sort of involutive solutions to the Kaup-Newell hierarchy are exhibited through the obtained finite dimensional integrable systems and the general involutive system engendered by binary nonlinearization is reduced to a specific involutive system generated by mono-nonlinearization.

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1. Introduction

It is very important (and interesting) to search for integrable reductions of soliton equations. Similarity reductions lead to equations of Painlevé type [1], and algebraic pole expansion solutions to soliton equations engender finite dimensional integrable N-body Hamiltonian systems [2]. These are two main examples of integrable reductions.

Recently a systematic approach called the nonlinearization (more precisely, we call it monononlinearization) of Lax pairs is introduced for generating finite dimensional completely integrable systems from soliton hierarchies by Cao and Geng et al. [3-5]. Based upon the idea of nonlinearity, one of the authors proposed binary nonlinearization technique for soliton hierarchies [6,7], which was successfully applied to a few well-known soliton hierarchies [6,7]. Nonlinearization method may yield a large variety of finite dimensional completely integrable Hamiltonian systems whose independent integrals of motion can be explicitly given out, and it provides a way of solving soliton equations by separation of spatial and temporal variables.

The integrable reductions in the mono-nonlinearization and binary nonlinearization are related to a kind of specific symmetry constraints [6,7]. Bargmann symmetry constraints (usually the first order with respect to derivative) lead to the most interesting reductions in the nonlinearization because of their simplicity. These constraints require conserved covariants to be linear functions
but not differential functions with respect to the potentials. The higher order symmetry constraints need to introduce new dependent variables, the so-called Jacobi-Ostrogradsky coordinates. The spatial parts of the resulting nonlinearized Lax pairs by nonlinearization are exactly the restricted flows named by Antonowicz and Wojciechowski etc. [8].

In this paper, we would like to discuss binary nonlinearization for Kaup-Newell hierarchy [9] of soliton equations. In Section 2, following the standard procedure, we shall reconstruct Kaup-Newell soliton hierarchy from a particular spectral problem other than the original Kaup-Newell spectral problem [9] and give rise to some algebraic properties connected with Kaup-Newell hierarchy. In Section 3, we shall in detail exhibit binary nonlinearization procedure corresponding to the Bargmann constraint with zero boundary conditions and without zero boundary conditions and vigorously constrain the proposed Lax pairs and adjoint Lax pairs of Kaup-Newell hierarchy into a hierarchy of commutative, finite dimensional integrable Hamiltonian systems in the Liouville sense. By the resulting finite dimensional integrable system, we shall further exhibit a sort of involutive solutions to Kaup-Newell hierarchy. Finally in Section 4, we reduce the general involutive system derived from binary nonlinearization to a specific involutive system generated by mono-nonlinearization and give some other remarks.

2. New Lax pairs of Kaup-Newell hierarchy and some related properties

Let us consider the following specific spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi = \begin{pmatrix} \lambda & q \\ \lambda r & -\lambda \end{pmatrix} \phi, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, u = \begin{pmatrix} q \\ r \end{pmatrix}. \quad (2.1)$$

This spectral problem is a little different from the original Kaup-Newell spectral problem [9]. We find by a large amount of computation that it is a possible spectral problem leading to an integrable hierarchy. Although we don’t know if there exists a gauge transform between (2.1) and the original spectral problem, we shall see later that its related integrable systems may be transformed into Kaup-Newell hierarchy.

In order to derive isospectral systems associated with (2.1), similar to Ref. [10] we first solve the adjoint representation equation (see Ref. [11]) $V_x = [U, V]$ of $\phi_x = U\phi$. Take

$$V = \begin{pmatrix} a & \lambda^{-1}b \\ c & -a \end{pmatrix}.$$ 

Noting that

$$[U, V] = \begin{pmatrix} qc - rb & 2b - 2qa \\ 2\lambda ra - 2\lambda c & rb - qc \end{pmatrix},$$

we find that the adjoint representation equation $V_x = [U, V]$ becomes

$$\begin{cases}
a_x = qc - rb, \\
\lambda^{-1}b_x = 2b - 2qa, \\
c_x = 2\lambda ra - 2\lambda c.
\end{cases} \quad (2.2)$$
On setting \( a = \sum_{i \geq 0} a_i \lambda^{-i}, b = \sum_{i \geq 0} b_i \lambda^{-i}, c = \sum_{i \geq 0} c_i \lambda^{-i}, \) (2.2) yields equivalently

\[
\begin{align*}
2b_0 - 2qa_0 &= 2ra_0 - 2c_0 = 0, \\
a_{ix} &= qc_i - rb_i, \\
b_{ix} &= 2b_{i+1} - 2qa_{i+1}, \quad i \geq 0, \\
c_{ix} &= 2ra_{i+1} - 2c_{i+1},
\end{align*}
\]

(2.3)

We choose \( a_0 = 1, \ b_0 = q, \ c_0 = r \) and assume that \( a_i[u]=b_i[u]=c_i[u]=0, \ i \geq 1, \) where \( [u] = (u, u_x, \cdots) \) (this means to select zero constants for integration). Because we have

\[
2a_{i+1,x} = 2qc_{i+1} - 2rb_{i+1}
\]

\[
= q(2ra_{i+1} - c_{ix}) - r(b_{ix} + 2qa_{i+1})
\]

\[
= -qc_{ix} - rb_{ix},
\]

the equality (2.3) gives rise to a recursion relation for determining \( a_i, b_i, c_i: \)

\[
\begin{align*}
a_{i+1} &= -\frac{1}{2} \partial^{-1}(qc_{ix} + rb_{ix}), \\
b_{i+1} &= \frac{1}{2} b_{ix} - \frac{1}{2} q \partial^{-1}(qc_{ix} + rb_{ix}), \quad i \geq 0. \\
c_{i+1} &= -\frac{1}{2} c_{ix} - \frac{1}{2} r \partial^{-1}(qc_{ix} + rb_{ix}),
\end{align*}
\]

(2.4)

Therefore, for instance, we can obtain

\[
\begin{align*}
a_1 &= -\frac{1}{2} qr, \ b_1 = \frac{1}{2} (q_x - q^2 r), \ c_1 = -\frac{1}{2} (r_x + qr^2), \\
a_2 &= \frac{3}{8} q^2 r^2 + \frac{1}{4} qr_x - \frac{1}{4} q_x r, \ b_2 = \frac{1}{4} qxx - \frac{3}{4} qq_x r + \frac{3}{8} q^3 r^2, \ c_2 = \frac{1}{4} rxx + \frac{3}{4} qrr_x + \frac{3}{8} q^2 r^3.
\end{align*}
\]

From \( (V^2)_x = [U, V^2], \) we see that \( (\frac{1}{2} \text{tr}V^2)_x = (a^2 + \lambda^{-1}bc)_x = 0. \) Thus we have \( a^2 + \lambda^{-1}bc = 1 \) by observing \( (\frac{1}{2} \text{tr}V^2)[u]=1. \) Further we may obtain

\[
2a_m = -\sum_{i=1}^{m-1} a_i a_{m-i} - \sum_{i=0}^{m-1} b_i c_{m-1-i}, \quad m \geq 2.
\]

(2.5)

This equality shows by an argument of mathematical induction that \( a_i, b_i, c_i \) are all differential polynomial functions of \( u. \)

Now we introduce the following auxiliary problem associated with the spectral problem (2.1)

\[
\phi_{\epsilon_m} = V^{(m)} \phi, \ V^{(m)} = (\lambda^{m+1} V)_+ - \begin{pmatrix} a_{m+1} & 0 \\ c_{m+1} & -a_{m+1} \end{pmatrix} = \begin{pmatrix} \lambda(\lambda^m a)_+ & (\lambda^m b)_+ \\ \lambda(\lambda^m c)_+ & -\lambda(\lambda^m a)_+ \end{pmatrix}, \quad m \geq 0,
\]

(2.6)
where the symbol + stands for the selection of the polynomial part of $\lambda$. Then cross differentiation (i.e. taking $\partial/\partial t_m$ (2.1), $\partial/\partial x$ (2.6) and setting them equal) leads to isospectral integrable systems

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}, \quad K_t = \begin{pmatrix} b_{mx} \\ c_{mx} \end{pmatrix} = \Phi^m u_x, \quad m \geq 0. \quad (2.7)$$

Here the operator $\Phi$ reads

$$\Phi = \left( \frac{1}{2} \partial - \frac{1}{2} \partial q \partial^{-1} q - \frac{1}{2} \partial q \partial^{-1} q \right), \quad \partial = \partial / \partial x,$$

which may be verified to be a hereditary symmetry operator [12], i.e. to satisfy

$$\Phi^2 [K, S] + [\Phi K, \Phi S] - \Phi \{ [K, \Phi S] + [\Phi K, S] \} = 0,$$

for arbitrary vector fields $K, S$. Obviously, under $\partial / \partial x \rightarrow -i \partial / \partial x$, $\partial / \partial t_m \rightarrow -i \partial / \partial t_m$, (2.7) is transformed into Kaup-Newell hierarchy [9]. We call (2.7) as a kind of real form of Kaup-Newell hierarchy because (2.7) itself is a hierarchy of real systems. The first two nonlinear systems in the hierarchy (2.7) are as follows

$$\begin{align*}
q_{t_1} &= \frac{1}{2} [q_{xx} - (q^2 r)_x], \\
q_{t_2} &= \frac{1}{4} q_{xxx} - \frac{3}{4} (q q x r)_x + \frac{3}{8} (q r^3)_x, \\
r_{t_1} &= -\frac{1}{2} [r_{xx} + (q r^2)_x], \\
r_{t_2} &= \frac{1}{4} r_{xxx} + \frac{3}{4} (q r r x)_x + \frac{3}{8} (q^2 r^3)_x.
\end{align*}$$

The former system may be rewritten as the derivative nonlinear Schrödinger equation [9]

$$iq_{t_1} = -\frac{1}{2} q_{xx} + \frac{1}{2} i (q^* q^2)_x$$

under $\partial / \partial x \rightarrow -i \partial / \partial x$, $\partial / \partial t_1 \rightarrow -i \partial / \partial t_1$, $q = -r^*$. 

We below want to tersely exhibit bi-Hamiltonian structures of Kaup-Newell hierarchy (2.7) by applying a powerful tool, i.e. the so-called trace identity proposed in Ref. [13,14]. Towards this end, we need, as is usual, the following quantities which are easy to work out:

$$< V, \frac{\partial U}{\partial \lambda} > = 2a + \lambda^{-1} r b, \quad < V, \frac{\partial U}{\partial q} > = c, \quad < V, \frac{\partial U}{\partial r} > = b,$$

where $< \cdot, \cdot >$ denotes the Killing form: $< A, B > = \text{tr}(AB)$. Then by means of the trace identity [13]

$$\delta < V, \frac{\partial U}{\partial \lambda} > = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (< V, \frac{\partial U}{\partial q} >, < V, \frac{\partial U}{\partial r} >)^T, \quad \gamma = \text{const.},$$

we obtain immediately

$$\delta (2a + \lambda^{-1} r b) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (c, b)^T.$$
Comparing the coefficients of $\lambda^{-m-1}$ on two sides of the above equality and noticing the equality (2.3), we have

$$\frac{\delta}{\delta u}[2a_{m+1} + \frac{1}{2}(rb_m + qc_m)] = \frac{\delta}{\delta u}(2a_{m+1} + rb_m) = (\gamma - m)(c_m, b_m)^T, \ m \geq 0. \tag{2.8}$$

The equality (2.8) with $m = 0$ yields that the constant $\gamma = 0$. Therefore (2.8) gives rise to an important formula

$$\frac{\delta}{\delta u}[2a_{m+1} + \frac{1}{2}(rb_m + qc_m)] = -m(c_m, b_m)^T, \ m \geq 0. \tag{2.9}$$

At this stage, Kaup-Newell hierarchy (2.7) may be transformed into the Hamiltonian form

$$u_{tm} = \begin{pmatrix} q \\ r \end{pmatrix}, \ K_m = \begin{pmatrix} b_{mx} \\ c_{mx} \end{pmatrix} = JG_m = J\Psi^m \begin{pmatrix} c_0 \\ b_0 \end{pmatrix} = J\frac{\delta H_m}{\delta u}, \ m \geq 0. \tag{2.10}$$

Here the Hamiltonian operator $J$, the recursive operator $\Psi$ and the Hamiltonian functions $H_m$ are defined by

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \ \Psi = \Phi^* = \begin{pmatrix} -\frac{1}{2}\partial - \frac{1}{2}r\partial^{-1}q\partial & -\frac{1}{2}r\partial^{-1}r\partial \\ -\frac{1}{2}q\partial^{-1}q\partial & \frac{1}{2}\partial - \frac{1}{2}q\partial^{-1}r\partial \end{pmatrix}, \tag{2.11a}$$

$$H_0 = -2a_1, \ H_m = -\frac{1}{m}[2a_{m+1} + \frac{1}{2}(rb_m + qc_m)], \ m \geq 1. \tag{2.11b}$$

A direct calculation can show that $J$ and $J\Psi$ constitutes a Hamiltonian pair. Therefore the flows determined by Kaup-Newell hierarchy commute with each other, and every Kaup-Newell system has an infinite number of symmetries $\{K_n\}_{n=0}^\infty$ and conserved quantities $\{H_n\}_{n=0}^\infty$. It is known that Magri [15] showed the existence of bi-Hamiltonian formulation of Kaup-Newell hierarchy, however here we also give the explicit expressions (2.11b) of the Hamiltonian functions $H_m$.

Further, similar to Ref. [16], we can get the Zakharov-Shabat equation

$$V^{(m)}_{t_m} - V^{(n)}_{t_m} + [V^{(m)}, V^{(n)}] = (V^{(m)})'[K_n] - (V^{(n)})'[K_m] + [V^{(m)}, V^{(n)}] = 0,$$

which also shows the commutability of the flows of (2.10). In addition, we can directly verify

$$V_{t_m} = [V^{(m)}, V], \ m \geq 0, \tag{2.12}$$

when $u_{tm} = K_m$, i.e. $U_{tm} - V_x^{(m)} + [U, V^{(m)}] = 0, \ m \geq 0$. In fact, we may deduce that $V_{t_m} - [V^{(m)}, V]$ satisfies the adjoint representation equation of $\phi_x = U\phi$ and that $(V_{t_m} - [V^{(m)}, V])[|u|=0 = 0$. Hence (2.12) follows from the uniqueness property [6] (i.e. if $V_x = [U, V]$ and $V|[|u|=0 = 0$, then $V$ itself vanishes) of the adjoint representation equation. The equality (2.12) shall be used to elucidate the commutability of the flows generated by nonlinearization in the next section.
3. Binary nonlinearization of Lax pairs

In this section, we would like to exhibit a binary nonlinearization procedure, as in Refs. [6,7], for the Lax pairs and adjoint Lax pairs of Kaup-Newell hierarchy (2.7) or (2.10). Note that $U_t - V^{(m)} + [U, V^{(m)}] = 0$ if and only if $(-U^T)_t - (-V^{(m)})^T_x + [-U^T, -(V^{(m)})^T] = 0$. Kaup-Newell systems (2.10) have another kind of Lax pairs called the adjoint Lax pairs

\begin{align*}
\psi_x &= -U^T \psi = -U^T(u, \lambda) \psi, \\
\psi_{t_m} &= -(V^{(m)})^T \psi = -(V^{(m)})^T(u, \lambda) \psi,
\end{align*}

where $T$ means the transpose of matrix and $\psi = (\psi_1, \psi_2)^T$. The Lax pairs and adjoint Lax pairs are a basic object of binary nonlinearization.

3.1. Bargmann symmetry constraint:

Let us first calculate the variational derivatives of $\lambda = \lambda(u)$ with respect to the potentials. Taking Gateaux derivative of $\phi_x = U(u, \lambda) \phi$ at $K = (K_1, K_2)^T$ engenders that

\[ (\phi'[K])_x = \left( \frac{\partial U}{\partial q} K_1 + \frac{\partial U}{\partial r} K_2 + \frac{\partial U}{\partial \lambda} \lambda'[K] \right) \phi + U \phi'[K]. \]

Now by using $\psi_x = -U^T \psi$, we obtain

\[ \int_{-\infty}^{\infty} \psi^T \left( \frac{\partial U}{\partial q} K_1 + \frac{\partial U}{\partial r} K_2 + \frac{\partial U}{\partial \lambda} \lambda'[K] \right) \phi \, dx = 0, \]

which leads to

\[ \lambda'[K] = \frac{1}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} <\phi \psi^T, \frac{\partial U}{\partial q}> \, dx} \int_{-\infty}^{\infty} \left( <\phi \psi^T, \frac{\partial U}{\partial q}> K_1 + <\phi \psi^T, \frac{\partial U}{\partial r}> K_2 \right) \, dx. \]

In this way, according to the definition, we get the variational derivatives of $\lambda$ with respect to the potentials $q, r$:

\[ \frac{\delta \lambda}{\delta q} = -\int_{-\infty}^{\infty} <\phi \psi^T, \frac{\partial U}{\partial q}> \, dx, \quad \frac{\delta \lambda}{\delta r} = -\int_{-\infty}^{\infty} <\phi \psi^T, \frac{\partial U}{\partial r}> \, dx. \]  

(3.2)

It follows that

\[ \frac{\delta \lambda}{\delta u} = \frac{1}{E} (\phi_2 \psi_1, \lambda \phi_1 \psi_2)^T, \quad E = -\int_{-\infty}^{\infty} (\phi_1 \psi_1 + r \phi_1 \psi_2 - \phi_2 \psi_1) \, dx. \]

(3.3)

When the zero boundary conditions: $\lim_{|x| \to +\infty} \phi = \lim_{|x| \to +\infty} \psi = 0$ are imposed, we can give out a very useful characterization of the variational derivatives of $\lambda$:

\[ \Psi \frac{\delta \lambda}{\delta u} = \lambda \frac{\delta \lambda}{\delta u}, \]

(3.4)

where $\Psi$ and $\delta \lambda / \delta u$ are defined by (2.11a) and (3.3), respectively. Introducing $N$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, we obtain the following spatial and temporal systems
nonlinearized Lax pairs and adjoint Lax pairs

substituted (3.9) into the Lax pairs and the adjoint Lax pairs: (3.5) and (3.6), we acquire the

Here for economic writing, we have used the notation:

This kind of constraints requires the

In what follows, we will denote by \( \bar{P} \) the expression of \( P(u) \) under the constraint (3.9). Having

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nonlinearized Lax pairs and adjoint Lax pairs

\[
\begin{align*}
\left( \begin{array}{c}
\phi_{1j} \\
\phi_{2j}
\end{array} \right)_x &= U(u, \lambda_j) \left( \begin{array}{c}
\phi_{1j} \\
\phi_{2j}
\end{array} \right), \quad j = 1, 2, \cdots, N, \quad (3.5a) \\
\left( \begin{array}{c}
\psi_{1j} \\
\psi_{2j}
\end{array} \right)_x &= -U^T(u, \lambda_j) \left( \begin{array}{c}
\psi_{1j} \\
\psi_{2j}
\end{array} \right), \quad j = 1, 2, \cdots, N; \quad (3.5b) \\
\left( \begin{array}{c}
\phi_{1j} \\
\phi_{2j}
\end{array} \right)_{tm} &= V^{(m)}(u, \lambda_j) \left( \begin{array}{c}
\phi_{1j} \\
\phi_{2j}
\end{array} \right), \quad j = 1, 2, \cdots, N, \quad (3.6a) \\
\left( \begin{array}{c}
\psi_{1j} \\
\psi_{2j}
\end{array} \right)_{tm} &= -(V^{(m)})^T(u, \lambda_j) \left( \begin{array}{c}
\psi_{1j} \\
\psi_{2j}
\end{array} \right), \quad j = 1, 2, \cdots, N. \quad (3.6b)
\end{align*}
\]
Under the zero boundary condition, the characterization (3.4) asserts that functions and adjoint eigenfunctions in order to derive such an involutive and independent system. We will use the nonlinearized Lax pairs and adjoint Lax pairs in the case of zero boundary condition on the eigen-

\[
\begin{align*}
\left( \phi_{1j} \right)_{x} &= \left( \lambda_{j} - \tilde{q} \right) \left( \phi_{1j} \right), \quad j = 1, 2, \cdots, N, \\
\left( \phi_{2j} \right)_{x} &= \left( \lambda_{j} \right) \left( \phi_{2j} \right), \quad j = 1, 2, \cdots, N; \\
\left( \psi_{1j} \right)_{x} &= \left( -\lambda_{j} \right) \left( \psi_{1j} \right), \quad j = 1, 2, \cdots, N; \\
\left( \psi_{2j} \right)_{x} &= \left( -\tilde{q} \right) \left( \psi_{2j} \right), \quad j = 1, 2, \cdots, N;
\end{align*}
\]

(3.10a)  

\[
\begin{align*}
\left( \phi_{1j} \right)_{t_{m}} &= \left( \sum_{i=0}^{m} \tilde{a}_{i} \lambda_{j}^{m+1-i} - \sum_{i=0}^{m} \tilde{c}_{i} \lambda_{j}^{m+1-i} \right) \left( \phi_{1j} \right), \quad j = 1, 2, \cdots, N, \\
\left( \phi_{2j} \right)_{t_{m}} &= \left( \sum_{i=0}^{m} \tilde{b}_{i} \lambda_{j}^{m+1-i} - \sum_{i=0}^{m} \tilde{d}_{i} \lambda_{j}^{m+1-i} \right) \left( \phi_{2j} \right), \quad j = 1, 2, \cdots, N.
\end{align*}
\]

(3.11a)  

\[
\begin{align*}
\left( \psi_{1j} \right)_{t_{m}} &= \left( -\sum_{i=0}^{m} \tilde{a}_{i} \lambda_{j}^{m+1-i} - \sum_{i=0}^{m} \tilde{c}_{i} \lambda_{j}^{m+1-i} \right) \left( \psi_{1j} \right), \quad j = 1, 2, \cdots, N; \\
\left( \psi_{2j} \right)_{t_{m}} &= \left( -\sum_{i=0}^{m} \tilde{b}_{i} \lambda_{j}^{m+1-i} - \sum_{i=0}^{m} \tilde{d}_{i} \lambda_{j}^{m+1-i} \right) \left( \psi_{2j} \right), \quad j = 1, 2, \cdots, N.
\end{align*}
\]

(3.11b)

We easily see that the spatial part of the nonlinearized Lax pairs and adjoint Lax pairs, namely, the system (3.10) is a finite dimensional system with regard to $x$, but the temporal parts of the nonlinearized Lax pairs and adjoint Lax pairs, namely, the systems (3.11) for $m \geq 0$ are all systems of evolution equations with regard to $t_{m}, x$. Obviously the system (3.10) reads

\[
\begin{align*}
P_{1x} &= AP_{1} + \langle AP_{1}, Q_{2} \rangle > P_{2}, \\
P_{2x} &= \langle P_{2}, Q_{1} \rangle > AP_{1} - AP_{2}, \\
Q_{1x} &= -AQ_{1} - \langle P_{2}, Q_{1} \rangle > AQ_{2}, \\
Q_{2x} &= -\langle AP_{1}, Q_{2} \rangle > Q_{1} + AQ_{2},
\end{align*}
\]

and it may be represented as the following Hamiltonian form

\[
P_{ix} = \frac{\partial H}{\partial Q_{i}}, \quad Q_{ix} = -\frac{\partial H}{\partial P_{i}}, \quad i = 1, 2,
\]

(3.12a)  

where the Hamiltonian function

\[
H = \langle AP_{1}, Q_{1} \rangle > -\langle AP_{2}, Q_{2} \rangle > + \langle P_{2}, Q_{1} \rangle > < AP_{1}, Q_{2} > .
\]

(3.12b)

In the following, we shall prove that the system (3.10) is an integrable finite dimensional Hamiltonian system in the Liouville sense [17] and that under the control of the system (3.10), the systems (3.11) for $m \geq 0$ are all transformed into integrable finite dimensional Hamiltonian systems in the Liouville sense [17].

### 3.3 **An involutive and independent system:**

To show the Liouville integrability of the nonlinearized systems (3.10) and (3.11), we need an involutive and independent system to generate integrals of motion for them. We will use the nonlinearized Lax pairs and adjoint Lax pairs in the case of zero boundary condition on the eigenfunctions and adjoint eigenfunctions in order to derive such an involutive and independent system. Under the zero boundary condition, the characterization (3.4) asserts that

\[
\begin{align*}
\left( \tilde{c}_{m} \right) &= \tilde{c}_{0} m \left( \tilde{b}_{m} \right) = \left( < A^{m}P_{2}, Q_{1} > \right), \quad m \geq 0,
\end{align*}
\]

(3.13a)
and further from (2.3), we obtain

$$\tilde{a}_0 = 1, \quad \tilde{a}_m = \partial^{-1}(\tilde{q} \tilde{c}_m - \tilde{r} \tilde{b}_m) = \frac{1}{2} \left( < A^m P_1, Q_1 > - < A^m P_2, Q_2 > \right), \quad m \geq 1. \quad (3.13b)$$

It is easy to find that $\tilde{V}_x = [\tilde{U}, \tilde{V}]$ still holds. Thus an obvious equality $(\tilde{V}^2)_x = [\tilde{U}, \tilde{V}^2]$ leads to

$$F_x = \left( \frac{1}{2} \text{tr}\tilde{V}^2 \right)_x = \frac{d}{dx}(\tilde{a}^2 + \lambda^{-1}\tilde{b}\tilde{c}) = 0.$$  

This allows us to conclude that $F$ is a generating function of integrals of motion for (3.12). Due to that $F = \sum_{n \geq 0} F_n \lambda^{-n}$, we obtain the following expressions of integrals of motion

$$F_n = \sum_{i=0}^{n} (\tilde{a}_i \tilde{a}_{n-i} + \sum_{i=0}^{n-1} \tilde{c}_i \tilde{b}_{n-1-i}).$$

Substitution (3.13) into the above equality gives us the explicit formulas of $F_n$:

$$F_0 = 1, \quad F_1 = < AP_1, Q_1 > - < AP_2, Q_2 > + < P_1, Q_1 > < AP_1, Q_2 > = H, \quad (3.14a)$$

$$F_n = \sum_{i=1}^{n-1} \frac{1}{4} ( < A^i P_1, Q_1 > - < A^i P_2, Q_2 > ( < A^{n-i} P_1, Q_1 > - < A^{n-i} P_2, Q_2 > )$$

$$+ \sum_{i=0}^{n-1} < A^i P_2, Q_1 > < A^{n-i} P_1, Q_2 > + < A^n P_1, Q_1 > - < A^n P_2, Q_2 >, \quad n \geq 2. \quad (3.14b)$$

Note that we still have a similar equality $\tilde{V}_{n, m} = [\tilde{V}^{(m)}, \tilde{V}], \quad m \geq 0$, to (2.12). With the same argument, we may find that $F = \frac{1}{2} \text{tr}\tilde{V}^2$ is also a generating function of integrals of motion for (3.11). Moreover, a direct calculation can give rise to

$$Q_{it} = -\frac{\partial (F_{m+1})}{\partial P_i}, \quad P_{it} = \frac{\partial (F_{m+1})}{\partial Q_i}, \quad i = 1, 2. \quad (3.15)$$

Actually, we have

$$\left( \frac{\partial F}{\partial P_i}, \frac{\partial F}{\partial Q_i} \right)^T = \left( \text{tr}(\tilde{V} \frac{\partial}{\partial P_i} \tilde{V}), \text{tr}(\tilde{V} \frac{\partial}{\partial Q_i} \tilde{V}) \right)^T, \quad i = 1, 2. \quad (3.16)$$

Further it is carried out that

$$\text{tr}(\tilde{V} \frac{\partial}{\partial P_1} \tilde{V}) = \text{tr} \sum_{i=0}^{\infty} \left( \tilde{a}_i \tilde{c}_i \tilde{b}_{i-1} \right) \lambda^{-i} \sum_{j=0}^{\infty} \frac{\partial}{\partial P_1} \left( \tilde{a}_j \tilde{c}_j \tilde{b}_{j-1} \right) \lambda^{-j}$$

$$= \text{tr} \sum_{i \geq 0, j \geq 1} \left( \tilde{a}_i \tilde{c}_i \tilde{b}_{i-1} \right) \left( \begin{array}{cc} \frac{1}{2} A^j Q_1 & A^j Q_2 \\ 0 & -\frac{1}{2} A^j Q_1 \end{array} \right) \lambda^{-(i+j)}$$

$$= \sum_{i \geq 0, j \geq 1} (\tilde{a}_i A^j Q_1 + \tilde{c}_i A^j Q_2) \lambda^{-(i+j)}. \quad (3.17)$$
Similarly we can give
\[
\text{tr}(\tilde{V} \frac{\partial}{\partial P_2} \tilde{V}) = \left( \sum_{i \geq 1} b_{i-1} A^i Q_1 - \sum_{i \geq 0} a_i A^i Q_2 \right) \lambda^{-(i+j)}, \quad (3.18)
\]

It follows from (3.16), (3.17) and (3.18) that
\[
\left( \frac{\partial F_{m+1}}{\partial P_1}, \frac{\partial F_{m+1}}{\partial P_2} \right)^T = - \left( \tilde{V}^{(m)} \right)^T \left( \begin{array}{c} Q_1 \\ Q_2 \end{array} \right), \quad m \geq 0.
\]

A complete similar argument can lead to
\[
\left( \frac{\partial F_{m+1}}{\partial Q_1}, \frac{\partial F_{m+1}}{\partial Q_2} \right)^T = \tilde{V}^{(m)} \left( \begin{array}{c} P_1 \\ P_2 \end{array} \right), \quad m \geq 0.
\]

The above two equalities enables us to write all the flows of (3.11) in the Hamiltonian form (3.15).

Now we can easily show the involution \( F_n, \ n \geq 1 \). It is known that for the standard symplectic structure on \( R^{4N} \)
\[
\omega^2 = dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2, \quad (3.19)
\]
the corresponding Poisson bracket is defined as follows
\[
\{ f, g \} = \omega^2(Idf, Idg), \quad (3.20)
\]
where \( Idh \) expresses a Hamiltonian vector field of a smooth function \( h \) on \( R^{4N} \), defined by \( Idh \lhd \omega^2 = -dh \), with \( \lhd \) being the left interior product. Because \( F \) is also a generating function of integrals of motion for (3.11), it is found that
\[
\{ F_{n+1}, F_{m+1} \} = \frac{\partial}{\partial t_{m+1}} F_{n+1} = 0, \quad m, n \geq 0, \quad (3.21)
\]
by using the Hamiltonian structure (3.15) of (3.11). This implies that \( \{ F_n \}_{n=0}^{\infty} \) constitutes an involutive system with regard to (3.20). It may also directly be shown that \( \{ F_m, F_n \} = 0, \ m, n \geq 1 \).

**Theorem 3.1.** **Theorem 3.1:** Let \( \bar{F}_k, 1 \leq k \leq N, \) be defined by
\[
\bar{F}_k = \psi_{1k} \phi_{1k} + \psi_{2k} \phi_{2k}, \ 1 \leq k \leq N.
\]
Then \( \bar{F}_k, 1 \leq k \leq N, \ F_n, n \geq 1, \) constitute an involutive system, of which \( \bar{F}_k, F_n, 1 \leq k, n \leq N, \) are functionally independent on some region of \( R^{4N} \).

**Proof:** **Proof:** Noting that we already have \( \{ F_m, F_n \} = 0, \ m, n \geq 1 \), it suffices for the involution of \( \bar{F}_k, 1 \leq k \leq N, \ F_n, n \geq 1, \) that we prove \( \{ \bar{F}_k, F_n \} = 0, \ 1 \leq k \leq N, \ n \geq 1 \). This needs just a simple calculation due to the particular form of \( \bar{F}_k, 1 \leq k \leq N. \)
In what follows, we show that there exists some region of $R^{4N}$ over which the functions $\bar{F}_k, F_n, 1 \leq k, n \leq N$, can become an independent system. Suppose that this result is not true. Namely, there doesn’t exist any region of $R^{4N}$ over which the functions $\bar{F}_k, F_n, 1 \leq k, n \leq N$, could be independent. Hence there exist $2N$ constants $\alpha_k, \beta_n, 1 \leq k, n \leq N$, satisfying

$$\sum_{k=1}^{N} \alpha_k^2 + \sum_{n=1}^{N} \beta_n^2 \neq 0,$$

so that we have for all points $(P_1^T, P_2^T, Q_1^T, Q_2^T)$ in $R^{4N}$

$$\sum_{k=1}^{N} \alpha_k \left( \left( \frac{\partial \bar{F}_k}{\partial P_1} \right)^T, \left( \frac{\partial \bar{F}_k}{\partial P_2} \right)^T \right) + \sum_{n=1}^{N} \beta_n \left( \left( \frac{\partial F_n}{\partial P_1} \right)^T, \left( \frac{\partial F_n}{\partial P_2} \right)^T \right) = 0. \quad (3.22)$$

From the above equality, we would like to derive that all constants $\alpha_k, \beta_n, 1 \leq k, n \leq N$, must be zero. We will utilize the following equalities:

$$\frac{\partial \bar{F}_k}{\partial P_1} \bigg|_{P_1=P_2=0} = (0, \ldots, 0, \psi_{ik}, 0, \ldots, 0)^T, \quad i = 1, 2, \ 1 \leq k \leq N,$n
$$\frac{\partial F_1}{\partial P_1} \bigg|_{P_1=P_2=0} = AQ_1, \quad \frac{\partial F_1}{\partial P_2} \bigg|_{P_1=P_2=0} = -AQ_2,$$n
$$\frac{\partial F_n}{\partial P_1} \bigg|_{P_1=P_2=0} = A^nQ_1, \quad \frac{\partial F_n}{\partial P_2} \bigg|_{P_1=P_2=0} = -A^nQ_2, \ n \geq 2.$n

The equality (3.22) upon choosing $P_1 = P_2 = 0$ engenders

$$\alpha_k = \sum_{n=1}^{N} \beta_n \lambda_k^n = -\sum_{n=1}^{N} \beta_n \lambda_k^n, \ 1 \leq k \leq N.$$n

Therefore we have $\sum_{n=1}^{N} \beta_n \lambda_k^n = 0, \ 1 \leq k \leq N$. Let $f(\lambda) = \sum_{n=1}^{N} \beta_n \lambda^n$, which is a polynomial of degree $N$. Since it possesses $N$ distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_N$, we have $f(\lambda) = 0$. This means that $\beta_n = 0, \ 1 \leq n \leq N$, and further we obtain $\alpha_k = 0, \ 1 \leq k \leq N$. In this way, we arrive at all zero constants, which contradicts to our assumption. Hence there must exists at least a region $\Omega \subseteq R^{4N}$ over which the $2N$ 1-forms $d\bar{F}_k, dF_n, 1 \leq k, n \leq N$, are linearly independent. The proof is finished.

The theorem provides us with an involutive and independent system of polynomials, which is completely new as far as we know.

**3.4 Liouville integrability of the nonlinearized Lax pairs and adjoint Lax pairs:**

We now proceed to show the Liouville integrability of the nonlinearized Lax pairs and adjoint Lax pairs, i.e. (3.10) and (3.11). We point out that the above manipulation with the explicit expression (3.15) of (3.11) requires the zero boundary conditions: $\lim_{|x| \to +\infty} Q_i = \lim_{|x| \to +\infty} P_i =$
0, \ i = 1, 2. Now let us deal with a general case, i.e. the case where the zero boundary conditions are not imposed. In this case, it follows from \( \phi_x = U(u, \lambda)\phi, \ \psi_x = -U(u, \lambda)^T\psi \) that

\[
\Psi \begin{pmatrix} \phi_2 \psi_1 \\ \lambda \phi_1 \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_2 \psi_1 \\ \lambda \phi_1 \psi_2 \end{pmatrix} + I \begin{pmatrix} r \\ q \end{pmatrix},
\]

where \( I \) is an integral of motion of \( \phi_x = U(u, \lambda)\phi, \ \psi_x = -U(u, \lambda)^T\psi \). By applying (3.23) \( n \) times, we easily work out that

\[
\left( \frac{\tilde{c}_m}{\tilde{b}_m} \right) = \tilde{\Psi}^m \left( \frac{c_0}{b_0} \right) = \sum_{i=0}^{m} I_i \begin{pmatrix} < A^{m-i}P_2, Q_1 > \\ < A^{m+1-i}P_1, Q_2 > \end{pmatrix}, \ m \geq 0,
\]

where \( I_0 = 1 \) and \( I_i, \ 1 \leq i \leq m, \) are all integrals of motion of (3.10). Further we can compute that

\[
\tilde{a}_m = \partial^{-1}(\tilde{q}c_m - \tilde{r}b_m)
\]

\[
= \frac{1}{2} \sum_{i=0}^{m-1} I_i (< A^{m-i}P_1, Q_1 > - < A^{m-i}P_2, Q_2 >) + T_m, \ m \geq 1,
\]

where \( T_m \) is an integral of motion of (3.10), too. The latter two equalities in (2.3) lead to that

\[
T_m = -I_m, \ m \geq 1. \]

Therefore by using (2.5), we obtain

\[
\sum_{i=0}^{m-1} I_i (< A^{m-i}P_1, Q_1 > - < A^{m-i}P_2, Q_2 >) - 2I_m
\]

\[
= - \sum_{i=1}^{m-1} \frac{1}{4} \left[ \sum_{k=0}^{i-1} I_k (< A^{i-k}P_1, Q_1 > - < A^{i-k}P_2, Q_2 >) - 2I_i \right] \times
\]

\[
\left[ \sum_{l=0}^{m-1-i} I_l (< A^{m-1-i-l}P_1, Q_1 > - < A^{m-1-i-l}P_2, Q_2 >) - 2I_{m-l} \right]
\]

\[
- \sum_{i=1}^{m-1} \sum_{k=0}^{i-1} I_k < A^{i+k}P_2, Q_2 > \sum_{l=0}^{m-1-i} I_l < A^{m-1-i-l}P_2, Q_1 >, \ m \geq 2.
\]

By interchanging the summing in the above equality and noting the expressions of \( F_n' \)’s, a direct but quite lengthy calculation may give rise to

\[
I_m = \frac{1}{2} \sum_{k+l \leq m-1 \atop k, l \geq 0} I_k I_l F_{m-(k+l)} - \frac{1}{2} \sum_{i=1}^{m-1} I_i I_{m-i}, \ m \geq 2.
\]

This equality and \( I_1 = \frac{1}{2} F_1 \), determined by \( \tilde{a}_1 = -\frac{1}{2} \tilde{q}r \), readily yield the following expressions of \( I_m' \)’s:

\[
I_m = \sum_{n=1}^{m} d_n \sum_{i_1 + \cdots + i_n = m \atop i_1, \ldots, i_n \geq 1} F_{i_1} \cdots F_{i_n}, \ m \geq 1.
\]
Here the constants $d_n$ are defined recursively by
\[
d_1 = \frac{1}{2}, \quad d_2 = \frac{3}{8}, \quad d_n = d_{n-1} + \frac{1}{2} \sum_{s=1}^{n-2} d_s d_{n-s-1} - \frac{1}{2} \sum_{s=1}^{n-1} d_s d_{n-s}, \quad n \geq 3.
\] (3.29)

At this point, we can give the expressions of $\tilde{a}_m$, $\tilde{b}_m$, $\tilde{c}_m$ in terms of $F_n$ and further a direct computation may show that the temporal parts (3.11) of the nonlinearized Lax pairs and adjoint Lax pairs are represented as the following Hamiltonian systems
\[
Q_{itm} = -\frac{\partial H_m}{\partial P_i}, \quad P_{itm} = \frac{\partial H_m}{\partial Q_i}, \quad i = 1, 2
\] (3.30a)

with the Hamiltonian functions
\[
H_m = \sum_{n=0}^{m} \frac{d_n}{n+1} \sum_{i_1 + \cdots + i_{n+1} = m+1 \atop i_1, \ldots, i_{n+1} \geq 1} F_{i_1} \cdots F_{i_{n+1}} (d_0 = 1), \quad m \geq 0.
\] (3.30b)

The Hamiltonian phase flows $g_{itm}^{H_m}$ of the systems (3.30) commute with each other due to the commutability of $F_n$. Because $H$, $H_m$, $m \geq 1$, are all functions of $F_n$, $n \geq 1$, we can now conclude the Liouville integrability on the nonlinearized Lax pairs and adjoint Lax pairs.

**Theorem 3.2.** Theorem 3.2: The spatial part (3.10) of the nonlinearized Lax pairs and adjoint Lax pairs and the temporal parts (3.11) of the nonlinearized Lax pairs and adjoint Lax pairs under the control of the spatial part are all finite dimensional Liouville integrable Hamiltonian systems with the corresponding Hamiltonian functions $H$ and $H_m$ defined by (3.13b) and (3.30b). Moreover they all possess involutive and independent integrals of motion: $\bar{F}_k$, $F_n$, $1 \leq k, n \leq N$.

### 3.5. Involutive solutions to Kaup-Newell hierarchy:

The above manipulation also allows us to establish the following involutive solutions of the $m$th Kaup-Newell system $u_{itm} = K_m$:

\[
\begin{align*}
q(x, t_m) &= \langle Ag_H^m g_{H_m}^{t_m} P_1(0, 0), g_H^m g_{H_m}^{t_m} Q_2(0, 0) \rangle \quad (3.31a) \\
r(x, t_m) &= \langle g_H^x g_{H_m}^{t_m} P_1(0, 0), g_H^x g_{H_m}^{t_m} Q_2(0, 0) \rangle \quad (3.31b)
\end{align*}
\]

where $g_H^x$, $g_{H_m}^{t_m}$ are the Hamiltonian phase flows [17] associated with the Hamiltonian systems (3.12) and (3.30), respectively, but $P_i(0, 0)$, $Q_i(0, 0)$, $i = 1, 2$, may be arbitrary. This provides us with a way to present solutions to Kaup-Newell systems by solving two separate finite dimensional integrable systems with the variables $x$ and $t_m$. In fact, in view of the Liouville integrability of the nonlinearized Lax pairs and adjoint Lax pairs, this kind of involutive solutions to integrable systems not only gives rise to the interrelation between $1 + 1$ dimensional integrable systems and finite dimensional integrable systems, but also exhibits the characteristic of integrability by quadratures for integrable systems in $1 + 1$ dimensions.
4. Conclusions and remarks

We have introduced a different spectral problem from the original one for Kaup-Newell systems and discussed the corresponding Bargmann symmetry constraint. We have also exhibited a new explicit involutive system \( \{ \bar{F}_k, 1 \leq k \leq N, F_n, n \geq 1 \} \) on the symplectic manifold \( (R^{4N}, \omega^2) \), of which \( \{ \bar{F}_k, F_n, 1 \leq k, n \leq N \} \) constitute an independent system. The Lax pairs and adjoint Lax pairs are nonlinearized into a hierarchy of finite dimensional Liouville integrable Hamiltonian systems and Bargmann symmetry constraint leads to a sort of involutive solutions to Kaup-Newell systems, which may be presented by solving two commutative finite dimensional Hamiltonian systems, i.e. the spatial part and the temporal part of nonlinearized Lax pairs and adjoint Lax pairs.

We remark that under the reduction \( Q_1 = -P_2, Q_2 = P_1 \), the involutive system \( \{ F_n \}_{n=1}^{\infty} \) defined by (3.14) are reduced to

\[
\begin{align*}
F_1 &= -2 < AP_1, P_2 > - < AP_1, P_1 > < P_2, P_2 >, \\
F_n &= \sum_{i=0}^{n} \left( < A^i P_1, P_2 > < A^{n-i} P_1, P_2 > - < A^i P_1, P_1 > < A^{n-i} P_2, P_2 > \right) \\
&\quad - 2 < A^n P_1, P_2 > - < P_1, P_2 > < A^n P_1, P_2 > + < P_1, P_1 > < A^n P_2, P_2 >, \quad n \geq 2.
\end{align*}
\]

This involutive system corresponds to mono-nonlinearization for the Kaup-Newell systems (2.10) (see Ref. [18]). It involves \( 2N \) dependent variables but not \( 4N \) dependent variables.

It should be noted that binary nonlinearization procedure is theoretically very systematic (refer to [19] for more information). It paves a way to show Liouville integrability for both the spatial part (sometimes called constrained flows or restricted flows) and the temporal parts of nonlinearized Lax pairs and adjoint Lax pairs, which is short of discussion in the works of restricted flows (see for instance [8]). We only need the distinction of \( \lambda_1, \lambda_2, \ldots, \lambda_N \) for proving Liouville integrability but may have the condition \( \prod_{i=1}^{N} \lambda_i = 0 \), which is not allowed in the case of restricted flows. The resulting binary nonlinear constraints involve two classes of eigenfunctions and thus might greatly extend explicit finite dimensional integrable Hamiltonian systems in the Liouville sense.

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**References**

[1] Ablowitz M. J., Ramani A. and Segur H., *Lett. Nuovo Cim.*, 23 (1978) 333; *J. Math. Phys.* 21 (1980) 715, 1006.
[2] Airault H., McKean H. P. and Moser J., *Commun. Pure Appl. Math.*, **30** (1977) 95; Case K. M. *Proc. Nat’l. Acad. Sci.*, **75** (1978) 3562; *Proc. Nat’l. Acad. Sci.*, **76** (1979) 1.

[3] Cao C. W., *Chin. Quart. J. Math.*, **3** (1988) 90; *Sci. China A*, **33** (1990) 528.

[4] Cao C. W. and Geng X. G., *J. Phys. A: Math. Gen.*, **23** (1990) 4117; *J. Math. Phys.*, **32** (1991) 2323.

[5] Zeng Y. B. and Li Y. S., *J. Math. Phys.*, **30** (1989) 1679; Zeng Y. B., *Phys. Lett. A*, **160** (1991) 541.

[6] Ma W. X. and Strompp W., *Phys. Lett. A*, **185** (1994) 277.

[7] Ma W. X., *J. Phys. Soc. Jpn.*, **64** (1995) 1085; *Physica A*, **219** (1995) 467; Binary nonlinearization for the Dirac systems, to appear in *Chinese Annals of Math. B* [solv-int/9512002].

[8] Antonowicz M. and Wojciechowski S., *Phys. Lett. A*, **147** (1990) 455; *J. Phys A: Math. Gen.*, **24** (1991) 5043; *J. Math. Phys.*, **33** (1992) 2115; Ragnisco O. and Wojciechowski S., *Inverse Problems*, **8** (1992) 245; Blaszak M., *Phys. Lett. A*, **174** (1993) 85; Tondo G., *J. Phys. A: Math. Gen.*, **28** (1995) 5097.

[9] Kaup D. J. and Newell A. C., *J. Math. Phys.*, **19** (1978) 798.

[10] Tu G. Z., *J. Phys. A: Math. Gen.*, **22** (1989) 2375; Ma W. X., *J. Phys. A: Math. Gen.*, **26** (1993) 2573.

[11] Fordy A. P. and Gibbons J., *J. Math. Phys.*, **21** (1980) 2508; ibid, **22** (1981) 1170.

[12] Fuchssteiner B., *Nonlinear Anal. TMA*, **3** (1979) 849; *Prog. Theor. Phys.*, **65** (1981) 861.

[13] Tu G. Z., *Scientia Sinica A*, **24** (1986) 138; *J. Math. Phys.*, **30** (1989) 330.

[14] Tu G. Z. and Meng D. Z., *Nonlinear Evolutions*, ed. Leon J. J. P. (World Scientific, Singapore) 1988 p425; *Acta Math. Appl. Sinica*, **5** (1989) 89.

[15] Magri F., *Nonlinear Evolution Equations and Dynamical Systems*, eds. Boiti M., Pempinelli F. and Soliani G. (Springer-Verlag, Berlin) 1980 p233

[16] Ma W. X., *J. Math. Phys.*, **33** (1992) 2464; *J. Phys. A: Math. Gen.*, **25** (1992) 5329.

[17] Arnold V. I., *Mathematical Methods of Classical Mechanics* (Springer-Verlag, Berlin) 1978; Abraham R. and Marsden J., *Foundations of Mechanics*, 2nd edition (Addison-Wesley, Massachusetts, Reading) 1978.

[18] Liu C. P., *Appl. Math.–J. Chin. Universities Ser. A*, **8** (1993) 157.

[19] Ma W. X. and Fuchssteiner B., Binary nonlinearization of Lax pairs, to appear in *Proceedings of International Conference of Nonlinear Physics, Gallipoli, Italy* (World Scientific, Singapore).