On eigenfunctions corresponding to a small resurgent eigenvalue.

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Abstract

The paper is devoted to some foundational questions in resurgent analysis. As a main technical result, it is shown that under appropriate conditions the infinite sum of endlessly continuable majors commutes with the Laplace transform. A similar statement is proven for compatibility of a convolution and of an infinite sum of majors. We generalize the results of Candelpergher-Nosmas-Pham and prove a theorem about substitution of a small (extended) resurgent function into a holomorphic parameter of another resurgent function. Finally, we discuss an application of these results to the question of resurgence of eigenfunctions of a one-dimensional Schrödinger operator corresponding to a small resurgent eigenvalue.

1 Introduction

A resurgent function $f(h)$ is, roughly, a function admitting a nice enough hyperasymptotic expansion

$$\sum_k e^{-c_k/h}(a_{k,0} + a_{k,1}h + a_{k,2}h^2 + ...), \quad h \to 0,$$

where
and resurgent analysis is a way of working with such expansions by means of representing $f(h)$ as a Laplace integral of a ramified analytic function in the complex plane.

Applications of the theory of resurgent functions to quantum-mechanical problems ([V83], [DDP97], [DP99]) have demonstrated its elegance and computational power, which brings forward the question of a fully rigorous mathematical justification of these methods.

This article is devoted to some foundational questions of resurgent analysis as applied to the Schrödinger equation in one dimension. Suppose, as we explicitly mention in [G] or as it implicitly happens, say, in [DDP97], that using methods of [ShSt] one can construct a resurgent solution of the Schrödinger equation in one dimension

$$[-h^2 \partial_q^2 + V(q, h)]\psi = 0$$

for the potential $V(q, h)$ analytic with respect to $q$ and polynomial in $h$. This situation includes, in particular, the eigenvalue problem

$$[-h^2 \partial_q^2 + V(q, h)]\psi = hE_1 \psi \quad (1.1)$$

Then, using methods of asymptotic matching (see e.g. [DP99, DDP97]) one can write the quantization conditions and obtain by solving them some resurgent energy level $E_1$. This $E_1$ will in general not be a polynomial in $h$, but some arbitrary resurgent function. In this note we show that if $E_1$ is a small resurgent function (see below for a precise definition), then one can substitute $E_1$ into a resurgent solution $\psi(E_1, q, h)$ of (1.1) and obtain another resurgent function of $h$ that would solve the corresponding differential equation (see section 9).

In order to work out the detail of substituting a small resurgent function of $h$ into a holomorphic parameter of another resurgent function, the following more technical issues are addressed.

A resurgent function is given as a Laplace integral of a ramified analytic function (called “major”) along some infinite contour. In order for the Laplace integral to converge, one needs to correct that major by an entire function so that it stays bounded along the infinite branches of the contour; the details of this procedure are discussed in [CNP]. If we are to deal with an infinite sum of majors, we need to choose the corresponding corrections by entire functions in a coherent way, in the precise sense of section 3.2.
Then one can show that the Laplace transform is compatible, under certain conditions, with an infinite sum of majors.

A similar statement is shown for a convolution and for the reconstruction homomorphism (the reconstruction of a major from its decomposition into microfunctions).

We also consider the question of substituting of a small resurgent function into a holomorphic function and generalize the argument of [CNP] to the case of what they would call an extended resurgent function. This case is reduced to the case considered in [CNP] by means of interchanging the infinite sums with convolutions and Laplace transforms as above. The [CNP]'s proof will also be recalled in a slightly generalized form that allows to accommodate the case of substituting of $k$ small resurgent functions into a holomorphic functions of $k$ variables.

Although the results of this article are perfectly natural, they have not been, to our knowledge, properly documented in the literature and we felt it is important to write up their detailed proofs.

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2 Preliminaries from Resurgent Analysis

We need to combine the setup of [CNP] (the resurgence is with respect to the semiclassical parameter $h$ rather than the coordinate $q$) and mathematical clarity of [ShSt] and therefore have to mix their terminology. Also, we are somewhat changing their notation to make sure that typographical peculiarities of fonts do not interfere with clear understanding of symbols.
2.1 Laplace transform and its inverse. Definition of a resurgent function.

Morally speaking, we will be studying analytic functions \( \varphi(h) \) admitting asymptotic expansions \( e^{-c/h}(a_0 + a_1 h + a_2 h^2 + \ldots) \) for \( h \to 0 \) and \( \arg h \) constrained to lie in an arc \( A \) of the circle of directions on the complex plane; respectively, the inverse asymptotic parameter \( x = 1/h \) will tend to infinity and \( \arg x \) will belong to the complex conjugate arc \( A^* \). Such functions can be represented as Laplace transforms of functions \( \Phi(\xi) \) where the complex variable \( \xi \) is Laplace-dual to \( x = 1/h \) and \( \Phi(\xi) \) is analytic in \( \xi \) for \( |\xi| \) large enough and \( \arg \xi \in \bar{A} \), the copolar arc to \( A \). The concept of a resurgent function will be defined by imposing conditions on analytic behavior of \( \Phi \).

2.1.1 "Strict" Laplace isomorphism

For details see [CNP Pré I.2].

Let \( A \) be a small (i.e. of aperture \( < \pi \)) arc in the circle of directions \( S^1 \). Denote by \( \bar{A} \) its copolar arc, \( \bar{A} = \cup_{\alpha \in A} \bar{\alpha} \), where \( \bar{\alpha} \) is the open arc of legth \( \pi \) consisting of directions forming an obtuse angle with \( \alpha \).

Denote by \( \mathcal{O}^\infty(A) \) the space of sectorial germs at infinity in direction \( A \) of holomorphic functions and by \( \mathcal{E}(A) \) the subspace of those that are of exponential type in the direction \( A \), i.e. bounded by \( e^{K|t|} \) as the complex argument \( t \) goes to infinity in the direction \( A \).

We want to construct the isomorphism

\[ \mathcal{L} : \mathcal{E}(\bar{A})/\mathcal{O}(\mathbb{C})^{\exp} \longleftrightarrow \mathcal{E}(A^*) : \bar{\mathcal{L}} \]

where \( \mathcal{O}(\mathbb{C})^{\exp} \) denotes the space of functions of exponential type in all directions.

**Construction of \( \mathcal{L} \).** Let \( \Phi \) be a function holomorphic in a sectorial neighborhood \( \Omega \) of infinity in direction \( \bar{A} \). For any small arc \( A' \subset A \) we can choose a point \( \xi_0 \in \mathbb{C} \) such that \( \Omega \) contains a sector \( \xi_0 \bar{A}' \) with the vertex \( \xi_0 \) opening in the direction \( \bar{A}' \). Define the Laplace transform

\[ \Phi_\gamma(x) := \int_\gamma e^{-x\xi}\Phi(\xi)d\xi \]

with \( \gamma = -\partial(\xi_0 \bar{A}') \). Then \( \Phi_\gamma \) is holomorphic of exponential type in a sectorial neighborhood of infinity in direction \( A^* \). Cauchy theorem shows that is \( \Phi \) is
entire of exponential type, then $\Phi_\gamma = 0$, which allows us to define

$$\mathcal{L}(\Phi \mod \mathcal{O}(\mathbb{C})^{\exp}) = \Phi_\gamma.$$  

The construction of $\bar{\mathcal{L}} = \mathcal{L}^{-1}$ will not be used in this paper.

### 2.1.2 ”Large” Laplace isomorphism

The Laplace transform $\mathcal{L}$ defined in the previous subsection can be applied only to a function $\Phi(\xi)$ satisfying a growth condition at infinity. At a price of changing the target space of $\mathcal{L}$ this restriction can be removed as follows.

For details see [CNP, Pré I.3].

Let $\hat{A} = (\alpha_0, \alpha_1)$ be the copolar of a small arc, where $\alpha_0, \alpha_1 \in S^1$ are two directions in the complex plane, and let $\gamma : \mathbb{R} \to \mathbb{C}$ be an endless continuous path. We will say that $\gamma$ is adapted to $\hat{A}$ if $\lim_{t \to -\infty} \gamma(t)/|\gamma(t)| \to \alpha_0$, $\lim_{t \to -\infty} \gamma(t)/|\gamma(t)| \to \alpha_1$, and if the length of the part of $\gamma$ contained in a ring $\{z : R \leq |z| \leq R + 1\}$ is bounded by a constant independent of $R$.

Let us now construct for a small arc $\hat{A}$ two mutually inverse isomorphisms

$$\mathcal{L} : \mathcal{O}^\infty(\hat{A})/\mathcal{O}(\mathbb{C}) \leftrightarrow \mathcal{E}(A^*)/\mathcal{E}^{-\infty}(A^*) : \bar{\mathcal{L}},$$

where $\mathcal{E}^{-\infty}(A^*)$ is the set of sectorial germs at infinity that decay faster than any function of exponential type (cf. [CNP, Pré I.0]).

**Construction of $\mathcal{L}$**. Let $\Psi$ be holomorphic in $\Omega$, a sectorial neighborhood of infinity of direction $\hat{A}$. Let $\gamma$ be a path adapted to $\hat{A}$ in $\Omega$. As we will see in lemma 3.1 below, there is a function $\Phi$ bounded on $\gamma$ such that $\Phi - \Psi \in \mathcal{O}(\mathbb{C})$ and define

$$\mathcal{L}(\Psi \mod \mathcal{O}(\mathbb{C})) := \int_\gamma e^{-\xi \Phi(\xi)}d\xi \mod \mathcal{E}^{-\infty}(A^*)$$

**Definition.** Any of the functions $\Psi(\xi)$ satisfying $\mathcal{L}(\Psi \mod \mathcal{O}(\mathbb{C})) = \psi(x)$ is called a major of the function $\psi(x)$.

An equivalence class of functions defined on a subset of $\mathbb{C}$ modulo adding an entire function is also called an integrality class.

### 2.1.3 Resurgent functions.

Resurgent functions are usually understood to be functions of a large parameter $x$. For brevity we will speak of resurgent functions of $h$ to mean resurgent functions of $1/h$. 

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Definition. A germ \( f(\xi) \in \mathcal{O}_{\xi_0} \) is endlessly continuable if for any \( L > 0 \) there is a finite set \( \Omega_L \subset \mathbb{C} \) such that \( f(\xi) \) has an analytic continuation along any path of length \( < L \) avoiding \( \Omega_L \).

Definition. (cf. [ShSt, p.122]) Let \( A \subset S^1 \subset \mathbb{C} \) be a small arc and let \( A^* \) be obtained from \( A \) by complex conjugation. A resurgent function \( f(x) \) (of the variable \( x \to \infty \)) in direction \( A^* \) is an element of \( \mathcal{E}(A^*)/\mathcal{E}^{-\infty} \) such that \( (\mathcal{L}f)(\xi) \) (called its major) is endlessly continuable.

Remark. [CNP] calls the same kind of object an ”extended resurgent function”.

Let (cf. [CNP, Rés I]) \( \mathcal{R}(A) \) denote the set of endlessly continuable sectorial germs of analytic functions \( \Phi(\xi) \) defined in a neighborhood of infinity in the direction \( \tilde{A} \). Then a resurgent function of \( x \) in the direction \( A^* \) has a major in \( \mathcal{R}(A) \).

When we mean a resurgent function \( g(h) \) of a variable \( h \to 0 \), under the correspondence \( x = 1/h \) the sectorial neighborhood of infinity in the direction \( A^* \) becomes a sectorial neighborhood of the origin in the direction \( A \), and we will talk about a resurgent function \( g(h) \) (for \( h \to 0 \)) in the direction \( A \).

2.1.4 Examples of resurgent functions.

Example 1. \( h^\alpha \)

The major corresponding to \( h^\nu \) for \( \nu \neq 1, 2, 3, .. \) is \( \frac{-1}{2i \sin(\pi \nu)} \frac{(-\xi)^{-\nu-1}}{\Gamma(\nu)} \), and \( \xi^{-\nu-1} \frac{\log \xi}{2\pi i \Gamma(\nu)} \) for \( \nu = 1, 2, .... \)

Example 2. \( \log h. \)

Example 3. \( \Phi(h) := e^{1/h}, \Phi(h) := e^{-1/\sqrt{h}}. \) (CNP, Rés II.3.4]

Example 4. \( e^{-1/h^2} \) is zero as a resurgent function on the arc \( (-\pi/4, \pi/4) \), but does not give a resurgent function on any larger arc because there it is no longer bounded by any function \( e^{c/h} \) for \( c \in \mathbb{R} \).

2.2 Decomposition theorem for a resurgent function

Making rigorous sense of a formula of the type

\[
\phi(h) \sim \sum_k e^{-ck/h} (a_{k,0} + a_{k,1}h + a_{k,2}h^2 + ...), \quad h \to 0,
\]


may be done by explicitly writing an estimate of an error that occurs if we truncate the $k$-th power series on the right at the $N_k$-th term. Resurgent analysis offers the following attractive alternative: if $\phi(h)$ is resurgent, then the numbers $c_k$ can be seen as locations of the first sheet singularities of the major $\Phi(\xi)$, which is expressed in terms of a decomposition of $\Phi(\xi)$ in a formal sum of microfunctions that we are going to discuss now. Microfunctions, or the singularities of $\Phi(\xi)$ at $c_k$ are related to power series $a_{k,0} + a_{k,1}h + a_{k,2}h^2 + \ldots$ through the concept of Borel summation, see section 2.3.

### 2.2.1 Microfunctions and resurgent symbols

**Definition.** (see [CNP, Pré II.1]) A microfunction at $\omega \in \mathbb{C}$ in the direction $\hat{A} \subset S^1$ is the datum of a sectorial germ at $\omega$ in direction $\hat{A}$ modulo holomorphic germs in $\omega$; the set of such microfunctions is denoted by $\mathcal{O}^\omega(\hat{A}) = \mathcal{O}^\omega(\omega) / \mathcal{O}_\omega$.

A microfunction is said to be resurgent if it has an endlessly continuable representative. The set of resurgent microfunctions at $\omega$ in direction $\hat{A}$ is denoted in [CNP, Rés I.3.0, p.178] by $\mathcal{R}^\omega(\hat{A})$.

**Definition.** ([CNP, Rés I.3.3, p.183]) A resurgent symbol in direction $\hat{A}$ is a collection $\hat{\phi} = (\phi^\omega)_{\omega \in \mathbb{C}}$ such that $\phi^\omega$ is nonzero only for $\omega$ in a discrete subset $\Omega \subset \mathbb{C}$ called the support of $\hat{\phi}$, and for any $\alpha \in A$ the set $\mathbb{C} \setminus \Omega \alpha$ is a sectorial neighborhood of infinity in direction $\hat{A}$.

The set of such resurgent symbols is denoted $\mathcal{R}(A)$, and resurgent symbols themselves can be written as $\hat{\phi} = (\phi^\omega)_{\omega \in \Omega} \in \mathcal{R}(A)$ or as $\hat{\phi} = \sum_{\omega \in \Omega} \phi^\omega \in \mathcal{R}(A)$.

**Definition.** A resurgent symbol is elementary if its support $\Omega$ consists of one point. It is elementary simple if that point is the origin.

### 2.2.2 Decomposition isomorphism.

The correspondence between resurgent symbols in the direction $A$ and majors of resurgent functions in direction $A$ depends on a resummation direction $\alpha \in A$, which we will fix once and for all. The direction $\alpha$ can more concretely be thought of as $\arg h$ or as the the direction of the cuts in the $\xi$-plane that we are going to draw.
Let $\dot{\phi} = (\phi^\omega)_{\omega \in \Omega} \in \mathcal{R}(A)$ be a resurgent symbol with the singular support $\Omega$, and $\alpha \in A$. Let $\Omega_\alpha = \bigcup_{\omega \in \Omega} \omega \alpha$ be the union of closed rays in the direction $\alpha$ emanating from the points of $\Omega$.

![Figure 1: Singularities of a major and corresponding cuts](image)

Suppose (\cite[p.182-183]{CNP}) $\Omega$ is a discrete set (of singularities) in the complement of a sectorial neighborhood of infinity in direction $\hat{A}$, and a holomorphic function $\Phi$ is defined on $\mathbb{C}\setminus\Omega$ and is endlessly continuable. Take $\omega \in \Omega_\alpha$. Let $D_\omega$ be a small disc centered at $\omega$. Its diameter in the direction $\alpha$ cuts $D_\omega$ into the left and right open half-discs $D^-_\omega$ and $D^+_\omega$ (or top and bottom if $\alpha$ is the positive real direction). If $D_\omega$ is small enough, the function $\Phi|D^+_\omega$, resp. $\Phi|D^-_\omega$, can be analytically continued to the whole split disc $D_\omega \setminus \omega \alpha$.

Denote by $\text{sing}^{\omega+}_\alpha \Phi$, resp. $\text{sing}^{\omega-}_\alpha \Phi$, the microfunction at $\omega$ of direction $\hat{\alpha}$ defined by the class modulo $O_\omega$ of this analytic continuation.

**Theorem 2.1.** There is a function $\Phi \in \mathcal{O}(\mathbb{C}\setminus\Omega_\alpha)$, endlessly continuable, such that

$$\text{sing}^{\omega+}_\alpha \Phi = \begin{cases} 
\phi^\omega & \text{if } \omega \in \Omega \\
0 & \text{if not}
\end{cases}$$

In this case we write

$$\Phi = \mathbf{s}_{\alpha+} \dot{\phi}$$

and call the inverse map $(\mathbf{s}_{\alpha+})^{-1}$ the **decomposition isomorphism**

An endlessly continuable function $\mathbf{s}_{\alpha-} \dot{\phi}$ can be defined analogously.

The maps $\mathbf{s}_{\alpha+}, \mathbf{s}_{\alpha-}$ respect sums and convolution products (cf. \cite[p.185, Rés I, section 4]{CNP}).

The map defined in \cite[Rés I]{CNP} as

$$\mathbf{s}_{\alpha+} \circ (\mathbf{s}_{\alpha-})^{-1} : \mathcal{R}(A) \rightarrow \hat{\mathcal{R}}(A)$$

...
is not identity and gives rise to the concept of alien derivatives which is central in resurgent analysis but will not be discussed in this article.

Further, if a resurgent function \( \varphi(h) = \varphi(h,t) \) and its major \( \Phi(\xi) = \Phi(\xi,t) \) depend, say, continuously in some appropriate sense, on an auxiliary parameter \( t \), the decomposition into microfunctions \((s_{\alpha+})^{-1}\Phi \) will depend on \( t \) “discontinuously” – an effect referred to as **Stokes phenomenon** and discussed, e.g., in [DP99].

### 2.2.3 Mittag-Leffler sum

The concept of a Mittag-Leffler sum formalizes the idea of an infinite sum of resurgent functions \( \sum_j \varphi_j(h) \) where \( \varphi_j(h) \) have smaller and smaller exponential type, e.g., \( \varphi = O(e^{-c_j/h}) \) for \( c_j \to \infty \) as \( j \to \infty \).

Following [CNP Pré I.4.1], let \( \Phi_j, j = 1, 2, \ldots \), be endlessly continuable holomorphic functions, \( \Phi_j \in \mathcal{O}(\Omega_j) \), where \( \Omega_j \) are sectorial neighborhoods of infinity satisfying \( \Omega_j \subset \Omega_{j+1} \) and \( \bigcup_j \Omega_j = \mathbb{C} \). Then [CNP] show that there is a function \( \Phi \in \Omega_1 \) such that

\[
\Phi - \sum_{j=1}^{n} \Phi_j \in \mathcal{O}(\Omega_{n+1}), \quad n = 1, 2, \ldots
\]

In this case we will call \( \Phi \) the **Mittag-Leffler sum** of \( \Phi_1, \Phi_2, \ldots \) and write

\[
\Phi = \text{ML} \Sigma_j \Phi_j.
\]

### 2.3 Borel summation. Resurgent asymptotic expansions.

**Definition.** A **resurgent hyperasymptotic expansion** is a formal sum

\[
\sum_k e^{-c_k/h} (a_{k,0} + a_{k,1}h + a_{k,2}h^2 + \ldots),
\]

where:

i) \( c_k \) form a discrete subset in \( \mathbb{C} \) in the complement to some sectorial neighborhood of infinity in direction \( \tilde{A} \);

ii) the power series of every summand satisfies the Gevrey condition, and
iii) each infinite sum \( a_{k,0} + a_{k,1}h + a_{k,2}h^2 + \ldots \) defines, under formal Borel transform

\[
\mathcal{B} : e^{-c_k/h}h^\ell \mapsto (\xi - c_k)^{\ell-1} \frac{\log(\xi - c_k)}{2\pi i\Gamma(\ell)} \quad \text{if } \ell \in \mathbb{N},
\]

\[
\mathcal{B} : e^{-c_k/h} \mapsto \frac{1}{2\pi i(\xi - c_k)},
\]

an endlessly continuable microfunction centered at \( c_k \).

The authors of [CNP] denote by \( \mathcal{R}(A) \) (regular, as opposed to the bold-faced, \( \mathcal{R} \)) the algebra of resurgent hyperasymptotic expansions.

The right and left summations of resurgent asymptotic expansions are defined in [DP99] or [CNP] as follows. Given a Gevrey power series \( \sum_{k=0}^{\infty} a_k h^k \), replace it by a function (the corresponding “minor”) \( f(\xi) = \sum_{k=1}^{\infty} a_k \xi^{k-1} \), assume that \( f(\xi) \) has only a discrete set of singularities, and consider the Laplace integrals \( \int_{[0,\alpha]} e^{-\xi/h}f(\xi) d\xi \) along a ray from 0 to infinity in the direction \( \alpha \) deformed to avoid the singularities from the right or from the left, as on the figure 2.

![Figure 2:](image)

After some technical discussion, this procedure defines a resurgent function of \( h \) which [CNP] denote \( S_{\alpha \pm} \left( \sum_{k=0}^{\infty} a_k h^k \right) \) and we, for typographical reasons, prefer to denote \( B_{\alpha \pm} \left( \sum_{k=0}^{\infty} a_k h^k \right) \). Comparing the results of the left and right resummations lead to the notion of the “homomorphisme de passage” discussed earlier.

The following diagram helps to visualize the logical relationship of the concepts that have been introduced.

3 Majors exponentially decreasing along a path.

As we have seen, the definition of the Laplace isomorphism \( \mathcal{L} \) involves choosing a representative of a class \( \mathcal{O}(\mathbb{C}) \) of a major that is bounded along
Hyperasymptotics
\[ \sum e^{-\mathcal{H}(a_{k0}+a_{k1}h+...)} \]
formal Borel transform
\[ \rightarrow \]
Resurgent symbols
\[ \sum \varphi_k(\zeta - c_k) \] for microfunctions \( \varphi_k \)

Steepest descent for Laplace integral
\[ \searrow \nearrow \nearrow \]
Major of a resurgent function
\[ \searrow \nearrow \] Decomposition theorem

infinite branches of a contour. Let us begin by recalling how such a representative is constructed in [CNP].

### 3.1 Case of a single major.

Let \( D_n = \{ z \in \mathbb{C} : |z| \leq n \} \).

**Lemma 3.1.** ([CNP, Pré I.3, Lemma 3.0]) Let \( \Gamma \subset \mathbb{C} \) be an embedded curve (i.e. a closed submanifold of dimension 1) transverse to circles \(|\zeta| = R\) for all \( R \geq R_0 \). Let \( \Phi \) be a holomorphic function in a neighborhood of \( \Gamma \). Then for any function \( m : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying \( \inf_{x \leq N} m(x) > 0 \) for any \( N > 0 \), there is an entire function \( E \) such that \(|(\Phi + E)(\zeta)| \leq m(|\zeta|)\) for any \( \zeta \in \Gamma \).

**Proof.** Without loss of generality \( R_0 = 1 \).

**Lemma reformulated.** Let \( m_n \) be a sequence of positive numbers. Then there is an entire function \( E(\xi) \) such that \(|\Phi + E| \leq m_n \) on \((D_{n+1} \setminus D_n) \cap \Gamma\).

Without loss of generality \( m_{n+k} \leq \frac{m_n}{2^k} \) for all \( k \geq 1 \). Using the lemma below, we will inductively construct \( E_n \) so that \( E_n \leq \frac{m_n}{2^k} \) on \( D_{n-1} \) and such that
\[ |\Phi + E_2 + ... + E_n| \leq \frac{m_n}{2} \quad \text{on} \quad (D_n \setminus D_{n-1}) \cap \Gamma. \]

Then the series \( E_2 + ... + E_n + ... \) of holomorphic functions will converge uniformly on \( D_n \). Furthermore, since \(|E_{n+k}| \leq \frac{m_n}{2^{k+1}}\) on \( D_{n-1} \), we see that \( \sum_{k=1}^{\infty} E_{n+k} | \leq \frac{m_n}{2} \) and hence the conditions of the lemma will be satisfied with \( E = \sum_{n=1}^{\infty} E_n \).

Let \( \Gamma \) now be the intersection of the curve \( \Gamma \) of the above lemma with the annulus \( R \leq |\zeta| \leq R' \), where \( R \) is large enough.
**Lemma 3.2.** Let $\Gamma \subset \mathbb{C}$ be a finite disjoint union of curvilinear segments $\Gamma_i = [\zeta_i, \zeta'_i]$ with $i = 1, \ldots, r$, joining transversally the boundaries of the annulus $R \leq |\zeta| \leq R'$. Let $\Phi$ be a function holomorphic in a neighborhood of $\Gamma$ and zero at $\zeta_1, \ldots, \zeta_r$. Then for any $\varepsilon > 0$ there is an entire (and even a polynomial) function $E$ such that

i) $|E| \leq \varepsilon$ on $D_R = \{|\zeta| < R\}$;

ii) $|\Phi + E| \leq \varepsilon$ on $\Gamma$;

iii) $(\Phi + E)(\zeta'_i) = 0$ for $i = 1, \ldots, r$.

**Proof.** For a number $\delta > 0$ to be fixed later, put

$$\tilde{E}(\zeta) = \sum_{j=1}^{r} \frac{1}{2\pi i} \int_{\lambda_i} \frac{\Phi(\eta)}{\zeta - \eta} d\eta$$

where $\lambda_i$ is the path encircling $\Gamma_i$ whose endpoints are chosen at a distance $< \delta$ from $\zeta_i$ and distance $> \delta/2$ from $D_R \cup \Gamma$.

Figure 3: Notation in Lemma 3.2.

Figure 4: Integration path $\lambda_1$ in the proof of Lemma 3.2.
Let $\Gamma_\rho$ denote the $\rho$-neighborhood of the path $\Gamma$.
Then $\tilde{E}$ is holomorphic in a neighborhood of $D_R \cup \Gamma$, and for any $\varepsilon' > 0$ one can choose $\delta > 0$ so small that the following inequalities are satisfied:

$$
|\tilde{E}(\zeta)| \leq \varepsilon'/3, \quad \forall \zeta \in D_R,
$$
$$
|\Phi(\zeta) + \tilde{E}(\zeta)| \leq \varepsilon'/3, \quad \forall \zeta \in \Gamma_{\delta/4}.
$$

To prove these inequalities, one estimates the above Cauchy integrals on the paths:

It is known (cf., e.g., [Ga]) that there is a polynomial $E'$ such that

$$
|E' - \tilde{E}| \leq \varepsilon'/3 \quad \text{on} \quad D_R \cup \Gamma_{\delta/4},
$$
then $E'$ satisfies (i) and (ii) with $2\varepsilon'/3$ instead of $\varepsilon$.

Let $P(\zeta)$ be the interpolation polynomial such that $P(\zeta_i) = E'(\zeta_i) + \Phi(\zeta_i)$ (which is bounded by $2\varepsilon'/3$.) Take $E(\zeta) = E'(\zeta) - P(\zeta)$. The lemma below with, say, $K = D_R \cup \Gamma_1$ and $\epsilon$ as in the statement of the lemma we are proving, gives me $\varepsilon''$.

Take $\varepsilon' := \min(\varepsilon, \frac{1}{2}\varepsilon'')$. Then

$$
|E(\zeta)| = |E'(\zeta) - \tilde{E}(\zeta) - P(\zeta) + \tilde{E}(\zeta)| \leq |E' - \tilde{E}| + |P| + |\tilde{E}| \leq \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3} \leq \varepsilon \quad \text{on} \quad D_R
$$

and analogously for $|\Phi + E|$ on $\Gamma$.

**Lemma 3.3.** Given $r$ points $\zeta_1, ..., \zeta_r$ on a compact set $K \subset \mathbb{C}$, then for all $\varepsilon > 0$ there is $\varepsilon'' > 0$ such that for any choice of the interpolation data $a_1, ..., a_r$ satisfying $|a_i| \leq \varepsilon''$, the Lagrange interpolation polynomial defined by $Q(\zeta_i) = a_i$ is estimated by $\varepsilon$ on $K$.

$\square$

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3.2 Case of a convergent series of majors.

In order to be able to work with a Laplace integral of an infinite sum of majors, it is useful to choose a representative of each summand of that series that would be bounded on the infinite branches of the integration path, and to do it in a way that such choices would be consistent with taking the infinite sum. The main issue is to show that the entire correction functions to each of the summand form themselves a series convergent on compact subsets of \( \mathbb{C} \). More precisely,

**Proposition 3.4.** Suppose \( \sum_{j=1}^{\infty} \Phi^{(j)} \) is an infinite series of majors defined on the same (unbounded) domain \( S \subset \mathbb{C} \) which converges uniformly and faster than some geometric series with a ratio \( q \) on compact subsets to a function \( \Phi \), \( \Gamma \) a contour transversal to circles \( \partial D_R \) for \( R \geq R_0 \) and \( m : \mathbb{R}_+ \to \mathbb{R}_+ \) a function as in Lemma 3.1. Then one can choose entire functions \( F^{(j)} \) and a number \( Q, q < Q < 1 \), so that:

i) \( \sum_{j=1}^{\infty} F^{(j)} \) converges uniformly on compact sets of \( \mathbb{C} \) faster than geometric series with the ratio \( Q \) to an entire function \( F \);

ii) for all \( j \) the function \( |\Phi^{(j)}(\xi) + F^{(j)}(\xi)| \leq \frac{1}{2^j} m(|\xi|) \) along \( \Gamma \);

iii) the function \( |\Phi(\xi) + F(\xi)| \leq m(|\xi|) \) along \( \Gamma \);

iv) \( |\Phi^{(j)}(\xi) + F^{(j)}| \) can be, on every compact subset of \( S \), estimated by a geometric series with the ratio \( Q \).

**Proof.** Only the parts i) and ii) need to be proven in detail, the parts iii) and iv) will then follow as easy consequences.

Suppose, to simplify notation, that \( R_0 = 1 \), that \( m(|\xi|) \leq 2^{-|\xi|} \) and \( m(|\xi|) = m_k \) (constant) for \( k - 1 < |z| \leq k \).

There are positive numbers \( b_{k\ell} \) satisfying the following property:

For any interpolation data on the finite set \( \Gamma \cap \partial D_k \), namely, any function \( d : \Gamma \cap \partial D_k \to \mathbb{C} \) with \( \max_{p \in \Gamma \cap \partial D_k} |d(p)| \leq 1 \), the supremum of the corresponding degree \( |\Gamma \cap \partial D_k| - 1 \) interpolation polynomial is \( < b_{k\ell} \) on \( D_{k\ell} \).

Clearly we can choose \( b_{k\ell} \leq b_{kk} \) for \( \ell \leq k \) and \( b_{k\ell} \geq 1 \) for all \( k, \ell \).

Fix a number \( Q \) such that \( q < Q < 1 \).

Step 1. Find a number \( N_1 \) so that for \( \forall j > N_1 \) it holds \( |\Phi^{(j)}(\xi)| < \frac{(1-Q)Q^j}{2b_{11}m_1} \) on \( \Gamma \cap D_1 \). Choose functions \( E^{(j)}_1 \) so that \( |\Phi^{(j)} + E^{(j)}_1| < \frac{m_1(1-Q)Q^j}{b_{11}2^j} \) on \( D_1 \cap \Gamma \) for \( j \leq N_1 \) and \( E^{(1)}_j = 0 \) for \( j > N_1 \). Notice that the series \( \sum_{j=1}^{\infty} (\Phi^{(j)} + E^{(j)}_1) \) still converges uniformly and absolutely and faster

\[ 1 \text{ i.e., for every disc } D_\ell \text{ there is a constant } M_\ell \text{ so that } |\Phi^{(j)}| \leq M_\ell q^j \text{ on } D_\ell \]
some geometric series of ratio \( q \) on compact subsets. Now choose \( G^{(j)}_1 \) as interpolation polynomials so that \( \Phi^{(j)} + E^{(j)}_1 + G^{(j)}_1 = 0 \) on \( \Gamma \cap \partial D_1 \). Then \( \sup_{D_1} |G^{(j)}_1| < \frac{b_{12}m_1(1-Q)^{j}}{2b_{11}} \) for small \( j \) and decreases faster than some geometric series of ratio \( Q \) for larger \( j \).

Put \( F^{(j)}_1 := E^{(j)}_1 + G^{(j)}_1 \). Then on \( D_1 \cap \Gamma \)

\[
|\Phi^{(j)} + F^{(j)}_1| \leq |\Phi^{(j)} + E^{(j)}_1| + |G^{(j)}_1| \leq \frac{m_1(1-Q)^{j}}{2b_{11}} + \frac{m_1(1-Q)^{j}}{2} \leq m_1(1-Q)^{j}
\]

**Step 2.** Let \( N_2 \) be such that \( |F^{(j)}_1 + \Phi^{(j)}(\zeta)| < \frac{m_2(1-Q)^{j}}{2b_{22}} \) on \( \Gamma \cap (D_2 \setminus D_1) \) for \( \forall j > N_2 \). Then for \( j \leq N_2 \) choose entire \( E^{(j)}_2 \) so that:

a) \( |E^{(j)}_2| < \frac{m_2(1-Q)^{j}}{2b_{22}} \) on \( D_1 \);

b) \( |E^{(j)}_2 + F^{(j)}_1 + \Phi^{(j)}| < \frac{m_2(1-Q)^{j}}{2b_{22}} \) on \( (D_2 \setminus D_1) \cap \Gamma \).

Now choose \( G^{(j)}_2 \) as the interpolation polynomials so that \( \Phi^{(j)} + F^{(j)}_1 + E^{(j)}_2 + G^{(j)}_2 = 0 \) on \( \Gamma \cap \partial D_2 \). Then \( \sup_{D_1} |G^{(j)}_2| < \frac{b_{22}m_2(1-Q)^{j}}{2b_{22}} \) for small \( j \) and decreases faster than some geometric series of ratio \( Q \) for larger \( j \). Put \( F^{(j)}_2 := E^{(j)}_2 + G^{(j)}_2 \).

Analogously to the step 1, \( |\Phi^{(j)} + F^{(j)}_1 + F^{(j)}_2| \leq m_2(1-Q)^{j} \) on \( \Gamma \cap (D_2 \setminus D_1) \).

The series \( \sum_j (\Phi^{(j)} + F^{(j)}_1 + F^{(j)}_2) \) still converges uniformly and faster than some geometric series of ratio \( Q \) on compact sets.

**Step 3.**

Choose \( N_3 \) so that on \( \Gamma \cap (D_3 \setminus D_2) \) we have \( \forall j > N_3 \) the inequality \( |\Phi^{(j)} + F^{(j)}_1 + F^{(j)}_2| < \frac{m_3(1-Q)^{j}}{2b_{33}} \). Then there are entire functions \( E^{(j)}_3 \) such that

a) \( |E^{(j)}_3| < \frac{m_3(1-Q)^{j}}{2b_{33}} \) on \( D_2 \) for \( j \leq N_3 \)

b) \( |E^{(j)}_3 + F^{(j)}_2 + F^{(j)}_1 + \Phi^{(j)}| < \frac{m_3(1-Q)^{j}}{2b_{33}} \) on \( (D_3 \setminus D_2) \cap \Gamma \).

Choose \( G^{(j)}_3 \) as the interpolation polynomial such that \( F^{(j)}_2 + F^{(j)}_1 + \Phi^{(j)} + E^{(j)}_3 + G^{(j)}_3 = 0 \) on \( \Gamma \cap \partial D_3 \) and put \( F^{(j)}_3 := E^{(j)}_3 + G^{(j)}_3 \).

Finally, for every fixed \( \ell \) we get \( \sum_{j,k} \sup_{D_\ell} |F^{(j)}_k| < \infty \) for the following reasons:

a) If \( k \leq \ell \), then

\[
\sum_{j=N_k+1}^{\infty} \sup_{D_\ell} |F^{(j)}_k| \leq \sum_{j=N_k+1}^{\infty} b_{k\ell}m_k(1-Q)^{j} \leq b_{k\ell};
\]
b) If \( k \geq \ell + 1 \), then
\[
\sum_{j=1}^{\infty} \sup_{D_{z}} |F_{k}(j)| \leq \sum_{j=1}^{\infty} m_{k}(1 - Q)Q^{j} = m_{k} \leq \frac{1}{2^{k}}
\]

The fast convergence of \( F^{(j)} \) follows from:
\[
\sup_{D_{z}} |F_{k}(j)| \leq b_{k \ell}m_{k}(1 - Q)Q^{j} \quad \text{for} \quad k \leq \ell, \quad j > \max\{N_{1}, ..., N_{\ell}\}
\]
and
\[
\sup_{D_{z}} |F_{k}(j)| \leq m_{k}(1 - Q)Q^{j} \quad \text{for} \quad k \geq \ell + 1,
\]
because then for \( j > \max\{N_{1}, ..., N_{\ell}\} \) we get
\[
\sup_{D_{z}} \sum_{k=1}^{\infty} |F_{k}(j)| \leq \sum_{k=1}^{\ell} \sup_{D_{z}} |F_{k}(j)| + \sum_{k=\ell+1}^{\infty} \sup_{D_{z}} |F_{k}(j)| \leq \sum_{k=1}^{\ell} b_{k \ell}m_{k}(1 - Q)Q^{j} + \sum_{k=\ell+1}^{\infty} m_{k}(1 - Q)Q^{j} \leq \left[ \sum_{k=1}^{\ell} b_{k \ell}m_{k}(1 - Q) + \sum_{k=\ell+1}^{\infty} m_{k}(1 - Q) \right] Q^{j}.
\]

The figure\(^{5}\) helps understand these estimates. □

### 3.3 Interchangeability of infinite sum and \( \mathcal{L} \).

**Proposition 3.5.** Suppose \( \Phi^{(j)} \) is a sequence of majors defined on a common sectorial neighborhood of infinity and suppose that on each compact subset of this neighborhood the sum \( \sum_{j} |\Phi^{(j)}| \) uniformly converges faster than a geometric series with a ratio \( q < 1 \). Then
\[
\mathcal{L} \left\{ \left( \sum_{j} \Phi^{(j)} \right) \mod \mathcal{O}(\mathbb{C}) \right\} = \sum_{j} \mathcal{L} \left( \Phi^{(j)} \mod \mathcal{O}(\mathbb{C}) \right).
\]

**Proof.** Taking the representatives \( \Phi^{(j)} \) of the corresponding integrality classes and the number \( Q < 1 \) provided by proposition\(^{3,4}\) for \( m_{\ell} = 1/2^{\ell} \), we can write \( \mathcal{L} \left( \Phi^{(j)} \mod \mathcal{O}(\mathbb{C}) \right) \) as \( \int_{\gamma} e^{-\xi/h} \Phi^{(j)}(\xi)d\xi \), and similarly for \( \mathcal{L} \left\{ \left( \sum_{j} \Phi^{(j)} \right) \mod \mathcal{O}(\mathbb{C}) \right\} \). Then the question reduces to showing that
\[
\int_{\gamma} \sum_{j} e^{-\xi/h} \Phi^{(j)}(\xi)d\xi = \sum_{j} \int_{\gamma} e^{-\xi/h} \Phi^{(j)}(\xi)d\xi.
\]
By Fubini’s theorem we need to check:

i) For any $\xi$ the sum $\sum_j e^{-\xi/h} |\Phi^{(j)}(\xi)|$ converges – because if $\xi \in D_\ell$, then $|\Phi^{(j)}(\xi)| < m_\ell/Q^j \leq 1/Q^{j+\ell}$, and so the sum is $\leq 1/Q^\ell$;

ii) the integral of such a sum clearly converges;

iii) For any $j$ the integral $\int_{\gamma} e^{-\xi/h} |\Phi^{(j)}(\xi)| d\xi$ converges – in fact it is less than $\frac{1}{Q^\ell} \int_{\gamma} |e^{-\xi/h}| d\xi$;

iv) the sum of these integrals is then clearly convergent, too. $\square$

4 Majors bounded in a neighborhood of infinity.

Let $A$ be a small arc in the circle of directions which for simplicity of language will be assumed symmetric with respect to the real axis. Let us study the convolution product of majors that will be assumed holomorphic on some
sectorial neighborhood of infinity in the direction $\hat{A}$. This convolution of majors is known to correspond to multiplication of resurgent functions.

Recall [CNP] that the convolution of two integrality classes of majors $[\Phi]$ and $[\Psi]$ along a path $\Gamma$ is defined by choosing two representatives: $\Phi$ that is exponentially decreasing along the infinite branches of $\Gamma$, and $\Psi$ that is bounded on $\Gamma + \mathbb{R}_{\leq 0}$ (see lemma 4.1 below), and considering the integral

$$(\Phi \ast_\Gamma \Psi)(\xi) = \int_{\Gamma} d\eta \Phi(\eta)\Psi(\xi - \eta)d\eta.$$ 

This integral defines a sectorial germ of an analytic function at infinity and can be analytically continued to some Riemann surface by deforming the integration contour.

It is discussed in [CNP] that the result of the convolution is independent of choices modulo $O(\mathbb{C})$.

Let us look a little closer at the deformation of the contour. Suppose for simplicity that $\Gamma$ consists of two rays starting at $\eta = 0$. Then for $\xi$ to the left of $\Gamma$ the above formula defines an analytic function “on the nose”; this is the case a) on the figure 6.

In order to analytically continue the convolution to other values of $\xi$, to the right of the contour $\Gamma$, we need to continuously deform the convolution contour (cases b), c) of the figure 6) to obtain a contour $\Gamma_\xi$ so that $\Gamma_\xi$ avoids singularities of $S_\Phi$ and $\xi - \Gamma_\xi$ avoids singularities of $S_\Psi$. The singularities of the convolution appear for those $\xi$ for which this deformation becomes impossible.

It is shown in [CNP] that if $\Phi$ and $\Psi$ are defined on endless Riemann surfaces $S_\Phi$ and $S_\Psi$ respectively, then $\Phi \ast \Psi$ can be analytically continued to an endless Riemann surface denoted $S_\Phi \ast S_\Psi$.

Note that if $\xi$ stays within a compact subset $K$ ($K$ is a subset of $S_\Phi \ast S_\Psi$, but we thought it helpful to superimpose it on the Riemann surface $S_\Phi$ for the purpose of the drawing), then the deformation of the contour $\Gamma_\xi$ will be confined to a compact subset $K'$ of $S_\Phi$.

As has been mentioned, the definition of the convolution product involves a choice in an integrality class of a major, $\Psi \mod O(\mathbb{C})$, of a representative that is bounded on sectorial neighborhood of infinity in the direction $\hat{A}$. Let us begin by recalling how this choice is made for a single major, and then prove that this choice can be made compatible with an infinite sum of majors (section 4.2), that will allow us to state when convolution is interchangeable with an infinite sum of majors.
4.1 Case of a single major

Lemma 4.1. (cf. [CNP, Pré I.3.4.2]) Every integrality class $[\Phi]$ in the direction $\hat{A}$ for a small arc $A$ has a representative $\Psi$ so that $\Psi$ is bounded in a (possibly smaller) sectorial neighborhood in the direction $\hat{A}$.

Proof. Let $\Omega$ be a sectorial neighborhood of infinity in the direction $\hat{A}$ and suppose $\Phi$ is defined in $\Omega$. Let $\gamma$ be an infinite path contained in $\Omega$ and adapted to $\hat{A}$ consisting of two rays coming together, for simplicity of notation, at the origin, fig.7. Without loss of generality assume $\Phi$ to be exponentially decreasing along the branches of $\gamma$. 

Figure 6: Deformation of the convolution contour and analytic continuation of the convolution product.
Then

$$\Phi(\xi) = \Psi(\xi) + E(\xi),$$

where

$$\Psi(\xi) = \frac{1}{2\pi i} \int_{\gamma} \frac{\Phi(\eta)}{\xi - \eta} d\eta$$

and where $E \in O(\mathbb{C})$ can be defined as follows:
Given $\xi \in D_R \subset \mathbb{C}$, construct the contour $\gamma_R$ consisting of an arc of radius $R + 1$ and two infinite branches of $\gamma$ as shown on the fig.7 and put

$$E(\xi) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{\Phi(\eta)}{\xi - \eta} d\eta.$$

Finally, notice that $\Psi$ is bounded in any subset of $\Omega$ where $\sup_{\eta \in \gamma} \frac{1}{|\xi - \eta|}$ is bounded. □

![Integration contours in the proof of Lemma 4.1](image)

The following observation about the construction of $\Psi$ and $E$ will be useful in the future:
If $\xi \in D_R$, and if $\Phi$ is bounded by $C \cdot 2^{-|\xi|}$ on $\gamma$, then make the integration contour go along the arc of radius $R + 1$, and by $M_{R+1}$ on $\text{Sec}(0, \hat{A}) \cap D_{R+1}$, get $|E(\xi)| \leq \frac{1}{2\pi} \int_{\text{arc}} |\Phi(\xi)||c\xi| + C_1 2^{-R}$

That means that if $\Phi^{(j)}$ are bounded by $1/2^j m(\xi)$ on the infinite branches of $\gamma$ and converge faster than some geometric series with ratio $q$ on $D_{R+1}$, then on $D_R$ the values of $E(\xi)$ are bounded by some geometric series with ratio $Q$, $\max\{\frac{1}{2}, q\} < Q < 1$.

4.2 Case of a convergent series of majors

Proposition 4.2. Given a series of majors $\Phi^{(j)}$, all of them analytic on $\text{Sec}(0, \hat{A})$, converging uniformly and faster than a geometric series of ratio
$q < 1$ on compact subsets of their common Riemann surface $\mathcal{S}$. Let $\gamma$ be a contour contained in $\text{Sec}(0, \bar{A})$ and adapted to $\bar{A}$, let $\varepsilon > 0$ and the contour $\Gamma = \gamma - \varepsilon$. Then we can choose entire functions $F^{(j)}$ and a number $q < Q < 1$ so that:

i) Outside of $\Gamma$ and on the compact subsets of the Riemann surface $|\Phi^{(j)} - F^{(j)}| < M_K Q^j$.

ii) $\sum_j F^{(j)}$ converges uniformly on compact subsets.

**Proof.** Indeed, begin by choosing $\Phi^{(j)}_1$ such that $\Phi^{(j)}_1(\xi) \leq \frac{m(|\xi|)}{2^j}$ along $\gamma$ with $m(|\xi|) = e^{-|\xi|}$ and such that $\Phi^{(j)} - \Phi^{(j)}_1$ are holomorphic and converge faster than some geometric series with a fixed ratio $q < 1$ on compact subsets.

Choose

$$F^{(j)} := \frac{1}{2\pi i} \int_\gamma \frac{\Phi^{(j)}_1(\eta)}{\xi - \eta} \, d\eta - \Phi^{(j)}_1(\xi).$$

First we need to show that $\sum F^{(j)}$ converges uniformly on compact subsets of $\mathbb{C}$ and hence defines a holomorphic function. We want the series

$$\sum_j \left[ \frac{1}{2\pi i} \int_\gamma \frac{\Phi^{(j)}_1(\eta)}{\xi - \eta} \, d\eta - \Phi^{(j)}_1(\xi) \right]$$

to converge on compact subsets. But each summand will equal the integral along a contour $\gamma_R$ as on the figure [7] for $\xi \in B(0, R - 1)$. The pieces of this integral along the circular arc converge because the arc is compact and $\Phi^{(j)}$ are bounded there by a geometric series. The integrals along the infinite branches converge because the integrands are estimated as $< \frac{1}{2^j} m(|\xi|)$.

Then we need to show that $|\Phi^{(j)}_1 - F^{(j)}| < \tilde{M}_K Q^j$ on a compact subset $K$. (Since, as noted before, a similar inequality holds for $|\Phi^{(j)} - \Phi^{(j)}_1|$, the differences $|\Phi^{(j)} - F^{(j)}|$ will also be estimated by a geometric series of ratio $Q$.) I.e., we need to show that $\int_\gamma \frac{\Phi^{(j)}_1(\eta)}{\xi - \eta} \, d\eta$ with $\gamma$ chosen as in the statement, converges uniformly of compact subsets.

Let $U_-, U_+$ be subsets of $\mathcal{S}$ so that $U_- = \Gamma + \mathbb{R}_{\leq 0}$ (recall that $\Gamma = \gamma - \varepsilon$) and $U_+ = (\mathcal{S} \setminus U_-) \cup \Gamma$. For a compact subset $K \subset \mathcal{S}$ let $K_+ = K \cap U_+$.

Let us show that $\sum_{n=0}^\infty \int_\gamma \frac{\Phi^{(n)}_1(\eta)}{\xi - \eta} \, d\eta$ converges uniformly on compact subsets. Fix $K \subset \mathcal{T}$ and let us check for $\xi \in K$ the inequality $\left| \int_\gamma \frac{\Phi^{(n)}_1(\eta)}{\xi - \eta} \, d\eta \right| < C\alpha^n$. 

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Without loss of generality $\gamma$ consists of two rays coming together at the origin. We know that $|\Phi_1(n)(\xi)| < C_1 \alpha_1 e^{-\beta|\xi|}$ on $\gamma$.

Suppose first that $\xi \in U_-$. When $\xi \in U_-$ and $\eta \in \gamma$, then $\xi - \eta \in U_-$ and so $\frac{1}{\xi - \eta} < C_2$. Then $\int_{\xi - \eta} \Phi_1(n) d\eta < C_3(\alpha_1)^n$ for yet another constant $C_3$.

Suppose now $\xi \in K^+$. In this case $\gamma$ gets deformed to a path $\gamma_\xi$ so that both $\gamma_\xi$ and $\xi - \eta$ are fully contained in $K' \cup U_-$ for some compact set $K'$ and the length of $\gamma_\xi \cap K'$ is $\leq L_K$. Use the following estimates: for $\eta \in K'$ we have $|1/\xi - \eta| < C_4$ (note that $\xi$ is never equal to $\eta$ because $\xi \notin \gamma_\xi$), $|\Phi_1(n)(\eta)| < C_5 \beta_5^n$ (because $\Phi_1(n)$ converges faster than some geometric series on compact subsets, cf. Proposition 3.4 iv), so $\int_{\gamma_\xi \cap K'} \frac{\Phi_1(n)(\eta)}{\xi - \eta} d\eta < L_K C_4 C_5 \beta_5^n$.

For the part of the integral along $\gamma_\xi \setminus K'$ proceed as in the case of $\xi \in U_-$. Note that neither $\alpha_1$ nor $\beta_5$ depend on the compact set $K$ and define $Q = \max\{\alpha_1, \beta_5\}$. □

### 4.3 Interchanging infinite sum and a convolution.

**Proposition 4.3.** Suppose $\Phi^{(j)}$ is a sequence of majors defined on a common Riemann surface containing $\text{Sec}(0, \tilde{A})$ and converging faster than a geometric series on compact subsets. Let $\gamma$ be a contour adapted to this sector and $\Gamma = \gamma - \varepsilon$ for some $\varepsilon > 0$. Suppose also $\sum \Phi^{(j)}$ converges faster than a geometric series on $\Gamma + \mathbb{R}_{\leq 0}$. Then

$$
\Psi *_{\Gamma} \left( \sum_j \Phi^{(j)} \right) = \sum_j \Psi *_{\Gamma} \Phi^{(j)} \quad \text{mod } O(\mathbb{C}).
$$

**Proof.** We are studying the integrals $\int_{\Gamma} \Phi^{(n)}(\xi - \eta) \Psi(\eta) d\eta$. Choose representatives $\Phi_n$ of $\Phi^{(n)}$ that are bounded by geometric series on compact sets and to the left of $\Gamma$. Without loss of generality $\Psi$ exponentially decays along $\Gamma$.

Let us show that with this choice of representatives the equality holds exactly, not just modulo entire functions. By Fubini’s theorem, need to show that $\int, \int \sum, \sum, \int \sum$ are absolutely convergent. Only the convergence of $\sum \int$ is not completely trivial.

In fact, $\sum_{n=0}^{\infty} \int_{\Gamma} \Psi(\eta) \Phi_n(\xi - \eta) d\eta$ converges uniformly on any compact subsets $K$ of the common Riemann surface $\mathcal{T} = S_{\Psi} * S_{\Phi^{(j)}}$ of the summands. Then one can prove the inequality $|\int_{\Gamma} \Psi(\eta) \Phi_n(\xi - \eta) d\eta| < C\alpha^n$ using that
$|\Phi_n(\xi)| < C_1 \alpha_1^n$ on $U_-$ and that $|\Psi(\eta)| < C_2 e^{-\beta|n|}$ for $\eta \in \Gamma$. The argument proceeds along the lines of proposition 4.2, the reader can also consult [G] for details. □

5 Interchanging infinite sums and the reconstruction isomorphism.

In 2.2 we have reminded the correspondence between resurgent symbols and endlessly continuable majors. We will now prove that that correspondence respects infinite sums.

Proposition 5.1. Given an infinite series of microfunctions at 0 whose representatives $\phi_j(\zeta)$ converge uniformly and faster than geometric series with ratio $q$ on compact subsets of their common Riemann surface, then the same is true for the majors $s_+ \phi_j$, and $s_+ \sum_j \phi_j = \sum_j s_+ \phi_j$ (modulo $O(C)$).

Proof. Choose $\Gamma$ as in the construction of $s_+$ and choose representatives $\Phi_j$ of microfunctions $\phi_j$ satisfying $\Phi_j(\zeta) < e^{-|\zeta|}$ along the infinite branches of $\Gamma$ and bounded by some geometric series of ratio $q < 1$ on every compact set of their common Riemann surface.

Then also the integrals $\Psi_j = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi_j(\eta)}{\zeta-\eta} \, d\eta$ are bounded by some geometric series of ratio $Q < 1$ on every compact set of their common Riemann surface. This is shown by an obvious modification of the argument from sections 4.2-4.3, where the integration contour now needs to avoid the singularities of the chosen representative of a microfunction, see figure 8. □

![Figure 8: Integration contour in the proof of Proposition 5.1](image-url)
6 Small resurgent functions.

**Definition.** ([CNP Pré II.4, p.157]) A microfunction \( \varphi \in \mathcal{C}(A) \) is said to be a small microfunction if it has a representative \( \Phi \) such that \( \Phi = o\left(\frac{1}{|z|}\right) \) uniformly in any sectorial neighborhood of direction \( \tilde{A} \) for \( A' \subset A \).

E.g., \( h^\alpha \) for \( \alpha > 0 \) satisfies that property.

The following definition has been somewhat modified compared to ([CNP Rés II.3.2, p.219]).

**Definition.** For a given arc of direction \( A \), a small resurgent function is such a resurgent function that all singularities of its major \( \omega_\alpha \) satisfy \( \text{Re} \, \omega_\alpha > 0 \), except maybe for one \( \omega_0 = 0 \), and if \( \omega_0 = 0 \) then the corresponding microfunction is small in the direction of a large (i.e. \( > 2\pi \)) arc \( B \) with \( \hat{B} \supset A \), see figure 9.

![Figure 9: Arcs in the definition of a small resurgent microfunction.](image)

**Lemma 6.1.** A small resurgent function can be represented by a major that is \( o(1/|\xi|) \) around the origin.

**Proof.** It is enough to check that for a small resurgent function whose decomposition consist of only one small resurgent microfunction \( \phi \) at the origin, i.e. to construct a major \( \Phi(\xi) \) of \( s_{\alpha+\phi} \) satisfying \( \Phi(\xi) = o(1/|\xi|) \) for \( \xi \to 0 \). Here \( \alpha \) is any direction in the complex plane which we will for definiteness take to be the positive real direction.

The major \( \Phi \) will be given as an integral involving the analytic continuation of \( \phi \) along \( \Gamma \), figure 10.a).

\[
\Phi(\xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\eta)}{\xi - \eta} \, d\eta.
\]
Let $\xi$ be close to zero, but let $\Gamma$ be chosen so that $\xi$ is outside it. Then since $\Phi$ is $o(1/|\zeta|)$ around zero, by making the loop around 0 tighter and tighter, can change the integration path to the one shown on figure 10,b).

From this we see that the major is $o(1/|\xi|)$ around the origin. Indeed, if $\xi$ is close to 0, deform the contour of integration as on the figure 10,c), and then $\left| \int_0^1 \frac{\phi(\eta)}{\xi-\eta} d\eta \right| \leq \frac{1}{|\xi|} \cdot \left| \int_0^1 \phi(\eta) d\eta \right|$, and the integral from 1 to $\infty$ is estimated by a constant. □

Using the previous lemma, one easily obtains:

**Lemma 6.2.** Let $\varphi(h)$ be any representative of a small resurgent function in the direction $A$. Then, for any arc $A' \subset A$ and any $\varepsilon > 0$ there is a sectorial neighborhood $U$ of 0 in the direction $A'$ such that $|\varphi(h)| < \varepsilon$, $\forall h \in U$. □

**6.1 Minors**

Following [CNP, Pré II.4], denote $\mathcal{C}(A)$ the subalgebra (with respect to convolution) of small microfunctions in $\mathcal{C}(A)$. (It is indeed a subalgebra with respect to convolution. )

From now on consider microfunctions defined on a big arc $B$. For an arc $A$ denote by $\mathcal{O}^0(A)$ the space sectorial germs of analytic functions at 0 in the direction $A$. We can define a variation \(\text{var} : \mathcal{C}(B) \rightarrow \mathcal{O}^0(\hat{B})\) as follows: take a microfunction $\phi$ at 0, analytically continue it once counterclockwise around zero, obtain a microfunction $\tilde{\phi}$, and put $\text{var} \phi := \phi - \tilde{\phi}$.

We will show that small microfunctions are specified by their variation (or its “minor”).

Denote by $\min\mathcal{O}^0(\hat{B})$ the space of germs of holomorphic functions in a sectorial neighborhood $V$ of direction $\hat{B}$ whose primitive tend to a finite limit when $\zeta \rightarrow 0$. (It follows then that the convergence is uniform in any sectorial neighborhood of strictly smaller direction)
Proposition 6.3. ([CNB, Pré II.4.2.1]) The map $\text{var}$ is an isomorphism $\mathcal{C}(B) \to \text{min} \mathcal{O}^0(\hat{B})$. Denote by bémol its inverse $g \mapsto \flat g$.

**Proof.** Part I.\(^2\) Let us show that $\text{var}$ send $\mathcal{C}(B)$ to $\text{min} \mathcal{O}^0(\hat{B})$. Indeed, let $\varphi$ be represented by a function $\Phi$ holomorphic in a sectorial neighborhood of 0 in direction $B$ and satisfying $\Phi(\xi) = o(1/|\xi|)$ for $\xi \to 0$.

Let $V$ be a sectorial neighborhood of 0 in direction $\hat{B}$, let $\zeta_0 \in V$, let $\gamma$ be a contour starting at $\zeta_0$ and going around 0, as on the figure 11.

\[\text{Figure 11: Contours in the proof of Proposition 6.3}\]

We have
\[
\int_{\gamma} \Phi(\xi) d\xi = \int_{C_\zeta} \Phi(\xi) d\xi - \int_{[\zeta_0, \xi]} (\text{var } \varphi)(\xi) d\xi
\]
where $C_\zeta$ is a small circular contour starting from a point $\zeta$ close to 0.

As $\int_{\gamma} \Phi(\xi) d\xi$ is independent of $\zeta$ and the integral over $C_\zeta$ tends to zero when $\zeta \to 0$, the function $\zeta \mapsto \int_{C_\zeta} \Phi(\xi) d\xi$ is a primitive of $\text{var } \varphi$ that is finite at zero.

**Part II.** Construction of the inverse map “bémol”.

Let $g \in \text{min} \mathcal{O}^0(\hat{B})$ which in particular means that $g$ is integrable at the origin. To construct a representative $\Phi$ of a microfunction $\flat g$ choose in the direction $\hat{B}$ a segment of integration $[0, \eta_1]$\(^3\) and put

\[\Phi(\zeta) = \frac{1}{2\pi i} \int_{[0, \eta_1]} \frac{g(\eta)}{\eta - \zeta} \, d\eta.\]

Let $G$ be the primitive of $g$ that tends to 0 when $\eta \to 0$. Integration by parts gives:

\[\Phi(\zeta) = \frac{1}{2\pi i} \frac{G(\eta_1)}{\eta_1 - \zeta} + \frac{1}{2\pi i} \int_{[0, \eta_1]} \frac{G(\eta)}{(\eta - \zeta)^2} \, d\eta.\]

\(^2\)This part of the proof has been clarified using suggestions of this paper’s anonymous referee.

\(^3\)The point $\eta_1 \neq 0$ is chosen arbitrarily, a different choice will change $\Phi$ by a germ of a function analytic in a neighborhood of 0.
One can show that the integral of the second member converges and that \( \zeta \Phi(\zeta) \to 0 \) when \( \zeta \to 0 \), uniformly in every angular sector not adherent to \([0, \eta_1]\).

The function \( \Phi \) defined in this way on \( \mathbb{C} - [0, \eta_1] \) can be analytically continued across two banks of the cut \([0, \eta_1]\), and one checks using the Cauchy theorem that \( \text{var } \Phi = g \).

It is easy to see from the above formulas that \( \Phi \) is defined Riemann surface obtained from the Riemann surface of \( g \) by adding a branch point on its every sheet over \( \eta_1 \). The function \( \Phi \) is called the adapted major in [CNP], or more to the author’s taste, the adapted representative of our microfunction.

Suppose \([\phi], [\psi]\) are resurgent microfunctions, and their variation has no singularities on the line segment \([0, \eta]\).

Then it is known that for \( t \in (0, \eta) \)

\[
[\text{var } (\phi \ast \psi)](t) = \int_{(0,t)} (\text{var } \phi(\tau))(\text{var } \psi(t - \tau))d\tau
\]

and for \( t \) farther away from 0 we might need to use analytic continuation.

## 7 Substitution of a small resurgent function into a holomorphic function

The goal of this section is to prove the following theorem:

**Theorem 7.1.** If \( g(z_1, \ldots, z_k) = \sum a_{j_1 \ldots j_k}z_1^{j_1} \ldots z_k^{j_k} \) is a complex analytic function given around the origin by a convergent power series, and \( \varphi_1(h), \ldots, \varphi_k(h) \) are small resurgent functions, then the composition \( g(\varphi_1, \ldots, \varphi_k) \) is a resurgent function.

In [CNP] (see also our short sketch in [G]) it is discussed how to construct the major of a composition \( g(\varphi(h)) \), where \( g \) is a single-variable analytic function and \( \varphi(h) \) is a resurgent function representable by a major with a single singularity at the origin. Here we will limit ourselves to document changes that one has to make in the \( k \)-variable case.

The second step is to generalize to the case of small resurgent functions whose majors have singularities in the half-plane \( \text{Re } \xi > 0 \). This case will be reduced to the case of small resurgent functions given by a single singularity in the origin.

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[CNP] say it is defined on the same Riemann surface as \( g \).
7.1 Generalization of \[\text{[CNP]}\]’s construction of a composite function to the case of \(k\) variables.

By the previous section, in order to fully establish Theorem 7.1, we need to prove it in the case of resurgent functions whose majors each have only one singularity at the origin. In that case most of the work has been done in \[\text{[CNP]}\] (in case \(k = 1\)) and it remains to document the necessary changes in the case of a general \(k\).

First let us mention that \[\text{[CNP]}\] work with a weaker definition of an endlessly continuable function. For them the function is endlessly continuable if it can be analytically continued along any path of length \(L\) and angle variation \(\delta\) avoiding a finite set \(\Omega_{L,\delta}\). To our knowledge, resurgent functions in this weaker sense possess all useful properties of resurgent functions in the stronger sense.

Following \[\text{[CNP], Rés II.3}\], for a resurgent function consider the Riemann surface \(S\) of its major and the discrete filtered set \(\Omega_\ast = \{\Omega_L\}\) of its singularities, accessible along paths \(\gamma\) of length \(L\) with fixed starting point \(\gamma(0)\) and tangent vector \(\dot{\gamma}(0)\).

Conversely, by a discrete filtered set \(\Omega_\ast\), \[\text{[CNP]}\] construct a Riemann surface \(S(\Omega)\).

The sum of two discrete filtered set \(\Omega'_\ast\) and \(\Omega''_\ast\) denoted \((\Omega' + \Omega'')_\ast\) is defined as follows:
\[(\Omega' + \Omega'')_L = \{\omega + \omega'\} \text{ where } \omega' \in \Omega'_L, \omega'' \in \Omega''_L \text{ and } L' + L'' = L.
\]
Let \(n\Omega_\ast\) denote the sum of \(n\) copies of \(\Omega_\ast\); let \(\infty \Omega_\ast := \bigcup n\Omega_\ast\).

If two resurgent functions have \(\Omega'_\ast\) and \(\Omega''_\ast\) as filtered discrete sets of their singularities, the singularities of their convolution are included in \((\Omega' + \Omega'')_\ast\), cf. \[\text{[CNP, Rés II.3.1.5]}\].

If \(\flat f\) is a small microfunction, denote by \(f^{(-1)}\) the primitive of \(f\) that vanishes at 0.

Given small resurgent function \(\flat f_1, \ldots, \flat f_n\), the function \(\flat f_1 \ast \ldots \ast \flat f_n =: \flat g\) is also small, and for \(\zeta\) close to 0 we have
\[g^{(-n-1)}(\zeta) = \int f_1^{(-1)}(s_1)f_2^{(-1)}(s_2)\ldots f_n^{(-1)}(s_n)ds_1ds_2\ldots ds_n,\]
where the integral is taken over the \(n\)-simplex defined by
\[\arg s_1 = \ldots = \arg s_n = \arg \zeta, \quad |s_1| + |s_2| + \ldots + |s_n| \leq |\zeta|.
\]
In order to obtain some estimates on the growth of \( g^{(-n-1)} \) it will be shown that it is possible to define the continuation of \( g^{(-n-1)} \) along any allowed path as an integral over an \( n \)-simplex obtained by a deformation of the initial simplex.

Let \( S \) be an endless Riemann surface and \( \Omega_\ast \) the discrete filtered set of its singularities. A sectorial neighborhood of 0 in \( S \) is said to be small if all its points are \( L \)-accessible from 0, with \( L \) so small that \( \Omega_L = \{0\} \); a sectorial neighborhood of 0 is bounded if it is the union of a small neighborhood of 0 and a relatively compact subset of \( S \).

If \( f \) is a minor of a small resurgent function, then its primitive \( f^{(-1)} \) is bounded on any bounded neighborhood of 0.

Let \( S := S(\Omega_\ast) \) be the Riemann surface on which \( f_1, ..., f_k \) are simultaneously defined. Let \( \infty S := S(\infty \Omega_\ast) \).

The following corresponds to [CNP]'s Key lemma 2.

**Lemma 7.2.** There is an exhaustive family of bounded neighborhoods \((V_\alpha, \infty \subset \infty S)\), a family of bounded neighborhoods \((V_\alpha \subset S)\), constants \( C_\alpha > 0 \), and for any minor \( f \) of a small resurgent microfunction which is analytic on \( S \), there is a family of functions \( \epsilon_\alpha \) defined on \( \mathbb{N} \), s.th. \( \epsilon_\alpha(n) > 0 \) and \( \epsilon_\alpha(n) \to 0 \) as \( n \to +\infty \), so that if \( g_{j_1...j_k} \) denotes the minor of \((f_1)^{*j_1} \ast ... \ast (f_1)^{*j_k} \) with \( j_1 + ... + j_k = n \), one has an estimate

\[
|g_{j_1...j_k}^{(-n-1)}|_{V_\alpha, \infty} \leq \frac{1}{n!} |C_\alpha (\max_j |f_j^{(-1)}|_{V_\alpha})^{1/2} \epsilon_\alpha(n)|^n \tag{7.1}
\]

The proof is an easy modification of the one given in [CNP]. Once we have established the counterpart of Key lemma, the rest of the proof goes along the same lines as in [CNP].

### 7.2 Reduction to the case of a small resurgent function with only one singularity of the major

To simplify our notation, we will restrict ourselves to the case \( k = 1 \) in this subsection.

The proposition [5.1] together with lemma [7.2] imply that if \( \Phi \) is an (appropriate) major of a small resurgent function having the origin as its only singularity, than the representatives of \( \Phi^{*j} \mod O \) can be chosen so as to be all defined on the same Riemann surface and converge faster than some geometric series on its compact subsets.
Let \( g(x) = \sum_{j=0}^{\infty} a_j x^j \) be a convergent series representing a holomorphic function \( g(x) \) near the origin, and let \( \varphi(h) \) be a small resurgent function represented by a major \( F(\xi) + R(\xi) \), where \( F(\xi) \) has its only singularity at \( \xi = 0 \) (or no singularities at all) and all singularities of \( R(\xi) \) are contained in the half-plane \( \Re \xi > 0 \).

Now consider the sum
\[
\sum_{j=0}^{\infty} a_j (F + R)^{*j}
\]

and formally rewrite it as
\[
\sum_{j=0}^{\infty} a_j F^{*j} + \left[ \sum_{j=0}^{\infty} j a_j F^{*(j-1)} \right] * R + \left[ \sum_{j=0}^{\infty} C_j^2 a_j F^{*(j-2)} \right] * R^{*2} + ...
\]

The proposition 5.1 together with lemma 7.2 imply that if \( F \) is an (appropriate) major of a small resurgent function having the origin as its only singularity, than the representatives of \( F^{*j} \) mod \( \mathcal{O} \) can be chosen so as to be all defined on the same Riemann surface and converge faster than some geometric series on its compact subsets. With these choices of representatives of \( F^{*j} \) every sum \( \sum C_j^k a_j F^{*(j-k)} \) is convergent on compact sets. If \( R \) has no singularities at a distance \(< L_0 \) form the origin, any function in the integrality class of \( \sum C_j^k a_j F^{*(j-k)} \) * \( R^{*k} \) has no singularities distance \(< kL_0 \) from the origin and therefore this integrality class has a representative bounded by \( 1/k! \) on \( B(0, 1/2kL_0) \). This yields a series of majors that converge uniformly and faster than any geometric series on every compact subset of its Riemann surface, and the sum of this series is our candidate for the major of \( g(\varphi(h)) \).

However, since we have been manipulating with infinite sums and integral signs formally, we now need to take that major, calculate its Laplace integral and show that the resulting function is equal to the sum of the power series \( \sum a_j \varphi(h)^j \), i.e. we have to prove the following equality of resurgent functions
\[
\sum_j a_j (\mathcal{L}[F + R])^j = \mathcal{L} \left[ \text{ML}\Sigma_k \left( \sum_j C_j^k a_j F^{*(j-k)} \right) * R^{*k} \right],
\]

where \( \text{ML}\Sigma \) stands for the Mittag-Leffler sum of majors as introduced in section 2.2.3.
1. **LEFT-HAND SIDE** The left-hand side is the sum of the double series
\[ \sum_{j,k} a_j C_j^k \left( \int_{\gamma} d\xi e^{-\xi/h} F(\xi) \right)^k \left( \int_{\gamma} d\xi e^{-\xi/h} R(\xi) \right)^{j-k} \]

By Weierstrass’ argument (cf, e.g., [Hi, p.22]), this is a doubly convergent series since both integrals tend to 0 when \( h \to 0^+ \).

2. **RIGHT-HAND SIDE**: The right-hand side gives the same double sum. Indeed,
\[
\mathcal{L} \left[ M \sum_k \left( \sum_j C_j^k a_j F^*(j-k) \right) * R^k \right] \overset{(1)}{=} \sum_k \mathcal{L} \left[ \left( \sum_j C_j^k a_j F^*(j-k) \right) * R^k \right] \overset{(2)}{=} \sum_k \mathcal{L} \left[ \sum_j C_j^k a_j F^*(j-k) \right] \cdot (\mathcal{L} R)^k \overset{(3)}{=} \sum_k \sum_j C_j^k a_j \mathcal{L} [F^*(j-k)] \cdot (\mathcal{L} R)^k
\]

The steps (1) and (3) are justified by proposition 3.5 and there is no issue in (2).

This argument can be easily generalized to the case of \( k \) variables.

### 7.3 Parameter-dependent version.

The following is an easy parameter-dependent version of theorem 7.1.

**Theorem 7.3.** Suppose \( r(E, h) \) is a small resurgent function such that its major \( R(E, \xi) \) is defined on one and the same Riemann surface and analytically dependent on \( E \) for \( E \in U \) a neighborhood of 0. Suppose \( f \in \mathcal{O}(D_\rho) \). Then we can choose a major \( \Phi(E, \xi) \) of \( f(r(E, h)) \) for which the same is true.

**Proof** is analogous to the one given for the parameter-independent case. The only modification is that in 7.1 one has to replace \( |f_j^{-1}|_{V_{\alpha}} \) by \( |f_j^{-1}|_{V_{\alpha} \times D_{\rho'}} \) for any \( \rho' < \rho \). \( \square \)

Here is an application of this theorem. A major of a solution of an \( h \)-differential equation \( P\psi = E_r h\psi \) can be chosen to analytically depend on \( E_r \), and the same is therefore true for microfunctions – formal solutions of the above equation. We can apply the above theorem to conclude that quotients of such microfunctions, in particular, formal monodromies can be
have represented by majors that holomorphically depend on holomorphically on $E_r$, and the same is then also true for the formal monodromy exponents. So, we have:

**Corollary 7.4.** Formal monodromy exponents for an $h$-differential equation can be represented by majors holomorphically dependent on $E_r$.

### 8 Substitution of a small resurgent function for a holomorphic parameter of another resurgent function.

**Proposition 8.1.** Suppose $\varphi(E, h)$ is a resurgent function for every $E \in \mathbb{C}$ satisfying the following properties:

i) For all $E \in \mathbb{C}$ the majors $\Phi(E, \xi)$ are defined on the same Riemann surface $S$;

ii) The Taylor series of $\Phi(E, \xi)$ converges absolutely and uniformly on compact subsets of $S$;

Then for every small resurgent function $E(h)$ the function $\varphi(E(h), h)$ is resurgent.

**Proof.** Construct the Riemann surface $T$ for the major of $\varphi(E(h), h)$ as $S \ast \infty S_E$ where $S_E$ is the Riemann surface of $\tilde{E}(\xi)$.

Let us construct the major for $\varphi(E(h), h)$. Let $\tilde{E}(\xi) = G + R$ where $G$ is represented by only one small resurgent microfunction $g$ and $R$ has all its singularities to the right of the imaginary axis.

Let us construct the major as follows:

a) choose representatives of $\Phi(E, \xi)$ that is exponentially decreasing along $\gamma$ with a uniform estimate on some compact set of $\mathbb{C}_E$, then $\frac{1}{n!} \frac{\partial^n \Phi(E, \xi)}{\partial E^n}$ will be bounded by a geometric series. Make the ratio of the series less than $1/2$.

So suppose $\left| \frac{1}{n!} \frac{\partial^n \Phi(E, \xi)}{\partial E^n} \right| < C2^{-n-\ell}$ on $D_\ell \cap \gamma$.

The path $\Gamma$ splits the Riemann surface $T = S \ast \infty S_E$ into two parts: $U_- = \Gamma + \mathbb{R}_{\leq 0}$, $U_+ = S \setminus \Gamma$. Without loss of generality $U_- \cap U_+ = \Gamma$. For a compact set $K \subset T$ denote $K \cap U_{\pm} = K_{\pm}$.

b) Choose representatives $G_n$ of $G^{*\infty}$ so that they will be bounded “outside of $\Gamma$” by a geometric series, i.e. $|G_n| \leq C2^{-n}$ in $U_-$. Indeed, on compact sets $G^{*\infty}$ goes to zero faster than a geometric series, hence we can choose $G_{1,n}$.
exponentially decreasing along $\gamma$, and then take $G_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{G_{1,n}(\eta) d\eta}{\xi - \eta}$ and use estimates from section 4.2. The contour $\Gamma$ will be a little farther to the left from $\gamma$.

We are given that $\sup_{\xi \in K} |G_n|$ decreases faster than any geometric series as $n \to \infty$ and faster than an exponential function as $\xi \to \infty$ along $\Gamma$. Further, $\sup_{\xi \in K_{+} \cup U_{-}} |\Phi_n| < C K_{n}$.

c) Let us show that $\sum_{n=0}^{\infty} C_n \int_{\Gamma} G_n(\xi - \eta) \Phi_n(\eta) d\eta$ converges uniformly on compact subsets. Fix $K \subset T$ and let us check for $\xi \in K$ the inequality $|\int_{\Gamma} G_n(\xi - \eta) \Phi_n(\eta) d\eta| < C \alpha^n$.

Assuming, without loss of generality, that $\Gamma$ consists of two rays coming together at the origin, we can imitate an argument from the proof of Proposition 4.2; see (G) for more details.

d) Now we need to show that this major is really the major of the composite function, i.e. we need to verify the equality

$$\mathcal{L} \left[ ML \sum_{j=0}^{\infty} R^{j} \ast \left( \sum_{n=0}^{\infty} C_n \int_{\Gamma} G_n(\xi - \eta) * \Phi_n(\eta) d\eta \right) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \phi}{\partial E^n} [E(h)]^n.$$

In the first sum $ML \sum$ and $\mathcal{L}$ can be interchanged and we get

$$\sum_{j=0}^{\infty} \mathcal{L}[R]^j \cdot \mathcal{L} \left[ \sum_{n=0}^{\infty} C_n^{j} \int_{\Gamma} G_n(\xi - \eta) * \Phi_n(\eta) d\eta \right].$$

Since the majors inside the sum converge uniformly and faster than geometric series with some fixed ratio on compact sets, $\mathcal{L}$ and the infinite sum can be interchanged, after which the proposition is proven. \(\square\)

9 Application to the existence theorem.

Consider the Schrödinger operator

$$P = -h^2 \partial_q^2 + V(q, h)$$

where $V(q, h)$ is analytic in $q$ on the whole complex plane and polynomial in $h$. Denote by $\tilde{P}$ the Laplace-transformed operator, i.e.,

$$\tilde{P} = -\partial_{\xi}^2 \partial_{q}^2 + V(q, \partial_{\xi}^{-1}).$$
Suppose we know that $\psi$ is a resurgent solution of the differential equation

$$P\psi(E_1, q) = hE_1\psi(E_1, q) \quad \text{mod } \mathcal{E}^{-\infty}$$

for every $E_1 \in \mathbb{C}$ a number, obtained by Shatalov-Sternin method. I.e., the majors $\Psi(E_1, \xi, q)$ are holomorphic with respect to $E_1$, are defined on the same Riemann surface and satisfy

$$\tilde{P}\Psi(E_1, q, \xi) = \hat{h} * E_1\Psi(E_1, q, \xi) \quad \text{mod } \mathcal{O}(U_1 \times U_2 \times \mathbb{C}),$$

where $U_1$ is an open neighborhood of 0 in $\mathbb{C}$ and $q \in U_2$, an open subset of the universal cover $\tilde{\mathbb{C}}$ of $\mathbb{C}\backslash\{\text{turning pts}\}$. (Turning points are those $q$ for which $V(q, h) = O(h)$.)

Differentiating both sides of this equality with respect to $E_1$ for $E_1 = 0$, obtain

$$\tilde{P} \frac{\partial^n \Psi}{\partial E_1^n} \bigg|_{E_1=0} = \hat{h} * n \frac{\partial^{n-1} \psi}{\partial E_1^{n-1}} \bigg|_{E_1=0} \quad \text{mod } \mathcal{O}(U_1 \times U_2 \times \mathbb{C}).$$

As $\Psi(E_1, q, \xi) = \sum E_1^n \frac{1}{n!} \frac{\partial^n \Psi}{\partial E_1^n} \bigg|_{E_1=0} \quad \text{converges on compact sets of the Riemann surface of } \Psi \text{ and holomorphic with respect to } E_1$, we see that $\frac{1}{n!} \frac{\partial^n \Psi}{\partial E_1^n}$ can be estimated on compact sets of that Riemann surface by geometric series with the same ratio. Hence, by the previous section, we may substitute a small resurgent function $E(h)$ for $E_1$ and obtain a resurgent function.

We know that $P\psi(E_1, h) = hE_1\psi(E_1, h) \quad \text{mod } \mathcal{E}^{-\infty}$ for $E_1 \in \mathbb{C}$. If they were equal as functions, not as classes modulo $\mathcal{E}^{-\infty}$, then there would be no issue substituting $E(h)$ into this equality. As it is, we need an additional argument in order to show:

$$P\psi(E(h), q) = hE(h)\psi(E(h), q) \quad \text{mod } \mathcal{E}^{-\infty}.$$

(Here $\mathcal{E}^{-\infty}$ stands for functions of $h$ that are $< C_{a,K}e^{-a/h}$ for small $|h|$ for $(E, q)$ ranging over some compact subset of $\mathbb{C} \times \tilde{\mathbb{C}}$.)

On the level of majors it boils down to showing (here $\hat{h}$ is a major of $h$)

$$\tilde{P} \sum \tilde{E}^n * \frac{1}{n!} \frac{\partial^n \Psi}{\partial E_1^n} \bigg|_{E_1=0} = \hat{h} * \tilde{E} * \sum \tilde{E}^n * \frac{1}{n!} \frac{\partial^n \Psi}{\partial E_1^n} \bigg|_{E_1=0} \quad \text{mod } \mathcal{O}(U \times \mathbb{C}).$$

We can interchange $\tilde{P}$ and the infinite sum on the left. Indeed, for the convolution with $\hat{h}^2$ and $V(x, \hat{h})$ this has been shown in the section about
interchangeability of an infinite sum and a convolution. Infinite sum and $\frac{\partial}{\partial x}$ can be interchanged because on compacts with respect to $x$ the terms of the series can be assumed to be uniformly bounded by $\frac{1}{\mu(|\xi|)}$ along the infinite branches of $\Gamma$, and since everything is analytic in $x$, so are the $x$-derivatives of majors.

We can interchange $\hat{E}*$ and the infinite sum on the right. Indeed, it is shown in the proof of proposition 8.1 that majors of $\hat{E}^{*n} \ast \frac{1}{n!} \frac{\partial^n \Psi}{\partial E_1^n}$ have representative converging faster than a geometric series on compact subsets. Therefore, by section 4.3, we can interchange $\hat{E}*$ and the infinite sum.

So, to show (9.1), since interchanging $\hat{P}$ and $\hat{E}*$ with the infinite sums in question is legal, it is enough to show that
\[
\sum \hat{E}^{*n} \ast \frac{1}{n!} \hat{P} \frac{\partial^n \Psi}{\partial E_1^n} = \sum \hat{h} \ast \hat{E}^{*(n+1)} \ast \frac{1}{n!} \frac{\partial^n \Psi}{\partial E_1^n} \mod \mathcal{O}(U \times \mathbb{C})
\]
which follows because
\[
\hat{P} \frac{\partial^n \Psi}{\partial E_1^n} = n \hat{h} \ast \frac{\partial^{n-1} \Psi}{\partial E_1^{n-1}} \mod \mathcal{O}(U \times \mathbb{C}).
\]

□

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