Abstract. We extend the Garkusha-Panin and Voevodsky strict $\mathbb{A}^1$-invariance theorems in motivic homotopy theory to one-dimensional base schemes with perfect residue fields. In the proof, we utilize the trivial fiber topology and localization for motivic homotopy categories relative to closed immersions. This approach allows us to reduce strict $\mathbb{A}^1$-invariance to the case of fields. Furthermore, we compare motivic and Nisnevich local fibrant replacement functors and study $\mathbb{G}_m$-loops.

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2010 Mathematics Subject Classification. 14F05, 14F35, 14F42, 19E15.

Key words and phrases. Motivic homotopy theory, tf-topology, strict $\mathbb{A}^1$-invariance, framed correspondences.
1. Introduction

1.1. Strictly $A^1$-invariant sheaves. Voevodsky’s notion of strictly $A^1$-invariant Nisnevich sheaves has been central in the development of motives and motivic homotopy theory [25]. Recall that every field $k$ has an associated linear category of finite correspondences $\text{Cor}_k$ in the sense of Friedlander-Suslin-Voevodsky [37]. A presheaf with transfers, i.e., an additive functor

$$\mathcal{F} : \text{Cor}_{k}^{\text{op}} \to \text{Ab}$$

is called $A^1$-invariant if, for any smooth separated $k$-scheme $X$, the projection $X \times A^1 \to X$ from the affine line induces an isomorphism of abelian groups

$$\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X \times A^1).$$

Moreover, $\mathcal{F}$ is called Nisnevich strictly $A^1$-invariant if, for all $i \geq 0$, the cohomology presheaf

$$X \mapsto H_i^{\text{Nis}}(X, \mathcal{F}_{\text{Nis}})$$

is an $A^1$-invariant presheaf with transfers. The same definitions apply verbatim to presheaves of abelian groups on the category $\text{Sm}_k$ of smooth separated $k$-schemes of finite type.

In this paper, we establish a generalization of strict $A^1$-invariance to base schemes such as the integers. First, we motivate our interest in this topic by recalling the fundamental results.

1.1.1. Voevodsky’s strict $A^1$-invariance theorem for presheaves with transfers over fields. A key technical input in Voevodsky’s construction of motives is his result on strict $A^1$-invariance of additive presheaves with transfers in [35, §3.2].

**Theorem 1.1.** If $k$ is a perfect field then every $A^1$-invariant presheaf with transfers

$$\mathcal{F} : \text{Cor}_{k}^{\text{op}} \to \text{Ab}$$

is Nisnevich strictly $A^1$-invariant.

Let $\Delta^*_k$ denote the standard cosimplicial $k$-scheme. A notable consequence of Theorem 1.1 says the associated Suslin complex

$$C^A_\ast(\mathcal{F}_{\text{Nis}}) := \mathcal{F}_{\text{Nis}}(\Delta^*_k \times -) = (\cdots \to \mathcal{F}_{\text{Nis}}(\Delta^1_k \times -) \to \mathcal{F}_{\text{Nis}}(\Delta^2_k \times -) \to \mathcal{F}_{\text{Nis}} \to 0)$$

of $\mathcal{F}_{\text{Nis}}$ is $A^1$-local, and hence an object in Voevodsky’s category of effective motives $\text{DM}^{\text{eff}}(k)$. More generally, the endofunctor on the derived category of bounded above chain complexes of Nisnevich sheaves with transfers given by the totalization

$$\mathcal{F}^\ast \mapsto \text{Tot}(C^A_\ast(\mathcal{F}^\ast)) \quad (1.1)$$

computes the $A^1$-localization functor with values in $\text{DM}^{\text{eff}}(k)$, see also [4, §4.2].

**Remark 1.2.** Theorem 1.1 holds for every field after inverting the exponential characteristic, see Suslin [31].

1.1.2. Morel’s homotopy $t$-structure on the motivic stable homotopy category of fields. Let $\text{SH}_s(k)$ and $\text{SH}(k)$ denote the $S^1$-stable and stable motivic homotopy categories of $k$, respectively, see [29]. The heart $\text{SH}_s(k)^{\text{st}}$ of the homotopy $t$-structure consists of strictly $A^1$-invariant Nisnevich sheaves of abelian groups on $\text{Sm}_k$, and the heart $\text{SH}(k)^{\text{st}}$ is equivalent to the category of homotopy modules, see [26, Theorem 5.2.6]. Morel’s result can be expressed in terms of framed homotopy modules, see Ananyevskiy-Neshitov [1], and in terms of Milnor-Witt cycle modules, see Feld [15].
1.1.3. **Garkusha-Panin’s strict $A^1$-invariance theorem for framed presheaves over fields.** Voevodsky introduced framed correspondences in [36] to establish a more computational friendly approach to $\text{SH}(k)$. An appealing feature is that framed correspondences encode transfer maps for cohomology theories represented by motivic spectra, see [2, Lemma 7.9] and [12]. Framed presheaves restrict to presheaves with transfers because framed correspondences form in a precise way the universal correspondence category. In [17], Garkusha-Panin proves a strict $A^1$-invariance theorem for $A^1$-invariant quasi-stable additive framed presheaves over perfect fields, see also Definition 2.1. The said result is fundamental for their construction of framed motives, which in turn provides explicit geometric models for fibrant replacements of motivic spectra [16]. Recall from [16, §2] the category of framed correspondences $\text{Fr}_+(k)$ with objects smooth finite type $k$-schemes, and morphisms pointed sets $\bigcup_n \text{Fr}_n(X, Y)$ of all level $n$ framed correspondences.

**Theorem 1.3** ([17, §17]). If $k$ is a perfect field then every $A^1$-invariant quasi-stable additive framed presheaf $F: \text{Fr}_+(k)^{op} \to \text{Ab}$ is Nisnevich strictly $A^1$-invariant.

**Remark 1.4.** For the proof of Theorem 1.3 see [17, §17] for infinite fields of odd characteristic and fields of even characteristic (assuming $\mathbb{Z}[1/2]$-coefficients). See [10] for fields of characteristic two. The case of finite fields is established independently in [8] and [13, Appendix B]. We also provide a short proof in Theorem A.27.

Recall from [16, Definition 5.2 (3)] that the framed motive of $X \in \text{Sm}_k$ is the $S^1$-spectrum of motivic spaces with framed transfers given by

$$\mathcal{M}_{fr}(X) := (\text{Fr}(\Delta^1_k \times - , X), \text{Fr}(\Delta^1_k \times - , X \otimes S^1), \ldots, \text{Fr}(\Delta^i_k \times - , X \otimes S^i), \ldots).$$

Strict $A^1$-invariance and additivity for framed presheaves [16, Theorem 6.4] are used in the proof of [16, Corollary 7.6] to show the levelwise Nisnevich fibrant replacement $\mathcal{M}_{fr}(X)_{\text{Nis}}$ is a motivic fibrant $\Omega$-spectrum in positive degrees. We refer to [13] for an $\infty$-categorical discussion of framed motives. In that setting, the motivic localization of $\mathcal{M}_{fr}(X)$ coincides with the group completion of $\mathcal{M}_{fr}(X)_{\text{Nis}}$.

1.2. **What is done in this paper?** We generalize the strict $A^1$-invariance theorems due to Voevodsky and Garkusha-Panin to one-dimensional base schemes with perfect residue fields. We note that discrete valuation rings with perfect residue fields, the integers $\mathbb{Z}$, and, more generally, rings of integers in number fields, are examples of such base schemes. Any scheme of finite type over $\mathbb{Z}$, or a localization of the integers, with perfect residue fields has dimension at most one. We refer to Theorems 1.6, 2.3 and 11.2 for precise statements of our main results.

1.3. **The tf-topology.** It is known that strict $A^1$-invariance in the sense of Theorem 1.1 fails for every positive dimensional scheme, see [4, Remark 4.9] and Example 13.2 for details. To remedy the situation we introduce the tf-topology, where tf is shorthand for “trivial fiber.”

Let $B$ be a finite dimensional noetherian base scheme and let $\text{Sm}_B$ denote smooth separated $B$-schemes of finite type. A diagram in $\text{Sch}_B$ of the form

$$\begin{array}{ccc}
X' \times_B (B - Z) & \longrightarrow & X' \\
\downarrow & \phi \\
X \times_B (B - Z) & \longrightarrow & X
\end{array}$$

(1.2)
is called a tf-square if it is a Nisnevich square in the sense of \[27, \text{Definition 1.3, p. 96}\], \(\varphi\) is affine, and the closed immersion \(Z \hookrightarrow B\) induces an isomorphism of \(B\)-schemes

\[X' \times_B Z \cong X \times_B Z.\]

The tf-squares form a cd-structure on \(\mathbf{Sm}_B\) in the sense of \[33, \text{Definition 2.1}\]. The associated tf-topology on \(\mathbf{Sm}_B\) is the Grothendieck topology generated by tf-squares as in \[12\] and the empty sieve covering \(\emptyset\). The tf-topology on \(\mathbf{Sm}_B\) is intermediate between the trivial topology and the Nisnevich topology. Over a field, the tf-topology coincides with the trivial topology, see Proposition \[37, \text{(iii)}\]. On the other hand, every Nisnevich covering of \(B\) is also a tf-covering. Thus the tf-topology is finer than the Nisnevich base covering topology \(\text{Nis}_B\), where the coverings are given by pullbacks of Nisnevich coverings of \(B\) along some structure morphism in \(\mathbf{Sm}_B\). While the tf- and \(\text{Nis}_B\)-topologies are different, they have common features, e.g., one can show the corresponding cohomological dimensions agree for all \(X \in \mathbf{Sm}_B\). For our purposes, the relevance of these topologies will become clear in the context of homotopical descent.

### 1.4. Main results.

If \(\tau\) is a topology on \(\mathbf{Sm}_B\), a presheaf of abelian groups \(F\) on \(\mathbf{Fr}_+(B)\) is \(\tau\)-strictly \(A^1\)-invariant if the cohomology presheaf \(H^i_\tau(\cdot, F)\) is \(A^1\)-invariant for all \(i \geq 0\). We refer to Appendix \[\mathbf{A}\] for a discussion of framed correspondences \(\mathbf{Fr}_+(B)\) over \(B\) and quasi-stable radditive framed presheaves. Over fields this specializes to \[16, \text{Definition 2.3}\].

**Definition 1.5.** A base scheme \(B\) satisfies tf-Nisnevich strict \(A^1\)-invariance for abelian groups if every tf-strictly \(A^1\)-invariant quasi-stable radditive framed presheaf

\[F: \mathbf{Fr}_+(B)^{\text{op}} \to \mathbf{Ab}\]

is Nisnevich strictly \(A^1\)-invariant.

We show the following permanence result for strict \(A^1\)-invariance:

**Theorem 1.6.** Suppose the residue fields of the base scheme \(B\) satisfy strict \(A^1\)-invariance for abelian groups. Then \(B\) satisfies tf-Nisnevich strict \(A^1\)-invariance for abelian groups.

As noted in Section \[1.3\] the tf-topology of any field is trivial. Hence for fields, the notion of tf-Nisnevich strict \(A^1\)-invariance for abelian groups agrees with the notion of strict \(A^1\)-invariance for abelian groups. By combining Theorems \[1.3\] and \[1.6\] we deduce:

**Corollary 1.7.** Every base scheme with perfect residue fields satisfies tf-Nisnevich strict \(A^1\)-invariance for abelian groups.

**Remark 1.8.** If \(B\) is a one-dimensional scheme, then for any presheaf of abelian groups \(F\) on \(\mathbf{Sm}_B\) and \(X \in \mathbf{Sm}_B\), there is similarly to Lemma \[12.3\] an isomorphism

\[H^i_\text{tf}(X, F) \cong \begin{cases} 
\ker(d) & i = 0 \\
\coker(d) & i = 1 \\
0 & i \geq 2,
\end{cases} \quad (1.3)\]

where \(d\) is the naturally induced restriction map

\[d: \bigoplus_{z \in B^{(1)}} F(X^h_z) \oplus \bigoplus_{\eta \in B^{(0)}} F(X_\eta) \to \bigoplus_{z \in B^{(1)}, \eta \in B^{(0)}} F((X^h_z)_\eta).\]

Here \(B^{(p)}\) denotes the set of codimension \(p\) points of \(B\), \(X^h_z\) denotes the henselization of \(X\) along the closed subscheme \(X_z = X \times_B z\); \(X_\eta = X \times_B \eta\), and \((X^h_z)_\eta = (X^h_z) \times_B \eta\). Suppose \(F\) is a quasi-stable radditive framed abelian presheaf on \(\mathbf{Sm}_B\) and \(B\) has perfect residue fields. With these assumptions, Corollary \[1.7\] is equivalent to saying that if the presheaves in \[1.3\] are \(A^1\)-invariant, then the Nisnevich sheafification of \(F\) is Nisnevich strictly \(A^1\)-invariant.
1.5. **Scholium.** To elucidate the role of the tf-topology, we note that a base scheme \( B \) satisfies \( \text{tf-Nisnevich} \) strict \( \mathbb{A}^1 \)-invariance for abelian groups if and only if for every \( X \in \text{Sm}_B, \ x \in X, \) and \( \mathcal{F} : \text{Fr}_+(B)^{op} \rightarrow \text{Ab} \) as in Definition 1.5, there is similarly to Corollary 8.14 an isomorphism

\[
H^i_{\text{Nis}}(X^h_x \times \mathbb{A}^1, \mathcal{F}_{\text{Nis}}) \cong \begin{cases} \mathcal{F}(X^h_x) & i = 0 \\ 0 & i > 0. \end{cases}
\]  

Here \( X^h_x \) denotes the henselization of the local scheme \( X \) at \( x \in X \). By the same reasoning, the assumption that \( \mathcal{F} \) is \( \text{tf-strictly} \) \( \mathbb{A}^1 \)-invariant implies the isomorphism

\[
H^i_{\text{tf}}(X^h_x \times \mathbb{A}^1, \mathcal{F}_{\text{tf}}) \cong \begin{cases} \mathcal{F}(X^h_x) & i = 0 \\ 0 & i > 0. \end{cases}
\]

The formula (1.5) holds trivially over fields for any \( \mathbb{A}^1 \)-invariant presheaf \( \mathcal{F} \). To deduce (1.4) we consider the Nisnevich topology on the small \( \acute{e}tale \) site \( \text{\acute{e}t}_{\mathbb{A}^1} \times X^h_x \) of the affine line \( \mathbb{A}^1_{X^h_x} \) for all \( x \in X \). The latter topology is generated by the Nisnevich squares

\[
\begin{array}{ccc}
W \times_V V' & \longrightarrow & V' \\
\downarrow & & \downarrow \\
W & \longrightarrow & V
\end{array}
\]

with a corresponding contractible chain complex of presheaves of abelian groups

\[
\mathbb{Z}F(-, W \times_V V') \rightarrow \mathbb{Z}F(-, W) \oplus \mathbb{Z}F(-, V') \rightarrow \mathbb{Z}F(-, V).
\]

Here \( \mathbb{Z}F(-, V) \) is the quotient of the presheaf of free abelian groups \( \mathbb{Z}F(-, V) \) by \( \mathbb{A}^1 \)-homotopy, additivity, and quasi-stabilization, see Definition A.13. We refer to (1.6) as an \( \left( \mathbb{A}^1, \mathbb{Z}F \right) \)-contractible Nisnevich square. For every positive dimensional base scheme \( B \) there exists a Nisnevich square

\[
\begin{array}{ccc}
\mathbb{A}^1_{U - Z(f)} - Z(r) & \longrightarrow & \mathbb{A}^1_{U} - Z(r) \\
\downarrow & & \downarrow \\
\mathbb{A}^1_{U - Z(f)} & \longrightarrow & \mathbb{A}^1_U
\end{array}
\]

Here \( U \subset B \) is an open affine subscheme, \( f \in \mathcal{O}(U) \) is any regular function with a nonempty vanishing locus \( Z(f) \), and \( r = ft - 1 \in \mathcal{O}(\mathbb{A}^1_U) \). In Section 13 we show that (1.7) is an example of a non-(\( \mathbb{A}^1, \mathbb{Z}F \))-contractible Nisnevich square. Hence the Nisnevich topology on \( \mathbb{A}^1_{X^h_x} \) cannot be generated by (\( \mathbb{A}^1, \mathbb{Z}F \))-contractible Nisnevich squares. The tf-squares allow us to fix this problem when \( B \) has perfect residue fields. This is decisive for showing that (1.5) implies (1.4). In order to make this precise we use localization techniques, see Remark 11.5 for a more detailed discussion.

**Remark 1.9.** In the proof of Theorem 1.6, we show that \( \mathcal{F}_{\text{Nis}} \) and its Nisnevich cohomology groups are quasi-stable additive framed presheaves. This does not require any assumptions on \( B \).

Section 2 gives a more precise account of the results in the paper.

1.6. **Acknowledgments.** We acknowledge the support of the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo, Norway, which funded and hosted our research project “Motivic Geometry” during the 2020/21 academic year, and the RCN Frontier Research Group Project no. 250399 “Motivic Hopf Equations.” Druzhinin was supported by a Young Russian Mathematics award. Østvær gratefully acknowledges support from the Radboud Excellence Initiative.
2. **Strict $\mathbb{A}^1$-invariance for framed presheaves of $S^1$-spectra**

In this section we give a more detailed outline of the paper and discuss applications. The main result Theorem 2.3 is an enhanced version of Theorem 1.6 for framed presheaves of $S^1$-spectra.

We refer to Bousfield-Friedlander [6] for $S^1$-spectra $\mathbf{Spt}$, and to Hovey [20], Jardine [23], [24] for the category $\mathbf{Spt}_s(B)$ of presheaves of $S^1$-spectra on $\text{Sm}_B$. We write equivalence for a stable weak equivalence in $\mathbf{Spt}$, and $\Omega_\ast$-spectra for the fibrant objects in the stable model structure on $\mathbf{Spt}$. For a topology $\tau$ on $\text{Sm}_B$ we have the following model structures on $\mathbf{Spt}_s(B)$:

1. The levelwise (resp. stable) $\tau$-local model structure $\mathbf{Spt}_{s,\tau}(B)$ (resp. $\mathbf{Spt}_{st,\tau}(B)$).

2. The levelwise (resp. stable) motivic $\tau$-local model structure $\mathbf{Spt}^{\mathbb{A}^1}_{s,\tau}(B)$ (resp. $\mathbf{Spt}^{\mathbb{A}^1}_{st,\tau}(B)$).

For a quick introduction we refer to [11, §2]. Let $\mathbf{SH}_{s,\tau}(B)$ and $\mathbf{SH}^{\mathbb{A}^1}_{s,\tau}(B)$ denote the corresponding stable homotopy categories. The fibrant objects in (1) resp. (2) are the levelwise fibrant presheaves of $S^1$-spectra that are $\tau$-local resp. $\tau$-local and $\mathbb{A}^1$-local, see [28, Lemma 20] for a characterization of the stable fibrations. By [20, Definition 3.1, Theorem 3.4], the fibrant objects in the stable projective model structure on $\mathbf{Spt}_s(B)$ are the levelwise fibrant presheaves of $\Omega_\ast$-spectra. If $\tau$ is a completely decomposable topology, an object in $\mathbf{SH}_{s,\tau}(B)$ is $\tau$-local if it satisfies (stable) excision with respect to $\tau$-squares in the sense of [23, Corollary 1.4], [24, Theorem 5.39]. We say $F \in \mathbf{SH}_s(B)$ is $\mathbb{A}^1$-local with respect to $\tau$ if, for all $X \in \text{Sm}_B$, the canonical morphism $L_\tau F(X) \to L_\tau F(X \times \mathbb{A}^1)$ is an equivalence. Here $L_\tau$ is the $\tau$-localization endofunctor on $\mathbf{SH}_s(B)$. In this terminology, $\mathbb{A}^1$-local means $\mathbb{A}^1$-local with respect to the trivial topology.

Since the Nisnevich topology is finer than the tf-topology, the discussion in [14, §2] shows that there exists a derived tf-Nisnevich localization functor

$$L^{tf}_{Nis} : \mathbf{SH}_{s,tf}(B) \to \mathbf{SH}_{s,Nis}(B).$$

(2.1)

Thus for every $\mathcal{E} \in \mathbf{SH}_{s,tf}(B)$ there is a canonical morphism $\mathcal{E} \to L^{tf}_{Nis} \mathcal{E}$ that witnesses $L^{tf}_{Nis} \mathcal{E}$ as the initial Nisnevich-local presheaf of $S^1$-spectra receiving a morphism from $\mathcal{E}$.

**Definition 2.1.** A framed presheaf of $S^1$-spectra

$$F : \text{Fr}_+(B)^{\text{op}} \to \mathbf{Spt}$$

is quasi-stable if, for all $X \in \text{Sm}_B$, the level one framed correspondence $\sigma_X$ in Example A.3 induces an equivalence

$$\sigma_X : F(X) \overset{\sim}{\to} F(X).$$

Moreover, $F$ is additive if for all $X, Y \in \text{Sm}_B$ there is a naturally induced equivalence

$$F(X \amalg Y) \overset{\sim}{\to} F(X) \times F(Y).$$

A presheaf of framed $S^1$-spectra can be viewed as an object of $\mathbf{Spt}_s(B)$ via the functor from $\text{Sm}_B$ to $\text{Fr}_+(B)$. Here we use that every morphism of schemes defines a framed correspondence.

**Definition 2.2.** A base scheme $B$ satisfies strict $\mathbb{A}^1$-invariance for framed presheaves of $S^1$-spectra if the tf-Nisnevich localization functor $L^{tf}_{Nis}$ in (2.1) preserves $\mathbb{A}^1$-local quasi-stable additive framed objects in $\mathbf{SH}_{s,tf}(B)$.

We are ready to state the main result in this paper. Although we show a slightly stronger result, see Theorem 11.2 and Corollary 11.4 it seems reasonable to point out the validity of the following result first.

**Theorem 2.3.** Suppose the base scheme $B$ has perfect residue fields. Then $B$ satisfies strict $\mathbb{A}^1$-invariance for framed presheaves of $S^1$-spectra.
2.1. The tf-motivic localization theorem. In the proof of Theorem 2.3 we appeal to a new tf-motivic localization theorem akin to results in [3, 7, 21], and [27] for the Nisnevich topology. Here we state the stable tf-motivic localization theorem and refer to Theorem 14.7 for the unstable result. For a closed immersion \( i: Z \to B \) with open complement \( j: U \to B \) there are naturally induced adjunctions \((i_*, j^*)\) are the left adjoints):

\[
i_*: \SH_{s,\tau}^{A^1}(B) \rightleftarrows \SH_{s,\tau}^{A^1}(Z) \:
\]

\[j^*: \SH_{s,\tau}^{A^1}(B) \rightleftarrows \SH_{s,\tau}^{A^1}(U) : j_*
\]

Theorem 2.4. The functors \( i_* \) and \( j_* \) in (2.2) are fully faithful. Moreover, the counit \( i_*i^! \to id \) and the unit \( id \to j_*j^* \) induce, for every \( \mathcal{F} \in \SH_{s,\tau}^{A^1}(B) \), a homotopy fiber sequence

\[
i_*i^!(\mathcal{F}) \to \mathcal{F} \to j_*j^*(\mathcal{F}).
\]

for any topology \( \tau \) on \( \Sm_B \) that is finer then the tf-topology.

We use localization as a tool to reduce questions about \( \SH_{s,\tau}^{A^1}(B) \) to \( \SH_{s,\tau}^{A^1}(Z) \) and \( \SH_{s,\tau}^{A^1}(U) \). To prove (2.3) we work with the subcategory \( \Sm_{B,Z} \) of \( \Sm_B \) spanned by essentially smooth schemes \( X^B := X^B_{s,\tau} \) given by (2.2). Following the usual script we use \( \Sm_{B,Z} \) to construct the stable \( \tau \)-local homotopy category \( \SH_{s,\tau}(B, Z) \) and the adjunctions

\[
\tilde{i}_*: \SH_{s,\tau}(B, Z) \rightleftarrows \SH_{s,\tau}(B) : \tilde{i}^! \text{ and } j^*: \SH_{s,\tau}(B) \rightleftarrows \SH_{s,\tau}(U) : j_*
\]

(2.4)

Proposition 2.5. The functors \( \tilde{i}_* \) and \( j_* \) in (2.4) are fully faithful. Moreover, the counit \( \tilde{i}_*\tilde{i}^! \to id \) and the unit \( id \to j_*j^* \) induce, for every \( \mathcal{F} \in \SH_{s,\tau}(B) \), a homotopy fiber sequence

\[
\tilde{i}_*\tilde{i}^!(\mathcal{F}) \to \mathcal{F} \to j_*j^*(\mathcal{F}).
\]

for any topology \( \tau \) on \( \Sm_B \) that is finer then the tf-topology.

We deduce (2.3) from (2.5) and an equivalence between stable motivic \( \tau \)-local homotopy categories

\[
\SH_{s,\tau}^{A^1}(B, Z) \simeq \SH_{s,\tau}^{A^1}(Z).
\]

2.2. Outline of the main argument. After setting up the adjunctions

\[
\SH_{s,\tau}(Z) \rightleftarrows \SH_{s,\tau}(B, Z) \rightleftarrows \SH_{s,\tau}(B)
\]

and showing the analogous version of Theorem 2.4, we divide the proof into two steps: (1) strict \( \mathbb{A}^1 \)-invariance for \( \SH_{s,\tau}(B, Z) \) and \( \SH_{s,\tau}(U) \) implies strict \( \mathbb{A}^1 \)-invariance for \( \SH_{s,\tau}(B) \), and (2) strict \( \mathbb{A}^1 \)-invariance for \( \SH_{s,\tau}(Z) \) implies strict \( \mathbb{A}^1 \)-invariance for \( \SH_{s,\tau}(B, Z) \).

To deduce (1) we study the adjunctions in (2.3) for \( \tau = \text{tf} \). The tf-squares in (1.2) form a key geometrical input in this step. To deduce (2) we study the adjunction

\[
\SH_{s,\tau}(Z) \rightleftarrows \SH_{s,\tau}(B, Z).
\]

The key geometrical input in this step is the lifting property saying that for every closed immersion \( Z \to X \) in \( \Sm_{B,\text{Aff}} \) and framed correspondence \( Z \to W \) of smooth affine \( B \)-schemes, there exists a framed correspondence \( X^B_Z \to W \) and lifting as indicated in the diagram

\[
\begin{array}{ccc}
W & \to & \\
\downarrow & & \\
X^B_Z & \leftarrow & Z.
\end{array}
\]

We view (2.6) as a diagram of framed correspondences between essentially smooth \( B \)-schemes, see Appendix A.
2.3. Applications. The Garkusha-Panin strict homotopy invariance theorem [17, §17] is used to compute the stable motivic fibrant replacements of suspension spectra of smooth schemes over infinite perfect fields [16]. We use strict $\mathcal{A}^1$-invariance over a base $B$ in the sense of Theorem \ref{thm:2.6} to study motivic fibrant replacements of quasi-stable framed presheaves of $S^1$-spectra.

We write $\mathcal{L}_r$ (resp. $\mathcal{L}_{\mathcal{A}^1, r}$) for the fibrant replacement functor on $\mathbf{Spt}_{s, r}(B)$ (resp. $\mathbf{Spt}_{\mathcal{A}^1, s, r}(B)$), and $\mathcal{L}_{\text{mot}} = \mathcal{L}_{\mathcal{A}^1, \text{Nis}}$ for the levelwise fibrant replacement functor on $\mathbf{Spt}_{\mathcal{A}^1, s, \text{Nis}}(B)$. If $F$ is a presheaf of $\Omega_s$-spectra, then $\mathcal{L}_{\mathcal{A}^1, r}F$ is given by applying levelwise the fibrant replacement functor in the unstable motivic $r$-local model structure. The framed motive $M_B(X)$ of $X \in \text{Sm}_B$ is the framed presheaf of $S^1$-spectra $\{\text{Fr}(-, X \otimes S^1)\}$ comprised of quasi-stable presheaves of framed correspondences, see [16, Definition 5.2]. In Section 14.3 we show:

**Theorem 2.6.** Let $B$ be a one-dimensional scheme with perfect residue fields. For any quasi-stable radditive framed presheaf of $\Omega_s$-spectra $F$ on $\text{Sm}_B$ there is a canonical equivalence of presheaves of $S^1$-spectra

$$\mathcal{L}_{\mathcal{A}^1, \text{Nis}}(F) \simeq \mathcal{L}_{\text{Nis}}\mathcal{L}_{\mathcal{A}^1, \text{tf}}(F).$$

Moreover, $\mathcal{L}_{\text{Nis}}\mathcal{L}_{\mathcal{A}^1, \text{tf}}(F)$ is fibrant in $\mathbf{Spt}_{s, \text{Nis}}(B)$. Thus, for any $X \in \text{Sm}_B$, the canonical morphism

$$\mathcal{L}_{\text{mot}}(M_B(X)) \simeq \mathcal{L}_{\text{Nis}}\mathcal{L}_{\mathcal{A}^1, r}(M_B(X))$$

is a levelwise equivalence in positive degrees. The same identity holds for the Nisnevich motivic localization and tf-motivic localization endofunctors on $\mathbf{SH}_s(B)$.

When $B$ is a field, the tf-topology on $\text{Sm}_B$ is trivial and $\mathcal{L}_{\mathcal{A}^1, \text{triv}}(F)$ for any $F \in \mathbf{Spt}_s(B)$ is weak equivalent to $F(\Delta^1 \times -)$. In this case (2.7) specializes to the equivalence in [16]

$$\mathcal{L}_{\mathcal{A}^1, \text{Nis}}(F) \simeq \mathcal{L}_{\text{Nis}}C^s(F)$$

based on the Garkusha-Panin strict homotopy invariance theorem [17, §17].

**Remark 2.7.** Any presheaf with transfers defines a presheaf with framed transfers, and any chain complex of (additive) presheaves with transfers defines a radditive presheaf of Eilenberg-MacLane $S^1$-spectra. Theorem 2.6 implies 2.7 holds for chain complexes of presheaves with transfers; over fields, Voevodsky’s strict homotopy invariance theorem [35, §3.2] implies (2.8).

Some of the deepest results in motivic homotopy theory are only valid or only known to hold over fields. The tf-topology gives us a new tool for generalizing such results to more general base schemes. To illustrate the techniques used in this paper, we show the motivic localization functor commutes with $G_\infty$-loops. Further applications are beyond the scope of this paper and will appear in future works.

2.4. Conventions and notation. Throughout, we follow the same conventions as in the Stacks Project [32]. A base scheme $B$ refers to a finite dimensional separable noetherian scheme. To a closed immersion $Z \rightarrow B$ and $B$-scheme $X$, we form the fiber product $X_Z := X \times_B Z$ and the scheme $X^h_Z$ defined in [32]. If $X$ is affine, then $X^h_Z$ is the henselization of $X$ along $X_Z$. We write $\text{Sm}_B$ for the site associated with the $\tau$-topology on $\text{Sm}_B$; the category of smooth separated finite type $B$-schemes. We let $\text{Aff}_B$ denote the subcategory of $\text{Sm}_B$ spanned by schemes that admit a closed immersion into some finite dimensional affine space $\mathbb{A}^n_B$, and write $\text{EssSm}_B$ for essentially smooth $B$-schemes, see Definition [31]. Moreover, we will make use of the following constructions:

\begin{align*}
\text{Sm}_{B,Z} & \quad \langle X_Z^h \mid X \in \text{Sm}_B \rangle \\
\text{SmAff}_{B,Z} & \quad \langle X^h_Z \mid X \in \text{SmAff}_B \rangle \\
\text{Sm}_{B,Z_*} & \quad \langle X_Z, X^h_Z \mid X \in \text{Sm}_B \rangle \\
\text{Sm}_{B,\text{ess}} & \quad \langle X \mid X \in \text{SmAff}_B, TX \cong \mathcal{O}_X^n \text{ for some } n \rangle
\end{align*}
For example, the category \( \text{Sm}_B^{cci} \) is spanned by all \( X \in \text{SmAff}_B \) with trivial tangent bundle. These categories are related via evident functors, where "\( \hookrightarrow \)" denotes a fully faithful embedding, \( U = B - Z \), and \( X_U = X \times_B U \):

\[
\begin{array}{cccccc}
\text{Sm}^{cci}_U & \hookrightarrow & \text{SmAff}_U & \hookrightarrow & \text{Sm}_U & \hookrightarrow & \text{EssSm}_U \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Sm}^{cci}_B & \hookrightarrow & \text{SmAff}_B & \hookrightarrow & \text{Sm}_B & \hookrightarrow & \text{EssSm}_B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Sm}^{cci}_{B,Z} & \hookrightarrow & \text{SmAff}_{B,Z} & \hookrightarrow & \text{Sm}_{B,Z} & \hookrightarrow & \text{EssSm}_B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Sm}^{cci}_Z & \hookrightarrow & \text{SmAff}_Z & \hookrightarrow & \text{Sm}_Z & \hookrightarrow & \text{Sch}_B \\
\end{array}
\] (2.9)

Let \( S \) be any of the categories in (2.9). We write \( \text{Spc}(S) \) (resp. \( \text{Spt}_s(S) \)) for the category of presheaves of simplicial sets (resp. \( S^1 \)-spectra) on \( S \). There are the injective model structures

\[
\text{Spc}_s(S), \quad \text{Spc}_\tau(S), \quad \text{Spc}^{A_1}_s(S),
\]

on \( S \) where the weak equivalences are the sectionwise equivalences, \( \tau \)-local equivalences, and motivic \( \tau \)-local equivalences, respectively. Here \( \tau \) is a bounded, complete, regular cd-structure on \( S \). Similarly, in the stable setting, there are the sectionwise, stable \( \tau \)-local, and stable motivic \( \tau \)-local model structures

\[
\text{Spt}_{st}(S), \quad \text{Spt}_{st,\tau}(S), \quad \text{Spt}^{A_1}_{st,\tau}(S).
\]

We denote the homotopy categories associated to the model structures in (2.10) and (2.11) by:

\[
\text{H}_s(S), \quad \text{H}_\tau(S), \quad \text{H}^{A_1}_s(S), \quad \text{SH}_s(S), \quad \text{SH}_s(\tau)(S), \quad \text{SH}^{A_1}_s(\tau)(S).
\]

In the \( \tau \)-local setting we use the following notations for the \( \tau \)-localization functor, the \( \tau \)-localization endofunctor, and the \( \tau \)-fibrant replacement functor:

\[
L_\tau : \quad \text{SH}_s(S) \to \text{SH}_s(\tau)(S), \quad \mathcal{L}_\tau : \quad \text{SH}_s(\tau)(S) \to \text{SH}_s(\tau)(S), \quad \mathcal{L}_\tau : \quad \text{Spt}(S) \to \text{Spt}(S).
\]

We will use similar notation for the functors derived from the other model structures in (2.10), (2.11).

3. The trivial fiber topology

In this section, we introduce the tf-topology on \( \text{Sch}_B \) and discuss its basic properties.

3.1. tf-coverings.

**Definition 3.1.** A commutative square in \( \text{Sch}_B \)

\[
\begin{array}{ccc}
X' & \to & X' \\
\downarrow & & \downarrow \\
X & \to & X
\end{array}
\] (3.1)
is called a trivial fiber square, or a tf-square for short, if the following holds.

(i) There exists a closed immersion \(Z \to B\) such that \(Y \cong X \times_B Z\), \(Y' \cong X' \times_X Y\).

(ii) The morphism \(\varphi\) is affine étale and induces an isomorphism \(\varphi|_{(Y')} : i'(Y') \cong i(Y)\)
for the closed immersions \(i : Y \to X\), \(i' : Y' \to X'\) in \(\text{Sch}_B\).

That is, a tf-square is a Nisnevich square of the form (3.1) for which \(\varphi\) is affine and \(Y \cong X \times_B Z\).

The tf-squares form a Nisnevich square of the form (3.1) for which \(\varphi\) is affine and \(Y \cong X \times_B Z\).

The tf-squares form a Nisnevich square of the form (3.1) for which \(\varphi\) is affine and \(Y \cong X \times_B Z\).

Definition 3.2. The tf-topology on \(\text{Sm}_B\) is the topology generated by the tf-squares in \(\text{Sm}_B\).

Proposition 3.3. The tf-squares form a complete and regular cd-structure on \(\text{Sm}_B\). If \(B\) is of finite Krull dimension then the same cd-structure is bounded.

Proof. If \(X = \emptyset\) in the tf-square (3.1), then \(X - Y = \emptyset\) and \(X' = \emptyset\). For a morphism \(X_1 \to X\)
we set \(X'_1 = X' \times_X X_1\), \(Y_1 = Y \times_X X_1\), and \(Y'_1 = Y' \times_X X_1\). Then the schemes \(X'_1\), \(X'_1 - Y'_1\),
\(X_1 - Y_1\), and \(X_1\) form a tf-square. This shows the tf cd-structure is complete.

To show regularity we check the following conditions for any tf-square (3.1):

(i) The square (3.1) is cartesian.

(ii) The morphism \(X - Y \to X\) is a monomorphism.

(iii) The square

\[
\begin{array}{ccc}
X - Y & \to & X \\
\downarrow & & \downarrow \\
(X' - Y') \times^2_Y & \to & X' \times^2_X
\end{array}
\]

is a tf-square.

Parts (i) and (ii) follow immediately from Definition 3.1. Since \(X' \to X\) is étale, the diagonal morphism \(X' \to X' \times_X X'\) is a clopen immersion; in particular, it is affine and étale. Since \(X' \times_X Y \cong Y' \cong Y\), it follows that \((X' - Y') \times_{X - Y} (X' - Y') \cong X' \times_X X' - Y' \times_Y Y' \cong X' \times_X X' \times_B (B - Z)\). This shows (3.2) is a tf-square.

For boundedness we define a density structure \(D_d(\_\_\_\_)\) on \(\text{Sm}_B\) by associating to \(X \in \text{Sm}_B\) the family \(D_d(X)\) comprised of open immersions of the form \(X \times_B (B - W) \to X\), where \(\text{codim}_B W \geq d\). With respect to the said density structure, every \(X \in \text{Sm}_B\) has dimension less or equal to \(\text{dim} B\). Next we prove that every tf-square (3.1) is reducing with respect to \(D_d(\_\_\_\_)\). Suppose \(W_{\text{orf}}\), \(W_o\), \(W_\varepsilon\) are closed subschemes of \(B\) such that \(\text{codim}_B W_{\text{orf}} \geq d\), \(\text{codim}_B W_o\), \(\text{codim}_B W_\varepsilon \geq d + 1\).

There are open immersions in the density structure

\[
\begin{align*}
X'_W & := X' \times_B (B - W_\varepsilon) \to X' \\
(X' - Y')_W & := (X' - Y') \times_B (B - W_{\text{orf}}) \to X' - Y' \\
(X - Y)_W & := (X - Y) \times_B (B - W_o) \to X - Y.
\end{align*}
\]

We set \(T = W_\varepsilon \cup W_o \cup W_{\text{orf}} \cap (B - Z) \cap Z\), where \(W_{\text{orf}} \cap (B - Z)\) is the closure of \(W_{\text{orf}} \cap (B - Z)\) in \(B\). We have that \(\text{codim}_B T \geq d\). Consider the base change of (3.1) along the open immersion \(B - T \to B\), and let \(X_1 = X \times_B (B - T)\), \(X'_1 = X' \times_B (B - T)\), \(Y_1 = X_1 \times_B Z\), \(Y'_1 = X'_1 \times_B Z\).

The canonically induced morphisms \(X'_1 - Y'_1 \to X' - Y'\), \(X'_1 \to X'\), \(X_1 - Y_1 \to X - Y\) factor through \((X' - Y')_W, X'_W, (X - Y)_W\), respectively. This implies (3.1) is reducing.

Example 3.4. We give some examples to elucidate the notion of tf-coverings.

(i) For any \(X \in \text{Sm}_B\) and any affine Nisnevich covering \(\tilde{B} \to B\), the morphism \(X \times_B \tilde{B} \to X\)
is a tf-covering. Thus every Nisnevich covering of the base scheme \(B\) has a refinement that is a tf-covering.
(ii) The Zariski covering of the affine line $\mathbb{A}^1_B$ by its open subschemes $\mathbb{A}^1_B - 0_B$ and $\mathbb{A}^1_B - 1_B$ is not a tf-covering in $\text{Sm}_B$.

(iii) Let $f \in \mathcal{O}(B)$ be a regular function and let $t$ denote the coordinate on $\mathbb{A}^1_B$. The Zariski covering of the affine line $\mathbb{A}^1_B$ by $\mathbb{A}^1_B - Z(f)$ and $\mathbb{A}^1_B - Z(f - 1)$ is a tf-covering.

**Lemma 3.5.** Suppose $Z \not\subseteq B$ is a closed immersion and $X \in \text{Sch}_B$. Then a morphism $F \to G$ of tf-sheaves on $\text{Sm}_B$ induces an isomorphism $F(X) \to G(X)$ if it induces isomorphisms

$$F(X \times_B (B - Z)) \cong G(X \times_B (B - Z)) \text{ and } F(X_X) \cong G(X_X).$$

(3.3)

**Proof.** Suppose $F \to G$ induces the isomorphisms in (3.3) for every $X \in \text{Sm}_B$. Since $F(X_X) \cong G(X_X)$ it follows that $F(X') \cong G(X')$ for an étale neighborhood $X'$ of $Z$ in $X$. Moreover, we have $F(X \times_B (B - Z)) \cong G(X \times_B (B - Z))$ and $F(X' \times_B (B - Z)) \cong G(X' \times_B (B - Z))$. Now $X, X', X \times_B (B - Z), X' \times_B (B - Z)$ form a tf-square as in (3.1). Since $F$ and $G$ are tf-sheaves we are done. \qed

We follow the same conventions on sites and topoi as in [32, Tag 00UZ]. Let $X_{\text{ét}}$ denote the small étale site of $X$, see [32, Tag 021A]. Its underlying category of étale $X$-schemes with coverage given by tf-coverings form a site we denote by $X_{\text{ét}}^\text{tf}$.

If $S$ is a site we denote its associated topos by $\mathbf{S}$. Note that every object $X \in S$ defines a morphism of topoi $X : \text{Sets} \to \mathbf{S}$ by sending a sheaf $F$ to $F(X)$.

**Proposition 3.6.** Let $B$ be a separated scheme of finite Krull dimension. The following properties hold for the tf-topology on $\text{Sm}_B$.

(i) Any covering on the small Zariski site of $B$ is a tf-covering.

(ii) The tf-topology is generated by étale coverings $\tilde{X} \to X$ satisfying: (1) for every $\sigma \in B$ there exists a lifting in the diagram

$$\xymatrix{ \tilde{X} \ar[d] \ar[rd] & \ar[d] \ar[r] \ar[l] & X, \\
X_{\sigma} := X \times_B \sigma & \ar[r] & X, }$$

and (2) there exists a Zariski open neighborhood $U$ of $\sigma$ in $B$ such that the induced morphism $\tilde{X} \times_B U \to X \times_B U$ is affine.

(iii) If $B$ is a field, then the tf-topology on $\text{Sm}_B$ is trivial.

(iv) For $X \in \text{Sm}_B$ and $\sigma \in B$, the stalk of a continuous presheaf $F$ on $\text{EssSm}_B$ at the tf-point $X_\sigma$ equals $F(X_\sigma)$.

(v) For $X \in \text{SmAff}_B$ and $\sigma \in B$, the filtering systems of Nisnevich and tf-neighbourhoods of the local scheme $X_\sigma$ coincide.

(vi) For $X \in \text{Sm}_B$, a morphism of tf-sheaves $F \to G$ on $\text{Sm}_B$ is an isomorphism on $X$-sections if for every $\sigma \in B$, the tf-point $X_\sigma$ induces an isomorphism of stalks $F_{X_\sigma} \to G_{X_\sigma}$.

(vii) There is a naturally induced conservative family of morphisms of topoi

$$\{\text{Sm}_\sigma \to \text{Sm}_B\}_{\sigma \in B}.$$ 

Moreover, the naturally induced family of morphisms of topoi

$$\text{Sets} \xrightarrow{X_\sigma} \text{Sm}_\sigma \xrightarrow{\text{tf}} \text{Sm}_B$$

indexed by all $X \in \text{Sm}_B$ and $\sigma \in B$ forms a conservative set of points for $\text{Sm}_B^\text{tf}$.

The same holds for $\text{SmAff}_\sigma$ and $\text{SmAff}_B^\text{tf}$.

\[ (3.4) \]
(viii) For every $X \in \text{Sm}_{B}$, the naturally induced family of morphisms of topoi

\[ \text{Sets}^{|U|} \xrightarrow{U \mapsto (X_\sigma)_{\text{Et}}} \hat{X}_{\text{Et}}^\text{tf} \]  

indexed by all $U \in X_{\text{Et}}$ and $\sigma \in B$ forms a conservative set of points for $\hat{X}_{\text{Et}}^\text{tf}$.

The same conclusion holds for every $X \in \text{Sm}_{\text{Aff}B}$ and the tf-site of affine étale $X$-schemes.

Proof. We begin with (i). If $X$ is an open subscheme of $B$ it suffices to show that every Zariski covering $\coprod_{i=0}^{l} U_i \to X$ by affine open subschemes $U_i$ is a tf-covering. We set $X_{l-1} := \coprod_{i=0}^{l-1} U_i$ and proceed by induction on $l$. Since $B$ (and $X$) are separated, the morphism $U_1 \to X$ is affine. Thus $X_{l-1} \cap U_1$, $X_{l-1}$, $U_1$ and $X$ form a tf-square, i.e., $X_{l-1} \amalg U_1 \to X$ is a tf-covering. Since $\coprod_{i=0}^{l-1} U_i \to X_{l-1}$ is a tf-covering by assumption, this finishes the proof of (i).

To prove (ii) we let $\text{tf}'$ be the topology generated by the given étale coverings $\bar{X} \to X$. For any tf-square (3.1) the morphism $(X - Y)_{\Sigma} X' \to X$ is a tf'-covering, and hence $\text{tf}' \supset \text{tf}$. Conversely, we claim that every tf'-covering $\bar{X} \to X$ is a tf-covering. By definition, $\text{tf}'$ contains the Zariski topology on the small Zariski site over $B$, which by (i) is contained in the tf-topology. Hence we may assume $B$ is a local scheme of finite Krull dimension.

The claim holds trivially if $B = \emptyset$, since then $\bar{X} \cong X = \emptyset$. Suppose inductively the claim holds for all base schemes of dimension less than dim $B$, and let $\mathcal{B} \subset B$ be a closed point. The lifting $X^h_{\mathcal{B}} \to \bar{X}$, afforded by the definition of a tf'-covering, gives rise to a lifting $X' \to \bar{X}$ for an étale neighborhood of $X^h_{\mathcal{B}}$ in $X$ given by an affine morphism $X' \to X$. Note that $(X - X_{\mathcal{B}}) \amalg X' \to X$ is a tf-covering. Since $\bar{X} \times_B X^h_{\mathcal{B}} \to X^h_{\mathcal{B}}$ has a right inverse, it is a covering in the trivial topology and thus also in the tf-topology. The inductive assumption implies that $\bar{X} \times_B (B - \mathcal{B}) \to X \times (B - \mathcal{B}) \cong (X - X_{\mathcal{B}})$ is a tf-covering. In summary, $\bar{X} \times_X ((X - X_{\mathcal{B}}) \amalg X^h_{\mathcal{B}}) \to (X - X_{\mathcal{B}}) \amalg X^h_{\mathcal{B}} \to X$ is a composition of tf-coverings, and we are done with (ii).

Part (iii) follows since if $B$ is a field, then any closed subscheme $Z$ of $B$ is either empty or equals $B$. That is, for any tf-square (3.1), either $X - Y = X$ or $X' = X \amalg (X' - X)$. Hence any tf-covering over $B$ admits a section.

We proceed with the characterization of points in the tf-topology. For all $X \in \text{Sm}_{B}$ and $\sigma \in B$ we claim the only tf-covering of $X^h_{\mathcal{B}}$ is the identity, and moreover for every tf-covering $\bar{X} \to X$ there exists an étale neighborhood of $X^h_{\mathcal{B}}$ in $X$ given by an affine morphism $X' \to X$ that factors through $\bar{X}$. In effect, we consider a tf-square as in (3.1). If $\mathcal{B} \not\subset Z$ then $Y = \emptyset$ and there is a canonically induced morphism $X^h_{\mathcal{B}} \to X - Y$. If $z \in Z$, then $X^h_{\mathcal{B}} \to X$ is an affine étale morphism $X^h_{\mathcal{B}} \to X - Y$. In this case there exists a lifting $X^h_{\mathcal{B}} \to X'$. Since the tf-topology is a cd-topology generated by tf-squares, it follows that any tf-covering $\bar{X} \to X^h_{\mathcal{B}}$ admits a lifting $X^h_{\mathcal{B}} \to \bar{X}$. Hence there exists a lifting $X' \to \bar{X}$ for an étale neighborhood of $X^h_{\mathcal{B}}$ given by an affine morphism $X' \to X$. It follows that the functor $\mathcal{F} \mapsto \mathcal{F}(X^h_{\mathcal{B}})$ defines a point in the tf-topology for any presheaf $\mathcal{F}$ on $\text{EssSm}_{B}$. Moreover, the functor $\mathcal{F} \mapsto \lim_{\to X, \mathcal{F}(X')}$ defines a point in the tf-topology on $\text{Sm}_{B}$. This proves part (iv).

Part (v) follows from (iv) since any étale neighborhood $\bar{X} \to X$ over an affine scheme $X$ admits a refinement by affine schemes.

We assume (vi) holds for all base schemes of dimension less than $\text{dim} B$ and proceed by induction. Since any affine Zariski covering $\bar{B} \to B$ is a tf-covering by Example (3.1), a morphism $\mathcal{F}(X) \to \mathcal{G}(X)$ is an isomorphism if and only if for every $\sigma \in B$, the morphism $\mathcal{F}(X \times_B B_{\sigma}) \to \mathcal{G}(X \times_B B_{\sigma})$ is an isomorphism. So we may assume $\bar{B}$ is local. By the inductive assumption
\( \mathcal{F}(X \times B(B - z)) \cong \mathcal{G}(X \times B(B - z)) \), where \( z \in B \) is the closed point of \( B \). Since \( \mathcal{F}(X^h_B) \cong \mathcal{G}(X^h_B) \) we deduce the desired isomorphism \( \mathcal{F}(X) \cong \mathcal{G}(X) \) from Lemma 3.5.

Parts (vii) and (viii) follow from (iv) and (vi) since for any \( X \in \text{Sm}_B \) the functor in (3.4) is given by \( \mathcal{F} \mapsto \mathcal{F}(X^h_B) \), and for \( U \in \text{Ét}_X \) the functor in (3.5) is given by \( \mathcal{F} \mapsto \mathcal{F}(U^h_B) \).

**Remark 3.7.** The trivial fiber topology on the category of affine schemes over a base scheme \( B \) is the strongest subtopology of the Nisnevich topology, which is trivial over fields. Moreover, it is probably the weakest subtopology of the Nisnevich topology for which localization holds in the sense of Theorem 9.13. The strongest subtopology of the Nisnevich topology on \( \text{Sch}_S \) which is trivial over fields is generated by squares as in Definition 3.1 without the affine assumption. According to our definition, the latter topology is stronger than the trivial fiber topology. Our definition of the trivial fiber topology, with the additional affine condition, is chosen so that the \( \text{tf} \)-points of the form \( X^h_B \).

### 3.2. Base change for points.

Any morphism of base schemes \( f: B' \to B \) induces a canonical base change functor

\[
f^*: \text{SH}_s(B) \to \text{SH}_s(B').
\] (3.6)

For each point \( \sigma \in B \) we let \( B_\sigma \) denote the local scheme of \( B \) at \( \sigma \).

**Lemma 3.8.** The canonical functor

\[
\text{SH}_s(B) \to \prod_{\sigma \in B} \text{SH}_s(B_\sigma)
\]
detects \( \mathbb{A}^1 \)-local objects on the subcategory of \( \text{tf} \)-local objects.

**Proof.** For any \( \text{tf} \)-local \( \mathcal{F} \in \text{SH}_s(B) \) the presheaf of \( S^1 \)-spectra \( \mathcal{F}(\mathbb{A}^1 \times -) \) is \( \text{tf} \)-local. Thus to conclude \( \mathcal{F} \) is \( \mathbb{A}^1 \)-local, it suffices to prove there is a canonically induced isomorphism

\[
\mathcal{F}(X^h_B) \cong \mathcal{F}(\mathbb{A}^1 \times X^h_B)
\] (3.7)

for all \( X \in \text{Sm}_B, \sigma \in B \). We note that \( X^h_B \) is a \( B^h \)-scheme, see (B.2). Hence, if the image of \( \mathcal{F} \) under \( \text{SH}_s(B) \to \text{SH}_s(B^h) \) is \( \mathbb{A}^1 \)-local, then (3.7) follows. \( \square \)

**Lemma 3.9.** Suppose \( B \) is a base scheme and \( f: B' \to B \) is an open immersion. Then \( f^* \) in (3.6) commutes with \( L_{\mathbb{A}^1}, L_{\text{Nis}}, L_{\text{tf}} \). Consequently, the canonical functor

\[
\text{SH}_s(B) \to \prod_{\sigma \in B} \text{SH}_s(B_\sigma)
\] (3.8)

commutes with \( L_{\text{Nis}}, L_{\text{tf}}, L_{\mathbb{A}^1} \). In addition, \( f^* \) and (3.8) preserve quasi-stable additive framed objects.

**Proof.** The functor \( f^* \) preserves Nisnevich local equivalences and \( \text{tf} \)-local equivalences because it is a left Quillen adjoint with right adjoint \( f_* \). Since \( f \) is an open immersion, \( f^* \) is also a right Quillen adjoint with left adjoint \( f_\# \) for the Nisnevich local and \( \text{tf} \)-local projective model structures on \( \text{Spt}_s(\text{Sm}_B) \) and \( \text{Spt}_s(\text{Sm}_B^\text{tf}) \). Hence \( f^* \) preserves local projective fibrant objects. Thus \( f^* \) commutes with \( L_{\text{Nis}} \) and \( L_{\text{tf}} \). In more details, since \( f \) is an open immersion, the functor

\[
\text{Sm}_{B'} \to \text{Sm}_B; (X \to B') \mapsto (X \to B')^f
\] (3.9)

preserves Nisnevich- and \( \text{tf} \)-squares. Thus \( f_\# \), which is determined by (3.9), preserves Nisnevich local and \( \text{tf} \)-local equivalences. For \( L_{\mathbb{A}^1} \), we note that (3.9) commutes with the endofunctors on \( \text{Sm}_{B'} \) and \( \text{Sm}_B \) given by \( X \mapsto X \times_B \Delta^+_B \cong X \times_B \Delta^+_B \). Lemma 11.2 implies the claim for (3.8).

The final claim follows because (3.9) is well defined on framed correspondences, preserves coproducts, and sends the framed correspondence \( \sigma_X \in \text{Fr}_+(B') \), for \( X \in \text{Sm}_{B'} \), to \( \sigma_X \in \text{Fr}_+(B) \). Here we use that \( f \) is a smooth map. \( \square \)
In particular, the functor in (3.8) preserves $A^1$-local objects.

Remark 3.10. Using similar arguments one can show that if $f: B' \to B$ is a Zariski covering, then $f^*: \text{SH}_*(B) \to \text{SH}_*(B')$ detects $A^1$-local objects on the subcategory of $\text{tf}$-local objects.

4. The Nisnevich- and $\text{tf}$-topology for closed immersions

For a closed immersion $Z \not\subset B$ recall that $\text{Sm}_{B,Z}$ is the full subcategory of $\text{EssSm}_B$ spanned by schemes of the form $X_Z^h = X^h \times_Z$ in the sense of (B.2). We note that $\text{Sm}_{B,Z}$ admits a monoidal product given by

$$Y_Z^h \times_{B,Z} W_Z^h := (Y \times_B W)^h.$$ (4.1)

For later reference, we introduce the Nisnevich- and the $\text{tf}$-topology on $\text{Sm}_{B,Z}$.

Definition 4.1. A Nisnevich covering of $X^h_Z \in \text{Sm}_{B,Z}$ is a morphism $\tilde{X}^h_Z \to X^h_Z$ such that $\tilde{X}_Z \to X_Z$ is a Nisnevich covering in $\text{Sm}_Z$. A weak Nisnevich covering of $X^h_Z \in \text{Sm}_{B,Z}$ is a morphism $\tilde{X}^h_Z \to X^h_Z$ induced by a Nisnevich covering $\tilde{X} \to X$ in $\text{Sm}_B$.

Alternatively, the Nisnevich topology on $\text{Sm}_{B,Z}$ is the strongest topology $\tau$ for which $\text{Sm}_Z^{\text{Nis}} \to \text{Sm}_{B,Z}$ defines a morphism of sites. Similarly, the weak Nisnevich topology ($= \text{wNis}$ topology) on $\text{Sm}_{B,Z}$ is the weakest topology $\tau$ for which we have a morphism of sites

$$\text{Sm}_{B,Z}^\tau \to \text{Sm}_B^{\text{Nis}}$$

$$X^h_Z \longleftarrow X.$$  

Proposition 4.2. The Nisnevich and weak Nisnevich topologies on $\text{Sm}_{B,Z}$ coincide. Hence there is a canonical isomorphism of sites

$$\text{Sm}_{B,Z}^{\text{Nis}} \cong \text{Sm}_{B,Z}^{\text{wNis}}.$$  

Proof. Let $\tilde{X}^h_Z \to X^h_Z$ be a weak Nisnevich covering in $\text{Sm}_{B,Z}$. Since the functor $\text{Sm}_B \to \text{Sm}_Z$ given by $X \mapsto X_Z$ preserves Nisnevich coverings, $\tilde{X}_Z \to X_Z$ is a Nisnevich covering in $\text{Sm}_Z$. By definition this means $\tilde{X}^h_Z \to X^h_Z$ is a Nisnevich covering in $\text{Sm}_{B,Z}$.

Conversely, every Nisnevich covering $\tilde{X}^h_Z \to X^h_Z$ in $\text{Sm}_{B,Z}$ is obtained from a morphism $f: \tilde{X} \to X$ in $\text{Sm}_B$ such that $f_Z: \tilde{X}_Z \to X_Z$ is a Nisnevich covering in $\text{Sm}_Z$. Since $\tilde{X}, X \in \text{Sm}_B$ and $f_Z$ is étale, the morphism $f$ is étale over $X_Z$. Moreover, since $f_Z$ is a Nisnevich covering it follows that $\tilde{X} \amalg (X_Z)_{X-B-Z} \to X$ is a Nisnevich covering. Thus $\tilde{X}^h_Z \amalg (X^h_Z)_{X-B-Z}^h \to X^h_Z$ is a Nisnevich covering in $\text{Sm}_{B,Z}$. This implies $\tilde{X}^h_Z \to X^h_Z$ is a weak Nisnevich covering in $\text{Sm}_{B,Z}$ since $(X^h_Z)_{X-B-Z}^h = \emptyset$. □

Corollary 4.3. The functor $\text{Sm}_B \to \text{Sm}_{B,Z}$ given by $X \mapsto X^h_Z$ preserves coverings and points for the Nisnevich topology.

Proof. By definition the functor $\text{Sm}_B \to \text{Sm}_{B,Z}$ given by $X \mapsto X^h_Z$ takes Nisnevich coverings and points in $\text{Sm}_B$ to weak Nisnevich coverings and points in $\text{Sm}_{B,Z}$, so the claim follows by Proposition 4.2. □
Owing to Corollary 4.3 there are induced site morphisms given by the assignments
\[
\begin{align*}
\Sm_{B,Z}^{\Nis} & \longrightarrow \Sm_B^{\Nis} \quad \longrightarrow \Sm_{B,Z}^{\Nis} \\
X^h_2 & \longleftarrow X \longleftarrow X \times_B (B - Z).
\end{align*}
\]

Remark 4.4. The results in this section hold for the Zariski topology on \(\Sm_{B,Z}\) (defined analogously to the Nisnevich topology on \(\Sm_{B,Z}\)).

Definition 4.5. A tf-covering of \(X^h_2 \in \Sm_{B,Z}\) is a morphism \(\tilde{X}^h_2 \longrightarrow X^h_2\) induced by a tf-covering \(\tilde{X} \longrightarrow X\) in \(\Sm_B\).

Alternatively, the tf-topology on \(\Sm_{B,Z}\) is the weakest topology \(\tau\) for which we have a morphism of sites
\[
\Sm_{B,Z}^{\tau} \longrightarrow \Sm_{B,Z}^{tf} \longrightarrow \Sm_B.
\]
For \(\sigma \in B\), we write \(B_\sigma\) for the local scheme of \(B\) at \(\sigma\) and \(\Sm_{B,\sigma}\) for \(\Sm_B \cap B_\sigma\).

Lemma 4.6. For \(\sigma \in B\), the tf-topology on \(\Sm_{B,\sigma}\) is trivial.

Proof. Applying the functor \(\Sm_B \rightarrow \Sm_{B,Z}\) to the tf-square (3.1) yields a cartesian square:
\[
\begin{array}{ccc}
X'^h_\sigma & \longrightarrow & X^h_\sigma \\
\downarrow & & \downarrow \\
X'^h_\sigma \times_B, (B - Z) & \longrightarrow & X^h_\sigma \\
\end{array}
\]
We are going to construct a section of the morphism
\[
X'^h_\sigma \sqcup X^h_\sigma \times_B, (B - Z) \rightarrow X^h_\sigma. \tag{4.2}
\]
If \(\sigma \not\in Z\), then \(X_\sigma \cap X_Z = \emptyset\) and consequently \((X^h_\sigma)^h \times_B, (B - Z) \cong X^h_\sigma\). Thus (4.2) admits a section. If \(\sigma \in Z\), then \(X_\sigma \subset X_Z\), and consequently \((X^h_\sigma)^h \cong X^h_\sigma\). Since \(X' \rightarrow X\) is affine étale and induces an isomorphism \(X' \times_B Z \cong X \times_B Z\), there is a canonical morphism \(X^h_\sigma \rightarrow X'\). Now the composite
\[
X^h_\sigma \cong (X^h_\sigma)^h \rightarrow (X'^h_\sigma)^h
\]
induces a section to (4.2). This shows every tf-covering in \(\Sm_{B,\sigma}\) admits a section, and hence the tf-topology is trivial. \(\square\)

Proposition 4.7. The functor \(\Sm_B \rightarrow \Sm_{B,Z}\) given by \(X \mapsto X^h_2\) preserves coverings and points for the tf-topology.

Proof. The first claim holds by definition. To prove the second claim we use that every tf-point of \(\Sm_B\) is of the form \(X^h_\sigma\) for some \(X \in \Sm_{B,\sigma}, \sigma \in B\), see Proposition 3.4(vii). If \(\sigma \not\in Z\), then \(X_\sigma \cap X_Z = \emptyset\) and \((X^h_\sigma)^h \cong X^h_\sigma\). It remains to check that \(X^h_\sigma\) defines a tf-point of \(\Sm_{B,Z}\). Consider the functor \(\Sm_{B,Z} \rightarrow \Sm_{B,\sigma}\) given by \(X^h_2 \mapsto X^h_\sigma\). By definition of the tf-topology, the latter functor defines a morphism of tf-sites, and consequently it detects tf-points. By Lemma 4.6, the tf-topology on \(\Sm_{B,\sigma}\) is trivial. Hence, on \(\Sm_{B,\sigma}\), any scheme defines a tf-point; in particular, \(X^h_\sigma\) defines a tf-point. Thus \(X^h_\sigma\) is a tf-point on \(\Sm_{B,Z}\). \(\square\)
5. Reduction to smooth affine schemes with a trivial vector bundle

In what follows, we consider the injective model structure on the category $\text{Spc}(\text{Sm}_B)$ of simplicial presheaves on $\text{Sm}_B$. The cofibrations are the monomorphisms, and the equivalences are sectionwise equivalences \([24, \S 5.1]\). On the associated homotopy category $\text{H}_\text{a}(\text{Sm}_B)$ we have the Nisnevich, tf, and $A^1$-localization endofunctors

$$L_{\text{Nis}}, L_{\text{tf}}, L_{A^1} : \text{H}_\text{a}(\text{Sm}_B) \to \text{H}_\text{a}(\text{Sm}_B)$$

induced by the fibrant replacement functors in $\text{Spc}_{s,Nis}(\text{Sm}_B)$, $\text{Spc}_{s,tf}(\text{Sm}_B)$, $\text{Spc}_{s,\text{triv}}(\text{Sm}_B)$. The subcategories $\text{H}_{\text{Nis}}(\text{Sm}_B)$, $\text{H}_{\text{tf}}(\text{Sm}_B)$, $\text{H}_{A^1}(\text{Sm}_B)$ of $\text{H}_\text{a}(\text{Sm}_B)$ are spanned by Nisnevich local, tf-local, and $A^1$-local objects. These categories arise as Bousfield localizations of $\text{Spc}(\text{Sm}_B)$, with respect to Nisnevich equivalences, tf-equivalences, and $A^1$-equivalences. The Morel-Voevodsky motivic homotopy category $\text{H}_{\text{mot}}(\text{Sm}_B)$ is equivalent to $\text{H}_{\text{Nis}}(\text{Sm}_B) \cap \text{H}_{A^1}(\text{Sm}_B)$ \([24, \S 7.2]\), \([27]\). Similarly, we have $\text{H}_\text{a}(\text{Sm}_B, Z)$ and $\text{H}_\text{a}(\text{Sm}^{cci}_B, Z)$ and the said Bousfield localizations. Our aim in this section is to establish an equivalence between motivic homotopy categories

$$\text{H}_{\text{mot}}(\text{Sm}_B, Z) \simeq \text{H}_{\text{mot}}(\text{Sm}^{cci}_B, Z).$$

The $A^1$-localization endofunctor $L_{A^1}$ on $\text{H}_\text{a}(\text{Sm}_B, Z)$ maps $F$ to $F(\Delta^n_{B, Z} \times_{B, Z} -)$. Here $\times_{B, Z}$ is defined in (5.1), $A^n_{B, Z}$ is shorthand for $A^n_{B, Z}^{\text{aff}}$ and $\Delta^n_{B, Z}$ is the cosimplicial scheme with terms

$$\Delta^n_{B, Z} = (\Delta^n_{B, Z}^{\text{aff}}) \otimes A^n_{B, Z}^{\text{aff}} = (\Delta^n_{B, Z}^{\text{aff}}) \simeq A^n_{B, Z} \times_{B, Z} \cdots \times_{B, Z} A^n_{B, Z}.$$  

We want to compare $A^n_{B, Z}$-local objects along the adjunctions

$$\text{H}_\text{a}(\text{Sm}^{cci}_B, Z) \overset{r^{cci}}{\longrightarrow} \text{H}_\text{a}(\text{Sm}^{\text{aff}}_B, Z) \overset{l^{aff}}{\longrightarrow} \text{H}_\text{a}(\text{Sm}_B, Z).$$  

In (5.2) the right adjoints $r^{aff}$ and $r^{cci}$ are restriction functors. Moreover, $l^{aff}$ and $l^{cci}$ are defined as left Kan extensions along the full embeddings

$$\text{Sm}^{cci}_B \to \text{Sm}^{\text{aff}}_B \to \text{Sm}_B.$$  

This implies that the unit maps in (5.2) are natural isomorphisms. Owing to Lemma [5.1] whose proof is left to the reader, it remains to show that the right adjoints $r^{aff}$ and $r^{cci}$ are conservative.

**Lemma 5.1.** Suppose

$$F : C \rightleftarrows D : G$$

is an adjunction, where the unit $\eta : id_C \to G \circ F$ is a natural isomorphism and the right adjoint functor $G$ is conservative. Then (5.4) is an equivalence of categories.

**Lemma 5.2.** If $B$ is an affine scheme then every $X \in \text{Sm}_B, Z$ admits a Zariski covering by schemes in $\text{Sm}^{cci}_B, Z$. Thus if $X \in \text{Sm}^{\text{aff}}_B, Z$, then every Nisnevich covering of $X$ in $\text{Sm}_B, Z$ admits a refinement by schemes in $\text{Sm}^{\text{aff}}_B, Z$ (resp. $\text{Sm}^{cci}_B, Z$).

**Proof.** First, we consider the case when $\text{Sm}_B, Z = \text{Sm}_B$. By the assumption on $B$ every $X \in \text{Sm}_B$ admits a Zariski covering by affine $B$-schemes. Since the tangent bundle of a smooth scheme is locally trivial, we can refine the Zariski covering by schemes in $\text{Sm}^{cci}_B$. In the general case, let $X' \in \text{Sm}_B, Z$, where $X \in \text{Sm}_B$. The previous case implies there exists a Zariski covering and hence a Nisnevich covering $Y \to X$, where $Y \in \text{Sm}^{cci}_B$. The naturally induced morphism $Y^h \to X^h$ is a Zariski covering and hence a Nisnevich covering by definition. \qed

**Lemma 5.3.** If $B$ is an affine scheme then the right adjoint functors $r^{aff}$ and $r^{cci}$ commute with the localization endofunctors $L_{\text{Nis}}$, $L_{\text{tf}}$, $L_{A^1}$.
The functors in (5.3) preserve Nisnevich and tf-coverings, and fiber products. Hence the right adjoints $r^\text{aff}$ and $r^\text{cci}$ preserve Nisnevich local objects. Owing to Lemma 5.2 if $X \in \text{Sm}^{\text{aff}}_{B,Z}$, then every Nisnevich covering $Y \to X$ in $\text{Sm}^{\text{aff}}_{B,Z}$ admits a refinement in $\text{Sm}^{\text{aff}}_{B,Z}$. This shows that $r^\text{aff}$ commutes with $\mathcal{L}^{\text{Nis}}_{f}$. If $f \in \mathcal{O}(B)$ is a regular function, then $X \times_B (B - Z(f)) \in \text{Sm}^{\text{aff}}_{B,Z}$ and every étale neighborhood of $X \times_B Z(f)$ admits a refinement in $\text{Sm}^{\text{aff}}_{B,Z}$. This shows every tf-covering of $X$ admits a refinement in $\text{Sm}^{\text{aff}}_{B,Z}$. It follows that $r^\text{aff}$ commutes with $\mathcal{L}^{\text{tf}}_{f}$. If $Y \to X$ is an étale morphism in $\text{Sm}^{\text{aff}}_{B,Z}$ and $X \in \text{Sm}^{\text{cci}}_{B,Z}$, then $Y \in \text{Sm}^{\text{cci}}_{B,Z}$. In particular, this applies to Nisnevich and tf-coverings, and thus $r^\text{cci}$ commutes with $\mathcal{L}^{\text{Nis}}_{f}$ and $\mathcal{L}^{\text{tf}}_{f}$.

### Remark 5.5

Any affine scheme with a stably trivial tangent bundle is $\mathbb{A}^1$-equivalent to a scheme with a trivial tangent bundle.

If $f$ is an endomorphism of $X \in \text{Spc}(\text{Sm}_{B,Z})$, a simplicial presheaf on $\text{Sm}_{B,Z}$, we can form the homotopy colimit

$$X[f^{-1}] := \text{hocolim}(X \xrightarrow{f} X \xrightarrow{f} \cdots).$$

Note that $f$ induces an equivalence on $X[f^{-1}]$. If $f$ is an equivalence on $X$, then $X \simeq X[f^{-1}]$.

### Lemma 5.6

Suppose $i : Y \hookrightarrow X$ is a closed immersion in $\text{Sm}_{B,Z}$ and $p : X \to Y$ is a morphism such that $p \circ i = \text{id}_Y$. Then for $f := i \circ p : X \to X$ there is a canonically induced isomorphism

$$i_\infty : Y \xrightarrow{\simeq} X[f^{-1}]$$

in $H_*(\text{Sm}_{B,Z})$.

### Proof

Since $f$ is an idempotent and $f \circ i = i$ there are naturally induced morphisms

$$i_\infty : Y \simeq Y[\text{id}_Y^{-1}] \to X[f^{-1}], \quad p_\infty : X[f^{-1}] \to Y[\text{id}_Y^{-1}] \simeq Y.$$

The composite $p_\infty \circ i_\infty$ is the identity on $Y$. Moreover, the composite $i_\infty \circ p_\infty$ is induced by $f$, and hence it is an equivalence on $X[f^{-1}]$.

### Lemma 5.7

If $B$ is an affine scheme then the restriction functor

$$r^\text{cci} : H_*(\text{Sm}^{\text{aff}}_{B,Z}) \to H_*(\text{Sm}^{\text{cci}}_{B,Z})$$

is conservative.

### Proof

Suppose $F \to G$ is a morphism in $H_*(\text{Sm}^{\text{aff}}_{B,Z})$ such that $r^\text{cci}$ induces an isomorphism

$$r^\text{cci}(F) \xrightarrow{\simeq} r^\text{cci}(G) \in H_*(\text{Sm}^{\text{cci}}_{B,Z}).$$

Lemma 5.4 shows that $X \in \text{Sm}_{B,Z}$ is a retract of some $V \in \text{Sm}^{\text{cci}}_{B,Z}$. Let $z : X \to V$ and $p : V \to X$ be morphisms with compositions $p \circ z = \text{id}_X$ and $f = z \circ p$. From (5.6) we deduce an isomorphism between hom groups

$$[V[f^{-1}], r^\text{cci}(F)]_{H_*(\text{Sm}^{\text{cci}}_{B,Z})} \cong [V[f^{-1}], r^\text{cci}(G)]_{H_*(\text{Sm}^{\text{aff}}_{B,Z})}. \quad (5.7)$$
Note that $V$ represents a constant simplicial presheaf on $\text{Sm}_{B,Z}^{\text{aff}}$ and $\text{SmAff}_{B,Z}$. Applying the left adjoint $\mathcal{L}^{\text{aff}}: \mathcal{H}_s(\text{Sm}_{B,Z}^{\text{aff}}) \to \mathcal{H}_s(\text{SmAff}_{B,Z})$, which commutes with homotopy colimits, yields

$$\mathcal{L}^{\text{aff}}(V) \simeq V \in \mathcal{H}_s(\text{SmAff}_{B,Z}), \quad \mathcal{L}^{\text{aff}}(V^{[\ell - 1]}) \simeq V^{[\ell - 1]} \in \mathcal{H}_s(\text{SmAff}_{B,Z}).$$

Combined with (5.7) we deduce the isomorphism

$$[V^{[\ell - 1]}, \mathcal{J}]_{\mathcal{H}_s(\text{SmAff}_{B,Z})} \simeq [V^{[\ell - 1]}, \mathcal{G}]_{\mathcal{H}_s(\text{SmAff}_{B,Z})}.$$

Finally, using the isomorphism $X \simeq V^{[\ell - 1]} \in \mathcal{H}_s(\text{SmAff}_{B,Z})$ from Lemma 5.9, we get

$$[X, \mathcal{J}]_{\mathcal{H}_s(\text{SmAff}_{B,Z})} \simeq [V^{[\ell - 1]}, \mathcal{J}]_{\mathcal{H}_s(\text{SmAff}_{B,Z})} \simeq [V^{[\ell - 1]}, \mathcal{G}]_{\mathcal{H}_s(\text{SmAff}_{B,Z})} \simeq [X, \mathcal{G}]_{\mathcal{H}_s(\text{SmAff}_{B,Z})}.$$

It follows that $\mathcal{F} \to \mathcal{G}$ is an isomorphism in $\mathcal{H}_s(\text{SmAff}_{B,Z})$. \hfill $\square$

**Proposition 5.8.** If $B$ is an affine scheme then the restriction functor

$$r^{\text{aff}}: \mathcal{H}_s(\text{SmAff}_{B,Z}) \to \mathcal{H}_s(\text{Sm}_{B,Z}^{\text{aff}})$$

is an equivalence. Moreover, $r^{\text{aff}}$ commutes with $\mathcal{L}_{\text{Nis}}$, $\mathcal{L}_{\text{aff}}$, and $\mathcal{L}_{\mathbb{A}^1}$. In particular, $r^{\text{aff}}$ preserves and detects Nisnevich local and $\mathbb{A}^1$-local objects.

**Proof.** The first claim follows from Lemmas 5.1 and 5.7. To conclude for $\mathcal{L}_{\text{Nis}}$, $\mathcal{L}_{\text{aff}}$, and $\mathcal{L}_{\mathbb{A}^1}$ we use Lemma 5.3. \hfill $\square$

**Lemma 5.9.** If $B$ is an affine scheme then the restriction functor

$$r^{\text{aff}}: \mathcal{H}_s(B, Z) \to \mathcal{H}_s(\text{SmAff}_{B,Z})$$

preserves $\mathbb{A}^1$-local objects and it is essentially surjective on $\mathbb{A}^1$-local objects.

**Proof.** The first claim follows since for every $X \in \text{SmAff}_{B,Z}$ the cylinder $\mathbb{A}^1 \times X \in \text{SmAff}_{B,Z}$. Suppose $\mathcal{F} \in \mathcal{H}_s(\text{SmAff}_{B,Z})$ is $\mathbb{A}^1$-local. Via left Kan extension we deduce the adjunction

$$l^{\text{aff}}: \mathcal{H}_s(\text{SmAff}_{B,Z}) \rightleftarrows \mathcal{H}_s(B, Z): r^{\text{aff}} \quad (5.8)$$

Then $\mathcal{L}_{\mathbb{A}^1}l^{\text{aff}}(\mathcal{F}) \in \mathcal{H}_s(B, Z)$ is $\mathbb{A}^1$-local, and there are isomorphisms

$$r^{\text{aff}}(\mathcal{L}_{\mathbb{A}^1}l^{\text{aff}}(\mathcal{F})) \simeq \mathcal{L}_{\mathbb{A}^1}r^{\text{aff}}(l^{\text{aff}}(\mathcal{F})) \simeq \mathcal{L}_{\mathbb{A}^1}(\mathcal{F}) \simeq \mathcal{F}.$$

The first isomorphism follows from Lemma 5.3, and the second follows because the unit map of (5.8) is a natural isomorphism. \hfill $\square$

**Lemma 5.10.** If $B$ is an affine scheme then the restriction functors

$$\mathcal{H}_{\text{Nis}}(B, Z) \to \mathcal{H}_{\text{Nis}}(\text{SmAff}_{B,Z}) \to \mathcal{H}_{\text{Nis}}(\text{Sm}_{B,Z}^{\text{aff}})$$

are conservative. The same holds for $\mathcal{H}_{\text{Zar}}(B, Z)$.

**Proof.** Suppose $\mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{H}_{\text{Nis}}(B, Z)$ or $\mathcal{H}_{\text{Nis}}(\text{SmAff}_{B,Z})$ that maps to an isomorphism in $\mathcal{H}_{\text{Nis}}(\text{Sm}_{B,Z}^{\text{aff}})$, i.e., $\mathcal{F}(X) \simeq \mathcal{G}(X)$ for all $X \in \text{Sm}_{B,Z}^{\text{aff}}$. Lemma 5.2 implies that the same holds for all $X \in \text{Sm}_{B,Z}$ or $X \in \text{SmAff}_{B,Z}$. \hfill $\square$

**Proposition 5.11.** If $B$ is an affine scheme then the restriction functor

$$\mathcal{H}_{\text{Nis}}(B, Z) \to \mathcal{H}_{\text{Nis}}(\text{SmAff}_{B,Z})$$

is an equivalence. Moreover, the same functor preserves and detects $\mathbb{A}^1$-local objects. The same results hold for $\mathcal{H}_{\text{Zar}}(B, Z)$, $\mathcal{H}_{\text{Nis}}(\text{Fr}_+(B, Z))$, and $\mathcal{H}_{\text{Zar}}(\text{Fr}_+(B, Z))$. 
Proof. The unit map of the adjunction \((5.8)\) is a natural isomorphism. Since \(H_*(B, Z) \to H_*(\text{SmAff}_B, Z)\) is Nisnevich exact, the same holds for the unit map of the adjunction
\[
H_{\text{Nis}}(\text{SmAff}_B, Z) \xrightarrow{\sim} H_{\text{Nis}}(B, Z). \tag{5.9}
\]

Lemma 5.10 implies \((5.9)\) is an equivalence of categories. Lemma 5.3 shows the right adjoint in \((5.9)\) commutes with \(L_{A^1}\); hence it preserves and detects \(A^1\)-local objects. The remaining cases are similar and left to the reader. \(\square\)

**Lemma 5.12.** If \(f: \tilde{B} \to B\) is a Nisnevich covering then the base change functor
\[
H_{\text{Nis}}(\text{Sm}_B, Z) \to H_{\text{Nis}}(\text{Sm}_B, \tilde{Z})
\]
is conservative and preserves \(A^1\)-local objects. Here we write \(\tilde{Z}\) for the fiber product \(Z \times_B \tilde{B}\).

The same holds for \(H_{\text{Nis}}(\text{SmAff}_B, Z)\) and \(H_{\text{Nis}}(\text{Sm}_{cci} B, Z)\).

Proof. Suppose the morphism \(F \to G\) in \(H_{\text{Nis}}(\text{Sm}_B, Z)\) maps to an isomorphism in \(H_{\text{Nis}}(\text{Sm}_B, \tilde{Z})\). For every \(X \in \text{Sm}_B, Z\) and \(x \in X\) there exists a lifting \(X^h \to \tilde{B}\) of \(X^h \to B\) along \(f\). Hence \(X^h\) is a local henselian essentially smooth scheme over \(B\). Thus by assumption there is an isomorphism \(F(X^h) \cong G(X^h)\). This shows the base change functor is conservative. The second assertion follows since \(\text{Sm}_B, \tilde{Z} \to \text{Sm}_B, Z\) maps \(X \times_B \tilde{Z}\) to \(X\) and \(X \times_B (A^1_B, Z)\) to \(X\).

The arguments for \(H_{\text{Nis}}(\text{SmAff}_B, Z)\) and \(H_{\text{Nis}}(\text{Sm}_{cci} B, Z)\) are similar. \(\square\)

**Corollary 5.13.** For every base scheme \(B\) and closed immersion \(Z \not\subseteq B\) there are canonical equivalences between motivic homotopy categories
\[
H_{\text{mot}}(\text{Sm}_B, Z) \simeq H_{\text{mot}}(\text{SmAff}_B, Z) \simeq H_{\text{mot}}(\text{Sm}_{cci} B, Z). \tag{5.10}
\]

Proof. If \(B\) is affine, Propositions 5.8 and 5.11 show the canonical equivalences
\[
H_{\text{Nis}}(\text{Sm}_B, Z) \simeq H_{\text{Nis}}(\text{SmAff}_B, Z) \simeq H_{\text{Nis}}(\text{Sm}_{cci} B, Z) \tag{5.11}
\]
preserve and detect \(A^1\)-local objects. This allows us to conclude \((5.10)\).

For an arbitrary base scheme \(B\) consider a Zariski covering \(\tilde{B} \to B\) by affine schemes. By base change to \(\tilde{Z} \cong Z \times_B \tilde{B}\) we obtain the diagram
\[
\begin{array}{ccc}
H_{\text{mot}}(\text{Sm}_{cci} B, \tilde{Z}) & \xrightarrow{\sim} & H_{\text{mot}}(\text{SmAff}_B, \tilde{Z}) & \xrightarrow{\sim} & H_{\text{mot}}(\text{Sm}_B, \tilde{Z}) \\
\uparrow & & \uparrow & & \uparrow \\
H_{\text{mot}}(\text{Sm}_{cci} B, Z) & \xrightarrow{\sim} & H_{\text{mot}}(\text{SmAff}_B, Z) & \xrightarrow{\sim} & H_{\text{mot}}(\text{Sm}_B, Z),
\end{array} \tag{5.12}
\]
where \(\tilde{Z} = \tilde{B} \times_B Z\). The unit and counit maps in the upper row of \((5.12)\) are natural isomorphisms according to the previous case. By Lemma 5.12 the vertical functors in \((5.12)\) are conservative. Hence the unit and counit maps in the lower row of \((5.12)\) are also natural isomorphisms. \(\square\)

6. \(A^1\)-LOCALITY AND RIGIDITY FOR CLOSED IMMERSIONS

To a fixed closed immersion \(Z \not\subseteq B\), we associate a full subcategory \(\text{Sm}_{B*Z}\) of \(\text{Sch}_B\). The main innovation is to establish a close connection between \(\text{Sm}_{B*Z}\) and \(\text{Sm}_Z\) by means of \(A^1\)-local and rigid objects. This input is pivotal for the proof of our main result.
6.1. The category \( \text{Sm}_{B,Z} \) and its Nisnevich site.

**Definition 6.1.** Let \( \text{Sm}_{B,Z} \) be the full subcategory of \( \text{Sch}_B \) spanned by objects of the form \( X \times_B Z \) and \( X^h_B \) for all \( X \in \text{Sm}_B \). The category \( \text{SmAff}_{B,Z} \) (resp. \( \text{Sm}^{\text{cci}}_{B,Z} \)) is defined similarly subject to the condition \( X \in \text{SmAff}_B \) (resp. \( X \in \text{Sm}^{\text{cci}}_B \)).

We note that the assumption in the next result holds for closed points.

**Lemma 6.2.** Assume there exists an open affine neighborhood \( B' \subset B \) of \( Z \). Then for every \( Y \in \text{Sm}^{\text{cci}}_Z \) there exists an object \( \tilde{Y} \in \text{Sm}^{\text{cci}}_{B,Z} \) such that \( Y \cong \tilde{Y} \times_B Z \).

**Proof.** Consider a closed immersion \( Y \hookrightarrow \mathbb{A}^1_Z \) with trivial normal bundle, and choose polynomials \( f_1, \ldots, f_r \) with vanishing locus

\[
Z(f_1, \ldots, f_r) = Y \amalg Y' \subset \mathbb{A}^1_Z.
\]

Since \( B' \) is affine, the restriction morphism \( \mathcal{O}(\mathbb{A}^1_{B'}) \to \mathcal{O}(\mathbb{A}^1_Z) \) is surjective on global sections. Say \( \tilde{f}_i \in \mathcal{O}(\mathbb{A}^1_{B'}) \) maps to \( f_i \in \mathcal{O}(\mathbb{A}^1_Z) \). Then

\[
Z(\tilde{f}_1, \ldots, \tilde{f}_r) \cong Z \times_{B'} Z(\tilde{f}_1, \ldots, \tilde{f}_r).
\]

where \( Z(\tilde{f}_1, \ldots, \tilde{f}_r) \subset \mathbb{A}^1_{B'} \) is the vanishing locus. Owing to Corollary 6.3 we have a decomposition

\[
Z(\tilde{f}_1, \ldots, \tilde{f}_r) \cong \tilde{Y} \amalg \tilde{Y}',
\]

where \( Y \cong \tilde{Y} \times_B Z \). Since the normal bundle \( N_{Y/\mathbb{A}^1_Z} \) is trivial and the tangent bundle \( T_Y \) is stably trivial, \( N_{Y/(\mathbb{A}^1_Z)^h} \) is trivial and \( T_Y \) is stably trivial. This shows that \( \tilde{Y} \in \text{Sm}^{\text{cci}}_{B,Z} \).

**Corollary 6.3.** There is a fully faithful functor \( \text{Sm}^{\text{cci}}_Z \to \text{Sm}^{\text{cci}}_{B,Z} \).

**Proof.** The functor is the identity morphism on objects. Lemma 6.2 shows that every \( Y \in \text{Sm}^{\text{cci}}_Z \) is isomorphic to \( \tilde{Y} \times_B Z \) for some \( \tilde{Y} \in \text{Sm}^{\text{cci}}_B \).

**Definition 6.4.** A morphism \( \tilde{X} \to X \) in \( \text{Sm}_{B,Z} \) is a Nisnevich covering if it is an étale morphism of \( B \)-schemes and \( \tilde{X} \to X_Z \) is a Nisnevich covering in \( \text{Sm}_Z \).

**Lemma 6.5.** The site morphism \( \text{Sm}^{\text{Nis}}_Z \to \text{Sm}^{\text{Nis}}_B \) factors as

\[
\text{Sm}^{\text{Nis}}_Z \xrightarrow{\iota_{B,Z}} \text{Sm}^{\text{Nis}}_{B,Z} \xrightarrow{u} \text{Sm}^{\text{Nis}}_{B,Z} \xrightarrow{v} \text{Sm}^{\text{Nis}}_B \quad (6.1)
\]

\[
Y, X_Z \quad \xleftarrow{\quad} \quad Y, Z \quad \xleftarrow{\quad} \quad Z.
\]

**Remark 6.6.** The Nisnevich topology on \( \text{Sm}_{B,Z} \) is the weakest topology \( \tau \) for which the following functors define site morphisms

\[
\text{Sm}^{\text{Nis}}_Z \xleftarrow{\quad} \text{Sm}_{B,Z} \xrightarrow{\quad} \text{Sm}^{\text{Nis}}_{B,Z} \quad (6.1)
\]

\[
Y \quad \xleftarrow{\quad} \quad Y, Z \quad \xleftarrow{\quad} \quad Z.
\]

6.2. \( \mathbb{A}^1 \)-local and rigid presheaves on \( \text{Sm}_{B,Z} \).

**Definition 6.7.** An object \( F \in \mathcal{H}_s(\text{Sm}_{B,Z}) \) is rigid if for all \( X \in \text{Sm}_B \) there is a naturally induced equivalence of simplicial sets

\[
\mathcal{F}(X^h_Z) \cong \mathcal{F}(X \times_B Z).
\]

**Definition 6.8.** An object \( F \in \mathcal{H}_s(\text{Sm}_{B,Z}) \) is \( \mathbb{A}^1 \)-local if for all \( X \in \text{Sm}_B \) there are naturally induced equivalences of simplicial sets

\[
\mathcal{F}(X_Z) \cong \mathcal{F}(\mathbb{A}^1 \times X_Z), \quad \mathcal{F}(X^h_Z) \cong \mathcal{F}((\mathbb{A}^1 \times X)^h_Z).
\]
Consider the functor $u: \text{Sm}_{B,Z}^{\text{cst}} \to \text{Sm}_{B,Z}^{\text{cst}}$ and the adjunction
\[ u^*: \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}}) \rightleftarrows \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}}): u_* \]
where $u_* F(X{\frac{1}{2}}) = F(X{\frac{1}{2}})$ for any $F \in \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ and $X{\frac{1}{2}} \in \text{Sm}_{B,Z}^{\text{cst}}$, and $u^*$ is the left adjoint.

If $F \in \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ then $u^*(F) \in \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ is the left Kan extension of $F$ along $u$. Similarly, we have the adjunction
\[ u^*_H: \mathsf{H}_s(\text{Fr}_+(\text{Sm}_{B,Z}^{\text{cst}})) \rightleftarrows \mathsf{H}_s(\text{Fr}_+(\text{Sm}_{B,Z}^{\text{cst}})): u^*_H. \]

Note that $u_*$ agrees with the composition of $u^*$ with the forgetful functor $\mathsf{H}_s(\text{Fr}_+(\text{Sm}_{B,Z}^{\text{cst}})) \to \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$.

**Lemma 6.9.** The functors $u^*$ and $u^*_H$ preserve $\aleph_1$-local objects.

**Proof.** For every $Y \in \text{Sm}_{B,Z}^{\text{cst}}$ there are filtering categories given by morphisms of $B$-schemes
\[ (A_1 \times Y)/\text{Sm}_{B,Z}^{\text{cst}} = \{ f^*: A_1 \times Y \to (X{\frac{1}{2}}), (X{\frac{1}{2}}) \in \text{Sm}_{B,Z}^{\text{cst}} \}, \]
\[ Y/\text{Sm}_{B,Z}^{\text{cst}} = \{ f: Y \to X{\frac{1}{2}}, X{\frac{1}{2}} \in \text{Sm}_{B,Z}^{\text{cst}} \}. \]

For every $F \in \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ we have
\[ u^*(F)(A_1 \times Y) = \operatorname{hocolim}_{(A_1 \times Y)/\text{Sm}_{B,Z}^{\text{cst}}}(F((X'\frac{1}{2})), u^*(F)(Y) = \operatorname{hocolim}_{Y/\text{Sm}_{B,Z}^{\text{cst}}}(F(X{\frac{1}{2}})). \]

Moreover, there exists a functor
\[ Y/\text{Sm}_{B,Z}^{\text{cst}} \to (A_1 \times Y)/\text{Sm}_{B,Z}^{\text{cst}}: f \mapsto (A_1 \times Y, f), \]
\[ A_1 \times B, Z f: A_1 \times Y \to A_1 \times B, Z X{\frac{1}{2}} \cong (A_1 \times Y, f). \]

Lemma 6.2 and Lemma 6.10 imply $(Y/\text{Sm}_{B,Z}^{\text{cst}})$ is a cofinal subcategory of $(A_1 \times Y)/\text{Sm}_{B,Z}^{\text{cst}}$. It readily follows that
\[ \operatorname{hocolim}_{(f, X) \in (A_1 \times Y)/\text{Sm}_{B,Z}^{\text{cst}}}(F((X'\frac{1}{2}))) \simeq \operatorname{hocolim}_{(f, X) \in Y/\text{Sm}_{B,Z}^{\text{cst}}}(F((A_1 \times Y)\frac{1}{2})). \]

Since $F$ is $\aleph_1$-local, we also have
\[ \operatorname{hocolim}_{(f, X) \in Y/\text{Sm}_{B,Z}^{\text{cst}}}(F((A_1 \times X)\frac{1}{2})) \simeq \operatorname{hocolim}_{(f, X) \in Y/\text{Sm}_{B,Z}^{\text{cst}}}(F(X{\frac{1}{2}})), \]
and hence
\[ u^*(F)(A_1 \times Y) \simeq u^*(F)(Y). \]

This shows $u^*(F)$ is $\aleph_1$-local.

The case of $u^*_H$ follows by a similar argument using Lemma A.11.

We note the $\aleph_1$-localization endofunctor $\mathcal{L}_{\aleph_1}$ on $\mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ is given by
\[ G(-) \mapsto G((- \times_B \Delta{\frac{1}{2}})), \]
and similarly for $\mathsf{H}_s(\text{Fr}_+(\text{Sm}_{B,Z}^{\text{cst}}))$. Let $\mathcal{F}_{\aleph_1}$ be shorthand for $\mathcal{L}_{\aleph_1} u^*(F)$, $F \in \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$, and let $\mathcal{F}_{\aleph_1}$ be shorthand for $\mathcal{L}_{\aleph_1} u^*_H(F)$, $F \in \mathsf{H}_s(\text{Fr}_+(\text{Sm}_{B,Z}^{\text{cst}}))$.

**Proposition 6.10.** For any $F \in \mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ the simplicial presheaf $\mathcal{F}_{\aleph_1}$ is $\aleph_1$-local and rigid. The same holds for $F \in \mathsf{H}_s(\text{Fr}_+(\text{Sm}_{B,Z}^{\text{cst}}))$ and $\mathcal{F}_{\aleph_1}$ in the framed setting.

**Proof.** We may assume $F = \text{Sm}_{B,Z}^{\text{cst}}(-, X{\frac{1}{2}})$, $X \in \text{Sm}_{B,Z}^{\text{cst}}$, since every object of $\mathsf{H}_s(\text{Sm}_{B,Z}^{\text{cst}})$ is a homotopy colimit of representable presheaves. We have
\[ u^*(F) = \text{Sm}_{B,Z}^{\text{cst}}(-, X{\frac{1}{2}}), \]
and
\[ \mathcal{F}_{\aleph_1}(-) = \text{Sm}_{B,Z}^{\text{cst}}((\Delta{\frac{1}{2}} \times -), X{\frac{1}{2}}). \tag{6.2} \]
We claim that $F_{\mathcal{X}}(U_Z^h) \to F_{\mathcal{X}}(U \times Z)$ is a trivial fibration of simplicial sets for any $U \in \text{SmAff}_B$.

By \cite[Corollary 3.9]{24}, the pushout
\[(\partial \Delta^n_B \times U)_Z^h \amalg (\partial \Delta^n_B \times U_Z) \to (\Delta^n_B \times U)_Z^h\]
of the diagram of closed immersions
\[(\partial \Delta^n_B \times U)_Z^h \amalg (\partial \Delta^n_B \times U_Z) \not\to \to (\Delta^n_B \times U)_Z^h\]
every object of $\text{Sch}_B$ exists in the category of schemes. There is a naturally induced closed immersion
\[(\partial \Delta^n_B \times U)_Z^h \amalg (\partial \Delta^n_B \times U_Z) \to (\Delta^n_B \times U)_Z^h. \tag{6.3}\]

Since \((6.3)\) is a henselian pair, Lemma \([3, 10]\) shows there exists a lifting in every diagram in $\text{Sch}_B$ of the form:
\[
\begin{array}{ccc}
(\partial \Delta^n_B \times U)_Z^h \amalg (\partial \Delta^n_B \times U_Z) & \to & (\Delta^n_B \times U)_Z^h \\
\downarrow & & \downarrow \\
(\Delta^n_B \times U)_Z^h & \to & X_Z^h
\end{array}
\]

Owing to \((6.2)\), a lifting in \((6.3)\) is equivalent to a lifting in the diagram
\[
\begin{array}{ccc}
\partial \Delta^n & \to & F_{\mathcal{X}}(U_Z^h) \\
\downarrow & & \downarrow \\
\Delta^n & \to & F_{\mathcal{X}}(U_Z).
\end{array}
\]

Indeed, by \((6.2)\), we have
\[
s\text{Set}_\cdot(\Delta^n, F_{\mathcal{X}}(U_Z^h)) \cong \text{Sm}_{B,Z}(\Delta^n_B \times B, Z \times U_Z^h, X_Z^h) \cong \text{Sch}_B((\Delta^n_B \times U)_Z^h, X_Z^h),
\]
and commutative squares of the form \((6.5)\) are in bijection with morphisms $p$ in \((6.4)\). Since the left vertical morphism in \((6.5)\) is a generating cofibration of simplicial sets, the right vertical morphism is a trivial Kan fibration. It follows that there is a naturally induced equivalence
\[
F_{\mathcal{X}}(U_Z^h) \cong F_{\mathcal{X}}(U_Z).
\]

This shows that $F_{\mathcal{X}}$ is a rigid presheaf on $\text{Sm}_{B,Z}$.

The same proof applies in the framed setting for $F_{\mathcal{X}}^{\text{fr}}$, by reference to Lemma \([3, 11]\) instead of Lemma \([3, 10]\).

\[\Box\]

**Lemma 6.11.** The functor $i_*^{\text{fr}}: H_*(\text{Sm}_{B,Z}^{\text{cri}}) \to H_*(\text{Sm}_{Z}^{\text{cri}})$ is equivalent to the restriction $r_*$ along the embedding $r: \text{Sm}_{Z}^{\text{cri}} \to \text{Sm}_{B,Z}^{\text{cri}}$.

**Proof.** We may assume $F = \text{Sm}_{B,Z}^{\text{cri}}(-, X_Z^h)$, $X \in \text{Sm}_{B,Z}^{\text{cri}}$, or $F = \text{Sm}_{B,Z}^{\text{cri}}(-, Y)$, $Y \in \text{Sm}_{Z}^{\text{cri}}$, since every object of $H_*(\text{Sm}_{B,Z}^{\text{cri}})$ is a homotopy colimit of representable presheaves. Since $i_*^{\text{fr}}$ is defined as a left Kan extension, we have
\[
i_*^{\text{fr}}(\text{Sm}_{B,Z}^{\text{cri}}(-, X_Z^h)) \cong \text{Sm}_{Z}^{\text{cri}}(-, X_Z), i_*^{\text{fr}}(\text{Sm}_{B,Z}^{\text{cri}}(-, Y)) \cong \text{Sm}_{Z}^{\text{cri}}(-, Y).
\]

The claim follows now from the isomorphisms
\[
\text{Sm}_{Z}^{\text{cri}}(-, X_Z) \cong r_*\text{Sm}_{B,Z}^{\text{cri}}(-, X_Z^h), \text{Sm}_{Z}^{\text{cri}}(-, Y) \cong r_*\text{Sm}_{B,Z}^{\text{cri}}(-, Y).
\]

\[\Box\]
Lemma 6.12. For every \( \mathbb{A}^1 \)-local object \( \mathcal{F} \in \mathcal{H}_*(\text{Sm}_{B,Z}^{CCI}) \) the unit of the adjunction
\[
i_B^*: \mathcal{H}_*(\text{Sm}_{B,Z}^{CCI}) \rightleftarrows \mathcal{H}_*(\text{Sm}^{CCI}_Z): i_*^{B,Z}
\]
induces a natural equivalence
\[
\mathcal{F} \xrightarrow{\simeq} i_*^{B,Z}i_B^*(\mathcal{F}).
\]
The same holds for every \( \mathbb{A}^1 \)-local object \( \mathcal{F} \in \mathcal{H}_*(\text{Fr}_+(\text{Sm}_{B,Z}^{CCI})) \) in the framed setting.

Proof. The functors \( i_B^* \) and \( i_*^{B,Z} \) admit factorizations:
\[
\xymatrix{ H_*(\text{Sm}_{B,Z}^{CCI}) \ar[r]^{u^*} & H_*(\text{Sm}^{CCI}_{B+Z}) \ar[r]^{i_B^{B,Z}} & H_*(\text{Sm}^{CCI}_Z) \\
H_*(\text{Sm}^{CCI}_{B+Z}) \ar[r]^{u_*} & H_*(\text{Sm}^{CCI}_Z) \ar[r]^{i_*^{B,Z}} & H_*(\text{Sm}^{CCI}_Z) \\
i_B^* \ar[u] & & i_B^* \ar[u]
}\]
Here \( u^* \) and \( u_* \) are obtained from the embedding \( \text{Sm}_{B,Z}^{CCI} \to \text{Sm}^{CCI}_{B+Z} \), while \( i_B^{B,Z} \) and \( i_*^{B,Z} \) are obtained from the functor \( 
\text{Sm}^{CCI}_{B+Z} \to \text{Sm}^{CCI}_Z \) given by \( X_{B+Z}^Z, Y \mapsto X_B, Y \) for \( X \in \text{Sm}^{CCI}_B, Y \in \text{Sm}^{CCI}_Z \).
By Lemma 6.11 we have \( i_B^{B,Z} \simeq i_*^{B,Z} \) obtained from the embedding \( \text{Sm}^{CCI}_{B+Z} \to \text{Sm}_Z^{CCI} \). The canonical morphism \( \mathcal{F} \to u_*u^*(\mathcal{F}) \) is an equivalence. Since \( \mathcal{F} \) is \( \mathbb{A}^1 \)-local, the presheaf \( u^*(\mathcal{F}) \) is rigid according to Lemma 6.9 and Proposition 6.10. Hence there are equivalences
\[
u^*(\mathcal{F}) \xrightarrow{\simeq} i_B^{B,Z}\nu_*(\mathcal{F}) \xrightarrow{\simeq} i_*^{B,Z}i_B^*(\mathcal{F}),
\]
and we deduce
\[
i_B^*i_B^*(\mathcal{F}) \simeq \nu_*i_B^{B,Z}\nu_*(\mathcal{F}) \simeq \nu_*\nu^*(\mathcal{F}) \simeq \mathcal{F}.
\]
The proof for framed correspondences is similar. \( \square \)

Lemma 6.13. For every motivic local object \( \mathcal{F} \in \mathcal{H}_*(\text{Sm}_{B,Z}) \) the unit of the adjunction
\[
i_B^*: \mathcal{H}_*(\text{Sm}_{B,Z}) \rightleftarrows \mathcal{H}_*(\text{Sm}^{CCI}_Z): i_*^{B,Z}
\]
induces an equivalence
\[
\mathcal{F} \xrightarrow{\simeq} i_*^{B,Z}i_B^*(\mathcal{F}).
\]
The same holds for every motivic local object \( \mathcal{F} \in \mathcal{H}_*(\text{Fr}_+(\text{Sm}_{B,Z})) \).

Proof. Owing to Lemma 6.12 the unit of the adjunction induces an equivalence
\[
\mathcal{F}|_{\text{Sm}_{B,Z}^{CCI}} \xrightarrow{\simeq} i_*^{B,Z}i_B^*(\mathcal{F})|_{\text{Sm}_{B,Z}^{CCI}}. \tag{6.6}
\]
Our claim follows since \( \mathcal{F} \) is Nisnevich local, and every scheme in \( \text{Sm}_{B,Z} \) (resp. \( \text{Sm}^{CCI}_Z \)) has a Nisnevich covering in \( \text{Sm}_{B,Z}^{CCI} \) (resp. \( \text{Sm}^{CCI}_Z \)). \( \square \)

Lemma 6.14. For every \( \mathbb{A}^1 \)-local object \( \mathcal{F} \in \mathcal{H}_*(\text{Sm}_{B,Z}^{AFF}) \) the unit of the adjunction
\[
(i_{B,Z}^{AFF})^*: \mathcal{H}_*(\text{Sm}_{B,Z}^{AFF}) \rightleftarrows \mathcal{H}_*(\text{Sm}^{AFF}_Z): (i_*^{B,Z})_*
\]
induces an equivalence
\[
\mathcal{F} \xrightarrow{\simeq} i_*^{B,Z}(i_{B,Z}^{AFF})^*(\mathcal{F}).
\]
The same holds for every \( \mathbb{A}^1 \)-local object \( \mathcal{F} \in \mathcal{H}_*(\text{Fr}_+(\text{Sm}_{B,Z}^{AFF})) \).
Proof. This follows from (6.6) since the canonical embedding $\text{Sm}^{cci}_{B,Z} \to \text{SmAff}_{B,Z}$ induces an equivalence between the subcategories of $\mathbb{A}^1$-local objects in $H_s(\text{Sm}_{B,Z})$ and $H_s(\text{Sm}^{cci}_{B,Z})$, see Proposition 5.8. We write $H_{\mathbb{A}^1}(\text{Sm}_{B,Z})$ and $H_{\text{rig}}(\text{Sm}_{B,Z})$ for the subcategories of $H_s(\text{Sm}_{B,Z})$ spanned by $\mathbb{A}^1$-local and rigid presheaves, respectively.

Lemma 6.15. The functor $i^*_{B,Z}$ induces an equivalence of categories

$$i^*_{\text{rig}}: H_{\text{rig}}(\text{Sm}^{cci}_{B,Z}) \xrightarrow{\sim} H_s(\text{Sm}^{cci}_{B,Z}).$$

Moreover, both $i^*_{\text{rig}}$ and its inverse preserve and detect Nisnevich local objects and Nisnevich local equivalences.

Proof. Lemma 6.2 shows there is a fully faithful embedding $r: \text{Sm}_Z \to \text{Sm}_{B,Z}$. Due to (6.1) there are functors

$$\begin{align*}
\text{Sm}_Z & \xrightarrow{r} \text{Sm}_{B,Z} \\
Y & \xleftarrow{u} Y' \\
Y, X_Z & \xleftarrow{\iota} Y, X'_{Z}.
\end{align*}$$

(6.8)

Part (6.1) in Lemma 6.5 shows $r$ is right inverse to $i_{B,Z}$, $i d_{\text{Sm}_Z} \simeq i_{B,Z} \circ r$. Lemma 6.11 shows $i_{B,Z}$ is equivalent to $r_*$. Hence the identity functor on $H_s(\text{Sm}^{cci}_{Z})$ is naturally isomorphic to

$$H_s(\text{Sm}^{cci}_{Z}) \xrightarrow{(i_{B,Z})^*} H_s(\text{Sm}^{cci}_{B,Z}) \xrightarrow{(i_{B,Z})^*} H_s(\text{Sm}^{cci}_{Z}).$$

(6.9)

Owing to the isomorphisms of schemes

$$(X^h_Z) \times_B Z \cong (X_Z) \times_B Z \cong X_Z,$$

the functor $(i_{B,Z})_*$ takes values in $H_{\text{rig}}(\text{Sm}^{cci}_{B,Z})$. By (6.9) there are induced functors

$$H_s(\text{Sm}^{cci}_{Z}) \xrightarrow{i^*_{\text{rig}}} H_{\text{rig}}(\text{Sm}^{cci}_{B,Z}) \xrightarrow{i^*_{\text{rig}}} H_s(\text{Sm}^{cci}_{B,Z}).$$

(6.10)

Here $i^*_{\text{rig}}$ is the restriction of $i^*_{B,Z}$, $i^*_{\text{rig}}$ is induced by $(i_{B,Z})^*$, and the composite is naturally isomorphic to the identity on $H_s(\text{Sm}^{cci}_{Z})$. Moreover, the composite

$$H_{\text{rig}}(\text{Sm}^{cci}_{B,Z}) \xrightarrow{i^*_{\text{rig}}} H_s(\text{Sm}^{cci}_{B,Z}) \xrightarrow{i^*_{\text{rig}}} H_{\text{rig}}(\text{Sm}^{cci}_{B,Z})$$

(6.11)

is naturally isomorphic to the identity on $H_{\text{rig}}(\text{Sm}^{cci}_{B,Z})$, since for all $F \in H_{\text{rig}}(\text{Sm}^{cci}_{B,Z})$ we have

$$i^*_{\text{rig}} i^*_{\text{rig}} F(X^h_Z) = F(r(i_{B,Z}(X^h_Z))) = F(X_Z) \cong F(X^h_Z).$$

for $X \in \text{Sm}^{cci}_{B,Z}$ and similarly for $Y \in \text{Sm}^{cci}_{Z}$.

Since $i_{B,Z}$ and $r$ in (6.8) preserve Nisnevich coverings and fibered products, both $(i_{B,Z})_*$ and $r_*$, and consequently $i^*_{\text{rig}}$, $i^*_{\text{rig}}$, preserve Nisnevich local objects. Since $i^*_{\text{rig}}$ and $i^*_{\text{rig}}$ are inverses, the same functors preserve Nisnevich equivalences.

The functors $i_{B,Z}$ and $u$ from (6.8) induce the adjunctions

$$\begin{align*}
H_s(\text{Sm}^{cci}_{B,Z}) & \xrightarrow{u^*} H_s(\text{Sm}^{cci}_{B,Z}) \xrightarrow{(i_{B,Z})^*} H_s(\text{Sm}^{cci}_{B,Z}) \\
& \xrightarrow{(i_{B,Z})^*} H_s(\text{Sm}^{cci}_{B,Z}).
\end{align*}$$

(6.12)

We shall consider the homotopy categories $SH_s(\text{Sm}^{cci}_{B}), SH_s(\text{Sm}^{cci}_{Z}), SH_s(\text{Sm}^{cci}_{B,Z})$, and the subcategory $SH_{s,\text{rig}}(\text{Sm}^{cci}_{B,Z})$ of $SH_s(\text{Sm}^{cci}_{B,Z})$ spanned by levelwise rigid objects. The considerations
above for simplicial presheaves extend to pointed simplicial presheaves. By levelwise application of \((6.12)\), there are naturally induced functors
\[
\begin{align*}
  u_\ast & : \text{SH}(\text{Sm}^{\text{cci}}_{B,Z}) \to \text{SH}(\text{Sm}^{\text{cci}}_{B,Z}) \\
  (i_{B,Z})^\ast & : \text{SH}(\text{Sm}^{\text{cci}}_{B,Z}) \to \text{SH}(\text{Sm}^{\text{cci}}_{Z}) \\
  (i_{B,Z})_\ast & : \text{SH}(\text{Sm}^{\text{cci}}_{B,Z}) \to \text{SH}(\text{Sm}^{\text{cci}}_{Z}).
\end{align*}
\]

We obtain the following corollaries.

\textbf{Corollary 6.16.} For every \(F \in \text{SH}(\text{Sm}^{\text{cci}}_{B,Z})\) the object \(\mathcal{F}^{A_1}\) is \(A_1\)-local and rigid. The same holds in the framed setting.

\textbf{Corollary 6.17.} The functors \((i^{B,Z})_\ast\) and \((i_{B,Z})^\ast\) induce an equivalence of categories
\[
i^{\ast}_{\text{rig}} : \text{SH}(\text{Sm}^{\text{cci}}_{B,Z}) \xrightarrow{\sim} \text{SH}(\text{Sm}^{\text{cci}}_{Z}) : i^\ast.
\]
Moreover, \(i^{\ast}_{\text{rig}}\) and \(i^\ast_{\text{rig}}\) preserve and detect Nisnevich local objects and equivalences.

\section{7. tf-Nisnevich strict \(A_1\)-invariance}

The strict \(A_1\)-invariance theorems of Garkusha-Panin [17, §17] and Voevodsky [35, §3.2] concern the effect of the Nisnevich localization functor \(L_{\text{Nis}}\) on \(A_1\)-local objects. The verbatim generalization to a base scheme \(B\) and \(S^1\)-spectra takes the following form.

\textbf{Definition 7.1.} Let \(\mathcal{F}\) be an \(A_1\)-local quasi-stable radditive framed presheaf of \(S^1\)-spectra on \(\text{Sm}_B\). We say that Nisnevich strict \(A_1\)-invariance holds on \(\text{Sm}_B\) if, for every \(\mathcal{F}\) as above, \(L_{\text{Nis}}(\mathcal{F})\) is an \(A_1\)-local quasi-stable radditive framed presheaf of \(S^1\)-spectra.

As shown in Section 12 the main result in [17, §17] implies Nisnevich strict \(A_1\)-invariance on \(\text{Sm}_k\) when the field \(k\) is perfect (see Theorem 1.27 for the case of finite fields). In Section 10 we prove that Nisnevich strict \(A_1\)-invariance holds on \(\text{Sm}_{B,Z}\) when \(Z\) is a zero-dimensional scheme with perfect residue fields. In Section 13, however, we show that Nisnevich strict \(A_1\)-invariance fails on \(\text{Sm}_B\) for any positive dimensional base scheme. The same holds for \(\text{Sm}_{B,Z}\) when \(Z\) is positive dimensional. In Definition 7.4 and below we can replace \(\text{Sm}_B\) with \(\text{Sm}_{B,Z}\), \(\text{Sm}_{\text{Aff},B}, \text{Sm}_{\text{Aff},B,Z}, \text{Sm}^{\text{cci}}_{B}, \text{Sm}^{\text{cci}}_{B,Z}\). This motivates our next definition.

\textbf{Definition 7.2.} Let \(\mathcal{F}\) be a quasi-stable radditive framed presheaf of \(S^1\)-spectra on \(\text{Sm}_B\) such that \(L_{\text{tf}}(\mathcal{F})\) is \(A_1\)-local. We say that tf-Nisnevich strict \(A_1\)-invariance holds on \(\text{Sm}_B\) if, for every \(\mathcal{F}\) as above, \(L_{\text{Nis}}(\mathcal{F})\) is an \(A_1\)-local quasi-stable radditive framed presheaf of \(S^1\)-spectra.

\textbf{Remark 7.3.} tf-Nisnevich strict \(A_1\)-invariance implies \(L_{\text{Nis}}\) preserves \(A_1\)-local tf-local quasi-stable radditive framed presheaves of \(S^1\)-spectra, i.e., \(L_{\text{Nis}}^{\ast}\) preserves \(A_1\)-local quasi-stable radditive framed presheaves of \(S^1\)-spectra, see Definition 2.2. Theorem 11.2 shows tf-Nisnevich strict \(A_1\)-invariance implies tf-Nisnevich strict \(A_1\)-invariance for presheaves of abelian groups, see Definition 1.3.

\textbf{Remark 7.4.} By Proposition 3.6 (iii) (resp. Lemma 4.6), the tf-topology is trivial on \(\text{Sm}_B\) (resp. \(\text{Sm}_{B,Z}\)) if and only if \(\dim B = 0\) (resp. \(\dim Z = 0\)). In this case, the notions of Nisnevich and tf-Nisnevich strict \(A_1\)-invariance coincide.

In Section 11 we prove tf-Nisnevich strict \(A_1\)-invariance for one-dimensional schemes with perfect residue fields. Next, we discuss some reduction steps and establish the equivalence between different forms of Nisnevich and tf-Nisnevich strict \(A_1\)-invariance.

\textbf{Lemma 7.5.} Suppose tf-Nisnevich strict \(A_1\)-invariance holds on \(\text{Sm}_{B_{s}}\) for all \(s \in B\), where \(B_{s}\) denotes the local scheme at \(s\). Then tf-Nisnevich strict \(A_1\)-invariance holds on \(\text{Sm}_B\).
Proof. Lemma 5.9 shows the base change morphisms $f^* : \mathbf{SH}(B) \to \prod_{x \in B} \mathbf{SH}_x(B)$ and $f^* : \mathbf{SH}_x(\mathbf{Fr}_x(\mathbf{Sm}_B)) \to \prod_{x \in B} \mathbf{SH}_x(\mathbf{Fr}_x(\mathbf{Sm}_B))$ along $\prod_{x \in B} B_x \to B$ commute with $L_{\text{Nis}}, L_{\text{tf}}$, and $L_{\text{ht}}$. In particular, $f^*$ preserves $\mathbb{A}^1$-local objects. Since $f^*$ detects $\mathbb{A}^1$-local objects on the subcategory of $\text{tf}$-local objects, see Lemma 5.8 we are done.

Lemma 7.6. Suppose $B$ is an affine scheme and $Z \not\to B$ is a closed immersion. Then $\text{tf}$-Nisnevich strict $\mathbb{A}^1$-invariance holds on $\mathbf{Sm}^{\text{rci}}_{B,Z}$ if and only if it holds on $\mathbf{Sm}_{B,Z}$.

Proof. By Proposition 5.8 the restriction functor $\mathfrak{r}_{\text{rci}} : H_x(\mathbf{Sm}_{B,Z}) \to H_x(\mathbf{Sm}^{\text{rci}}_{B,Z})$ preserves and detects $\mathbb{A}^1$-local objects, and it commutes with the localization endofunctors $L_{\text{Nis}}$ and $L_{\text{tf}}$. The desired stable result follows by arguing levelwise.

Lemma 7.7. Suppose $B$ is an affine scheme and $Z \not\to B$ is a closed immersion. Then Nisnevich strict $\mathbb{A}^1$-invariance holds on $\mathbf{Sm}_{B,Z}$ if and only if it holds on $\mathbf{Sm}^{\text{rci}}_{B,Z}$.

Proof. Proposition 5.8 allows us to replace $\mathbf{Sm}^{\text{rci}}_{B,Z}$ with $\mathbf{Sm}_{B,Z}$. Moreover, via Lemma 5.9 and Proposition 5.11 we arrive at $\mathbf{Sm}_{B,Z}$.

8. The Nisnevich Cohomology of Generic Fibers

Suppose $B$ is a one-dimensional base scheme with generic point $\eta$ such that strict $\mathbb{A}^1$-invariance holds for $\eta$. Let $F$ be an $\mathbb{A}^1$-local quasi-stable additive framed presheaf of $S^1$-spectra on $\mathbf{Sm}_{\eta}$. Theorem 8.13 shows that for all $x \in \mathbf{Sm}_B$, $x \in X$, there is an equivalence of $S^1$-spectra

$$L_{\text{Nis}}(F)(X^h_x \times_B \eta) \simeq F(X^h_x \times_B \eta),$$

where $L_{\text{Nis}}$ is the Nisnevich localization endofunctor. An equivalent statement, see Section 12, says that for any $\mathbb{A}^1$-invariant framed quasi-stable additive presheaf of abelian groups $F$ on $\mathbf{Sm}_{\eta}$ we have

$$H^i_{\text{Nis}}(X^h_x \times_B \eta, F_{\text{Nis}}) \simeq \begin{cases} F(X^h_x \times_B \eta) & i = 0 \\ 0 & i > 0. \end{cases}$$

Our idea for proving (8.1) is to find a framed $\mathbb{A}^1$-homotopy between $V_\eta = X^h_x \times_B \eta$ and its generic point. This suffices because the Nisnevich topology on the generic point of $V_\eta$ is trivial, so that the values of $L_{\text{Nis}}(F)$ and $F$ at $\eta$ coincide up to equivalence. Our assumption on the residue fields of the generic points of $B$ is used to ensure $L_{\text{Nis}}(F)$ is $\mathbb{A}^1$-local. We use our assumption $\dim(B) = 1$ for constructing the contracting framed $\mathbb{A}^1$-homotopy. A technical point is that $V_\eta$ and its generic point are not objects of $\mathbf{Sm}_{\eta}$. For an essentially smooth local henselian scheme $V = X^h_x$, where $X \in \mathbf{Sm}_B$, $x \in X$, we construct a linear framed $\mathbb{A}^1$-correspondence over $\eta$

$$\mathbb{A}^1 \times V'_\eta \to X_\eta.$$ (8.3)

Here $V'$ is an étale neighborhood of $x$ in $X$ and $V'_\eta = V' \times_B \eta$. Using (8.3) we obtain for a closed immersion $Y \not\to X$ the $\mathbb{A}^1$-homotopy

$$\mathbb{A}^1 \times V'_\eta \to X_\eta/(X_\eta - Y_\eta)$$ (8.4)

between the canonical morphism $V'_\eta \to X_\eta/(X_\eta - Y_\eta)$ and the constant pointed morphism. Passing to the limit along all possible choices of $X$ and $Y_\eta$ for a given $V$ concludes the argument. The injectivity theorems in [13 §8] for affine lines and essentially smooth local schemes over a field are important precursors for the equivalence (8.1). Contrary to [13 §8], we cannot assume the Picard groups of all closed subschemes of $\mathbb{A}^1$ and $X_x$ are trivial.

The Krull dimension of a topological space is defined in [32, Tag 0055]. A scheme is of pure dimension if all its irreducible components have the same Krull dimension. The codimension of an irreducible closed subset in a topological space is defined in [32, Tag 0213]. The codimension
of a closed subscheme $Y$ in $X$ at $y \in Y$ equals the codimension of local scheme $Y_{(y)}$ of $Y$ at $y$ in $X_{(y)}$. We say $Y$ has pure codimension $c$ if the codimension of $Y$ in $X$ equals $c$ at every point $y \in Y$. Moreover, $X$ has (pure) relative dimension $d$ over $B$ if the (pure) dimension of all the fibers of the structure morphism $f: X \to B$ equals $d$. An equidimensional scheme, a.k.a. pure dimensional scheme, is a scheme all of whose irreducible components are of the same dimension. We say $Y$ has (pure) relative codimension $c$ in $X$ if for each $\sigma \in B$, $\text{codim}_{X_{\sigma}}(Y_{\sigma}) = c$. For a constructible subset $W$ of $X$, if $\text{codim}_{X_{\sigma}}(W_{\sigma}) < c$ for each $\sigma \in B$, we denote its closure by $\overline{W}$.

**Lemma 8.1.** Suppose $B$ is a one-dimensional scheme, and let $z \in B$ be a closed point with open complement $U = B - z$. Let $X$ be an irreducible $B$-scheme of finite type over $B$ such that $X_z = X \times_B z \neq \emptyset$ and $X_U = X \times_B U$ is dense in $X$. If $X_U$ is equidimensional over $U$, then $X$ is equidimensional over $B$.

**Proof.** Without loss of generality, we may assume $X$ is affine, and $B$ is local with generic point $U$. If $\dim X_z = 0$, then $X$ is quasi-finite over some Zariski neighborhood of $z$ in $B$, and the claim follows. Thus we may assume $\dim X_z = d > 0$ and that the claim holds for all $Y$ with $\dim Y_z < d$. Since $X_z$ is affine and $d > 0$ there exists a nonzero regular function $f_z$ on $X_z$ that is not invertible on any of the irreducible components of $X_z$. Indeed, since each irreducible component of $X_z$ has positive dimension there exist disjoint finite sets of closed points $C_0, C_1 \subset X_z$ each of which contains at least one point from every irreducible component of $X_z$. Now, using the Chinese remainder theorem, choose $f_z \in \mathcal{O}_{X_z}(X_z)$ such that $f_z|_{C_0} = 0$ and $f_z|_{C_1} = 1$. Then $Z(f_z)$ is of pure codimension one in $X_z$. Since $X$ is affine, $f_z$ lifts to a regular function $f$ on $X$. Since $Z(f_z) \neq X_z$, it follows that $Z(f) \neq X$. Since $X$ is irreducible, $Z(f)$ is of pure codimension one in $X$. Let $X_1$ be an irreducible component of $Z(f)$. This is a closed subscheme in $X$ of pure codimension one, and $X_1 z := X_1 \times_B z$ is of pure codimension one in $X_z$. Since $X_U$ is dense in $X$, and $X_1 \neq X$, it follows that $X_{1, U} := X_1 \times_B U \neq X_U$. Note that $X_U$ is irreducible since it is dense in the irreducible scheme $X$. Thus $X_{1, U}$ has positive codimension in $X_U$. Further, $X_1 U$ is dense in $X$, so $X_1$ has positive codimension in $X$, and $X_1 z$ has codimension at least 2 in $X$. Hence $X_1 \neq X_1 z$ and $X_1 U \neq \emptyset$. Since $X_1$ is irreducible it follows that $X_{1, U}$ is dense in $X_1$. Summarizing the above we conclude the vanishing locus $X_{1, U} = Z(f|_{X_{1, U}})$ is nonempty and of pure codimension one in $X_U$. Since $X_{1, U}$ is dense in the irreducible scheme $X_1$, $\dim X_{1, z} = \dim X_z$ and $X_{1, z} \neq \emptyset$. By the inductive assumption $X_1$ is equidimensional over $B$. Using that $\text{codim}_{X_z} X_{1, z} = \text{codim}_{X_U} X_{1, U} = 1$ we deduce the same statement for $X$. □

**Lemma 8.2.** Suppose $B$ is a one-dimensional scheme with closed point $z \in B$ and complement $U = B - z$. Let $X$ be a scheme over $B$ such that $X_z = X \times_B z$ and $X_U = X \times_B U$ are nonempty. Assume $Z_U$ is a closed subscheme in $X_U$ of positive codimension over $U$, and let $Z$ be the closure of $Z_U$ in $X$. Then $Z$ has positive codimension over $z$ in $X$.

**Proof.** Without loss of generality we may assume $B$ is local, $Z \times_B z \neq \emptyset$, and $X$, $Z_U$ are irreducible. Lemma 8.1 implies $X$ and $Z$ are equidimensional over $B$. Since $Z_U$ has positive codimension in $X_U$, the same holds for $Z$ in $X$. □

### 8.1. Automorphisms and framed correspondences.

In the following, we introduce framed morphisms and analyze the action of such framed correspondences on Nisnevich local $\mathbb{A}^1$-local quasi-stable framed presheaves of $S^1$-spectra. We refer to Appendix A for our conventions on framed correspondences. Fix a base scheme $B$ and $X \in \text{Sm}_{B}$. If $E \in \text{GL}_n(X)$ we use the same symbol $E \in \text{Fr}_n(X, X)$ to denote the framed correspondence

$$(0 \times X, E(t_1, \ldots, t_n), \text{pr}: (\mathbb{A}^n_X)_0 \times X \to X).$$
Here $E(t_1, \ldots, t_n)$ is the vector of regular functions on $\mathbb{A}_X^n$ obtained from multiplication by $E$ on the coordinates $(t_1, \ldots, t_n)$ of $\mathbb{A}_X^n$. We let $E_n(X) \subset \text{GL}_n(X)$ denote the subgroup generated by all elementary matrices. In particular, the $(n \times n)$-identity matrix $\text{id}_n \in E_n(X)$.

**Lemma 8.3.** If $E \in E_n(X)$ then $E = \text{id}_n \in \mathcal{F}_n(X, X)$. Thus, for every $X \in \text{Sm}_B$ and $\mathbb{A}^1$-local framed presheaf $\mathcal{F}$ of $S^1$-spectra on $\text{Sm}_B$, the morphism $E^* : \mathcal{F}(X) \to \mathcal{F}(X)$ is an equivalence.

**Proof.** We write $E = E_1 \cdots E_m$ for elementary matrices $E_i$ and set

$$E_i(\lambda) = (1 - \lambda)E_i + \lambda \text{id}_n \in \mathcal{F}_n(X \times \mathbb{A}^1, X).$$

The framed correspondences $E_i(\lambda) \in \mathcal{F}_n(X \times \mathbb{A}^1)$ for $1 \leq i \leq m$ yields the $\mathbb{A}^1$-homotopies $E = E_1 \cdots E_m \sim\sim E_1 \cdots E_{m-1} \sim\sim \cdots \sim\sim E_1 \sim\sim \text{id}_n$.

Proposition \[A.13\] implies the claim for $E^* : \mathcal{F}(X) \to \mathcal{F}(X)$.

**Lemma 8.4.** Let $\mathcal{F}$ be an $\mathbb{A}^1$-local quasi-stable framed presheaf of $S^1$-spectra on $\text{Sm}_B$. Suppose $X$ is an essentially smooth local scheme over $B$. If $E \in \text{GL}_n(X)$, then the framed correspondence $E \in \mathcal{F}_n(X, X)$ induces an equivalence $E^* : \mathcal{F}(X) \to \mathcal{F}(X)$.

**Proof.** When $E \in \text{SL}_n(X)$, then $E \in E_n(X)$ since $X$ is local and the claim follows from Lemma 8.3. Let $(E^{-1}, E) \in \text{GL}_{2n}(X)$ denote the block-diagonal matrix

$$
\begin{pmatrix}
E^{-1} & 0 \\
0 & E
\end{pmatrix}.
$$

With this definition, we have equalities of framed correspondences

$$(E^{-1}) \circ E = (E, E^{-1}), E \circ (E^{-1}) = (E^{-1}, E) \in \mathcal{F}_{2n}(X, X).$$

It follows that

$$(E^{-1})^* \circ E^* = (E^{-1}, E)^*, E^* \circ (E^{-1})^* = (E^{-1}, E)^*.$$

Since $(E^{-1}, E), (E, E^{-1}) \in \text{SL}_{2n}(X)$ we conclude that $E^*$ and $(E^{-1})^*$ are equivalences.

**Lemma 8.5.** Let $\mathcal{F}$ be an $\mathbb{A}^1$-local quasi-stable framed presheaf of $S^1$-spectra on $\text{Sm}_B$. Suppose $c = (Z, \varphi_1, \ldots, \varphi_n, \text{pr}) \in \mathcal{F}_n(X, X)$, where $\text{pr}$ is induced by the projection $\mathbb{A}_X^n \to X$ and induces an isomorphism $Z \cong X$. Then $c^* : \mathcal{L}_{\text{Nis}}(\mathcal{F})(X) \to \mathcal{L}_{\text{Nis}}(\mathcal{F})(X)$ is an equivalence.

**Proof.** First we reduce to the case when $Z = 0 \times X$. Since $c$ is a framing of $\text{id}_X$, the canonical projection induces an equivalence $Z \cong X$. There is an $\mathbb{A}^1$-homotopy $c \sim\sim c'$ between $c$ and $c' = (0 \times X, T^*(\varphi), \text{pr}) \in \mathcal{F}_n(X, X)$.

Here $l = (l_1, \ldots, l_n) : X \cong Z \hookrightarrow \mathbb{A}_X^n$, and

$$T = \text{id}_{\mathbb{A}^1_X} + l \text{pr} : \mathbb{A}^n_X \to \mathbb{A}^n_X; (t_1, \ldots, t_n) \mapsto (t_1 + l_1, \ldots, t_n + l_n).$$

The $\mathbb{A}^1$-homotopy is given by the framed correspondence $c_{\lambda} = (T_{\lambda}^{-1} \circ (Z \times \mathbb{A}^1_X), T_{\lambda}(\varphi), (\text{pr} \times \text{id}_{\mathbb{A}^1_X} \circ T_{\lambda}) \in \mathcal{F}_n(X \times \mathbb{A}^1, X)$,

where

$$T_{\lambda} = \text{id}_{\mathbb{A}^n_X} + \lambda \text{pr} : \mathbb{A}^n_X \times \mathbb{A}^1 \to \mathbb{A}^n_X; (t_1, \ldots, t_n) \mapsto (t_1 + \lambda l_1, \ldots, t_n + \lambda l_n).$$

Note that $(\text{pr} \times \text{id}_{\mathbb{A}^1_X}) = (\text{pr} \times \text{id}_{\mathbb{A}^1_X}) \circ T_{\lambda}$, where $(\text{pr} \times \text{id}_{\mathbb{A}^1_X}) : \mathbb{A}^n_X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ is the canonical projection. Then $T_0 = \text{id}_{\mathbb{A}^n_X}$, $T_1 = T$, and

$$
\begin{align*}
0 & = (T_0^{-1}(Z), T_0^*(\varphi), (\text{pr} \circ T_0)) = (Z, \varphi, \text{pr}) = c, \\
c_1 & = (T_1^{-1}(Z), T_1^*(\varphi), (\text{pr} \circ T_1)) = (0 \times X, T^*(\varphi), \text{pr}) = c'.
\end{align*}
$$

Thus we may assume $c = c'$ and $Z = 0 \times X$. 

\[28\]
For any polynomial $f = f_0 + f_1 + \cdots + f_d \in \mathcal{O}_X([t_1, \ldots, t_n])$, where $f_i$ is homogeneous of degree $i$, we have $f - (f_0 + f_1) \in I_{k^n \times X}(0 \times X)$ for the linear polynomial $f_0 + f_1$. We may choose linear polynomials $\varphi_1, \ldots, \varphi_n \in \mathcal{O}_X([t_1, \ldots, t_n]) \cong \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n_X)$ such that $\varphi_i - \varphi_i \in I_{k^n \times X}(0 \times X)$ for $i = 1, \ldots, n$. Then $\varphi_i \in I_{k^n \times X}(0 \times X)$, and hence $(\varphi_1, \ldots, \varphi_n) \cong E(t_1, \ldots, t_n)$ for some $E \in GL_n(X)$. Here $I_{k^n \times X}(0 \times X)$ is the ideal sheaf of the closed immersion $0 \times X \hookrightarrow \mathbb{A}^n \times X$.

Now the framed correspondence

$$(0 \times X \times \mathbb{A}^1, (1 - \lambda)\varphi + \lambda\varphi', (pr \times id_{\mathbb{A}^1})) \in \mathcal{F}_n(X \times \mathbb{A}^1, X)$$

gives an $\mathbb{A}^1$-homotopy $c \sim c''$ between $c$ and

$$c'' = (X \times \mathbb{A}^1, \varphi', pr) \in \mathcal{F}_n(X, X).$$

Hence we may assume $c = c''$ and $(\varphi_1, \ldots, \varphi_n) \cong E(t_1, \ldots, t_n)$ for some $E \in GL_n(X)$.

To each $\varphi \in X$ we can associate the framed correspondence $c_{\varphi} = E_{\varphi} \in \mathcal{F}_n(X_{\varphi}^h, X_{\varphi}^h)$, where $E_{\varphi} \in GL(X_{\varphi}^h)$ is the stalk of $E$. Lemma 8.5 shows there is an induced equivalence of stalks

$$c_{\varphi}^* : \mathcal{F}(X_{\varphi}^h) \xrightarrow{\sim} \mathcal{F}(X_{\varphi}^h).$$

This is, $c^* : \mathcal{F}(X) \to \mathcal{F}(X)$ is a Nisnevich local equivalence. This finishes the proof.

**Corollary 8.6.** Suppose $B$ is affine and let $\mathcal{F}$ be an $\mathbb{A}^1$-local quasi-stable framed presheaf of $S^1$-spectra on $\mathcal{S}_{\mathcal{N} B}$. For every framing $c = (Z, \varphi_1, \ldots, \varphi_n, \gamma) \in \mathcal{F}_n(X, Y)$ of $r : X \to Y$, we have $c^* = r^* : \mathcal{L}_{\mathcal{N}B}(\mathcal{F})(Y) \to \mathcal{L}_{\mathcal{N}B}(\mathcal{F})(X)$.

**Proof.** We note that $id^c = (Z, \varphi_1, \ldots, \varphi_n, pr) : (\mathbb{A}^n_X)^h \to X \in \mathcal{F}_n(X, X)$ is a framing of $id_X$. Since $c$ and $g \circ id^c$ define the same intermediate framed correspondence, see Definition A.17, we conclude using Lemma 8.3 and Lemma 8.15.

**8.2. Contracting framed $\mathbb{A}^1$-homotopies.** We fix a smooth irreducible scheme $X$ over a one-dimensional base scheme $B$. Suppose $x \in X$ maps to some closed point $z \in B$, and set $V = X_x^h$, $U = B - z$, $V_U = V \times_B U$, $X_U = X \times_B U$. Proposition 8.11 shows that for every closed immersion $Y_U \hookrightarrow X_U$ of relative positive codimension there exists a linear framed $\mathbb{A}^1$-homotopy $c : V_U \times \mathbb{A}^1 \to X_U/(X_U - Y_U)$ between the canonical morphism $V_U \to X_U/(X_U - Y_U)$ and the constant pointed morphism.

To achieve this, we construct diagrams in $\mathcal{S}_{\mathcal{N}B}$ of the form

$$
\begin{array}{ccc}
  & & P \\
  & r & \\
 X_x^h & \downarrow j & C \\
 V_U & \downarrow i & C \cup \! \! \cup C \\
 \end{array}
$$

subject to the following properties, see Lemmas 8.8 to 8.10 for further refinements.

1. $\varphi$ is a projective equidimensional morphism, $p$ is affine, $p$ is smooth at $r(x)$, $C_U \in \mathcal{S}_{\mathcal{N}B}$, where $P = P \times_B U$.
2. $i$ is a closed immersion such that $i(C_{\infty})$ has positive relative codimension over $P$, $j$ is an open immersion. Moreover, there exists an ample bundle $\mathcal{O}(1)$ on $\mathcal{C}$ over $P$ with a section $x_{\infty}$ such that $C_{\infty} \cong Z(x_{\infty})$ and $C_U \cong (\mathcal{C} - C_{\infty}) \times_B U$.
3. The closure $\overline{W}$ of $W_U = v^{-1}(Y_U)$ in $\mathcal{C}$ has positive relative codimension in $\mathcal{C}$ over $P$. 

**Strict $\mathcal{A}^1$-Invariance Over the Integers**
Theorem on sections of ample bundles \cite[Theorem 5.2]{19} implies that for

This implies (0) when setting

and

While the two composites in \((8.6)\) and \((8.7)\) agree, \((8.6)\) moves the canonical morphism via an \(\mathbb{A}^1\)-homotopy whereas \((8.7)\) moves it via a morphism \(C \times_P V_U \to X_U\) which extends to the compactification \(\overline{C}\). For the desired framed \(\mathbb{A}^1\)-homotopy over \(U\) we equip \(C_U\) with a framing over \(P_U\) such that \(C_U \in \text{Sch}_\mathbb{C}^{\text{ct}}\). In the remaining of this section we refine the properties (0)-(3) for \((8.5)\). To that end we construct in total \(\dim_B X + 1\) diagrams of this form. In particular, we study the relative dimension of \(C\) over \(P\). Lemma \ref{8.10} constructs \(C \to P\) as a relative curve.

Next we define \(\text{Sch}_S^{\text{ct}} \subset \text{Aff}_S\) for any scheme \(S\), see Section \ref{2.3} for the smooth case.

**Definition 8.7.** Let \(S\) be a scheme. We say \(X\) is a cci-scheme over \(S\) and write \(X \in \text{Sch}_S^{\text{ct}}\) if \(Z(f) = X \amalg X'\), where \(f = (f_1, \ldots, f_m)\) consists of regular functions on \(\mathbb{A}^n_S\) and \(\text{codim}_{\mathbb{A}^n_S} Z = m\).

**Lemma 8.8.** Suppose \(B\) is a one-dimensional scheme and \(z \in B\) is a closed point with complement \(U = B \setminus z\). Fix \(X \in \text{Sm}_B\), \(x \in X \times_B z\), \(V, V_U, X_U, Y_U\) as above, and let \(n = \dim_B X_x\) be the relative dimension of \(X\) at \(x\) over \(B\), \(X_x\) be the local scheme of \(X\) at \(x\). Then there is a diagram of the form \((8.5)\) such that \(\dim_B \overline{X} = n = \dim_B X_x\), \(P = B_z\), and \(p_U\) is smooth.

**Proof.** By shrinking \(X\) to an étale neighborhood of \(x\) and using \(X^h_B \cong X^h_B \times_B B^h\) we may assume \(B = B^h\) is local and \(X\) is irreducible. Since \(X \in \text{Sm}_B\), there exists an open affine neighborhood \(X'\) of \(x\) in \(X\) such that the relative tangent bundle of \(X'\) over \(B\) is trivial. Note that \(X' \in \text{Sm}_B^{\text{ct}}\). Since \(X'\) is affine, its schematic closure inside some projective space yields an open immersion \(X' \hookrightarrow \overline{X} \subset \overline{X}\). Lemma \ref{8.4} shows \(\overline{X} := \overline{X}\) is equidimensional over \(B\).

To construct \(C\) we consider \(\overline{C} - X'\) and let \(X'_\infty\) be the closure of \((\overline{C} - X') \times_B U\) in \(\overline{C}\). Then \(X'_\infty\) has positive codimension over \(z\), see Lemma \ref{8.2} and \(x \in X' \subset \overline{C} - X'_\infty\). Let \(\mathcal{O}(1)\) denote the canonical ample invertible sheaf on \(\overline{C}\) over \(B\). We may choose a finite set \(F\) of closed points in \(\overline{C}\) that has a nonempty intersection with each irreducible component of \(\overline{C} - X'_\infty\). Serre’s Theorem on sections of ample bundles \cite[Theorem 5.2]{19} implies that for \(d \gg 0\) there exists a section \(x_\infty \in \Gamma(\overline{C}, \mathcal{O}(d))\) such that \(x_\infty|_{X'_\infty} = 0\), and \(x_\infty|_{F \cup x}\) is invertible. Then \(Z(x_\infty)\) has positive relative codimension in \(\overline{C}\) over \(z\), and we set

\[
C_\infty := Z(x_\infty), \quad C := \overline{C} - C_\infty.
\]

Using the open immersion \(C = \overline{C} - Z(x_\infty) \hookrightarrow \overline{C} - X'_\infty\) and the fiber product \(X'_\infty \times_B U \cong (\overline{C} - X') \times_B U\) we obtain the composite

\[
v: C_U = C \times_B U \hookrightarrow (\overline{C} - X'_\infty) \times_B U \cong X' \times_B U \to X_U.
\]

Since \(C\) is an open neighborhood of \(x\) in \(\overline{C}\) and \(X'\) is an open neighborhood of \(x\) in \(\overline{C}\), the canonically induced composite

\[
\tau: V = X^h_x \to C \to \overline{C},
\]

coincides with

\[
V = X^h_x \to C \to \overline{C}.
\]

This implies (0) when setting \(P = B_z\). Moreover, the \(B\)-scheme \(C\) is smooth over \(r(x)\) since the localization of \(C\) at \(r(x)\) agrees with the localization of \(X\) at \(x\). Lemma \ref{8.10} implies that the projective morphism \(\tau: C \to \overline{C}\) is equidimensional since \(X'\) and consequently \(C\) are irreducible. Since \(C_\infty = Z(x_\infty)\) is the vanishing divisor of a section of some ample sheaf, \(C = \overline{C} - C_\infty\) is
affine over $B$. In addition, $C_{\infty}$ has positive relative codimension in $\mathcal{C}$ by (8.8) and the choice of $x_{\infty}$. Since $x_{\infty}\big|_{X'_{\infty}} = 0$ and $X''_{\infty} \not\to C_{\infty}$, there is an open immersion
\[(\mathcal{C} - C_{\infty})) \times_B U \hookrightarrow (\mathcal{C} - X''_{\infty}) \times_B U \cong X' \times_B U,
\]
and since $C_U$ is affine over $P$, it follows that $C_{U}$ is smooth with a trivial tangent bundle over $P_U = U$. This completes the proofs of (1) and (2). Further (1) implies that $W_U$ has positive codimension in $C_U$ since $\mathcal{C}$ and consequently $C$ and $C_U$ are irreducible. Part (3) follows by Lemma 8.2. Finally, by construction, $\dim_B \mathcal{C} = \dim_B X_\infty$, and $\mu_U$ is smooth since $C_U \in \text{Sm}^C_{\mathcal{C}_1}$.

**Lemma 8.9.** In the setting of Lemma 8.8, assume more generally that $1 \leq n \leq \dim_B X_\infty$. Then in (8.8) we have $\dim_P \mathcal{C} = n$ and $P$ is a local henselian scheme.

**Proof.** We may assume $X$ is irreducible. The proof uses induction on $n$. Lemma 8.8 verifies the case $n = \dim_B X_\infty$ with $C$ and $\mathcal{C}$. We shall construct $C_1$ and $\mathcal{C}_1$ for $n = 1$. Since $C$ is smooth at $r(x)$ over $P$ there exists an étale morphism $f = (f_1, \ldots, f_n): C \to \mathbb{A}^n_B$. Note that $p(r(x))$ is the closed point of $P$. We set $T_X = Z(x_{\infty}) \cap (\mathcal{C} - r(x))$ and choose a finite set of closed points $F_{\infty}$ in $T$ that contains at least one irreducible component of $T \times_P p(r(x))$.

Let $W_\infty$ be the union of the irreducible components of $W$ with positive dimension over $P$, and set $T_W = W_\infty \cap (C - (r(x) \cup F))$. Let $F_{W,0}, F_{W,1}$ be disjoint finite sets of closed points in $W_\infty \times_P p(r(x))$. By construction, $(F_{W,0} \cup F_{W,1}) \cap F_{\infty} = \emptyset$ and $r(x) \not\in F_{\infty}, F_{W,0}, F_{W,1}$. For $d > 0$ there exists a section $s \in \Gamma(\mathcal{C}, \mathcal{O}(d))$ such that $s|_{F_{W,1}}$ and $s|_{F_{W,0}}$ are invertible, $s|_{F_{W,0}} = 0$, and $s|_{Z((p_x(x_{\infty})))} = x_{\infty}^d f_n$. (8.9)

Here $I_{Z(r(x))} \subset \mathcal{O}_{\mathcal{C}}$ is the ideal sheaf of $r(x)$ contained in the structure sheaf of $\mathcal{C}$. With this definition we have

\[
\dim_P (Z(s) \cap Z(x_{\infty})) \leq \dim_P Z(x_{\infty}) - 1 \leq \dim_P \mathcal{C} - 2 = n - 2.
\]

We write $[t_0 : t_\infty]$ for the coordinates on $\mathbb{P}^1$ and form the section
\[
g = st_\infty + x_{\infty}^d f_0 \in \Gamma(\mathcal{C} \times \mathbb{P}^1, \mathcal{O}(d, 1)),
\]
where $\mathcal{O}(d, 1)$ is the tensor product of the pullback of $\mathcal{O}(d)$ to $\mathcal{C}$ with $\mathcal{O}(1)$ on $\mathbb{P}^1$. Let $P_1$ be shorthand for $(\mathbb{P}^1_P)^b_{(r(x))}$ where $f_0(r(x)) \in \mathbb{A}^1_B$ is the image of $r(x)$ along $f_0: C \to \mathbb{A}^1_B$. We set
\[
\mathcal{C}_1 = Z(g) \times_{\mathbb{P}^1} P_1, C_{\infty,1} = \mathcal{C}_1 \times_{\mathcal{C}} C_{\infty}, C_1 = \mathcal{C}_1 \times_{\mathcal{C}} C, r_1 = (r_1, s : x_{\infty}^d) : V \to C_1.
\]

Moreover, we set $C_{1,U} = C_1 \times_B U$ and form the composite
\[
v_1: C_{1,U} \to C_U \to X_U.
\]

Since $x_{\infty}$ in invertible on $C$ we can identify $C_1$ with $C \times_{\mathbb{A}^1_B} P_1$ under the morphism $C \to \mathbb{A}^1_B$, given by the regular function $s/x_{\infty}^d$ on $C$. We can identify $C_{\infty,1}$ with $(Z(s) \cap Z(x_{\infty})) \times_P P_1$. Note that $C_{\infty} = Z(x_{\infty})$ has positive codimension, and it has pure relative codimension one in $\mathcal{C}$ over $P$. The same holds for $C_{\infty,1}$ in $\mathcal{C}_1$. Setting $\max \dim_T (\mathcal{C}_1) := \max_{c \in \mathcal{C}_1} \dim_T (\mathcal{C}_1)_c$ and similarly for $C_{1,\infty}$, we have
\[
\max \dim_T (\mathcal{C}_1) \leq \max \dim_T C_{1,\infty} + 1 = \dim_P (Z(s) \cap Z(x_{\infty})) \leq \dim_B \mathcal{C} - 2 = n - 2.
\]

In the second inequality, we use (8.10). On the other hand, we have
\[
\min \dim_P (\mathcal{C}_1) := \min_{c \in \mathcal{C}_1} \dim_P (\mathcal{C}_1)_c \geq \dim_P \mathcal{C}_1 - \dim_P P_1 = \dim_P \mathcal{C} - 1 = n - 1.
\]

This shows $\mathcal{C}_1$ has relative dimension $n - 1$ over $P_1$. By (8.2), it follows that $\mathcal{C}_1$ is smooth at $r_1(x)$ over $P_1$. Note that $C_1 \times_B U$ is the vanishing locus of the regular function $s/x_{\infty}^d - t_0/t_\infty$ on
Lemma 8.8. Then in can such that for the canonical morphisms $r$ since $s|_{Fv,o} = 0$ and $s|_{Fw,o}$ is invertible, the maximal dimension of $W'$ over $P_1$ is less then the maximal dimension of $W$ over $P$. Since $\dim_{P_1} C_1 = \dim_{P} C_1 - 1$, $W'$ has positive relative codimension in $C_1$ over $P_1$. Consequently, the closure $W'_1$ and $W_1$ are of positive relative codimension in $C_1$ over $P_1$. □

Lemma 8.10. Let $B, z \in B, U = B - z, X \in Sm_B, x \in X \times_B z, V, V_U, and X_U$ be as in Lemma 8.8. Then in (8.5) we can arrange that $\dim_P C_1 = 1$, $P \cong V = X_U^b$, the closure $W$ of $v^{-1}(Y_U)$ in $C$ is finite over $P$, and $W \cap C_\infty = \emptyset$.

Proof. We apply Lemma 8.9 for $n = 1$. By base change along $V \to P$ we may assume $P \cong V$. We shall modify $C$ in such a way that $C_\infty$ and $W_\infty$ become finite over $P$ and $W_\cap C_\infty = \emptyset$. Note that $C_\infty$ is a closed subscheme of positive relative codimension in $C$ over $P$. The same holds for $W$, see Lemma 8.9 and property (3) of (8.5). Hence $C_\infty$ and $W_\infty$ are finite over the local henselian scheme $P$, i.e., both schemes are disjoint unions of local henselian schemes. We have $W = W_1 \amalg W_\infty$, where $r(x) \in W_1$, $r(x) \notin W_\infty$. Note that $C_\infty$ and $W_\infty$ are of positive codimension in $C$. Hence there exists a finite set of closed points $F$ in $C - (C_\infty \cup W_\infty)$ that contains at least one point on each irreducible component of $C$. Now choose a section $x'_\infty \in \Gamma(C, \mathcal{O}(d))$ for $d \gg 0$ such that $x'_\infty|_{C_\infty \cup W_\infty} = 0$ and $x'_\infty|_{W_\cup F}$ is invertible.

Next we set $\mathcal{O}_{new}(1) := \mathcal{O}(d), x_{new} := x'_\infty, C_{new} := C - Z(x_{new}), C_{\infty,new} := Z(x_{new}).$ Here $C_{\infty,new}$ is finite over $P$ since $x_{new}|_F$ is invertible. Then $C_{U,new} \cong C_{new} \times_B U$, and we let $v_{new}: C_{U,new} \to X$ be the restriction of $v$. The closure $W_{new}$ of $v_{new}^{-1}(Y_U)$ in $C$ equals $W_\cap C = W_1$. It follows that $W_{new} \cap C_{\infty,new} \cong W_1 \cap Z(x_{new}) = \emptyset$, and we are done. □

Proposition 8.11. In the setting of Lemma 8.10 there exists linear framed correspondences

\[ c \in \mathcal{Z}_{F_n,U}(\mathbb{A}^1 \times V_U, X_U), \tilde{c}_1 \in \mathcal{Z}_{F}(V_U, X_U - Y_U), \tag{8.11} \]

such that for the canonical morphisms $c: V_U \to X_U$ and $c: X_U - Y_U \to X_U$ we have

\[ c \circ i_0 = c \circ i_1 = j \circ \tilde{c}_1. \]

Here $c'$ is a level $n$ framing of $c$.

Proof. We use the properties of (8.5) shown in Lemma 8.10. Since $C_U \in Sch^{\text{ng}}$, there are regular functions $\varphi = (\varphi_1, \ldots, \varphi_N - 1) \in \mathcal{O}_{\mathbb{A}^N}((A_1^N))$ such that $Z(\varphi) = X_U \amalg \hat{X}_U$ for some $\hat{X}^U$. Both $W_\infty$ and $C_\infty$ are finite over the local scheme $V$ since $W_\cap C_\infty = \emptyset$. The closed subscheme $\Delta = r(V)$ of $C$ is local and $\Delta \cap C_\infty = \emptyset$. Moreover, since $C_\infty$ and $\Delta \cup W_\infty$ are semi-local affine schemes, all line bundles on $C_\infty$ and $\Delta \cup W_\infty$ are trivial. The constant non-zero sections of the trivial bundles on $C_\infty$ and $W_\cup \Delta$ define invertible sections

\[ u_\infty \in \Gamma(C_\infty, \mathcal{O}(1)), \quad u_{Y,\Delta} \in \Gamma(W_\cup \Delta, \mathcal{O}(1)). \]

Since $C \to V$ is smooth over $\Delta$, there exists an invertible sheaf $\mathcal{L}(\Delta)$ on $\mathcal{C}$ with a section $\delta \in \Gamma(C, \mathcal{L}(\Delta))$ such that $Z(\delta) = \Delta$. It follows that $\delta|_{C_\infty}$ and $x_{\infty}|_{\Delta}$ are invertible. Similarly, since $W \cup \Delta$ is an affine semi-local scheme, there exists an invertible section

\[ \tilde{\delta}_{\Delta} \in \Gamma(W_\cup \Delta, \mathcal{O}(\Delta)). \]

Recall that $W \cap C_\infty = \emptyset$ and $\Delta \cap C_\infty = \emptyset$. Serre’s theorem on ample bundles implies that for $d \gg 0$, there exists sections satisfying the properties:
Using the above, we define the section
\[ s = \delta s_0^+ \lambda + s_1 (1 - \lambda) \in \Gamma (\overline{C} \times \mathbb{A}^1, \mathcal{O}(d)). \]

Since \( s_{(\overline{C} \times \mathbb{A}^1)} = u_{d}^d \) is invertible on \( \overline{X}_\infty \times \mathbb{A}^1 \), we see that \( Z(s) \) is finite over \( \overline{C} \times \mathbb{A}^1 \). Thus we obtain the framed correspondence
\[ c' = (Z(s) \times_B U, (\varphi_1, \ldots, \varphi_{N-1}, s/x_{\infty}^d, g)) \in \text{Fr}_N((X^h_U)_U \times \mathbb{A}^1, X_U), \]
where, by Lemma [3.10] the morphism \( g: (\mathbb{A}^N_U)_U \times_B U \to X_U \) lifts the composite
\[ Z(s) \times_B U \to C_U \overset{\varphi}{\to} X_U. \]

The sections \( s_{0\Delta}^+ \) and \( s_{1\Delta} \) are invertible. Thus \( Z(s_0^+) \) and \( Z(s_1) \) are open in \( g^{-1}(X_U - Y_U) \) and there are framed correspondences
\[ \tilde{c}_U^+ = (Z(s_0^+), (\phi_1, \ldots, \phi_{N-1}, (\delta s_0^+/x_{\infty}^d), \tilde{g}_0^+)) \in \text{Fr}_N((X^h_U)_U, X_U - Y_U), \]
\[ \tilde{c}_U^+ = (Z(s_1), (\phi_1, \ldots, \phi_{N-1}, s_1/x_{\infty}^d), \tilde{g}_1) \in \text{Fr}_N((X^h_U)_U, X_U - Y_U). \]

Here \( \tilde{g}_0^+ \) and \( \tilde{g}_1 \) are induced by \( g \). Since \( s_0^+ \) is invertible, we deduce the equalities
\[ c' \circ i_0 = \text{can}^r + j \circ c_0^+, \]
\[ c' \circ i_1 = j \circ \tilde{c}_U, \]
(8.12)

where \( \text{can}^r = (\Delta, (\phi_1, \ldots, \phi_{N-1}, (\delta s_0^+/x_{\infty}^d), g_0, \Delta) \in \text{Fr}_n((X^h_U)_U, X_U), g_0, \Delta: (\mathbb{A}^N_U)_U \to X_U \)
is induced by \( g \), and \( V_U = (X^h_U)_U \). Setting \( c' = c' - j \circ \tilde{c}_U \) and using (8.12) finishes the proof. \qed

### 8.3. Triviality of the cohomology on \( (X^h_U)_U \)

Let \( F \in \text{Spt}_*(\text{Fr}_+(B)) \) be a radditive framed presheaf of \( S^1 \)-spectra on \( \text{Sm}_B \). For a morphism \( \tilde{X} \to X \) in \( \text{Sm}_B \) and \( U \in \text{Sm}_B \), the Čech construction \( \check{C}_{\tilde{X}}(U, F) \) is the homotopy limit of the diagram of \( S^1 \)-spectra
\[ F(\tilde{X} \times_X U) \rightarrow F(\tilde{X} \times_X \tilde{X} \times_X U) \rightarrow F(\tilde{X} \times_X \tilde{X} \times_X \tilde{X} \times_X U) \rightarrow \cdots. \]

We write \( \check{C}_{\tilde{X}}(F) \in \text{Spt}_*(\text{Fr}_+(B)) \) for the radditive framed presheaf of \( S^1 \)-spectra given by
\[ U \mapsto \check{C}_{\tilde{X}}(U, F). \]

Suppose \( \tilde{X}_1 \to X \) and \( \tilde{X}_2 \to X \) are Nisnevich coverings, where \( \tilde{X}_1 \) is a refinement of \( \tilde{X}_2 \). Then the inverse image functor induces a natural morphism
\[ \check{C}_{\tilde{X}_1}(F) \to \check{C}_{\tilde{X}_2}(F). \]

**Lemma 8.12.** For any \( F \in \text{Spt}_*(\text{Sm}_B) \) and \( X \in \text{Sm}_B \) there is a canonical equivalence
\[ \mathcal{L}_{\text{Nis}}(F)(X) \simeq \text{hocolim}_X \check{C}_{\tilde{X}}(F)(X). \]
(8.14)

The homotopy colimit is taken over the filtering system of Nisnevich coverings \( \tilde{X} \) of \( X \).

**Proof.** Let \( \mathcal{L}'_{\text{Nis}}(F)(X) \) be shorthand for the homotopy colimit in (8.14). Since the Nisnevich squares generate the Nisnevich topology, it follows that \( \mathcal{L}'_{\text{Nis}}(F) \) is Nisnevich local. Hence there is a canonical equivalence
\[ \mathcal{L}'_{\text{Nis}}(F) \simeq \mathcal{L}_{\text{Nis}} \mathcal{L}'_{\text{Nis}}(F) \]
(8.15)
If \(F\) is Nisnevich local, there is an equivalence \(\mathcal{F}(X) \simeq \mathcal{C}_X(\mathcal{F})(X)\) for every Nisnevich covering \(\tilde{X} \to X\) (this holds more generally for bounded, complete, regular cd-structures). It follows that \(\mathcal{F} \simeq \mathcal{L}_{\text{Nis}}'(\mathcal{F})\). Hence there is a canonical equivalence

\[
\mathcal{L}_{\text{Nis}}(\mathcal{F}) \xrightarrow{\simeq} \mathcal{L}_{\text{Nis}}' \mathcal{L}_{\text{Nis}}(\mathcal{F}). \tag{8.16}
\]

By definition \(\mathcal{L}_{\text{Nis}}'\) commutes with \(\mathcal{L}_{\text{Nis}}\) on \(H_*\left(\text{Sm}_B\right)\). To conclude we combine \[8.15\], \[8.16\]. □

In the proof of the next result, we will make use of the definitions

\[
\tilde{\mathcal{C}}(\mathcal{F}) := \text{hocofib}(\mathcal{F} \to \tilde{\mathcal{C}}(\mathcal{F})), \quad \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F}) := \text{hocofib}(\mathcal{F} \to \mathcal{L}_{\text{Nis}}(\mathcal{F})).
\]

**Theorem 8.13.** Suppose that \(B\) is a one-dimensional scheme such that strict \(\mathbb{A}^1\)-invariance holds over its generic points. For a closed point \(x \in B, X \in \text{Sm}_B\), and \(x \in X \times_B z\), we set \(U = B - z, X_U = X \times_B U\), and \((X^h_U) = X^h \times_B U\). Then for any \(\mathbb{A}^1\)-local quasi-stable additive framed presheaf \(\mathcal{F}\) of \(S^1\)-spectra over \(U\), there is a natural equivalence

\[
\mathcal{F}((X^h_U)) \xrightarrow{\simeq} \mathcal{L}_{\text{Nis}}(\mathcal{F})((X^h_U)).
\]

**Proof.** We may assume \(B\) is a local domain with generic point \(U = \eta\). For every Nisnevich covering \(\tilde{X}_U \to X_U\) there exists a closed immersion \(Y_U \hookrightarrow X_U\) of positive relative codimension and a lifting \(X_U - Y_U \to \tilde{X}_U\) of the open immersion \(j: X_U - Y_U \hookrightarrow \tilde{X}_U\). We set \(V_U = X^h \times_B U, V_U = \tilde{X}_U \times \times V_U\), and \(Y_U = Y_U \times \times V\). Proposition \[8.11\] furnishes a framed correspondence \(c \in \mathbb{Z}\mathcal{F}_N(V_U \times \mathbb{A}^1, X_U)\) such that for the zero section \(i_0\) and the unit section \(i_1\) of \((X^h_U) \to (X^h_U) \times \mathbb{A}^1\) we have

\[
c \circ i_0 = c_0 = \text{can}^\nu, c \circ i_1 = c_1 = j \circ c_1. \tag{8.17}
\]

Recall that \(\text{can}^\nu \in \mathbb{F}_{\mathcal{N}}((X^h_U), X_U)\) is a framing of the canonical morphism \(\text{can}: (X^h_U) \to X_U\) and \(c_1 \in \mathbb{F}_{\mathcal{N}}((X^h_U), X_U - Y_U)\). By strict \(\mathbb{A}^1\)-invariance over \(U\), \(\mathcal{F}\) and \(\mathcal{L}_{\text{Nis}}(\mathcal{F})\) are \(\mathbb{A}^1\)-local quasi-stable framed presheaves of \(S^1\)-spectra on \(\text{Sm}_B\). Owing to Corollary \[8.6\] there is a commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{C}}_{X_U}(\mathcal{F})(X_U) & \xrightarrow{c_0} & \tilde{\mathcal{C}}_{X_U}(\mathcal{F})(X_U) \\
\downarrow c & & \downarrow c \\
\tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(V_U) & \xrightarrow{\simeq} & \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(V_U) \\
\end{array}
\]

By the second equality in \[8.14\], \(c_1\) induces the trivial morphism

\[
\tilde{\mathcal{C}}_{X_U}(\mathcal{F})(X_U) \to \tilde{\mathcal{C}}_{X_U}(\mathcal{F})(X_U - Y_U) \xrightarrow{c_1} \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(V_U) \tag{8.18}
\]

since \(\tilde{\mathcal{C}}_{X_U}(\mathcal{F})(X_U - Y_U) \simeq 0\). By the first equality in \[8.17\] it follows that \(c_0^* = (\text{can}^\nu)^*\). This implies the canonical morphism

\[
\tilde{\mathcal{C}}_{X_U}(\mathcal{F})(X_U) \to \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(V_U) = \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})((X^h_U))
\]

is trivial. By passing to the colimit over the filtering system of Nisnevich coverings \(\tilde{X}_U\) of \(X_U\), the equivalence

\[
\tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(X_U) \simeq \text{hocofib}_{\tilde{X}_U} \tilde{\mathcal{C}}_{\tilde{X}_U}(\mathcal{F})(X_U)
\]

from Lemma \[8.12\] implies the morphism \(\tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(X_U) \to \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})((X^h_U))\) is trivial on homotopy groups. The same argument applies to every étale neighborhood \(X'\) of \(x\) in \(X\). Combined with the equivalence

\[
\tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})((X^h_U)) \simeq \text{hocofib}_{X_U} \tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})(X_U)
\]

we conclude \(\tilde{\mathcal{L}}_{\text{Nis}}(\mathcal{F})((X^h_U)) \simeq 0\). □
Corollary 8.14. The identification of Nisnevich cohomology in \([8.2]\) holds for every generic point \(\eta \in B\) and quasi-stable additive framed presheaf of abelian groups \(F\) on \(\text{Sm}_B\).

Proof. Without loss of generality we may assume \(B\) is a local irreducible scheme which equals the closure of \(\eta\). Applying Theorem [8.13] to the Eilenberg-MacLane object \(H(F) \in \text{SH}_s(B)\) of \(F\) finishes the proof. \(\square\)

9. tf-LOCALIZATION BY CLOSED SUBSCHEMES

In this section we prove our localization theorem in \(\text{SH}_{s,tf}(B, Z)\) for a fixed closed immersion \(Z \hookrightarrow B\). This allows us to reduce the problem of tf-Nisnevich strict \(A^1\)-invariance for \(\text{SH}_{s,tf}(\text{Sm}_B)\), where \(B\) is a one-dimensional base scheme, to \(\text{SH}_{s,tf}(B, Z)\) and \(\text{SH}_{s,tf}(\text{Sm}_{B-Z})\).

9.1. tf-localization. Pertinent to localization is the geometric observation that every \(X \in \text{Sm}_B\) allows a tf-covering by \(X^b_2 \in \text{Sm}_{B,Z}\) and \(X \times_B (B - Z) \in \text{Sm}_{B-Z}\). If \(B\) is one-dimensional, then with the exception of \(X\) all the terms in the tf-square

\[
\begin{array}{c}
(X^b_2) \times_B (B - Z) \\
\downarrow \\
X^b_Z \\
\downarrow \\
X
\end{array}
\]

afford only trivial tf-coverings. The second critically important ingredient is the triviality of the Nisnevich cohomology of \(X \times_B (B - Z)\) when \(X\) is an essentially smooth local henselian scheme. In contrast to the localization theorem for \(\text{SH}(B)\), we work with \(\text{Sm}_{B,Z}\) instead of \(\text{Sm}_B\). This allows us to extend the localization theorem for motivic equivalences to the level of local Nisnevich equivalences.

The pointed homotopy categories we study are related via the functors

\[
\begin{array}{ccc}
\text{H}_{s,\bullet}(\text{Sm}_B) & & \text{H}_{s,\bullet}(\text{Sm}_{B-Z}) \\
\text{i}^* & & \text{j}^* \\
\text{i}_! & & \text{j}_!
\end{array}
\]

\[\text{(9.1)}\]

For \(X \in \text{Sm}_B, X^b_Z \in \text{Sm}_{B,Z}, V \in \text{Sm}_{B-Z}\), the functors \(\text{i}^*, \text{j}^*, \text{i}_!, \text{j}_!\) are given by

\[
\begin{align*}
\text{i}^!(F)(X^b_2) &= \text{hofib}(F(X^b_2) \to F(X^b_Z - X_Z)), \\
\text{i}^*(F)(V) &= F(V), \\
\text{i}_!(F)(X) &= F(X^b_Z), \\
\text{j}^!(F)(X) &= F(X - X_Z), \\
\text{j}^*(F)(V) &= F(V), \\
\text{j}_!(F)(X) &= F(X - X_Z).
\end{align*}
\]

\[\text{(9.2)}\]

Here \(i^!\) is a left Kan extension and \(i_*\) is the restriction along the functor \(\text{Sm}_{B,Z} \to \text{Sm}_Z, X^b_2 \mapsto X_Z\). The tf-localization of \((9.1)\) induces the same adjunctions as in \((2.4)\) for \(\tau = \text{tf}\) and the pointed homotopy category \(\text{H}_{s,\bullet}(\text{Sm}_B^!)\). We shall use similar notation for \(\text{SH}_{s,\bullet}(\text{Sm}_B), \text{SH}_{s,\bullet}(\text{Sm}_{B,Z}), \text{SH}_{s,\bullet}(\text{Sm}_{B-Z})\). In this context, the motivic localization endofunctor \(\text{L}_{\text{mot}}\) preserves the adjunctions in \((2.4)\) and their properties.
Proposition 9.1. For every $\mathsf{tf}$-local object $\mathcal{F} \in \mathsf{H}_\bullet(\mathsf{Sm}_B)$ there is a homotopy pullback square

$$
\begin{array}{ccc}
\tilde{i}_*\tilde{i}^!(\mathcal{F}) & \to & \mathcal{F} \\
\downarrow & & \downarrow \\
\ast & \to & j_*j^*(\mathcal{F}).
\end{array}
$$

(9.3)

The same holds for every $\mathsf{tf}$-local object $\mathcal{F} \in \mathsf{H}_\bullet(\mathsf{Sm}_{Aff}_B)$.

Proof. Let $\mathcal{H}$ be shorthand for the homotopy fiber $\mathsf{hofib}(\mathcal{F} \to j_*j^*(\mathcal{F}))$. By the definitions of $\tilde{i}_*$, $\tilde{i}^!$, $j_*$, $j^*$, there are equivalences

$\tilde{i}_*\tilde{i}^!(\mathcal{F})(X) \simeq \mathsf{hofib}(\mathcal{F}(X_B^h) \to \mathcal{F}(X_B \times_B (B - Z)))$,

$\mathcal{H}(X) \simeq \mathsf{hofib}(\mathcal{F}(X) \to \mathcal{F}(X \times_B (B - Z)))$.

For every $\mathsf{tf}$-local simplicial presheaf $\mathcal{F}$ there is a homotopy pullback square

$$
\begin{array}{ccc}
\mathcal{F}(X) & \to & \mathcal{F}(X \times_B (B - Z)) \\
\downarrow & & \downarrow \\
\mathcal{F}(X_B^h) & \to & \mathcal{F}(X_B \times_B (B - Z)).
\end{array}
$$

Hence there is a canonically induced equivalence

$\mathcal{H}(X) \simeq \tilde{i}_*\tilde{i}^!(\mathcal{F})(X)$. (9.4)

In more detail, since $\mathcal{F}$ is $\mathsf{tf}$-local simplicial presheaf on $\mathsf{Sm}_B$, it follows that for any étale neighborhood $\tilde{X}$ of $Y = X \times_B Z$ in $X$, there is an equivalence

$\mathsf{hofib}(\mathcal{F}(X) \to \mathcal{F}(X \times_B (B - Z))) \xrightarrow{\simeq} \mathsf{hofib}(\mathcal{F}(\tilde{X}) \to \mathcal{F}(\tilde{X} \times_B (B - Z)))$.

Thus for the henselization $X_B^h$, there is an equivalence

$\mathsf{hofib}(\mathcal{F}(X) \to \mathcal{F}(X \times_B (B - Z))) \xrightarrow{\simeq} \mathsf{hofib}(\mathcal{F}(X_B^h) \to \mathcal{F}(X_B \times_B (B - Z)))$.

Clearly (9.3) implies (9.3).

Corollary 9.2. For every $\mathsf{tf}$-local object $\mathcal{F} \in \mathsf{SH}_\bullet(\mathsf{Sm}_B)$ there is a distinguished triangle

$$
\tilde{i}_*\tilde{i}^!(\mathcal{F}) \to \mathcal{F} \to j_*j^*(\mathcal{F}) \to \tilde{i}_*\tilde{i}^!(\mathcal{F})[1].
$$

(9.5)

Over a one-dimensional local base scheme, the distinguished triangle (9.5) allows us to describe the $\mathsf{tf}$-localization endofunctor on $\mathsf{SH}_\bullet(\mathsf{Sm}_B)$.

Proposition 9.3. Suppose $B$ is a one-dimensional local base scheme with closed point $z$. Then for every $\mathcal{F} \in \mathsf{SH}_\bullet(\mathsf{Sm}_B)$ there is a canonical equivalence

$$
\mathcal{L}_{\mathsf{tf}}(\mathcal{F}) \simeq \mathsf{hofib}(j_*j^*(\mathcal{F})[-1] \to \tilde{i}_*\tilde{i}^!(\mathcal{F})).
$$

That is, for every $X \in \mathsf{Sm}_B$, there is an equivalence

$$
\mathcal{L}_{\mathsf{tf}}(\mathcal{F})(X) \simeq \mathsf{hofib}(\mathcal{F}(X \times_B (B - z)) \oplus \mathcal{F}(X_B^h) \to \mathcal{F}(X_B \times_B (B - z))).
$$

Proof. By the assumption on $B$, we can view $z$ and $B - z$ as presheaves of $S^1$-spectra over the residue field and the fraction field of $B$, respectively. The $\mathsf{tf}$-topologies on $\mathsf{Sm}_{B,z}$ and $\mathsf{Sm}_{B,z}$ are trivial by Proposition 3.3 (iii). Thus the outer terms in the distinguished triangles

$\tilde{i}_*\tilde{i}^!(\mathcal{F}) \to \mathcal{F} \to j_*j^*(\mathcal{F})$

and

$\mathcal{L}_{\mathsf{tf}}(\tilde{i}_*\tilde{i}^!(\mathcal{F})) \to \mathcal{L}_{\mathsf{tf}}(\mathcal{F}) \to \mathcal{L}_{\mathsf{tf}}(j_*j^*(\mathcal{F}))$
are pairwise equivalent, see Corollary 9.2. It follows that $\mathcal{F} \to \mathcal{L}_{tf}(\mathcal{F})$ is an equivalence. □

9.2. Framed tf-localization. To show $\tilde{i}^!, \tilde{i}_*, j^*, j_*$ preserve quasi-stable radditive framed presheaves we work with $\mathbf{Spc}_c(\mathbb{F}_{+}(B))$ and $\mathbf{H}_c(\mathbb{F}_{+}(B))$, and similarly for $\mathbb{F}_{+}(B - Z)$ and $\mathbb{F}_{+}(B, Z)$. Due to Lemma A.8 see also Remark A.7 there are well defined functors

$$
\begin{align*}
\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2}) & \to \mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2}); X \mapsto X^B, \\
\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B-2}) & \to \mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B-2}); X \mapsto X \times_B (B - Z).
\end{align*}
(9.6)
$$

As in (9.2) there exist base change functors

$$
\begin{align*}
\tilde{i}^! & : \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2})) \to \mathbf{H}_c(\mathbb{F}
_{+}(\mathbb{S}
\mathbb{M}_{B2}));, \\
\tilde{i}_* & : \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2})) \to \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2})), \\
j^* & : \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2})) \to \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B-2})), \\
j_* & : \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B-2})) \to \mathbf{H}_c(\mathbb{F}_{+}(\mathbb{S}
\mathbb{M}_{B2})).
\end{align*}
(9.7)

For simplicity we let $\mathbb{S}
\mathbb{M} \in \{\mathbb{S}
\mathbb{M}_{B2}, \mathbb{S}
\mathbb{M}_{B-2}\}$ and $\mathbb{F}_{+} \in \{\mathbb{F}_{+}(B, Z), \mathbb{F}_{+}(B), \mathbb{F}_{+}(B - Z)\}$. An object $\mathcal{F} \in \mathbf{H}_c(\mathbb{F}_{+})$ is Nisnevich local (resp. tf-local) if it maps to a Nisnevich local object under the forgetful functor $\mathbf{H}_c(\mathbb{F}_{+}) \to \mathbf{H}_c(\mathbb{S}
\mathbb{M})$.

**Proposition 9.4.** The functors $\tilde{i}^!, \tilde{i}_*, j^*, j_*$ commute with the forgetful functor in (9.7) and preserve quasi-stable radditive framed presheaves.

**Proof.** The first claim follows from the definitions of $\tilde{i}^!, \tilde{i}_*, j^*, j_*$. The functors in (9.6) preserve coproducts of schemes, and for every $X$, the framed correspondence $\sigma_X$ maps to $\sigma_X^{B2}$ in $\mathbb{S}
\mathbb{M}_{B2}$ and $\sigma_X^{B-2}$ in $\mathbb{S}
\mathbb{M}_{B-2}$. □

Similarly to Proposition 9.1, Corollary 9.2, and Proposition 9.3 we have:

**Proposition 9.5.** For every tf-local object $\mathcal{F} \in \mathbf{H}_c(\mathbb{F}_{+})$ there is a homotopy pullback square

$$
\begin{align*}
\tilde{i}_* \tilde{i}^!(\mathcal{F}) \to \mathcal{F} \\
\tilde{i}_* \tilde{i}^!(\mathcal{F}) \downarrow \quad \downarrow \quad j_* j^!(\mathcal{F}).
\end{align*}
$$

Thus for every tf-local object $\mathcal{F} \in \mathbf{S}
\mathbb{H}_c(\mathbb{F}_{+})$ there is a distinguished triangle

$$
\tilde{i}_* \tilde{i}^!(\mathcal{F}) \to \mathcal{F} \to j_* j^!(\mathcal{F}) \to \tilde{i}_* \tilde{i}^!(\mathcal{F})[1].
$$

**Corollary 9.6.** Suppose $B$ is a one-dimensional local base scheme. There exists a functor $\mathcal{L}_{tf}^i : \mathbf{S}
\mathbb{H}_c(\mathbb{F}_{+}(B)) \to \mathbf{S}
\mathbb{H}_c(\mathbb{F}_{+}(B))$ that preserves quasi-stable radditive framed presheaves of $S^1$-spectra. Moreover, the diagram

$$
\begin{align*}
\mathbf{S}
\mathbb{H}_c(\mathbb{F}_{+}(B)) & \xrightarrow{\mathcal{L}_{tf}^i} \mathbf{S}
\mathbb{H}_c(\mathbb{F}_{+}(B)) \\
\mathbf{S}
\mathbb{H}_c(B) & \xrightarrow{\mathcal{L}_{tf}^i} \mathbf{S}
\mathbb{H}_c(B)
\end{align*}
(9.8)
$$
commutes. It follows that $\mathcal{L}_{tf}$ preserves quasi-stable radditive framed presheaves of $S^1$-spectra.
Proof. Let \( z \) be the closed point of \( B \). We define the \( \mathcal{L}^i_{fr} \) by setting
\[
\mathcal{L}^i_{fr}(\mathcal{F}) := \text{hofib}(j_I^*(\mathcal{F})[-1] \to \tilde{i}^*_I(\mathcal{F})).
\] (9.9)
That is, for each \( X \in \text{Sm}_B \), we have
\[
\mathcal{L}^i_{fr}(\mathcal{F})(X) \simeq \text{hofib}(\mathcal{F}(X \times_B (B - z)) \oplus \mathcal{F}(X^h_z) \to \mathcal{F}(X^h_z \times_B (B - z))).
\]
We note that the functors \( \mathcal{L}^i_{fr} \) preserve quasi-stable Nisnevich framed presheaves of \( S^1 \)-spectra: This follows since the functors \( \text{Sm}_B \to \text{Sm}_{B-Z}; \ X \mapsto X \times_B (B - z), \text{Sm}_B \to \text{Sm}_{B-Z}; \ X \mapsto X^h_z \) are well defined on the level of framed correspondences, as in Lemma [A.3] and preserve coproducts and framed correspondences of the form \( \sigma_X \). Proposition 9.3 implies the commutativity of 9.3.

9.3. \( \mathbb{A}^1 \), Nisnevich- and tf-localization. In the following, we show the functors \( \tilde{i}^*_I, \tilde{i}_*, j^*, j_* \) preserve the various locality conditions on radditive framed presheaves.

Proposition 9.7. The functors \( \tilde{i}^*_I, \tilde{i}_*, j^*, j_* \) preserve \( \mathbb{A}^1 \)-local objects.

Proof. For \( \tilde{i}^*_I \) there are equivalences of \( S^1 \)-spectra
\[
\tilde{i}^*_I(\mathcal{F}((\mathbb{A}^1 \times X)^h_Z)) \simeq \mathcal{F}((\mathbb{A}^1 \times X)/((\mathbb{A}^1 \times X)^h_Z \times (B - Z)))
\]
\[
\simeq \mathcal{F}((\mathbb{A}^1 \times X)/((\mathbb{A}^1 \times X) \times (B - Z)))
\]
\[
\simeq \mathcal{F}(X/(X \times (B - Z)))
\]
\[
\simeq \mathcal{F}(X^h_Z/(X^h_Z \times (B - Z)))
\]
\[
\simeq \tilde{i}^*_I(\mathcal{F}(X^h_Z)).
\]
The claims for \( \tilde{i}_*, j^*, j_* \) follow because, in suggestive notation, the functors
\[
\text{Sm}_B \to \text{Sm}_{B,Z}; \ X \mapsto W = X^h_Z,
\]
\[
\text{Sm}_{B-Z} \to \text{Sm}_B; \ V \mapsto X = V,
\]
\[
\text{Sm}_B \to \text{Sm}_{B-Z}; \ X \mapsto V = X \times_B (B - Z),
\]
commute with the corresponding endofunctors
\[
\text{Sm}_{B,Z} \to \text{Sm}_{B,Z}; \ W \mapsto W \times_{B,Z} \mathbb{A}^1_{B,Z},
\]
\[
\text{Sm}_B \to \text{Sm}_B; \ X \mapsto X \times_B \mathbb{A}^1_B,
\]
\[
\text{Sm}_{B-Z} \to \text{Sm}_{B-Z}; \ V \mapsto V \times_{B-Z} \mathbb{A}^1_{B-Z}.
\]

Proposition 9.8. The functors \( \tilde{i}_*, j^* \) preserve Nisnevich local objects and Nisnevich equivalences. The functor \( j_* \) preserve Nisnevich local objects. The same results hold for tf-local objects and tf-equivalences.

Proof. Corollary 1.3 Proposition 4.7 and Lemmas 3.3 and 3.4 imply the claim for \( \tilde{i}_* \). The case of \( j^* \) follows similarly because the functor \( \text{Sm}_{B-Z} \to \text{Sm}_B \) preserves fiber products, and Nisnevich coverings and points. The latter also hold for tf-coverings and points. For \( j_* \), we use the first part of Lemma 3.3.

Proposition 9.9. The functor \( \tilde{i}^*_I \) preserves Nisnevich local objects and tf-local objects.

Proof. Suppose \( \mathcal{F} \in \mathbf{H}^*_i(\text{Sm}_B) \) is Nisnevich local (resp. tf-local). By Definitions 4.4 and 4.5 any Nisnevich covering (resp. tf-covering) \( X^h_Z \to X^h_Z \) in \( \text{Sm}_{B,Z} \) is induced by a Nisnevich covering...
Moreover, if \( j \) is a one-dimensional base scheme then \( \tilde{\eta} \) and \( j_* \) preserve tf-equivalences on \( \mathbf{SH}(\text{Sm}_B) \).

**Proof.** By Proposition 3.6(vii), the claim for \( j_* \) follows because the functor \( \text{Sm}_{B,Z} \to \text{EssSm}_B; \ X_2^B \mapsto X_2^B \times_B (B - Z) \), preserves tf-points when \( B \) is one-dimensional. The claim for \( \tilde{\eta} \) follows since \( \text{Sm}_{B,Z} \to \text{EssSm}_B; \ X_2^B \mapsto X_2^B \), preserves tf-points for any \( B \). We note that \( X_2^B \) is a tf-point.

Furthermore, \( X_2^B \times_B (B - Z) \) is a tf-point since
\[
X_2^B \times_B (B - Z) \cong X_2^B \times_B \prod_{\eta \in (B - Z)^{(1)}} \eta,
\]
where \( \eta \) runs over the set of generic points of \( B - Z \). To conclude we note that the tf-topology on \( \text{EssSm}_B \) is trivial.

**Corollary 9.11.** The functors \( \tilde{i}_*, j^* \) commute with the localization endofunctors \( \mathcal{L}_{\text{Nis}} \) and \( \mathcal{L}_{\text{tf}} \). Moreover, if \( \dim B = 1 \) with closed point \( z \), then \( \tilde{\eta} \) and \( j_* \) commute with \( \mathcal{L}_{\text{tf}} \).

**Proof.** Proposition 9.8 implies the claims for \( \tilde{i}_* \) and \( j^* \). The equivalence \( \tilde{\eta} \mathcal{L}_{\text{tf}} \cong \mathcal{L}_{\text{tf}} \tilde{\eta} \) follows from Propositions 9.9 and 9.10 and \( j_* \mathcal{L}_{\text{tf}} \cong \mathcal{L}_{\text{tf}} j_* \) follows from Propositions 9.8 and 9.10.

**Proposition 9.12.** Suppose \( B \) is a one-dimensional base scheme with closed point \( z \). If \( \mathcal{F} \) is an \( \mathbb{A}^1 \)-local quasi-stable additive framed presheaf of \( S^1 \)-spectra on \( \text{Sm}_B \), there are equivalences
\[
\mathcal{L}_{\text{Nis}}(\tilde{i}^!(\mathcal{F})) \cong \tilde{i}^!(\mathcal{L}_{\text{Nis}}(\mathcal{F})), \quad \mathcal{L}_{\text{Nis}}(j_*(\mathcal{F})) \cong j_*(\mathcal{L}_{\text{Nis}}(\mathcal{F})).
\]

**Proof.** Propositions 9.8 and 9.9 show \( j_*(\mathcal{L}_{\text{Nis}}(\mathcal{F})) \) and \( i^!(\mathcal{L}_{\text{Nis}}(\mathcal{F})) \) are Nisnevich local, and the same hold for \( \mathcal{L}_{\text{Nis}}(\tilde{i}^!(\mathcal{F}))(X_2^B) \) and \( \mathcal{L}_{\text{Nis}}(j_*(\mathcal{F})) \) by construction. If \( X_2^B \) is an essentially smooth local scheme in \( \text{EssSm}_{B,z} \), then there are equivalences
\[
\mathcal{L}_{\text{Nis}}(\tilde{i}^!(\mathcal{F}))(X_2^B) \cong \mathcal{L}_{\text{Nis}}(\tilde{i}^!(\mathcal{F}))(X_2^B) \cong \mathcal{L}_{\text{Nis}}(j_*(\mathcal{F}))(X_2^B)
\]
\[
\cong \mathcal{L}_{\text{Nis}}(\mathcal{F})(X_2^B) = \mathcal{L}_{\text{Nis}}(\tilde{i}^!(\mathcal{F}))(X_2^B).
\]

For the middle equivalence we appeal to Theorem S.13. It follows that \( \mathcal{L}_{\text{Nis}}(\tilde{i}^!(\mathcal{F})) \cong \tilde{i}^!(\mathcal{L}_{\text{Nis}}(\mathcal{F})) \), since both objects are Nisnevich local.

For every \( X \in \text{Sm}_B, \ x \in X \times_B z \), we can similarly use Theorem S.13 to show
\[
i_*(\mathcal{L}_{\text{Nis}}(\mathcal{F}))(X_2^B) \cong \mathcal{L}_{\text{Nis}}(i_*(\mathcal{F}))(X_2^B).
\]

Moreover, if \( x \in X - X \times_B z \), there are equivalences
\[
j_*(\mathcal{L}_{\text{Nis}}(\mathcal{F}))(X_2^B) \cong \mathcal{L}_{\text{Nis}}(j_*(\mathcal{F}))(X_2^B) \cong \mathcal{L}_{\text{Nis}}(j_*(\mathcal{F}))(X_2^B).
\]

This concludes the proof because \( j_*(\mathcal{L}_{\text{Nis}}(\mathcal{F})) \) is Nisnevich local.
9.4. Localization and tf-Nisnevich strict $\mathbb{A}^1$-invariance.

**Theorem 9.13.** Suppose $B$ is a one-dimensional base scheme. Let $Z$ be a closed subscheme of dimension zero with open complement $B - Z$. Then tf-Nisnevich strict $\mathbb{A}^1$-invariance on $\text{Sm}_{B,Z}$ and $\text{Sm}_{B-Z}$ implies tf-Nisnevich strict $\mathbb{A}^1$-invariance on $\text{Sm}_B$.

**Proof.** Let $\mathcal{F} \in \mathbf{SH}_*(\text{Fr}_+(B))$ be a quasi-stable radditive framed presheaf such that $\mathcal{L}_{\text{tf}}(\mathcal{F})$ is $\mathbb{A}^1$-local. Owing to Corollary 9.2 there is a distinguished triangle

$$
\tilde{i}^* \tilde{i}_!(\mathcal{L}_{\text{Nis}}(\mathcal{F})) \to \mathcal{L}_{\text{Nis}}(\mathcal{F}) \to j_*j^*(\mathcal{L}_{\text{Nis}}(\mathcal{F})).
$$

(9.10)

Proposition 9.4 and Corollary 9.11 show that $\tilde{i}^!(\mathcal{F})$ and $j^*(\mathcal{F})$ are quasi-stable radditive framed presheaves. Since $\mathcal{L}_{\text{tf}}(\mathcal{F})$ is $\mathbb{A}^1$-local, Proposition 9.7 and Corollary 9.11 imply that $\mathcal{L}_{\text{tf}}(\tilde{i}^!(\mathcal{F}))$ and $\mathcal{L}_{\text{tf}}(j^*(\mathcal{F}))$ are $\mathbb{A}^1$-local quasi-stable radditive framed presheaves. Thus, by Propositions 9.7 and 9.12 the outer terms in (9.10) are $\mathbb{A}^1$-local. Moreover, since all the terms in (9.10) are tf-local, it follows that $\mathcal{L}_{\text{Nis}}(\mathcal{F})$ is $\mathbb{A}^1$-local by viewing (9.10) as a distinguished triangle in $\mathbf{SH}_*(\text{Sm}_B)$.

\[ \Box \]

10. Reduction from $\text{Sm}_{B,Z}$ to $\text{Sm}_Z$

In this section, we reduce the problem of tf-Nisnevich strict $\mathbb{A}^1$-invariance over $\text{Sm}_{B,Z}$ to the same problem over $\text{Sm}_Z$. The key geometric input we use is that smooth morphisms admit liftings along henselian pairs $X_Z \twoheadrightarrow X_Z^h$ in the category of schemes. In Lemma A.11 we show a similar property for framed correspondences.

Recall from Section 2.4 the categories $\text{Sm}^\text{cci}_{B,Z}$, $\text{Sm}^\text{cci}_{B,Z}$, and from \([6.10], (6.12)\) the functors

$$
\begin{align*}
& H_\ast(\text{Sm}^\text{cci}_{B,Z}) \xleftarrow{i^*_\text{rig}} H_\ast(\text{Fr}_+(\text{Sm}^\text{cci}_{B,Z})) \xrightarrow{R_{\text{rig}}} H_\ast(\text{Sm}^\text{cci}_{B,Z}) \xrightarrow{u^*} H_\ast(\text{Sm}^\text{cci}_{B,Z}) \xrightarrow{Y \times Z} Y \xleftarrow{i^*_\text{rig}} H_\ast(\text{Sm}^\text{cci}_{B,Z}) \\
& X^h_Z \leftarrow X^h_Z.
\end{align*}
$$

(10.1)

Here $R_{\text{rig}}$ is the canonical embedding so that $i^*_\text{rig} \simeq (i_{B,Z})^* \circ R_{\text{rig}}$, where $(i_{B,Z})^* : H_\ast(\text{Sm}^\text{cci}_{B,Z}) \to H_\ast(\text{Sm}^\text{cci}_{B,Z})$ is equivalent to the restriction along the canonical functor $r : \text{Sm}^\text{cci}_Z \to \text{Sm}^\text{cci}_B$ by Lemma 6.11. We will use similar notation in the setting of framed correspondences.

**Proposition 10.1.** Suppose for each $\mathbb{A}^1$-local quasi-stable radditive framed presheaf of $S^1$-spectra $\mathcal{F}$ on $\text{Sm}^\text{cci}_Z$ the Nisnevich localization endofunctor $\mathcal{L}_{\text{Nis}}(\mathcal{F})$ is $\mathbb{A}^1$-local. Then the same holds for $\mathbb{A}^1$-local quasi-stable radditive framed presheaves of $S^1$-spectra on $\text{Sm}^\text{cci}_{B,Z}$.

**Proof.** Let $\mathcal{F}$ be an $\mathbb{A}^1$-local quasi-stable radditive framed presheaf of $S^1$-spectra on $\text{Sm}^\text{cci}_{B,Z}$. We need to show that $\mathcal{L}_{\text{Nis}}(\mathcal{F})$ is an $\mathbb{A}^1$-local. To that end we employ (10.1) and break the proof into the following steps.

(i) By Lemma 10.3(1) there is an $\mathbb{A}^1$-local rigid framed presheaf of $S^1$-spectra $\mathcal{F}^{\text{rig}}$ on $\text{Sm}^\text{cci}_{B,Z}$ such that $\mathcal{F} \simeq u_*R_{\text{rig}}(\mathcal{F}^{\text{rig}})$.

(ii) By Lemma 10.3(2), $\mathcal{F}^{\text{rig}}$ is a quasi-stable radditive framed presheaf of $S^1$-spectra.

(iii) Consider the object $\mathcal{F}_Z \simeq i^*_\text{rig}(\mathcal{F}^{\text{rig}}) \in H_\ast(\text{Sm}^\text{cci}_Z)$. By Lemma 10.10(2), it is $\mathbb{A}^1$-local, and by Lemma 10.10(1), it is a quasi-stable radditive framed presheaf of $S^1$-spectra.

(iv) If the endofunctor $\mathcal{L}_{\text{Nis}}$ on $H_\ast(\text{Fr}_+(\text{Sm}^\text{cci}_Z))$ takes $\mathbb{A}^1$-local quasi-stable radditive framed presheaves to $\mathbb{A}^1$-local presheaves, then $\mathcal{L}_{\text{Nis}}(\mathcal{F}_Z)$ is $\mathbb{A}^1$-local. According to Lemma 10.10(2) we have $i^*_\text{rig}(\mathcal{L}_{\text{Nis}}(\mathcal{F}^{\text{rig}})) \simeq \mathcal{L}_{\text{Nis}}(\mathcal{F}_Z)$, and hence it is an $\mathbb{A}^1$-local quasi-stable radditive framed presheaf of $S^1$-spectra by Lemma 10.10(1).

(v) Lemma 10.5(2) and Lemma 10.8(2) imply the equivalence $\mathcal{L}_{\text{Nis}}(\mathcal{F}) \simeq u_*R_{\text{rig}}(\mathcal{L}_{\text{Nis}}(\mathcal{F}^{\text{rig}}))$. Lemma 10.5(1) and Lemma 10.8(1) show it is an $\mathbb{A}^1$-local radditive quasi-stable framed presheaf of $S^1$-spectra.
The argument for Proposition 6.10 establishes the following result.

**Proposition 10.2.** If $F \in H_*(Fr_+(Sm^\text{cci}_{B,Z}))$ then $F_{BZ}$ is $\mathbb{A}^1$-local and rigid.

We will use the following reformulation of Lemma A.11 and Lemma B.11.

**Lemma 10.3.** For $X_i \in Sm^\text{cci}_{B,Z}$ we set $Y_i = X_i \times_B Z$ for $i = 0, 1$. Then for the canonical closed immersions $Y_i \hookrightarrow X_i$ and every morphism $Y_0 \to Y_1$ in $\text{Sch}_B$ there exists a commutative diagram

$$
\begin{array}{ccc}
Y_0 & \longrightarrow & X_0 \\
\downarrow & & \downarrow \\
Y_1 & \longrightarrow & X_1.
\end{array}
$$

The same result holds for morphisms in $Fr_+(\text{Sch}_B)$.

**Lemma 10.4.** The following hold for the composite functor $u_* \circ R_{\text{rig}}$ in (10.1).

1. For any $\mathbb{A}^1$-local framed presheaf of $S^1$-spectra $F$ on $Sm^\text{cci}_{B,Z}$ there is an $\mathbb{A}^1$-local rigid framed presheaf $F_{\text{rig}}$ on $Sm^\text{cci}_{B,Z}$ such that $u_*(R_{\text{rig}}(F_{\text{rig}})) \simeq F$.
2. It detects quasi-stable radditive framed presheaves of $S^1$-spectra on the subcategory of framed presheaves of $S^1$-spectra.
3. It is conservative.

**Proof.** Part (1) follows from Proposition 10.2 for $F_{\text{rig}} \simeq F_{BZ}$. Suppose $u_*, R_{\text{rig}}(F)$ is quasi-stable. Then $\sigma_{X_Z}$ induces an auto-equivalence on $F(X_Z)$ for all $X \in Sm^\text{cci}_{B}$, hence $\sigma_{X_Z}$ induces an auto-equivalence on $F(X_Z)$, since $F(X_Z) \cong F(X)$ by rigidity. Moreover, if $u_*, R_{\text{rig}}(F)$ is radditive, then $F$ is radditive by Corollary B.4 and rigidity as above. This proves part (2). To prove (3), suppose $F \to G$ is an equivalence in $H_{\text{rig}}(Sm^\text{cci}_{Z})$. Then for all $X_Z \in Sm^\text{cci}_{B,Z}$ we have the equivalences

$$
F(X_Z) \cong G(X_Z) \cong F(X_Z) \cong G(X_Z) \cong F(X_Z) \cong G(X_Z).
$$

**Lemma 10.5.** The following hold for the functor $u_*$ in (10.1).

1. It preserves $\mathbb{A}^1$-local and quasi-stable radditive framed presheaves of $S^1$-spectra.
2. It commutes with $\mathcal{L}_{\text{Nis}}$.

**Proof.** To conclude (1) we use that $Sm^\text{cci}_{B,Z} \to Sm^\text{CCI}_{B,Z}; X_Z \mapsto X_Z^h$, preserves morphisms of the form $(\mathbb{A}^1 \times X)_Z^h \to X_Z^h$. To conclude (2) we use that the same functor preserves Nisnevich coverings, fiber products, and Nisnevich points owing to Lemmas B.3 and B.4.

**Lemma 10.6.** The endofunctor $\mathcal{L}_{\text{Nis}}$ on $H_*(Sm^\text{cci}_{B,Z})$ preserves rigid presheaves of $S^1$-spectra.

**Proof.** This follows since every Nisnevich covering in the category $Sm^\text{cci}_{B,Z}$ of a henselian pair $X_Z \to X_Z^h$ is a henselian pair of the form $\tilde{X}_Z \to \tilde{X}_Z^h$ for some Nisnevich covering $\tilde{X} \to X$.

**Corollary 10.7.** The functor $\mathcal{L}_{\text{Nis}}$ restricts to an endofunctor on $H_{\text{rig}}(Sm^\text{cci}_{B,Z})$.

**Lemma 10.8.** The following hold for $R_{\text{rig}} : H_{\text{triv.rig}}(Sm^\text{cci}_{B,Z}) \to H_*(Sm^\text{cci}_{B,Z})$ in (10.1).

1. It preserves and detects $\mathbb{A}^1$-local quasi-stable radditive framed presheaves of $S^1$-spectra.
2. It commutes with $\mathcal{L}_{\text{Nis}}$. 
Proof. Part (1) follows because \( \mathbb{R}_{rig} \) is the embedding of the subcategory spanned by rigid presheaves of \( S^1 \)-spectra in \( \mathbb{H}_s(\text{Sm}^{cci}_{B,Z}) \). For part (2), Lemma 10.6 implies that the endofunctor \( \mathcal{L}_{Nis} \) on \( \mathbb{H}_s(\text{Sm}^{cci}_{B,Z}) \) coincides with the restriction of the endofunctor \( \mathcal{L}_{Nis} \) on \( \mathbb{H}_s(\text{Sm}^{cci}_{B,Z}) \).

Lemma 10.9. The following hold for the functor \( i^*_B(Z) \) in Lemma 6.11:

1. It preserves \( \mathbb{A}^1 \)-local quasi-stable radditive framed presheaves of \( S^1 \)-spectra.
2. It commutes with \( \mathcal{L}_{Nis} \).

Proof. By Lemma 6.11 the functor \( i^*_B(Z) \) is induced by restriction along the canonical embedding

\[
\text{Sm}^{cci}_Z \to \text{Sm}^{cci}_{B,Z}; Y \mapsto Y.
\]

(10.2)

Part (1) follows since (10.2) gives rise to the embedding \( \text{Fr}_+(\text{Sm}^{cci}_Z) \to \text{Fr}_+(\text{Sm}^{cci}_{B,Z}) \) given by \( Y \mapsto Y \). This functor preserves coproducts and morphisms of the form \( \mathbb{A}^1 \times Y \to Y \) and \( \sigma_V \).

Part (2) is equivalent to the statement that \( i^*_B(Z) \) preserves Nisnevich local objects and Nisnevich equivalences. This follows by Lemmas 10.3 and 10.4 because the embedding in (10.2) preserves coverings and points in the Nisnevich topology, and preserves fiber products.

Lemma 10.10. The following hold for the functor \( i^*_{rig} \) in (10.1):

1. It preserves \( \mathbb{A}^1 \)-local and quasi-stable radditive framed presheaves of \( S^1 \)-spectra.
2. It commutes with \( \mathcal{L}_{Nis} \).

Proof. The first claim in (1) follows by Lemma 10.8(1) and Lemma 10.9(1). For the second claim we consider \( i_{B,Z}: \text{Sm}^{cci}_{B,Z} \to \text{Sm}^{cci}_Z; X^h_Z \mapsto X_Z \). The restriction \( (i_{B,Z})_*: \mathbb{H}_s(\text{Sm}^{cci}_Z) \to \mathbb{H}_s(\text{Sm}^{cci}_{B,Z}) \) takes values in rigid objects. Hence we obtain \( i^*_{rig}: \mathbb{H}_s(\text{Sm}^{cci}_Z) \to \mathbb{H}_{rig}(\text{Sm}^{cci}_{B,Z}) \).

Lemma 6.13 shows \( i^*_{rig} \) and \( i^*_{rig} \) are inverses, and Lemma 6.14 shows \( i^*_{rig} \) is isomorphic to the functor \( r_* \) obtained from restriction along \( r: \text{Sm}^{cci}_Z \to \text{Sm}^{cci}_{B,Z} \). Moreover, \( i_{B,Z} \) and \( r \) lift to functors between categories of framed correspondences

\[
(i_{B,Z})^{fr}_*: \text{Fr}_+(\text{Sm}^{cci}_{B,Z}) \to \text{Fr}_+(\text{Sm}^{cci}_Z); X^h_Z \mapsto X_Z,
\]
\[
r^{fr}_*: \text{Fr}_+(\text{Sm}^{cci}_{B,Z}) \to \text{Fr}_+(\text{Sm}^{cci}_{B,Z}); X_Z \mapsto X^h_Z.
\]

To conclude for (1) we use that \( (i_{B,Z})^{fr} \) and \( r^{fr} \) preserve coproducts and morphisms of the form \( \mathbb{A}^1 \times V \to V \) and \( \sigma_V \) for any scheme \( V \in \text{Sm}^{cci}_{B,Z} \). Part (2) follows from Lemma 6.15, Lemma 10.8(2), Lemma 10.9(2).

Theorem 10.11. Let \( B \) be an affine scheme with closed subscheme \( Z \). If Nisnevich strict \( \mathbb{A}^1 \)-invariance holds on \( \text{Sm}_Z \) then it holds on \( \text{Sm}_{B,Z} \).

Proof. Owing to Lemma 7.7 we may replace \( \text{Sm}_Z \) and \( \text{Sm}_{B,Z} \) with \( \text{Sm}^{cci}_Z \) and \( \text{Sm}^{cci}_{B,Z} \), respectively. We conclude by appealing to Proposition 10.1.

11. PROOF OF THE MAIN THEOREM

In this section, we prove our main theorem concerning tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance, see Definition 7.2.

Proposition 11.1. Suppose \( Z \not
= B \) is a closed subscheme of dimension zero and the base scheme \( B \) is one-dimensional. If tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance holds on \( \text{Sm}_Z \) and \( \text{Sm}_{B-Z} \), then tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance holds on \( \text{Sm}_B \).

Proof. In view of Remark 7.3 since tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance holds on \( \text{Sm}_Z \) by assumption, Theorem 10.11 implies tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance on \( \text{Sm}_{B,Z} \). Using that tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance holds on \( \text{Sm}_{B-Z} \) by assumption, we conclude tf-Nisnevich strict \( \mathbb{A}^1 \)-invariance on \( \text{Sm}_B \) by appealing to Theorem 9.13.

\]
We are ready to prove our final permanence result about strict $\mathbb{A}^1$-invariance.

**Theorem 11.2.** Let $B$ be a base scheme of dimension one. If Nisnevich strict $\mathbb{A}^1$-invariance holds over every residue field of $B$, then tf-Nisnevich strict $\mathbb{A}^1$-invariance holds on $\text{Sm}_B$.

**Proof.** Owing to Lemma 7.5 we may assume $B$ is a local scheme. If $z \in B$ is the unique closed point, our assumption shows that Nisnevich strict $\mathbb{A}^1$-invariance holds on $\text{Sm}_z$ and $\text{Sm}_{B,z}$. Since $z$ and $B = z$ are zero-dimensional the tf-topologies on $\text{Sm}_z$, $\text{Sm}_{B,z}$, $\text{Sm}_{B-z}$ are trivial by Proposition 7.0(iii). Thus tf-Nisnevich strict $\mathbb{A}^1$-invariance is equivalent to Nisnevich strict $\mathbb{A}^1$-invariance for these categories. To conclude the proof, we use Proposition 11.1. □

**Corollary 11.3.** Suppose $B$ is a one-dimensional scheme. If strict $\mathbb{A}^1$-invariance holds for quasi-stable radditive framed presheaves of abelian groups over every residue field of $B$, then $B$ satisfies tf-Nisnevich strict $\mathbb{A}^1$-invariance for framed presheaves of abelian groups in the sense of Definition 1.5.

**Proof.** Follows from Theorems 11.2, 12.1 and 12.2. □

**Corollary 11.4.** Suppose $B$ is a one-dimensional scheme and the endofunctor $\mathcal{L}_{\text{Nis}}$ on $\text{SH}_*(\text{Sm}_B)$ preserves $\mathbb{A}^1$-local quasi-stable framed presheaves of $S^1$-spectra for all $\sigma \in B$. Then $B$ satisfies strict $\mathbb{A}^1$-invariance in the sense of Definition 2.2, i.e., the endofunctor $\mathcal{L}_{\text{Nis}}^\mathcal{L}$ on $\text{SH}_{*\mathcal{L}}(\text{Sm}_B)$ preserves $\mathbb{A}^1$-local quasi-stable framed presheaves of $S^1$-spectra.

**Proof.** Our assumption is that Nisnevich strict $\mathbb{A}^1$-invariance holds over all the residue fields of $B$. Theorem 11.2 shows that tf-Nisnevich strict $\mathbb{A}^1$-invariance holds on $\text{Sm}_B$. An $\mathbb{A}^1$-local quasi-stable framed object of $\text{SH}_{*\mathcal{L}}(\text{Sm}_B)$ is synonymous with an $\mathbb{A}^1$-local tf-local quasi-stable object $F \in \text{Spt}_*(\text{Fr}_*(B))$. Then $\mathcal{L}_{\mathcal{L}}(F) \simeq F$ is an $\mathbb{A}^1$-local quasi-stable framed presheaf of $S^1$-spectra, and $\mathcal{L}_{\mathcal{L}}(\mathcal{L}_{\text{Nis}}(F)) \simeq \mathcal{L}_{\text{Nis}}(F)$. By tf-Nisnevich strict $\mathbb{A}^1$-invariance over $B$, $L_{\text{Nis}}(F)$ is an $\mathbb{A}^1$-local quasi-stable framed presheaf of $S^1$-spectra. Thus $L_{\mathcal{L}}(F)$ preserves $\mathbb{A}^1$-local quasi-stable framed presheaves of $S^1$-spectra. □

**Remark 11.5.** Let us elaborate on the discussion in Section 1.5 by outlining an alternate approach to Theorem 11.2. If $V$ is a local henselian essentially smooth scheme over a scheme $B$, it suffices to show every Nisnevich covering $\tilde{U} \to \mathbb{A}^1 \times V$ admits a refinement $\tilde{U}' \to \mathbb{A}^1 \times V$, in the sense of [32, Tag 00VT], obtained from tf-squares and $(\mathbb{A}^1, \text{ZF})$-contractible Nisnevich squares, see [1.6].

Over fields, an analysis of the proof for Nisnevich strict $\mathbb{A}^1$-invariance in [17, §17] shows every Nisnevich covering of $\mathbb{A}^1 \times V$ admits a refinement obtained from $(\mathbb{A}^1, \text{ZF})$-contractible Nisnevich squares. Owing to Lemma 6.14 and Corollary 6.17, the same holds in $\text{Sm}_{B,z}$. The proof of Theorem 8.14 shows every Nisnevich covering of $(X^h) \times_B \eta$, where $B$ is one-dimensional and $z \in B^{(1)}$, $\eta \in B^{(0)}$, admits a refinement obtained from tf-squares and $(\mathbb{A}^1, \text{ZF})$-contractible Nisnevich squares. In the general case, we can use tf-squares to reduce to the case of $\text{Sm}_{B,z}$, for various $z \in B$, and proceed as above to find $(\mathbb{A}^1, \text{ZF})$-contractible squares for each $z$. However, controlling the steps in this process is technically more demanding than simply incorporating the localization theorem in Section 8.

12. tf-Nisnevich strict $\mathbb{A}^1$-invariance for abelian groups

We deduce Theorem 11.6 on tf-Nisnevich strict $\mathbb{A}^1$-invariance for abelian groups from the following comparison result.

**Theorem 12.1.** If tf-Nisnevich strict $\mathbb{A}^1$-invariance for presheaves of $S^1$-spectra holds on $\text{Sm}_B$ in the sense of Definition 7.3 then $B$ satisfies tf-Nisnevich strict $\mathbb{A}^1$-invariance for presheaves of abelian groups in the sense of Definition 11.5.
Proof. We write $H(F) \in \text{SH}_s(B)$ for the Eilenberg-MacLane object associated to a presheaf of abelian groups $F$ on $\text{Sm}_B$. By [24, Theorem 8.26] there are canonical isomorphisms
\[
\pi_n L_{\text{Nis}} H(F) \cong H^{tf}_{\text{Nis}}(-, F)_{\text{Nis}}, \quad \pi_n L_{\text{tf}} H(F) \cong H^{tf}_{\text{Nis}}(-, F_{\text{tf}}).
\] (12.1)

Assume tf-Nisnevich strict $\mathbb{A}^1$-invariance holds for presheaves of $S^1$-spectra. Let $F : \text{Fr}_+(B) \to A$ be a $\text{tf}$-strictly $\mathbb{A}^1$-invariant quasi-stable radditive framed presheaf. The equivalences $\sigma^* : F(X) \cong F(X_1 \sqcup X_2)$ hold for $H(F)$, i.e., it is a quasi-stable radditive framed presheaf of $S^1$-spectra. By [12.1] it follows that $L_{\text{tf}} H(F)$ is $\mathbb{A}^1$-local. By assumption $L_{\text{Nis}} H(F)$ is $\mathbb{A}^1$-local. Thus $H^{tf}_{\text{Nis}}(-, F) \cong \pi_n L_{\text{Nis}} H(F)$ is $\mathbb{A}^1$-invariant for all $n \geq 0$ by (12.1).

Theorem 12.2. If the field $k$ satisfies $\text{tf}$-Nisnevich strict $\mathbb{A}^1$-invariance for presheaves of abelian groups in the sense of Definition 7.4 then $\text{tf}$-Nisnevich strict $\mathbb{A}^1$-invariance for presheaves of $S^1$-spectra holds on $\text{Sm}_k$ in the sense of Definition 7.2 (or equivalently $\text{tf}$-Nisnevich strict $\mathbb{A}^1$-invariance in the sense of Definition 7.7).

Proof. Let $F$ be an $\mathbb{A}^1$-local quasi-stable radditive framed presheaf of $S^1$-spectra. Note that $\pi_l(F)$, $l \in \mathbb{Z}$, is an $\mathbb{A}^1$-invariant quasi-stable radditive framed presheaf of abelian groups. Thus $H^{tf}_{\text{Nis}}(-, F)$ is $\mathbb{A}^1$-local by the assumption on $k$. By (12.1) also $L_{\text{Nis}} H(F)$ is $\mathbb{A}^1$-local, i.e., there is an equivalence
\[
L_{\mathbb{A}^1} L_{\text{Nis}} H(\pi_l F) \cong L_{\text{Nis}} H(\pi_l F).
\] (12.2)

We refer to [24, Section 10.6] for the yoga of Postnikov towers with respect to $t$-structures. Let $\text{SH}_{s \geq t}(\text{Fr}_+(B))$ be the subcategory of $\text{SH}_s(\text{Fr}_+(B))$ spanned by objects $F$ with trivial homotopy presheaf $\pi_n(F)$ is the range $n < l$. Likewise, we write $\text{SH}_{s \leq t}(\text{Fr}_+(B))$ for the subcategory spanned by objects $F$ for which $\pi_n(F) = 0$ when $n > l$. Let $F_{\geq t}$ and $F_{\leq t}$ be the truncations of $F$ in $\text{SH}_{s \geq t}(\text{Fr}_+(B))$ and $\text{SH}_{s \leq t}(\text{Fr}_+(B))$, respectively. With these definitions there are equivalences
\[
\text{holim}_t F_{\geq t} \simeq F, \quad \text{hocolim}_t F_{\leq t} \simeq F.
\] (12.3)

For $l, b \in \mathbb{Z}$ we can form the truncation
\[
F_{\geq l, \leq b} \in \text{SH}(\text{Fr}_+(B)) \cap \text{SH}_{s \leq b}(\text{Fr}_+(B)),
\]
and for $b \geq l$ there is an equivalence
\[
\text{hofib}(F_{\geq l, \leq b} \to F_{\geq l, \leq b-1}) \simeq H(\pi_b(F)).
\] (12.4)

By (12.3) and induction we conclude that for all $b \geq l \in \mathbb{Z}$ there is an equivalence
\[
L_{\mathbb{A}^1} L_{\text{Nis}}(F_{\geq l, \leq b}) \cong L_{\text{Nis}}(F_{\geq l, \leq b}).
\] (12.5)

This shows $\pi_n(L_{\text{Nis}} F_{\geq l, \leq b})(X)$ is $\mathbb{A}^1$-invariant for every $n \in \mathbb{Z}$.

If $X \in \text{Sm}_B$ is $d$-dimension and $b > n + d$, Lemma 12.4 implies $\pi_n(L_{\text{Nis}} F_{\geq b})(X) = 0$. Hence, for all $l \in \mathbb{Z}$, we have
\[
\pi_n(L_{\text{Nis}} F_{\geq l})(X) \cong \pi_n(L_{\text{Nis}} F_{\geq l, \leq n+d})(X),
\]
and
\[
\pi_n(L_{\text{Nis}} F_{\geq l})(X \times \mathbb{A}^1) \cong \pi_n(L_{\text{Nis}} F_{\geq l})(X).
\] (12.6)

Since (12.6) holds all $X \in \text{Sm}_B$ and $n \in \mathbb{Z}$ there is an equivalence
\[
L_{\mathbb{A}^1} L_{\text{Nis}}(F_l) \simeq L_{\text{Nis}}(F_l)
\] (12.7)
for every $l \in \mathbb{Z}$. Owing to Lemma 12.3 we deduce the equivalence
\[
L_{\mathbb{A}^1} L_{\text{Nis}}(F) \simeq L_{\text{Nis}}(F),
\] (12.8)
and thus $L_{\text{Nis}} L_{\text{Nis}}(F)$ is $\mathbb{A}^1$-local. □
Lemma 12.3. If \( F \simeq \hocolim_d \mathcal{F}_l \) in \( \text{SH}_\ast(\text{Fr}_\ast(B)) \) then there are equivalences
\[
\mathcal{L}_{\text{Nis}}(F) \simeq \hocolim_d \mathcal{L}_{\text{Nis}}(\mathcal{F}_d), \quad \mathcal{L}_{\Delta}^\ast(F) \simeq \hocolim_d \mathcal{L}_{\Delta}^\ast(\mathcal{F}_d).
\]

Proof. We may assume \( F \in \text{Sm}_\ast(\text{Sm}_B) \) since the forgetful functor \( \text{SH}_\ast(\text{Fr}_\ast(B)) \to \text{SH}_\ast(\text{Sm}_B) \) preserves homotopy colimits. To prove the first claim, we show that the sequential homotopy colimit in question preserves equivalences and Nisnevich local objects in \( \text{SH}_\ast(\text{Sm}_B) \).

Suppose \( \mathcal{F}_\ast \to \mathcal{G}_\ast \) is a morphism of sequential diagrams such that \( \mathcal{F}_l \to \mathcal{G}_l \) is an equivalence in \( \text{SH}_\ast(\text{Sm}_B) \) for all \( l \geq 0 \). Then, for all \( X \in \text{Sm}_B \) and \( x \in X \), there are equivalences
\[
(\hocolim_d \mathcal{F}_l)(X^h_x) \simeq \hocolim_d \mathcal{F}_l(X^h_x) \simeq \hocolim_d \mathcal{G}_l(X^h_x) \simeq (\hocolim_d \mathcal{G}_l)(X^h_x).
\]
Moreover, if \( \mathcal{F}_\ast \) is comprised of Nisnevich local objects, then for any Nisnevich square given by an open immersion \( U \to X \) and étale morphism \( X' \to X \) we have equivalences
\[
F(X' \times X U) \simeq \hocolim_d \ho e q(F_l(X) \Rightarrow F_l(U U X')) \simeq \ho e q(F(X) \Rightarrow F(U U X')).
\]
Here we use the assumption \( F \simeq \hocolim_d \mathcal{F}_l \). To finish the proof we use the equivalences
\[
\mathcal{L}_{\Delta}^\ast(F) \simeq \mathcal{L}_{\Delta}^\ast(\Delta^\ast \times -) \simeq \hocolim_d \mathcal{F}_l(\Delta^\ast \times -) \simeq \hocolim_d \mathcal{L}_{\Delta}^\ast(\mathcal{F}_l).
\]

Lemma 12.4. Let \( F \) be a presheaf of \( S^1 \)-spectra on \( \text{Sm}_B \) and suppose \( X \in \text{Sm}_B \) is \( d \)-dimensional. Then for integers \( l, n \in \mathbb{Z} \) such that \( n < l - d \), we have \( \pi_n(\mathcal{L}_{\text{Nis}}\mathcal{F}_l))(X) = 0 \).

Proof. Note that \( F \) extends continuously to a presheaf of \( S^1 \)-spectra on \( \text{EssSm}_B \). By forming \( (\mathcal{F}_\geq l)[l] \) we may assume that \( l = 0 \) and \( F \in \text{SH}_{s \geq 0}(\text{Sm}_B) \). Let \( U \) be an open subscheme of \( X \in \text{EssSm}_B \). In the range \( n < -d \) we will show the vanishing
\[
[X/U \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] = 0. \tag{12.9}
\]
We proceed by induction on \( d = \dim X \) and \( c = \dim(X - U) \). The claim for \( d = 0 \) holds since there are no nontrivial Nisnevich coverings of a zero-dimensional scheme.

Assume \( c = 0 \), \( d \geq 0 \), and set \( Y = X - U \). Then, since \( \mathcal{L}_{\text{Nis}}\mathcal{F} \) is Nisnevich local, we have
\[
[X/U \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] \cong [X^h_Y/(X^h_Y \times X U) \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}],
\]
Here, \( X^h_Y \) is a disjoint union of local henselian schemes because \( c = 0 \). Moreover, since \( F \) is additive, we have \( F \in \text{SH}_{s \geq 0}(\text{Sm}_B) \). Recall that Nisnevich coverings on a local henselian scheme are trivial; for \( n < 0 \), it follows that
\[
[X^h_Y \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] \cong [X^h_Y \otimes S^n, F_{\text{SH}_\ast(B)}] = 0.
\]
Thus, in the range \( n < -d \), the inequalities \( \dim(X^h_Y \times X U) < \dim X^h_Y \leq \dim X \) imply
\[
[(X^h_Y \times X U) \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] = 0.
\]
For \( n < -d \), we deduce the vanishing
\[
[X^h_Y/(X^h_Y \times X U) \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] = 0.
\]
Assume (12.9) holds for all pairs \( X', U' \) such that \( \dim X' < d \), or \( \dim X' = d \), \( \dim(X' - U') < c \).

Let \( \nu \) be the union of the generic points of \( Y = X - U \). For all \( n < -d \) the inductive assumption reads
\[
[X_{\nu'}/(X_{\nu' \times X U}) \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] = 0.
\]
Hence for every \( e \in [X/\nu \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] \) there exists an open subscheme \( V \subset X \) such that the closure of \( V \) contains \( Y \) and the image of \( e \) in \([V/(V \times X U) \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] = 0 \) is trivial. Since \( \dim(X - (U \cup V)) < \dim Y \), it follows by the inductive assumption that
\[
[X/(V \cup U) \otimes S^n, \mathcal{L}_{\text{Nis}}\mathcal{F}_{\text{SH}_\ast(B)}] = 0.
\]
for all $n < -d$. Using the naturally induced exact sequence

$$[X/(V \cup U) \otimes S^n, \mathcal{L}_{\text{Nis}} \mathcal{F}]_{\text{SH}_{B}} \to [X/U \otimes S^n, \mathcal{L}_{\text{Nis}} \mathcal{F}]_{\text{SH}_{B}} \to [V/(V \cap U) \otimes S^n, \mathcal{L}_{\text{Nis}} \mathcal{F}]_{\text{SH}_{B}}$$

we conclude $e = 0$.

**Lemma 12.5.** Let $B$ be a one-dimensional local irreducible scheme with closed point $z \in B$ and generic point $\eta$. If $\mathcal{F}$ is a presheaf of abelian groups on $\text{Sm}_B$, then for $X \in \text{Sm}_B$ we have

$$H^i_{\text{tf}}(X, \mathcal{F}) \cong \begin{cases} \ker(\mathcal{F}(X^h_z) \otimes \mathcal{F}(X_\eta) \to \mathcal{F}((X^h_z)_\eta)) & i = 0, \\ \text{coker}(\mathcal{F}(X^h_z) \otimes \mathcal{F}(X_\eta) \to \mathcal{F}((X^h_z)_\eta)) & i = 1, \\ 0 & i \geq 2. \end{cases}$$

*Proof.* Consider the Eilenberg-MacLane object $H_{\mathcal{F}}$ on $\text{Sm}_B$. Proposition 9.3 implies

$$\mathcal{L}_{\text{tf}}(H_{\mathcal{F}})(X) \cong \text{holib}(H_{\mathcal{F}}(X^h_z) \otimes H_{\mathcal{F}}(X_\eta) \to H_{\mathcal{F}}((X^h_z)_\eta)) \cong H(\mathcal{L}_{\text{tf}}(\mathcal{F})).$$

where $\mathcal{L}_{\text{tf}}(\mathcal{F})$ denotes the complex

$$[\mathcal{F}(X^h_z) \otimes \mathcal{F}(X_\eta) \to \mathcal{F}((X^h_z)_\eta)].$$

Thus the tf-cohomology of $\mathcal{F}$ agrees with the cohomology of the complex (12.10). \hfill \Box

### 13. A counterexample to Nisnevich strict $\mathbb{A}^1$-invariance

Following Voevodsky’s Theorem 1.1 one may ask whether the Nisnevich sheafification of any $\mathbb{A}^1$-invariant presheaf with transfers over $B$ is strictly $\mathbb{A}^1$-invariant. An equivalent statement is that for any such presheaf $\mathcal{F}$ and essentially smooth local henselian scheme $U$ over $B$, we have

$$H^i_{\text{Nis}}(\mathbb{A}^1_U, \mathcal{F}_{\text{Nis}}) \cong \begin{cases} \mathcal{F}(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

(13.1)

We give a counterexample to (13.1) for every positive dimensional base scheme $B$. Owing to Lemma 13.1 we may assume $B$ is a local henselian scheme.

**Lemma 13.1.** The strict $\mathbb{A}^1$-invariance theorem in the form of (13.1) holds over a base scheme $B$ if and only if it holds over $B^h_\sigma$ for every $\sigma \in B$.

*Proof.* Any essentially smooth local henselian scheme $U$ over $B^h_\sigma$ is an essentially smooth local henselian scheme over $B$. Conversely, any essentially smooth local henselian scheme $U$ over $B$ is a scheme over $B^h_\sigma$, where $\sigma$ is the image of the closed point of $U$. Our claim for (13.1) follows readily from these observations. \hfill \Box

**Example 13.2.** Let $B$ be a local scheme. Assume $\dim B > 0$, and let $f \in \mathcal{O}(B)$ be a regular function such that the vanishing locus $Z(f)$ is a proper closed subscheme of $B$. Letting $t$ denote the coordinate on $\mathbb{A}^1_B$ we have the rational function $r = ft - 1 \in \mathcal{O}(\mathbb{A}^1_B \times B)$, and the open complement

$$V = \mathbb{A}^1_B - Z(fr).$$

Let $\mathcal{E}$ be the $\mathbb{A}^1$-invariant presheaf with transfers defined as the cokernel

$$\mathcal{E} := \text{coker} \left( \text{Cor}_B(\mathbb{A}^1_B \times_B (-), V) \xrightarrow{\text{i}_0^* - \text{i}_1^*} \text{Cor}_B(-, V) \right)$$
for the canonical sections of the affine line over $B$. Consider the class $id \in H^0_{Nis}(V, \mathcal{E})$ given by the identity morphism on $V$. We claim that $id \in H^0_{Nis}(V, \mathcal{E})$ maps to a nontrivial class in $H^1_{Nis}(\mathbb{A}^1_B, \mathcal{E})$ under the boundary morphism $\delta$ in the Mayer-Vietoris sequence

$$H^0_{Nis}(\mathbb{A}^1_B - Z(f)) \amalg (\mathbb{A}^1_B - Z(r)), \mathcal{E}) \to H^0_{Nis}(V, \mathcal{E}) \xrightarrow{\delta} H^1_{Nis}(\mathbb{A}^1_B, \mathcal{E}) \to \cdots$$

First we note that $\text{Cor}_B(\mathbb{A}^1_B - Z(r), V)$ is the trivial group, as the following argument shows. By assumption, $Z(f)$ is a nonempty proper closed subscheme of $B$, and moreover $Z(f) \times_B V = \emptyset$. Suppose that $W$ is a nonempty irreducible closed subscheme of $(\mathbb{A}^1_B - Z(r)) \times_B V$ which is finite and surjective over $\mathbb{A}^1_B - Z(r)$; in other words, assume that $W \in \text{Cor}_B(\mathbb{A}^1_B - Z(r), V)$ is nontrivial. Then the fiber of $W$ over $Z(f)$ would be nonempty. On the other hand, since $Z(f) \times_B V = \emptyset$ and $W \subset (\mathbb{A}^1_B - Z(r)) \times_B V$, it follows that $Z(f) \times_B W = \emptyset$. This shows the vanishing $H^0_{Nis}(\mathbb{A}^1_B - Z(r), \mathcal{E}) = 0$.

Thus, if $\delta(id) = 0$, then the element $id \in H^0_{Nis}(V, \mathcal{E})$ is in the image of $H^0_{Nis}(\mathbb{A}^1_B - Z(f), \mathcal{E})$. But the latter would imply that for every $\mathbb{A}^1$-invariant presheaf with transfers $\mathcal{F}$ over the nonempty scheme $B - Z(f)$, the canonically induced morphism

$$\mathcal{F}(\mathbb{A}^1_B - Z(f) \cong \mathbb{A}^1_{B-Z(f)}) \to \mathcal{F}(V \cong \mathbb{A}^1_{B-Z(f)} - Z(r)) \quad (13.2)$$

is surjective. However, this fails for $O^\times(-)$, the $\mathbb{A}^1$-invariant presheaf with transfers given by the global units on $\text{Sm}_{B-Z(f)}$.

For a general base scheme, Lemma 13.3 shows strict $\mathbb{A}^1$-invariance for additive quasi-stable framed presheaves of abelian groups implies strict $\mathbb{A}^1$-invariance for presheaves with transfers. By Theorems 12.1 and 12.2, strict $\mathbb{A}^1$-invariance for framed presheaves of abelian groups is equivalent to Nisnevich strict $\mathbb{A}^1$-invariance in the sense of Definition 7.1. In summary, Example 13.2 disproves all the three naive variants of strict $\mathbb{A}^1$-invariance discussed in this section.

**Lemma 13.3.** Every presheaf with transfers of abelian groups on $\text{Sm}_B$ has the structure of a framed additive presheaf of abelian groups on $\text{Sm}_B$. Moreover, every such framed presheaf is quasi-stable.

**Proof.** If $c = (S, V, \varphi, g) \in \text{Fr}_n(X, Y)$ then its (non-reduced) support $S$ is isomorphic to the vanishing locus $Z(\varphi)$ in $V$, where $V$ is an étale neighborhood of $S$ in $\mathbb{A}^n_X$. Note that $S$ is a closed subscheme in $\mathbb{A}^n_X$ and finite over $X$. It follows that $S$ is either finite surjective over $X$ or empty. The same holds for the image $\mathcal{S}_g$ of $S$ in $X \times Y$ under the canonical projection $S \to X$ and $g: V \to Y$. The irreducible components $\mathcal{S}_g$ of $\mathcal{S}$ yield the finite correspondence

$$c^{tr} = \sum_i c_i \mathcal{S}_g \in \text{Cor}(X, Y). \quad (13.3)$$

The sum runs over the generic point of $\eta_i$ of $S$ and $c_i = \dim_K \mathcal{O}_{S(\eta_i)}(S(\eta_i))/\dim_K \mathcal{O}_{\eta_i}(\eta_i)$, where $K$ is the residue field of the image of $\eta_i$ in $X$, $S(\eta_i)$ is the local scheme of $S$ at $\eta_i$, and $\mathcal{S}_g$ is the closure of the image of $\eta_i$ in $X \times Y$. Now the first claim follows by using the functor

$$\text{Fr}_n(B) \to \text{Cor}(B); c \mapsto c^{tr}. \quad (13.4)$$

Finally, the equality $(\sigma c)^{tr} = \sigma(c^{tr})$ implies the second claim. \hfill \square

**Remark 13.4.** We note that (13.4) preserves the composition by reduction to fields. Indeed, it suffices to consider fields since the multiplicities of the cycles in (13.3) are defined via the fibers over the generic points of $B$. 

14. Applications of tf-Nisnevich strict $\mathbb{A}^1$-invariance

In this section we discuss a few applications of strict $\mathbb{A}^1$-invariance. More precisely, we discuss the stable motivic fibrant replacement functor and localization for motivic local objects, and show that $G_m$-loops commute with the motivic localization functor.

14.1. Computing motivic fibrant replacements. As in Section 14.2 we write $\text{Spt}_{\ast,\text{Nis}}(B)$ for the levelwise Nisnevich local model structure, and $\text{Spt}_{\ast,\text{tf,Nis}}^{\mathbb{A}^1}(B)$ (resp. $\text{Spt}_{\ast,\text{tf,Nis}}^{\mathbb{A}^1}(B)$) for the stable motivic model structure (resp. stable motivic tf-local model structure) on $\text{Spt}_{\ast}(B)$. Let $\mathcal{L}_{\text{Nis}}, \mathcal{L}_{\ast,\text{mot}}, \mathcal{L}_{\ast,\text{tf,}\mathbb{A}^1}$ denote the corresponding fibrant replacement functors. Similarly, we write $\mathcal{L}_{\text{Nis}}, \mathcal{L}_{\ast,\text{mot}}, \mathcal{L}_{\ast,\text{tf,}\mathbb{A}^1}$ for the associated localization endofunctors on $\text{SH}_s(B)$.

To every $X \in \text{Sm}_B$ we associate the framed presheaf of $S^1$-spectra

$$\mathcal{M}_f(X) := \{\text{Fr}(-, X \otimes S^1)\}_{i \geq 0}$$

with terms the simplicial quasi-stable framed presheaves $\text{Fr}(-, X \otimes S^1) = \lim_{\rightarrow n} \text{Fr}_n(-, X \otimes S^1)$ introduced in Definition 14.1.

**Theorem 14.1.** Suppose $B$ satisfies tf-Nisnevich strict $\mathbb{A}^1$-invariance. If $F$ is a tf-motivic fibrant quasi-stable additive framed object of $\text{Spt}_{\ast,\text{tf,Nis}}^{\mathbb{A}^1}(\text{Sm}_B)$, then $\mathcal{L}_{\text{Nis}}(F)$ is motivic fibrant in $\text{Spt}_{\ast,\text{Nis}}^{\mathbb{A}^1}(\text{Sm}_B)$.

**Proof.** The functor $\mathcal{L}_{\text{Nis}}$ preserves presheaves of $\Omega_+$-spectra, so $\mathcal{L}_{\text{Nis}}(F)$ is a presheaf of $\Omega_+$-spectra. Moreover, $F$ is fibrant in the tf-local model structure on $\text{Spt}_{\ast,\text{tf,Nis}}(\text{Sm}_B)$, and $\mathcal{L}_{\text{tf}}(F)$ is $\mathbb{A}^1$-local in $\text{SH}_{\text{tf}}(\text{Sm}_B)$ since $F$ is $\mathbb{A}^1$-local. By tf-Nisnevich strict $\mathbb{A}^1$-invariance on $\text{Sm}_B$ we see that $\mathcal{L}_{\text{Nis}}(F)$ is $\mathbb{A}^1$-local in $\text{SH}_{\text{Nis}}(\text{Sm}_B)$. This implies our claim for $\mathcal{L}_{\text{Nis}}(F)$. \qed

**Corollary 14.2.** If $F$ is a tf-motivic fibrant quasi-stable additive framed presheaf of $\Omega_+$-spectra on $\text{Fr}_+(B)$, then we have $\mathcal{L}_{\ast,\text{mot}}(F) \simeq \mathcal{L}_{\text{Nis}}(F)$ and $\mathcal{L}_{\ast,\text{mot}}(F) \simeq \mathcal{L}_{\text{Nis}}(F)$.

**Proof.** The canonical morphisms $F \rightarrow \mathcal{L}_{\text{Nis}}(F) \rightarrow \mathcal{L}_{\ast,\text{mot}}(F)$ are stable motivic equivalences. By Theorem 14.1, $\mathcal{L}_{\text{Nis}}(F)$ is (stably) motivic local and hence it is equivalent to $\mathcal{L}_{\ast,\text{mot}}(F)$. Moreover, since $\mathcal{L}_{\text{Nis}}(F)$ is stably motivic fibrant, it is equivalent to $\mathcal{L}_{\ast,\text{mot}}(F)$. \qed

**Corollary 14.3.** If $F$ is quasi-stable additive framed presheaf of $S^1$-spectra on $\text{Fr}_+(B)$, then the stable motivic replacement $\mathcal{L}_{\ast,\text{mot}}(F) \simeq \mathcal{L}_{\text{Nis}}\mathcal{L}_{\ast,\text{tf,}\mathbb{A}^1}(F)$. In particular, there is an equivalence

$$\mathcal{L}_{\ast,\text{mot}}\mathcal{M}_f(X) \simeq \mathcal{L}_{\text{Nis}}\mathcal{L}_{\ast,\text{tf,}\mathbb{A}^1,\mathcal{M}_f}(X).$$

**Remark 14.4.** Theorem 14.1 and Corollary 14.3 reduce the problem of computing the motivic fibrant replacement of $\mathcal{M}_f(X)$ to the stable tf-motivic fibrant replacement in $\text{Spt}_{\ast}(B)$. Moreover, by the proofs of the additivity theorems for framed motives, see [16, Theorem 6.4] and [13, Proposition 2.2.11, Lemma 2.3.15], the question reduces to computing the tf-motivic localization endofunctor $\mathcal{L}_{\ast,\mathbb{A}^1,\mathcal{M}_f}(X)$ on the homotopy category of $\text{Spt}_{\ast}(B)$ with respect to levelwise equivalences. Since tf-strict $\mathbb{A}^1$-invariance does not hold for positive dimensional schemes, see Section 14.3, $\mathcal{L}_{\ast,\mathbb{A}^1,\mathcal{M}_f}(X)$ is not equivalent to $\mathcal{L}_{\ast,\mathbb{A}^1,\mathcal{M}_f}(X)$.

When $B$ is a one-dimensional scheme there is an equivalence $\mathcal{L}_{\mathbb{A}^1,\mathcal{M}_f}(X) \simeq \mathcal{L}_{\mathbb{A}^1,\mathcal{M}_f}(X)$ by definition there exists a canonical morphism $\mathcal{L}_{\mathbb{A}^1,\mathcal{M}_f}(X) \rightarrow \mathcal{L}_{\mathbb{A}^1,\mathcal{M}_f}(X)$. Preliminary work on constructing an inverse, together with a generalization to higher dimensional base schemes, occupies a large part of [R]. The said equivalence implies $\mathcal{L}_{\ast,\mathbb{A}^1,\mathcal{M}_f}(X)$ is $(-1)$-connective for every connective $F \in \text{SH}_s(B)$ since the tf-cohomological dimension of any $X \in \text{Sm}_B$ equals 1 when $B$ has Krull dimension one. We leave further details to a future work.
Remark 14.5. Let $B$ be a noetherian and separated base scheme. The category of presheaves with transfers $\text{Pre}^{tr}(B)$ is defined in [22]. We define the tf-motivic homotopy category $\text{DM}^{eff}_{tf}(B)$ as the Bousfield localization of $\text{Ch}(\text{Pre}^{tr}(B))$ with respect to the tf-local and $A^1$-local equivalences. If tf-Nisnevich strict $A^1$-invariance in the sense of Definition 7.2 holds on $\text{Sm}_B$, then the motivic localization endofunctor $L_{\text{mot}}$ on $\text{DM}^{eff}_{tf}(B)$ is induced by the Nisnevich localization endofunctor $L_{\text{Nis}}$ on $\text{Ch}(\text{Pre}^{tr}(B))$ in the sense that there is a commutative square

\[
\begin{array}{ccc}
\text{Ch}(\text{Pre}^{tr}(B)) & \xrightarrow{L_{\text{Nis}}} & \text{Ch}(\text{Pre}^{tr}(B)) \\
\downarrow & & \downarrow \\
\text{DM}^{eff}_{tf}(B) & \xrightarrow{L_{\text{mot}}} & \text{DM}^{eff}_{tf}(B).
\end{array}
\]

Remark 14.6. In the setting of [13] we can formulate tf-Nisnevich strict $A^1$-invariance on $\text{Sm}_B$ as a computation of the localization functor $\text{SH}^{fr}_{s,tf}(B) \to \text{SH}^{fr}_{s,Nis}(B)$ from the $\infty$-category of framed tf-motivic presheaves of $S^1$-spectra to the $\infty$-category of framed presheaves of $S^1$-spectra. The $S^1$-stable tf-motivic homotopy category $\text{SH}^{fr}_{s,tf}(B)$ is the left Bousfield localization of the $\infty$-category of framed presheaves of $S^1$-spectra on $\text{Sm}_B$ with respect to tf-local equivalences and $A^1$-local equivalences. Nisnevich-tf strict $A^1$-invariance in the sense of Definition 7.2 asserts that, for every $F \in \text{SH}^{fr}_{s,tf}(B)$, the motivic localization $L^{fr}_{\text{mot}}(F)$ coincides with the Nisnevich localization $L^{fr}_{\text{Nis}}(F)$.

14.2. Localization for motivic local objects.

Theorem 14.7. For every motivic local object $F \in H_{s,*}(\text{Sm}_B)$ (resp. every tf-motivic local object $F \in H_{s,*}(\text{SmAff}_B)$) there is a homotopy pullback square

\[
\begin{array}{ccc}
i_*i^!(F) & \xrightarrow{i_*} & F \\
\downarrow & & \downarrow \\
* & \xrightarrow{j_*j^*} & j_*j^!(F).
\end{array}
\]  

(14.1)

The same holds for every motivic local object $F \in H_{s,*}(\text{Fr}_+(B))$ (resp. every tf-motivic local object $F \in H_{s,*}(\text{Fr}_+(\text{SmAff}_B))$).

Proof. We note there exist factorizations:

\[
\begin{array}{ccc}
H_{s,*}(\text{SmZ}) & \xrightarrow{i_{B,Z}^!} & H_{s,*}(\text{Sm}_B,\text{Z}) & \xrightarrow{i_*} & H_{s,*}(\text{Sm}_B) \\
\downarrow & & \downarrow & & \downarrow \\
H_{s,*}(\text{Sm}_B) & \xrightarrow{i_* \circ i^!} & H_{s,*}(\text{Sm}_B,\text{Z}) & \xrightarrow{i_*} & H_{s,*}(\text{Sm}_B).
\end{array}
\]
Proposition 9.1 implies that for every motivic local object $F \in H_{s,\bullet}(\text{Sm}_B)$, and similarly for $H_{s,\bullet}(\text{SmAff}_B)$, there is a homotopy pullback square

$$
\begin{align*}
\tilde{i}_s^* i_!^!(F) & \to F \\
\downarrow & \\
\tilde{i}_s^! & \to j_* j^!(F).
\end{align*}
$$

To conclude the proof we appeal to Lemma 6.13 and Lemma 6.14 for $H_{s,\bullet}(\text{SmAff}_B)$, which shows there is an equivalence

$$
\tilde{i}_s^! i_!^!(F) \cong i^B_\ast i^B_\ast \tilde{i}_s^! i_!^!(F).
$$

14.3. $\mathbb{G}_m$-loops and motivic localization. As another application of tf-Nisnevich strict $\mathbb{A}^1$-invariance, we will show that the motivic localization functor commutes with $\mathbb{G}_m$-loops. This result is of independent interest and it suggests that $\mathbb{G}_m$-cancellation [34] holds for tf-local $\mathbb{A}^1$-local quasi-stable radditive framed bispectra over one-dimensional base schemes with perfect residue fields. We write $\text{Spt}_{s,t}(B)$ for the category of $(s,t)$-bispectra on $\text{Sm}_B$, see [11, §2.3]. On $\text{Spt}_{s,t}(B)$ we have the injective model structure $\text{Spt}_{s,\mathbb{G}_m}(B)$ with respect to the $S^1$-stable equivalences, and its Bousfield localization

$$
\text{Spt}_{s,\mathbb{G}_m,\text{Nis}}(B) \quad (14.2)
$$

with respect to the levelwise motivic equivalences. In what follows, $\text{SH}_{s,\mathbb{G}_m}(B)$ denotes the homotopy category of $\text{Spt}_{s,\mathbb{G}_m}(B)$, $\mathcal{L}_{\text{mot}}$ denotes the fibrant replacement functor on (14.2), which in turn induces $\mathcal{L}_{\text{mot}} : \text{SH}_{s,\mathbb{G}_m}(B) \to \text{SH}_s(B)$. Our main aim is to show that the natural transformation

$$
\mathcal{L}_{\text{mot}} \Omega^\infty_{\mathbb{G}_m} \to \Omega^\infty_{\mathbb{G}_m} \mathcal{L}_{\text{mot}} : \text{SH}_{s,\mathbb{G}_m}(B) \to \text{SH}_s(B) \quad (14.3)
$$

is an equivalence on the subcategory of quasi-stable radditive framed $(s,t)$-bispectra for any one-dimensional base scheme $B$ with perfect residue fields.

In Sections 6, 9 and 10 we studied $\mathbb{A}^1$-local (tf-local) stable radditive framed presheaves of $S^1$-spectra and the Nisnevich localization endofunctor on $\text{SH}_s(B)$. An $(s,t)$-bispectrum $F$ is an $\Omega_{s,t}$-bispectrum if $F_{i,j} \simeq \Omega_i(F_{i+1,j})$ and $F_{i,j} \simeq \Omega_t(F_{i,j+1})$. We refer to $\Omega_{s,t}$-bispectra of $\mathbb{A}^1$-local (tf-local) quasi-stable radditive framed presheaves as $\mathbb{A}^1$-local (tf-local) quasi-stable radditive framed $\Omega_{s,t}$-bispectra. In this subsection, we apply similar arguments to $\mathbb{A}^1$-local (tf-local) quasi-stable radditive framed $\Omega_{s,t}$-bispectra and the $\mathbb{G}_m$-levelwise Nisnevich localization endofunctor $\mathcal{L}_{\text{Nis}}$ on $\text{SH}_{s,\mathbb{G}_m}(B)$.

Lemma 14.8. Suppose $B$ is a one-dimensional base scheme and the endofunctor $\mathcal{L}_{\text{Nis}}$ on $\text{SH}_{s,\mathbb{G}_m}(\sigma)$ preserves $\mathbb{A}^1$-local quasi-stable radditive framed $\Omega_{s,t}$-bispectra for all $\sigma \in B$. Then the same holds for the endofunctor $\mathcal{L}_{\text{Nis}}$ on $\text{SH}_{s,\mathbb{G}_m}(B)$.

Proof. To any closed immersion $Z \to B$ we can associate

$$
\text{SH}_{s,\mathbb{G}_m}(Z), \quad \text{SH}_{s,\mathbb{G}_m}(B, Z), \quad \text{SH}_{s,\mathbb{G}_m}(B - Z).
$$

Applying the arguments in Section 9 to the Nisnevich localization on $\text{SH}_{s,\mathbb{G}_m}(B)$ and $\mathbb{A}^1$-local quasi-stable radditive framed $\Omega_{s,t}$-bispectra shows that if the Nisnevich localization endofunctor $\mathcal{L}_{\text{Nis}}$ preserves $\mathbb{A}^1$-local quasi-stable radditive framed in $\text{SH}_{s,\mathbb{G}_m}(B, Z)$ and $\text{SH}_{s,\mathbb{G}_m}(B - Z)$, then the same holds for $\text{SH}_{s,\mathbb{G}_m}(B)$. Here, as in Section 9 we use that $\tilde{\iota}$ preserves Nisnevich local equivalences; this requires that $B$ is one-dimensional. Arguing as in Sections 6 and 10 for the Nisnevich localization endofunctor $\mathcal{L}_{\text{Nis}}$ on $\text{SH}_{s,\mathbb{G}_m}(B)$ and $\mathbb{A}^1$-local
quasi-stable radditive framed $\Omega_{s,t}$-bispectra shows that if $\mathcal{L}_{\text{Nis}}$ preserves $A^1$-local quasi-stable radditive framed $\Omega_{s,t}$-bispectra in $\text{SH}_{s,\mathbb{G}_m}(Z)$, then the same holds for $\text{SH}_{s,\mathbb{G}_m}(Z)$. \hfill \square

We write $\text{SH}_{s,t}(B)$ for the localization of $\text{SH}_{s,\mathbb{G}_m}(B)$ with respect to $\mathbb{G}_m$-stable equivalences. Note that $\text{SH}_{s,t}(B)$ is equivalent to the homotopy category of $\text{Spt}_{s,t}(B)$ with respect to the stable motivic equivalences. Let $\mathcal{L}_{\mathbb{G}_m}: \text{SH}_{s,\mathbb{G}_m}(B) \to \text{SH}_{s,\mathbb{G}_m}(B)$ denote the corresponding localization endofunctor.

**Proposition 14.9.** Suppose $B$ is a one-dimensional base scheme. If the natural transformation

$$\mathcal{L}_{\mathbb{G}_m} \mathcal{L}_{\text{mot}} \to \mathcal{L}_{\text{mot}} \mathcal{L}_{\mathbb{G}_m}: \text{SH}_{s,\mathbb{G}_m}(\sigma) \to \text{SH}_{s,\mathbb{G}_m}(\sigma) \quad (14.4)$$

is an equivalence on the subcategory of quasi-stable radditive framed $(s,t)$-bispectra for all $\sigma \in B$, then the same holds over $B$.

**Proof.** The functors $\mathcal{L}_{\mathbb{G}_m}$ and $\mathcal{L}_{A^1}$ commute since the endofunctors on $\text{Sm}_B$ given by $X \mapsto X \times \mathbb{G}_m$ and $X \mapsto X \times A^1$ commute, and $\mathcal{L}_{A^1} \simeq \text{Hom}(\Delta^*,-)$. We let $\mathcal{L}_{\mathbb{G}_m,A^1}$ denote the localization endofunctor on $\text{SH}_{s,\mathbb{G}_m}(B)$ with respect to the $A^1$-equivalences and $\mathbb{G}_m$-stable equivalences. Note that $\mathcal{L}_{\mathbb{G}_m,A^1} \simeq \mathcal{L}_{\mathbb{G}_m,A^1} \simeq \mathcal{L}_{\mathbb{G}_m,A^1}$ on the subcategory of quasi-stable radditive framed objects.

For every quasi-stable radditive framed object $\mathcal{F} \in \text{SH}_{s,\mathbb{G}_m}(B)$, Lemma [11.8] shows that $\mathcal{L}_{\text{Nis}} \mathcal{L}_{\mathbb{G}_m,A^1}(\mathcal{F})$ is a Nisnevich local $A^1$-local quasi-stable radditive framed $\Omega_{s,t}$-bispectrum. It follows that $\mathcal{L}_{\text{Nis}} \mathcal{L}_{\mathbb{G}_m,A^1}(\mathcal{F}) \simeq \mathcal{L}_{\text{Nis},\mathbb{G}_m,A^1}(\mathcal{F})$,

where $\mathcal{L}_{\text{Nis},\mathbb{G}_m,A^1}$ denotes the localization endofunctor with respect to Nisnevich local equivalences, $A^1$-equivalences, and $\mathbb{G}_m$-stable equivalences. This shows $\mathcal{L}_{\text{mot}} \mathcal{L}_{\mathbb{G}_m,A^1}(\mathcal{F})$ is a Nisnevich local $A^1$-local quasi-stable radditive framed $\Omega_{s,t}$-bispectrum and $\mathcal{L}_{\text{mot}} \mathcal{L}_{\mathbb{G}_m,A^1}(\mathcal{F}) \simeq \mathcal{L}_{\text{Nis},\mathbb{G}_m,A^1}(\mathcal{F})$.

Since the endofunctors $X \mapsto X \times \mathbb{G}_m$ and $X \mapsto X \times A^1$ preserve Nisnevich squares, it follows that $\mathcal{L}_{\mathbb{G}_m,A^1}$ preserves Nisnevich local objects and hence $\mathcal{L}_{\mathbb{G}_m,A^1} \mathcal{L}_{\text{mot}} \simeq \mathcal{L}_{\mathbb{G}_m,A^1} \mathcal{L}_{\text{Nis}} \simeq \mathcal{L}_{\text{Nis},\mathbb{G}_m,A^1}$.

Thus for every quasi-stable radditive framed $\Omega_\tau$-bispectrum $\mathcal{F}$, there are equivalences

$$\mathcal{L}_{\text{mot}} \mathcal{L}_{\mathbb{G}_m}(\mathcal{F}) \simeq \mathcal{L}_{\text{mot}} \mathcal{L}_{A^1} \mathcal{L}_{\mathbb{G}_m}(\mathcal{F}) \simeq \mathcal{L}_{\text{mot}} \mathcal{L}_{\mathbb{G}_m,A^1}(\mathcal{F}) \simeq \mathcal{L}_{\text{Nis},\mathbb{G}_m,A^1}(\mathcal{F}),$$

$$\mathcal{L}_{\text{Nis},\mathbb{G}_m,A^1}(\mathcal{F}) \simeq \mathcal{L}_{\mathbb{G}_m,A^1} \mathcal{L}_{\text{mot}}(\mathcal{F}) \simeq \mathcal{L}_{\mathbb{G}_m} \mathcal{L}_{A^1} \mathcal{L}_{\text{mot}}(\mathcal{F}) \simeq \mathcal{L}_{\mathbb{G}_m} \mathcal{L}_{\text{mot}}(\mathcal{F}).$$

\hfill \square

**Corollary 14.10.** Suppose $B$ is a one-dimensional base scheme. If the natural transformation

$$\mathcal{L}_{\text{mot}} \Omega_\mathbb{G}_m^\infty \to \Omega_\mathbb{G}_m^\infty \mathcal{L}_{\text{mot}}: \text{SH}_{s,\mathbb{G}_m}(\sigma) \to \text{SH}_{s}(\sigma) \quad (14.5)$$

is an equivalence on the subcategory of quasi-stable radditive framed $(s,t)$-bispectra for all $\sigma \in B$, then the same holds over $B$. The same holds for the fibrant replacement functor $\mathcal{L}_{\text{mot}}$ and $\Omega_\tau^\infty$.

**Proof.** For any $\mathcal{F} \in \text{SH}_{s,\mathbb{G}_m}(B)$ there is a canonical equivalence

$$\mathcal{L}_{\mathbb{G}_m}(\mathcal{F}) \simeq (\Omega_\mathbb{G}_m^\infty \mathcal{F}, \Omega_\mathbb{G}_m^\infty \mathcal{F}(1), \ldots, \Omega_\mathbb{G}_m^\infty \mathcal{F}(l), \ldots),$$

where $\mathcal{F}(1)$ denotes the $\mathbb{G}_m$-suspension of $\mathcal{F}$. Thus for every quasi-stable radditive framed $(s,t)$-bispectrum the natural transformation $[11.3]$ is an equivalence if and only if $[11.3]$ is an equivalence, and we are done by Proposition [11.9]. The case of $\mathcal{L}_{\text{mot}}$ follows since

$$\Omega_\tau^\infty: \text{Spt}_{s,\mathbb{G}_m}(B) \to \text{Spc}_{s,\text{Nis}}(B)$$

preserves fibrant objects.

\hfill \square

**Corollary 14.11.** If every residue field of a one-dimensional base scheme $B$ satisfies strict $A^1$-invariance, then $[11.3]$ is an equivalence on any quasi-stable radditive framed $(s,t)$-bispectrum. In particular, this holds if $B$ has perfect residue fields.
Proof. By the proof of [3, Theorem A] strict $A^1$-invariance implies $\mathbb{G}_m$-cancellation for any field. In turn, $\mathbb{G}_m$-cancellation over the residue fields implies (14.3) is an equivalence. To conclude we apply Corollary 14.10.

For any chain complex of presheaf with transfers $F$ over $B$, we can associate the Eilenberg-MacLane object $H(F) \in \mathbb{SH}(B)$. In this way, a $\mathbb{G}_m$-spectrum of chain complexes of presheaves with transfers on $\text{Sm}_B$ gives rise to an $(s,t)$-bispectrum on $\text{Sm}_B$.

Corollary 14.12. If every residue field of a one-dimensional base scheme $B$ satisfies strict $A^1$-invariance for presheaves with transfers, then (14.3) is an equivalence on any $(s,t)$-bispectrum obtained from a $\mathbb{G}_m$-spectrum of chain complexes of presheaves with transfers. In particular, this holds if $B$ has perfect residue fields.

Proof. Arguing like in Section 12, we deduce strict $A^1$-invariance in the derived category of chain complexes of presheaves with transfers. As in Corollary 14.11, this allows us to conclude.

Let $\text{Spt}_t(B)$ denote the category of $t$-spectra of presheaves of simplicial sets on $\text{Sm}_B$, and let

$$\Omega_\infty : \text{Spt}_t(B) \to \text{Spc}(B)$$

denote the infinite $t$-loops functor. Let $\mathcal{L}_{\text{mot}}$ be the endofunctor on $\text{Spt}_t(B)$ given by applying levelwise the motivic fibrant replacement functor on $\text{Spc}(B)$. There is a canonical natural transformation

$$\mathcal{L}_{\text{mot}} \Omega_\infty^\infty \to \Omega_\infty^\infty \mathcal{L}_{\text{mot}} : \text{Spt}_t(B) \to \text{Spc}(B).$$

(14.6)

Let $\Gamma \text{Spt}_t(B)_{\text{gp}}$ denote the category of group-like framed motivic $\Gamma$-spaces over $B$ in the sense of [13, Axiom 1.1(7)]. There is a forgetful functor $\Gamma \text{Spt}_t(B)_{\text{gp}} \to \text{Spt}_t(B)$.

Corollary 14.13. Suppose $B$ is a one-dimensional base scheme. If the natural transformation

$$\mathcal{L}_{\text{mot}} \Omega_\infty^\infty \to \Omega_\infty^\infty \mathcal{L}_{\text{mot}} : \text{Spt}_t(\sigma) \to \text{Spc}(\sigma)$$

(14.7)

is an equivalence on the subcategory of quasi-stable additive grouplike framed motivic $\Gamma$-spaces for all $\sigma \in B$, then the same holds over $B$.

Proof. Segal’s $\Gamma$-space technology [30] provides a functor

$$\Gamma^{\text{op}} S \to \text{Spt}_t; F \mapsto F(S)$$

(14.8)

from the category of $\Gamma$-spaces to the category of $S^1$-spectra with values in the subcategory consisting of $\Omega_\infty$-spectra in positive degrees. Moreover, by [30, Prop. 1.4, 1.5], (14.8) takes a group-like $\Gamma$-space to an $\Omega_\infty$-spectrum, and the composite of (14.8) with the canonical morphism

$$\text{Spt}_t \to \text{Spc}; (E_0, \ldots, E_l, \ldots) \mapsto E_0$$

(14.9)

coinsides with the forgetful functor $\Gamma^{\text{op}} S \to \text{Spc}$ to simplicial sets. Using [18, 19] we obtain the functor

$$\Gamma \text{Spt}_t(B)_{\text{gp}} \to \text{Spt}_t(\sigma),$$

(14.10)

with values in $t$-spectra of framed presheaves of $\Omega_\infty$-spectra, i.e., $\Omega_\infty$-bispectra of framed presheaves of simplicial sets. Furthermore, (14.10) preserves quasi-stable additive objects. Since both $\Omega_\infty$ and $\mathcal{L}_{\text{mot}}$ commute with the functor

$$\text{Spt}_t(\sigma) \to \text{Spt}_t; (F_0, \ldots, F_l, \ldots) \mapsto F_0,$$

and with (14.10), the claim follows from Corollary 14.10.
Appendix A. Framed correspondences

In this appendix we collect a few definitions and results from the theory of framed correspondences which are used in the main body of the paper.

**Definition A.1.** Let $X$ and $Y$ be schemes over a base scheme $B$. A framed correspondence of level $n$ from $X$ to $Y$ is a triple $c = (S, \phi, g)$, where $S \not
\to \mathbb{A}^n_X$ is a closed immersion, $V = (\mathbb{A}^n_X)_S$ is the scheme defined in \[\text{(B.2)}, \; \phi: V \to \mathbb{A}^n_B \] and $g: V \to Y$ are regular morphisms such that $S \cong V \times_\phi \mathbb{A}^n_{B,i}$ (0 × $B$). Here $i: 0 \times B \to \mathbb{A}^n_B$ is the 0-section. We write Fr$_n(X, Y)$ for the set of framed correspondences of level $n$ from $X$ to $Y$, and refer to Supp$(c) := S$ as the support of $c$.

**Remark A.2.** In Definition [**A.1**] we use the scheme $(\mathbb{A}^n_X)_S$ instead of étale neighborhoods of $S$ as in [16, Definition 2.1]. These two definitions of framed correspondences agree if $X$ is affine. Consequently, the notions of strict $\mathbb{A}^1$-invariance in the sense of Definitions [7.1] and [7.2] are equivalent for both types of framed correspondences.

**Example A.3.** If $X \in \text{Sch}_B$ we set $\sigma_X := (0 \times X, \text{id}_{\mathbb{A}^1_X}, \text{pr}: \mathbb{A}^1_X \to X) \in \text{Fr}_1(X, X)$.

**Definition A.4.** The composition of two framed correspondences $(S, \phi, g) \in \text{Fr}_n(X, X')$ and $(S', \phi', g') \in \text{Fr}_m(X', X'')$ is the level $n + m$ framed correspondence from $X$ to $X''$ given by $(S'', \phi'', g'')$, where

- $(1) \quad S'' = S \times_X S' \subset \mathbb{A}^{n+m}_X \cong \mathbb{A}^n_X \times_{g, X'} \mathbb{A}^m_{Y'}$, $\text{pr}_{X'}: \mathbb{A}^m_{Y'} \to X'$,
- $(2) \quad V'' \to \mathbb{A}^{n+m}_X$ is the henselization of $\mathbb{A}^{n+m}_X$ at $S''$. There is a canonical morphism $V'' \to V$ (resp. $V'' \to V'$) to the henselization of $\mathbb{A}^n_X$ at $S$ (resp. $\mathbb{A}^m_Y$ at $S'$).
- $(3) \quad \phi'': V'' \to \mathbb{A}^{n+m}_B$ is given by the composites $V'' \to V \cong \mathbb{A}^n_B$, $V'' \to V' \cong \mathbb{A}^m_B$.
- $(4) \quad g'': V'' \to X''$ is the composite of the morphism $V'' \to V'$, induced by the base change $(g')^*(g)$ of $g$ along $g': V' \to X'$ and $g'': V'' \to X''$.

If $Z \not\to B$ is a closed immersion, recall from Section [2.3] that $X^h_Z$ denotes $X^h_{X \times Z}$. We write $\text{Sch}_{B,Z}$ for the full subcategory of $\text{Sch}_B$ spanned by $X^h_Z$ for $X \in \text{Sch}_B$. The monoidal product $\text{Sch}_{B,Z} \times \text{Sch}_{B,Z} \to \text{Sch}_{B,Z}$ is given by $(X^h_Z, Y^h_Z) \mapsto (X \times Y)^h_Z$. If $X \in \text{Sch}_{B,Z}$ we write $\mathbb{A}^n_X$ for the fiber product $\mathbb{A}^n \times_{B,Z} X \in \text{Sch}_{B,Z}$, and note that it is isomorphic to $(\mathbb{A}^n \times_B X)^h_Z$.

Next we define framed correspondences in $\text{Sch}_{B,Z}$ by tweaking Definition [**A.1**].

**Definition A.5.** For $X, Y \in \text{Sch}_{B,Z}$, a framed correspondence of level $n$ from $X$ to $Y$ is a triple $(S, \phi, g)$, where $S \not\to (\mathbb{A}^n_X)^h_Z$ is a closed subscheme, $V = (\mathbb{A}^n_X)^h_Z \cong (\mathbb{A}^n_X)_S$ is given by \[\text{(B.2)}, \; \phi: V \to \mathbb{A}^n_B \] and $g: V \to Y$ are regular morphisms such that $S \cong V \times_\phi \mathbb{A}^n_{B,i}$ (0 × $B$). Here $i: 0 \times B \to \mathbb{A}^n_B$ is the 0-section.

Framed correspondences in $\text{Sch}_{B,Z}$ admit a composition given as in Definition [**A.1**]. We write $\text{Fr}_+ (\text{Sch}_B)$ for the category with objects $X \in \text{Sch}_B$ and morphisms

$$\text{Fr}_+ (X, Y) := \bigvee_n \text{Fr}_n(X, Y).$$

The smooth $B$-schemes spans a full subcategory $\text{Fr}_+ (B) = \text{Fr}_+ (\text{Sm}_B)$ of $\text{Fr}_+ (\text{Sch}_B)$. Similarly, the subcategory $\text{Sm}_{B,Z} \subset \text{Sch}_{B,Z}$ spanned by $X^h_Z$, $X \in \text{Sm}_B$, gives rise to a full subcategory $\text{Fr}_+ (B, Z)$ of $\text{Fr}_+ (\text{Sch}_{B,Z})$.

**Lemma A.6.** If $Z \not\to W \not\to X$ are closed immersions then $X^h_Z \cong (X^h_W)^h_Z$.

*Proof.* This follows from the universal property of the cofiltered limit in \[\text{(B.2)}].

**Remark A.7.** Lemma [**A.6**] shows there is an isomorphism $(\mathbb{A}^n_X)^h_Z \cong (\mathbb{A}^n_X)_S$ for every closed immersion $S \not\to (\mathbb{A}^n_X)^h_Z$. Thus the set of framed correspondences $\text{Fr}_n(X^h_Z, Y^h_Z)$ in $\text{Sm}_{B,Z}$ is in bijection with the set of framed correspondences $\text{Fr}_n(X^h_Z, Y^h_Z)$ in $\text{Sch}_{B,Z}$. 

Lemma A.8. \textit{If }$Z \not\rightarrow B$\textit{ is a closed immersion then the functors}
\[ \text{Sch}_B \rightarrow \text{Sch}_B: X \mapsto X^h_B, X \mapsto X \times_B (B - Z), \]
and
\[ \text{Sch}_B \rightarrow \text{Sch}_{B,Z}: X \mapsto X^h_{B,Z}, \text{ Sch}_B \rightarrow \text{Sch}_{B-Z}: X \mapsto X \times_B (B - Z), \]
preserve framed correspondences, see Definition A.7. \textit{Hence there are functors}
\[ \text{Fr}_+(\text{Sch}_B) \rightarrow \text{Fr}_+(\text{Sch}_B) : X \mapsto X^h_B, X \mapsto X \times_B (B - Z), \]
\[ \text{Fr}_+(\text{Sch}_B) \rightarrow \text{Fr}_+(\text{Sch}_{B,Z}) : X \mapsto X^h_{B,Z}, \]
\[ \text{Fr}_+(\text{Sch}_B) \rightarrow \text{Fr}_+(\text{Sch}_{B-Z}) : X \mapsto X \times_B (B - Z). \]

\textbf{Proof.} Consider the functors
\[ \text{Sch}_B \rightarrow \text{Sch}_B : X \mapsto X^h_B, X \mapsto X \times_B (B - Z), \]
\[ \text{Sch}_B \rightarrow \text{Sch}_{B,Z} : X \mapsto X^h_{B,Z}, \]
\[ \text{Sch}_B \rightarrow \text{Sch}_{B-Z} : X \mapsto X \times_B (B - Z). \tag{A.1} \]
The data of an explicit framed correspondence over $B$ is encoded by diagrams of the form
\[ \begin{array}{ccc}
\mathbb{A}^n_X & \xrightarrow{(\varphi,g)} & V \\
S & \downarrow & \nearrow \\
X & \xrightarrow{(0)} & Y.
\end{array} \tag{(A.2)} \]

To conclude we note that the functors in \ref{A.1} preserve diagrams of the form \ref{A.2}. \qed

Lemma A.9. \textit{There is an embedding of categories $\text{Sm}_B^\text{cci} \rightarrow \text{Sch}_B^\text{cci},$ i.e., for every }$X \in \text{Sm}_B^\text{cci}$\textit{ there exists a closed immersion }$X \not\rightarrow \mathbb{A}^n_B$\textit{ and regular functions }$e_1, \ldots, e_l \in \mathcal{O}(\mathbb{A}^n_B)$\textit{ such that $Z(e_1, \ldots, e_l) = X \cup \bar{X}$}. \textbf{Proof.} By assumption $X \in \text{SmAff}_B$ and the relative tangent bundle of $X$ over $B$ is stably trivial. Hence there is a closed immersion $X \not\rightarrow \mathbb{A}^n_B$ and $T_{X/B} \oplus \mathcal{O}_X^{\oplus l} \cong \mathcal{O}_X^{\oplus n+l}$. By increasing $n_1$ and $n_2$ we may assume $n = n_1 = n_2$. We claim there exists a trivialization of the normal bundle $N_{X/\mathbb{A}^n_B}$. In effect, choose regular functions $e_1, \ldots, e_l$ on $\mathbb{A}^n_B$ such that $e_i|_X = 0$, $i = 1, \ldots, l$. Then the differential of $(e_1, \ldots, e_l)$ yields the desired trivialization of $N_{X/\mathbb{A}^n_B}$. \qed

Lemma A.10. \textit{For every }$X \in \text{Sm}_B^\text{cci}$\textit{ or }$X \in \text{Sch}_B^\text{cci}$\textit{ there exists a closed immersion }$X \not\rightarrow \mathbb{A}^n_B$\textit{ and regular functions }$e_1, \ldots, e_l \in \mathcal{O}(\mathbb{A}^n_B)$\textit{ such that }$Z(e_1, \ldots, e_l) = X$. \textbf{Proof.} Owing to Lemma A.9 we may assume $X \in \text{Sch}_B^\text{cci}$ is a closed subscheme of the vanishing locus $Z(e'_1, \ldots, e'_l)$ for some regular functions $e'_i \in \mathcal{O}(\mathbb{A}^n_B)$ such that $\dim_B Z(e'_1, \ldots, e'_l) = n - l$. It follows that $Z(e'_1, \ldots, e'_n) = X \cup \bar{X}$. Let $s$ be a regular function on $\mathbb{A}^n_B$ such that $s|_X = 0$, $s|_{\bar{X}} = 1$, and set $n = n' + 1$, $l = l' + 1$. Define $e_1, \ldots, e_l \in \mathcal{O}(\mathbb{A}^n_B)$ as the inverse images of $e'_i$ along the projection $\mathbb{A}^n_B \rightarrow \mathbb{A}^{n'}_B$. Next we set $e_l := x_n - \bar{s} \in \mathcal{O}(\mathbb{A}^n_B)$, where $x_n$ is the coordinate on $\mathbb{A}^n$ such that $Z(x_n) = \mathbb{A}^{n'}_B$ and $\bar{s}$ is the inverse image of $s$. Then the subscheme $X \times_B 0$ of $\mathbb{A}^n_B$ agrees with the vanishing locus $Z(e_1, \ldots, e_l)$. \qed

Lemma A.11. \textit{Suppose }$B$\textit{ is affine and let }$X \in \text{SmAff}_B$, $U \in \text{AffSch}_B$. \textit{If }$Y \not\rightarrow U$\textit{ is a closed immersion, there is a naturally induced surjection}
\[ \text{Fr}_+(U^h_Y, X) \rightarrow \text{Fr}_+(Y, X). \]
Proof. Let \( c = (S, V, \phi, g) \in \text{Fr}_n(Y, X) \) for some \( S \subset \mathbb{A}_n^v \) and \( \phi \in O(V) \). Let \( T \) be the first-order thickening of \( Z(\phi) \subset \mathbb{A}_n^v \), see [32, Tag 04EW], with restrictions \( \phi|_T \in O(T) \) for \( 1 \leq i \leq n \). Let \( \overline{\phi}_i \in O(\mathbb{A}_U^v) \) be a lifting of \( \phi_i|_T \) along the closed immersion of affine schemes \( T \to \mathbb{A}_U^v \), and set \( \overline{S} = Z(\overline{\phi}) \). By shrinking \( V \) we may assume there is a closed immersion \( V \to \mathbb{A}_n^{1 \times X} \), for some \( l \), such that the image of \( S \) equals \( 0 \times X \). Moreover, since \( V \) is étale over \( \mathbb{A}_n^{1 \times X} \), Lemma A.10 lets us assume \( V = Z(\epsilon_1, \ldots, \epsilon_l) \) for regular functions \( \epsilon_j \in O(\mathbb{A}_n^{1 \times X}), 1 \leq j \leq l \).

Now choose a lifting \( \tilde{\epsilon}_i \in O(\mathbb{A}_n^{1 \times X}) \) of \( \epsilon_j \) such that \( \epsilon_j|_{O \times \overline{S}} = 0 \) and set \( \tilde{V} = Z(\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_l) \); this is a subscheme of \( \mathbb{A}_n^{1 \times X} \). The first order thickening \( \tilde{T} \) of \( \overline{S} \) in \( \mathbb{A}_U^v \) and the closed immersions

\[
V \to \mathbb{A}_n^{1 \times X} \to \mathbb{A}_n^{1 \times X} \times \mathbb{A}_n^{1 \times X} \to \mathbb{A}_n^{1 \times X} \times \mathbb{A}_n^{1 \times X} \to \mathbb{A}_n^{1 \times X}
\]

yield \( V \rightrightarrows \mathbb{A}_n^{1 \times X} \to \mathbb{A}_n^{1 \times X} \times \mathbb{A}_n^{1 \times X} \) \( 1 \leq i \leq n \). Note that \( \overline{S} \) is finite over \( U \); hence \( S \to \overline{S} \) is a henselian pair since the same holds for \( Y \to U \).

Thus the henselization of \( S \) in \( \mathbb{A}_n^v \) equals the henselization of \( \overline{S} \) in \( \mathbb{A}_U^v \). Since \( X \) is smooth over \( B \), Lemma B.10 shows there exists a lifting \( \tilde{g}: (\mathbb{A}_U^v)^h \to X \) of \((\mathbb{A}_n^v)^h \to X \).

The framed correspondence \( \tilde{c} = (\tilde{S}, \tilde{V}, \tilde{\phi}, \tilde{g}) \) is the desired lifting of \( c \). \( \square \)

**Definition A.12.** For \( c_0, c_1 \in \text{Fr}_n(X, Y) \), \( X, Y \in \text{Sm}_B \), we write \( c_1 \sim_{\mathbb{A}_1} c_2 \) if there exists a framed correspondence \( c \in \text{Fr}_n(X \times \mathbb{A}_1, Y) \) such that for the canonical sections \( i_0, i_1 : X \to X \times \mathbb{A}_1 \) we have \( c \circ i_0 = c_0 \) and \( c \circ i_1 = c_1 \). Let \( \text{Fr}_n(X, Y) \) denote the quotient set of \( \text{Fr}_n \) with respect to the equivalence relation generated by \( \sim_{\mathbb{A}_1} \).

**Definition A.13.** For \( Y \in \text{Sm}_B \) let \( \text{ZF}_n(-, Y) \) denote the presheaf of free abelian groups associated to \( \text{Fr}_n(-, Y) \). Following the notations of [17, Definition 2.11], we write \( \text{ZF}_n(-, Y) \) for the quotient of \( \text{ZF}_n(-, Y) \) with respect to the relations

\[
c = c_1 + c_2, \quad (S_1, \phi, g) \in \text{Fr}_n(X, Y), \quad X \in \text{Sm}_B,
\]

\[
S = S_1 \amalg S_2, \quad c_1 = (S_1, \phi, g) \in \text{Fr}_n(X, Y),
\]

\[
c_2 = (S_2, \phi, g) \in \text{Fr}_n(X, Y).
\]

For \( X \in \text{Sm}_B \) and the canonical sections \( i_0, i_1 : X \to \mathbb{A}_1 \times X \), we define \( \text{ZF}_n(-, Y) \) by

\[
\text{ZF}_n(X, Y) := \ker(\text{ZF}_n(X \times \mathbb{A}_1, Y) \xrightarrow{i_0 - i_1} \text{ZF}_n(X, Y)).
\]

**Definition A.14.** If \( X, Y \in \text{Sm}_B \) consider the pointed sets and abelian groups

\[
\text{Fr}_n(X, Y) := \bigvee_{n \geq 0} \text{Fr}_n(X, Y), \quad \text{Fr}(X, Y) := \varinjlim_n \text{Fr}_n(X, Y),
\]

\[
\text{ZF}_n(X, Y) := \bigoplus_{n \geq 0} \text{ZF}_n(X, Y), \quad \text{ZF}(X, Y) := \varinjlim_n \text{ZF}_n(X, Y),
\]

\[
\text{ZF}_*(X, Y) := \bigoplus_{n \geq 0} \text{ZF}_n(X, Y), \quad \text{ZF}(X, Y) := \varinjlim_n \text{ZF}_n(X, Y).
\]

The transition morphisms in the colimits are given by precomposition with \( \sigma_Y \), see Example A.3.

Let \( \text{ZF}_{n}(B), \text{ZF}_{n}(B) \), and \( \text{ZF}_{*}(B) \) denote the categories with objects smooth \( B \)-schemes and with morphisms as above. Let \( \text{Fr}(Y), \text{ZFr}(Y), \text{ZF}(Y), \text{ZF}(Y) \) denote the quasi-stable framed presheaves on \( \text{Sm}_B \) defined sectionwise as above.

Any framed presheaf of \( S^1 \)-spectra \( F \in \text{Spt}_*(\text{Fr}_+(B)) \) defines a functor \( F: \text{Fr}_+(B) \to \text{SH} \), which we denote by the same symbol. Since \( \text{SH} \) is additive, any additive functor \( F: \text{Fr}_+(B) \to \text{SH} \) induces a functor

\[
ZF_{*}(B) \to \text{SH}. \quad \text{(A.3)}
\]
Proposition A.15. Suppose \( \mathcal{F} : \text{Fr}_+(B) \to \text{SH} \) is radditive and \( \mathbb{A}^1 \)-local. For \( c_1, c_2 \in \text{Fr}_n(X, Y) \) assume there exists an element \( c \in \mathcal{Z}\text{F}_n(X \times \mathbb{A}^1, Y) \) such that

\[
c_1 = c \circ i_0, c_2 = c \circ i_1 \in \mathcal{Z}\text{F}_n(X, Y)
\]

for the canonical sections \( i_0, i_1 : X \to X \times \mathbb{A}^1 \). Then we have the equality

\[
c_1^* = c_2^* : \mathcal{F}(Y) \to \mathcal{F}(X).
\]

In particular, the above applies to any radditive object \( F \in \text{Spt}^+(\text{Fr}_+(B)) \).

Proof. From (A.3) we obtain the diagram

\[
\begin{array}{ccc}
\mathcal{F}(Y) & \xrightarrow{c^*} & \mathcal{F}(X \times \mathbb{A}^1) \\
\downarrow & & \downarrow \\
\mathcal{F}(Y) & \xrightarrow{c_i^*} & \mathcal{F}(X).
\end{array}
\]

Here we have \( i_0 \circ c^* = c^*_2 \) and \( i_1^* \circ c^* = c^*_1 \). Since \( \mathcal{F} \) is \( \mathbb{A}^1 \)-local the projection \( p : X \times \mathbb{A}^1 \to X \) induces an inverse \( p^* : \mathcal{F}(X) \to \mathcal{F}(X^1 \times X) \) to \( i_0^* \) and \( i_1^* \) in \( \text{SH} \). Hence the classes of \( i_0^* \) and \( i_1^* \), and thus also the classes of \( c_i^* \) and \( c^*_2 \), agree in \( \text{SH} \). \( \square \)

Corollary A.16. Every \( \mathbb{A}^1 \)-local radditive functor \( \mathcal{F} : \text{Fr}_+(B) \to \text{SH} \) induces canonically a functor \( \overline{\mathcal{F}}^+ : \text{Fr}_+(B) \to \text{SH} \).

Next, we introduce the notion of intermediate framed correspondences. Such correspondences are somewhat easier to construct than framed correspondences, see Definition A.4

Definition A.17. If \( X, Y \in \text{Sch}_B \), an intermediate framed correspondence of level \( n \) from \( X \) to \( Y \) is a triple \( c = (S, \phi, g) \), where

1. \( S \not\cong \mathbb{A}^n_X \) is a closed immersion,
2. \( \phi : V \to \mathbb{A}^n_B \) is a regular morphism such that \( S \cong V \times_{\phi, \mathbb{A}^n_B, i} (0 \times B) \), where \( V = (\mathbb{A}^n_X)_0^\phi \) and \( i : 0 \times B \to \mathbb{A}^n_B \) is the 0-section,
3. \( g : S \to Y \) is a regular morphism.

We refer to \( S \) as the support of \( c \) and write \( \text{Supp}(c) := S \).

Lemma A.18. Suppose \( B \) is an affine scheme, \( X \in \text{AffSch}_B \), and \( Y \in \text{SmAff}_B \). Then for every intermediate framed correspondence \( S, \varphi, g \) there exists a framed correspondence \( (S, \varphi, \tilde{g}) \) such that \( g : S \to Y \) and \( \tilde{g} : (\mathbb{A}^n_X)_0 \to Y \) lifts \( g \).

Proof. This follows from Lemma [4.10] \( \square \)

Remark A.19. Intermediate framed correspondences mediate between framed correspondences and normally framed correspondences, see [1, §4], [13]. For \( Y \in \text{Sm}_B \) there are canonical motivic equivalences \( \text{Fr}(Y) \to \text{Fr}^\text{in}(Y) \to \text{Fr}^\text{nl}(Y) \) for the said types of framed correspondences.

Theorem A.20. Suppose \( \mathcal{F} \) is an \( \mathbb{A}^1 \)-invariant quasi-stable radditive framed presheaf of \( \text{S}^1 \)-spectra over a field \( k \). For \( X \in \text{Sm}_k \) and \( x \in X \), assume that \( \mathcal{F}(\eta) \simeq 0 \), where \( \eta \in X_{(x)} \) is the generic point of the local scheme \( X_{(x)} \) of \( X \) at \( x \). Then we have \( \mathcal{F}(X_{(x)}) \simeq 0 \).

Proof. The presheaf of homotopy groups \( \pi_i \mathcal{F} : \text{Sm}_k \to \text{Ab} \), \( i \geq 0 \), equals the composite

\[
\text{Sm}_k \to \text{Fr}_+(k) \xrightarrow{\mathcal{F}} \text{SH} \xrightarrow{\pi_i} \text{Ab}.
\]

Thus \( \pi_i \mathcal{F} \) is an \( \mathbb{A}^1 \)-invariant quasi-stable radditive framed presheaf of abelian groups (the same holds over any base scheme when replacing the \( \mathbb{A}^1 \)-invariance condition with \( \mathbb{A}^1 \)-locality, see Section 2). We are done since the proof of [17, Theorem 3.9] works for all fields. \( \square \)
Proposition A.21. Suppose strict $\mathbb{A}^1$-invariance holds on $\text{Sm}_k$ and $F$ is an $\mathbb{A}^1$-local radditive framed presheaf of $S^1$-spectra on $\text{Sm}_k$ in the sense of Definition 7.1. Then there is a canonical equivalence

$$\mathcal{L}_{\text{Zar}}(F) \xrightarrow{\sim} \mathcal{L}_{\text{Nis}}(F).$$

Proof. Strict $\mathbb{A}^1$-invariance implies $\mathcal{L}_{\text{Nis}}(F)$ is $\mathbb{A}^1$-local, quasi-stable, and radditive. Theorem A.20 implies, for any $X \in \text{Sm}_k$ and $x \in X$, an equivalence of $S^1$-spectra

$$\mathcal{L}_{\text{Nis}}(F)(X(x)) \xrightarrow{\sim} F(X(x)).$$

Moreover, there is a naturally induced equivalence

$$\mathcal{L}_{\text{Zar}}(F)(X(x)) \xrightarrow{\sim} F(X(x)).$$

Hence the canonical morphism $\mathcal{L}_{\text{Zar}}(F) \to \mathcal{L}_{\text{Nis}}(F)$ is a Zariski local equivalence. Since both $\mathcal{L}_{\text{Nis}}(F)$ and $\mathcal{L}_{\text{Zar}}(F)$ are Zariski local, this ends the proof. \hfill $\square$

Definition A.22. For an invertible function $u \in O^*_B(B)$ we define

$$\langle u \rangle := (0 \times B, \mathbb{A}^1 \times B, ut, \text{pr}: \mathbb{A}^1 \times B \to B) \in F_1(B, B),$$

where $t$ denotes the coordinate function on $\mathbb{A}^1$. Moreover, for $n \geq 1$, we set

$$n_\varepsilon := \sum_{i=0}^{n-1} (-1)^i \in \mathbb{Z}F_1(B, B).$$

For units $u_1, u_2 \in O^*_B(B)$ we have the equality

$$[\langle u_1 \rangle \circ \langle u_2 \rangle] = [\langle u_1 u_2 \rangle] \in \mathbb{Z}F_2(B, B),$$

and for $n, l \geq 1$ we have

$$[\sigma(nl)_l] = [nl \circ l] \in \mathbb{Z}F_2(B, B). \quad (A.4)$$

Proposition A.23. For $U \in \text{Sm}_B$ suppose the regular functions $f_1, f_2 \in O(\mathbb{A}^1_U)$ are defined by monic separable polynomials of $O(U)[t] \cong O(\mathbb{A}^1_U)$ of coprime degrees $l_1$ and $l_2$, respectively. Then for every $\mathbb{A}^1$-local quasi-stable radditive framed presheaf of $S^1$-spectra $F$ on $\text{Sm}_B$, the identity $\text{id}_{F(U)}$ factors through the morphism $F(U) \to F(Z(f_1) \amalg Z(f_2))$ induced by $Z(f_1) \amalg Z(f_2) \to U$.

Proof. The vanishing loci $Z(f_1)$ and $Z(f_2)$ are closed subschemes in $\mathbb{A}^1_U$ that are finite and étale over $U$. Since $l_1$ and $l_2$ are coprime, there exist non-zero integers $n_1, n_2$ such that

$$n_1 l_1 - n_2 l_2 = 1. \quad (A.5)$$

We may assume $n_1, n_2 > 0$. Proposition A.28 shows the endomorphism on $F(U)$ induced by $(l_j)_l$ admits a factorization

$$(l_j)_l: F(U) \xrightarrow{pr^j} F(Z(f_j)) \xrightarrow{c^*_j} F(U),$$

for some morphisms $c^*_j$, $j = 1, 2$. We set

$$c^* := (n_1)_l \circ c^*_1 + (n_2)_l \circ c^*_2: F(Z(f_1) \amalg Z(f_2)) \to F(U).$$

Using (A.4) and (A.5) we deduce

$$(l_1)_l \circ (n_1)_l + (l_2)_l \circ (n_2)_l = [\text{id}_U] \in \mathbb{Z}F(U, U).$$

Since $F$ is quasi-stable, Corollary A.16 implies the composite morphism

$$c^* \circ pr^* = ((n_1)_l \circ c^*_1 \circ pr^* + ((n_2)_l \circ c^*_2 \circ pr^*)$$

$$= (n_1)_l \circ (l_1)_l^* + (n_2)_l \circ (l_2)_l^*$$
is an auto-equivalence of $\mathcal{F}(U)$. Hence there is a lift $\mathcal{F}(U) \to \mathcal{F}(U(f_1) \amalg U(f_2))$ of the identity morphism on $\mathcal{F}(U)$ along $\text{pr}^*$. 

**Corollary A.24.** Suppose $B$ is a local scheme whose closed point maps to $\gamma \in \text{Spec}(\mathbb{Z})$, and 

$$f = x^i + b_{i-1}x^{i-1} + \cdots + b_0 \in \mathbb{Z}[t]$$

is separable over the residue field of $\gamma$. For every $U \in \text{Sm}_B$ we set $U_f^i = Z(f_U)$, where $f_U \in \mathcal{O}(\mathbb{A}^1_U)$ is obtained from $f$. Suppose $\mathcal{F}$ is an $\mathbb{A}^1$-local additive framed presheaf of $S^1$-spectra on $\text{Sm}_B$ such that $\mathcal{F}(U_f^i) \simeq 0$ for $l > 0$. Then we have $\mathcal{F}(U) \simeq 0$.

**Proof.** Assume that $\mathcal{F}(U_f^i) \simeq 0$ for all $l > n$. Choose a closed point $p$ in $\text{Spec}(\mathbb{Z})$ that is contained in the closure of $\gamma$; its residue field is $\mathbb{F}_p$. For $j = 1, 2$ we may choose coprime integers $l_j \in \mathbb{Z}$, $l_j > n$, and a separable monic polynomial $r_j$ over $k$ of degree $l_j$. Such a polynomial exists for $l_j > 0$. Let $f_j$ be a monic polynomial with integral coefficients of degrees $l_j$ that lifts $r_j$. Then $f_j$ is separable over $\mathbb{F}_p$, and consequently over the residue field of $\gamma$. We denote by the same symbol the inverse image of $f_j$ in $\mathcal{O}_{\mathbb{A}^1 \times U}(\mathbb{A}^1 \times U)$. Note that $U'_j := Z(f_j)$ is finite and separable over $U$. Proposition [A.23] shows id$_{\mathcal{F}(U)}$ admits a factorization

$$\mathcal{F}(U) \to \mathcal{F}(U'_1 \amalg U'_2) \to \mathcal{F}(U).$$

We have $\mathcal{F}(U'_1 \amalg U'_2) \simeq 0$ since $l_1, l_2 > n$. It follows that $\mathcal{F}(U) \simeq 0$. 

**Corollary A.25.** Let $l_{1,i}, l_{2,i}$, for $i \geq 0$, be integers such that $(l_{1,i}, l_{2,i}) = 1$ for all $i, j \geq 0$. For $U \in \text{Sm}_B$, suppose the finite surjective étale morphisms in the diagrams

$$U'_{1,i} \to U'_{1,i+1} \to \cdots \to U'_{1,0} = U$$

$$U'_{2,i} \to U'_{2,i+1} \to \cdots \to U'_{2,0} = U$$

are defined by separable monic polynomials in $\mathcal{O}_U(U)$ of degrees $l_{1,i}$ and $l_{2,i}$, respectively. Suppose $\mathcal{F}$ is an $\mathbb{A}^1$-local quasi-stable additive framed presheaf of $S^1$-spectra on $\text{Sm}_B$ such that

$$\lim_{i \geq 0} \mathcal{F}(U'_{1,i}) \simeq 0 \text{ and } \lim_{i \geq 0} \mathcal{F}(U'_{2,i}) \simeq 0.$$ 

Then we have $\mathcal{F}(U) = 0$.

**Proof.** Using the projections $\text{pr}_{1,i} : U'_{1,i} \to U$ and $\text{pr}_{2,i} : U'_{2,i} \to U$ we set

$$F_i := \text{hofib}(\mathcal{F}(U) \to \mathcal{F}(U'_{1,i} \amalg U'_{2,i})).$$

Proposition [A.23] shows $F_i \to \mathcal{F}(U)$ is trivial, and by the assumption on $\mathcal{F}$ we have

$$\lim_{i} F_i \simeq \mathcal{F}(U).$$

**Lemma A.26.** For every finite field $k$ there exists a sequence of irreducible polynomials $(f_i)_{i \geq 0}$ with coprime degrees.

**Proof.** We prove the claim by induction. Let $f_1, \ldots, f_d \in k[t]$ be irreducible polynomials of coprime degrees $l_i > 1$. For $N > 0$ and $L = Nl_1 \cdots l_d + 1$, there exists a polynomial $f$ in $k[t]$ of degree $L$ such that $f(\alpha) = 1$ for every $\alpha \in k$. Since $L$ is coprime to $l_i$ for all $i = 1, \ldots, d$, there is an irreducible polynomial $f_{d+1}$ that divides $f$ and its degree $l_{d+1}$ is coprime to $l_i$ for all $i = 1, \ldots, d$. Since $l_{d+1} \neq 1$ we are done.

**Theorem A.27.** Let $k$ be a finite field. Then strict $\mathbb{A}^1$-invariance holds on $\text{Sm}_k$ in the sense of Definition [7.1].
Proof. Let \( \mathcal{F} \) be an \( \mathbb{A}^1 \)-local quasi-stable additive framed presheaf of \( S^1 \)-spectra on \( \text{Sm}_k \). For \( X \in \text{Sm}_k \) and \( x \in X \) we set \( U = X^k_x \). Let \( \tilde{V} \to V = \mathbb{A}^1_k \) be a Nisnevich covering. We use \( \text{V.13} \) to define the presheaf \( \text{Fib}(\_)_1 \) of \( S^1 \)-spectra on \( \text{Fr}_1(V) \) by setting
\[
\text{Fib}(W) := \text{hofib}(\tilde{C}_{\tilde{V} \times V, W}((W, \mathcal{F}) \to \mathcal{F}(W))).
\]

Let \( K \) be an infinite perfect field over \( k \). By strict \( \mathbb{A}^1 \)-invariance for \( K \) \( \text{[A.17, §17]} \) the presheaf \( L_{\text{Nis}}(\mathcal{F}_K) \) is \( \mathbb{A}^1 \)-local, where \( \mathcal{F}_K \) denotes the base change to \( K \). Hence there are equivalences
\[
L_{\text{Nis}}(\mathcal{F}_K)(\mathbb{A}^1 \times U \times \text{Spec } K) \simeq \mathcal{F}_K(\mathbb{A}^1 \times U \times \text{Spec } K),
\]
and
\[
\tilde{C}_{\tilde{V} \times \text{Spec } K}(V \times_k \text{Spec } K, \mathcal{F}) \simeq \mathcal{F}_K(V \times_k \text{Spec } K).
\]
This yields the equivalence
\[
\text{Fib}(V_K) \simeq 0.
\]
Furthermore we claim that
\[
\text{Fib}(\mathbb{A}^1_k) \simeq 0.
\]
Note that \( \text{A.7} \) implies \( L_{\text{Nis}}(\mathcal{F})(\mathbb{A}^1_k) \simeq \mathcal{F}(\mathbb{A}^1_k) \), and hence \( L_{\text{Nis}}(\mathcal{F})(\mathbb{A}^1 \times -) \simeq L_{\text{Nis}}(\mathcal{F})(-) \). This shows that \( L_{\text{Nis}}(\mathcal{F}) \) is \( \mathbb{A}^1 \)-local. It remains to prove \( \text{A.7} \). Since \( k \) is finite, every finite extension of \( k \) is separable. By Lemma \( \text{A.26} \) there exist separable extensions \( k(\alpha_1, i) \) and \( k(\alpha_{2, i}), i \geq 0 \), of coprime extension degrees over \( k \). For the infinite perfect field \( K_j = \lim_{j \geq 0} k(\alpha_{j, i}) \), \( \text{A.6} \) implies \( \text{Fib}(V_{K_j}) \simeq 0 \) for \( j = 1, 2 \). Corollary \( \text{A.26} \) implies that \( \text{Fib}(V) \simeq 0 \).

\( \square \)

Proposition A.28. Let \( U \in \text{Sm}_B \) and suppose
\[
f = x^l + b_{l-1}x^{l-1} + \cdots + b_0 \in \mathcal{O}_U(U)
\]
is a separable monic polynomial over \( \mathcal{O}_U(U) \) with vanishing locus \( Z(f) \neq \mathbb{A}^1_k \). Let \( \text{pr}: Z(f) \to U \) be the canonical étale morphism, and note that \( Z(f) \in \text{Sm}_B \). Then for every \( \mathbb{A}^1 \)-local additive framed presheaf of \( S^1 \)-spectra \( \mathcal{F} \) on \( \text{Sm}_B \), the endomorphism \( l^*_\epsilon \) on \( \mathcal{F} \) induced by the framed correspondence \( l_\epsilon \in \text{ZF}_1(U, U) \), see Definition \( \text{A.25} \) admits a factorization of the form
\[
\mathcal{F}(U) \xrightarrow{\text{pr}^*} \mathcal{F}(Z(f)) \to \mathcal{F}(U).
\]

Proof. Note that \( c = (Z(f), \mathbb{A}^1_k, g) \in \text{Fr}_1(U, Z(f)) \), where \( g: (\mathbb{A}^1_k)^h_{Z(f)} \to Z(f) \) exists by Lemma \( \text{B.10} \). We set \( \tilde{f} = f(1 - \lambda) + x^l \lambda \in \mathcal{O}_{\mathbb{A}^1 \times U \times \mathbb{A}^1}(\mathbb{A}^1 \times U \times \mathbb{A}^1) \), for the coordinate \( \lambda \) on the second copy of \( \mathbb{A}^1 \). Now consider the framed correspondence \( \tilde{c} = (\tilde{Z}(f), \mathbb{A}^1_{U \times \mathbb{A}^1}, \tilde{g}) \), where \( \tilde{g}: (\mathbb{A}^1_k)^h_{Z(f)} \to U \) is the canonical projection. The \( \mathbb{A}^1 \)-homotopy given by \( \tilde{c} \) implies the equality
\[
[\text{pr} \circ c] = [l_\epsilon] \in \text{ZF}_1(U, U).
\]

Since \( \mathcal{F} \) is an \( \mathbb{A}^1 \)-local quasi-stable additive framed presheaf, Proposition \( \text{A.15} \) and \( \text{A.8} \) imply the endomorphism
\[
(\text{pr} \circ c)^* : \mathcal{F}(U) \to \mathcal{F}(Z(f)) \to \mathcal{F}(U)
\]
defined by the composite framed correspondence \( \text{pr} \circ c \) coincides in \( \text{SH} \) with the composition
\[
\mathcal{F}(U) \xrightarrow{l^*_\epsilon} \mathcal{F}(U) \xrightarrow{\tau} \mathcal{F}(U)
\]
for some auto-equivalence \( \tau \). That is, \( l^*_\epsilon = \tau^{-1} \circ c^* \circ \text{pr}^* : \mathcal{F}(U) \to \mathcal{F}(U) \) in \( \text{SH} \), and the claim follows. \( \square \)
APPENDIX B. ESSENTIALLY SMOOTH SCHEMES AND HENSELIAN PAIRS

Let $X_\alpha$ be a directed inverse system of schemes with affine transitions morphisms $X_\alpha \to X_\beta$. The cofiltered limit $\lim_{\alpha} X_\alpha$ is a scheme [32, Tag 01YX]. If $U \to X$ is an open immersion and each $X_\alpha$ is an $X$-scheme, we set $U_\alpha = X_\alpha \times_X U$. Then the canonically induced morphism $\lim_{\alpha} U_\alpha \to \lim_{\alpha} X_\alpha$ is also an open immersion.

**Definition B.1.** An essentially smooth $B$-scheme $X$ is a cofiltered limit

$$X = \lim_{\alpha} X_\alpha$$

of smooth $B$-schemes with affine and dominant transition morphisms.

**Lemma B.2.** The canonical functor $\mathbf{SH}_s(\text{Sm}_B) \to \mathbf{SH}_s(\text{EssSm}_B)$ preserves Nisnevich local and $\text{tf}$-local objects, and it preserves Nisnevich local equivalences and $\text{tf}$-local equivalences.

**Proof.** Suppose $X \in \text{EssSm}_B$ and $Z$ is a closed subscheme of $X$. Then $X \cong \lim_{\alpha} X_\alpha$ for a filtering system of the schemes $X_\alpha \in \text{Sm}_B$ with affine transition morphisms, and $Z \cong \lim_{\alpha} Z_\alpha$, where $Z_\alpha$ is the closure of $Z$ in $X_\alpha$. If $F \in \mathbf{SH}_s(\text{Sm}_B)$ is Nisnevich local, then for the continuation of $F$ to essentially smooth schemes we have

$$F_{\cont}(X) := \text{hocollim}_a F(X_\alpha) \cong \text{hocollim}_a \text{hofib}(F((X_\alpha)^{h}_{Z_\alpha}) \vee F_{\cont}(X_\alpha - Z_\alpha))$$

$$\to F((X_\alpha)^{h}_{Z_\alpha} - Z_\alpha))$$

$$\cong \text{hofib(hocollim}_a F((X_\alpha)^{h}_{Z_\alpha}) \vee \text{hocollim}_a F_{\cont}(X_\alpha - Z_\alpha))$$

$$\to \text{hocollim}_a F((X_\alpha)^{h}_{Z_\alpha} - Z_\alpha)),$$

or equivalently

$$F_{\cont}(X) \cong \text{hofib}(F_{\cont}(X^{h}_Z) \vee F_{\cont}(X - Z) \to F_{\cont}(X^{h}_Z - Z)).$$

This shows that $F_{\cont} \in \mathbf{SH}_s(\text{EssSm}_B)$ is Nisnevich local. A similar argument applies in the $\text{tf}$-topology since $X_Z$ is isomorphic to $\lim_{\alpha} (X_\alpha \times_B Z)$. The claim for the local equivalences follows since a point in the Nisnevich topology (resp. $\text{tf}$-topology) is an essentially smooth local henselian scheme $X^{h}_x$, $x \in X$ (resp. $X^{h}_\sigma$, $\sigma \in B$, by Proposition 3.0(vii)).

For further reference we record two base change results for simplicial presheaves. We refer to [32, Tags 00X1, 00YK, 05V1] for our standard terminology on sites.

**Lemma B.3.** Suppose $(S', \tau') \to (S, \tau)$ is a morphism of sites given by a continuous morphism $f: S \to S'$. Then the base change functor $f_*: s\text{Pre}(S') \to s\text{Pre}(S)$ preserves local objects. If $(S, \tau)$, $(S', \tau')$ have enough points, then $f_*$ preserves local equivalences.

**Proof.** The first claim follows by [24, Corollary 5.24] because $f_*$ is a right Quillen adjoint for the local injective model structure. Indeed, if $F \in s\text{Pre}(S')$ is $\tau'$-local and $\tilde{X} \to X$ is a $\tau$-covering in $S$, then for the Čech construction $\tilde{C}_f(\tilde{X}, (f(X), F))$, see [8.13], there are equivalences

$$\tilde{C}_f(X, f_*, F) \cong \tilde{C}_f(\tilde{X}, f(X), F) \cong F(f(X)) \cong f_* F(X).$$

For preservation of local objects we note the canonically induced morphism $f_* \to RF_*$ is an equivalence, see also [27, Proposition 1.27, p.105]. If $F \to G$ is a local equivalence in $s\text{Pre}(S')$, then for any $\tau'$-point $V'$, there is an equivalence $F(V') \cong G(V')$. Hence, since $f$ preserves points by assumption, for any $\tau$-point $V$, we have $f_* F(V) \cong f_* G(V)$.

**Lemma B.4.** Let $(\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ be a morphism of sites with underlying functor $f: C' \to C$, where $\mathcal{C}$, $\mathcal{C}'$ are subcategories of $\text{Sch}_B$. Suppose $(\mathcal{C}, \tau)$, $(\mathcal{C}', \tau')$ have enough points given by objects in $\mathcal{C}$ and $\mathcal{C}'$, respectively, and for any $\tau'$-point $U$ in $C'$, $f(U)$ is a $\tau$-point in $\mathcal{C}$. Then $f$
is a continuous morphism of sites, and on simplicial presheaves, \( f_* \) takes \( \tau' \)-local equivalences to \( \tau \)-local equivalences.

**Proof.** Let \( F \to G \) be a \( \tau \)-local equivalence between simplicial presheaves on \( C \). Then \( F(V) \cong G(V) \) for each \( \tau \)-point \( V \) in \( C \). For any \( \tau' \)-point \( U \) in \( C' \), \( f(U) \) is a \( \tau \)-point by assumption, so that \( f_* F(U) = F(f(U)) \cong G(f(U)) = f_* G(U) \). Thus \( f_* F \to f_* G \) is a \( \tau' \)-local equivalence. \( \square \)

Recall from [32, Tag 09XD] the notion of an affine henselian pair. Next we recall the notion of henselization of pairs used in Section 4, see [32, Tag 0A02, 0EM7].

**Lemma B.5.** The henselization of an affine scheme \( X \) along a closed subscheme \( Y \) is the cofiltered limit

\[
X^h_Y \cong \varprojlim X_\alpha
\]

indexed by affine étale morphisms \( X_\alpha \to X \) that admit a lifting \( Z \to X_\alpha \). The henselization \( X^h_Y \) is an affine scheme. If \( X \) is noetherian, then so is \( X^h_Y \).

More generally, for a closed immersion \( Y \hookrightarrow X \) of \( B \)-schemes, we form the cofiltered category of affine étale morphisms \( X_\alpha \to X \) equipped with liftings

\[
\begin{array}{ccc}
X_\alpha & \to & X \\
\downarrow & & \\
Z & \hookrightarrow & X.
\end{array}
\]

Using the diagrams (B.1) we form the cofiltered limit and \( B \)-scheme

\[
X^h_Y := \varprojlim X_\alpha.
\]

The canonically induced closed immersion \( Y \hookrightarrow X^h_Y \) lifts the morphism \( Y \hookrightarrow X \) in \( \text{Sch.} \)

In what follows, a pro-étale morphism refers to a cofiltered limit

\[
\varphi: \tilde{X} = \varprojlim X_\alpha \to X
\]

obtained from étale morphisms \( \varphi_\alpha: X_\alpha \to X \) and affine étale transition morphisms \( X_\alpha \to X_\beta \). We note the morphism \( \varphi \) is affine.

**Lemma B.6.** Suppose that \( \varphi: \tilde{X} \to X \) is a pro-étale morphism and let \( i: Y \hookrightarrow X \) be a closed immersion that admits a lifting \( i: Y \hookrightarrow \tilde{X} \). There is a canonically induced isomorphism \( \tilde{X}^h_Y \cong X^h_Y \).

Consequently, \( i \) lifts to a morphism \( \tilde{X}^h_Y \to \tilde{X} \).

**Proof.** Write \( X^h_Y \cong \varprojlim X_\alpha, f_\alpha: X_\alpha \to X \), and \( s_\alpha: Y \to X_\alpha \) following (B.1). Similarly, we have \( \tilde{X}^h_Y \cong \varprojlim \tilde{X}_\alpha, \tilde{f}_\alpha: \tilde{X}_\alpha \to \tilde{X}, \) and \( \tilde{s}_\alpha: Y \to \tilde{X}_\alpha \). The morphism in question is induced by

\[
(f_\alpha \circ \tilde{f}_\alpha: \tilde{X}_\alpha \to X, s_\alpha: Y \to X_\alpha) \mapsto (\varphi \circ f_\alpha: \tilde{X}_\alpha \to X, s_\alpha: Y \to \tilde{X}_\alpha).
\]

To construct an inverse we note that any étale neighborhood of the form

\[
(f_\alpha: X_\alpha \to X, s_\alpha: Y \to X_\alpha),
\]

yields an étale neighborhood

\[
(f_\alpha: \tilde{X}_\alpha := \tilde{X} \times_X X_\alpha \to \tilde{X}, \tilde{s}_\alpha := (\tilde{i}, s_\alpha): Y \to \tilde{X}_\alpha).
\]

\( \square \)

**Corollary B.7.** Suppose \( X \in \text{Sm}_B \) and \( i: Z \hookrightarrow X \) is a closed immersion. Then \( X^h_Z \) is essentially smooth over \( B \).
Proof. Let \( \tilde{X} \) be the union of the connected components of \( X \) that have nonempty intersection with \( Z \). Lemma B.10 shows the open immersion \( \tilde{X} \to X \) induces an isomorphism \( \tilde{X}_Y^h \cong X_Y^h \). For any \( \acute{e} \text{tale} \) neighborhood \( X_\alpha \) of \( Y \) in \( \tilde{X} \), the morphism \( X \to \tilde{X} \) is dominant. Hence \( \tilde{X}_Y^h \) is an essentially smooth scheme.

Corollary B.8. Suppose \( i: Y \not\approx X \) and \( i': Y \not\approx X' \) are closed immersions such that \( X \cong X_Y^h \), \( X' \cong (X')_{Y'}^h \). If \( \varphi: X' \to X \) is a \( \acute{e} \text{tale} \) morphism and \( \varphi \circ i' = i \), then \( \varphi \) is an isomorphism.

Proof. The claim follows since \( X' \cong (X')_{Y'}^h \), where the last isomorphism follows from Lemma B.9.

Corollary B.9. Let \( Y \not\approx X \) be a closed immersion such that \( X \cong X_Y^h \). If \( Y = Y_1 \amalg Y_2 \), then \( X \cong X_1 \amalg X_2 \), where \( X_1 = X_{Y_1}^h \) and \( X_2 = X_{Y_2}^h \).

Proof. The \( \acute{e} \text{tale} \) morphism \( t: X_{Y_1}^h \amalg X_{Y_2}^h \to X_Y^h \cong X \) restricts to the decomposition \( Y_1 \amalg Y_2 \). Hence, by Corollary B.8 it follows that \( t \) is an isomorphism.

Lemma B.10. Suppose \( B \) is affine and let \( X \in \text{SmAff}_B \), \( U \in \text{AffSch}_B \). If \( T \not\approx U \) is a closed immersion, there is a naturally induced surjection

\[
\text{SmAff}_B(U_T^h, X) \to \text{SmAff}_B(T, X).
\]

Proof. Since \( X \) is smooth affine, there exists a vector bundle \( \xi \) on \( X \) such that \( \xi \oplus TX \) is trivial. Note that \( X \) is a retract of the total space \( \tilde{X} \) of \( \xi \) and the tangent bundle of the \( X \)-scheme \( \tilde{X} \) is trivial. Hence we may assume \( X \) has a trivial tangent bundle.

Choose a closed immersion \( X \not\approx A_B^n \) for some \( n \gg 0 \). Since the tangent bundle of \( X \) is trivial, the associated normal bundle \( N_{X/A_B^n} \) of \( X \) in \( A_B^n \) is trivial. By increasing \( n \) we may assume \( N_{X/A_B^n} \) is trivial. Hence there exist polynomials \( f_1, \ldots, f_l \in \mathcal{O}(A_B^n) \) such that the vanishing locus \( Z(f_1, \ldots, f_l) \) splits as \( X \amalg X' \) for some \( X' \). We use the same notation for the pullbacks of the \( f_i \)'s in \( \mathcal{O}(A^n_T) \) and \( \mathcal{O}(A^n_T) \).

The graph of any scheme morphism \( T \to X \) yields a closed immersion \( \Gamma \not\approx X \times_B T \subset A_B^n \) such that the canonical projection induces an isomorphism \( \Gamma \cong T \). Since the cotangent bundle of \( X \) is trivial there exist polynomials \( f_{i+1}, \ldots, f_n \in \mathcal{O}(A^n_T) \) such that the differentials of the \( f_i \)'s, \( i = 1, \ldots, n \), furnish a trivialization of the conormal bundle of \( \Gamma \) in \( X \times_B T \). We choose liftings of \( f_{i+1}, \ldots, f_n \) to regular functions \( \tilde{f}_{i+1}, \ldots, \tilde{f}_n \in \mathcal{O}(A^n_{U_T}) \) and use the trivialization of the conormal bundle of \( \Gamma \) in \( A^n_T \) to conclude the vanishing locus

\[
V = Z(f_1, \ldots, f_l, \tilde{f}_{i+1}, \ldots, \tilde{f}_n) \subset (X \times_B U) \subset A^n_U
\]

is \( \acute{e} \text{tale} \) over \( \Gamma \). Now, since \( T \not\approx U_T^h \) is a henselian pair, Lemma B.6 shows the morphism \( T \cong \Gamma \to V \) lifts to \( U_T^h \to V \). The desired lifting \( U_T^h \to X \) of \( T \to X \) is given by the composite

\[
U_T^h \to V \to X \times_B U \to X.
\]

\[ \square \]

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