RNA, LOCAL MOVES ON PLANE TREES, AND TRANSPOSITIONS
ON TABLEAUX

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Abstract. We define a collection of functions \( s_i \) on the set of plane trees (or Young tableaux). The functions are adapted from transpositions in the representation theory of the symmetric group and almost form a group action. They were motivated by local moves in combinatorial biology, which are maps that represent a certain unfolding and refolding of RNA strands. One main result of this manuscript identifies a subset of local moves that we call \( s_i \)-local moves, and proves that \( s_i \)-local moves correspond to our maps. We also prove that the graph of \( s_i \)-local moves is a connected, graded poset with unique minimal and maximal elements. We then extend this discussion to functions \( s_i^C \) that mimic reflections in the Weyl group of type \( C \). The corresponding graph is no longer connected, but we prove it has two connected components, one of symmetric and the other of asymmetric plane trees. We give open questions and possible biological interpretations.

1. Introduction

This paper analyzes a combinatorial question inspired by biology, specifically the mathematical structure of RNA. RNA has primary structure (a sequence of letters A, U, C, and G), secondary structure (a partial matching of the letters in the primary structure, indicating how the RNA strand has folded and bonded to itself), and a tertiary structure (how this folding occurs in 3-dimensional space). All of these structures contribute to the function of the RNA strand in ways that are still being uncovered.

There are many combinatorial models for the secondary structure of RNA, including plane trees and standard Young tableaux of shape \((n,n)\). We will compare two important operations on these combinatorial objects, one from biological applications and the other from representation theory.

The first operation is called a local move. Defined by Condon, Heitsch, and Hoos (and in Definition 3.2), local moves model unfolding an RNA strand and refolding it differently [3]. Heitsch described key combinatorial statistics of the graph whose vertices are plane trees

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on $n$ edges and whose edges are local moves; Heitsch also showed how this graph is related to other important graphs like those with noncrossing partitions as vertices [3].

The second operation comes from constructions of representations of the symmetric group $S_n$. One classical construction of representations of $S_n$ uses Young diagrams, which are staircase-shaped collections of boxes. The symmetric group acts naturally on the set of all fillings of a Young diagram with the integers $1, 2, \ldots, n$ (without repeating numbers), just by permuting the numbers. It turns out that this action on filled Young diagrams gives rise to irreducible representations of $S_n$ (see, e.g. [1, 8] for more).

It is natural to ask whether this symmetric group action can be modified to act on standard Young tableaux, which are fillings so that both rows and columns increase—and which are known to index bases for the irreducible representations of $S_n$, as well as other quantities of combinatorial interest. The answer is yes and no. In Section 2 we define a collection of maps that act on standard Young tableaux and agree as much as possible with the action of the simple transpositions $(i, i + 1)$ on arbitrary fillings of Young diagrams. More precisely, the map corresponding to the simple reflection $(i, i + 1)$ simply exchanges $i$ and $i + 1$ in the tableau $Y$ when this makes sense. The maps do not induce a group action of $S_n$ because composition of functions does not agree with multiplication in $S_n$. Thus these maps cannot directly give information about $S_n$-representations. However the maps are involutions, as we confirm in Theorem 2.4. Moreover similar maps arise in other parts of combinatorial representation theory, including Vogan’s generalized tau invariants [10, 5].

We then restrict our study to the Young tableaux corresponding to the partition $(n, n)$. This partition is an especially important one in applications from geometry [2] to knot theory [4], as well as the biological applications discussed here. In Theorem 3.6 we prove that our maps actually correspond to certain local moves, whose defining conditions are shown in Figure 3. We call the local moves that arise in this way $s_i$-local moves.

Note that not all local moves correspond to the action of permutations. In particular the graph $G^A$ whose vertices are plane trees and whose edges correspond to $s_i$-local moves is different from the graph whose edges are all local moves. The graph of all local moves is a connected graded poset for which the cardinalities of the ranks form a symmetric, unimodal sequence (see, e.g., [3]). Section 4 proves that the graph $G^A$ is still a

- connected (Proposition 4.1)
- graded poset (Proposition 4.4)
- with a unique minimal element and a unique maximal element (Proposition 4.7).

However the grading of the graph of $s_i$-local moves does not coincide with that of the graph of local moves, nor does the graph of $s_i$-local moves satisfy symmetry of ranks that the graph of local moves does (see Remark 4.6).

Our $s_i$-local moves were constructed by analogy with the symmetric group $S_n$. Thus we finish by extending the analogy to Weyl groups of other classical types, which we can do
by considering these groups as subgroups of $S_n$. Like Heitsch, we find that the description of type C local moves is particularly natural when we use the plane tree model. Indeed we prove in Corollary 5.7 that the graph $G^C$ of plane trees under type C local moves contains exactly two connected components: one consisting of symmetric plane trees and the other consisting of asymmetric plane trees.

We conclude with a brief discussion of biological interpretations of the type C local moves, as well as the analogies to types D and B. We give open questions throughout the paper.

Throughout this manuscript $Y$ denotes Young tableaux and $T$ denotes plane trees.

2. Maps on tableaux corresponding to simple transpositions

In this section we describe a set of involutions on the set of standard Young tableaux of shape $(n,n)$ that are indexed by simple reflections. Our maps are inspired by a well-known $S_n$-action from classical representation theory that gives all irreducible representations of the symmetric group. Our maps do not generate a group action, as we show in Remark 2.7. However because they are involutions, our maps induce a graph whose vertices are the set of standard Young tableaux of shape $(n,n)$ and whose edges correspond to the image under each map. We define this graph in this section. In subsequent sections we study combinatorial properties of the graph, prove that these maps agree with operations on planar trees from combinatorial biology, and discuss how to modify our maps analogously to changing Lie types.

To begin we recall the definition of Young tableaux and sketch their relationship to the representation theory of the symmetric group $S_n$.

**Definition 2.1.** Let $\lambda$ be a partition of $n$. A Young diagram of shape $\lambda$ is a collection of $\lambda_1$ boxes in the top row, $\lambda_2$ boxes in the second row, and so on, aligned on the top and the left. A standard Young tableau $Y$ of shape $\lambda$ is a filling of the Young diagram with the integers $\{1, 2, \ldots, n\}$ without repetition so that each row increases left-to-right and each column increases top-to-bottom.

The Specht module for a partition $\lambda$ is generated as a complex vector space by vectors $v_Y$ indexed by standard tableaux $Y$ of shape $\lambda$. The dimension of the irreducible representation of $S_n$ corresponding to $\lambda$ is also the number of standard Young tableaux of shape $\lambda$. The reasonable question is: is there an action of $S_n$ on standard Young tableaux under which the Young tableaux themselves can be the basis for the irreducible representation? Sadly the answer is generally no: the vectors $v_Y$ in the Specht module are linear combinations of terms corresponding to different fillings of $\lambda$. (See Fulton’s text [1] or Sagan’s text [8] for more.) The problem is that $S_n$ “should” act by permuting the entries of $Y$ but permuting the entries of $Y$ usually doesn’t produce another standard tableau.
In the following family of maps, we modify the permutation action on all fillings so that it produces standard tableaux. We define the maps on standard Young tableaux for arbitrary partitions; in later sections we only use the case $\lambda = (n, n)$.

**Definition 2.2.** Suppose that $Y$ is a Young tableaux and $s_i = (i, i+1)$ is a simple reflection. If $i, i+1$ are not in the same row or in the same column of $Y$ then define $s_i(Y)$ to be the tableau with $i$ and $i+1$ exchanged. If $i, i+1$ are in the same row or in the same column of $Y$ then define $s_i(Y)$ to be $Y$. For an arbitrary word $s_{i_1}s_{i_2}\cdots s_{i_k}(Y)$ define $s_{i_1}s_{i_2}\cdots s_{i_k}(Y)$ to be the tableau obtained by composition of maps.

Others have considered an analogous action on 3-row tableaux [5] that comes from Vogan’s generalized tau invariant [10].

The next result shows that these operations always give well-defined maps on standard tableaux (of arbitrary but fixed shape).

**Proposition 2.3.** For each $i = 1, \ldots, n - 1$ the map $s_i$ is well-defined and the image $s_i(Y)$ is a standard Young tableau of the same shape as $Y$.

**Proof.** By construction $s_i$ preserves the shape of $Y$. The boxes containing $i$ and $i+1$ inside the Young tableau $Y$ have numbers less than $i$ to the left and above and have numbers greater than $i+1$ to the right and below by definition of Young tableaux. Hence if $s_i$ exchanges $i$ and $i+1$ then the result $s_i(Y)$ is also a standard Young tableau. \hfill $\Box$

Moreover these maps are involutions, namely $s_i^2 = e$ for each $s_i$.

**Theorem 2.4.** Definition 2.2 produces a well-defined involution on the set of standard Young tableaux of shape $\lambda$.

**Proof.** We must check that for all $i$ we have $s_i^2 = e$. There are two cases:

1. **$i$ and $i+1$ are in the same row:** If $i$ and $i+1$ are in the same row then by our definition $s_i(Y) = Y$. That means $s_i(s_i(Y)) = s_i(Y) = Y$.

2. **$i$ and $i+1$ are in different rows:** If $i$ and $i+1$ are in different rows then by our definition the map $s_i$ swaps the positions of $i$ and $i+1$. That means when we apply $s_i$ twice to a Young tableau, we exchange $i$ and $i+1$ two times, bringing them back to their original positions.

In both cases $s_i^2 = e$ which proves the claim. \hfill $\Box$

This leads us to construct a graph whose vertices are standard Young tableaux of shape $\lambda$ and whose edges describe the maps $s_i$. The edges are undirected precisely because the maps $s_i$ are involutions for each $i$. 
Definition 2.5. Let $G_\lambda = (V, E)$ be the edge-labeled graph whose vertices $V$ are the set of standard Young tableaux of shape $\lambda$ and with an edge labeled $s_i$ between two tableaux $Y, Y'$ exactly when $s_i(Y) = Y'$. We call $G_\lambda$ the group of $s_i$-local moves for $\lambda$.

Remark 2.6. We typically omit all edges corresponding to fixed points $s_i(Y) = Y$ from our drawings of $G_\lambda$. In later sections we restrict to the case $\lambda = (n, n)$ and so omit $\lambda$ from our notation. We will also modify the maps $s_i$ that define the edges, so we often write $G^A$ to denote the graph with the precise edges in Definition 2.5 or $G^C$ to denote the modified graph in Section 5.

For example consider the graph $G_{(2,2)}$ corresponding to the partition $(2, 2)$ in Figure 1.

![Figure 1](image1.png)

**Figure 1.** The graph of $s_i = (i, i + 1)$ on $(2, 2)$ Young tableaux and their corresponding plane trees

Remark 2.7. Note that the maps $s_i$ do not induce a group action of the symmetric group $S_n$ on the standard Young tableaux even for the shape $(n, n)$. For a counterexample, inspect Figure 1. On the one hand $s_2s_3s_2(Y) = Y$ for each standard tableau $Y$ of shape $(2, 2)$. On the other hand $s_3s_2s_3(Y)$ is the opposite tableau of shape $(2, 2)$. Since $s_2s_3s_2 = s_3s_2s_3$ in the symmetric group, we conclude that these maps do not satisfy the associativity condition of a group action.

Question 2.8. In subsequent sections we analyze the graph $G_{(n,n)}$. What can be said about the graph $G_\lambda$ for arbitrary partitions?

3. $S_n$-action and local moves on plane trees

This section relates the functions defined in the previous section to an operation on plane trees called local moves. Condon, Heitsch, and Hoos defined local moves to represent an unfolding-and-refolding process on a strand of RNA. Heitsch then proved many combinatorial properties of a graph whose vertices are plane trees and whose edges come from local moves, for instance that the graph is symmetric and unimodal [3]. She also showed that under one natural modification to the edges, we obtain the graph whose vertices are noncrossing partitions and whose edges come from Kreweras complementation [3].
We extend these results in a different direction, showing that many local moves correspond naturally to the action of the maps $s_i$ on Young tableaux.

We begin by recalling the definition of plane trees and local moves.

**Definition 3.1.** A plane tree is a rooted tree whose subtrees at any vertex are linearly ordered.

Our convention is that the root is at the top and that the subtrees are linearly ordered from left to right. In figures, ordinary vertices are drawn with solid circles and the root is drawn with an open circle.

Plane trees are related to Young diagrams, noncrossing matchings, and other fundamental combinatorial objects by considering each edge as a pair of two half-edges, each of which is indexed with one of the integers from 1 to $2n$. The half-edges are labeled in increasing order counterclockwise from the root. We write $e(i,j)$ to denote the edge whose left half-edge is labeled $i$ and whose right half-edge is labeled $j$. Given this set-up, the half-edges $i$ and $j$ in the edge $e(i,j)$ satisfy many constraints, including $i < j$.

The next definition describes local moves, which are operations on plane trees that are central to this manuscript.

**Definition 3.2.** A local move on a plane tree $T \in T_n$ converts a pair of adjacent edges in one of two ways:

1. If $i < i' < j' < j$ then replace $e(i,j)$ and $e(i',j')$ with $e(i,i')$ and $e(j,j')$. This is a local move of type [1].
2. If $i < j < i' < j'$ then replace $e(i,j)$ and $e(i',j')$ with $e(i,j')$ and $e(j,i')$. This is a local move of type [2].

The following map provides a natural bijection between plane trees with $n$ edges and standard Young tableaux of shape $(n,n)$.

**Definition 3.3.** Let $T_n$ denote the collection of plane trees with $n$ edges and let $YT_{(n,n)}$ denote the set of standard Young tableaux of shape $(n,n)$. Define a map $\phi : T_n \to YT_{(n,n)}$ by the rule that for each $T \in T_n$ the Young tableau $\phi(T)$ has the labels of the left-half-edges of $T$ on its top row and the labels of the right-half-edges of $T$ on its bottom row.

The following proposition confirms that the map $\phi$ is bijective. Both image and domain are sets that are known to index the Catalan numbers [9 Chapter 6, Problem 19(e) and (ww)]; we include this proof to confirm that the specific map $\phi$ is a direct bijection.

**Proposition 3.4.** The map $\phi : T_n \to YT_{(n,n)}$ is a well-defined bijection.

**Proof.** The half-edges of a plane tree are labeled counter-clockwise, so for each $k$ there are at least as many left-edges $i$ with $i \leq k$ as right edges $j$ with $j \leq k$. Thus if $i$ is above $j$ in a column of the Young tableau $\phi(T)$ then $i < j$. It follows that $\phi$ is well-defined.
If $\phi(T) = \phi(T')$ then both $T$ and $T'$ have the same set of left half-edges and the same set of right half-edges. But the indexing of the half-edges determines the plane tree, since every subtree of a plane tree is linearly ordered by definition. So $\phi$ is injective.

The sets $T_n$ and $YT_{(n,n)}$ have the same cardinality [9, Chapter 6, Problem 19(e) and (ww)], so the map $\phi$ is a bijection, as desired. □

In order to prove our main result, we need more precise information about the fragments of a plane tree that correspond to the boxes filled with $i$ and $i+1$ in a Young tableau. The next lemma compiles this information.

**Lemma 3.5.** Consider a standard Young tableau $Y$ and its preimage $\phi^{-1}(Y)$ under the bijection in Definition 3.3. The half-edges corresponding to $i$ and $i+1$ are in one of the following relative positions:

- The numbers $i$ and $i+1$ are on the same row in $Y$ if and only if $i$ and $i+1$ label one of the following pairs of edges in $\phi^{-1}(Y)$:

  - $i$ and $i+1$ are both left half-edges
  - $i$ and $i+1$ are both right half-edges

  \[ \begin{array}{c}
  i & j \\
  \bullet & \bullet \\
  \bullet & \bullet \\
  i+1 & j'
  \end{array} \quad \begin{array}{c}
  j & i+1 \\
  \bullet & \bullet \\
  \bullet & \bullet \\
  j' & i
  \end{array} \]

  (a) $i$ and $i+1$ are both left half-edges (b) $i$ and $i+1$ are both right half-edges

  \[ \text{Figure 2. Edges that indicate } i \text{ and } i+1 \text{ are on the same row} \]

- The numbers $i$ and $i+1$ are on opposite rows in $Y$ if and only if in $\phi^{-1}(Y)$ either $i$ and $i+1$ label a leaf (Figure 3a) or $i$ and $i+1$ label the interior of a peak (Figure 3b).

  \[ \begin{array}{c}
  j & j' \\
  \bullet & \bullet \\
  \bullet & \bullet \\
  i & i+1
  \end{array} \quad \begin{array}{c}
  j & j' \\
  \bullet & \bullet \\
  \bullet & \bullet \\
  i & i+1
  \end{array} \]

  (a) $i$ and $i+1$ label a leaf (b) $i$ and $i+1$ label the interior of a peak

  \[ \text{Figure 3. Edges that indicate } i \text{ and } i+1 \text{ are on different rows} \]

- The numbers $i$ and $i+1$ are in the same column of $Y$ if and only if $i$ and $i+1$ label a leaf incident to the root in $\phi^{-1}(Y)$, as shown in Figure 4.
In no case is there an additional edge or half-edge incident to the vertex between \( i \) and \( i+1 \).

Proof. By convention, plane trees are labeled in counterclockwise order from the root so that there can be no edges or half-edges on the vertex incident to both \( i \) and \( i+1 \). We think of each edge \( e(i, j) \) as having a left half-edge labeled \( i \) and a right half-edge labeled \( j \).

Consider the case where the numbers \( i \) and \( i+1 \) are on the same row in \( Y \). By definition of \( \phi \) the top row of the Young tableau has the labels on the left half-edges of the corresponding plane tree while the bottom row has the labels on the right half-edges. Suppose \( i \) and \( i+1 \) are on the top row of the Young tableau. Then \( i \) and \( i+1 \) are left half-edges and must be in the configuration shown in Figure 2a.

Suppose \( i \) and \( i+1 \) are on the bottom row of the Young tableau. Then \( i \) and \( i+1 \) are right half-edges and must be in the configuration shown in Figure 2b.

Consider the case where the numbers \( i \) and \( i+1 \) are on different rows in \( Y \). Suppose \( i \) is on the top row and \( i+1 \) is on the bottom row. Then \( i \) is a left half-edge and \( i+1 \) is a right half-edge. That means these two numbers will label a leaf in the tree, as seen in Figure 3a.

Now suppose \( i+1 \) is in the top row and \( i \) is in the bottom row of \( Y \). Then \( i \) labels a right half-edge and \( i+1 \) labels a left half-edge. In a plane tree, this configuration must be a peak with \( i \) and \( i+1 \) on the interior labels, as shown in Figure 3b.

The numbers \( i \) and \( i+1 \) are in the same column of \( Y \) if and only if the first \( \frac{i-1}{2} \) columns of \( Y \) form a standard Young tableau of size \((\frac{i-1}{2}, \frac{i-1}{2})\) filled with the numbers 1, 2, \ldots, \( i-1 \). By restricting \( \phi \) to plane trees on \( \frac{i-1}{2} \) edges we note that the first \( \frac{i-1}{2} \) edges of the plane tree \( \phi^{-1}(Y) \) form a subtree with the same root as \( \phi^{-1}(Y) \). This is equivalent to saying that \( i-1 \) labels the right half of an edge incident to the root, as shown in Figure 4 which is true if and only if \( i \) and \( i+1 \) label the half-edges of a leaf incident to the root. \( \square \)

We use this lemma to show that if \( i \) and \( i+1 \) are in different rows (but not in the same column!) of a standard Young tableau then the action of the map \( s_i \) on the tableau corresponds to a local move on the corresponding plane tree. The following theorem makes this precise.

**Theorem 3.6.** Consider a plane tree \( T \) and its image \( Y = \phi(T) \) under the bijection in Definition 3.3. The half-edges labeled \( i \) and \( i+1 \) are in one of the two relative positions in \( T \) shown in Figure 3 if and only if the local move on edges with half-edges \( j < i < i+1 < j' \) produces the plane tree \( \phi^{-1}(s_i Y) \).
Proof. Lemma 3.5 showed that $i$ and $i + 1$ are on different rows and different columns exactly when $i$ and $i + 1$ are in the configurations from Figure 3 also given in Figure 5. In fact local moves exchange these two configurations because $j < i < i + 1 < j'$. Let $T'$ denote the image of $T$ under the allowed local move on half-edges $j < i < i + 1 < j'$ and let $Y' = \phi(T')$. Comparing $T$ and $T'$ in Figure 5 shows that $i$ and $i + 1$ change from a left half-edge to a right half-edge or vice versa. Thus $i$ is on the opposite row in $Y$ as it is in $Y'$ and similarly for $i + 1$. Both $j$ and $j'$ stay on the same half of the edge in $T$ and $T'$ by inspection of Figure 5. Local moves change only the two edges involved with the local move by definition. Thus all other numbers remain on the same rows in the corresponding Young tableau, and so every other integer is in the same row in $Y$ as it is in $Y'$. Finally $i$ and $i + 1$ are in opposite rows in $Y$ by the hypotheses of the theorem together with Lemma 3.5. Thus $Y' = s_i Y$.

Conversely suppose there is a local move involving the half-edges $j < i < i + 1 < j'$. The configurations in Figure 5 are the only possibilities listed in Lemma 3.5 that satisfy $j < i < i + 1 < j'$. The claim follows. □

Remark 3.7. Not every local move corresponds to one of the maps $s_i$. In particular when $i$ and $i + 1$ are on the same row of a tableau then $s_i$ fixes the tableau while the corresponding plane tree has a nontrivial local move involving the half-edges labeled $i$ and $i + 1$. Figure 6 gives an example: the top row shows the local move corresponding to $s_1$ though $s_1$ fixes each tableau in the bottom row.
To avoid this problem we have the following definition.

**Definition 3.8.** An $s_i$-local move is a local move that sends a plane tree $T$ to $\phi^{-1}(s_iT)$ for some $s_i$.

We conclude this section with an open question.

**Question 3.9.** Can other transpositions $(i,j)$ also be interpreted as local moves on plane trees?

### 4. The graph of $s_i$-local moves in type A

Theorem 3.6 showed that the graph whose vertices are plane trees and whose edges are pairs of plane trees related by $s_i$-local moves is isomorphic to the graph in Definition 2.5 for the partition $(n,n)$. Remark 3.7 demonstrated that this graph is a proper subgraph of the graph of plane trees under *all* local moves.

Heitsch studied the graph of plane trees under *all* local moves and compared it to similar graphs for other combinatorial objects enumerated by Catalan numbers [3]. However when we remove edges, many of those properties no longer hold. We explore these statistics in this section. In later sections we will look at local moves corresponding to Weyl groups other than the symmetric group. In this section we restrict to the partition $(n,n)$ and...
denote the graph from Definition 2.5 by $G^A$. We refer to $G^A$ as the graph of $s_i$-local moves in type $A$.

We begin by proving that the graph of $s_i$-local moves is still connected in type $A$.

**Proposition 4.1.** The graph $G^A$ is connected.

**Proof.** We describe a way to construct a path between any two standard Young tableaux $Y$ and $Y'$ that both have shape $(n, n)$. If $Y = Y'$ then the path is trivial. We now induct on the minimum number $i$ that lies in a different row in $Y$ than in $Y'$. Suppose that $i$ is the smallest number whose row in $Y$ is different from $Y'$. Suppose further that $i, i+1, i+2, \ldots, i+k$ are all on the same row in $Y$ and that $i+k+1$ is on the opposite row of $Y$. Then consider the standard Young tableau $s_is_{i+1}s_{i+2}\cdots s_{i+k-2}s_{i+k-1}s_{i+k}Y$. It is connected to $Y$ in the graph $G^A$ by construction. The numbers $1, 2, \ldots, i-1$ are all in the same positions in $s_is_{i+1}\cdots s_{i+k}Y$ as in $Y$. Furthermore the number $i$ is in the opposite row in $s_is_{i+1}\cdots s_{i+k}Y$ as in $Y$. Thus the first $i$ numbers are all in the same rows in $s_is_{i+1}\cdots s_{i+k}Y$ as they are in $Y'$. If $1, 2, \ldots, n-1$ are all on the same rows in $Y$ as in $Y'$ then $n$ must also be on the same row in $Y$ and $Y'$. By induction we can find a path from $Y$ to $Y'$ in $G^A$ as desired. $\square$

The graph of plane trees under local moves has the structure of a graded poset. This is true for $G^A$ as well, but for a different rank function. The next two results describe total distance and total number of descendants, two functions that rank $G^A$. Like Heitsch, we find that the language of plane trees characterizes the ranking more naturally than tableaux do. In particular, we show that $s_i$-local moves change both the total distance and the total number of descendants by exactly one.

**Proposition 4.2.** Fix a plane tree $T$ with root $v_0$.

The total distance of the plane tree $d_T$ is defined as $d_T = \sum_{v \in V(T)} dist(v, v_0)$.

If $T'$ is obtained from $T$ by a local move of type (1) then $d_T + 1 = d_T'$. If $T'$ is obtained from $T$ by a local move of type (2) then $d_T' - 1 = d_T$.

**Proof.** The proof follows by comparing the distances in the schematics in Figure 7. The local moves do not change the distance between the root and the vertices in the subtrees in locations a, b, c, d, e. The vertex between half-edges $i$ and $i+1$ has no descendants, as shown in the schematic. Moreover the vertex between half-edges $i$ and $i+1$ is one edge farther from the root than the vertices at the end of the peak, changing the total distance by exactly one. $\square$
Proposition 4.3. Fix a plane tree $T$ with root $v_0$.

The total number of descendants in $T$ is defined as $\text{des}_T = \sum_{v \in V(T)} |\text{descendants of } v|$. If $T'$ is obtained from $T$ by a local move of type (1) then $\text{des}_T - 1 = \text{des}_{T'}$. If $T'$ is obtained from $T$ by a local move of type (2) then $\text{des}_T + 1 = \text{des}_{T'}$.

Proof. Consider again the schematic in Figure 7. The number of descendants of the root as well as all vertices in the subtrees a, b, c, d, e remains the same after each local move. However the length-two path itself has three descendants while the peak has only two. □

The following corollary is a direct result of the previous propositions.

Proposition 4.4. Both total distance and total number of descendants partition the vertices of $G^A$ into the same set. The sets of plane trees with total distance $k$ (respectively total number of descendants $k$) characterize the ranks of the graded poset obtained from $G^A$ by only considering directed edges of type (1).

Proof. Directed edges in $G^A$ corresponding to local moves of type (1) decrease both the total number of descendants by one and the total distance by one; similarly local moves of type (2) increase each number by one.

Consider the tree in Figure 8a. The total number of its descendants is $n - 1$ and its total distance is $n - 1$. Since 1) the graph of $s_i$-local moves is connected, 2) total distance and total number of descendants agree at one tree, and 3) total distance and total number of descendants change in the same way across each edge, we conclude that $d_T = \text{des}_T$ for each plane tree $T$. □
**Remark 4.5.** We can prove directly that the total distance equals the total number of descendants $d_{vT} = d_T$. (The vertex $v$ has distance $k$ from the root exactly when the unique path between $v$ and the root has $k + 1$ vertices on it. The $k$ vertices on this path other than $v$ are precisely the vertices in $T$ with $v$ as a descendant. Thus each vertex $v$ contributes exactly $k$ to $d_T$ and exactly $k$ to $d_{vT}$.)

However we find both viewpoints useful in different contexts. For one example, see Proposition 4.7 below. For another, note that each descendent in a plane tree corresponds to a nesting of arcs in the associated noncrossing matching. Thus the total number of descendants of a plane tree corresponds to the total nesting within a noncrossing matching. (We do not discuss noncrossing matchings in detail in this manuscript; for more, see e.g. [7, 6].)

**Remark 4.6.** The graph $G^A$ does not satisfy the kind of symmetries as the graph for all local moves. For instance Heitsch proved that the number of plane trees of rank $k$ agrees with those of rank $n - k + 1$ for each $k = 1, \ldots, n$. That is clearly false here: for instance the sequence of the number of tableaux of shape $(3, 3)$ of each rank is $(1, 2, 1, 1)$.

However we can prove the following.

**Proposition 4.7.** There is a unique element of maximal rank and a unique element of minimal rank.

*Proof.* The root must have $n - 1$ descendants since every other vertex in the graph is a descendant of the root. So the minimal number of descendants is $n - 1$. This is achieved by the star graph.

Plane trees are connected so there is at least one vertex of each possible distance from the root. The path graph has just one vertex at each distance from the root and so it maximizes the total distance.
Corollary 4.8. The number of ranks in the ranked poset determined by $G^A$ and the total distance function $d_T$ is $\binom{n+1}{2} - n + 1$.

Proof. The total distance of the path is the binomial coefficient $\binom{n+1}{2}$. The total distance of the star graph is $n - 1$. There is at least one plane tree of each rank between these because $G^A$ is connected and each edge changes rank by exactly one.

Again we close with an open question.

Question 4.9. Is the rank sequence of $G^A$ unimodal for every $n$?

5. $s_i$-Local moves of type $C$

In our description of $s_i$-local moves, we relied on an analogy with the generators of the symmetric group. In this section we extend the analogy to define maps $s^C_i$ corresponding to the generators of the Weyl group of type C. Biologically these maps correspond to certain mutations of RNA: the map $s^C_n$ replaces stacked base pairs with their Watson-Crick complement while the map $s^C_i$ can be thought of as exchanging adjacent sets of stacked base pairs while preserving their bonds.
Mathematically we prove that restricting to type-C $s_i$-local moves identifies symmetry within the plane trees. More precisely the main theorem of this section shows that within the graph whose vertices are plane trees and whose edges are type-C $s_i$-local moves, there are precisely two connected components: symmetric and asymmetric plane trees.

We define functions analogous to the maps $s_i$ for type C instead of type A. When the maps $s_i^C$ are considered as elements of the symmetric group $S_n$ they generate a subgroup of $S_n$ isomorphic to the Weyl group of type C.

**Definition 5.1.** The maps of type C are the involutions defined by:

- $s_1^C = s_1 s_{2n-1}$ corresponding to the reflection $(1,2)(2n-1,2n)$
- $s_2^C = s_2 s_{2n-2}$ corresponding to the reflection $(2,3)(2n-2,2n-1)$
- $s_{n-1}^C = s_{n-1} s_{n+1}$ corresponding to the reflection $(n-1,n)(n+1,n+2)$
- $s_n^C = s_n$ corresponding to the reflection $(n,n+1)$

We use the same notation $s_i^C$ for the involutions on tableaux as well as on plane trees.

The maps $s_i^C$ define a graph $G^C$ just like the maps $s_i$ defined a graph $G^A$.

**Definition 5.2.** The graph $G^C$ is the graph whose vertices are plane trees and with edges $T, T'$ precisely when $T' = s_i^C T$ for a map $s_i^C$ of type C. We call this the graph of plane trees under $s_i^C$-local moves (or $s_i$-local moves of type C).

Note that except for $s_n^C$ the maps $s_i^C$ are in fact pairs of $s_i$-local moves.

The following definition formalizes our notion of symmetric and asymmetric plane trees.

**Definition 5.3.** The tree $T$ is symmetric if for every $1 \leq i < j \leq n$

- $e(i,j)$ is an edge in $T$ if and only if $e(2n-j+1,2n-i+1)$ is an edge in $T$.

The tree is asymmetric if it is not symmetric.

We will prove that the graph of plane trees $G^C$ under the $s_i^C$-local moves has two connected components: one corresponding to symmetric plane trees and one corresponding to asymmetric plane trees. Our proof proceeds in several steps. First we show that no connected component contains both a symmetric plane tree and an asymmetric plane tree.

**Lemma 5.4.** Each connected component of $G^C$ consists either entirely of symmetric plane trees or entirely of asymmetric plane trees.

**Proof.** We will show that if $s_i^C$ is a generator of the Weyl group of type C and $T$ is symmetric then $s_i^C T$ is symmetric. It follows that the connected component of any symmetric plane tree consists entirely of other symmetric plane trees. It follows further that the connected
component of any asymmetric plane tree must consist entirely of other asymmetric plane trees (since every tree is either symmetric or asymmetric).

Given a subtree $T'$ of $T$ we call the edges in $T$ that are symmetric to $T'$ its “mirror image”.

Consider the edges labeled by $i$ and $i+1$. A priori there are four possibilities: they could both be left-half-edges, they could both be right-half-edges, they could form a leaf, or they could form the interior of a peak.

Table 1 shows these four possibilities, the mirror image of each of these four configurations, the $s_i^C$-local move on each of the previous two configurations, and the mirror image of the $s_i^C$-local move on the original configuration. We inspect columns three and five in Table 1 and observe that they are the same. So if two edges were part of a symmetric tree before we perform an $s_i^C$-local move on them, then they are part of a symmetric tree after the $s_i^C$-local move.
Since these are the only edges changed by the local move, all other edges still satisfy the symmetry condition. We conclude that $s^C_i T$ is symmetric whenever $T$ is symmetric. The result follows. \qed

Next we prove there is exactly one connected component of asymmetric plane trees in $G^C$.

**Theorem 5.5.** If $T$ and $T'$ are asymmetric plane trees then there is a finite sequence of $s^C_i$-local moves that transform $T$ into $T'$.
Proof. The proof is by induction on the total number of edges $n$. The inductive hypothesis is that if $T$ and $T'$ have at most $n$ edges then there is a sequence of $s_i^C$-local moves that transform $T$ into $T'$. The base case of $n = 3$ has two asymmetric plane trees, which are related by $s_2^C = s_2s_4$ as shown in Figure 9.

![Figure 9. The case when $n = 3$](image)

Suppose that $T$ and $T'$ have $n + 1$ edges. We will show that both can be transformed via a finite sequence of $s_i^C$-local moves to trees $T_1$ and $T_1'$ each of which has only one edge $e(1, 2n)$ coming out of the root. Erasing that edge produces an asymmetric tree $T_2$ and respectively $T'_2$ with $n - 1$ edges whose new root is the vertex previously adjacent to the old root and whose half-edges are labeled $2, 3, \ldots, 2n - 1$. The induction hypothesis then applies to the trees $T_2$ and $T'_2$ giving a sequence of $s_i^C$-local moves that uses only $s_2^C, s_3^C, \ldots, s_n^C$. Combining this sequence with the sequences of moves from $T$ to $T_1$ and from $T' \rightarrow T_1'$ gives a sequence of $s_i^C$-local moves from $T$ to $T'$.

We now show how to obtain a sequence of $s_i^C$-local moves from an arbitrary tree $T$ to a tree with the single edge $e(1, 2n)$ incident to the root. Figure 10 shows the edges incident to the root in $T$. 
where \( j_1 + 1 < j_2 < j_2 + 1 < j_3 < \ldots < j_{k-1} + 1 < j_k \). (By convention adjacent half-edges are labeled consecutively, which explains the half-edges labeled \( j_1, j_1 + 1, j_2, j_2 + 1, \ldots \) in Figure 10.) Thus we can perform an \( s_{j_1}^C \)-local move on edges \( e(1, j_1) \) and \( e(j_1 + 1, j_2) \) to get the edge \( e(1, j_2) \). (We also get an edge \( e(j_1, j_1 + 1) \) which we ignore.) We can repeat this process until the only edge out of the root is \( e(1, 2n) \). Thus the sequence of local moves corresponding to \( s_{j_{k-1}}^C \cdots s_{j_2}^C s_{j_1}^C \) transforms \( T \) to \( T_1 \). This proves the desired claim. \( \square \)

We now show that one connected component of \( G^C \) contains all symmetric plane trees, in a proof similar to the previous.

**Theorem 5.6.** If \( T \) and \( T' \) are symmetric plane trees, then there is a finite sequence of \( s_i \)-local moves of type C that transforms \( T \) into \( T' \).

**Proof.** We prove this theorem by induction with two base cases. The base case \( n = 2 \) was addressed in Figure 1 since \( s_{j_1}^C = s_2 \) in that case. The three symmetric plane trees with \( n = 3 \) are shown in Figure 9; the top and the middle are connected by the edge \( s_{j_2}^C = s_2 s_4 \) while the middle and the bottom are connected by \( s_{j_3}^C = s_3 \).

For the induction step assume that any two symmetric trees with at most \( n - 1 \) edges can be transformed to one another by a sequence of \( s_i^C \)-local moves and let \( T, T' \in T_n \). We must show that \( T \) and \( T' \) can still be transformed into each other using a sequence of \( s_i^C \)-local moves.

We first show that we may assume that \( e(1, 2) \) is an edge in both \( T \) and \( T' \). Suppose not, namely that \( e(1, j') \) and \( e(2, j) \) are edges in at least one of the trees, without loss of generality \( T \). Apply an \( s_1^C \)-local move to \( T \). By definition of \( s_1^C \)-local moves, both \( e(1, 2) \) and \( e(2n - 1, 2n) \) are edges in \( s_1^C(T) \). We conclude that \( T \) is in the same connected component
of $G^C$ as a tree with the edge $e(1,2)$ and similarly for $T'$. The trees are symmetric and so also contain $e(2n-1,2n)$.

Thus we assume that $T$ and $T'$ both contain the edges $e(1,2)$ and $e(2n-1,2n)$. Now consider the subtrees $T_1$ and $T'_1$ consisting of all edges except $e(1,2)$ and $e(2n-1,2n)$. These trees are still symmetric but only have $n-2$ edges. By the inductive hypothesis we can transform $T_1$ into $T'_1$ with a sequence of $s^C_i$-local moves. The same sequence of $s^C_i$-local moves transforms $T$ into $T'$. By induction the claim is proven.

The main result is a simple corollary of these key steps.

**Corollary 5.7.** The graph $G^C$ has exactly two connected components, one containing exactly the symmetric plane trees and the other containing exactly the asymmetric plane trees.

**Question 5.8.** Do the other properties of $G^A$ hold for the components of $G^C$ as well? For instance is each component of $G^C$ graded by a function with a nice description?

## 6. Remarks on Classical Types Other than A

We begin this section with comments on possible biological interpretations of type C local moves, which act by twisting RNA strands at a particular location. Messenger RNA, or mRNA, is the product of DNA transcription and carries genetic information contained in the DNA from the cell nucleus to the cytoplasm where protein synthesis takes place. During the process of translation, the mRNA strand gets read in the 5’ to 3’ direction, meaning that we number the mRNA bases starting from the 5’ end of the base sequence. Normally translation into amino acids proceeds in this order along the strand, translating three bases into one amino acid molecule at a time.

We think of a $s^C_i$-local move as exchanging two triples of base pairs at one moment in the translation process. This may completely change the sequence of amino acids: depending on how many bases there are in the untwisted RNA component and in the twisted one, this might change amino acids not only for the twisted component, but also for several bases closest to the twisted bases on every end of the non-twisted RNA strand.

We wonder if any processes like this are observed biologically.

### 6.1. Type D

The maps $s_i$ do not generate a group action of the symmetric group $S_n$ because composition of functions does not correspond to multiplication in the group, as discussed in Remark 2.7. This makes functions $s^D_i$ corresponding to the Weyl group of type D problematic: when described as a subgroup of the symmetric group $S_n$, the Weyl group of type D has one generator that is not a product of disjoint simple transpositions $(i, i+1)$. For instance one generator of the Weyl group of type $D_5$ is $(2,5)(3,4)$ and Remark 2.7 demonstrated that the action of $(2,5)$ need not be well-defined.
For these reasons we believe extending these methods to type D will not produce meaningful results, either in biology or combinatorics.

6.2. Type B. Analyzing the Weyl group of type B may lead to meaningful biological and combinatorial implications. The maps of type B are the involutions defined by:

\[
s_1^B = s_1 s_{2n} \quad \text{corresponding to the reflection } (1, 2)(2n, 2n + 1)
\]
\[
s_2^B = s_2 s_{2n-1} \quad \text{corresponding to the reflection } (2, 3)(2n - 1, 2n)
\]
\[
\vdots
\]
\[
s_{n-1}^B = s_{n-1} s_{n+2} \quad \text{corresponding to the reflection } (n - 1, n)(n + 2, n + 3)
\]
\[
s_n^B = s_n \quad \text{corresponding to the reflection } (n, n + 2)
\]

The careful reader has noticed that these involutions do not act on plane trees since plane trees must have an even number of half-edges. Rather these involutions act on objects like plane trees that have \( n \) whole edges and one “half-edge” labeled \( n + 1 \) and forming a small bump between the edges containing \( n \) and \( n + 2 \).

Like the Weyl group of type C, the elements of the Weyl group of type B correspond to mutations on an RNA strand. The type-B model is different because stacked base pairs can also contain a bulge, namely unmatched nucleotides corresponding to the integer \( n + 1 \).

Similarly the Weyl group of type B defines involutions on Young tableaux of shape \( (n+1, n) \) that fix one of the integers filling the tableau.

We leave these investigations for future researchers, for instance in the following open questions.

**Question 6.1.** What are the orbits of the action of involutions \( s_i^B \)? What is a natural collection of involutions to represent mutations on strands with several bulges (namely fixing several integers)?
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