RATIONAL CURVES ON THE SPACE OF DETERMINANTAL NETS OF CONICS

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Abstract. We describe the Hilbert scheme components parametrizing lines and conics on the space of determinantal nets of conics, $\mathbf{N}$. As an application, we use the quantum Lefschetz hyperplane principle to compute the instanton numbers of rational curves on a complete intersection Calabi-Yau threefold in $\mathbf{N}$. We also compute the number of lines and conics on some Calabi-Yau sections of non-decomposable vector bundles on $\mathbf{N}$. The paper contains a brief summary of the A-model theory leading up to Givental-Kim’s quantum Lefschetz hyperplane principle. This work makes up my doctoral dissertation, defended Oktober 31, 1997.

Date: September 29, 1997.

1991 Mathematics Subject Classification. 14N10, 14J30, 14M12.

Key words and phrases. Quantum Cohomology, Calabi-Yau, Instantons.
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RATIONAL CURVES ON $N(3, 2, 3)$

Introduction

In recent years, observations in theoretical physics have directed enormous attention to problems related to the enumerative geometry of algebraic curves on manifolds. Ideas from string theory and topological field theory have led to the development of quantum cohomology, and to the mirror symmetry conjecture.

To a manifold $M$ that satisfies certain positivity assumptions, one may associate a topological field theory $A(M)$, where the correlation functions carry enumerative information (in the form of Gromov-Witten (GW)-invariants) about morphisms of algebraic curves to the manifold. In \cite{KM} it was shown that this puts severe restrictions on the GW-invariants, sometimes so strong that all GW-invariants (at least in the genus zero case) can be determined from a few simple ones. Subsequently, a big effort was made to build a mathematical framework for this formalism. This has now been completed within the realms of symplectic geometry \cite{RT}, and algebraic geometry \cite{KM, BM, BF, B, LiT}. This theory has provided enumerative geometers with a whole new tool. An example of its power is the celebrated recursion formula obtained by Kontsevich for the number of rational curves of degree $d$ in $P^2$ meeting $3d - 1$ given points (See Example 2 in Section 2.2).

In string theory, the assumption is that space-time is a ten-dimensional manifold which is locally a product $M_4 \times M$, where $M_4$ is the usual Minkowski space-time, and $M$ is a Calabi-Yau complex threefold. Initially, any Calabi-Yau is eligible. By a formal dimension count, algebraic curves on $M$ are expected to be rigid, and the $A$-model correlation functions should carry invariants which reflect the number of curves of given genus in a given cohomology class. Accordingly, the study of algebraic curves on Calabi-Yau manifolds is a very natural and interesting mathematical problem of “physical” interest. A further feature which enhances the appeal of these manifolds, is the following intriguing physical observation: To a Calabi-Yau manifold $M$ one can associate two different topological field theories $A(M)$ and $B(M)$, in such a way that there should exist another Calabi-Yau manifold $W$ with $A(M) = B(W)$ and $A(W) = B(M)$. Mathematically, it indicates an unexpected relationship between the symplectic geometry of $M$ and the complex geometry of $W$, and vice versa. This is the mirror symmetry conjecture.

Interest into this field exploded with the advent of a sensational computation by a group of physicists \cite{CdOGP}. In their paper, the mirror symmetry conjecture was used to compute a generating function for the numbers $n_d$ of rational curves of degree $d$ on the quintic threefold in $P^4$. The essential outcome of the computation was that two differential operators

$$D^4 - 5q(5D + 1)(5D + 2)(5D + 3)(5D + 4) \quad \text{and} \quad D^2 \frac{1}{K(q)} D^2,$$

with $K(q) = 5 + \sum_{d>0} n_d q^d$ and $D = q \frac{\partial}{\partial q}$ were equivalent up to a transformation (See Example 8 in Section 8 for a more precise statement). This enabled the (conjectural) determination of $n_d$ for arbitrary large $d$. A result of this order was far beyond reach of conventional mathematical methods. Up to then, mathematicians had only been able to compute the numbers $n_1$ and $n_2$, and soon afterwards $n_3$ and $n_4$. These computations confirmed the conjectured numbers. A crucial point in the computation of \cite{CdOGP} was that a mirror candidate of the quintic had been identified. Simply stated: A difficult computation related to the $A$-model of the quintic, became simple once it could be translated to the $B$-model of its mirror. Soon afterwards, the computations pioneered in \cite{CdOGP} were further developed and successfully applied to a host of new cases where mirror candidates had been identified, e.g., \cite{M1, LT, GMP, BvS, Me}. One limitation to this procedure is
that only for a relatively small class of Calabi-Yau manifolds does one know of good mirror candidates. However, by studying some combinatorial aspects of mirror computations, it was essentially conjectured in \[BvS\], that in some cases the $A$-model contains enough structure to explain the computations. In a series of remarkable papers \[G\], Givental proved the above mirror computations by finding a way to perform them rigorously on the $A$-model as indicated by \[BvS\]. The core of the proof is a quantum Lefschetz hyperplane principle, which relates the quantum cohomology of a manifold to the quantum cohomology of its complete intersections. For a deeper background into the subject of mirror symmetry, the reader is recommended \[M2, Vo2, CK\].

In this paper, we study rational curves on the six-dimensional variety $N$ which parametrizes determinantal nets of conics. Alternatively, it may be viewed as the variety parametrizing trisecant planes of the Veronese surface in $P^5$. Our main reason for this study is that there are three vector bundles on $N$ whose general global sections are Calabi-Yau threefolds. In fact, one of the bundles is decomposable, hence the associated Calabi-Yau section is a complete intersection in $N$. As usual, the problem of computing (by conventional methods) rational curves on the sections leads to the problem of describing the curves in $N$. We shall describe the Hilbert schemes of lines and conics on $N$, and use this to compute the number of lines and conics on the Calabi-Yaus. The highlight of the paper is an application of the quantum Lefschetz hyperplane principle to the computation of the instanton numbers of rational curves on the complete intersection. A generating function for these numbers will be obtained from a simple knowledge of the cohomology of $N$ and the lines on $N$. Again, this illustrates the power of quantum cohomology. To the best of our knowledge, this is the first “mirror computation” performed without knowing the mirror.

The paper is organized as follows: Sections 1 and 2 contain a basic review of the theory of quantum cohomology following \[KM, BM, G, Kim\]. No originality is claimed for this, except for perhaps a few odd parts in Section 2.7. The cohomology of $N$ has essentially been computed in \[ES2\]. We give a review of this computation in Section 3. In Section 4 we describe the Hilbert scheme of lines on $N$ as a projective bundle over $P^2 \times P^2$. This enables us to describe the Chow ring of the Hilbert space of lines. As an application we compute the number of lines on the Calabi-Yau sections. The instantons for the complete intersection Calabi-Yau are computed in Section 5. In Section 6, we give a description of the Hilbert scheme of conics which suffices for computational purposes. This is used to compute the (virtual) number of conics on the Calabi-Yau sections. We have collected some technical aspects of the computations in Section 7 and in the appendices.

**Preliminaries.** An important reason for the success of \[G\] is the use of a localization principle in equivariant cohomology \[AD\]. The idea to use such a method in the context of enumerative geometry first appeared in \[ES3\]. In our work, this principle will also play a role. Below, we state a simple version that is suitable for our purpose.

Let $M$ be a smooth orbifold with a $C^*$-action whose fixpoints are isolated. Let $M^{C^*}$ denote the set of fixpoints. Let $B_1, \ldots , B_s$ be $C^*$-bundles on $M$ of rank $r_1, \ldots , r_s$. If $f \in M$ is a fixpoint, there is an induced orbifold $C^*$-action action on the fibers $B_{ij}$. Let $\omega_1(f), \ldots , \omega_m(f)$ and $\tau_1(B_{ij}, f), \ldots , \tau_r(B_{ij}, f)$ be the weights of the $C^*$-action on $T_f M$ and $B_{ij}$ respectively. Since $M$ is an orbifold we must allow the weights to be in $Q$.

Suppose $P(\ldots , c_j(B_i), \ldots )$ is a polynomial in the Chern classes $c_j(B_i)$. Since the Chern classes $c_j(B_i)$ are symmetric polynomials in the Chern roots $b_{i,1}, \ldots , b_{i,r_j}$ of $B_i$, we can find

---

\[1\] Most notably, complete intersections in toric varieties \[BB\], but see also \[Vo1, Bo, R, BCKvS\] for other mirror candidates and recent development.
a polynomial $\tilde{P}$ such that 

$$P(\ldots , c_j(B_i) , \ldots ) = \tilde{P}(\ldots , b_i,1 , \ldots , b_i,r,\ldots ) .$$

**Theorem.** (Bott’s formula) 

$$\int_M P(\ldots , c_j(B_i) , \ldots ) = \frac{1}{|\text{Aut}(f)|} \sum_{f \in M^{C^*}} \frac{\tilde{P}(\ldots , \tau_1(B_i,f) , \ldots , \tau_r(B_i,f) , \ldots )}{\omega_1(f) \ldots \omega_m(f)} .$$

Another result of the same flavor is the classical Bialynicki-Birula theorem [3]. Let $M$ be a manifold with a $C^*$-action as above. Assume $M^{C^*} = \{ f_1 , \ldots , f_r \}$, and for each $i$, $1 \leq i \leq p$, let $M_i = \{ m \in M \mid \lim_{m \to 0} tm = m_i \}$. Consider the induced action on $T_f M$, and let $T_f M^+$ be the subspace where $C^*$ acts with positive weights.

**Theorem.** (Bialynicki-Birula) The $M_i$ form a cellular decomposition of $M$. Further, each $M_i$ is an affine space with $\dim M_i = \dim T_f M^+$.

From [3] (Example 1.9.1, 19.1.11) it follows that homological and rational equivalence on $M$ coincide, and that the Chow ring is the free abelian group generated by the classes of $M_i$.

**Notation.** In this paper, we only work with even cohomology. For a complex variety $X$ we use the notation $A^k(X) = H^{2k}(X, \mathbb{Z})$. If $R$ is a $\mathbb{Z}$-algebra, we let $A^k_R(X) = A^k(X) \otimes R$. In many instances our varieties will be smooth with a cellular decomposition, hence $A^*(X)$ corresponds to the Chow ring. If $V$ is a closed subvariety of $X$, then $V$ determines a cohomology class in $A^*(X)$, which we also denote by $V$. The bracket notation $[V]$ will be reserved for virtual fundamental classes. If $X$ is smooth of dimension $n$, we may identify cohomology classes with homology classes via the Poincaré duality isomorphism $A^k(X) \to A^{n-k}_Q(X)$. This will be done without warning throughout the text. Hence, if $\alpha, \beta \in A^*(X)$, we use both $<\alpha,\beta>$ and $\int_\beta \alpha$ to denote the Poincaré pairing.

The notational conventions used are mostly as in [11]. If $\mathcal{F}$ is a sheaf on a scheme $X$, we write $H^i(\mathcal{F})$ for the $i$-th cohomology group unless there is an ambiguity. If $f : Y \to X$ is a morphism of schemes, we often use $f_* \mathcal{F}$ to denote the pullback $f^* \mathcal{F}$. For a coherent sheaf $\mathcal{F}$, we use Grothendieck’s $\mathbf{P}(\mathcal{F}) = \text{Proj}(\text{Sym} \mathcal{F})$, and $\text{Grass}_r(\mathcal{F})$ denotes the Grassmanian of rank $r$ subbundles. If $\mathcal{L}$ is a line with first Chern class $\tau$, we shall often use the convenient notation $\mathcal{L} = \mathcal{O}(\tau)$.

**Acknowledgments.** It is a pleasure to thank I. Ciocan-Fontanine, G. Ellingsrud, G. Fløystad, T. Graber, P. Meurer, K. Ranestad, E. Rødland, and D. van Straten for valuable conversations on the subject matter. A special thanks is due to B. Kim for explaining me the work of Givental and himself. More than anyone, I am grateful to my supervisor S. A. Strømme, for introducing me to a stimulating problem, and for always being available for discussions.

A major part of this work came together during a tremendously active year at the Mittag-Leffler Institute. I would like to thank the Institute, and in particular D. Laksov, for support and for making the stay possible.

I also benefited much from a stay at Institut Henri Poincaré. I am thankful to C. Peskine for this invitation.

Finally, I would like to thank the Department of Mathematics at the University of Bergen for a friendly working environment.
This work has been supported by the Norwegian Research Council for Science and Humanities, and by Nordisk Forskerutdanningsakademi (NorFA).

1. Moduli spaces of stable curves and maps

The theory of quantum cohomology requires a good compactification of the space of maps of curves into a variety. One such compactification, at least when the target variety is nice enough, is the moduli space of stable maps introduced by Kontsevich [K]. The moduli space of stable maps specializes to the well known moduli space of curves $\overline{M}_{g,n}$ if the target variety is simply a point. In this section we define, and state some relevant results about, these moduli spaces. We only consider the genus zero situation, but most results are true for higher genera. For a nice introduction to this topic, and further details, the reader is recommended [FP].

In Sections 1 and 2, $X$ will always denote a smooth complex projective variety.

1.1. Definitions. An $n$-pointed stable curve of genus 0 $(C, s_1, \ldots, s_n)$ is a projective, connected, (at worst) nodal algebraic curve $C$ with $n$ marked points $s_1, \ldots, s_n$ such that:

i) $\dim H^1(C, O_C) = 0$

ii) $s_1, \ldots, s_n$ are non-singular points on $C$

iii) each component of $C$ contains at least three special points, where special means singular or marked.

A stable map $(C, f, s_1, \ldots, s_n)$ from an $n$-marked genus 0 curve $C$ to $X$ is a morphism $f : C \to X$ such that all components of $C$ which are mapped to a point are stable. Let $S$ be an algebraic scheme over $C$. A family of stable maps $(C \to S, f, s_1, \ldots, s_n)$ of $n$-marked genus 0 curves to $X$ consists of a flat projective morphism $C \to S$ with sections $s_i$ and a morphism $f : C \to X$ such that each geometric fiber $(C_s, f_s, s_1(s), \ldots, s_n(s))$ is a stable map.

Two families of maps over $S$,

$$(C \to S, f, s_1, \ldots, s_n) \quad \text{and} \quad (C' \to S, f', s'_1, \ldots, s'_n)$$

are isomorphic if there exists an $S$-isomorphism $\varphi : C \to C'$ such that for each $i$ the following diagram commutes:

$$\begin{array}{ccc}
C & \overset{\sim}{\longrightarrow} & C' \\
\uparrow & & \uparrow \\
\downarrow \varphi & & \downarrow \\
S & & S \\
\downarrow s_i & & \downarrow s'_i \\
\end{array}$$

Let $\beta \in A_1(X)$. Define a contravariant functor:

$$\overline{M}_{0,n}(X, \beta) : \text{Sch}^0 \to \text{Sets}$$

by letting $\overline{M}_{0,n}(X, \beta)(S)$ be the set of isomorphism classes of families of stable maps $(C \to S, f, s_1, \ldots, s_n)$ such that $f_*[C_s] = \beta$ for all geometric fiber $C_s$. Whenever $X$ is a point, we shall simply write $\overline{M}_{0,n}$.

Remark 1. The stability conditions i)–iii) ensure that the automorphism groups of the maps $(C, f, s_1, \ldots, s_n)$ are finite.
Remark 2. We shall allow labelling of marked points by other indexing sets than \( \{1, \ldots, n\} \), and we will write \( \overline{M}_{0,A}(X, \beta) \) if the indexing set is \( A \).

1.2. Natural maps. The functor of stable maps comes equipped with the following natural structures:

**Evaluation.** For \( 1 \leq i \leq n \) define morphisms
\[
e_i : \overline{M}_{0,n}(X, \beta) \to \text{Hom}(\cdot, X)
\]
by
\[
e_i(C, f, s_1, \ldots, s_n) = f(s_i).
\]

**Forgetting point (Contraction).** For \( 1 \leq i \leq n + 1 \) define morphisms
\[
\pi_i : \overline{M}_{0,n+1}(X, \beta) \to \overline{M}_{0,n}(X, \beta)
\]
by forgetting the \( i \)-th marking and contracting eventual nonstable components of the resulting map.

**Forgetting the map.** Define a morphism
\[
\eta : \overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,n}
\]
by forgetting the map and contracting eventual nonstable components of the resulting marked curve.

**Gluing.** Define a morphism
\[
\varphi_{AB}^{\beta',\beta''} : \overline{M}_{0,A \cup \{\bullet\}}(X, \beta') \times_X \overline{M}_{0,B \cup \{\bullet\}}(X, \beta'') \to \overline{M}_{0,A \cup B}(X, \beta' + \beta'')
\]
by gluing the maps in \( \bullet \).

1.3. Representability. The moduli problem of stable curves and maps has been studied in detail in [DM, Kn, K]. Unfortunately, the functor \( \overline{M}_{0,n}(X, \beta) \) is in general not representable.

**Theorem 1.1.** There exists a projective, coarse moduli space \( \overline{M}_{0,n}(X, \beta) \) for the functor \( \overline{M}_{0,n}(X, \beta) \).

This will suffice for most of what we are going to consider. However, we shall occasionally need a universal object for the moduli problem, something that forces us to work in the category of stacks [DM]. A satisfactory intersection theory for Deligne-Mumford stacks has been developed in [V].

**Theorem 1.2.** The moduli stack of stable maps \( \overline{M}_{0,n}(X, \beta) \) is a proper, separated Deligne-Mumford stack of finite type with universal family \( (\overline{M}_{0,n+1}(X, \beta), e_{n+1}, s_1, \ldots, s_n) \)

\[
\begin{array}{ccc}
\overline{M}_{0,n+1}(X, \beta) & \xrightarrow{e_{n+1}} & X \\
\pi \downarrow & & \\
\overline{M}_{0,n}(X, \beta) & & \\
\end{array}
\]

(1.1)
where \( \pi \) forgets the \((n+1)\)th point, and the sections \( s_i, \ i = 1, \ldots, n \), are defined by the map

\[
s_i : \overline{M}_{0,n}(X, \beta) \simeq \overline{M}_{0,n}(X, \beta) \times \overline{M}_{0,3} \to \overline{M}_{0,n+1}(X, \beta),
\]

which glues along the \( i \)-th point of first factor.

Note that we abuse notation and write \( \overline{M}_{0,n}(X, \beta) \) regardless of whether we are working with it as a stack, or as a scheme.

**Remark 1.** By deformation arguments (See [FP]), if \( \overline{M}_{0,n}(X, \beta) \) is non-empty, then

\[
\dim \overline{M}_{0,n}(X, \beta) \geq \dim X + \langle c_1(TX), \beta \rangle + n - 3.
\]

The right-hand side of the equation is called the expected dimension, which we will denote \( \text{ed}_\beta \). If \( \dim H^1(C, f^*TX) = 0 \) for all points \([C, f, s_1, \ldots, s_n]\) in \( \overline{M}_{0,n}(X, \beta) \), then \( \overline{M}_{0,n}(X, \beta) \) is a smooth Deligne-Mumford stack (i.e., an orbifold) of expected dimension. Or, as a scheme, \( \overline{M}_{0,n}(X, \beta) \) has only finite quotient singularities.

**Remark 2.** If \( X \) is a point and \( n \geq 3 \), we use \( \overline{M}_{0,n} \) to denote the moduli space instead of \( \overline{M}_{0,n}(pt, 0) \). If \( \beta = 0 \), then \( \overline{M}_{0,n}(X, 0) = \overline{M}_{0,n} \times X \). When \( n \leq 2 \), it will be convenient for us to let \( \overline{M}_{0,n} \) be a point, even if the expected dimension is negative. For \( n \geq 3 \), \( \overline{M}_{0,n} \) is smooth of dimension \( n - 3 \).

### 1.4. Boundary divisors.

In this section, we describe some divisors on the moduli space of stable curves. Relations among these divisors are the core of quantum cohomology.

Let \( A \cup B \) be a partition of \([1, \ldots, n]\). The image of the gluing maps \( \overline{M}_{0,A\cup\{\bullet\}} \times \overline{M}_{0,B\cup\{\bullet\}} \to \overline{M}_{0,A\cup B} \) define divisors which we denote \( D(A|B) \). If \( \overline{M}_{0,n} \to \overline{M}_{0,4} \) is the contraction map, then the inverse image of \( D(12|34) \) is the divisor \( \sum D(A|B) \), where the sum is over all partitions \( A \cup B \) of \([1, \ldots, n]\) such that \( 1, 2 \in A \) and \( 3, 4 \in B \). Since \( \overline{M}_{0,4} \simeq \mathbb{P}^1 \), the divisors \( D(12|34) \) and \( D(13|24) \) are linearly equivalent, hence we have the fundamental relations

\[
(1.2) \quad \sum_{1,2 \in A \atop 3,4 \in B} D(A|B) = \sum_{1,3 \in A \atop 2,4 \in B} D(A|B)
\]

in the Chow ring of \( \overline{M}_{0,n} \). In fact, this describes the Chow ring of \( \overline{M}_{0,n} \):

**Theorem 1.3.** ([K4]) The Chow ring \( A^*(\overline{M}_{0,n}) \) is generated by the divisors \( D(A|B) \). The ideal of relations among the divisors is generated by the relations \( (1.2) \) and the relations \( D(A|B) \cdot D(C|D) = 0 \), for which there are no inclusions among the sets \( A, B, C, D \).

## 2. Gravitational quantum cohomology

In this section we give a rapid review of some basics of the quantum theory found in [W], [KM], [B], [G]. Our main focus is to explain the quantum Lefschetz hyperplane principle. In Sections 2.1–2.4 we explain some aspects of the WDVV-theory for gravitational descendants, which are correlators generalizing the Gromov-Witten classes. In Sections 2.5–2.8 we have extracted the relevant parts of [C] and [Kim], necessary to state the quantum Lefschetz hyperplane principle.

Throughout the remainder of this paper, we will use the following notation. Let \( T_0, \ldots, T_m \) and \( T^0, \ldots, T^n \) be homogeneous bases for \( A^*_Q(X) \) such that \( T_0 = 1 \), \( \{T_1, \ldots, T_r\} \) is a basis for \( A^1_Q(X) \), and \( \langle T_i, T_j \rangle = \delta_{ij} \). We shall also use \( p_i = T_i \) to denote the divisor classes.
The coordinates of \( T_i \) on \( A^*_C(X) \) are denoted by \( t_i \). Also let \( q_i \) be formal parameters for \( i = 1, ..., r \).

### 2.1. Gravitational descendents.

For \( n \geq 0 \) consider the natural maps

\[
\begin{align*}
\overline{M}_{0,n}(X, \beta) & \xrightarrow{(e, \eta)} X^n \times \overline{M}_{0,n} \\
\eta & \downarrow \\
X^n & \rightarrow \overline{M}_{0,n}
\end{align*}
\]

(2.1)

where \( e = (e_1, \ldots, e_n) \) is the evaluation map, and \( \eta \) forgets the map. By a slight abuse of notation, we shall also use \( e \) and \( \eta \) to denote the projection maps. Let \( \gamma_i \) denote arbitrary classes in \( A^*(X) \). In \([KM, B, BF, BM]\) a virtual fundamental class \([\overline{M}_{0,n}(X, \beta)]\) \( \in A_{ed,\beta}(\overline{M}_{0,n}(X, \beta)) \) has been constructed such that the (tree level) Gromov-Witten (GW) classes

\[
(\gamma_1, \ldots, \gamma_n)_{0,\beta} = \eta_* \left( \prod_{i=1}^{n} e_i^* (\gamma_i) \cdot [\overline{M}_{0,n}(X, \beta)] \right)
\]

(2.2)

satisfy the axioms introduced by Kontsevich and Manin. Here \([\overline{M}_{0,n}(X, \beta)]\) has been identified with the Poincaré dual of its image in \( X^n \times \overline{M}_{0,n} \).

More generally, we shall consider the classes of (tree level) gravitational descendents

\[
(\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n)_{0,\beta} = \eta_* \left( \prod_{i=1}^{n} c_1(L_i)^{d_i} e_i^* (\gamma_i) \cdot [\overline{M}_{0,n}(X, \beta)] \right)
\]

and the invariants

\[
<\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n>_{0,\beta} = \int_{[\overline{M}_{0,n}(X, \beta)]} \prod_{i=1}^{n} c_1(L_i)^{d_i} e_i^* (\gamma_i),
\]

where \( L_i \) is the line bundle \( s_i^* \omega_{\pi} \), and \( d_i \) are non-negative integers. We extend this definition to include negative \( d_i \)'s by defining \( (\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n)_{0,\beta} = 0 \) if some \( d_i < 0 \). Here, \( \omega_{\pi} \) is the relative canonical line bundle of the universal map, hence the fibers of \( L_i \) over a point \([C, f, s_1, \ldots, s_n]\) are \( T_i, CV \). Occasionally, we abbreviate the notation and write \((\otimes_{i=1}^{n} \tau_{d_i} \gamma_i)_{0,\beta}\) instead of \((\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n)_{0,\beta}\). Gravitational descendents satisfy the following analogues of the KM-axioms:

---

**Fundamental class.** If \( \beta \neq 0 \), then

\[
<\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n, 1>_{0,\beta} = \sum_{i=1}^{n} <\tau_{d_1} \gamma_1, \ldots, \tau_{d_i-1} \gamma_i, \ldots, \tau_{d_n} \gamma_n>_{0,\beta}.
\]

**Divisor.** If \( \beta \neq 0 \) and \( D \in A^1(X) \), then

\[
\pi_*(\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n, D)_{0,\beta} =
\]

\[
\left( \int_{\beta} D \right)(\tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n)_{0,\beta} + \sum_{i=1}^{n} (\tau_{d_1} \gamma_1, \ldots, \tau_{d_i-1} D \cdot \gamma_i, \ldots, \tau_{d_n} \gamma_n)_{0,\beta}.
\]

---

Note that we only work with even dimensional cohomology, as opposed to \([KM]\) which is formulated with odd cohomology as well.
**Splitting.** Consider the commutative diagram, where the upper left and lower right squares are fibred:

\[
\begin{array}{ccc}
\overline{M}_{0,n} & \xleftarrow{\varphi_{AB}} & \overline{M}_{0,A\cup\{\bullet\}} \times \overline{M}_{0,B\cup\{\bullet\}} \\
\eta \uparrow & & \eta_{A\times\eta_B} \uparrow \\
\prod_{\beta} & \xrightarrow{\psi_1 \times \psi_2} & \prod_{\beta}' \\
\overline{M}_{0,n}(X,\beta) & \xleftarrow{G} & \Delta \\
\end{array}
\]

\begin{equation}
\tag{2.3}
\overline{M}_{0,n}(X,\beta) \xleftarrow{G} \prod_{\beta} \xrightarrow{\psi_1 \times \psi_2} \prod_{\beta}' \xrightarrow{\Delta} \overline{M}_{0,n}(X,\beta_1) \times \overline{M}_{0,n}(X,\beta_2).
\end{equation}

Here, \(\Delta\) is the diagonal map on the last factor, \(G\) is the gluing map, \(\psi_1\) and \(\psi_2\) are projection maps,

\[
\prod_{\beta} = \prod_{\beta_1 + \beta_2 = \beta} \overline{M}_{0,n}(X,\beta_1) \times X \overline{M}_{0,n}(X,\beta_2),
\]

and

\[
\prod_{\beta}' = \prod_{\beta_1 + \beta_2 = \beta} \overline{M}_{0,n}(X,\beta_1) \times \overline{M}_{0,n}(X,\beta_2).
\]

In [BM] and [B] it is shown that the virtual fundamental class defines an orientation for \(\overline{M}\). An important consequence of this is:

\[
\varphi_{AB}^!(\overline{M}_{0,n}(X,\beta)) = \sum_{\beta_1 + \beta_2 = \beta} \Delta_!([\overline{M}_{0,n}(X,\beta_1)] \times [\overline{M}_{0,n}(X,\beta_2)]).
\]

Multiply each side of this equation with \(\prod_{i=1}^n c_1(L_i)^{d_i} e_i^*(\gamma_i)\), push down by \(\eta_A \times \eta_B\), and use the projection formula and commutativity of the diagram to find

\[
\varphi_{AB}^!(\tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n|_{0,\beta}) = \sum_{\beta_1 + \beta_2 = \beta} \sum_f (T_f, \otimes \tau_{d_a}\gamma_a|_{0,\beta_1} (T_f', \otimes \tau_{d_b}\gamma_b|_{0,\beta_2}).
\]

**Mapping to a point.** If \(\beta = 0\) then \(\overline{M}_{0,n}(X,\beta) = \overline{M}_{0,n} \times X\) and

\[
<\tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n|_{0,\beta} = \int_X \prod_{i=1}^n \gamma_i \cdot \int_{\overline{M}_{0,n}} \prod_{i=1}^n c_1(L_i)^{d_i},
\]

where

\[
\prod_{i=1}^n c_1(L_i)^{d_i} = \begin{cases} 
\frac{(n-3)!}{d_1! \cdots d_n!} & \text{if } \sum_{i=1}^n d_i = n - 3 \\
0 & \text{else}.
\end{cases}
\]

See [W] and [K] for the last claim.

**Remark 1.** If \(\overline{M}_{0,n}(X,\beta)\) is of expected dimension, then \(\overline{M}_{0,n}(X,\beta)\) is just the fundamental class of \(\overline{M}_{0,n}(X,\beta)\).

**Remark 2.** With \(d_i = 0\) for all \(i\), we recover the axioms in [KM].

**Remark 3.** Note that if all \(\gamma_i\) are homogeneous, then \(<\tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n|_{0,\beta} = 0\) unless \(\sum_{i=1}^n \text{codim}(\gamma_i) + d_i\) is equal to the expected dimension of \(\overline{M}_{0,n}(X,\beta)\). The same claim is also true unless \(\beta = 0\) or \(\beta\) is effective, since \(\overline{M}_{0,n}(X,\beta)\) is then empty.
Remark 4. The fundamental class and divisor properties follow from the relations

\[ c_1(L_i) - \pi^*c_1(L_i) = s_i(\overline{M}_{0,n}(X, \beta)) \quad \text{for} \quad i = 1, \ldots, n. \]

Rigorous proofs of these claims can be found in [F] when \( X \) is convex. These proofs hold in a more general setting, since the virtual fundamental classes satisfy \( \pi^*[\overline{M}_{0,n}(X, \beta)] = [\overline{M}_{0,n+1}(X, \beta)]. \)

Example 1. If \( ed_\beta = 0 \) and \( n = 0 \), then \( \langle >_{0,\beta} \rangle \) is simply the degree of \( [\overline{M}_{0,0}(X, \beta)] \).

An interesting example is the class of Calabi-Yau threefolds. A variety is Calabi-Yau if its canonical bundle is trivial. Hence, if \( X \) is Calabi-Yau of dimension 3, the expected dimension of \( \overline{M}_{0,0}(X, \beta) \) is 0 for all effective \( \beta \). Let \( N^X_\beta \) denote the degree of \( [\overline{M}_{0,0}(X, \beta)] \).

The numbers \( N^X_\beta \) are related to the virtual numbers\(^3\) \( n^X_\beta \) of rational curves of class \( \beta \) in \( X \).

The Aspinwall-Morrison formula \([AM]\) describes this relationship. For simplicity, we shall assume that \( \text{Pic}(X) \simeq \mathbb{Z} \), and identify \( \beta \) with the degree \( d = <p, \beta> \), where \( p \) is the ample generator of \( \text{Pic}(X) \). Then the Aspinwall-Morrison formula reads

\[ N_d^X = \sum_{k|d} k^{-3} n_{d/k}^X. \]

In [K], this formula was explained as follows: If \( d > 1 \), then \( \overline{M}_{0,0}(X, d) \) is rarely of expected dimension since the locus of multiple coverings of lower degree curves will be of positive dimension. Hence we need to compute the contributions to \( [\overline{M}_{0,0}(X, d)] \) from this locus.

Suppose \( C \simeq \mathbb{P}^1 \) is a smooth rational curve of degree \( d/k \) in \( X \), and \( C \) is rigid, i.e., \( N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \). Then the locus of \( k : 1 \) coverings of \( C \) in \( \overline{M}_{0,0}(X, d) \) is a \( 2k - 2 \) dimensional component which is isomorphic to \( \overline{M}_{0,0}(\mathbb{P}^1, k) \). By cohomology and base change theorems in [H], the obstruction sheaf \( R^1\pi_*\mathcal{E}^*_1N_{C/X} \) is a vector bundle of rank \( 2k - 2 \) on \( \overline{M}_{0,0}(\mathbb{P}^1, k) \). The contribution to \( N_d^X \) from the component \( \overline{M}_{0,0}(\mathbb{P}^1, k) \) is the integral

\[ \int_{\overline{M}_{0,0}(\mathbb{P}^1, k)} c_{\text{top}}(R^1\pi_*\mathcal{E}^*_1N_{C/X}). \]

An evaluation of this integral using Bott’s formula yields \( 1/k^3 \). See also [GrP] for more general multiple cover computations.

2.2. The potential. For non-negative integers \( i, j \), and \( j = 0, \ldots, m \), let \( t^0_i = t_j \), and let \( T = \sum_{i,j} t^0_i T_j \). The gravitational potential is defined as

\[ \Phi = \sum_{n \geq 0} \frac{1}{n!} \sum_{\beta} q^\beta <T^{\otimes n}>_{0,\beta}. \]

Using the linearity of the gravitational descendents we obtain a formal power series in \( \mathbb{Q}[[q_i, q_i^{-1}, t^0_i]] \),

\[ \Phi = \sum_{n \geq 0} \sum_{\beta} q^\beta <\otimes_{i,j}(T_i)^{\otimes n_i^0}>_{0,\beta} q^\beta \prod_{i,j} \frac{(t^0_i)^{n_i^0}}{n_i^0!}, \]

\(^3\)The virtual number \( n^X_\beta \) is defined in such a way that if the rational curves of class \( \beta \) in \( X \) are rigid, then \( n_\beta \) is the actual number of such curves. See [Katz] for a discussion.
where the second summation is over all partitions $\sum_{i,j} n_{ij}^l = n$. We use Witten’s notation to denote the partial derivatives of $\Phi$:

$$\langle \langle \tau_{d_1} T_{j_1}, \ldots, \tau_{d_k} T_{j_k} \rangle \rangle_0 = \partial_{d_1}^{j_1} \cdots \partial_{d_k}^{j_k} \Phi,$$

where $\partial_k^{j_k} = \frac{\partial}{\partial \tau_{d_k}}$. This is extended linearly to define $\langle \langle \tau_{d_1} \gamma_{j_1}, \ldots, \tau_{d_k} \gamma_{j_k} \rangle \rangle_0$ for arbitrary classes $\gamma_i = \sum a_i j_i T_j$ in $A^*_Q(X)$. It is easy to verify that

$$\langle \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k} \gamma_k \rangle \rangle_0 = \sum_{n \geq 0} \frac{1}{n!} \sum_\beta q^\beta \langle \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k} \gamma_k, T^{\otimes n} \rangle \rangle_0.$$ 

Also, let

$$\langle \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k} \gamma_k \rangle \rangle_0 = \langle \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k} \gamma_k \rangle \rangle_0 \mid \{ t_i = 0 \mid i > 0 \}.$$ 

**Remark.** Several variations of the potential appear in the literature on GW-invariants. If we restrict $\Phi$ to $\{ q_i = 1 \} \cup \{ t_i = 0 \mid i > 0 \}$ we recover the potential in [FP]. On the other hand, restricting $\Phi$ to $\{ t_i = 0 \mid 1 \leq i \leq r \} \cup \{ t_i = 0 \mid i > 0 \}$ yields the potential defined in [GP]. Using the fundamental and divisor properties one can deduce that these are equivalent if we identify $q_i = e^{t_i}$.

The entire structure of (gravitational) quantum cohomology is entailed in the fact that the potential is a solution to the following equations:

**Theorem 2.1.** (WDVV equations)

$$\sum_f \langle \langle \tau_{d_1} \gamma_1, \tau_{d_2} \gamma_2, T_f \rangle \rangle_0 < \langle \langle T_f, \tau_c \gamma_c, \tau_d \gamma_d \rangle \rangle_0 = \sum_f \langle \langle \tau_{d_1} \gamma_1, \tau_{d_2} \gamma_2, T_f \rangle \rangle_0 < \langle \langle T_f, \tau_c \gamma_c, \tau_d \gamma_d \rangle \rangle_0.$$ 

**Proof.** Recall the fundamental relations (1.2) on $\overline{M}_{0,n+4}$,

$$\sum A \cup \{ \partial \{ B \cup \{ c \} \} \} = \sum D(A \cup \{ ac \} \cup \{ bd \}),$$

where the summations run over all partitions $A \cup B = \{ 1, \ldots, n \}$. Multiply each side of (2.7) by $(T^{\otimes n}, \tau_0 \gamma_0, \tau_0 \gamma_0, \tau_c \gamma_c, \tau_d \gamma_d)_{0, \beta}$ and use splitting to establish

$$\sum_{\beta_1 + \beta_2 = \beta} \sum_f \langle \langle T^{\otimes |A|}, \tau_0 \gamma_0, \tau_0 \gamma_0, T_f \rangle \rangle_{0, \beta_1} < \langle \langle T^{\otimes |B|}, \tau_c \gamma_c, \tau_d \gamma_d, T_f \rangle \rangle_{0, \beta_2} = \sum_{\beta_1 + \beta_2 = \beta} \sum_f \langle \langle T^{\otimes |A|}, \tau_0 \gamma_0, \tau_0 \gamma_0, T_f \rangle \rangle_{0, \beta_1} < \langle \langle T^{\otimes |B|}, \tau_c \gamma_c, \tau_c \gamma_c, T_f \rangle \rangle_{0, \beta_2},$$

where the first summations are as in (2.7). Since there are $\binom{n}{n_1}$ partitions of $\{ 1, \ldots, n \}$ such that $|A| = n_1$, we have

$$\sum_{n_1 + n_2 = n, \beta_1 + \beta_2 = \beta} \frac{n!}{n_1! n_2!} \langle \langle T^{\otimes n_1}, \tau_0 \gamma_0, \tau_0 \gamma_0, T_f \rangle \rangle_{0, \beta_1} < \langle \langle T^{\otimes n_2}, \tau_c \gamma_c, \tau_d \gamma_d, T_f \rangle \rangle_{0, \beta_2} = \sum_{n_1 + n_2 = n, \beta_1 + \beta_2 = \beta} \frac{n!}{n_1! n_2!} \langle \langle T^{\otimes n_1}, \tau_0 \gamma_0, \tau_0 \gamma_0, T_f \rangle \rangle_{0, \beta_1} < \langle \langle T^{\otimes n_2}, \tau_c \gamma_c, \tau_c \gamma_c, T_f \rangle \rangle_{0, \beta_2}. $$

Dividing by $n!$, multiplying by $q^\beta$, and summing over all $n$ and $\beta$ yields the desired result. \qed
Proof. This result is a consequence of the following well-known relation:
\[ c_1(L) = \sum_{K \cup L = \{1\} \cup \{2, 3\}} \eta^* D(K \cup \{1\} \cup \{2, 3\}). \]
Applying it to \( \langle \tau_{a+1} \gamma_a, \tau_{b} \gamma_b, \tau_{c} \gamma_c \rangle_0 \) we find
\[ \sum_{n \geq 0} \sum_{\beta} \frac{1}{n!} \sum_{K \cup L = \{4, \ldots, n+3\}} q^\beta \int_{\mathbb{P}^{0,n+3}} \left( D(K \cup \{1\} \cup \{2, 3\})(\tau_{a} \gamma_a, \tau_{b} \gamma_b, \tau_{c} \gamma_c; T^{\otimes n})_{0,\beta} \right). \]
The result follows by splitting and a combinatorial argument as in the proof of Theorem 2.1.

We now give an example due to Kontsevich of the remarkable enumerative implications of the WDVV-equations.

Example 2. (Quantum cohomology of \( \mathbb{P}^2 \)) Let \( T_i = p_i, i = 0, 1, 2, \) where \( p \) is the ample generator of \( \text{Pic}(\mathbb{P}^2) \). For \( d \geq 1 \), let
\[ K_d = \langle T_2, \ldots, T_2 \rangle_{0,d}. \]
Then \( K_d \) is the number of rational curves of degree \( d \) through \( 3d - 1 \) points, and the GW-potential of \( \mathbb{P}^2 \) is \( \Phi \}_{t_0, t_1, t_2} = \frac{1}{2}(t_0 t_1^2 + t_2 t_1) + \Gamma, \) where
\[ \Gamma = \sum_{d=1}^{\infty} \frac{K_d q^d}{(3d-1)!}. \]
A direct application of Theorem 2.1 yields
\[ \partial_1 \partial_2 \partial_2 \Gamma = (\partial_1 \partial_1 \partial_2 \Gamma)^2 - (\partial_1 \partial_1 \partial_1 \Gamma) \cdot (\partial_1 \partial_2 \partial_2 \Gamma), \]
and equating the coefficients on both sides, we find Kontsevich’s recursion formula:
\[ K_d = \sum_{d_1 + d_2 = d} K_{d_1} K_{d_2} d_1^2 d_2 \left[ d_2 \left( \frac{3d - 4}{3d_1 - 2} \right) - d_1 \left( \frac{3d - 4}{3d_1 - 1} \right) \right]. \]

2.3. Quantum cohomology rings. Let us consider the free \( \mathbb{Q}[[q, q^{-1}, t]] \)-module \( A^*(X) \otimes \mathbb{Q}[[q, q^{-1}, t]] \) generated by \( T_0, \ldots, T_m \). The quantum product is defined as
\[ T_i \ast T_j = \sum_f <T_i, T_j, T_f >_0 T^f. \]
Extending this product \( \mathbb{Q}[[q, q^{-1}, t]] \)-linearly puts a \( \mathbb{Q}[[q, q^{-1}, t]] \)-algebra structure on \( A^*(X) \otimes \mathbb{Q}[[q, q^{-1}, t]] \). This ring is called the (big) quantum cohomology ring \( \mathbb{Q}H^*_q(X) \).

Proposition 2.3. \( \mathbb{Q}H^*_q(X) \) is an associative, commutative ring with identity \( T_0 = 1 \).

Proof. Commutativity is obvious. That \( T_i \ast T_0 = T_i \) for all \( i \) follows from the fundamental class and mapping to a point property. Finally, associativity is equivalent to the WDVV equations for the correlators \(< >_0 \).
We shall mainly work with the (small) quantum cohomology ring $QH^*(X)$ which is the specialization of $QH^*_b(X)$ obtained by restricting $t = (t_1, \ldots, t_m) = (0, \ldots, 0)$. Using (2.6), one can see that the product on $QH^*(X)$ is

$$T_i * T_j = \sum_\beta q^\beta \sum_f <T_i, T_j, T_f> \cdot T_f.$$  

We make the ring $QH^*(X)$ graded by putting $\deg q_i = \langle T_i, c_1(TX) \rangle$, and letting $T_i$ keep its grading from $A^*(X)$.

**Remark.** If the gravitational potential $\Phi$ is a formal power series we can also define

$$T_i * T_j = \sum_f \langle T_i, T_j, T_f \rangle \cdot T_f.$$  

This puts a $\mathbb{Q}[[t]]$-algebra structure on $A^*(X) \otimes \mathbb{Q}[[t]]$. This ring is still commutative and associative, but due to the more complicated nature of the fundamental class property, $T_0$ is no longer the identity.

**Example 3.** The quantum cohomology ring of $\mathbb{P}^n$ is $QH^*(\mathbb{P}^n) = \mathbb{Q}[p, q]/(p^{n+1} - q)$, where $p$ is the ample generator of $\text{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.

**2.4. Dubrovin’s formalism.** In this section we describe yet another way one may view the WDVV-equations, as developed in [D]. First we review some general theory about integrability conditions and flat connections.

Let $M$ be a vector bundle on $\mathbb{C}^m$ with coordinates $x_i$ on $\mathbb{C}^m$ and $m_j$ on the fibers, and let $\tilde{m}_j = (x_1, \ldots, x_m, 0, \ldots, 0)$ be sections of $M$.

Suppose we have a connection $\nabla: M \to M \otimes \Omega_{\mathbb{C}^m}$ with covariant derivatives

$$\nabla_i(g_j \tilde{m}_j) = \frac{\partial g_j}{\partial x_i} \tilde{m}_j + \sum_k \Gamma_{ij}^k g_j \tilde{m}_k,$$

where $\Gamma_{ij}^k$ are holomorphic functions on $\mathbb{C}^m$. The connection defines a distribution

$$\mathcal{D}: T\mathbb{C}^m \to TM$$

on $TM$ by letting

$$\frac{\partial}{\partial x_p} \mapsto \frac{\partial}{\partial x_p} - \sum_{j,k} \Gamma_{ij}^k m_j \frac{\partial}{\partial m_k}.$$  

By Frobenius’ theorem, a distribution is integrable if it is closed under the Lie bracket operation $[\cdot, \cdot]$. It is straightforward to verify the following facts:

$\mathcal{D}$ is closed under $[\cdot, \cdot]$ if and only if $\nabla$ is flat.

A section $F: \mathbb{C}^m \to M$ is an integral of $\mathcal{D}$ if and only if it is flat, i.e., $\nabla(F) = 0$.

We now apply this theory to the affine space $V = A^*_\mathbb{C}(X)$ and the trivial vector bundle $M = V \times V$. Consider the “connection” defined by the covariant derivatives

(2.9)  

$$\nabla_i(F) = \hbar \frac{\partial F}{\partial t_i} - T_i * F.$$  

Here, $\hbar$ is just a formal parameter. If $\hbar = 1$, then (2.9) is a formal connection. It is straightforward to verify:

**Proposition 2.4.** $\nabla$ is flat if and only if the quantum product $*$ is associative.
2.5. Flat sections. From the discussion in the previous section, we can conclude that there should exist formal solutions to the system of first order differential equations

\[ \hbar \frac{\partial}{\partial t_i} = T_i * \]

where \( i = 0, \ldots, m \). Via gravitational descendents we can actually obtain closed formulas for these solutions.

For \( a = 0, \ldots, m \), let

\[ \vec{S}_a = T_a + \sum_n \sum_b h^{-n-1} < \tau_n T_a, T_b >_0 T^b. \]

Claim. \( \vec{S}_a \) is a solution to the system of first order differential equations (2.10).

Proof. This is a consequence of the equations in Theorem 2.2, specialized to the partial derivatives \( < >_0 \). On one side of (2.10) we find

\[ \hbar \frac{\partial \vec{S}_a}{\partial t_i} = \sum_n \sum_b h^{-n} < \tau_n T_a, T_b, T_i >_0 T^b, \]

and on the other side

\[ T_i * \vec{S}_a = \sum_b < T_i, T_a, T_b >_0 T^b + \sum_n \sum_{b,c} h^{-n-1} < \tau_n T_a, T_b >_0 < T^b, T_i, T_c >_0 T^c, \]

which by Theorem 2.2 equals

\[ \sum_b < T_i, T_a, T_b >_0 T^b + \sum_n \sum_c h^{-n-1} < \tau_n T_a, T_i, T_c >_0 T^c. \]

\[ \square \]

We shall restrict to only consider the system (2.10) with \( i = 0, \ldots, r \) and the small quantum product. A formal calculation using the fundamental class, divisor, and mapping to a point properties yields:

The system of differential equations

\[ \hbar \frac{\partial}{\partial t_0} = T_0*, \quad \hbar \frac{\partial}{\partial t_i} = p_i* \quad \text{for} \quad i = 1, \ldots, r, \]

has solutions

\[ \vec{s}_a = e^{t_0/\hbar} \left( e^{p_i/\hbar} T_a + \sum_{\beta \neq 0} q^\beta \sum_b T^b \int_{[\Sigma_{0,2}]} \frac{e^x_1(e^{p_I/\hbar} T_a)}{h-c} e^x_2(T_b) \right), \]

where \( a = 0, \ldots, m \), \( pt = \sum p_i t_i \), \( c = c_1(L_1) \), and \( q_i = e^{t_i} \).

2.6. Quantum \( \mathcal{D} \)-modules. The quantum \( \mathcal{D} \)-module \( Q\mathcal{D}(X) \) of \( X \) is the \( \mathcal{D} \)-module generated by all \( < \vec{s}, 1 > \) for flat sections \( \vec{s} \). In other words, if \( \mathcal{D} \) is the ring of linear differential operators \( \mathbb{C}[h \frac{\partial}{\partial t_i}, q_i, h | i = 0, \ldots, r] \), with

\[ \left[ \hbar \frac{\partial}{\partial t_i}, q_i \right] = \hbar \frac{\partial}{\partial t_i} \circ q_i - q_i \circ \hbar \frac{\partial}{\partial t_i} = h q_i, \]
then the quantum $D$-module is the set of $P \in D$ such that $P(<s,1>) = 0$ for all flat sections $s$. Using the formula (2.13), we can write up a set of generators for the quantum $D$-module in vector form:

$$s^X = \sum_{a=0}^{m} <\tilde{s}_a,1> T^a = e^{(t_0+p)t}/\hbar \left( 1 + \sum_{\beta \neq 0} q^\beta e_{1*} \left( \frac{1}{\hbar - c} \right) \right)$$

where $e_{1*}: \mathcal{M}_{0,2}(X,\beta) \to X$.

**Remark.** The $D$-module can be useful in finding relations in $QH^*(X)$. If $P(h\frac{\partial}{\partial t},q,\hbar)$ is in $QD(X)$, then $P(p,q,0) = 0$ in $QH^*(X)$. A similar result holds in $QH^*_c(X)$.

We close this section with a description of the quantum $D$-module in two cases.

**Example 4.** $\mathbb{P}^4$. Let $p$ be the ample generator of $\text{Pic}^{\mathbb{P}^4}$ then the multiplication matrix $p* -$ with respect to the basis $1,p,\ldots,p^4$ is

$$p*1 = p, \quad p*p = p^2, \quad p*p^2 = p^3, \quad p*p^3 = p^4, \quad p*p^4 = q.$$ 

Hence a section $s = s_0 + s_1 p + \cdots + s_4 p^4$ is flat if and only if

$$\frac{\hbar d}{dt} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & q \\ 1 & 0 & 0 & 0 & s_1 \\ 0 & 1 & 0 & 0 & s_2 \\ 0 & 0 & 1 & 0 & s_3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} qs_1 \\ s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}.$$

Solving for $s_4$, we find

$$\frac{\hbar d}{dt} s_4 = s_3, \quad (\frac{\hbar d}{dt})^2 s_4 = s_2, \quad (\frac{\hbar d}{dt})^3 s_4 = s_1, \quad (\frac{\hbar d}{dt})^4 s_4 = s_0, \quad (\frac{\hbar d}{dt})^5 s_4 = qs_4,$$

hence $(\frac{\hbar d}{dt})^5 - q$ generates the quantum $D$-module of $\mathbb{P}^4$. Note that $p^5 - q = 0$ in $QH^*(\mathbb{P}^4)$.

**Example 5.** Suppose $Y$ is a Calabi-Yau threefold with $\text{Pic}(Y) \simeq \mathbb{Z}$. Let $p$ be the ample generator of $\text{Pic}(Y)$. Let $N_d^Y$ and $n_d^Y$ be as in Example 4 of Section 2.1. From the definition of the quantum product in $QH^*_c(Y)$, we find

$$p*1 = p, \quad p*p = K(q)p^2, \quad p*p^2 = p^3, \quad p*p^3 = 0,$$

where

$$K(q) = 1 + \frac{1}{\deg(Y)} \sum_{d>0} <p,p>_{0,d} q^d = 1 + \frac{1}{\deg(Y)} \sum_{d>0} d^3 n_d^Y q^d.$$ 

A section $s = s_0 + s_1 p + s_2 p^2 + s_3 p^3$ is flat if and only if

$$\frac{\hbar d}{dt} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & K(q) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}.$$ 

Solving for $s_3$, we find that $(\frac{\hbar d}{dt})^2 \frac{1}{K(q)} (\frac{\hbar d}{dt})^2$ generates the quantum $D$-module of $Y$. Note that by the Aspinwall-Morrison formula

$$K(q) = 1 + \frac{1}{\deg(Y)} \sum_{d>0} d^3 n_d^Y q^d \frac{q^d}{1 - q^d}.$$
2.7. Generalized quantum structures. Let $B$ be a vector bundle of rank $b$ on $X$ that is generated by its global sections and suppose $Y \subseteq X$ is the zero scheme of a general global section of $B$. The idea is to find a way to relate the quantum theory of $Y$ to the quantum theory of $X$.

First, we look at the virtual fundamental class $\overline{M}_{0,n}(Y, \beta)$. Note that by the normal bundle sequence and the adjunction formula $c_1(TY) = c_1(TX) - c_1(B)$, hence the expected dimension of $\overline{M}_{0,n}(Y, \beta)$ is 

$$\dim X - b + \langle \beta, c_1(TX) - c_1(B) \rangle + n - 3.$$ 

The inclusion map $i: Y \to X$ induces a map 

$$j: \overline{M}_{0,n}(Y, \beta) \to \overline{M}_{0,n}(X, i_* \beta)$$ 

such that

$$\begin{array}{ccc}
\coprod_{i_* \beta = d} \overline{M}_{0,n}(Y, \beta) & \xrightarrow{j} & \overline{M}_{0,n}(X, d) \\
\downarrow e & & \downarrow e \\
Y^n & \xrightarrow{i} & X^n
\end{array}$$

(2.17)

commutes. Let $B$ be the universal map. Tensoring $\mathcal{O}_{\overline{M}_{0,n}(X, d) \times X} \to \mathcal{O}_{\overline{M}_{0,n+1}(X, d)}$ by $e^{n+1}_* B$ and pushing down yields a map

$$H^0(X, \mathcal{B})_{\overline{M}_{0,n}(X, d)} \xrightarrow{\nu} \pi_* e^{n+1}_* \mathcal{B}.$$ 

(2.18)

For any point $[C, f, s_i]$ in $\overline{M}_{0,n}(X, d)$, we have $H^1(C, f^* \mathcal{B}) = 0$, hence base change theorems in [1] imply that $B_d = \pi_* e^1_1 \mathcal{B}$ is a vector bundle with fibers $H^0(C, f^* \mathcal{B})$. Note that $B_d$ is independent of the marked points, in the sense that $\pi^*_i B_d = B_d$, where $\pi_i$ forgets the $i$-th point. The map $\nu$ sends a section $s \in H^0(X, \mathcal{B})$ to the restriction $s \circ f \in H^0(C, f^* \mathcal{B})$, so the stable map $f: C \to X$ factors through $i$ if and only if $[C, f, s_i]$ lies in the zero locus of $\nu(s)$. In other words, if $o$ is the zero section, then the diagram

$$\begin{array}{ccc}
\coprod_{i_* \beta = d} \overline{M}_{0,n}(Y, \beta) & \xrightarrow{o} & \overline{M}_{0,n}(X, d) \\
\downarrow \nu(s) & & \downarrow o \\
\overline{M}_{0,n}(X, d) & \to & B_d
\end{array}$$

(2.20)

is fibred. The construction of the classes $[\overline{M}_{0,n}(Y, \beta)]$ is such that

$$[\overline{M}_{0,n}(Y, \beta)] = (o! [\overline{M}_{0,n}(X, d)])_{\overline{M}_{0,n}(Y, \beta)}.$$
where \((\ )_{\overline{M}_{0,n}(Y,\beta)}\) indicates the proper component of \(\bigsqcup_{i*\beta=d} \overline{M}_{0,n}(Y,\beta)\). From the excess intersection formula \(E\),

\[
(2.21) \quad j_* \sum_{i*\beta=d} [\overline{M}_{0,n}(Y,\beta)] = B_d \cdot [\overline{M}_{0,n}(X, d)],
\]

with \(B_d = c_{\text{top}}(B_d)\). Note that if \(C \simeq \mathbb{P}^1\), then by Riemann-Roch

\[
\dim H^0(\mathbb{P}^1, f^*\mathcal{B}) = b + \int_d c_1(\mathcal{B}),
\]

hence the rank of \(B_d\) is \(b + \int_d c_1(\mathcal{B})\), and \((2.21)\) is a class of expected dimension.

We shall use this construction to define a new associative quantum product on \(A^*(X) \otimes Q[[q, q^{-1}]]\). The product will account for GW-invariants on \(Y\) involving classes from \(i^* A(X)\).

Define new correlators on \(A^*(X)\)

\[
(\tau_{d_1\gamma_1}, \ldots, \tau_{d_n\gamma_n})^B_{0, d} = \eta_* \left( \prod_{i=1}^n c_1(L_i)^{d_i} e_i^* (\gamma_i) \cdot B_d \cdot [\overline{M}_{0,n}(X, d)] \right).
\]

As before let \(<>^B_{0,d} = \int_{\overline{M}_{0,n}(X,d)} \cdot [\ )_{0,d}.\) Note that, by \((2.21)\),

\[
(2.22) \quad (\tau_{d_1\gamma_1}, \ldots, \tau_{d_n\gamma_n})^B_{0,d} = \sum_{i*\beta=d} (\tau_{d_1} i^*(\gamma_1), \ldots, \tau_{d_n} i^*(\gamma_n))_{0,\beta}.
\]

Hence if for some \(j\), \(i^*(\gamma_j) = 0\), then \((\tau_{d_1\gamma_1}, \ldots, \tau_{d_n\gamma_n})^B_{0,d} = 0.\)

It is easy to see that the new correlators satisfy the fundamental class and divisor properties of \(\Sigma\). We shall also see that there is a splitting principle. Let \(\rho_j : B_d \to e_j^* B\) be the map which on the fibers \(H^0(C, f^* \mathcal{B})\) of \(B_d\) evaluates the sections at the \(j\)th marked point. Associated to the gluing map

\[
\varphi : \overline{M}_{0,A\cup\{\bullet\}}(X, \beta_1) \times_X \overline{M}_{0,B\cup\{\bullet\}}(X, \beta_2) \to \overline{M}_{0,A\cup B}(X, \beta_1 + \beta_2),
\]

there is an exact sequence

\[
0 \to \varphi^* B_{\beta_1 + \beta_2} \to \psi_{12}^* B_{\beta_1} \oplus \psi_{21}^* B_{\beta_2} \stackrel{\rho_1^* - \rho_2^*}{\longrightarrow} e_0^* B \to 0,
\]

where \(\psi_i\) are projection maps. Hence, if \(B = c_{\text{top}}(B)\) and \(B_{j,d} = c_{\text{top}}(\ker \rho_j)\), we have

\[
(2.23) \quad c_{\text{top}}(\varphi^* B_{\beta_1 + \beta_2}) \cdot e_0^* B = B_{\beta_1} \cdot B_{\beta_2} \quad \text{and} \quad B_d = B_{j,d} \cdot e_0^* B.
\]

Let \(A \cup B\) be a partition of \(\{1, \ldots, n\}\). By the commutativity of the diagram \((2.3)\),

\[
(2.24) \quad \varphi_{AB}(\tau_{d_1\gamma_1}, \ldots, \tau_{d_n\gamma_n})^B_{0, d} = (\eta_A \times \eta_B)^* \varphi_{AB}^1 \left( \prod_{i=1}^n c_1(L_i)^{d_i} e_i^* (\gamma_i) \cdot [\overline{M}_{0,n}(X, d)] \cdot B_d \right)
\]

\[
= (\eta_A \times \eta_B)^* (\psi_1 \times \psi_2)^* (G^*(B_d))(e_0^* \cdot e_0^*)^*(\Delta) \sum
\]

\[
\sum = \sum_{\beta_1 + \beta_2 = d} \left( \prod_{a \in A} c_1(L_a)^{d_a} e_a^* (\gamma_a) \cdot [\overline{M}_{0,A\cup\{\bullet\}}(X, \beta_1)] \right) \times \prod_{b \in B} c_1(L_b)^{d_b} e_b^* (\gamma_b) \cdot [\overline{M}_{0,B\cup\{\bullet\}}(X, \beta_2)].
\]

Since

\[
(\psi_1 \times \psi_2)^* (G^*(B_d))(e_0^* \cdot e_0^*)^*(\Delta) = \sum_f B_{\beta_1} e_0^*(T_f) \times B_{\beta_2} e_0^*(T_f)
\]

\[
= \sum_f B_{\beta_1} e_0^*(B \cdot T_f) \times B_{\beta_2} e_0^*(T_f),
\]

\[
(2.25)
\]
we only have to sum over a basis for $A^*(X)/\ker i^*$.

Define a new pairing $<,>_B$ on $A^*(X)$ by $<\gamma,\gamma'>_B=<\gamma,B\cdot\gamma'>$. This pairing is non-degenerate on $A^*(X)/\ker i^*$, so we can choose bases $\{T_j\}$ and $\{e^T\}$ for $A^*(X)/\ker i^*$ such that $<T_f,e^T>_B=\delta_{ef}$. Then (2.25) becomes

$$
\sum_f B_{\beta_1} e^*_{\beta_1}(T_f) \times B_{\beta_2} e^*_{\beta_2}(fT),
$$

and with (2.24) we have the following splitting principle for the new correlators:

$$
\varphi_{AB}(T_{\alpha_1},\ldots,T_{\alpha_n})_0 = \sum_{\beta_1+\beta_2=d} \sum_f (\otimes_{a\in A} T_{\alpha_a}T_f)_0 \otimes_{b\in B} (fT)_0 \otimes_{c\in C} (T_c)_0.
$$

This means that the results of Sections 2.1–2.6 are true if we replace $(\ )_0,<_B$ and $<=$ by $(\ )_0,<_B$. In particular, if we define the partial derivatives $<<>_0^B$ and $>>_0^B$ as in Section 2.2, we have the following structures:

**WDVV equations.** Theorem 2.1 and Theorem 2.2 hold for $<<>_0^B$ and $>>_0^B$.

**QH*(B).** Define products $*_B$ on $A^*(X) \otimes \mathbb{Q}[[q,q^{-1},t]]$ and $A^*(X) \otimes \mathbb{Q}[[q,q^{-1}]]$ by

$$
T_i*_BT_j = \sum_k <T_i,T_j,T_k>_0^B kT
$$

and

$$
T_i*_BT_j = \sum_{d,k} <T_i,T_j,T_k>_0^B q^d kT,
$$

respectively, where $q^d$ still denotes $\prod q_{i_{\alpha}}^{a_{\alpha}}$ with the old pairing. This makes $QH_0(B) = A^*(X) \otimes \mathbb{Q}[[q,q^{-1},t]]$ and $QH^*(B) = A^*(X) \otimes \mathbb{Q}[[q,q^{-1}]]$ into commutative and associative rings. The ring $QH^*(B)$ is graded if we let $\deg q_i = <T_i,c_1(X) - c_1(B)>_X$.

**Remark.** $QH^*(Y)^\perp$. We can split

$$
A^*(Y) = i^*A^*(X) \oplus i^*A^*(X)^\perp,
$$

where $\perp$ is with respect to the Poincaré pairing $<,>_Y$ on $A^*(Y)$. Let $p'_i$ be generators of $A_0(Y)$ with exponential coordinates $q'$. Let $\perp: A^*(Y) \rightarrow i^*A^*(X)$ denote the projection map. The product $*_{Y} = \perp \circ *_{Y}$ defines a ring structure on $QH^*(Y)^\perp = i^*A^*(X) \otimes \mathbb{Q}[[q',q'^{-1}]]$. The arguments that produced associativity of $QH^*(B)$ also show that $QH^*(Y)^\perp$ is associative. If $i^*: A_1(Y) \rightarrow A_1(Y)$ is an isomorphism, then $i^*$ induces an isomorphism between $QH^*(B)/\ker i^*$ and $QH^*(Y)^\perp$.

A word of caution: There is no reason to believe that $QH^*(Y)^\perp$ is associative in general when $Y \subseteq X$ without $Y$ being a zero scheme of a bundle.
Quantum D-modules. As in 2.4–2.6 there is a flat connection $\nabla^B$ defined by $*_B$, with flat sections

$$s^B_a = e^{i_0/h} \left( e^{pt/h} T_a + \sum_{d \neq 0} q^d \sum_b bT \int_{[\overline{M}_{0,2}(X,\beta)]} \frac{e_1^* (e^{pt/h} T_a) \ b_d}{\hbar - c} - e_2^* (T_b) B_d \right),$$

and a quantum D-module $QD(B)$ generated by

$$s^B = \sum_a <s^B_a, 1>_B aT = e^{(t_0 + pt)/h} \left( B + \sum_{d \neq 0} q^d e_1^* \left( \frac{B_d}{\hbar - c} \right) \right),$$

where $e_1: \overline{M}_{0,2}(X,\beta) \to X$ and $c = c_1(L_1)$.

Remark. Similar, corresponding to the product $*_Y^i$ on $QH^*(Y)^\perp$ there is the quantum D-module $QD^\perp(Y)$ generated by

$$s^\perp_Y = e^{(t'_0 + pt')/h} \left( 1 + \sum_{d \neq 0} q^d e_{1}\left( \frac{1}{\hbar - c} \right) \right),$$

where $e_1: \overline{M}_{0,2}(Y,\beta) \to Y$ and $c = c_1(L_1)$. Note that $QD^\perp(Y) \supseteq QD(Y)$.

2.8. Quantum Lefschetz hyperplane principle. Suppose $B = \oplus O(L_i)$ is a rank $k$ decomposable vector bundle generated by global sections, and with Chern roots $L_i \in A^1(X)$. The classical Lefschetz hyperplane theorem states that $i^*: A^l_Q(X) \to A^l_Q(Y)$ is an isomorphism for $l \leq \dim X - (k + 1)$ and injective for $l = \dim X - k$.

The quantum Lefschetz hyperplane principle relates $QD(X)$ and $QD^\perp(Y)$. Let

$$(2.26) \qquad I = e^{(t_0 + pt)/h} \left( 1 + \sum_{\beta \neq 0} q^\beta \prod_{j=1}^k \prod_{m=1} (L_j + m\hbar) e_1^* \left( \frac{1}{\hbar - c} \right) \right),$$

and let $q_i$ be of degree $\deg q_i = < T^i, c_1(TX) - c_1(B)>$ in this expression. The quantum Lefschetz hyperplane principle states that the D-module generated by $i^* I$ and $s^\perp_Y$ are equivalent. More precisely: $i^* I$ and $s^\perp_Y$ coincide up to a unique weighted homogeneous change of variables

- $t'_0 \mapsto t_0 + f_0 h + f_{-1}$
- $t'_i \mapsto t_i + f_i$

where $f_{-1}, \ldots, f_r$ are weighted homogeneous power series of $q_1, \ldots, q_r$, $\deg f_i = 0$ for all $i \geq 0$ and $\deg f_{-1} = 1$. The change of variables is determined by the coefficients of $1 = (\frac{1}{\hbar})^0$ and $\frac{1}{\hbar}$ in the expansion of $i^* I$ and $s^\perp_Y$ as power series of $\frac{1}{\hbar}$.

A version of this principle was first conjectured in [BvS]. This principle has been proven by Givental [G] for convex toric manifolds and by Kim [Kim] for homogeneous manifolds. In fact the proofs hold for all manifolds acted upon by a torus with a finite number of fixpoints and one-dimensional orbits, but then only up to the existence of a virtual equivariant fundamental class as defined in [E].

We conclude this part by illustrating how this formalizes the mirror calculation of [CdOOGP] for the quintic in $\mathbb{P}^4$. 
Example 6. (Mirror calculation)
Suppose $Y \subseteq \mathbb{P}^4$ is a general quintic hypersurface. The solutions to the differential equations $\frac{d}{dt} - 1 = 0$ and $(\frac{d}{dt})^5 - q = 0$ are explicitly known [3] as

$$s_{\mathbb{P}^4} = e^{(t_0 + pt)/\hbar} \left( 1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{m=1}^{5d} (p + mh)^5} \right).$$

For the quintic, (2.26) becomes

$$i^* I = e^{(t_0 + p't)/\hbar} \left( 1 + \sum_{d=1}^{\infty} q^d \frac{\prod_{m=1}^{5d} (5p' + mh)}{\prod_{m=1}^{5d} (p' + mh)^5} \right) = e^{t_0/\hbar} \left( I_0 + \frac{I_1}{\hbar} p' + \frac{I_2}{\hbar^2} p'^2 + \frac{I_3}{\hbar^3} p'^3 \right),$$

where $i^* p = p'$ and $I_i \in \mathbb{C}[[q]][t]$. On the other hand, $(\frac{d}{dt})^2 K(q') = 1$ has explicit solutions

$$s_{\mathbb{P}^4}^* = e^{t'/\hbar} \left( e^{p't'/\hbar} + \frac{p'^2}{\hbar^2} \sum_{d=1}^{\infty} n_d q^d \sum_{k=1}^{\infty} (e^{p't'/\hbar} q^d)^2 \right),$$

(2.28)

$$= e^{t'/\hbar} \left( 1 + \frac{t'}{\hbar} p' + \cdots \right),$$

where $K(q') = 1 + \frac{1}{5} \sum_{d=1}^{\infty} n_d q^d \frac{q^d}{1 - q^d}$. The quantum Lefschetz hyperplane principle identifies (2.28) with $e^{t'/\hbar} \left( I_0 + \frac{I_1}{I_0} p' + \frac{I_2}{I_0^2} p'^2 + \frac{I_3}{I_0^3} p'^3 \right)$ if we put $t' = \frac{I_1}{I_0}$. But (2.27), with $\hbar = 1$ is a complete set of solutions to the Picard-Fuchs equation

$$\left( \frac{d}{dt} \right)^4 I = 5q \left( 5 \frac{d}{dt} + 1 \right) \left( 5 \frac{d}{dt} + 2 \right) \left( 5 \frac{d}{dt} + 3 \right) \left( 5 \frac{d}{dt} + 4 \right).$$

This proves the mirror computation of [CdOGP].

3. Trisecant planes of the Veronese surface in $\mathbb{P}^5$

In this section we introduce our main object of study, $N$, the space of trisecant planes to the Veronese surface in $\mathbb{P}^5$, which is the same as the space of determinantal nets of conics. A related space has been studied in [EPS, ES1, ES2]. We give a summary of results found there, modified to our case. In Section 3.1, $N$ is constructed as a geometric quotient, and in Sections 3.2, 3.3, 3.4 we look at the geometry of $N$. In particular, Ellingsrud and Strømme’s derivation of the Chow ring of $N$ is covered.
3.1. The quotient construction. Let \( V \) be a \( \mathbb{C} \)-vector space of dimension 3. We shall study the space of \( 3 \times 2 \) matrices with entries in \( V \), modulo row and column operations. Let \( E \) and \( F \) be \( \mathbb{C} \)-vector spaces of dimension 3 and 2 respectively. Let the group \( GL(E) \times GL(F) \) act on \( \text{Hom}(F,E \otimes V) \) by

\[
(g,h)\alpha = (g \otimes id_V) \circ \alpha \circ h^{-1},
\]
i.e., so that,

\[
\begin{array}{ccc}
F & \xrightarrow{\alpha} & E \otimes V \\
h \downarrow & & \downarrow g \otimes id_V \\
F & \xrightarrow{(g,h)\alpha} & E \otimes V
\end{array}
\]

commutes for \((g,h) \in GL(E) \times GL(F)\) and \( \alpha \in \text{Hom}(F,E \otimes V) \). The normal subgroup

\[
\Gamma = \{ k(id_E, id_F) \mid k \in \mathbb{C}^* \}
\]
acts trivially on \( \text{Hom}(F,E \otimes V) \). Let \( G = GL(E) \times GL(F)/\Gamma \) and consider the induced action of \( G \) on \( \text{Hom}(F,E \otimes V) \). A point of \( \text{Hom}(F,E \otimes V) \) will be called stable (resp. semistable) if the corresponding point in \( \mathbb{P}(\text{Hom}(F,E \otimes V)^\vee) \) is stable (resp. semistable) in the sense of [MT] for the induced action of \( G \cap SL(\text{Hom}(F,E \otimes V)) \).

Suppose \( A = (L_{ij}) \) is a \( 3 \times 2 \) matrix representing \( \alpha \), with \( L_{ij} \in \mathbb{C} \). Let \( E_\alpha \subseteq S_2 V \) be the span of all \( 2 \times 2 \) minors of \( A \). Then \( E_\alpha \) does not depend on the matrix representative \( A \).

**Proposition 3.1.** [EPS]. For any \( \alpha \in \text{Hom}(F,E \otimes V) \) the following are equivalent:

i) \( \alpha \) is stable  
ii) \( \alpha \) is semistable  
iii) \( \dim E_\alpha = 3 \)  
iv) \( \alpha \) cannot be represented by a matrix of type \( \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \) or \( \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \).

By geometric invariant theory there exists a projective geometric quotient

\[
N = \text{Hom}(F,E \otimes V)^{ss}/G.
\]

**Proposition 3.2.** [EPS]. The group \( G \) acts freely on \( \text{Hom}(F,E \otimes V)^{ss} \), hence \( N \) is smooth.

Proposition 3.1 implies that there is a morphism

\[
\iota : N \to \text{Grass}_3(S_2 V)
\]
given by \( \iota(\alpha) = E_\alpha \), where \( (\alpha) \) is the class of \( \alpha \in \text{Hom}(F,E \otimes V)^{ss} \).

Let \( \text{Hilb}_3^{P(V)} \) be the Hilbert scheme of subschemes of length 3 in \( \mathbb{P}(V) \). There is a morphism

\[
b : \text{Hilb}_3^{P(V)} \to \text{Grass}_3(S_2 V)
\]
given by \( b([Z]) = H^0(\mathbb{P}(V), \mathcal{I}_Z(2)) \) for \( Z \subseteq \mathbb{P}(V) \) a zero-dimensional subscheme of length 3. Using the Hilbert-Burch theorem [EB], one may show that \( \iota \) is an embedding and that the image of \( b \) can be identified with \( N \subseteq \text{Grass}_3(S_2 V) \). In fact

\[
b : \text{Hilb}_3^{P(V)} \to N
\]
blows up the locus of nets of type \( \langle x_0^2, x_0x_1, x_0x_2 \rangle \subseteq S_2 V \), where \( \{x_0, x_1, x_2\} \) is a basis for \( V \).

**Remark 1.** The variety \( N \) is often denoted \( N(3;2,3) \). As the notation indicates it belongs to a bigger class of varieties \( N(q;n,m) \), which parametrize \( n \times m \) matrices with entries from
a $q$-dimensional vector space, modulo row and column operations. Topological properties of $N(q; n, m)$ have been studied in [Dr]. In [EPS, ES1] the variety $N(4; 2, 3)$ is considered. It is related to the component of the Hilbert scheme containing the twisted cubics in an analogous manner to $(3.2)$. 

Remark 2. If Grass$_3(S_2V)$ is seen as the variety of 2-planes in $P(S_2V)$ then $(3.1)$ identifies $N$ as the variety parametrizing trisecting planes to the Veronese surface in $P(S_2V)$.

3.2. Universal vector bundles. Let $E'$ and $F'$ denote the trivial vector bundles

$$E \otimes \frac{\det F}{\det E} \times \text{Hom}(F, E \otimes V)^{ss} \text{ and } F \otimes \frac{\det F}{\det E} \times \text{Hom}(F, E \otimes V)^{ss}$$
on the variety $V$. The bundles $E'$ and $F'$ along with the tautological map $F' \rightarrow E' \otimes V$ are $G$-equivariant hence descend to bundles $E$ and $F$ on $N$ and a map $A : F \rightarrow E \otimes V$ of vector bundles.

The embedding $t : N \rightarrow \text{Grass}_3(S_2V)$ identifies the bundles $E = t^*U$, where $U$ is the tautological subbundle on Grass$_3(S_2V)$, and the map $A$ sits inside the complex

$$(3.3) \quad 0 \rightarrow F \xrightarrow{A} E \otimes V \xrightarrow{m} S_3V,$$

where $m$ is the multiplication map. In fact, $(3.3)$ is exact outside the locus of nets of type $<x_0^2, x_0x_1, x_0x_2>$, hence, by analytic continuation, the complex uniquely determines $F$, in the sense that if $G$ is a rank two bundle on $N$, and if $G \rightarrow E \otimes V$ is a map of vector bundles such that $0 \rightarrow G \rightarrow E \otimes V \xrightarrow{m} S_3V$ is a complex, then $F$ and $G$ are isomorphic.

The differential of the quotient map $\text{Hom}(F, E \otimes V)^{ss} \rightarrow N$ gives rise to an exact sequence

$$(3.4) \quad 0 \rightarrow \mathcal{O}_N \rightarrow \text{End}(E) \oplus \text{End}(F) \rightarrow \text{Hom}(F, E \otimes V) \rightarrow TN \rightarrow 0,$$

where local sections $(g, h)$ of $\text{End}(E) \oplus \text{End}(F)$ are sent to $(g \otimes id_V) \circ A - A \circ h$.

3.3. A torus action. Let $T \subset GL(V)$ be a maximal torus and consider the action of $T$ on $V$. It induces an action on $N$ and the bundles $E$ and $F$. Let $\{x_0, x_1, x_2\}$ be a basis for $V$ such that the action of $T$ on $V$ is diagonal, and let $\lambda_i$ the corresponding characters so that $t \cdot x_i = \lambda_i(t)x_i$ for all $t \in T$. The action on $P(V)$ has isolated fixpoints $\{(x_0, x_1), (x_0, x_2), (x_1, x_2)\}$ and isolated fixlines $\{(x_0), (x_1), (x_2)\}$. The blow-up map $(3.2)$ is $T$-equivariant, hence if $\alpha \in N$ is a fixpoint, the locus of base points $B(E_\alpha)$ of the net $E_\alpha$ must be supported on the triangle of fixpoints and fixlines in $P(V)$. If $B(E_\alpha)$ is not a line, then it is projectively equivalent to one of the following types: three distinct points, the union of a point doubled on a line and another point on the line, or the full first-order neighborhood of a point.

In Table II a list of the fixpoints in $N$ is presented (up to a permutation of $(x_0, x_1, x_2)$). For each fixpoint $\alpha$ the table contains a matrix representative, generators for the vector spaces $E_\alpha$ and $F_\alpha$, a description of $B(E_\alpha)$, and the isotropy group of the $S_3$-action on the fixpoints induced from permutation of $(x_0, x_1, x_2)$.

Remark. Up to projective equivalence there are five type of points in $N$: the four in Table II, and the net $<x_1^2, x_1x_2, x_2^2 + x_0x_1>$, whose locus of base points is a point with a second-order direction.

Let $\gamma : C^* \rightarrow T$ be a one-parameter subgroup such that $\lambda_i(\gamma(t)) = t^{\omega_i}$ for integer weights $\{\omega_0, \omega_1, \omega_2\}$, and consider the induced $C^*$-action on $N$. The weights can be chosen such that the fixpoints of the $C^*$-action are the same as the fixpoints of the $T$-action. Since the fixpoints are isolated we can apply the Bialynicki-Birula theorem to $N$. Counting the
Table 1. Fixpoints

| Matrix | $\mathcal{E}_\alpha$ | $\mathcal{F}_\alpha$ | Isotropy | $B(\mathcal{E}_\alpha)$ |
|--------|-----------------------|-----------------------|----------|--------------------------|
| $(x_1 \ 0)$ \ $(x_2 \ x_2)$ \ $(0 \ x_0)$ | $x_1 x_2$ \ $x_0 x_1$ \ $x_0 x_2$ | $x_1 x_2 \otimes x_0 - x_0 x_1 \otimes x_2$ \ $x_0 x_1 \otimes x_2 - x_0 x_2 \otimes x_1$ | $S_3$ | $\cdot \cdot$ |
| $(x_1 \ 0)$ \ $(x_0 \ x_1)$ \ $(0 \ x_2)$ | $x_2^2$ \ $x_1 x_2$ \ $x_0 x_2$ | $x_2^2 \otimes x_2 - x_1 x_2 \otimes x_1$ \ $x_1 x_2 \otimes x_0 - x_0 x_2 \otimes x_1$ | $\{id\}$ | $\cdot \rightarrow$ |
| $(x_1 \ 0)$ \ $(x_1 \ x_2)$ \ $(0 \ x_2)$ | $x_2^2$ \ $x_1 x_2$ \ $x_0 x_2$ | $x_2^2 \otimes x_2 - x_1 x_2 \otimes x_1$ \ $x_1 x_2 \otimes x_0 - x_0 x_2 \otimes x_1$ | $x_1 \leftrightarrow x_2$ | $\circ$ |
| $(x_1 \ 0)$ \ $(0 \ x_1)$ \ $(x_0 \ x_2)$ | $x_1^2$ \ $x_1 x_2$ \ $x_0 x_1$ | $x_1^2 \otimes x_2 - x_1 x_2 \otimes x_1$ \ $x_1 x_2 \otimes x_0 - x_0 x_1 \otimes x_1$ | $x_0 \leftrightarrow x_2$ | $\rightarrow$ |

In Table 1 we find 13, and the weights of the $T$-action on the tangent spaces at fixpoints can be found by using (3.4). In fact, in Section 7 we shall look closer at how such computations are performed.

**Theorem 3.3.**

i) $\chi(N) = 13$

ii) The Betti numbers $b_p = \dim A^p(N)$ are

$$(b_0, b_1, \ldots, b_6) = (1, 1, 3, 3, 3, 1, 1).$$

**Remark.** In [Dr] a recurrence formula can be found for the Betti numbers of $N(q; n, m)$ when $n$ and $m$ are relatively prime.

3.4. **Multiplicative structure of $A^*(N)$.** In [ES2] a detailed computation of $A^*(N(4; 2, 3))$ can be found. The same method works for our case.

**Proposition 3.4.** The Chow ring $A^*(N)$ is generated as a $\mathbb{Z}$-algebra by the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. Their monomial values in $A^6(N)$ are (with $\gamma_i = c_i(\mathcal{E})$ and $\delta_i = c_i(\mathcal{F})$) as in Table 2.

**Remark.** Since $\wedge^3 E' \simeq \wedge^2 F'$ as $G$-bundles, $c_1(\mathcal{E}) = c_1(\mathcal{F})$. 
Table 2. Monomial values

| \( \gamma_1 \gamma_2 \) | \( \gamma_1 \delta_2 \) | \( \gamma_2 \gamma_3 \) | \( \gamma_1^2 \gamma_2 \) | \( \gamma_1^2 \delta_2 \) | \( \gamma_1 \delta_2 \) | \( \gamma_1 \gamma_2 \gamma_3 \) |
|----------------|-------|----------------|----------------|----------------|-------|----------------|
| 27             | 18    | 5             | 14             | 9              | 6     | 3             |
| 2             | 9     | 5             | 3              | 1              | 57    | 2             |

Proof. First, we find some generators of \( A^*(N) \), then we look at relations among them, and finally we describe the class of a point.

Generators. Let

\[ \eta: P = \text{Isom}(E, E) \times_N \text{Isom}(F, F) \to N \]

be the principal \( GL(E) \times GL(F) \)-bundle associated to \( E \) and \( F \). It is a result of Grothendieck [Gr] that \( \eta^*: A^*(N) \to A^*(P) \) is surjective with kernel generated by \( c_i(E) \) and \( c_i(F) \), \( i > 0 \). As \( \eta \) factors through the quotient map \( \text{Hom}(F, E \times V)^{ss} \to N \), and \( A^*(\text{Hom}(F, E \otimes V)^{ss}) \simeq \mathbb{Z} \), this shows that \( c_i(E), c_i(F) \) generate \( A^*(N) \) over \( \mathbb{Z} \).

Relations. We find relations among the generators by displaying \( A^*(N) \) as the quotient of a polynomial ring.

First let \( F = F(E) \times_N P(F) \xrightarrow{\rho} N \) be the fiber product of the flag bundles with universal flags

\[ \rho^*E = E^3 \twoheadrightarrow E^2 \twoheadrightarrow E^1 \twoheadrightarrow E^0 = 0 \]

and

\[ \rho^*F = F^2 \twoheadrightarrow F^1 \twoheadrightarrow F^0 = 0. \]

Let \( R = \mathbb{Z}[e_1, e_2, e_3, f_1, f_2] \) and \( R^S = \mathbb{Z}[\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2] \) be polynomial rings where \( \gamma_i \) and \( \delta_i \) are the elementary symmetric polynomials in the \( e_i \) and \( f_i \), respectively. Note that \( R \) is a free \( R^S \)-module with basis

\[ B = \{ e_1^i e_2^j f_1^k \mid 0 \leq i \leq 2, 0 \leq j, k \leq 1 \}. \]

Consider the following commutative diagram of ring homomorphisms:

\[
\begin{array}{ccc}
R^S & \xrightarrow{\lambda} & R \\
\downarrow & & \downarrow \mu \\
A^*(N) & \xrightarrow{\rho^*} & A^*(F)
\end{array}
\]

where:

\( \rho^* \) is the injective pullback map.
\( \iota \) is the inclusion map.
\( \lambda \) maps \( \gamma_i, \delta_j \) to the Chern classes \( c_i(E), c_j(F) \), respectively.
\( \mu \) maps \( e_i, f_j \) to the Chern roots \( c_1(E^i) - c_1(E^{i-1}), c_1(F^i) - c_1(F^{i-1}) \), respectively.

For \( r \in R \) let \( c(r) \subseteq R^S \) be the ideal generated by the coordinates of \( r \) with respect to the basis in \( B \). For \( J \subseteq R \) an ideal, let \( c(J) \) be the ideal generated by \( \{ c(r) \mid r \in J \} \). A short argument using the surjectivity of \( \mu \) and the commutativity of (3.3) yields \( \ker(\lambda) = c(\ker(\mu)) \).

Claim 1. \( \ker(\mu) \) contains the following elements:
i) \( r_0 = e_1 + e_2 + e_3 - f_1 - f_2 \)

ii) \( r_1 = (e_1 - f_1)^3(e_1 - f_2)^3 \)

iii) \( r_2 = (e_1 - f_2)^3(e_2 - f_2)^3 \).

Proof. i) follows from \( c_1(E) = c_1(F) \).

For ii) and iii) consider the maps

\[
\begin{align*}
\rho^*F & \xrightarrow{A} \rho^*F \\
\rho^*E \otimes V & \xrightarrow{A} E^1 \otimes V \\
\rho^*E \otimes V & \xrightarrow{A} E^2 \otimes V
\end{align*}
\]

where \( F^{(1)} = \ker(\rho^*F \rightarrow F^1) \). The stability condition iv) of Proposition 3.1 prevents the maps \( \rho^*F \rightarrow E^1 \otimes V \) and \( F^{(1)} \rightarrow \rho^*E^2 \otimes V \) to have any zeros, hence

\[
0 = c_6(\rho^*F^\vee \otimes E^1 \otimes V) = \mu((e_1 - f_1)^3(e_1 - f_2)^3)
\]

\[
0 = c_6(F^{(1)} \vee \otimes \rho^*E^2 \otimes V) = \mu((e_1 - f_2)^3(e_2 - f_2)^3)
\]

\[\square\]

Claim 2. Let \( a \) be the ideal generated by \( \{r_i\} \), \( i = 0, 1, 2 \). Then

\[
A^*_Q(N) = (R^5/a) \otimes z Q.
\]

Proof. See [ES2], Theorem 6.9.

Class of a point. We need to express the class of a point in terms of the generators \( c_i(E) \), \( c_i(F) \). By Gauss-Bonnet, \( 13 = \chi(N) = \deg c_6(TN) \), hence the class of a point is \( (1/13)c_6(TN) \). A formal Chern class computation, using the exact sequence (3.4), yields an expression of desired kind for \( c_6(TN) \).

The numbers in Table 2 are obtained from the above description using the Schubert package in MAPLE [KS].

3.5. Calabi-Yau sections. From the resolution (3.4) it follows that \( c_1(TN) = 3p \) where \( p = -c_1(E) \) is ample, hence \( N \) is Fano of index 3.

Consider the following bundles:

\[
3O(p), \quad F^\vee \oplus O(2p), \quad S_2 F^\vee.
\]

All three are generated by their global sections and have 3p as their first Chern class. Hence zero schemes of general global sections of the above bundles are Calabi-Yau threefolds. We shall denote these by \( X, Y, \) and \( Z \), respectively.

Questions on the enumerative geometry of rational curves on these Calabi-Yaus, lead to questions about \( \overline{M}_{0,0}(N, d) \). For the time being we simply record that the expected dimension of this moduli space is \( 3 + 3d \).

4. Lines on \( N \)

We show that the functor of stable maps \( \overline{M}_{0,0}(N, 1) \) is represented by a \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}(V) \times \mathbb{P}(V^\vee) \). This enables us to find an explicit representation of the Chow ring \( A^*(\overline{M}_{0,0}(N, 1)) \) in terms of generators and relations. As an application we calculate the number of lines on the Calabi-Yau threefold sections.

---

4 This is of course the same as the Hilbert functor of lines in \( N \).
4.1. The moduli space. The idea is as follows: A line \( L \subseteq \text{Grass}_2(S_2V) \) is determined by a pencil \( P \subseteq S_2V \) and a web \( W \subseteq S_2V \) such that \( P \subseteq W \). The line \( L \) parametrizes the family of nets \( P \subseteq E_t \subseteq W \). From the relationship between \( N \) and \( \text{Hilb}_2^P(V) \), it follows that the line \( L \) is contained in \( N \) if and only if \( \{ B(E_t) \mid t \in L \} \) describes a (non-constant) family of lines/three points in \( P(V) \). As \( B(W) \subseteq B(E_t) \subseteq B(P) \) for all \( t \in L \), this implies that \( L \subseteq N \) if and only if both i) and ii) hold:

i) \( B(P) \) is one dimensional. As \( B(P) \) is the intersection of two conics, \( B(P) \) must be the union of a point \( p \) and a line \( l \), with the point possibly embedded onto the line.

ii) \( W/P \) is a pencil of conics through \( p \), so \( W/P \) corresponds to a point in \( P(H^0(I_p(2))/H^0(I_{p,l}(2))) \).

We show that this correspondence can be relativized.

Let \( PP^\vee = P(V) \times P(V^\vee) \) and let \( \mathcal{P} = P(V) \times PP^\vee \xrightarrow{pr} PP^\vee \) be the projection map. Let \( \mathcal{O}_{pr}(1) \) denote the tautological line bundle. Consider the universal line and point in \( P(V) \)

\[
\tilde{\ell} = \{(x, p, l) \in P \mid x \in l\} \quad \text{and} \quad \tilde{p} = \{(x, p, l) \in P \mid x = p\},
\]

and the exact sequence

\[
0 \to \mathcal{I}_{\tilde{\ell}}(2) \to \mathcal{I}_{\tilde{p}}(2) \to \mathcal{I}_{\tilde{p}} \otimes \mathcal{O}_l(2) \to 0.
\]

Since the sheaves in this sequence are flat, and \( H^1(\mathcal{I}_{\tilde{\ell}}(2)) = H^1(\mathcal{I}_{\tilde{p}}(2)) = H^1(\mathcal{I}_{\tilde{p}} \otimes \mathcal{O}_l(2)) = 0 \) for arbitrary point-line pairs \((p, l)\) in \( P(V) \), base change theorems in [1] imply that

\[
0 \to pr_*\mathcal{I}_{\tilde{\ell}}(2) \to pr_*\mathcal{I}_{\tilde{p}}(2) \to \mathcal{V} \to R^1pr_*\mathcal{I}_{\tilde{p}}(2),
\]

with \( \mathcal{V} = pr_*(\mathcal{I}_{\tilde{p}} \otimes \mathcal{O}_l(2)) \), is an exact sequence of vector bundles. Further, the fibers of \( \mathcal{V} \) over point-line pairs \((p, l)\) are \( H^0(\mathcal{I}_{\tilde{p}} \otimes \mathcal{O}_l(2)) = H^0(\mathcal{I}_{\tilde{p}}(2))/H^0(\mathcal{I}_{\tilde{p},l}(2)) \).

**Theorem 4.1.** The functor of stable maps \( \overline{M}_{0,0}(N, 1) \) is represented by \( P(V) \). In particular, \( \overline{M}_{0,0}(N, 1) \) is smooth and of dimension 6.

We shall actually prove a more general version which will be useful later. If \( f : P^1 \to C \) is a parametrization of a smooth curve \( C \) of degree \( d \) in \( N \), then the pullback of \( \mathcal{E} \) decomposes into parts

\[
\mathcal{E}_{P^1} \simeq \mathcal{O}_{P^1}(-a_1) \oplus \mathcal{O}_{P^1}(-a_2) \oplus \mathcal{O}_{P^1}(-a_3)
\]

such that \( d = \sum a_i \) and \( a_i \geq 0 \) for all \( i \). Let \( H_d \) denote the components of \( \text{Hilb}_N^{d+1} \) containing the set of rational curves of degree \( d \), and let \( H^0,d \) denote the closure in \( H_d \) of the locus of smooth curves with splitting type \((0, 0, d)\). The scheme \( H^{0,d} \) comes equipped with a universal family of degree \( d \) curves \( C_d \subseteq H^{0,d} \times N \). In particular, \( H^{0,1} = \overline{M}_{0,0}(N, 1) \) and \( C_1 = \overline{M}_{0,1}(N, 1) \).

Consider the Hilbert scheme \( P(S_dV) \) parametrizing flat families of degree \( d \) curves in \( P(V^\vee) \) over \( PP^\vee \) and let \( C'_d \subseteq P(S_dV) \times PP^\vee P(V^\vee) \) be the universal family. In the proof of the theorem we show that \( P(S_dV) \) and \( H^{0,d} \) are naturally isomorphic when \( d = 1, 2 \).

Before we start, we introduce some more notations: Let

\[
\rho : P(S_dV) \to PP^\vee \quad \text{and} \quad \rho^\vee : P(V^\vee) \to PP^\vee
\]

denote the projection maps, and let \( \mathcal{O}(\tilde{\sigma}) \) denote the tautological line bundle on \( P(V^\vee) \).

**Proof.** The morphism

\[
P(V^\vee) \to N,
\]
which maps $V^\vee_T \to \mathcal{O}_T(\hat{\sigma})$ to nets $pr_*\mathcal{L}_p\mathcal{L}_i(2) \oplus \mathcal{O}_T(-\hat{\sigma})$ is surjective, embeds each fiber $\mathbb{P}(V_{(p,l)}^\vee)$ onto planes in $N$, and induces morphisms

\begin{equation}
\mathbb{P}(S_dV) \to H^{0,d}
\end{equation}

for $d = 1, 2$. From our initial analysis of the lines in $N$, it follows that the morphism (4.5) is surjective for $d = 1$. This argument can easily be extended to the case $d = 2$.

Consider the induced morphisms $\mathbb{P}(S_dV) \times_{\mathbb{P}^1} \mathbb{P}(V^\vee) \to H^{0,d} \times N$. To prove that (4.5) is an isomorphism we must show that the universal family $C_d$ lifts to the universal family $C_d'$. We shall prove that the images $P_l \subseteq N$ and $P_2 \subseteq N$ of two different fibers $\mathbb{P}(V_{(p_1,l_1)}^\vee)$ and $\mathbb{P}(V_{(p_2,l_2)}^\vee)$ are such that either $P_1 \cap P_2$ is empty or $P_1 \cap P_2$ is a point. It then follows that $C_d'$ is the unique component of dimension $\dim C_d$ in the preimage of $C_d$.

If $[N]$ is a point in $P_l$ then either $B(N)$ is the line $l_i$, or $B(N)$ is supported on $p_i$ and a length two subscheme of $l_i$. Keeping this scenario in mind, it is easy to deduce the following (assuming $(p_1,l_1) \neq (p_2,l_2)$):

- If $l_1 = l_2$, then $P_1 \cap P_2$ is empty unless $p_1 \in l_1$ and $p_2 \in l_2$. If so, $[N] \in P_1 \cap P_2$ if and only if $B(N)$ is the line $l_1 = l_2$, hence $P_1 \cap P_2$ is a point.
- If $l_1 \neq l_2$, then $P_1 \cap P_2$ is empty unless $p_1 \in l_2$ and $p_2 \in l_1$. If so, there are three cases to consider:
  - i) $p_1 \notin l_1$ and $p_2 \notin l_2$,
  - ii) $p_1 \notin l_1$ and $p_2 \in l_2$,
  - iii) $p_1 = p_2 = l_1 \cap l_2$.

In all three cases, if $[N] \in P_1 \cap P_2$, then $B(N)$ must contain the point $l_1 \cap l_2$. Without loss of generality we may assume that $l_1 = (x_1)$, $l_2 = (x_2)$, $p_1, p_2 \in (x_0)$ in i), and $p_1 \in (x_0)$ in ii). It is straightforward to verify that in each case $P_1 \cap P_2$ consists of only one point. These represent the nets:

i) $<x_0x_1, x_1x_2, x_0x_2>$,  
ii) $<x_0x_1, x_1x_2, x_2^2>$,  
iii) $<x_1^2, x_1x_2, x_2^2>$.

\[ \square \]

**Remark 1.** Note that if $P_1$ and $P_2$ is as in the proof, and their intersection is non-empty, then the union $P_1 \cup P_2$ is determined up to projective equivalence by the intersection point, which again must be a point of type 1-4) in Table I.

**Remark 2.** On $\mathbb{P}(V^\vee)$ there is a natural $T$-equivariant complex

\begin{equation}
0 \to pr_*\mathcal{I}_l(1) \otimes (\wedge^2 pr_*\mathcal{I}_p(1) \oplus \mathcal{O}(-\hat{\sigma})) \to (pr_*\mathcal{I}_p\mathcal{L}_i(2) \oplus \mathcal{O}(-\hat{\sigma})) \otimes V \xrightarrow{m} S_3V,
\end{equation}

so the morphism (1.4) identifies the bundles

\begin{equation}
\mathcal{E}_{\mathbb{P}(V^\vee)} = pr_*\mathcal{I}_p\mathcal{L}_i(2) \oplus \mathcal{O}(-\hat{\sigma})
\end{equation}

\begin{equation}
\mathcal{F}_{\mathbb{P}(V^\vee)} \simeq pr_*\mathcal{I}_l(1) \otimes (\wedge^2 pr_*\mathcal{I}_p(1) \oplus \mathcal{O}(-\hat{\sigma})).
\end{equation}

4.2. **The Chow ring** $A^\ast(\overline{M}_{0,0}(N,1))$. Let $\mathcal{O}(\tau)$ and $\mathcal{O}(\hat{\tau})$ be the tautological line bundles on $\mathbb{P}(V)$ and $\mathbb{P}(V^\vee)$, and let $\mathcal{O}(\sigma)$ be the tautological bundle on $\mathbb{P}(V)$. Then

\[ A^\ast(\mathbb{P}(V) \times \mathbb{P}(V^\vee)) = \mathbb{Z}[\tau, \hat{\tau}/(\tau^3, \hat{\tau})]. \]

Since $\overline{M}_{0,0}(N,1)$ is a projective bundle we have, by Grothendieck’s theorem [I],

\begin{equation}
A^\ast(\overline{M}_{0,0}(N,1)) = \mathbb{Z}[\tau, \hat{\tau}, \sigma]/(\tau^3, \hat{\tau}, r),
\end{equation}

where $r = \sum_{i=0}^{3} (-1)^i \rho^i c_i(\mathcal{V}) \sigma^{3-i}$. 

We shall determine the total Chern class $c(V)$ in terms of the generators $\tau$ and $\tilde{\tau}$.

Let $p_1: \mathbf{P} \to \mathbf{P}(V)$ and $p_2: \mathbf{P} \to \mathbf{P}(V)$ be the projection maps on the first and second components. On $\mathbf{P}$ there are exact sequences

$$
0 \longrightarrow p_2^*\Omega(\tau) \stackrel{\alpha}{\longrightarrow} V_{\mathbf{P}} \longrightarrow \mathcal{O}(\tau) \longrightarrow 0
$$

(4.9)

$$
\begin{array}{c}
0 \\
\downarrow \\
p_1^*\Omega(1) \\
\downarrow \\
\tau \\
\downarrow \\
\mathcal{O}_{pr}(1) \\
\downarrow \\
0
\end{array}
$$

where $\Omega$ is the sheaf of Kähler differentials on $\mathbf{P}(V)$. The map $s = \beta \circ \alpha: p_2^*\Omega(\tau) \to \mathcal{O}_{pr}(1)$ induces a section of $\Omega(\tau)^{\vee} \otimes \mathcal{O}_{pr}(1)$ whose zero scheme is $\tilde{p}$. Since $s$ is regular, the twisted Koszul complex

$$
0 \longrightarrow \wedge^2 p_2^*\Omega(\tau) \otimes \mathcal{O}_{pr}(-1) \rightarrow p_2^*\Omega(\tau) \rightarrow I_{\tilde{p}}(1) \rightarrow 0
$$

(4.10)

is exact. Pushing down this sequence we find $pr_*I_{\tilde{p}}(1) \simeq \Omega(\tilde{\tau})$, and twisting it by $\mathcal{O}_{pr}(1)$ before pushing down yields an exact sequence

$$
0 \rightarrow \wedge^2 \Omega(\tau) \rightarrow \Omega(\tau) \otimes V \rightarrow pr_*I_{\tilde{p}}(2) \rightarrow 0 .
$$

(4.11)

Using a similar diagonal argument as in (4.9), one can show that $I_{\tilde{f}}(1) \simeq \mathcal{O}(-\tilde{\tau})$. Consequently $I_{\tilde{p}}I_{\tilde{f}}(2) \simeq I_{\tilde{p}}(1) \otimes \mathcal{O}(-\tilde{\tau})$, so

$$
pr_*I_{\tilde{f}}(1) \simeq \mathcal{O}(-\tilde{\tau}) \quad \text{and} \quad pr_*I_{\tilde{p}}I_{\tilde{f}}(2) \simeq pr_*I_{\tilde{p}}(1) \otimes \mathcal{O}(-\tilde{\tau}).
$$

(4.12)

The exact sequence (4.12), along with (4.11) and (4.12) results in

$$
c(V) = c(pr_*I_{\tilde{p}}(2))c(pr_*I_{\tilde{p}}I_{\tilde{f}}(2))^{-1}
$$

(4.13)

$$
= c(\Omega(\tau))^3c(\wedge^2 \Omega(\tau))^{-1}c(\Omega(\tau) \otimes \mathcal{O}(-\tilde{\tau}))^{-1},
$$

where $c(\Omega(\tau)) = 1 - \tau + \tau^2$.

4.3. Lines on the Calabi-Yau threefolds. Recall the construction of Section 2.7. Let

$$
\begin{array}{c}
\overline{\mathcal{M}}_{0,1}(N,1) \\
\downarrow \\
\pi \\
\overline{\mathcal{M}}_{0,0}(N,1)
\end{array}
$$

be the universal map, and let $\mathcal{B}$ denote any of the three bundles $3\mathcal{O}(p)$, $\mathcal{F}^{\vee} \oplus \mathcal{O}(2p)$ and $S_2\mathcal{F}^{\vee}$. Then $\mathcal{B}_1 = \pi_*c_1^*\mathcal{B}$ is locally free with fibers $H^0(\mathbf{P}^1,f^*\mathcal{B})$ over degree one maps $f: \mathbf{P}^1 \to N$, and the virtual numbers of lines on the Calabi-Yau sections $X, Y, Z$ are given by the integrals

$$
\int_{\overline{\mathcal{M}}_{0,0}(N,1)} c_6(3\mathcal{O}(p)_1), \quad \int_{\overline{\mathcal{M}}_{0,0}(N,1)} c_6(\mathcal{F}^{\vee}_1 \oplus \mathcal{O}(2p)_1), \quad \int_{\overline{\mathcal{M}}_{0,0}(N,1)} c_6(S_2\mathcal{F}^{\vee}_1),
$$

(4.14)
Lemma 4.3. The actual numbers of lines.

Theorem 4.2. The desired expressions for the integrands can now be obtained from formal Chern class computations using the identifications (4.12), (4.13), and (4.17). These computations and the evaluations of the integrals are effectively done using the Schubert package in MAPLE package in MAPLE.

Theorem 4.2. The number of lines on the Calabi-Yau threefolds $X$, $Y$, and $Z$ which are given by the integrals (4.14) are 147, 216, and 144, respectively.

There remains to show that the virtual numbers given by the integrals (4.14) are in fact the actual numbers of lines.

Lemma 4.3. If $B$ is one of the bundles $3O(p)$, $F^\vee \oplus O(2p)$ or $S_2 F^\vee$ then the map

$$H^0(N, B)_{\overline{M}_{0,0}(N,1)} \xrightarrow{\nu} B_1$$

is surjective.

The proof of Theorem 4.2 follows from this. Indeed, if $s \in H^0(N, B)$, then $V(\nu(s)) \subseteq \overline{M}_{0,0}(N,1)$ describes the locus of lines on the zero scheme $V(s) \subseteq N$. From the lemma we have that $B_1$ is generated by $H^0(N, B)$, hence, by Kleiman’s Bertini theorem [K], it follows that for a general global section $s$, $V(\nu(s))$ is nonsingular of codimension $\text{rank}(B_1)$. Since $\text{rank}(B_1) = \dim \overline{M}_{0,0}(N,1)$, the number of points is finite and given by

$$\int_{\overline{M}_{0,0}(N,1)} c_0(B_1).$$

Further, the section $s$ can be chosen such that $\nu(s)$ has zeros outside any closed subset, for instance the subset $P(V) \subseteq \overline{M}_{0,0}(N,1)$, where $I \subseteq P(V) \times P(V^\vee)$ is the incidence correspondence.
Proof. Let \( f: \mathbb{P}^1 \to \mathbb{N} \) be a map of degree one. It is enough to show surjectivity on the fibers of \((4.18)\)
\[
H^0(\mathbb{N}, \mathcal{E}) \to H^0(\mathbb{P}^1, f^*\mathcal{E}).
\]
Let \( pl: \mathbb{N} \to \mathbb{P}^{19} \) be the Plücker embedding. For all \( m \), \( pl^*\mathcal{O}_{\mathbb{P}^{19}}(m) = \mathcal{O}(mp) \), and the map
\[
H^0(\mathbb{P}^{19}, \mathcal{O}_{\mathbb{P}^{19}}(m)) \to H^0(\mathbb{P}^1, f^*\mathcal{O}(mp))
\]
factors through the map
\[
H^0(\mathbb{N}, \mathcal{O}(m)) \to H^0(\mathbb{P}^1, f^*\mathcal{O}(mp)).
\]
The map \((4.21)\) is surjective for \( m = 1 \) and \( 2 \), hence so is \((4.21)\).

Next, since \( \deg f^*\mathcal{F}^\vee = 1 \) and \( f^*\mathcal{F}^\vee \) is generated by its global sections
\[
(4.22) \quad S_2V^\vee \otimes V^\vee \to H^0(\mathbb{P}^1, f^*\mathcal{F}^\vee)
\]
is surjective, where \((4.22)\) is induced from the inclusion \( \mathcal{F} \to S_2V \otimes V_\mathbb{N} \). The map \((4.22)\) factors through \( H^0(\mathbb{N}, \mathcal{F}^\vee) \to H^0(\mathbb{P}^1, f^*\mathcal{F}^\vee) \), so this map is also surjective.

Finally, \( H^0(\mathbb{P}^1, f^*S_2\mathcal{F}^\vee) = S_2H^0(\mathbb{P}^1, f^*\mathcal{F}^\vee) \), so right exactness of \( S_2 \) implies that
\[
(4.23) \quad S_2(V^\vee \otimes V^\vee) \to H^0(\mathbb{P}^1, f^*S_2\mathcal{F}^\vee)
\]
is surjective. The map \((4.23)\) factors through \( H^0(\mathbb{N}, S_2\mathcal{F}^\vee) \to H^0(\mathbb{P}^1, f^*S_2\mathcal{F}^\vee) \), and so the claim follows. \( \square \)

4.4. Fixlines. The torus action \( T \) on \( \mathbb{N} \) induces an action on \( \overline{M}_{0,0}(\mathbb{N}, 1) \). The projection
\[
\rho: \overline{M}_{0,0}(\mathbb{N}, 1) \to \mathbb{P}(V) \times \mathbb{P}(V^\vee)
\]
is \( T \)-equivariant. Let \( L \subseteq \mathbb{N} \) be a line in \( \mathbb{N} \). If \( [L] \in \overline{M}_{0,0}(\mathbb{N}, 1) \) is a fixpoint, then \( \rho([L]) = (p, l) \) must correspond to a fixpoint and a fixline in \( \mathbb{P}(V) \). There are two projective equivalence classes for the point-line pair, depending on whether the point is incident to the line or not. Up to the \( S_3 \)-action we may assume \( p = (x_1, x_2) \) and \( l = (x_0) \), or \( l = (x_1) \). In addition, the line \( L \subseteq \mathbb{P}(V^\vee_{\{p,l\}}) \) must be fixed. For each choice of \((p, l)\) there are three fixlines in \( \mathbb{P}(V^\vee_{\{p,l\}}) \). Since the lines \( <x_0x_1, x_0x_2, x_1^2 + tx_0x_2> \) and \( <x_0x_1, x_0x_2, x_2^2 + tx_0x_1> \) are swapped by the permutation \( x_1 \leftrightarrow x_2 \), this leaves us with five \( S_3 \)-classes of fixlines. Table 3 contains a complete list of the fixlines in \( \overline{M}_{0,0}(\mathbb{N}, 1) \), up to the \( S_3 \)-action on \( V \), along with the isotropy subgroups of this action.

Remark. Up to projective equivalence there are six types of lines on \( \mathbb{N} \): The five described above and the line
\[
\text{Type 6)} \quad <x_1^2, x_1x_2, x_2^2 + x_0x_1 + tx_0x_2> .
\]

4.5. Positivity. Let \( L \) be a line in \( \mathbb{N} \). Since \( L \cong \mathbb{P}^1 \), the vector bundle \( TN_L \) decomposes into a sum of line bundles. In Appendix 3 we show that
\[
TN_L = \begin{cases} \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus 4\mathcal{O} & \text{if } L \text{ is of type 1)-2) } \\ \mathcal{O}(2) \oplus 2\mathcal{O}(1) \oplus 4\mathcal{O} \oplus \mathcal{O}(-1) & \text{if } L \text{ is of type 3)-6) } \end{cases}
\]
In particular, we see that \( \mathbb{N} \) is not convex, since \( H^1(f^*TN) \neq 0 \) if \( f: C \to \mathbb{N} \) is any multiple cover of a line of type 3)-6).
Table 3. Fixlines

| Type | $\mathcal{E}_t$ | Isotropy |
|------|----------------|-----------|
| 1    | $<x_0 x_1, x_0 x_2, x_1^2 + tx_2^2>$ | $x_1 \leftrightarrow x_2$ |
| 2    | $<x_0 x_1, x_0 x_2, x_1 x_2 + tx_1^2>$ | $\{id\}$ |
| 3    | $<x_1^2, x_1 x_2, x_0 x_1 + tx_2^2>$ | $\{id\}$ |
| 4    | $<x_1^2, x_1 x_2, x_2^2 + tx_0 x_2>$ | $\{id\}$ |
| 5    | $<x_1^2, x_1 x_2, x_0 x_2 + tx_0 x_1>$ | $\{id\}$ |

5. **Quantum $D$-module of $N$**

We now put the machinery of Section 2 to use. Our goal is to compute the instanton numbers of rational curves for the Calabi-Yau section $X$. This enables us (in principle) to determine all virtual numbers of rational curves on $X$. Remarkably, the only geometrical input this procedure requires is the Chow ring $A^*(N)$, the resolution (3.4), and the description of the fixlines, Table 3.

In what follows $T_0, \ldots, T_{12}$ will refer to the basis of $A^*_Q(N)$ where $T_0 = 1$, and the rest is as in Table 4, with $\gamma_i = (-1)^i c_i(\mathcal{E})$ and $\delta_2 = c_2(\mathcal{F})$. The sign $(-1)^i$ makes the classes effective. Note that $T_{11}$ is the class of a line, and $T_{12}$ is the class of a point. As before, $p = \gamma_1$, $T^j$ denotes the dual generators, and $t_0$ (resp. $t$) are coordinates on $A^0_C(N)$ (resp. $A^1_C(N)$).

Table 4. Basis

| $T_1 = \gamma_1$ | $T_2 = \gamma_1^2$ | $T_3 = \gamma_2$ | $T_4 = \delta_2$ | $T_5 = \gamma_1^3$ | $T_6 = \gamma_1 \gamma_2$ |
|-----------------|-------------------|-----------------|-----------------|-------------------|-----------------|
| $T_7 = \gamma_1 \delta_2$ | $T_8 = \gamma_1^4$ | $T_9 = \gamma_1^2 \gamma_2$ | $T_{10} = \gamma_1^2 \delta_2$ | $T_{11} = \frac{1}{\delta_2} \gamma_1^5$ | $T_{12} = \frac{1}{\delta_2} \gamma_1^6$ |

5.1. **The $A$-connection.** We begin by determining the system of first order differential equations

\[
\hbar \frac{\partial}{\partial t_0} F = p \ast F, \quad \hbar \frac{\partial}{\partial t} F = p \ast F
\]
for vector valued functions $F: A_C^0(N) \oplus A_C^1(N) \to A_C^1(N)$. The operator $p * -$ will be represented as a matrix with respect to the basis $T_0, T_1, \ldots, T_{12}$.

Recall

$$p * T_a = \sum_d q^d \sum_b <p, T_a, T_b>_{0,d} T^b = \sum_d q^d \sum_b <p, T_a, T_b>_{0,d} g^{bc} T_c,$$

where the matrix $(g^{bc})$ is the inverse of the matrix $(g_{bc})$ with $g_{bc} = <T_b, T_c>$. By the mapping to a point and the divisor properties,

$$p * T_a = p \cdot T_a + \sum_{q>0} q^d \sum_{b,c} d <T_a, T_b>_{0,d} g^{bc} T_c.$$ (5.2)

The intersection product $p \cdot T_a = \sum_{b,c} (\int_N p T_a T_b) g^{bc} T_c$ is determined from the list of monomial values in Table 2. Hence, to determine (5.2) there remains to compute all two-point numbers $<T_a, T_b>_{0,d}$ for all $d$. However, $<T_a, T_b>_{0,d} = 0$ unless

$$\text{codim } T_a + \text{codim } T_b = \exp \dim \overline{M}_{0,2}(N, d) = 3d + 5.$$ (5.3)

**Theorem 5.1.** There are twelve two-point line numbers satisfying (5.3), with values as in Table 2, and one two-point conic number satisfying (5.3), with value

$$<T_{11}, T_{12}>_{0,2} = 1.$$ (5.4)

**Table 5. Two-point line numbers**

| $<T_2, T_{12}>_{0,1}$ | $<T_5, T_{11}>_{0,1}$ | $<T_8, T_8>_{0,1}$ | $<T_9, T_9>_{0,1}$ |
|-----------------------|-----------------------|---------------------|---------------------|
| 3                     | 8                     | 603                 | 121                 |

| $<T_3, T_{12}>_{0,1}$ | $<T_6, T_{11}>_{0,1}$ | $<T_8, T_9>_{0,1}$ | $<T_9, T_{10}>_{0,1}$ |
|-----------------------|-----------------------|---------------------|-----------------------|
| 0                     | 3                     | 270                 | 81                    |

| $<T_4, T_{12}>_{0,1}$ | $<T_7, T_{11}>_{0,1}$ | $<T_8, T_{10}>_{0,1}$ | $<T_{10}, T_{10}>_{0,1}$ |
|-----------------------|-----------------------|-----------------------|--------------------------|
| 0                     | 2                     | 180                  | 54                       |

The line numbers are computed using Bott’s formula on the moduli space $\overline{M}_{0,2}(N, 1)$. In Section 7.1 we explain this computation. The number $<T_{11}, T_{12}>_{0,2} = 1$ comes out as a solution of the WDVV-relations (see the remark below) once the two-point line numbers are known. Another way to compute this number is to use Bott’s formula over the space $\overline{M}_{0,0}(N, 2)$. In Section 6 we give a description of the space $\overline{M}_{0,0}(N, 2)$ which contains enough information for such a computation to be carried out.
Putting this data into (5.2) we find that $p^\ast -$ is represented by the matrix

$$
\begin{pmatrix}
3q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2q^2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2q^2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21q & 9q & 6q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -33q & -12q & -9q & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3q & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 57 & 27 & 18 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

with respect to the basis $\{T_0, T_1, \ldots, T_{12}\}$.

*Remark.* There is a computer program farsta written by Andrew Kresch [Kr] which can be quite useful for solving WDVV relations on the GW-level if the rank of the intersection ring is not too large. Suppose we have a scheme $S$ and a good description of $A^\ast (S)$. The program will derive all WDVV relations among the GW-invariants of $S$, and, once enough numbers are specified, start to solve the equations for new numbers. The number $<T_{11}, T_{12}>=1$ was first obtained by farsta.

### 5.2. The Picard-Fuchs operator

Suppose

$$
F = \sum_{i=0}^{12} F_i T_i = \begin{pmatrix} F_0 \\ \vdots \\ F_{12} \end{pmatrix}
$$

is a solution to the system of differential equations (5.1). Solving for $F_{12}$ as in Example 4 of Section 2.6 (with $\hbar = 1$), we find that $F_{12}$ is a solution to $QD_N = 0$, where $QD_N$ is the Picard-Fuchs operator

$$
\begin{align*}
D^7(299D^2 - 1035D + 907)(D - 1)^3 \\
- qD^3(10166D^6 + 5474D^5 - 7135D^4 - 5855D^3 + 1148D^2 + 2109D + 513) \\
+ q^2(-83122D^6 - 377246D^5 - 675645D^4 - 607063D^3 - 289727D^2 - 70962D - 7668) \\
- 243q^3(D + 1)(299D^2 + 1357D + 1551),
\end{align*}
$$

with $D = \partial/\partial t$.

*Remark 1.* The actual computation of $QD_N$ was carried out in MAPLE, using a procedure written by Ciocan-Fontanine and van Straten.

*Remark 2.* By the remark in Section 2.6, it follows that the homogeneous degree 12 part of $QD_N$, with $D = p$, is a relation in $QH^\ast (N)$. A closer inspection of the factors in this expression reveals that in fact

$$
p(q + p^3)(p^6 - 35qp^3 - 243q^2)
$$

is a relation.
5.3. Instantons. We use the quantum Lefschetz hyperplane principle to compute the instantons for the complete intersection Calabi-Yau threefold $X \subseteq \mathbb{N}$. It will be convenient to use a basis $\{p^2, k_1, k_2\}$ for $A_{\mathbb{C}}^2(\mathbb{N})$ such that $k_1 \cdot p^3 = 0 = k_2 \cdot p^3$. In other words $k_1$ and $k_2$ span ker$(i^*: A_{\mathbb{C}}^2(\mathbb{N}) \to A_{\mathbb{C}}^2(X))$. Let $p^{2\nu} \in A_{\mathbb{C}}^2(\mathbb{N})$ be such that $<p^2, p^{2\nu}>=1$ and $<k_1, p^{2\nu}>=0$ for $i = 1, 2$. Then $p^{2\nu} = (1/57)T_8$. Let $p' = i^*p$ denote the ample generator of Pic$(X)$.

First, we need to determine $I_0, I_1, I_2$ such that

$$
i I = i^* e^{(t_0 + pt)/\hbar}\left(1 + \sum_{d \neq 0} q^d \prod_{m=1}^d (p + mh)^3 e_{1*} \left(\frac{1}{\hbar - c}\right)\right)$$

$$= e^{t_0/\hbar}(I_0 + I_1 p' + I_2 p'^2 + I_3 p'^3).$$

The crucial point in what follows, is that the expression (with $\hbar = 1$)

$$s_{\mathbb{N}} = e^{(t_0 + pt)/\hbar}\left(1 + \sum_{d \neq 0} q^d e_{1*} \left(\frac{1}{\hbar - c}\right)\right)$$

is a vector solution to $QD_{\mathbb{N}}$. The first few components of $s_{\mathbb{N}}$ with respect to the new basis for $A_{\mathbb{C}}^2(\mathbb{N})$ are

$$s_{\mathbb{N}} = e^{t_0/\hbar}\left(\psi_0 + \left(\frac{t}{\hbar} \psi_0 + \psi_1\right)p + \left(\frac{t^2}{2\hbar^2} \psi_0 + \frac{t}{\hbar} \psi_1 + \psi_2\right)p^2 + <s_{\mathbb{N}}, k_1> k_1$$

$$+ <s_{\mathbb{N}}, k_2> k_2 + \cdots\right),$$

where

$$\psi_0 = \sum_{d > 0} q^d a_d \frac{d}{\hbar^{3d}}; \quad \psi_1 = \sum_{d > 0} q^d b_d \frac{d}{\hbar^{3d+1}}; \quad \text{and} \quad \psi_2 = \sum_{d > 0} q^d c_d \frac{d}{\hbar^{3d+2}},$$

with

$$a_d = \int_{[\mathbb{M}_{0,2}(\mathbb{N}, d)]} c^{3d-1} e_1^*(T_{12}), \quad b_d = \int_{[\mathbb{M}_{0,2}(\mathbb{N}, d)]} c^{3d} e_1^*(T_{11}), \quad c_d = \int_{[\mathbb{M}_{0,2}(\mathbb{N}, d)]} c^{3d+1} e_1^*(p^{2\nu}).$$

Write $QD_{\mathbb{N}} = P_0(D) + qP_1(D) + q^2 P_2(D) + q^3 P_3(D)$. Since $P_0(n) \neq 0$ for $n > 1$, it follows from standard theory about Picard-Fuchs equations (see Appendix B) that the coefficients of the hypergeometric series are determined recursively from the relations

$$\sum_{i=0}^{3} a_{n-i} P_i(n-i) = 0$$

$$\sum_{i=0}^{3} a_{n-i} P_i'(n-i) + \sum_{i=0}^{3} b_{n-i} P_i(n-i) = 0$$

$$\sum_{i=1}^{3} \frac{1}{2} a_{n-i} P_i''(n-i) + \sum_{i=1}^{3} b_{n-i} P_i'(n-i) + \sum_{i=1}^{3} c_{n-i} P_i(n-i) = 0$$

for all $n > 1$, and

$$a_1 = \int_{[\mathbb{M}_{0,2}(\mathbb{N}, 1)]} c^2 e_1^*(T_{12}), \quad b_1 = \int_{[\mathbb{M}_{0,2}(\mathbb{N}, 1)]} c^3 e_1^*(T_{11}), \quad c_1 = \int_{[\mathbb{M}_{0,2}(\mathbb{N}, 1)]} c^4 e_1^*(p^{2\nu}).$$
These integrals can be evaluated using Bott’s formula. We explain the computation in Section 7.1.

**Lemma 5.2.** \(a_1 = 3, b_1 = -1, \text{ and } c_1 = -\frac{65}{19}\).

Let \(R_i(d)\) be the coefficient of \(p^i\) in \(\prod_{m=1}^{d}(p + m)^3\). Then

\[
I_0 = 1 + \sum_{d>0} q^d R_0(d) a_d
\]

\[
I_1 = \frac{t}{h} I_0 + \sum_{d>0} q^d R_1(d) \frac{a_d}{h} + \sum_{d>0} q^d R_0(d) \frac{b_d}{h}
\]

\[
I_2 = \frac{t^2}{2h^2} I_0 + \frac{t}{h} I_1 + \sum_{d>0} q^d R_2(d) \frac{a_d}{h^2} + \sum_{d>0} q^d R_1(d) \frac{b_d}{h^2} + \sum_{d>0} q^d R_0(d) \frac{c_d}{h^2}.
\]

Now, since \(A^*(X)\) is generated by \(p'\), \(QD(X) = QD^\perp(X)\), and since \(X\) is a Calabi-Yau threefold as in Example 3,

\[s^\perp_X = e^{(t'_0 + tp'/\hbar)} \left( 1 + \sum_{d>0} q^d (\perp \circ e_1)_* \left( \frac{1}{h - c} \right) \right) \]

is a vector solution of the differential equation

\[
(h - \frac{d}{dt'})^2 \frac{1}{K(q')}(h \frac{d}{dt'})^2 = 0,
\]

where

\[K(q') = 1 + \frac{1}{57} \sum_{d>0} d^3 n_d X d q^d \frac{q^d}{1 - q^d}.
\]

Therefore,

\[
s^\perp_X = e^{t'_0/\hbar} \left( 1 + \frac{t'}{h} p' + \left( \frac{t'^2}{2h^2} + \frac{1}{57} \sum_{d>0} q^d n_d X d q^d \frac{q^k d}{k^2} \right) p'^2 + \cdots \right).
\]

The quantum Lefschetz hyperplane principle identifies (with \(\hbar = 1\))

\[
\frac{I_1}{I_0} = t' \quad \text{and} \quad \frac{I_2}{I_0} = \frac{t'^2}{2} + \frac{1}{57} \sum_{d>0} q^d n_d X d \sum_k q^{kd} \frac{q^d}{k^2}.
\]

Solving this for \(n_d^X\) gives the desired numbers. Table 6 contains a list of the first ten numbers.

**Remark 1.** Note that, amazingly, these numbers were determined just from knowledge of the lines on \(N\) through:

i) associativity relations to find \(<T_{11}, T_{12} >_{0,2}\)

ii) the recursion formulas (5.5)

iii) the formulas (5.6) and (5.7).
Table 6. Numbers

| n | Value |
|---|------|
| 1 | 147  |
| 2 | 756  |
| 3 | 5283 |
| 4 | 56970|
| 5 | 738477|
| 6 | 10964412|
| 7 | 177916032|
| 8 | 3091158090|
| 9 | 56551583952|
| 10| 1077954415692|

Remark 2. The Picard-Fuchs operator governing the $I_i$’s (with $\hbar = 1$) is

$$QD_X = D^4 + q\left(-\frac{700}{19}D^4 - \frac{1238}{19}D^3 - \frac{999}{19}D^2 - 20D - 3\right)$$
$$+ q^2\left(-\frac{64745}{361}D^4 - \frac{609133}{361}D^3 - \frac{21724}{19}D - \frac{5382}{19}\right)$$
$$+ q^3\left(\frac{172719}{19}D^4 + \frac{321921}{361}D^3 - \frac{38880}{19}D - \frac{16038}{19}\right)$$
$$+ q^4\left(\frac{46656}{361}D^4 + \frac{1478266}{361}D^3 + \frac{1347192}{361}D + \frac{354294}{361}\right)$$
$$- \frac{177147}{361}q^5(D + 1)^4.$$  

6. Conics on $N$

In this section we compute the virtual number of conics on the Calabi-Yau sections $X$, $Y$, and $Z$ using classical methods, i.e., intersection theory on a parameter space of conics. The number obtained for $X$ coincides with $n_X^2$ in Section 5.3. Moreover, we show that the conics on $X$ are rigid, so the number $n_X^2$ is the actual number of conics on $X$.

6.1. The Hilbert scheme of conics on $N$. Let $C \subseteq N$ be a smooth conic on $N$ and let $f : P^1 \to C$ be a parametrization. There are two possibilities for the splitting type of $f^*E_C$:

$$f^*E_C \simeq 2\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-2) \quad \text{and} \quad f^*E_C \simeq \mathcal{O}_{P^1} \oplus 2\mathcal{O}_{P^1}(-1).$$

We say a smooth conic is of type $(0,2)$ or $(1,1)$ according to how $f^*E_C$ decomposes.

Let $H = H_2$ denote the components of the Hilbert scheme of $N$ containing the set of conics, and let $H^{0,2}$ and $H^{1,1}$ denote the closure in $H$ of the locus of smooth conics of type $(0,2)$ and $(1,1)$ respectively.

Theorem 6.1.

i) $H^{0,2}$ and $H^{1,1}$ are schemes of dimension 9.

ii) $H^{0,2}$ is smooth and irreducible.

iii) $H = H^{0,2} \cup H^{1,1}$ and the scheme-theoretical intersection $H^{0,2} \cap H^{1,1}$ is smooth of dimension 8. In particular, $H$ is of dimension 9, as expected.

iv) $H^{1,1} - H^{0,2} \cap H^{1,1}$ is smooth.
The proof follows from a series of lemmas.

**Lemma 6.2.** $H^{0,2} \simeq \mathbf{P}(S_2V)$.  

This was proved in Section 4.1.

**Lemma 6.3.** $H = H^{0,2} \cup H^{1,1}$.  

*Proof.* Let $C$ be a singular conic in $\mathbf{N}$. We must show that $[C] \in H^{0,2} \cup H^{1,1}$. A singular conic is either the union of two lines intersecting at a point, or a line doubled in a plane. Recall the morphism (4.4), $\mathbf{P}(V^\vee) \to \mathbf{N}$. If $C$ lies in the image of $\mathbf{P}(V^\vee_{(p,l)})$ for some $(p,l)$ in $PP^\vee$, the result follows by Lemma 6.2; hence we only need to consider singular conics not contained in the image of any $\mathbf{P}(V^\vee_{(p,l)})$. For each such conic, we shall display a 1-parameter family of smooth conics of (1,1)-type which specialize to it.

First, we consider the reduced singular conics. Suppose $L_1$ and $L_2$ are two lines in $\mathbf{N}$ that meet in a point, and $\rho([L_i]) = (p_i, l_i)$. Then $L_i$ lies in the image $P_i$ of $\mathbf{P}(V^\vee_{(p_i,l_i)})$. If $(p_1, l_1) = (p_2, l_2)$, then $[L_1 \cup L_2] \in H^{0,2}$. Therefore we assume $(p_1, l_1) \neq (p_2, l_2)$. Since the lines meet in a point, the planes $P_1$ and $P_2$ must also meet. In the proof of Theorem 4.1 we saw that when this occurs, then $P_1 \cap P_2$ is a point of projective equivalence type 1–4) in Table 1, and that the union $P_1 \cup P_2$ is determined up to projective equivalence by this point. Hence for each point $P_1 \cap P_2$ of type 1–4), there is a $\mathbf{P}^1 \times \mathbf{P}^1$-choice for the line pair $(L_1, L_2)$ such that $P_1 \cap P_2 \in L_i \subseteq P_i$. Table 2 contains a list of all such line pairs up to projective equivalence. There $[a,b], [c,d] \in \mathbf{P}^1$, and the line pairs intersect at the point $t_1 = 0 = t_2$.

Consider the family of matrices $M_k(t)$ in Table 3. For general $k \in \mathbf{C}^*$, $M_k(t)$ is stable, and $C_k = \{(M_k(t)) \in \mathbf{N} \mid t \in \mathbf{P}^1\}$ is a (1,1)-curve. It is straightforward to obtain the equations for (the Plücker embedding of) $C_k$ in $\mathbf{P}(\wedge^3 S_2 V^\vee)$, and from these one may check, for each case in Table 3, that $C_k$ specializes to $L_1 \cup L_2$ when $k \to 0$. This shows that all reduced singular conics are contained in $H^{0,2} \cup H^{1,1}$.

We now look at the double lines in $\mathbf{N}$. Let $L$ be a line in $\mathbf{N}$. The functor of genus 0 double lines in $\mathbf{N}$ with support on $L$ is represented by $\mathbf{P}(H^0(N_L/\mathbf{N}(1)))^\vee$. If $L$ is a line of type 1 or 2) in Table 3, then $\dim H^0(N_L/\mathbf{N}(1)) = 1$ so there is only one genus 0 double line with support on $L$. This double line must be the one spanning the image of $\mathbf{P}(V^\vee_{\rho(L)})$, hence it is represented by a point in $H^{0,2}$. If $L$ is of type 3–6), then $\dim H^0(N_L/\mathbf{N}(1)) = 2$, so there exists a one-parameter family of genus 0 double lines with support on $L$. For each type of line, we exhibit a family of (1,1)-curves $C_{m,k}$, with $(m,k) \in \mathbf{P}^1 \times \mathbf{C}^*$, such that $C_{m,k}$ specializes to a double line represented by $m$, as $k \to 0$. The parametrization $m$ of $\mathbf{P}^1$ is chosen such that $C_{0,k}$ specializes to the double line spanning $\mathbf{P}(V^\vee_{\rho(L)})$.

For general $k \in \mathbf{C}^*$, the family of matrices $M_{m,k}(t)$ in Table 3 are stable, and $C_{m,k} = \{(M_{m,k}(t)) \in \mathbf{N} \mid t \in \mathbf{P}^1\}$ are (1,1)-curves. Again using the Plücker embedding, it is straightforward to verify that $C_{m,k}$ specialize to lines doubled in the planes $\mathbf{P}^2_m \subseteq \mathbf{P}(\wedge^3 S_2 V^\vee)$, generated by the vectors listed in Table 3. This concludes the proof of Lemma 6.3. \hfill \Box
Table 7. Line pairs in $M^{1,1} - M^{0,2} \cap M^{1,1}$

| $l_1 = l_2$ | $M_k(t)$ | Line pairs | Isotropy |
|------------|----------|------------|----------|
| $l_1 \neq l_2$ | $\left( \begin{array}{c} x_1 \\ k(ax_2 + bx_0) \\ x_2 \\ x_0 + kbt x_2 \end{array} \right)$ | $<x_1^2, x_1x_0, x_1x_2 + t_1(ax_0x_2 + bx_0^2) + t_2(cx_0x_2 + dx_0^2)> <x_1^2, x_1^2, x_1x_0 + t_2(cx_0x_2 + dx_0^2)>$ | $x_0 \leftrightarrow x_2$ |

Table 8. Families of double lines

| $M_{m,k}(t)$ | $P_m^2$ |
|--------------|---------|
| 3) $\left( \begin{array}{c} x_1 \\ k(k+m)x_0 \\ x_2 + kx_0 \\ tx_0 \end{array} \right)$ | $x_1^2 \land x_1x_2 \land x_2^2$ $x_1^2 \land x_1x_2 \land x_0x_2$ $x_1^2 \land (x_1x_2 \land x_0x_1 + mx_2 \land x_0x_2)$ |
| 4) $\left( \begin{array}{c} x_1 \\ 0 \\ x_0 + x_1 + kx_2 \\ x_2 + k(k+m)x_0 + tx_2 \end{array} \right)$ | $x_1^2 \land x_1x_2 \land x_0x_2$ $x_1^2 \land x_1x_2 \land x_0x_1$ $x_1^2 \land (x_1x_2 \land x_0x_2 + m(x_0x_1 \land x_0x_2 + x_1x_2 \land x_0^2))$ |
| 5) $\left( \begin{array}{c} x_1 \\ 0 \\ x_2 + x_1 + kmx_0 \\ x_2 + k(k+m)x_0 + tx_2 \end{array} \right)$ | $x_1^2 \land x_1x_2 \land x_2^2$ $x_1^2 \land x_1x_2 \land x_0x_1$ $x_1^2 \land (x_1x_2 \land x_0x_2 + mx_0x_1 \land x_2^2)$ |
| 6) $\left( \begin{array}{c} x_1 \\ k(k+m)x_0 \\ x_2 + kx_0 \\ k(k+m)x_0 + tx_2 \end{array} \right)$ | $x_1^2 \land x_1x_2 \land (x_2^2 + x_0x_1)$ $x_1^2 \land x_1x_2 \land x_0x_2$ $x_1^2 \land (x_1x_2 \land x_2^2 + m(x_1x_2 \land x_0^2 + x_2^2 \land x_0x_2))$ |
Lemma 6.4.

i) \( \dim TH_{[C]} = 9 \) if \([C] \in H - P(S_2 V_I)\).

ii) \( \dim TH_{[C]} = 10 \) if \([C] \in P(S_2 V_I)\).

Note that this concludes the proof of Theorem 6.1.

Proof. By standard deformation arguments, \( TH_{[C]} = H^0(N_{C/N}) \). The normal bundle \( N_{C/N} \) sits inside an exact sequence

\[
0 \to TC \to TN_C \to N_{C/N} \to l_C \to 0
\]
on \( C \), where \( l_C \) is a sheaf with support on the singularities of \( C \), and \( H^1(TC) = 0 = H^1(l_C) \). This implies \( H^1(TN_C) = H^1(N_{C/N}) \). Since \( H \) parametrizes flat families, \( \chi(N_{C/N}) = \dim H^0(N_{C/N}) - \dim H^1(N_{C/N}) \) is constant for \([C] \in H\). By Riemann-Roch for \( C \cong P^1 \), \( \chi(N_{C/N}) = 9 \). Hence, to prove the lemma, it is enough to prove

\[
(6.1) \quad \dim H^1(TN_C) = \begin{cases} 0 & \text{if } [C] \in H - P(S_2 V_I) \\ 1 & \text{if } [C] \in P(S_2 V_I). \end{cases}
\]

First assume \([C] \in H^{0,2} \). Then \( C \subseteq P^2 \subseteq N \), where \( P^2 = P(V_{(p,l)}) \) for some \((p,l)\) in \( PP^\vee \). Since \( E_{p_2} \cong 2O_{p_2} \oplus O_{p_2}(-1) \) and \( F_{p_2} \cong O_{p_2} \oplus O_{p_2}(1) \), the resolution \([3.4] \) implies that the first and second cohomology groups of \( TN_{p_2} \) and \( TN_{p_2}(-1) \) vanish. Consequently, if \( L \subseteq P^2 \) is any line, the exact sequences

\[
0 \to TN_{p_2}(-2) \to TN_{p_2} \to TN_C \to 0
\]

and

\[
0 \to TN_{p_2}(-2) \to TN_{p_2}(-1) \to TN_L(-1) \to 0
\]
yields \( H^1(TN_C) = H^2(TN_{p_2}(-2)) = H^1(TN_L(-1)). \) In Section 4.5 we saw that

\[
\dim H^1(TN_L(-1)) = \begin{cases} 0 & \text{if } p \notin l \\ 1 & \text{if } p \in l. \end{cases}
\]

This proves the lemma for \([C] \in H^{0,2} \).

Now, let \([C] \in H^{1,1} - H^{0,2} \cap H^{1,1} \). By semi-continuity, it is enough to show \([6.1] \) for degenerated conics. If \( C \) is the union of two lines \( L_1 \) and \( L_2 \), then \( \rho([L_1]) \neq \rho([L_2]) \). Consider the component sequence

\[
(6.2) \quad 0 \to TN_{L_1 \cup L_2} \to TN_{L_1} \oplus TN_{L_2} \to TN_{L_1 \cap L_2} \to 0.
\]

Let \( \rho([L_i]) = (p_i, l_i) \). If \( p_i \notin l_i \), then \( TN_{L_i} \) is generated by global sections, hence

\[
(6.3) \quad H^0(TN_{L_1}) \oplus H^0(TN_{L_2}) \to TN_{L_1 \cap L_2}
\]
is surjective, whenever this is the case for one of the lines. If \( p_i \in l_i \) for both lines, then \([6.3] \) is still surjective, since the \( O(-1) \)-component of \( TN_{L_i} \) is canonically isomorphic to \( H^0(I_{\ell_i}(1)) \cap H^0(I_{p_i}(1)) \) \( \cong V/H^0(I_{p_i}(1)) \cap O_{\ell_i}(-1) \) (see Appendix 3), and we are assuming \((p_1, l_1) \neq (p_2, l_2)\). This implies \( H^1(TN_{L_1 \cup L_2}) = 0 \).

If \( C \) is a double line, then it is projectively equivalent to one of the double lines \( D_m \) which spans \( P^2_m, \ m \neq 0, \) in Table 3. Since lines of type 3)-6) in Table 3 specialize to lines of type 5), and a double line \( D_m, \ m \neq 0, \) specializes to \( D_\infty, \) it is enough to show \([6.1] \) for the double line \( D_\infty \) which spans the plane

\[
(6.4) \quad P^2_\infty = \langle x_0^2 \land x_0 x_1 \land x_0 x_2, x_0^3 \land x_0 x_1 \land x_1^2, x_0^2 \land x_0 x_2 \land x_1^2 \rangle / C.
\]
in $\mathbf{P}(\wedge^3 S_2 V^\vee)$. All points in $\mathbf{P}^2_{\infty}$ are decomposable, hence $\mathbf{P}^2_{\infty}$ is contained in the image of Grass$_3(S_2V)$. Consider the restriction of the tautological exact sequence to $\mathbf{P}^2_{\infty}$,

$$0 \to \mathcal{U}_{\mathbf{P}^2_{\infty}} \to S_2 V_{\mathbf{P}^2_{\infty}} \to \mathcal{Q}_{\mathbf{P}^2_{\infty}} \to 0.$$  

From (6.3) one sees that $0 \to \mathcal{U}_{\mathbf{P}^2_{\infty}} \to S_2 V_{\mathbf{P}^2_{\infty}}$ factors through the trivial rank four bundle $<x_0^2, x_0x_1, x_0x_2, x_1^2>_P$. Since deg $\mathcal{Q}_{\mathbf{P}^2_{\infty}} = 1$, this implies that $\mathcal{Q}_{\mathbf{P}^2_{\infty}} \cong 2\mathcal{O}_{\mathbf{P}^2_{\infty}} \oplus \mathcal{O}_{\mathbf{P}^2_{\infty}}(1)$. This means that the restriction of (6.5) to $D_{\infty}$ is exact on global sections, so $H^1(\mathcal{E}_{D_{\infty}}) = 0$. To show $H^1(T\mathcal{N}_{D_{\infty}}) = 0$ it suffices, by (3.4), to show that $H^1(\mathcal{F}^\vee \otimes \mathcal{E} \otimes V_{D_{\infty}}) = 0$. But this is now clear, since

$$H^1((S_2 V \otimes V)^\vee \otimes \mathcal{E} \otimes V_{D_{\infty}}) = 0.$$  

is surjective, and $H^1((S_2 V \otimes V)^\vee \otimes \mathcal{E}_{D_{\infty}} \otimes V) = 0$. Here (6.6) is the map induced from the inclusion $\mathcal{F}_{D_{\infty}} \to \mathcal{E}_{D_{\infty}} \otimes V$. This completes the proof of Lemma 6.4. $\square$

**Remark 1.** We come up just short of proving that $H^{1,1}$ is also smooth. However, the families in Table 8 show that if $[C] \in H^{1,1} \cap H^{0,2}$, then the tangent direction $TH_{[C]} - TH_{[C]}^{0,2}$ is unobstructed. Hence, for every point $[C] \in H^{1,1}$, there exists an immersion of $\mathbf{C}^9$ onto a neighborhood of $[C]$. We expect that $H^{1,1}$ is smooth, and that the components $H^{0,2}$ and $H^{1,1}$ intersect transversally.

**Remark 2.** We have described the Hilbert scheme of conics on $\mathbf{N}$ instead of $\overline{M}_{0,0}(\mathbf{N}, 2)$. A similar result holds for $\overline{M}_{0,0}(\mathbf{N}, 2)$. In fact, if we let $M^{0,2}$ and $M^{1,1}$ be the compactifications in $\overline{M}_{0,0}(\mathbf{N}, 2)$ of the loci of $(0, 2)$-curves and $(1, 1)$-curves respectively, then the theorem holds for $\overline{M}_{0,0}(\mathbf{N}, 2) = M^{0,2} \cup M^{1,1}$. In particular we make the following observations:

i) There is a fiber square

$$M^{0,2} \cap M^{1,1} \longrightarrow M^{0,2} \quad \begin{array}{c} \downarrow \rho \rho \downarrow \end{array} \quad \begin{array}{c} I \quad \longrightarrow \quad \mathbf{P}(V) \times \mathbf{P}(V^\vee) \end{array}$$

where the vertical maps are $\overline{M}_{0,0}(\mathbf{P}^2, 2)$-fibrations.

ii) $M^{1,1} - M^{0,2} \cap M^{1,1}$ contains no double coverings of lines.

**Remark 3.** It will be convenient for us to work with $H$ in the next section. This will enable us to show that all conics in $X$ are rigid. When we carry out our computations, however, we will use Bott’s formula over $M^{1,1}$, instead of $H^{1,1}$. This is slightly easier because the fixpoints on $M^{1,1}$ are isolated, whereas on $H^{1,1}$, the family $D_m$ of double lines of type 5) in Table 8 results in a $\mathbf{P}^1$-component of fixpoints on $H^{1,1}$.

**Remark 4.** Since $H^{0,2}$ is a projective bundle we can use Grothendieck’s theorem again to determine the Chow ring. If we let $\xi$ be the first Chern class of the tautological line bundle on $\mathbf{P}(S_2 V) \to PP^\vee$, then

$$A^*(H^{0,2}) = \mathbf{Z}[\tau, \bar{\tau}, \xi]/(\tau^3, \bar{\tau}^3, r),$$

where

$$r = \sum_{i=0}^{6} (-1)^i r^i c_i(S_2 V)\xi^{6-i}.$$
The total Chern class \( c(S_2 V) \) can be expressed in terms of \( \tau \) and \( \tilde{\tau} \) by using a standard formula \([\mathbb{F}]\) which relates \( c(S_2 V) \) to \( c(V) \).

6.2. Fixpoints on \( M^{1,1} \). We present a complete description of fixpoints of the torus action on \( M^{1,1} \). If \([f: C \to \mathbb{N}] \in M^{1,1}\), then either \( f \) is an embedding, or \( f \) is a \( 2:1 \) covering of a line. If \( f \) is an embedding, we identify \( C \) with its image \( f(C) \).

We begin by describing the fixpoints in \( M^{1,1} - M^{0,2} \cap M^{1,1} \). As noted in Remark 2 of Section 6.1, all points in \( M^{1,1} - M^{0,2} \cap M^{1,1} \) represent embeddings \( f: C \to \mathbb{N} \).

First the smooth conics of (1, 1)-type. Let \( \{ \mathcal{E}_t \mid t \in \mathbb{P}^1 \} \) be a family of nets represented by a conic of (1, 1)-type. If the conic is fixed, then \( \{ B(\mathcal{E}_t) \mid t \in \mathbb{P}^1 \} \) must describe a fixed curve in \( \mathbb{P}(V) \), hence a union of lines (possibly with multiplicity). Since the conic is of (1, 1)-type there exists a conic \( Q \subseteq \mathbb{P}(V) \) (the generator of \( \mathcal{O}_{\mathbb{P}^1} \)) in \( f^* \mathcal{E}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus 2\mathcal{O}_{\mathbb{P}^1}(-1) \) that contains \( \{ B(\mathcal{E}_t) \mid t \in \mathbb{P}^1 \} \), hence \( Q \) must be either i) a union of two lines, say \( x_0x_1 \), or ii) a double line, say \( x_0^2 \). In each case there are matrix representatives of type

\[
(6.7) \quad \begin{cases} 
\text{i) } & \begin{pmatrix} x_0 & 0 \\ x_2 & x_1 \\ 0 & x_0 + tx_2 \end{pmatrix} \\
\text{ii) } & \begin{pmatrix} x_0 & 0 \\ L & x_0 \\ M_t & x_1 + tx_2 \end{pmatrix}
\end{cases}
\]

for the nets \( \mathcal{E}_t \), where \( L = ax_1 + bx_2 \), and \( M_t = (cx_1 + dx_2) + t(ex_1 + gx_2) \), with \( a, b, c, d, e, f, g \) in \( \mathbb{C} \).

Each smooth fixconic must contain two fixpoints. Without loss of generality, we can assume that these occur at \( t = 0 \) and \( t = \infty \). It is easy to verify that the matrices in (6.7) yield fixed conics of (1, 1)-type, with fixpoints at \( t = 0 \) and \( t = \infty \), only when they take the form as in Table 9. Besides the matrix representative, Table 9 contains a list of the nets generated by the \( 2 \times 2 \) minors of the matrices, along with the isotropy subgroups of the smooth fixconics.

### Table 9. Smooth fixconics

| Type | Matrix | \( \mathcal{E}_t \) | Isotropy |
|------|--------|------------------|----------|
| i)   | \( \begin{pmatrix} x_0 & 0 \\ x_2 & x_1 \\ 0 & x_0 + tx_2 \end{pmatrix} \) | \( <x_0x_1, x_0(x_0 + tx_2), x_2(x_0 + tx_2)> \) | \{id\} |
| ii)  | \( \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 + tx_2 \end{pmatrix} \) | \( <x_0^2, x_0(x_1 + tx_2), x_1(x_1 + tx_2)> \) | \{id\} |

If \([C] \in M^{1,1} - M^{0,2} \cap M^{1,1} \) is a union of two lines, then \( C \) is projectively equivalent to one of the line pairs in Table 9. A line pair is fixed if and only if both lines and the intersection point are fixed. Hence the line pairs in Table 9 are fixed only for the four values \( \{(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)\} \) of \([a, b], [c, d]\). Table 9 contains a list of the line pairs, and
the isotropy subgroup of $S_3$ acting on the set $\{(0,0), (0,\infty), (\infty,0), (\infty,\infty)\}$. This takes care of all fixpoints in $M^{1,1} \setminus M^{0,2} \cap M^{1,1}$.

We now look at fixpoints in $M^{0,2} \cap M^{1,1}$. Suppose $[f : C \to Y] \in M^{0,2} \cap M^{1,1}$ is fixed. Since $M^{0,2} \cap M^{1,1}$ is a $\mathcal{M}_{0,0}(\mathbb{P}^2, 2)$ fibration over $I$, the map $f$ factors through $\mathbf{P}(\mathcal{V}_{(p,l)}) \to Y$ for some $(p,l) \in I$, which must be fixed. Without loss of generality, we may assume that $(p,l)$ is the point-line pair $(x_1^2, x_1x_2)$. The plane $\mathbf{P}(\mathcal{V}_{(p,l)})$ contains no smooth fixconics, hence $f(C)$ must be supported on the triangle of fixlines 3)–5) in Table 3. If $f$ is a double covering, then the branch points must be fixed. For each of the fixpoints in 3)–5) there are three possibilities for a 2:1 covering which is fixed: If $C \to Y$ is a double covering, then the branch points must be fixed.

6.3. Conics on the Calabi-Yau threefolds. Suppose $\mathcal{B}$ is one of the bundles $3\mathcal{O}(p), \mathcal{F}^\vee \oplus \mathcal{O}(2p), S_2\mathcal{F}^\vee$, and $W$ is a general Calabi-Yau sections of $\mathcal{B}$. Let

\[
\begin{align*}
B & \quad \downarrow \\
C & \quad e' \quad \to \quad N \\
\pi' & \quad \downarrow \\
H & 
\end{align*}
\]

be the universal curve over $H$, let $B'_2 = \pi'_e e'^* B$, and let $B'_2 = c_9(B'_2)$. The virtual number of conics on $W$ is by definition

\[
n^W_2 = \int_H B'_2 = \int_{H^{1,1}} B'_2 + \int_{H^{0,2}} B'_2,
\]

where $\int_{H^{1,1}} B'_2$ and $\int_{H^{0,2}} B'_2$ are virtual numbers of $(1,1)$-conics and $(0,2)$-conics, respectively.

Let $j^d : \mathcal{M}_{0,0}(W,d) \to \mathcal{M}_{0,0}(\mathbb{N},d)$. Recall from the Kleiman-Bertini argument in Section 6.3 that if $L \subseteq W$, then $j^e_!(L) \in \mathbf{P}(\mathcal{V}) - \mathbf{P}(\mathcal{V}_1)$. Suppose $[f : C \to L] \in \mathcal{M}_{0,0}(W,2)$ is a double covering of $L$. Remark 2 of Section 6.1 implies that $j^e_!(f : C \to L) \in \mathcal{M}_{0,0}(\mathbb{N},2) - M^{1,1}$. In other words, there is no contribution from the double coverings to $\int_{M^{1,1}} B_2$, where $B_2 = c_9(\pi_e e^* B)$, hence

\[
\int_{M^{1,1}} B_2 = \int_{H^{1,1}} B'_2.
\]

**Theorem 6.5.**

i) The number of conics on $X$ is 756. These are all smooth of $(1,1)$-type.

ii) The virtual number of conics on $Y$ is 1674. All of these are of $(1,1)$-type.

iii) The virtual number of conics on $Z$ is 504. Of these are 468 of $(1,1)$-type and 36 of $(0,2)$-type.

**Proof.** We must evaluate

\[
\int_{M^{1,1}} B_2 \quad \text{and} \quad \int_{H^{0,2}} B'_2.
\]
for each of the three vector bundles $3\mathcal{O}(p)$, $\mathcal{F}^\vee \oplus \mathcal{O}(2p)$, and $S_2\mathcal{F}^\vee$. We shall use Bott's formula to determine $\int_{M_{1,1}} B_2$. This computation is explained in the next section. The integral $\int_{\mathcal{H}_{0,2}} B_2^\vee$ will be evaluated in the Chow ring $A^*(\mathcal{H}_{0,2})$. This goes exactly as our computation in Section 4.3. We need to express the top Chern classes of (the restrictions of) the vector bundles

$$\mathcal{O}(p)'_2, \quad \mathcal{O}(2p)'_2, \quad \mathcal{F}'_2^\vee, \quad \text{and} \quad (S_2\mathcal{F}^\vee)'_2$$

in terms of the generators $\tau$, $\tilde{\tau}$, and $\xi$. Let $\mathcal{C}_2$ denote the restriction of the universal curve $\mathcal{C}$ to $\mathcal{H}_{0,2}$. As seen in Section 4.3, the map $\mathcal{C}_2 \to \mathbb{N}$ factors through the map $P(V^\vee) \to \mathbb{N}$, hence

$$\mathcal{C} \longrightarrow P(V^\vee) \longrightarrow \mathbb{N}$$

(6.11)

$$H^{0,2} \longrightarrow PP^\vee$$

is commutative. This identifies

$$\mathcal{E}_{\mathcal{C}_2} = pr_*\mathcal{I}_2\mathcal{I}_p\mathcal{I}_2 \mathcal{C}^\vee \oplus \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma})$$

(6.12)

$$\mathcal{F}_{\mathcal{C}_2} \simeq pr_*\mathcal{I}_2 \mathcal{C}_2^\vee \left( \wedge^2 pr_*\mathcal{I}_p \mathcal{I}_2 \mathcal{C}_2^\vee \oplus \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma}) \right),$$

hence, on $H^{0,2}$, we have

(6.13)

$$\mathcal{O}(p)'_2 = \pi'_* \wedge^3 \mathcal{E}_{\mathcal{C}_2} = \wedge^2 pr_*\mathcal{I}_2\mathcal{I}_p\mathcal{I}_2 \mathcal{C}^\vee \oplus \pi'_* \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma})$$

$$\mathcal{O}(2p)'_2 = \pi'_* \wedge^3 \mathcal{E}_{\mathcal{C}_2} \wedge \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma})$$

$$\mathcal{F}'_2 = pr_*\mathcal{I}_2 \mathcal{C}_2^\vee \left( \wedge^2 pr_*\mathcal{I}_p \mathcal{I}_2 \mathcal{C}_2^\vee \oplus \pi'_* \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma}) \right)$$

$$(S_2\mathcal{F}^\vee)'_2 = pr_*\mathcal{I}_2 \mathcal{C}_2^\vee \left( \wedge^2 pr_*\mathcal{I}_p \mathcal{I}_2 \mathcal{C}_2^\vee \oplus \pi'_* \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma}) \right).$$

There remains to determine $\pi'_* \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma})$ and $\pi'_* \mathcal{O}_{\mathcal{C}_2}(2\tilde{\sigma})$. Consider the twisted ideal sequence

(6.14)

$$0 \to \mathcal{I}_{\mathcal{C}_2}(m\tilde{\sigma}) \to \mathcal{O}_P(m\tilde{\sigma}) \to \mathcal{O}_{\mathcal{C}_2}(m\tilde{\sigma}) \to 0,$$

where $P = \mathbb{P}(S_2V^\vee) \times_{\mathbb{P}V^\vee} \mathbb{P}(V^\vee)$. Again, by a diagonal argument as in (1.9), $\mathcal{I}_{\mathcal{C}_2} \simeq \mathcal{O}(-\xi) \otimes \mathcal{O}(-2\tilde{\sigma})$, hence pushing down (6.14) with $m = 1$ and $m = 2$, we find

(6.15)

$$V^\vee = \pi'_* \mathcal{O}_{\mathcal{C}_2}(\tilde{\sigma})$$

and an exact sequence

(6.16)

$$0 \to \mathcal{O}(-\xi) \to S_2V^\vee \to \pi'_* \mathcal{O}_{\mathcal{C}_2}(2\tilde{\sigma}) \to 0.$$

Along with the expressions in Section 4.2, this enables us to express the Chern classes of

(6.13) in terms of the generators $\sigma$, $\tilde{\sigma}$, and $\xi$.

Finally, the last assertions in i) follow (as in Section 4.3) from Kleiman’s Bertini theorem and from the surjectivity of

$$H^0(\mathbb{P}^{19}, \mathcal{O}_{\mathbb{P}^{19}}(1)) \to H^0(C, \mathcal{O}_C(p))$$

for all conics $C \subseteq \mathbb{P}^{19}$. \qed
7. Computations. Bott’s formula on the space of maps

The integrands in the integrals of Theorem 5.1, Lemma 5.2, and Theorem 6.5 are all polynomials in the Chern classes of $T$-equivariant bundles. In Section 4.4 and Section 6.2 we saw that fixpoints on $\overline{M}_{0,0}(N,1)$ and $M^{1,1}$ were isolated. In Section 7.1 we shall see that fixpoints on $\overline{M}_{0,2}(N,1)$ are also isolated.

Let $x_i$ denote the basis of $V$ with characters $\lambda_i$ as in Section 3.3. If $U$ is a finite dimensional representation of $T$ we shall let $[U]$ denote the class of $U$ in the (orbifold) representation ring generated by $\lambda_i^s$ for non-zero rational numbers $s$. Hence we write $[U] = \sum a_n \lambda_1^n \lambda_2^n$ with $a_n \in \mathbb{Z}$ and $n_i \in \mathbb{Q}$. We also use the notation $[U]^{(k)} = \sum a_n (\lambda_1^n \lambda_2^n)^k$ for $k \in \mathbb{Q}$. Let $C^* \subseteq T$ be a one-parameter subgroup of $T$ with weights $\omega_i$. The induced weights of the $C^*$-action on $U$ are $n_i \omega_0 + n_1 \omega_1 + n_2 \omega_2$. Choose weights $\omega_i$ such that the induced weights on the tangent spaces of $\overline{M}_{0,2}(N,1)$ (resp. $M_{1,1}$) at fixpoints are all non-zero. Then the fixpoints of the induced $C^*$-action on $\overline{M}_{0,2}(N,1)$ (resp. $M_{1,1}$) are the same as the fixpoints of the $T$-action. In particular, they are isolated, hence we can apply the version of Bott’s formula in the Introduction to evaluate the integrals.

Suppose $\int_M P$ is any of the relevant integrals, where $P$ is a polynomial in the Chern classes of vector bundles $A, B$, etc.. To evaluate the integral using Bott’s formula we must determine the weights of the vector spaces $T_f M, A_f, B_f, \ldots$, at each fixpoint $f \in M$. Our method will be as follows: For each fixpoint $f \in M$, we shall obtain formulas (in terms of the $\lambda_i$) for the tangent representation $[T_f M]$ and the integrand representations $[A_f], [B_f], \ldots$. Once these are in store, they can readily be translated into the relevant lists of weights. In Section 7.1 we explain how to obtain formulas for the tangent and integrand representations of the integrals over $\overline{M}_{0,2}(N,1)$ (Theorem 7.1, and Lemma 7.2). In Section 7.2 we do the same for the integrals over $M^{1,1}$ (Theorem 7.3). The actual computation is carried out in MAPLE (see Appendix ??), where lists of fixpoints are constructed from our descriptions in Sections 4.4 and 6.2, along with the corresponding representations of Sections 7.1 and 7.2.

Determining the tangent space representations is the most laborious. Let $(C, f, s_i)$ be a fixed stable $n$-marked map of degree $d$ to $N$. We shall abuse notation and use $f$ to also denote the corresponding point in $\overline{M}_{0,n}(N, d)$, i.e., $f = [C, f, s_i]$. Suppose $C = \cup C^\alpha$ and all $C^\alpha \simeq \mathbb{P}^1$. By standard deformation arguments [3], [5],

$$(7.1) \quad [T_f \overline{M}_{0,n}(N, d)] = [H^0(TN_C)] - \sum_{\alpha} [H^0(TC^\alpha)] + \sum_{i=1}^n [T_{s_i} C]$$
$$+ \sum_{y \in C^\alpha \cap C^\beta} [T_y C^\alpha] + [T_y C^\beta] + [T_y C^\beta].$$

We shall only need to consider cases with $C \simeq \mathbb{P}^1$, and $C = \mathbb{P}^1 \cup \mathbb{P}^1$. To determine the terms of (7.1), we shall use the resolution (6.4), along with the component sequence

$$(7.2) \quad 0 \to TN_C \to TN_{C_1} \oplus TN_{C_2} \to T_y N \to 0,$$

when $C \simeq C_1 \cup C_2$ and $y = C_1 \cap C_2$. Also, if $L$ is a line bundle of degree one on $\mathbb{P}^1$, then there is an exact sequence

$$(7.3) \quad 0 \to \mathcal{O}_{\mathbb{P}^1} \to H^0(L)^\vee \otimes L \to TP^1 \to 0.$$
Remark 1. The number $<T_{11}, T_{12}>_{0,2}$, and the number of lines and $(0,2)$-conics on the Calabi-Yau sections can also be computed using Bott’s formula. However, fixpoints on $H^0$ and $M^{0,2}$ are not isolated, hence a more general version of Bott's formula than the one stated in the introduction is required.

Remark 2. Recall that we are using Bott’s formula over the space $M^{1,1}$ under the assumption that it is smooth. Even if this is not true, our computation still holds. There is a theorem due to Graber and Pandharipande [GrP] which gives a localization formula for integrals over virtual fundamental classes. Applied to the space $\bar{M}_{0,0}(N, 2)$, the formula says that the numerator in the contributions to Bott’s formula from singular points $f$ must be multiplied by the weight of the $C^*$-action on the obstruction space $H^1(f^*TN)$. A computation using this formula reduces to the computation obtained by assuming $M^{1,1}$ is smooth.

Remark 3. We should not forget about the automorphism group of a stable map. If $[C, f, s_1, s_2] \in \bar{M}_{0,2}(N, 1)$ then $\text{Aut}(f)$ is trivial. If $[C, f] \in M^{1,1}$, then $\text{Aut}(f)$ is trivial unless $f$ is a $2:1$-covering, and if so $|\text{Aut}(f)| = 2$.

7.1. Representations on $\bar{M}_{0,2}(N, 1)$. First we describe the fixpoints on $\bar{M}_{0,2}(N, 1)$. Suppose $f = [C, f, s_1, s_2] \in \bar{M}_{0,2}(N, 1)$ is a fixpoint. Then the line $f(C)$ and the points $f(s_1)$ and $f(s_2)$ must be fixed. There are essentially two possibilities:

i) $f(s_1) \neq f(s_2)$. Then $C \simeq \mathbb{P}^1$ and $f(s_1), f(s_2)$ are two fixpoints on $f(C)$.

ii) $f(s_1) = f(s_2)$. Then $C$ is the union of two $\mathbb{P}^1$-components $C^1$ and $C^2$, such that $f(C^1)$ is a fixline, and $C^2$ contains the two marked points. Hence $f(C^2)$ is a fixpoint on $f(C^1)$.

In other words, for each fixline $L$ in $N$ with fixpoints $f_1, f_2 \in L$, there are four fixpoints in $\bar{M}_{0,2}(N, 1)$ corresponding to the configurations $(L, f_1, f_2)$, $(L, f_2, f_1)$, $(L, f_1)$, and $(L, f_2)$. In particular, the fixpoints on $\bar{M}_{0,2}(N, 1)$ are isolated.

The integrals in Theorem 5.1 and Lemma 5.2 are of type

$$\int_{\bar{M}_{0,2}(N, 1)} e^*_T(T_r)e^*_T(T_s) \quad \text{and} \quad \int_{\bar{M}_{0,2}(N, 1)} e^*_T(T_r) \frac{1}{1 - c},$$

where $0 \leq r, s \leq 12$. The integrands are polynomials in the Chern classes of the bundles $L_1$, $e^*_k(\mathcal{E})$, and $e^*_k(\mathcal{F})$, for $k = 1, 2$, whose fibers over $f$ are $T_1 C^1$, $\mathcal{E}_{f(s_1)}$, and $\mathcal{F}_{f(s_2)}$, respectively. Since $f(s_k)$ is a fixpoint in $N$, the representations $[\mathcal{E}_{f(s_k)}]$ and $[\mathcal{F}_{f(s_k)}]$ are immediately obtained from Table 1. To determine the representations $[C^1]$ and $[T_f\bar{M}_{0,2}(N, 1)]$, we must consider the fixpoint types i) and ii) separately. Suppose $f(C) = L$, a fixline with fixpoints $f_1$ and $f_2$.

Case i). Then $[T_1 C^1] = [T_{f(s_1)} L]^{-1}$, and (7.1) yields

$$[T_f\bar{M}_{0,2}(N, 2)] = [H^0(TNL)] - [H^0(TL)] + [T_{f_1}(L)] + [T_{f_2}(L)].$$

Case ii). Since $C^2$ is contracted by $f$, the induced action on $C^2$ is trivial and $H^0(TN_{C^2}) = H^0(TNy)$, where $y = C^1 \cap C^2$. Hence $[T_1 C^1] = 1 = [T_y C^2]$, and (7.1), (7.2) yields

$$[T_f\bar{M}_{0,2}(N, 1)] = [H^0(TNL)] - [H^0(TL)] + 2[T_y L].$$

Note that since $[T_1 C^1] = 1$, the contributions from these points to the integral $\int_{\bar{M}_{0,2}(N, 1)} e^*_T(T_r) \frac{1}{1 - c}$ is zero.
We shall express the different parts of (7.4) and (7.5) in terms of the elementary representations
\[ H^0(I_p(1)), \quad H^0(I_l(1)), \quad [O_{f_i} (\sigma)], \quad \text{and} \quad H^0(O_L(\sigma)), \]
where \( \rho([L]) = (p,l) \). From Remark 2 in Section 4.1, there are \( T \)-equivariant isomorphisms:
\[
\begin{align*}
\mathcal{E}_L &= H^0(I_p I_l(2)) \oplus O(-\hat{\sigma}) \\
\mathcal{F}_L &\simeq H^0(I_l(1)) \otimes (\wedge^2 H^0(I_p(1)) \oplus O_L(-\hat{\sigma})).
\end{align*}
\]
For each fixline, the representations (7.6) can be read off Table 3.
As \( O_L(\sigma) \) is of degree one on \( L \), there is a \( T \)-equivariant exact sequence
\[ 0 \to O_L \to H^0(O_L(\sigma)) \otimes O_L(\sigma) \to TL \to 0, \]
hence
\[ [T_{f_i} L] = [O_{f_i}(\sigma)]^{-1} \cdot [O_{f_i}(\sigma)], \]
with \( i \neq j \), and
\[ [H^0(TL)] = [H^0(O_L(\sigma))]^{-1} \cdot [H^0(O_L(\sigma))] - 1. \]
Since \( H^1(\mathcal{E}_L^\vee \otimes \mathcal{E}_L) = 0 = H^1(\mathcal{F}_L^\vee \otimes \mathcal{F}_L) \), the exact sequence (7.4) implies
\[ [H^0(TN_L)] = [H^0(\mathcal{F}_L^\vee \otimes \mathcal{E}_L)] \cdot (\lambda_0 + \lambda_1 + \lambda_2) - [H^0(\mathcal{E}_L^\vee \otimes \mathcal{E}_L)] - [H^0(\mathcal{F}_L^\vee \otimes \mathcal{F}_L)] + 1. \]
The terms in this expression can be determined by using the identifications (7.7) and
\[ [H^0(I_p(1))] = [H^0(I_l(1))] \cdot [H^0(O_L(\sigma))]. \]
This yields
\[
\begin{align*}
[H^0(\mathcal{F}_L^\vee \otimes \mathcal{E}_L)] &= [H^0(I_p(1))]^{-1} + [H^0(I_l(1))]^{-1} + [H^0(O_L(\sigma))] \cdot [H^0(I_p(1))] \\
[H^0(\mathcal{E}_L^\vee \otimes \mathcal{E}_L)] &= [H^0(I_p(1))]^{-1} \cdot [H^0(I_l(1))] + [H^0(O_L(\sigma))] \cdot [H^0(I_p I_l(2))] + [H^0(O_L(\sigma))] \cdot [\wedge^2 H^0(I_p(1))].
\end{align*}
\]
In the next section we shall need the representations \([H^0(\wedge^3 \mathcal{E}_L^\vee)], [H^0(\mathcal{F}_L^\vee)], [H^0(\wedge^3 \mathcal{E}_L^\vee \otimes 2)], \)
and \([H^0(S_2 \mathcal{F}_L^\vee)] \). Again, these are obtained from (7.7):
\[
\begin{align*}
[H^0(\wedge^3 \mathcal{E}_L^\vee)] &= [\wedge^2 H^0(I_p(1))]^{-1} \cdot [H^0(I_l(1))]^{-2} \cdot [H^0(O_L(\sigma))] \\
[H^0(\mathcal{F}_L^\vee)] &= [H^0(I_l(1))]^{-1} \cdot ([\wedge^2 H^0(I_p(1))]^{-1} + [H^0(O_L(\sigma))]) \\
[H^0(\wedge^3 \mathcal{E}_L^\vee \otimes 2)] &= [S_2 H^0(\wedge^3 \mathcal{E}_L^\vee)] \\
[H^0(S_2 \mathcal{F}_L^\vee)] &= [S_2 H^0(\mathcal{F}_L^\vee)].
\end{align*}
\]
\[\text{Here } H^0(I_l(1)) = H^0(P(V), I_l(1)), \text{ where } l \in P(V^\vee) \text{ is identified with the line } l \subseteq P(V).\]
7.2. Representations on $M^{1,1}$. The integrals in Theorem 6.3 can be expressed as

$$\int_{M^{1,1}} c_3(\mathcal{O}(p)_2)^3, \quad \int_{M^{1,1}} c_4(\mathcal{F}_2^*) c_5(\mathcal{O}(2p)_2), \quad \text{and} \quad \int_{M^{1,1}} c_9(S_2\mathcal{F}_2^*)_2,$$

hence for each fixpoint $f = [f: C \to \mathbb{N}] \in M^{1,1}$, we must determine $[TfM^{1,1}]$, and the integrand representations

$$\int_{M^{1,1}} \mathcal{O}_C(p)], \quad [H^0(\mathcal{F}_C)], \quad [H^0(\mathcal{O}_C(2p))], \quad \text{and} \quad [H^0(S_2\mathcal{F}_C^*)].$$

The fixpoints of $M^{1,1}$ are described in Section 6.2. We determine the representations (7.14) for each type separately. First, we consider the fixpoints in $M^{1,1} - M^{0,2} \cap M^{1,1}$. As seen, if $f \in M^{1,1} - M^{0,2} \cap M^{1,1}$, then $f$ is an embedding and $C \simeq \mathbb{P}^1$, or $C \simeq \mathbb{P}^1 \cup \mathbb{P}^1$, and $TfM^{1,1} = TfM_{0,2}(\mathbb{N}, 2)$.

$M^{1,1} - M^{0,2} \cap M^{1,1} = \mathbb{P}^1$. If $f \in M^{1,1} - M^{0,2} \cap M^{1,1}$ and $C \simeq \mathbb{P}^1$, then we may assume $f(C)$ is one of the smooth conics in Table 3. For each type, consider its matrix representation. Let $a_{ij}$ denote the $(i, j)$-entry and let $\mathcal{L}$ be the line bundle on $C$ with fibers $\mathcal{L}_i = a_{32}(t)$. Then the bundles $\mathcal{E}_C$ and $\mathcal{F}_C$ decomposes as:

$$(7.15) \quad \mathcal{E}_C = a_{11}a_{22} \oplus a_{11} \mathcal{L} \oplus a_{21} \mathcal{L},$$

$$\mathcal{F}_C \simeq a_{11}a_{22} \mathcal{L} \oplus a_{11}a_{22} \mathcal{L},$$

where we abuse notation and write $a_{11}a_{22}$, $a_{11} \mathcal{L}_i$, etc., for the bundles with fibers $a_{11}a_{22}$, $a_{11} \mathcal{L}_i$, etc.. The isomorphism follows by observing that the map

$$a_{11}a_{22} \mathcal{L} \oplus a_{11}a_{22} \mathcal{L} \longrightarrow \mathcal{E}_C \otimes V,$$

given on the fibers by

$$a_{11}a_{22} \mathcal{L}_i \longrightarrow a_{11} \mathcal{L}_i \otimes a_{22} - a_{21} \mathcal{L}_i \otimes a_{11}$$

$$a_{11}a_{22} \mathcal{L}_i \longrightarrow a_{11}a_{22} \mathcal{L}_i - a_{11} \mathcal{L}_i \otimes a_{22}$$

is injective, and lies in the kernel of the multiplication map $S_2V \otimes V \to S_3V$.

From (7.1) we have

$$[TfM^{1,1}] = [H^0(T\mathcal{N}_C)] - [H^0(T\mathcal{C})].$$

The line bundle $\mathcal{L}^\vee$ is of degree one on $C \simeq \mathbb{P}^1$, hence the exact sequence of (7.3) yields

$$[H^0(T\mathcal{C})] = [H^0(\mathcal{L}^\vee)]^{(-1)} \cdot [H^0(\mathcal{L}^\vee)] - 1.$$

Since $H^1(\mathcal{E}_C \otimes \mathcal{E}_C) = 0 = H^1(\mathcal{F}_C \otimes \mathcal{F}_C)$, (7.11) still holds with $L = C$. Using (7.15), we find the following representations for the terms in (7.11):

$$[H^0(\mathcal{F}_C \otimes \mathcal{E}_C)] = [H^0(\mathcal{L}^\vee)](1 + [a_{22}]^{-1}[a_{22}]) + [a_{21}]^{-1} + [a_{11}]^{-1} + [a_{22}]^{-1}$$

$$+ [a_{11}]^{-1}[a_{22}]^{-1}[a_{21}]$$

(7.18) $$[H^0(\mathcal{F}_C \otimes \mathcal{E}_C)] = [H^0(\mathcal{L}^\vee)]([a_{22}] + [a_{11}][a_{22}][a_{21}]^{-1}) + [a_{21}]^{-1}[a_{11}]$$

$$+ [a_{11}]^{-1}[a_{21}] + 3$$

$$[H^0(\mathcal{F}_C \otimes \mathcal{E}_C)] = [a_{21}]^{-1}[a_{22}] + [a_{22}]^{-1}[a_{21}] + 2.$$
Since $O_C(p) = \wedge^3 E_C$, $O_C(2p) = \wedge^3 E_C^{\otimes 2}$, and $S_2F_C^\vee \cong S_2(a_{21} \oplus a_{22})^\vee \oplus a_1^2 \oplus L^{\otimes 2}$, the representations (7.14) are:

\[
\begin{align*}
[H^0(O_C(p))] &= [a_{11}]^{-2} [a_{22}]^{-1} [a_{21}]^{-1} [S_2H^0(L^\vee)] \\
[H^0(O_C(2p))] &= [a_{11}]^{-4} [a_{22}]^{-2} [a_{21}]^{-2} [S_1H^0(L^\vee)]
\end{align*}
\]

\[
\begin{align*}
[H^0(F_C^\vee)] &= [a_{11}]^{-1} [a_{21} \oplus a_{22}]^{(-1)} [H^0(L^\vee)] \\
[H^0(S_2F_C^\vee)] &= [S_2(a_{21} \oplus a_{22})]^{(-1)} [a_{11}]^{-2} \cdot [S_2H^0(L^\vee)].
\end{align*}
\]

In these formulas the representations $[a_{ij}]$ and $[H^0(L^\vee)]$ can be read off from Table 3.

$M^{11} - M^{0,2} \cap M^{11}$. $C \simeq P^1 \cup P^1$. If $f \in M^{11} - M^{0,2} \cap M^{11}$ and $C \simeq C^1 \cup C^2$ with $C^1 \simeq P^1 \simeq C^2$, then we may assume that $f(C)$ is one of the fixed line pairs in Table 4. Let $f^i$ be the restriction of $f$ to $C^i$. Applying (7.2), we find

\[
(7.19) \quad [T_f M^{11}] = [T_f \overline{M}_{0,0}(N, 1)] + [T_f \overline{M}_{0,0}(N, 1)] + [T_y C^1 : T_y C^2] \\
+ [T_y C^1] + [T_y C^2] - [T_y N],
\]

Formulas for all terms, except $[T_y N]$, in (7.19) were essentially found in Section 7.1. The exact sequence (3.4) yields

\[
(7.20) \quad [T_y N] = [F_y]^{-1} [E_y]((\lambda_0 + \lambda_1 + \lambda_2) - [E_y]^{-1} [E_y] - [F_y]^{-1} [F_y] + 1.
\]

The representations in these expressions obtained from Table 1.

The integrand representations (7.14) are found using component sequences similar to (7.2) on the integrand bundles:

\[
\begin{align*}
[H^0(O_C(p))] &= [H^0(O_{C^1}(p))] + [H^0(O_{C^2}(p))] - [O_y(p)] \\
[H^0(O_C(2p))] &= [H^0(O_{C^1}(2p))] + [H^0(O_{C^2}(2p))] - [O_y(p)]^2
\end{align*}
\]

\[
\begin{align*}
[H^0(F_C^\vee)] &= [H^0(F_{C^1}^\vee)] + [H^0(F_{C^2}^\vee)] - [F_y]^{(-1)} \\
[H^0(S_2F_C^\vee)] &= [H^0(S_2F_{C^1}^\vee)] + [H^0(S_2F_{C^2}^\vee)] - [S_2F_y]^{(-1)},
\end{align*}
\]

where all representations on the right-hand sides can be obtained from (7.13) and Table 3.

We now turn our attention to fixpoints $f$ in $M^{0,2} \cap M^{11}$. By our reasoning in Section 4.1, we may assume without loss of generality that $f : C \to N$ factors through $P(V^\vee_{(p,l)}) \to N$, where $(p, l)$ is the point-line pair $(x_1^2, x_1x_2)$. Since we are assuming that $M^{0,2}$ and $M^{1,1}$ intersect transversally,

\[
(7.22) \quad [T_f M^{1,1}] = [T_f \overline{M}_{0,0}(N, 2)] - [N_f],
\]
where \(N\) is the normal bundle of \(M^{0.2} \cap M^{1.1}\) in \(M^{0.2}\). From the fiber square in Remark 2, Section \([3]\), we obtain a \(T\)-equivariant diagram of exact sequences on \(M^{0.2} \cap M^{1.1}\)

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \longrightarrow TR \longrightarrow T(M^{0.2} \cap M^{1.1}) \longrightarrow \rho^*TI \longrightarrow 0 \\
\downarrow \\
0 \longrightarrow TR \longrightarrow TM^{0.2}_{M^{0.2} \cap M^{1.1}} \longrightarrow \rho^*TPP^{\vee}_{M^{0.2} \cap M^{1.1}} \longrightarrow 0 \\
\downarrow \\
N_f \longrightarrow \rho^*N_{I/PP} \\
\downarrow \\
0 \longrightarrow 0
\end{array}
\]

where \(TR\) is the relative tangent bundle of \(\rho\). The fibers of \(\rho^*N_{I/PP}\) are \(\text{Hom}(\mathcal{O}_p(-\tau), \mathcal{O}_l(\bar{\tau}))\), so with our choice of \((p, l)\),

\[
(7.23) \quad [N_f] = \lambda_0 \lambda_1^{-1}.
\]

We shall discuss the representations separately for the three types of fixpoints in \(M^{0.2} \cap M^{1.1}\). However, before we start, note that if \(L\) is a line of type 3)–5) in Table \([3]\) then \(TN_L \simeq \mathcal{O}_{p^1}(2) \oplus 2\mathcal{O}_{p^1}(1) \oplus 2\mathcal{O}_{p^1}(-1)\). In Appendix \([A]\) we show that the line bundle of degree \(-1\) in the split is \(T\)-equivariant to the line bundle

\[
(7.24) \quad H^0(I(1))^\vee \otimes \wedge^2 H^0(I(p^1(1)))^\vee \otimes V/H^0(I(p^1(1))) \otimes \mathcal{O}_L(-\bar{\sigma})
\]

\(M^{0.2} \cap M^{1.1}\), \(C \simeq C^1 \cup C^2\) and \(f\) an embedding. Assume \(f\) maps \(C \simeq C^1 \cup C^2\) onto two of the three lines 3)–5) in Table \([3]\). Then \((7.2)\) and \((7.1)\) yields

\[
(7.25) \quad [H^0(TN_C)] = [H^0(TN_{C^1})] + [H^0(TN_{C^2})] - [T_yN] + [H^1(TN_C)],
\]

\[
[T_f \overline{M}_{0,0}(N, 2)] = [T_f, \overline{M}_{0,0}(N, 1)] + [T_f \overline{M}_{0,0}(N, 1)] + [T_yC^1] + [T_yC^2]
\]

\[
+ [T_yC^1] + [T_yC^2] - [T_yN] + [H^1(TN_C)],
\]

where \(f^i: C^i \rightarrow N\) are restriction maps. Only \([H^1(TN_C)]\) remains to be determined. Since \(H^1(TN_C)\) is the cokernel of \(H^0(TN_{C^1}) \oplus H^0(TN_{C^2}) \rightarrow T_yN\), it follows from \((7.24)\) that, with our choice of \((p, l)\),

\[
(7.26) \quad [H^1(TN_C)] = \lambda_0 \lambda_1^{-2} \lambda_2^{-1}[O_y(-\bar{\sigma})].
\]

The integrand representations for these fixpoints receive the same formulas as in \((7.21)\).

\(M^{0.2} \cap M^{1.1}\) \(2:1\) covering with \(C \simeq C^1 \cup C^2\). This is similar to the previous case, except that the restriction maps \(f^1\) and \(f^2\) are the same. Hence we obtain the formulas \((7.25)\) and \((7.21)\) with \(C^1 = C^2\) and \(f^1 = f^2\).
$M^{0,2} \cap M^{11}$, a $2 : 1$ covering with $C \simeq \mathbb{P}^1$. Let $\mathcal{O}(1)$ be the tautological line bundle on $C$.

If $f : C \to L$ is a $2 : 1$ covering, then $f^* \mathcal{O}_L(\hat{\sigma}) = \mathcal{O}(2)$, and $f$ induces a $T$-action on $\mathcal{O}(1)$ with $[H^0(\mathcal{O}(1))] = [H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)}$. By (7.1), we have

$$[T_f M_{0,0}(\mathbb{N}, 2)] = [H^0(TN_C)] - [H^0(TC)].$$

Since $\mathcal{O}(1)$ is of degree 1 on $C$,

$$[H^0(TC)] = [H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)}[H^0(\mathcal{O}_L(\hat{\sigma}))]^{(-1/2)} - 1$$

The representation $H^0(TN_C)$ and the integrand representations are found by using the pullbacks of (7.7) and (7.24) to $C$. One may check that

$$[H^1(\mathcal{F}_C^\vee \otimes \mathcal{E}_C)](\lambda_0 + \lambda_1 + \lambda_2) = [H^1(TN_C)] + [H^1(\mathcal{E}_C^\vee \otimes \mathcal{E}_C)],$$

so the exact sequence (3.4) yields

$$(7.27) \quad [H^0(TN_C)] = [H^0(\mathcal{F}_C^\vee \otimes \mathcal{E}_C)](\lambda_0 + \lambda_1 + \lambda_2) - [H^0(\mathcal{E}_C^\vee \otimes \mathcal{E}_C)] - [H^0(\mathcal{F}_C^\vee \otimes \mathcal{F}_C)] + 1 + [H^1(\mathcal{F}_C^\vee \otimes \mathcal{F}_C)].$$

Here $[H^1(\mathcal{F}_C^\vee \otimes \mathcal{F}_C)] = \lambda_1^{-1} \lambda_2^{-1} [\wedge^2 H^0(\mathcal{O}_L(\hat{\sigma}))]^{-1/2}$. All the other terms in (7.27) are obtained by replacing $[H^0(\mathcal{O}_L(\hat{\sigma}))]$ with $[H^0(\mathcal{O}(2))] = [S_2 H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)}$ in (7.12). For the integrand representations, we obtain:

$$[H^0(\mathcal{O}_C(p))] = [\wedge^2 H^0(\mathcal{I}_p(1))]^{-1} \cdot [H^0(\mathcal{I}_l(1))]^{-2} \cdot [S_2 H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)}$$

$$[H^0(\mathcal{F}_C)] = [H^0(\mathcal{I}_l(1))]^{-1} \cdot ([\wedge^2 H^0(\mathcal{I}_p(1))]^{-1} + [S_2 H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)})$$

$$[H^0(\mathcal{O}_C(2p))] = [\wedge^2 H^0(\mathcal{I}_p(1))]^{-2} \cdot [H^0(\mathcal{I}_l(1))]^{-4} \cdot [S_4 H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)}$$

$$[H^0(S_2 \mathcal{F}_C)] = [H^0(\mathcal{I}_l(1))]^{-2} \cdot ([\wedge^2 H^0(\mathcal{I}_p(1))]^{-2} + [\wedge^2 H^0(\mathcal{I}_p(1))]^{-1} \cdot [S_2 H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)} + [S_4 H^0(\mathcal{O}_L(\hat{\sigma}))]^{(1/2)}).$$

**Appendix A. Tangent bundles over lines**

In this appendix, we compute the restriction of the tangent bundle $TN$ to lines $L$ on $N$. If $L$ is a line in $N$ with homogeneous coordinates $[u, t]$, then the differential map

$$\text{Hom}(\mathcal{F}_L, \mathcal{E}_L \otimes V) \to TN_L$$

can be represented by a $6 \times 6$-matrix with entries in $V^\vee = \text{Hom}(V, C)$ and $C[u, t]$. For lines 1)–6) in Section 4.4, we can decompose the vector bundles

$$(A.1) \quad \mathcal{E}_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L \oplus \mathcal{O}_L(-1) \quad \text{and} \quad \mathcal{F}_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(-1),$$

such that the universal maps $\mathcal{A}_L : \mathcal{F}_L \to \mathcal{E}_L \otimes V$ are represented by the matrices\footnote{Note that, up to a reparametrization of $L$, the $2 \times 2$-minors of the matrices in Table \ref{Table1} generate the nets in Section 4.4.} in Table \ref{Table1}.\footnote{Note that, up to a reparametrization of $L$, the $2 \times 2$-minors of the matrices in Table \ref{Table1} generate the nets in Section 4.4.}
Type | 1 | 2 | 3 | 4 | 5 | 6
--- | --- | --- | --- | --- | --- | ---
\( A_L \) | \( \begin{bmatrix} x_2 & ux_1 \\ x_1 & tx_2 \end{bmatrix} \) | \( \begin{bmatrix} x_2 & ux_2 + tx_1 \\ x_1 & tx_2 \end{bmatrix} \) | \( \begin{bmatrix} x_2 & ux_0 \\ x_1 & tx_2 \end{bmatrix} \) | \( \begin{bmatrix} x_2 & 0 \\ x_1 & ux_2 + tx_0 \end{bmatrix} \) | \( \begin{bmatrix} x_2 & tx_0 \\ x_1 & ux_0 \end{bmatrix} \) | \( \begin{bmatrix} x_2 & ux_0 \\ x_1 & ux_2 + tx_0 \end{bmatrix} \)

Let \( e_1, e_2, e_3, \) and \( f_1, f_2, \) respectively, be bases for the graded \( C[u, t] \)-modules in \( \text{(A.1)} \), with \( \deg e_1 = \deg e_2 = \deg f_1 = 0 \) and \( \deg e_3 = \deg f_2 = 1 \). If we represent elements in the \( C[u, t] \)-module \( \text{Hom}(F_L, E_L) \simeq F_L \otimes E_L \) as

\[
(A.2) \quad \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \sum a_{ij} f_i^\vee \otimes e_j,
\]

then the differential maps over lines 1)–6) can be represented by the matrices in Table 11. The procedure for finding the matrices in Table 11 is as follows:

1) For each line \( L \), use the matrix representation in Table 10 to write down a 6 × 13-matrix \( M \), with entries in \( V \) and \( C[u, t] \), representing the map

\[
\text{End}(E_L) \oplus \text{End}(F_L) \to F_L^\vee \otimes E_L \otimes V
\]

\[
(g, h) \mapsto (g \otimes \text{id}_V) \circ A_L - A_L \circ h
\]

with respect to our choice of basis \( \text{(A.2)} \) for \( F_L^\vee \otimes E_L \).

2) For each matrix \( M \), determine a 6 × 6-matrix \( D \), with entries in \( V^\vee \) and \( C[u, t] \), such that

i) \( D \cdot M = 0 \)

ii) rank \( D = 6 \) for all \([u, t] \in L\).

Remark. From Table 11, we can read off the decomposition of \( TN_L \). Since \( \deg f_1^\vee \otimes e_1 = \deg f_2^\vee \otimes e_2 = \deg f_2^\vee \otimes e_3 = 0 \), \( \deg f_1^\vee \otimes e_3 = 1 \), and \( \deg f_2^\vee \otimes e_1 = \deg f_2^\vee \otimes e_2 = -1 \), we find that for lines 1)–2)

\[
TN_L \simeq O_L(2) \oplus O_L(1) \oplus 4O_L,
\]

and for lines 3)–6)

\[
TN_L \simeq O_L(2) \oplus 2O_L(1) \oplus 2O_L \oplus O_L(-1),
\]

where the \( O_L(-1) \)-component is the image of \( f_1^\vee \otimes e_3 \otimes x_0 \).

More intrinsically, if \( \rho([L]) = (p, l) \), then there are canonical isomorphisms (see remark in Section 4.1)

\[
E_L = H^0(I_pI_l(2)) \oplus O(-\delta)
\]

\[
F_L \simeq H^0(I_p(1)) \otimes \left( \wedge^2 H^0(I_p(1)) \oplus O_L(-\delta) \right).
\]

This implies that the \( O_L(-1) \)-component of \( TN_L \) is canonically isomorphic to

\[
H^0(I_l(1))^\vee \otimes \wedge^2 H^0(I_p(1))^\vee \otimes V/H^0(I_p(1)) \otimes O_L(-\delta).
\]
Appendix B. Picard-Fuchs operators

This is an elaboration of sections 4.1–4.3 in [BvS].

Let $\mathbf{D} = \mathbb{C}[D,q]$ and $\mathbf{D}[t] = \mathbb{C}[D,q,t]$ be algebras of linear differential operators acting on $\mathbb{C}[[q]]$ and $\mathbb{C}[[q]][t]$, where $t$ is a formal variable, $q = e^t$, and $D = \frac{\partial}{\partial q} = q\frac{\partial}{\partial q}$. Hence $[D,q] = q$ and $[D,t] = 1$. A differential operator $\mathcal{P} \in \mathcal{D}$ of order $d$ and degree $m$ can be

| Type | $D$ |
|------|-----|
| 1)   | $\begin{pmatrix} x_0^\vee & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0^\vee & 0 & 0 & 0 & 0 \\ 0 & 0 & ux_2^\vee & 0 & 0 & -x_1^\vee \\ -tx_1^\vee & ux_2^\vee & 0 & x_2^\vee & -x_1^\vee & 0 \\ utx_2^\vee & -utx_1^\vee & 0 & -tx_1^\vee & ux_2^\vee & 0 \\ 0 & 0 & -tx_1^\vee & 0 & 0 & x_2^\vee \end{pmatrix}$ |
| 2)   | $\begin{pmatrix} x_0^\vee & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0^\vee & 0 & 0 & 0 & 0 \\ 0 & 0 & tx_2^\vee & 0 & 0 & -x_1^\vee \\ 0 & 0 & t^2x_2^\vee & 0 & tx_2^\vee - ux_1^\vee & -tx_1^\vee & 0 \\ 0 & x_1^\vee & 0 & tx_2^\vee - ux_1^\vee & -tx_1^\vee & 0 \\ 0 & 0 & -tx_1^\vee & 0 & 0 & tx_2^\vee - ux_1^\vee \end{pmatrix}$ |
| 3)   | $\begin{pmatrix} utx_2^\vee & -utx_1^\vee & 0 & -tx_0^\vee & ux_2^\vee & 0 \\ 0 & x_0^\vee & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0^\vee & 0 & 0 & 0 \\ tx_0^\vee & -ux_2^\vee & 0 & 0 & x_0^\vee & 0 \\ -ux_2^\vee & -utx_1^\vee & 0 & x_0 & ux_2^\vee \\ 0 & 0 & ux_2^\vee & 0 & 0 & x_0 \end{pmatrix}$ |
| 4)   | $\begin{pmatrix} x_0^\vee & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0^\vee & tx_1^\vee & 0 & 0 & -x_0^\vee \\ 0 & 0 & x_0^\vee & 0 & 0 & 0 \\ -tx_1^\vee & 0 & 0 & x_0^\vee & 0 & 0 \\ 0 & 0 & 0 & tx_2^\vee - ux_0^\vee & 0 & 0 \\ 0 & 0 & 0 & 0 & tx_2^\vee - ux_0^\vee & 0 \end{pmatrix}$ |
| 5)   | $\begin{pmatrix} utx_0^\vee & -tx_0^\vee & 0 & 0 & 0 & 0 \\ 0 & x_0^\vee & ux_1^\vee + tx_2^\vee & 0 & 0 & -x_0^\vee \\ 0 & 0 & x_0^\vee & 0 & 0 & 0 \\ -u^2x_1^\vee & 0 & 0 & ux_0^\vee & 0 & -tx_0^\vee \\ 0 & 0 & 0 & 0 & x_2^\vee & 0 \\ 0 & 0 & 0 & 0 & x_2^\vee & 0 \end{pmatrix}$ |
| 6)   | $\begin{pmatrix} tx_0^\vee & -ux_0^\vee & 0 & 0 & 0 & 0 \\ x_0^\vee & 0 & ux_1^\vee & 0 & 0 & -x_2^\vee \\ 0 & 0 & x_0^\vee & 0 & 0 & 0 \\ -ux_2^\vee - tx_1^\vee & ux_1^\vee & 0 & x_0^\vee & -x_2^\vee & 0 \\ -u^2x_0^\vee & u^2x_2^\vee & 0 & 0 & tx_2^\vee - ux_0^\vee & 0 \\ 0 & 0 & u^2x_2^\vee & 0 & 0 & tx_2^\vee - ux_0^\vee \end{pmatrix}$ |
written in the form
\[ \mathcal{P} = P_0(D) + qP_1(D) + \cdots + q^m P_m(D) \]
where \( P_0, \ldots, P_m \) are polynomials of degree \( \leq d \) with at least one equality. It is Picard-Fuchs if \( \deg P_0 = d \).

Let \( \psi_0 = \sum a_i q^i \) in \( \mathbb{C}[[q]] \). If \( P \) is a polynomial then \( P(D)q^n = q^n P(n) \), hence the coefficient of \( q^n \) in \( \mathcal{P}\psi_0 \) is
\[ a_n P_0(n) + a_{n-1} P_1(n-1) + \cdots + a_{n-m} P_m(n-m). \]
Thus \( \psi_0 = \sum a_i q^i \) is a solution to \( \mathcal{P} = 0 \) if and only if
\[ (B.1) \quad a_n P_0(n) + a_{n-1} P_1(n-1) + \cdots + a_{n-m} P_m(n-m) = 0 \]
for all \( n \). Note that the \( a_i \)'s are determined recursively by \( (B.1) \) once we have specified all the \( a_n \) where \( P_0(n) = 0 \).

**Lemma B.1.** \( [\mathcal{P}, t] = \mathcal{P}' \) where \( \mathcal{P}' \) is the formal derivative of \( \mathcal{P} \) with respect to \( D \).

**Proof.** See section 4 in [BvS]. \( \square \)

Next, suppose we are looking for \( \psi_1 = \sum b_i q^i \) such that \( t \cdot \psi_0 + \psi_1 \) and \( \psi_0 \) are solutions to \( \mathcal{P} \). If \( \mathcal{P}\psi_0 = 0 \) we have
\[ \mathcal{P}(t \cdot \psi_0 + \psi_1) = (\mathcal{P} \circ t)\psi_0 + \mathcal{P}\psi_1 = \mathcal{P}'\psi_0 + \mathcal{P}\psi_1. \]
Hence \( \psi_1 \) is determined recursively by the relations
\[ (B.2) \quad a_n P'_0(n) + \cdots + a_{n-m} P'_m(n-m) + b_n P_0(n) + \cdots + b_{n-m} P_m(n-m) = 0 \]
for all \( n \) where the \( a_i \)'s satisfy \( (B.1) \). To get started, we must specify \( b_n \) for all \( n \) such that \( P_0(n) = 0 \).

Finally, suppose we are looking for \( \psi_2 = \sum c_i q^i \) such that \( \frac{t^2}{2} \psi_0 + t\psi_1 + \psi_2, \ t\psi_0 + \psi_1 \) and, \( \psi_0 \) are solutions. If \( \mathcal{P}\psi_0 = 0 \) and \( \mathcal{P}(t\psi_0 + \psi_1) = 0 \), then
\[ \mathcal{P}\left(\frac{t^2}{2} \psi_0 + t\psi_1 + \psi_2\right) = \frac{1}{2}(\mathcal{P} \circ t)(t\psi_0 + \psi_1) + \frac{1}{2}(\mathcal{P} \circ t)\psi_1 + \mathcal{P}\psi_2 \]
\[ = \frac{1}{2}\mathcal{P}'(t\psi_0 + \psi_1) + \frac{1}{2}(\mathcal{P} \circ t)\psi_1 + \mathcal{P}\psi_2 \]
\[ = \frac{1}{2}(\mathcal{P} \circ t)\psi_0 + \frac{1}{2}\mathcal{P}'\psi_1 + \frac{1}{2}(\mathcal{P} \circ t)\psi_1 + \mathcal{P}\psi_2 \]
\[ = \frac{1}{2}\mathcal{P}'\psi_0 + \frac{1}{2}(t \circ \mathcal{P})\psi_0 + \mathcal{P}'\psi_1 + \frac{1}{2}(t \circ \mathcal{P})\psi_1 + \mathcal{P}\psi_2 \]
\[ = \frac{1}{2}\mathcal{P}'\psi_0 + \mathcal{P}'\psi_1 + \mathcal{P}\psi_2. \]
Hence \( \psi_2 \) is determined recursively by the relations
\[ (B.3) \quad \frac{1}{2} \sum_{i=1}^{m} a_{n-i} P'_i(n-i) + \sum_{i=1}^{m} b_{n-i} P'_i(n-i) + \sum_{i=1}^{m} c_{n-i} P_i(n-i) = 0 \]
for all \( n \) where the \( a_i \)'s and \( b_i \)'s satisfy \( (B.1) \) and \( (B.2) \). Again, \( c_n \) when \( P_0(n) = 0 \) must be specified.
Appendix C. The MAPLE code

Available from the author upon request.

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