Multi-mass collisional stellar systems

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In this paper the standard theory of collisional stellar systems is improved by considering the presence of a continuous mass distribution. The calculus of the diffusion coefficients is generalized and a new expression of the Fokker-Planck equation is found for multi-mass systems. A King-like distribution function, which validates the basic assumptions of most multi-mass models for Globular Clusters existing in literature, is obtained.

Keywords: Collisional systems, gravitational encounters, stellar dynamics, mass distribution

I. INTRODUCTION

Globular Clusters (GCs) are stellar systems belonging to the collisional ones, where the motion of stars is caused by the mean field of the other stars and perturbed by stellar encounters. Evolution of GCs is characterized by the relaxation time, the timescale over which stellar encounters become important in the dynamics of the system. Collisional systems presents a relaxation time less than their age and therefore the importance of stellar encounters has a great impact on the dynamical evolution of the clusters. They also present star evaporation \[15\] \[16\] \[4\] \[26\], a phenomenon which subtracts stars from the system, due to the presence of tidal forces induced by Galaxy, which provides a finite escape velocity and a finite boundary \[33\] \[16\].

The great importance of stellar encounters and the existence of an escape velocity reject a description based on the Gaussian velocity distribution, and require a velocity distribution produced by the effects of the stellar encounters (as highlighted by many authors, see references \[7\] \[31\] \[23\]). This distribution drops to zero at a finite limiting velocity. On the other hand, there exist in literature many works which distinguish about the characteristics of the relaxation process taking into account the so-called "resonant relaxation", which applies principally in multi-star systems in presence of a central Black Hole \[2\] \[3\] \[27\] \[1\]. Nevertheless, in our treatment, we consider the phenomenon of relaxation of stars in systems like GCs, where non-resonant relaxation appears to be the main effect in studying the evolution of such systems. In fact, several papers concerning stellar dynamics in GCs since many decades definitely consider King-like distribution functions, which arise from the Fokker-Planck equation in the formulation introduced by Chandrasekhar \[5\], because the conditions of validity of this approximation are fully satisfied.

In this view, the behavior of collisional systems is generally described by the collisional Boltzmann equation, as pointed out by Chandrasekhar in 1943 \[5\]

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \varphi \frac{\partial f}{\partial \mathbf{v}} = \Gamma(f),
\]

(1.1)

where \(f(r, v)\) is the star distribution function, \(\varphi\) is the gravitational potential, whose gradient gives the acceleration only due to the mean field, while \(\Gamma(f)\) is the collisional term, which gives the perturbations due to stellar encounters. In self-gravitating systems like GCs, binary encounters between stars make challenging the evaluation of the collisional term. However, with the Fokker-Planck approximation, which assumes local approximation and low energy exchanges, it is possible to write the collisional term in the following form

\[
\Gamma(f) = -\frac{\partial}{\partial v_i} [f(x, v) \langle \Delta v_i \rangle] + \frac{1}{2} \frac{\partial^2}{\partial v_j \partial v_i} [f(x, v) \langle \Delta v_i \Delta v_j \rangle],
\]

(1.2)

as shown by Chandrasekhar, in 1943 \[5\]. Here is used the convention that repeated indices are summed. The terms \(\langle \Delta v_i \rangle\) and \(\langle \Delta v_i \Delta v_j \rangle\) are called diffusion coefficients and describe the variation with time of the velocity and dispersion velocity due to stellar encounters. The diffusion coefficients can be calculated from the cumulus of binary stellar encounters in a homogeneous region between a test star with mass \(m\) and the field stars with mass \(m_a\) \[3\].

The work done by Chandrasekhar was also used for the studies on ionized gas, in particular from Cohen, Spitzer and Routly \[8\].

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Following Binney and Tremaine (see reference [4] and also [8] [28]), the diffusion coefficients must be inserted in the expression of $\Gamma(f)$ to rewrite the collisional term, for obtaining the Fokker-Planck equation in the Spitzer-Härm form

$$\frac{df}{dt} = \Gamma(f) = \frac{1}{t_R} \frac{\partial}{\partial x} \left[ 2xG(x) \left( 2x \frac{m}{m_a} f + \frac{\partial f}{\partial x} \right) \right], \quad (1.3)$$

where $x = v/(\sqrt{2} \sigma)$ is the dimensionless velocity and $\sigma$ the one-dimensional velocity dispersion. The function $G(x)$ and the relaxation time $t_R$ are given by

$$G(x) = \frac{2}{\sqrt{\pi} x^2} \int_0^x y^2 e^{-y^2} dy, \quad (1.4)$$

$$\frac{1}{t_R} = \frac{n_{0,m} \Gamma}{4\sqrt{2}\sigma^3}, \quad (1.5)$$

where $n_{0,m}$ is the numerical density of the field stars near the center (which is assumed to be constant) and $\Gamma = 4\pi G^2 m_a^2 \ln \Lambda$. The expression $\ln \Lambda = \ln \{b_{\max} V_0^2/[G(m + m_a)]\}$ is the Coulomb Logarithm, where $b_{\max}$ the maximum impact parameter and $V_0$ is the initial relative velocity magnitude.

A solution for equation (1.3) can be obtained by assuming that $f(x,t) = \exp (-\lambda t/t_R) g(x)$ where $\lambda$ is the evaporation rate [31]. Writing $g(x) = Ag(x)$, where $A$ is a normalization constant, the equation becomes

$$\frac{d}{dx} \left[ 2xG(x) \left( 2x \frac{m}{m_a} g + \frac{dg}{dx} \right) \right] + \lambda x^2 g = 0, \quad (1.6a)$$

$$\lambda = -\frac{t_R}{f} \frac{df}{dt}. \quad (1.6b)$$

The evaporation rate $\lambda$ gives the velocity at which the stars leave the system in time, as a consequence of the existence of a limited escape velocity [17]. Following King and expanding $g(x)$ in power series of $\lambda$, since the evaporation rate is typically very small [17], we found an approximated solution for the distribution function $g(x)$ and the evaporation rate, given by

$$\bar{g}(x) = \frac{e^{-x^2(m/m_a)} - e^{-x^2(m_2/m_a)}}{1 - e^{-x^2(m_2/m_a)}}, \quad (1.7a)$$

$$\lambda = \frac{8}{\sqrt{\pi}} \left( \frac{m}{m_a} \right)^\frac{5}{2} \frac{1}{e^{x^2(m/m_a)} - 1}, \quad (1.7b)$$

where $x_e = v_e/(\sqrt{2}\sigma)$ is the dimensionless escape velocity.

In the framework of single mass models we can assume that $m = m_a$. The escape velocity can easily be found by considering the kinetic energy necessary to reach the boundary of the configuration $r = R$, which leads to $v_e^2(r) = 2[\varphi_R - \varphi(r)]$.

Requiring that $g(x)$, in the limit of infinite escape velocity, returns a Boltzmann distribution function for the test stars with the correct normalization, such that the integration over velocities gives the (central) number density $n_{0,m}$ for the test stars, we found that

$$A = n_{0,m}/(2\pi\sigma^2)^{3/2}. \quad (1.8)$$

Note that in 1965 King [17] did not require this normalization and in 1966 he introduced a normalization factor $k$ (equation 3, reference [18]) which also includes the denominator in equation (1.7a). We found that the constant $k$ is related to our normalization as follows

$$k = \frac{A}{1 - e^{-x_e^2}} = \frac{n_{0,m}}{(2\pi\sigma^2)^{3/2}} \frac{1}{1 - e^{-(\varphi_R - \varphi_0)/\sigma^2}}. \quad (1.9)$$

where $\varphi_0 = \varphi(0)$. In order to extend the validity of the distribution function in the outer regions of the cluster, starting from the form valid near the center at $r = 0$ [18]

$$g(v)|_{r=0} = k(e^{-v^2/(2\sigma^2)} - e^{-(\varphi_R - \varphi_0)/\sigma^2})$$

$$= k e^{\varphi_0/\sigma^2}(e^{-E/(m\sigma^2)} - e^{-\varphi_R/\sigma^2}), \quad (1.10)$$

where the energy $E = m v^2/2 + m \varphi$ is calculated at the center. Using the Jeans theorem [13], we obtain

$$g(r,v) = k e^{-|\varphi(r) - \varphi_0|/\sigma^2}(e^{-v^2/(2\sigma^2)} - e^{-v^2(r)/(2\sigma^2)}), \quad (1.11)$$
where $\Delta m$ now can be written as $f_{\text{test star}}$ with mass $m_{\text{previous}}$. The equation must not depend on mass. Then, it is possible to write explicitly $f_{\text{test star}}$ and mass $m_{\text{previous}}$. The temporal variation of the distribution function of a collisional system, then it must be true that $\Gamma(m_{\text{tot}})$. This phenomenon is called mass segregation and has implication in the dynamical evolution and stability of these systems.

II. STELLAR DYNAMICS IN A MULTI-MASS COLLISIONAL SYSTEM

The presence of a mass distribution in stellar systems has brought some authors to study the structure of GCs by the King models. The existing multi-mass models of globular clusters (e.g., discrete multi-mass models by DaCosta & Freeman) with a measured mass function, e.g., continuous models by Merafina using, however, an initial mass distribution always present the hypothesis that stars follow a King-like distribution function and the total distribution function can be obtained integrating or summing on the different masses. However, in order to describe a multi-mass collisional system, remarking that the King distribution function is an approximated solution of the Boltzmann equation and insert all the possible differences due to the presence of a mass spectrum in the collisional Fokker-Planck equation taking into account stars with the same mass, one should consider stellar encounters between the test star and the field stars, that now have a mass spectrum. The first indication in this direction has been suggested by Michie. He considered the total distribution function as $f = \sum f_i(r, v; t)$, where $f_i$ is the distribution function of stars with a unique mass $m_i$. Each distribution function $f_i$ must satisfy the Boltzmann equation

$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + v \cdot \nabla f_i - \nabla \varphi \cdot \frac{\partial f_i}{\partial v} = \Gamma(f_i).$$

Michie defined the term on the right hand side as the temporal variation of the distribution function $f_i$, due to encounters between the star with mass $m_i$ and all the other ones. It should be noticed that not even Michie gives an explanation of the validity of the Boltzmann equation for each $f_i$. This assumption tends to keep the validity of the Boltzmann equation and insert all the possible differences due to the presence of a mass spectrum in the collisional term $\Gamma(f_i)$, which should be evaluated.

In this work we present a continuous approach to this topic. We consider the distribution function $f_m = f(r, v, m)$, that describes the number of stars with position between $r$ and $r + d^3r$, velocity between $v$ and $v + d^3v$ and mass between $m$ and $m + dm$. The total distribution function (i.e., the distribution function of the system), defined as the number of stars in an infinitesimal volume $d^3r d^3v$ of the phase space, can be obtained integrating all over the masses

$$f_{\text{tot}}(r, v) = \int_{\Delta m} f(r, v, m) dm,$$

where $\Delta m = m_{\text{max}} - m_{\text{min}}$ defines an arbitrary interval of masses, with finite extremes $m_{\text{min}}$ and $m_{\text{max}}$. As described before for the discrete case with $f_i$, the function $f_m$ satisfies the Boltzmann equation in the Michie description, that now can be written as

$$\frac{df_m}{dt} = \frac{\partial f_m}{\partial t} + v \cdot \nabla f_m - \nabla \varphi \cdot \frac{\partial f_m}{\partial v} = \Gamma(f_m),$$

where has been introduced $\Gamma(f_m)$ which is the collisional term due to the cumulus of stellar encounters between the test star with mass $m$ and the field stars, that have a mass spectrum defined in the interval $\Delta m$. The equation can be proved considering the total distribution function $f_{\text{tot}}(r, v)$. Since the Boltzmann equation describes the temporal variation of the distribution function of a collisional system, then it must be true that

$$\frac{df_{\text{tot}}}{dt} = \frac{\partial f_{\text{tot}}}{\partial t} + v \cdot \nabla f_{\text{tot}} - \nabla \varphi \cdot \frac{\partial f_{\text{tot}}}{\partial v} = \Gamma(f_{\text{tot}}),$$

where $\Gamma(f_{\text{tot}})$ is the collisional term that described the variation of $f_{\text{tot}}$ due to the cumulus of stellar encounters. The previous equation must not depend on mass. Then, it is possible to write explicitly $f_{\text{tot}}$ in its integral form:

$$\frac{d}{dt} \int_{\Delta m} f_m dm = \frac{\partial}{\partial t} \int_{\Delta m} f_m dm + v \cdot \nabla \int_{\Delta m} f_m dm - \nabla \varphi \cdot \frac{\partial}{\partial v} \int_{\Delta m} f_m dm = \Gamma(f_{\text{tot}}),$$

$$\int_{\Delta m} \frac{df_m}{dt} dm = \int_{\Delta m} \left( \frac{\partial f_m}{\partial t} + v \cdot \nabla f_m - \nabla \varphi \cdot \frac{\partial f_m}{\partial v} \right) dm = \int_{\Delta m} \Gamma(f_m) dm = \Gamma(f_{\text{tot}}).$$
The collisional term $\Gamma(f_m)$ of the single mass $m$, when integrated all over the masses belonging to the interval $\Delta m$, gives the total collisional term
\[
\int_{\Delta m} \left[ \frac{\partial f_m}{\partial t} + v \cdot \nabla f_m - \nabla \varphi \cdot \frac{\partial f_m}{\partial v} - \Gamma(f_m) \right] dm = 0 .
\]
(2.6)

This equation should be valid for every $\Delta m$, which is an arbitrary bounded range. Furthermore, the integrating function should be zero, bringing to the Boltzmann equation for $f_m$, in equation (2.3).

Like in the standard single mass theory, the collisional term is developed assuming local approximation and low energy exchanges (Fokker-Planck approximation). It is calculated from the probability that a star in a given position and velocity is subjected to a close encounter which changes only its velocity, while the change in position is negligible. Here, the assumptions are the same and the presence of a mass distribution does not affect the calculus of the collisional term expression. Then, the description by Chandrasekhar [6] can still be followed, bringing to the Boltzmann equation for $f_m$, which are described by a distribution function $f_a(v, m_a)$. The Rosenbluth potentials (see reference [28]) can be defined for each mass $m_a$ as $h(v, m_a)$ and $g(v, m_a)$
\[
h(v, m_a) = \int f_a(v, m_a) d^3v_a , \quad (3.1a)
\]
\[
g(v, m_a) = \int f_a(v, m_a) |v - v_a| d^3v_a . \quad (3.1b)
\]

Consequently, the multi-mass extension of the diffusion coefficients is given by
\[
\langle \Delta v_i \rangle_m = \int_{\Delta m_a} \Gamma \left( 1 + \frac{m}{m_a} \right) \frac{\partial h(v, m_a)}{\partial v_i} dm_a , \quad (3.2a)
\]
\[
\langle \Delta v_i \Delta v_j \rangle_m = \int_{\Delta m_a} \Gamma \frac{\partial^2 g(v, m_a)}{\partial v_i \partial v_j} dm_a , \quad (3.2b)
\]
where $\Gamma = \Gamma(m, m_a) = 4\pi G^2 m_a^2 \ln \Lambda$, also the Coulomb Logarithm $\ln \Lambda$, which has the same former expression, depends on $m$ and $m_a$. The subscript indicates the dependence on the mass $m$. Here isotropy is assumed again. We can define the diffusion coefficients for binary encounters as $\langle \Delta v_i \rangle_{m,m_a}$ and $\langle \Delta v_i \Delta v_j \rangle_{m,m_a}$, and then, integrating all over the masses $m_a$, we obtain
\[
\langle \Delta v_i \rangle_m = \int_{\Delta m_a} \langle \Delta v_i \rangle_{m,m_a} dm_a , \quad (3.3a)
\]
\[
\langle \Delta v_i \Delta v_j \rangle_m = \int_{\Delta m_a} \langle \Delta v_i \Delta v_j \rangle_{m,m_a} dm_a , \quad (3.3b)
\]
with
\[
\langle \Delta v_i \rangle_{m,m_a} = \Gamma \left( 1 + \frac{m}{m_a} \right) \frac{\partial h(v, m_a)}{\partial v_i} , \quad (3.4a)
\]
\[
\langle \Delta v_i \Delta v_j \rangle_{m,m_a} = \Gamma \frac{\partial^2 g(v, m_a)}{\partial v_i \partial v_j} . \quad (3.4b)
\]
This result basically preserve the standard procedure. Considering particular direction for the velocity vector (with respect to the reference frame) allows to find that

\[
\langle \Delta v_i \rangle_{m,m_a} = \Gamma \left(1 + \frac{m}{m_a}\right) \frac{\partial h(v, m_a)}{\partial v} , \quad (3.5a)
\]

\[
\langle \Delta v_i^2 \rangle_{m,m_a} = \Gamma \frac{\partial^2 g(v, m_a)}{\partial v^2} , \quad (3.5b)
\]

\[
\langle \Delta v_i^2 \rangle_{m,m_a} = 2 \Gamma \frac{1}{v} \frac{\partial g(v, m_a)}{\partial v} , \quad (3.5c)
\]

while \( \langle \Delta v_\perp \rangle_{m,m_a} = 0 \). These expressions can be used to write the diffusion coefficients

\[
\langle \Delta v_i \rangle_{m,m_a} = \frac{\nu_i}{v} \langle \Delta v_i \rangle_{m,m_a} , \quad (3.6a)
\]

\[
\langle \Delta v_i \Delta v_j \rangle_{m,m_a} = \frac{\delta_{ij}}{2} \langle \Delta v_i^2 \rangle_{m,m_a} + \frac{\nu_i \nu_j}{v^2} \left(\langle \Delta v_i \rangle_{m,m_a} - \frac{1}{2} \langle \Delta v_i^2 \rangle_{m,m_a}\right) , \quad (3.6b)
\]

where \( \delta_{ij} \) is the Kronecker delta.

The field stars distribution function \( f_a(v_a, m_a) \) can now be written as

\[
f_a(v_a, m_a) = \frac{A(m_a)}{(2\pi \sigma_a^2)^{3/2}} e^{-v_a^2/(2\sigma_a^2)} , \quad (3.7)
\]

where \( A(m_a) \) is a normalization term that also brings the physical dimensions, since \( f(v_a, m_a) d^3 v_a dm_a \) has the dimensions of a number density. The evaluation of \( A(m_a) \) is still an open problem to be addressed in a forthcoming paper. However, we noticed that it is related to the central numerical density of the field stars, defined as \( n_0(m_a) dm_a \), from

\[
n_0(m_a) = \int_0^\infty f_a(v_a, m_a) d^3 v_a = A(m_a) . \quad (3.8)
\]

In this way, the total central numerical density for the field stars is given by

\[
n_0 = \int_{\Delta m_a} n_0(m_a) dm_a = \int_A A(m_a) dm_a . \quad (3.9)
\]

The factor \( A(m_a) \) is a kind of weight function for the distribution function of stars with mass \( m_a \). It also contains information about the mass function of the system (see the discussion at end of section [V]).

In equation \( (3.7) \), the one dimension velocity dispersion \( \sigma_a \) is linked to the mass and the thermodynamic temperature \( \theta \), since \( \sigma_a^2 = k\theta/m_a \). The same is valid for \( \sigma \), because we have that \( \sigma^2 = k\theta/m \). Moreover, we noticed that in the standard case \( \sigma \) is used both in the distribution function of field stars with mass \( m_a \) (instead of a more appropriate \( \sigma_a \)) and also as a scale for the velocity \( v \) of the test star with mass \( m \). Here, we also distinguish the thermodynamic temperature \( \theta \) from the kinetic one, connected with the mean velocity of stars, due to truncated distribution function which limits the available phase space.

The distribution function of the field stars allows to evaluate the Rosenbluth potentials, leading to

\[
\langle \Delta v_i \rangle_{m,m_a} = -\frac{4\pi G^2 m_a (m + m_a) A(m_a) \ln \Lambda}{\sigma_a^2} G(x_a) , \quad (3.10a)
\]

\[
\langle \Delta v_i^2 \rangle_{m,m_a} = \frac{4\sqrt{2\pi} G^2 m_a^2 A(m_a) \ln \Lambda}{\sigma_a} \frac{G(x_a)}{x_a} , \quad (3.10b)
\]

\[
\langle \Delta v_i^2 \rangle_{m,m_a} = \frac{4\sqrt{2\pi} G^2 m_a^2 A(m_a) \ln \Lambda}{\sigma_a} \left[ \frac{\text{erf} (x_a) - G(x_a)}{x_a} \right] , \quad (3.10c)
\]

where \( x_a = v/(\sqrt{2}\sigma_a) \) and, consequently,

\[
G(x_a) = \frac{2}{\sqrt{\pi} x_a^3} \int_0^{x_a} y^2 e^{-y^2} dy , \quad (3.11a)
\]

\[
\text{erf} (x_a) = \frac{2}{\sqrt{\pi}} \int_0^{x_a} e^{-y^2} dy . \quad (3.11b)
\]

Then, the form for the diffusion coefficients become

\[
\langle \Delta v_i \rangle_{m,m_a} = -\frac{\nu_i}{v} \beta G(x_a) , \quad (3.12a)
\]

\[
\langle \Delta v_i \Delta v_j \rangle_{m,m_a} = \frac{\nu_i \nu_j}{v^2} \gamma A(x_a) - \delta_{ij} \gamma B(x_a) , \quad (3.12b)
\]
with \(A(x_a), B(x_a), \beta\) and \(\gamma\) given by

\[
A(x_a) = \frac{3G(x_a) - \text{erf}(x_a)}{2x_a},
\]

\[
B(x_a) = \frac{G(x_a) - \text{erf}(x_a)}{2x_a},
\]

\[
\beta = \frac{4\pi G^2 m_a (m + m_a) \ln \Lambda}{\sigma_a^2} A(m_a),
\]

\[
\gamma = \frac{4\sqrt{\pi} G^2 m_a^2 \ln \Lambda}{\sigma_a} A(m_a).
\]

**IV. THE MULTI-MASS FOKKER-PANCK EQUATION**

The diffusion coefficients \(\langle \Delta v_i \rangle_m\) and \(\langle \Delta v_i \Delta v_j \rangle_m\) must be inserted in the expression of the collisional term given in equation (2.7), considering their integral form in equation (3.3), leading to

\[
\Gamma(f_m) = -\frac{\partial}{\partial v_i} \left( f_m \int_{\Delta m_a} \langle \Delta v_i \rangle_{m,m_a} dm_a \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left( f_m \int_{\Delta m_a} \langle \Delta v_i \Delta v_j \rangle_{m,m,a} dm_a \right).
\]

(4.1)

Since only the diffusion coefficients \(\langle \Delta v_i \rangle_{m,m_a}\) and \(\langle \Delta v_i \Delta v_j \rangle_{m,m_a}\) depend on \(m_a\), being \(f_m = f(r, v, m)\), it can be written that

\[
\Gamma(f_m) = \int_{\Delta m_a} \Gamma(f_m, m_a) dm_a,
\]

(4.2)

where the integrating function is a partial collisional term that considers variation of \(f_m\) only due to encounters with a given mass \(m_a\), and has the following expression:

\[
\Gamma(f_m, m_a) = -\frac{\partial}{\partial v_i} \left( f_m \langle \Delta v_i \rangle_{m,m_a} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left( f_m \langle \Delta v_i \Delta v_j \rangle_{m,m_a} \right).
\]

(4.3)

In fact, like in the standard theory, equation (4.3) describes the temporal variation of the distribution function of the test stars with mass \(m\), due to encounters with the mass \(m_a\). However, differently from the standard case, making the partial derivative will lead to the use of \(x_a\) for the dimensionless expressions. We obtain

\[
\Gamma(f_m, m_a) = \beta G(x_a) \left( \frac{\partial f_m}{\partial v} + \frac{f_m}{v} \right) + \beta \frac{G(x_a)}{\sqrt{2\sigma_a}} \frac{f_m}{x_a} + \gamma \frac{1}{2} \left( \frac{\partial^2 f_m}{\partial v^2} - A(x_a) - B(x_a) \right) + \frac{1}{2} \frac{\partial^2 f_m}{\partial v} \left( \frac{2A(x_a) - B(x_a)}{v^2} \right) + \frac{2}{\sqrt{2\sigma_a}} \frac{\partial f_m}{\partial v}
\]

\[
\times \left( \frac{dA(x_a)}{dx_a} - \frac{dB(x_a)}{dx_a} \right) \left( \frac{2dA(x_a)}{dx_a} - \frac{dB(x_a)}{dx_a} \right) \frac{f_m}{v} + \frac{1}{2\sigma_a^2} \frac{\partial^2 A(x_a)}{dx_a^2} + \frac{\partial^2 B(x_a)}{dx_a^2} f_m \right) \right).
\]

(4.4)

Simplifying the expressions and writing the velocity \(v\) in terms of \(x_a\), is possible to demonstrate that

\[
\Gamma(f_m, m_a) = \frac{1}{t_R(m_a,m)} \frac{1}{x_a^2} \frac{\partial}{\partial x_a} \left[ 2x_a G(x_a) \left[ 2x_a f_m + \frac{\partial f_m}{\partial x_a} \right] \right],
\]

(4.5)

where

\[
\frac{1}{t_R(m_a,m)} = \frac{\gamma}{8\sigma_a^2} = \frac{\Gamma A(m_a)}{4\sqrt{\pi} \sigma_a^3}.
\]

(4.6)

Equation (4.5) is the Spitzer-Harm form of the Fokker-Planck equation in the standard treatment. The parameter \(t_R(m_a,m)\) is the relaxation time only due to gravitational encounters between the two masses and has the same meaning of the standard case. This is an expected result since, without integrating all over the masses, the collisional term is exactly the same.

Let now introduce \(x = v/\sqrt{2\sigma}\). Using the definition of \(x_a = v/\sqrt{2\sigma_a}\) is possible to write that

\[
\Gamma(f_m, m_a) = \frac{1}{t_R(m_a,m)} \left( \frac{\sigma_a}{\sigma} \right)^3 \frac{1}{x^2} \frac{\partial}{\partial x} \left[ 2x G(x_a) \left[ 2x \left( \frac{\sigma}{\sigma_a} \right)^2 m_a f_m + \frac{\partial f_m}{\partial x} \right] \right].
\]

(4.7)
If now we express how $\sigma$ depends on the mass, such that $\sigma^2 = k\theta/m$ and also $\sigma_a^2 = k\theta/m_a$, the equation becomes

$$
\Gamma(f_m, m_a) = \frac{1}{t_R(m_a, m)} \left( \frac{m}{m_a} \right)^{\frac{3}{2}} \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ 2xG(x_a) \left[ 2xf_m + \frac{\partial f_m}{\partial x} \right] \right\}.
$$

(4.8)

It should be noticed that $G(x_a)$ also depends on the mass $m_a$. However, since $x$ does not depend on $m_a$ and the function $G(x_a)$ is a multiplied factor, it can be defined the following ratio

$$
\frac{G(x, m)}{t_R(m)} = \int_{\Delta_m} \frac{1}{t_R(m_a, m)} \left( \frac{m}{m_a} \right)^{\frac{3}{2}} G(x_a) dm_a
$$

$$
= \int_{\Delta_m} \frac{1}{t_R(m_a, m)} \left( \frac{m}{m_a} \right)^{\frac{3}{2}} \frac{2}{\sqrt{\pi x^2}} \int_0^{\sqrt{m/m_a}} y^2 e^{-y^2} dy dm_a,
$$

(4.9)

where we introduced the function $G(x, m)$ and the relaxation time $t_R(m)$ valid for collisions of the mass $m$ with all the other ones. In the second equality has been made explicit $G(x_a)$ and its dependence on $x$.

The collisional term of the Fokker-Planck equation for the distribution function $f_m$ can now be written as

$$
\Gamma(f_m) = \frac{1}{t_R(m)} \frac{1}{x^2} \frac{\partial}{\partial x} \left\{ 2xG(x, m) \left[ 2xf_m + \frac{\partial f_m}{\partial x} \right] \right\},
$$

(4.10)

which is finally the Fokker-Planck equation for a multi-mass collisional system. It is similar to the standard one, despite the presence of the function $G(x, m)$ and the absence of the multiplied factor $m/m_a$. It should be recall that as usual, this equation is valid only in the center, at $r \approx 0$.

Furthermore, as said before, the relaxation time $t_R(m)$ is related only to the relaxation process on the mass $m$ with the field stars.

V. THE SOLUTION OF THE MULTI-MASS FOKKER-PLANCK EQUATION

It has been found that reproducing the King approach [17] lead to solve the equation (4.10).

Let assume that $f_m(x, t) = \exp \left[ -\lambda t/t_R(m) \right] g(x, m)$ where $\lambda$ is the evaporation rate of the stars with mass $m$. We write again that $g(x, m) = A(m) \bar{g}(x, m)$, where $A(m)$ is the normalization term that brings the dimensions. Then the multi-mass Fokker-Planck equation becomes

$$
\frac{d}{dx} \left\{ 2xG(x, m) \left[ 2x\bar{g}(x) + \frac{d\bar{g}(x)}{dx} \right] \right\} + \lambda x^2 \bar{g}(x) = 0,
$$

$$
\lambda = \frac{t_R(m)}{f_m} \frac{df_m}{dt},
$$

(5.1)

where the dependence on mass for $\bar{g}(x, m)$ has been omitted. With the use of the same expansion in series of $\lambda$, such that $\bar{g}(x) = \bar{g}_0(x) + \lambda \bar{g}_1(x) + \lambda^2 \bar{g}_2(x) + \ldots$ with the boundary conditions $\bar{g}(0) = 1$, $\bar{g}'(0) = 0$ and $\bar{g}(x_e) = 0$, considering the terms with the same power of $\lambda$, is possible to show that the following equations must be solved

$$
\frac{d\bar{g}_i(x)}{dx} + 2x\bar{g}_i(x) = 0,
$$

$$
\frac{d\bar{g}_{i+1}(x)}{dx} + 2x\bar{g}_{i+1}(x) = -\frac{1}{2xG(x, m)} \int_0^x \bar{g}_i(y) y^2 dy,
$$

(5.2)

where $i = 0, 1, 2, \ldots$ etc.

The first equation due to the boundary condition $\bar{g}_0(0) = 1$ leads to the solution $\bar{g}_0(x) = e^{-x^2}$. This is a particular results, since reproduce the Boltzmann dimensionless solution because $x^2 = v^2/(2\sigma^2)$. It is also interesting that having distinguished between $\sigma$ and $\sigma_a$ led to the vanishing of the presence of both masses $m$ and $m_a$ in the King mono-mass case. However, if this distinction was made in the standard case, the result would be the same: at the order zero of the King solution, the variable $x$ in his $g_0$ is in fact our $x_a$ and can easily be proved that $(m/m_a)x_a^2 = x^2$.

Requiring that at the zero order the time-independent part of the distribution function, that is $g(x, m)$, reproduce the Boltzmann distribution in equation (5.7) for the mass $m$ (by considering the dimensional factors), we obtain that

$$
A(m) = A(m) \left( \frac{m}{2\pi k\theta} \right)^{\frac{3}{2}},
$$

(5.3)
which therefore is similar to equation 113. The second equation can be described as $\tilde{g}_{i+1}(x) + 2x\tilde{g}_{i+1}(x) = Q_i(x)$, where the boundary condition $\tilde{g}_i(0) = 0$ leads to $\tilde{g}_{i+1}(x) = e^{-x^2} \int_0^x Q_i(t)e^{t^2} dt$, where

$$Q_i(t) = -\frac{1}{2t\tilde{g}_i(t, m)} \int_0^t \tilde{g}_i(y) y^2 dy = -\frac{\sqrt{\pi}t}{4} \int_0^t \tilde{g}_i(y) y^2 dy \left[ \int_{\Delta m_a} \left( \frac{m}{m_a} \right)^{\frac{5}{2}} \frac{t_R(m)}{t_R(m_a, m)} \left( \int_0^t \frac{y^2 e^{-y^2} dy}{m} \right) dm_a \right]^{-1}.$$  (5.4)

At the first order $\tilde{g}(x) = \tilde{g}_0(x) + \lambda \tilde{g}_1(x)$ is possible to calculate the previous ratio for $i = 1$ using the Michie approximation (also used by King [17]). We have

$$R = \frac{\int_0^t \int_{\Delta m_a} \frac{y^2 e^{-y^2} dy}{m}}{\int_0^t y^2 e^{-y^2} dy} \approx \frac{\int_0^\infty y^2 e^{-y^2} dy}{\int_0^\infty y^2 e^{-y^2} dy} = 1.$$  (5.5)

This approximation consider that the function $Q_i(t)$ is weighted with an exponential in the integration over $t$. This means that great values of $t$ are more important, while low values can be neglected. Note the importance of $A(m_a)$ in the integration over masses, whose knowledge may lead to future improvements, even in the approximations made. Consequently $\tilde{g}_1(x)$ becomes

$$\tilde{g}_1(x) \approx -\frac{\sqrt{\pi} e^{-x^2}}{4} \int_0^x e^{t^2} dt \left[ \int_{\Delta m_a} \left( \frac{m}{m_a} \right)^{\frac{5}{2}} \frac{t_R(m)}{t_R(m_a, m)} dm_a \right]^{-1} = -\frac{\sqrt{\pi}}{8} (1 - e^{-x^2}) \left[ \int_{\Delta m_a} \left( \frac{m}{m_a} \right)^{\frac{5}{2}} \frac{t_R(m)}{t_R(m_a, m)} dm_a \right]^{-1}.$$  (5.6)

From the global boundary condition $\tilde{g}(x_e) = \tilde{g}_0(x_e) + \lambda \tilde{g}_1(x_e) = 0$ we obtain the expression of the evaporation rate

$$\lambda = \frac{8}{\sqrt{\pi}} \left[ \int_{\Delta m_a} \left( \frac{m}{m_a} \right)^{\frac{5}{2}} \frac{t_R(m)}{t_R(m_a, m)} dm_a \right] \frac{1}{e^{x_e^2} - 1},$$  (5.7)

and, finally, the expression of $\tilde{g}(x, m)$, such that

$$\tilde{g}(x, m) = \frac{e^{-x^2} - e^{-x_e^2}}{1 - e^{-x_e^2}},$$  (5.8)

where the dependence on the mass is in $x$ and $x_e$. This distribution function has formally the same expression of the King solution in the single mass case, which directly shows that considering the cumulus of stellar encounters with a mass spectrum lead to the same expression of the distribution function.

What is really changed is the expression of the evaporation rate $\lambda$, which presents a complicated dependence on its variables, in particular the relaxation time $t_R(m)$, for which an analytical expression is under development and will be presented in a forthcoming paper. However, one can calculate $\lambda/t_R(m)$ which appear in the distribution function $f_m(x; t)$, if is able to evaluate the integral over $m_a$ and explicit the dependence in $t_R(m_a, m)$, as $A(m_a)$ which is still unknown.

The obtained results demonstrate the validity of the basic assumption of many multi-mass models in literature, that is considering a King-like distribution function for the generic mass $m$.

Different from the single mass model, the normalization term $A(m)$ depends on the mass as already assumed. Then, some multi-mass models in literature can be linked together by using a relation between the constants $A(m)$ which many authors define differently, but the physical meaning is always the same, that is a kind of weight function for each mass $9,11,20,25,29$.

Gathering the mass-dependent factors inside $k(m)$, such that

$$k(m) = \frac{A(m)}{1 - e^{-x_e^2}} = \frac{A(m)}{1 - e^{-m(x_e - \phi_0)/k\theta}} \left( \frac{m}{2\pi k\theta} \right)^{\frac{5}{2}},$$  (5.9)

we apply the Jeans theorem to the obtained solution for $g(x, m)$, bringing to a similar expression of the equation $11,11$ in function of the kinetic energy $\varepsilon$

$$g(r, \varepsilon, m) = k(m)e^{-m[\varepsilon(r) - \phi_0]/k\theta} \left( e^{-\varepsilon/k\theta} - e^{-\varepsilon_e(r)/k\theta} \right),$$  (5.10)
where \( \varepsilon = \frac{mv^2}{2} \), the cutoff kinetic energy \( \varepsilon_c(r) = \frac{mv^2}{2}(r)/2 \) and, as usual, \( \sigma^2 = k\theta/m \).

This distribution function can be used to study the equilibrium configuration of the system. It also brings information about the mass function of the system \( \xi(m) \) because, as said before, \( f(r, v, m; t) d^3 r d^3 v dm \) is dimensionless. Then, considering a single equilibrium configuration described by \( g(r, v, m) \), the mass function can be obtained marginalizing over positions and velocities. We have

\[
\xi(m) = \int_0^R \int_0^{v_{\text{esc}}} g(r, v, m) d^3 r d^3 v, \tag{5.11}
\]

where \( R \) is the radial edge of the system. In this way, the number of stars with mass between \( m \) and \( m + dm \) is \( dN(m) = \xi(m) dm \), which directly shows that the total number of stars for each configuration is obtained integrating the distribution function over masses, velocities and positions. Here, the normalization factor \( k(m) \) plays an important role and, with observations of the mass function, it is possible to constraint the knowledge about \( A(m) \).

### VI. CONCLUSIONS

In this paper are presented the improvements to the theory of stellar dynamics in multi-mass collisional systems with a continuous mass distribution. A new expression for the diffusion coefficients and the Fokker-Planck equation is derived, referred to the distribution function of a generic test star with mass \( m \). Following the approach by King in 1965 [17], a steady-state solution for the distribution function in a King-like form [18] is found, while the expression of the evaporation rate is slightly different. This prove the basic assumption of most multi-mass models for GCs in literature, that is describing the distribution function of stars as a King-like one, with the total distribution function obtained by integrating or summing over the different masses.

In the presented formulation, the distribution function of the test stars also brings information about the mass function of the system, which therefore changes slightly the meaning of the distribution function, which now is the number of particle in the seven-dimensional space of positions, velocities and masses.

A definite comprehension of the dynamical evolution of GCs with multi-mass models is in continuous development and many aspects are still open problems. In particular, the meaning of the factor \( A(m) \), its dependence on mass and its impact on the quantities in which is involved requires further studies.

The presence of different masses produce the phenomenon known as mass segregation, which tends to sink heavy stars and bring the lightest ones toward the outer regions. This affects the observational properties, such as the surface brightness profile, and also favours the evaporation of light stars, bringing to an evolution of the mass function.

Moreover, the studies by Merafina and Vitantoni [22] pointed out the relation between the King single mass model, the thermodynamic instability known as gravothermal catastrophe and the observations of Galactic GCs.

A better comprehension on the role of the mass function in the dynamical evolution and the thermodynamic instability of GCs is required and is still under discussion.

Moreover, the recent possibility of measuring the transverse velocity of the stars in GCs allows to analyze the internal kinematic and test multi-mass models with dynamical quantities, instead of only structural parameters and brightness profiles. This possibility enable to verify the possible phenomenon of the equipartition in presence of mass segregation [24].

Finally, the described theory must be improved by taking into account the presence of binary systems and their role in GCs evolution.

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