Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system

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Abstract

This paper contains two parts. In the first part, we shall show that the result given in the Żołądek’s example [1], which claims the existence of 11 small-amplitude limit cycles around a singular point in a particular cubic vector filed, is not correct. Mistakes made in the paper [1] leading to the erroneous conclusion have been identified. In fact, only 9 small-amplitude limit cycles can be obtained from this example after the mistakes are corrected, which agrees with the result obtained later by using the method of focus value computation [2]. In the second part, we present an example by perturbing a quadratic Hamiltonian system with cubic polynomials to obtain 10 small-amplitude limit cycles bifurcating from an elementary center, for which up to 5th-order Melnikov functions are used. This demonstrates a good example in applying higher-order Melnikov functions to study bifurcation of limit cycles.

Keywords: Bifurcation of limit cycles, Żołądek’s example, higher-order Melnikov function, Hamiltonian system, focus value.

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1. Introduction

The well-known Hilbert’s 16th problem [3] has been studied for more than one century, and the research on this problem is still very active today. To
be more specific, consider the following planar system:

\[
\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y),
\]  

where \(P_n(x, y)\) and \(Q_n(x, y)\) represent \(n\)th-degree polynomials of \(x\) and \(y\). The second part of Hilbert’s 16th problem is to find the upper bound, called Hilbert number \(H(n)\), on the number of limit cycles that system (1) can have. The progress in the solution of the problem is very slow. Even the simplest case \(n = 2\) has not been completely solved, though in early 1990’s, Ilyashenko & Yakovenko [4], and Écalle [5] proved that \(H(n)\) is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is \(H(2) \geq 4\), obtained more than 30 years ago [6, 7]. Recently, this result was also obtained for near-integrable quadratic systems [8]. However, whether \(H(2) = 4\) is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is \(H(3) \geq 13\) [9, 10]. Note that the 13 limit cycles obtained in [9, 10] are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems. A comprehensive review on the study of Hilbert’s 16th problem can be found in a survey article [11].

In order to help understand and attack Hilbert’s 16th problem, the so-called weak Hilbert’s 16th problem was posed by Arnold [12]. The problem is to ask for the maximal number of isolated zeros of the Abelian integral or Melnikov function:

\[
M(h, \delta) = \oint_{H(x,y)=h} Q(x, y) \, dx - P(x, y) \, dy,
\]  

where \(H(x, y), P(x, y)\) and \(Q(x, y)\) are all real polynomials of \(x\) and \(y\), and the level curves \(H(x, y) = h\) represent at least a family of closed orbits for \(h \in (h_1, h_2)\), and \(\delta\) denotes the parameters (or coefficients) involved in \(Q\) and \(P\). The weak Hilbert’s 16th problem itself is a very important and interesting problem, closely related to the study of limit cycles in the following near-Hamiltonian system [13]:

\[
\dot{x} = H_y(x, y) + \varepsilon P(x, y), \quad \dot{y} = -H_x(x, y) + \varepsilon Q(x, y),
\]  

where \(0 < \varepsilon \ll 1\) is a small perturbation. Studying the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the first-order Melnikov function \(M(h, \delta)\), given in (2).
When we focus on the maximum number of small-amplitude limit cycles, $M(n)$, bifurcating from an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was proved by Bautin in 1952 [14]. For $n = 3$, several results have been obtained (e.g. see [1, 2, 15]). Among them, in 1995 Žoladek [1] first constructed a rational Darboux integral to show 11 small-amplitude limit cycles, which was considered the best and was cited by many researchers in this field. The rational Darboux integral proposed by Žoladek [1] is given by

$$H_0 = \frac{f_1^5}{f_2^2} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4}, \quad (4)$$

which in turn generates the dynamical system in the form of

$$\dot{x} = x^3 + xy + 5x/2 + a,$$
$$\dot{y} = -ax^3 + 6x^2y - 3x^2 + 4y^2 + 2y - 2ax, \quad (5)$$

with the integrating factor $M = 20f_1^4f_2^{-5}$.

For $a < -2^{5/4}$, system (5) has a center $C_0 = (-a/2, -a^2/4 - 1/2)$ and five (real or complex) critical points $(r, -r^2 - 5/2 - a/r)$, where $r$ satisfies the polynomial equation $r^5 - 10r - 4a = 0$. In addition, system (5) has a saddle point and a non-elementary critical point at infinity. Let $h_0 = H_0(C_0) = -2/a$. Around $C_0$, there exists a family of periodic orbits given by $\gamma_h : H_0(x, y) = h$, $0 < h - h_0 \ll 1$. $\gamma_h$ approaches $C_0$ as $h \to h_0^+$.

Recently, Yu & Han [2] applied a different method to system (5) and only got 9 small-amplitude limit cycles. This difference motivated us to reconsider the Žoladek’s example and find that the result in [1] is not correct. In the next section, we shall present a detailed analysis on the Žoladek’s example and point out where the mistakes were made in the paper [1].

In the second part, we will present an example to demonstrate the use of higher-order Melnikov functions to obtain 10 small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with 3rd-degree polynomial functions. In general, a perturbed quadratic Hamiltonian system can be described by

$$\dot{x} = y + a_1xy + a_2y^2 + \varepsilon P(x, y, \varepsilon),$$
$$\dot{y} = -x + x^2 - \frac{1}{2}a_1y^2 + \varepsilon Q(x, y, \varepsilon), \quad (6)$$
where $P$ and $Q$ are $n$th-degree polynomials of $x$ and $y$ with coefficients depending analytically on the small parameter $\varepsilon$. When $\varepsilon = 0$, system (6) has a cubic Hamiltonian,

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + \frac{1}{2}a_1 xy^2 + \frac{1}{3}a_2 y^3,$$

(7)

and its parameters $a_1$ and $a_2$ take values from the set,

$$\Omega = \{-1 \leq a_1 \leq 2, \ 0 \leq a_2 \leq (1 - \frac{a_1}{2})\sqrt{1 + a_1}\}.$$

The Hamiltonian given in (7) is actually the so-called normal form [16] for all quadratic Hamiltonian systems which have a center at the origin. There exists a family of closed ovals around the origin given by $\Gamma_h$:

$$H(x, y) = h, \ h \in (0, \frac{1}{6}).$$

For any $h \in (0, \frac{1}{6})$ the displacement function $d(h, \varepsilon)$ of system (6) has a representation

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots,$$

(8)

where $M_i(h)$ is called the $i$th-order Melnikov function, particularly the higher-order Melnikov functions if $i \geq 2$. Then, we may determine the number of the limit cycles of system (6) emerging from the closed ovals $\{\Gamma_h\}$ by studying the zeros of the first non-vanishing Melnikov function $M_i(h)$ in $h \in (0, \frac{1}{6})$.

Suppose $M_1(h) \neq 0$ in (8). Denote $Z(n)$ the sharp upper bound of the number of zeros of $M_1(h)$. Gavrilov [17] proved $Z(2) = 2$ for the Hamiltonian $H$ with four distinct critical values (in a complex domain). Horozov & Iliev [18] obtained a linear estimate $Z(n) \leq 5(n + 3)$. Also, some sharp upper bounds were given for some particular cubic Hamiltonians: $n - 1$ for the Bogdanov-Takens Hamiltonian, $H = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3$ (see [19]), and $[\frac{2}{3}(n - 1)]$ for the Hamiltonian triangle, $H = \frac{1}{3}(x^2 + y^2) - \frac{1}{3}x^3 + xy^2$ (see [20]).

Moreover, for the Bogdanov-Takens Hamiltonian, there are some results on the upper bound of the number of zeros of the first nonvanishing higher-order Melnikov functions $M_k(h)$. Li & Zhang [21] got a sharp upper bound $2n - 2$ if $n$ is even or $2n - 3$ if $n$ is odd for $k = 2$. Iliev [22] obtained a sharp upper bound $3n - 4$ for $k = 3$, by applying the Françoise’s procedure [23] for computing higher-order Melnikov functions. The higher-order Melnikov functions can be also easily expressed via iterated integrals, which will be seen in the next section.
In this paper, we study the number of small-amplitude limit cycles in (6) bifurcating from the origin, using higher-order Melnikov functions. Hereafter we suppose $P$ and $Q$ are cubic polynomials with the following forms,

$$P(x, y, \varepsilon) = \sum_{m=1}^{\infty} \varepsilon^{m-1} P_m(x, y) \quad \text{with} \quad P_m(x, y) = \sum_{i+j=1}^{3} a_{ijm} x^i y^j,$$

$$Q(x, y, \varepsilon) = \sum_{m=1}^{\infty} \varepsilon^{m-1} Q_m(x, y) \quad \text{with} \quad Q_m(x, y) = \sum_{i+j=1}^{3} b_{ijm} x^i y^j.$$  

(9)

Our main result is given below, and its proof will be given in Section 4.

**Theorem 1.** Let the functions $P$ and $Q$ in (6) be given by (9). Then system (6) can have \[\frac{4k+11}{3}\] small-amplitude limit cycles around the origin, when $M_k(h)$ is the first non-vanishing Melnikov functions in (8), $1 \leq k \leq 5$.

**Remark 1.** It follows from Theorem 1 that 10 small-amplitude limit cycles exist in the vicinity of the origin of system (6) when $k = 5$, i.e., $M(3) \geq 10$.

The rest of the paper is organized as follow. In the next section, we consider the Žoladek’s example [1], and show that the result given in [1] is not correct. In Section 3, we present some preliminary results for polynomial one-forms with respect to the Hamiltonian (7). Then, in Section 4 by choosing special forms for the polynomials $P$ and $Q$ without loss of generality, we prove Theorem 1. Finally, conclusion is drawn in Section 5.

### 2. Žoladek’s example

In this section, we consider the Žoladek’s example, described by [1] and [3], and briefly describe the method used by Žoladek. Suppose the polynomial perturbed system (5) is described by

$$\dot{x} = M^{-1} H_{0y} + \varepsilon p(x, y, \varepsilon),$$

$$\dot{y} = -M^{-1} H_{0x} + \varepsilon q(x, y, \varepsilon),$$

(10)

where $p(x, y, \varepsilon)$ and $q(x, y, \varepsilon)$ are polynomials of $x$ and $y$ with coefficients depending analytically on the small parameter $\varepsilon$ and $\max(\deg(p), \deg(q)) \leq 3$.

Let $S$ be a section transversal to the closed orbit $\gamma_h$. Using $H_0 = h$ as a parameter, $0 < h - h_0 \ll 1$, we define the first return map $P(h, \varepsilon)$ of system
and thus the corresponding displacement function, \( d(h, \varepsilon) = P(h, \varepsilon) - h \) has the form

\[
d(h, \varepsilon) = \varepsilon \int_{L(h, \varepsilon)} M(qdx - pdy) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + O(\varepsilon^3),
\]

(11)

where \( L(h, \varepsilon) \) is a trajectory of the perturbed system (10). We can use the first non-vanishing Melnikov function \( M_k(h) \) in (11) to investigate the number of the limit cycles around the center \( C_0 \). Generally, the zeros of \( M_k(h) \) correspond to the number of the limit cycles of system (10).

Let \( \varpi = qdx - pdy, \deg(\varpi) = \max(\deg(p), \deg(q)) \). Then, the first-order Melnikov function \( M_1(h) \) can be expressed in the form of

\[
M_1(h) = \oint_{\gamma_h} M_{\varpi} \bigg|_{\varepsilon=0} = h \oint_{\gamma_h} \frac{\varpi}{f_1 f_2} \bigg|_{\varepsilon=0}.
\]

When \( M_1(h) \equiv 0 \), we may use an iterated integral to express the second-order Melnikov function \( M_2(h) \). The first integral of system (10) can be approximated as

\[
H_{\varepsilon} = H_0 - \varepsilon H_1,
\]

where the function \( H_1(B) \) is the integral

\[
H_1(B) = \int_A M_{\varpi} \big|_{\varepsilon=0}, \text{ computing along the } \gamma_h, \text{ with } A = \gamma_h \cap S \text{ and } B \in \gamma_h.
\]

Thus, for system (10) we have the second-order Melnikov function, given by

\[
M_2(h) = \left\{ \frac{d}{d\varepsilon} \left( \int_{H_{\varepsilon}=h} M_{\varpi} \right) \right|_{\varepsilon=0}.
\]

(12)

Suppose that the polynomials \( p \) and \( q \) are expanded as

\[
p(x, y, \varepsilon) = p_1(x, y) + \varepsilon p_2(x, y) + O(\varepsilon^2),
\]

\[
q(x, y, \varepsilon) = q_1(x, y) + \varepsilon q_2(x, y) + O(\varepsilon^2).
\]

Further, let \( \varpi_i = q_i dx - p_i dy, i = 1, 2 \). Then, (12) can be rewritten as

\[
M_2(h) = \left\{ \frac{d}{d\varepsilon} \left( \int_{H_{\varepsilon}=h} M_{\varpi_1} \right) \right|_{\varepsilon=0} + \oint_{\gamma_h} M_{\varpi_2}
\]

\[
= \oint_{\gamma_h} \frac{d(M_{\varpi_1})}{dH_0} H_1 + h \oint_{\gamma_h} \frac{\varpi_2}{f_1 f_2},
\]

(13)

where \( \frac{d(M_{\varpi_1})}{dH_0} \) is the Gelfand-Leray form (see [24]).

In [1], Żołdek studied small-amplitude limit cycles of system (10), bifurcating from the center \( C_0 \), by using the second-order Melnikov function
M_2(h). More precisely, for \( 13 \), the author chose twelve Abelian integrals \( I_{\omega_j}(h) = \oint_{\gamma_h} \omega/(f_1f_2) \), \( i = 1, \ldots, 12 \), where one-forms \( \omega_i \) are given as follows:

\[
\omega_k = x^{k-1}dx, \quad k = 1, 2, 3, 4, \quad \omega_5 = (18x^2 + 18y)dx, \quad \omega_6 = xydx,
\]

\[
\omega_7 = x^2ydx, \quad \omega_8 = xy^2dx, \quad \omega_9 = y^3dx, \quad \omega_{10} = xy^2dy, \quad \omega_{11} = y^3dy,
\]

\[
\omega_{12} = y^2(5 - 3x^2)dx + xy(x^2 + 1)dy.
\]

Then, by showing the independency of the integrals \( I_{\omega_j}(h) \), the author claimed that 11 small-amplitude limit cycles could bifurcate from the center \( C_0 \) after suitable cubic perturbations.

Later, system \( 10 \) was re-investigated by using the method of focus values computation \( 2 \). Based on the computation of \( \varepsilon \)-order and \( \varepsilon^2 \)-order focus values, the authors of \( 2 \) showed that system \( 10 \) has maximal 9 small-amplitude limit cycles bifurcating from the center \( C_0 \). This obvious difference raises a question: which conclusion is correct? If the result of 11 limit cycles is not correct, then what possible mistakes were made in the article \( 1 \)? In the following, we will answer these questions.

In \( 1 \), a key part in the proof of the existence of 11 limit cycles is the lemma in Section 5.1, which states that the eleven integrals, \( I_{\omega_j}(h), \quad j = 1, \ldots, 11, \) form a basis of the linear space of integrals \( I_\omega(h) = \oint_{\gamma_h} \omega/(f_1f_2) \), \( \text{deg}(\omega) \leq 3 \). We will show that this is not true. Firstly, we find the relation \( aI_{\omega_4}(h) - I_{\omega_5}(h) = 0 \), showing that \( I_{\omega_4}(h) \) and \( I_{\omega_5}(h) \) are linearly dependent. This can be seen from the proof of the lemma \( 1 \), where the author obtained nine one-forms \( \eta_j, \quad j = 1, \ldots, 9 \) such that \( I_{\eta_j}(h) = 0 \), where

\[
\eta_1 = (x^3 + 2x)dx + dy,
\]

\[
\eta_2 = (-3ax^2 + 12xy - 6x - 2a)dx - (3x^2 + y + 5/2)dy,
\]

\[
\eta_3 = (6x^2 + 8y + 2)dx - xdy,
\]

\[
\eta_4 = (-3ax^3 + 12x^2y - 6x^2 - 2ax)dx - (2x^2 - a)dy,
\]

\[
\eta_5 = (ax^3 + 3x^2 + 4y^2 + 2ax)dx - xydy,
\]

\[
\eta_6 = (-ax^3 + 6x^2y - 3x^2 + 4y^2 + 2y - 2ax)dx - (x^3 + x + 5/2x + a)dy,
\]

\[
\eta_7 = (3ax^2y - 12xy^2 + 6xy + 2ay)dx - (3x^2y - ax^2 - 3x^2 + 3y^2 - 1/2y - 2ax)dy,
\]

\[
\eta_8 = (-5x^3 - 7xy + 1/2x + a)dx + x^2dy,
\]

\[
\eta_9 = (\frac{21}{2}xy - 7xy^2 + ay)dx + (2x^2y - \frac{3}{2}x^2 + ax + y)dy.
\]

It is easy to show that \( (3a\eta_1 - 5\eta_3 - \eta_4)/2 - \eta_5 + \eta_6 = a\omega_4 - \omega_5 \), which yields
\[ aI_{\omega_4}(h) - I_{\omega_5}(h) = 0. \]

Secondly, we find another one-form \( \eta_{10} \), given by
\[
\eta_{10} = \left[ -\frac{29}{3}ax^3 - \frac{5}{3}y^3 - (2a^2 - \frac{5}{2})x^2 - 9axy + 6y^2 + \frac{13}{6}ax + a^2 \right]dx + xy^2dy,
\]
such that the integral \( I_{\eta_{10}}(h) \) vanishes near \( h_0 \). Therefore, based on these ten one forms \( \eta_j, 1 \leq j \leq 10 \), we can remove \( I_{\omega_5}(h) \) from the basis without adding another integral. Thus, the number of integrals in the basis claimed in [1] should be one less.

Next, consider the integral \( I_{\omega_{12}}(h) \), where \( \omega_{12} = y^2(5 - 3x^2)dx + xy(x^2 + 1)dy \), which was used in [1] when the second-order Melnikov function was considered. Obviously, \( \deg(\omega_{12}) = 4 \). In Remark 7 of [1], the author showed that \( I_{\omega_{12}}(h) \) could appear in the second-order Melnikov function \( M_2(h) \) of system (10), with a suitable perturbation.

However, \( I_{\omega_{12}}(h) \) has no contribution to generate small-amplitude limit cycles in the vicinity of \( C_0 \), since for \( x^2y^2dx \) and \( x^3ydy \) in \( \omega_{12} \), we can show that there are two one-forms \( \xi_1 \) and \( \xi_2 \), given by
\[
\xi_1 = \left[ x^2y^2 - \frac{1}{36}a(a^2 + 144)x^3 + (a^2 + 3)x^2y + \frac{13}{6}axy^2 - \frac{55}{3}y^3 \right.
+ \frac{1}{24}(3a^4 - 28a^2 + 46)x^2 + \frac{1}{6}a(6a^2 - 7)xy + \frac{1}{6}(3a^2 + 4)y^2
+ \frac{1}{72}a(9a^4 + 36a^2 + 77)x + \frac{1}{12}(3a^4 - 10a^2 + 12)y
\]
\[
+ \frac{1}{288}(9a^6 + 36a^4 + 100a^2 + 80)]dx
\]
\[
\xi_2 = \left[ \frac{1}{48}a(a^2 - 60)x^3 - \frac{3}{2}(a^2 + 4)x^2y - \frac{19}{4}axy^2 + \frac{5}{3}y^3 \right.
- \frac{1}{16}(3a^4 + 70a^2 - 16)x^2 - \frac{1}{2}a(3a^2 - 4)xy - (3a^2 + 2)y^2
\]
\[
+ \frac{1}{48}a(3a^4 - 36a^2 + 137)x - \frac{1}{16}(15a^4 + 2a^2 + 48)y + \frac{1}{32}a^4 + \frac{1}{48}a^2 - \frac{5}{6}]dx
\]
\[
+ \left[ x^3y + \frac{1}{4}(a^2 + 2)x^3 + \frac{3}{2}axy^2 + \frac{3}{8}a(a^2 + 2)x^2 + \frac{3}{4}a^2xy \right.
\]
\[
+ \frac{3}{16}a^2(a^2 + 2)x + \frac{1}{8}a^3y + \frac{1}{32}a^3(a^2 + 2)]dy,
\]
satisfying \( I_{\xi_1}(h) = I_{\xi_2}(h) = 0 \) for \( 0 < h - h_0 \ll 1 \). This implies that \( I_{\omega_{12}}(h) \) can be expressed as a linear combination of integrals \( I_\omega(h) \), \( \deg(\omega) \leq 3 \), for \( h \) near \( h_0 \).

Summarizing the above results show that \( I_{\omega_5}(h) \) and \( I_{\omega_{12}}(h) \) can be removed from the basis, since they can be expressed as linear combinations of other basis. Thereby, now there are only ten of the integrals chosen in [1] left. This clearly indicates that at most 9 (not 11) small-amplitude limit cycles may appear in the vicinity of the center \( C_0 \).

In [1], having obtained the twelve integrals \( I_{\omega_j}(h), 1 \leq j \leq 12 \), in order to show the existence of 11 limit cycles, the author tried to prove the independence of the twelve integrals. First, the author showed that \( I_{\omega_{11}}(h) \) is
independent of the other integrals \( I_{\omega j}(h), j \neq 11 \), by considering the behavior of the integrals at \( h = 1 \). In order to prove the independency of the integrals \( I_{\omega j}(h), j \neq 11 \), the author considered the integrals \( I_\omega \) as functions of two variables \( h \in \mathbb{C} \) and \( a \in \mathbb{C} \). With prolonging \( I_\omega(h, a) \) to the point \( a = 0 \), the author used the independency of the integrals, \( I_{\omega j}(h, a), j \neq 11 \), for \( a \) close to 0, to determine their independence for generic \( a \in \mathbb{C} \). The closed orbit \( \gamma_h \) has the form

\[
\gamma_h = \{(x, y) : x = \epsilon e^{i\theta}, y = -\frac{1}{2} + \frac{5}{2} \epsilon e^{-i\theta} + O(\epsilon^2), \theta \in [0, 2\pi]\}
\]

for \( a = 0 \) and \( h \) is close to the critical value \( h_0 = -2/a = \infty \).

In addition, the author introduced new variables \( u \) and \( v \) in the following form,

\[
u = \frac{1}{H_0^{1/4} x}, \quad H_0 v^4 = 1 + \frac{4}{x^2} + \frac{4y}{x^4},
\]

Let \( K_{i,j} = \int_{\delta_h} u^i v^j dv \), where \( \delta_h \) is the image of the closed orbit \( \gamma_h \) defined in (14) under the variables change. In Lemma 3 of [1], the integrals \( I_{\omega j}(h, 0) \) and some partial derivatives with respect to \( a \), for \( h \) close to \( h_0 \), are expressed in terms of \( h \) and \( K_{i,j} \). Using these expressions and eleven independent functions: 1, \( \tau - 1 \), \((\tau - 1)/\tau\), \( g_{i,j} = h^{1/4} K_{i,j}(h) \), \( (i,j) = (-1,0), (-1,-1), (-2,0), (-2,-1), (1,0), (1,-1) \) and \( h^{1/2} K_{-2,2}, h^{1/4} K_{-3,2} \), the author claimed the independency of integrals \( I_{\omega j}(h,a), j \neq 11 \) for \( a \) close to 0. Especially, the expressions for \( I_{\omega 5} \) and its derivatives with respect to \( a \) in Lemma 3 were given by

\[
\begin{align*}
I_{\omega 5}(h, 0) &= 0, \\
\frac{\partial I_{\omega 5}}{\partial a}(h, 0) &= I_{\omega 4}(h, 0), \\
\frac{\partial}{\partial a} (\frac{\partial I_{\omega 5}}{\partial a} - I_{\omega 4})(h, 0) &= \frac{9}{56} h^{1/2} K_{-2,2} + \sum_{j=-2}^{0} \alpha_j h^{-1/2} K_{-2,j}.
\end{align*}
\]

Based on the third equation of (15), the author claimed the independence of \( I_{\omega 5} \) from other integrals. But this is not correct since we have already shown that \( a I_{\omega 4}(h) - I_{\omega 5}(h) = 0 \), implying that the third equation of (15) should be replaced by

\[
\frac{\partial}{\partial a} (\frac{\partial I_{\omega 5}}{\partial a} - I_{\omega 4} - a \frac{\partial I_{\omega 4}}{\partial a})(h, 0) \equiv 0,
\]
which is the correct second-order derivative of the function \( F(h, a) = I_{\omega_3}(h) - aI_{\omega_4}(h) \). For the integral \( I_{\omega_2}(h) \), the author made a similar error in the proof of its independence from other integrals.

3. Cubic Hamiltonians with cubic perturbations

In order to prove Theorem 1, we need some preliminary results for cubic Hamiltonians with cubic perturbations. Let \( \omega_{ij} = x^i y^j \) and \( \sigma_{ij} = x^i y^j \).

Lemma 1. For the cubic Hamiltonian given in (7), the following identities hold.

(a) \( \sigma_{ij} = \frac{1}{j+1} d(x^i y^j) - \frac{i}{j+1} \omega_{i-1,j+1} \);
(b) \( \omega_{ij} = \omega_{i-1,j} + \frac{i-2}{2j+4} a_1 \omega_{i-2,j+2} - \frac{i-2}{j+3} a_2 \omega_{i-3,j+3} - x^{i-2} y^j dH + d(\frac{1}{j+2} x^{i-2} y^j + \frac{a_1}{j+2} x^{i-1} y^j + \frac{a_2}{j+3} x^{i-2} y^j + a_3), \ i \geq 2; \)
(c) \( \omega_{0,j} = \frac{3j}{a_2(j+1)} \left[ H \omega_{0,j-3} + \frac{1}{6} \omega_{1,j-3} - \frac{a_1(j-3) + 6j - 2}{12(j-1)} \omega_{0,j-1} - \frac{a_1(j+1) + 6j + 2}{3(j-1)} \omega_{1,j-1} + r_{0,j}(x, y) dH + dR_{0,j}(x, y) \right], \ j \geq 3; \)
(d) \( \omega_{1,j} = \frac{3j}{a_2(j+2)} \left[ H \omega_{1,j-3} - \frac{(j+2)a_1}{6(j+1)} \omega_{0,j+1} + \frac{a_2}{6} \omega_{0,j} - \frac{a_1(j+3) + 6j + 2}{12(j-1)} \omega_{1,j-1} - \frac{a_1(j-3) + 6j - 2}{12(j-1)} \omega_{0,j-1} + r_{1,j}(x, y) dH + dR_{1,j}(x, y) \right], \ j \geq 3; \)

where \( r_{i,j}(x, y) \) and \( R_{i,j}(x, y) \) are polynomials in \( x \) and \( y \) with degrees \( i + j - 2 \) and \( i + j + 1 \), respectively.

Proof. A direct calculation using integration by parts yields formula (a).

From the Hamiltonian, we have the equation \( \frac{1}{3} x^3 = \frac{1}{2} (x^2 + y^2) + \frac{1}{2} a_1 x y^2 + \frac{1}{3} a_2 y^3 - H \), giving
\[
x^2 dx = xdx + ydy + \frac{a_1}{2} y^2 dx + a_1 xy dy + a_2 y^2 dy - dH,
\]
which in turn yields
\[
\omega_{i,j} = \omega_{i-1,j} + \sigma_{i-2,j+1} + \frac{a_1}{2} \omega_{i-2,j+2} + a_1 \sigma_{i-1,j+1} + a_2 \sigma_{i-2,j+2} - x^{i-2} y^j dH, \ i \geq 2.
\]

Then, combining (16) with the formula (a) results in the formula (b).

Similarly, the equation \( \frac{1}{3} a_2 y^3 = H - \frac{1}{2} (x^2 + y^2) + \frac{1}{2} x^3 - \frac{1}{2} a_1 x y^2 \) generates
\[
\frac{1}{3} a_2 \omega_{i,j} = H \omega_{i,j-3} - \frac{1}{2} \omega_{i+2,j-3} - \frac{1}{2} \omega_{i,j-1} + \frac{1}{3} \omega_{i+3,j-3} - \frac{1}{2} a_1 \omega_{i+1,j-1}, \ j \geq 3.
\]

Finally, the formulas (c) and (d) follow the formula (b) and (17). □
From Lemma 1, we know that any polynomial one-form $\omega$, $\deg(\omega) = m$, can be expressed in the form of

$$\omega = r(x, y) dH + dR(x, y) + \sum_{i=0,1}^{m-i} \sum_{j=0}^{1} \alpha_{i,j} \omega_{i,j}.$$ 

The next lemma shows that there also exist some relationships among the one-forms $\omega_{i,j}$, $i = 0, 1$.

**Lemma 2.** For any non-negative integer $m \mod 3 \neq 2$, there exist $\beta_{i,j,m}$, $\tilde{r}_m(x, y)$ and $\tilde{R}_m(x, y)$ satisfying the following identity

$$\sum_{i=0,1}^{m-i} \sum_{j=0}^{1} \beta_{i,j,m} \omega_{i,j} = \tilde{r}_m(x, y) dH + d\tilde{R}_m(x, y),$$

where $\tilde{R}_m(x, y)$ and $\tilde{r}_m(x, y)$ are polynomials of degrees $m + 1$ and $m - 1$ in $x$ and $y$, respectively; and $\beta_{i,j,m}$ are polynomials in $a_1$ and $a_2$, with $\beta_{0,0,0} = \beta_{1,0,1} = 1$, $\beta_{0,1,1} = 0$, and

$$\beta_{0,m+3,m+3} = \frac{m + 4}{3(m + 3)} (a_2 \beta_{0,m,m} + a_1^2 \beta_{1,m-1,m}),$$

$$\beta_{1,m+2,m+3} = \frac{m + 4}{3(m + 2)} (a_1 \beta_{0,m,m} + a_2 \beta_{1,m-1,m}),$$

if $\beta_{1,-1,0}$ is defined as $\beta_{1,-1,0} = 0$.

**Proof.** We use the method of mathematics induction to prove this lemma. It is easy to see that the conclusion is true for $m = 0, 1$. Now, suppose (18) holds for $m \mod 3 \neq 2$. Then, we prove that (18) also holds for $m + 3$. Multiplying (18) by $H$ on both sides yields

$$\sum_{i=0,1}^{m-i} \sum_{j=0}^{1} \beta_{i,j,m} H \omega_{i,j} = H \tilde{r}_m dH + H d\tilde{R}_m.$$  

The right-hand side of (20) can be rewritten as

$$H \tilde{r}_m dH + H d\tilde{R}_m = (H \tilde{r}_m - \tilde{R}_m) dH + d(H \tilde{R}_m).$$
For the left-hand side of (20), it follows from the formulas (c) and (d) in Lemma 1 that

\[ H_{\omega_{i,j}} = \xi_{i,j} + \eta_{i,j}, \quad i + j < m, \]

\[ H_{\omega_{0,m}} = \frac{a_2(m + 4)}{3(m + 3)} \omega_{0,m+3} + \frac{a_1(m + 4)}{3(m + 2)} \omega_{1,m+2} + \eta_{0,m+3}, \]

\[ H_{\omega_{1,m-1}} = \frac{a_1^2(m + 4)}{6(m + 3)} \omega_{0,m+3} + \frac{a_2(m + 4)}{3(m + 2)} \omega_{1,m+2} + \eta_{1,m+2}, \quad m > 0, \]

where \( \eta_{i,j} = r_{i,j}dH + dR_{i,j} \), and \( \xi_{i,j} \) is a one-form with \( \deg(\xi_{i,j}) \leq i + j \). Then, substituting (22) into the left-hand side of (20) yields

\[ \sum_{i=0}^{m-i} \sum_{j=0}^{1 \leq j \leq m-i} \beta_{i,j} H_{\omega_{i,j}} = \frac{m + 4}{3(m + 3)} (a_2 \beta_{0,m,m} + \frac{a_1^2}{2} \beta_{1,m-1,m}) \omega_{0,m+3} \]

\[ + \frac{m + 4}{3(m + 2)} (a_1 \beta_{0,m,m} + a_2 \beta_{1,m-1,m}) \omega_{1,m+2} \]

\[ + \sum_{i=0}^{m-i} \sum_{j=0}^{1 \leq j \leq m-i} \beta_{i,j} (\xi_{i,j} + \eta_{i,j+3}). \]

Finally, combining (23) with (20) and (21) shows that the conclusion is also true for \( m + 3 \).

The proof of the lemma is complete. \( \Box \)

Noting that \( \beta_{0,1,0} = \beta_{1,0,1} = 1, \beta_{1,-1,0} = \beta_{0,1,1} = 0 \), we know from (19) that \( \beta_{k,m-k,m} \) in Lemma 2 are polynomials in \( a_1 \) and \( a_2 \) with positive coefficients for \( m \mod 3 = k, k < 2 \). Thus, \( \omega_{k,m-k} \) can be expressed in terms of other one-forms \( \omega_{i,j}, i + j \leq m \) and \( r_m dH + dR_m \). This gives the following lemma.

**Lemma 3.** Any polynomial one-form \( \omega \) of degree \( m \) can be expressed as

\[ \omega = r(x,y)dH + dR(x,y) + \sum_{i=0}^{1 \leq j \leq m-i} \sum_{j \mod 3 \neq 0} \alpha_{ij} \omega_{ij}, \]

where \( R(x,y) \) and \( r(x,y) \) are polynomials of degrees \( m + 1 \) and \( m - 1 \) in \( x \) and \( y \), respectively.

Now, it follows from (24) that

\[ M(h) = \oint_{\Gamma_h} \omega = \sum_{i=0}^{1 \leq j \leq m-i} \sum_{j \mod 3 \neq 0} \alpha_{ij} \oint_{\Gamma_h} \omega_{ij}, \]
that is, any Melnikov function \( M(h) = \oint_{\Gamma_h} \omega, \deg(\omega) = m \), can be expressed as a linear combination of integrals \( I_{ij}(h) = \oint_{\Gamma_h} \omega_{ij}, i = 0, 1, j \mod 3 \neq 0 \). A reasonable expectation is that the integrals \( I_{ij}(h) \) form a basis for the space of Melnikov functions \( M(h) = \oint_{\Gamma_h} \omega \). Actually, it will be seen in the next section that the space of Melnikov functions \( M(h) \) could be Chebyshev with accuracy at least 2. So the number of limit cycles in system (6) is not determined by the number of the basis. Further, the coefficients \( \alpha_{i,j} \) in (25) could became very complicated when \( M(h) \) is a higher-order Melnikov function of system (6). In this case, it is really not easy to prove the independency of \( \alpha_{i,j} \)'s, which is the second big obstacle in the use of the independency of the integrals \( I_{i,j}(h) \) to determine the number of limit cycles.

To overcome the above mentioned difficulty, we turn to an alternative, which decreases the complexity in computing \( M(h) \) by (24), but it still does not solve the problem of independency of basis. Let \( \omega_j = Q_j(x,y)dx - P_j(x,y)dy \). Then, for higher-order Melnikov functions, we have the following result.

**Lemma 4.** (cf. [23], [22]) Let (9) hold. Assume that for some \( k \geq 2 \), system (6) has

\[
M_m(h) = \oint_{\Gamma_h} r_m dH + dR_m \equiv 0, \ 1 \leq m \leq k - 1. \tag{26}
\]

Then,

\[
M_k(h) = \oint_{\Gamma_h} (\omega_k + \sum_{i+j=k} r_i \omega_j),
\]

\[
r_m dH + dR_m = \omega_m + \sum_{i+j=m} r_i \omega_j, \ 1 \leq m \leq k - 1. \tag{27}
\]

**Proof.** We prove this lemma by using the method of mathematical induction.

First, write system (6) in the Pfaffian form,

\[
dH - \varepsilon \omega_1 - \varepsilon^2 \omega_2 - \cdots = 0. \tag{28}
\]

Multiplying (28) by \( 1 + \varepsilon r_1 + \cdots + \varepsilon^{k-1} r_{k-1} \) and combing the like terms yield

\[
dH + \varepsilon (r_1 dH - \omega_1) + \varepsilon^2 (r_2 dH - r_1 \omega_1 - r_2) + \cdots + \varepsilon^k (-r_{k-1} \omega_1 - \cdots - r_1 \omega_{k-1} - \omega_k) + O(\varepsilon^{k+1}) = 0,
\]

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which, by using (27), can be written as
\[ dH - \varepsilon dR_1 - \cdots - \varepsilon^{k-1} dR_{k-1} - \varepsilon^k (r_{k-1} \omega_1 + \cdots + r_1 \omega_{k-1} + \omega_k) + O(\varepsilon^{k+1}) = 0. \]

Then, we integrate the above equation along the phase curve \( \gamma \) from point \( A \) to point \( B \), which are used to define the first return map. Note that
\[
d(h, \varepsilon) = \int_\gamma dH = H(B) - H(A) = O(|A - B|)
\]
and
\[
\left| \int_\gamma (\varepsilon dR_1 + \varepsilon^2 dR_2 + \cdots + \varepsilon^{k-1} dR_{k-1}) \right| = \varepsilon O(|A - B|).
\]

In addition, it follows from (8) and (26) that \( d(h, \varepsilon) = O(\varepsilon^k) \). Therefore, \( |A - B| = O(\varepsilon^k) \) and we finally obtain
\[
d(h, \varepsilon) = \varepsilon^k \int_\gamma (r_{k-1} \omega_1 + \cdots + r_1 \omega_{k-1} + \omega_k) + O(\varepsilon^{k+1}),
\]
which yields
\[
M_k(h) = \oint_{\Gamma_h} (\omega_k + \sum_{i+j=k} r_i \omega_j).
\]
The proof is finished.

4. Proof of Theorem 1

Now we are ready to prove our main result – Theorem 1.

Proof. We return to system (6) with \( P(x, y) \) and \( Q(x, y) \) defined in (9), and want to use higher-order Melnikov functions to prove the existence of 10 small-amplitude limit cycles around the origin. Due to the difficulty in the proof of independency of basis, we use the computation of focus values to prove the theorem. However, the computation becomes very demanding or almost impossible for computing higher-order focus values if all the coefficients are retained in the computation, and in fact many terms are not necessarily needed. Thus, before computing the focus values of system (6), without loss of generality, we want to simplify this system, by choosing a group of coefficients \( a_{ijm}, b_{ijm} \) in the polynomials \( P(x, y) \).
and $Q(x, y)$, which does not affect the number of limit cycles bifurcating from the origin.

In the following, we shall show how to choose a group of coefficients which are necessary for the first non-vanishing Melnikov function $M_k(h)$ in (28). Based on the results presented in the previous section (in particular, Lemmas 1, 3 and 4), we provide an algorithm as follows.

Consider $M_1(h)$ in system (6), we know $M_1(h) = \oint_{\Gamma_h} \omega_1$. Using Lemma 3, we have

$$\omega_1 = Q_1 dx - P_1 dy = \sum_{i=0}^{1} \sum_{j=1}^{2} \alpha_{ij1} x^i y^j dx + r_1 dH + dR_1,$$

with $r_1 = -(b_{211} + 3a_{301}) y$. Then,

$$M_1(h) = \oint_{\Gamma_h} (\alpha_{011} ydx + \alpha_{111} xydx + \alpha_{021} y^2 dx + \alpha_{121} xy^2 dx).$$

It is seen that $M_1(h)$ depends on $\alpha_{ij1}$, $i = 0, 1$, $j = 1, 2$. So only four coefficients in the polynomials $P_1(x, y)$ and $Q_1(x, y)$ are needed in order to keep $\alpha_{ij1}$, $i = 0, 1$, $j = 1, 2$ being independent without decreasing the number of zeros of $M_1(h)$. We choose these four coefficients as $b_{ij1}$, $i = 0, 1$, $j = 1, 2$. (Certainly, this is not a unique choice.) Then, we have polynomials

$$P_1(x, y) = 0, \quad Q_1(x, y) = b_{011} x + b_{111} xy + b_{021} y^2 + b_{121} xy^2. \quad (30)$$

Next, let us consider $M_2(h)$ when $M_1(h) = \oint_{\Gamma_h} r_1 dH + dR_1 \equiv 0$, i.e., all $\alpha_{ij1} = 0$ in (29). Lemma 4 gives $M_2(h) = \oint_{\Gamma_h} \tilde{\omega}_2$, where $\tilde{\omega}_2 = \omega_2 + r_1 \omega_1$. Thus, by using Lemma 3, we obtain

$$\tilde{\omega}_2 = \sum_{i=0}^{1} \sum_{j=1}^{2} \alpha_{ij2} x^i y^j dx + \alpha_{042} y^4 dx + r_2 dH + dR_2,$$

which shows that $M_2(h)$ depends on $\alpha_{ij2}$, $i = 0, 1$, $j = 1, 2$ and $\alpha_{042}$. Obviously, the coefficient $\alpha_{042}$ is derived from $r_1 \omega_1$ by Lemma 3 because the one-form $y^4 dx$ of degree 4 comes from $r_1 \omega_1$. For $\varepsilon$-order perturbations, $b_{ij1}$, $i = 0, 1$ $j = 1, 2$ are needed to get all $\alpha_{ij1} = 0$ in (29). For $r_1$ we may simply take $b_{211} = 1$ and $a_{301} = 0$, yielding $r_1 = -y$. We also see that the one-form $y^4 dx$ can be derived from $x^3 y dx$ by using the formula (b) in Lemma 1. Hence,
we may choose $b_{301}$ for $\alpha_{042}$ so that $b_{301}x^3ydx$ could appear in $r_1\omega_1$. For $\alpha_{ij2}$, $i = 0, 1$, $j = 1, 2$, by an argument similar to that for $M_1(h)$, we choose $b_{012}$, $b_{112}$, $b_{022}$ and $b_{122}$. Hence, we obtain the following polynomials,

$$P_1(x, y) = 0, \quad Q_1(x, y) = b_{011}x + b_{111}xy + b_{021}y^2 + b_{121}xy^2 + b_{301}x^3 + x^2y,$$

$$P_2(x, y) = 0, \quad Q_2(x, y) = b_{012}x + b_{112}xy + b_{022}y^2 + b_{122}xy^2.$$

(31)

Following the above procedure, we can choose the coefficients for $M_3(h)$, and so on. We list the polynomials for up to $M_5(h)$ in the following (the detailed arguments are omitted here for brevity)

$$P_j(x, y) = a_{21j}x^2y + a_{11j}xy^2, \quad j = 1, 2, 3, \quad P_4(x, y) = P_5(x, y) = 0,$$

$$Q_1(x, y) = b_{011}y + b_{111}xy + b_{021}y^2 + b_{121}xy^2 + b_{301}x^3 + b_{031}y^3 + b_{211}x^2y,$$

$$Q_2(x, y) = b_{012}y + b_{112}xy + b_{022}y^2 + b_{122}xy^2 + b_{302}x^3 + b_{032}y^3,$$

$$Q_3(x, y) = b_{013}y + b_{113}xy + b_{023}y^2 + b_{123}xy^2 + b_{303}x^3,$$

$$Q_4(x, y) = b_{014}y + b_{114}xy + b_{024}y^2 + b_{124}xy^2 + b_{304}x^3,$$

$$Q_5(x, y) = b_{015}y + b_{115}xy + b_{025}y^2 + b_{125}xy^2.$$

(32)

Here, the difficult part is to compute the functions $r_i$, $i = 1, 2, 3, 4$ in $\tilde{\omega}_i$.

Having determined the coefficients we need in $P$ and $Q$ of system (6), we now use the computation of focus values to prove the existence of 10 small-amplitude limit cycles. We compute the focus values up to $\varepsilon^5$ order as follows:

$$V = \sum_{i=0}^5 \varepsilon^i V_i, \quad \text{where} \quad V_i = \{v_{i0}, v_{i1}, v_{i2}, \ldots\}.$$  \hspace{1cm} (33)

We call $v_{ij}$ the $j$th $\varepsilon^i$-order focus value of system (6), and note that $v_{0ij} = 0$, $j = 0, 1, 2, \ldots$ since at $\varepsilon = 0$ system (6) is a Hamiltonian system. The computation of $V_i$ is equivalent to the computation of $i$th-order Melnikov function $M_i(h)$. But the computation of focus values is much easier than that of the higher-order Melnikov functions. The disadvantage of the focus value computation is that conditions obtained from the first few focus values can not be used to prove an infinite number of focus values to equal zero. But this can be easily verified by the above formulas $\tilde{\omega}_i$.

The focus values $v_{ij}$ can be obtained by using many different symbolic programs (e.g., the Maple program developed in [25]). Firstly, note that
\[v_{i0} = \frac{1}{5}b_{01i}, \ i = 1, 2, \ldots\] In order to execute the Maple program, set \(b_{01i} = 0, \ i = 1, 2, \ldots\) In addition, set \(b_{211} = 1\). Now, we start from \(V_1\) and obtain
\[v_{11} = \frac{1}{8}(a_{121} + 3b_{031} + b_{111} - \frac{1}{2}a_1b_{111} - 2a_2b_{021} + 1).\]
Setting \(v_{11} = 0\) yields \(b_{031} = -\frac{1}{3}(a_{121} + b_{111} - \frac{1}{2}a_1b_{111} - 2a_2b_{021} + 1)\). Further, setting \(v_{12} = 0\) results in
\[b_{121} = a_1b_{021} - a_{211} + \frac{1}{4a_2(5a_1 - 2)}(3a_1^2 + 20a_2^2 + 4a_1 - 20)(b_{111} + 1).\]
Then, we have:
\[v_{13} = \frac{35}{3072(5a_1 - 2)}(b_{111} + 1)(a_3^3 - 3a_2^3 + 4 - 4a_2^3)F_{11},\]
\[v_{14} = \frac{7}{32768(5a_1 - 2)}(b_{111} + 1)(a_3^3 - 3a_2^3 + 4 - 4a_2^3)F_{12},\]
\[v_{15} = \frac{1}{819246560(5a_1 - 2)}(b_{111} + 1)(a_3^3 - 3a_2^3 + 4 - 4a_2^3)F_{13},\]
where
\[F_{11} = 3a_1^2 + 12a_1 - 4 - 4a_2^2,\]
\[F_{12} = 27a_1^4 - 90a_1^3 - 1308a_2^2 + 1608a_1 - 256 + (420a_1^2 + 1608a_1 - 1376 - 256a_2^2)a_2^3,\]
\[F_{13} = 19683a_1^6 + 343116a_1^5 - 124524a_1^4 - 6168672a_1^3 + 7612368a_2^2 + 1585344a_1 - 1071424 + 4[3(140715a_1^4 + 622536a_3^3 + 39980a_2^2 - 1689568a_1 + 421808) + (404508a_1^2 - 396336a_1 + 267856a_2^3 - 12652424a_3^3)]a_2^2.\]
It is easy to see that setting \(b_{111} = -1\) results in \(v_{13} = v_{14} = v_{15} = \cdots = 0\), as discussed above. In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin, we have to use the coefficients \(a_1\) and \(a_2\) to solve \(F_{11} = F_{12} = 0\) (i.e., \(v_{13} = v_{14} = 0\)). If the solution of \(F_{11} = F_{12} = 0\) yields \(F_{13} \neq 0\), i.e., we have parameter values such that \(v_{10} = v_{11} = \cdots = v_{14} = 0\), but \(v_{15} \neq 0\), then we obtain 5 small-amplitude limit cycles by properly perturbing \(b_{011}, b_{031}, b_{021}, a_1\) and \(a_2\), respectively. In fact, by using the Groebner basis reduction procedure, we can reduce \(F_{12}\) and \(F_{13}\) to
\[\tilde{F}_{12} = F_{12}|_{F_{11}=0} = 18(a_1 + 2)(11a_1^3 + 46a_2^2 - 84a_1 + 24),\]
\[\tilde{F}_{13} = F_{13}|_{F_{11}=\tilde{F}_{12}=0} = -\frac{1279712}{121}(a_1 + 2)(3073a_1^2 - 5272a_1 + 1500) \neq 0.\]
In addition, we obtain for which we have applied the Groebner basis reduction procedure to obtain values to solve the polynomial equations $\nu_2 = \nu_3 = \nu_4 = 0$, yielding the solutions for $b_{032}, b_{122}$ and $b_{112}$. Under these solutions, we further obtain

$$
\nu_2 = \left\{ \frac{1}{1850196(3a_1^3 + 12a_1 - 4a_2)} F_{20} F_{21},
\nu_3 = \left\{ \frac{1}{10729722931(3a_1^3 + 12a_1 - 4a_2)} F_{20} F_{23},
\nu_4 = \left\{ \frac{1}{50} F_{20} F_{23},
\right. \}
$$

for which we have applied the Groebner basis reduction procedure to obtain

$$
F_{20} = \left\{ 2(3a_1^3 - 4a_2^2)b_{021} - 3(a_1^3 - 4a_2^2)b_{301} - 6a_2^2a_{211} + 4a_2a_{121} - 4a_1a_{211} \right\} b_{211} + 12a_1(a_1a_{121} - a_2a_{211}) b_{021},
F_{21} = 8a_1^4 - 64a_1^2 - 64a_1^2 + 1632a_1 - 880 - (504a_1^2 - 1632a_1 - 1696 + 880a_2^2)a_2,
\tilde{F}_{22} = F_{22} |_{F_{21} = 0} = -50688(63a_1^3 + 56a_1^2 - 148a_1 + 80) + 1408 [243a_1^3 - 522a_1^2 + 5172a_1 + 6664 + (1053a_1^2 - 2424a_1 - 5572 + 1300a_1^2)a_1^2] a_1^2,
\tilde{F}_{23} = F_{23} |_{F_{21} = F_{22} = 0} = 72(675121644a_1^3 + 475639745a_1^2 - 1491227668a_1 + 849702020) + \left\{ 3893155245a_1^3 + 22056197796a_1^2 - 131201934348a_1 - 117343356608 + 20(303274623a_1 + 3083354476 - 26(55458a_1 - 130879)a_2^2) a_2^2 \right\} a_2^2 \neq 0,
$$

In addition, we obtain

$$
\det \left[ \begin{array}{ccc}
\frac{\partial F_{21}}{\partial a_1} & \frac{\partial F_{21}}{\partial a_2} \\
\frac{\partial F_{22}}{\partial a_1} & \frac{\partial F_{22}}{\partial a_2} \\
\frac{\partial F_{23}}{\partial a_1} & \frac{\partial F_{23}}{\partial a_2}
\end{array} \right] |_{F_{21} = F_{22} = 0} = \left\{ \frac{1}{3a_1^3} \right\} a_1^2 \left\{ 36(1571445a_1^3 + 860083a_1^2 - 3207848a_1 + 1911580) + a_2^2(977612a_1 + 24045705a_1^2 - 138196596a_1 - 132836684 + 20a_2^2(-119877a_1 + 2945227 + 169a_2^2(459a_1 + 1799))) \right\} \neq 0.
$$
Thus, in order to obtain maximal number of limit cycles, we let $a_2 = a_1 = \cdots = a_{25} = 0$, but $v_{26} \neq 0$. Then, taking proper perturbations on the coefficients $b_{015}, b_{032}, b_{122}, b_{112}, a_1$ and $a_2$ yields 6 small-amplitude limit cycles around the origin of system (6) when the $\varepsilon^2$-order focus values (or the second-order Melnikov function $M_2(h)$) are used.

In order to get more number of limit cycles, we let $F_{20} = 0$ and solve this equation for $b_{301}$, yielding all the $\varepsilon^2$-order focus values $v_{2j} = 0$. Under the conditions obtained above, we then use the $\varepsilon^3$-order focus values $v_{3j}$ to determine the number of small-amplitude limit cycles. Similarly, we may linearly solve the polynomial equations $v_{31} = v_{32} = v_{33} = v_{34} = 0$ for the coefficients $b_{023}, b_{123}, b_{113}$ and $b_{032}$. After this, no coefficients can be solved linearly. Then, eliminating $a_{211}$ from the equations, $v_{35} = v_{36} = 0$ yields a solution $a_{211}$ and a resultant $F_{30} F_{31}$; eliminating $a_{211}$ from the equations, $v_{35} = v_{37} = 0$ gives a solution $a_{211}$ and a resultant $F_{30} F_{32}$; and eliminating $a_{211}$ from the equations, $v_{35} = v_{38} = 0$ gives a solution $a_{311}$ and a resultant $F_{30} F_{33}$. $F_{30}$ is a common factor, and it is found that $a_{211}^1 = a_{211}^2 = a_{211}^3$. Thus, in order to obtain maximal number of limit cycles, we let $a_{211} = a_{211}^1$, and then solve for $a_1$ and $a_2$ from the equations $F_{31} = F_{32} = 0$. $F_{31}$ and the reduced $\tilde{F}_{32}$ and $\tilde{F}_{33}$ by using the Groebner reduction procedure are given below.

$$F_{31} = 405 a_1^4 + 6264 a_1^3 + 6264 a_1^2 - 5664 a_1 + 1360 - 8(99 a_1^2 + 708 a_1 + 524 - 170 a_2^2) a_2^2,$$

$$\tilde{F}_{32} = F_{32}|_{F_{31}=0} = 4(261117 a_1^3 + 307422 a_1^2 - 260532 a_1 + 60680) - [9(1035 a_1^3 + 13266 a_1^2 +111492 a_1 + 84376 + 5(513 a_1^2 - 4824 a_1 - 57156 + 2660 a_2^2) a_2^2) a_2^2],$$

$$\tilde{F}_{33} = F_{33}|_{F_{31}=\tilde{F}_{32}=0} = 4(152348063679 a_1^3 + 175217936814 a_1^2 - 151386504684 a_1 +35757329960) + \{742838685 a_1^3 - 38896637238 a_1^2 -56826427476 a_1 - 439876872808 - 20[714254595 a_1 - 6998804702 -380(11970 a_1 + 132193) a_2^2] a_2^2\} a_2^2 \neq 0,$$

This seemingly suggests that we may have parameter values such that $v_{3i} = 0$, $i = 0, 1, 2, \ldots, 7$ and so the system could have at most 8 small-amplitude limit cycles. However, when we substituted the solutions of $a_1$ and $a_2$ solved from $F_{31} = \tilde{F}_{32} = 0$ to the original focus values $v_{35}, v_{36}$ and $v_{37}$, we found that none of these solutions satisfies $v_{35} = 0, v_{36} = 0$ or $v_{37} = 0$, which means that these solutions are extra solutions due to the operation of eliminate.
Therefore, the best result we can have is to let \( a_1 \) (or \( a_2 \)) free and so get 7 limit cycles. Indeed, we can find parameters values such that \( v_{3i} = 0, i = 0, 1, \ldots, 6 \), but \( v_{37} \neq 0 \), and thus at most 7 small-amplitude limit cycles can bifurcate form the origin. Further, properly applying perturbations on the coefficients, \( b_{013}, b_{023}, b_{123}, b_{113}, b_{321}, a_{221} \) and \( a_2 \) yields 7 limit cycles.

Now, we want all \( \varepsilon^2 \)-order focus values to vanish (i.e., \( M_3(h) \equiv 0 \)). This can be achieved by solving the coefficient \( a_{121} \) from a polynomial equation. Having obtained the conditions for which all the \( \varepsilon^1 \), \( \varepsilon^2 \) - and \( \varepsilon^3 \)-order focus values vanish, we now use the \( \varepsilon^4 \)-order focus values to linearly solve for \( b_{024}, b_{124}, b_{114}, b_{303}, a_{212} \) and \( a_{122} \) one by one from the equations \( v_{41} = v_{42} = v_{43} = v_{44} = v_{45} = v_{46} = 0 \). Then, the higher-order focus values are given by

\[
v_{47} = \frac{13}{1179648}F_{40}F_{41}, \quad v_{48} = \frac{-13}{127401984}F_{40}F_{41}, \quad v_{49} = \frac{13}{244611809280}F_{40}F_{41},
\]

where \( F_{40} \) is a common factor, and \( F_{41}, F_{42} \) and \( F_{43} \) are functions of \( a_1 \) and \( a_2 \), given by

\[
F_{41} = 37179a_1^3 - 524880a_1^7 - 4747248a_1^6 - 12436416a_1^5 + 7737120a_1^4
+ 13042944a_1^3 - 17299200a_1^2 + 6945792a_1 - 578816 - 16a_2^2\{12393a_1^6
- 802548a_1^4 - 102708a_1^2 - 1317600a_1^3 - 40464a_1^5 + 232128a_1 + 144704
- 2a_2^2\{3(11475a_1^4 - 35496a_1^3 - 271896a_1^2 - 38688a_1 + 129712)
+ 18088a_2^2\{3a_1^2 + 12a_1 - 4 - a_2^2})\}\}
\]

\[
F_{42} = 267688a_1^3 - 52205877a_1^5 - 223716978a_1^6 + 3206239200a_1^5
+ 200795760a_1^4 - 5054946912a_1^3 - 3905952192a_1^2 + 10386531072a_1^3
- 720573696a_1^4 + 2022961920a_1 - 144704000 + 2a_2^2\{780759a_1^8
- 9325368a_1^7 - 641496672a_1^6 - 290989056a_1^5 + 55897760a_1^4
+ 5294374272a_1^5 - 2824768512a_1^4 + 1366334976a_1 - 486784256
- 16a_2^2\{945999a_1^6 - 2537325a_1^5 - 136313118a_1^3 - 39702520a_1^2
+ 244645056a_1^4 + 201986928a_1 - 100792544 + 2a_2^2\{4716225a_1^4
- 23135436a_1^3 - 141272064a_1^2 + 4269976a_1 + 50396272
+ 904a_2^2\{3495a_1 + 810a_1^2 - 1682 - 250a_2^2})\}\}
\]

\[
F_{43} = 5(1366216713a_1^2 - 33939081972a_1^3 + 44272893168a_1^4
+ 493387025040a_1^9 - 628994289672a_1^8 - 1403627519808a_1^7
+ 18889326323712a_1^6 + 18007656030720a_1^5 - 42717415378176a_1^4
+ 28083532021216a_1^3 - 6517758455808a_1^2 - 261522960384a_1
+ 82636402688) - 16a_2^2\{3(3255393240a_1^10 - 44439681807a_1^9
- 361043394498a_1^8 - 1113177716208a_1^7 - 333653885856a_1^6
}
\[ +4370955883488a_1^5 + 418550391360a_1^4 - 5262495843072a_1^3 \\
+6238547741060a_1^3 - 2585249949440a_1 + 164748398080 \\
+2a_2^2 [3(17368810155a_1^8 - 138413665080a_1^7 - 974515821120a_1^6 \\
-2142581030008a_1^5 - 1380949222176a_1^4 + 8851316920704a_1^3 \\
+2793260427776a_1^2 - 2239742773760a_1 - 792175055104) \\
-16a_2^2 (7413637185a_1^6 - 9049012605a_1^5 - 244787495850a_1^4 \\
-257911746696a_1^3 + 922435001664a_1^2 + 20997588504a_1 \\
-361224302752) + a_2^2 (3(9653815755a_1^4 + 43625458140a_1^3 \\
-8316724720a_1^2 - 161578121840a_1 + 49510940944) \\
-180880a_2^2 (2603a_1^4 - 28239a_1 - 170778 + 8923a_1^2))} \]

Similarly, we can show that \( F_{41} = F_{42} = 0 \) have solutions for \( a_1 \) and \( a_2 \), which satisfy \( F_{43} \neq 0 \), and

\[
 \det \begin{bmatrix}
 \frac{\partial F_{41}}{\partial a_1} & \frac{\partial F_{41}}{\partial a_2} & \frac{\partial F_{41}}{\partial \varepsilon} \\
 \frac{\partial F_{42}}{\partial a_1} & \frac{\partial F_{42}}{\partial a_2} & \frac{\partial F_{42}}{\partial \varepsilon} \\
 \frac{\partial F_{43}}{\partial a_1} & \frac{\partial F_{43}}{\partial a_2} & \frac{\partial F_{43}}{\partial \varepsilon}
 \end{bmatrix}
 \bigg|_{\varepsilon = 0} = 0.
\]

This suggests that with the \( \varepsilon^4 \)-order focus values, we can obtain 9 small-amplitude limit cycles by properly perturbing the coefficients, \( b_{014}, b_{024}, b_{124}, b_{114}, b_{303}, a_{212}, a_{122}, a_1 \) and \( a_2 \).

Finally, in order to have all the \( \varepsilon^4 \)-order focus values to become zero, we let \( b_{021} = -\frac{2a_2}{a_1^4} \). Then, we have the following simplified conditions, under which all the \( \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4 \)-order focus values equal zero.

\[
\begin{align*}
 b_{121} &= b_{112} = b_{301} = 0, \quad b_{031} = -\frac{1}{10a_1} (a_1^3 + 8a_2^2), \quad b_{111} = -1, \\
 b_{032} &= \frac{12}{25} a_2 b_{022} - \frac{2}{125a_2 a_1^2 [a_1 - 2a_2]} \left\{ (5a_1 + 31)a_1^6 - 2a_2^2 [8a_1 + 13)a_1^3 + 4(a_1 + 14a_2^2) \right\}, \\
 b_{122} &= \frac{3}{2} a_1 b_{022} - \frac{2}{25a_1^2 [a_1 - 2a_2]} \left\{ (2a_1 - 2)a_1^6 - a_2^2 (17a_1 - 50)a_1^3 + 4(a_1 + 38)a_2^2 \right\}, \\
 b_{113} &= b_{302}, \quad a_{211} = -\frac{2a_2}{a_1^4}, \\
 a_{121} &= -\frac{a_1^3 + 8a_2^2}{5a_1^2}, \\
 b_{023} &= \frac{1}{2a_2} a_{123} - \frac{3+a_1}{10a_2} b_{022} - \frac{2}{25a_2 a_1^2 [a_1 - 2a_2]} \left\{ (2a_1^6 + a_2^2 [(3a_1 - 22)a_1^3 + 4(a_1 + 14a_2^2) \right\}, \\
 b_{123} &= -a_{213} + \frac{a_1}{2a_2} a_{123} - \frac{a_2^2}{50a_2} [5(a_1 + 3)a_1^3 + 4(a_1 - 34a_2^2)] b_{022} \\
 &\quad - \frac{1}{125a_2 a_1^2 [a_1 - 2a_2]} \left\{ (10(a_1 - 2)a_1^6 + a_2^2 [5a_1^2 - 142a_1 + 208)a_1^6 \\
 &\quad - 8a_2^2 ((a_1^2 - 118a_1 + 96)a_1^3 + 8a_2^2 (a_1 - 1)(a_1 + 21))^2 \right\}, \\
 b_{302} &= \frac{2}{5} b_{022} + \frac{4}{25a_1^2 [a_1 - 2a_2]} \left\{ 2a_1^6 + a_2^2 [(3a_1 - 22)a_1^3 + 4(a_1 + 14a_2^2) \right\}, \\
 b_{114} &= b_{303}, \quad b_{021} = -\frac{2a_2}{a_1^4}, \\
 a_{212} &= \frac{2a_2}{5} b_{022} - \frac{2}{25a_1^2 [a_1 - 2a_2]} \left\{ (3a_1 + 7)a_1^7 - a_2^2 [(3a_1 + 20)a_1^3 - 24(a_1 + 3a_2^2) \right\}, \\
 a_{122} &= \frac{144}{25} b_{022} + \frac{2}{125a_1^2 [a_1 - 2a_2]} \left\{ (15a_1 - 32)a_1^6 - 4a_2^2 [(12a_1 - 43)a_1^3 + 6(a_1 + 14)a_2^2) \right\}.
\end{align*}
\]
\[ b_{024} = \frac{a_1 - 2}{2a_1 a_2} a_{213} - \frac{1}{2a_1^2 a_2^2} \left\{ \left( 3a_1 - 1 \right) a_1^3 - 4(9a_1 - 13)a_2^2 \right\} a_{123} - \frac{a_1^2 (a_1 - 2)}{25(a_1^3 - 4a_2^3)} \left\{ \left( 15a_1^2 + 40a_1 - 15 \right) a_1^9 - 2a_2^2 \left( [113a_1^6 + 366a_1^3 - 609] a_1^6 - 4a_2^2 ((16a_1^7 + 571a_1 - 781) a_1^3 - 8a_2^2 (3a_1^2 + 102a_1 - 166)) \right) \right\} \]

\[ b_{124} = \frac{a_1 - 2a_1 - 4a_2}{2a_1 a_2} a_{213} - \frac{1}{2a_1^2 a_2^2} \left\{ \left( 3a_1 - 1 \right) a_1^3 - 4a_2^2 \left( [12a_1 - 5] a_1^3 \right) - 4(9a_1 + 14)a_2^2 \right\} a_{123} - \frac{1}{25(a_1^3 - 4a_2^3)} \left\{ \left( 5(3a_1^2 + 8a_1 - 3) a_1^7 - 2a_2^2 (((53a_1^2 + 436a_1 - 459) a_1^0 + 4a_2^2 (((26a_1^2 - 114a_1 + 624) a_1^6 - 4a_2^2 ((53a_1^2 - 758a_1 + 474) a_1^3 - 16a_2^2 (a_1^7 - 48a_1 + 21))) \right) \right\} b_{022} + \frac{1}{12a_1^2 a_2^2 (a_1^2 - 2a_2^2) (a_1^3 - 4a_2^3)} \left\{ (10(3a_1^2 - 7a_1 + 2)a_1^{14} - a_1^2 (((6a_1^4 + 617a_1^2 - 904a_1 - 2140)a_1^4 - 4a_2^2 (((218a_1^3 + 1205a_2 + 994a_1 - 5000) a_1^0 + 4a_2^2 ((156a_1^3 - 1731a_1 + 934a_1 - 4140) a_1^5 - 4a_2^2 ((15a_1^3 + 1392a_1^2 + 54a_1 + 20a_1^3 - 48a_2^2 (a_1^3 + 20a_1 - 56)))) \right) \right\} \]

\[ b_{303} = -\frac{2}{a_1} a_{213} + \left\{ \frac{3(a_1^2 - 12a_2^2)}{2a_2 (a_1 - 4a_2)} \right\} a_{123} + \frac{16a_1^2 a_2^2}{25(a_1^3 - 4a_2^3)} b_{022} - \frac{1}{125a_1^2 a_2^2 (a_1^2 - 2a_2^2) (a_1^3 - 4a_2^3)} \left\{ 30(a_1 - 2) a_1^{12} - a_1^2 (((6a_1^2 + 1062a_1 - 40) a_1^{10} - 8a_2^2 ((9a_1^2 + 967a_1 + 370) a_1^6 - 4a_2^2 (9a_1^2 + 935a_1 + 450) a_1^3 - 4a_2^2 (11a_1^2 + 204a_1 - 50))) \right\} \]

Under the above conditions, we use the \( \varepsilon^5 \)-order focus values to find 10 small-amplitude limit cycles. Linearly solving the seven polynomial equations, \( v_{51} = v_{52} = \cdots = v_{57} = 0 \) one by one for the seven coefficients, \( b_{025}, b_{125}, b_{115}, b_{904}, a_{213}, a_{123} \) and \( b_{022} \). Then, \( v_{58}, v_{59} \) and \( v_{510} \) are given in terms of \( a_1 \) and \( a_2 \):

\[ v_{58} = \frac{187}{6193152000} F_{50} F_{51}, \quad v_{59} = \frac{-187}{99909432000} F_{50} F_{52}, \quad v_{510} = \frac{17}{118908531840000} F_{50} F_{53}, \]

where the common factor \( F_{50} \) is a rational function of \( a_1 \) and \( a_2 \), and \( F_{5i} \), \( i = 1, 2, 3 \) are polynomials of \( a_1 \) and \( a_2 \) with degrees 6, 7 and 8 with respect to \( a_2^2 \), respectively. \( F_{51} \) and \( F_{52} \) are given below \( (F_{53} \) is omitted here).

\[ F_{51} = 3365793a_1^{12} + 60938568a_1^{11} - 774250488a_1^{10} + 196620480a_1^{9} + 1313617176a_1^{8} - 8029124352a_1^{7} - 42401159424a_1^{6} + 61639418880a_1^{5} + 11348709120a_1^{4} - 85053265920a_1^{3} + 68653025280a_1^{2} - 20425531392a_1 + 2343047168 - 8a_2^2 \{ 3(1620567a_1^{10} + 26340228a_1^{9} - 214842132a_1^{8} + \cdots) \}
\]

\[ F_{52} = \cdots \]
It can be shown that there are in total 12 real solutions for \((a_1, a_2)\) such that \(F_{51} = F_{52} = 0\), but \(F_{53} \neq 0\). For example, one of the 12 solutions is given by \(a_1 = -2.3956026741 \cdots, a_2 = 1.9553136257 \cdots\), at which

\[
\det \begin{bmatrix}
\frac{\partial v_{51}}{\partial a_1} & \frac{\partial v_{51}}{\partial a_2} \\
\frac{\partial v_{52}}{\partial a_1} & \frac{\partial v_{52}}{\partial a_2}
\end{bmatrix}_{v_{51}=v_{52}=0} = 0.6888969278 \cdots \neq 0,
\]

implying that we can apply perturbations on the 10 parameters, \(b_{015}, b_{025}, \ldots\).
$b_{125}, b_{115}, b_{304}, a_{213}, a_{123}, b_{022}, a_1$ and $a_2$ to obtain 10 small-amplitude limit cycles around the origin.

5. Conclusion

In this paper, we have shown that the result of 11 small-amplitude limit cycles found in the Žołądek’s example is wrong, and proved that there are maximal nine limit cycles when the two mistakes are corrected. Further, we have given an example of 10 small-amplitude limit cycles obtained by perturbing a quadratic Hamiltonian system. This demonstrates how to use higher-order Melnikov functions combined with the method of focus value computation to obtain more limit cycles.

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