Recent Progress in the Heavy Quark Theory

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Abstract

Some recent developments in the heavy quark theory are briefly reviewed. The main emphasis is put on interrelation between HQET and OPE. The notion of duality and deviations from duality are discussed in detail.
1 Instead of Introduction

QCD is a messy theory in the strong coupling regime and with no obvious expansion parameter. If we could choose for Nature we would definitely pick up something more elegant, preferably in less dimensions, with more symmetry and so on. Alas, the choice is not ours. This is the theory of matter, and we have to learn how to deal with it as it is. Although there has been no decisive breakthrough in QCD in the recent years, a steady progress takes place in different directions. My topic today is the progress in the heavy quark theory.

The reasons why the heavy quark is the best probe of the strong interactions (out of a rather scarce supply we have at the moment) are well known. The road which led people to the understanding of this fact was not straightforward, though. The history of development of the pertinent ideas and formalisms is briefly summarized in my recent talk [1] and in excellent reviews [2], and I will save time, sparing the audience the n-th repetition of (interesting and important) topics that have become routine. Instead, I will concentrate on some very recent results.

The development of the theory goes in the direction of new applications on the one hand, and a deeper understanding of foundations, on the other. Among the former let me mention a few which might have a significant impact on phenomenology. The first example of this type is the issue of the higher-order perturbative corrections in the inclusive semileptonic width of $B$ mesons where a noticeable progress has been reported [3, 4, 5]. Another example is a new calculation [6] of the power corrections to the $b \to c$ transition form factor at zero recoil.

On the theoretical side, we are witnessing how the focus is shifting away from building particular $1/m_Q$ expansions towards more general aspects. Surprising though it is, it was not fully realized that HQET [7] is merely a special version of Wilson’s operator product expansion (OPE) [8]. As this fact gains recognition more and more elements of Wilson’s OPE are being adapted in the HQET environment leading to powerful, and sometimes unexpected, consequences. For instance, a conspiracy between the infrared renormalons and condensates is common knowledge in all problems where the operator product expansion can be used [8]. The issue of the infrared renormalons in the context of HQET was raised in Refs. [10, 11]. It took some effort to reveal peculiarities of HQET and to figure out how concretely the same phenomenon of conspiracy works for heavy quark expansions [10, 12] (see also [13]). The research did not stop at this point, however.

As was already mentioned, practical needs of phenomenology require inclusion of higher-order perturbative corrections. In such problems as the weak inclusive decays of the heavy flavors complete calculation of even $\mathcal{O}(\alpha_s^2)$ terms is a notoriously difficult task. People tried playing with different approximate approaches; the so called ‘$b$ graph dominance’ prescription due to Brodsky, Lepage and Mackenzie (BLM) [14] is one of the most popular. Originally the BLM hypothesis was engineered as a scale setting procedure in the $\mathcal{O}(\alpha_s)$ terms allowing one to minimize the coefficients of the $\mathcal{O}(\alpha_s^2)$ terms without their complete calculation. This idea was married with
the heavy quark theory \[1\]. At the next stage the renormalon studies and the BLM calculations fuse giving rise to an approximation which is sometimes called ‘resummation of the renormalon chains’ \[15\]. Using an extended BLM hypothesis one isolates and sums up an infinite sequence of graphs which possibly gives an estimate of higher-order effects. The results obtained so far \[4, 15\] are only the beginning of the story. The extended BLM resummation procedure has not yet been properly formulated in the context of OPE. Terms of a very high order in the ‘renormalon chain’ come from large distances and are not calculable in this way. The large-distance contribution should be isolated and treated separately, as a condensate. The separation is readily done in Euclidean calculations but is technically hard to implement in the problems with the Minkowski kinematics, like in the inclusive $B$ decays. Until this is done the resummation of the renormalon chains is hardly more than an academic exercise and of a limited practical value. Still, this is obviously a new promising direction deserving further efforts that will, perhaps, eventually find its place in the OPE-based theory.

First attempts of the instanton calculations in the inclusive heavy flavor decays have been reported recently \[16\]. The instanton contribution manifests itself in the coefficient functions, as a deviation from the so-called practical version of OPE \[17\]. This is the only known example of this type which, unfortunately, is very poorly controllable numerically. At the present stage one can speak, rather, about theoretical aspects of the instanton contribution as a laboratory or a testing ground for different effects not seen within the practical version of OPE. The topic is not mature enough to warrant a detailed discussion at this Symposium – the existing publications leave more questions than answers and should be, rather, viewed as an invitation for further work. I will comment, though, on some general aspects of the instanton calculations later on.

The ideas people play with now existed in QCD, in this or that form, for years. The heavy quark theory, being one of a few branches of QCD still in active growth, gave a new life to them. Being unable to dwell on all these developments I have picked up two samples from the current flow: one problem is applied, the other is more fundamental.

\section{Deviations from the Heavy Quark Symmetry at Zero Recoil and Determination of $V_{cb}$}

The discussion below will hopefully show how fruitful the intrusions of the OPE-based methods are (familiar from the QCD sum rules) for heavy quark theory.

Needless to say determining $V_{cb}$ is one of the most important current tasks in particle physics. Since isolated quarks do not exist it is impossible to measure $V_{cb}$ directly, without heavily exploiting a theoretical kitchen. Strictly speaking, there are two different kitchens; one of them deals with extracting $V_{cb}$ from experimental data through analysis of the exclusive $B \to D^* \ell \nu$ decays in the so called small velocity
limit (slow $D^*$'s). Extrapolation of the amplitude to the point of zero recoil yields $|V_{cb}|F_{B \rightarrow D^*}(\text{zero recoil})$, where $F_{B \rightarrow D^*}$ is an effective $B \rightarrow D^*$ transition form factor. In the small velocity (SV) limit this form factor is close to unity as a consequence of the heavy quark symmetry [18, 19, 20]; deviations from unity are quadratic in the inverse heavy quark mass, $1/m_{b,c}^2$ [19, 21]. Our task is to calculate these deviations.

To predict $(F_{B \rightarrow D^*} - 1)$ at zero recoil we derive a sum rule for the transitions $B \rightarrow D^*$ and $B \rightarrow$ vector excitations generated by the axial-vector current, $A_\mu = \bar{b}\gamma_\mu\gamma_5 c$. If the momentum carried by the lepton pair is denoted by $q$, the zero recoil point is achieved if $\vec{q} = 0$. To obtain the sum rule we consider the $T$ product

$$h_{\mu\nu} = i \int d^4xe^{-iqx} \frac{1}{2M_B} \langle B|T\{A_\mu^\dagger(x)A_\nu(0)\}|B\rangle$$

where the hadronic tensor $h_{\mu\nu}$ can be systematically expanded in $\Lambda_{QCD}/m_{b,c}$. For our purposes it is sufficient to keep the terms quadratic in this parameter and to consider only one out of five possible kinematical structures, namely $h_1$, the only structure surviving for the spatial components of the axial-vector current, see e.g. [22, 23].

Next we use the standard technology of the QCD sum rule approach. Let us define

$$\epsilon = M_B - M_{D^*} - q_0 = \Delta M - q_0.$$  

If $\epsilon$ is positive we sit right on the cut. The imaginary part of the amplitude (1) is the sum of the form factors squared (taken at zero recoil). The sum runs over all possible intermediate states, $D^*$ and excitations. We want to know the first term in the sum, $|F_{B \rightarrow D^*}|^2$. Alas, the present-day QCD does not allow us to make calculations directly in this domain.

On the other hand, if $\epsilon$ is negative we are below the cut, in the Euclidean domain. Here the amplitude (1) can be calculated as an expansion in $1/m_{b,c}$ provided that $|\epsilon| \gg \Lambda_{QCD}$. To get a well-defined expansion in $1/m_{b,c}$ we must simultaneously assume that $\epsilon \ll m_{b,c}$.

The non-perturbative corrections we are interested in are due to the fact that both, the $c$ quark propagator connecting the points 0 and $x$ in Eq. (1) and the external $b$ quark lines, are not in the empty space but are, rather, submerged into a soft-gluon medium, a light cloud of the $B$ meson. Two parameters characterizing the properties of this soft medium are relevant for our analysis. A chromomagnetic parameter

$$\mu_2^G = \frac{1}{2M_B} \langle B|\bar{b}\frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu}b|B\rangle \approx \frac{-1}{2M_B} \langle B|\bar{b}\gamma_5\vec{B}b|B\rangle$$

measures the correlation between the spin of the $b$ quark inside $B$ and the chromomagnetic field $\vec{B}$ created by the light cloud. The second parameter is $\mu_2^\pi = (2M_B)^{-1} \langle B|\bar{b}(i\vec{D})^2b|B\rangle$ measuring the average spatial momentum squared of the $b$ quark. Both parameters are proportional to $\Lambda_{QCD}^2$. That’s all we need for the leading non-perturbative term.
If the amplitude (1) is considered in the Euclidean domain far below the cut (i.e. $-\epsilon \gg \Lambda_{\text{QCD}}$) the distance between the points 0 and $x$ is short and we can expand $h_1$ in $\Lambda_{\text{QCD}}^2/m_c^2$. Actually, the whole amplitude contains more information than we need; the sum rule sought for is obtained by considering the coefficient in front of $1/\epsilon$ in $h_1$. In this way we arrive at the following prediction:

$$F_{B \rightarrow D^*}^2 + \sum_{i=1,2,...} F_{B \rightarrow \text{excit}}^2 = 1 - \frac{1}{3} \frac{\mu_\pi^2}{m_c^2} - \frac{\mu_G^2}{4} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_cm_b} \right),$$

where the sum on the left-hand side runs over excited states with the appropriate quantum numbers, up to excitation energies $\sim \epsilon$. (In other words, $\epsilon$ plays the role of the normalization point. Higher excited states are dual to the graphs with the perturbative hard gluon in the intermediate state are neglected together with the latter). All form factors in Eq. (4) are taken at the point of zero recoil.

Let us now transfer the contribution of the excited states to the right-hand side and account for the fact [24, 6] that $\mu_\pi^2 > \mu_G^2$. Then we get a lower bound on the deviation of $F_{B \rightarrow D^*}$ from unity,

$$\eta_A - F_{B \rightarrow D^*} > \frac{\mu_G^2}{6m_c^2}.$$

Here we included the perturbative one-loop correction [23, 19] so that $1 \rightarrow \eta_A$,

$$\eta_A = 1 + \frac{\alpha_s}{\pi} \left( \frac{m_b + m_c}{m_b - m_c} \log \frac{m_b}{m_c} - \frac{8}{3} \right) \approx 0.975.$$

Using the known value of $\mu_G^2$ and $m_c = 1.3$ GeV we conclude that $F_{B \rightarrow D^*} < 0.94$.

Including the $\mu_\pi^2 - \mu_G^2$ term and the contribution from the excited states lowers the prediction for $F_{B \rightarrow D^*}$ making deviation from unity more pronounced. If $\mu_\pi^2$ is taken from the QCD sum rule calculation [20] the estimate of $F_{B \rightarrow D^*}$ is reduced to 0.92. As far as the excited states are concerned a rough estimate of the $D\pi$ intermediate state can be given [6] implying that

$$F_{B \rightarrow D^*} = 0.89 \pm 0.03.$$

The error bars here reflect only the uncertainty in the excited states. The parameters $\mu_\pi^2$, $\mu_G^2$, $m_c$ and $\eta_A$ have their own error bars which I can not discuss here due to time/space limitations. Some relevant numerical estimates can be found in Ref. [27].

The corrections $O(1/m_{b,c}^2)$ to the form factors at zero recoil have been discussed previously [28, 29] within an HQET expansion. In this version, instead of the excited state contribution, one deals with certain non-local correlation functions which are basically unknown. The advantage of the sum rule approach presented above seems obvious: here we deal with only local operators; all non-locality is hidden in the excited state contribution which (i) has definite sign; (ii) its magnitude is known to be relatively small numerically; (iii) it can be relatively reliably estimated in models.
3 OPE and Deviations from Duality

Many applications of the heavy quark expansions are based, apart from the expansion per se, on duality. All problems associated with the inclusive heavy flavor decays belong to this class. Needless to say, they are important. As usual in QCD, a legitimate formulation of the expansion can be given in the Euclidean domain. The physically measurable quantities (e.g. the lepton spectrum in the inclusive $B \rightarrow l\nu X_u$ decays) belong to the Minkowski domain. We want to extract consequences of the Euclidean expansion in the Minkowski domain. A bridge between these two is provided by duality, a concept which is very frequently used but is very poorly studied. The notion is not even clearly defined in the literature, so that different people ascribe to it different meaning.

Now I will dwell on some general features of the operator product expansion in QCD with the intent to at least initiate discussion of this God-forgotten topic, deviations from duality. Many of the arguments presented below are part of ongoing research [30].

To establish an appropriate setting and introduce necessary terminology let me start from a classic example, the total cross section of $e^+e^- \rightarrow$ hadrons. More specifically, assume, for simplicity, that only $u$ and $d$ quarks exist, they are massless, and the “electromagnetic current” has the form

$$j_\mu = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d).$$

Inclusion of other quarks or the quark masses is a technical issue which does not affect the essence of the construction. Also, the particular choice of the current is not of importance.

The central object of our analysis is the $T$ product of two currents

$$T_{\mu\nu} = i \int e^{iqx}d^4xT\{j_\mu(x)j_\nu^+(0)\} \equiv (q_\mu q_\nu - q^2 g_{\mu\nu}) T.$$ (9)

For large Euclidean $q^2$ ($q^2 < 0$)

$$T = \sum C_n(q; \mu)\mathcal{O}_n(\mu)$$ (10)

where the normalization point $\mu$ is indicated explicitly. (Strictly speaking, one should deal with $q^2$ derivatives of Eq. (10)). In what follows we will also use the notation $Q^2 \equiv -q^2$. The sum in Eq. (10) runs over all possible Lorentz and gauge invariant local operators built from the gluon and quark fields. The operator of lowest (zero) dimension is the unit operator $I$, then comes the gluon condensate $G_{\mu\nu}$, of dimension four; the four-quark condensate represents the example of dimension-six operators (for further details see Ref. [31]).

The coefficient functions $C_n$ (absorbing the short-distance contribution) contain both, perturbative and non-perturbative contributions of different types. It is convenient to start from the unit operator whose coefficient accounts for the entire perturbation theory.
Consider a typical Feynman graph for the vacuum expectation value of $T_{\mu\nu}$ depicted on Fig. 1. Assume that the momentum flowing through the graph is Euclidean and large, $Q \to \infty$. All subgraphs in this graph are assumed to be renormalized at $Q$. The final result, being expressed in terms of the running coupling constant $\alpha_s(Q)$ is finite. In the standard calculation of the diagram of Fig. 1 all virtual momenta saturating the loop integrals scale with $Q$ as the first power of $Q$. For large $Q$ we use the perturbative expressions for the quark and gluon Green functions and one might say, naively, that we arrive at the standard perturbation theory for the coefficient of the unit operator,

$$C_I = \sum_l a_l \alpha_s^l,$$

(11)

where $a_l$ are numerical coefficients. As a matter of fact, the above expression is not correct theoretically. One should not forget the following: in doing the loop integrations in $C_I$ we must discard the domain of virtual momenta below $\mu$, by definition of $C(\mu)$. Subtracting this domain from the perturbative loop integrals we introduce in $C_I$ power corrections of the type $(\mu/Q)^n$ by hand. (If it is possible to choose $\mu$ sufficiently small these corrections may be insignificant numerically and can be omitted; such a situation will be realized in what is called a practical version of OPE, see below). Needless to say that the parameter $\mu$ is auxiliary – the final result for $\langle T \rangle$ must be $\mu$ independent.

At any finite order the perturbative contribution, eq. (11), is well-defined. At the same time, if the number of loops becomes of order $1/\alpha_s$ and each loop carries roughly speaking one and the same momentum the total momentum $Q$ is shared between many lines. This might be called a semi-hard contribution. One can visualize it in terms of, say, ‘direct instantons’ [32]. Integration over the sizes $\rho$ of the direct instantons is implied; the integral is saturated at $\rho \sim Q^{-1}$. It should be emphasized that the correspondence between the Feynman graphs with $1/\alpha_s$ lines and the semi-hard contributions described above is qualitative. Still, if we rely on this correspondence we may say that the characteristic loop momentum scales like $Q/\ln Q$. It is quite clear that the contribution of this type has the form

$$\Delta C_I \sim \exp \left( -C/\alpha_s(Q) \right) \sim \left( \frac{\Lambda_{QCD}}{Q} \right)^\gamma$$

(12)

where $C$ is some positive constant and the exponent $\gamma$ need not be integer. Typically, the numerical value of $\gamma$ is large. In this way we get a non-perturbative contribution to the coefficient function $C_I$, a non-condensate non-perturbative term. Such terms may or may not be important numerically depending on a particular value of $Q$ under consideration. (They are disregarded in the practical version of OPE which, thus, applies to the values of $Q$ above a critical point).

This is not the only source of the non-perturbative terms, however. Consider a situation in which one, or a few lines in the diagram of Fig. 1 are in a special regime – their momenta are soft in the sense that the characteristic off-shellness does not
scale with $Q$. These lines are marked by crosses on Fig. 2. Graphically one can cut these lines and treat the product as a local operator. In this way we arrive to the sub-leading operators $O_n$, $n \neq 0$, in the expansion (11). For instance, cutting the gluon line we get $G^{\mu\nu}_{\mu\nu}$. The remaining part of the graph where the loop momenta scale as $Q$ will yield the coefficient function in front of the corresponding operator $O_n$ in the form of a series in $\alpha_s(Q)$. Taking the matrix element of the sub-leading operator produces the condensate non-perturbative correction. (Analogously to $C_I$, the coefficients $C_n$ also receive, generally speaking, a semi-hard non-perturbative contribution of the type presented in eq. (12). By the same token, in calculating the perturbative part of $C_n$ one must discard the domain of virtual momenta $k < \mu$.

Finally, it is conceivable that there are no hard or semi-hard lines at all. The external momentum $Q$ is transferred from the initial to the final vertex through a large number of quanta (growing like a power of $Q$) and none of the quanta carries momentum scaling with $Q$ (Fig. 3). Of course, in this situation one can not speak of individual quanta, one should rather use the language of typical field fluctuations transferring the momentum $Q$ without having any Fourier components with frequencies of order $Q$. This type of contribution is not seen in OPE truncated at any finite order. It is reflected in the rate of divergence of high orders in the power series. It is intuitively clear that the corresponding part of the correlation function must be exponentially small, $\sim \exp(-Q)$. A transparent example is again provided by instantons. This time one has to fix the size of the instanton $\rho$ by hand, $\rho \sim \rho_0$. Then this contribution is $\mathcal{O}(\exp(-Q\rho_0))$. The relation between $\exp(-Q\rho_0)$ piece and the high-order terms of the power (condensate) series is akin the connection between $\exp(-1/\alpha_s)$ terms and $l \sim 1/\alpha_s$ orders in the perturbative expansion. Indeed, proceeding to higher-dimension operators in OPE we force more and more lines to become soft, and the limit of only soft lines is correlated with the operators built from a large number of the gluon and quark fields.

I hasten to make the following reservation. It may happen (and actually happens) that some of the loop integrations are not saturated at $Q$ or $\mu$ but are rather logarithmic. Logarithms $\ln(Q/\mu)$ occurring in this way are associated either with the running coupling constant $\alpha_s(Q)$ or with the anomalous dimensions of the operators at hand. They will play an important role in the analysis following below.

Summarizing, Wilson’s OPE (11) yields $\langle T \rangle$ in the deep Euclidean domain as an expansion in different parameters. Purely logarithmic terms $(\ln Q)^{-l}$ are due to ordinary perturbation theory. Terms of the type $(Q^2)^{-k}(\ln Q)^{-l}$ reflect higher-dimension operators and semi-hard contributions (‘direct instantons’). In the former case the values of $k$ are integer, the latter case may produce non-integer values of $k$. In the practical version of OPE we neglect the semi-hard contributions calculating the coefficient functions perturbatively. All non-perturbative terms come from condensates within this approximation.

The perturbative calculation of the coefficient function yields a series in $\alpha_s$. Everybody knows that the series is factorially divergent. First, the number of graphs grows factorially with the number of loops. This factorial behavior can be corre-
lated with the instanton-antiinstanton contribution \[33\]. Second, there exist specific graphs which are factorially large by themselves, the renormalons \[34\]. As I have already mentioned the latter received much attention lately, in connection with various applications of HQET. In the framework of the operator product expansion the infrared renormalons *per se* play no role since they are completely hidden in the condensate corrections. By construction, in calculating the coefficient functions we cut off the infrared contribution at \(\mu\), never come close to the Landau pole and, hence, the coefficient functions, calculated properly, are free from the infrared renormalons. (The ultraviolet renormalons are not absorbed in the condensate corrections, but they are harmless anyway.)

Passing to the next level, from the perturbative series to the condensate power series, we face the same problem since this power series is also asymptotic and is truncated in practical calculations (only those operators whose dimension is smaller than some number are retained). The uncertainty introduced in this way in the Euclidean calculation is expected to be less than the last term retained. As has been already mentioned the effect of the high order ‘tail’ of the power terms is exponential. The exponential terms are conceptually connected with the divergence of OPE for high-dimension operators and do not show up in the truncated OPE. One should proceed, however, with extreme caution estimating the uncertainty due to the high order power terms in the Minkowski domain (see below). The existing theory tells us nothing about when the exponential terms become negligibly small. At this point we have to rely on indirect methods and phenomenological information existing in this or that channel.

Let us discuss a serious theoretical question – whether it is legitimate at all to retain in the expansion of the correlation functions logarithmic, power and exponential terms simultaneously. Indeed, formally any \(1/(\ln Q)^l\) term is parametrically larger that, say, \(1/Q^4\), and, moreover, any power term is parametrically larger than exponential ones. If so, the theoretical uncertainty due to truncation of the logarithmic series – and we, of course, have to truncate it at some finite order – is formally larger than even the lowest-order power correction.

There are two arguments which seemingly justify the procedure of retaining power terms simultaneously with the truncated perturbative series. First, in many instances power terms have distinct origin, and the corresponding effects do not show up in perturbation theory. This is valid, for instance, in all cases where the chiral quark condensate is involved. Another example is due to four-fermion operators responsible for the Pauli interference in the non-leptonic weak decays of heavy flavors \[35\]. Second, even if the power terms are not different in their structure from those occurring in perturbation theory, there is a strong numeric enhancement of the power terms inherent to QCD whose origin is not completely clear at the moment. Thus, estimating the uncertainty due to infrared renormalons we get \(Q^{-4}\) with the coefficient much smaller than that appearing in Wilsonian OPE.
3.1 Practical version of OPE

In analytic QCD it is rather difficult to carry out in practice the consistent Wilsonian separation procedure outlined above. Therefore, in applications, one usually settles for what is called the ‘practical version of OPE’. In the vast majority of applications the practical version works very well numerically [31]; notable exceptions are also known, however [32].

What is the essence of the practical version of OPE? Let us elucidate this question using the example of the previous section. Assume that we want to calculate the vacuum expectation value of \( T \) (to be denoted by \( \Pi \)) limiting ourselves to some finite number of loops. For pedagogical purposes let us consider the two-loop graphs of Fig. 1. The corresponding expression for \( \Pi \) has a generic form

\[
\Pi(Q^2) \equiv \langle T \rangle = \alpha_s \int \frac{dk^2}{k^2} F(k^2/Q^2)
\]

where \( k \) is the virtual gluon momentum (the Wick rotation to the Euclidean space is implied) and \( F(k^2/Q^2) \) is a dimensionless function describing the fermion lines in the graphs of Fig. 1 and emerging after integration over the angle variables of the vector \( k \). Of course, at \( k^2 < \mu_0^2 \) the genuine expression for \( \Pi \) has nothing to do with \( 1/k^2 \). Therefore, in the genuine OPE one should cut off the integral from below at \( \mu_0^2 \) excluding the domain \( k^2 < \mu_0^2 \) from the perturbative calculation. This domain must be referred to the vacuum matrix elements of different gluon condensates which certainly can not be calculated perturbatively. These gluon operators are normalized at \( \mu_0^2 \), by construction, and their coefficients are given, to the leading order in \( \alpha_s \), by the \( k^2 \) derivatives of the function \( F(k^2/Q^2) \) at \( k^2 = 0 \). The Taylor expansion of this function starts from \( k^2 \) [31], generating the coefficient of \( G_{\mu\nu}^2 \); the next term of the Taylor expansion of \( F(k^2/Q^2) \) will give rise to the coefficient of \( (D\alpha G_{\mu\nu})^2 \), and so on. Then the coefficient function \( C_I \) is given by the integral \( \langle T \rangle \) with \( k^2 > \mu_0^2 \).

In the practical version of OPE, instead, one does the following. Let us add the integral (13) in the domain \( 0 < k^2 < \mu_0^2 \) to the coefficient \( C_I \). One should not think that this added integral correctly represents the low \( k^2 \) part of the graphs at hand. The idea is just to add this integral, by definition, and to subtract it. Then the coefficient \( C_I \), to order \( \mathcal{O}(\alpha_s) \), is given by full perturbative graphs of Fig. 1. To avoid double counting, however, we subtract the same contribution from the condensates. More exactly, we subtract \textit{almost} the same contribution. Since the sum over condensates is truncated at some finite order \( n_0 \), the best we can do is to subtract from the condensate part the integral

\[
\alpha_s \int_{\mu_0^2} \frac{dk^2}{k^2} \frac{1}{n!} F^{(n)}(k^2/Q^2)
\]

In this way we subtract from each condensate its ‘one-loop perturbative value’.

It is quite clear that this strategy of converting the genuine Wilson OPE into a practical version (i) leads to a loss of factorization of short and large distance
contributions inherent to Wilson’s OPE; (ii) can not be systematically generalized to any number of loops since there is no way to unambiguously define the integrand to all orders in the small $k^2$ domain (iii) only approximately avoids double counting due to the necessity of truncating the condensate series at a finite order (the lower the order we truncate the larger error is introduced; likewise, the larger value of $\mu_0^2$ is chosen the larger error is introduced).

Therefore, the practical version described above is useful (i.e. sufficiently accurate numerically) only provided $\mu_0^2$ can be chosen small enough to ensure that the ‘one-loop perturbative’ contributions to the condensates are much smaller than their genuine values and, at the same time, $\alpha_s(\mu_0^2)/\pi$ is small enough for the expansion to make sense. The existence of such a window is not granted apriori and is a very fortunate feature of QCD. The practical version obviously ignores the semihard contributions to the coefficient functions discussed above. Working within this version we automatically limit ourselves to a finite order in $\alpha_s$ forbidding any questions concerning any aspects of the high order behavior.

### 3.2 Consequences of OPE in the Minkowski domain

Now, let us recall that in the problems at hand we are not interested in the correlation functions at the Euclidean values of $q^2$ per se. The physically observable quantity is the imaginary part. The question is what can be predicted about the imaginary part from the Euclidean expansion of the type discussed above. In other words, we would like to reinterpret the expansion in terms of the imaginary parts it corresponds to.

If in the Euclidean expansion the notion of the soft and hard lines means small and large off-shellness, respectively, one can not extend this classification to the Minkowski domain. This is especially clear for the perturbative contribution which, by means of the Cutkosky rule, is obtained by putting all quarks and gluons in the intermediate state on shell.

Of course, quarks and gluons are confined and have no mass shell, literally speaking. The genuine imaginary part is saturated by hadrons, not quarks and gluons. At large $s$ (where $s = q^2$) the numerical value of the genuine imaginary part coincides with that obtained by calculating with quarks and gluons only approximately. How large is the actual deviation between the genuine imaginary part and that inferred from the quark-gluon calculation in the Euclidean domain? This is the key question the QCD practitioners have to address sooner or later. I will suggest some tentative answers.

Let us first explain what is meant when one speaks about the quark-gluon imaginary part. The starting point is the Euclidean calculation. Consider first the perturbative series for $\Pi(Q^2)$ truncated at some finite order below the critical order where the perturbation theory becomes senseless. The successive terms in this series have the form

$$
\frac{1}{\ln Q^2}, \frac{1}{(\ln Q^2)^2}, ..., \text{and } \frac{\ln \ln Q^2}{(\ln Q^2)^2} \text{ and so on.}
$$
Now, in each separate term of this series we substitute $Q^2 \to -s$ and then take the imaginary part.

We repeat the procedure with the condensate terms, again truncated at some finite order below the point where the asymptotic divergence shows up. In the Euclidean domain a generic contribution due to higher dimensional operators is

$$\frac{1}{(Q^2)^k} \times [(\ln Q^2)^\gamma + ...]$$

where $k$ is an integer number related to the (canonical) dimension of the operator at hand. It translates into

$$\frac{\gamma \pi}{(s)^k} \times [(\ln s)^{\gamma - 1} + ...]$$

in the imaginary part. Notice that the imaginary parts concentrated at $s = 0$ (i.e. $\delta(s)$ and its derivatives) do not manifest themselves at large $s$ in the truncated series. Notice also that in the absence of the logarithmic factors (i.e. for $\gamma = 0$) we get no imaginary part at large $s$. This feature is specific to the example at hand and does not extend, generally speaking, to other problems.

Summarizing, we do a straightforward analytical continuation, term by term. The sum of the imaginary parts obtained in this way will be referred to as the quark-gluon cross section. This quantity will serve as a reference quantity in formulating the duality relations. When one says that the hadron cross section is dual to the quark-gluon one the latter must be calculated by virtue of the procedure described above.

Defining the analytic continuation to the Minkowski domain in this way it is not difficult to see that those lines which were far off shell in the Euclidean calculation remain hard in the sense that now they are either still far off shell or on shell but carry large components of the four-momenta, scaling like $\sqrt{s}$.

The contributions left aside in the above procedure are related, at least at a conceptual level, to high-order tails of the both series, the logarithmic $\alpha_s$ series and the power (condensate) one. These tails generate negligibly small terms in the Euclidean domain, falling off at large $Q^2$ as a high (and, generically, non-integer) power of $1/Q^2$ or exponentially, $\sim \exp(-Q\rho_0)$.

Both can be visualized through instantons. In the first case we deal with the direct instanton contribution integrated over $\rho$ (the $\rho$ integration is saturated at $\rho \sim Q^{-1}$). In the second case $\rho$ is to be considered as a fixed parameter not scaling with $Q^{-1}$. As we will see below, the fixed-size instanton is actually relevant in both cases, so we start our discussion assuming that $\rho$ is fixed. The corresponding contribution to $\Pi(Q^2)$ was found long ago [36, 37]. In the Euclidean domain it is representable as an integral over an auxiliary parameter over a McDonald function. We do not intend here to rely on specific details of the instanton expression; rather, we would like to abstract general features inherent to mechanisms of this type. To this end we will follow the suggestion of Ref. [38]. The following observation was made in this work: if the condensate terms correspond, in the coordinate space, to
singularities of $\Pi(x)$ at $x = 0$, the instanton-type contributions are connected with the singularities of $\Pi(x)$ located at a finite distance from the origin; the distance from the origin plays the same role as the instanton radius. For our purposes it is reasonable to assume that these singularities have the simplest possible structure, namely,

$$\frac{1}{x^2 + \rho^2}, \quad \ln (x^2 + \rho^2), \quad (x^2 + \rho^2) \ln (x^2 + \rho^2), \quad \text{and so on.}$$

The Fourier transforms of these expressions have the generic form

$$(Q\rho)^{-n} K_n(Q\rho), \quad n = 1, 2, ...$$

where $K_n$ is the McDonald function.

As was expected, at large Euclidean $Q^2$ the corresponding contribution dies off exponentially,

$$\Delta \Pi = (Q\rho)^{-n} K_n(Q\rho) \propto (Q\rho)^{-n-1/2} e^{-Q\rho}, \quad (15)$$

in full accord with intuition regarding transmitting a large momentum through a soft field fluctuation. We are interested, however, in the Minkowski values of the momentum. The question is what is the impact of the contribution above on $\text{Im} \Pi$.

Analytically continuing $\Delta \Pi$ given in Eq. (15) to the Minkowski space ($Q \rightarrow iQ$) we arrive at

$$\text{Im} \Delta \Pi = (-1)^{n+1} \frac{\pi}{2} (\sqrt{s}\rho)^{-n} J_n(\sqrt{s}\rho) \propto (-1)^{n+1} \frac{\pi}{2} (\sqrt{s}\rho)^{-n-1/2} \cos\left(\sqrt{s}\rho - \delta_n\right), \quad (16)$$

where $J_n$ is the Bessel function. We see that the imaginary part falls off rather slowly, as a modest power of $1/\sqrt{s}$, and oscillates.

This result may or may not seem surprising. It tells us that with no further dynamical input, on general grounds alone, one can not rule out large violations of duality. If this were the end of the story the operator product expansion would be, to a large extent, useless in all problems where one needs to predict quantities referring to the Minkowski kinematics, such as decay probabilities, spectra, etc.

Fortunately, the situation is not as bad as it might seem at first sight. Indeed, so far we were discussing transmitting a large momentum through a fixed-size fluctuation. In QCD the size of the fluctuation is not fixed; rather, we integrate it over with a weight function which depends on $\rho$.

The result of this integration depends on the choice of the weight function $w(\rho)$. Leaving aside inessential details we are basically left with two distinct possibilities – one will represent the tail of the $\alpha_s$ series, the other the tail of the condensate series. The first option is related to the small-size direct instantons $\rho \sim 1/\sqrt{s}$; it generates a duality violating contribution falling off as a high power of $1/\sqrt{s}$. The second type of the duality-violating contribution, falling off exponentially with $\sqrt{s}$, is due to instantons whose size is smeared around some $\rho_0$ (i.e. parametrically small $\rho$ are excluded).

Let us start from the small-size instantons. If $\sqrt{s}$ is parametrically large and $\rho \sim 1/\sqrt{s}$ is parametrically small one can use the dilute instanton gas approximation
Then the instanton density \( d(\rho) \) (related to the weight function mentioned above) has the form

\[
d(\rho) = (\Lambda_{\text{QCD}} \rho)^b
\]  

where \( b \) is the first coefficient in the Gell-Mann-Low function. For three (almost) massless quarks \( b = 9 \). For small \( \rho \) the weight function \( w(\rho) \) has the same form as \( d(\rho) \),

\[
w(\rho) = (\Lambda_{\text{QCD}} \rho)^\gamma f(\rho),
\]

but the exponent \( \gamma \) depends on dynamical details. Say, if we treat three light quarks as massless (i.e. neglect their mechanical masses) then \( w(\rho) \) is proportional to the cube of the quark condensate \([40]\) and \( \gamma = 18 \). If the strange quark mass is considered as a parameter \( \ll \Lambda_{\text{QCD}} \) but the mechanical masses of \( u \) and \( d \) are still neglected \( w(\rho) \) is proportional to the square of the quark condensate and \( m_s \), and and \( \gamma = 16 \). If all three light quarks are treated as massive \( \gamma = 12 \). Even following the latter (quite unrealistic) treatment we can safely say that \( \gamma \) is large numerically.

Let us remind that generically \( \gamma \) need not be integer. For instance, if one considers very small values of \( \rho \) where the \( c \) quark can be treated as light (i.e. \( \rho m_c \ll 1 \)) one should use \( \gamma = 55/3 \) in the chiral limit with respect to \( u, d, s \) quarks. This example is quite academic, though, since at such small values of \( \rho \) the weight function is so strongly suppressed that the corresponding contribution can hardly be observable in hadronic processes in the foreseeable future.

Since our purpose is mostly illustrative we will not restrict ourselves to a specific value of \( \gamma \); rather it will be treated as a generic numerical parameter lying somewhere in between 10 and 20.

The function \( f(\rho) \) in Eq. (18) is a cut off function suppressing large values of \( \rho \). It is equal to unity in the dilute instanton gas approximation. In fact, one does not need any cut off if calculations are done at large Euclidean momenta \( Q^2 \). The integral is automatically convergent at \( \rho \sim 1/Q \) (see Eq. (14)). Since we deal now with the imaginary parts that fall off at large \( \sqrt{s} \) only slowly (see Eq. (16)), to be on the safe side, we introduce a cut off function. The final result will be essentially independent of the choice of \( f(\rho) \) provided the cut off is soft enough. For instance, \( f(\rho) = \exp(-\rho/\rho_0) \) is quite suitable.

With this information in hands we proceed to smearing \( \text{Im}\Delta \Pi \) given in Eq. (13),

\[
\langle \text{Im}\Delta \Pi \rangle_s \propto \int (\sqrt{s}\rho)^{-n} J_n(\sqrt{s}\rho) w(\rho) \frac{d\rho}{\rho}
\]  

where the angle brackets denote smearing and the subscript \( s \) indicates the we average over the small size fluctuations. Doing the integral we arrive at

\[
\langle \text{Im}\Delta \Pi \rangle_s \propto \left( \sin \frac{\pi \gamma}{2} \right) 2^{\gamma-n} \Gamma(\gamma 2) \Gamma\left( \frac{\gamma}{2} - \frac{2n}{2} \right) \times \left( \frac{\Lambda_{\text{QCD}}}{\sqrt{s}} \right)^\gamma
\]  

plus subleading in \( 1/\sqrt{s} \) terms. This expression assumes that \( \sqrt{s}\rho_0 \gg \gamma \), and characteristic values of \( \rho \) saturating the integral (13) are of order \( \rho \sim \gamma/\sqrt{s} \). The
very same expression for the imaginary part could be obtained by integrating the
Euclidean $\Delta \Pi$ over $\rho$ with the subsequent analytic continuation to the Minkowski
domain and separation of the imaginary part.

What lessons can we learn from this simple exercise that describes, at a qual-
itative level, transmitting large momenta through a small-size fluctuation? First,
this particular duality-violating contribution to the imaginary part falls of as a high-
power of $1/\sqrt{s}$ at large $s$. In order to be able to single out the small-size fluctuations
one must choose $\sqrt{s} > \gamma/\rho_0$. Even if $\rho_0^{-1}$ is set at 0.5 GeV (in fact, this parameter is
probably larger [41]) the small-size direct instantons can show up at energies $\sqrt{s} > 5$ GeV. At these energies

$$\langle \text{Im}\Delta \Pi \rangle_s \sim \left( \frac{\gamma \Lambda_{\text{QCD}}}{\sqrt{s}} \right)^\gamma \lesssim (1/2)^\gamma$$  \hspace{1cm} (21)

and this effect is unimportant for all practical purposes. Parameter $\Lambda_{\text{QCD}}$ in Eq.
(21) can be hardly pinned down better than up to a factor of two at present but
this is quite irrelevant – the nature of the $s$ dependence in Eq. (21) is such that
we can safely approximate it by the step function. If $\sqrt{s}$ is at least factor of two
higher than $\gamma \Lambda_{\text{QCD}}$ the result is practically zero; shifting slightly towards lower
values of $\sqrt{s}$ we formally observe an almost immediate explosion in Eq. (21). This
explosion happens, however, outside the domain where one can speak about the
small-size fluctuations. The conclusion is as follows: immediately above a critical
point $\langle \text{Im}\Delta \Pi \rangle$, practically vanishes. Below the critical point no reliable predictions
exist in the Minkowski domain.

Let us discuss now another contribution which is formally much smaller than
that of the small-size instantons, namely the contribution coming from the vacuum
medium where the size of the fluctuations can not be arbitrarily small. This is
presumably the dominant component of the vacuum fields responsible for the most
salient features of the hadronic processes. Being formally exponentially suppressed
this mechanism at the moment seems to be the leading source of the duality viola-
tions in all processes of practical interest. To have something particular in mind one
can think of the instanton liquid model [41], although this model is certainly not
singled out in the context under discussion. The general features will be inherent
to any mechanism of this type.

As was already demonstrated, the contribution of the fixed-size instanton falls
off very slowly in the Minkowski domain. Certainly, it is not realistic to saturate by
fixed-size fluctuations; the ensemble of fluctuations in the genuine QCD vacuum is
characterized by a distribution in the sizes centered near some typical size $\rho_0$. Let
us assume that this distribution falls off steeply at smaller and larger values of $\rho$,
with a width $\Delta$. A typical distribution of this type can be parametrized as follows:

$$w(\rho) = N \exp\left\{ -\frac{\alpha}{\rho} - \beta\rho \right\}$$  \hspace{1cm} (22)
where \( \mathcal{N} \) is a normalization constant,
\[
\alpha = \frac{\rho_0^3}{\Delta^2}, \quad \beta = \frac{\rho_0}{\Delta^2},
\]
(23)
\( \rho_0 \) is the center of the distribution and \( \Delta \) is its width.

Averaging \( \text{Im } \Delta \Pi \) over this weight function one obtains (for \( n = 1 \))
\[
\langle \text{Im } \Delta \Pi \rangle_l = \mathcal{N}^2 E^{-1} J_1 \left\{ \sqrt{2\alpha} \left[ \sqrt{\beta^2 + E^2 - \beta} \right]^{1/2} \right\} K_1 \left\{ \sqrt{2\alpha} \left[ \sqrt{\beta^2 + E^2 + \beta} \right]^{1/2} \right\} (24)
\]
(similar expressions for \( n > 1 \) can be derived by differentiating with respect to \( \alpha \); the subscript \( l \) indicates that we average over the large size fluctuations). If
\[
E \gg \frac{1}{\Delta} \frac{\rho_0}{\Delta}
\]
(25)
the smeared imaginary part reduces to
\[
\langle \text{Im } \Delta \Pi \rangle_l \to 2\mathcal{N} E^{-1} J_1 (\sqrt{2E\rho_0} \frac{\rho_0}{\Delta}) K_1 (\sqrt{2E\rho_0} \frac{\rho_0}{\Delta}) .
\]
(26)

It is quite clear that the exponential suppression of \( \langle \text{Im } \Delta \Pi \rangle_l \) that emerged after the smearing is due to the oscillations of the original imaginary part generated by the fixed-size fluctuation. If the weight function is narrow (\( \Delta \ll \rho_0 \)) this suppression starts at high energies, see Eq. (23). In the limit when \( \Delta \to 0 \) with \( E \) fixed the exponential suppression disappears from Eq. (24), and we return back to the original oscillating imaginary part. Notice also that the exponent at \( E \gg \frac{1}{\Delta} \frac{\rho_0}{\Delta} \) is different from that one deals with in Euclidean domain (\( \sqrt{E} \text{ versus } Q \)). Moreover, apart from the exponential suppression we observe residual oscillations as well. Although the particular results (25), (26) for the duality violating contributions were obtained by using instanton-inspired calculations (and are dependent on the choice of the weight function) the general features of the results – an exponential suppression starting at a point correlated with the width of the smearing function plus residual oscillations – are presumably common to all mechanisms and will be confirmed in the future solution of QCD (if it is found) and/or phenomenologically. I make a conjecture that the exponentially suppressed duality violating contributions have a generic form
\[
\exp(-f(E)), \quad f(E) \to k E^\sigma \text{ at large } E
\]
(27)
where the critical index \( \sigma < 1 \). Shortly we will discover the same qualitative picture in a totally different context.

It is worth emphasizing once more that although at academically high energies the exponentially suppressed contribution dies off faster than that of the small-size instantons in all practical problems the latter is totally negligible once we cross the critical point while the former, being a less steeper function, can play a role.
3.3 Further explorations of deviations from duality

Let us consider now a model spectral density suggested in Ref. [1]. Although this model is not derived as a solution of QCD it satisfies all general properties of QCD we are aware of today and seems to be very instructive in studying the duality violations. It will be seen that qualitatively it nicely matches with the discussion above.

To begin with we remind the model and its motivations. The model can be formulated, in a most straightforward way, for the spectral density associated with one heavy and one light quark in the limit when the heavy quark mass tends to infinity. Simplifications occur due to the fact that the heavy quark Green function then is trivially known. Specifically, let us consider heavy-to-light quark currents

\[ J_S = \bar{Q}q, \quad J_P = \bar{Q}\gamma_5 q; \]

\[ J_1 = \frac{1}{2}(J_S + J_P) = \bar{Q}\frac{1}{2}(1 + \gamma_5)q, \quad J_2 = \frac{1}{2}(J_S^\dagger - J_P^\dagger) = \bar{q}\frac{1}{2}(1 + \gamma_5)Q. \]  

(28)

assuming that \( m_Q \to \infty \) and \( m_q \to 0 \). Note that \( J_2 \) is not hermitean conjugate to \( J_1 \); rather \( J_2 \) is a chiral partner to \( J_1^\dagger \). Therefore, the correlation function

\[ \Pi = i \int e^{ikx} dx \langle \text{vac} | T \{ J_1(x), J_2(0) \} | \text{vac} \rangle \]  

(29)

vanishes in perturbation theory if \( m_q \) is put to zero.

First of all, we should choose the external momentum \( k \) in a most advantageous way. If \( m_Q \to \infty \) it is clear that the optimal reference point lies “slightly” below threshold,

\[ k_0 = m_Q - \epsilon, \quad \vec{k} = 0, \]

where \( \epsilon = O(\Lambda_{\text{QCD}}) \).

Taking the limit \( m_Q \to \infty \) and neglecting the hard gluon exchanges one can rewrite the correlation function \( \Pi(\epsilon) \) at positive (Euclidean) values of \( \epsilon \) in the following way \[1\]:

\[ \Pi(\epsilon) = \frac{1}{4} \int_0^\infty e^{-\epsilon \tau} d\tau \langle \bar{q}q(\tau) \exp \{ \int_0^\tau igA_0(t) dt \} q(0) \rangle \]  

(30)

where the angle brackets denote the vacuum averaging. The correlation function of interest is thus nothing else than the Laplace transform of the Wilson line (in the time direction) with the light quark fields at the end points.

If \( \epsilon \gg \Lambda_{\text{QCD}} \) one can expand Eq. (30) in \( 1/\epsilon \),

\[ \Pi(\epsilon) = \frac{1}{4\epsilon} \left[ \langle \bar{q}q \rangle - \frac{1}{\epsilon^2} \langle \bar{q}P^2_0 q \rangle + \frac{1}{\epsilon^4} \langle \bar{q}P^4_0 q \rangle - \frac{1}{\epsilon^6} \langle \bar{q}P^6_0 q \rangle + \ldots \right], \] 

(31)

where \( P_0 \) is the time component of the \((\text{Euclidean})\) momentum operator.

From Eq. (31) it is perfectly clear that the operator product expansion for \( \Pi(\epsilon) \) must be asymptotic; otherwise it would define an odd function of \( \epsilon \), which is obvious
nonsense. Indeed, at positive $\varepsilon$ we are below the cut, and $\Pi(\varepsilon)$ is analytic in $\varepsilon$. At negative $\varepsilon$, however, we sit right on the cut generated by the intermediate physical states produced by the currents $J_{1,2}$, and the correlation function $\Pi(\varepsilon)$ develops an imaginary part, a discontinuity across the cut. Qualitatively we have a pretty good idea of how $\text{Im}\Pi$ looks like. We will return to this issue later on and now observe only that if the series in (31) is truncated at any finite order the only imaginary part we obtain from Eq. (31) is concentrated at $\varepsilon = 0$. In accordance with our formulation of duality we conclude that at large positive values of $E (E \equiv -\varepsilon)$, $E \gg \Lambda_{\text{QCD}}$, the physical spectral density is predicted to vanish.

While the exact spectral density is unknown, of course, a model which should be close to it, at least qualitatively, has been suggested in Ref. [1]. On general grounds we know that at large $\tau$ the Wilson line $\langle \bar{q}(\tau) \exp\{\int_0^\tau i g A_0(t) dt\} q(0)\rangle$ must be proportional to $\exp\{-\Lambda\tau\}$; moreover, this function should have no singularities in the complex plane except on the imaginary axis. At small $\tau$ it must be expandable in powers of $\tau$ and this expansion must run over only even powers of $\tau$ (this expansion generates Eq. (31)). The simplest choice satisfying the above criteria is

$$\langle \bar{q}(\tau) \exp\{\int_0^\tau i g A_0(t) dt\} q(0)\rangle = \langle \bar{q}q \rangle \times \frac{1}{\cosh \Lambda \tau}. \quad (32)$$

This expression for the Wilson line generates a correlation function $\Pi(\varepsilon)$,

$$\Pi(\varepsilon) = \frac{1}{4\Lambda} \langle \bar{q}q \rangle \beta \left( \frac{\varepsilon + \Lambda}{2\Lambda} \right) \quad (33)$$

where $\beta$ is related to Euler's $\psi$ function,

$$\beta(x) = \frac{1}{2} \left[ \psi \left( \frac{x + 1}{2} \right) - \psi \left( \frac{x}{2} \right) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{x + k}.$$

We pause here to make a digression illustrating the degree to which realistic details can be expected to be reproduced by the model above. Accepting Eq. (32) we made a specific assumption about the behavior of the matrix elements appearing in Eq. (31) – as dimension of the operators grows their matrix elements grow factorially,

$$\langle \bar{q}(P_0)^{2n} q \rangle \sim \langle \bar{q}q \rangle \left( \Lambda^2_{\text{QCD}} \right)^n C^{2n} (2n)!,$$

where $C$ is a numerical constant. Although very little is known for fact for high-dimension operators one can confront the dimension-5 operator with information that already exists. Indeed,

$$\langle \bar{q}P_0^2 q \rangle = \frac{1}{8} \langle \bar{q}igG_{\mu\nu}^a \sigma_{\mu\nu} t^aq \rangle = \frac{1}{8} m_0^2 \langle \bar{q}q \rangle \quad (35)$$

\[\text{1Warning: in many publications it is erroneously assumed that } \langle \bar{q}(\tau) \exp\{\int_0^\tau i g A_0(t) dt\} q(0)\rangle \propto \exp(-C\tau^2) \text{ which is incompatible with general principles.}\]
where \( m_0^2 \) is a parameter introduced in [42], and we accounted for the fact that \( P_0 \) is defined in Euclidean. By comparing the first subleading term in the large \( \epsilon \) expansion of Eq. (33) with the general expansion (31) we establish that (i) the sign is correct; (ii) \( \bar{\Lambda}^2 = m_0^2/8 \). Phenomenologically \( m_0^2 \) is close to 0.8 GeV\(^2\) [42], and we conclude that in our model

\[
\bar{\Lambda} \approx 0.31 \text{GeV}.
\]

This number is in the right ballpark; perhaps, a factor of \( \sim 1.5 \) smaller than the expected phenomenological value of \( \bar{\Lambda} \).

The correlation function (33) we end up with is a sum of simple poles at

\[
E \equiv -\epsilon = 1, 3, 5, 7, \ldots
\]

(in the remainder of this section we put \( \bar{\Lambda} = 1 \); thus all energies will be measured in these units), all residues are equal in the absolute value and are sign-alternating. We remind that \( \text{Im}\Pi(E) \) is actually the difference between the spectral densities in the scalar and pseudoscalar channels and, therefore, need not be positive.

The above model perfectly matches a picture one expects to get in the multicolor QCD, with \( N_c \to \infty \): two combs of infinitely narrow peaks sitting, back-to-back, on top of each other. Let us defer for a while the discussion of how the spectral density evolves from \( N_c = \infty \) to \( N_c = 3 \) and study now the implications of this picture for duality.

The spectral density at \( E \gg \Lambda_{\text{QCD}} \) stemming from Eq. (33)

\[
\text{Im}\Pi = \frac{\pi}{2} \langle \bar{q}q \rangle \sum_{k=0}^{\infty} (-1)^k \delta(E - (1 + 2k)),
\]

taken at its face value, does not vanish. We understand, however, that under the circumstances duality should be applied to the average spectral density rather than to \( \text{Im}\Pi(E) \) itself. If we merely average \( \text{Im}\Pi(E) \) in the interval from \( E_1 \) to \( E_2 \) we get a sign-alternating result of order \( 1/(E_2 - E_1) \) – i.e. a huge deviation from duality which does not even die off as \( E \to \infty \) as long as the size of the smearing interval is kept fixed!

If this were the actual situation in QCD any OPE-based expansions in the Minkowski domain would be hopeless. For instance, why then bother about calculating the \( 1/m_Q^2 \) corrections to the lepton spectrum in \( B \to l\nu X_u \) due to \( \mu_G^2 \) and \( \mu_\pi^2 \) [43] if uncontrollable deviations may die off slower than \( 1/m_Q^2 \)? Fortunately, at large energies QCD with \( N_c = 3 \) does not look at all like its multicolor limit; the limits \( E \to \infty \) and \( N_c \to \infty \) are not commutative. If the number of colors decreases from infinity to 3 the spectral density \( \text{Im}\Pi(E) \) at large \( E \) experiences a dramatic evolution. A comb of delta functions is converted into a smooth function. The details of the smearing mechanism are foggy but qualitatively one may conjecture the

\[\footnote{For a review see the Reprint Volume cited in Ref. [31]; note that Eq. (2.4) on page 22 of the above Volume contains a misprint, \( -g \) must be substituted by \( ig \); moreover, \( \sigma_{\mu\nu} \) is understood as \( (1/2)[\gamma_\mu\gamma_\nu] \).} \]

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following. At $N_c = \infty$ the correlation functions in QCD are saturated by a master-field \[44\], a classical field configuration which does not fluctuate. This freezes the scale and makes the calculation in a sense similar to that with the fixed-size instanton. With $N_c$ decreasing, the field configurations saturating the functional integral start fluctuating; correspondingly, a distribution of scales emerges. The correlation function corresponding to $N_c = 3$ can be obtained, qualitatively, by substituting $\epsilon \to \epsilon\rho$ in Eq. (33) and integrating over $\rho$ with a weight function similar to (22) centered at $\rho = 1$. In numerical exercises below we will consider the weight function (22) as a particular example, with

$$\alpha = \beta = \frac{1}{\Delta^2}, \quad N' = \frac{1}{2K_1(2/\Delta^2)},$$

where $\Delta$ is the width of the distribution, $\Delta = \mathcal{O}(1/N_c)$.

Thus, our model spectral density becomes

$$\text{Im}\Pi(E) = \int_0^\infty d\rho w(\rho) \frac{\pi}{2} \langle \bar{q}q \rangle \sum_{k=0}^\infty (-1)^k \delta (E\rho - (1 + 2k))$$

$$= \frac{\pi}{2E} \langle \bar{q}q \rangle \sum_{k=0}^\infty (-1)^k w \left( \frac{1 + 2k}{E} \right). \quad (37)$$

Analytically it is not difficult to prove that this imaginary part falls off faster than any given power of $1/E$. Indeed, let us focus on the function

$$\sigma(E) = \sum_{k=0}^\infty (-1)^k w \left( \frac{1 + 2k}{E} \right) = \sum_{k=0}^\infty \left[ w \left( \frac{1 + 4k}{E} \right) - w \left( \frac{3 + 4k}{E} \right) \right]. \quad (38)$$

To show that at large $E$ terms $\mathcal{O}(1/E)$ are absent in $\sigma(E)$ we represent

$$w \left( \frac{3 + 4k}{E} \right) = \frac{1}{2} \left[ w \left( \frac{1 + 4k}{E} \right) J + w \left( \frac{5 + 4k}{E} \right) \right] + \mathcal{O}(1/E^2).$$

Then

$$\sigma(E) = \frac{1}{2} \sum_{k=0}^\infty \left[ w \left( \frac{1 + 4k}{E} \right) - w \left( \frac{5 + 4k}{E} \right) \right] + \mathcal{O}(1/E^2)$$

$$= w(1/E) + \mathcal{O}(1/E^2).$$

Moreover, in the next order one uses

$$w \left( \frac{3 + 4k}{E} \right) = \frac{3}{8} \left[ w \left( \frac{1 + 4k}{E} \right) J + 2w \left( \frac{5 + 4k}{E} \right) - \frac{1}{3} w \left( \frac{9 + 4k}{E} \right) J \right] + \mathcal{O}(1/E^3).$$

and so on.

Hence, the suppression is exponential, modulated by oscillations. These facts are quite general and do not depend on the particular choice of the weight function.
provided that the weight function falls off faster than any power of $\rho$ at small $\rho$. For instance, to prove that an infinite number of oscillations must necessarily take place in the imaginary part we write the dispersion relation for $\Pi(\epsilon)$, expand it in $1/\epsilon$ and compare the result of this expansion with Eq. (31). The absence of all even powers in $1/\epsilon$ requires an infinite number of oscillations in $\text{Im}\Pi(E)$.

I was unable to find analytically the type of the exponential suppression. Numerically it seems that Eq. (27) goes through, with the index $\sigma$ depending on the width $\Delta$. (Thus, at $\Delta = 1/3$ we get $\sigma \approx 0.7$). The plots of $\text{Im}\Pi(E)$ for $\Delta = 0$, $\Delta = 1/3$ and $\Delta = 1/2$ are presented on Fig. 4. We see that the heights of the oscillations rapidly fall off starting with some boundary energy which, in turn, depends on $\Delta$. Duality is restored with the exponential accuracy at $E \gg 1/\Delta$. Below this boundary value the resonance peaks are conspicuous. Notice also that the length of oscillations grows with energy. At $\Delta \to 0$ the boundary value of energy shifts to infinity, and we return back to the infinite comb of the infinitely narrow peaks.

Theoretically the precise shape of the fall off of $\text{Im}\Pi(E)$ at large $E$ is correlated with the divergence of the high-order terms in Eq. (31). This divergence depends, in turn, on the behavior of the matrix elements $\langle \bar{q}P_{\alpha}^{2n}q \rangle$ which is not known. In our comb-like model (see Eq. (22)) these matrix elements grow as $(2n)!$. After smearing this formula changes, the result being dependent on the particular choice of the smearing weight function. The crucial parameter is $n\Delta$. As long as this parameter is smaller than unity the leading behavior remains intact, $(2n)!$. For $n\Delta \gg 1$ the regime changes and $(2n)!$ is substituted by $\Delta^{2n}((2n)!)^2$. We remind that $\Delta \sim 1/N_c$. This means that the terms in the operator product expansion (31) which are subleading in $1/N_c$ are more singular in $n$, i.e. their divergence in high orders of OPE must be stronger. This aspect is not new, though; the very same situation takes place in the ordinary perturbation theory [45].

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