On systems of commuting matrices, Frobenius Lie algebras and Gerstenhaber’s Theorem

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Abstract

This work relates to three problems, the classification of maximal Abelian subalgebras (MASAs) of the Lie algebra of square matrices, the classification of 2-step solvable Frobenius Lie algebras and the Gerstenhaber’s Theorem. Let \(M\) and \(N\) be two commuting square matrices of order \(n\) with entries in an algebraically closed field \(K\). Then the associative commutative \(K\)-algebra, they generate, is of dimension at most \(n\). This result was proved by Murray Gerstenhaber in 1961. The analog of this property for three commuting matrices is still an open problem, its version for a higher number of commuting matrices is not true in general. In the present paper, we give a sufficient condition for this property to be satisfied, for any number of commuting matrices and arbitrary field \(K\). Such a result is derived from a discussion on the structure of 2-step solvable Frobenius Lie algebras and a complete characterization of their associated left symmetric algebra structure. We discuss the classification of 2-step solvable Frobenius Lie algebras and show that it is equivalent to that of \(n\)-dimensional MASAs of the Lie algebra of square matrices, admitting an open orbit for the contragradient action associated to the multiplication of matrices and vectors. Numerous examples are discussed in any dimension and a complete classification list is supplied in low dimensions. Furthermore, in any finite dimension, we give a full classification of all 2-step solvable Frobenius Lie algebras corresponding to nonderogatory matrices.

1 Introduction

Let \(K\) be a field, considered here to have characteristic zero. In the \(K\)-vector space \(\mathcal{M}(n, K)\) of all \(n \times n\) matrices with entries in \(K\), we consider the following two algebraic structures: (1) the \(K\)-algebra structure given by the ordinary multiplication and addition of matrices and (2) the Lie algebra structure corresponding to the Lie bracket \([\, , \,]\) given by the commutator \([M, N] := MN - NM\)

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of matrices $M, N \in \mathcal{M}(n, \mathbb{K})$. We will let $\mathfrak{gl}(n, \mathbb{K})$ stand for $\mathcal{M}(n, \mathbb{K})$ together with the latter Lie algebra structure. Let $\{M_1, \ldots, M_k\}$ be a set of $k$ pairwise commuting elements of $\mathcal{M}(n, \mathbb{K})$. On the one hand, we consider the commutative associative $\mathbb{K}$-subalgebra $\mathbb{K}[M_1, \ldots, M_k]$ of $\mathcal{M}(n, \mathbb{K})$ made of all matrices of the form $p_k(M_1, \ldots, M_k)$, for all polynomials $p_k(X_1, \ldots, X_k)$ in $k$ variables. On the other hand, we denote by $\mathfrak{B}(M_1, \ldots, M_k)$ the vector subspace of $\mathcal{M}(n, \mathbb{K})$ spanned by $\{M_1, \ldots, M_k\}$. If the $M_i$'s are linearly independent, it obviously simply reads $\mathfrak{B}(M_1, \ldots, M_k) = \mathbb{K}M_1 \oplus \cdots \oplus \mathbb{K}M_k$, where $\mathbb{K}M_i$ is the line of matrices of the form $\lambda M_i$, with $\lambda \in \mathbb{K}$. Of course, $\mathfrak{B}(M_1, \ldots, M_k)$ is an Abelian Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$, but in general it is not a $\mathbb{K}$-subalgebra of $\mathcal{M}(n, \mathbb{K})$, although it is a vector subspace of $\mathbb{K}[M_1, \ldots, M_k]$. In [12], Gerstenhaber proved the following result, hereafter named the Gerstenhaber’s theorem. If $M$ and $N$ are two commuting elements of $\mathcal{M}(n, \mathbb{K})$, the vector space underlying $\mathbb{K}[M, N]$ is of dimension at most $n$, when $\mathbb{K}$ is algebraically closed. The version for four or a higher number of commuting matrices is not true in general, as shown by Gerstenhaber himself in [12] using the following example. Let $E_{i,j}$ be the square matrix whose coefficients are all zero except the $(i, j)$ entry which is equal to 1. One sees that for $n = 4$, the system $(E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4})$ generates a 5-dimensional $\mathbb{K}$-algebra with $(I_{2 \times 2}, E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4})$ as a basis, where $I_{2 \times 2}$ hereafter stands for the identity matrix of order $n$. The analog of this property for three commuting matrices is still an open problem. It has been proved true up to order $n = 10$, in [13, 14, 15, 18, 30, 31, 32]. In [25, 27], amongst other results, the authors supply cases where the analog of Gerstenhaber’s theorem for three commuting matrices holds true. A lot of other works have been dedicated to the subject and its applications amongst which [3, 23, 17, 20, 35]. In [26], it is suggested that the problem might have a negative answer in general. But no counterexample has been found so far. A comprehensive account of the state of the art, up to the year 2011, can be found in [29].

In the present work, we supply a general sufficient condition for the $k$-matrix analog of Gerstenhaber’s theorem to hold true, for any $k \geq 1$. Note that in [12], Gerstenhaber further showed that, if $M$ and $N$ are two commuting elements of $\mathcal{M}(n, \mathbb{K})$, then $\mathbb{K}[M, N]$ is always contained in an $n$-dimensional Abelian Lie subalgebra of $\mathcal{M}(n, \mathbb{K})$. In general the maximal dimension of a maximal Abelian subalgebra (MASA, for short) of $\mathcal{M}(n, \mathbb{K})$ is $\left\lceil \frac{n}{4} \right\rceil + 1$, as shown by I. Schur in [28] for any algebraically closed field $\mathbb{K}$ and further generalized to any arbitrary field by N. Jacobson [19].

Our result can be summarized as follows. Let $\{M_1, \ldots, M_k\}$ be a set of $k$ pairwise commuting elements of $\mathcal{M}(n, \mathbb{K})$. If one can complete this set into a set $\{M_1, \ldots, M_k, M_{k+1}, \ldots, M_n\}$ of $n$ linearly independent pairwise commuting matrices, such that the Abelian subalgebra $\mathfrak{B}(M_1, \ldots, M_n)$ of $\mathcal{M}(n, \mathbb{K})$, has an open orbit in $(\mathbb{K}^n)^*$ for the contragredient action corresponding to the ordinary action of matrices on vectors of $\mathbb{K}^n$, then $\dim \mathbb{K}[M_1, \ldots, M_k] \leq n$. In fact, under such an assumption, the equality $\mathbb{K}[M_1, \ldots, M_n] = \mathfrak{B}(M_1, \ldots, M_n)$ holds true, hence any subset of $p$ matrices satisfy the Gerstenhaber’s theorem, for any $p = 1, \ldots, n$. Let us remind here that an action $\rho$ of a Lie algebra $G$ on a vector space, say $\mathbb{K}^n$, is naturally lifted into an action $\rho^*$ of $G$ on the dual space $(\mathbb{K}^n)^*$ of $\mathbb{K}^n$, called the corresponding contragredient action given by $\rho^*(a)f = -f \circ \rho(a)$, for any $a \in G$ and $f \in (\mathbb{K}^n)^*$. The approach so far used is mainly based on the study of the variety of $k$-tuples of commuting matrices of $\mathcal{M}(n, \mathbb{K})$. In the case of 3-tuples of
commuting matrices, such a variety is no longer irreducible, so the methods used so far do not apply any longer. Our approach is different and heavily relies on techniques akin to left invariant affine geometry on Lie groups with a left invariant symplectic structure. It can be summarized as follows. We embed $\mathfrak{B}(M_1,\ldots,M_n)$ in the Lie algebra of a 2-step solvable Lie group with a left invariant exact symplectic structure (Theorem 3.1) and show that the induced left symmetric algebra (LSA, for short) preserves $\mathfrak{B}(M_1,\ldots,M_n)$ and furthermore it coincides with the ordinary product of matrices (up to a sign) (Theorem 4.1), hence implying the equality $K[M_1,\ldots,M_n] = \mathfrak{B}(M_1,\ldots,M_n)$. See Theorem 5.1 We show that if an $n$-dimensional Abelian subalgebra $\mathfrak{B}$ of $\mathfrak{gl}(n,K)$, has an open orbit on $(K^n)^*$, for the contragredient action $\mathfrak{B} \times (K^n)^* \to (K^n)^*$, $(a,f) \mapsto -f \circ a$, then $\mathfrak{B}$ is a maximal Abelian subalgebra (MASA) of $\mathfrak{gl}(n,K)$. We also discuss the classification of 2-step solvable Frobenius Lie algebras and further show that it is equivalent to that of $n$-dimensional MASAs of $\mathfrak{gl}(n,K)$ admitting an open orbit for the above contragredient action on $(K^n)^*$. See Theorem 6.1. As applications, several examples are discussed in any dimension, see Examples 5.3, 6.1.1, 6.1.2. The study of MASAs has been a vibrant and vivid subject these last decades, in relation with several subjects in mathematics and physics such as the classification of Lie algebras, the study of dynamical systems, especially the symmetry breaking, complete sets of commuting operators in a quantum-mechanical system, the generalized cusps in singularity theory and their applications to computer graphics, ... The paper is organized as follows. Section 2 is devoted to some preliminaries and a few notions. In Section 3 we give a construction of all 2-step solvable Frobenius Lie algebras and thereby a description of their algebraic structure. A complete characterization of the corresponding left symmetric algebra (LSA) structure is given in Section 4. Section 5 discusses our extension of Gerstenhaber’s theorem and its proof, together with several examples in any dimension. In Section 6 we discuss the classification of 2-step solvable Frobenius Lie algebras. The paper ends by some concluding remarks in Section 9.

2 Preliminaries

Throughout this work, if $\mathfrak{g}$ is a vector space, we will denote its (linear) dual by $\mathfrak{g}^\ast$. The symmetric bilinear form $\langle \cdot , \cdot \rangle$ stands for the duality pairing between vectors and linear forms. More precisely, for any $u \in \mathfrak{g}$ and $f \in \mathfrak{g}^\ast$, we define $\langle u,f \rangle$ as $\langle u,f \rangle := f(u)$. Let $\mathfrak{B}$ be a Lie subalgebra of $\mathfrak{gl}(n,K)$. We consider the action $\mathfrak{B} \times K^n \to K^n$, $(a,x) \mapsto \rho(a)x := ax$ of $\mathfrak{B}$ on $K^n$ via the ordinary action of square matrices on vectors. For any $(a,f) \in \mathfrak{B} \times (K^n)^*$, let $\rho^*(a)f$ be the element of the dual $(K^n)^*$ defined on elements $x$ of $K^n$ by $\langle \rho^*(a)f, x \rangle := -\langle f, \rho(a)(x) \rangle = -f(ax)$. We recall that $\rho^*$ is called the contragredient representation (or just action) of $\rho$.

**Definition 2.1.** We say that the orbit of $\alpha \in (K^n)^*$, under the action $\rho^*$, is open if it is equal to the whole $(K^n)^*$. That is $(K^n)^* = \{ \rho^*(a)\alpha, \ a \in \mathfrak{B} \}$. We say that $\alpha$ has a trivial isotropy if there is no nonzero $a \in \mathfrak{B}$ satisfying $\rho^*(a)\alpha = 0$. 


The vector space $\mathcal{B} \oplus \mathbb{K}^n$ endowed with the Lie bracket defined, for every $a, b \in \mathcal{B}$ and $x, y \in \mathbb{K}^n$, as $[a, b] = [x, y] = 0$, and $[a, x] = \rho(a)x = ax$, will be termed the semidirect sum of $\mathcal{B}$ and $\mathbb{K}^n$ via $\rho$ and denoted by $\mathcal{B} \ltimes \mathbb{K}^n$. A symplectic Lie algebra (also called a quasi Frobenius Lie algebra) is a Lie algebra $\mathcal{G}$ together with a nondegenerate skew-symmetric bilinear form $\omega$, which is closed, that is,

$$\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0, \quad \forall u, v, w \in \mathcal{G},$$

(1)

where $[,]$ is the Lie bracket of $\mathcal{G}$. Exact symplectic (also termed Frobenius) structures are special examples of symplectic structures on Lie algebras. They are given by the Chevalley-Eilenberg coboundary $\omega = \partial\alpha$ of a linear form $\alpha \in \mathcal{G}^*$, for the adjoint action of $\mathcal{G}$. More precisely, $(\mathcal{G}, \omega)$ is a Frobenius Lie algebra if there exists $\alpha \in \mathcal{G}^*$, called a Frobenius functional, such that the skew-symmetric bilinear form $\omega$ defined, for any $u, v \in \mathcal{G}$,

$$\omega(u, v) = \partial\alpha(u, v) = -\langle \alpha, [u, v] \rangle,$$

(2)

is nondegenerate. Every Frobenius Lie algebra is a codimension 1 subalgebra of some contact Lie algebra and could be used to construct the latter. The converse is true under some conditions [5, 9, 2].

Recall that a left symmetric algebra (LSA) structure on a vector space $\mathfrak{F}$ is a product $\mathfrak{F} \times \mathfrak{F} \rightarrow \mathfrak{F}$, $(u, v) \mapsto u \ast v$ such that the corresponding associator $\mathfrak{a}(u, v, w) := (u \ast v) \ast w - u \ast (v \ast w)$ is left symmetric, that is, $\mathfrak{a}(u, v, w) = \mathfrak{a}(v, u, w)$, $\forall u, v, w \in \mathfrak{F}$. Any symplectic structure, say given by a symplectic form $\omega$, on a Lie algebra $\mathcal{G}$, induces an LSA, denoted hereafter by $\ast$, on $\mathcal{G}$. It is defined, for any $u, v \in \mathcal{G}$, by

$$\omega(u \ast v, w) := -\omega(v, [u, w]).$$

(3)

Moreover, the LSA $\ast$ is compatible with the Lie bracket of $\mathcal{G}$, which is equivalent to the following equality: for any $u, v \in \mathcal{G}$,

$$u \ast v - v \ast u = [u, v].$$

(4)

Consider the vector space isomorphism $q : \mathcal{G} \rightarrow \mathcal{G}^*$, $u \mapsto i_u\omega = \omega(u, \cdot)$. In the case where $\omega = \partial\alpha$, the vector $v_0 \in \mathcal{G}$ such that $q(v_0) = \alpha$ is called the principal element, corresponding to $\alpha$.

Any Lie group $G$ with Lie algebra $\mathcal{G}$, has a left invariant symplectic form $\omega^+$ whose value at the neutral element $\varepsilon$ of $G$, is precisely $\omega$. Here $\mathcal{G}$ is identified with the tangent space to $G$ at $\varepsilon$. There is also, in $\mathcal{G}$, a left invariant affine structure induced by a left invariant locally flat torsion free connection $\nabla$, defined on left invariant vector fields $u^+, v^+$ as

$$\nabla_{u^+}v^+ := (u \ast v)^+, \quad \text{where } u^+_\varepsilon = u \text{ and } v^+_\varepsilon = v$$

(5)

and naturally extended to all smooth vector fields on $G$. See for example [7, 10].

**Definition 2.2.** A Lie algebra $\mathcal{G}$ is said to be 2-step solvable if its derived ideal $[\mathcal{G}, \mathcal{G}]$ is Abelian, where $[,]$ stands for its Lie bracket, in other words, $[[u, v], [u', v']] = 0$ for any $u, v, u', v' \in \mathcal{G}$. A Lie algebra is said to be indecomposable if it cannot be written as the direct sum of two of its ideals.
Definition 2.3. An Abelian subalgebra $\mathcal{A}$ of a Lie algebra $\mathcal{G}$ is called a maximal Abelian subalgebra (MASA) of $\mathcal{G}$ if it is contained in no bigger Abelian subalgebra of $\mathcal{G}$, or equivalently, if $\mathcal{A}$ coincides with its centralizer $\{ b \in \mathcal{G}, [b,a] = 0, \forall a \in \mathcal{A} \}$ in $\mathcal{G}$. Recall that an $n \times n$ matrix $M$ (resp. linear map) is said to be nonderogatory (or cyclic), if its characteristic and minimal polynomials coincide.

For $M \in \mathcal{M}(n, \mathbb{K})$, let $\mathbb{K}[M]$ and $C(M)$ respectively stand for the ring of polynomials in $M$ with coefficients in $\mathbb{K}$ and the space of $n \times n$ matrices (with coefficients in $\mathbb{K}$) that commute with $M$. For a linear map $\psi : \mathbb{K}^n \rightarrow \mathbb{K}^n$, we will use the convention $\psi^0 = \mathbb{I}_{\mathbb{K}^n}$ and $\psi^{p+1}(x) = \psi(\psi^p(x))$, for every $x \in \mathbb{K}^n$ and $p$ an integer.

Lemma 2.1 (see e.g. [5], [24]). Let $E$ be a vector space of dimension $n$ over a field $\mathbb{K}$ with characteristic zero and $M$ the matrix of a linear transformation of $E$, in a given basis of $E$. Denote $E^*$ the dual space of $E$. The following assertions are equivalent:
1) there exists $\alpha \in E^*$ such that $(\alpha, \alpha \circ M, ..., \alpha \circ M^{n-1})$ is a basis of $E^*$,
2) there exists $\tilde{x} \in E$ such that $(\tilde{x}, M(\tilde{x}), ..., M^{n-1}(\tilde{x}))$ is a basis of $E$,
3) $\dim(C(M)) = n$ and $C(M)$ is commutative,
4) $C(M) = \mathbb{K}[M]$,
5) the characteristic and the minimal polynomials of $M$ are the same,
6) In every extension $\tilde{\mathbb{K}}$ of $\mathbb{K}$ where $M$ admits a Jordan form, this latter has only one Jordan bloc for each eigenvalue.

3 On the structure of 2-step solvable Frobenius Lie algebras

For simplicity, if $\{a_1, \ldots, a_n\}$ is a set of linearly independent pairwise commuting $n \times n$ matrices, we denote by $\mathcal{B}$ the vector space
$$\mathcal{B} = \mathcal{B}(a_1, \ldots, a_n) := \mathbb{K}a_1 \oplus \cdots \oplus \mathbb{K}a_n$$
endowed with the Abelian Lie algebra structure $[a_i, a_j] = 0$, for any $i, j = 1, \ldots, n$. We consider the natural action $\mathcal{B} \times \mathbb{K}^n \rightarrow \mathbb{K}^n$, $(a, x) \mapsto \rho(a)x$, given by the ordinary multiplication $\rho(a)x := ax$ of matrices $a \in \mathcal{B}$ and vectors $x \in \mathbb{K}^n$, together with the corresponding contragredient action $\mathcal{B} \times (\mathbb{K}^n)^* \rightarrow (\mathbb{K}^n)^*$, $(a, f) \mapsto -f \circ a$.

Theorem 3.1. (A) Let $\{a_1, \ldots, a_n\}$ be a set of $n$ linearly independent mutually commuting $n \times n$ matrices with entries in a field $\mathbb{K}$. Let $\mathcal{B}$ stand for the vector space over $\mathbb{K}$ spanned by $a_1, \ldots, a_n$, looked at as an Abelian Lie subalgebra of $\mathfrak{gl}(n, \mathbb{K})$. Suppose the action $\mathcal{B} \times (\mathbb{K}^n)^* \rightarrow (\mathbb{K}^n)^*$, $(a, f) \mapsto -f \circ a$, has an open orbit in $(\mathbb{K}^n)^*$. Consider the Lie algebra $\mathcal{G}$ obtained by performing the semi-direct sum $\mathcal{G} := \mathcal{B} \ltimes \mathbb{K}^n$. That is, $\mathcal{G}$ is the vector space $\mathcal{B} \oplus \mathbb{K}^n$ endowed with the Lie bracket defined, for every $a, b \in \mathcal{B}$ and $x, y \in \mathbb{K}^n$, as
$$[a, b] = [x, y] = 0, \quad [a, x] = \rho(a)x = ax.$$ 
Then $\mathcal{G}$ is a 2-step solvable Frobenius Lie algebra.

(B) Conversely, any 2-step solvable Frobenius Lie algebra is isomorphic to a Lie algebra constructed as in (A).
Proof. (A) Consider the Lie algebra \( \mathcal{G} := \mathfrak{B} \rtimes \rho \mathbb{K}^n \), its underlying Lie bracket reads, for every \( a, b \in \mathfrak{B} \) and \( x, y \in \mathbb{K}^n \), as \( [a, b] = [x, y] = 0 \), and \( [a, x] = \rho(a)x := ax \). By hypothesis, for the contragredient action \( \rho^* \), there exists a linear form \( \alpha \in (\mathbb{K}^n)^* \), whose orbit is equal to the whole \((\mathbb{K}^n)^* \). So, there exists a basis \((a_1, \ldots, a_n)\) of \( \mathfrak{B} \) such that \( (\rho^*(a_1)\alpha, \ldots, \rho^*(a_n)\alpha) \) is a basis of \((\mathbb{K}^n)^* \). Now extend \( \alpha \) into the element \( \tilde{\alpha} \) of the dual \( \mathcal{G}^* \) of \( \mathcal{G} \) defined by \( \tilde{\alpha}(a) = 0 \) for any \( a \in \mathfrak{B} \) and \( \tilde{\alpha}(x) = \alpha(x) \), for any \( x \in \mathbb{K}^n \). From now on, we simply write \( \alpha \) instead of \( \tilde{\alpha} \). Let \((a_1^*, \ldots, a_n^*)\) be the dual basis of \((a_1, \ldots, a_n)\), so that \((a_1^*, \ldots, a_n^*, \rho^*(a_1)\alpha, \ldots, \rho^*(a_n)\alpha) \) is a basis of \( \mathcal{G}^* \), where each \( a_i^* \) is regarded as the element of \( \mathcal{G}^* \) whose restriction to \( \mathbb{K}^n \) is identically equal to zero and which coincides with \( a_i^* \) on \( \mathfrak{B} \). The Chevalley-Eilenberg coboundary \( \partial \alpha \), of \( \alpha \), is given, for any \( a, b \in \mathfrak{B} \) and any \( x, y \in \mathbb{K}^n \), by

\[
\partial \alpha(a, b) := -\langle \alpha, [a, b] \rangle = 0, \quad \partial \alpha(x, y) := -\langle \alpha, [x, y] \rangle = 0
\]  

(8)

and

\[
\partial \alpha(a, x) := -\langle \alpha, [a, x] \rangle = -\sum_{i=1}^n a_i^*(\alpha)\langle \alpha, [a_i, x] \rangle = -\sum_{i=1}^n a_i^*(\rho^*(a_i)\alpha, x) = \left(\sum_{i=1}^n a_i^* \wedge \rho^*(a_i)\alpha\right)(a, x).
\]  

(9)

Hence we get

\[
\partial \alpha := \sum_{i=1}^n a_i^* \wedge \rho^*(a_i)\alpha
\]  

(10)

so that

\[
\left(\partial \alpha\right)^n := (-1)^{\frac{n(n-1)}{2}} n! a_1^* \wedge \cdots \wedge a_n^* \wedge \rho^*(a_1)\alpha \wedge \cdots \wedge \rho^*(a_n)\alpha
\]  

(11)

which is a volume form on \( \mathcal{G} \). Hence \( \mathcal{G} \) is an exact symplectic (that is, a Frobenius) Lie algebra. By construction, \( \mathcal{G} \) is 2-step solvable.

(B) Conversely let \( \mathcal{G} \) be a 2-step solvable Frobenius Lie algebra over a field \( \mathbb{K} \) of characteristic 0. As \( \mathcal{G} \) must have a trivial center, then it splits as a direct sum of vector subspaces \( \mathcal{G} = \mathcal{B}_1 \oplus \mathcal{B}_2 \) where \( \mathcal{B}_2 \) is the derived ideal \( \mathcal{B}_2 := [\mathcal{G}, \mathcal{G}] \) of \( \mathcal{G} \) and \( \mathcal{B}_1 \) is an Abelian Cartan subalgebra of \( \mathcal{G} \). Set \( p_i := \dim(\mathcal{B}_i), \ i = 1, 2 \). For \( a \in \mathcal{B}_1 \) let \( \rho(a) \) be the restriction to \( \mathcal{B}_2 \) of the adjoint \( \text{ad}(a) \). As the center of \( \mathcal{G} \) is trivial, \( \rho \) then defines a faithful representation of \( \mathcal{B}_1 \) by linear transformations of \( \mathcal{B}_2 \). In this decomposition, the Lie bracket of \( \mathcal{G} \) reads

\[
[a, b] = [x, y] = 0 \text{ and } [a, x] = \rho(a)x,
\]  

(12)

for any \( a, b \in \mathcal{B}_1 \) and any \( x, y \in \mathcal{B}_2 \). A linear form on \( \mathcal{G} \) is closed if and only if its restriction to \( \mathcal{B}_2 \) is identically zero. Let \( \rho^* \) be the associated contragredient action of \( \mathcal{B}_1 \) on \( \mathcal{B}_2^* \) defined by \( \rho^*(a)(\alpha) := -\alpha \circ \rho(a) \), for \( a \in \mathcal{B}_1 \) and \( \alpha \in \mathcal{B}_2^* \). From the decomposition \( \mathcal{G} = \mathcal{B}_1 \oplus \mathcal{B}_2 \), the dual space \( \mathcal{G}^* \) of \( \mathcal{G} \) can be viewed as \( \mathcal{G}^* = \mathcal{B}_2^* \oplus \mathcal{B}_1^* \), where \( \mathcal{B}_1^* \) is the subspace consisting of all linear forms on \( \mathcal{G} \), whose restriction to \( \mathcal{B}_1 \) is identically zero.
Suppose now that $G$ is of even dimension $m = 2n$, with $n \geq 1$ and suppose further that $\eta = \alpha + \alpha'$ is a Frobenius functional on $G$, where $\alpha$ is in $\mathfrak{B}_1 = \mathfrak{B}_2$ and $\alpha'$ in $\mathfrak{B}_2' = \mathfrak{B}_1^*$. Then for any basis $(a_1, \ldots, a_p)$ of $\mathfrak{B}_1$ with dual basis $(a_1^*, \ldots, a_p^*)$, the expression of $\partial \eta$ reads

$$
\partial \eta = \partial \alpha = -\sum_{i=1}^{p_1} \rho^*(a_i)(\alpha) \wedge a_i^*.
$$

(13)

If $k$ is the dimension of the orbit of $\alpha$ under the action $\rho^*$, such a sum merely factorizes into

$$
\partial \eta = \partial \alpha = \sum_{i=1}^{p} \alpha_i \wedge \beta_i,
$$

(14)

where $p = \inf(k, p_1)$, the linear 1-forms $\alpha_i$ and $\beta_i$ are respectively in $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Due to the property $(\alpha_i \wedge \beta_i)^2 = 0$ for each $i = 1, \ldots, p$, the $2(p + j)$-form $(\partial \eta)^{p+j}$ is identically zero, if $j$ is greater than or equal to 1. But of course $p$ is less than or equal to $\inf(p_1, p_2)$, as $k$ is less than or equal to $p_1$. Thus as $p_1 + p_2 = 2n$, the non vanishing condition on $(\partial \eta)^n$ imposes that $n = p = p_1 = p_2$. Hence $\rho(\mathfrak{B}_1)$ is an $n$-dimensional Abelian subalgebra of the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$ of all linear transformations of $\mathfrak{B}_2$ and $(\rho^*(a_1)(\alpha), \rho^*(a_2)(\alpha), \ldots, \rho^*(a_n)(\alpha))$ is a basis of the orbit of $\alpha$, for any basis $(a_1, \ldots, a_n)$ of $\mathfrak{B}_1$. So the orbit of $\alpha$ under $\rho^*$ is open.

We now write $\mathbb{K}^n = \mathfrak{B}_2 = [G, G]$ so that $G = \mathfrak{B} \ltimes \mathbb{K}^n$, where $\mathfrak{B} := \mathfrak{B}_1$. Fixing a basis on $\mathbb{K}^n$, then $\rho(\mathfrak{B})$ can be seen as an $n$-dimensional Abelian Lie algebra of $n \times n$ matrices acting on $\mathbb{K}^n$ via the canonical action $\rho$ of matrices on vectors, so that, for the corresponding contragredient action, the orbit of $\alpha$ is open. Furthermore, the linear map

$$
\psi : G = \mathfrak{B} \ltimes \mathbb{K}^n \to \rho(\mathfrak{B}) \ltimes \mathbb{K}^n, \quad \psi(a, x) = (\rho(a), x),
$$

is an isomorphism between the Lie algebras $G$ and $\rho(\mathfrak{B}) \ltimes \mathbb{K}^n$, where the Lie bracket in the latter is the canonical one

$$
\left[\left((\rho(a), x), (\rho(b), y)\right)\right] = \left((\rho(a)\rho(b) - \rho(b)\rho(a), \rho(a)y - \rho(b)x)\right) = \left(0, \rho(a)y - \rho(b)x\right).
$$

This can indeed be seen from the equalities

$$
\psi \left(\left[\left(a, x\right), \left(b, y\right)\right]_G\right) = \psi \left(0, \rho(a)y - \rho(b)x\right) = \left(0, \rho(a)y - \rho(b)x\right),
$$

$$
\left[\psi(a, x), \psi(b, y)\right]_{\rho(G)} = \left[(\rho(a), x), (\rho(b), y)\right]_{\rho(G)} = \left(0, \rho(a)y - \rho(b)x\right).
$$

Remark 3.1. Note that the equalities $[\mathfrak{B}]$ mean that $\mathfrak{B}$ and $\mathbb{K}^n$ are both Lagrangian Abelian Lie subalgebras of $G$, for the exact symplectic form $\omega := \partial \alpha$. So $\mathfrak{B}$ and $\mathbb{K}^n$ integrate to two left invariant foliations, on any 2-step solvable Frobenius Lie group $G$ with Lie algebra $G$, whose leaves are everywhere transverse Lagrangian submanifolds of $G$. Furthermore, the leaves through the unit $e$ are Abelian subgroups of $G$ whose Lie algebras are $\mathfrak{B}$ and $\mathbb{K}^n$, respectively.
4 On the left symmetric algebra induced by the symplectic structure

Let \( G \) be a 2-step solvable Frobenius Lie algebra. By a convenient choice of a complementary subspace \( B \) of the derived ideal \( \mathbb{K}^n = [G, G] \) of \( G \), we can always rewrite \( G \) as the semi-direct sum \( B \ltimes \mathbb{K}^n \). From Theorem 3.1, we can choose \( B \) to be an \( n \)-dimensional subspace of commuting \( n \times n \) matrices. In this section, we completely characterize the LSA (3) induced by exact symplectic forms in the case of 2-step solvable Frobenius Lie algebras.

**Theorem 4.1.** Let \( (G = B \ltimes \mathbb{K}^n, \partial\alpha) \) be a 2-step solvable Frobenius Lie algebra, where \( \mathbb{K}^n \) is the derived ideal \( \mathbb{K}^n = [G, G] \) and \( B \) a complementary subspace of \( \mathbb{K}^n \).

Then, up to a choice of the Frobenius functional \( \alpha \), the LSA \( \star \) in (3) preserves \( B \).

In other words, \( a \star b \in B, \) for any \( a, b \in B \).

Furthermore when \( B \) is chosen to be an \( n \)-dimensional subspace of commuting \( n \times n \) matrices with entries in \( \mathbb{K} \), let us denote by \( ab \) the ordinary product of the matrices \( a, b \in M(n, \mathbb{K}) \) and by \( ax \), the ordinary multiplication of the matrix \( a \) and the vector \( x \in \mathbb{K}^n \). Then the LSA \( \star \) in (3) is completely characterized, for any \( a, b \in B \) and any \( x, y \in \mathbb{K}^n \), by

\[
a \star b = -ab, \quad a \star x = 0, \quad x \star y = 0, \quad x \star a = -ax.
\]

This means that the product \( ab \) of two matrices \( a \) and \( b \) of \( B \), lies again in \( B \). The principal element \( v_0 \) coincides with \( -I_{\mathbb{K}^n} \), where \( I_{\mathbb{K}^n} \) is the \( n \times n \) identity matrix.

**Proof.** Set \( \omega := \partial\alpha \). For any \( a, b \in B \), we have \( \omega(a \star b, c) = -\omega(b, [a, c]) = 0 \). Hence we have

\[
a \star b \in B, \quad \text{for any } a, b \in B.
\]

Indeed, if \( a \star b \) were not an element of \( B \), then there would exist some \( c_0 \in B \) and a nonzero \( y_0 \in \mathbb{K}^n \) such that \( a \star b = c_0 + y_0 \). So for every \( c \in B \), we would have \( 0 = \omega(a \star b, c) = \omega(y_0, c) \) and as \( \mathbb{K}^n \) is isotropic (Remark 3.1), we would also have \( \omega(y_0, x) = 0 \) for every \( x \in \mathbb{K}^n \). That means that \( \omega(y_0, u) = 0 \) for every \( u \in G \), which is in contradiction with the fact that \( y_0 \) is nonzero, given that \( \omega \) is nondegenerate.

Note that, as \( B \) is Abelian, Equation (4) implies that the restriction of \( \star \) to \( B \times B \) is commutative

\[
a \star b = b \star a
\]

for any \( a, b \in B \). Below, we now show that \( a \star b \) actually coincides with the matrix multiplication \( ab \), up to a sign. In order to do so, we only need to prove that \( a \star b \) and \( -ab \) coincide when applied on vectors \( x \in \mathbb{K}^n \). Indeed, for any \( a, b, c \in B \) and any \( x \in \mathbb{K}^n \), we have

\[
\omega(c, (a \star b)x) = \omega(c, [a \star b, x]) = -\omega((a \star b) \star c, x).
\]
We apply here the commutativity (18) of $\star$ to get $(a \star b) \star c = c \star (a \star b) = c \star (b \star a)$, so that

$$
\omega(c, (a \star b)x) = -\omega(c \star (b \star a), x) = \omega((b \star a), [c, x]) = -\omega(a, [b, [c, x]])
$$

$$= \langle \alpha, [a, b, [c, x]] \rangle. \tag{19}
$$

The last equality above comes from Equation (2), with $u = a$ and $v = [b, [c, x]]$. Now the Jacobi identity, coupled with the fact that $\mathfrak{B}$ is Abelian, give the property $[a, [b, [c, x]]] = [a, c, [b, x]] = [c, [a, b, x]]$, which we plug into the above equalities (19) to get

$$
\omega(c, (a \star b)x) = \langle \alpha, [c, [a, b, x]] \rangle = -\omega(c, [a, b, x]) = -\omega(c, abx). \tag{20}
$$

On the other hand, as $\mathbb{K}^n$ is totally isotropic with respect to $\omega$, we obviously have

$$\omega(y, (a \star b)x) = 0 = \omega(y, abx), \text{ for any } y \in \mathbb{K}^n. \tag{21}
$$

Altogether, this is equivalent to the equality $a \star b = -ab = -ba$, for any $a, b \in \mathfrak{B}$. Now using the commutativity of $\mathfrak{B}$ and again the fact that $\mathbb{K}^n$ is totally isotropic, we get

$$\omega(a \star x, c) := -\omega(x, [a, c]) = 0, \quad \omega(a \star x, y) := -\omega(x, ay) = 0, \tag{22}
$$

which imply the equality $a \star x = 0$, for any $a \in \mathfrak{B}$ and any $x \in \mathbb{K}^n$. In the same way,

$$\omega(x \star a, b) := -\omega(a, [x, b]) = \omega(a, [b, x])) = -\omega(b \star a, x) = -\omega(a \star b, x)
$$

$$= \omega(b, [a, x]) = -\omega(ax, b) \tag{23}
$$

and

$$\omega(x \star y, a) := -\omega(a, [x, y]) = 0 = -\omega(ax, y), \tag{24}
$$

for any $a, b \in \mathfrak{B}$ and any $x, y \in \mathbb{K}^n$. This proves the equality $x \star a = -ax$, for any $a \in \mathfrak{B}$ and any $x \in \mathbb{K}^n$. It is also easy to see that $x \star y = 0$, due to the equalities

$$\omega(x \star y, a) := -\omega(y, ax) = 0, \quad \text{and} \quad \omega(x \star y, z) = -\omega(y, [x, z]) = 0.
$$

The Frobenius functional $\alpha$ can be chosen in $(\mathbb{K}^n)^*$, so that $\langle \alpha, a \rangle = 0$, for any $a$ in $\mathfrak{B}$. Consider the vector space isomorphism $q : \mathcal{G} \to \mathcal{G}^*$, $u \mapsto q(u) := i_u \omega = \omega(u, \cdot)$. Let $v_0$ be the principal element, corresponding to $\alpha$, that is, $q(v_0) = \alpha$. For every $a \in \mathfrak{B}$, we have $0 = \langle \alpha, a \rangle = \omega(v_0, a)$. Hence, using the same reasoning as above, we conclude that $v_0$ is an element of $\mathfrak{B}$. It is well known that $v_0$ is a right unit, in the general case, for the LSA $\star$ defined in (3), induced on $\mathcal{G}$ by $\omega$ (see e.g. [7]). Now we are going to prove that, in the particular case of 2-step solvable Frobenius Lie algebras at hand, $v_0$ coincides with $-\mathbb{I}_{\mathbb{K}^n}$. Let $x \in \mathbb{K}^n$. For any $a \in \mathfrak{B}$, we have

$$\omega(a, v_0x) = -\omega(v_0 \star a, x) = -\omega(a \star v_0, x)
$$

$$= \omega(v_0, [a, x]) = \langle \alpha, [a, x] \rangle = -\omega(a, x). \tag{25}
$$

Hence $v_0x = -x$, for any $x \in \mathbb{K}^n$ and so $v_0 = -\mathbb{I}_{\mathbb{K}^n}$. \hfill \square
5 A generalization of Gerstenhaber’s theorem

In the following, we prove the Gerstenhaber’s theorem for the general case of any set \( \{a_1, \ldots, a_k\} \) of \( k \) pairwise commuting \( n \times n \) matrices, for any \( 1 \leq k \leq n \). This is done under the assumption that such a set can be completed into a set \( \{a_1, \ldots, a_k, a_{k+1}, \ldots, a_n\} \) of \( n \) linearly independent \( n \times n \) matrices such that, under the canonical action of matrices on vectors of \( \mathbb{K}^n \), the corresponding Abelian Lie algebra has an open orbit in the space \((\mathbb{K}^n)^*\) of linear forms on \( \mathbb{K}^n \).

Theorem 5.1. Let \( S := \{a_1, \ldots, a_n\} \) be a set of \( n \) linearly independent mutually commuting \( n \times n \) matrices with entries in a field \( \mathbb{K} \). Suppose that, as an \( n \)-dimensional Abelian Lie subalgebra of \( \mathfrak{gl}(n, \mathbb{K}) \), the vector space \( \mathcal{B} \) over \( \mathbb{K} \) spanned by \( a_1, \ldots, a_n \), acts on \((\mathbb{K}^n)^*\) with an open orbit, via the action \( \rho^* : \mathcal{B} \times (\mathbb{K}^n)^* \rightarrow (\mathbb{K}^n)^* \), \((a, f) \mapsto \rho^*(a)f := -f \circ a\). For any integer \( p \), with \( 1 \leq p \leq n \) and for any subset \( \{a_{i_1}, \ldots, a_{i_p}\} \) of \( p \) elements of \( S \), denote by \( \mathbb{K}[a_{i_1}, \ldots, a_{i_p}] \) the commutative associative \( \mathbb{K} \)-algebra of all polynomials in \( a_{i_1}, \ldots, a_{i_p} \), with coefficients in \( \mathbb{K} \). Then we have \( \dim \mathbb{K}[a_{i_1}, \ldots, a_{i_p}] \leq n \), in particular, \( \mathbb{K}[a_1, \ldots, a_n] = \mathcal{B} \).

Proof. Theorem 5.1 is a consequence of Theorem 3.1 and Theorem 4.1. Indeed, if \( S := \{a_1, \ldots, a_n\} \) is a set of \( n \) linearly independent mutually commuting \( n \times n \) matrices with entries in a field \( \mathbb{K} \) of characteristic zero, we consider the vector subspace \( \mathcal{B} := \mathbb{K} a_1 + \cdots + \mathbb{K} a_n \subset \mathcal{M}(n, \mathbb{K}) \) and naturally endow it with the structure of an Abelian Lie algebra. First, using Theorem 3.1 we embed \( \mathcal{B} \) as an Abelian Cartan subalgebra of a 2-step solvable Frobenius Lie algebra \( (\mathcal{G}, \partial \alpha) \). Then we use Theorem 4.1 to show that the LSA \( \ast \) of \( (\mathcal{G}, \partial \alpha) \) preserves \( \mathcal{B} \). Better yet, we show that the restriction of \( \ast \) to \( \mathcal{B} \times \mathcal{B} \) coincides with the usual product of matrices of \( \mathcal{B} \), up to a sign. In particular, the product of any two matrices \( a, b \in \mathcal{B} \), lies again in \( \mathcal{B} \). That is precisely equivalent to the equality \( \mathbb{K}[a_1, \ldots, a_n] = \mathcal{B} \), or, equivalently, for any \( p = 1, \ldots, n \) and for every subset \( \{a_{i_1}, \ldots, a_{i_p}\} \) of \( p \) elements of \( S \), the vector space underlying the \( \mathbb{K} \)-algebra \( \mathbb{K}[a_{i_1}, \ldots, a_{i_p}] \), is a vector subspace of \( \mathcal{B} \), which implies the needed inequalities \( \dim \mathbb{K}[a_{i_1}, \ldots, a_{i_p}] \leq \dim \mathcal{B} = n \).

Definition 5.1. We say that a system \( S := \{M_1, \ldots, M_p\} \) of matrices is polynomially independent if none of the matrices \( M_j \in S \) can be written as a linear combination of polynomials of the remaining elements of \( S \). That is \( M_j \) is not an element of the algebra \( \mathbb{K}[M_1, \ldots, M_{j-1}, M_{j+1}, \ldots, M_p] \) of polynomials of elements of \( S \setminus \{M_j\} \). The degree of polynomial freedom of the system \( S \) or of the algebra \( \mathbb{K}[M_1, \ldots, M_p] \) is the minimum number \( q \leq p \) of polynomially independatants \( M_{j_1}, \ldots, M_{j_q} \in S \), such that \( \mathbb{K}[M_1, \ldots, M_p] = \mathbb{K}[M_{j_1}, \ldots, M_{j_q}] \).

Remark 5.1. Although its proof has been made simple, thanks to tools from affine geometry, Theorem 5.1 is yet powerful in application, as can be seen in Examples 5.1, 5.2, 5.3. In particular, Theorem 5.1 combined with Theorem 6.1 give a receipt of a simple algorithm for constructing MASAs (resp. maximal Abelian nilpotent subalgebras (MANS)) of dimension \( n \) (resp. of dimension \( \leq n - 1 \)) of \( \mathfrak{gl}(n, \mathbb{K}) \) or \( \mathfrak{sl}(n, \mathbb{K}) \) with a given degree of polynomial freedom.

Remark 5.2. From Lemma 2.1, if \( M \) is a nonderogatory \( n \times n \) matrix, then the algebra \( \mathcal{B} := \mathbb{K}[M] \) of polynomials in \( M \) is an \( n \)-dimensional MASA of \( \mathfrak{gl}(n, \mathbb{K}) \). The 2-step solvable Lie algebra \( \mathcal{G}_M := \mathbb{K}[M] \times \mathbb{K}^n \) is a Frobenius Lie algebra (Theorem 3.1). Such Frobenius Lie algebras are studied in details and fully classified in
Section 5.1

Obviously, if $M$ and $N$ are $n \times n$ matrices such that $M = PNP^{-1}$, for some invertible matrix $P$ then $\mathbb{K}[M]$ and $\mathbb{K}[N]$ are conjugate and $G_M$ and $G_N$ are isomorphic via the linear map $\xi : G_N \to G_M$, $\xi(a) = PaP^{-1}$, $\xi(x) = Px$, for any $a \in \mathbb{K}[N]$, $x \in \mathbb{K}^n$.

5.1 Example 5.1

The commutative $\mathbb{R}$-subalgebra of $\mathcal{M}(3, \mathbb{R})$ spanned by the system of matrices $\{e_2 := E_{1,2}, e_3 := E_{1,3}\}$, coincides with the 3-dimensional vector space $\mathfrak{B}_{3,1}$ spanned by $\{e_1 := I_3, e_2, e_3\}$ which is also an Abelian Lie subalgebra of $\mathfrak{gl}(3, \mathbb{R})$. In the canonical basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ of $\mathbb{R}^3$, with dual basis $(\tilde{e}_1^*, \tilde{e}_2^*, \tilde{e}_3^*)$, the linear form $\tilde{e}_1^*$ satisfies $\tilde{e}_1^* \circ e_1 = \tilde{e}_1^* \circ e_2 = \tilde{e}_2^* \circ e_1 = \tilde{e}_3^* \circ e_3 = \tilde{e}_3^* \circ e_3$. Thus $\tilde{e}_1^*$ has an open orbit for the action $\mathfrak{B}_{3,1} \times (\mathbb{R}^3)^* \to (\mathbb{R}^3)^*$, $(a, f) \mapsto -f \circ a$. So the system $\{E_{1,2}, E_{1,3}\}$ has been completed into a system $\{e_1, e_2, e_3\}$ of linearly independent matrices satisfying the condition of Theorem 5.1. One sees that $\mathfrak{B}_{3,1}$ does not contain any nonderogatory matrix, as the characteristic polynomial $p_M(\lambda) = (x - \lambda)^3$ of any of its elements $M := xe_1 + ye_2 + ze_3$, is different from its minimal polynomial. Indeed, the equality $(x - M)^3 = 0$ implies that the minimal polynomial of $M$ divides $(x - \lambda)^3$. More precisely, $\mathfrak{B}_{3,1}$ is not the algebra of polynomials of any nonderogatory matrix. In the basis $(e_1, e_2, e_3, e_4 = e_1, e_5 = e_2, e_6 = e_3)$, the Lie bracket of the Lie algebra $G_{3,1} := \mathfrak{B}_{3,1} \ltimes \mathbb{R}^3$, reads $[e_1, e_{3+j}] = e_{3+j}$, $j = 1, \ldots, 3$, $[e_2, e_3] = e_4$, $[e_3, e_6] = e_4$. So $e_4$ is a Frobenius functional, as $\partial e_4^* = -e_4^* \wedge e_4^* - e_4^* \wedge e_5^* - e_4^* \wedge e_6^*$ is a symplectic form on $G_{3,1}$. One also notes that $G_{3,1}$ is a 1D extension of the Heisenberg Lie algebra $\mathcal{H}_5$ spanned by $e_2, e_3, e_4, e_5, e_6$ and $\mathcal{H}_5$ is hence its nilradical.

5.2 Example 5.2

Let $\mathfrak{B}_{4,2}$ be the commutative subalgebra of $\mathfrak{gl}(4, \mathbb{R})$ spanned by the $4 \times 4$ matrices $e_1 := I_4$, $e_2 := M_1 := E_{1,2} + E_{2,3}$, $e_3 := (M_1)^2 = E_{1,3}$, $e_4 := M_2 := E_{1,4}$. If we let $(\tilde{e}_1, \ldots, \tilde{e}_4)$ stand for the canonical basis of $\mathbb{R}^4$, with dual basis $(\tilde{e}_1^*, \ldots, \tilde{e}_4^*)$, we clearly see that $\tilde{e}_1^*$ satisfies $\tilde{e}_1^* \circ e_1 = \tilde{e}_1^* \circ e_2 = \tilde{e}_2^* \circ e_1 = \tilde{e}_3^* \circ e_3 = \tilde{e}_4^* \circ e_4$. Thus $\tilde{e}_1^*$ has an open orbit for the action $\mathfrak{B}_{4,2} \times (\mathbb{R}^4)^* \to (\mathbb{R}^4)^*$, $(a, f) \mapsto -f \circ a$. So the system $\{M_1, M_2\}$ has been completed into a system $\{e_1, e_2, e_3, e_4\}$ of linearly independent matrices satisfying the condition of Theorem 5.1. One has $M_1^3 = M_1 M_2 = M_2^2 = 0$. So any polynomials in elements of $\mathfrak{B}_{4,2}$ is a linear combination of $e_1, e_2, e_3, e_4$. Precisely the equality $\mathfrak{B}_{4,2} = \mathbb{R}[M_1, M_2]$ holds. Note that $\mathfrak{B}_{4,2}$ does not contain any nonderogatory matrix. Indeed the characteristic polynomial of any matrix $M := xe_1 + ye_2 + ze_3 + te_4$ is $p_M(\lambda) = (x - \lambda)^4$. But we also have $(x - M)^3 = 0$. Thus the minimal polynomial of $M$ divides $(x - \lambda)^3$ and is hence different from its characteristic polynomial. So $\mathfrak{B}_{4,2}$ is not the algebra of polynomials of any nonderogatory matrix. In the corresponding 2-step solvable Frobenius Lie algebra $\mathfrak{G} := \mathfrak{B}_{4,2} \ltimes \mathbb{R}^4$, with basis $(e_j, e_{4+j} = \tilde{e}_j, j = 1, 2, 3, 4)$, the Lie bracket reads $[e_1, e_{4+j}] = e_{4+j}$, $j = 1, \ldots, 4$, $[e_2, e_6] = e_5$, $[e_2, e_7] = e_6$, $[e_3, e_7] = e_5$, $[e_4, e_8] = e_5$ and $\partial e_5^* = -e_5^* \wedge e_5^* - e_5^* \wedge e_6^* - e_5^* \wedge e_5^* - e_5^* \wedge e_5^*$ is a symplectic form on $\mathfrak{G}$. So $e_5^*$ is a Frobenius functional.
5.3 Example 5.3

Examples 5.1 and 5.2 generalize to any dimension $n \geq 3$, into a family of $(n-1)$ Frobenius Lie algebras $G_{n,p} \coloneqq \mathfrak{B}_{n,p} \times \mathbb{R}^n$, $p = 1, \ldots, n-1$, all 2-step solvable. The $\mathfrak{B}_{n,p}$’s are pairwise non-isomorphic commutative algebras of polynomials in $n-1, n-2, \ldots, 2, 1$ matrices, respectively, satisfying the condition of Theorem 5.1. The space of all polynomials in elements of $\mathfrak{B}_{n,p}$ be the Abelian subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ spanned by the matrices $e_j$, $j = 1, \ldots, n$, where $e_j := M_j^{-1}$, $j = 1, \ldots, p+1$, and for $j = p+2, \ldots, n$, $e_j := E_{1,j}$. In the canonical basis $(\tilde{e}_1, \ldots, \tilde{e}_n)$ of $\mathbb{R}^n$, with dual basis $(\tilde{e}^1, \ldots, \tilde{e}^n)$, we have

$$M_{n,p} \tilde{e}_q = 0, \quad q \geq p + 2,$$

and for $1 \leq j \leq \min(k, q-1)$, we have

$$M_{n,p}^{k-1} \tilde{e}_{q-1} = M_{n,p}^{k-j} \tilde{e}_{j-1}, \quad 2 \leq q \leq p + 1.$$ 

So if $k \geq q$, then $M_{n,p}^k \tilde{e}_q = 0$, and if $k \leq q - 1 \leq p$, then $M_{n,p}^k \tilde{e}_q = \tilde{e}_{q-k}$. As one can see, we have $\tilde{e}_1 \circ e_j = \tilde{e}_j \circ M_j^{-1} = \tilde{e}_j$, $j = 1, \ldots, p+1$ and $\tilde{e}_1 \circ e_j = \tilde{e}_j \circ E_{1,j} = \tilde{e}_j$, for $j = p+2, \ldots, n$. This shows that $\tilde{e}_1^*$ has an open orbit for the contragredient action of $\mathfrak{B}_{n,p}$ on $(\mathbb{R}^n)^*$. So the set $\{M_{n,p}, E_{1,p+2}, \ldots, E_{1,n}\}$ of $n-p$ matrices, has been completed into a system $S = \{e_1, \ldots, e_n\}$ of $n$ linearly independent matrices satisfying the condition of Theorem 5.1. The space of all polynomials in elements of $S$ is thus equal to $\mathbb{R}[M_{n,p}, E_{1,p+2}, \ldots, E_{1,n}] = \mathfrak{B}_{n,p}$, so the degree of polynomial freedom of $S$ is $n - p$. Again, $\mathfrak{B}_{n,p}$ does not contain any nonderogatory matrix, as long as the degree of polynomial freedom is $n - p$ is greater or equal to 2. Thus, $\mathfrak{B}_{n,p}$ is not the space of polynomials of a nonderogatory matrix, if $p \neq n-1$. In the basis $(e_j, e_{n+j} = e_j, \ j = 1, \ldots, n)$ the Lie bracket of the 2-step solvable Frobenius Lie algebra $G_{n,p} := \mathfrak{B}_{n,p} \times \mathbb{R}^n$, is given by the following table

$$[e_1, e_{n+j}] = e_{n+j}, \quad j = 1, \ldots, n,$$

$$[e_p, e_{n+q}] = 0, \quad p \geq q + 1,$$

$$[e_q, e_{n+q}] = e_{n+1}, \quad q = p + 2, \ldots, n. \quad (26)$$

The form $e_{n+1}^*$ is a Frobenius functional. More precisely, $\partial e_{n+1}^* = -\sum_{i=1}^n e_j^* \wedge e_{n+j}$ is a symplectic form on $G_{n,p}$. Note that, for $n \geq 3$, in the family $(G_{n,p})_{1 \leq p \leq n-1}$, two Lie algebras $G_{n,p}$ and $G_{n,q}$ are isomorphic if and only if $p = q$. This is due to the fact that the degree of polynomial degree of $\mathfrak{B}_{n,p}$ is $n - p$ which is obviously different from $n - q$ if $p \neq q$, so $\mathfrak{B}_{n,p}$ and $\mathfrak{B}_{n,q}$ cannot be conjugate, unless $p = q$. One may also argue that, although each $G_{n,p}$ has a codimension 1 nilradical $N_{n,p}$ spanned by $(e_2, \ldots, e_{2n})$, the derived ideal $[N_{n,p}, N_{n,p}]$ has dimension $p$ and is spanned by $(e_{n+1}, \ldots, e_{n+p})$. So whenever $p \neq q$, the nilradicals $N_{n,p}$ and $N_{n,q}$ are not isomorphic and hence, neither are the Lie algebras $G_{n,p}$ and $G_{n,q}$. The family $(G_{n,p})_{1 \leq p \leq n-1}$, has two special cases. (1) The case $p = n - 1$ is the only one where the degree of polynomial freedom is 1, so $\mathfrak{B}_{n,n-1}$ is the algebra $\mathbb{R}[M_{n,n-1}]$ of polynomials of the nonderogatory matrix $M_{n,n-1}$. The 2-step-solvable Lie algebras given by nonderogatory matrices are discussed in Section 7 where a full classification is supplied, among other results.
In order to keep the same notations as in Section 7, we let \( D_p \) stand for \( G_{n,n-1} \).

Note that, although we have considered the case \( p = n - 1 \) when \( n = 2 \), we get the \( 2 \times 2 \) nonderogatory real matrix \( M_{2,1} := E_{1,2} \).

The Lie bracket of the Lie algebra \( G_{2,1} := \mathbb{R}[M_{2,1}] \ltimes \mathbb{R}^2 \) is expressed in the basis \( (e_1 := 1_{\mathbb{R}^2}, e_2 := M_{2,1}, e_3 := \tilde{e}_1, e_4 := \tilde{e}_2) \) as follows, \( [e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_4] = e_3 \). The linear form \( \psi \) is a Frobenius functional and \( \partial e_3 = -e_1 \wedge e_4 - e_2 \wedge e_4 \).

We also set \( D_0^2 := G_{2,1} \). (2) As regards the case \( p = 1 \), the algebra \( B_{n,1} \) has \( n - 1 \) degrees of polynomial freedom. The nilradical \( N_{n,1} \) of \( G_{n,1} \) is the \( (2n-1) \)-dimensional Heisenberg Lie algebra \( H_{2n-1} \). It is spanned by \( (e_j := E_{1,j}, j = 2, \ldots, n, e_{n+k}, k = 1, \ldots, n) \), with Lie brackets \( [e_j, e_{n+j}] = e_{n+1}, j = 2, \ldots, n \). Note also that the \( n - 1 \) spaces \( L_{n,p} = \text{span}(e_2, e_3, \ldots, e_n) \) are all \((n - 1)\)-dimensional Abelian subalgebras of \( \mathfrak{s}(n, \mathbb{R}) \), which according to Theorem 6.1, are pairwise non-conjugate MASAs of \( \mathfrak{s}(n, \mathbb{R}) \).

6 On the classification of 2-step solvable Frobenius Lie algebras

6.1 Classification of \( n \)-dimensional MASAs and Frobenius Lie algebras

The following Lemma 6.1 is important in many aspects. It deepens the link between 2-step solvable Lie algebras and MASAs. In general, it is not obvious to tell if two MASAs are conjugate or not. Invariants (cohomologies, derivations, subalgebras, series of ideals, ...) of the corresponding 2-step solvable Lie algebras may help to tell them apart in a simple way, as we do e.g. in Example 5.3. In physics, MASAs are often studied in relation with symmetries of dynamical systems, in this regard Lemma 6.1 suggests that invariants of 2-step solvable Lie algebras may encode informations on such dynamical systems and their governing equations.

**Lemma 6.1.** Let \( G_1 := \mathfrak{B}_1 \ltimes \mathbb{K}^n \) and \( G_2 := \mathfrak{B}_2 \ltimes \mathbb{K}^n \) be 2-step solvable Lie algebras over \( \mathbb{K} \), where \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) are two Abelian subalgebras of \( M(n, \mathbb{K}) \). Then \( G_1 \) and \( G_2 \) are isomorphic if and only if \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) are conjugate. That is, if and only if there exists a linear invertible map \( \phi : \mathbb{K}^n \to \mathbb{K}^n \) such that \( \mathfrak{B}_2 = \phi \mathfrak{B}_1 \phi^{-1} \), or equivalently, every element of \( \mathfrak{B}_2 \) is of the form \( \phi \circ a \circ \phi^{-1} \) for some \( a \in \mathfrak{B}_1 \). More precisely, a linear map \( \psi : G_1 \to G_2 \) is a Lie algebra isomorphism if and only if there exist \( x_0 \in \mathbb{K}^n \) and some invertible linear map \( \phi : \mathbb{K}^n \to \mathbb{K}^n \) such that \( \mathfrak{B}_2 = \phi \mathfrak{B}_1 \phi^{-1} \) and for any \( (a, x) \in \mathfrak{B}_1 \ltimes \mathbb{K}^n \)

\[
\psi(a + x) = \phi \circ a \circ \phi^{-1} + \phi \circ a \circ \phi^{-1}(x_0) + \phi(x). \tag{27}
\]

**Proof.** Suppose \( G_1 \) and \( G_2 \) are isomorphic under some Lie algebra isomorphism \( \psi : G_1 \to G_2 \). As the derived ideal \( [G_1, G_1] = \mathbb{K}^n \) must be mapped to \( [G_2, G_2] = \mathbb{K}^n \), the components of \( \psi \) are linear maps \( \psi_{1,1} : \mathfrak{B}_1 \to \mathfrak{B}_2, \psi_{1,2} : \mathfrak{B}_1 \to \mathbb{K}^n, \phi : \mathbb{K}^n \to \mathbb{K}^n \), with \( \psi_{1,1} \) and \( \phi \) invertible, such that \( \psi(a) = \psi_{1,1}(a) + \psi_{1,2}(a) \) and \( \psi(x) = \phi(x) \) for any \( a \in \mathfrak{B}_1 \) and \( x \in \mathbb{K}^n \). We deduce the equality \( \psi_{1,1}(a) = \phi \circ a \circ \phi^{-1} \), from the identity \( \phi(ax) = \psi([a, x]) = \psi_{1,1}(a) + \psi_{1,2}(a) + \phi(x) = (\psi_{1,1}(a) + \phi)(x) \). In particular \( \phi \circ a \circ \phi^{-1} \in \mathfrak{B}_2 \), for any \( a \in \mathfrak{B}_1 \), which is equivalent also to the equality

13
Taking \( b = e_1 := 1_{K^n} \in \mathcal{B}_1 \) and \( x_0 := \psi_1(2e_1) \) in the identity 0 = \( [\psi(\alpha), \psi(\beta)] = \psi([-\alpha, \beta]) \). Suppose that, for the canonical action \( \rho \) of matrices is an invariant of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are conjugate, or equivalently, if the Lie algebras \( \mathcal{B}_1 \times K^n \) and \( \mathcal{B}_2 \times K^n \) are isomorphic, then \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) have the same index of polynomial freedom.

**Proof.** Suppose the Lie algebras \( \mathcal{B}_1 \times K^n \) and \( \mathcal{B}_2 \times K^n \) are isomorphic. Using Lemma \( \ref{prop} \) let \( \phi : K^n \rightarrow K^n \) be a linear invertible map such that \( \mathcal{B}_2 = \phi \mathcal{B}_1 \phi^{-1} \). A system \( (o_1, \ldots, o_p) \) of elements of \( \mathcal{B}_1 \) is polynomially independent if and only if \( (\phi a_1 \phi^{-1}, \ldots, \phi a_p \phi^{-1}) \) is polynomially independent in \( \mathcal{B}_2 \).

We have the following:

**Theorem 6.1.** Let \( \mathcal{B} \) be an \( n \)-dimensional Abelian Lie subalgebra of \( \mathfrak{gl}(n, K) \). Write \( \mathcal{B} \) as \( \mathcal{B} = I_{K^n} \oplus L \), where \( L \) is an \((n-1)\)-dimensional Abelian subalgebra of \( \mathfrak{sl}(n, K) \). Suppose that, for the canonical action \( \rho \) of \( \mathcal{B} \) on \( K^n \) given by the ordinary multiplication \( a \in \mathcal{B} \) and vectors \( x \in K^n \), the corresponding contragredient action \( \rho^* : \mathcal{B} \times (K^n)^* \rightarrow (K^n)^*, \) \( (a, f) \mapsto \rho^*(a)f := -f \circ a \), has an open orbit. Then \( \mathcal{B} \) (resp. \( L \)) is a maximal Abelian subalgebra (MASA) of \( \mathfrak{gl}(n, K) \) (resp. of \( \mathfrak{sl}(n, K) \)).

**Proof.** Suppose \( \rho^* : \mathcal{B} \times (K^n)^* \rightarrow (K^n)^* \), \( (a, f) \mapsto \rho^*(a)f := -f \circ a \), has an open orbit and consider some \( \alpha \in (K^n)^* \) with an open orbit. Thus, any basis \( (a_1, \ldots, a_n) \) of \( \mathcal{B} \) gives rise to a basis \( (\rho^*(a_1)\alpha, \ldots, \rho^*(a_n)\alpha) \) of \((K^n)^*\) and the (linear) orbital map \( Q : \mathcal{B} \rightarrow (K^n)^* \), \( Q(a) = \rho^*(a)\alpha \) is an isomorphism between the vector spaces \( \mathcal{B} \) and \((K^n)^*\). Just for convenience, we will let \( (\hat{e}_1, \ldots, \hat{e}_n) \) stand for the basis of \( K^n \) whose dual basis is \( (\hat{e}_1^*, \ldots, \hat{e}_n^*) \). Suppose \( \hat{a} \in \mathfrak{gl}(n, K) \) is such that \( [\hat{a}, \hat{a}] = 0 \) for any \( a \in \mathcal{B} \) and \( \hat{a} \neq 0 \). Assume \( \hat{a} \) is not an element of \( \mathcal{B} \), then \( \mathfrak{B} := K\hat{a} \oplus \mathcal{B} \) is an \((n+1)\)-dimensional Abelian subalgebra of \( \mathfrak{gl}(n, K) \). Because \( \dim \mathfrak{B} = \dim(K^n)^* + 1 \), the orbital map \( \hat{Q} : \mathfrak{B} \rightarrow (K^n)^* \), also given by \( \hat{Q}(a) = \hat{\rho}^*(a)\alpha = -\alpha \circ a, \) must have a 1-dimensional kernel. So, there exists some \( \hat{b} = k\hat{a} + a_0 \neq 0 \), with \( k \in K \) and \( a_0 \in \mathcal{B} \) such that \( \hat{\rho}^*(\hat{b})\alpha = -\alpha \circ \hat{b} = 0 \). Since \( \hat{\rho}^*(a_0)\alpha = \rho^*(a_0)\alpha = Q(a_0) \neq 0 \) if \( a_0 \neq 0 \), we must have \( k = 0 \). Then we must have \( \hat{b}x = 0 \), for any \( x \in K^n \), or equivalently \( \hat{b} = 0 \). Indeed, for any \( x \in K^n \), the expression of \( \hat{b}x \) in the above basis is \( \hat{b}x = \sum_{j=1}^n \langle \hat{e}_j^*, \hat{b}x \rangle \hat{e}_j \). But the components \( \langle \hat{e}_j^*, \hat{b}x \rangle \) are all equal to zero for any \( j = 1, \ldots, n, \) as

\[ \langle \hat{e}_j^*, \hat{b}x \rangle = \langle \hat{\rho}^*(a_j)\alpha, \hat{b}x \rangle = -\langle \alpha, a_j \hat{b}x \rangle = -\langle \alpha, \hat{b}a_jx \rangle = \langle \hat{\rho}^*(\hat{b})\alpha, a_jx \rangle = 0. \]

Now, the equality \( \hat{b} = k\hat{a} + a_0 = 0 \), contradicts the assumption that \( \hat{a} \) is not in \( \mathcal{B} \). So \( \hat{a} \) must necessary be in \( \mathcal{B} \). This proves that \( \mathcal{B} \) is a MASA of \( \mathfrak{gl}(n, K) \). The last claim automatically follows, due to the fact that a subalgebra \( L \) of \( \mathfrak{sl}(n, K) \) is a MASA of \( \mathfrak{sl}(n, K) \), if and only if \( I_{K^n} \oplus L \) is a MASA of \( \mathfrak{gl}(n, K) \).
6.1.1 Example 6.1.1

On the space $\mathfrak{B}_{n,n} := L_{n,n} \oplus \mathbb{R} \mathbb{R}^n$, where $L_{n,n}$ is the space of matrices $L_{n,n} = \{ M := m_1 E_{1,1} + \cdots + m_p E_{p,p} : m_j \in \mathbb{R}, j = 2, \ldots, n \}$, we set $e_1 := I_{\mathbb{R}^n}$ and $e_j := E_{1,j} + E_{j,1}$, $j = 2, \ldots, n - 1$. Non-zero Lie bracket of $\mathfrak{B}_{n,n}$ is given by

$$[M, A] = \sum_{j=2}^{n-1} m_j a_{j+2} [E_{1,j} + E_{j,1}, E_{1,p} + E_{p,1}] = \sum_{j=2}^{n-1} m_j a_{j+2} (\delta_{j,p} - \delta_{j,p}) E_{1,1} = 0.$$ 

The characteristic polynomial of any $M = m_1 e_1 + \cdots + m_n e_n$ is $ch(M) = (m_1 - X)^n$ and $(m_1 - M)^2 = (m_2^2 + \cdots + m_{n-1}^2) E_{1,1} + (m_1 - M)^3 = 0$. So, up to a scaling, the minimal polynomial of $M$ is $(m_1 - X)^3$. Thus for any $n \geq 4$, the algebra $\mathfrak{B}_{n,n}$ contains no nonderogatory matrix. Moreover, $\mathfrak{B}_{n,n}$ is not the algebra of polynomials in any nonderogatory matrix, for any $n \geq 4$. We have $e_i \circ e_j = e_j^*$, $j = 1, \ldots, n$. So $e_i^*$ has an open orbit. Hence $\mathfrak{B}_{n,n}$ satisfies the condition of Theorem 5.1 and $\mathfrak{B}_{n,n}$ and $L_{n,n}$ are MASAs of $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$, respectively, according to Theorem 6.1. In the basis $(e_j, e_{n+j} := \tilde{e}_j), j = 1, \ldots, n$, of the 2-step solvable Lie algebra $\mathfrak{G}_{n,n} := \mathfrak{B}_{n,n} \oplus \mathbb{R}^n$, with dual basis $(e^*_1, \ldots, e^*_n)$, the non-zero Lie bracket of $\mathfrak{G}_{n,n}$ are

$$[e_1, e_{n+j}] = e_{n+j}, \quad j = 1, \ldots, n,$$

$$[e_j, e_{n+j}] = e_{n+j}, \quad j = 2, \ldots, n - 1, \quad [e_n, e_{2n}] = e_{n+1},$$

so we have $\partial e^*_{n+1} = - \sum_{j=1}^{n} e^*_{j} \wedge e^*_{n+j}$. In other words, $\mathfrak{G}_{n,n}$ is a Frobenius Lie algebra and $e^*_{n+1}$ is a Frobenius functional. The nilradical $\mathcal{N}_{n,n} := \text{span}(e_2, \ldots, e_{2n})$ of $\mathfrak{G}_{n,n}$ is of codimension 1 and is the semi-direct sum $\mathcal{N}_{n,n} = \mathbb{R} e_{2n} \ltimes (\mathcal{H}_{2n-3} \oplus \mathbb{R} e_n)$ of the line $\mathbb{R} e_{2n}$ and the Abbena Lie algebra $\mathcal{H}_{2n-3} \oplus \mathbb{R} e_n$ where the former acts on the latter by nilpotent derivations. Recall that the $(2n - 2)$-dimensional Abbena Lie algebra $\mathcal{H}_{2n-3} \oplus \mathbb{R} e_n$ is the direct sum of the line $\mathbb{R} e_n$ and the $(2n - 3)$-dimensional Heisenberg Lie algebra $\mathcal{H}_{2n-3} = \text{span}(e_2, \ldots, e_{n-1}, e_{n+1}, \ldots, e_{2n-1})$. The derived ideal $[\mathcal{N}_{n,n}, \mathcal{N}_{n,n}]$ is $(n - 1)$-dimensional and spanned by $(e_{n+1}, \ldots, e_{2n-1})$. So $\mathcal{N}_{n,n}$ is isomorphic to none of the nilradicals $\mathcal{N}_{n,p}$, $1 \leq p \leq n - 2$ of Example 5.3, as $[\mathcal{N}_{n,p}, \mathcal{N}_{n,p}]$ is $p$-dimensional. Recall that for $p = n - 1$, the Lie algebra $\mathcal{G}_{n,n-1}$ is obtained from a nonderogatory matrix. Hence, altogether, $\mathcal{G}_{n,n}$ is not isomorphic to any of the $n - 1$ pairwise non-isomorphic Lie algebras $\mathcal{G}_{n,p}$ of Example 5.3. Thus from Theorem 6.1 none of the MASAs $\mathfrak{B}_{n,p}$ of Example 5.3 is conjugate to $\mathfrak{B}_{n,n}$.
6.1.2 Example 6.1.2

Consider the following space $L'_{n,n}$ of $n \times n$ matrices

$$L'_{n,n} = \{ M := \sum_{j=2}^{n} m_j (E_{1,j} + E_{n-j+1,n}) , \ m_j \in \mathbb{R}, j = 2, \ldots, n \}. \quad (29)$$

On the Abelian algebra $\mathfrak{B}'_{n,n} := \mathbb{R}^{\otimes n} \oplus L'_{n,n}$, we set $e_1 := \mathbb{I}^{n}, e_j := E_{1,j} + E_{n-j+1,n}, j = 2, \ldots, n$. The Lie algebra $L'_{n,n}$ is Abelian. Indeed, the commutator of any elements $M = m_1 e_1 + \cdots + m_n e_n$ and $A = a_1 e_1 + \cdots + a_n e_n$ of $L'_{n,n}$ is

$$[M, A] = \sum_{j=2, p=2}^{n} m_j a_p [E_{1,j} + E_{n-j+1,n}, E_{1,p} + E_{n-p+1,n}]$$

$$= \sum_{j=2, p=2}^{n} m_j a_p (\delta_{j,n-p+1} - \delta_{p,n-j+1}) E_{1,n} = 0.$$ 

The characteristic polynomial of any element $M = m_1 e_1 + \cdots + m_n e_n$ of $\mathfrak{B}$ is $\chi(X) = (m_1 - X)^n$ and $(m_1 - M)^3 = 0$. So, up to a scaling, the minimal polynomial of $M$ is $(m_1 - X)^3$. So for any $n \geq 4$, the algebra $\mathfrak{B}'_{n,n}$ contains no nonderogatory matrix. More precisely, $\mathfrak{B}'_{n,n}$ is not the algebra of polynomials in any nonderogatory matrix, for any $n \geq 4$. We have $\tilde{e}_1^* \circ e_j = \tilde{e}_j^*, j = 1, \ldots, n - 1, \tilde{e}_1^* \circ e_n = 2 \tilde{e}_n^*$. So $\tilde{e}_1$ has an open orbit. Hence $\mathfrak{B}'_{n,n}$ and $L'_{n,n}$ are MASAs of $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$, respectively, according to Theorem 6.1.

So in the dual basis $(e_1^*, \ldots, e_{2n})$ of the basis $(e_j, e_{n+j} := \tilde{e}_j), j = 1, \ldots, n$, of the 2-step solvable Lie algebra $\mathcal{G}'_{n,n} := \mathfrak{B}'_{n,n} \ltimes \mathbb{R}^n$, the non-zero Lie bracket of $\mathcal{G}'_{n,n}$ are

$$[e_1, e_{n+j}] = e_{n+j}, \ j = 1, \ldots, n,$$

$$[e_j, e_{n+j}] = e_{n+1}, \ [e_j, e_{2n}] = e_{2n-j+1}, \ j = 2, \ldots, n - 1, \ [e_n, e_{2n}] = 2e_{n+1},$$

(30)

so we have $\partial e_{n+1}^* = - \sum_{j=1}^{n-1} e_j^* \wedge e_{n+j}^* - 2 e_n^* \wedge e_{2n}^*$. In other words, $\mathcal{G}'_{n,n}$ is a Frobenius Lie algebra and $e_{n+1}^*$ is a Frobenius functional. As in Example 6.1.1, the Lie algebra $\mathcal{G}'_{n,n}$ also has a codimension 1 nilradical $\mathcal{N}'_{n,n}$ spanned by $e_2, \ldots, e_{2n}$, which is the semi-direct sum $\mathcal{N}'_{n,n} = \mathbb{R} e_{2n} \ltimes \left( \mathcal{H}_{2n-3} \oplus \mathbb{R} e_n \right)$ of $\mathbb{R} e_{2n}$ and $\mathcal{H}_{2n-3} \oplus \mathbb{R} e_n$ where the former acts on the latter by nilpotent derivations. So here again, $\mathcal{G}'_{n,n}$ is not isomorphic to any of the $n - 1$ pairwise non-isomorphic Lie algebras $\mathcal{G}_{n,p}$ of Example 5.3 and, from Theorem 6.1, none of the MASAs $\mathfrak{B}_{n,p}$ of Example 5.3 is conjugate to $\mathfrak{B}'_{n,n}$. We can check that $\mathcal{N}'_{n,n}$ is not isomorphic to the nilradical $\mathcal{N}_{n,n}$ in Example 6.1.1 so $\mathcal{G}'_{n,n}$ and $\mathcal{G}_{n,n}$ are not isomorphic, $\mathfrak{B}'_{n,n}$ and $\mathfrak{B}_{n,n}$ are not conjugate.

Remark 6.1. From Theorem 6.1 and Lemma 6.7, the classification of 2-step solvable Frobenius Lie algebras is equivalent to that of $n$-dimensional MASAs of $\mathfrak{gl}(n, \mathbb{K})$ acting on $(\mathbb{K}^n)^*$ with an open orbit. However, not all $n$-dimensional MASAs have open orbit on $(\mathbb{K}^n)^*$. Indeed, for $n \geq 3$, the algebra $\mathfrak{B}_n := \mathbb{R}^{\otimes n} \oplus L_n$, is a MASA of $\mathfrak{gl}(n, \mathbb{R})$, where $L_n := \left\{ \sum_{i=1}^{n-1} k_i E_i, \ k_i \in \mathbb{R}, i = 1, \ldots, n-1 \right\}$ is a
MASA of \(\mathfrak{sl}(n, \mathbb{R})\). To see that, consider \(b = \sum_{p,q=1}^{n} t_{p,q}E_{p,q} \in \mathfrak{gl}(n, \mathbb{R})\), with \(t_{p,q} \in \mathbb{R}\), \(p,q = 1, \ldots, n\). We have \([E_{i,n}, b] = \sum_{l=1}^{n-1} t_{n,l}E_{i,l} + (t_{n,n} - t_{i,i})E_{i,n} - \sum_{1 \leq k \leq n, k \neq i} t_{k,i}E_{k,n}\), for any \(i = 1, \ldots, n - 1\). So the relation \([b, a] = 0, \forall a \in \mathfrak{B}_n\), is equivalent to the following, valid for any \(i\) with \(1 \leq i \leq n - 1\): \(t_{i,i} = t_{n,n}\) and for any \(k\) with \(1 \leq k \leq n\) and \(k \neq i\), one has \(t_{k,i} = 0\). This simply means that for any \(1 \leq i \leq n - 1\), the coefficients of the \(i\)-th column of \(b\) are all equal to zero except the \((i,i)\) entry which is equal to \(t_{n,n}\). In other words \(b = t_{n,n}I_n + t_{1,n}E_{1,n} + \cdots + t_{n-1,n}E_{n-1,n}\), or equivalently, \(b\) is an element of \(\mathfrak{B}_n\). This simply means that \(\mathfrak{B}_n\) is a MASA of \(\mathfrak{gl}(n, \mathbb{R})\). The orbit \(\{\alpha \circ a, a \in L_n\}\) of any \(\alpha \in (\mathbb{R}^n)^*\) is at most 2-dimensional and spanned by \(\alpha\) and \(\partial_n\). More precisely, let \((\partial_1, \ldots, \partial_n)\) be the canonical basis of \(\mathbb{R}^n\) and let \(\alpha = s_1\partial_1^* + \cdots + s_n\partial_n^* \in (\mathbb{R}^n)^*\), where \(s_1, \ldots, s_n \in \mathbb{R}\), then for any \(k_1, k_1, \ldots, k_{n-1} \in \mathbb{R}\) and \(a = k_1\partial_1 + k_{1,n}E_{1,n} + \cdots + k_{n-1,n}E_{n-1,n} \in \mathfrak{B}_n\), one has \(\partial_i \circ a = k_i\partial_i + k_{i,n}\partial_n\), \(i = 1, \ldots, n - 1\) and \(\partial_n^* \circ a = k_n\partial_n\), so that
\[
\alpha \circ a = k_1\alpha + (k_{1,n}s_1 + k_{2,n}s_2 + \cdots + k_{n-1,n}s_{n-1})\partial_n^*.
\]
For \(n = 3\), \(L_3\) coincides with \(L_{2,4}\) in the list of MASA of \(\mathfrak{sl}(n, \mathbb{R})\) supplied in [30].

6.2 Cartan subalgebras of \(\mathfrak{sl}(n, \mathbb{R})\)

Recall that a Cartan subalgebra \(\mathfrak{h}\) of a Lie algebra \(\mathfrak{g}\) is a nilpotent subalgebra which is equal to its own normalizer \(\mathcal{N}_\mathfrak{g}(\mathfrak{h}) := \{x \in \mathfrak{g}, [x, y] \in \mathfrak{h}, \forall y \in \mathfrak{h}\}\) in \(\mathfrak{g}\). In other words, a Cartan subalgebra of \(\mathfrak{g}\) is a subalgebra \(\mathfrak{h}\) which is nilpotent and satisfies the condition that if \(x \in \mathfrak{g}\) is such that \([x, \mathfrak{h}] \subset \mathfrak{h}\), then \(x\) must belong to \(\mathfrak{h}\). Cartan subalgebras of semi-simple Lie algebras must necessarily be Abelian, more precisely they are MASAs which contain only semisimple elements. Cartan subalgebras of simple or semi-simple Lie algebras have been extensively studied by several authors amongst which E. Cartan, Harish-Chandra [10], B. Kostant [21], M. Sugiura [31], etc. It is natural that Theorem 6.1 brings them into play in the study of 2-step solvable Frobenius Lie algebras. Recall that a Cartan subalgebra \(\mathfrak{h}\) of a semisimple Lie algebra \(\mathfrak{g}\), splits into a direct sum \(\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^−\) of two subalgebras \(\mathfrak{h}^+\) and \(\mathfrak{h}^−\), respectively called its toroidal and its vector parts, such that \(\mathfrak{h}^+\) is only made of elements of \(\mathfrak{h}\) whose adjoint operator (as a linear operator of \(\mathfrak{g}\)) only has purely imaginary eigenvalues and adjoint operators of elements of \(\mathfrak{h}^−\) have only real eigenvalues. See e.g. [31]. We propose the following characterization of Cartan subalgebras of \(\mathfrak{sl}(n, \mathbb{R})\).

**Theorem 6.2.** Let \(\mathfrak{h}\) be an Abelian subalgebra of \(\mathfrak{sl}(n, \mathbb{R})\) and set \(\mathfrak{B}_\mathfrak{h} := \mathbb{R}I_{\mathbb{R}^n} \oplus \mathfrak{h}\). The following are equivalent,

(a) \(\mathfrak{h}\) is a Cartan subalgebra of \(\mathfrak{sl}(n, \mathbb{R})\),
(b) \(\mathfrak{B}_\mathfrak{h}\) is the algebra \(\mathbb{R}[M]\) of polynomials of some nonderogatory \(n \times n\) real matrix \(M\) with \(n\) distinct (real or complex) eigenvalues,
(c) the Lie algebra \(\mathfrak{B}_\mathfrak{h} \times \mathbb{R}^n\) is the direct sum \(\mathfrak{aff}(\mathbb{C}) \oplus \cdots \mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \cdots \mathfrak{aff}(\mathbb{R})\) of \(p\) copies of the Lie algebra \(\mathfrak{aff}(\mathbb{C})\) and \(q\) copies of \(\mathfrak{aff}(\mathbb{R})\), where \(p\) and \(q\) are the respective dimensions of the toroidal and the vector parts of \(\mathfrak{h}\).

The proof of Theorem 6.2 is given in Section 7.6. A complementary characterization is given in Theorem 7.2 where the part (c) is also restated and proved.
6.3 Lagrangian subalgebras of the Heisenberg Lie algebra as MASAs of $\mathfrak{sl}(n, \mathbb{R})$

The MASAs $L_{n,n}$, $L'_{n,n}$ and $L_n$ of $\mathfrak{sl}(n, \mathbb{R})$ in Example 6.1.1 and Remark 6.1.1 are subspaces of the space of $n \times n$ strictly upper triangular matrices spanned by the $n \times n$ matrices $E_{1,j}$, $E_{j,n}$, $j = 2, \ldots, n-1$ and $E_{1,n}$. That is, the space of $n \times n$ strictly upper triangular real matrices whose entries are all equal to zero everywhere, except on the first row and on the last column, the diagonal being also entirely made of zeros. Note that, such a space together with the commutator $[M, N] := MN - NM$ of matrices, is a subalgebra of $\mathfrak{sl}(n, \mathbb{R})$, which is isomorphic to the Heisenberg Lie algebra $\mathcal{H}_{2n-3}$ of dimension $2n - 3$. Indeed, this is the most frequently used representation of $\mathcal{H}_{2n-3}$ by $n \times n$ matrices. Recall that, the Heisenberg Lie algebra is also the central extension $\mathbb{R}^{2n-4} \rtimes \mathbb{R}^{2n-3}$ of the Abelian Lie algebra $\mathbb{R}^{2n-4}$ along its standard symplectic form $\omega_0(x, y) = \sum_{j=2}^{n-1} (x_j y_{n+j} - x_{n+j} y_j)$.

The Lie bracket being $[x + s \hat{e}_{2n-3}, y + t \hat{e}_{2n-3}] = \omega_0(x, y)\hat{e}_{2n-3}$ and the above representation by matrices is obtained by letting the canonical basis $(\hat{e}_2, \ldots, \hat{e}_{2n-4}, \hat{e}_{2n-3})$ of $\mathbb{R}^{2n-3}$ undergo the identification $\hat{e}_j \mapsto E_{1,j}, \hat{e}_{n+j} \mapsto E_{j,n}$, $j = 2, \ldots, n-1$ and $\hat{e}_{2n-3} \mapsto E_{1,n}$. The 2-form $\omega_0'$ on $\mathcal{H}_{2n-3}$ such that $\omega_0'(E_{1,j}, E_{1,k}) = \omega_0(\hat{e}_j, \hat{e}_k)$, $\omega_0'(E_{1,j}, E_{k,n}) = \omega_0(\hat{e}_j, \hat{e}_{n+k})$, $\omega_0'(E_{j,n}, E_{k,n}) = \omega_0(\hat{e}_{n+j}, \hat{e}_{n+k})$, $\omega_0'(M, E_{1,n}) = 0$ for any $M \in \mathcal{H}_{2n-3}$, is again denoted by $\omega_0$. The Lie bracket of two matrices $M = \sum_{j=2}^{n-1} (m_j E_{1,j} + m_{n+j} E_{j,n}) + m_{2n-3} E_{1,n}$ and $N = \sum_{j=2}^{n-1} (n_j E_{1,j} + n_{n+j} E_{j,n}) + n_{2n-3} E_{1,n}$ is expressed as

$$[M, N] = \sum_{j=2}^{n-1} \sum_{l=2}^{n-1} (m_j m_{n+l} [E_{1,j}, E_{1,l}] + m_{n+j} n_l [E_{j,n}, E_{1,l}])$$

$$\quad = \sum_{j=2}^{n-1} (m_j m_{n+j} - m_{n+j} n_j) E_{1,n} = \omega_0(M, N) E_{1,n}. \quad (31)$$

So, a subspace of $\mathcal{H}_{2n-3}$ is Abelian if and only if it is totally isotropic with respect to $\omega_0$. Thus, a subspace of $\mathfrak{sl}(n, \mathbb{R})$ which is included in $\mathcal{H}_{2n-3}$ as above, is a MASA of $\mathfrak{sl}(n, \mathbb{R})$, if and only if it is a Lagrangian subspace of $\mathcal{H}_{2n-3}$, that is, a subspace $\mathcal{H}$ of dimension $n - 1$ such that $\omega_0(M, N) = 0$, for any $M, N \in \mathcal{H}$. Thus the classification of the MASAs of $\mathfrak{sl}(n, \mathbb{R})$ which are also subspaces of $\mathcal{H}_{2n-3}$, is equivalent to the classification of the Lagrangian subspaces of $\mathcal{H}_{2n-3}$, up to conjugaison.

7 Classification of 2-step solvable Frobenius Lie algebras given by nonderogatory matrices

From Theorems 3.1 and 6.1 every 2-step solvable Frobenius Lie algebra is a semidirect sum $\mathcal{G} = \mathcal{B} \rtimes \mathbb{R}^n$ of two Abelian Lagrangian subalgebras $\mathcal{B}$ and $\mathbb{R}^n$, where $\mathbb{R}^n$ is the derived ideal $\mathbb{R}^n = [\mathcal{G}, \mathcal{G}]$ and $\mathcal{A}$ is a maximal Abelian subalgebra of the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of $n \times n$ matrices, that acts on $\mathbb{R}^n$ such that, the corresponding contragredient action has an open orbit. Moreover, $\mathcal{A} \rtimes \mathbb{R}^n$ and $\mathcal{B} \rtimes \mathbb{R}^n$
are isomorphic if and only if \( A \) and \( B \) are conjugate. That is, \( B = \phi A \phi^{-1} \), for some invertible linear map \( \phi : \mathbb{R}^n \to \mathbb{R}^n \). In this section, we discuss an important family of 2-step solvable Frobenius Lie algebras, those given by nonderogatory matrices.

As in Remark 5.2, when \( \mathfrak{B} \) is the algebra \( \mathbb{K}[M] \) of all the polynomials in a nonderogatory matrix \( M \), we will simply write \( \mathcal{G}_M = \mathbb{K}[M] \ltimes \mathbb{R}^n \) instead of \( \mathfrak{B} \ltimes \mathbb{R}^n \). The same holds when we use the setting of nonderogatory linear maps.

### 7.1 Some key examples

#### 7.1.1 The Lie algebra \( \mathfrak{D}_0^1 := \aff(\mathbb{R}) \)

Consider the simplest nonderogatory linear map \( \psi : \mathbb{R} \to \mathbb{R}, x \mapsto x \). The associated 2-step solvable Lie algebra is the Lie algebra \( \mathcal{G}_\psi = \aff(\mathbb{R}) \) of the group of affine motions of the real line \( \mathbb{R} \). It has a basis \( (e_1, e_2) \) in which the Lie bracket reads \([e_1, e_2] = e_2 \) and \( \partial e_2 = \partial e_2 = 0 \). The characteristic and the minimal polynomials of \( \psi \) are both equal to 1.\(^2\) So \( M_{-1} \) has the 2 complex eigenvalues \( i, -i \). Of course, \( \mathbb{R}[M_{-1}] = \mathbb{R}[\mathbb{R}^2] \oplus \mathbb{R}M_{-1} \) is a MASA of \( \mathfrak{gl}(2, \mathbb{R}) \) and \( \hat{e}_1 \circ (M_0)^0 = \hat{e}_1, \hat{e}_1 \circ (M_{-1}) = -\hat{e}_2 \). In the basis \( (e_1, \cdots, e_4) \) of the 2-step solvable Lie algebra \( \mathcal{G}_{M_{-1}} \), with \( e_1 = (M_{-1})^0, e_2 = M_{-1}, e_3 := \hat{e}_1, e_4 := \hat{e}_2 \), the Lie bracket reads \([e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, [e_2, e_4] = -e_3 \). The exact form \( \partial \hat{e}_3 = -e_1 \wedge e_3^* + e_2 \wedge e_4^* \) is a symplectic form on \( \mathcal{G}_{M_{-1}} \). Note that \( \mathcal{G}_{M_{-1}} \) is the Lie algebra \( \aff(\mathbb{C}) = \{ (z_1, z_2), z_2 = x_j + iy_j, x_j, y_j \in \mathbb{R}, j = 1, 2, \} \) of the group of affine motions of the complex line \( \mathbb{C} \), looked at as a real Lie algebra, with the identifications \( e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \). We set \( \mathfrak{D}_{0,1}^2 := \aff(\mathbb{C}) \).

#### 7.1.2 The Lie algebra \( \mathfrak{D}_{0,1}^2 := \aff(\mathbb{C}) \)

Let \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear map with matrix \( M_{-1} := E_{2,1} - E_{1,2} \) in the canonical basis \( (\hat{e}_1, \hat{e}_2) \), with dual basis \( (e_1^*, e_2^*) \). The characteristic and the minimal polynomials of \( \psi \) are both equal to 1.\(^2\). So \( M_{-1} \) has the 2 complex eigenvalues \( i, -i \). Of course, \( \mathbb{R}[M_{-1}] = \mathbb{R}[\mathbb{R}^2] \oplus \mathbb{R}M_{-1} \) is a MASA of \( \mathfrak{gl}(2, \mathbb{R}) \) and \( e_1 \circ (M_0)^0 = e_1, e_1 \circ (M_{-1}) = -e_2 \). In the basis \( (e_1, \cdots, e_4) \) of the 2-step solvable Lie algebra \( \mathcal{G}_{M_{-1}} \), with \( e_1 = (M_{-1})^0, e_2 = M_{-1}, e_3 := \hat{e}_1, e_4 := \hat{e}_2 \), the Lie bracket reads \([e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, [e_2, e_4] = -e_3 \). The exact form \( \partial \hat{e}_3 = -e_1 \wedge e_3^* + e_2 \wedge e_4^* \) is a symplectic form on \( \mathcal{G}_{M_{-1}} \). Note that \( \mathcal{G}_{M_{-1}} \) is the Lie algebra \( \aff(\mathbb{C}) = \{ (z_1, z_2), z_2 = x_j + iy_j, x_j, y_j \in \mathbb{R}, j = 1, 2, \} \) of the group of affine motions of the complex line \( \mathbb{C} \), looked at as a real Lie algebra, with the identifications \( e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \). We set \( \mathfrak{D}_{0,1}^2 := \aff(\mathbb{C}) \).

#### 7.1.3 The Lie algebras \( \mathfrak{D}_0^n \)

In the canonical basis \( (\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n) \) of \( \mathbb{R}^n \), consider the principal nilpotent matrix \( M_0 = \sum_{i=1}^{n-1} E_{i,i+1} \). It is a nonderogatory matrix, as its minimal polynomial \( m_0(X) = X^n \) coincides with its characteristic polynomial. Of course, 0 is the unique eigenvalue of \( M_0 \) and it has multiplicity \( n \). We use the following notation \( e_1 := I_2 = M_0^0, \cdots, e_j := M_0^{j-1}, j = 1, \ldots, n \). Note that the vector \( \hat{x} := \hat{e}_n \) is such that \( e_{n+1} := \hat{x}, e_{n+2} := M_0 \hat{x} = \hat{e}_{n-1}, \ldots, e_{n+j} := M_0^{j-1} \hat{x} = \hat{e}_{n-j+1}, \ldots, e_{2n} := M_0^{n-1} \hat{x} = \hat{e}_1, \ldots \)
form a basis of $\mathbb{R}^n$. In the basis $(e_1, e_2, \ldots, e_{2n})$ of the Lie algebra $\mathcal{G}_{M_0}$, up to skew-symmetry, the nonzero Lie brackets are

$$[e_i, e_{n+j}] = e_{n+j+i-1}, \quad i,j = 1, \ldots, n,$$

where we use the convention $e_{2n+k} = 0$, whenever $k \geq 1$. In particular,

$$[e_1, e_{n+j}] = e_{n+j}, \quad j = 1, \ldots, n, \quad \text{and} \quad [e_j, e_{2n}] = 0, \quad j = 2, \ldots, n.$$

The following exact form on $\mathcal{G}_{M_0}$, is non-degenerate,

$$\partial e^*_{2n} = -\sum_{j=1}^{n} e^*_j \wedge e^*_{2n-j+1} = -\sum_{j=1}^{n} e^*_{n-j+1} \wedge e^*_{n+j}. \quad (33)$$

Throughout the present paper, we write $\mathcal{D}_n^0$ instead of $\mathcal{G}_{M_0}$. The Lie algebra $\mathcal{D}_n^0$ has an $n$-step nilpotent co-dimension 1 nilradical $\mathcal{N}_n = \text{span}(e_2, \ldots, e_{2n})$. Indeed, if we write $C^0(\mathcal{N}) := \mathcal{N}$, $C^{p+1}(\mathcal{N}) := [\mathcal{N}, C^p(\mathcal{N})]$, $p \geq 0$, we have $C^{n-1}(\mathcal{N}) = \mathbb{R}e_{2n}$ and $C^n(\mathcal{N}) = 0$.

7.1.4 The Lie algebra $\mathcal{D}_{0,1}^n$

Consider the real nonderogatory $4 \times 4$ matrix $M_{0,1} = E_{2,1} - E_{1,2} + E_{4,3} - E_{3,4} + E_{1,3} + E_{2,4}$. Its characteristic and minimal polynomials are both equal to $\chi(X) = (X^2 + 1)^2$. So $i$ and $-i$ are the only (repeated complex conjugate) eigenvalues of $M_{0,1}$. So $\mathbb{R}[M_{0,1}] = \text{span}(M_{0,1}^0, M_{0,1}, M_{0,1}^2, M_{0,1}^3)$ is a MASA of $\mathfrak{gl}(4, \mathbb{R})$. In the basis $(e_1, e_2, \ldots, e_8)$ of $\mathcal{G}_{M_{0,1}}$, with $e_1 := I_{4,4}$, $e_2 := M_{0,1}$, $e_3 := M_{0,1}^2$, $e_4 := M_{0,1}^3$, $e_{4+j} = \tilde{e}_j$, $j = 1, \ldots, 4$, the Lie bracket reads

$$[e_1, e_l] = e_l, \quad l = 5, 6, 7, 8,$$

$$[e_2, e_5] = e_5, \quad [e_2, e_6] = -e_5, \quad [e_2, e_7] = e_5 + e_8, \quad [e_2, e_8] = e_6 - e_7,$$

$$[e_3, e_5] = -e_5, \quad [e_3, e_6] = \tilde{e}_6, \quad [e_3, e_7] = 2e_6 - e_7, \quad [e_3, e_8] = -2e_5 - e_7,$$

$$[e_4, e_5] = -e_6, \quad [e_4, e_6] = e_5, \quad [e_4, e_7] = -3e_5 - e_8, \quad [e_4, e_8] = -3e_6 + e_7.$$

Note that both $(e_7, M_{0,1}e_7 = e_5 + e_8, M_{0,1}^2e_7 = 2e_6 - e_7, M_{0,1}^3e_7 = -3e_5 - e_8)$ and $(e_8, M_{0,1}e_8 = e_6 - e_7, M_{0,1}^2e_8 = -2e_5 - e_8, M_{0,1}^3e_8 = -3e_5 + e_7)$ are bases of $\mathbb{R}^4$. The 2-form $\partial e^*_5 = -e^*_5 \wedge e^*_6 + e^*_5 \wedge (e^*_6 + e^*_7) + e^*_5 \wedge (e^*_6 - 3e^*_7)$ is nondegenerate, so $e^*_5$ is a Frobenius functional. We set $\mathcal{D}_{0,1}^4 := \mathcal{G}_{M_{0,1}}$. This generalizes (see Section 4.4.3) to a $2n$-dimensional 2-step solvable Frobenius Lie algebra denoted by $\mathcal{D}_{0,1}^n := \mathcal{G}_{M_{0,1}}$, for any even $n \geq 4$, as follows. Let $M_{0,1}$ be the nonderogatory $n \times n$ matrix

$$M_{0,1} = M_S + M_n \text{ with } M_S = -\sum_{j=0}^{n/2-1} (E_{2j+1,3j+2} - E_{2j+2,2j+1}), \quad M_n = \sum_{j=1}^{n/2} E_{2j,2j+1},$$

in the canonical basis $(\tilde{e}_1, \ldots, \tilde{e}_n)$ of $\mathbb{R}^n$. Its characteristic and minimal polynomials are both equal to $\chi(X) = (X^2 + 1)^{\frac{n}{2}}$. So $i$ and $-i$ are the only ($\frac{n}{2}$ times repeated complex conjugate) eigenvalues of $M_{0,1}$. We say that each of $i$ and $-i$ is of multiplicity $\frac{n}{2}$. So $\mathbb{R}[M_{0,1}] = \text{span}(M_{0,1}^0, M_{0,1}, \ldots, M_{0,1}^{n-1})$ is a MASA of $\mathfrak{gl}(n, \mathbb{R})$. In the basis $(e_1, e_2, \ldots, e_{2n})$ of $\mathcal{D}_{0,1}^n$, with $e_j := (M_{0,1})^{j/2}$, $e_{n+j} = \tilde{e}_j$, $j = 1, \ldots, n$, the Lie bracket reads

$$[e_j, e_{n+l}] = (M_{0,1})^{j-1}e_l, \quad j, l = 1, \ldots, n.$$  \quad (35)
7.2 The classification Theorem

Here we concentrate on the case $\mathbb{K} = \mathbb{R}$. We summarize our main results of this section in Theorem 7.3, in which we give a complete classification of all 2-step solvable Frobenius Lie algebras of the form $G_M := \mathbb{R}[M] \ltimes \mathbb{R}^n$, given by nonderogatory real matrices $M$. In particular, we show that the Lie algebras $D^0_0$, $D^p_{0,1}$, where $p \geq 1$ is an integer, discussed in Section 7.1, are the building blocks that make up, in a trivial way (direct sums), the Lie algebras $G_M$, whenever $M$ is a nonderogatory $n \times n$ real matrix. As in Examples 7.1, 7.2 the notations $D^1 := \text{aff}(\mathbb{R})$ and $D^2_{0,1} := \text{aff}(\mathbb{C})$, will be implicitly adopted throughout this work. It is obvious that, if $z, \bar{z}$ are two complex conjugate eigenvalues of a square matrix $M$, then the polynomial $(X - \text{Re}(z))^2 + \text{Im}(z)^2$ divides the characteristic polynomial $\chi(X)$ of $M$ (Lemma 7.1).

**Definition 7.1.** Let $M$ be an $n \times n$ real matrix and $\chi(X)$ its characteristic polynomial. We say that the complex eigenvalues $z, \bar{z}$ are of multiplicity $m$, if $m$ is the greatest integer such that $f_z := \left((X - \text{Re}(z))^2 + \text{Im}(z)^2\right)^m$ is a factor of $\chi(X)$. In other words, if $P_X \equiv \left((X - \text{Re}(z))^2 + \text{Im}(z)^2\right)^q$ divides $\chi(X)$ for some integer $q$, then $P_X$ divides $f_z$.

We call the following, the Factorization Lemma.

**Lemma 7.1** (Factorization Lemma). Let $M$ be a nonderogatory $n \times n$ real matrix with only two eigenvalues which are both complex and hence conjugate $\lambda, \bar{\lambda}$. Then, the characteristic polynomial of $M$ is $\chi(X) = \left((\text{Re}(\lambda) - X)^2 + \text{Im}(\lambda)^2\right)^2$. More generally, if a real $n \times n$ matrix has $p$ distinct real eigenvalues $\lambda_1, \ldots, \lambda_p$, with respective multiplicities $k_1, \ldots, k_p$ and $2q$ complex eigenvalues $\lambda_{p+1}, \bar{\lambda}_{p+1}, \ldots, \lambda_{p+q}, \bar{\lambda}_{p+q}$, with respective multiplicities $k_{p+1}, \ldots, k_{p+q}$, with $n = k_1 + \cdots + k_p + 2(k_{p+1} + \cdots + k_{p+q})$.

Then, the characteristic polynomial of $M$ coincides with the following product

$$\chi(X) = \Pi_{j=1}^p (X - \lambda_j)^{k_j} \Pi_{j=1}^q \left((X - \text{Re}(\lambda_{p+j}))^2 + \text{Im}(\lambda_{p+j})^2\right)^{k_{p+j}}.$$

**Proof.** Let $M$ be a nonderogatory $n \times n$ real matrix with only two eigenvalues which are both complex and hence conjugate $\lambda, \bar{\lambda}$. Then obviously the polynomial $(X - \lambda)(X - \bar{\lambda}) = (X - \text{Re}(\lambda))^2 + \text{Im}(\lambda)^2$ divides the characteristic polynomial $\chi(X)$ of $M$. We factorize $\chi(X)$ as $\chi(X) = \left((X - \text{Re}(\lambda))^2 + \text{Im}(\lambda)^2\right) P_1(X)$ where $P_1(X)$ is a polynomial of degree $n - 2$, with real coefficients. As $\mathbb{C}$ is a closed field, $P_1(X)$ admits some complex zeros, bound to be $\lambda, \bar{\lambda}$ as they are the only zeros of $\chi(X)$, by hypothesis. We re-write $\chi(X)$ as $\chi(X) = \left((X - \text{Re}(\lambda))^2 + \text{Im}(\lambda)^2\right)^2 P_2(X)$ where $P_2(X)$ is a polynomial of degree $n - 4$, with real coefficients. The result follows by inductively reapplying the same process to $P_2$. The proof of the general case where $M$ is not necessarily nonderogatory, immediately follows by applying the Primary Decomposition Theorem to $\chi(X)$ to reduce the problem to the cases where $M$ admits a unique eigenvalue or only two eigenvalues which are both complex and conjugate, as above.

**Theorem 7.1.** Let $M$ be a nonderogatory $n \times n$ real matrix. Suppose $M$ has $p$ real distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, with respective multiplicities $k_1, \ldots, k_p$ and $2q$ distinct
complex eigenvalues \(z_1, \bar{z}_1, \ldots, z_q, \bar{z}_q\) with respective multiplicities \(m_1, \ldots, m_q\), where \(n = k_1 + \cdots + k_p + 2(m_1 + \cdots + m_q)\). Then the Lie algebra \(G_M := \mathbb{R}[M] \ltimes \mathbb{R}^n\) is isomorphic to the direct sum \(D_0^k \oplus \cdots \oplus D_0^k \oplus D_{0,1}^m \oplus \cdots \oplus D_{0,1}^m\) of the Lie algebras \(D_0^k, \ldots, D_0^k, D_{0,1}^m, \ldots, D_{0,1}^m\).

In particular, let \(M\) be a real \(n \times n\) nonderogatory matrix,

- (a) if \(M\) admits \(n\) distinct real eigenvalues, then the Lie algebra \(G_M\) is isomorphic to the direct sum \(\text{aff} (\mathbb{R}) \oplus \cdots \oplus \text{aff} (\mathbb{R})\) of \(n\) copies of the Lie algebra \(\text{aff} (\mathbb{R})\) of the group of affine motions of the real line \(\mathbb{R}\),
- (b) if \(M\) admits a unique real eigenvalue \(\lambda\), then \(G_M\) is isomorphic to \(D_0^k\),
- (c) if \(M\) admits \(n\) distinct complex eigenvalues, then \(G_M\) is isomorphic to the direct sum \(\text{aff} (\mathbb{C}) \oplus \cdots \oplus \text{aff} (\mathbb{C})\) of \(n\) copies of the Lie algebra \(\text{aff} (\mathbb{C})\) of the group of affine motions of the complex line \(\mathbb{C}\), seen as a real Lie algebra,
- (d) if \(M\) admits only two eigenvalues which are both complex (nonreal) and hence conjugate, then \(G_M\) is isomorphic to \(D_{0,1}^m\).

The following is a direct corollary of Theorem \(7.1\).

**Corollary 7.1.** Let \(M\) be a nonderogatory \(n \times n\) real matrix. The Lie algebra \(G_M\) is indecomposable if and only if one of the following holds true: (a) \(M\) has a unique eigenvalue (which is necessarily real), in which case \(G_M\) is isomorphic to \(D_0^k\) and is hence completely solvable, or (b) \(M\) has only 2 eigenvalues which are both complex and conjugate, in which case \(G_M\) is isomorphic to \(D_{0,1}^m\).

The rest of this section and Sections \(7.3, 7.4\) are mainly concerned with discussions and the proof of Theorem \(7.1\). Lemma \(7.2\) plays a fundamental role in the classification of Frobenius Lie algebras of the form \(G_M\), where \(M\) is a nonderogatory matrix. It allows us to split Theorem \(7.1\) into two main cases discussed in Propositions \(7.1, 7.2, 7.3, 7.4\). The particular case of Theorem \(7.1\) (a) is obtained by taking, in the general case, \(p = n, q = 0\) and by identifying \(D_0^k\) with \(\text{aff} (\mathbb{R})\). The proof of the particular case of Theorem \(7.1\) (b) can be directly found in Lemma \(7.6\) whereas Theorem \(7.1\) (c) and (d) are directly proved in Propositions \(7.2, 7.3\) respectively.

**Lemma 7.2.** Let \(M\) be a nonderogatory \(n \times n\) real matrix. Suppose \(\mathbb{R}^n\) splits as a direct sum \(\mathbb{R}^n = E_1 \oplus E_2\) of two subspaces \(E_1\) and \(E_2\) which are both invariant under \(M\), that is, the image \(Mv_j\) of any vector \(v_j \in E_j\), still lies in \(E_j\), for any \(j = 1, 2\). Denote by \(M_j\) the restriction of \(M\) to \(E_j\) and let \(G_{M_j}\) stand for the Lie algebra \(G_M := \mathbb{R}[M_j] \ltimes E_j\). Then each \(G_{M_j}\), \(j = 1, 2\), is an ideal of \(G_M\). More precisely, \(G_M\) splits as the direct sum \(G_M = G_{M_1} \oplus G_{M_2}\).

**Proof.** Under the assumptions of Lemma \(7.2\), let \(M_j\) be the restriction of \(M\) to the invariant subspace \(E_j, j = 1, 2\). We extend \(M_j\) to a linear map \(\tilde{M}_j\) on \(\mathbb{R}^n\) in such a way that \(\tilde{M}_j\) identically vanishes on \(E_p\) whenever \(p \neq j\). So we have \([\tilde{M}_j, M] = 0\) for every \(j = 1, 2\). That is, \(M_j\) and, in fact, every polynomial in \(M_j, j = 1, 2\), are all polynomials in \(M\), hence they are all elements of \(\mathbb{R}[M]\). More precisely \(\mathbb{R}[M_j]\) is a subalgebra of \(\mathbb{R}[M]\). But \(E_j\) being obviously an ideal of the Abelian Lie algebra \(\mathbb{R}^n\), we thus see that the Lie algebra \(G_{M_j}\), identified with \(G_{M_j} = \mathbb{R}[M_j] \ltimes E_j\), is a Lie
subalgebra of the Lie algebra \( \mathcal{G}_M := \mathbb{R}[M] \ltimes \mathbb{R}^n \). As a matter of fact, each \( \mathcal{G}_{M_j} \) is an ideal of \( \mathcal{G}_M \). This is due to \( \mathcal{E}_j \) being stable by \( M \) and \( M_j \) identically vanishing on \( \mathcal{E}_p \), whenever \( p \neq j \), in addition to the sums \( M = M_1 + M_2 \) and \( \mathbb{R}^n = \mathcal{E}_1 \oplus \mathcal{E}_2 \). Thus, as the ideals \( \mathcal{G}_{M_j} \) form a direct sum (they only meet at \( \{0\} \), unless they are identical), the Lie algebra \( \mathcal{G} \) splits as the direct sum \( \mathcal{G} := \mathcal{G}_{M_1} \oplus \mathcal{G}_{M_2} \).

Consider the general case where a real matrix \( M \) admits \( p \) real and \( 2q \) complex eigenvalues. We write the characteristic polynomial identical), the Lie algebra \( G \) of the zeros of \( Q \) the Primary Decomposition Theorem and Cayley-Hamilton theorem, we have

\[
\mathbb{R}^n = \ker(\chi_M(M)) = \ker(Q_1(M)) \oplus \ker(Q_2(M)).
\]

As \( \ker Q_1(M) \) and \( \ker Q_2(M) \) are both stable by \( M \), Lemma \[7.2\] reduces the proof of Theorem \[7.1\] to two cases: the case where all the eigenvalues of \( M \) are real and the case where all the eigenvalues of \( M \) are complex.

### 7.3 Nonderogatory matrices with only real eigenvalues

Let \( M \) be a nonderogatory \( n \times n \) real matrix with \( p \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_p \), all of which are real and of respective multiplicity \( k_1, \ldots, k_p \), with \( k_1 + \cdots + k_p = n \), so that its characteristic polynomial factorizes as \( \chi_M(X) = \chi_1(X)\chi_2(X) \cdots \chi_p(X) \), where \( \chi_j(X) = (X - \lambda_j)^{k_j} \), for \( j = 1, \ldots, p \). The polynomials \( \chi_j(X) \) are pairwise relatively prime. Indeed, any divisor of \( \chi_j(X) \) is of the form \( (X - \lambda_j)^{r_j} \) for some integer \( r_j \leq k_j \), for every \( j = 1, \ldots, p \). So any common divisor of \( \chi_i(X) \) and \( \chi_j(X) \) would simultaneously be of the forms \( (X - \lambda_i)^{r_i} \) and \( (X - \lambda_j)^{r_j} \), so that the equality \( (X - \lambda_i)^{r_i} = (X - \lambda_j)^{r_j} \) entails \( r_i = r_j = 0 \) or \( \lambda_i = \lambda_j \) and hence \( i = j \). By the Primary Decomposition Theorem and Cayley-Hamilton theorem, we have

\[
\mathbb{R}^n = \ker(\chi_M(M)) = \ker(\chi_1(M)) \oplus \ker(\chi_2(M)) \oplus \cdots \oplus \ker(\chi_p(M)).
\]

Of course the subspaces \( \mathcal{E}_j := \ker(M - \lambda_j)^{k_j}, j = 1, \ldots, p \), are all stable by \( M \), due to the fact that the endomorphisms \( M \) and \( (M - \lambda_j)^{k_j} \) commute.

**Lemma 7.3.** The restriction \( M_j \) of \( M \) to \( \mathcal{E}_j := \ker(M - \lambda_j)^{k_j} \) is again a nonderogatory endomorphism of \( \mathcal{E}_j \) with a unique eigenvalue, for every \( j = 1, \ldots, p \).

**Proof.** The characteristic polynomial of the restriction \( M_j \) of \( M \) to the subspace \( \mathcal{E}_j := \ker(M - \lambda_j)^{k_j} \) is \( \chi_j(X) = (X - \lambda_j)^{k_j} \). If the minimal polynomial of \( M_j \), hereafter denoted by \( m_j(X) \), were different from \( \chi_j(X) \), then there would exist some integer \( r_j \) with \( 1 \leq r_j < k_j \), such that \( m_j(X) = (X - \lambda_j)^{r_j} \). Furthermore, the polynomial \( T(X) = \chi_1(X)\chi_2(X) \cdots \chi_{j-1}(X)m_j(X)\chi_{j+1}(X) \cdots \chi_{p-1}(X)\chi_p(X) \) would be of lower degree than \( \chi(X) \) and yet would satisfy \( T(M) = 0 \). Thus the minimal polynomial of \( M \) would divide \( T(M) \) and would hence be different from the characteristic polynomial of \( M \). This would contradict the fact that \( M \) is nonderogatory.

The above shows that, if a nonderogatory matrix \( M \) has only real eigenvalues, \( \lambda_1, \ldots, \lambda_p \) of respective multiplicity \( k_1, \ldots, k_p \), then its restriction \( M_j \) to each subspace \( \mathcal{E}_j := \ker(M - \lambda_j)^{k_j} \) is again a nonderogatory endomorphism with a unique
eigenvalue $\lambda_j$ of multiplicity $k_j$ and $\mathbb{R}^n$ splits as a direct sum $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_p$. To conclude that $M$ has a Jordan form, we now need to show that each $M_j$ has the following well known Jordan form.

**Lemma 7.4 (Jordan form).** If a real nonderogatory $n \times n$ matrix $M$ has a unique eigenvalue $\lambda$ with multiplicity $n$, then there exists a basis $(\vec{e}_1', \cdots , \vec{e}_n')$ of $\mathbb{R}^n$ in which $M$ has the form $M\lambda = \lambda I_n + \sum_{i=1}^{n-1} E_{i,i+1}$

**Proof.** Let $M$ be a real $n \times n$ nonderogatory matrix with a unique real eigenvalue $\lambda$ with multiplicity $n$. So, up to a sign, its characteristic polynomial is $P_M(X) = (X-\lambda)^n = \sum_{i=0}^{n-1} C^i_n(-1)^{n-i} \lambda^{n-i} X^i$, with $C^i_p = \frac{p!}{q!(p-q)!}$, for $p \geq q$. By Cayley-Hamilton’s Theorem $M^n = \sum_{i=0}^{n-1} C^i_n(-1)^{n-i+1} \lambda^n M^i$.

Choose $\vec{x} \in \mathbb{R}^n$ such that $(\vec{e}_1, \ldots , \vec{e}_n) = (M^{n-1} \vec{x}, M^{n-2} \vec{x}, \ldots , M \vec{x}, \ldots , \vec{x})$ is a basis of $\mathbb{R}^n$. We have $M \vec{e}_1 = M^n \vec{x} = \sum_{i=0}^{n-1} C^i_n(-1)^{n-i+1} \lambda^n M^i \vec{x} = \sum_{j=1}^{n} C^j_n(-1)^{j+1} \lambda^j \vec{e}_j$ and $M \vec{e}_j = \vec{e}_{j-1}$, for $j = 2, \ldots , n$. In the basis $(\vec{e}_j)$, the matrix $M$ has the following form $M\lambda = \sum_{j=1}^{n} \vec{k}_j E_{j,1} + \sum_{j=1}^{n-1} E_{j,j+1}$, with $\vec{k}_j = C^j_n(-1)^{j+1} \lambda^j$. If $\lambda = 0$, then we are done. If $\lambda \neq 0$, we use a direct approach by looking for the coefficients $p_{i,j}$ of a matrix $P$ such that $M\lambda P = PM\lambda$. Using the explicit expressions of $PM\lambda$ and of $M\lambda P$ and by a direct identification of the coefficients, we get

$$p_{k,r} = \sum_{j=0}^{r-1} (-1)^{k-r+j} \lambda^{k-r+j} C^{k-r+j}_{n-r+j} p_{1,j+1},$$

where the numbers $p_{1,j+1}$, $j = 0, \ldots , n-1$, are seen as parameters. In particular, the matrix $P$ whose coefficients in the above basis $(\vec{e}_s)$ are

$$p_{ij} = \begin{cases} (-1)^{i-j} \lambda^{i-j} C^{i-j}_{n-j} & \text{if } i \geq j, \\ 0 & \text{if } i < j \end{cases}$$

is solution, where $C^p_q = \frac{p!}{q!(p-q)!}$, $p \geq q$.  

**Definition 7.2 (Notation).** When $M$ is an $n \times n$ nonderogatory real matrix with a unique real eigenvalue $\lambda$ of multiplicity $n$, we denote by $\mathfrak{D}_\lambda^n$, the corresponding 2-step solvable Lie algebra $\mathcal{G}_M$.

**Lemma 7.5.** Suppose $M$ is a nonderogatory $n \times n$ real matrix with $p$ distinct eigenvalues $\lambda_1, \ldots , \lambda_p$, all of which are real and of respective multiplicity $k_1, \ldots , k_p$, where $k_1 + \cdots + k_p = n$. Then the corresponding Lie algebra $\mathcal{G}_M$ is isomorphic to the direct sum $\mathfrak{D}_{\lambda_1}^{k_1} \oplus \cdots \oplus \mathfrak{D}_{\lambda_p}^{k_p}$ of the Lie algebras $\mathfrak{D}_{\lambda_1}^{k_1}$, $i = 1, \ldots , p$.

**Proof.** Lemma 7.2 to Equality (37), where $\ker \chi_j(M) = \ker(M - \lambda_j I_{\mathbb{R}^n})^{k_j} =: E_j$, $j = 1, \ldots , p$, leads to $\mathcal{G}_M = \mathcal{G}_{M_1} \oplus \cdots \oplus \mathcal{G}_{M_p}$, where $M_j$ is the restriction of $M$ to $E_j$. Taking Lemma 7.3 into account, each $M_j$ is a nonderogatory $k_j \times k_j$ real matrix with a unique eigenvalue $\lambda_j$ of multiplicity $k_j$, so that $\mathcal{G}_{M_j}$ is exactly $\mathfrak{D}_{\lambda_j}^{k_j}$, by Definition 7.2, and thus $\mathcal{G}_M = \mathfrak{D}_{\lambda_1}^{k_1} \oplus \cdots \oplus \mathfrak{D}_{\lambda_p}^{k_p}$. 

24
Lemma 7.6. If a real \( n \times n \) matrix \( M \) admits a unique real eigenvalue \( \lambda \), then the Lie algebra \( \mathcal{G}_M =: \mathcal{D}_\lambda^0 \) is isomorphic to \( \mathcal{D}_\lambda^0 \) as in Example 7.1.3.

Proof. Following Lemma 7.4 consider a basis in which \( M \) has the following form

\[
M_\lambda = \lambda \mathbb{I}_{\mathbb{R}^n} + \sum_{i=1}^{n-1} E_{i,i+1} \quad \text{and set } M_0 := \sum_{i=1}^{n-1} E_{i,i+1}.
\]

Note that \( M_0 \) is given in the same basis as \( M_\lambda \) and it has the same form as the one in Example 7.1.3. One notes that the two nonderogatory matrices \( M_0 \) and \( M_\lambda \) commute. More precisely, \( M_\lambda \) is a polynomial in \( M_0 \) as it can be written as \( M_\lambda = \lambda(M_0)^0 + M_0. \) Hence \( C(M_0) = C(M_\lambda) = \mathbb{K}[M_0] \). Thus we have \( \mathcal{D}_\lambda^0 = \mathcal{G}_{M_0} = \mathcal{G}_{M_\lambda} = \mathcal{D}_\lambda^0 \). \( \square \)

Lemmas 7.5 and 7.6 prove the following.

Proposition 7.1. Suppose \( M \) is a nonderogatory \( n \times n \) real matrix with \( p \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_p \), all of which are real and of respective multiplicity \( k_1, \ldots, k_p \), where \( k_1 + \cdots + k_p = n \). Then the Lie algebra \( \mathcal{G}_M \) is isomorphic to the direct sum \( \mathcal{D}_{\lambda_1}^{k_1} \oplus \cdots \oplus \mathcal{D}_{\lambda_p}^{k_p} \) of the Lie algebras \( \mathcal{D}_{\lambda_j}^{k_j} \), \( j = 1, \ldots, p \).

Proof. Indeed, if \( M \) is a nonderogatory \( n \times n \) real matrix with \( p \) real eigenvalues \( \lambda_1, \ldots, \lambda_p \), of respective multiplicity \( k_1, \ldots, k_p \), where \( k_1 + \cdots + k_p = n \), then Lemma 7.5 ensures that \( \mathcal{G}_M \) is isomorphic to the direct sum \( \mathcal{D}_{\lambda_1}^{k_1} \oplus \cdots \oplus \mathcal{D}_{\lambda_p}^{k_p} \) and Lemma 7.6 further proves that each \( \mathcal{D}_{\lambda_j}^{k_j} \) is isomorphic to \( \mathcal{D}_{\lambda_j}^{k_j} \). \( \square \)

This concludes the proof of Theorem 7.1 in the case where \( M \) is a nonderogatory \( n \times n \) real matrix with only real eigenvalues.

7.4 Nonderogatory matrices all of whose eigenvalues are complex

7.4.1 Real matrices diagonalizable in \( \mathbb{C} \)

Let \( M \) be a nonderogatory \( n \times n \) real matrix all of whose eigenvalues are complex (and not real), in this case \( n \) is even. We suppose that \( M \) is diagonalizable in \( \mathbb{C} \). Thus, \( M \) being nonderogatory imposes that all the eigenvalues are of multiplicity 1 and hence pair-wise distinct. That is, \( M \) has \( n \) distinct (complex) eigenvalues \( \lambda_j, \bar{\lambda}_j, j = 1, \ldots, \frac{n}{2} \). Let \( \lambda = \lambda_R - i \lambda_I \) be an eigenvalue of \( M \), where \( \lambda_R, \lambda_I \) are real numbers and \( \lambda_I \neq 0 \). Consider an eigenvector \( v \) (with complex components) of \( M \) with corresponding eigenvalue \( \lambda \). Further set \( \mathcal{E}_\lambda := \{ zv + \overline{zv}, z \in \mathbb{C} \} \). Note that, as \( \bar{v} \) is also an eigenvector of \( M \) with eigenvalue \( \bar{\lambda} \), we also have \( \mathcal{E}_\lambda = \mathcal{E}_{\bar{\lambda}} \).

Furthermore, \( \mathcal{E}_\lambda \) is a real 2-dimensional vector subspace of \( \mathbb{R}^n \) which is stable by \( M \) and the vectors \( \text{Re}(v) \) and \( \text{Im}(v) \) form a basis of \( \mathcal{E}_\lambda \) in which the matrix of the restriction of \( M \) is \( M_{\lambda} := \begin{pmatrix} \lambda_R & -\lambda_I \\ \lambda_I & \lambda_R \end{pmatrix} \). Indeed, any element of \( \mathcal{E}_\lambda \) is of the form \( zv + \overline{zv} = 2\text{Re}(z) \text{Re}(v) - 2i\text{Im}(z) \text{Im}(v) \) and taking the real and imaginary parts of both sides of the equation

\[
Mv = \lambda v = \begin{pmatrix} \lambda_R - i \lambda_I \\ \lambda_I \end{pmatrix} \begin{pmatrix} \text{Re}(v) + i \text{Im}(v) \\ i(\lambda_R \text{Re}(v) + \lambda_I \text{Im}(v)) \end{pmatrix} = \begin{pmatrix} \lambda_R \text{Re}(v) + \lambda_I \text{Im}(v) \\ \lambda_I \text{Re}(v) \end{pmatrix} + i \begin{pmatrix} \lambda_R \text{Im}(v) - \lambda_I \text{Re}(v) \end{pmatrix}, \tag{38}
\]
one gets
\[ MRe(v) = \lambda_R Re(v) + \lambda_I Im(v), \]
\[ MIm(v) = -\lambda_I Re(v) + \lambda_R Im(v). \]  

(39)

Note that \( M_\lambda \) is a nonderogatory \( 2 \times 2 \) real matrix whose characteristic polynomial is \( \chi_\lambda(X) = (X - \lambda_R)^2 + \lambda_I^2 \) and both \( Re(v) \) and \( Im(v) \) are in \( \ker \chi_\lambda(M) \). We denote by \( A_\lambda = \mathbb{R}I_{E_\lambda} \oplus \mathbb{R}M_\lambda \) the 2-dimensional real vector space generated by the endomorphisms \( I_{E_\lambda} \) and \( M_\lambda \) of \( E_\lambda \). So \( \mathbb{R}[M_\lambda] = A_\lambda \) and the 4-dimensional space \( A_\lambda \oplus E_\lambda \), inherits the Lie bracket of the semi-direct sum \( G_{M_\lambda} := A_\lambda \rtimes E_\lambda \). More precisely, in the basis \( e_1 = I_{E_\lambda} = M_\lambda^0, e_2 = M_\lambda, e_3 = Re(v), e_4 = Im(v) \), such a Lie bracket reads \([e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = Re(\lambda)e_3 + Im(\lambda)e_4 \) and \([e_2, e_3] = -Im(\lambda)e_3 + Re(\lambda)e_4 \). The following holds.

**Lemma 7.7.** Let \( M \) be a nonderogatory \( n \times n \) real matrix all of whose eigenvalues, say \( \lambda_j, \bar{\lambda}_j, j = 1, \ldots, q \), are complex. Suppose \( M \) is diagonalizable in \( \mathbb{C} \), in which case the eigenvalues are pairwise distinct and \( 2q = n \). For each couple \( \lambda_j, \bar{\lambda}_j \), let \( G_{M_{\lambda_j}} \) be the 4-dimensional Lie algebra with Lie bracket \([e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = Re(\lambda_j)e_3 + Im(\lambda_j)e_4, [e_2, e_3] = -Im(\lambda_j)e_3 + Re(\lambda_j)e_4 \). Then the Lie algebra \( G_M \) is isomorphic to the direct sum
\[ G_M = G_{M_{\lambda_1}} \oplus \cdots \oplus G_{M_{\lambda_q}}. \]

(40)

**Proof.** As above, consider the 2-dimensional real subspaces \( E_{\lambda_j} \). Note that, in the basis \((Re(v_j), Im(v_j)), j = 1, \ldots, q \) of \( \mathbb{R}^n \), we have the Jordan form of \( M \) and \( M = P \text{diag}(M_{\lambda_1}, \ldots, M_{\lambda_q}) \) \( P^1 \), where \( P \) is the bloc diagonal matrix \( P = \text{diag}(P_1, \ldots, P_q) \) each \( P_j \) being a \( 2 \times 2 \) real matrix whose first and second columns are the vectors \( Re(v_j) \) and \( Im(v_j) \) respectively. The result follows by applying Lemma 7.2 to the direct sum \( \mathbb{R}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_q} \), where all the \( E_{\lambda_j} \) are stable by \( M \) as explained above. Thus, \( G_M \) splits as a direct sum \( G_M = G_{M_{\lambda_1}} \oplus \cdots \oplus G_{M_{\lambda_q}} \), of the ideals \( G_{M_{\lambda_j}} \), \( j = 1, \ldots, q \).

Of course, each Lie ideal \( G_{M_{\lambda_j}} \) is isomorphic to \( \text{aff}(\mathbb{C}) \). This can be easily deduced from the classification of 4-dimensional Lie algebras. For self-containedness purposes, we supply here an easy proof. Identifying \( E_{\lambda_j} \) with \( \mathbb{R}^2 \), we have

**Lemma 7.8.** Let \( M_\lambda \) be a \( 2 \times 2 \) real matrix with complex eigenvalues \( \lambda, \bar{\lambda}, \) where \( \lambda = \lambda_R - i\lambda_I \) and \( \lambda_R, \lambda_I \) are real numbers, with \( \lambda_I \neq 0 \). Then \( G_{M_\lambda} \) is isomorphic to \( \text{aff}(\mathbb{C}) \), as in Example 7.1.2.

**Proof.** Without loss of generality, we consider \( M_\lambda = \begin{pmatrix} \lambda_R & -\lambda_I \\ \lambda_I & \lambda_R \end{pmatrix} \) in the canonical basis \((\hat{e}_1, \hat{e}_2)\) of \( \mathbb{R}^2 \). In the basis \( e_1 = I_{\mathbb{R}^2}, e_2 = M_\lambda, e_3 = \hat{e}_1, e_4 = \hat{e}_2 \), the nonzero Lie brackets of \( G_{M_\lambda} \) are \([e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = \lambda_R e_3 + \lambda_I e_4, [e_2, e_4] = -\lambda_I e_3 + \lambda_R e_4 \). Now in the new basis \( X_1 := e_1, X_2 := -\frac{\lambda_I}{\lambda_R} e_1 + \frac{1}{\lambda_R} e_2, X_3 := pe_3 - q e_4, X_4 := q e_3 + pe_4, \) with \( p^2 + q^2 \neq 0 \), the Lie bracket reads \([X_1, X_3] = X_3, [X_1, X_4] = X_4, [X_2, X_3] = X_4, [X_2, X_4] = -X_3 \) as for that of \( \text{aff}(\mathbb{C}) \). In other words, the invertible linear map \( \phi : \text{aff}(\mathbb{C}) \to G_{M_\lambda}, \phi(e_j) = X_j, j = 1, 2, 3, 4, \) is an isomorphism between the Lie algebras \( \text{aff}(\mathbb{C}) \) and \( G_{M_\lambda} \).
**Proposition 7.2.** Suppose a nonderogatory $n \times n$ real matrix $M$ has $2q = n$ distinct complex eigenvalues $\lambda_j, \bar{\lambda}_j$, $j = 1 \ldots, q$. Then the Lie algebra $G_M$ is isomorphic to the direct sum $\text{aff}(\mathbb{C}) \oplus \cdots \oplus \text{aff}(\mathbb{C})$ of $q$ copies of the Lie algebra $\text{aff}(\mathbb{C})$.

**Proof.** Suppose $M$ is a nonderogatory $n \times n$ real matrix with $2q$ distinct complex (nonreal) eigenvalues $\lambda_j, \bar{\lambda}_j$, $j = 1 \ldots, q$. From Lemma 7.7, the Lie algebra $G_M$ is isomorphic to the direct sum $G_M = G_{M_{\lambda_1}} \oplus \cdots \oplus G_{M_{\lambda_q}}$, of the ideals $G_{M_{\lambda_j}}, j = 1 \ldots, q$ and according to Lemma 7.8 each $G_{M_{\lambda_j}}$, is isomorphic to $\text{aff}(\mathbb{C})$. $\square$

7.4.2 Example: the circular permutation of the vectors of a basis

Here is a typical example of a (real) nonderogatory linear map with $n$ real and complex eigenvalues, hence diagonalizable in $\mathbb{C}$. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by the circular permutation of the canonical basis $\psi(\hat{e}_i) = \hat{e}_{i+1}$, for $i = 1, \ldots, n-1$ and $\psi(\hat{e}_n) = \hat{e}_1$. The vector $\hat{e}_1$ is raised to $(\hat{e}_1, \psi(\hat{e}_1), \ldots, \psi^{n-1}(\hat{e}_1))$ coincides with the basis $(\hat{e}_1, \ldots, \hat{e}_n)$. As a matter of fact, any vector $\hat{e}_i$ of this basis is such that $(\hat{e}_1, \psi(\hat{e}_1), \ldots, \psi^{n-1}(\hat{e}_1))$ is again a basis of $\mathbb{R}^n$. The map $\psi$ is non-derogatory and its matrix in the above basis reads $[\psi] = E_{1,n} + \sum_{i=1}^{n-1} E_{i+1,i}$. Its characteristic polynomial is $\chi(X) = X^n - 1$, up to a sign. So the eigenvalues are the complex $n$th roots of 1. They are $z_k = e^{i\frac{2\pi k}{n}}$, where $k = 1, 2, \ldots, n$. When $n = 2$, then it reads $[\psi] = E_{1,2} + E_{2,1}$ and has the two distinct real eigenvalues $z_1 = -1$ and $z_2 = 1$. So $G_\psi$ is isomorphic to $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R})$. For $n = 3$, the eigenvalues are $z_1 = 1, z_2 = -e^{i\frac{\pi}{3}}, z_3 = -e^{-i\frac{\pi}{3}}$. So $G_\psi$ is isomorphic to $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{C})$. For $n = 4$, the eigenvalues are $z_1 = -1, z_2 = 1, z_3 = i, z_4 = -i$, thus $G_\psi$ is isomorphic to $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{R})$. When $n = 5$, there are one real and 4 complex eigenvalues and hence we get $G_\psi = \text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{C})$. For $n = 7$, we have one real and six complex, namely $1, -e^{i\frac{\pi}{7}}, -e^{-i\frac{3\pi}{7}}, e^{i\frac{2\pi}{7}}, e^{-i\frac{2\pi}{7}}, -e^{i\frac{3\pi}{7}}, -e^{-i\frac{3\pi}{7}}$. Hence we have $G_\psi = \text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{C})$.

7.4.3 Real matrices non-diagonalizable in $\mathbb{C}$

In this section, we discuss the case of $n \times n$ nonderogatory real matrices $M$ all of whose eigenvalues are complex (nonreal), but which are not diagonalizable in $\mathbb{C}$. In this case, $n$ is even. Lemma 7.9 is a well known result (attributed to Hirsch and Smale).

**Lemma 7.9** (Jordan form). *Suppose a nonderogatory $n \times n$ real matrix $M$ has only two eigenvalues which are both complex $\lambda = r + is$ and $\bar{\lambda}$. If $M$ is not diagonalizable in $\mathbb{C}$, then there exists a basis $(\hat{e}_1, \ldots, \hat{e}_n)$ of $\mathbb{R}^n$ in which $M$ has the form*

\[
M_\lambda = \begin{pmatrix}
M_{r,s} & I_2 & 0 & 0 & \cdots & 0 \\
0 & M_{r,s} & I_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & M_{r,s} & I_2 & 0 \\
0 & 0 & \cdots & 0 & M_{r,s} & I_2 \\
0 & 0 & \cdots & 0 & 0 & M_{r,s}
\end{pmatrix},
\]

where $M_{r,s} := \begin{pmatrix} r & -s \\ s & r \end{pmatrix}$, $I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
One easily sees that the characteristic and the minimal polynomials of \( M_\lambda \) are both equal to \( \chi(X) = \left( (r - X)^2 + s^2 \right)^{\frac{n}{2}} \), as in the factorization lemma 7.1. In the same basis in which \( M_\lambda \) is in the form (41), consider the nonderogatory matrix

\[
M_{0,1} = \begin{pmatrix}
M_{0,1} & I_2 & 0 & 0 & \cdots & 0 \\
0 & M_{0,1} & I_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & M_{0,1} & I_2 & 0 \\
0 & 0 & \cdots & 0 & \tilde{M}_{0,1} & I_2 \\
0 & 0 & \cdots & 0 & 0 & M_{0,1}
\end{pmatrix},
\]

(42)

where \( \tilde{M}_{0,1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and write \( M_{0,1} = M_s + M_n \), where

\[
M_s = -\sum_{j=0}^{\frac{n}{2}-1} (E_{2j+1,2j+2} - E_{2j+2,2j+1}), \quad M_n = \sum_{j=1}^{n-2} E_{j,j+2}.
\]

(43)

We note that \( M_s \) and \( M_n \) commute \([M_s, M_n] = 0\), more precisely, we have

\[
M_s M_n = -\sum_{j=0}^{\frac{n}{2}-2} \left( E_{2j+1,2j+4} - E_{2j+2,2j+3} \right) = M_n M_s,
\]

(44)

and so \([M_{0,1}, M_s] = [M_n, M_s] = [M_s, M_{0,1}] = 0\). In particular, \( M_s \) and \( M_n \) are respectively the semisimple and the nilpotent parts of \( M_{0,1} \). The matrices \( M_s \) and \( M_n \) are both polynomials in \( M_{0,1} \). Thus, the matrix \( M_\lambda = rI_{R^n} + sM_s + M_n \) is also polynomial in \( M_{0,1} \). This induces the following equalities

\[
\mathbb{R}[M_{0,1}] = \mathbb{R}[M_\lambda] \text{ and } \mathcal{G}_{M_{0,1}} = \mathcal{G}_{M_\lambda}.
\]

(45)

We have thus proved the

**Proposition 7.3.** Let \( M \) be a real nonderogatory \( n \times n \) matrix. If \( M \) has just two eigenvalues which are complex and conjugate, and if \( M \) is non-diagonalizable in \( \mathbb{C} \), then \( \mathcal{G}_M = \mathcal{G}_{M_{0,1}} \). In this case, we write \( \mathcal{D}_{0,1}^n \) instead of \( \mathcal{G}_{M_{0,1}} \).

Finally, we deduce the following.

**Proposition 7.4.** Let \( M \) be a real nonderogatory \( n \times n \) matrix. If all the eigenvalues \( \lambda_1, \lambda_1, \ldots, \lambda_p, \tilde{\lambda}_p \) of \( M \) are complex, then \( \mathcal{G}_M \) is isomorphic to the direct sum \( \mathcal{D}_{0,1}^{k_1} \oplus \cdots \oplus \mathcal{D}_{0,1}^{k_p} \) of copies of \( \mathcal{D}_{0,1}^{k_j} \), where \( k_j \) is the multiplicity of the eigenvalues \( \lambda_j, \tilde{\lambda}_j \).

**Proof.** As above, applying the Primary Decomposition Theorem to Lemma 7.2 and further applying Lemma 7.2 together with Proposition 7.3 yield the result.

\[ \square \]

### 7.4.4 More on the Lie algebra \( \mathcal{D}_{0,1}^n \)

Without loss of generality (from Proposition 7.3, Lie algebras of the form \( \mathcal{G}_{M_\lambda} \) are all isomorphic, anyway), we suppose that the form (42) is in the canonical basis \((\tilde{e}_1, \ldots, \tilde{e}_n)\) of \( \mathbb{R}^n \). Note that \( \tilde{x} = \tilde{e}_n \) is such that \((\tilde{x}, M_{0,1} \tilde{x}, \ldots, M_{0,1}^{n-2} \tilde{x}, M_{0,1}^{n-1} \tilde{x})\) is a
basis of $\mathbb{R}^n$. The matrices $M_s, M_n$ satisfy $M_s^2 = -I_{\mathbb{R}^n}, M_n^2 = 0$, $(M_s M_n)^{\frac{n}{2}} = 0$ and for any $j = 1, \ldots, \frac{n}{2} - 1$,

$$\left(M_n\right)^j = \sum_{p=1}^{n-2j} E_{p,p+2j} \text{ and } M_s \left(M_n\right)^j = -\sum_{p=0}^{\frac{n}{2}-j-1} \left(E_{2p+1,2p+2j+2} - E_{2p+2,2p+2j+1}\right).$$

Note that each $M^p_{0,1} = (M_s + M_n)^p = \sum_{j=0}^p \binom{p}{j} (M_s)^j (M_n)^{p-j}$, $p = 0, 1, 2, \ldots, n - 1$, is a linear combination of $I_{\mathbb{R}^n}$, $M_s, (M_n)^j, M_s(M_n)^j, j = 1, \ldots, \frac{n}{2} - 1$, where as above, $\binom{p}{j} = \frac{p!}{j!(p-j)!}$, $p \geq j$. Thus $\left(I_{\mathbb{R}^n}, M_s, (M_n)^j, M_s(M_n)^j, j = 1, \ldots, \frac{n}{2} - 1\right)$ is another basis of the vector space underlying $\mathbb{R}[M_{0,1}]$. The codimension 2 subspace $N := \text{span}\left((M_n)^j, M_s(M_n)^j, j = 1, \ldots, \frac{n}{2} - 1\right) \times \mathbb{R}^n$ is an ideal of $\mathfrak{d}^n_{0,1}$, for it contains the derived ideal $\mathbb{R}^n = [\mathfrak{d}^n_{0,1}, \mathfrak{d}^n_{0,1}]$. The equalities $(M_s)^{\frac{n}{2}} = (M_s M_n)^{\frac{n}{2}} = 0$, show that $N$ is a non-nilpotent Frobenius Lie algebra with a codimension 2 non-Abelian nilradical $N$. Indeed, if we denote by $\mathfrak{f} := \mathbb{R} I_{\mathbb{R}^n} \oplus \mathbb{R} M_{0,1}$, the plane spanned by $(M_{0,1})^0 = I_{\mathbb{R}^n}$ and $M_{0,1}$, then the vector space underlying $\mathfrak{d}^n_{0,1}$ splits as the direct sum $\mathfrak{d}^n_{0,1} = \mathfrak{f} \oplus N$, so that, any subspace of dimension higher than $n - 2$, must meet $\mathfrak{f}$ non-trivially and hence must not be a nilpotent subalgebra of $\mathfrak{d}^n_{0,1}$. Thus $N$ is the biggest nilpotent ideal of $\mathfrak{d}^n_{0,1}$. Altogether, $\mathfrak{d}^n_{0,1}$ is a nondecomposable, non-completely solvable (the adjoint of $M_s$ has complex eigenvalues) 2-step solvable Frobenius Lie algebra with a codimension 2 non-Abelian nilradical $N$. One sees that $\mathfrak{d}^n_{0,1}$ is not isomorphic to any of the Lie algebras of the form $G_M$ discussed so far. Indeed $\mathfrak{d}^n_0$ is completely solvable and has a codimension 1 non-Abelian nilradical (except for $\mathfrak{aff}(\mathbb{R})$ whose nilradical is Abelian). As for $\mathfrak{aff}(\mathbb{C})$, it has an Abelian nilradical.

### 7.5 Derivations and automorphisms of $G_M$

Let $M$ be a nonderogatory $n \times n$ matrix with coefficients in a field $\mathbb{K}$. The following describes the derivations of $G_M$.

**Proposition 7.5.** Let $\mathfrak{R}$ stand for the normalizer of $\mathbb{K}[M]$ in $\mathcal{M}(n, \mathbb{K})$, that is

$$\mathfrak{R} := \{N \in \mathcal{M}(n, \mathbb{K}), \text{ such that } [N, \mathbb{K}[M]] \subset \mathbb{K}[M]\},$$

then the Lie algebra of derivations of $G_M$ is the semi-direct sum of $\mathfrak{R}$ and $\mathbb{K}^n$,

$$\text{Der}(G_M) = \mathfrak{R} \ltimes \mathbb{K}^n.$$  

More precisely, $D$ is a derivation of $G_M$ if and only if there exist a vector $x_D \in \mathbb{K}^n$ and a linear map $h : \mathbb{K}^n \rightarrow \mathbb{K}^n$ satisfying the condition $[h, M^j] \in \mathbb{K}[M], j = 1, \ldots, n - 1$, such that $D$ is of the form $D(a + x) = [h, a] + ax_D + h(x)$ for every $a \in \mathbb{K}[M]$ and $x \in \mathbb{K}^n$.

**Proof.** Let $D$ be a derivation of $G_M$. As the derived ideal $\mathbb{K}^n = [G_M, G_M]$ of $G_M$ is preserved by $D$, we can write $D(x) = h(x)$ and $D(a) = D_{1,1}(a) + D_{1,2}(a)$, for every $a \in \mathbb{K}[M], x \in \mathbb{K}^n$, where $D_{1,1} : \mathbb{K}[M] \rightarrow \mathbb{K}[M], D_{1,2} : \mathbb{K}[M] \rightarrow \mathbb{K}^n$, $h : \mathbb{K}^n \rightarrow \mathbb{K}^n$ are some linear maps. Now, setting $e_1 := I_{\mathbb{K}^n}$ and $x_D := D_{12}(e_1)$, the equality $D_{12}(a) = ax_D$, for any $a \in \mathbb{K}[M]$, follows:

$$0 = D([e_1, a]) = [D(e_1), a] + [e_1, D(a)] = [x_D, a] + [e_1, D_{12}(a)] = D_{12}(a) - ax_D.$$
We also have \( h(ax) = D[a, x] = [D_{11}(a), x] + [a, h(x)] = D_{11}(a) x + ah(x) \), thus entailing \([h, a] = D_{11}(a) \in \mathbb{K}[M]\), for any \( a \in \mathbb{K}[M]\). So the equality \( D_{11}(a) := [h, a]\) stands as the definition of \( D_{11}(a)\) and this means that the commutator \([h, a]\) of the linear map \( h \) and any \( a \) must be a polynomial in \( M \). In conclusion, every derivation of \( G_M \) is given by a vector \( x_D \) and a linear map \( h : \mathbb{K}^n \to \mathbb{K}^3\) satisfying the condition \([h, \mathbb{K}[M]] \subset \mathbb{K}[M]\). Let \( \mathfrak{N} \) stand for the normalizer of \( \mathbb{K}[M]\) in \( \mathcal{M}(n, \mathbb{K})\), that is

\[
\mathfrak{N} := \{ N \in \mathcal{M}(n, \mathbb{R}) , \text{ such that } [N, \mathbb{K}[M]] \subset \mathbb{K}[M]\},
\]

then the above means \( Der(G_M) = \mathfrak{N} \times \mathbb{K}^n \).

Note that in Proposition 7.5.2 the inner derivations of \( G_M \) are those for which the component \( h \) is itself a polynomial in \( M \).

### 7.5.1 Example: derivations of \( \mathfrak{D}_0^n \)

For \( n = 2 \), the normalization of \( \mathbb{R}[E_{1,2}] \) in \( \mathfrak{gl}(2, \mathbb{R}) \), is the 3-dimensional algebra of matrices spanned by \( E_{1,1}, E_{1,2}, E_{2,2} \), which is 1 dimension higher than \( \mathbb{R}[E_{1,2}] \). For example, \( E_{2,2} \) is not in \( \mathbb{R}[E_{1,2}] \) and will thus act as an outer derivation on \( \mathfrak{D}_0^2 \).

Considering elements of \( \mathbb{R}^2 \) as inner derivations, the space of derivations of \( \mathfrak{D}_0^2 \) is 5-dimensional and is spanned by \( E_{1,1}, E_{1,2}, E_{2,2}, \bar{e}_1, \bar{e}_2 \). For \( n = 3 \), the normalization of \( \mathbb{R}[E_{1,2} + E_{2,3}] \) in \( \mathfrak{gl}(3, \mathbb{R}) \), is the 5-dimensional algebra of matrices spanned by \( E_{1,1} - E_{3,3}, \ E_{2,2} + 2E_{33}, E_{1,2}, E_{2,3}, E_{1,3} \). For example, \( E_{1,1} - E_{3,3}, E_{2,2} + 2E_{33} \), are not in \( \mathbb{R}[E_{1,2} + E_{2,3}] \), so they represent outer derivations of \( \mathfrak{D}_0^2 \). Counting elements of \( \mathbb{R}^3 \) in as inner derivations, the space of derivations of \( \mathfrak{D}_0^3 \) is 8-dimensional. More generally, for a given \( n \geq 4 \), if as above, we set \( M_0 := E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} \), then the normalization of \( \mathbb{R}[M_0] \) in \( \mathfrak{gl}(n, \mathbb{R}) \), is of dimension \( 2n - 1 \) and is spanned by

\[
D_k := E_{1,k} - \sum_{j=3}^{n-k+1} (j-2)E_{j,j+k-1}, \ D'_k := E_{2,k+1} + \sum_{j=3}^{n-k+1} (j-1)E_{j,j+k-1}, \ k = 1, \ldots, n-2
\]

and \( D_{n-1} := E_{1,n-1} \), \( D'_{n-1} := E_{2,n} \), \( D_n := E_{1,n} \). For example, \( D_1 \) and \( D'_1 \) are not in \( \mathbb{R}[M_0] \) and will act as non-trivial outer derivations of \( \mathfrak{D}_0^0 \). In particular \( \mathbb{R}[M_0] \) is not a Cartan subalgebra of \( \mathfrak{gl}(n, \mathbb{R}) \). Taking into account elements of \( \mathbb{R}^n \) as inner derivations, the space of derivations of \( \mathfrak{D}_0^n \) is thus of dimension \( 3n - 1 \).

### 7.5.2 Example: on derivations of \( \mathfrak{D}_{0,1}^n \)

For \( n = 4 \), consider the \( n \times n \) matrix \( M_{0,1} = E_{2,1} - E_{1,2} + E_{4,3} - E_{3,4} + E_{1,3} + E_{2,4} \) as in Example 7.1.3. The normalization of \( \mathbb{R}[M_{0,1}] \) in \( \mathfrak{gl}(4, \mathbb{R}) \), is of dimension 6 and is spanned by the matrices \( E_{1,1} + E_{2,2}, E_{3,3} + E_{4,4}, E_{1,2} - E_{2,1}, E_{3,4} - E_{4,3}, E_{1,3} + E_{2,4}, E_{2,3} - E_{1,4} \). In particular the two matrices \( E_{1,1} + E_{2,2}, E_{3,3} + E_{4,4} \), are not elements of \( \mathbb{R}[M_{0,1}] \). So they both represent non-trivial outer derivations of \( \mathfrak{D}_{0,1}^4 \). More generally, for any \( n \geq 4 \), if we let again \( M_{0,1} \) stand for the \( n \times n \) matrix \( M_{0,1} \), the normalization of \( \mathbb{R}[M_{0,1}] \) in \( \mathfrak{gl}(n, \mathbb{R}) \), contains \( \mathbb{R}[M_{0,1}] \) properly. For example for \( n \geq 6 \), the \( n \times n \) matrix \( Z_1 := E_{1,1} + E_{2,2} - \sum_{j=2}^{n-1} (j - 1)(E_{2j+1,2j+1} + E_{2j+2,2j+2}) \) is in the normalization of \( \mathbb{R}[M_{0,1}] \) in \( \mathfrak{gl}(n, \mathbb{R}) \), but not in \( \mathbb{R}[M_{0,1}] \).
Indeed, for any s, we have \([Z_1, \left( M_n \right)^s] = s \sum_{j=1}^{n-2s} E_{j,j+2s} = s \left( M_n \right)^s\), \([Z_1, M_s] = 0\)
and \([Z_1, M_s \left( M_n \right)^s] = M_s [Z_1, \left( M_n \right)^s]\) + \([Z_1, M_s] \left( M_n \right)^s\) = \(s M_s \left( M_n \right)^s\).

So the linear map \(M \mapsto [Z_1, M]\) preserves \(\mathbb{R}[M_{0,1}]\) and has the following diagonal matrix \(\text{diag}(0, 0, 1, 2, \ldots, \frac{n}{2} - 1, 1, 2, \ldots, \frac{n}{2} - 1)\) in the following new basis \((I_{\mathbb{R}^2}, M_s, M_n, (M_n)^2, \ldots, (M_n)^{-1}, M_s M_n, M_s (M_n)^2, \ldots, M_s (M_n)^{-1})\) and \(Z_1\) is not an element of \(\mathbb{R}[M_{0,1}]\), given that \(\mathbb{R}[M_{0,1}]\) is Abelian. In particular \(\mathbb{R}[M_{0,1}]\) is not a Cartan subalgebra of \(\mathfrak{g}(n, \mathbb{R})\). Also, \(Z_1\) will act as an outer derivation of \(\mathcal{D}_{0,1}^n\).

We have the following.

**Theorem 7.2.** Let \(M\) be an \(n \times n\) nonderogatory real matrix. The following are equivalent.

1. Every derivation of \(\mathcal{G}_M\) is an inner derivation.
2. \(\mathbb{R}[M]\) is a Cartan subalgebra of \(\mathfrak{gl}(n, \mathbb{R})\).
3. The matrix \(M\) has \(n\) distinct (real or complex) eigenvalues.
4. The nilradical of \(\mathcal{G}_M\) is Abelian.
5. \(\mathcal{G}_M\) is the direct sum \(\mathcal{G}_M = \mathfrak{aff}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{aff}(\mathbb{C})\) of only copies of \(\mathfrak{aff}(\mathbb{R})\) and \(\mathfrak{aff}(\mathbb{C})\).

**Proof.** The equivalence between (1) and (2) is a direct consequence of Proposition 7.5. Indeed, every derivation of \(\mathcal{G}_M\) is inner, if and only if the normalizer \(\mathfrak{N}\) of \(\mathbb{R}[M]\) in \(\mathcal{M}(n, \mathbb{R})\), coincides with \(\mathbb{R}[M]\). The equivalence between (3) and (4) and (5) is shown as follows. From Theorem 7.1, the matrix \(M\) has \(n\) distinct eigenvalues, say \(p\) distinct real eigenvalues and \(2q\) distinct complex eigenvalues, with \(p + 2q = n\), if and only if \(\mathcal{G}_M\) is isomorphic to the direct sum \(\mathcal{G}_M = \mathfrak{aff}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{aff}(\mathbb{C})\) of \(p\) copies of \(\mathfrak{aff}(\mathbb{R})\) and \(q\) copies of \(\mathfrak{aff}(\mathbb{C})\). Thus (3) and (5) are equivalent. Let us remind here that, by a direct sum, we mean that each copy of either \(\mathfrak{aff}(\mathbb{R})\) or \(\mathfrak{aff}(\mathbb{C})\), is an ideal of the Lie algebra \(\mathcal{G}_M\). But of course, as both \(\mathfrak{aff}(\mathbb{R})\) and \(\mathfrak{aff}(\mathbb{C})\) have Abelian nilradical, then so does \(\mathcal{G}_M\). So (3) implies (4).

Conversely, from the classification Theorem 7.1, the only Lie algebras of the form \(\mathcal{G}_M\) that have an Abelian nilradical, with \(M\) a nonderogatory matrix, are \(\mathfrak{aff}(\mathbb{R})\), \(\mathfrak{aff}(\mathbb{C})\) and all Lie algebras made of direct sums of copies of them. Hence (4) implies (5) and thus (3) and (4) are equivalent, too. Let us prove that (3) implies (1). As \(\mathcal{G}_M\) has no center, each copy of either \(\mathfrak{aff}(\mathbb{R})\) or \(\mathfrak{aff}(\mathbb{C})\), is preserved by every derivation of \(\mathcal{G}_M\). So a derivation of \(\mathcal{G}_M\) is an inner derivation if and only if its restriction to each copy of either \(\mathfrak{aff}(\mathbb{R})\) or \(\mathfrak{aff}(\mathbb{C})\) is inner. From 7.1, for any \(m \geq 1\), the Lie algebra \(\mathfrak{aff}(m, \mathbb{R})\) only has inner derivations. On the other hand, as above, the \(2 \times 2\) nonderogatory matrix \(M_{-1} = E_{2,1} - E_{1,2}\) satisfies \(\mathcal{G}_{M_{-1}} = \mathfrak{aff}(\mathbb{C})\). For a \(2 \times 2\) matrix \(M\) with coefficients \(m_{i,j}\) , the matrix \([M, M_{-1}] = (m_{2,2} - m_{1,1})(E_{1,2} + E_{2,1}) + (m_{1,2} + m_{2,1})(E_{1,1} - E_{2,2})\) is in \(\mathbb{R}[M_{-1}]\) if and only if \(m_{1,2} = -m_{2,1}\) and \(m_{1,1} = m_{2,2}\), that is, if and only if \(M\) itself lies in \(\mathbb{R}[M_{-1}]\). Hence all derivations of \(\mathfrak{aff}(\mathbb{C})\) are inner derivations. In order to complete the proof, we need to show that (1) implies (5). First, recall that \(\mathcal{D}_{0,1}^n\) and \(\mathcal{D}_{0}^n\) both possess outer derivations, for any \(n \geq 2\), as seen in Examples 7.5.1 and 7.5.2. So if \(M\) is a real nonderogatory matrix, in order for the Lie algebra \(\mathcal{G}_M\) to only have inner derivations, it must not be isomorphic to \(\mathcal{D}_{0,1}^n\) or \(\mathcal{D}_{0}^n\) if it is indecomposable and in the case where it is decomposable, it must not contain a copy of \(\mathcal{D}_{0,1}^n\) or \(\mathcal{D}_{0}^n\), for any \(p \geq 2\), as a component of its decomposition.
into a direct sum of ideals. Hence the matrix $M$ has $n$ distinct eigenvalues. Thus (1) implies (5).

Note that Proposition 7.2 is in agreement with the classification of Cartan subalgebras of $\mathfrak{g}(n, \mathbb{R})$ supplied by Kostant [21] and Sugiura [34].

**Proposition 7.6.** Let $M$ be a nonderogatory real $n \times n$ matrix. Any Lie algebra automorphism of $\mathcal{G}_M$ is of the form

$$\psi(a, x) = (\phi \circ a \circ \phi^{-1}, \phi \circ a \circ \phi^{-1}(x_0) + \phi(x))$$

for any $(a, x) \in \mathbb{R}[M] \ltimes \mathbb{R}^n =: \mathcal{G}_M$, for some $x_0 \in \mathbb{R}^n$ and some invertible linear map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi \circ M \circ \phi^{-1} \in \mathbb{R}[M]$. The inverse of $\psi$ is given by

$$\psi^{-1}(a, x) = (\phi^{-1} \circ a \circ \phi, \phi^{-1}(x - ax_0)).$$

**Proof.** As the derived ideal $[\mathcal{G}_M, \mathcal{G}_M] = \mathbb{R}^n$ is preserved by any automorphism $\psi$, we set $\psi(a) = \psi_{1,1}(a) + \psi_{1,2}(a)$ and $\psi(x) = \phi(x)$ for any $a \in \mathbb{R}[M]$ and $x \in \mathbb{R}^n$, where $\psi_{1,1} : \mathbb{R}[M] \to \mathbb{R}[M], \psi_{1,2} : \mathbb{R}[M] \to \mathbb{R}^n, \phi : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps, with $\psi_{1,1}$ and $\phi$ invertible. From the equality $\phi(ax) = \psi([a, x]) = [\psi_{1,1}(a) + \psi_{1,2}(a), \phi(x)] = (\psi_{1,1}(a) \circ \phi)(x)$, we deduce $\psi_{1,1}(a) = \phi \circ a \circ \phi^{-1}$. In particular $\phi \circ a \circ \phi^{-1} \in \mathbb{R}[M]$, for any $a \in \mathbb{R}[M]$, which is equivalent to $\phi \circ M \circ \phi^{-1} \in \mathbb{R}[M]$. Taking $b = e_1$ and $x_0 := \psi_{1,2}(e_1)$ in the equality $0 = \psi([a, b]) = [\psi_{1,1}(a) + \psi_{1,2}(a), \psi_{1,1}(b) + \psi_{1,2}(b)] = \psi_{1,1}(a)\psi_{1,2}(b) - \psi_{1,1}(b)\psi_{1,2}(a)$ yields $\psi_{1,2}(a) = \phi \circ a \circ \phi^{-1}x_0$, for any $a \in \mathbb{R}[M]$.

### 7.6 Proof of Theorem 6.2

From Proposition 7.2, for a given $n$, the number of isomorphism classes of 2-step solvable Frobenius Lie algebras of dimension $2n$ of the form $\mathcal{G}_M := \mathbb{R}[M] \ltimes \mathbb{R}^n$, where $\mathbb{R}[M]$ is a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, is exactly $[\frac{n}{2}] + 1$. Indeed, one can look at $[\frac{n}{2}] + 1$ as the number (counting from zero) of possible copies of $\mathfrak{aff}(\mathbb{C})$ that can count in a decomposable Lie algebra containing only copies of either $\mathfrak{aff}(\mathbb{C})$ or $\mathfrak{aff}(\mathbb{R})$. On the other hand, from e.g. [34], there are exactly $[\frac{n}{2}] + 1$ non-conjugate Cartan subalgebras of $\mathfrak{gl}(n, \mathbb{R})$. So we have derived Theorem 6.2 in a simple and direct way. More precisely we prove it as follows.

**Proof.** Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ with a $k$-dimensional toroidal part. From [34], up to conjugacy under an element of the Weyl group, there is a basis $(\hat{e}_1, \ldots, \hat{e}_n)$, considered here as the canonical basis of $\mathbb{R}^n$, in which $\mathfrak{h}$ is of the form

$$\mathfrak{h} = \left\{ \begin{pmatrix} D_1 & -D_2 & 0 \\ D_2 & D_1 & 0 \\ 0 & 0 & D_3 \end{pmatrix} \right\},$$

where $D_1 = \text{diag}(h_1, \ldots, h_k), D_2 = \text{diag}(h_{k+1}, \ldots, h_{2k})$, $D_3 = \text{diag}(h_{2k+1}, \ldots, h_n)$ with $h_j \in \mathbb{R}, j = 1, \ldots, n$, so that $\mathfrak{h}$ is conjugate to the algebra $\mathfrak{h}' = \{ \text{diag}(D'_1, \ldots, D'_k, D_3), \text{with } D'_j = \begin{pmatrix} h_j & -h_{k+j} \\ h_{k+j} & h_j \end{pmatrix} \}$ and $D_3 = \text{diag}(h_{2k+1}, \ldots, h_n), h_j \in \mathbb{R}, j = 1, \ldots, n \}$. Obtained by reordering the canonical basis of $\mathbb{R}^n$ into $(\hat{e}_1, \hat{e}_{k+1}, \hat{e}_2, \hat{e}_{k+2}, \ldots, \hat{e}_k, \hat{e}_{2k}, \hat{e}_{2k+1}, \hat{e}_{2k+2}, \ldots, \hat{e}_n)$. A matrix in $\mathfrak{h}'$ is nonderogatory if and only if $(h_i, h_{k+i}) \neq (h_j, h_{k+j}), \text{whenever } i \neq j \text{ and } h_{2k+s} \neq h_{2k+l} \text{ whenever } s \neq l$. But the existence of a matrix (a regular element)
satisfying these conditions are guaranteed by the fact that $\mathfrak{h}$ is a Cartan subalgebra. Hence, there exists a nonderogatory matrix $M$ with the $n$ distinct eigenvalues $h_1 + i h_{k+1}, h_1 - i h_{k+1}, \ldots, h_k + i h_{2k}, h_k - i h_{2k}, h_{2k+1}, \ldots, h_n$, such that, up to a conjugation, $I_{R^k} \oplus h = R[M] = \mathfrak{B}_h$. So (a) implies (b). Furthermore, Theorem 7.1 ensures that $\mathfrak{B}_h \times R^n$ is isomorphic to the direct sum $\text{aff}(\mathbb{C}) \oplus \cdots \oplus \text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{R}) \oplus \cdots \oplus \text{aff}(\mathbb{R})$ of $k$ copies of $\text{aff}(\mathbb{C})$ and $(n - k)$ of $\text{aff}(\mathbb{R})$. In fact, the equivalence between (b) and (c) has already been proved by Theorem 7.1. Conversely, suppose $M$ is a $n \times n$ nonderogatory real matrix with $n$ distinct eigenvalues, $2k$ of which are complex. From Proposition 7.2, $\mathbb{R}[M]$ is a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ and so (b) implies (a). Furthermore, using the Primary Decomposition Theorem and results from Section 7.4 then $\mathbb{R}[M]$ can be put in the form $\mathfrak{h}'$. □

8 On low dimensional 2-step solvable Frobenius Lie algebras

Applying Theorem 7.1 to low dimensions, we obtain the following. In dimension 2, there is a unique such Lie algebra, namely $\mathfrak{D}_0^1 = \text{aff}(\mathbb{R})$. In dimension 4, the Lie algebras of the form $\mathcal{G}_M$ are $\mathfrak{D}_0^2$, $\text{aff}(\mathbb{C})$ and $\text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{R})$. They correspond to the family PHC7 of [3] and to S11, S8 and S10 respectively in [33]. In dimension 6, they are $\mathfrak{D}_0^3$, $\mathfrak{D}_0^3 \oplus \text{aff}(\mathbb{R})$, $\text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{R})$ and $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R})$. Theorem 8.1 below provides, up to isomorphism, a complete list of all 2-step solvable Frobenius Lie algebras of dimension 4 or 6.

**Theorem 8.1.** A 2-step solvable Frobenius Lie algebra of dimension $2n \leq 6$ is either of the form $\mathcal{G}_\psi = \mathbb{R}[\psi] \times R^n$ for some nonderogatory endomorphism $\psi$ of $\mathbb{R}^n$, or isomorphic to $\mathcal{G}_{\mathfrak{h},1}$ as in Example 7.1. Consequently, a 2-step solvable Frobenius Lie algebra of dimension less or equal to 6 is either isomorphic to $\text{aff}(\mathbb{R})$, $\mathfrak{D}_0^2$, $\text{aff}(\mathbb{C})$, $\mathfrak{D}_0^3$, $\mathcal{G}_{\mathfrak{h}}$, or to one of the direct sums $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R})$, $\mathfrak{D}_0^2 \oplus \text{aff}(\mathbb{R})$, $\text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{R})$, $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R})$ of their copies.

**Proof.** The case $n = 1$ is trivial, as every 2-dimensional non-Abelian Lie algebra is isomorphic to $\text{aff}(\mathbb{R}) = \mathcal{G}_\psi$, where $\psi = \text{id}_R$, as in Example 7.1. Let $\mathcal{G}$ be a 2-step solvable Lie algebra of dimension $2n$, with $2 \leq n \leq 3$. Write $\mathcal{G} = \mathcal{A} \times R^n$ where $\mathcal{A}$ is an $n$-dimensional Abelian subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Note that, when $n \leq 3$, then $\left\lceil \frac{n^2}{4} \right\rceil + 1 = n$, so that from Jacobson’s theorem [19], every $n$-dimensional Abelian subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ is a MASA. Furthermore, as $\mathcal{A}$ contains the identity matrix $I_{R^n}$, we use the decomposition $\mathcal{A} = R[I_{R^n}] \oplus L$ of $\mathcal{A}$ into a direct sum of the line $R[I_{R^n}]$ and a MASA $L$ of $\mathfrak{sl}(n, \mathbb{R})$. When $n = 2$, then $L$ is 1-dimensional, namely $L$ is the line $L = R \cdot M$, for some nonzero element $M$ of $\mathfrak{sl}(2, \mathbb{R})$. But every nonzero element of $\mathfrak{sl}(2, \mathbb{R})$ is nonderogatory. Thus, $\mathcal{A} = R[I_{R^2}] \oplus R[M] = R[M]$ and $\mathcal{A} \times R^2 = \mathcal{G}_M$. According to Theorem 7.1 a 4-dimensional Frobenius Lie algebra $\mathcal{G}_M$ given by a nonderogatory matrix $M$, is isomorphic to $D_0^3$, $\text{aff}(\mathbb{C})$, or $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R})$. In the case where $n = 3$, referring to the classification list quoted in [30], there are six classes of non-mutually conjugate MASAs of $\mathfrak{sl}(3, \mathbb{R})$. With the same notation as in [30], we denote them by $L_{2,i}$, $i = 1, \ldots, 6$. The first one is the Lie algebra $L_{2,1}$ of diagonal matrices of the form $\text{diag}(k_1 + k_2, -k_1 + k_2, -2k_2)$, where $k_1, k_2 \in \mathbb{R}$. Any matrix in $L_{2,1}$ is a polynomial in the nonderogatory matrix $S_{2,1} = \text{diag}(1, 0, -1)$. More precisely $\text{diag}(k_1 + k_2, -k_1 + k_2, -2k_2) = (k_2 - k_1)S_{2,1}^0 + \frac{1}{2}(k_1 + 3k_2)S_{2,1} + \frac{3}{2}(k_1 - k_2)S_{2,1}^2$. So...
$L_{2,1}$ is a 2-dimensional subalgebra of the algebra $\mathbb{R}[S_{2,1}]$ of polynomials in $S_{2,1}$ and we have the equality $\mathcal{A}_{2,1} = \mathbb{R}^n \oplus L_{2,1} = \mathbb{R}[S_{2,1}]$. Theorem 7.1 asserts that, since $S_{2,1}$ has three distinct eigenvalues, the corresponding Lie algebra $\mathcal{A}_{2,1} \ltimes \mathbb{R}^3 = \mathcal{G}_{S_{2,1}}$ is isomorphic to the direct sum $\text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R}) \oplus \text{aff}(\mathbb{R})$. Every element of the second class $L_{2,2} := \{k_1(E_{1,1} + E_{2,2} - 2E_{3,3}) + k_2(E_{1,2} - E_{2,1}), k_1, k_2 \in \mathbb{R}\}$, is a nonderogatory matrix with one real eigenvalue, namely the real $-2k_1$ and two complex conjugate eigenvalues $k_1 + ik_2$ and $k_1 - ik_2$, except when $k_2 = 0$. For instance, let $S_{2,2} := E_{1,2} - E_{2,1}$ be the nonderogatory matrix in $L_{2,2}$ corresponding to $k_1 = 0, k_2 = 1$; Every matrix in $L_{2,2}$ is of the form $-2k_1S_{2,2}^0 + k_2S_{2,2} - 3k_1S_{2,2}^2$. So $L_{2,2}$ is a 2-dimensional subalgebra of $\mathbb{R}[S_{2,2}]$ and the space $\mathcal{A}_{2,2} = \mathbb{R}^n \oplus L_{2,2}$ is equal to $\mathbb{R}[S_{2,2}]$ and the Lie algebra $\mathcal{A}_{2,2} \ltimes \mathbb{R}^3$ is isomorphic to $\mathcal{G}_{S_{2,2}}$. Theorem 7.1 ensures that such a Lie algebra is isomorphic to the direct sum $\text{aff}(\mathbb{C}) \oplus \text{aff}(\mathbb{R})$. Next, we note that every element of $L_{2,3} := \{k_1(E_{1,1} + E_{2,2} - 2E_{3,3}) + k_2E_{1,2}, k_1, k_2 \in \mathbb{R}\}$ is of the form $k_1S_{2,3}^0 + k_2S_{2,3} - (3k_1 + k_2)S_{2,3}^2$ where $S_{2,3}$ stands for the nonderogatory matrix $S_{2,3} := E_{1,2} + E_{3,3}$ with the double eigenvalue 0 and the simple eigenvalue 1. Hence, the algebra $L_{2,3}$ is a 2-dimensional subalgebra of $\mathbb{R}[S_{2,3}]$ and thus, the equality $\mathcal{A}_{2,3} = \mathbb{R}^n \oplus L_{2,3} = \mathbb{R}[S_{2,3}]$ entails the needed equality $\mathcal{A}_{2,3} \ltimes \mathbb{R}^3 = \mathcal{G}_{S_{2,3}}$. According to Theorem 7.1, the latter Lie algebra is isomorphic to $\text{aff}(\mathbb{R}) \oplus \mathcal{D}^2_0$.

As regards $L_{2,5} := \{k_1E_{1,1} + k_2E_{1,3}, k_1, k_2 \in \mathbb{R}\}$, the space $\mathcal{A}_{2,5} = \mathbb{R}^3 \oplus L_{2,5}$ corresponds to $\mathcal{B}_{3,1}$ in Examples 5.1 5.3, it contains no nonderogatory matrix and the Lie algebra $\mathcal{A}_{2,5} \ltimes \mathbb{R}^3 = \mathcal{B}_{3,1} \ltimes \mathbb{R}^3 = \mathcal{G}_{3,1}$ is indecomposable. Every element of the algebra $L_{2,6} := \{k_1(E_{1,2} + E_{2,3}) + k_2E_{1,3}, k_1, k_2 \in \mathbb{R}\}$ is a polynomial in the nonderogatory matrix $M_0 = E_{1,2} + E_{2,3}$ with 0 as a unique eigenvalue of multiplicity 3. Thus, as above, we conclude that $\mathcal{A}_{2,6} = \mathbb{R}[M_0]$, so that $\mathcal{A}_{2,6} \ltimes \mathbb{R}^3 = \mathcal{G}_{M_0} = \mathcal{D}^2_0$ (see Theorem 7.1). It is easy to see that, for the remaining algebra $L_{2,4} := \{k_1, k_2, k_3, k_4, k_5, k_6 \in \mathbb{R}\}$ in the list in [30], if we set $\mathcal{A}_{2,4} = \mathbb{R}^n \oplus L_{2,4}$, then there is no Frobenius functional on the Lie algebra $\mathcal{A}_{2,4} \ltimes \mathbb{R}^3$, as every linear form $\alpha$ satisfies $(\partial \alpha)^3 = 0$. Indeed, in the basis $e_1 := 1, e_2 := 1, e_3 := 1, e_3 + j = \delta_j$, $j = 1, 2, 3$, the Lie bracket reads

$[e_1, e_3 + j] = e_1 + j, j = 1, 2, 3, [e_2, e_6] = e_4, [e_3, e_6] = e_5, \partial e_6 = \partial e_5 = \partial e_4 = 0$

and for all $\alpha \in \mathbb{R}$, $j = 1, 2, 3$, $e_j \in \mathbb{R}$, setting $c_1 = e_1^* + \cdots + e_6^*$, we have

$$\partial \alpha = \partial(s_3e_4^* + s_5e_5^* + s_6e_6^*) = -s_3(e_1^* + e_2^* + e_3^* - e_4^* - e_5^* - e_6^*),$$

$$-s_5e_1^* - s_6e_6^* = -s_1e_1^* + \cdots + s_6e_6^*, \quad (s_3e_4^* + s_5e_5^* + s_6e_6^*) = \partial \alpha. \quad (50)$$

This is of the form $\partial \alpha = e_1^* \wedge \beta + \gamma \wedge e_6^*$. Hence $(\partial \alpha)^3 = 0$. See also Remark 6.1. □

9 Some concluding remarks

Our discussions in this paper relates to three problems, the classification of 2-step solvable Frobenius Lie algebras, the classification of maximal Abelian subalgebras (MASAs) of the Lie algebra of square matrices and the Gerstenhaber’s Theorem. We have proposed a condition under which Gerstenhaber’s Theorem is valid for any number of commuting matrices. In particular, under such a condition, Gerstenhaber’s Theorem is valid for a set of 3 commuting matrices. We have
also shown that every 2-step solvable Frobenius Lie algebra $\mathcal{G}$ is a semidirect sum $\mathcal{G} = \mathcal{B} \ltimes \mathbb{R}^n$ of two Abelian Lagrangian subalgebras $\mathcal{B}$ and $\mathbb{R}^n$, where $\mathbb{R}^n$ is the derived ideal $\mathbb{R}^n = [\mathcal{G}, \mathcal{G}]$ and $\mathcal{B}$ is an $n$-dimensional MASA of $\mathfrak{gl}(n, \mathbb{R})$, such that the action $\mathcal{B} \times (\mathbb{R}^n)^* \rightarrow (\mathbb{R}^n)^*, (a, f) \mapsto -f \circ a$, has an open orbit. Yet interestingly, we have also shown that if an $n$-dimensional Abelian subalgebra $\mathcal{B}$ of $n \times n$ matrices over $\mathbb{K}$, has an open orbit on $(\mathbb{K}^n)^*$, for the above action, then $\mathcal{B}$ is a MASA of $\mathfrak{gl}(n, \mathbb{K})$. Moreover, $\mathcal{A} \ltimes \mathbb{R}^n$ and $\mathcal{B} \ltimes \mathbb{R}^n$ are isomorphic if and only if $\mathcal{A}$ and $\mathcal{B}$ are conjugate. That is, $\mathcal{B} = \phi \mathcal{A} \phi^{-1}$, for some invertible linear map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$. So the classification of 2-step solvable Frobenius Lie algebras is equivalent to that of $n$-dimensional MASAs of $\mathfrak{gl}(n, \mathbb{R})$, for which the above action has some open orbit. We have also given a complete characterization of the corresponding LSA. New characterizations of Cartan subalgebras of $\mathfrak{sl}(n, \mathbb{R})$ are also given. In any dimension, we have performed a complete classification of all 2-step solvable Frobenius Lie algebras corresponding to nonderogatory real matrices. We have also discussed examples corresponding to $n$ non-isomorphic 2-step solvable Frobenius Lie algebras which are not given by nonderogatory matrices, in any dimension $n$. In low dimensions, we have classified all 2-step solvable Frobenius Lie algebras up to dimension 6.

References

[1] Alvarez, M. A.; Rodríguez-Vallarte, M. C. and Salgado, G.: Contact and Frobenius solvable Lie algebras with abelian nilradical. Comm. Algebra 46, no. 10, 4344-4354 (2018).

[2] Barajas, T.; Roque, E. and Salgado, G.: Principal derivations and codimension one ideals in contact and Frobenius Lie algebras. Comm. Algebra 47, no. 12, 5380-5391 (2019).

[3] Barría, J. and Halmos, P. R.: Vector bases for two commuting matrices. Linear and Multilinear Algebra 27, no. 3, 147-157 (1990).

[4] Blazić, N. and Vukmirović, S.; Four-dimensional Lie algebras with a parahypercomplex structure. Rocky Mountain J. Math. 40 (2010), no. 5, 1391-1439.

[5] Bordemann, M.; Medina, A. and Ouadfel, A.; Le groupe affine comme variété symplectique. Tohoku Math. J. (2) 45, no. 3, 423-436 (1993).

[6] Campoamor-Stursberg, R.; Symplectic forms on six-dimensional real solvable Lie algebras. I. Algebra Colloq. 16, no. 2, 253-266 (2009).

[7] Diatta, A. and Manga, B.: On properties of principal elements of Frobenius Lie algebras. J. Lie Theory 24, no. 3, 849-864 (2014).

[8] Diatta, A.: Left invariant contact structures on Lie groups. Differential Geom. Appl. 26, no. 5, 544-552 (2008).

[9] Diatta, A.: Riemannian geometry on contact Lie groups. Geom. Dedicata 133, 83-94 (2008).
[10] Diatta, A. and Medina, A.: Classical Yang-Baxter equation and left invariant affine geometry on Lie groups. Manuscripta Math. 114, no. 4, 477-486 (2004).

[11] Dixmier, J.: Sous-anneaux abéliens maximaux dans les facteurs de type fini. Ann. of Math. (2) 59, 279-286 (1954).

[12] Gerstenhaber, M.: On dominance and varieties of commuting matrices. Ann. of Math. (2) 73, 324-348 (1961).

[13] Guralnick, R. M. and Sethuraman, B. A.: Commuting pairs and triples of matrices and related varieties. Linear Algebra Appl. 310, no. 1-3, 139-148 (2000).

[14] Guralnick, R. M.: A note on commuting pairs of matrices. Linear and Multilinear Algebra 31, no. 1-4, 71-75 (1992).

[15] Han, Y.: Commuting triples of matrices. Electron. J. Linear Algebra 13, 274-343 (2005).

[16] Harish-Chandra, The characters of semisimple Lie groups. Trans. Amer. Math. Soc., 83 (1956), 98-163.

[17] Holbrook, J. and O'Meara, K. C.: Some thoughts on Gerstenhaber’s theorem. Linear Algebra Appl. 466, 267-295 (2015).

[18] Holbrook, J. and Omladic, M.: Approximating commuting operators. Linear Algebra Appl. 327, no. 1-3, 131-149 (2001).

[19] Jacobson, N.: Schur’s theorems on commutative matrices. Bull. Amer. Math. Soc. 50, no. 6, 431-436 (1944).

[20] Kandić, M. and Sivic, K.: On the dimension of the algebra generated by two positive semi-commuting matrices. Linear Algebra Appl. 512, 136-161 (2017).

[21] Kostant B.: On the conjugacy of real Cartan subalgebras. I. Proc. Nat. Acad. Sci. U.S.A. 41, 967-970 (1955).

[22] Lichnerowicz, A. and Medina, A.: On Lie groups with left-invariant symplectic or Kählerian structures. Lett. Math. Phys. 16, no. 3, 225-235 (1988).

[23] Kreuzer, M. and Robbiano, L.: Computational linear and commutative algebra. Springer, Cham, 2016. xviii+321 pp

[24] Lucon D.: Commutation avec un endomorphisme. Revue de Math. Spé, n. 8, Vuibert 1983.

[25] Neubauer, M. G. and Sethuraman, B. A.: Commuting pairs in the centralizers of 2-regular matrices. J. Algebra 214, no. 1, 174-181 (1999).

[26] O’Meara, K. C.: The Gerstenhaber problem for commuting triples of matrices is "decidable". Comm. Algebra 48, no. 2, 453-466 (2020).

[27] Rajchgot, J. and Satriano, M.: New classes of examples satisfying the three matrix analog of Gerstenhaber’s theorem. J. Algebra 516, 245-270 (2018).
Schur I.: *Zur Theorie vertauschbaren Matrizen.* J. Reine Angew. Math. 130, 66-76 (1905).

Sethuraman, B. A.: *The algebra generated by three commuting matrices.* Math. Newsl. 21, no. 2, 62-67 (2011).

Sivic, K.: *On varieties of commuting triples III.* Linear Algebra Appl. 437, no. 2, 393-460 (2012).

Sivic, K.: *On varieties of commuting triples II.* Linear Algebra Appl. 437, no. 2, 461-489 (2012).

Sivic, K.: *On varieties of commuting triples.* Linear Algebra Appl. 428, 2006-2029 (2008).

Snow, J. E.: *Invariant complex structures on four dimensional solvable real Lie groups.* Manuscripta Math. 66, 397-412 (1990).

Sugiura, M.: *Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras.* J. Math. Soc. Japan 11, 374-434 (1959).

Wadsworth, A. R.: *The algebra generated by two commuting matrices.* Linear and Multilinear Algebra 27, no. 3, 159-162 (1990).

Winternitz, P.: *Subalgebras of Lie algebras. Example of sl(3,R).* Symmetry in physics, 215-227, CRM Proc. Lecture Notes, 34, Amer. Math. Soc., Providence, RI, 2004.