Asymptotic bit frequency in Fibonacci words

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Abstract

It is known that binary words containing no $k$ consecutive 1s are enumerated by $k$-step Fibonacci numbers. In this note we discuss the expected value of a random bit in a random word of length $n$ having this property.

1 Introduction

For $n \geq 0$ and $k \geq 2$, we denote by $B_n(1^k)$ the set of length $n$ binary words avoiding $k$ consecutive 1s. For example, we have

$B_4(11) = \{0000, 0001, 0010, 0100, 1000, 1001, 1010\}$, and
$B_4(111) = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101\}$.

It is well known, see Knuth [12, p. 286], that $B_n(1^k)$ is enumerated by the $k$-step Fibonacci numbers, precisely $|B_n(1^k)| = f_{n+k,k}$, where $f_{n,k}$ is defined, following Miles [14] as

$$ f_{n,k} = \begin{cases} 0 & \text{if } 0 \leq n \leq k - 2, \\ 1 & \text{if } n = k - 1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases} $$

Denote by $v_{n,k}$ the frequency (also called popularity) of 1s in $B_n(1^k)$, i.e. the total number of 1s in all words of $B_n(1^k)$. For instance, $v_{4,2} = 10$ and $v_{4,3} = 22$. The ratio of frequency of 1s to the overall number of bits in words of $B_n(1^k)$ is

$$ \alpha_{n,k} = \frac{v_{n,k}}{n \cdot |B_n(1^k)|}, $$

and it equals the expected value of a random bit in a random word from $B_n(1^k)$. In [2], the authors left without proof the fact that, for any $k \geq 2$, $\lim_{n \to \infty} \alpha_{n,k}$ converges to a
non-zero value as $n$ grows. This note is devoted to proving this fact, which apart from its interest en soi has practical counterparts. Indeed, words in $B_n(1^k)$ play a critical role in some telecommunication frame synchronization protocols, see for example [1, 3, 5], or in particular Fibonacci-like interconnection networks [8].

Our discussion is based on the bivariate generating function

$$F_k(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{n-\lfloor \frac{n}{k} \rfloor} a_{n,m} x^n y^m$$

whose coefficient $a_{n,m}$ equals the number of words from $B_n(1^k)$ containing exactly $m$ 1s. For $k = 2$ and $k = 3$, Table 1 presents some values of $a_{n,m}$ for small $n$ and $m$.

| $m \setminus n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----------------|---|---|---|---|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1               | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2               | 1 | 3 | 6 | 10 | 15 | 21 | 28 |   |   |
| 3               | 1 | 4 | 10 | 20 | 35 |   |   |   |   |
| 4               | 1 | 5 | 15 |   |   |   |   |   |   |
| 5               |   |   |   |   |   |   |   |   |   |

| $m \setminus n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----------------|---|---|---|---|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1               | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2               | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 |   |
| 3               | 2 | 7 | 16 | 30 | 50 | 77 |   |   |   |
| 4               | 1 | 6 | 19 | 45 | 90 |   |   |   |   |
| 5               | 3 | 16 | 51 |   |   |   |   |   |   |

Table 1: First few values of $a_{n,m}$ for $k = 2$ (left) and $k = 3$.

## 2 Main result

Proposition 1 gives the expression of the generating function $F_k(x, y)$. Even though this result is already obtained in [2], in order to make the paper self-contained we give an alternative proof of it. Then we calculate the generating functions for the frequency of 1s and for the overall number of bits in $B_n(1^k)$ by means of classic generating functions manipulations (Propositions 2). Applying Theorem 4.1 from [16], after ensuring that its conditions are satisfied, we obtain the main result of this note, Theorem 1. The evolution of the random bit expectation for $k = 2$ and $k = 3$ is presented on Figure 1 for small values of $n$. And numerical estimations for the limit value ($n \to \infty$) of the random bit expectation, for small values of $k$ are given in Table 2.

**Proposition 1 ([2]).**

$$F_k(x, y) = \frac{y \left( 1 - (xy)^k \right)}{y - xy^2 - xy + (xy)^{k+1}}.$$  

**Proof.** The set $B(1^k) = \bigcup_{n=0}^{\infty} B_n(1^k)$ respects the following recursive decomposition

$$B(1^k) = 1_{k-1} \cup \left( \bigcup_{i=0}^{k-1} \left( 1^i \cdot B(1^k) \right) \right)$$
where $\mathbf{I}_{k-1} = \bigcup_{i=0}^{k-1} \{1^i\}$ is the set of words in $\mathcal{B}(1^k)$ containing no 0s, and $\cdot$ denotes the concatenation. Note that the empty word also lies in $\mathbf{I}_{k-1}$. The claimed generating function is the solution of the following functional equation

$$F_k(x, y) = \sum_{i=0}^{k-1} x^i y^i + F_k(x, y) \sum_{i=0}^{k-1} x^{i+1} y^i.$$  

In the proof of Theorem 1 we need the following easy to derive results.

**Proposition 2.**

- $P_k(x) = \left. \frac{\partial F_k(x, y)}{\partial y} \right|_{y=1}$ is the generating function where the coefficient of $x^n$ is the frequency of 1s in $\mathcal{B}_n(1^k)$. We have
  $$P_k(x) = \frac{x \cdot \sum_{i=0}^{k-2} (i + 1) x^i}{(x^k + x^{k-1} + \cdots + x^2 + x + 1)^2}.$$  

- $T_k(x) = x \frac{\partial F_k(x, 1)}{\partial x}$ is the generating function where the coefficient of $x^n$ equals the total number of all bits in $\mathcal{B}_n(1^k)$. We have
  $$T_k(x) = x \frac{\left( \sum_{i=0}^{k-2} (2i + 2) x^i + \sum_{i=k-1}^{2k-2} (2k - i - 1) x^i \right)}{(x^k + x^{k-1} + \cdots + x^2 + x + 1)^2}.$$  

Every root $r$ of a polynomial $h(x)$ of degree $n$ with a non-zero constant term corresponds to the root $1/r$ of its negative reciprocal $-x^n h(1/x)$. The denominator of both $P_k(x)$ and $T_k(x)$ involves $x^k + x^{k-1} + \cdots + x^2 + x - 1$ and its negative reciprocal is $x^k - x^{k-1} - \cdots - x^2 - x - 1$ which is known in the literature as Fibonacci polynomial, see for instance [6, 7, 9, 10, 11, 13, 14, 15, 19] and references therein. In particular, Dubeau
Table 2: Numerical estimations for the limit of the expected value of a random bit in a random word from $B_n(1^k)$, $n \to \infty$.

| $k$ | Limit of the expected bit value |
|-----|---------------------------------|
| 2   | 0.276393202250021               |
| 3   | 0.381580077680607               |
| 4   | 0.433657112297348               |
| 5   | 0.46207383180840                |
| 6   | 0.478227505713290               |
| 7   | 0.487545982771861               |
| 8   | 0.492928265543398               |
| 9   | 0.496019724266083               |
| 10  | 0.497779940783496               |
| 11  | 0.498772398758879               |
| 12  | 0.499326557312936               |
| 13  | 0.499633184444604               |

proved [7, Theorem 1] that its root of the largest modulus is $\varphi_k = \lim_{n \to \infty} f_{n+1,k}/f_{n,k}$, the generalized golden ratio, and $\varphi_k$ approaches 2 when $k \to \infty$ [7, Theorem 2]. Wolfram [19, Lemma 3.6] showed that any other root $r$ of the Fibonacci polynomial satisfies $3^{-1/k} < |r| < 1$. See Figure 2 for an illustration of this fact. Moreover, Corollary 3.8 in [19] proves that Fibonacci polynomial is irreducible over $\mathbb{Q}$. In order to refer later to them we summarize these results in the next proposition.

**Proposition 3.** The polynomial $g_k(x) = x^k + x^{k-1} + \cdots + x^2 + x - 1$ is irreducible over $\mathbb{Q}$, its root of the smallest modulus is unique and equal to $1/\varphi_k$.

The next lemma says that both fractions representing $P_k(x)$ and $T_k(x)$ are irreducible.

**Lemma 1.** The polynomials $\sum_{i=0}^{k-2} (i+1)x^i$ and $x^k + x^{k-1} + \cdots + x^2 + x - 1$ are relatively prime; and so are $\sum_{i=0}^{k-2} (2i+2)x^i + \sum_{i=k-1}^{2k-2} (2k-i-1)x^i$ and $x^k + x^{k-1} + \cdots + x^2 + x - 1$.

**Proof.** The polynomial $x^k + x^{k-1} + \cdots + x^2 + x - 1$ is irreducible due to Proposition 3. It does not divide $\sum_{i=0}^{k-2} (i+1)x^i$ as it has a greater degree. And it also cannot divide
Figure 2: Roots of the polynomial $x^k - x^{k-1} - \cdots - x^2 - x - 1$ (the negative reciprocal of $g_k(x)$) for certain values of $k$.

$$\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i)x^i$$ as the latter does not have any positive real roots.

From Propositions 2, 3, Dubeau’s results [7], and Lemma 1 we have:

**Lemma 2.** Both generating functions $P_k(x)$ and of $T_k(x)$ have the same and unique pole of the smallest modulus with multiplicity 2. The pole equals $1/\varphi_k$, where $\varphi_k$ is the generalized golden ratio.

For our main result of this note we need the Theorem 4.1 from [16]:

**Theorem 4.1 from [16].** Assume that a rational generating function $\frac{f(x)}{g(x)}$, with $f(x)$ and $g(x)$ relatively prime and $g(0) \neq 0$, has a unique pole $1/\beta$ of the smallest modulus. Then, if the multiplicity of $1/\beta$ is $\nu$, we have

$$[x^n] \frac{f(x)}{g(x)} \sim \nu \frac{(\beta \nu)^n f(1/\beta)}{g^{(\nu)}(1/\beta)} \beta^n n^{\nu-1}. $$
Both $P_k(x)$ and $T_k(x)$ are rational generating functions, and by Lemmas 1 and 2 they fulfill the conditions in the above theorem, so

$$[x^n]P_k(x) \sim 2n\varphi_k^{n+2} \cdot \frac{x\left(\sum_{i=0}^{k-2}(i+1)x^i\right)}{((x^k + x^{k-1} + \cdots + x^2 + x - 1)^n)'|_{x=1/\varphi_k}}$$

$$[x^n]T_k(x) \sim 2n\varphi_k^{n+2} \cdot \frac{x\left(\sum_{i=0}^{k-2}(2i+2)x^i + \sum_{i=k-1}^{2k-2}(2k-i-1)x^i\right)}{((x^k + x^{k-1} + \cdots + x^2 + x - 1)^n)'|_{x=1/\varphi_k}}.$$

The expected value of a random bit in a random word from $B_n(1^k)$ is $[x^n]P_k(x)/[x^n]T_k(x)$. Taking the limit, we obtain:

**Theorem 1.** The expected value of a random bit in a random word from $B_n(1^k)$ tends to

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{n}{|B_n(1^k)|} \frac{v_{n,k}}{v_{n,k}} = \frac{1}{2},$$

where $\varphi_k = \lim_{n \to \infty} f_{n+1,k}/f_{n,k}$ is the generalized golden ratio, in particular $\varphi_2$ is the golden ratio.

See Table 2 for some numerical estimations of the result obtained in the previous theorem. This result involves the generalized golden ratio. More than 20 years ago it was conjectured by Wolfram [19] that the Galois group of the polynomial $x^k - x^{k-1} - \cdots - x^2 - x - 1$ is the symmetric group $S_k$, and so there is no algebraic expression for $\varphi_k$ (the root of the largest modulus of this polynomial) when $k \geq 5$. In case of even or prime $k$ the conjecture was settled by Martin [13]. Cipu and Luca [6] showed that $\varphi_k$ cannot be constructed by ruler and compass for $k \geq 3$. Nevertheless, good approximations are available, for instance Hare, Prodinger and Shallit [11] expressed $\varphi_k$ and $1/\varphi_k$ in terms of rapidly converging series.

The generalized golden ratio $\varphi_k$ tends to 2 as $k$ grows, and we deduce the following.

**Corollary 1.** The limit of the expected bit value of binary words avoiding $k$ consecutive 1s, whose length tends to infinity, approaches 1/2 as $k$ grows:

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{v_{n,k}}{n \cdot |B_n(1^k)|} = \frac{1}{2}.$$
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\(^1\)See the original discussions here https://math.stackexchange.com/questions/4120185 and here https://math.stackexchange.com/questions/4125568.
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