CORE FORGING AND LOCAL LIMIT THEOREMS FOR THE $k$-CORE OF RANDOM GRAPHS

AMIN COJA-OGLHAN*, OLIVER COOLEY**, MIHYUN KANG** AND KATHRIN SKUBCH

ABSTRACT. We establish a multivariate local limit theorem for the order and size as well as several other parameters of the $k$-core of the Erdős-Rényi random graph. The proof is based on a novel approach to the $k$-core problem that replaces the meticulous analysis of the 'peeling process' by a generative model of graphs with a core of a given order and size. The generative model, which is inspired by the Warning Propagation message passing algorithm, facilitates the direct study of properties of the core and its connections with the mantle and should therefore be of interest in its own right.

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1. INTRODUCTION

1.1. The $k$-core problem. The $k$-core of a graph $G$ is the largest subgraph of minimum degree at least $k$. It can be determined algorithmically by the peeling process that removes an arbitrary vertex of degree less than $k$ while there is one. In one of the most influential contributions to the theory of random graphs Pittel, Spencer and Wormald analysed the peeling process on the Erdős-Rényi random graph via the method of differential equations [26]. They determined the precise threshold $d_k$ from where the $k$-core is non-empty w.h.p. as well as the asymptotic order (number of vertices) and size (edges) of the $k$-core for $d > d_k$, $k \geq 3$. The case $k \geq 3$ is very different from the case $k = 2$, as the 2-core simply emerges continuously along with the giant component. By contrast, a most remarkable feature of the case $k \geq 3$, first observed by Łuczak [20, 21], is that the order of the $k$-core leaps from 0 to a linear number of vertices at the very moment that the $k$-core becomes non-empty.

Since the seminal work of Pittel, Spencer and Wormald several alternative derivations of the $k$-core threshold have been put forward [5, 10, 11, 16, 18, 23, 28, 29]. Some of these extend to hypergraphs and/or given degree sequences. Additionally, establishing a bivariate central limit theorem, Janson and Łuczak [17] studied the joint limiting distribution of the order and size of the $k$-core. Further aspects of the problems that have been studied include the 'depth' of the peeling process as well as the width of the critical window [7, 13, 14].

The great interest in the $k$-core problem is due not least to the many connections that the problem has with other questions in combinatorics and computer science. For example, coinciding with the largest $k$-connected subgraph w.h.p., the $k$-core problem is a natural generalisation of the 'giant component' problem [3]. Cores also play a very important role in the study of random constraint satisfaction problems such as random $k$-SAT or random graph colouring. In these problems the emergence of a core-like structure causes freezing, a particular kind of long-range correlations that has been associated with the algorithmic difficulty of finding solutions [1, 24]. In addition, the hypergraph version of the $k$-core holds the key to understanding problems such as random XOR-SAT, hypergraph orientability and cuckoo hashing [8, 12, 25]. The problem plays an important role in coding theory as well [19].

While most of the previous work on the $k$-core problem has been based on tracing the peeling process, the only exception being [28], reliant on branching processes, in the present paper we develop a very different approach. We devise a generative model for random graphs with a $k$-core of a given order and size. Formally, we develop a randomised sampling algorithm Forge that produces a graph with a core of a given desired order and size (under certain reasonable assumptions on the input parameters). The output distribution of Forge converges in total variation to the distribution of an Erdős-Rényi random graph given the order and size of the $k$-core. Because the randomised construction employed by Forge is surprisingly simple, we can immediately read off the asymptotic number of graphs with a $k$-core of a given order and size. As an application, we obtain a bivariate local limit theorem for the distribution of the order and size of the $k$-core of the Erdős-Rényi random graph. This result
substantially sharpens the central limit theorem of Janson and Luczak [17]. Additionally, the sampling algorithm completely elucidates the way the \( k \)-core is embedded into the random graph, a question on which we obtained partial results in an earlier paper via the formalism of local weak convergence [4]. We expect that this structural insight will facilitate the future study of the \( k \)-core and of similar structures arising in random constraint satisfaction problems.

The paper is almost entirely self-contained and most of the proofs are elementary. The only (mildly) advanced ingredient that we use is a local limit theorem for sums of independent random variables [6]. In particular, we do not rely on any of the previous results on the \( k \)-core, not even the one on the location of the \( k \)-core threshold.

1.2. A local limit theorem. Let \( G = G(n, m) \) be the random graph with \( n \) vertices and \( m = \lfloor dn/2 \rfloor \) edges, where \( d > 0 \) is independent of \( n \). Moreover, for an integer \( k \geq 3 \) consider the function

\[
\phi_{d,k} : [0, 1] \to [0, 1], \quad x \mapsto P(\text{Po}(dx) \geq k - 1) = 1 - \exp(-dx) \sum_{j=0}^{k-2} \frac{(dx)^j}{j!}.
\]  

(1.1)

Clearly, \( \phi_{d,k} \) is continuous and \( \phi_{d,k}(0) = 0 \). Let \( p = p(d, k) \in [0, 1] \) be the largest fixed point of \( \phi_{d,k} \) and set

\[
d_k = \inf \{ d > 0 : p(d, k) > 0 \}.
\]

(1.2)

In addition, define

\[
q = q(d, k) = P(\text{Po}(dp) = k - 1|\text{Po}(dp) \geq k - 1) = \frac{d^{k-1}p^{k-2} \exp(-dp)}{(k-1)!}.
\]

(1.3)

Theorem 1.1. Suppose that \( k \geq 3, d > d_k \) and fix any number \( \xi > 0 \). Then \( 1 - (k-1)q > 0 \) and the \( 2 \times 2 \) matrix

\[
\mathcal{Q} = (1 - (k-1)q)^{-2} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}
\]

(1.4)

is regular. Further, let \( X \) be the order of the \( k \)-core of \( G \) and let \( Y \) be its size. Then uniformly for all integers \( x, y \) such that \( |x - np(1-q)| + |y - mp^2| \leq \xi \sqrt{n} \) we have

\[
P\left[ X = x, Y = y \right] \sim \frac{\sqrt{\det \mathcal{Q}}}{\pi d n} \exp \left\{ -\frac{n}{2} \begin{pmatrix} x/n - p(1-q) \\ y/m - p^2 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} x/n - p(1-q) \\ y/m - p^2 \end{pmatrix}^T \right\}.
\]

(1.5)

The formula (1.5) determines the asymptotic probability that the order and size \( X, Y \) of the \( k \)-core attain specific values within \( O(\sqrt{n}) \) of their expectations. Hence, Theorem 1.1 provides a bivariate local limit theorem for the order and size of the \( k \)-core. This result is significantly stronger than a mere central limit theorem stating that \( X, Y \) converge jointly to a bivariate Gaussian because (1.5) actually yields the asymptotic joint probabilities. Still it is worthwhile pointing out that Theorem 1.1 immediately implies a central limit theorem.

Corollary 1.2. Suppose that \( k \geq 3 \) and \( d > d_k \), let \( \mathcal{Q} \) be the matrix from (1.4) and let \( X, Y \) be the order and size of the \( k \)-core of \( G \). Then \( n^{-1/2}(X - np(1-q), 2(Y - mp^2)/d) \) converges in distribution to a bivariate Gaussian with mean 0 and covariance matrix \( \mathcal{Q}^{-1} \).

A statement similar to Corollary 1.2 was previously established by Janson and Luczak [17] via a careful analysis of the peeling process. However, they did not obtain an explicit formula for the covariance matrix. Indeed, although the formula for \( \mathcal{Q} \) is a bit on the lengthy side, the only non-algebraic quantity is \( p = p(d, k) \), the solution to the fixed point equation. By contrast, the formula of Janson and Luczak implicitly characterises the covariance matrix in terms of another stochastic process, and they do not provide a local limit theorem.

The number \( d_k \) from (1.2) does, of course, coincide with the \( k \)-core threshold first derived in [26]. The formula given in that paper looks a bit different but we pointed out the equivalence in [4]. In fact, it is very easy to show...
that the $k$-core is empty w.h.p. if $d < d_k$. On the other hand, Corollary 1.2 implies immediately that for $d > d_k$ the $k$-core contains $n(p(1 - q) + o(1)) = \Omega(n)$ vertices w.h.p. Since the proofs of Theorem 1.1 and Corollary 1.2 do not assume knowledge of the $k$-core threshold, we thus obtain a new derivation of the threshold result.

1.3. Warning Propagation. A key idea of the present paper is to investigate not merely the $k$-core itself but also the “surrounding structure” of the graph from the right angle. As it turns out, the necessary additional parameters can be set out concisely by way of the Warning Propagation message passing algorithm introduced in non-rigorous physics work on random constraint satisfaction problems [22]. The link between Warning Propagation and the $k$-core problem is well known [4, 15, 22]. The important feature that we highlight and exploit here is that the Warning Propagation messages allow us to describe succinctly how the $k$-core is embedded into the rest of the random graph, the mantle. More precisely, as we pointed out in [4] Warning Propagation gives rise naturally to a few further parameters apart from the order and size of the $k$-core that are of combinatorial significance but that, unfortunately, get lost in the peeling process. The main result of the paper, Theorem 1.4 below, provides a local limit theorem for the joint distribution of all these parameters.

Warning Propagation assigns messages to edges, one in either direction, and marks to vertices. The messages and the marks are $\{0, 1\}$-valued. Initially all messages are set to 1. Thus, for a graph $G = (V(G), E(G))$ we let $\mu_{v \to w}(0(G)) = 1$ for all pairs $(v, w) \in V(G) \times V(G)$ such that $(v, w) \in E(G)$. Subsequently the messages get updated in parallel rounds. That is, writing $\partial v = \partial_G v$ for the neighbourhood of vertex $v$ and abbreviating $\partial v \setminus w = \partial v \setminus \{w\}$, we inductively define

$$
\mu_{v \to w}(t + 1)(G) = 1\left\{\sum_{u \in \partial v \setminus w} \mu_{u \to v}(t)(G) \geq k - 1\right\}
$$

for integers $t \geq 0$. (1.6)

We emphasise that the messages are directed and quite possibly $\mu_{v \to w}(t)(G) \neq \mu_{w \to v}(t)(G)$. Additionally, the mark of $v \in [n]$ at time $t \geq 0$ is defined as

$$
\mu_v(t)(G) = 1\left\{\sum_{u \in \partial v} \mu_{u \to v}(t)(G) \geq k\right\}.
$$

Clearly, $\mu_{v \to w}(t + 1)(G) \leq k$ for all $t \geq 0$ and all $v, w$. Hence, $\mu_v(t)(G) \leq k$ for all $v$ and the limits

$$
\mu_v(G) = \lim_{t \to \infty} \mu_v(t)(G), \quad \mu_{v \to w}(G) = \lim_{t \to \infty} \mu_{v \to w}(t)(G)
$$

exist for all $v, w$. Denote by

$$
\mu(G) = (\mu_v(G), \mu_{v \to w}(G))_{v \in V(G), (v, w) \in E(G)}.
$$

The following observation is immediate from the construction.

**Fact 1.3** (Lemma 3.1). Let $G$ be a graph.

1. A vertex $u$ belongs to the $k$-core of $G$ iff $\mu_u(G) = 1$.
2. An edge $(v, w)$ links two vertices of the $k$-core iff $\mu_{v \to w}(G) = \mu_{w \to v}(G) = 1$.

The messages encode how the $k$-core is embedded into the mantle. To see this, we introduce

$$
\mathcal{N}_0(G) = \{v: \sum_{u \in \partial v} \mu_{u \to v}(G) \leq k - 2\},
$$

$$
\mathcal{N}_k(G) = \{v: \sum_{u \in \partial v} \mu_{u \to v}(G) \leq k - 1\},
$$

$$
\mathcal{N}_1(G) = \{v: \sum_{u \in \partial v} \mu_{u \to v}(G) \geq k\},
$$

$$
\mathcal{M}_{xy}(G) = \{(v, w) \in V(G)^2 : (v, w) \in E(G), \mu_{v \to w}(G) = x,\mu_{v \to w}(G) = y\}.
$$

Fact 1.3 shows that $\mathcal{N}(G)$ is just the vertex set of the $k$-core. Moreover, the vertices in $\mathcal{N}_k(G)$ miss out on core membership by just one incoming 1-message. In effect, if they receive a 0 message from a neighbour, they send back a 1, and vice versa. By contrast, the vertices in $\mathcal{N}_0(G)$ send out 0 messages to all their neighbours, although they may receive up to $k - 2$ many 1-messages. Further, Fact 1.3 implies that $(v, w) \in \mathcal{M}_{11}(G)$ iff the edge $(v, w)$ connects two vertices inside the $k$-core. Similarly, if $(v, w) \in \mathcal{M}_{10}(G)$, then $v \in \mathcal{N}_k(G)$ and $w \in \mathcal{N}_0(G)$ and vice versa. Finally, $(v, w) \in \mathcal{M}_{01}(G)$ iff $(w, v) \in \mathcal{M}_{00}(G)$.

Given this Warning Propagation-inspired decomposition of the vertices and edges, the key parameters of the $k$-core problem are

$$
n_0(G) = |\mathcal{N}_0(G)|, \quad n_k(G) = |\mathcal{N}_k(G)|, \quad n_1(G) = |\mathcal{N}_1(G)|, \quad m_{xy}(G) = |\mathcal{M}_{xy}(G)|.
$$

Of course, by Fact 1.3 the order of the $k$-core equals $n_1(G)$ and its size is equal to $m_{11}(G)/2$. Further, both $m_{00}(G)$ and $m_{11}(G)$ are even and

$$
n_0(G) + n_1(G) + n_k(G) = |V(G)|, \quad m_{00}(G) = m_{10}(G), \quad m_{00}(G) + m_{01}(G) + m_{10}(G) + m_{11}(G) = 2|E(G)|.
$$

(1.8)
Then we have the following local limit theorem for

**Theorem 1.4.**

\[ Q_{11} = -\frac{1}{d} \left( (dk^2 - 2dk + d) p^2 q^4 + (2d^2 k - d^2) p^3 - (2d^2 k - d^2 + 2d^2 k - 2d) k p^2 + (dk^2 - 2dk + d) p q^3 \right. \\
\left. - dpq + \left( (d^2 + 2d) p^4 - (d^3 + 2d^2 + 2d - 4d) p^3 + ((d+2)k^2 - d^2 + 2d^2 k + 2) p^2 \right. \\
\left. - (dk^2 - 2dk + d) p q^2, \right. \\
\left. Q_{12} = (k^2 - 2k + 1)p^2 q^4 + (2d + 2k^2 - d^2) p^3 - (2d + 2k^2 - d^2 + 2d + 2k^2 - 2d - 1) p^2 \right. \\
\left. - (k^2 - 2k + 1)p q^2 \right. \\
\left. + 2d p^3 - (d + k)p^2 + (k - 1) p q, \right. \\
\left. Q_{13} = -\frac{1}{d} \left[ (2(dk - d) p^4 + 2d p + 2(d + k + 2k - d - 1) p^3 - 3(dk - d) p^2 + (dk - k - 2k - d + 1)p q^2 \right. \\
\left. + 2d^2 + d p^4 - 3d^2 + 2(d + 1)k + 2d + 2k^2 - d)^2 p^3 + (d^2 + 3d + 2k^2 - 2d + 1)p^2 - (d + 1)k - 1)p q, \right. \\
\left. Q_{14} = \frac{2}{d} \left[ (d^2 + 2d + 2k + 1) p^2 q^4 + (2d + 2k + 2k - 1) p^4 + (dk - d^2 + 1) p^3 + (dk - d^2 + 1) p^2 \right. \\
\left. + (d^2 + 2d - 2d) p^4 - (d^2 + 2d + 2k + 1) p^3 + (2d + 2k^2 - 2d + 2d^2) p^2 - (k^2 - 1)p q^2 - p^2 \right. \\
\left. - (2d^3 - 2d + k)p^2 + (2k + 1)p q + p, \right. \\
\left. Q_{23} = 2d^3 + 2(d + 1)p^4 - 2d + 3k^2 - 3(k - 1)p^2 + (k - 1)p q^2 - 3p^2 \right. \\
\left. + (2d + 1)p^4 - 3d^2 + 2k^2 + 3d^2 p^2 + (d + 3k)p^2 - kp q + p, \right. \\
\left. Q_{24} = -2d^3 - 2(d + 1)p^4 + (k - 1)p^3 \right. \\
\left. q^3 + 2p^2 - 2(d + 1)p^4 - (d + k)p^3 + (k - 1)p^2 q, \right. \\
\left. Q_{33} = -\frac{1}{d} \left[ (2d + 1)p^4 - 2d + 4p^2 + (2k^2 - 2k - 1)p^3 + (2 + 2k - 2) p^2 + (k^2 - 2k - 1)p q^2 \right. \\
\left. + (k - 1)p^4 - 4(k - 1)p^3 + (2k^2 - 2k + 1)p q^2, \right. \\
\left. Q_{34} = \frac{2}{d} \left[ (k^2 - 2k + 1)p^4 q^2 + (2d + 1)p^4 - 3d^2 + 4dp^2 + (2k^2 - 2k + 1)p q^2 \right. \\
\left. - (2d + 1)p^4 + 3d - 2k - 1)p^3 + (k^2 - 2k + 1)p q^2 + (k^2 - 2k + 1)p^2 q^2 - p^2 \right. \\
\right)

**Figure 1.** The matrix entries \( Q_{ij} \).

In effect, the seven parameters

\[ n(G) = (n_0(G), n_0(G), n_1(G)) \quad \text{and} \quad m(G) = (m_{00}(G), m_{01}(G), m_{10}(G), m_{11}(G)) \]

boil down to the four variables

\[ N(G) = (n_*(G), n_1(G)) \quad \text{and} \quad M(G) = (m_{10}(G), m_{11}(G)). \]

Then we have the following local limit theorem for \( N(G), M(G) \).

**Theorem 1.4.** Suppose that \( k \geq 3, d > d_k \) and \( \xi > 0 \). Then the symmetric 4 \( \times \) 4-matrix

\[ Q = \frac{1}{(1 - (k - 1)p)^2} \left[ Q_{ij} \right]_{1 \leq i, j \leq 4} \]

with \( Q_{ij} \) from Figure 1 is regular and uniformly for all integer vectors \( N = (n_*, n_1), M = (m_{10}, m_{11}) \) such that \( m_{11} \) is even and

\[ |n_* - nv_*| + |n_1 - nv_1| + |m_{10} - 2m_010| + |m_{11} - 2m_{111}| \leq \xi \sqrt{n} \]

we have

\[ \mathbb{P} \left[ N(G) = N, M(G) = M \right] = \frac{1}{2\pi d n^2} \frac{\exp \left( \frac{n}{2} \left( Q^{-1} \Delta(N, M), \Delta(N, M) \right) \right)}{\det Q} + o(n^{-2}) \]
where

\[
\Delta(N, M) = \begin{pmatrix}
\frac{n_x}{n} & p q \\
\frac{n_1}{n} & p (1 - q) \\
\frac{m_{10}}{(2m)} & p (1 - p) \\
\frac{m_{11}}{(2m)} & p^2
\end{pmatrix}.
\]

(1.11)

Theorem 1.1 is immediate from Theorem 1.4 by just projecting on \(n_1(G)\) and \(m_{11}(G)/2\).

1.4. Techniques, outline and further related work. We do not prove Theorem 1.4 by analysing Warning Propagation on \(G\). Instead, we are going to employ the seven parameters supplied by Warning Propagation in order to establish a corresponding local weak convergence result. Thus, the offspring matrix of this 5-type branching process off the generative model. We believe that this approach is much simpler than the direct analysis of the peeling process as performed, e.g., in [2] for the hypergraph 2-core, and that it will find future applications, e.g., in the theory of random constraint satisfaction problems.

In Section 2 we present Warning Propagation and the sampling algorithm Forge. In Section 3 we outline the analysis of Forge and the counting argument that yields the asymptotic number of graphs with a given outcome of \(N, M\). The details of that analysis follow in the remaining sections.

1.5. Notation and preliminaries. With respect to general notation, we let \(G[S]\) denote the subgraph of a graph \(G = (V(G), E(G))\) induced on \(S \subset V(G)\). Moreover, the transpose of a matrix \(A\) is denoted by \(A^\top\) and for reals \(a_1, \ldots, a_s\) we let \(\text{diag}(a_1, \ldots, a_s)\) be the \(s \times s\) diagonal matrix with diagonal entries \(a_1, \ldots, a_s\).

In addition to the parameters \(p = p(d, k)\), which we defined as the largest fixed point of the function \(\phi_{d,k}\) from (1.1), and \(q\) from (1.3) we introduce

\[
\tilde{q} = \tilde{q}(d, k) = \mathbb{P}[\text{Po}(dp) = k - 2|\text{Po}(dp) \leq k - 2].
\]

(1.12)

The definitions of \(p\) and \(q\) ensure that

\[
\tilde{q} = \frac{(k - 1) q}{(1 - p)d}.
\]

(1.13)

Furthermore, a bit of calculus reveals the following.

Fact 1.5 ([4, Lemma 2.3.]). Let \(k \geq 3\) and \(d > d_k\) and let \(p\) be the largest fixed point of \(\phi_{d,k}\). Then

1. \(p \geq \frac{k - 2 + \sqrt{k - 2}}{d}\);
2. \(\frac{d}{dx} \phi_{d,k}(x)|_{x=p} = q(k-1) = \tilde{q}(1-p)d < 1\).
Throughout the paper we will frequently encounter truncated Poisson distributions. To be precise, for real numbers \( y, z > 0 \) we let \( \text{Po}_{z, y} \) denote the Poisson distribution \( \text{Po}(y) \) conditioned on the event that the outcome is at least \( z \). Thus,

\[
P[\text{Po}_{z, y} = \ell] = \frac{1 [\ell \geq z] y^\ell \exp(-y)}{\ell! P[\text{Po}(y) \geq z]} \quad \text{for any integer } \ell \geq 0.
\]

The distributions \( \text{Po}_{z, y}, \text{Po}_{z, x}, \text{Po}_{z, y} \) are defined analogously. We will also occasionally encounter the function

\[
\varphi_{\ell} : [0, 1] \rightarrow [0, 1], \quad y \mapsto P[\text{Po}(y) \geq \ell - 1] \quad (\ell \geq 3),
\]

whose derivatives work out to be

\[
\frac{\partial}{\partial y} \varphi_{\ell}(y) = \frac{y^{\ell-2}}{(\ell - 2)! \exp(y)}, \quad \frac{\partial^2}{\partial y^2} \varphi_{\ell}(y) = \frac{(\ell - y - 2) y^{\ell-3}}{(\ell - 2)! \exp(y)}.
\]

In particular, recalling \( \varphi_{d, k} \) from (1.1), we see that \( \varphi_{d, k}(x) = \varphi_{k}(d x) \) for all \( x \in [0, 1] \) and

\[
\frac{\partial}{\partial x_1} \varphi_{d, k}(x) = d! \frac{\partial}{\partial y} \varphi_{\ell}(y) \bigg|_{y = d - x} \quad (i \geq 0, \ k \geq 3).
\]

The following standard result shows that joint convergence to a family of independent Poisson variables can be established by way of calculating joint factorial moments.

**Theorem 1.6** ([3]). Let \( (X_n^{(1)})_{i \geq 1} \) be a family of random variables. If \( \lambda_i, i \geq 0 \) are such that for all \( r_1, \ldots, r_m \geq 0 \),

\[
\lim_{n \to \infty} E \left[ (X_n^{(1)})_{r_1} \cdots (X_n^{(m)})_{r_m} \right] = \lambda_1^{r_1} \cdots \lambda_m^{r_m},
\]

then \( (X_n^{(i)})_{i \geq 1} \to (Z_i)_{i \geq 1} \) in distribution, where \( Z_i \) are independent with distribution \( \text{Po}(\lambda_i) \).

Furthermore, in Section 5 we will need the following local limit theorem for sums of independent random variables.

**Theorem 1.7** ([6, Theorem 2.1]). Let \( \ell \geq 1 \). For \( n \geq 1 \) let \( X_{1,n}, \ldots, X_{n,n} \) be a sequence of independent \( \mathbb{N}^\ell \)-valued random variables. Let \( 1_r \in \mathbb{N}^\ell \) denote the vector whose \( r \)-th component is 1 and whose other components are 0. Assume that there is a constant \( c > 0 \) such that for all \( r \leq \ell \) and \( n \geq 1 \),

\[
\max_{k \in \mathbb{N}^\ell} \min \{ P\{X_{i,n} = k\}, P\{X_{i,n} = k + 1_r\} \} \geq c.
\]

Then for \( S_n = \sum_{i=1}^n X_{i,n} \) the following holds. Suppose that there is a vector \( \alpha \in \mathbb{R}^\ell \) such that \( n^{-1/2} (S_n - \alpha) \) converges in distribution to a multivariate normal distribution with mean 0 and covariance matrix \( D \). Then uniformly for all vectors \( k \in \mathbb{N}^\ell \),

\[
P(S_n = k) = \frac{1}{\sqrt{(2\pi n)^\ell \det D}} \exp \left( -\frac{n}{2} \left( \begin{bmatrix} k \n - a \end{bmatrix}, \begin{bmatrix} k \n - a \end{bmatrix} \right) D^{-1} \right) + o(n^{-\ell/2}).
\]

Additionally, we need a few basic combinatorial counting results. We recall that for an integer \( \ell \) the number of perfect matchings of the complete graph of order \( 2\ell \) is equal to

\[
(2\ell - 1)!! = \frac{(2\ell)!}{2^\ell \ell!}.
\]

Further, for \( s, t \in \mathbb{N} \) let \( \mathcal{S}(s, t) \) denote the Stirling number of the second kind.

**Theorem 1.8** ([27, Theorem 3]). For all \( s, t \in \mathbb{N} \) we have \( \mathcal{S}(s, t) \leq \frac{1}{t^{\ell-1}} \binom{t}{\ell} \).

We need the following upper bound on the number of labelled forests that comes in terms of the Stirling number.

**Theorem 1.9** ([9, Corollary 3.1]). The number of labelled forests on \( v \) vertices with exactly \( \ell \) leaves and exactly \( c \) components is upper bounded by

\[
\frac{v! \left( \begin{bmatrix} v - 1 \n - c - \ell \end{bmatrix} \right)}{\ell!} \mathcal{S}(v - c, v - \ell).
\]
The entropy of a probability distribution $\rho$ on a finite set $\Omega \neq \emptyset$ is defined as

$$H(\rho) = - \sum_{\omega \in \Omega} \rho(\omega) \ln(\rho(\omega)).$$

(1.17)

Further, we recall that for two probability distributions $\rho, \rho'$ on the same finite set $\Omega \neq \emptyset$ the Kullback-Leibler divergence is defined as

$$D_{\text{KL}}(\rho \parallel \rho') = \sum_{\omega \in \Omega} \rho(\omega) \ln \frac{\rho(\omega)}{\rho'(\omega)},$$

(1.18)

with the convention that $0 \ln 0 = 0 \ln \frac{0}{0} = 0$ and $D_{\text{KL}}(\rho \parallel \rho') = \infty$ if there is $\omega \in \Omega$ such that $\rho(\omega) = 0 < \rho'(\omega)$. The derivatives of a generic summand on the right hand side of (1.18) work out to be

$$\frac{\partial}{\partial x} x \ln \frac{x}{y} = 1 + \ln \frac{x}{y}, \quad \frac{\partial^2}{\partial x^2} x \ln \frac{x}{y} = \frac{1}{x}.$$  (1.19)

From here on we tacitly assume that $k \geq 3$ and $d > d_k$. We continue to use the notation from Sections 1.2 and 1.5 throughout the paper.

2. Core Forging

The key insight of the present paper is that the extra information provided by the Warning Propagation algorithm can easily be turned into a generative process for creating random graphs with a core of a given order and size (under certain reasonable assumptions). To set up this generative process, we need a few further parameters: let

$$\mu_0 = 1 - p^2, \quad \mu_0 = \mu_1 = p(1 - p), \quad \mu_1 = p^2,$$

$$\nu_0 = 1 - p, \quad \nu_0 = \nu_1 = p, \quad \nu_1 = p(1 - q),$$

$$\lambda = (\mu_0, \mu_1, \mu_1), \quad \mu = (\mu_0, \mu_0, \mu_1, \mu_1).$$

(2.1)

In light of Theorem 1.4 the (intended) semantics of $\nu, \mu$ is clear: $\nu_0$ is going to emerge as the expectation of $n_z(G)/n$ for $z \in \{0, 1, \ast\}$ and $\mu_{yz}$ as that of $m_{yz}(G)/(2m)$ for $y, z \in \{0, 1\}$.

Further, let us write $d_G(v)$ for the degree of vertex $v$ in a graph $G$ and let $d_{G,ab}(v)$ be the number of vertices $w \in \partial_G v$ such that $\mu_{w-v}(G) = a$ and $\mu_{v-w}(G) = b$ for $a, b \in \{0, 1\}$. Then it is immediate from the definitions (1.6), (1.7) of the Warning Propagation marks and messages that the sets $\mathcal{N}_0(G), \mathcal{N}_4(G), \mathcal{N}_1(G)$ can be characterised in terms of the degrees $d_{G,ab}$ as follows.

Fact 2.1. Let $G$ be a graph.

1. $v \in \mathcal{N}_0(G)$ iff $d_{G,10}(v) \leq k - 2$ and $d_{G,11}(v) = d_{G,01}(v) = 0.$
2. $v \in \mathcal{N}_4(G)$ iff $d_{G,11}(v) = k - 1$ and $d_{G,01}(v) = d_{G,00}(v) = 0.$
3. $v \in \mathcal{N}_1(G)$ iff $d_{G,11}(v) \geq k$ and $d_{G,01}(v) = d_{G,00}(v) = 0.$

Finally, introducing

$$\lambda_{00} = \lambda_{01} = d(1 - p), \quad \lambda_{10} = \lambda_{11} = dp,$$

(2.2)

we will see that the parameters $\lambda_{ab}$ govern the distributions of the degrees $d_{G,ab}(v)$, subject to the conditions listed in Fact 2.1.

We can now describe the randomised algorithm Forge that generates a graph $\hat{G}$ along with a set of ‘supposed’ Warning Propagation messages $\hat{\mu}$, see Figure 2. In the first step Forge randomly assigns each vertex a type $0, \ast, 1$ independently according to the distribution $\nu$. The second step generates a sequence $(\hat{d}_{ab}(v))_{a,b,v}$ of ‘pseudo-degrees’ by independently sampling from the conditional Poisson distributions with parameters $\lambda_{ab}$. Of course, in order to ultimately generate a graph with $m$ edges it had better be the case that the total degree sum come to $2m$, which step (3) checks. In addition, we require that the total 00 and 11-degree sums be even and that $\hat{\mu}_{10} = \hat{\mu}_{01}$. Hence, if $\hat{\mu}_{00}, \hat{\mu}_{01}, \hat{m}_{10}, \hat{m}_{11}$ fail to satisfy any of the conditions from (1.8), then the algorithm aborts. Since the $\hat{m}_{ab}$ are sums of independent random variables, we verify easily that the success probability of step (3) is $\Theta(n^{-1}).$

The next two steps of Forge use the $(\hat{d}_{ab}(v))_{a,b,v}$ to generate a random graph from an enhanced version of the configuration model of graphs with given degree distributions. More precisely, for each vertex $v$ we create $\hat{d}_{ab}(v)$ half-edges of type $ab$ for every $a, b \in \{0, 1\}$. Then we create a random matching of the half-edges that respects the types. That is, a half-edge of type 11 has to be matched to another one of type 11, a half-edge of type 00 gets
Algorithm \textit{Forge}(n, m).

(1) Partition the vertex set \([n]\) randomly into three sets \(\mathcal{N}_0, \mathcal{N}_*, \mathcal{N}_1\), with vertex \(v\) being placed into set \(\mathcal{N}_i\) with probability \(\nu_i\) for \(x \in \{0, *\}\) independently. Let \(n_0 = |\mathcal{N}_0|, n_* = |\mathcal{N}_*|, n_1 = |\mathcal{N}_1|\) and \(\hat{n} = (n_0, n_*, n_1)\).

(2) For each vertex \(v\) independently let
\[
\chi_{00}(v) = \text{Po}(\lambda_{00}), \quad \chi_{01}(v) = \text{Po}(\lambda_{01}), \quad \chi_{10}(v) = \text{Po}_{\leq k}(\lambda_{10}), \quad \chi_{11}(v) = \text{Po}_{\leq k}(\lambda_{11})
\]
and
\[
\hat{d}_{00}(v) = \chi_{00}(v)1\{v \in \mathcal{N}_0\}, \quad \hat{d}_{10}(v) = (k - 1)1\{v \in \mathcal{N}_*\} + \chi_{10}(v)1\{v \in \mathcal{N}_1\},
\]
\[
\hat{d}_{01}(v) = \chi_{01}(v)1\{v \in \mathcal{N}_0 \cup \mathcal{N}_*\}, \quad \hat{d}_{11}(v) = \chi_{11}(v)1\{v \in \mathcal{N}_1\}.
\]
Let
\[
\hat{m}_{00} = \sum_{v \in [n]} \hat{d}_{00}(v), \quad \hat{m}_{01} = \sum_{v \in [n]} \hat{d}_{01}(v), \quad \hat{m}_{10} = \sum_{v \in [n]} \hat{d}_{10}(v), \quad \hat{m}_{11} = \sum_{v \in [n]} \hat{d}_{11}(v).
\]
and \(\hat{m} = (\hat{m}_{00}, \hat{m}_{01}, \hat{m}_{10}, \hat{m}_{11})\).

(3) If either \(\hat{m}_{00} \neq \hat{m}_{10}\) or \(\hat{m}_{01} \neq \hat{m}_{11}\) then output \textit{failure} and abort.

(4) Else let
\[
V_{00} = \bigcup_{v \in \mathcal{N}_0} \{(v, 0, 0)\} \times \{\hat{d}_{00}(v)\}, \quad V_{01} = \bigcup_{v \in \mathcal{N}_0 \cup \mathcal{N}_*} \{(v, 0, 1)\} \times \{\hat{d}_{01}(v)\},
\]
\[
V_{10} = \bigcup_{v \in \mathcal{N}_* \cup \mathcal{N}_1} \{(v, 1, 0)\} \times \{\hat{d}_{10}(v)\}, \quad V_{11} = \bigcup_{v \in \mathcal{N}_1} \{(v, 1, 1)\} \times \{\hat{d}_{11}(v)\}.
\]

Independently generate uniformly random perfect matchings \(\mathcal{M}_{00}\) of the complete graph \(K_{V_{00}}\) of \(\mathcal{K}_{V_{10}}\) and \(\mathcal{K}_{V_{11}}\) of the complete bipartite graph \(K_{V_{01}, V_{10}}\).

(5) Let \(\hat{G}\) be the multi-graph obtained from \(\mathcal{M}_{00} \cup \mathcal{M}_{10} \cup \mathcal{M}_{11}\) by contracting the sets \(\{(v, x, y, z) : x, y \in \{0, 1\}, z \in \{\hat{d}_xy(v)\}\}\) to the single vertex \(v\). If \(\hat{G}\) fails to be simple, then output \textit{failure} and stop.

(6) Let \(\mu_\nu = 1\{v \in \mathcal{N}_i\}\) for all \(v \in [n]\). Moreover, for \((v, w) \in [n] \times [n]\) set
\[
\mu_{\nu \rightarrow w} = 1\{v \in \mathcal{N}_i, w \in \partial \hat{G} v\} + 1\{v \in \mathcal{N}_i, 3i, j : ((v, 0, 1), (w, 1, 0, j)) \in \mathcal{M}_{10}\}.
\]
Let \(E(\hat{G})\) be the edge set of \(\hat{G}\) and
\[
\hat{\mu} = (\mu_\nu, \mu_{\nu \rightarrow w})_{v \in [n], (v, w) \in E(\hat{G})}.
\]
(7) If \(\hat{\mu} \neq \mu(\hat{G})\), then output \textit{failure}. Otherwise output \(\hat{G}\) and declare success.

\begin{figure}[h]
\centering
\caption{The algorithm \textit{Forge}.}
\end{figure}

matched to another 00 half-edge and the 10 half-edges get matched to the 01 ones. The conditions on \(\hat{m}_{00}, \ldots, \hat{m}_{11}\) from step (3) guarantee that such a matching exists. We check right away whether the resulting graph \(\hat{G}\) is simple (i.e. contains no loops or multiple edges) and abort if it is not.

Step (6) sets up pseudo-messages \(\hat{\mu}_{\nu \rightarrow w} \in \{0, 1\}\) for every pair \((v, w)\). These reflect the intuition that guided the construction of the graph. That is, we set \(\hat{\mu}_{\nu \rightarrow w}\) to the value that we believe the actual Warning Propagation messages \(\mu_{\nu \rightarrow w}(\hat{G})\) ought to take. The final step of the algorithm checks whether the actual Warning Propagation on \(\hat{G}\) meet these expectations. If \(\hat{\mu}_{\nu \rightarrow w}(\hat{G}) \neq \mu_{\nu \rightarrow w}(\hat{G})\) for some vertex pair \(v, w\), the algorithm aborts. Otherwise it outputs \(\hat{G}\).

The following theorem shows that the success probability of \textit{Forge} is not too small and that given success the output distribution is close to the Erdős-Rényi random graph in total variation.

\begin{theorem}
If \(k \geq 3\) and \(d > d_k\), then the success probability of \textit{Forge}(n, m) is \(\Omega(n^{-1})\) and the total variation distance of \(\hat{G}\) and \(G\) given success is \(o(1)\).
\end{theorem}

Theorem 2.2 makes it easy to analyse properties of the core of the Erdős-Rényi graph, the mantle and the connections between them. Indeed, all we need to do is to investigate \textit{Forge}, which samples from a fairly accessible random graph model composed of nothing but independent random variables and random matchings. There are ample techniques for studying such models. In particular, Theorem 2.2 shows that any property that the pair \((\hat{G}, \hat{\mu})\) enjoys with probability \(1 - o(1/n)\) holds for the pair \((G, \mu(G))\) w.h.p. In fact, the \(1/n\)-factor in the success probability comes exclusively from the harmless conditioning in step (3). Thus, if \((\hat{G}, \hat{\mu})\) has a property w.h.p. given that step (3) does not abort, then the same property holds for \((G, \mu(G))\) w.h.p.
We proceed to state an enhanced version of Theorem 2.2 that allows us to condition on the order and size of the \(k\)-core. To this end, given integer vectors \(N = (n_0, n_1)\) and \(M = (m_{10}, m_{11})\) such that \(m_{11}\) is even let \(\mathcal{F}(N, M)\) be the event that \(\text{Forge}\) succeeds and \(\hat{n}_* = n_*, \hat{n}_1 = n_1, m_{10} = m_{10}, \hat{m}_{11} = m_{11}\). Further, consider the event
\[
\mathcal{F}(N, M) = \{\hat{n}_* = n_*, \hat{n}_1 = n_1, m_{10} = m_{10}, \hat{m}_{11} = m_{11}, \hat{m}_{00} = 2m - 2m_{10} - m_{11}\}.
\]
Additionally, set
\[
\zeta = \zeta(d, k) = (1 - (k-1)q)^{3/2} \exp(-d/2 - d^2/4).
\] (2.3)
Finally, let \(\Gamma_{n,m}(N, M)\) be the set of all graphs \(G\) on vertex set \([n]\) with \(m\) edges such that \(N(G) = N\) and \(M(G) = M\).

**Theorem 2.3.** Let \(k \geq 3, \sigma > d\) and let \(\zeta > 0\). Then uniformly for all integer vectors \(N = (n_0, n_1)\) and \(M = (m_{10}, m_{11})\) such that \(m_{11}\) is even and (1.0) holds, we have
\[
P\left[\mathcal{F}(N, M) \mid \hat{\mathcal{F}}(N, M)\right] \sim \zeta > 0.
\] (2.4)
Furthermore, given \(\mathcal{F}(N, M), \hat{G}\) is uniformly distributed on \(\Gamma_{n,m}(N, M)\).

Since \(\hat{n}_*, \hat{n}_1\) and \(\hat{m}_{ab}, a, b \in \{0, 1\}\) are sums of independent random variables, it is easy to work out that under the assumption (1.10) we have \(P\left[\hat{\mathcal{F}}(N, M)\right] = \Theta(n^{-1})\). Further, Theorem 2.3 shows that given the event \(\hat{\mathcal{F}}(N, M)\) the algorithm \(\text{Forge}\) succeeds with a probability \(\zeta + o(1)\) that is bounded away from 0 and, crucially, given success the resulting random graph is perfectly uniformly distributed over the set of all graphs with \(k\)-core parameters \(N, M\). In effect, Theorem 2.3 makes it easy to study the random graph \(G\) given the order and size of its \(k\)-core.

In addition, since \(G\) is uniform on \(\Gamma_{n,m}(N, M)\) given \(\mathcal{F}(N, M)\), in order to calculate the size of the set \(\Gamma_{n,m}(N, M)\) we just need to compute the entropy of the output distribution of \(\text{Forge}\) given \(\mathcal{F}(N, M)\). This is fairly straightforward because the construction involves a great degree of independence. As we shall see in the next section this argument directly yields Theorem 1.4, the multivariate local limit theorem.

3. Proof Strategy

The main task is to prove Theorem 2.3, whence Theorems 2.2 and 1.4 follow fairly easily. Although some diligence is required, the proofs are completely elementary and none of the arguments are particularly difficult. Let us begin by verifying that \(\hat{G}\) is uniform on \(\Gamma_{n,m}(N, M)\) given success, i.e. that the second statement of Theorem 2.3 holds.

**Proposition 3.1.** Given \(\mathcal{F}(N, M), \hat{G}\) is uniformly distributed on \(\Gamma_{n,m}(N, M)\).

**Proof.** Fix \(N, M\), let \(n_0 = n - n_* - n_1, m_{10} = m_{10}\) and \(m_{00} = 2m - 2m_{10} - m_{11}\), set
\[
n = (n_0, n_*, n_1), \quad m = (m_{00}, m_{10}, m_{11})
\]
and let \(\hat{n} = (\hat{n}_0, \hat{n}_*, \hat{n}_1)\) and \(\hat{m} = (\hat{m}_0, \hat{m}_0, \hat{m}_1, \hat{m}_1)\) as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{ Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{Forge}\). Further, fix \(G \in \Gamma_{n,m}(N, M)\) and let \(d = (d_{G,ab}(v))_{v,a,b}\) be as in \(\text{For}
Moreover, by double counting
\[
P \left[ \hat{G} = G | \hat{d} = d, \mathcal{F}(N, M) \right] = \frac{P \left[ \hat{G} = G | \hat{d} = d \right]}{P \left[ \mathcal{F}(N, M) | \hat{d} = d \right]} = \frac{\Pi}{P \left[ \mathcal{F}(N, M) | \hat{d} = d \right] (m_{00} - 1)!!(m_{11} - 1)!!m_{10}!}. \tag{3.5}
\]
Combining (3.4) and (3.5), we find
\[
P \left[ \hat{G} = G | \mathcal{F}(N, M) \right] = \frac{P}{n! (m_{00} - 1)!!(m_{11} - 1)!!m_{10}! P \left[ \mathcal{F}(N, M) | \hat{n} = n \right]}. \tag{3.6}
\]
Crucially, in the expression (3.1) that defines \( P \) the factorials cancel, whence \( P \) depends on \( N, M \) but not on \( d \). Therefore, so does the right hand side of (3.6), which means that the expression is independent of \( G \).

As a next step, in Section 4 we calculate the success probability of Forge, confirming the first statement of Theorem 2.3, which is thus immediate from Propositions 3.1 and 3.2.

**Proposition 3.2.** Suppose that \( k \geq 3, d > d_k \) and let \( \zeta > 0 \). Assume that \( N, M \) are such that (1.10) holds and that \( m_{11} \) is even. Then uniformly
\[
P \left[ \mathcal{F}(N, M) | \hat{F}(N, M) \right] \sim \zeta.
\]

The proof of Proposition 3.2 is based on the insight that given \( \hat{F}(N, M) \) the algorithm is very likely to succeed unless the random graph \( \hat{G} \) contains certain small substructures. For example, in order to calculate the probability that \( \hat{G} \) is simple we just need to calculate the probability that the random matchings from step (4) produces loops or multiple edges, a standard computation. Similarly, it emerges that the most likely reason for step (7) to fail is the existence of certain bounded-sized subgraphs within the subgraph of \( \hat{G} \) induced on \( \hat{N}_0 \cup \hat{N}_* \), an event whose probability we calculate by the method of moments. The only aspect that requires a bit of technical work is ruling out troublesome sub-structures of intermediate sizes (unbounded but of lower order than \( n \)).

Further, in Section 5 we use Propositions 3.1 and 3.2 to determine \( |\Gamma_{n,m}(N, M)| \) asymptotically.

**Proposition 3.3.** Suppose that \( k \geq 3, d > d_k \). Let \( \xi > 0 \) and let \( Q \) be the matrix from (1.9). Then \( Q \) is regular. Moreover, let \( N, M \) be such that (1.10) holds and that \( m_{11} \) is even. Then uniformly
\[
|\Gamma_{n,m}(N, M)| \sim \frac{1}{2\pi^2 d^2 n^2 \sqrt{\det Q}} \exp \left( -\frac{n}{2} \left( Q^{-1} \Delta(N, M), \Delta(N, M) \right) \right) \left( \frac{n}{m} \right)^{\frac{m}{2}}.
\]

The proof of Proposition 3.3 requires not much more than writing out the number of possible outcomes of \( \hat{G} \) given the event \( \hat{F}(N, M) \) and applying Stirling’s formula to obtain an asymptotic formula. Theorem 1.4 is immediate from Proposition 3.3.

## 4. Proof of Proposition 3.2

**Throughout this section we keep the assumptions of Proposition 3.2.**

### 4.1. Overview.

We prove Proposition 3.2 by calculating the success probability of steps (5) and (7) of Forge. To determine the success probability of step (7), we need to calculate the probability that running Warning Propagation on \( \hat{G} \) results in messages \( \mu(\hat{G}) \) that match the “pseudo-messages” \( \hat{\mu} \). In Section 4.2 we will identify certain minimal structures, called flipping structures, which may cause this to fail. Indeed, we show that w.h.p. any flipping structure present is of a particular form, called a forbidden cycle. Hence, the success probability is asymptotically the same as the probability that no forbidden cycles are present. Finally in Section 4.4 we calculate the probability that \( \hat{G} \) is simple and contains no forbidden cycle.

The construction of \( \hat{G} \) is nothing but an enhanced configuration model. Specifically, each vertex \( v \in [n] \) receives \( d_{ab}(v) \) half-edges of type \( ab \) for \( a, b \in \{0, 1\} \) and step (4) of Forge is a uniform matching of these half-edges that respects the types. To be precise, half-edges of type 00 get matched to other half-edges of type 00, and analogously for half-edges of type 11. Moreover, half-edges of type 01 are matched to half-edges of type 10 and vice versa. Each pair of matched half-edges induces an edge of the random multi-graph \( \hat{G} \). We orient the edges of \( \hat{G} \) that result from the matching of 01 and 10 half-edges from 01 to 10. Thus, \( \hat{G} \) contains some undirected edges (resulting from 00 and 11 half-edges) and some directed ones. Further, let
\[
\hat{N}_+ = \left\{ v \in \hat{N}_0 : \hat{d}_{10}(v) = k - 2 \right\}, \quad \hat{n}_+ = |\hat{N}_+|.
\]
In addition, we define the events
\[ \mathcal{E}_1 = \{ \hat{G} \text{ is simple (i.e. contains no loops or multiple edges) \}, \quad \mathcal{E}_2 = \{ \hat{G}[\hat{N}_v] \text{ contains no directed cycle \}, \quad \mathcal{E}_3 = \{ \hat{G}[\hat{N}_v] \text{ contains no cycles \}, \quad \mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3. \]
Moreover, we recall from Section 2 that for any integer vectors \( N = (n_1, n_1) \) and \( M = (m_1, m_1) \) such that \( m_1 \) is even, \( \mathcal{F}(N, M) \) denotes the event that\( \hat{F} \) succeeds and \( \hat{n}_v = n_1, \hat{m}_v = m_1 \). While
\[ \hat{F}(N, M) = \{ \hat{n}_v = n_1, \hat{m}_v = m_1 \right\}. \]
We break the proof of Proposition 3.2 down into the two steps summarised by the following two propositions.

Proposition 4.1. Let \( \delta > 0 \) be any constant. Uniformly for all \( N, M \) such that \( m_1 \) is even and (1.10) holds, we have
\[ \Pr[\mathcal{E}_1] \sim (1 - (k - 1)q) \quad \text{and} \quad \Pr[\mathcal{E}_2] \sim (1 - (k - 1)q), \]

Furthermore, conditioned on \( \mathcal{F}(N, M) \), the events \( \mathcal{E}_2 \) and \( \mathcal{E}_3 \) are independent, so
\[ \Pr[\mathcal{E} | \mathcal{F}(N, M)] \sim (1 - (k - 1)q)^{3/2} \exp \left( -\frac{d}{2} - \frac{d^2}{4} \right). \]

Proposition 4.2. Uniformly for all \( N, M \) such that \( m_1 \) is even and (1.10) holds, we have
\[ \Pr[\mathcal{F}(N, M) | \mathcal{F}(N, M)] \sim \Pr[\mathcal{E} | \mathcal{F}(N, M)]. \]

After formally introducing flipping structures in Section 4.2 and investigating the subgraph \( \hat{G}[\hat{N}_0] \) in Section 4.3, we will prove Proposition 4.1 in Section 4.4 and Proposition 4.2 in Section 4.5. Proposition 3.2 follows immediately from Propositions 4.1 and 4.2.

4.2. Flipping structures. Recall that \( N_0, N_1, N_1 \) denote the random partition of \( [n] \) constructed in step (1) of \( \hat{F} \). Further recall that given success in step (5), in step (6) for \( (v, w) \in [n] \times [n] \) we defined pseudo-messages
\[ \hat{\mu}_{v \rightarrow w} = \mathbf{1}(v \in N_1, w \in \hat{G}[v]) + \mathbf{1}(v \in N_1, \exists i, j : ((v, 0, 1, i), (w, 1, 0, j)) \in \hat{M}_1 \), \]

and our aim is to calculate the probability that \( \mu(\hat{G}) = \hat{\mu} \). We begin with some basic observations.

Fact 4.3. If \( \hat{G} \) is simple, then \( \hat{\mu}_{v \rightarrow w} \preceq \mu_{v \rightarrow w}(\hat{G}) \) for all \( (v, w) \in [n] \times [n] \).

Proof. A straightforward induction shows that \( \hat{\mu}_{v \rightarrow w} \preceq \mu_{v \rightarrow w}(t(\hat{G})) \) for all \( t \geq 0 \).

In contrast to \( N_0, N_1, N_1 \), which are defined in terms of the pseudo-messages \( \hat{\mu} \), the partition \( N_0(\hat{G}), N_1(\hat{G}), N_1(\hat{G}) \) is induced by the actual Warning Propagation messages on \( \hat{G} \).

Fact 4.4. If \( \hat{G} \) is simple, then we have \( \hat{\mu} = \mu(\hat{G}) \) if and only if \( \hat{N}_x = N_x(\hat{G}) \) for all \( x \in \{0, \ast, 1\} \).

Proof. The construction of \( \hat{G} \) guarantees that \( \hat{d}_{xy}(v) \) equals the number of neighbours \( w \) of \( v \) in \( \hat{G} \) such that \( \hat{\mu}_{w \rightarrow v} = x \) and \( \hat{\mu}_{v \rightarrow w} = y \). Hence,
\[ \hat{N}_0 = \{ v : \sum_{w \in \partial v} \hat{\mu}_{u \rightarrow v} \leq k - 2 \}, \quad \hat{N}_1 = \{ v : \sum_{w \in \partial v} \hat{\mu}_{u \rightarrow v} = k - 1 \}, \quad \hat{N}_1 = \{ v : \sum_{w \in \partial v} \hat{\mu}_{u \rightarrow v} \geq k \}, \]
and thus the assertion is immediate from Fact 4.3.

Suppose that \( \hat{G} \) is simple but \( \hat{\mu} \neq \mu(\hat{G}) \). By Fact 4.4 there is \( x \in \{0, \ast, 1\} \) with \( \hat{N}_x \neq N_x(\hat{G}) \). We would like to identify a minimal structure that is “responsible” for the discrepancy. To this end we introduce a modified version of Warning Propagation. Let us write \( \hat{E}(\hat{G}) \) for the set of ordered pairs of adjacent vertices in \( \hat{G} \), i.e., \( \hat{E}(\hat{G}) \) contains the pairs \( (v, w), (w, v) \) if \( v, w \) are connected by an edge in \( \hat{G} \). For a subset \( S \subseteq \hat{E}(\hat{G}) \) we define the modified Warning Propagation with messages \( \mu_{v \rightarrow w}(t(\hat{G}), S) \) and marks \( \mu_{v}(t(\hat{G}), S) \) as follows. Initially, we set
\[ \mu_{v \rightarrow w}(0)(\hat{G}, S) = \begin{cases} 1 & \text{if } \hat{\mu}_{v \rightarrow w} = 1 \text{ or } (v, w) \in S, \\ 0 & \text{otherwise.} \end{cases} \]
In other words, we initialise according to the pseudo-messages, except possibly on \( S \), where all messages are initially \( 1 \). Further, we use the same update rules (1.6) as in Section 1.3, namely

\[
\mu_{v \rightarrow w}(t + 1|\hat{G}, S) = 1 \left\{ \sum_{u \in \delta_v \setminus \{ w \}} \mu_{u \rightarrow v}(t|\hat{G}, S) \geq k - 1 \right\} \quad \text{for integers } t \geq 0.
\]

Additionally, the mark of \( v \in [n] \) is defined as

\[
\mu_v(t|\hat{G}, S) = 1 \left\{ \sum_{u \in \delta_v \setminus \{ w \}} \mu_{u \rightarrow v}(t|\hat{G}, S) \geq k \right\} \quad \text{for integers } t \geq 0.
\]

As in the original Warning Propagation algorithm, all messages are monotonically decreasing and we set

\[
\mu_{v \rightarrow w}(\hat{G}, S) = \lim_{t \to \infty} \mu_{v \rightarrow w}(t|\hat{G}, S).
\]

Furthermore, let

\[
\hat{N}_0(S) = \left\{ v : \sum_{u \in \delta_v} \mu_{u \rightarrow v}(\hat{G}, S) \leq k - 2 \right\},
\]

\[
\hat{N}_1(S) = \left\{ v : \sum_{u \in \delta_v} \mu_{u \rightarrow v}(\hat{G}, S) = k - 1 \right\},
\]

\[
\hat{N}_2(S) = \left\{ v : \sum_{u \in \delta_v} \mu_{u \rightarrow v}(\hat{G}, S) \geq k \right\}.
\]

We make three simple but important observations.

**Fact 4.5.**

1. \( \hat{N}_1(\emptyset) = \hat{N}_2(\emptyset) \) for all \( x \in \{0, *, 1\} \).
2. \( \hat{N}_1(\hat{E}(\hat{G})) = \hat{N}_3(\hat{G}) \) for all \( x \in \{0, *, 1\} \).
3. \( \hat{N}_2 \subset \hat{N}_1(\hat{G}) \subset \hat{N}_2(\hat{G}) \) and \( \hat{N}_1 \cup \hat{N}_2 \subset \hat{N}_1(\hat{G}) \cup \hat{N}_2(\hat{G}) \) for any \( S \subset \hat{E}(\hat{G}) \).

**Proof.** To obtain the first claim we observe that \( \mu_{v \rightarrow w}(0|\hat{G}, \emptyset) = \hat{\mu}_{v \rightarrow w} \) and that by construction \( \hat{\mu} \) is a fixed point of the modified Warning Propagation algorithm for \( S = \emptyset \), i.e. \( \mu_{v \rightarrow w}(\hat{G}, \emptyset) = \hat{\mu}_{v \rightarrow w} \) for all \( v, w \). With respect to the second assertion, since \( \mu_{v \rightarrow w}(0|\hat{G}, \hat{E}(\hat{G})) = 1 \) for all \( v, w \), we have \( \mu_{v \rightarrow w}(\hat{G}, \hat{E}(\hat{G})) = \hat{\mu}_{v \rightarrow w}(\hat{G}) \) for all \( v, w \). The third assertion is immediate from Fact 4.3. \( \square \)

**Definition 4.6.** A flipping structure of \( \hat{G} \) is an inclusion-minimal set \( S \subset \hat{E}(\hat{G}) \) such that there exists \( x \in \{0, *, 1\} \) such that \( \hat{N}_x \neq \hat{N}_x(S) \).

Facts 4.5 shows that, unless \( \hat{N}_x \neq \hat{N}_x(\hat{G}) \) for all \( x \in \{0, *, 1\} \), there exists a flipping structure.

Hence, we are left to calculate the probability that \( \hat{G} \) contains a flipping structure. To this end we point out a few (deterministic) properties of a flipping structure. Let \( \hat{E}_1(\hat{G}) \) be the set of all pairs \( (v, w) \in \hat{E}(\hat{G}) \) with \( \hat{\mu}_{v \rightarrow w} = 1 \).

Recall that we oriented the edges within \( \hat{G}[\hat{N}_2] \). For a set \( S \subset \hat{E}(\hat{G}) \) let \( V(S) \) be the set of vertices \( v \in [n] \) such that there is a neighbour \( w \) of \( v \) in \( \hat{G} \) with \( (v, w) \in S \) or \( (w, v) \in S \). We denote by \( \hat{G}(S) \) the directed graph on vertex set \( V(S) \) and edge set \( S \) and let \( \delta^-(\hat{G}(S)), \delta^+(\hat{G}(S)) \) be the minimum in- and out-degree of this directed graph. Similarly, denote by \( G(S) \) the undirected graph on \( V(S) \) with edge set \( \{(v, w) : (v, w) \in S\} \).

**Proposition 4.7.** Given that \( \hat{G} \) is simple, any flipping structure \( S \) of \( \hat{G} \) enjoys the following eight properties.

(i) \( \hat{E}_1(\hat{G}) \cap S = \emptyset \).

(ii) For any edge \( (u, v) \) we have \( \mu_{u \rightarrow v}(\hat{G}, S) = 1 \) for \( (v, w) \in \hat{E}_1(\hat{G}) \cup S \). In other words, the initialisation of the modified Warning Propagation algorithm with input \( S \) is already a fixed point.

(iii) \( \hat{G}(S) \) is strongly connected – in particular, \( \delta^- (\hat{G}(S)), \delta^+ (\hat{G}(S)) \geq 1 \).

(iv) Either \( S \subset \hat{N}_0 \times \hat{N}_0 \) or \( S \subset \hat{N}_2 \times \hat{N}_2 \).

(v) If \( S \subset \hat{N}_2 \times \hat{N}_2 \), then \( G(S) \) forms a directed cycle in \( \hat{G}[\hat{N}_2] \).

(vi) If \( S \subset \hat{N}_2 \times \hat{N}_2 \), then \( G(S) \) forms a cycle in \( \hat{G}[\hat{N}_2] \).

(vii) Any vertices of \( G(S) \) in \( \hat{N}_3 \setminus \hat{N}_2 \) have at least 3 distinct neighbours in \( G(S) \).

(viii) Any vertices of \( G(S) \) have at least 2 distinct neighbours in \( G(S) \).

**Proof.** For \( S \subset \hat{E}(\hat{G}) \) let

\[
d^-_S(v) = |\{w : \mu_{w \rightarrow v}(\hat{G}, S) = 1\}|, \quad d^+_S(v) = |\{w : \mu_{v \rightarrow w}(\hat{G}, S) = 1\}|.
\]

(i) This simply follows from the minimality of \( S \), since an edge of \( \hat{E}_1(\hat{G}) \) would be initialised with a message of 1 in the modified Warning Propagation algorithm regardless of whether it lies in \( S \) or not.
(ii) Since the messages of the modified Warning Propagation algorithm are monotonically decreasing, we have $\mu_{v \rightarrow w}(\tilde{G}, S) \leq 1((v, w) \in \tilde{E}_1(\tilde{G}) \cup S)$. Further, by construction $\tilde{\mu}$ is a fixed point of the modified Warning Propagation algorithm for $S = \emptyset$. Therefore, for $(v, w) \in \tilde{E}_1(\tilde{G})$ we have $\mu_{v \rightarrow w}(\tilde{G}, S) \geq \mu_{v \rightarrow w}(\tilde{G}, \emptyset) = \tilde{\mu}_{v \rightarrow w} = 1$. Let $S'$ consist of those directed edges $(v, w) \notin \tilde{E}_1(\tilde{G})$ such that $\mu_{v \rightarrow w}(\tilde{G}, S) = 1$. Then $S' \subset S$ and for any $v, w$, $\mu_{v \rightarrow w}(\tilde{G}, S') = \mu_{v \rightarrow w}(\tilde{G}, S)$. By the minimality of $S$ we have $S = S'$.

(iii) Suppose there is a partition $X \cup Y$ of the vertex set of $\tilde{G}(S)$ such that $X$ and $Y$ are both non-empty and there are no edges in $\tilde{G}(S)$ from $X$ to $Y$. Then let $S' = \{(v, w) \in S : v, w \in Y\}$. For any $y \in Y$ and $v \in V(\tilde{G})$ we have $\mu_{v \rightarrow y}(\tilde{G}, S) = \mu_{v \rightarrow y}(\tilde{G}, S')$, and therefore also $\mu_{v \rightarrow \hat{y}}(\tilde{G}, S) = \mu_{v \rightarrow \hat{y}}(\tilde{G}, S')$. In other words, $X$ has no effect on the messages sent out by $Y$. But then $S'$ would be a smaller flippable structure, contradicting the minimality of $S$.

(iv) By (i) no edge $(v, w)$ where $v \in \mathcal{N}_d$ lies in $S$, for such a directed edge lies in $\tilde{E}_1(\tilde{G})$. But since $\delta^+(\tilde{G}(S)) \geq 1$ by (iii), no vertex of $\mathcal{N}_d$ can lie in $S$. Similarly, for any $u \in \mathcal{N}_s$ and $v \in \mathcal{N}_0$ we have $(u, v) \in \tilde{E}_1(\tilde{G})$ and therefore $(u, v) \notin S$. Thus the result follows by (iii).

(v) By construction a vertex $v \in \mathcal{N}_s$ has $d^+_{\tilde{G}}(v) = k - 1$. By (iii), $\tilde{G}(S)$ contains a directed cycle. On the other hand, if $S' \subset S$ is such that $S'$ forms a directed cycle within $\mathcal{N}_s$, then for each $v \in S'$ we have $d^+_{\tilde{G}}(v) \geq k$, meaning $v \in \mathcal{N}_d(S')$. Therefore by the minimality of $S$ we have $S = S'$.

(vi) By (i), $\tilde{G}(S)$ must contain a directed cycle. On the other hand, if $S' \subset S$ forms a directed cycle, then for $v \in S'$ we have $d^+_{\tilde{G}}(v) = k - 1$. Therefore such vertices are in $\mathcal{N}_d(S')$ and by the minimality of $S$ we have $S = S'$ and the assertion follows since $S'$ forms a cycle in $G(S)$.

(vii) Let $v \in \mathcal{N}_d \setminus \mathcal{N}_s$ be a vertex in $G(S)$, then it holds that $d^+_{\tilde{G}}(v) \leq k - 3$. If $v$ has only one in-neighbour in $G(S)$, then by (ii) we have $d^+_{\tilde{G}}(v) \leq k - 2$ and $\mu_{v \rightarrow w}(\tilde{G}, S) = 0$ for all neighbours $w$ of $v$ in $\tilde{G}$, i.e. $d^+_{\tilde{G}}(v) = 0$ so by (ii), we obtain that $v$ has no out-neighbour in $\tilde{G}(S)$ and therefore $\delta^+(\tilde{G}(S)) = 0$. But this contradicts (iii). Therefore, $v$ has at least 2 in-neighbours in $\tilde{G}(S)$. By (iii), $v$ has at least one out-neighbour in $\tilde{G}(S)$. Now we just need to exclude the possibility that equality holds in both cases and one of the in-neighbours of $v$ in $\tilde{G}(S)$ is also the out-neighbour. For if equality holds, i.e. $v$ has exactly two in-neighbours, then we have $d^+_{\tilde{G}}(v) \leq k - 1$. But this means that if $w$ is such that $\mu_{w \rightarrow v}(\tilde{G}, S) = 1$, then $\mu_{v \rightarrow w}(\tilde{G}, S) = 0$. That is, no vertex $w$ can simultaneously be in- and out-neighbour of $v$, as required.

(viii) Let $v \in \mathcal{N}_d$ be a vertex in $G(S)$, so $d^+_{\tilde{G}}(v) = k - 2$. Assume that $v$ does only have one neighbour $w$ in $G(S)$. By (i) we have that $d^+_{\tilde{G}}(v) = k - 1$, so again we can never have $\mu_{v \rightarrow w}(\tilde{G}, S) = \mu_{v \rightarrow w}(\tilde{G}, S) = 1$. In light of Proposition 4.7 (v) and (vi) we call a flippable structure $S$ a forbidden cycle if either $S \subset \mathcal{N}_d \times \mathcal{N}_s$ or $S \subset \mathcal{N}_d \times \mathcal{N}_s$.

4.3. The subgraph $\tilde{G}[\mathcal{N}_0]$. We proceed to analyse the structure of the induced subgraphs $\tilde{G}[\mathcal{N}_0]$ and $\tilde{G}[\mathcal{N}_s]$ to facilitate the proofs of Propositions 4.1 and 4.2. We condition on the event $\mathcal{E}_1 \supset \mathcal{E}$ that $\tilde{G}$ is simple. The following lemma determines the precise distribution of $\tilde{G}[\mathcal{N}_0]$ given $\tilde{\mathcal{E}}(N, M) \cap \mathcal{E}_1$.

**Lemma 4.8.** Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Given $\tilde{\mathcal{E}}(N, M) \cap \mathcal{E}_1$ the induced subgraph $\tilde{G}[\mathcal{N}_0]$ is a uniform random graph on $n_0$ vertices with $\tilde{n}_{00}/2$ edges.

**Proof.** Given $\tilde{\mathcal{E}}(N, M)$, $\tilde{G}[\mathcal{N}_0]$ clearly has $n_0$ vertices. Further, by step (2) of Forge we have $\hat{d}_{00}(v) = 0$ for all $v \notin \mathcal{N}_0$. That is, all $\tilde{n}_{00}$ half-edges of type 00 are assigned to vertices in $\mathcal{N}_0$. Given $\tilde{n}_{00}$ each such half-edge is assigned to a vertex in $\mathcal{N}_0$ uniformly at random, and subsequently $\tilde{G}[\mathcal{N}_0]$ is formed by matching the half-edges randomly. In effect, given $\mathcal{E}_1$ the random graph $\tilde{G}[\mathcal{N}_0]$ is uniformly distributed.

**Corollary 4.9.** For any $\delta > 0$ there exists $\epsilon = \epsilon(\delta, d, k) > 0$ such that for all $N, M$ such that $m_{11}$ is even and (1.10) holds the following is true.

Given $\tilde{\mathcal{E}}(N, M) \cap \mathcal{E}_1$, w.h.p. $\tilde{G}[\mathcal{N}_0]$ does not contain a subgraph on fewer than $\epsilon n$ vertices with average degree at least $2(1 + \delta)$.

**Proof.** Since a sparse uniformly random graph is well-known to feature no small subgraphs of average degree strictly greater than two, the assertion is immediate from Lemma 4.8.

**Corollary 4.10.** For any $d, k$ there exists $\delta(d, k) > 0$ such that for all $N, M$ such that $m_{11}$ is even and (1.10) holds, the following is true.
Given $\hat{\mathcal{G}}(N, M) \cap E_1$, w.h.p. $\hat{\mathcal{G}}[\mathcal{N}_0]$ does not contain a pair of disjoint non-empty subsets $S, T \subset \mathcal{N}_0$ such that $|S| \leq \delta |T|$ and such that every vertex in $T$ has at least two neighbours in $S$.

**Proof.** We claim that the probability that there exist such sets $S, T$ of sizes $s, t$ is bounded by

$$\binom{n}{s} \binom{n}{t} \frac{O(s)}{n}^{2t},$$

with the $O(\cdot)$-term depending on $d$. Indeed, the binomial coefficients bound the number of ways of choosing $S, T$. Due to monotonicity we may bound the probability term via the binomial random graph of bounded average degree, and thus the probability that a given $v \in T$ has two neighbours in $S$ is bounded by $(O(s)/n)^2$. Further,

$$\binom{n}{s} \frac{O(s)}{n}^{2t} \leq \left( \frac{en}{s} \right)^s \left( \frac{en}{t} \right)^t \frac{O(s)}{n}^{2t} \leq \exp(s + O(t)) \left( \frac{s}{t} \right)^{t-s} \leq (O(\delta))^{\frac{(t-s)}{2}}.$$

Summing over all $s, t$, we obtain

$$\sum_{s \leq t} \frac{O(\delta^2)^{t-s}}{n^{t-s}} \leq \sum_{t \leq \ln n} \delta \ln n \frac{1}{\sqrt{n}} + \sum_{t \leq \ln n} n \left( \frac{O(\delta^2)}{\ln n} \right)^{\ln n} = o(1),$$

as desired. \qed 

As a next step we establish that the subgraph induced on $\hat{\mathcal{N}}_+$ is subcritical, i.e. has average degree less than 1. In effect, there is no large component w.h.p.

**Lemma 4.11.** Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Given $\hat{\mathcal{G}}(N, M) \cap E_1$ the average degree of $\hat{\mathcal{G}}[\hat{\mathcal{N}}_+]$ converges in probability to $\gamma_+ = \tilde{q}(1 - p) d = (k - 1)q < 1$.

We proceed to prove Lemma 4.11. We recall that $\hat{n}_+ = |\hat{\mathcal{N}}_+|$ and further let $\hat{m}_+$ be the number of edges spanned by $\hat{\mathcal{N}}_+$. Let $\hat{\mathcal{G}}(N, M, n_+) = \hat{\mathcal{G}}(N, M) \cap (\hat{\mathcal{N}}_+ = n_+)$ and $\hat{\mathcal{G}}(N, M, n_+, m_+) = \hat{\mathcal{G}}(N, M) \cap (\hat{\mathcal{N}}_+ = n_+, \hat{\mathcal{M}}_+ = m_+)$. The following two claims facilitate the proof of Lemma 4.11.

**Claim 4.12.** Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Then $\hat{n}_+$ has distribution $\text{Bin}(n_0, \tilde{q})$. Moreover, given $\hat{\mathcal{G}}(N, M, n_+), \hat{m}_+$ has distribution $\text{Bin}(m_{00}/2, (\tilde{n}_0/\tilde{n})^2)$. Further, given $\hat{\mathcal{G}}(N, M, n_+, m_+) \cap E_1 \in \hat{\mathcal{G}}[\hat{\mathcal{N}}_+]$ is a uniformly random graph on $\hat{n}_+$ vertices with $\hat{m}_+$ edges.

**Proof.** We recall that $\hat{\mathcal{N}}_+$ is the set of all $v \in \mathcal{N}_0$ such that $\hat{d}_{10}(v) = k - 2$. By the definition of $\hat{d}_{10}$,

$$\mathbb{P} \left[ v \in \hat{\mathcal{N}}_+ \mid v \in \mathcal{N}_0 \right] = \mathbb{P} \left[ \hat{d}_{10}(v) = k - 2 \mid \hat{d}_{10}(v) \leq k - 2 \right] = \tilde{q}$$

independently for all $v \in [n]$. Hence, given $\hat{n}_0$, the parameter $\hat{n}_+$ has distribution $\text{Bin}(\hat{n}_0, \tilde{q})$.

Since $\hat{\mathcal{N}}_+ \subset \mathcal{N}_0$, all edges spanned by $\hat{\mathcal{N}}_+$ are of type 00. Moreover, the construction in steps (2)–(3) of Forz on ensures that given $\hat{n}_+$ and $\hat{n}_0$, for each of the $\hat{m}_{00}$ half-edges of type 00 the probability of being assigned to a vertex in $\hat{\mathcal{N}}_+$ is just $\tilde{n}_+ / \hat{n}_0$. Further, each of the $\hat{m}_{00}/2$ edges constructed from the matching of half-edges of type 00 forms a edge within $\hat{\mathcal{G}}[\hat{\mathcal{N}}_+]$ iff both of the corresponding half-edges were assigned to a vertex from $\hat{\mathcal{N}}_+$. Therefore, the number $\hat{m}_+$ of edges within $\hat{\mathcal{N}}_+$ is distributed as $\text{Bin}(\hat{m}_{00}/2, (\tilde{n}_0/\tilde{n})^2)$. Finally, given $\hat{m}_+$, steps (5) and (6) of Forz on generate a random multi-graph on $\hat{\mathcal{N}}_+$ and given the event $E_1$, this graph is uniformly distributed given its order and size by the same token as in the proof of Lemma 4.8. \qed 

**Claim 4.13.** Suppose that $\omega = \omega(n) \to \infty$. Uniformly for all $N, M$ such that $m_{11}$ is even and (1.10) holds, given $\hat{\mathcal{G}}(N, M)$ w.h.p. we have $|\hat{n}_+ - (1 - p) \tilde{q} n| \leq \omega \sqrt{n}$.

**Proof.** To estimate $\hat{n}_+$ denote by $\hat{\mathcal{A}}(\omega)$ the event that $|\hat{n}_+ - (1 - p) \tilde{q} n| \leq \omega \sqrt{n}$ and let $\hat{\mathcal{G}}(N) = (\hat{n}_+ = n_+, \hat{n}_1 = n_1)$. By Claim 4.12 given $\hat{n}_0$, the parameter $\hat{n}_+$ has distribution $\text{Bin}(\hat{n}_0, \tilde{q})$. Hence,

$$\mathbb{P} \left[ \hat{\mathcal{A}}(\omega/2) \mid \hat{\mathcal{G}}(N) \right] = 1 + o(1).$$

(4.1)

To prove the desired bound given $\hat{\mathcal{G}}(N, M)$, consider the event

$$\hat{\mathcal{E}}(\xi) = \{|\hat{n}_+ - n v_1| + |\hat{n}_1 - n v_1| + |\hat{m}_{10} - 2 m_{10}| + |\hat{m}_{11} - 2 m_{11}| \leq \xi \sqrt{n} \}.$$
To estimate its probability, we calculate

\[
E[\chi_{10}(v)] = \frac{1}{1 - p} \sum_{i \leq k} \frac{i(dp)^i}{i!} \exp(dp) = \frac{dp}{1 - p} \mathbb{P}[\operatorname{Po}(dp) \leq k - 3] = dp(1 - \bar{q}),
\]

\[
E[\chi_{11}(v)] = \frac{1}{p(1 - q)} \sum_{i \leq k} \frac{i(dp)^i}{i!} \exp(dp) = \frac{dp}{p(1 - q)} \mathbb{P}[\operatorname{Po}(dp) \geq k - 1] = \frac{dp}{1 - q}.
\]

Recalling the definitions of \(\mu_{10}, \mu_{11}, \nu_1, \nu_0\) we obtain that \(E[\hat{m}_{10} | \hat{F}(N)] = 2m_{10}\) and \(E[\hat{m}_{11} | \hat{F}(N)] = 2m_{11}\). Given \(\hat{F}(N)\), the parameters \(\hat{m}_{10}\) and \(\hat{m}_{11}\) are sums of independent random variables with a bounded second moment by the construction in step (2) of Forge. Thus, the central limit theorem shows that \(\mathbb{P}[\hat{F}(\xi) | \hat{F}(N)] = \Omega(1)\) for any fixed \(\xi > 0\). Therefore, (4.1) implies that

\[
\mathbb{P}[\hat{F}(\omega) | \hat{F}(N) \cap \hat{F}(\xi)] = 1 + o(1).
\]

Furthermore, conditioned on \(\hat{F}(N)\), perturbing \(M\) by at most \(O(\sqrt{m})\) in each coordinate will change \(\hat{n}_*\) by at most \(O(\sqrt{m})\). This implies that for \(N, M\) such that (1.10) holds we have

\[
\mathbb{P}[\hat{F}(\omega) | \hat{F}(N, M)] = 1 + o(1)
\]

by (4.2).

**Proof of Lemma 4.11.** Let \(\omega = o(n) \to \infty\) sufficiently slowly. Let \(\hat{F}(\omega)\) be the event that \(|\hat{n}_* - (1 - p)\bar{q}n| \leq o(\sqrt{n})\). By Claim 4.12, the number \(\hat{m}_+\) of edges within \(\hat{G}[\mathcal{N}]\) is distributed as \(\operatorname{Bin}(\hat{m}_{00}/2, (\hat{n}_*/\hat{n}_0)^2)\). Hence,

\[
E[\hat{m}_+, \hat{F}(N, M) \cap \mathcal{E}_1 \cap \hat{F}(\omega)] = \frac{\hat{m}_{00}}{2} \left(\frac{\hat{n}_*}{\hat{n}_0}\right)^2 \sim \frac{(1 - p)\bar{q}d}{2} \hat{n}_+.
\]

Claim 4.13 shows that given \(\hat{F}(N, M)\), the event \(\hat{F}(\omega)\) occurs w.h.p. The Chernoff bound therefore shows that conditioned on \(\hat{F}(N, M) \cap \mathcal{E}_1\) we have \(\hat{m}_+ \sim \frac{(1 - p)\bar{q}d}{2} \hat{n}_+\) w.h.p. Therefore w.h.p. the average degree of \(\hat{G}[\mathcal{N}]\) is \((1 - p)\bar{q}d + o(1)\). The assertion thus follows from Fact 1.5 (2).

**Corollary 4.14.** Let \(N, M\) be such that \(m_{11}\) is even and (1.10) holds. Then there exists \(\varepsilon = \varepsilon(d, k)\) such that \(\hat{F}(N, M) \cap \mathcal{E}_1\) w.h.p. there is no set \(T \subset \mathcal{N}_0\) with the following properties:

1. \(t = |T| \leq \varepsilon n\),
2. there are \(0.99|T| \leq y \leq 1.01|T|\) edges in \(\hat{G}[T]\),
3. there are \(s \geq 0.1|T|\) vertex-disjoint paths of length at least 2 whose internal vertices lie in \(\hat{G}[\mathcal{N}] \setminus T\) and that each join two vertices in \(T\).

**Proof.** Let us define \(\nu_+ = (1 - p)\bar{q}\) and \(\nu_- = (1 - p)(1 - \bar{q})\) and pick a slowly growing \(\omega = o(n) \to \infty\). By Claim 4.13 and Proposition 4.1, because \(\hat{n}_* + \hat{n}_* = \hat{n}_0\) we have

\[
\mathbb{P}[|\hat{n}_* - \nu_+ n| + |\hat{n}_* - \nu_- n| \leq 3\omega \sqrt{n} | \hat{F}(N, M) \cap \mathcal{E}_1] = 1 - o(1).
\]

Let \(\hat{A}(3\omega)\) denote the event that \(|\hat{n}_* - \nu_+ n| + |\hat{n}_* - \nu_- n| \leq 3\omega \sqrt{T}\) holds. By Claim 4.12, the number \(\hat{m}_+\) of edges within \(\hat{G}[\mathcal{N}]\) is distributed as \(\operatorname{Bin}(\hat{m}_{00}/2, (\hat{n}_*/\hat{n}_0)^2)\). Hence,

\[
E[\hat{m}_+, \hat{F}(N, M) \cap \mathcal{E}_1 \cap \hat{A}(3\omega)] = \frac{\hat{m}_{00}}{2} \left(\frac{\hat{n}_*}{\hat{n}_0}\right)^2 \sim \frac{\gamma n}{2} \hat{n}_+.
\]

Further, the Chernoff bound implies that conditioned on \(\hat{F}(N, M) \cap \mathcal{E}_1 \cap \hat{A}(3\omega)\) w.h.p. we have

\[
|2\hat{m}_+ - \gamma \hat{n}_+| \leq \omega \sqrt{n}
\]

(4.3)

Let \(Y(k_1, \ldots, k_s)\) denote the number of subsets \(T \subset \mathcal{N}_0\) with properties (1)-(3) of size \(t\) with paths of lengths \(k_1, \ldots, k_s\). We aim to use the first moment method for \(Y(k_1, \ldots, k_s)\) conditioned on \(\hat{F} = \hat{A}(3\omega) \cap \hat{F}(N, M) \cap \mathcal{E}_1\).

Since the appearance of the given subgraph is a monotone graph property, by Lemma 4.8 it suffices to estimate the probability of the existence of a subgraph with properties (1)-(3) in the binomial random graph on \(\hat{n}_0\) vertices with average degree \(\gamma_0 = \hat{m}_{00}/\hat{n}_0\); we will merely lose a constant factor. Therefore, conditioned on \(\hat{F}(N, M) \cap \mathcal{E}_1\) the expected number of sets \(T \subset \mathcal{N}_0\) of size \(t\) that span \(y\) edges is approximated up to a constant factor by

\[
D = D(t, y, s) = \binom{\hat{n}_0}{t} \binom{\gamma_0}{y} \left(\frac{\hat{n}_0}{\gamma_0}\right)^y \leq \left(\frac{e\gamma_0}{2y}\right)^t \left(\frac{e\gamma_0}{2y}\right)^y \left(\frac{\gamma_0}{\hat{n}_0}\right)^y.
\]

(4.4)
Similarly, by Claim 4.12 conditioned on $\hat{\mathcal{B}}(\mathbf{N}, \mathbf{M}, \nu_n, \nu) \land \mathcal{C}_1$ the expected number of paths of lengths $k_1, \ldots, k_s$ in $\tilde{\mathcal{N}} \setminus T$ whose endpoints are adjacent to a vertex in $T$ is upper bounded up to a constant by

$$B(k_1, \ldots, k_s) \leq \left( \frac{\hat{n}_+}{s} \right) \prod_{i=1}^{s} \left( \frac{\hat{n}_+}{k_i} k_i! \left( \frac{2\hat{n}_+}{n^2} \right)^k \left( \frac{\gamma_0 t}{\hat{n}_0} \right)^2 \right) \leq \left( \frac{\hat{n}_0}{s} \right) \prod_{i=1}^{s} \left( \frac{2\hat{n}_+}{\hat{n}_0} k_i \left( \frac{\gamma_0 t}{\hat{n}_0} \right)^2 \right).$$

Let

$$B_+(k_1, \ldots, k_s) = \prod_{i=1}^{s} \gamma_i^{k_i}.$$ 

For $\mathbf{N}, \mathbf{M}$ such that (1.10) holds and $\omega \to \infty$ slowly enough, we have $(\nu_0 n / \hat{n}_0)^l \leq \exp(O(\omega t / \sqrt{n}))$. Therefore, from (4.4) we obtain that conditioned on $\hat{\mathcal{B}}$

$$D \leq \left( \frac{\nu_0 n}{\hat{n}_0} \right)^y \left( \frac{\hat{n}_0}{\nu_0 n} \right)^y \exp\left( O \left( \frac{\omega t}{\sqrt{n}} \right) \right). \quad (4.5)$$

Similarly, by (4.3), conditioned on $\hat{\mathcal{B}}$ we have $(2\hat{n}_+ / \gamma_+ \hat{n}_+)k_i \leq \exp(O(\omega k_i / \sqrt{n}))$. Therefore, conditioned on $\hat{\mathcal{B}}$ we have

$$B(k_1, \ldots, k_s) \leq \left( \frac{\nu_0 n}{s} \right)^y \left( \frac{\gamma_0 t}{\nu_0 n} \right)^{2s} B_+(k_1, \ldots, k_s) \exp\left( O \left( \frac{\omega \sum_i k_i}{\sqrt{n}} \right) \right). \quad (4.6)$$

Note also that conditioned on $\hat{\mathcal{B}}$, we have $\frac{\gamma_0}{\hat{n}_0} \sim d > 1$. To apply the first moment method for $Y(k_1, \ldots, k_s)$ we consider two cases.

**Case 1:** $s \geq 2t$: Denote by $Y(k_1, \ldots, k_s)$ the number of subsets $T \subset \tilde{\mathcal{N}}$ with properties (1)–(3) and $s \geq 2t$. Using $y \leq 1.01 t \leq 3s$, from (4.5) and (4.6) we obtain

$$\mathbb{E}[Y(k_1, \ldots, k_s) | \hat{\mathcal{B}}] \leq e^{s+y+t} \left( \frac{\nu_0}{\nu_0} \right)^{2s+y} \left( \frac{\gamma_0 t}{\nu_0 n} \right)^y \left( \frac{\gamma_0 t}{s} \right)^{t} \left( \frac{\gamma_0 t}{n} \right)^{t+y-t} \exp\left( O \left( \frac{\omega \sum_i k_i + \omega t}{\sqrt{n}} \right) \right).$$

Further, since $\gamma_+ < 1$, for $\omega \to \infty$ slowly enough we have

$$\sum_{k_1, \ldots, k_s} B_+(k_1, \ldots, k_s) \exp\left( O \left( \frac{\omega \sum_i k_i}{\sqrt{n}} \right) \right) = O(1).$$

Therefore using $s - t + y \geq s - t \geq 0.5s$ we obtain that for $\varepsilon > 0$ small enough,

$$\mathbb{E}[Y'(k_1, \ldots, k_s) | \hat{\mathcal{B}}] = o(1). \quad (4.7)$$

**Case 2:** $s < 2t$: Denote by $Y''(k_1, \ldots, k_s)$ the number of subsets $T \subset \tilde{\mathcal{N}}$ with properties (1)–(3) and $s < 2t$. Using $y \leq 1.01 t$, from (4.5) and (4.6) we obtain

$$\mathbb{E}[Y''(k_1, \ldots, k_s) | \hat{\mathcal{B}}] \leq e^{s+y} \left( \frac{\nu_0}{\nu_0} \right)^{2s} \left( \frac{\gamma_0 t}{n} \right)^{s+y-t} \exp\left( O \left( \frac{\omega \sum_i k_i + \omega t}{\sqrt{n}} \right) \right).$$

Similarly as in Case 1, from $\gamma_+ < 1$ and $s + y - t \geq 0.09 t$, we obtain that for $\omega \to \infty$ slowly enough and $\varepsilon > 0$ small enough,

$$\mathbb{E}[Y''(k_1, \ldots, k_s) | \hat{\mathcal{B}}] = o(1). \quad (4.8)$$

Finally, from (4.7) and (4.8) we obtain

$$\mathbb{E}[Y(k_1, \ldots, k_s) | \hat{\mathcal{B}}] = \mathbb{E}[Y'(k_1, \ldots, k_s) | \hat{\mathcal{B}}] + \mathbb{E}[Y''(k_1, \ldots, k_s) | \hat{\mathcal{B}}] = o(1)$$

as desired. \qed
4.4. **Proof of Proposition 4.1.** Our aim is to determine the probability that we have no forbidden cycles in \( \mathcal{N} \) or \( \mathcal{N}_+ \), and no loops or multiple edges. We do this by proving that the number of such structures is approximately Poisson distributed with the appropriate mean. For this we use the method of moments, that is, Theorem 1.6.

To this end, let \( X_{+,\ell} \) be the number of directed cycles of length \( \ell \) in \( \mathcal{N}_+ \), \( X_{+,\ell} \) the number of cycles of length \( \ell \) in \( \mathcal{N} \), and define \( X_+ = \sum_{\ell=1}^{\infty} X_{+,\ell} \) and \( X_+ = \sum_{\ell=1}^{\infty} X_{+,\ell} \). Furthermore, define \( Y, Z \) to be the number of loops and multiple edges in \( G \) respectively. Our aim is to determine the (conditional) probability of the event that \( X_+ = X_+ = Y = Z = 0 \). Let \( \hat{\mathcal{F}}(N,M,n_+) = \mathcal{F}(N,M) \cap \{ n_+ = n_+ \} \). For \( \omega \to \infty \), by Claim 4.13 assumption (1.10) implies that 

\[ |n_+ - (1-p)\bar{q}n| \leq \omega \sqrt{n} \text{ w.h.p.} \]

**Lemma 4.15.** Let \( \omega \to \infty \). Further, let \( n_+ \) be such that \( |n_+ - (1-p)\bar{q}n| \leq \omega \sqrt{n} \). Then, uniformly for all \( N, M \) such that \( m_{11} \) even and (1.10) holds, we have

\[
\mathbb{E} \left[ X_+ | \hat{\mathcal{F}}(N,M,n_+) \right] = -(1 + o(1)) \ln(1 - (k-1)q);
\]

\[
\mathbb{E} \left[ X_+ | \hat{\mathcal{F}}(N,M,n_+) \right] = -\frac{1}{2} (1 + o(1)) \ln(1 - (k-1)q);
\]

\[
\mathbb{E} \left[ Y | X_+ = 0, \hat{\mathcal{F}}(N,M,n_+) \right] = (1 + o(1)) \frac{d}{2},
\]

\[
\mathbb{E} \left[ Z | X_+ = 0, \hat{\mathcal{F}}(N,M,n_+) \right] = (1 + o(1)) \frac{d^2}{4}.
\]

**Proof.** We begin with \( X_+ \), and will consider \( \mathbb{E} [ X_{+,\ell} | \hat{\mathcal{F}}(N,M,n_+) ] \) for bounded \( \ell \geq 1 \) – this expectation tends to 0 exponentially as \( \ell \to \infty \), justifying our choice of only considering \( \ell \) bounded. We first calculate, for bounded \( \ell \geq 1 \), the expected number of collections of \( \ell \) cyclically ordered vertices and \( 2\ell \) ordered half-edges which could conceivably form a directed cycle in \( \mathcal{N}_+ \): we have \( \hat{n}_+ / \ell \) choices for the cyclically ordered vertices. By construction, each such vertex has \( k-1 \) half-edges of type 10. The number of half-edges of type 01 at each vertex is asymptotically Poisson distributed with the appropriate mean. For this we use the method of moments, that is, Theorem 1.6.

Now given such a choice of vertices and half-edges, the probability that they form a directed cycle (with this ordering) is the probability that the relevant half-edges are matched to each other, which is \( 1 / (m_{01})_\ell \). Thus, by (1.16) the expected number of directed cycles of length \( \ell \) is

\[
\mathbb{E} [ X_{+,\ell} | \hat{\mathcal{F}}(N,M,n_+) ] = (1 + o(1)) \frac{1}{\ell} (k-1) \frac{m_{01}}{n_1 + n_+} \eta_+ (d)^{\ell}. \]

Note that \( (k-1)q < 1 \) by Fact 1.5 (2), and so (approximating the sum over all bounded \( \ell \) by the sum to infinity) the expected total number of directed cycles in \( \mathcal{N}_+ \) is

\[
\mathbb{E} [ X_+ | \hat{\mathcal{F}}(N,M,n_+) ] = \sum_{\ell=1}^{\infty} \mathbb{E} [ X_{+,\ell} | \hat{\mathcal{F}}(N,M,n_+) ] = -(1 + o(1)) \ln(1 - (k-1)q). \]

The arguments for \( X_+ \) are similar, although the calculations are slightly different. Conditioned on \( \hat{\mathcal{F}}(N,M,n_+) \), each vertex of \( \mathcal{N}_+ \) has asymptotically Poisson \( m_{00} / \eta_0 \) half-edges of type 00, and therefore for \( v \in \mathcal{N}_+ \) we have

\[
\mathbb{E} [ d_{00}(v) | \hat{\mathcal{F}}(N,M,n_+) ] = \frac{m_{00}^2}{\eta_0}. \]

Now the expected number of sequences of \( \ell \) cyclically ordered (in either direction) vertices and \( 2\ell \) half-edges that could conceivably form a cycle is approximately

\[
\frac{1}{2\ell} (n_+ \eta_+ d_{00}(v) | \hat{\mathcal{F}}(N,M,n_+) | \mathcal{N}_+) \ell = (1 + o(1)) \frac{1}{2\ell} n_+ (m_{00})_\ell^{(m_{00}^2 / \eta_0)^{2\ell}},
\]

while the probability that such a potential cycle is present (i.e. that the appropriate half-edges are matched together) is

\[
\frac{1}{(m_{00} - 1)(m_{00} - 3) \ldots (m_{00} - 2\ell + 1)} = (1 + o(1)) m_{00}^{\ell}. \]
Thus, conditioned on \( \hat{\mathcal{G}}(N, M, n_+) \) we obtain
\[
E \left[ X_+, \ell \mid \hat{\mathcal{G}}(N, M, n_+) \right] = (1 + o(1)) \frac{1}{2\ell} \left( \frac{\hat{n} + \hat{n}_{00}}{\hat{n}_0} \right)^\ell = (1 + o(1)) \left( (1 - p) \hat{q} d \right)^\ell.
\]
Since (1.13) and Fact 1.5 imply that \( (1 - p) \hat{q} d < 1 \), as in the previous case we have
\[
E \left[ X_+ \mid \hat{\mathcal{G}}(N, M, n_+) \right] = \sum_{\ell = 1}^\infty E \left[ X_+, \ell \mid \hat{\mathcal{G}}(N, M, n_+) \right] = -(1 + o(1)) \ln(1 - (1 - p) \hat{q} d)
\]
as claimed.

It remains to determine the expected number of loops and multiple edges given \( X_+ = X_0 = 0 \cap \hat{\mathcal{G}}(N, M, n_+) \). Conditioned on this event there are no loops or multiple edges in \( \mathcal{N}_+ \) or \( \hat{\mathcal{N}}_+ \). We therefore consider the probability of having other loops or multiple edges. Let \( Y_0, Y_1 \) denote the number of loops in \( \mathcal{N}_0 \setminus \hat{\mathcal{N}}_+ \) and \( \hat{\mathcal{N}}_1 \) respectively. Conditioned on \( \hat{\mathcal{G}}(N, N, n_+) \), for \( v \in \hat{\mathcal{N}}_+ \) we have that \( \hat{d}_{00}(v) \) is asymptotically distributed as \( Po(\hat{m}_{00}/\hat{n}_0) \), and so the expected number of loops is
\[
E \left[ Y_0 \mid X_+ = 0, \hat{\mathcal{G}}(N, M, n_+) \right] = (1 + o(1)) \hat{n}_- \left( \frac{\hat{d}_{00}(v)}{2} \right) \left( \hat{\mathcal{G}}(N, M, n_+), v \in \hat{\mathcal{N}}_0 \setminus \hat{\mathcal{N}}_+ \right) \frac{1}{\hat{n}_0 - 1} = (1 + o(1))(1 - p)(1 - \hat{q}) n \frac{d^2(1 - p)^2}{2} \frac{1}{(1 - p)^2 d n} = (1 + o(1))(1 - p)(1 - \hat{q}) d / 2. (4.9)
\]
To determine the expected number of loops in \( \hat{\mathcal{N}}_1 \) we aim to determine the asymptotic distribution of \( \hat{d}_{11}(v) \) for \( v \in \hat{\mathcal{N}}_1 \). We have
\[
\hat{n}_1 E[Po_{\geq k}(\lambda_{11})] = \hat{n}_1 \sum_{x \geq k} \frac{x^{d(p)^x}}{x! \exp(d(p)p(1 - q))} = \hat{n}_1 \frac{d(p)}{p(1 - q)} = (1 + o(1)) p(1 - q) n \frac{d p^2}{p(1 - q)} = (1 + o(1)) \hat{n}_{11}. (4.10)
\]
Conditioned on \( \hat{\mathcal{G}}(N, M, n_+) \) step (2) of Forge can be described by the following balls and bins experiment. Each of the \( \hat{n}_{11} \) half-edges is distributed uniformly among \( \hat{n}_1 \) vertices subject to the constraint that each vertex receives at least \( k \) half-edges. By (4.10), we have that \( E[Po_{\geq k}(\lambda_{11})] \sim \hat{n}_{11}/\hat{n}_1 \). Since this is the distribution with highest entropy and this expectation, for \( v \in \hat{\mathcal{N}}_1 \) we have that \( \hat{d}_{11}(v) \) asymptotically distributed as \( Po_{\geq k}(\lambda_{11}) \). Therefore, we have
\[
E \left[ Y_1 \mid X_+ = 0, \hat{\mathcal{G}}(N, M, n_+) \right] = (1 + o(1)) E \left[ Y_1 \right] = (1 + o(1)) \hat{n}_1 \left( \frac{\hat{d}_{11}(v)}{2} \right) v \in \hat{\mathcal{N}}_1 \frac{1}{\hat{n}_{11} - 1} = (1 + o(1)) \frac{p(1 - q) n}{2 p^2 d n} \sum_{x \geq k} x(x - 1) \frac{(d p)^x}{x! \exp(d(p)p(1 - q))} = (1 + o(1)) \frac{1}{2 p^2 d} (d(p)^2 p \left[ Po(d(p) \geq k - 2) \right] = (1 + o(1)) \frac{d}{2} \left( p + (1 - p) \hat{q} \right). (4.11)
\]
Summing up the two contributions from (4.9) and (4.11) we obtain
\[
E \left[ Y \mid X_+ = 0 \right] = (1 + o(1)) \frac{d}{2}
\]
as claimed.

We now calculate the expected number of multiple edges. Assume that there is a multiple edge joining two vertices in \( \hat{G} \). Then the types of the edges are determined by the end-vertices. By construction, it either holds that both edges must result from the same matching in step (4) of Forge. Along these lines, we will say that a multiple edge is of type 11, 00 or 01/10 respectively for each possible case. Conditioned on \( X_+ = 0 \) there are no multiple edges of type 00 such that both end-vertices lie in \( \mathcal{N}_+ \). Further, conditioned on \( X_+ = 0 \) there is no multiple edge of type 01/10 such that both edges are oriented in the same direction. Denote by \( Z_{00} \) the number of multiple edges of type 00 which lie within \( \hat{\mathcal{N}}_+ \), by \( Z_{11} \) the number of multiple edges of type 11 and by \( Z_{01} \) the number of multiple
Multiple edges of type 00 can only exist within \( \hat{\mathcal{N}} \). Therefore, the distribution of \( \hat{X} \) on \( \{ \text{edges of type 01/10 in which the two edges are oriented in the same direction} \} \) then implies that conditioned on \( \hat{\mathcal{N}} \), \( \hat{d}_{00}(v) \) is asymptotically distributed as \( \text{Po}(\hat{m}_{00}/\hat{n}_0) \). Therefore,

\[
\mathbb{E} [Z_{00} \mid X_+ = X_+ = 0, \hat{\mathcal{N}}(\mathbf{N}, \mathbf{M}, n_+)] = (1 + o(1)) \mathbb{E} [Z_{00} \mid \hat{\mathcal{N}}(\mathbf{N}, \mathbf{M}, n_+)]
\]

\[
= (1 + o(1)) \left( \frac{\hat{n}_0 - \bar{n}_+}{2} + \bar{n}_0 \bar{n}_+ \right) \mathbb{E} \left[ \left( \frac{\hat{d}_{00}(v)}{2} \right) v \in \hat{\mathcal{N}}_0 \right]^2 \frac{2}{(\hat{m}_{00} - 1)(\hat{m}_{00} - 3)}
\]

\[
= (1 + o(1)) \frac{(1-p)^2(1-\tilde{q})^2 n^2 + 2(1-p)^2 \tilde{q} n^2 (1-p)^4 d^4}{4 (1-p)^4 d^2 n^2}
\]

\[
= (1 + o(1)) \frac{d^2}{4 (1-p)^2 (1-\tilde{q})^2}.
\] (4.12)

Similarly, multiple edges of type 11 can only exist within \( \hat{\mathcal{N}}_1 \). For \( v \in \hat{\mathcal{N}}_1 \) we have that conditioned on \( \hat{\mathcal{N}}(\mathbf{N}, \mathbf{M}, n_+) \), \( \hat{d}_{11}(v) \) is asymptotically distributed as \( \text{Po}_{\geq k}(\lambda_{11}) \). Therefore,

\[
\mathbb{E} [Z_{11} \mid X_+ = X_+ = 0] = (1 + o(1)) \mathbb{E} [Z_{11}]
\]

\[
= (1 + o(1)) \left( \frac{\hat{n}_1}{2} \right) \mathbb{E} \left[ \left( \frac{\hat{d}_{11}(v)}{2} \right) v \in \hat{\mathcal{N}}_1 \right]^2 \frac{2}{(m_{11} - 1)(m_{11} - 3)}
\]

\[
= (1 + o(1)) \frac{p^2(1-q)^2 n^2}{2 p(1-q)} \mathbb{P} \left( \text{Po}(dp) \geq k - 2 \right)^2 \frac{2}{p^4 d^2 n^2}
\]

\[
= (1 + o(1)) \frac{d^2}{4} \left( p + (1-p)\tilde{q} \right)^2.
\] (4.13)

Finally we calculate the number of multiple edges of type 01/10. To this end, we aim to determine the asymptotic distribution of \( \hat{d}_{10}(v) \) for \( v \in \hat{\mathcal{N}}_0 \). By (1.13) we have

\[
\hat{n}_0 \mathbb{E} [\text{Po}_{\geq k-2}(\lambda_{10})] + \hat{n}_+ (k-1) = \hat{n}_0 \sum_{x=0}^{k-2} \frac{(dp)^x}{x! \exp(dp)(1-p)} + \hat{n}_+ (k-1)
\]

\[
= (1 + o(1)) \{(1-p)npd(1-\tilde{q}) + pn(k-1)\}
\]

\[
= (1 + o(1)) \{(1-p)npd(1-\tilde{q}) + pn(1-p)\tilde{q}d\}
\]

\[
= (1 + o(1)) p(1-p)dn = (1 + o(1)) \hat{m}_{10}.
\] (4.14)

Conditioned on \( \hat{\mathcal{N}}(\mathbf{N}, \mathbf{M}, n_+) \), step (2) of Forge can be described by the following balls and bins experiment. Each of the \( \hat{m}_{10} \) half-edges of type 10 are distributed uniformly at random over \( \hat{n}_+ + \hat{n}_0 \) vertices subject to the condition that \( \hat{n}_+ \) vertices receive exactly \( k - 1 \) and the remaining \( \hat{n}_0 \) vertices all receive at most \( k - 2 \). By (4.14), we have that \( \mathbb{E} [\text{Po}_{\geq k-2}(\lambda_{10})] \sim \hat{m}_{10}/\hat{n}_0 \). Since this is the distribution with highest entropy and this expectation, conditioned on \( \hat{\mathcal{N}}(\mathbf{N}, \mathbf{M}, n_+) \) for \( v \in \hat{\mathcal{N}}_0 \) we have that \( \hat{d}_{10}(v) \) is asymptotically distributed as \( \text{Po}_{\geq k-2}(\lambda_{10}) \). For \( v \in \hat{\mathcal{N}}_+ \cup \hat{\mathcal{N}}_1 \), \( \hat{d}_{01}(v) \) is
asymptotically distributed as $\text{Po}(\hat{m}_{01}/(\hat{n}_s+\hat{n}_1))$. Therefore we have
\[
\mathbb{E} \left[ Z_{01} \mid X_s = X_s = 0, \hat{\mathcal{F}}(N, M, n_s) \right] = (1 + o(1))\mathbb{E} \left[ Z_{01} \mid \hat{\mathcal{F}}(N, M, n_s) \right]
\]
\[
= (1 + o(1)) \left( \hat{m}_{01} \mathbb{E} \left[ \left( \frac{d_{10}(v)}{2} \right)^{x_{\lambda-1}} \mid v \in \mathcal{N}_0 \right] + \hat{n}_s \left( \frac{k-1}{2} \right) \right) \left( \frac{d_{01}(v)}{2} \right) \left( \frac{k}{2} \right) \left( \hat{n}_1 + \hat{n}_s \right) \mathbb{E} \left[ \left( \frac{d_{01}(v)}{2} \right) \left( \frac{k}{2} \right) \left( \hat{n}_1 + \hat{n}_s \right) \right] \left( \frac{2}{\hat{m}_{01}(\hat{m}_{01} - 1)} \right)
\]
\[
= (1 + o(1)) \left( (1 - p)n \sum_{i=0}^{k-2} \frac{(dp)^x}{x!} \exp(dp) + p\bar{q}n \left( \frac{k-1}{2} \right) \right) \left( \frac{2}{\hat{m}_{01} + \hat{n}_s} \right) \left( \frac{\hat{m}_{01}}{\hat{m}_{01}^2} \right) \left( \frac{2}{\hat{m}_{01}} \right)
\]
\[
= (1 + o(1)) \left( (1 - p)n \frac{(dp)^2}{2(1 - p)} \mathbb{P} \left[ \text{Po}(dp) \leq k - 4 \right] + p\bar{q}n \frac{(k-1)(k-2)}{2} \right) \frac{1}{pn}
\]
\[
= (1 + o(1)) \frac{d^2}{4} \left( 2p \left( 1 - (1 - p)\bar{q} - (1 - p)\bar{q}^2 \right) + \frac{2q}{d^2} \frac{k-2}{k-1}(k-2) \right).
\]
(4.15)

Summing (4.12), (4.13), (4.15), and using (1.13), we obtain
\[
\frac{\mathbb{E} \left[ Z \mid X_s = X_s = 0, \hat{\mathcal{F}}(N, M, n_s) \right]}{(1 + o(1))d^2/4} = (1 - p) \left( (1 - p)(1 - q^2) + (1 - p)\bar{q}^2 + 2p \left( 1 - \bar{q}^2 - \frac{k-2}{dp} \right) + \frac{2q}{d^2} (k-2) + 2p\bar{q} \right) + p^2
\]
\[
= (1 - p) \left( 1 + p + \bar{q} \left( -2p - \frac{2}{d}(k-2) + \frac{2q}{d^2}(k-2) + 2p \right) \right) + p^2
\]
\[
= (1 - p)(1 + p) + p^2 = 1.
\]

This completes the proof of the claim. \( \square \)

We also need to estimate higher factorial moments, which correspond to the expected number of ordered tuples of cycles, loops or multiple edges. We will give the argument only for the higher moments of $X_s$, since those of the other variables can be argued analogously.

So consider the expected number of ordered $r$-tuples of cycles of length $\ell_1, \ldots, \ell_r$ in $\mathcal{N}_s$. Recall that the expected number of cycles of length $\ell$ was asymptotically $\frac{1}{\ell} (k-1)q$. Thus the contribution made by $r$ pairwise disjoint cycles is asymptotically
\[
\prod_{i=1}^{r} \left( \frac{1}{\ell_i} (k-1)q \right)^{\ell_i}.
\]

Summing over all choices of the $\ell_i$ we obtain
\[
\sum_{\ell_1, \ldots, \ell_r} \prod_{i=1}^{r} \left( \frac{1}{\ell_i} (k-1)q \right)^{\ell_i} = \prod_{i=1}^{r} \sum_{\ell_i} \left( \frac{1}{\ell_i} (k-1)q \right)^{\ell_i} = (\mathbb{E} [X_s | \hat{\mathcal{F}}(N, M, n_s)])^r.
\]

We would like to argue that the contribution made by tuples of cycles which are not pairwise disjoint is negligible. For this we prove a more general claim.

**Claim 4.16.** Let $N, M$ be such that $m_{11}$ is even and (1.10) hold. Then conditioned on $\hat{\mathcal{F}}(N, M)$ w.h.p. there are no sets of $s = O(1)$ vertices in $\hat{G}$ which contain at least $s + 1$ edges.

**Proof.** We first crudely bound the degree distribution of any vertex of $\hat{G}$ from above by $k - 1 + \text{Po}(d)$. Now given any pair of half-edges, the probability that they are matched is $O(1/n)$. Thus for a constant $s$, the expected number of sets of size $s$ containing at least $s + 1$ edges is at most
\[
\binom{n}{s} (k - 1 + d)s^{2s+2} O(1/n)^s = O(1/n).
\]
Thus by Markov’s inequality, with high probability there is no such set, even taking a union bound over all $s = O(1)$.

In particular, if an $r$-tuple of cycles is not pairwise disjoint, then it forms a subgraph with fewer vertices than edges. By Claim 4.16, the contribution to the expected number of $r$-tuples of cycles made by those which are not pairwise disjoint is negligible.

This shows that
\[
\mathbb{E} \left[ X_s | \hat{\mathcal{F}}(N, M, n_s) \right] = (1 + o(1)) (\mathbb{E} [X_s | \hat{\mathcal{F}}(N, M, n_s)])^r.
\]
for any bounded $r$, and therefore by Theorem 1.6, $X_\ast$ is asymptotically Poisson distributed with mean $E[X_\ast]$. Therefore the probability that there is no directed cycle in $\mathcal{N}_\ast$ is asymptotically 

$$\exp(-E[X_\ast]|F(N, M, n_\ast)) = (1 + o(1))(1 - (k - 1)q).$$

A similar argument works for each of the other expectations, and we obtain the results of Proposition 4.1.

### 4.5. Proof of Proposition 4.2.

To prove Proposition 4.2 we will then show that $\hat{G}$ is very unlikely to contain a flipping structure other than a forbidden cycle. By Proposition 4.7 (iv) any flipping structure that is not a forbidden cycle lies completely within $\hat{N}_0 \times \hat{N}_0$ and contains at least one vertex from $\hat{N}_0 \setminus \hat{N}_\ast$. The following two lemmas establish that given $\mathcal{E} \cap \mathcal{S}(N, M)$, there are no such flipping structures w.h.p. We consider two cases separately, depending on the the order of the flipping structure, i.e., the number of vertices in $G(S)$.

#### Lemma 4.17.

There exists $\epsilon_1 = \epsilon_1(d, k) > 0$ such that the following is true. Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Then conditioned on $\mathcal{S}(N, M) \cap \mathcal{E}$ w.h.p. there is no flipping structure of order at most $\epsilon_1 n$ in $\hat{N}_0 \times \hat{N}_0$ that contains at least one vertex from $\hat{N}_0 \setminus \hat{N}_\ast$.

#### Lemma 4.18.

Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Then conditioned on $\mathcal{S}(N, M) \cap \mathcal{E}$ w.h.p. there are no flipping structures of order at least $\epsilon_1 n$ in $\hat{N}_0$.

We prove Lemmas 4.17 and 4.18 in Sections 4.5.1 and 4.5.2. But let us first point out that Proposition 4.2 is an immediate consequence of Proposition 4.7 and Lemmas 4.17 and 4.18.

#### Proof of Proposition 4.2.

We have

$$P[\mathcal{S}(N, M) | \mathcal{S}(N, M) \cap \mathcal{E}] = P[\mathcal{S}(N, M) | \mathcal{E}] \cdot P[\mathcal{E} | \mathcal{S}(N, M) \cap \mathcal{E}] \cdot P[\mathcal{E} | \mathcal{S}(N, M)] \cdot P[\mathcal{S}(N, M)].$$

That is, our aim is to show that $P[\mathcal{S}(N, M) | \mathcal{E}] = 1 + o(1)$. Certainly, given $\mathcal{E}$ it holds that $\hat{G}$ is simple. Further, given $\mathcal{E}_2$, Proposition 4.7 (v) and (iv) imply that a possible flipping structure must lie completely within $\hat{N}_0$. Similarly, given $\mathcal{E}_5$, Proposition 4.7 (vi) implies that there is no flipping structure completely within $\hat{N}_\ast$. Therefore invoking Lemmas 4.17 and 4.18 we conclude that given $\mathcal{E} \cap \mathcal{S}(N, M)$ w.h.p. $\mathcal{S}(N, M)$ holds, as required. 

#### 4.5.1. Proof of Lemma 4.17.

Let $\hat{N}_\ast = \hat{N}_0 \setminus \hat{N}_\ast$ and for a set $S \subseteq [n]^2$ let $V_(S) = V(S) \cap \hat{N}_\ast$ and $V_i(S) = V(S) \cap \hat{N}_\ast$. Further, denote by $G_\ast(S)$ and $G_i(S)$ the subgraphs of $G(S)$ induced on $V_(S)$ and $V_i(S)$ respectively. Additionally, let $a = a(S) = |V_(S)|$ and $b = b(S) = |V_i(S)|$ and let $i = i(S)$ be the number of vertices that are isolated in $G_\ast(S)$. We assume throughout that $a + b \leq \epsilon_1 n$.

Let $\ell = \ell(S)$ be the number of leaves (i.e., vertices of degree one) in $G_\ast(S)$. Let $c = c(S)$ denote the number of components of order at least two in $G_\ast(S)$. Let $x = x(S)$ denote the number of edges in $G_\ast(S)$. Throughout this section we assume that $0 < \epsilon_1 \ll \epsilon_2 \ll \epsilon_3 \ll \epsilon_4(d, k)$.

#### Fact 4.19.

Given that $\hat{G}$ is simple, the following statements hold for any flipping structure $S \subseteq \hat{N}_0 \times \hat{N}_0$ with $V(S) \cap \hat{N}_\ast \neq \emptyset$.

1. $G_\ast(S)$ is acyclic.
2. Every leaf of $G_\ast(S)$ has a $G(S)$-neighbour in $V_(S)$.
3. Every isolated vertex of $G_\ast(S)$ has at least two $G(S)$-neighbours in $V_(S)$.
4. Every vertex in $G_i(S)$ has at least three $G(S)$-neighbours.

#### Proof.

If $G_\ast(S)$ contains a cycle, then this cycle is itself a flipping structure, and thus $S$ is not minimal. This shows (1) and (2), (3) follow from Proposition 4.7 (viii). Finally, (4) follows from Proposition 4.7 (vii).

#### Claim 4.20.

Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Given $\mathcal{S}(N, M) \cap \mathcal{E}_1$, w.h.p. $\hat{G}$ does not contain a flipping structure $S \subseteq \hat{N}_0 \times \hat{N}_0$ with $a + b \leq \epsilon_1 n$ such that $a \geq \epsilon_2 b$.

#### Proof.

Fact 4.19 implies that the induced subgraph $G(S)$ of $\hat{G}(\hat{N}_0)$ has average degree at least

$$\frac{3a + 2b}{a + b} \geq 2 + \frac{\epsilon_2}{2}.$$

But by Corollary 4.9, for $\epsilon_1 = \epsilon_1(\epsilon_2, d, k) > 0$ small enough $\hat{G}(\hat{N}_0)$ does not contain such a subgraph w.h.p. 

\[\Box\]
Claim 4.21. Let $N, M$ be such that $m_{i1}$ is even and (1.10) holds. Given $\tilde{W}(N, M) \cap e_1$, w.h.p. $\tilde{G}$ does not contain a flipping structure $S \subset \tilde{\mathcal{N}}_0 \times \tilde{\mathcal{N}}_0$ with $a + b \leq \varepsilon_1 n$ such that $a \leq \varepsilon_2 b$ and $i \geq \varepsilon_3 b$.

Proof. Every isolated vertex of $G_+ (S)$ has at least two $G(S)$-neighbours in $V_-(S)$. Therefore, Corollary 4.10 applies. □

Claim 4.22. Let $N, M$ be such that $m_{i1}$ is even and (1.10) holds. Given $\tilde{W}(N, M) \cap e_1$, w.h.p. $\tilde{G}$ does not contain a flipping structure $S \subset \tilde{\mathcal{N}}_0 \times \tilde{\mathcal{N}}_0$ with $a + b \leq \varepsilon_1 n$ such that $a \leq \varepsilon_2 b$, $i \leq \varepsilon_3 b$ and $\ell \geq \varepsilon_4 (b - i)$.

Proof. We aim to determine the average degree in the induced subgraph $G(S)$ of $\tilde{G} \setminus \tilde{\mathcal{N}}_0$. By Proposition 4.7 (viii) each vertex in $G(S)$ has degree at least 2 in $G(S)$. That is, the total degree among the vertices of $G(S)$ in $\tilde{\mathcal{N}}_+$ is at least 2b. It remains to determine the total degree among the vertices of $G(S)$ in $\tilde{\mathcal{N}}_+$. By Fact 4.19 every leaf of $G_+(S)$ has a $G(S)$-neighbour in $V_-(S)$, and each isolated vertex in $G_+(S)$ has at least two $G(S)$-neighbours in $V_-(S)$. That is, there are at least $2i + \ell$ edges between $V_+(S)$ and $V_-(S)$ in $G(S)$ and so the total degree among the vertices of $G(S)$ in $\tilde{\mathcal{N}}_+$ is at least $2i + \ell$. Since $\ell \geq \varepsilon_4 (1 - \varepsilon_3) b$ and $a \leq \varepsilon_2 b$, the average degree in $G(S)$ is at least

$$\frac{2b + 2i + \ell}{a + b} \geq \frac{2b + \ell}{a + b} \geq \frac{2 + \varepsilon_4 (1 - \varepsilon_3)}{1 + \varepsilon_2}.$$ 

But by Corollary 4.9, for $\varepsilon_3 < 1$, $\varepsilon_2 = \varepsilon_2 (d, k, \varepsilon_3, \varepsilon_4)$ and $\varepsilon_1 = \varepsilon_1 (d, k, \varepsilon_2) > 0$ small enough $\tilde{G} \setminus \tilde{\mathcal{N}}_0$ does not contain such a subset w.h.p. □

Claim 4.23. Let $N, M$ be such that $m_{i1}$ is even and (1.10) holds. Given $\tilde{W}(N, M) \cap e_1$ w.h.p. $\tilde{G}$ does not contain a flipping structure $s \subset \tilde{\mathcal{N}}_0 \times \tilde{\mathcal{N}}_0$ with $a + b \leq \varepsilon_1 n$ and $i \geq 0.1 \alpha$.

Proof. If $i \geq 0.1 \alpha$, the induced subgraph $G_-(S)$ has average degree 2.02. By Corollary 4.9 for $\varepsilon_1 = \varepsilon (0.01, d, k)$ no such subgraph exists in $\tilde{G} \setminus \tilde{\mathcal{N}}_0$. □

Claim 4.24. Let $N, M$ be such that $m_{i1}$ is even and (1.10) holds. Given $\tilde{W}(N, M) \cap e_1$ w.h.p. $\tilde{G}$ does not contain a flipping structure $S \subset \tilde{\mathcal{N}}_0 \times \tilde{\mathcal{N}}_0$ with the following properties.

1. $a + b \leq \varepsilon_1 n$,
2. $x > 0.99 a$,
3. $i \geq 0.1 \alpha$.

Proof. We aim to determine the average degree in the induced subgraph $G(S)$ on $V_-(S)$ and the isolated vertices of $G_+(S)$. By Fact 4.19 every isolated vertex in $G_+(S)$ has at least two $G(S)$-neighbours in $V_-(S)$. By assumption there are $x$ edges in $G_-(S)$. Therefore, the average degree is

$$\frac{2x + 4i}{a + i} \geq \frac{1.98 a + 4i}{a + i} \geq \frac{2.38}{1.1}.$$ 

By Corollary 4.9 for $\varepsilon_1 = \varepsilon_1 (d, k)$ no such subgraph exists in $\tilde{G} \setminus \tilde{\mathcal{N}}_0$. □

Claim 4.25. Let $N, M$ be such that $m_{i1}$ is even and (1.10) holds. Given $\tilde{W}(N, M) \cap e_1$ w.h.p. $\tilde{G}$ does not contain a flipping structure $S \subset \tilde{\mathcal{N}}_0 \times \tilde{\mathcal{N}}_0$ with the following properties.

1. $a + b \leq \varepsilon_1 n$,
2. $x > 0.99 a$,
3. $\ell - c + i \leq a \leq \frac{100}{99} (c + i)$.

Proof. By Claim 4.24 w.h.p. there are no flipping structures with $x > 0.99 a$ and $i \geq 0.1 \alpha$. Now, assume that there is a flipping structure $S$ with (3) and $i \leq 0.1 \alpha$. For such a flipping structure, from the assumption that $\ell - c + i \leq a \leq \frac{100}{99} (c + i)$ and $c \leq \ell / 2.25$, each component in $S$ is not an isolated vertex has at least two leaves. Therefore, letting $c' = c'(S)$ be the number of components of order at least 2 in $S$ with exactly two leaves, we conclude that $\ell \geq 2 c' + 3 (c - c')$, and thus $c' \geq 0.75 c$. This implies that there are at least $c'$ paths contained in $\tilde{\mathcal{N}}_+$ whose endpoints are adjacent to vertices in $V_-(S)$. Consequently, Corollary 4.14 completes the proof. □

The rest of the proof is based on the first moment method. Let $v_+ = (1 - p) \tilde{q}$ and $v_+ = (1 - p)(1 - \tilde{q})$ and pick a slowly growing $\omega = o(n) \to \infty$. By Claim 4.13 and Proposition 4.1, because $\tilde{n}_+ + \tilde{n}_- = \tilde{n}_0$ we have

$$\mathbb{P}(|\tilde{n}_+ + \tilde{n}_-| + |\tilde{n}_- + \tilde{n}_-| \leq 3\omega v + n | \tilde{W}(N, M) \cap e_1) = 1 - o(1).$$

(4.16)
Let $\mathcal{A}(3\omega)$ denote the event that $|\hat{n}_+ - \nu_+ n| + |\hat{n}_- - \nu_- n| \leq 3\omega \sqrt{n}$ holds. By Claim 4.12, the number $\hat{m}_+$ of edges within $\hat{G}[\mathcal{N}]$ is distributed as Bin($\hat{m}_0 / 2, (\hat{n}_+/\hat{n}_0)^2$). Hence, setting $\mathcal{B} = \mathcal{F}(N, M) \cap \mathcal{E}_1 \cap \mathcal{A}(3\omega)$

$$E[\hat{m}_+ | \mathcal{B}] = \frac{\hat{m}_0}{2} \left( \frac{\hat{n}_+}{\hat{n}_0} \right)^2 - \frac{\gamma_+}{2} \hat{n}_+.$$ 

Further, a Chernoff bound implies that conditioned on $\mathcal{B}$ w.h.p. we have

$$|2\hat{m}_+ - \gamma_+ \hat{n}_+| \leq \omega \sqrt{n}. \quad (4.17)$$

We begin with deriving an auxiliary proposition bounding the following quantity, which will appear in the rest of the proof. Let

$$C = C(a, b, c, \ell, i) = \binom{\hat{n}_+ - (b - i)}{b - i} \left( \binom{\hat{n}_+ - (b - i)}{i} \right) \left( \frac{2\hat{m}_+}{\hat{n}_+^2} \right)^{b - i - c} \left( \frac{b - i - 1}{c - 1} \right) \mathcal{J}(b - i - c, b - i - \ell).$$

Let

$$B = B(a, b, c, i) = \binom{\hat{n}_+ - (b - i)}{a} \left( \frac{\hat{n}_+}{c} \right)^c \leq e^{a - i} \left( \frac{n^a}{a} \right) \left( \frac{\ell^i}{i} \right) \left( \frac{m^c}{c} \right)$$

and let $f(x) = -x \ln(x)$.

**Proposition 4.26.** If $c \leq \epsilon_4 b$, then conditioned on $\mathcal{B}$ we have

$$C \leq \epsilon_4 B \gamma_+^{b - i} \left( \frac{b - i - 1}{\ell} \right) \exp \left[ 2\ell + b \left( f \left( \frac{\ell}{b - i} \right) + 2f \left( \frac{\ell}{2(b - i)} \right) \right) + O \left( \frac{\omega b}{\sqrt{n}} \right) + (c - 1) \ln \left( \frac{c}{c - 1} \right) \right]. \quad (4.18)$$

**Proof.** Using Theorem 1.8 and upper bounding

$$\frac{(b - i)!}{\ell^i} \leq e^{\ell - b + i + 1} \left( \frac{b - i}{\ell} \right)^{b - i + 1/2},$$

we obtain

$$C \leq e^{\ell - b + i + 1} \left( \frac{\hat{n}_+}{\ell} \right)^{b - i - 1} \left( \frac{\hat{n}_+}{b - i} \right)^{b - i} \left( \frac{\hat{n}_+}{c - 1} \right)^{c - 1} \left( \frac{b - i e}{\ell - c} \right)^{\ell - c} \left( \frac{b - i}{\ell - c} \right)^{\ell - c} \left( \frac{\gamma_+}{\hat{n}_+} \right)^{b - i - c} \exp \left( O \left( \frac{\omega b}{\sqrt{n}} \right) \right)$$

From (4.16) and (4.17) we obtain that conditioned on $\mathcal{B}$ we have $(2\hat{m}_+/\gamma_+ \hat{n}_+) \leq \exp(O(\omega b/\sqrt{n}))$ and $(\hat{n}_+/\nu_+ n)^{2b - 2i} \leq \exp(O(\omega b/\sqrt{n}))$. Therefore, conditioned on $\mathcal{B}$

$$C \leq e^{\ell - b + i + 1} \left( \frac{\hat{n}_+}{\ell} \right)^{b - i - 1} \left( \frac{\hat{n}_+}{b - i} \right)^{b - i - 1} \left( \frac{\hat{n}_+}{c - 1} \right)^{c - 1} \left( \frac{b - i e}{\ell - c} \right)^{\ell - c} \left( \frac{b - i}{\ell - c} \right)^{\ell - c} \left( \frac{\gamma_+}{\hat{n}_+} \right)^{b - i - c} \exp \left( O \left( \frac{\omega b}{\sqrt{n}} \right) \right)$$

The bound on $C$ follows directly from the assumption that $c \leq \epsilon_4 b$. 

**Claim 4.27.** Let $N, M$ be such that $m_{12}$ is even and (1.10) holds. Given $\mathcal{F}(N, M) \cap \mathcal{E}_1$ w.h.p. $G$ does not contain a flipping structure $S \subset N \times N_0$ with the following properties.

1. $a + b \leq \epsilon_1 n$,
2. $a \leq \epsilon_2 b$,
3. $i \leq \epsilon_3 b$,
4. $\ell \leq \epsilon_4 (b - i)$,
5. $a < \ell - c + i$
Proof. Let $Z'$ denote the number of such flipping structures $S$. Recall that each leaf of $G_+(S)$ must have a $G(S)$-neighbour among the $a$ vertices in $V_- (S)$, and every isolated vertex must have two $G(S)$-neighbours in $V_- (S)$. Since the existence of these edges is a monotone graph property, by Lemma 4.8 the probability that all necessary edges are present is upper bounded up to a constant by

$$R' = R'(a, \ell, i) = \left( \frac{a \gamma_0}{n_0} \right)^{\ell - i}.$$

Therefore

$$\mathbb{E}[Z' \mid \mathcal{H}] \leq O(C \cdot R').$$

Conditioned on $\mathcal{H}$ we have

$$R' \leq \left( \frac{a \gamma_0}{v_0 n} \right)^{\ell + i} \exp \left( O \left( \frac{b}{\sqrt{n}} \right) \right).$$

Since $a \leq b$, we obtain that conditioned on $\mathcal{H}$

$$B \cdot R' \leq e^{a+i} \left( \frac{\gamma_0}{v_0} \right)^{\ell + i} \exp \left( O \left( \frac{b}{\sqrt{n}} \right) \right).$$

The map $f$ is continuous and monotonically increasing on $[0, 1/e)$ with $f(x) \to 0$ as $x \to 0$. Therefore using $a \leq \varepsilon_2 b$, $i \leq \ell/2$, $c \leq \ell/2 \leq \varepsilon_4 (1 - \varepsilon_3) b/2$ and $a \leq \ell - c + i$, from (4.18) and (4.20) we obtain that for $0 < \varepsilon_4 < 1$, $\varepsilon_3 = \varepsilon_3 (d, k, \varepsilon_4)$, $\varepsilon_2 = \varepsilon_2 (d, k, \varepsilon_3) > 0$ small enough it holds that

$$C \cdot R' \leq B \cdot R' \leq e^{a+i} \gamma_{\varepsilon_3} \exp \left( 2\ell + a + i + b \left( f \left( \frac{\ell}{b - 1} \right) + 2 f \left( \frac{\ell}{2(b - 1)} \right) \right) + O \left( \frac{b}{\sqrt{n}} \right) \right)$$

where the last line follows since $\gamma_+ < 1$. For $\omega \to \infty$ slowly enough by (4.19) and (4.21) we obtain

$$\mathbb{E}[Z' \mid \mathcal{H}] = O(C \cdot R') = o(1)$$

as required.

Claim 4.28. Let $N, M$ be such that $m_{11}$ is even and (1.10) holds. Given $\mathcal{H} (N, M) \cap E_1$ w.h.p. $\hat{G}$ does not contain a flipping structure $S \subset \mathcal{N}_0 \times \mathcal{N}_0$ with the following properties.

1. $a + b \leq \varepsilon_1 n$,
2. $a \leq \varepsilon_2 b$,
3. $i \leq \varepsilon_3 b$,
4. $\ell \leq \varepsilon_4 (b - i)$,
5. $x \leq 1.01 a$,
6. $a \geq \ell - c + i$.
7. Either $a \geq \frac{10}{99} (c + i)$ or $x \leq 0.99 a$.

Proof. Recall that in such a flipping structure $S$, every vertex in $G_-(S)$ must have at least three neighbours in $G(S)$. Since $x$ is the number of edges within $G_-(S)$, there must be $3a - 2x$ other edges and we obtain the probability that all necessary edges are present is bounded up to a constant by

$$R'' = R''(a, b, x) = \left( \frac{a}{2} \right) \left( \frac{ab}{x} \right) \left( \frac{\gamma_0}{n_0} \right)^{3a - x}.$$
Conditioned on \( \mathcal{B} \), from (2.3) we obtain
\[
R'' \leq \binom{\gamma_0}{x} \left( \frac{ab}{3a - 2x} \right) \frac{\gamma_0}{v_0} n^{3a-x} \exp \left( O \left( \frac{b}{\sqrt{n}} \right) \right) \\
\leq e^{3a \gamma_0} v_0^{3a-x} \left( \frac{a}{n} \right)^x \left( \frac{1}{c-1} \right)^c \left( \frac{b}{3a - 2x} \right)^{3a-2x} \exp \left( O \left( \frac{b}{\sqrt{n}} \right) \right).
\]

Hence, for \( a \leq b \),
\[
B \cdot R'' \leq e^{3a+i} \left( \frac{\gamma_0}{v_0} \right)^{3a-x} \left( \frac{a}{n} \right)^{2a-x-c-i} \exp \left[ 4a + i + b \left( \frac{x}{b} + \frac{i}{b} + f \left( \frac{3a-2x}{b} \right) \right) \right] + (c-1) \ln \left( \frac{c}{c-1} \right) + O \left( \frac{b}{\sqrt{n}} \right). \tag{4.23}
\]

**Case 1**: \( a > (100/99)(c+i) \). Let \( Z''(a, b, c, i, \ell, x) \) be the number of flipping structures satisfying the conditions of the Claim and also \( a > (100/99)(c+i) \), which implies \( 2a - x - c - i > 0 \). From (4.18) and (4.23) we obtain that for \( \varepsilon_4 > 0 \), \( \varepsilon_3 = \varepsilon_3(d, k, \varepsilon_4) \), \( \varepsilon_2 = \varepsilon_2(d, k, \varepsilon_4) > 0 \) small enough
\[
E[Z''(a, b, c, i, \ell, x)] = O(C_{\mathcal{R}}') = o(1).[4.24]
\]

**Case 2**: \( \ell - c + i \leq a \leq (100/99)(c+i), x \leq 0.99 a \). Finally, denote by \( Z''(a, b, c, i, \ell, x) \) the number of flipping structures satisfying the conditions of the claim and \( \ell - c + i \leq a \leq 100/99(c+i), x \leq 0.99 a \). Again we obtain \( 2a - x - c - i > 0 \) and

\[
E[Z''(a, b, c, i, \ell, x)] = O(C_{\mathcal{R}}') = o(1). \tag{4.25}
\]

The assertion follows from combining (4.24) and (4.25). \( \square \)

**Proof of Lemma 4.17.** From Claims 4.20–4.25 and Claims 4.27 and 4.28 we obtain that conditioned of \( \mathcal{F}(N, M) \cap \mathcal{B}_1 \) w.h.p. there is no flipping structure of order at most \( \varepsilon_1 n \). The assertion follows since from Proposition 4.1 we have \( \mathbb{P}(\mathcal{B}_2 \cap \mathcal{B}_1 \cap \mathcal{F}(N, M) \cap \mathcal{B}_1) = \Theta(1) \). \( \square \)

4.5.2. **Proof of Lemma 4.18.** Assume that there is a flipping structure on at least \( \varepsilon_1 n \) vertices of \( \mathcal{N}_0 \), then by Proposition 4.7 (i) for every pair of vertices \( (v, w) \) in \( \mathcal{S} \) we have that \( \mu_{v-w}(\mathcal{G}) \geq \mu_{v-w}(\mathcal{G}, \mathcal{S}) = 1 \). That is, there has to be a set of \( \varepsilon_1 n \) vertices \( v \in \mathcal{N}_0 \) such that applying Warning Propagation on \( \mathcal{G} \) would result in a message of type \( \mu_{v-w}(\mathcal{G}) = 1 \), whereas \( \mu_{v-w}(\mathcal{G}) = 0 \).

We aim to show that given \( \mathcal{F}(N, M) \cap \mathcal{B}_1 \) w.h.p. such a set does not exist in \( \mathcal{N}_0 \) by exploring the component of \( v \in \mathcal{N}_0 \) in \( \mathcal{G} \) and describing the local neighbourhood of \( v \) by a two-type branching process. By construction \( v \) can have neighbours incident to half-edges of type 00 and 10 only. Further conditioned on \( \mathcal{B}_1 \), for each half-edge of type 00 the matching in step (5) of Forage will result in an edge from \( v \) to another vertex \( w \in \mathcal{N}_0 \). Similarly, each half-edge of type 10 the matching will result in an edge from \( v \) to vertex \( w \in \mathcal{N}_0 \).

Conditioned on \( \mathcal{F}(N, M) \), the number \( X \) of neighbours of \( v \) in \( \mathcal{N}_0 \cup \mathcal{N}_1 \) is asymptotically distributed as \( \text{Po}_{s+k-2}(d) \), and the number \( Y \) of neighbours in \( \mathcal{N}_0 \) is asymptotically distributed as \( \text{Po}(d(1 - p)) \) independently of \( X \). We define a 2-type branching process with these parameters, i.e. we start from a vertex \( v \) of type \( \mathcal{N}_0 \) and each vertex of type \( \mathcal{N}_0 \) has \( \text{Po}_{s+k-2}(d) \) children of type \( \mathcal{N}_0 \cup \mathcal{N}_1 \) and \( \text{Po}(d(1 - p)) \) children of type \( \mathcal{N}_0 \) independently. Vertices of type \( \mathcal{N}_0 \cup \mathcal{N}_1 \) have no children in this branching process.

To prove Lemma 4.18 we show that applying Warning Propagation to this branching process would result in a message of type 1 at \( v \). We may assume that a child of \( v \) in \( \mathcal{N}_0 \cup \mathcal{N}_1 \) will always send message 1 towards \( v \) in the tree. This is necessary because we ignored any children of such vertices. Let \( Y_1 \) be the number of children of \( v \) in \( \mathcal{N}_0 \) that send a 1 towards \( v \) after \( r \) iterations of Warning Propagation.

Now, let \( u_1 = 1 \) \((X + Y_1 \geq k-1)\). Our aim is to bound \( \mathbb{P}(u_1 = 1) \) from above. By the recursive structure of the tree, \( Y_1 \) has \( \text{Po}(d(1 - p)\mathbb{E}u_{t-1}) \) distribution independently of \( X \). Now, recall (1.14). Setting \( \tilde{u}_1 = \mathbb{E}u_1 \), by the assumptions

\[ \]
In light of Proposition 3.1 we basically need to study the entropy of the output distribution of $\mathcal{F}$. We keep the notation and assumptions from Proposition 3.3. The following lemma provides an asymptotic formula for $\mathbb{E}[u_t] \leq \mathbb{P}[X + Y_t \geq k - 1] = \sum_{j=0}^{k-2} \frac{(dp)^j}{(1-p)j! \exp(dp)} \mathbb{P} [\text{Po}(d(1-p)\tilde{u}_{t-1}) \geq k - 1 - j] \leq 
abla^2 \mathbb{E}[u_t]
abla^2 \mathbb{P} [\text{Po}(d(1-p)\tilde{u}_{t-1}) \geq k - 1 - j] = \mathbb{P} [\text{Po}(d(1-p)\tilde{u}_{t-1}) = f_k(\tilde{u}_{t-1})].$

We will prove that $f_k(x) < x$ for all $x \in (0, 1]$ by showing that $f_k$ has derivative strictly less than 1 on $[0, 1]$. By definition, $f_k(x) \geq 0$ with equality iff $x = 0$, and

$$f_k(1) \leq \mathbb{P} \{ \text{Po}(d(1-p)) = 1 - \exp(-d(1-p)) < 1. \}

Using (1.15) we obtain

$$\frac{\partial}{\partial x} f_k(x) = \frac{d(1-p)}{1-p} \sum_{j=0}^{k-2} \frac{(dp)^j}{j! \exp(dp)} \frac{(d(1-p)x)^{k-2-j}}{(k-2-j)! \exp(d(1-p)x)}$$

= $d\mathbb{P} \{ \text{Po}(d(p) + \text{Po}(d(1-p)x) = k-2 \} = \frac{\partial}{\partial y} \varphi_k(y) |_{y=d(p+(1-p)x)}$

(4.26) and therefore

$$\frac{\partial^2}{\partial x^2} f_k(x) = d^2(1-p) \frac{\partial^2}{\partial y^2} \varphi_k(y) |_{y=d(p+(1-p)x)}.$$

Since $\frac{\partial}{\partial y} \varphi_k(y)$ is positive for $y \geq 0$, so is $\frac{\partial}{\partial x} f_k(x)$ for all $x \in [0, \infty)$, i.e. $f_k$ is monotonically increasing on $[0, \infty)$. Similarly since

$$\text{sign} \left[ \frac{\partial^2}{\partial y^2} \varphi_k(y) \right] (1.15) = \text{sign}(k-2-y),$$

we have that $\frac{\partial}{\partial x} f_k(x) \leq 0$ for all $x \geq (k - 2 - d) / (d(1-p)) \cap 0$. By Fact 1.5 (1) we have that $d(p \geq k - 2$ i.e. $f_k$ is concave on the entire interval $[0, \infty)$.

Recalling the definition of $\varphi_{d,k}$ in (1.1), we obtain that $\frac{\partial}{\partial x} \varphi_{d,k}(x) |_{x=p} = \frac{d}{\partial y} \varphi_k(y) |_{y=d(p+(1-p)x)}$. Therefore (4.26) implies that

$$\frac{\partial}{\partial x} f_k(x) |_{x=0} = \frac{\partial}{\partial x} \varphi_{d,k}(x) |_{x=p}.$$

Hence, by Fact 1.5 (2) we obtain that $\frac{\partial}{\partial x} f_k(x) |_{x=0} < 1$. Since $f_k$ is monotonically increasing and concave on $[0, \infty)$ this implies that $f_k$ has derivative strictly less than one on $[0, \infty)$ and therefore $f_k(x) < x$ for all $x > 0$.

We may thus conclude that 0 is the only non-negative fixed point of the function $f_k$, and therefore $\tilde{u}_t \to 0$. Thus also $u_t \to 0$ w.h.p. In other words, each vertex has probability $o(1)$ of lying in any flipping structure. Thus the expected number of vertices in any flipping structure is $o(n)$ and by Markov’s inequality, conditioned on $\mathcal{F}(N, M) \cap \mathcal{E}_1$ w.h.p. there is certainly no flipping structure of order at least $\varepsilon_1 n$. Again the result follows since by Proposition 4.1 we have $\mathbb{E}[\mathcal{E}_2 \cap \mathcal{E}_3 | \mathcal{F}(N, M) \cap \mathcal{E}_1] = \Theta(1)$.

5. PROOF OF PROPOSITION 3.3

We keep the notation and assumptions from Proposition 3.3. In light of Proposition 3.1 we basically need to study the entropy of the output distribution of the given $\mathcal{F}(N, M)$. Given $N = (n_*, n_1), M = (m_{01}, m_{11})$ let

$$n_0 = n - n_1 - n_*,$$

$$m_{01} = m_{10},$$

$$m_{00} = 2m - 2m_{10} - m_{11},$$

$$m = (m_{00}, m_{01}, m_{10}, m_{11}).$$

The following lemma provides an asymptotic formula for $|\Gamma_{n,m}(N, M)|$. 

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Lemma 5.1. Uniformly in \( N, M \) we have

\[
|\Gamma_{n,m}(N,M)| \sim \frac{\zeta \exp(dn)\eta(n)\kappa(m)u(n,m)}{\Lambda(m)}
\]

where

\[
\eta(n) = \binom{n}{n_0^n_1^n_2}, \quad \kappa(m) = (m_{00} - 1)!!(m_{11} - 1)!!m_{01}!,
\]

\[
\Lambda(m) = \lambda_{00}^{m_{00}}\lambda_{01}^{m_{01}}\lambda_{10}^{m_{10}}\lambda_{11}^{m_{11}}, \quad u(n,m) = \P(\hat{m} = m|\hat{n} = n).
\]

Proof. For a sequence \( d = (d_{ab}(v))_{v \in [n],a,b \in \{0,1\}} \) let

\[
\mathcal{N}_0(d) = \{v \in [n]: d_{10}(v) \leq k-2, d_{01}(v) = d_{11}(v) = 0\},
\]

\[
\mathcal{N}_a(d) = \{v \in [n]: d_{10}(v) = k-1, d_{00}(v) = d_{11}(v) = 0\},
\]

\[
\mathcal{N}_1(d) = \{v \in [n]: d_{11}(v) \geq k, d_{00}(v) = d_{10}(v) = 0\}.
\]

Let \( \mathcal{D}(n,m) \) be the set of all \( d \) such that \( |\mathcal{N}_0(d)| = n_0, |\mathcal{N}_a(d)| = n_a, |\mathcal{N}_1(d)| = n_1 \) and \( \sum_{v \in [n]} d_{ab}(v) = m_{ab} \) for all \( a, b \in \{0,1\} \). In addition, let \( \mathcal{D}_0(n,m) \) be the set of all \( d \) such that \( \mathcal{N}_0(d) = \{1, \ldots, n_0\}, \mathcal{N}_a(d) = \{n_0 + 1, \ldots, n_0 + n_a\} \) and \( \mathcal{N}_1(d) = \{n\} \setminus (\mathcal{N}_0(d) \cup \mathcal{N}_a(d)) \). Further, let \( s(d) \) be the probability that the random graph \( \hat{G} \) constructed in step (5) of Forge is simple and that \( \hat{\mu} = \mu(\hat{G}) \). We claim that

\[
|\Gamma_{n,m}(N,M)| = \sum_{d \in \mathcal{D}(n,m)} \frac{\kappa(m)s(d)}{\Lambda(m)} = \sum_{d \in \mathcal{D}_0(n,m)} \frac{\kappa(m)s(d)}{\Lambda(m)} = \frac{(n!}{n^d} \sum_{d \in \mathcal{D}_0(n,m)} \frac{\kappa(m)s(d)}{\Lambda(m)}.
\]

Indeed, by Proposition 3.1 \( |\Gamma_{n,m}(N,M)| \) is equal to the number of graphs \( \hat{G} \) that Forge can create given the event \( \mathcal{F}(N,M) \). Step (2) of Forge ensures that given \( \mathcal{F}(N,M) \) the sequence \( d = (d_{ab}(v))_{v \in [n],a,b \in \{0,1\}} \) belongs to the set \( \mathcal{D}(n,m) \). Furthermore, given \( d \) the number of possible matchings that step (4) can create is equal to \( \kappa(m) \), and every possible simple graph can be obtained from exactly \( \prod_{v,a,b} d_{ab}(v) \) matchings. Thus, we obtain (5.2).

Proceeding from (5.2) and observing that \( \sum_{a,b \in \{0,1\}} m_{ab} = 1 \) by the definition (2.2) of the \( \lambda_{ab} \), we obtain

\[
|\Gamma_{n,m}(N,M)| = \sum_{d \in \mathcal{D}} \exp(dn)\kappa(m)\sum_{d \in \mathcal{D}_0(n,m)} \frac{s(d)}{\Lambda(m)} \prod_{v,a,b} \P(\Lambda_{ab} = d_{ab}(v)).
\]

The definition of \( p = p(d,k) \) as the largest fixed point of \( \phi_{d,k} \) from (1.1) and the definition (1.3) of \( q \) ensure that

\[
P(\Lambda_{10} \leq k-2) = 1 - p, \quad P(\Lambda_{10} = k-1) = pq, \quad P(\Lambda_{11} \geq k) = p(1-q).
\]

Therefore, letting \( V = \{\mathcal{N}_0 = [n_0], \mathcal{N}_1 = [n] \setminus [n_a]\} \), we can rewrite the product on the right hand side of (5.3) in terms of the random variables \( d_{ab}(v) \) from step (2) of Forge as

\[
\prod_{v,a,b} \P(\Lambda_{ab} = d_{ab}(v)) = \prod_{v,1 \leq v \leq n_0} \P[\hat{d}_{00}(v) = d_{00}(v)|V] \P[\hat{d}_{10}(v) = d_{10}(v)|V] \P[\Lambda_{10} \leq k-2]
\]

\[
\cdot \prod_{n_0 < v \leq n_0 + n_a} \P[\hat{d}_{01}(v) = d_{01}(v)|V] \P[\hat{d}_{11}(v) = d_{11}(v)|V] \P[\Lambda_{10} = k-1]
\]

\[
\cdot \prod_{n_0 + n_a < v \leq n} \P[\hat{d}_{01}(v) = d_{01}(v)|V] \P[\hat{d}_{11}(v) = d_{11}(v)|V] \P[\Lambda_{11} \geq k]
\]

\[
= (1-p)^{n_0}(pq)^{n_a}(p(1-q)^{n_1} \prod_{v,a,b} \P[\hat{d}_{ab}(v) = d_{ab}(v)|V].
\]

Hence, remembering the definition of \( \nu_0, \nu_a \nu_1 \) from (2.1) and plugging (5.4) into (5.3), we obtain

\[
|\Gamma_{n,m}(N,M)| = \frac{\eta(n)\kappa(m)\exp(dn)}{\Lambda(m)} \sum_{d \in \mathcal{D}_0(n,m)} s(d) \prod_{v,a,b} \P[\hat{d}_{ab}(v) = d_{ab}(v)|V].
\]

Moreover, by symmetry with respect to vertex permutations and by Proposition 3.2,

\[
\sum_{d \in \mathcal{D}_0(n,m)} s(d) \prod_{v,a,b} \P[\hat{d}_{ab}(v) = d_{ab}(v)|V] = E[\zeta(d)] \mathcal{P}(\hat{m} = m|\hat{n} = n) \mathcal{P}(\hat{m} = m|\hat{n} = n) \sim \zeta \mathcal{P}(\hat{m} = m|\hat{n} = n).
\]

Finally, the assertion follows from (5.5) and (5.6).

\[\square\]

As a next step we use Stirling's formula to bring the expression from (5.1) into a more manageable form.
Corollary 5.2. Uniformly in \( N,M \),
\[
|\Gamma_{n,m}(N,M)| \sim \frac{\sqrt{2}d!w(n,m)}{\sqrt{pq(1-q)}} \exp \left[ -n \left( D_{KL}(n^{-1}n\|\nu) - \frac{d}{2} D_{KL}((dn)^{-1}m\|\mu) \right) + \frac{d}{2} + \frac{d^2}{4} \right] \frac{n!}{m!}. \quad (5.7)
\]

Proof. Let us begin by approximating the very last factor. Invoking Stirling's formula, we find
\[
\left( \frac{n!}{m!} \right) \sim \sqrt{2\pi n} \left( \frac{e}{n} \right)^n \exp \left( -\frac{n}{2} \right). \quad (5.8)
\]
Since \( m = \lfloor dn/2 \rfloor \) we obtain
\[
\left( \frac{n(n-1)}{2m} \right)^m \sim \left( \frac{n^2}{2m} \right)^m \exp \left( -\frac{d}{2} \right). \quad (5.9)
\]
Further, the approximation \( \ln(1+x) = x - \frac{1}{2} x^2 + O(x^3) \) shows that
\[
\left( 1 + \frac{m}{n} \right)_{\lfloor n/2 \rfloor - m} \sim \exp \left( m - \frac{d^2}{4} \right). \quad (5.10)
\]
Plugging (5.9) and (5.10) into (5.8) we obtain
\[
\left( \frac{n!}{m!} \right) \sim (2\pi m)^{-1/2} \left( \frac{ne}{d} \right)^m \exp \left( -\frac{d}{2} - \frac{d^2}{4} \right). \quad (5.11)
\]
One more application of Stirling's formula and the fact that \( m = \lfloor dn/2 \rfloor \) yield
\[
\sqrt{(2m)!} \sim \sqrt{2\pi m^{1/4}} \left( \frac{d}{e} \right)^m. \quad (5.12)
\]
Moreover, combining (5.12) and (5.11) we find
\[
\sqrt{(2m)!} \left( \frac{n!}{m!} \right)^{-1} \sim 2\pi m^{3/4} \exp \left( \frac{d}{2} + \frac{d^2}{4} \right) \exp(-dn). \quad (5.13)
\]
We proceed to expand \( |\Gamma_{n,m}(N,M)| \) asymptotically. Let \( H \) denote the entropy function defined in (1.17). By Stirling's formula, our assumption on \( N \) and the definitions (2.1) of \( \nu_0,\nu_* \),
\[
\left( \frac{n}{n^H} \right) \sim (2\pi n)^{-1} \exp(-nH) \sim \frac{n^{H(n^-1)n}}{2\pi n \sqrt{\nu_0\nu_*}} \sim \frac{\exp(-nH(n^-1)n)}{2\pi n \sqrt{\nu_0\nu_*}} \quad (5.14)
\]
Hence,
\[
\eta(n) \sim \frac{\exp(-nD_{KL}(n^{-1}n\|\nu))}{2\pi n \sqrt{\nu_0\nu_*}q(1-p)(1-q)}. \quad (5.15)
\]
Further, (1.16) and Stirling's formula yield
\[
\frac{(m_{ab}-1)!}{\sqrt{m_{ab}!}} = (2\pi m_{ab})^{-1/4} (1 + O(n^{-1})) \quad \text{for all } a,b \in \{0,1\}.
\]
Thus, by (2.1) and the assumption on \( M \)
\[
\kappa(m) \sim (m_{00}-1)!!(m_{11}-1)!!m_{01}! \sim \sqrt{\frac{2}{\pi}} \sqrt{m_{00}!m_{01}!m_{10}!m_{11}!} (m_{00}m_{11})^{-1/4} \quad (5.15)
\]
Since \( \Lambda(m) = d^{dn} \prod_{a,b} m_{ab}^{n_{ab}^2} \), the definition (2.1) of the \( \mu_{ab} \) and (5.15) yield
\[
\kappa(m) \Lambda(m) \sim d^{dn} \frac{\sqrt{(2m)!}}{\sqrt{\pi mp(1-p)}} \frac{2m}{m} \prod_{a,b} \mu_{ab}^{-n_{ab}/2}. \quad (5.16)
\]
Further, applying Stirling's formula and using the assumption on $M$, we obtain

$$
\binom{2m}{m} \prod_{a,b} \mu_{ab}^{m_{ab}} \sim \frac{\exp\left(-2mD_{KL} \left((2m)^{-1} m \| \mu\right)\right)}{(4\pi m)^{3/2} p^2(1-p)^2}.
$$

(5.17)

Thus, combining (5.13), (5.16) and (5.17), we obtain

$$
\frac{\kappa(m)}{\Lambda(m)} \sim \frac{1}{2\pi m \sqrt{p(1-p)}} \exp\left(-dn + mD_{KL} \left((2m)^{-1} m \| \mu\right) + \frac{d}{2} \frac{d^2}{4}\right) \left(\frac{d}{m}\right).
$$

(5.18)

Plugging in (5.18) and (5.14) into (5.1) completes the proof.

Corollary 5.2 provides an explicit formula for $|\Gamma_{n,n}(N,M)|$, apart from the conditional probability $u(n,m) = \mathbb{P} [\hat{m} = m | \hat{n} = n]$. As a next step we will derive an explicit expression for $u(n,m)$. To this end we introduce the matrices

$$
\Sigma = \frac{1}{d} \begin{pmatrix}
(1-p)^2 & 0 & 0 & 0 \\
0 & p(1-p) & 0 & 0 \\
0 & 0 & p(1-p) \left(1 + \tilde{q} \left(d \tilde{p} \left(1 - \tilde{q} \right) - (k-1)\right)\right) & 0 \\
0 & 0 & 0 & p^2 \left[1 - \frac{d \tilde{p}}{\tilde{q}} + d \left(p + (1-p) \tilde{q}\right)\right]
\end{pmatrix}
$$

(5.19)

and

$$
L = \begin{pmatrix}
1-p & 0 & 0 & 0 \\
0 & 1-p & 0 & 0 \\
p(1-\tilde{q}) & (k-1)/d & 0 & 0 \\
0 & 0 & 0 & p(1-q)
\end{pmatrix}.
$$

(5.20)

**Lemma 5.3.** Let $k \geq 3, d > d_k$ and let $\xi > 0$. Then $\Sigma$ is regular. Moreover, let $n = (n_0, n_\star, n_1)$ be such that $n_0 + n_\star + n_1 = n$ and $|n_0 - n_0 \star| + |n_1 - n_1 \star| \leq \sqrt{\xi} n$. Then uniformly for all $m \in \mathbb{N}^d$,

$$
u(n,m) = \frac{1}{(2\pi n)^d d! \sqrt{\det \Sigma}} \exp\left(-n \frac{1}{2} \left(\begin{pmatrix} L^* \Sigma^{-1} L & -L^* \Sigma^{-1} \Delta(n) \\
-\Sigma^{-1} L & \Delta(m)\end{pmatrix}\right) + o(n^{-2})\right)
$$

where

$$
\Delta(n) = \left(\frac{n_0}{n} - v_0, \frac{n_\star}{n} - v_\star, \frac{n_1}{n} - v_1\right)^t,
\Delta(m) = \left(m_{00} - m_{00}, m_{01} - m_{01}, m_{10} - m_{10}, m_{11} - m_{11}\right)^t.
$$

(5.21)

**Proof.** Given $\hat{\lambda}_0, \hat{\lambda}_\star, \hat{\lambda}_1$, we can characterise the distributions of the random variables $\hat{d}_{a,b}(v)$ from step (1) of Forge as follows in terms of the $\lambda_0, \ldots, \lambda_1$ from (2.2):

$$
\hat{d}_{00}(v) \overset{d}{=} \text{Po}(\lambda_00), \quad \hat{d}_{01}(v) = 0, \quad \hat{d}_{10}(v) \overset{d}{=} \text{Po}_{<k-2}(\lambda_{10}), \quad \hat{d}_{11}(v) = 0 \quad \text{given } v \in \hat{\lambda}_0,
$$

$$
\hat{d}_{00}(v) = 0, \quad \hat{d}_{01}(v) \overset{d}{=} \text{Po}(\lambda_{01}), \quad \hat{d}_{10}(v) = k-1, \quad \hat{d}_{11}(v) = 0 \quad \text{given } v \in \hat{\lambda}_\star,
$$

$$
\hat{d}_{00}(v) = 0, \quad \hat{d}_{01}(v) \overset{d}{=} \text{Po}(\lambda_{01}), \quad \hat{d}_{10}(v) = 0, \quad \hat{d}_{11}(v) \overset{d}{=} \text{Po}_{k}(\lambda_{11}) \quad \text{given } v \in \hat{\lambda}_1.
$$

Hence, for an arbitrary $v \in \{0, 1\}$ and $x \in \{0, *, 1\}$ let

$$
\tilde{a}_x = (\mathbb{E}[\hat{d}_{00}(v)|v \in \hat{\lambda}_0], \mathbb{E}[\hat{d}_{01}(v)|v \in \hat{\lambda}_0], \mathbb{E}[\hat{d}_{01}(v)|v \in \hat{\lambda}_0], \mathbb{E}[\hat{d}_{11}(v)|v \in \hat{\lambda}_1])^t.
$$

and $\tilde{a} = \sum_{x \in \{0, *, 1\}} \tilde{a}_x$. Further, let

$$
\hat{D}_x = \begin{pmatrix}
\text{Var}[\hat{d}_{00}(v)|v \in \hat{\lambda}_0] & 0 & 0 & 0 \\
0 & \text{Var}[\hat{d}_{01}(v)|v \in \hat{\lambda}_0] & 0 & 0 \\
0 & 0 & \text{Var}[\hat{d}_{11}(v)|v \in \hat{\lambda}_1] & 0 \\
0 & 0 & 0 & \text{Var}[\hat{d}_{11}(v)|v \in \hat{\lambda}_1]
\end{pmatrix}
$$

and $\hat{D} = \sum_{x \in \{0, *, 1\}} x \cdot \hat{D}_x$. By definition of $\hat{d}_{a,b}(v), a,b \in \{0, 1\}$ we obtain that $\hat{D}$ is regular. Further, because the random variables $(\hat{d}_{a,b}(v))_{a,b}^0,1$ are mutually independent, given $\hat{\lambda} = n$ the sequence $n^{-1/2} (\hat{m} - n \hat{a})$ converges in distribution to a multivariate normal distribution with covariance matrix $\hat{D}$ and mean $(0, 0, 0, 0)$. Indeed, Theorem 1.7
implies that uniformly for all $m \in \mathbb{N}^d$,

$$
\mathbb{P}[\mathbf{m} = m|\mathbf{n} = n] = \exp\left(-\frac{n}{2} \left(\mathbf{\hat{D}}^{-1}(m/n - \hat{\mathbf{a}}), (m/n - \hat{\mathbf{a}})\right)\right) + o(n^{-2}).
$$

(5.22)

Hence, to complete the proof we just need to calculate $\hat{\mathbf{a}}$ and $\hat{\mathbf{D}}$ explicitly. We claim that

$$
\hat{\mathbf{a}}_0 = \begin{pmatrix}
    d(1-p) \\
    d(1-p) \\
    k-1 \\
    0 \\
\end{pmatrix}, \quad \hat{\mathbf{a}}_* = \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
\end{pmatrix}, \quad \hat{\mathbf{a}}_1 = \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    d(1-p)/(1-q) \\
\end{pmatrix}.
$$

(5.23)

Indeed, remembering (2.2), we see that

$$
\mathbb{E}[\text{Po}(\lambda_{00})] = \lambda_{00} = d(1-p), \quad \mathbb{E}[\text{Po}(\lambda_{01})] = \lambda_{01} = d(1-p).
$$

(5.24)

Furthermore, remembering (1.3) and (1.12),

$$
\mathbb{E}[\text{Po}_{\leq k-2}(\lambda_{10})] = \frac{1}{1-p} \sum_{i=1}^{k-2} \frac{i(d/p)^i}{i! \text{exp}(p)} = \frac{d/p}{1-p} \mathbb{P}[\text{Po}(d/p) \leq k-3] = \frac{d/p}{1-p} \text{Po}(d/p) \leq k-3 = \frac{d/p}{1-q} \mathbb{P}[\text{Po}(d/p) \geq k-1] = \frac{d/p}{1-q}.
$$

(5.25)

and (5.23) is immediate from (5.24)–(5.26). Moving on to the covariance matrix $\hat{\mathbf{D}}$, we clearly have

$$
\text{Var}[\text{Po}(\lambda_{00})] = \lambda_{00} = d(1-p), \quad \text{Var}[\text{Po}(\lambda_{01})] = \lambda_{01} = d(1-p).
$$

(5.27)

Moreover, by the definition (1.12) of $\hat{q}$,

$$
\mathbb{P}[\text{Po}(d/p) = k-2] = (1-p)q.
$$

(5.28)

Furthermore,

$$
\mathbb{P}[\text{Po}(d/p) = k-3] = \frac{k-2}{dp} \mathbb{P}[\text{Po}(d/p) = k-2].
$$

(5.29)

Hence, using (5.28) we obtain

$$
\mathbb{E}[\hat{d}_{10}(v)(\hat{d}_{10}(v) - 1)|v \in \mathbb{N}_0] = \frac{1}{1-p} \sum_{i=1}^{k-2} \frac{i(d/p)^i}{i! \text{exp}(d/p)} = \frac{d/p}{1-p} \mathbb{P}[\text{Po}(d/p) \leq k-4] = \frac{d/p}{1-q} \frac{1}{1-q} \mathbb{P}[\text{Po}(d/p) = k-2] = \frac{d/p}{1-q} \mathbb{P}[\text{Po}(d/p) = k-2] + (1-p)\hat{q}. 
$$

(5.30)

Similarly, by (5.29)

$$
\mathbb{E}[\hat{d}_{11}(v)(\hat{d}_{11}(v) - 1)|v \in \mathbb{N}_1] = \frac{d/p}{p/(1-q)} \mathbb{P}[\text{Po}(d/p) = k-2] = \frac{d/p}{1-q} + (1-p)\hat{q}. 
$$

(5.31)

Combining (5.25) and (5.30) as well as (5.26) and (5.31) and using that $\text{Var}(X) = \mathbb{E}(X)^2 + \mathbb{E}(X^2) + \mathbb{E}(X)\mathbb{E}(X-1)$, we obtain

$$
\text{Var}(\hat{d}_{10}(v)|v \in \mathbb{N}_0) = d/p(1-\hat{q}(k-1)) + (d/p)^2\hat{q}(1+\hat{q}), \quad \text{Var}(\hat{d}_{11}(v)|v \in \mathbb{N}_1) = \frac{d/p}{1-q} + (1-p)\hat{q}. 
$$

(5.32)

Combining (5.27), (5.32) and (5.33), we obtain

$$
\hat{D}_0 = \begin{pmatrix}
    d(1-p) & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & d/p(1-\hat{q}(k-1)) + (d/p)^2\hat{q}(1-\hat{q}) & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \hat{D}_* = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & d(1-p) & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \hat{D}_1 = \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & d/p(1-\hat{q}(k-1)) + (d/p)^2\hat{q}(1-\hat{q}) & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$

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Finally, we verify that the matrices $\Sigma, L$ from (5.19) and (5.20) satisfy $m/n - \hat{a} = d (\Delta(m) - L \Delta(n))$ and $\hat{D} = \Sigma_x \psi_x \hat{D}_x = d^2 \Sigma$. Since $\hat{D}$ is regular, we obtain that $\Sigma$ is regular. Hence,

$$\langle \hat{D}^{-1} (m/n - \hat{a}), (m/n - \hat{a}) \rangle = \left \langle \begin{pmatrix} L^* \Sigma^{-1} L & -L^* \Sigma^{-1} \\ -\Sigma^{-1} L & \Sigma^{-1} \end{pmatrix} \begin{pmatrix} \Delta(m) \\ \Delta(m) \end{pmatrix}, \begin{pmatrix} \Delta(m) \\ \Delta(m) \end{pmatrix} \right \rangle. \tag{5.34}$$

Plugging in (5.34) in (5.22), we obtain the assertion because $\det \hat{D} = d^2 \det \Sigma$.

**Proof of Proposition 3.3.** We are going to prove Proposition 3.3 by combining Corollary 5.2 and Lemma 5.3. To this end, we remember the Taylor expansion of the Kullback-Leibler divergence $D_{KL}(\cdot; \cdot)$ from (5.7). Using (1.19), we see that the first derivative of $D_{KL}(\cdot; \cdot)$ vanishes at the point $\nu$, where the global minimum of 0 is attained, and similarly $D_{KL}(\cdot; \mu)$ attains its global minimum of 0 at $\mu$. Expanding the Kullback-Leibler divergence to the second order, we obtain with $\Delta(m), \Delta(n)$ from (5.21) that

$$D_{KL}(n^{-1}m, n) = \frac{1}{2} \langle \text{diag}(\nu)^{-1} \Delta(n), \Delta(n) \rangle + O(n^{-3/2}), \tag{5.35}$$

$$D_{KL}(m^{-1}n, m) = \frac{1}{2} \langle \text{diag}(\mu)^{-1} \Delta(m), \Delta(m) \rangle + O(n^{-3/2}). \tag{5.36}$$

Further,

$$\langle \begin{pmatrix} L^* \Sigma^{-1} L & -L^* \Sigma^{-1} \\ -\Sigma^{-1} L & \Sigma^{-1} \end{pmatrix} \begin{pmatrix} \Delta(m) \\ \Delta(m) \end{pmatrix}, \begin{pmatrix} \Delta(m) \\ \Delta(m) \end{pmatrix} \rangle + \langle \text{diag}(\nu)^{-1} \Delta(n), \Delta(n) \rangle - \frac{d}{2} \langle \text{diag}(\mu)^{-1} \Delta(m), \Delta(m) \rangle

= \langle \begin{pmatrix} L^* \Sigma^{-1} L + \text{diag}(\nu)^{-1} & -L^* \Sigma^{-1} \\ -\Sigma^{-1} L & \Sigma^{-1} - \frac{d}{2} \text{diag}(\mu)^{-1} \end{pmatrix} \begin{pmatrix} \Delta(m) \\ \Delta(m) \end{pmatrix}, \begin{pmatrix} \Delta(m) \\ \Delta(m) \end{pmatrix} \rangle. \tag{5.37}$$

Combining (5.35), (5.36) and (5.37) with Corollary 5.2 and Lemma 5.3, we obtain

$$\frac{|\Gamma_{n,m}(N, M)}{|N|} = \frac{C}{n^2} \exp \left( -\frac{n}{2} \langle \begin{pmatrix} L^* \Sigma^{-1} L + \text{diag}(\nu)^{-1} & -L^* \Sigma^{-1} \\ -\Sigma^{-1} L & \Sigma^{-1} - \frac{d}{2} \text{diag}(\mu)^{-1} \end{pmatrix} \begin{pmatrix} \Delta(n) \\ \Delta(n) \end{pmatrix}, \begin{pmatrix} \Delta(n) \\ \Delta(n) \end{pmatrix} \rangle \right), \tag{5.38}$$

where

$$C = \frac{\sqrt{2} \zeta}{(2\pi)^{3d/2} \sqrt{pq(1-q) \det \Sigma}}. \tag{5.39}$$

To proceed, let

$$T = \begin{pmatrix} -1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -2 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.40}$$

Then the vector $(\frac{\Delta(n)}{\Delta(m)})$ can be written as $T \Delta(N, M)$, with $\Delta(N, M)$ from (1.11). By means of a computer algebra system \footnote{We use the free open-source mathematics software system SageMath. An executable code file and PDF version of the source code are provided at http://www.uni-frankfurt.de/53778787. SageMath worksheets can be executed using the online platform CoCalc, see https://cocalc.com/} we verify that

$$C = \frac{1}{2\pi^2 d^2 \sqrt{\det(q)}}. \tag{5.41}$$

Using Lemma 5.3 this implies that $Q$ is a regular matrix. Finally, calculating the entries of the matrix on the right hand side explicitly (for which once more we use a computer algebra system), we see that the matrix $Q$ from (1.9) satisfies

$$Q^{-1} = T^* \begin{pmatrix} L^* \Sigma^{-1} L + \text{diag}(\nu)^{-1} & -L^* \Sigma^{-1} \\ -\Sigma^{-1} L & \Sigma^{-1} - \frac{d}{2} \text{diag}(\mu)^{-1} \end{pmatrix} T. \tag{5.42}$$
Hence, (5.38) can be written as

$$\frac{\left| \Gamma_{n,M}(N,M) \right|}{\left( \begin{array}{c} N \\ M \end{array} \right)} \sim \frac{1}{2\pi^2 d^2 n^2 \sqrt{\det(Q)}} \exp \left( -\frac{n}{2} \left( Q^{-1} \Delta(N,M), \Delta(N,M) \right) \right),$$

as desired. \qed

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AMIN COJA-OGLHAN, acoghlan@math.uni-frankfurt.de, GOETHE UNIVERSITY, MATHEMATICS INSTITUTE, 10 ROBERT MAYER STR, FRANKFURT 60325, GERMANY.

OLIVER COOLEY, cooley@math.tugraz.at, GRAZ UNIVERSITY OF TECHNOLOGY, INSTITUTE OF DISCRETE MATHEMATICS, STEYRERGASSE 30, 8010 GRAZ, AUSTRIA

MIHYUN KANG, kang@math.tugraz.at, GRAZ UNIVERSITY OF TECHNOLOGY, INSTITUTE OF DISCRETE MATHEMATICS, STEYRERGASSE 30, 8010 GRAZ, AUSTRIA

KATHRIN SKUBCH, skubch@math.uni-frankfurt.de, GOETHE UNIVERSITY, MATHEMATICS INSTITUTE, 10 ROBERT MAYER STR, FRANKFURT 60325, GERMANY.