EXACT DECAY RATE OF A NONLINEAR ELLIPTIC EQUATION RELATED TO THE YAMABE FLOW

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Abstract. Let \(0 < m < \frac{n-2}{n}, n \geq 3\), \(\alpha = \frac{2\beta + \rho}{1-m}\) and \(\beta > \frac{m\rho}{n-2-mn}\) for some constant \(\rho > 0\). Suppose \(v\) is a radially symmetric solution of \(\frac{n-1}{m} \Delta v^m + \alpha v + \beta x \cdot \nabla v = 0\), \(v > 0\), in \(\mathbb{R}^n\). When \(m = \frac{n-2}{n+2}\), the metric \(g = v^{\frac{1}{1-m}} dx^2\) corresponds to a locally conformally flat Yamabe shrinking gradient soliton with positive sectional curvature. We prove that the solution \(v\) of the above nonlinear elliptic equation has the exact decay rate \(\lim_{r \to \infty} r^2 v(r)^{1-m} = \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m)-2\beta)}\).

1. Introduction

Recently, there has been a lot of study of the equation

\[
\frac{n-1}{m} \Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, \quad v > 0, \quad \text{in} \ \mathbb{R}^n
\]

where

\[
0 < m < \frac{n-2}{n}, \quad n \geq 3,
\]

and

\[
\alpha = \frac{2\beta + \rho}{1-m}
\]

for some constant \(\rho \in \mathbb{R}\) by P. Daskalopoulos and N. Sesum [DS2]; S.Y. Hsu [H1], [H2]; M.A. Peletier and H. Zhang [PZ]; and J.L. Vázquez [V1]. In the paper [DS2] P. Daskalopoulos and N. Sesum (cf. [CSZ], [CMM]) proved the important result that any locally conformally flat non-compact gradient Yamabe soliton \(g\) with positive sectional curvature on an \(n\)-dimensional manifold, \(n \geq 3\), must be radially symmetric and have the form \(g = v^{\frac{1}{1-m}} dx^2\), where \(dx^2\) is the Euclidean metric on \(\mathbb{R}^n\) and \(v\) is a radially symmetric solution of (1.1) with \(m = \frac{n-2}{n+2}\), and \(\alpha, \beta\) satisfy (1.3) for some constant \(\rho > 0\), \(\rho = 0\) or \(\rho < 0\), depending on whether \(g\) is a shrinking, steady, or expanding Yamabe soliton.

On the other hand, as observed by B.H. Gilding, M.A. Peletier and H. Zhang [GP], [PZ], and others ([DS1], [DS2], [V1], [V2]), (1.1) also arises in the study of the self-similar solutions of the degenerate diffusion equation

\[
u_t = \frac{n-1}{m} \Delta u^m \quad \text{in} \ \mathbb{R}^n \times (0, T).
\]
For example (cf. [H1], [V1]) if $v$ is a radially symmetric solution of (1.1) with
\[ \alpha = \frac{2\beta + 1}{1 - m} > 0, \]
then for any $T > 0$ the function
\[ u(x, t) = (T - t)^\alpha v(x(T - t)^\beta) \]
is a solution of (1.4) in $\mathbb{R}^n \times (-\infty, T)$. We refer the reader to the book [V1] and the paper [H1] for the relation between solutions of (1.1) and the other self-similar solutions of (1.4) for the other parameter ranges of $\alpha$, $\beta$.

Note that when $v$ is a radially symmetric solution of (1.1), then $v$ satisfies
\[ n - 1 \quad \frac{m}{m} \left( (v^m)'' + \frac{n - 1}{r} (v^m)' \right) + \alpha v + \beta rv' = 0, \quad v > 0, \quad \text{in } (0, \infty) \]
and
\[ \begin{cases} v(0) = \eta, \\ v'(0) = 0, \end{cases} \]
for some constant $\eta > 0$. Existence of solutions of (1.6), (1.7), for the case $n \geq 3$, $0 < m \leq (n - 2)/n$, $\beta > 0$ and $\alpha \leq \beta(n - 2)/m$ is proved by S.Y. Hsu in [H1]. On the other hand, by the result of [PZ] and Theorem 7.4 of [V1] if (1.2) holds, then there exists a constant $\bar{\beta}$ with $\bar{\beta} = 0$ when $m = \frac{n - 2}{n + 2}$ such that for any $\alpha = \frac{2\beta + 1}{1 - m}$ and $\beta > \bar{\beta}$, there exists a unique solution of (1.6), (1.7). Moreover, if $0 < \alpha = \frac{2\beta + 1}{1 - m}$ and $\beta < \bar{\beta}$, then (1.6), (1.7) have no global solution.

Since the asymptotic behavior of solutions of (1.4) is usually similar to the behavior of the radially symmetric self-similar solutions of (1.4), in order to understand the asymptotic behavior of solutions of (1.4) and the asymptotic behavior of locally conformally flat non-compact gradient Yamabe solitons, it is important to study the asymptotic behavior of the solutions of (1.6), (1.7).

Exact decay rate of the solutions of (1.6), (1.7) for the case
\[ \alpha = \frac{2\beta}{1 - m} > 0 \]
and the case
\[ \frac{2\beta}{1 - m} > \max(\alpha, 0), \]
with $m, n$ satisfying (1.2), was obtained by S.Y. Hsu in [H1]. When (1.2) and (1.3) hold for some constant $\rho > 0$, although it is known ([DS2], [V1]) that solution $v$ of (1.6), (1.7) satisfies $v(r) = O(r^{-\frac{\rho}{n - 2}})$ as $r \to \infty$, nothing is known about the exact decay rate of $v$. In [H2] S.Y. Hsu proved, by using estimates for the scalar curvature of the metric $g = v^{-\frac{4}{n - 2}} dx^2$ where $v$ is a radially symmetric solution of (1.1), that when $m = \frac{n - 2}{n + 2}$, $\beta > \frac{\rho}{n - 2} > 0$,
\[ \lim_{r \to \infty} r^2 v(r) = \frac{(n - 1)(n - 2)}{\rho}. \]

In this paper we will extend the above result and prove the exact decay rate of radially symmetric solution $v$ of (1.1) when (1.2) and (1.3) hold for some constant $\rho > 0$. More precisely we will prove the following theorem.
**Theorem 1.1.** Let \( \eta > 0, \rho > 0, m, n, \alpha, \beta \), satisfy (1.2), (1.3), and
\[
\beta > \frac{m\rho}{n - 2 - mn}.
\]
Suppose \( v \) is a solution of (1.6), (1.7). Then
\[
\lim_{r \to \infty} r^2 v(r)^{1-m} = \frac{2(n-1)(n(1-m) - 2)}{(1-m)(\alpha(1-m) - 2\beta)}.
\]

**Remark 1.2.** The function
\[
v_0(x) = \left( \frac{2(n-1)(n(1-m) - 2)}{\alpha(1-m) - 2\beta} \right)^{\frac{1}{1-m}}
\]
is a singular solution of (1.1) in \( \mathbb{R}^n \setminus \{0\} \). If \( v \) is a solution of (1.1), then for any \( \lambda > 0 \) the function
\[
v_\lambda(x) = \lambda^{\frac{2}{1-m}} v(\lambda x)
\]
is also a solution of (1.1).

**Corollary 1.3.** Let \( \rho, m, n, \alpha, \beta \) satisfy (1.2), (1.3), (1.9). Suppose \( v \) is a radially symmetric solution of (1.1), and \( v_0, v_\lambda \) are given by (1.11) and (1.12), respectively. Then \( v_\lambda(x) \) converges uniformly on \( \mathbb{R}^n \setminus B_R(0) \) to \( v_0(x) \) for any \( R > 0 \) as \( \lambda \to \infty \).

**Corollary 1.4** (cf. [H2]). The metric \( g_{ij} = v^{\frac{4}{n+2}} dx^2 \), \( n \geq 3 \), of a locally conformally flat non-compact gradient shrinking Yamabe soliton where \( v \) is radially symmetric and satisfies (1.1) with \( m = \frac{n-2}{n+2} \), and \( \beta > \frac{\rho}{2} > 0 \), \( \alpha \), satisfying (1.3) has the exact decay rate (1.8).

Since the scalar curvature of the metric \( g_{ij} = v^{\frac{4}{n+2}} dx^2 \), \( n \geq 3 \), where \( v \) is a radially symmetric solution of (1.1) with \( m = \frac{n-2}{n+2} \) is given by ([DS2], [H2])
\[
R(r) = (1-m) \left( \alpha + \beta \frac{rv'(r)}{v(r)} \right),
\]
by Corollary 1.4 and an argument similar to the proof of Lemma 3.4 and Theorem 1.3 of [H2], we obtain the following extensions of Theorem 1.2 and Theorem 1.3 of [H2].

**Theorem 1.5.** Let \( m = \frac{n-2}{n+2} \), \( n \geq 3 \), \( \beta > \frac{\rho}{2} > 0 \), \( \alpha \), satisfy (1.3). Let \( v \) be a radially symmetric solution of (1.1). Then
\[
\lim_{r \to \infty} \frac{rv'(r)}{v(r)} = -\frac{2}{1-m}
\]
and the scalar curvature \( R(r) \) of the metric \( g_{ij} = v^{\frac{4}{n+2}} dx^2 \) satisfies
\[
\lim_{r \to \infty} R(r) = \rho.
\]
If \( K_0 \) and \( K_1 \) are the sectional curvatures of the 2-planes perpendicular to and tangent to the spheres \( \{x\} \times S^{n-1} \), respectively, then
\[
\lim_{r \to \infty} K_0(r) = 0
\]
and
\[
\lim_{r \to \infty} K_1(r) = \frac{\rho}{(n-1)(n-2)}.
\]
**Corollary 1.6.** Let \( \eta > 0 \), \( \rho > 0 \), \( m \), \( n \), \( \alpha \), \( \beta \) satisfy (1.2), (1.3), and (1.9). Suppose \( v \) is a solution of (1.6), (1.7). Then (1.13) holds.

The plan of the paper is as follows. We will prove the boundedness of the function (1.14)\[ w(r) = r^2 v(r)^{1-m} \]
where \( v \) is the solution of (1.1) in section two. We will also find the lower bound of \( w \) in section two. In section three we will prove Theorem 1.1 and Corollary 1.3. We will assume that (1.2), (1.3) hold for some constant \( \rho > 0 \) and let \( v \) be a radially symmetric solution of (1.1) or equivalently the solution of (1.6), (1.7), for some \( \eta > 0 \), and
\[
 w_\infty = \frac{2(n-1)(n(1-m)-2)}{(1-m)(\alpha(1-m) - 2\beta)}
\]
for the rest of the paper. Note that when \( \alpha = n\beta \) and \( \alpha = \frac{2\beta + 1}{1-m} \), the solution of (1.1) is given explicitly by (cf. [DS2])
\[
 v_\lambda(x) = \left( \frac{2(n-1)(n-2-\lambda m)}{(1-m)(\lambda^2 + |x|^2)} \right)^{\frac{1}{1-m}}, \quad \lambda > 0,
\]
which satisfies (1.10).

2. \( L^\infty \) Estimate of \( w \)

**Lemma 2.1.** Let \( \rho > 0 \), \( m \), \( n \), \( \alpha \), \( \beta \) satisfy (1.2) and (1.3) and let \( v \) be a radially symmetric solution of (1.1). Let \( w \) be given by (1.14). Suppose there exists a constant \( C_1 > 0 \) such that
\[
 w(r) \leq C_1 \quad \forall r \geq 1.
\]
Then any sequence \( \{w(r_i)\}_{i=1}^\infty \), \( r_i \to \infty \) as \( i \to \infty \), has a subsequence \( \{w(r'_i)\}_{i=1}^\infty \) such that
\[
 \lim_{r \to \infty} w(r'_i) = \begin{cases} 
 0 & \text{or} \ w_\infty \text{ if } v \notin L^1(\mathbb{R}^n), \\
 0 & \text{or} \ w_1 \text{ if } v \in L^1(\mathbb{R}^n) \text{ and } \beta > 0, \\
 0 & \text{if } v \in L^1(\mathbb{R}^n) \text{ and } \beta \leq 0,
\end{cases}
\]
where
\[
 w_1 = \frac{2(n-1)}{(1-m)\beta} \quad \text{if } \beta > 0.
\]

**Proof.** Let \( \{r_i\}_{i=1}^\infty \) be a sequence such that \( r_i \to \infty \) as \( i \to \infty \). By (2.1) the sequence \( \{w(r_i)\}_{i=1}^\infty \) has a subsequence that we may assume, without loss of generality, to be the sequence itself that converges to some constant \( a \in [0,C_1] \) as \( i \to \infty \). Integrating (1.6) over \( (0,r) \) and simplifying,
\[
 -\frac{n-1}{m} (v^m)'(r) = \beta rv(r) + \frac{\alpha - n\beta}{r^{n-1}} \int_0^r z^{n-1} v(z) \, dz \quad \forall r > 0.
\]
Integrating (2.4) over \( (r,\infty) \), by (2.1) we get
\[
 \frac{n-1}{m} v(r)^m = \beta \int_r^\infty sv(s) \, ds + \int_r^\infty \frac{\alpha - n\beta}{s^{n-1}} \left( \int_0^s z^{n-1} v(z) \, dz \right) \, ds \quad \forall r > 0.
\]
Let \( b = \alpha^{1/m} = \lim_{i \to \infty} r_i^{-m} v(r_i) \). Then by \((2.1), (2.5)\), and the l'Hospital rule,
\[
\frac{(n - 1)}{m} b^m = \frac{(n - 1)}{m} \lim_{i \to \infty} \left( r_i^{-2m} v(r_i) \right)^m
\]
\[
= \beta \lim_{i \to \infty} \int_{r_i^{-2m}}^{r_i^{-2m}} sv(s) ds + \lim_{i \to \infty} \int_{r_i^{-2m}}^{r_i^{-2m}} \frac{\alpha - n \beta}{n - \frac{2}{1 - m}} \left( \int_{0}^{r_i} z^{n-1} v(z) dz \right) ds
\]
\[
= \frac{(1 - m)}{2m} \left( \beta \lim_{i \to \infty} r_i v(r_i) + (\alpha - n \beta) \lim_{i \to \infty} \frac{1}{r_i^{n-1}} \int_{0}^{r_i} z^{n-1} v(z) dz \right)
\]
\[
= \frac{(1 - m)}{2m} \left( \beta b + (\alpha - n \beta) \lim_{i \to \infty} \frac{1}{r_i^{n-2}} \int_{0}^{r_i} z^{n-1} v(z) dz \right).
\]

(2.6)

We now divide the proof into two cases.

Case 1: \( v \notin L^1(\mathbb{R}^n) \). By (2.6) and the l'Hospital rule,
\[
\frac{(n - 1)}{m} b^m = \frac{(1 - m)}{2m} \left( \beta b + \frac{\alpha - n \beta}{n - \frac{2}{1 - m}} \lim_{i \to \infty} r_i^{n-1} v(r_i) \right)
\]
\[
= \frac{(1 - m)}{2m} \left( \beta b + \frac{\alpha - n \beta}{n - \frac{2}{1 - m}} \right)
\]
\[
= \frac{(1 - m)[\alpha(1 - m) - 2\beta]}{2m[n(1 - m) - 2]} b
\]

(2.7) \( \Rightarrow \) \( a = b = 0 \) or \( a = b^{1-m} = w_\infty \).

Case 2: \( v \in L^1(\mathbb{R}^n) \). By (2.6),
\[
\frac{(n - 1)}{m} b^m = \frac{(1 - m)\beta}{2m} b \Rightarrow \left\{ \begin{array}{ll}
    a = b = 0 & \text{or} \ a = b^{1-m} = w_1 \text{ if } \beta > 0,
    \\
    a = b = 0 & \text{if } \beta \leq 0,
\end{array} \right.
\]

By (2.7) and (2.8) the lemma follows.

\[ \Box \]

Remark 2.2. When \( \beta > 0, w_1 > w_\infty \) if and only if \( \alpha > n \beta \).

Corollary 2.3. Suppose there exist constants \( C_1 > C_2 > 0 \) such that
\[
C_2 \leq w(r) \leq C_1 \quad \forall r \geq 1.
\]

Then \((1.10)\) holds.

Lemma 2.4. Let \( \eta > 0, \rho > 0, \beta > 0, m, n, \alpha \leq n \beta \) satisfy \((1.2)\) and \((1.3)\). Then
\[
v(r) \geq \left( \eta^{m-1} + \frac{(1 - m)\beta}{2(n - 1)} \rho^2 \right)^{-\frac{1}{1-m}} \quad \forall r \geq 0.
\]

Hence, there exists a constant \( C_2 > 0 \) such that
\[
w(r) \geq C_2 \quad \forall r \geq 1.
\]
Proof. \((2.9)\) is proved on page 22 of \([DS2]\). For the sake of completeness, we will give a simple different proof here. By \((2.4)\),
\[
-\frac{n-1}{m} (v^m)'(r) \leq \beta r v(r) \quad \forall r > 0
\]
\[
\Rightarrow - (n-1)v^{m-2}(r) \leq \beta r \quad \forall r > 0
\]
\[
\Rightarrow - \frac{n-1}{1-m} \left( v(r)^{m-1} - \eta^{m-1} \right) \leq \frac{\beta}{2} r^2 \quad \forall r > 0
\]
and \((2.9)\) follows. By \((2.9)\), we get \((2.10)\) and the lemma follows. \(\square\)

We now recall a result of \([H2]\). Let \(\eta > 0, \rho > 0, m, n, \alpha \geq n\beta > 0\) satisfy \((1.2)\) and \((1.3)\). Then there exists a constant \(C_1 > 0\) such that \((2.1)\) holds.

**Lemma 2.5** (cf. Lemma 2.3 of \([H2]\)). Let \(\eta > 0, \rho > 0, m, n, 0 < \alpha \leq n\beta\) satisfy \((1.2)\) and \((1.3)\). Then there exists a constant \(C_1 > 0\) such that \((2.1)\) holds.

**Proof.** This result is proved in \([H2]\). For the sake of completeness, we will repeat the proof here. By \((2.4)\), \(v'(r) < 0\) for all \(r > 0\). Then by \((2.4)\),
\[
\frac{n-1}{m} r^{n-1}(v^m)'(r) \leq -\beta r^n v(r) - (\alpha - n\beta) \int_0^r z^{n-1} v(z) \, dz
\]
\[
= -\frac{\alpha}{n} r^n v(r) \quad \forall r > 0
\]
\[
\Rightarrow v^{m-2}(r)v'(r) \leq -\frac{\alpha}{n(n-1)} r \quad \forall r > 0
\]
\[
\Rightarrow v(r) \leq \left( \eta^{m-1} + \frac{\alpha(1-m)}{2n(n-1)} r^2 \right)^{-\frac{1}{m-1}} \leq \left( \frac{2n(n-1)}{\alpha(1-m)} r^{-2} \right)^{\frac{1}{m}} \quad \forall r > 0.
\]
Hence, \((2.1)\) holds with \(C_1 = \frac{2n(n-1)}{\alpha(1-m)}\) and the lemma follows. \(\square\)

**Lemma 2.6.** Let \(\eta > 0, \rho > 0, m, n, 0 < \alpha \leq n\beta\) satisfy \((1.2)\) and \((1.3)\). Then there exists a constant \(C_1 > 0\) such that \((2.1)\) holds.

**Proof.** Let \(A = \{ r \in [1, \infty) : w'(r) \geq 0 \}\). We now divide the proof into two cases.

**Case 1:** \(A \cap [R_0, \infty) \neq \emptyset \quad \forall R_0 > 1\). We will use a modification of the proof of Lemma 3.2 of \([H2]\) to prove this case. By Lemma 2.4 there exists a constant \(C_2 > 0\) such that \((2.10)\) holds. Hence, by \((2.10)\),
\[
\int_0^1 z^{n-1} v(z) \, dz \leq \frac{1}{n(1-m) - 2}.
\]
We now claim that
\[
\limsup_{r \to \infty} \int_A \frac{z^{n-1} v(z) \, dz}{r^n v(r)} \leq \frac{1}{n(1-m) - 2}.
\]
We divide the proof of the above claim into two cases.

**Case (1a):** \(\int_0^\infty z^{n-1} v(z) \, dz < \infty\). By \((2.11)\) we get \((2.12)\).

**Case (1b):** \(\int_0^\infty z^{n-1} v(z) \, dz = \infty\). Since
\[
\frac{d}{dr} (r^n v(r)) = \left( n - \frac{2}{1-m} \right) r^{n-1} v(r) + \frac{1}{1-m} r^{n-2} w^{m-1} (r) w'(r)
\]
\[
> \left( n - \frac{2}{1-m} \right) r^{n-1} v(r) \quad \forall r \in A,
\]
by (2.11) and the l’Hospital rule,
\[
\limsup_{r \to \infty} \frac{\int_0^r z^{n-1}v(z) \, dz}{r^n v(r)} = \limsup_{r \to \infty} \frac{r^{n-1}v(r)}{(n - \frac{2}{1-m}) r^{n-1}v(r) + \frac{1}{1-m} r^{n-2} v(r) w_m(r) w'(r)} \leq \left( n - \frac{2}{1-m} \right)^{-1}
\]
and (2.12) follows. Let 0 < \delta < \frac{\rho}{n(1-m)-2}. By (2.12) there exists a constant \( R_1 > 1 \) such that
\[
\frac{\int_0^r z^{n-1}v(z) \, dz}{r^n v(r)} < \frac{(1-m)}{n(1-m) - 2} + \frac{\delta}{1 + n\beta - \alpha} \quad \forall r \geq R_1, r \in A,
\]
(2.13)
\[
\Rightarrow \int_0^r z^{n-1}v(z) \, dz \leq \left( \frac{(1-m)}{n(1-m) - 2} + \frac{\delta}{1 + n\beta - \alpha} \right) r^n v(r) \quad \forall r \geq R_1, r \in A.
\]
By (2.4) and (2.13),
\[
\frac{n-1}{m} r^{n-1} (v_m)^'(r) \leq -\beta r^n v(r) + \left( \frac{(n\beta - \alpha)(1-m)}{n(1-m) - 2} + \delta \right) r^n v(r)
\leq -\left( \frac{\rho}{n(1-m) - 2} - \delta \right) r^n v(r) \quad \forall r \geq R_1, r \in A,
\]
\[
\Rightarrow (n-1)v^{n-2}v'(r) \leq -\left( \frac{\rho}{n(1-m) - 2} - \delta \right) r \quad \forall r \geq R_1, r \in A.
\]
Hence, there exists a constant \( C_3 > 0 \) such that
\[
\frac{r v'(r)}{v(r)} \leq -C_3 r^2 v(r)^{1-m} = -C_3 w(r) \quad \forall r \geq R_1, r \in A,
\]
(2.14)
\[
\Rightarrow 0 \leq w'(r) = \frac{2w(r)}{r} \left( 1 + \frac{1-m}{2} \cdot \frac{r v'(r)}{v(r)} \right) \leq \frac{2w(r)}{r} \left( 1 - \frac{(1-m)C_3}{2} \cdot \frac{w(r)}{w(r)} \right) \quad \forall r \geq R_1, r \in A,
\]
\[
\Rightarrow w(r) \leq \frac{2}{(1-m)C_3} \quad \forall r \geq R_1, r \in A.
\]
Let \( r_1 \in A \cap [R_1, \infty) \). Then for any \( r' \in (r_1, \infty) \setminus A \), there exists \( r_2 \in A \cap [r_1, \infty) \) such that
\[
w'(r) < 0 \quad \forall r_2 < r \leq r' \quad \text{and} \quad w'(r_2) = 0
\]
(2.15)
\[
\Rightarrow w(r') \leq w(r_2) \leq \frac{2}{(1-m)C_3} \quad \forall r' > r_1, r' \not\in A \quad \text{(by (2.14))}.
\]
By (2.13) and (2.15),
\[
w(r) \leq \frac{2}{(1-m)C_3} \quad \forall r \geq r_1
\]
and (2.11) holds with \( C_1 = \max \left( \frac{2}{(1-m)C_3}, \max_{1 \leq r \leq r_1} w(r) \right) \).
Case 2: There exists a constant $R_0 > 1$ such that $A \cap [R_0, \infty) = \emptyset$. Then $w'(r) < 0$ for all $r \geq R_0$. Hence, (2.1) holds with $C_1 = \max_{1 \leq r \leq R_0} w(r)$ and the lemma follows. □

3. PROOF OF THEOREM 1.1

We first recall a result of [H1]:

Lemma 3.1 (cf. Lemma 2.1 of [H1]). Let $\eta > 0$, $m$, $n$, $\alpha > 0$, $\beta \neq 0$ satisfy (1.2) and

$$\frac{m \alpha}{\beta} \leq n - 2.$$  

Let $v$ be the solution of (1.0), (1.1). Then

$$v(r) + \frac{\beta}{\alpha} rv'(r) > 0 \quad \forall r \geq 0$$  

and

$$v'(r) < 0 \quad \forall r > 0.$$  

Lemma 3.2. Let $\rho > 0$, $m$, $n$, $\alpha > n \beta$ satisfy (1.2), (1.3) and (1.9). Then

$$\lim_{r \to \infty} r^{n-2} v^m(r) = \infty.$$  

Proof. Suppose (3.3) does not hold. Then there exists a sequence $\{r_i\}_{i=1}^{\infty}$, $r_i \to \infty$ as $i \to \infty$, such that $r_i^{n-2} v^m(r_i) \to a_1$ as $i \to \infty$ for some constant $a_1 \geq 0$. By Lemma 2.1, the sequence $\{r_i\}_{i=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself such that $w(r_i) \to a_2$ as $i \to \infty$ where $a_2 = 0$, $w_\infty$, or $w_1$ with $w_1$ being given by (2.3). By (2.5), Lemma 2.6 and the l’Hospital rule,

$$\frac{(n-1)}{m} a_1 = \frac{(n-1)}{m} \lim_{i \to \infty} r_i^{n-2} v(r_i)^m$$  

$$= \beta \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} s v(s) ds}{r_i^{2-n}} + \lim_{i \to \infty} \frac{\int_{r_i}^{\infty} \frac{\alpha-n\beta}{z^{n-1}} \left(\int_0^z \frac{v(z)}{dz}\right) dz}{r_i^{2-n}}$$  

$$= \frac{\beta}{n-2} \lim_{i \to \infty} r_i^n v(r_i) + \frac{\alpha-n\beta}{n-2} \lim_{i \to \infty} \int_0^{r_i} z^{n-1} v(z) dz$$  

$$= \frac{\beta}{n-2} \lim_{i \to \infty} r_i^{n-2} v(r_i)^m \cdot \lim_{i \to \infty} r_i^2 v(r_i)^{1-m} + \frac{\alpha-n\beta}{n-2} \int_0^{\infty} z^{n-1} v(z) dz$$  

$$= \frac{\beta}{n-2} a_1 a_2 + \frac{\alpha-n\beta}{n-2} \int_0^{\infty} z^{n-1} v(z) dz.$$  

Hence,

$$\frac{\alpha-n\beta}{a_1} \int_0^{\infty} z^{n-1} v(z) dz = \frac{(n-1)(n-2)}{m} - \beta a_2.$$  

By (2.4) and (3.3),

$$-(n-1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = \beta \lim_{i \to \infty} r_i^2 v(r_i)^{1-m} + \lim_{i \to \infty} \frac{(\alpha-n\beta)}{r_i^{n-2} v(r_i)^m} \int_0^{r_i} z^{n-1} v(z) dz$$  

$$= \frac{(n-1)(n-2)}{m}.$$  

By (2.4) and (3.3),

$$-(n-1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = \beta \lim_{i \to \infty} r_i^2 v(r_i)^{1-m} + \lim_{i \to \infty} \frac{(\alpha-n\beta)}{r_i^{n-2} v(r_i)^m} \int_0^{r_i} z^{n-1} v(z) dz$$  

$$= \frac{(n-1)(n-2)}{m}.$$
Hence,
\[
(3.6) \quad \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = -\frac{(n - 2)}{m}.
\]

By (1.2), (1.3) and (1.9),
\[
\frac{m \alpha}{\beta} < n - 2
\]
holds. Hence, there exists a constant \(\varepsilon > 0\) such that
\[
(3.7) \quad \frac{m \alpha}{\beta} < n - 2 - \varepsilon.
\]

By (3.7) and Lemma 3.1, (3.1) and (3.2) hold. Then by (3.1), (3.2) and (3.7),
\[
0 > rv'(r) > -\frac{\alpha \beta}{\beta} > -\frac{n - 2}{m} + \frac{\varepsilon}{m} \quad \forall r > 0
\]
\[
\Rightarrow \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} \geq -\frac{n - 2}{m} + \frac{\varepsilon}{m},
\]
which contradicts (3.6). Hence, no such sequence \(\{r_i\}_{i=1}^{\infty}\) exists, and the lemma follows. \(\square\)

**Lemma 3.3.** Let \(\rho > 0, m, n, \alpha > n \beta\) satisfy (1.2), (1.3) and (1.9). Then there exists a constant \(\varepsilon \in (0, \min(1, w_{\infty}/2))\) such that for any \(R_0 > 1\) there exists \(r' > R_0\) such that
\[
w(r') \geq \varepsilon.
\]

**Proof.** Suppose the lemma is false. Then
\[
(3.9) \quad \lim_{r \to \infty} w(r) = 0.
\]

We claim that
\[
(3.10) \quad \lim_{r \to \infty} \frac{r v'(r)}{v(r)} = 0.
\]

By the proof of Lemma 3.2 there exists a constant \(\varepsilon > 0\) such that (3.8) holds. Suppose (3.10) does not hold. Then by (3.3) and (3.9) there exists a sequence \(\{r_i\}_{i=1}^{\infty}\), \(r_i \to \infty\) as \(i \to \infty\), such that \(r_i v'(r_i)/v(r_i) \to a_3\) as \(i \to \infty\) for some constant \(a_3\) satisfying
\[
(3.11) \quad -\frac{n - 2}{m} + \frac{\varepsilon}{m} \leq a_3 < 0
\]
and (3.5) holds. By Lemma 3.2 (3.5), (3.9) and (3.11), we get
\[
-(n - 1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = 0
\]
if \(v \in L^1(\mathbb{R}^n)\),
and if \(v \notin L^1(\mathbb{R}^n)\), then by the l'Hospital rule,
\[
-(n - 1) \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = (\alpha - n \beta) \lim_{i \to \infty} \frac{r_i^{n-1}v(r_i)}{(n - 2)r_i^{n-3}v(r_i)m + mr_i^{n-2}v(r_i)m-1v'(r_i)}
\]
\[
= (\alpha - n \beta) \lim_{i \to \infty} \frac{r_i^2v(r_i)^{1-m}}{n - 2 + m(r_i v'(r_i)/v(r_i))}
\]
\[
= \frac{\alpha - n \beta}{n - 2 + ma_3} \cdot \lim_{i \to \infty} r_i^2v(r_i)^{1-m}
\]
\[
= 0.
\]
Hence,
\[ a_3 = \lim_{i \to \infty} \frac{r_i v'(r_i)}{v(r_i)} = 0, \]
which contradicts (3.11). Thus, no such sequence \( \{r_i\}_{i=1}^{\infty} \) exists and (3.10) follows. Since
\[ w'(r) = \frac{2w(r)}{r} \left( 1 + \frac{1-m}{2} \cdot \frac{rv'(r)}{v(r)} \right), \]
by (3.10) there exists a constant \( R_0 > 0 \) such that
\[ w'(r) > 0 \quad \forall r \geq R_0, \]
which contradicts (3.9) and the lemma follows. 

We are now ready for the proof of Theorem 1.1. 

Proof of Theorem 1.1 We divide the proof into two cases. 

Case 1: \( \alpha \leq n\beta \). By Corollary 2.3, Lemma 2.4 and Lemma 2.6, we get (1.10). 

Case 2: \( \alpha > n\beta \). By Lemma 2.5 there exists a constant \( C_1 > 0 \) such that (2.1) holds. Let \( 0 < \varepsilon < \min(1, w_\infty/2) \) be as in Lemma 3.3. Suppose there exists a sequence \( \{r_i\}_{i=1}^{\infty} \) such that \( w(r_i) < \varepsilon \) for all \( i \in \mathbb{Z}^+ \). Then by Lemma 3.3 there exists a subsequence of \( \{r_i\}_{i=1}^{\infty} \) which we may assume without loss of generality to be the sequence itself and a sequence \( \{r_i'\}_{i=1}^{\infty} \) such that \( r_i < r_i' < r_{i+1} \) for all \( i = 1, 2, \ldots \) and
\[ w(r_i) < \varepsilon < w(r_i') \quad \forall i = 1, 2, \ldots. \]
By (3.12) and the intermediate value theorem, for any \( i = 1, 2, \ldots \), there exists \( a_i \in (r_i, r_i') \) such that
\[ w(a_i) = \varepsilon \quad \forall i = 1, 2, \ldots. \]
Hence, \( a_i \to \infty \) as \( i \to \infty \) and
\[ \lim_{i \to \infty} w(a_i) = \varepsilon. \]
This contradicts Lemma 2.1 and Remark 2.2. Hence no such sequence \( \{r_i\}_{i=1}^{\infty} \) exists. Thus there exists a constant \( R_1 > 1 \) such that \( w(r) < \varepsilon \) for all \( r \geq R_1 \). Hence (2.10) holds with \( C_2 = \min(\varepsilon, \min_{1 \leq i \leq R} w(r)) > 0 \). By Corollary 2.3 we get (1.10) and the theorem follows. 

Proof of Corollary 1.3 By Theorem 1.1
\[ |x|^2 v_\lambda(x)^{1-m} = (\lambda|x|^2 v(\lambda x))^{1-m} \]
\[ \to \frac{2(n-1)(n(1-m) - 2)}{(1-m)(\alpha(1-m) - 2\beta)} \quad \text{uniformly on } \mathbb{R}^n \setminus B_R(0) \]
as \( \lambda \to \infty \) for any \( R > 0 \) and the corollary follows.
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