DELIGNE PAIRINGS AND DISCRIMINANTS OF ALGEBRAIC VARIETIES

H. MANILAL KAPADIA

Abstract. Let $V$ be a finite dimensional complex vector space, $V^*$ its dual and let $X \subset \mathbb{P}(V)$ be a smooth projective variety of dimension $n$ and degree $d \geq 2$. For a generic $n$-tuple of hyperplanes $(H_1, \ldots, H_n) \in \mathbb{P}(V^*)^n$, the intersection $X \cap H_1 \cap \cdots \cap H_n$ consists of $d$ distinct points. We define the “discriminant of $X$”, to be the set $D_X$ of $n$-tuples for which the set-theoretic intersection is not equal to $d$ points. Then $D_X \subset \mathbb{P}(V^*)^n$ is a hypersurface and the set of defining polynomials, which is a one-dimensional vector space, is called the “discriminant line”. We show that this line is canonically isomorphic to the Deligne pairing $\langle KL^n, \ldots, L \rangle$ where $K$ is the canonical line bundle of $X$ and $L \to X$ is the restriction of the hyperplane bundle. As a corollary, we obtain a generalization of Paul’s formula [2] which relates the Mabuchi K-energy on the space of Bergman metrics to $\Delta_X$, the “hyperdiscriminant of $X$”.

1. Introduction

Let $V$ be a finite dimensional complex vector space, $V^*$ its dual, and $X \subset \mathbb{P}(V)$ be a smooth projective manifold of dimension $n$ and degree $d \geq 2$.

Definition 1. The discriminant variety of $X$ is the projective variety $D(X)$ defined by

$$D(X) = \{(H_1, \ldots, H_n) \in \mathbb{P}(V^*)^n : \# \{X \cap H_1 \cap \cdots \cap H_n \} \neq d\}$$

Thus, $(H_1, \ldots, H_n) \in D(X)$ if and only if there exists a point $x \in X \cap H_1 \cdots \cap H_n$ such that $\dim (H_1 \cap \cdots \cap H_n \cap ET_x X) \geq 1$, where $ET_x X \subset \mathbb{P}(V)$ is the imbedded tangent space $x$.

An essential property of $D(X) \subset \mathbb{P}(V^*)^n$ is that it has codimension one. Thus we may consider the one-dimensional vector space $\hat{D}(X)$ of polynomials of degree $(d,d,\ldots,d)$ which vanish on $D_X$. We call $\hat{D}(X)$ the discriminant line.

If $\pi : \mathcal{X} \to S$ is a flat family of varieties of dimension $n$ and degree $d$ over a base $S$, then one can similarly define $\hat{D}_X \to S$, a line bundle whose fiber at $s \in S$ equals the one dimensional vector space $\hat{D}(X_s)$ where $X_s = \pi^{-1}(x)$.

It is useful to compare the definition of $D(X)$ to that of $C(X)$, the Chow variety:

Definition 2. The Chow variety of $X$ is the projective variety $C(X)$ defined as follows

$$C(X) = \{(H_0, \ldots, H_n) \in \mathbb{P}(V^*)^{n+1} : \# \{X \cap H_0 \cdots \cap H_n \} \neq 0\}$$

The variety $C(X) \subset \mathbb{P}(V^*)^{n+1}$ also has codimension one and the one-dimensional vector space of polynomials of degree $(d,d,\ldots,d)$ which vanish on $C(X)$ is called the Chow line.
In the terminology of the work by Gelfand-Kapranov-Zelevinsky [1], the variety $D_X$ is essentially the “first higher associated hypersurface of $X$”.

Let $K \to X$ the canonical line bundle, and $L \to X$ the restriction of the hyperplane line bundle. Zhang’s theorem [7] says that the Deligne pairing $\langle L, ..., L \rangle$ of $n+1$ copies of $L$ is canonically isomorphic to the Chow line.

In this paper we show that the Deligne pairing $\langle KL^n, L, L, ..., L \rangle$ of $KL^n$ with $n$ copies of $L$ is canonically isomorphic to the discriminant line. As an application, we conclude, via the theorem of Phong-Ross-Sturm [6], that the sub-dominant term in the Mumford-Knudsen expansion is equal to the discriminant of $X$. A second application is to the asymptotics of the K-energy on the space of Bergman metrics: The Mabuchi bundle $\mathcal{M}$, which was defined and studied by Phong-Sturm [3, 4, 5]. They showed its associated Deligne metric is precisely the Mabuchi K-energy. Combining this with our result, we show that the K-energy on the space of Bergman metrics equals the log of the Deligne norm of the Discriminant Point minus the log of the Deligne norm of the Chow Point (see Corollary 2 for the statement). This generalizes, and makes more precise, the theorem of Paul [2] (in which the same formula is proved under a restrictive hypothesis).

2. Deligne Pairings

We outline the basic theory, following closely the paper of Zhang [7]. Let $\pi : \mathcal{X} \to S$ be a flat projective morphism of integral schemes defined over $\mathbb{C}$, of pure relative dimension $n$. Let $L_0, ..., L_n$ be line bundles on $X$. Then the Deligne pairing $\langle L_0, ..., L_n \rangle(\mathcal{X}/S) \to S$ is a line bundle on $S$ which is defined as follows: A section of $\langle L_0, ..., L_n \rangle$ over a small open set $U \subseteq S$ is a symbol $\langle l_0, ..., l_n \rangle$ where the $l_j : U \to \mathcal{X}$ are rational sections whose divisors have empty intersection. The relation between the symbols is given as follows: If $f$ is a generic rational function on $U$ and if

$$\prod_{i \neq j} \text{div}(l_i) = \sum_k n_k Y_k$$

is flat over $S$, then

$$\langle l_0, ..., fl_j, ..., l_n \rangle = \prod_k \text{Norm}_{Y_k/S}(f)^{n_k} \langle l_0, ..., l_n \rangle$$

We summarize below some of the properties of Deligne pairings which will be needed.

2.1. Projection Formulas.

2.1.1. With $n$ pullbacks.

Let $\phi : X \to Y$ and $\pi : Y \to S$. Let $m = \text{dim}(X/Y)$ and $n = \text{dim}(Y/S)$.

Let $\mathcal{K}_0, ..., \mathcal{K}_m$ be line bundles on $X$. Let $\mathcal{L}_1, ..., \mathcal{L}_n$ be line bundles on $Y$.

Then

$$\langle \mathcal{K}_0, ..., \mathcal{K}_m, \phi^* \mathcal{L}_1, ..., \phi^* \mathcal{L}_n \rangle(\mathcal{X}/S) = \langle \langle \mathcal{K}_0, ..., \mathcal{K}_m \rangle, \mathcal{L}_1, ..., \mathcal{L}_n \rangle(\mathcal{Y}/S)$$
The map is given by $F : \langle k_0, \ldots, k_m, \phi^* l_1, \ldots, \phi^* l_n \rangle \mapsto \langle \langle k_0, \ldots, k_m \rangle, l_1, \ldots, l_n \rangle$.

2.1.2. With $n + 1$ pullbacks.

Let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be line bundles on $X$. Let $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n$ be line bundles on $Y$.

Then

$$\langle \mathcal{K}_1, \ldots, \mathcal{K}_m, \phi^* \mathcal{L}_0, \ldots, \phi^* \mathcal{L}_n \rangle(X/S) = \langle \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle(Y/S)^D$$

where $D = \text{deg}[c_1(\mathcal{K}_{1\eta}) \cdot c_1(\mathcal{K}_{2\eta}) \cdots c_1(\mathcal{K}_{m\eta})]$ and $\eta$ is a generic point on $Y$. In other words, $D$ is the number of points in $\text{div}(\mathcal{K}_1) \cap \cdots \cap \text{div}(\mathcal{K}_m)$ in a generic fiber.

2.1.3. With $n + 2$ pullbacks.

Let $\mathcal{K}_1, \ldots, \mathcal{K}_{m-1}$ be line bundles on $X$. Let $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{n+1}$ be line bundles on $Y$.

Then

$$\langle \mathcal{K}_1, \ldots, \mathcal{K}_{m-1}, \phi^* \mathcal{L}_0, \ldots, \phi^* \mathcal{L}_{n+1} \rangle(X/S) = \mathcal{O}_S$$

2.2. Induction Formula.

Let $\pi : X \to S$ and $\mathcal{L}_0, \ldots, \mathcal{L}_n$ as before. Let $l$ be a rational section of $\mathcal{L}_n$. Assume all components of $\text{div}(l)$ are flat over $S$. Then

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle(X/S) = \langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1} \rangle(\text{div}(l)/S)$$

3. Metrics on Deligne Pairing

Assume that $h_j$ is a smooth metric on $\mathcal{L}_j$. If $\mathcal{X}$ is not smooth, then this means that there is a smooth manifold $\mathcal{X}'$, a smooth line bundle $\mathcal{L}_j' \to \mathcal{X}'$, and a smooth metric $h_j'$ on $\mathcal{L}_j$ whose restriction to $\mathcal{X}$ equals $h_j$. Then the Deligne metric $\langle h_0, \ldots, h_n \rangle$ is a metric on the line bundle $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle$ which is defined inductively by the formula

$$\log \|\langle l_0, \ldots, l_n \rangle\| = \log \|\langle l_0, \ldots, l_{n-1} \rangle\| + \int_{\mathcal{X}/S} \log |l_n| \omega_0 \wedge \cdots \wedge \omega_{n-1}$$

where $\omega_j = -\frac{i}{2\pi} \partial \bar{\partial} \log |l_j|^2$.

The inductive formula (3.7) implies that (2.6) is an isometry

$$\langle h_0, \ldots, h_n \rangle = \langle h_0, \ldots, h_{n-1} \rangle \exp \left( -\int_{\mathcal{X}/S} \log |l_n| \omega_0 \wedge \cdots \wedge \omega_{n-1} \right)$$

If $\phi_0, \ldots, \phi_n$ are smooth functions on $\mathcal{X}$, formula (3.7) also implies

$$\langle h_0 e^{-\phi_0}, \ldots, h_n e^{-\phi_n} \rangle = \langle h_0, \ldots, h_n \rangle \exp(-E(\phi_0, \ldots, \phi_n))$$

where

$$E(\phi_0, \ldots, \phi_n) = \sum_{j=0}^{n} \int_{\mathcal{X}/S} \phi_j \left( \bigwedge_{k<j} \omega_{\phi_k} \right) \wedge \left( \bigwedge_{k>j} \omega_k \right)$$
and \( \omega_{\phi_k} = \omega_k + \frac{i}{2} \partial \overline{\partial} \phi_k \). In particular, if \( L_0 = \cdots = L_n \) and \( h_0 = \cdots = h_n \), then setting \( E_X(\phi) = E(\phi, \ldots, \phi) \) we obtain

\[
E_X(\phi) = \sum_{j=0}^{n} \int_{X/S} \phi_j \omega_j^\phi \omega^{n-j}
\]

which coincides with the well known Aubin-Yau functional.

A simple consequence of these formulas which will later be useful is the following:

**Proposition 1.** Let \( X \) be a smooth projective variety of dimension \( n \), \( L_0, \ldots, L_n \to X \) holomorphic line bundles, and \( h_j \) a hermitian metric on \( L_j \). Let \( G \) be a semi-simple Lie group acting on \( L \to X \), and define \( \phi^*_j \) by the following formula.

\[
\sigma^* h_j = h e^{-\phi^*_j}
\]

Then \( E(\phi^*_0, \ldots, \phi^*_n) = 0 \).

**Proof.** Define \( \rho : G \to \mathbb{C}^\times \) by the formula

\[
\langle \sigma^* s_0, \ldots, \sigma^* s_n \rangle = \rho(\sigma) \langle \sigma_0, \ldots, \sigma_n \rangle
\]

Then \( ||\langle \sigma^* s_0, \ldots, \sigma^* s_n \rangle|| = ||\rho(\sigma)|| \cdot ||\langle \sigma_0, \ldots, \sigma_n \rangle|| \) so \( E(\phi^*_0, \ldots, \phi^*_n) = -\log ||\rho(\sigma)|| \). On the other hand, \( \rho : G \to \mathbb{C}^\times \) is a homomorphism which must be trivial since \( G \) is semi-simple.

Example. Let \( X = \mathbb{P}^N \), \( L_j = O(1) \) and \( h = h_{FS} \). Then for every \( \sigma \in SL(N+1) \) we have

\[
E_{\mathbb{P}^N}(\phi^\sigma) = 0
\]

4. The Mabuchi Line Bundle

Let \( \pi : \mathcal{X} \to S \) be as above, and assume \( K_{\mathcal{X}/S} \), the relative canonical bundle, is well defined and let \( h \) be a positively curved metric on \( \mathcal{L} \) with curvature \( \omega \). Define \( h^{-1}_K \), which as a metric on \( K^{-1} \), by the formula \( h^{-1}_K = \omega^n \). Phong-Sturm [3, 4] introduced the Mabuchi line bundle \( \mathcal{M}_h \), which is the hermitian bundle

\[
\mathcal{M}_h = \langle K, \mathcal{L}, \ldots, \mathcal{L} \rangle_{\mathcal{L}^{-1}} \langle L, \ldots, L \rangle_{c_1(L)^\mu}
\]

where \( c_1(L)^n \) is computed on a generic fiber, and \( \mu \in \mathbb{Q} \) is uniquely determined by requiring that the metric is scale invariant, that is, invariant under \( h \mapsto \lambda h \) for \( \lambda \) a positive real number. If follows from the definitions that \( nc_1(K)c_1(L)^{n-1} - \mu(n+1)c_1(L)^n = 0 \) so

\[
\mu = \frac{n}{n+1} \frac{c_1(K)c_1(L)^{n-1}}{c_1(L)^n}
\]

Then, as shown in [4],

\[
\mathcal{M}_{h e^{-\phi}} = \mathcal{M}_h \exp(-\nu(\phi_s))
\]

where for \( s \in X \) with \( X_s \) a smooth fiber, \( \nu(s, \phi) \) is the Mabuchi K-energy \( \phi_s = \phi|_{X_s} \).

*Zhang had also deduced this bundle earlier in a 1993 letter to Deligne.*
For our purposes, it is more convenient to rewrite as follows:

\[(4.14) \quad M_{c_1(L)^n} = \langle K \mathcal{L}^n, \mathcal{L}, ..., \mathcal{L} \rangle \langle \mathcal{L}, ..., \mathcal{L} \rangle^{-\mu - n}\]

The theorem of S. Zhang shows that \(\langle \mathcal{L}, ..., \mathcal{L} \rangle\) is canonically isomorphic to the Chow bundle. We shall use Zhang’s approach to prove \(\langle K \mathcal{L}^n, \mathcal{L}, ..., \mathcal{L} \rangle\) is canonically isomorphic to the Discriminant bundle.

5. Tangent bundle for projective space

Let \(V\) be a complex vector space of dimension \(N + 1\) and \(\mathbb{P}(V) = \{ x \subseteq V : \dim(x) = 1 \}\).

Let \(O(1) \rightarrow \mathbb{P}(V)\) be the hyperplane line bundle.

If \(0 \neq \hat{x} \in V\) and \(x = \mathbb{C} \cdot \hat{x} \in \mathbb{P}(V)\) then we have a canonical map \(I_{\hat{x}} : V = T_{\hat{x}}V \rightarrow T_{\hat{x}}\mathbb{P}(V)\).

Since \(\ker(I_{\hat{x}}) = x\) we see \(I_{\hat{x}} : V/x \rightarrow T_{\hat{x}}\mathbb{P}(V)\) is an isomorphism, or equivalently, the map \(I_{\hat{x}}^* : (V/x)^* \rightarrow T_{\hat{x}}^*\mathbb{P}(V)\) is an isomorphism.

Let \(J_{\hat{x}} : T_{\hat{x}}\mathbb{P}(V) \rightarrow V/x\) be the inverse. Then for \(\alpha \in \mathbb{C}^*\) we have \(J_{\alpha \hat{x}} = \alpha J_{\hat{x}}\) so we see \(J : O(1) \rightarrow (V/x)^* \otimes T_{\hat{x}}\mathbb{P}(V)\) is a vector bundle map and \(O(1) \otimes V/x \rightarrow T_{\hat{x}}\mathbb{P}(V)\) is a canonical isomorphism:

\[(5.15) \quad O(-1) \otimes T_{\hat{x}}\mathbb{P}(V) = V/x\]

Alternatively,

\[T_{\hat{x}}^*\mathbb{P}(V) = (V/x)^* \otimes O_{\hat{x}}(-1)\]

6. The Deligne metric

Let \(Y\) be a projective manifold of dimension \(m\), \(L \rightarrow Y\) an ample line bundle, and \(d\) a positive integer. Let \(V = c_1(L)^n\) and \(\mathcal{N} \rightarrow \mathbb{P}(H^0(Y, L^d))\) be the hyperplane line bundle.

Let \(h\) be a positively curved hermitian metric on \(L\) and \(\omega = -i\partial \bar{\partial} \log h > 0\) its curvature. Let \(D(h)\) be the norm on the vector space \(H^0(Y, L^d)\) defined by the following formula. If \(0 \neq f \in H^0(Y, L^d)\) then

\[(6.16) \quad \log \|f\|_{D(h)}^2 = \frac{1}{V} \int_Y \log |f|^2_{h^{\omega^m}}\]

where \(V = \int_X 1 \omega^n\). In particular, \(D(h)\) makes \(\mathcal{N}\) into a hermitian line bundle.

Remark: Note that \(D(h)\) is not equal to \(\text{Hilb}(h)\), but \(D(h) = e^{\psi} \text{Hilb}(h)\) for some bounded smooth function \(\psi\) on \(H^0(Y, L^d)\) with the property: \(\psi(\lambda v) = \psi(v)\) for all positive real numbers \(\lambda\).

Let \(0 \neq f \in H^0(X, L^d)\) and assume \(Z = \{ f = 0 \} \subseteq Y\) is smooth.

The map

\[I_f : \langle L, ..., L \rangle_Z \rightarrow \langle L, ..., L \rangle_Y\]
In this section we give a slightly modified version of Zhang’s proof. Let
\[ N_{[f]}^{-1} = \langle L, ..., L^d \rangle_Y \otimes \langle L, ..., L \rangle_Z^{-1} \]
Equivalently there is a canonical isomorphism
\[ J_{[f]} : N_{[f]}^{-1} \rightarrow \langle L, ..., L \rangle_Z \otimes \langle L, ..., L \rangle_Y^{-1} \]
which is easily seen to be an isometry (with metric \( D(h) \) on the left and the Deligne metric on the right). Moreover, \( J \) is \( G \subseteq \text{GL}(H^0(Y, L^d)) \) equivariant, where \( G = \text{Aut}(Y, L) \).

Let \( L \rightarrow Y \) be a holomorphic line bundle on a projective manifold \( Y \) and \( h \) a smooth metric on \( L \). Suppose \( G \subseteq \text{Aut}(Y, L) \) is a semi-simple Lie group, and write \( \sigma^*h = he^{-\phi_\sigma} \) for \( \sigma \in G \).

**Corollary 1.** Let \( f \in H^0(Y, L^d) \) be such that \( Z = \{ f = 0 \} \) is a smooth sub manifold. If \( E_Z \) is the Aubin-Yau functional on \( Z \) \( (3.10) \) then for all \( \sigma \in G \) we have
\[ E_Z(\phi^\sigma) = \frac{1}{V} \log \left( \frac{\|f^\sigma\|^2_{D(h)} }{\|f\|^2_{D(h)}} \right) \]

7. **The theorem of Zhang**

In this section we give a slightly modified version of Zhang’s proof. Let \( X(n) \subseteq \mathbb{P}(V) \) of degree \( d \), and write \( L \rightarrow \mathbb{P}(V) \) and \( M \rightarrow \mathbb{P}(V^*) \) for the hyperplane bundles. Let \( \mathbb{P} = \mathbb{P}(V^*)^{n+1} \) and \( M_i = \pi_i^* M \rightarrow \mathbb{P} \). Let
\[ \mathcal{M} = \pi_1^* M \otimes \cdots \otimes \pi_{n+1}^* M = M_1 \otimes \cdots \otimes M_{n+1} \rightarrow \mathbb{P} \]
Let \( \pi_X : X \times \mathbb{P} \rightarrow X \) and \( \pi_P : X \times \mathbb{P} \rightarrow \mathbb{P} \) be the projection maps and consider
\[ B = \langle \pi_P^* \mathcal{M}, ..., \pi_P^* \mathcal{M}, \pi_X^* L \otimes \pi_1^* M_1, ..., \pi_X^* L \otimes \pi_{n+1}^* M_{n+1} \rangle_{X \times \mathbb{P}/s} \]
where \( m = \dim \mathbb{P} \) and \( s \) is a point. We evaluate \( B \) in two different ways. First, let
\[ \Gamma = \{(x, H_1, ..., H_{n+1}) \in X \times \mathbb{P} : x \in H_1 \cap \cdots \cap H_{n+1} \} \]
and let \( Z = \pi_\mathbb{P}(\Gamma) \). Then \( Z = C(X) \subseteq \mathbb{P} \) is the Chow hypersurface of \( X \). The line \( \langle \mathcal{M}, ..., \mathcal{M} \rangle_{Z/s} = \langle \mathcal{M}, ..., \mathcal{M} \rangle_Z \) is called the Chow line.

Next we define a section \( s_i \) of \( \pi_X^* L \otimes \pi_i^* M_i \) as follows: \( s_i(x, H) \in x^* \otimes H^* = \text{Hom}(x \otimes H, \mathbb{C}) \) is the restriction of the canonical paring \( \text{Hom}(V \otimes V^*, \mathbb{C}) \), in other words
\[ s_i(x, H)(z, \lambda) = \lambda(z) \]
for all \( z \in x \) and \( \lambda \in H \). Note that \( s_i(x, H) = 0 \) if and only if \( x \in H \). Thus, applying \( (2.6) \) a total of \( n + 1 \) times we obtain:
\[ B = \langle \mathcal{M}, \mathcal{M}, ..., \mathcal{M} \rangle_{Z/s} \]
On the other hand, expanding the last \( n + 1 \) terms on (7.19) we have \( B = B_1 \otimes B_2 \) where

\[
B_1 = \langle \pi_1^* M, \ldots, \pi_n^* M, \pi_X^* L, \ldots, \pi_X^* L \rangle_{X \times \mathbb{P}}
\]

and

\[
B_2 = \prod_{i=1}^{n+1} \langle \pi_i^* M, \ldots, \pi_i^* M, \pi_X^* L \otimes \pi_1^* M_1, \ldots, \pi_i^* M_i, \ldots \pi_X^* L \otimes \pi_n^* M_{n+1} \rangle_{X \times \mathbb{P}}
\]

Now (2.4) gives

\[
B_1 = \langle L, \ldots, L \rangle_{X}^{\deg(M)}
\]

If we expand the last \( n + 1 \) terms in \( B_2 \), using (2.5) the only terms which survive are those of the form

\[
\langle \pi_i^* M_i, \pi_X^* L, \ldots \pi_X^* L \rangle_{X \times \mathbb{P}}
\]

since, by (2.5), the other terms vanish. Applying (2.4) once again:

\[
B_2 = \prod_{i=1}^{n+1} \langle M, \ldots, M, M_i \rangle_{\mathbb{P}}^d = \langle M, \ldots, M \rangle_{\mathbb{P}}^d
\]

where \( d = c_1(L)^n \). Since \( \deg(M) = 1 \) we conclude

\[
\langle L, \ldots, L \rangle_X = \langle M, \ldots, M \rangle_{\mathbb{P}} \otimes \langle M, \ldots, M \rangle_{\mathbb{P}}^{-d}
\]

Combining with (6.17) we obtain Zhang’s theorem.

8. Deligne pairings and discriminants

Let \( V \) be a vector space over \( \mathbb{C} \) of dimension \( N + 1 \) and \( X \subseteq \mathbb{P}(V) \) a smooth projective variety of degree \( d > 1 \). Recall that \( D(X) \subseteq \mathbb{P} = \mathbb{P}(V^*) \) is a hypersurface. Let \( L \to \mathbb{P}(V) \) and \( M \to \mathbb{P}(V^*) \) be the hyperplane line bundles, let \( M_j = p_j^* M \to \mathbb{P} \) and let \( M \to \mathbb{P} \) be the line bundle \( M = M_1 \otimes \cdots \otimes M_n \).

**Theorem 1.**

\[
\langle KL^n, L, \ldots, L \rangle_X = \langle M, \ldots, M \rangle_{D(X)} \otimes \langle M, \ldots, M \rangle_{\mathbb{P}}^{-d}
\]

**Proof.** Let \( m = \dim(\mathbb{P}) \) and

\[
B = \langle \pi_1^* M, \ldots, \pi_n^* M, \pi_X^* (K \otimes L^n) \otimes \pi_1^* M, \pi_X^* L \otimes \pi_1^* M_1, \ldots, \pi_X^* L \otimes \pi_1^* M_{n+1} \rangle_{X \times \mathbb{P}} \to X \times \mathbb{P}
\]
On the one hand,

\[ \mathcal{B} = \langle \mathcal{M}, \mathcal{M}, \ldots, \mathcal{M} \rangle_d \otimes \langle KL^n, L, \ldots, L \rangle_X, \]

where

\[ d = (n + 1)c_1(L)^n + c_1(K)c_1(L)^{n-1}. \]

On the other hand,

\[ \mathcal{B} = \langle \pi_0^* \mathcal{M}, \ldots, \pi_0^* \mathcal{M}, \pi_X^*(K \otimes L^n) \otimes \pi_0^* \mathcal{M} \rangle_{\Gamma'} \]

where

\[ \Gamma' = \{ (x, H_1, \ldots, H_n) \in X \times \mathbb{P} : x \in H_1 \cap \cdots \cap H_n \} \]

Now we wish to define a holomorphic section

\[ s : \Gamma' \rightarrow \pi_0^* X \otimes (K \otimes L^n) \otimes \pi_0^* \mathcal{M}. \]

We define

\[ s(x, H_1, \ldots, H_n) = (\Lambda^n(T_X^*) \otimes O_x(\mathbb{P}) \otimes H_1 \otimes \cdots \otimes H_n^*) \]

as follows: Since \( O_x(-1) \otimes H_j \subseteq T_X^* \mathbb{P}, \) if \( \eta_j \in O_x(-1) \otimes H_j \) then \( \eta_j \in T_x \mathbb{P}^* \) so \( \eta_1 \wedge \cdots \wedge \eta_n \in \Lambda^n T_x^* \mathbb{P}^* \). Hence if \( \omega \in \Lambda^n T_x^* \mathbb{P}^* \) we have a canonical multilinear map

\[ [\Lambda^n(T_X^*)] \times [O_x(-1) \otimes H_1] \times \cdots [O_x(-1) \otimes H_n] \rightarrow \mathbb{C} \]

given by

\[ (\omega, \eta_1, \ldots, \eta_n) \mapsto [\eta_1 \wedge \cdots \wedge \eta_n](\omega) \]

Thus

\[ \mathcal{B} = \langle \mathcal{M}, \ldots, \mathcal{M} \rangle_D \]

where \( D = \{ s = 0 \} \subseteq \Gamma' \). This proves the theorem.

Combining Theorem 1 and (4.13) we obtain:

**Corollary 2.** Let \( X \subseteq \mathbb{P}^N \) be a smooth variety of dimension \( n \) and degree \( d \geq 2 \). Then

\[ (8.26) \quad \nu_\omega(\phi_\sigma) = \text{deg}(C_X) \log \frac{\|D_X^\sigma\|}{\|D_X\|^2} - \text{deg}(D_X) \log \frac{\|C_X\|^2}{\|D_X\|^2} \]

where \( \nu_\omega \) is the Mabuchi K-energy and the norm is the Deligne norm defined by (6.16).

Remark: Suppose that \( X \subseteq \mathbb{P}^N \) is a smooth variety as above whose dual defect vanishes (which holds in the case where \( X \) is linearly normal). Then Paul [2] defines \( \Delta_X \), the \( X \)-hyperdiscriminant of \( X \) to be the dual variety of \( X \times \mathbb{P}^{n-1} \) viewed as a sub variety of \( \mathbb{P}^{nN+n} \) via the Segre imbedding. Using different methods, he proves formula (8.26) with \( D_X \) replaced by \( \Delta_X \) and with the Deligne norm replaced by an inexplicit norm. It is not hard to show that \( D_X \) and \( \Delta_X \) are canonically isometric in the case where the dual defect vanishes. Thus Corollary [2] may be viewed as a generalization of Paul’s theorem: we don’t place any requirement on the dual defect of \( X \). Also, our norm is explicitly given by formula (6.16).

Remark: In order to simplify notation, we have restricted to the case where the base \( S \) is a single point. One can generalize our results to the case where the base is arbitrary: The
Chow line and the Discriminant line, which are one dimensional vector spaces when $S$ is a point, become line bundles on $S$ when $S$ has positive dimension.

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**References**

[1] Gelfand, I.M., M.M. Kapranov and A.V. Zelevinsky, “Discriminants, Resultants and Multidimensional Determinants”, Mathematics: Theory and Applications, Birkäuser (1994)

[2] Paul, S., “Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics”, Ann. of Math. 75 (2012), 255–296

[3] Phong, D.H. and J. Sturm, “Stability, Energy Functionals and Kähler-Einstein metrics”, Communications in Analysis and Geometry 11 (2003), 563–595

[4] Phong, D.H. and J. Sturm, “The Futaki Invariant and the Mabuchi Energy of a Complete Intersection”, Comm. in Analysis and Geometry 104 (2004), 77–105

[5] Phong, D.H. and J. Sturm, “Scalar Curvature, Moment Maps and the Deligne Pairing”, Amer. J. Math. 126 (2004), 693–712

[6] Phong, D.H., J. Ross, and J. Sturm, “Deligne pairings and the Knudsen-Mumford expansion”, J. Differential Geom. 78 (2008), no. 3, 475-496

[7] Zhang, S., “Heights and reductions of semi-stable varieties”, Comp. Math. 12 (1996), 323–345

*e-mail address: hetalk@rutgers.edu*

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEWARK, NJ 07102