A TECHNIQUE FOR CONSTRAINING THE DRIVING SCALE OF TURBULENCE AND A MODIFIED CHANDRASEKHAR–FERMI METHOD

JUNGYEON CHO AND HYUNJU YOO
Department of Astronomy and Space Science, Chungnam National University, Daejeon, Korea; jcho@cnu.ac.kr, hyunju527@gmail.com

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ABSTRACT

The Chandrasekhar–Fermi (CF) method is a powerful technique for estimating the strength of the mean magnetic field projected on the plane of the sky. In this paper, we present a technique for improving the CF method in which we take into account the averaging effect arising from independent eddies along the line of sight (LOS). In the conventional CF method, the strength of fluctuating magnetic field divided by \( \sqrt{4\pi \bar{\rho}} \), where \( \bar{\rho} \) is average density, is assumed to be comparable to the LOS velocity dispersion. However, this is not true when the driving scale of turbulence \( L_p \), i.e., the outer scale of turbulence, is smaller than the size of the system along the LOS \( L_{los} \). In fact, the conventional CF method overestimates the strength of the mean plane-of-the-sky magnetic field by a factor of \( \sim \sqrt{L_{los}/L_p} \). We show that the standard deviation of centroid velocities divided by the average LOS velocity dispersion is a good measure of \( \sqrt{L_{los}/L_p} \), which enables us to propose a modified CF method.

Key words: ISM: magnetic fields – magnetohydrodynamics (MHD) – techniques: polarimetric – turbulence

1. INTRODUCTION

Magnetic fields play important roles in many astrophysical environments. However, measuring their strength is a very challenging problem. The Chandrasekhar–Fermi method (Chandrasekhar & Fermi 1953; hereinafter the CF method) is a simple and powerful technique that can measure the strength of a regular magnetic field perpendicular to the line of sight (LOS), i.e., the component of the mean magnetic field projected on the plane of the sky.

The idea behind the CF method is simple. Let us consider a fluid filled with Alfvén waves or Alfvénic turbulence. In Alfvénic disturbances, the rms fluctuations of the magnetic field (\( \delta b \)) and the rms velocity (\( \delta v \)) are related by

\[
\frac{\delta b}{\sqrt{4\pi \bar{\rho}}} \sim \delta v \quad \text{or} \quad 1 \sim \sqrt{4\pi \bar{\rho}} \frac{\delta v}{\delta b},
\]

where \( \bar{\rho} \) is average density. If we multiply both sides by the mean plane-of-the-sky magnetic field \( B_{0,sky} \), we obtain

\[
B_{0,sky} \sim \sqrt{4\pi \bar{\rho}} \frac{\delta v}{\delta b}/B_{0,sky}.
\]

If the velocity fluctuation (\( \delta v \)) and the magnetic field fluctuation (\( \delta b \)) are isotropic, then we may write \( \delta v_{los}/\delta b_{los,sky} \sim \delta v/\delta b \), where \( \delta v_{los} \) is the LOS velocity dispersion and \( \delta b_{los,sky} \) is the rms fluctuation of the plane-of-the-sky magnetic field that is perpendicular to \( B_{0,sky} \). Therefore, the equation above becomes

\[
B_{0,sky} = \xi \sqrt{4\pi \bar{\rho}} \frac{\delta v_{los}}{\delta b_{los,sky}/B_{0,sky}} \sim \xi \sqrt{4\pi \bar{\rho}} \frac{\delta v_{los}}{\delta \phi},
\]

where \( \delta \phi \) is the variation of the angle between the plane-of-the-sky magnetic field and the mean plane-of-the-sky magnetic field \( B_{0,sky} \), and we use

\[
\delta \phi \sim \tan(\delta \phi) = \delta b_{los,sky}/B_{0,sky}.
\]

The factor \( \xi \) is a correction factor, which is usually taken as \( \sim 0.5 \) (Heitsch et al. 2001; Ostriker et al. 2001; Padoan et al. 2001). We can obtain \( \delta \phi \) from observations of star-light polarization or polarized far-infrared emission from magnetically aligned dust grains and we can measure \( \delta v_{los} \) from the width of an optically thin molecular emission line (see, for example, Gonatas et al. 1990; Di Francesco et al. 2001; Lai et al. 2001; Crutcher et al. 2004; Girart et al. 2006; Curran & Chrysostomou 2007; Heyer et al. 2008; Mao et al. 2008; Tang et al. 2009; Sugitani et al. 2011). Further elaboration of the CF method has been made by many researchers (Zweibel 1990, 1996; Myers & Goodman 1991; Heitsch et al. 2001; Ostriker et al. 2001; Padoan et al. 2001; Kudoh & Basu 2003; Wiebe & Watson 2004; Falcke-Goçalves et al. 2008; Hildebrand et al. 2009; Houde et al. 2009).

Reduction of \( \delta \phi \) due to averaging effects is of great importance in our paper. Roughly speaking, two main averaging effects exist. First, averaging along the LOS can reduce \( \delta \phi \). That is, if there is more than one independent turbulent eddy along the LOS, the measured value of \( \delta \phi \) will be reduced (see Myers & Goodman 1991; Zweibel 1996; Houde et al. 2009). Second, averaging the polarization angle within the telescope beam can also reduce \( \delta \phi \) (see Heitsch et al. 2001; Wiebe & Watson 2004; Falcke-Goçalves et al. 2008; Houde et al. 2009). The averaging effects in general make the step from Equation (2) to Equation (3) inaccurate. In the presence of the averaging effects, the CF method tends to overestimate \( B_{0,sky} \).

In this paper, we focus on the averaging effect along the LOS and propose a simple method to compensate for the effect. Using three-dimensional direct magnetohydrodynamic (MHD) turbulence simulations, we test the proposed technique. In Section 2, we describe our numerical method and explain in detail how averaging along the LOS makes the conventional CF method overestimate \( B_{0,sky} \). At the end of Section 2, we describe our new technique. In Section 3, we present the results of our numerical simulations. In Section 4, we discuss our findings and summarize.
2. NUMERICAL METHOD AND THEORETICAL CONSIDERATIONS

2.1. Numerical Code

To obtain the turbulence data, we solve the following compressible MHD equations in a periodic box of size $2\pi$ using an essentially non-oscillatory scheme (see Cho & Lazarian 2002):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \rho^{-1} \nabla (C_s^2 \rho) &= 0, \\
-(\nabla \cdot \mathbf{B}) \mathbf{B} / 4\pi \rho &= \mathbf{f}, \\
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) &= 0,
\end{align*}
\]

with $\nabla \cdot \mathbf{B} = 0$ and an isothermal equation of state $P = C_s^2 \rho$, where $C_s$ is the sound speed and $\rho$ is density. Here, $\mathbf{v}$ is the velocity, $\mathbf{B}$ is the magnetic field, and $\mathbf{f}$ is the driving force. We use $512^3$ grid points in our simulations, $C_s = 0.1$, $\rho = 1$, and $B_0/\sqrt{4\pi \rho} = 1$. In all simulations, the rms velocity $v_{\text{rms}}$ is between $\sim 0.7$ and $\sim 0.8$, and the sonic Mach number is $M_s \equiv v_{\text{rms}}/C_s \sim 7$. Since the Alfvén speed of the mean field ($V_A = B_0/\sqrt{4\pi \rho}$) is 1, the Alfvén Mach number is $M_A \equiv v_{\text{rms}}/V_A \sim 0.7$, which means that the turbulence considered in this paper is sub-Alfvénic.

2.2. Forcing

In this work, we drive turbulence in Fourier space and consider only solenoidal ($\nabla \cdot \mathbf{f} = 0$) forcing. We use $\sim 100$ forcing components isotropically distributed in the range $k_f/1.3 \lesssim k \lesssim 1.3 k_f$, where $k$ is the wavenumber and $k_f$ varies from simulation to simulation (see Table 1). Therefore, the peak of energy injection occurs at $k \sim k_f$. More detailed descriptions of forcing can be found in Yoo & Cho (2014).

2.3. Theoretical Considerations: The Effects of Averaging along an LOS

Our main concern is to investigate the effect of the driving scale on the CF method. For this purpose, we change the driving scale by changing the driving wavenumber $k_f$.

If the driving scale is $L_f$, then there are $\sim L_{\text{los}}/L_f$ ($\equiv N$) large-scale eddies (i.e., the largest energy-containing eddies) along an LOS, where $L_{\text{los}}$ is the size of the system along the LOS (see Figure 1(a)). In this case, what will be the strength of the observed magnetic field projected on the plane of the sky? If we take a coordinate system as shown in Figure 1(a), the plane of the sky is parallel to the $xy$-plane. Let us consider the $x$ and $y$ components of the magnetic field separately. Here the $x$ and $y$ directions are parallel and perpendicular to the mean plane-of-the-sky magnetic field, respectively. We can write

\[
\begin{align*}
B_x,\text{obs} &\propto \int_0^{L_{\text{los}}} b_x dz \sim B_{0,\text{sky}} L_{\text{los}}, \\
b_y,\text{obs} &\propto \int_0^{L_{\text{los}}} b_y dz \sim b_y L_f \frac{L_{\text{los}}}{L_f} = b_y \sqrt{L_f/L_{\text{los}}},
\end{align*}
\]

where we ignore the contribution of the random magnetic field in the integration of $B_y (=B_{0,\text{sky}} + b_y)$ and assume that each large-scale eddy contributes randomly in the integration of $b_y$ (see Figure 1(b)). Therefore, the variation of the angle between the magnetic field projected on the plane of the sky and the mean plane-of-the-sky magnetic field is given by

\[
\delta \phi \sim \tan(\delta \phi) \sim \frac{b_y,\text{obs}}{B_x,\text{obs}} \frac{b_y}{B_0} \sqrt{L_f/L_{\text{los}}},
\]

which is $(L_{\text{los}}/L_f)^{1/2}$ times smaller than the conventional estimate of $\delta \phi$ (see Equation (4)).

\footnote{The length-scale $L_f$ in Equation (9) should be the coherence scale of the magnetic field (see Cho & Ryu 2009). In this paper, we assume that the coherence scale coincides with the driving scale of turbulence, which is a very good approximation for trans-Alfvénic or sub-Alfvénic turbulence, i.e., turbulence in which the rms velocity is similar to or less than the Alfvén speed of the mean magnetic field. In sub-Alfvénic turbulence, the magnetic energy spectrum peaks at the driving scale and therefore the coherence scale of the magnetic field should be very close to the driving scale of turbulence.}
Indeed, our simulations confirm that the larger the number of independent eddies along a line of sight, the smaller the variation $\delta \varphi$. Using our MHD turbulence data (see Table 1) and the numerical method in Fiege & Pudritz (2000; see also Heitsch et al. 2001), we obtain synthetic polarization maps arising from magnetically aligned dust grains at a far-infrared/sub-millimeter wavelength. The polarization maps are taken after saturation of the turbulence. The left and the right panels are from Runs KF5 and KF20, respectively. The projections are done along a line of sight perpendicular to the mean field $B_0$.

Figure 2. Reduction of variation in polarization angle $\delta \varphi$ due to the averaging effect along the line of sight. The driving wavenumbers, and hence the number of independent eddies along a line of sight, are different in the left and right panels. Left: the driving wavenumber is $\sim 5$ and therefore there are approximately 5 independent eddies along a line of sight. Right: the driving wavenumber is $\sim 20$ and therefore there are approximately 20 independent eddies along a line of sight. Variation in polarization angle is smaller in the right panel due to a larger number of independent eddies along the line of sight. The contours represent the intensity of FIR/sub-millimeter emission from magnetically aligned dust. The polarization maps are taken after saturation of the turbulence. The left and the right panels are from Runs KF5 and KF20, respectively. The projections are done along a line of sight perpendicular to the mean field $B_0$.

Figure 3. Left: time evolution of $\langle \delta v \rangle^2$ and $B^2/(4\pi \rho)$. Right: time evolution of $k_f B$ times the standard deviation of velocity centroids $\delta V_c$ divided by the average line of sight velocity dispersion $\delta v_{\text{los}}$, where $k_f$ is the driving wavenumber (see Equation (13) for details). In both plots, the x-axes denote time normalized by $L_f/\delta v$, where $L_f$ is the driving scale of turbulence and $\delta v$ is the rms velocity.

Indeed, our simulations confirm that the larger the number of independent eddies along the LOS $L_{\text{los}}/L_f$, the smaller the variation $\delta \varphi$. Using our MHD turbulence data (see Table 1) and the numerical method in Fiege & Pudritz (2000; see also Heitsch et al. 2001), we obtain synthetic polarization maps arising from magnetically aligned dust grains at a far-infrared/sub-millimeter wavelength. The polarization maps are taken after saturation of the turbulence. The left and the right panels are from Runs KF5 and KF20, respectively. The projections are done along a line of sight perpendicular to the mean field $B_0$.

2.4. Observational Estimation of $(L_f/L_{\text{los}})$

From the discussion in the previous subsection, it is clear that the conventional CF method overestimates the strength of the mean plane-of-the-sky magnetic field by a factor of $\sqrt{L_{\text{los}}/L_f}$. Therefore it is necessary to know $L_{\text{los}}/L_f$ to obtain a correct estimate of $B_{0,\text{sky}}$. 
We propose that the standard deviation of centroid velocities \( \delta V_c \) normalized by the average LOS velocity dispersion \( \delta v_{\text{los}} \) is a good measure of \( \sqrt{L_{\text{los}}/L_f} \): \[
(11)
\]
where \( V_c = \int v_{\text{los}} I(v_{\text{los}}) dv_{\text{los}} \) \int I(v_{\text{los}}) dv_{\text{los}} \) and \( I(v_{\text{los}}) \) is the observed line profile for the LOS.

Centroid velocity \( V_c \) is a kind of average velocity. If we have several independent eddies along the LOS, then each eddy has its own mean velocity. Then, roughly speaking, the observed centroid velocity for the LOS is the average of the mean velocities of individual eddies along the LOS. Note that the mean velocities of independent eddies are likely to be random. Therefore, if we obtain centroid velocities for many different lines of sight and calculate their standard deviation, then the standard deviation should be proportional to \( 1/\sqrt{N} \), where \( N \) is the number of independent eddies along an LOS. In Figure 1(a) we draw four independent eddies along the LOS and in Figure 1(c), we plot velocity profiles (or emission line profiles) from four independent eddies in solid lines and the observed line profile in a dashed line. Then the observed centroid velocity \( V_c \) is approximately the average of the mean velocities of four eddies. If we have more eddies along the LOS, the variation in velocity centroids normalized by the average width of an optically thin emission line will become smaller.

3. RESULTS

The left panel of Figure 3 shows the time evolution of \( \langle \delta v \rangle^2 \) and \( B^2/(4\pi \rho) \), where \( B^2 = B_0^2 + \langle \delta b \rangle^2 \). At \( t = 0 \), \( \langle \delta v \rangle^2 \approx 0 \) and only the uniform magnetic field, with units of Alfvén speed (i.e., \( B_0/\sqrt{4\pi \rho} = 1 \)), exists. Due to driving, \( \langle \delta v \rangle^2 \) and \( B^2/(4\pi \rho) \) grow initially. After \( t \approx 2(L_f/\delta v) \), turbulence seems to show saturation in all simulations.

The right panel of Figure 3 shows the time evolution of \[
\sqrt{k_f \delta V_c / \delta v_{\text{los}}} \approx \sqrt{N} \delta V_c / \delta v_{\text{los}},
(13)
\]
where \( \delta v_{\text{los}} \) is the average LOS velocity dispersion. The standard deviation of centroid velocities \( \delta V_c \) and the average of LOS velocity dispersion \( \delta v_{\text{los}} \) are calculated over 512\(^2 \) lines of sight perpendicular to the mean field \( B_0 \). The figure confirms that the quantity \( \delta V_c / \delta v_{\text{los}} \) is indeed proportional to \( 1/\sqrt{k_f} \) (\( \approx \sqrt{L_f/L_{\text{los}}} \)), as we proposed in Section 2.4 (see Equation (11)).

The left panel of Figure 4 shows our main result. Since the conventional CF method tends to overestimate \( B_{0,\text{sky}} \) by a factor of \( \sqrt{L_{\text{los}}/L_f} \) and \( \delta V_c / \delta v_{\text{los}} \propto \sqrt{L_f/L_{\text{los}}} \), we can write \[
B_{0,\text{sky}} = \xi' \sqrt{4\pi \rho} \delta V_c / \delta v_{\text{los}},
(14)
\]
where \( \xi' \) is a constant of order unity that can be determined by numerical simulations. In the left panel of Figure 4 we plot estimates of \( B_{0,\text{sky}} / \sqrt{4\pi \rho} \) from this modified CF method:

\[
\delta V_c / \delta v_{\text{los}}.
(15)
\]
In the panel, we can see that the estimates are fluctuating between \( \sim 1.0 \) and \( \sim 1.5 \). Therefore, since \( B_{0,\text{sky}} / \sqrt{4\pi \rho} = 1 \) in our simulations, the constant \( \xi' \) in Equation (14) is between \( \sim 0.7 \) and \( \sim 1 \).

The right panel of Figure 4 shows estimates of \( B_{0,\text{sky}} / \sqrt{4\pi \rho} \) from the conventional CF method:

\[
\delta v_{\text{los}} / \delta v_{\text{los}}.
(16)
\]
We can see that the conventional CF method indeed overestimates \( B_{0,\text{sky}} \) when the number of independent eddies along the LOS (\( \sim k_f \) in our simulations) is large. Note, however, that the conventional CF method seems to work fine for small \( N \)'s.

4. DISCUSSIONS AND SUMMARY

In this paper, we have considered the effects of driving scale on the estimates of the mean plane-of-the-sky magnetic field \( B_{0,\text{sky}} \) from the CF method. The method we propose in Equation (14) with \( 0.7 \lesssim \xi' \lesssim 1.0 \) does not require new observations. That is, the method is readily applicable for
present observational data. Apart from numerical constants, the only difference between our method and the conventional CF method is that our method requires the standard deviation of velocity centroids $\delta V_c$, while the conventional method requires the average width of the emission line profiles $\delta v_{los}$. The standard deviation of velocity centroids $\delta V_c$ can be easily obtained from existing optically thin emission line profiles. If such emission line profiles $(I(v_{los})$’s) are available for $n_{obs}$ lines of sight, then we need the following two steps to obtain $\delta V_c$:

1. We calculate the centroid velocity $V_c$ (see Equation (12)) for each LOS. Let $V_{c,i}$ be the centroid velocity for LOS $i$:

$$V_{c,i} = \frac{\int v_{los} I(v_{los})dv_{los}}{\int I(v_{los})dv_{los}},$$

where $I(v_{los})$ is the optically thin emission line profile for the LOS.

2. We calculate $\delta V_c$ from the formula

$$\delta V_c^2 = \frac{1}{n_{obs}} \sum_{i=1}^{n_{obs}} V_{c,i}^2 - \left( \frac{1}{n_{obs}} \sum_{i=1}^{n_{obs}} V_{c,i} \right)^2.$$

The CF method is useful for obtaining the strengths of the plane-of-the-sky magnetic fields in molecular clouds. Since observations suggest the existence of supersonic motions and strong magnetic fields in molecular clouds, we have considered only supersonic ($M_s \sim 7$) and marginally sub-Alfvénic ($M_s \lesssim 1$) MHD turbulence in this paper. It is possible that the constant $\zeta'$ in Equation (14) depends on the the sonic Mach number $M_s$. But we expect that the dependence is weak because the sonic Mach number does not play an important role in our discussion in Section 2.4. Nevertheless, more parameter studies are needed to determine the dependence of $\zeta'$ on $M_s$. Another limitation of our current work is that we have considered only the case in which the mean magnetic field is perpendicular to the LOS, which means that the inclination angle (with respect to the plane of the sky) of the mean magnetic field is zero. In principle, our method, as well as the conventional CF method, should work for an arbitrary inclination angle, unless the inclination angle is very close to $90^\circ$ (see discussions in Heitsch et al. 2001; Ostriker et al. 2001; Falceta-Gonçalves et al. 2008). We will address these issues elsewhere.

In this paper we have demonstrated that the conventional CF method indeed overestimates the mean plane-of-the-sky magnetic field $B_{0,sky}$ by a factor of $\sqrt{N}$, where $N$ is the number of independent eddies along the LOS. We have found that the standard deviation of centroid velocities divided by the average LOS velocity dispersion $(\delta V_c/\delta v_{los})$ is proportional to $1/\sqrt{N}$ (Equation (11) and the right panel of Figure 3). Therefore Equation (14) with $\xi' \approx 0.7 \sim 1$ provides a better estimate for $B_{0,sky}$.

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