ON SEMITOPOLOGICAL BICYCLIC EXTENSIONS OF LINEARLY ORDERED GROUPS

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ABSTRACT. For a linearly ordered group $G$ let us define a subset $A \subseteq G$ to be a shift-set if for any $x, y, z \in A$ with $y < x$ we get $x \cdot y^{-1} \cdot z \in A$. We describe the natural partial order and solutions of equations on the semigroup $\mathcal{B}(A)$ of shifts of positive cones of $A$. We study topologizations of the semigroup $\mathcal{B}(A)$. In particular, we show that for an arbitrary countable linearly ordered group $G$ and a non-empty shift-set $A$ of $G$ every Baire shift-continuous $T_1$-topology $\tau$ on $\mathcal{B}(A)$ is discrete. Also we prove that for an arbitrary linearly non-densely ordered group $G$ and a non-empty shift-set $A$ of $G$, every shift-continuous Hausdorff topology $\tau$ on the semigroup $\mathcal{B}(A)$ is discrete, and hence $(\mathcal{B}(A), \tau)$ is a discrete subspace of any Hausdorff semitopological semigroup which contains $\mathcal{B}(A)$ as a subsemigroup.

1. Introduction and preliminaries

We shall follow the terminology of [14, 18, 20, 24, 33, 40, 41].

A semigroup is a non-empty set with a binary associative operation. A semigroup $S$ is called inverse if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element $y$ in $S$ is called the inverse of $x$ and is denoted by $x^{-1}$. The map defined on an inverse semigroup $S$ which maps every element $x$ of $S$ to its inverse $x^{-1}$ is called the inversion.

For a semigroup $S$ by $E(S)$ we denote the set of idempotents in $S$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as the band of $S$. A semilattice is a commutative semigroup of idempotents.

Let $I_X$ denote the set of all partial one-to-one transformations of an infinite set $X$ together with the following semigroup operation: $x(\alpha \beta) = (x\alpha)\beta$ if $x \in \text{dom}(\alpha \beta) = \{y | y \in \text{dom} \alpha : y\alpha \in \text{dom} \beta\}$, for $\alpha, \beta \in I_X$. The semigroup $I_X$ is called the symmetric inverse semigroup over the set $X$ (see [18]). The symmetric inverse semigroup was introduced by Wagner [44] and it plays a major role in the theory of semigroups.

The bicyclic monoid $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subjected only to the condition $pq = 1$. The bicyclic monoid is a combinatorial bisimple $F$-inverse semigroup and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known O. Andersen’s result [1] states that a $(0-) simple semigroup is completely $(0-)$ simple if and only if it does not contain the bicyclic monoid. The bicyclic monoid does not embed into stable semigroups [35].

Recall [24] that a partially ordered group is a group $(G, \cdot)$ equipped with a translation-invariant partial order $\leq$; in other words, the binary relation $\leq$ has the property that, for all $a, b, g \in G$, if $a \leq b$ then $a \cdot g \leq b \cdot g$ and $g \cdot a \leq g \cdot b$.

Later by $e$ we denote the identity of a group $G$. The set $G^+ = \{x \in G : e \leq x\}$ in a partially ordered group $G$ is called the positive cone, or the integral part, of $G$ and it satisfies the properties:

1) $G^+ \cdot G^+ \subseteq G^+$; 2) $G^+ \cap (G^+)^{-1} = \{e\}$; and 3) $x^{-1} \cdot G^+ \cdot x \subseteq G^+$ for each $x \in G$.

Any subset $P$ of a group $G$ that satisfies the conditions 1)–3) induces a partial order on $G$ ($x \leq y$ if and only if $x^{-1} \cdot y \in P$) for which $P$ is the positive cone. Elements of the set $G^+ \setminus \{e\}$ are called positive.

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A *linearly ordered* or *totally ordered group* is an ordered group $G$ whose order relation $\leq$ is total (see [13] and [17]).

From now on we shall assume that $G$ is a non-trivial linearly ordered group.

For every $g \in G$ the set

$$G^+(g) = \{ x \in G : g \leq x \}. $$

The set $G^+(g)$ is called a *positive cone on element* $g$ in $G$.

For arbitrary elements $g, h \in G$ we consider a partial map $\alpha^g_h : G \rightarrow G$ defined by the formula

$$(x)\alpha^g_h = x \cdot g^{-1} \cdot h, \quad \text{for } x \in G^+(g).$$

We observe that Lemma XIII.1 from [13] implies that for such partial map $\alpha^g_h : G \rightarrow G$ the restriction $\alpha^g_h : G^+(g) \rightarrow G^+(h)$ is a bijective map.

We consider the semigroups

$$\mathcal{B}(G) = \{ \alpha^g_h : G \rightarrow G : g, h \in G \} \text{ and } \mathcal{B}^+(G) = \{ \alpha^g_h : G \rightarrow G : g, h \in G^+ \},$$

endowed with the operation of the composition of partial maps. Simple verifications show that

$$(1) \quad \alpha^a_h \cdot \alpha^b_i = \alpha^a_i, \quad \text{where } a = (h \vee k) \cdot h^{-1} \cdot g \text{ and } b = (h \vee k) \cdot k^{-1} \cdot l,$$

for $g, h, k, l \in G$, and by $h \vee k$ we denote the join of $h$ and $k$ in the linearly ordered set $(G, \leq)$. Therefore, property 1) of the positive cone and condition (1) imply that $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ are subsemigroups of $\mathcal{I}_G$.

By Proposition 1.2 in [28] for a linearly ordered group $G$ the following assertions hold:

(i) elements $\alpha^g_h$ and $\alpha^h_g$ are inverses of each other in $\mathcal{B}(G)$ for all $g, h \in G$ (resp., $\mathcal{B}^+(G)$ for all $g, h \in G^+$);

(ii) an element $\alpha^g_h$ of the semigroup $\mathcal{B}(G)$ (resp., $\mathcal{B}^+(G)$) is an idempotent if and only if $g = h$;

(iii) $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ are inverse subsemigroups of $\mathcal{I}_G$;

(iv) the semigroup $\mathcal{B}(G)$ (resp., $\mathcal{B}^+(G)$) is isomorphic to the set $S_G = G \times G$ (resp., $S^+_G = G^+ \times G^+$) with the following semigroup operation:

$$(2) \quad (a, b)(c, d) = \begin{cases} 
(c \cdot b^{-1} \cdot a, d), & \text{if } b < c; \\
(a, d), & \text{if } b = c; \\
(a, b \cdot c^{-1} \cdot d), & \text{if } b > c,
\end{cases}$$

where $a, b, c, d \in G$ (resp., $a, b, c, d \in G^+$).

It is obvious that:

(1) if $G$ is isomorphic to the additive group of integers $(\mathbb{Z}, +)$ with usual linear order $\leq$ then the semigroup $\mathcal{B}^+(G)$ is isomorphic to the bicyclic monoid $\mathcal{C}(p, q)$ and the semigroup $\mathcal{B}(G)$ is isomorphic to the extended bicyclic semigroup $\mathcal{C}_\mathbb{Z}$ (see [21]);

(2) if $G$ is the additive group of real numbers $(\mathbb{R}, +)$ with usual linear order $\leq$ then the semigroup $\mathcal{B}(G)$ is isomorphic to $B^1_{(-\infty, \infty)}$ (see [36] [37]) and the semigroup $\mathcal{B}^+(G)$ is isomorphic to $B^1_{[0, \infty)}$ (see [2] [3] [4] [5] [6]), and

(3) the semigroup $\mathcal{B}(G)$ is isomorphic to the semigroup $S(G)$ which is defined in [22] [23].

In the paper [28] semigroups $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ are studied for a linearly ordered group $G$. That paper describes Green’s relations on $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ and their bands, and shows that these are bisimple. Also in [28] it’s proved that for a commutative linearly ordered group $G$ all non-trivial congruences on the semigroups $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ are group congruences if and only if the group $G$ is Archimedean; and the structure of group congruences on the semigroups $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ is described.

In this paper we present more general construction than the semigroups $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$. Namely, for a linearly ordered group $G$ let us define a subset $A \subseteq G$ to be a *shift-set* if for any $x, y, z \in A$ with $y < x$ we get $x \cdot y^{-1} \cdot z \in A$. For any shift-set $A \subseteq G$ let

$$\mathcal{B}(A) = \{ \alpha^a_b : G^+(a) \rightarrow G^+(b) : a, b \in A \}$$
be the semigroup of partial bijections defined by the formula
\[(x)\alpha_b^a = x \cdot a^{-1} \cdot b, \quad \text{for} \ x \in G^+(a).\]
The semigroup \(\mathcal{B}(A)\) is isomorphic to the semigroup \(S_A = A \times A\) endowed with the binary operation
defined by formula (2). For \(A = G\) the semigroup \(\mathcal{B}(A)\) coincides with \(\mathcal{B}(G)\) and for \(A = G^+\) it
coincides with the semigroup \(\mathcal{B}^+(G)\).

Later in this paper for a non-empty shift-set \(A \subseteq G\) we identify the semigroup \(\mathcal{B}(A)\) with the
semigroup \(S_A\) endowed with the multiplication defined by formula (2). We observe that \(\mathcal{B}(A)\) is an
inverse subsemigroup of \(\mathcal{B}(G)\) for any non-empty shift-set \(A\) of a linearly ordered group \(G\). Moreover,
the results of [28] imply that an element \((a, b)\) of \(\mathcal{B}(A)\) is an idempotent iff \(a = b\), and \((b, a)\) is inverse
of \((a, b)\) in \(\mathcal{B}(G)\).

We recall that a topological space \(X\) is said to be
- **locally compact**, if every point \(x \in X\) has an open neighbourhood with the compact closure;
- **Čech-complete**, if \(X\) is Tychonoff and \(X\) is a \(G_δ\)-set in its the Čech-Stone compactification;
- **Baire**, if for each sequence \(A_1, A_2, \ldots, A_i, \ldots\) of open dense subsets of \(X\) the intersection \(\bigcap_{i=1}^{\infty} A_i\)
is dense in \(X\).

Every Hausdorff locally compact space is Čech-complete, and every Čech-complete space is Baire (see [20]).

A **semitopological (topological) semigroup** is a topological space with a separately continuous (jointly
continuous) semigroup operation.

A topology \(\tau\) on a semigroup \(S\) is called:
- **semigroup** if \((S, \tau)\) is a topological semigroup;
- **shift-continuous** if \((S, \tau)\) is a semitopological semigroup.

The bicyclic monoid admits only the discrete semigroup Hausdorff topology and if a topological
semigroup \(S\) contains it as a dense subsemigroup then \(C(p, q)\) is an open subset of \(S\) [19]. We observe that
the openness of \(C(p, q)\) in its closure easily follows from the non-topologizability of the bicyclic
monoid, because the discrete subspace \(D\) is open in its closure \(\overline{D}\) in any \(T_1\)-space containing \(D\). Bertman
and West in [12] extended this result for the case of Hausdorff semitopological semigroups. Stable and \(Γ\)-compact
topological semigroups do not contain the bicyclic monoid [7, 34]. The problem of embedding the bicyclic monoid into compact-like topological semigroups was studied in [8, 9, 30].

Independently Taimanov in [12] constructed a semigroup \(A_κ\) of cardinality \(κ\) which admits only the
discrete semigroup topology. Also, Taimanov [13] gave sufficient conditions on a commutative semigroup
to have a non-discrete semigroup topology. In the paper [20] it was showed that for the Taimanov
semigroup \(A_κ\) from [42] the following conditions hold: every \(T_1\)-topology \(τ\) on the semigroup \(A_κ\) such
that \((A_κ, τ)\) is a topological semigroup is discrete; for every \(T_1\)-topological semigroup which contains
\(A_κ\) as a subsemigroup, \(A_κ\) is a closed subsemigroup of \(S\); and every homomorphic non-isomorphic
image of \(A_κ\) is a zero-semigroup. Also, in the paper [21] it is proved that the discrete topology is the
unique shift-continuous Hausdorff topology on the extended bicyclic semigroup \(C_Z\). Also, for many
(0-) bisimple semigroups of transformations \(S\) the following statement holds: **every shift-continuous Hausdorff Baire (in particular locally compact) topology on \(S\) is discrete** (see [15, 16, 29, 31, 32]). In the paper [39] Mesyan, Mitchell, Morayne and Péresse showed that if \(E\) is a finite graph, then the only locally compact Hausdorff semigroup topology on the graph inverse semigroup \(G(E)\) is the discrete topology.

In [11] it was proved that the conclusion of this statement also holds for graphs \(E\) consisting of one
vertex and infinitely many loops (i.e., infinitely generated polycyclic monoids). Amazing dichotomy for
the bicyclic monoid with adjoined zero \(C^0 = C(p, q) \sqcup \{0\}\) was proved in [23]: every Hausdorff
locally compact semitopological bicyclic monoid \(C^0\) with adjoined zero is either compact or discrete.
The above dichotomy was extended by Bardyla in [10] to locally compact \(λ\)-polycyclic semitopological
monoids and to locally compact semitopological interassociates of the bicyclic monoid [27].

For a linearly ordered group \(G\) and a non-empty shift-set \(A\) of \(G\), the natural partial order and
solutions of equations on the semigroup \(\mathcal{B}(A)\) are described. We study topologizations of the semigroup
This implies that for an arbitrary countable linearly ordered group $G$ and a non-empty shift-set $A$ of $G$, every Baire shift-continuous $T_1$-topology $\tau$ on $B(A)$ is discrete. Also we prove that for an arbitrary linearly non-densely ordered group $G$ and a non-empty shift-set $A$ of $G$, every shift-continuous Hausdorff topology $\tau$ on the semigroup $B(A)$ is discrete, and hence $(B(A), \tau)$ is a discrete subspace of any Hausdorff semitopological semigroup which contains $B(A)$ as a subsemigroup.

2. Solutions of some equations and the natural partial order on the semigroup $B(A)$

It is well known that every inverse semigroup $S$ admits the natural partial order:

$$s \preceq t \text{ if and only if } s = ct \text{ for some } c \in E(S).$$

This order induces the natural partial order on the semilattice $E(S)$, and for arbitrary $s, t \in S$ the following conditions are equivalent:

$$(3) \quad \begin{align*}
(\alpha) \ s \preceq t; \quad (\beta) \ s = ss^{-1}t; \quad (\gamma) \ s = ts^{-1}s,
\end{align*}$$

(see [38] Chapter 3.)

**Proposition 2.1.** Let $G$ be a linearly ordered group and $A$ be a non-empty shift-set in $G$. Then the following assertions hold:

(i) if $(g, g), (h, h) \in E(B(A))$ then $(g, g) \preceq (h, h)$ if and only if $g \geq h$ in $A$;

(ii) the semilattice $E(B(A))$ is isomorphic to $A$, considered as a $\vee$-semilattice under the isomorphisms $i: E(B(A)) \to A$, $i: (g, g) \mapsto g$;

(iii) $(g, h) \mathcal{H}(k, l)$ in $B(A)$ if and only if $g = k$ in $A$;

(iv) $(g, h) \mathcal{L}(k, l)$ in $B(A)$ if and only if $h = l$ in $A$;

(v) $(g, h) \mathcal{H}(k, l)$ in $B(A)$ if and only if $g = k$ and $h = l$ in $A$, and hence every $\mathcal{H}$-class in $B(A)$ is a singleton;

(vi) $B(A)$ is a bisimple semigroup and hence it is simple.

**Proof.** Assertions (i) and (ii) are trivial, (iii) – (v) follow from Proposition 2.1 of [28] and Proposition 3.2.11 of [38] and (vi) follows from Proposition 3.2.5 of [38].

Later we need the following lemma, which describes the natural partial order on the semigroup $B(A)$:

**Lemma 2.2.** Let $G$ be a linearly ordered group and $A$ be a non-empty shift-set in $G$. Then for arbitrary elements $(a, b), (c, d) \in B(A)$ the following conditions are equivalent:

(i) $(a, b) \preceq (c, d)$ in $B(A)$;

(ii) $a^{-1} \cdot b = c^{-1} \cdot d$ and $a \geq c$ in $A$;

(iii) $b^{-1} \cdot a = d^{-1} \cdot b$ and $b \geq d$ in $A$.

**Proof.** $(i) \Rightarrow (ii)$ The equivalence of conditions $(\alpha)$ and $(\beta)$ in (3) implies that $(a, b) \preceq (c, d)$ in $B(A)$ if and only if $(a, b) = (a, b)(a, b)^{-1}(c, d)$. Therefore we have that

$$(a, b) = (a, b)(a, b)^{-1}(c, d) = (a, b)(b, a)(c, d) = (a, a)(c, d) = \begin{cases} 
(c \cdot a^{-1} \cdot a, d), & \text{if } a < c; \\
(c, d), & \text{if } a = c; \\
(a, a \cdot c^{-1} \cdot d), & \text{if } a > c.
\end{cases}$$

This implies that

$$(a, b) = \begin{cases} 
(c, d), & \text{if } a < c; \\
(c, d), & \text{if } a = c; \\
(a, a \cdot c^{-1} \cdot d), & \text{if } a > c,
\end{cases}$$

and hence the condition $(a, b) \preceq (c, d)$ in $B(A)$ implies that $a^{-1} \cdot b = c^{-1} \cdot d$ and $a \geq c$ in $A$.

$(ii) \Rightarrow (i)$ Fix arbitrary $(a, b), (c, d) \in B(A)$ such that $a^{-1} \cdot b = c^{-1} \cdot d$ and $a \geq c$ in $A$. Then we have that

$$(a, b)(a, b)^{-1}(c, d) = (a, b)(b, a)(c, d) = (a, a)(c, d) = (a, a \cdot c^{-1} \cdot d) = (a, b),$$

and hence $(a, b) \preceq (c, d)$ in $B(A)$.

The proof of the equivalence $(ii) \Leftrightarrow (iii)$ is trivial.
The definition of the semigroup operation in \( B(A) \) implies that \( (a, b) = (a, c)(c, d)(d, b) \) for arbitrary elements \( a, b, c, d \) of \( A \). The following two propositions give amazing descriptions of solutions of some equations in the semigroup \( B(A) \).

**Proposition 2.3.** Let \( G \) be a linearly ordered group, \( A \) be a non-empty shift-set in \( G \) and \( a, b, c, d \) be arbitrary elements of \( A \). Then the following conditions hold:

\[
(i) \quad (a, b) = (a, c)(x, y) \text{ for } (x, y) \in B(A) \text{ if and only if } (c, b) \preceq (x, y) \text{ in } B(A);
(ii) \quad (a, b) = (x, y)(d, b) \text{ for } (x, y) \in B(A) \text{ if and only if } (a, d) \preceq (x, y) \text{ in } B(A);
(iii) \quad (a, b) = (a, c)(x, y)(d, b) \text{ for } (x, y) \in B(A) \text{ if and only if } (c, d) \preceq (x, y) \text{ in } B(A).
\]

**Proof.**

(i) \( \Rightarrow \) Suppose that \( (a, b) = (a, c)(x, y) \) for some \( (x, y) \in B(A) \). Then we have that

\[
(a, c)(x, y) = \begin{cases} 
(a, c \cdot x^{-1} \cdot y), & \text{if } c > x; \\
(a, y), & \text{if } c = x; \\
(x \cdot c^{-1} \cdot a, y), & \text{if } c < x.
\end{cases}
\]

Then in the case when \( c > x \) we get that \( b = c \cdot x^{-1} \cdot y \) and hence Lemma 2.2 implies that \( (c, b) \preceq (x, y) \) in \( B(A) \). Also, in the case when \( c = x \) we have that \( b = y \), which implies the inequality \( (c, b) \preceq (x, y) \) in \( B(A) \). The case \( c < x \) does not hold because the group operation on \( G \) implies that \( x \cdot c^{-1} \cdot a < a \).

\( \Leftarrow \) Suppose that the relation \( (c, b) \preceq (x, y) \) holds in \( B(A) \). Then by Lemma 2.2 we have that \( c^{-1} \cdot b = x^{-1} \cdot y \) and \( c \geq x \) in \( A \), and hence the semigroup operation of \( B(A) \) implies that

\[
(a, c)(x, y) = (a, c \cdot x^{-1} \cdot y) = (a, c \cdot c^{-1} \cdot b) = (a, b).
\]

The proof of statement (ii) is similar to statement (i).

(iii) \( \Rightarrow \) Suppose that \( (a, b) = (a, c)(x, y)(d, b) \) for some \( (x, y) \in B(A) \). Then we have that

\[
(a, c)(x, y) = \begin{cases} 
(a, c \cdot x^{-1} \cdot y), & \text{if } c > x; \\
(a, y), & \text{if } c = x; \\
(x \cdot c^{-1} \cdot a, y), & \text{if } c < x.
\end{cases}
\]

Therefore,

(a) if \( c > x \) then

\[
(a, c)(x, y)(d, b) = (a, c \cdot x^{-1} \cdot y)(d, b) = \begin{cases} 
(a, c \cdot x^{-1} \cdot y \cdot d^{-1} \cdot b), & \text{if } c \cdot x^{-1} \cdot y > d; \\
(a, b), & \text{if } c \cdot x^{-1} \cdot y = d; \\
(d \cdot y^{-1} \cdot x \cdot c^{-1} \cdot a, b), & \text{if } c \cdot x^{-1} \cdot y < d,
\end{cases}
\]

(b) if \( c = x \) then

\[
(a, c)(x, y)(d, b) = (a, y)(d, b) = \begin{cases} 
(a, y \cdot d^{-1} \cdot b), & \text{if } y > d; \\
(a, b), & \text{if } y = d; \\
(d \cdot y^{-1} \cdot a, b), & \text{if } y < d,
\end{cases}
\]

(c) if \( c < x \) then

\[
(a, c)(x, y)(d, b) = (x \cdot c^{-1} \cdot a, y)(d, b) = \begin{cases} 
(x \cdot c^{-1} \cdot a \cdot y \cdot d^{-1} \cdot b), & \text{if } y > d; \\
(x \cdot c^{-1} \cdot a, b), & \text{if } y = d; \\
(d \cdot y^{-1} \cdot x \cdot c^{-1} \cdot a, b), & \text{if } y < d.
\end{cases}
\]

Then the equality \( (a, b) = (a, c)(x, y)(d, b) \) implies that

(a) if \( c > x \) then \( c \cdot x^{-1} \cdot y \cdot d^{-1} = e \) in \( G \);
(b) if \( c = x \) then \( y = d; \)
and case (c) does not hold. Hence by Lemma 2.2 we get that \( (c, d) \preceq (x, y) \) in \( B(A) \).
\begin{proof}
Suppose that the relation \((c, d) \preceq (x, y)\) holds in \(\mathcal{B}(A)\). Then by Lemma 2.2 we have that \(c^{-1} \cdot d = x^{-1} \cdot y\) and \(c \geq x\) in \(A\), and hence the semigroup operation of \(\mathcal{B}(A)\) implies

\[
(a, c)(x, y)(d, b) = (a, c)(x, y)(c \cdot x^{-1} \cdot y, b) = (a, c)(c \cdot x^{-1} \cdot y \cdot y^{-1} \cdot x, b) = (a, c)(c \cdot x^{-1} \cdot x, b) = (a, c)(c, b) = (a, b),
\]

because \(c \cdot x^{-1} \cdot y \geq y\) in \(A\).
\end{proof}

**Proposition 2.4.** Let \(G\) be a linearly ordered group, \(A\) be a non-empty shift-set in \(G\) and \(a, b, c, d\) be arbitrary elements of \(A\). Then the following statements hold:

\begin{enumerate}
  \item if \(a < c\) in \(G\) then the equation \((a, b) = (c, d)(x, y)\) has no solutions in \(\mathcal{B}(A)\);
  \item if \(a > c\) in \(A\) then the equation \((a, b) = (c, d)(x, y)\) has the unique solution \((x, y) = (a \cdot c^{-1} \cdot d, b)\) in \(\mathcal{B}(A)\);
  \item \((a, b) = (a, d)(x, y)\) has the unique solution \((x, y) = (d, b)\) in \(\mathcal{B}(A)\);
  \item if \(b < d\) in \(A\) then the equation \((a, b) = (x, y)(c, d)\) has no solutions in \(\mathcal{B}(A)\);
  \item if \(b > d\) in \(A\) then the equation \((a, b) = (x, y)(c, d)\) has the unique solution \((x, y) = (a, b \cdot d^{-1} \cdot c)\) in \(\mathcal{B}(A)\);
  \item the equation \((a, b) = (x, y)(c, b)\) has the unique solution \((x, y) = (a, c)\) in \(\mathcal{B}(A)\).
\end{enumerate}

**Proof.** (i) Assume that \(a < c\). Then formula (2) implies that \(d < x\) in \(G\) and hence \((a, b) = (x \cdot d^{-1} \cdot c, y)\). This implies that \(a = x \cdot d^{-1} \cdot c\) and \(b = y\). Since \(d < x\), the equality \(a = x \cdot d^{-1} \cdot c\) implies \(a > c\), which contradicts the assumption of statement (i).

(ii) Assume that \(a > c\). Then formula (2) implies that \(d < x\) in \(G\) and hence we have that \((a, b) = (x \cdot d^{-1} \cdot c, y)\). This implies the equalities \(x = a \cdot c^{-1} \cdot d\) and \(y = b\).

(iii) follows from formula (2).

The proofs of statements (iv), (v) and (vi) are dual to the proofs of (i), (ii) and (iii), respectively. \(\square\)

Later we need the following proposition which follows from formula (2) and describes right and left principal ideals in the semigroup \(\mathcal{B}(A)\) for a non-empty shift-set \(A\) in \(G\).

**Proposition 2.5.** Let \(G\) be a linearly ordered group and \(A\) be a non-empty shift-set in \(G\). Then the following conditions hold:

\begin{enumerate}
  \item \((a, a)\mathcal{B}(A) = \{(x, y) \in \mathcal{B}(A) : x \geq a\ \text{in}\ A\}\);
  \item \(\mathcal{B}(A)(a, a) = \{(x, y) \in \mathcal{B}(A) : y \geq a\ \text{in}\ A\}\).
\end{enumerate}

\section{3. On topologizations of the semigroup \(\mathcal{B}(A)\)}

It is obvious that every left (right) topological group \(G\) with an isolated point is discrete. This implies that every countable \(T_1\)-Baire left (right) topological group is a discrete space, too. Later we shall show that the similar statement holds for a Baire semitopological semigroup \(\mathcal{B}(A)\) over a non-empty shift-set \(A\) of a countable linearly ordered group \(G\).

For an arbitrary element \((a, b)\) of the semigroup \(\mathcal{B}(A)\) we denote

\[
\uparrow_{\preceq}(a, b) = \{(x, y) \in \mathcal{B}(A) : (a, b) \preceq (x, y)\}.
\]

**Lemma 3.1.** Let \(G\) be a linearly ordered group, \(A\) be a non-empty shift-set in \(G\), and \(\tau\) be a shift-continuous topology on \(\mathcal{B}(A)\) such that \((\mathcal{B}(A), \tau)\) contains an isolated point. Then the space \((\mathcal{B}(A), \tau)\) is discrete.
Proof. Suppose that \((a, b)\) is an isolated point of the topological space \((\mathcal{B}(A), \tau)\). Assume that for an arbitrary \(u \in A\) there exists \(c \in A\) such that \(u > c\). Since \(A\) is a shift-set, \(d = c \cdot u^{-1} \cdot b < b\) in \(A\). By Proposition 2.4(v), the equation \((a, b) = (x, y)(c, d)\) has the unique solution
\[
(x, y) = (a, b \cdot d^{-1} \cdot c) = (a, b \cdot (c \cdot u^{-1} \cdot b)^{-1} \cdot c) = (a, b \cdot b^{-1} \cdot u \cdot c^{-1} \cdot c) = (a, u)
\]
in \(\mathcal{B}(A)\). If \(u\) is the smallest element of \(A\), then by Proposition 2.4(vi) the equation \((a, b) = (x, y)(u, b)\) has the unique solution \((x, y) = (a, u)\). In both cases the continuity of right translations in \((\mathcal{B}(A), \tau)\) implies that for arbitrary \(u \in A\) the pair \((a, u)\) is an isolated point of the topological space \((\mathcal{B}(A), \tau)\) for arbitrary \(u \in A\).

Fix an arbitrary element \(v\) of \(A\). Assume that there exists \(d \in A\) such that \(d < v\). Since \(A\) is a shift-set, \(c = d \cdot v^{-1} \cdot a < a\) in \(A\). Then by Proposition 2.4(ii), the equation \((a, u) = (c, d)(x, y)\) has the unique solution
\[
(x, y) = (a \cdot v^{-1} \cdot d, u) = (a \cdot (d \cdot v^{-1} \cdot a)^{-1} \cdot d, u) = (a \cdot a^{-1} \cdot v \cdot d^{-1} \cdot d, u) = (v, u)
\]
in \(\mathcal{B}(A)\). If \(v\) is the smallest element of \(A\), then by Proposition 2.4(iii), the equation \((a, u) = (a, v)(x, y)\) has the unique solution \((x, y) = (v, u)\). Since \((a, u)\) is an isolated point of \((\mathcal{B}(G), \tau)\), in both cases the continuity of left translations in \(\mathcal{B}(G)\) implies that for arbitrary \(u \in A\) the pair \((v, u)\) is an isolated point of the topological space \((\mathcal{B}(G), \tau)\) for arbitrary \(u \in G\). This completes the proof of the lemma. \(\square\)

**Theorem 3.2.** Let \(A\) be a countable non-empty shift-set in a linearly ordered group \(G\), and \(\tau\) be a \(T_1\)-Baier shift-continuous topology on \(\mathcal{B}(A)\). Then the topological space \((\mathcal{B}(A), \tau)\) is discrete.

Proof. By Proposition 1.30 of [33] every countable Baire \(T_1\)-space contains a dense subspace of isolated points, and hence the space \((\mathcal{B}(A), \tau)\) contains an isolated point. Then we apply Lemma 3.1. \(\square\)

Theorem 3.2 implies the following corollary:

**Corollary 3.3.** Let \(A\) be a countable non-empty shift-set in a linearly ordered group \(G\), and \(\tau\) be a shift-continuous \(\check{\text{C}}\)ech complete (or locally compact) \(T_1\)-topology on \(\mathcal{B}(A)\). Then the topological space \((\mathcal{B}(A), \tau)\) is discrete.

**Remark 3.4.** Let \(\mathbb{R}\) be the set of reals with usual topology. It is obvious that \(S_\mathbb{R} = \mathbb{R} \times \mathbb{R}\) with the semigroup operation
\[
(a, b) \cdot (c, d) = \begin{cases} 
(a - b + c, d), & \text{if } b < c; \\
(a, d), & \text{if } b = c; \\
(a, b - c + d), & \text{if } b > c,
\end{cases}
\]
is isomorphic to the semigroup \(\mathcal{B}(\mathbb{R})\), where \(\mathbb{R}\) is the additive group of reals with usual linear order. Then simple verifications show that \(S\) with the product topology \(\tau_p\) is a topological inverse semigroup (also, see [36] [37]). Then the subspace \(S_\mathbb{Q} = \{(x, y) \in S_\mathbb{R} : x \text{ and } y \text{ are rational}\}\) with the induced semigroup operation from \(S\) is a countable non-discrete non-Baire topological inverse subsemigroup of \((S, \tau_p)\). Also, the same we get in the case of subsemigroup \(S_\mathbb{Q}^+ = \{(x, y) \in S_\mathbb{Q} : x \geq 0 \text{ and } y \geq 0\}\) of \((S, \tau_p)\) (see [2] [3] [1] [5] [6]). The above arguments show that the condition in Theorem 3.2 that \(\tau\) is a \(T_1\)-Baier topology is essential.

Recall that a linearly ordered group \(G\) is said to be densely ordered if for every positive element \(g \in G\) there exists a positive element \(h \in G\) such that \(h < g\).

**Remark 3.5.** It is obviously that for a linearly ordered group \(G\) the following conditions are equivalent:
\begin{itemize}
\item[(i)] \(G\) is not densely ordered;
\item[(ii)] for every \(g \in G\) there exists a unique \(g^+ \in G\) such that \(G^+(g) \setminus G^+(g^+) = \{g\}\);
\item[(iii)] for every \(g \in G\) there exists a unique \(g^- \in G\) such that \(G^-(g) \setminus G^-(g^-) = \{g\}\), where \(G^-(g)\) is the negative cone on the element \(g\), i.e., \(G^-(g) = \{x \in G : x \leq g\}\).
\end{itemize}

Later for a linearly ordered group \(G\) which is not densely ordered and an arbitrary element \(g\) of a non-empty shift-set \(A\) in \(G\) by \(g^+\) (resp., \(g^-\)) we denote the minimum (resp., maximum) element of the set \(G^+(g) \setminus \{g\} \cap A\) (resp., \(G^-(g) \setminus \{g\} \cap A\)).
Theorem 3.6. Let $G$ be a linearly ordered group which is not densely ordered and $A$ be a non-empty shift-set in $G$. Then every shift-continuous Hausdorff topology $\tau$ on the semigroup $B(A)$ is discrete, and hence $B(A)$ is a discrete subspace of any semitopological semigroup which contains $B(A)$ as a subsemigroup.

Proof. We fix an arbitrary idempotent $(a, a)$ of the semigroup $B(A)$ and suppose that $(a, a)$ is a non-isolated point of the topological space $(B(A), \tau)$. Since the maps $\lambda_{(a, a)} : B(A) \to B(A)$ and $\rho_{(a, a)} : B(A) \to B(A)$ defined by the formulae $((x, y)) \lambda_{(a, a)} = (a, a)(x, y)$ and $((x, y)) \rho_{(a, a)} = (x, y)(a, a)$ are continuous retractions, we conclude that $(a, a)B(A)$ and $B(A)(a, a)$ are closed subsets in the topological space $(B(A), \tau)$ (see [20] Exercise 1.5.C). For an arbitrary element $b$ of the shift-set $A$ in the linearly ordered group $G$ we put

$$DL_{(b,b)} [(b, b)] = \{(x, y) \in B(A) : (x, y)(b, b) = (b, b)\}.$$

Lemma 2.2 and Proposition 2.3 imply that

$$DL_{(b,b)} [(b, b)] = \uparrow_{\tau} (b, b) = \{(x, x) \in B(A) : x \leq b \text{ in } A\},$$

and since right translations are continuous maps in $(B(A), \tau)$ we get that $DL_{(b,b)} [(b, b)]$ is a closed subset of the topological space $(B(A), \tau)$ for every $b \in A$. Then there exists an open neighbourhood $W_{(a,a)}$ of the point $(a, a)$ in the topological space $(B(A), \tau)$ such that

$$W_{(a,a)} \subseteq B(A) \setminus \{(a^+, a^+)B(A) \cup B(A)(a^+, a^+) \cup DL(a^−, a^−))\}.$$

Since $(B(A), \tau)$ is a semitopological semigroup, we conclude that there exists an open neighbourhood $V_{(a,a)}$ of the idempotent $(a, a)$ in the topological space $(B(A), \tau)$ such that the following conditions hold:

$$V_{(a,a)} \subseteq W_{(a,a)}, \quad (a, a) \cdot V_{(a,a)} \subseteq W_{(a,a)} \quad \text{and} \quad V_{(a,a)} \cdot (a, a) \subseteq W_{(a,a)}.$$

Hence at least one of the following conditions holds:

(a) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x, y) \in B(A)$ such that $x < y \leq a$ in the set $A$; or

(b) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x, y) \in B(A)$ such that $y < x \leq a$ in the set $A$.

In case (a) by Proposition 2.3 we have that

$$(a, a)(x, y) = (a, a \cdot x^{-1} \cdot y) \notin W_{(a,a)}$$

because $x^{-1} \cdot y \geq e$ in $G$, and in case (b) by Proposition 2.3 we have that

$$(x, y)(a, a) = (a \cdot y^{-1} \cdot x, a) \notin W_{(a,a)}$$

because $y^{-1} \cdot x \geq e$ in $G$ which contradicts the separate continuity of the semigroup operation in $(B(A), \tau)$. The obtained contradiction implies that the set $V_{(a,a)}$ is a singleton, and hence the idempotent $(a, a)$ is an isolated point of the topological space $(B(A), \tau)$.

Now, we apply Lemma 3.1 and get that the topological space $(B(A), \tau)$ is discrete.

Theorem 3.6 implies the following three corollaries:

Corollary 3.7. Let $G$ be a linearly ordered group which is not densely ordered and $A$ be a non-empty shift-set in $G$. Then every semigroup Hausdorff topology $\tau$ on the semigroup $B(A)$ is discrete.

Corollary 3.8 ([21]). Every shift-continuous Hausdorff topology $\tau$ on the bicyclic extended semigroup $C_E$ is discrete.

Corollary 3.9 ([12, 13]). Every shift-continuous Hausdorff topology $\tau$ on the bicyclic monoid $C(p, q)$ is discrete.

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