Boundary controllability of two coupled wave equations with space-time first-order coupling in 1-D

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Abstract. This paper is devoted to study exact controllability of two one-dimensional coupled wave equations with first-order coupling terms with coefficients depending on space and time. We give a necessary and sufficient condition for both exact controllability in high frequency in the general case and the unique continuation in the cascade case.

1. Introduction

We are interested in the boundary controllability of the following system of two strongly coupled 1-D wave equations

\[
\begin{aligned}
\begin{cases}
y_{tt} = y_{xx} + M((ay)_t + (by)_x), & \text{in } Q_T := (0, T) \times (0, 1), \\
y(t, 0) = Bu(t), & \text{in } (0, T), \\
y(0, x) = y_0(x), & \text{in } (0, 1), \\
y_t(0, x) = y_1(x), & \text{in } (0, 1),
\end{cases}
\end{aligned}
\]

where \( y = (y_1, y_2)' \) is a vector function and

\[
M = (m_{ij})_{1 \leq i, j \leq 2} \in \mathcal{L} (\mathbb{R}^2), \quad B = (b_1, b_2)' \in \mathbb{R}^2, \quad a, b \in C^1(Q_T; \mathbb{R}),
\]

and \( u \) a is scalar control function acting at \( x = 0 \).

This work is motivated by some previous papers. One of them is the result of Zhang [27]: a single wave equation in any space dimension with lower-order terms is proved to be exactly controllable by a control acting on part of the boundary under a suitable geometric condition and independently from the lower terms. The author extended earlier Carleman inequalities proved by Fursikov–Imanuvilov [18] for the wave equation without these lower-order terms.

The same issue arises for systems of \( n \) (\( \geq 2 \))-coupled wave equations with boundary or distributed controls. In [1], Alabau-Boussouira studied the controllability of 2-coupled wave equations with zero-order coupling operator with constant coefficients. Later, this result has been generalized by Alabau-Boussouira and Léautaud in [2], [3], for coupling coefficients depending on the space variable under the geometric

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control condition introduced in [10]. In these works, one of the coupling coefficients is supposed to be small.

Dehman, Le Rousseau and Léautaud [15] studied distributed controllability of 2-coupled wave equations on a Riemannian manifold without boundary (periodic boundary conditions in the 1-D case) with a particular zero-order coupling operator of cascade type. They proved that exact controllability holds provided that the Geometric Control Condition is satisfied. Further, they gave a characterization of the minimal time of control. An abstract result on the exact controllability of cascade systems is due to Alabau-Boussouira [4] (abstract setting with application to various coupled second-order PDEs). In all these cited works, it has been assumed that the coupling functions are of constant sign.

A boundary controllability result has been established without the sign or the smallness conditions by Bennour et al. [11] for 2-coupled wave equations in 1-D with cascade type coupling through velocity.

Concerning the constant case, Avdonin and De Tereza gave a complete answer for the exact boundary controllability issue for 2-coupled wave equations by zero-order operator in 1-D. The same authors came back in [9] and generalized their result to $n \geq 2$-coupled wave equations in $1 - D$ but always with constant coupling coefficients under a Kalman rank condition. The same condition appears for distributed controllability of $n \geq 2$-coupled multidimensional wave equations with zero-order coupling matrix with constant coefficients. It has been proved by Liard and Lissy in [20] that it is necessary and sufficient for the exact controllability in more regular energy space. An extension of this result can be found in Duprez and Olive [16] for cascade systems with zero-order coupling operator whose coefficients depend on the space variable. However, boundary controllability has not been treated yet.

Recently, in [13], Cui et al. studied distributed controllability of $n \geq 2$-coupled wave equations with zero and first-order coupling operator whose coefficients depend on both space and time variables on a compact Riemannian manifold without boundary (periodic boundary conditions in the 1-D case). It has been shown that the exact controllability issue can be reduced to the controllability of a finite-dimensional system along the associated Hamiltonian flow. The authors also gave a unique continuation results in the autonomous case under classical support and sign assumptions. The same idea appears in [5] by Alabau-Boussouira et al. where distributed controllability of 1-D first-order system with periodic boundary conditions is considered. The authors proved that exact controllability is reduced to the controllability of parameterized nonautonomous finite-dimensional system. We would also like to mention the recent paper by Coron and Nguyen [14] where they proved exact boundary controllability result for a hyperbolic system with space-time zero-order term in 1-D. We emphasize that in this work the control matrix is invertible (the control acts on all the components with negative speeds).

In light of all of the cited works, we can see that the main issue that has to be solved is to figure out the optimal assumptions the coupling coefficients (or operators) of
such systems must satisfy so that exact or approximate controllability holds with less number of controls.

In this article, and by using the characteristics method and a perturbation argument introduced in the pioneer work of Russell [25], we give a necessary and sufficient condition for the boundary exact controllability of System (1) in high frequency for a general matrix \( M \). We shall also propose a criterion for the unique continuation property in the cascade case. Actually, we will prove that the unique continuation property is equivalent to solving a system a 2-coupled Fredholm integral equations of the third kind. We apply this criterion to nontrivial examples.

This paper is organized as follows: after some preliminaries and fixing some notations, well-posedness and equivalence with a first-order symmetric hyperbolic system gathered in Sect. 2, we present the main results of exact controllability of System (1) in high frequency (weak observability) in Sect. 3. Section 4 is devoted to the unique continuation issue for System (1). Appendix 5 contains the proof of some technical lemmas used in the previous sections.

2. Preliminaries

In this section, we recall some results about well-posedness which can be proved exactly as in the scalar case. For the proof of these results in the scalar case, we refer for instance to [27] and the references therein.

**Proposition 1.** Let \( T > 0 \). Under the assumption (2), suppose that:

\[
(y_0, y_1, u) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2 \times L^2(0, T).
\]

Then, there exists a unique weak solution \( y \) to System (1) such that

\[
(y, y_t) \in C \left([0, T], L^2(0, 1)^2 \times H^{-1}(0, 1)^2\right).
\]

Moreover, there exists a constant \( C = C(T, a, b) > 0 \) such that:

\[
\|y\|_{C([0, T], L^2(0, 1)^2 \times H^{-1}(0, 1)^2)} \leq C \left(\|Bu\|_{L^2(0, T)^2} + \|(y_0, y_1)\|_{L^2(0, 1)^2 \times H^{-1}(0, 1)^2}\right).
\]

The adjoint problem associated with (1) writes:

\[
\begin{align*}
\varphi_{tt} &= \varphi_{xx} - M^*(a\varphi_t + b\varphi_x), \quad \text{in } (0, T) \times (0, 1), \\
\varphi|_{x=0,1} &= 0, \quad \text{in } (0, T), \\
(\varphi, \varphi_t)|_{t=T} &= (\varphi_0^T, \varphi_1^T), \quad \text{in } (0, 1).
\end{align*}
\]

**Proposition 2.** Let \( T > 0 \) and assume (2). Then,

1. For any

\[
\left(\varphi_0^T, \varphi_1^T\right) \in H^1_0(0, 1)^2 \times L^2(0, 1)^2,
\]

there exists a unique weak solution \( \varphi \) to System (3) such that

\[
\varphi \in C \left([0, T], H^1_0(0, 1)^2\right) \cap C^1 \left([0, T], L^2(0, 1)^2\right).
\]
Moreover, $\varphi_{x|_{x=0,1}} \in L^2(0, T)^2$ and there exists a constant $C = C (T, a, b) > 0$ such that:

$$
\| (\varphi, \varphi_t) \|_{C([0,T], H_0^1(0,1))^2 \times L^2(0,1)^2)} + \| \varphi_{x|_{x=0,1}} \|_{L^2(0,T)^2}
\leq C \left\| \left( \varphi_T^0, \varphi_T^1 \right) \right\|_{H_0^1(0,1)^2 \times L^2(0,1)^2}.
$$

2. For any

$$
\left( \varphi_T^0, \varphi_T^1 \right) \in \left( H^2 \cap H_0^1(0, 1) \right)^2 \times H_0^1(0, 1)^2
$$

there exists a unique strong solution $\varphi$ to System (3) such that

$$
\varphi \in C \left( [0, T], \left( H^2 \cap H_0^1(0, 1) \right)^2 \right) \cap C^1 \left( [0, T], H_0^1(0, 1)^2 \right) \cap C^2 \left( [0, T], L^2(0, 1)^2 \right),
$$

We are interested in the controllability issue for System (1). Recall that System (1) is said to be

1. **exactly controllable at time $T > 0$** if for any

$$
(y_0, y_1), (\tau_0, \tau_1) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2,
$$

there exists $u \in L^2(0, T)$ such that the associated solution $y$ to (1) satisfies

$$
(y, y_t)_{|t=T} = (\tau_0, \tau_1), \quad \text{in} \quad (0, 1).
$$

2. **approximately controllable at time $T > 0$** if for any

$$
(y_0, y_1), (\tau_0, \tau_1) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2,
$$

and any $\varepsilon > 0$, there exists $u \in L^2(0, T)$ such that the associated solution $y$ to (1) satisfies:

$$
\| (y, y_t)_{|t=T} - (\tau_0, \tau_1) \|_{L^2(0,1)^2 \times H^{-1}(0,1)^2} < \varepsilon.
$$

These controllability concepts are known to be connected with the observability properties of the adjoint system (3) (see [26, Part 4, Chapter 2]). Namely:

- System (1) is exactly controllable at time $T > 0$ if, and only if, there exists $C = C_T > 0$ such that for any $(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2$, the associated solution $\varphi$ to (3) satisfies the **observability inequality**:

$$
\left\| \left( \varphi_T^0, \varphi_T^1 \right) \right\|_{H_0^1(0,1)^2 \times L^2(0,1)^2} \leq C \int_0^T |B^* \varphi_x(t, 0)|^2 \, dt.
$$

In this case, the adjoint system is said exactly observable.
We point out that under assumption (5) of the same proposition, we get:

\[ \left( B^* \varphi_x (t, 0) = 0, \ t \in (0, T) \right) \Rightarrow \varphi \equiv 0 \ in \ Q_T. \]  

To study the observability inequality (6) for solutions to (3), we transform this system into a hyperbolic system of order one. Introduce the Riemann invariants:

\[ a = q, \ b = \varphi. \]

Thus,

\[ (a, b, \varphi) \left\| \begin{array}{c} qT \\ pT \end{array} \right\| = T \varphi \in (0, 1)^2, \]

\[ \| \varphi \|_{0} = L^2 (0, 1)^2, \]

where:

\[ (p, q)_{|x=0} = 0, \ (p, q)_{|t=T} = (qT, pT), \]

\[ \| \varphi \|_{T} \leq C \parallel \varphi \parallel_{T} \]

\[ \parallel \varphi \parallel_{T} \leq C \parallel qT, pT \parallel_{L^2 (0, 1)^2}, \]

\[ \parallel \varphi \parallel_{T} \leq C \parallel qT, pT \parallel_{L^2 (0, 1)^2}. \]
Choosing $C(\mathbb{R})$ since in $\mathbb{R}$, we have:

$$
\{(0, 1)^2 \cap C([0, T], L^2(0, 1)^2) \cap C^1([0, T], L^2(0, 1)^2) \cap L^2(0, 1)^2),
(q - p)|_{x=0, 1} \in L^2(0, T)^2, \\
\|(p, q)\|_{C([0, T], L^2(0, 1)^2 \times L^2(0, 1)^2)} + \|(q - p)|_{x=0, 1}\|_{L^2(0, T)^2} \\
\leq C \|(p T, q T)\|_{L^2(0, 1)^2 \times L^2(0, 1)^2}.
$$

Conversely, if $(p, q)$ is a solution to (12) associated with $(p T, q T) \in H^1(0, 1)^2 \times H^1(0, 1)^2$ satisfying (10), then there exists $\varphi \in H^1(Q_T)^2$ such that:

$$
\left(\begin{array}{c}
\varphi_t \\
\varphi_x
\end{array}\right) = \left(\begin{array}{c}
\frac{q + p}{2} \\
\frac{q - p}{2}
\end{array}\right),
$$

since in $Q_T$, from System (12), the scalar curl of $(q + p, q - p)^t$ is:

$$(q + p)_x - (q - p)_t = (p_x + p_t) - (q_t - q_x) = 0.$$

The assumed regularity of $(p, q)$ implies that:

$$
\varphi \in C\left([0, T], H^2(0, 1)^2\right) \cap C^2\left([0, T], L^2(0, 1)^2\right).
$$

Moreover, taking into account the definition of $\alpha_1$ and $\alpha_2$ in (13), it is straightforward that:

$$
\varphi_{tt} - \varphi_{xx} = -M^x (a\varphi_t + b\varphi_x), \text{ in } Q_T.
$$

We note moreover that

$$
\varphi_x = \frac{q - p}{2} \Rightarrow \varphi(t, x) = \int_0^x \frac{q - p}{2} (t, \xi) \, d\xi + C.
$$

where $C$ is an arbitrary real constant. From (12), it appears that

$$(q - p)_t = (q + p)_x \Rightarrow \left(\int_0^1 (q - p) (t, \xi) \, d\xi\right)_t = 0 \Rightarrow \int_0^1 (q - p) (t, \xi) \, d\xi = 0,
$$

the last equality coming from (10) and the continuity in time of $(p, q)$. It follows that:

$$
\varphi|_{x=0, 1} = C, \text{ in } (0, T).
$$

Choosing $C = 0$ in the previous equality leads to a function $\varphi(t, .) \in H^1_0(0, 1)^2$ for $t \in (0, T)$.

To summarize, let us introduce the space:

$$
H = \left\{(f, g) \in L^2(0, 1)^2 \times L^2(0, 1)^2, \int_0^1 (f - g) (\xi) \, d\xi = 0\right\}.
$$

This is clearly a closed subspace of $L^2(0, 1)^2 \times L^2(0, 1)^2$ and thus a Hilbert space with the usual norm (and scalar product) of $L^2(0, 1)^2 \times L^2(0, 1)^2$. In view of the previous considerations, we have:
Proposition 3. Let $T > 0$ and $H$ defined in (15).

1. For any $(p_T, q_T) \in H$, there exists a unique weak solution $(p, q)$ to (12) such that $(p, q) \in C([0, T], H)$. Moreover $(p - q)|_{x=0,1} \in L^2(0, T)^2$ and there exists a constant $C = C(T, \alpha_1, \alpha_2) > 0$ such that:
\[
\| (p, q) \|_{C([0, T], H)} + \| (p - q)|_{x=0,1} \|_{L^2(0, T)^2} \leq C \| (p_T, q_T) \|_H.
\]

2. The observability inequality (6) is equivalent to
\[
\| (p_T, q_T) \|_H^2 \leq C_T \int_0^T \left| B^* p(t, 0) \right|^2 dt.
\] (16)

We will need in an essential way the block diagonal system associated with System (12):
\[
\begin{cases}
  p_t + p_x + M^* \alpha_1 p = 0, & \text{in } Q_T, \\
  q_t - q_x + M^* \alpha_2 q = 0, & \text{in } Q_T, \\
  (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\
  (p, q)|_{t=T} = (p_T, q_T), & \text{in } (0, 1).
\end{cases}
\] (17)

System (12) is a perturbation of System (17) by the multiplication operator defined on $H$ by:
\[
\mathcal{P} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0_{2 \times 2} & \alpha_2 I_{2 \times 2} \\
\alpha_1 I_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.
\] (18)

The plan now is the following:

1. In a first step (Sect. 3), we will give necessary and sufficient condition for the solution to the diagonal system (17) to satisfy the observability inequality (16).

2. In a second step and in the same section (Sect. 3.5), and in the same spirit of [21], we will prove that if the solutions of (17) satisfy (16), then up to a finite-dimensional subspace of initial data in $H$, the same is true for solutions to System (12). More precisely, we will prove that there exists a compact operator $N : H \to L^2(0, T)$, such that the following weak observability inequality
\[
\| (p_T, q_T) \|_H^2 \leq C_T \int_0^T \left| B^* p(t, 0) \right|^2 dt + \| N(p_T, q_T) \|_{L^2(0, T)}^2,
\] (19)

holds. Actually, we will see that $N := p(t, 0) - p_d(t, 0)$ where $p$ and $p_d$ are the solutions of Systems (12) and (17), respectively. Note that by the weak observability inequality entails exact controllability up to finite-dimensional space of target states (which are known as the invisible states) (see [24, Lemma 3]).

Let us point out that in view of the equivalence between the $(p, q)$ system and the $\varphi$ system, weak observability inequality in $(p, q)$ readily transports to a weak observability inequality in the associated $\varphi$.

3. The last step (Sect. 4) will provide sufficient (and necessary in some cases) for the unique continuation property to be satisfied (or the Fattorini criterion) for some particular matrix $M$ and functions $\alpha_1$ and $\alpha_2$. Nontrivial examples will be developed at the end of the section.
3. Weak observability

In the block diagonal system (17), the change of variables \( t = T - t \) (we keep the same notations) leads to a system of the from

\[
\begin{align*}
p_t + p_x - M^* \eta_1 p &= 0, \quad \text{in } QT, \\
q_t - q_x - M^* \eta_2 q &= 0, \quad \text{in } QT, \\
(p + q)|_{x=0,1} &= 0_{\mathbb{R}^2}, \quad \text{in } (0, T), \\
(p, q)|_{t=0} &= (p_0, q_0), \quad \text{in } (0, 1),
\end{align*}
\]

where

\[
\eta_1(t, x) = \alpha_2(T - t, x), \quad \eta_2(t, x) = \alpha_1(T - t, x), \quad (t, x) \in QT.
\]

(20)

Note that the observed component does not change since \((p + q)|_{x=0} = 0_{\mathbb{R}^2}\).

In this subsection, for any numbers \( s, T \) such that \( 0 < s < T \), we compute the explicit solution \( Z = (p, q) \) to the system

\[
\begin{align*}
p_t + p_x - M^* \eta_1 p &= 0, \quad \text{in } QT, \\
q_t - q_x - M^* \eta_2 q &= 0, \quad \text{in } QT, \\
(p + q)|_{x=0,1} &= 0_{\mathbb{R}^2}, \quad \text{in } (0, T), \\
(p, q)|_{t=s} &= (p_s, q_s), \quad \text{in } (0, 1).
\end{align*}
\]

(22)

For given real numbers \( s, t \) such that \( 0 \leq s < t \), the function \( Z(t, x) = Z(t, x; s, Z_s) = (p, q)(t, x; s, Z_s) \) for \( t \in (s, T) \) and \( x \in (0, 1) \) will denote the solution to (22) with its dependence on the starting time \( s \geq 0 \) and the initial data \( Z_s = (p_s, q_s) \in H \).

When \( s = 0 \), we simply write \( Z(t, x) = Z(t, x; Z_0) \) but unless necessary, all along this section, \( Z = (p, q) \) will denote the solution to (20).

The following assumption is fixed and is assumed in all the results of this section:

\[
\eta_i \in C^1(\overline{QT}, \mathbb{R}), \quad i = 1, 2.
\]

(21)

It is simply derived from the assumption on \( a, b \) in (2).

Notice that the exact observability property of System (20) amounts to the observability inequality:

\[
\exists C_T > 0, \quad \|(p_0, q_0)\|_H^2 \leq C_T \int_0^T \|B^* p(t, 0)\|^2 dt, \quad \forall (p_0, q_0) \in H.
\]

(23)

To express more compactly the formulas for the solutions to (22), we introduce the function \( \phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) defined by

\[
\phi(t, s) = \begin{cases} 
0, & \text{if } t \leq s, \\
\int_{\max\{s, t-2\}}^{\max\{s, t-1\}} \eta_1(\tau, \tau - (t - 2)) d\tau + \int_t^{\max\{s, t-1\}} \eta_2(\tau, t - \tau) d\tau, & \text{if } t > s,
\end{cases}
\]

(24)
and the sequence of functions:

\[ f_n(t, s) = \sum_{k=0}^{n} \phi(t - 2k, s), \quad n \geq 0. \]  
(25)

When \( s = 0 \), we simply write:

\[ \phi(t, 0) = \phi(t), \quad f_n(t, 0) = f_n(t), \quad t \in \mathbb{R}, \quad n \geq 0. \]

At this level, it is useful to clarify the geometric meaning of the function \( \phi \). Actually the characteristic curves associated with the hyperbolic systems (17), (20) are the lines \( x + t = c_1 \), \( x - t = c_2 \), \( c_1, c_2 \in \mathbb{R} \).

Introduce the vector field \( F = \left( \frac{\eta_1 + \eta_2}{2}, \frac{\eta_1 - \eta_2}{2} \right) \) and let \( \gamma_j \) (\( j = 1, 2 \)) the two directions \( \gamma_1 = (1, 1) \) and \( \gamma_2 = (1, -1) \) of the characteristic lines. For the canonical scalar product in \( \mathbb{R}^2 \), one has \( F \cdot \gamma_j = \eta_j, \quad (j = 1, 2) \) and for \( t > 0 \), \( \phi(t) \) is then the line integral of the vector field \( F \) along the line \( \Gamma_t \) defined by the function:

\[
\gamma_1(\tau) = \begin{cases} 
(\tau, \tau - (t - 2)), & \text{if } \max\{0, t - 2\} \leq \tau \leq \max\{0, t - 1\}, \\
(\tau, t - \tau), & \text{if } \max\{0, t - 1\} \leq \tau \leq t.
\end{cases}
\]  
(26)

As a consequence, for \( t > 0 \) and \( n \geq 0 \), the function \( f_n(t) \) is the line integral of the vector field \( F \) along the lines \( \bigcup_{1 \leq k \leq n+1} \Gamma_{t-2k} \) with the convention that if \( t - 2k < 0 \), \( \Gamma_{t-2k} = \emptyset \) (see the figure below for the representation of these lines).

3.1. Main results

As pointed out above, observability inequality for System (12) will hold modulo compact operator. A classical functional analysis result shows that the space of invisible target states is finite codimension and it might be reduced to zero if approximate controllability (or the unique continuation property for the adjoint system) holds (see Sect. 4).

We start by a negative controllability result:
Theorem 1. If $T < 4$, the weak observability inequality (19) doesn’t hold. More precisely, there is an infinite-dimensional space of unreachable target states.

For the proof, see Proposition 6 and Remark 5.

Now, we present a positive controllability result. Denote by $\lambda_1, \lambda_2$ the eigenvalues of $M^*$ if it is diagonalizable and by $\mu$ the multiple eigenvalue of $M^*$ if it is not. We have the following controllability result:

**Theorem 2.** Let $n \geq 2$ be an integer.

- If $2n \leq T < 2n + 1$. Then, System (12) is weakly observable (see 19) if, and only if, the following three conditions are satisfied:
  1. $\text{rank } [B \mid MB] = 2$.
  2. For any $x \in [0, 1]$, there exists $1 \leq k \leq n - 1$ such that:
     \[
     \phi (2k - 2 - x) \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\},
     \]
     \[
     \phi (2k - 2 - x) \notin \frac{\pi}{\sqrt{\lambda_1}} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
     \]
  3. For any $x \in [0, T - 2n)$ and $x^* \in [T - 2n, 1)$, there exist $1 \leq k \leq n$ and $1 \leq k^* \leq n - 1$, respectively, such that:
     \[
     \phi (x + 2k) \neq 0, \phi (x^* + 2k^*) \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\},
     \]
     \[
     \phi (x + 2k), \phi (x^* + 2k^*) \notin \frac{\pi}{\sqrt{\lambda_1}} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
     \]

- If $2n + 1 \leq T < 2n + 2$. Then, System (12) is weakly observable (see 19) if, and only if, the following three conditions are satisfied:
  1. $\text{rank } [B \mid MB] = 2$.
  2. For any $x \in [2n + 2 - T, 1)$ and $x^* \in [0, 2n + 2 - T)$, there exist $1 \leq k \leq n$ and $1 \leq k^* \leq n - 1$, respectively, such that:
     \[
     \phi (2k - 2 - x) \neq 0, \phi (2k^* + 2 - x^*) \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\},
     \]
     \[
     \phi (2k - 2 - x), \phi (2k^* + 2 - x^*) \notin \frac{\pi}{\sqrt{\lambda_1}} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
     \]
  3. For any $x \in [0, 1]$, there exists $1 \leq k \leq n$ such that:
     \[
     \phi (x + 2k) \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\},
     \]
     \[
     \phi (x + 2k) \notin \frac{\pi}{\sqrt{\lambda_1}} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
     \]

The proof is rather long, it relies on studying the exact controllability of the diagonal block system (20) combined with a compactness argument. See Propositions 8 and 9 and Remark 6.

Let us make several observations:
Remark 1. To illustrate geometrically the assertions of the above theorem, we recall that from each point \((0, x)\) (with \(x \in [0, 1]\)) come two characteristics which stop at some point of the line \(t = T\). If for example \(2n \leq T < 2n + 1\) with \(n \geq 2\), these characteristics touch the observability boundary \([0, T] \times [0]\) at least two times at points of the form \((2k - x, 0)\) for one of them and of the form \((2l + x, 0)\) for the other. The conditions on \(\phi\) means that there exist at least two consecutive points of the form \((2k - x, 0)\) and two consecutive points of the form \((2l + x, 0)\) such that the line integrals of the vector field \(\mathbf{F} = \left(\frac{n_1 + n_2}{2}, \frac{n_1 - n_2}{2}\right)\), namely \(\int_{\Gamma_{(2l+1)+x,0}} \mathbf{F} \cdot \gamma\) and \(\int_{\Gamma_{(2l-x,0)}} \mathbf{F} \cdot \gamma\), are not, depending on the coupling matrix nature, zero or are not in some discrete set.

Remark 2. Let us first recall that for any \(t \geq 2\), the function \(\phi\) defined in (24) is given by

\[
\phi(t) = \int_{t-2}^{t-1} \eta_1(\tau, \tau - \tau - 2) \, d\tau + \int_{t-1}^{t} \eta_2(\tau, \tau - \tau) \, d\tau.
\]

Then, by (21) and (14) we get

\[
\phi(t) = \frac{1}{2} \int_{0}^{1} (a + b) (T - \tau + \tau - 2, \tau) \, d\tau + \frac{1}{2} \int_{0}^{1} (a - b) (T - \tau + \tau, \tau) \, d\tau.
\]

Observe that in the autonomous case \((a \text{ and } b \text{ are time independent})\) the above formulas becomes

\[
\phi(t) = \int_{0}^{1} a(s) \, ds.
\]

Hence, the coupling with first-order derivative in space doesn’t have any influence on the controllability of System (20) in high frequency unless \(b\) depends on time. More precisely, if we let \(a = 0\) and \(b = b(x)\), then the weak observability inequality (19) doesn’t hold in any time and for any \(b\) since \(\phi\) will be zero. The situation is not the same for parabolic systems. In [17], boundary controllability of a cascade system of two parabolic equations in \(1 - D\) has been studied with coupling acting on first-order component. It has been shown that the underlying system is exactly controllable if the coupling function satisfies a moment assumption for the low frequency part and an average assumption like (27) for the high frequency. This shows that differences between hyperbolic and parabolic systems are not limited to the geometric control condition introduced in [10] or the minimal time of control.

3.2. Construction of the solution to the diagonal system

Given \((t, x) \in Q_T\), the value of \(p(t, x)\) and \(q(t, x)\) is determined either by \((p_s, q_s)\) or by their values at \(x = 0\) or \(x = 1\). More precisely, we have by the characteristics method:

\[
p(t, x) = \begin{cases} 
\exp\left(M^* \int_{1-x}^{t} \eta_1(\tau, \tau - (t - \tau)) \, d\tau\right) p(t - x, 0), & \text{if } t - x > s, \\
\exp\left(M^* \int_{s}^{t} \eta_1(\tau, \tau - (t - \tau)) \, d\tau\right) p_s(x - t + s), & \text{if } t - x < s,
\end{cases}
\]

(28)
and
\[
q(t, x) = \begin{cases} 
\exp \left( M^* \int_{x+t-1}^{x} \eta_2(\tau, t + x - \tau) d\tau \right) q(t + x - 1, 1), & \text{if } t + x - 1 > s, \\
\exp \left( M^* \int_{x}^{s} \eta_2(\tau, t + x - \tau) d\tau \right) q_s(x + t - s), & \text{if } t + x - 1 < s.
\end{cases}
\]
(29)

Thus, computing \( p(t, x) \) and \( q(t, x) \) amounts to evaluate \( p(t, 0) \) and \( q(t, 1) \) (respectively) as functions of the initial data \( (p_0, q_0) \), keeping in mind the boundary conditions. The following lemma can be proved by induction:

**Lemma 1.** Let \( n \geq 0 \) be an integer and \( Z_s = (p_s, q_s) \in H \). Then, if \( Z = (p, q) \) is the solution to System (22), one has:
\[
p(t, 0) = -e^{\int_{0}^{t} \eta_1(\tau, t-\tau) d\tau} p_s(t - 2n), \quad \text{if } 2n \leq t < 2n + 1, \quad (30)
p(t, 0) = e^{\int_{0}^{t} \eta_1(\tau, t-\tau) d\tau} p_s(2n + 2 - t + s), \quad \text{if } 2n + 1 \leq t - s < 2n + 2. \quad (31)
\]

As a consequence:
\[
q(t, 1) = e^{M^* \int_{0}^{t} \eta_1(\tau, t-\tau) d\tau} p_s(t - 2n), \quad \text{if } 0 \leq t - s < 1, \quad (32)
q(t, 1) = -e^{M^* \int_{0}^{t} \eta_1(\tau, t-\tau) d\tau} p(t - 1, 0), \quad \text{if } t - s > 1. \quad (33)
\]

**Proof.** We give the proof for \( s = 0 \), and a simple change of variable \( t \mapsto t - s \) leads to the formulas of the lemma.

Assume \( n = 0 \) in (30)–(31). For \( 0 < t < 1 \), the characteristics method, the boundary conditions and (24)–(25) give:
\[
q(t, 0) = e^{M^* \int_{0}^{t} \eta_2(\tau, t-\tau) d\tau} q_0(t) = e^{f_0(t)M^*} q_0(t).
\]

For \( 1 \leq t < 2 \), as previously:
\[
q(t, 0) = e^{M^* \int_{-1}^{t} \eta_2(\tau, t-\tau) d\tau} q(t - 1, 1) = -e^{M^* \int_{-1}^{t} \eta_2(\tau, t-\tau) d\tau} p(t - 1, 1) = -e^{M^* \phi(t)} p_0(2 - t) = -e^{f_0(t)M} p_0(2 - t).
\]

Thus, (30)–(31) are verified for \( n = 0 \).

Given \( n \geq 0 \), let us assume (30) and (31). Let \( 2n + 2 \leq t < 2n + 3 \). Then, by the same computations using the characteristics method:
\[
q(t, 0) = -e^{\phi(t)M^*} p(t - 2, 0). \quad (34)
\]

Since \( 2n \leq t - 2 < 2n + 1 \), formula (30) applies and gives
\[
p(t - 2, 0) = e^{\int_{0}^{t-2} \eta_1(\tau, t-\tau) d\tau} q_0(t - 2(n + 1)). \quad (35)
\]
Now, from (25)
\[ f_n(t - 2) = \sum_{k=0}^{n} \phi(t - 2 - 2k) \]
\[ = \sum_{k=1}^{n+1} \phi(t - 2k) \]
\[ = f_{n+1}(t) - \phi(t). \] (36)
Thus, inserting (35)–(36) in (34) leads to:
\[ 2n + 2 \leq t < 2n + 3 \Rightarrow q(t, 0) = e^{f_{n+1}(t)M^*}q_0(t - 2(n + 1)), \]
and (30) is proved with \( n \) replaced by \( n + 1 \).

The proof by induction of (31) can be performed in the same way. \( \square \)

Remark 3. 1. System (22) defines an evolution family \( (U_{\text{diag}}(t, s))_{0 \leq s \leq t} \) on \( H \) (see [23] for instance) which is explicitly computed by mean of formulas (28)–(29)–(30)–(31):
\[ U_{\text{diag}}(t, s)Z_0 = (p, q)(t, \cdot; s, Z_0), \quad 0 \leq s \leq t, \quad Z_0 \in H. \] (37)
2. From (30)–(31), the following formula follows: if \( Z = (p, q) \) is a solution to System (20) associated with an initial data \( Z_s = (p_s, q_s) \), then
\[ B^* p(t, 0; s, Z_s) = \begin{cases} 
-B^* e^{f_n(t, s)M^*}q_s(t - s - 2n), & \text{if } 2n \leq t - s < 2n + 1, \\
B^* e^{f_n(t, s)M^*}p_s(2n + 2 - t + s), & \text{if } 2n + 1 \leq t - s < 2n + 2.
\end{cases} \] (38)
for \( n \geq 0 \). Formula (38) will be used in the next subsection in the study of the observability issue for System (20).

3.3. Some technical results on multiplication operators

In order to make clear the proof of our exact observability results, we will need some preliminary results on multiplications operators defined from \( L^2(0, 1)^k \) in \( L^2(0, 1)^n \) for some positive integers \( k, n \).

Let \( M = (m_{ij})_{1 \leq i, j \leq n} \) a \( n \times n \) matrix whose entries satisfies \( m_{ij} \in C([0, 1], \mathbb{R}) \).

The multiplication operator \( M : L^2(0, 1)^k \to L^2(0, 1)^n \) associated with \( M \) is defined by:
\[ (Mh)(x) = M(x)h(x), \quad x \in (0, 1), \quad h \in L^2(0, 1)^n. \]
Clearly \( M \) is a bounded operator. When \( k = n \), the following characterization of the invertibility of \( M \) is derived from [19, Proposition 2.2]:

\textbf{Proposition 4.} The operator \( M : L^2(0, 1)^k \to L^2(0, 1)^k \) is invertible if, and only if:
\[ \inf_{x \in [0,1)} |\det M(x)| > 0. \]

We are now interested by the case \( k < n \).
Proposition 5. Let $k < n$ and $\mathbb{M} : L^2(0, 1)^k \to L^2(0, 1)^n$. The following properties are equivalent:

1. There exists a constant $C > 0$ such that
   \[ \|h\|_{L^2(0, 1)^k} \leq C \|\mathbb{M}h\|_{L^2(0, 1)^n}, \forall h \in L^2(0, 1)^k. \]

2. For all $x \in [0, 1]$, there exists a $s \times s$ matrix $M_{\text{ext}}$, extracted from $M$, such that $\det M_{\text{ext}}(x) \neq 0$.

Proof. For the proof, see Appendix 5. □

3.4. Observability results

Let us start with the following Lemma:

Lemma 2. Let $T > 0$ and $(p, q)$ be the solution to (20) associated with $Z_0 = (p_0, q_0) \in H$. Then,

- If $2n \leq T < 2n + 1$ for some $n \geq 0$, one has:
  \[ \int_0^T \left| B^* p(t, 0) \right|^2 dt = \int_0^T \left| B^* e^{f_0(x)M^*} q_0(x) \right|^2 dx, \text{ for } n = 0, \]
  and for any $n \geq 1$
  \[ \int_0^T \left| B^* p(t, 0) \right|^2 dt = \sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx + \sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx + \int_{T-2n}^T \left| B^* e^{f_n(x+2n)M^*} q_0(x) \right|^2 dx. \]

- If $2n + 1 \leq T < 2n + 2$ for some $n \geq 0$, then
  \[ \int_0^T \left| B^* p(t, 0) \right|^2 dt = \int_0^{T-1} \left| B^* e^{f_0(2-x)M^*} p_0(x) \right|^2 dx + \int_0^1 \left| B^* e^{f_0(x)M^*} q_0(x) \right|^2 dx, \text{ for } n = 0, \]
  and for any $n \geq 1$
  \[ \int_0^T \left| B^* p(t, 0) \right|^2 dt = \sum_{k=0}^n \int_0^1 \left| e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx + \sum_{k=0}^{n-1} \int_0^1 \left| e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx + \int_{2n+2-T}^1 \left| e^{f_n(2n+2-x)M^*} p_0(x) \right|^2 dx. \]
Proof. Let \( n \geq 0 \) be an integer and suppose that \( 2n \leq T < 2n + 1 \). Then, if \( n = 0 \), we have from (30):

\[
\int_0^T \left| B^* p(t, 0) \right|^2 dt = \int_0^T \left| B^* e^{f_0(t)} M^* q_0(t) \right|^2 dt.
\]

If \( n \geq 1 \):

\[
\int_0^T \left| B^* p(t, 0) \right|^2 dt = \sum_{k=0}^{n-1} \left( \int_{2k}^{2k+1} + \int_{2k+1}^{2k+2} \right) \left| B^* p(t, 0) \right|^2 dt + \int_{2n}^T \left| B^* p(t, 0) \right|^2 dt
\]

\[
:= \sum_{k=1}^n (I_k + J_k) + \int_{2n}^T \left| B^* p(t, 0) \right|^2 dt. \tag{43}
\]

Using (38) leads to:

\[
I_k = \int_{2k}^{2k+1} \left| B^* p(t, 0) \right|^2 dt
\]

\[
= \int_{T-2k}^{T-2k+1} \left| B^* e^{f_k(t)} M^* q_0(t - 2k) \right|^2 dt
\]

\[
= \int_0^1 \left| B^* e^{f_k(x+2k)} M^* q_0(x) \right|^2 dx.
\]

For the second integral, in the same way:

\[
J_k = \int_{2k+1}^{2k+2} \left| B^* p(t, 0) \right|^2 dt
\]

\[
= \int_{2k+1}^{2k+2} \left| B^* e^{f_k(t)} M^* p_0(2k + 2 - t) \right|^2 dt
\]

\[
= \int_0^1 \left| B^* e^{f_k(2k+2-x)} M^* p_0(x) \right|^2 dt.
\]

And last:

\[
\int_{2n}^T \left| B^* p(t, 0) \right|^2 dt = \int_{2n}^T \left| B^* e^{f_n(t)} M^* q_0(t - 2n) \right|^2 dt
\]

\[
= \int_0^{T-2n} \left| B^* e^{f_n(x+2n)} M^* q_0(x) \right|^2 dx.
\]

Inserting the last formula in (43), we get (40).

If \( 2n + 1 \leq T < 2n + 2 \), exactly as in the previous computations, if \( n = 0 \), we get:

\[
\int_0^T \left| B^* p(t, 0) \right|^2 dt = \int_0^1 \left| B^* p(t, 0) \right|^2 dt + \int_1^T \left| B^* p(t, 0) \right|^2 dt
\]

\[
= \int_0^1 \left| B^* e^{f_0(x)} M^* q_0(x) \right|^2 dx + \int_0^{T-1} \left| B^* e^{f_0(2-x)} M^* p_0(x) \right|^2 dx,
\]
and if \( n \geq 1 \):

\[
\int_0^T \left| B^* p(t, 0) \right|^2 \, dt = \left( \sum_{k=0}^{n-1} \int_{2k}^{2k+1} + \sum_{k=0}^{n-1} \int_{2k+2}^{2k+1} \right) \left| B^* p(t, 0) \right|^2 \, dt
\]

\[
+ \int_{2n+1}^T \left| B^* p(t, 0) \right|^2 \, dt = \sum_{k=0}^{n-1} \int_0^1 \left| e^{\int_{x+2k}^{2k+2} M^*} q_0(x) \right|^2 \, dx
\]

which is exactly (42). This ends the proof of the lemma.

As an immediate consequence, we have:

**Corollary 1.** Let \( n \geq 1 \).

- If \( 2n \leq T < 2n + 1 \), a necessary and sufficient condition for exact observability of System (20) is that:

\[
\exists C_T > 0 : \int_0^1 \left| p_0(x) \right|^2 \, dx \\
\leq C_T \sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{\int_{x+2k}^{2k+2} M^*} p_0(x) \right|^2 \, dx, \quad \forall p_0 \in L^2(0,1)^2 .
\]  

(44)

and

\[
\sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{\int_{x+2k}^{2k+2} M^*} q_0(x) \right|^2 \, dx + C_T \int_0^{T-2n} \left| B^* e^{\int_{x+2n}^{x+2n+n} M^*} q_0(x) \right|^2 \, dx
\]

\[
\geq C_T \int_0^1 \left| q_0(x) \right|^2 , \quad \forall q_0 \in L^2(0,1)^2 .
\]  

(45)

- If \( 2n + 1 \leq T < 2n + 2 \), a necessary and sufficient condition for exact observability of System (20) is:

\[
\sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{\int_{x+2k}^{2k+2} M^*} p_0(x) \right|^2 \, dx + \int_{2n+2}^{T-2} \left| B^* e^{\int_{x+2n+2}^{x+2n+2} M^*} p_0(x) \right|^2 \, dx
\]

\[
\geq C_T \int_0^1 \left| p_0(x) \right|^2 \, dx, \quad \forall p_0 \in L^2(0,1)^2 ,
\]  

(46)

and

\[
\int_0^1 \left| q_0(x) \right|^2 \, dx \leq C_T \sum_{k=0}^{n} \int_0^1 \left| B^* e^{\int_{x+2k}^{x+2k+2} M^*} q_0(x) \right|^2 \, dx, \quad \forall q_0 \in L^2(0,1)^2 .
\]  

(47)
**Remark 4.** Let $n \geq 2$. For $2n \leq T < 2n + 1$, introduce the matrices:

\[
P_{2n}(x, T) = \begin{bmatrix}
B^* e^{f_0(2-x)M^*} \\
B^* e^{f_1(4-x)M^*} \\
\vdots \\
B^* e^{f_{n-1}(2n-x)M^*}
\end{bmatrix}
\]

\[
Q_{2n}(x, T) = \begin{bmatrix}
B^* e^{f_0(x)M^*} \\
\vdots \\
B^* e^{f_{n-1}(x+2(n-1))M^*} \\
\mathbb{I}_{(0,T-2n)}(x) B^* e^{f_n(x+2n)M^*}
\end{bmatrix},
\]

and their associated multiplication operators $P_{2n} : L^2(0,1)^2 \to L^2(0,1)^n$ and $Q_{2n} : L^2(0,1)^2 \to L^2(0,1)^{n+1}$. With these notations, (44) and (45), respectively, write:

\[
\exists C_T > 0, \int_0^1 |p_0(x)|^2 \, dx \leq C_T \|p_0\|_{L^2(0,1)^n}^2, \forall p_0 \in L^2(0,1)^2, \quad (49)
\]

\[
\exists C_T > 0, \int_0^1 |q_0(x)|^2 \, dx \leq C_T \|q_0\|_{L^2(0,1)^{n+1}}^2, \forall q_0 \in L^2(0,1)^2. \quad (50)
\]

For $2n + 1 \leq T < 2n + 2$, introduce the matrices:

\[
P_{2n+1}(x, T) = \begin{bmatrix}
B^* e^{f_0(2-x)M^*} \\
\vdots \\
B^* e^{f_{n-1}(2n-x)M^*} \\
\mathbb{I}_{(1,2n+2-T)}(x) B^* e^{f_n(2n+2-x)M^*}
\end{bmatrix}
\]

\[
Q_{2n+1}(x, T) = \begin{bmatrix}
B^* e^{f_0(x)M^*} \\
\vdots \\
B^* e^{f_{n-1}(x+2(n-1))M^*} \\
B^* e^{f_n(x+2n)M^*}
\end{bmatrix},
\]

and their associated multiplication operators $P_{2n+1} : L^2(0,1)^2 \to L^2(0,1)^{n+1}$ and $Q_{2n} : L^2(0,1)^2 \to L^2(0,1)^{n+1}$. With these notations, (46) and (47), respectively, write:

\[
\exists C_T > 0, \int_0^1 |p_0(x)|^2 \, dx \leq C_T \|p_0\|_{L^2(0,1)^{n+1}}^2, \forall p_0 \in L^2(0,1)^2, \quad (52)
\]

\[
\exists C_T > 0, \int_0^1 |q_0(x)|^2 \, dx \leq C_T \|q_0\|_{L^2(0,1)^{n+1}}^2, \forall q_0 \in L^2(0,1)^2. \quad (53)
\]

We are ready to state our first (negative) results on the controllability of System (20).
Proposition 6. For $T < 4$, there exists an infinite-dimensional subspace of initial data $(p_0, q_0) \in H$ for which the exact observability inequalities (44)–(47) are not satisfied by the associated solution $(p, q)$ to System (20).

Proof. The goal is to prove (23) does not hold for any $0 \leq T < 4$. If $0 \leq T < 1$, from (39), one has:

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^T |B^* e^{f_0(x)} M^* q_0(x)|^2 dx,$$

and clearly the observability inequality (23) does not hold for all $(p_0, q_0) \in H \times V_T$ where

$$V_T = \left\{ q_0 \in H : q_0 \cdot e^{f_0 M} B = 0 \text{ in } (0, T) \right\}.$$

If $1 \leq T < 2$, from (41):

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^{T-1} |B^* e^{f_0(2-x)} M^* p_0(x)|^2 dx + \int_0^1 |B^* e^{f_0(x)} M^* q_0(x)|^2 dx.$$

The observability inequality (23) does not hold for all nontrivial $(p_0, q_0) \in U_T \times H$ (for instance) where

$$U_T = \left\{ p_0 \in H : p_0 \cdot e^{f_0(2-x)} M B = 0 \text{ in } (0, T - 1) \right\}.$$

If $2 \leq T < 3$, from (40) follows the equality:

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^1 |B^* e^{f_0(2-x)} M^* p_0(x)|^2 dx$$

$$+ \int_0^1 |B^* e^{f_0(x)} M^* q_0(x)|^2 dx +$$

$$+ \int_0^{T-2} |B^* e^{f_1(x+2)} M^* q_0(x)|^2 dx,$$

and again (23) does not hold for all $(p_0, q_0) \in U_{T=1} \times H$.

Last, for $3 \leq T < 4$, from (42):

$$\int_0^T |B^* p(t, 0)|^2 dt = \sum_{k=0}^{n-1} \int_0^1 |B^* e^{f_0(2-x)} M^* p_0(x)|^2 dx$$

$$+ \int_0^{T-3} |B^* e^{f_1(4-x)} M^* p_0(x)|^2 dx$$

$$+ \sum_{k=0}^{1} \int_0^1 |B^* e^{f_1(x+2k)} M^* q_0(x)|^2 dx,$$

and (23) does not hold for all $(p_0, q_0) \in W \times H$ where

$$W = U_{T=1} \cap \{ p_0 \in H : \text{supp}(p_0) \subset (T - 3, 1) \}.$$
All the introduced subspaces of nonobservable initial data are actually infinite dimensional.

In these spaces can be found initial data \((p_0, q_0)\) for which approximate observability does not hold too: if \(0 < T < 4\), there exists \((p_0, q_0) \in H \times H\) such that \(\|(p_0, q_0)\|_{H \times H} = 1\) and for which the associated solution \((p, q)\) satisfies:

\[
B^* p(t, 0) = 0, \ t \in (0, T).
\]

□

Remark 5. The above proposition shows that if \(T < 4\) then the diagonal system \((20)\) is not exactly controllable. Combining this and the fact that the difference of the input maps is compact (see Theorem 3) entails that System \((12)\) is not weakly controllable; more precisely, there exists an infinite-dimensional space of unreachable target states.

It has to be pointed out that the previous negative observability result does not depend of the choice of \(\eta_1, \eta_2\) and \(M\).

Before going one in the analysis, let us give a necessary condition for the exact observability to hold:

Lemma 3. Let \(T > 0\). A necessary condition for the exact observability of System \((20)\) is

\[
\text{rank} \ [B \ | \ MB] = 2. \quad (54)
\]

Proof. If \((54)\) does not hold, there exists \(\lambda \in \mathbb{R}\) such that \(MB = \lambda B\) (in other worlds, \(B\) is an eigenvector to \(M\)). It follows that for any \(r \in \mathbb{R}\), \(e^{rM}B = e^{r\lambda}B \enskip (\Leftrightarrow B^* e^{rM^*} = e^{r\lambda}B^*)\). Thus, in this case, \((40)\) writes:

\[
\int_0^T \left| B^* p(t, 0) \right|^2 \, dt = \int_0^1 \left( \sum_{k=0}^{n-1} e^{2f_k(x+2k\lambda)} \right) \left| B^* p_0(x) \right|^2 \, dx
\]

\[
+ \int_0^1 \left( \sum_{k=0}^{n-1} e^{2f_k(2k+2-x)\lambda} \right) \left| B^* q_0(x) \right|^2 \, dx
\]

\[
+ \int_0^{T-2n} e^{2f_n(x+2n\lambda)} \left| B^* q_0(x) \right|^2 \, dx.
\]

Clearly, the exact observability property will not hold for initial data of the form \(p_0 = \alpha B^\perp\) and \(q_0 = \beta B^\perp\) where \(\alpha, \beta \in L^2(0, 1)\) and \(B^\perp\) denotes any orthogonal vector to \(B\). The same conclusion is achieved starting from \((42)\).

□

The purpose in the sequel is to give answers for the exact controllability when \(T \geq 4\). We begin by the limit case \(T = 4\). As a first step, we have the following necessary and sufficient condition for observability:

Proposition 7. For \(T = 4\), System \((20)\) is exactly observable if, and only if:

\[
\inf_{t \in [2, 4]} \left| \det \left[ B^* \mid B^* e^{\phi(t)M^*} \right] \right| > 0. \quad (55)
\]
Proof. Let \((p_0, q_0) \in H\) and \((p, q)\) the associated solution to System (20). For \(n = 2\), the matrices \(P_4\) and \(Q_4\) defined in (48) write:

\[
P_4(x) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ B^* e^{f_1(4-x)M^*} \end{bmatrix}, \quad Q_4(x) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ B^* e^{f_1(x+2)M^*} \end{bmatrix}.
\]

From Remark 4, System (20) is exactly observable if, and only if, the conditions (49)–(50) are satisfied with \(n = 2\) and \(T = 4\). From Proposition 4, (49)–(50) are equivalent to

\[
\inf_{x \in [0,1]} |\det P_4(x)| > 0 \quad \text{and} \quad \inf_{x \in [0,1]} |\det Q_4(x)| > 0.
\]

But since the multiplication operator on \(L^2((0,1))^2\) whose matrix is \(e^{f_0(2-x)M^*}\) (resp. \(e^{f_0(x)M^*}\) \((x \in (0, 1))\)) is invertible, the two last conditions are equivalent to the following:

\[
\inf_{x \in [0,1]} \left| \det \left( P_4(x) e^{-f_0(2-x)M^*} \right) \right| > 0
\]

and

\[
\inf_{x \in [0,1]} \left| \det \left( Q_4(x) e^{-f_0(x)M^*} \right) \right| > 0.
\]

Now:

\[
P_4(x) e^{-f_0(2-x)M^*} = \begin{bmatrix} B^* & B^* e^{(f_1(4-x) - f_0(2-x))M^*} \end{bmatrix}
\]

\[
= \begin{bmatrix} B^* & B^* e^{\phi(4-x)M^*} \end{bmatrix},
\]

and

\[
Q_4(x) e^{-f_0(x)M^*} = \begin{bmatrix} B^* & B^* e^{(f_1(x+2) - f_0(x))M^*} \end{bmatrix}
\]

\[
= \begin{bmatrix} B^* & B^* e^{\phi(x+2)M^*} \end{bmatrix}.
\]

This leads to the desired inequalities (55) after noting that \(3 \leq 4 - x \leq 4\) and \(2 \leq x + 2 \leq 3\) for \(0 \leq x \leq 1\). \(\square\)

Denote by \(\lambda_1, \lambda_2\) the eigenvalues of \(M^*\) if it is diagonalizable and by \(\mu\) the multiple eigenvalue if it is not. The next lemma will provide an equivalent condition to (55).

Lemma 4. Let \(r \in \mathbb{R}\). Then, \(\det \begin{bmatrix} B^* & B^* e^{rM^*} \end{bmatrix} \neq 0\) if, and only if:

\[
\text{rank} \begin{bmatrix} B & MB \end{bmatrix} = 2 \quad \text{and} \quad \begin{cases} r \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R}, \\
\frac{\pi}{\lambda_1} \notin \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
\end{cases}
\]

(56)

Proof. Let \(P\) a \(2 \times 2\) invertible matrix and set \(\tilde{B} = P^{-1}B\). Then,

\[
\begin{bmatrix} B & e^{rM}B \end{bmatrix} = \begin{bmatrix} P\tilde{B} & e^{rM}P\tilde{B} \end{bmatrix}
\]
\[ P \left[ \begin{array}{c|c} \tilde{B} & P^{-1}e^r P \tilde{B} \end{array} \right] = P \left[ \begin{array}{c|c} \tilde{B} & e^r P^{-1} M P \tilde{B} \end{array} \right]. \]

But
\[ \det \left[ \begin{array}{c|c} B^* & B^* e^r M^* \end{array} \right] \neq 0 \iff \det P \times \det \left[ \begin{array}{c|c} \tilde{B} & e^r P^{-1} M P \tilde{B} \end{array} \right] \neq 0. \]

If \( M \) is diagonalizable in \( \mathbb{R} \) then it admits a basis \( \{V_1, V_2\} \) of real eigenvectors associated with the real eigenvalues \( \{\lambda_1, \lambda_2\} \). If \( P = [V_1 \mid V_2] \) is the eigenvectors matrix, we get
\[ e^r P^{-1} M P \tilde{B} = \left( \begin{array}{cc} e^{\lambda_1 r} & 0 \\ 0 & e^{\lambda_2 r} \end{array} \right) \tilde{B}. \]

So that, if \( \tilde{B} = \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right) \) (so that \( B = \beta_1 V_1 + \beta_2 V_2 \)), then:
\[ \det \left[ \begin{array}{c|c} \tilde{B} & e^r P^{-1} M P \tilde{B} \end{array} \right] = \beta_1 \beta_2 \left( e^{\lambda_1 r} - e^{\lambda_2 r} \right). \]

Thus, in this case:
\[ \det \left[ \begin{array}{c|c} B^* & B^* e^r M^* \end{array} \right] \neq 0 \iff \beta_1 \beta_2 \left( e^{\lambda_1 r} - e^{\lambda_2 r} \right) \neq 0 \iff \begin{cases} \beta_1 \beta_2 \neq 0, \\ e^{\lambda_1 r} - e^{\lambda_2 r} \neq 0. \end{cases} \]

The condition \( \beta_1 \beta_2 \neq 0 \) expresses that \( B \) is not an eigenvector for \( M \) and this is equivalent to rank \( [B \mid MB] = 2 \). For the second condition, one has:
\[ e^{\lambda_1 r} - e^{\lambda_2 r} \neq 0 \iff \begin{cases} r \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R}, \\ r \neq \frac{\pi}{3(\lambda_1)} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases} \]

If \( M \) is not diagonalizable, then there exists a \( 2 \times 2 \) invertible matrix \( P \) such that
\[ M = P \left( \begin{array}{cc} \mu & 1 \\ 0 & \mu \end{array} \right) P^{-1}. \]

Then
\[ e^r P^{-1} M P \tilde{B} = \left( \begin{array}{cc} e^{\mu r} e^{r r} & e^{\mu r} \\ 0 & e^{\mu r} \end{array} \right) \tilde{B} \]

and in this case:
\[ \det \left[ \begin{array}{c|c} \tilde{B} & e^r P^{-1} M P \tilde{B} \end{array} \right] = -\beta_2^2 r e^{\mu r}. \]

The proof follows immediately. (\( \beta_2 \neq 0 \) says that \( B \) is not an eigenvector to \( M \)). \( \square \)

An immediate consequence of Proposition 7 and Lemma 4 is the following
Corollary 2. For $T = 4$, System (20) is exactly observable if, and only if, for any $t \in [2, 4]$:

$$\text{rank } [B | MB] = 2 \quad \text{and} \quad \begin{cases} \phi(t) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \sigma(M^*) = \{\mu\}, \\ \phi(t) \notin \frac{\pi}{3(\lambda_1)} \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

Now, we deal with the case $T \geq 4$:

Proposition 8. Let $n \geq 2$ be an integer and $2n \leq T < 2n + 1$. Then, System (20) is exactly observable if, and only if the following three conditions are satisfied:

1. $\text{rank } [B | MB] = 2$.
2. For any $x \in [0, 1]$, there exists $2 \leq k \leq n$ such that:

$$\begin{cases} \phi(2k - x) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \sigma(M) = \{\mu\}, \\ \phi(2k - x) \notin \frac{\pi}{3(\lambda_1)} \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. For any $x \in [0, T - 2n)$ and $x^* \in [T - 2n, 1)$, there exist $1 \leq k \leq n$ and $1 \leq k^* \leq n - 1$ such that:

$$\begin{cases} \phi(x + 2k) \neq 0, \phi(x^* + 2k^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \sigma(M) = \{\mu\}, \\ \phi(x + 2k), \phi(x^* + 2k^*) \notin \frac{\pi}{3(\lambda_1)} \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

Proof. From Remark 4, System (20) is exactly observable if, and only if the conditions (49)–(50) are satisfied where we recall that for $x \in [0, 1]$

$$P_{2n}(x, T) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ B^* e^{f_1(4-x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(2n-x)M^*} \end{bmatrix} ; \quad Q_{2n}(x, T) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(x+2(n-1))M^*} \\ 1_{(0, T-2n)}(x) B^* e^{f_n(x+2n)M^*} \end{bmatrix}.$$

From Proposition 5 with $s = 2$ and $n$ given in the lemma, (49)–(50) amount to say that for any $x \in [0, 1]$, there exist $2 \times 2$ matrices $P_{2n}^{\text{ext}}$ and $Q_{2n}^{\text{ext}}$, respectively, extracted from $P_{2n}$ and $Q_{2n}$ such that

$$\det P_{2n}^{\text{ext}}(x, T) \neq 0 \quad \text{and} \quad \det Q_{2n}^{\text{ext}}(x, T) \neq 0.$$ 

Fix $x \in [0, 1]$ and let us first deal with $P_{2n}(x, T)$. We are going to prove that $P_{2n}$ satisfies the required property if, and only if, the following matrix satisfies it too:

$$\tilde{P}_{2n}(x, T) = \begin{bmatrix} B^* \\ B^* e^{f(4-x)M^*} \\ \vdots \\ B^* e^{f(2n-x)M^*} \end{bmatrix}.$$
The proof of this last point is based on the identity:
\[ f_k (2(k + 1) - x) - f_{k-1} (2k - x) = \phi (2(k + 1) - x), \quad k \geq 1, \quad x \in [0, 1], \quad (57) \]
which is easily derived from the definitions of the function \( \phi \) and the sequence \((f_n)\) in (24) and (25).

Assume first that there exists \( x_0 \in [0, 1] \) such that for any \( 2 \times 2 \) matrices \( \tilde{P}_{2n}^{\text{ext}} \) extracted from \( \tilde{P}_{2n} \) one has:
\[
\det \tilde{P}_{2n}^{\text{ext}} (x_0, T) = 0.
\]
It follows that
\[
\det \begin{bmatrix} B^* & B^* e^{\phi(4-x_0)M^*} \end{bmatrix} = 0 \iff \begin{cases} \phi(4-x_0) = 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \cap \sigma(M) = \{\mu\}, \\ \phi(4-x_0) \in \frac{\pi}{3(\lambda_1)} \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases},
\]
by the equivalence given by Lemma 4. By induction, we get that if for \( k \geq 1, \)
\[ \phi (2k - x_0) = 0 \]
then again using (57), it is easily deduced that for any \( 1 \leq k \leq n : \)
\[
\begin{cases} \phi (2k - x_0) = 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \cap \sigma(M) = \{\mu\}, \\ \phi (2k - x_0) \in \frac{\pi}{3(\lambda_1)} \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}
\]
and thus for this \( x_0 \), any \( 2 \times 2 \) matrices \( P_{2n}^{\text{ext}} \) extracted from \( P_{2n} \) satisfies: \( \det P_{2n}^{\text{ext}} (x_0, T) = 0. \)

Conversely, if we assume there exists \( x_0 \in [0, 1] \) such that for any \( 2 \times 2 \) matrices \( P_{2n}^{\text{ext}} \) extracted from \( P_{2n} \) one has:
\[
\det P_{2n}^{\text{ext}} (x_0, T) = 0,
\]
then again using (57), it is easily deduced that for any \( 1 \leq k \leq n : \)
\[
\begin{cases} \phi (2k - x_0) = 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \cap \sigma(M) = \{\mu\}, \\ \phi (2k - x_0) \in \frac{\pi}{3(\lambda_1)} \mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}
\]
The same considerations hold for \( Q_{2n} \) and this proves the proposition. \( \square \)
Proposition 9. Let \( n \geq 2 \) be an integer and \( 2n + 1 \leq T < 2n + 2 \). System (20) is exactly observable if, and only if, the following three conditions are satisfied:

1. \( \text{rank } [B \mid MB] = 2 \).
2. For any \( x \in [2n + 2 - T, 1) \) and \( x^* \in [0, 2n + 2 - T) \), there exist \( 2 \leq k \leq n + 1 \) and \( 2 \leq k^* \leq n \), respectively, such that:
   \[
   \begin{align*}
   \phi (2k - x), \phi (2k^* - x^*) & \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \sigma(M) = \{\mu\}, \\
   \phi (2k - x), \phi (2k^* - x^*) & \notin \frac{\pi}{3(\lambda_1)} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
   \end{align*}
   \]
3. For any \( x \in [0, 1) \), there exists \( 1 \leq k \leq n \) such that:
   \[
   \begin{align*}
   \phi (x + 2k) & \neq 0, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \sigma(M) = \{\mu\}, \\
   \phi (x + 2k) & \notin \frac{\pi}{3(\lambda_1)} \mathbb{Z}, \quad \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}.
   \end{align*}
   \]

Proof. Exactly as previously, it suffices to develop the same arguments for the matrices

\[
P_{2n+1} (x, T) = \begin{bmatrix}
B^* e^{f_0(2-x)M^*} \\
\vdots \\
B^* e^{f_{n-1}(2n-x)M^*} \\
\mathbf{1}_{(2n+2-T,1)}(x) B^* e^{f_n(2n+2-x)M^*}
\end{bmatrix};
\]

\[
Q_{2n+1} (x, T) = \begin{bmatrix}
B^* e^{f_0(x)M^*} \\
\vdots \\
B^* e^{f_{n-1}(x+2(n-1))M^*} \\
B^* e^{f_n(x+2n)M^*}
\end{bmatrix}.
\]

Remark 6. Up to now, we have dealt only with the block diagonal system (22), and by Propositions 8 and 9, we have obtained a necessary and sufficient conditions that guarantee the exact controllability of System (22). In the next subsection, we will prove a compactness result which will allow to extend the obtained results to the complete system (12) up to finite-dimensional space of target states. Indeed, we have proved that there exists a \( C_T > 0 \) such that

\[
\| (p_0, q_0) \|_H^2 \leq C_T \int_0^T |B^* p_d (t, 0)|^2 \, dt,
\]

where \((p_d, q_d)\) is the solution of the diagonal system (22). Now, by using the triangular inequality we obtain

\[
\| (p_0, q_0) \|_H^2 \leq C_T \int_0^T |B^* p (t, 0)|^2 \, dt + C_T \int_0^T |B^* (p - p_d) (t, 0)|^2 \, dt,
\]
where \((p, q)\) is the solution of the complete system (12). So, if we can prove that the map \((p_0, q_0)^t \mapsto B^* (p - p_d)|_{x=0}\) is compact, we can then conclude that the complete system (12) is exactly controllable for any target state in the orthogonal in \(H\) of the operator \((p_0, q_0)^t \mapsto B^* p(t, 0)\) (see for instance [24, Lemma 3]). Notice that if its kernel is reduced to zero then exact controllability holds for the complete system (12).

3.5. Compactness

In Sect. 3, we have, after the change of variable \(t \rightarrow T - t\), considered the System (20) that we recall here:

\[
\begin{align*}
  p_t + p_x - M^* \eta_1 p &= 0, \quad \text{in } Q_T, \\
  q_t - q_x - M^* \eta_2 q &= 0, \quad \text{in } Q_T, \\
  (p + q)|_{x=0} = 0 &\in \mathbb{R}^2, \quad \text{in } (0, T), \\
  (p, q)|_{t=0} = (p_0, q_0), &\in (0, 1).
\end{align*}
\]  

(58)

The associated whole system obtained by the same change of variable from (12) is:

\[
\begin{align*}
  p_t + p_x - M^* \eta_1 p - M^* \eta_2 q &= 0, \quad \text{in } Q_T, \\
  q_t - q_x - M^* \eta_1 p - M^* \eta_2 q &= 0, \quad \text{in } Q_T, \\
  (p + q)|_{x=0,1} = 0 &\in \mathbb{R}^2, \quad \text{in } (0, T), \\
  (p, q)|_{t=0} = (p_0, q_0), &\in (0, 1).
\end{align*}
\]  

(59)

In the sequel, we denote by \(Z = Z(t, \cdot ; s; p_0, q_0) = (p, q)(t, \cdot ; s; p_0, q_0)\) the solution to (58) and by \(Z_d = Z_d(t, \cdot ; s; p_0, q_0) = (p_d, q_d)(t, \cdot ; s; p_0, q_0)\) the solution to the diagonal system (59).

This section is devoted to the proof of the compactness of the following operator:

\[D_T : H \rightarrow L^2(0, T), \quad (p_0, q_0)^t \mapsto B^* (p - p_d)|_{x=0}.\]

In fact, we have:

**Theorem 3.** Let \(T > 0\). Then, the operator \(D_T\) is compact.

The proof of Theorem 3 will need some preliminaries. Recall that the solution to System (58) can be expressed in terms of the evolution family \((U_d(t, s))_{s\leq t}\) as

\[U_d(t, s)Z_0 = (p_d, q_d)(t, \cdot ; s, Z_0), \quad 0 \leq s \leq t, \quad Z_0 \in H.\]

Therefore, there exist two operators \((S_d^\pm(t, s))_{s\leq t}\in H \rightarrow L^2(0, 1)^2\) such that

\[p_d(t, \cdot ; s, Z_0) = S_d^-(t, s)Z_0(\cdot), \quad q_d(t, \cdot ; s, Z_0) = S_d^+(t, s)Z_0(\cdot), \quad 0 \leq s \leq t, \quad Z_0 \in H.\]

Since System (59) is a bounded perturbation of System (58), then by [23, Chapter 5, Theorem 2.3], there exists a unique evolution family associated with System (59) defined by

\[U(t, s)Z_0 = (p, q)(t, \cdot ; s, Z_0), \quad 0 \leq s \leq t, \quad Z_0 \in H.\]
Similarly, there exist two operators \( \left( S^\pm(t, s) \right)_{s \leq t} \in H \rightarrow L^2(0, 1)^2 \) such that
\[
p(t, \cdot; s, Z_0) = S^- (t, s) Z_0, \quad q(t, \cdot; s, Z_0) = S^+ (t, s) Z_0, \quad 0 \leq s \leq t, \quad Z_0 \in H.
\]
With these new notations, the operator \( D_T \) takes the form
\[
D_T Z_0 = C \left( S^- (t, s) - S^- (t, t) \right) Z_0, \quad 0 \leq s \leq t, \quad Z_0 \in H,
\]
where \( C \) is the operator
\[
C : C(0, T; L^2(0, 1)^2) \rightarrow L^2(0, T)
\]
\[
\nu \mapsto B^* \nu \big|_{x=0}.
\]
Since \( (S(t, s))_{s \leq t} \) is a perturbation of \( (S_d(t, s))_{s \leq t} \) by a multiplication operators with multiplier \( P_T(s) = P_T(T - s) \), it is clear that the two evolutions families are linked by the Duhamel formula:
\[
U(t, 0) Z_0 = U_d(t, 0) Z_0 + \int_0^t U_d(t, s) P_T(s) U(s, 0) Z_0 ds, \quad Z_0 \in H.
\]
Therefore,
\[
(U(t, 0) - U_d(t, 0)) Z_0 = \int_0^t U_d(t, s) P_T(s) (U(s, 0) - U_d(s, 0)) Z_0 ds
\]
\[
+ \int_0^t U_d(t, s) P_T(s) U_d(s, 0) Z_0 ds.
\]
Consequently,
\[
(S^- (t, s) - S^- (t, t)) Z_0 = \int_0^t S^- (t, s) P_T(s) (U(s, 0) - U_d(s, 0)) Z_0 ds
\]
\[
+ \int_0^t S^- (t, s) P_T(s) U_d(s, 0) Z_0 ds
\]
\[
= \Psi_1(t, \cdot, Z_0) + \Psi_2(t, \cdot, Z_0).
\]
Thus,
\[
D_T(p_0, q_0) = C \Psi_1(t, \cdot, Z_0) + C \Psi_2(t, \cdot, Z_0), \quad Z_0 \in H.
\]
So, proving that \( D_T \) is compact amounts to prove the compactness of the operators \( C \Psi_i(t, \cdot, Z_0) \), \( i = 1, 2 \).

Since \( (U_d(t, 0))_{0 \leq t \leq T} \) is completely known, the compactness will be just a consequence of the explicit formula of the operator \( C \Psi_2(t, \cdot, Z_0) \). To deal with \( C \Psi_1(t, \cdot, Z_0) \), we use the following lemma inspired from ([16]) which has been used also in ([22]) in the same context to deal with more general autonomous hyperbolic systems.

**Lemma 5.** For any \( f \in C([0, T], H) \), there exists \( C_T > 0 \) such that
\[
\left| C \int_0^T S^- (t, s) f(s) ds \right|^2 dt \leq C_T \| f \|_{L^2(0, T; H)}^2.
\]
We have by (30) and (31) for any $t$.

Consider first the case $T$.

**Proof.** For the time being, let $f = (f_1, f_2) \in C([0, T] \times D(A^*)) (D(A^*)$ is defined in (76)) which is a dense subspace of $C([0, T], H)$. By using the characteristics method, we have by (30) and (31) for any $n \geq 0$:

\[
(S_d^- (t, s) f(s))(x) = \begin{cases} 
-R_n(t, x; s) f_2(s, t - x - s - 2n), & \text{if } t - x - s \in [2n, 2n + 1), \\
R_n(t, x; s) f_1(s, 2n + 2 - t + x + s), & \text{if } t - x - s \in [2n + 1, 2n + 2),
\end{cases}
\]

where

\[
R_n(t, x; s) = e^{f_n(t-x,s)M^*} + \int_{t-s}^{t} \eta_1(\tau, \tau - t - x) d\tau M^*, \quad n \geq 0.
\]

In particular, if $t - x \in (0, 1)$:

\[
(S_d^- (t, s) f(s))(x) = \begin{cases} 
-e^{f_n(t-x,s)M^*} + \int_{t-s}^{t} \eta_1(\tau, \tau - t - x) d\tau f_2(s, t - x - s), & \text{if } s \in [0, t - x), \\
e^{f_n(t-x,s)M^*} \int_{t-s}^{t} \eta_1(\tau, \tau - t - x) d\tau f_1(s, s - t + x), & \text{if } s \in [t - x, t).
\end{cases}
\]

Consider first the case $t - x \in [0, 1)$. Since $f$ is a continuous function, the trace operator makes sense and thus we obtain

\[
C \int_0^t S_d^- (t, s) f(s) ds = B^* \left( \int_0^t S_d^- (t, s) f(s) ds \right)(0)
= -B^* e^{f_n(t-x,s)M^*} \int_{t-s}^{t} \eta_1(\tau, \tau - t - x) d\tau f_2(s, t - s) ds.
\]

Therefore,

\[
\left| B^* \left( \int_0^t S_d^- (t, s) f(s) ds \right)(0) \right|^2 \leq C \int_0^t |f_2(t - s, s)|^2 ds.
\]

By integrating over $(0, T)$ the above inequality, we obtain for any $T \leq 1$:

\[
\int_0^T \left| C \int_0^t S_d^- (t, s) f(s) ds \right|^2 dt \leq C \int_0^T \int_0^t |f_2(t - s, s)|^2 ds dt
\leq C \int_0^T \int_0^t |f_2(t - s, s)|^2 + |f_1(t - s, s)|^2 ds dt
\leq C \int_0^T \int_0^t |f_2(t, s)|^2 + |f_1(t, s)|^2 ds dt
= C \|f\|_{L^2(0,T;H)}^2.
\]

Now, we deal with the case $T \geq 1$. Let $t - x \in [2n - 1, 2n)$, $n \geq 1$. We write

\[
\left( \int_0^t S_d^- (t, s) f(s) ds \right)(x)
\]
A simple variable substitution yields
\[
\left( \int_{t-x}^{t} S_{a}^{-}(t, s) f(s) ds \right) (x)
= \int_{t-x}^{t} e^{\int_{t-x}^{t} \eta_{1}(\tau, t-x-\tau) d\tau M^{*}} f_{1}(s, s - t + x) ds
+ \sum_{k=1}^{n-1} \int_{0}^{1} R_{k-1}(t, s) f_{1}(s, 2k - t + x + s) ds
- \sum_{k=0}^{n-1} \int_{0}^{1} R_{k}(t, s) f_{2}(s, t - x - s - 2k) ds
+ \int_{0}^{1} R_{n-1}(t, s) f_{1}(s, 2n - t + x + s) ds.
\]

Again, since \( f \) is a continuous function, the trace operator makes sense and we obtain:
\[
C \left( \int_{0}^{t} S_{a}^{-}(t, s) f(s) ds \right)
= B^{*} \left( \int_{0}^{t} S_{a}^{-}(t, s) f(s) ds \right) (0)
= \sum_{k=1}^{n-1} \int_{0}^{1} B^{*} R_{k-1}(t, 0; s + t - 2k) f_{1}(s + t - 2k, s) ds
- \sum_{k=0}^{n-1} \int_{0}^{1} B^{*} R_{k}(t, 0; t - s - 2k) f_{2}(t - s - 2k, s) ds
- \int_{2n-t}^{1} B^{*} R_{n-1}(t, 0; s + t - 2n) f_{1}(s + t - 2n, s) ds.
\]

Since \( \eta_{1}, \eta_{2} \) are bounded, we get, using Cauchy–Schwarz inequality:
\[
\left| C \int_{0}^{t} S_{a}^{-}(t, s) f(s) ds \right|^{2} \leq C_{n,t} \sum_{k=1}^{n-1} \int_{0}^{1} |f_{1}(s + t - 2k, s)|^{2} ds
\]
\[ + C_{n,t} \sum_{k=0}^{n-1} \int_0^1 |f_2(t - s - 2k, s)|^2 \, ds \]
\[ + C_{n,t} \int_{2n-1}^1 |f_1(s + t - 2n, s)|^2 \, ds, \]
\[ (63) \]

where \( C_{n,t} \) is a positive constant depending on \( t \) and \( n \). Taking the integral of (63) over \((0, T)\) for \( T \in [2n - 1, 2n), n \geq 1\), and using the fact
\[ \int_0^T \int_0^1 |f_1(s + t - 2k, s)|^2 \, ds \, dt \leq \int_0^T \int_0^1 |f_1(t, s)|^2 \, ds \, dt, \forall k \in \{1, \ldots, n - 1\}, \]
\[ \int_0^T \int_0^1 |f_2(t - s - 2k, s)|^2 \, ds \, dt \leq \int_0^T \int_0^1 |f_2(t, s)|^2 \, ds \, dt, \forall k \in \{0, \ldots, n - 1\}, \]
yields
\[ \int_0^T \left| \int_0^1 S_d^-(t, s) f(s) \, ds \right|^2 \, dt \leq n \| f \|^2_{L^2(0, T; H)}. \]

Similarly, we obtain the following estimate for \( T \in (2n, 2n + 1), n \geq 1 \)
\[ \int_0^T \left| \int_0^1 S_d^-(t, s) f(s) \, ds \right|^2 \, dt \leq (n + 1) \| f \|^2_{L^2(0, T; H)}, \]

which ends the proof. The estimates (62), (64) and (65) can be extended for any \( f \in C(0, T; H) \) by using a standard density argument. \( \square \)

**Proposition 10.** The operator \( Z_0 \mapsto \mathcal{C} \Psi_1(t, \cdot, Z_0) \) acting from \( H \) to \( L^2(0, T) \) is compact.

**Proof.** The proof of Proposition 10 is a direct consequence of the result proved in [21] (and an extension of this result in [6] to the case where some wave speeds are equal) which asserts that the difference of the evolution operators defined by systems (59) and (58) are compact) and Lemma 5. More precisely, by letting
\[ f(s) = \mathcal{P}_T(s)(U(s, 0) - U_d(s, 0)) Z_0, \quad Z_0 \in H, \quad s \leq t, \]
in (60), we obtain
\[ \| \mathcal{C} \Psi_1(t, \cdot, Z_0) \|_{L^2(0, T)} = \int_0^T C \left| \int_0^1 S_d^-(t, s) \mathcal{P}_T(s)(U(s, 0) - U_d(s, 0)) Z_0 \, ds \right|^2 \, dt \]
\[ \leq C_T \| (U(\cdot, 0) - U_d(\cdot, 0)) Z_0 \|^2_{L^2(0, T; H)}, \]

which is a compact operator by the result in [6]. It remains to deal with the operator \( \mathcal{C} \Psi_2(t, \cdot, Z_0) \). \( \square \)

**Proposition 11.** The operator \( Z_0 \mapsto \mathcal{C} \Psi_2(t, \cdot, Z_0) \) acting from \( H \) to \( L^2(0, T) \) is compact.
Proof. The proof is purely constructive. First, we find the solution \( q(t, \cdot ; s, Z_0) = S^+(t, s)Z_0 \).

\((D(A^*))\) is defined in (76))

Let \((p_0, q_0) \in D(A^*)\) (see (76)). By using the characteristics method, we compute (66) for \( x = 1 \) by (32) and (33) we obtain:

\[
(S^+_d(t, s)Z_0)(x) = \begin{cases} 
N_n(t, x; s)f_2(s, x + t - s - 2n), & \text{if } x + t - s \in [2n, 2n + 1), \\
-N_n(t, x; s)f_1(s, 2n + 2 - x - t + s), & \text{if } x + t - s \in [2n + 1, 2n + 2).
\end{cases}
\]  

(66)

where

\[ N_n(t, x; s) = R_{n-1}(x + t - 1, 1; s) + e^{\int_{s+t-1}^{t} \eta_2(\tau, -\tau+t+x) d\tau}, \quad n \geq 0. \]

The aim now is to compute \( \Psi_2(t, \cdot , Z_0) \) explicitly. We recall that

\[ \Psi_2(t, \cdot , Z_0) = \int_0^t S^{-}_{d}(t, s)\mathcal{P}_T(s)U_d(s, 0)Z_0 ds, \quad t \leq T, \]

where \( U_d(s, 0)Z_0 = (S^+_{d}(s, 0)Z_0, S^+_{d}(s, 0)Z_0)^t, 0 \leq s \leq t, \) and \( (S^\pm_{d}(t, s))_{s \leq t \leq T} \) are given in (66) and (66).

Applying \( \mathcal{P}_T(\cdot) \) yields

\[ \mathcal{P}_T(s)U_d(s, 0) = (\eta_2(s)S^+_d(s, 0)Z_0, \eta_1(s)S^-_d(s, 0)Z_0)^t, s \leq t. \]

Therefore,

\[
\mathcal{C}\Psi_2(t, \cdot, Z_0) = \mathcal{C} \int_0^t S^-_d(t, s) (\eta_2(s)S^+_d(s, 0)Z_0, \eta_1(s)S^-_d(s, 0)Z_0)^t Z_0 ds, \quad t \leq T. \]  

(67)

Let us start by computing the integrand in (67). We have for any \( n \geq 0 \)

\[
S^-_d(t, s) (\eta_2(s)S^+_d(s, 0)Z_0, \eta_1(s)S^-_d(s, 0)Z_0)^t(x) = \begin{cases} 
M_n^1(t, x; s) (S^-_d(s, 0)Z_0) (t - x - s - 2n), & \text{if } t - x - s \in [2n, 2n + 1), \\
M_n^2(t, x; s) (S^+_d(s, 0)Z_0) (2n + 2 - t + x + s), & \text{if } t - x - s \in [2n + 1, 2n + 2),
\end{cases}
\]

where

\[
M_n^1(t, x; s) = -R_n(t, x; s)\eta_1(s, t - x - s - 2n), \quad t - x - s \in [2n, 2n + 1), n \geq 0, \\
M_n^2(t, x; s) = R_n(t, x; s)\eta_2(s, 2n + 2 - t + x + s), \quad t - x - s \in [2n + 1, 2n + 2), n \geq 0.
\]

Now, by using (66) and (66) for \( s = 0 \) we get for any \( n, k \geq 0 \)

\[
(S^-_d(\tau, 0)Z_0)(t - x - \tau - 2n)
\]
Therefore, we obtain for any $k, n \geq 0$

\[
\Psi_2(t, \cdot, Z_0) = -\sum_{k,n \geq 0} \int_{\frac{2(k-n)+t-x+1}{2}}^{\frac{2(k-n)+t-x+1}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^1(t, x; \tau) q_0(2\tau - t + x + 2(n-k))d\tau \\
+ \sum_{k,n \geq 0} \int_{\frac{2(k-n)+t-x+2}{2}}^{\frac{2(k-n)+t-x+2}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^1(t, x; \tau) p_0(2(k-n+1) - 2\tau + t - x)d\tau \\
+ \sum_{k,n \geq 0} \int_{\frac{2(k-n)-t-x-1}{2}}^{\frac{2(k-n)-t-x-1}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^2(t, x; \tau) q_0(2n-k+1) + t + x + 2\tau)d\tau \\
- \sum_{k,n \geq 0} \int_{\frac{2(k-n)-t-x-1}{2}}^{\frac{2(k-n)-t-x-1}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^2(t, x; \tau) p_0(2(k-n) + t - x - 2\tau)d\tau,
\]

where

\[
P_{k,n}^1(t, x; \tau) = M_n^1(t, x; \tau) R_k(s, t - x - s - 2n; 0), \\
P_{k,n}^1(t, x; \tau) = M_n^2(t, x; \tau) N_k(s, 2n + 2 - t + x + \tau; 0).
\]

Consequently,

\[
C \Psi_2(t, \cdot, Z_0) \\
= -B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)+t-x+1}{2}}^{\frac{2(k-n)+t-x+1}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^1(t, 0; \tau) q_0(2\tau - t + 2(n-k))d\tau \\
+ B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)+t+x+2}{2}}^{\frac{2(k-n)+t+x+2}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^1(t, 0; \tau) p_0(2(k-n+1) - 2\tau + t)d\tau \\
+ B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)-t+x+1}{2}}^{\frac{2(k-n)-t+x+1}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^2(t, 0; \tau) q_0(2n-k+1) + t + 2\tau)d\tau \\
- B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)-t+x+1}{2}}^{\frac{2(k-n)-t+x+1}{2}} \mathbb{I}(\tau)(0,t) P_{k,n}^2(t, 0; \tau) p_0(2(k-n) + t - 2\tau)d\tau.
\]
The proof follows by [21, Lemma 4] which allows to conclude that \( C\Psi_2(t, \cdot, Z_0) \) is a compact operator from \( H \) to \( L^2(0, T) \) since it is a finite sum of such operators. \( \square \)

4. Unique continuation

In this section, we deal with the unique continuation property for System (1). We give a necessary and sufficient condition for the constant case, and a semi-explicit condition in the autonomous case. When the coefficients depend on time, we give also a necessary and sufficient condition for the cascade case with providing some nontrivial examples at the end.

4.1. The constant case

Here, we assume that \( a, b \in \mathbb{R} \). Recall that the adjoint system of System (1) is given by

\[
\begin{align*}
\varphi_{tt} &= \varphi_{xx} - M^*(a\varphi_t + b\varphi_x), \text{ in } (0, T) \times (0, 1), \\
\varphi|_{x=0,1} &= 0, \text{ in } (0, T), \\
(\varphi, \varphi_t)|_{t=T} &= (\varphi_0^T, \varphi_1^T), \text{ in } (0, 1).
\end{align*}
\]

(70)

The correspond unique continuation property reads:

\[
B^*\varphi_x(t, 0) = 0, \; \forall t \in (0, T) \Rightarrow (\varphi_0, \varphi_1) = (0, 0), \; \forall \left( \varphi_0^T, \varphi_1^T \right) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2.
\]

(71)

For the sake of simplicity, we set \( e^{\frac{b_t}{2}M^*} \tilde{\varphi}(x) = \varphi(x) \). Then, \( \tilde{\varphi} \) is the solution of the following system

\[
\begin{align*}
\tilde{\varphi}_{tt} &= \tilde{\varphi}_{xx} - \frac{1}{4}b^2 (M^*)^2 \tilde{\varphi} - aM^* \tilde{\varphi}_t, \text{ in } (0, T) \times (0, 1), \\
\tilde{\varphi}|_{x=0,1} &= 0, \text{ in } (0, T), \\
(\tilde{\varphi}, \tilde{\varphi}_t)|_{t=T} &= (\tilde{\varphi}_0^T, \tilde{\varphi}_1^T), \text{ in } (0, 1).
\end{align*}
\]

(70)

By using a standard spectral decomposition of the solution to System (70) (see for instance [8,11] in the hyperbolic context or [7,17] in the parabolic one), we can see that proving that (71) holds in a time \( T \geq 4 \) amounts to proving that all the eigenvalues of the corresponding spectral problem

\[
\begin{align*}
\lambda^2 \tilde{\varphi}(x) &= \tilde{\varphi}'''(x) - \left( \frac{1}{4}b^2 (M^*)^2 + aM^*\lambda \right) \tilde{\varphi}(x), \; x \in (0, 1), \\
\tilde{\varphi}(0) &= \tilde{\varphi}(1) = 0, \; \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2),
\end{align*}
\]

(72)

are simple. First, let us assume first that \( M^* \) is diagonalizable.

**Proposition 12.** Assume that \( M^* \) has 2 distinct eigenvalues \( \mu_1, \mu_2 \). Then, all the eigenvalues of the Sturm–Liouville problem (72) are simple if, and only if

\[
a^2 (\mu_1 - \mu_2) \left( \mu_2 \xi_{n_1}^2 - \mu_1 \xi_{n_2}^2 \right) \neq \left( \xi_{n_1}^2 - \xi_{n_2}^2 \right)^2, \; \forall n_1, n_2 \in \mathbb{Z},
\]

(73)

where \( \xi_{n_i} = \frac{1}{4}b^2 \mu_i^2 + (n_i\pi)^2, \; i = 1, 2.\)
Proof. Since $M^*$ is diagonalizable, there exists a diagonal matrix $D = \text{diag}(\mu_1, \mu_2)$ and $2 \times 2$ invertible matrix $P$ such that $M^* = PDP^{-1}$. Letting $z = P^{-1}\tilde{\phi}$ with $z = (z_1, z_2)$ in (72) yields the following Sturm–Liouville problem

$$
\begin{cases}
  z''_1(x) = \left(\lambda^2 + a\lambda \mu_i + \frac{1}{4}b^2\mu_i^2\right)z_1(x), & x \in (0, 1), \\
  z_i(0) = z_i(1) = 0, & i = 1, 2.
\end{cases}
$$

In order to prove that the eigenvalues of the above problem are simple we have to check that the following polynomial equations

$$
\lambda^2 + \lambda a \mu_i + \frac{1}{4}b^2\mu_i^2 + (n_i\pi)^2 = 0, \quad n_i \in \mathbb{Z}, \quad i = 1, 2,
$$

don’t have a common roots which is equivalent to check that the following Sylvester matrix is invertible

$$
S_{k,n} = \begin{pmatrix}
  1 & a\mu_1 & \frac{1}{4}b^2\mu_1^2 + (n_1\pi)^2 & 0 \\
  0 & a\mu_1 & 0 & 0 \\
  1 & a\mu_2 & \frac{1}{4}b^2\mu_2^2 + (n_2\pi)^2 & 0 \\
  0 & a\mu_2 & 0 & 0
\end{pmatrix}, \quad n_1, n_2 \in \mathbb{Z},
$$

which is the case if, and only if (73) is satisfied. \qed

Remark 7. In particular, if $\mu_1 = -\mu_2 = \mu$, assumption (73) becomes

$$
-2a^2\mu^2 \left(\xi_{n_1}^2 + \xi_{n_2}^2\right) \neq \left(\xi_{n_1}^2 - \xi_{n_2}^2\right)^2, \quad \forall n_1, n_2 \in \mathbb{Z},
$$

which might occur only if, and only if $\mu \in i\mathbb{R}$.

Now, we consider the case where $M^*$ is not diagonalizable.

**Proposition 13.** Assume that $M^*$ is not diagonalizable and let $\mu$ be its eigenvalue. Then, all the eigenvalues of the Sturm–Liouville problem (72) are simple if, and only if

$$
\frac{1}{2}b^2\mu + a \neq 0.
$$

**Proof.** In this case, we write $M^*$ in the Jordan form: there exists a matrix $J$ and a $2 \times 2$ invertible matrix $P$ such that $M^* = PJP^{-1}$, where

$$
J = \begin{pmatrix}
  \mu & 1 \\
  0 & \mu
\end{pmatrix}.
$$

Letting $z = P^{-1}\tilde{\phi}$ in (72) yields the following coupled Sturm–Liouville problem

$$
\begin{cases}
  z''_1(x) = \left(\lambda^2 + a\lambda \mu + \frac{1}{4}b^2\mu^2\right)z_1(x) + (\frac{1}{2}b^2\mu + a)z_2, \\
  z''_2(x) = \left(\lambda^2 + a\lambda \mu + \frac{1}{4}b^2\mu^2\right)z_2(x), \\
  y(0) = y(1) = 0, \\
  z(0) = z(1) = 0.
\end{cases}
$$
By the same reasoning as in [7, 17, Propositions 2.1], it can be seen that the above system has nontrivial solution if, and only if
\[ \frac{1}{2} b^2 \mu + a \neq 0 \]
with \( \lambda \) fulfills the following second-order polynomial equation for some \( n \in \mathbb{Z} \):
\[
\lambda^2 + a \mu \lambda + \frac{1}{4} b^2 \mu^2 + (n \pi)^2 = 0.
\]
It is clear that the above equation has simple roots for any \( n \in \mathbb{Z} \). This finishes the proof.

**Remark 8.** As it is shown, if we let \( a = 0 \), the coupling parameter \( b \) affects controllability in low frequency. Actually, conditions (73) and (74) become, respectively,
\[
b^2 \neq 2 \pi^2 \frac{n_2^2 - n_1^2}{\mu_1^2 - \mu_2^2}, \quad \forall n_1, n_2 \in \mathbb{Z} \text{ and } b \neq 0.
\]
However, \( b \) doesn’t affect controllability in high frequency unless it is time dependent (see Remark 2).

### 4.2. The autonomous case

When the coefficients don’t depend on time, we can give a characterization to cover the invisible target states by applying the Fattorini criterion on system (12) which is possible since the difference between the control maps is compact by Theorem 3 (see [16, Remarks 2.4 and 1.5]). Note that the Fattorini criterion with the fact that the two control maps is compact implies exact controllability of the complete system (12) in the same minimal time \( T = 4 \). So, in order to cover the invisible target states we have to ensure that
\[
\ker (sl - A^*) \cap \ker (C^*) = \{0\}, \quad \forall s \in \mathbb{C}, \tag{75}
\]
where \( C^* : (p, q) \mapsto (B^* p|_{x=0}, 0) \) and \( A^* \) is the operator
\[
A^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -p_x - M^* (\eta_1 p + \eta_2 q) \\ q_x - M^* (\eta_1 p + \eta_2 q) \end{pmatrix},
\]
with domain
\[
D(A^*) = \{(f, g) \in H^1(0, 1)^2 \times H^1(0, 1)^2, \ (f + g)|_{x=0,1} = 0, \ \int_0^1 (f - g) (\xi) \, d\xi = 0\}.
\]
(76)

Introduce the matrices \( Q_i, i = 1, 2 \) defined by
\[
Q_0 = \begin{pmatrix} I_{2 \times 2} & I_{2 \times 2} \\ 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 \\ I_{2 \times 2} & I_{2 \times 2} \end{pmatrix},
\]
and let \( R_s (\cdot, \cdot) \) the fundamental matrix of the finite-dimensional system
\[
\begin{pmatrix} p \\ q \end{pmatrix}_x = \begin{pmatrix} -sI_{2 \times 2} - \eta_1 M^* & -\eta_2 M^* \\ \eta_1 M^* & \eta_2 M^* + sI_{2 \times 2} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{77}
\]
Then, we have:
Proposition 14. The Fattorini criterion is satisfied if, and only if
\[
\operatorname{rank} \left( Q_0 + Q_1 \mathcal{R}_s(1, 0) \right) (B^*, 0, 0) = 4, \text{ for all } s \in \mathbb{C}. \tag{78}
\]

Proof. Let \((p, q) \in D(A^*)\). So \((p, q) \in \ker(sI - A^*)\) if and only if
\[
\begin{pmatrix} p \\ q \end{pmatrix}_x = \begin{pmatrix} -sI_{2 \times 2} - \eta_1 M^* & -\eta_2 M^* \\ \eta_1 M^* & \eta_2 M^* + sI_{2 \times 2} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \tag{79}
\]
with boundary conditions \((p + q)|_{x=0,1} = 0\) which can be rewritten as
\[
\begin{pmatrix} 0_2 \\ 0_2 \end{pmatrix} = Q_0 \begin{pmatrix} p(0) \\ q(0) \end{pmatrix} + Q_1 \begin{pmatrix} p(1) \\ q(1) \end{pmatrix}. \tag{80}
\]
If \(\mathcal{R}_s(\cdot, \cdot)\) denotes the fundamental matrix solution to System (79), then we have
\[
\begin{pmatrix} p(x) \\ q(x) \end{pmatrix} = \mathcal{R}_s(x, 0) \begin{pmatrix} p(0) \\ q(0) \end{pmatrix}.
\]
By using the boundary conditions (80), we arrive at
\[
0_4 = (Q_0 + Q_1 \mathcal{R}_s(1, 0)) \begin{pmatrix} p(0) \\ q(0) \end{pmatrix}.
\]
On the other hand, \((p, q) \in \ker C^*\) if and only if \(B^* p(0) = 0_2\). So the property (75) is satisfied if and only if (78) holds which implies that \(p(0) = q(0) = 0\) which gives \(p = q = 0\). \qed

Remark 9. Generally, knowing \(\mathcal{R}_s(\cdot, \cdot)\) is not possible. However, the above characterization could work for some particular classes of systems. For instance, for \(M\) of the form
\[
M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
4.3. Cascade coupling

In this subsection, we prove the unique continuation property for a particular class of System (20). We assume in the sequel that the matrix \(M\) and the vector \(B\) have the following form:
\[
M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
With this in mind, and by decomposing the system by writing \( p = (p^-, p^+) \), \( q = (q^-, q^+) \), the unique continuation problem of System (12) reads

\[
\begin{align*}
  p_t^- + p_x^- &= 0, & \text{in } Q_T, \\
  q_t^- - q_x^- &= 0, & \text{in } Q_T, \\
  p_t^+ + p_x^+ - \eta_1 p^- - \eta_2 q^- &= 0, & \text{in } Q_T, \\
  q_t^+ - q_x^+ - \eta_1 p^- - \eta_2 q^- &= 0, & \text{in } Q_T, \\
  (p^+ + q^-)|_{x=0,1} &= 0, & \text{(82)} \\
  p^+|_{x=0} &= 0, & \text{in } (0, T), \\
  (p^+, q^-)|_{t=0} = (p_0^- - q_0^-), & \text{in } (0, 1), \\
  q^+|_{x=0,1} &= 0, & \text{(81)}
\end{align*}
\]

for any \( (p_0^-, p_0^+, q_0^-, q_0^+) \in D(\mathcal{A}^*) \) where \( D(\mathcal{A}^*) \) is defined in (76).

Observe that the first and the third equations of the above system are free. The idea is to solve these equations explicitly and then considering their solution as a second member for the second and the fourth equations. Let us start by solving the homogeneous part of System (81), i.e.,

\[
\begin{align*}
  p_t^- + p_x^- &= 0, & \text{in } Q_T, \\
  q_t^- - q_x^- &= 0, & \text{in } Q_T, \\
  (p^+ + q^-)|_{x=0,1} &= 0, & \text{(82)} \\
  (p^-, q^-)|_{t=0} = (p_0^- - q_0^-), & \text{in } (0, 1).
\end{align*}
\]

**Lemma 6.** The solution \((p^-, q^-)\) to System (82) is given by

\[
p^-(t, x) = \begin{cases} 
  p_0^- (x - t + 2n), & \text{if } 2n - 1 \leq t - x \leq 2n, \ n \geq 0, \\
  -q_0^- (x - t - 2n), & \text{if } 2n \leq t - x \leq 2n + 1, 
\end{cases} \quad (83)
\]

\[
q^-(t, x) = \begin{cases} 
  q_0^- (x + t - 2n), & \text{if } 2n \leq x + t \leq 2n + 1, \\
  -p_0^- (2n + 2 - x - t), & \text{if } 2n + 1 \leq x + t \leq 2n + 2, \ n \geq 0.
\end{cases} \quad (84)
\]

**Proof.** The proof follows immediately by using the characteristics method. \( \square \)

Now, we focus on the nonhomogeneous part of System (81) where boundary condition \( q^+_|_{x=0} = 0 \) has been removed, i.e.,

\[
\begin{align*}
  p_t^+ + p_x^+ &= f, & \text{in } Q_T, \\
  q_t^+ - q_x^+ &= f, & \text{in } Q_T, \\
  (p^+ + q^+)|_{x=1} &= p^+_|_{x=0} = 0, & \text{in } (0, T), \\
  (p^+, q^+)|_{t=0} = (p_0^+, q_0^+), & \text{in } (0, 1).
\end{align*}
\]

Observe that the function \( f = \eta_1 p^- + \eta_2 q^- \) plays the role of a second member. The explicit solution to (85) is given in the following lemma:

**Lemma 7.** The solution \((p^+, q^+)\) to System (85) is given by

\[
p^+(t, x) = \begin{cases} 
  p_0^+ (x - t) + \int_0^t f(\tau, \tau + x - t) \, d\tau, & \text{if } 0 \leq x - t \leq 1 \\
  \int_{t-x}^t f(\tau, \tau - t + x) \, d\tau, & \text{if } t - x \geq 1,
\end{cases}
\]
and

\[
q^+(t, x) = \begin{cases} 
q_0^+(x + t) + \int_0^t f(\tau, x + t - \tau) \, d\tau, & \text{if } 0 \leq x + t \leq 1, \\
\int_{x+t-1}^t f(\tau, x + t - \tau) \, d\tau - p_0^+(2 - x - t), & \text{if } 1 \leq x + t \leq 2, \\
\int_{x+t-1}^t f(\tau, -\tau + x + t) \, d\tau \\
- \int_{x+t-2}^{t} f(\tau, \tau + 2 - t - x) \, d\tau, & \text{if } x + t \geq 2,
\end{cases}
\]  

(86)

**Proof.** By using the characteristics method, it follows

\[
p^+(t, x) = p_0^+(x - t) + \int_0^t f(\tau, \tau + x - t) \, d\tau, \quad \text{if } 0 \leq x - t \leq 1,
\]

and

\[
q^+(t, x) = q_0^+(x + t) + \int_0^t f(\tau, x + t - \tau) \, d\tau, \quad \text{if } 0 \leq x + t \leq 1.
\]

Now, at \( x = 1 \) we obtain

\[
p^+(t, 1) = p_0^+(1 - t) + \int_0^t f(\tau, \tau + 1 - t) \, d\tau, \quad \text{if } 0 \leq 1 - t \leq 1.
\]

Using the boundary condition, \( q^+(s, 1) = -p^+(s, 1), \ s \geq 0 \), entails

\[
q^+(s, 1) = -p_0^+(1 - s) - \int_0^s f(\tau, \tau + 1 - s) \, d\tau, \quad \text{if } 0 \leq 1 - s \leq 1.
\]

Solving the second equation of System (85) along the characteristic \( x(t) = -t + s + 1 \), we get

\[
q^+(t, -t + s + 1) = q^+(s, 1) + \int_s^t f(\tau, s + 1 - \tau) \, d\tau \\
= -p_0^+(1 - s) - \int_0^s f(\tau, \tau + 1 - s) \, d\tau + \int_s^t f(\tau, s + 1 - \tau) \, d\tau.
\]

Letting \( x = -t + s + 1 \) yields for any \( 0 \leq 2 - x - t \leq 1 \):

\[
q^+(t, x) = -p_0^+(2 - x - t) - \int_0^{x+t-1} f(\tau, \tau + 2 - x - t) \, d\tau \\
+ \int_{x+t-1}^t f(\tau, x + t - \tau) \, d\tau.
\]

Now, since \( p(s, 0) = 0, \ s \geq 0 \), we have for any \( t - x \geq 1 \)

\[
p^+(t, x) = \int_{t-x}^t f(\tau, \tau - t + x) \, d\tau.
\]  

(87)
By taking \( x = 1 \) in (87), the using the boundary conditions \( q^+(s, 1) = -p^+(s, 1), \ s \geq 0, \) we get
\[
q^+(t, x) = \int_{x+t-1}^t f(\tau, x + t - \tau) \, d\tau.
\]
Similarly, we obtain for any \( x + t \geq 2 \)
\[
q^+(t, x) = -\int_{x+t-2}^{x+t-1} f(\tau, \tau + 2 - t - x) \, d\tau + \int_{x+t-1}^t f(\tau, -\tau + x + t) \, d\tau,
\]
which ends the proof.

To satisfy the remained boundary condition \( q^+|_{x=0} = 0 \), it suffices to replace \( x \) by zero in (86). This gives the following system of equations

\[
\begin{align*}
0 &= q^+_0(t) + \int_0^t f(\tau, t - \tau) \, d\tau, \quad \text{if } 0 \leq t \leq 1, \quad (88) \\
0 &= -p^+_0(2 - t) - \int_0^{t-1} f(\tau, \tau + 2 - t) \, d\tau + \int_{t-1}^t f(\tau, t - \tau) \, d\tau = 0, \quad \text{if } 1 \leq t \leq 2, \quad (89) \\
0 &= \int_{t-2}^{t-1} f(\tau, \tau + 2 - t) \, d\tau + \int_{t-1}^t f(\tau, -\tau + t) \, d\tau = 0, \quad \text{if } t \geq 2. \quad (90)
\end{align*}
\]

As a consequence, we have:

**Proposition 15.** The unique continuation property (81) holds true if, and only if the solution \((p_0, q_0)\) to System (88)--(90) is the null one.

The strategy is the following: We solve Equation (90) which depends only on the initial states \( p^-_0, q^-_0 \) since \( f \) does. Then, we prove that \( p^-_0 \equiv q^-_0 \equiv 0 \) which entails that \( f \equiv 0 \) since it depends linearly on \( p^-_0 \) and \( q^-_0 \). This leads to \( p^+_0 \equiv q^+_0 \equiv 0 \) by (88) and (89).

Let \( t \geq 2 \). Recall that \( f = \eta_1 p^- + \eta_2 q^- \). Equation (90) becomes
\[
0 = -\int_{t-2}^{t-1} (\eta_1 p^-)(\tau, \tau + 2 - t) \, d\tau + \int_{t-1}^t (\eta_2 q^-)(\tau, -\tau + t) \, d\tau
\]
\[
\quad - \int_{t-2}^{t-1} (\eta_2 q^-)(\tau, \tau + 2 - t) \, d\tau + \int_{t-1}^t (\eta_1 p^-)(\tau, -\tau + t) \, d\tau
\]
\[
:= I_1(t) + I_2(t) + I_3(t) + I_4(t).
\]

Since \( p^- \) (resp. \( q^- \)) is defined on the characteristics of slope 1 (resp. -1), \( I_1(\cdot) \) (resp. \( I_2(\cdot) \)) can be easily computed. Indeed, by using the expressions of \( p^- \) and \( q^- \) given in (83) and (86), respectively, we obtain
\[
I_1(t) = -\int_{t-2}^{t-1} (\eta_1 p^-)(\tau, \tau + 2 - t) \, d\tau
\]
\[
I_2(t) = \int_{t-1}^{t} (\eta_1 q^-)(\tau, -\tau + t) d\tau \\
= \begin{cases} 
- \left( \int_{t-1}^{t} \eta_1(\tau, -\tau + t) d\tau \right) q_0^-(t - 2n), & \text{if } 2n \leq t \leq 2n + 1, \\
\left( \int_{t-1}^{t} \eta_1(\tau, -\tau + t) d\tau \right) p_0^-(2n - 2 - t), & \text{if } 2n + 1 \leq t \leq 2n + 2, \quad n \geq 1.
\end{cases}
\]

(92)

It can be seen that \( I_1(\cdot) \) and \( I_2(\cdot) \) are the high frequency part of the solution. Now, we deal with the compact terms \( I_3(\cdot) \) and \( I_4(\cdot) \).

By using the expression of \( q^- \) given in (86), we obtain for any \( n \geq 0 \)

\[
I_3(t) = - \int_{t-2}^{t-1} (\eta_2 q^-)(\tau, \tau + 2 - t) d\tau \\
= \begin{cases} 
- \left( \int_{t-2}^{t-1} \eta_2(\tau, \tau + 2 - t) d\tau \right) q_0^-(2\tau + 2 - t - 2n) d\tau, & \text{if } \tau \in \left( \frac{2n-3+4t}{2}, \frac{2n-2+4t}{2} \right) \cap (0, t), \\
\left( \int_{t-2}^{t-1} \eta_2(\tau, \tau + 2 - t) d\tau \right) p_0^-(2n - 2 - t + 2\tau) d\tau, & \text{if } \tau \in \left( \frac{2n-2+4t}{2}, \frac{2n-1+4t}{2} \right) \cap (0, t).
\end{cases}
\]

(93)

- If \( t \in [2n - 2, 2n] \), \( n \geq 2 \) : In this case, the interval \((t - 2, t - 1)\) can be decomposed as

\[
(t - 2, t - 1) = (t - 2, \frac{2n - 4 + t}{2}) \cup \left( \frac{2n - 4 + t}{2}, \frac{2n - 3 + t}{2} \right) \cup \left( \frac{2n - 3 + t}{2}, t - 1 \right).
\]

Therefore, by using (93) we get

\[
I_3(t) = \int_{t-2}^{\frac{2n-4+t}{2}} \eta_2(\tau, \tau + 2 - t) p_0^-(2n - 4 + t - 2\tau) d\tau \\
- \int_{\frac{2n-3+t}{2}}^{\frac{2n-4+t}{2}} \eta_2(\tau, \tau + 2 - t) q_0^-(2\tau + 4 - t - 2n) d\tau \\
+ \int_{\frac{2n-3+t}{2}}^{t-1} \eta_2(\tau, \tau + 2 - t) p_0^-(2n - 2 - t + 2\tau) d\tau.
\]

And after a change of variables, we obtain

\[
I_3(t) = \frac{1}{2} \int_{0}^{2n-t} \eta_2(\frac{2n - 4 + t - s}{2}, \frac{2n - t - s}{2}) p_0^-(s) ds \\
- \frac{1}{2} \int_{0}^{1} \eta_2(\frac{s + t - 4 + 2n}{2}, \frac{s - t + 2n}{2}) q_0^-(s) ds \\
+ \frac{1}{2} \int_{2n-t}^{1} \eta_2(\frac{2n - 2 + t - s}{2}, \frac{2n + 2 - t - s}{2}) p_0^-(s) ds. \quad (94)
\]
I

Consider the first case:

\( (t - 2, t - 1) = (t - 2, -\frac{2n - 3 + t}{2}) \cup (-\frac{2n - 3 + t}{2}, \frac{2n - 2 + t}{2}) \cup (\frac{2n - 2 + t}{2}, t - 1). \)

Similarly, we obtain

\[
I_3(t) = -\int_{t - 2}^{\frac{2n - 3 + t}{2}} \eta_2(\tau, \tau + 2 - t)q_0^- (2\tau + 4 - t - 2n) d\tau
+ \int_{\frac{2n - 3 + t}{2}}^{\frac{2n - 3 + t}{2}} \eta_2(\tau, \tau + 2 - t)p_0^- (2n - 2 + t - 2\tau) d\tau
- \int_{\frac{2n - 2 + t}{2}}^{t - 1} \eta_2(\tau, \tau + 2 - t)q_0^- (2\tau + 2 - t - 2n) d\tau.
\]

And after a change of variables, we get

\[
I_3(t) = -\frac{1}{2} \int_{t - 2n}^{1} \eta_2(\frac{s + 2n + t - 4}{2}, \frac{s + 2n - t}{2})q_0^- (s) ds
+ \frac{1}{2} \int_{0}^{1} \eta_2(\frac{2n + t - 2 - s}{2}, \frac{2n - t + 2 - s}{2})p_0^- (s) ds
- \frac{1}{2} \int_{0}^{t - 2n} \eta_2(\frac{2n + t - 2 + s}{2}, \frac{2n - t + 2 + s}{2})q_0^- (s) ds.
\]

(95)

Now, we deal with \( I_4(\cdot) \). In the same way, we have for any \( n \geq 0 \):

\[
I_4(t) = \int_{-1}^{t} (\eta_1 p^-)(\tau, -\tau + t) d\tau
\]

\[
= \begin{cases} 
\int_{-1}^{t} \eta_1(\tau, -\tau + t) p_0^-(2\tau - t + 2n) d\tau, & \text{if } \tau \in (\frac{2n - 1 + t}{2}, \frac{2n + t}{2}) \cap (0, t), \\
- \int_{-1}^{t} \eta_1(\tau, -\tau + t) q_0^-(2\tau - t - 2n) d\tau, & \text{if } \tau \in (\frac{2n + t}{2}, \frac{2n + t + 1}{2}) \cap (0, t).
\end{cases}
\]

(96)

Consider the first case:

- If \( t \in [2n, 2n + 1), n \geq 1 \): In this case, we write

\[
(t - 1, t) = (t - 1, \frac{t + 2n - 1}{2}) \cup (\frac{t + 2n - 1}{2}, \frac{t + 2n}{2}) \cup (\frac{t + 2n}{2}, t),
\]

\[
I_4(t) = -\int_{-1}^{\frac{t + 2n - 1}{2}} \eta_1(\tau, -\tau + t) q_0^- (2\tau - t - 2n) d\tau
+ \int_{\frac{t + 2n}{2}}^{\frac{t + 2n - 1}{2}} \eta_1(\tau, -\tau + t) p_0^- (2\tau + t + 2n) d\tau
- \int_{\frac{t + 2n}{2}}^{t} \eta_1(\tau, -\tau + t) q_0^- (2\tau - t - 2n) d\tau,
\]
which yields after a change of variables

\[
I_4(t) = -\frac{1}{2} \int_{t-2n}^{t} \eta_1\left(\frac{s+t+2n-2}{2}, \frac{t-2n-s+2}{2}\right)q_0^- (s) \, ds \\
+ \frac{1}{2} \int_{0}^{1} \eta_1\left(\frac{2n+t-s}{2}, \frac{t+s-2n}{2}\right)p_0^- (s) \, ds \\
- \frac{1}{2} \int_{t-2n}^{-} \eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right)q_0^- (s) \, ds.
\]

(97)

- If \( t \in [2n + 1, 2n + 2) \), \( n \geq 1 \):

In the same way,

\[(t-1, t) = (t-1, \frac{t+2n}{2}) \cup (\frac{t+2n}{2}, \frac{t+2n+1}{2}) \cup (\frac{t+2n+1}{2}, t),\]

so, we obtain

\[
I_4(t) = \int_{t-1}^{t+2n} \eta_1(\tau, -\tau + t)p_0^- (-2\tau + t + 2n) \, d\tau \\
- \int_{t+2n}^{t+2n+1} \eta_1(\tau, -\tau + t)q_0^- (2\tau - t - 2n) \, d\tau \\
+ \int_{t+2n+1}^{t} \eta_1(\tau, -\tau + t)p_0^- (-2\tau + t + 2n + 2) \, d\tau,
\]

and after a change of variables we obtain

\[
I_4(t) = \frac{1}{2} \int_{0}^{2n+2-t} \eta_1\left(\frac{2n+t+s}{2}, \frac{-2n+t+s}{2}\right)p_0^- (s) \, ds \\
- \frac{1}{2} \int_{0}^{1} \eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right)q_0^- (s) \, ds \\
+ \frac{1}{2} \int_{2n+2-t}^{1} \eta_1\left(\frac{2n+2+t-s}{2}, \frac{t+s-2n-2}{2}\right)p_0^- (s) \, ds.
\]

(98)

To summarize, the initial states \( p_0^- \), \( q_0^- \) are solutions of the following two equations:

- If \( t \in [2n, 2n + 1) \), \( n \geq 1 \):

By gathering (91), (92), (95), and (97), we get

\[
0 = \left(\int_{t-2}^{t-1} \eta_1(\tau, \tau + 2 - t) \, d\tau + \int_{t-1}^{t} \eta_1(\tau, -\tau + t) \, d\tau\right)q_0^- (t-2n) \\
- \frac{1}{2} \int_{t-2n}^{t-1} \left[\eta_1\left(\frac{s+t+2n-2}{2}, \frac{t-2n-s+2}{2}\right) + \eta_2\left(\frac{s+2n+t-4}{2}, \frac{s+2n-t}{2}\right)\right]q_0^- (s) \, ds \\
+ \frac{1}{2} \int_{0}^{1} \left[\eta_1\left(\frac{2n+t-s}{2}, \frac{t+s-2n}{2}\right) + \eta_2\left(\frac{2n+t-2-s}{2}, \frac{2n-t+2-s}{2}\right)\right]p_0^- (s) \, ds \\
- \frac{1}{2} \int_{t-2n}^{-} \left[\eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right) + \eta_2\left(\frac{2n+t-2+s}{2}, \frac{2n-t+2+s}{2}\right)\right]q_0^- (s) \, ds.
\]
Letting $x = t - 2n$ yields

$$
\phi(x + 2n)q_0^-(x) - \int_0^1 \left( K_n^{21}(s, x) p_0^-(s) + K_n^{22}(s, x) q_0^-(s) \right) ds = 0, \; x \in (0, 1),
$$

(99)

where $\phi$ is defined in (24) and the kernels $K_n^{21}(\cdot, \cdot)$, $K_n^{22}(\cdot, \cdot)$, are given by

$$
K_n^{21}(s, x) = -\frac{1}{2} \eta_1 \left( \frac{4n + x - s}{2}, \frac{x + s}{2} \right) - \frac{1}{2} \eta_2 \left( \frac{4n + x - 2 - s}{2}, \frac{2 - s - x}{2} \right),
$$

$(s, x) \in (0, 1)^2$,  
(100)

$$
K_n^{22}(s, x) = \frac{1}{2} \left\{ \eta_1 \left( \frac{4n + x + s}{2}, \frac{x - s}{2} \right) + \eta_2 \left( \frac{4n - 2 + x + s}{2}, \frac{2 + s - x}{2} \right), \quad \text{if } 0 \leq s \leq x,
\right.

(101)

Observe that when $t$ varies in $[2n, 2n + 1)$ the $x$ varies in $[0, T - 2n]$.

- If $t \in [2n + 1, 2n + 2)$, $n \geq 1$:

Gathering (91), (92), 94, and (98) yields

$$
0 = - \left( \int_{-\tau + 2}^{t-1} \eta_1(\tau, \tau + 2 - t) d\tau + \int_{t-1}^{t} \eta_1(\tau, -\tau + t) d\tau \right) p_0(2n + 2 - t) 
+ \frac{1}{2} \int_0^{2n+2-t} \left[ \eta_1 \left( \frac{2n + t - s}{2}, \frac{2n + t + s}{2} \right) + \eta_2 \left( \frac{2n - 2 + t - s}{2}, \frac{2n + 2 - t - s}{2} \right) \right] p_0(s) ds
+ \frac{1}{2} \int_{2n+2-t}^{t} \left[ \eta_1 \left( \frac{2n + 2 + t - s}{2}, \frac{2n + t - s}{2} \right) + \eta_2 \left( \frac{2n + 4 - t - s}{2}, \frac{2n + 4 - t - s}{2} \right) \right] p_0(s) ds
- \frac{1}{2} \int_0^{2n+2-t} \left[ \eta_1 \left( \frac{2n + t + s}{2}, \frac{2n + 2 - t - s}{2} \right) + \eta_2 \left( \frac{2n + 2 + t - s}{2}, \frac{2n + 2 - t - s}{2} \right) \right] q_0(s) ds.
$$

Letting $2n + 2 - t = x$ yields

$$
\phi(2n + 2 - x)q_0^-(x) - \int_0^1 \left( K_n^{11}(s, x) p_0^-(s) + K_n^{12}(s, x) q_0^-(s) \right) ds = 0, 
$$

$(s, x) \in (0, 1)^2$,

(102)

where $\phi$ is defined in (24) and the kernels $K_n^{11}(\cdot, \cdot)$, $K_n^{12}(\cdot, \cdot)$, are given by

$$
K_n^{11}(s, x) = \frac{1}{2} \left\{ \eta_1 \left( \frac{4n + 2 - x - s}{2}, \frac{2x + s}{2} \right) + \eta_2 \left( \frac{4n - x - s}{2}, \frac{x - s}{2} \right), \quad \text{if } 0 \leq s \leq x,
\right.

(103)

$$
$$
$$

K_n^{12}(s, x) = \frac{1}{2} \eta_1 \left( \frac{4n + 2 - x + s}{2}, \frac{2 - x - s}{2} \right) - \frac{1}{2} \eta_2 \left( \frac{s + 4n - x}{2}, \frac{s + x}{2} \right),
$$

$(s, x) \in (0, 1)^2$.

(104)

Similarly, when $t$ varies in $[2n + 1, 2n + 2)$ then $x$ varies in $(2n + 2 - T, 1]$.

Observe that Eqs. (99) and (102) form a system of Fredholm Integral equations of third kind. Indeed, if $t \in [2n, 2n + 1)$, the interval $[2, t)$ can be written as

$$
[2, t) = \left( \bigcup_{i=1}^{n-1} [2i, 2i + 1) \right) \cup \left( \bigcup_{k=1}^{n-1} [2k + 1, 2k + 2) \right) \cup [2n, t),
$$
In this case, we have to solve Eq. (99) (resp. Eq. 102) in intervals of the form $[2l, 2l+1)$ (resp. $[2k + 1, 2l + 2)$) for some $k, l \geq 1$. In the same way, for $t \in [2n + 1, 2n + 2)$, the interval $[2, t)$ can be written as

$$[2, t) = \left( \bigcup_{i=1}^{n} [2l, 2l+1) \cup \left( \bigcup_{k=1}^{n-1} [2k + 1, 2k + 2) \right) \cup [2n + 1, T) \right).$$

Similarly, we have to solve Eq. (99) (resp. Eq. 102) in the intervals of the form $[2l, 2l+1)$ (resp. $[2k + 1, 2l + 2)$) for some $k, l \geq 1$. More precisely, introduce the kernel $\mathbb{K}_{n,k,l}(\cdot, \cdot)$, $1 \leq i \leq 2$, $n \geq 2$, $k, l \geq 1$, by

- If $T \in [2n, 2n + 1)$

$$\mathbb{K}_{n,k,l}(s, x) = \begin{cases} K_{k,l}(s, x), & 1 \leq k \leq n - 1, \ 1 \leq l \leq n, \text{ if } x \in [0, T - 2n), \\ K_{k,l}(s, x), & 1 \leq k \leq n - 1, \ 1 \leq l \leq n - 1, \text{ if } x \in [T - 2n, 1], \end{cases}$$

- If $T \in [2n + 1, 2n + 2)$

$$\mathbb{K}_{n,k,l}(s, x) = \begin{cases} K_{k,l}(s, x), & 1 \leq k \leq n, \ 1 \leq l \leq n, \text{ if } x \in [0, 2n + 2 - T), \\ K_{k,l}(s, x), & 1 \leq k \leq n + 1, \ 1 \leq l \leq n, \text{ if } x \in [2n + 2 - T, 1], \end{cases}$$

where

$$A_{k,l}(x) = \begin{pmatrix} \phi(2k + 2 - x) & 0 \\ 0 & \phi(2l + x) \end{pmatrix}, \ x \in [0, 1], \ k, l \geq 1,$$

$$K_{k,l}(s, x) = \begin{pmatrix} K_{k}^{11}(s, x) & K_{k}^{12}(s, x) \\ K_{k}^{21}(s, x) & K_{k}^{22}(s, x) \end{pmatrix}, \ (s, x) \in [0, 1]^2, \ k, l \geq 1,$$

associated with the third kind Fredholm integral equations

$$A_{k,l}(x) \begin{pmatrix} p_{0}^-(x) \\ q_{0}^-(x) \end{pmatrix} = \int_{0}^{1} \mathbb{K}_{n,k,l}(s, x) \begin{pmatrix} p_{0}^-(s) \\ q_{0}^-(s) \end{pmatrix} \, ds, \quad (105)$$

Now, we come to the main theorem of this section:

**Theorem 4.** Let $n \geq 2$ be an integer. The unique continuation property for (81) holds true at time $T$ if, and only if, there exist $k, l \geq 1$ such that the unique solution $(p_{0}^-, q_{0}^-)$ to Eq. (105) is the null one.

Now, let us assume that the weak observability holds, i.e.,

- If $2n \leq T < 2n + 1$:
  1. There exist $1 \leq k \leq n - 1, \ 1 \leq l \leq n$, such that
     $$\phi(2k + 2 - x) \neq 0, \ \phi(2l + x) \neq 0, \ \forall x \in [0, T - 2n). \quad (106)$$
  2. There exist $1 \leq k \leq n - 1, \ 1 \leq l \leq n - 1$, such that
     $$\phi(2k + 2 - x) \neq 0, \ \phi(2l + x) \neq 0, \ \forall x \in [T - 2n, 1]. \quad (107)$$
− If $2n + 1 \leq T < 2n + 2$ :
  1. There exist $1 \leq k \leq n - 1, 1 \leq l \leq n$, such that
     \[ \phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [2n + 2 - T, 0). \] (108)
  2. There exist $1 \leq k \leq n, 1 \leq l \leq n$, such that
     \[ \phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [2n + 2 - T, 1]. \] (109)

Under assumptions (106) and (107) (resp. (108) and (109)), Equations (105) writes
\[
\begin{pmatrix}
  p_0^{-}(x) \\
  q_0^{-}(x)
\end{pmatrix} = \int_0^1 A_{k,l}^{-1}(x)K_{n,k,l}(s, x) \begin{pmatrix}
  p_0^{-}(s) \\
  q_0^{-}(s)
\end{pmatrix} \, ds := K_{k,l} \begin{pmatrix}
  p_0^{-} \\
  q_0^{-}
\end{pmatrix}(x), \quad i = 1, 2,
\]
which are a second kind Fredholm integral equations. The following corollary is a straightforward consequence of the above theorem:

**Corollary 3.** Let $n \geq 2$ be an integer. Assume that (106) and (109) hold for some $k, l \geq 1$. The unique continuation property for (81) holds at time $T$ if, and only if $1 \notin \sigma(K_{k,l})$.

**Proof.** By assumptions (106)–(109), Equation (105) is Fredholm integral equation of second kind. By the Fredholm alternative, it possesses a unique solution if, and only if $1 \notin \sigma(K_{k,l})$, for some $l, k \geq 1$.

**Remark 10.** Since the component of the kernel $A_{k,l}^{-1}(\cdot)K_{n,k,l}(\cdot, \cdot)$, $n \geq k, l \geq 1$, are completely known, we can always prove that $1 \notin \sigma(K_{k,l})$, by assuming that $\|K_{k,l}\|_{L(L^2(0,1;M_{2\times 2}(\mathbb{R})))} < 1$. Notice that this is not necessarily a smallness assumption on the coupling coefficients since the kernel involves the matrix $A_{k,l}^{-1}(\cdot)$.

**Remark 11.** It is not difficult to see that if $\eta_1$ and $\eta_2$ are time independent, the compact operator $K_{k,l} = K$, is symmetric. Therefore, its spectrum consists of real eigenvalues. However, giving a characterization of the spectrum needs more care.

**Remark 12.** In [11], it has been shown that the unique continuation property for $a(t, x) = a(x)$ and $b = 0$ holds at time $T > 0$ for System (81) if, and only if $T \geq 4$, and
\[
\int_0^1 a(s) \sin^2(\pi n s) \, ds \neq 0, \forall n \geq 1.
\] (111)

It is natural to expect that condition (111) is equivalent to the uniqueness of the solution to Equation (105) for $K_{n,k,l} = K$.

**Remark 13.** With a simple change of variable (see [4, Remark 15]), we can deduce that all the results proved for cascade system with velocity coupling ($a \neq 0$) hold true for zero-order coupling (replace $a\varphi$ by $a\varphi$ in System (3)) in the space
\[ H^1_0(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1). \]
4.4. Examples

Here, we will provide some illustrations of Theorem 4. For the sake of simplicity, we will use both coupling functions \( a \) and \( b \), also, we will choose the time of control to be \( T = 2n \) for some \( n \geq 2 \).

4.4.1. The case \( \eta_1(t, x) = \alpha(t - x), \eta_2(t, x) = \beta(t + x) \)

Set \( \eta_1(t, x) = \alpha(t - x) \) and \( \eta_2(t, x) = \beta(t + x) \) for some smooth functions \( \alpha \) and \( \beta \) in \( \mathcal{Q}_T \). After performing a simple computations, the components \( K_{n}^{i,j} (\cdot, \cdot) \), \( 1 \leq i, j \leq 2, n \geq 1 \), given in (100), (101), (103) and (104) and the function \( \phi \) defined in (24) take the form

\[
2K_{k}^{12}(s, x) = -\alpha(2k + s) - \beta(2k + s), \quad s \in (0, 1),
\]

\[
2K_{l}^{21}(s, x) = -\alpha(2l - s) - \beta(2l - s), \quad s \in (0, 1),
\]

\[
2K_{k}^{11}(s, x) = \begin{cases} \alpha(2k - s) + \beta(2k - s), & \text{if } 0 \leq s \leq x, \\ \alpha(2k + 2 - s) + \beta(2k + 2 - s), & \text{if } x \leq s \leq 1, \end{cases}
\]

\[
2K_{l}^{22}(s, x) = \begin{cases} \alpha(2l + s) + \beta(2l + s), & \text{if } 0 \leq s \leq x, \\ \alpha(2l + 2 - s) + \beta(2l + 2 - s), & \text{if } x \leq s \leq 1. \end{cases}
\]

\[
\phi(t) = \int_{t-2}^{t-1} \alpha(t-2)d\tau + \int_{t-1}^{t} \beta(t)d\tau = \alpha(t-2) + \beta(t), \quad t \geq 2.
\]

Next, we define the functions \( S_{j}^{i}, i = 1, 2 \) on \([0, 1]\) for any \( j \geq 1 \) by

\[
S_{j}^{1}(s) = (2\phi(2j + 2 - s))^{-1} \left( \begin{array}{c} -2\phi'(2j + 2 - s) - \alpha(2j + 2 - s) \\ +\beta(2j - s) - \beta(2j + 2 - s) + \alpha(2j - s) \end{array} \right),
\]

and

\[
S_{j}^{2}(s) = (2\phi(2j + s))^{-1} \left( \begin{array}{c} -\phi'(2j + s) + \alpha(2j + s) + \beta(2j + s) \\ -\alpha(2j - 2 + s) - \beta(2j - 2 + s) \end{array} \right).
\]

We have the following unique continuation result:

**Proposition 16.** Let \( n \geq 2 \). Assume that \( A_{k,l}^{-1}(\cdot) \) exists on \([0, 1]\) for some \( 1 \leq k, l \leq n - 1 \). Then, the unique continuation property (81) holds true in time \( T = 2n \) if

\[
\int_{0}^{1} S_{k}^{1}(s)ds \neq \int_{0}^{1} S_{l}^{2}(s)ds,
\]

**Proof.** For \( T = 2n \), the system of integral equations (105) writes:

\[
2\phi(2k + 2 - x)p_{0}^{-}(x) = \int_{0}^{x} (\alpha(2k - s) + \beta(2k - s)) p_{0}^{-}(s)ds
\]

\[
+ \int_{x}^{1} (\alpha(2k + 2 - s) + \beta(2k + 2 - s)) p_{0}^{-}(s)ds
\]
Taking the derivative of (112) and (113) yields
\[ 2\phi(2l + x)q_0^{-}(x) = \int_{0}^{x} (\alpha(2l + s) + \beta(2l + s)) q_0^{-}(s)ds \]
\[ + \int_{x}^{1} (\alpha(2l - 2 + s) + \beta(2l - 2 + s)) q_0^{-}(s)ds \]
\[ - \int_{0}^{1} (\alpha(2l - s) + \beta(2l - s)) p_0^{-}(s)ds. \] (113)

Taking the derivative of (112) and (113) yields
\[ 2\phi(2k + 2 - x) (p_0^{-}(x))' = \left( -2\phi'(k + 2 - x) - \alpha(2k + 2 - x) + \beta(2k - x) - \beta(2k + 2 - x) + \alpha(2k - x) \right) p_0^{-}(x), \] (114)
\[ 2\phi(2l + x) (q_0^{-}(x))' = \left( -\phi'(2l + x) + \alpha(2l + x) + \beta(2l + x) - \alpha(2l - 2 + x) - \beta(2l - 2 + x) \right) q_0^{-}(x). \] (115)

Now, we divide by $2\phi(2k + 2 - x)$ and $2\phi(2l + x)$ equations (114) and (115) and integrating the latter two systems to get
\[ p_0^{-}(x) = \exp\left( \int_{0}^{x} S_k^1(s)ds \right) p_0^{-}(0), \quad q_0^{-}(x) = \exp\left( \int_{0}^{x} S_k^2(s)ds \right) q_0^{-}(0), \]
where
\[ S_k^1(s) = (2\phi(2k + 2 - s))^{-1} \left( -2\phi'(k + 2 - s) - \alpha(2k + 2 - s) + \beta(2k - s) - \beta(2k + 2 - s) - \alpha(2k - s) \right) \]
and
\[ S_k^2(s) = (2\phi(2l + s))^{-1} \left( -2\phi'(2l + s) + \alpha(2l + s) + \beta(2l + s) - \alpha(2l - 2 + s) - \beta(2l - 2 + s) \right). \]

Since $p_0^{-}$ and $q_0^{-}$ lie in $D(A^*)$, they are linked by the boundary conditions $(p_0^{-} + q_0^{-})|_{x=0,1}$, we get the system
\[ \begin{cases} 
    p_0^{-}(0) + q_0^{-}(0) = 0, \\
    p_0^{-}(x) = \exp\left( \int_{0}^{1} S_k^1(s)ds \right) p_0^{-}(0) + \exp\left( \int_{0}^{1} S_k^2(s)ds \right) q_0^{-}(0) = 0,
\end{cases} \]
which has a unique solution if, and only if
\[ \int_{0}^{1} S_k^1(s)ds \neq \int_{0}^{1} S_k^2(s)ds. \]
\[ \square \]
4.4.2. The case $\eta_1(t, x) = \alpha(t + x)$, $\eta_2(t, x) = \beta(t - x)$

This time set $\eta_1(t, x) = \alpha(t + x)$ and $\eta_2(t, x) = \beta(t - x)$ for some smooth functions $\alpha$ and $\beta$ in $Q_T$. The components $K^i_{i,j}(\cdot, \cdot)$, $1 \leq i, j \leq 2$, $n \geq 1$, given in (100), (101), (103) and (104) and the function $\phi$ defined in (24) take the form

\[
2K^1_k(x) = -2K^2_k(x) = \alpha(2k + 2 - x) + \beta(2k - x), \quad x \in [0, 1],
\]

\[
2K^2_l(x) = -2K^1_l(x) = \alpha(2l + x) + \beta(2l - 2 + x), \quad x \in [0, 1],
\]

\[
\phi(t) = \int_{t-2}^{t-1} a(\tau, \tau - (t - 2))d\tau + \int_{t-1}^t \beta(\tau, t - \tau)d\tau
\]

\[
= \int_{t-2}^{t-1} \alpha(2\tau - (t - 2))d\tau + \int_{t-1}^t \beta(2\tau - t)d\tau
\]

\[
= \frac{1}{2} \int_{t-2}^t (\alpha(s) + \beta(s))ds, \quad t \geq 2.
\]

At time $T = 2n$, the integral equation (110) turns to

\[
(K_{k,l}(\phi))(x) = A^{-1}_{k,l}(x)K_{n,k,l}(x) \int_{0}^{1} \Phi(s)ds, \quad \Phi \in L^2(0, 1)^2, \quad k, l \geq 1, \quad x \in [0, 1],
\]

where

\[
A^{-1}_{k,l}(x)K_{n,k,l}(x) = \begin{pmatrix}
K^1_k(x) - K^1_l(x) & K^1_k(x) \\
\phi(2k+2-x) & \phi(2k+2-x)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
K^2_l(x) & K^2_l(x) \\
\phi(2l+x) & \phi(2l+x)
\end{pmatrix}, \quad 1 \leq k, l \leq n - 1, \quad x \in [0, 1].
\]

We have the following unique continuation result:

**Proposition 17.** Let $n \geq 2$. Assume that $A^{-1}_{k,l}(\cdot)$ exists on $[0, 1]$ for some $1 \leq k, l \leq n - 1$. Then, the unique continuation property (81) holds true in time $T = 2n$ if, and only if

\[
1 \notin \sigma \left( \int_{0}^{1} A^{-1}_{k,l}(x)K_{n,k,l}(x)dx \right).
\]

The proof of the above proposition is an immediate consequence of combining Corollary 3 and the following lemma:

**Lemma 8.** Let $J : L^2(0, 1)^n \to L^2(0, 1)^n$ be the compact operator

\[
(Jf)(x) = M(x) \int_{0}^{1} f(s)ds, \quad x \in (0, 1),
\]

where $M \in C([0, 1], M_{n \times n}(\mathbb{R}))$. Then, $1 \in \sigma(J)$ if, and only if $1 \in \sigma \left( \int_{0}^{1} M \right)$.

**Proof.** See Appendix 5.2. \qed

**Remark 14.** In the above two examples, we have used both $a$ and $b$ to simplify the computations. Dealing with the integral equation (105) with only one of them seems to be not accessible unless they are in a very particular class of simple functions (for instance $a(t, x) = \kappa_2 t + \kappa_1 x + \kappa_0$, $\kappa_i \in \mathbb{R}, \quad i = 1, 2, 3$).
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5. Appendix

5.1. Proof of Proposition 5

We start by proving that $1 \Rightarrow 2$.

Suppose there exists $x_0 \in [0, 1]$ such for any $s \times s$ matrix $M_{\text{ext}}$, extracted from $M$, we have $\det M_{\text{ext}} (x_0) = 0$. Denote by $M_1, M_2, ..., M_n$ the $s$-dimensional row vector functions of the matrix $M$ so that:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}.$$

Then, from our assumption, the vector space spanned by the $M_i (x_0)$ is at most of dimension $s - 1$. Thus, there exists $V \in \mathbb{R}^s$ such that

$$|V|_{\mathbb{R}^s} = 1 \text{ and } M_i (x_0) \cdot V = 0, \ 1 \leq i \leq n.$$

Let $u \in C_0^\infty (\mathbb{R})$ with $\int_{\mathbb{R}} u^2 (x) \, dx = 1$ and (for example) $\text{supp}(u) = [-1, 1]$. We are going to prove that the sequence:

$$h_j (x) = \sqrt{j} u (j (x - x_0)) \, V, \ j \geq j_0, \ x \in \mathbb{R},$$

is a singular sequence for the multiplication operator $\mathbb{M}_{\text{ext}}$ on $L^2 (0, 1)^s$ whose matrix is some $s \times s$ matrix $M_{\text{ext}}$, extracted from $M$. If $x_0 \in (0, 1)$ (we leave to the reader to check that the proof works with $x_0 = 0$ by choosing $h_j (x) = \sqrt{j} u (j x - 1) \, V$ and with $x_0 = 1$ by choosing $h_j (x) = \sqrt{j} u (j (x - 1) + 1) \, V$), we see from the definition of $u$ that for $j_0$ sufficiently large

$$\text{supp} \left( h_j \right) = \left[ x_0 - \frac{1}{j}, x_0 + \frac{1}{j} \right] \subset [0, 1], \ j \geq j_0.$$

Moreover, for all $j \geq j_0$.

$$\int_0^1 h_j^2 (x) \, dx = j \int_0^1 u^2 (j (x - x_0)) |V|_{\mathbb{R}^s}^2 \, dx.$$
\[ = j \int_{x_0 - \frac{1}{j}}^{x_0 + \frac{1}{j}} u^2 \left( j (x - x_0) \right) \, dx \]
\[ = \int_{-1}^{1} u^2 (\xi) \, d\xi = 1. \]

Since \( \text{supp}(h_j) \to \{x_0\} \) as \( j \to \infty \), \( \{h_j\} \) has no convergent subsequence in \( L^2(0, 1)^n \).

On the other hand:

\[ \| M_{\text{ext}} h_j \|_{L^2(0, 1)^s}^2 = \int_0^1 |M_{\text{ext}}(x) \cdot h_j(x)|^2 \, dx \]
\[ = j \int_{x_0 - \frac{1}{j}}^{x_0 + \frac{1}{j}} u^2 \left( j (x - x_0) \right) |M_{\text{ext}}(x) V|^2 \, dx \]
\[ = \int_{-1}^{1} u^2 (\xi) \left| M_{\text{ext}} \left( x_0 + \frac{1}{j} \xi \right) V \right|^2 \, dx. \]

Thus, from Lebesgue’s dominated convergence theorem, we get:

\[ \lim_{j \to \infty} \| M_{\text{ext}} h_j \|_{L^2(0, 1)^s}^2 = 0. \]

Since, in particular, the choice of \( V \) and the continuity of \( M_{\text{ext}} \) give:

\[ \lim_{j \to \infty} M_{\text{ext}} \left( x_0 + \frac{1}{j} \xi \right) V = M_{\text{ext}}(x_0) V = 0. \]

If \( n = ds + q \) with \( d \geq 1 \) and \( 0 \leq q \leq s - 1 \), we form the \( s \times s \) extracted matrices:

\[
M_{\text{ext}}^1 = \begin{bmatrix} M_1 \\ \vdots \\ M_s \end{bmatrix}, \ldots, M_{\text{ext}}^d = \begin{bmatrix} M_{(d-1)s+1} \\ \vdots \\ M_{ds} \end{bmatrix}, \quad M_{\text{ext}}^{d+1} = \begin{bmatrix} M_{ds+1} \\ \vdots \\ M_n \\ M_1 \\ \vdots \\ M_{s-q} \end{bmatrix}.
\]

We then have:

\[ \| M h_j \|_{L^2(0, 1)^n}^2 = \sum_{k=1}^{n} \| M_k \cdot h_j \|_{L^2(0, 1)}^2 \leq \sum_{k=1}^{d+1} \| M_{\text{ext}} h_j \|_{L^2(0, 1)^s}^2. \]

It readily follows that:

\[ \lim_{j \to \infty} \| M h_j \|_{L^2(0, 1)^n}^2 = 0. \]
This proves that $1 \Rightarrow 2$.
To prove that $2 \Rightarrow 1$, we assume that for all $x \in [0, 1]$, there exists a $s \times s$ matrix $M_{\text{ext}}$, extracted from $M$, such that
\[
\det M_{\text{ext}}(x) \neq 0.
\]
Each one of the functions $|\det M_{\text{ext}}(x)|$ is uniformly continuous on $[0, 1]$:
\[
\forall \varepsilon > 0, \exists \eta_{M_{\text{ext}}} > 0, \ |x - y| < \eta_{M_{\text{ext}}} \Rightarrow |\det M_{\text{ext}}(x)| - |\det M_{\text{ext}}(y)|| < \varepsilon,
\forall (x, y) \in [0, 1]^2.
\]
In the sequel, we set $\eta = \min \{ \eta_{M_{\text{ext}}}, M_{\text{ext}} \text{ extracted } s \times s \text{ matrix} \}$. Let $0 \leq j \leq m - 1$ and $\xi_j \in \left[\frac{j}{m}, \frac{j + 1}{m}\right]$. There exists a $s \times s$ matrix $M_{\text{ext}}^j$
\[
\left| \det M_{\text{ext}}^j(\xi_j) \right| := \delta_j > 0.
\]
Choosing $m$ such that $\frac{1}{m} < \eta$, we get with $\varepsilon < \min_{0 \leq j \leq m - 1} \delta_j = \delta$
\[
\left| \det M_{\text{ext}}^j(x) \right| > \delta - \varepsilon, \ \forall x \in \left[\frac{j}{m}, \frac{j + 1}{m}\right].
\]
From Proposition 4, this ensures that for each $0 \leq j \leq m - 1$, $M_{\text{ext}}^j$ is invertible on $L^2 \left(\frac{j}{m}, \frac{j + 1}{m}\right)^s$. Now, we can write for any $h \in L^2(0, 1)^s$ :
\[
\|Mh\|^2_{L^2(0,1)^n} = \sum_{k=1}^{n} \|M_k \cdot h\|^2_{L^2(0,1)}
\]
\[
= \sum_{j=0}^{m-1} \sum_{k=1}^{n} \int_{\frac{j}{m}}^{\frac{j + 1}{m}} |M_k(x) \cdot h(x)|^2 \, dx
\]
\[
\geq \sum_{j=0}^{m-1} \int_{\frac{j}{m}}^{\frac{j + 1}{m}} \left| M_{\text{ext}}^j(x) \cdot h(x) \right|^2 \, dx
\]
\[
\geq C \sum_{j=0}^{m-1} \int_{\frac{j}{m}}^{\frac{j + 1}{m}} |h(x)|^2 \, dx
\]
\[
\geq C \int_{0}^{1} |h(x)|^2 \, dx.
\]
This ends the proof.

5.2. Proof of Lemma 8

Proof of $\Rightarrow$: If $1 \in \sigma(J)$, there will exist a nonzero $f \in L^2(0, 1)^n$ such that
\[
f(x) = M(x) \int_{0}^{1} f(s) \, ds, \ x \in (0, 1).
\]
Integrating (116) over (0, 1) yields
\[ \int_0^1 f(s)ds = \int_0^1 M(s)ds \int_0^1 f(s)ds. \]
This shows that \( f(0, 1) \) is an eigenvector of the matrix \( \int_0^1 M \) associated with the eigenvalue 1.

Proof of \( \Leftarrow \): If \( 1 \in \sigma \left( \int_0^1 M \right) \), then there exists an eigenvector of \( \int_0^1 M \) denoted by \( V \in \mathbb{R}^n \) such that
\[ V = \left( \int_0^1 M(x)dx \right) V. \]
Applying the matrix \( M(\cdot) \) yields
\[ M(x)V = M(x) \left( \int_0^1 M(x)Vdx \right), \quad x \in [0, 1]. \]
This shows that the vector \( M(\cdot)V \) is an eigenvector of \( J \) associated with the eigenvalue 1.

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