ππ SCATTERING AND PION FORM FACTORS

L. Beldjoudi and Tran N. Truong

Centre de Physique Théorique *
Ecole Polytechnique, 91128 Palaiseau, France

ABSTRACT

The S and P wave ππ phase shifts are recalculated in terms of two phenomenological parameters using the one loop CPTH and the elastic unitarity condition. Using these phase shifts, the vector and scalar form factors are calculated and shown to be in a good agreement with experimental data. It is found that the simpler and more phenomenological approach, where the left hand cut contributions to the partial wave amplitude are neglected, yields approximately the same result.

* Laboratoire Propre du CNRS UPR A.0014
Chiral Perturbation Theory (CPTh) is a low energy expansion in the Nambu-Goldstone boson momenta which takes into account in a systematic manner the chiral symmetry breaking effect [1]. The matrix elements obtained from CPTh can be considered as approximate low energy theorems. They have been successful in correlating a number of low energy phenomena involving soft pions.

Unfortunately, these low energy theorems are only valid near the threshold region and cannot take into account of the resonance phenomenon or of a very strong interaction. The basic reason why it cannot handle these situations is because it is a perturbative calculation and can only satisfy perturbatively the unitarity relation. In contrast, the low energy effective range theory of the nucleon nucleon scattering, which satisfies the elastic unitarity exactly can handle both the bound state and resonance problems. We therefore want also to impose the exact elastic unitarity relation for the \( \pi\pi \) partial wave amplitudes. It is then not difficult to make a resummation of the one loop CPTh into a geometric like series so that the elastic unitarity condition is exactly satisfied. This geometric like series can be constructed from the one loop calculation and is known as the Padé [1,1] approximant method. We use here the Padé method as a most convenient way to satisfy exactly the elastic unitarity relation, but not in the mathematical sense of summing a diverging series. The use of the Padé method in order to satisfy the elastic unitarity relation is certainly not unique. It has been shown, for the special case of the P wave \( \pi\pi \) scattering, all methods lead to essentially the same physical consequences [2], namely, given the correct scattering length and the effective range, the \( \rho \) resonance exists once the elastic unitarity condition is implemented.

The purpose of this note is to recalculate the well known one loop CPTh for \( \pi\pi \) scattering. The partial wave amplitudes are then resummed by the Padé approximant method in order to satisfy the elastic unitarity condition. We then calculate the S and P waves phase shifts to be used in the Omnès representation [3] in order to calculate the corresponding pion form factors.

We want to point out that this more exact calculation of the form factors yields essentially the same result as the simplified approximation of the N/D method where the discontinuity of the left hand cut contribution is neglected.

Because we do not use the standard CPTh to calculate the form factors [4], our calculation of the form factors, using Omnès representation, involves the same parameters as those used in the scattering amplitudes and therefore no new parameters are needed. In particular we can directly calculate the pion r.m.s radii in terms of the calculated phase shifts and compared them with the experimental
I) Calculation of pion pion scattering phase shifts

The most general $\pi^a(p_1)\pi^b(p_2) \rightarrow \pi^c(p_3)\pi^d(p_4)$ scattering amplitude is given by:

$$A(s, t, u)\delta^{ab}\delta^{cd} + A(t, s, u)\delta^{ac}\delta^{bd} + A(u, t, s)\delta^{ad}\delta^{bc}$$  \hspace{1cm} (1)

In the $\pi\pi$ scattering we have three isospin amplitude $I = 0, 1, 2$ which are simple functions of $A(s, t, u)$:

$$T_0(s, t, u) = 3A(s, t, u) + A(t, s, u) + A(u, t, s)$$

$$T_1(s, t, u) = A(t, s, u) - A(u, t, s)$$

$$T_2(s, t, u) = A(t, s, u) + A(u, t, s)$$ \hspace{1cm} (2)

Where $s, t, u$ are Mandelstam variables with $s + t + u = 4m^2\pi$. Without loss of generality, they can be decomposed into the partial waves:

$$T^I_l(s, t) = 32\pi \sum_l (2l + 1) P_l(cos\theta) t^I_l(s)$$

Below the inelastic thresholds such as $K\bar{K}$, $\pi\omega$, the unitarity of the S matrix implies $\text{Im} t^I_l(s) = \sigma(s)|t^I_l(s)|^2$ for $s \geq 4m^2\pi$ where $\sigma(s) = \sqrt{1 - \frac{4m^2\pi}{s}}$ is the phase space factor. Because of the unitarity relation, the partial wave amplitude can be written as $t^I_l(s) = \frac{e^{+i\delta^I_l}}{\sigma(s)}$.

At one loop chiral perturbation theory [5,6], we have:

$$A(s, t, u) = \frac{s - m^2\pi}{f^2\pi} + B(s, t, u) + C(s, t, u) + 0(p^6)$$ \hspace{1cm} (3 - a)

where

$$B(s, t, u) = \frac{1}{6f^2\pi} [3(s^2 - m^4\pi)\bar{J}(s) + (t(t - u) - 2m^2\pi t + \frac{4m^2u}{s} - 2m^4\pi)\bar{J}(t)$$

$$+ (u(u - t) - 2m^2\pi u + 4m^2t - 2m^4\pi)\bar{J}(u)]$$ \hspace{1cm} (3 - b)

and

$$C(s, t, u) = \frac{1}{96\pi^2f^2\pi} [2(\bar{l}_1 - 4/3)(s - 2m^2\pi)^2 + (\bar{l}_2 - 5/6)(s^2 + (t - u)^2)$$

$$+ 12m^2\pi s(\bar{l}_4 - 1) - 3(\bar{l}_3 + 4\bar{l}_4 - 5)]$$ \hspace{1cm} (3 - c)
These results are obtained from the $0(p^4)$ calculation using $SU(2)_L \times SU(2)_R$ chiral lagrangian:

$$L = \frac{F^2}{4} tr \partial_\mu U \partial^\mu U^\dagger + \frac{F^2}{4} m^2 tr(U + U^\dagger) + \frac{l_1}{4} tr(\partial_\mu U \partial^\mu U^\dagger)^2$$

$$+ \frac{l_2}{4} tr(\partial_\mu U \partial^\nu U^\dagger) tr(\partial_\nu U \partial^\mu U^\dagger) + \frac{l_3 + l_4}{16} m^4 (tr(U + U^\dagger))^2$$

$$+ \frac{l_4}{8} m^2 tr(\partial_\mu U \partial^\mu U^\dagger) tr(U + U^\dagger)$$

(4)

where $U = \exp \frac{i\pi^a c^a}{F}$ is the exponential representation of the pion field, $m$ and $F$ are the bare mass and decay constant of the pion.

In Eq.(3) $f_\pi = 93.3$ Mev is the physical pion decay constant, up to a scale factor $\bar{l}_i$ are function of $l_i^r(\mu)$ plus a quantity which make them scale independent as indicated in ref. [6]. The constants $\bar{l}_i$ are scale independent parameters. $\bar{J}(t)$ is the once substracted scalar two point function.

$$\bar{J}(t) = \frac{1}{16\pi^2} (2 + \{ \begin{array}{ll} \sigma(\log(\frac{1-\sigma}{1+\sigma}) + i\pi) & \text{for } t \geq 4m^2_\pi \\
\sigma \log(\frac{-1+\sigma}{1+\sigma}) & \text{for } t \leq 0 \\
-2|\sigma| \arctan \frac{1}{|\sigma|} & \text{otherwise} \end{array} )$$

(5)

Below $K\bar{K}$ production the one loop chiral perturbation theory satisfies the perturbative unitarity: $\text{Im } t^{(1)} = \sigma(s) t^{(0)} s$ where $t = t^{(0)} + t^{(1)}$ and the superscripts stand for the tree graph and one loop calculation; the isospin index are omitted for convenience. It is straightforward to show that the reconstructed amplitude

$$t(s) = \frac{t^{(0)}}{1 - t^{(1)} / t^{(0)}}$$

(6)

satisfies exactly the elastic unitarity. Our determination of $\bar{l}_i$ parameters is different than ref.[6], where one can deduce $\bar{l}_1$ and $\bar{l}_2$ from I=0,I=2 D wave scattering length. It was found that $\bar{l}_1 = -0.6$, and $\bar{l}_2 = 6.3$. In our calculation the quantity $\bar{l}_2 - \bar{l}_1$ is sensitive to the $\rho$ resonance and it is directly fixed by the $\rho$ mass. The $\rho$ mass is defined as the energy where the I=1, l=1 phase shift passing through 90 degrees. To establish completely the unknown parameters, we use the experimental S wave I=0 phase shift at 500 Mev [7]. $\bar{l}_3$ and $\bar{l}_4$ measure the chiral symmetry breaking effect; their contributions to the scattering amplitude are proportional to the pion
mass squared. In this calculation they are given by the reference [6] \( \bar{l}_3 = 2.9 \) \( \bar{l}_4 = 4.3 \), which are determined by the SU(3) mass formula and the ratio \( f_K/f_\pi \). Our best values are \( \bar{l}_1 = -0.45 \), and \( \bar{l}_2 = 5.51 \), which are slightly different from those determined by[8]. The \( \rho \) mass is taken to be its experimental value \( m_\rho = 770 \text{MeV} \).

Once we fix the \( \bar{l}_i \) parameters \( \pi\pi \) scattering amplitude is known. The predictions to the scattering length are: \( a_0^1 = 0.22 \), and \( a_1^1 = 0.038 \). These values are consistent with those given by experiments. In Fig[1-2], S and P wave phase shifts are compared with experimental data. It can be seen that the agreement between the theoretical predictions and experimental data are excellent. The deviation of the theoretical prediction and experimental data for I=0 S wave above 700 Mev is due to the opening of the inelastic channel \( a^0(980) \) which is not included in our calculation.

The diagonal [1,1] Padé method for S waves is not without a problem: because \( t^{(0)} \) has a zero at \( s = \frac{m_\pi^2}{2} \), from Eq (6), it is clear that the denominator of \( t(s) \) has a zero near \( \frac{m_\pi^2}{2} \) or \( t(s) \) has a spurious pole near \( \frac{m_\pi^2}{2} \). It is simple to show that it has a very small residue. We must, in principle, substract this pole from \( t(s) \) in order to have a correct analytical property. Such a substraction has a little on the calculated amplitude and would result in a tiny violation of the unitarity relation in the physical region. We shall ignore, in the following this substraction procedure.

II) Calculation of the pion form factors

In the remainder of this paper, we calculate the vector and the scalar form factors. Using the CVC hypothesis and Lorentz invariance, one can write straightforwardly the following matrix elements:

\[
\langle \pi^a(p_1)\pi^b(p_2)|V^c_\mu(0)|0\rangle = i\epsilon^{abc}f(s)(p_2 - p_1)_\mu \\
\langle \pi^a\pi^b|\hat{m}(\bar{u}u + \bar{d}d)|0\rangle = \delta^{ab}\Gamma(s)
\]

At zero momentum transfer we have the following normalization: \( f(0) = 1 \) and \( \Gamma(0) = m_\pi^2 \). Assuming the elastic unitarity condition, we deduce the following relations:

\[
Imf(s) = f(s) \exp -i\delta_1^1 \sin \delta_1^1 \quad (8-a) \\
Im\Gamma(s) = \Gamma(s) \exp -i\delta_0^0 \sin \delta_0^0 \quad (8-b)
\]
Hence \( f(s) \) must have the phase of the P wave phase shift, and \( \Gamma(s) \) the S wave phase shift.

The general solutions to this equation are well known, they are of the Muskhelishvili Omnes type:

\[
\begin{align*}
    f(s) &= P_f(s)\Omega_1(s) \\
    \Gamma(s) &= \Gamma(0)P_\Gamma(s)\Omega_0(s)
\end{align*}
\]  

(9)

where

\[
\begin{align*}
    \Omega_1(s) &= \exp\left(\frac{s}{\pi} \int_{4m^2}^{+\infty} \frac{\delta_1^1 dz}{z(z - s - i\epsilon)}\right) \\
    \Omega_0(s) &= \exp\left(\frac{s}{\pi} \int_{4m^2}^{+\infty} \frac{\delta_0^0 dz}{z(z - s - i\epsilon)}\right)
\end{align*}
\]  

(10)

\( P_f \) and \( P_\Gamma \) are polynomials which determine the high energy behavior of the form factors. They could also represent the low energy contribution of the higher mass intermediate states to the form factors. In the following we assume the dominance of the elastic unitarity relation and hence we set \( P_f(s) = P_\Gamma(s) = 1 \).

Using the S and P wave phase shifts as calculated above, the scalar and vector form factors are calculated numerically, using Eqs (9). Because there are no experimental informations on the scalar form factor, we only compare the vector form factor with the experimental data. It is seen that the agreement between theory and experimental data is satisfactory although the peak values of the experimental form factor squared at the \( \rho \) mass is about 40 as compared with the theoretical calculation value 32 or an error of the order of 20%. This discrepancy is probably due to the inelastic effect of the \( \omega\pi \) channel as was previously pointed out [9].

We can also calculate the vector and scalar pion rms radii using the following formula:

\[
\begin{align*}
    \langle r^2_V \rangle &= \frac{6}{\pi} \int_{4m^2}^{+\infty} \frac{\delta_1^1 dz}{z^2} \\
    \langle r^2_S \rangle &= \frac{6}{\pi} \int_{4m^2}^{+\infty} \frac{\delta_0^0 dz}{z^2}
\end{align*}
\]

Numerical integration gives \( \langle r^2_V \rangle = 0.40 fm^2 \), and \( \langle r^2_S \rangle = 0.47 fm^2 \) compared to the experimental value \( \langle r^2_V \rangle = 0.439 \pm 0.03 fm^2 \), and the \( \langle r^2_S \rangle = 0.5 fm^2 \) which is
obtained from the experimental $\pi K$ scalar radius and from SU(3) symmetry. The agreement between experimental data and theoretical calculations is satisfactory.

III) Phenomenological approximations

The following approximation schemes have been used in the literature [10-11-12]. Decomposing the partial wave amplitudes as $t_I^l = N_I^l / D_I^l$ where for convenience we normalize $D_I^l(0) = 1$. In this case $D_I^l$ is identical to the function $\Omega_I^l$ defined in Eq(10). Approximating $N_I^l$ by the tree amplitude $N_I^l(s) = t^0(s)$, we have:

$$D_I^l(s) = 1 + b_I^l s - \frac{s^2}{\pi} \int_{4m_r^2}^{+\infty} \frac{\sigma(z)N_I^l(z)dz}{z^2(z - s - i\epsilon)}$$

This N/D construction satisfies the elastic unitarity condition exactly but it neglects the contribution from the one loop graph in t and u channel. $b_I^l$’s are phenomenological parameters to be determined from the experimental data.

How good are these approximations and what are their relation with the unitarized Chiral Perturbation Theory?

By explicit numerical evaluation of the expression corresponding to the left hand cut contribution in the denominator of Eq[6] one can show that it can be approximated with a good accuracy by a polynomial for $4m_r^2 \leq s \leq 1 Gev^2$. The validity of this approximation explains why the left hand cut contribution can be absorbed into the adjustable parameter $b_I^l$ in the $D_I^l$ function defined by Eq(11).

The P wave result in this approximation is well known as first given by Brown and Goble [10]. Fitting the unknown $b_1^1$ with the $\rho$ mass, defined as the energy where the P wave phase shift passing through 90 degrees, we get the usual KSRF relation giving the $\rho$ width $\Gamma_\rho = 142$ Mev which is smaller than the experimental value $\Gamma_\rho = 153$ Mev. Our exact calculation, taking into account the left hand cut yields $\Gamma_\rho = 157$ Mev which is in excellent agreement with the data. The modulus squared of the pion form factor calculated in this approximation is shown in Fig[3]. It is seen that it is in a good agreement with that given by the exact calculation.

Because the difference between these two calculations is small, we conclude that the neglect of the left hand cut discontinuity coming from t and u channel one loop graph, and treating $b_I^l$ as an adjustable parameter is a good approximation.

Similarly we make the same approximation for the I=0 scalar form factor. In Fig [1-4] we compare the phase and modulus of the approximate calculation, where the t and u one loop contribution is neglected with those where their effects are taken into account. It is seen that there is little difference between the approximate and the exact results.
The expression for the I=0 S wave D function $D_0^0$ without the contribution of the left hand cut to the scattering amplitude was previously used in connection with the $K \rightarrow 2\pi$, $K \rightarrow 3\pi$, $K_S \rightarrow 2\gamma$, $\gamma\gamma \rightarrow 2\pi$ and $K_L \rightarrow \pi^0\gamma\gamma$ problems. Various results given in ref[12] remain of course valid when the more exact calculation of $1/D_0^0$ function is used.

We show in this article the interrelation between the scattering amplitudes and form factors. Instead of using CPTh in the low energy region near to $2\pi$ threshold, we extend its validity to a much wider energy region by imposing the unitarity relation to take into account of the resonance effect (in P wave). It has recently shown that CPTh for the scalar form factor, using some prescriptions, can be extended to 400-500 Mev, and can describe $K_S \rightarrow 2\pi$ amplitude with the correct phase[13]. It remains to be seen whether the same prescriptions can be used without ambiguity for the $\gamma\gamma \rightarrow 2\pi$, $K_S \rightarrow 2\gamma$, and $K_L \rightarrow \pi\gamma\gamma$ problems.

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FIGURE CAPTIONS

Fig.1 The solid line I=0,l=0 pion scattering phase shift calculated from unitarized CPTh. Dashed line corresponds to the similar phase shift where the left hand cut is neglected. The dot dashed line is the CPTh prediction. The experimental data are those of Estabrook et al. Ref.[7].

Fig.2 The solid line I=1,l=1 pion scattering phase shift calculated from the unitarized CPTh. Dashed line corresponds to the similar phase shift where the left hand cut is neglected. The dot dashed line is the CPTh prediction. The experimental data are those of Estabrook et al. Ref.[7]. The difference between the solid and the dashed line is not distinguishable.

Fig.3 The solid line is the pion vector form factor squared calculated from the Omnés representation. The dashed line represents the same quantity calculated by the approximate solution where the left cut contribution is neglected. The experimental data are those of L.M. Barkov et al Ref[14]. The difference between the solid and the dashed line is not distinguishable.

Fig.4 The solid line is the pion scalar form factor squared calculated from the Omnés representation. The dashed line represents the same quantity calculated by the approximate solution where the left cut contribution is neglected.
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