Asymptotically Best Possible Lebesgue-Type Inequalities for the Fourier Sums on Sets of Generalized Poisson Integrals

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**Abstract.** In this paper we establish Lebesgue-type inequalities for $2\pi$-periodic functions $f$, which are defined by generalized Poisson integrals of the functions $\varphi$ from $L_p$, $1 \leq p < \infty$. In these inequalities uniform norms of deviations of Fourier sums $\|f - S_n\|_C$ are expressed via best approximations $E_n(\varphi)_{L_p}$ of functions $\varphi$ by trigonometric polynomials in the metric of space $L_p$. We show that obtained estimates are asymptotically best possible.

1. Introduction

Let $L_p$, $1 \leq p < \infty$, be the space of $2\pi$-periodic functions $f$ summable to the power $p$ on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \left(\int_0^{2\pi} |f(t)|^p dt\right)^{1/p}$; $L_\infty$ be the space of measurable and essentially bounded $2\pi$-periodic functions $f$ with the norm $\|f\|_\infty = \text{ess sup}_t |f(t)|$; $C$ be the space of continuous $2\pi$-periodic functions $f$, in which the norm is specified by the equality $\|f\|_C = \max_t |f(t)|$.

Denote by $C_0^\infty L_p$, $\alpha > 0$, $r > 0$, $\beta \in \mathbb{R}$, $1 \leq p \leq \infty$, the set of all $2\pi$-periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [20, Ch.3, 7-8])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt, \ a_0 \in \mathbb{R}, \ \varphi \perp 1,$$

where $\varphi \in L_p$ and $P_{\alpha,r,\beta}(t)$ are the following generated kernels

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\[ P_{\alpha, r, \beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k} \cos \left( kt - \frac{\beta \pi}{2} \right), \quad \alpha, r > 0, \quad \beta \in \mathbb{R}. \] (2)

The kernels \( P_{\alpha, r, \beta} \) of the form (2) are called generalized Poisson kernels. For \( r = 1 \) and \( \beta = 0 \) the kernels \( P_{\alpha, r, \beta} \) are usual Poisson kernels of harmonic functions.

If the functions \( f \) and \( \varphi \) are related by the equality (1), then the function \( f \) in this equality is called generalized Poisson integral of the function \( \varphi \) and is denoted by \( \mathcal{P}_{\beta}^{n}(\varphi)(t) = \mathcal{P}_{\beta}^{n}(\varphi, t) \). The function \( \varphi \) in equality (1) is called generalized derivative of the function \( f \) and is denoted by \( f_{\beta}^{n}(\varphi')(t) = f_{\beta}^{n}(\varphi') \).

The set of functions \( f \) from \( \mathcal{C}_{\beta}^{n} L_{p}, 1 \leq p \leq \infty \), such that \( f_{\beta}^{n} \in B_{p} \), where

\[ B_{p} = \{ \varphi : ||\varphi||_{p} \leq 1 \}, \]

we will denote by \( \mathcal{C}_{\beta}^{n} \).

The sets of generalized Poisson integrals \( \mathcal{C}_{\beta}^{n} L_{p} \) are closely related with the well–known Gevrey classes (see, e.g. [21]).

Let \( \tau_{2n-1} \) be the space of all trigonometric polynomials of degree at most \( n - 1 \) and let \( E_{n}(f)_{\tau} \) be the best approximation of the function \( f \in L_{p} \) in the metric of space \( L_{p} \), \( 1 \leq p \leq \infty \), by the trigonometric polynomials \( t_{n-1} \) of degree \( n - 1 \), i.e.,

\[ E_{n}(f)_{\tau} = \inf_{t_{n-1} \in \tau_{2n-1}} ||f - t_{n-1}||_{p}. \]

Analogously, by \( E_{n}(f)_{C} \) we denote the best uniform approximation of the function \( f \) from \( C \) by trigonometric polynomials of degree \( n - 1 \), i.e.,

\[ E_{n}(f)_{C} = \inf_{t_{n-1} \in \tau_{2n-1}} ||f - t_{n-1}||_{C}. \]

Let \( \rho_{n}(f; x) \) be the following quantity

\[ \rho_{n}(f; x) := f(x) - S_{n-1}(f; x), \] (3)

where \( S_{n-1}(f; \cdot) \) are the partial Fourier sums of degree \( n - 1 \) of a function \( f \).

One can estimate the norms \( ||\rho_{n}(f; \cdot)||_{C} \) via \( E_{n}(f)_{C} \) by Lebesgue inequalities

\[ ||\rho_{n}(f; \cdot)||_{C} \leq (1 + L_{n-1})E_{n}(f)_{C}, \quad n \in \mathbb{N}, \] (4)

where quantites \( L_{n-1} \) are Lebesgue constants of the Fourier sums of the form

\[ L_{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_{n-1}(t)| dt = \frac{2}{\pi} \int_{0}^{\pi} \frac{|\sin(2n-1)t|}{\sin t} dt, \]

\[ D_{n-1}(t) := \frac{1}{2} + \sum_{k=1}^{\infty} \cos kt = \frac{\sin(n - \frac{1}{2})t}{2 \sin \frac{t}{2}}. \]

Fejer [3] established the asymptotic equality for Lebesgue constants \( L_{n} \)

\[ L_{n} = \frac{4}{\pi^{2}} \ln n + O(1), \quad n \to \infty. \]

More exact estimates for the differences \( L_{n} - \frac{4}{\pi^{2}} \ln(n + a), a > 0, \) as \( n \in \mathbb{N} \) were found in works [1], [2], [4], [8], [18] and [28].
In particular, it follows from [20] (see also [8, p.97]) that
\[
|L_{n-1} - \frac{4}{\pi^2} \ln n| < 1,271, \ n \in \mathbb{N}.
\]

Then, the inequality (4) can be written in the form
\[
\|\rho_n(f;\cdot)\|_C \leq \left(\frac{4}{\pi^2} \ln n + R_n\right) E_n(f)_C,
\]
where \(|R_n| < 2,271\).

On the whole space \(C\) the inequality(5) is asymptotically exact. At the same there exist subsets of functions from \(C\) and for elements of these subsets the inequality (5) is not exact even by order (see, e.g., [24, p. 435]).

In the paper [10] the following estimate was established
\[
\|\rho_n(f;\cdot)\|_C \leq \sum_{v=0}^{2n-1} \frac{E_v(f)_C}{v-n+1}, \ f \in C, \ n \to \infty,
\]
(here \(K\) is some absolute constant) and it was proved that this constant is exact by the order on the classes \(C(\varepsilon) := \{f \in C : E_n(f)_C \leq \varepsilon_v, \ v \in \mathbb{N}\}\), where \(\varepsilon_v \downarrow 0\) as \(v \to \infty\).

In [6], [7], [14], [22] and [24] the analogs of the Lebesque inequalities for functions \(f \in C_{\beta}^{\alpha}L_p\) have been found in the case \(r \in (0,1)\) and \(p = \infty\), and also in the case \(r \geq 1\) and \(1 \leq p \leq \infty\), where the estimates for the deviations \(\|f(\cdot) - S_{n-1}(f;\cdot)\|_C\) are expressed in terms of the best approximations \(E_n(f^{\alpha r})_C\). Namely, in [24] it was proved that for arbitrary \(f \in C_{\beta}^{\alpha}L_p, r \in (0,1), \beta \in \mathbb{R}\), the following inequality holds
\[
\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \leq \left(\frac{4}{\pi^2} \ln n^{-\tau} + O(1)\right)e^{-\alpha r} E_n(f^{\alpha r})_C,
\]
where \(O(1)\) is a quantity uniformly bounded with respect to \(n, \beta\) and \(f \in C_{\beta}^{\alpha}C\). It was also shown that for any function \(f \in C_{\beta}^{\alpha}C\) and for every \(n \in \mathbb{N}\) one can find a function \(F(\cdot) = F(f; n;\cdot)\) in the set \(C_{\beta}^{\alpha}C\), such that \(E_n(F^{\alpha r})_C = E_n(f^{\alpha r})_C\) and for this function the relation (6) becomes an equality.

Least upper bounds of the quantity \(\|\rho_n(f;\cdot)\|_C\) over the classes \(C_{\beta}^{\alpha}\), we denote by \(E_n(C_{\beta}^{\alpha})_C\), i.e.,
\[
E_n(C_{\beta}^{\alpha})_C = \sup_{f \in C_{\beta}^{\alpha}} \|\rho_n(f;\cdot)\|_C, \ r > 0, \ \alpha > 0, \ 1 \leq p \leq \infty.
\]

Asymptotic behaviour of the quantities \(E_n(C_{\beta}^{\alpha})_C\) of the form (7) was studied in [9], [11], [15]-[17], [19], [20], [23], [25], [27].

The present paper is a continuation of [6], [7], [14], [22] and [24] and is devoted to obtaining of asymptotically best possible analogs of Lebesque-type inequalities on the sets \(C_{\beta}^{\alpha}L_p, r \in (0,1)\) and \(p \in [1, \infty)\). This case has not been considered yet.

It should be also noticed that asymptotically best possible Lebesque inequalities on classes of generalized Poisson integrals \(C_{\beta}^{\alpha}L_p\) for \(r \in (0,1), p = \infty\) and \(r \geq 1, 1 \leq p \leq \infty\) were also established for approximations by Lagrange trigonometric interpolation polynomials with uniform distribution of interpolation nodes (see, e.g., [12], [13], [26]).

2. Main results

Let us formulate the results of the paper.
By \( F(a, b; c; z) \) we denote Gauss hypergeometric function

\[
F(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!},
\]

(8)

\((x)_k := x(x+1)(x+2)...(x+k-1)\).

For arbitrary \( \alpha > 0, r \in (0, 1) \) and \( 1 \leq p < \infty \) we denote by \( n_0 = n_0(\alpha, r, p) \) the smallest integer \( n \) such that

\[
\frac{1}{n} \sum_{k=1}^{n} \frac{1}{r} \leq \lambda \left( \frac{1}{n} \right), \quad 1 < p < \infty.
\]

(9)

The following theorem takes place.

**Theorem 2.1.** Let \( 0 < r < 1, \alpha > 0, \beta \in \mathbb{R} \) and \( n \in \mathbb{N} \). Then in the case \( 1 < p < \infty \) for any function \( f \in C^{a,r}_\beta L_p \) and \( n \geq n_0(\alpha, r, p) \), the following inequality holds

\[
\|f() - S_{n-1}(f; \cdot)\|_C \leq e^{-ar} n^{\frac{1}{2r}} F\left( \frac{1}{2}, 1 - \frac{1}{p}; 1; \frac{3}{2}, 1 \right)
\]

\[
+ \gamma_n(\alpha, r, p) \left( 1 + \frac{1}{n^{\frac{1}{2r}} \left( \frac{1}{p} - 1 \right) + \frac{1}{n^{\frac{1}{2r}}} \right) E_n(\beta, r)_p \left( \frac{1}{p} + \frac{1}{p'} = 1 \right).
\]

(10)

where \( F(a, b; c; z) \) is Gauss hypergeometric function.

Moreover, for any function \( f \in C^{a,r}_\beta L_p(\mathbb{R}) \) one can find a function \( F(x) = \mathcal{F}(f; n; x) \), such that \( E_n(\beta, r)_p = E_n(\beta, r)_p \) and for \( n \geq n_0(\alpha, r, p) \) the following equality holds

\[
\|\mathcal{F}(\cdot) - S_{n-1}(F; \cdot)\|_C \leq e^{-ar} n^{\frac{1}{2r}} F\left( \frac{1}{2}, 1 - \frac{1}{p}; 1; \frac{3}{2}, 1 \right)
\]

\[
+ \gamma_n(\alpha, r, p) \left( 1 + \frac{1}{n^{\frac{1}{2r}} \left( \frac{1}{p} - 1 \right) + \frac{1}{n^{\frac{1}{2r}}} \right) E_n(\beta, r)_p \left( \frac{1}{p} + \frac{1}{p'} = 1 \right).
\]

(11)

In (10) and (11) the quantity \( \gamma_n(\alpha, r, p) \) is such that \( |\gamma_n(\alpha, r, p)| \leq (14\pi)^2 \).

**Proof.** The first part of Theorem 2.1 was proved by the authors in the work [14]. That is why here we will prove only the equality (11).

Denote

\[
p^{(n)}_{\alpha, \beta}(t) := \sum_{k=n}^{\infty} e^{-a \pi} \cos \left( k \frac{\beta \pi}{2} \right), \quad 0 < r < 1, \quad \alpha > 0, \quad \beta \in \mathbb{R}.
\]

(12)

The function \( p^{(n)}_{\alpha, \beta}(t) \) is orthogonal to any trigonometric polynomial \( t^{n-1} \) of degree not greater than \( n - 1 \). Hence, for \( f \in C^{a,r}_\beta L_p(\mathbb{R}) \) and \( 1 \leq p \leq \infty \) for any polynomial \( t^{n-1} \in \tau_{2n-1} \) at every point \( x \in \mathbb{R} \) the following equality holds

\[
\rho_n(f; x) = f(x) - S_{n-1}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta_n(t) p^{(n)}_{\alpha, \beta}(x-t) dt,
\]

(13)

where

\[
\delta_n(x) = \delta_n(\alpha, r, \beta; x) := f^{n-1}(x) - t^{n-1}(x).
\]

(14)
To prove the second part of Theorem 2.1, according to the equality (13), for arbitrary \( \varphi \in L_p \) we should find the function \( \Phi(\cdot) = \Phi(\varphi, n; \cdot) \in L_p \), such that \( E_n(\Phi)_{L_p} = E_n(\varphi)_{L_p} \), and for all \( n \geq n_0(\alpha, r, p) \) the following equality holds

\[
\frac{1}{n} \int_{-\pi}^{\pi} \left| \Phi(t) - t_{n-1}^r(t) \right| dt = e^{-n^{1-r} \pi} \left( \frac{\alpha}{(n-\pi)^{1+r}} \right) \left( \frac{1}{2} \left( 1 - \frac{3}{2} + \frac{3}{1} \right) \right) + y_{n,r} \left( \frac{(1 + \frac{(ar)^{r-1}}{p-1}) \frac{1}{n^{1-r}} \frac{1}{p} + \frac{1}{p} \right)
\]

where \( t_{n-1}^r \) is the polynomial of the best approximation of the degree \( n - 1 \) of the function \( \Phi \) in the space \( L_p, |y_{n,r}| \leq (14\pi)^2 \).

In this case for an arbitrary function \( f \in C_{p}^{\alpha,r} L_p, 1 < p < \infty \), there exists a function \( \Phi(\cdot) = \Phi(f_{\beta}^{\alpha,r}, \cdot) \), such that \( E_n(\Phi)_{L_p} = E_n(f_{\beta}^{\alpha,r})_{L_p} \), and for \( n \geq n_0(\alpha, r, p) \) the formula (15) holds, where as function \( \varphi \) we take the function \( f_{\beta}^{\alpha,r} \).

Let us assume

\[ \mathcal{F}(\cdot) = \mathcal{F}_{\beta}(\cdot - \frac{a_0}{2}) \]

where

\[ a_0 = a_0(\Phi) := \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t) dt. \]

The function \( \mathcal{F} \) is the function, which we have looked for because \( \mathcal{F} \in C_{p}^{\alpha,r} L_p \) and

\[ E_n(f_{\beta}^{\alpha,r})_{L_p} = E_n(\Phi - \frac{a_0}{2})_{L_p} = E_n(\Phi)_{L_p} = E_n(f_{\beta}^{\alpha,r})_{L_p}, \]

so (13), (10) and (15) imply (11).

At last, let us prove (15). Let \( \varphi \in L_p, 1 < p < \infty \). Then as a function \( \Phi(\cdot) \) we consider the function

\[ \Phi(t) = \left| P_{\alpha,r,p}^{(n)}(t) \right|^{-1} \left| P_{\alpha,r,p}^{(n)}(t)^{r-1} \right| \text{sign}(P_{\alpha,r,p}^{(n)}(t)) E_n(\varphi)_{L_p} \]

(16)

For this function we have that

\[ \|\Phi\|_p = \left| P_{\alpha,r,p}^{(n)}\right|^{-1} \left| P_{\alpha,r,p}^{(n)}\right|^{r-1} \left| E_n(\varphi)_{L_p}\right| \]

Now we show that the polynomial \( t_{n-1}^r \) of best approximation of the degree \( n - 1 \) in the space \( L_p \) of the function \( \Phi(t) \) equals identically to zero: \( t_{n-1}^r \equiv 0 \).

For any \( t_{n-1}^r \in \tau_{2n-1} \)

\[
\int_{-\pi}^{\pi} t_{n-1}^r(t) \Phi(t)^{p-1} \text{sign}(\Phi(t)) dt = \left| P_{\alpha,r,p}^{(n)} \right|^{-1} \left| E_n(\varphi)_{L_p}\right| \int_{-\pi}^{\pi} t_{n-1}^r(t) P_{\alpha,r,p}^{(n)}(t) dt = 0.
\]

Then, according to Proposition 1.4.12 of the work [5, p. 29] we can make conclusion that the polynomial \( t_{n-1}^r \equiv 0 \) is the polynomial of the best approximation of the function \( \Phi(t) \) in the space \( L_p, 1 < p < \infty \).
For the function $\Phi(t)$ of the form (16) we can write
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t) - t_{n-1}'(t)P_{n,\beta}^{(\alpha)}(-t)dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t)P_{n,\beta}^{(\alpha)}(-t)dt
\]
\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t)P_{n,\beta}^{(\alpha)}(-t)dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t)P_{n,\beta}^{(\alpha)}(t)dt
\]
\[
= \frac{1}{\pi} \|P_{n,\beta}^{(\alpha)}\|_{p}^{1-p'} E_n(\varphi) \gamma_{n,p}^2 \int_{-\pi}^{\pi} |P_{n,\beta}^{(\alpha)}(t)|^{p'} dt = \frac{1}{\pi} \|P_{n,\beta}^{(\alpha)}\|_{p}^{1-p'} E_n(\varphi) \gamma_{n,p}^2.
\] (17)

It follows from the relation (18) of the work [14] that for $n \geq n_0(\alpha, r, p)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, the following equality holds
\[
\frac{1}{\pi} \|P_{n,\beta}^{(\alpha)}\|_{p}^{1-p'} e^{-an^2} n^{\frac{3p'}{2p} - 1} F_{\pi} \left( \frac{1}{2}, 3 - \frac{p'}{2}, \frac{3}{2}, 1 \right) + \frac{n^2}{(ar)^4} \left( \frac{1}{n} + \frac{\gamma_{n,p}^{(2)}}{p' - 1} \right)
\] (18)

where the quantities $\gamma_{n,p}^{(2)} = \gamma_{n,p}^{(2)}(\alpha, r, \beta)$, satisfy the inequality $|\gamma_{n,p}^{(2)}| \leq (14\pi)^2$.

Thus, from (18) and (17) we arrive at the equality (11). Theorem 2.1 is proved. □

**Theorem 2.2.** Let $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$, $n \in \mathbb{N}$. Then, for any $f \in C^{0,r}_\beta L_1$ and $n \geq n_0(\alpha, r, 1)$ the following inequality holds:
\[
\|f(-) - S_{n-1}(f; \cdot)\|_C \leq e^{-an^2} n^{1-r} \left( \frac{1}{\pi ar} + \gamma_{n,1}(\frac{1}{(ar)^2} n^r + \frac{1}{n}) \right) E_n(f^{0,r}_\beta)_{L_1}.
\] (19)

Moreover, for any function $f \in C^{0,r}_\beta L_1$ one can find a function $F(x) = F(f; n, x)$ in the set $C^{0,r}_\beta L_1$, such that $E_n(F)_{L_1} = E_n(f^{0,r}_\beta)_{L_1}$ and for $n \geq n_0(\alpha, r, 1)$ the following equality holds
\[
\|F(-) - S_{n-1}(F; \cdot)\|_C \leq e^{-an^2} n^{1-r} \left( \frac{1}{\pi ar} + \gamma_{n,1}(\frac{1}{(ar)^2} n^r + \frac{1}{n}) \right) E_n(F^{0,r}_\beta)_{L_1}.
\] (20)

In (19) and (20) the quantity $\gamma_{n,1} = \gamma_{n,1}(\alpha, r, \beta)$ is such that $|\gamma_{n,1}| \leq (14\pi)^2$.

**Proof.** The first part of Theorem 2.2 was proved in [14].

So let us prove the second part of Theorem 2.2. For this we need for any function $\varphi \in L_1$ to find the function $\Phi(\cdot) = \Phi(\varphi; \cdot) \in L_1$, such that $E_n(\Phi)_{L_1} = E_n(\varphi)_{L_1}$ and for all $n \geq n_0(\alpha, r, 1)$ the following equality holds
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \left( \Phi(t) - t_{n-1}'(t)P_{n,\beta}^{(\alpha)}(0-t) \right) dt = e^{-an^2} n^{1-r} \left( \frac{1}{\pi ar} + \gamma_{n,1}(\frac{1}{(ar)^2} n^r + \frac{1}{n}) \right) E_n(\varphi)_{L_1},
\] (21)

where $t_{n-1}'$ is the polynomial of the best approximation of degree $n - 1$ of the function $\Phi$ in the space $L_1$ and $|\gamma_{n,1}| \leq (14\pi)^2$.

In this case for any function $f \in C^{0,r}_\beta L_1$ there exists a function $\Phi(\cdot) = \Phi(f^{0,r}_\beta; \cdot)$, such that $E_n(\Phi)_{L_1} = E_n(f^{0,r}_\beta)$, and for $n \geq n_0(\alpha, r, 1)$ the formula (21) holds, where as function $\varphi$ we will take the function $f^{0,r}_\beta$. 

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Let us consider the function
\[ F(t) = \mathcal{J}^{\alpha, r}_\beta (\Phi(t) - \frac{a_0}{2}), \]
where
\[ a_0 = a_0(\Phi) := \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t) dt. \]

The function \( F \) is the function, which we look for, because \( F \in C^{\alpha, r}L_1 \) and
\[ E_n(F^{\alpha, r})_{\ell_1} = E_n(\Phi - \frac{a_0}{2})_{\ell_1} = E_n(\Phi)_{\ell_1} = E_n(f_n^{\alpha, r})_{\ell_1}, \]
and on the basis (13), (19) and (21) the formula (20) holds.

Let us prove (21). Let \( t^* \) be the point from the interval \( T = \left[ \frac{\pi(1-\beta)}{2n}, 2\pi + \frac{\pi(1-\beta)}{2n} \right) \), where the function \( |p^{(n)}_{a, \alpha, r, \beta}| \) attains its largest value, i.e.,
\[ |p^{(n)}_{a, \alpha, r, \beta}(t^*)| = \|p^{(n)}_{a, \alpha, r, \beta}\|_C = \|p^{(n)}_{a, \alpha, r, \beta}\|_C. \]

Let put \( \Delta^n_k := \left[ \frac{(k-1)n}{n}, \frac{\pi(1-\beta)}{2n} + \frac{(k-1)n}{n} \right), k = 1, ..., 2n. \) By \( k^* \) we denote the number, such that \( t^* \in \Delta^n_{k^*}. \)
Taking into account, that function \( p^{(n)}_{a, \alpha, r, \beta} \) is absolutely continuous, so for arbitrary \( \varepsilon > 0 \) there exists a segment \( \ell^* = [\xi^*, \xi^* + \delta] \subset \Delta^n_{k^*}, \) such that for arbitrary \( t \in \ell^* \) the following inequality holds
\[ |p^{(n)}_{a, \alpha, r, \beta}(t)| > \|p^{(n)}_{a, \alpha, r, \beta}\|_C - \varepsilon. \] It is clear that mes \( \ell^* = |\ell^*| = \delta < \frac{\pi}{n}. \)

For arbitrary \( \varphi \in L_1 \) and \( \varepsilon > 0 \) we consider the function \( \Phi_\varepsilon(t), \) which on the segment \( T \) is defined with a help of equalities
\[ \Phi_\varepsilon(t) = \begin{cases} E_n(\varphi)_{\ell_1}, & t \in \ell^*, \\ E_n(\varphi)_{\ell_1} \varepsilon \sin \cos \left( nt + \frac{\beta\pi}{2} \right), & t \in T \setminus \ell^*. \end{cases} \]

For the function \( \Phi_\varepsilon(t) \) for arbitrary small values of \( \varepsilon > 0 \) (\( \varepsilon \in (0, \frac{1}{2\pi}) \)) the following equality holds
\[
\|\Phi_\varepsilon\|_1 = E_n(\varphi)_{\ell_1} \frac{1 - \varepsilon(2\pi - \delta)}{\delta} \int_{\ell^*} \left| \sin \cos \left( nt + \frac{\beta\pi}{2} \right) \right| dt \\
+ E_n(\varphi)_{\ell_1} \varepsilon \int_{T \setminus \ell^*} \left| \sin \cos \left( nt + \frac{\beta\pi}{2} \right) \right| dt \\
= E_n(\varphi)_{\ell_1} \left( \frac{1 - \varepsilon(2\pi - \delta)}{\delta} \delta + \varepsilon(2\pi - \delta) \right) = E_n(\varphi)_{\ell_1}. \quad (22)
\]

It should be noticed that
\[ \text{sign} \Phi_\varepsilon(t) = \text{sign} \cos \left( nt + \frac{\beta\pi}{2} \right). \quad (23) \]

Since for arbitrary trigonometric polynomial \( t_{n-1} \in \tau_{2n-1} \)
\[ \int_{0}^{2\pi} t_{n-1}(t) \text{sign} \cos \left( nt + \frac{\beta\pi}{2} \right) dt = 0, \]
so, taking into account (23)
\[ \int_0^{2\pi} t_{n-1}(t) \text{sign} \left( \Phi_n(t) - 0 \right) dt = 0, \quad t_{n-1} \in \tau_{2n-1}. \]

According to Proposition 1.4.12 of the work [5, p.29], the polynomial $t'_{n-1} \equiv 0$ is a polynomial of the best approximation of the function $\Phi_n$ in the metric of the space $L_1$, i.e., $E_n(\Phi_n)_{L_1} = \|\Phi_n\|_1$, so (22) yields $E_n(\Phi_n)_{L_1} = E_n(\varphi)_{L_1}$.

Moreover, for the function $\Phi_n$

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} (\Phi_n(t) - t'_{n-1}(t)) p_{a,\pi,\beta}^{(n)}(-t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_n(t) p_{a,\pi,\beta}^{(n)}(t) dt \]
\[ = \frac{1 - \varepsilon(2\pi - \delta)}{\pi \delta} E_n(\varphi)_{L_1} \int_{\pi} \text{sign} \cos \left( nt + \frac{\beta \pi t}{2} \right) p_{a,\pi,\beta}^{(n)}(t) dt \]
\[ + \frac{\varepsilon}{\pi} E_n(\varphi)_{L_1} \int_{\pi} \text{sign} \cos \left( nt + \frac{\beta \pi t}{2} \right) p_{a,\pi,\beta}^{(n)}(t) dt. \]

(24)

Taking into account that $\text{sign} \Phi_n(t) = (-1)^k$, $t \in \Delta_k^{(n)}$, $k = 1, \ldots, 2n$, and also the embedding $t' \subset \Delta_k^{(n)}$, we get

\[ \left| \frac{1 - \varepsilon(2\pi - \delta)}{\pi \delta} E_n(\varphi)_{L_1} \int_{\pi} \text{sign} \cos \left( nt + \frac{\beta \pi t}{2} \right) p_{a,\pi,\beta}^{(n)}(t) dt \right| \]
\[ = \left| (-1)^k \frac{1 - \varepsilon(2\pi - \delta)}{\pi \delta} E_n(\varphi)_{L_1} \int_{\pi} p_{a,\pi,\beta}^{(n)}(t) dt \right| \]
\[ \geq \frac{1 - \varepsilon(2\pi - \delta)}{\pi} E_n(\varphi)_{L_1} \left( \| p_{a,\pi,\beta}^{(n)} \|_C - \varepsilon \right) \]
\[ > \frac{1 - 2\pi \varepsilon}{\pi} E_n(\varphi)_{L_1} \left( \| p_{a,\pi,\beta}^{(n)} \|_C - \varepsilon \right) \]
\[ = \frac{1}{\pi} E_n(\varphi)_{L_1} \left( \| p_{a,\pi,\beta}^{(n)} \|_C - 2\pi \varepsilon \| p_{a,\pi,\beta}^{(n)} \|_C - \varepsilon + 2\pi \varepsilon^2 \right) \]
\[ > E_n(\varphi)_{L_1} \left( \frac{1}{\pi} \| p_{a,\pi,\beta}^{(n)} \|_C - \varepsilon \left( 2\| p_{a,\pi,\beta}^{(n)} \|_C + \frac{1}{\pi} \right) \right). \]

(25)

Also, it is not hard to see that

\[ \left| \frac{\varepsilon}{\pi} E_n(\varphi)_{L_1} \int_{\pi} \text{sign} \cos \left( nt + \frac{\beta \pi t}{2} \right) p_{a,\pi,\beta}^{(n)}(t) dt \right| \leq \frac{\varepsilon}{\pi} E_n(\varphi)_{L_1} \| p_{a,\pi,\beta}^{(n)} \|_C. \]

(26)

Formulas (24)–(26) yield the following inequality

\[ \left| \int_{-\pi}^{\pi} \frac{1}{\pi} (\Phi_n(t) - t'_{n-1}(t)) p_{a,\pi,\beta}^{(n)}(-t) dt \right| \]
\[ > E_n(\varphi)_{L_1} \left( \frac{1}{\pi} \| p_{a,\pi,\beta}^{(n)} \|_C - \varepsilon \left( 2 + \frac{1}{\pi} \right) \| p_{a,\pi,\beta}^{(n)} \|_C + \frac{1}{\pi} \right). \]

(27)
It should be noticed that asymptotic estimate for the quantity \( \| P_{a,\beta}^{(n)} \|_\infty \) was obtained in [17]. Let us show that this estimate can be improved, if we decrease the diapason for the remainder.

Formulas (34), (50)–(52) of the work [17], and also Remark 1 from [17] allow us to write that for any \( n \in \mathbb{N} \)

\[
\| P_{a,\beta}^{(n)} \|_\infty = \| P_{a,\beta,0} \|_\infty \left( 1 + \delta_n^{(1)} \frac{M_n}{n} \right),
\]

(28)

where

\[
P_{a,\beta,0}(t) := \sum_{k=0}^{\infty} e^{-\alpha(t+n') \varphi},
\]

\[
M_n := \sup_{s \in \mathbb{R}} \| P_{a,\beta,0}(t) \|,
\]

and for \( \delta_n^{(1)} = \delta_n^{(1)}(a, r, \beta) \) the following estimate takes place \( |\delta_n^{(1)}| \leq 5 \sqrt{2} \pi \).

Then, as it follows from the estimates (87) and (99) of the work [17] for \( n \geq n_0(a, r, 1) \)

\[
\| P_{a,r,\beta,0} \|_\infty = \frac{e^{-\alpha}}{\alpha} n^{1-r} \left( 1 + \theta_{a,\beta,0} \left( \frac{1-r}{\alpha} + \frac{\alpha}{n^{1-r}} \right) \right), \quad |\theta_{a,\beta,0}| \leq \frac{14}{13}
\]

(29)

and

\[
M_n \leq \frac{784 \pi^2}{117} \left( \frac{1}{\alpha} + \frac{\alpha}{n^{1-r}} \right).
\]

(30)

Combining formulas (28)–(30), we obtain that for \( n \geq n_0(a, r, 1) \)

\[
\frac{1}{\pi} \| P_{a,\beta}^{(n)} \|_\infty = \frac{e^{-\alpha}}{\alpha} n^{1-r} \left( 1 + \theta_{a,\beta,0} \left( \frac{1-r}{\alpha} + \frac{\alpha}{n^{1-r}} \right) \right) \left( 1 + \delta_n^{(1)} \frac{M_n}{n} \right)
\]

\[
= e^{-\alpha} n^{1-r} \left( \frac{1}{\alpha} + \gamma_{n,1} \left( \frac{1}{(\alpha)\beta} + \frac{\alpha}{n^{1-r}} \right) \right),
\]

(31)

where

\[
|\gamma_{n,1}| \leq \frac{1}{\pi} \left( \frac{14}{13} \right) \frac{784 \pi^2}{117} \frac{\sqrt{2} \pi}{\alpha} + \frac{14 \cdot 5 \sqrt{2} \pi \cdot 784 \pi^2}{13 \cdot 117 \cdot 14} = \frac{14}{13 \pi} \left( 1 + \frac{3920 \sqrt{2} \pi^3}{117} \right).
\]

(32)

Let us choose \( \varepsilon \) small enough that

\[
\varepsilon < \frac{\left( (14\pi)^2 - \frac{14}{13 \pi} \left( 1 + \frac{3920 \sqrt{2} \pi^3}{117} \right) \right) e^{-\alpha} n^{1-r} \left( \frac{1}{\alpha} + \frac{\alpha}{n^{1-r}} \right)}{(2 + \frac{1}{\pi}) \| P_{a,\beta}^{(n)} \|_\infty + \frac{1}{\pi}}
\]

(33)

and for this \( \varepsilon \) we put

\[
\Phi(t) = \Phi_{\varepsilon}(t).
\]

(34)

The function \( \Phi(t) \) is the function, which we have looked for, because \( E_n(\Phi)_{L_1} = E_n(\varphi)_{L_1} \) and according to (27), (31)–(33) for \( n \geq n_0(a, r, 1) \)

\[
\left| \frac{1}{\pi} \left( \Phi(t) - t_{n-1}^*(t) \right) P_{a,\beta}^{(n)}(-t) dt \right|
\]

\[
> E_n(\varphi)_{L_1} \left( \frac{1}{\pi} \| P_{a,\beta}^{(n)} \|_{L_1} - (14\pi)^2 \left( \frac{14}{13 \pi} \frac{1 + 3920 \sqrt{2} \pi^3}{117} \right) e^{-\alpha} n^{1-r} \left( \frac{1}{\alpha} + \frac{\alpha}{n^{1-r}} \right) \right)
\]

\[
\geq e^{-\alpha} n^{1-r} \left( \frac{1}{\alpha} - (14\pi)^2 \left( \frac{1}{(\alpha)^2 n^{1-r}} + \frac{\alpha}{n^{1-r}} \right) \right) E_n(\varphi)_{L_1}.
\]

(35)
On the other hand, according to (13) for $f \in C_{\beta}^{s,r} L_1$ we get

$$\|f(-S_{n-1}(f))\|_C = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( f_{\alpha}^{s,r}(t) - t^*_n(t) \right) p_{\alpha,\beta}(x - t) dt \leq \frac{1}{\pi} \|p_n\|_\infty E_n(f_{\alpha}^{s,r})_{\ast,1},$$

(36)

where $t^*_n$ is the polynomial of the best approximation of the function $f_{\alpha}^{s,r}$ in the space $L_1$.

Formulas (35), (36), (31) and (32) imply (21). Theorem 2.2 is proved.

As it was already mentioned, the inequalities (10) and (19) were proved in [14]. At the same time the problem about asymptotically best possible upper estimates of uniform norms of deviations of partial Fourier sums of the function $f$ from $C_{\beta}^{s,r} L_1$, $1 \leq p < \infty$, remains open. Theorems 2.1 and 2.2 give the full answer on this question: the asymptotic equalities (11) and (20) prove that the estimates (10) and (19) are asymptotically best possible for functions from $C_{\beta}^{s,r} L_1$ in the cases $1 < p < \infty$ and $p = 1$ respectively. At the very end, we notice that inequalities (10) and (19) are asymptotically best possible on such important subsets from $C_{\beta}^{s,r} L_1$ as sets $C_{\beta,p}^{s,r}$, $1 \leq p < \infty$.

Indeed, if $f \in C_{\beta,p}^{s,r}$ then $\|f_{\alpha}^{s,r}\|_p \leq 1$ and $E_n(f_{\alpha}^{s,r})_{\ast,1} \leq 1$, $1 \leq p < \infty$. Considering the least upper bounds of both sides of inequality (10) over the classes $C_{\beta,p}^{s,r}$, $1 < p < \infty$, we arrive at the inequality

$$E_n(C_{\beta,p}^{s,r}) \leq e^{-\pi n^{\mu'}} \left( \frac{\|f\|_p}{n^{\frac{1}{p} + \frac{1}{p'}}} \left( \frac{1}{2} + \frac{3}{2} \right) \right) + \gamma n^p \left( \frac{1}{p'} - 1 \right) \left( \frac{1}{n^{\frac{1}{p'}}} + \left( \frac{p'}{\|p\|_1} \right)^{\frac{1}{r}} \right) \frac{1}{p} + \frac{1}{p'} = 1.$$

(37)

Comparing this relation with the estimate of Theorem 4 from [16] (see also [17]), we conclude that inequality (10) on the classes $C_{\beta,p}^{s,r}$, $1 < p < \infty$, is asymptotically best possible.

In the same way, the asymptotic sharpness of the estimate (19) on the classes $C_{\beta,1}^{s,r}$ follows from comparing inequality

$$E_n(C_{\beta,1}^{s,r}) \leq e^{-\pi n^{\mu'}} \left( \frac{1}{n^{\frac{1}{p} + \frac{1}{p'}}} \left( \frac{1}{2} + \frac{3}{2} \right) \right) + \gamma n^p \left( \frac{1}{p'} - 1 \right) \left( \frac{1}{n^{\frac{1}{p'}}} + \left( \frac{p'}{\|p\|_1} \right)^{\frac{1}{r}} \right) \frac{1}{p} + \frac{1}{p'} = 1.$$

(38)

and formula (18) from [17].

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