Research Article

Dohoon Choi and Subong Lim*

Schneider–Siegel theorem for a family of values of a harmonic weak Maass form at Hecke orbits

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Abstract: Let \( j(z) \) be the modular \( j \)-invariant function. Let \( \tau \) be an algebraic number in the complex upper half plane \( \mathbb{H} \). It was proved by Schneider and Siegel that if \( \tau \) is not a CM point, i.e., \( [\mathbb{Q}(\tau) : \mathbb{Q}] \neq 2 \), then \( j(\tau) \) is transcendental. Let \( f \) be a harmonic weak Maass form of weight 0 on \( \Gamma_0(N) \). In this paper, we consider an extension of the results of Schneider and Siegel to a family of values of \( f \) on Hecke orbits of \( \tau \). For a positive integer \( m \), let \( T_m \) denote the \( m \)-th Hecke operator. Suppose that the coefficients of the principal part of \( f \) at the cusp \( i\infty \) are algebraic, and that \( f \) has its poles only at cusps equivalent to \( i\infty \). We prove, under a mild assumption on \( f \), that, for any fixed \( \tau \), if \( N \) is a prime such that \( N \geq 23 \) and \( N \notin \{23, 29, 31, 41, 47, 59, 71\} \), then \( f(T_m.\tau) \) are transcendental for infinitely many positive integers \( m \) prime to \( N \).

Keywords: Harmonic weak Maass form, CM point, meromorphic differential

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1 Introduction

Let \( j(z) \) be the modular \( j \)-invariant function on the complex upper half plane \( \mathbb{H} \). Let \( \tau \) be an algebraic number in \( \mathbb{H} \). It was proved by Kronecker [10] and Weber [18] that if \( \tau \) is a CM point, i.e., \( [\mathbb{Q}(\tau) : \mathbb{Q}] = 2 \), then \( j(\tau) \) is algebraic. Schneider [15] and Siegel [17] proved that if \( \tau \) is not a CM point, then \( j(\tau) \) is transcendental. By combining these two results, we state the following.

**Theorem A** (Kronecker, Schneider, Siegel, Weber). Assume that \( \tau \) is an algebraic number in \( \mathbb{H} \). Then \( \tau \) is a CM point if and only if \( j(\tau) \) is algebraic.

Let \( m \) be a positive integer, and let \( T_m \) denote the \( m \)-th Hecke operator. The operators \( T_m \) act on both of modular forms \( f \) and divisors \( D \) of a modular curve, and they are denoted by \( j|T_m \) and \( T_m.D \), respectively. Then \( j(T_m.\tau) = (j|T_m)(\tau) \), and \( (j|T_m)(z) \) is a polynomial of \( j(z) \) with rational coefficients. Thus, \( j(T_m.\tau) \) is algebraic for every \( m \) if and only if \( j(\tau) \) is algebraic. Therefore, Theorem A is equivalent to the following theorem.

**Theorem B** (Kronecker, Schneider, Siegel, Weber). Assume that \( \tau \) is an algebraic number in \( \mathbb{H} \). Then \( \tau \) is a CM point if and only if \( j(T_m.\tau) \) is algebraic for every positive integer \( m \).

*Corresponding author: Subong Lim, Department of Mathematics Education, Sungkyunkwan University, Jongno-gu, Seoul 03063, Republic of Korea, e-mail: subong@skku.edu. http://orcid.org/0000-0003-2768-6172
Dohoon Choi, Department of Mathematics, Korea University, Seoul 02841; and School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea, e-mail: dohoonchoi@korea.ac.kr
In this vein, we consider an extension of the results of Kronecker, Schneider, Siegel and Weber to a family of values of a harmonic weak Maass form \( f \) on Hecke orbits of \( \tau \). Let \( N \) be a positive integer and \( f \) a harmonic weak Maass form of weight 0 on \( \Gamma_0(N) \). In contrast to the case for the \( j \)-invariant function, the value of \( f \) at a CM point \( \tau \) is not algebraic in general. Thus, first we obtain the period of \( f(\tau) \) for a CM point \( \tau \), which is expressed as the regularized Petersson inner product of a cusp form and a meromorphic modular form. Next, by using this result, we obtain an extension of the results of Kronecker, Schneider, Siegel and Weber to a family of values of \( f \) on Hecke orbits of \( \tau \).

Let \( Y_0(N) \) be the modular curve of level \( N \) defined by \( \Gamma_0(N) \setminus \mathbb{H} \), and let \( X_0(N) \) denote the compactification of \( Y_0(N) \) by adjoining the cusps. Let us note that \( X_0(N) \) is a curve defined over \( \mathbb{Q} \). We fix an algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \). Let \( K \) be a subfield of \( \overline{\mathbb{Q}} \) or the field \( \mathbb{C} \) of complex numbers. Let \( C \) be a curve defined over \( K \). For an extension \( E \) of \( K \), we denote by \( \text{Div}_C(E) \) the group of divisors of \( C \) defined over \( E \). Let \( \mathcal{E} \) be a function on \( C(\mathbb{C}) \setminus S \) for a finite subset \( S \) of \( C(\mathbb{C}) \). If \( D = \sum_{P \in C} n_P P \in \text{Div}_C(\mathbb{C}) \) and the support of \( D \) does not contain any point in \( S \), then we define \( f(D) := \sum n_P f(P) \). The \( m \)-th Hecke operator \( T_m \) acts on \( \text{Div}_{X_0(N)}(\mathbb{C}) \), and it is denoted by \( T_m \).

Let \( k \) be a non-negative even integer. Let \( S_k(\Gamma_0(N)) \) denote the space of cusp forms of weight \( k \) on \( \Gamma_0(N) \). We denote by \( H_k(\Gamma_0(N)) \) the space of harmonic weak Maass forms of weight \( k \) on \( \Gamma_0(N) \). For the differential operator \( \xi_k \) defined by \( \xi_k(f)(z) := 2iy^{-k} \frac{\partial}{\partial y} f(z) \), the assignment \( f(z) \mapsto \xi_k(f)(z) \) gives an anti-linear mapping \( \xi_k : H_k(\Gamma_0(N)) \to M_{k+2}(\Gamma_0(N)) \), where \( M_k(\Gamma_0(N)) \) denotes the space of weakly holomorphic modular forms of weight \( k \) on \( \Gamma_0(N) \). Here, \( y \) denotes the imaginary part of \( z \in \mathbb{H} \). Let \( H^+_k(\Gamma_0(N)) \) be the inverse image of the space \( S_{k+2}(\Gamma_0(N)) \) of cusp forms under the mapping \( \xi_k \).

**Definition 1.1.** Let \( f \) be a harmonic weak Maass form of weight 0 on \( \Gamma_0(N) \). We say that \( f \) is arithmetic if \( f \) satisfies the following conditions:

1. the principal part of \( f \) at the cusp \( i\infty \) belongs to \( \overline{\mathbb{Q}}[q^{-1}] \), and its constant term is zero;
2. the principal part of \( f \) at each cusp not equivalent to \( i\infty \) is constant.

Here, \( q := e^{2\pi i \tau} \) for a complex number \( \tau \in \mathbb{H} \).

This definition is similar to that of being good in [6]; however, the conditions in this definition are weaker than those in the definition of being good. Furthermore, a harmonic weak Maass form \( f \) is called an arithmetic Hecke eigenform if \( f \) is arithmetic and \( \xi_0(f) \) is a Hecke eigenform.

For \( \tau \in \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q} \), let \( Q_\tau \) be the image of \( \tau \) under the canonical map from \( \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q} \) to \( X_0(N) \) and \( D_\tau := Q_{i\infty} - Q_\tau \in \text{Div}_{X_0(N)}(\mathbb{C}) \).

If \( \tau \) is a CM point, then \( D_\tau \) is defined over \( \overline{\mathbb{Q}} \). Thus, there exists a differential \( \psi_{D_\tau}^{\text{reg}} \) of the third kind associated to \( D_\tau \) such that \( \psi_{D_\tau}^{\text{reg}} \) is defined over \( \overline{\mathbb{Q}} \). Note that \( \psi_{D_\tau}^{\text{reg}} \) can be written as \( \psi_{D_\tau}^{\text{reg}} = 2\pi i f_{\psi_{D_\tau}^{\text{reg}}}(z) \ dz \) for some meromorphic modular form \( f_{\psi_{D_\tau}^{\text{reg}}} \) of weight 2 on \( \Gamma_0(N) \) (see Section 2.3 for details). Let \( (\xi_0(f), f_{\psi_{D_\tau}^{\text{reg}}})_{\text{reg}} \) be the regularized Petersson inner product of \( \xi_0(f) \) and \( f_{\psi_{D_\tau}^{\text{reg}}} \) (see Section 3 for the definition of the regularized Petersson inner product). The following theorem shows that, for each positive integer \( m \) prime to \( N \), the period of \( f(T_m \cdot Q_\tau) \) can be expressed as the multiplication of \( (\xi_0(f), f_{\psi_{D_\tau}^{\text{reg}}})_{\text{reg}} \) and the eigenvalue of \( \xi_0(f) \) for \( T_m \).

**Theorem 1.2.** Let \( N \) be a prime and \( f \) a harmonic weak Maass form of weight 0 on \( \Gamma_0(N) \). Assume that \( f \) is an arithmetic Hecke eigenform and that \( \tau \) is a CM point. Let \( \psi_{D_\tau}^{\text{reg}} := 2\pi i f_{\psi_{D_\tau}^{\text{reg}}}(z) \ dz \) be a differential of the third kind associated to \( D_\tau \) defined over \( \overline{\mathbb{Q}} \). Then

\[
f(T_m \cdot Q_\tau) - m^{-1} \lambda_m(\xi_0(f), f_{\psi_{D_\tau}^{\text{reg}}})_{\text{reg}}
\]

is algebraic for every positive integer \( m \) prime to \( N \), where \( \lambda_m \) is the eigenvalue of \( \xi_0(f) \) for \( T_m \).

Theorem 1.2 is applied to study the transcendence of \( f(T_m \cdot Q_\tau) \) for an algebraic number \( \tau \) in \( \mathbb{H} \). Then we have the following theorem concerning an extension of the above results of Kronecker, Schneider, Siegel and Weber to a family of values of \( f \) on Hecke orbits of \( \tau \).
Theorem 1.3. Let \( N \) and \( f \) be given as in Theorem 1.2. Assume that \((g, \zeta_0(f)) \neq 0\) for each Hecke eigenform \( g \in S_2(\Gamma) \). Let \( \tau \) be an algebraic number in \( \mathbb{H} \). Then \( f(T_m, Q_\tau) \) is algebraic for every positive integer \( m \) prime to \( N \) and if only if \( \tau \) is not a CM point and \( nD \) is rational on \( X_0(N) \) for some positive integer \( n \).

Remark 1.4. Assume that \( f \) and \( \tau \) are given as in Theorem 1.3. In fact, if there is a positive integer \( m \) prime to \( N \) such that \( f(T_m, Q_\tau) \) is transcendental, then there are infinitely many such positive integers \( m \) prime to \( N \) (see the proof of Theorem 1.3 in Section 4).

Let \( J_{\Gamma_0(N)} \) be the Jacobian variety of the \( X_0(N) \) defined over \( \mathbb{Q} \). An Albanese embedding \( i_{Q_\infty} : X_0(N) \to J_{\Gamma_0(N)} \) can be defined by sending \( Q \) to \( Q_{\infty} - Q \). Let us note that \( m(Q_{\infty} - Q) \) is rational on \( X_0(N) \) for some positive integer \( m \) if and only if \( i_{Q_\infty}(Q) \) is a torsion point in \( J_{\Gamma_0(N)} \). Let

\[
T_{Q_\infty}(X_0(N)) := \{ Q \in X_0(N)(\mathbb{Q}) \mid i_{Q_\infty}(Q) \text{ is a torsion point in } J_{\Gamma_0(N)} \}.
\]

If the genus of \( X_0(N) \) is larger than or equal to 2, then, by the Mumford–Manin conjecture (proved by Raynaud [13, 14]), \( T_{Q_\infty}(X_0(N)) \) is a finite set.

Let \( X'_0(N) \) denote the quotient of \( X_0(N) \) by the Atkin–Lehner involution \( w_N \). For primes \( N \), Coleman, Kaskel and Ribet [8] conjectured the following statement: for all prime numbers \( N \geq 23 \),

\[
T_{Q_\infty}(X_0(N)) = \begin{cases} 
\{ 0, \infty \} & \text{if } g^* > 0, \\
\{ 0, \infty \} \cup \{ \text{hyperelliptic branch points} \} & \text{if } g^* = 0,
\end{cases}
\]

where \( g^* \) denotes the genus of \( X_0(N)^* \). Baker [2] proved this conjecture. Furthermore, for \( N \geq 23 \), \( g^* \) is zero if and only if \( N \in \{ 23, 29, 31, 41, 47, 59, 71 \} \). Thanks to these results on torsion points on the Jacobian of a modular curve, we obtain the following theorem from Theorem 1.3.

Theorem 1.5. Under the assumption as in Theorem 1.3, assume that

\[
N \geq 23 \text{ and } N \notin \{ 23, 29, 31, 41, 47, 59, 71 \}.
\]

Then \( f(T_m, Q_\tau) \) are transcendental for infinitely many positive integers \( m \) prime to \( N \).

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminaries for harmonic weak Maass forms, residues of meromorphic differentials on a modular curve, and differentials of the third kind on a complex curve. In Section 3, we review the definition of a regularized Petersson inner product and prove that the regularized Petersson inner product of a meromorphic modular form, associated with a canonical differential of the third kind of some divisor on \( X_0(N) \), with every cusp form of weight 2 on \( \Gamma_0(N) \) is zero. In Section 4, we prove Theorem 1.2 and 1.3.

2 Preliminaries

In this section, we recall definitions and basic facts about harmonic weak Maass forms, residues of meromorphic differentials on a modular curve, and properties for differentials of the third kind on a complex curve.

2.1 Harmonic weak Maass forms

For details of harmonic weak Maass forms, we refer to [4, 12]. Let \( k \) be an even integer. We recall the weight \( k \) slash operator \((f_k)_y(y) := (cz + d)^{-k}f(yz)\) for any function \( f \) on \( \mathbb{H} \) and \( y = (a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z}) \). Let \( \Delta_k \) denote the weight \( k \) hyperbolic Laplacian defined by

\[
\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + \frac{i}{y} \frac{\partial}{\partial y} \right),
\]

where \( x \) (resp. \( y \)) denotes the real (resp. imaginary) part of \( z \). Now, we give the definition of a harmonic weak Maass form.
**Definition 2.1.** Let $N$ be a positive integer. A smooth function $f$ on $\mathbb{H}$ is a harmonic weak Maass form of weight $k$ on $\Gamma_0(N)$ if it satisfies the following conditions:

1. $f|_k y = f$ for all $y \in \Gamma_0(N),$
2. $\Delta_k f = 0,$
3. a linear exponential growth condition in terms of $y$ at every cusp of $\Gamma_0(N).

We denote by $H_k(\Gamma_0(N))$ the space of harmonic weak Maass forms of weight $k$ on $\Gamma_0(N)$.

Assume that $t$ is a cusp of $\Gamma_0(N).$ Let $\sigma_t \in \text{SL}_2(\mathbb{Z})$ be a matrix such that $\sigma_t(i\infty) = t,$ and let $\Gamma_0(N)_t$ denote the stabilizer of cusp $t$ in $\Gamma_0(N).$ We define a positive integer $a_t$ by

$$a_t^{-1}\Gamma_0(N)_t \sigma_t = \left\{ \begin{pmatrix} 1 & \ell \alpha_t \\ 0 & 1 \end{pmatrix} : \ell \in \mathbb{Z} \right\}.$$ 

Recall that $f|_k \sigma_t$ has the Fourier expansion of the form $f|_k \sigma_t = f^+_t + f^-_t,$ where

$$f^+_t(z) = \sum_{n \in \mathbb{Z}, \sigma_t \in \mathbb{Z}} a^+_f(n) e^{2\pi i n z / a_t},$$

$$f^-_t(z) = b^+_f(0) y^{1-k} + \sum_{n \in \mathbb{Z}, n \neq 0} b^-_f(n) \Gamma(\pi n y / a_t, -k + 1) e^{2\pi i n z / a_t},$$

where $\Gamma(x, s)$ denotes the incomplete gamma function defined as an analytic continuation of the function $\int_0^\infty t^s - 1 e^{-t} dt.$ This Fourier series is called the **Fourier expansion of $f$ at a cusp $t.$** The function

$$\sum_{n \leq 0} a^+_f(n) e^{2\pi i n z / a_t}$$

is called the **principal part of $f$ at the cusp $t.$**

For a positive integer $n,$ let $T_n$ denote the $n$-th Hecke operator. Then the Hecke operator $T_n$ commutes with the differential operator $\xi_{-k}$ in the following way:

$$\xi_{-k}(f|_k T_n) = n^{-k-1}(\xi_{-k}(f)|_{k+2} T_n)$$

for a harmonic weak Maass form $f$ of weight $-k.$

### 2.2 Residues of a meromorphic differential on $X_0(N)$

Let $\psi$ be a meromorphic differential on $X_0(N).$ Then there exists a unique meromorphic modular form $g$ of weight 2 on $\Gamma_0(N)$ such that $\psi = g(z) dz.$ Assume that $t$ is a cusp of $\Gamma_0(N).$ Assume that, for each cusp $t,$ $g$ has the Fourier expansion of the form $(g|_2 \sigma_t)(z) = \sum a^+_g(n) q^{n/a_t},$ where $q := e^{2\pi i z}$ for $z \in \mathbb{H}.$

For $\tau \in \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q},$ let $Q_\tau$ be the image of $\tau$ under the canonical map from $\mathbb{H} \cup \{i\infty\} \cup \mathbb{Q}$ to $X_0(N).$ Let $\text{Res}_{Q_\tau} g dz$ denote the residue of the differential $g(z) dz$ at $Q_\tau$ on $X_0(N),$ and let $\text{Res}_\tau g$ be the residue of $g$ at $\tau$ on $\mathbb{H}.$ We describe $\text{Res}_{Q_\tau} g dz$ in terms of $\text{Res}_\tau g$ as follows. Let $\mathcal{C}_N$ be the set of inequivalent cusps of $\Gamma_0(N).$ For $\tau \in \mathbb{H},$ let $e_\tau$ be the order of the isotropy subgroup of $\text{SL}_2(\mathbb{Z})$ at $\tau.$ Then we have

$$\text{Res}_{Q_\tau} g dz = \begin{cases} \frac{1}{2\pi i} \text{Res}_\tau g & \text{if } \tau \in \mathbb{H}, \\ \frac{1}{2\pi i} \alpha_\tau a^+_g(0) & \text{if } \tau \in \mathcal{C}_N. \end{cases}$$

### 2.3 Differentials of the third kind

In this subsection, we review properties for differentials of the third kind on a complex curve. For details, we refer to [5, 9, 16]. A differential of the third kind on $X_0(N)$ means a meromorphic differential on $X_0(N)$ such that its poles are simple and its residues are integers. Let $\phi$ be a differential of the third kind on $X_0(N)$ such that $\phi$ has a pole at $P_j$ with residue $m_j$ and is holomorphic elsewhere.
Then we define a linear map “res” for the space of differentials of the third kind on \( X_0(N) \) to \( \text{Div}_{X_0(N)}(\mathbb{C}) \) by
\[
\text{res}(\phi) = \sum_j m_j P_j.
\]
The image of \( \phi \) under the map “res” is called the residue divisor of \( \phi \). By the residue theorem, the residue divisor \( \text{res}(\phi) \) has degree zero.

Conversely, if \( D \) is a divisor on \( X_0(N) \) whose degree is zero, then there is a differential \( \psi_D \) of the third kind with \( \text{res}(\psi_D) = D \) by the Riemann–Roch theorem and the Serre duality. The differential \( \psi_D \) is unique up to addition of a cusp form of weight 2 on \( \Gamma_0(N) \).

Let \( D \) be a divisor of \( X_0(N) \) with degree zero. Then there is a unique differential \( \Phi_D \) of the third kind such that \( \text{res}(\Phi_D) = D \) and \( \Phi_D = \partial_z h \), where \( h \) is a real harmonic function on \( X_0(N) \) with some log-type singularities. Here, the differential \( \Phi_D \) of the third kind is called the canonical differential of the third kind associated to \( D \) (for example, see [16, Section 1] for more details). Scholl proved the following theorem by using Waldschmidt’s result on the transcendence of periods of differentials of the third kind.

**Theorem 2.2** ([16, Theorem 1]). With the above notation, assume that \( D \) is defined over a number field \( F \). Then \( \Phi_D \) is defined over \( \overline{\mathbb{Q}} \) if and only if some non-zero multiple of \( D \) is a principal divisor.

For a differential \( \psi \) of the third kind, we may write \( \psi = 2\pi i f(z) \, dz \), where \( f \) is a meromorphic modular form of weight 2 on \( \Gamma_0(N) \). All poles of \( f \) are simple poles and lie on \( Y_0(N) \), and their residues are integers. The residue of \( \psi \) at the cusp \( t \) is a constant term of the Fourier expansion of \( f \) at the cusp \( t \). By the \( q \)-expansion principle, \( \psi \) is defined over a number field \( F \) if and only if all Fourier coefficients of \( f \) at the cusp \( \infty \) are contained in \( F \).

Therefore, the following theorem [5, Theorem 3.3] follows from Theorem 2.2.

**Theorem 2.3** ([5, Theorem 3.3]). Let \( F \) be a number field. Let \( D \) be a divisor of degree 0 on \( X_0(N) \) defined over \( F \). Let \( \Phi_D \) be the canonical differential of the third kind associated to \( D \), and write \( \Phi_D = 2\pi i f(z) \, dz \). If some non-zero multiple of \( D \) is a principal divisor, then all the coefficients \( a(n) \) of \( f \) at the cusp \( \infty \) are contained in \( F \). Otherwise, there exists an integer \( n \) such that \( a(n) \) is transcendental.

### 3 Regularized Petersson inner product

Petersson introduced an inner product on the space of cusp forms, which is called the Petersson inner product. Borcherds [3] used a regularized integral to extend the Petersson inner product to the case that one of two modular forms is a weakly holomorphic modular form. In this section, we recall the definition of a regularized Petersson inner product of a cusp form and a meromorphic modular form with the same weight by following [3, 7]. Furthermore, we prove that if \( g \) is a meromorphic modular form on \( \Gamma_0(N) \) such that \( 2\pi i g(z) \, dz \) is the canonical differential of the third kind associated to some divisor, then the regularized Petersson inner product of \( g \) with every cusp form of weight 2 on \( \Gamma_0(N) \) is zero.

Let \( g \) be a meromorphic modular form of weight \( k \) on \( \Gamma_0(N) \). Let \( \text{Sing}(g) \) be the set of singular points of \( g \) on \( \mathcal{F}_N \), where \( \mathcal{F}_N \) denotes the fundamental domain for the action of \( \Gamma_0(N) \) on \( \mathbb{H} \). For a positive real number \( \varepsilon \), an \( \varepsilon \)-disk \( B_\varepsilon(\tau) \) at \( \tau \) is defined by
\[
B_\varepsilon(\tau) := \begin{cases} 
\{ z \in \mathbb{H} : |z - \tau| < \varepsilon \} & \text{if } \tau \in \mathbb{H}, \\
\{ z \in \mathcal{F}_N : \text{Im}(\sigma_j z) > 1/\varepsilon \} & \text{if } \tau \in \{ \infty \} \cup \mathbb{Q}.
\end{cases}
\]

Let \( \mathcal{F}_N(\varepsilon) \) be a punctured fundamental domain for \( \Gamma_0(N) \) defined by
\[
\mathcal{F}_N(\varepsilon) := \mathcal{F}_N - \bigcup_{\tau \in \text{Sing}(g) \cap \mathbb{C}_N} B_\varepsilon(\tau).
\]

Let \( f \) be a cusp form of weight \( k \) on \( \Gamma_0(N) \). The regularized Petersson inner product \( (f, g)_{\text{reg}} \) of \( f \) and \( g \) is defined by
\[
(f, g)_{\text{reg}} := \lim_{\varepsilon \to 0} \int_{\mathcal{F}_N(\varepsilon)} f(z) \overline{g(z)} \frac{dx \, dy}{y^{2-k}}.
\]
The following lemma was proved by the Stokes theorem (see [4, Proposition 3.5] or [7, Lemma 3.1]).

**Lemma 3.1.** Suppose that $f$ is a harmonic weak Maass form in $H^s_\kappa(\Gamma_0(N))$ with singularities only at cusps equivalent to $i\infty$ and that $g$ is a meromorphic modular form of weight $2 - k$. Then

$$
(\xi_k(f), g)_{\text{reg}} = \sum_{m+n=0} \sum_{\kappa \in \mathcal{C}_N} a^m_n(\kappa) a^k_n(\kappa) + \sum_{\kappa \in \mathcal{C}_N} \frac{2\pi i}{\kappa} \text{Res}_\kappa(g) f(\kappa),
$$

where $a^m_n(\kappa)$ and $a^k_n(\kappa)$ are $n$-th Fourier coefficients of $g$ and $f$ at the cusp $\kappa$, respectively.

To use Lemma 3.1 in the proof of Theorem 1.2, we prove the following proposition, which states that the regularized Petersson inner product of a meromorphic modular form associated with a canonical differential of the third kind of some divisor on $X_0(N)$ with every cusp form of weight 2 on $\Gamma_0(N)$ is zero.

**Proposition 3.2.** Let $g$ be a meromorphic modular form of weight 2 on $\Gamma_0(N)$ associated with a canonical differential of the third kind. Then, for every cusp form $f$ of weight 2 on $\Gamma_0(N)$, we have $(f, g)_{\text{reg}} = 0$

**Proof.** Let $G$ be a harmonic function on $\mathbb{H}$ with log-type singularities such that $\partial_z G = g$. Then we have

$$
d(f(z)G(z)) = f(z) \partial_z G(z) \, dz = f(z)G(z)(2i) \, dx \, dy.
$$

To apply the Stokes theorem, we give the description of the boundary of $\mathcal{F}_N(g, \varepsilon)$. For a subset $D$ of $\mathbb{C}$, let $\partial D$ denote the boundary of $D$. For a positive real number $\varepsilon$, we define

$$
y_\varepsilon(\tau) := \begin{cases} 
\{z \in \mathbb{H} : |z - \tau| = \varepsilon\} & \text{if } \tau \in \mathbb{H}, \\
\{z \in \mathcal{F}_N : \text{Im}(\sigma_r z) = 1/\varepsilon\} & \text{if } \tau \in \{i\infty\} \cup \mathbb{Q}.
\end{cases}
$$

Assume that $\varepsilon$ is sufficiently small. If we let $\partial^* \mathcal{F}_N(g, \varepsilon)$ be the closure of the set $\partial \mathcal{F}_N(g, \varepsilon) - \partial \mathcal{F}_N$ in $\mathbb{C}$, then

$$
\partial^* \mathcal{F}_N(g, \varepsilon) = \bigcup_{\text{re}(\text{Sing}(g)) \in \mathbb{C}_N} y_\varepsilon(\tau).
$$

From (3.1), the Stokes theorem implies

$$
\mathcal{K}_N^* (g, \varepsilon) = \int_{\mathcal{K}_N(g, \varepsilon)} \frac{1}{2i} f(z)G(z) \, dz = \int_{\text{re}(\text{Sing}(g)) \in \mathbb{C}_N} \frac{1}{2i} f(z)G(z) \, dz.
$$

For each $\gamma \in \text{SL}_2(\mathbb{Z})$, the absolute value $|f(z)|$ exponentially decays as $\text{Im}(z) \to \infty$ since $f$ is a cusp form. Thus, if $\tau \in \mathbb{C}_N$, then $\lim_{\varepsilon \to 0} \int_{y_\varepsilon(\tau)} \frac{1}{2i} f(z)G(z) \, dz = 0$.

To complete the proof, we assume that $\tau \in \text{Sing}(g)$. Then

$$
\left| \int_{y_\varepsilon(\tau)} \frac{1}{2i} f(z)G(z) \, dz \right| \leq \max \{ |G(z)| : z \in y_\varepsilon(\tau) \} M_1 \int_{y_\varepsilon(\tau)} |dz| (\text{some constant } M_1)
$$

The function $G$ can be expressed around $\tau$ as

$$
G(z) = - \log_\varepsilon |z - \tau| + G_0(z),
$$

where $G_0(z)$ is a smooth function around $\tau$. In the definition of $\log_\varepsilon$, we use the principal branch. If $\varepsilon$ is sufficiently small, then, for any $z \in y_\varepsilon(\tau)$, we have

$$
|G(z)| \leq |\log_\varepsilon |z - \tau| + G_0(z)|
$$

$$
\leq |\log_\varepsilon |z - \tau| + |G_0(z)|
$$

$$
\leq 2|\log_\varepsilon \varepsilon| + \pi + M_2. 
$$
Thus, for sufficiently small $\epsilon$, we obtain
\[ \left| \int \frac{1}{2i} f(z) \overline{G(z)} \, dz \right| \leq (2 \log_e |\epsilon| + \pi + M_2) M_1(2\pi \epsilon). \]
This implies that, for $\tau \in \text{Sing}(g)$,
\[ \lim_{\epsilon \to 0} \int \frac{1}{2i} f(z) \overline{G(z)} \, dz = 0. \]
Thus, we complete the proof.

\section*{4 Proofs}

In this section, we prove Theorems 1.2 and 1.3. In this section, we always assume that $N$ is a prime. Let $D$ be a divisor of $X_0(N)$. Let $\psi_D$ be a differential of the third kind associated to $D$. Let $A_D$ be the space of meromorphic differentials, with only simple poles, such that their poles are only at the support of $D$. If $D$ is defined over $\mathbb{Q}$, then there is a basis of $A_D$ consisting of meromorphic differentials defined over $\mathbb{Q}$. Let us note that $D$ is defined over $\mathbb{Q}$ if and only if there exists $\psi_D$ defined over $\mathbb{Q}$. Thus, for a divisor $D$ defined over $\mathbb{Q}$, let $\psi_D^{\text{alg}}$ be a differential of the third kind associated to $D$ defined over $\mathbb{Q}$. Let $\Phi_D$ be the canonical differential of the third kind associated to $D$ and
\[ F_D := \Phi_D - \psi_D^{\text{alg}}. \]
Note that $F_D$ (resp. $\Phi_D$ and $\psi_D^{\text{alg}}$) can be written as $2\pi if_D(z) \, dz$ (resp. $2\pi if_{\Phi_D}(z) \, dz$ and $2\pi if_{\psi_D^{\text{alg}}}(z) \, dz$) for some meromorphic modular forms $f_{F_D}, f_{\Phi_D}$, and $f_{\psi_D^{\text{alg}}}$ of weight 2 on $\Gamma_0(N)$. Then $f_{F_D}$ is a holomorphic modular form.

Let us consider special divisors $D_\tau$, where
\[ D_\tau = Q_{\text{loc}} - Q_\tau \in \text{Div}_{X_0(N)}(\mathbb{C}) \quad \text{for } \tau \in \mathbb{H}. \]
Let us note that $Q_{\text{loc}}$ is defined over $\overline{\mathbb{Q}}$. A modular curve $Y_0(N)$ is defined by an equation $\Phi_N(X, Y) = 0$ such that $\Phi_N(X, Y) \in \mathbb{Q}[X, Y]$ and $\Phi_N(j(z), j(z)) = 0$ for all $z \in \mathbb{H}$. Thus, by Theorem A, $\tau \in \mathbb{H}$ is a CM point if and only if $Q_\tau$ is defined over $\overline{\mathbb{Q}}$. Thus, there exists $\psi_D^{\text{alg}}$ for $D_\tau$ if and only if $\tau$ is a CM point.

With these notations, we prove the following lemma.

**Lemma 4.1.** With the above notation, assume that $f$ is an arithmetic harmonic weak Maass form in $H^*_k(\Gamma_0(N))$ and that $D := Q_{\text{loc}} - Q_\tau$ is defined over $\overline{\mathbb{Q}}$. Let $m$ be a positive integer prime to $N$. Then the following statements are true.

1. For each $m$, $f(T_m, Q_\tau) - (\xi_0(fT_m), f_{\psi_D^{\text{alg}}})_{\text{reg}}$ is algebraic. Especially, if $f$ is a Hecke eigenform, then
   \[ f(T_m, Q_\tau) - m^{-1} \lambda_m(\xi_0(f), f_{\psi_D^{\text{alg}}})_{\text{reg}} \]
   is algebraic. Here, $\lambda_m$ is the eigenvalue of $\xi_0(f)$ for $T_m$.
2. For each $m$, $f(T_m, Q_\tau)$ is algebraic if and only if $(\xi_0(fT_m), f_{F_D})$ is algebraic.

**Proof.** (1) By the definition of $\psi_D^{\text{alg}}$, the constant term of $f_{\psi_D^{\text{alg}}}$ at each cusp is an integer. Let
\[ E_N(z) := E_2(z) - NE_2(Nz), \]
where $E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(d) q^n$ is the Eisenstein series of weight 2. Here, $\sigma_1(n)$ is a function over $\mathbb{Q}$ defined by
\[ \sigma_1(n) := \begin{cases} \sum_{d|n} d & \text{if } n \text{ is a positive integer}, \\ 0 & \text{elsewhere}. \end{cases} \]
Since $N$ is a prime, $\Gamma_0(N)$ has only two inequivalent cusps. Thus, there is a rational number $c_0$ such that the constant term of $f_{\psi_D^{\text{alg}}} - c_0E_N$ is zero at each cusp inequivalent to $\infty$. Let us note that $E_N$ is orthogonal to
If $\xi_0(f|T_m)$, $\xi_0(f|T_m)_\text{reg}$ is a harmonic weak Maass form, then it is enough to show that $\xi_0(f|T_m)_\text{reg}$ is equal to (4.1). Thus, we need to show that $\xi_0(f|T_m)_\text{reg}$ is a meromorphic modular form of weight $\infty$ on $\Gamma_0(N)$. If $\tau$ is not a CM point, then there are infinitely many positive integers $m$ for which $\xi_0(f|T_m)_\text{reg}$ is not a CM point. Let $\xi_0(f|T_m)_\text{reg}$ be an arithmetic Hecke eigenform. Let $\tau$ be an algebraic number on $\mathbb{H}$. Assume that $D := Q_{i\infty} - Q_{i\tau}$. If $\tau$ is not a CM point, then there are infinitely many positive integers $m$ prime to $N$. $\Prin(f)$ such that

$$\sum_{n \geq 1} a_f(-n)c_{\Phi_0}(mn)$$

is transcedental, where $a_f(n)$ (resp. $c_{\Phi_0}(n)$) is the $n$-th Fourier coefficient of $f^+$ (resp. $f_{\Phi_0}$) at the cusp $i\infty$.

Proof. Let $A$ be the set of positive integers $n$ such that $a_f(-n) \neq 0$. Note that $A$ is a finite set. Then we define

$$F(z) := \sum_{n \in A} a_f(-n)(f_{\Phi_0}|U_n)(z).$$

The $m$-th Fourier coefficient $a_f(m)$ of $F$ at the cusp $i\infty$ is equal to (4.1). Thus, we need to show that $F$ has infinitely many transcendental Fourier coefficients $a_f(m)$ with $(m, N \cdot \Prin(f)) = 1$.

Let $B = \{p_1, \ldots, p_k\}$ be the set of primes $p$ such that $p|N \cdot \Prin(f)$. Using the operators $U_n$ and $V_n$, we will remove Fourier coefficients $a_f(m)$ with $(m, N \cdot \Prin(f)) \neq 1$. We define $F_0 := F$ and $F_{i+1} := F_i - F_i|U_{p_i} - V_{p_i}$ for $0 \leq i \leq k - 1$. Then it is enough to show that $F_k$ has infinitely many transcedental Fourier coefficients at the cusp $i\infty$. 


By the definition of $f_{\phi_0}$, we see that $f_{\phi_0}$ has a singularity at the non-CM point $r$ in $\mathbb{H}$. We will prove that $F_k$ also has a singularity at a non-CM point in $\mathbb{H}$. We define

$$f(n,i,\mu_1,\ldots,\mu_k)(z) := \phi_n^\theta \left( \frac{z}{n} + \frac{i}{p_1} + \cdots + \frac{\mu_k}{p_k} \right)$$

for $n \in A$, $0 \leq i \leq n - 1$, $0 \leq \mu_1 \leq p_1 - 1$, $\ldots$, $0 \leq \mu_k \leq p_k - 1$. Note that

$$F_k(z) = F(z) - (F|U_p|V_p_1)(z) - \cdots - (F|U_p|^n|V_p_1|^n)(z) + \cdots + (-1)^k(F|U_p|^n|V_p_1|^n)(z)$$

$$= \sum_{n \in A} \sum_{0 \leq i \leq n - 1} \sum_{0 \leq \mu_1 \leq p_1 - 1} \cdots \sum_{1 \leq j \leq k} \alpha_{n,i,\mu_1,\ldots,\mu_k}(z)$$

for some non-zero constants $\alpha_{n,i,\mu_1,\ldots,\mu_k}$. We fix $n_0 \in A$. Let

$$\beta := n_0 \tau - \frac{1}{p_1} - \cdots - \frac{1}{p_k}.$$

Then $\beta$ is not a CM point since $\tau$ is not a CM point, and $\beta$ is a singular point of the function $f_{n_0,0,1,\ldots,1}$. To prove that $F_k$ has a singularity at $\beta$, it is enough to show that $\beta$ is not a singular point of $f(n,i,\mu_1,\ldots,\mu_k)$ if $(n, i, \mu_1, \ldots, \mu_k) \neq (n_0, 0, 1, \ldots, 1)$. Note that the set of singular points of $f(n,i,\mu_1,\ldots,\mu_k)$ is a subset of

$$T(n,i,\mu_1,\ldots,\mu_k) := \left\{ (n(y\tau) - i - \frac{\mu_1}{p_1} - \cdots - \frac{\mu_k}{p_k} : y \in \Gamma_0(N) \right\}.$$

Suppose that $\beta \in T(n,i,\mu_1,\ldots,\mu_k)$. Then, for some $y = (a \ b \ c \ d) \in \Gamma_0(N)$, we have

$$n_0 \tau - \frac{1}{p_1} - \cdots - \frac{1}{p_k} = (n(y\tau) - i) - \frac{\mu_1}{p_1} - \cdots - \frac{\mu_k}{p_k}.$$

This implies that

$$n_0 \tau - n\frac{ar + b}{ct + d} = -i - \frac{\mu_1 - 1}{p_1} - \cdots - \frac{\mu_k - 1}{p_k}.$$

If $c \neq 0$, then $\tau$ satisfies a quadratic equation; this is not possible since $\tau$ is not a CM point. Thus, $c = 0$. Then we may assume that $a = d = 1$. From this, we have

$$n_0 \tau - n(\tau + b) = -i - \frac{\mu_1 - 1}{p_1} - \cdots - \frac{\mu_k - 1}{p_k}.$$

Since $\tau$ is not a rational number, we see that $n = n_0$. Then we obtain

$$-nb + i = -\frac{\mu_1 - 1}{p_1} - \cdots - \frac{\mu_k - 1}{p_k}.$$

This holds only if $\mu_1 = \cdots = \mu_k = 1$ and $b = i = 0$. Thus, $\beta$ is a singular point only for $f_{n_0,0,1,\ldots,1}$.

Let $p$ be a prime. In a similar argument, we see that $F_k|U_p$ also has a singularity at a non-CM point for every prime $p$. By the Siegel–Schneider theorem, $F_k|U_p$ is not defined over $\mathbb{Q}$. Then the $q$-expansion principle implies there is a Fourier coefficient of $F_k|U_p$ at the cusp $i\infty$ which is transcendental. Therefore, $F_k$ has infinitely many transcendental Fourier coefficients.

**Lemma 4.3.** Let $f \in H^1_0(\Gamma_0(N))$ be an arithmetic Hecke eigenform. Assume that $(g, \xi(f)) \neq 0$ for every Hecke eigenform $g \in S_2(\Gamma_0(N))$. Let $D$ be a divisor of $X_0(N)$ defined over $\overline{\mathbb{Q}}$. Assume that $N$ is a prime and that $\Phi_D$ is not defined over $\overline{\mathbb{Q}}$. Then there exist infinitely many positive integers $m$ prime to $N$ such that $f(T_mD)$ are transcendental.

**Proof.** Let $(f_1, \ldots, f_k)$ be the set of all normalized Hecke eigenforms in $S_2(\Gamma_0(N))$. Assume that $f_i$ has a Fourier expansion of the form

$$f_i(z) = \sum_{n=1}^{\infty} a_{f_i}(n)e^{2\pi inz}.$$
Then we have
\[(\xi_0(f), f_i) = \sum_{m \geq 0} a_f(m)a_f(n),\]
where \(a_f(n)\) is the \(n\)-th Fourier coefficient of \(f^n\) at the cusp \(i\infty\). Thus, \((\xi_0(f), f_i)\) is algebraic.

Note that \(f_{\mathbb{Q}}\) is a cusp form in \(S_2(\Gamma_0(N))\). Assume that \(f_{\mathbb{Q}}=\sum_{i=1}^{k} \beta_i f_i\) for some \(\beta_i\) and that \(\xi_0(f) = \sum_{i=1}^{k} \alpha_i f_i\) for some \(\alpha_i\). By the assumption, \(\alpha_i \neq 0\) for all \(i\). Let \(m\) be a positive integer prime to \(N\). Then we have
\[
\xi_0(f|T_m) = m^{-1} \xi_0(f)|T_m = m^{-1} \sum_{i=1}^{k} \alpha_i \lambda_{i,m} f_i,
\]
where \(\lambda_{i,m}\) is the eigenvalue of \(f_i\) for \(T_m\). From this, we obtain
\[
(\xi_0(f|T_m), f_{\mathbb{Q}}) = m^{-1} \sum_{i=1}^{k} \alpha_i \lambda_{i,m} = \sum_{j=1}^{\ell} \beta_j \bar{w}_j,
\]
We define \(\beta_i := \alpha_i \bar{\beta}_i(f_i, f_i)\) for \(i = 1, \ldots, k\). Since \(\Phi_D\) is not defined over \(\overline{\mathbb{Q}}\), we see that \(F_{\mathbb{Q}}\) is not defined over \(\overline{\mathbb{Q}}\). This implies that at least one of \(\beta_i\) is transcendental. Note that \(\alpha_i(f_i, f_i)\) is a non-zero algebraic number for all \(i\) since \(\alpha_i(f_i, f_i) = (\xi_0(f), f_i)\) and \(\alpha_i \neq 0\) for all \(i\). Thus, at least one of \(\beta_i\) is transcendental. As a vector space over \(\overline{\mathbb{Q}}\), let \(W\) be the subspace of \(\mathbb{C}\) generated by \(\beta_1, \ldots, \beta_k\). Let \(\{w_1, \ldots, w_{\ell}\}\) be a basis of \(W\), where \(w_1 = 1\). We may assume that
\[
\beta_i = \sum_{j=1}^{\ell} \beta_{ij} w_j \quad (4.2)
\]
for \(i = 1, \ldots, k\) and for some \(\beta_{ij} \in \overline{\mathbb{Q}}\). From this, we have
\[
(\xi_0(f|T_m), f_{\mathbb{Q}}) = m^{-1} \sum_{i=1}^{k} \beta_i \lambda_{i,m} = m^{-1} \sum_{i=1}^{k} \left( \sum_{j=1}^{\ell} \beta_{ij} w_j \right) \lambda_{i,m} = m^{-1} \sum_{j=1}^{\ell} \left( \sum_{i=1}^{k} \beta_{ij} \lambda_{i,m} \right) w_j. \quad (4.3)
\]
By Lemma 4.1 (2) and (4.3), \(f(T_m \cdot D)\) is algebraic if and only if
\[
\sum_{i=1}^{k} \beta_{ij} \lambda_{i,m} = 0 \quad \text{for every } j \geq 2. \quad (4.4)
\]
Suppose that \(f(T_m \cdot D)\) is algebraic for all positive integers \(m\) prime to \(N\). Since at least one of \(\beta_i\) is transcendental, at least one of \(\beta_{ij}\) with \(j \geq 2\) is non-zero by (4.2). From this, there is a positive integer \(j_0 \geq 2\) such that \((\beta_{1,j_0}, \ldots, \beta_{\ell,j_0}) \neq (0, \ldots, 0)\). We define a cusp form \(g\) in \(S_2(\Gamma_0(N))\) by
\[
g := \sum_{i=1}^{k} \beta_{i,j_0} f_i. \quad (4.5)
\]
Note that, by (4.4), \(a_g(m) = 0\) for all positive integers \(m\) prime to \(N\), where \(a_g(m)\) denotes the \(m\)-th Fourier coefficient of \(g\). This implies that \(g|U_N|V_N = g \in S_2(\Gamma_0(N))\) since \(N\) is a prime. By [1, Lemma 16], \(g|U_N\) is a cusp form in \(S_2(\Gamma_0(1))\). Since \(S_2(\Gamma_0(1)) = \{0\}\), this implies that \(g|U_N = 0\). From this, we have \(g = 0\) since \(a_g(m) \neq 0\) only when \(N \mid m\). This is a contradiction due to the fact that \(\{f_1, \ldots, f_k\}\) is a basis of \(S_2(\Gamma_0(N))\) and \((\beta_{1,j_0}, \ldots, \beta_{\ell,j_0}) \neq (0, \ldots, 0)\). Thus, there is a positive integer \(m_0\) prime to \(N\) such that \(f(T_{m_0} \cdot D)\) is transcendental.

By (4.4), there exists \(j_0 \geq 2\) such that \(\sum_{i=1}^{k} \beta_{i,j_0} \lambda_{i,m_0} \neq 0\). If we define \(g\) by (4.5), then \(a_g(m_0) \neq 0\). Thus, the function
\[
g - g|U_N|V_N(z) = \sum_{N \mid m} a_g(m)e^{2\pi imz}
\]
is a non-zero cusp form of weight 2. Therefore, there exist infinitely many positive integer \(m\) prime to \(N\) such that \(a_g(m) \neq 0\). With (4.4), this completes the proof.
Now we prove Theorem 1.3.

Proof of Theorem 1.3. First, we prove that if \( \tau \) is not a CM point or \( nD \) is not rational on \( X_0(N) \) for any positive integer \( n \), then \( f(T_m, Q_\tau) \) is not algebraic for some positive integer \( m \) prime to \( N \). Suppose that \( \tau \) is not a CM point. Then, by Lemma 4.2, there exists a positive integer \( m \) prime to \( N \cdot \text{Prin}(f) \) such that (4.1) is transcendental.

We take a positive integer \( m \) which is prime to \( N \cdot \text{Prin}(f) \). As in the proof of Lemma 4.1, it follows from Lemma 3.1 and Proposition 3.2 that

\[
0 = (\xi_0(f|T_m), f_{D_0})_{\text{reg}} = \sum_{n \geq 1} a_{f|T_m}(-n)c_{D_0}(n) + f(T_m, Q_\tau).
\]

Since \( m \) is prime to \( N \cdot \text{Prin}(f) \), we have \( a_{f|T_m}(-n) = a_f(-n/m) \), and hence, we obtain

\[
f(T_m, Q_\tau) = -\sum_{n \geq 1} a_{f|T_m}(-n)c_{D_0}(n) = -\sum_{n \geq 1} a_f(-n/m)c_{D_0}(n) = -\sum_{n \geq 1} a_f(-n)c_{D_0}(mn).
\]

Therefore, we see that \( f(T_m, Q_\tau) \) is transcendental for some positive integer \( m \) prime to \( N \).

Suppose that \( \tau \) is a CM point and that \( nD \) is not rational on \( X_0(N) \) for any positive integer \( n \). This implies that \( D \) is defined over \( \overline{\mathbb{Q}} \) and that \( D \) is not a principal divisor. By Theorem 2.2, \( \Phi_D \) is not defined over \( \overline{\mathbb{Q}} \). Therefore, there exists a positive integer \( m \) prime to \( N \) such that \( f(T_m, Q_\tau) \) is not algebraic by Lemma 4.3.

Conversely, suppose that \( \tau \) is a CM point and that \( nD \) is rational on \( X_0(N) \) for some positive integer \( n \). Then \( D \) is defined over \( \mathbb{Q} \), and \( D \) is a principal divisor. By Theorem 2.2, \( \Phi_D \) is defined over \( \mathbb{Q} \). Thus, all the Fourier coefficients of \( f_{F_0} \) at the cusp \( i\infty \) are algebraic. Let \( m \) be a positive integer prime to \( N \). Then, by Lemma 3.1, we have

\[
(\xi_0(f|T_m), f_{F_0}) = \sum_{n \geq 1} a_{f|T_m}(-n)c_{F_0}(n),
\]

where \( a_{f|T_m}(n) \) (resp. \( c_{F_0}(n) \)) is the \( n \)-th Fourier coefficient of \( (f|T_m)^+ \) (resp. \( f_{F_0} \)) at the cusp \( i\infty \). This implies that \( (\xi_0(f|T_m), f_{F_0}) \) is algebraic. Therefore, by Lemma 4.1 (2), \( f(T_m, Q_\tau) \) is algebraic.  

\[ \square \]

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