SCL AND MIXED SCL ARE NOT EQUIVALENT FOR SURFACE GROUPS

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Abstract. For the pair of the surface group $G = \pi_1(\Sigma_\ell)$ of genus at least 2 and its commutator subgroup $N = [\pi_1(\Sigma_\ell), \pi_1(\Sigma_\ell)]$, we prove that the stable commutator length $\text{scl}_G$ and the stable mixed commutator length $\text{scl}_{G,N}$ are not equivalent. We also show the non-equivalence for a pair $(G, N)$ such that $G$ is the fundamental group of a 3-dimensional closed hyperbolic mapping torus. These pairs serve as the first family of examples of such $(G, N)$ in which $G$ is finitely generated.

1. Introduction

1.1. Main results. In this paper, we provide the first family of examples of finitely generated group $G$ and its normal subgroup $N$ for which $\text{scl}_G$ and $\text{scl}_{G,N}$ are not equivalent. The following theorem is the initial goal; this answers the latter question in Problem 9.9 in [KKM+21] in the negative.

Theorem 1.1 (non-equivalence for surface groups). Let $G_\ell$ be the fundamental group of a closed connected oriented surface $\Sigma_\ell$ of genus $\ell \geq 2$ and $G'_{\ell}$ its commutator subgroup. Then $\text{scl}_{G_\ell}$ and $\text{scl}_{G_\ell,G'_{\ell}}$ are not bi-Lipschitzly equivalent on $[G_\ell,G'_{\ell}]$.

Here, $\text{scl}_{G_\ell}$ and $\text{scl}_{G_\ell,G'_{\ell}}$ denote the stable commutator length and the stable mixed commutator length, respectively, which are defined in the next subsection (Definition 1.3; see also Definition 1.4). We note that $\text{scl}_{G}$ is equivalent to $\text{scl}_{G,G'}$ when $G$ is a free group or the fundamental group of a non-orientable closed connected surface; see [KKM+21] or Remark 6.3. In this paper, for a group $H$, we use the symbol $H'$ to indicate the commutator subgroup $[H,H]$ of $H$.

Furthermore, we provide a pair $(G, N)$ of a 3-manifold group $G$ and its normal subgroup $N$ with solvable $G/N$ such that $\text{scl}_G$ and $\text{scl}_{G,N}$ are not bi-Lipschitzly equivalent on $[G, N]$. The precise setting goes as follows: let $\ell \geq 2$. Let $X$ be a hyperbolic 3-manifold that fibers over the circle with fiber $\Sigma_\ell$. Then the fundamental group $\pi_1(X)$ has the presentation

$$G_\psi = G_\ell \times_\psi \mathbb{Z} = \langle G_\ell, c \mid \psi(\gamma)c\gamma^{-1}c^{-1} \rangle$$

for some $\psi \in \text{Aut}_+(G_\ell)$. More precisely, $X$ is a mapping torus of some pseudo-Anosov surface diffeomorphism $f_\psi$; $f_\psi$ induces an action on $G_\ell$, which corresponds to the automorphism $\psi$ above. Then the abelianization map $\text{Ab}_{G_\ell} : G_\ell \to \mathbb{Z}^{2\ell}$ induces a surjection

$$p_\psi : G_\psi \to \mathbb{Z}^{2\ell} \times_{s_\psi} \mathbb{Z}.$$
Here, \( s_\ell : \text{Aut}_+(G_\ell) \to \text{Sp}(2\ell, \mathbb{Z}) \) denotes the symplectic representation. Set \( N_\psi = \text{Ker}(p_\psi) \), which equals \( i(G'_\ell) \). Here, \( i: G_\ell \to G_\psi = G_\ell \times_\psi \mathbb{Z} \) is the natural inclusion.

**Theorem 1.2** (non-equivalence for hyperbolic mapping tori). For the pair \((G_\psi, N_\psi)\) in the setting above, \( \text{scl}_{G_\psi} \) and \( \text{scl}_{G_\psi, N_\psi} \) are not bi-Lipschitzly equivalent on \([G_\psi, N_\psi]\).

### 1.2. Equivalence problem of stable commutator lengths and stable mixed commutator lengths.

Let \( G \) be a group and \( N \) its normal subgroup of \( G \). A (single) mixed commutator is an element of the form \([g, w] = gwg^{-1}w^{-1}\) with \( g \in G \) and \( w \in N \). Let \([G, N]\) be the subgroup of \( G \) generated by mixed commutators, which we call the mixed commutator subgroup. For an element \( x \in [G, N] \), the mixed commutator length \( \text{cl}_{G, N}(x) \) is the least number of mixed commutators needed to express \( x \) as their product.

**Definition 1.3** (stable mixed commutator length). Let \( G \) be a group and \( N \) its subgroup. The stable mixed commutator length \( \text{scl}_{G, N} \) is defined as the function

\[
\text{scl}_{G, N} : [G, N] \to \mathbb{R}_{\geq 0}; \quad x \mapsto \lim_{m \to \infty} \frac{\text{cl}_{G, N}(x^m)}{m}.
\]

It follows from Fekete’s lemma that the limit above always exists. If \( N = G \), then \([G, G]\), \( \text{cl}_{G,G} \) and \( \text{scl}_{G,G} \) coincide with the commutator subgroup \( G' \), the commutator length \( \text{cl}_G \) and the stable commutator length \( \text{scl}_G : G' \to \mathbb{R}_{\geq 0} \), respectively. We study the equivalence problem of \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \), which asks whether \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \) are equivalent in the following sense.

**Definition 1.4** (equivalence of \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \)). Let \( G \) be a group and \( N \) its normal subgroup. We say that \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \) are equivalent if there exists a positive real constant \( C \) such that for every \( x \in [G, N] \),

\[
\text{scl}_{G, N}(x) \leq C \cdot \text{scl}_G(x)
\]

holds.

By definition, the inequality \( \text{scl}_{G} \leq \text{scl}_{G, N} \) always holds on \([G, N]\). Hence, \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \) are equivalent (in the sense of Definition 1.4) if and only if they are bi-Lipschitzly equivalent in the standard sense on \([G, N]\). (We note that \( \text{scl}_{G, N} \) is not defined on \( G' \setminus [G, N] \).) It is known that in several cases \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \) are equivalent. See Remark 6.3 for details.

In [KK19], Kawasaki and Kimura showed the following: let \( \Sigma_\ell \) be a closed connected oriented surface of genus \( \ell \geq 2 \) and \( \omega \) a symplectic form on \( \Sigma_\ell \). Let \( G = \text{Symp}_0(\Sigma_\ell, \omega) \) be the identity component of the symplectomorphism group and \( N = \text{Ham}(\Sigma_\ell, \omega) \) the Hamiltonian diffeomorphism group (these are infinite dimensional Lie groups). Then \( \text{scl}_{G} \) and \( \text{scl}_{G, N} \) are not equivalent. This is the first example of such pairs \((G, N)\). Based on the work in [KKMM21], we have variants \((G, N)\) of this example (with smaller \( G \)) in [KKM+21, Example 7.15].

### 1.3. Outlined proofs of Theorems 1.1 and 1.2.

Here we describe the outline of the proofs of Theorems 1.1 and 1.2; this description simultaneously provides the organization of the present paper. Throughout this paper, we use the following symbol for group conjugation

\[
h_z = hzh^{-1}
\]
for a group $H$ and for $h, z \in H$. We use the symbol

$$\mathcal{H} = \text{Homeo}_+(S^1),$$

which is the group of orientation preserving homeomorphisms of the circle $S^1$. This group $\mathcal{H}$ plays a key role for our constructions of quasimorphisms (Subsection 2.4 and Section 3).

Let $G$ be a group and $N$ its normal subgroup. To show the non-equivalence of $\text{scl}_G$ and $\text{scl}_{G,N}$, it suffices to find a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in $[G, N]$ that fulfills the following two conditions:

(a) $\sup_{n \in \mathbb{N}} \text{scl}_G(x_n) < \infty$;

(b) $\sup_{n \in \mathbb{N}} \text{scl}_{G,N}(x_n) = \infty$.

To obtain such a sequence $(x_n)_{n \in \mathbb{N}}$, we employ one auxiliary lemma (Lemma 2.6), one of whose conditions is stated in terms of a $G$-invariant homogeneous quasimorphism on $N$ (Subsection 2.2). Section 3 is devoted to construct a $G$-invariant homogeneous quasimorphisms on $N$ for an arbitrary group $G$ and its normal subgroup $N$. The construction works greatly when $G$ admits a representation $\rho$ to the group $\mathcal{H}(= \text{Homeo}_+(S^1))$ whose Euler class is non-zero. In this case, for an element $x \in [G, N]$, we can compute the value $\mu(x)$ in terms of the Poincaré translation number and the representation $\rho: G \to \mathcal{H}$ (Lemma 3.4).

To find an element of the mixed commutator subgroup which applicable to the auxiliary lemma, we use the group $R_\ell = (a, b | [a, b]^\ell)$ for $\ell \geq 2$. We consider the elements $y = ba^2b^{-1} = ba^2b^{-1}(ba^2)^{-1}$ and $z = ba^2a^{-\ell} = ba^2a^{-\ell}(ba^2)^{-1}$. The commutator $[y, z]$ is an element of $[R_\ell, R_\ell]$, and in fact, it is written as a product of $\ell - 1$ mixed commutators (Corollary 2.9). These $y$ and $z$ are the source of elements to which we apply the auxiliary lemma.

Due to the theorem of Eisenbud–Hirsch–Neumann [EHN81] (Theorem 2.10), we can construct a representation $\rho_\ell: R_\ell \to \mathcal{H}$ with non-zero Euler class. By using this representation, we obtain an $R_\ell$-invariant homogeneous quasimorphism $\mu = \mu_{\rho_\ell}$ on $R_\ell'$ by Lemma 3.4. This quasimorphism $\mu$ and the commutator expression $[y, z]$ fit in the auxiliary lemma (Lemma 2.6); this implies that the sequence $([y^n, z])_{n \in \mathbb{N}}$ fulfills conditions (a) and (b). Thus, we obtain the non-equivalence of $\text{scl}_{R_\ell}$ and $\text{scl}_{R_\ell, R_\ell'}$ (Theorem 4.1). In Section 4, we provide the precise construction of $\mu_{\rho_\ell}$ and present the proof of Theorem 4.1.

In Section 5, we show Theorem 1.1. Recall that $G_\ell$ denotes the fundamental group of a closed connected oriented surface of genus $\ell \geq 2$. The natural projection $q: G_\ell \to R_\ell$ induces a surjection $q': G_\ell' \to R_\ell'$ on their commutator subgroups. Then the pullback $q'^*\mu_\ell$ is a $G_\ell$-invariant homogeneous quasimorphism on $G_\ell'$. By taking lifts $y_i, z_i \in G_\ell$ of $y, z \in R_\ell$ appropriately (the precise form is given in (5.1)), we can apply the auxiliary lemma to $q'^*\mu_\ell$ and the commutator expression $[y_1, z_1] \cdots [y_\ell, z_\ell]$. This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ fulfills conditions (a) and (b), where we set

$$\quad (x_n)_{n \in \mathbb{N}} = ([y_1^n, z_1] \cdots [y_\ell^n, z_\ell])_{n \in \mathbb{N}}.$$

Thus we obtain Theorem 1.1.

In Section 6, we prove Theorem 1.2. Let $G_\psi$ be the fundamental group of a hyperbolic 3-manifold that fibers over the circle with fiber $\Sigma_\ell$. We note that there exists an inclusion $i: G_\ell \to G_\psi$. We use a universal circle representation $\overline{\rho}: G_\psi \to \mathcal{H}$ whose restriction to $i(G_\ell) \simeq G_\ell$ is a Fuchsian representation $\rho: G_\ell \to \mathcal{H}$. Let $\mu: N_\psi \to \mathbb{R}$ be the $G_\psi$-invariant
homogeneous quasimorphism defined via the representation 7. Take the sequence \((x_n)_{n \in \mathbb{N}}\) in \([G_t, G_t']\) defined by (1.2). By comparing the restriction \(\mu|_{G_t'}\) with \(q'^*\mu_t\) (Lemma 6.2), we can apply the auxiliary lemma: thus we conclude that the sequence \((\iota(x_n))_{n \in \mathbb{N}}\) fulfills (a) and (b). This proves Theorem 1.2.

In Section 7, we make concluding remarks. In the proof of Theorem 1.2, we use the most basic case of the universal circle representation. In Subsection 7.1, we remark a relation between \(G_\psi\)-invariant homogeneous quasimorphisms on \(N_\psi\) and the universal circle representations of taut foliations on the hyperbolic mapping torus. More precisely, we prove that for every non-extendable \(G_\psi\)-invariant homogeneous quasimorphism \(\mu\) on \(N_\psi\), there exist taut foliations on the hyperbolic mapping torus such that \(\mu\) is written as, modulo trivial and extendable quasimorphisms, a linear combination of quasimorphisms induced from the universal circle representation of the taut foliations (Proposition 7.1). In Subsection 7.2, we state our overflow argument (Lemma 7.2), which provides a useful sufficient condition to apply the auxiliary lemma (Lemma 2.6).

2. Preliminaries

Throughout the present paper, we use the following symbol: for every \(r, s \in \mathbb{R}\) and \(D \geq 0\), we write \(r \sim s\) to mean \(|r - s| \leq D\).

2.1. Group cohomology and bounded cohomology. For a group \(G\) and \(n \geq 0\), let \(C_n(G)\) be the free \(\mathbb{Z}[G]\)-module on \(G^n\) and set \(C_{-1}(G) = 0\). Let \(\partial: C_n(G) \to C_{n-1}(G)\) be the map defined by

\[
\partial(g_1, \ldots, g_n) = (g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_ig_{i+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1}).
\]

The homology \(H_1(G)\) of the chain complex \((C_*(G), \partial)\) is called the group homology of \(G\).

Let \(A = \mathbb{Z}\) or \(\mathbb{R}\). The group cohomology \(H^*(G; A)\) with coefficients in \(A\) is the homology of the dual complex of \((C_*(G), \partial)\). An explicit cochain complex \((C^*(G; A); \delta)\) is given by

\[
C^n(G; A) = \{c: G^n \to A\}
\]

and

\[
\delta c(g_1, \ldots, g_{n+1}) = c(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} c(g_1, \ldots, g_n)
\]

for \(c \in C^n(G; A)\) and \(g_i \in G\). The subset \(C^n_b(G; A)\) of all bounded functions defines a subcomplex \((C_b^*(G; A), \delta)\). The cohomology \(H^*_b(G; A)\) of this subcomplex is called the bounded cohomology of \(G\) with coefficients in \(A\). We note that \(H^n_0(G; \mathbb{R}) = 0\) for all \(n \geq 1\) provided that \(G\) is amenable (see [Fr17, Theorem 3.6]). The inclusion \(C^*_b(G; A) \to C^*(G; A)\) induces a homomorphism \(c_G: H^*_b(G; A) \to H^*(G; A)\) called the comparison map. If \(G\) is Gromov-hyperbolic, the comparison map \(c_G: H^*_b(G; \mathbb{R}) \to H^*(G; \mathbb{R})\) is surjective ([Gro87]).

Remark 2.1. A group cocycle \(c \in C^n(G; A)\) is said to be normalized if

\[
c(g_1, \ldots, g_n) = 0
\]
whenever \( g_i = 1_G \) for some \( i \). It is known that every element of \( H^n(G; A) \) (resp. \( H^n_B(G; A) \)) can be represented by a normalized cocycle (resp. normalized bounded cocycle). ([Mac67, Section 6]; see also [Heu20, Proposition 2.1].)

We note that a one-cochain \( c \in C^1(G; A) \) is a one-cocycle if and only if it is a homomorphism from \( G \) to \( A \) by definition, and hence the first cohomology group \( H^1(G; A) \) is isomorphic to the vector space of \( A \)-valued homomorphisms \( \text{Hom}(G; A) \). The second cohomology group \( H^2(G; A) \) classifies the central \( A \)-extensions of \( G \) up to isomorphism, that is,

\[
H^2(G; A) \cong \{ \text{central } A \text{-extensions of } G \}/\{ \text{split extensions} \}
\]

(see [Bro82, (3.12)Theorem] for example). For a central \( A \)-extension \( 0 \to A \to E \to G \to 1 \), the corresponding cohomology class under (2.1) is called the Euler class of the central extension \( E \) and denoted by \( e(E) \). It is known that the Euler class \( e(E) \) is an obstruction to the existence of section homomorphisms \( G \to E \). In fact, the following holds:

**Lemma 2.2** (see [Fri17, Lemma 2.4]). Let \( 0 \to A \to E \xrightarrow{p} G \to 1 \) be a central extension, \( K \) a group, and \( \varphi: K \to G \) a group homomorphism. Then the pullback \( \varphi^*e(E) \) is equal to zero if and only if there exists a homomorphism \( \psi: K \to E \) such that \( p \circ \psi = \varphi \).

A function \( f: N \to A \) is said to be \( G \)-invariant if \( f(gwg^{-1}) = f(w) \) for every \( g \in G \) and \( w \in N \). Let \( H^1(N; A)^G \) denote the vector space of \( G \)-invariant \( A \)-valued homomorphisms. For an exact sequence of groups \( 1 \to N \xrightarrow{i} G \xrightarrow{p} \Gamma \to 1 \), the following five-term exact sequence holds:

\[
0 \to H^1(\Gamma; \mathbb{R}) \xrightarrow{p^*} H^1(G; \mathbb{R}) \xrightarrow{p^*} H^1(N; \mathbb{R})^G \to H^2(\Gamma; \mathbb{R}) \xrightarrow{p^*} H^2(G; \mathbb{R}).
\]

There also exists a five-term exact sequence of bounded cohomology (see [Mon01, Theorem 12.0.2]):

\[
0 \to H^2_b(\Gamma; \mathbb{R}) \xrightarrow{p^*} H^2_b(G; \mathbb{R}) \xrightarrow{i^*} H^2_b(N; \mathbb{R})^G \to H^3_b(\Gamma; \mathbb{R}) \xrightarrow{p^*} H^3_b(G; \mathbb{R}).
\]

Here \( H^2_b(N; \mathbb{R})^G \) is the invariant part of \( G \)-action on \( H^2_b(N; \mathbb{R}) \) induced from the conjugation \( G \)-action on the bounded cochain group \( C^2_b(N; \mathbb{R}) \).

### 2.2. Invariant quasimorphisms and the Bavard duality theorem for stable mixed commutator lengths.

A real-valued function \( \mu: G \to \mathbb{R} \) on a group \( G \) is called a homogeneous quasimorphism if there exists a non-negative real number \( D \) such that

\[
\mu(gh) \sim_D \mu(g) + \mu(h)
\]

for every \( g, h \in G \) and if it is a homomorphism on every cyclic subgroup of \( G \). The minimal value \( D(\mu) \) of such \( D \) is called the defect of \( \mu \): \( D(\mu) = \sup_{g, h \in G} |\mu(gh) - \mu(g) - \mu(h)| \). Every homogeneous quasimorphism \( \mu: G \to \mathbb{R} \) is \( G \)-invariant (recall the definition of \( G \)-invariance from Subsection 2.1), and hence satisfies

\[
|\mu([g, h])| \leq D(\mu)
\]

for every \( g, h \in G \) (see [Cal09, Section 2.3.3] for example). Let \( Q(G) \) denote the vector space of homogeneous quasimorphisms on \( G \). Clearly \( H^1(G; \mathbb{R}) \) is a subspace of \( Q(G) \).
In the celebrated paper [Bav91], Bavard established the relation between homogeneous quasimorphisms on $G$ and the stable commutator length $\text{scl}_G$, which is called the Bavard duality theorem.

**Theorem 2.3 ([Bav91]).** Let $G$ be a group and $y \in [G,G]$. Then the following holds:

$$\text{scl}_G(y) = \sup_{[\mu] \in Q(G)/H^1(G;\mathbb{R})} \frac{|\mu(y)|}{2D(\mu)}.$$ 

Note that the right-hand side of the equality in Theorem 2.3 is regarded as zero when $Q(G) = H^1(G;\mathbb{R})$.

Let $G$ be a group, $N$ a normal subgroup of $G$, and $Q(N)^G$ the vector space of $G$-invariant homogeneous quasimorphisms on $N$. Kawasaki–Kimura–Matsushita–Mimura proved the following Bavard duality theorem for stable mixed commutator lengths, which connects $\text{scl}_{G,N}$ and $G$-invariant homogeneous quasimorphisms on $N$.

**Theorem 2.4 ([KKMM20, Theorem 1.2]).** Let $G$ be a group, $N$ a normal subgroup, and $x \in [G,N]$. Then the following equality

$$\text{scl}_{G,N}(x) = \sup_{[\mu] \in Q(N)^G/H^1(N;\mathbb{R})^G} \frac{|\mu(x)|}{2D(\mu)}.$$ 

holds true.

2.3. The auxiliary lemma. Recall our symbol (1.1) for group conjugation. We note that the equality

(2.5) \[ [g^n, h] = g^{n-1}[g, h] \cdot g^{n-2}[g, h] \cdot \ldots \cdot g[g, h] \cdot [g, h] \]

holds for every $g, h \in G$ and for every positive integer $n > 0$.

**Lemma 2.5.** Let $G$ be a group, $N$ a normal subgroup, and $\mu \in Q(N)^G$. Let $k$ be a positive integer and $y_1, \ldots, y_k, z_1, \ldots, z_k \in G$. Assume that the following two conditions are satisfied:

(i) $[y_i, z_i] \in N$ for every $i = 1, \ldots, k$;
(ii) $[y_1, z_1] \cdot \ldots \cdot [y_k, z_k] \in [G, N]$.

Then for every $n \in \mathbb{N}$

$$[y_1^n, z_1] \cdot \ldots \cdot [y_k^n, z_k] \in [G, N]$$

and

$$[y_1^n, z_1] \cdot \ldots \cdot [y_k^n, z_k] \in [G, N]$$

hold.

**Proof.** We first show $[y_1^n, z_1] \cdot \ldots \cdot [y_k^n, z_k] \in [G, N]$. We set $\overline{g} = g[G, N] \in G/[G, N]$ for every $g \in G$. Note that, for every $g \in G$ and $w \in N$, we have $\overline{g} \cdot \overline{w} = \overline{w} \cdot \overline{g}$, and hence $\overline{g} \cdot \overline{w} \cdot \overline{g}^{-1} = \overline{w}$. Condition (i), together with (2.5), implies that

$$[y_1^n, z_1] = \overline{y_1}^{-1} \overline{y_1} \overline{y_1}^{-2} \overline{y_1} \overline{z_1} \overline{y_1} \overline{z_1} \ldots \overline{y_1} \overline{y_1} \overline{z_1} \overline{y_1} \overline{z_1} = ([y_1, z_1])^n.$$ 

Since $[\overline{y}_i, \overline{z}_i]$'s commute with each other by (i), we obtain

$$[\overline{y}_1^n, \overline{z}_1] \cdot [\overline{y}_k^n, \overline{z}_k] = ([\overline{y}_1, \overline{z}_1])^n \cdot ([\overline{y}_k, \overline{z}_k])^n = ([\overline{y}_1, \overline{z}_1] \cdot [\overline{y}_k, \overline{z}_k])^n$$

$$= ([y_1, z_1] \cdot [y_k, z_k])^n = 1_{G/[G, N]}.$$
Here the last equality comes from (ii). We can prove the latter in a similar manner, using

\[ [y_i, z_i^n] = [y_i, z_i] \cdot z_i [y_i, z_i] \cdot \ldots z_i^{n-1} [y_i, z_i]. \]

\[ \square \]

The following auxiliary lemma is one key to the proofs of Theorems 1.1 and 1.2.

**Lemma 2.6** (the auxiliary lemma). Let \( G \) be a group, \( N \) a normal subgroup, and \( \mu \in \mathbb{Q}(N)^G \). Let \( k \) be a positive integer, and \( y_1, \ldots, y_k, z_1, \ldots, z_k \in G \). Assume that the following three conditions are satisfied:

(i) same as condition (i) in Lemma 2.5;
(ii) same as condition (ii) in Lemma 2.5;
(iii) \( \lim_{n \to \infty} |\mu([y_1^n, z_1] \cdots [y_k^n, z_k])| = \infty \).

Then, for the sequence

\[ (x_n)_{n \in \mathbb{N}} = ([y_1^n, z_1] \cdots [y_k^n, z_k])_{n \in \mathbb{N}}, \]

we have

\[ \sup_{n \in \mathbb{N}} \text{scl}_G(x_n) < \infty \quad \text{but} \quad \lim_{n \to \infty} \text{scl}_{G,N}(x_n) = \infty. \]

In particular, \( \text{scl}_G \) and \( \text{scl}_{G,N} \) are not equivalent.

**Proof.** First, note that by (i) and (ii), Lemma 2.5 shows that \( (x_n)_{n \in \mathbb{N}} \) is a sequence in \([G, N]\). We have

\[ \sup_{n \in \mathbb{N}} \text{scl}_G(x_n) \leq \sup_{n \in \mathbb{N}} \text{cl}_G(x_n) \leq k < \infty. \]

Contrastingly, by condition (iii), Theorem 2.4 implies that

\[ \lim_{n \to \infty} \text{scl}_{G,N}(x_n) = \infty, \]

as desired (here we only use the estimate of \( \text{scl}_{G,N} \) by Theorem 2.4 from below: it is the easy direction).

\[ \square \]

In Lemma 7.2, we provide a sufficient condition to apply Lemma 2.6 that seems useful.

We next provide an example of commutators which satisfies conditions (i) and (ii) in Lemma 2.5.

**Definition 2.7.** We define words \( g_i \) and \( w_i \) on the alphabet \( \{a, b\} \) as follows:

\[ g_i = \begin{cases} \text{bab}^{-1} & i = 1 \\ \text{ba}^{2-i} b^{-1} a^{i-1} & i > 1 \end{cases} \]

and \( w_i = \begin{cases} [b, a] & i = 1 \\ [b, a^{2-i} b^{-1}] [b, a^{-1}] & i > 1. \end{cases} \)

By induction, we obtain the following.

**Lemma 2.8.** In the free group \( F_2 = \langle a, b \mid - \rangle \) of rank 2, the equality

\[ [g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}] = \text{ba}^{2-\ell} b^{-1} a^{-\ell} [a, b]^{\ell} a^{2-\ell} b a^{\ell-2} b^{-1} \]

holds for every \( \ell \geq 2 \).

We set \( y = \text{ba}^{2} b^{-1} \) and \( z = \text{ba}^{2} a^{-\ell} \). Since \([a, b]^{\ell} \) is trivial in the group \( R_{\ell} = \langle a, b \mid [a, b]^{\ell} \rangle \), we have the following:
Corollary 2.9. In $R_\ell$, the equality
\[ [g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}] = [y, z] \]
holds for every $\ell \geq 2$. In particular, $[y^n, z] \in [R_\ell, R_\ell']$ for every positive integer $n$.

2.4. The group of orientation preserving homeomorphisms of the circle. Recall from Subsection 1.3 that we have set $H = \text{Homeo}_+(S^1)$, the group of orientation preserving homeomorphisms of the circle $S^1$. We regard $S^1$ as $\mathbb{R}/\mathbb{Z}$ throughout this paper. For every $r \in \mathbb{R}$, let $T_r : \mathbb{R} \to \mathbb{R}$ be the homeomorphism defined by $T_r(x) = x + r$ for $x \in \mathbb{R}$. We set
\[ \tilde{H} = \{ \tilde{f} : \mathbb{R} \to \mathbb{R} \mid \tilde{f} \circ T_1 = T_1 \circ \tilde{f} \}. \]
Let $p : \tilde{H} \to H$ be the canonical projection. This projection gives rise to a central $\mathbb{Z}$-extension
\[ (2.6) \quad 0 \to \mathbb{Z} \to \tilde{H} \xrightarrow{\pi} H \to 1. \]
Eisenbud, Hirsch, and Neumann [EHN81] completely determined the commutator lengths of the elements of $\tilde{H}$ as follows: For every $\tilde{f} \in \tilde{H}$, we set
\[ m(\tilde{f}) = \min_{x \in \mathbb{R}} (\tilde{f}(x) - x) \quad \text{and} \quad m(\tilde{f}) = \max_{x \in \mathbb{R}} (\tilde{f}(x) - x). \]

Theorem 2.10 ([EHN81]). Let $\tilde{f}$ be an element of $\tilde{H}$. Let $n \geq 1$ be an integer. Then, the following two conditions on $\tilde{f}$ are equivalent:

1. $\text{cl}_{\tilde{H}}(\tilde{f}) \leq n$.
2. $m(\tilde{f}) < 2n - 1$ and $m(\tilde{f}) > 1 - 2n$.

In particular, $\tilde{f}$ may be expressed as a single commutator of $\tilde{H}$ if both of the inequalities $m(\tilde{f}) < 1$ and $m(\tilde{f}) > -1$ hold.

Poincaré [Poi82] introduced a homogeneous quasimorphism $\tau \in Q(\tilde{H})$ called the translation number. The translation number $\tau \in Q(\tilde{H})$ is defined by
\[ \tau(\tilde{f}) = \lim_{n \to \infty} \frac{\tilde{f}^n(0)}{n}. \]
This limit exists and defines a homogeneous quasimorphism with defect $D(\tau) = 1$ (see [Cal09, Lemma 2.40 and Proposition 2.92]).

In [Mat86], Matsumoto introduced the canonical Euler cocycle $\chi \in C^2(H; \mathbb{R})$. The cocycle $\chi$ is defined by
\[ (2.7) \quad \chi(f, g) = \tau(\tilde{f}g) - \tau(\tilde{f}) - \tau(\tilde{g}), \]
where $\tilde{f}$ and $\tilde{g}$ in $\tilde{H}$ are arbitrary lifts of $f$ and $g$ in $H$, respectively. Since $\tau$ is a homogeneous quasimorphism, the right-hand side of (2.7) also defines a bounded cocycle $\chi_b \in C^2_b(H; \mathbb{R})$, which is called the canonical bounded Euler cocycle.

It is known that the cohomology class $[\chi] \in H^2(H; \mathbb{R})$ corresponds to the Euler class $e_\mathbb{R} = e(\tilde{H}) \in H^2(H; \mathbb{Z})$ of central extension (2.6) under the change of coefficients homomorphism $H^2(H; \mathbb{Z}) \to H^2(H; \mathbb{R})$ (see [Fri17, Proposition 10.26] for example). We set $e_\mathbb{R} = [\chi] \in H^2(H; \mathbb{R})$ and call it the real Euler class of $H$. 
Hence there exists a one-cocycle $h$. By condition (1), we have $u$ is a quasimorphism (2.8) $\tilde{\tau}(f, r) = \tilde{\tau}(f, s)$ for every $(f, r, s) \in R$. This is written as $τ(f, r) · (g, s) = τ(fg, r + s)$. By abuse of notation, $(f, r)$ also denotes the element of $H_R$ represented by $(f, r) \in H \times R$. This group $H_R$ gives rise to a central $R$-extension

$$0 \to R \xrightarrow{i} H_R \xrightarrow{\tau} H \to 1,$$

where $i(r) = (id_R, r) \in H_R$ and $π((f, s)) = π(f)$ for every $r \in R$ and $(f, s) \in H_R$. It is verified that central $R$-extension (2.8) corresponds to the real Euler class $e_R \in H^2(H; R)$ (see [Mor16, Remark 1] for example).

Let us define a map $τ_R : H_R \to R$ by

$$(2.9) \quad τ_R((f, r)) = τ(f) + r.$$ 

This $τ_R$ is well-defined since $τ(f \circ T_1) = τ(f) + 1$. Moreover, this is a homogeneous quasimorphism since

$$τ_R((f, r)^n) = τ_R((f^n, nr)) = τ(f^n) + nr = n · τ_R((f, r))$$

and

$$|τ_R((f, r) · (g, s)) - τ_R((f, r)) - τ_R((g, s))| = |τ(fg) - τ(f) - τ(g)| \leq D(τ) = 1$$

for every $(f, r, g, s) \in H_R$ and for every integer $n$. In particular, we have $D(τ_R) = D(τ) = 1$. Note that for every $f, g \in H$, the equality

$$(2.10) \quad χ(f, g) = χ_{b}(f, g) = τ_R((f, r) · (g, s)) - τ_R((f, r)) - τ_R((g, s))$$

holds, where $(f, r, g, s) \in H_R$ are lifts of $f, g$, respectively.

3. A CONSTRUCTION OF INVARIANT HOMOGENEOUS QUASIMORPHISMS

Let $G$ be a group and $N$ a normal subgroup. In this section, we construct a $G$-invariant homogeneous quasimorphism on $N$. Set $Γ = G/N$.

**Lemma 3.1.** Let $G$ be a group, $N$ a normal subgroup, $i : N \to G$ the inclusion, and $p : G \to Γ$ the projection. Set $Γ = G/N$. Let $c_b \in C^2_b(G; R)$ be a bounded two-cocycle satisfying the following two conditions.

1. There exists a homogeneous quasimorphism $μ \in Q(N)$ such that $i^*c_b = δμ$.
2. There exist a normalized cocycle $A \in C^2(Γ; R)$ and a cochain $u \in C^1(G; R)$ such that $c_b - p^*A = δu$.

Then the restriction $u|_N : N \to R$ is a $G$-invariant homogeneous quasimorphism. This quasimorphism $u|_N$ is written as $u|_N = μ + h$ for some homomorphism $h \in H^1(N; R)$.

**Proof.** By condition (1), we have

$$δ(u|_N) = δ(i^*u) = i^*(δu) = i^*c_b = δμ.$$ 

Hence there exists a one-cocycle $h \in C^1(N; R)$ such that $u|_N = μ + h$. Since every one-cocycle is a homomorphism, $u|_N$ is a homogeneous quasimorphism on $N$. 

Now we show the $G$-invariance of $u|_N$. For every $g \in G$ and $w \in N$, we consider group two-chains $(g^{-1}, wg)$, $(w, g)$, and $(g^{-1}, g)$ in $C_2(G)$. Substituting them for $c_b - p^*A = \delta u$ by condition (2), we obtain
\begin{equation}
\begin{aligned}
  u(g^{-1}wg) &= u(wg) + u(g^{-1}) - c_b(g^{-1}, wg) + p^*A(g^{-1}, wg), \\
  u(wg) &= u(g) + u(w) - c_b(w, g) + p^*A(w, g), \\
  0 &= u(1_G) = u(g) + u(g^{-1}) - c_b(g^{-1}, g) + p^*A(g^{-1}, g).
\end{aligned}
\end{equation}
(3.1)
Since $N = \ker(p)$ and $A$ is normalized, we have
\begin{equation}
\begin{aligned}
  p^*A(g^{-1}, wg) &= p^*A(g^{-1}, g) \quad \text{and} \quad p^*A(w, g) = 0.
\end{aligned}
\end{equation}
(3.2)
Equalities (3.1) and (3.2) imply
\begin{equation}
\begin{aligned}
  u(g^{-1}wg) - u(w) &= c_b(g^{-1}, g) - c_b(w, g) - c_b(g^{-1}, wg).
\end{aligned}
\end{equation}
(3.3)
Since $w \in N$ is arbitrary and $u|_N$ is homogeneous, we have
\begin{equation}
\begin{aligned}
  |u(g^{-1}wg) - u(w)| = |u(g^{-1}w^n g) - u(w^n)| &= |c_b(g^{-1}, g) - c_b(w^n, g) - c_b(g^{-1}, w^n g)|
\end{aligned}
\end{equation}
for every $n \in \mathbb{Z}$. Since $c_b$ is a bounded cochain, the very right-hand side of (3.3) is uniformly bounded. This implies that $u(g^{-1}wg) - u(w) = 0$. \hfill \square

**Remark 3.2.** As an element of $Q(N)^G/(H^1(N; \mathbb{R})^G + i^*Q(G))$, the resulting $G$-invariant homogeneous quasimorphism $u|_N$ in Lemma 3.1 depends only on the class $c_G([c_b]) \in H^2(G; \mathbb{R})$. To see this, let $d_b \in C^2_b(G; \mathbb{R})$ be a bounded two-cocycle satisfying the equality $c_G([d_b]) = c_G([c_b]) \in H^2(G; \mathbb{R})$. We take $\nu \in Q(N), B \in C^2(\Gamma; \mathbb{R})$, and $v \in C^1(\Gamma; \mathbb{R})$ satisfying conditions (i) and (ii) in Lemma 3.1. Since $c_G([c_b - d_b]) = 0$, there exist a homogeneous quasimorphism $f \in Q(G)$ and a bounded cochain $z \in C^1_b(G; \mathbb{R})$ such that $\delta f + \delta z = c_b - d_b$. Then we have
\begin{equation}
\begin{aligned}
\delta(u|_N - v|_N) = i^*(\delta u - \delta v) = i^*(c_b - d_b) = \delta(i^*f + i^*z).
\end{aligned}
\end{equation}
Hence the difference $(u|_N - v|_N) - (i^*f + i^*z)$ is a homomorphism from $N$ to $\mathbb{R}$. In particular, the boundedness of $z$ implies that $i^*z$ is the zero map. Therefore, $u|_N - v|_N$ belongs to $H^1(N; \mathbb{R})^G + i^*Q(G)$.

**Remark 3.3.** Under the setting in Lemma 3.1, we further assume that $H^2_b(\Gamma; \mathbb{R}) = 0$ and the class $c_G([c_b]) \in H^2(G; \mathbb{R})$ is non-zero. Then, the resulting $G$-invariant homogeneous quasimorphism $u|_N$ is not contained in $H^1(N; \mathbb{R})^G + i^*Q(G)$. Indeed, if there exist an element $h' \in H^1(N; \mathbb{R})^G$ and an element $\mu' \in Q(G)$ such that $u|_N = h' + i^*\mu'$, then we have
\begin{equation}
\begin{aligned}
i^*(\delta \mu') = i^*(\delta u) = i^*c_b.
\end{aligned}
\end{equation}
Since $H^2_b(\Gamma; \mathbb{R}) = 0$, the homomorphism $i^*: H^2_b(G; \mathbb{R}) \to H^2_b(N; \mathbb{R})$ is injective by (2.3). Hence the bounded cohomology class $[c_b] \in H^2_b(G; \mathbb{R})$ coincides with $[\delta \mu']$. This contradicts the assumption since $c_G([\delta \mu']) = 0$.

Lemma 3.1 greatly works when the group $G$ has a representation $\rho: G \to \mathcal{H}(= \text{Homeo}_+(S^1))$, and in this case the resulting $G$-invariant homogeneous quasimorphism has a nice formula.

Recall that the real Euler class $e_\mathbb{R} \in H^2(\mathcal{H}; \mathbb{R})$ is the Euler class of extension (2.8).
Lemma 3.4. Let $G$ be a group and $N$ its normal subgroup. Set the group quotient $\Gamma = G/N$ and the projection $p: G \to \Gamma$. Let $\rho: G \to H$ be a homomorphism. Assume that the pullback $\rho^* e_R$ is contained in the image of the map $p^*: H^2(\Gamma; \mathbb{R}) \to H^2(G; \mathbb{R})$. Then there exists a map $\mu_\rho: N \to \mathbb{R}$ such that the following hold true.

1. $\mu_\rho$ is a $G$-invariant homogeneous quasimorphism of defect $D(\mu_\rho) \leq 1$.
2. Let $k$ be a positive integer. Then for every $g_1, \ldots, g_k \in G$ and $w_1, \ldots, w_k \in N$,

$$\mu_\rho([g_1, w_1] \cdots [g_k, w_k]) = -\tau_R \left([\rho(g_1), \rho(w_1)] \cdots [\rho(g_k), \rho(w_k)]\right).$$

Here $\tau_R: \mathcal{H} \to \mathbb{R}$ is the map defined by (2.9) and $\rho(g_i), \rho(w_i) \in \mathcal{H}_R$ are lifts of $\rho(g_i), \rho(w_i) \in \mathcal{H}$, respectively, for every $i = 1, \ldots, k$.

Proof. Since $\rho^* e_R$ is contained in the image of $p^*: H^2(\Gamma; \mathbb{R}) \to H^2(G; \mathbb{R})$, we have that $(\rho \circ i)^* e_R = 0$. Hence, by Lemma 2.2, there exists a homomorphism $\tilde{\rho}: N \to \mathcal{H}_R$ such that the following commutes:

$$\begin{array}{ccc}
N & \xrightarrow{\tilde{\rho}} & \mathcal{H}_R \\
\downarrow{i} & & \downarrow{\pi} \\
G & \xrightarrow{\rho} & \mathcal{H}.
\end{array}$$

Recall that $\chi_b \in C^2_b(\mathcal{H}; \mathbb{R})$ denotes the canonical bounded Euler cocycle. The cocycle $\rho^* \chi_b$ satisfies condition (1) in Lemma 3.1 because

$$i^* \rho^* \chi_b = \tilde{\rho}^* \pi^* \chi_b = \tilde{\rho}^*(-\delta \tau_R) = \delta(-\tilde{\rho}^* \tau_R)$$

by (2.10) and (3.4). Remark 2.1, together with the assumption, implies that condition (2) in Lemma 3.1 holds, that is, there exist a normalized cocycle $A \in C^2(\Gamma; \mathbb{R})$ and a cochain $u \in C^1(G; \mathbb{R})$ such that $\rho^* \chi_b - p^* A = \delta u$. We set $\nu_{\rho,A,u} = u|_N$. In what follows, we will see that this $\nu_{\rho,A,u}$ satisfies the conditions of $\mu_\rho$ in the lemma. By Lemma 3.1, $\nu_{\rho,A,u}$ is a $G$-invariant homogeneous quasimorphism on $N$ and $\nu_{\rho,A,u} = -\tilde{\rho}^* \tau_R + h$ for some homomorphism $h \in H^1(\mathbb{N}; \mathbb{R})$. Since $D(\tau_R) = 1$, we obtain (1).

Next we show (2). For $g_1, \ldots, g_k \in G$ and $w_1, \ldots, w_k \in N$, we define a group two-chain $\sigma \in C^2_2(G)$ by

$$\sigma = \sum_{i=1}^k (g_i, g_i^{-1}) + \sum_{i=1}^k (w_i, w_i^{-1}) + 2k(1_G, 1_G)$$

$$-((g_1, w_1) + (g_1 w_1, g_1^{-1}) + (g_1 w_1 g_1^{-1}, w_1^{-1}) + ([g_1, w_1], g_2)$$

$$+ ([g_1, w_1] g_2, w_2) + ([g_1, w_1] g_2 w_2, g_2^{-1}) + \cdots$$

$$+ ([g_1, w_1] \cdots [g_k-1, w_{k-1}] g_k w_k, g_k^{-1}) + ([g_1, w_1] \cdots [g_{k-1}, w_{k-1}] g_k w_k g_k^{-1}, w_k^{-1})).$$

This cochain satisfies $d\sigma = [g_1, w_1] \cdots [g_k, w_k]$. Hence we have

$$\nu_{\rho,A,u}([g_1, w_1] \cdots [g_k, w_k]) = u(d\sigma) = \delta u(\sigma) = (\rho^* \chi_b - p^* A)(\sigma) = \chi_b(\rho_\ast \sigma) - A(p_\ast \sigma).$$

Since $A$ is a normalized cochain, $A(p_\ast \sigma) = 0$ holds. Let $\widetilde{\rho(g_i)}, \widetilde{\rho(w_i)} \in \mathcal{H}_R$ be lifts of $\rho(g_i), \rho(w_i) \in \mathcal{H}$, respectively, for every $i = 1, \ldots, k$. Let $\tilde{\sigma} \in C^2_2(\mathcal{H}_R)$ be the two-chain
defined by replacing $g_i, w_i$, and $1_G$ in each term of $\sigma$ by $\rho(g_i), \rho(w_i)$, and $1_{\tilde{H}_R}$, respectively, for every $i = 1, \ldots, k$. Then $\pi_* \sigma = \rho_* \sigma$. Hence, by $\chi_b = -\pi^*(\delta \tau_R)$ and $\delta \tau_R(f, f^{-1}) = 0$ for every $f \in \tilde{H}_R$, we have

$$\chi_b(\rho_* \sigma) = -\delta \tau_R(\sigma) = -\tau_R \left( \left[ \rho(g_1), \rho(w_1) \right] \cdots \left[ \rho(g_k), \rho(w_k) \right] \right).$$

This completes the proof of (2). Now we have checked that $\mu_\rho = \nu_{\rho, A, u}$ does the job. \hfill \Box

Remark 3.5. In the proof of Lemma 3.4, strictly speaking, the homogeneous quasimorphism $\mu_\rho = \nu_{\rho, A, u}$ does depend also on $A$ and $u$. However, if $N$ is contained in $G'$, then for two choices $(A_1, u_1)$ and $(A_2, u_2)$ of $A$ and $u$ the difference $\nu_{\rho, A_1, u_1} - \nu_{\rho, A_2, u_2}$ lies in $H^1(N; \mathbb{R})^G$. Therefore, under this additional assumption, the class $[\mu_\rho]$ in $\text{Q}(N)^G/H^1(N; \mathbb{R})^G$ is determined by $\rho$ and does not depend on the choice of $A$ or $u$.

### 4. One-relator groups with torsion

Let $\ell$ be an integer greater than 1. Recall from Subsection 1.3 that we have set $R_\ell = \langle a, b | [a, b]^{\ell} \rangle$. In this section, we show the following:

**Theorem 4.1.** In the setting above, $\text{scl}_{R_\ell}$ and $\text{scl}_{R_\ell', R'_\ell}$ are not equivalent.

Let $p: R_\ell \to R_\ell/R'_\ell \simeq \mathbb{Z}^2$ be the projection.

**Lemma 4.2.** The map $p^*: H^2(R_\ell/R'_\ell; \mathbb{R}) \to H^2(R_\ell; \mathbb{R})$ is an isomorphism.

**Proof.** Let $\langle \langle a, b \rangle^\ell \rangle$ be the normal closure of $[a, b]^{\ell}$ in $F_2$. By five-term exact sequence (2.2) applied to $1 \to \langle \langle a, b \rangle^\ell \rangle \to F_2 \to R_\ell \to 1$ and the triviality of $H^2(F_2; \mathbb{R})$, the map

$$H^1(\langle \langle a, b \rangle^\ell \rangle; \mathbb{R})^F_2 \to H^2(R_\ell; \mathbb{R})$$

is an isomorphism. Since an $F_2$-invariant homomorphism from $\langle \langle a, b \rangle^\ell \rangle$ to $\mathbb{R}$ is determined by its value on $[a, b]^{\ell}$, we have $H^2(R_\ell; \mathbb{R}) \simeq H^1(\langle \langle a, b \rangle^\ell \rangle; \mathbb{R})^F_2 \simeq \mathbb{R}$.

Five-term exact sequence (2.2) applied to $1 \to R'_\ell \to R_\ell \to R_\ell/R'_\ell \to 1$ asserts that

$$\cdots \to H^1(R'_\ell; \mathbb{R})^{R_\ell} \to H^2(R_\ell/R'_\ell; \mathbb{R}) \to H^2(R_\ell; \mathbb{R})$$

is exact. Since $H^1(R'_\ell; \mathbb{R})^{R_\ell} = 0$ and $H^2(R_\ell/R'_\ell; \mathbb{R}) \cong \mathbb{R}$, the map $H^2(R_\ell/R'_\ell; \mathbb{R}) \to H^2(R_\ell; \mathbb{R})$ is an injection from $\mathbb{R}$ to $\mathbb{R}$. This implies the lemma. \hfill \Box

For the proof of Theorem 4.1, we employ the words $g_1, \ldots, g_{\ell-1}, w_1, \ldots, w_{\ell-1}$ on the alphabet $\{a, b\}$ defined in Definition 2.7. We employ two more words $y = ba^2b^{-1}$ and $z = ba^2a^{-\ell}$. Corollary 2.9 states that the equality

$$[y, z] = [g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}]$$

holds in $R_\ell$.

**Proof of Theorem 4.1.** In this proof, we regard $y = ba^2b^{-1}$ and $z = ba^2a^{-\ell}$ as elements in $R_\ell$. Recall that the map $T_r: \mathbb{R} \to \mathbb{R}$ is defined by $T_r(x) = x + r$ for every $r \in \mathbb{R}$. By Theorem 2.10, there exist $\tilde{f}, \tilde{g} \in \tilde{H}$ such that

$$T_{r-\ell} = [\tilde{f}, \tilde{g}]$$
holds. We set $\tilde{\alpha} = (\tilde{f}, 0)$ and $\tilde{\beta} = (\tilde{g}, 0) \in \mathcal{H}_R$. Then we have
\[
[\tilde{\alpha}, \tilde{\beta}] = ((\tilde{f}, \tilde{g}), 0) = (T_{\ell-1}, 0).
\]
We set $\alpha = \pi(\tilde{\alpha})$ and $\beta = \pi(\tilde{\beta})$ in $\mathcal{H}$. Then the map $\rho_\ell : R_\ell \rightarrow \mathcal{H}$ defined by $\rho_\ell(a) = \alpha$ and $\rho_\ell(b) = \beta$ is a well-defined homomorphism. Indeed,
\[
\rho_\ell([a, b]^\ell) = [\alpha, \beta]^\ell = [\pi(\tilde{\alpha}), \pi(\tilde{\beta})]^\ell = \pi([(\tilde{\alpha}, \tilde{\beta})^\ell] = \pi((T_{\ell-1}, 0)) = 1_\mathcal{H}.
\]
By Lemma 4.2, we can apply Lemma 3.4 to the group pair $(G, N) = (R_\ell, R'_\ell)$ and to this homomorphism $\rho_\ell : R_\ell \rightarrow \mathcal{H}$. (In fact, for this pair, Lemma 3.4 applies to an arbitrary homomorphism $\rho : R_\ell \rightarrow \mathcal{H}$.) Thus, we obtain a map $\mu_{\rho_\ell} : R'_\ell \rightarrow \mathbb{R}$ satisfying (1) and (2) of Lemma 3.4. In particular, $\mu_{\rho_\ell} \in Q(R'_\ell)^{R_\ell}$ and $D(\mu_{\rho_\ell}) \leq 1$. In what follows, we will verify that Lemma 2.6 applies to the case where $k = 1$, $y_1 = y$, $z_1 = z$ and $\mu = \mu_{\rho_\ell}$. Condition (i) trivially holds since we take $N = R'_\ell$. Condition (ii) holds by Corollary 2.9. Therefore, it only remains to show that condition (iii) is satisfied, namely,
\[
\lim_{n \rightarrow \infty} |\mu_{\rho_\ell}([y^n, z])| = \infty
\]
holds.

We define elements $\widetilde{\rho_\ell}(g_i)$ and $\widetilde{\rho_\ell}(w_i)$ of $\mathcal{H}_R$ by replacing each $a$ and $b$ in the words $g_i$ and $w_i$ in Definition 2.7 with elements $\tilde{\alpha}$ and $\tilde{\beta}$ of $\mathcal{H}_R$, respectively, for every $i = 1, \ldots, \ell - 1$. We set $\widetilde{\rho_\ell}(y) = \tilde{\beta}^{-1}\tilde{\alpha}^{-1}$. These $\widetilde{\rho_\ell}(g_i)$, $\widetilde{\rho_\ell}(w_i)$, and $\widetilde{\rho_\ell}(y)$ are lifts of $\rho_\ell(g_i)$, $\rho_\ell(w_i)$, and $\rho_\ell(y)$, respectively, for every $i = 1, \ldots, \ell - 1$. By Lemma 2.8 and the fact that $[\tilde{\alpha}, \tilde{\beta}]^\ell = (T_{\ell-1}, 0)$ is contained in the center of $\mathcal{H}_R$, we obtain
\[
[\rho_\ell(g_1), \rho_\ell(w_1)] \cdots [\rho_\ell(g_{\ell-1}), \rho_\ell(w_{\ell-1})] = \tilde{\beta}^{-1}\tilde{\alpha}^{-1} \cdot [\tilde{\alpha}, \tilde{\beta}]^\ell = \tilde{\beta}^{-1}\tilde{\alpha}^{-1} [\tilde{\alpha}, \tilde{\beta}]^\ell.
\]
Let $n \in \mathbb{N}$. By (2.5) and the equality $[y^n, z] = [g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}]$, we have
\[
[y^n, z] = y^{-n}([g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}]) \cdot y^{n-2}([g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}]) \cdots y([g_1, w_1] \cdots [g_{\ell-1}, w_{\ell-1}]).
\]
By (4.2), we obtain
\[
\begin{align*}
\rho_\ell(y)^{-n}([\rho_\ell(g_1), \rho_\ell(w_1)] & \cdots [\rho_\ell(g_{\ell-1}), \rho_\ell(w_{\ell-1})]) \cdot \rho_\ell(y)^{n-2}([\rho_\ell(g_1), \rho_\ell(w_1)] \cdots [\rho_\ell(g_{\ell-1}), \rho_\ell(w_{\ell-1})]) \\
\cdots \rho_\ell(y) & ([\rho_\ell(g_1), \rho_\ell(w_1)] \cdots [\rho_\ell(g_{\ell-1}), \rho_\ell(w_{\ell-1})]) \cdot ([\rho_\ell(g_1), \rho_\ell(w_1)] \cdots [\rho_\ell(g_{\ell-1}), \rho_\ell(w_{\ell-1})]) \\
= & \rho_\ell(y)^{-n}([\tilde{\beta}^{-1}, \tilde{\alpha}^{-1}]) \cdot \rho_\ell(y)^{n-2}([\tilde{\beta}^{-1}, \tilde{\alpha}^{-1}]) \cdots \rho_\ell(y)([\tilde{\beta}^{-1}, \tilde{\alpha}^{-1}]) \cdot ([\tilde{\beta}^{-1}, \tilde{\alpha}^{-1}]^n \\
= & \tilde{\beta}^2([\tilde{\beta}^{-1}, \tilde{\alpha}^{-1}]^n) \\
= & \tilde{\beta}^2([\tilde{\beta}^{-1}, \tilde{\alpha}^{-1}]^n).
\end{align*}
\]
Here the last equality comes from (2.5). Hence, by Lemma 3.4, we obtain
\[ \mu_{\rho_\ell}([y^n, z]) = -\tau_\mathbb{R} \left( \tilde{\alpha}^\ell \tilde{\beta}^{-n}, \tilde{\alpha}^{-\ell} \right) \cdot (\tilde{\alpha}, \tilde{\beta})^n = -\tau_\mathbb{R} \left( \tilde{\alpha}^\ell \tilde{\beta}^{-n}, \tilde{\alpha}^{-\ell} \right) - n(\ell - 1). \]

By (2.4), we have \(|\tau_\mathbb{R} \left( \tilde{\alpha}^\ell \tilde{\beta}^{-n}, \tilde{\alpha}^{-\ell} \right)| \leq D(\tau_\mathbb{R}) = 1\). Hence we conclude that
\[ |\mu_{\rho_\ell}([y^n, z])| \geq n(\ell - 1) - 1. \]

Then, (4.1) immediately follows from (4.3).

Therefore, Lemma 2.6 applies: the sequence \(([y^n, z])_{n \in \mathbb{N}}\) witnesses the non-equivalence of \(\text{scl}_{R_\ell}^{G_\ell}\) and \(\text{scl}_{R_\ell, R_\ell'}\). \(\square\)

5. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. Let \(G_\ell\) be the fundamental group of a closed oriented surface of genus \(\ell \geq 2\). The group \(G_\ell\) has the following presentation:
\[ G_\ell = \langle a_1, \ldots, a_\ell, b_1, \ldots, b_\ell \mid [a_1, b_1] \cdots [a_\ell, b_\ell] \rangle. \]

Let \(q: G_\ell \to R_\ell\) be a map defined by
\[ q(a_i) = a \quad \text{and} \quad q(b_i) = b \]
for each \(i = 1, \ldots, \ell\). Since \(q([a_1, b_1] \cdots [a_\ell, b_\ell]) = [a, b]^\ell = 1_{R_\ell}\), the map \(q\) is a well-defined homomorphism. Let \(G_\ell'\) be the commutator subgroup and \(q': G_\ell' \to R_\ell'\) the induced map.

Let \(\rho_\ell: R_\ell \to \mathcal{H}\) be the homomorphism and \(\mu_{\rho_\ell}\) the \(R_\ell\)-invariant homogeneous quasimorphism on \(R_\ell'\) used in the proof of Theorem 4.1. From the proof of Theorem 4.1, we continue to use the representation \(\rho_\ell: R_\ell \to \mathcal{H}\) and the resulting quasimorphism \(\mu_{\rho_\ell} \in Q(R_\ell')^{R_\ell}\). Then we obtain a quasimorphism \(q'^* \mu_{\rho_\ell} \in Q(G_\ell')^{G_\ell}\). This approach of employing \(q: G_\ell \to R_\ell\) was suggested to the authors by Morimichi Kawasaki.

**Proof of Theorem 1.1.** We set
\[ (5.1) \quad y_i = b_i a_i^2 b_i^{-1}, \quad z_i = b_i a_i^2 a_i^{-\ell} \in G_\ell \]
for every \(i = 1, 2, \ldots, \ell\). In what follows, we will verify that conditions (i), (ii) and (iii) in Lemma 2.6 are satisfied for \(k = \ell, y_1, \ldots, y_\ell, z_1, \ldots, z_\ell\) and \(\mu = q'^* \mu_{\rho_\ell}\). Condition (i) trivially holds since \(N = G_\ell'\). To verify condition (ii), we first claim that
\[ (5.2) \quad [y_1, b_i a_i^2 a_i^{-1}] \cdots [y_\ell, b_i a_i^2 a_i^{-1}] \in [G_\ell, G_\ell'] \]

To show (5.2), set \(g = g[G_\ell, G_\ell'] \in G_\ell/[G_\ell, G_\ell']\) for \(g \in G_\ell\). Then, as we argued in the proof of Lemma 2.5, we have \(\overline{gw} = \overline{w}g\) and (in particular) \(\overline{gw} = \overline{w}\) for every \(g \in G_\ell\) and \(w \in G_\ell'\). Also, recall the surface group relation \([a_1, b_1] \cdots [a_\ell, b_\ell] = 1_{G_\ell}\). Thus, we obtain the following
applies, and the sequence

\[ y_1, b_1 a_1^{-1}, \ldots, y_k, b_k a_k^{-1} \]  

\[ y_1, b_1 a_1^{-1}, \ldots, y_k, b_k a_k^{-1} \]

\[ b_1 a_1^{-1}, \ldots, b_k a_k^{-1} \]

\[ a_1, b_1^{-1}, \ldots, a_k, b_k^{-1} \]

\[ a_1, b_1, \ldots, a_k, b_k^{-1} \]

\[ 1_G^{-1} = 1_G, \]

hence obtaining (5.2). Then by Lemma 2.5, we conclude that condition (ii) holds.

Finally, we verify condition (iii). Let \( n \in \mathbb{N} \). Since

\[ q^* \mu_{\rho_\ell}([y_1^n, z_1] \cdots [y_k^n, z_k]) = \mu_{\rho_\ell}([y^n, z]^{\ell}) = \ell \cdot \mu_{\rho_\ell}([y^n, z]), \]

we obtain

\[ |q^* \mu_{\rho_\ell}([y_1^n, z_1] \cdots [y_k^n, z_k])| \geq \ell(n - 1) \]

by (4.3). Inequality (5.3) in particular yields

\[ \lim_{n \to \infty} |q^* \mu_{\rho_\ell}([y_1^n, z_1] \cdots [y_k^n, z_k])| = \infty, \]

verifying condition (iii). Therefore Lemma 2.6 applies, and the sequence

\[ (x_n)_{n \in \mathbb{N}} = ([y_1^n, z_1] \cdots [y_k^n, z_k])_{n \in \mathbb{N}} \]

witnesses the non-equivalence of scl\( G_\ell \) and scl\( G_\ell, G'_\ell \).

\[ \square \]

**Remark 5.1.** The \( G_\ell \)-invariant homogeneous quasimorphism \( q^* \mu_{\rho_\ell} \in \text{Q}(G'_\ell)^{G_\ell} \) defines a non-zero element of \( \text{Q}(G'_\ell)^{G_\ell}/(\text{H}^1(G'_\ell)^{G_\ell} + i^* \text{Q}(G_\ell)) \). Indeed, if not, then we may take \( h \in \text{H}^1(G'_\ell)^{G_\ell} \) and \( \mu \in \text{Q}(G_\ell) \) such that \( q^* \mu_{\rho_\ell} = h + i^* \mu \). Then we have

\[ \sup_{n \in \mathbb{N}} |q^* \mu_{\rho_\ell}(x_n)| = \sup_{n \in \mathbb{N}} |\mu(x_n)| \leq (2\ell - 1)D(\mu) < \infty, \]

contradicting \( \lim_{n \to \infty} |q^* \mu_{\rho_\ell}(x_n)| = \infty \). Here, note that \( h(x_n) = 0 \) since \( x_n \in [G_\ell, G'_\ell] \).

**6. Proof of Theorem 1.2**

We recall our setting from the introduction as follows. Let \( X \) be a hyperbolic 3-manifold that fibers over the circle with fiber \( \Sigma_\ell \). Then there exists \( \psi \in \text{Aut}_+(G_\ell) \) such that \( \pi_1(X) = G_\ell \ltimes \psi \mathbb{Z} \). Using this \( \psi \), we set \( G_\psi = \pi_1(X) \). The abelianization map \( \text{Ab}_{G_\ell} : G_\ell \to \mathbb{Z}^{2\ell} \) induces a surjection \( p_\psi : G_\psi \to \mathbb{Z}^{2\ell} \ltimes_{s_\psi} \mathbb{Z} \), where \( s_\psi : \text{Aut}_+(G_\ell) \to \text{Sp}(2\ell, \mathbb{Z}) \) is the symplectic representation, which is induced by \( \text{Ab}_{G_\ell} \). Set \( N_\psi = \text{Ker}(p_\psi) \); we have \( N_\psi = \iota(G'_\ell) \), where \( \iota : G_\ell \to G_\psi = G_\ell \ltimes \psi \mathbb{Z} \) is the natural inclusion. Set the group quotient \( \Gamma_\psi = G_\psi/N_\psi \simeq \mathbb{Z}^{2\ell} \ltimes_{s_\psi} \mathbb{Z} \). It was essentially shown in the proof of Theorem 1.2 of [KKM+21] that the pullback \( \text{H}^2(\Gamma_\psi; \mathbb{R}) \to \text{H}^2(G_\psi; \mathbb{R}) \) of the quotient map is surjective. Hence every representation \( G_\psi \to \mathcal{H} \) satisfies the condition in Lemma 3.4.
Let \( \iota: G_\ell \rightarrow G_\psi \) be the natural inclusion. It is known that there exists a representation \( \overline{\rho}: G_\psi \rightarrow \mathcal{H} \) whose restriction \( \rho = \overline{\rho} \circ \iota: G_\ell \rightarrow \text{PSL}(2, \mathbb{R}) \rightarrow \mathcal{H} \) is a Fuchsian representation (see [KKM19, Section 7.1] for example). We apply Lemma 3.4 to this representation \( \overline{\rho} \) and take a \( G_\psi \)-invariant homogeneous quasimorphism \( \mu_\overline{\rho}: N_\psi \rightarrow \mathbb{R} \).

Since the pullback \( \text{H}^2(G_\ell/G_\ell'; \mathbb{R}) \rightarrow \text{H}^2(G_\ell; \mathbb{R}) \) is also surjective, the Fuchsian representation \( \rho: G_\ell \rightarrow \mathcal{H} \) fulfills the assumption in Lemma 3.4. We take the \( G_\ell \)-invariant homogeneous quasimorphism \( \mu_\rho: G_\ell' \rightarrow \mathbb{R} \) as in Lemma 3.4. By condition (ii) of Lemma 3.4, the pullback \( \iota^* \mu_\overline{\rho} \) coincides with \( \mu_\rho \) on \( [G_\ell, G_\ell'] \). Remark 3.3, together with the fact that the class \( \rho \in \mathbb{E} \) is non-zero, the quasimorphism \( \mu_\rho \) defines a non-zero element of \( Q(G_\ell'^\ell)/\text{H}^1(G_\ell'^\ell + \iota^* Q(G_\ell)) \). Here \( i: G_\ell' \rightarrow G_\ell \) is the inclusion.

The quasimorphism \( q^* \mu_\rho \) used in the proof of Theorem 1.1 also defines a non-zero element of \( Q(G_\ell'^\ell)/\text{H}^1(G_\ell'^\ell + \iota^* Q(G_\ell)) \) by Remark 5.1. By using the following theorem, we can compare these two quasimorphisms up to \( \text{H}^1(G_\ell'^\ell + \iota^* Q(G_\ell)) \).

**Theorem 6.1 ([KKM+21, Theorem 1.1]).** We have

\[
\text{dim}_\mathbb{R} \left( Q(G_\ell'^\ell)/\text{H}^1(G_\ell'^\ell + \iota^* Q(G_\ell)) \right) = 1.
\]

By Theorem 6.1, there exist \( h \in \text{H}^1(G_\ell'^\ell, \mu \in Q(G_\ell) \), and a non-zero constant \( a \) such that

\[
\mu_\rho = a \cdot (q^* \mu_\rho) + h + \iota^* \mu.
\]

**Lemma 6.2.** For the non-zero constant \( a \) above, the inequality

\[
|\mu_\rho(x_n)| \geq |a| \cdot (\ell(n(\ell - 1)) - (2\ell - 1)D(\mu))
\]

holds for every positive integer \( n \). Here \( x_n \) is the element of \( G_\ell \) defined by (5.4) and (5.1).

**Proof.** Let \( n \in \mathbb{N} \). We have shown that \( x_n \in [G_\ell, G_\ell'] \) in the proof of Theorem 1.1. Since \( h \) is \( G_\ell \)-invariant homomorphism, we have \( h(x_n) = h([y_1^n, z_1] \cdots [y_\ell^n, z_\ell]) = 0 \). Hence we obtain

\[
\mu_\rho(x_n) = a \cdot q^* \mu_\rho(x_n) + \mu(x_n).
\]

Observe that \( |\mu(x_n)| = |\mu([y_1^n, z_1] \cdots [y_\ell^n, z_\ell])| \leq (2\ell - 1)D(\mu) \). Therefore, we obtain

\[
|\mu_\rho(x_n)| \geq |a| \cdot |q^* \mu_\rho(x_n)| - (2\ell - 1)D(\mu).
\]

Hence (5.3) completes the proof. \( \square \)

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( y_1, \ldots, y_\ell \) and \( z_1, \ldots, z_\ell \) are the elements of \( G_\ell \) defined in (5.1). Let \( x_n \in G_\ell \) be defined in (5.4). We set \( \eta_i = \iota(y_i) \) and \( \zeta_i = \iota(z_i) \) for every \( i = 1, \ldots, \ell \) and

\[
\xi_n = [\eta_1^n, \zeta_1] \cdots [\eta_\ell^n, \zeta_\ell] \quad (= \iota(x_n))
\]

for every \( n \in \mathbb{N} \). Take the quasimorphism \( \mu_\overline{\rho} \in Q(N_\psi)^G_\psi \) constructed in the first part of this section. In what follows, we will prove that conditions (i), (ii) and (iii) in Lemma 2.6 are fulfilled for \( k = \ell, y_1 = \eta_1, \ldots, y_\ell = \eta_\ell, z_1 = \zeta_1, \ldots, z_\ell = \zeta_\ell, \) and \( \mu = \mu_\overline{\rho} \). Condition (i) is clear: recall that \( N_\psi = \iota(G_\ell') \). Since \( x_n \in [G_\ell, G_\ell'] \), we have \( \xi_n \in [\iota(G_\ell), N_\psi] \subset [G_\psi, N_\psi] \).
Hence condition (ii) holds. Finally, we discuss condition (iii). Note that 
\( \nu^* \mu_{\overline{\Psi}} = \mu_{\rho} \) on 
\([G_\ell, G'_\ell]\) by Lemma 3.4 (2). Hence Lemma 6.2 implies that 
\[
|\mu_{\overline{\Psi}}(\xi_n)| = |\mu_{\rho}(x_n)| \\
\geq |a| \cdot (\ell(n(\ell - 1)) - (2\ell - 1)D(\mu)).
\]
By recalling that \( a \neq 0 \), we conclude that 
\[
\lim_{n \to \infty} |\mu_{\overline{\Psi}}(\xi_n)| = \infty,
\]
thus verifying condition (iii). Therefore, Lemma 2.6 applies, and the sequence \((\xi_n)_{n \in \mathbb{N}}\) witnesses the non-equivalence of \( \text{scl}_{G, \psi} \) and \( \text{scl}_{G, \psi, N_\psi} \).

\[\square\]

Remark 6.3. In [KKM+21], Kawasaki, Kimura and the authors showed that in several cases \( \text{scl}_G \) and \( \text{scl}_{G, N} \) are equivalent. Their results are summarized as follows: if
\[
(6.1) \quad Q(N)^G = H^1(N)^G + i^* Q(G),
\]
then \( \text{scl}_G \) and \( \text{scl}_{G, N} \) are equivalent; moreover if \( \Gamma = G/N \) is solvable, then \( \text{scl}_G \) and \( \text{scl}_{G, N} \) coincide on \([G, N]\) ([KKM+21, Theorem 2.1]). In [KKM+21], they also showed that (6.1) holds for the pair \((G, N) = (G, G')\), where \( G \) is a free group or the fundamental group of a non-orientable closed connected surface. Therefore, for these examples, \( \text{scl}_G \) and \( \text{scl}_{G, N} \) coincide on \([G, N]\). As we stated as Theorem 6.1 ([KKM+21, Theorem 1.1]), the dimension of \( Q(G')^G/(H^1(G')^G + i^* Q(G)) \) equals 1 for \( \ell \geq 2 \). Therefore Theorem 1.1 implies that \( \text{scl}_G \) and \( \text{scl}_{G, N} \) can be non-equivalent even if the space \( Q(N)^G/(H^1(N)^G + i^* Q(G)) \) is only 1-dimensional.

We say that a surjective group homomorphism \( p: G \to \Gamma \) \textit{virtually splits} if there exists a subgroup \( \Lambda \) of finite index of \( \Gamma \) and a group homomorphism \( s: \Lambda \to G \) satisfying \( p(s(\gamma)) = \gamma \) for every \( \gamma \in \Lambda \). In [KKM20, Theorem 1.5], Kawasaki–Kimura–Matsushita–Mimura showed that \( \text{scl}_{G, N} \leq 2 \cdot \text{scl}_G \) on \([G, N]\) if the projection \( G \to G/N \) virtually splits. In this case, we have \( Q(N)^G = i^* Q(G) \); see also [KK19, Proposition 1.4].

7. Concluding remarks

7.1. Non-extendable quasimorphisms and taut foliations. Let \( X \) be a hyperbolic 3-manifold that fibers over the circle with fiber \( \Sigma_\ell \). Let \( G_\psi, N_\psi, \) and \( \psi \in \text{Aut}_+(G_\ell) \) be as in Theorem 1.2. Let \( i: N_\psi \to G_\psi \) be the inclusion. It was shown in [KKM+21, Theorem 1.2] that the space \( Q(N_\psi)^G_\psi/(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^* Q(G_\psi)) \) is isomorphic to \( H^2(G_\psi; \mathbb{R}) \), and the dimension of them is equal to \( 1 + \dim(\text{Ker}(I_{2\ell} - s(\psi))) \). Here \( I_{2\ell} \) denotes the identity matrix of size \( 2\ell \).

We used the element \( \mu_{\overline{\Psi}} \) of \( Q(N_\psi)^G_\psi \) in the proof of Theorem 1.2, which is non-zero in \( Q(N_\psi)^G_\psi/(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^* Q(G_\psi)) \). This \( G_\psi \)-invariant homogeneous quasimorphism is constructed by Lemma 3.4 applied to the representation \( \overline{\Psi}: G_\psi \to \mathcal{H} \). This \( \overline{\Psi} \) is (the most basic) one of the universal circle representations due to Thurston, that is, the representation corresponds to the taut foliation on \( X \) whose leaves are the fibers. We refer to [CD03], [Cal07], and [BK20] for taut foliations and universal circle representations.

For a taut foliation \( \mathcal{F} \) on \( X \), let \( p_\mathcal{F}: G_\psi \to \mathcal{H} \) be the universal circle representation defined via \( \mathcal{F} \). As was mentioned in Section 6, the pullback \( H^2(\Gamma_\psi; \mathbb{R}) \to H^2(G_\psi; \mathbb{R}) \) is surjective.
(Recall that $\Gamma_\psi = G_\psi / N_\psi$ and that it is solvable.) Hence every representation $\rho_\mathcal{F}$ gives rise to a $G_\psi$-invariant homogeneous quasimorphism $\mu_{\rho_\mathcal{F}}$ on $N_\psi$ via Lemma 3.4. Here, note that $\mu_{\rho_\mathcal{F}} (= \nu_{\rho_\mathcal{F}, A, u})$ itself does depend also on the choices of cochains $A$ and $u$. However, as we discussed in Remark 3.5, the class $[\mu_{\rho_\mathcal{F}}]$ in $Q(N_\psi)^{G_\psi}/H^1(N_\psi; \mathbb{R})^{G_\psi}$ is uniquely determined by $\rho_\mathcal{F}$. Now we define $\mu_\mathcal{F}$ as the class $[\mu_{\rho_\mathcal{F}}]$ in $Q(N_\psi)^{G_\psi}/(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^*Q(G_\psi))$, which depends only on $\mathcal{F}$. We set $\mu_\mathcal{F} = [\mu_{\rho_\mathcal{F}}] \in Q(N_\psi)^{G_\psi}/(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^*Q(G_\psi))$.

The goal of this section is to show the following:

**Proposition 7.1.** The space $Q(N_\psi)^{G_\psi}/(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^*Q(G_\psi))$ is $\mathbb{R}$-spanned by 

$$\{ \mu_\mathcal{F} \mid \mathcal{F} \text{ is a taut foliation on } X \}.$$ 

**Proof.** Let us consider the following diagram whose rows are exact:

$$
\begin{array}{cccc}
H^1(N_\psi; \mathbb{R})^{G_\psi} & \longrightarrow & Q(N_\psi)^{G_\psi} & \xrightarrow{d} & H^2_b(N_\psi; \mathbb{R})^{G_\psi} \\
& & i^* & & i^*
\end{array}
$$

Here $d: Q(N_\psi)^{G_\psi} \rightarrow H^2_b(N_\psi; \mathbb{R})^{G_\psi}$ is the map defined by $d(\mu) = [\delta \mu] \in H^2_b(N_\psi; \mathbb{R})$. Note that $d(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^*Q(G_\psi)) \subset i^*(H^2_b(N_\psi; \mathbb{R}))$ and that $i^*: H^2_b(N_\psi; \mathbb{R}) \rightarrow H^2_b(N_\psi; \mathbb{R})^{G_\psi}$ is an isomorphism since $\Gamma_\psi$ is amenable. Since $G_\psi$ is Gromov-hyperbolic, the map $c_{G_\psi}$ is surjective. Hence the map $c_{G_\psi} \circ (i^*)^{-1} \circ d$ induces an injection 

$$Q(N_\psi)^{G_\psi}/(H^1(N_\psi; \mathbb{R})^{G_\psi} + i^*Q(G_\psi)) \rightarrow H^2(G_\psi).$$ 

By the construction of $\mu_{\rho_\mathcal{F}}$, we have 

$$c_{G_\psi} \circ (i^*)^{-1} \circ d(\mu_{\rho_\mathcal{F}}) = \rho_\mathcal{F}^*(e_\mathbb{R}).$$ 

Hence it suffices to show that the cohomology group $H^2(G_\psi; \mathbb{R})$ is $\mathbb{R}$-spanned by 

$$\{ \rho_\mathcal{F}^*(e_\mathbb{R}) \mid \mathcal{F} \text{ is a taut foliation on } X \}.$$ 

This then turns out to be a direct consequence of Gabai’s theorem [Gab97, Remark 7.3] (see also [GY20, Theorem 1.4]). More precisely, let $x^*: H^2(X; \mathbb{R}) \rightarrow \mathbb{R}$ be the dual Thurston norm. Since $X$ is hyperbolic, the unit norm ball $B_{x^*}$ of $x^*$ is a polyhedron in $H^2(X; \mathbb{R})$ [Thu86, Theorem 2]. Gabai’s theorem asserts that every vertex of the dual Thurston norm ball for a compact oriented irreducible 3-manifold can be realized as the Euler class of some taut foliation. Hence, for every vertex $v \in B_{x^*}$, there exists a taut foliation $\mathcal{F}$ such that the Euler class $e(T\mathcal{F})$ of the plane field of $\mathcal{F}$ is equal to $v$. Hence the cohomology group $H^2(X; \mathbb{R})$ is $\mathbb{R}$-spanned by 

$$\{ e(T\mathcal{F}) \mid \mathcal{F} \text{ is a taut foliation on } X \}.$$ 

It is known that $\rho_\mathcal{F}^*(e_\mathbb{R}) \in H^2(G_\psi)$ is equal to $e(T\mathcal{F}) \in H^2(X; \mathbb{R})$ under the canonical isomorphism $H^2(G_\psi; \mathbb{R}) \cong H^2(X; \mathbb{R})$ (see [BH19, Proposition 7.1]). This completes the proof. \qed
7.2. **Overflow argument.** In this short subsection, we provide an argument to show the non-equivalence of $\text{scl}_G$ and $\text{scl}_{G,N}$. In the proofs of Theorems 1.1, 1.2 and 4.1, we have employed the auxiliary lemma (Lemma 2.6) to show the non-equivalence of $\text{scl}_G$ and $\text{scl}_{G,N}$. Conditions (i) and (ii) in Lemma 2.6 are purely algebraic conditions. Contrastingly, condition (iii) is formulated in terms of the behavior of values $\mu(x_n)$ as $n \to \infty$, where $(x_n)_{n \in \mathbb{N}} = ([y_1^n, z_1] \cdots [y_k^n, z_k])_{n \in \mathbb{N}}$. Given $\mu$, it seems to be a hard task to find elements $y_1, \ldots, y_k, z_1, \ldots, z_k$ that fulfill condition (iii). Nevertheless, the following argument, which we call the *overflow argument*, allows us to derive condition (iii) by estimating the value of a quasimorphism at a single element.

**Lemma 7.2** (overflow argument). Let $G$ be a group, $N$ a normal subgroup, and let $\mu \in \text{Q}(N)^G$. Let $k$ be a positive integer and let $y_1, \ldots, y_k, z_1, \ldots, z_k \in G$. Assume that the following three conditions are fulfilled:

- (i) $[y_i, z_i] \in N$ for every $i = 1, \ldots, k$;
- (ii) $[y_1, z_1] \cdots [y_k, z_k] \in [G, N]$;
- (iii)' $|\mu([y_1, z_1] \cdots [y_k, z_k])| > (2k - 1)D(\mu)$.

Then we have

$$\lim_{n \to \infty} \mu([y_1^n, z_1] \cdots [y_k^n, z_k]) = \infty.$$  \hspace{1cm} (7.1)

In particular, $\text{scl}_G$ and $\text{scl}_{G,N}$ are not equivalent.

**Proof.** By the definition of homogeneous quasimorphisms and $G$-invariance of $\mu$, we have

$$\mu([y_1^n, z_1] \cdots [y_k^n, z_k]) \sim_{(k-1)D(\mu)} \mu([y_1^n, z_1]) + \cdots + \mu([y_k^n, z_k]) \sim_{k(n-1)D(\mu)} n\mu([y_1, z_1] + \cdots + [y_k, z_k]) \sim_{n(k-1)D(\mu)} n\mu([y_1, z_1] \cdots [y_k, z_k]).$$

Hence we obtain

$$|\mu([y_1^n, z_1] \cdots [y_k^n, z_k]) - n\mu([y_1, z_1] \cdots [y_k, z_k])| \leq (n(2k - 1) - 1)D(\mu).$$

This, together with (iii)', implies (7.1). Now the latter assertion follows from Lemma 2.6 since (7.1) is exactly condition (iii) in Lemma 2.6. 

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