The Black-Scholes Equation and Certain Quantum Hamiltonians

Juan M. Romero\textsuperscript{1,*} O. González-Gaxiola\textsuperscript{2†} J. Ruíz de Chávez\textsuperscript{3‡} R. Bernal-Jaquez\textsuperscript{4§}

\textsuperscript{1,2,4} Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa México, D.F 01120, México
\textsuperscript{3} Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa Av. San Rafael Atlixco No. 186, Col. Vicentina, A.P. 55–534, 09340 Iztapalapa, México, D.F.

Abstract

In this paper a quantum mechanics is built by means of a non-Hermitian momentum operator. We have shown that it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians using this momentum operator. We can construct a generalized supersymmetric quantum mechanics that has a dual based on these Hamiltonians. In addition, it is shown that the non-Hermitian Hamiltonians of this theory can be related to Hamiltonians that naturally arise in the so-called quantum finance.

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\textsuperscript{*}jromero@correo.cua.uam.mx
\textsuperscript{†}ogonzalez@correo.cua.uam.mx
\textsuperscript{‡}jrch@xanum.uam.mx
\textsuperscript{§}rbernal@correo.cua.uam.mx
1 Introduction

In the origins of quantum mechanics P.A.M Dirac [6] observed that commutation relations

\[ [x_i, x_j] = [P_i, P_j] = 0, \quad [x_i, P_j] = i \delta_{ij}, \]

(1)

are satisfied by the operators \( x_i, P_j = -i \partial_j \) and also by the set

\[ x_i, \quad P_{(f)j} = -i \partial_j + i \partial_j f, \]

(2)

with \( f \) been an arbitrary function [6]. For different reasons, operators [2] were discarded, for example, the Hamiltonian \( H_f = \frac{P_f^2}{2} \) is non-Hermitian and it could have a non-real spectrum. However, it has recently been shown that there are non-Hermitian operators with real spectrum [3]. Studies of non-Hermitian Hamiltonians and their applications in physics can be found in the papers [7], [4] and [10].

In this paper, we will show that, using the operator \( P_{(f)j} \), four different types of Hamiltonians can be built, two of these Hermitians and the other two, non-Hermitians. We will show that, from two Hermitian Hamiltonians in one dimension, it is possible to construct a supersymmetric mechanics, and that using one of the two non-Hermitian Hamiltonians a generalized supersymmetric mechanic can be constructed. Moreover, we will show that this new supersymmetric quantum mechanics has a dual and the ground state of the corresponding Hamiltonians will be found.

As a second point of this work, it is shown how the operator \( P_{(f)j} \) can also be used to build some of non-Hermitian Hamiltonians that naturally arise in the so-called quantum finance.

2 Non-Hermitian Hamiltonians

In this section, we will study the quantum mechanics that emerges when the operator \( P_{(f)j} \) is considered. As an starting point, we have to notice that the operator \( P_{(f)i} \) is given by the transformation

\[ P_{(f)i} = e^f P_i e^{-f}, \]

(3)
and also that it is not Hermitian.

With $\vec{P}(f)$ we can construct four Hamiltonians, two of them Hermitians

\begin{align}
H_1 &= \alpha^2 \vec{P}^\dagger(f) \cdot \vec{P}(f) = \alpha \left( \vec{P}^2 + \nabla^2 f + (\vec{\nabla} f)^2 \right), \quad (4) \\
H_2 &= \alpha^2 \vec{P}(f) \cdot \vec{P}^\dagger(f) = \alpha \left( \vec{P}^2 - \nabla^2 f + (\vec{\nabla} f)^2 \right) \quad (5)
\end{align}

and two non-Hermitians

\begin{align}
H_3 &= \beta^2 \vec{P}^\dagger(f) \cdot \vec{P}^\dagger(f) \\
&= \beta^2 \left( \vec{P}^2 - 2i \vec{\nabla} f \cdot \vec{P} - \nabla^2 f - (\vec{\nabla} f)^2 \right), \quad (6) \\
H_4 &= \beta^2 \vec{P}(f) \cdot \vec{P}(f) \\
&= \beta^2 \left( \vec{P}^2 + 2i \vec{\nabla} f \cdot \vec{P} + \nabla^2 f - (\vec{\nabla} f)^2 \right). \quad (7)
\end{align}

These Hamiltonians are obtained naturally in different contexts. In the following subsection, we will see that, they can be used to obtain a generalized version of the supersymmetric quantum mechanics.

### 3 Supersymmetric Quantum Mechanics

In the one dimensional case, the Hamiltonians $H_1$ and $H_2$ are given by

\begin{align}
H_1 &= \alpha^2 \left( P^2 + \frac{d^2 f}{dx^2} + \left( \frac{df}{dx} \right)^2 \right), \quad (8) \\
H_2 &= \alpha^2 \left( P^2 - \frac{d^2 f}{dx^2} + \left( \frac{df}{dx} \right)^2 \right). \quad (9)
\end{align}

Moreover, if

\[ f(x) = \int_0^x W(u)du, \]

then

\[ H_1 = \alpha^2 \left( P^2 + \frac{dW}{dx} + W^2 \right), \quad (11) \]
\[ H_2 = \alpha^2 \left( P^2 - \frac{dW}{dx} + W^2 \right). \]  
\[ (12) \]

This Hamiltonians can be used to form the matrix

\[ h = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \]  
\[ (13) \]

Now, defining

\[ Q = \begin{pmatrix} 0 & \alpha P_{(f)} \\ 0 & 0 \end{pmatrix}, \]  
\[ (14) \]

we have

\[ h = \{Q, Q^\dagger\}, \quad Q^2 = 0, \quad \{Q, H\} = 0. \]  
\[ (15) \]

According to the supersymmetric quantum mechanics [5], \( h \) represents a superhamiltonian and \( Q \) a supercharge. Therefore, the quantum mechanics built using \( P_{(f)} \) contains the usual supersymmetric quantum mechanics.

Moreover, \( P_{(f)} \) allows us to generalize supersymmetric quantum mechanics. In fact, we can define the matrices

\[ Q_1 = \begin{pmatrix} 0 & \alpha P_{(f)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta P_{(f)}^\dagger \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha P_{(f)}^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P_{(f)}^\dagger & 0 \end{pmatrix}, \]  
\[ (16) \]

and then \( Q_1^2 = Q_2^2 = 0 \). Using \( Q_1^2 = Q_2^2 = 0 \) we can construct the Hamiltonian

\[ H = \{Q_1, Q_2\} = \begin{pmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_3 \end{pmatrix}, \]  
\[ (17) \]

with the conserved charges

\[ \dot{Q}_1 = [Q_1, H] = 0, \quad \dot{Q}_2 = [Q_2, H] = 0; \]  
\[ (18) \]
now, if $\beta = 0$, we have $Q_1 = Q_2 = Q$ and this quantum mechanics reduces to the usual supersymmetric quantum mechanics.

Besides, we have another quantum mechanics that reduces to the usual supersymmetric quantum mechanics. In fact, if

\[ Q_3 = \begin{pmatrix} 0 & \alpha P_{(f)}^\dagger & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P_{(f)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha P_{(f)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta P_{(f)} & 0 \end{pmatrix}, \quad (20) \]

then $Q_3^2 = Q_4^2 = 0$ and we can construct the Hamiltonian

\[ \tilde{H} = \{Q_3, Q_4\} = \begin{pmatrix} H_2 & 0 & 0 & 0 \\ 0 & H_1 & 0 & 0 \\ 0 & 0 & H_4 & 0 \\ 0 & 0 & 0 & H_4 \end{pmatrix}. \quad (21) \]

Note that if $\beta = 0$ this Hamiltonian is just the usual superhamiltonian $h$.

If we make the transformation $f \to -f$, we have

\[ (H_1, H_2, H_3, H_4) \to (H_2, H_1, H_4, H_3), \quad (22) \]

i.e.

\[ H \to \tilde{H}. \quad (23) \]

Then, there is a duality transformation between Hamiltonians $H$ and $\tilde{H}$. Therefore these generalized quantum mechanics are duals.

Now, if we consider the functions $\psi_0 = A_1 e^f, \phi_0 = A_2 e^{-f}$ that satisfy

\[ P_{(f)} \psi_0 = 0, \quad P_{(f)}^\dagger \phi_0 = 0. \quad (24) \]

Then, the wave function

\[ \psi = \begin{pmatrix} A_1 e^{-f} \\ A_2 e^f \\ A_3 e^-f \\ A_3 e^{-f} \end{pmatrix}. \quad (25) \]
satisfies
\[ H\psi = 0. \] (26)

Moreover, the wave function
\[ \tilde{\psi} = \begin{pmatrix} e^f \\ e^{-f} \\ e^f \\ e^f \end{pmatrix} \] (27)
satisfies
\[ \tilde{H}\tilde{\psi} = 0. \] (28)

Thus, \( \psi \) is the ground state of \( H \) and \( \tilde{\psi} \) is the ground state of \( \tilde{H} \).

### 4 The Black-Scholes model

This model is a partial differential equation whose solution describes the value of an European Option. See [2], [3]. Nowadays, it is widely used to estimate the pricing of options other than the European ones. Let \( (\Omega, F, P, F_{t\geq 0}) \) be a filtered probability space and let \( W_t \) be a brownian motion in \( \mathbb{R} \). We will consider the stochastic differential equation (s.d.e.)
\[ dX(t) = a(t, X(t))dt + \sigma(t, X(t))dW(t), \] (29)
with \( a \) and \( \sigma \) continuous in \((t, x)\) and Lipschitz in \( x \). The price processes given by the geometric brownian motion \( S(t), S(0) = x_0 \), solution of the s.d.e.
\[ dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \] (30)
with \( \mu \) and \( \sigma \) constants. It is well know the solution of this s.d.e. it is given by:
\[ dS(t) = x_0 \exp\{\sigma(W(t) - W(t_0)) + (r - \frac{1}{2}\sigma^2)(t - t_0)\} \] (31)

Let \( 0 \leq t < T \) and \( h \) be a Borel measurable function, \( h(X(T)) \) denote the contingent claim, let \( E^{x,t}h(X(T)) \) be the expectation of \( h(X(T)) \), with the initial condition \( X(t) = x \).
Now we recall the Feynman–Kac theorem \[9\]. Let \( v(t, x) = E^{x,t}h(X(T)) \) be, \( 0 \leq t < T \), where \( dX(t) = a(X(t))dt + \sigma(X(t))dW(t) \). Then
\[
v_t(t, x) + a(x)v_x(t, x) + \frac{1}{2}\sigma^2(x)v_{xx}(t, x) = 0, \quad v(T, x) = h(x).
\] (32)

Now, if we consider the discounted value
\[
u(t, x) = e^{-r(T-t)}E^{x,t}h(X(T)) = e^{-r(T-t)}v(t, x).
\]
Then if at time \( t \), \( S(t) = x \), if we proceed in standard way,
\[
v(t, x) = e^{r(T-t)}u(t, x), \\
v_t(t, x) = -re^{r(T-t)}u(t, x) + e^{r(T-t)}u_t(t, x), \\
v_x(t, x) = e^{r(T-t)}u_x(t, x), \\
v_{xx}(t, x) = e^{r(T-t)}u_{xx}(t, x).
\]
The Black-Scholes equation is obtained substituting the above equalities in the equation (32) and multiplying by the factor \( e^{-r(T-t)} \):
\[
-r u(t, x) + u_t(t, x) + rxu_x(t, x) + \frac{1}{2}\sigma^2x^2u_{xx}(t, x) = 0, \quad 0 \leq t < T, x \geq 0.
\] (33)

5 The Relation with the Black-Scholes Equation

The operators \( \vec{P}_f \cdot \vec{P}_f \) and \( \vec{P}_f^\dagger \cdot \vec{P}_f^\dagger \) are non-Hermitians, using them only non-Hermitians Hamiltonians such as \( H_3 \) and \( H_4 \), can be constructed. However, we will show that these operators may have applications in some other areas such as quantum finance. In order to see this, we define the potentials
\[
U_1(x, y, z) = -\beta^2 \left( \nabla^2 f - \left( \vec{\nabla} f \right)^2 \right) + V_1(x, y, z),
\] (34)
\[
U_2(x, y, z) = \beta^2 \left( \nabla^2 f + \left( \vec{\nabla} f \right)^2 \right) + V_2(x, y, z),
\] (35)
and the non-Hermitians Hamiltonians
\[
H_I = \beta^2 \vec{P}_f \cdot \vec{P}_f + U_1(x, y, z)
= \beta^2 \left( \vec{P}^2 + 2i\vec{\nabla} f \cdot \vec{P} \right) + V_1(x, y, z),
\] (36)
\[
H_{II} = \beta^2 \vec{P}_f^\dagger \cdot \vec{P}_f^\dagger + U_2(x, y, z)
= \beta^2 \left( \vec{P}^2 - 2i\vec{\nabla} f \cdot \vec{P} \right) + V_2(x, y, z).
\] (37)
On the other hand, let us consider the fundamental equation in quantum finance, the so-called Black-Scholes equation (33)
\[
\frac{\partial C}{\partial t} = -\frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} - rS \frac{\partial C}{\partial S} + rC,
\]
where \( C \) is the option price, \( \sigma \) is a constant called the volatility and \( r \) is the interest rate \([1]\). With the change of variable \( S = e^x \) we obtain
\[
\frac{\partial C}{\partial t} = H_{BS}C,
\]
\[
H_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} + r
\]
this non-Hermitian Hamiltonian is called Black-Scholes Hamiltonian. Now, considering the one dimensional case of (37) and identifying
\[
\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \frac{1}{\sigma^2} \left( \frac{\sigma^2}{2} - r \right) x, \quad V_2(x) = r
\]
we obtain \( H_{II} = H_{BS} \).

One generalized Black-Scholes equation, (see \([1]\)) is given by
\[
H_{BSG} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{\sigma^2}{2} - V(x) \right) \frac{\partial}{\partial x} + V(x).
\]
In this case, considering again the one dimensional case of equation (37) and with
\[
\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \int_0^x du \frac{1}{\sigma^2} \left( \frac{\sigma^2}{2} - V(u) \right), \quad V_2(x) = V(x)
\]
we have \( H_{II} = H_{BSG} \).

Moreover, the so-called barrier option case has Hamiltonian
\[
H_{BSB} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} + V(x).
\]
Again, considering (37) in one dimension and with
\[
\beta^2 = \frac{\sigma^2}{2}, \quad f(x) = \frac{1}{\sigma^2} \left( \frac{\sigma^2}{2} - r \right) x, \quad V_2(x) = V(x)
\]
we have $H_{II} = H_{BSB}$.

As we have seen, several important Hamiltonians appearing in quantum finance are particular cases of this new version of quantum mechanics.

6 Summary

A quantum mechanics is built by means of a non-Hermitian momentum operator. Moreover, it is shown that using this momentum operator it is possible to construct two Hermitian and two non-Hermitian type of Hamiltonians. Using these Hermitian Hamiltonians we have built, a generalized supersymmetric quantum mechanics with a dual that can be constructed. It also shown that, the non-Hermitian Hamiltonians of this theory may be related to so-called quantum finance Hamiltonian.

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