Sublevel-Set Estimates Over Global Domains

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Abstract
Since Varchenko’s seminal paper, the asymptotics of oscillatory integrals and related problems have been elucidated through the Newton polyhedra associated with the phase $P$. The supports of those integrals are concentrated on sufficiently small neighborhoods. The aim of this paper is to investigate the estimates of sublevel sets and oscillatory integrals whose supports are global domains $D$. A basic model of $D$ is $\mathbb{R}^d$. For this purpose, we define the Newton polyhedra associated with $(P, D)$ and establish analogues of Varchenko’s theorem in global domains $D$, under nondegeneracy conditions of $P$.

Keywords  Newton polyhedron · Oscillatory integral · Sublevel-set estimate

Mathematics Subject Classification  Primary 42B20 · 42B25

1 Introduction
Asymptotic estimates of the sublevel sets $\{x \in D : |\lambda P(x)| \leq 1\}$ and the oscillatory integrals $\int e^{i\lambda P(x)} \psi(x)dx$ with $\psi \in C^\infty(D)$ arise in many areas of mathematics. For systematic estimates for global $D$, we consider the regions $D$ defined as:

$$D_B := \{x \in \mathbb{R}^d : |x|^b \leq 1 \text{ for all } b \in B\}, \text{ given finite subsets } B \subset \mathbb{Q}^d.$$

Here $x^b := x_1^{b_1} \ldots x_d^{b_d}$ for $b = (b_1, \ldots, b_d)$ and $x_ν^0 := 1$ for every real $x_ν$. For instance, let $\{e_ν\}_ν=1^d$ be the set of the standard unit vectors in $\mathbb{R}^d$, then

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When it converges, under a minimal nondegeneracy type condition

Question 1.2

As a phase function, we shall take a real valued polynomial \( P(x) \) in \( \mathbb{R}^d \):

\[
P(x) = \sum_{m \in \Lambda(P)} c_m x^m \text{ where } \Lambda(P) = \{ m \in \mathbb{Z}^d_+: c_m \neq 0 \}.
\]

1.1 Main Questions

Choose a model polynomial \( P(x) = x_1 x_2 \) on the two regions \( D_{e_1, e_2} = [-1, 1]^2 \) and \( D_{[0]} = \mathbb{R}^2 \). Then one can compute \(|\{ x \in D_{e_1, e_2} : |\lambda x_1 x_2| \leq 1 \}| = 4\lambda^{-1}(1 + |\log \lambda|) \) and \(|\{ x \in D_{[0]} : |\lambda x_1 x_2| \leq 1 \}| = \infty \). This simple computation leads us to study the following questions regarding the sublevel-set estimate in this paper:

Question 1.1 Find a condition of \((P, D_B)\) that determines whether a sublevel-set measure \(|\{ x \in D_B : |\lambda P(x)| \leq 1 \}|\) converges or diverges.

Question 1.2 When it converges, under a minimal nondegeneracy type condition of \((P, D_B)\), determine the indices \( \rho \) and \( a \) satisfying the sublevel-set estimate \(|\{ x \in D_B : |\lambda P(x)| \leq 1 \}| \approx \lambda^{-\rho}(|\log \lambda| + 1)^a\), according as \( \lambda \in [1, \infty) \) or \( \lambda \in (0, 1) \).

For the corresponding oscillatory integrals, we are not asking their asymptotics with a fixed individual amplitude function \( \Psi_{D_B} \), but we are concerned with convergences and upper bounds, universal to all suitable cutoff functions \( \Psi \in C_\infty(D_B) \).

Question 1.3 Under a minimal nondegeneracy type condition of \((P, D_B)\),

- check if \( \int e^{i\lambda P(x)} \Psi(x) dx \) converge for all appropriate \( \Psi \in C_\infty(D_B) \), or not.
- find the best indices \( \rho \) and \( a \): \( \int e^{i\lambda P(x)} \Psi(x) dx \lesssim \lambda^{-\rho}(|\log \lambda| + 1)^a \) for all appropriate \( \Psi \in C_\infty(D_B) \), according as \( \lambda \in [1, \infty) \) or \( \lambda \in (0, 1) \).

The constants involved in \( \approx \) and \( \lesssim \), depend on \((P, D_B)\), but are independent of \( \lambda \).

1.2 Local Estimates

We first go over these questions in a sufficiently small neighborhood \( D \) of the origin. Write the usual Newton polyhedron as \( N(P) \), which is defined by \( \text{conv}(\Lambda(P) + \mathbb{R}^d_+) \) where \( \text{conv}(E) \) denotes the convex hull of a set \( E \). Call the non-negative number \( d(N(P)) \) displayed below, the Newton distance:

\[
N(P) \cap \text{cone}(1) = [d(N(P)), \infty) \text{ 1 with 1} = (1, \ldots, 1) \in \mathbb{R}^d
\]

where \( \text{cone}(1):=\{t1 : t \geq 0\} \) is the diagonal ray. In 1976, Varchenko [21] studied the asymptotics of the oscillatory integrals in Question 1.3 with \( \Psi \) supported near the...
origin, associated with real analytic phase functions \( P \) with \( \nabla P(0) = 0 \). He assumed that \( d(N(P)) > 1 \) and imposed the face-nondegenerate hypothesis:

\[
\nabla P_{\mathcal{F}}|_{(\mathbb{R}\setminus\{0\})^d}\text{ are non-vanishing for all compact faces of } \mathcal{F} \text{ of } N(P) \quad (1.1)
\]

where \( P_{\mathcal{F}}(x) := \sum_{m \in \Lambda(P) \cap \mathcal{F}} c_m x^m. \) Then he calculated the oscillation index \( \rho \) to be \( 1/d(N(P)) \) and the multiplicity \( a \) to be \( d - 1 - k \) for \( k = \dim(\mathcal{F})_{\text{main}}. \) Here \( \mathcal{F}_{\text{main}} \) is the lowest dimensional face of \( N(P) \) containing \( d(N(P)) \). In 1977, Vassiliev [22] proved that the sublevel-set growth index of Question 1.2 for the local domain, \( \rho \), is \( 1/d(N(P)) \) and the multiplicity \( a \) is \( d - 1 - k \), under the normal-crossing assumption:

there is \( c > 0 \) so that \( |P(x)| \geq c \sum_{m \in \Lambda(P)} |x^m| \quad \forall \, x \in D \cap (\mathbb{R} \setminus \{0\})^d. \quad (1.2) \)

### 1.3 Model Result

To obtain the corresponding indices of Varchenko and Vassiliev [21, 22] in the whole domain \( D_B = \mathbb{R}^d \), define the Newton polyhedron \( N(P, \mathbb{R}^d) \) as the convex hull of \( \Lambda(P) \). Then \( \text{cone}(\Lambda(P) \cap \{1\}) \) determines the convergence in Questions 1.1, and the line segment \([\delta_{\text{for}}, \delta_{\text{bac}}] = N(P, \mathbb{R}^d) \cap \text{cone}(1)\) determines the growth rates in Questions 1.2, as well as those of Question 1.3:

**A 1.1.** If \( \text{cone}(\Lambda(P) \cap \{1\}) \neq \mathbb{R}^d \), then \( |\{x \in \mathbb{R}^d : |\lambda P(x)| \leq 1\}| \) diverges.

**A 1.2.** If \( \text{cone}(\Lambda(P) \cap \{1\}) = \mathbb{R}^d \), then the growth rate \( \rho \) and its multiplicity \( a \) in the sublevel-set estimate in Question 1.2 under the condition (1.2) are

\[
(\rho, a) = \begin{cases} 
(1/\delta_{\text{for}}, d - 1 - \dim(\mathcal{F})_{\text{for}}) & \text{if } \lambda \geq 1 \\
(1/\delta_{\text{bac}}, d - 1 - \dim(\mathcal{F})_{\text{bac}}) & \text{if } 0 < \lambda < 1 
\end{cases}
\]

with \( \mathcal{F}_{\text{for}} \) and \( \mathcal{F}_{\text{bac}} \) smallest faces of \( N(P, \mathbb{R}^d) \) containing \( \delta_{\text{for}} \) and \( \delta_{\text{bac}} \).

**A 1.3.** If \( \text{cone}(\Lambda(P) \cap \{1\}) \neq \mathbb{R}^d \), then there is an appropriate \( \Psi \) supported in \( D_B \) such that \( \int e^{1/\lambda P(x)}\Psi(x)dx = \infty. \) If \( \text{cone}(\Lambda(P) \cap \{1\}) = \mathbb{R}^d \), then \( \int e^{1/\lambda P(x)}\Psi(x)dx \leq \lambda^{-\rho}(\log \lambda + 1)^{d} \) with the same indices \( \rho \) and \( a \) above.

Not like a local domain, in the global domain, the indices \( \rho \) and \( a \) turn out to be different according as \( 0 < \lambda \ll 1 \) or \( \lambda \gg 1 \).

**Remark 1.1.** From the computations of the model phase \( P(x_1, x_2) = x_1x_2 \):

\[
\int_{D(0)} e^{2\pi i\lambda x_1x_2}dx_1dx_2 := \lim_{R \to \infty} \int_{|x_1| < R, |x_2| \leq R} e^{2\pi i\lambda x_1x_2}dx_1dx_2 = \frac{2}{\lambda}
\]

\[
\int_{D(0)} e^{2\pi i\lambda x_1x_2} \psi_{D(0)}(x)dx_1dx_2 := \lim_{R \to \infty} \int_{|x_1| < R} e^{2\pi i\lambda x_1x_2} \psi\left(\frac{x_1}{R}\right)\psi\left(\frac{x_2}{R}\right)dx_1dx_2 = O\left(\frac{1}{\lambda}\right)
\]

we can observe that the criterion of the convergence of the oscillatory integral for a fixed amplitude \( \psi_{D_B} \) would be different from that of the sublevel-set estimates above. However, we do not deal with these main issues in this paper.
1.4 Resolution of Singularities in the Local Region

When (1.2) breaks down, one needs appropriate resolutions of the singularities. For the study of the classical resolution of singularities of analytic functions, we refer selectively [1, 12, 15] with its evolution [3, 6, 11, 20, 23] in the context of algebraic geometry. When the nondegeneracy hypothesis (1.1) (or (1.2)) fails, Varchenko [21] established, via toric geometry, the resolution of singularity algorithm in $\mathbb{R}^2$ for finding an adaptable local coordinate system $\Phi$ satisfying

$$d(N(P \circ \Phi)) = \sup_{\phi \text{ local coordinates}} d(N(P \circ \phi)).$$

Later, Phong, Stein, and Sturm [19] utilized the Weierstrass preparation theorem and the Puiseux series expansions of the roots $r_i$ of $P(x_1, r_i(x_1)) = 0$ for constructing pullbacks $\phi_i(x) = (x_1, x_2 + r_i(x_1))$ of the horns $D_i$ for $D \subset \bigcup_{i=1}^M D_i$ making $P \circ \phi_i$ satisfying (1.1) and (1.2) on $\phi_i^{-1}(D_i)$. Moreover, Ikromov and Muller [13] accomplished the Varchenko’s algorithm for the adaptable local coordinate systems of the form $\phi(x_1, x_2) = (x_1, x_2 + r_1(x_1))$ or $(x_1 + r(x_2), x_2)$ with $r$ analytic and $r(0) = 0$ in $\mathbb{R}^2$ by using the elementary analysis of the Newton polyhedron. They can handle a class of smooth functions. Moreover, Greenblatt [8] computed the leading terms of asymptotics of related integrals for the smooth phase after constructing Varchenko’s adaptable local coordinate system in $\mathbb{R}^2$ by performing only an elementary analysis such as an implicit function theorem. With only analysis tools, Greenblatt [10] utilized the induction argument, as in a spirit of Hironaka’s, to establish an elementary local resolution of singularities in $\mathbb{R}^d$ for all $d \geq 1$. More recently, Collins, Greenleaf, and Pramanik [4] developed the classical resolution of singularities to obtain a higher dimensional resolution of singularity algorithm applicable to the above integrals in a local domain of $\mathbb{R}^d$. In a small neighborhood $D$ of the origin, the aforementioned oscillatory integral estimates yield the oscillation index of $P$ at the origin given by the infimum $\rho_0$ below over all $\psi$ of the asymptotics

$$\int e^{i\lambda P(x)} \psi(x) dx \sim \sum \sum_{i=0}^{\rho_i} c_{i,n}(\psi) \lambda^{\rho_i}(\log \lambda)^n$$

where $\rho_i < \rho_{i+1}$ and $\sum_{n=0}^{d-1} |c_{0,n}(\psi)| \neq 0$ as $\lambda \to \infty$ (its existence follows from Hironaka’s resolution of singularities in [12]). In this paper, we do not establish the global resolution of singularity nor exact initial coefficient $c_{0,n}(0)$. But, we fix a coordinate system under a nondegeneracy hypothesis as a global variant of (1.1) or (1.2) and focus only on finding the leading indices $\rho$ and $a$ of Varchenko or Vassiliev in Main Theorems 1 and 2. Next, we partition the domain $D \subset \bigcup_{i=0}^M D_i$ so that $P \circ \phi_i$ are normal-crossing on $\phi_i^{-1}(D_i)$ for all $i = 0, \ldots, M$, in Main Theorem 3.

Notation. Denote the set of non-negative real numbers (integers, rationals) by $\mathbb{R}_+ (\mathbb{Z}_+, \mathbb{Q}_+)$. For $j = (j_1, \ldots, j_d) \in \mathbb{R}^d$, we write $2^{-j} = (2^{-j_1}, \ldots, 2^{-j_d})$. Moreover, by $\pm 2^{-j}$, we denote the $2^d$ number of all possible vectors of the forms
(±2−j1, . . . , ±2−jK). Set those vectors with their exponents in $K \subset \mathbb{R}^d$ as

$$2^{-K} := \{ \pm 2^{-j} : j \in K \}. \quad (1.3)$$

Let $[d] := \{1, \ldots , d\}$. Given $j = (j_1, \ldots , j_d) \in \mathbb{Z}^d$ and $x = (x_1, \ldots , x_d) \in \mathbb{R}^d$, we write the dilation $2^{-j}x = (2^{-j_1}x_1, \ldots , 2^{-j_d}x_d)$ and denote $x \sim 2^{-j}$ if

$$2^{-j}x = \{ x \in \mathbb{R}^d : \|x\| \leq 1 \} \quad (1.4)$$

Thus $x \sim 2^{-0}$ iff $1/2 \leq |x_v| \leq 2$ for $v \in [d]$. Sometimes, we shall use the notation $x \sim_h 2^0$ representing $1/h \leq |x_v| \leq h$ for a fixed number $1 \leq h < \infty$. For $K \subset \mathbb{Z}^d$ and $B \subset \mathbb{Z}^d \cap [-r, r]^d$, we denote the set $K + B$ by $K + O(r)$. We employ the following smooth non-negative cutoff functions

1. $\psi$ supported in $\{ u \in \mathbb{R}^d : |u| \leq 1 \}$ for $\psi(u) \equiv 1$ in $|u| < \frac{1}{2}$ and $\psi^c = 1 - \psi$,

2. $\chi$ supported in $\{ u \in \mathbb{R} : 1/2 \leq |u| \leq 2 \}$ or $\bigcap_{v=1}^d \{ u \in \mathbb{R}^d : \frac{1}{2} \leq |u_v| \leq 2 \}$,

allowing slight line-by-line modifications of $\chi$ and $\psi$. In this paper, we let $D$ be a Borel set and let $\psi_D$ indicate a cutoff function supported in $D$. Given two scalars $a, b$, write $a \lesssim b$ if $a \leq Cb$ for some $C > 0$ depending only on $(P, D_B)$ in (1.1)–(1.2). The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. Notice the bounds involved in $\lesssim$ of (I.1) and (I.2) are independent of $\lambda$ and $x$. In additions, denote $0 \leq a \ll b$ if $a/b$ is a sufficiently small number compared with 1. Note that our positive constants $\epsilon (\epsilon \ll 1)$ and $c, C$ may be different line by line. Finally, $\text{rank}(A)$ is the number of linearly independent vectors in $A$.

**Organization.** In Sect. 2, we define the Newton polyhedron $\mathbf{N}(P, D_B)$ associated with a general domain $D_B$ and its balancing condition that determines a divergence of our integral. In Sect. 3, we state the two main theorems regarding Questions 1.1 and 1.2 under a normal-crossing hypothesis derived from (1.2) on $(P, D_B)$. In Sect. 4, we introduce basic properties of polyhedra and its supporting planes. In Sects. 5, we prove some combinatorial lemmas regarding the distances and orientations of $\mathbf{N}(P, D_B)$. In Sect. 6, we decompose our integrals according to the oriented and simplicial dual faces of $\mathbf{N}(P, D_B)$. In Sects. 7–9, we give a bulk of proofs for the two main theorems stated in Sect. 3. In Sect. 10, by partitioning domains $D$ into finite pieces, we restate the main results under a type of face-nondegeneracy (1.1). In the last two sections, we prove the dual face decomposition (Theorem 6.1), and the equivalence of two nondegeneracy conditions (Theorem 10.1).

### 2 Global Integrals and Newton Polyhedra

#### 2.1 Two Aspects of Global Integrals

The following examples illustrate the two main features of global integral estimates.

(F1) Divergence due to unbalanced $\text{Ch}(\Lambda(P))$. Compare:
(i) \( \lim_{R \to \infty} \int \psi(\lambda(x_1^2 + x_2^2)) \psi \left( \frac{x_1}{R} \right) \psi \left( \frac{x_2}{R} \right) \, dx_1 \, dx_2 \approx \lambda^{-1} \),

(ii) \( \lim_{R \to \infty} \int \psi(\lambda x_1^2) \psi \left( \frac{x_1}{R} \right) \psi \left( \frac{x_2}{R} \right) \, dx_1 \, dx_2 \approx \lim_{R \to \infty} \lambda^{-1/2} R = \infty. \)

Observe that the divergence of (ii) is owing to the deviation \( \text{Ch}(\Lambda(x_1^2)) = \{(2,0)\} \) from the cone (I) causing the biased integration \( dx_2 \).

(F2) Different decays according to \( \lambda \gg 1 \) or \( \lambda \ll 1 \). Consider the 1D estimates

\[
\int_{\mathbb{R}} \psi(\lambda t^4 + t^6) \, dt \approx \begin{cases} 
\int_{|t| < 1} \psi(\lambda t^4) \, dt \approx \lambda^{-1/4} \text{ if } \lambda \in [1, \infty) \\
\int_{|t| > 1} \psi(\lambda t^6) \, dt \approx \lambda^{-1/6} \text{ if } \lambda \in (0, 1). 
\end{cases}
\]

This 1D estimate yields the distinct decay rates of the 2D sublevel-set measure for \( P(x_1, x_2) = x_1^4 + x_2^4 + x_1^6 + x_2^6 \) according as \( \lambda \gg 1 \) or \( \lambda \ll 1 \):

\[
\int_{\mathbb{R}^2} \psi(\lambda P(x)) \, dx_1 \, dx_2 \approx \prod_{i=1}^{2} \int_{\mathbb{R}} \psi(\lambda(x_i^4 + x_i^6)) \, dx_i \approx \begin{cases} 
\lambda^{-1/2} \text{ if } \lambda \in [1, \infty) \\
\lambda^{-1/3} \text{ if } \lambda \in (0, 1). 
\end{cases}
\]

Observe the exponents 1/2 and 1/3 are in \([2, 3] = \text{Ch}(\Lambda(P)) \cap \text{cone}(I)\).

The geometric intuition in (F1) suggests us to define the balancing condition of Newton polyhedron along cone(I) in Sect. 2.4. The regions \( |t| < 1 \) and \( |t| \geq 1 \) in (F2) suggest us to define forward and backward polyhedra in Sect. 5.1.

### 2.2 Domain \( D_B \) and its Dual Cone Representation

**Definition 2.1** Given a finite \( B \subset \mathbb{Q}^d \), we set \( \text{cone}(B) \) and its dual cone \( \text{cone}^\vee(B) \):

\[
\text{cone}(B) := \left\{ \sum_{b \in B} \alpha_b \, b : \alpha_b \geq 0 \right\} \quad \text{and} \quad \text{cone}^\vee(B) := \bigcap_{b \in \text{cone}(B)} \{ q : \langle b, q \rangle \geq 0 \}.
\]

**Example 2.1** If \( B = \{ e_1 \}_{i=1}^d \), \( \text{cone}(B) = \text{cone}^\vee(B) = \mathbb{R}_+^d \), and \( \text{cone}^\vee(\{0\}) = \mathbb{R}^d \).

**Lemma 2.1** For \( K \subset \mathbb{R}^d \), recall \( 2^{-K} = \{ \pm 2^{-j} : j \in K \} \) defined in (1.3). Then,

\[
D_B = D_{\text{cone}(B)} = 2^{-\text{cone}^\vee(B)} \text{ except set of measure } 0. \tag{2.1}
\]

**Hence, for evaluating integrals \( \int_{D_B} \), we can regard \( D_B = 2^{-\text{cone}^\vee(B)} \) from (2.1).**

**Proof** To show \( D_B = D_{\text{cone}(B)} \), it suffices to claim \( D_B \subset D_{\text{cone}(B)} \) for \( B = \{ b_1, b_2 \} \). If \( x \in D_B \), then \( |x| \leq 1 \) for all \( \alpha \geq 0 \). Thus \( |x|^{\alpha_1 b_1 + \alpha_2 b_2} = |x|^{\alpha_1} |x|^{\alpha_2} \leq 1 \) for all \( \alpha \geq 0 \). So \( x \in D_{\text{cone}(B)} \). Next, set \( D'_{\text{cone}(B)} := \{ x \in (\mathbb{R} \setminus \{0\})^d : |x| \leq 1 \} \) for all \( b \in \text{cone}(B) \) which is \( \{ \pm 2^{-q} \, b : \langle b, q \rangle \leq 1 \} \) for all \( b \in \text{cone}(B) \) which is \( \{ \pm 2^{-q} : q \in \text{cone}(B) \} \). Hence, the second part of (2.1) holds as \( D_{\text{cone}(B)} \setminus D'_{\text{cone}(B)} \subset \bigcup_v \{ x : x_v = 0 \} \). \( \square \)
2.3 Cutoff Functions on $D_B$ and Strongly Convexity of Cone($B$)

**Definition 2.2** (Amplitude on $D_B$) Let $\psi \in C^\infty([-1, 1])$ such that $\psi \equiv 1$ on $[-1/2, 1/2]$ and let $D_{B,R} := D_B \cap [-R, R]^d$ for $R > 0$. Then put

$$
\psi_{D_B}(x) := \prod_{b \in B} \psi\left(\frac{x^b}{R}\right) \quad \text{and} \quad \psi_{D_{B,R}}(x) := \psi_{D_B}(x) \psi\left(\frac{x_1}{R}\right) \cdots \psi\left(\frac{x_d}{R}\right)
$$

satisfying $\text{supp}(\psi_{D_B}) \subset D_B$ and $\text{supp}(\psi_{D_{B,R}}) \subset D_{B,R}$. More generally, denote by $A(D_B)$ the set of smooth cutoff functions $\psi_1$ supported in $D_B$, satisfying a zero symbol condition:

$$
\sup_{x \in D_B} \left| x^\alpha \partial^\alpha_x \psi(x) \right| \leq C_\alpha \quad \text{for } \alpha \in \mathbb{Z}_+^d.
$$

(2.2)

For example, $\psi_{D_B}, \psi_{D_{B,R}}$ defined above belong to the class $A(D_B)$.

By using $\psi_{D_B}, \psi_{D_{B,R}}$, we express the sublevel-set measure $|\{x \in D_B : |\lambda P(x)| \leq 1\}|$ and the oscillatory integral $\int e^{i\lambda P(x)} \psi_{D_B}(x) dx$ as the limits:

$$
\lim_{R \to \infty} \int \psi(\lambda P(x)) \psi_{D_{B,R}}(x) dx, \quad \text{and} \quad \lim_{R \to \infty} \int e^{i\lambda P(x)} \psi_{D_{B,R}}(x) dx
$$

(2.3)

respectively. Into these integrals, insert the dyadic decomposition $\sum_{j \in \mathbb{Z}_+^d} \chi\left(\frac{x}{2^{-j}}\right) \equiv 1$ with $\chi\left(\frac{x}{2^{-j}}\right) := \prod_{v=1}^d \chi\left(\frac{x_v}{2^{-j_v}}\right)$. Then the convergence of the limit in $R$ follows from the boundedness for the sum over $j$ of the absolute values below:

$$
\sum_{j \in \mathbb{Z}_+^d} \int \psi(\lambda P(x)) \psi_{D_B}(x) \chi\left(\frac{x}{2^{-j}}\right) dx \quad \text{and} \quad \sum_{j \in \mathbb{Z}_+^d} \int e^{i\lambda P(x)} \psi_{D_B}(x) \chi\left(\frac{x}{2^{-j}}\right) dx.
$$

Write the two summands as $T_{j}^{\text{sub}}(\lambda)$ and $T_{j}^{\text{osc}}(\lambda)$ respectively, and observe $2^{-j} \sim x$ for $x \in D_B = 2^{-\text{cone}^\circ(B)}$ in (2.1). Henceforth, we shall rewrite (2.3) as

$$
T_{j}^{\text{sub}}(P, D_B, \lambda) = \sum_{j \in \text{cone}^\circ(B) \cap \mathbb{Z}_+^d} T_{j}^{\text{sub}}(\lambda),
$$

$$
T_{j}^{\text{osc}}(P, D_B, \lambda) = \sum_{j \in \text{cone}^\circ(B) \cap \mathbb{Z}_+^d} T_{j}^{\text{osc}}(\lambda).
$$

(2.4)

**Definition 2.3** Note that cone($B$) is said to be strongly convex if cone($B$) $\cap (-\text{cone}(B)) = \{0\}$. If cone($B$) is not strongly convex, then there exist nonzero $\tau$ and $-\tau$ contained in cone($B$). This implies that

$$
D_B = \{x : |x^b| \leq 1 \text{ for all } b \in B\} \subset \{x : |x^\tau| \leq 1 \text{ and } |x^{-\tau}| \leq 1\} = \{x : |x^\tau| = 1\},
$$
whose measure is zero, so that (2.3) and (2.4) vanish. Hence, we shall state our main theorems in Sect. 3.2, assuming that cone$\left(B\right)$ is strongly convex. But, we shall define a class of domains generalizing $D_B$ and remove the strong convexity of cone$\left(B\right)$ in Sect. 9.4, which enables us to treat a class of Laurent polynomials.

### 2.4 Newton Polyhedra

Viewing $\mathbb{R}^2_{+}$ as $\text{cone}(e_1, e_2)$ in the original definition $N(P) \coloneqq \text{conv}(\Lambda(P) + \mathbb{R}^2_{+})$ in the local region $D_{\{e_1, e_2\}}$, we extend the notion of the Newton polyhedron to all pairs of polynomial $P$ and domain $D_B$ (with $B \subset \mathbb{Q}^d$).

**Definition 2.4 [Newton Polyhedron and Balancing Condition]** Recall $\Lambda(P)$ the exponent set of $P(x)$ with $x \in \mathbb{R}^d$. We define the Newton polyhedron for $(P, D_B)$:

$$N(P, D_B) \coloneqq \text{conv}(\Lambda(P) + \text{cone}(B)).$$

(2.5)

We say that $N(P, D_B) \subset \mathbb{R}^d$ is **balanced** if $\text{cone}(B \cup \Lambda(P) \cup \{-1\}) = \mathbb{R}^d$ and unbalanced if $\text{cone}(B \cup \Lambda(P) \cup \{-1\}) \neq \mathbb{R}^d$. See Fig. 1 for $B = \{0\}$.

**Example 2.2** Given a polynomial $P$, the regions $\mathbb{R}^3 = D_{\{0\}}, [-1, 1]^3 = D_{\{e_1, e_2, e_3\}}$ and $(\mathbb{R}\backslash(-1, 1))^3 = D_{\{-e_1, -e_2, -e_3\}}$ in (2.2), have the following Newton polyhedra:
1) \( \mathbf{N}(P, D(0)) = \text{conv}[\Lambda(P)] \): Convex hull of \( \Lambda(P) \).

2) \( \mathbf{N}(P, D_{[e_1, e_2, e_3]}) = \text{conv}[\Lambda(P) + \text{cone}(e_1, e_2, e_3)] \): Originally defined \( \mathbf{N}(P) \).

3) \( \mathbf{N}(P, D_{[-e_1, -e_2, -e_3]}) = \text{conv}[\Lambda(P) + \text{cone}(-e_1, -e_2, -e_3)] \).

**Example 2.3** Observe that \( \mathbf{N}(x_1^2 + x_2^2, D(0)) = \{(2, 0), (0, 2)\} \) are balanced since \( \text{cone}(B \cup \Lambda(P) \cup \{-1\}) = \text{cone}((2, 0), (0, 2), -1) = \mathbb{R}^2 \). But \( \mathbf{N}(x_1^2, D(0)) = \{(0, 2)\} \) are unbalanced as \( \text{cone}((0, 0), (0, 2), -1) \neq \mathbb{R}^2 \), and \( \mathbf{N}(x_1^2, x_2^2, D(0)) = \{(2, 2), (4, 4)\} \) are unbalanced since \( \text{cone}((0, 0), (2, 2), (4, 4), -1) \neq \mathbb{R}^2 \). Note that \( \mathbf{N}(P, D_{[e_1, e_2]}) \subset \mathbb{R}^2 \) is balanced for any polynomial \( P \) since \( \text{cone}(e_1, e_2, -1) = \mathbb{R}^2 \).

### 3 Statements of Main Theorems

#### 3.1 Normal-Crossing Condition of \((P, D)\)

**Definition 3.1** Given a polynomial \( P(x) = \sum_{m \in \Lambda(P)} c_m x^m \) and a Borel set \( D \subset \mathbb{R}^d \), call \((P, D)\) normal-crossing of type \([\sigma, \tau]\) if \( \tau \in \mathbb{Z}_+ \) is the minimal number:

\[
\sum_{\sigma \leq |\mu| \leq \tau} |x^\sigma_a \partial_x^\sigma P(x)| \geq c \sum_{m \in \Lambda(P)} |x^m| \text{ for all } x \in D \cap (\mathbb{R} \setminus \{0\})^d \tag{3.1}
\]

where \( c > 0 \), independent of \( x \), can depend on \((P, D)\). Given \( \tau \geq 1 \), type \([1, \tau]\) implies type \([0, \tau]\). Denote the number \( \tau \) above by \( \tau_\sigma(P, D) \) or \( \tau(P, D) \) for simplicity.

**Example 3.1** Let \( P(x) = \sum_{\mu=1}^d c_{\mu} x^{m_{\mu}} \) with all \( c_{\mu} \neq 0 \). From \( x^{m_{\mu}} = \prod_{v=1}^d x_v^{m_{\mu,v}} \), it follows that \( x_v \partial_{x_v} P(x) = \sum_{\mu=1}^d c_{\mu} m_{\mu,v} x^{m_{\mu}} \). Regarding \( c_{\mu} m_{\mu} \) as column vectors,

\[
\left( x_v \partial_{x_v} P(x) \right)_{v=1}^d \bigg|_{x_1^3}^x = \sum_{\mu=1}^d c_{\mu} x^{m_{\mu}} m_{\mu} = (c_1 m_1, c_2 m_2, \ldots, c_d m_d) (x_1^m, \ldots, x_d^m)^T
\]

where \((c_1 m_1, c_2 m_2, \ldots, c_d m_d)\) is the \( d \times d \) matrix. Hence \( \tau_1(P, \mathbb{R}^d) = 1 \) if and only if \( \text{rank}(m_1, \ldots, m_d) = d \). So \( \tau_1(x_1^3 - x_1 x_2^2, \mathbb{R}^2) = \tau_1(x_1 x_2 + x_2 x_3 + x_3 x_1, \mathbb{R}^3) = 1 \).

#### 3.2 Main Results

**Definition 3.2** Let \( P \cap \text{cone}(1) \neq \emptyset \) for \( P = \mathbf{N}(P, D_B) \), then there are \( \delta_{\text{for}}, \delta_{\text{bac}} \geq 0 \):

\[
P \cap \text{cone}(1) = [\delta_{\text{for}}, \delta_{\text{bac}}] 1. \tag{3.2}
\]

Call the face \( F \) of \( P \), of the minimal dimension, containing \( \delta_{\text{for}} 1 \) (\( \delta_{\text{bac}} 1 \)), the main forward (backward) face. Denote the face by \( F_{\text{for}}^{\text{main}} (F_{\text{bac}}^{\text{main}}) \), and its dimension by \( k_{\text{for}} = \dim(F_{\text{for}}^{\text{main}}) \) (\( k_{\text{bac}} = \dim(F_{\text{bac}}^{\text{main}}) \)) respectively. See Figs. 1 and 2.

**Main Theorem 1** (Sublevel-Set) Let \( P(x) \) be a polynomial in \( \mathbb{R}^d \). Suppose that \( \text{cone}(B) \) is strongly convex in Definition 2.3 and \( 0 \in P(D_B) = \{P(x) : x \in D_B\} \).
(A) Suppose that $N(P, D_B)$ is balanced. If $(P, D_B)$ is normal-crossing of type $[0, \tau]$ for $\tau < \delta_{for}$, then it holds that

$$\left| \{ x \in D_B : |\lambda P(x)| \leq 1 \} \right| \approx \begin{cases} \lambda^{1/\delta_{for}}(|\log |\lambda| + 1|) d^{1-k_{for}} & \text{if } \lambda \in [1, \infty), \\ \lambda^{1/\delta_{bac}}(|\log |\lambda| + 1|) d^{1-k_{bac}} & \text{if } \lambda \in (0, 1) \end{cases}$$

where the constants involving $\approx$ depend on $(P, D_B)$, independent of $\lambda$.

(B) Suppose that $N(P, D_B)$ is unbalanced. Then there exists $c > 0$ such that

$$\left| \{ x \in D_B : |\lambda P(x)| \leq 1 \} \right| = \infty \text{ for all } \lambda \in (0, c)$$

where the range $(0, c)$ is $(0, \infty)$ if $D_B$ contains a neighborhood of the origin.

Remark 3.1 We assume $0 \in P(D_B)$ for generalizing the condition $P(0) = 0$.

Corollary 3.1 (Powers, Integrability) Let $(P, D_B)$ be in Main Theorem 1.

(A) Under the hypothesis of (A) of Main Theorem 1,

$$\left\{ \rho \in (0, \infty) : \int_{D_B} |P(x)|^{-\rho} dx < \infty \right\} = \left(1/\delta_{bac}, 1/\delta_{for} \right).$$

(B) Under the hypothesis of (B) of Main Theorem 1,

$$\left\{ \rho \in (0, \infty) : \int_{D_B} |P(x)|^{-\rho} dx < \infty \right\} = \emptyset.$$

Main Theorem 2 (Oscillatory Integral) Let $P(x)$ be a polynomial in $\mathbb{R}^d$ and let cone$(B)$ be strongly convex with $0 \in P(D_B)$.

(A) Suppose that $N(P, D_B)$ is balanced. If $(P, D_B)$ is normal-crossing of type $[1, \tau]$ for $\tau < \delta_{for}$, then it holds that

$$\left| \int e^{i\lambda P(x)} \psi_{D_B}(x) dx \right| \leq C \begin{cases} \lambda^{1/\delta_{for}}(|\log |\lambda| + 1|) d^{1-k_{for}} & \text{if } \lambda \in [1, \infty), \\ \lambda^{1/\delta_{bac}}(|\log |\lambda| + 1|) d^{1-k_{bac}} & \text{if } \lambda \in (0, 1), \end{cases}$$

and there is $c > 0$ with $p = d - k_{for}$ and $q = d - k_{bac}$ such that

$$\limsup_{|\lambda| \to \infty} \left| \frac{\int e^{i\lambda P(x)} \psi_{D_B}(x) dx}{\lambda^{1/\delta_{for}}(|\log |\lambda| + 1|) \rho^{-1}} \right| \geq c \quad \text{and} \quad \limsup_{|\lambda| \to 0} \left| \frac{\int e^{i\lambda P(x)} \psi_{D_B}(x) dx}{\lambda^{1/\delta_{bac}}(|\log |\lambda| + 1|) \rho^{-1}} \right| \geq c.$$

(B) Suppose $N(P, D_B)$ is unbalanced. Then there is $\Psi_{D_B} \in \mathcal{A}(D_B)$ in (2.2): 

$$\left| \int_{\mathbb{R}^d} e^{i\lambda P(x)} \Psi_{D_B}(x) dx \right| = \infty \text{ for almost every } \lambda > 0.$$

Once $\tau = 1$ in the estimate $\lesssim$ of (A), the restriction $\tau < \delta_{for}$ is not needed.
Remark 3.2 For the estimate of \( \lesssim \) in (A) of Main Theorem 2, one can replace the amplitudes \( \psi_{DB} \) with all functions in \( \mathcal{A}(DB) \).

Remark 3.3 The unbalanced condition of \( N(P, DB) \) in (B) of Main Theorem 2, does not always imply a divergence of \( \int e^{i\lambda P(x)}\psi_{DB}(x)dx \), but guarantees an existence of an amplitude \( \Psi_{DB} \in \mathcal{A}(DB) \) such that \( |\int e^{i\lambda P(x)}\Psi_{DB}(x)dx| = \infty \). For instance, despite the unbalanced condition of \( N \) of Main Theorems 1 and 2 together with Corollary 3.1 to have that \( \psi_{DB} \) are in the unit circle, then Remark 3.3 does not always imply a divergence of \( N \).

Example 3.2 Let \( D = \mathbb{R}^2 \) and \( P(x_1, x_2) = x_1^4 x_2^4 (1 - x_1^2 - x_2^2)^2 \) whose nontrivial zeros are in the unit circle. Then \( \tau_1(P, D) = 2 < 4 = \delta_{for} \) and \( \delta_{bac} = 6 \) with \( k_{for} = 0 \) and \( k_{bac} = 1 \). By applying (A) of Main Theorems 1 and 2,

\[
\int_{\mathbb{R}^2} \psi(\lambda P(x))dx \approx \begin{cases} \lambda^{-1/4}(|\log \lambda| + 1) & \text{if } \lambda \in [1, \infty) \\ \lambda^{-1/6} & \text{if } \lambda \in (0, 1) \end{cases}
\]

while \( |\{(x_1, x_2) \in \mathbb{R}^2 : |x_1x_2\lambda| < 1\}| = \infty \) straightforwardly from Main Theorem 1.

Example 3.3 Let \( D = \mathbb{R}^2 \) and \( P(x_1, x_2) = x_1^4 x_2^4 (x_2 - x_1^2)^2 \) whose nontrivial zeros are in a parabola. Then \( \tau_1(P, D) = 2 < 16/3 = \delta_{for} \) and \( \delta_{bac} = 6 \). Apply (A) of Main Theorems 1 and 2 together with Corollary 3.1 to have

\[
\int_{\mathbb{R}^2} \psi(\lambda P(x))dx \approx \lambda^{-3/16} \text{ and } |\int_{\mathbb{R}^2} e^{i\lambda P(x)}dx| \lesssim \lambda^{-3/16} \text{ for all } \lambda \in (0, \infty)
\]

whereas \( \int_{\mathbb{R}^2} |P(x)|^{-\rho}dx = \infty \) for all \( \rho \geq 0 \).

If the order of zeros of \( P(x) \) is less than \( \delta_{for} \), that is, \( \tau_0(P, D) < \delta_{for} \), then one can apply the Main Theorems 1 to obtain the exact growth indices of the sublevel-set measure as well as its convergence.

3.3 The Case \( \tau < \delta_{for} \) Breaks

Suppose that \( \delta_{for} \leq \tau = \tau(P, D_B) < \delta_{bac} \). Then, once \( 0 < \lambda < 1 \), the estimates of Main Theorems 1 and 2 still hold. If \( \lambda \geq 1 \), without resolution of singularity, we shall obtain at least a non-sharp estimate:

\[
|\mathcal{I}(P, D_B, \lambda)| \lesssim \frac{(|\log \lambda| + 1)^p}{\lambda^{1/\tau}} \text{ if } p = \begin{cases} 0 & \text{if } \tau > \delta_{for} \\ d - k_{for} & \text{if } \tau = \delta_{for} \end{cases}
\]
where $I(P, DB, \lambda)$ stands for both $I_{\text{sub}}(P, DB, \lambda)$ and $I_{\text{osc}}(P, DB, \lambda)$ in (2.4). This with (A) of Main Theorems 1 and 2, implies that

$$\delta \in (\delta_{\text{for}}, \delta_{\text{bac}}) \cap (\tau, \delta_{\text{bac}}) \Rightarrow |I(P, DB, \lambda)| \leq C_\delta \lambda^{-\frac{1}{2}} \text{ for all } \lambda \in (0, \infty) \quad (3.4)$$

since $|\lambda|^{-1/\delta_{\text{for}}} \leq |\lambda|^{-1/\delta_{\text{bac}}}$ if $\lambda \geq 1$ and $|\lambda|^{-1/\delta_{\text{bac}}} \leq |\lambda|^{-1/\delta_{\text{for}}}$ if $\lambda \leq 1$. Next, we consider the worse case $\tau \geq \delta_{\text{bac}}$. Then not only, it can break (3.3), but also diverge. For example, $|\{x \in \mathbb{R}^2 : \lambda|x_2 - x_1|^2 \leq 1\}| = \infty$ for $\delta_{\text{bac}} = 1 < 2 = \tau$ and $|\{x \in \mathbb{R}^2 : \lambda|x_2^2 - x_1^2| \leq 1\}| = \infty$ for $\delta_{\text{bac}} = 1 = \tau$, because $N(x_2^2, D_{(-1,1)})$, $N(x_2(x_2 + 2x_1), D_{(-1,1)})$ after coordinate changes, are unbalanced. In Sect. 10, we split $DB = \bigcup DB_i$ so as to treat the cases $\tau(P, DB) \geq \delta_{\text{for}}$.

4 Polyhedra and Balancing Conditions

4.1 Two Representations of Polyhedra

Definition 4.1 (Polyhedron) Let $V$ be an inner product space of dimension $d$. For $q \in V \setminus \{0\}$ and $r \in \mathbb{R}$, set a hyperplane and its upper half-space,

$$\pi_{q,r} = \{y \in V : \langle q, y \rangle = r\} \text{ with } \pi_{q,r}^+ = \{y \in V : \langle q, y \rangle \geq r\}.$$  

Denote its interior $\{y \in V : \langle q, y \rangle > r\}$ by $(\pi_{q,r}^+)^\circ$. Given a finite set $\Pi(P) = \{\pi_{q_i,r_i}\}_{i=1}^M$ of hyperplanes, define a convex polyhedron $P$ (convex polytope) as the intersection of the upper half-spaces of the elements in $\Pi(P)$:

$$P = \bigcap_{\pi_{q,r} \in \Pi(P)} \pi_{q,r}^+. \quad (4.1)$$

If all $r = 0$, then $P$ is called a convex polyhedral cone. If $P \cap (-P) = \{0\}$, then $P$ is said to be strongly convex.

Definition 4.2 (Supporting Plane) Let $P$ be a polyhedron in $V$. We say that a hyperplane $\pi_{q,r}$ (which needs not belong to $\Pi(P)$) is a supporting plane of $P$ if

$$\pi_{q,r} \cap P \neq \emptyset \text{ and } \pi_{q,r}^+ \supset P.$$  

Call $\pi_{q,r}^+$ a supporting upper half-space of $P$. Let $\overline{\Pi}(P)$ stand for the set of all supporting planes $\pi_{q,r}$ of $P$. Then the inner normal vectors $q$ of the all elements in $\overline{\Pi}(P)$ form a convex polyhedral cone:

$$P^\vee := \{q \in V : \pi_{q,r} \in \overline{\Pi}(P)\}. \quad (4.2)$$
We call \( \mathbb{P}^{\vee} \) the **dual cone** of \( \mathbb{P} \). In view of (4.1), we can observe that

\[
\mathbb{P}^{\vee} = \left( \bigcap_{i=1}^{M} \pi_{q_i,0}^{+} \right)^{\vee} = \text{cone}(q_1, \ldots, q_M),
\]

where the second \( \vee \) indicates a set of inner normal vectors of the supporting planes.

We shall show that \( \mathbb{N}(P, DB) \) can be expressed as \( \mathbb{P} \) of (4.1) with \( \mathbb{P}^{\vee} = \text{cone}^{\vee}(B) \). The restriction of the bilinear form \( \langle \cdot, \cdot \rangle : \mathbb{P} \times \mathbb{P}^{\vee} \to \mathbb{R} \) to \( \Lambda(P) \times \text{cone}^{\vee}(B) \) enables us to control all \( |x^m| \sim 2^{-\langle m, q \rangle} \) for \( x \sim 2^{-q} \in 2^{-\text{cone}^{\vee}(B)} = DB \) with \( q = j \) in (2.4).

**Lemma 4.1** \( \mathbb{P} = \bigcap_{\pi_{q,r} \in \Pi(\mathbb{P})} \pi_{q,r}^{+} \) in (4.1) if and only if \( \mathbb{P} = \mathbb{N}(P, DB) \) in (2.5) for some \( P, B \). Here \( q, \Lambda(P), B \in \mathbb{Q}^d \) and \( r \in \mathbb{Q} \).

**Proof** In Definition 2.1, \( b \in \text{cone}(B) \) if and only if \( \langle b, q \rangle \geq 0 \) for all \( q \in \text{cone}^{\vee}(B) \):

\[
\text{cone}(B) = \bigcap_{q \in \text{cone}^{\vee}(B)} \pi_{q,0}^{+} = \bigcap_{q \in U} \pi_{q,0}^{+} \text{as in (4.1) for } U = \{q_i\}_{i=1}^{N} \subset \text{cone}(B)^{\vee}.
\]

Thus \( m + \text{cone}(B) = \bigcap_{q \in U} \pi_{q,(m,q)}^{+} \) with \( \langle n(q), q \rangle := \min\{\langle m, q \rangle : m \in \Lambda(P)\} \) implies

\[
\mathbb{N}(P, DB) = \text{conv}(\Lambda(P) + \text{cone}(B)) = \text{conv} \left\{ \bigcap_{q \in U} \pi_{q,m,q}^{+} : m \in \Lambda(P) \right\} = \bigcup_{m \in \text{conv}(\Lambda(P))} \bigcap_{q \in U} \pi_{q,(m,q)}^{+} = \bigcup_{q \in U} \bigcap_{m \in \Lambda(P)} \pi_{q,1}^{+} = \bigcap_{q \in U} \pi_{q,\langle n(q), q \rangle}^{+}
\]

showing \( \Leftarrow \). Next \( \Rightarrow \) follows from \( \bigcap_{\pi_{q,r} \in \Pi(\mathbb{P})} \pi_{q,r}^{+} = \mathbb{N}(P, DB) \) where \( \Lambda(P) = \{r/q : r \in \pi_{q,r}^{+} \in \Pi(\mathbb{P}) \} \) and \( \text{cone}(B)^{\vee} = \{q : \pi_{q,r} \in \Pi(\mathbb{P})\} \).

**4.2 Balancing Condition of Supporting Planes**

We state a balancing condition of a polyhedron \( \mathbb{N}(P, DB) = \bigcap \pi_{q,r}^{+} \) in terms of \( \pi_{q,r}^{+} \). See Fig. 1.

**Definition 4.3** An upper half-space \( \pi_{q,r}^{+} \) is **across-diagonal** if \( (\pi_{q,r}^{+})^{\circ} \cap \text{cone}(1) \neq \emptyset \), or **off-diagonal** if \( (\pi_{q,r}^{+})^{\circ} \cap \text{cone}(1) = \emptyset \) as in Fig. 1. This implies

\[
\pi_{q,r}^{+} \text{ is off-diagonal if and only if } \langle q, t1 \rangle \leq r \forall t \geq 0.
\]

**Lemma 4.2** From RHS of (4.3), it follows that

1. \( \pi_{q,r}^{+} \) is off-diagonal if and only if (i) \( r \geq 0 \) and (ii) \( \langle q, 1 \rangle \leq 0 \),
2. \( \pi_{q,r}^{+} \) is across-diagonal if and only if (i) \( r > 0 \) or (ii) \( \langle q, 1 \rangle > 0 \).
(3) \( \pi_{q,r}^+ \) is across-diagonal if and only if \( \pi_{q,r}^+ \) as a Newton polyhedron is balanced.

**Proof** Put \( t = 0 \) and \( t \gg 1 \) in (4.3) to get (1) that gives (2). To show (3), observe \( \pi_{q,r}^+ = N(P, B) \) for \( \Lambda(P) = \{rq/|q|\} \) and \( B = \text{cone}^\vee(q) \). Then \( \pi_{q,r}^+ \) balanced if and only if cone\((rq/|q| \cup \text{cone}^\vee(q) \cup \{-1\}) = \mathbb{R}^d \) if and only if \( r < 0 \) or \( \langle q, 1 \rangle > 0 \). □

Recall the set \( \overline{\Pi}(\mathbb{P}) \) of all supporting planes of \( \mathbb{P} \).

**Proposition 4.1** Let \( \mathbb{P} = N(P, D_B) \). Then all \( \pi^+ \) of \( \pi \in \overline{\Pi}(\mathbb{P}) \) are across-diagonal (balanced) if and only if \( \mathbb{P} \) is balanced.

**Proof** We claim its contraposition. We can see that there is \( \pi_{q,r} \in \overline{\Pi}(N(P, D_B)) \) such that \( \pi_{q,r} \) is off-diagonal, i.e., (i) \( r \geq 0 \) and (ii) \( \langle q, 1 \rangle \leq 0 \) if and only if there is a nonzero \( q \in \text{cone}^\vee(B) \) such that \( \langle q, m \rangle \geq 0 \) for all \( m \in \Lambda(P) \) and \( \langle q, -1 \rangle \geq 0 \) if and only if there is a nonzero \( q \in \text{cone}^\vee(B \cup \Lambda(P) \cup \{-1\}) \) and only if cone\((B \cup \Lambda(P) \cup \{-1\}) \neq \mathbb{R}^d \). □

**Remark 4.1** Geometrically, \( \mathbb{P} \) is balanced if and only if cone\((\mathbb{P}) \) contains a conical neighborhood cone\((\{1 + \epsilon e_v\}_{v \in \{d\}}) \) of cone\((1) \).

## 5 Combinatorial Lemmas

### 5.1 Forward and Backward Orientation

Let \( \mathbb{P} = N(P, D_B) \) and consider the integral \( \sum_{q \in \text{cone}^\vee(B) \cap \mathbb{Z}^d} \int \psi(\lambda P(x)) \chi(\frac{x}{\mathbb{Z}^d}) \psi_{D_B}(x)dx \) with \( q = j \) in (2.4). As \( |q| \to \infty \), the volume \( |\{x : x \sim 2^{-q}\}| \approx 2^{-|q,1|} \) of its support is to vanish or to blow up as \( \langle q, 1 \rangle > 0 \) or \( \langle q, 1 \rangle < 0 \). This observation leads us to bisect cone\((\mathbb{P}^\vee) = \mathbb{P}^\vee\) according to the signs of \( \langle q, 1 \rangle \) and split the domain \( D_B = 2^{-\text{cone}^\vee(B)} \) of the integral.

**Definition 5.1** (Forward and Backward Orientation of Polyhedron) Let \( \pi_{q,r} \in \overline{\Pi}(\mathbb{P}) \). We call \( q, \pi_{q,r} \) and \( \pi_{q,r}^+ \) **forward** if \( \langle q, 1 \rangle \geq 0 \), and **backward** if \( \langle q, 1 \rangle \leq 0 \). We split the dual cone \( \mathbb{P}^\vee \) of \( \mathbb{P} \) in (4.2) into the two sectors as \( \mathbb{P}^\vee = \mathbb{P}_{\text{for}}^\vee \cup \mathbb{P}_{\text{bac}}^\vee \) where

\[
\mathbb{P}_{\text{for}}^\vee := \mathbb{P}^\vee \cap \text{cone}^\vee(1) \quad \text{and} \quad \mathbb{P}_{\text{bac}}^\vee := \mathbb{P}^\vee \cap \text{cone}^\vee(-1).
\]

Given \( \mathbb{P} = \bigcap_{q \in \mathbb{P}^\vee} \pi_{q,r}^+ \), we can define a forward and a backward polyhedron of \( \mathbb{P} \) as

\[
\mathbb{P}_{\text{for}} := \bigcap_{q \in \mathbb{P}_{\text{for}}^\vee} \pi_{q,r}^+ \quad \text{and} \quad \mathbb{P}_{\text{bac}} := \bigcap_{q \in \mathbb{P}_{\text{bac}}^\vee} \pi_{q,r}^+.
\]

These are illustrated in Fig. 2. When computing \( I(P, D_B, \lambda) \) with \( \mathbb{P} = N(P, D_B) \), it turns out that \( \mathbb{P}_{\text{for}} \) determines the decay rate of \( \lambda \geq 1 \), whereas \( \mathbb{P}_{\text{bac}} \) determines not only the decay rate of \( \lambda < 1 \), but also the convergence of the integral.
Fig. 2 The pentagon $P$ has its forward and backward polyhedra satisfying $P = P_{\text{for}} \cap P_{\text{bac}}$ and $P_{\text{for}} \cup P_{\text{bac}} = P^\vee = \mathbb{R}^2$. The vertex $F$ has the dual face $F^\vee$ splitting $P_{\text{for}} \cup P_{\text{bac}}$ with $P_{\text{for}}^\vee = \text{cone}(q(1), ((1, -1)))$ and $P_{\text{bac}}^\vee = \text{cone}(q(2), ((1, -1)))$ in the last part.

5.2 Oriented Distances of Upper Half Spaces

**Definition 5.2** (Distance) Let $\pi_{q,r} \in \overline{\Pi}(P)$. To each across-diagonal upper half-space $\pi_{q,r}^+$, from $(\pi_{q,r}^+)^o \cap \text{cone}(1) \neq \emptyset$, one can assign the distance $d(\pi_{q,r}^+)$ satisfying

$$(\pi_{q,r}^+)^o \cap \mathbb{R}1 = \begin{cases} (d(\pi_{q,r}^+), \infty)1 & \text{if } q \text{ is forward,} \\ (-\infty, d(\pi_{q,r}^+))1 & \text{if } q \text{ is backward.} \end{cases} \quad (5.2)$$

In case of confusion, we write $d(\pi_{q,r}^+)$ as $d_{\text{for}}(\pi_{q,r}^+)$ or $d_{\text{bac}}(\pi_{q,r}^+)$ according as $q \cdot 1 \geq 0$ or $q \cdot 1 \leq 0$ respectively. See the second and third pictures of Fig. 2. Once $\pi_{q,r}^+$ is across-diagonal, the distance in (5.2) exists from the observation:

1. If $\langle q, 1 \rangle \neq 0$, $-\infty < d(\pi_{q,r}^+) < \infty$, since $\pi_{q,r} \cap \mathbb{R}1 = d(\pi_{q,r}^+)1$ is a singleton.
2. If $\langle q, 1 \rangle = 0$, then $d_{\text{for}}(\pi_{q,r}^+) = -\infty$ and $d_{\text{bac}}(\pi_{q,r}^+) = \infty$ due to $(\pi_{q,r}^+)^o \supset \mathbb{R}1$.

If $P$ is balanced, we define the distances $d(P_{\text{for}})$ and $d(P_{\text{bac}})$ by the numbers $^1$

$$[d(P_{\text{for}}), \infty)1 = P_{\text{for}} \cap \mathbb{R}1 \text{ and } (-\infty, d(P_{\text{bac}})]1 = P_{\text{bac}} \cap \mathbb{R}1. \quad (5.3)$$

$^1$ If $P_{\text{for}} \supset \mathbb{R}1$ or $P_{\text{bac}} \supset \mathbb{R}1$, then $d(P_{\text{for}}) = -\infty$ or $d(P_{\text{bac}}) = \infty$ respectively.
Lemma 5.1 Let \( \mathbb{P} \) be balanced. Then \([d(\mathbb{P}_{\text{for}}), d(\mathbb{P}_{\text{bac}})]_{1} = \mathbb{P} \cap \mathbb{R} \neq \emptyset \) with \( d(\mathbb{P}_{\text{bac}}) > 0 \). If \( \pi_{q_{1}, r_{1}}, \pi_{q_{2}, r_{2}} \in \overline{\Pi}(\mathbb{P}) \) with \( q_{1} \in \mathbb{P}_{\text{for}} \) and \( q_{2} \in \mathbb{P}_{\text{bac}} \), then

\[
d(\pi_{q_{1}, r_{1}}) \leq d(\mathbb{P}_{\text{for}}) \leq \delta \text{max} \{0, d(\mathbb{P}_{\text{for}})\} \leq \delta_{\text{bac}} = d(\mathbb{P}_{\text{bac}}) \leq d(\pi_{q_{2}, r_{2}}). \tag{5.4}
\]

Recall that if \( \mathbb{P} = \mathbb{N}(P, D_{B}) \) for a polynomial \( P \), then \( \mathbb{P} \cap \text{cone}(1) = [\delta_{\text{for}}, \delta_{\text{bac}}]_{1} \) as in (3.2). See the first three pictures in Fig. 2.

**Proof of (5.4)** Observe that \( \mathbb{P} \cap \{t \mathbf{1} : t > 0\} \neq \emptyset \) as \( \mathbb{P} \) is balanced. Take \( t^{*} \mathbf{1} \in \mathbb{P} \cap \{t \mathbf{1} : t > 0\} \subset \mathbb{P} \cap \mathbb{R} \mathbf{1} \). This with (5.3) and \( \mathbb{P} = \mathbb{P}_{\text{for}} \cap \mathbb{P}_{\text{bac}} \) yield \([d(\mathbb{P}_{\text{for}}), d(\mathbb{P}_{\text{bac}})]_{1} = \mathbb{P} \cap \mathbb{R} \mathbf{1} \neq \emptyset \) and \( d(\mathbb{P}_{\text{bac}}) > 0 \). By this, we have the middle part \( d(\mathbb{P}_{\text{for}}) \leq \delta_{\text{for}} \leq \delta_{\text{bac}} = d(\mathbb{P}_{\text{bac}}) \) in (5.4). From (5.1), (5.2) and (5.3), it follows that

\[
[d(\mathbb{P}_{\text{for}}), \infty)_{1} = \bigcap_{q \in \mathbb{P}_{\text{for}}} \pi_{q_{r}}^{+} \cap \mathbb{R} \mathbf{1} = \bigcap_{q \in \mathbb{P}_{\text{for}}} [d(\pi_{q_{r}}^{+}), \infty)_{1},
\]

\[
(-\infty, d(\mathbb{P}_{\text{bac}})]_{1} = \bigcap_{q \in \mathbb{P}_{\text{bac}}} \pi_{q_{r}}^{+} \cap \mathbb{R} \mathbf{1} = \bigcap_{q \in \mathbb{P}_{\text{bac}}} (-\infty, d(\pi_{q_{r}}^{+})]_{1}
\]

which give the first and last inequalities of (5.4).

Lemma 5.2 Let \( \pi_{q_{r}} \) be an hyperplane containing \( m \in \pi_{q_{r}} \). If \( \langle q, \mathbf{1} \rangle \neq 0 \), then

\[
\langle m, q \rangle = \langle d(\pi_{q_{r}}^{+})_{1}, q \rangle. \tag{5.5}
\]

**Proof** As \( d(\pi_{q_{r}}^{+})_{1}, m \in \pi_{q_{r}} \), it holds \( (d(\pi_{q_{r}}^{+})_{1} - m) \perp q \) in the first of Fig. 2. \( \square \)

**5.3 Model Estimates**

Take a supporting plane \( \pi_{q_{r}} \in \overline{\Pi}(\mathbb{P}) \) such that \( \langle q, \mathbf{1} \rangle \neq 0 \). We apply Lemmas 4.2 and 5.2 for the model sum in (2.4), given by

\[
\mathcal{I}_{\text{sub}}^{\text{sub}}(\lambda, q) := \sum_{\alpha \in \mathbb{Z}_{+}} \int_{x \sim 2^{-\alpha q}} \psi(\lambda, P(x)) dx \text{ with } d(\pi_{q_{r}}^{+}) = \delta.
\]

(A) Let \( \pi_{q_{r}}^{+} \) be across-diagonal and \( \tau = \tau_{0}(P, D_{B}) \). If \( \tau < \delta \), we show that

(A-1) \( \int_{x \sim 1} \psi(\lambda, P(2^{-\alpha q} x)) dx = O(|\lambda 2^{-\alpha q} m|^{-1/\tau}) \) for \( m \in \pi_{q_{r}} \),

(A-2) \( q \cdot m = \delta \langle q, \mathbf{1} \rangle \) as in Lemma 5.2, to obtain that

\[
\mathcal{I}_{\text{sub}}^{\text{sub}}(\lambda, q) \gtrsim \sum_{\alpha \in \mathbb{Z}_{+}} \frac{2^{-\alpha(q, \mathbf{1})}}{(1 + |\lambda 2^{-\alpha q} m|)^{1/\tau}} \approx \frac{2^{-\alpha(q, \mathbf{1})}}{(1 + |\lambda 2^{-\alpha q} m|)^{1/\tau}} \lesssim \lambda^{-1/\delta}.
\]

(B) Let \( \pi_{q_{r}}^{+} \) be off-diagonal for \( q \) in Lemma 4.2. Then \( \int_{x \sim 2^{-\alpha q}} \psi(\lambda, P(x)) dx \) has

(B-1) a big support \(|\{x : x \sim 2^{-\alpha q}\}| \approx 2^{-\alpha(q, \mathbf{1})} \geq 1 \) and

(B-2) a small phase \(|P(x)| \lesssim |x^{m}| \sim 2^{-\alpha(m, q)} = 2^{-\alpha \delta(q, \mathbf{1})} \ll 1 \) for \( m \in \pi_{q_{r}} \),

\( \square \) Springer
showing that $I_{sub}(\lambda, q) \approx \sum_{j=\alpha q} 2^{-\alpha(q,1)} \approx \infty$ for all real $\lambda$.

6 Oriented Simplicial Cone Decomposition

6.1 Basic Dual Face Decomposition

**Definition 6.1** *(Face and Dual Faces)* Let $P$ be a polyhedron in $V$. A subset $F \subset P$ is called a face of $P$ (denoted by $F \preceq P$) if there is a supporting plane $\pi_{q,r} \in \mathbb{P}(P)$ such that $F = \pi_{q,r} \cap P$. Denote the set of $k$-dimensional faces of $P$ by $\mathcal{F}^k(P)$ and the set of all faces by $\mathcal{F}(P)$. Define the dual face $F^\vee$ of $F \in \mathcal{F}(P)$ as the set of all normal vectors $q$ of supporting planes $\pi_{q,r}$ containing $F$:

$$F^\vee = \{ q \in P^\vee : \pi_{q,r} \cap P \supset F \}$$

with $(F^\vee)^c = \{ q \in P^\vee : \pi_{q,r} \cap P = F \}$.

See $F$ and $F^\vee$ in the first and last of Fig. 2. Finally, call $F, F^\vee$ oriented, that is, forward or backward if $F^\vee \subset \text{cone}^\vee(1)$ or $F^\vee \subset \text{cone}^\vee(-1)$ respectively.

**Proposition 6.1** *(Dual Face decomposition)* If $\text{rank}(P^\vee) = d - k_0$, then,

$$P^\vee = \bigcup_{F \in \mathcal{F}(P)} F^\vee = \bigcup_{F \in \mathcal{F}^{k_0}(P)} F^\vee$$

where $k_0$ is the minimal possible dimension\(^2\) of faces in $P$. The second part of (6.1) follows from the fact $G \preceq F \Rightarrow F^\vee \preceq G^\vee$. See [7, 18]

6.2 Oriented Simplicial Cone Decomposition of Integral

By $P = N(P, DB)$ with $P^\vee = \text{cone}^\vee(B)$ in the second part of (6.1), we can write (2.4) as

$$I_{sub}^{\text{sub}}(P, DB, \lambda) = \sum_{F \in \mathcal{F}^{k_0}(P)} (\sum_{j \in F^\vee \cap \mathbb{Z}^d} I_{sub}^{\text{sub}}(\lambda)).$$

**Remark 6.1** Reset $F_n^\vee$ as $F_n^\vee \setminus (\bigcup_{i=1}^{n-1} F_i^\vee)$ in $\mathcal{F}^{k_0} = \{ F_n \}_{n=1}^M$ to make all $F^\vee$ in (6.2) mutually disjoint, though they originally may overlap on their boundaries.

**Definition 6.2** *(Essential Disjointness)* We say polyhedra $G_1, \ldots, G_m$ in $V$ are essentially disjoint (simply ess disjoint) if $G_i^\circ \cap G_j^\circ = \emptyset$ for all pairs with $G^\circ = G \setminus \partial G$.

**Definition 6.3** Let $K$ be a polyhedral cone of dimension $n$ in $V$. We say that $K$ is simplicial if $K = \text{cone}(\{ q_i \}_{i=1}^n)$ for some linearly independent vectors $q_i$’s in $V$.

\(^2\) Every minimal (under $\subset$) face of $P$ has dimension $k_0 = d - \text{rank}(P^\vee)$.
For a convenient computation, we shall make a dual face $F$ in (6.2):

(i) simplicial cone $(q_1, \ldots, q_{d_0})$ with $d_0 = d - k_0$,

(ii) contained in an oriented dual cone $P_\lor$ or $P_{\lor\text{bac}}$.

**Theorem 6.1** (Oriented Simplicial Cone Decomposition) Let $P = N(P, D_B)$ and let $d_0 = \dim(\text{cone}^\lor(B)) = d - k_0$. Then, we can make the integral of (6.2) as

$$I_{\text{sub}}(P, D_B, \lambda) = \sum_{F \in F_0(P_\lor) \cup F_0(P_{\lor\text{bac}})} \sum_{j \in F_\lor \cap \mathbb{Z}^d} I_{\text{sub}}^j(\lambda)$$

where $F_\lor = \text{cone}(q_1, \ldots, q_{d_0})$ are oriented and disjoint simplicial cones. Moreover, $F_\lor \cap \mathbb{Z}^d$ is equipped with the rational coordinates of the basis $\{q_i\}_{i=1}^{d_0} \subset \mathbb{Q}^d$:

$$\{ \sum_{i=1}^{d_0} \alpha_i q_i : (\alpha_i) \in (M_0 \mathbb{Z}_+)^{d_0} \} \subset F_\lor \cap \mathbb{Z}^d \subset \{ \sum_{i=1}^{d_0} \alpha_i q_i : (\alpha_i) \in \left( \frac{\mathbb{Z}_+}{M_1} \right)^{d_0} \}$$

where $M_0, M_1 \in \mathbb{N}$ and $1 \leq |q_i| \leq 2$. One can replace $F_k$ with $\bigcup_{k=k_0}^{d-1} F_k$ in (6.3).

We use (6.2) with some geometric argument to prove Theorem 6.1 in Sect. 11.

**Remark 6.2** To show (6.3), we take $F_0$ as a newly formed set (denoted by $F_0^{\text{osc}}$) by modifying its element $F_0$ to have an oriented and simplicial dual face. In (6.3), we need to correct $F_\lor$ to be disjoint as in Remark 6.1.

**Remark 6.3** There is a similar decomposition of $I_{\text{osc}}(P, D_B, \lambda)$ as in (6.3).

**Remark 6.4** It suffices to take $\alpha_i \in \mathbb{Z}_+$ rather than $\alpha_i \in (1/M_1)\mathbb{Z}_+$ in (6.4).

### 7 Estimate of One Dyadic Piece

Under the normal-crossing condition (3.1), we estimate $I_{\text{sub}}^j(\lambda)$ and $I_{\text{osc}}^j(\lambda)$ of Theorem 6.1. The estimate is based on the control of derivatives of $P$ in the following lemma.

**Lemma 7.1** (Monomialization) Suppose that $F \in \mathcal{F}(P)$ in Theorem 6.1, $m \in F \cap \Lambda(P)$ and $j \in F_\lor \cap \mathbb{Z}^d$. Then for $x \sim 2^{-\theta}$, it holds that

$$|P(2^{-j}x)| \lesssim \sum_{n \in \Lambda(P)} 2^{-j \cdot n} \approx 2^{-j \cdot m} \text{ with } j \cdot m = j \cdot \hat{m} \text{ for } \hat{m} \in F \cap \Lambda(P),  \quad (7.1)$$

$$2^{c|j|} \sum_{n \in \Lambda(P) \setminus \pi_{q,r}} 2^{-j \cdot n} \lesssim 2^{-j \cdot m} \text{ for some } c > 0 \text{ if } F = \pi_{q,r} \cap \mathbb{P}.  \quad (7.2)$$

As a consequence, if $(P, D_B)$ is normal-crossing of type $[\sigma, \tau]$, then

$$\sum_{\sigma \leq |\alpha| \leq \tau} |\partial^\alpha(P(2^{-j}x))| \approx 2^{-j \cdot m}  \quad (7.3)$$
with constants in \( \lesssim \), depend on the coefficients of \( P \), but independent of \( j \) and \( x \). One can perturbate \( x \sim 2^{-0} \) as \( x \sim h \) \( 2^{-0} \), i.e., \( 1/h \leq |x_v| \leq h \) for a fixed \( h \geq 1 \).

**Proof of (7.1)** By the triangle inequality, we have

\[
|P(2^{-j}x)| \leq \sum_{n \in \Lambda(P)} |c_n|2^{-j-n}|x^n| \leq \max\{|c_n|\} \sum_{n \in \Lambda(P)} 2^{-j-n}. \tag{7.4}
\]

Let \( F \in \mathcal{F}^k(\mathbb{P}) \). Then, there are \( \ell \) supporting planes \( \pi_{q_v,r_v} \) such that

\[
F = \bigcap_{\nu=1}^{\ell} \pi_{q_v,r_v} \cap \mathbb{P} \quad \text{and} \quad F^\nu = \text{cone}(q_1, \ldots, q_\ell).
\]

Let \( m \in F \cap \Lambda(P) \) and \( n \in \Lambda(P) \subset \mathbb{P} \). Then \( m \in \pi_{q_v,r_v} \) and \( n \in \pi_{q_v,r_v}^+ \), that is, \( q_v \cdot (n - m) \geq 0 \) for all \( v \in [\ell] \). By this with \( j = \alpha_1q_1 + \cdots + \alpha_\ell q_\ell \in \mathbb{F}^\nu \cap \mathbb{Z}^d \) for \( \alpha_1, \ldots, \alpha_\ell \geq 0 \), one obtain \( j \cdot (n - m) \geq 0 \). Thus \( 2^{-j-m} \geq 2^{-j-n} \) in (7.4), which implies \( \approx \) of (7.1) because \( \Lambda(P) \) is finite. Since \( m, m \in \mathbb{F} \cap \pi_{q_v,r_v} \) and \( q_v \cdot (m - m) = 0 \) for all \( v = 1, \ldots, \ell \). So, \( j \cdot (m - m) = 0 \) in (7.4). This completes the proof of (7.1).

**Proof of (7.2)** As \( j \in \mathbb{F}^\nu = \text{cone}(q) \), \( j = \alpha q \) for \( \alpha > 0 \). If \( m \in \pi_{q,r} \) and \( n \in (\pi_{q,r})^0 \), there is \( c(m/n) > 0 \) such that \( \frac{q}{\alpha} \cdot (n - m) > c(m/n) \). Take \( c \) as the minimal \( c(m/n) \) over \( n \in \Lambda(P) \setminus \pi_{q,r} \). Then \( j \cdot (n - m) \geq c(m/n) \geq c \), that is, \( 2^{c|j|} 2^{-j-n} \leq 2^{-j-m} \).

**Proof of (7.3)** By the chain rule in differentiation and \( x \sim 1 \),

\[
\text{LHS of (7.3)} \approx \sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-j}x)\sigma(\partial_x^\alpha P)(2^{-j}x)| \approx \sum_{n \in \Lambda(P)} 2^{-j-n} \approx 2^{-j-m}
\]

where the second \( \approx \) is owing to (3.1) and the last \( \approx \) is due to (7.1).

**Lemma 7.2** (Decay Estimates) Let \( \mathbb{P} = \mathbb{N}(P, D_B) \) and \( \mathbb{F} \in \mathcal{F}(\mathbb{P}) \). Take \( m \in \mathbb{F} \cap \Lambda(P) \) and \( j \in \mathbb{Z}^d \cap \mathbb{F}^\nu \) in Theorem 6.1. Then

\[
|T_{j}^{\text{sub}}(\lambda)| \lesssim 2^{-j-1} \min \left\{ 1, \frac{1}{|\lambda 2^{-j-m}|^{1/\tau}} \right\} \text{ if } \tau_0(P, D_B) = \tau \text{ as in (3.1)}, \tag{7.5}
\]

\[
|T_{j}^{\text{osc}}(\lambda)| \lesssim 2^{-j-1} \min \left\{ 1, \frac{1}{|\lambda 2^{-j-m}|^{1/\tau}} \right\} \text{ if } \tau_1(P, D_B) = \tau. \tag{7.6}
\]

If \( \tau_0(P, D_B) = 0 \) in (7.5), or \( \tau_1(P, D_B) = 1 \) in (7.6), one can take \( 0 < \tau \ll 1 \).

**Proof of (7.5) and (7.6)** By the change of variable,

\[
T_{j}^{\text{sub}}(\lambda) = 2^{-j-1} \int \psi(\lambda P(2^{-j}x))\psi_{D_B,R}(2^{-j}x)\chi(x)dx,
\]

\[
T_{j}^{\text{osc}}(\lambda) = 2^{-j-1} \int e^{i\lambda P(2^{-j}x)}\psi_{D_B,R}(2^{-j}x)\chi(x)dx.
\]
From (7.3) for \( m \in \mathbb{F} \cap \Lambda(P) \) and \( j \in \mathbb{Z}^d \cap \mathbb{F}^\vee \) with the normal-crossing assumption,
\[
\sum_{\sigma \leq |\alpha| \leq \tau} |\partial^\alpha (P(2^{-j}x))| \approx 2^{-j \cdot m} \quad \text{with} \ x \sim 1.
\]

This with the rapid decreasing property of \( \psi \) yields the desired bound of \( I^\text{sub}_j(\lambda) \). From the hypothesis \( \Psi_{DB, R} \in \mathcal{A}(D_B) \) in (2.2), it follows that
\[
|\partial^\alpha (\Psi_{DB, R}(2^{-j}x) \chi(x))| \lesssim 1 \quad \text{in the support of the integral} \ I_{j}^{\text{osc}}(\lambda).
\]

With this, we apply the van der Corput lemma to obtain the bounds of \( I_{j}^{\text{osc}}(\lambda) \).

### 8 Summation Over a Dual Face

We utilize Lemma 7.2 for computing \( I_j(\lambda) \) for \( j \in \mathbb{F}^\vee \) in (6.3). Next, we need to sum \( I_j(\lambda) \) over \( \alpha_1, \ldots, \alpha_{d_0} \in \mathbb{Z}_+ \) where \( j = \alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0} \) in (6.4).

#### 8.1 Summation Formula

We use the following lemma for summing over \( \alpha \).

**Lemma 8.1** Let \( 0 < \tau < \delta \). Then for all \( \lambda > 0 \), we have
\[
\sum_{(\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p} 2^{- (\alpha_1 + \cdots + \alpha_p)} \min \left\{ 1, \frac{1}{\lambda 2^{- (\alpha_1 + \cdots + \alpha_p) \delta} 1/\tau} \right\} \lesssim \frac{(| \log \lambda | + 1)^{p-1}}{\lambda^{1/\delta}}, \quad (8.1)
\]
\[
\sum_{(\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}_+^p} 2^{(\alpha_1 + \cdots + \alpha_p)} \min \left\{ 1, \frac{1}{\lambda 2^{(\alpha_1 + \cdots + \alpha_p) \delta} 1/\tau} \right\} \lesssim \frac{(| \log \lambda | + 1)^{p-1}}{\lambda^{1/\delta}}. \quad (8.2)
\]

Let \( \tau \geq \delta \). Then it holds
\[
LHS \ of \ (8.1) \lesssim \frac{(| \log \lambda | + 1)^{p(\tau)}}{\lambda_1^{\tau}} \quad \text{for} \ p(\tau) = \begin{cases} \ p & \text{if} \ \tau = \delta \\ 0 & \text{if} \ \tau > \delta. \end{cases} \quad (8.3)
\]

**Proof of (8.1)** If \( p = 1 \), one can obtain (8.1) as
\[
\sum_{\alpha \in \mathbb{Z}_+} 2^{-\alpha} \min \left\{ 1, \frac{1}{(\lambda 2^{-\alpha \delta})^{1/\tau}} \right\} \approx \sum_{\lambda_2^{-\alpha \delta} \leq 1} 2^{-\alpha} + \sum_{\lambda_2^{-\alpha \delta} \geq 1} \frac{2^{\alpha (\frac{\delta}{\tau} - 1)}}{\lambda^{1/\tau}} \approx \lambda^{-1/\delta}. \quad (8.4)
\]
Next, to show the case $p \geq 2$ of (8.1), split the sum in (8.1) into the two parts

\[
\sum_{2^{\alpha_1+\cdots+\alpha_p} \geq (\lambda + \lambda^{-1})^{2/\delta}} 2^{-(\alpha_1+\cdots+\alpha_p)} - \frac{1}{(\lambda^{2-\alpha_1-\cdots-\alpha_p})^{1/\tau}} \min \{1, (\lambda^{2-\alpha_1-\cdots-\alpha_p})^{1/\tau}\} \leq \lambda^{-1/\delta} \left\{ \frac{1}{|\log \lambda| + 1} \right\}^{p-1}.
\]

over the indices $\alpha_i \in \mathbb{Z}_+$ for $i \in [p]$. The first sum in (8.5) is bounded by

\[
\sum_{\alpha_i \in \mathbb{Z}_+ \text{ for } i=1,\ldots,p} 2^{-(\alpha_1+\cdots+\alpha_p)} (\lambda + \lambda^{-1})^{-1/\delta} \leq \lambda^{-1/\delta} \left\{ \frac{1}{|\log \lambda| + 1} \right\}^{p-1}.
\]

The second sum (8.5), after coordinate change $\alpha_1 + \cdots + \alpha_p = \alpha$, becomes

\[
\sum_{0 \leq \alpha_1 + \cdots + \alpha_p \leq 2\tau \log_2 (\lambda + \lambda^{-1})} 2^{-\alpha} \min \{1, (\lambda^{2-\alpha})^{1/\tau}\} \approx \frac{|\log \lambda| + 1}{\lambda^{1/\delta}}^{p-1}.
\]

because of (8.4) and $\log(\lambda + \lambda^{-1}) \approx |\log \lambda| + 1$. This with (8.4) yields (8.1).

**Proof of (8.2)** Let $\alpha = \alpha_1 + \cdots + \alpha_p$ and $A := 2^\delta - 1 > 1$ and write the sum

\[
\lambda^{-1/\tau} \min \left[ 2^\alpha \lambda^{1/\tau}, 2^{-\alpha} \left( \frac{2}{\tau} \right) - \lambda^{-1/\tau} A^{-\alpha} \min \left\{ \left( \lambda^{-1} (A^{-1} A^{-\frac{1}{\tau} - 1})^{-1/\tau} \right)^{-1/\tau}, 1 \right\} = \lambda^{-1/\tau} A^{-\alpha} \min \left\{ \left( \lambda^{-1} A^{-\frac{1}{2}} \right)^{-1/\tau}, 1 \right\}
\]

with $\delta = \tau \left( 1 + \frac{1}{\delta - 1} \right)$. Replace 2 in (8.1) by $A$ and sum the last line over $\alpha_i$ in $\mathbb{Z}_+$ to have the bound $\lambda^{-1/\tau} (\lambda^{-1})^{-1/\delta} (|\log \lambda^{-1}| + 1)^{p-1} = \lambda^{-1/\delta} (|\log \lambda| + 1)^{p-1}$.

**Proof of (8.3)** The case $\tau > \delta$ follows from the geometric sum, and the case $\tau = \delta$ from $\sum_{2^{\alpha_1+\cdots+\alpha_p} < \lambda 1 \approx (\log \lambda)^p$. Next, use $\sum_{\alpha \in \mathbb{Z}_+} 2^\alpha \min\{1, (\lambda^{2\alpha})^{-1}\} = \infty$.

### 8.2 Forward Face Sum

We estimate $\sum_{j \in F} T_{\lambda}^{\text{sub}}(\lambda)$ in (6.3) for $F \in F(\mathbb{P}\text{for})$.

**Proposition 8.1** (Forward Face Estimate) Let $\mathbb{P} = \bigcap_{i=1}^{d_0} \pi_{q_i, r_i} \in F^{k_0}(\mathbb{P}\text{for})$ in Theorem 6.1. Suppose that all $\pi_{q_i, r_i}$ are across-diagonal and satisfying

\[
\mathcal{U} \{ \pi_{q_i, r_i} : d(\pi_{q_i, r_i}) = d(\mathbb{P}\text{for}) \}_{i=1}^{d_0} = p \geq 0.
\]

(1) Let $\lambda > 1$. If $\delta_{\lambda} > 0$ and $P$ is of type $[0, \tau]$ with $\tau < \delta_{\lambda} \in (7.5)$, then,

\[
\sum_{j \in F} T_{\lambda}^{\text{sub}}(\lambda) \leq C \lambda^{-1/\delta_{\lambda}} (|\log \lambda| + 1)^{p-1} \text{ for } C \text{ independent of } \lambda.
\]
One can show it has the lower bound
\[
\sum_{j \in \mathbb{F}^\vee \cap \mathbb{Z}^d} T_j^\text{sub}(\lambda) \geq C^{-1} \lambda^{1-1/\delta_{\text{for}}}(\log \lambda + 1)^{p-1} \text{ if } p \geq 1.
\]

(2) Let \(0 < \lambda \leq 1\). Then there is \(C_{\epsilon} > 0\),
\[
\sum_{j \in \mathbb{F}^\vee \cap \mathbb{Z}^d} T_j^\text{sub}(\lambda) \leq C_{\epsilon} \lambda^{-\epsilon} \text{ for an arbitrary small } \epsilon > 0.
\]

**Proof of Proposition 8.1** Since \(\mathbb{F}^\vee = \text{cone}((q_i d_0)_{i=1}^{d_0})\) is simplicial with \(\text{rank}((q_i d_0)_{i=1}^{d_0}) = d_0\), one can express \(j \in \mathbb{F}^\vee \cap \mathbb{Z}^d\) in \(\sum_{j \in \mathbb{F}^\vee \cap \mathbb{Z}^d} T_j^\text{sub}(\lambda)\) in Theorem 6.1 as
\[
j = \alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0} \text{ with } \alpha_i \in \mathbb{Z}_+.
\]

If \(P\) is a polynomial, then \(-\infty \leq d(\pi_{q_i, r_i}^+) \leq d(P_{\text{for}}) = \delta_{\text{for}}\) in (5.3) and (5.4). With \(0 \leq p \leq d_0\), we rearrange \((\pi_{q_i, r_i}) d_0_{i=1}\) of the forward supporting planes:
\[
\begin{align*}
q_i \cdot 1 &> 0 \text{ and } d(\pi_{q_i, r_i}^+) = \delta_{\text{for}} \text{ for } i = 1, \ldots, p, \\
q_i \cdot 1 &> 0 \text{ and } d(\pi_{q_i, r_i}^+) < \delta_{\text{for}} \text{ for } i = p + 1, \ldots, n, \\
q_i \cdot 1 &= 0 \text{ for } i = n + 1, \ldots, d_0 \text{ where } 0 \leq p \leq n \leq d_0.
\end{align*}
\]

Use (7.5) to have the upper bound of \(\sum_{j \in \mathbb{F}^\vee \cap \mathbb{Z}^d} T_j^\text{sub}(\lambda)\) in terms of the above \(\alpha\):
\[
\sum_{(\alpha_1, \ldots, \alpha_{d_0}) \in \mathbb{Z}_+^{d_0}} 2^{-(\alpha_1 q_1 + \cdots + \alpha_n q_n) - 1} \min \left\{1, \frac{1}{\lambda^{1/2} 2^{-(\alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0}) - 1/2}} \right\} = 2^{-(\alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0}) - 1/2}
\]

where \(m \in \mathbb{F} \cap \Lambda(P)\), that is, \(m \in \bigcap_{i=1}^{d_0} \pi_{q_i, r_i}\) in (8.6). Then it follows that
\[
q_i \cdot m = \begin{cases} d(\pi_{q_i, r_i}^+) q_i \cdot 1 \text{ from (5.5) if } i = 1, \ldots, n \text{ and,} \\
r_i < 0 \text{ from (2) of Lemma 4.2 if } i = n + 1, \ldots, d_0.
\end{cases}
\]

By using the second case \(q_i \cdot m < 0\) for \(i = n + 1, \ldots, d_0\) in (8.8), we put
\[
\lambda(\alpha) := \lambda 2^{-\sum_{i=n+1}^{d_0} q_i \cdot m \alpha_i} = \lambda 2^{-\sum_{i=n+1}^{d_0} r_i \alpha_i} = \lambda 2^{\sum_{i=n+1}^{d_0} r_i |\alpha_i|} \text{ for } |r_i| > 0.
\]

Then by inserting (8.8), (8.9) into (8.7), one can rewrite it as
\[
\sum_{\alpha_1, \ldots, \alpha_{d_0}} 2^{-(\alpha_1 q_1 + \cdots + \alpha_n q_n) - 1} \min \left\{1, \frac{2^{\sum_{i=p+1}^{d_0} \alpha_i q_i \cdot d(\pi_{q_i, r_i}^+) \over r} \lambda^2}{\lambda(\alpha)^{1/2}} \right\} = 2^{-(\alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0}) - 1/2}
\]
For $i = p + 1, \ldots, n$, put $\delta_i = \max\{0, d(\pi_+^{q_i, r_i})\}$. By (8.6) and $\tau < \delta_{\text{for}}$, assume that
\[
\max\{\delta_i\}_{i=p+1}^n < \tau
\]  
keeping $\tau < \delta_{\text{for}}$ in (7.5). Then one can see that (8.10) is bounded by
\[
\sum_{\alpha_1, \ldots, \alpha_p} 2^{-(\alpha_1 q_1 + \cdots + \alpha_p q_p) \cdot 1} \sum_{\alpha_{p+1}, \ldots, \alpha_n} 2^{-(\alpha_{p+1} q_{p+1} + \cdots + \alpha_n q_n) \cdot 1(1 - \frac{\delta_p}{\tau})}
\times \sum_{\alpha_{n+1}, \ldots, \alpha_d_0} \min\left\{ 1, \frac{1}{\left(\lambda(\alpha) 2^{-\sum_{i=1}^{p} \alpha_i q_i \cdot 1 \delta_{\text{for}}} \right)^{\frac{1}{\tau}}} \right\}.
\]
Summing $\sum_{\alpha_{p+1}, \ldots, \alpha_n}$ due to $\frac{\delta_i}{\tau} < 1$ in (8.11), majorize the above by $S(\lambda)$ where
\[
S(\lambda) := \sum_{\alpha_1, \ldots, \alpha_p, \alpha_{n+1}, \ldots, \alpha_d_0} 2^{-(\sum_{i=1}^{p} \alpha_i q_i \cdot 1)} \min\left[ 1, \frac{1}{\left(\lambda(\alpha) 2^{-\sum_{i=1}^{p} \alpha_i q_i \cdot 1 \delta_{\text{for}}} \right)^{\frac{1}{\tau}}} \right].
\]  
Proof of (1) for the case $\lambda \geq 1$ Regard $\alpha_i q_i \cdot 1$ and $\lambda(\alpha)$ in (8.12) as $\alpha_i$ and $\lambda$ in (8.1). Then, apply (8.1) for the sum $\sum_{\alpha_1, \ldots, \alpha_p}$ of (8.12) with (8.9) to obtain that
\[
S(\lambda) \lesssim \sum_{\alpha_{n+1}, \ldots, \alpha_d_0} \frac{(|\log \lambda(\alpha)| + 1)^{p-1}}{\lambda(\alpha)^{1/\delta_{\text{for}}}} \lesssim \frac{(|\log \lambda| + 1)^{p-1}}{\lambda^{1/\delta_{\text{for}}}}
\]
where the second follows from $|\log \lambda(\alpha)| \leq |\log \lambda| + \sum_{i=n+1}^{d_0} \alpha_i$ and $|r_i| > 0$ in (8.9). If $p = 0$, then in (8.12), it holds that $S(\lambda) \leq \sum_{\alpha_{n+1}, \ldots, \alpha_d_0} |\alpha(\alpha)|^{-\frac{1}{\tau}} \lesssim \lambda^{1/\delta_{\text{for}}} \leq \lambda^{-1/\delta_{\text{for}} - \epsilon}$ for $\lambda > 1$ giving a better bound than $\lambda^{-1/\delta_{\text{for}}} (|\log \lambda| + 1)^{-1}$.

Next, we show (2) and the reverse inequality of (1).

Proof of (2) for the case $0 < \lambda < 1$ One can assume that $1/\tau$ in (8.12) satisfies $0 < 1/\tau \ll 1$. This combined with $q_i \cdot 1 > 0$ for $1 \leq i \leq p$ enables us to estimate
\[
S(\lambda) \lesssim \sum_{\alpha_1, \ldots, \alpha_p, \alpha_{n+1}, \ldots, \alpha_d_0} 2^{-(\alpha_1 q_1 + \cdots + \alpha_p q_p) \cdot 1} \frac{2^{\delta_{\text{for}}/(\alpha_1 q_1 + \cdots + \alpha_p q_p) \cdot 1}}{\lambda(\alpha)^{1/\tau}} \lesssim \frac{1}{\lambda^{1/\tau}}
\]
because $1 \ll \tau$ makes $\delta_{\text{for}} < \tau$. This gives the desired bound of (2).

Proof of the lower bound in (1) We shall find a lower bound of
\[
\sum_{j \in \mathbb{F} \cap \mathbb{Z}^d} \mathcal{I}^{\text{sub}}_j(\lambda) = \sum_{j \in \mathbb{F} \cap \mathbb{Z}^d} 2^{-j-1} \int \psi(\lambda P(2^{-j} x)) \psi_D(2^{-j} x) \chi(x) dx
\]
under the condition $\delta_{\text{for}} > 0$ and $p \geq 1$ in (8.6). By (7.1), it holds that

$$|P(2^{-j}x)| \leq C2^{-jm} \text{ for } j \in \mathbb{F}^\vee \cap \mathbb{Z}^d \text{ and } m \in \mathbb{F} \cap \Lambda(P)$$

which implies $\psi(\lambda P(2^{-j}x)) = 1$ whenever $|C\lambda 2^{-jm}| < 1/10$. Hence, one has

$$\sum_{j \in \mathbb{F}^\vee \cap \mathbb{Z}^d} T_j^{\text{sub}}(\lambda) \geq \sum_{(\alpha_i) \in (M_{\mathbb{Z}_+})^p \cap A} 2^{-(\alpha_1 q_1 \cdot 1 + \cdots + \alpha_p q_p \cdot 1)}. \quad (8.13)$$

where $A := \{(\alpha_i) : C\lambda 2^{-(\alpha_1 q_1 \cdot 1 + \cdots + \alpha_p q_p \cdot 1)} \delta_{\text{for}} \leq 1/10\}$ because one can restrict $j = \alpha_1 q_1 + \cdots + \alpha_p q_p$ by taking $\alpha_{p+1} = \cdots = \alpha_{d_0} = 0$ so that

$$2^{-j \cdot m} = 2^{-(\alpha_1 q_1 \cdot 1 + \cdots + \alpha_p q_p \cdot 1)} \delta_{\text{for}} \text{ with } q_i \cdot 1 > 0 \text{in (8.6) and (8.8).}$$

Reset $\gamma_i = \alpha_i q_i \cdot 1 > 0$ for $i \in [p]$, and rewrite RHS of (8.13) in terms of $\gamma_i$’s:

$$\sum_{(\gamma_1, \ldots, \gamma_p) \in (N \mathbb{Z}_+)^p \cap A'} 2^{-(\gamma_1 + \cdots + \gamma_p)} \text{ with } A' = \{(\gamma_i) : 2^{-(\gamma_1 + \cdots + \gamma_p)} \leq (10C\lambda)^{-1/\delta_{\text{for}}}\}$$

for some $N \in \mathbb{N}$, giving the lower bound $\lambda_{\text{for}}^{-1} (|\log \lambda| + 1)^{p-1}$.

Therefore, we finish the proof of Proposition 8.1.

We next show (3.3) saying the case $\tau \geq \delta_{\text{for}}$ (which includes $\tau = \delta_{\text{for}} = 0$).

**Proof of (3.3)** Suppose that $\tau := \tau_0(P, D) \geq \delta_{\text{for}} \geq 0$ with $\lambda \geq 1$. Then

$$\sum_{j \in \mathbb{F}^\vee \cap \mathbb{Z}^d} T_j^{\text{sub}}(\lambda) \leq C\lambda^{-1/\tau} (|\log \lambda| + 1)^{p(\tau)} \text{ if } p(\tau) = \begin{cases} 0 & \text{if } \tau > \delta_{\text{for}} \\ p & \text{if } \tau = \delta_{\text{for}}. \end{cases} \quad (8.14)$$

This with (2) of Proposition 8.2 yields (3.3).

**Proof of (8.14)** **Case 1** $\delta_{\text{for}} > 0$. We have (8.11), (8.12) even if $\tau \geq \delta_{\text{for}} = d(\mathbb{P}_{\text{for}}) > 0$. Regard $\alpha_i q_i \cdot 1$ and $\lambda(\alpha)$ in (8.12) as $\alpha_i$ and $\lambda$ in (8.3). Then, we can apply (8.3) for $\sum_{\alpha_1, \ldots, \alpha_p}$ of (8.12) to obtain

$$S(\lambda) \lesssim \sum_{\alpha_{n+1}, \ldots, \alpha_{d_0}} \frac{(|\log \lambda(\alpha)| + 1)^{p(\tau)}}{\lambda(\alpha)^{1/\tau}} \lesssim \frac{(|\log \lambda| + 1)^{p(\tau)}}{\lambda^{1/\tau}}.$$ 

where the second inequality follows from (8.9).

**Case 2** $\delta_{\text{for}} = 0$. Note $d(\mathbb{P}_{\text{for}}) \leq \delta_{\text{for}} = 0$. We still have (8.11) for $\tau \geq \delta_{\text{for}} = 0$. Then if $\tau > \delta_{\text{for}} = 0$, we have the same estimate as above. But if $\tau = \delta_{\text{for}} = 0$, then we can take $\tau \ll 1$ in (7.5) and (8.12), showing $S(\lambda) \lesssim \frac{1}{\lambda^{1/\tau}}$. 

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8.3 Backward Face Sum

We treat the backward faces similarly.

**Proposition 8.2** (Backward Face) Let \( F = \bigcap_{i=1}^{d_0} \pi_{q_i, r_i} \in \mathcal{F}^{k_0}(\mathbb{P}_{bac}) \) in (6.3). Suppose that all \( \pi_{q_i, r_i}^+ \) are across-diagonal and \#\[ \pi_{q_i, r_i} : d(\pi_{q_i, r_i}^+) = \delta_{bac} \] \( i=1 \) \( = p \).

1. Let \( 0 < \lambda \leq 1 \). If \( \tau_0(P, D) = \tau \in [0, \delta_{bac}) \) in (7.5), then there is \( C > 0 \):

\[
\sum_{j \in F \cap \mathbb{Z}^d} T^\text{sub}_j(\lambda) \leq C\lambda^{-1/\delta_{bac}}(|\log \lambda| + 1)^{p-1}.
\]

If \( p \geq 1 \), then there is \( 0 < b \leq 1 \) such that for all \( \lambda \in (0, b] \),

\[
\sum_{j \in F \cap \mathbb{Z}^d} T^\text{sub}_j(\lambda) \geq C^{-1}\lambda^{-1/\delta_{bac}}(|\log \lambda| + 1)^{p-1}.
\]

2. Let \( \lambda > 1 \). If \( \tau_0(P, D) = \tau \in [0, \delta_{bac}) \) in (7.5), then there is \( C > 0 \):

\[
\sum_{j \in F \cap \mathbb{Z}^d} T^\text{sub}_j(\lambda) \leq C\lambda^{-\frac{1}{\tau}} \text{ which is } O(\lambda^{-\left(1/\delta_{for}+\epsilon\right)}) \text{ if } \tau \in [0, \delta_{for}).
\]

**Proof of Proposition 8.2** Observe that \( 0 < \delta_{bac} = d(\mathbb{P}_{bac}) \leq d(\pi_{q_i, r_i}^+) \) as in (5.4)\(^3\) for \( i \in [d_0] \). Then, rearrange \( \pi_{q_i, r_i} \) forming a backward face \( F = \bigcap_{i=1}^{d_0} \pi_{q_i, r_i} \) as

\[
\begin{align*}
q_i \cdot 1 &< 0 \text{ and } d(\pi_{q_i, r_i}^+) = \delta_{bac} \text{ for } i = 1, \ldots, p, \\
q_i \cdot 1 &< 0 \text{ and } d(\pi_{q_i, r_i}^+) > \delta_{bac} \text{ for } i = p + 1, \ldots, n, \\
q_i \cdot 1 &> 0 \text{ for } i = n + 1, \ldots, d_0.
\end{align*}
\] (8.15)

With \( \tau \) in (7.5) and \( m \in F \cap \Lambda(P) \), estimate \( \sum_{j=\alpha_1, \ldots, \alpha_{d_0} \in \mathbb{Z}^d} T^\text{sub}_j(\lambda) \) by

\[
\sum_{\alpha_1, \ldots, \alpha_{d_0} \in \mathbb{Z}^d} 2^{-(\alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0})} \min \left\{ \frac{1}{\lambda^{1/\tau} 2^{-(\alpha_1 q_1 + \cdots + \alpha_{d_0} q_{d_0})} m/\tau}, 1 \right\}. \tag{8.16}
\]

From (5.5) and (2) of Lemma 4.2, we have for each \( q_i \) in (8.15),

\[
\begin{cases}
q_i \cdot m = q_i \cdot 1 d(\pi_{q_i, r_i}^+) \text{ if } i = 1, \ldots, n, \text{ for the case } q_i \cdot 1 < 0 \\
q_i \cdot m = r_i < 0 \text{ if } i = n + 1, \ldots, d_0, \text{ for the case } q_i \cdot 1 = 0.
\end{cases} \tag{8.17}
\]

\(^3\) If \( \delta_{bac} = \infty \), then \( \delta_{bac} \leq d(\pi_{q_i, r_i}^+) = \infty \) for all \( i \) of \( F = \bigcap_{i=1}^{d_0} \pi_{q_i, r_i} \). This with Definition 5.2 implies \( q_i \cdot 1 = 0 \) for all \( i \leq d_0 \). So, \( n = 0 \) in (8.15)–(8.17) so that (8.16) = \( \sum_{(\alpha_1, \ldots, \alpha_{d_0})} \leq \frac{1}{\lambda^{1/\tau}} \).
In (8.16), (8.17), we rewrite \( -\alpha_i q_i \cdot 1 \geq 0 \) as \( \alpha_i \in \frac{1}{\lambda} \mathbb{Z}_+ \) for \( i = 1, \ldots, n \) and set
\[
\lambda(\alpha) := \lambda 2^{-\left(\alpha_{p+1} q_{p+1} + \cdots + \alpha_d q_d\right) m} = \lambda 2^{\alpha_{p+1} d(\pi_{q_{p+1}, r_{p+1}}^+) + \cdots + \alpha_n d(\pi_{q_n, r_n}^+) \frac{1}{\tau} \alpha_{n+1} |r_{n+1}| + \cdots + \alpha_d |r_d|}.
\] (8.18)

Then we rewrite (8.16) as
\[
S(\lambda) := \sum_{\alpha_{p+1}, \ldots, \alpha_d} 2^{\alpha_{p+1} + \cdots + \alpha_d} \sum_{\alpha_1, \ldots, \alpha_p} 2^{\alpha_1 + \cdots + \alpha_p} \min \left\{ \frac{1}{(\lambda(\alpha) 2^{(\alpha_1 + \cdots + \alpha_p) \delta_{bac}})^\frac{1}{\tau}}, 1 \right\}.
\] (8.19)

**Proof of (2) in Proposition 8.2** Let \( \lambda \geq 1 \). Then the condition \( 0 < \tau < \delta_{bac} < d(\pi_{q_i, r_i}^+) \) for \( i = p + 1, \ldots, n \) in (8.18) implies \( S(\lambda) \lesssim \lambda^{-1/\tau} \) in (8.19) which yields (2). If \( \tau = 0 \), then take \( 0 < \tau \ll 1 \) in \( S(\lambda) \).

**Proof of \( \lesssim (1) \) in Proposition 8.2** Let \( 0 < \lambda < 1 \). If \( p \geq 1 \), by regarding \( \lambda(\alpha) \) in \( S(\lambda) \) as \( \lambda \), apply (8.2) with \( \tau < \delta_{bac} \) for the inner sum over \( (\alpha_1, \ldots, \alpha_p) \) in (8.19),
\[
S(\lambda) \lesssim \sum_{\alpha_{p+1}, \ldots, \alpha_d} 2^{\alpha_{p+1} + \cdots + \alpha_d} \lambda(\alpha)^{-1/\delta_{bac}} (|\log \lambda(\alpha)| + 1)^{p-1}.
\]

For \( \lambda(\alpha) \) in (8.18), put \( \delta_i = d(\pi_{q_i, r_i}^+) > \delta_{bac} \) for \( i = p + 1, \ldots, n \). Then in the above,
\[
RHS \lesssim \sum_{\alpha_{p+1}, \ldots, \alpha_d} \lambda^{-1/\delta_{bac}} \left( |\log \lambda|^{p-1} + |\sum_{i=p+1}^d \alpha_i|^{p-1} \right) \frac{\lambda^{1/\delta_{bac}}} {2^{\alpha_{p+1} \left( \frac{d_p \delta_{bac} - 1}{d_{bac}} \right) + \cdots + \alpha_n \left( \frac{d_n \delta_{bac} - 1}{d_{bac}} \right)} 2^{\alpha_{n+1} |r_{n+1}| + \cdots + \alpha_d |r_d|}}.
\]

If \( p = 0 \), take \( \tau \) as \( \delta_{bac} < \tau < \delta_i \) for \( i = p + 1, \ldots, n \) of \( S(\lambda) \) in (8.19) to have
\[
S(\lambda) \lesssim \sum_{\alpha_{p+1}, \ldots, \alpha_d \in \mathbb{Z}_+} 2^{\alpha_{p+1} + \cdots + \alpha_n} \frac{\lambda^{1/\delta_{bac}} 2^{\alpha_{n+1} |r_{n+1}| + \cdots + \alpha_d |r_d|}} {2^{\alpha_{p+1} \left( \frac{d_p \delta_{bac} - 1}{d_{bac}} \right) + \cdots + \alpha_n \left( \frac{d_n \delta_{bac} - 1}{d_{bac}} \right)} 2^{\alpha_{n+1} |r_{n+1}| + \cdots + \alpha_d |r_d|}} \lesssim \lambda^{-1/\tau}
\]
with \( 1/\tau = 1/\delta_{bac} - \epsilon \). We proved \( \lesssim \) of (1) in Proposition 8.2.

**Proof for the reverse inequality in (1) of Proposition 8.2** Let \( \lambda < b := \frac{1}{100C} \). As (8.13), with \( q_i \cdot 1 < 0 \) for \( i = 1, \ldots, p \) in (8.15),
\[
\sum_{j \in F \cap \mathbb{Z}^d} T_j^{\text{sub}}(\lambda) \gtrsim \sum_{(\alpha_i) \in (\mathbb{M}_0 \mathbb{Z}_+)^p \cap A} 2^{-\left(\alpha_1 q_1 + \cdots + \alpha_p q_p\right) \cdot 1} \approx \lambda^{-1/\delta_{bac}} (|\log \lambda| + 1)^{p-1}
\] (8.20)
due to \( A = \{(\alpha_i) : \lambda 2^{-\left(\alpha_1 q_1 + \cdots + \alpha_p q_p\right) \cdot 1} \delta_{bac} < 1/(10C)\} \).

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Therefore, we have finished the proof of Proposition 8.2.

**Proposition 8.3** (Oscillatory integrals) In Propositions 8.1 and 8.2, if the hypothesis (7.5) is replaced with (7.6), then \( \sum_{j \in F'} \sum_{d \in \mathbb{Z}^d} |I_{j,\delta}^\text{osc}(\lambda)| \) has the same upper bounds where \( (\tau, \delta_{\text{for}}) = (0, 0) \) in (8.14) switched with \( (\tau, \delta_{\text{for}}) = (1, 0) \) and \( (1, 1) \).

### 9 Proof of Main Theorems 1–2

#### 9.1 Part (A) of Main Theorems 1 and 2

**Proof of (A) of Main Theorems 1** The hypothesis of balanced \( N(P, D_B) \) and the normal-crossing \( (P, D_B) \) of type \([0, \tau]\) for \( \tau < \delta_{\text{for}} \) yield Propositions 8.1 and 8.2. Under these propositions, it suffices to claim that

\[
\sum_{I \in F_0} \sum_{j \in F' \cap \mathbb{Z}^d} I_{j,\delta}^\text{sub}(\lambda) \approx \begin{cases} 
\lambda^{-1/\delta_{\text{for}}} (|\log \lambda| + 1)^{d-1-k_{\text{for}}} & \text{if } \lambda \geq 1, \\
\lambda^{-1/\delta_{\text{bac}}} (|\log \lambda| + 1)^{d-1-k_{\text{bac}}} & \text{if } 0 < \lambda < 1 
\end{cases} \quad (9.1)
\]

where \( F_0 := F_0(\mathbb{P}_{\text{for}}) \cup F_0(\mathbb{P}_{\text{bac}}) \) in Theorem 6.1.

**Case 1.** Let \( \lambda \geq 1 \) and \( \dim(\mathbb{P}_{\text{for}}) = k_{\text{for}} \). Suppose that \( \delta_{\text{for}} > 0 \) and \( \tau \in (0, \delta_{\text{for}}) \).

Then by using (1) of Proposition 8.1 (the forward case) and (2) in Proposition 8.2 (the backward case), we obtain that

\[
\sum_{j \in F' \cap \mathbb{Z}^d} I_{j,\delta}^\text{sub}(\lambda) \begin{cases} 
\approx \lambda^{-1/\delta_{\text{for}}} (|\log \lambda| + 1)^{p-1} & \text{if } I \in F_0(\mathbb{P}_{\text{for}}) \\
\lesssim \lambda^{-1/(\delta_{\text{for}}+\varepsilon)} & \text{if } I \in F_0(\mathbb{P}_{\text{bac}}) 
\end{cases} \quad (9.2)
\]

Here the case \( p = 0 \) has a better major term. Sum the RHS of (9.2) over all (finitely many) faces \( F \) in (9.1). Then the largest bound is \( \lambda^{-1/\delta_{\text{for}}} (|\log \lambda| + 1)^{d-k_{\text{for}}-1} \), since \( d - \dim(\mathbb{P}_{\text{for}}) = d - k_{\text{for}} \) is the largest \( p \) in Proposition 8.1. Therefore, one has the desired bound \( \lambda^{-1/\delta_{\text{for}}} (|\log \lambda| + 1)^{d-k_{\text{for}}-1} \) for \( \lambda \geq 1 \) in (9.1).

**Case 2.** Let \( 0 < \lambda < 1 \) and \( \dim(\mathbb{P}_{\text{main}}) = k_{\text{bac}} \). By taking (1) of Proposition 8.2 (the backward case) and (2) of Proposition 8.1 (the forward case) as

\[
\sum_{j \in F' \cap \mathbb{Z}^d} I_{j,\delta}^\text{sub}(\lambda) \begin{cases} 
\approx \lambda^{-1/\delta_{\text{bac}}} (|\log \lambda| + 1)^{p-1} & \text{if } I \in F_0(\mathbb{P}_{\text{bac}}) \\
\lesssim \lambda^{-\varepsilon} & \text{for } 0 < \varepsilon \ll 1 \text{ if } I \in F_0(\mathbb{P}_{\text{for}}) 
\end{cases} \quad (9.3)
\]

The largest bound on RHS of (9.3), among all faces \( F \) in (9.1), is \( \lambda^{-1/\delta_{\text{bac}}} (|\log \lambda| + 1)^{d-k_{\text{bac}}-1} \) because \( d - \dim(\mathbb{P}_{\text{bac}}) = d - k_{\text{bac}} \) is the largest possible \( p \) in Proposition 8.2. The reverse inequality \( \lesssim \) holds for only \( 0 < \lambda \leq b \) as in (1) of Proposition 8.2. If \( b < \lambda \leq 1 \), then the lower bound \( \lambda^{-1/\delta_{\text{bac}}} (|\log \lambda| + 1)^{d-k_{\text{bac}}-1} \approx 1 \) in (9.1) follows from \( 0 \in P(D_B) \) in the hypothesis of Main Theorem 1.

**Proof of \( \lesssim \) in (A) of Main Theorem 2** Under the normal-crossing hypothesis of type \([1, \tau]\), we can claim \( \lesssim \) of (9.1) for \( |I_{j,\delta}^\text{osc}(\lambda)| \) by applying Proposition 8.3 for the estimates of \( \sum_{j \in F' \cap \mathbb{Z}^d} |I_{j,\delta}^\text{osc}(\lambda)| \) analogous to (9.2) and (9.3).
**Proof of \( \gtrsim \) in (A) of Main Theorem 2** Let \( p = d - k_{\text{for}} \). We show the case \( \lambda \geq 1 \):

\[
\limsup_{|\lambda| \to \infty} |\mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda)| \geq c \text{ for } \mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda) = \int e^{i\lambda P(x)} \psi_{D_{B,R}}(x) dx.
\]

It suffices to find \( c(P) > 0 \) such that for any large \( M \) there is \( |\lambda| \geq M \) satisfying

\[
\lim_{R \to \infty} |\mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda)| > c(P) \lambda^{-1/\delta_{\text{for}}} |\log \lambda|^{p-1}.
\]

(9.4)

Assume the contrary. Then for an arbitrary small \( \epsilon > 0 \), there is \( M_{\epsilon} > 1 \) such that

\[
|\mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda)| < 2\epsilon \lambda^{-1/\delta_{\text{for}}} |\log \lambda|^{p-1} \text{ for all } |\lambda| > M_{\epsilon} \text{ and all } R > R_{1}
\]

(9.5)

for some \( R_{1} > 0 \). Let \( m \in \mathbb{Z}_{+} \). Apply the Fourier inversion and the Fubini theorem,

\[
\int \psi \left( \frac{P(x)}{2-m} \right) \psi_{D_{B,R}}(x) dx = \int \left[ \int 2^{-m} \hat{\psi}(2^{-m} \lambda) e^{i\lambda P(x)} d\lambda \right] \psi_{D_{B,R}}(x) dx
\]

\[
= \int \left[ \int e^{i\lambda P(x)} \psi_{D_{B,R}}(x) dx \right] 2^{-m} \hat{\psi}(2^{-m} \lambda) d\lambda
\]

\[
\leq \int |\mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda)| 2^{-m} |\hat{\psi}(2^{-m} \lambda)| d\lambda.
\]

(9.6)

Insert the lower bound in (A) of Main Theorem 1 into the LHS of (9.6). Then there is \( c, R_{1} > 0 \) such that for \( R > R_{1} \),

\[
c (2^{-m})^{1/\delta_{\text{for}}} (1 + |m|)^{p-1} \leq \int |\mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda)| 2^{-m} |\hat{\psi}(2^{-m} \lambda)| d\lambda.
\]

(9.7)

By using \( M_{\epsilon} \) in (9.5), split the RHS of (9.7) into the three intervals

\[
\int_{|\lambda| > M_{\epsilon}} + \int_{|\lambda| \leq 1} + \int_{1 \leq |\lambda| \leq M_{\epsilon}} |\mathcal{T}^{\text{osc}}(P, D_{B,R}, \lambda)| 2^{-m} |\hat{\psi}(2^{-m} \lambda)| d\lambda.
\]

(9.8)

Utilize (9.5) to majorize the first integral over \( |\lambda| > M_{\epsilon} \) in (9.8) by

\[
\epsilon (2^{-m})^{1/\delta_{\text{for}}} \int_{|\lambda| > M_{\epsilon}} |\log \lambda|^{p-1} (2^{-m})^{1/\delta_{\text{for}}} 2^{-m} |\hat{\psi}(2^{-m} \lambda)| d\lambda \leq C \epsilon (2^{-m})^{1/\delta_{\text{for}}} |m|^{p-1}
\]

(9.9)

as \( |\log \lambda|^{p-1} \leq (\log(2^{-m} \lambda) + log 2^{m})^{p-1} \leq |\log(2^{-m} \lambda)|^{p-1} + |m|^{p-1} \). Majorize the remaining integrals in (9.8) by the upper bounds in (A) of Main Theorem 2:

\[
\int_{|\lambda| \leq 1} \frac{(1 + |\log \lambda|)^{d-\delta_{\text{bac}}-1}}{\lambda^{1/\delta_{\text{bac}}}} 2^{-m} d\lambda + \int_{1 \leq |\lambda| \leq M_{\epsilon}} \frac{(1 + |\log \lambda|)^{p-1}}{\lambda^{1/\delta_{\text{for}}}} 2^{-m} d\lambda
\]
which is smaller than \( C(M_\epsilon + 1)2^{-m} \) where \( 1 < \delta_{for} \leq \delta_{bac} \). With this and (9.9),
\[
c(2^{-m})^{1/\delta_{for}} (1 + |m|)^{p-1} \leq \text{RHS of (9.7)} \leq C[2^{-m} (1/\delta_{for}) |m|^{p-1} + (M_\epsilon + 1)2^{-m}]
\]
which is smaller than \( 2C \epsilon (2^{-m})^{1/\delta_{for}} |m|^{p-1} \) if we take \( 2^{-m} \) satisfying \( 2^{-m(1-\frac{1}{\delta_{for}})} M_\epsilon \ll \epsilon \) due to \( \delta_{for} > 1 \). This with \( \epsilon \ll 1 \) makes a contradiction to \( c \leq 2C \epsilon \). Hence (9.4) is true. One can similarly obtain \( \gtrsim \) in (A) of the second case \( \lambda \leq 1 \).

### 9.2 The Divergence Part in Main Theorems 1 and 2

**Proof of (B) of Main Theorem 1** Assume that \( \mathcal{N}(P, D_B) \) is unbalanced. Then we show \( \int \psi(\lambda P(x)) \psi_{P_B}(x) dx = \infty \). Take \( q \in \text{cone}^\vee(B \cup \Lambda(P) \cup \{-1\}) \) where \( \pi^+_q \) with \( r \geq 0 \) is an off-diagonal supporting upper half-space of \( \mathcal{N}(P, D_B) \):

\[
q \in \text{cone}^\vee(B) \text{ such that } q \cdot n \geq 0 \forall n \in \Lambda(P) \text{ and } q \cdot 1 \leq 0 \text{ where } |q| \approx 1. \quad (9.10)
\]

If \( D_B \) does not contain a neighborhood of \( 0 \), then take \( p = 0 \). If \( D_B \) contains a neighborhood of \( 0 \), then take \( p \in (\text{cone}^\vee(B))^0 \) satisfying that \( 2^{-p} \sim (y_1, \ldots, y_d) \in D_B \) with \( p_v \gg 1 \) for all \( v \in (p_v) = p \) such that for all \( n \in \Lambda(P) \),

\[
2^{-p} n \leq \frac{1}{2^{10}C(P)\lambda} \text{ with } C(P) := 2|\Lambda(P)| \max(|c_n|) \quad (9.11)
\]

for \( P(x) = \sum_{n \in \Lambda(P)} c_n x^n \). For \( q \) in (9.10) and \( p \) in (9.11), define

\[
\mathbb{Z}^d(q, p, R) := \left\{ j = \alpha q + p \in \text{cone}^\vee(B) \cap \mathbb{Z}^d : \log_2 R \leq \alpha \leq 2 \log_2 R \right\} \quad (9.12)
\]

where \( \alpha \in M_0\mathbb{Z}_+ \) with \( M_0 \in \mathbb{N} \) in (6.4) and \( R \gg 1 \). From \( j = \alpha q + p \in \text{cone}^\vee(B) \),

\[
\mathcal{I}^{\text{sub}}(P, D_B, \lambda) \geq \sum_{j \in \mathbb{Z}^d(q, p, R)} 2^{-j-1} \int \psi(\lambda P(2^{-j}x)) \psi_{D_B,R}(2^{-j}x) \chi(x) dx. \quad (9.13)
\]

By \( \Lambda(P) \subset \pi^+_q \), we have \( q \cdot n \geq r \) for all \( n \in \Lambda(P) \). This with \( 2^\alpha \sim R \) for \( R \gg 1 \) in (9.12) implies in (9.13),

\[
|\lambda P(2^{-j}x)| \leq \lambda C(P) \sup_{n \in \Lambda(P)} 2^{-p} n 2^{-\alpha} q \cdot n \leq \lambda C(P) \sup_{n \in \Lambda(P)} 2^{-p} n 2^{-\alpha r}
\]

\[
\leq 2^{-10} \quad \text{if } r > 0,
\]

\[
\leq 2^{-11} \quad \text{if } r = 0 \text{ and } \lambda \in (0, c) \text{ with } c := 1/(2^{11}C(P)) \quad . \quad (9.14)
\]

\[
\text{if } r = 0 \text{ and } 0 \in D_B, \text{ due to (9.11).}
\]
As $p$ is fixed and $q \cdot 1 \leq 0$ in (9.10), we obtain that for the above three cases of $r$,

$$\text{RHS of (9.13)} \geq \sum_{\alpha \in M_0Z_+; \log_2 \alpha \leq 2 \log_2 R} 2^{-p} 12^{-\alpha q} 1 \gtrsim \log R$$

as $R \to \infty$. This yields $T_{sub}^\nu(P, DB, \lambda) = \infty$ in (B) of Main Theorem 1. \hfill \Box

**Proof of (B) of Main Theorem 2** Let $N(P, DB)$ be unbalanced. Then, choose $q = (q_1, \ldots, q_d)$ in (9.10). Set

$$\Psi(x) = \sum_{\alpha \in Z_+} \chi \left( \frac{x_1}{2^{-\alpha q_1}} \right) \cdots \chi \left( \frac{x_d}{2^{-\alpha q_d}} \right).$$

Then $x \in \text{supp}(\Psi)$ if and only if $x \sim 2^{-\alpha d}$ for some $\alpha$. We see that $\Psi \in A(DB)$. To show (B) of the main theorem 2, we claim that

$$\lim_{R \to \infty} \left| \int e^{i\lambda P(x)} \sum_{\alpha \in Z_+} \chi \left( \frac{x_1}{2^{-\alpha q_1}} \right) \cdots \chi \left( \frac{x_d}{2^{-\alpha q_d}} \right) \psi \left( \frac{x_1}{R} \right) \cdots \psi \left( \frac{x_d}{R} \right) dx \right| = \infty.$$ 

By the change of variables $x_i \to 2^{-\alpha q_i} x_i$ and $\chi(x) = \chi (x_1) \cdots \chi (x_d)$, split the above integral into the two terms

$$\sum_{0 \leq \alpha \leq \log_2 R} 2^{-\alpha (q, 1)} \int e^{i\lambda P(2^{-\alpha q} x)} \chi(x) dx = A(R) + B(R).$$

**Case 1.** Let $r > 0$ in (9.10). By (9.14) with $p = 0$, $|\lambda P(2^{-\alpha q} x)| \ll 1$. This with $(q, 1) \leq 0$ in (9.10) yields that

$$|B(R)| \geq 2^{(\log_2 R)|\alpha|} |C/2| \log_2 R,$$

which is much bigger than $|A(R)|$ because $|A(R)| \leq 2^{(\log_2 R)|\alpha|} \epsilon \log_2 R$. Therefore, it holds that $\lim_{R \to \infty} |A(R) + B(R)| = \infty$.

**Case 2.** Let $r = 0$ in (9.10). If $m \in \mathbb{F} = \pi_{q, r} \cap N(P, DB)$, then $q \cdot m = 0$. In (7.2),

$$1 = 2^{-\alpha q \cdot m} \geq 2^{c_\alpha} \sum_{n \in A(P) \setminus \mathbb{F}} |c_n 2^{-\alpha q \cdot n}| \text{ for } c > 0 \text{ and } \alpha \geq 1.$$

This implies that for $\alpha \gg 1$,

$$P(2^{-\alpha q} x) = 2^{-\alpha q \cdot m} P_m(x) + P_{\Lambda(P) \setminus \mathbb{F}}(2^{-\alpha q} x) = P_m(x) + O(2^{-c_\alpha}).$$

By this with the mean value property and $\epsilon \log_2 R \leq \alpha \leq C \log_2 R$ in $B(R)$,

$$e^{i\lambda P(2^{-\alpha q} x)} = e^{i\lambda P_m(x)} + O(2^{-c_\alpha} \lambda) \text{ with } O(2^{-c_\alpha} \lambda) = O(R^{-\epsilon/2}).$$

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for sufficiently large $R \gg 1$. Thus for this $R$, it holds that

$$B(R) = \sum_{\epsilon \log_2 R \leq \alpha \leq C \log_2 R} 2^{-\alpha(q,1)} \left[ \int e^{i\lambda P(x)} \chi(x) dx + O(R^{-\epsilon/2}) \right].$$

(9.15)

A function $Q$ defined by $\lambda \to Q(\lambda) := \int e^{i\lambda P(x)} \chi(x) dx$ is an analytic function in $\mathbb{R}$, not identically zero. The identity theorem implies that $\{\lambda \in \mathbb{R} : Q(\lambda) = 0\}$ is a measure-zero set. For $\lambda$ with $Q(\lambda) > 0$ and $R \gg 1$, we have that $|\text{RHS of (9.15)}|$ is

$$\sum_{\epsilon \log_2 R \leq \alpha \leq C \log_2 R} 2^{-\alpha(q,1)} |Q(\lambda) + O(R^{-\epsilon/2})| \geq \sum_{\epsilon \log_2 R \leq \alpha \leq C \log_2 R} 2^{-\alpha(q,1)} Q(\lambda) \geq |Q(\lambda)2^{(\log_2 R)(q,1)}(C/2) \log_2 R|,$$

which is $\geq \epsilon 2^{(\log_2 R)(q,1)} \log_2 R \geq |A(R)|$. So, $\lim_{R \to \infty} |A(R) + B(R)| = \infty$. □

### 9.3 Proof of Corollary 3.1

**Proof of (A) in Corollary 3.1** Suppose that $N(P, D_B)$ is balanced. Then

$$\int_{D_B} |P(x)|^{-\rho} dx \approx \sum_{k \in \mathbb{Z}} 2^{-\rho k} \left( |\{x \in D_B : |P(x)| \leq 2^k\}| - |\{x \in D_B : |P(x)| \leq 2^{k-1}\}| \right)$$

$$=(1 - 2^{-\rho}) \sum_{k \in \mathbb{Z}} 2^{-\rho k} |\{x \in D_B : |P(x)| \leq 2^k\}|$$

$$\approx \sum_{2^k < 1} 2^{-\rho k} \sum_{x \in D_B : |P(x)| \leq 2^k} + \sum_{2^k \geq 1} 2^{-\rho k} |\{x \in D_B : |P(x)| \leq 2^k\}|.$$

From (A) of the main theorem 1, there are $C_1, C_2 > 0$ independent of $k$ such that

$$C_1 2^{k/\delta_{for}} (|k| + 1)^a \leq |\{x \in D_B : |P(x)| \leq 2^k\}| \leq C_2 2^{k/\delta_{for}} (|k| + 1)^a$$

if $2^k < 1$, $C_1 2^{k/\delta_{bac}} (|k| + 1)^b \leq |\{x \in D_B : |P(x)| \leq 2^k\}| \leq C_2 2^{k/\delta_{bac}} (|k| + 1)^b$ if $2^k \geq 1$

for $a = d - k_{for} - 1$ and $b = d - k_{bac} - 1$. This yields that

$$\int_{D_B} |P(x)|^{-\rho} dx \approx \sum_{2^k < 1} 2^{(1/\delta_{for} - \rho)k} (|k| + 1)^a + \sum_{2^k \geq 1} 2^{(1/\delta_{bac} - \rho)k} (|k| + 1)^b$$

which converges if and only if $1/\delta_{bac} < \rho < 1/\delta_{for}$. This proves (A).

**Proof of (B) in Corollary 3.1** Suppose that $N(P, D_B)$ is unbalanced. Then, the part (B) of the main theorem 1 implies $|\{x \in D_B : |P(x)| \leq 2^k\}| = \infty$ for some fixed $k$. This shows $\int_{D_B} |P(x)|^{-\rho} dx \geq 2^{-k \rho} |\{x \in D_B : |P(x)| \leq 2^k\}| = \infty.$
9.4 General Class of Phase Functions and Domains

We shall extend the Main Theorems 1 and 2 to a larger class of smooth functions \( Q \) and regions \( D \).

**Definition 9.1** Let \( P \) be a polynomial and let \( \mathbb{P} = N(P, D_B) \). Set

\[
\mathbb{P}^{\text{special}} = \left\{ \begin{array}{ll}
\mathbb{P}_{\text{for}}^{\text{main}} \cup \mathbb{P}_{\text{bac}}^{\text{main}} & \text{if } \mathbb{P} \text{ is balanced} \\
\mathbb{P}^{\text{off}} := \text{cone}^{\text{off}}(B \cup \Lambda(P) \cup \{-1\}) & \text{if } \mathbb{P} \text{ is unbalanced}
\end{array} \right.
\]

as in Fig. 1. Consider a region \( D \subset \mathbb{R}^d \) and a smooth function \( Q \) on \( D \cap (\mathbb{R} \setminus \{0\})^d \). Then \( (Q, D) \) is equivalent to \( (P, D_B) \), provided (1) and (2) below hold.

1. \( D \cong D_B \) if \( 2^{-\mathbb{P}^{\text{special}} \cap B(0, r)^c} \subset D \cap (\mathbb{R} \setminus \{0\})^d \subset 2^{-(\text{cone}^{\text{off}}(B) + O(1))} \) for some \( r \).
2. \( Q \cong_{[\sigma, \tau]} P \) if \( \sum_{\sigma \leq |\alpha| \leq \tau} |x^\alpha \partial^\alpha Q(x)|_{D \cap (\mathbb{R} \setminus \{0\})^d} \approx \sum_{m \in \Lambda(P)} |x^m| \),

Denote (1) and (2) at once by \( (Q, D) \cong_{[\sigma, \tau]} (P, D_B) \). For this case, define

\[
N(Q, D) := N(P, D_B).
\]

**Example 9.1** Thanks to \( O(1) \) in (1) of Definition 9.1, one can treat the perturbed domain \( D = \{ x \in \mathbb{R}^d : |x^b| \leq 5 \text{ for all } b \in B \} \) of \( D_B \) satisfying \( D \cong D_B \). For instance, take \( D = \{ x : |x^{e_1}|, |x^{-e_1}| \leq 5 \} \). Then \( D \cong D_{\{e_1,-e_1\}} \) though \( \text{cone}(e_1, -e_1) \) is not strongly convex.

**Example 9.2** (Fractional Laurent Polynomial) Let \( \Lambda(P) \subset \left( \frac{1}{K_1} \mathbb{Z} \right) \times \cdots \times \left( \frac{1}{K_d} \mathbb{Z} \right) \) be a finite set with \( K_v \in \mathbb{N} \) for \( v \in [d] \). Then we call \( P(x) = \sum_{m = (m_v) \in \Lambda(P)} c_m x^m \) a Laurent polynomial. For \( m_v = p/q \) with \( q \in \mathbb{N} \) and \( p \in \mathbb{Z} \), let

\[
x^{m_v}_{p/q} = x^{p/q} := \left\{ \begin{array}{ll}
(\sqrt[p]{|x_v|})^p & \text{if } x_v \geq 0 \\
\text{one of } \pm (\sqrt[p]{|x_v|})^p & \text{if } x_v < 0.
\end{array} \right.
\]

Given \( P, D_B \), one can set \( N(P, D_B) := \text{Ch}(\Lambda(P) + \text{cone}(B)) \) as in Definition 2.4

**Example 9.3** One can exclude the middle region \( M_h = \{ x \in \mathbb{R}^d : 1/h \leq |x_v| \leq h \text{ for all } v \in [d] \} \) from \( D_B \), keeping \( D_B \setminus M(h) \cong D_B \) due to \( B(0, r)^c \) in (1).

**Corollary 9.1** Let \( (Q, D) \cong_{[0, \tau]} (P, D_B) \) for \( \tau < \delta_{\text{for}} \) and \( 0 \in Q(D) \).

(A) If \( N(Q, D) \) is balanced, then it holds that

\[
|[x \in D : |\lambda Q(x)| \leq 1]| \approx \left\{ \begin{array}{ll}
\lambda^{-1/\delta_{\text{for}}} (| \log \lambda | + 1)^{d-1-k_{\text{for}}} & \text{if } \lambda \in [1, \infty), \\
\lambda^{-1/\delta_{\text{bac}}} (| \log \lambda | + 1)^{d-1-k_{\text{bac}}} & \text{if } \lambda \in (0, 1).
\end{array} \right.
\]

(B) If \( N(Q, D) \) is unbalanced, then there exists \( c > 0 \) such that

\[
|[x \in D : |\lambda Q(x)| \leq 1]| = \infty \text{ for all } \lambda \in (0, c).
\]
Remark 9.1 In Corollary 9.1, we do not assume that cone \( B \) is strongly convex. One can include \( \tau = \delta_{\text{for}} = 0 \) in (A).

Proof of Corollary 9.1 We restrict the region \( D \) to \( D \cap (\mathbb{R} \setminus \{0\})^d \). Replace \( D_B \) and \( P(x) \) with \( D \subset 2^{-(\text{cone}^{\nu}(B)+B(0,r))} \) and a smooth function \( Q(x) \) in Theorem 6.1. Then we obtain the same decay rate for each piece of integral in Propositions 8.1 and 8.2. This enables us to have the upper bound of (9.1), which gives the desired upper bound for \( I_{\text{sub}}(Q,D,\lambda) \). The lower bounds (8.13) and (8.20) are obtained from (9.13) and \( 2^{-\text{bac}} \) in Definition 9.1. The divergence follows from (9.1). Thus, Corollary 9.1 yields \( I_{\text{sub}}(F(x),\mathbb{R}^3,\lambda) = \infty \).

Example 9.4 A decay of a sublevel-set measure can be slower than that of a corresponding oscillatory integral in a local region. But this can occur in an extreme manner in the global region \( \mathbb{R}^d \). For example, let \( F(x) = x_3^2 - (x_1^2 + x_2^2) \). Then the local estimates are \( I_{\text{sub}}(F(x),[-1,1]^3,\lambda) = O(\lambda^{-1}) \) and \( I_{\text{osc}}(F(x),[-1,1]^3,\lambda) = O(\lambda^{-3/2}) \). But in the global domain, one can compute

\[
\begin{align*}
I_{\text{sub}}(F(x),\mathbb{R}^3,\lambda) &= \infty \\
I_{\text{osc}}(F(x),\mathbb{R}^3,\lambda) &= O(\lambda^{-3/2}).
\end{align*}
\]

The above estimate of the oscillatory integral follows from the iterated integration, or from Main Theorem 2 with \( \tau_1(P,\mathbb{R}^3) = 1, \delta_{\text{for}} = 2/3 \). For the above sublevel-set estimate, use \( \Phi(x) = (x_1, x_2, x_3 + \sqrt{x_1^2 + x_2^2}) \) to have \( F \circ \Phi(x) = x_3(x_3 + 2\sqrt{x_1^2 + x_2^2}) \) on \( D = \{x : |x_0^{-1}x_3| \leq 10^{-1} \text{ for } v = 1, 2\} \). Then \( (F \circ \Phi, D) \cong_{[0,1]} (x_3x_1 + x_3x_2, D_{[(-1,0,1),(0,-1,1)}) \) in Definition 9.1 and \( N(x_3x_1 + x_3x_2, D_{[(-1,0,1),(0,-1,1)}) \) unbalanced. Thus, Corollary 9.1 yields

\[
I_{\text{sub}}(F(x),\mathbb{R}^3,\lambda) = |\{x \in D : |\lambda F \circ \Phi(x)| \leq 1\}| = \infty.
\]

Example 9.5 Let \( P = N(P, D_B) \). If \( P \) is a polynomial, then \( \delta_{\text{for}} = d(P_{\text{for}}) \geq 0 \). If \( P \) is a Laurent polynomial, then \( d(P_{\text{for}}) = 2 \) in Fig.2 can be negative and \( \delta_{\text{for}} = \max\{0, d(P_{\text{for}})\} = 0 \). For example, if \( P(x) = \frac{1}{x_1^2} + \frac{1}{x_1^2} \), then \( d(P_{\text{for}}) = -2 \) and \( \delta_{\text{for}} = 0 \). Since \( N(P, (\mathbb{R} \setminus \{0\})^2) = \text{conv}((-1,-3),(-3,-1)) \) is unbalanced, \( I_{\text{sub}}(P, (\mathbb{R} \setminus \{0\})^2,\lambda) = \infty \).

10 Partition of Domain

10.1 Statement of Global Theorems After Partition of Domain

Let \( P(x) = ((x_1^2 + x_2^2) - 1)^2 \) and \( D = \mathbb{R}^2 \). Then as \( \tau_0(P, D) = 2 > 0 = \delta_{\text{for}}(P, D) \), one cannot apply the Main Theorems 1 for \( |x \in D : \lambda P(x)| \leq 1| \). However, we can find a partition \( D = \bigcup_{i=0}^{M} D_i \) so as to compute \( |\{x \in D_i : \lambda P(x)| \leq 1| \) for each \( i \). In this section, we restate the Main Theorems 1 and 2 by partitioning the domains.
**Main Theorem 3** Let $P$ be a polynomial $P(0) = 0$ in a domain $D \subset \mathbb{R}^d$. Suppose that a partition $\{D_i\}_{i=0}^M$ of $D$ with coordinate maps $\phi_i: \phi_i^{-1}(D_i) \to D_i$ decomposes

$$
\int_D \psi(\lambda P(x))dx \approx \sum_{i=0}^M \int \psi(\lambda P(x))\psi_{D_i}(x)dx \text{ for } \psi_{D_i} \in C^\infty(D_i)
$$

satisfying $(P \circ \phi_i, \phi_i^{-1}(D_i)) \cong [0, \tau_i]$ $(P_i, D_{B_i})$ are normal-crossing of type $[0, \tau_i]$. \hfill (10.1)

Then with the distance and multiplicity derived from $\mathbb{P}_i := N(P_i, D_{B_i})$ in (3.2):

$$
[\delta'_{for}, \delta''_{bac}]_1 = \bigcap_{i=0}^M \mathbb{P}_i \cap \text{cone(1)} \text{ and }
\left\{ \begin{array}{l}
k'_{for} = \min\{k_{for}(\mathbb{P}_i) : \delta_{for}(\mathbb{P}_i) = \delta'_{for}\}_{i=0}^M
\\k'_{bac} = \min\{k_{bac}(\mathbb{P}_i) : \delta_{bac}(\mathbb{P}_i) = \delta''_{bac}\}_{i=0}^M,
\end{array} \right.
$$

where $[\delta_{for}(\mathbb{P}_i), \delta_{bac}(\mathbb{P}_i)] = \mathbb{P}_i \cap \text{cone(1)}$, one can have the estimates:

(A) If $\mathbb{P}_i$ are balanced and $\tau_i < \delta'_{for}$ for all $i$, then it holds that

$$
\int \psi(\lambda P(x))\psi_{D}(x)dx \approx \begin{cases}
\lambda^{-1/\delta'_{for}}(|\log \lambda| + 1)^{d-1-k'_{for}} & \text{if } \lambda \geq 1 \\
\lambda^{-1/\delta_{bac}}(|\log \lambda| + 1)^{d-1-k'_{bac}} & \text{if } \lambda < 1.
\end{cases} \hfill (10.2)
$$

If the type $[0, \tau_i]$ is replaced with $[1, \tau_i]$, then the oscillatory integral estimates hold. Here $\delta'_{for}$, $\delta''_{bac}$ and $k'_{for}$, $k'_{bac}$ are independent of choices $\{D_i\}_{i=0}^M$.

(B) If at least one of $\mathbb{P}_i$ is unbalanced, then LHS of (10.2) diverges.

**Proof Main Theorem 3** Applying (A) of Main Theorem 1 and Corollary 9.1,

$$
\int \psi(\lambda P(x))\psi_{D_1}(x)dx = \int \psi(\lambda P \circ \phi_1(x))\psi_{D_1}(\phi_1(x))dx \approx \int \psi(\lambda P_1(x))\psi_{D_{B_1}}(x)dx
\approx \begin{cases}
\lambda^{-1/\delta_{for}(\mathbb{P}_1)}(|\log \lambda| + 1)^{d-1-k_{for}(\mathbb{P}_1)} & \text{if } \lambda \geq 1 \\
\lambda^{-1/\delta_{bac}(\mathbb{P}_1)}(|\log \lambda| + 1)^{d-1-k_{bac}(\mathbb{P}_1)} & \text{if } \lambda < 1.
\end{cases}
$$

So, the decay rates of LHS of (10.1), according to $\lambda \geq 1$ or $\lambda < 1$ are reciprocals of

$$
\max\{\delta_{for}(\mathbb{P}_i)\}_{i=0}^M \text{ or } \min\{\delta_{bac}(\mathbb{P}_i)\}_{i=0}^M
$$

which coincide with the above $1/\delta'_{for}$ or $1/\delta''_{bac}$, respectively. Thus, we obtain (10.2). Assume that there is another partition having $\delta''_{for}$, $\delta''_{bac}$ and $k''_{for}$, $k''_{bac}$. By applying (10.2), it holds that with the constants involved in $\approx$ below independent of $\lambda$,

$$
\lambda^{-1/\delta'_{for}}(|\log \lambda| + 1)^{d-1-k'_{for}}
\approx \int \psi(\lambda P(x))\psi_{D_B}(x)dx \approx \lambda^{-1/\delta''_{for}}(|\log \lambda| + 1)^{d-1-k''_{for}},
$$

showing $\delta''_{for} = \delta'_{for}$, $k''_{for} = k'_{for}$. Similarly, $\delta''_{bac} = \delta'_{bac}$, $k''_{bac} = k'_{bac}$. Finally, the oscillatory integral estimates follow from Main Theorem 2.
Remark 10.1 The above theorem is the first step toward a global resolution of singularity. But, we do not establish the resolution of singularity in this paper.

10.2 Three Types of Singular Set

Set $V(P) := \{ x \in \mathbb{R}^d : P(x) = 0 \text{ or } \nabla P(x) = 0 \}$ of singular points. To treat non-local $V(P)$ with a partition $\{D_i\}$, we apply Main Theorem 3 for the following model cases:

(i) $V(P)$ is a compact irreducible curve (circle) in Example 10.1,
(ii) $V(P)$ is a non-compact irreducible curve (parabola) in Example 10.2,
(iii) $V(P)$ is the union of the cases (i) and (ii) in Example 10.3.

Example 10.1 [Circle] Let $P_{\text{circ}}(x) = (x_1^2 + x_2^2 - 1)^2$ and $D_B = \mathbb{R}^2$. Then

$$\int \psi(\lambda P_{\text{circ}}(x)) \psi_{D_B}(x) dx \approx \lambda^{-1/2} \text{ if } \lambda \in (0, \infty),$$

$$\int e^{i\lambda P_{\text{circ}}(x)} \psi_{D_B}(x) dx = O(\lambda^{-1/2}) \text{ if } \lambda \in (0, \infty).$$

Due to the compactness of $V(P_{\text{circ}}) = S^1$, we can find $\{c_i\}_{i=1}^M \subset V(P_{\text{circ}})$ such that

$$V(P_{\text{circ}}) + [-h/2, h/2]^d \subset \bigcup_{i=1}^M c_i + [-h, h]^d \text{ with } |h| \ll 1. \quad (10.5)$$

Set $D_i = c_i + [-h, h]^2$ for $i \in [M]$ and $D_0 = D_B \setminus \bigcup_{i \in [M]} D_i$. Then

$$\int \psi(\lambda P_{\text{circ}}(x)) \psi_{D_B}(x) dx = \sum_{i=0}^M \int \psi(\lambda P_{\text{circ}}(x)) \psi_{D_i}(x) dx.$$

(1) Take $\phi_0 = Id$ on $D_0$ to define $P_0 = P_{\text{circ}} \circ \phi_0$ and $\phi_0^{-1}(D_0) = D_0$. Then $(P_0, \phi_0^{-1}(D_0)) \cong (P_{\text{circ}}, D_B)$ is normal-crossing of type $[0, \tau_0]$ for $\tau_0 = 0$.

(2) Fix $i \in [M]$ and let $c_i = (c_1, c_2) \in V(P) = S^1$ in (10.5). Then

$$\int \psi(\lambda P_{\text{circ}}(x)) \psi_{D_i}(x) dx = \int \psi(\lambda P_{\text{circ}}(x)) \psi \left( \frac{x - c_i}{h} \right) dx$$

$$= \int \psi(\lambda P_{\text{circ}}(x + c_i)) \psi \left( \frac{x}{h} \right) dx.$$

Let $|c_1| \leq |c_2|$ for $(c_1, c_2) \in S^1$. As $|x| \leq h \ll 1$, express $P(x + c_i) = (x_1^2 + x_2^2 + 2c_1x_1 + 2c_2x_2)^2 = [(x_2 - a(x_1))(x_2 - b(x_1))]^2$ with

$$\begin{align*}
 a(x_1) &= -c_2 + \sqrt{c_2^2 - (2c_1x_1 + x_1^2)} = -(c_1/c_2)x_1 + O(x_1^2) \\
 b(x_1) &= -c_2 - \sqrt{c_2^2 - (2c_1x_1 + x_1^2)} = -2c_2 + O(x_1) \approx -2c_2. \quad (10.6)
\end{align*}$$
Using a coordinate map $\phi_i(x_1, x_2) = x + c_i + (0, a(x_1))$, define $P_i(x)$:

$$P_{\text{circ}} \circ \phi_i(x_1, x_2) = [x_2(x_2 + a(x_1) - b(x_1))]^2 = (4c_2^2 + O(x_1^2))x_2^2 \approx x_2^2 \tag{10.7}$$

on $\phi_i^{-1}(D_i) = \{|x_1| \leq h, |x_2 + a(x_1)| \leq h\} \cong [-1, 1]^2$. This leads

$$\int \psi(\lambda P_{\text{circ}}(x + c_i)) \psi \left(\frac{x}{h}\right) dx = \int \psi(\lambda P_i(x)) \psi_{\phi_i^{-1}(D_i)}(x) dx$$

for $(P_i, \phi_i^{-1}(D_i)) \cong (x_2^2, [-1, 1]^2)$ of type $[0, \tau_i]$ with $\tau_i = 0$.

(3) Set $\mathbb{P}_i := \begin{cases} N((1 - (x_1^2 + x_2^2))^2, \mathbb{R}^2) & \text{if } i = 0 \\ N(x_2^2, [-1, 1]^d) & \text{if } i \in [M]. \end{cases}$ Then $\bigcap_{i=0}^{M} \mathbb{P}_i \cap \text{cone(1)} = [\delta'_\text{for}, \delta'_\text{bac}](1) = 21$ due to $[\delta_{\text{for}}(\mathbb{P}_0), \delta_{\text{bac}}(\mathbb{P}_0)] = [0, 2]$ and $[\delta_{\text{for}}(\mathbb{P}_i), \delta_{\text{bac}}(\mathbb{P}_i)] = [2, \infty]$ where $k'_{\text{for}} = 1$ and $k'_{\text{bac}} = 1$.

Hence Main Theorem 3 gives (10.3). The type condition $[1, 1]$ yields (10.4).

**Example 10.2** (Parabola) Let $P_{\text{para}}(x) = (x_2 - x_1^2)^2$ in $D_B = \mathbb{R}^2$. If $\lambda > 0$,

$$\int \psi(\lambda P_{\text{para}}(x)) \psi_D(x) dx = \infty$$

and

$$\int e^{i\lambda P_{\text{para}}(x)} \psi_D(x) dx = \infty$$

for some $\Psi_D \in \mathcal{A}(D_B)$.

**Proof** Split $\mathbb{R}^2 = D_1 \cup D_2$ where

$$D_1 = \{x : |x_2 - x_1^2| \geq \epsilon|x_1^2|\} \text{ and } D_2 = \{x : |x_2 - x_1^2| < \epsilon|x_1^2|\}.$$ 

Set $\phi_i(x) = (x_1, x_2 + x_1^2)$ and $P_i(x) = P_{\text{para}} \circ \phi_i(x) = x_2^2$ for $i = 1, 2$. Then

$$\phi_1^{-1}(D_1) := \{x : |x_1^2 x_2^2 - 1| \leq \epsilon^{-1}\} \cong D_{1(-2, -1)} \text{ and } \mathbb{P}_1 = N(P_1, D_{1(-2, -1)})$$

$$\phi_2^{-1}(D_2) := \{x : |x_1^{-2} x_2^2| \leq \epsilon\} \cong D_{1(-2, -1)} \text{ and } \mathbb{P}_2 = N(P_2, D_{1(-2, -1)})$$

is unbalanced.

Thus apply Main Theorem 3 to obtain the above estimates. \(\square\)

**Example 10.3** (Circle \cup Parabola) Let $P(x) = (x_1^2 + x_2^2 - 1)^2(x_2 - x_1^2)^2$ and $D_B = \mathbb{R}^2$. Then Main Theorem 3 shows

$$\int \psi(\lambda P(x)) \psi_D(x) dx \approx \begin{cases} \lambda^{-1/2}(\log \lambda + 1) & \text{if } \lambda \geq 1 \\
\lambda^{-1/4}(\log \lambda + 1) & \text{if } \lambda < 1 \end{cases} \tag{10.8}$$

$$\int e^{i\lambda P(x)} \psi_D(x) dx \approx \begin{cases} \lambda^{-1/2}(\log \lambda + 1) & \text{if } \lambda \geq 1 \\
\lambda^{-1/4}(\log \lambda + 1) & \text{if } \lambda < 1. \end{cases} \tag{10.9}$$
Proof of (10.8) and (10.9) Let $\psi + \psi^c \equiv 1$ on $\mathbb{R}^2$. Define the singular regions of the circle and the parabola as

$$D_{\text{circ}} = \left\{ |x_1^2 + x_2^2 - 1| \leq \epsilon \right\} \text{ and } D_{\text{para}} = \left\{ \frac{|x_2 - x_1^2|}{x_1^2} \leq \epsilon \right\}$$

and the non-singular regions as

$$D_{\text{circ}}^{\text{away}} = \left\{ |x_1^2 + x_2^2 - 1| \geq \epsilon/2 \right\} \text{ and } D_{\text{para}}^{\text{away}} = \left\{ \frac{|x_2 - x_1^2|}{x_1^2} \geq \epsilon/2 \right\}.$$ 

Decompose $Id_{\mathbb{R}^2} = \sum_{i=1}^4 \psi_i$ with $\psi_i \in \mathcal{A}(D_i)$ defined by

$$\psi_1(x) := \psi^c \left( \frac{x_1^2 + x_2^2 - 1}{\epsilon} \right) \frac{x_2 - x_1^2}{\epsilon x_1^2} \text{ supported } D_1 := D_{\text{circ}}^{\text{away}} \cap D_{\text{para}}^{\text{away}}$$

$$\psi_2(x) := \psi \left( \frac{x_1^2 + x_2^2 - 1}{\epsilon} \right) \psi^c \left( \frac{x_2 - x_1^2}{\epsilon x_1^2} \right) \text{ supported } D_2 := D_{\text{circ}} \cap D_{\text{para}}^{\text{away}}$$

$$\psi_3(x) := \psi^c \left( \frac{x_1^2 + x_2^2 - 1}{\epsilon} \right) \psi \left( \frac{x_2 - x_1^2}{\epsilon x_1^2} \right) \text{ supported } D_3 := D_{\text{circ}}^{\text{away}} \cap D_{\text{para}}$$

$$\psi_4(x) := \psi \left( \frac{x_1^2 + x_2^2 - 1}{\epsilon} \right) \psi^c \left( \frac{x_2 - x_1^2}{\epsilon x_1^2} \right) \text{ supported } D_4 := D_{\text{circ}} \cap D_{\text{para}}$$

Case $(P, D_1)$. This is the non-singular region. Set $P_1 = P$ and $\phi_1(x) = Id$. Then $(P_1 \circ \phi_1, \phi_1^{-1}(D_1)) \cong ((1 + x_1^4 + x_2^4)(x_1^4 + x_2^2), \mathbb{R}^2)$ is normal-crossing of type $[0, \tau_1]$ with $\tau_1 = 0$. Define $\mathbb{P}_1 = \mathbb{N}((1 + x_1^4 + x_2^4)(x_1^4 + x_2^2), \mathbb{R}^2)$ where

$$\delta_{\text{for}}(\mathbb{P}_1) = 4/3 \text{ and } \delta_{\text{bac}}(\mathbb{P}_1) = 4 \text{ with } k_{\text{for}}(\mathbb{P}_1) = 1 \text{ and } k_{\text{bac}}(\mathbb{P}_1) = 0.

Case $(P, D_2)$. Cover $D_2 = D_{\text{circ}} \cap D_{\text{para}}^{\text{away}} = \bigcup_{\ell=1}^M D_{2, \ell}$ where $D_{2, \ell} := D_2 \cap (c_\ell + [-h, h]^2)$ with $h \ll \epsilon \ll 1$ and

$$c_\ell = (c_1(\ell), c_2(\ell)) \in S^1 \setminus D_{\text{para}} \text{ where } h^2 \ll |c_2(\ell) - c_1(\ell)^2| \approx 1. \quad (10.10)$$

Then, decompose

$$\int_{\mathbb{R}^2} \psi(\lambda P(x))\psi_2(x)dx = \sum_{\ell=1}^M \int_{\mathbb{R}^2} \psi(\lambda P(x))\psi^c \left( \frac{x_2 - x_1^2}{\epsilon x_1^2} \right) \psi \left( \frac{x - c_\ell}{h} \right)dx.$$

As (10.6), the pullback of $D_{2, \ell}$ is the coordinate map $\phi_\ell : \phi_\ell^{-1}(D_{2, \ell}) \to D_{2, \ell}$:

$$\phi_\ell(x) = (x_1 + c_1(\ell), x_2 + c_2(\ell) + a(x_1)) \quad (10.11)$$
changing the above integrals as
\[
\int \psi(\lambda P \circ \phi_\ell(x)) \psi \left( \frac{x_1 + a(x_1)}{\epsilon/(x_1 + c_1(\ell))} \right) \psi \left( \frac{a(x_1) + c_2(\ell) - (x_1 + c_1(\ell))^2}{h} \right) dx.
\]

Define \( P_{2,\ell}(x) := P \circ \phi_\ell(x) = P_{\text{circ}} \circ \phi_\ell(x) \circ P_{\text{para}} \circ \phi_\ell(x) \approx x^2_2 \) due to \( P_{\text{circ}} \circ \phi_\ell(x) \approx x^2_2 \) in (10.7) and \( P_{\text{para}} \circ \phi_\ell(x) \approx \left[ x_2 + a(x_1) + c_2(\ell) - (x_1 + c_1(\ell))^2 \right] \approx |c_2(\ell) - c_1(\ell)|^2 \approx 1 \). Thus the above integrals become
\[
\int \psi(\lambda P_{2,\ell}(x)) \psi \left( \frac{x_1 + a(x_1)}{h} \right) dx = \int \psi(\lambda P_{2,\ell}(x)) \psi \left( \frac{x_1 + a(x_1)}{h} \right) dx.
\]

The support of the integrals is \( \phi^{-1}_\ell(D_{2,\ell}) \approx \{ x : |x| \leq h \} \). Thus \( (P_{2,\ell}(x), \phi^{-1}_\ell(D_{2,\ell})) \approx (x^2_2, [-1, 1]^2) \) is normal-crossing of type \([0, \tau]\) with \( \tau = 0 \). From \( \mathbb{P}_{2,\ell} = N(x^2_2, [-1, 1]^2) \),

\[
\delta_{\text{for}}(\mathbb{P}_{2,\ell}) = 2 \text{ and } \delta_{\text{bac}}(\mathbb{P}_{2,\ell}) = \infty \text{ with } k_{\text{for}}(\mathbb{P}_{2,\ell}) = 1 \text{ and } k_{\text{bac}}(\mathbb{P}_{2,\ell}) = 1.
\]

**Case** \((P, D_3)\). On \( D_3 = D_{\text{circ}}^{\text{away}} \cap D_{\text{para}} \), as \( P_{\text{para}}(x) = (x_2 - x^2_1)^2 \) is degenerate and \( P_{\text{circ}}(x) = (x^2_1 + x^2_2 - 1)^2 \) is normal-crossing, treat the parabola with \( \phi_3(x_1, x_2) = (x_1, x_2 + x^2_1) \) and take \( P_3(x) = P \circ \phi_3(x) = x^2_2 [x^2_1 + (x_2 + x^2_1)^2 - 1]^2 \). Then
\[
\int \psi(\lambda P(x)) \psi(x_3(x)) dx = \int \psi(\lambda P_3(x)) \psi \circ \phi_3(x) dx
\]
\[
= \int \psi(\lambda P_3(x)) \psi \left( \frac{x_2}{\epsilon x^2_1} \right) \psi \left( \frac{x_1^2 + (x_2 + x^2_1)^2 - 1}{\epsilon} \right) dx
\]
\[
= \int \psi(\lambda P_3(x)) \psi \phi^{-1}_3(D_3) dx.
\]

From \( \phi^{-1}_3(D_3) = \{ |x_1| \leq \epsilon x^2_1 \text{ and } |x^2_1 + (x_2 + x^2_1)^2 - 1| \geq \epsilon/2 \} \), it follows that \( (P_3(x), \phi^{-1}_3(D_3)) \approx (x^2_2(x^4_1 + x^8_1 + 1), D_{\{(-2,1)\}}) \) with \( \mathbb{P}_3 = N(x^2_2(x^4_1 + x^8_1 + 1), D_{\{(-2,1)\}}) \):

\[
\delta_{\text{for}}(\mathbb{P}_3) = 2 \text{ and } \delta_{\text{bac}}(\mathbb{P}_3) = 4 \text{ with } k_{\text{for}}(\mathbb{P}_3) = k_{\text{bac}}(\mathbb{P}_3) = 1.
\]

**Case** \((P, D_4)\). There exists \( \epsilon = (c_1, c_2) \in S^1 \cap \{c_2 = c^2_1\} \) such that \( D_4 = D_{\text{circ}} \cap D_{\text{para}} \subset \epsilon \subset [-h, h]^2 \). Thus one can replace \( \psi_4 \) supported on \( D_4 \) with \( \psi \left( \frac{x_2 - x^2_1}{\epsilon x^2_1} \right) \psi \left( \frac{x - c}{h} \right) \). In view of \( a(x_1) = -c_1 x_1 + O(|x^2_1|) \) in (10.6),(10.11), change coordinates via \( \phi^1_4(x) = (x_1 + c_1, x_2 + c_2 + a(x_1)) \) as
\[
\int_{\mathbb{R}^2} \psi(\lambda P(x)) \psi_4(x) dx \approx \int \psi(\lambda P \circ \phi^1_4(x)) \psi \left( \frac{x_2 - k x_1 - x^2_1}{\epsilon (x_1 + c_1)} \right) \psi \left( \frac{(x_1, x_2 + a(x_1))}{h} \right) dx.
\]
\[ \approx \int \psi(\lambda P \circ \phi_4^1(x)) \psi \left( \frac{(x_1, x_2 + a(x_1))}{h} \right) \, dx \]

for \( k = 2c_1 + 1/c_2 \) and \( x \) supported in \(|x| \lesssim h \ll 1\). From \( P_{\text{cir}} \circ \phi_1^1(x) \approx x^2 \) and \( P_{\text{para}} \circ \phi_4^1(x) = (x_2 + c_2 + a(x_1) - (x_1 + c_1))^2 \approx (x_2 - (kx_1 + x_1^2))^2 \), split the integral:

\[ \int \psi(\lambda P_4^1(x)) \left( \psi \left( \frac{x_2 - kx_1}{\epsilon x_1} \right) + \psi \left( \frac{x_2 - kx_1}{\epsilon x_1} \right) \right) \psi \left( \frac{(x_1, x_2 + a(x_1))}{h} \right) \, dx. \]

where \( P_4^1 = P \circ \phi_4^1 \). The first part supported on \( D_4^1 : = \{ |x_2 - kx_1| \geq |x_1| \text{ and } |x| \ll 1 \} \) corresponds to \((P_4^1, D_4^1) \cong (|x_2|^2, [1, 1]^2)\). We next apply another coordinate change via \( \phi_4^2(x) = (x_1, x_2 + kx_1) \) and \( P_4^2 = P_4^1 \circ \phi_4^2 \) for the second part:

\[ \int \psi(\lambda P_4^2(x)) \psi \left( \frac{x_2}{\epsilon x_1} \right) \psi \left( \frac{(x_1, x_2 + kx_1 + a(x_1))}{h} \right) \, dx \]

supported on \( D_4^2 : = \{ |x_2| \ll |x_1| \text{ and } |x| \ll 1 \} \) so that \((P_4^2, D_4^2) \cong (|x_2|, [-1, 1]^2)\). Moreover, \( P_4^\nu \) is of type \([0, \tau]\) with \( \tau = 0 \) on \( D_4^\nu \). Take \( \mathbb{P}_4^\nu = N(P_4^\nu, [-1, 1]^2) \). Then

\[ \delta_{\text{for}}(\mathbb{P}_4^\nu) = 2 \text{ and } \delta_{\text{bac}}(\mathbb{P}_4^\nu) = \infty \text{ with } k_{\text{for}}(\mathbb{P}_4^\nu) = 0 \text{ and } k_{\text{bac}}(\mathbb{P}_4^\nu) = 1 \text{ for } \nu = 1, 2. \]

**Conclusion.** By applying the Main Theorem 3 with \( \cap_{i=1}^4 (\mathbb{P}_i \cap \text{cone}(1)) \) given by

\[ [2, 4]I = [4/3, 4] \cap [2, \infty] \cap [2, 4] \cap [2, \infty]I \text{ and 0} \]

\[ \begin{cases} k_{\text{for}}^i = \min\{k_{\text{for}}(\mathbb{P}_i) : \delta_{\text{for}}(\mathbb{P}_i) = 2\} \\ k_{\text{bac}}^i = \min\{k_{\text{bac}}(\mathbb{P}_i) : \delta_{\text{bac}}(\mathbb{P}_i) = 4\} \end{cases} \]

to obtain (10.8). Similarly, we have (10.9).

### 10.3 Face-Nondegeneracy in Global Domains

**Definition 10.1** Call \((P, D_B)\) **face-nondegenerate** of type \([\sigma, \tau]\) if \( \tau \) is minimal:

\[ \sum_{\sigma \leq |a| \leq \tau} \left| \partial_x^\alpha \tilde{P}_\mathbb{F} \right|_{(\mathbb{R} \setminus \{0\})^d} \text{ are non-vanishing for all faces } \mathbb{F} \text{ of } \mathbf{N}(P, D_B). \tag{10.12} \]

In [9], Greenblatt weakened the assumption (1.1) of Varchenko [21] by restricting the orders \( \tau \) of zeros of \( P_\mathbb{F} \) less than \( \delta := d(\mathbf{N}(P, [-1, 1]^d)) \), which is equivalent to the face-nondegeneracy in (10.12) of type \( \tau < \delta \) and \( \sigma = 1 \). In a small neighborhood \( D \cong D_B \) with \( B = \{ e_v \}_{v=1}^d \), one can see that \((P, D)\) is normal-crossing of type \([\sigma, \tau]\) in (3.1) if and only if \((P, D_B)\) is face-nondegenerate of type \([\sigma, \tau]\) in (10.12), which had already appeared in Theorem 1.5 of [22]. This equivalence does not always hold in a global domain \( D \cong D_B \). But, it does hold, once \( D \) is away from a middle region \( M_h := \{ x \in \mathbb{R}^d : h^{-1} \leq |x_v| \leq h \text{ for all } v \in [d] \} \) for some \( h \geq 1 \).
Theorem 10.1 Let $D \cong D_B$. Then $(P, D_B)$ is face-nondegenerate of type $[\sigma, \tau]$ if and only if $(P, D \setminus M_h)$ for some $h \geq 1$ is normal-crossing of type $[\sigma, \tau]$.

We shall prove Theorem 10.1 in Sect. 12. Consequently, one can replace the normal-crossing hypothesis of Main Theorem 3 by face-nondegeneracy:

$$(P \circ \phi_i, \phi_i^{-1}(D_i)) \cong (P_i, D_{B_i})$$ are face - nondegenerate of type $[0, \tau_i]$ after choosing the decomposition $D = \left( \bigcup_{i=1}^{M+1} D_i \right) \cup D_{\text{nonsing}}$:

- $D_i = B_\epsilon(c_i)$ for $c_i \in P^{-1}(0) \cap [-h, h]^d$ so that $\bigcup_{i=1}^{M} D_i \supset P^{-1}(0) \cap [-h, h]^d$ where $\phi_i^{-1}(D_i) = B_\epsilon(0)$ with $\phi_i(x) = x + c_i$ (or a further coordinate change).
- $D_{M+1} = \bigcup_{v=1}^{d} \{ x \in D : |x_v| \geq h \}$ and $D_{\text{nonsing}} = D \cap [-h, h]^d \cap (P^{-1}(0))^c$.

Here we need to choose $\epsilon, 1/h \ll 1$. See Examples 10.1 through 10.3.

11 Oriented Simplicial Dual Faces

We shall prove Theorem 6.1. We start with Observations 11.1 and 11.2.

Observation 11.1 (Representation of Face and Dual face) Given $\mathbb{P}$, there is $\Pi(\mathbb{P})$ such that $\mathbb{P} = \bigcap_{\pi \in \Pi(\mathbb{P})} \pi^+$ as in (4.1). For $F \in \mathcal{F}^k(\mathbb{P})$, set $\Pi(\mathbb{P}^+) := \{ \pi_{q,r} \in \Pi(\mathbb{P}) : F \subset \pi_{q,r} \}$. Then it holds that

$$F = \bigcap_{\pi_{q,r} \in \Pi(\mathbb{P}^+)} \pi_{q,r} \cap \mathbb{P} \text{ and } \mathbb{F}^\vee = \text{cone}\left( \{ q : \pi_{q,r} \in \Pi(\mathbb{P}^+) \} \right) \quad (11.1)$$

where the dual face $\mathbb{F}^\vee$ is a $(d - k)$ dimensional cone. See Propositions 4.1 and 4.2 of [16]. For further studies, we refer [5, 7, 18] for readers.

Observation 11.2 (Dual face and Dual cone) The dual face $\mathbb{F}^\vee$ in (11.1) is the dual cone $(\mathbb{F}^+)^\vee$ of the polyhedron $\mathbb{F}^+ = \bigcap_{\pi_{q,r} \in \Pi(\mathbb{P}^+)} \pi_{q,r}^+(\text{conical extension of } \mathbb{F})$. See the first picture of Fig. 3.

We prove the two propositions making all dual faces $\mathbb{F}^\vee$ simplicial and oriented.

Simplicial duals. See an idea of proof in Fig. 3 and the triangulization lemma.

Lemma 11.1 (Simplicial Cones) Every $n$-dimensional cone $\mathbb{K} = \text{cone}(\{ q_i \}_{i=1}^m)$ in $V$ can be expressed as the union $\bigcup_{\ell=1}^{\ell} \mathbb{K}_\ell$ of ess disjoint $n$-dimensional simplicial cones $\mathbb{K}_\ell = \text{cone}(q_1^\ell, \ldots, q_n^\ell)$ where $\{ q_1^\ell, \ldots, q_n^\ell \} \subset \{ q_i \}_{i=1}^m$. Moreover, every face of simplicial cone is simplicial.

We omit its proof. By Lemma 11.1, we have a simplicial decomposition of $\mathbb{F}^\vee$. 
Fig. 3 The polyhedron $\mathbb{P}$ has the vertex $F$ represented as $\bigcap \pi_{q_1} \cap \pi_{q_2} \cap \pi_{q_3} \cap \pi_{q_4}$. Its dual face $F^\vee$ is the cone $\text{cone}(q_1, q_2, q_3, q_4)$. We can regard $F_1 = \pi_{q_1} \cap \pi_{q_2} \cap \pi_{q_3}$ and $F_2 = \pi_{q_1} \cap \pi_{q_3} \cap \pi_{q_4}$ as two faces, different from $F$, because $F_1, F_2$ have the dual faces $F_1^\vee, F_2^\vee$ different from $F^\vee$ while $F_1$ and $F_2$ themselves are same to $F$ as sets.

**Proposition 11.1** (Dual face Splits to Simplicial Cones) Recall $F^\vee$ in (11.1) where $F = \bigcap_{\pi_{q, r} \in \Pi(F^+)} \pi_{q, r} \in \mathcal{F}^k(\mathbb{P})$. Then, the dual face $F^\vee$ splits

$$ F^\vee = \bigcup_{\ell=1}^L F_\ell^\vee $$

with all $F_\ell^\vee$ ess disjoint simplicial dual cones with

$$ F_\ell := \bigcap_{\pi_{q, r} \in \Pi_\ell} \pi_{q, r} \cap \mathbb{P} \text{ are same to } F \text{ as a set where } \bigcup_{\ell=1}^L \Pi_\ell = \Pi(F^+) \text. $$

**Proof** Let $\Pi(F^+) = \{\pi_{q_i, r_i}\}_{i=1}^m$ in (11.1). Apply the simplicial decomposition of the cone $K$ of Lemma 11.1 for the $(d-k)$-dimensional cone $F^\vee = \text{cone}(\{q_i\}_{i=1}^m)$ as

$$ F^\vee = \bigcup_{\ell=1}^L \text{cone}(\{q_i(\ell)\}_{i=1}^{d-k}) \text{ for } \{q_i(\ell)\}_{i=1}^{d-k} \subset \{q_i\}_{i=1}^m $$

$\square$ Springer
where \( \text{cone}((q_i(\ell))_{i=1}^{d-k}) \) is ess disjoint simplicial cone. Then for each \( \ell \), choose

\[
\Pi \ell := \{\pi_{q_i(\ell), r_i(\ell)}\}_{i=1}^{d-k} \subset \Pi (\mathbb{R}^+) \quad \text{and} \quad \begin{cases}
\mathbb{F}^\ell := \bigcap_{i=1}^{d-k} \pi_{q_i(\ell), r_i(\ell)} \cap \mathbb{P} \\
\mathbb{F}^+ := \bigcap_{i=1}^{d-k} \pi_{q_i(\ell), r_i(\ell)}^+ .
\end{cases}
\]

Then (11.4) with \((\mathbb{F}^\ell)^\vee = \text{cone}((q_i(\ell))_{i=1}^{d-k})\) shows (11.2). Next, \( \text{rank}((q_i(\ell))_{i=1}^{d-k}) = d - k \) implies that \( \mathbb{F}^\ell \) is a \( k \)-dimensional boundary object of \( \mathbb{P} \) containing \( \bigcap_{i=1}^{m} \pi_{q_i, r_i} \) which is \( \mathbb{F} \). Thus \( \mathbb{F}^\ell \) and \( \mathbb{F} \) coincide, showing (11.3). See Fig. 3.

**Orientation.** Call \( \mathbb{F} \in \mathcal{F}(\mathbb{P}) \) and \( \mathbb{F}^\vee \) forward or backward oriented if \( \mathbb{F}^\vee \subset \mathbb{P}^\vee_{\text{for}} \) or \( \subset \mathbb{P}^\vee_{\text{bac}} \). In Fig. 2, we demonstrate \( \mathbb{F}^\vee = \mathbb{F}^\text{for} \cup \mathbb{F}^\text{bac} \) showing how to reset \( \mathcal{F}(\mathbb{P}) \) as \( \mathcal{F}(\mathbb{P}^\text{for}) \cup \mathcal{F}(\mathbb{P}^\text{bac}) \). Indeed, one can switch the single vertex \( \mathbb{F} = \pi_{q(1)} \cap \pi_{q(2)} \) with the two newly formed vertices \( \mathbb{F}^\text{for} := \pi_{q(1)} \cap \pi_{(-1,1)} \) of \( \mathbb{P}^\text{for} \) and \( \mathbb{F}^\text{bac} := \pi_{(-1,1)} \cap \pi_{q(2)} \) of \( \mathbb{P}^\text{bac} \), which are same to the original \( \mathbb{F} \) as a set, but have distinct duals \( \mathbb{F}^\text{for} \subset \mathbb{P}^\text{for} \) and \( \mathbb{F}^\text{bac} \subset \mathbb{P}^\text{bac} \). We state this intuition as the proposition below.

**Proposition 11.2** (Dichotomy to Oriented Faces) Let \( \mathbb{F} \in \mathcal{F}^k(\mathbb{P}) \). Suppose that

\[
(\mathbb{F}^\vee)^\circ \cap (\mathbb{P}^\text{for})^\circ \neq \emptyset \quad \text{and} \quad (\mathbb{F}^\vee)^\circ \cap (\mathbb{P}^\text{bac})^\circ \neq \emptyset . \tag{11.5}
\]

Then we can split the dual face \( \mathbb{F}^\vee = \mathbb{F}^\text{for} \cup \mathbb{F}^\text{bac} \) as

\[
\begin{cases}
\mathbb{F}^\text{for} := \mathbb{F}^\vee \cap \mathbb{P}^\text{for} = \text{cone}((p_i)_{i=1}^m) \\
\mathbb{F}^\text{bac} := \mathbb{F}^\vee \cap \mathbb{P}^\text{bac} = \text{cone}((q_j)_{j=1}^n)
\end{cases} \tag{11.6}
\]

for \( \text{dim}(\mathbb{F}^\text{for}) = \text{dim}(\mathbb{F}^\text{bac}) = d - k \), which are dual faces of

\[
\begin{cases}
\mathbb{F}^\text{for} := \bigcap \pi_{p_i, r_i} \cap \mathbb{P} \in \mathcal{F}^k(\mathbb{P}^\text{for}) \quad \text{with} \quad \Pi (\mathbb{P}^\text{for}^+) = (\pi_{p_i, r_i})_{i=1}^m \\
\mathbb{F}^\text{bac} := \bigcap \pi_{q_j, s_j} \cap \mathbb{P} \in \mathcal{F}^k(\mathbb{P}^\text{bac}) \quad \text{with} \quad \Pi (\mathbb{P}^\text{bac}^+) = (\pi_{q_j, s_j})_{j=1}^n
\end{cases} \tag{11.7}
\]

which are identical to \( \mathbb{F} \) as a set. Here \( \pi_{p_i, r_i} \in \mathbb{F}^\text{for} \) and \( \pi_{q_j, s_j} \in \mathbb{F}^\text{bac} \). If \( \mathbb{F}^\vee \subset \mathbb{P}^\text{for} \) or \( \subset \mathbb{P}^\text{bac} \), rewrite \( \mathbb{F} \) as \( \mathbb{F}^\text{for} \in \mathcal{F}^k(\mathbb{P}^\text{for}) \) or \( \mathbb{F}^\text{bac} \in \mathcal{F}^k(\mathbb{P}^\text{bac}) \), respectively.

**Proof of Proposition 11.2** Since \( \mathbb{F} \in \mathcal{F}^k \), the cone \( \mathbb{F}^\vee \) of dim \( d - k \) is imbedded in the \( d - k \) dimensional subspace \( U \subset \mathbb{V} \). From (11.5), both \( (\mathbb{F}^\vee)^\circ \cap (\mathbb{P}^\text{for})^\circ \) and \( (\mathbb{F}^\vee)^\circ \cap (\mathbb{P}^\text{bac})^\circ \) contain a non-empty open sets in \( U \). Thus, both \( \mathbb{F}^\vee \cap \mathbb{P}^\text{for} \) and \( \mathbb{F}^\vee \cap \mathbb{P}^\text{bac} \) in \( U \) are \( d - k \) dimensional polyhedral cones. Therefore we have (11.6) such that \( \text{rank}((p_i)_{i=1}^m) = d - k \) and \( \text{rank}((q_j)_{j=1}^n) = d - k \). Since \( (p_i)_{i=1}^m \subset \mathbb{F}^\vee \cap \mathbb{P}^\text{for} \subset \mathbb{F}^\vee \) in (11.6), there are \( \pi_{p_1, r_1}, \ldots, \pi_{p_m, r_m} \in \mathbb{F}^\vee \) containing \( \mathbb{F} \). Hence \( \mathbb{F}^\text{for} \), defined in (11.7), is at most \( k \)-dimensional object, containing \( \mathbb{F} \). This implies that \( \mathbb{F}^\text{for} \) in (11.7) coincides with \( \mathbb{F} \) as a set. Similarly, \( \mathbb{F}^\text{bac} \subset \mathbb{F} \).

**Decomposition by Oriented Simplicial Dual Faces** Let \( \text{dim}(\mathbb{P}^\vee) = d - k_0 \). Insert \( \mathbb{F} \in \mathcal{F}(\mathbb{P}^\text{for}) \cup \mathcal{F}(\mathbb{P}^\text{bac}) \) in Proposition 11.2 into (6.1) as

\[
\mathbb{P}^\vee = \bigcup_{\mathbb{F} \in \mathcal{F}(\mathbb{P}^\text{for}) \cup \mathcal{F}(\mathbb{P}^\text{bac})} \mathbb{F}^\vee = \bigcup_{\mathbb{F} \in \mathcal{F}^k(\mathbb{P}^\text{for}) \cup \mathcal{F}^k(\mathbb{P}^\text{bac})} \mathbb{F}^\vee .
\]
Define $\mathcal{F}_k^k(\mathbb{F}) = \{\mathbb{F}_\ell\}$ as the set of $k$-dimensional newly formed faces $\mathbb{F}_\ell$ in (11.3) having ess disjoint simplicial duals $\mathbb{F}_\ell^\vee$ in (11.2) forming $\bigcup_{\mathbb{F}_\ell \in \mathcal{F}_k^k(\mathbb{F})} \mathbb{F}_\ell^\vee = \mathbb{F}^\vee$. By inserting $\mathcal{F}_k^k := \bigcup_{\mathbb{F} \in \mathcal{F}_k^k} \mathcal{F}_k^k(\mathbb{F})$ and $\mathcal{F}_\text{os} := \bigcup_{k \geq k_0} \mathcal{F}_k^k(\mathbb{F})$ into the above decomposition,

$$
\mathbb{F}^\vee = \bigcup_{\mathbb{F} \in \mathcal{F}_\text{os}(\mathbb{P}_{\text{for}}) \cup \mathcal{F}_\text{os}(\mathbb{P}_{\text{bac}})} \mathbb{F}^\vee = \bigcup_{\mathbb{F} \in \mathcal{F}_k^k(\mathbb{P}_{\text{for}}) \cup \mathcal{F}_k^k(\mathbb{P}_{\text{bac}})} \mathbb{F}^\vee. \quad (11.8)
$$

We can apply (11.8) to (6.2). Fix $\mathbb{P} = \mathbb{N}(P, D_B)$ to prove Theorem 6.1.

**Proof of Theorem 6.1** One can apply (11.8) for (6.2) to have (6.3) where $\{q_i\}_{i=1}^{d_0} \subset \mathbb{Q}^d$ is the set of linearly independent vectors. We next claim (6.4). First take $M_0$ as the product of the denominators of all entries in $\{q_i\}_{i=1}^{d_0}$. This implies the first inclusion of (6.4). For the $d \times d_0$ matrix $A := (q_1 | \cdots | q_{d_0})$, it holds $j \in \mathbb{F}^\vee \cap \mathbb{Z}^d$ if and only if $j = \sum_{i=1}^{d_0} \alpha_i q_i = A(\alpha_i) \in \mathbb{Z}^d$ for $\alpha_i \geq 0$. Since the first $d_0 \times d_0$ submatrix $A_0$ of $A$ is non-singular,

$$
\mathbb{F}^\vee \cap \mathbb{Z}^d \subset \left\{ \sum_{i=1}^{d_0} \alpha_i q_i : (\alpha_i)_{i=1}^{d_0} \in A_0^{-1}(\mathbb{Z}_0) \cap \mathbb{R}_+^{d_0} \right\}. \quad (11.9)
$$

Take $M_1$ as the product the denominators of all entries in $A_0^{-1}$ in (11.9). Then $A_0^{-1}(\mathbb{Z}_0) \subset \left( \frac{1}{M_1} \mathbb{Z}_+ \right)^{d_0}$ implies the second inclusion of (6.4).

### 12 Equivalence with Face-Nondegeneracy

We prove Theorem 10.1. We need a notion of a neighborhood of a dual face.

#### 12.1 Neighborhoods of Dual Faces

Suppose that $D \cong D_B$ and $D$ is away from the middle region $M_h = \{x : 1/h \leq |x_v| \leq h \text{ for all } v\}$ for $h \gg 1$. Thus $D \subset \{x : |x_v| < 1/h \text{ or } |x_v| > h \text{ for some } v\}$. Take $h = 2^{d_0 d + 100}$ with $r \geq 1$. Then

$$
D \subset 2^{-N_r^c} \text{ where } N_r = B(0, dr^{d+100}).
$$

Hence, $D \subset 2^{-\text{cone}^\vee(B) \cap N_r^c + O(1)}$. Thus we shall work $\text{cone}^\vee(B) \cap N_r^c$ rather than $\text{cone}^\vee(B)$. One can split by the faces in (11.8),

$$
\text{cone}(B)^\vee \cap N_r^c = \bigcup_{\mathcal{F}(\mathbb{P})} \mathbb{F}^\vee \cap N_r^c \text{ where } \mathcal{F}(\mathbb{P}) = \mathcal{F}_{\text{os}}(\mathbb{P}_{\text{for}}) \cup \mathcal{F}_{\text{os}}(\mathbb{P}_{\text{bac}}). \quad (12.1)
$$

Next, consider a neighborhood of a dual face $\mathbb{F}^\vee \cap N_r^c$. 
Definition 12.1 (Neighborhood of $\mathbb{F}^\vee$ in $\mathbb{G}^\vee$) Let $\mathbb{F} \in \mathcal{F}^k$ with $\mathbb{G} \in \mathcal{F}^{k_0}$ such that $\mathbb{G} \leq \mathbb{F}$ in (12.1). Since $\mathbb{F}^\vee$, $\mathbb{G}^\vee$ simplicial, one can take such that
\[
\text{cone}((q_i)_{i=1}^{d-k}) = \mathbb{F}^\vee \text{ and } \text{cone}((q_i)_{i=1}^d) = \mathbb{G}^\vee \text{ where } q_i \in \Pi(\mathbb{P}). \tag{12.2}
\]

Denote basis$(\mathbb{F}^\vee) = \{q_i\}_{i=1}^{d-k}$ and basis$(\mathbb{G}^\vee) = \{q_i\}_{i=1}^d$ to define a neighborhood $N_r^k(\mathbb{F}^\vee|\mathbb{G}^\vee)$ of $\mathbb{F}^\vee \cap N_r^c$ in $\mathbb{G}^\vee \cap N_r^c$:
\[
\left\{ \sum_{q_i \in \text{basis}(\mathbb{F}^\vee)} \alpha_i q_i + \sum_{q_j \in \text{basis}(\mathbb{G}^\vee) \setminus \text{basis}(\mathbb{F}^\vee)} \alpha_j q_j \in N_r^c : \alpha_i \geq r^{k+1} \text{ and } 0 \leq \alpha_j < r^k \right\}.
\]

The vector $j \in N_r^k(\mathbb{F}^\vee|\mathbb{G}^\vee)$ is the sum of the first term (main term) in $\mathbb{F}^\vee$ of size $\geq d r^{d+99}$ and the second term (error term) in $\mathbb{G}^\vee$ of size $\leq r^k d$. Therefore, $N_r^k(\mathbb{F}^\vee|\mathbb{G}^\vee)$ is a perturbation of the cone $\mathbb{F}^\vee$ in $\mathbb{G}^\vee$ located away from both $\partial(\mathbb{F}^\vee)$ and the origin. For simplicity, write it as $N_r^k(\mathbb{F}^\vee|\mathbb{G}^\vee)$. Notice that if $k = k_0$, owing to $\mathbb{F} = \mathbb{G}$, we can write $N_r^k(\mathbb{F}^\vee|\mathbb{G}^\vee)$ as
\[
N_r^{k_0}(\mathbb{G}^\vee|\mathbb{G}^\vee):=\left\{ \sum_{q_i \in \text{basis}(\mathbb{G}^\vee)} \alpha_i q_i \in N_r^c : \alpha_i \geq r^{k_0+1} \right\}. \tag{12.3}
\]

Lemma 12.1 (Strict Dual Face Decomposition) Let $\mathbb{P}$ be a polyhedron in $\mathbb{R}^d$ and let $\dim(\mathbb{P}) = d - k_0$ and $\mathcal{F} = \mathcal{F}^\text{cos}$ in (11.8) and (12.1). Then,
\[
\text{cone}(B)^\vee \cap N_r^c = \bigcup_{\mathbb{G} \in \mathcal{F}^{k_0}(\mathbb{P})} \bigcup_{\mathbb{F} \in \mathcal{F} : \mathbb{G} \leq \mathbb{F}} N_r(\mathbb{F}^\vee|\mathbb{G}^\vee). \tag{12.4}
\]

Proof We prove $k_0 = 0$. Note $\supseteq$ is true from $N_r(\mathbb{F}^\vee|\mathbb{G}^\vee) \subseteq \text{cone}(B)^\vee$. To claim $\subseteq$, let $p \in \text{cone}(B)^\vee \cap N_r^c$. By (12.1), find $\mathbb{G} \in \mathcal{F}^0(\mathbb{P})$:
\[
p = \alpha_1 q_1 + \cdots + \alpha_d q_d \in \bigcup_{\mathbb{F} \in \mathcal{F} : \mathbb{G} \leq \mathbb{F}} \mathbb{F}^\vee \cap N_r^c \text{ with basis}(\mathbb{G}^\vee) = (q_j)_{j=1}^d.
\]

It suffices to show that $p \in N_r^k(\mathbb{F}^\vee|\mathbb{G}^\vee)$ for some $\mathbb{F}^\vee \leq \mathbb{G}^\vee$ with $0 \leq k \leq d - 1$. We can assume that $0 \leq \alpha_1 \leq \cdots \leq \alpha_d$ above. Set $d + 1$ number of disjoint intervals $I_k := [r^k, r^{k+1})$ where $k = 1, \ldots, d - 1$, and $I_d := [r^d, \infty)$ and $I_0 := [0, r)$.

- Observe $\alpha_d \in I_d$ because $|p| \geq d r^{d+100}$ for $p \in N_r^c = \{j | j| \geq d r^{d+100}\}$.
- Next $\alpha_{d-1} \in I_{d-1} \cup I_d$. If not, $\alpha_{d-1} \in I_0 \cup \cdots \cup I_{d-2}$, namely, $\alpha_{d-1} < r^{d-1}$ and $\alpha_d \geq r^d$, leading $p \in N_r^{d-1}(\mathbb{F}^\vee|\mathbb{G}^\vee)$ for $\mathbb{F}^\vee = \text{cone}(q_d)$.
- Next $\alpha_{d-2} \in I_{d-2} \cup I_{d-1} \cup I_d$. If not, $\alpha_{d-2} \in I_0 \cup \cdots \cup I_{d-3}$, namely, $\alpha_{d-2} < r^{d-2}$ with $\alpha_{d-1} \geq r^{d-1}$, leading $p \in N_r(\mathbb{F}^\vee|\mathbb{G}^\vee)$ with $\mathbb{F}^\vee = \text{cone}(\{q_d, q_{d-1}\})$.
- Repeat until $\alpha_1 \in I_1 \cup \cdots \cup I_d$. So $\alpha_d \geq \cdots \geq \alpha_1 \geq r^1$ and $p \in N_r^0(\mathbb{G}^\vee|\mathbb{G}^\vee)$.
Therefore, we are done with $\supset$ in (12.4).

\textbf{Proposition 12.1} (Strict Dual Face Decomposition) Let $P$ be a polynomial and $B \subset \mathbb{Q}^d$ with $\dim(\text{cone}^\vee(B)) = d - k_0$. Then, one can decompose $\int \psi(\lambda P(x)) \psi_{D_R}(x) dx$:

$$
\sum_{G \in \mathcal{F}^k(P)} \sum_{F \in \mathcal{F} : G \subseteq F} \sum_{j \in \mathbb{N}^r(F \vee |G \vee \cap \mathbb{Z}^d)} \int \psi(\lambda P(x)) \psi_{DB}(x) \chi \left( \frac{x}{2^{-j}} \right) dx
$$

where $\mathcal{F}^\vee$ are oriented simplicial cones of the form $\text{cone}(q_1, \ldots, q_{d-k})$ with $k \geq k_0$ and $q_i \in \mathbb{Q}^d \cap \{1/2 < |q| < 1\}$ having the rational coordinates as in (6.4). Moreover, if $\mathcal{F}$ is of dimension $k$, then there exists $r_P > 0$ such that for $r > r_P$,

$$
2^{Cr^{k+1}} \left( \sum_{n \in \Lambda(P) \setminus \mathcal{F}} |c_n|2^{-j-n} \right) \leq |c_m|2^{-j-m} \forall m \in \mathcal{F} \text{ and } j \in \mathcal{N}_r^k(\mathcal{F}^\vee |G^\vee) \quad (12.5)
$$

that is a general version of (7.2). If $\sum_{\sigma \leq |\alpha| \leq \tau} |\partial^\alpha_x P_F|_{(\mathbb{R} \setminus \{0\})^d}$ is non-vanishing, then there is $C > 0$ and $r_P$ such that for $x \sim 2^{-j}$ and $r > r_P$ as in (12.5),

$$
\frac{2^{-j-m}}{2^{6Cr^k}} \leq \sum_{\sigma \leq |\alpha| \leq \tau} |x^\alpha \partial^\alpha_x P(x)| \approx \sum_{\sigma \leq |\alpha| \leq \tau} |x^\alpha \partial^\alpha_x P_F(x)| \leq \frac{2^{-j-m}}{2^{6Cr^k}}. \quad (12.6)
$$

The decomposition of the integral follows from the application of (12.4).

\textbf{12.2 Proof of (12.5) and (12.6)}

To show (12.5), we need the following lemmas and the definition of some constants involving the coefficients of $P$.

\textbf{Lemma 12.2} (Łojasiewicz) Let $U \subset \mathbb{R}^d$ be an open set containing a compact set $K$. Suppose $g$ and $G$ are real analytic functions (polynomials) in $U$ such that

$$
\{u \in U : g(u) = 0\} \subset \{u \in U : G(u) = 0\}
$$

Then there is constants $\mu, C > 0$ such that

$$
|g(u)| \geq C|G(u)|^\mu \text{ for all } u \in K.
$$

\textbf{Proof} See its proof in [10] and [17].

\textbf{Lemma 12.3} Suppose that $Q : \mathbb{R}^d \to \mathbb{R}$ is a polynomial, non-vanishing on $(\mathbb{R} \setminus \{0\})^d$. Then there are constants $\gamma, B > 0$ independent of $x$ such that,

$$
|Q(x)| \geq B \min\{|x|^{-\gamma}, |x_1 \ldots x_d|^{\gamma}\}. \quad (12.7)
$$
Proof of Lemma 12.3} Set \( K_i = \{ u : |u_1|, \ldots, |u_d| \leq 1 \text{ and } |u_i| \leq 1/100 \} \) and
\[
g_i(u) := u_i^n Q \left( \frac{u_1}{u_i}, \ldots, \frac{u_{i-1}}{u_i}, \frac{u_{i+1}}{u_i}, \ldots, \frac{u_d}{u_i} \right) \text{ for } n = \deg(Q). \tag{12.8}
\]

Then \( g \) is a polynomial, because \( u_i^n \) cancels the all \( u_i^{-s} \) with \( 0 \leq s \leq n \) arising from \( Q(\cdot) \). Since \( Q \) is non-vanishing in \( (\mathbb{R} \setminus \{0\})^d \), we can observe that for \( U := \mathbb{R}^d \),
\[
\{ u \in U : g_i(u) = 0 \} \subset \{ u \in U : u_1 \ldots u_d = 0 \}.
\]

Then, one can apply Lemma 12.2 for \( G(u) = u_1 \ldots u_d \) to have \( C, \mu_i \geq 1 \):
\[
|g_i(u_1, \ldots, u_d)| \geq C|u_1 \ldots u_d|^{\mu_i} \text{ for all } u \in K_i.
\]

Set \( W_i = \{ x : |x_i| \geq |x_1|, \ldots, |x_d| \text{ and } |x_i| \geq 100 \} \) and \( \Phi_i : W_i \to \Phi(W_i) = K_i \) by
\[
\Phi(x) := \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_d}{x_i} \right) = (u_1, \ldots, u_d).
\]

This coordinate change in (12.8) implies that
\[
|x_i^{-n} Q(x_1, \ldots, x_d)| = |g_i(u)| \geq C|u_1 \ldots u_d|^{\mu_i} \text{ for } x \in W_i.
\]

This with \( 100 \leq |x_i| \leq |x| \leq d|x_i| \) in \( W_i \) yields that for \( \nu = \max(\mu_i(d+1) - n) \),
\[
|Q(x)| \geq (x_i)^n (x_i)^{-\mu_i(d+1)} |x_1 \ldots x_d|^{\mu_i} \geq C|x|^{-\nu} |x_1 \ldots x_d|^\nu \text{ if } x \in W_i
\]

which holds true for all \( i \in [d] \). Hence
\[
|Q(x)| \geq C \min\{|x|^{-2\nu}, |x_1 \ldots x_d|^{2\nu}\} \text{ for } x \in \bigcup_{i=1}^d W_i \supset \{ x : |x| \geq 100d \}. \tag{12.9}
\]

On the other hand, by applying Lemma 12.2 again for \( \{ x \in \mathbb{R}^d : Q(x) = 0 \} \subset \{ x \in \mathbb{R}^d : x_1 \ldots x_d = 0 \} \) due to non-vanishing condition of \( Q \) on \( (\mathbb{R} \setminus \{0\})^d \),
\[
|Q(x)| \geq C|x_1 \ldots x_d|^{\mu} \text{ on the compact set } K := \{ x : |x| \leq 100d \}.
\]

By this with (12.9), we obtain (12.7) for \( \gamma = \max(2\nu, \mu) \).

Lemma 12.4 Let \( Q \) be a non-vanishing polynomial on \( (\mathbb{R} \setminus \{0\})^d \) and let \( U_{2^{100d} \delta^k} := \{ y : 2^{-100d\delta^2} \leq |y_v| \leq 2^{100d\delta^2} \text{ for } v = 1, \ldots, d \} \) where \( r \geq 1 \). Then there is \( b > 0 \) depending on \( Q \) such that
\[
2^{-b(Q)\delta^k} \leq |Q(y)| \leq 2^{b(Q)\delta^k} \text{ for all } y \in U_{2^{100d} \delta^k}. \tag{12.10}
\]
**Definition 12.2** (Constants Associated with $q$ for all $m$)

Let $P = N(P, D_B)$. Given $P(x) = \sum_{m \in \Lambda(P)} c_m x^m$, we define the maximal ratio of coefficients of $P$ as

$$C_P := \frac{\sum_{n \in \Lambda(P)} |c_n|}{\min\{1, |c_n| : n \in \Lambda(P)\}}.$$  

From (12.10), we take $b = \max\{b(P_F) : (P_F)_{[0, t]}\}$ non-vanishing $F \in \mathcal{F}(P)$. Set the two constants as

$$H := (b + 10d + \deg(P))^{10} \sum_{m,n \in \Lambda(P)} |n - m|,$$

$$L := \min\{1, (n-m) \cdot q : n \in \Lambda(P) \setminus \pi_q \text{ and } m \in \Lambda(P) \cap \pi_q\}_{q \in \mathbb{P}}.$$

where $1/2 \leq |q| \leq 1$. Then $L > 0$ because $(n-m) \cdot q > 0$ for $n \in \pi_q \setminus \pi_q$ and $m \in \pi_q$.

**Proof of (12.5)** Let $r_P := \max\{C_P / L, H / L\}$ and $j \in \mathcal{N}_k^\text{max}(\mathbb{F}^\vee |G)$ for $0 \leq k \leq d - 1$. Then, we claim (12.5) for $C = \frac{8L}{10}$. By Definition 12.1, we can write $j$ as

$$\sum_{q_i \in \text{basis}(\mathbb{F}^\vee)} \alpha_i q_i + \sum_{q_i \in \text{basis}(\mathbb{G}^\vee) \setminus \text{basis}(\mathbb{F}^\vee)} \alpha_\ell q_\ell : 0 \leq \alpha_\ell < r^k \text{ and } r^{k+1} \leq \alpha_i$$  \hspace{1cm} (12.11)

where the second sum is zero if $k = 0$. Let $m, \tilde{m} \in \mathbb{F}$. As $(m - \tilde{m}) \cdot q_i = 0$ for $q_i \in \text{basis}(\mathbb{F}^\vee)$, we have

$$|j \cdot (m - \tilde{m})| \leq \left| \sum_{q_i \in \text{basis}(\mathbb{G}^\vee) \setminus \text{basis}(\mathbb{F}^\vee)} \alpha_\ell q_\ell \cdot (m - \tilde{m}) \right| \leq r^k H.$$  \hspace{1cm} (12.12)

Let $m \in \mathbb{F} = \bigcap_{q_i \in \text{basis}(\mathbb{F}^\vee)} \pi_{q_i} \cap \mathbb{P}$ and $n \in \Lambda(P) \setminus \mathbb{F}$. Then $m \in \pi_{q_i}$ and $n \in \pi_{q_i}^+$ for all $q_i \in \text{basis}(\mathbb{F}^\vee)$, whereas $n \in (\pi_{q_i}^+)^o$ for some $q_s \in \text{basis}(\mathbb{F}^\vee)$. Thus

$$q_s \cdot (n - m) > 0 \text{ and } q_i \cdot (n - m) \geq 0 \text{ for all } q_i \in \text{basis}(\mathbb{F}^\vee).$$

From this with the constants in Definition 12.2, $\alpha_i \geq r^{k+1}$ and $\alpha_\ell < r^k$ in (12.11),

$$\sum_{q_i \in \text{basis}(\mathbb{F}^\vee)} \alpha_i q_i \cdot (n - m) \geq r^{k+1} L \text{ and } \left| \sum_{q_\ell \in \text{basis}(\mathbb{G}^\vee) \setminus \text{basis}(\mathbb{F}^\vee)} \alpha_\ell q_\ell \cdot (n - m) \right| < r^k H.$$

This with (12.12) implies that $j \cdot (n - m) \geq \frac{q^{k+1} L}{10}$ and $2 r^{k+1} L / 10 > r^{k+1} L \geq C_P$ due to $r > r_P$. Therefore,

$$2^{-j \cdot m} \geq 2^{r^{k+1} L / 10} 2^{-j \cdot n} \geq 2^{r^{k+1} L / 10} 2^{-j \cdot n} C_P.$$
This together with $C_P$ defined in Definition 12.2 yields that

$$2^{-j \cdot m} \geq \frac{2^{(r^k + 1)b_j}}{\min\{|c_m| : m \in \Lambda(P) \cap \mathbb{F}\}} \sum_{n \in \Lambda(P) \setminus \mathbb{F}} |c_n| 2^{-j \cdot n}$$

which yields (12.5).

**Proof of (12.6)** We show (12.6) for $\sigma = \tau = 0$. Let $j = N_r^c(F^\vee | G)$ in Definition 12.1. Then $j = q + u$ for $q \in F^\vee$ with $|q| \geq d r^{d+99}$ and $u \in G^\vee$ with $|u| \leq r^k d$. Let $x \sim 2^{-b}$. Then

$$P(2^{-j} x) = P_F(2^{-j} x) + \sum_{n \in \Lambda(P) \setminus \mathbb{F}} 2^{-j \cdot n} c_n x^n \quad (12.13)$$

with $P_F(2^{-j} x) = 2^{-q \cdot m} P_F(2^{-u} x)$. Note that $2^{-(d+1)r^k} \leq |2^{-u} x| \leq 2|u| + 1 \leq 2^{(d+1)r^k}$ where $|u| \leq r^k d$ in the above. Thus, we use (12.10) to obtain that

$$2^{-b r^k} \leq |P_F(2^{-u} x)| \leq 2^{br^k}.$$

This multiplied by $2^{-q \cdot m} = 2^{-j \cdot m} 2^{-u \cdot m} \in 2^{-j \cdot m}[2^{-\deg(P)d r^k}, 2^{\deg(P)d r^k}]$ leads that there exists $C = \deg(P)d(b + 1) > 0$ such that for $r > r_P$,

$$2^{-j \cdot m} 2^{-3 Cr^k} \leq 2^{-q \cdot m} |P_F(2^{-u} x)| \leq 2^{3 Cr^k} 2^{-j \cdot m}.$$

Thus, $|P_F(2^{-j} x)| = 2^{-q \cdot m} |P_F(2^{-u} x)|$ in (12.13) with (12.5) implies (12.6) for $\sigma = \tau = 0$. To consider the general case, replace $|P_F(2^{-j} x)|$ and $P(2^{-j} x)$ by

$$\sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-j \cdot \alpha} x^\alpha) \partial_x^\alpha P_F(2^{-j} x)| \text{ and } \sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-j \cdot \alpha} x^\alpha) \partial_x^\alpha P(2^{-j} x)|$$

where

$$\sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-j \cdot \alpha} x^\alpha) \partial_x^\alpha P_F(2^{-j} x)| = 2^{-q \cdot m} \sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-u \cdot \alpha} x^\alpha) \partial_x^\alpha P_F(2^{-u} x)|.$$

Here $2^{-q \cdot m} = 2^{-j \cdot m}[2^{-\deg(P)d r^k}, 2^{\deg(P)d r^k}]$ with the non-vanishing condition

$$2^{-b r^k} \leq \sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-u \cdot \alpha} x^\alpha) \partial_x^\alpha P(2^{-u} x)| \leq 2^{b r^k}$$

leads (12.6) for $P_F$. Finally, this with the difference $\sum_{\sigma \leq |\alpha| \leq \tau} |(2^{-j \cdot \alpha} x^\alpha) \partial_x^\alpha (P - P_F)(2^{-j} x)| \lesssim \sum_{n \in \Lambda(P) \setminus \mathbb{F}} 2^{-j \cdot n} \lesssim 2^{-Cr^k+1}$ gives the desired estimate for $P$.  

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12.3 Proof of Theorem 10.1 (Normal-Crossing ⇔ Face-nondegeneracy)

Proof (Normal-Crossing ⇒ Face-Nondegeneracy) Suppose that there exists $h \geq 1$ satisfying (3.1) on $D = D \cap M^c_h$. Let $F$ be a face with $F^c = \text{cone}([q_i])_{i=1}^s$ and let $y \in (\mathbb{R}\setminus\{0\})^d$. We claim that $\sum_{\sigma \leq |\sigma| \leq r} |(\partial^\sigma_x P_F)(y)| \neq 0$. Choose $q := (q_1 + \cdots + q_s) \in (F^c)^o$. Then there is $\rho > 0$ such that for all $r > 0$,

$$2^{-r \rho} r^{-2} \leq 2^{-rq} \leq 2^{-r \rho} r^{-m} \text{ for all } m \in \Lambda(P) \cap F \text{ and } n \in \Lambda(P) \setminus F.$$  \hfill (12.14)

Take $r \gg 1$ and $q = (q_1, \ldots, q_d)$ above so that

$$x := 2^{-r q} y = (2^{-r q_1} y_1, \ldots, 2^{-r q_d} y_d) \in D_B \cap M^c_h \subset D,$$

because $y_i \neq 0$ for all $i$ with $q \in (F^c)^o$. Thus, by (12.14) and (3.1), for $m \in \Lambda(P) \cap F$,

$$2^{-r q} \cdot |x^m| = |x^m| \leq \sum_{\sigma \leq |\sigma| \leq r} |x^\sigma \partial^\sigma_x P(x)| \leq \sum_{\sigma \leq |\sigma| \leq r} \left| \sum_{\sigma \leq |\sigma| \leq r} \sum_{n \in \Lambda(P) \setminus F} |x^m| \right| \leq 2^{-r q} \cdot \sum_{\sigma \leq |\sigma| \leq r} |x^m|.$$  \hfill (12.15)

Divide by $2^{-r q} \cdot |x^m|$ and take $r \to \infty$. Then $|y^m| \leq \sum_{\sigma \leq |\sigma| \leq r} |x^\sigma \partial^\sigma_x P(x)| \neq 0$. From this with $y^m \neq 0$ for $y \in (\mathbb{R}\setminus\{0\})^d$, we obtain $\sum_{\sigma \leq |\sigma| \leq r} |x^\sigma \partial^\sigma_x P(x)| \neq 0$. \hfill \Box

Face-Nondegeneracy ⇒ Normal-Crossing Choose $h := 2^{-d r^{d+100}}$ where $r = r_P$ in Lemma 12.2. Let $x \in D \subset 2^{-\text{cone}^c(B) + O(1)} \cap M^c_h$. Then from (12.4), we can express $x = 2^{-j} y$ for some $y \sim 2^{-r}$ and $j \in N_r(F^c \|G\) in Definition 12.1. Thus, the non-vanishing condition of $P_F^{y}(\mathbb{R}\setminus\{0\})^d(x)$ implies (12.6) with $2^{j \cdot m} = |x^m|$. \hfill \Box

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