On collinear sets in straight line drawings

Alexander Ravsky and Oleg Verbitsky

Institute for Applied Problems of Mechanics and Mathematics
Naukova St. 3-п, Lviv 79060, Ukraine

Abstract

Given a planar graph $G$ on $n$ vertices, let $\text{fix}(G)$ denote the maximum $k$ such that any straight line drawing of $G$ with possible edge crossings can be made crossing-free by moving at most $n - k$ vertices to new positions. Let $\bar{v}(G)$ denote the maximum possible number of collinear vertices in a crossing-free straight line drawing of $G$. In view of the relation $\text{fix}(G) \leq \bar{v}(G)$, we are interested in examples of $G$ with $\bar{v}(G) = o(n)$. For each $\epsilon > 0$, we construct an infinite sequence of graphs with $\bar{v}(G) = O(n^{\sigma + \epsilon})$, where $\sigma < 0.99$ is a known graph-theoretic constant, namely the shortness exponent for the class of cubic polyhedral graphs.

Let $S$ be a set of vertices in a crossing-free straight line drawing, all lying on a line $\ell$. We call $S$ free if, after any displacement of the vertices in $S$ along $\ell$ without violating their mutual order, the drawing can be kept crossing-free and straight line by moving the vertices outside $S$. Let $\tilde{v}(G)$ denote the largest size of a free collinear set maximized over all drawings of $G$. Noticing relation $\text{fix}(G) \geq \sqrt{\tilde{v}(G)}$, we get interested in classes of planar graphs with $\tilde{v}(G) = \Omega(n)$. We show that outerplanar graphs form one of such classes and prove this way that $\text{fix}(G) \geq \sqrt{n/2}$ for every outerplanar $G$. This slightly improves the lower bound of $\sqrt{n/3}$ by Spillner and Wolff and makes the untangling procedure for outerplanar graphs somewhat simpler.

1 Introduction

Let $G$ be a planar graph. By a drawing of $G$ we mean an arbitrary injective map $\pi : V(G) \to \mathbb{R}^2$, where $V(G)$ denotes the vertex set of $G$. Given a drawing $\pi$, we just suppose that each edge $uv$ of $G$ is drawn as the straight line segment with endpoints $\pi(u)$ and $\pi(v)$. Thus, it is quite possible that in $\pi$ we encounter edge crossings and even overlaps. Let $\text{fix}(G, \pi) = \max_{\pi'} | \{ v \in V(G) : \pi'(v) = \pi(v) \} |$

where the maximum is taken over all crossing-free drawings $\pi'$ of $G$. Furthermore, let $\text{fix}(G) = \min_{\pi} \text{fix}(G, \pi)$. 

(1)
In other words, $\text{fix}(G)$ is the maximum number of vertices which can be fixed in any drawing of $G$ while untangling it.

No efficient way for evaluating the parameter $\text{fix}(G)$ is known. Note that computing $\text{fix}(G, \pi)$ is NP-hard \cite{2 10}. Essential efforts are needed to estimate $\text{fix}(G)$ even for cycles, for which we know bounds

$$2^{-5/3} n^{2/3} - O(n^{1/3}) \leq \text{fix}(C_n) \leq O((n \log n)^{2/3})$$

due to, respectively, Cibulka \cite{2} and Pach and Tardos \cite{8}. In the general case Bose et al. \cite{1} establish a lower bound

$$\text{fix}(G) \geq (n/3)^{3/4},$$

where $n$ denotes the number of vertices in a graph under consideration. A better bound

$$\text{fix}(G) \geq \sqrt{n/3}$$

is proved for all trees (Goaoc et al. \cite{3}) and, more generally, outerplanar graphs (Spillner and Wollf \cite{9}). For trees this bound is improved in \cite{1} to

$$\text{fix}(G) \geq \sqrt{n/2}.$$ 

On the other hand, \cite{1 3 6 9} provide examples of planar graphs (even acyclic ones) with

$$\text{fix}(G) = O(\sqrt{n}).$$

Here we consider the role played in this circle of questions by sets of collinear vertices in straight line graph drawings. Suppose that $\pi$ is a crossing-free drawing of a graph $G$. Denote $V(\pi) = \pi(V(G))$. A set of vertices $S \subseteq V(\pi)$ in $\pi$ is collinear if all of them lie on a line $\ell$. By a conformal displacement of $S$ we mean a relocation $\delta : S \rightarrow \ell$ preserving the relative order in which the vertices in $S$ go along $\ell$. We call $S$ free if every conformal displacement $\delta : S \rightarrow \ell$ is extendable to a mapping $\delta : V(\pi) \rightarrow \mathbb{R}^2$ so that $\delta \circ \pi$ is a crossing-free drawing of $G$ (i.e., whenever we displace $S$ along $\ell$ without breaking their relative order, the crossing-free drawing $\pi$ can be kept in a modified form by appropriately shifting the other vertices of $G$). Introduce the following notation.

$$\bar{v}(G, \pi) = \text{maximum size of a collinear set in } \pi,$$

$$\bar{v}(G, \pi) = \text{maximum size of a free collinear set in } \pi,$$

$$\bar{v}(G) = \max_{\pi} \bar{v}(G, \pi),$$

$$\bar{v}(G) = \max_{\pi} \bar{v}(G, \pi),$$

where the maximum is taken over all crossing-free drawings of $G$.

Obviously, $\bar{v}(G) \leq \bar{v}(G)$. These parameters have a direct relation to $\text{fix}(G)$, namely

$$\sqrt{\bar{v}(G)} \leq \text{fix}(G) \leq \bar{v}(G).$$

(6)
The latter inequality follows immediately from the definitions. The former inequality is proved as Theorem 4.1 below.

A quadratic gap between $\fix(G)$ and $\bar{v}(G)$ is given, for instance, by the fan graph $F_n$, for which we obviously have $\bar{v}(F_n) = n - 1$ and, by [9], $\fix(F_n) = O(\sqrt{n})$. In fact, we have $\bar{v}(G) = \Omega(n)$ for all the graphs used in [1, 3, 6, 9] to show the upper bound [3]. Therefore, it would be interesting to find examples of planar graphs with $\bar{v}(G) = o(n)$, which could be considered a qualitative strengthening of [5]. We solve this problem in Section 3, constructing for each $\epsilon > 0$ an infinite sequence of graphs with $\bar{v}(G) = O(n^{\sigma+\epsilon})$ where $\sigma = \frac{\log 26}{\log 27}$ is a known graph-theoretic constant, namely the shortness exponent for the class of cubic polyhedral graphs.

By the lower bound in [6], we have $\fix(G) = \Omega(\sqrt{n})$ whenever $\bar{v}(G) = \Omega(n)$. Therefore, identification of classes of planar graphs with linear $\bar{v}(G)$ is of big interest. In Section 4 we prove that, for every outerplanar graph $G$, we have $\bar{v}(G) \geq n/2$ and hence $\fix(G) \geq \sqrt{n}/2$. This slightly improves Spillner-Wolff’s bound [3]. As already mentioned, in the case of trees such an improvement was made by Bose et al. [1]. We actually use an important ingredient of their untangling procedure, namely [1, Lemma 1], which is behind the equality (13) in the proof of our Theorem 4.1. More important than an improvement in a constant factor, our approach makes the untangling procedure for outerplanar graphs somewhat simpler and, hopefully, can be applied for some other classes of planar graphs.

2 Preliminaries

Given a planar graph $G$, we denote the number of vertices, edges, and faces in it, respectively, by $v(G)$, $e(G)$, and $f(G)$. The latter number does not depend on a particular plane embedding of $G$ and hence is well defined. Moreover, for connected $G$ we have

$$v(G) - e(G) + f(G) = 2$$

by Euler’s formula.

A graph is $k$-connected if it has more than $k$ vertices and stays connected after removal of any $k$ vertices. 3-connected planar graphs are called polyhedral as, according to Steinitz’s theorem, these graphs are exactly the 1-skeletons of convex polyhedra. By Whitney’s theorem, all plane embeddings of a polyhedral graph $G$ are equivalent, that is, obtainable from one another by a plane homeomorphism up to the choice of outer face. In particular, the set of facial cycles (i.e., boundaries of faces) of $G$ does not depend on a particular plane embedding.

A planar graph $G$ is maximal if adding an edge between any two non-adjacent vertices of $G$ violates planarity. Maximal planar graphs on more than 3 vertices are 3-connected. Clearly, all facial cycles in such graphs have length 3. By this reason maximal planar graphs are also called triangulations. Note that for every triangulation $G$ we have $3f(G) = 2e(G)$. Combined with (7), this gives us

$$f(G) = 2v(G) - 4.$$
The dual of a polyhedral graph $G$ is a graph $G^*$ whose vertices are the faces of $G$ (represented by their facial cycles). Two faces are adjacent in $G^*$ iff they share a common edge. $G^*$ is also a polyhedral graph. If we consider $(G^*)^*$, we obtain a graph isomorphic to $G$. In a cubic graph every vertex is incident to exactly 3 edges. As easily seen, the dual of a triangulation is a cubic graph. Conversely, the dual of any cubic polyhedral graph is a triangulation.

The circumference of a graph $G$, denoted by $c(G)$, is the length of a longest cycle in $G$. The shortness exponent of a class of graphs $G$ is the limit inferior of quotients $\log c(G) / \log v(G)$ over all $G \in G$. Let $\sigma$ denote the shortness exponent for the class of cubic polyhedral graphs. It is known that

$$0.694 \ldots = \log_2 (1 + \sqrt{5}) - 1 \leq \sigma \leq \frac{\log 26}{\log 27} = 0.988 \ldots$$

(see [7] for the lower bound and [5] for the upper bound).

### 3 Graphs with small collinear sets

We here construct a sequence of triangulations $G$ with $\bar{v}(G) = o(v(G))$. For our analysis we will need another parameter of a straight line drawing. Given a crossing-free drawing $\pi$ of a graph $G$, let $\bar{f}(G, \pi)$ denote the maximum number of collinear points in the plane such that each of them is an inner point of some face of $\pi$ and no two of them are in the same face. Let $\bar{f}(G) = \max_\pi \bar{f}(G, \pi)$. In other words, $\bar{f}(G)$ is equal to the maximum number of faces in some straight line drawing of $G$ whose interiors can be cut by a line. Further on, saying that a line cuts a face, we mean that the line intersects the interior of this face.

For the triangulations constructed below, we will show that $\bar{v}(G)$ is small with respect to $v(G)$ because small is $\bar{f}(G)$ with respect to $f(G)$ (though we do not know any relation between $\bar{v}(G)$ and $\bar{f}(G)$ in general). Our construction can be thought of as a recursive procedure for essentially decreasing the ratio $\bar{f}(G)/f(G)$ at each recursion step provided that we initially have $\bar{f}(G) < f(G)$.

Starting from an arbitrary triangulation $G_1$ with at least 4 vertices, we recursively define a sequence of triangulations $G_1, G_2, \ldots$. To define $G_k$, we will describe a spherical drawing $\delta_k$ of this graph. Let $\delta_1$ be an arbitrary drawing of $G_1$ on a sphere. Furthermore, $\delta_{i+1}$ is obtained from $\delta_i$ by triangulating each face of $\delta_i$ so that this triangulation is isomorphic to $G_1$. An example is shown on Fig. [1]. In general, upgrading $\delta_i$ to $\delta_{i+1}$ can be done in different ways, that may lead to non-isomorphic versions of $G_{i+1}$. We make an arbitrary choice and fix the result.

**Lemma 3.1.** Denote $f = f(G_1)$, $\bar{f} = \bar{f}(G_1)$, and $\alpha = \frac{\log(\bar{f} - 1)}{\log(f - 1)}$.

1. $f(G_k) = f(f - 1)^{k-1}$.
2. $\bar{f}(G_k) \leq \bar{f}(\bar{f} - 1)^{k-1}$.
3. \( \bar{v}(G_k) < cv(G_k)^\alpha \), where \( c \) is a constant depending only on \( G_1 \).

**Proof.** The first part follows from the obvious recurrence

\[ f(G_{i+1}) = f(G_i)(f - 1). \]

We have to prove the other two parts.

Consider an arbitrary crossing-free straight line drawing \( \pi_k \) of \( G_k \). Recall that, by construction, \( G_1, \ldots, G_{k-1} \) is a chain of subgraphs of \( G_k \) with

\[ V(G_1) \subset V(G_2) \subset \ldots \subset V(G_{k-1}) \subset V(G_k). \]

Let \( \pi_i \) be the part of \( \pi_k \) which is a drawing of the subgraph \( G_i \). By the Whitney theorem, \( \pi_k \) can be obtained from \( \delta_k \) (the spherical drawing defining \( G_k \)) by an appropriate stereographic projection of the sphere to the plane combined with a homeomorphism of the plane onto itself. It follows that, like \( \delta_{i+1} \) and \( \delta_i \), drawings \( \pi_{i+1} \) and \( \pi_i \) have the following property: the restriction of \( \pi_{i+1} \) to any face of \( \pi_i \) is a drawing of \( G_1 \). Given a face \( F \) of \( \pi_i \), the restriction of \( \pi_{i+1} \) to \( F \) (i.e., a plane graph isomorphic to \( G_1 \)) will be denoted by \( \pi_{i+1}[F] \).

Consider now an arbitrary line \( \ell \). Let \( \bar{f}_i \) denote the number of faces in \( \pi_i \) cut by \( \ell \). By definition, we have

\[ \bar{f}_1 \leq \bar{f}. \] (9)

For each \( 1 \leq i < k \), we have

\[ \bar{f}_{i+1} \leq \begin{cases} \bar{f} & \text{if } \bar{f}_i = 1, \\ \bar{f}_i(\bar{f} - 1) & \text{if } \bar{f}_i > 1. \end{cases} \] (10)

Indeed, let \( K \) denote the outer face of \( \pi_i \). Equality \( \bar{f}_i = 1 \) means that, of all faces of \( \pi_i \), \( \ell \) cuts only \( K \). Within \( K \), \( \ell \) can cut only faces of \( \pi_{i+1}[K] \) and, therefore, \( \bar{f}_{i+1} \leq \bar{f} \).
Assume that \( f_i > 1 \). Within \( K \), \( \ell \) can now cut at most \( f - 1 \) faces of \( \pi_{i+1} \) (because \( \ell \) cuts \( \mathbb{R}^2 \setminus K \), a face of \( \pi_{i+1}[K] \) outside \( K \)). Within any inner face \( F \) of \( \pi_i \), \( \ell \) can cut at most \( \bar{f} - 1 \) faces of \( \pi_{i+1} \) (the subtrahend 1 corresponds to the outer face of \( \pi_{i+1}[F] \), which surely contributes to \( \bar{f} \) but is outside \( F \)). The number of inner faces \( F \) cut by \( \ell \) is equal to \( \bar{f}_i - 1 \) (again, the subtrahend 1 corresponds to the outer face of \( \pi_i \)). We therefore have \( \bar{f}_i + 1 \leq (\bar{f} - 1) + (\bar{f}_i - 1)(\bar{f} - 1) = \bar{f}_i(\bar{f} - 1) \), completing the proof of (10).

Using (9) and (10), a simple inductive argument gives us

\[
\bar{f}_i \leq \bar{f}(\bar{f} - 1)^{i-1} \tag{11}
\]

for each \( i \leq k \). As \( \pi_k \) and \( \ell \) are arbitrary, the part 2 is proved by setting \( i = k \) in (11).

To prove part 3, we have to estimate from above \( \bar{v} = |\ell \cap V(\pi_k)| \), the number of vertices of \( \pi_k \) on the line \( \ell \). Put \( \bar{v}_i = |\ell \cap V(\pi_i)| \) and \( \bar{v}_i = |\ell \cap (V(\pi_i) \setminus V(\pi_{i-1}))| \) for \( 1 < i \leq k \). Clearly, \( \bar{v} = \sum_{i=1}^k \bar{v}_i \). Abbreviate \( v = v(G_1) \). It is easy to see that

\[
\bar{v}_1 \leq v - 2
\]

and, for all \( 1 < i \leq k \),

\[
\bar{v}_i \leq \bar{f}_{i-1}(v - 3).
\]

It follows that

\[
\bar{v} \leq (v - 2) + (v - 3) \sum_{i=1}^{k-1} \bar{f}_i \leq \frac{(v - 3)\bar{f}}{f - 2}(\bar{f} - 1)^{k-1}, \tag{12}
\]

where we use (11) for the latter estimate. It remains to express the obtained bound in terms of \( v(G_k) \). By (8) and by part 1 of the lemma, we have \((f - 1)^{k-1} < 2v(G_k)/f \) and, therefore,

\[
(\bar{f} - 1)^{k-1} = (f - 1)^{\alpha(k-1)} < (2/f)^{\alpha} v(G_k)^{\alpha}.
\]

Plugging this in to (12), we arrive at the desired bound for \( \bar{v} \) and hence for \( \bar{v}(G_k) \). 

We now need an initial triangulation \( G_1 \) with \( \bar{f}(G_1) < f(G_1) \). The following lemma shows a direction where one can seek for such triangulations.

**Lemma 3.2.** For every triangulation \( G \) with more than 3 vertices, we have

\[
\bar{f}(G) \leq c(G^*). \]

**Proof.** Given a crossing-free drawing \( \pi \) of \( G \) and a line \( \ell \), we have to show that \( \ell \) crosses no more than \( c(G^*) \) faces of \( \pi \). Shift \( \ell \) a little bit to a new position \( \ell' \) so that \( \ell' \) does not go through any vertex of \( \pi \) and still cuts all the faces that are cut by \( \ell \). Thus, \( \ell' \) crosses boundaries of faces only via inner points of edges. Each such crossing corresponds to transition from one vertex to another along an edge in the dual graph \( G^* \). Note that this walk is both started and finished at the outer face of \( \pi \). Since all faces are triangles, each of them is visited at most once. Therefore, \( \ell' \) determines a cycle in \( G^* \), whose length is at least the number of faces of \( \pi \) cut by \( \ell \). 

\( \square \)
Lemma 3.2 suggests the following choice of $G_1$: Take a cubic polyhedral graph $H$ approaching the infimum of the set of quotients $\log(c(G) - 1)/\log(v(G) - 1)$ over all cubic polyhedral graphs $G$ and set $G_1 = H^*$. In particular, we can approach arbitrarily close to the shortness exponent $\sigma$ defined in Section 2. By Lemma 3.1.3, we arrive at the main result of this section.

**Theorem 3.3.** Let $\sigma$ denote the shortness exponent of the class of cubic polyhedral graphs. Then for each $\alpha > \sigma$ there is a sequence of triangulations $G$ with $\bar{v}(G) = O(v(G)^\alpha)$.

**Remark 3.4.** Theorem 3.3 can be translated to a result on convex polyhedra. Given a convex polyhedron $\pi$ and a plane $\ell$, let $\bar{v}(\pi, \ell) = V(\pi) \cap \ell$ where $V(\pi)$ denotes the vertex set of $\pi$. Given a polyhedral graph $G$, we define

$$\bar{v}(G) = \max_{\pi,\ell} \bar{v}(\pi, \ell),$$

where $\pi$ ranges over convex polyhedra with 1-skeleton isomorphic to $G$. Using our construction of a sequence of triangulations $G_1, G_2, G_3, \ldots$, we can prove that $\bar{v}(G) = O(v(G)^\alpha)$ for each $\alpha > \sigma$ and infinitely many polyhedral graphs $G$.

Grübaum [4] investigated the minimum number of planes which are enough to cut all edges of a convex polyhedron $\pi$. Given a polyhedral graph $G$, define

$$\bar{e}(G) = \max_{\pi,\ell} \bar{e}(\pi, \ell),$$

where $\bar{e}(\pi, \ell)$ denotes the number of edges that are cut by a plane $\ell$ in a convex polyhedron $\pi$ with 1-skeleton isomorphic to $G$. Using the relation $\bar{e}(G) \leq c(G^*)$, Grübaum showed (implicit in [4] pp. 893–894) that $\bar{e}(G) = O(v(G)^{\beta})$ for each $\beta > \log_3 2$ and infinitely many $G$ (where $\log_3 2$ is the shortness exponent for the class of all polyhedral graphs).

### 4 Graphs with large free collinear sets

Let $\pi$ be a crossing-free drawing and $\ell$ be a line. Recall that a set $S \subset V(\pi) \cap \ell$ is called free if, whenever we displace the vertices in $S$ along $\ell$ without violating their mutual order and therewith introduce edge crossings, we are able to untangle the modified drawing by moving the vertices not in $S$. By $\bar{v}(G)$ we denote the largest size of a free collinear set maximized over all drawings of a graph $G$.

**Theorem 4.1.** $\bar{v}(G) \geq \sqrt{\bar{v}(G)}$.

**Proof.** Let $\bar{v}^-(G)$ be defined similarly to [4] but with minimization over all $\pi$ such that $V(\pi)$ is collinear. Obviously, $\bar{v}(G) \leq \bar{v}^-(G)$. As proved in [4], we actually have

$$\bar{v}(G) = \bar{v}^-(G).$$ (13)
We use this equality here.

Suppose that \((k - 1)^2 < \tilde{v}(G) \leq k^2\). By \([13]\), it suffices to show that any drawing \(\pi : V(G) \rightarrow \ell \) of \(G\) on a line \(\ell\) can be made crossing-free with keeping \(k\) vertices fixed. Let \(\rho\) be a crossing-free drawing of \(G\) such that, for some \(S \subset V(G)\) with \(|S| > (k - 1)^2\), \(\rho(S)\) is a free collinear set on \(\ell\). By the Erdős-Szekeres theorem, there exists a set \(F \subset S\) of \(k\) vertices such that \(\pi(F)\) and \(\rho(F)\) lie on \(\ell\) in the same order. By the definition of a free set, there is a crossing-free drawing \(\rho'\) of \(G\) with \(\rho'(F) = \pi(F)\). Thus, we can come from \(\pi\) to \(\rho'\) with \(F\) staying fixed.

Theorem 4.1 sometimes gives a short way of proving bounds of the kind \(\text{fix}(G) = \Omega(\sqrt{n})\). For example, for the wheel graph \(W_n\) we immediately obtain \(\text{fix}(W_n) > \sqrt{n} - 1\) from an easy observation that \(\tilde{v}(W_n) = n - 2\) (in fact, this repeats the argument of Pach and Tardos for cycles \([8]\)). The classes of graphs with linear \(\tilde{v}(G)\) are therefore of big interest in the context of disentanglement of drawings. One of such classes is addressed below.

We call a drawing \(\pi\) almost layered if there are parallel lines, called layers, such that every vertex of \(\pi\) lies on one of the layers and every edge either lies on one of the layers or connects endvertices lying on two consecutive layers. We call a graph almost layered\(^1\) if it has a crossing-free almost layered drawing.

An obvious example of an almost layered graph is a grid graph \(P_s \times P_s\). It is also easy to see that any tree is an almost layered graph: two vertices are to be aligned on the same layer iff they are at the same distance from an arbitrarily assigned root. The latter example can be considerably extended.

Call a drawing outerplanar if all the vertices lie on the outer face. An outerplanar graph is a graph admitting an outerplanar drawing. Note that this definition does not depend on whether straight line or curved drawings are considered. The set of edges on the boundary of the outer face does not depend on a particular outerplanar drawing. We will call such edges outer. All outerplanar drawings of a 2-connected outerplanar graph \(H\) are equivalent. In particular, this means that the set \(F(H)\) of the facial cycles bounding inner faces does not depend on a particular outerplanar drawing of \(H\). Let \(H^\gamma\) be a graph on the vertex set \(V(H^\gamma) = F(H)\) where two facial cycles are adjacent iff they share an edge. \(H^\gamma\) is a tree.

Given a graph \(G\) and its vertex \(v\), by \(G - v\) we denote the graph obtained from \(G\) by removal of \(v\). Suppose that \(G\) is connected. A vertex \(v\) is called a cutvertex if \(G - v\) is disconnected. Any maximal connected subgraph of \(G\) with no cutvertex is called a block of \(G\). Each block is either a 2-connected graph or a single-edge graph. Blocks are attached to each other in a tree-like fashion. \(G\) is outerplanar if outerplanar are all its blocks. Moreover, \(F(G)\) is the union of \(F(B)\) over all blocks \(B\) of \(G\) (where \(F(B) = \emptyset\) if \(B\) is a single-edge graph).

**Lemma 4.2.** Outerplanar graphs are almost layered.

**Proof.** Let \(\ell_1, \ell_2, \ell_3, \ldots\) be horizontal lines, listed in the upward order. Using induction on \(n\), we prove that for every \(n\)-vertex connected outerplanar graph \(G\) the

\(^1\)This extends the class of layered planar graphs considered in the literature. Layered planar graphs are not allowed to have edges on layers.
following is true:

1. For every $v \in V(G)$ and triangle $T \subset \mathbb{R}^2$ with vertices $a \in \ell_1$ and $b, c \in \ell_{n+1}$, $G$ has an almost layered outerplanar drawing $\pi : V(G) \to T$ with $\pi(v) = a$;

2. For every outer edge $uv$ and triangle $T \subset \mathbb{R}^2$ with vertices $a, b \in \ell_1$ and $c \in \ell_{n+1}$, $G$ has an almost layered outerplanar drawing $\pi : V(G) \to T$ with $\pi(v) = a$ and $\pi(u) = b$.

The proof of Claim 1 is split into three cases.

Case 1: $v$ is a cutvertex. Let $G_1, \ldots, G_s$ be the subgraphs into which $v$ splits $G$, where we include $v$ in each $G_i$. Split $T$ into $s$ parts $T_1, \ldots, T_s$ by rays from $a$. As $v(G_i) < v(G)$ for all $i$, we can use the induction assumption and draw each $G_i$ within its own $T_i$ in an almost layered fashion.

Case 2: $v$ has degree 1. Let $T' \subset T$ be a triangle with vertices $b, c$, and $a' \in \ell_2 \cap T$. Draw $G - v$ within $T'$ by induction, putting the neighbor $v'$ of $v$ at $a'$.

Case 3: $G - v$ is connected and $v$ has degree at least 2. This means that $v$ and all its neighbors belong to the same 2-connected block $H$ of $G$. Let $F_v(H)$ denote the set of the inner facial cycles of $H$ containing $v$. We put the vertices of all such cycles, except $v$, on $\ell_2 \cap T$. The order in which these vertices go along $\ell_2$ is determined by the path induced by $F_v(H)$ in $H^\gamma$. A crucial fact is that the remaining part of $G$ consists of outerplanar graphs attached to the edges and the vertices already drawn on $\ell_2$ and any two of these graphs are either vertex-disjoint or share one vertex, lying on $\ell_2$. All these fragments of $G$ can be drawn in an almost layered fashion above $\ell_2$ within $T$ by Claims 1 and 2 for graphs with smaller number of vertices.

To prove Claim 2, we distinguish two cases.

Case 1: $v$ or $u$ is a cutvertex. Without loss of generality, suppose that $v$ is a cutvertex. Let $G_1, \ldots, G_s$ be the subgraphs into which $v$ splits $G$ ( $v$ is included in each $G_i$). Suppose that $u$ is contained in $G_s$. Split $T$ into $s$ subtriangles $T_1, \ldots, T_s$ with a common vertex $a$ so that the other two vertices of $T_s$ are $b$ and the intersection
point of the line $\ell_n$ and the segment $bc$. We are able to draw each $G_i$ within $T_i$ in an almost layered fashion by induction.

Case 2: Neither $v$ nor $u$ is a cutvertex. This means that $v$ and $u$ belong to the same 2-connected block $H$ of $G$ together with their neighborhoods. We hence can proceed similarly to Case 3 above. More precisely, we use the fact that $F_v(H) \cup F_u(H)$ induce a path in $H^\gamma$ and, tracing through it, put the vertices of all cycles in $F_v(H) \cup F_u(H)$, except $v$ and $u$, on $\ell_2 \cap T$. The remaining parts of $G$ grow above $\ell_2$ by induction.

Fig. 2 shows an example of making an almost layered drawing by the recursive procedure resulting from the proof.

**Lemma 4.3.** For any almost layered graph $G$ we have $\tilde{v}(G) \geq v(G)/2$.

**Proof.** Let $\pi$ be an almost layered drawing of $G$ with layers $\ell_1, \ldots, \ell_s$, lying in the plane in this order. Let $\sigma_1, \ldots, \sigma_s$ be straight line segments of the layers containing all the vertices. Let $\ell$ be a horizontal line. Consider two redrawings of $\pi$.

To make a redrawing $\pi'$, we put $\sigma_1, \sigma_3, \sigma_5, \ldots$ on $\ell$ one by one. For each even index $2i$, we drop a perpendicular $p_{2i}$ to $\ell$ between the segments $\sigma_{2i-1}$ and $\sigma_{2i+1}$. We then put each $\sigma_{2i}$ on $p_{2i}$ so that $\sigma_{2i}$ is in the upper half-plane if $i$ is odd and in the lower half-plane if $i$ is even. It is clear that such a relocation can be done so that $\pi'$ is crossing-free (the whole procedure can be thought of as sequentially unfolding each strip between consecutive layers to a quadrant of the plane, see Fig. 3).

It is clear that the vertices on $\ell$ form a free collinear set: if the neighboring vertices of $\sigma_{2i-1}$ and $\sigma_{2i+1}$ are displaced, then $p_{2i}$ is to be shifted appropriately.

In the redrawing $\pi''$ the roles of odd and even indices are interchanged, that is, $\sigma_2, \sigma_4, \sigma_6, \ldots$ are put on $\ell$ and $\sigma_1, \sigma_3, \sigma_5, \ldots$ on perpendiculars (see Fig. 3). Thus, we have either $\tilde{v}(G, \pi') \geq v(G)/2$ or $\tilde{v}(G, \pi'') \geq v(G)/2$.
By Lemmas 4.2 and 4.3, we obtain the following corollary of Theorem 4.1.

**Corollary 4.4.** For any outerplanar graph $G$ we have $\hat{v}(G) \geq \sqrt{v(G)/2}$.

## 5 Further questions

1. How far or close are parameters $\hat{v}(G)$ and $\bar{v}(G)$? It seems that a priori we even cannot exclude equality. In particular, is it generally true that every collinear set in any straight line drawing is free?

2. We constructed examples of graphs with $\hat{v}(G) \leq \bar{v}(G) \leq O(n^{\sigma+\varepsilon})$ for a graph-theoretic constant $\sigma$, for which it is known that $0.69 < \sigma < 0.99$. Are there graphs with $\bar{v}(G) = O(\sqrt{n})$? If so, this could be considered a strengthening of the examples of graphs with $fix(G) = O(\sqrt{n})$ given in [1, 3, 6, 9]. Are there graphs with, at least, $\bar{v}(G) = O(\sqrt{n})$? If not, by Theorem 4.1 this would lead to an improvement of Bose et al.’s bound (2).

3. By Lemmas 4.2 and 4.3 we have $\hat{v}(G) \geq n/2$ for any outerplanar $G$. For which other classes of graphs do we have $\hat{v}(G) = \Omega(n)$ or, at least, $\bar{v}(G) = \Omega(n)$? In particular, is $\hat{v}(G)$ linear for series-parallel graphs (= partial 2-trees = graphs of treewidth at most 2)? What about planar graphs with bounded vertex degrees? Note that the graphs constructed in the proof of Theorem 3.3 have vertices with degree more than $n^\delta$ for some $\delta > 0$.

4. Since grid graphs are almost layered, we have for them $fix(P_s \times P_s) \geq \sqrt{n/2}$ where $n = s^2$. How tight is this lower bound? From [2, Corollary 4.1] we know that $fix(P_s \times P_s) = O((n \log n)^{2/3})$.

## References

[1] P. Bose, V. Dujmovic, F. Hurtado, S. Langerman, P. Morin, D.R. Wood. A polynomial bound for untangling geometric planar graphs. E-print: http://arxiv.org/abs/0710.1641 (2007).

[2] J. Cibulka. Untangling polygons and graphs. E-print: http://arxiv.org/abs/0802.1312 (2008).

[3] X. Goaoc, J. Kratochvíl, Y. Okamoto, C.S. Shin, A. Wolff. Moving vertices to make a drawing plane. In: Proc. of the 15-th International Symposium Graph Drawing. Lecture Notes in Computer Science, vol. 4875, pages 101–112. Springer-Verlag, 2007.

[4] B. Grünbaum. How to cut all edges of a polytope? The American Mathematical Monthly 79(8):890–895 (1972).

[5] B. Grünbaum, H. Walther. Shortness exponents of families of graphs. J. Combin. Theory A 14:364–385 (1973).
[6] M. Kang, O. Pikhurko, A. Ravsky, M. Schacht, O. Verbitsky. Obfuscated drawings of planar graphs. E-print: [http://arxiv.org/abs/0803.0858](http://arxiv.org/abs/0803.0858), 16 pages (2008).

[7] B. Jackson. Longest cycles in 3-connected cubic graphs. *J. Combin. Theory B* 41:17–26 (1986).

[8] J. Pach, G. Tardos. Untangling a polygon. *Discrete and Computational Geometry* 28:585–592 (2002).

[9] A. Spillner, A. Wolff. Untangling a planar graph. In: *Proc. of the 34-th International Conference on Current Trends Theory and Practice of Computer Science*. Lecture Notes in Computer Science, vol. 4910, pages 473–484. Springer-Verlag, 2008.

[10] O. Verbitsky. On the obfuscation complexity of planar graphs. *Theoretical Computer Science* 396(1–3):294–300 (2008).