GROWTH RATE OF ENDOMORPHISMS OF HOUGHTON’S GROUPS

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Abstract. A Houghton’s group $H_n$ consists of translations at infinity of a $n$ rays of discrete points on the plane. In this paper we study the growth rate of endomorphisms of Houghton’s groups. We show that if the kernel of an endomorphism $\phi$ is not trivial then the growth rate $\text{GR}(\phi)$ equals either 1 or the spectral radius of the induced map on the abelianization. It turns out that every monomorphism $\phi$ of $H_n$ determines a unique natural number $\ell$ such that $\text{GR}(\phi)$ equals $\ell$ for all $2 \leq n$.

1. Introduction

Let $G$ be a finitely generated group with a generating set $A$. Let $\phi : G \to G$ be an endomorphism. For any $g \in G$, let $|g|$ denote the word length of $g$, that is, the minimum length of a word over $A \cup A^{-1}$ which represents $g$. Then the growth rate of $\phi$ is defined to be (11)

$$\text{GR}(\phi) := \sup \left\{ \limsup_{k \to \infty} \left( |\phi^k(g)|^{1/k} \right) : g \in G \right\}.$$ 

The growth rate is well-defined, i.e., independent of the choice of a set of generators (12, p. 114).

The problem of determining the growth rate of a group endomorphism was initiated by R. Bowen in (1). The growth rate of an endomorphism is related to algebraic entropy and topological entropy (1, 12). Algebraic entropy of $\phi$ is defined by $h_{\text{alg}}(\phi) = \log \text{GR}(\phi)$. Note that $h_{\text{alg}}(\phi)$ provides a lower bound for the topological entropy of a continuous self map $f$ on a compact connected manifold $M$ which induces the endomorphism $\phi$ of $\pi_1(M)$.

A group theoretic approach is discussed in [7] including the result that $\text{GR}(\phi)$ is finite and bounded by the maximum length of the image of a generator. In case $\phi$ is an automorphism of a nilpotent group it is shown in [13] that $\text{GR}(\phi)$ coincides with the growth rate of the induced automorphism on its abelianization. In [8], the first author extends this to all endomorphisms of nilpotent groups. In the same article it was proven that $\text{GR}(\phi)$ is an algebraic number if $\phi$ is an endomorphism of a torsion-free nilpotent or lattices of Sol.

For $n \in \mathbb{N}$, a Houghton’s group $H_n$ consists of eventual translations on a disjoint union of $n$ copies of $\mathbb{N}$, each arranged along a ray emanating from the origin in the plane. The group $H_2$ is finitely generated but not finitely presented. In general, by the work of K. Brown ([2], $H_n$ has finiteness type $F_{n-1}$ but not $F_n$. For each $n$, $H_n$ fits into the short exact sequence

$$1 \to \text{FSym}_n \to H_n \to \mathbb{Z}^{n-1} \to 1,$$

where $\text{FSym}_n$ consists of permutations on the underlying set with finite supports. Note that every $H_n$ contains all finite groups. Our main result is the following.

**Theorem 1.1.** Suppose $\phi$ is an endomorphism of $H_n$, $2 \leq n$. Every monomorphism $\phi$ of $H_n$ determines $\ell \in \mathbb{N}$ such that GR($\phi$) = $\ell$. If $\phi$ is not a monomorphism then GR($\phi$) equals either 1 or the spectral radius of the induced map on the abelianization.
In Section 2, we define Houghton’s groups and review basic facts including explicit presentations. In Section 3, we deal with automorphisms of $\mathcal{H}_n$. We calculate $\text{GR}(\phi)$ by using the structure of $\text{Aut}(\mathcal{H}_n)$ which is well understood. If $\phi$ is an endomorphism with non trivial kernel we can reduce the calculation of $\text{GR}(\phi)$ to the problem of growth rate of endomorphisms on finitely generated free abelian groups. In Section 4, we use the result of [8] to prove the second part of our main theorem above. The rest of paper is used to understand monomorphisms of $\mathcal{H}_n$. For $3 \leq n$, $\mathcal{H}_n$ is generated by $g_2, \cdots, g_n$ where $g_i$ translates points on the first ray toward $i^{th}$ ray by 1. We use the relations of $\mathcal{H}_n$ together with the ray structure of the underlying set to characterize the behavior of monomorphisms in several steps carefully. It turns out that each monomorphism $\phi$ determines $\ell \in \mathbb{N}$ such that $\phi$ maps each generator $g_i$ of $\mathcal{H}_n$ to a translation of length $\ell$. This character of a monomorphism is essential for us to understand iterations $\phi^k$ applied to an element $f$ with a finite support, which is the main obstruction in calculating $\text{GR}(\phi)$. Perhaps Houghton’s groups are first examples of groups where relations were used extensively to calculate the growth rate of endomorphisms.

The following provides an effective calculation for $\text{GR}(\phi)$ ([7]), which will be used throughout the paper. For an endomorphism $\phi : G \to G$, let $|\phi^k|$ denote the maximum of $|\phi^k(a_i)|$ over a generating set $A = \{a_1, \cdots, a_m\}$ of $G$, then

$$
\text{GR}(\phi) = \lim_{k \to \infty} \left( |\phi^k| \right)^{1/k} = \inf_k \left\{ \left( |\phi^k| \right)^{1/k} \right\}.
$$

2. Houghton’s Groups $\mathcal{H}_n$

Let us use the following notational conventions. All bijections (or permutations) act on the right unless otherwise specified. Consequently $gh$ means $g$ followed by $h$. The conjugation by $g$ is denoted by $\mu(g), h^g = g^{-1}hg =: \mu(g)(h)$, and the commutator is defined by $[g, h] = ghg^{-1}h^{-1}$.

Our basic references are [9, 14] for Houghton’s groups and [3] for their automorphism groups. Fix an integer $n \geq 1$. For each $k$ with $1 \leq k \leq n$, let

$$
R_k = \left\{ me^{i \theta} \in \mathbb{C} \mid m \in \mathbb{N}, \ \theta = \frac{\pi}{2} + (k-1)\frac{2\pi}{n} \right\}
$$

and let $X_n = \bigcup_{k=1}^n R_k$ be the disjoint union of $n$ copies of $\mathbb{N}$, each arranged along a ray emanating from the origin in the plane. We shall use the notation $\{1, \cdots, n\} \times \mathbb{N}$ for $X_n$, letting $(k, p)$ denote the point of $R_k$ with distance $p$ from the origin.

A bijection $g : X_n \to X_n$ is called an eventual translation if the following holds:

There exist an $n$-tuple $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ and a finite set $K_g \subset X_n$ such that

$$
(2-1) \quad (k, p) \cdot g := (k, p + m_k) \quad \forall (k, p) \in X_n \setminus K_g.
$$

An eventual translation acts as a translation on each ray outside a finite set. For each $n \in \mathbb{N}$ the Houghton’s group $\mathcal{H}_n$ is defined to be the group of all eventual translations of $X_n$.

Let $g_i$ be the translation on the ray of $R_1 \cup R_i$ by 1 for $2 \leq i \leq n$. Namely,

$$(j, p) \cdot g_i = \begin{cases} 
(1, p - 1) & \text{if } j = 1 \text{ and } p \geq 2, \\
(i, 1) & \text{if } (j, p) = (1, 1), \\
i, p + 1 & \text{if } j = i, \\
(j, p) & \text{otherwise.}
\end{cases}
$$

Johnson provided a finite presentation for $\mathcal{H}_3$ in [11] and the second author gave a finite presentation for $\mathcal{H}_n$ with $n \geq 3$ in [14] as follows:

**Theorem 2.1** ([14] Theorem C). For $n \geq 3$, $\mathcal{H}_n$ is generated by $g_2, \cdots, g_n, \alpha$ with relations

$$
\alpha^2 = 1, \ (\alpha g_2)^3 = 1, \ [\alpha, \alpha g_2] = 1, \ \alpha = [g_i, g_j], \ \alpha^{g_i^{-1}} = \alpha^{g_j^{-1}}
$$

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for $2 \leq i \neq j \leq n$.

From the definition of Houghton’s groups, the assignment $g \in H_n \mapsto (m_1, \ldots, m_n) \in \mathbb{Z}^n$ defines a homomorphism $\pi = (\pi_1, \ldots, \pi_n) : H_n \to \mathbb{Z}^n$. Then we have:

**Lemma 2.2** ([14, Lemma 2.3]). For $n \geq 3$, we have $\ker \pi = [H_n, H_n]$.

Note that $\pi(g_i) \in \mathbb{Z}^n$ has only two nonzero values $-1$ and $1$,

$$\pi(g_i) = (-1,0,\ldots,0,1,0,\ldots,0)$$

where $1$ occurs in the $i^{th}$ component. Since the image of $H_n$ under $\pi$ is generated by those elements, we have that

$$\pi(H_n) = \left\{ (m_1, \ldots, m_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n m_i = 0 \right\},$$

which is isomorphic to the free Abelian group of rank $n - 1$. Consequently, $H_n$ ($n \geq 3$) fits in the following short exact sequence

$$1 \longrightarrow H'_n = [H_n, H_n] \longrightarrow H_n \xrightarrow{\pi} \mathbb{Z}^{n-1} \longrightarrow 1.$$

The above abelianization, first observed by C. H. Houghton in [9], is the characteristic property of $\{H_n\}$ for which he introduced those groups in the same paper. We may regard $\pi$ as a homomorphism $H_n \to \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ given by

$$\pi : g_i \mapsto (-1,0,\ldots,0,1,0,\ldots,0) \mapsto (0,\ldots,0,1,0,\ldots,0).$$

In particular, $\pi(g_2), \ldots, \pi(g_n)$ form a set of free generators for $\mathbb{Z}^{n-1}$.

As definition, $\mathcal{H}_1$ is the symmetric group itself on $X_1$ with finite support, which is not finitely generated. Furthermore, $\mathcal{H}_2$ is

$$\mathcal{H}_2 = \langle g_2, \alpha \mid \alpha^2 = 1, (\alpha \alpha^{g_2})^3 = 1, [\alpha, \alpha^{g_k}] = 1 \text{ for all } |k| > 1 \rangle,$$

which is finitely generated, but not finitely presented. It is not difficult to see that $\mathcal{H}_2 = F\text{Alt}_2$.

**Notation.**

- $\text{Sym}_n = \text{ the full symmetric group of } X_n$,
- $F\text{Sym}_n = \text{ the symmetric group of } X_n$ with finite support,
- $F\text{Alt}_n = \text{ the alternating group of } X_n$ with finite support.

**Theorem 2.3** ([3, Theorem 2.2]). For $n \geq 2$, we have

$$\text{Aut}(H_n) \cong H_n \rtimes \Sigma_n$$

where $\Sigma_n$ is the symmetric group that permutes $n$ rays isometrically.

Here are some other known results for the Houghton’s groups $\mathcal{H}_n$:

- K. S. Brown ([2]) showed that $\mathcal{H}_n$ has type $F_{n-1}$ but not $F_n$.
- By [14, Theorem 2.18], $\mathcal{H}_n$ is an amenable group.
- Röver [15] showed that for all $n \geq 1, r \geq 2, m \geq 1$, $\mathcal{H}_n$ embeds in Higman’s groups $G_{r,m}$ (defined in [10]), and in particular all Houghton’s groups are subgroups of Thompson’s group $V$. 

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3. Growth rates for automorphisms of \( H_n \)

In this section, we will study the growth rates for automorphisms of the Houghton’s groups \( H_n \). We refer to [1] for general information on growth rates for endomorphisms of finitely generated groups. Since \( H_1 = \text{FSym}_1 \) is not finitely generated, we must assume \( n \geq 2 \).

For \( n \geq 3 \), we fix a generating set \( \Gamma = \{ g_2, \ldots, g_n \} \) for \( H_n \). Let \(|g|\) denote the word length \(|g|_\Gamma \) of \( g \in H_n \) with respect to \( \Gamma \). With \( g_1 = \text{id} \), for any automorphism \( \phi = \mu(\sigma \gamma) \) we have

\[ g_i^\sigma = g_{\sigma(i)} g_\sigma(i) \text{ for } i = 2, \ldots, n, \]

\[ g_i^\gamma = g_i^* = \text{an eventual } g_i. \]

The second line follows from the fact that \( \gamma \) is a word of \( g_j^{1+1} \)'s and an observation that \( g_i^{-1} g_i g_j \) is the translation on \( R_1 \cup \{(1)\} \cup R_i \) by 1. Observe also that

(1) \((g_i^*)^{-1} = (g_i^{-1})^*, (g_i^{1+1} g_j)^* = (g_j^{1+1})^* g_j^*, \)

(2) \((g_i^*)^\sigma = (g_i^*)^*, (g_i^*)^\gamma = (g_i^*)^* = g_i^*. \)

In particular, we have \( \phi(g_i) = \phi(g_i^*) = \phi(g_i)^* \). Thus we have

\[ \phi^m(g_i^*) = g_{\sigma^m(i)} (1 - g_{\sigma^m(i)}^*). \]

This shows that if \( \sigma \) is of order \( m \), then \( \phi^m(g_i) = \phi^m(g_i^*) = g_i^* \). Remark also that if \( \gamma = 1 \), then \( \phi^m(g_i) = g_i^* \) for all \( 2 \leq i \leq n \), i.e., \( \phi^m = \text{id} \). Hence \( \text{GR}(\phi^m) = 1 \) and so \( \text{GR}(\phi) = 1 \). In fact, we have:

**Theorem 3.1.** For any automorphism \( \phi \) of \( H_n \) \((n \geq 3)\), we have \( \text{GR}(\phi) = 1 \).

**Proof.** Let \(|\gamma| = \ell \). Then we have \(|\mu(\gamma)^k| \leq 1 + 2\ell k \). Indeed, since \( \mu(\gamma)^k(g_i) = \gamma^{-k} g_i \gamma^k \), we have

\[ |\mu(\gamma)^k(g_i)| \leq |\gamma^{-k}| + |g_i| + |\gamma^k| \]

\[ = 1 + 2|\gamma|^k \leq 1 + 2k|\gamma| = 1 + 2k\ell. \]

Thus we have \(|\mu(\gamma)^k| \leq 1 + 2k\ell \). This implies that

\[ \text{GR}(\mu(\gamma)) = \lim_{k \to \infty} |\mu(\gamma)^k|^{1/k} \leq \lim_{k \to \infty} (1 + 2k\ell)^{1/k} = 1. \]

Let \( \phi = \mu(\sigma \gamma) \) with \( \sigma \) of order \( m \). Then

\[ \phi^m = \mu(\sigma \gamma)^m = \mu(\sigma^m \gamma \sigma^m \cdots \gamma \gamma) = \mu(\gamma \sigma^{m^{-1}} \cdots \gamma \sigma). \]

Consequently, we have that

\[ \text{GR}(\phi^m) = \text{GR}(\mu(\gamma \sigma^{m^{-1}} \cdots \gamma \sigma)) \leq 1. \]

Since \( \text{GR}(\phi)^m = \text{GR}(\phi^m) \) (this follows from definition), we have \( \text{GR}(\phi) \leq 1 \). If \( \text{GR}(\phi) < 1 \), then by [3] Lemma 2.3), \( \phi \) is an eventually trivial endomorphism. This is impossible because \( \phi \) is an automorphism, and so we must have \( \text{GR}(\phi) = 1 \). \( \square \)

Next, we consider \( H_2 \). By Theorem 2.3, again, every automorphism \( \phi \) of \( H_2 \) is induced from the conjugation by some element of \( \sigma \gamma \in H_2 \sigma_2 \subset \text{Sym}_2 \). We remark that the proof of the above theorem relies on only the fact that automorphisms are induced from conjugations. This makes possible to repeat the above proof verbatim for the set \( \Gamma = \{ g_2, \alpha \} \) of generators for \( H_2 \). Therefore we have:

**Theorem 3.2.** For any automorphism \( \phi \) of \( H_2 \), we have \( \text{GR}(\phi) = 1 \).
We recall the following results.

**Theorem 4.1 ([6], Theorem 8.1A]).** Let \( \Omega \) be any set with \(|\Omega| > 4\). Then the nontrivial normal subgroups of \( \text{Sym}(\Omega) \) are precisely: \( \text{FAlt}(\Omega) \) and the subgroups of the form \( \text{Sym}(\Omega, c) \) with \( n_0 \leq c \leq |\Omega| \). Here,

\[
\text{Sym}(\Omega, c) := \{ x \in \text{Sym}(\Omega) \mid |\text{supp}(x)| < c \}.
\]

In particular, \( \text{Sym}(\Omega, n_0) = \text{FSym}(\Omega) \), and so the nontrivial normal subgroups of \( \text{Sym}_n \) are precisely \( \text{FAlt}_n \) and \( \text{FSym}_n \).

**Theorem 4.2 ([16], 11.3.3]).** The nontrivial normal subgroups of \( \text{Sym}(\Omega, B) \) are the groups \( \text{Sym}(\Omega, D) \) with \( n_0 \leq D < B \leq 2^{|\Omega|} \) and \( \text{FAlt}(\Omega) \). In particular, we have:

1. The only nontrivial normal subgroup of \( \text{FSym}_n \) is \( \text{FAlt}_n \) (when \( B = n_0 \)).
2. The only nontrivial normal subgroups of \( \text{Sym}_n \) are \( \text{FSym}_n \) and \( \text{FAlt}_n \) (when \( B = 2^{n_0} \)).

**Corollary 4.3.** If \( N \) is a nontrivial normal subgroup of \( \mathcal{H}_n \), then \( N \cap \text{FSym}_n \) is either \( \text{FAlt}_n \) or \( \text{FSym}_n \). In particular \( N \) contains \( \text{FAlt}_n \).

**Proof.** If \( N \) is a normal subgroup of \( \mathcal{H}_n \), then \( N \cap \text{FSym}_n \) is a normal subgroup of \( \text{FSym}_n \). By Part (1) of Theorem 4.2 \( N \cap \text{FSym}_n \) is either 1, \( \text{FAlt}_n \), or \( \text{FSym}_n \).

If \( N \cap \text{FSym}_n = \text{FAlt}_n \), then \( \text{FAlt}_n \subset N \); if \( N \cap \text{FSym}_n = \text{FSym}_n \), then \( \text{FAlt}_n \subset \text{FSym}_n \subset N \).

We now consider the case \( N \cap \text{FSym}_n = 1 \). This implies that the surjection \( \mathcal{H}_n \to \mathbb{Z}^{n-1} \) maps \( N \) isomorphically into \( \mathbb{Z}^{n-1} \). If \( N \neq 1 \), then \( N \) contains an element \( g \in \mathcal{H}_n \) of infinite order. There exists a point \( x \in \text{supp}(g) \) whose orbit under \( g \) is infinite. Observe that for any transposition \( \beta \) exchanging two points in the orbit of \( x \), \( [\beta, g] = g^3 \) is a 3-cycle. In particular, for any transposition \( \beta \) exchanging two consecutive points \( x \) and \( y \) in \( \text{supp}(g) \), there exists a finite index normal subgroup \( N_x \) of \( \mathcal{H}_n \) such that \( x \notin N_x \). Since \( N_x \neq 1 \), Corollary 4.3 implies that \( \text{FAlt}_n \subset N_x \) for all \( x \in \mathcal{H}_n - \{1\} \); in particular, for \( x \in \text{FAlt}_n - \{1\} \). This is a contradiction. \( \square \)

**Corollary 4.4 ([5]).** The Houghton’s group \( \mathcal{H}_n \) is not residually finite.

**Proof.** Suppose that \( \mathcal{H}_n \) is residually finite. By definition, for any \( x \in \mathcal{H}_n - \{1\} \) there exists a homomorphism \( \phi \) from \( \mathcal{H}_n \) to a finite group such that \( \phi(x) \neq 1 \). Equivalently, for any \( x \in \mathcal{H}_n - \{1\} \) there exists a finite index normal subgroup \( N_x \) of \( \mathcal{H}_n \) such that \( x \notin N_x \). Since \( N_x \neq 1 \), Corollary 4.3 implies that \( \text{FAlt}_n \subset N_x \) for all \( x \in \mathcal{H}_n - \{1\} \); in particular, for \( x \in \text{FAlt}_n - \{1\} \). This is a contradiction. \( \square \)

**Remark 4.5.** The Houghton’s group \( \mathcal{H}_n \) is not co-Hopfian for \( n \geq 3 \) ([3]). That is, there is an injection which is not an isomorphism. Indeed, \( \phi : g_i \mapsto g_i^2 \), \( 2 \leq i \leq n \), defines such an injection. It is direct to check that \( \phi(r) = 1 \) for all relators \( r \) of the presentation in Theorem 2.1. For the injectivity of \( \phi \) one can realize \( \phi(\mathcal{H}_n) \) as disjoint union of two subgroups each of which is isomorphic to \( \mathcal{H}_n \). (See Example 5.6)

**Remark 4.6.** Recall that for a short exact sequence of groups \( 0 \to A \to G \to Q \to 1 \) with \( A \) abelian, the following are equivalent:

1. \( Q \) acts trivially on \( A \).
2. \( A \) lies in the center of \( G \).

In this case, we say that the short exact sequence is central. Remark that if \( A \cong \mathbb{Z}_2 \) then the exact sequence is always central. In fact, for any \( g \in G \) and \( a \neq 1 \) in \( A \), we must have that \( gag^{-1} = a \).

**Corollary 4.7 ([5]).** The Houghton’s group \( \mathcal{H}_n \) is Hopfian.
Proof. Let \( \phi : \mathcal{H}_n \to \mathcal{H}_n \) be an epimorphism, but not an isomorphism. Put \( K = \ker(\phi) \); then \( K \neq 1, \mathcal{H}_n/K \cong \mathcal{H}_n \) and from the proof of Corollary 4.3, \( K \cap \text{FSym}_n \) is either \( \text{FAlt}_n \) or \( \text{FSym}_n \).

If \( K \cap \text{FSym}_n = \text{FSym}_n \), then \( \text{FSym}_n \subset K \) and thus we have

\[
\mathcal{H}_n/\text{FSym}_n(\cong \mathbb{Z}^{n-1}) \longrightarrow \mathcal{H}_n/K \mathrel{\overset{\cong}{\longrightarrow}} \mathcal{H}_n.
\]

This implies that \( \mathcal{H}_n \) is abelian, which is absurd.

Consider next the case \( K \cap \text{FSym}_n = \text{FAlt}_n \). Then we have a commuting diagram

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\uparrow & \uparrow & \uparrow \\
1 & \longrightarrow & \text{FSym}_n/K \cap \text{FSym}_n \longrightarrow \mathcal{H}_n/K \longrightarrow \mathcal{H}_n/K \cdot \text{FSym}_n \longrightarrow 1 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
1 & \longrightarrow & \text{FSym}_n \longrightarrow \mathcal{H}_n \longrightarrow \mathbb{Z}^{n-1} \longrightarrow 1 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
1 & \longrightarrow & K \cap \text{FSym}_n \longrightarrow K \longrightarrow K/K \cap \text{FSym}_n \longrightarrow 1 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
1 & 1 & 1 & 1
\end{array}
\]

Since \( \text{FSym}_n/K \cap \text{FSym}_n = \text{FSym}_n/\text{FAlt}_n \cong \mathbb{Z}_2 \), the top row exact sequence is a central extension by Remark 4.6. This implies that \( \mathcal{H}_n/K \cong \mathcal{H}_n \) has a nontrivial center. But Lemma 4.8 below shows that \( \mathcal{H}_n \) has trivial center. □

**Lemma 4.8.** For all \( n \), the center of \( \mathcal{H}_n \) is trivial.

**Proof.** Assume that \( Z(\mathcal{H}_n) \) is nontrivial. By Corollary 4.3 \( \text{FAlt}_n \subset Z(\mathcal{H}_n) \). This implies that \( \text{FAlt}_n \) is abelian, a contradiction. □

Let \( \varphi \) be an endomorphism of \( \mathcal{H}_n \). In this section, we shall assume \( K = \ker(\varphi) \neq 1 \). [Since \( \mathcal{H}_n \) is Hopfian by Corollary 4.7 \( \varphi \) cannot be epic.] By Corollary 4.3 \( K \cap \text{FSym}_n \) is either \( \text{FAlt}_n \) or \( \text{FSym}_n \).

Consider first the case \( K \cap \text{FSym}_n = \text{FSym}_n \). Since \( \text{FSym}_n \subset K \), \( \varphi \) induces the following diagram

\[
\begin{array}{ccc}
\mathcal{H}_n & \longrightarrow & \mathcal{H}_n/\text{FSym}_n \longrightarrow \mathcal{H}_n/K \\
\downarrow \varphi & & \downarrow \hat{\varphi} \quad \downarrow \hat{\varphi} \\
\mathcal{H}_n & \longrightarrow & \mathcal{H}_n/\text{FSym}_n \longrightarrow \mathcal{H}_n/K
\end{array}
\]

where the horizontal maps are canonical surjections. It is immediate from the definition of the growth rate that

\[ \text{GR}(\varphi) \leq \text{GR}(\hat{\varphi}) \leq \text{GR}(\hat{\varphi}). \]

We claim now that \( \text{GR}(\varphi) = \text{GR}(\hat{\varphi}) = \text{GR}(\hat{\varphi}) \). This follows from the fact that the endomorphism \( \hat{\varphi} : \mathcal{H}_n/K \to \mathcal{H}_n/K \) induced by \( \varphi \) is simply the restriction of \( \varphi \) on the \( \varphi \)-invariant subgroup \( \varphi(\mathcal{H}_n) \) of \( \mathcal{H}_n \), \( \varphi|_{\varphi(\mathcal{H}_n)} : \varphi(\mathcal{H}_n) \to \varphi(\mathcal{H}_n) \). It is known that \( \text{GR}(\varphi) = \text{GR}(\varphi|_{\varphi(\mathcal{H}_n)}) = \text{GR}(\hat{\varphi}) \), see for example [4, Sect. 4]. Therefore, we have

\[ \text{GR}(\varphi) = \text{GR}(\hat{\varphi}). \]

Recall also that \( \text{GR}(\hat{\varphi}) \) is the spectral radius of the integer matrix determined by the endomorphism \( \hat{\varphi} \) of \( \mathcal{H}_n/\text{FSym}_n = \mathbb{Z}^{n-1} \).
Next we consider the case \( K \cap \text{FSym}_n = \text{FAlt}_n \). Since \( \text{FAlt}_n \subset K \), \( \varphi \) induces the following diagram
\[
\begin{array}{c}
\text{H}_n \longrightarrow \text{H}_n/\text{FAlt}_n \longrightarrow \text{H}_n/K \\
\downarrow \varphi \hspace{1cm} \downarrow \hat{\varphi} \hspace{1cm} \downarrow \tilde{\varphi} \\
\text{H}_n \longrightarrow \text{H}_n/\text{FAlt}_n \longrightarrow \text{H}_n/K
\end{array}
\]
where the horizontal maps are canonical surjections. Repeating the same argument as above, we have
\[
\text{GR}(\varphi) = \text{GR}(\hat{\varphi}).
\]
Since \( \text{FSym}_n \) is the subgroup of \( \text{H}_n \) consisting of all elements that have finite order, \( \text{FSym}_n \) is \( \phi \)-invariant. Consequently, \( \varphi \) induces the following diagram
\[
\begin{array}{c}
\text{FSym}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FSym}_n \\
\downarrow \phi' \hspace{1cm} \downarrow \tilde{\phi} \hspace{1cm} \downarrow \bar{\phi} \\
\text{FSym}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FSym}_n
\end{array}
\]
By \([8, \text{Lemma 2.5}]\), we have
\[
\text{GR}(\tilde{\phi}) = \max \{ \text{GR}(\phi'), \text{GR}(\bar{\phi}) \}.
\]
Remark also that \( \text{FSym}_n/\text{FAlt}_n = \mathbb{Z}_2 \), hence \( \text{GR}(\phi') = 0 \) or \( 1 \) depending on \( \phi' \) is trivial or not trivial (then \( \tilde{\phi} \) is the identity). If \( \text{GR}(\tilde{\phi}) = \text{sp}[\tilde{\phi}] < 1 \) then \( \phi \) is eventually trivial or \( \text{GR}(\bar{\phi}) = 0 \). This implies that \( \text{GR}(\bar{\phi}) \geq 1 \) if \( \tilde{\phi} \) is not eventually trivial. So we can summarize our discussion in this section as follows.

**Theorem 4.9.** Let \( \phi \) be an endomorphism of \( \text{H}_n \) with nontrivial kernel. Then \( \text{FSym}_n \) is \( \phi \)-invariant and so there results in the following commutative diagram
\[
\begin{array}{c}
\text{FSym}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FSym}_n \\
\downarrow \phi' \hspace{1cm} \downarrow \tilde{\phi} \hspace{1cm} \downarrow \bar{\phi} \\
\text{FSym}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FAlt}_n \longrightarrow \text{H}_n/\text{FSym}_n
\end{array}
\]
If \( \phi' \) is nontrivial and \( \bar{\phi} \) is eventually trivial then \( \text{GR}(\phi) = 1 \), and otherwise we have
\[
\text{GR}(\phi) = \text{sp}[\bar{\phi}].
\]

5. **Monomorphisms of \( \text{H}_n \)**

Let \( 2 \leq n \) and let \( \phi \) denote a monomorphism of \( \text{H}_n \) throughout this section unless otherwise stated. Let \( \text{supp} \, g \) denote the support of \( g \in \text{H}_n \). Recall from Theorem \ref{thm:presentation} that if \( n \geq 3 \) then \( \text{H}_n \) has the following presentation
\begin{align}
(5–1) \quad \alpha^2 &= 1, \ (\alpha \alpha^g)^3 = 1, \ [\alpha, \alpha^g] = 1, \ \alpha = [g_i, g_j], \ \alpha g_i^{-1} = \alpha g_j^{-1}
\end{align}
for \( 2 \leq i \neq j \leq n \).

For \( i \neq j \) and \( k \in \mathbb{N} \), the support of the transposition \( \alpha^{g_{i+1}} \) belongs to \( R_i \) and does not intersect \( \text{supp} \, g_j \). Consequently we have the following identities
\begin{align}
(5–2) \quad [\alpha^{g_{i+1}}, g_j] &= 1
\end{align}
for \( k \in \mathbb{N} \) and \( i \neq j \). By the same reason, we also have
\begin{align}
(5–3) \quad [\alpha, \alpha^{g_{i+1}}] &= 1
\end{align}
for all \( k \in \mathbb{N} \). Note that the actions of \( g_i^{-1} \) and \( g_j^{-1} \) coincide on \( R_1 \). So we have identities

\[
\alpha g_i^{-k} = \alpha g_j^{-k}
\]

for all \( 2 \leq i, j \leq n \) and \( k \in \mathbb{N} \).

We say \( P \in \text{supp} g \) is an essential point of \( g \) if its orbit under \( g \) is infinite. Let \( E \text{supp} g \) denote the set of essential points of \( g \). For two elements \( g, h \in \mathcal{H}_n \), we say \( g \) intersects \( h \) if \( \text{supp} g \cap \text{supp} h \neq \emptyset \). Similarly we say \( g \) intersects \( X \subset X_n \) if \( \text{supp} g \cap X \neq \emptyset \).

For each pair \( 2 \leq i \neq j \leq n \), \( [\phi(g_i), \phi(g_j)] = \phi(\alpha) \) is nontrivial and has order 2 for a monomorphism \( \phi \). So \( \phi(\alpha) \) can be written

\[
\phi(\alpha) = \tau_1 \cdots \tau_t
\]

as a product of commuting transpositions. Let \( T_0 \) denote the set of transpositions in \( [5–5] \), and let \( S_\alpha = \text{supp} \phi(\alpha) \). Note that each \( P \in S_\alpha \) determines a unique transposition in \( T_0 \) which moves \( P \). Let \( T_i \) be the collection of transpositions \( \tau \) in \( [5–5] \) with \( \text{supp} \tau \cap \text{supp} \phi(g_i) \neq \emptyset \), \( 2 \leq i \leq n \).

**Lemma 5.1.** Suppose \( \phi \) is a monomorphism of \( \mathcal{H}_n \) with \( n \geq 2 \). For each \( i \), \( T_i \neq \emptyset \).

**Proof.** We first consider a monomorphism when \( n \geq 3 \). Fix \( i \). The conjugations \( \alpha g_i^k \) are all distinct for \( k \in \mathbb{Z} \). A monomorphism \( \phi \) induces a bijection between infinite sets \( \{ \alpha g_i^k \mid k \in \mathbb{Z} \} \leftrightarrow \{ \phi(\alpha g_i^k) \mid k \in \mathbb{Z} \} = A \). Let us denote by \( \beta_k \) conjugation

\[
\beta_k = \phi(\alpha) g_i^k \tau_1 \cdots \tau_t g_i^{-k} = (\tau_1 \cdots \tau_t)^{\phi(g_i)^k}
\]

If all points of \( \text{supp} \phi(\alpha) \) are not essential points of \( \phi(g_i) \), then the set \( A \) must be finite since \( \phi(g_i)^k \) conjugates \( \phi(\alpha) \) to itself for some integer \( k \neq 0 \). Suppose that all points of \( S_\alpha \) are not essential points of \( \phi(g_i) \). If \( S_\alpha \cap \text{supp} \phi(g_i) = \emptyset \), then it is clear that \( \phi(\alpha)^{\phi(g_i)^k} = \phi(\alpha) \). Next consider \( S_\alpha \cap \text{supp} \phi(g_i) = \{ P_1, \ldots, P_s \} \neq \emptyset \). By the supposition, each \( P_j \) is not an essential point of \( \phi(g_i) \), i.e., \( (P_j)\phi(g_i)^{k_j} = P_j \) for some \( k_j > 0 \). Let \( k_0 = \text{lcm}(k_1, \ldots, k_s) \). Hence \( (P_j)\phi(g_i)^{k_0} = P_j \) for all \( j = 1, \ldots, s \) and so \( (P)\phi(g_i)^{k_0} \sigma = P_j \) for all \( P \in S_\alpha \). This shows that \( \beta_k = \phi(\alpha) \).

Let \( P \in \text{supp} \phi(\alpha) \) be an essential point of \( \phi(g_i) \). Let \( \tau' \) denote the unique transposition in \( [5–5] \) which moves \( P \) to \( Q = (P)\tau' \). We claim that \( Q \) is also an essential point of \( \phi(g_i) \). Assume the contrary that

\[
(Q)\phi(g_i)^{k_0} = Q
\]

for some integer \( k_0 \neq 0 \) \( (k_0 = 1 \text{ when } Q \text{ is fixed by } \phi(g_i), \text{ and } |k_0| \geq 2 \text{ when } Q \in \text{supp } \phi(g_i)) \). We need to examine \( \beta_k \) when \( k = 0 \). Fix \( m \), \( \phi(g_i)^{mk_0} \) fixes \( Q \) for each nonzero \( m \in \mathbb{Z} \), \( \phi(g_i)^{mk_0} \) conjugates \( \tau' = (P)Q \) to the transposition, denoted by \( \tau'_m \), which exchanges \( (P)\phi(g_i)^{mk_0} \) and \( Q \). See Figure 1. Observe that

\[
Q \in \text{supp} \tau' \cap \text{supp} \tau'_m \subset S_\alpha \cap \text{supp} \beta_{mk_0}
\]

for each nonzero \( m \in \mathbb{Z} \). To draw a contradiction, we use identities \( 5–3 \) which say in particular that \( \phi(\alpha) \) commutes with \( \beta_k \) for all \( k = mk_0 \) with \( |mk_0| \geq 2 \). We apply Lemma 5.2 to see that the intersection \( I_m = S_\alpha \cap \text{supp} \beta_{mk_0} \) satisfies that

\[
(I_m)\phi(\alpha) = I_m = (I_m)\beta_{mk_0}
\]

for all \( m \) with \( |mk_0| \geq 2 \). In particular, \( S_\alpha \) contains \( (Q)\beta_{mk_0} = (Q)\tau'_m = (P)\phi(g_i)^{mk_0} \) for infinitely many \( m \in \mathbb{Z} \). Since \( P \) is an essential point of \( \phi(g_i) \), \( S_\alpha \) must be an infinite set. This contradicts that an element \( \phi(\alpha) \in \mathcal{F}\text{Sym}_n \) has a finite support. Therefore \( Q \) is also an essential point of \( \phi(g_i) \), and hence the transposition \( \tau' \) exchanges two essential points \( P \) and \( Q \) of \( \phi(g_i) \).

For \( n = 2 \), recall that \( \mathcal{H}_2 \) has a presentation

\[
\mathcal{H}_2 = \langle g_2, \alpha \mid \alpha^2 = 1, (\alpha a g_2)^3 = 1, [\alpha, \alpha g_2^j] = 1 \text{ for all } |k| > 1 \rangle
\]
It follows from analogous arguments applied to $g_2$ with commutation relations above that $T_1 \neq \emptyset$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$\tau_m'$ moves $Q$ to $(P)\phi(g_i)^{mk_0}$ for infinitely many $m \in \mathbb{Z}$.}
\end{figure}

**Lemma 5.2.** Suppose two involutions $\beta, \gamma \in \text{FSym}_n$ commute. Then $I = (I)\beta = (I)\gamma$ where $I = \text{supp} \beta \cap \text{supp} \gamma$.

**Proof.** We may assume the intersection $I = \text{supp} \beta \cap \text{supp} \gamma$ is non-empty. Since $\beta$ is an involution, $B = I \cup (I)\beta$ is a $\beta$-invariant subset of $\text{supp} \beta$. Similarly $C = I \cup (I)\gamma$ satisfies $(C)\gamma = C$. Let $\beta_B$ and $\gamma_C$ denote the restrictions of $\beta$ and $\gamma$ on $B$ and $C$ respectively. We denote by $\beta_{B^c}$ and $\gamma_{C^c}$ the restrictions on the complements $B^c = \text{supp} \beta \setminus B$ and $C^c = \text{supp} \gamma \setminus C$ respectively. We can write

$$
\beta = \beta_B \cdot \beta_{B^c} \quad \text{and} \quad \gamma = \gamma_C \cdot \gamma_{C^c}.
$$

Note that $B^c \cap \text{supp} \gamma = \emptyset$ and $C^c \cap \text{supp} \beta = \emptyset$.

Since $\beta^\gamma = \beta$ and

$$
\beta^\gamma = (\beta_B \cdot \beta_{B^c})^\gamma = (\beta_B)^\gamma \cdot (\beta_{B^c})^\gamma = (\beta_B)^\gamma \cdot (\beta_B)^{\gamma_C} = (\beta_B)^{\gamma_C} \cdot \beta_{B^c},
$$

we have $(\beta_B)^{\gamma_C} \cdot \beta_{B^c} = \beta = \beta_B \cdot \beta_{B^c}$. So, $\gamma_C$ conjugates $\beta_B$ to itself, and $B$ is an invariant set of $\gamma_C$. Similarly $\gamma^\beta = \gamma$ implies that $(\gamma_C)^{\beta_B} = \gamma_C$.

We first show $B \subset C$. If $P \in B \setminus C \subset (I)\beta \setminus (I)\gamma$, there exists $Q \in I$ such that $P = (Q)\beta_B$. Observe that $P$ and $Q$ determine a unique transposition in the cycle decomposition of $\beta_B$ such that $P = (Q)\tau$.

Since $\gamma_C$ moves $Q$ to $Q' \neq P$ while fixing $P \notin C$, $\gamma_C$ conjugates $\tau$ to $\tau^{\gamma_C} \neq \tau$ which exchanges $P$ and $Q'$. However this contradicts the uniqueness of the transposition $\tau$ of $\beta_B$. In other words $\gamma_C$ fails to conjugate $\beta_B$ to itself. Therefore $(I)\beta \subset (I)\gamma$ and so $B \subset C$.

For the converse $(I)\gamma \subset (I)\beta$ or $C \subset B$, one can use analogous arguments applied to $\gamma^\beta = \gamma$. In all,

$$(I)\beta = (I)\gamma, \quad B = I \cup (I)\beta = I \cup (I)\gamma = C.$$

Furthermore, if $P \in (I)\beta = (I)\gamma$ then $P = (Q)\beta = (Q')\gamma$ for some $Q, Q' \in I$, thus $P \in \text{supp} \beta \cap \text{supp} \gamma = I$. The reverse inclusion follows since $\beta$ and $\gamma$ are involutions, hence $(I)\beta = (I)\gamma = I$. \qed

**Remark 5.3. (Squares on $I$)** An interesting consequence of Lemma 5.2 is that a point $P \in I$ with $(P)\beta \neq (P)\gamma$ determines a certain square (or 4-cycle of transpositions) on $I$. Let $\tau_1$ and $\tau'_1$ be the transpositions of $\beta$ and $\gamma$ respectively which moves $P$ to distinct points. The Lemma 5.2 says $Q = (P)\tau_1 \in I \subset \text{supp} \gamma$ and $R = (P)\tau'_1 \in I \subset \text{supp} \beta$. Let $\tau_2$ and $\tau'_2$ be the unique transpositions of $\beta$ and $\gamma$ respectively so that $R \in \text{supp} \tau_2$ and $Q \in \text{supp} \tau'_2$. The intersection sup $\tau_2 \cap \text{supp} \tau'_2$ contains a point $S$ since $\beta \gamma = \gamma \beta$ applied to $P$ implies

$$
(P)\beta \quad (P)\gamma \\
\beta \quad \gamma
$$

Applying alternate compositions of $\beta$ and $\gamma$, we have a square on the four points $P \xrightarrow{\tau_1} R \xrightarrow{\tau_2} S \xrightarrow{\tau'_2} Q \xrightarrow{\tau_1} P$. Reading off four transpositions of $\beta$ and $\gamma$ in order we have a 4-cycle $\tau_1 - \tau_2 - \tau'_2 - \tau_1$.
of transpositions. (This should not be confused with a 4-cycle of $FSym_n$.) Figure 2 illustrates the square on four points and corresponding 4-cycle of transpositions.

\[
P \quad \tau_1 \quad Q
\]
\[
R \quad \tau'_1 \quad \tau' \quad S
\]

**Figure 2.**

Next we consider a useful necessary condition when two involutions $\beta, \gamma \in FSym_n$ form $\beta \gamma$ of order 3. Suppose $\beta$ and $\gamma$ have cycle decompositions

\[
\beta = \tau_1 \cdots \tau_k \quad \text{and} \quad \gamma = \tau'_1 \cdots \tau'_\ell.
\]

Let $S = \{\tau_1, \ldots, \tau_k\}$ and $T = \{\tau'_1, \ldots, \tau'_\ell\}$, and let $\Psi(S)$ and $\Psi(T)$ denote the power sets of $S$ and $T$ respectively. We define assignments

\[
\xi : S \rightarrow \Psi(T) \quad \text{and} \quad \chi : T \rightarrow \Psi(S)
\]

so that $\text{supp } \tau_i$ has non-trivial intersection with each support of $\tau'_j \in \xi(\tau_i)$. The assignment $\chi(\tau_i)$ is defined by the same manner after swapping $S$ and $T$. As we shall see below the cardinalities of $\xi(*)$ and $\chi(*)$ are either one or two. Let $S_1 \subseteq S$ and $T_1 \subseteq T$ be defined by

\[
S_1 = \{\tau_i \in S : |\xi(\tau_i)| = 1\}, \quad T_1 = \{\tau'_j \in T : |\chi(\tau'_j)| = 1\}.
\]

**Lemma 5.4.** With the notations for two involutions $\beta$ and $\gamma$ such that $\beta \gamma$ has order 3 as above, $1 \leq |\xi(\tau_i)| \leq 2$ and $1 \leq |\chi(\tau'_j)| \leq 2$ for each $\tau_i \in S$ and $\tau'_j \in T$. There exists a bijection $S_1 \leftrightarrow T_1$ where correspondence is given by $\xi(\tau_i) = \{\tau'_j\} \Leftrightarrow \chi(\tau'_j) = \{\tau_i\}$.

**Proof.** Fix a transposition $\tau_i \in S$ which exchanges two points $P$ and $Q$. It is obvious that $T$ contains at most two transpositions which move $P$ or $Q$. So the cardinality of $\xi(\tau_i)$ is at most 2. To show $|\xi(\tau_i)| \neq 0$, assume that every transposition of $T$ fixes $P$ and $Q$. For convenience we write $\beta = \tau_i \cdot \beta'$ where $\beta' = \tau_1 \cdots \tau_i \cdots \tau_k$ fixes $P$ and $Q$. If $\gamma$ fixes $P$ and $Q$ then the product $(\beta \gamma)^3$ can be written as

\[
(\beta \gamma)^3 = (\tau_i \beta' \cdot \gamma)^3 = \tau_i^3 (\beta' \gamma)^3 = \tau_i (\beta' \gamma)^3
\]

since $\tau_i$ commutes with $\beta'$ and $\gamma$. The support of $\tau_i (\beta' \gamma)^3$ contains $P$ and $Q$ and so it is not empty. This means that $\beta \gamma$ is not an element of order 3. We have shown that $|\xi(\tau_i)|$ equals 1 or 2. For the cardinality of $\chi(*)$ one applies analogous arguments after swapping $\beta$ and $\gamma$.

To establish the correspondence $S_1 \leftrightarrow T_1$, suppose $\xi(\tau_i) = \{\tau'_j\}$, i.e., $\tau'_j \in T$ is the unique transposition which moves $P$ or $Q$. If $\tau_i = \tau'_j$ it is obvious that $\xi(\tau_i) = \{\tau'_j\} \Leftrightarrow \chi(\tau'_j) = \{\tau_i\}$. We assume $\tau_i \neq \tau'_j$ and show that $\chi(\tau'_j) = \{\tau_i\}$ as follows. It suffices to show that $|\chi(\tau'_j)| = 1$ since $\tau_i \in \chi(\tau'_j)$. If $|\chi(\tau'_j)| = 2$ then $S$ contains a transposition $\tau \neq \tau_i$, whose support intersects supp $\tau'_j$ nontrivially. Since $\tau'_j$ has to move $P$ or $Q$, so we may assume $\tau'_j$ moves $P$. Let $P' = (P') \tau$. Then $\tau$ exchanges $P'$ and $R = (P') \tau$, and the point $Q$ is fixed by $\gamma$. Applying the product $\beta \gamma$ repeatedly

...
to the point $R$ we have $(R)(\beta\gamma)^3 = P_1' \neq R$ since

$$R \xrightarrow{\beta} (R)\beta = (R)\tau = P' \xrightarrow{\gamma} (P')\gamma = (P')\tau_j' = P,$$

$$P \xrightarrow{\beta} (P)\beta = (P)\tau_i = Q \xrightarrow{\gamma} (Q)\gamma = Q,$$

$$Q \xrightarrow{\beta} (Q)\beta = (Q)\tau_i = P \xrightarrow{\gamma} (P)\gamma = (P)\tau_j' = P'.$$

So $R \in \text{supp} (\beta\gamma)^3 \neq \emptyset$. This contradicts that the product $\beta\gamma$ has order 3. Figure 3 illustrates this situation.

So far we have shown that $\xi(\tau_i) = \{\tau_j\}$ implies $\chi(\tau_j) = \{\tau_i\}$. The converse follows because a symmetric argument shows that $\gamma\beta = (\beta\gamma)^{-1}$ fails to have order 3 if $\chi(\tau_j) = \{\tau_i\}$ but $|\xi(\tau_i)| = 2$. Therefore $\xi(\tau_i) = \{\tau_j\} \Leftrightarrow \chi(\tau_j) = \{\tau_i\}$ determines the correspondence $S_1 \leftrightarrow T_1$. □

**Figure 3.**

**Remark 5.5. (Hexagons on $I$)** To understand the relation between complements of $S_1$ and $T_1$ consider assignments $\xi$ and $\chi$ on

$$S_2 = S \setminus S_1 \text{ and } T_2 = T \setminus T_1.$$

Lemma 5.4 implies that $|\xi(\tau_i)| = 2$ for all $\tau_i \in S_2$, and conversely $|\chi(\tau_j)| = 2$ for all $\tau_j \in T_2$. It is interesting to see that each element $\tau_i \in S_2$ involves a certain hexagon (or 6-cycle of transpositions) on $I = \text{supp} \beta \cap \text{supp} \gamma$. Suppose $\tau_i \in S_2$ exchanges $P$ and $Q$. Let us describe the other 5 transpositions. Let $\xi(\tau_i) = \{\tau_j'\}$ where $\tau_j'$ exchanges $P$ and $(P)\tau_j' = \tau_i$ and $\tau_j'$ exchanges $Q$ and $(Q)\tau_j' = Q_1$. Since $|\chi(\tau_j')| = 2$ and $\tau_i \in \chi(\tau_j')$, $\chi(\tau_j')$ contains $\tau_{P_1} \in S_2$ which exchanges $P_1$ and $P_2 = (P_1)\tau_{P_1}$. Similarly $\chi(\tau_j')$ contains two transpositions $\tau_i$ and $\tau_{Q_1} \in S_2$ which exchanges $Q_1$ and $Q_2 = (Q_1)\tau_{Q_1}$. One can check that $\tau_{P_1} \neq \tau_{Q_1}$. If $\tau_{P_1} = \tau_{Q_1}$ the four transpositions $\tau_i$, $\tau_j'$, $\tau_{P_1} = \tau_{Q_1}$, and $\tau_j'$ form a square as described in Remark 5.3. It is direct to see that $\{P_1, Q_1\} \subset \text{supp} (\beta\gamma)^3$, which contradicts that $\beta\gamma$ has order 3. So far we have 5 transpositions $\tau_i, \tau_j', \tau_j', \tau_{P_1}, \tau_{Q_1}$ and $\tau_{Q_1}$. Next we check that $\tau_{P_1}$ and $\tau_{Q_1}$ determine a unique transposition $\tau_j' \in T_2$, so that $\tau_j' \in \xi(\tau_{P_1}) \cap \xi(\tau_{Q_1})$, namely $\tau_j'$ exchanges $P_2$ and $Q_2$. Applying $\beta\gamma$ to the point $Q_2$ twice we have

$$Q_2 \xrightarrow{\beta} (Q_2)\beta = (Q_2)\tau_{Q_1} = Q_1 \xrightarrow{\gamma} (Q_1)\gamma = (Q_1)\tau_j' = Q,$$

$$Q \xrightarrow{\beta} (Q)\beta = (Q)\tau_i = P \xrightarrow{\gamma} (P)\gamma = (P)\tau_j' = P_1.$$

Since $\beta\gamma$ has order 3 we must have $(Q_2)(\beta\gamma)^3 = (P_1)\beta\gamma = Q_2$. The later identity means that

$$(P_1)\tau_{P_1}\gamma = (P_2)\gamma = (P_2)\tau_j' = Q_2.$$

Thus the transposition $\tau_j'$ exchanges $P_2$ and $Q_2$, and so it is the unique transposition of $T_2$ such that $\tau_j' \in \xi(\tau_{P_1}) \cap \xi(\tau_{Q_1})$ as desired. Alternate composition $\beta$-and-then-$\gamma$ applied to $P$ determines
6 points

\[ P \xrightarrow{\tau_i} Q \xrightarrow{\tau'_Q} Q_1 \xrightarrow{\tau Q_1} Q_2 \xrightarrow{\tau'_j} P_2 \xrightarrow{\tau P_2} P_1 \xrightarrow{\tau'_P} P. \]

The six transpositions in order form a hexagon or 6-cycle of transpositions: \( \tau_i - \tau'_Q - \tau Q_1 - \tau'_j - \tau P_1 - \tau'_P \).

(Again, one should distinguish this from a 6-cycle of \( \mathcal{H}_n \).) Figure 4 illustrates the hexagon on four points and corresponding 6-cycle of transpositions.

**Partial translations.** For \( g \in \mathcal{H}_n \), it is useful to decompose \( g \) into essential part and finitary part. Let \( E(g) \) and \( F(g) \) denote the restrictions of \( g \) on \( \text{Esupp} \ g \) and \( g \setminus \text{Esupp} \ g \) respectively. As an element of \( \text{Sym}_n \), the restriction of \( g \) on \( \text{Esupp} \ g \) has a cycle decomposition. The set \( \text{Esupp} \ g \) is partitioned into orbits of essential points of \( g \). The element \( g \) restricts on each orbit to define an infinite cycle, which will be called a partial translation of \( g \). Denoting partial translations of \( g \) by \( p_1, \cdots, p_\ell \), we can write \( g \) as

\[
(5–9) \quad g = p_1 \cdots p_\ell \cdot f
\]

where \( \text{supp} \ f \subseteq g \setminus \text{Esupp} \ g \). Partial translations of \( g \) commute with each other. In case \( g \) has no essential points, \( (5–9) \) becomes \( g = f \). Note that a partial translation is not an element of \( \mathcal{H}_n \) in general. For example, \( g_2 \) has two partial translations, which are not eventual translations.

As an infinite cycle on \( X_n \), each partial translation \( p \) can be realized as an embedding \( \mathbb{Z} \hookrightarrow X_n \). Picking a base point \( x_0 \in \text{supp} \ p \) we identify \( \text{supp} \ p = \{(x_0)p^k | k \in \mathbb{Z}\} \) with \( \mathbb{Z} \). Let \( [k]_p \) denote the point of \( \text{supp} \ p \) corresponding to \( k \in \mathbb{Z} \):

\[
(5–10) \quad (x_0)p^k \leftrightarrow [k]_p.
\]

Under the identification \( (5–10) \), a partial translation \( p \) of \( g \) translates points on \( \text{supp} \ p \) by +1 in the above identification, i.e.,

\[
(5–11) \quad [k]_p g^m = [k]_p p^m \leftrightarrow [k + m]_p
\]

for each \( m \in \mathbb{Z} \).

By the definition \( (2–1) \) an eventual translation \( g \in \mathcal{H}_n \) acts as a translation by \( m_i \) on \( R_i \) outside a finite set \( F_g \) where \( \pi(g) = (m_1, \cdots, m_n) \). We call a ray \( R_i \) a source of \( g \) if \( m_i < 0 \), and we call a ray \( R_i \) a target of \( g \) if \( m_i > 0 \). As a restriction of \( g \) on one of its orbits, a partial translation \( p \) of \( g \) has unique source and target so that \( p \) moves points from the source towards the target. More precisely there exist exactly two rays \( R_p^– \) and \( R_p^+ \) in \( \{R_1, \cdots, R_n\} \), and \( k_0 \in \mathbb{N} \) such that

\[
(5–12) \quad [-k]_p \in R_p^– \text{ and } [k]_p \in R_p^+\]

for all \( k \geq k_0 \).

Note that the generators \( g_2, \cdots, g_n \) of \( \mathcal{H}_n \) are all partial translations. Generators \( g_i \)’s share a unique source \( R_1 \) on which the actions of \( g_i^{-1} \)’s coincide as translations by +1. Moreover those generators have all distinct targets. Intuitively this behavior can be characterized as, for each pair \( i \neq j \),
(1) two actions of $g_i$ and $g_j$ are identical on $R_1 \setminus (1,1)$,
(2) once $g_i$ and $g_j$ ‘diverge’ from $(1,1)$ then they never meet again,
(3) the commutator $[g_i, g_j] = \alpha$ is the transposition which exchanges $(1,1)$ and $(1,2)$.

Note that $supp \phi$ consists of two points which are the last two points before $g_i$ and $g_j$ diverge. It turns out that generators $\phi(g_2), \ldots, \phi(g_n)$ of $\phi(\mathcal{H}_n)$ follow the same rule provided $\phi$ is a monomorphism. See Lemmas 5.7 and 5.8.

**Example 5.6.** Consider the monomorphism $\phi: \mathcal{H}_3 \to \mathcal{H}_3, g_i \mapsto g_i^2$ for $i = 2, 3$. The generators $g_2^2$ and $g_3^2$ of $\phi(\mathcal{H}_3)$ share a common source $R_1$ and their targets, $R_2$ and $R_3$ respectively, are distinct. Both $g_2^2$ and $g_3^2$ are products of two partial translations. Let $p_i$ and $p_i'$ denote the two partial transpositions of $g_i^2$ moving $(1,1)$ and $(1,2)$ respectively for $i = 2, 3$. Note that $R_1$ is the unique source for all four partial translations. The ray $R_2$ is the target of $p_2$ and $p_2'$, and $R_3$ is the target of $p_3$ and $p_3'$. As in (5–10), we label $supp p_2$ and $supp p_3$ by $\mathbb{Z}$ with a base point $(1,3)$:

$$(1,3) \leftrightarrow [0]_{p_2} \text{ and } (1,3) \leftrightarrow [0]_{p_3}.$$ It is direct to check that $[k]_{p_2} = [k]_{p_3}$ for all $k \leq 0$ and so $p_2$ and $p_3$ agree on $\{[k]_{p_2} | k \leq 1\}$. Note that $[p_2, p_3]$ is the transposition which exchanges $(1,3) \leftrightarrow [0]_{p_2}$ and $(1,1) \leftrightarrow [1]_{p_2}$, the last two consecutive points before they diverge. The partial translations $p_2'$ and $p_3'$ follow the same rule: they agree on the source $R_1$ except $(1,2)$, and $[p_2', p_3']$ exchanges $(1,2)$ and $(1,4)$, the last two consecutive points before they diverge.

We also remark that two pairs of partial translations induce decomposition of the underlying set $X_3$. Let $X$ denote the orbit of $(1,1)$ under $\phi(\mathcal{H}_3)$. In other words, $X$ is the union of two supports of $p_2$ and $p_3$. Similarly $p_2'$ and $p_3'$ determine a subset $X'$ which is the orbit of $(1,2)$ under $\phi(\mathcal{H}_3)$. Obviously two invariant subsets provide a partition $X_3 = X \sqcup X'$. The corresponding decomposition of $\phi(\mathcal{H}_3)$ says that $\phi(\mathcal{H}_3)$ is the disjoint union of two identical subgroups $H_3$ and $H'_3$ where $H_3$ and $H'_3$ are generated by $p_i$ and $p_i'$ respectively, $i = 2, 3$. The isomorphism between $H_3$ and $\mathcal{H}_3$ can be described by the obvious bijection between $X$ and $X_3$. In particular, $\phi$ is injective.

**Lemma 5.7.** A monomorphism $\phi: \mathcal{H}_n \to \mathcal{H}_n$ determines $\ell \in \mathbb{N}$ and a ray $R_i$ such that $R_i$ is a unique common source of $\phi(g_i)$, and $\phi(g_i)$ has translation length $–\ell$ on $R_i$ for all $2 \leq i \leq n$. Moreover targets of $\phi(g_i)$ and $\phi(g_j)$ do not share a ray if $2 \leq i \neq j \leq n$.

**Proof.** The group $H_2$ has only one generator $g_2$ of infinite order. So $\phi(g_2)$ satisfies the above automatically. To deal with a monomorphism $\phi$ of $\mathcal{H}_n$ with $3 \leq n$ recall notations: $S_\alpha = supp \phi(\alpha), T_\alpha = \text{the set of all transpositions in (5–3)}, T_i \subset T_\alpha$ consisting of transpositions which intersect $Esupp(\phi(g_i))$. Being a monomorphism, $\phi$ implies that the order of every $\phi(g_i)$ is infinite. So $\phi(g_i)$ contains at least one partial translation when written as in (5–9). Let $P_i$ denote the set of all partial translations of $\phi(g_i)$.

Fix $i$. Lemma 5.1 says that $T_i \neq \emptyset$, and so $P_i$ contains a partial translation $p$ such that

$$(5–13) \quad supp p \cap S_\alpha \neq \emptyset.$$ Consider the source $R_i$ of $p$ with the property (5–12). Since $R_i$ is also a source of $\phi(g_i)$, $\phi(g_i)$ has translation length $–\ell < 0$ on $R_i$.

**Step 1.** In this step we show $\phi(g_j)$ has the same translation length $–\ell < 0$ on $R_i$ for all $j \neq i$. The identities (5–4) imply that two involutions $\phi(\alpha)^{\phi(g_i)^{-k}}$ and $\phi(\alpha)^{\phi(g_j)^{-k}}$ are identical for all $k \in \mathbb{N}$. So the two involutions must share their supports,

$$(5–14) \quad (S_\alpha)\phi(g_i)^{-k} = (S_\alpha)\phi(g_j)^{-k}$$ for all $k \in \mathbb{N}$. Since $p$ intersects $S_\alpha$, one can take $A \in supp p \cap S_\alpha \neq \emptyset$. By (5–12), we have

$$\quad (A)p^{-k} \in supp p \cap R_i$$
for all $k \geq k_0$ where $k_0 \in \mathbb{N}$. Hence $(A)p^{-k} \in (S_\alpha)p^{-k} \cap R_t$. Moreover, $(S_\alpha)p^{-k} \subset (S_\alpha)\phi(g_i)^{-k}$ for each integer $k$ because $p$ is the restriction of $\phi(g_i)$ on $\text{supp } p$. Therefore $(S_\alpha)\phi(g_i)^{-k} \cap R_t$ is non-empty for all but finitely many $k$. The identities (5–14) imply

$$
(5–15) \quad (S_\alpha)\phi(g_i)^{-k} \cap R_t = (S_\alpha)\phi(g_j)^{-k} \cap R_t
$$

for all $k \in \mathbb{N}$. The identities (5–15) force $\phi(g_j)$ to have the same translation length $-\ell < 0$ on $R_t$. Consider the two smallest balls on $X_\alpha$ (centered at the origin) which contain two sets in (5–15) respectively. If $\phi(g_j)$ moves points on $R_t$ (up to a finite set) by $\ell' \neq -\ell$, the two balls have to have different sizes for infinitely many $k$. This means $\phi(g_i)$ and $\phi(g_j)$ fail to satisfy the identities (5–15). Therefore $\phi(g_i)$ and $\phi(g_j)$ have the same translation length $-\ell < 0$ on $R_t$. Since $i$ and $j$ were arbitrary the first assertion of the Lemma is verified. The uniqueness of the ray will be verified in the last step.

**Step 2.** Identify $\text{supp } p$ with $\{[k]_\rho : k \in \mathbb{Z}\}$ as described in (5–10). We will show that there exists $k_0 \in \mathbb{Z}$ such that for each $j \neq i$, $p$ intersects only one partial translation $q \in \mathcal{P}_j$ satisfying

$$
(5–16) \quad p = q \text{ on } \{[k]_\rho : k \leq k_0\}
$$

and

$$
(5–17) \quad \text{supp } p \cap \text{supp } q = \{[k]_\rho : k \leq k_0 + 1\}.
$$

From Step 1 we see that actions of $\phi(g_j)$ and $\phi(g_i)$ are identical on $R_t$ as a translation by $-\ell < 0$ up to a finite set. In particular, there exists $k_1 \in \mathbb{Z}$ such that

$$
(5–18) \quad (P)\phi(g_i) = (P)\phi(g_j)
$$

for all $P \in \{[k]_\rho : k \leq k_1\}$. The orbit of (any such point) $P$ under $\phi(g_j)$ determines a partial translation $q$ as $p$ is the restriction of $\phi(g_i)$ on the orbit of $P$ under $\phi(g_i)$. From the identity (5–18) it is immediate to check that $q$ satisfies

$$
(5–19) \quad q = p \text{ on } \{[k]_\rho : k \leq k_1\}.
$$

We first claim that $p \neq q$. If $p = q$ then $\phi(\alpha)$ restricted to $\text{supp } p$ is the identity map since

$$(P)\phi(g_i)\phi(g_j) = (P)pq = (P)p^2 = (P)qp = (P)\phi(g_j)\phi(g_i)$$

for all $P \in \text{supp } p$. So $p = q$ implies that $p$ does not intersect $\phi(\alpha)$. However this contradicts the condition (5–13).

From the condition (5–19) together with $p \neq q$ we can consider the largest $k_2 \geq 0$ such that

$$
([k_1 + k_2]_\rho)p = ([k_1 + k_2]_\rho)q,
$$

or equivalently $k_2$ is the smallest number such that

$$
([k_1 + k_2 + 1]_\rho)p \neq ([k_1 + k_2 + 1]_\rho)q.
$$

It is obvious that $k_0 = k_1 + k_2$ satisfies (5–16) and $\text{supp } p \cap \text{supp } q \supset \{[k]_\rho : k \leq k_0 + 1\}$. For the reverse inclusion we assume $[k]_\rho \in \text{supp } q$ with $k \geq k_0 + 2$ and then draw a contradiction using the identities (5–2). Observe that $[k_0]_\rho \in S_\alpha$. By definition of $k_0$, $[k_0]_\rho$ satisfies

$$
([k_0]_\rho)\phi(g_i)\phi(g_j) = ([k_0]_\rho)p\phi(g_j) = ([k_0 + 1]_\rho)\phi(g_j) = ([k_0 + 1]_\rho)q
$$

which is distinct from

$$
([k_0 + 1]_\rho)p = ([k_0 + 1]_\rho)\phi(g_i) = ([k_0]_\rho)q\phi(g_i) = ([k_0]_\rho)\phi(g_j)\phi(g_i).
$$

So there exists a transposition $\tau \in T_\alpha$ which moves $[k_0]_\rho$, and hence

$$
(5–20) \quad [k_0]_\rho \in S_\alpha.
$$
Moreover the condition \((5-16)\) implies that \([k]_p \notin S_\alpha\) for all \(k < k_0\), i.e., \(k_0\) is the smallest number in
\[(5-21)\]
\[N_p = \{ k \in \mathbb{Z} : [k]_p \in S_\alpha \}.\]

The identities in \((5-2)\) imply that \([\beta_k, \phi(g_j)] = 1\) for all \(k \geq k_0 + 2\) where \(\beta_k\) is the conjugation defined by
\[(5-22)\]
\[\beta_k := \phi(\alpha)^{\phi(g_i)k-k_0}.\]

To complete the proof we show that \(\phi(g_j)\) does not commute with \(\beta_k\) under the assumption \([k]_p \in \text{supp } q\) for some \(k \geq k_0 + 2\). Our claim follows basically from that the involution \(\beta_k \in \text{FSym}_n\) intersects \(q\). Since the cycle decomposition of \(\beta_k\) contains \(\tau_k := \tau_k^{\phi(g_i)k-k_0}\), \(\beta_k\) intersects \(q\) at the point
\[\(([k_0]_p)^\phi(g_i)k-k_0 = ([k_0]_p)_{p^k-k_0} = [k_0 + (k - k_0)]_p = [k]_p\]
by the observation \((5-11)\).

If \(\text{supp } \tau_k \subset \text{supp } q\) then we can apply a similar argument used in Corollary \(4.3\) which shows that an infinite order element \(g\) does not commute with a transposition \(\beta\) if \(\text{supp } \beta \subset \text{ESupp } g\). Identify \(\text{supp } q\) with \(\mathbb{Z}\) with an appropriate base point so that \([k]_p = [k]_q\). From \([k]_q \in \text{supp } \tau_k\) and the minimality of \(k_0 \in N_p\) we need to consider two cases: \(\tau_k = ([k]_q, [k+1]_q)\) or \(\tau_k = ([k]_q, [k']_q)\) with \(k' \geq k + 2\). For the first case we check that \((([k]_q)\beta_k \phi(g_j) \neq ([k]_q)\phi(g_j)\beta_k\) by
\[\begin{align*}
[k]_q &\xrightarrow{\beta_k} ([k]_q)\tau_k = [k+1]_q \xrightarrow{\phi(g_j)} ([k+1]_q)\phi(g_j) = [k+2]_q, \\
[k]_q &\xrightarrow{\phi(g_j)} ([k]_q)\phi(g_j) = [k+1]_q \xrightarrow{\beta_k} ([k]_q)\tau_k = [k]_q.
\end{align*}\]
So \(\text{supp } [\beta_k, \phi(g_j)] \ni [k]_q\) is nontrivial, and hence \([\beta_k, \phi(g_j)] \neq 1\). In case \(\tau_k\) exchanges \([k]_q\) and \([k']_q\) with \(k' \geq k + 2\) we check \([\beta_k, \phi(g_j)]\) applied to \([k]_q\):
\[\begin{align*}
[k]_q &\xrightarrow{\beta_k} ([k]_q)\tau_k = [k']_q \xrightarrow{\phi(g_j)} ([k]_q)\phi(g_j) = [k+1]_q \xrightarrow{\beta_k^{-1}} ([k']_q)\beta_k^{-1} \phi(g_j)^{-1} \tau_k = [k']_q \phi(g_j)^{-1} \phi(g_j)^{-1}.
\end{align*}\]
If \(\beta_k\) commutes with \(\phi(g_j)\) then we have \((k+1]_q)\beta_k^{-1} \phi(g_j)^{-1} = [k]_q\) or equivalently
\[\begin{align*}
([k']_q)\beta_k = ([k]_q)\phi(g_j) = [k+1]_q.
\end{align*}\]
In other words \([\beta_k, \phi(g_j)] = 1\) implies that \(\beta_k\) contains a transposition exchanging \([k+1]_q\) and \([k' + 1]_q\). Inductively one applies the same argument to show that \(\beta_k\) contains infinitely many transpositions in its cycle decomposition. Consequently \(\text{supp } \beta_k\) is an infinite set provided \([\beta_k, \phi(g_j)] = 1\). However every element of \(\text{FSym}_n\) must have a finite support.

We can apply an analogous argument when \(\tau_k\) exchanges \([k]_q\) and \(P \notin \text{supp } q\). Our claim is that \(\beta_k\) has an infinite support if it commutes with \(\phi(g_j)\). From \(\beta_k \phi(g_j) = \phi(g_j)\beta_k\) and
\[\begin{align*}
[k]_q &\xrightarrow{\beta_k} ([k]_q)\tau_k = P \xrightarrow{\phi(g_j)} (P)\phi(g_j), \\
[k]_p &\xrightarrow{\phi(g_j)} ([k+1]_q) \xrightarrow{\beta_k} ([k+1]_q)\beta_k
\end{align*}\]
we see that \((P)\phi(g_j) = ([k+1]_q)\beta_k\). Since \(\text{supp } q\) is an \(\phi(g_j)\)-invariant subset of \(\text{supp } \phi(g_j)\), \((P)\phi(g_j) \notin \text{supp } q\). In particular \([k+1]_q \in \text{supp } \beta_k\). By induction argument as before, we can show that \(\text{supp } \beta_k\) is an infinite set, which is a contradiction.

So far we have found a partial translation \(q \in P_j\) and \(k_0 \in \mathbb{Z}\) with the properties \((5-16)\) and \((5-17)\). For the uniqueness of \(q\) suppose \(p\) intersects \(q' \in P_j\) with \(q' \neq q\). Since \(q\) does not intersect \(q'\), if \([k]_p \in \text{supp } q'\) then \(k \geq k_0 + 2\) by \((5-16)\). One can apply the same argument as above to show that \(\beta_k\) defined in \((5-22)\) does not commute with \(q'\), which contradicts \((5-2)\).

We remark that the smallest number \(k_0 \in N_p\) does not depend on \(j\). This follows from the relation \(\alpha = [g_i, g_j]\) of \((5-4)\) which implies that \(\phi(\alpha) = \phi([g_i, g_j]) = \phi([g_i, g_j])\) for pairwise distinct
i, j and j’ in \{2, \cdots, n\}. The minimality of \(k_0\) in \(N_p\) implies that \(\phi(g_{j'})\) determines a unique partial translation \(q_{j'} \in P_{j'}\) which agrees with \(p\) on \([k_0]_p: k \leq k_0\) and that \((q_{j'})^2 = ([k_0]_p)^2\). In other words \(k_0\) satisfies identities (5–16) and (5–17) where \(q\) is replaced by \(q_{j'}\).

**Step 3.** In this step we verify the second part of the Lemma. For \(2 \leq i \leq n\), let \(\mathcal{T}_i\) denote the set of all targets of \(\phi(g_i)\), i.e., \(\phi(g_i)\) has a positive translation length on a ray \(R_k \in \mathcal{T}_i\). If \(\mathcal{T}_i\) and \(\mathcal{T}_j\) share a ray \(R_k\) then \(\phi(g_i)\) and \(\phi(g_j)\) have positive translation lengths on \(R_k\). Using the (positive) least common multiple of the two translation lengths one can find \(p \in P_i\) and \(q \in P_j\) such that \(\text{supp } p \cap \text{supp } q\) contains infinitely many points on the target \(R_k\). However this pair of partial translations fail to satisfy the conditions (5–16) and (5–17) of Step 2. Therefore \(\phi(g_i)\) and \(\phi(g_j)\) can not share any ray in their targets provided \(i \neq j\).

**Step 4.** From Step 1, we know that there exists a ray \(R_l\) which is a source of \(\phi(g_i)\) for all \(2 \leq i \leq n\). So each target \(\mathcal{T}_i\) is a subset of \(\{R_1, \cdots, R_n\} \setminus \{R_l\}\). The only way for the targets \(\mathcal{T}_2, \cdots, \mathcal{T}_n\) to be pairwise disjoint is when they are all distinct singleton sets. A monomorphism \(\phi\) defines a bijection \(\gamma: \{2, \cdots, n\} \to \{1, \cdots, l, \cdots, n\}\) such that
\[
\mathcal{T}_i = \{R_{\gamma(i)}\}
\]
for \(2 \leq i \leq n\). Now it is obvious that \(R_l\) is only ray on which \(\phi(g_i)\) has a negative translation length for all \(i\). Consequently \(\phi\) determines a unique ray \(R_l\) which is a common source of \(\phi(g_i)\) for all \(i\).

As we checked in Step 2 in the proof of Lemma [5,7] if two partial translations \(p \in P_i\) and \(q \in P_j\) \((i \neq j)\) intersect then there exists a unique point \([k_0]_p\) satisfying conditions (5–16) and (5–17). Intuitively those two conditions can be interpreted as (1) for each \(p \in P_i\) there exists \(q \in P_j\) such that \(p\) and \(q\) are identical all the way up to \([k_0]_p\), (2) once \(p\) develops a different orbit from \([k_0 + 1]_p\), \(p\) never intersects \(q\) again. Compare the above with the characterization (1) and (2) of the generators of \(H_n\) in page 12. With this intuition, let us call the point \([k_0 + 1]_p\) the diverging point of \(p\) and denote it by \(D_p\). For all partial translations in \(P = \cup_{i=2}^n P_i\) we label those supports with \(\mathbb{Z}\) appropriately such that
\[
D_p = [0]_p
\]
for each \(p \in P\). From now on let us use the above default labeling for \(\text{supp } p\) for all \(p \in P\).

**Lemma 5.8.** Let \(i \in \{2, \cdots, n\}\). Each partial translation \(p\) of \(\phi(g_i)\) intersects exactly one transposition \(\tau_p \in T_i\) which exchanges \([-1]_p\) and \([0]_p\).

**Proof.** Any transposition \(\tau \in T_i\) is one of the three types:
- Type I: \(\tau\) exchanges two consecutive points of one partial transposition of \(P_i\).
- Type II: \(\tau\) exchanges two non-consecutive points of one partial transposition of \(P_i\).
- Type III: \(\tau\) intersects two distinct partial transpositions of \(P_i\).

We show that each \(p \in P_i\) can intersect only one transposition \(\tau \in T_i\) of type I. From (5–20) we see that \(k_0 = -1\) is indeed the smallest integer in \(N_p\) defined in (5–21). We first focus on the case \(n \geq 3\).

**Claim I:** If \(p \in P_i\) intersects \(\tau\) of type I, then \(p\) does not intersect any other transposition of \(T_i\).

The first case to consider is when \(\tau\) exchanges \([-1]_p\) and \([0]_p\) where \(-1\) is the smallest number in \(N_p\). Suppose \(p\) intersects \(\tau' \in T_a\) of any type whose support contains \([q]_p\). Since \(\tau\) and \(\tau'\) does not intersect, \(q \geq 1\).

In case \(q = 1\), we use the relation \(r_2\) of (5–1) and the hexagon argument described in Remark 5.5. Two involutions \(\phi(\alpha)\) and \(\beta := \phi(\alpha) \phi(g_i)\) satisfy
\[
(\phi(\alpha) \beta)^3 = 1.
\]
Observe that $\beta$ contains a transposition $\tau_{\phi(g_i)} = \tau^p = ([0]_p, [1]_p)$ which intersects two transpositions $\tau$ and $\tau'$ of $\phi(\alpha)$. Therefore there exists a hexagon on 6 points
\[
[-1]_p \xrightarrow{\tau} [0]_p \xrightarrow{\tau^p} [1]_p \xrightarrow{\tau'} Q \xrightarrow{\beta} (Q)\beta \xrightarrow{\phi(\alpha)} S \xrightarrow{\beta} [-1]_p
\]
where $Q = ([1]_p)\tau'$ and $S = ([1]_p)\beta$. In particular, $\text{supp} \beta$ contains $[-1]_p$. This means that $S_\alpha$ contains $[-2]_p$, a contradiction to the minimality of $-1 \in N_p$.

Next, we may assume $q \geq 2$ and $q$ is the smallest number in $N_p \setminus \{-1, 0\}$. Consider the conjugation defined by
\[
\beta = \phi(\alpha)^{\phi(g_i)g^q}.
\]
Since $q \geq 2$ the identity \text{(5–3)} says that $\phi(\alpha)$ and $\beta$ commute. Two involutions $\phi(\alpha)$ and $\beta$ move $[q]_p$ to distinct points since
\[
[q]_p\phi(\alpha) = ([q]_p)\tau' \neq [q - 1]_p = ([q]_p)\beta = ([q]_p)\tau^p q = [q - 1]_p
\]
where the inequality follows from the minimality of $q$ in $N_p \setminus \{-1, 0\}$. We are in good position to apply the ‘square argument’ described in Remark \text{5.3}. The square on $I = S_\alpha \cap \text{supp} \beta$ involving 4 points
\[
[q - 1]_p \xrightarrow{\beta} [q]_p \xrightarrow{\tau'} ([q]_p)\tau' \xrightarrow{[q - 1]_p} [q]_p \xrightarrow{\phi(\alpha)} [q - 1]_p
\]
implies that $[q - 1]_p$ must belong to $N_p \setminus \{-1, 0\}$. Again, this contradicts to the minimality of $q$.

\textbf{Claim II:} If $p \in P_i$ intersects $\tau \in T_i$ then $\tau$ can not be of type II. First we argue that $\text{supp} \tau$ dose not contain $[-1]_p$ if $\tau$ is of type II. Assume $\tau = ([1]_p, [q]_p)$ for some $q \geq 1$. The involution $\beta$ defined by
\[
\beta = \phi(\alpha)^{\phi(g_i)g^{q+1}}
\]
contains $\tau_{\phi(g_i)}^{q+1}$ in its cycle decomposition. So $\beta$ moves $[q]_p$ to $[q + (q + 1)]_p$ but $\phi(\alpha)$ moves $[q]_p$ to $[q - (q + 1)]_p = [-1]_p$. Therefore we can apply the square argument to see that $[-1]_p$ belongs to $\text{supp} \beta$, or equivalently $([-1] - (q + 1)]_p \in N_p$ as in the proof of Claim I. However, the minimality of $-1$ in $N_p$ says that this can not happen.

To complete the proof, we need to rule out the case when $p$ intersects $\tau = ([q]_p, [r]_p)$ for some $q \geq 0$ and $r \geq q + 2$. Recall that $-1$ is the smallest number in $N_p$, and so $T_i$ contains a transposition $\tau'$ which moves $[-1]_p$. From the discussion so far we may assume that $\tau'$ is of type III. Consequently the point $[-1]_p$ gets mapped to distinct points by $\phi(\alpha)$ and $\beta$ defined by
\[
\beta = \phi(\alpha)^{\phi(g_i)g^{-(r+1)}}
\]
since $([1]_p)\beta = [-1 - (r - q)]_p \in \text{supp} \ p$ but $([-1]_p)\phi(\alpha) = ([1]_p)\tau' \notin \text{supp} \ p$. The square argument applied to two involutions $\phi(\alpha)$ and $\beta$ implies that the point $[1 - (r - q)]_p$ belongs to $N_p$, which can not happen.

\textbf{Claim III:} If $p$ intersects $\tau \in T_i$ of type III then $\tau$ fixes the point $[-1]_p$ where $-1$ is the smallest number in $N_p$. Observe that Claim III together with Claim II implies that the transposition $\tau_p$ of $T_i$, which moves $[-1]_p$, must be of type I. Consequently $p$ intersects only transposition $\tau_p$ which exchanges $[-1]_p$ and $[0]_p$, completing the proof for the Lemma.

To this end we draw a contradiction from the assumption that $p$ intersects $\tau \in T_i$ of type III with $[-1]_p \in \text{supp} \tau$. Since $\tau$ is of type III, it relates $p$ to a partial translation $p_1 \in P_i$ that $\tau$ intersects. From Step II of Lemma \text{5.7} we know that there exists a smallest number in
\[
N_{p_1} = \{ k \in \mathbb{Z} : [k]_{p_1} \in S_\alpha \}.
\]
By the convention \text{5–24} we can further say that the smallest number is $-1$ (after pre-composing an appropriate translation on $\mathbb{Z}$ to $\mathbb{Z} \mapsto \text{supp} p_1$ if necessary). We can show that if $([-1]_p)\tau = [k_1]_{p_1}$ then $k_1 \geq 0$. If $k_1 = -1$, we have to have
\[
([-1]_p)pq = ([1]_p)\phi(g_i)\phi(g_j) = ([1]_p)\phi(g_j)\phi(g_i) = ([1]_{p_1})q_1 p_1
\]
where \( q \) and \( q_1 \) are unique partial translations of \( P_j \) which intersect \( p \) and \( p_1 \) respectively. Consequently \( q \) intersects both of \( p \) and \( p_1 \), a contradiction to the uniqueness of \( q \). Therefore \( \tau \) fixes \([-1]_{p_1}\) and so \( k_1 \geq 0 \).

The partial translation \( p_1 \) intersects \( \tau \) of type III and so Claim I implies that \( p_1 \) can only intersect transpositions of type III. In particular the transposition \( \tau_1 \in T_i \) with \([-1]_{p_1} \in \text{supp} \tau_1 \) must be of type III. Observe that \( \tau_1 \) relates \( p_1 \) to \( p_2 \in P_j \setminus \{ p_1 \} \) as \( \tau \) relates \( p \) to \( p_1 \). As argued above one can check that \( \tau_1 \) intersects \( p_2 \) at a point \([k]_{p_2}\) with \( k_2 > -1 \) where \(-1 \) is the smallest number in \( N_{p_2} = \{ k \in \mathbb{Z} : [k]_{p_2} \in S_\alpha \} \). Inductively we can consider a cycle of partial translations

\[(5–25) \quad \tau_0 = \tau \rightarrow \tau_1 \rightarrow \cdots \rightarrow \tau_{m-1} \rightarrow \tau_m \rightarrow \tau_0.
\]

Let \( k_1, \ldots, k_m \) be integers so that \( \tau_r = ([-1]_{p_r}, [k_r + 1]_{p_r} \phi) \) for \( 0 \leq r \leq m-1 \) and \( \tau_m = ([-1]_{p_m}, [k_m]_{p_0}) \).

Next we show that \( k_r = 0 \) for \( 1 \leq r \leq m \). Assume the contrary that, for example, \( k_1 \geq 1 \). We can apply the square argument to two commuting involutions \( \phi(\alpha) \) and \( \beta \) defined by

\[ \beta = \phi(\alpha)\phi(\alpha)^{k_1+1} \]

since \( ([k_1]_{p_1})\phi(\alpha) = [-1]_{p_0} \neq [k_2 + 1]_{p_2} = ([k_1]_{p_1})\beta \). Consequently \([-1]_{p_0} \in \text{supp} \beta \) or equivalently \(-1 - (k_1 + 1) \in N_{p_0} \), contradicting to the minimality of \(-1 \in N_{p_0} \). A similar argument shows that \( k_r = 0 \) for all \( r \).

So far we have shown that if \( p \) intersects \( \tau \in T_i \) of type III then there exists a cycle of partial translations \((5–25)\) which are related by transpositions \( \tau_r = ([-1]_{p_r}, [0]_{p_{r+1}}) \) for \( 0 \leq r \leq m-1 \) and \( \tau_m = ([-1]_{p_m}, [0]_{p_0}) \). Next we show that \( \phi(g_i) \) restricts on the set \( C_1 = \{ [1]_{q_0}, \ldots, [1]_{q_p} \} \) to define a \((m+1)\)-cycle

\[ \sigma_1 : [1]_{q_0} \rightarrow [1]_{q_0-1} \rightarrow \cdots \rightarrow [1]_{q_0} \rightarrow [1]_{q_0} \]

For each \( 0 \leq r \leq m-1 \), the translation \( \tau_r = ([1]_{q_r}, [0]_{q_{r+1}}) \) of \([\phi(g_i), \phi(g_j)]\) implies that

\[ [1]_{q_r} \phi(g_i) \rightarrow [0]_{q_r}, \quad \phi(g_j) \rightarrow [1]_{q_r}, \quad \phi(g_i)^{-1} \rightarrow [1]_{q_{r+1}} \phi(g_j)^{-1} \rightarrow [0]_{q_{r+1}}. \]

From the last arrow we have \( ([1]_{q_r})\phi(g_i)^{-1} = [1]_{q_{r+1}} \) or \( ([1]_{q_{r+1}})\phi(g_i) = [1]_{q_r} \) for \( 0 \leq r \leq m-1 \).

Finally we show that \( \phi(g_i) \) contains infinitely many copies of \( \sigma_1 \) in its cycle decomposition. Observe that the conjugation \( \phi(\alpha)\phi(g_j)^2 \) contains a factor \( \beta_2 \) given by

\[(5–26) \quad \beta_2 = (\tau_0 \tau_1 \cdots \tau_m)^{\phi(g_j)^2}. \]

The action of \( \phi(g_j)^2 \) on \( \text{supp} \beta \) is the translation by \(+2\) along partial translations \( q_0, q_1, \ldots, q_m \). So \( \beta_2 \) can be written as a product of transpositions

\[ ([1]_{q_0}, [2]_{q_1}) \cdots ([1]_{q_{m-1}}, [2]_{q_m})([1]_{q_m}, [2]_{q_0}) \]

Therefore the conjugation \( \sigma_1 \beta_2 \) is the \((m+1)\)-cycle

\[ \sigma_2 : [2]_{q_0} \rightarrow [2]_{q_0-1} \rightarrow \cdots \rightarrow [2]_{q_0} \rightarrow [2]_{q_0} \]

Intuitively speaking, taking the conjugation of \( \sigma_1 \) by \( \beta_2 \) results in translating the \((m+1)\)-cycle \( \sigma_1 \) globally by \(+1\) along the partial translations \( q_0, \ldots, q_m \). Now the identity \((5–4)\) says that \( \phi(g_i) \) commutes with \( \phi(\alpha)\phi(g_j)^2 \). In particular \( \phi(g_i)^{\beta_2} = \phi(g_i) \). This means that \( \phi(g_i) \) already contains \( \sigma_2 \), a translated copy of \( \sigma_1 \), in its cycle decomposition. Applying identity \((5–4)\) repeatedly to \( \phi(g_i) \) and \( \phi(\alpha)\phi(g_j)^k \), we see that \( \phi(g_i)^{\beta_k} = \phi(g_i) \) for \( k \geq 2 \) where \( \beta_k \) is defined in an analogous way as in \( (5–26) \) with the power \( k \) instead of \( 2 \). Thus \( \phi(g_i) \) contains infinitely many copies of \( \sigma_2 \) in its cycle decomposition, which is absurd. In all, a partial translation \( p \) which intersects \( \tau \in T_i \) of type III forces that \( \phi(g_i) \notin H_n \). Claim III is verified and so we are done.
If \( n = 2 \), we can show the same result for a partial translation of \( \phi(g_2) \) as a special case of the above discussion. The braid relation and commutation relation of \((5–7)\) can be used for Claim 1 and Claim 2 respectively. One can verify Claim 3 immediately with braid relation again.

**Corollary 5.9.** Let \( 3 \leq n \). Suppose that a partial translation \( p \) of \( \phi(g_i) \) intersects a partial translation \( q \) of \( \phi(g_j) \). Then \( p \) does not intersect any other cycles of \( \phi(g_j) \) but \( q \).

**Proof.** We first check that \([1]_p \) is fixed under \( \phi(g_j) \). Lemma 5.8 states that the unique transposition \( \tau_p \), which intersects \( p \), exchanges \([-1]_p \) and \([0]_p \). From the identity \(((0)_p)[\phi(g_i), \phi(g_j)] = ([0]_p)\tau_p = \tau_p = \tau_p \) we have \((0)_p\phi(g_i)\phi(g_j) = \tau_p\phi(g_i)\phi(g_j), \) and so

\[
([1]_p)\phi(g_j) = ([0]_p)p\phi(g_j) = ([0]_p)\phi(g_i)\phi(g_j) = ([1]_p)\phi(g_j)\phi(g_i),
\]

so 

\[
(1\ldots1)p\phi(g_j) = ([0]_p)p\phi(g_j) = ([0]_p)\phi(g_i)\phi(g_j) = ([1]_p)p\phi(g_i)\phi(g_j) = ([1]_p)p \]

since \((-1)_p = ([0]_p)p = [0]_p \) by \((5–6)\). Suppose \( p \) intersects a component \( f \neq q \) in the cycle decomposition of \( \phi(g_j) \). We may further assume that if \( k_1 \) is the smallest integer in \( \{ k \in \mathbb{Z} : [k]_p \in \text{supp} f \} \) then \([k]_p \) is fixed under \( \phi(g_j) \) for all \( 1 \leq k \leq k_1 - 1 \). To show \([k_1 - 1]_p \) is a partial translation \( \phi(\alpha) \) we check that

\[
[k_1 - 1]_p \phi(g_j) = ([k_1 - 1]_p) \phi(g_j) = ([k_1 - 1]_p) \phi(g_j) = \phi(g_j) = [k_1 - 1]_p \phi(g_j).
\]

which is distinct from

\[
[k_1 - 1]_p \phi(g_j) = ([k_1 - 1]_p) \phi(g_j) = ([k_1 - 1]_p) \phi(g_j) = ([k_1 - 1]_p) \phi(g_j).
\]

Since \( k_1 - 1 > 0, \phi(\alpha) \) contains a transposition \( \beta \neq \tau_p \) which intersects \( p \). However this contradicts Lemma 5.8.

The unique positive integer \( \ell \) in Lemma 5.7 is called the **eventual length** of a monomorphism \( \phi \), and denoted by \( \ell(\phi) \). Lemma 5.7 implies that \( \mathcal{P}_\ell \) contains precisely \( \ell = \ell(\phi) \) partial translations \( p_1, \ldots, p_\ell \) for each \( 2 \leq i \leq n \). Since they are only infinite cycles of \( \phi(g_i) \) we have \( \text{ES} \phi(g_i) = \cup_{j=1}^n \text{supp} p_j \). Lemma 5.7 also implies that each ray of \( X_n \) is either a source or a target of \( \phi(g_i) \) for some \( i \). So the set \( \cup_{i=2}^n \text{ES} \phi(g_i) \) contains all points of \( X_n \) but finitely many. Let \( \text{ES}(\phi) \subset X_n \) be the set

\[
\text{ES}(\phi) = \bigcup_{g \in \mathcal{H}_n} \text{ES} \phi(g).
\]

**Proposition 5.10.** With the notation defined above, \( \text{ES}(\phi) = \cup_{i=2}^n \text{ES} \phi(g_i) \).

**Proof.** One side inclusion is clear. For the reverse inclusion we claim that \( F = X_n \setminus \cup_{i=2}^n \text{ES} \phi(g_i) \) is invariant under \( \phi(\cdot) \) for all \( g \in \mathcal{H}_n \). Note that \( P \in F \) if and only if \( P \) has a finite orbit under \( \phi(g_i) \) for all \( 2 \leq i \leq n \). In other words each \( \phi(g_i) \) contains a finite cycle which moves \( P \). We argue by induction on the word length of \( g \). If a finite cycle \( f \) of \( \phi(g_i) \) moves \( P \) then obviously \((P)\phi(g_i) = (P)f \) has the same finite orbit under \( \phi(g_i) \). If \((P)\phi(g_i) \) is fixed under \( \phi(g_j) \) for some \( j \) then we have nothing to prove because the orbit under \( \phi(g_j) \) is trivial. To show \((P)\phi(g_i) \subset F \) for every \( i \), we need to check that if \( \phi(g_j) \) has a finite cycle \( f \) such that \((P)f \in \text{supp} \phi(g_j) \) for some \( j \neq i \), then \( \phi(g_j) \) also has a finite cycle which moves \((P)\phi(g_i) \). This follows from the last assertion of Corollary 5.9 which implies that if \( f \) intersects \( \phi(g_i) \) then it can only intersect finite cycles of \( \phi(g_j) \) for all \( j \neq i \). The same argument shows that if \((P)\phi(g_i) \) belongs to \( \text{supp} \phi(g_j) \) then it has a finite orbit under \( \phi(g_j) \) for every \( j \neq i \). So \((P)\phi(g_i) \subset F \). This establishes the base case.

Suppose \( g \in \mathcal{H}_n \) with \( |g| = k + 1 \). Say the last letter of \( g \) is \( g_i^{\pm 1} \), i.e., \( g = g' \cdot g_i^{\pm 1} \) with \( |g'| = k \). By induction assumption \((P)\phi(g') \subset F \). So, for each \( i \), if \((P)\phi(g') \subset \text{supp} \phi(g_i) \) then there exists a finite cycle of \( \phi(g_i) \) which moves the point \((P)\phi(g') \). A similar argument applied to \((P)\phi(g') \phi(g_i)^{\pm 1} \), instead of \((P)\phi(g_i)^{\pm 1} \) as in the base case, shows that the point \((P)\phi(g') \phi(g_i)^{\pm 1} \) has a finite orbit under \( \phi(g_j) \) for all \( 2 \leq j \leq n \). Therefore \((P)\phi(g) \phi(g_i)^{\pm 1} \subset F \), and hence \( F \) is an invariant set under
\(\phi(g)\) for all \(g \in \mathcal{H}_n\). Since \(F\) is finite \(P \in F\) must have a finite orbit under \(\phi(g)\). This means that \(P \not\in \text{ES}(\phi)\). \(\square\)

Consider the decomposition of \(X_n = \text{ES}(\phi) \sqcup F\) where \(F\) is the finite set as in the above proof. Let \(E(\phi(g))\) denote the restriction of \(\phi(g)\) on the set \(\text{ES}(\phi)\). From the decomposition we have \(E(\phi(g))E(\phi(h)) = E(\phi(gh))\) for all \(g, h \in \mathcal{H}_n\). One crucial observation is that, for each \(2 \leq i \leq n\), \(E(\phi_{i,j})\) is nothing but the product of its commuting partial translations:

\[
(5–27) \quad E(\phi_{i,j}) = \prod_{p \in \mathcal{P}_i} p.
\]

Consequently we have

\[
(5–28) \quad \text{Esupp } (\phi_{i,j}) = \bigcup_{p \in \mathcal{P}_i} \text{supp } p.
\]

So Lemma 5.8 implies that \(T_i\) consists of \(\ell\) transpositions each of which exchanges \([-1]_p\) and \([0]_p\) for some \(p \in \mathcal{P}_i\). Two conditions \((5–16)\) and \((5–17)\) show that \(T_i = T_j\) for each pair \(2 \leq i \neq j \leq n\).

It follows from that \(E(\phi(\alpha))\) is the product of \(\ell\) transpositions in \(T_i\) for any \(i\). Indeed we have the following.

**Corollary 5.11.** Let \(D\) be the set of all diverging points of partial translations in \(\mathcal{P} = \bigcup_{i=2}^n \mathcal{P}_i\).

There exists bijections

\[\mathcal{P}_i \leftrightarrow D \leftrightarrow T_i\]

for each \(2 \leq i \leq n\).

**Proof.** Fix \(i\) and consider the sequence of maps

\[\mathcal{P}_i \xrightarrow{\iota_1} D \xrightarrow{\iota_2} T_i \xrightarrow{\iota_3} \mathcal{P}_i\]

defined by

\[\iota_1 : p \mapsto [0]_p, \quad \iota_2 : [0]_p \mapsto \tau_p, \quad \iota_3 : \tau_p \mapsto p.\]

We claim that all three maps are injective whose composition in a row yields the identity. Partial translations of \(\mathcal{P}_i\) do not intersect each other, which explains \(\iota_1\) is injective. Every diverging point is given by \([0]_q\) for some \(q \in \mathcal{P}_j\). If \(j = i\), we have nothing to show that \(\iota_2\) is well-defined due to Lemma 5.8. If not, from Step 2 of Lemma 5.7 one can find a partial translation \(p \in \mathcal{P}_i\) with conditions \((5–16)\) and \((5–17)\). So we have \([0]_q = [0]_p\). The assignment \([0]_p \mapsto \tau_p\) is well-defined and injective due to Lemma 5.8. For \(\iota_3\) it suffices to check if \(\tau \in T_i\) then \(\tau\) intersects a unique partial translation in \(\mathcal{P}_i\). By definition, if \(\tau \in T_i\) then \(\tau\) intersects \(\text{Esupp } \phi_{i,j}\). The union \((5–28)\) forces \(\tau\) to intersect at least one partial translation of \(\mathcal{P}_i\). By Lemma 5.8 again, \(\iota_3\) is well-defined and injective. It follows immediately from definitions of three maps that the composition is the identity map. We have established bijections as desired. \(\square\)

**Expanding map \(\tilde{\phi}\).** Recall that \(\mathcal{H}_n\) acts on the underlying set \(X_n = \{1, \ldots, n\} \times \mathbb{N}\) transitively. So \(X_n\) can be considered as a single orbit of a base point \((1, 1)\). Let us to express points \((j, m) \in X_n\) \((1 \leq j \leq n, m \in \mathbb{N})\) as

\[
(j, m) = \begin{cases} 
(1, 1)g_2^{-(m-1)} & \text{if } j = 1, \\
(1, 1)g_j^m & \text{if } 2 \leq j \leq n.
\end{cases}
\]

In case \((j, m) \in R_1\) we simply choose \(g_2^{-1}\) among inverses of \(n-1\) generators whose actions coincide on \(R_1\) as translations by +1. By taking \(g_2^{-1}\) as a default translation on \(R_1\) we can make the
above expression unique. Take subscripts and exponents of the generators to obtain the following coordinate system for \((j, m) \in X_n\)

\[
(5–29) \quad (j, m) \leftrightarrow \begin{cases} [2, -(m-1)] & \text{if } j = 1, \\ [j, m] & \text{if } 2 \leq j \leq n. \end{cases}
\]

Corollary 5.11 implies that a monomorphism \(\phi\) determines \(\ell = \ell(\phi)\) diverging points \(D_1, \ldots, D_\ell\). Let \(Q_l = \{p \in P : |0|_p = D_l\}\) for \(1 \leq l \leq \ell\). For each \(2 \leq i \leq n\), \(P_i\) contains precisely one partial translation \(p\) such that \(|0|_p = D_l\). Let \(p_{l,i}\) denote such a unique partial translation \(p \in P_i\), i.e., \(p_{l,i} = \iota_1^{-1}(D_l)\) where \(\iota_1 : P_i \to D\) is the bijection defined in proof of Corollary 5.11. We label partial translations of \(Q_l\) as

\[
Q_l = \{p_{l,2}, \ldots, p_{l,n}\}
\]

With this new labeling, \(P_i = \{p_{l,i} : 1 \leq l \leq \ell\}\) for each \(2 \leq i \leq n\), and the set of all partial translations \(P\) determined by \(\phi\) has decompositions \(P = \bigsqcup_{i=2}^{n} P_i = \bigsqcup_{i=1}^{\ell} Q_l\). The last identity follows from the equivalence relation: partial translations \(p, q \in P\) belong to \(Q_l\) for some \(l\) if and only if \(p\) intersects \(q\). For better notation, let \([m]_{l,i}\) denote the point \([m]_{p_{l,i}}\) from now on. Let \(O(D_l) \subset ES(\phi)\) denote the set

\[
O(D_l) = \bigcup_{2 \leq i \leq n} \text{supp } p_{l,i}.
\]

Since \(D_l = [0]_{l,i}\) for all \(2 \leq i \leq n\), every point in \(O(D_l)\) can be written as \([m]_{l,i} = (D_l)p_{l,i}^m\) for some \(p_{l,i} \in Q_l\) and \(m \in \mathbb{Z}\). Note that this expression is not unique when \(m \leq 0\). Each pair of partial translations of \(Q_l\) satisfies conditions \((5–16)\) and \((5–17)\). The first condition implies that the point \([m]_{l,i}\) with \(m \leq 0\) and \(3 \leq i \leq n\) is identified to \([m]_{l,2}\). The second condition implies that \([m]_{l,i} = [m]_{l,i'}\) if and only if \(i = i'\) when \(m \geq 1\). So every point \(P \in O(D_l)\) can be expressed uniquely as

\[
(5–30) \quad P = \begin{cases} [m]_{l,2} & m \leq 0, \\ [m]_{l,i} & 2 \leq i \leq n, 1 \leq m. \end{cases}
\]

Comparing \((5–29)\) and \((5–30)\) we obtain a canonical bijection \(\tilde{\phi}_l : X_n \to O(D_l)\), for each \(l = 1, \ldots, \ell\), defined by

\[
(5–31) \quad ([i, m]) \tilde{\phi}_l = [m]_{l,i}.
\]

The expanding map \(\tilde{\phi} : X_n \to (X_n)^{\ell}\) is defined by

\[
\tilde{\phi} = \tilde{\phi}_1 \times \cdots \times \tilde{\phi}_\ell.
\]

Note that \(\tilde{\phi}\) depends on the choice of diverging points. However components of \(\tilde{\phi}\) are all distinct. This follows from that \(O(D_l)\) does not intersect \(O(D_{l'})\) if \(l \neq l'\).

**Remark 5.12.** A monomorphism \(\phi\) of \(\mathcal{H}_n\) determines a group \(G_l \leq \text{Sym}_n\) which consists of restrictions \(g \in \phi(\mathcal{H}_n)\) on the orbit of \(D_l\) under \(\phi(\mathcal{H}_n)\), \(1 \leq l \leq \ell\). We remark that \(G_l \cong \mathcal{H}_n\) for all \(l\). Observe that each \(O(D_l)\) coincides with the orbit of \(D_l\). This is because all partial translations in \(P \setminus Q_l\) fix \(D_l\). Indeed \(G_l\) is generated by \(n - 1\) partial translations of \(Q_l\). The map \(\tilde{\phi}_l\) conjugates \(\mathcal{H}_n\) to \(G_l\) in the ambient group \(\text{Sym}_n\).

**Proposition 5.13.** Suppose \(\tau = (P, Q)\) is the transposition exchanging \(P\) and \(Q\). Then \(E(\phi(\tau))\) coincides with a product of \(\ell\) commuting transpositions \(\prod_{j=1}^{\ell} \left((P)\tilde{\phi}_j, (Q)\tilde{\phi}_j\right)\).
Proof. Assume \( n \geq 3 \). We consider the following four cases depending on which rays \( P \) and \( Q \) lie.

**Case I.** Suppose both \( P \) and \( Q \) lie on \( R_1 \). Let \( P = [2, m] = (1, -m + 1) \) and \( Q = [2, m'] = (1, -m' + 1) \) with \( m < m' \leq 0 \). From the definition of \( \phi \) we have

\[
(P)\bar{\phi}_l = ([2, m])\bar{\phi}_l = [m]_{l,2}, \quad (Q)\bar{\phi}_l = ([2, m'])\bar{\phi}_l = [m']_{l,2}
\]

for each \( l \). Since components of \( \bar{\phi} \) are all distinct, \( \prod_{l=1}^\ell \left( (P)\bar{\phi}_l, (Q)\bar{\phi}_l \right) = \prod_{l=1}^\ell \left( [m]_{l,2}, [m']_{l,2} \right) \) is a product of \( \ell \) commuting transpositions.

On the other hand, one can check that \( \tau \) can be written as \( \tau = \alpha^h \) where

\[
h = g_2 g_3^{-m-m'+1} g_2^{m'-1}.
\]

So \( E(\phi(\tau)) = E(\phi(\alpha^h)) = E(\phi(\alpha))^E(\phi(h)) = E(\phi(\alpha))\phi(h) \). Recall that \( \phi(g_2) \) contains partial translations \( p_1, p_2, \ldots, p_{\ell} \) with \( D_l = [0]_{l,2} \) for \( l = 1, \ldots, \ell \). So the involution \( E(\phi(\alpha)) \) is the product of \( \ell \) transpositions \( \tau_1, \ldots, \tau_{\ell} \) where \( \tau_l \) exchanges \( [0]_{l,2} \) and \( [-1]_{l,2} \). We examine the effect of \( E(\phi(h)) \) on those points. Since \( \phi(g_3) \) fixes \( [m]_{l,2} \) for all \( m \geq 1 \), we have, for each \( l \),

\[
[0]_{l,2} \xrightarrow{\phi(g_3)} ([0]_{l,2})_2 = [1]_{l,2} \xrightarrow{\phi(g_3^{m-m'+1})} [1]_{l,2} \xrightarrow{\phi(g_2^{m'-1})} ([1]_{l,2})_2 = [m]_{l,2},
\]

\[
[-1]_{l,2} \xrightarrow{\phi(g_3)} [0]_{l,2} \xrightarrow{\phi(g_3^{m-m'+1})} ([0]_{l,2})_3 = [m-m'+1]_{l,2} \xrightarrow{\phi(g_2^{m'-1})} ([m-m'+1]_{l,2})_2 = [m]_{l,2}
\]

where \( q_{l,3} \) is the partial translation of \( \phi(g_3) \) such that \( q_{l,3} = p_{l,2} \) on \( \{ [m]_{l,2} : m \leq -1 \} \). Therefore \( E(\phi(\tau)) \) is the product of \( \ell \) transpositions exchanging \( [m']_{l,2} \) and \( [m]_{l,2} \) for \( l = 1, \ldots, \ell \), as expected.

To complete the proof, one can repeat similar calculation for \( E(\phi(h)) \) in the following cases:

**Case II.** \( P = [2, m] \) and \( Q = [i, m'] \), with \( m \leq 0, m' \geq 1 \). Then \( \tau = \alpha^h \) where

\[
h = \begin{cases} g_2 g_3 g_2^{m'-1} g_3^{m-1} & i = 2 \\ g_2 g_3^{m'-1} g_2^{m-1} & 3 \leq i \leq n. \end{cases}
\]

**Case III.** \( P = [i, m] \) and \( Q = [i, m'] \) with \( 1 \leq m' < m \). Then \( \tau = \alpha^h \) where

\[
h = g_2 g_3^{-(m-m'-1)} g_2^{-1} g_i^{m'}.
\]

**Case IV.** \( P = [i, m] \) and \( Q = [j, m'] \) with \( i \neq j \). Then \( \tau = \alpha^h \) where

\[
h = g_i g_j g_i^{m-1} g_j^{m'-1}.
\]

If \( n = 2 \) we can apply similar argument with \( \alpha = ((1, 1), (1, 2)) \). Suppose \( P = [2, m] \) and \( Q = [2, m'] \) with \( 1 \leq m < m' \). The transposition \( \tau = (P, Q) \) can be written as a conjugation \( \tau = \alpha^h \) where

\[
h = (g_2 \alpha)^{m'-m-1} g_2^{m+1}.
\]

Since \( \phi(\alpha) \) exchanges \( [-1]_{l,2} \) and \( [0]_{l,2} \), but fixes all other points for each \( l \) we have

\[
[0]_{l,2} \xrightarrow{\phi(g_2 \alpha)} [1]_{l,2} \xrightarrow{\phi(g_2 \alpha)} ([2]_{l,2})_2 \rightarrow \cdots \rightarrow [m' - m - 1]_{l,2} \xrightarrow{\phi(g_2 \alpha^{m'+1})} [m']_{l,2},
\]

\[
[-1]_{l,2} \xrightarrow{\phi(g_2 \alpha)} [-1]_{l,2} \rightarrow \cdots \rightarrow [-1]_{l,2} \xrightarrow{\phi(g_2^{m'+1})} [m]_{l,2}
\]

Therefore \( E(\phi(\tau)) = \prod_{l=1}^\ell \left( [m]_{l,2}, [m']_{l,2} \right) = \prod_{l=1}^\ell \left( (P)\bar{\phi}_l, (Q)\bar{\phi}_l \right) \). The other cases can be taken care of by similar calculation because a transposition of \( H_2 \) can be expressed as a conjugation of \( \alpha \) by \( (g_2 \alpha)^{m_1} g_2^{m_2} \) for some \( m_1, m_2 \).

\[\square\]

**Lemma 5.14.** Suppose that \( \phi(g_i) = g_i f_i \) for all \( 2 \leq i \leq n \) where \( f_i \in \text{FSym}_n \). If \( \tau \) is a transposition, there exists a constant \( A_2 \), which do not depend on \( k \), such that \( |\phi^k(\tau)| \leq A_2 k \) for all \( k \in \mathbb{N} \).
Proof. By the observation [5–27], each $\phi(g_i)$ is nothing but a partial translation $p_i$. So $p_i$ acts as a translation by $-1$ on $R_1$ and by $1$ on $R_i$ up to a finite set for $2 \leq i \leq n$. We first show that $|\phi^k(\tau)|$ for a transposition $\tau = (P, Q)$. By Proposition 5.13 we see that

$$\phi^k(\tau) = ((P)\tilde{\phi}^k, (Q)\tilde{\phi}^k)$$

where $\tilde{\phi} = \tilde{\phi}_1$ is the expanding map $X_n \to \text{ES}(\phi)$ defined in [5–31]. We claim that both $(P)\tilde{\phi}^k$ and $(Q)\tilde{\phi}^k$ belong to $B_{n,r}$ with $r \leq m + ks$ for all $k \in \mathbb{N}$ for some constants $m$ and $s$ which are determined by $\phi$. Let $[p]_i$ denote the point $[p]_{|p|}$. For each $2 \leq i$ one can find the smallest integer $0 < k_i$ such that

$$[p]_i = (i, m_i + p - k_i)$$

for all $k_i \leq p$ where $[k_i]_i = (i, m_i)$. We take the largest number $k_1 \leq 0$ so that

$$[p]_2 = (1, m_1 + k_1 - p)$$

for all $p \leq k_1$ where $[k_1]_2 = (1, m_1)$. Note that $k_i$ is well defined since the actions of $\phi(g_2)^{-1}, \cdots, \phi(g_n)^{-1}$ are identical on $\{[p]_2 : p \leq 0\}$. Let $B_{n,m}$ denote the ball of $X_n$ with radius $m = \max\{m_1, \cdots, m_n\}$. Since each $\phi(g_i)$ acts as a translation by $\pm 1$ on each ray outside $B_{n,m}$, $P \in B_{n,s}$ implies that $(P)\tilde{\phi} \in B_{n+1,s}$. Observe that if $P = (i, p) \notin B_{n,m}$ then $(P)\tilde{\phi} \in B_{n,p+s}$ where $s = \max\{|m_2 - k_2|, \cdots, |m_n - k_n|, m_1 + k_1 - 1\}$. One can check that if $k_i \leq p$ then, by the coordinate system [5–29],

$$(i, p)\tilde{\phi} = [p]_i = (i, m_i + p - k_i) = [i, p + (m_i - k_i)]$$

for each $2 \leq i$. Similarly if $p \leq k_1$ then

$$(2, p)\tilde{\phi} = [p]_2 = (1, m_1 + k_1 - p) = [2, p - (m_1 + k_1 - 1)]$$

by [5–29]. Thus our claim is verified, and so the transposition $\phi^k(\tau) = ((P)\tilde{\phi}^k, (Q)\tilde{\phi}^k)$ has support in $B_{n,r}$ with $r \leq m + ks$ for all $k \in \mathbb{N}$. It is not difficult to show that a transposition $\tau_0$ with $\text{supp} \tau_0 \subset B_{n,r}$ has length $< 10r$. Observe that $\tau_0$ can be written as a conjugation $\tau_0 = \alpha^h$ where $h \in \mathcal{H}_n$ can be taken so that $2|h| + |\alpha| < 10r$ as in the proof of Proposition 5.13 Therefore we have

$$|\phi^k(\tau)| = |((P)\tilde{\phi}^k, (Q)\tilde{\phi}^k)| < 10(m + ks) < 10(m + s)k = A_k$$

for all $k$.

From now on we assume that a monomorphism $\phi$ with $2 \leq \ell(\phi)$ satisfies that $\phi(g_i)$ has a source $R_1$ and a target $R_i$ for all $2 \leq i \leq n$. (In view of Proposition 5.24 this assumption is legitimate) We also assume that $P \in \text{ES}(\phi)$ unless otherwise stated.

Rooted trees induced by $\tilde{\phi}$. The iteration of $\tilde{\phi}$ applied to $P$ determines a labeled rooted $\ell$-ary tree, which we will denote by $\mathcal{T}_P$. The vertex set $V_P$ of $\mathcal{T}_P$ is equipped with label, level and $\ell$-ary sequences. The root of $\mathcal{T}_P$ is the unique vertex at level 0, labeled by $P$, which corresponds to the empty sequence. Inductively the label $L : V_P \to \text{ES}(\phi)$ and corresponding sequence $W : V_P \to \Omega_\ell$ are defined such that a vertex $v \in V_P$ at level $k \in \mathbb{N}$ is labeled by

$$L(v) = (P)\tilde{\phi}_{i_1} \tilde{\phi}_{i_2} \cdots \tilde{\phi}_{i_k}$$

and corresponds to a sequence

$$W(v) = l_1 l_2 \cdots l_k$$

where $\Omega_\ell$ consists of $\ell$-ary sequences on $\{1, \cdots, \ell\}$ and $1 \leq l_j \leq \ell$ for all $1 \leq j \leq k$. Note that $W(v)$ can be realized as the unique edge path from the root to $v$ with edge labels $\{1, \cdots, \ell\}$. Consequently we have a bijective correspondence between $V_P^k$, the set of vertices at level $k$, and $\Omega_{\ell,k}$, the set of all $\ell$-ary sequences of length $k$. 

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**Example 5.15.** Consider a monomorphism $\phi$ of $\mathcal{H}_3$ defined by $\phi(g_i) = (g_i)^2 \cdot f_i$ for $i = 2, 3$ where $f_2$ and $f_3$ are cycles given by

$$f_2 : (1, 1) \to (1, 2) \to (2, 1) \to (1, 1), \quad f_3 : (1, 1) \to (1, 2) \to (2, 1) \to (3, 2) \to (1, 1)$$

From direct computation we see that $\phi(\alpha) = E(\phi(\alpha))$ is the product of two transpositions $\tau_1$ and $\tau_2$ where

$$\tau_1 : ((2, 1), (1, 4)), \quad \tau_2 : ((1, 2), (1, 3)).$$

Set $D_1 = (2, 1)$ and $D_2 = (1, 2)$. Each of $\phi(g_2)$ and $\phi(g_2)$ has $\ell(\phi) = 2$ partial translations which are

$$p_{1,2} : \cdots \to (1, 2m+4) \to (1, 2m+2) \to \cdots \to (1, 4) \to (2, 1) \to \cdots \to (2, 2m-1) \to (2, 2m+1) \to \cdots$$

$$p_{2,2} : \cdots \to (1, 2m+3) \to (1, 2m+1) \to \cdots \to (1, 3) \to (1, 2) \to (1, 1) \to \cdots \to (2, 2m) \to (2, 2m+2) \to \cdots$$

$$p_{1,3} : \cdots \to (1, 2m+4) \to (1, 2m+2) \to \cdots \to (1, 4) \to (2, 1) \to \cdots \to (3, 2m) \to (3, 2m+2) \to \cdots$$

$$p_{2,3} : \cdots \to (1, 2m+3) \to (1, 2m+1) \to \cdots \to (1, 3) \to (1, 2) \to \cdots \to (3, 2m-1) \to (3, 2m+1) \to \cdots$$

where $m \in \mathbb{N}$. With $D_l = [0]_{l,2} = [0]_{l,3}$ for $l = 1, 2$, let us apply $\tilde{\phi}$ repeatedly to describe rooted trees of points in $\{(1, 1), (1, 2), (2, 1)\} = \text{supp} f_2$.

```
(1, 1) = [2, 0]  \tilde{\phi}_1
      \       \         \   \tilde{\phi}_2
      [0]_{1,2} = D_1 = (2, 1) = [2, 1] [0]_{2,2} = D_2 = (1, 2) = [2, -1]

\tilde{\phi}_1

[1]_{1,2} = (2, 3) = [2, 3] [1]_{2,2} = (1, 1) = [2, 0] [1]_{1,2} = (1, 4) = [2, -3]

\tilde{\phi}_2

[3]_{1,2} = [2, 7] [3]_{2,2} = [2, 4] [2, 1] [2, -1] [3]_{1,2} = [2, 7] [3]_{2,2} = [2, -6]

The above illustrates the tree $T_{(1,1)}$ with labels up to level 3. Each vertex $v$ has $\ell = 2$ children; the left child is $(v)\tilde{\phi}_1$ and the right child is $(v)\tilde{\phi}_2$. Accordingly all left edges are labelled by 1 ($\tilde{\phi}_1$) and all right edges are labeled by 2 ($\tilde{\phi}_2$). Each $v \in V_{(1,1)}^k$ corresponds to a unique sequence $W(v) \in \Omega_{2,k}$. For example the vertex $[2, 0]$ at level 2 corresponds to the sequence $12 \in \Omega_{2,2}$. One important observation is that $T_{(1,1)}$ contains infinitely many copies of itself. The point $(1, 1)$ appears as a label for all vertex $v$ with $W(v) = 1212 \cdots 12$ (as emphasized in blue color!) because the subtree spanned by $v$ and its descendants is identical to $T_{(1,1)}$.

The following figure illustrates the tree $T_{(1,2)}$ up to level 3, which happens to be a subtree of $T_{(1,1)}$. Observe that $T_{(1,2)}$ never contains a copy of itself. This is because the vertices of the tree are all distinct. Yet another crucial fact is that each pair of children has labels which are *translations* of the labels $(1, 4)$ and $(1, 3)$ at level 1. For example, $T_{(1,2)}$ has four pairs of labels at level 3, each of which is a translation of the pair $(1, 4)$ and $(1, 3)$ by some power of $g_2$. Intuitively this is because $p_{1,2}$ and $p_{2,2}$ act as a translation by $\pm 2$ on all points in their supports but finitely many. More precisely

$$([2, m])\tilde{\phi}_1 = [m]_{1,2} = \begin{cases} [2, 2m - 1] = (1, -2m + 2) & m \leq -1, \\ [2, 2m + 1] = (2, 2m + 1) & 1 \leq m. \end{cases}$$
\(([2, m]) \phi_2 = [m]_{2, 2} = \begin{cases} [2, 2m] = (1, -2m + 1) & m \leq -1 \\ [2, 2m - 2] = (2, 2m - 2) & 2 \leq m \end{cases}\)

where the last equalities in each case follow from

\([2, m] = \begin{cases} (1, -m + 1) & m \leq 0 \\ (2, m) & 1 \leq m \end{cases}\)

by \([5-29]\). See Proposition \([5.19]\).

\[D_2 = (1, 2) = [2, -1]\]

\[\tilde{\phi}_1 \quad \tilde{\phi}_2\]

\([-1]_{1, 2} = (1, 4) = [2, -3]\]

\([-1]_{2, 2} = (1, 3) = [2, -2]\]

\([-3]_{1, 2} = [2, -7]\]

\([-3]_{2, 2} = [2, -6]\]

\([-2]_{1, 2} = [2, -5]\]

\([-2]_{2, 2} = [2, -4]\]

\([-7]_{1, 2} = (1, 16)\] \([-7]_{2, 2} = (1, 15)\]

\((1, 14) \quad (1, 13) \quad (1, 12) \quad (1, 11) \quad (1, 10) \quad (1, 9)\)

The tree \(T_{(2, 1)}\) below also contains infinitely many copies of itself. The subtrees determined by vertices with sequences \(2121 \cdots 21\) are identical to \(T_{(2, 1)}\). On the other hand, the subtree \(T_{(2, 3)}\) illustrates the opposite behavior; it does not contain a copy of itself. Moreover any children at level \(k \geq 2\) in \(T_{(2, 3)}\) has labels which are translations of the labels \((2, 7)\) and \((2, 4)\).

\[(2, 1) = [2, 1]\]

\[\tilde{\phi}_1 \quad \tilde{\phi}_2\]

\([1]_{1, 2} = (2, 3) = [2, 3]\]

\([1]_{2, 2} = (1, 1) = [2, 0]\]

\([3]_{1, 2} = (2, 7) = [2, 7]\]

\([3]_{2, 2} = (2, 4) = [2, 4]\]

\((2, 1) = [2, 1]\)

\(D_2 = (1, 2) = [2, -1]\)

\([2, 15] = (2, 15)\] \([2, 12] = (2, 12)\] \([2, 9] = (2, 9)\] \([2, 6] = (2, 6)\]

\((2, 3) \quad (1, 1) \quad (1, 4) \quad (1, 3)\)

For a vertex \(v \in V_P\) with \(\omega = W(v)\), let \(P_\omega\) denote the label \(L(v)\). From Proposition \([5.13]\) we have the following.

**Corollary 5.16.** Suppose \(\tau = (P, Q)\) is the transposition on two points \(P, Q \in ES(\phi)\). Then \(E(\phi^k(\tau))\) coincides with a product of \(k\) commuting transpositions \(\prod_{\omega \in \Omega_{k, l}} (P_\omega, Q_\omega)\) for each \(k \in \mathbb{N}\).

**Proof.** The base case follows directly from Proposition \([5.13]\) since \(\Omega_{k, 1} = \{1, \cdots, l\}\). Suppose that \(E(\phi^k(\tau)) = \prod_{\omega \in \Omega_{k, l}} \tau_\omega\) is a product of \(k\) commuting transpositions for \(k \geq 1\) where \(\tau_\omega = (P_\omega, Q_\omega)\).
By Proposition 5.13 again, we have
\[ E(\phi(\tau_\omega)) = \prod_{j=1}^{\ell} \left( (P_\omega)\tilde{\phi}_j, (Q)\omega \tilde{\phi}_j \right) \]
for all \( \omega \in \Omega_{\ell,k} \). So \( E(\phi(\tau_\omega)) = \prod_{\omega'} (P_{\omega'}, Q_{\omega'}) \) where \( \omega' = \omega \ell \in \Omega_{\ell,k+1} \) for \( l = 1, \cdots, \ell \). Therefore
\[ E(\phi^{k+1}(\tau)) = E(\phi^k(\tau)) = E(\phi(\prod_{\omega \in \Omega_{\ell,k}} \tau_\omega)) = \prod_{\omega \in \Omega_{\ell,k}} \left( E(\phi(\tau_\omega)) \right) \]
\[ = \prod_{\omega \in \Omega_{\ell,k}} \prod_{j=1}^{\ell} \left( (P_\omega)\tilde{\phi}_j, (Q)\omega \tilde{\phi}_j \right) \]
\[ = \prod_{\omega \in \Omega_{\ell,k}} \prod_{j=1}^{\ell} \left( (P_{\omega'})\tilde{\phi}_j, \omega \tilde{\phi}_j \right) \]
where \( \omega' = \omega \ell \). The commutativity of transpositions above follows from Proposition 5.17.
\[ \square \]
Recall the notation \( V^k_P \) which denotes the set of all vertices of \( T_P \) at level \( k \).

**Proposition 5.17.** The label map \( L : V^k_P \to \text{ES}(\phi) \) is injective for each \( k \in \mathbb{N} \).

**Proof.** Induction on \( k \). The decomposition \( \text{ES}(\phi) = \bigcup_{l=1}^{\ell} \mathcal{O}(D_l) \) shows that if \( l \neq l' \), \( (X_n)\tilde{\phi}_l = \mathcal{O}(D_l) \) does not intersect \( (X_n)\tilde{\phi}_{l'} = \mathcal{O}(D_{l'}) \). This establishes the base case. Suppose \( v \in V^{k+1}_P \). Observe that the last letter of \( W(v) \) determines which \( \mathcal{O}(D_l) \) the label \( L(v) \) belongs to. More precisely, \( L(v) \in \mathcal{O}(D_l) \) if and only if the last letter of \( W(v) \) is \( l \). If \( L(v) = L(v') = Q \in \mathcal{O}(D_l) \) for vertices \( v \) and \( v' \) at level \( k+1 \), then \( W(v) \) and \( W(v') \) share the same last letter \( l \), that is, \( W(v) = \eta l \) and \( W(v') = \eta' l \) for some \( \eta, \eta' \in \Omega_{\ell,k} \). Since \( \tilde{\phi}_l : X_n \to \mathcal{O}(D_l) \) is a bijection, the two vertices which correspond to \( \eta \) and \( \eta' \) respectively share the same label \( Q\tilde{\phi}_l^{-1} \). However this contradicts induction assumption.

**Stable points.** In Example 5.15 we saw that the trees \( T_{(2,1)} \) and \( T_{(1,1)} \) contain copies of themselves. However the tree \( T_{(1,2)} \) illustrates the opposite behavior. Intuitively this is because each \( \tilde{\phi}_l \) applied to \((i,k)\) ‘doubles’ the second coordinate for all points of its support but finitely many. We need to formulate this rigorously.

Let \( \ell \geq 2 \). We want to define *stable points* for \( \phi(g_i) \) with the assumption that \( \phi(g_i) \) has a unique target \( R_i \) for \( i = 2, \cdots, n \). Fix \( i \). Recall that \( \phi(g_i) \) translates points of \( R_i \) by \(+\ell\) up to a finite set. In other words, the action of \( \phi(g_i) \) on \( R_i \) is eventually stabilized as a translation by \(+\ell\). Being a restriction of \( \phi(g_i) \), \( p_{l,i} \) is also eventually stabilized for all \( 1 \leq l \leq \ell \). From property (5–12) one can take an integer \( 0 \leq k_{l,i} \) for each \( l \) such that if \([k]_{l,i} = (i, m)\) then \([k+1]_{l,i} = (i, m + \ell)\) for all \( k_{l,i} \leq k \). More precisely, for each \( l \), there exists a smallest integer \( k_{l,i} \) such that
\[ (5–32) \quad [k]_{l,i} = (i, m_{l,i} + \ell(k - k_{l,i})) \]
for all \( k_{l,i} \leq k \) where \([k+1]_{l,i} = (i, m_{l,i})\). The \( i^{th} \) *threshold* is the positive integer \( s_i \) defined by
\[ (5–33) \quad s_i = \max_{1 \leq l \leq \ell} \left\{ \left\lfloor \frac{k_{l,i} - m_{l,i}}{\ell - 1} \right\rfloor, k_{l,i} \right\} \]
where \([\ast]\) stands for the smallest integer function. A point \((i, k)\) is called a *stable point* if \( s_i < k \) where \( 2 \leq i \leq n \). Recall the coordinate system (5–29); \([i, k] = (i, k)\) for \( 2 \leq i \leq n \) and \( k \in \mathbb{N} \). One crucial observation is that if \([i, k] \) is a stable point, \((i, k)\tilde{\phi}_l \) is again a stable point for all \( l \). A stable point \([i, k] \) satisfies \( k_{l,i} < k \) for all \( l \). By (5–32) we have
\[ (5–34) \quad ([i, k])\tilde{\phi}_l = [k]_{l,i} = (i, m_{l,i} + \ell(k - k_{l,i})). \]
for all $k > s_i$. Since
\[ k < m_{l,i} + \ell(k - k_{l,i}) \iff \frac{\ell k_{l,i} - m_{l,i}}{\ell - 1} < k, \]
satisfies $s_i < k < m_{l,i} + \ell(k - k_{l,i})$ provided $s_i < k$. So $([i, k])\tilde{\phi}_l$ is a stable point if $[i, k]$ is a stable point for $2 \leq i \leq n$.

We can define stable points on $R_1$ in a similar manner because all the actions of $\phi(g_i)$'s on $R_1$ are also eventually stabilized as a translation by $-\ell$. However, we need to change signs and take reverse inequalities accordingly. It suffices to consider one partial translation, say $p_{l,2}$, of $\phi(g_2)$ since actions of $p_{l,1}$ and $p_{l,2}$ are identical on $\{[k]_{l,2} : k \leq 0\}$ for $3 \leq i \leq n$. There exists a largest integer $k_l \leq 0$ such that
\[
[k]_{l,2} = (1, m_l + \ell(k_l - k))
\]
for all $k \leq k_l$ where $[k_{l,2}] = (1, m_l)$. Now points $(1, k) \in R_1$ are called stable points for all $s_1 < k$ where $s_1$ is the positive integer defined by
\[
s_1 = \max\left\{ \left\lceil \frac{\ell - \ell k_l - m_l}{\ell - 1} \right\rceil, -k_l + 1 \right\}
\]
The above definition implies that the image of $(1, k)$ under $\tilde{\phi}_l$ is a stable point if $(i, k) = [2, -k + 1]$ (by the coordinate system of $\phi(g_2)$) is a stable point. Since $s_1 < k \iff -k + 1 < -s_1 + 1 < k_l$ we have, by $[5–35]$, 
\[
([2, -k + 1])\tilde{\phi}_l = [-k + 1]_{l,2} = (1, m_l + \ell(k_l - (-k + 1))) = (1, m_l + \ell(k_l + k - 1))
\]
for all stable points on $R_1$. So we have $s_1 < k < m_l + \ell(k_l + k - 1)$ from
\[
k < m_l + \ell(k_l + k - 1) \iff \frac{\ell - \ell k_l - m_l}{\ell - 1} < k.
\]

Let $S$ denote the set of all stable points. We summarize the discussion above as

**Remark 5.18.** If $P = (i, p)$ is a stable point then $(P)\tilde{\phi}_l = (i, q)$ with $p < q$ for each $1 \leq l \leq \ell$, $1 \leq i \leq n$. In particular, $S$ is invariant under $\tilde{\phi}_l$ for each $l$.

**Translations on a ray.** Let $v \in V_P$ with $\omega = W(v)$. The descendants of $v$ at a depth $m \in \mathbb{N}$ consists of vertices $u$ such that $W(u) = \omega_{v,1}$ for some $\omega_{v,1} \in \Omega_{v,m}$. Let $D_{v,m}$ denote the set of descendants of $v$ with depth $m$, and let $D_v = \cup_m D_{v,m}$. The children of $v$ is the descendants of $v$ at depth 1. Note that $V_P^k = \cup_{v \in V_P} V_{v,1}^{k-1}$ for all $k$. In case an ordered $\ell$-tuple $L(D_{v,1}) = ((i, p_1), \ldots, (i, p_\ell))$ we suppress the first coordinate and take a vector $\nu = [p_1, \ldots, p_\ell] \in \mathbb{N}^\ell$ to express $L(D_{v,1})$. The translation of an $\ell$-tuple $\nu$ by $t \in \mathbb{Z}$ is the $\ell$-tuple
\[
\nu + t = [p_1 + t, \ldots, p_\ell + t].
\]

We require that all points of $\nu$ and its translation $\nu + t$ stay in the same ray $R_i$. Let $B_{n,r}$ denote the ball of $X_n$ with radius $r$,
\[
B_{n,r} = \{(i, p) | p \leq r\}.
\]
Recall that $\phi(g_i)$ acts as a translation on $\{(i, p) | p > s_i\}$ and $\{(1, q) | q > s_i\}$ by $\ell$ and $-\ell$ respectively for all $2 \leq i \leq n$ where $s_i$ is the $i^{th}$ threshold defined in $\{5–33\}$ and $\{5–36\}$. Consequently $\phi(g_i)$ acts by the same manner on $R_i \setminus B_{n,s}$ and $R_1 \setminus B_{n,s}$ for
\[
s = \max\{s_1, \ldots, s_n\}.
\]

**Proposition 5.19.** Suppose $P \in S \cap R_i$, $1 \leq i \leq n$. The label map $L : V_P \rightarrow ES(\phi)$ is injective and $L(V_P) \subset S \cap R_i$. For each $v \in V_P$, $L(D_{v,1})$ is a translation of $L(V_P)$. Moreover there exists a constant $A_0 = A_0(P)$ such that $L(V_P^k) \subset B_{n,r}$ with $r = A_0 \ell^k + s$
Proof. First we consider the case when \( P = (i, p) \) is a stable point for \( 2 \leq i \). By Remark \ref{remark:stable_points}, we have \( L(v) = (i, q) \) with \( p < q \) for all \( v \in V_P \), and so \( L(V_P) \subseteq S \cap R_i \). Suppressing \( i \) in the first coordinate we can write \( L(V_P^i) = [p_1, \ldots, p_\ell] \) where

\[
(i, p_l) = ([i, p])\tilde{\phi}_l = [p]_{l,i} = (i, m_{l,i} + \ell(p - k_{l,i})
\]

for \( l = 1, \ldots, \ell \) by \((\ref{translation_on_root})\). The label map \( L \) restricted on the root vertex and \( V_P^1 \) is injective since \( p < p_l \) for all \( l \) and each \( p_l \) belongs to \( O(D_1) \) which are all disjoint for \( l = 1, \ldots, \ell \). With this base case assume that the map \( L \) is injective on the set of vertices up to level \( k \). Since no vertex at level \( k + 1 \) attains the label \((i, p) = P \) it suffices to check whether two vertices \( v \) and \( v' \) (other than the root vertex) at levels \( \leq k + 1 \) share the same label \( L(v) = L(v') \). We can apply similar argument as in Proposition \ref{proposition:isospectral} to draw a contradiction; \( L(v) = L(v') \) implies that the ascendants of \( v \) and \( v' \) share the same label at levels \( \leq k \).

For the second assertion suppose \( v \in V_P \) with \( L(v) = (i, q) \). Since \((1, q) \) is a stable point the identity \((\ref{is_first})\) implies that \( L(D_{v,1}) = [q_1, \ldots, q_\ell] \) where

\[
(i, q_l) = \left(i, m_{l,i} + \ell(q - k_{l,i})\right) = \left(i, m_{l,i} + \ell(p - k_{l,i}) + \ell(q - p)\right)
\]

for \( l = 1, \ldots, \ell \). Since \( p < q \)

\begin{equation}
(\ref{translation_on_root})
\end{equation}

is the translation of \( L(V_P^1) \) by \( \ell(q - p) > 0 \).

Since \( L(v) = (i, q) \) for all \( v \in V_P^k \), we can consider a sequence of natural numbers \( \{a_k\} \) so that \( a_k \) denotes the maximum of such \( q \)'s. We want to show \( a_k \leq s\ell^k + s \) for all \( k \). From the rewriting

\begin{equation}
(\ref{rewrite_translation})
\end{equation}

we have \( a_1 \leq A_0 \ell + s \) where

\[
A_0 = \max_{1 \leq i \leq \ell} \left\{ p - \frac{\ell k_{l,i} - m_{l,i}}{\ell - 1} \right\}.
\]

(since \( P = (i, p) \) is a stable point the above maximum is taken over positive numbers, and so \( A_0 > 0 \)). One can rewrite \( a_{k+1} = m_{l,i} + \ell(a_k - k_{l,i}) \), which follows from the second step above, as in \((\ref{rewrite_translation})\) with \( p \) replaced by \( a_k \) to check that

\begin{equation}
(\ref{rewrite_translation})
\end{equation}

\[
A_0 = \max_{1 \leq i \leq \ell} \left\{ p - \frac{\ell k_{l,i} - m_{l,i}}{\ell - 1} \right\}.
\]

for all \( k \in \mathbb{N} \). Therefore \( L(V_P^k) \subseteq B_{n,a_k} \) and \( a_k \leq A_0 \ell^k + s \) for all \( k \in \mathbb{N} \).

In case \( P = (1, p) \) one can apply analogous arguments. Remark \ref{remark:stable_points} together with 'cancelling argument' shows that \( L(v) = (1, q) \) is a stable point for all \( v \in V_P \). By \((1, q) = [2, -q + 1] \) and \((\ref{translation_on_root})\), we have that \((1, q)\tilde{\phi}_l \) becomes

\[
([2, -q + 1])\tilde{\phi}_l = [-q + 1]_{l,2} = \left(1, m_l + \ell(k_l + q - 1)\right) = \left(1, m_l + \ell(k_l + p - 1) + \ell(q - p)\right)
\]

for all \( l \). Therefore \( L(D_{v,1}) \) is a translation of \( L(V_P^1) \) by \( \ell(q - p) > 0 \). From the rewriting for the root \( P = (1, p) = [2, -p + 1] \),

\[
([2, -p + 1])\tilde{\phi}_l = m_l + \ell(k_l + p - 1) = \ell\left(p - \frac{\ell - m_l - \ell k_l}{\ell - 1}\right) + \frac{\ell - m_l - \ell k_l}{\ell - 1}
\]

we find \( a_1 = A_0 \ell + s \) such that \( L(V_P^1) \subseteq B_{n,a_1} \) where

\[
A_0 = \max_{1 \leq l \leq \ell} \left\{ p - \frac{\ell - m_l - \ell k_l}{\ell - 1} \right\}.
\]

Finally the identity \( a_{k+1} = m_l + \ell(k_l + a_k - 1) \) implies that \( L(V_P^k) \subseteq B_{n,a_k} \) where the sequence \( \{a_k\} \) satisfies \((\ref{rewrite_translation})\) for all \( k \in \mathbb{N} \). □
Remark 5.20. We remark that the radii of balls that contain $L(V^k_P)$ can be bound by a linear term of $\ell^k$ even when $P$ is not a stable point. Intuitively this happens because the images of $P$ under expanding map $\mathcal{E}_k$ travel inside the ball $B_{n,a}$ for first finite steps. Only after does $L(V^k_P)$ contain stable points the radii of the balls contain $V^k_P$ follow the growth as described in Proposition 5.19. More precisely, a ball $B_{n,a}$ that contains $L(V^k_P)$ has radius

$$a_k \leq \begin{cases} s & k \leq k_0 \\ A_0 \ell^{k-k_0} + s & k_0 + 1 \leq k \end{cases}$$

for some constant $A_0$. In all, every $P \in \mathcal{S}(\phi)$ determines a constant $A_0$ such that $L(V^k_P) \subset B_{n,r}$ with $r \leq A_0 \ell^k + s$.

Intervals of $V^k_P$. We want to decompose $V^k_P$ into intervals which provide a coarser decomposition than $V^k_P = \bigcup_{v \in V^{k-1}_P} D_{v,1}$. Motivating examples come from stable points. Proposition 5.19 states that $L(D_{v,1})$ is a translation of $L(V^1_P)$ for all $v \in V^{k-1}_P$ if $P$ is a stable point. The set $\{L(D_{v,1}) : v \in V^{k-1}_P\}$ can be viewed as a single orbit of $V^1_P$ under translation. In this case, we want to take $V^k_P$ as a single interval.

Let $v \in V^{k_1}_P$ with $k_1 < k$. We say that $D_{v,k-k_1} \subset V^k_P$ is an interval of $V^k_P$ if either $v \in V^{k_1}_P$ or $L(v)$ is a stable point. (The naming ‘interval’ comes from that if $D_{v,k-k_1}$ is an interval, $\Lambda(D_{v,k-k_1})$ forms an interval of $\mathbb{N}$ with size $\ell^{k-k_1}$ where $\Lambda : V^k_P \to \Omega_{\ell,k} \to \{1, \ldots, \ell^k\}$ is the unique map such that $\Lambda(v) = j$ if $W(v)$ is the $j$th sequence in the lexicographic order on $\Omega_{\ell,k}$.) Note that an interval may contain smaller intervals. For example, a descendant $u$ of $v$ determines an interval $D_u \cap V^k_P$ if $u \in V^{k_2}_P$ with $k_2 < k$. However we can take maximal intervals so that $V^k_P$ consists of minimum number of intervals. For each $k \in \mathbb{N}$, let $N_P(k)$ denote the minimum number of intervals which cover $V^k_P$. We remark that $V^k_P$ can be expressed as a disjoint union of $N_P(k)$ intervals.

In view of Proposition 5.19, we say a tree $T_P$ is stabilized if $L : V_P \to \mathcal{S}(\phi)$ is injective. Let $\mathcal{U} \subset \mathcal{S}(\phi)$ denote the set of all points $P$ such that $L : V_P \to \mathcal{S}(\phi)$ is not injective. Proposition 5.19 implies that the cardinality of $\mathcal{U}$ is finite since $\mathcal{U} \subset \mathcal{S}(\phi) \setminus \mathcal{S}$. (example)

Lemma 5.21. There exists a constant $A_1 = A_1(\phi)$ such that $N_P(k) \leq (k - 1)(\ell - 1)A_1 + 1$ for all $P \in \mathcal{S}(\phi)$ and $k \in \mathbb{N}$.

Proof. Proposition 5.19 implies that $N_P(k) = 1$ for all $P \in \mathcal{S}$. Indeed $N_P(k)$ is bounded by a constant for all $P \notin \mathcal{U}$. If $L : V_P \to \mathcal{S}(\phi)$ is injective, one can find $k_0 \in \mathbb{N}$ such that $L(V^k_P)$ consists of stable points for all $k \geq k_0$ since $\mathcal{S}(\phi) \setminus \mathcal{S}$ is finite. From the decomposition

$$V^k_P = \bigcup_{u \in V^{k_0}_P} D_{u,k-k_0},$$

where each component is an interval of $V^k_P$, we have $N_P(k) \leq |V^{k_0}_P| = \ell^{k_0}$ for all $k \geq k_0$. So $N_P(k) \leq \ell^{k_0}$ for all $k \in \mathbb{N}$ since the function $N_P(k)$ is monotone increasing on $k$. Taking maximum of upper bounds $\ell^{k_0}$ over all $P \in \mathcal{S}(\phi) \setminus \mathcal{S}$, which is a finite set, one obtains a constant $A_1 \geq 1$ such that $N_P(k) \leq A_1$ for all $P \in \mathcal{S}(\phi) \setminus \mathcal{S}$. To establish desired bound for points of $\mathcal{U}$ we need following steps together with induction on $k$.

Step 1. A vertex $v$ of $T_P$ and its descendants determine a unique subtree $T_v$. Note that a pair of subtrees $T_v$ and $T_v'$ satisfies the following dichotomy: they do not intersect or one contains the other. In this step we show that if $L(v) = L(v')$ then either $T_v \subset T_{v'}$ or $T_{v'} \subset T_v$.

Suppose $L(v) = L(v')$ for $v \neq v'$. Then $W(v) = \omega l$ and $W(v') = \omega' l$ for some $l$. As in proof of Proposition 5.17, we can cancel the last same letters of two sequences. We have either $\omega l$ is a subword of $\omega' l$ or vice versa since $v \neq v'$. This precisely means that one subtree contains the other.
Step 2. We claim that $P \in U$ if and only if $\mathcal{T}_P$ contains a vertex $v$ other than the root with $L(v) = P$. We need to show that if the label map $L : \mathcal{T}_P \to \text{ES}(\phi)$ is not injective then $V_P$ contains $v$ at level $k \geq 1$ with $L(v) = P$. Suppose $V_P$ contains $v_1$ and $v_2$ with $L(v_1) = L(v_2) = Q$. From Step 1 we know that one of subtrees contains the other. We may further assume that one of two trees, say $\mathcal{T}_{v_1}$, is maximal, which means that $\mathcal{T}_{v_1}$ contains all subtrees determined by $v$ with $L(v) = Q$. Consequently $\omega_1 = W(v_1)$ is a subsequence of $\omega_2 = W(v_2)$, i.e., $\omega_2 = \omega_1 \eta$ for some sequence $\eta$. The first case to consider is when $|\omega_1| \leq |\eta|$. Using the same canceling argument as in Step 1, one can show that $\eta = \eta_0 \omega_1$ for some sequence $\eta_0$. The sequence $\omega_1 = l_1 \cdots l_k$ determines a composition of injective maps such that $(P)\tilde{\phi}_{l_1} \cdots \tilde{\phi}_{l_k} = Q$. Observe that $\omega_1$ also determines a unique path from the vertex $u$, which corresponds to $\omega_1 \eta_0$, to $v_2$. This path transforms into the same composition of maps such that $(L(u))\tilde{\phi}_{l_1} \cdots \tilde{\phi}_{l_k} = L(v_2) = Q$. So $u$ is a vertex with $L(u) = P$. The next case to consider is when $|\omega_1| > |\eta|$. The cancelling argument implies that $\omega_1$ is the concatenation $\omega_1 = \omega_0 \eta$ for some sequence $\omega_0$. Since the sequence $\eta = l_1' \cdots l_m'$ determines identity map $(Q)\tilde{\phi}_{l_1'} \cdots \tilde{\phi}_{l_m'} = (L(v_1))\tilde{\phi}_{l_1} \cdots \tilde{\phi}_{l_k} = L(v_2) = Q$ we have $(L(u))\tilde{\phi}_{l_1'} \cdots \tilde{\phi}_{l_m'} = L(v_1) = Q$ for the vertex $u$ corresponding to $\omega_0$. So $L(u) = Q$. However this contradicts the maximality of the tree $\mathcal{T}_{v_1}$.

Step 3. Next we show that $V_P^1$ contains only one vertex $v$ with $L(v) \in U$. Suppose there exist two vertices $v_1$ and $v_2$ at level 1 such that $L(v_1) \in U$, $i = 1, 2$. By Step 2, $\mathcal{T}_{v_i}$ contains a vertex $u_i$ with $L(u_i) = L(v_i)$, $i = 1, 2$. Suppose $L(v_1) = (P)\tilde{\phi}_i$ for some $l$. Since $L(u_1) = L(v_1)$ the parent of $u_1$ has label $P$. So $\mathcal{T}_{v_1}$ contains a vertex $w_1$ such that $L(w_1) = P$. By the same reason $\mathcal{T}_{v_2}$ contains a vertex $w_2$ with $L(w_2) = P$. Since two subtrees $\mathcal{T}_{v_1}$ and $\mathcal{T}_{v_2}$ do not intersect, $\mathcal{T}_P$ contains two disjoint copies of itself which are determined by $w_1$ and $w_2$. Obviously this can not occur by Step 1.

Step 4. We complete the proof by induction on $k$. The base case is obvious since $N_\ell^1 = 1$ for any $P \in \text{ES}(\phi)$. From Step 3, we know that $V_P^1 = \{v_1, \ldots, v_\ell\}$ contains one vertex $v_j$ with $L(v_j) = Q_j \in U$. Since all other subtrees $\mathcal{T}_{v_i}$ are stabilized there exists a constant $A_1$ such that $N_{Q_j}(k-1) \leq A_1$ for all $Q_j = L(v_j)$ and $k \in \mathbb{N}$, $i \neq j$. We complete the proof by

$$N_P(k) \leq \sum_{1 \leq l \leq \ell} A_1 Q_l(k-1) \leq A_1 Q_j(k-1) + (\ell - 1)A_1 \leq (k-1)(\ell - 1)A_1 + 1.$$ 

\hfill \Box

Lemma 5.22. Suppose $\tau = (P,Q)$ is the transposition on two points $P, Q \in \text{ES}(\phi)$. There exists a polynomial $\rho$ on $k \in \mathbb{N}$ whose degree does not depend on $k$ such that $|\phi^k(\tau)| \leq \rho(k)\ell^k$ for all $k \in \mathbb{N}$.

Proof. First we show the existence of such a polynomial when $P$ and $Q$ are stable points, and then we extend the discussion to the general case. Suppose $P = (i,p)$ is a stable point. Observe that explicit expressions for $L(v)$ can be obtained for $v \in V_P^1$ whenever $W(v)$ is given. We want to find such a expressions using $L(V_P^1)$. For each $1 \leq l \leq \ell$, let $V_P^1 = \{v_1, \ldots, v_\ell\}$ with $L(v_1) = (i,p_1) = (P)\tilde{\phi}_1$, and let $d_l = p_l - p$. We claim that if $W(v) = l_1l_2 \cdots l_{k+1} \in \Omega_{\ell,k+1}$ then $L(v) = (i,p_v)$ where

$$p_v = d_{l_1} \ell^k + d_{l_2} \ell^{k-1} + \cdots + d_{l_k} \ell + p_{k+1}$$

for each $k \in \mathbb{N}$. The base case for $k = 1$ follows immediately from (5-37). Assume $u \in V_P^{k+2}$ with $W(u) = l_1l_2 \cdots l_{k+2}$. The unique parent vertex $v$ of $u$ with $W(v) = l_1l_2 \cdots l_{k+1}$ has label $L(v) = (i,p_v)$ which satisfies (5-40). By the identity (5-37), $L(u) = (i,p_u)$ satisfies

$$p_u = \ell(p_v - p) + p_{k+2} = \ell(d_{l_1} \ell^k + \cdots + d_{l_k} \ell + p_{k+1} - p) + p_{k+2} = d_{l_1} \ell^{k+1} + \cdots + d_{l_k} \ell + p_{k+2}$$

since $u$ is the $(l_{k+1})^{th}$ child of $v$.

Next we want to describe the relationship between $L(D_{v_1,k})$ and $L(V_P^1)$ for $1 \leq l \leq \ell$ and $k \in \mathbb{N}$. A vertex $v_l \in V_P^1$ determines a set of vertices $D_{v_l,k} \subset V_P^{k+1}$ with $|D_{v_l,k}| = \ell^k = |V_P^1|$. Observe
that each vertex \( u \in D_{v_i,k} \) corresponds to \( W(u) = l\omega \) for some \( \omega \in \Omega_{t,k} \). The concatenation \( W(v) \mapsto lW(v) \) induces a bijection between \( V_P^k \) and \( D_{v_i,k} \); \( V_P^k \ni v \iff u \in D_{v_i,k} \) if \( W(u) = lW(v) \).

We can further show that \( L(D_{v_i,k}) \) is a translation of \( L(V_P^k) \) by \( d_l\ell^k \) for each \( l \) and \( k \). Using the expression we compare \( L(v) \) and \( L(u) \) for a corresponding pair under the above bijection. If \( v \in V_P^k \) with \( W(v) = l_1 \cdots l_k \), the \( L(v) = (i, p_v) \) satisfies

\[
p_v = d_{l_1}\ell^k - 1 + \cdots + d_{l_{k-1}}\ell + p_{l_k}.
\]

The corresponding vertex \( u \in D_{v_i,k} \) with \( W(u) = ll_1 \cdots l_k \) has label \( L(u) = (i, p_u) \) where

\[
p_u = d_l\ell^k + d_{l_1}\ell^k - 1 + \cdots + d_{l_{k-1}}\ell + p_{l_k}.
\]

The we have the desired difference \( p_u - p_v = d_l\ell^k \). Therefore the above bijection between \( V_P^k \) and \( D_{v_i,k} \) transforms into a translation of \( L(V_P^k) \) by \( d_l\ell^k \) for all \( l \) and \( k \), i.e., the following diagram commute

\[
\begin{array}{ccc}
V_P^k & \xrightarrow{L} & D_{v_i,k} \\
\downarrow & & \downarrow \\
L(V_P^k) \subset R_i & +d_l\ell^k & L(D_{v_i,k}) \subset R_i
\end{array}
\]

We are ready to find a polynomial \( \rho(k) \) such that \( |\phi^k(\tau)| \leq \rho(k)\ell^k \) for all \( k \) when \( P = (i, p) \) and \( Q = (j, q) \) are stable points where \( 1 \leq i, j \leq n \). Note that we want to make sure that the degree of \( \rho(k) \) does not depend on \( k \). Let \( V_Q^1 = \{u_1, \cdots, u_\ell\} \) with \( L(u_l) = (j, q_l) = (Q)\phi_i \), and let \( d'_l = q_l - q \) for \( 1 \leq i \leq \ell \). To deal with the base case \( k = 1 \), we shall take \( \rho(k) \) with \( p(1) = |\phi(\tau)| \). Corollary 5.16 implies that

\[
E(\phi^{k+1}(\tau)) = \prod_{\omega \in \Omega_{t,k+1}} (P_\omega, Q_\omega).
\]

To establish desired bounds for \( |\phi^{k+1}(\tau)| \) let us regroup those \( \ell^{k+1} \) transpositions into \( \ell \) subcollections as follows. The sets \( V_P^1 = \{v_1, \cdots, v_\ell\} \) and \( V_Q^1 = \{u_1, \cdots, u_\ell\} \) provide canonical decompositions

\[
V_{P,k}^{k+1} = \bigsqcup_{1 \leq l \leq \ell} D_{v_i,k} \quad \text{and} \quad V_{Q,k}^{k+1} = \bigsqcup_{1 \leq l \leq \ell} D_{u_i,k}.
\]

Let \( \sigma_l \) be the product

\[
\sigma_l = \prod_{\omega \in W(D_{v_i,k})} (P_\omega, Q_\omega) = \prod_{\omega = \omega'} (P_\omega, Q_\omega)
\]

for \( 1 \leq l \leq \ell \). Note that each \( \sigma_l \) is the restriction of \( E(\phi^{k+1}(\tau)) \) on a \( \phi^{k+1}(\tau) \)-invariant subset \( L(D_{v_i,k}) \cup L(D_{u_i,k}) \). So \( E(\phi^{k+1}(\tau)) = \sigma_1 \cdots \sigma_\ell \). Since \( L(D_{v_i,k}) \subset R_i \) and \( L(D_{u_i,k}) \subset R_j \) are translations of \( L(V_P^k) \) and \( L(V_Q^k) \) by \( d_l\ell^k \) and \( d'_l\ell^k \) respectively for each \( l \), \( \sigma_l \) can be expressed as a conjugation of \( E(\phi^k(\tau)) \). We need to consider three cases depending on \( i \) and \( j \).

**Case I.** \( 2 \leq i \neq j \leq n \). Define \( \beta_l \) by

\[
\beta_l = g_l d_l\ell^k g_l'.
\]

for \( 1 \leq l \leq \ell \). Then \( \sigma_l \) is the conjugation of \( \phi^k(\tau) \) by \( \beta_l \). Setting \( d = 1/4 \max\{d_1, \cdots, d_\ell, d_1', \cdots, d'_\ell\} \) we can expect

\[
|\sigma_l| = |E(\phi^k(\tau))| + 2|\beta_l| \leq \rho(k)\ell^k + 2(d_l + d'_l)\ell^k \leq \left(\rho(k) + d\right)\ell^k
\]
for each \( l \). Obviously there exists a polynomial \( \rho(k) \) of degree \( d \) with the condition \( p(1) = |\phi(\tau)| \) such that

\[
|E(\phi^{k+1}(\tau))| \leq \sum_{1 \leq \ell} |\sigma_l| = \left( \rho(k) + d \right) \ell^{k+1} < \rho(k + 1) \ell^{k+1}.
\]

For example one can take \( \rho(k) = k^d + |\phi(\tau)| \).

**Case II.** \( j = 1, 2 \leq i \leq n \). Take \( i' \neq i \) to define \( \beta_i \) as

\[
\beta_i = (g_{i'}^{-1} g_i) d_l \ell^k = g_i' \ell^k
\]

for \( 1 \leq \ell \leq \ell \). Since each \( \sigma_l \) is the conjugation of \( \phi^k(\tau) \) by \( \beta_i \) for each \( l \), we have

\[
|\sigma_l| = |E(\phi^k(\tau))| + 2|\beta_l| \leq \rho(\ell \ell^k) + 2(2d_l + d'_l) \ell^k \leq \left( \rho(k) + d \right) \ell^k
\]

where \( d = 1/6 \max\{d_1, \ldots, d_\ell, d_1', \ldots, d'_\ell\} \). Therefore we can choose a polynomial \( \rho(k) \) which satisfies the inequality \((5-43)\) as well as the condition on \( p(1) \).

**Case III.** \( i = j \). In this case it may not be useful to express \( \sigma_l \) as a conjugation of \( E(\phi^k(\tau)) \) by \( \beta_i \). It seems that \( |\beta_l| \) can be larger than we expected as \( \beta_i \) translates \( L(D_{l1}, k) \) and \( L(D_{l1} k) \) independently by distinct amounts \( d_l \ell^k \) and \( d'_l \ell^k \) on the same ray. Instead of induction on \( k \), we can use a pattern that the transpositions of \( E(\phi^k(\tau)) \) follow. Observe that if \( P = (i, p) \) and \( Q = (i, q) \) are stable points, then \( L(V_P^q) \) is a translation of \( L(V_Q^q) \) on the ray \( R_l \) for each \( k \in \mathbb{N} \). Assume that \( q < p \) and \( 2 \leq k \). For \( k = 1 \), it is each to check that

\[
p_v - q_u = m_{t,i} + \ell(p - k_{t,i}) - \left( m_{t,i} + \ell(q - k_{t,i}) \right) = \ell(p - q)
\]

for each \( l \). So \( L(V_P^q) \) is the translation of \( L(V_Q^q) \) by \( \ell(p - q) \). Since

\[
d_l - d'_l = (p_v - p) - (q_u - q) = (p_v - q_u) - (p - q) = (\ell - 1)(p - q)
\]

for all \( l \), we can use the expression \((5-40)\) to compare \( L(v) = (i, p_v) \) and \( L(u) = (i, q_u) \) for each pair of vertices \( v \in V_P^q \) and \( u \in V_Q^q \) with \( W(v) = W(u) \). If \( W(v) = l_1 l_2 \cdots l_k \), then we have

\[
p_v - q_u = \left( d_{l_1} \ell^{k-1} + \cdots + d_{l_{k-1}} \ell + p_{l_k} \right) - \left( d'_{l_1} \ell^{k-1} + \cdots + d'_{l_{k-1}} \ell + q_{l_k} \right)
\]

\[
= (d_{l_1} - d'_{l_1}) \ell^{k-1} + \cdots + (d_{l_{k-1}} - d'_{l_{k-1}}) \ell + (p_{l_k} - q_{l_k})
\]

\[
= (\ell - 1)(p - q)(\ell^{k-1} + \cdots + \ell) + (p - q),
\]

which does not depend on the choice of \( v \) and \( u \). Therefore the whole set \( L(V_P^q) \) is a translation of \( L(V_Q^q) \) by \( y = p_v - q_u \). In particular, all \( \ell^k \) transpositions of \( \phi^k(\tau) \) are translations of one transposition on \( R_i \). So we can write \( \phi^k(\tau) \) as

\[
E(\phi^k(\tau)) = \prod_{1 \leq m \leq \ell^k} (x_m, x_m + y)
\]

where \((x, x')\) stands for the transposition exchanging \((i, x)\) and \((i, x')\). Since all transpositions in \((5-42)\) commute with each other we can arrange them so that \( x_m < x_{m'} \) for all \( m < m' \). Set \( \tau_1 = (x_1, x_1 + y) \) to express \((x_m, x_m + y)\) as a conjugation of \( \tau_1 \) by \( g_i x_m - x_1 \) for all \( m \). Cancelling subwords \( g_i g_i^{-1} \) we obtain

\[
\begin{align*}
|E(\phi^k(\tau))| &= |\tau_1 g_i x_2 - x_1 \cdots g_i x_{m-1}| \\
&= |\tau_1 g_i^{-x_2 - x_1} \cdots g_i^{-x_{m-1}}| \\
&\leq |\tau_1| + 2(x_m - x_1)
\end{align*}
\]
To bound $|\tau|$ we can use the fact that a transposition $\tau_0$ with $\text{supp} \tau_0 \subset B_{n,r}$ has length at most $5r$. Proposition 5.19 implies that $\text{supp} \tau_1 \subset B_{n,r}$ with $r \leq A_0 \mathcal{e}^k + s$ for some constant $A_0$ which does not depend on $k$. Thus $|\tau_1| \leq 5(A_0 \mathcal{e}^k + s)$. Proposition 5.19 also implies that $x_m + y \leq A_0 \mathcal{e}^k + s$ for all $m$. So

$$|E(\phi^k(\tau))| \leq 5\ell(A_0 \mathcal{e}^k + s) + 2(A_0 \mathcal{e}^k + s)$$

for all $k$. Therefore there exists a polynomial $\rho(k) = 5A_0 \ell + 2A_0 + 5s\ell + 2s$ such that $|E(\phi^k(\tau))| \leq \rho(k)\mathcal{e}^k$ for all $k \in \mathbb{N}$.

In case $P$ and $Q$ are stable points of $R_1$ one can apply similar argument and calculation after expressing transpositions of (5–43) as conjugations of $\tau_1$ by negative powers of $g_i$. So far we have shown that if $\tau$ exchanges two stable points there exists a polynomial $\rho$ such that

$$|E(\phi^k(\tau))| \leq \rho(k)\mathcal{e}^k$$

for all $k$ where the degree of $\rho(k)$ does not depend on $k$.

Suppose $P, Q \in \text{ES}(\phi)$. For each $k \in \mathbb{N}$, $V_P^k$ and $V_Q^k$ can be expressed as disjoint unions of $N_P(k)$ and $N_Q(k)$ intervals respectively. The two collections of intervals determine two partitions on $\{1, \ldots, \ell^k\}$ via the map $A : V_P^k \rightarrow \{1, \ldots, \ell^k\}$. The common refinement of two partitions determine a set of common intervals for both $V_P^k$ and $V_Q^k$. Let $\mathcal{I}_k = \{I_1, \ldots, I_N\}$ and $\mathcal{I}_k' = \{I_1', \ldots, I'_N\}$ denote the common intervals for $V_P^k$ and $V_Q^k$ respectively, i.e., $W(I_j) = W(I'_j)$ for all $j$. Lemma 5.21 guarantees that $N \leq 2A_1 k\ell$ where $A_1$ is a constant determined by $\phi$. Using those common intervals we can write $E(\phi^k(\tau)) = \prod_{1 \leq j \leq N} \sigma_j$ where

$$\sigma_j = \prod_{\omega \in W(I_j)} (P_\omega, Q_\omega)$$

for all $k$. Note that an interval of has size $\ell^m$ for some $m \in \mathbb{N}$ by definition. If an interval $I_j$ has size $\ell$ then $\sigma_j$ is a product of $\ell$ transpositions on $L(I_j) \cup L(I'_j)$ whose supports belong to a ball of radius $A_0 \mathcal{e}^k + s$. So we have a bound

$$|\sigma_j| \leq 5(A_0 \mathcal{e}^k + s) \leq (5A_0 + s)\ell^k = \rho_j(k)\ell^k.$$  

If an interval $I_j$ has size $\ell^m$ with $2 \leq m \leq k$ then $I_j$ and $I'_j$ were common subintervals of original intervals for $V_P^k$ and $V_Q^k$ that we started with. Thus there exist $u \in V_P$ and $v \in V_Q$ such that $I_j = D_{u,m} \cap V_P^k$ and $I'_j = D_{u,m} \cap V_Q^k$. Since both $L(u)$ and $L(v)$ are stable points, we have a polynomial as in (5–43) to bound $|\sigma_j|$ by

$$|\sigma_j| \leq \rho_j(m)\ell^m.$$  

So far we have found upper bounds for individual $\sigma_j$ depending on the size of the interval $I_j$. To obtain a universal bound consider the set of all polynomials $\{\rho_1, \cdots, \rho_N\}$ that we need for (5–44) and (5–45). Let $\rho$ denote the polynomial in the above set with the largest degree (or fastest growth). Assuming that $\rho$ is increasing on $\mathbb{N}$, we have

$$|\sigma_j| \leq \rho(k)\mathcal{e}^k$$

for all $j$. Thus we have found a polynomial $\ell\rho(k)$ such that

$$|E(\phi^k(\tau))| = \sum_{1 \leq j \leq N} |\sigma_j| \leq \ell\rho(k)\mathcal{e}^k$$

for all $k \in \mathbb{N}$. \qed

We remark that the expression (5–40) can be used to show that $V_P^k \subset B_{n,r}$ with $r$ bounded by a linear term of $\mathcal{e}^k$ as in Proposition 5.19.

**Theorem 5.23.** Let $\phi$ be a monomorphism of $\mathcal{H}_n$ with $2 \leq n$. Then $\text{GR}(\phi) = \ell$ where $\ell = \ell(\phi)$.  

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Proof. If \( \ell(\phi) = 1 \) it suffices to establish an upper bound since \( \GR(\phi) < 1 \) means that \( \phi \) is an eventually trivial map. In view of Proposition \[5.24\] let us assume \( \phi(g_i) = g_i f_i \) for each \( i \) with \( f_i \in \FSym_n \). Each \( f_i \) is a product of transpositions; \( f_i = \tau_1, \cdots, \tau_{F_i} \). Lemma \[5.14\] implies that there exists a constant \( A_2 \) such that

\[
|\phi^k(f_i)| = |\phi^k(\tau_1) \cdots \phi^k(\tau_{F_i})| \leq A_2 F_i k
\]

for all \( 2 \leq i \leq n \) and \( k \in \mathbb{N} \). Let \( F \) denote the maximum of \( F_i \)'s. From \( \phi^{k+1}(g_i) = \phi^k(g_i) \phi^k(f_i) \), we have

\[
|\phi^{k+1}(g_i)| \leq |\phi^k(g_i)| + |\phi^k(f_i)| \leq |\phi^k(g_i)| + A_2 F k
\]

for all \( i \) and \( k \). Take a polynomial \( \rho \) such that

\[
\rho(k) + A_2 F k \leq \rho(k+1) \quad \text{and} \quad G \leq \rho(1)
\]

where \( G = \max\{|\phi(g_i)| : 2 \leq i \leq n \} \). By induction on \( k \), we check that

\[
|\phi^{k+1}(g_i)| \leq \rho(k) + A_2 F k \leq \rho(k+1)
\]

for all \( i \). Thus we have a desired upper bound

\[
\GR(\phi) \leq \lim_{k \to \infty} \rho(k)^{1/k} = 1.
\]

Therefore every monomorphism \( \phi \) with \( \ell(\phi) = 1 \) satisfies \( \GR(\phi) \leq 1 \) since \( \GR(\phi)^d = \GR(\phi^d) \leq 1 \) by Proposition \[5.24\].

Let us assume that \( \ell(\phi) \geq 2 \) and that \( \phi \) satisfies \( \phi(g_i) = g_i^\ell f_i \) for all \( 2 \leq i \leq n \) with \( f_i \in \FSym_n \). We first establish an upper bound for \( \GR(\phi) \). Each \( f_i \) can be written as a product of \( F_i \) transpositions, \( f_i = \tau_1, \cdots, \tau_{F_i} \). By Lemma \[5.22\] we can find polynomials \( \rho_1, \cdots, \rho_{F_i} \) such that

\[
|\phi^k(f_i)| = |\phi^k(\tau_1) \cdots \phi^k(\tau_{F_i})| \leq \left( \rho_1(k) + \cdots + \rho_{F_i}(k) \right)^{\ell k}
\]

for all \( 2 \leq i \leq n \) and \( k \in \mathbb{N} \). Taking \( \rho \) to be the polynomial with largest degree over all \( F_2 + \cdots + F_n \) polynomials, we have

\[
|\phi^k(f_i)| \leq F \rho(k)^{\ell k}
\]

for all \( i \) and \( k \) where \( F \) denotes the maximum over all \( F_i \)'s. Since \( \phi^{k+1}(g_i) = (\phi^k(g_i))^\ell \phi^k(f_i) \), we have

\[
|\phi^{k+1}(g_i)| \leq \ell |\phi^k(g_i)| + |\phi^k(f_i)| \leq \ell |\phi^k(g_i)| + F \rho(k)^{\ell k}
\]

for all \( k \). We seek a polynomial \( \tilde{\rho}_i \) such that \( |\phi^k(g_i)| \leq \tilde{\rho}_i(k)^{\ell k} \) for all \( k \) and \( i \). There is no obstruction to take a polynomial \( \tilde{\rho}_i \) such that

\[
\tilde{\rho}_i(k) + F \rho(k) \leq \tilde{\rho}_i(k+1) \quad \text{and} \quad |\phi(g_i)| \leq \tilde{\rho}_i(1)
\]

for each \( i \). Take a polynomial \( \tilde{\rho} \) such that \( \tilde{\rho}_i(k) \leq \rho(k) \) for all \( i \) and \( k \). Now \( (5-47) \) becomes

\[
|\phi^{k+1}(g_i)| \leq \tilde{\rho}(k)^{\ell k+1} + F \rho(k)^{\ell k} \leq \left( \tilde{\rho}(k) + F \rho(k) \right)^{\ell k+1} \leq \tilde{\rho}(k+1)^{\ell k+1}
\]

for all \( i \). Therefore we have a universal bound \( |\phi^k(g_i)| \leq \tilde{\rho}(k)^{\ell k} \) for all \( i \) and \( k \) where \( \tilde{\rho} \) is a polynomial on \( k \). We have a desired upper bound

\[
\GR(\phi) \leq \lim_{k \to \infty} \left( \tilde{\rho}(k)^{\ell k} \right)^{1/k} = \ell
\]

The induced homomorphism \( \tilde{\phi} \) on the abelianization of \( H_n \) maps \( g_i \) to \( g_i^\ell \) for all \( i \). So \( \GR(\phi) \geq \GR(\tilde{\phi}) = \ell \). So far we have shown that \( \GR(\phi) = \ell(\phi) \) if \( \phi \) satisfies the above assumption. Now Proposition \[5.24\] completes the proof because \( \GR(\phi)^d = \GR(\phi^d) = \ell^d. \)
Recall that $R_1$ is the common source of all generators $g_2, \ldots, g_{n-1}$ and that each of them has a unique target such that targets are pairwise distinct. Lemma 5.7 tells us that the behavior of $\phi(g_i)$’s is similar to this up to an element of $\Sigma_n$, the group of outer automorphisms of $H_n$ described in Theorem 2.3. Observe that a monomorphism $\phi$ of $H_n$ defines a permutation $\delta_\phi$ on $\{1, \ldots, n\}$. For each $i$, $\phi(g_i)$ has a unique target, which defines the bijection $\gamma$ as in (5–23); the target of $\phi(g_i)$ is $R_{\gamma(i)}$. We extend $\gamma : \{2, \ldots, n\} \rightarrow \{1, \ldots, n\}$ to get a bijection $\delta_\phi$ on $\{1, \ldots, n\}$ by setting $R_{\delta_\phi(1)}$ to be the unique source of $\phi(g_i)$’s. Note that $\delta_\phi$ determines an arrangement of $n$ rays for $2 \leq i \leq n$. Lemma 5.7 guarantees that $\delta_\phi$ is a bijection.

Being an element of the symmetric group $\Sigma_n$, $\delta = \delta_\phi$ determines a permutation matrix of $A_\delta \in \text{GL}(n, \mathbb{Z})$, which has exactly one entry of 1 in each row and each column and 0’s elsewhere, with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$. The matrix $A_\delta$ has the form

$$
\delta(i) \begin{pmatrix} i \\ \vdots \\ \vdots \end{pmatrix}
$$

Under the correspondence $R_i \leftrightarrow e_i$, $A_\delta$ realizes $\delta$ in (5–48) as $e_{\delta(i)} = A_\delta(e_i)$ for each $i$. Recall that $\pi : H_n \rightarrow \mathbb{Z}^n$ measures translation lengths on the rays; $\pi(g_i) = e_i - e_1$ for all $i$. The image $\pi(H_n) = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n | \sum m_i = 0\}$ is freely generated by $\bar{g}_2, \ldots, \bar{g}_n$ where $\bar{g} = \pi(g)$. The map $A_\delta$ restricts on the $A_\delta$-invariant subgroup $\pi(H_n)$ to define a matrix $\bar{A}_\delta \in \text{GL}(n - 1, \mathbb{Z})$ with respect to the ordered basis $\{\bar{g}_2, \ldots, \bar{g}_n\}$.

On the other hand, the induced map $\bar{\phi}$ in the following commutative diagram can be expressed as a matrix $A_{\bar{\phi}}$ with respect to the ordered basis $\{\bar{g}_2, \ldots, \bar{g}_n\}$ of $\mathbb{Z}^{n-1}$.

$$
\begin{array}{ccc}
1 & \rightarrow & [H_n, H_n] \\
\downarrow & & \downarrow \phi' \\
1 & \rightarrow & [H_n, H_n]
\end{array}
\begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
\phi & \rightarrow & \bar{\phi}
\end{array}
\begin{array}{ccc}
H_n & \rightarrow & \mathbb{Z}^{n-1} \\
\pi & \rightarrow & \bar{\pi} \\
H_n & \rightarrow & \mathbb{Z}^{n-1} \\
\pi & \rightarrow & \bar{\pi}
\end{array}
\begin{array}{c}
1 \\
1
\end{array}
$$

Since $\phi(g_i)$ has a unique target $R_{\delta(i)}$ and a unique source $R_\delta(1)$, where $\phi(g_i)$ acts as a translation by $\ell$ and $-\ell$ respectively, we have

$$
\bar{\phi}(\bar{g}_i) = \phi(\bar{g}_i) = \ell(e_{\delta(i)} - e_1) = \ell A_\phi(e_i - e_1) = \ell \bar{A}_\phi(\bar{g}_i)
$$

for each $i$ where $\ell = \ell(\phi)$. Thus we have

$$
A_{\bar{\phi}} = \ell \bar{A}_\phi.
$$

We also have

$$
A_{\bar{\phi}}^k = (\bar{A}_\phi)^k = \ell^k (\bar{A}_\phi)^k
$$

for all $k \in \mathbb{N}$. In particular, $A_{\bar{\phi}}^d = (\bar{A}_\phi)^d = \ell^k I$ where $d = d(\delta_\phi)$ denotes the order of $\delta_\phi$ and $I \in \text{GL}(n - 1, \mathbb{Z})$ is the identity matrix. This proves Proposition 5.24

**Proposition 5.24.** A monomorphism $\phi$ of $H_n$ satisfies that for each $i$, $2 \leq i \leq n$,

$$
\phi^d(g_i) \sim (g_i)^{\ell^d}
$$

where $d = d(\delta_\phi)$ and $\ell = \ell(\phi)$.
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