On the derived Picard group of the Brauer star algebra

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Abstract

In this paper we show that the derived Picard group $TrPic(A)$ of the Brauer star algebra of type $(n,t)$ is generated by shift, $Pic(A)$ and equivalences $\{H_i\}_{i=1}^n$ in the case $t > 1$, where $H_i$ were shown to satisfy the relations of the braid group on the affine diagram $\tilde{A}_{n-1}$ by Schaps and Zakay-Illouz. In the multiplicity free case we show that $TrPic(A)$ is generated by a slightly bigger set.

1 Introduction

The derived Picard group $TrPic(A)$ of an algebra $A$ is the group of isomorphism classes of two-sided tilting complexes in $D^b(A \otimes A^{op})$, with the product of the classes of $X$ and $Y$ given by the class $X \otimes_A Y$. Equivalently $TrPic(A)$ is the group of the standard autoequivalences of $D^b(A)$ modulo natural isomorphisms.

Rouquier and Zimmermann started the study of the derived Picard group of Brauer tree algebras [16]. In the case of multiplicity one they constructed a morphism from Artin’s braid group on $n + 1$ strings ($n$ is the number of simple modules of $A$) to $TrPic(A)$ and showed it to be an isomorphism modulo some central subgroup when $n = 2$. Khovanov and Seidel defined an action of the Artin’s braid group on the bounded derived category of a certain algebra similar to the Brauer tree algebra, according to their results the action of the Artin’s braid group on the bounded derived category of a Brauer tree algebra with multiplicity one is faithful [10].

Schaps and Zakay-Illouz constructed an action of the braid group on the affine diagram $\tilde{A}_{n-1}$ on the bounded derived category of a Brauer tree algebra with arbitrary multiplicity [17]. Muchtadi-Alamsyah showed this action to be faithful in the case of multiplicity one [12].

Schaps and Zakay-Illouz also raised the question whether in the case of multiplicity $\neq 1$ the braid generators, together with shifts and $Pic(A)$, generate the entire derived Picard group and whether this homomorphism from the braid group is one-to-one. Using the technique of tilting mutations developed by Ai-hara and Iyama [3], [2], we answer the first question positively, namely:

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Theorem 1. Let $A$ be a Brauer star algebra of type $(n,t)$, $t > 1$. Then $\text{TrPic}(A)$ is generated by shift, $\text{Pic}(A)$ and equivalences induced by $H_i.\\\\H_i(P_j) = \begin{cases} 0 \to 0 \to P_j, & j \neq i, i - 1 \\ 0 \to 0 \to P_{i-1}, & j = i \\ P_i \beta \to P_{i-1} \xrightarrow{\text{soc}} P_{i-1}, & j = i - 1. \end{cases}\\\\$ In the multiplicity free case we show that $\text{TrPic}(A)$ is generated by a slightly bigger set.

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2 Preliminaries

2.1 Derived equivalences

Let $A$ and $B$ be algebras over a commutative ring $R$ and let $A$ and $B$ be projective as modules over $R$. Denote by $D^b(A)$ the bounded derived category of $A$, by $\text{per} - A$ the category of perfect complexes, by $K^b(\text{proj} - A)$ the homotopy category of bounded complexes of finitely generated projective modules, by $C(A)$ the category of complexes of $A$-modules. The following theorem by Rickard and Keller gives a necessary and sufficient condition for $A$ and $B$ to be derived equivalent [15], [14], [9].

Theorem. The following are equivalent:

1. The categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories.
2. The categories $K^b(\text{proj} - A)$ and $K^b(\text{proj} - B)$ are equivalent as triangulated categories.
3. There is a complex $T \in \text{per} - A$ such that
   - $\text{Hom}(T, T[i]) = 0$ for $i \neq 0$,
   - $\text{per} - A$ is generated by $T$,
   - $End_{D^b(A)}(T) \simeq B$.
4. There is a bounded complex $X$ of $(A \otimes B^{op})$-modules whose restrictions to $A$ and to $B^{op}$ are projective and a bounded complex $Y$ of $(B \otimes A^{op})$-modules whose restrictions to $B$ and to $A^{op}$ are projective such that $X \otimes_B Y \simeq A$ in $D^b(A \otimes A^{op})$ and $Y \otimes_A X \simeq B$ in $D^b(B \otimes B^{op})$.\\

The complex $T$ is called a tilting complex, $X$ and $Y$ are called two-sided tilting complexes inverse to each other. Complex $X$ viewed as a complex of $A$-modules or as a complex of $B^{op}$-modules is a tilting complex. The inverse equivalences between $D^b(A)$ and $D^b(B)$ are given by $X \otimes_B -$ and $Y \otimes_A -$. Such equivalences are called standard.
Let $T$ be a tilting complex, $\text{End}_{D^b(A)}(T) \simeq B$. There exists a two-sided tilting complex $X$ of $(A \otimes B^{op})$-modules whose restriction to $A$ is isomorphic to $T$ in $D^b(A)$. If $X'$ is another two-sided tilting complex of $(A \otimes B^{op})$-modules whose restriction to $A$ is isomorphic to $T$, then there exists $\alpha \in \text{Aut}(A)$ such that $X' = \alpha A \otimes_A X$, where $\alpha A$ is an $(A \otimes A^{op})$-module isomorphic to $A$ but with the left action twisted by $\alpha$.

**Definition 1.** Let $\Gamma$ be a tree with $n$ edges and a distinguished vertex, which has an assigned multiplicity $t \in \mathbb{N}$ (this vertex is called exceptional, $t$ is called the multiplicity of the exceptional vertex). Let us fix a cyclic ordering of the edges adjacent to each vertex in $\Gamma$ (if $\Gamma$ is embedded into plane, we will assume that the cyclic ordering is clockwise). In this case $\Gamma$ is called a Brauer tree of type $(n,t)$.

The case when the tree is a star and the exceptional vertex is in the middle is called the Brauer star.

To a Brauer tree of type $(n,t)$ one can associate an algebra $A(n,t)$. The algebra $A(n,t)$ is a path algebra of a quiver with relations. Let us construct a Brauer quiver $Q_\Gamma$ using the Brauer tree $\Gamma$. The vertices of $Q_\Gamma$ are the edges of $\Gamma$. If two edges $i$ and $j$ are incident to the same vertex in $\Gamma$ and $j$ follows $i$ in the cyclic order of the edges incident to their common vertex, then there is an arrow from the vertex $i$ to the vertex $j$ in $Q_\Gamma$. $Q_\Gamma$ has the following property: $Q_\Gamma$ is the union of oriented cycles corresponding to the vertices of $\Gamma$, each vertex of $Q_\Gamma$ belongs to exactly two cycles. The cycle corresponding to the exceptional vertex is called exceptional. The arrows of $Q_\Gamma$ can be divided into two families $\alpha$ and $\beta$ in such a manner that the arrows belonging to intersecting cycles are in different families. By a slight abuse of notation arrows belonging to the family $\alpha$ will be denoted $\alpha$ and arrows belonging to the family $\beta$ will be denoted $\beta$ respectively.

**Definition 2.** Let $k$ be a field. The basic Brauer tree algebra $A(n,t)$, corresponding to a tree $\Gamma$ of type $(n,t)$ is isomorphic to $kQ_\Gamma/I$, where the ideal $I$ is generated by the relations:

1. $\alpha \beta = 0 = \beta \alpha$;

2. for any vertex $x$, not belonging to the exceptional cycle, $\alpha^{x_\alpha} = \beta^{x_\beta}$, where $x_\alpha$, resp. $x_\beta$ is the length of the $\alpha$, resp. $\beta$-cycle, containing $x$;

3. for any vertex $x$, belonging to the exceptional $\alpha$-cycle (resp. $\beta$-cycle), $(\alpha^{x_\alpha})^t = \beta^{x_\beta}$ (resp. $\alpha^{x_\alpha} = (\beta^{x_\beta})^t$).

An algebra $A(n,t)$ is called a Brauer tree algebra of type $(n,t)$.

Note that the ideal $I$ is not admissible. From now on for convenience all algebras are supposed to be basic.

Rickard showed that two Brauer tree algebras corresponding to the trees $\Gamma$ and $\Gamma'$ are derived equivalent if and only if their types $(n,t)$ and $(n',t')$ coincide
and it follows from the results of Gabriel and Riedtmann that this class is closed under derived equivalence [6].

Let $A$ be a Brauer star algebra with $n$ edges and multiplicity $t$. The quiver of $A$ is of the form:

Following Schaps and Zakay-Illouz [18] we will call a tilting complex two-restricted if its indecomposable direct summands are all shifts of the following complexes, with initial nonzero term in degree zero

where $P_i$ and $P_j$ are indecomposable projective $A$-modules and the morphism $P_i \to P_j$ is the morphism of maximal rank (as a morphism over $k$). Two-restricted tilting complexes correspond to some additional combinatorial data associated to the Brauer tree called "pointing". If one goes out by one pointing and returns by another, one gets an autoequivalence of a Brauer star algebra given by a tilting complex, which is called "refolded". Schaps and Zakay-Illouz studied the subgroup of the derived Picard group generated by refolded tilting complexes and showed that it is generated by tilting complexes $H_i$ which satisfy the relations of the braid group on the affine diagram $\tilde{A}_{n-1}$. The autoequivalences of $D^b(A)$ corresponding to these complexes will be also denoted by $H_i$.

They act on the projective modules as follows:

$$H_i(P_j) = \begin{cases} 
0 \to 0 \to P_j, & j \neq i, i-1 \\
0 \to 0 \to P_{i-1}, & j = i \\
P_i \beta \to P_{i-1} \to P_{i-1}, & j = i-1, 
\end{cases}$$

where $soc$ is the morphism whose image is isomorphic to the socle of $P_{i-1}$.

2.2 Mutations

Let $k$ be a field. Let $\mathcal{T}$ be a Krull-Schmidt, $k$-linear, Hom-finite triangulated category. A morphism $X \to M' \in \mathcal{T}$ is called left minimal if any morphism $g : M' \to M'$ satisfying $g f = f$ is an isomorphism. Let $\mathcal{M}$ be a subcategory of $\mathcal{T}$, $X$ an object of $\mathcal{T}$, a morphism $X \to M'$ is called a left approximation of $X$ with respect to $\mathcal{M}$ if $\text{Hom}_\mathcal{T}(M', M) \to \text{Hom}_\mathcal{T}(X, M)$ is surjective for any
\( M \in \mathcal{M} \). Right minimal morphism and right \( \mathcal{M} \)-approximations are defined dually.

\( T \in \mathcal{T} \) is called silting if \( \text{Hom}_\mathcal{T}(T, T[i]) = 0 \) for any \( i > 0 \) and \( \mathcal{T} \) is generated by \( \text{add}(T) \) as a triangulated category. We say that a silting object \( T \) is basic if \( T \) is isomorphic to direct sum of indecomposable objects which are mutually non-isomorphic.

Let \( T \) be a basic silting object in \( \mathcal{T} \), \( T = M \oplus X \), \( M = \text{add}(M) \). Consider a triangle

\[
X \xrightarrow{f} M' \xrightarrow{g} Y \xrightarrow{h},
\]

where \( f \) is a minimal left approximation of \( X \) with respect to \( M \), note that \( f \) is unique up to isomorphism. \( \mu^+_X(T) := M \oplus Y \) is called a left mutation of \( T \) with respect to \( X \). By results of Aihara and Iyama \([3]\) \( \mu^+_X(T) \) is again a basic silting object. If \( X \) is indecomposable the mutation is called irreducible. Right mutations are defined dually and are denoted by \( \mu^-_X(T) \). Note that \( \mu^-_Y(\mu^+_X(T)) = T \).

Let \( T, U \) be basic silting objects in \( \mathcal{T} \). \( T \geq U \) if \( \text{Hom}_\mathcal{T}(T, U[i]) = 0 \) for any \( i > 0 \). \( \geq \) gives a partial order on the set of the isomorphism classes of basic silting object of \( \mathcal{T} \) \([3]\). We say that \( U \) is connected (left-connected) to \( T \) if \( U \) can be obtained from \( T \) by iterated irreducible (left) mutation. A triangulated category \( \mathcal{T} \) is called silting-connected if all basic silting objects in \( \mathcal{T} \) are connected to each other. \( \mathcal{T} \) is strongly silting-connected if for any silting objects \( T, U \) such that \( T \geq U \) \( U \) is left-connected to \( T \). It is well known that in the case of a symmetric algebra any silting object in \( \text{Kb}(\text{proj}-A) \) is a tilting complex.

**Theorem.** (Aihara, \([2]\)) \( \text{Kb}(\text{proj}-A) \) is tilting-connected if \( A \) is a representation-finite symmetric algebra.

Note that it also follows from \([2]\) (Theorem 5.6 and Corollary 3.9) that in the case of representation-finite symmetric algebra \( \text{Kb}(\text{proj}-A) \) is strongly tilting connected. So any tilting complex concentrated in nonpositive degrees can be obtained from \( A \) by iterated left mutations.

A well known example of tilting mutation is the mutation of Brauer graphs or equivalently SSB-algebras. We will define it in the context of Brauer trees, i.e. we will not consider the case of loops.

Consider a Brauer tree algebra \( A \) as a tilting complex over itself. \( A = \bigoplus_{i=1, i \neq j}^n P_i \oplus P_j \). Consider a left tilting mutation

\[
\mu^+_{P_j}(A) = (\bigoplus_{i=1, i \neq j}^n P_i) \oplus (P_j \xrightarrow{f} P_m \oplus P_l)
\]

of \( A \) at \( P_j \) with respect to \( \text{add}(\bigoplus_{i=1, i \neq j}^n P_i) \), where \( P_m \) and \( P_l \) are the projective modules corresponding to the edges in the Brauer tree which are next to \( j \) in the cyclic ordering of the edges adjacent to the same vertices and \( f = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \),

where \( \alpha \) and \( \beta \) correspond to the arrows from \( j \) to \( m \) and from \( j \) to \( l \) respectively. The Brauer tree of \( A \) is on the left-hand side and the Brauer tree of the
endomorphism ring of $\mu_{P_j}^+(A)$ is on the right-hand side, the edge corresponding to $P_j \xrightarrow{f} P_m \oplus P_l$ will also be denoted by $j$.

In the case where $j$ is incident to a leaf in the Brauer tree of $A$

$$\mu_{P_j}^+(A) = \bigoplus_{i=1, i \neq j}^n P_i \oplus (P_j \xrightarrow{\beta} P_l)$$

and on the level of the Brauer trees we have:

As far as we know these moves were first introduced by Kauer [8] but were also studied in [4], [1], [5], [11]. $\mu_{P_j}^-(A)$ is defined dually and corresponds to the move in the opposite direction, i.e. from the right-hand side to the left-hand side. The tilting complexes of the form $\mu_{P_j}^-(A)$, where $A$ is an arbitrary Brauer tree algebra, will sometimes be called elementary. Mutations involving only one additional edge $l$ will be called the mutations of type I and the mutations involving both $m$ and $l$ will be called the mutations of type II. Note also that these mutations involve only edges: the exceptional vertex stays unchanged.

3 Mutations and the derived Picard group

Denote by $\text{Pic}(A)$ the Picard group of an algebra $A$, i.e. the group of isomorphism classes of invertible $A \otimes A^{op}$-modules or equivalently the group of Morita autoequivalences of $A$. The group $\text{Out}(A)$ of outer autoequivalences of $A$ coincides with $\text{Pic}(A)$ in the case of a basic algebra $A$ and is clearly a subgroup of $\text{TrPic}(A)$. If $\alpha \in \text{Out}(A)$, then there is an invertible $A \otimes A^{op}$-module $\alpha A$, where $\alpha A$ is an $(A \otimes A^{op})$-module isomorphic to $A$ but with the left action twisted by $\alpha$. $\alpha A$ is isomorphic to $\alpha' A$ if and only if $\alpha$ coincides with $\alpha'$. Consider an equivalence $F : D^b(B) \to D^b(A)$. Assume that it is given by a two-sided tilting complex $X$ whose restriction to $A$ is a tilting complex $T$, assume that $F' : D^b(B) \to D^b(A)$ is another equivalence given by a two-sided tilting complex $X'$ whose restriction to $A$ is a tilting complex $T$, we know that $X' = \alpha A \otimes_A X$ for some $\alpha \in \text{Out}(A)$. If additionally we assume that $F$ and $F'$ send the same
direct summands of \( B \) to the same summands of \( T \), then \( \alpha \in \widehat{\text{Out}}(A) \), where \( \widehat{\text{Out}}(A) \) is a subgroup of outer automorphisms acting trivially on idempotents.

**Lemma 1.** Let \( \mathcal{A}, \mathcal{B} \) be two triangulated categories, \( F : \mathcal{A} \to \mathcal{B} \) a triangular equivalence, \( \mathcal{M} \) a subcategory of \( \mathcal{A} \), \( X \in \mathcal{A} \) and let \( X \xrightarrow{\sim} M' \) (\( M' \xrightarrow{\sim} X \)) be a minimal left (resp. right) approximation of \( X \) with respect to \( \mathcal{M} \), then \( F(X) \xrightarrow{\sim} F(M') \) (\( F(M') \xrightarrow{\sim} F(X) \)) is a minimal left (resp. right) approximation of \( F(X) \) with respect to \( F(\mathcal{M}) \).

Let \( A \) be a symmetric algebra, \( T \) a basic tilting complex. Denote by \( F \) the equivalence induced by \( T \), where \( F : D^b(B = \text{End}_{D^b(A)}(T)) \to D^b(A) \). Let \( T = M \oplus X \), where \( X \) is indecomposable, \( \mu_X^\pm(T) \) a tilting complex obtained from \( T \) by right or left mutation. Denote by \( C = \text{End}_{D^b(A)}(\mu_X^b(T)) \) and by \( G : D^b(C) \to D^b(A) \) the equivalence induced by \( \mu_X^b(T) \). Consider \( B \) as a tilting complex over itself. Summands of \( T \) and of \( B \) are in one to one correspondence under \( F \), so denote by \( P_X \) the indecomposable projective \( B \)-module corresponding to \( X \), a tilting complex \( \mu_X^\pm(B) \) is obtained from \( B \) by mutation with respect to \( P_X \).

**Lemma 2.** \( \text{End}_{D^b(B)}(\mu_{P_X}^\pm(B)) \simeq C \).

**Proof.** Consider the case of right mutations. \( C = \text{End}_{D^b(A)}(M \oplus Y), \) \( \text{End}_{D^b(B)}(\mu_{P_X}^R(B)) = \text{End}_{D^b(B)}(F^{-1}(M) \oplus Y') \), where \( Y' \to N \to P_X \to \) is a triangle, and \( N \to P_X \) is a minimal right approximation of \( P_X \) with respect to \( \text{add}(F^{-1}(M)) \). \( F^{-1} \) is fully faithful and triangulated, and due to the previous lemma \( F^{-1}(Y) \simeq Y' \), so \( \text{End}_{D^b(B)}(\mu_{P_X}^R(B)) \simeq \text{End}_{D^b(B)}(F^{-1}(M \oplus Y)) \) and the assertion follows.

Let’s denote by \( H : D^b(C) \to D^b(B) \) the equivalence induced by \( \mu_{P_X}^R(B) \).

**Lemma 3.** There exists \( \alpha \in \widehat{\text{Out}}(A) \) such that \( G \simeq (\alpha A \otimes_A -) \circ F \circ H \).

**Proof.** By the previous lemma we have the following diagram:

\[
\begin{array}{ccc}
D^b(C) & \xrightarrow{H} & D^b(B) \\
\downarrow{G} & & \downarrow{F} \\
D^b(A) & & 
\end{array}
\]

We need to check that the action of \( G \) and \( F \circ H \) on indecomposable projective modules coincides. We will deal only with the case of right mutation again. Let \( P \) be an indecomposable projective module over \( C \), such that its image under \( G \) is a summand of \( M \). It is clear that its image under \( H \) is a projective \( B \)-module different from \( P_X \) and that its image under \( F \circ H \) is the same summand of \( M \). Let \( P \) be an indecomposable projective module over \( C \), such that its image under \( G \) is \( Y \), then its image under \( H \) is the shift of the cone of the approximation of \( P_X \) with respect to other projective \( B \)-modules. And so its image under \( F \circ H \)
is isomorphic to the shift of the cone of the approximation of $X$ with respect to other summands of $T$, which is clearly $Y$. □

We are going to need a couple of technical lemmas.

**Lemma 4.** Let $A = X \xrightarrow{g} Y$, $B = Z \xrightarrow{h} Y$ be two objects of $D^b(A)$ concentrated in the same degrees. Let

\[
\begin{array}{cc}
X & \xrightarrow{g} Y \\
\downarrow^{w} & \downarrow^{Id} \\
Z & \xrightarrow{h} Y
\end{array}
\]

be a morphism $f \in D^b(A)$, then $\text{Cone}(f) \simeq X \xrightarrow{w} Z$ with $Z$ concentrated in the same degree as in $B$.

**Lemma 5.** Let $A = X \xrightarrow{v} Y \xrightarrow{g} Y'$, $B = Z' \xrightarrow{w} Z \xrightarrow{h} X$ be two objects of $D^b(A)$. Let

\[
\begin{array}{cccccccccc}
0 & \xrightarrow{v} & 0 & \xrightarrow{X} & Y & \xrightarrow{g} & Y' \\
\downarrow{Z'} & \xrightarrow{w} & Z & \xrightarrow{h} & X & \xrightarrow{Id} & 0 & \xrightarrow{0} & 0
\end{array}
\]

be a morphism $f \in D^b(A)$, then $\text{Cone}(f) \simeq Z' \xrightarrow{w} Z \xrightarrow{v \circ h} Y \xrightarrow{g} Y'$ with $Z'$ concentrated in the same degree as in $B$.

Note that in this lemma one can set $Y$, $Y'$ and $f$ equal to zero.

**Lemma 6.** Let $A = X \xrightarrow{v} Y$, $B = X \xrightarrow{w} Z \xrightarrow{h} Z'$ be two objects of $D^b(A)$. Let

\[
\begin{array}{cc}
X & \xrightarrow{v} Y \\
\downarrow^{Id} & \downarrow^{g} \\
X & \xrightarrow{w} Z & \xrightarrow{h} Z'
\end{array}
\]

be a morphism $f \in D^b(A)$, then $\text{Cone}(f) \simeq Y \xrightarrow{g} Z \xrightarrow{h} Z'$ with $Z'$ concentrated in the same degree as in $B$.

### 3.1 The standard construction of a tree

In this section we will fix a standard way to build a tree from a star using mutations. The procedure will only involve mutation of type I.

Let $\Gamma$ be a Brauer tree of type $(n,t)$. Let us assume that the root of $\Gamma$ is chosen in the exceptional vertex, and that $\Gamma$ is embedded in the plane in such a manner that all nonroot vertices are situated on the plane lower than the root according to their level (the further from the root, the lower, all vertices of the same level lie on a horizontal line). The edges around vertices are ordered clockwise.
Let $A$ be a Brauer star algebra if the corresponding tree is embedded in the plane as described above, let its edges be labelled from left to right. $A = \bigoplus_{i=1}^n P_i$, where $P_i$ are indecomposable projective modules. We are going to perform a series of irreducible mutations of $A$, after each mutation we are going to obtain a tilting complex $T^r = \bigoplus_{i=1}^n T^r_i$, $T_0 = A$, $T^0_i = P_i$. Each mutation changes only one summand of the tilting complex $T^r$, say $T^r_i$. Denote by $T^{r+1}_i$ the summand which was changed by the $r+1$-st mutation, $T^{r+1}_i := T^r_i$, $i \neq j$. All this allows us to write a composition of mutations. Denote by $\mu_1^+$ the mutation which changes $T^r_i$. The Brauer tree of the endomorphism ring of $T^{r+1}$ can be obtained by the mutation of the Brauer tree of $T^r$, for example by Lemma 3.

Note also, that since the endomorphism ring of $T^r$ is a Brauer tree algebra it is easy to compute minimal right approximation of $T^r_i$ with respect to other summands of $T^r$. If the edge $j$ corresponding to $T^r_j$ is not incident to a leaf, let $T^r_m$ and $T^r_l$ be the edges corresponding to $m$ and $l$, the edges next to $j$ in the cyclic ordering and let $f$ be a morphism corresponding to two arrows in $\text{End}_{D^n(A)}(T^r)$, then $T^r_j \overset{f}{\longrightarrow} T^r_m \oplus T^r_l$ is a minimal left approximation of $T^r_j$ with respect to other summands of $T^r$. If $j$ is incident to a leaf, then $T^r_j \overset{f}{\longrightarrow} T^r_l$ is a minimal left approximation $T^r_j$, where $l$ is the only edge next to $j$ in the cyclic ordering and $f$ is the corresponding arrow. We will say that we mutate $j$ along the edges $m$ and $l$ or along the the edge $l$ respectively.

Let $\Gamma$ be a Brauer tree of type $(n, t)$, assume $n > 1$. Let us number the edges of the tree $\Gamma$ as follows: put 1 on the left-hand edge incident to the root. If the edge with label 1 is not incident to a leaf, put 2 on the edge incident to its nonexceptional end which is the previous edge coming before the edge with label 1 in the cyclic ordering; if the edge with label 1 is incident to a leaf, put 2 on the only edge which is the previous edge coming before the edge with label 1 in the cyclic ordering. Assume the label $i$ is assigned to some edge, if it is not incident to a leaf, put the label $i + 1$ on the edge of the lower level which is the previous edge coming before the edge with label $i$ in the cyclic ordering; if the edge with the label $i$ is incident to a leaf, put the label $i + 1$ on the only edge which is the previous edge coming before the edge with label $i$ in the cyclic ordering; if this edge has a label already, find the edge with the biggest label which has an unlabelled edge incident to its upper end $x$, put the label $i + 1$ on the previous edge coming before this edge in the cyclic ordering around $x$. Note that the same labelling can be obtained using Green walks.

Let $\phi_T : \{1, 2, ..., n\} \rightarrow \{0, 1, ..., n - 1\}$ be a function which assigns to a label $i$ the length of the shortest path from this edge to the exceptional vertex, or equivalently the level of the edge with label $i$ (assuming that the edges incident to the exceptional vertex belong to the level 0).

**Lemma 7.** Let $\Gamma$ be a Brauer tree of type $(n, t)$ labelled as described above. Let $T = (\mu^+_n)^{\phi_T(n)} \circ (\mu^+_{n-1})^{\phi_T(n-1)} \circ ... \circ (\mu^+_1)^{\phi_T(1)}(A)$. Then the Brauer tree of the endomorphism ring of $T$ is $\Gamma$.

**Proof.** Clear from the construction. \qed
Let us compute the tilting complex $T$ from the lemma. Denote by $\psi_T : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ the function which assigns to a label $i$ the label of the edge from a higher level which shares a common vertex with the edge $i$. Let us assume that in the complexes $0 \to P_i \to 0$, $0 \to P_i \to P_j \to 0$, $P_i$ is concentrated in 0, (note that we are using the cohomological notation).

Lemma 8. Let $\Gamma$ be a Brauer tree of type $(n, t)$ labelled as described above. Let $T$ be from Lemma 7, then $T$ is of the form $T = \bigoplus_{i=1}^n T_i$.

\[
T_i = \begin{cases} 
P_i, & \psi_T(i) = 0, \\
(P_i \xrightarrow{\beta_{\psi_T(i)}} P_{\psi_T(i)})(\psi_T(i)), & \psi_T(i) \neq 0.
\end{cases}
\]

Proof. Recall that the arrows in the Brauer star algebra were denoted by $\beta$. Since $T = (\mu_n^+)^{\psi_T(n)} \circ (\mu_{n-1}^+)^{\psi_T(n-1)} \circ \ldots \circ (\mu_1^+)^{\psi_T(1)}(A)$, first we mutate the summand with label 1, then with label 2 and so on, hence we can compute $T$ by induction on the label. \(\psi_T(1) = 0\), hence $T_1 = P_1$. If the edge with label 1 is not incident to a leaf, 2 is on the edge incident to its nonexceptional end which is the previous edge coming before the edge with label 1 in the cyclic ordering, $\psi_T(2) = 1$ and we should apply $\mu_2^+$ to $A$. $T_2 = P_2 \xrightarrow{\beta} P_1$ concentrated in $-1$ and 0, as desired. If the edge with label 1 is incident to a leaf, 2 is on the edge incident to the exceptional vertex, $\psi_T(2) = 0$, hence $T_2 = P_2$.

If the edge with label $i$ is incident to the exceptional vertex and $T_i = P_i$, then $T_{i+1} = P_{i+1}$ or $T_{i+1} = P_{i+1} \xrightarrow{\beta} P_i$ concentrated in $-1$ and 0 depending on whether the edge with label $i$ is incident to a leaf or not and we are done.

Assume the label $i$ is assigned to some edge not incident to the exceptional vertex which is not incident to a leaf, $i + 1$ is assigned to the edge of the lower level which is the previous edge coming before the edge with label $i$ in the cyclic ordering, then $\psi_T(i + 1) = \psi_T(i) + 1$. So by assumption

\[
T' = (\mu_i^+)^{\psi_T(i)} \circ (\mu_{i-1}^+)^{\psi_T(i-1)} \circ \ldots \circ (\mu_1^+)^{\psi_T(1)}(A) = \bigoplus_{k=1}^i T_k \oplus \bigoplus_{k=i+1}^n P_k.
\]

Denote by $x_1, x_2, \ldots, x_{\psi_T(i)} = i$ the edges from the shortest path from the exceptional vertex to $j$ indexed by the value of $\psi_T$. By assumption

\[
T_{x_1} = P_{x_1}, \quad T_{x_k} = (P_{x_k} \xrightarrow{\beta_{x_k-x_{k-1}}} P_{x_{k-1}})(\psi_T(x_k))].
\]

We want to apply $(\mu_{i+1}^+)^{\psi_T(i+1)}$ to $T'$, or equivalently mutate $P_{i+1}$ along $T_{x_1}$, $T_{x_2}, \ldots, T_{x_{\psi_T(i)}}$. Clearly $T_{i+1}^{i+1} = P_{i+1} \xrightarrow{\beta_{i+1-x_1}} P_{x_1}$. The minimal left approximation of $T_{i+1}^{i+1}$ with respect to other summands of $T'^{i+1}$ is

\[
\begin{array}{c}
P_{i+1} \\
\xrightarrow{\beta_{i+1-x_1}} \\
\xrightarrow{\beta_{x_2-x_1}} \\
P_{x_2} \\
\end{array} \quad \xrightarrow{\beta_{i-x_2}} \quad \begin{array}{c}
P_{x_1} \\
\xrightarrow{Id} \\
\end{array}
\]

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By Lemma 4 $T_{r+1}^{r+2} = (P_{r+1} \xrightarrow{\beta_{r+1} - x_2} P_{r+2})[2]$, and by iterated application of Lemma 4 we get the desired result. The case where $i$ is incident to a leaf is done similarly. □

**Remark 1.** The resulting complex is two-restricted. In fact, similar complexes have already been studied in the works of Schaps and Zakay-Ilouz (see for example [18]), they correspond to a special choice of pointing.

**Example 1.** Let $\Gamma$ be a Brauer Tree

$$
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}
$$

$$
T = (\mu_4^+)^2 \circ (\mu_3^+)^2 \circ \mu_2^+(A)
$$

And the summands of $T$ are:

- $P_1$
- $P_2 \xrightarrow{\beta} P_1$
- $P_3 \xrightarrow{\beta} P_2$
- $P_4 \xrightarrow{\beta^2} P_2$
- $P_5$.

### 3.2 Main result

Let $A$ be a Brauer star algebra of type $(n,t)$. Let $\mathcal{R}$ denote the subgroup of $TrPic(A)$ generated by shift, $Pic(A)$ and equivalences induced by $H_i$.

$$
H_i(P_j) = \begin{cases} 
0 \rightarrow 0 \rightarrow P_j, & j \neq i, i-1 \\
0 \rightarrow 0 \rightarrow P_{i-1}, & j = i \\
P_i \xrightarrow{\beta} P_{i-1} \xrightarrow{soc} P_{i-1}, & j = i - 1.
\end{cases}
$$

**Remark 2.** (a) The subgroup $\mathcal{R}$ coincides with the subgroup considered in [17], it was also shown there that this subgroup has an action by braid group on the
diagram $\tilde{A}_{n-1}$, the homomorphism is defined sending half-twists to $H_i$’s. In [12] this action is shown to be faithful for $t = 1$.

(b) There is an outer automorphism in $\text{Pic}(A)$ corresponding to the rotation of the Brauer star and sending $H_i$ to $H_{i+1}$, so one can define $R$ as a subgroup of $\text{TrPic}(A)$ generated by shift, $\text{Pic}(A)$ and equivalence induced by $H_1$.

(c) $H_i \simeq (\mu_i^+)^2(A)$.

**Theorem 1.** If $t > 1$, then $\text{TrPic}(A) \simeq R$.

**Sketch of the Proof.** Since $R$ embeds into $\text{TrPic}(A)$ we only need to show that this embedding is surjective. Any element $a$ from $\text{TrPic}(A)$ restricts to some tilting complex $T$ such that $\text{End}_{\text{Pic}(A)}(T) \simeq A$; any other element from $\text{TrPic}(A)$ which restricts to the same complex, differs from $a$ by an element from $\text{Pic}(A)$, hence by an element from $R$. So it is sufficient to prove that for any tilting complex $T$ there is an element from $R$ which sends the projective modules of $A$ to the summands of $T$.

Assume that $T$ is concentrated in nonpositive degrees. By results of Aihara [2] $T = \mu_1^+ \circ \ldots \circ \mu_{i-1}^+ \circ \mu_i^+ \circ \mu_{i-1}^+ \circ \ldots \circ \mu_1^+$ for some $\{i, i_1, i_2, \ldots, i_s\}$. By Lemma 3 up to $\text{Pic}(A)$ the equivalence given by $T$ is a product of equivalences induced by elementary tilting complexes. Denote by $T^r := \mu_1^+ \circ \mu_{i-1}^+ \circ \ldots \circ \mu_2^+ \circ \mu_1^+$ and by $\Gamma^r$ the Brauer tree of its endomorphism ring. Note that if the Brauer star is labelled in a standard way, $\Gamma^r$ has a natural labelling of edges, which may not coincide with the standard one.

Recall from Lemma 7 that if $A$ was a Brauer star algebra with a standard labelling, then the endomorphism ring of $(\mu_1^+)_{\phi_{\Gamma^+(n)}} \circ (\mu_{n-1}^+)_{\phi_{\Gamma^-(n-1)}} \circ \ldots \circ (\mu_1^+)_{\phi_{\Gamma^+(1)}}(A)$ is a Brauer tree algebra associated to $\Gamma^r$ with a standard labelling. If the labelling $\rho$ of $\Gamma^r$ is not standard, there is some permutation $\tau$ one needs to apply to the standard labelling of $\Gamma^r$ to obtain $\rho$. Applying $\tau$ to the standard labelling of the Brauer star we obtain the Brauer star with the labelling, which will also be denoted by $\rho$. And applying $\tau$ to the indices of $(\mu_1^+)_{\phi_{\Gamma^+(n)}} \circ (\mu_{n-1}^+)_{\phi_{\Gamma^-(n-1)}} \circ \ldots \circ (\mu_1^+)_{\phi_{\Gamma^+(1)}}(A)$ one gets a series of mutations needed to obtain the Brauer tree $\Gamma^r$ with the labelling $\rho$ from the Brauer star with the labelling $\rho$. We will denote this series of mutations $\tilde{\rho}^r$ for the natural labelling $\rho$ of $\Gamma^r$. By $\tilde{\rho}^r := (\tilde{\rho}^r)^{-1}$ we will denote the series of mutations $(\mu_1^-)_{\phi_{\Gamma^+(1)}}(\mu_1^-)_{\phi_{\Gamma^-(1)}} \circ \ldots \circ (\mu_{n-1}^-)_{\phi_{\Gamma^+(n-1)}}(\mu_{n-1}^-)_{\phi_{\Gamma^-(n-1)}}$. If $A'$ is a Brauer star algebra with the labelling $\rho$ and $B'$ is the Brauer tree algebra of $\Gamma^r$ with natural labelling, then $\tilde{\rho}^r(A') = B'$, $\tilde{\rho}^r \circ \tilde{\rho}^r(A') = A'$ and $\tilde{\rho}^r \circ \tilde{\rho}^r(B') = B'$.

$$T = \mu_1^+ \circ \ldots \circ \mu_{i-1}^+ \circ \mu_i^+ \circ \mu_{i-1}^+ \circ \ldots \circ \mu_1^+$$

$$= \mu_1^+ \circ \ldots \circ \tilde{\rho}^r \circ \mu_{i-1}^+ \circ \mu_i^+ \circ \mu_{i-1}^+ \circ \ldots \circ \mu_1^+ \circ \mu_i^+ \circ \mu_{i-1}^+ \circ \ldots$$

So by Lemma 3 up to $\text{Pic}(A)$ the equivalence induced by $T$ is a product of equivalences induced by complexes of the form $\tilde{\rho}^r \circ \mu_{i-1}^+ \circ \mu_i^+ \circ \mu_{i-1}^+ \circ \mu_1^+$. The endomorphism ring of these
tilting complexes is the Brauer star algebra. Clearly, \( \tilde{\mu}^{-1} \circ \mu_{i+1}^+(A) = A \in \mathcal{R} \) and \( \mu_{i+1}^+ \circ \tilde{\mu}^r(A^\rho) = H_i \in \mathcal{R} \), here we use the fact that \( t > 1 \) and that the exceptional vertex is distinguished by multiplicity. So it is sufficient to show that \( \tilde{\mu}^{-(r+1)} \circ \mu_{i+1}^+ \circ \tilde{\mu}^r(A^\rho) \in \mathcal{R} \) for any tree \( \Gamma^r \) and for any mutation \( \mu_{i+1}^r \).

Applying \( \tau^{-1} \) we are going to perform a suitable series of mutations with the standard Brauer star. This is done in the next sections. □

**Remark 3.** The question of the relations in \( \mathcal{R} \) remains open.

### 3.3 Mutation of type I

Note that these computations are also valid for \( t = 1 \).

Without loss of generality assume that \( \Gamma^r \) is a tree with a standard labelling, so \( \tilde{\mu}^r(A) \) is a series of mutations described in section 3.1. Assume also that \( j := i_{r+1} \) is an edge incident to a vertex \( x \) of degree one in \( \Gamma^r \), so \( \mu_j^+ \) is a mutation of type I. In order to compute \( \tilde{\mu}^{-(r+1)} \circ \mu_j^+ \circ \tilde{\mu}^r(A) \) we are going to consider the following three cases: 1) \( x \) is not the root and there is an edge of the same level as \( j \), following \( j \) in the cyclic ordering; 2) \( x \) is not the root and there is no edge of the same level as \( j \), following \( j \) in the cyclic ordering; 3) \( x \) is the root.

1) \( x \) is not the root and there is an edge \( l \) of the same level as \( j \), following \( j \) in the cyclic ordering in \( \Gamma^r \).

**Claim:** \( \Gamma^{r+1} \) is a tree with the standard labelling, \( \mu_j^+ \circ \tilde{\mu}^r(A) \) is isomorphic to the standard tilting complex from Lemma 8, associated to this tree, and hence \( \tilde{\mu}^{-(r+1)} \circ \mu_j^+ \circ \tilde{\mu}^r(A) \simeq A \).

**Proof.** It is clear that \( \Gamma^{r+1} \) is a tree with the standard labelling. Assume that \( j \) is not incident to the root, denote by \( q \) the edge of the higher level incident to both \( j \) and \( l \) in \( \Gamma^r \). Since \( \phi_{\Gamma^r}(j) = \phi_{\Gamma^r}(l) \) and \( \psi(j) = \psi(l) = q \) in \( \Gamma^r \) the minimal left approximation of \( T_j^r \) has the form

\[
\begin{array}{c}
P_j \xrightarrow{\beta_j^r} P_q \\
\downarrow \beta_j^{r-1} \quad \downarrow 1d \\
\beta_j^{r-1} \quad \downarrow 1d \\
P_l \xrightarrow{\beta_j^{r-1}} P_q.
\end{array}
\]

\( T_j^{r+1} = (P_j \xrightarrow{\beta_j^{r-1}} P_l)[\phi_{\Gamma^r}(j) + 1] \) by Lemma 4, as desired, since \( \phi_{\Gamma^{r+1}}(j) = \phi_{\Gamma^r}(j) + 1 \) and \( \psi(j) = l \) in \( \Gamma^{r+1} \).

If \( j \) is incident to the root in \( \Gamma^r \), \( l \) is also incident to the root in \( \Gamma^r \), \( T_j^{r+1} = (P_j \xrightarrow{\beta_j^{r-1}} P_l)[1] \).

**Example 2.** Mutate edge 4 in Example 1.
And the summands of $T^{r+1}$ are:

- $P_1$
- $P_2 \xrightarrow{\beta} P_1$
- $P_3 \xrightarrow{\beta} P_2$
- $P_4 \xrightarrow{\beta} P_3$
- $P_5$

$\text{Claim:}$ It is enough to consider the case when $\Gamma^r$ is of the following model form, with $j = n$.

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (m) at (3.5,0) {$m$};
  \node (m+1) at (4.5,0) {$m+1$};
  \node (m+2) at (4.5,-0.5) {$m+2$};
  \node (n) at (4.5,-1) {$n$};
  \draw (1) to (m);
  \draw (m) to (m+1);
  \draw (m+1) to (n);
\end{tikzpicture}
\end{center}

Proof. Assume that $j \neq n$. Recall that when we perform one irreducible mutation we change only one summand of the tilting complex and only one edge in the Brauer tree. The beginning of the series of mutations $\tilde{\mu}^{-(r+1)}$ is 

\[(\mu_{\tau(j+1)}^{-})^{\rho_{r+1}(\tau(j+1))} \circ ... \circ (\mu_{\tau(n-1)}^{-})^{\rho_{r}(\tau(n-1))} \circ (\mu_{\tau(n)}^{-})^{\rho_{r}(\tau(n))}.\]

But $\tau$ acts as identity on these indices and since we do not mutate these edges along $j$, the resulting tilting complex is $\bigoplus_{i=1}^{j} T_i^{r+1} \oplus \bigoplus_{i=j+1}^{n} P_n$, and the cyclic ordering of \{\textcolor{red}{$P_i$}\}_{i=j+1}^{n} is standard. The series of mutations $\tilde{\mu}^{-(r+1)}$ does not involve mutations of the edge $j$ along the edges with bigger indices, so we can assume that $j = n$.

Assume that \{\textcolor{red}{$m + 1, ..., n$}\} is not a line. The mutations of the edges not labelled $j$ in the series of mutations $\tilde{\mu}^{-(r+1)}$ do not involve $j$ in the computations of minimal right approximation so the resulting tilting complex is $\bigoplus_{i \neq j} P_i \oplus T_j$. The series of mutations $\tilde{\mu}^{-(r+1)}$ involves mutations of the edge $j$ only along the shortest path from $j$ to the exceptional vertex, which is a line. □

Claim: Assume $\Gamma^r$ is of the model form, then $\tilde{\mu}^{-(r+1)} \circ \mu_j^+ \circ \tilde{\mu}^r(A) \simeq H_n(A)$. 

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Proof. $T^r = \tilde{\mu}^r(A) = \bigoplus_{i=1}^{n+1} P_i \oplus \bigoplus_{i=m+2}^n (P_i \xrightarrow{\beta} P_{i-1})[i - (m + 1)]$. If $n - 1 = m + 1$, there is nothing to prove. Assume $n - 1 > m + 1$. $T^r_{n+1}$ is a cone of the morphism

$$
\begin{array}{ccc}
P_n & \xrightarrow{\beta} & P_{n-1} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{soc} & P_{n-1} \xrightarrow{\beta} P_{n-2},
\end{array}
$$

where $soc$ is the morphism, whose image is isomorphic to the socle of $P_{n-1}$. So $T^r_{n+1} \simeq P_n \xrightarrow{\beta} P_{n-1} \xrightarrow{soc} P_{n-1} \xrightarrow{\beta} P_{n-2}$, with $P_{n-2}$ concentrated in the same degree as in $T^r$. Let us apply $\tilde{\mu}^{-(r+1)}$. Applying $(\mu_{n-1}^{-1})^{n-1-(m+1)}$ we clearly get $\bigoplus_{i=1}^{m+1} P_i \oplus \bigoplus_{i=m+2}^n (P_i \xrightarrow{\beta} P_{i-1}[i - (m + 1)] \oplus (P_n \xrightarrow{\beta} P_{n-1} \xrightarrow{soc} P_{n-1} \xrightarrow{\beta} P_{n-2}) \oplus P_{n-1}$. Apply $(\mu_n^{-1})$: $T^{r+1+n-1-(m+1)}$ is a cone of the morphism shifted by $-1$

$$
\begin{array}{ccc}
P_n & \xrightarrow{\beta} & P_{n-1} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{soc} & P_{n-1} \xrightarrow{\beta} P_{n-2} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{soc} & P_{n-1} \xrightarrow{\beta} P_{n-2} \xrightarrow{id} 0.
\end{array}
$$

By Lemma 5 $T^{r+1+n-1-(m+1)} \simeq P_n \xrightarrow{\beta} P_{n-1} \xrightarrow{soc} P_{n-1} \xrightarrow{\beta^2} P_{n-3}$, with $P_{n-3}$ concentrated in the same degree as in $T^r$. By iterated application of Lemma 5 $T_n \simeq P_n \xrightarrow{\beta} P_{n-1} \xrightarrow{soc} P_{n-1}$ with $P_n$ concentrated in $-2$.

Applying the rest of the mutations from the series $\tilde{\mu}^{-(r+1)}$ we clearly get $H_n(A)$. □

3) $x$ is the root, $j = 1$. Let $l$ be the next edge in the cyclic ordering around the nonroot vertex of $j$ in $\Gamma$. $T^r_1 = P_1$ and $T^r_1 = P_1 \xrightarrow{\beta^{-1}} P_1$. The minimal left approximation of $P_1$ is

$$
\begin{array}{ccc}
0 & \xrightarrow{id} & P_1 \\
\downarrow & & \downarrow \\
P_l & \xrightarrow{\beta^{-1}} & P_1.
\end{array}
$$

Its cone is $P_l[1]$, hence $T^{r+1}$ is two-restricted and by results of Schaps and Zakay-Illouz $\tilde{\mu}^{-(r+1)} \circ \mu_j^{\pm} \circ \tilde{\mu}^r(A) \in R$ [17]. □

Note that $\tilde{\mu}^{-(r+1)} \circ \mu_j^{\pm} \circ \tilde{\mu}^r(A)$ can be computed directly quite easily. Keeping in mind that $H_i^{-1}(A) \simeq (\mu_j^{-1}(A))^2(\mu_j^{\pm}) \circ \mu_j^{\pm} \circ \tilde{\mu}^r(A) \simeq (H_2^{-1} \circ H_3^{-1} \circ ... \circ H_l^{-1}(A))(1)$.
3.4 Mutation of type II

Assume that both ends of \( j := i_{r+1} \) are vertices of degree \( > 1 \) in \( \Gamma^r \), so \( \mu_j^+ \) is a mutation of type II. In order to compute \( \tilde{\mu}^{-\langle r+1 \rangle} \circ \mu_j^+ \circ \tilde{\mu}^r(A) \) we are going to consider the following three cases: 1) \( j \) is not incident to the root and there is an edge of the same level as \( j \), following \( j \) in the cyclic ordering; 2) \( j \) is not incident to the root and there is no edge of the same level as \( j \), following \( j \) in the cyclic ordering; 3) \( j \) is incident to the root.

1) \( j \) is not incident to the root and there is an edge \( l \) of the same level as \( j \), following \( j \) in the cyclic ordering. The edge of the next level, following \( j \) in the cyclic ordering is denoted by \( m \). Denote by \( q \) the edge of the higher level incident to both \( j \) and \( l \) in \( \Gamma^r \). Since \( \phi_{\Gamma^r}(j) = \phi_{\Gamma^r}(l) \), \( \phi_{\Gamma^r}(m) = \phi_{\Gamma^r}(j) + 1 \) and \( \psi(j) = \psi(l) = q \) in \( \Gamma^r \) the minimal left approximation \( f \) of \( T^r_j \) has the form:

\[
\begin{array}{c}
\begin{array}{c}
0 \to P_j \overset{\beta^{-q}}{\to} P_q
\\
P_m \overset{(\beta^{-q})}{\to} P_j \oplus P_l \overset{(0,\beta^{-q})}{\to} P_q.
\end{array}
\end{array}
\]

The following diagram is commutative:

\[
\begin{array}{c}
\begin{array}{c}
P_m \overset{-\beta^{-l}}{\to} P_l \overset{0}{\to} 0
\\
P_m \oplus P_j \overset{(\beta^{-l})}{\to} P_j \oplus P_l \overset{(0,\beta^{-l})}{\to} P_q \oplus P_q \overset{(-\beta^{-l},1d)}{\to} P_q
\\
P_m \overset{-\beta^{-l}}{\to} P_l \overset{(-\beta^{-l},1d,0)}{\to} 0.
\end{array}
\end{array}
\]

\[
\text{Cone}(f) =
\begin{array}{c}
\begin{array}{c}
(P_m) \overset{(Id,0)}{\to} P_l \overset{0}{\to} 0
\\
(P_m \oplus P_j) \overset{\beta^{-l}}{\to} P_l \oplus P_q \overset{(0,\beta^{-l},1d)}{\to} P_q
\\
(P_m \oplus P_j) \overset{(-\beta^{-l},1d,0)}{\to} 0.
\end{array}
\end{array}
\]

hence \( P_m \overset{-\beta^{-l}}{\to} P_l \cong P_m \overset{-\beta^{-l}}{\to} P_l \) is a summand of \( \text{Cone}(f) \). Clearly, \( \text{Cone}(f) \) is homotopic to \( P_m \overset{-\beta^{-l}}{\to} P_l \), because it is an indecomposable summand of a tilting complex. We see that, \( T^{r+1} \) is two-restricted and by results of Schaps and Zakay-Illouz \( \tilde{\mu}^{-\langle r+1 \rangle} \circ \mu_j^+ \circ \tilde{\mu}^r(A) \in \mathcal{R} \). \( \square \)
Note that a direct computation shows that 
\[ \hat{\mu}^{-(r+1)} \circ \mu_j^+ \circ \hat{\mu}^r(A) \simeq H^{-1}_{j+1} \circ H^{-1}_{j+2} \circ \ldots \circ H^{-1}_m(A). \]

2) \( j \) is not incident to the root and there is no edge of the same level as \( j \), following \( j \) in the cyclic ordering.

**Claim:** It is enough to consider the case when \( \Gamma^r \) is of the following model form.

Proof. We have left only the edges which are mutated along \( j \) in the series of mutations \( \hat{\mu}^{-(r+1)} \) and the edges along which \( \mu_j \) is mutated. If \( \{j+1, \ldots, m-1\} \) or \( \{m+1, \ldots, n\} \) do not form stars but form more complicated trees, then in the series of mutations \( \hat{\mu}^{-(r+1)} \) each edge is first brought to its model position and only then is mutated along \( j \) so these mutations from \( \hat{\mu}^r \) and \( \hat{\mu}^{-(r+1)} \) cancel each other. □

**Claim:** Assume \( \Gamma^r \) is of the model form, then \( \hat{\mu}^{-(r+1)} \circ \mu_j^+ \circ \hat{\mu}^r(A) \simeq F_j(A) \), where \( F_j(A) \) is defined in the Appendix (the cyclic ordering of the edges corresponding to the summands of \( F_j(A) \) in the Brauer star is defined by the linear ordering of these summands from the bottom to the top).

Proof. In these calculations we are going to omit the degrees of the complexes but we will take them into account, of course. Let us compute \( T_j^{r+1} \). The minimal left approximation of \( T_j^r \) is

\[
\begin{array}{ccccccc}
0 & \rightarrow & P_j & \beta^{j-1} & P_l & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_m & \beta^{m-j} & P_j & \text{Id} & P_l & \text{soc} & P_l & \beta & P_{l-1}.
\end{array}
\]

And by an analog of Lemma 5 \( T_j^{r+1} \simeq P_m \beta^{m-1} \rightarrow P_l \text{soc} \rightarrow P_l \beta \rightarrow P_{l-1} \).

Apply \( \hat{\mu}^{-(r+1)} = \mu_2^{-1} \circ \ldots \circ (\mu_{j-1})^{-1} \circ (\mu_j)^{l-2} \circ (\mu_{j+1})^{l-1} \circ (\mu_m)^{l-1} \circ (\mu_{m+1})^{l+1} \circ \ldots \circ (\mu_{m-1})^{l+1} \circ (\mu_{m-1})^l \circ \ldots \circ (\mu_n)^l \circ (\mu_1)^{l-1} \). Applying \( (\mu_i)^{l-1} \) we clearly get \( T_l \simeq P_l \).
Applying \((\mu_n^{-})^i\): The minimal right approximation of \(P_n \xrightarrow{\beta^{n-m}} P_m\) is a morphism from \(P_m \xrightarrow{\beta^{m-1}} P_1 \xrightarrow{soc} P_1 \xrightarrow{\beta} P_{l-1}\) induced by \(P_m \xrightarrow{Id} P_m\) and by Lemma 5 its cone is isomorphic to \(P_n \xrightarrow{\beta^{n-1}} P_1 \xrightarrow{soc} P_1 \xrightarrow{\beta} P_{l-1}\). Its minimal right approximation is a morphism from \(P_{l-1} \xrightarrow{\beta} P_{l-2}\) induced by \(P_{l-1} \xrightarrow{Id} P_{l-1}\) and by Lemma 5 the cone is isomorphic to \(P_n \xrightarrow{\beta^{n-1}} P_1 \xrightarrow{soc} P_1 \xrightarrow{\beta} P_{l-2}\). By iterated application of Lemma 5 we get that \(T_n \simeq P_n \xrightarrow{\beta^{n-1}} P_l \xrightarrow{soc} P_l\).

Application of \((\mu_{n-1}^{-})^i\), ..., \((\mu_{m+1}^{-})^i\) is completely analogous and we get

\[
T_{n-1} \simeq P_{n-1} \xrightarrow{\beta^{n-1-i}} P_1 \xrightarrow{soc} P_l
\]

... \[
T_{m+1} \simeq P_{m+1} \xrightarrow{\beta^{m+1-i}} P_1 \xrightarrow{soc} P_l.
\]

Applying \((\mu_{m-1}^{-})^{i+1}\): The minimal right approximation of \(P_{m-1} \xrightarrow{\beta^{m-1-i}} P_j\) is a morphism from \(P_m \xrightarrow{\beta^{m-j}} P_j\) to \(P_{m-1} \xrightarrow{\beta^{m-1-j}} P_j\) induced by \((\beta, Id)\), by Lemma 4 its cone is \(P_m \xrightarrow{\beta} P_{m-1}\). The minimal right approximation of \(P_m \xrightarrow{\beta} P_{m-1}\) is a morphism from \(P_m \xrightarrow{\beta^{m-1-i}} P_1 \xrightarrow{soc} P_1 \xrightarrow{\beta} P_{l-1}\) to \(P_m \xrightarrow{\beta} P_{m-1}\) induced by \((soc, 0, 0, 0)\), its cone is \(P_m \xrightarrow{\beta} P_{m-1}\). By iterated application of Lemma 5 we get that \(T_{m-1} \simeq P_m \xrightarrow{\beta^{m-1-i}} P_l \xrightarrow{soc} P_l\). Similarly,

\[
T_{m-2} \simeq P_m \xrightarrow{\beta^{m-1-i}} P_1 \xrightarrow{soc} P_1 \xrightarrow{soc} P_{l-2}
\]

... \[
T_{j+1} \simeq P_m \xrightarrow{\beta^{m-1-i}} P_1 \xrightarrow{soc} P_1 \xrightarrow{soc} P_{l+1}.
\]

Applying \((\mu_{m-1}^{-})^i\): the minimal right approximation of \(P_m \xrightarrow{\beta^{m-j}} P_j\) is a morphism from \(P_m \xrightarrow{\beta^{m-j}} P_1 \xrightarrow{soc} P_1 \xrightarrow{\beta} P_{l-1}\) to \(P_m \xrightarrow{\beta^{m-j}} P_j\) induced by \((soc, 0, 0, 0)\) and analogously to the previous case we have

\[
T_m \simeq P_m \xrightarrow{\beta^{m-j}} P_l \xrightarrow{soc} P_l \xrightarrow{soc} P_j.
\]
Applying $(\mu_j^{-})^{l-1}$, the minimal right approximation of $P_m \xrightarrow{\beta^{n-1}} P_l \xrightarrow{soc} P_l$ is a morphism from $P_{l-1} \xrightarrow{\beta} P_{l-2}$ induced by $P_{l-1} \xrightarrow{Id} P_{l-1}$ and by iterated application of Lemma 5 we get that $T_j \simeq P_m \xrightarrow{\beta^{n-1}} P_l \xrightarrow{soc} P_l$.

Application of $\mu_2^{-} \circ ... \circ (\mu_{l-1}^{-})^{l-2}$ clearly yields $T_{l-1} \simeq P_{l-1}, ..., T_1 \simeq P_1$. □

Claim: $F_j(A) = H_n \circ ... \circ H_{m+1} \circ H_j^{-1} \circ H_{j+1}^{-1} \circ ... \circ H_{m-1}^{-1} \circ H_m \circ H_{m-1} \circ ... \circ H_j$.

and hence $F_j(A) \in R$.

$H_{n-1}^{-1}(A) \simeq (\mu_{n-1}^{-})^2(A)$, so $H_{n-1}^{-1}(F_j(A))$ can be computed mutating $T_n$ twice along $T_i$. The minimal right approximation of $T_n$ is a map from $T_i$ to $T_n$ induced by $P_i \xrightarrow{Id} P_i$, by Lemma 5 its cone is isomorphic to $P_n \xrightarrow{\beta^{n-1}} P_l$ and after the shift $P_i$ is concentrated in the same degree as in $T_i$, applying Lemma 5 again $[H_{n-1}^{-1}(F_j(A))]_n \simeq P_n$.

Similarly,

$$[H_{n-1}^{-1} \circ H_{n-1}^{-1}(F_j(A))]_{n-1} \simeq P_{n-1}$$

... 

$$[H_{m+1}^{-1} \circ ... \circ H_{n-1}^{-1} \circ H_{n-1}^{-1}(F_j(A))]_{m+1} \simeq P_{m+1}.$$

Note that although we have not mutated $T_j$, its position in the cyclic ordering has changed and $[H_{m+1}^{-1} \circ ... \circ H_{n-1}^{-1} \circ H_{n-1}^{-1}(F_j(A))]_m \simeq P_l$. Denote $H_{m+1}^{-1} \circ ... \circ H_{n-1}^{-1} \circ H_{n-1}^{-1}(F_j(A))$ as $T^r$.

Recall that the homotopy category can be defined as the stable category of the Frobenius category of complexes with respect to degree-wise split exact sequences. So the distinguished triangles in the homotopy category come from degree-wise split exact sequences [7].

$H_j(A) \simeq (\mu_j^{-})^2(A)$, so $H_j(T^r)$ can be computed mutating $T_m$ twice along $T_j$. Consider the following degree-wise split exact sequence in the category of complexes:

\[
\begin{array}{ccc}
X = & 0 & P_m \xrightarrow{\beta^{m-1}} P_j \\
0 & P_m \xrightarrow{-\beta^{m-1}} P_l \oplus P_m \xrightarrow{soc} P_l \oplus P_j \\
T_m = & P_m \xrightarrow{soc} P_l \oplus P_m \xrightarrow{\beta^{m-1}} P_l \oplus P_j \\
0 & P_m \xrightarrow{Id} P_l \oplus P_m \xrightarrow{(I_0 \cdot 0)} P_l \oplus P_j \\
T_j = & P_m \xrightarrow{-\beta^{m-1}} P_l \oplus P_m \xrightarrow{-soc} P_l \oplus P_j
\end{array}
\]

where $f$ is clearly a minimal left approximation of $T_m$ with respect to other summands of $T^r$. So $Cone(f) \simeq (P_m \xrightarrow{\beta^{m-1}} P_j)[1]$. The minimal left approximation of $Cone(f)$ with respect to other summands of $T^{r+1}$ is a morphism
3.5 Multiplicity free case

Let \( A \) be a Brauer star algebra of type \((n,1)\). Consider a subgroup \( \hat{\mathcal{R}} \) of the derived Picard group of \( A \) generated by shift, \( \text{Pic}(A) \), equivalences induced by \( H_i \) and equivalences induced by \( Q_i \).

\[
Q_i(P_j) = \begin{cases} 
0 \to P_i \to 0, & j = i \\
0 \to P_i \beta^{-j} \to P_j, & j \neq i.
\end{cases}
\]

So we have

\[
H_{m-1} \circ \ldots \circ H_{j+1} \circ H_{j} \circ H_{m-1} (P_{m-1} \circ \ldots \circ P_{j+1}) \sim P_{m-1}
\]

Similarly,

\[
H_{m-1} \circ \ldots \circ H_{j+1} \circ H_{j} \circ H_{m-1} (P_{m-1} \circ \ldots \circ P_{j+1}) \sim P_{j}.
\]

So \( H_{j} \circ H_{j+1} \circ \ldots \circ H_{m-1} \circ H_{m} (G_{j}) \sim A \) and the assertion follows. \( \square \)

Note that in the resulting copy of \( A \) the cyclic ordering of the projective modules is standard, but the labelling is not: \( j \) and \( m \) have switched their places.

3) \( j \) is incident to the root. This case is similar to case (1). If we set \( q = 0 \) it is clear that \( T'_{r+1} \sim P_{m} \beta_{m-i} \), hence \( T'_{r+1} \) is two-restricted and by results of Schaps and Zakay-Illouz \( \tilde{\mu}^{-1} \circ \mu_{j}^{r} \circ \tilde{\mu}' \in \mathcal{R} \). \( \square \)

Note that in this case the direct computation also shows that \( \tilde{\mu}^{-1} \circ \mu_{j}^{r} \circ \tilde{\mu}'(A) \sim H_{j+1} \circ H_{j+2} \circ \ldots \circ H_{m}(A) \).

This finishes the proof of Theorem 1.
Remark 4. The group generated by $Q_i$ was considered by Muchtadi-Alamsyah in [12]. It has an action of the braid group whose associated Coxeter group is given by the complete graph on $n$ vertices, but this action is not faithful.

Remark 5. It is clear that $Q_i(A) \simeq \mu_{i+1}^+ \circ \mu_{i+2}^+ \circ ... \circ \mu_{i-2}^+ \circ \mu_{i-1}^- (A)$ and that this series of mutations changes the central vertex of the Brauer star to the outer vertex of the edge labelled $i$. It is also clear that one can obtain $Q_i$ from $Q_j$ by rotation.

Theorem 2. If $t = 1$, then $\text{TrPic}(A) \simeq \tilde{\mathcal{R}}$.

Proof. As before, it is sufficient to prove that for any tilting complex $T$ there is an element from $\tilde{\mathcal{R}}$ which sends the projective modules of $A$ to the summands of $T$. Assume that $T$ is concentrated in nonpositive degrees. By results of Aihara [2] $T = \mu_{i_1}^+ \circ ... \circ \mu_{i_r}^+ \circ \mu_{i_r}^+ \circ \mu_{i_{r-1}}^- \circ ... \circ \mu_{i_2}^+ \circ \mu_{i_1}^- (A)$ for some $\{i_1, i_2, ..., i_s\}$. If the series of mutations $\mu_{i_1}^+ \circ ... \circ \mu_{i_r}^+ \circ \mu_{i_{r-1}}^- \circ ... \circ \mu_{i_2}^+ \circ \mu_{i_1}^- (A)$ does not change the central vertex of the Brauer star, then by calculations of the previous section the equivalence induced by $T$ belongs to $\mathcal{R}$ and hence to $\tilde{\mathcal{R}}$. If the series of mutations changes the central vertex of the Brauer star to the outer vertex of the edge labelled $i$, then applying $\mu_{i_{r-1}}^- \circ \mu_{i_{r-2}}^+ \circ ... \circ \mu_{i_2}^+ \circ \mu_{i_1}^- (A)$ we obtain an equivalence $T' \in \mathcal{R}$. So we have $T \simeq T' \circ Q_i (A)$, hence the equivalence induced by $T$ belongs to $\tilde{\mathcal{R}}$. □

Remark 6. As in the case $t > 1$, the question of the relations in $\tilde{\mathcal{R}}$ remains open.
4 Appendix

\[ T_1 \quad P_1 \]

\[ T_2 \quad P_2 \]

\[ \vdots \]

\[ T_{l-1} \quad P_{l-1} \]

\[ T_j \quad P_m \rightarrow_{\beta^{m-1}} P_l \rightarrow_{soc} P_l \]

\[ T_m \quad P_m \rightarrow_{soc} P_l \oplus P_m \rightarrow_{\beta^{m-1}} P_l \oplus P_j \]

\[ F_j = T_{j+1} = P_m \rightarrow_{soc} P_l \oplus P_m \rightarrow_{\beta^{m-j-1}} P_l \oplus P_{j+1} \]

\[ \vdots \]

\[ T_{m-1} \quad P_m \rightarrow_{soc} P_l \oplus P_m \rightarrow_{\beta^{m-1}} P_l \oplus P_{m-1} \]

\[ T_{m+1} \quad P_{m+1} \rightarrow_{\beta^{m+1-1}} P_l \rightarrow_{soc} P_l \]

\[ \vdots \]

\[ T_n \quad P_n \rightarrow_{\beta^{n-1}} P_l \rightarrow_{soc} P_l \]

\[ T_l \quad P_l \]
\[ G_j = T_{j+1} = P_{j+1} \rightarrow P_l \rightarrow P_l \]

\[ T_{m-1} \quad P_{m-1} \rightarrow P_l \rightarrow P_l \]

\[ T_j \quad P_m \rightarrow P_l \rightarrow P_l \]

\[ T_l \quad P_l \]

\[ T_{m+1} \quad P_{m+1} \]

\[ T_n \quad P_n \]
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