QCD penguins are responsible for about 2/3 of the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decays, as inferred from a combined analysis of $K \to \pi\pi$ and $K_L \to \gamma\gamma$. Further tests based on the decays $K_S \to \pi^0\gamma\gamma$ and $K^+ \to \pi^+\gamma\gamma$ are proposed. New insights into the treatment of $\pi^0$, $\eta$, $\eta'$ pole amplitudes are also reported.

1 Introduction

Recently, a systematic analysis of $\eta_0$ pole contributions to radiative $K$ decays was performed in the context of large $N_c$ ChPT, in order to better understand the role of gluonic penguin operators in $K \to \pi\pi$ transitions. In this note, we emphasize some aspects of this study, in view of the forthcoming new experimental information on $K^+ \to \pi^+\gamma\gamma$ by the NA48 Collaboration. A number of issues, like the correspondence between the $SU(3)$ and $U(3)$ chiral expansions, the impact of our analysis for $K_L \to \gamma\gamma^*$, $K_L \to \pi^0\pi^0\gamma\gamma$ and $K_L \to \pi^+\pi^-\gamma$, or the fate of the weak mass operator, are left to the original paper.

2 General framework

The effective weak Hamiltonian relevant to describe (CP-conserving) hadronic $K$ decays reads:

$$\mathcal{H}_{\Delta S=1}^{\Delta S=1} (\mu < m_c) \simeq \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left[ z_1 (\mu) Q_1 (\mu) + z_2 (\mu) Q_2 (\mu) + z_6 (\mu) Q_6 (\mu) \right],$$

(1)
with the familiar current-current operators

\[ Q_1 = 4 \langle \bar{s}_L \gamma_\alpha d_L \rangle \langle \bar{u}_L \gamma^\alpha u_L \rangle, \quad Q_2 = 4 \langle \bar{s}_L \gamma_\alpha u_L \rangle \langle \bar{u}_L \gamma^\alpha d_L \rangle, \]

and the density-density dominant penguin operator

\[ Q_6 = -8 \langle \bar{s}_L q_R \rangle \langle \bar{q}_R d_L \rangle. \]

In our notations, \( q^R_L \equiv \frac{1}{2}(1 \pm \gamma_5)q \) and the light flavours \( q = u, d, s \) are summed over. The effective coupling constants \( z_i (\mu) \) contain QCD effects above the renormalization scale \( \mu \), kept high enough to allow the use of perturbation theory. In order to investigate the effects of long-distance strong interactions, we will make use of ChPT (Chiral Perturbation Theory) techniques.

ChPT relies on the SU(3)_L \times SU(3)_R symmetry of the QCD Lagrangian in the massless limit to build a dual representation, in terms of meson fields. If one formally considers the number of QCD colours \( N_c \) as large, SU(3) can be extended to U(3) and the spontaneous symmetry breaking \( U(3)_L \times U(3)_R \rightarrow U(3)_V \) gives rise to a nonet \( \Pi \) of pseudoscalar Goldstone bosons, which are written \( U \equiv \exp(i\sqrt{2}\Pi/F) \) in the standard parametrization. This extension to U(3) will prove crucial afterwards. The corresponding leading nonlinear Lagrangian reads

\[ \mathcal{L}_S^{(p^2,\infty)+(p^0,1/N_c)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle + \frac{F^2}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle + \frac{F^2}{16N_c} m_0^2 (\ln U - \ln U^\dagger)^2 \]

where \( \langle \rangle \) denotes a trace over flavours, the external source \( \chi \) is frozen at \( \chi = rM \) with \( M = \text{diag}(m_u, m_d, m_s) \) to account for meson masses, \( F \) is identified with the pion decay constant \( F_\pi = 92.4 \text{ MeV} \) at this order and \( m_0 \) represents the anomalous part of the \( \eta_0 \) mass. Note that the leading SU(3) chiral Lagrangian is recovered in the limit \( m_0 \rightarrow \infty \), when the \( \eta_0 \) decouples.

The meson realization of Eq. (1) can be obtained from the chiral representations of the corresponding quark currents and densities, i.e., preserving the colour and flavour structures:

\[ \mathcal{H}_{\text{eff},Q(p^2)} (\mu \sim m_{\pi,K}) \simeq \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^{*} \left[ x_1 \hat{Q}_1 + x_2 \hat{Q}_2 + x_6 \hat{Q}_6 \right], \]

with

\[ \hat{Q}_1 = 4 \langle L_{\mu} \rangle_{23} \langle \bar{L}^{\mu} \rangle_{11}, \quad \hat{Q}_2 = 4 \langle L_{\mu} \rangle_{13} \langle \bar{L}^{\mu} \rangle_{21}, \quad \hat{Q}_6 = 4 \langle L_{\mu} \bar{L}^{\mu} \rangle_{23}, \]

and the left-handed currents \( \langle L_{\mu} \rangle_{lk}^{lk} = \frac{iF^2}{2} \langle \partial_\mu U U^\dagger \rangle_{lk} \). The weak coefficients \( x_i \) are not fixed by symmetry arguments, and contain both short-distance and long-distance strong interaction effects. The latter are known to be important in explaining the \( \Delta I = 1/2 \) rule observed in \( K \rightarrow \pi\pi \) decays. Still, the genuine mechanism responsible for the \( \Delta I = 1/2 \) enhancement, i.e., the relative strength of the penguin and current-current operators, has not been completely settled yet. In this work, we propose a phenomenological extraction of the \( x_i \) parameters, and thus of the penguin fraction \( \mathcal{F}_P = 3x_6/(\sim x_1 + 2x_2 + 3x_6) \).

To reach this goal, it is clear that one has to go beyond the standard SU(3) ChPT which only contains two independent weak operators \( (Q_8 \text{ and } Q_{27}) \) such that current-current and penguin operators cannot be disentangled. On the other hand, in U(3), the presence of \( \eta_0 \) as a dynamical degree of freedom allows for an extra \( \mathcal{O}(p^2) \) weak operator

\[ Q_8^* = 4 \langle L_{\mu} \rangle_{23} \langle \bar{L}^{\mu} \rangle \sim 4 \langle L_{\mu} \rangle_{23} \partial^\mu \eta_0, \]

which, together with the straightforward extensions of \( Q_8 \) and \( Q_{27} \) to U(3)

\[ Q_8 = 4 \langle L_{\mu} \bar{L}^{\mu} \rangle_{23}, \quad Q_{27} = 4 \left[ \langle L_{\mu} \rangle_{23} \langle \bar{L}^{\mu} \rangle_{11} + \frac{2}{3} \langle L_{\mu} \rangle_{13} \langle \bar{L}^{\mu} \rangle_{21} - \frac{1}{3} \langle L_{\mu} \rangle_{23} \langle \bar{L}^{\mu} \rangle \right], \]
permits now to write the effective Hamiltonian in a way equivalent to Eq. (5):
\[ H_{\text{eff},O(p^2)}^{\Delta S=1} (\mu \sim m_{\pi,K}) \simeq G_8 Q_8 + G_{27} Q_{27} + G_{8}^8 Q_{8}^8. \] (9)

Explicitly, this change of basis reads (\( G_W \equiv G_F V_{ud} V_{us}/\sqrt{2} \)):
\[ G_8/G_W = -\frac{2}{5} x_1 + \frac{2}{5} x_2 + x_6, \quad G_{8}^8/G_W = \frac{2}{5} x_1 - \frac{2}{5} x_2, \quad G_{27}/G_W = \frac{3}{5} (x_1 + x_2). \] (10)

G_8 and G_{27} are still extracted from \( K \to \pi \pi \). The knowledge of \( G_8^8 \) would thus give access to the \( x_i \) parameters, and consequently to \( F_P \). Because of Eq. (7), natural candidates for its extraction are anomaly-driven radiative \( K \) decays, that receive a \( \eta_0 \) pole contribution.

3 Penguin fraction in \( K \to \pi \pi \) vs \( \eta_0 \) effects in \( K_L \to \gamma \gamma \)

Due to the well-known pole cancellations at work in \( K_L \to \gamma \gamma \), we propose a two-step analysis for this mode:

**Step 1**: work with the theoretical masses \( m_{\pi}, m_{\eta}, m_{\eta}', \) i.e., consistently at a given order in ChPT, in order to identify the vanishing pole contributions (Fig.1a). It turns out that \( Q_8 \) does not contribute at \( O(p^4) \), just like in \( SU(3) \) ChPT. The leading contribution, of \( O(p^4) \), comes from the \( \overline{u}u \) intermediate state generated by \( \hat{Q}_1 \) (Fig.1b), and is thus proportional to \( G_W x_1 = G_8^8 + 2 G_{27}/3 \), i.e., the non-symmetry breaking couplings.

**Step 2**: freeze the \( \pi^0, \eta, \eta' \) poles at the physical values \( M_{\pi}, M_{\eta}, M_{\eta}' \) to ensure correct analytical properties for the remaining contributions (\( \hat{Q}_1 \)) only. This is done through the following prescription for the \( \eta-\eta' \) propagator:
\[ i P_{\text{phys}} (q^2)_{\eta_0}^{-1} = \left( \begin{array}{cc} \cos \theta_P & \sin \theta_P \\ -\sin \theta_P & \cos \theta_P \end{array} \right) \left( \begin{array}{cc} q^2 - M_{\eta}'^2 & 0 \\ 0 & q^2 - M_{\eta^0}^2 \end{array} \right) \left( \begin{array}{cc} \cos \theta_P & -\sin \theta_P \\ \sin \theta_P & \cos \theta_P \end{array} \right), \] (11)

where the parametrisation in terms of one mixing angle is allowed as we work at lowest order in the chiral expansion, cf. Eq. (10). A discussion of two-angle pole formulas may be found in our original paper.

The resulting pole amplitude (\( c_\theta \equiv \cos \theta_P, s_\theta \equiv \sin \theta_P \)),
\[ A^{\mu \nu} (K_L \to \gamma \gamma) = \frac{2 F_A}{\pi} \left[ G_8^8 + \frac{2}{3} G_{27} \right] M_{K^*}^2 i \varepsilon^{\mu \nu \rho \sigma} k_{1\rho} k_{2\sigma} \times \left( \frac{1}{M_{K^*}^2 - M_{\eta'}^2} + \frac{(c_\theta - 2 \sqrt{2} s_{\theta})(c_\theta - \sqrt{2} s_{\theta})}{3(M_{K^*}^2 - M_{\eta^0}^2)} + \frac{(s_\theta + 2 \sqrt{2} c_{\theta})(s_\theta + \sqrt{2} c_{\theta})}{3(M_{K^*}^2 - M_{\eta^0}^2)} \right), \] (12)

turns out to be dominated by the \( \eta \):
\[ A^{\mu \nu} (K_L \to \gamma \gamma) = \left[ G_8^8 + \frac{2}{3} G_{27} \right] \left[ (0.46)_\pi - (1.83 \pm 0.30)_{\eta^0} - (0.12 \pm 0.02)_{\eta'} \right] i \varepsilon^{\mu \nu \rho \sigma} k_{1\rho} k_{2\sigma}, \] (13)
and is quite stable with respect to the $\eta_8$-$\eta_0$ mixing angle $\theta_P$, allowed to vary in the large range $[-25^\circ, -15^\circ]$ to get a hold on the typical size of NLO effects. From the experimental $K_L \to \gamma\gamma$ branching ratio, we obtain $(G_s^8/G_8)_{ph} \simeq \pm 1/3$, in agreement with the QCD-inspired value $(G_s^8/G_8)_{th} = -0.38 \pm 0.12$, leading to $(F_P)_{th} \approx 60\%$. 

4 $K_S \to \pi^0\gamma\gamma$ - the simplest probe 

The simplest mode to test $(G_s^8/G_8)_{th+ph} \simeq -1/3$ is $K_S \to \pi^0\gamma\gamma$. Indeed, at leading order in the chiral expansion, i.e., $O(p^4)$, it proceeds entirely through pole diagrams (Fig.2a). It receives contributions from $\hat{Q}_1$ and $\hat{Q}_6$, but not $\hat{Q}_2$, or correlated contributions from $Q_8^s$, $Q_9^s$ and $Q_8$ in the natural $U(3)$ basis. The latter dominates the decay via the pion pole. When $\eta_0$ effects are integrated out, the standard $SU(3)$ result \cite{5} is recovered:

$$B(K_S \to \pi^0\gamma\gamma)_{SU(3) O(p^4)}^{m_{\gamma\gamma}>220 \text{ MeV}} = 3.8 \times 10^{-8}. \quad (15)$$

However, the contribution of the $\eta_0$ meson, despite non-leading, can significantly enhance the branching fraction (Fig.2b). For our preferred value \cite{14} and $\theta_P \in [-25^\circ, -15^\circ]$, we obtain:

$$B(K_S \to \pi^0\gamma\gamma)_{U(3) O(p^4)}^{m_{\gamma\gamma}>220 \text{ MeV}} = (4.8 \pm 0.5) \times 10^{-8}, \quad (16)$$

where the theoretical error only reflects the ranges assigned to $G_s^8/G_8$ and $\theta_P$. The current experimental value is \cite{6}:

$$B(K_S \to \pi^0\gamma\gamma)_{m_{\gamma\gamma}>220 \text{ MeV}}^{exp} = (4.9 \pm 1.8) \times 10^{-8}. \quad (17)$$

Note that a more precise measurement could already fix the sign of $G_s^8/G_8$.

5 $K^+ \to \pi^+\gamma\gamma$ - a promising probe 

The case of $K^+ \to \pi^+\gamma\gamma$ is slightly more involved as it proceeds through both loop and pole diagrams at leading order in the chiral expansion, i.e., again, $O(p^4)$. Still, these two types of contributions correspond to photons in different $CP$ eigenstates, and do not interfere in the rate. The usual $SU(3)$ analysis, including unitarity corrections, can thus be applied to the loops while the poles (Fig.3a), sensitive to $\eta_0$ effects, are better treated within the $U(3)$ framework.

Unlike for $K_S \to \pi^0\gamma\gamma$, the pion pole contribution from $Q_8$ plays a minor role here as $K^+ \to \pi^+\pi^0$ is purely $\Delta I = 3/2$ when on-shell. The pole amplitude is thus quite sensitive to
Figure 3: a) Pole diagrams for $K^+ \to \pi^+\gamma\gamma$. b) $\mathcal{B}(K^+ \to \pi^+\gamma\gamma)$ as a function of $G_s^s/G_s$ for $\theta_P = -15^\circ, -20^\circ, -25^\circ$. c) $\mathcal{B}(K^+ \to \pi^+\gamma\gamma)$ for $m_{\gamma\gamma} < 108$ MeV, $\times 10^9$. Assuming non-negligible loop contributions\textsuperscript{8}, the recent upper bound\textsuperscript{10} hints towards negative values for $G_s^s/G_s$. The stars refer to Eq.(14).

$Q_s^8$ and $Q_{27}$. Already at the $SU(3)$ level, when $\eta_0$ effects are discarded, one can see that the 27 operator actually accounts for about half of the pole-induced branching fraction:

$$\mathcal{B}(K^+ \to \pi^+\gamma\gamma)^{P,SU(3),\mathcal{O}(p^4)}_{m_{\gamma\gamma}>220 \text{ MeV}} = 1.17 \times 10^{-7},$$

(18)

instead of 0.51 $\times$ 10$^{-7}$ without $Q_{27}$\textsuperscript{7}. The contribution of the $\eta_0$ meson may substantially suppress or enhance this value, depending on $G_s^s/G_s$ (Fig.3b).

In particular, for $G_s^s/G_s = -0.38 \pm 0.12$ and $\theta_P \in [-25^\circ, -15^\circ]$, poles can be safely neglected with respect to loops:

$$\mathcal{B}(K^+ \to \pi^+\gamma\gamma)^{P,U(3),\mathcal{O}(p^4)}_{m_{\gamma\gamma}>220 \text{ MeV}} \lesssim 0.3 \times 10^{-7},$$

(19)

while, for $G_s^s/G_s > 0$, they could increase the total rate by more than 20%. In that case, they should be taken into account in the extraction of the $\mathcal{O}(p^4)$ combination of counterterms $\hat{c}$ to reach consistency between the total and differential rate\textsuperscript{7,8,9}.

Finally, restricting the analysis to the low energy end of the $\gamma\gamma$ spectrum, negative values of $G_s^s/G_s$ are already favoured (cf. Fig3c).

6 Implication for $\Delta M_K$

Pole diagrams also play a central role in the long distance contribution to the $K_L$-$K_S$ mass difference $\Delta M_K$ (Fig.4a). The situation here is quite similar to the one of $K_L \to \gamma\gamma$, in that the contribution of $Q_8$ vanishes both in $SU(3)$ and $U(3)$ ChPT at $\mathcal{O}(p^4)$, the leading effect being driven by the $\hat{Q}_1$ operator, i.e., a $u\bar{u}$ pair (Fig.4b). The resulting pole formula was worked out in our original paper\textsuperscript{11}. Its contribution to $\Delta M_K$ is summarized in Fig.4c. For the preferred value Eq.(14), the negative contribution of poles partially cancels the positive contribution of $\pi\pi$ loops, leaving to short-distance effects\textsuperscript{11} the task of reproducing the bulk of the observed mass difference.
7 Conclusion

The $\Delta S = 1$ effective operator $Q_8^s$, which describes pure $\eta_0$ effects, holds the key to a phenomenological extraction of the penguin fraction in $K \to \pi\pi$ amplitudes via the change of chiral basis $(\hat{Q}_1, \hat{Q}_2, \hat{Q}_6) \leftrightarrow (Q_8, Q_{27}, Q_8^s)$.

From $\mathcal{B}(K_L \to \gamma\gamma)$, we found $G_8^s/G_8 \simeq -1/3$, which corresponds to a rather smooth non-perturbative current-current operator evolution and a penguin contribution to the $\Delta I = 1/2$ rule around $2/3$ at the hadronic scale. Better measurements of the decays $K_S \to \pi^0\gamma\gamma$ and $K^+ \to \pi^+\gamma\gamma$ would provide important tests of this picture.

The recourse to (broken) $U(3)$ chiral symmetry also allowed us to identify correctly the leading contribution to $K_L \to \gamma\gamma$, namely the transition $K_L \to u\bar{u}$ generated by $\hat{Q}_1$. This results in a new pole formula, based on $\hat{Q}_1$ instead of $\hat{Q}_6$. For $G_8^s/G_8 < 0$, the sign of the interference between the short-distance and dispersive $\gamma\gamma$ amplitudes in $K_L \to \mu^+\mu^-$ is flipped.

Along the same lines, the pole contribution to $\Delta M_K^{LD}$ was shown to be essentially due to $\hat{Q}_1$, pleading again for a better knowledge of the low-energy constants $x_i$, that is to say, of $G_8^s$.

Acknowledgments: S.T. would like to thank the organizers of the XLIrst Rencontres de Moriond. J.-M.G. acknowledges support by the Belgian Federal Office for Scientific, Technical and Cultural Affairs through IAP P5/27; C.S. is supported by the Schweizerischer Nationalfonds; S.T. is supported by the DFG grant No. NI 1105/1-1; this work has also been supported in part by IHP-RTN, EC contract No. HPRN-CT-2002-00311 (EURIDICE).

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