A note on tilted Sperner families with patterns

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Abstract

Let \( p \) and \( q \) be two nonnegative integers with \( p + q > 0 \) and \( n > 0 \). We call \( \mathcal{F} \subset \mathcal{P}([n]) \) a \((p,q)\)-tilted Sperner family with patterns on \([n]\) if there are no distinct \( F, G \in \mathcal{F} \) with:

(i) \( p|F \setminus G| = q|G \setminus F| \), and

(ii) \( f > g \) for all \( f \in F \setminus G \) and \( g \in G \setminus F \).

E. Long in [10] proved that the cardinality of a \((1,2)\)-tilted Sperner family with patterns on \([n]\) is

\[
O(c^{120\sqrt{\log n}} \frac{2^n}{\sqrt{n}}).
\]

We improve and generalize this result, and prove that the cardinality of every \((p,q)\)-tilted Sperner family with patterns on \([n]\) is

\[
O(\sqrt{\log n} \frac{2^n}{\sqrt{n}}).
\]

Keywords: Sperner family, tilted Sperner family, permutation method

1 Introduction

A family \( \mathcal{F} \) of subsets of \([n]\) (where for \( n > 0 \) we will use the \([n]\) notation for \( \{1,2,...,n\} \) and \( \mathcal{P}([n]) \) for the power set) is called a Sperner family if \( F \not\subset G \) for all distinct \( F, G \in \mathcal{F} \). A classic result in extremal combinatorics is Sperner’s theorem [12], which states that the maximal cardinality of a Sperner family is \( \left(\frac{n}{2}\right) \). This result has a huge impact on combinatorics and has many generalizations (see e.g. [2]).

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Recently Sperner’s theorem played some role in the Polymath project to discover a new proof of the density Hales-Jewett theorem [11]. Motivated by its role in the proof Kalai asked whether one can achieve ‘Sperner-like theorems’ for ‘Sperner like families’ [8].

One direction to generalize the notion of Sperner families is the so called tilted Sperner families (see Definition 1.1). As written in [8]: Kalai noted that the ‘no containment’ condition can be rephrased as follows: \( F \) does not contain two sets \( F \) and \( G \) such that, in the unique subcube of \( \mathcal{P}([n]) \) spanned by \( F \) and \( G \), the bottom point is \( F \) and \( G \) is the top point. He asked: what happens if we forbid \( F \) and \( G \) to be at a different position in this subcube? In particular, he asked how large \( F \subset \mathcal{P}([n]) \) can be if we forbid \( F \) and \( G \) to be at a fixed ratio \( p:q \) in this subcube. That is, we forbid \( F \) to be \( p/(p+q) \) of the way up this subcube and \( G \) to be \( q/(p+q) \) of the way up this subcube. Equivalently we can say:

**Definition 1.1.** Let \( p,q \) be two nonnegative integers. We call \( F \subseteq \mathcal{P}([n]) \) a \((p,q)\)-tilted Sperner family if for all distinct \( F, G \in F \) we have

\[
p|F \setminus G| \neq q|G \setminus F|.
\]

Note that we can restrict ourselves to coprime \( p \) and \( q \). Also note the a Sperner family is just a \((1,0)\)-tilted Sperner family. In [8] Leader and Long proved the following theorem, which gives an asymptotically tight answer for the maximal cardinality of a \((p,q)\)-tilted Sperner family:

**Theorem 1.2.** Let \( p,q \) be coprime nonnegative integers with \( q \geq p \). Suppose \( F \subset \mathcal{P}([n]) \) is a \((p,q)\)-tilted Sperner family. Then

\[
|F| \leq (q - p + o(1))\left(\frac{n}{2}\right).
\]

Note that up to the \( o(1) \) term, this is the best possible, since the union of \( p - q \) consecutive levels is a \((p,q)\)-tilted Sperner family.

In [10] Long started to investigate the cardinality of tilted Sperner families with patterns (see Definition 1.3), which was also asked by Kalai ([9]).

**Definition 1.3.** Let \( p \) and \( q \) be nonnegative integers with \( p + q > 0 \). We call \( F \) a \((p,q)\)-tilted Sperner family with patterns, if there are no distinct \( F, G \in F \) with:

(i) \( p|F \setminus G| = q|G \setminus F| \), and

(ii) \( f > g \) for all \( f \in F \setminus G \) and \( g \in G \setminus F \).

In [10] he gave an upper bound on the cardinality of a \((1,2)\)-tilted Sperner family with patterns:

**Theorem 1.4.** ([10], Theorem 1.3) Let \( F \subset \mathcal{P}([n]) \) be a \((1,2)\)-tilted Sperner family with patterns. Then

\[
|F| \leq O(e^{120\sqrt{\log n}} \frac{2^n}{\sqrt{n}}).
\]
Actually in [10] he gives a proof of a weaker result with the density Hales-Jewett theorem, and proves Theorem [14] with a randomized generalization of Katona’s cycle method (see [5]).

In this note we generalize and improve his result by applying another generalization of Katona’s cycle method, the so called permutation method. We will apply the permutation method in a somewhat similar way like the authors of [3] and prove the following:

**Theorem 1.5.** Let $p$ and $q$ be non negative integers with $p + q > 0$ and let $F$ be a $(p,q)$-tilted Sperner family with patterns. Then

$$|F| \leq O\left(\sqrt{\log n} \frac{2^n}{\sqrt{n}}\right).$$

The paper is organized as follows: in Section 2 we prove our main theorem and in Section 3 we pose some questions.

## 2 Proof of Theorem [1.5]

**Proof.** If either $p$ or $q$ is zero, then we get back the usual Sperner family for which we know that the statement is true. In the following we fix $p,q > 0$ and furthermore we assume that $p \leq q$. The proof works similarly in case $p > q$.

### 2.1 The $(p,q)$-cut point

First we introduce a notion that will have crucial role in the proof.

**Definition 2.1.** We say that $x \in [n]$ is a $(p,q)$-cut point of $A \subseteq [n]$, if

$$0 \leq \frac{n - x - |([n] \setminus [x]) \cap A|}{q} - \frac{|A \cap [x]|}{p} < \frac{1}{p}. \quad (1)$$

We remark that $x$ is a $(p,q)$-cut point means that $\frac{p}{q}$ times the number of points of $A$ less than $x$ is ‘approximately’ equal to the number of points not belonging to $A$ that are larger than $x$.

**Lemma 2.2.** Every $A \subseteq [n]$ has a $(p,q)$-cut point.

**Proof.** Let us introduce the following functions: for $u \in \{0\} \cup [n]$ and $A \subseteq [n]$ let

$$f(A, u) := \frac{|A \cap [u]|}{p} \quad \text{and} \quad g(A, u) := \frac{n - u - |([n] \setminus [u]) \cap A|}{q},$$

with $|A \cap [0]| = 0$. Observe that if $|A| \neq 0$, then we have

$$0 = f(A, 0) < g(A, 0) = \frac{n - |A|}{q} \quad \text{and} \quad \frac{|A|}{p} = f(A, n) > g(A, n) = 0. \quad (2)$$

Also note that for all $i \in [n]$ if

- $i \in A$, then
\[ f(A, i - 1) + \frac{1}{p} = f(A, i) \quad \text{and} \quad g(A, i - 1) = g(A, i) \]

• 2 \( i \not\in A \), then
\[ f(A, i - 1) = f(A, i) \quad \text{and} \quad g(A, i - 1) - \frac{1}{q} = g(A, i). \]

By • 1, • 2 and (2) we have \( f(A, 0) < g(A, 0) \) and going towards \( n \), \( f \) is increasing, \( g \) is decreasing, but both of them changes with at most \( \frac{1}{p} \) and we have \( f(A, n) > g(A, n) \).

We are done with the proof of Lemma 2.2.

2.2 Using the permutation method

Let us introduce two pieces of notation:

1) for all \( F \in \mathcal{F} \) choose a \((p, q)\)-cut point \( x_F \) (we can do it by Lemma 2.2), and let
\[ \mathcal{F}_x := \{ F \in \mathcal{F} : x = x_F \} \quad \text{for} \quad x \in [n], \]

2) for \( x + k \leq n \) let \( j(x, k) := \lfloor \frac{q}{p}(n - x - k) \rfloor \).

Note that if \( x \) is a \((p, q)\)-cut point for \( A \subseteq [n] \), then
\[ |A \cap [x]| = j(x, |([n] \setminus [x]) \cap A|). \]

In this section we will prove an upper bound on \(|\mathcal{F}_x|\) using the permutation method.

Let us consider the following permutation group of \([n]\): for any \( x \in [n] \) let us denote by \( S_x \) the symmetric group on \( x \) elements, and let \( \Pi_x := S_x \times S_{n-x} \), the direct product of \( S_x \) and \( S_{n-x} \) (for definition of direct product of groups see e.g. [7]). An element \((\pi_1, \pi_2) = \pi \in \Pi_x\) acts on \([n]\) the following way:
\[ \pi(i) = \begin{cases} \pi_1(i) & \text{if } i \leq x, \\ \pi_2(i - x) + x & \text{if } i > x. \end{cases} \]

For \( A \subseteq [n] \) and \( \pi \in \Pi_x \) we will use the notation \( \pi(A) \) for \( \{\pi(a) : a \in A\} \).

Let us define the following families of sets for \( x \in [n] \), \( 0 \leq k \leq n - x \) if \( j(x, k) < x \):
\[ C(x, k) := \{1, 2, ..., j(x, k), x + 1, x + 2, ..., x + k\}. \]

Observe two things:
°1 For any \(x \in [n]\) and \(r < q\) we have
\[
|\{C(x, tq + r) : 0 \leq t \leq \frac{n}{q}\} \cap F| \leq 1
\]
by the assumptions that \(F\) is a \((p, q)\)-tilted Sperner family with patterns and two such sets for different \(t'\)s are forbidden. Note here that \(C(x, tq + r)\) does not even exist for some \(t\). We also have that for all \(\pi \in \Pi_x\)
\[
|\{\pi(C(x, tq + r)) : 0 \leq t \leq \frac{n}{q}\} \cap F_x| \leq 1.
\]
Indeed, if \(F\) and \(G\) are both in this family, it is easy to calculate that \(p|F \setminus G| = q|G \setminus F|\), and elements of \(F \setminus G\) are smaller than \(x\) while elements of \(G \setminus F\) are larger than \(x\).
°2 For any \(F \in F_x\) there are \(k \leq n - x\) and \(\pi \in \Pi_x\) with
\[
F = \pi(C(x, k)).
\]
Now let us do the following computation: fix \(x \in [n]\). Using °1 we have the following
\[
\sum_{\pi \in \Pi_x} \sum_{r=0}^{q-1} \sum_{t=0}^{\left\lfloor \frac{n}{q} \right\rfloor} |\pi(C(x, tq + r)) \cap F_x| \leq q(n - x)!x!.
\]
After changing the order summations using °2 we get
\[
\sum_{F \in F_x} |F \cap [x]|!(x - |F \cap [x]|)!(|F \setminus [x]|)!(n - x - |F \setminus [x]|)! \leq q(n - x)!x!,
\]
and finally, dividing both sides by \((n - x)!x!\) we have
\[
\sum_{F \in F_x} \frac{1}{\binom{x}{|F \cap [x]|} \binom{n-x}{|F \setminus [x]|}} \leq q. \tag{3}
\]
Using the fact that \(\binom{n}{t} \leq 2^t / \sqrt{t}\), from (3) we have that for all \(x \in [n]\):
\[
|F_x| \leq O\left(\frac{2^n}{\sqrt{x(n-x)}}\right). \tag{4}
\]

2.3 Finishing the proof of Theorem 1.5
We finish the proof of Theorem 1.5 by a standard application of the Chernoff-Hoeffding bound ([1], [4]):
**Chernoff-Hoeffding bound:** Let $X_i$ be independent random variables in the $[0, 1]$ interval and let 

$$X(n) := \sum_{i=1}^{n} X_i.$$ 

Then for $t \leq \mathbb{E}[X(n)]$ we have 

$$\mathbb{P}(|X(n) - \mathbb{E}[X(n)]| \geq t) \leq 2 \exp\left(-\frac{2t^2}{n}\right).$$

The next lemma is probably well known, however for the sake of completeness we present a proof here. Let 

$$\mathcal{G} := \{G \subseteq [n] : \text{there is } x \in [n] \text{ with } |[x] \cap G| - \frac{x}{2} > \sqrt{n \log n}\}.$$ 

**Lemma 2.3.** We have 

$$|\mathcal{G}| \leq O\left(\frac{2^n}{n}\right).$$

**Proof.** Note that $\mathcal{G} = \bigcup_{x \in [n]} \mathcal{G}_x$, where 

$$\mathcal{G}_x := \{G \in \mathcal{G} : |[x] \cap G| - \frac{x}{2} > \sqrt{n \log n}\}.$$ 

Observe that 

$$|\mathcal{G}_x| \frac{1}{2^n} \leq \left(\sum_{y=0}^{\left\lfloor \frac{x}{2} - \sqrt{n \log n}\right\rfloor} \binom{x}{y} + \sum_{y=\left\lceil \frac{x}{2} + \sqrt{n \log n}\right\rceil}^{x} \binom{x}{y} \right) \frac{1}{2^x} \quad (5)$$

Applying the Chernoff-Hoeffding bound on the right hand side of (5) with $t = \sqrt{n \log n}$ (which is less than $\frac{x}{2}$ for $n \geq 10$) we have 

$$|\mathcal{G}_x| \frac{1}{2^n} \leq 2 \exp\left(-\frac{2n \log n}{x}\right). \quad (6)$$

Using $x \leq n$ on the right hand side of (6), we have 

$$|\mathcal{G}_x| \leq O\left(\frac{2^n}{n^2}\right),$$

which easily implies the statement of the lemma.

Let $\mathcal{F}' := \mathcal{F} \setminus \mathcal{G}$. Using Lemma 2.3 we prove that a $(p, q)$-cut point of any $F \in \mathcal{F}'$ is in a $O(\sqrt{n \log n})$ neighborhood of $\frac{p}{p+q} n$. 

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Lemma 2.4. For $n \geq 2$ and all $F \in \mathcal{F}'$ we have

$$|x_F - \frac{p}{p+q}n| \leq 8 \sqrt{n \log n}.$$  

Proof. By the fact that $F \in \mathcal{F}'$ we have both

$$\left||[x_F] \cap F| - \left|\frac{x_F}{2}\right|\right| \leq \sqrt{n \log n} \quad \text{(7)}$$

and

$$\left||[n] \cap F| - \left|\frac{n}{2}\right|\right| \leq \sqrt{n \log n} \quad \text{(8)}$$

By (7) and (8) we have (loosing at most 1 in putting together two inequalities and using that $1 \leq \sqrt{n \log n}$ for $n \geq 2$.)

$$\left||[n] \setminus [x_F] \cap F| - \left|\frac{n - x_F}{2}\right|\right| \leq 4 \sqrt{n \log n} \quad \text{(9)}$$

However $x_F$ is a $(p, q)$-cut point for $F$, so by (7), (8) and (9) we have

$$\left|(n - x_F - \left|\frac{n - x_F}{2}\right|)\frac{1}{q} - \left|\frac{x_F}{2}\right|\frac{1}{p}\right| \leq 8 \sqrt{n \log n},$$

and we are done with Lemma 2.4.

By (4) and Lemma 2.4 we have

$$|\mathcal{F}'| \leq O\left(\frac{\sqrt{n \log n} \cdot 2^n}{n}\right),$$

and by Lemma 2.3 we are done with the proof of Theorem 1.5.

3 Concluding remarks

We proved in Theorem 1.5 that the cardinality of a $(p, q)$-tilted Sperner family with patterns on $[n]$ is $O(\sqrt{n \log n} \cdot \frac{2^n}{\sqrt{n}})$, however we do not have much better constructions than the ones in [8]. We conjecture that for different $p$ and $q$ the order of a maximal size $(p, q)$-tilted Sperner family with patterns on $[n]$ is $\Theta(\frac{2^n}{\sqrt{n}})$.

For $p = q$ we are not able to give really good constructions, we only know that the $(0, 0)$-tilted Sperner family with patterns on $[n]$ (which we define just with property $(ii)$ in Definition 1.3) is $O(\frac{2^n}{n})$, and we do not know what should be the right order.

It is worth mentioning that the whole topic from a more general viewpoint is investigated in the recent paper [6].
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