The space of all $p$-th roots of a nilpotent complex matrix is path-connected

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December 7, 2018

Abstract

Let $p$ be a positive integer and $A$ be a nilpotent complex matrix. We prove that the set of all $p$-th roots of $A$ is path-connected.

AMS Classification: 15A24; 54D05.

Keywords: Matrices, Nilpotency, Path-connectedness, Jordan normal form.

1 Introduction

Let $U$ be an open subset of the field $\mathbb{C}$ of complex numbers, $f : U \rightarrow \mathbb{C}$ be an analytic function and $n$ be a positive integer. Given a matrix $A \in M_n(\mathbb{C})$, it is natural to ask whether the matrix equation $f(X) = A$, with unknown $X \in M_n(\mathbb{C})$, has at least one solution. By using the fact that $X$ commutes with $f(X)$, and by using the characteristic subspaces of $A$, this problem can be reduced to the one of deciding whether the equation $g(X) = N$ has a solution, where $N$ is a given nilpotent matrix, and $g$ is a given analytic function.

There is a (not very satisfying) answer to that question, and we shall recall it in short notice. Given a nilpotent matrix $A \in M_n(\mathbb{C})$ and a positive integer $k$, we denote by $m_k(A)$ the number of Jordan cells of size $k$ in the Jordan normal form of $A$. The sequence $(m_k(A))_{k \geq 1}$ is called the (Jordan) profile of $A$. It belongs to the additive semigroup $\mathbb{N}^\ast$ of all sequences of non-negative integers

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with finite support and indexed over the positive integers (here, $\mathbb{N}$ denotes the set of all non-negative integers, and $\mathbb{N}^*$ the one of all positive integers). More generally, any element of $\mathbb{N}^{(\mathbb{N}^*)}$ is called a profile. Two nilpotent matrices are similar if and only if they have the same Jordan profile. Throughout the article, profiles will be seen as elements of the abelian group $\mathbb{Z}^{(\mathbb{N}^*)}$ of all sequences of integers with finite support.

Given $k \in \mathbb{N}^*$, we denote by $J_k \in M_k(\mathbb{C})$ the Jordan cell of size $k$ (i.e. the matrix of $M_k(\mathbb{C})$ in which the entry at the $(i, i+1)$-spot equals 1 for all $i \in [1, k-1]$, and all the other entries equal 0), and we denote its profile by $e_k$ (so that $(e_k)_i = 1$ if $i = k$, and $(e_k)_i = 0$ otherwise). We convene that $J_0$ is the 0-by-0 matrix and that $e_0$ is the zero sequence in $\mathbb{N}^{(\mathbb{N}^*)}$. Classically, given positive integers $k$ and $p$, the matrix $J_k^p$ is similar to the direct sum of $k - pa$ copies of $J_{a+1}$ and of $p(a+1) - k$ copies of $J_a$, for every non-negative integer $a$ such that $pa \leq k \leq p(a+1)$ (in particular, this holds when $a$ is the quotient of $k$ modulo $p$). From there, one proves that, given a nilpotent matrix $A \in M_n(\mathbb{C})$, the equation $f(X) = A$ has a solution if and only if the profile of $A$ belongs to the sub-semigroup of $\mathbb{N}^{(\mathbb{N}^*)}$ generated by the profiles of the form $r.e_{a+1} + (p - r).e_a$ – where $a$ is a non-negative integer, $p$ is the finite multiplicity of some zero of $f$, and $r \in [0, p]$ – and the profile $e_1$ if some zero of $f$ has infinite multiplicity (i.e. $f$ is constant on the connected component of that zero). In particular, if $f$ has at least one simple zero then the equation $f(X) = A$ has a solution for every nilpotent matrix $A$.

The above characterization is not very convenient however. In very special cases, one can formulate an equivalent one that can easily be tested: a nilpotent matrix $A$ has a $p$-th root if and only if, for all $k \in \mathbb{N}^*$, the integer $p - m_k(A)$ is less than or equal to the remainder of $\sum_{j=k+1}^{+\infty} m_j(A)$ modulo $p$ provided that this remainder is non-zero (for example, if $p = 2$ this means that $m_k(A) > 0$ whenever $\sum_{j=k+1}^{+\infty} m_j(A)$ is odd). Moreover this result holds not only over the field of complex numbers, but over any skew field. If $f$ has exactly two zeroes, one with multiplicity 2 and one with multiplicity 3 (e.g. if $f : z \mapsto z^3(z-1)^2$), then, given a nilpotent matrix $A \in M_n(\mathbb{C})$, the equation $f(X) = A$ has a solution if and only if there is no pair $(k, l)$ of positive integers for which $m_k(A) = m_k+2l(A) = 0$ and $m_{k+l}(A) = 1$ for all $i \in [1, 2l-1]$.

Here, we will stick to the equation $X^p = A$ for a fixed nilpotent complex
matrix $A$ and a fixed positive integer $p$. When this equation has a solution, we are interested in the topological structure of its solution set $A^{1/p}$, i.e. the set of all $p$-th roots of $A$. Note that all the matrices in $A^{1/p}$ are nilpotent.

A very ambitious goal is to understand the homotopy type of $A^{1/p}$. As a first step towards that goal, we will consider here its path-connectedness. Here is our main theorem:

**Theorem 1.** Let $p$ be a positive integer and $A$ be a nilpotent complex matrix. Then, the set $A^{1/p}$ is path-connected.

The case $p = 1$ is straightforward. In the remainder of this section, we fix an integer $p > 1$ and a nilpotent matrix $A \in M_n(\mathbb{C})$. Given $m \in \mathbb{N}(\mathbb{N}^*)$, we denote by $A^{1/p}_m$ the subset of all $N \in A^{1/p}$ with profile $m$ (of course this subset may be empty). We denote by $P_p(A)$ the set of all profiles $m$ such that $A^{1/p}_m$ is non-empty. Hence, the family $(A^{1/p}_m)_{m \in P_p(A)}$ yields a partition of $A^{1/p}$.

Two profiles $m$ and $m'$ are called $p$-adjacent, and we write $m \sim_p m'$, when there exist non-negative integers $a, k, l$ such that

$$pa \leq k < l \leq p(a + 1)$$

and

$$m - m' = \pm(e_k + e_l - e_{k+1} - e_{l-1})$$

Finally, we denote by $A_p(A)$ the set of all pairs $\{m, m'\}$ of distinct $p$-adjacent elements of $P_p(A)$. Thus, we have defined a non-oriented graph $(P_p(A), A_p(A))$.

The definition of $p$-adjacency is motivated by the following basic result:

**Lemma 2.** Let $a, k, l$ be integers such that $0 \leq pa \leq k < l \leq p(a + 1)$. Then, the matrices $(J_k \oplus J_l)^p$ and $(J_{k+1} \oplus J_{l-1})^p$ are similar.

**Proof.** Denote respectively by $r$ and $s$ the remainders of $k$ and $l - 1$ modulo $p$. By a previous remark, we find that $(J_k \oplus J_l)^p$ is similar to the direct sum of $r + (s + 1)$ copies of $J_{a+1}$ and of $(p - r) + (p - s - 1)$ copies of $J_a$. Likewise, $(J_{k+1} \oplus J_{l-1})^p$ is similar to the direct sum of $(r + 1) + s$ copies of $J_{a+1}$ and of $(p - r - 1) + (p - s)$ copies of $J_a$. The claimed result ensues. \(\square\)

We are now able to state the three steps of our proof of Theorem 1:

**Lemma 3.** Let $m \in P_p(A)$. Then, the space $A^{1/p}_m$ is path-connected.

**Lemma 4.** Let $m, m'$ be adjacent profiles in $P_p(A)$. Then, there exist $N \in A^{1/p}_m$ and $N' \in A^{1/p}_{m'}$ together with a path from $N$ to $N'$ in $A^{1/p}$.

**Lemma 5.** The graph $(P_p(A), A_p(A))$ is connected.

Combining those three results readily yields Theorem 1.
2 Proof of Theorem

Throughout this part, we let $A \in M_n(\mathbb{C})$ be a nilpotent matrix and $p$ be a positive integer.

2.1 Proof of Lemma

Let $m \in P(A)$. Let $X$ and $Y$ belong to $A_1^m$. The matrices $X$ and $Y$ are nilpotent with the same profile, and hence they are similar. Thus we have some $P \in GL_n(\mathbb{C})$ such that $Y = PXP^{-1}$. Since $X^p = Y^p = A$, we obtain that $P$ belongs to the centralizer $C(A)$ of $A$ in the algebra $M_n(\mathbb{C})$. As $C(A) \cap GL_n(\mathbb{C})$ is a Zariski-open subset of the complex finite-dimensional vector space $C(A)$, it is path-connected (see Lemma 7.2 in [4]). Choose a path $Q : t \in (0, 1) \mapsto Q(t) \in C(A) \cap GL_n(\mathbb{C})$ from $I_n$ to $P$. Then, one checks that $q : t \in (0, 1) \mapsto Q(t)XQ(t)^{-1}$ is a path from $X$ to $Y$, and $q(t)^p = Q(t)AQ(t)^{-1} = A$ for all $t \in (0, 1)$. Finally, $q(t)$ is similar to $X$ for all $t \in (0, 1)$, and hence its profile is $m$. Hence, there is a path from $X$ to $Y$ in $A_1^m$. This completes the proof of Lemma.

2.2 Proof of Lemma

As we will see, the proof of Lemma boils down to the following basic result:

Lemma 6. Let $a, k, l$ be integers such that $0 \leq pa \leq k < l \leq p(a+1)$. Set $N := k + l$. Then, there exists a path $\gamma : (0, 1) \mapsto M_N(\mathbb{C})$ such that:

(i) $\gamma(0) = J_k \oplus J_l$;
(ii) $\gamma(1)$ is similar to $J_{k+1} \oplus J_{l-1}$;
(iii) the mapping $t \in (0, 1) \mapsto \gamma(t)^p$ is constant.

Proof. We shall think in terms of endomorphisms of $\mathbb{C}^N$: denote by $u$ the endomorphism of $\mathbb{C}^N$ represented by $J_k \oplus J_l$ in the standard basis $(x_k, \ldots, x_1, y_l, \ldots, y_1)$ of $\mathbb{C}^N$. We convene that $y_j = 0$ for all $j > l$, and that $x_i = 0$ for all $i > k$. Hence, $u$ maps $x_i$ to $x_{i+1}$ for all $i > 0$, and it maps $y_j$ to $y_{j+1}$ for all $j > 0$. Given $t \in (0, 1)$, define $u_t$ as the endomorphism of $\mathbb{C}^N$ on the standard basis by $u_t(y_1) = (1-t)y_2 + tx_1$, and by mapping any other vector $z$ of that basis to $u_z$. Clearly, $t \in (0, 1) \mapsto u_t$ is a path in the space of all endomorphisms of $\mathbb{C}^N$, and $u_0 = u$.  

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Next, one sees that $u_1$ is represented by the matrix $J_{k+1} \oplus J_{l-1}$ in the basis $(x_k, \ldots, x_1, y_1, y_1, \ldots, y_2)$.

Next, let $t \in [0, 1)$. One checks that $(x_k, \ldots, x_1, (1-t)y_1 + tx_{l-1}, \ldots, (1-t)y_2 + tx_1, y_1)$ is a basis of $\mathbb{C}^N$, and the matrix of $u_t$ in that basis is $J_k \oplus J_l$. Hence, $u_t$ is similar to $u_0$, and it follows that $u_t^p$ is similar to $u_0^p$. Besides, Lemma 2 shows that $u_t^p$ is also similar to $u_0^p$.

Now, for $t \in (0, 1)$, denote by $U_t$ the matrix of $u_t$ in the standard basis of $\mathbb{C}^N$. It follows from the above that $t \in (0, 1) \mapsto U_t$ is a path, in the space $M_N(\mathbb{C})$, from $J_k \oplus J_l$ to a matrix that is similar to $J_{k+1} \oplus J_{l-1}$, and that the path $t \in (0, 1) \mapsto (U_t)^p$ takes its values in the similarity class $S(U_0^p)$ of the matrix $U_0^p$.

It is folklore that the mapping $P \in GL_N(\mathbb{C}) \mapsto PU_0^pP^{-1} \in S(U_0^p)$ is a fibration (it is a principal fibre bundle whose structural group is the group of all invertible elements of the centralizer of $U_0^p$): see the appendix for a short elementary proof, and the combination of Propositions 1.4.3 and 1.4.6 of [1] and Proposition 8.3 of [2] for a more sophisticated one. Hence, there is a path $q : (0, 1) \to GL_N(\mathbb{C})$ such that

$$\forall t \in (0, 1), \ U_t^p = q(t)U_0^p q(t)^{-1} \quad \text{and} \quad q(t) = I_N.$$  

Finally, we consider the path $\gamma : t \in (0, 1) \mapsto q(t)^{-1}U_t q(t) \in M_N(\mathbb{C})$. The above properties of $q$ show that $t \mapsto \gamma(t)^p$ is constant. Next, $\gamma(0) = U_0 = J_k \oplus J_l$. Finally, $\gamma(1)$ is similar to $U_1$ and hence to $J_{k+1} \oplus J_{l-1}$. □

Now, we can prove Lemma 4. Let $m, m'$ be distinct adjacent profiles in $P_p(A)$. We wish to prove that some element of $A_{m}^{1/p}$ is path-connected to some element of $A_{m'}^{1/p}$. Without loss of generality, we can assume that there is a non-negative integer $a$ together with elements $k < l$ of $[pa, p(a+1)]$ such that $m - m' = e_k + e_l - e_{k+1} - e_{l-1}$. If $m \neq m'$, then we must have $l > k + 1$, and it follows that $m_k > 0$ and $m_l > 0$. Consequently, $N$ has at least one Jordan cell of each size $k$ and $l$. Hence, $N = P(B \oplus J_k \oplus J_l)P^{-1}$ for some nilpotent matrix $B$ and some $P \in GL_n(\mathbb{C})$. The profile of $B$ is obviously $m - e_k - e_l$.

Let us take a path $\gamma$ that satisfies the conclusion of Lemma 6 for the pair $(k, l)$: then, $q : t \in (0, 1) \mapsto P(B \oplus \gamma(t))P^{-1}$ is a path in $M_n(\mathbb{C})$, and we see from condition (iii) in Lemma 6 that $t \mapsto q(t)^p$ is constant with value $q(0)^p = N^p = A$. In other words, $q$ is a path in $A^{1/p}$. Finally, $q(1)$ is similar to $B \oplus \gamma(1)$, and hence to $B \oplus J_{k+1} \oplus J_{l-1}$, whose profile equals $(m - e_k - e_l) + e_{k+1} + e_{l-1} = m'$. Hence, $q(1) \in A_{m'}^{1/p}$. This completes the proof of Lemma 4.

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2.3 Proof of Lemma \[5\]

We start with some preliminary notation. Given an element \( \mathbf{m} \in \mathbb{Z}^{(N^*)} \), we set

\[
S(\mathbf{m}) := \sum_{k=1}^{+\infty} km_k
\]

(called the size of \( \mathbf{m} \)), and

\[
\mathbf{m}[p] := \left( \sum_{-p<k<p} (p - |k|) m_{pa+k} \right)_{a \geq 1},
\]

which is an element of \( \mathbb{Z}^{(N^*)} \). Note that both maps \( S : \mathbb{Z}^{(N^*)} \to \mathbb{Z} \) and \( \mathbf{m} \in \mathbb{Z}^{(N^*)} \mapsto \mathbf{m}[p] \in \mathbb{Z}^{(N^*)} \) are group homomorphisms.

Using the results recalled in the introduction, one sees that if \( \mathbf{m} \) is the profile of some nilpotent matrix \( N \), then \( \mathbf{m}[p] \) is the profile of \( N^p \), while \( S(\mathbf{m}) = \) obviously the number of rows of \( N \), and hence \( S(\mathbf{m}[p]) = S(\mathbf{m}) \). Besides, using Lemma \[2\], we find that \( \mathbf{m}[p] = (\mathbf{m}')[p] \) for any two \( p \)-adjacent profiles \( \mathbf{m} \) and \( \mathbf{m}' \).

Given profiles \( \mathbf{m} \) and \( \mathbf{m}' \), a \( p \)-chain of profiles from \( \mathbf{m} \) to \( \mathbf{m}' \) is a list \((a(0), \ldots, a(N))\) of profiles such that \( a(i) \sim_p a(i+1) \) for all \( i \in [0, N-1] \), and \( \mathbf{m} = a(0) \) and \( \mathbf{m}' = a(N) \).

From there, Lemma \[5\] can be seen as a reformulation of the following result:

**Lemma 7.** Let \( \mathbf{m}, \mathbf{m}' \) be two profiles such that \( \mathbf{m}[p] = \mathbf{m}'[p] \). Then, there is a \( p \)-chain of profiles from \( \mathbf{m} \) to \( \mathbf{m}' \).

**Proof.** Note that the assumptions yield \( S(\mathbf{m}) = S(\mathbf{m}[p]) = S((\mathbf{m'})[p]) = S(\mathbf{m}') \). We will prove the result by induction on the size of \( \mathbf{m} \).

The result is obvious if \( S(\mathbf{m}) = 0 \): in that case both \( \mathbf{m} \) and \( \mathbf{m}' \) equal the zero sequence, and we simply take the trivial chain \((\mathbf{m})\). Assume now that \( S(\mathbf{m}) > 0 \).

Assume first that there exists an integer \( k \geq 1 \) such that \( m_k > 0 \) and \( m'_k > 0 \). Then, \( \mathbf{m} - e_k \) and \( \mathbf{m}' - e_k \) obviously satisfy the assumptions, and their size equals \( S(\mathbf{m}) - k \). By induction, there is a \( p \)-chain \((a(0), \ldots, a(N))\) of profiles from \( \mathbf{m} - e_k \) to \( \mathbf{m}' - e_k \). Clearly, \((a(0) + e_k, \ldots, a(N) + e_k)\) is a \( p \)-chain of profiles from \( \mathbf{m} \) to \( \mathbf{m}' \).

Hence, in the remainder of the proof we assume that \( m_k m'_k = 0 \) for all \( k \geq 1 \).

Denote by \( q \) the greatest positive integer such that \( m_q + m'_q > 0 \). Without loss of generality, we can assume that \( m_q > 0 \) (and hence \( m_q = 0 \)). Denote by \( a \) the least (non-negative) integer such that \( q \in [pa, p(a+1)] \), so that \( q > pa \). Hence,
\[ m_{a+1}^{[p]} = (m')_{a+1}^{[p]} \geq q - pa. \] In particular, \( m_k > 0 \) for some \( k \in \lceil pa + 1, p(a + 1) \rceil \), and we consider the greatest such integer \( k \). Note that \( pa < k < q \). If \( m_k = 1 \), then having \( m_{a+1}^{[p]} \geq q - pa \) we must also have \( m_l > 0 \) for some \( l \in [pa + 1, k - 1] \), and then we note that \( m - e_k - e_l + e_{k+1} + e_{l-1} \) is a profile. In any case, we have found a profile \( a^{(k+1)} \) that is \( p \)-adjacent to \( m \) and for which \( k + 1 \) is the greatest integer \( i \) such that \( a^{(k+1)}_i > 0 \). Continuing by finite induction, we create a \( p \)-chain \((a^{(k)}, a^{(k+1)}, \ldots, a^{(q)})\) of profiles from \( m \) to some profile \( a^{(q)} \) such that \( (a^{(q)})_q > 0 \). Hence \( (a^{(q)})^{[p]} = \cdots = (a^{(k)})^{[p]} = m^{[p]} = (m')^{[p]} \). As \( (a^{(q)})_q > 0 \), the first case tackled in the above yields a \( p \)-chain of profiles from \( a^{(q)} \) to \( m' \). Linking those \( p \)-chains yields a \( p \)-chain of profiles from \( m \) to \( m' \).

Lemmas 3 to 5 are now proved, and hence Theorem 1 is established.

## 3 Further questions

Now that Theorem 1 has been proved, we wish to suggest several related open problems. First, given an analytic function \( f : U \to \mathbb{C} \), what are the nilpotent complex matrices \( A \) for which the set of all solutions of the equation \( f(X) = A \) is path-connected? More precisely, is there a simply characterization of such matrices in terms of the profile of \( A \) and the zeroes of \( f \) (and their multiplicities)?

Next, given a positive integer \( p \), we wonder about the homotopy type of \( A^{1/p} \). For example, if \( A = 0 \) then \( A^{1/p} \) is contractible (since it is star-shaped around 0). However, for \( E := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \), one checks that \( E^{1/2} \) is the set of all matrices of the form \( \begin{bmatrix} 0 & x & y \\ 0 & 0 & x^{-1} \\ 0 & 0 & 0 \end{bmatrix} \), a space that is homeomorphic to \( (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \) and hence homotopy equivalent to the circle \( S^1 \) (and not contractible!). Is there a simple way to compute the homotopy type of \( A^{1/p} \) as a function of \( p \) and the profile of \( A \)? Computing the fundamental group of \( A^{1/p} \) would be interesting, for a start.

There are other interesting open questions related to the real and quaternionic cases. The set of all square roots of \( E \) with real entries is homeomorphic to \((\mathbb{R} \setminus \{0\}) \times \mathbb{R} \), and hence it has exactly two path-connected components. Is there a sensible way to compute the number of path-connected components of
the set of all \( p \)-th roots of \( A \) (with real entries) as a function of \( p \) and of the profile of \( A \)? In that prospect, it is worthwhile to note that the real equivalent of Lemmas 4 and 5 holds (with the same proof): the only step that fails is the real equivalent of Lemma 3. Nevertheless, the set of all real \( p \)-th roots of \( A \) is an affine algebraic variety, and hence it has finitely many path-connected components (alternatively, one can adapt the proof of Lemma 3 to yield that \( A_{m/p} \) has finitely many path connected components, being a Zariski open subset of a finite-dimensional real vector space, see [3]). Finally, there are similar issues in the quaternionic case: in that one however we have not succeeded in finding an example of a nilpotent quaternionic matrix \( A \) and of a positive integer \( p \) such that the set of all \( p \)-th roots of \( A \) is not path-connected.

Appendix: the fibration \( P \mapsto P A P^{-1} \)

Here, \( \mathbb{F} \) denotes one of the fields \( \mathbb{R} \) or \( \mathbb{C} \). Let \( A \in M_n(\mathbb{F}) \). Denote by \( C(A) \) the centralizer of \( A \) in the algebra \( M_n(\mathbb{F}) \), by \( C(A)^{\times} \) its group of invertible elements, and by \( S(A) \) the similarity class of \( A \). We wish to prove that the mapping \( \pi : P \in GL_n(\mathbb{F}) \mapsto P A P^{-1} \in S(A) \) defines a \( C(A)^{\times} \)-principal bundle. For the continuous left-action \((P, M) \mapsto P M P^{-1} \) of \( GL_n(\mathbb{F}) \) on \( M_n(\mathbb{F}) \), the stabilizer of \( A \) is \( C(A)^{\times} \), and hence classically it suffices to prove that the mapping \( \pi \) admits a local cross-section around \( A \).

The proof is based upon the following elementary lemma:

**Lemma 8.** Let \( V \) be a finite-dimensional vector space over \( \mathbb{F} \). Let \( u \in \text{End}(V) \), and let \( x_0 \in V \) be a non-zero vector such that \( u(x_0) = 0 \). Then, there exists a neighborhood \( U \) of \( u \) in \( \text{End}(V) \), together with a continuous mapping \( f : U \to V \) such that \( v[f(v)] = 0 \) for all \( v \in U \) with the same rank as \( u \), and \( f(u) = x_0 \).

**Proof.** Denote by \( n \) the dimension of \( V \), and by \( p \) the rank of \( u \). Let us extend \( x_0 \) first into a basis \((e_{n-p}, \ldots, e_n) \) of the kernel of \( u \), with \( e_n = x_0 \), and then into a basis \( B := (e_1, \ldots, e_n) \) of \( V \). We extend the linearly independent \( p \)-tuple \((u(e_1), \ldots, u(e_p)) \) into a basis \( C := (u(e_1), \ldots, u(e_p), f_{p+1}, \ldots, f_n) \) of \( V \). In the bases \( B \) and \( C \), the matrix of \( u \) reads

\[
\begin{bmatrix}
I_p & 0_{p \times (n-p)} \\
0_{(n-p) \times p} & 0_{(n-p) \times (n-p)}
\end{bmatrix}.
\]
For any \( v \in \text{End}(V) \), let us write its matrix in the bases \( B \) and \( C \) as

\[
M(v) = \begin{bmatrix}
A(v) & C(v) \\
B(v) & D(v)
\end{bmatrix}
\]

along the same pattern. The mapping \( v \in \text{End}(V) \mapsto A(v) \in M_p(F) \) is linear, and hence continuous. It follows that

\[
U := \{ v \in \text{End}(V) : A(v) \in \text{GL}_p(F) \}
\]

is an open subset of \( \text{End}(V) \) that contains \( u \).

Next, let \( v \in U \). Consider the invertible matrix

\[
N(v) := \begin{bmatrix}
I_p & -A(v)^{-1}C(v) \\
0_{(n-p)\times p} & I_{n-p}
\end{bmatrix} \in \text{GL}_n(F),
\]

so that \( M(v)N(v) = \begin{bmatrix}
A(v) & 0_{p\times (n-p)} \\
B(v) & ?
\end{bmatrix} \) has the same rank as \( N(v) \). Assume that \( v \) has rank \( p \). Since \( A(v) \) has rank \( p \), it follows that the last \( n-p \) columns of \( M(v)N(v) \) equal zero, and in particular \( M(v) \) annihilates the last column of \( N(v) \).

For \( v \in U \), denote by \( f(v) \) the vector of \( V \) whose matrix in \( B \) is the last column of \( N(v) \); obviously \( f : U \rightarrow V \) is continuous, and the previous study shows that \( v[f(v)] = 0 \) for all \( v \in U \) with rank \( p \). Finally, \( f(u) = e_n = x_0 \).

\textbf{Remark 1.} Set \( p := \text{rk } u \) and define \( \text{End}_p(V) \) as the set of all endomorphisms of \( V \) with rank \( p \). Then

\[
\begin{cases}
\{(v,x) \in \text{End}_p(V) \times V : v(x) = 0\} & \rightarrow \text{End}_p(V) \\
(w,y) & \rightarrow w
\end{cases}
\]

is known to be a vector bundle, and the above result can be obtained by using a local trivialization of it.

We are now ready to construct the claimed local cross-section. Consider the endomorphism \( \text{ad}_A : M \mapsto AM - MA \) of the vector space \( M_n(F) \). Denote by \( p \) its rank. Applying the above lemma, we find a neighborhood \( U \) of \( \text{ad}_A \) in \( \text{End}(M_n(F)) \) together with a continuous mapping \( f : U \rightarrow M_n(F) \) such that \( f(\text{ad}_A) = I_n \) and \( v(f(v)) = 0 \) for all \( v \in \text{End}(M_n(F)) \) with rank \( p \). The mapping

\[
\Phi : B \in M_n(F) \mapsto [M \mapsto BM - MA] \in \text{End}(M_n(F))
\]

is known as the local cross-section.
is affine, and hence continuous: thus $U_0 := \Phi^{-1}(U)$ is a neighborhood of $A$ in $M_n(\mathbb{F})$. We set

$$g : B \in U_0 \cap S(A) \mapsto f(\Phi(B)) \in M_n(\mathbb{F}),$$

so that $g(A) = I_n$. Since $g$ is continuous, $U'_0 := g^{-1}(\text{GL}_n(\mathbb{F}))$ is a neighborhood of $A$ in $S(A)$.

We will conclude the proof by showing that the restriction $g|U'_0$ is a local cross-section for the mapping $P \in \text{GL}_n(\mathbb{F}) \mapsto PA^{-1} \in S(A)$.

Let $B \in U'_0$. Since $B \in S(A)$, there is a matrix $Q \in \text{GL}_n(\mathbb{F})$ such that $B = QAQ^{-1}$. It follows that $\Phi(B) = L_Q \circ \text{ad}_A \circ L_Q^{-1}$ where $L_N : M \mapsto NM$ for all $N \in M_n(\mathbb{F})$. Hence, $\text{rk } \Phi(B) = \text{rk } \text{ad}_A = p$. It follows that $\Phi(B)[g(B)] = 0$, that is $Bg(B) = g(B)A$. Moreover, $g(B)$ is invertible, and hence $B = g(B)Ag(B)^{-1}$, as claimed.

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