Probabilistic Recovery of Multiple Subspaces in Point Clouds by Geometric $l_p$ Minimization

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Abstract: We assume data independently sampled from a mixture distribution on the unit ball of $\mathbb{R}^D$ with $K+1$ components: the first component is a uniform distribution on that ball representing outliers and the other $K$ components are uniform distributions along $K$ $d$-dimensional linear subspaces restricted to that ball. We study both the simultaneous recovery of all $K$ underlying subspaces and the recovery of the best $l_0$ subspace (i.e., with largest number of points) by minimizing the $l_p$-averaged distances of data points from $d$-dimensional subspaces of $\mathbb{R}^D$. Unlike other $l_p$ minimization problems, this minimization is non-convex for all $p > 0$ and thus requires different methods for its analysis. We show that if $0 < p \leq 1$, then both all underlying subspaces and the best $l_0$ subspace can be precisely recovered by $l_p$ minimization with overwhelming probability. This result extends to additive homoscedastic uniform noise around the subspaces (i.e., uniform distribution in a strip around each of them) and near recovery with an error proportional to the noise level. On the other hand, if $K > 1$ and $p > 1$, then we show that both all underlying subspaces and the best $l_0$ subspace cannot be recovered and even nearly recovered. The uniform distributions of both subspaces and outliers can be easily replaced by spherically symmetric distributions, e.g., Gaussian distributions. Similarly, the uniform noise can be easily relaxed by assuming a distribution from a parametric family with the expectation of the absolute value approaching zero when its parameter approaches zero. The results of this paper partially justify recent effective algorithms for modeling data by mixtures of multiple subspaces as well as for discussing the effect of using variants of $l_p$ minimizations in RANSAC-type strategies for single subspace recovery.

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1. Introduction

The most common tool in high-dimensional data analysis has been Principal Component Analysis (PCA), which approximates a given data set by a low-dimensional affine subspace. More recent works extend PCA to approximation by several subspaces. However, many popular methods for such modeling problems are not robust to outliers. Furthermore, methods whose robustness has been numerically demonstrated for special cases, often lack theoretical guarantees. In particular, robustness to outliers of any strategy for recovering multiple subspaces has not yet been proved beyond some experimental evidence.

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In practice, some of the most successful methods for recovery of subspaces use the \( l_1 \) distance. In the context of multiple \( K \) subspaces, the \( l_p \) distance of a data set from the mixture of those subspaces is gotten by first computing for a given point the minimal Euclidean distances of data points to all \( K \) subspaces and then taking the \( l_p \)-averaged sum over all points. Thus, one can try to recover \( K d \)-dimensional subspaces by minimizing the \( l_1 \) distance of the underlying data over all possible \( K d \)-dimensional subspaces (if \( K = 1 \), then we recover this way a single subspace). The robustness of the \( l_1 \) norm has been rigorously quantified in various important settings, such as \( l_1 \) regression and principal components pursuit. However, we are not aware of rigorous justification of the \( l_1 \) subspace recovery even when \( K = 1 \). Indeed, a crucial distinction of the \( l_1 \) subspaces recovery from other \( l_1 \) recovery problems is that it involves a non-convex optimization and thus requires very different methods for its analysis.

The purpose of this paper is to explore the effectiveness of subspace recovery by \( l_p \) minimization for all \( p > 0 \) under the assumptions of uniform outliers and “uniform sampling” along the underlying subspaces (or a strip around them). We typically refer to this situation of uniform outliers as a point cloud. In the case of clean subspaces (i.e., without additive noise), we address two different questions. One question is whether it is possible to simultaneously recover all underlying subspaces via \( l_p \) minimization for some values of \( p > 0 \). The other question is whether the best \( l_0 \) subspace, that is, the subspace containing the largest number of points, can be recovered via \( l_p \) minimization for some \( p > 0 \). While this question is about a single subspace recovery, the underlying model still assumes multiple subspaces and consequently the \( l_p \) optimization is not so trivial to analyze. If one can positively answer this question for a more general setting, then one can prove that iterative repetition of \( l_p \) recovery of the best \( l_0 \) subspace \( K \) times results in recovery of all underlying \( K \) subspaces.

We extend the solutions of these questions to the case of additive noise around the underlying subspaces, while allowing the recovery error to be controlled by the noise level. We also suggest relaxations of the uniform model described here.

\subsection{Background and Related Work}

The \( l_1 \) norm has been widely used to form robust statistics. For example, the geometric median is the point in a data set minimizing the sum of distances from the rest of data points, i.e., the \( l_1 \)-averaged distance. For points on the real axis, it coincides with the usual median. Its robustness is most commonly quantified by showing that it has a breakdown point of 0.5 (i.e., the estimator will obtain arbitrarily large values only when the proportion of large observations is at least a half) \cite{32}. The \( l_1 \) norm has also been successfully applied to robust regression \cite{24, 22, 38, 35}.

Basis pursuit\cite{11} uses \( l_1 \) minimization to search for the sparsest solutions (i.e., solutions minimizing the \( l_0 \) norm) of an undercomplete system of linear equations. It is used for decomposing a signal as a linear combination of few representative elements from a large and redundant dictionary of functions. In this application one often preprocesses the data by normalizing the columns of the underlying matrix by their \( l_2 \) norm. Donoho and Elad\cite{17} have shown that “sufficiently sparse” solutions can be completely recovered by minimizing the \( l_1 \) norm instead of the \( l_0 \) norm. However, this result restricts
the size of the mutual incoherence $M$ of the dictionary and consequently the size of the sparse solution (which is inversely controlled by $M$). Other works [7, 16, 15, 8] show that for the overwhelming majority of matrices representing undercomplete systems, the minimal $l_1$ solution of each system coincides with the sparsest one as long as the solution is sufficiently sparse. Moreover, this fact holds even when noise is added to the decomposed signal (with a slight modification of the problem formulation).

Candès et al. [6] proposed and analyzed the principal component pursuit algorithm for robust PCA, which minimizes a weighted combination of the nuclear norm and a different $l_1$ norm (allowing convex optimization) among all decompositions matching the available data. A simpler use of the $l_1$ norm between given data points and representative points in a lower dimensional model (though without using the nuclear norm term to infer this model) has appeared in several other works [3, 20, 30, 28]. Alternatively, Ding et al. [14] as well as Brooks and Dulá [5] used the geometric and non-convex $l_1$ recovery of a single subspace defined earlier (i.e., minimizing the $l_1$ sum of Euclidean distances of data points from all possible subspaces). However, we are not aware of any quantitative study of the robustness of this best $l_1$ subspace to outliers in the setting of both multiple underlying subspaces and a point cloud (whose data points are not necessarily far away from the subspaces).

The latter $l_1$ (or $l_p$) subspace minimization can also be applied to Hybrid Linear Modeling (HLM), i.e., the modeling of data by mixtures of affine subspaces. This kind of modeling finds diverse applications in many areas, such as motion segmentation in computer vision, hybrid linear representation of images, classification of face images and temporal segmentation of video sequences (see e.g., [46, 34]). There are already many algorithms for HLM [25, 12, 42, 41, 4, 45, 26, 27, 23, 46, 48, 49, 34, 33, 10, 1, 50, 51]. Among these, the ones suggesting robust strategies to deal with many outliers are RANSAC (for HLM) [49], Robust GPCA [34], SCC [10], Sparse ALC [37], MKF [50] (or any $l_p$ variant of $K$-subspaces [25, 4, 45, 23] when $0 < p \leq 1$) and LBF [51]. Both MKF and LBF apply (in different ways) the $l_1$ subspace minimization discussed in this paper, whereas RANSAC (for HLM) can be successfully modified utilizing such $l_1$ minimization (in the spirit of [43, 44] who use other norms). Sparse ALC also applies an $l_1$ minimization, which is different than the one discussed here (in particular, it involves convex optimization).

Despite the many HLM algorithms and strategies for robustness to outliers, there has been little investigations into performance guarantees of such algorithms. Accuracy of segmentation of HLM algorithms under some sampling assumptions is only analyzed in [9] and [2], whereas tolerance to outliers of an HLM algorithm under some sampling assumptions is only analyzed in [2] (in fact, [2] analyzes the more general problem of modeling data by multiple manifolds, though it assumes an asymptotically zero noise level, unlike [9]).

1.2. Contribution of This Paper

This paper studies the effectiveness of recovering subspaces in point clouds by $l_p$ subspace minimization for different values of $0 < p < \infty$. In particular, we study the recovery of the best $l_0$ subspace by the best $l_p$ subspace, that is, feasibility of using
the best $l_p$ subspace as the best $l_0$ subspace. We also study full recovery of all $K$ underlying subspaces by the collection of $K$ subspaces minimizing an $l_p$ energy for multiple subspaces. We restrict the discussion to linear subspaces, which we refer to as $d$-subspaces.

We assume an underlying data set $X \subseteq \mathbb{R}^D$ of $N$ points independently sampled from the mixture measure defined as follows (while distinguishing between two cases according to the presence of noise).

**Definition 1.1.** We say that a probability measure $\mu$ on the unit ball $B(0, 1)$ of $\mathbb{R}^D$ is a uniform mixture measure if $\mu = \sum_{i=0}^{K} \alpha_i \mu_i$, where $\{\alpha_i\}_{i=0}^{K}$ are nonnegative numbers summing to 1, $\mu_0$ is the uniform probability measure (i.e., scaled Lebesgue) in the unit ball and $\{\mu_i\}_{i=1}^{K}$ are uniform probability measures along the restriction to the unit ball of distinct $d$-subspaces of $\mathbb{R}^D$, $\{L_i\}_{i=1}^{K}$. For $\epsilon > 0$, we say that $\mu_{\epsilon}$ is a uniform mixture measure with noise level $\epsilon$ if $\mu_{\epsilon} = \alpha_0 \mu_0 + \sum_{i=1}^{K} \alpha_i \mu_{i,\epsilon}$, where $\{\alpha_i\}_{i=0}^{K}$ and $\mu_0$ are the same as above and $\{\mu_{i,\epsilon}\}_{i=1}^{K}$ are uniform on the cylinders $(L_i \cap B(0, 1)) \times (L_i^\perp \cap B(0, \epsilon))$, $i = 1 \ldots K$, around the $d$-subspaces $\{L_i\}_{i=1}^{K}$.

To simplify this introduction we discuss here the clean case with underlying uniform mixture measure $\mu$. We first explain the $l_p$ recovery of the best $l_0$ subspace, i.e., the subspace containing the largest number of points of $X$ (so that its complement minimizes the $l_0$ distance). When addressing this problem, we will always assume the following condition (using notation of Definition 1.1):

$$\alpha_1 > \sum_{i=2}^{K} \alpha_i. \quad (1)$$

This condition implies that $L_1$ is the best $l_0$ subspace for $X$ with overwhelming probability. By saying “with overwhelming probability”, or in short “w.o.p.”, we mean that the underlying probability is at least $1 - Ce^{-N/C}$, where $C$ is a constant independent of $N$, but possibly depending on other parameters of the underlying uniform mixture measure.

The $l_p$ recovery of the best $l_0$ subspace for $X$ minimizes the quantity

$$e_{l_p}(X, L) = \sum_{x \in X} \operatorname{dist}(x, L)^p, \quad (2)$$

where $\operatorname{dist}(x, L)$ denotes the Euclidean distance between a data point $x$ and the subspace $L$. This is not a convex optimization, since it takes place on the Grassmannian. We refer to the minimizer of (2) as the best $l_p$ $d$-subspace.

Our main result for exact $l_p$ recovery w.o.p. of the best $l_0$ subspace from multiple clean subspaces in point clouds is formulated in the following theorem. We remark that in this theorem as well as throughout the rest of the paper the distance between two subspaces is the geodesic one, which is specified later in (8).

**Theorem 1.1.** If $\mu$ is a uniform mixture measure on $\mathbb{R}^D$ with $K$ $d$-subspaces $\{L_i\}_{i=1}^{K} \subseteq G(D, d)$ and mixture coefficients $\{\alpha_i\}_{i=0}^{K}$ satisfying (1), $X$ is a data set of $N$ points independently sampled from $\mu$ and $0 < p \leq 1$, then the probability that $L_1$ is a best $l_p$
subspace is at least $1 - C \exp(-N/C)$, where $C$ is a constant depending on $D$, $d$, $K$, $p$, $\alpha_0$, $\alpha_1$ and $\min_{2 \leq i \leq K} (\text{dist}(L_1, L_i))$.

Next, we address the second problem of simultaneous recovery of all $K$ subspaces via $l_p$ minimization of the following energy, defined for the data set $X$ and any subspaces $L_1, \ldots, L_K$:

$$e_{l_p}(X, L_1, \ldots, L_K) = \sum_{x \in X} \min_{1 \leq i \leq K} (\text{dist}(x, L_i))^p. \quad (3)$$

The following theorem states that when $0 < p \leq 1$ the minimization of this energy exactly recovers w.o.p. the underlying clean $K$ subspaces within a point cloud.

**Theorem 1.2.** If $\mu$ is a uniform mixture measure on $\mathbb{R}^D$ with $K$ $d$-subspaces $\{L_i\}_{i=1}^K \subseteq G(D, d)$ and mixture coefficients $\{\alpha_i\}_{i=0}^K$, $X$ is a data set independently sampled from $\mu$ and $0 < p \leq 1$, then there exists a positive constant $\nu_0 = \nu_0(d, K, p)$, such that whenever

$$\alpha_0 < \frac{\nu_0}{2} \cdot \min_{i=1,\ldots,K} \alpha_i \cdot \min_{1 \leq i,j \leq K} \text{dist}(L_i, L_j), \quad (4)$$

then the set $\{L_1, L_2, \ldots, L_K\}$ minimizes the energy (3) among all $d$-subspaces in $\mathbb{R}^D$ with overwhelming probability.

For the noisy setting, we assume a uniform mixture measure with noise level $\epsilon$ and show later in Section 5 that the above two $l_p$ minimization procedures with $0 < p \leq 1$ nearly recover w.o.p. (up to an error of order $\epsilon$) the $l_0$ subspace and the $K$ underlying subspaces. In fact, we also extend there these results to $p > 1$ and $K = 1$. That is, we will show that a single underlying subspace in a point cloud can be nearly recovered (with error proportional to the noise level) by $l_p$ minimization for any $p > 0$. On the other hand, we later establish in Section 6 a phase transition phenomenon for multiple underlying subspaces. That is, if $K > 1$ and $p > 1$, then the $l_p$ recovery as well as near-recovery of the best $l_0$ subspace will not work well. We will also provide there some indication why we expect a similar negative result for $l_p$ recovery of all $K$ underlying subspaces, when $p > 1$ and $K > 1$. The uniformity assumption (along subspaces, of outliers and of noise) will be relaxed in Section 8.

The theory developed here is a quantitative study of robustness of $l_p$ subspace approximations in point clouds. We are not aware of other informative quantifications of robustness. Indeed, the notion of a breakdown point of robust statistics [24, 22, 38, 35] does not directly apply to best $l_p$ subspaces, since they are contained in a compact space, i.e., the Grassmannian, and thus the discussion of arbitrarily far element is irrelevant. On the other hand, measuring the influence function, which is also common in robust statistics [24, 22, 38, 35] is not informative for our probabilistic model (as opposed to sufficiently far outliers).

This quantitative study of robustness has direct implications for both single subspace modeling and hybrid linear modeling in point clouds. We will use Theorem 1.1, its extension to noise (Theorem 5.1) and the breakdown of both theorems when $p > 1$ (Theorem 6.1) in order to analyze the effectiveness of $l_p$-based loss functions in a RANSAC framework (in the spirit of [43, 44]). We will also use Theorem 1.2 and its extension to noise (Theorem 5.2) to partially justify two different robust algorithms for HLM [50, 51].
1.3. More Results and Structure of the Paper

Additional theory is developed throughout the paper in the following order. In Section 2 we describe basic notation and review frequently used concepts. In Section 3 we specify general algebraic conditions for a best $l_0$ subspace to be a local minimum of the energy (2) for various $0 < p < \infty$. We also demonstrate natural instances, distinct from point clouds, where the best $l_0$ subspace is neither a local $l_p$ subspace (even for $p = 1$) nor global one (even for $0 < p < 1$). Section 4 involves data sampled from a mixture composed of a uniform distribution along a single $d$-subspace and a uniform background of outliers. It studies when the best $l_0$ subspace for such data is either a local or global minimum of the energy (2) for $0 < p \leq 1$ (for example, if one samples $N_0$ outliers and $N_1$ inliers and if both $N_0 = o(N_1^2)$ and $p = 1$ or both $N_0 = \Omega(1)$ and $0 < p < 1$, then the best $l_0$ subspace is a local $l_1$ minimum). Theorems 1.1 and 1.2 above extended part of this study for data sampled from several $d$-subspaces with an outlier component. Section 5 extends the latter two theorems to near-recovery in noisy setting, whereas Section 6 discusses failures of $l_p$ recovery or near-recovery when $p > 1$ and $K > 1$. Section 7 uses some of the theory developed here to partially justify two effective algorithms for robust HLM as well as an approach for single subspace recovery. Section 8 discusses some immediate extensions of the results of this paper as well as open directions. We separately include all mathematical details verifying the main theory in Section 9, while leaving some auxiliary verifications to the appendix.

2. Preliminaries

2.1. Main Setting and Basic Notation

The noiseless setting of the paper is obtained by independently sampling a data set $X$ of $N$ points from a uniform mixture measure $\mu$ (see Definition 1.1). We often partition $X$ into the subsets $\{X_i\}_{i=0}^K$ with $\{N_i\}_{i=0}^K$ points sampled according to the measures $\{\mu_i\}_{i=0}^K$ used in the definition of $\mu$. We remark that in Theorem 4.1 we will directly sample from $\mu_0$ and $\mu_1$, instead of the uniform mixture measure $\mu$.

We will inquire whether the best $l_0$ subspace for $X$ is a local $l_p$ subspace or a global $l_p$ subspace w.o.p. By global and local $l_p$ subspaces we mean local or global minimum of the energy expressed in (2). We use both terminologies of global $l_p$ subspace and best $l_p$ subspace to describe the same thing.

We sometimes apply the energies (2) and (3) to a single point $x$, while using the notation: $e_l_p(x, L) \equiv e_l_p(\{x\}, L)$ and $e_{l_p}(x, L_1, L_2, \cdots, L_K) \equiv e_{l_p}(\{x\}, L_1, L_2, \cdots, L_K)$.

We denote possibly large scalars by upper-case plain letters (e.g., $N, C$) and scalars with relatively small values by lower-case Greek letters (e.g., $\alpha, \epsilon$); vectors by boldface lower-case letters (e.g., $u, v$); matrices by boldface upper-case letters (e.g., $A$); sets by upper-case Roman (e.g., $L$) or calligraphic letters (e.g., $X$) and measures by lower-case Greek letters (e.g., $\mu, \theta_D$ and $\gamma_{D,d}$). We often distinguish between different constants within the same proof, but may use the same notation for different constants of different proofs.
In addition to the shorthand w.o.p., we use the following ones: w.p. for “with probability”, w.r.t. for “with respect to” and WLOG for “without loss of generality”.

We denote the Euclidean norm of \( x \in \mathbb{R}^D \) by \( \| x \| \) and the ball centered at \( x \in \mathbb{R}^D \) with radius \( r \) by \( B(x, r) := B(x, c \cdot r) \).

The \((i, j)\)-element of a matrix \( A \) is denoted by \( A_{ij} \). The transpose of \( A \) by \( A^T \) and that of a vector \( v \) by \( v^T \). The Frobenius and nuclear norms of \( A \) are denoted by \( \| A \|_F \) and \( \| A \|_* \) respectively (the former one is the square root of the sum of squares of singular values of \( A \) and the latter one is the sum of singular values). The \( n \times n \) identity matrix is written as \( I_n \). We designate the orthogonal group of \( n \times n \) matrices by \( O(n) \) and the semigroup of \( n \times n \) nonnegative scalar matrices by \( S_+(n) \). We denote the subset of \( S_+(n) \) with Frobenius norm 1 by \( NS_+(n) \). If \( m > n \) we let \( O(m, n) = \{ X \in \mathbb{R}^{m \times n} : X^T X = I_n \} \), whereas if \( n > m \), \( O(m, n) = \{ X \in \mathbb{R}^{m \times n} : XX^T = I_m \} \).

If \( L \) is a subspace of \( \mathbb{R}^D \), we denote by \( L_\perp \) its orthogonal complement. We designate the projection from \( \mathbb{R}^D \) onto \( L \) and \( L_\perp \) by \( P_L \) and \( P_{L_\perp} \) respectively. If \( x \in \mathbb{R}^D \), we use \( \text{dist}(x, L) \) to denote the orthogonal distance from \( x \) to \( L \). We define the scaled outlying “correlation” matrix \( B_{L, x} \) of a data set \( X \) and a \( d \)-subspace \( L \) as follows

\[
B_{L, x} = \sum_{x \in X \setminus L} P_L(x)P_{L_\perp}(x)^T/\text{dist}(x, L). \tag{5}
\]

We will also use the following operator:

\[
D_{L, x, p} = P_L(x)P_{L_\perp}(x)^T \text{dist}(x, L)^{(p-2)}. \tag{6}
\]

### 2.2. Principal Angles, Principal Vectors and Related Notation

We denote the principal angles [21] between two \( d \)-subspaces \( F \) and \( G \) by \( \pi/2 \geq \theta_1 \geq \theta_2 \geq \cdots \geq \theta_d \geq 0 \), where we order them decreasingly, unlike common notation. We denote by \( k = k(F, G) \) the largest number such that \( \theta_k \neq 0 \), so that \( \theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} = \cdots = \theta_d = 0 \). We refer to this number as interaction dimension and reserve the index \( k \) for denoting it (the subspaces \( F \) and \( G \) will be clear from the context). We recall that the principal vectors \( \{v_i\}_{i=1}^d \) and \( \{v'_i\}_{i=1}^d \) of \( F \) and \( G \) respectively are two orthogonal bases for \( F \) and \( G \) satisfying

\[
\langle v_i, v'_j \rangle = \cos(\theta_i), \quad \text{for } i = 1, \ldots, d,
\]

and

\[
v_i \perp v'_j, \quad \text{for all } 1 \leq i \neq j \leq k.
\]

We define the complementary orthogonal system \( \{u_i\}_{i=1}^d \) for \( G \) with respect to \( F \) by the formula:

\[
\begin{aligned}
v'_i &= \cos(\theta_i)v_i + \sin(\theta_i)u_i, \quad i = 1, 2, \cdots, k, \\
u_i &= v_i, \quad i = k+1, \cdots, d.
\end{aligned} \tag{7}
\]

We note that

\[
u_i \perp v_j \quad \text{for all } 1 \leq i, j \leq k.
\]
We thus orthogonally decomposed $F+G$ into the 2-dimensional subspaces $\text{Sp}(v_i, u_i)$, $i = 1, \ldots, k$, of mutually orthogonal systems and the residual subspace $F \cap G$. The interaction between $F$ and $G$ can then be described only within these subspaces via the principal angles. This idea is also motivated by purely geometric intuition in [47, Section 2].

2.3. Grassmannian, Invariant Metric and Geodesics

The Grassmannian space $G(D, d)$ is the set of all $d$-subspaces of $\mathbb{R}^D$ with a manifold structure. Throughout the paper we implicitly use principal vectors to represent $G(D, d)$ by $O(d) \times O(d, D - d) \times S_{+}(d)$. Indeed, we fix a $d$-subspace $L_1 \in G(D, d)$ and for any $L \in G(D, d)$ we form the principal vectors $\{v_i\}_{i=1}^d$ and $\{v'_i\}_{i=1}^d$ for $L_1$ and $L$ respectively; the projection of $\{v_i\}_{i=1}^d$ onto $L_1$ corresponds to an element of $O(d)$; the projection of $\{v'_i\}_{i=1}^d$ (or the complementary vectors $\{u_i\}_{i=1}^d$ of $L$ w.r.t. $L_1$) onto $L_1$ gives rise to an element of $O(d, D - d)$; The principal angles in $S_{+}$ then relate elements projected onto $L_1^+$ and $L_1$. Our representation is rather different than the common representation in numerical computation [18, Table 2.1], which uses either of the quotient spaces: $O(D, d)/O(d)$ or $O(D)/(O(d) \times O(D - d))$.

We will measure distances between $F$ and $G$ in $G(D, d)$ by the following metric

$$
\text{dist}(F, G) = \sqrt{\sum_{i=1}^{d} \theta_i^2} = \sqrt{\sum_{i=1}^{k} \theta_i^2}.
$$

(8)

This distance was suggested in [47] as invariant metric since it measures the geodesic distance in $G(D, d)$ between the corresponding subspaces [47] as one can see from (9) below.

It follows from [47, Theorem 9] that if the largest principal angle between $F$ and $G$ is less than $\pi/2$, then there is a unique geodesic line between them. Following [18, Theorem 2.3], we can parametrize this line from $F$ to $G$ by the following function $L : [0, 1] \rightarrow G(D, d)$, which is expressed in terms of the principal angles $\{\theta_i\}_{i=1}^d$ of $F$ and $G$, the principal vectors $\{v_i\}_{i=1}^d$ of $F$ and the complementary orthogonal system $\{u_i\}_{i=1}^d$ of $G$ with respect to $F$:

$$
L(t) = \text{Sp}(\{\cos(t\theta_i)v_i + \sin(t\theta_i)u_i\}_{i=1}^d).
$$

(9)

If $L \in G(D, d)$, we denote by $B(L, r)$ the closed ball in $G(D, d)$ around $L$ with radius $r$. We also denote by $B_E(B(L, r_1), r_2)$ the Euclidean ball around $B(L, r_1)$, i.e., the set of all points in $\mathbb{R}^D$ whose distance from the set $\cup_{L' \in B(L, r_1)}L'$ is at most $r_2$.

We will use the natural probability measure on the Grassmannian, commonly denoted by $\gamma_{D,d}$ [36]. We recall that for any fixed $F \in G(D, d)$ and any $A \subseteq G(D, d)$:

$$
\gamma_{D,d}(A) = \theta_D(\{B \in O(D) : BF \in A\}),
$$

where $\theta_D$ is the Haar measure on $O(D)$, so that for any $x \in S^{D-1}$ (where $S^{D-1}$ is the $(D - 1)$-dimensional unit sphere with uniform probability measure $\sigma^{D-1}$) and $E \subseteq S^{D-1}$:

$$
\theta_D(\{B \in O(D) : Bx \in E\}) = \sigma^{D-1}(E).
$$
3. Counterexamples and Conditions for Robustness of $l_p$ Subspaces

3.1. Counterexamples for Robustness of Best $l_p$ Subspaces

We show here that there are many natural situations, though different than our underlying model of uniform outliers, where best $l_p$ $d$-subspaces are not robust to outliers for all $0 < p < \infty$. More precisely, we show how a single outlier can completely change the underlying subspace.

A typical example includes $N_1$ points sampled independently and uniformly from a $d$-dimensional ball in $\mathbb{R}^D$ centered around the origin with radius $\epsilon$ and an additional outlier located on a unit vector orthogonal to that $d$-subspace. By choosing $\epsilon$ sufficiently small, e.g., $\epsilon \lesssim (1/N_1)^{1/p}$, the best $l_p$ subspace passes through the single outlier and is thus orthogonal to the initial $d$-subspace for all $p > 0$.

If $p = 1$, then the best $l_0$ $d$-subspace in this example is still a local $l_1$ subspace. Nevertheless, if the outlier is located instead on a unit vector having elevation angle with the original $d$-subspace less than $\pi/2$, then $\epsilon$ can be chosen so that the best $l_0$ subspace is neither a local nor global $l_1$ subspace. However, if $0 < p < 1$, then the best $l_0$ subspace is still a local $l_p$ subspace in both examples as well as almost any other scenario (see e.g., Proposition 3.1 below).

Similarly, it is not hard to produce an example of data points on the unit sphere of $\mathbb{R}^D$ where the best $l_0$ subspace is still not a best $l_1$ subspace. This is in contrast to the case of sparse representation of signals, where normalization of the column vectors of a matrix representing an undercomplete linear system of equations ensures that the solution minimizing the $l_1$ norm is also the sparsest solution as long as it is sufficiently sparse [17, Theorem 2]). For simplicity we give a counterexample for $d = 2$ by letting $N_1$ data points be uniformly sampled along an arc of length $\epsilon$ of a great circle of the sphere $S^2 \subseteq \mathbb{R}^3$. We then place an outlier on another great circle, which passes through the center of the $\epsilon$-arc and has a small angle with it. Taking $\epsilon$ sufficiently small and the outlier furthest from the intersection of the two great circles, then the best $l_0$ subspace is not a local $l_1$ subspace and consequently not a global one. We remark that in this example both assumptions of this paper requiring uniformity of outliers (or more generally symmetry around the origin; see Section 8) and symmetry around the origin of inliers (see again Section 8) are not satisfied.

3.2. Combinatorial Conditions for $l_0$ Subspaces being Local $l_p$ Subspaces

We formulate conditions for the best $l_0$ subspace to be a local $l_p$ subspace, while distinguishing between three cases: $p = 1$, $0 < p < 1$ and $p > 1$. We prove these results in Section 9.2. The most interesting condition is when $p = 1$, which we describe as follows. It uses notation introduced in Section 2, in particular, the scaled outlying “correlation” matrix $B_{L,X}$ of (5).

**Theorem 3.1.** If $L_1 \in G(D,d)$, $X_1 = \{x_i\}_{i=1}^{N_1} \in L_1$, $X_0 = \{y_i\}_{i=1}^{N_0} \in \mathbb{R}^D \setminus L_1$ and $X = X_0 \cup X_1$, then a sufficient condition for $L_1$ to be a local minimum of $e_1(\mathcal{X}, L)$
among all $d$-subspaces $L \in G(D,d)$ is that for any $V \in O(d)$ and $C \in S_+(d)$:
\[
\sum_{i=1}^{N_L} \|CV P_{L_1}(x_i)\| > \|CV B_{L_1,x}\|_*. 
\] (10)

The next proposition shows that for $p < 1$ the best $l_0$ subspace is almost always a local $l_p$ subspace.

**Proposition 3.1.** If $L_1 \in G(D,d)$, $X_1 = \{x_i\}_{i=1}^{N_L_1} \in L_1$, $X_0 = \{y_i\}_{i=1}^{N_0} \in \mathbb{R}^D \setminus L_1$, 
$Sp(\{x_i\}_{i=1}^{N_L_1}) = L_1$ and $p < 1$, then $L_1$ is a local minimum of $e_{l_p}(X, L)$ among all $L \in G(D,d)$.

At last, for $p > 1$ we establish a necessary condition for the best $l_0$ subspace to be a local $l_p$ subspace. This condition is rather degenerate and often cannot be satisfied.

**Proposition 3.2.** If $L_1 \in G(D,d)$, $X_1 = \{x_i\}_{i=1}^{N_L_1} \in L_1$, $X_0 = \{y_i\}_{i=1}^{N_0} \in \mathbb{R}^D \setminus L_1$ and $p > 1$, then a necessary condition for $L_1$ to be a local minimum of $e_{l_p}(X, L)$ among all $L \in G(D,d)$ is
\[
\sum_{i=1}^{N_0} P_{L_1}(y_i) P_{L_1}(y_i)^T \text{dist}(y_i, L_1)^{p-2} = 0. 
\] (11)

The above results manifest a phase transition phenomenon. Indeed, the best $l_0$ subspace is almost always a local $l_p$ subspace for $p < 1$, whereas for $p > 1$ this is often not the case (except for an underlying measure which is symmetric in the complement of $L_1$; for example, in the case of an underlying uniform mixture with $K = 1$, the best $l_0$ subspace is asymptotically a best $l_p$ subspace for all $p > 0$). The combinatorial condition implying when it is a local $l_1$ subspace is more complicated and we exemplify its application throughout the paper.

4. Best $l_0$ Subspaces as Local or Global $l_p$ Subspaces for Uniform Sampling

We assume here the probabilistic setting of uniform mixture measure with a single underlying subspace $L_1$, i.e., $K = 1$. Clearly, $L_1$ is the best $l_0$ subspace for the sampled data w.o.p. For any $p > 0$, we ask whether $L_1$ is also a local or even global $l_p$ subspace w.o.p. We prove the corresponding results described below in Section 9.3.

We first claim that for $p = 1$ the best $l_0$ subspace is a local $l_p$ subspace w.o.p. as long as the fraction of inliers is sufficiently large.

**Theorem 4.1.** If $L_1 \in G(D,d)$ and $X$ is a data set in $\mathbb{R}^D$ of $N_0 + N_1$ points, where $N_0$ of them are uniformly and independently sampled from the unit ball $B(0, 1)$ in $\mathbb{R}^D$ and $N_1$ of them are independently and uniformly sampled from $B(0, 1) \cap L_1$; then $L_1$ is a local $l_1$ subspace of $X$ w.p. at least
\[
1 - 2d^2 \exp \left( - \frac{N_1 \eta^2}{8d^2} \right) - 2dD \exp \left( - \frac{N_0 \epsilon^2}{2d^2D} \right), \text{ where } \eta + \frac{N_0}{N_1} \epsilon < 2/(2d + 3).
\]
In particular, if \( N_0 = o(N_1^2) \), then \( L_1 \) is a local \( l_1 \) subspace of \( \mathcal{X} \) w.p. at least
\[
1 - 2d^2 \exp \left( -\frac{N_1}{72(d^2 + 2d)^2} \right) - 2dD \exp \left( -\frac{N_1^2}{8(d^2 + 2d)^2DN_0} \right).
\] (12)

For \( 0 < p < 1 \), Proposition 3.1 implies that if \( N_1 = \Omega(1) \) then \( L_1 \) is a local \( l_p \) subspace w.o.p. On the other hand we claim next that if \( p > 1 \) and \( N_1 = \Omega(1) \), then the subspace \( L_1 \) is a local \( l_p \) subspace w.p. 0.

**Proposition 4.1.** Consider \( L_1 \in G(D, d) \), \( \mu_0 \) a uniform distribution in \( B(0, 1) \subseteq \mathbb{R}^D \), \( \mu_1 \) a uniform distribution on \( L_1 \cap B(0, 1) \), \( \mu = \alpha_0\mu_0 + \alpha_1\mu_1 \), where \( 0 \leq \alpha_0 < 1 \), \( \alpha_1 = 1 - \alpha_0 \) and \( \mathcal{X} \) is a data set sampled independently from \( \mu \). If \( p > 1 \), then the probability that \( L_1 \) is a local \( l_p \) subspace of \( \mathcal{X} \) is 0.

The proof of this proposition is rather immediate. Indeed, the outliers, denoted by \( \{y_i\}_{i=1}^{N_0} \), have uniform distribution \( \mu_1 \), which has a bounded and nonzero probability density function for vectors in the unit \( D \)-dimensional ball. Therefore for any \( L' \in G(D, d) \) the joint probability density function of \( \sum_{i=1}^{N_0} P_{L'}(y_i)P_{L'}(y_i)^T \text{dist}(y_i, L')^{p-2} \) is also bounded and nonzero on the corresponding range and thus (11) has probability 0.

Another question is whether the best \( l_0 \) subspace is also the global \( l_p \) subspace. Proposition 4.1 and Theorem 1.1 already answered this question in our setting. Indeed, if \( p > 1 \), then by Proposition 4.1 the best \( l_0 \) subspace is a global \( l_p \) subspace with probability 0; whereas if \( 0 < p \leq 1 \), then Theorem 1.1 with \( K = 1 \) implies that for \( N_0 = O(N_1) \) the best \( l_0 \) subspace is also the best \( l_p \) subspace w.o.p.

We formulate this special case of Theorem 5.1 below and prove it separately. We believe that it is easier to digest the whole proof of Theorem 1.1 by first following it for this special case and later generalizing it.

**Theorem 4.2.** If \( L_1 \in G(D, d) \), \( \mu_0 \) a uniform distribution in \( B(0, 1) \subseteq \mathbb{R}^D \), \( \mu_1 \) a uniform distribution on \( L_1 \cap B(0, 1) \), \( \mu = \alpha_0\mu_0 + \alpha_1\mu_1 \), where \( \alpha_0, \alpha_1 \) are nonnegative numbers summing to 1 and \( \mathcal{X} \) is a data set independently sampled from \( \mu \), then \( L_1 \) is a best \( l_p \) subspace for \( \mathcal{X} \) w.o.p. for any \( 0 < p \leq 1 \).

At last, we remark that the phase transition phenomenon demonstrated above at \( p = 1 \) is rather artificial in the current setting. Indeed, this phase transition is based on the fact that (11) holds w.p. 0 for \( p > 1 \) and any finite sample; however, the LHS of (11) divided by \( N \) is 0 w.p. 1 as \( N \) approaches infinity. Moreover-, when \( p > 1 \) the positive distance between the best \( l_0 \) subspace and the best \( l_p \) subspace approaches 0 as \( N \) approaches infinity. We will show in Theorem 5.1 that this formal phase transition also breaks down with noise. Nevertheless, as we show in Theorem 6.1, there is a clear phase transition for a uniform mixture model with \( K > 1 \). This is rather intuitive since the underlying measure of the latter case is not symmetric on the complement of \( L_1 \), unlike the case where \( K = 1 \).

5. Extension of the Theory to Noisy Setting

We present here extensions of previous results, in particular, Theorems 1.1 and 1.2, to the setting of independent samples from uniform mixture measure of noise level \( \epsilon > 0 \).
We prove those extensions in Section 9.5.

In this noisy setting, Theorem 1.1 is still valid up to a recovery error proportional to the noise level $\epsilon$. In fact, if $K = 1$, then such a near-recovery generalizes to all $0 < p < \infty$.

**Theorem 5.1.** If $\epsilon > 0$, $\mu_{\epsilon}$ is a uniform mixture measure of noise level $\epsilon$ on $\mathbb{R}^D$ with $K$ $d$-subspaces $\{L_i\}_{i=1}^K \subseteq \mathbb{R}^D$ and mixture coefficients $\{\alpha_i\}_{i=0}^K$. $\mathcal{X}$ is a data set of $N$ points sampled independently from $\mu_{\epsilon}$ and $0 < p \leq 1$, then the best $l_p$ subspace for $\mu_{\epsilon}$ is in the ball $B(L_1, f)$, where

$$f \equiv f(\epsilon, K, d, p, \alpha_0, \alpha_1) = \frac{2^{\frac{3+p}{p}d \frac{3}{2} \epsilon}}{(\alpha_1 - \sum_{i=2}^K \alpha_i)^{\frac{1}{p}}},$$

w.p. at least $1 - C \exp(-N/C)$, where $C = C(\epsilon, p, d, \alpha_0, \alpha_1, \min_{2 \leq i \leq K} (\text{dist}(L_1, L_i)))$.

If $K = 1$, then the above statement extends for all $0 < p < \infty$ with

$$f \equiv f(\epsilon, K, d, p, \alpha_0, \alpha_1) = 2^{\frac{3+p}{p}d \frac{3}{2} \epsilon} \left(\frac{\alpha_1}{\alpha_1}\right)^{\frac{1}{p}} \epsilon^{\frac{1}{p}}.$$

**Remark 5.1.** If $0 < p \leq 1$ and

$$\epsilon > \frac{\pi (\alpha_1 - \sum_{i=2}^K \alpha_i)^{\frac{1}{p}}}{2^{\frac{3+p}{p}d}},$$

or $p > 1$, $K = 1$ and

$$\epsilon > \frac{\pi^p \alpha_1}{2^{\frac{3+p}{p}d}} \left(\frac{d}{p}\right)^{\frac{1}{p}},$$

then $f > \frac{\pi \sqrt{d}}{2}$, which implies that $B(L_1, f) = G(D, d)$ (since all principle angles are at most $\pi/2$). It thus makes sense to restrict the level of noise to be at least lower than the right hand sides of (13) or (14).

Theorem 1.2 also extends to uniform mixture measures with restricted noise level. This restriction on $\epsilon$ is expressed in the theorem below, while using the following constant:

$$\tau_0 = \frac{1}{21+p K^p d^{\frac{1}{p}}}.$$

**Theorem 5.2.** Let $\epsilon > 0$, $\mu_{\epsilon}$ a uniform mixture measure of noise level $\epsilon$ on $\mathbb{R}^D$ with $K$ $d$-subspaces, $\{L_i\}_{i=1}^K \subseteq \mathbb{R}^D$ as well as mixture coefficients $\{\alpha_i\}_{i=0}^K$ and $\mathcal{X}$ a data set of $N$ points sampled independently from $\mu_{\epsilon}$. If $0 < p \leq 1$ and

$$\epsilon < 3^{\frac{1}{p}} \left(\tau_0 \min_{1 \leq j \leq K} \alpha_j \min_{1 \leq i, j \leq K} \text{dist}^p(L_i, L_j) / 2^p - \alpha_0\right)^{\frac{1}{p}},$$

then the minimizer of (3) in $G(D, d)^K$ has a distance smaller than

$$f \equiv f(\epsilon, K, d, p, \{\alpha_i\}_{i=1}^K) = 3^{\frac{1}{p}} \left(\tau_0 \min_{1 \leq j \leq K} \alpha_j - \alpha_0\right)^{-\frac{1}{p}} \epsilon$$

from one of the permutations of $(L_1, L_2, \cdots, L_K)$.  

6. The Phase Transition at $p = 1$

Theorems 1.1, 1.2, 5.1 and 5.2 established the recovery and near recovery of the best $l_0$ subspace as well as all underlying subspaces by $l_p$ minimization whenever $0 < p \leq 1$. We also showed that for a single subspace, i.e., $K = 1$, near recovery extends to $p > 1$ (see Theorem 5.1) and exact recovery asymptotically extends to $p > 1$, but is never realized (see Section 4). Here we establish the impossibility of such $l_p$ recoveries when $p > 1$ and $K > 1$ and thus demonstrate a phase transition at $p = 1$ when $K > 1$. We prove all statements in Section 6.

We first claim that the best $l_0$ subspace cannot be effectively recovered or nearly recovered by $l_p$ minimization when $p > 1$ and $K > 1$. That is, we establish a phase transition of the $l_p$ recovery of the best $l_0$ subspace at $p = 1$.

**Theorem 6.1.** Assume that $\{L_i\}_{i=1}^K$ are $K$ $d$-subspaces in $\mathbb{R}^D$, which are independently distributed according to $\gamma_{D,d}$. For each $\epsilon \geq 0$ and a random sample of $\{L_i\}_{i=1}^K$, let $\mu_\epsilon$ be a uniform mixture measure of noise level $\epsilon$ (or without noise when $\epsilon = 0$) on $\mathbb{R}^D$ w.r.t. $\{L_i\}_{i=1}^K \subseteq \mathbb{R}^D$ and let $X$ be a data set of $N$ points sampled independently from $\mu_\epsilon$. If $K > 1$ and $p > 1$, then for almost every $\{L_i\}_{i=1}^K$ (w.r.t. $\gamma_{D,d}^K$), there exist positive constants $\delta_0$ and $\kappa_0$, independent of $N$, such that for any $0 \leq \epsilon < \delta_0$ the best $l_p$ subspace of $X$ is not in the ball $B(L_1, \kappa_0)$ with overwhelming probability.

**Remark 6.1.** The above constants $\delta_0$ and $\kappa_0$ depend on other parameters of the underlying uniform mixture model in particular the underlying subspaces $\{L_i\}_{i=1}^K$. For example, in the case of $p \geq 2$ one can estimate from below both $\kappa_0$ and $\delta_0$ by the following number:

$$\frac{\| \sum_{i=2}^d \alpha_i E_{\tilde{\mu}_{i,\epsilon}}(D_{L_1 \times \cdot\cdot\cdot \times \cdot \cdot\cdot \times L_{n}}) \|_2^2}{dD^{2p+5}},$$

where $D_{L_1 \times \cdot\cdot\cdot \times \cdot \cdot\cdot \times L_{n}}$ is defined in (6) and for any $i = 1, \ldots, K$, $\tilde{\mu}_{i,\epsilon}$ is obtained by projecting $\mu_{i,\epsilon}$ onto the subspace $L_i$ (that is, for any set $E \subseteq B(0, 1) \cap L_i$: $\tilde{\mu}_{i,\epsilon}(E) = \mu_{i,\epsilon}(P_{L_i}^{-1}(E))$).

Next, we formulate the impossibility of recovering all underlying $d$-subspaces by $l_p$ minimization for $p > 1$ and thus demonstrate a phase transition in this case.

**Theorem 6.2.** Assume that $\{L_i\}_{i=1}^K$ are $K$ $d$-subspaces in $\mathbb{R}^D$, which are independently distributed according to $\gamma_{D,d}$. For each $\epsilon \geq 0$ and a random sample of $\{L_i\}_{i=1}^K$, let $\mu_\epsilon$ be a uniform mixture measure of noise level $\epsilon$ (or without noise when $\epsilon = 0$) on $\mathbb{R}^D$ w.r.t. $\{L_i\}_{i=1}^K \subseteq \mathbb{R}^D$ and let $X$ be a data set of $N$ points sampled independently from $\mu_\epsilon$. If $p > 1$ and $K > 1$, then for almost every $\{L_i\}_{i=1}^K$ (w.r.t. $\gamma_{D,d}^K$) there exist positive constants $\delta_0$ and $\kappa_0$, independent of $N$, such that for any $0 < \epsilon < \delta_0$ the minimizer of (3), $\hat{L}_1, \hat{L}_2, \cdot\cdot\cdot, \hat{L}_K$, satisfies w.o.p.:

$$\text{dist}((\hat{L}_1, \hat{L}_2, \cdot\cdot\cdot, \hat{L}_K), (L_1, L_2, \ldots, L_K)) > \kappa_0.$$

7. Implications of the Theory for Subspace Modeling

We discuss the implications of the theory described above for robust HLM and even for the simpler case of robust modeling by a single subspace. Since we study here only
particular distributions, we cannot fully explain the general behavior of the algorithms mentioned below. Nevertheless, we can still provide some quantitative explanations of their performance and clarify situations where it is necessary to use the values $0 < p \leq 1$ for efficient $l_p$ minimizations.

A very common algorithm for recovering a $d$-subspace in a point cloud is the RANSAC algorithm\[19]\). Its simplest version repeatedly applies the following two steps: 1. randomly select a set of $d$ independent vectors; 2. count the number of data points within a strip of width $\epsilon$ around the $d$-subspace spanned by those $d$ vectors (both $\epsilon$ and the number of iterations of these two steps are parameters set by the user). The final output of this algorithm is the $d$-subspace maximizing the quantity computed in step 2. Almost all other variants of RANSAC assess the best $d$-subspace by the same quantity, which depends on the unknown parameter $\epsilon$.

Torr and Zisserman \[43, 44\] have suggested a RANSAC-type strategy which minimizes a variant of the $l_2$ distance from a subspace. This variant uses the square function until a fixed threshold and a constant function for larger values. Theorems 1.1, 5.1 and 6.1 provide some insights on the effectiveness of recovering the best $l_0$ $d$-subspace (or best $l_0$ strip of width $\epsilon$) in a uniform mixture setting by minimizing $l_p$ distances in the spirit of \[43, 44\]. In particular, they imply that if $K > 1$ then only $l_p$ distances with $0 < p \leq 1$ should be considered in the latter setting. Even distances that coincide with the $l_2$ distance for sufficiently small values, such as \[43, 44\] or Huber’s loss function \[24\], will not recover the underlying subspaces as the proofs of those theorems show. On the other hand, for a single underlying subspace in point clouds with possibly additive noise, $l_p$ recovery should succeed in theory for any $0 < p < \infty$, though the bounding constants worsen as $p$ increases. The idea of \[43, 44\] making the loss function constant for large values is expected to help with significantly far and nonuniform outliers that are not covered by our model. Such outliers are discussed e.g., in Section 3.1.

For the recovery of multiple subspaces, RANSAC has been repeatedly applied in \[49\], while removing the points around the subspace found at the current iteration and providing the reduced data for the next one. Numerical results in \[51\] show that this strategy is both accurate and fast for some artificial data when setting the RANSAC parameter $\epsilon$ to be the model’s noise level. However, in practice, the noise level is unknown. Da Silva and Costeiranuno \[13\] have suggested an alternative numerical optimization over the Grassmannian to iteratively estimate subspaces, while avoiding the RANSAC procedure. However, their method seems to be sensitive to local minima and there is no obvious interpretation for their objective function. On the other hand, for the particular setting of uniform mixture measures with noise, Theorem 5.1 provides a clear interpretation for the $l_p$ minimization and also guarantees its stability. However, in practice, we may apply such setting only when recovering the best $l_0$ subspace among all underlying $K$ subspaces.

Rigorous application of Theorem 5.1 for iterative recovery of the rest of the underlying subspaces requires the extension of this theorem to more general scenarios; such an extension depends on the precise way of removing the part of the data around a subspace (see some relevant though not sufficient extensions in Section 8).

On the contrary, Theorems 1.2 and 5.2 explain the simultaneous minimization of subspaces via the energy (3). Zhang et al. \[50\] suggested a stochastic gradient descent
approach for approximating this minimization problem (only \( p = 1 \) is discussed there, but their method applies to any \( 0 < p < \infty \)). They have demonstrated robustness to outliers for artificial data sets. More recently, [51] described multiscale geometric strategies for forming candidate \( d \)-subspaces. They then select the best \( K \) \( d \)-subspaces by minimizing the energy (3) among all such candidates (or many of them). Their choice of candidates is justified in [51, Theorem 1], i.e., they show that among the large set of candidate subspaces there are \( K \) subspaces closely approximating the true underlying subspaces. On the other hand, their use of \( l_1 \) minimization to find the best approximating candidates to the true subspace is justified by Theorems 1.2 and 5.2 for particular sampling rules.

8. Discussion

We studied the effectiveness of \( l_p \) minimization for recovering both the best \( l_0 \) subspace and all underlying \( K \) subspaces with overwhelming probability when independently sampling from a uniform mixture measure. A probabilistic setting was necessary since we also described some typical cases where best \( l_p \) subspaces are different than best \( l_0 \) subspaces for all \( 0 < p < \infty \). We also showed how to generalize this study in order to nearly recover the subspaces in the case of additive uniform noise. Furthermore, we demonstrated a phase transition phenomenon around \( p = 1 \) for \( l_p \) recovery of the best \( l_0 \) subspace when \( K > 1 \). Our analysis has provided some guarantees for the robustness to point clouds of some recent HLM algorithms as well as single subspace recovery.

There are many possibilities to extend this work and we would like to discuss some of these directions here.

**More general distributions.** It will be interesting to extend our probabilistic results to more general distributions, i.e., distributions that are not purely uniform. We discuss here some of these generalizations, which are apparent from the proofs of the theory. We first note that our results extend with weaker bounds to approximately uniform distributions, i.e., distributions whose pdf’s are bounded away from 0 and \( \infty \) on the corresponding regions. By weaker bounds, we mean for example that the lower bound on \( \alpha_0 \) in Theorem 4.2 (i.e., \( \alpha_0 > 0 \)) and more generally the ratio between the LHS and RHS of (1) in Theorem 1.1 need to increase (depending on the upper and lower bounds of the underlying pdf’s).

Moreover, it is clear that the uniformity along subspaces can be generalized to uniformity (or approximate uniformity) along spheres around the origin. More precisely, we may assume that all \( \{ \mu_i \}_{i=1}^K \) have the same distribution (up to rotation) with a radially symmetric pdf (or approximately so). For example, one can use the same spherical Gaussian distribution along subspaces. Similarly, the assumption of uniform outliers can be relaxed by assuming that the pdf of \( \mu_0 \) is spherically symmetric around the origin. When exploring when the best \( l_0 \) subspace is a local \( l_p \) subspace, e.g., as in Theorem 4.1, then it is sufficient to ask that the pdf of \( \mu_0 \) is symmetric with respect to \( L_1 \). More precisely, such a symmetry requires that \( E_{\mu_0} (D_{L_1, x, p}) = 0 \) for all \( p > 0 \). For example, this pdf may obtain the same values on all points in the unit ball with the same distance to \( L_1 \). Alternatively, it may obtain the same values on all points within
the unit ball on the boundary of cones in $\mathbb{R}^D$ centered on $L_1$ at the origin (such cones are defined e.g., in [31, Section 2.1]).

By a further weakening of Theorem 1.1, it is possible to replace the uniformity (or approximate uniformity) of outliers in the unit ball by uniformity (or approximate uniformity) of the projection of outliers onto the best $l_0$ subspace $L_1$. This will require a sufficiently large lower bound on $\alpha_1$, in particular, larger than 0.5. This lower bound has to depend on the maximal distance of outliers from $L_1$.

The noisy setting can be extended with weaker bounds to densities of the form $f(x) \equiv f(P_{L}(x), P_{L^\perp}(x)) = g(P_{L^\perp}(x))h(P_{L}(x))$, where $h$ is uniform (or radially symmetric) and $g$ decays sufficiently fast, e.g., $g$ is the pdf of a normal distribution. In fact, if $g$ is chosen from a parametric family with parameter $\epsilon$, then it is sufficient to assume that the $L_1$ norm of the random variable with density $g$ approaches zero as $\epsilon$ approaches zero.

The case of affine subspaces. Our analysis was restricted to linear subspaces, though it can be formally extended to affine subspaces intersecting a fixed ball fully contained in $B(0, 1)$, e.g., the ball $B(0, 1/2)$. Indeed, we can consider the affine Grassmannian [36, 29], which distinguishes between subspaces according to both their offsets with respect to the origin (i.e., distances to closest linear subspaces of the same dimension) and their orientations (based on principal angles of the shifted linear subspaces). The assumption above on the affine subspaces (i.e., their offsets are less than $1/2$) restricts them to be in a compact subspace of the affine Grassmannian as necessary to our analysis. Nevertheless, it is not obvious whether the metric on the affine Grassmannian is relevant for our applications, since it mixes two different quantities of different units (i.e., offset values and orientations) so that one can arbitrarily weigh their contributions. We remark that the common strategy of using homogenous coordinates which transform $d$-dimensional affine subspaces in $\mathbb{R}^D$ to $(d + 1)$-dimensional linear subspaces in $\mathbb{R}^{D+1}$ is not useful to us since it distorts the structure of both noise and outliers.

A related problem of interest to us is to explain why different variants of both the $K$-subspaces and iterative RANSAC (for HLM) do not perform well with affine subspaces as they do with linear ones. It is clear though that the analysis of the $K$-subspaces algorithm is different in the two cases. Indeed, the required analysis needs to deal with sets of points closer to a given subspace among all underlying subspaces, namely the regions $\{Y_j\}_{j=1}^K$ of (96). For linear subspaces the boundaries of such regions are polyhedral surfaces, whereas for affine subspaces they are piecewise quadratic.

Further performance guarantees for $l_p$-based HLM Algorithms.

The MKF algorithm [50] attempts to minimize the energy (3) for $p = 1$. The theory described here advocates such minimization. However, in practice, the MKF applies a stochastic gradient descent for approximating the minimum value. We are interested in direct study of convergence as well as robustness to outliers of this iterative approximation.

Another iterative method based on $l_p$ subspace minimization is the $K$-subspaces algorithm [25, 4, 45, 23]. It minimizes a function of both the $K$ $d$-subspaces and the $K$ clusters. Consequently, it can be more sensitive to initializations of the clusters. In
particular, it seems hard to generalize Theorems 1.1 and 5.1 to provide performance guarantees for the $l_p$-based $K$-subspaces algorithm with underlying linear subspaces. We are also curious about even partial analysis for this algorithm in the case of mixed dimensions.

9. Verification of Theory

We describe here the complete proofs of the various theorems and propositions of this paper.

9.1. Auxiliary Lemmata

We formulate several technical lemmata, which will be used throughout the proofs of the following sections. Their proofs appear in Appendices A.1-A.4.

**Lemma 9.1.** Suppose that $L_1, \hat{L}_1, L_2, \cdots, \hat{L}_K \in G(D, d)$, $p \geq 0$ and $\mu_1$ is a uniform distribution in $B(0, 1) \cap L_1$. If $\min_{1 \leq j \leq K} \text{dist}(L_1, \hat{L}_j) > \epsilon$, then

$$E_{\mu_1} (\epsilon_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K)) > \frac{\epsilon^p}{2^{1+p} K^p d_{\mu_1}^p}.$$ 

**Lemma 9.2.** For any $x \in B(0, 1)$ and $L_1, L_2 \in G(D, d)$:

$$|\text{dist}(x, L_1) - \text{dist}(x, L_2)| \leq ||x|| \text{dist}(L_1, L_2).$$

**Lemma 9.3.** If $L_1, L_2 \in G(D, d)$, $x_1$, $x_2$ are uniformly distributed random variables in $B(0, 1) \cap L_1$, $B(0, 1) \cap L_2$ respectively and $p \leq 1$, then for any $x \in G(D, d)$:

$$E(\text{dist}(x_1, \hat{L})^p) + E(\text{dist}(x_2, \hat{L})^p) \geq E(\text{dist}(x_1, L_i)^p) + E(\text{dist}(x_2, L_i)^p) \quad \text{for } i = 1, 2.$$ 

(18)

The next lemma uses the constant $\tau_0$ of (15) and the following notation w.r.t. the fixed $d$-subspaces $L_1, L_2, \cdots, L_K, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K : G(D, d)$:

$$I(i) = \arg \min_{1 \leq j \leq K} \text{dist}(L_i, \hat{L}_j) \quad \forall 1 \leq i \leq K$$

(19)

and

$$d_0 = \min_{i_1, i_2, \cdots, i_K \in \mathcal{P}_K} \text{dist}((L_{i_1}, L_{i_2}, \cdots, L_{i_K}), (\hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K)).$$

(20)

**Lemma 9.4.** Suppose that $L_1, L_2, \cdots, L_K, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K \in G(D, d)$ and $0 < p \leq 1$. If $(I(1), \cdots, I(K))$ is a permutation of $(1, \cdots, K)$, then

$$E_{\mu} \epsilon_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) - E_{\mu} \epsilon_{l_p}(x, L_1, L_2, \cdots, L_K) \geq \left( \tau_0 \min_{1 \leq j \leq K} \alpha_j - \alpha_0 \right) d_0^p.$$ 

On the other hand, if $(I(1), \cdots, I(K))$ is not a permutation of $(1, \cdots, K)$, then

$$E_{\mu} \epsilon_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) - E_{\mu} \epsilon_{l_p}(x, L_1, L_2, \cdots, L_K) \geq \left( \min_{1 \leq j \leq K} \alpha_j \right) \left( \min_{1 \leq i, j \leq K} \text{dist}(L_i, L_j)/2 \right)^p - \alpha_0.$$
9.2. Theory of Section 3

9.2.1. Proof of Theorem 3.1

In order to show that $L_1$ is a local minimum of $e_{l_1}(\mathcal{X}, L)$ among all $d$-subspaces in $G(D, d)$, we arbitrarily fix a $d$-subspace $\hat{L} \in B(L_1, 1)$ and show that the derivative of the $l_1$ energy when restricted to the geodesic line from $L_1$ to an arbitrary subspace $\hat{L}$ is positive at $L_1$.

The restriction of $\hat{L}$ to $B(L_1, 1)$ implies that $\theta_1 \leq 1$ and thus by [47, Theorem 9] this geodesic line (connecting $L_1$ and $\hat{L}$) is unique. We parametrize it by the function $L: [0,1] \rightarrow G(D, d)$ of (9), where here $\{\theta_i\}_{i=1}^d$ are the principal angles between $L_1$ and $\hat{L}$, $\{v_i\}_{i=1}^d$ are the principal vectors of $L_1$ and $\{u_i\}_{i=1}^d$ are the complementary orthogonal system for $\hat{L}$ with respect to $L_1$. Using this parametrization we need to prove that the function $e_{l_1}(\mathcal{X}, L(t))$ for $i = 1, \ldots, N_1$:

$$
\text{dist}(y_i, L(t)) = \sqrt{\sum_{j=1}^d \text{dist}^2(P_j(y_i), L(t)) + \text{dist}^2(P_j^\perp(y_i), L(t)).}
$$

For $1 \leq j \leq d$, we let $\phi_j \in [0,2\pi]$ denote the angle such that $P_j(y_i) = ||P_j(y_i)|| \cdot (\cos(\phi_j)v_j + \sin(\phi_j)u_j)$ and consequently express each term of the sum in (21) as follows:

$$
\text{dist}^2(P_j(y_i), L(t)) = ||P_j(y_i)||^2 \sin^2(\phi_j - t\theta_j), \ j = 1, \ldots, d.
$$

Applying (22) in (21) and differentiating, we obtain the following expression for the derivative of $\text{dist}(y_i, L(t))$ for all $1 \leq i \leq N_0$:

$$
\frac{d}{dt} \text{dist}(y_i, L(t)) = -\sum_{j=1}^d \theta_i ||P_j(y_i)||^2 \sin(\phi_j - t\theta_j) \cos(\phi_j - t\theta_j) \text{dist}(y_i, L(t)) / \text{dist}(y_i, L(t))
$$

$$
= -\sum_{j=1}^d \theta_j (\cos(t\theta_j)v_j + \sin(t\theta_j)u_j) \cdot y_i (\sin(t\theta_j)v_j + \cos(t\theta_j)u_j) \cdot y_i
$$

At $t = 0$ it becomes

$$
\left. \frac{d}{dt} \text{dist}(y_i, L(t)) \right|_{t=0} = -\sum_{j=1}^d \theta_j (v_j \cdot y_i) (u_j \cdot y_i) / \text{dist}(y_i, L(0))
$$

$$
= -\sum_{j=1}^k \theta_j (v_j \cdot y_i) (u_j \cdot y_i) / \text{dist}(y_i, L(0)),
$$

where the interaction dimension $k = k(L_1, \hat{L})$ has been introduced in Section 2.2.
We form the following matrices: $C = \text{diag}(\theta_1, \theta_2, \cdots, \theta_d)$, $V \in O(d, D)$ with $j$-th row $v_j^T$ and $U \in O(k, D)$ with $j$-th row $u_j^T$. We then reformulate (24) using these matrices as follows:
\[
\frac{d}{dt} \left( \text{dist}(y_i, L(t)) \right) \bigg|_{t=0} = -\frac{\text{tr}_k(C Vy_i y_i^T U^T)}{\text{dist}(y_i, L)} ,
\]
(25)
where $\text{tr}_k$ denotes the trace of the first $k$ rows of the corresponding $d \times k$ matrix, whose last $d - k$ rows are zeros. Similarly, for all $x_i \in L_1$, $i = 1, 2, \cdots, N_1$,
\[
\text{dist}(x_i, L(t)) = \sqrt{\sum_{j=1}^{d} |(v_j \cdot x_i)|^2 \sin^2(t \theta_j)},
\]
and
\[
\frac{d}{dt} \left( \text{dist}(x_i, L(t)) \right) = \frac{\sum_{j=1}^{d} \theta_j |v_j \cdot x_i|^2 \sin(t \theta_j) \cos(t \theta_j)}{\text{dist}(x_i, L(t))}.
\]
(26)
At $t = 0$, this derivative becomes
\[
\frac{d}{dt} \left( \text{dist}(x_i, L(t)) \right) \bigg|_{t=0} = \sqrt{\sum_{j=1}^{d} |(v_j \cdot x_i)|^2 \theta_j^2} = ||CVx_i||.
\]
(27)
Combining (25) and (27) and using
\[
A := \sum_{i=1}^{N_0} y_i^T y_i / \text{dist}(y_i, L_1),
\]
we obtain the following expression for the derivative of the $l_1$ energy of (2):
\[
\frac{d}{dt} \left( e_1(\mathcal{X}, L(t)) \right) \bigg|_{t=0} = \sum_{i=1}^{N_1} ||CVx_i|| - \text{tr}_k(CVAU^T).
\]
(28)
Since $V$ is a projection onto $L_1$ and $U$ is a projection onto $L_1^\perp$, we may rewrite this expression by the matrix $V \in O(d)$, whose $j$-th row is $P_{L_1}(v_j)^T$ and the matrix $U \in O(k, D - d)$, whose $j$-th row is $P_{L_1}^+(v_j)^T$:
\[
\frac{d}{dt} \left( e_1(\mathcal{X}, L(t)) \right) \bigg|_{t=0} = \sum_{i=1}^{N_1} ||CVP_{L_1}x_i|| - \text{tr}_k(C^\ast V^\ast B_{L_1}U^T).
\]
(29)
At last, we note that
\[
\max_{U^T} \text{tr}_k(C^\ast V^\ast B_{L_1}U^T) = ||C^\ast V^\ast B_{L_1}U^T||_1.
\]
Indeed, denoting the SVD decomposition of $C^\ast V^\ast B_{L_1}U^T$ by $U_0 \Sigma_0 V_0^T$ we have that
\[
\text{tr}_k(C^\ast V^\ast B_{L_1}U^T) = \text{tr}_k(U_0 \Sigma_0 V_0^T U_0^T) = \text{tr}_k(U_0 \Sigma_0 V_0^T U_0^T) \leq \sum (\text{diag}(\Sigma_0)) = ||C^\ast V^\ast B_{L_1}U^T||_1,
\]
and this equality can be achieved when $U^T$ consists of the first $k$ columns of $V_0 U_0^T$. The theorem is thus concluded by combing (29) and (30). \qed
9.2.2. Simultaneous Proof for Both Propositions 3.1 and 3.2

For the $d$-subspace $L_1$ and an arbitrary $d$-subspace $\hat{L} \in B(L_1, 1)$, we form the geodesic line parametrization $L(t)$ and the corresponding matrices $C, V, U, \hat{V}$ and $\hat{U}$ as in the proof of Theorem 3.1. Similarly to verifying (25) and (27) in the latter proof, we obtain that
\[
\frac{d}{dt} \left( \text{dist}(y, L(t))^p \right) \bigg|_{t=0} = -p \text{dist}(y, L_1)^{p-2} \text{tr}_k(CVy, y^T U^T) \tag{31}
\]
and
\[
\frac{d}{dt} \left( \text{dist}(x, L(t))^p \right) \bigg|_{t=0} = p \text{dist}(x, L_1)^{p-1} ||CVx||. \tag{32}
\]
Consequently
\[
\frac{d}{dt} \left( e_{t^p}(X, L(t)) \right) \bigg|_{t=0} = p \sum_{i=1}^{N_1} \text{dist}(x_i, L_1)^{p-1} ||CVx_i|| \tag{33}
\]
\[- p \sum_{i=1}^{N_0} \text{dist}(y_i, L_1)^{p-2} \text{tr}_k(CVy_i, y_i^T U^T) = p \sum_{i=1}^{N_1} \text{dist}(x_i, L_1)^{p-1} ||CV P_{L_1}(x_i)|| \]
\[- p \sum_{i=1}^{N_0} \text{dist}(y_i, L_1)^{p-2} \text{tr}_k(CV P_{L_1}(y_i) P_{L_1}^T(y_i)^T \hat{U}^T). \]

Assume first that $p < 1$. Then
\[
\frac{d}{dt} \left( e_{t^p}(X, L(t)) \right) \bigg|_{t=0} = pt^{1-p} \sum_{i=1}^{N_1} \text{dist}(x_i, L_1)^{p-1} ||CV P_{L_1}(x_i)|| \tag{34}
\]
\[- pt^{1-p} \sum_{i=1}^{N_0} \text{dist}(y_i, L_1)^{p-2} \text{tr}_k(CV P_{L_1}(y_i) P_{L_1}^T(y_i)^T \hat{U}^T) \]
\[= p \sum_{i=1}^{N_1} \left( \lim_{t \to 0} \text{dist}(x_i, L(t))/t \right)^{p-1} ||CV P_{L_1}(x_i)|| = \sum_{i=1}^{N_0} ||CV P_{L_1}(x_i)||. \]

It follows immediately from the definitions of $C$ and $V$ that
\[
||CVx_i|| \geq \theta_1 ||v_1^T x_i||. \tag{35}
\]

Now, the assumption $\text{Sp}(x_i x_j^{N_1}) = L_1$ implies that there exists $1 \leq j \leq N_1$ such that $v_1^T x_j \neq 0$ and thus $||CV P_{L_1}(x_i)|| = ||CVx_i|| > 0$. Therefore, (34) is positive, $L_1$ is a local minimum of $e_{t^p}(X, L(t))$ and Proposition 3.1 is proved.

Next, assume that $p > 1$ and note that
\[
p \sum_{i=1}^{N_1} \text{dist}(x_i, L_1)^{p-1} ||CV P_{L_1}x_i|| = 0. \tag{36}
\]
Since $L_1$ is a local minimum of $e_{l_p}(X, L)$, the derivative in (33) is nonnegative and in view of (36), the subtracted term in (33) is thus nonpositive. Now, for a subspace $L \in G(D, d)$ such that $C = V = I_d$ we obtain that

$$0 \geq \max_{D} \sum_{i=1}^{N_0} \text{dist}(y_i, L_1)^{p-2} \text{tr}_k(P_{L_1}(y_i)P_{L_1}^\perp(y_i)^T \hat{U}^T)$$

$$= p \left\| \sum_{i=1}^{N_0} \text{dist}(y_i, L_1)^{p-2} P_{L_1}(y_i)P_{L_1}^\perp(y_i)^T \right\|,$$

where the last equality follows from (30). Therefore, (11) holds and Proposition 3.2 is thus proved.

9.3. Theory of Section 4

9.3.1. Proof of Theorem 4.1

To find the probability that $L_1$ is a local $l_1$ subspace we will estimate the probabilities of large LHS and small RHS of (10) for arbitrary $L \in B(L_1, 1)$. We use the similar notation as in the proof of Theorem 3.1, in particular, we denote the $N_0$ outliers and $N_1$ inliers by $\{y_i\}_{i=1}^{N_0}$ and $\{x_i\}_{i=1}^{N_1}$ respectively. Due to the homogeneity of (10) in $C$, we will assume WLOG that $||C||_2 = 1$, i.e., $\theta_1 = 1$.

We start with estimating the probability that the RHS of (10) is small. Applying the above assumption that $||C||_2 = 1$ we have that

$$\|CVB_{L_1,x}\|_F \leq \|VB_{L_1,x}\|_F = \|B_{L_1,x}\|_F$$

and consequently

$$\Pr\left(\|CVB_{L_1,x}\|_2 < \epsilon\right) \geq \Pr\left(\|CVB_{L_1,x}\|_F < \epsilon\sqrt{d}\right)$$

$$\geq \Pr\left(\|B_{L_1,x}\|_F < \epsilon\sqrt{d}\right) \geq \Pr\left(\max_{p,l} \|B_{L_1,x}\|_{p,l} < \epsilon\sqrt{d}\right).$$

We further estimate this probability by Hoeffding’s inequality as follows: we view the matrix $B_{L_1,x}$ as the sum of random variables $P_{L_1}(y_i)P_{L_1}^\perp(y_i)^T/||P_{L_1}(y_i)||$, $i = 1, \ldots, N_0$. The coordinates of both $P_{L_1}(y_i)$ and $P_{L_1}^\perp(y_i)^T/||P_{L_1}(y_i)||$ take values in $[-1, 1]$ and their expectations are 0. We can thus apply Hoeffding’s inequality to the sum defining $B_{L_1,x}$ and consequently obtain that

$$\Pr\left(\max_{p,l} \|B_{L_1,x}\|_{p,l} < \epsilon \sqrt{d}\right) \geq 1 - 2dD \exp\left(-\frac{N_0 \epsilon^2}{2d^2 D}\right). \tag{37}$$

Next, we estimate the probability that the LHS of (10) is sufficiently large. For this purpose we make the following observations. First of all,
\[ \sum_{i=1}^{N_1} |x_i^T P_{L_1}(x_i)|^2 \geq \min_t \left( \sum_{i=1}^{N_1} P_{L_1}(x_i) P_{L_1}(x_i)^T \right). \]  

(38)

Second of all, as proved in Appendix A.5:

\[ E_{\mu_1}(P_{L_1}(x) P_{L_1}(x)^T) = \delta_* I_d, \text{ where } \delta_* = 1/(d+2). \]  

(39)

Last of all, as verified in Appendix A.6:

\[ \text{If } \max_t \sigma_t \left( \sum_{i=1}^{N_1} P_{L_1}(x_i) P_{L_1}(x_i)^T - \delta_* I_d \right) < \eta, \]  

(40)

\[ \text{then } \min_t \sigma_t \left( \sum_{i=1}^{N_1} P_{L_1}(x_i) P_{L_1}(x_i)^T \right) > \delta_* - \eta. \]  

(41)

We combine (38)-(40) and Hoeffding’s inequality to obtain the following probabilistic estimate for the LHS of (10):

\[ \Pr \left( \frac{\sum_{i=1}^{N_1} \|CV P_{L_1}(x_i)\|}{N_1} > \delta_* - \eta \right) \geq \Pr \left( \min_t \sigma_t \left( \frac{\sum_{i=1}^{N_1} P_{L_1}(x_i) P_{L_1}(x_i)^T}{N_1} \right) > \delta_* - \eta \right) \]  

(42)

\[ \geq \Pr \left( \max_t \sigma_t \left( \frac{\sum_{i=1}^{N_1} P_{L_1}(x_i) P_{L_1}(x_i)^T}{N_1} - \delta_* I_d \right) < \eta \right) \]  

\[ \geq \Pr \left( \left\| \frac{\sum_{i=1}^{N_1} P_{L_1}(x_i) P_{L_1}(x_i)^T}{N_1} - \delta_* I_d \right\|_{p,d} < \eta \right) \geq 1 - 2d^2 \exp \left( -\frac{N_1 \eta^2}{2d^2} \right). \]

From (37) and (42), (10) is valid with probability at least

\[ 1 - 2d^2 \exp \left( -\frac{N_1 \eta^2}{2d^2} \right) - 2dD \exp \left( -\frac{N_0 \epsilon^2}{2d^2 D} \right) \text{ for any } \epsilon, \eta \text{ such that } \eta + \frac{N_0}{N_1} \epsilon < \delta_. \]  

(43)

We can choose \( \epsilon = N_1 \delta_*/(2N_0) = N_1/(2N_0(d+2)), \eta = 1/(3(d+2)) \) and obtain that if \( N_0 = o(N_1^2) \) then (10) is valid with the probability specified in (12).

9.3.2. Proof of Theorem 4.2

We first prove that there exists a constant \( \gamma_1 > 0 \) such that w.o.p. \( L_1 \) is the best \( l_p \) subspace in \( B(L_1, \gamma_1) \). We arbitrarily choose \( \hat{L} \in G(D, d) \) such that \( \text{dist}(\hat{L}, L_1) = 1 \).
and parameterize a geodesic line from \( L_1 \) to \( \hat{L} \) by a function \( L : [0, 1] \to G(D, d) \), where \( L(0) = L_1 \) and \( L(1) = \hat{L} \). We then observe that there exists \( \gamma_1 > 0 \) such that the function \( e_\tau(x, L(t)) : [0, 1] \to \mathbb{R} \) of (2) has a positive derivative w.o.p. at any \( t \in [0, \gamma_1] \), that is,

\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{x \in X} \text{dist}(x, L(t))^p \right) > 0 \quad \text{for all } t \in [0, \gamma_1] \quad \text{w.o.p.} \tag{44}
\]

We will deduce (44) from the following two equations:

\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{x \in X} \text{dist}(x, L(t))^p \right) \bigg|_{t=0} > \gamma_2 \quad \text{w.o.p. for some } \gamma_2 > 0. \tag{45}
\]

and

\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{x \in X} \text{dist}(x, L(t))^p \right) \bigg|_{t=0} - \frac{d}{dt} \left( \frac{1}{N} \sum_{x \in X} \text{dist}(x, L(t))^p \right) \bigg|_{t=t_0} < \frac{\gamma_2}{2}, \tag{46}
\]

\( \forall t_0 \in [0, \gamma_1] \) w.o.p.

When \( 0 < p < 1 \), equation (45) follows from (34) and Hoeffding’s inequality. When \( p = 1 \), equation (45) practically follows from the proof of Theorem 4.1 by arbitrarily fixing \( \epsilon \) and \( \eta \) such that \( \epsilon \alpha_0 + \eta + \gamma_2/\alpha_1 < \delta_* \) and noting that when sampling from the mixture measure specified in the current theorem (unlike Theorem 4.1) the ratio of sampled outliers to inliers, \( N_0/N_1 \), goes w.o.p. to \( \alpha_0/\alpha_1 \). We also observe that

\[ \gamma_2 \equiv \gamma(\alpha_0, \alpha_1, d). \]

We first verify (46) for the sum of elements in \( X_1^* = X \cap L_1 \). In view of (26) and (34), for any \( x \in X_1^* \), the single term in that sum (i.e., \( \text{dist}(x, L(t))^p \)) has a bounded second derivative with respect to \( t \); hence, we can find constants \( \gamma_1 \) and \( \gamma_2 \) satisfying

\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{x \in X_1^*} \text{dist}(x, L(t))^p \right) \bigg|_{t=0} - \frac{d}{dt} \left( \frac{1}{N} \sum_{x \in X_1^*} \text{dist}(x, L(t))^p \right) \bigg|_{t=t_0} < \frac{\gamma_2}{6}, \tag{47}
\]

\( \forall t_0 \in [0, \gamma_1] \).

We derive a similar estimate by replacing the summation of \( x \in X_1^* \) by the summation of \( x \in X \backslash X_1^* \). Using the constant \( \gamma_3 \), which we clarify later, we separate the latter sum into two components: \( X' := \{ x \in X \backslash X_1^* : \text{dist}(x, L_1) \leq 2 \gamma_3 \} \) and \( (X \backslash X_1^*) \backslash X' \).

In order to deal with the first sum, we define

\[ \gamma_4 := \mu(x : 0 < \text{dist}(x, L_1) \leq 2 \gamma_3), \]

where we note that we can choose \( \gamma_3 \equiv \gamma_3(D, \gamma_2) \equiv \gamma_3(D, d, \alpha_0, \alpha_1) \) sufficiently small such that \( \gamma_4 \equiv \gamma_4(d, \alpha_0, \alpha_1) \leq \gamma_2/24 \). We use \( \gamma_4 \) to bound the ratio of sampled points from \( X' \) and \( X \) as follows:

\[
\frac{\#(X')}{\#(X)} \leq 2 \gamma_4 \leq \frac{\gamma_2}{12} \quad \text{w.o.p.} \tag{48}
\]
Indeed, we note that \( \#(\hat{\mathcal{X}}) = \sum_{x \in \mathcal{X}} I_{\hat{\mathcal{X}}}(x) \), \( E(I_{\hat{\mathcal{X}}}(x)) = \mu(x : x \in \hat{\mathcal{X}}) = \gamma_4 \) and \( I_{\hat{\mathcal{X}}}(x) \) takes values in \([0, 1]\), therefore by applying Hoeffding’s inequality to \( I_{\hat{\mathcal{X}}}(x) \), where \( x \in \mathcal{X} \), we conclude (48).

Now, the derivative expressed in (23) takes values in \([-1, 1]\) for any \( y_1 \in \hat{\mathcal{X}} \). Thus, by combining this observation with (48) we obtain that for any \( t_0 \in [0, \gamma_1] \):

\[
\left. \frac{d}{dt} \left( \frac{\sum_{x \in \hat{\mathcal{X}}} \text{dist}(x, L(t))^p}{N} \right) \right|_t = 0 - \left. \frac{d}{dt} \left( \frac{\sum_{x \in \hat{\mathcal{X}}} \text{dist}(x, L(t))^p}{N} \right) \right|_t = t_0 < 4\gamma_4 < \gamma_2 \tag{49} \]

w.o.p.

Differentiating (23) one more time, we obtain that for every \( x \in (\mathcal{X} \setminus \mathcal{X}_1) \setminus \hat{\mathcal{X}} \), the second derivative of \( \text{dist}(x, L(t)) \) is bounded by \( C(d)/\gamma_3^3 \). Thus we can choose \( \gamma_1 \equiv \gamma_1(\gamma_2, \gamma_3, d) \equiv \gamma_1(\alpha_0, \alpha_1, d, D) \) sufficiently small such that for any \( t_0 \in [0, \gamma_1] \):

\[
\left. \frac{d}{dt} \sum_{x \in (\mathcal{X} \setminus \mathcal{X}_1) \setminus \hat{\mathcal{X}}} \text{dist}(x, L(t))^p \right|_t = 0 - \left. \frac{d}{dt} \sum_{x \in (\mathcal{X} \setminus \mathcal{X}_1) \setminus \hat{\mathcal{X}}} \text{dist}(x, L(t))^p \right|_t = t_0 < \gamma_2 \tag{50} \]

Equation (46) and consequently (44) are thus verified by combing (47), (49) and (50). That is, we showed that \( L_1 \) is the best \( l_p \) subspace in \( B(L_1, \gamma_1) \) for sufficiently small \( \gamma_1 \).

At last, we will show that for all \( L \in G(D, d) \setminus B(L_1, \gamma_1) \) and any fixed \( p \leq 1 \), there exists some \( \gamma_7 > 0 \) such that

\[
e_{l_p}(\mathcal{X}, L) - e_{l_p}(\mathcal{X}, L_1) > \gamma_7 N, \text{ w.o.p.} \tag{51} \]

Indeed, we first conclude from Lemma 9.1 (applied with \( K = 1 \)) that

\[
E_{\mu} \left( e_{l_p}(x, L) \right) - E_{\mu} \left( e_{l_p}(x, L_1) \right) > \alpha_0 \left( E_{l_p} \left( e_{l_p}(x, L) \right) - E_{l_p} \left( e_{l_p}(x, L_1) \right) \right) + \alpha_1 \left( E_{l_p} \left( e_{l_p}(x, L) \right) - E_{l_p} \left( e_{l_p}(x, L_1) \right) \right) > 0 \tag{52} \]

Setting \( \gamma_7 = \alpha_1 \gamma^p / (2^{1 + p} d^{d + 1}) \) and combining (52) with Hoeffding’s inequality, we obtain (51).

Now, (51) extends for a small neighborhood of \( L \). That is, for any \( L \in G(D, d) \) we can find a ball \( B(L, t) \) for some \( t > 0 \) such that w.o.p. the subspace \( L_1 \) is a better \( l_p \) subspace than any of the subspaces in that ball. By covering the compact space \( G(D, d) \setminus B(L_1, \gamma_1) \) with finite number of such balls we obtain that w.o.p. \( L_1 \) is the best \( l_p \) subspace in \( G(D, d) \setminus B(L_1, \gamma_1) \). Combining this observation with the first part of the proof, we conclude that w.o.p. \( L_1 \) is the best \( l_p \) subspace in \( G(D, d) \).

\[ \square \]

### 9.4. Theory Presented in the Introduction (Section 1)

#### 9.4.1. Proof of Theorem 1.1

Several ideas of this proof have already appeared when verifying Theorem 4.2. We will thus maintain the same notation, in particular for denoting similar constants.
As in proving the latter theorem, we will first prove the theorem locally. That is, we will show that w.o.p. $L_1$ is a best $l_p$ subspace in the ball $B(L_1, \gamma_1)$, where $\gamma_1$ is a sufficiently small constant.

In order to do so, we arbitrarily fix $\hat{L} \in G(D, d)$ such that $\text{dist}(\hat{L}, L_1) = 1$ (so that $C \in \text{NS}_+(d)$) and parameterize a geodesic line from $L_1$ to $\hat{L}$ by a function $L: [0, 1] \to G(D, d)$, where $L(0) = L_1$ and $L(1) = \hat{L}$. We will then estimate the probability that for any such $\hat{L}$ the function $e_{L_1}(\mathcal{X}, L(t))$: $[0, 1] \to \mathbb{R}$ has a positive derivative at any $t \in (0, \gamma_1)$, that is

$$\frac{d}{dt} \left( \frac{\sum_{x \in \mathcal{X}} \text{dist}(x, L(t))^p}{N} \right) > 0 \quad \text{for all } t \in (0, \gamma_1). \quad (53)$$

First of all, we estimate the probability that the LHS of (53) is larger than some constant $\gamma_2 > 0$ at $t = 0$. When $0 < p < 1$, it follows from (34) and Hoeffding’s inequality that this probability is overwhelming. When $p = 1$, it follows from (10) that this probability is the same as the probability of the event

$$\sum_{x \in \mathcal{X}_1} \frac{||CV_{P_{L_1}}(x)|| - ||CV_{B_{L_1}, \mathcal{X} \setminus \mathcal{X}_1}||}{N} > \gamma_2$$

$$\forall C \in \text{NS}_+(d) \text{ and } V \in O(d). \quad (54)$$

We notice that for all $C \in \text{NS}_+(d)$ and $V \in O(d)$:

$$||CV_{B_{L_1}, \mathcal{X} \setminus \mathcal{X}_1}|| = ||CV \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} P_{L_1}(x) P_{L_1}^+(x) \|/\text{dist}(x, L_1)||_* \\
\leq \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||CV_{P_{L_1}}(x) P_{L_1}^+(x) \|/||P_{L_1}^+(x)|| \|_* \leq \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||CV_{P_{L_1}}(x)||.$$

Consequently, in order to estimate the probability of (54) it is sufficient to estimate the probability that

$$\frac{\sum_{x \in \mathcal{X}_1} ||CV_{P_{L_1}}(x)|| - \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||CV_{P_{L_1}}(x)||}{N} > \gamma_2 \forall C \in \text{NS}_+(d) \text{ and } V \in O(d). \quad (55)$$

We arbitrarily fix $C_0 \in \text{NS}_+(d)$, $V_0 \in O(d)$ and verify (55) by Hoeffding’s inequality in the following way. We define the random variable $J(x) = (2I(x \in \mathcal{X}_1) - 1)||C_0 V_0 P_{L_1}(x)||$ and note that

$$E_\mu(J(x)) = E_\mu \left( \frac{\sum_{x \in \mathcal{X}_1} ||C_0 V_0 P_{L_1}(x)|| - \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||C_0 V_0 P_{L_1}(x)||}{N} \right) \quad \text{and note that}$$

$$E_\mu(J(x)) = E_\mu \left( \sum_{x \in \mathcal{X}_1} ||C_0 V_0 P_{L_1}(x)|| - \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||C_0 V_0 P_{L_1}(x)|| \right) \quad (56)$$

$$= \alpha_1 E_{\mu_1} ||C_0 V_0 P_{L_1}(x)|| - \alpha_0 E_{\mu_0} ||C_0 V_0 P_{L_1}(x)|| - \sum_{j=2}^{K} \alpha_j E_{\mu_j} ||C_0 V_0 P_{L_1}(x)|| \geq \alpha_1 E_{\mu_1} ||C_0 V_0 P_{L_1}(x)|| - \sum_{j=2}^{K} \alpha_j E_{\mu_j} ||C_0 V_0 P_{L_1}(x)|| = \beta_0 E_{\mu_1} ||C_0 V_0 P_{L_1}(x)||,$$
where \( \beta_0 = \alpha_1 - \sum_{j=2}^{K} \alpha_j \).

Now, let \( \gamma_2 := \beta_0 E_{\mu_1} ||C_0 V_0 P_{L_1}(x)||/4 \), so that the random variable \( J(x) \) has expectation larger than \( 4\gamma_2 \) while taking values in \([−1, 1]\); thus by Hoeffding’s inequality:

\[
\sum_{x \in \mathcal{X}_1} \frac{||C_0 V_0 P_{L_1}(x)|| - \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||C_0 V_0 P_{L_1}(x)||}{N} > 2\gamma_2 \quad (57)
\]

w.p. \( \geq 1 - \exp(-2N\gamma_2^2) \).

We have thus proved that (54) is valid with sufficiently high probability for fixed matrices \( C_0 \in NS_+(d) \) and \( V_0 \in O(d) \). Next we estimate the probability of (54) for all matrices \( C \in NS_+(d) \) and \( V \in O(d) \), when restricted to a ball with sufficiently small radius. We let

\[
\text{dist}((C_1, V_1), (C_2, V_2)) := \max(||C_1 - C_2||_2, ||V_1 - V_2||_2) \quad (58)
\]

and note that whenever \( \text{dist}((C_1, V_1), (C_2, V_2)) < \gamma_2/2 \) and \( x \in B(0, 1) \) we have that

\[
||C_1 V_1 P_{L_1}(x)|| - ||C_2 V_2 P_{L_1}(x)|| = \langle ||C_1 V_1 P_{L_1}(x)|| - ||C_2 V_2 P_{L_1}(x)|| \rangle + (||C_2 V_1 P_{L_1}(x)|| - ||C_2 V_2 P_{L_1}(x)||)
\]

\[
\leq ||C_1 - C_2||_2 + ||C_2||_2 ||V_1 - V_2||_2 \leq \gamma_2. \quad (59)
\]

Combining (57) and (59) we obtain that for any ball in \( G(D, d) \) of radius \( \gamma_2/2 \) and center \((C_0, V_0)\):

\[
\sum_{x \in \mathcal{X}_1} \frac{||CV P_{L_1}(x)|| - \sum_{x \in \mathcal{X} \setminus \mathcal{X}_1} ||CV P_{L_1}(x)||}{N} > \gamma_2 \quad \text{w.p.} \quad 1 - \exp(-2N\gamma_2^2). \quad (60)
\]

We easily extend (60) for all pairs of matrices \((C, V)\) in the compact space \( NS_+ \times O(d) \) (with the distance specified in (58)). Indeed, it follows from (40) together with some basic estimates that the latter space can be covered by \( C_{1(d+1)/2} \gamma_2^{d(d+1)/2} \) balls of radius \( \gamma_2/2 \). Therefore,

\( (54) \) is valid for any \( C \in NS_+(d) \) and \( V \in O(d) \) \( \text{w.p.} \ 1 - C_{1(d+1)/2} \gamma_2^{d(d+1)/2} \gamma_2 \).

Equation (53) follows w.o.p. from (54) in exactly the same way of deriving (44) from (45) and (46). We remark that (46), which is deterministic, easily extends to the current case. While we did not estimate the overwhelming probability for (44), it is easy to show that in the current case, (54) implies (53) w.p. \( 1 - \exp(-N\gamma_8)/\gamma_8 \). Carrying this analysis, one notices that both \( \gamma_1 \) and \( \gamma_8 \) depend on \( p, d, K, \alpha_0, \alpha_1 \) and \( \min_{2 \leq i \leq K} (\text{dist}(L_1, L_i)) \). Combining this with (61), we obtain that

\( L_1 \) is a best \( l_p \) subspace in \( B(L_1, \gamma_1) \) w.p. \( 1 - C_{1(d+1)/2} \gamma_2^{d(d+1)/2} \gamma_2 \) - \( \exp(-N\gamma_4)/\gamma_4 \).
We have just proved that $L_1$ is a best $l_p$ subspace w.o.p. in $B(L_1, \gamma_1)$. We now extend this result to subspaces in $G(D, d) \setminus B(L_1, \gamma_1)$. Applying Lemma \ref{lem:9.3} we obtain that
\begin{equation}
E_{\mu_i} ((\text{dist}(x, L) - \text{dist}(x, L_1))^p + E_{\mu_i} ((\text{dist}(x, L) - \text{dist}(x, L_1))^p) \geq 0 \quad \text{for all } 2 \leq i \leq K.
\end{equation}

Further application of Lemma \ref{lem:9.1} with $L \in G(D, d) \setminus B(L_1, \gamma_1)$, results in the inequality:
\begin{equation}
E_{\mu_i} ((\text{dist}(x, L))^p) > \frac{\gamma_1^p}{2^{1+p}d + \gamma_1}. \quad \text{for all } 2 \leq i \leq K.
\end{equation}

Now, combining (63) and (64) we have that
\begin{align*}
E_{\mu_i} ((\text{dist}(x, L))^p - \text{dist}(x, L_1)^p) &\leq \sum_{i=2}^{K} \alpha_i (E_{\mu_i} ((\text{dist}(x, L))^p - \text{dist}(x, L_1)^p) + E_{\mu_i} ((\text{dist}(x, L)^p - \text{dist}(x, L_1)^p)) \\
&\quad + \beta_0 E_{\mu_i} ((\text{dist}(x, L))^p - \text{dist}(x, L_1)^p) \\
&\geq 0 + \beta_0 E_{\mu_i} ((\text{dist}(x, L))^p) \geq \gamma_9 = \frac{\beta_0 \gamma_1^p}{2^{1+p}d + \gamma_1},
\end{align*}

where $\gamma_9$ depends on $d$, $K$, $p$, $\alpha_0$, $\alpha_1$ and $\min_{2 \leq i \leq K} \text{dist}(L_1, L_i)$. Noting further that $\text{dist}(x, L) - \text{dist}(x, L_1)$ takes values in $[{-1, 1}]$ and applying Hoeffding’s inequality we obtain that for any $L \in G(D, d) \setminus B(L_1, \gamma_1)$:
\begin{equation}
equality(\text{dist}(x, L) - \text{dist}(x, L_1)) > \gamma_9 N / 2 \text{ w.p. } 1 - \exp(-N \gamma_9^2 / 8).
\end{equation}

By Lemma \ref{lem:9.2} we obtain that for any $L' \in G(D, d)$ satisfying $\text{dist}(L, L') < (\gamma_9 / 4)^2$ and any $x \in B(0, 1)$:
\begin{equation}
|\text{dist}(x, L')^p - \text{dist}(x, L)^p| < \gamma_9 / 4.
\end{equation}

Consequently, for any $L \in G(D, d) \setminus B(L_1, \gamma_1)$ and all $L' \in B(L, (\gamma_9 / 4)^{2})$:
\begin{equation}
\text{dist}(x, L') - \text{dist}(x, L_1) > 0 \text{ w.p. } 1 - \exp(-N \gamma_9^2 / 8).
\end{equation}

Following [Remark 8.4][39] we can cover $G(D, d) \setminus B(L_1, \gamma_1)$ by $C_{\epsilon}^{2(d-d)} / \gamma_9^{d(D-d)}$ balls of radius $\epsilon$. Now, for each such ball we have that (65) is valid for its center w.p. $1 - \exp(-N \gamma_9^2 / 8)$ and consequently (66) is valid for subspaces in that ball with the same probability. We thus conclude that (66) is valid for all $L' \in G(D, d) \setminus B(L_1, \gamma_1)$ w.p. $1 - \exp(-N \gamma_9^2 / 8) C_{\epsilon}^{2(d-d)} / \gamma_9^{d(D-d)}$. Combining this with (62), we obtain that the probability that $L_1$ is a best $l_p$ subspace in $G(D, d)$ is
\begin{equation}
1 - C_1^d \exp(-2N \gamma_9^2 / (\gamma_4 / 2)^d - \exp(-N \gamma_4 / \gamma_4 - \exp(-N \gamma_9^2 / 8) C_{\epsilon}^{2(d-d)} / \gamma_9^{d(D-d)}),
\end{equation}
or equivalently, $1 - C \exp(-N/C)$ for some $C$ depending on $D$, $d$, $K$, $p$, $\alpha_0$, $\alpha_1$ and $\min_{2 \leq i \leq K} \text{dist}(L_1, L_i)$. 

\hfill \Box
9.4.2. Proof of Theorem 1.2

We will prove the theorem for \( p = 1 \) only, since its extension to \( 0 < p < 1 \) follows from the proof of Theorem 4.2. Other parts of the proof will also be shortened due to their similarity to the proofs of Theorems 1.1 and 4.2.

Throughout the proof we view the energy \( e_{i_1}(\mathcal{X}, L_1, L_2, \ldots, L_K) \) as a function defined on \( G(D, d)^K \) while being conditioned on the fixed data set \( \mathcal{X} \). On the other hand we view \( e_{i_1}(x, L_1, L_2, \ldots, L_K) \) as a function on \( \mathbb{R}^D \times G(D, d)^K \). We distinguish elements in \( G(D, d)^K \) by the \( l_\infty \) norm on the product space, that is

\[
\text{dist}(\{L_1, L_2, \ldots, L_K\}, (\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_K)) = \max_{i=1,\ldots,K} (\text{dist}(L_i, \hat{L}_i)).
\]

We note that it is enough to prove that the set \( \{L_1, L_2, \ldots, L_K\} \) minimizes w.o.p. the energy \( e_{i_1}(\mathcal{X}_0 \cup \mathcal{X}_1, L_1, L_2, \ldots, L_K) \), where \( \{\mathcal{X}_i\}_{i=0}^K \) have been defined in Section 2.1. Indeed, it follows from the immediate observation:

\[
e_{i_1} \left( \sum_{i=2}^K \mathcal{X}_i, L_1, L_2, \ldots, L_K \right) = 0.
\]

We will first show that there exists a constant \( \gamma_1 > 0 \) such that the set \( \{L_1, L_2, \ldots, L_K\} \) is a minimizer w.o.p. of \( e_{i_1}(\mathcal{X}_0 \cup \mathcal{X}_1, L_1, L_2, \ldots, L_K) \) in \( B((L_1, L_2, \ldots, L_K), \gamma_1) \). In order to simplify notation in this part of the proof, we will adopt WLOG the convention that the RHS of (67) occurs at \( i = 1 \), i.e.,

\[
\text{dist}(L_1, \hat{L}_1) = \max_{i=1,\ldots,K} (\text{dist}(L_i, \hat{L}_i)).
\]

For all \( 1 \leq i \leq k \), we parameterize the geodesic lines from \( L_i \) to \( \hat{L}_i \), where

\[
\text{dist} \left( \{L_1, L_2, \ldots, L_K\}, (\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_K) \right) = 1,
\]

by functions \( L_i(t) \) on the interval \( [0, \text{dist}(L_1, \hat{L}_1)] \) such that \( L_i(0) = L_i \) and \( L_i(\text{dist}(L_1, \hat{L}_1)) = \hat{L}_i \). Applying Lemma 9.2 and assuming \( j = \arg\min_{1 \leq i \leq K} \text{dist}(x, L_i) \), we derive the following estimate:

\[
\frac{d}{dt} (e_{i_1}(x, L_1(t), L_2(t), \ldots, L_K(t))) \bigg|_{t=0} = \lim_{t \to 0} \frac{\text{dist}(x, L_j(t)) - \text{dist}(x, L_j(0))}{t} \geq -\|x\| \lim_{t \to 0} \frac{\text{dist}(L_j(t), L_j(0))}{t} = -\|x\| \text{dist}(L_j(1), L_j(0)) \geq -\|x\|.
\]

Combining (69) with Hoeffding’s inequality, we obtain that

\[
\frac{d}{dt} (e_{i_1}(x_0, L_1(t), L_2(t), \ldots, L_K(t))) \bigg|_{t=0} \geq -\sum_{x \in \mathcal{X}_0} \|x\| \geq -\alpha_0 N \text{ w.o.p.} \tag{70}
\]

Now, following the arguments of the proof of (27), we conclude the equality:

\[
\frac{d}{dt} (e_{i_1}(x_1, L_1(t), L_2(t), \ldots, L_K(t))) \bigg|_{t=0} = \sum_{x \in \mathcal{X}_1} \|CVP_L(x)\| \text{ w.o.p.,} \tag{71}
\]
where $C \in \text{NS}_+(d)$ and $V \in O(d)$ as in the latter proof. Thus, by Hoeffding’s inequality, there exists $\lambda_1 \equiv \lambda_1(d) > 0$ such that
\[
\frac{d}{dt} \left( e_{t_1}(\mathcal{X}_1, L_1(t), L_2(t), \ldots, L_K(t)) \right) \bigg|_{t=0} \geq \alpha_1 \lambda_1 N \text{ w.o.p.} \quad (72)
\]
Using this constant $\lambda_1$, we set
\[
\nu_0 := \min \left( \lambda_1, \frac{1}{4Kd^2} \right). \quad (73)
\]
It follows from (4) and (73) that $\alpha_1 \lambda_1 - \alpha_0 > 0$. Now, combining (70) and (72) we obtain that there exists a constant $0 < \gamma_2 < \alpha_1 \lambda_1 - \alpha_0$ such that
\[
\frac{d}{dt} \left( e_{t_1}(\mathcal{X}_0 \cup \mathcal{X}_1, L_1(t), L_2(t), \ldots, L_K(t)) \right) \bigg|_{t=0} \geq \gamma_2 N \text{ w.o.p.} \quad (74)
\]
Consequently, $\{L_1, L_2, \ldots, L_K\}$ is a minimizer w.o.p. of $e_{t_1}$ in the ball $B((L_1, L_2, \ldots, L_K), \gamma_1)$. Since $e_{t_1}$ is symmetric on $G(D, d, \gamma_1)$, it is also the minimizer w.o.p. of $e_{t_1}$ in $\cup_{i_1, i_2, \ldots, i_K \in P_K} B((L_{i_1}, L_{i_2}, \ldots, L_{i_K}), \gamma_1)$, where $P_K$ is the set of all permutations of $(1, 2, \ldots, K)$.

Next, we note that the set $\{L_1, L_2, \ldots, L_K\}$ is also a global minimizer outside this ball, that is, for any $\hat{L} = (\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_K) \in G(D, d, \gamma_1)$
\[
\hat{L} = G(D, d)^K \setminus \cup_{i_1, i_2, \ldots, i_K \in P_K} B((L_{i_1}, L_{i_2}, \ldots, L_{i_K}), \gamma_1) \quad (75)
\]
\[
eq e_{t_1}(\mathcal{X}, \hat{L}_1, \hat{L}_2, \ldots, \hat{L}_K) - e_{t_1}(\mathcal{X}, L_1, L_2, \ldots, L_K) > C_2 N \text{ w.o.p.} \quad (76)
\]
Indeed, (76) follows by choosing $\gamma_1 < d_0$ (where $d_0$ was defined in (20)) and combining (73), Hoeffding’s inequality, Lemma 9.4 and the assumption specified in (4).

In order to conclude the theorem we extend (76) w.o.p. for all $K$ subspaces in the set $G(D, d, \gamma_1)$ defined in (75) (and not for a fixed subspace in that set). This is done as in the proof of Theorem 4.2 by covering $G(D, d, \gamma_1)$ with balls and similarly concluding that $L_1, L_2, \ldots, L_K$ and any of its permutations minimizes $e_{t_1}(\mathcal{X}_0 \cup \mathcal{X}_1, L_1, L_2, \ldots, L_K)$ and consequently $e_{t_1}(\mathcal{X}, L_1, L_2, \ldots, L_K)$ w.o.p.

\[ \square \]

9.5. Theory of Section 5

9.5.1. Reduction of Theorem 5.1 and Theorem 5.2

We first explain how to reduce the proof of Theorem 5.1 when $0 < p \leq 1$ to the verification of a simpler statement. We then adapt this idea for proving the same theorem when both $p > 1$ and $K = 1$, as well as for proving Theorem 5.2.
In order to prove Theorem 5.1 when $0 < p \leq 1$, i.e., prove that the global minimum of $e_{l_p}(X, L)$ is in $B(L_1, f)$ w.o.p., we only need to show that there exists a constant $\gamma_1 > 0$ such that for any $L \notin B(L_1, f)$:

$$E_{\mu_\epsilon}(e_{l_p}(x, L)) > E_{\mu_\epsilon}(e_{l_p}(x, L_1)) + \gamma_1.$$  

(77)

Indeed, we cover the compact space $\mathcal{G}(D, d) \setminus B(L_1, f)$ by small balls with radius $\gamma_1/2$. Then by using (77) and Hoeffding’s inequality, we obtain that $e_{l_p}(X, L) > e_{l_p}(X, L_1)$ for any $L$ in each such ball w.o.p. Therefore, $e_{l_p}(X, L) > e_{l_p}(X, L_1)$ for $L \in \mathcal{G}(D, d) \setminus B(L_1, f)$ w.o.p. Equivalently, $\mathcal{G}(D, d) \setminus B(L_1, f)$ does not contain the global minimum of $e_{l_p}(X, L)$ w.o.p.

For $i = 1, \ldots, K$, let $\tilde{\mu}_{i, \epsilon}$ be the measure obtained by projecting $\mu_{i, \epsilon}$ onto its corresponding subspace $L_i$ (that is, for any set $E \subseteq B(0, 1) \cap L_i$: $\tilde{\mu}_{i, \epsilon}(E) = \mu_{i, \epsilon}(P_{L_i}^{-1}(E))$). We also let $\tilde{\mu}_\epsilon := \alpha_0\mu_0 + \sum_{i=1}^K \alpha_i\tilde{\mu}_{i, \epsilon}(E)$. By the triangle inequality:

$$|E_{\mu_\epsilon}(e_{l_p}(x, L)) - E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, L))| < \epsilon^p.$$  

Hence, in order to prove (77) and thus Theorem 5.1 for $p \leq 1$, the following equation is sufficient:

$$E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, L)) > E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, L_1)) + \gamma_1 + 2\epsilon^p, \quad \text{for any } L \in \mathcal{G}(D, d) \setminus B(L_1, f).$$  

(78)

Similarly, we reduce Theorem 5.1 when $K = 1$ and $p > 1$ to the following condition:

$$E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, L)) > E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, L_1)) + \gamma_1 + 2p\epsilon, \quad \text{for any } L \in \mathcal{G}(D, d) \setminus B(L_1, f).$$  

(79)

Indeed, we note that for any $x_1, x_2 \in B(0, 1)$ with $\text{dist}(x_1, x_2) < \eta < 1$ and any $\bar{L}_1, \bar{L}_2 \in \mathcal{G}(D, d)$ with $\text{dist}(\bar{L}_1, \bar{L}_2) < \eta$:

$$\text{dist}(x_1, \bar{L}_1)^p - \text{dist}(x_2, \bar{L}_1)^p < 1 - (1 - \eta)^p < p\eta,$$  

(80)

and

$$\text{dist}(x_1, \bar{L}_1)^p - \text{dist}(x_1, \bar{L}_2)^p < 1 - (1 - \eta)^p < p\eta.$$  

(81)

When $p = 1$, (80) follows from the triangle inequality and (81) follows from Lemma 9.2, whereas both equations extend to $p > 1$ by the following property of the $p$-th power: if $0 \leq y_1, y_2 \leq 1$, $y_1 - y_2 < \eta$ and $p > 1$, then $y_1^p - y_2^p < 1 - (1 - \eta)^p$.

Following a similar argument, we reduce the verification of Theorem 5.2 to proving that for all permutations $i_1, i_2, \cdots, i_K \in \mathcal{P}_K$ with $\text{dist}(\{L_{i_1}, L_{i_2}, \cdots, L_{i_K}\}, \{L_1, \bar{L}_2, \cdots, \|L_K\|\}) > f$:

$$E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, \bar{L}_1, \cdots, \bar{L}_K)) > E_{\tilde{\mu}_\epsilon}(e_{l_p}(x, L_1, \cdots, L_K)) + \gamma_1 + 2\epsilon^p.$$  

(82)

We conclude with the proofs of (78), (79) and (82).
9.5.2. Proof of (78) and (79) and conclusion of Theorem 5.1

We arbitrarily fix \( L \in G(D, d) \setminus B(L_1, f) \). We assume first that \( 0 < p \leq 1 \) and apply Lemma 9.3 to obtain that

\[
E_{\tilde{\mu}_e} e_l_p(x, L) - E_{\tilde{\mu}_e} e_l_p(x, L_1) = \sum_{i=2}^{K} \alpha_i (E_{\tilde{\mu}_1} e_l_p(x, L) - E_{\tilde{\mu}_1} e_l_p(x, L_1)) \geq 0.
\]

Consequently, we prove (78) with \( \gamma_1 := 2e^p \) as follows:

\[
E_{\tilde{\mu}_e} e_l_p(x, L) - E_{\tilde{\mu}_e} e_l_p(x, L_1) \geq \left( \alpha_1 - \sum_{i=2}^{K} \alpha_i \right) f^p \geq 4e^p,
\]

where the second inequality applies Lemma 9.1.

Equation (79) follows from the same argument of (83), where \( e^p \) is now replaced by \( pe \).

**Proof of (82) and conclusion of Theorem 5.2**

In view of Lemma 9.4 it is sufficient to prove that

\[
\left( \tau_0 \min_{1 \leq j \leq K} \alpha_j - \alpha_0 \right) f^p > \gamma_1 + 2e^p
\]

and

\[
\tau_0 \min_{1 \leq j \leq K} \alpha_j \min_{1 \leq i,j \leq K} \text{dist}^p(L_i, L_j)/2^p - \alpha_0 > \gamma_1 + 2e^p.
\]

Setting \( \gamma_1 = e^p \), (84) follows from (17) and (85) follows from (16).

\[ \square \]

9.6. Theory of Section 6

9.6.1. Reduction of Theorem 6.1

We explain how to reduce Theorem 6.1. We use the same notation of Section 9.5.1, in particular, \( \tilde{\mu}_e \).

Theorem 6.1 states that the best \( l_p \) subspace is not in \( B(L_1, \kappa_0) \) w.o.p. for almost every \( \{L_i\}_{i=1}^K \in G(D, d)^K \). We claim that it reduces to the following simple equation:

\[
\gamma_{D, d}^L \left( \{L_i\}_{i=1}^K \subset G(D, d) : L_1 = \arg\min_{L} E_{\tilde{\mu}_e} e_l_p(x, L) \right) = 0.
\]
Indeed, if (86) is not satisfied, then for any $K$-subspaces $\{L_i\}_{i=1}^K$ in a subset of $G(D,d)^K$ with nonzero $\gamma_{D,d}^K$ measure there exists $L_0 \in G(D,d)$ such that

$$\gamma_1 := E_{\hat{\mu}_s}(e_{l_p}(x, L_1)) - E_{\hat{\mu}_s}(e_{l_p}(x, L_0)) > 0.$$ 

Letting $\delta_0 = \kappa_0 = \gamma_1/4p\epsilon$, we obtain from (80) and (81) that for any $L^* \in B(L_1, \kappa_0)$:

$$E_{\hat{\mu}_s}(e_{l_p}(x, L^*)) - E_{\hat{\mu}_s}(e_{l_p}(x, L_0)) > E_{\hat{\mu}_s}(e_{l_p}(x, L_0)) - 2\delta_0 p$$

$$> E_{\hat{\mu}_s}(e_{l_p}(x, L_1)) - E_{\hat{\mu}_s}(e_{l_p}(x, L_0)) - 2\delta_0 p - \kappa_0 p = \frac{\gamma_1}{4}.$$ 

Therefore, by Hoeffding’s inequality:

$$e_{l_p}(X, L^*) - e_{l_p}(X, L_0) > \frac{\gamma_1 N}{8} \text{ w.o.p.}$$

In order to have

$$e_{l_p}(X, L^*) - e_{l_p}(X, L_0) > 0 \quad \text{for all } L^* \in B(L_1, \kappa_0) \text{ w.o.p.,}$$

we cover $B(L_1, \kappa_0)$ by small balls with radius $\gamma_1/16$, so that $e_{l_p}(X, L) > e_{l_p}(X, L_0)$ for all $L$ in each such ball w.o.p. Therefore, $e_{l_p}(X, L) > e_{l_p}(X, L_0)$ for all $L \in B(L_1, \kappa_0)$ w.o.p. Equivalently, $B(L_1, \kappa_0)$ will not contain the global minimum of $e_{l_p}(X, L)$ w.o.p. This contradicts Theorem 6.1 and therefore (86) implies this theorem.

9.6.2. Proof of (86) and conclusion of Theorem 6.1

In view of Proposition 3.2, we only need to prove that

$$\gamma_{D,d}^K (\{L_i\}_{i=1}^K \subset G(D,d) : E_{\hat{\mu}_s}(D_{L_1,x,p}) = 0) = 0,$$  

(87)

where $D_{L_1,x,p}$ is the operator defined in (6). Using the notation

$$h(L_1, L_i) = E_{\hat{\mu}_s}(D_{L_1,x,p}) \quad 2 \leq i \leq K,$$

we rewrite (87) as follows:

$$\gamma_{D,d}^K (\{L_i\}_{i=1}^K \subset G(D,d) : E_{\hat{\mu}_s}(D_{L_1,x,p}) = 0)$$

$$= \gamma_{D,d}^K (\{L_i\}_{i=1}^K \subset G(D,d) : E_{\sum_{i=2}^K \alpha_i \hat{\mu}_s}(D_{L_1,x,p}) = 0)$$

$$= \gamma_{D,d}^K \left( \{L_i\}_{i=1}^K \subset G(D,d) : \sum_{i=2}^K \alpha_i h(L_1, L_i) = 0 \right) = 0.$$  

(88)

Since $\{L_i\}_{i=1}^K$ are independently distributed according to $\gamma_{D,d}$, Fubini’s theorem implies that (88) follows from the equation:

$$\gamma_{D,d} (L_2 \in G(D,d) : h(L_1, L_2) = C(L_1, L_3, \cdots, L_K)) = 0,$$  

(89)

where $C(L_1, L_3, \cdots, L_K) = - \sum_{i=3}^K \alpha_i h(L_1, L_i)/\alpha_2$. 

We follow by proving (89) and consequently concluding (86). We denote the principal angles between $L_2$ and $L_1$ by $\{\theta_j\}_{j=1}^d$, the principal vectors of $L_2$ and $L_1$ by $\{\mathbf{v}_j\}_{j=1}^d$ and $\{\mathbf{v}_j\}_{j=1}^d$ respectively and the complementary orthogonal system for $L_2$ w.r.t. $L_1$ by $\{\mathbf{u}_j\}_{j=1}^d$. Note that as an operator, $h(L_1, L_2)$ maps $\text{Sp}(\{\mathbf{v}_i\}_{i=1}^d)$ to $\text{Sp}(\{\mathbf{v}_i\}_{i=1}^d)$. Now, transforming $\mathbf{x} \in L_2 \cap B(0, 1)$ to $\{\mathbf{a}_i\}_{i=1}^d$ in a $d$-dimensional unit ball by $\mathbf{x} = \sum_{i=1}^d a_i \mathbf{v}_i$, we have that for any $1 \leq i_1, i_2 \leq d$:

$$v_{i_1}^T h(L_1, L_2) u_{i_2} = E_{\mu_2}(v_{i_1}^T P_{L_1}(\mathbf{x}) P_{L_2}^\perp(\mathbf{x})^T u_{i_2} \text{dist}(\mathbf{x}, L_1)^{p-2})$$

$$= \int_{\sum_{i=1}^d a_i^2 \leq 1} \cos \theta_{i_1} a_{i_1} \sin \theta_{i_2} a_{i_2} \left( \sum_{i=1}^d a_i^2 \sin^2 \theta_i \right)^{p-2} dV,$$

where $dV$ denotes the scaled volume element on the $d$-dimensional ball $\sum_{i=1}^d a_i^2 \leq 1$. When $i_1 \neq i_2$, the function

$$\cos \theta_{i_1} a_{i_1} \sin \theta_{i_2} a_{i_2} \left( \sum_{i=1}^d a_i^2 \sin^2 \theta_i \right)^{p-2}$$

is odd w.r.t. $a_{i_1}$ and consequently

$$v_{i_1}^T h(L_1, L_2) u_{i_2} = \int_{\sum_{i=1}^d a_i^2 \leq 1} \cos \theta_{i_1} a_{i_1} \sin \theta_{i_2} a_{i_2} \left( \sum_{i=1}^d a_i^2 \sin^2 \theta_i \right)^{p-2} = 0.$$ 

Therefore, when we form $V$ and $U$ as in (25), the $d \times d$ matrix

$$V \ E_{\mu_2}(P_{L_1}(\mathbf{x}) P_{L_2}^\perp(\mathbf{x})^T \text{dist}(\mathbf{x}, L_1)^{p-2}) U^T$$

is diagonal with the elements

$$\int_{\sum_{i=1}^d a_i^2 \leq 1} \cos \theta_j a_j^2 \left( \sum_{i=1}^d a_i^2 \sin^2 \theta_i \right)^{p-2}, \quad j = 1, \ldots, d.$$ 

We denote

$$\lambda_j(h(L_1, L_2)) = \int_{\sum_{i=1}^d a_i^2 \leq 1} \cos \theta_j a_j^2 \left( \sum_{i=1}^d a_i^2 \sin^2 \theta_i \right)^{p-2}, \quad j = 1, \ldots, d,$$

where we note that $\{\lambda_i(h(L_1, L_2))\}_{i=1}^d$ are the singular values of $h(L_1, L_2)$. We arbitrarily fix $L_1, L_3, L_4, \ldots, L_K$ and denote the singular values of $C \equiv C(L_1, L_3, L_4, \ldots, L_K)$ by $\{\sigma_i\}_{i=1}^D$ and note that (89) is implied by the following equation:

$$\gamma_{D,d}(L_2 \in G(D,d) : \lambda_1(h(L_1, L_2)) \in \{\sigma_i\}_{i=1}^D) = 0,$$

which we express as:

$$\gamma_{D,d} \left( \int_{\sum_{i=1}^d a_i^2 \leq 1} \cos \theta_1 \sin \theta_1 a_1^2 \left( \sum_{i=1}^d a_i^2 \sin^2 \theta_i \right)^{p-2} dV \in \{\sigma_i\}_{i=1}^D \right).$$
\[ \int_{\sum_{i=1}^{d} a_{i}^2 \leq 1} \cos \theta_1 \sin \theta_1 a_{1}^2 \left( \sum_{i=1}^{d} a_{i}^2 \sin^2 \theta_i \right)^{\frac{p-2}{2}} \equiv \int_{\sum_{i=1}^{d} a_{i}^2 \leq 1} \cos \theta_1 \sin \theta_1 a_{1}^2 \]

is a monotone function of \( \theta_1 \) on \([0, \pi/4]\) as well as \([\pi/4, \pi/2]\). That is, the requirement that \( \lambda_1(h(L_1, L_2)) \in \{ \sigma_i \}_{i=1}^{D} \) can occur only at discrete values of \( \theta_1 \) and consequently has \( \gamma_{D, d} \) measure 0, that is, (91) (and consequently (86)) is verified in this case.

If \( p \neq 2 \) and \( \{ \theta_i \}_{i=1}^{d-1} \) are fixed, then

\[ \int_{\sum_{i=1}^{d} a_{i}^2 \leq 1} \cos \theta_1 \sin \theta_1 a_{1}^2 \left( \sum_{i=1}^{d} a_{i}^2 \sin^2 \theta_i \right)^{\frac{p-2}{2}} \]

is a monotone function of \( \theta_d \). Following a similar argument we conclude that

\[ \gamma_{D, d} \left( h(L_1, L_2) \in \{ \sigma_i \}_{i=1}^{D} | \{ \theta_i \}_{i=1}^{d-1} \right) = 0. \] (94)

Combining (94) and Fubini’s theorem, we conclude (91) (and consequently (86)) in this case.

9.6.3. First Reduction of Theorem 6.2

Following the argument of Section 9.6.1, we note that Theorem 6.2 will follow by proving the following equation:

\[ \gamma_{D, d}^{K}(\{L_i\}_{i=1}^{K}) \subseteq G(D, d) : (L_1, L_2, \cdots, L_K) = \arg\min_{(L_1, L_2, \cdots, L_K)} E_{p}(e_{L_i}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K)) \equiv 0. \] (95)

We further reduce (95) by using the operator \( D_{L, x, p} \) of (6) and the regions \( \{ Y_i \}_{i=1}^{K} \), which are obtained by a Voronoi diagram (restricted to the unit ball) of the \( d \)-subspaces \( \{ L_i \}_{i=1}^{K} \subseteq G(D, d) \) as follows:

\[ Y_i \equiv Y_i(L_1, L_2, \cdots, L_K) = \{ x \in B(0, 1) : \text{dist}(x, L_i) < \text{dist}(x, L_j) \forall j : 1 \leq j \neq i \leq K \}. \] (96)

We will show that the following equation implies (95) and thus Theorem 6.2:

\[ \gamma_{D, d}^{K}(\{L_i\}_{i=1}^{K}) \subseteq G(D, d) : E_{p}(I(x \in Y_j(L_1, L_2, \cdots, L_K)) D_{L_j, x, p}) = 0 \forall 1 \leq j \leq K \equiv 0. \] (97)

We first show that if the condition

\[ (L_1, L_2, \cdots, L_K) = \arg\min_{(L_1, L_2, \cdots, L_K)} E_{p}(e_{L_i}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K)) \]
of (95) is true, then
\[
L_1 = \arg\min_{\ell \in G(D, d)} E_{\hat{\mu}_\ell}(e_{\ell_\theta}(x, L) I(x \in Y_1)) .
\] (98)

Indeed, for any \( \tilde{L}_1 \in G(D, d) \), we have that
\[
0 \leq E_{\hat{\mu}_\ell}(e_{\ell_\theta}(x, \tilde{L}_1, L_2, \cdots, L_K)) - E_{\hat{\mu}_\ell}(e_{\ell_\theta}(x, L_1, L_2, \cdots, L_K)) \\
\leq E_{\hat{\mu}_\ell}(I(x \in Y_1)e_{\ell_\theta}(x, \tilde{L}_1)) + \sum_{2 \leq i \leq K} E_{\hat{\mu}_\ell}(I(x \in Y_i)e_{\ell_\theta}(x, L_i)) \\
- \sum_{1 \leq i \leq K} E_{\hat{\mu}_\ell}(I(x \in Y_i)e_{\ell_\theta}(x, L_i)) \\
= E_{\hat{\mu}_\ell}(I(x \in Y_1)e_{\ell_\theta}(x, \tilde{L}_1)) - E_{\hat{\mu}_\ell}(I(x \in Y_1)e_{\ell_\theta}(x, L_1)).
\]

Therefore, (95) immediately follows from the equation:
\[
\gamma^K_{D,d}(\{L_i\}_{i=1}^K \subset G(D, d) : \\
L_j = \arg\min_{L \in G(D, d)} E_{\hat{\mu}_\ell}(e_{\ell_\theta}(x, L) I(x \in Y_j)) \quad \forall \ 1 \leq j \leq K) = 0. \quad (99)
\]

Proposition 3.2 can be extended to probability measures instead of data sets, where its sum is replaced by an expectation (indeed, this is inferred by the proof of Proposition 3.2). In view of this modified version, (99) implies the following equation:
\[
\gamma^K_{D,d}(\{L_i\}_{i=1}^K : E_{\hat{\mu}_\ell}(I(x \in Y_j) D_{L_j, x, p}) = 0 \ \forall \ 1 \leq j \leq K) = 0. \quad (100)
\]

At last we easily note that (100) is equivalent with (97) and thus a proof of (97) will conclude Theorem 6.2.

9.6.4. Second Reduction of Theorem 6.2

In order to motivate a further reduction of (97), we formulate a proposition demonstrating under some conditions the sensitivity of the region \( Y_1 \) (or WLOG any of the regions \( \{Y_i\}_{i=1}^K \)) to perturbations in the subspace \( L_2 \) (or WLOG any of the other subspaces but \( L_1 \), when fixing \( Y_1 \)). This proposition uses the following notation, which will also be used throughout the rest of Section 9.6: \( d^* = \min(d, D - d) \), \( \theta_{d^*}(L_1, L_j) \) is the \( d^* \)-th largest principal angle between the \( d \)-subspaces \( L_1 \) and \( L_j \), \( L_k \) is the \( k \)-th dimensional Lebesgue measure, \( a \lor b \) denotes the maximum of \( a \) and \( b \) and for \( L_2, L_1, L_2, \cdots, L_K \in G(D, d) \) and \( 1 \leq i \leq K \), we use the notation: \( Y_i = Y_i(L_1, L_2, \cdots, L_K) \), \( \bar{Y}_i = Y_i(L_1, \tilde{L}_2, \cdots, L_K) \), \( \bar{Y}_i = Y_i(L_1, \tilde{L}_2, \tilde{L}_3, \cdots, L_K) \) and \( \bar{Y}_i = Y_i(L_1, \tilde{L}_2, \tilde{L}_3, \cdots, L_K) \), where \( \bar{\cdot} \) is used to denote closure, e.g.,
\[
\bar{Y}_i = \{x \in B(0, 1) : \text{dist}(x, L_i) \leq \text{dist}(x, L_j) \ \forall j : 1 \leq j \neq i \leq K\}.
\] (101)

that is, \( \bar{Y}_i \) is the closure of \( Y_i \). Furthermore for \( \tilde{L}_2, L_1, L_2, \cdots, L_K \in G(D, d) \) and \( 1 \leq i \leq K \), we use the notation: \( Y_i = Y_i(\tilde{L}_1, L_2, \cdots, L_K) \), \( \bar{Y}_i = Y_i(L_1, \tilde{L}_2, \cdots, L_K) \), \( \bar{Y}_i = \bar{Y}_i(L_1, \tilde{L}_2, \tilde{L}_3, \cdots, L_K) \) and \( \bar{Y}_i = \bar{Y}_i(L_1, \tilde{L}_2, \tilde{L}_3, \cdots, L_K) \).

Given this notation we formulate the proposition mentioned above as follows.
Lemma 9.5. If \( \hat{L}_2, L_1, L_2, \ldots, L_K \) are subspaces in \( G(D, d) \) such that \( \hat{L}_2 \neq L_2 \),

\[
\min_{j \neq 2} (\theta_{d^*}(\hat{L}_2, L_j)) > 0, \quad \min_{1 \leq i \neq j \leq K} (\theta_{d^*}(L_i, L_j)) > 0, \quad \theta_{d^*}(L_2, L_1) \cup \theta_{d^*}(L_2, L_1) \leq \min_{3 \leq i \leq K} \theta_{d^*}(L_i, L_1),
\]

then

\[
\mathcal{L}_D \left( \hat{Y}_1 \setminus Y_1 \right) \cup (Y_1 \setminus \hat{Y}_1) > 0.
\]

Using the conditions of Lemma 9.5 regarding the subspaces \( \{L_i\}_{i=1}^K \), we can reduce (97) to the following equation:

\[
\gamma_{D,d} \left( L_2 \in G(D, d) : \min_{1 \leq i \neq j \leq K} \theta_{d^*}(L_i, L_j) > 0, \quad \arg \min_{2 \leq i \leq K} \theta_{d^*}(L_1, L_i) = 2, \quad E_{\mu_0}(I(x \in Y_1) D_{t_1,x,p}) = 0 \right) = 0.
\]

Indeed, we first note that WLOG (105) can be equivalently formulated by replacing \( L_2 \) with any \( L_k, k = 3, \ldots, K \), and letting \( \arg \min_{2 \leq i \leq K} \theta_{d^*}(L_1, L_i) = k \) respectively. Using this observation and elementary properties of measures we have that

\[
\gamma^K_{D,d} \left( \{L_i\}_{i=1}^K : E_{\mu_0}(I(x \in Y_j(L_1, L_2, \ldots, L_K))) D_{t_1,x,p} = 0 \right) \leq K \int_{G(D,d)^{K-1}} \gamma_{D,d} \left( L_k : \min_{1 \leq i \neq j \leq K} \theta_{d^*}(L_i, L_j) > 0, \quad \arg \min_{2 \leq i \leq K} \theta_{d^*}(L_1, L_i) = k, \quad E_{\mu_0}(I(x \in Y_1) D_{t_1,x,p}) = 0 \right) \cdot d \left( \gamma^K_{D,d} \left( \{L_i\}_{i=1}^K : \min_{1 \leq i \neq j \leq K} \theta_{d^*}(L_i, L_j) = 0 \right) \right).
\]

It is thus sufficient to prove (105) in order to conclude Theorem 6.2. We end this section by proving Lemma 9.5.

Proof of Lemma 9.5. We will first show that there exists \( x_0 \in B(0, 1) \) such that

\[
\text{dist}(x_0, L_1) = \text{dist}(x_0, L_2) < \min_{3 \leq i \leq K} \text{dist}(x_0, L_i).
\]

We will then prove that (106) implies (104).

We verify (106) in the two cases: \( d^* = d \) and \( d^* = D - d \) and consequently the conclusion. For both cases, we denote the principal vectors of \( L_2 \) and \( L_1 \) by \( \{v_{i}\}_{i=1}^{d^*} \) and \( \{v_{i}\}_{i=1}^{d^*} \) respectively. We assume first that \( d^* = d \) and denote \( x_0 = \hat{v}_{d^*} + v_{d^*}/\|v_{d^*} + v_{d^*}\| \). Equation (102) implies that the intersections of the \( d \)-subspaces \( \{L_i\}_{i=1}^K \) are empty. Applying this observation as well as elementary geometric estimates we obtain that for any \( i_0 > 3 \) and any unit vector \( v_0 \in L_{i_0} \):

\[
\text{ang}(x_0, v_0) \geq \text{ang}(v_{d^*}, v_0) - \text{ang}(v_{d^*}, x_0) \geq \theta_{d^*}(L_{i_0}, L_1) - \theta_{d^*}(L_2, L_1)/2 \geq \theta_{d^*}(L_2, L_1) - \theta_{d^*}(L_2, L_1)/2 = \theta_{d^*}(L_2, L_1)/2.
\]
We claim that (107) cannot be an equality. Indeed, if the first inequality in (107) holds, then \( v_0, v_{d^*} \) and \( x_0 \) are on a geodesic line within the sphere \( S^{D-1} \). Combining this with the assumption that (107) is an equality, we obtain that \( \text{ang}(x_0, v_0) = \theta_{d^*} (L_2, L_1)/2 = \text{ang}(x_0, v_{d^*}) = \text{ang}(x_0, v_{d^*}) \). We thus conclude that either \( v_0 = v_{d^*} \) or \( v_0 = v_{d^*} \), which contradict (102) as well as the definition of principal vectors. Using the concluded strict inequality in (107), we prove (106) in this case as follows:

\[
\text{dist}(x_0, L_i) \geq \text{ang}(x_0, v_0) > \theta_{d^*} (L_2, L_1)/2 = \text{dist}(x_0, L_1) = \text{dist}(x_0, L_2).
\]

If \( d^* = D-d \), then \( x_1 = (\hat{v}_{d^*} + v_{d^*})/\|\hat{v}_{d^*} + v_{d^*}\| \). It follows from basic dimension equalities of subspaces and (102) that for all \( 2 \leq i \leq K \): \( \dim(L_1 \cup L_i) = D \) and \( \dim(L_1 \cap L_i) = 2d - D \). We denote by \( K_0 \) the integer in \( \{0, \ldots, K\} \) such that for any \( 3 \leq i \leq K_0 \), \( L_1 \cap L_i = L_1 \cap L_2 \) and for any \( i > K_0 \), \( L_1 \cap L_i \neq L_1 \cap L_2 \) (the existence of \( K_0 \) may require reordering of the indices of the subspaces \( \{L_i\}_{i=1}^K \)). Let \( x_2 \) be an arbitrarily fixed unit vector in \( L_1 \cap (L_2 \setminus \cup_{K_0 \leq i \leq K} L_i) \). Denote \( \epsilon_0 = \text{dist}(x_2, \cup_{K_0 \leq i \leq K} L_i) \) and let \( x_0 = x_2/2 + \epsilon_0 x_1/5 \). We first claim that

\[
\text{dist}(x_0, L_1) = \text{dist}(x_0, L_2) < \min_{3 \leq j \leq K_0} \text{dist}(x_0, L_j). \tag{108}
\]

Indeed, we can remove \( L_1 \cap L_2 \) from the subspaces \( \{L_i\}_{i=1}^K \) and obtain subspaces of dimension \( D - d \) intersecting each other at the origin. We note that (108) is equivalent to a similar equations for the new subspaces, which replaces \( x_0 \) with \( x_1 \). The latter follows from the proof of the case \( d = d^* \).

On the other hand, we note that

\[
\text{dist}(x_0, L_1) = \epsilon_0 \text{dist}(x_1, L_1)/5 \leq \epsilon_0/5 < \text{dist}(x_2/2, \cup_{K_0 \leq i \leq K} L_j) - \epsilon_0/5 \leq \text{dist}(x_2/2 + \epsilon_0 x_1/5, \cup_{K_0 \leq i \leq K} L_j) = \min_{K_0 \leq i \leq K} \text{dist}(x_0, L_i). \tag{109}
\]

Equation (106) thus follows by combining (108) and (109).

We note that (106) implies that

\[
x_0 \in Y_1 \cup Y_2 \cup (\tilde{Y}_1 \cap \tilde{Y}_2) \quad \text{and} \quad x_0 \in \tilde{Y}_1 \cup \tilde{Y}_2 \cup (\tilde{Y}_1 \cap \tilde{Y}_2). \tag{110}
\]

The continuity of the distance function implies a stronger version of (110): There exists \( \epsilon > 0 \) such that

\[
B(x_0, \epsilon) \subset Y_1 \cup Y_2 \cup (\tilde{Y}_1 \cap \tilde{Y}_2) \quad \text{and} \quad B(x_0, \epsilon) \subset \tilde{Y}_1 \cup \tilde{Y}_2 \cup (\tilde{Y}_1 \cap \tilde{Y}_2). \tag{111}
\]

We will deduce (104) from (111) by considering two different cases. Assume first that \( \tilde{Y}_1 \cap \tilde{Y}_2 \cap B(x_0, \epsilon) \neq \tilde{Y}_1 \cap \tilde{Y}_2 \cap B(x_0, \epsilon) \). Using (111) and the fact that \( \mathcal{L}_D(\tilde{Y}_1 \cap \tilde{Y}_2) = 0 \), we may choose \( y \in \tilde{Y}_1 \cap \tilde{Y}_2 \cap B(x_0, \epsilon) \) and also \( y \in Y_1 \cup Y_2 \); WLOG we assume instead of the later condition that \( y \in Y_1 \). By perturbing \( y \) slightly we can choose another point \( y_0 \) such that \( y_0 \in Y_2 \) and \( y_0 \in Y_1 \setminus \tilde{Y}_1 \). It follows from the continuity of the distance function that there exists a small \( \eta > 0 \) such that \( (\tilde{Y}_1 \setminus Y_1) \cup (Y_1 \setminus \tilde{Y}_1) \supset Y_1 \setminus \tilde{Y}_1 \supset B(y_0, \eta) \), which proves (104).
Next, assume the complimentary case where \( \bar{\text{Y}}_1 \cap \bar{\text{Y}}_2 \cap B(x_0, \epsilon) = \text{Y}_1 \cap \text{Y}_2 \cap B(x_0, \epsilon) \). We will exclude this case by showing that it leads to either one of the following contradictions: \( L_1 = L_2 \) and \( L_2 = L_2 \). We note that the solutions sets of \( x^T_0 (P_{L_1} - P_{L_2}) x_0 = 0 \) and \( x^T_0 (P_{L_1} - P_{L_2}) x_0 = 0 \) in \( B(x_0, \epsilon) \) are \( \bar{\text{Y}}_1 \cap \bar{\text{Y}}_2 \cap B(x_0, \epsilon) \) and \( \text{Y}_1 \cap \text{Y}_2 \cap B(x_0, \epsilon) \) respectively. In view of (111), these two manifolds (i.e., the solution sets) coincide. Therefore, their \( (D - 1) \)-dimensional tangent spaces at \( x_0 \), i.e., \( x^T_0 (P_{L_1} - P_{L_2}) x_0 = 0 \) and \( x^T_0 (P_{L_1} - P_{L_2}) x_0 = 0 \), also coincide. Consequently we have that \( x^T_0 (P_{L_1} - P_{L_2}) = t x^T_0 (P_{L_1} - P_{L_2}) \) for some \( t \neq 0 \). Similarly, for any \( x_1 \in \bar{\text{Y}}_1 \cap \bar{\text{Y}}_2 \cap B(x_0, \epsilon) \), we have \( x^T_0 (P_{L_1} - P_{L_2}) = t_1 x^T_0 (P_{L_1} - P_{L_2}) \) for some \( t_1 \neq 0 \). Note that \( t_1 = t \) by the following argument: \( t_1 x^T_0 (P_{L_1} - P_{L_2}) x_0 = x^T_0 (P_{L_1} - P_{L_2}) x_0 = t x^T_0 (P_{L_1} - P_{L_2}) x_0 \). Therefore, for any \( x_1 \in \bar{\text{Y}}_1 \cap \bar{\text{Y}}_2 \cap B(x_0, \epsilon) \), we have

\[
x^T_0 (P_{L_1} - P_{L_2}) = t x^T_0 (P_{L_1} - P_{L_2}). \quad (112)
\]

Since the tangent space of \( \bar{\text{Y}}_1 \cap \bar{\text{Y}}_2 \cap B(x_0, \epsilon) \) at \( x_0 \) has dimension \( D - 1 \), the subspace \( L_0 = \text{Sp} \{ \bar{\text{Y}}_1 \cap \bar{\text{Y}}_2 \cap B(x_0, \epsilon) \} \) has dimension at least \( D - 1 \) (it is possible to show that its dimension is \( D \), though we have a simpler argument avoiding this technical detail). In view of (112), \( L_0 \) satisfies

\[
P_{L_0} (P_{L_1} - P_{L_2}) = t P_{L_0} (P_{L_1} - P_{L_2}). \quad (113)
\]

Using the fact that \( (P_{L_1} - P_{L_2}) \) and \( (P_{L_1} - P_{L_2}) \) can be represented as symmetric \( D \times D \) matrices, we have the following equivalent formulation of (113):

\[
(P_{L_1} - P_{L_2}) P_{L_0} = (P_{L_1} - P_{L_2}) P_{L_0}. \quad (114)
\]

Furthermore, using the fact that \( (P_{L_1} - P_{L_2}) \) and \( (P_{L_1} - P_{L_2}) \) have trace 0, we obtain that

\[
\text{tr}(P_{L_0}^T (P_{L_1} - P_{L_2}) P_{L_0}) = - \text{tr}(P_{L_0} (P_{L_1} - P_{L_2}) P_{L_0}) = - t \cdot \text{tr}(P_{L_0} (P_{L_1} - P_{L_2}) P_{L_0})
\]

\[
= t \cdot \text{tr}(P_{L_0}^T (P_{L_1} - P_{L_2}) P_{L_0}). \quad (115)
\]

Since \( P_{L_0}^T \) is at most one-dimensional (it is actually zero-dimensional, but it will take us more time to establish it), (115) can be rewritten as

\[
P_{L_0} (P_{L_1} - P_{L_2}) P_{L_0} = t \cdot (P_{L_0} (P_{L_1} - P_{L_2}) P_{L_0}). \quad (116)
\]

Combining (113), (114) and (116), we obtain that \( (P_{L_1} - P_{L_2}) = t (P_{L_1} - P_{L_2}) \) and

\[
P_{L_1} = t P_{L_1} + t P_{L_2}.
\]

We conclude the desired contradiction in two different cases. Assume first that \( t < 1 \) and let \( v_0 \) be an arbitrary unit vector in \( L_2 \). We note that \( v_0^T P_{L_2} v_0 = 1 \) as well as \( (1 - t) v_0^T P_{L_1} v_0 = 1 - t v_0^T P_{L_2} v_0 \geq 1 - t \). Consequently, \( v_0^T P_{L_1} v_0 = 1 \), i.e., \( v_0 \in L_1 \) and thus we obtain the following contradiction with (102): \( L_1 = L_2 \). Next, assume that \( t \geq 1 \) and as before \( v_0 \) is an arbitrary unit vector in \( L_2^\perp \). In this case, \( v_0^T P_{L_2} v_0 = (1 - t) v_0^T P_{L_1} v_0 + t v_0^T P_{L_2} v_0 \leq 0 + 0 = 0 \). Therefore, \( v_0 \in L_2 \) and we obtain the following contradiction with (102): \( L_2 = L_2 \). Equation (104) is thus proved.
9.6.5. Proof of (105) when $d = 1$ or $d = D - 1$

We first prove (105) when $d = 1$. We fix $v_1$ to be one of the two unit vectors spanning $L_1$ and denote by $u_1$ the unit vector spanning $(L_1 + L_2) \cap L_1^\perp$ having orientation such that for any point $x \in L_2$: $(x^T u_1)(x^T v_1) \geq 0$. We will prove that

$$
\gamma_{D,d} \left( L_2 : \min_{1 \leq i \neq j \leq K} \theta_{d^*}(L_i, L_j) > 0, \ \arg \min_{2 \leq i \leq K} \theta_{d^*}(L_i, L_i) = 2, \ E_{\mu_0}^\gamma(I(x \in Y_1) D_{L_1, x, p}) \right) = \left( (L_1 + L_2) \cap L_1^\perp = \text{Sp}(u_1) \right) = 0. \tag{117}
$$

We first show that (117) implies (105). We define $\Omega_0 = \{x \in S^{D-1} : x \perp v\}$ (recalling that $S^{D-1}$ is the unit sphere in $\mathbb{R}^D$) and the measure $\omega$ on $\Omega_0$ such that for any $A \subseteq \Omega_0$: $\omega(A) = \gamma_{D,d} \left( L_2 \in G(D, d) : (L_1 + L_2) \cap L_1^\perp \in \text{Sp}(A) \right)$. Using this notation, (117) implies (105) as follows:

\begin{align*}
\gamma_{D,d} \left( L_2 \in G(D, d) : \min_{1 \leq i \neq j \leq K} \theta_{d^*}(L_i, L_j) > 0, \ \arg \min_{2 \leq i \leq K} \theta_{d^*}(L_i, L_i) = 2, \ E_{\mu_0}^\gamma(I(x \in Y_1) D_{L_1, x, p}) = 0 \right) &= \int_{\Omega_0} \gamma_{D,d} \left( L_2 : \min_{1 \leq i \neq j \leq K} \theta_{d^*}(L_i, L_j) > 0, \ \arg \min_{2 \leq i \leq K} \theta_{d^*}(L_i, L_i) = 2, \ E_{\mu_0}^\gamma(I(x \in Y_1) D_{L_1, x, p}) = 0 \right) d(\omega(u_1)) = 0.
\end{align*}

We will prove (117) by showing that at most one element satisfies its underlying condition (i.e., is a member of the set for which $\gamma_{D,d}$ is evaluated). Assume on the contrary that there are two subspaces $\hat{L}_2$ and $\tilde{L}_2$ satisfying this underlying condition with corresponding angles $\hat{\theta}$ and $\tilde{\theta}$ in $[0, \pi/2]$, where WLOG $\hat{\theta} > \tilde{\theta}$.

We define

$$
\hat{Y}_1 = Y_1(L_1, \hat{L}_2, L_3, \cdots, L_K) \quad \text{and} \quad \tilde{Y}_1 = Y_1(L_1, \tilde{L}_2, L_3, \cdots, L_K). \tag{118}
$$

Since both $\hat{L}_2$ and $\tilde{L}_2$ satisfy the underlying condition of (117), we have that

$$
E_{\mu_0} \left( I(x \in \hat{Y}_1 \setminus \tilde{Y}_1) D_{L_1, x, p} \right) - E_{\mu_0} \left( I(x \in \tilde{Y}_1 \setminus \hat{Y}_1) D_{L_1, x, p} \right) = 2 \cdot \left( E_{\mu_0} \left( I(x \in \hat{Y}_1) D_{L_1, x, p} \right) - E_{\mu_0} \left( I(x \in \tilde{Y}_1) D_{L_1, x, p} \right) \right) = 0 - 0 = 0. \tag{119}
$$

Consequently,

$$
E_{\mu_0} \left( I(x \in \hat{Y}_1 \setminus \tilde{Y}_1) v_1^T D_{L_1, x, p} u_1 \right) - E_{\mu_0} \left( I(x \in \tilde{Y}_1 \setminus \hat{Y}_1) v_1^T D_{L_1, x, p} u_1 \right) = 0. \tag{120}
$$

Defining

$$
\theta_{u_1, v_1}(x) = \arctan \frac{u_1 \cdot x}{v_1 \cdot x}
$$

and

$$
\hat{Y}_0 = \{x \in B(0, 1) : \text{dist}(x, L_1) < \min_{3 \leq i \leq K} \text{dist}(x, L_i)\},
$$

we have that

$$
E_{\mu_0} \left( I(x \in \hat{Y}_0) v_1^T D_{L_1, x, p} u_1 \right) - E_{\mu_0} \left( I(x \in \tilde{Y}_0) v_1^T D_{L_1, x, p} u_1 \right) = 0.
$$

However, this contradicts (119), and hence we conclude that (117) holds.
we express the regions $\hat{Y}_1$ and $\tilde{Y}_1$ as follows:

$$\hat{Y}_1 = Y_0 \cap \{ x \in B(0, 1) : \hat{\theta}/2 - \pi/2 < \theta_{u_1, v_1}(x) < \hat{\theta}/2 \}, \quad (121)$$

$$\tilde{Y}_1 = Y_0 \cap \{ x \in B(0, 1) : \tilde{\theta}/2 - \pi/2 < \theta_{u_1, v_1}(x) < \tilde{\theta}/2 \}. \quad (122)$$

Figure 1 demonstrates those regions and clarifies (121) and (122) in the special case where $d = 1$ and $K = 2$.

Combining (121) and (122) with the definition of $D_{L, x, p}$ in (6), we obtain that

$$\hat{Y}_1 \setminus \tilde{Y}_1 \subset \{ x \in B(0, 1) : v_1^T x x^T u_1 \equiv \text{dist}(x, L_1)^{(2-p)} v_1^T D_{L_1, x, p} u_1 > 0 \}, \quad (123)$$

and

$$\tilde{Y}_1 \setminus \hat{Y}_1 \subset \{ x \in B(0, 1) : v_1^T x x^T u_1 \equiv \text{dist}(x, L_1)^{(2-p)} v_1^T D_{L_1, x, p} u_1 < 0 \}. \quad (124)$$

It follows from Lemma 9.5 that $\mathcal{L}(\hat{Y}_1 \setminus \tilde{Y}_1) \cup (\tilde{Y}_1 \setminus \hat{Y}_1)) > 0$. This observation combined with (120), (123) and (124) lead to a contradiction, which proves (117) and consequently (105).

We note that the same proof can be generalized to the case $d = D - 1$, by fixing $v_1$ as one of the two unit vectors spanning $L_1 \cap (L_1 \cap L_2) \perp v_1$ (note that $\dim(L_1) = D - 1$ and $\dim(L_1 \cap L_2) = d - 2$ so that $\dim(L_1 \cap (L_1 \cap L_2) \perp ) = 1$) and $u_1$ the unit vector of $(L_1 + L_2) \cap L_1^\perp$ with a similar orientation as in the case where $d = 1$.

9.6.6. Concluding the Proof of (105) and Theorem 6.2

We conclude the proof of (105) and consequently Theorem 6.2 by considering the case not discussed above, i.e., $d \neq 1$ and $d \neq D - 1$. We first reduce (105) to a simpler
form using the following notation. For \( L, L^* \in G(D, d) \), we define the “orthogonal subtraction” \( \ominus \) as follows:

\[
L^* \ominus L = L^* \cap (L \cap L^*)^\perp.
\]

We let

\[
\Omega_1 = \{(P_1, P_2) \in \mathbb{R}^{D \times D} \times \mathbb{R}^{D \times D} : \exists L \in G(D, d), \text{ not orthogonal to } L_1,
\text{ s.t. } \dim(L \ominus L) > 1, \ P_{L_1}^T P_L P_{L_1} = P_1, P_{L_1}^T P_L P_{L_1} = P_2\}
\]

and define the measure \( \omega_1 \) on \( \Omega_1 \) as follows: For any set \( A \subseteq \Omega_1 \)

\[
\omega_1(A) = \gamma_{D,d} (L \in G(D, d) : (P_{L_1}^T P_L P_{L_1}, P_{L_1}^T P_L P_{L_1}) \in A).
\]

Using some of this notation we reduce (105) as follows:

\[
\begin{align*}
\gamma_{D,d} (L_2 : L_1 \text{ is not orthogonal to } L_2, \ \dim(L_1 \cap L_2^\perp) > 1, \\
\min_{1 \leq i \neq j \leq K} \theta_d(L_i, L_j) > 0, \ \arg \min_{2 \leq i \leq K} \theta_d(L_1, L_i) = 2, \ E_{\mu_0}(I(x \in Y_1) D_{L_1,x,p}) = 0 \ \\
(P_{L_1}^T P_L P_{L_1}, P_{L_1}^T P_L P_{L_1}) = (P_1, P_2) \in X_0) = 0.
\end{align*}
\]

Indeed,

\[
\begin{align*}
\gamma_{D,d} (L_2 : L_1 \text{ is not orthogonal to } L_2, \ \dim(L_1 \ominus L_2) > 1, \\
\min_{1 \leq i \neq j \leq K} \theta_d(L_i, L_j) > 0, \ \arg \min_{2 \leq i \leq K} \theta_d(L_1, L_i) = 2, \ E_{\mu_0}(I(x \in Y_1) D_{L_1,x,p}) = 0 \ \\
(P_{L_1}^T P_L P_{L_1}, P_{L_1}^T P_L P_{L_1}) = (P_1, P_2) \in X_0) \ d(\tilde{\mu}(P_1, P_2)) &+ \gamma_{D,d} (L_2 \in G(D, d) : \dim(L_1 \ominus L_2) \leq 1, \text{ or } L_2 \perp L_1) = 0 + 0 = 0.
\end{align*}
\]

We prove (105) by using the following two lemmata.

**Lemma 9.6.** If \( \dim(L_1 \ominus L_2) \geq 2 \) and \( L_1 \) is not orthogonal to \( L_2 \), then the set

\[
Z = \{L \in G(D, d) : P_{L_1}(P_{L_2} - P_{L_1})P_{L_1} = 0, \ P_{L_1}^T (P_{L_2} - P_{L_1})P_{L_1} = 0\}
\]

is infinite.

**Lemma 9.7.** If \( \tilde{L}_2, \hat{L}_2 \in G(D, d) \) satisfy \( \tilde{L}_2 \neq \hat{L}_2, \ \theta_d(L_2, L_1) \vee \theta_d(L_1, L_1) \leq \min_{3 \leq i \leq K} \theta_d(L_1, L_i), \ P_{L_1}(P_{L_2} - P_{L_1})P_{L_1} = 0, \ \text{and } P_{L_1}^T (P_{L_2} - P_{L_1})P_{L_1} = 0, \) then either \( \tilde{L}_2 \) or \( \hat{L}_2 \) will not satisfy the condition in (105).

Lemma 9.6 implies that there are infinite subspaces \( L \in G(D, d) \) satisfying \( P_{L_1}^T P_L P_{L_1} = P_1 \) and \( P_{L_1}^T P_L P_{L_1} = P_2 \). On the other hand, Lemma 9.7 implies that only one subspace out of these infinite subspaces satisfies the underlying condition of (125) and thus the latter equality is proved.

We conclude the proof by verifying Lemmata 9.6 and 9.7.
**Proof of Lemma 9.6.** We denote \( \hat{L}_1 = L_1 \ominus (L_1 \cap L_2) \) and \( \hat{L}_2 = L_2 \ominus (L_1 \cap L_2) \). The idea of the proof is to construct a one-to-one function \( g : S^{D-1} \cap L_2 \rightarrow Z \). Using this function and the fact that \( \dim(\hat{L}_2) = \dim(L_1) - \dim(L_2 \cap L_1) \geq 2 \) we conclude that \( Z \), which contains \( g(S^{D-1} \cap L_2) \), is infinite.

For any \( u_0 \in S^{D-1} \cap \hat{L}_2 \), we arbitrarily fix \( v_0 = v_0(u_0) \) as one of the two unit vectors spanning \( \hat{L}_1 \cap (\hat{L}_2 \oplus \text{Sp}(u_0)) \). The vector \( v_0 \) exists since
\[
\dim\left( \hat{L}_1 \cap (\hat{L}_2 \oplus \text{Sp}(u_0)) \right) = \dim(\hat{L}_1) + \dim\left( (\hat{L}_2 \oplus \text{Sp}(u_0)) \right) - D = d + (D - d + 1) - D = 1.
\]

We define the function \( g \) as follows:
\[
g(u_0) = \text{Sp}(u_0 - 2(v_0^T u_0)v_0, L_2 \oplus \text{Sp}(u_0)).
\]

We first claim that the image of \( g \) is contained in \( Z \). Indeed, we note that
\[
P_g(u_0) - P_{L_2} = (u_0 - 2(v_0^T u_0)v_0)^T(u_0 - 2(v_0^T u_0)v_0) - u_0^T u_0
\]
\[
= -2(v_0^T u_0)(v_0^T(u_0 - (v_0^T u_0)v_0) + (u_0 - (v_0^T u_0)v_0)^T v_0).
\]
Combining (126) with the following two facts: \( v_0 \in L_1 \) and \( u_0 - (v_0^T u_0)v_0 \in L_1^\perp \) we obtain that \( g(u_0) \in Z \).

At last, we prove that \( g \) is one-to-one and thus conclude the proof. If on the contrary there exist \( u_1, u_2 \in S^{D-1} \cap \hat{L}_2 \) such that \( u_1 \neq u_2 \) and \( g(u_1) = g(u_2) \), then \( g(u_1) = \text{Sp} g(u_1), g(u_2) \cong (L_2 \oplus \text{Sp}(u_1)) + (L_2 \oplus \text{Sp}(u_2)) \cong L_2 \). Since \( \dim(g(u_1)) = \dim(\hat{L}_2) \) we conclude that \( g(u_1) = L_2 \). On the other hand we claim that for any \( u_0 \in S^{D-1} \cap L_2 \), \( g(u_0) \neq L_2 \) and thus lead to contradiction. Indeed, since \( u_0 \in \hat{L}_2, v_0 \in \hat{L}_1 \) and \( L_1 \) is not orthogonal to \( L_2 \) we have that \( v_0^T u_0 \neq 0 \) and consequently \( u_0 - (v_0^T u_0)v_0 \neq u_0 \). Applying the latter observation in (126), we obtain that \( P_g(u_0) \neq P_{L_2} \) and consequently \( g(u_0) \neq L_2 \).

**Proof of Lemma 9.7.** We assume by contrary that both \( \hat{L}_2 \) and \( \hat{L}_2 \) satisfy the underlying condition of (105) and conclude a contradiction. We use the notation \( \hat{Y}_1 \) and \( \hat{Y}_1 \) of (118).

We arbitrarily fix here \( x \in \hat{Y}_1 \setminus \hat{Y}_1 \). We note that \( \text{dist}(x, L_1) < \text{dist}(x, \hat{L}_2) \) and \( \text{dist}(x, L_1) < \arg \min_{3 \leq i \leq K} \text{dist}(x, L_i) \). Since \( x \notin \hat{Y}_1 \), \( \text{dist}(x, L_1) > \text{dist}(x, L_2) \) and thus
\[
\text{dist}(x, \hat{L}_2) < \text{dist}(x, L_1) < \text{dist}(x, \hat{L}_2).
\]
Consequently,
\[
x^T(P_{L_2} - P_{\hat{L}_2})x = \text{dist}(x, \hat{L}_2)^2 - \text{dist}(x, \hat{L}_2)^2 < 0.
\]

We partition \( P_{L_2} - P_{\hat{L}_2} \) into four parts: \( P_{L_1}(P_{L_2} - P_{\hat{L}_2})P_{L_1} \), \( P_{L_1}(P_{L_2} - P_{\hat{L}_2})P_{\hat{L}_1} \), \( P_{L_1}(P_{L_2} - P_{\hat{L}_2})P_{\hat{L}_1} \), and \( P_{L_1}(P_{L_2} - P_{\hat{L}_2})P_{L_1} \). The first two are zero, and the last two
are adjoint to each other; we thus only consider $P_{L_1}(P_{L_2} - P_{L_2})P_{L_1}^\perp$. Let its SVD be

$$P_{L_1}(P_{L_2} - P_{L_2})P_{L_1}^\perp = U\Sigma V = \sum_{i=1}^d \sigma_i u_i v_i^T.$$  

(129)

We can express the SVD of $P_{L_2} - P_{L_2}$ using (129) and the partition mentioned above as follows:

$$P_{L_2} - P_{L_2} = \sum_{i=1}^d \sigma_i (u_i v_i^T + v_i u_i^T).$$  

(130)

Combining (128) and (130), we obtain that

$$\sum_{i=1}^n \sigma_i u_i^T xx^T v_i = x^T (\sum_{i=1}^n \sigma_i (u_i v_i^T + v_i u_i^T)) x/2 < 0.$$  

(131)

We define a function $f : \mathbb{R}^{D \times D} \rightarrow \mathbb{R}$ such that for any $A \in \mathbb{R}^{D \times D}$: $f(A) = \sum_{i=1}^n \sigma_i u_i^T A v_i$. Using (131) and the facts that $\{u_i\}_{i=1}^d \in L_1$ and $\{v_i\}_{i=1}^d \in L_1^\perp$, we have

$$f(\mathcal{D}_{L_1}, x, p) = \text{dist}(x, L_1) (p-2) f(\mathcal{P}_{L_1}(x) P_{L_1}^\perp (x)^T)$$

$$= \text{dist}(x, L_1) (p-2) \sum_{i=1}^n \sigma_i u_i^T P_{L_1}(x) P_{L_1}^\perp (x)^T v_i$$

$$= \text{dist}(x, L_1) (p-2) \sum_{i=1}^n \sigma_i u_i^T xx^T v_i < 0.$$  

(132)

Similarly, for any point $x \in \tilde{Y}_1 \setminus \hat{Y}_1$:

$$f(\mathcal{D}_{L_1}, x, p) > 0.$$  

(133)

Combining (119), (132), (133), Lemma 9.5 and the linearity of $f$ we conclude the following contradiction establishing the current lemma:

$$0 = f \left( E_{\mu_0} (I(x \in \tilde{Y}_1 \setminus \hat{Y}_1) \mathcal{D}_{L_1, x, p}) - E_{\mu_0} (I(x \in \hat{Y}_1) \mathcal{D}_{L_1, x, p}) \right)$$

$$= f \left( E_{\mu_0} (I(x \in \tilde{Y}_1 \setminus \hat{Y}_1) \mathcal{D}_{L_1, x, p}) \right) - f \left( E_{\mu_0} (I(x \in \hat{Y}_1) \mathcal{D}_{L_1, x, p}) \right) > 0,$$  

(134)

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Appendix A: Supplementary Details

A.1. Proof of Lemma 9.1

We will use the following inequality, which we verify below in Section A.1.1:

\[
\mu_1 \left( x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta \text{dist}(L_1, \hat{L}_1) \right) \leq \beta d_2^2 \quad \forall \beta > 0. \tag{135}
\]

We denote $\beta_1 = 1/2Kd_2^2$ and apply (135) to obtain that

\[
\mu_1 \left( x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta_1 \epsilon \right)
\leq \mu_1 \left( x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta_1 \text{dist}(L_1, \hat{L}_1) \right) \leq \frac{1}{2K}
\]

for any $1 \leq i \leq K$.

Consequently, we derive the following estimate

\[
\mu_1 \left( x \in B(0, 1) \cap L_1 : \min_{1 \leq i \leq K} \text{dist}(x, \hat{L}_1) \geq \beta_1 \epsilon \right)
\geq 1 - \sum_{i=1}^{K} \mu_1 \left( x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta_1 \epsilon \right) \geq 1/2
\]

and thus by Chebyshev’s inequality the lemma is concluded as follows:

\[
E_{\mu_1} \left( e_p(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) \right) \geq \beta_1^p e_p/2 = \frac{e_p}{2^{1+p}K^p d_2^2}.
\]
A.1.1. Proof of (135)

We denote the principal angles between $L_1$ and $L_2$ by $\{\theta_i\}_{i=1}^d$, the principle vectors of $L_1$ and $L_1$ by $\{v_i\}_{i=1}^d$ and $\{\hat{v}_i\}_{i=1}^d$ respectively, the interaction dimension by $k \equiv k(L_1, L_2)$ (see Section 2.2), the volume of the $d$-dimensional unit ball by $v_d$ and

$$\gamma_i = \frac{\sin(\theta_i)^2}{\sum_{j=1}^d \sin(\theta_j)^2}, \quad i = 1, \ldots, d.$$  

Expressing the points in $L_1$ by their coordinates with respect to $\{v_i\}_{i=1}^d$, we obtain that

$$\{x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta \text{dist}(L_1, \hat{L}_1)\}$$

$$= \left\{ x = (x_1, x_2, \ldots, x_d) \in B(0, 1) \cap L_1 : \sum_{i=1}^d x_i^2 \sin^2 \theta_i < \beta \sum_{i=1}^d \theta_i \right\}$$

$$\subset \left\{ x = (x_1, x_2, \ldots, x_d) \in B(0, 1) \cap L_1 : \sum_{i=1}^d x_i^2 \sin^2 \theta_i < \frac{\pi}{2} \beta \right\}$$

$$= \left\{ x = (x_1, x_2, \ldots, x_d) \in B(0, 1) \cap L_1 : \sum_{i=1}^k x_i^2 < \frac{\pi}{2} \beta \right\}$$

$$\subset \left\{ x = (x_1, x_2, \ldots, x_d) \in B(0, 1) \cap L_1 : |x_1| < \frac{\pi}{2\sqrt{\gamma_1}} \beta \right\}.$$  

Since $\sum_{i=1}^k \gamma_i = 1$, WLOG we assume that $\gamma_1 \geq 1/k \geq 1/d$ and consequently get that

$$\{x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta \text{dist}(L_1, \hat{L}_1)\}$$

$$\subset \left\{ x = (x_1, x_2, \ldots, x_d) \in B(0, 1) \cap L_1 : |x_1| < \frac{\sqrt{d}}{2\beta}, |x_2| \leq 1, \sum_{i=3}^d x_i^2 \leq 1 \right\}.$$  

Therefore

$$\text{Vol} \left\{ x : x \in B(0, 1) \cap L_1, \text{dist}(x, \hat{L}_1) < \beta \text{dist}(L_1, \hat{L}_1) \right\} < 2\pi v_{d-2}\sqrt{d}/\beta.$$  

(136)

Combining (136) with the immediate observation: $v_d = \frac{2\pi}{d} v_{d-2}$, we conclude (135) as follows:

$$\mu_1 \left\{ x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta \text{dist}(L_1, \hat{L}_1) \right\}$$

$$= \text{Vol} \left\{ x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}_1) < \beta \text{dist}(L_1, \hat{L}_1) \right\} \Big/ \text{Vol} \left\{ x \in B(0, 1) \cap L_1 \right\}$$

$$< \frac{2\pi v_{d-2}\sqrt{d}/\beta}{v_d} = \beta d^2.$$  

□
A.2. Proof of Lemma 9.2

We denote the principal angles between the $d$-subspaces $L_1, L_2$ by $\theta_1 \geq \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d$. Arbitrarily choosing $Q_1, Q_2 \in O(D, d)$, representing $L_1, L_2$ respectively, we note that

$$\left| \text{dist}(x, L_1) - \text{dist}(x, L_2) \right| = \|x - xQ_1Q_1^T\| - \|x - xQ_2Q_2^T\| \leq \|x - xQ_2Q_2^T\| \leq \|x\| \text{dist}(L_1, L_2).$$

Second of all, we claim that it is sufficient to assume that $\hat{L} = \text{Sp}(\hat{v}_1^T, L_2^T)$, the principle vectors of $L_1$ and $L_2$ by $\{v_i\}_{i=1}^d$, and $\hat{v}_i$ the complementary orthogonal system for $L_2$ w.r.t. $L_1$ by $\{u_i\}_{i=1}^d$.

We denote the principal angles between $L_1$ and $L_2$ by $\{t_i\}_{i=1}^d$, the principle vectors of $L_1$ and $L_2$ by $\{v_i\}_{i=1}^d$ and $\{\hat{v}_i\}_{i=1}^d$ and the complementary orthogonal system for $L_2$ w.r.t. $L_1$ by $\{u_i\}_{i=1}^d$.

We notice that we can restrict the set of subspaces $\hat{L}$ satisfying (137). First of all, we only need to consider subspaces $\hat{L} \subset L_1 + L_2$. (138)

Indeed, the LHS of (137) is the same if we replace $\hat{L}$ by $\hat{L} \cap (L_1 + L_2)$.

Second of all, we claim that it is sufficient to assume that

$$\text{Sp}(\hat{v}_i, v_1) \not\subset \hat{L} \text{ for all } 1 \leq i \leq k. \quad (139)$$

Indeed, WLOG let $i = 1$ and suppose on the contrary to (139) that $v_1, \hat{v}_1 \in \hat{L}$. Since $\hat{L}$ is $d$-dimensional, there exists $2 \leq j \leq d$ (assume WLOG $j = 2$) such that it does not contain both $\hat{v}_j$ and $v_j$. For any pair of points $x = \sum_{i=1}^d a_i v_i \in L_1$ and $\hat{x} = \sum_{i=1}^d a_i \hat{v}_i \in L_2$:

$$\text{dist}(x, \hat{L}) = \sqrt{\sin(\theta_2)^2 a_2^2 + \nu_1^2} \quad \text{and} \quad \text{dist}(\hat{x}, \hat{L}) = \sqrt{\sin(\theta_1)^2 a_1^2 + \nu_2^2},$$

where

$$\nu_1 = \text{dist} \left( \sum_{i=3}^d a_i \hat{v}_i, \hat{L} \right) \quad \text{and} \quad \nu_2 = \text{dist} \left( \sum_{i=3}^d a_i v_i, \hat{L} \right).$$

Now, for $\tilde{L} = \text{Sp}(\hat{L} \setminus \{v_1, \hat{v}_1\}, v_1, v_2)$, we obtain that

$$\text{dist}(\hat{x}, \tilde{L}) = \sqrt{\sin(\theta_1)^2 a_1^2 + \sin(\theta_2)^2 a_2^2 + \nu_2^2} \quad \text{and} \quad \text{dist}(x, \tilde{L}) = \nu_1.$$
Therefore
\[ \text{dist}(\mathbf{x}, \hat{\mathbf{L}})^p + \text{dist}(\hat{\mathbf{x}}, \hat{\mathbf{L}})^p \leq \text{dist}(\mathbf{x}, \hat{\mathbf{L}})^p + \text{dist}(\hat{\mathbf{x}}, \hat{\mathbf{L}})^p \]
and by direct integration we have that
\[ \mathbb{E}(\text{dist}(\mathbf{x}_1, \hat{\mathbf{L}})^p) + \mathbb{E}(\text{dist}(\mathbf{x}_2, \hat{\mathbf{L}})^p) \leq \mathbb{E}(\text{dist}(\mathbf{x}_1, \hat{\mathbf{L}})^p) + \mathbb{E}(\text{dist}(\mathbf{x}_2, \hat{\mathbf{L}})^p). \]
We can thus replace the subspace \( \hat{\mathbf{L}} \) with the subspace \( \tilde{\mathbf{L}} \), which satisfies (139) (for \( i = 1 \), but can similarly be changed for all \( 1 < i \leq K \)).

It follows from (138) and (139) that \( \hat{\mathbf{L}} \) can be represented as follows:
\[ \hat{\mathbf{L}} = \text{Sp}(\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_d^*), \]
where
\[ \mathbf{v}_i^* = \cos \theta_i^* \mathbf{v}_i + \sin \theta_i^* \mathbf{u}_i. \]
Thus, for any pair of points \( \mathbf{x} = \sum_{i=1}^d a_i \mathbf{v}_i \in \mathbb{L}_1 \) and \( \hat{\mathbf{x}} = \sum_{i=1}^d a_i \hat{\mathbf{v}}_i \in \mathbb{L}_2 \):
\[ \text{dist}(\mathbf{x}, \hat{\mathbf{L}}) = \sqrt{\sum_{i=1}^d \sin^2 \theta_i^* a_i^2} \quad \text{and} \quad \text{dist}(\hat{\mathbf{x}}, \hat{\mathbf{L}}) = \sqrt{\sum_{i=1}^d \sin^2(\theta_i - \theta_i^*) a_i^2} \quad (140) \]
and
\[ \text{dist}(\mathbf{x}, \mathbb{L}_1) = 0 \quad \text{and} \quad \text{dist}(\hat{\mathbf{x}}, \mathbb{L}_1) = \sqrt{\sum_{i=1}^d \sin^2 \theta_i a_i^2}. \quad (141) \]
Combining (140), (141), the triangle inequality (for “sine vectors” in \( \mathbb{R}^d \)) and the subadditivity of the sine function, we conclude that
\[ \text{dist}(\mathbf{x}, \hat{\mathbf{L}}) + \text{dist}(\hat{\mathbf{x}}, \hat{\mathbf{L}}) \geq \sqrt{\sum_{i=1}^d (\sin \theta_i^* + \sin (\theta_i - \theta_i^*))^2 a_i^2} \]
\[ \geq \sqrt{\sum_{i=1}^d \sin^2 \theta_i a_i^2} = \text{dist}(\hat{\mathbf{x}}, \mathbb{L}_1) + \text{dist}(\mathbf{x}, \mathbb{L}_1). \]
Since \( p \leq 1 \), this inequality extends to
\[ \text{dist}(\mathbf{x}, \hat{\mathbf{L}})^p + \text{dist}(\hat{\mathbf{x}}, \hat{\mathbf{L}})^p \geq \text{dist}(\hat{\mathbf{x}}, \mathbb{L}_1)^p = \text{dist}(\hat{\mathbf{x}}, \mathbb{L}_1)^p + \text{dist}(\mathbf{x}, \mathbb{L}_1)^p. \quad (142) \]
Integrating (142) w.r.t. the uniform distribution we conclude (137) and thus prove the lemma.

A.4. Proof of Lemma 9.4

Assume first that \( (I(1), \cdots, I(K)) \) is a permutation of \( (1, \cdots, K) \), then \( I \) has an inverse function, \( I^{-1} \). We define
\[ M = \arg\max_{1 \leq i \leq K} \text{dist}(\mathbb{L}_i, \hat{\mathbb{L}}_{I(i)}) \]
and note that
\[
\min_{1 \leq j \leq K} \text{dist}(L_M, \hat{L}_j) = \text{dist}(L_M, \hat{L}_{I(M)}) = \text{dist}(L_M, \hat{L}_{I(1)}), \hat{L}_{I(2)}, \cdots, \hat{L}_{I(K)}) \geq d_0. \tag{143}
\]
Combining (143) with Lemma 9.1 we obtain that
\[
E_{\mu_M} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) = E_{\mu_M} e_{l_p}(x, L_1, L_2, \cdots, L_K) \tag{144}
\]
\[
= E_{\mu_M} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) = d_0^{l_p}. \tag{145}
\]
For any \( x \in \mathcal{X}_0 \), let \( n(x) = \arg\min_{1 \leq i \leq K} \text{dist}(x, L_i) \), \( \hat{m}(x) = \arg\min_{1 \leq i \leq K} \text{dist}(x, \hat{L}_i) \) and note that
\[
\begin{align*}
& e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) = e_{l_p}(x, L_1, L_2, \cdots, L_K) = \text{dist}(x, \hat{L}_{\hat{m}(x)})^p \\
& - \text{dist}(x, L_{\hat{m}(x)})^p \geq \text{dist}(x, \hat{L}_{\hat{m}(x)})^p - \text{dist}(x, L_{I^{-1}(\hat{m}(x))})^p \geq -\|x\|^p \text{dist}(\hat{L}_{\hat{m}(x)}, L_{I^{-1}(\hat{m}(x))})^p \geq -d_0^{l_p}.
\end{align*}
\]
where the second inequality in (145) uses Lemma 9.2. Therefore,
\[
E_{\mu_0} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) = E_{\mu_0} e_{l_p}(x, L_1, L_2, \cdots, L_K) > -d_0^{l_p}. \tag{146}
\]
At last, we observe that
\[
\begin{align*}
E_{\mu} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) & = E_{\mu} e_{l_p}(x, L_1, L_2, \cdots, L_K) \\
& \geq \alpha M \left( E_{\mu_M} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) - E_{\mu_M} e_{l_p}(x, L_1, L_2, \cdots, L_K) \right) \\
& + \alpha_0 \left( E_{\mu_0} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) - E_{\mu_0} e_{l_p}(x, L_1, L_2, \cdots, L_K) \right).
\end{align*}
\]
Combining (144), (146) and (147), the lemma is proved in this case.

Next, we assume that \( I(1), \cdots, I(K) \) is not a permutation of \( 1, 2, \cdots, K \), where we use some of the notation introduced above. In this case, there exist \( 1 \leq n_1, n_2 \leq K \) such that \( I(n_1) = I(n_2) \) and consequently
\[
2 \min_{1 \leq j \leq K} \text{dist}(L_{I_M}, \hat{L}_j) = 2\text{dist}(L_{I_M}, \hat{L}_{I(1)}), \hat{L}_{I(2)}, \cdots, \hat{L}_{I(K)}) \geq \text{dist}(L_{n_1}, \hat{L}_{I(n_1)}) + \text{dist}(L_{n_2}, \hat{L}_{I(n_2)}) \tag{148}
\]
Combining (148) and Lemma 9.1 (applied with \( \epsilon = \min_{1 \leq i, j \leq K} \text{dist}(L_i, L_j)/2 \)), we obtain that
\[
E_{\mu_M} e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) = E_{\mu_M} e_{l_p}(x, L_1, L_2, \cdots, L_K) \tag{149}
\]
\[
> \tau_0 \left( \min_{1 \leq i, j \leq K} \text{dist}(L_i, L_j)/2 \right)^p.
\]
Using the above notation for \( m(x) \) and \( \hat{m}(x) \) we get that for any \( x \in \mathcal{X}_0 \):
\[
e_{l_p}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) = \min_{1 \leq i, j \leq K} \text{dist}(L_i, L_j)/2 \tag{150}
\]
\[ \text{dist}(x, \hat{L}_m(x)) - \text{dist}(x, L_{m(x)}) \geq -1 \]

and consequently
\[ E_{\mu_0} e_{l_0}(x, \hat{L}_1, \hat{L}_2, \cdots, \hat{L}_K) - E_{\mu_0} e_{l_0}(x, L_1, L_2, \cdots, L_K) > -1. \] (151)

The lemma is concluded by combing (147), (149) and (151).

\[ \square \]

**A.5. Proof of (39)**

The fact that \( E_{\mu_1} (P_{L_1}(x)P_{L_1}(x)^T) \) is a scalar matrix follows from the uniformity of \( \mu_1 \) on \( L_1 \cup B(0, 1) \). We compute the underlying scalar, \( \delta_* \), as follows. We arbitrarily fix a vector \( v \in \mathbb{R}^d \) as well as a \((d - 1)\)-subspace \( \hat{L}_1 \subseteq L_1 \) orthogonal to \( v \) and observe that
\[ \delta_* = E_{\mu_1} \left( (P_{L_1}(x) v)^2 \right) = E_{\mu_1} \left( \text{dist}(x, \hat{L})^2 \right). \]

We further note that for any \( 0 < r \leq 1 \), the set \( \{ x \in B(0, 1) \cap L_1 : \text{dist}(x, \hat{L}) = r \} \) consists of two \((d - 1)\)-dimensional balls of radius \( \sqrt{1 - r^2} \). We consequently compute the constant \( \delta_* \) using the beta function \( B \) and the Gamma function \( \Gamma \) in the following way:
\[ \delta_* = E_{\mu_1} \left( \text{dist}^2(x, \hat{L}) \right) = \frac{\int_{r=0}^{1} r^2 (1 - r^2)^{\frac{d-1}{2}} \, dr}{\int_{r=0}^{1} (1 - r^2)^{\frac{d-1}{2}} \, dr} = \frac{\int_{\theta=0}^{\frac{\pi}{2}} \sin^2(\theta) \cos^{\frac{d+1}{2}}(\theta) \, d\theta}{\int_{\theta=0}^{\frac{\pi}{2}} \cos^{\frac{d+1}{2}}(\theta) \, d\theta} = \frac{B\left(\frac{d-1}{2}, \frac{d+1}{2}\right)}{B\left(\frac{d}{2}, \frac{d+2}{2}\right)} = \frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right) \Gamma\left(\frac{d}{2}\right)} = \frac{1}{d + 2}. \]

\[ \square \]

**A.6. Proof of (40)**

For simplicity we denote \( B = \sum_{i=1}^{N_1} P_{L_1}(x_i)P_{L_1}(x_i)^T \). We note that if \( \xi_i (B - \delta_* I_d) < \eta \), then
\[ \frac{\|Bv - \delta_* v\|}{\|v\|} < \eta \quad \text{for all } v \in \mathbb{R}^d \setminus \{0\}, \]
and consequently
\[ \delta_* - \eta < \frac{\|Bv\|}{\|v\|} \quad \text{for all } v \in \mathbb{R}^d \setminus \{0\}, \]
that is, \( \min_{i} \xi_i (B) > \delta_* - \eta. \)

\[ \square \]
References

[1] A. Aldroubi, C. Cabrelli, and U. Molter. Optimal non-linear models for sparsity and sampling. *Journal of Fourier Analysis and Applications*, 14(5-6):793–812, December 2008.

[2] E. Arias-Castro, G. Chen, and G. Lerman. Spectral clustering based on local linear approximations. Submitted, Jan. 2010. Available at http://arxiv.org/abs/1001.1323.

[3] A. Baccini, P. Besse, and A. de Falguerolles. A $L_1$-norm PCA and a heuristic approach. In E. Diday, Y. Lechevalier, and O. Opitz, editors, *Ordinal and symbolic data analysis*, pages 359–368, New York, 1996. Springer.

[4] P. Bradley and O. Mangasarian. k-plane clustering. *J. Global optim.*, 16(1):23–32, 2000.

[5] J. P. Brooks and J. H. Dulá. The $L_1$-norm best-fit hyperplane problem. http://www.optimizationonline.org/DBFILE/2009/05/2291.pdf, 2009.

[6] E. J. Candès, X. Li, Y. Ma, and J. Wright. Robust principal component analysis? Submitted, Dec. 2009, arXiv:0912.3599.

[7] E. J. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *Information Theory, IEEE Transactions on*, 52(2):489–509, 2006.

[8] E. J. Candès, J. K. Romberg, and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. *Communications on Pure and Applied Mathematics*, 59(8):1207–1223, 2006.

[9] G. Chen and G. Lerman. Foundations of a multi-way spectral clustering framework for hybrid linear modeling. *Found. Comput. Math.*, 9(5):517–558, 2009.

[10] G. Chen and G. Lerman. Spectral curvature clustering (SCC). *Int. J. Comput. Vision*, 81(3):317–330, 2009.

[11] S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursuit. *SIAM J. Sci. Comput.*, 20(1):33–61 (electronic), 1998.

[12] J. Costeira and T. Kanade. A multibody factorization method for independently moving objects. *International Journal of Computer Vision*, 29(3):159–179, 1998.

[13] N. P. da Silva and J. P. Costeira. Subspace segmentation with outliers: A grassmannian approach to the maximum consensus subspace. In *CVPR*. IEEE Computer Society, 2008.

[14] C. Ding, D. Zhou, X. He, and H. Zha. R1-PCA: rotational invariant l1-norm principal component analysis for robust subspace factorization. In *ICML ’06: Proceedings of the 23rd international conference on Machine learning*, pages 281–288, New York, NY, USA, 2006. ACM.

[15] D. L. Donoho. For most large underdetermined systems of equations, the minimal $l_1$-norm near-solution approximates the sparsest near-solution. *Comm. Pure Appl. Math.*, 59(7):907–934, 2006.

[16] D. L. Donoho. For most large underdetermined systems of linear equations the minimal $l_1$-norm solution is also the sparsest solution. *Comm. Pure Appl. Math.*, 59(6):797–829, 2006.

[17] D. L. Donoho and M. Elad. Optimally sparse representation in general (nonorthogonal) dictionaries via $l^1$ minimization. *Proc. Natl. Acad. Sci. USA*,}
A. Edelman, T. A. Arias, and S. T. Smith. The geometry of algorithms with orthogonality constraints. *SIAM J. Matrix Anal. Appl.*, 20(2):303–353 (electronic), 1999.

M. Fischler and R. Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. *Comm. of the ACM*, 24(6):381–395, June 1981.

J. Gao. Robust $L_1$ principal component analysis and its Bayesian variational inference. *Neural Comput.*, 20(2):303–353 (electronic), 1999.

G. Golub and C. V. Loan. *Matrix Computations*. John Hopkins University Press, Baltimore, Maryland, 1996.

F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, and W. A. Stahel. *Robust Statistics: The Approach Based on Influence Functions (Wiley Series in Probability and Statistics)*. Wiley-Interscience, New York, revised edition, April 2005.

J. Ho, M. Yang, J. Lim, K. Lee, and D. Kriegman. Clustering appearances of objects under varying illumination conditions. In *Proceedings of International Conference on Computer Vision and Pattern Recognition*, volume 1, pages 11–18, 2003.

P. J. Huber. *Robust Statistics*. John Wiley & Sons Inc., New York, 1981. Wiley Series in Probability and Mathematical Statistics.

N. Kambhatla and T. K. Leen. Fast non-linear dimension reduction. In *Advances in Neural Information Processing Systems 6*, pages 152–159. Morgan Kaufmann, 1994.

K. Kanatani. Motion segmentation by subspace separation and model selection. In *Proc. of 8th ICCV*, volume 3, pages 586–591. Vancouver, Canada, 2001.

K. Kanatani. Evaluation and selection of models for motion segmentation. In *7th ECCV*, volume 3, pages 335–349, May 2002.

Q. Ke and T. Kanade. Robust subspace computation using $L_1$ norm. Technical report, Carnegie Mellon, 2003.

D. A. Klain and G.-C. Rota. *Introduction to Geometric Probability*. Cambridge University Press, 1997.

N. Kwak. Principal component analysis based on L1-norm maximization. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 30(9):1672–1680, 2008.

G. Lerman and J. T. Whitehouse. On $d$-dimensional $d$-semimetrics and simplex-type inequalities for high-dimensional sine functions. *J. Approx. Theory*, 156(1):52–81, 2009.

H. P. Lopuhaä and P. J. Rousseeuw. Breakdown points of affine equivariant estimators of multivariate location and covariance matrices. *Ann. Statist.*, 19(1):229–248, 1991.

Y. Ma, H. Derksen, W. Hong, and J. Wright. Segmentation of multivariate mixed data via lossy coding and compression. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(9):1546–1562, September 2007.

Y. Ma, A. Y. Yang, H. Derksen, and R. Fossum. Estimation of subspace arrangements with applications in modeling and segmenting mixed data. *SIAM Review*, 50(3):413–458, 2008.
[35] R. A. Maronna, R. D. Martin, and V. J. Yohai. *Robust statistics*. Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester, 2006. Theory and methods.

[36] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press, 1995.

[37] S. Rao, R. Tron, R. Vidal, and Y. Ma. Motion segmentation in the presence of outlying, incomplete, or corrupted trajectories. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 99(1), 5555.

[38] P. J. Rousseeuw and A. M. Leroy. *Robust regression and outlier detection*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Inc., New York, 1987.

[39] S. J. Szarek. The finite-dimensional basis problem with an appendix on nets of Grassmann manifolds. *Acta Math.*, 151(3-4):153–179, 1983.

[40] S. J. Szarek. Metric entropy of homogeneous spaces. In *Quantum probability (Gda´nsk, 1997)*, volume 43 of *Banach Center Publ.*, pages 395–410. Polish Acad. Sci., Warsaw, 1998.

[41] M. Tipping and C. Bishop. Mixtures of probabilistic principal component analysers. *Neural Computation*, 11(2):443–482, 1999.

[42] P. H. S. Torr. Geometric motion segmentation and model selection. *Phil. Trans. Royal Society of London A*, 356:1321–1340, 1998.

[43] P. H. S. Torr and A. Zisserman. Robust computation and parametrization of multiple view relations. In *ICCV ’98: Proceedings of the Sixth International Conference on Computer Vision*, page 727, Washington, DC, USA, 1998. IEEE Computer Society.

[44] P. H. S. Torr and A. Zisserman. MLESAC: A new robust estimator with application to estimating image geometry. *Computer Vision and Image Understanding*, 78(1):138–156, 2000.

[45] P. Tseng. Nearest $q$-flat to $m$ points. *Journal of Optimization Theory and Applications*, 105(1):249–252, April 2000.

[46] R. Vidal, Y. Ma, and S. Sastry. Generalized principal component analysis (GPCA). *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 27(12), 2005.

[47] Y.-C. Wong. Differential geometry of Grassmann manifolds. *Proc. Nat. Acad. Sci. U.S.A.*, 57:589–594, 1967.

[48] J. Yan and M. Pollefeys. A general framework for motion segmentation: Independent, articulated, rigid, non-rigid, degenerate and nondegenerate. In *ECCV*, volume 4, pages 94–106, 2006.

[49] A. Y. Yang, S. R. Rao, and Y. Ma. Robust statistical estimation and segmentation of multiple subspaces. In *CVPRW ’06: Proceedings of the 2006 Conference on Computer Vision and Pattern Recognition Workshop*, page 99, Washington, DC, USA, 2006. IEEE Computer Society.

[50] T. Zhang, A. Szlam, and G. Lerman. Median $K$-flats for hybrid linear modeling with many outliers. In *Computer Vision Workshops (ICCV Workshops), 2009 IEEE 12th International Conference on Computer Vision*, pages 234–241, Kyoto, Japan.

[51] T. Zhang, A. Szlam, Y. Wang, and G. Lerman. Randomized hybrid linear model-
ing by local best-fit flats. pages 1927–1934, jun. 2010.