INTERSECTION PATTERNS IN SPACES WITH A FORBIDDEN HOMOLOGICAL MINOR

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ABSTRACT. In this paper we study generalizations of classical results on intersection patterns of set systems in $\mathbb{R}^d$, such as the fractional Helly theorem or the $(p, q)$-theorem, in the setting of arbitrary triangulable spaces with a forbidden homological minor.

Given a simplicial complex $K$ and an integer $b$, we say that a family $F$ of subcomplexes of some simplicial complex $U$ is a $(K, b)$-free cover if (i) $K$ is a forbidden homological minor of $U$, and (ii) the $j$th reduced Betti number $\tilde{\beta}_j(\bigcap_{S \in G} S, \mathbb{Z}_2)$ is strictly less than $b$ for all $0 \leq j < \dim K$ and all nonempty subfamilies $G \subseteq F$.

We show that for every $K$ and $b$, the fractional Helly number of a $(K, b)$-free cover is at most $\mu(K) + 1$, where $\mu(K)$ is the maximum sum of the dimensions of two disjoint faces in $K$. This implies that the assertion of the $(p, q)$-theorem holds for every $p \geq q > \mu(K)$ and every $(K, b)$-free cover $F$. For $b = 1$ and a suitable $K$ this recovers the original $(p, q)$-theorem and its generalization to good covers. Interestingly, our results show that the range of parameters $(p, q)$ for which the $(p, q)$-theorem holds is independent of $b$.

Our proofs use Ramsey-type arguments combined with the notion of stair convexity of Bukh et al. to construct (forbidden) homological minors in cubical complexes.

2020 Mathematics Subject Classification. Primary 52A35, 52C45; Secondary 05E45, 05D99

Keywords: Helly-type theorems, topological combinatorics, homological minors, stair convexity, cubical complexes, homological VC dimension, Ramsey-type theorems

1. INTRODUCTION

Helly’s theorem on the intersection of convex sets is one of the most well-known results of combinatorial convexity. Applications, generalizations, and variations of this theorem have been studied extensively for nearly a century, and now comprise a significant area of discrete geometry. We refer the reader to [10, 12, 16] for in-depth surveys and further references, and to the textbooks [6, 33] for an introduction to the area.

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Date: April 17, 2024.
1.1. Problem statement. In this paper we are concerned with generalizations of Helly’s theorem that allow for more flexible intersection patterns and relax the convexity assumption. A famous example is the celebrated $(p, q)$-theorem [3], which asserts that for a finite family of convex sets in $\mathbb{R}^d$, if the intersecting subfamilies are evenly distributed, in the sense that among every $p$ members some $q$ intersect, then a constant number of points suffice to intersect all the convex sets. The crucial part here is that the constant depends only on $p$, $q$ and $d$, but not on the size of family.

Two central problems in this line of research are to identify the weakest possible assumptions under which the classical theorems generalize, and to determine their key parameters, for instance the Helly number $(d + 1$ for convex sets in $\mathbb{R}^d$) or the range for which the $(p, q)$-theorem holds (every $p \geq q \geq d + 1$ for convex sets in $\mathbb{R}^d$). In this respect, the case of convex lattice sets, that is, sets of the form $C \cap \mathbb{Z}^d$ where $C$ is a convex set in $\mathbb{R}^d$, showcases an interesting phenomenon: the Helly number is $2^d$ [14, 36], an exponential dependency on the dimension that contributes to the computational intractability of integer programming [12, §6], but a $(p, q)$-theorem holds for every $p \geq q \geq d + 1$ [7]; in the words of Bárány and Matoušek [7, §1], “... this large Helly number can be regarded as a ‘local anomaly’ and that the relevant number for other, more global Helly-type properties is only $d + 1$.”.

In this paper, we show that the gap observed for convex lattice sets occurs in the broad topological setting of triangulable spaces with a forbidden homological minor, a notion introduced by Wagner [37] as a higher-dimensional analogue of the familiar notion of graph minors [34].

1.2. Our setting. Before stating our results, some definitions are in order. Let $K$ and $U$ be finite simplicial complexes. Throughout this paper, all simplicial complexes are finite, and homology is taken with $\mathbb{Z}$ and $\mathbb{K}$ means $\mathbb{Z}$ or $\mathbb{K}$-coefficients, that is $C_\bullet(K)$ means $C_\bullet(K; \mathbb{Z})$. Depending on the setting, we use singular homology, simplicial homology or cellular homology.

The support of a chain $\sigma$, denoted $\text{supp}(\sigma)$, in a simplicial complex is the set of simplices with nonzero coefficients in $\sigma$. We say that two chains $\sigma$ and $\tau$ have overlapping supports if there exists a simplex in the support of $\sigma$ that intersects a simplex in the support of $\tau$; if no such pair of simplices exist we say that $\sigma$ and $\tau$ have nonoverlapping supports. A chain map $f_\bullet: C_\bullet(K) \to C_\bullet(U)$ is nontrivial if the image of every vertex of $K$ is a 0-chain of $U$ supported on an odd number of vertices. The simplicial complex $K$ is a homological minor of $U$, written $K \prec_H U$, if there exists a nontrivial chain map $f_\bullet: C_\bullet(K) \to C_\bullet(U)$ such that disjoint simplices are mapped to chains with nonoverlapping supports. If no such chain map exists we say that that $K$ is a forbidden minor of $U$, and write $K \not\prec_H U$. \(^1\)

In our results, the usual ambient space $\mathbb{R}^d$ will be replaced by a simplicial complex $U$ with a forbidden homological minor $K$ (we elaborate on this in Section 1.4.1). This point of view requires a substitute for the ambient dimension $d$,

\(^1\)The notion of homological minor readily extends to any triangulable space: $K$ is a forbidden homological minor of a space $X$ if $K \not\prec_H T$ for every triangulation $T$ of $X$. For instance, it can be shown (see [18, Corollary 13]) that the complete graph on 5 vertices (viewed as a 1-dimensional simplicial complex) is a forbidden homological minor of every triangulation of a disk.
which turns out to be the parameter
\[ \mu(K) = \max_{\sigma, \tau \in K} \{ \dim \sigma + \dim \tau \}. \]

We note the obvious bounds \( \dim K \leq \mu(K) \leq 2 \dim K \).

Finally, we need to define what will replace (and relax!) the convexity assumption. To do so, let us denote by \( \hat{\beta}_j(\cdot) \) the \( j \)th reduced Betti number, that is, the rank of the reduced homology group \( \tilde{H}_j(\cdot) \). For a finite simplicial complex \( K \) of positive dimension and a positive integer \( b \) we define a \((K, b)\)-free cover in a simplicial complex \( \mathcal{U} \) to be a finite family \( \mathcal{F} \) of (not necessarily induced) subcomplexes of \( \mathcal{U} \) such that:

(i) \( K \) is a forbidden minor of \( \mathcal{U} \), and
(ii) the \( j \)th reduced Betti number \( \hat{\beta}_j(\bigcap_{S \in G} S) \) is strictly less than \( b \) for all \( 0 \leq j < \dim K \) and all nonempty subfamilies \( G \subseteq \mathcal{F} \).

1.3. Main results. It is known that the Helly number of a \((K, b)\)-free cover is bounded from above in terms of \( K \) and \( b \) [18] \(^2\), as is the Radon number [35, Proposition 3.7]. This implies that \((K, b)\)-free covers also enjoy a fractional Helly theorem [22] and a \((p, q)\)-theorem [2]. This approach, however, only yields a fractional Helly number \( m \), and a \((p, q)\)-theorem for \( p \geq q \geq m \), where for \( \dim K > 1 \), \( m \) is some very large number obtained by successive iterations of Ramsey’s theorem [35]. \(^3\) Our main result improves this:

**Theorem 1.1.** The assertion of the \((p, q)\)-theorem holds for \((K, b)\)-free covers for every finite simplicial complex \( K \) and integers \( p \geq q > \mu(K) \) and \( b \geq 1 \).

For some complexes \( K \) this range of pairs \((p, q)\) is sharp, for instance when \( K \) is the \( d \)-skeleton of the \((2d + 2)\)-simplex (which includes the classical \((p, q)\) theorem for convex sets in \( \mathbb{R}^{2d} \) as a special case). The bound on the piercing number given by Theorem 1.1 depends on \( p, q, K \) and \( b \) (but, as usual, is independent of the size of the cover). Moreover, while the Helly number of a \((K, b)\)-free cover can grow with \( b \) (it is at least \((b - 1)(\mu(K) + 2) \) [18, Example 2]), the range for which the \((p, q)\)-theorem holds is independent of \( b \), thus displaying a similar gap as observed for convex lattice sets.

Let us describe our main technical result and the cornerstone of the proof of Theorem 1.1. For a finite family \( \mathcal{F} \) of subsets of some (finite or infinite) ground set, let \( \pi_m(\mathcal{F}) \) denote the number of \( m \)-element subfamilies of \( \mathcal{F} \) with nonempty intersection. A straightforward counting argument shows that “positive density propagates downwards” in the sense that if \( \pi_m(\mathcal{F}) \geq \alpha(\frac{|\mathcal{F}|}{m}) \), then \( \pi_{m-1}(\mathcal{F}) \geq \alpha(\frac{|\mathcal{F}|}{m - 1}) \). (A more precise bound follows from the Kruskal–Katona theorem.) We show that, roughly speaking, positive density propagates upwards for \((K, b)\)-free covers:

**Theorem 1.2.** Fix a simplicial complex \( K \), a value \( \delta \in (0, 1] \), and integers \( b \geq 1 \) and \( m > \mu(K) \). If \( \mathcal{F} \) is a sufficiently large \((K, b)\)-free cover such that \( \pi_m(\mathcal{F}) \geq \)

\(^2\) The bound on Helly number of \((K,b)\)-free cover directly follows from a combination of Proposition 30 and Lemma 26 in [18].

\(^3\) For \( \dim K = 1 \), \( m \) is in fact linear in \( b \) and the bound is optimal for \( b = 1 \) [35].
One immediate application of Theorem 1.2 is the reduction of fractional Helly numbers. For instance, it easily improves a theorem of Paták [35, Theorem 2.3] into:

**Corollary 1.3.** Let $K$ be a finite simplicial complex and $b \geq 1$ an integer. If a positive fraction of the $(\mu(K) + 1)$-tuples of a $(K, b)$-free cover $F$ have nonempty intersection, then a positive fraction of $F$ has nonempty intersection.

In the terminology of combinatorial convexity, Corollary 1.3 states that the fractional Helly number for $(K, b)$-free covers is at most $\mu(K) + 1$. More specifically, for any $\alpha > 0$, if a $(K, b)$-free cover $F$ satisfies $\pi_{\mu(K)+1}(F) \geq \alpha \left( \frac{|F|}{\mu(K)+1} \right)$, then there is a subfamily $G \subset F$ with nonempty intersection of size at least $\beta |F|$, where $\beta > 0$ depends only on $K$, $b$, and $\alpha$. The results from [35] required a positive fraction of the $m$-tuples to have a nonempty intersection, where for $\dim K > 1$, $m$ is some hypergraph Ramsey number depending on $b$ and $K$. So in order to prove Corollary 1.3 it suffices to show that if a positive fraction of the $(\mu(K) + 1)$-tuples intersect, then a positive fraction of the $m$-tuples intersect. This follows from successive applications of Theorem 1.2. (Note that [35, Theorem 2.3] still needs to be proven independently to provide a stopping point for the successive applications of Theorem 1.2; also, the implicit bound given by the proof of [35, Theorem 2.3] on the constant $\beta$ changes in the process.)

Theorem 1.1 is an immediate consequence of Corollary 1.3 and the results of Alon et al. [2]. Roughly speaking, Alon et al. show that if an intersection closed set system has fractional Helly number at most $m$, then the set system also satisfies a $(p, q)$-theorem for the range $p \geq q \geq m$. (See Theorems 8(i) and 9 and the discussion in §2.1 in [2].) Thus Theorem 1.1 follows from Corollary 1.3 applied to the family $F^{\cap} \overset{\text{def}}{=} \{ \cap_{S \in G} S \colon G \subset F \}$, by observing that $F^{\cap}$ is a $(K, b)$-free cover if and only if $F$ is.

### 1.4. Relation to previous work

Let us comment on how our results relate to classical lines of research in combinatorial convexity and topological combinatorics.

#### 1.4.1. Topological covers

Relaxing the convexity assumption in Helly’s theorem is a classical question that goes back to the topological Helly theorem [21, 11], which asserts that Helly’s theorem holds for finite good covers, that is, for finite families of open sets in $\mathbb{R}^d$ for which every nonempty intersection is acyclic (in the sense of homology). The fractional Helly theorem and the $(p, q)$-theorem also hold for finite good covers [2].

Several of the topological generalizations of Helly’s theorem [32, 13, 29] were recently brought under a common umbrella by considering covers of “bounded topological complexity”. Let $F = \{ S_1, \ldots, S_n \}$ be a finite family of open subsets of $\mathbb{R}^d$, and let $h$ and $b$ be positive integers. We say that $F$ is a $(h, b)$-cover in $\mathbb{R}^d$ if $\beta_j(\bigcap_{I \subseteq [n]} S_I) < b$ for all $0 \leq j < h$ and $\emptyset \neq I \subseteq [n]$. In $\mathbb{R}^d$, $(d, 1)$-covers are the same as finite good covers in the traditional sense, and $(d, b)$-covers already generalize various assumptions considered in earlier works in this area [32, 29, 13].

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*Footnote:* [35, Theorem 2.3] was not phrased in terms of $(K, b)$-free covers but readily generalizes to that setting, see Section 1.4.1.
It turns out that for every $d \geq 1$ there exists a $\lceil d/2 \rceil$-dimensional simplicial complex $K_d$ with $\mu(K_d) = d$ such that such that any $\lceil d/2 \rceil$-cover in $\mathbb{R}^d$ has the intersection pattern of a $(K_d, b)$-free cover (see [18, section 2.2]). This follows from two observations: (i) any $\lceil d/2 \rceil$-cover in $\mathbb{R}^d$ can be represented as a family of subcomplexes of a sufficiently fine triangulation of the $d$-dimensional ball, and (ii) if a simplicial complex $U$ embeds in $\mathbb{R}^d$, then $K_d \not\preceq H U$.

Through a series of papers [18, 35, 22], the Helly numbers, Radon numbers, and fractional Helly numbers for $\lceil d/2 \rceil$-covers in $\mathbb{R}^d$ were bounded in terms of $d$ and $b$, which also implies a $(p, q)$-theorem for $p \geq q \geq m$, for sufficiently large $m$.

Recasting $\lceil d/2 \rceil$-covers as $(K, b)$-free covers allows to extend the range of parameters for which a $(p, q)$-theorem holds. Specifically, Theorem 1.1 implies:

**Corollary 1.4.** The assertion of the $(p, q)$-theorem holds for $\lceil d/2 \rceil$-covers in $\mathbb{R}^d$ for every $p \geq q \geq d + 1$ and $b \geq 1$.

Note that the constant number of points given by the $(p, q)$-theorem in this case depends not only on $p$, $q$, and $d$, but also on $b$. For the setting of $(1, b)$-covers in surfaces, studied in [35], Theorem 1.1 also improves [35, Theorem 2.6], for $b \geq 2$, into:

**Corollary 1.5.** The assertion of the $(p, q)$-theorem holds for $(1, b)$-covers in surfaces for every $p \geq q \geq 3$ and $b \geq 1$.

Now the constant numbers of points given by these $(p, q)$-theorems also depend on the surfaces considered.

We conjecture the following generalization of Corollaries 1.4 and 1.5:

**Conjecture 1.6.** For any $d$-dimensional manifold $M$, the assertion of the $(p, q)$-theorem holds for $\lceil d/2 \rceil$-covers in $M$ for every $p \geq q \geq d + 1$ and $b \geq 1$.

We believe that the approach of forbidden homological minors would be useful for tackling Conjecture 1.6. One important step would be to establish the following (see also [35, Problem 4]):

**Conjecture 1.7.** For every $d$-dimensional manifold $M$ there exists a simplicial complex $K_M$ with $\mu(K_M) = d$ such that for any arbitrary triangulation $U$ of $M$ we have $K_M \not\preceq U$.

### 1.4.2. Stepping-up in combinatorial convexity.

The original fractional Helly theorem of Katchalski and Liu [31] asserts that for any $d \geq 1$ there is a function $\beta_d : (0, 1) \to (0, 1)$ such that any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ with $\pi_{d+1}(\mathcal{F}) \geq \alpha(\mathcal{F}, \beta_d^{-1})$ must contain an intersecting subfamily of size at least $\beta_d(\alpha)|\mathcal{F}|$. They also observed that one may assume that $\beta_d(\alpha) \to 1$ when $\alpha \to 1$. They derived it from the observation, which they dubbed the stepping-up lemma, that for any $0 < \alpha \leq 1$, any $d < k < \ell$ and any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$,

$$1. \quad \pi_k(\mathcal{F}) \geq \alpha \cdot \binom{|\mathcal{F}|}{k} \quad \implies \quad \pi_\ell(\mathcal{F}) \geq \left(1 - (1 - \alpha) \left(\frac{\ell - 1}{k - 1}\right)\right) \binom{|\mathcal{F}|}{\ell}.$$

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5By a *surface* we mean a compact 2-dimensional real manifold.
In particular, if \( \alpha > 1 - 1/(k-1) \), then a positive proportion of the \( \ell \)-element subfamilies of \( \mathcal{F} \) have nonempty intersection. Theorem 1.2 asserts that this propagation of positive densities holds more generally for \((K,b)\)-free covers.

Kalai [25] later showed that one can take \( \beta_d(\alpha) \) defined as \( 1 - (1 - \alpha)^{1/d+1} \), which is best possible. His proof is based on the upper bound theorem that he [25] and Eckhoff [15] established independently. The upper bound theorem asserts that for any family \( \mathcal{F} \) of \( n \) convex sets in \( \mathbb{R}^d \), for any \( k \) such that \( d < k \leq d + r \),

\[
\pi_k(\mathcal{F}) > \sum_{i=0}^{d} \binom{n-r}{i} \binom{r}{k-i} \implies \pi_{d+r+1}(\mathcal{F}) > 0,
\]

This was recently extended to \((1,b)\)-covers in \( \mathbb{R}^2 \) and in surfaces for any fixed \( b \geq 1 \) [30, Theorem 2.2 and 2.3]. A further extension to \((K,b)\)-free covers would be interesting.

### 1.4.3. Collapsibility and Lerayness

The known proofs of the upper bound theorem (2) are typically more general than the geometric setting, and deal with certain properties of nerves of families of convex sets. The more elementary proofs apply to \( d \)-collapsible complexes [15, 25, 1], that is complexes that can be reduced by discrete homotopy moves (called collapses) to a \( d \)-dimensional complex [38, Lemma 1]. The more general proof, also due to Kalai (a presentation can be found in Hell’s PhD thesis [20, §5.2]), applies to \( d \)-Leray complexes, that is complexes in which all induced subcomplexes have vanishing homology in dimension \( d \) and above. Theorem 1.2 reveals that nerves of \((K,b)\)-free covers enjoy some of the consequences of bounded Lerayness. More generally, we conjecture:

**Conjecture 1.8.** For any simplicial complex \( K \) and positive integer \( b \), the nerve of any \((K,b)\)-free cover is \( L \)-Leray, where \( L \) depends only on \( K \) and \( b \).

We note that \( L \) must be at least \((b-1)\mu(K) + 2b - 3\) [18, Example 2]. A first step in this direction was done by Holmsen et al. [23], who considered the special case when \( \mathcal{F} = \{G_i\} \) is a family of connected subgraphs of a graph \( G \) such that any nonempty intersection of members of \( \mathcal{F} \) is also connected. They conjectured that for any \( r \geq 1 \), if \( K_{r+2} \) is not a minor of \( G \), then the nerve of \( \mathcal{F} \) is \( r \)-Leray, and verified this for \( r \leq 3 \). It should be noted that in [23] they deal with graph minors [34] and not homological minors.

### 1.4.4. Homological VC-dimension

Deeper connections between discrete geometry and topological combinatorics were suggested by Kalai and Meshulam in a program to develop a theory of homological VC dimension. For a positive integer \( h \) and a family \( \mathcal{F} \) of sets in \( \mathbb{R}^d \), let us call the function

\[
\phi_F^{(h)} : \begin{cases} \mathbb{N} &\rightarrow& \mathbb{N} \cup \{\infty\} \\ k &\mapsto& \sup \{ \beta_i(\cap \mathcal{G}) : \mathcal{G} \subseteq \mathcal{F}, 1 \leq |\mathcal{G}| \leq k, 0 \leq i < h \} \end{cases}
\]

the \((h)\)th homological shatter function of \( \mathcal{F} \). Combining two conjectures of Kalai and Meshulam, suggests that families of open sets in \( \mathbb{R}^d \) with polynomial homological shatter function should enjoy a fractional Helly theorem:

**Conjecture 1.9 (Following [26, Conjectures 6 and 7]).** For any integer \( 0 \leq m \leq d \) and any constant \( C > 0 \), there exists a function \( \beta : (0,1) \rightarrow (0,1) \) such that the following holds. For any \( \alpha > 0 \) and any sufficiently large finite family \( \mathcal{F} \) of
open sets in $\mathbb{R}^d$ with $\phi^{(d)}_F(k) \leq CK^m$, if $\pi_{d+1}(\mathcal{F}) \geq \alpha\left(\frac{|\mathcal{F}|}{d+1}\right)$ then some $\beta(\alpha)|\mathcal{F}|$ members of $\mathcal{F}$ have a point in common.

A combination of Conjectures 6 and 7 from [26] also appeared in [27, Conjecture 17]. Here we took upon ourselves to dissociate the dimension $d$ of the space and the degree $m$ of the polynomial bounding the homological shatter function. We give in to the temptation to generalize this conjecture to $K$-free covers, that is to finite families $\mathcal{F}$ of (not necessarily induced) subcomplexes of a simplicial complex $\mathcal{U}$ such that $K \not\prec \mathcal{U}$. We propose the following more general conjecture:

**Conjecture 1.10.** For any simplicial complex $K$, integer $t \geq 0$, and constant $C > 0$, there exists $h \geq 1$ and a function $\beta : (0,1) \to (0,1)$ such that the following holds. For any $\alpha > 0$ and any sufficiently large $K$-free cover $\mathcal{F}$ with $\phi^{(\dim K)}_F(k) \leq CK^t$, if $\pi_h(\mathcal{F}) \geq \alpha\left(\frac{|\mathcal{F}|}{h}\right)$, then some $\beta(\alpha)|\mathcal{F}|$ members of $\mathcal{F}$ have a point in common.

In other words, we conjecture a fractional Helly theorem for $K$-free covers whose $(\dim K)$th homological shatter function is bounded by a polynomial of degree $t$. Our Corollary 1.3, with $b > C$, settles the case $t = 0$ of this conjecture.

1.4.5. **Algorithmic consequences.** Fractional Helly theorems find applications in the area of property testing. Recall that Helly’s theorem relates to the size of witness sets for convex programming, so that its generalizations, the so-called Helly-type theorems, correspond to the combinatorial dimension of LP-type problems [4] (see also [18, §1.3]). Similarly, generalizations of the fractional Helly theorem lead to property testing algorithms for optimization under constraints, by relating the probability that a random choice of $k$ constraints is satisfiable to the size of the largest subset of constraints that can be simultaneously satisfied. (Here, $k$ denotes the fractional Helly number.) This relation was spelled out by Chakraborty et al. [9] in the convex settings and holds more generally. Again, notice that reducing a fractional Helly number also improves this property testing algorithm by reducing the size $k$ of the random subproblems to be tested.

## 2. Proof Outline and Preparation

Here we give an outline the proof Theorem 1.2 and introduce some terminology. We write $\mathbb{N} = \{1,2,\ldots\}$ for the set of positive integers and $\mathbb{N}_0 = \{0,1,\ldots\}$ for the set of non-negative integers. We write $[n] = \{1,2,\ldots,n\}$ and $\binom{[n]}{k}$ for the set of $k$-element subsets of $[n]$. We denote by $\Delta_s$ the $s$-dimensional simplex and by $K^{(t)}$ the $t$-skeleton of a simplicial (or cell) complex $K$.

### 2.1. Strategy.

At a high-level, we follow a two-stage approach that we learned from the work of Bárány and Matoušek, on the fractional Helly number for convex lattice sets [7]. The first stage is to establish a Helly-type theorem that turns a particular intersection pattern on the $m$-element subsets of some family of constant size into at least one intersection of $m + 1$ sets. We then use the positive density assumption $\pi_m(\mathcal{F}) > \delta\binom{n}{m}$ to find many occasions to apply that constant-size theorem.

For this approach to work, we need the particular intersection pattern from the first stage to be “massively unavoidable” when $\pi_m(\mathcal{F}) > \delta\binom{n}{m}$. Here, some extremal hypergraph theory, in particular the Erdős and Simonovits “supersaturation
2.2. Intersection patterns via constrained chain maps. Section 5.

a matter of supersaturation and double counting. These details are spelled out in

With Theorem 2.1 at our fingertips, our main result Theorem 1.2 is then merely

subfamily has nonempty intersection, then

Theorem 2.1 is quite involved and builds on the method of

The proof of Theo-

m 1.2 is merely

reducible to the construction of nontrivial chain maps

f : C_*(K) → C_*(U) constrained by F [18, Proposition 30].
The proof produces a chain map \( f_* : C_*(K(t)) \to C_*(\Delta) \) constrained by \( F \) by induction on \( t \), for \( t = 0, 1, \ldots, \dim K \). Assume the chain map \( f_* : C_*(K(t-1)) \to C_*(\Delta) \) has been constructed, and is constrained by some \( \Phi : K(t-1) \to 2^{|n|} \). Now for any \((t-1)\)-simplex \( \sigma \in K \), let \( h(\sigma) \) denote the homology class of \( f_*(\sigma) \) in \( H_{t-1}(\bigcap_{i \in X} S_i) \). For each \( X \subseteq [n] \) the space \( H_{t-1}(\bigcap_{i \in X} S_i) \) is a \( \mathbb{Z}_2 \)-vector space of dimension at most \( b \). After fixing a basis for each of these vector spaces, we may view \( h \) as a map that sends \((t-1)\)-simplices of \( K \) to vectors in \((\mathbb{Z}_2)^b \), and then linearly extend \( h \) into a group homomorphism \( h : C_{t-1}(K) \to (\mathbb{Z}_2)^b \). A simple application of Ramsey’s theorem then yields:  

**Claim 2.2.** For any integers \( b, s, t \) there exists an integer \( r \) such that for any group homomorphism \( h : C_{t-1}(\Delta_r) \to (\mathbb{Z}_2)^b \) there exists a chain map \( f_* : \Delta(t) \to \Delta(t) \) such that for every \( t \)-dimensional simplex \( \sigma \in \Delta_s \) we have \( h(f_*(\partial \sigma)) = 0 \).

This may seem enough to bootstrap the induction: since each \( f_*(\partial \sigma) \) has vanishing homology, it is a “boundary” and can thus be filled in homology, and this filling can be used to extend \( f_* \) to \( \sigma \). On closer inspection, each \( h(f_*(\partial \sigma)) \) is a sum of vectors, each representing the homology class of the image of a facet of \( \sigma \) under \( f_* \), but these vectors live in distinct vector spaces, that is, in \( H_{t-1}(\bigcap_{i \in X} S_i) \) for different subsets \( X \). The vanishing of the sum therefore has no clear homological meaning, and in \([18, 35]\) this leads to additional technical complications; in our current setting, this difficulty does not arise.

Returning to the proof of the Theorem 2.1, a major difference is that we now need to account for the \( m \)-partite structure of the color classes. To do so, we work with homological minors in chain complexes built out of cubical cells rather than simplices (we will call them grid complexes). Consider an \( m \)-colorful family \( F = \mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_m \), and suppose \(|\mathcal{F}_i| = n\), for every \( i \). By labelling the members of \( \mathcal{F}_i \) arbitrarily by the integers in \([n]^b\), we get a bijection between the maximal colorful subfamilies \( \{S_{i_1}, \ldots, S_{i_m}\} \), where \( S_{i_j} \in \mathcal{F}_j \), and the lattice points

\[
(i_1, \ldots, i_m) \in [n] \times \cdots \times [n] \subset \mathbb{N}^m.
\]

Similarly, an axis parallel line of the form

\[
\ell_j = \{(i_1, \ldots, i_{j-1}, x, i_{j+1}, \ldots, i_m)\},
\]

where \( i_1, \ldots, i_m \) are fixed and \( 1 \leq x \leq n \) is the free variable, will be associated with the colorful subfamily \( \{S_{i_1}, \ldots, S_{i_{j-1}}, S_{i_{j+1}}, \ldots, S_{i_m}\} \). Thus, the fixed coordinates of the line tell us the sets in the colorful subfamily. More generally, we associate an axis parallel \( k \)-flat in \([n]^m\) with a colorful subfamily of size \( m - k \).

2.3. **Grid complexes.** Let us recall how chain complexes can be built out of cubical cells. These are the cubical complexes in the sense, e.g., of Kaczynski et al. [24].

Let \( G[n] \) denote the 1-dimensional cell complex whose vertices (0-cells) are the singletons \( \{1\}, \{2\}, \ldots, \{n\} \) and whose closed 1-cells are the intervals \( \{i, j\} \), with \( i < j \). The \( \{i, j\} \) are indexed by ordered pairs, \( (i, j) \in \mathbb{N} \times \mathbb{N} \setminus \{\{i, i\}\} \); we do not spell out the proof as what we will actually need is the subgrid Lemma 4.2.

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\(^6\)This statement does not explicitly appear in \([18, 35]\). It follows from finding a homogeneous copy of the barycentric subdivision of \( \Delta(t) \) in \( \Delta(t) \); we do not spell out the proof as what we will actually need is the subgrid Lemma 4.2.
For $m \geq 1$, define the grid complex $G[n]^m$ as the $m$-fold product
\[
G[n]^m \overset{\text{def}}{=} \underbrace{G[n] \times \cdots \times G[n]}_{\text{$m$-fold}},
\]
equipped with the product topology. We can think of $G[n]^m$ abstractly, or geometrically realized in $\mathbb{R}^m$ as a collection of unit cubes whose union form an $m$-dimensional axis-parallel cube with sidelength $n - 1$.

**Cells.** For every integer $a \in [n]$ we use interchangeably the notations $[a, a] = \{a\}$ to denote the corresponding 0-cell in $G[n]$. For every integers $a, b \in [n]$ with $a < b$ we let $[a, b] = [b, a]$ denote the 1-chain with $\mathbb{Z}_2$ coefficients
\[
[a, b] \overset{\text{def}}{=} [a, a + 1] + [a + 1, a + 2] + \cdots + [b - 1, b].
\]
Notice that for any pairwise distinct integers $a, b, c \in [n]$ we have $[a, c] = [a, b] + [b, c]$, as we work with $\mathbb{Z}_2$-coefficients. Every $k$-cell $\sigma$ in $G[n]^m$ can be written as the product of exactly $(m - k)$ 0-cells and $k$ 1-cells
\[
\sigma = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m],
\]
where $1 \leq a_1 \leq b_1 \leq a_2 + 1 \leq n$. Note that $G[n]^m$ is a regular cell complex of dimension $m$.

The terminology introduced for simplicial complexes carries directly over to grid complexes. A $k$-chain is a sum of $k$-cells in $G[n]^m$ with coefficients in $\mathbb{Z}_2$. The support of a chain $\sigma$, denoted $\text{supp}(\sigma)$, is the set of cells with nonzero coefficients in $\sigma$, and two chains, $\sigma$ and $\tau$, have overlapping supports if there is a cell in $\text{supp}(\sigma)$ which intersects a cell in $\text{supp}(\tau)$. We formulate the following simple observation for future reference.

**Observation 2.3.** For any cells $\sigma, \tau$ in $G[n]^m$ such that $\text{dim } \sigma + \text{dim } \tau < m$, there is at least one coordinate that is constant for both. Moreover, if $\sigma$ and $\tau$ intersect, then they must be contained in a common axis-parallel hyperplane.

**Proof.** The observation easily follows from the fact that a $k$-dimensional cell of $G[n]^m$ has $m - k$ coordinates that are constant. Since $\text{dim } \sigma + \text{dim } \tau < m$, the conclusion follows. \qed

**Products and boundaries.** The usual (Cartesian) product $\times$ of a $k_1$-cell of $G[n]^{m_1}$ by a $k_2$-cell of $G[n]^{m_2}$ is a $(k_1 + k_2)$-cell of $G[n]^{m_1 + m_2}$. We extend it to chains by putting
\[
(\sigma_1 + \cdots + \sigma_{\ell_1}) \times (\tau_1 + \cdots + \tau_{\ell_2}) \overset{\text{def}}{=} \sum_{i=1}^{\ell_1} \sum_{j=1}^{\ell_2} \sigma_i \times \tau_j.
\]
We denote the null chain (with empty support) by $0$ and clarify that for any chain $\sigma$ we have $\sigma \times 0 = 0 \times \sigma = 0$. We can now define the boundary of a cell of $G[n]^m$ recursively, as follows:

- (0-cells) \( \partial\{a\} \overset{\text{def}}{=} 0 \) (the null chain)
- (1-cells) \( \partial[a, a + 1] \overset{\text{def}}{=} \{a\} + \{a + 1\} \)
- (\( \geq 2 \)-cells) \( \partial(\sigma \times \tau) \overset{\text{def}}{=} \partial\sigma \times \tau + \sigma \times \partial\tau \)
The definition of $\partial$ extends from $k$-cells to $k$-chains by linearity. A simple induction on the dimension yields that $\partial \circ \partial = 0$; in fact, $\partial$ coincides with the standard boundary operator on $G[n]^m$ seen as a regular cell complex. For a subcomplex $G$ of a grid complex, we write $C_\bullet (G)$ for the chain complex defined by the chains of $G$ together with $\partial$.

2.4. **Paper organization.** We first prove, in Section 3, that complexes with a forbidden simplicial homological minor also have a forbidden grid-like homological minor. The proof uses the *stair convexity* of Bukh et al. [8] to build, in a systematic way, chain maps from simplicial complexes to cubical complexes. We then adapt, in Section 4, the method of constrained chain maps and forbidden homological minors to study colorful intersection patterns. We conclude with the proof of Theorem 1.2 in Section 5.

3. **Homological minors in grid complexes**

The main goal of this section is to establish the following:

**Theorem 3.1.** Fix a $d$-dimensional simplicial complex $K$ on $n$ vertices and a simplicial complex $U$ such that $K \not\preceq_H U$. Let $Y$ be the $d$-skeleton of $G[n]^m$, where $m > \mu(K)$. For every nontrivial chain map $f_\bullet : C_\bullet (Y) \to C_\bullet (U)$ there exist disjoint cells $\sigma$ and $\tau$ in $Y$ which satisfy:

(i) $\sigma$ and $\tau$ are not contained in a common axis-parallel hyperplane.

(ii) The supports of $f_\bullet (\sigma)$ and $f_\bullet (\tau)$ are overlapping.

3.1. **Stair convex chains.** The stair convex hull was introduced by Bukh et al. [8] as a tool for analyzing point configurations related to extremal problems in discrete geometry such as lower bounds on the size of weak $\varepsilon$-nets. Here we define a particular class of chains in the grid complex $G[n]^m$ that resembles their recursive definition.

We fix some integer $n \geq 2$ and work, implicitly, in the grid complex $G[n]^m$. For any $m > 0$, $m \geq k \geq 0$, and for any integers $1 \leq a_1 < \cdots < a_{k+1} \leq n$ we define the stair convex $k$-chain $sc^m_k (a_1, \ldots, a_{k+1}) \in C_k (G[n]^m)$. The definition is recursive:

\[
\begin{align*}
(m > k = 0) & \quad sc^m_0 (a) \overset{\text{def}}{=} \underbrace{a, \ldots, a}_m \\
(m = k > 0) & \quad sc^m_k (a_1, \ldots, a_{m+1}) \overset{\text{def}}{=} [a_1, a_2] \times [a_2, a_3] \times \cdots \times [a_m, a_{m+1}] \\
(m > k > 0) & \quad sc^m_k (a_1, \ldots, a_{k+1}) \overset{\text{def}}{=} sc^{m-1}_{k-1} (a_1, \ldots, a_k) \times [a_k, a_{k+1}] \\
& \quad \quad + sc^{m-1}_k (a_1, \ldots, a_{k+1}) \times \{a_{k+1}\}
\end{align*}
\]

**Examples.** At one end, $sc^m_0 (a)$ is a vertex on the main diagonal of $G[n]^m$. At the other end, $sc^m_m (a_1, \ldots, a_{m+1})$ is an $m$-dimensional box. Let us examine some simple examples for intermediate values of $k$. For $m = 2$ and $k = 1$ we have

\[
sc^2_1 (a, b) = sc^0_1 (a) \times [a, b] + sc^1_1 (a, b) \times \{b\} = \{a\} \times [a, b] + [a, b] \times \{b\}
\]

which is a rectilinear path from $(a, a)$ to $(b, b)$ with a bend at $(a, b)$. Similarly, $sc^2_1 (a, b)$ is a rectilinear path from $(a, a, a)$ to $(b, b, b)$ with bends at $(a, a, a)$ and $(a, b, b)$:
\[ \text{Remark.} \] We will see that stair convex chains behave like simplices in several occasions (e.g. Proposition 3.4 and Lemma 3.6).

3.2. Non-recursive formulation. From the examples above, we start to glimpse a general pattern for “unwrapping” the recursive definition of \( \text{sc}_k^m \). Let us extend the definition of stair convex chain to the case \( m = k = 0 \) by putting, for any integer \( a \) and any chain \( \sigma \), \( \text{sc}_0^0(a) \times \sigma = \sigma \times \text{sc}_0^0(a) = \sigma \). Note that \( \text{sc}_0^0(a) \) behaves like the identity with respect to \( \times \).

**Lemma 3.2.** For every \( m \geq k \geq 1 \) and any \( 1 \leq a_1 < a_2 < \ldots < a_{k+1} \leq n \) we have

\[
\text{sc}_k^m(a_1, \ldots, a_{k+1}) = \sum_{t_1, t_2, \ldots, t_{k+1} \in \mathbb{N}_0 \atop t_1 + t_2 + \ldots + t_{k+1} = m-k} \text{sc}_0^{t_1}(a_1) \times \prod_{i=1}^{k} \left( [a_i, a_{i+1}] \times \text{sc}_0^{t_i}(a_{i+1}) \right).
\]

**Proof.** We define \( s_0^m(a) \overset{\text{def}}{=} \text{sc}_0^m(a) \), and for \( m \geq k \geq 1 \) we denote by \( s_k^m(a_1, \ldots, a_{k+1}) \) the right-hand term of the identity to prove. Note that \( s_k^m \) immediately satisfies the first two relations that define \( \text{sc}_k^m \):

\[
\begin{align*}
(m > k = 0) & \quad s_0^m(a) = \underbrace{(a, \ldots, a)}_{m\text{-fold}} \\
(m = k > 0) & \quad s_k^m(a_1, \ldots, a_{m+1}) = [a_1, a_2] \times [a_2, a_3] \times \ldots \times [a_m, a_{m+1}]
\end{align*}
\]

It therefore suffices to prove that \( s_k^m \) also satisfies the third relation:

\[
\begin{align*}
(m > k > 0) & \quad s_k^m(a_1, \ldots, a_{k+1}) = s_{k-1}^{m-1}(a_1, \ldots, a_k) \times [a_k, a_{k+1}] \\
& \quad \quad \quad + s_{k-1}^{m-1}(a_1, \ldots, a_{k+1}) \times \{a_{k+1}\}
\end{align*}
\]
For $m \geq k \geq 1$ let us define

$$I(m, k) \defeq \{(t_1, t_2, \ldots, t_{k+1}) \in \mathbb{N}_0^{k+1} : t_1 + t_2 + \ldots + t_{k+1} = m - k\},$$

and note that these sets of vectors are pairwise disjoint. For $(t_1, \ldots, t_{k+1}) \in I(m, k)$, let us define

$$\phi(t_1, \ldots, t_{k+1}) \defeq \text{sc}_0^1(a_1) \times \prod_{i=1}^k \left([a_i, a_{i+1}] \times \text{sc}_0^{t_i+1}(a_{i+1})\right).$$

In this notation, we have

$$s_k^m(a_1, \ldots, a_{k+1}) = \sum_{(t_1, t_2, \ldots, t_{k+1}) \in I(m, k)} \phi(t_1, \ldots, t_{k+1}).$$

Note that for $(t_1, t_2, \ldots, t_{k+1}) \in I(m, k)$, we have

$$\phi(t_1, \ldots, t_{k+1}) = \begin{cases} \phi(t_1, \ldots, t_k) \times [a_k, a_{k+1}] & \text{if } t_{k+1} = 0, \\ \phi(t_1, \ldots, t_{k+1} - 1) \times \{a_{k+1}\} & \text{otherwise}. \end{cases}$$

The desired identity now follows by splitting $I(m, k)$ into

$$I_0(m, k) \defeq \{(t_1, t_2, \ldots, t_{k+1}) \in I(m, k) : t_{k+1} = 0\}, \quad \text{and} \quad I_\succ (m, k) \defeq I(m, k) \setminus I_0(m, k),$$

and observing that $(t_1, t_2, \ldots, t_{k+1}) \mapsto (t_1, t_2, \ldots, t_{k+1} - 1)$ defines a bijection between $I_\succ (m, k)$ and $I(m - 1, k)$, and that $(t_1, t_2, \ldots, t_{k+1}) \mapsto (t_1, t_2, \ldots, t_k)$ defines a bijection between $I_0(m, k)$ and $I(m - 1, k - 1)$.

**Corollary 3.3.** Let $k < m$. An axis-parallel hyperplane $x_j = a$ contains a $k$-dimensional face of the support of $\text{sc}_k^m(a_1, \ldots, a_{k+1})$ if and only if $a \in \{a_1, \ldots, a_{k+1}\}$.

**Proof.** By Lemma 3.2 we have

$$\text{sc}_k^m(a_1, \ldots, a_{k+1}) = \sum_{t_1, t_2, \ldots, t_{k+1} \in \mathbb{N}_0} \text{sc}_{t_1}^1(a_1) \times \prod_{i=1}^k \left([a_i, a_{i+1}] \times \text{sc}_{t_i+1}^1(a_{i+1})\right).$$

Note that the support of $\text{sc}_k^m(a_1, \ldots, a_{k+1})$ is the union of the supports of the summands of the right-hand side. The support of a summand is contained in $x_j = a$ if and only if for some $i$ we have $t_i \neq 0$ and $a_i = a$.

### 3.3. Stair convex chains and the boundary operator

We start by proving that stair convex chains behave like $k$-dimensional simplices with respect to the boundary operator on grid complexes. Figure 1 illustrates this phenomenon in 2 dimensions.

To formalize this claim, let us define

$$(a_1, \hat{a}_i, \ldots, a_{k+1}) \defeq (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1}),$$

that is, $\hat{a}$ denotes the coordinate to be omitted. (Recall that all homology in this paper has coefficients in $\mathbb{Z}_2$.)
Proposition 3.4. For integers $m \geq k \geq 1$ and any sequence $a_1 < a_2 < \ldots < a_m$ of elements from $[n]$ we have $\partial sc^m_k(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} sc^m_{k-1}(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})$.

The proof is a somewhat lengthy, but straightforward, calculation that takes up most of this section. We set up an induction on $m$ by using the recursive definition of $sc^m_k$ (for $k < m$) or by applying the product rule after singling out the factor $[a_m, a_{m+1}]$ (for $k = m$). One important trick is to handle the factors $[a_{i-1}, a_{i+1}]$ arising from $sc^m_{k-1}(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})$ using the identity $[a_{i-1}, a_{i+1}] = [a_{i-1}, a_i] + [a_i, a_{i+1}]$ between 1-chains. Our first step is to establish a simpler identity.

Lemma 3.5. For any $m \geq 1$ and $1 \leq a_1 < a_2 < \ldots < a_{m+2} \leq n$ we have

$$\sum_{i=1}^{m+2} sc^m_m(a_1, \ldots, \hat{a}_i, \ldots, a_{m+2}) = 0.$$  

Proof. For $m = 1$, this is a reformulation of $[a_1, a_3] = [a_1, a_2] + [a_2, a_3]$ which holds by definition whenever $a_1, a_2$ and $a_3$ are pairwise distinct. For $m \geq 2$, we have

$$\sum_{i=1}^{m+2} sc^m_m(a_1, \ldots, \hat{a}_i, \ldots, a_{m+2}) = \left( \sum_{i=1}^{m} sc^m_{m-1}(a_1, \ldots, \hat{a}_i, \ldots, a_{m+1}) \times [a_{m+1}, a_{m+2}] \right)$$

$$+ sc^m_{m-1}(a_1, \ldots, a_m) \times [a_m, a_{m+2}]$$

$$+ sc^m_{m-1}(a_1, \ldots, a_m) \times [a_m, a_{m+1}]$$

$$= \left( \sum_{i=1}^{m+1} sc^m_{m-1}(a_1, \ldots, \hat{a}_i, \ldots, a_{m+1}) \right) \times [a_{m+1}, a_{m+2}],$$

since $[a_m, a_{m+2}] + [a_m, a_{m+1}] = [a_{m+1}, a_{m+2}]$ (here we use that the $a_i$ are pairwise distinct). The statement therefore follows by induction on $m$. \qed
Let us return to the proof of the claim that stair convex chains behave as simplices for the boundary operator on $G[n]^m$.

**Proof of Proposition 3.4.** The case $m = 1$ follows directly from the definition of boundary operator.

**Case $m = k \geq 2$.** Here, we proceed by induction on $m$, and assume that the statement holds for $m - 1$:

$$
\partial \mathcal{s}c_{m-1}^m(a_1, \ldots, a_m) = \sum_{i=1}^{m} \mathcal{s}c_{m-2}^{m-1}(a_1, \ldots, \hat{a}_i, \ldots, a_m).
$$

From the definition of $\partial$ we get

$$
\partial \mathcal{s}c_{m}^m(a_1, \ldots, a_{m+1}) = \partial (\mathcal{s}c_{m-1}^m(a_1, \ldots, a_m) \times [a_m, a_{m+1}])
$$

$$
= \left(\partial \mathcal{s}c_{m-1}^m(a_1, \ldots, a_m) \times [a_m, a_{m+1}] \right) + \mathcal{s}c_{m}^{m-1}(a_1, \ldots, a_m) \times (\{a_m\} + \{a_{m+1}\})
$$

We then use the induction hypothesis to rewrite $\alpha$ as

$$
\alpha = \sum_{i=1}^{m} \mathcal{s}c_{m-2}^{m-1}(a_1, \ldots, \hat{a}_i, \ldots, a_m) \times [a_m, a_{m+1}].
$$

and use Lemma 3.5 to partially expand $\beta$ into

$$
\beta = \mathcal{s}c_{m-1}^m(a_1, \ldots, a_m) \times \{a_m\} + \sum_{i=1}^{m} \mathcal{s}c_{m-1}^{m-1}(a_1, \ldots, \hat{a}_i, \ldots, a_{m+1}) \times \{a_{m+1}\}
$$

For $i \leq m - 1$, we have $\alpha_i + \beta_i = \mathcal{s}c_{m-1}^m(a_1, \ldots, \hat{a}_i, \ldots, a_{m+1})$ by the recursive definition of $\mathcal{s}c_{m-1}^m$. For the same reason, for any $a \in [n]$,

$$
(4) \quad \mathcal{s}c_{m-1}^m(a_1, \ldots, a_{m-1}, a) = \mathcal{s}c_{m-2}^{m-1}(a_1, \ldots, a_{m-1}) \times [a_{m-1}, a] + \mathcal{s}c_{m-1}^{m-1}(a_1, \ldots, a_{m-1}, a) \times \{a\}.
$$

Since the $a_i$ are pairwise distinct, we have $[a_m, a_{m+1}] = [a_{m-1}, a_m] + [a_{m-1}, a_{m+1}]$. Using Identity (4) once with $a = a_m$ and once with $a = a_{m+1}$, we obtain

$$
\alpha_m = \mathcal{s}c_{m-1}^m(a_1, \ldots, a_m) + \beta_0 + \mathcal{s}c_{m-1}^m(a_1, \ldots, a_{m-1}, a_{m+1}) + \beta_m.
$$

Altogether,

$$
\partial \mathcal{s}c_{m}^m(a_1, \ldots, a_{m+1}) = \alpha + \beta = \sum_{i=1}^{m+1} \mathcal{s}c_{m-1}^{m-1}(a_1, \ldots, \hat{a}_i, \ldots, a_{m+1})
$$

as claimed.
General case. We now prove the general case by induction on \( m \). So assume the statement holds for \( m - 1 \) and consider \( 1 \leq k \leq m \). We already handled the case \( k = m \), so let us consider the case \( k < m \), for which we can use the recursive definition of \( \text{sc}^m_k \):

\[
\text{sc}^m_k(a_1, \ldots, a_{k+1}) = \text{sc}^{m-1}_{k-1}(a_1, \ldots, a_k) \times [a_k, a_{k+1}] + \text{sc}^{m-1}_k(a_1, \ldots, a_{k+1}) \times \{a_{k+1}\}.
\]

We thus have

\[
\partial \text{sc}^m_k(a_1, \ldots, a_{k+1}) = (\partial \text{sc}^{m-1}_{k-1}(a_1, \ldots, a_k)) \times [a_k, a_{k+1}]
+ \text{sc}^{m-1}_{k-1}(a_1, \ldots, a_k) \times (\{a_k\} + \{a_{k+1}\})
+ (\partial \text{sc}^{m-1}_k(a_1, \ldots, a_{k+1})) \times \{a_{k+1}\}
\]

Using the induction hypothesis for \( \text{sc}^{m-1}_{k-1} \) and \( \text{sc}^{m-1}_k \), we obtain

\[
\partial \text{sc}^m_k(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k} \text{sc}^{m-1}_{k-2}(a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) \times [a_k, a_{k+1}]
+ \sum_{i=1}^{k+1} \text{sc}^{m-1}_{k-1}(a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) \times \{a_{k+1}\}
+ \text{sc}^{m-1}_{k-1}(a_1, \ldots, a_k) \times (\{a_k\} + \{a_{k+1}\})
= \sum_{i=1}^{k} \text{sc}^{m-1}_{k-2}(a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) \times [a_k, a_{k+1}]
+ \sum_{i=1}^{k} \text{sc}^{m-1}_{k-1}(a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) \times \{a_{k+1}\}
+ \text{sc}^{m-1}_k(a_1, \ldots, a_k) \times \{a_k\}
\]

For \( i \leq k - 1 \), we recognize the recursive definition of \( \text{sc}^{m}_{k-1} \) in

\[
\alpha_i + \beta_i = \text{sc}^m_{k-1}(a_1, \ldots, a_i, a_{k+1}).
\]

Since the elements \( a_1, \ldots, a_{k+1} \) are pairwise distinct, \( [a_k, a_{k+1}] = [a_{k-1}, a_k] + [a_{k-1}, a_{k+1}] \) and we can split \( \alpha_k \) and use the recursive definition of \( \text{sc}^m_{k-1} \) on each part:

\[
\alpha_k = \text{sc}^{m-1}_{k-2}(a_1, \ldots, a_{k-1}) \times [a_{k-1}, a_{k+1}] + \text{sc}^{m-1}_{k-2}(a_1, \ldots, a_k) \times [a_{k-1}, a_k]
= \left(\text{sc}^m_{k-1}(a_1, \ldots, a_{k-1}, a_{k+1}) + \beta_k\right) + \left(\text{sc}^m_{k-1}(a_1, \ldots, a_{k-1}, a_k) + \beta_0\right).
\]

Altogether,

\[
\text{sc}^m_k(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} \text{sc}^m_{k-1}(a_1, \ldots, a_i, \ldots, a_{k+1}),
\]

as claimed.
3.4. Generic chain maps into grid complexes. We are now ready to apply our machinery on stair convex chains to prove Theorem 3.1. The first step is to give a canonical construction of chain maps from simplicial complexes into grid complexes of sufficiently high dimension. To be applicable for the proof of Theorem 3.1 these maps should also have the property that disjoint simplices of the simplicial complex should be sent to chains of the grid complex which avoid certain hyperplane alignments.

Consider chains $\alpha \in C^k(G[n]^m)$ and $\beta \in C^\ell(G[n]^m)$. We say that the pair $\{\alpha, \beta\}$ is degenerate if there exist a $k$-cell $\alpha_k \in \text{supp}(\alpha)$ and an $\ell$-cell $\beta_\ell \in \text{supp}(\beta)$ such that $\alpha_k$ and $\beta_\ell$ are contained in a common axis-parallel hyperplane $x_j = a$. If the supports of $\alpha$ and $\beta$ contain no such axis-aligned cells $\alpha_k$ and $\beta_\ell$, then we say that the pair $\{\alpha, \beta\}$ is generic.

The notion of genericity extends to chain maps as follows. Let $K$ be a simplicial complex and consider a chain map $g_\bullet : C_\bullet(K) \to C_\bullet(G[n]^m)$. Now we say that $g_\bullet$ is generic if every pair of disjoint simplices of $K$ maps to a generic pair of chains of $C_\bullet(G[n]^m)$. In other words, the chain map $g_\bullet$ is generic if $\sigma, \tau \in K, \sigma \cap \tau = \emptyset \implies \{g_\bullet(\sigma), g_\bullet(\tau)\}$ is generic.

Roughly speaking, the following “Picasso Lemma” asserts that any simplicial complex can be realized within a cubical complex via a generic chain map. (See Figure 2.)

**Lemma 3.6.** Let $K$ be a simplicial complex on $n$ vertices. For any $m > \mu(K)$ there exists a generic nontrivial chain map from $C_\bullet(K)$ to $C_\bullet(G[n]^m)$.

**Proof.** Label the vertices of $K$ by $v_1, v_2, \ldots, v_n$. For every simplex $\{v_{i_1}, \ldots, v_{i_{k+1}}\} \in K$, with $i_1 < i_2 < \cdots < i_{k+1}$, we define $g(\{v_{i_1}, \ldots, v_{i_{k+1}}\}) \overset{\text{def}}{=} \text{sc}_k^m(i_1, \ldots, i_{k+1})$. 

![Figure 2. The graph $K_5$ (considered as a 1-dimensional simplicial complex) realized as a subcomplex of the grid complex $G[5]^3$ via the generic chain map given in Lemma 3.6.](image-url)
We extend \( g \) linearly into a map \( g^\# : C_\bullet(K) \rightarrow C_\bullet(G[n]^m) \), and note that \( g^\# \) is a nontrivial chain map, as the vertex \( v_a \in V(K) \) is mapped to a single vertex \( \{a\} \times \cdots \times \{a\} \in G[n]^m \), and Proposition 3.4 ensures that for any simplex \( \sigma \in C_\bullet(K) \) we have \( \partial(g^\#(\sigma)) = g^\#(\partial \sigma) \). To see that \( g^\# \) is generic, let \( \sigma \) and \( \tau \) be a pair of disjoint simplices in \( K \), and consider an arbitrary pair of cells \( \sigma_k, \tau_l \in \text{supp}(g^\#(\sigma)) \) and \( \tau_l \in \text{supp}(g^\#(\tau)) \). Since \( \dim \sigma + \dim \tau < m \), it follows that \( \dim \sigma_k + \dim \tau_l < m \) and so, by Observation 2.3, there is at least one coordinate, \( x_i \), that is constant for both. But in this coordinate they can not have the same constant value \( x_i = a \), since Corollary 3.3 would imply that \( \sigma \) and \( \tau \) both contain vertex \( v_a \) (here we use that \( \dim \sigma_k < m \) and \( \dim \tau_l < m \)). Thus \( g^\# \) is a generic chain map. \( \square \)

**Remark.** The reader may note (and it can be formalized using [8, Lemma 1.4]) that mapping simplices to stair convex chains generated by points on the main diagonal of the grid complex, bears close resemblance to the canonical geometric realization of a simplicial complex by putting its vertices on the \( d \)-dimensional moment curve.

We are ready to prove the main result of this section which we restate for convenience.

**Theorem 3.1.** Fix a \( d \)-dimensional simplicial complex \( K \) on \( n \) vertices and a simplicial complex \( U \) such that \( K \not\approx_U U \). Let \( Y \) be the \( d \)-skeleton of \( G[n]^m \), where \( m > \mu(K) \). For every nontrivial chain map \( f_\bullet : C_\bullet(Y) \rightarrow C_\bullet(U) \) there exist disjoint cells \( \sigma \) and \( \tau \) in \( Y \) which satisfy:

(i) \( \sigma \) and \( \tau \) are not contained in a common axis-parallel hyperplane \( x_j = a \).

(ii) The supports of \( f_\bullet(\sigma) \) and \( f_\bullet(\tau) \) overlap.

**Proof.** Fix a nontrivial chain map \( f_\bullet : C_\bullet(Y) \rightarrow C_\bullet(U) \). By Lemma 3.6 there is a generic nontrivial chain map \( g_\bullet : C_\bullet(K) \rightarrow C_\bullet(G[n]^m) \). Since \( K \) is \( d \)-dimensional the image of \( g_\bullet \) is contained in the \( d \)-skeleton of \( G[n]^m \), and so the composition

\[
f_\bullet \circ g_\bullet : C_\bullet(K) \rightarrow C_\bullet(U)
\]

is a well-defined, nontrivial chain map. By the hypothesis \( K \not\approx_U U \) there exist disjoint simplices \( \sigma' \) and \( \tau' \) in \( K \) such that the supports of \( f_\bullet \circ g_\bullet(\sigma') \) and \( f_\bullet \circ g_\bullet(\tau') \) overlap. Consequently, there are cells \( \sigma \in \text{supp}(g_\bullet(\sigma')) \) and \( \tau \in \text{supp}(g_\bullet(\tau')) \) such that the supports of \( f_\bullet(\sigma) \) and \( f_\bullet(\tau) \) overlap. Since the chain map \( g_\bullet \) is generic, it follows that the pair \( \{g_\bullet(\sigma'), g_\bullet(\tau')\} \) is generic, and so \( \sigma \) and \( \tau \) are not contained in a common axis-parallel hyperplane \( x_j = a \). Now, since \( \sigma' \) and \( \tau' \) are disjoint in \( K \),

\[
\dim \sigma + \dim \tau = \dim \sigma' + \dim \tau' \leq \mu(K) < m
\]

and by Observation 2.3, \( \sigma \) and \( \tau \) are disjoint. \( \square \)

4. COLORFUL INTERSECTION PATTERNS VIA HOMOLOGICAL MINORS

The machinery of [18, 35] for analyzing intersection patterns via homological minors was designed to analyze complete intersection patterns. We now adapt it to the complete multipartite intersection patterns that are required in a colorful setting.

4.1. Subgrids. We first establish a purely combinatorial Ramsey-type lemma about maps into grid complexes.
4.1.1. Subgrid definition. The structure that our Ramsey-type result identifies is a subgrid of $G[n]^m$. Given $1 \leq \ell \leq n$, a subgrid of size $\ell$ in $G[n]^m$ is a map
\[ \gamma : V(G[\ell]^m) \rightarrow V(G[n]^m) \] given by $(x_1, \ldots, x_m) \mapsto (\gamma_1(x_1), \ldots, \gamma_m(x_m))$, where each $\gamma_i : [\ell] \rightarrow [n]$ is a strictly increasing function. We will use the following consequence of the Gallai-Witt theorem [19, p. 40]:

**Proposition 4.1.** For any integers $m, \ell$ and $q$, $\ell \geq 2$, there exists $N = N(m, \ell, q)$ such that for every $q$-coloring of $V(G[N]^m)$, there exists a subgrid of size $\ell$ in $G[N]^m$ such that every vertex in the image has the same color.

We actually do not need the full power of the Gallai-Witt theorem as we do not require the subgrid to be “homothetic”. Iterated applications of pigeonhole principle show that Proposition 4.1 holds with $N(1, \ell, q) = (\ell - 1)q + 1$ and $N(m, \ell, q) = (\ell - 1)q \left( \frac{N(m - 1, \ell, q)}{\ell} \right)^{m-1} + 1$.

Any subgrid $\gamma$ of size $\ell$ in $G[n]^m$ induces a chain map $\gamma : C_\bullet(G[\ell]^m) \rightarrow C_\bullet(G[n]^m)$ defined as follows. For any $a, b \in [n]$ we let $\gamma_i(\{a\}) \defeq \{\gamma_i(a)\}$, $\gamma_i(\{a, b\}) \defeq \{\gamma_i(a), \gamma_i(b)\}$ and $\gamma_i([a, b]) \defeq [\gamma_i(a), \gamma_i(b)]$. We then define $\gamma#(\sigma)$ for a cell $\sigma = \sigma_1 \times \cdots \times \sigma_m$ in $G[\ell]^m$ by
\[
\gamma#(\sigma) = \gamma_1(\sigma_1) \times \cdots \times \gamma_m(\sigma_m)
\]
and extend it linearly. Indeed this is a chain map because for any $k$-cell $\sigma = \sigma_1 \times \cdots \times \sigma_m$ of $G[\ell]^m$, letting $S \defeq \{i : \dim \sigma_i = 1\}$, we have

\[
\gamma#(\partial \sigma) = \gamma# \left( \sum_{i=1}^m \sigma_1 \times \cdots \times \partial \sigma_i \times \cdots \times \sigma_m \right) = \gamma# \left( \sum_{i \in S} \sigma_1 \times \cdots \times \partial \sigma_i \times \cdots \times \sigma_m \right) = \sum_{i \in S} \gamma_i(\sigma_1) \times \cdots \times \gamma_i(\partial \sigma_i) \times \cdots \times \gamma_m(\sigma_m) = \sum_{i \in S} \gamma_i(\sigma_1) \times \cdots \times \partial \gamma_i(\sigma_i) \times \cdots \times \gamma_m(\sigma_m) = \partial \gamma#(\sigma).
\]

4.1.2. The subgrid lemma. Now, fix an integer $k \geq 1$ and consider a group homomorphism $h$ from the group $\left(C_k(G[n]^m), + \right)$ of $k$-chains into $(\mathbb{Z}_2)^b$ for some integer $b$. We say that a subgrid $\gamma$ of size $\ell$ in $G[n]^m$ lies in the kernel of $h$ if $h(\gamma#(c)) = 0$ for every $c \in C_k(G[\ell]^m)$. Here is our Ramsey-type statement:

**Lemma 4.2** (Subgrid lemma). For any $b, k, m, \ell \in \mathbb{N}$, $\ell \geq 2$, there exists $N = N(b, k, m, \ell)$ such that for every $n \geq N$ and any group homomorphism $h : C_k(G[n]^m) \rightarrow (\mathbb{Z}_2)^b$, there exists a subgrid $\gamma$ of size $\ell$ in $G[N]^m$ that lies in the kernel of $h$.

The rest of Section 4.1 is devoted to the proof of Lemma 4.2. The proof first handles the case $k = m$, and then uses it to prove the case $k < m$. Note that for $k > m$ the lemma is trivial, as the chain group contains only a trivial chain and we can take $N = \ell$. 
4.1.3. **The case** \(k = m\). Let us fix \(k = m, b\) and \(\ell \geq 2\). Let \(n \geq \ell\) be an integer and let \(1\) denote \((1, 1, \ldots, 1) \in V(G[n]^{m})\). Given \(x, y \in V(G[n]^{m})\) we write \(x \preceq y\) if \(x_i \leq y_i\) for \(i = 1, 2, \ldots, m\). For any two vertices \(x \preceq y\) we put 

\[
\text{box}_m(x, y) = \begin{cases} 
0 & \text{if } x_i = y_i \text{ for some } i, \\
[x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_m, y_m] & \text{otherwise.}
\end{cases}
\]

(When \(m\) is clear from the context, we omit it and write box instead of \(\text{box}_m\).)

Note that \(\text{box}_m(x, y)\) is an \(m\)-chain and it is nontrivial if and only if \(x\) and \(y\) do not lie in a common axis parallel hyperplane.

The vector space \(C_m(G[n]^{m})\) is generated by \(\{\text{box}_m(z, z + 1) : z \in V(G[n - 1]^{m})\}\), as for \(z \in V(G[n - 1]^{m})\) we have 

\[
\text{box}_m(z, z + 1) = [z_1, z_1 + 1] \times [z_2, z_2 + 1] \times \cdots \times [z_m, z_m + 1].
\]

Thus, for a subgrid \(\gamma\) of size \(\ell\) to lie in a kernel of a homomorphism \(h : C_k(G[n]^{m}) \to (\mathbb{Z}_2)^b\), it suffices to have 

\[
(5) \quad h(\text{box}_m(\gamma(z), \gamma(z + 1))) = 0 \quad \text{for all } z \in V(G[\ell - 1]^{m}),
\]

since \(\text{box}_m(\gamma(z), \gamma(z + 1)) = \gamma_{\#m}(\text{box}_m(z, z + 1))\).

Let us now associate to the group homomorphism \(h : C_m(G[n]^{m}) \to (\mathbb{Z}_2)^b\) the coloring \(\chi_h : V(G[n]^{m}) \to (\mathbb{Z}_2)^b\) given by \(\chi_h(y) \overset{\text{def}}{=} h(\text{box}(1, y))\).
Claim 4.3. Let γ be a subgrid of size ℓ in G[n]^m. If χ_h(γ(a_1, ..., a_m)) is the same for all (a_1, ..., a_m) ∈ V(G[ℓ]^m), then γ lies in the kernel of h.

Proof. By (5), it suffices to show that h(box(u, v)) = 0 for all u = γ(z) and v = γ(z + 1) with z ∈ V(G[ℓ − 1]^m). We can assume that u and v do not lie in a common axis parallel hyperplane, as otherwise box_m(u, v) = 0 and the claim immediately follows from the fact that h is a homomorphism. Put

W = \{(w_1, ..., w_m) : w_i ∈ \{u_i, v_i\} for every i ∈ [m]\}.

By the inclusion-exclusion principle, and since we work with coefficients in \mathbb{Z}_2, we have

box_m(u, v) = \sum_{w ∈ W} box_m(1, w).

In other words, box_m(u, v) can be written as the sum of the boxes spanned by 1 and each of the corners of the m-dimensional box with opposite vertices u and v.

Applying a homomorphism h on the previous equation, we get

h(box(u, v)) = \sum_{w ∈ W} h(box(1, w)) = \sum_{w ∈ W} χ_h(w) = χ_h(u)|W| = 0,

since all vertices in W have the same color χ_h(u) and |W| = 2^m is even since m ≥ 1.

To complete the proof of Lemma 4.2 for the case k = m, let N = N(b, m, m, ℓ) be the constant \(N(m, ℓ, 2^b)\) from Proposition 4.1. For any n ≥ N and any group homomorphism h : C(G[n]^m) → (\mathbb{Z}_2)^b, Proposition 4.1 ensures that G[n]^m contains a subgrid of size ℓ where the images of all vertices have the same color χ_h. This subgrid lies in the kernel of h by Claim 4.3.

4.1.4. The case 1 ≤ k < m. The proof essentially goes through the same steps as the case k = m. Let us fix k < m, b and ℓ ≥ 2. For any vertex y ∈ G[n]^m and every I ∈ \(\binom{[m]}{k}\) we define a k-chain

\[
\text{chn}_I(y) = \begin{cases} 
0 & \text{if } y_i = 1 \text{ for some } i ∈ I, \\
[x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_m, y_m] & \text{otherwise, where } \begin{cases} x_i = 1 & \text{for } i ∈ I \\
x_i = y_i & \text{for } i \notin I. \end{cases}
\end{cases}
\]

The chain chn_I(y) is nontrivial if and only if y_i ≠ 1 for all i ∈ I, in which case it is a k-chain, as it is a product of k 1-chains. (If we were to allow k = m, then chn_{[m]}(y) would be box_m(1, y).)

The vector space C_k(G[ℓ]^m) is generated by \{chn_I(z) : z ∈ V(G[ℓ]^m), I ∈ \(\binom{[m]}{k}\)\}. Indeed, for z ∈ V(G[ℓ − 1]^m) and I ∈ \(\binom{[m]}{k}\), letting I_I = (x_1, ..., x_m) where x_i = 1 if i ∈ I and x_i = 0 otherwise, we have

chn_I(z) + chn_I(z + 1_I) = [z_1, z_1'] × [z_2, z_2'] × \cdots × [z_m, z_m'],

where z_i' = z_i + 1 if i ∈ I and z_i' = z_i otherwise.

Observe that for any I ∈ \(\binom{[m]}{k}\) and any z ∈ V(G[ℓ]^m), we have chn_I(γ(z)) = γ#(chn_I(z)). It follows that for a subgrid γ of size ℓ in G[n]^m to lie in a kernel of a homomorphism h : C_k(G[n]^m) → (\mathbb{Z}_2)^b, it suffices to have h(chn_I(γ(z))) = 0 for all z ∈ V(G[ℓ]^m) and all I ∈ \(\binom{[m]}{k}\).
Let us now associate to the group homomorphism \( h : C_k(G[n]^m) \to (\mathbb{Z}_2)^b \) the coloring
\[
\rho_h : V(G[n]^m) \to (\mathbb{Z}_2)^b \left( \binom{m}{k} \right),
\]
where \( \rho_h(y) \) is a vector of \( \binom{m}{k} \) elements from \((\mathbb{Z}_2)^b\) and we index its coordinates by subsets of \( \binom{m}{k}\). We denote by \([\rho_h(\cdot)]_I\) the coordinate of \(\rho_h(\cdot)\) with index \(I\).

**Claim 4.4.** Let \( \gamma \) be a subgrid of size \( \ell \) in \( G[n]^m \). If \( \rho_h(\gamma(a_1, \ldots, a_m)) \) is the same for all \((a_1, \ldots, a_m) \in V(G[\ell]^m)\), then \( \gamma \) lies in the kernel of \( h \).

**Proof.** As argued above, it suffices to show that \( h(\text{chn}_I(\gamma(z))) = 0 \) for all vertices \( z \in G[\ell]^m \) and \( I \in \binom{\ell}{k} \). Let us fix \( I \in \binom{\ell}{k} \) and \( z \in V(G[\ell]^m) \) and put \( y = \gamma(z) \). If \( \text{chn}_I(\gamma(z)) = 0 \) then \( h(\text{chn}_I(\gamma(z))) = 0 \) trivially. Otherwise, let \( F \) be unique axis-parallel \( k \)-flat containing \( \text{chn}_I(y) \), that is, the flat whose non-constant coordinates are indexed by \( I \). The support of the \( k \)-chain \( \text{chn}_I(y) \) is contained in \( F \), so we do not loose any information if we restrict \( h \) and \( \gamma \) to \( F \). This brings us back to the the full-dimensional case \( k = m \). Since \( [\rho(y)]_I = [h(\text{chn}_I(y))]_I = h(\text{box}_k(1, y)) \), Claim 4.3 ensures that \( h(\text{chn}_I(\gamma(z))) = 0 \). \( \square \)

To complete the proof of Lemma 4.2, let \( N = N(b, k, m, \ell) \) be the constant \( N \left( m, \ell, 2^b \binom{m}{k} \right) \) from Proposition 4.1. Again, for any \( n \geq N \) and for every homomorphism \( h : C_k(G[n]^m) \to (\mathbb{Z}_2)^b \), there exists a subgrid of size \( \ell \) in \( G[n]^m \) such that every vertex in the image has the same color \( \rho_h \). Claim 4.4 ensures that \( \gamma \) is in the kernel of \( h \).

**4.2. Colorful constrained drawings.** We now present an adaptation of the constrained chain map method to analyze \( m \)-colored intersection patterns.

**4.2.1. The forbidden minor and its grid.** Let \( \mathcal{U} \) be a simplicial complex with a forbidden homological minor \( K \). Suppose \( K \) is \( d \)-dimensional and has \( n \) vertices,
and fix some integer $m > \mu(K)$. We define $\Gamma_K \equiv (G[n]^{m})^d$ and call it the $m$-grid of $K$. Now Theorem 3.1 asserts that for every nontrivial chain map

$$f_* : C_\bullet(\Gamma_K) \to C_\bullet(\mathcal{U}),$$

there exists disjoint cells $\sigma$ and $\tau$ in $\Gamma_K$, not contained in a common axis-parallel hyperplane, such that the supports of $f_*(\sigma)$ and $f_*(\tau)$ overlap.

4.2.2. The cover and its grid. Recall that, as defined in Section 2.1, an $m$-colored $(K, b)$-free cover in a simplicial complex $\mathcal{U}$ is a finite family

$$\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \cdots \sqcup \mathcal{F}_m$$

of subcomplexes of $\mathcal{U}$ such that:

(i) $K$ is a forbidden homological minor of $\mathcal{U}$, and
(ii) the $j$th reduced Betti number $\tilde{\beta}_j(\bigcap_{S \in G} S)$ is strictly less than $b$ for all $0 \leq j < \dim(K)$ and every nonempty colorful subfamily $\mathcal{G} \subseteq \mathcal{F}$.

Now, fix an integer $b \geq 0$ and let $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \cdots \sqcup \mathcal{F}_m$ be a $m$-colored $(K, b)$-free cover in $\mathcal{U}$ where every color class has the same size $t$. We define $\Gamma_\mathcal{F} \equiv G[t]^m$ and call it the grid of the cover $\mathcal{F}$.

We label the members of each $\mathcal{F}_i$ arbitrarily as $\mathcal{F}_i = \{S_{(1,1)}, \ldots, S_{(t,1)}\}$. We associate to each $S_{j,i}$ the hyperplane $x_i = j$, and for a subset $X \subseteq \Gamma_\mathcal{F}$ we put

$$\mathcal{G}(X) \equiv \{S_{j,i} : X \subset \{x_i = j\}\}.$$ 

This defines a bijection $A \mapsto \mathcal{G}(A)$ between the axis-parallel $k$-flats that intersect $V(\Gamma_\mathcal{F})$ and the colorful subfamilies of size $m-k$. Next, for any chain $\alpha \in C_\bullet(\Gamma_\mathcal{F})$, we define $\mathcal{G}(\alpha)$ to be $\mathcal{G}(\text{supp}(\alpha))$. Note that if $\alpha$ is a $k$-cell of $\Gamma_\mathcal{F}$, then $|\mathcal{G}(\alpha)| = m - k$. See Figure 6. For any nonempty subfamily $\mathcal{G} \subseteq \mathcal{F}$ we write $\bigcap \mathcal{G}$ to mean $\bigcap_{S \in \mathcal{G}} S$.

$$\mathcal{F}_1 = \{S_{(1,1)}, S_{(2,1)} , S_{(3,1)}, S_{(4,1)} \}$$

$$\mathcal{F}_2 = \{S_{(1,2)}, S_{(2,2)}, S_{(3,2)}, S_{(4,2)} \}$$

$$\mathcal{F}_3 = \{S_{(1,3)}, S_{(2,3)}, S_{(3,3)}, S_{(4,3)} \}$$

**Figure 6.** A 3-colored family $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3$ with $|\mathcal{F}_i| = 4$ and the corresponding grid $G[4]^3$. The point $(2, 3, 4)$ (in green), the line $(3, t, 2)$ (in red), and the plane $(s, 2, t)$ (in blue), correspond to the subfamilies $\{S_{(2,1)}, S_{(3,2)}, S_{(4,3)} \}$, $\{S_{(3,1)}, S_{(2,3)} \}$, and $\{S_{(2,2)} \}$, respectively.

The definition of $\mathcal{G}(\cdot)$ has three straightforward properties:

**Claim 4.5.** Let $\ell \leq t$ and let $\gamma$ be a subgrid of size $\ell$ in $\Gamma_\mathcal{F}$. For every cell $\sigma$ in $G[\ell]^m$ we have:
4.2.3. Colorful constrained chain maps. Note that for any subgrid $\gamma$ of size $n$ in $\Gamma_F$, by the inclusion $\Gamma_K \subset G[n]^m$, there is an induced chain map $\gamma_\# : C_\bullet(\Gamma_K) \to C_\bullet(\Gamma_F)$. Now we say that a chain map

$$f_\bullet : C_\bullet(\Gamma_K) \to C_\bullet(\mathcal{U})$$

is constrained by $\mathcal{F}$ if there exists a subgrid $\gamma$ of size $n$ in $\Gamma_F$ such that for every cell $\sigma \in \Gamma_K$, $\text{supp}\ f_\bullet(\sigma)$ is a subcomplex of $\bigcap \mathcal{G}(\gamma_\#(\sigma))$.

Our goal for the remainder of this section is to establish the following:

**Lemma 4.6.** Let $\mathcal{U}$ be a simplicial complex with a forbidden homological minor $K$. For any integers $b \geq 1$ and $m > \mu(K)$ there exists an integer $t = t(b, K, m)$ such that for every $m$-colored $(K, b)$-free cover $\mathcal{F}$ in $\mathcal{U}$ where

(i) each color class has $t$ members, and

(ii) every colorful subfamily has nonempty intersection,

there exists a nontrivial chain map $f_\bullet : C_\bullet(\Gamma_K) \to C_\bullet(\mathcal{U})$ that is constrained by $\mathcal{F}$.

4.2.4. The interpolating grids. Recall that $K$ is a $d$-dimensional simplicial complex on $n$ vertices. Let us define constants $t_0 > t_1 > \cdots > t_d$ by setting $t_d \overset{\text{def}}{=} n$ and, having defined $t_{\ell+1}$, set $t_\ell \overset{\text{def}}{=} N(b, \ell + 1, m, t_{\ell+1})$ where $N(\cdot, \cdot, \cdot, \cdot)$ is the function from the subgrid lemma (Lemma 4.2). We let $\Gamma_\ell \overset{\text{def}}{=} G[\ell]^m$ and note that the definition of the $t_\ell$’s ensures:

**Claim 4.7.** For any $0 \leq \ell \leq d - 1$ and any homomorphism $h : C_{\ell+1}(\Gamma_\ell) \to (\mathbb{Z}_2)^b$, there exists a subgrid $\gamma$ of size $t_{\ell+1}$ in $\Gamma_\ell$ that lies in the kernel of $h$.

We recursively use this claim to propagate the following property from $\ell = 0$ to $\ell = d$:

$(P_\ell) :$ there exists a subgrid $\gamma^{(\ell)}$ of size $t_\ell$ in $\Gamma_0$ and a nontrivial chain map

$$f^{(\ell)}_\bullet : C_\bullet(\Gamma^{(\ell)}_\ell) \to C_\bullet(\mathcal{U})$$

such that for any $k$-cell $\sigma \in \Gamma^{(\ell)}_\ell$, $\text{supp}\ f^{(\ell)}_\bullet(\sigma)$ is a subcomplex of $\bigcap \mathcal{G}(\gamma^{(\ell)}_\#(\sigma))$.

Observe that $\Gamma_K = \Gamma_d^{(d)}$ and for an $m$-colored $(K, b)$-free cover $\mathcal{F}$ where each color class has exactly $t_0$ members we have $\Gamma_\ell = \Gamma_0$. The subgrid $\gamma = \gamma^{(d)}$ and the chain map $f_\bullet = f^{(d)}_\bullet$ given by $(P_d)$ will prove Lemma 4.6 for $t(b, K, m) = t_0$.

4.2.5. Setting up the induction. It remains to prove that property $(P_\ell)$ holds under the hypotheses of Lemma 4.6 for $\ell = 0, 1, \ldots, d$. The proof is by induction, but before we start we fix (arbitrarily) for each colorful subfamily $\mathcal{G}$ of $\mathcal{F}$ a basis for $\hat{H}_j(\bigcap \mathcal{G})$, $0 \leq j \leq d$. These bases remain fixed for remainder of the proof. The fact that $\mathcal{F}$ is an $m$-colored $(K, b)$-free cover allows us to consider each homology group $\hat{H}_j(\bigcap \mathcal{G})$ as a subgroup of $(\mathbb{Z}_2)^b$. 

(i) $\mathcal{G}(\gamma_\#(\tau)) = \mathcal{G}(\tau)$ for every chain $\tau$ in the support of $\gamma_\#(\sigma)$,

(ii) $\mathcal{G}(\gamma_\#(\partial\sigma)) = \mathcal{G}(\gamma_\#(\sigma))$, and

(iii) if $\sigma'$ is another cell in $G[\ell]^m$ such that $\sigma, \sigma'$ do not lie in a common axis-parallel hyperplane, then the families $\mathcal{G}(\gamma_\#(\sigma))$ and $\mathcal{G}(\gamma_\#(\sigma'))$ are disjoint.
4.2.6. The induction step. Let $0 \leq \ell < d$ and suppose we are given a subgrid $\gamma^{(\ell)}$ of size $t_\ell$ in $\Gamma_0$ and a nontrivial chain map
\[ f^{(\ell)}_\bullet : C_\bullet \left( \Gamma^{(\ell)}_\ell \right) \rightarrow C_\bullet (U) \]
which satisfies property $(P_\ell)$. For any $(\ell + 1)$-cell $\sigma$ in $\Gamma_\ell$, the chain $\gamma^{(\ell+1)}_{\#}(\sigma)$ is well-defined and, by Claim 4.5, its support has the same affine span in $\Gamma_\ell$ as the support of the chain $\gamma^{(\ell)}_\#(\partial \sigma)$, so
\[ \text{supp } f^{(\ell)}_\#(\partial \sigma) \text{ is a subcomplex of } \bigcap \mathcal{G} \left( \gamma^{(\ell)}_\#(\partial \sigma) \right) = \bigcap \mathcal{G} \left( \gamma^{(\ell+1)}_\#(\sigma) \right). \]

Now we define a homomorphism $h : C_{\ell+1}(\Gamma_\ell) \rightarrow (\mathbb{Z}^2)^h$ by setting, for any $(\ell + 1)$-dimensional cell $\sigma$,
\[ h(\sigma) = \text{def} \left[ f^{(\ell)}_\#(\partial \sigma) \right] \in \tilde{H}_\ell \left( \bigcap \mathcal{G} \left( \gamma^{(\ell)}_\#(\partial \sigma) \right) \right), \]
and extending $h$ linearly. In other words, $h(\sigma)$ equals the homology class of the image $f^{(\ell)}_\#(\partial \sigma)$ in the $\ell$-dimensional (reduced) homology group of $\bigcap \mathcal{G} \left( \gamma^{(\ell)}_\#(\partial \sigma) \right)$, which we can view as an element in $(\mathbb{Z}^2)^h$.

By Claim 4.7, there exists a subgrid $\varphi : V(\Gamma_{\ell+1}) \rightarrow V(\Gamma_\ell)$ of size $t_{\ell+1}$ that lies in the kernel of $h$.

We set $\gamma^{(\ell+1)}_\# = \gamma^{(\ell)} \circ \varphi$ and note that $\gamma^{(\ell+1)}_\#$ is indeed a subgrid $V(\Gamma_{\ell+1}) \rightarrow V(\Gamma_0)$. Moreover, notice that the induced chain map $\gamma^{(\ell+1)}_\#$ satisfies
\[ \gamma^{(\ell+1)}_\# = \gamma^{(\ell)}_\# \circ \varphi_\#. \]

Since $\varphi$ lies in the kernel of $h$, for every $(\ell + 1)$-cell $\tau \in \Gamma_{\ell+1}$ we have $h(\varphi_\#(\tau)) = 0$. By (7), the value of $h(\varphi_\#(\tau))$ equals the homology class of $f^{(\ell)}_\#(\partial \varphi_\#(\tau))$. Hence, by Claim 4.5 (ii),
\[ \left[ f^{(\ell)}_\#(\partial \varphi_\#(\ell+1)(\tau)) \right] = 0 \in \tilde{H}_\ell \left( \bigcap \mathcal{G} \left( \gamma^{(\ell+1)}_\#(\varphi_\#(\ell+1)\tau) \right) \right), \]
which, using that $\varphi_\#$ is a chain map and the fact that $\gamma^{(\ell+1)}_\# = \gamma^{(\ell)}_\# \circ \varphi_\#$, rewrites as
\[ \left[ f^{(\ell)}_\#(\varphi_\#(\ell)(\partial \tau)) \right] = 0 \in \tilde{H}_\ell \left( \bigcap \mathcal{G} \left( \gamma^{(\ell+1)}_\#(\ell\tau) \right) \right). \]

For a cell $\sigma \in \Gamma_{\ell+1}$ of dimension at most $\ell$, we set
\[ f^{(\ell+1)}_\#(\sigma) = \text{def} \left[ f^{(\ell)}_\#(\varphi_\#(\sigma)) \right]. \]

For any $(\ell + 1)$-cell $\tau \in \Gamma_{\ell+1}$, Equation (9) reveals that $f^{(\ell+1)}_\#(\partial \tau)$ is a boundary in $C_\ell \left( \bigcap \mathcal{G} \left( \gamma^{(\ell+1)}_\#(\ell\tau) \right) \right)$. We pick some arbitrary $\alpha \in C_{\ell+1} \left( \bigcap \mathcal{G} \left( \gamma^{(\ell+1)}_\#(\ell\tau) \right) \right)$ such that $\partial \alpha = f^{(\ell+1)}_\#(\partial \tau)$, and set
\[ f^{(\ell+1)}_{\ell+1}(\tau) = \text{def} \alpha. \]
Thus defined, \( f_*^{(\ell+1)} \) is indeed a chain map, from \( C_* \left( \Gamma^{(\ell+1)} \right) \) to \( C_* (U) \). Note that by (10), it follows that \( f_*^{(\ell+1)} \) is nontrivial because \( f_*^{(\ell)} \) is nontrivial and \( \varphi \), being a subgrid, maps each vertex to a single vertex. Moreover, by construction \( f_*^{(\ell+1)} \) and \( \gamma^{(\ell+1)} \) satisfy \( (P_{\ell+1}) \). This concludes the proof of the induction step and of Lemma 4.6.

4.2.7. Weakening the assumptions. Lemma 4.6 assumes that the \( m \)-colored family \( \mathcal{F} \) has the property that for \( 0 \leq j < \dim K \) and for every colorful subfamily \( \mathcal{G} \) of \( \mathcal{F} \), the \( j \)-th reduced Betti number \( \hat{\beta}_j(\bigcap_{F \in \mathcal{G}} F) \) is strictly less than \( b \). A careful inspection of the proof reveals that this assumption is only used in the induction step, for the definition of the labeling \( h \) in Equation (7). When proving that \( (P_{\ell}) \) implies \( (P_{\ell+1}) \), the face \( \sigma \) appearing in Equation (7) is \( (\ell+1) \)-dimensional, so
\[
\left| \mathcal{G} \left( \gamma_{\#(\ell+1)}(\sigma) \right) \right| = m - \dim \sigma = m - \ell - 1.
\]
In particular, we do not need that \( \mathcal{F} \) is an \( m \)-colored \((K, b)\)-free cover, and instead we get:

**Lemma 4.8.** The conclusion of Lemma 4.6 holds for every finite, \( m \)-colored family of subcomplexes of \( \mathcal{G} \) such that the \( j \)-th reduced Betti number \( \hat{\beta}_j(\bigcap_{S \in \mathcal{G}} S) \) is strictly less than \( b \) for all \( 0 \leq j < \dim K \) and all colorful subfamilies \( \mathcal{G} \subseteq \mathcal{F} \) with \( |\mathcal{G}| = m - j - 1 \).

Note that the bound on \( t(b, K, m) \) given by the proof is the same for Lemmas 4.6 and 4.8.

5. Wrapping up

We now have all the ingredients to prove Theorems 2.1 and 1.2.

5.1. A weak colorful Helly theorem.

**Theorem 2.1.** For any finite simplicial complex \( K \) and integers \( b \geq 1 \) and \( m > \mu(K) \), there exists an integer \( t = t(b, K, m) \) with the following property: If \( \mathcal{F} \) is an \( m \)-colored \((K, b)\)-free cover where each color class has size \( t \) and every colorful subfamily has nonempty intersection, then \( \mathcal{F} \) contains some \( 2m - \mu(K) \) members with nonempty intersection.

**Proof.** Let \( t = t(b, K, m) \) be as given by Lemma 4.6. Let \( \mathcal{F} \) be an \( m \)-colored \((K, b)\)-free cover where each color class has size \( t \) and every colorful subfamily has nonempty intersection. By Lemma 4.6, there exists a nontrivial chain map \( f_* : C_* (\Gamma_K) \to C_* (U) \) that is constrained by \( \mathcal{F} \). Let \( n \) denote the number of vertices of \( K \) and let \( \gamma : V(\Gamma_K) \to V(\Gamma_{\mathcal{F}}) \) denote a subgrid that witnesses that \( f_* \) is constrained by \( \mathcal{F} \). Applying Theorem 3.1 to \( f_* \) we get two disjoint cells \( \sigma \) and \( \tau \) in \( \Gamma_K \) which satisfy:

(i) \( \sigma \) and \( \tau \) are not contained in a common axis-parallel hyperplane.
(ii) The supports of \( f_* (\sigma) \) and \( f_* (\tau) \) overlap.

Since \( \gamma \) witnesses that \( f_* \) is constrained by \( \mathcal{F} \), from (ii) we conclude that there is a point contained in every member of \( \mathcal{G} (\gamma_{\#}(\sigma)) \cup \mathcal{G} (\gamma_{\#}(\tau)) \). From (i) and Claim 4.5 (iii) it follows that \( \mathcal{G} (\gamma_{\#}(\sigma)) \) and \( \mathcal{G} (\gamma_{\#}(\tau)) \) are disjoint. Finally, we have
\[
\left| \mathcal{G} (\gamma_{\#}(\sigma)) \cup \mathcal{G} (\gamma_{\#}(\tau)) \right| = \left| \mathcal{G} (\gamma_{\#}(\sigma)) \right| + \left| \mathcal{G} (\gamma_{\#}(\tau)) \right|
\]
tuples subset of µ applies to every such subfamily \( F \) with nonempty intersection. By assumption, our hypergraph whose vertices are the members of \( F \) \((m > \mu )\) with

\[ |F| - (\rho ) = (m - \dim \sigma ) + (m - \dim \tau ) \geq 2m - \mu (K). \]

(Note that the last inequality uses that \( \sigma \) and \( \tau \) are disjoint.) □

5.2. Density propagates towards higher dimensions. We can finally prove our main technical result:

**Theorem 1.2.** Fix a simplicial complex \( K \), a value \( \delta \in (0, 1] \), and integers \( b \geq 1 \) and \( m > \mu (K) \). If \( \mathcal{F} \) is a sufficiently large \((K, b)\)-free cover such that \( \pi_m (\mathcal{F}) \geq \delta (\frac{|\mathcal{F}|}{m}) \), then \( \pi_{m+1} (\mathcal{F}) \geq \gamma (\frac{|\mathcal{F}|}{m+1}) \), where \( \gamma > 0 \) is a constant that depends only on \( \delta \), \( b \), \( m \), and \( K \).

5.2.1. Preparation. An \( m \)-uniform hypergraph is a pair \( H = (V, E) \) where \( V = V(H) \) is a finite set of vertices and \( E = E(H) \subset \binom{V}{m} \) is the edge set. A hypergraph \( H \) contains a hypergraph \( H' \) if there is an injection \( f : V(H') \to V(H) \) such that for every \( e' \in E(H') \), \( f(e') \in E(H) \). (In particular, we do not require that \( H' \) is an induced sub-hypergraph of \( H \).) An \( m \)-uniform hypergraph is \( m \)-partite if the vertex set can be partitioned into disjoint sets (vertex classes) \( V(H) = V_1 \cup \cdots \cup V_m \) such that every edge contains exactly one vertex from each \( V_i \). Given integers \( m \geq 2 \) and \( t \geq 1 \), let \( K^m (t) \) denote the complete \( m \)-partite \( m \)-uniform hypergraph on vertex classes \( V_1, \ldots, V_m \) where \( |V_i| = t \). That is, the edge set of \( K^m (t) \) consists of all \( m \)-tuples of \( V_1 \cup \cdots \cup V_m \) that contain exactly one element from each \( V_i \). We use the following “supersaturation” theorem of Erdős and Simonovits:

**Theorem** ([17, Corollary 2]). For any positive integers \( m \) and \( t \) and any \( \varepsilon > 0 \) there exists \( \rho = \rho (\varepsilon, m, t) > 0 \) such that any \( m \)-uniform hypergraph \( H = (V, E) \) with \( |E| \geq \varepsilon \binom{|V|}{m} \) contains at least \( \rho |V|^mt \) copies of \( K^m (t) \).

5.2.2. Proof of Theorem 1.2. Fix a simplicial complex \( K \) and integers \( b \geq 1 \) and \( m > \mu (K) \). Let \( t = t(b, K, m) \) be the constant from Theorem 2.1. Consider some \((K, b)\)-free cover \( \mathcal{F} \). For \( \mathcal{F}' \subseteq \mathcal{F} \), let \( H[\mathcal{F}'] \) be the \( m \)-uniform hypergraph whose vertices are the members of \( \mathcal{F}' \) and whose edges are the \( m \)-tuples of \( \mathcal{F}' \) with nonempty intersection. By assumption, our hypergraph \( H[\mathcal{F}] \) contains at least \( \delta (\frac{|\mathcal{F}|}{m}) \) edges.

By the Erdős–Simonovits theorem, for some constant \( \rho > 0 \) depending only on \( m, t, \) and \( \delta \), there are at least \( \rho (\frac{|\mathcal{F}|}{mt}) \) distinct \( mt \)-element subfamilies \( \mathcal{F}' \) of \( \mathcal{F} \) such that \( H[\mathcal{F}'] \) contains a copy of \( K^m (t) \). Our choice of \( t \) ensures that Theorem 2.1 applies to every such subfamily \( \mathcal{F}' \), and therefore each \( \mathcal{F}' \) contributes some \( 2m - \mu (K) \geq m + 1 \) members with non-empty intersection. Each \( (m + 1) \)-element subset of \( \mathcal{F} \) with non-empty intersection is contained in \( \binom{|\mathcal{F}|-(m+1)}{mt} \) distinct \( (mt) \)-tuples \( \mathcal{F}' \). There are therefore at least

\[
\frac{\rho (\frac{|\mathcal{F}|}{mt})}{\binom{|\mathcal{F}|-(m+1)}{mt}} = \rho (\frac{t}{m+1}) \binom{|\mathcal{F}|}{m+1} \]

\((m+1)\)-tuples of \( \mathcal{F} \) with nonempty intersection. In other words, \( \pi_{m+1} (\mathcal{F}) \) is at least \( \delta' = \rho (\frac{t}{m+1}) \), where \( \rho \) depends only on \( m, t, \) and \( \delta \), that is on \( m, b, K \) and \( \delta \). That concludes the proof.
5.3. A “sliding-window” relaxation and the Kalai-Meshulam conjecture. If we use Lemma 4.8 in place of Lemma 4.6 in the proof of Theorem 2.1, the hypothesis on the \( m \)-colored family \( \mathcal{F} \) can be weakened. This “improved” Theorem 2.1 can in turn be applied in the proof of Theorem 1.2, yielding the following:

**Theorem 5.1.** Fix a simplicial complex \( K \), a value \( \delta \in (0, 1] \), and integers \( b \geq 1 \) and \( m > \mu(K) \). If \( \mathcal{F} \) is a \( K \)-free cover such that

(i) \( \tilde{\beta}_j(\bigcap_{S \in G} S) < b \), for all \( 0 \leq j < \dim K \) and \( G \subseteq \mathcal{F} \) with \( |G| = m - j - 1 \), and

(ii) \( \pi_m(\mathcal{F}) \geq \delta \binom{|F|}{m} \),

then \( \pi_{m+1}(\mathcal{F}) \geq \gamma \binom{|F|}{m+1} \), where \( \gamma \) is a constant that depends only on \( \delta, b, m, \) and \( K \).

Recall Conjecture 1.10 (in subsection 1.4.3) which asserts that there is a fractional Helly theorem for \( K \)-free covers whose \( (\dim K) \)th homological shatter function is bounded by a polynomial of degree \( t \). Now if Conjecture 1.10 is true, then Theorem 5.1 implies that the fractional Helly number is at most \( \mu(K) + 1 \). Indeed, suppose the fractional Helly number of \( \mathcal{F} \) is bounded by some number \( h \). By successive applications of Theorem 5.1, with \( b = Ch^t \), it would follow that if a positive fraction of the \( (\mu(K) + 1) \)-tuples of \( \mathcal{F} \) are intersecting, then a positive fraction of the \( h \)-tuples of \( \mathcal{F} \) are also intersecting.

6. Acknowledgements

X.G. and A.F.H. were supported by the INRIA associate team FIP. A.F.H. was also supported by the National Research Foundation of Korea (NRF) grants funded by the Ministry of Science and ICT (NRF-2020R1F1A1A01048490) and the Institute for Basic Science (IBS-R029-C1). Z.P. was supported by the Charles University project PRIMUS/21/SCI/014 and also by the GAČR grant no. 22-19073S.

**References**

[1] N. Alon and G. Kalai. A simple proof of the upper bound theorem. Eur. J. Comb. 6, 211–214 (1985).
[2] N. Alon, G. Kalai, J. Matoušek, and R. Meshulam. Transversal numbers for hypergraphs arising in geometry. Adv. in Appl. Math. 29, 79 – 101 (2002).
[3] N. Alon, D. J. Kleitman Piercing convex sets and the Hadwiger–Debrunner \((p, q)\)-problem. Adv. Math. 96, 103–112 (1992).
[4] N. Amenta. Helly theorems and generalized linear programming. Discrete Comput. Geom. 12, 241—261 (1994).
[5] I. Bárány. A generalization of Carathéodory’s theorem. Discrete Math. 40, 141–152 (1982).
[6] I. Bárány. Combinatorial Convexity. AMS University Lecture Series (2021)
[7] I. Bárány and J. Matoušek. A fractional Helly theorem for convex lattice sets. Adv. Math. 174, 227–235 (2003).
[8] B. Bukh, J. Matoušek, and G. Nivasch. Lower bounds for weak epsilon-nets and stair-convexity. Israel J. Math. 182, 199–208 (2011).
[9] S. Chakraborty, R. Pratap, S. Roy, and S. Saraf. Helly-type theorems in property testing. Int. J. Comput. Geom. Appl. 28, 365–379 (2018).
[10] L. Danzer, B. Grünbaum, and V. Klee. Helly’s theorem and its relatives. In *Proc. Sympos. Pure Math., Vol. VII*, pages 101–180. Amer. Math. Soc., Providence, R.I., 1963.
[11] Debrunner, H. E.: Helly type theorems derived from basic singular homology. Amer. Math. Monthly 77, 375–380 (1970). https://doi.org/10.2307/2316144
[12] J. De Loera, X. Goaoc, F. Meunier, and N. Mustafa. The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg. Bull. Am. Math. Soc. 56, 415–511 (2019).
[13] É. C. De Verdière, G. Ginot, and X. Goaoc. Helly numbers of acyclic families. Adv. Math. 253, 163–193 (2014).
[14] J. P. Doignon. Convexity in crystallographical lattices. J. Geom. 3, 71–85 (1973).
[15] J. Eckhoff. An upper-bound theorem for families of convex sets. Geom. Dedicata 19, 217–227 (1985).
[16] J. Eckhoff. Helly, Radon, and Carathéodory type theorems. In Handbook of convex geometry, Vol. A, B, pages 389–448. North-Holland, Amsterdam, 1993.
[17] P. Erdős and M. Simonovits. Supersaturated graphs and hypergraphs. Combinatorica 3, 181–192 (1983). URL: https://doi.org/10.1007/BF02579292.
[18] X. Goaoc, P. Paták, Z. Patákrová, M. Tancer, and U. Wagner. Bounding Helly numbers via Betti numbers. In A journey through discrete mathematics, pages 407–447. Springer, Cham, 2017.
[19] R. L. Graham, B. L. Rothschild, and J. H. Spencer. Ramsey theory, volume 20. John Wiley & Sons, 1990.
[20] S. Hell. Tverberg-type theorems and the fractional Helly property. PhD thesis, 2006.
[21] Helly, E.: Über systeme von abgeschlossenem mengen mit gemeinschaftlichen punkten. Monatsh. f. Mathematik und Physik 37, 281–302 (1930).
[22] A. F. Holmsen and D. Lee. Radon numbers and the fractional Helly theorem. Isr. J. Math. 24, 433–447 (2021).
[23] A. F. Holmsen, M. Kim, and S. Lee. Nerves, minors, and piercing numbers. Trans. Am. Math. Soc. 371, 8755–8779 (2019).
[24] T. Kaczynski, K. Mischaikow, and M. Mrozek. Computational homology, volume 157. Springer Science & Business Media, 2006.
[25] G. Kalai. Intersection patterns of convex sets. Isr. J. Math. 48, 161–174 (1984).
[26] G. Kalai. Combinatorial expectations from commutative algebra. In I. Peeva and V. Welker, editors, Combinatorial Commutative Algebra, volume 1(3), pages 1729–1734. Oberwolfach Reports, 2004.
[27] G. Kalai. Problems for Imre Bárány’s birthday. https://gilkalai.wordpress.com/2017/05/23/problems-for-imre-baranys-birthday/, 2017.
[28] G. Kalai and R. Meshulam. A topological colorful Helly theorem. Adv. Math. 191, 305–311 (2005).
[29] G. Kalai and R. Meshulam. Leray numbers of projections and a topological Helly-type theorem. J. Topol. 1, 551–556 (2008).
[30] G. Kalai, Z. Patákrová. Intersection patterns of planar sets. Discrete Comput. Geom. 64, 304–323 (2020).
[31] M. Katchalski and A. Liu. A problem of geometry in \( \mathbb{R}^n \). Proc. Am. Math. Soc. 75, 284–288 (1979).
[32] J. Matoušek. A Helly-type theorem for unions of convex sets. Discrete Comput. Geom. 18, 1–12 (1997).
[33] J. Matoušek. Lectures on discrete geometry, volume 212. Springer Science & Business Media, 2013.
[34] B. Mohar. What is . . . a graph minor. Notices Am. Math. Soc. 53, 338–339 (2006).
[35] Z. Patáková. Bounding Radon Numbers via Betti Numbers. International Mathematics Research Notices, 2024 https://doi.org/10.1093/imrn/rnae056
[36] H. E. Scarf. An observation on the structure of production sets with indivisibilities. Proc. Nat. Acad. Sci. U.S.A. 74, 3637–3641 (1977).
[37] U. Wagner. Minors, embeddability, and extremal problems for hypergraphs. In Thirty essays on geometric graphs theory, pages 569–607. Springer, 2013.
[38] G. Wegner. d-collapsing and nerves of families of convex sets. Arch. Math. 26, 317–321 (1975).