COORDINATING THE SUPPLIER-RETAILER SUPPLY CHAIN
UNDER NOISE EFFECT WITH BUNDLING
AND INVENTORY STRATEGIES

ATA ALLAH TALEIZADEH
School of Industrial Engineering, College of Engineering, University of Tehran
Tehran, Iran

LEOPOLDO EDUARDO CÁRDENAS-BARRÓN*
School of Engineering and Sciences, Tecnológico de Monterrey
E. Garza Sada 2501 Sur, C.P. 64849, Monterrey, Nuevo León, México

ROYA SOHANI
Department of Industrial Engineering, Islamic Azad University
South Tehran Branch, Tehran, Iran

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ABSTRACT. In current competitive market, the products and their demand’s uncertainty are high. In order to reduce these uncertainties the coordination of supply chain is necessary. Supply chain can be managed under two viewpoints typically: 1) centralized supply chain and 2) decentralized supply chain, and the coordination can be done in both types of chains. In the centralized supply chain there exists a global decision maker who takes all the best decisions in order to maximize the profit of the whole supply chain. Here, the useful information required to make the best decisions is open to all members of the chain. On the other hand, in the decentralized supply chain all members decide in a separate and sequential way, how to maximize their profits. In order to coordinate efficiently the supply chain, both supplier and retailer are involved in a coordination contract that makes it possible for the decentralized decisions to maximize the profit of the entire supply chain. In this context, the situation that the supplier-retailer chain faces is a two-stage decision model. In the first stage the supplier, based on former knowledge about the market, decides the production capacity to reserve for the retailer. In the second stage, after that demand information is updated, the retailer determines the bundle price and the quantity of bundles to order. This paper considers a supply chain comprised of one supplier and one retailer in which two complementary fashion products are manufactured and sold as a bundle. The bundle has a short selling season and a stochastic price dependent on demand with a high level of uncertainty. Therefore, this research considers that the demand rates are uncertain and are dependent on selling prices and on a random noise effect on the market. Profit maximization models are developed for centralized and decentralized

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* Corresponding author: Tel. +52 81 83284235, Fax +52 81 83284153. E-mail address: lecarden@itesm.mx (L.E. Cárdenas-Barrón).
supply chains to determine decisions on production capacity reservation, order quantity of bundled products and the bundle-selling price. The applicability of the developed models and solution method are illustrated with a numerical example.

1. Introduction and literature review.

1.1. Motivation. In the last decade, two complementary products coming from a duopoly market like a supply chain, where relationship was either found to be cooperative or non-cooperative among channel members (manufacturer, distributor, supplier, retailer, buyer, etc.). Duopoly market does not allow the channel members of the supply chain to have the same power when compared to the retailers or manufacturers. In the case of non-cooperative market, the party with more power plays a leader and the other as a follower. Also in integrated businesses’ where complementary products like multimedia PC-sound system, conditioner and shampoo, operating system and computer are produced by manufacturers influencing consumer’s demands in the market such a way that the purchase of one product affects and increases the possibility of purchasing the other product. Thereby, small changes in demand resulting in large changes in the decision. Therefore, when the manager investigates factors influencing demand, purchasing behaviour of the customers’ may be affected by factors like selling price, seasonality, and inventory level. Two separate firms for pricing of complementary goods under information asymmetry were considered by Yue et al. (2006) and Mukhopadhyay et al. (2011).

Recently, the companies are practising commercial strategies like bundling, commonly known as tying. Selling of different products (or services) together is what bundling consists of. Practice of bundling comprises of a principal product and a secondary product, complementing each other, although, same number of products can also form a bundle. Some examples of bundling two or more services or products, such as shampoo with conditioner, toothbrush and toothpaste, computer and printer, flight with hotel, just to name a few illustrations. Basically, when a bundling strategy is used by a company for the selling of its products, the selling of a highly complementary product at the same time is an advantage that it gets. Many of the reasons behind companies implementing bundling strategies are: increasing of market share, reduction in packing costs, increase of sales, improving of customer service, thereby extending market power of one product to another. Bundling does not have a consistent, universally accepted definition but the integration and sale of two or more separate products at same price is what product bundling is. Categorized into pure bundling and mixed bundling, pure bundling a strategy where items are only sold as bundles and not separately whereas mixed bundling a strategy where products can be sold as both bundles and separately. Bundling price and its process influences consumer by offering two complementary products as a bundle price that generally increases the chain profit. A descriptive study on bundling of complementary products by Estelami (1999) clearly pointed out that minimize consumer costs from 18 to 57% through bundling all depending on amount of bundled items, the value of same items, and level of variations. Nalebuff (2004) provided an excellent discussion regarding motivations employing bundling strategy. Two types of bundling strategies exist: pure and mixed. Pure bundling strategy, products are only sold as a bundle. In contrast, to the mixed bundling where products can be sold as a bundle or individually.

Generally, the products and their demand’s uncertainty are high if the market for the same is unpredictable. Therefore, reduction of uncertainty is important through
the coordination of the supply chain. Consequently, coordination is a mandatory task in any supply chain. Supply chain can be managed under two viewpoints typically: 1) centralized supply chain and 2) decentralized supply chain, and the coordination can be done in both types of chains. In the centralized supply chain there is a global decision maker who makes all the best decisions with the aim to maximize the profit of the whole supply chain. Here, the useful information necessary to make the best decisions is open to all participants of the chain. In the decentralized supply chain all participants decide in an isolated and sequential way, how to maximize their profits. In order to coordinate efficiently the supply chain, both supplier and retailer are involved in a coordination contract that makes it possible for the decentralized decisions to maximize the profit of the entire supply chain. The situation that the supplier-retailer chain faces is a two-stage decision model. In the first stage the supplier, based on former knowledge about the market, decides the production capacity to reserve for the retailer. In the second stage, after that demand information is updated, the retailer determines the bundle price and the quantity of bundles to order. This paper considers a supply chain comprised of one supplier and one retailer in which two complementary fashion products are manufactured and sold as a bundle. The bundle has a short selling season and a stochastic price dependent on demand, with a high level of uncertainty. Furthermore, as the two products are selling together, there is random noise in the market due to the psychological effects of the bundling strategy that affects customers’ purchases. The research works more closely related to our paper are Chen et al. (2010) and Yan and Bandyopadhyay (2011). Chen et al. (2010) study and analyze a coordination contract for a retailer-supplier chain in which only one type of product is processed and sold in a short vending season. The problem is also modeled as a two-stage news-vendor model in which an initial decision on ordering is made in the first stage and additional ordering and pricing decisions are done in the second stage. Conversely, Yan and Bandyopadhyay (2011) develop a profit maximization model to study the benefits of a bundling strategy. This paper combines the main ideas related to coordination and bundling from Chen et al. (2010), and Yan and Bandyopadhyay (2011) into a new profit maximization model to determine the bundle price of two products that are sold together and the bundle order quantity.

1.2. Literature review. Bundling permits monopolist extracting additional surpluses by diminishing the variance of average valuations by using the law of large numbers that it was showed by Armstrong and Vickers (2010). Li et al. (2013) defined a measure of consumer heterogeneity, which increases with costs, and as they presented that, an increase in the measure of consumer heterogeneity affects pure bundling and its performance is poorly relative to individual sales. Stochastic modelling of a retail firm selling two types of perishable products in a single period both as bundle and independent items is considered by Gürlüer et al. (2009) where they showed that when more bundles are formed by retailer, or higher prices are charged for the bundle or both as the products convert less substitutable and more complimentary. Bhargava (2012) found that the reason behind the less attractiveness of bundling is due to the conflicts in supply chain that produce an overpricing of component products by manufacturers.

There exist several studies by researchers regarding products’ bundling allowing discount. For example see the works by Matutes and Regibeau (1992) and Gans, and King (2006). Price consumer evaluations of a discounted product coming from a bundle is considered by Sheng et al. (2007) where they showed the effects of price
discounts interplay with complementarity of bundle components. Yan and Bady-opadhyay (2011) investigate bundling of complementary products. They exposed how conditionally firms could be benefited from complementary bundling. On further investigation by Yan et al. (2014) regarding the advertising on the success of product bundling and strategic influence of product complementarity, showed that when bundled products are sold by a firm, both advertising and product complementarity significantly impacts its performance. Brito and Vasconcelos (2015) examined competitive effects of bundled discounts offered by pairs of independent firms in a setting with vertically differentiated goods.

The literature regarding bundling is also rich and vast. In this line, the following research works are relevant: Guiltinan (1987) proposes a normative framework to choose suitable kinds of services under different mixed-bundling forms. Later, Rosenthal et al. (1995) develop a mixed integer linear program to examine the relationship among various bundling strategies. Their model determines the optimal purchasing strategy for the buyer that minimizes the total purchase cost. Simultaneously, Simonin and Ruth, (1995) examine the impact of bundling policies on the reserve prices to the customers of the bundle and its components involving a new product and a tie-in product. Afterwards, Estelami (1999) investigate the savings to customers under complementary bundling products. Later, Bennett and Robson (2001) examine how associations balance their provision of distinct services, the potential for associations to offer new services, and the importance of bundling. Venkatesh and Kamakura (2003) discuss an analytical model of contingent valuations to maximize the profit of system. They survey different bundling strategies such as pure bundling and mixed bundling. Later, Oppewal and Holyoake (2004) study the impacts of retail accumulation and bundling on shopping behavior, especially on in-store purchase incidence and the sequencing of shopping activities. Subsequently, Vaubourg (2006) presents a theoretical argument for the market outcome when some enterprises use pure bundling strategies whereas others apply mixed bundling. One year later, Bitran and Ferrer (2007) analyze the problem of how to define the composition and price of a bundle such that it maximizes the total expected profit. At the same time, Hubbard et al. (2007) study the economic effects of pure bundling under the settings of duopoly and monopoly. They conclude that the bundled price is less than the summation of unbundled prices. McCardle et al. (2007) examine the properties of bundling products on retail trades. They determine the optimal bundle prices, order sizes, and profits under bundling strategy. Later, Arora (2008) study the efficacy of price bundling and message framing on intentions, beliefs and attitudes associated to characteristics of teeth whitening products. Afterwards, Bulut et al. (2009) develop a single period pricing model to determine the optimal bundling pricing policy of two perishable products that could be sold either as a bundle or individually, under a stochastic demand. Simultaneously, Gürler et al. (2009) present a stochastic modelling for a retail company, which vends two kinds of perishable items. Eckelbar, (2010) study the case of a monopoly selling two different products to a group of traders determined by their reserve prices. In addition, they propose closed-form solutions to determine the optimal quantities, prices, customers’ surplus and profit under the situations of pure-, mixed-bundling and individual sales. In the same year, Ferrer et al. (2010) develop a pricing model of bundles comprised of a service and a related product using a two-part tariff arrangement. Applying dynamic programming, they derive the optimal pricing policy that maximizes the firm’s profit. Wappling et al. (2010)
examine some product bundling strategies offered to customers by selected companies in Swedish automobile, travel and banking organizations. Later, Yan and Bandyopadhyay (2011) build a profit-maximization model to find the optimum pricing and bundling policies for complementary items. Recently, Chakravarty et al. (2013) investigate bundling decisions in a two-level supply chain comprised of one retailer and several suppliers. Concurrently, Girju et al. (2013) study how channel interactions affect the product bundling decisions of channel members. Sheikhzadeh and Elahi (2013) explore two facets of bundling. They analyze the effects of risk and product heterogeneity on bundling decisions. Finally, Yan et al. (2014) develop a profit maximization model to analyze the strategic effect of advertising and product complementarity on the success of bundling products. Other related research works are Taleizadeh et al. (2010), Taleizadeh et al. (2013), Taleizadeh and Pentico (2013), Taleizadeh et al. (2015), Taleizadeh and Noori-daryan (2016) and Taleizadeh et al. (2016).

There are several papers on coordination contracts and bundling. Regarding the coordination contracts in supply chain, the following works are relevant: Donohue (2000) discusses the problem of developing supply contracts that permit the coordination of forecast information and production decisions of a manufacturer and a distributor for seasonal products. Afterwards, Barnes-Schuster et al. (2002) consider a two-period model and study the role of options in a buyer-supplier system. They also explain how options give flexibility to a buyer in response to market changes during the second period. In the same year, Taylor (2002) studies the coordination of supply chains with sales effort and channel rebate considerations. Cachon (2003) investigates the supply chain coordination under different transaction contracts, such as the wholesale-price contract, the buyback contract, the revenue-sharing contract, the quantity-flexibility contract, the sale-rebate contract and the quantity-discount contract. Later, Cachon and Lariviere (2005) apply revenue-sharing contracts in a supply chain with revenues calculated by each retailer’s purchase quantity and price. Subsequently, Wang (2005) extend traditional quantity discount policies, which are merely based on order quantity of buyers, to discount policies based on both individual order quantity and the annual volume. Mathur and Shah (2008) consider a contract in a supply chain where a supplier needs to determine capacity under an offered-price regime before realizing the market demand. One year later, Yan and Pei (2009) focus on the strategic role played by the retail services in a dual-channel competitive market. Chen et al. (2010) study a coordination contract for a supply chain in which a fashionable product, with stochastic price-dependent demand, is produced and sold. Chen and Bell (2011) consider a decentralized supply chain, including a manufacturer and a retailer, in which the retailer determines the order quantity and the retail price at the same time that customer returns and price-dependent stochastic demand are occurring. Yan (2011) extends an analytical model to investigate impacts of profit sharing and differentiated branding strategies on the decisions of a multi-channel manufacturer-retailer supply chain’s members.

Li et al. (2013) introduced a bundling sale strategy of the goods with a prevalent strategy in a manufacturing firm. They studied numerically the maximum profit of three bundling strategies (individual sale of the product, pure bundling of the product and mixed bundling of the product) and concluded that mixed bundling strategy is more profitable than other strategies. In manufacturing firms, the objective is to fabricate the product smoothly and maintain the service according to the customers’ requirement. Due to this objective, manufacturing firms offer the good
Table 1. Some recent works related to bundling strategy

| Literature               | Strategies                      | Selling price       | Demand rate                  | Situation                                                                 |
|--------------------------|---------------------------------|---------------------|------------------------------|---------------------------------------------------------------------------|
| Chakravarti et al. (2013)| Bundling                        | Bundle price        | Selling price                | Decentralized supply chains                                              |
| Li et al. (2013)         | Mix bundling                     | Bundle price        | Selling price                | Bi-level programming                                                     |
| Yan et al. (2014)        | Bundle pricing and advertising   | Bundle price        | Selling price                | Product complementary and advertisement of bundle product                |
| Wang et al. (2015)       | Service bundling                 | —                   | Service and Price bundling   | Duopoly competitive environment                                          |
| Banciu and Ødegaard (2016)| Different bundling              | —                   | —                            | Simulation technique                                                     |
| Giri et al. (2017)       | Pricing                         | Bundling price      | Linearly dependent on price  | Duopoly market                                                           |
| Vamosiu (2018)           | Imperfect Competition           | Mixed bundling      | —                            | Pure bundling                                                            |
| This paper               | Bundling                        | Bundle selling price| Uncertain, selling price and random noise effect on market | Centralized and decentralized supply chains |

price of the product as well as an excellent service facility in bundles form; this is named as price and service bundles (P&S bundles). Wang et al. (2015) introduced the firms’ incentive and offer to customers’ for buying the products in bundling form with maintenance or repair facility in a duopoly competitive marketing situations. Using this concept, Wang et al. (2015) developed three game models and analyzed the market situations for one firm, both firms and no firm when there exist the offer of P&S bundles in form. Banciu and Ødegaard (2016) introduced the problem of pricing a bundle of products when the underlying valuations of the bundle components are dependent. They used copula theory to model and solved the problem for different bundle strategies. Giri et al. (2017) studied a duopoly market situation where two manufacturers separately fabricate two complementary products and sell the fabricated products into together through a common retailer. In their paper, they considered that the demand depends on price. Giri et al. (2017) proposed two different situations: without and with bundling and described the problem mathematically to maximize the profit of each situation. They concluded that in the supply chain profit for the bundling product is better than the profit when products are sold separately. Vamosiu (2018) studied the profitability of bundling for a two-product seller of differentiated products facing competition from a one-product rival. Table 1 shows some recent works related to bundling strategy.

The contribution to existing literature of this research is to investigate a coordination contract for the supplier-retailer chain. Basically, this research focuses on a supply chain including one retailer and one supplier who manufactures two complementary products; their demand rates are price dependent and uncertain. Indeed, a random noise affects the market demand. Moreover, this research proposes a
mathematical model to maximize the profit, and coordinates the supply chain considering bundling strategy, restriction on capacity and noise effect. In fact, this paper develops a two-stage optimization strategy in which the supplier decides the capacity amount in stage 1 and the retailer optimizes the order quantity and the selling price in stage 2. Furthermore, a risk and profit sharing contract under three parameters to coordinate the chain is developed. The proposed contract allows any agreed-upon division of the supply-chain profit, among the channel members, using controllable parameters. This is another contribution of this paper, and it should be noted that this structure has never been studied by existing research.

The remaining of this study is organized as follows. The section 2 describes briefly the problem, defines the notation and assumptions. Section 3 develops the profit maximization models for centralized and decentralized supply chains. Section 4 presents a numerical example. Section 5 provides the managerial insights. Finally, Section 6 gives some conclusions and future research directions.

2. Mathematical model and its description.

2.1. Problem description. This paper deals with a supply chain comprised of one supplier and one retailer. The supplier manufactures two complementary products, which will be sold only as a bundle at price $p_B$, while previously both products were sold separately at prices $p_1$ and $p_2$. The demands of products are dependent on price, and there is a random noise effect on the market. In the chain, three main decisions must be made on different events that occur in two stages. These decisions are: production capacity reservation, order quantity and bundle price. In the first stage, the supplier forecasts the demand and then reserve a production capacity $M$ for the retailer. After some time, the second stage occurs. Here, the retailer updates the forecast of the demand. Then, the retailer determines the bundling order quantity as well as the bundle price.

Yan and Bandyopadhyay (2011) proposed a linear demand function for the bundled products, and the random noise effect ($x_B$) is added to it. Thus, the demand function for the bundled products under noise effect is as follows:

$$D(p_1, p_2, p_B) = a_1 - a_2 p_B + \lambda(p_1 + p_2 - p_B) + x_B \tag{1}$$

Here, $a_1$ is the bundled product's acceptance by the market (potential demand), $a_2$ represents the response rate to the bundle price, $\lambda$ denotes the bundling discount price sensitivity, $p_1$ is the price for product 1, $p_2$ is the price for product 2, $p_B$ is the bundle price and $x_B$ is the noise effect on the market demand. So when the bundle price increases, the demand quantity should be decreased ($-a_2 p_B$), and increase in the complementary degree should the demand rate of bundle products increases($+\lambda(p_1 + p_2 - p_B)$), if $p_1 + p_2 > p_B$ which is the main assumption in bundle selling of items. Moreover, the demand quantity is affected by the noise effect, represented in the demand rate by $x_B$.

The demand function given by Equation (1) has two types of uncertainty which are the bundled products acceptance by market ($a_1$) and noise effect ($x_B$). It is assumed that $a_1$ has a probability density function $g(a_1)$ and the interval for $a_1$ is $[l,h]$. On the other hand, the noise effect on demand is also a random variable distributed in the interval $[L,H]$ and it has a probability density function $f(x_B)$, cumulative distribution $F(x_B)$ and mean $\mu_B$.

Only, at the end of the first stage, one can know the product’s demand reaction to the bundle price. Then in the second stage, the value for $a_1$ is updated with $\tilde{a}_1$. 

Therefore, the demand function now becomes deterministic and known. It is given by:

\[ y(p_1, p_2, p_B) = \tilde{a}_1 - a_2p_B + \lambda(p_1 + p_2 - p_B) + x_B \quad (2) \]

Here, notice that Equation (2) does not consider the noise effect on demand market.

2.2. Notation and assumptions. The following notation is used throughout this paper. Some symbols and assumptions are similar to those given by Chen et al. (2010) and Yan and Bandyopadhyay (2011).

2.2.1. Notation. In the notation, the superscripts c and d refer to centralized and decentralized models, respectively.

With regard to per unit salvage value and per unit penalty cost of stock-out of bundled products, conditions \( v_{2B} < v_{1B} + c_{2B} \) and \( \pi_B > v_{2B} \) are established.
2.2.2. Assumptions. The following assumptions are made.

1. A single-supplier single-retailer supply chain is considered.
2. The supplier manufactures two complementary products, which are sold only as a bundle.
3. The demand depends on selling price and a random noise effect on the market is assumed.
4. During the two stages, three decisions related to production capacity reservation, order quantity and bundle price must be made.
5. During the first stage, the supplier predicts demand and a production capacity for the retailer is reserved.
6. During the second stage, the retailer updates the prediction of the demand and determines the bundling order quantity as well as the bundle price.
7. The following two types of uncertainty the bundled products acceptance by market and noise effect are considered.

3. Coordination supply chain model formulation and its solution.

3.1. Strategy without bundling. The demand functions for product 1 and product 2 without bundle policy considering the complementarity (θ) and noise effect (x_i) are:

\[
\begin{align*}
D_1(p_1, p_2) &= a_1 - a_2p_1 - a_2\theta p_2 + x_i \\
D_2(p_1, p_2) &= a_1 - a_2p_2 - a_2\theta p_1 + x_i
\end{align*}
\]

where \(a_1 \in [\gamma_1, \beta_1] \), \(x_i \in [\gamma_1, \beta_1]\). If the reserved production capacity is enough for the order quantity of both products \((Q_1^*, Q_2^* < M_1^* \), \(Q_2^* < M_2^*)\) then the supply chain’s profit at stage 2 is:

\[
\Pi_{21} = \begin{cases} 
(p_1 - c_{21})(y_1(p_1, p_2) + x_1) - v_{11}(y_1(p_1, p_2) + s_1) \\
-\pi_1(x_1 - s_1) + v_{11}M_1^* + (p_2 - c_{22})(y_2(p_1, p_2) + x_2) \\
-\pi_2(x_2 - s_2) + v_{12}M_2^* - c_{22}s_2 \\
(x_1 \le s_1, x_2 \le s_2) \equiv (D_1 \le Q_1^*, D_2 \le Q_2^*) \\
(p_1 - c_{21})(y_1(p_1, p_2) + s_1) - v_{11}(y_1(p_1, p_2) + s_1) \\
-\pi_1(x_1 - s_1) + v_{11}M_1^* + (p_2 - c_{22})(y_2(p_1, p_2) + s_2) \\
-\pi_2(x_2 - s_2) + v_{12}M_2^* - c_{22}s_2 \\
(x_1 > s_1, x_2 > s_2) \equiv (D_1 > Q_1^*, D_2 > Q_2^*) \\
(p_1 - c_{21})(y_1(p_1, p_2) + s_1) - v_{11}(y_1(p_1, p_2) + s_1) \\
-\pi_1(x_1 - s_1) + v_{11}M_1^* + (p_2 - c_{22})(y_2(p_1, p_2) + x_2) \\
-\pi_2(x_2 - s_2) + v_{12}M_2^* - c_{22}s_2 \\
(x_1 > s_1, x_2 \le s_2) \equiv (D_1 > Q_1^*, D_2 \le Q_2^*)
\end{cases}
\]
Using algebra, the prices for products 1 and 2 when $Q < M$ are calculated using Equation (4), when there is no bundle selling.

$$p_1(s) = \left( \frac{\hat{a}_1(1 - \theta) + \mu_1 - \mu_2\theta}{2a_2(1 - \theta^2)} + \frac{c_{21} + v_{11}}{2} \right) - \frac{B_1(s) - \theta B_2(s)}{2a_2(1 - \theta^2)}$$

$$p_2(s) = \left( \frac{\hat{a}_1(1 - \theta) + \mu_2 - \mu_1\theta}{2a_2(1 - \theta^2)} + \frac{c_{22} + v_{12}}{2} \right) - \frac{B_2(s) - \theta B_1(s)}{2a_2(1 - \theta^2)}$$  

(5)

Where $B_i(s) = \int_{s_i}^{H_i} (x_i - s_i) f(x_i) dx_i$ and the detailed derivation of Equation (5) is given in Appendix A.

Moreover, the prices for product 1 and product 2 when $Q = M$ are determined using the following equations (See Appendix B).

$$\int_{M_1^c - y_1(p_1,p_2)}^{H_1} [M_1^c - x_1 - y_1(p_1,p_2) + a_2(p_1 - c_{21} + \pi_1 - v_{21})] f(x_1) dx_1$$

$$+ (y_1(p_1,p_2) + \mu_1 - a_2(p_1 - c_{21} - v_{21}) - a_2(p_2 - c_{22} - v_{22}))$$

$$+ \int_{M_2^c - y_2(p_1,p_2)}^{H_2} a_2\theta(p_2 - c_{22} + \pi_2 - v_{22}) f(x_2) dx_2 = 0$$

(6)

$$\int_{M_1^c - y_1(p_1,p_2)}^{H_1} a_2\theta(p_1 - c_{21} + \pi_1 - v_{21}) f(x_1) dx_1 = 0$$

### 3.2. Centralized policy model considering bundling.

First, the centralized model is considered for determining the joint decision on bundle price and order quantity of bundled products. In stage 2, the value of $a_1$ is already known as $\hat{a}_1$ therefore the demand function for the product under the bundle policy is:

$$D(p_1,p_2,p_B) = \hat{a}_1 - a_2p_B + \lambda(p_1 + p_2 - p_B) + x_B$$  

(7)

The safety stock of bundled products ($s_B$) is defined as follows: $s_B = Q_B^* - y(p_1,p_2,p_B)$. Here, $Q_B^*$ represents the ordering quantity of bundled products in the centralized model.

#### 3.2.1. When the reserved production capacity is enough.

If the reserved production capacity is enough for the order quantity of bundled products ($Q_B^* < M^*_B$) then the supply chain’s profit at stage 2 is:

$$\Pi_B = \begin{cases} 
(\hat{a}_1 - c_{2B})[y(p_1,p_2,p_1B) + x_B] - (v_{11} + v_{12})[y(p_1,p_2,p_1B) + s_B] \\
+ (s_B - x_B)(v_{21} + v_{22}) + M_B^c(v_{11} + v_{12}) - (c_{21} + c_{22})s_B & x_B \leq s_B \\
(\hat{a}_1 - c_{2B})[y(p_1,p_2,p_1B) + s_B] - (v_{11} + v_{12})[y(p_1,p_2,p_1B) + s_B] \\
- \pi_B(x_B - s_B) + M_B^c(v_{11} + v_{12}) - (c_{21} + c_{22})s_B & x_B > s_B 
\end{cases}$$

(8)

Calculating the expected value with respect to random noise variable ($x_B$) one obtains the expected value for the centralized supply chain profit under the bundle
policy, given by:

\[
E[\Pi_B^p] = \int_{\mathbb{R}} \left( (p_{1B} - c_{2B})[y(p_1, p_2, p_{1B}) + x_B] + (v_{21} + v_{22})(s_B - x_B) \right) f(x_B) dx_B \\
+ M_c^c(v_{11} + v_{12}) + \int_{\mathbb{R}} \left( (p_{1B} - c_{2B})[y(p_1, p_2, p_{1B}) + s_B] - \pi_B(x_B - s_B) \right) f(x_B) dx_B \\
-(c_{21} + c_{22})s_B - (v_{11} + v_{12})[y(p_1, p_2, p_{1B}) + s_B] 
\]

where \( c_{2B} = c_{21} + c_{22} \).

Now let \( U(s_B) = \int_{\mathbb{R}}^s (s_B - x_B) f(x_B) dx_B \), \( B(s_B) = \int_{s_B}^H (x_B - s_B) f(x_B) dx_B \), substituting the above relations into Equation (8) one obtains,

\[
E[\Pi_B^p] = (p_{1B} - c_{2B} - v_{11} - v_{12}) (y(p_1, p_2, p_{1B}) + \mu_B) \\
-(c_{2B} + v_{11} + v_{12} - v_{21} - v_{22})U(s_B) \\
-(p_{1B} + \pi_B - c_{2B} - v_{11} - v_{12})B(s_B) + (v_{11} + v_{12})M_c^c 
\]

If there is no capacity constraint for \( s_B \) then we have (See Appendix C):

\[
\frac{\partial E[\Pi_B^p]}{\partial p_{1B}} = a_1 - a_2 p_{1B} + \lambda(p_1 + p_2) - \lambda p_{1B} + \mu_B \\
+(a_2 + \lambda)(c_{22} + c_{21} + v_{11} + v_{12} - p_{1B}) - B(s_B) 
\]

Note that \( E[\Pi_B^p] \) is concave with respect to bundle price \( p_{1B} \). Then the bundle price under bundling strategy is:

\[
p_{1B} = \frac{1}{2(a_2 + \lambda)} (a_1 + \lambda(p_1 + p_2) + \mu_B + (a_2 + \lambda)(c_{22} + c_{21}) \\
+(a_2 + \lambda)(v_{11} + v_{12} - B(s_B)) 
\]

Assuming that the demand function at stage 2 is given by \( D(p_1, p_2, p_{1B}) = \tilde{a} - a_2 p_{1B} + \lambda(p_1 + p_2 - p_{1B}) + x_B \) and if there is no production capacity constraint, then the optimal joint decision is to vend the bundle at \( p_{1B}^c \) and the order quantity of bundled products is \( Q_B^c = y(p_1, p_2, p_{1B}) + s_B^c \). The \( s_B^c \) corresponds to the largest solution that satisfies Equation (13) where \( v_{2B} = v_{21} + v_{22} \).

\[
-(c_{2B} + v_{1B} - v_{2B}) + (p_{1B}^c + \pi_B - v_{2B})[1 - F(s_B^c)] = 0 
\]

The expected profit of the supply chain in the centralized model \( E[\Pi_B^p] \) is unimodal in \( s_B^c \) or \( Q_B^c \). Thus, there exists only one solution to Equation (13). This is because the following condition \( p_{1B}^c - c_{2B} \geq v_{1B} - \pi_B \) holds. Otherwise, no bundled products should be vended at stage 2.

3.2.2. When the reserved production capacity is not enough. If the reserved production capacity is not enough for the order quantity of bundled products then \( Q_B^c = M^c_B \) and the supply chain profit at stage 2 is:

\[
\Pi_B = \begin{cases} 
(p_{2B} - c_{2B})[y(p_1, p_2, p_{2B}) + x_B] \\
+(v_{21} + v_{22})[M^c_B - y(p_1, p_2, p_{2B}) - x_B] - c_{2B}M^c_B & D \leq M^c_B \\
(p_{2B} - c_{2B})M^c_B - \pi_B[y(p_1, p_2, p_{2B}) + x_B - M^c_B] - c_{2B}M^c_B & D > M^c_B 
\end{cases} 
\]

Where \( D = y(p_1, p_2, p_{2B}) + x_B \). In this case, the expected profit can be calculated as follows:

\[
E[\Pi_B^p] = \int_{\mathbb{R}}^{M^c_B - y(p_1, p_2, p_{2B})} \left( (p_{2B} - c_{2B})[y(p_1, p_2, p_{2B}) + x_B] \\
+(v_{21} + v_{22})[M^c_B - y(p_1, p_2, p_{2B}) - x_B] \right) f(x_B) dx_B \\
+ \int_{M^c_B - y(p_1, p_2, p_{2B})}^H \left( (p_{2B} - c_{2B})M^c_B - \pi_B[y(p_1, p_2, p_{2B}) + x_B - M^c_B] \right) f(x_B) dx_B \\
-c_{2B}M^c_B 
\]
Equation (20) implies that

\[ E[\Pi_B^*] = (p_{2B} - c_{2B} - v_{21} - v_{22})[y(p_1, p_2, p_{2B}) + \mu_B] + (v_{21} + v_{22} - c_{2B})M_B^h \]

\[ - (p_{2B} - c_{2B} - v_{21} - v_{22} + \pi_B) \int_{M_B^h - y(p_1, p_2, p_{2B})}^{H} \left[ y(p_1, p_2, p_{2B}) + x_B - M_B^h \right] \times f(x_B) dx_B \]

\[ \times \int_{M_B^h - y(p_1, p_2, p_{2B})}^{H} \left[ y(p_1, p_2, p_{2B}) + x_B - M_B^h \right] \times f(x_B) dx_B \]

(15)

It is easy to show that \( E[\Pi_B^*] \) is a concave function.

\[ \frac{\partial E[\Pi_B^*]}{\partial p_{2B}} = y(p_1, p_2, p_{2B}) + \mu_B - (a_2 + \lambda)(p_{2B} - c_{2B} - v_{21} - v_{22}) + \int_{M_B^h - y(p_1, p_2, p_{2B})}^{H} \left[ M_B^h - y(p_1, p_2, p_{2B}) - x_B \right] \times f(x_B) dx_B \]

(16)

\[ \frac{\partial^2 E[\Pi_B^*]}{\partial p_{2B}^2} = -2(a_2 + \lambda) \int_{M_B^h - y(p_1, p_2, p_{2B})}^{H} f(x_B) dx_B \]

\[ - (a_2 + \lambda)^2 (p_{2B} - c_{2B} + \pi_B - v_{21} - v_{22}) f(M_B^h - y(p_1, p_2, p_{2B})) < 0 \]

(17)

For the detailed derivation of Equation (17) see Appendix D.

If the reserved production capacity is less than the optimal level specified in the order, then the optimal selling price can be determined solving Equation (17),

\[ y(p_1, p_2, p_{2B}) + \mu_B - (a_2 + \lambda)(p_{2B} - c_{2B} - v_{21} - v_{22}) \]

\[ + \int_{M_B^h - y(p_1, p_2, p_{2B})}^{H} \left[ M_B^h - y(p_1, p_2, p_{2B}) - x_B \right] \times f(x_B) dx_B = 0 \]

(18)

3.3. Production capacity reservation decision at stage 1. In the centralized supply chain, the global decision maker needs to reserve production capacity at the beginning of stage 1. When production capacity constraint is not binding, the optimal order quantity of bundled products at stage 2 is defined as follows.

\[ h^c(\tilde{a}_1) = \tilde{a}_1 - a_2p_{1B} + \lambda(p_1 + p_2 - p_{1B}) + s_B^c = y(p_1, p_2, p_{1B}) + s_B^c \]

(19)

Therefore, from Equation (12) we have;

\[ 1 - F(s_B^c - (p_{1B} + \pi_B - (v_{21} + v_{22}))) f(s_B^c) \frac{\partial s_B^c}{\partial p_{1B}} = 0 \]

(20)

Equation (20) implies that \( \frac{\partial s_B^c}{\partial p_{1B}} > 0 \), hence, \( \frac{\partial s_B^c}{\partial p_{1B}} = \frac{\partial s_B^c}{\partial a_1} \). The first derivative of \( h^c(\tilde{a}_1) \) with respect to \( \tilde{a}_1 \) is given by;

\[ \frac{\partial h^c(\tilde{a}_1)}{\partial a_1} = 1 - a_2 \frac{\partial p_{1B}}{\partial a_1} - \lambda \frac{\partial p_{1B}}{\partial a_1} + \frac{\partial s_B^c}{\partial a_1} = \frac{1}{2} \left\{ 1 + [1 + F(s_B^c)] \frac{\partial s_B^c}{\partial a_1} \right\} > 0 \]

(21)

According to equation (21) it is easy to see that \( h^c(\tilde{a}_1) \) is increasing in \( \tilde{a}_1 \). The values for \( p_{1B}^c, s_B^c \) are jointly calculated by Equation (12). Now, we know that \( M_B^h > 0 \) because \( p_{1B} > c_{1B} + c_{2B} \). Therefore, the demand function is positive. The full benefit under the bundle policy is:

\[ E[\Pi_B^*] = -c_{1B}M_B^c + \int_{1}^{a(M_B^h)} E[\Pi_B^*] g(a_1) da_1 + \int_{a(M_B^h)}^{h} E[\Pi_B^*] g(a_1) da_1 \]

(22)

It is supposed that the reserved production capacity is in the interval \( h^c(l) \leq M_B^h \leq h^c(h) \) and \( |a(M_B^h) - M_B^c| = 2M_B^h + \mu + a_2(c_{2B} + v_{1B}) - B(s_B^c) - 2s_B^c \). At these values, the reserved production capacity matches the optimal order quantity of bundled products when there is no capacity limitation at stage 2. Therefore, the following theorem is proposed to determine the production capacity. This theorem affirms there is a unique solution for the optimal production capacity.
Theorem 3.1. The optimal reserved production capacity reservation is the solution of Equation (23).

\[ \int_{1}^{h}(v_1 - c_1 + (p^*_2 - c_2 + \pi - v_1)) \int_{h}^{H}(f(x)dx)[g(a_1)]da_1 \]

The derivation details of this theorem are shown in Appendix E.

3.4. Decentralized policy under the bundle policy. The decentralized supply chain is now considered in order to determine the best decisions. Here, the supplier decides the production capacity at stage 1 and the retailer decides the order quantity of the bundled products after the demand is updated at stage 2.

In the decentralized supply chain, both parties have sufficient information about market demand and costs. For the coordination of the supply chain, a three-parameter contract \((w_B, \alpha_B, d_B)\) is used, as proposed by Chen et al. (2010). In contract \((w_B, \alpha_B, d_B)\), \(w_B\) is the wholesale price that the supplier offers to its retailer. \(d_B\) is the unit buyback price that the supplier pays to its retailer for each unit of the unsold bundle at the end of stage 2, and \(\alpha_B\) is the risk subsidy. When \(\alpha_B\) is positive, there is a compensation that the retailer pays to its supplier to cover the loss due to the over reserved production capacity in stage 1. Conversely, when \(\alpha_B\) is negative, there is a bonus for the retailer from its supplier to avoid the over ordering of the retailer at stage 2. Following Chen et al. (2010)'s solution procedure, the following results are obtained.

\[
\begin{align*}
    w_B &= c_1B + c_2 + \rho \left[ p_B + \pi - c_1B - c_2B - \frac{\pi B(y(p_2, p_1, p_1B) + \mu_B)}{M^d_B} \right] + \frac{\beta}{M^d_B} \\
    \alpha_B &= (1 - \rho)(c_1B - v_1B) - \rho \pi B(y(p_2, p_1, p_1B) + \mu_B) + \frac{\beta}{M^d_B} \\
    d_B &= v_2B + \rho(p_B + \pi_B - v_2B)
\end{align*}
\]

Where \(\rho\) is on the interval \([0, 1]\) and \(\beta\) is any real number.

If the reserved production capacity is enough for the order quantity of product, \(Q_B^d < M^d_B\), the benefit for the retailer under the contract is obtained as follows:

\[
\Pi_B = \begin{cases} 
  p_1B(y(p_1, p_2, p_1B) + xB) - (w_B - \alpha_B)(y(p_1, p_2, p_1B) + sB) & x_B \leq s_B \\
  d_B(s_B - x_B) - \alpha_B M^d_B & x_B > s_B
\end{cases}
\]

Then the retailer's and supplier's expected profits are given by Equations (26) and (27), respectively.

\[
E[\Pi_B] = (p_1B - w_B + \alpha_B)(y(p_1, p_2, p_1B) + \mu_B) - (w_B - d_B - \alpha_B)U(s_B) - (p_1B + \pi_B - w_B + \alpha_B)B(s_B) - \alpha_B M^d_B
\]

\[
E[\Pi_B^d] = (w_B - c_2B - v_1B - \alpha_B)(y(p_1, p_2, p_1B) + \mu_B) + (v_1B + \alpha_B)M^d_B - (d_B - v_2B)U(s_B)
\]
If the supplier’s reserved production capacity fails to fulfil the retailer’s optimal order quantity ($Q^d_B = M^d_B$) then the retailer’s profit is determined as follows:

$$
\Pi_B^r = \begin{cases} 
    p_{2B} (y(p_1, p_2, p_{2B}) + x_B) - w_B M^d_B + d_B (M^d_B - y(p_1, p_2, p_{2B}) - x_B) & \text{if } x_B \leq M^d_B - y(p_1, p_2, p_{2B}) \\
    (p_{2B} - w_B) M^d_B - \pi_B (y(p_1, p_2, p_{2B}) + x_B - M^d_B) & \text{if } x_B > M^d_B - y(p_1, p_2, p_{2B})
\end{cases}
$$

(28)

Now, the retailer’s and supplier’s expected profits functions are given by Equations (29) and (30), respectively.

$$
E[\Pi_B^r] = \left( p_{2B} - d_B \right) \left( y(p_1, p_2, p_{2B}) + \mu_B \right) - \left( w_B - d_B \right) M^d_B + \left( p_{2B} + \pi_B - d_B \right) \int_{M^d_B - y(p_1, p_2, p_{2B})}^{H^B} (y(p_1, p_2, p_{2B}) - x_B) f(x_B) dx_B
$$

(29)

$$
E[\Pi_B^s] = \left( d_B - v_{2B} \right) \left( y(p_1, p_2, p_{2B}) + \mu_B \right) + \left( w_B - c_{2B} - d_B + v_{2B} \right) M^d_B + \left( d_B - v_{2B} \right) \int_{M^d_B - y(p_1, p_2, p_{2B})}^{H^B} (y(p_1, p_2, p_{2B}) + x_B - M^d_B) f(x_B) dx_B.
$$

(30)

To verify that the $(w_B, \alpha_B, d_B)$ shown in Equation (24) can coordinate the chain, it is necessary to check first the decision of the retailer in stage 2. Substituting contract parameters into Equation (26) yields the profit of the retailer in the first case as below:

$$
E[\Pi_B^r] = (1 - \rho) \times \left[ \left( p_{1B} - c_{2B} - v_{11} - v_{12} \right) \left( y(p_1, p_2, p_{1B}) + \mu_B \right) - \left( c_{1B} - v_{11} - v_{12} \right) M^d_B - \left( c_{2B} + v_{11} + v_{12} - v_{21} - v_{22} \right) U(s_B) - \left( p_{1B} + \pi_B - c_{2B} - v_{11} - v_{12} \right) B(s_B) \right] - \beta
$$

(31)

For the second case, by substituting $(w_B, \alpha_B, d_B)$ shown in Equation (24) into Equation (29), it is obtained:

$$
E[\Pi_B^r] = (1 - \rho) \left[ \left( p_{2B} - c_{2B} - v_{21} - v_{22} \right) \left| y(p_1, p_2, p_{2B}) + \mu_B \right| + \left( v_{21} + v_{22} - c_{2B} \right) M^d_B \right. - \left. \left( p_{2B} - c_{2B} - v_{21} - v_{22} + \pi_B \right) \int_{M^d_B - y(p_1, p_2, p_{2B})}^{H^B} \left| y(p_1, p_2, p_{2B}) + x_B - M^d_B \right| f(x_B) dx_B \right] - \beta
$$

(32)

Additionally for the supplier, by substituting contract parameters in Equations (27) and (30), the profit functions for both cases are as below:

$$
E[\Pi_B^s] = \rho \left[ \left( p_{1B} - c_{2B} - v_{11} - v_{12} \right) \left| y(p_1, p_2, p_{1B}) + \mu_B \right| - \left( c_{1B} - v_{11} - v_{12} \right) M^d_B \right. - \left. \left( c_{2B} + v_{11} + v_{12} - v_{21} - v_{22} \right) U(s_B) - \left( p_{1B} + \pi_B - c_{2B} - v_{11} - v_{12} \right) B(s_B) \right] + \beta + c_{1B} M^d_B
$$

(33)

$$
E[\Pi_B^s] = \rho \left[ \left( p_{2B} - c_{2B} - v_{21} - v_{22} \right) \left| y(p_1, p_2, p_{2B}) + \mu_B \right| + \left( v_{21} + v_{22} - c_{2B} \right) M^d_B \right. - \left. \left( p_{2B} - c_{2B} - v_{21} - v_{22} + \pi_B \right) \int_{M^d_B - y(p_1, p_2, p_{2B})}^{H^B} \left| y(p_1, p_2, p_{2B}) + x_B - M^d_B \right| f(x_B) dx_B \right] + \beta + c_{1B} M^d_B
$$

(34)

So the retailer and the supplier decisions are the same as the results of the centralized case (See (8) and (14)), therefore the supply chain is coordinated.

In order to implement a contract, it is necessary that the supply chain profit can be allocated between the members under contract. Now, it is investigated whether the proposed contract can allocate any division of profit among the supply chain members:
Table 2. Effects of basic demand size $a_1$ to the contract for product 1 when $Q_1^c < M_1^c = 487$

| $a_1$ | $p_1$ | $w_1$ | $d_1$ | $Q_1^c$ | $F(s_1)$ |
|-------|-------|-------|-------|---------|-----------|
| 500   | 237   | 151   | 133.50| 2.64    | 316       | 0.887     |
| 550   | 259   | 161   | 144.50| 1.85    | 343       | 0.896     |
| 600   | 282   | 172   | 156.00| 1.08    | 369       | 0.903     |
| 650   | 304   | 183   | 167.00| 0.29    | 397       | 0.910     |
| 700   | 326   | 192   | 178.00| -0.50   | 424       | 0.915     |
| 750   | 348   | 203   | 189.00| -1.29   | 450       | 0.920     |
| 800   | 371   | 213   | 200.00| -2.50   | 477       | 0.925     |
| 819.9 | 380   | 218   | 205.00| -2.36   | 487       | 0.927     |

Consider the case when $\beta=0$. It means the retailer takes up the entire profit of the supply chain. But if it is assumed that $\rho=1$, the supplier takes the entire profit of the supply chain. When $0 < \rho < 1$, then $\rho$ proportion of the supply chain’s expected profit goes to the supplier and the remaining portion, $(1 - \rho)$, goes to the retailer. Thus, the proposed contract in this research is flexible enough to allocate an arbitrary profit between the members.

From Equations (31) to (34), it is concluded that $\beta$ can be used as the transfer payment between the members. If $\beta = 0$, there is no transfer payment between retailer and supplier; but when $\beta > 0$, the retailer needs to pay the supplier. On the other hand, the supplier needs to pay the retailer when $\beta < 0$. Thus in addition to $\rho$, $\beta$ also reflects the relative powers between the supplier and the retailer.

For the proposed contract $(w_B, \alpha_B, d_B)$ satisfying Equation (24), $w_B$ is increasing in $\rho$ and $\beta$, while $\alpha_B$ is decreasing in $\rho$, but increasing in $\beta$. Moreover, $d_B$ is increasing in $\rho$ and independent of $\beta$. Thus, they have different effects on the contract $(w_B, \alpha_B, d_B)$ as well as the retailer’s and the supplier’s profit structures. Although $\rho$ and $\beta$ are in different scales, having different meanings, and affecting the profit structures differently, both can reflect the relative powers of the members.

4. **Computational and practical results.** In this part, a numerical example is constructed to illustrate the ideas developed in this research work. Here, we let the parameters as $H_i = 100$, $L_i = -100$, $H_B = 200$, $L_B = -200$, $h = 1000$, $\mu_i = 0$, $l = 500$, $c_{11} = 20$, $c_{12} = 30$, $c_{21} = 30$, $c_{22} = 40$, $a_2 = 1$, $\theta = 0.1$, $\lambda = 1$, $\rho = 0.5$, $\beta = 0$, $c_{1B} = 50$, $c_{2B} = 70$, $v_{1i} = v_{2i} = 0$, $\pi_1 = 30$, $\pi_2 = 40$, $\pi_B = 70$, $\mu_B = 0$, of which are used based on Chen et al. (2010). The random variables $s_B$ and $a_1$ follow a uniform distribution. Using these data, the optimal values for decision variables are obtained and expressed in Tables 2 to 7.

In Table 2 the effects of the basic demand size ($a_1$) on the decision variables are shown. Table 3 represents the effects of basic demand size ($a_1$) to the contract for product 1, when $Q_1^c = M_1^c = 487$. Table 4 reports comparison between coordination contract vs price-only contract for profit of product 1. Since the same results from the second product are obtained, then, in order to avoid repetition, we do not perform the same analysis. Table 5, shows the effects of the basic demand size of bundled products on the related decision variables when the reserved production capacity is enough for the order quantity of the bundled products. Table 6 shows the effects of basic demand size on the contract under bundling policy.
Table 3. Effects of basic demand size $a_1$ to the contract for product 1, $Q_1^c = M_1^c = 487$

| $a_1$ | $p_1$ | $w_1$ | $d_1$ | $\alpha_1$ | $F(s_1)$ |
|-------|-------|-------|-------|-----------|---------|
| 820   | 405   | 231   | 217.50| -1.50     | 0.975   |
| 850   | 427   | 242   | 228.50| -1.68     | 0.965   |
| 880   | 449   | 253   | 239.50| -1.86     | 0.950   |
| 910   | 472   | 264   | 251.00| -2.04     | 0.940   |
| 940   | 495   | 275   | 262.50| -2.16     | 0.905   |
| 970   | 518   | 287   | 274.00| -2.31     | 0.880   |
| 1000  | 541   | 298   | 285.50| -2.46     | 0.855   |

Table 4. Comparison between coordination contract vs price-only contract for profit of product 1, Coordination contract: $M_1^c = 487$ and total profit=279270

| $w_1$ | Capacity profit | Supplier profit | Retailer profit | Total profit |
|-------|----------------|----------------|-----------------|--------------|
| 163   | 449            | 142000         | 123210          | 265210       |
| 199   | 426            | 125720         | 105810          | 231530       |
| 250   | 401            | 10954          | 90740           | 191694       |
| 290   | 378            | 95801          | 75437           | 171238       |
| 320   | 356            | 81918          | 62549           | 144467       |

Table 5. Effects of basic demand size to the contract under bundling policy, $Q_B^c < M_B^c = 867$

| $a_1$ | $p_1B$ | $w_B$ | $d_B$ | $\alpha_B$ | $Q_B^c$ | $F(s_B)$ |
|-------|--------|-------|-------|------------|---------|---------|
| 500   | 279    | 216   | 174.50| 8.00       | 564     | 0.799   |
| 550   | 303    | 225   | 186.50| 6.10       | 592     | 0.812   |
| 600   | 327    | 234   | 198.50| 4.20       | 644     | 0.823   |
| 650   | 350    | 244   | 210.00| 2.27       | 696     | 0.833   |
| 700   | 374    | 255   | 222.00| 0.41       | 746     | 0.842   |
| 750   | 398    | 264   | 234.00| -1.48      | 796     | 0.850   |
| 800   | 421    | 273   | 245.50| -3.46      | 847     | 0.857   |
| 819.9 | 431    | 278   | 250.50| -4.18      | 867     | 0.860   |

Table 6. Effects of basic demand size on the contract under bundling policy when, $Q_B^c = M_B^c = 867$

| $a_1$ | $p_{2B}$ | $w_B$ | $d_B$ | $\alpha_B$ | $F(s_B)$ |
|-------|----------|-------|-------|------------|---------|
| 820   | 437      | 279   | 253.50| -5.92      | 0.752   |
| 850   | 474      | 298   | 272.0 | -5.90      | 0.750   |
| 880   | 510      | 315   | 290.00| -5.86      | 0.747   |
| 910   | 546      | 333   | 308.00| -5.60      | 0.742   |
| 940   | 585      | 353   | 327.00| -5.96      | 0.740   |
| 970   | 621      | 371   | 345.50| -6.25      | 0.739   |
| 1000  | 658      | 389   | 364.00| -6.35      | 0.736   |
Table 7. Profit with bundling policy, Proposed contract: 
\( M_B^* = 867 \) and total profit = 260621

| \( w_B \) | Capacity | Supplier profit | Retailer profit | Total profit |
|---|---|---|---|---|
| 234 | 838 | 142490 | 104540 | 247030 |
| 259 | 794 | 126730 | 87871 | 214601 |
| 299 | 745 | 112410 | 72037 | 184447 |
| 352 | 668 | 96453 | 60367 | 156820 |
| 386 | 630 | 85792 | 45607 | 131399 |

From Figures 1 and 2, it is easy to see that the impact of the basic demand \( a_1 \) on both pricing strategies with bundling and without bundling is positive; that is, the prices and wholesale prices always increase with an increase in the value of \( a_1 \).

![Figure 1. Impact \( a_1 \) between two products on the retailer’s pricing strategy.](image1)

Now, a sensitivity analysis is conducted. Principally, we vary the value of one parameter and fix the others. Then we see its impact on the optimal values for of decision variables and the total profit for the supplier and retailer. The initial data for the sensitivity analysis are \( a_1 = 750, H_i = 100, L_i = -100, \mu_i = 0, c_{11} = 20, c_{12} = 30, c_{21} = 30, c_{22} = 40, a_2 = 0.5, L_B = -200, H_B = 200, \mu_B = 0, l = 500, h = 1000, \lambda = 0.35, \rho = 0.5, \beta = 0, \pi_B = 70, v_1 = v_2 = 0, c_{1B} = 50, c_{2B} = 70, \pi_1 = 30, \pi_2 = 40, \theta = 0.25 \), where the basic optimal values of decision variables based on the mentioned initial data, are \( p_1 = 240, p_2 = 242, p_B = 424, F(s_B) = 0.834, w_B = 278, \alpha_B = -3.60, Q_B^* = 624, d_B = 247 \), retailer’s profit = 81592 and supplier’s profit = 115320.
Figure 2. Impact $a_1$ between two products on the wholesale pricing strategy.

Table 8. The results in numerical analysis

| Percent change | $p_1$  | $p_2$  | $p_B$  | $F(s_B)$ | $Q_B$ | $w_B$ | $\alpha_B$ | $d_B$ | Retailer profit | Supplier profit |
|----------------|--------|--------|--------|----------|-------|-------|-------------|-------|----------------|----------------|
| $a_2 = 0.5$    |        |        |        |          |       |       |             |       |                |                |
| +50            | 52.8   | 52.48  | 38.44  | 6.92     | -29.97| 33.09 | -279.44    | 33    | -7.97         | -4.66          |
| +25            | 25.83  | 26.03  | 18.63  | 3.12     | -12.82| 15.83 | -121.39    | 15.99 | 4.00           | 3.47           |
| +15            | 15.42  | 15.70  | 11.08  | 2.04     | -7.05 | 9.35  | -68.06     | 9.51  | 3.99           | 3.13           |
| -15            | -15.42 | -15.29 | -10.64 | -2.28    | 6.09  | -8.99 | 51.94      | -9.11 | -7.31          | -5.91          |
| -25            | -25.42 | -25.62 | -17.22 | -4.08    | 9.29  | -14.03| 90.83      | -14.98| -14.60         | -9.97          |
| -50            | Infeasible |       |        |          |       |       |             |       |                |                |
| $\theta = 0.25$|        |        |        |          |       |       |             |       |                |                |
| +50            | 10.42  | 10.33  | 1.65   | 0.36     | 2.08  | 1.08  | 17.78      | 1.42  | 4.89           | 3.53           |
| +25            | 5.00   | 4.96   | 0.94   | 0.24     | 0.96  | 0.72  | 8.34       | 0.81  | 2.29           | 1.86           |
| +15            | 2.92   | 3.31   | 0.47   | 0.12     | 0.64  | 0.36  | 6.39       | 0.40  | 1.34           | 1.55           |
| -15            | -3.33  | -3.31  | -0.71  | 0.00     | -0.32 | -0.36 | -3.06      | -0.64 | -1.91          | -0.08          |
| -25            | -5.42  | -4.96  | -0.94  | -0.12    | -0.96 | -0.72 | -6.39      | -0.81 | -2.32          | -1.05          |
| -50            | -10.00 | -10.53 | -1.65  | -0.24    | -1.06 | -1.08 | -14.72     | -1.42 | -4.72          | -3.42          |
| $\lambda = 0.35$|        |        |        |          |       |       |             |       |                |                |
| +50            | 0.00   | 0.00   | 13.21  | 2.40     | -7.05 | 11.15 | -66.39     | 11.34 | 6.83           | 5.81           |
| +15            | 0.00   | 0.00   | 8.25   | 1.56     | -4.01 | 6.83  | -39.44     | 6.88  | 5.11           | 4.10           |
| +15            | 0.00   | 0.00   | 8.25   | 1.56     | -4.01 | 6.83  | -39.44     | 6.88  | 5.11           | 4.10           |
| -15            | -8.25  | -1.68  | 4.17   | -6.47    | -36.67| -7.29 | -7.64      | -12.06| -8.74          | -10.74         |
| -25            | -13.44 | -3.00  | 3.61   | -11.15   | -35.83| -11.74| -12.06     | -8.74 |                |                |
| -50            | -10.00 | -10.53 | -1.65  | -0.24    | -1.06 | -1.08 | -14.72     | -1.42 | -4.72          | -3.42          |

The results of the sensitivity analysis based only in changes on the parameters $a_2$, $\lambda$ and $\theta$ are shown in Table 8.
5. **Managerial insights.** From Table 2 to Table 8 the following managerial insights are observed:

1. According to Table 2, increasing the basic demand size increases the selling price, wholesale price, order quantity and unit buyback price from the supplier to its retailer, while the risk subsidy decreases. Also by increasing the basic demand, the safety stock level increases. These results occur when the reserved production capacity is enough for the order quantity of the first product. When $\alpha_1$ is positive, there is a compensation that the retailer pays to its supplier to cover the loss due to the over reserved production capacity in stage 1. Conversely, when $\alpha_1$ is negative, there is a bonus for the retailer from its supplier to avoid the over ordering of the retailer at stage 2.

2. According to Table 3 results, when there is not enough capacity, the same behavior occurs for selling price, wholesale price, order quantity, unit buyback price and the risk subsidy. Conversely, increasing the basic demand decreases the safety stock level.

3. According to Table 4, one can observe that the total supply chain profits in the price-only supply chain case (transaction between the supplier and the retailer is solely based on the wholesale price and there is no risk sharing) are lower than under contract, with a range of loss between 5% and 48%. Also, notice that the supplier’s profit is ever higher than the retailer’s profit. It means retailer should try to set the contract parameters between himself and the supplier to gain more profit.

4. According to Table 5, increasing the basic demand size $a_1$ causes the bundle selling price to increase, as well as the wholesale price, order quantity and unit buyback price, but the risk subsidy decreases. Also by increasing the basic demand size of the bundled products, the safety stock level rises too.

5. According to Table 6, when there is not enough capacity, the same behavior occurs for the decision variables, but increasing the basic demand size decreases the safety stock level.

6. Table 7 shows the results for several wholesale prices. It can be noted that the total supply chain profits in the price-only supply chain are inferior to those of the centralized supply chain, with a range of loss between 5% and 50%.

7. According to Table 8, in the bundling policy, it should be noted that when $a_2$ and $\theta$ increase both $p_1$ and $p_2$ increase too. On the other hand, $p_1$ and $p_2$ are highly sensitive respect to the changes in the values of $a_2$ while are sensitive respect to the changes in the values of $\theta$. Since $p_1$ and $p_2$ are not function of $\lambda$ so they are not sensitive respect to the changes in the values of $\lambda$. However, $p_B$ is highly sensitive respect to the change in the values of $a_2$ and it increases when $a_2$ increases.

8. Also, $p_B$ is sensitive respect to the increase in the values of $\lambda$ but $p_B$ is highly sensitive respect to the decrease in the values of $\lambda$. On the other hand, $p_B$, $Q_B^c$, $w_B$, and $d_B$ are slightly sensitive respect to the change in the values of $\theta$. When $a_2$ decreases by 50%, there is no feasible solution.

9. On the other hand, $F(s_B)$ is sensitive respect to the increase of $a_2$ while it is slightly sensitive respect to the increase in the values of $\theta$ and $\lambda$. Additionally, $Q_B^c$ is highly sensitive respect to the increase in the values of $a_2$ while $Q_B^c$ is sensitive to the decrease in the values of $a_2$. Conversely, $Q_B^c$, $w_B$ are sensitive respects to the change in the values of $\lambda$. Furthermore, $w_B$ is highly sensitive respect to the change in the values of $a_2$. When $\lambda$ increases by 50%, there is no
feasible solution. Moreover, $\alpha_B$ is highly sensitive respect to the changes in the values of $a_2$ and $\lambda$. $\alpha_B$ is sensitive respect to the change in the values of $a_2$ while the profits are slightly sensitive respect to the change in the values of $\theta$. Likewise, the supplier’s profit is sensitive respect to the change in the values of $a_2$ while the retailer’s profit is sensitive respect to the decrease in the values of $\lambda$.

It is noted that the supplier’s profit and retailer’s profit are sensitive respect to the change in the values of $a_2$ while the profits are slightly sensitive respect to the change in the values of $\theta$. Likewise, the supplier’s profit is sensitive respect to the change in the values of $a_2$ while the retailer’s profit is sensitive respect to the increase in the values of $\lambda$.

The practical application of the proposed model in this paper is in the sense that in the real life-marketing situation, it is observed the bundle strategy for selling the products everywhere. Some manufacturing firms’ sells their products in the way of buy one get one free. Due to this offers, the manufacturer tries to attract more customers in order that they buy the more products. In addition, someone tie up with other manufacturing firms and sells the products in bundle form with different products. This kind of strategy, it is seen everywhere in the market all over world. Main aim of the manufacturing firms is how to get more profit for sales of the products and attract more and more customers. Therefore, one objective of this paper is how to represent and solve mathematically this type problem.

6. Conclusion. In this research work, a supply chain optimization via a coordination contract, considering that demand has random noise, is studied. We used the profit and risk-sharing contract that can coordinate the chain in a decentralized situation. In the chain studied, two complementary fashion products are sold. This paper analyzes the following two pricing strategies: 1) pricing of complementary products without bundling policy and 2) pricing of complementary products under bundling policy. In the first strategy, the products are sold individually, while in the second strategy both products are sold together. In this study, it is shown that in both pricing strategies, the retail price and the wholesale price of products increase with an increment in the demand. Also, with increment in the market demand, the order of bundled products also increases. It was found that the retailer does not tend to order extra products and the retailer considers the selling price lower when the demand is high.

In the bundling policy, the wholesale and retail prices are lower than the prices without bundling. Moreover, the capacity has important effects on the decision variables, especially on the service level constraint. When there is not enough capacity, increasing the basic demand decreases the safety stock level, while when the capacity is high enough, increasing the basic demand raises the safety stock level. This behavior can be seen both, when the products are sold individually and when they are sold as a bundle. As mentioned above, the price of the bundled products is lower than the prices of the individual products. These results indicate that a firm that uses bundling strategy to sell its products should offer a larger discount.

Moreover, the contract is shown to be flexible and can split the total supply chain profit in any portion between the supplier and the retailer, in a decentralized situation, under the proposed contract.

Finally, we conclude that the contract with bundling is applicable to the supply chain coordination and practitioners and academicians can use it. It is especially useful for businesses that produce and/or sells complementary products, because the proposed model fits those supply chains very well. If the product combination is
right, the decision to bundle can produce the following results: 1) increase unit sales volume; 2) increase margins; 3) offer new channel and cross-industry opportunities; and 4) offer exposure to new potential customers.

For future research directions, we would recommend to consider dynamic pricing or marketing efforts such as cooperative advertising. Also of interest, there are other types of demand functions with asymmetric or non-asymmetric information that can be explored. Moreover, consider different potential demand for both types of products could be of interest. These are undoubtedly research topics that can be investigated by researchers in the near future.

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Appendix A. Detailed derivation of Equation (5) From Equation (4), we have:

\[
E[\Pi_{21}]=\int_{L_i}^H ((p_1-c_2) (y_1 (p_1, p_2) + x_1) + v_21 (s_1-x_1)) f(x_1) dx_1 \\
+ \int_{L_i}^H ((p_2-c_2) (y_2 (p_1, p_2) + x_2) + v_22 (s_2-x_2)) f(x_2) dx_2 \\
- c_21 s_1 - v_11 (y_1 (p_1, p_2) + s_1) + v_12 M_2^c - c_22 s_2 - v_12 (y_2 (p_1, p_2) + s_2) \\
+ v_11 M_1^c + \int_{L_i}^H ((p_1-c_21) (y_1 (p_1, p_2) + x_1) + v_21 (s_1-x_1)) f(x_1) dx_1 \\
+ \int_{L_i}^H ((p_2-c_22) (y_2 (p_1, p_2) + x_2) + \pi_2 (x_2-s_2)) f(x_2) dx_2 \\
- c_21 s_1 - v_11 (y_1 (p_1, p_2) + s_1) + v_12 M_2^c - c_22 s_2 - v_12 (y_2 (p_1, p_2) + s_2) \\
+ v_11 M_1^c + \int_{L_i}^H ((p_1-c_21) (y_1 (p_1, p_2) + x_1) + \pi_1 (x_1-s_1)) f(x_1) dx_1 \\
+ \int_{L_i}^H ((p_2-c_22) (y_2 (p_1, p_2) + x_2) + v_22 (-s_2-x_2)) f(x_2) dx_2 \\
- c_21 s_1 - v_11 (y_1 (p_1, p_2) + s_1) + v_12 M_2^c - c_22 s_2 - v_12 (y_2 (p_1, p_2) + s_2) + v_11 M_1^c
\]

where

\[
U(s_i) = \int_{L_i}^s (s_i-x) f(x) dx_i, \quad B(s_i) = \int_s^H (x_i-s) f(x) dx_i
\]

\[
\Rightarrow E[\Pi_{21}] = ((p_1-c_21 - v_11)(y_1 (p_1, p_2) + \mu_1) - (c_21 + v_11 - v_12) U(s_1) \\
- (p_1-c_21 + \pi_1 - v_11) B(s_1)) \\
+ ((p_2-c_22 - v_12)(y_2 (p_1, p_2) + \mu_2) - (c_22 + v_12 - v_22) U(s_2) \\
- (p_2-c_22 + \pi_2 - v_22) B(s_2)) + v_11 M_1^c + v_12 M_2^c \\
+ \left( (p_1-c_21 - v_11)(y_1 (p_1, p_2) + \mu_1) - (c_21 + v_11 - v_12) U(s_1) \right) \\
- (p_1-c_22 + \pi_1 - v_11) B(s_1)) \\
+ (p_2-c_22 - v_12)(y_2 (p_1, p_2) + \mu_2) - (c_22 + v_12 - v_22) U(s_2) \\
- (p_2-c_22 + \pi_2 - v_22) B(s_2), + v_11 M_1^c + v_12 M_2^c
\]

If there is no capacity constraint for $s_i$ then we have:

\[
H(p_1, p_2) = \begin{bmatrix}
\frac{\partial^2 E[\Pi_{21}]}{\partial p_1^2} & \frac{\partial^2 E[\Pi_{21}]}{\partial p_1 \partial p_2} \\
\frac{\partial^2 E[\Pi_{21}]}{\partial p_2 \partial p_1} & \frac{\partial^2 E[\Pi_{21}]}{\partial p_2^2}
\end{bmatrix} = \begin{bmatrix}
-4a_2 & -4a_2 \theta \\
-4a_2 \theta & -4a_2
\end{bmatrix} = 16a_2^2(1-\theta^2) > 0
\]

Since $\frac{\partial^2 E[\Pi_{21}]}{\partial p_1^2} = -4a_2 < 0$ and $\|H(p_1, p_2)\| = 16a_2^2(1-\theta^2) > 0$ , $H(p_1, p_2)$ is negative definite matrix and the profit function is concave. Therefore $E[\Pi_{21}]$ is concave. Thus,

\[
\frac{\partial E[\Pi_{21}]}{\partial p_1} = 2 (-2a_2 p_1 + a_1 + 2a_2 c_21 + v_11 a_2 + \mu_1 - B(s_1) + v_12 a_2 + 2c_22 a_2 \theta) - 4a_2 \theta p_2 \\
\frac{\partial E[\Pi_{21}]}{\partial p_2} = 2 (2a_2 c_21 + a_1 + 2a_2 c_22 + v_11 a_2 \theta + \mu_2 - B(s_2) + v_12 a_2 + 2a_2 \theta) - 4a_2 \theta p_1
\]
Now, setting two above equations equal to zero and solving them simultaneously, we obtain the optimal values for \( p_1, p_2 \) as given below:

\[
\begin{align*}
p_1(s) &= \left( \frac{a_1(1-\theta)+\mu_1-\mu_2\theta}{2\theta_2(1-\theta^2)} + \frac{c_{21}+v_{11}}{2} - \frac{B_1(s)-\theta B_2(s)}{2\theta_2(1-\theta^2)} \right) \\
p_2(s) &= \left( \frac{a_1(1-\theta)+\mu_2-\mu_1\theta}{2\theta_2(1-\theta^2)} + \frac{c_{22}+v_{12}}{2} - \frac{B_2(s)-\theta B_1(s)}{2\theta_2(1-\theta^2)} \right)
\end{align*}
\]

Where \( B_i(s) = \int_{s_i}^{H_i} (x_i - s_i) f(x_i) dx_i \).

**Appendix B.** Detailed derivation of Equation (6)

If the reserved production capacity is not enough for the order quantity of product 1 and product 2 then \( Q_1^c = M_1^c \), \( Q_2^c = M_2^c \). Thus, the supply chain profit at stage 2 is:

\[
\Pi_2^c = \begin{cases} 
(p_1 - c_{21}) D_1 (p_1, p_2) - v_{21} (M_1^c - D_1 (p_1, p_2)) - c_{21} M_1^c + (p_2 - c_{22}) D_2 (p_1, p_2) \\
+ v_{22} (M_2^c - D_2 (p_1, p_2)) - c_{22} M_2^c \quad (D_1 \leq M_1^c, D_2 \leq M_2^c) \\
(p_1 - c_{21}) D_1 (p_1, p_2) - v_{21} (M_1^c - D_1 (p_1, p_2)) - c_{21} M_1^c + (p_2 - c_{22}) M_2^c \\
- \pi_2 (D_2 (p_1, p_2) - M_2^c) - c_{22} M_2^c \quad (D_1 \leq M_1^c, D_2 > M_2^c) \\
(p_1 - c_{21}) M_1^c - \pi_1 (D_1 (p_1, p_2) - M_1^c) - c_{21} M_1^c + (p_2 - c_{22}) M_2^c \\
- \pi_2 (D_2 (p_1, p_2) - M_2^c) - c_{22} M_2^c \quad (D_1 > M_1^c, D_2 > M_2^c) \\
(p_1 - c_{21}) M_1^c - \pi_1 (D_1 (p_1, p_2) - M_1^c) - c_{21} M_1^c + (p_2 - c_{22}) D_2 (p_1, p_2) \\
+ v_{22} (M_2^c - D_2 (p_1, p_2)) - c_{22} M_2^c \quad (D_1 > M_1^c, D_2 \leq M_2^c)
\end{cases}
\]

In this case, the expected profit is determined as follows:

\[
E[\Pi_2^c] = \int_{y_1(p_1, p_2)}^{H_1} \int_{y_1(p_1, p_2)}^{M_1^c - y_1(p_1, p_2)} ((p_1 - c_{21}) D_1 (p_1, p_2) - v_{21} (M_1^c - D_1 (p_1, p_2))) f(x_1) dx_1 - c_{21} M_1^c \\
+ \int_{H_1}^{M_1^c - y_1(p_1, p_2)} ((p_1 - c_{21}) M_1^c - \pi_1 (D_1 (p_1, p_2) - M_1^c)) f(x_1) dx_1 \\
+ \int_{y_1(p_1, p_2)}^{M_2^c - y_2(p_1, p_2)} ((p_2 - c_{22}) D_2 (p_1, p_2) - v_{22} (M_2^c - D_2 (p_1, p_2))) f(x_2) dx_2 - c_{22} M_2^c \\
+ \int_{H_2}^{M_2^c - y_2(p_1, p_2)} ((p_2 - c_{22}) M_2^c - \pi_2 (D_2 (p_1, p_2) - M_2^c)) f(x_2) dx_2 \\
+ \int_{y_1(p_1, p_2)}^{M_1^c - y_2(p_1, p_2)} ((p_1 - c_{21}) D_1 (p_1, p_2) + v_{21} (M_1^c - D_1 (p_1, p_2))) f(x_1) dx_1 - c_{21} M_1^c \\
+ \int_{H_1}^{M_1^c - y_2(p_1, p_2)} ((p_1 - c_{21}) M_1^c - \pi_1 (D_1 (p_1, p_2) - M_1^c)) f(x_1) dx_1 \\
+ \int_{y_2(p_1, p_2)}^{M_2^c - y_2(p_1, p_2)} ((p_2 - c_{22}) D_2 (p_1, p_2) + v_{22} (M_2^c - D_2 (p_1, p_2))) f(x_2) dx_2 - c_{22} M_2^c \\
+ \int_{H_2}^{M_2^c - y_2(p_1, p_2)} ((p_2 - c_{22}) M_2^c - \pi_2 (D_2 (p_1, p_2) - M_2^c)) f(x_2) dx_2 \\
+ \int_{y_2(p_1, p_2)}^{M_1^c - y_2(p_1, p_2)} ((p_1 - c_{21}) M_1^c - \pi_1 (D_1 (p_1, p_2) - M_1^c)) f(x_1) dx_1
\]

\[
\Rightarrow E[\Pi_2^c] = 2 \left( (p_1 - c_{21} - v_{21}) [y_1(p_1, p_2) + \mu_1] - (p_1 - c_{21} + \pi_1 - v_{21}) \right) \\
+ 2 \left( (p_2 - c_{22} - v_{22}) [y_2(p_1, p_2) + \mu_2] - (p_2 - c_{22} + \pi_2 - v_{22}) \right) \\
+ 2 \left( -a_2 (p_2 - c_{22} - v_{22}) + \int_{H_2}^{M_2^c - y_2(p_1, p_2)} \frac{a_2 \theta (p_2 - c_{22} + \pi_2 - v_{22}) f(x_2) dx_2}{2\theta_2(1-\theta^2)} \right)
\]

We show that the order affects the optimal price of the product. First, we take the first and second derivatives with respect to \( p_1 \) and \( p_2 \) as follows:

\[
\frac{\partial E[\Pi_2^c]}{\partial p_1} = 2 \left( \frac{y_1(P_1, P_2) + \mu_1 - a_2(p_1 - c_{21} - v_{21})}{2\theta_2(1-\theta^2)} + \frac{y_1(P_1, P_2) + \mu_1 - a_2(p_1 - c_{21} + \pi_1 - v_{21})}{2\theta_2(1-\theta^2)} \right) \\
+ 2 \left( -a_2 (p_2 - c_{22} - v_{22}) + \int_{H_2}^{M_2^c - y_2(p_1, p_2)} \frac{a_2 \theta (p_2 - c_{22} + \pi_2 - v_{22}) f(x_2) dx_2}{2\theta_2(1-\theta^2)} \right)
\]

\[
\frac{\partial^2 E[\Pi_2^c]}{\partial p_1^2} = \left\{ \begin{array}{ll}
2 \left( -2a_2 \int_{y_1(p_1, p_2)}^{-y_1(p_1, p_2)} f(x_1) dx_1 \right) \\
-2a_2 (p_1 - c_{21} + \pi_1 - v_{21}) f(M_1^c - y_1(p_1, p_2)) \\
+ 2 \left( -a_2 \theta^2 (p_2 - c_{22} + \pi_2 - v_{22}) f(M_2^c - y_2(p_1, p_2)) \right)
\end{array} \right\} < 0
\]
The derivation detail of Equation (11) is as follows;

\[
\frac{\partial^2 E[\pi_{22}]}{\partial p_1 \partial p_2} = 2 \left\{ -a_2 \theta - a_2 - a^2 \theta (p_2 - c_{22} + \pi_2 - v_{22}) f(M_2 - y_2(p_1, p_2)) \right\}
\]

\[
\frac{\partial E[\pi_{22}]}{\partial p_2} = 2 \left\{ -a_2 \theta (p_1 - c_{21} - v_{21}) - f_{M_2 - y_1(p_1, p_2)} \right\}
\]

\[
\frac{\partial^2 E[\pi_{22}]}{\partial p_1 \partial p_2} = 2 \left\{ -a_2 \theta (p_1 - c_{21} - v_{21}) - f_{M_2 - y_1(p_1, p_2)} \right\}
\]

\[
\frac{\partial^2 E[\Pi_{22}]}{\partial p_1 \partial p_2} = 2 \left\{ -a_2 \theta - a_2 - a^2 \theta (p_2 - c_{22} + \pi_2 - v_{22}) f(M_2 - y_2(p_1, p_2)) \right\}
\]

According to the above expressions, one obtains the Hessian (H):

\[
H(p_1, p_2) = \begin{bmatrix}
-2a_2 & \int_{L_1}^{M_1 - y_1(p_1, p_2)} \left( f(x_1) dx_1 - a^2 (p_1 - c_{21} + \pi_1 - v_{21}) f(M_1 - y_1(p_1, p_2)) \right) \\
-2a_2 & \int_{L_2}^{M_2 - y_2(p_1, p_2)} \left( f(x_2) dx_2 - a^2 (p_2 - c_{22} + \pi_2 - v_{22}) f(M_2 - y_2(p_1, p_2)) \right)
\end{bmatrix}
\]

Since

\[
\frac{\partial^2 E[\Pi_{22}]}{\partial p_1^2} = -2a_2 \int_{L_1}^{M_1 - y_1(p_1, p_2)} f(x_1) dx_1 - a^2 (p_1 - c_{21} + \pi_1 - v_{21}) f(M_1 - y_1(p_1, p_2)),
\]

- \(a^2 \theta (p_2 - c_{22} + \pi_2 - v_{22}) f(M_2 - y_2(p_1, p_2))\) < 0 and \(\|H(p_1, p_2)\| > 0\), \(H(p_1, p_2)\) is negative definite matrix and the profit function is concave. According to the above equation we have \(H(p_1, p_2) > 0\) then the expected profit \(E[\Pi_{22}]\) in new selling prices \(p_1^*, p_2^*\) is concave. Hence, the optimal pricing strategy is determined as follows.

\[
\begin{bmatrix}
2 \left( \int_{M_1 - y_1(p_1, p_2)}^{f_{M_1 - y_1(p_1, p_2)}} \left[ a_2 (p_1 - c_{21} + \pi_1 - v_{21}) - f_{p_1, p_2} \right] dx_1 + f_{y_1(p_1, p_2), \mu_1} \right) \\
+2 \left( -a_2 (p_2 - c_{22} + \pi_2 - v_{22}) + \int_{M_2 - y_2(p_1, p_2)}^{f_{M_2 - y_2(p_1, p_2)}} \left[ a_2 (p_2 - c_{22} + \pi_2 - v_{22}) - f_{p_1, p_2} \right] dx_2 \right)
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
2 \left( \int_{M_2 - y_2(p_1, p_2)}^{f_{M_2 - y_2(p_1, p_2)}} \left[ a_2 (p_2 - c_{22} + \pi_2 - v_{22}) - f_{p_1, p_2} \right] dx_2 + f_{y_2(p_1, p_2), \mu_2} \right) \\
+2 \left( -a_2 (p_1 - c_{21} + \pi_1 - v_{21}) - \int_{M_1 - y_1(p_1, p_2)}^{f_{M_1 - y_1(p_1, p_2)}} \left[ a_2 (p_1 - c_{21} + \pi_1 - v_{21}) - f_{p_1, p_2} \right] dx_1 \right)
\end{bmatrix} = 0
\]

**Appendix C.** Detailed derivation of Equation (11)

The derivation detail of Equation (11) is as follows;
\[ E[\Pi_B^c] = (p_1B - c_21 - c_22 - v_{11} - v_{12}) (y(p_1, p_2, p_1B) + \mu_B) - (c_21 + c_22 + v_{11} + v_{12} - v_{21} - v_{22})U(s_B) - (p_2B + \pi_B - c_21 - c_22 - v_{11} - v_{12})B(s_B) + M_B^c(v_{11} + v_{12}) \]  
(C1)

Substituting (2) into B1, one obtains

\[
E[\Pi_B^c] = (p_1B - c_21 - c_22 - v_{11} - v_{12}) \left(a_1 - a_2p_1B + \lambda(p_1 + p_2) + \mu_B\right) \\
- (c_21 + c_22 + v_{11} + v_{12} - v_{21} - v_{22})U(s_B) - (p_1B) \\
+ \pi_B - c_21 - c_22 - v_{11} - v_{12})B(s_B) \\
+ M_B^c(v_{11} + v_{12})
\]

Taking the first partial derivative with respect to \(p_1B\) one has Equation (11).

\[
\frac{\partial E[\Pi_B^c]}{\partial p_{1B}} = a_1 - a_2p_1B + \lambda(p_1 + p_2) - \lambda p_1B + \mu_B - a_2(p_1B - c_21 - c_22 - v_{11} - v_{12}) \\
- \lambda(p_1B - c_21 - c_22 - v_{11} - v_{12}) - B(s_B) \\
= a_1 - a_2p_1B + \lambda(p_1 + p_2) - \lambda p_1B + \mu_B - a_2p_1B \\
+ a_2(c_21 + c_22) + a_2(v_{11} + v_{12}) - \lambda p_1B + \lambda(c_21 + c_22) + \lambda(v_{11} + v_{12}) - B(s_B) \\
= a_1 - a_2p_1B + \lambda(p_1 + p_2) - \lambda p_1B + \mu_B + (a_2 + \lambda)(c_21 + c_22) \\
+ (a_2 + \lambda)(v_{11} + v_{12} - p_1B) - B(s_B)
\]

Appendix D. Detailed derivation of Equation (17)

The derivation detail of Equation (17) is as follows. Taking the first derivative of \(E[\Pi_B^c]\) with respect to \(p_{2B}\) we have:

\[
\frac{\partial E[\Pi_B^c]}{\partial p_{2B}} = y(p_1, p_2, p_2B) + \mu_B - (a_2 + \lambda)(p_2B - c_2B - v_{21} - v_{22}) \\
+ \int_{H}^{M_B^c - y(p_1, p_2, p_2B)} f(x_B)dx_B - \lambda y(p_1, p_2, p_2B) \\
+ (a_2 + \lambda)(p_2B - c_2B + \pi_B - v_{21} - v_{22})f(x_B)dx_B
\]  
(D1)

Substituting (2) into C1, one obtains

\[
\frac{\partial E[\Pi_B^c]}{\partial p_{2B}} = a_1 - a_2p_2B + \lambda(p_1 + p_2 - p_2B) - (a_2 + \lambda)(p_2B - c_2B - v_{21} - v_{22}) \\
+ \int_{H}^{M_B^c - y(p_1, p_2, p_2B)} (M_B^c - y(p_1, p_2, p_2B) - x_B) f(x_B)dx_B \\
+ \int_{H}^{M_B^c - y(p_1, p_2, p_2B)} (a_2 + \lambda)(p_2B - c_2B + \pi_B - v_{21} - v_{22})f(x_B)dx_B
\]

\[
\frac{\partial E[\Pi_B^c]}{\partial p_{2B}} = -2(a_2 + \lambda) \int_{L}^{M_B^c - y(p_1, p_2, p_2B)} f(x_B)dx_B \\
- (a_2 + \lambda)(M_B^c - y(p_1, p_2, p_2B) - M_B^c) \\
- y(p_1, p_2, p_2B)f(M_B^c - y(p_1, p_2, p_2B)) \\
+ (a_2 + \lambda)^2(p_2B - c_2B + \pi_B - v_{21} - v_{22})f(M_B^c - y(p_1, p_2, p_2B))
\]

Appendix E. Proof of Theorem

1. If \(M_B^c \geq h^c(h)\) then we have

\[ Max_{M_B^c \geq h^c(h)} E[\Pi_B^c] = -c_1BM_B^c + \int_{L}^{h} E[\Pi_B^c]|g(a_1)da_1 \]

2. If \(M_B^c \leq h^c(h)\) then we have

\[ Max_{M_B^c \leq h^c(h)} E[\Pi_B^c] = -c_1BM_B^c + \int_{L}^{h} E[\Pi_B^c]|g(a_1)da_1 \]
3. If \( h^c(l) \leq M^c_B \leq h^c(h) \) then we have:

\[
\max_{a(t^c) \leq M^c_B \leq a(h)} E[\Pi_B^c] = -c_1B M^c_B + \int_{l}^{a(M^c_B)} E[\Pi_B^c] g(a_1) da_1 + \int_{a(M^c_B)}^{h} E[\Pi_B^c] g(a_1) da_1.
\]

When \( h^c(l) \leq M^c_B \leq h^c(h) \) then we have:

\[
\frac{\partial E[\Pi_B^c]}{\partial M^c_B} = g(a(M^c_B)) \cdot \frac{\partial a(M^c_B)}{\partial M^c_B} \cdot (E[\Pi_B^c] - E[\Pi_B^0]) \bigg|_{a_1=a(M^c_B)} + \int_{l}^{a(M^c_B)} \frac{\partial E[\Pi_B^c]}{\partial a(M^c_B)} g(a_1) da_1 + \int_{a(M^c_B)}^{h} \frac{\partial E[\Pi_B^c]}{\partial a(M^c_B)} g(a_1) da_1 - c_1B.
\]

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