On noncommutative sinh-Gordon equation

U. Saleem\textsuperscript{1}, M. Siddiq\textsuperscript{2,3} and M. Hassan\textsuperscript{4}

\textit{Department of Physics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan.}

Abstract

We give a noncommutative extension of sinh-Gordon equation. We generalize a linear system and Lax representation of the sinh-Gordon equation in noncommutative space. This generalization gives a noncommutative version of the sinh-Gordon equation with extra constraints, which can be expressed as global conserved currents.

PACS: 11.10.Nx, 02.30.Ik

Keywords: Noncommutative geometry, Integrable systems, sinh-Gordon.

\begin{flushleft}
\textsuperscript{1}usman\_physics@yahoo.com
\textsuperscript{2}mohsin\_pu@yahoo.com
\textsuperscript{3}On study leave from PRD (PINSTECH) Islamabad, Pakistan
\textsuperscript{4}mhassan@physics.pu.edu.pk
\end{flushleft}
1 Introduction

Noncommutative geometry has widely been used in the study of integrable field theories (IFTs) since the last decade [1]-[13]. The noncommutative version of integrable field theories (nc-IFTs) is obtained by replacing ordinary product with $\star$-star product. In the commutative limit, these noncommutative theories reduce to ordinary field theories. Noncommutative version of different integrable models, such as principal chiral model with and without a Wess-Zumino term, sine-Gordon and sinh-Gordon equations, Korteweg de Vries (KdV) equation, Boussinesq equation, Kadomtsev-Petviashvili (KP) equation, Sawada-Kotera equation, nonlinear Schrodinger equation and Burgers equation, have been studied [1]-[13].

The noncommutativity of space is characterized by

$$\left[ x^i, x^j \right] = i \theta^{ij},$$

where $\theta^{ij}$ is a constant second rank tensor, called parameter of noncommutativity. The $\star$-star product of two functions in noncommutative spaces is given by

$$(f \star g)(x) = f(x)g(x) + \frac{i\theta^{ij}}{2} \partial_i f(x) \partial_j g(x) + \vartheta(\theta^2),$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

The $\star$-star product obeys following properties:

$$f \star I = f = I \star f,$$

$$f \star (g \star h) = (f \star g) \star h.$$

The integration of two functions is [3]

$$\int d^D x f(x) \star g(x) = \int d^D x f(x)g(x),$$

where the integration is taken in all noncommutative directions.

The noncommutativity in $(1+1)$-dimensional space-time is defined as [7]-[9]

$$[t, x] = i \theta.$$

The $\star$-star product of two functions in $(1+1)$-dimensional noncommutative space is given by [7]-[9]

$$(f \star g)(t, x) = f(t, x)g(t, x) + \frac{i\theta}{2} (\partial_t \partial_{x''} - \partial_{t''} \partial_{x'}) f(t', x')g(t'', x'')|_{t=t'=t''; x=x'=x''} + \vartheta(\theta^2),$$

with $\partial_t = \frac{\partial}{\partial t}$. In what follows, we present a noncommutative version of sinh-Gordon equation and investigate the noncommutative version of its zero-curvature and Lax representations. The integrability condition of the linear system and the Lax equation gives rise to a noncommutative sinh-Gordon equation with some extra constraints.

In section [2] we give noncommutative generalization of a linear system whose compatibility condition is the noncommutative sinh-Gordon equation. In section [3] we present a
noncommutative version of Lax representation of the noncommutative sinh-Gordon equation. In Section 4, we expand the fields perturbatively and obtain zeroth and first order sinh-Gordon equations, the associated linear system and a set of parametric Bäcklund transformation (BT) of the sinh-Gordon equation. It has been shown that the compatibility condition of the associated linear system and the Bäcklund transformation (BT) is the sinh-Gordon equation at the perturbative level. Section 5 contains our conclusions.

2 Linear System for Noncommutative sinh-Gordon Equation

In this section we discuss the integrability of noncommutative extension of the sinh-Gordon equation. We start with an associated linear system of the equation different from the one given in Ref. [3] and show that its compatibility condition is the noncommutative sinh-Gordon equation along with some constraints. The constraints obtained here are different from those obtained in Ref. [3]. These constraints are also shown to be expressed as conserved global currents.

In general, a nonlinear evolution equation solvable by inverse scattering method can be expressed as a compatibility condition of a set of linear differential equations. The associated linear system can be related to the isospectral problem and the Lax representation. We now write a linear system whose compatibility condition gives the noncommutative version of the sinh-Gordon equation. The linear system of the sinh-Gordon equation is

\[ \partial_{\pm} u = A^*_\pm \times u, \]

where \( A^*_\pm \) are

\[
A^*_+ = \begin{pmatrix}
-i\lambda & \frac{\beta}{2}\partial_+ \varphi \\
\frac{\beta}{2}\partial_+ \varphi & i\lambda
\end{pmatrix},
\]

\[
A^*_- = \frac{im^2}{4\lambda} \begin{pmatrix}
\cosh_\star \beta \varphi & -\sinh_\star \beta \varphi \\
\sinh_\star \beta \varphi & -\cosh_\star \beta \varphi
\end{pmatrix}.
\]

with \( \varphi \) a real valued function and \( \beta, m \) are some positive parameters. The compatibility condition of the linear system (2.1) in noncommutative space is the zero-curvature condition:

\[
[\partial_+ - A^*_+, \partial_- - A^*_-]_* \equiv \partial_- A^*_+ - \partial_+ A^*_- + [A^*_+, A^*_-]_* = 0,
\]

where \( [A^*_+, A^*_-] = A^*_+ \times A^*_- - A^*_- \times A^*_+ \) is a commutator in noncommutative space. The above compatibility condition gives rise to the noncommutative sinh-Gordon equation and some extra constraints

\[
\partial_- \partial_+ \varphi = \frac{m^2}{\beta} \sinh_\star \beta \varphi,
\]

\[
\partial_+ (\cosh_\star \beta \varphi) - \frac{\beta}{2} (\sinh_\star \beta \varphi \times \partial_+ \varphi + \sinh_\star \beta \varphi \times \partial_+ \varphi) = 0,
\]

\[
\partial_+ (\sinh_\star \beta \varphi) - \frac{\beta}{2} (\cosh_\star \beta \varphi \times \partial_+ \varphi + \cosh_\star \beta \varphi \times \partial_+ \varphi) = 0.
\]
These constraints become total derivatives or as global conserved currents. We note that in the limit $\theta \to 0$, the first equation becomes the ordinary sinh-Gordon equation and extra constraints vanish, the $\star-$ product becomes the usual product.

### 3 Lax Representation for Noncommutative sinh-Gordon Equation

In this section we present the Lax representation of the noncommutative sinh-Gordon equation and find the corresponding Lax equations.

The eigenvalue equation for a Lax operator $L_\pm$ is

$$L_\pm \Psi = \lambda \Psi,$$

where $L_\pm$ is given by

$$L_\pm = \left( \begin{array}{cc} -i \partial_\pm & \frac{\beta}{2} \partial_\pm \varphi \\ \frac{\beta}{2} \partial_\pm \varphi & i \partial_\pm \end{array} \right),$$

and

$$\Psi = \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right).$$

The corresponding Lax equations are

$$\partial_\pm L_\pm = [L_\pm, M]_\star,$$  \hspace{1cm} (3.1)

with

$$M = \frac{im^2}{4\lambda} \left( \begin{array}{cc} \cosh_\star \beta \varphi & - \sinh_\star \beta \varphi \\ \sinh_\star \beta \varphi & - \cosh_\star \beta \varphi \end{array} \right).$$

The Lax equation gives noncommutative sinh-Gordon equation along with some constraints

$$\partial_- \partial_+ \varphi = \frac{m^2}{\beta} \sinh_\star \beta \varphi,$$

$$\partial_\pm (\cosh_\star \beta \varphi) - \frac{\beta}{2} (\sinh_\star \beta \varphi \star \partial_\pm \varphi + \sinh_\star \beta \varphi \star \partial_\pm \varphi) = 0,$$

$$\partial_\pm (\sinh_\star \beta \varphi) - \frac{\beta}{2} (\cosh_\star \beta \varphi \star \partial_\pm \varphi + \cosh_\star \beta \varphi \star \partial_\pm \varphi) = 0.$$  

These constraints become total derivatives (global currents). The first equation in the limit when the noncommutativity parameter reduces to zero becomes an ordinary sinh-Gordon equation and the constraints of the model vanish.

### 4 Perturbative Expansion of sinh-Gordon Model

We can expand the field in power series of noncommutative parameter $\theta$. The expansion of the field $\varphi$ up to first order is

$$\varphi = \varphi^{(0)} + \theta \varphi^{(1)}.$$
With this expansion we can obtain two pairs of the sinh-Gordon equation
\[ \partial_+ \partial_- \phi^{(0)} = \frac{m^2}{\beta} \sinh \beta \phi^{(0)}, \]
\[ \partial_+ \partial_- \phi^{(1)} = m^2 \phi^{(1)} \cosh \beta \phi^{(0)}. \]

The linear system for equation (4.1) becomes
\[ \partial_\pm u^{(0)} = A_\pm^{(0)} u^{(0)}, \]
where \( A_\pm^{(0)} \) are given by
\[ A_+^{(0)} = \left( \begin{array}{c} -i \lambda \\ \frac{\beta}{2} \partial_+ \phi^{(0)} \\ i \lambda \end{array} \right), \]
\[ A_-^{(0)} = \frac{im^2}{4\lambda} \left( \begin{array}{cc} \cosh \beta \phi^{(0)} - \sinh \beta \phi^{(0)} \\ \sinh \beta \phi^{(0)} - \cosh \beta \phi^{(0)} \end{array} \right). \]

The compatibility condition for the linear system (4.3) is the zeroth order zero-curvature condition:
\[ \left[ \partial_+ - A_+^{(0)}, \partial_- - A_-^{(0)} \right] \equiv \partial_- A_+^{(0)} - \partial_+ A_-^{(0)} + \left[ A_+^{(0)}, A_-^{(0)} \right] = 0. \]

The linear system of the equation for first order reads
\[ \partial_+ u^{(1)} = A_+^{(0)} u^{(1)} + A_+^{(1)} u^{(0)}, \]
\[ \partial_- u^{(1)} = A_-^{(0)} u^{(1)} + A_-^{(1)} u^{(0)}, \]
where \( A_\pm^{(1)} \) are given by
\[ A_+^{(1)} = \left( \begin{array}{c} 0 \\ \frac{\beta}{2} \partial_+ \phi^{(1)} \\ 0 \end{array} \right), \]
\[ A_-^{(1)} = \frac{im^2}{4\lambda} \left( \begin{array}{cc} \beta \phi^{(1)} \sinh \beta \phi^{(0)} - \beta \phi^{(1)} \cosh \beta \phi^{(0)} \\ \beta \phi^{(1)} \cosh \beta \phi^{(0)} - \beta \phi^{(1)} \sinh \beta \phi^{(0)} \end{array} \right). \]

The compatibility condition for the linear system (4.4) is
\[ \left( \partial_- A_+^{(0)} - \partial_+ A_-^{(0)} + \left[ A_+^{(0)}, A_-^{(0)} \right] \right) u^{(1)} + \left( \partial_- A_+^{(1)} - \partial_+ A_-^{(1)} + \left[ A_+^{(1)}, A_-^{(0)} \right] + \left[ A_+^{(0)}, A_-^{(1)} \right] \right) u^{(0)} = 0. \]

The Bäcklund transformation for equation (4.1) is
\[ \partial_+ \left( \frac{\phi_1^{(0)} - \phi^{(0)}}{2} \right) = \frac{m \lambda}{\beta} \sinh \beta \left( \frac{\phi_1^{(0)} + \phi^{(0)}}{2} \right), \]
\[ \partial_- \left( \frac{\phi_1^{(0)} + \phi^{(0)}}{2} \right) = m \frac{\beta \lambda}{\beta} \sinh \beta \left( \frac{\phi_1^{(0)} - \phi^{(0)}}{2} \right), \]
and first order correction to equation (4.5) is

\[
\partial_+ \left( \frac{\phi_1^{(1)} - \phi_1^{(1)}}{2} \right) = m\lambda \left( \frac{\phi_1^{(1)} + \phi_1^{(1)}}{2} \right) \cosh \left( \frac{\phi_0^{(0)} + \phi_0^{(0)}}{2} \right),
\]

(4.6)

\[
\partial_- \left( \frac{\phi_1^{(1)} + \phi_1^{(1)}}{2} \right) = \frac{m}{\lambda} \left( \frac{\phi_1^{(1)} - \phi_1^{(1)}}{2} \right) \cosh \left( \frac{\phi_0^{(0)} - \phi_0^{(0)}}{2} \right).
\]

The integrability condition of equations (4.5) and (4.5) yields equations (4.1) and (4.2), respectively. To solve equation (4.1) we first reduce the problem to a one-dimensional problem by assuming that solution of equations of motion are independent of time. The first soliton solution of equation (4.1) is

\[
\phi_0^{(0)} = \frac{4}{\beta} \tanh^{-1} \exp(2mx),
\]

(4.7)

the corresponding first order correction term to the solution is

\[
\phi_0^{(1)} = \frac{1}{\sinh(2mx)}.
\]

This solves the equation of motion and constraints for the noncommutative sinh-Gordon equation to first order in \( \theta \).

5 Conclusions

In summary, we have investigated a noncommutative version of sinh-Gordon equation and discussed some of its properties as an integrable equation. This noncommutative sinh-Gordon equation reduces to an ordinary sinh-Gordon equation and constraints of the model vanish in the commutative limit. The noncommutative version of the linear system (or equivalently zero-curvature representation) and Lax representation give an integrable noncommutative sinh-Gordon equation. The constraints of the model appear as total derivatives. We have also analyzed the integrability of the equation at perturbative level. We have presented a set of Bäcklund transformation for the zeroth order sinh-Gordon equation and the first order correction to the zeroth order Bäcklund transformation. The soliton solution of the equation has been obtained. We have also shown that the 1— soliton solution of the noncommutative sinh-Gordon equation solves the equations of motion and its constraints.

Acknowledgements:
We acknowledge the enabling role of the Higher Education Commission, Pakistan and appreciate its financial support through “Merit Scholarship Scheme for PhD studies in Science & Technology (200 Scholarships)”. We also acknowledge CERN scientific information Service (publication requests).
References

[1] A. Connes, “Noncommutative Geometry”, Academic Press (1994).

[2] K. Furuta and T. Inami, Mod. Phys. Lett. A15 (2000) 997.

[3] M. Moriconi and I.C. Carnero, Nucl. Phys. B673 (2003) 437.

[4] S. Profumo, JHEP 0210 (2002) 035.

[5] M.T. Grisaru and S. Penati, Nucl. Phys. B655 (2003) 250.

[6] M. T. Grisaru, L. Mazzanti, S. Penati and L. Tamassia, JHEP 0404 (2004) 057.

[7] M. Hamanaka, “Commuting flows and conservation laws for noncommutative Lax hierarchies”, hep-th/0311206

[8] M. Hamanaka and K. Toda, J. Phys. A36 (2003) 11981.

[9] M. Hamanaka and K. Toda, Phys. Lett. A316 (2003) 77.

[10] A. Dimakis and F. M. Hoissen, Int. J. Mod. Phys. B14 (2000) 2455.

[11] A. Dimakis and F. M. Hoissen, “Noncommutative Kortewega-de-Vries equation”, hep-th/0007074

[12] A. Dimakis and F. M. Hoissen, J. Phys. A37 (2004) 4069.

[13] A. Dimakis and F. M. Hoissen, “A noncommutative version of the nonlinear Schrödinger equation”, hep-th/0007015