The Dark Energy Star and Stability analysis

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Abstract: We have proposed a new model of dark energy star consisting of three zones namely, an inhomogeneous interior region with anisotropic pressures, thin shell and the exterior vacuum region of Schwarzschild spacetime. We have discussed various physical properties. The model satisfies all the physical requirements. The stability condition under small linear perturbation has also been discussed.

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I. INTRODUCTION

The study of dark energy star has become a subject of interest due to the fact that expansion of the universe is accelerating which was suggested by High-z supernova Search Team in 1998 by observing type Ia supernova. Dark energy is the most acceptable hypothesis to explain this accelerating expansion of the universe. According to the work done of Plank mission team and based on the standard model of cosmology the total mass energy of the universe contains 4.9 percent ordinary matter, 26.8 percent dark matter and 68.3 percent dark energy. Dark matter is attractive in nature which can not be seen by a telescope and it does not absorbs or emits light or any gravitational waves. But its existence has been proved by gravitational effects on visible matter and gravitational lensing of background radiation. On the other hand the dark energy needs to have a strong negative pressure in order to explain the rate of accelerating expansion of the universe.

To construct a model of a relativistic star we generally assume that the underlying fluid distribution is homogeneous and isotropic. But it is proved by advance researches that the highly compact astrophysical objects like X-ray pulsar, Her-x-1, X-ray buster 4U 1820-30, millisecond pulsar SAXJ1804.4-3658 etc. whose density of core is expected to be beyond the nuclear density ($\sim 10^{15}gm/cc$) shows the anisotropy. Anisotropy may occurs in the existence of solid core, in presence of type P superfluid, phase transition, rotation, magnetic field, mixture of two fluid, existence of external field etc. In case of anisotropy distribution the pressure inside the fluid sphere is not homogeneous in nature, it can be decomposed into two parts radial pressure $p_r$ and the transverse pressure $p_t$. So obviously $p_r \neq p_t$. Where $p_r$ is in the orthogonal direction to $p_t$, $\Delta = p_t - p_r$ is defined as the anisotropic factor whereas $\hat{\Delta}$ is defined as anisotropic force which is repulsive in nature if $p_t > p_r$ and attractive if $p_t < p_r$.

In this paper, we are going to model of a anisotropic dark energy star characterized by the parameter $\omega = -\frac{\Delta}{\rho}$, where $p_r$ and $\rho$ respectively the radial pressure and energy density. For accelerating expansion the dark energy parameter $\omega < -\frac{1}{3}$ is required. $-1 < \omega < -\frac{1}{3}$ is referred to as quintessence. The region where $\omega < -1$ is named as phantom regime which has a peculiar property namely infinitely increasing energy density $\omega = -1$ corresponds to Einstein cosmological constant and this value is called cosmological constant barrier or phantom divide.

A two dimensional Brans-Dicke star model with exotic matter and dark energy was studied in [1]. In that paper, the author has taken the matter state equation as $p = \gamma \rho$, where $\gamma$ is the state parameter of exotic matter which satisfies $-\frac{1}{3} < \gamma < 0$ and has shown that the mass of the star decrease if $\gamma$ decrease. Anisotropic dark energy star has been discussed in [2]. Star model with dark energy has been proposed in [3]. In this paper the authors have proposed a model of dark energy star consisting of four region and by analyzing the model they conclude that for static solution at least one of the regions must be constituted by dark energy. Anisotropic dark energy star was studied by Ghezzi et al [4]. In this paper the authors have assumed variable dark energy which suffers a phase transition at a critical density and the anisotropy. The anisotropy is concentrated on a thin shell where the phase transition takes place, while the rest of the star remains isotropic. The solutions shows several features similar to the gravastar model. Lobo [5] has given a model of stable dark energy star by assuming two spatial type of mass function one is of constant energy density and the other mass function is Tolman-Whitker mass. All the features of the dark energy star has been discussed and the system is stable under small linear perturbation. The van der Waals quintessence stars have been studied in [6]. In that work, the construction of inhomogeneous compact spheres supported by a van der Waals equation of state is explored. van der Waals gravastar, van der Waals wormhole have also been discussed. Variable Equation of State for Generalised Dark Energy Model has been studied in [7]. Jadav et al have given a dark energy models with variable equation of state parameter in [8]. Some other works on dark energy energy star are in [9, 22].
The plan of the paper as follows.In section II basic field equations have been given. The model of dark energy star, exterior spacetime and junction condition, TOV equation, Energy condition, Mass-radius relation have been respectively discussed in section III-VII. The stability analysis under small radial perturbation has been studied in section VIII. Finally in section IX we have provided a short discussion and made some concluding remarks.

II. BASIC FIELD EQUATIONS

A static and spherically symmetry spacetime in curvature coordinates is given by the following metric

\[ ds^2 = -\exp \left[ -2 \int_{r}^{\infty} g(\tilde{r})d\tilde{r} \right] dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  \hspace{1cm} (1)

where \( g(r) \) and \( m(r) \) are arbitrary functions of the radial parameters \( r \). The function \( m(r) \) is the quasi local mass and is denoted as the mass function. The factor \( g(r) \) is termed as the ‘gravity profile’ which is used to measure the acceleration due to gravity by the factor.

The stress energy momentum tensor is given by the equation

\[ T_{\mu\nu} = (\rho + p_t)U_\mu U_\nu + p_t g_{\mu\nu} + (p_r - p_t)\chi_\mu \chi_\nu \]  \hspace{1cm} (2)

where \( U^\mu \) is the vector 4-velocity, \( \chi^\mu \) is the spacelike vector, \( \rho(r) \) is the energy density and \( p_r \) is the radial pressure measured in the direction of the spacelike vector. \( p_t \) is the transverse pressure in the orthogonal direction to \( p_r \) and \( \Delta = p_t - p_r \) is called the anisotropic factor.

Using the Einstein field equations \( G_{\mu\nu} = 8\pi T_{\mu\nu} \) we get the following relationship,

\[ m' = 4\pi r^2 \rho \]  \hspace{1cm} (3)

\[ g = \frac{m + 4\pi r^3 p_t}{r(r - 2m)} \]  \hspace{1cm} (4)

\[ p'_r = -\frac{(\rho + p_r)(m + 4\pi r^3 p_t)}{r(r - 2m)} + \frac{2}{r}(p_t - p_r) \]  \hspace{1cm} (5)

where \( G_{\mu\nu} \) is the Einstein tensor and ‘prime’ denotes the derivative with respect to radial parameter \( r \).

The dark energy equation of state is given by the following equation

\[ p_r = \omega \rho \]  \hspace{1cm} (6)

Where \( \omega \) is the equation of state parameter.

Now one can note that we have five unknown functions namely \( \rho, p_r, p_t, m(r), g(r) \) and four equations [3] - (6). To solve the set of equations let us assume a particular choice of the energy density \( \rho \). This particular choice of \( \rho \) was chosen earlier by Dev and Gleiser [9] to discuss anisotropic star model. Rahaman et al. have also used this density function in [10]. Using this particular choice of energy density we will find out the other parameters in explicit form.

III. MODEL OF THE DARK ENERGY STAR

Let us choose the energy density of the star as

\[ \rho = \frac{1}{8\pi} \left( \frac{a}{r^2} + 3b \right) \]  \hspace{1cm} (7)

where both \( a \) and \( b \) are constants, e.g. \( a = \frac{3}{7} \) and \( b = 0 \) corresponds to relativistic Fermi gas which can be seen in the ultradense cores of a neutron star [11] and for \( a = \frac{3}{7}, b \neq 0 \) we get relativistic fermi gas core in a constant density background.

Using (7) into (3) we obtain the expression of the mass function as,

\[ m = \frac{1}{2}r(a + br^2) \]  \hspace{1cm} (8)

Solving equation (4) - (6) we get,

\[ g(r) = \frac{a(1 + \omega) + br^2(1 + 3\omega)}{2r(1 - a - br^2)}, \]  \hspace{1cm} (9)

which has been plotted in Fig 1 and fig 2 for \(-1 < \omega < -\frac{1}{3}\) and \(\omega < -1\) respectively. Now it is clear from equation (9) that \( g(r) > 0 \) when \( \omega > -\frac{a + br^2}{a + 3br^2} \).

From the fig. 1 we see that \( g(r) > 0 \) when \(-1 < \omega < -\frac{1}{3}\) and fig. 2 shows that for \( \omega < -1 \) \( g(r) < 0 \) for arbitrary choice of \( a, b \).

The radial and transverse pressure can be obtained as,

\[ p_r = \frac{\omega}{8\pi} \left( \frac{a}{r^2} + 3b \right) \]  \hspace{1cm} (10)

\[ p_t = \frac{(1 + \omega)(a + 3br^2)}{32\pi r^2(1 - a - br^2)} \left[ a(1 + \omega) + br^2(1 + 3\omega) \right] + \frac{3b\omega}{8\pi} \]  \hspace{1cm} (11)
FIG. 1: "gravity profile", $g(r)$, has been plotted against $r$ when $-0.9 \leq \omega \leq -0.4$

The density parameter, radial and transverse pressure have been depicted in fig 3. The anisotropy factor $\Delta$ is given by,

$$\Delta = \frac{(1 + \omega)(a + 3br^2)}{32\pi r^2(1 - a - br^2)} \left[ a(1 + \omega) + br^2(1 + 3\omega) \right] - \frac{\omega a}{8\pi r^2},$$

(12)

which has been shown in fig.4. Now $\Delta^\pm$ represents a force due to the pressure anisotropy. Which has been shown in fig.5. The force will be repulsive in nature i.e.in the outward direction if $p_t > p_r$ and attractive if $p_t < p_r$ or alternatively $\Delta < 0$. For our stellar model configuration(see fig.4) $\Delta > 0$ for both the cases when $-1 < \omega < -\frac{1}{3}$ and phantom regime $\omega < -1$.

IV. EXTERIOR SPACETIME AND JUNCTION CONDITION

In this section we match our interior spacetime to the exterior Schwarzschild vacuum solution along the junction surface with the junction radius $R'$. The exterior spacetime is given by the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Here, event horizon lies at $r = 2M$. So obviously $R > 2M$.

Previously, we have matched our interior spacetime to the exterior Schwarzschild at the boundary $r = R$. Obviously the metric coefficients are continuous at $r = R$, but it does not ensure that their derivatives are also continuous at the junction surface. In other words the affine connections may be discontinuous there. To take care of this let us use the Darmois-Israel\[12, 13\] formation to determine the surface stresses at the junction boundary. The intrinsic surface stress energy tensor $S_{ij}$ is given by Lancozs equations in the following form

$$S_{ij} = -\frac{1}{8\pi} (\kappa_{ij} - \delta_{ij} \kappa_k^k)$$

(14)

The discontinuity in the second fundamental form is given by,

$$K_{ij}^+ = K_{ij}^- - K_{ij}^\pm$$

(15)

where the second fundamental form is given by,

$$K_{ij}^\pm = -n^\pm_{\nu} \left[ \frac{\partial^2 X_{\nu}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\nu} \frac{\partial X^\alpha}{\partial \xi^i} \frac{\partial X^\beta}{\partial \xi^j} \right] |S|$$

(16)

where $n^\pm_{\nu}$ are the unit normal vector defined by,

$$n^\pm_{\nu} = \pm \sqrt{g^{\alpha\beta} \frac{\partial f}{\partial X^\alpha} \frac{\partial f}{\partial X^\beta}} \frac{1}{2} \frac{\partial f}{\partial X^\nu}$$

(17)

with $n^\nu n_{\nu} = 1$. Where $\xi^i$ is the intrinsic coordinate on the shell.+ and− corresponds to exterior i.e., Schwarzschild spacetime and interior(our) spacetime respectively. The model of our dark energy star is consisting with three zones namely, an inhomogeneous interior region with anisotropic pressures, thin shell and the exterior vacuum region which has been shown in fig.6.
FIG. 4: The anisotropy parameter $\Delta = p_t - p_r$ has been shown against $r$.

FIG. 5: The anisotropy force $\frac{\Delta}{r}$ has been shown against $r$.

Considering the spherical symmetry of the spacetime surface stress energy tensor can be written as $S^i_{\ j} = \text{diag}(-\sigma, \mathcal{P}, \mathcal{P})$, where $\sigma$ and $\mathcal{P}$ are the surface energy density and surface pressure respectively.

$$K^+_{\tau} = \frac{\frac{M}{R} + \dot{R}}{\sqrt{1 - \frac{2M}{R} + R^2}}$$  \hspace{1cm} (18)$$

$$K^-_{\tau} = \frac{(1+\omega)a+bR^2(1+3\omega)}{2R} + \frac{\frac{Rg(R)}{R} - \frac{(1+\omega)\dot{R}^2(a+3bR^2)}{2(R-a-bR^2)}}{\sqrt{1 - a - bR^2 + R^2}}$$  \hspace{1cm} (19)$$

$$K^0_{\theta} = \frac{1}{R} \sqrt{1 - \frac{2M}{R} + \dot{R}^2}$$  \hspace{1cm} (20)$$

$$K^-_{\theta} = \frac{1}{R} \sqrt{1 - a - bR^2 + R^2}$$  \hspace{1cm} (21)$$

The expressions of $\sigma$ and $\mathcal{P}$ are given by,

$$\sigma = -\frac{1}{4\pi R} \left[ \sqrt{1 - \frac{2M}{R} + \dot{R}^2} - \sqrt{1 - (a + bR^2) + \dot{R}^2} \right]$$  \hspace{1cm} (22)$$

Using conservation identity $S^i_{\ j,i} = -\left[ \dot{\sigma} + 2\frac{\dot{R}}{R}(\mathcal{P} + \sigma) \right]$, one can obtain

$$\sigma' = -\frac{2}{R} (\mathcal{P} + \sigma) + \Xi$$  \hspace{1cm} (24)$$

where $\Xi$ is given by,

$$\Xi = -\frac{1}{4\pi R} \frac{m - m'R}{R - 2m} \sqrt{1 - a - bR^2 + \dot{R}^2}$$  \hspace{1cm} (25)$$

The surface mass of the thin shell is given by

$$m_s = 4\pi R^2 \sigma$$  \hspace{1cm} (26)$$

Using the expression of $\sigma$ given in equation (22) (considering the static case) we get,

$$m_s = R \left[ \sqrt{1 - (a + bR^2)} - \sqrt{1 - \frac{2M}{R}} \right]$$  \hspace{1cm} (27)$$

After some little manipulation of equation (27) the total mass of the dark energy star can be obtained as,

$$M = \frac{1}{2} R(a + bR^2) - \frac{m_s^2}{2R} + m_s \sqrt{1 - (a + bR^2)}$$  \hspace{1cm} (28)$$

From equation (27) one can obtain

$$\left(\frac{m_s}{2R}\right)' = \Upsilon - \frac{4\pi}{R} \sigma' \eta$$  \hspace{1cm} (29)$$

(for details calculation see the appendix) where,

$$\eta = \frac{\mathcal{P}'}{\sigma'}, \hspace{1cm} \Upsilon = \frac{4\pi}{R} (\sigma + \mathcal{P}) + 2\pi R \Xi'$$  \hspace{1cm} (30)$$

where the 'prime' denotes derivative with respect to 'R'.

We will use the parameter $\eta$ to discuss the stability analysis of the system. This $\sqrt{\eta}$ is generally interpreted as
the velocity of the sound. So, for the physical acceptability one must have $0 < \eta \leq 1$. The profile of $\eta$ has been shown in Fig.7 and Fig.8.

Next we will discuss about the evolution identity given by,

$$[T_{\mu\nu}n^{\mu}n^{\nu}]_{\perp} = \tilde{K}^{j}_{j}S^{j}_{i}$$

(31)

where $\tilde{K}^{j}_{j} = \frac{1}{2}(K^{i+}_{j} + K^{i-}_{j})$. From equation (31) using the equation (18) - (21) one can obtain

$$p_{r} + \frac{(\rho + p_{r})\tilde{R}^{2}}{1 - a - b\tilde{R}^{2}} = -\frac{1}{2\tilde{R}} \left( \sqrt{1 - \frac{2M}{\tilde{R}} + \tilde{R}^{2}} + \sqrt{1 - a - b\tilde{R}^{2} + \tilde{R}^{2}} \right) \mathcal{P}$$

$$+ \frac{1}{2} \left( \frac{\tilde{M}}{\tilde{R}^{2}} + \tilde{R} + \frac{(1 + \omega)\alpha + b\tilde{R}^{2}(1 + 3\omega)}{2\tilde{R}} + \tilde{R} - \frac{(1 + \omega)\tilde{R}^{2}\alpha + 3\tilde{R}^{2}}{2R(\tilde{R}^{2} - \alpha - b\tilde{R}^{2})} \right) \sigma$$

(32)

Considering static solution at $R = R_{0}$ with $\tilde{R} = \dot{R} = 0$, we get,

$$p_{r} = \frac{1}{2R_{0}} \left( \sqrt{1 - \frac{2M}{R_{0}}} + \sqrt{1 - a - bR_{0}^{2}} \right) \mathcal{P} + \frac{1}{2} \left( \frac{\tilde{M}}{R_{0}^{2}} + \frac{(1 + \omega)\alpha + bR_{0}^{2}(1 + 3\omega)}{2R_{0}} \right) \sigma$$

(33)

Here, $\sigma < 0$, therefore, $p_{r} < 0$, i.e. tension is in radial direction. Hence a positive tangential surface pressure $\mathcal{P} > 0$ is required to keep the shell stable i.e. to hold the shell against collapsing.

V. TOV EQUATION

The generalized Tolman-Oppenheimer-Volkov (TOV) equation is given by the equation [14]

$$- \frac{M_{G}(\rho + p_{r})}{r^{2}} e^{\frac{\lambda}{2}} - \frac{dp_{r}}{dr} + \frac{2}{r}(p_{t} - p_{r}) = 0$$

(34)

Where $M_{G} = M_{G}(r)$ is the effective gravitational mass inside a sphere of radius $r$ given by the Tolman-Whittaker formula which can be derived from the equa-

$$M_{G}(r) = \frac{1}{2} r^{2} e^{\frac{\nu - \lambda}{2}}$$

(35)

The above equations describes the equilibrium conditions of the fluid sphere subject to gravitational, hydrostatics and anisotropy forces.

The equation (34) can be modified in the form

$$F_{g} + F_{h} + F_{a} = 0$$

(36)

where

$$F_{g} = -\frac{\nu'}{2}(\rho + p_{r})$$

(37)

$$F_{h} = -\frac{dp_{r}}{dr}$$

(38)
VI. ENERGY CONDITION

In this section we are going to verify whether our particular model of dark energy star satisfies all the energy conditions namely null energy condition (NEC), weak energy condition (WEC), strong energy condition (SEC) and dominant energy condition (DEC) as stated as follows:

\[(i)\] NEC : \(\rho + p_r \geq 0\)  \hspace{1cm} (40)

\[(ii)\] WEC : \(\rho + p_r \geq 0, \quad \rho \geq 0\)  \hspace{1cm} (41)

\[(iii)\] SEC : \(\rho + p_r \geq 0, \quad \rho + p_r + 2p_t \geq 0\)  \hspace{1cm} (42)

\[(iv)\] DEC : \(\rho > |p_r|, \quad \rho > |p_t|\)  \hspace{1cm} (43)

We will prove all the inequalities by plotting the L.H.S of the above inequalities. From Fig.10 we see that our model satisfies all the energy conditions. The SEC satisfies by the model ensures that the spacetime does not contains a black hole region.

VII. MASS RADIUS RELATION

The mass of the dark energy star has been given in equation (8).

The compactness of the star is defined as

\[u(r) = \frac{m(r)}{r} = \frac{1}{2}(a + br^2)\]  \hspace{1cm} (44)

and the surface redshift is defined by

\[Z_s = (1 - 2u)^{\frac{1}{2}} - 1 = (1 - a - br^2)^{\frac{1}{2}} - 1\]  \hspace{1cm} (45)

The profile of mass function, compactness and surface redshift of the dark energy star have been given in Fig.11, 12 and 13 respectively.
VIII. STABILITY ANALYSIS

In this section we are going to analyze the stability of our model.

Rearranging the equation (22) we get,

\[ \dot{R}^2 + V(R) = 0 \]  

(46)

Where \( V(R) \) is given by,

\[ V(R) = 1 - \frac{M - m}{R} - \left( \frac{m_s}{2R} \right)^2 - \left( \frac{M - m}{m_s} \right)^2 \]  

(47)

(For details derivation see Appendix:1)

To discuss the linearized stability analysis let us take a linear perturbation around a static radius \( R_0 \). Expanding \( V(R) \) by Taylor series around the radius of the static solution \( R = R_0 \) one can obtain

\[ V(R) = V(R_0) + (R - R_0)V'(R_0) + \frac{(R - R_0)^2}{2}V''(R_0) + O[(R - R_0)^3] \]  

(48)

where 'prime' denotes derivative with respect to 'R'.

Since we are linearizing around static radius \( R = R_0 \) we must have \( V(R_0) = 0, V'(R_0) = 0 \). The configuration will be stable if \( V(R) \) has a local minimum at \( R_0 \) i.e, if \( V''(R_0) > 0 \).

Now from the relation \( V'(R_0) = 0 \) we get,

\[ \left( \frac{m_s(R_0)}{2R_0} \right)' = A \left[ F'(R_0) - 2 \left( \frac{M - m(R_0)}{m_s} \right) \left( \frac{M - m(R_0)}{m_s} \right)' \right] \]  

(49)

where \( A \) is given in (*)

Now the configuration will be stable if \( V''(R_0) > 0 \). i.e if

\[ \eta \frac{d}{dR}(\sigma^2) > \frac{1}{2\pi} \left[ \sigma \Upsilon - \frac{1}{2\pi R_0}(H^2 - G^2) \right] \]  

(50)

For details derivation see Appendix:3

\[ G(R_0) = A \left[ F'(R_0) - 2 \left( \frac{M - m(R_0)}{m_s(R_0)} \right) \left( \frac{M - m(R_0)}{m_s(R_0)} \right)' \right] \]  

(51)
Now from equation (50) we get
\[ \eta_0 \frac{d \sigma^2}{d R} \bigg|_{R_0} > \Omega \]
(53)
where \( \Omega = \frac{1}{2\pi} \left[ \sigma \gamma - \frac{1}{2\pi R_0} (H^2 - G^2) \right] \). From equation (53) the stability regions are dictated by the following inequalities
\[ \eta_0 > \Omega \left( \frac{d \sigma^2}{d R} \bigg|_{R_0} \right)^{-1} \quad \text{if} \quad \frac{d \sigma^2}{d R} \bigg|_{R_0} > 0 \]
(54)
\[ \eta_0 < \Omega \left( \frac{d \sigma^2}{d R} \bigg|_{R_0} \right)^{-1} \quad \text{if} \quad \frac{d \sigma^2}{d R} \bigg|_{R_0} < 0 \]
(55)
From the plot of \( \frac{d \sigma^2}{d R} \) (See fig.14) we see that \( \frac{d \sigma^2}{d R} < 0 \). So the stability region for our model is given by equation (55).

**IX. DISCUSSIONS AND CONCLUDING REMARKS**

In this work we have obtained a new class of exact interior solution by choosing a special form of energy density which describes a model of dark energy star parameterized by \( \omega = \frac{p_t}{\rho} < -\frac{1}{3} \). The obtained solutions are well behaved inside the stellar model. From the figures 1 and 2, we see that gravity profile \( g(r) > 0 \) when \( \omega \) lies in the dark energy region \((-1 < \omega < -\frac{1}{3}) \) and \( g(r) < 0 \) when \( \omega \) lies in the phantom regime. The energy density \( \rho \), radial pressure \( (p_r) \), transverse pressure \( (p_t) \) all are monotonic decreasing function of \( r \). Since anisotropy force \( \frac{\Delta p}{\rho} > 0 \) it gives \( p_t > p_r \) i.e. the force is attractive in nature. We have matched our interior spacetime to the exterior Schwarzschild spacetime in presence of thin shell. The mass of the dark energy star in terms of the thin shell has been proposed as well as the relationship among \( p_r, \sigma, \rho \) has been given. By keeping \( \omega \) fixed and choosing different values of \( M \), we have shown that \( 0 < \eta < 1 \). From the profile we notice that the profile of \( \eta \) are parallel to each other and by keeping the mass \( M \) fixed and for \(-1 < \omega < -\frac{1}{3}\) we have velocity of sound less than 1. All the energy conditions are also satisfied by the model. The mass function is monotonic increasing and regular at the center. For \((3 + 1)D\) astrophysical object Buchdahl[8] has shown that \( \frac{2M}{R} = 0.564 < \frac{8}{9} \). The stability analysis under small radial perturbation has also been discussed.

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using the expression of $\sigma$ we get,

or \[ \frac{m_s}{4\pi R^2} = \frac{1}{4\pi R} \left[ \sqrt{1 - \frac{2m}{R} + \dot{R}^2} - \sqrt{1 - \frac{2M}{R} + \dot{R}^2} \right] \]

or \[ \frac{m_s}{a} = \sqrt{1 - \frac{2m}{R} + \dot{R}^2} - \sqrt{1 - \frac{2M}{R} + \dot{R}^2} \]

or \[ \frac{m_s}{a} - \sqrt{1 - \frac{2m}{R} + \dot{R}^2} = - \sqrt{1 - \frac{2M}{R} + \dot{R}^2} \]

Squaring bothside we get,

\[ (\frac{m_s}{R})^2 - 2 \frac{m_s}{R} \sqrt{1 - \frac{2m}{R} + \dot{R}^2} = \frac{2}{R} (m - M) \]

or \[ \frac{m_s}{R} \left[ \frac{m_s}{R} - 2 \sqrt{1 - \frac{2m}{R} + \dot{R}^2} \right] = \frac{2}{R} (m - M) \]

or \[ \frac{m_s}{R} - 2 \sqrt{1 - \frac{2m}{R} + \dot{R}^2} = \frac{2}{m_s} (m - M) \]

or \[ \frac{m_s}{R} - \frac{2}{m_s} (m - M) = \sqrt{1 - \frac{2m}{R} + \dot{R}^2} \]

or \[ \frac{m_s}{2R} + \frac{M - m}{m_s} = \sqrt{1 - \frac{2m}{R} + \dot{R}^2} \]

again squaring bothside we get,

\[ (\frac{m_s}{2R})^2 + \left( \frac{M - m}{m_s} \right)^2 + 2 \frac{M - m}{m_s} = 1 - \frac{2m}{R} + \dot{R}^2 \]

which gives,

\[ \dot{R}^2 = (\frac{m_s}{2R})^2 + \left( \frac{M - m}{m_s} \right)^2 + \frac{M - m}{R} - 1 \]

Now, \[ \dot{R}^2 = -V(R) \]

which gives,

\[ V(R) = 1 - \frac{M - m}{R} - (\frac{m_s}{2R})^2 - \left( \frac{M - m}{m_s} \right)^2 \]

or \[ \frac{m_s}{2R} = 2\pi R \sigma \]

Differentiating bothside with respect to $R$ we get, \[ (\frac{m_s}{2R})' = 2\pi (R\sigma' + \sigma) \]

\[ = 2\pi R \left\{ -\frac{2}{R} (\sigma + \mathcal{P}) + \Xi \right\} + 2\pi \sigma \]

\[ = -4\pi \mathcal{P} + 2\pi R \Xi' - 2\pi \sigma \]

Differentiating bothside with respect to $R$ we get, \[ (\frac{m_s}{2R})'' = -4\pi \mathcal{P}' + 2\pi (R\Xi' + \Xi) - 2\pi \sigma' \]

Using the value of $\sigma'$ we get,

\[ (\frac{m_s}{2R})'' = -4\pi \mathcal{P}' + 2\pi (R\Xi' + \Xi) - 2\pi \left\{ -\frac{2}{R} (\sigma + \mathcal{P}) + \Xi \right\} \]

\[ = \frac{4\pi}{R} (\sigma + \mathcal{P}) + 2\pi R \Xi' - 4\pi \eta \sigma' \]

therefore, \[ (\frac{m_s}{2R})'' = \Upsilon - 4\pi \eta \sigma' \]

where, \[ \Upsilon = \frac{4\pi}{R} (\sigma + \mathcal{P}) + 2\pi R \Xi' \]

\[ V(R) = F(R) - \left( \frac{m_s}{2R} \right)^2 - \left( \frac{M - m}{m_s} \right)^2 \]

\[ V'(R) = F'(R) - 2 \left( \frac{m_s}{2R} \right) \left( \frac{m_s}{2R} \right)' \]

\[ -2 \left( \frac{M - m}{m_s} \right) \left( \frac{M - m}{m_s} \right)' \]

Now, $V'(R_0) = 0$ gives,
\[
\left( \frac{m_s(R_0)}{2R_0} \right)' = \left( \frac{R_0}{m_s(R_0)} \right) \left[ F'(R_0) - 2 \left( \frac{M - m(R_0)}{m_s} \right) \left( \frac{M - m(R_0)}{m_s} \right) \right]
\]

let,
\[
\left( \frac{m_s(R_0)}{2R_0} \right)' = G(R_0) = \left( \frac{R_0}{m_s(R_0)} \right) \left[ F'(R_0) - 2 \left( \frac{M - m(R_0)}{m_s(R_0)} \right) \left( \frac{M - m(R_0)}{m_s(R_0)} \right) \right]
\]

now
\[
V''(R) = F''(R) - 2 \left[ \left( \frac{m_s(R)}{2R} \right) \left( \frac{m_s(R)}{2R} \right)' + \left\{ \left( \frac{m_s(R)}{2R} \right)' \right\}^2 \right] - 2 \left[ \left( \frac{M - m(R)}{m_s(R)} \right) \left( \frac{M - m(R)}{m_s(R)} \right)' + \left\{ \left( \frac{M - m(R)}{m_s(R)} \right)' \right\}^2 \right]
\]

\[
V''(R_0) = F''(R_0) - 2 \left[ \left( \frac{m_s(R_0)}{2R_0} \right) \left( \frac{m_s(R_0)}{2R_0} \right)' + G(R_0) \right] - 2 \left[ \left( \frac{M - m(R_0)}{m_s(R_0)} \right) \left( \frac{M - m(R_0)}{m_s(R_0)} \right)' + \left\{ \left( \frac{M - m(R_0)}{m_s(R_0)} \right)' \right\}^2 \right]
\]

Now \(V''(R_0) > 0\) gives,
\[
H(R_0)^2 > \left[ G(R_0) \right]^2 + \left( \frac{m_s(R_0)}{2R_0} \right) \left( \frac{m_s(R_0)}{2R_0} \right)''
\]

where
\[
H(R_0)^2 = \frac{1}{2} F''(R_0) - \left( \frac{M - m(R_0)}{m_s(R_0)} \right)^2
\]

\[
- \left( \frac{M - m(R_0)}{m_s(R_0)} \right) \left( \frac{M - m(R_0)}{m_s(R_0)} \right)''
\]

or,
\[
H^2 - G^2 > 2\pi R_0 \sigma \left( \Upsilon - 4\pi \eta \sigma' \right)
\]

or,
\[
\frac{1}{2\pi R_0} (H^2 - G^2) > \sigma \Upsilon - 2\pi \eta \frac{d}{da} (\sigma^2)
\]

or,
\[
\eta \frac{d}{da} (\sigma^2) > \frac{1}{2\pi} \left[ \sigma \Upsilon - \frac{1}{2\pi R_0} (H^2 - G^2) \right]
\]

or,
\[
H^2 - G^2 > 2\pi R_0 \sigma \left[ \frac{4\pi}{R_0} (\sigma + \mathcal{P}) + 2\pi R_0 \Sigma' - 4\pi \eta \sigma' \right]
\]