Are Nonrenormalizable Gauge Theories Renormalizable?

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Abstract — We raise the issue whether gauge theories, that are not renormalizable in the usual power-counting sense, are nevertheless renormalizable in the modern sense that all divergences can be cancelled by renormalization of the infinite number of terms in the bare action. We find that a theory is renormalizable in this sense if the a priori constraints that we impose on the form of the bare action correspond to the cohomology of the BRST transformations generated by the action. Recent cohomology theorems of Barnich, Brandt, and Henneaux are used to show that conventionally nonrenormalizable theories of Yang-Mills fields (such as quantum chromodynamics with heavy quarks integrated out) and/or gravitation are renormalizable in the modern sense.

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\footnote{Research supported in part by the Robert A. Welch Foundation and NSF Grants PHY 9009850 and PHY 9511632.}
1. Introduction

There are two senses in which we may say that a theory is perturbatively renormalizable. The first is that the theory satisfies the old Dyson criterion, that the Lagrangian density should contain only operators of dimensionality four or less. This condition is a necessary (though not sufficient) requirement for infinities to be cancelled with only a finite number of terms in the Lagrangian. Even with this condition violated, it still may be possible that all divergences are cancelled by renormalization of the terms in the Lagrangian, but that an infinite number of terms are needed. Despite the presence of an infinite number of free parameters, such theories have a good deal of predictive power — specifically, all the predictive power in the $S$-matrix axioms of unitarity, analyticity, etc., together with whatever symmetries are imposed on the theory — and can be used to carry out useful perturbative calculations.

Today it is widely believed that all our present realistic field theories are actually accompanied by interactions that violate the Dyson criterion. The standard model is presumably what we get when we integrate out modes of very high energy from some unknown theory, perhaps a string theory, and like any other effective field theory its Lagrangian density contains terms of arbitrary dimensionality, though the terms in the Lagrangian density with dimensionality greater than four are suppressed by negative powers of very large masses. Likewise for general relativity; there is no reason to believe
that the Einstein-Hilbert action is the whole story, but all terms in the action with more than two derivatives are suppressed by negative powers of a very large mass, perhaps the Planck mass. Even if we were to take seriously the idea that, say, the strong interactions are described by a fundamental gauge theory whose Lagrangian contains only terms of dimensionality four or less, nevertheless in calculations of processes at a few GeV we would use an effective field theory with heavier quarks integrated out, and such an effective theory necessarily involves terms in the Lagrangian of unlimited dimensionality. Similarly, although modern string theories have been generally based on two-dimensional field theories that are renormalizable in the Dyson sense, there is some interest in including terms in the action that violate this condition. 3

The second, ‘modern,’ sense in which a theory may be said to be renormalizable is that the infinities from loop graphs are constrained by the symmetries of the bare action in such a way that there is a counterterm available to absorb every infinity. Unlike the Dyson criterion, this condition is absolutely necessary for a theory to make sense perturbatively. It is automatically satisfied if the only limitations imposed on the terms in the bare action arise from global, linearly realized symmetries. The difficulty in satisfying this condition appears when we impose nonlinearly realized symmetries or gauge symmetries on the bare action. Nonlinearly realized symmetries of the bare action are in general not symmetries of the quantum effective action, while
gauge symmetries must be eliminated in quantizing the theory. A BRST symmetry\(^4\) does survive the gauge fixing, but it is nonlinearly realized, so that even though the quantum effective action respects a BRST symmetry, it is not the same as the BRST symmetry of the bare action.

The question of whether gauge theories are renormalizable in the modern sense was originally answered only in the context of theories that are renormalizable in the Dyson sense.\(^5\) These proofs relied on a brute force enumeration of the possible terms in the quantum effective action of dimensionality four or less, and it was not obvious that these proofs of renormalizability could be extended to Lagrangian densities that contain terms of unlimited dimensionality. This is what is meant by the question asked in the title of this article.\(^\text{[\textcolor{red}{???}]}\)

Section 2 discusses the ‘structural constraints’ that are imposed on the bare action in specifying a gauge symmetry. Section 3 outlines our method for addressing the question of renormalizability by the use of the antibracket formalism.\(^7,8\) We find there that renormalizability in the modern sense is guaranteed if the structural constraints imposed on the action are chosen in

\(^{\text{[\textcolor{red}{???}]}}\) To avoid possible confusion, we should distinguish between our aims in this paper and earlier efforts\(^6\) to make general relativity and other theories renormalizable in the Dyson sense by including higher derivative terms (such as terms bilinear in the curvature) in the unperturbed Lagrangian. Such efforts lead to problems with unitarity at the energies at which the renormalized momentum-space integrals begin to converge. In contrast, we accept the conventional way of splitting the Lagrangian into unperturbed and interaction terms, so that the unperturbed Lagrangian correctly describes the particle content of the theory, and no problems with unitarity arise in perturbation theory. Our aim here is not to restore renormalizability in the Dyson sense, but to learn how to live without it.
correspondence with the cohomology of the antibracket transformation generated by the bare action. (The renormalizability of theories with nonlinearly realized global symmetries can be dealt with by the same formalism, but with spacetime-independent ghost fields.) In section 4 we use recently proved cohomology theorems to show that theories of Yang-Mills fields and/or gravitation are renormalizable in the modern sense, even though we allow terms in the Lagrangian of arbitrary dimensionality. But we shall see that the matching of structural constraints with antibracket cohomologies is only a sufficient, not a necessary, condition for renormalizability. Cohomology theorems give the candidates for ultraviolet divergences or anomalies; a perturbative calculation is needed to see whether the divergences or anomalies actually occur. In fact, in Section 4 we shall encounter terms in the cohomology of the antibracket operator that do not correspond to actual infinities.

There are other cohomology theorems that can be applied to ‘first-quantized’ string theories. The question of the renormalizability of supergravity and superstring theories remains open, but can be studied by the methods of antibracket cohomology. It would be reassuring to prove that all these theories are renormalizable in the modern sense, but even more interesting if some were not, for then renormalizability could again be used, as we used to think that the Dyson power-counting condition could be used, as a criterion for selecting physically acceptable theories.

Our discussion does not pretend to be mathematically rigorous. In par-
ticular we work with infinite quantities without explicit consideration of pos-
sible regulators, and simply assume that there is some way of introducing a
regulator that does not produce anomalies that would invalidate our argu-
ments. This is no problem in Yang-Mills theories that are free of anomalies
in one-loop order because of the nature of the gauge group rather than be-
cause of cancellations among different fermion multiplets. In such theories
the cohomology theorem of reference 9 shows that the gauge symmetries are
free of anomalies to all orders, without regard to the dimensionality of the
Lagrangian. Theories with $U(1)$ factors may present special difficulties.\textsuperscript{11}

Before proceeding, we wish to comment on earlier work on the renor-
malization of general gauge theories, most of which were brought to our atten-
tion after the circulation of an earlier version of this paper. Dixon\textsuperscript{12} and
then Voronov, Tyutin, and Lavrov\textsuperscript{13} generalized the ideas of Zinn-Justin\textsuperscript{7} by
introducing a canonical transformation of fields and antifields as well as an
order-by-order renormalization of coupling constants. They emphasized the-
ories that are renormalizable in the Dyson sense, but Voronov, Tyutin, and
Lavrov briefly considered more general theories. More recently, Anselmi\textsuperscript{14} has
further analyzed the issue of renormalization in gauge theories that are not
renormalizable in the Dyson sense. He also uses a canonical transformation
as well as coupling constant renormalization to cancel infinities, and notes the
possibility that cohomological restrictions might force a weakening of what
we here call ‘structural constraints,’ but his motivation is different from ours;
he expresses the view that theories with infinite numbers of free parameters are not ‘predictive,’ and explains that his purpose is to find a framework for reducing the infinite number of free parameters in such theories to a finite number. Also, Harada, Kugo, and Yamawaki\textsuperscript{15} have recently studied certain aspects of the renormalization of a conventionally non-renormalizable gauge theory (a gauge-invariant formulation of a non-linear sigma model), using a generalization of the Zinn-Justin algorithm. In contrast with these earlier references, we aim here at showing how to use gauge theories with infinite numbers of free parameters as realistic field theories. Apart from our different motivation, we also give a more explicit discussion of the necessity of the possible structural constraints imposed on the bare action, which are used here to deal with the obstructions that arise, for example, for gauge groups with $U(1)$ factors. Our demonstration that renormalizability follows from cohomology is not limited to any specific choice of structural constraints, but only assumes that these are chosen in correspondence with the infinite terms in the BRST-cohomology of the theory, whatever that might be. Where some other assumptions make this impossible, the theory must be regarded as truly unrenormalizable.

2 Structural Constraints

Our first step is to consider how to constrain the bare action to implement local symmetries. The bare action is taken to be a local functional$^\star\star\star S_0[\Phi, \Phi^*]

$\textsuperscript{\star\star\star}$In a sense the bare action is not local, because it is the integral of an infinite power
of a set of fields $\Phi^n$, including some set of ‘classical’ (matter and gauge) fields $\phi^r$, ghosts $\omega^A$, and perhaps ghosts for ghosts, etc., as well as ‘non-minimal’ fields (antighosts $\bar{\omega}^A$, auxiliary fields $h^A$, and perhaps extraghasts), and of a corresponding set of antifields $\Phi^*_n$, which have statistics opposite to $\Phi^n$. The bare action is assumed to satisfy the quantum master equation \[ (S_0, S_0) - 2i\hbar \tilde{\Delta} S_0 = 0, \] (1) which incorporates all local symmetries as well as the associated commutation relations, Jacobi identities, etc.\(^8\) Here $(F, G)$ is the antibracket \[ (F, G) \equiv \frac{\delta F}{\delta R \Phi^n} \frac{\delta G}{\delta L \Phi^*_n} - \frac{\delta F}{\delta R \Phi^*_n} \frac{\delta G}{\delta L \Phi^n}, \] (2) with $L$ and $R$ denoting differentiation from the left and right, respectively, and $\tilde{\Delta} S_0$ is the differential operator \[ \tilde{\Delta} \equiv \frac{\delta^2 S_0}{\delta L \Phi^n \delta R \Phi^*_n}. \] (3) (This is usually called $\Delta$; the tilde is added to distinguish this from a symbol $\Delta$ introduced later.) We further suppose that various global, linearly realized symmetries are imposed, including Lorentz invariance and ghost number conservation. From now on it should be understood that we also impose the series in the fields and their derivatives, rather than of a polynomial in fields and field derivatives. Bare actions of this sort may be regarded as perturbatively local, in the sense that, to any given order of perturbation theory (whether in small couplings or small energies), only a finite number of terms in the bare action contribute.

\(^1\)In the original version of this work, we made the stronger assumption that both terms in Eq. (1) vanish. Both Lavrov and Tyutin\(^13\) and Anselmi\(^14\) considered theories that satisfy only the quantum master equation (1).
usual conditions on the antibrackets of the action with the non-minimal fields $\bar{\omega}^A$ and $h^A$ and their antifields.

If these were the only constraints imposed on the action then the theory would automatically be renormalizable in the modern sense, because as we shall see in the next section the infinite part of the quantum effective action in any order would satisfy the same constraints as the allowed changes in the counterterms in the bare action. But not all theories are renormalizable in this sense. One very familiar example of a theory that is not renormalizable in the modern sense is one in which we arbitrarily set some parameter (such as the $(\phi^\dagger \phi)^2$ coupling in the electrodynamics of a charged scalar $\phi$) equal to zero or any finite value. We are concerned here rather with what we shall call ‘structural constraints’ — the constraints that tell us what gauge symmetries are respected by the theory.

The structural constraints can be of various types:

(a) The usual structural constraints require the bare action $S_0$ to consist of a term $I[\phi]$ that depends only on the ‘classical’ (gauge and matter) fields and is invariant under some prescribed set of local symmetry transformations, plus appropriate terms depending also on a limited number of antifield field factors, whose number and structure are constrained by the master equation. For instance, for a theory with a closed irreducible gauge algebra like Yang-Mills theory or general relativity the action would be linear in antifields with
one ghost $\omega^A$ and antighost $\bar{\omega}^A$ for each gauge symmetry:

$$S_0[\Phi, \Phi^*] = I[\phi] + \omega^A C^r_A[\phi] \phi^*_r + \frac{1}{2} \omega^A \omega^B C^C_{AB}[\phi] \omega^*_{C} - \bar{\omega}^A h^A, \quad (4)$$

where $I[\phi]$ is invariant under the infinitesimal transformation $\phi^r \rightarrow \phi^r + \epsilon^A C^r_A[\phi]$, and $C^C_{AB}[\phi]$ is the structure constant for these transformations. (We are using a ‘De Witt notation,’ in which indices like $A$ and $r$ include a spacetime coordinate which is integrated in sums over these indices.) For supergravity without auxiliary fields the action would be quadratic in antifields.

b) Instead of imposing a fixed gauge symmetry on a theory, we can instead impose a symmetry with a fixed number of generators and fixed commutation relations, but with the effect of the symmetry transformations on the classical fields left arbitrary. For instance, in the case of an irreducible closed gauge symmetry the action would take the form (4), but with the transformation functions $C^r_A[\phi]$ otherwise arbitrary.†† This case provides an illustration of the fact that when we make a change $\Delta S_0$ in the bare action, the structural

††For instance, instead of the usual isospin matrices $t_i$ representing the algebra of $SU(2)$ we can take the generators of the $SU(2)$ gauge transformations to be linear combinations $O_{ij} t_j$. As long as the matrix $O_{ij}$ is real, orthogonal, and unimodular, this will not change the $SU(2)$ structure constants. In this case, the change in the gauge transformations is the same as would be produced by a redefinition of the gauge fields. The cohomology theorem used in Section 4 shows that in all semisimple Yang-Mills theories and gravitational theories any infinitesimal change in the transformation functions $C^r_A[\phi]$ is the same as would be produced by a redefinition of fields and antifields together with a corresponding change in $I[\phi]$, but this is not the case in general. For instance, changing the ratios of the coupling constants of various particles to a $U(1)$ gauge field would change the $U(1)$ transformation rules in a way that could not be absorbed into a renormalization of the gauge field, while of course leaving the structure constants zero.
constraints apply to $S_0 + \Delta S_0$ rather than to $\Delta S_0$ itself. In particular, $\Delta I[\phi]$ is not necessarily invariant under the original gauge transformation $\phi^r \to \phi^r + \epsilon^A C^r_A[\phi]$, but $I[\phi] + \Delta I[\phi]$ is always required to be invariant under the transformation $\phi^r \to \phi^r + \epsilon^A (C^r_A[\phi] + \Delta C^r_A[\phi])$.

c) We might weaken the structural constraints further, assuming only that the bare action is a polynomial of a given order in the antifields. For instance, if we required that the action is linear in antifields and involves only the fields $\phi^r, \omega^A, \bar{\omega}^A$, and $h^A$ and their antifields, then it would have to take the general form (4), but with unspecified coefficients $C^A_r[\phi]$ and $C^{CA}_r[\phi]$. In this case the master equation would require that the action $I[\phi]$ is invariant under the transformation $\phi^r \to \phi^r + \epsilon^A C^r_A[\phi]$ which form a closed irreducible algebra with structure constants $C^{CA}_r[\phi]$, but we would not be specifying in advance what this gauge symmetry algebra is or how it is represented on the matter fields, except in so far as we specify the transformation of $C^A_r[\phi]$ and $C^{CA}_r[\phi]$ under global linear symmetries.

One convenient aspect of structural constraints of types (a) and (b) is that we can reverse the connection between the master equation and the gauge symmetry: an action of the form (4) will automatically satisfy the quantum master equation as long as (1) $I[\phi]$ is invariant under the transformations $\phi^r \to \phi^r + \epsilon^A C^r_A[\phi]$ with structure constants $C^{CA}_r[\phi]$, and (2) a gauge-invariant regulator is used to define integrals over fields, so that $\tilde{\Delta} S_0 = 0$. The same is true when we consider the deformed action $I[\phi] + \Delta I[\phi]$ and
require invariance under the deformed gauge transformations \( \phi^r \rightarrow \phi^r + \epsilon^A(C^r_A[\phi] + \Delta C^r_A[\phi]) \). This is not true of structural constraints of type (c); merely assuming that the action is of some definite order in antifields does not lead to the master equation. We will not need to assume here that the structural constraints imply the master equation. We will however assume that (as is true of all the constraints discussed above) that the structural constraints are chosen to be linear conditions on possible changes in the action; if \( S_0 + A \) and \( S_0 + B \) both satisfy the structural constraints, then so does \( S_0 + \alpha A + \beta B \) for arbitrary constants \( \alpha \) and \( \beta \). Until Section 4 we will not be otherwise specific about the structural constraints to be adopted.

It is these structural constraints that create a potential problem for renormalizability, for in general they will not be respected by ultraviolet divergent terms in the quantum effective action. The quantum effective action will not even always satisfy restrictions on the number of antifield factors, so that, for example, a bare action with a closed gauge algebra may yield a quantum effective action with an open gauge algebra. Structural constraints arise from our fundamental assumptions about the sort of theory we wish to study, but to be physically sensible they must not constrain a theory so severely that they prevent the cancellation of ultraviolet divergences. Our problem is to decide what structural constraints satisfy this condition. As we shall see in the next section, this is a matter of matching the cohomology of the antibracket operation generated by the bare action. Structural constraints of
type (a) turn out to be adequate to deal with general relativity and semisimple gauge theories. We would need structural constraints of type (b) to deal with the candidate divergences that arise when the gauge group has $U(1)$, but as we shall see these candidate divergences do not correspond to actual infinities. On the other hand, first-quantized string theories require structural constraints weaker than those of type (a). In considering structural constraints other than those of type (a) and (b), it is intriguing that here we confront the possibility that gauge symmetries may be less fundamental than the antibracket formalism from which they can be derived.

3. Renormalization in General Gauge Theories

We begin with an outline of the antibracket approach to the renormalization of theories with local symmetries, presented here in a way that is independent of the specific structural constraints imposed on the theory.

A) In analogy with the renormalization of fields in conventionally renormalizable theories like quantum electrodynamics, in order for infinities to cancel here we need to perform a general canonical transformation $\Phi \to \Phi'(\Phi, \Phi^*)$, $\Phi^* \to \Phi'^*(\Phi, \Phi^*)$ of fields and antifields. By an canonical transformation is meant any transformation that preserves the antibracket structure

$$
(\Phi'^n, \Phi'^m_*) = \delta^m_*, \quad (\Phi'^n, \Phi'^m) = (\Phi'^*_n, \Phi'^*_m) = 0 , \quad (5)
$$

which insures that antibrackets of general functionals can be calculated in
terms of $\Phi^n$ and $\Phi^*_n$, in the same way as in terms of $\Phi^n$ and $\Phi^*_n$. The action $S_0[\Phi, \Phi^*]$ if expressed in terms of the transformed fields becomes a different functional $S_0'[\Phi', \Phi'^*] \equiv S_0[\Phi, \Phi^*]$, given by $S_0'[\Phi', \Phi'^*] = S_0[\Phi', \Phi'^*; 1]$, where $S_0[\Phi, \Phi^*; t]$ is defined by the differential equation

$$\frac{d}{dt}S_0[\Phi, \Phi^*; t] = (F[\Phi, \Phi^*; t], S_0[\Phi, \Phi^*; t])$$

with initial condition

$$S_0[\Phi, \Phi^*; 0] = S_0[\Phi, \Phi^*],$$

where $F[\Phi, \Phi^*; t]$ is an arbitrary fermionic functional of ghost number $-1$. Since the generator $F$ of the canonical transformation contains terms of arbitrary dimensionality, the bare action $S_0'[\Phi', \Phi'^*]$ will not generally have any simple dependence on the transformed antifields $\Phi'^*$. 

B) As a basis for perturbation theory, we must separate out a finite ‘renormalized’ zeroth-order action $S$ from the transformed bare action $S_0'$, with the remainder regarded as a sum of corrections proportional to powers of a ‘loop-counting’ parameter $\bar{h}$, with divergent coefficients. The correction term $\Delta S' = S_0' - S$ receives contributions both from the counterterm $\Delta S \equiv S_0 - S$ in the original bare action, and also from the field-antifield-renormalization canonical transformation in step A. To be specific, suppose we write the original bare action as a power series in $\bar{h}$:

$$S_0 = S + \bar{h}\Delta_1 + \frac{1}{2}\bar{h}^2\Delta_2 + \cdots.$$
The generator \( F(t) \) of the canonical transformation (4) may similarly be written as a power series

\[
F(t) = \hbar t F_1 + \frac{1}{2} \hbar^2 t^2 F_2 + \cdots.
\]

(9)

Eqs. (6) and (7) then give the transformed bare action as

\[
S'_0 = S + \hbar \left[ \Delta_1 + (F_1, S) \right] + \frac{1}{2} \hbar^2 \left[ \Delta_2 + 2(F_1, \Delta_1) + (F_2, S) + (F_1, (F_1, S)) \right] + \cdots.
\]

(10)

The renormalized action \( S \) is taken to have the same form as the original bare action \( S_0 \), satisfying the same structural constraints (including the same limitations on its dependence on antifields), only with finite instead of infinite coefficients. Also, since it can be regarded as the limit of \( S_0 \) for \( \hbar = 0 \), it satisfies the classical master equation

\[
(S, S) = 0,
\]

(11)

with the antibracket calculated in terms of either the original or the canonically transformed fields and antifields.

C) To carry out quantum mechanical calculations of expectation values, Greens functions, etc., it is necessary to fix a gauge by taking the antifields as functions of the fields. This is usually done by taking the antifields in the form

\[
\Phi^*_n = \frac{\delta \Psi(\Phi)}{\delta \Phi^n} + K_n
\]

(12)
where $\Psi$ is a local fermionic functional of $\Phi$, and $K_n$ is an external field, held constant in the path integral. It is important to recognize that the same relation then applies to the transformed antifields

$$
\Phi_n^* = \frac{\delta \Psi'(\Phi', K)}{\delta \Phi'^n} + K_n ,
$$

but with a different (and $K$-dependent) gauge-fixing fermionic functional $\Psi'$. We do not know whether a proof of this result has been published, so a proof is given in an appendix to this paper. An observable $O$ will be unaffected by small changes in $\Psi$, provided it is gauge invariant, in the sense that

$$(O, S) - i\hbar \Delta O = 0.$$  

D) Following the same reasoning as used originally by Zinn-Justin, the quantum effective action $\Gamma(\Phi, K)$ satisfies the master equation

$$(\Gamma, \Gamma) = 0 ,$$

with antibrackets calculated using $K_n$ in place of the antifield of $\Phi'^n$. But the variables $\Phi'^n$ and $\Phi'^n_*$ are related to $\Phi'^n$ and $K_n$ by a canonical transformation, so we can just as well regard $\Gamma$ as a functional of $\Phi'^n$ and $\Phi'^n_*$, satisfying a master equation (14) with the antibracket calculated in terms of these variables.

In lowest order, $\Gamma$ is the same as $S$, and is therefore finite. Suppose that through cancellations of infinities between loop diagrams and the counterterm $S_0 - S$, all infinities in $\Gamma$ cancel up to some given order $N - 1$ in coupling
parameters. Then in order $N$, the infinite part of the master equation constrains the infinite part $\Gamma_{N,\infty}$ of the $N$-th order term in $\Gamma$ by

$$ (S, \Gamma_{N,\infty}) = 0. \tag{15} $$

Because $(S, S) = 0$, the mapping $X \mapsto (S, X)$ is nilpotent, so that the nature of the solutions of Eq. (15) can be determined with the help of appropriate cohomology theorems.

E) We shall now suppose that for some given choice of the structural constraints discussed in Section 2, we can prove a cohomology theorem, that any local functional $X$ which is $S$-closed (in the sense that $(S, X) = 0$), and is invariant under the same linearly realized global symmetries (including ghost number conservation and Lorentz invariance) as $S$, may be expressed as

$$ X = G + (S, H) \tag{16} $$

where $G$ is a local functional for which $S + G$ satisfies the same structural constraints as $S$, and $H$ is a local fermionic functional, with both $G$ and $H$ satisfying the same linearly realized global symmetries as $S$. Eq. (15) tells us that $\Gamma_{N,\infty}$ is $S$-closed, and it automatically is invariant under the same linearly realized global symmetries as $S$, so it satisfies the conditions of this theorem. The cohomology theorem will be applied below not to $\Gamma_{N,\infty}$ itself, but to a term in $\Gamma_{N,\infty}$ that also satisfies these conditions.

Eq. (10) shows that in $N$-th order $S'_0$ will contain terms $(F_N, S)$ and $\Delta_N$, which make additive contributions to $\Gamma_{N,\infty}$, and which do not depend on the
terms in $F$ and $S_0$ that appear in $\Gamma_M$ for $M < N$. We must now inquire whether $\Delta_N$ and $F_N$ can be chosen to cancel the infinities in $\Gamma_N$.

Because the structural constraints are supposed to be satisfied by $S_0$ for all $h$, and are assumed to be linear, they are also satisfied by $S + \Delta_N$. Now, apart from these constraints, and invariance under linearly realized global symmetries, the only limitation on our freedom to chose the $N$-th order counterterm $\Delta_N$ in the original bare action is that it should not invalidate the master equation. For the structural constraints of type (a) and (b) discussed in Section 2, this is not much of a limitation, since the quantum master equation (1) automatically follows from these structural constraints, provided we use a gauge-invariant regulator. But for future use we also wish to consider the more general case, where the master equation must be imposed on $S_0$ independently of the structural constraints. Since $S_0$ is supposed to satisfy the master equation for all values of the loop-counting parameter $h$, the counterterms $\Delta_N$ are required to satisfy a sequence of equations

$$\langle S, \Delta_N \rangle = -\frac{1}{2} \sum_{M=1}^{N-1} (\Delta_M, \Delta_{N-M}) + 2i \hat{\Delta} \Delta_{N-1}. \quad (17)$$

These conditions on $\Delta_N$ are not the same as the condition $\langle S, \Gamma_\infty, N \rangle = 0$ on the infinite part of $\Gamma_N$.

This is no problem. Suppose we find a solution of the equations (17) up to order $N$, which satisfies the structural constraints. We may write the\footnotetext{The reader may be bothered by the question of how we know that these equations can be solved. It is true that if these equations are satisfied up to order $N-1$, then}
$N$-th order term in the general solution as

$$
\Delta_N = \Delta^0_N + \Delta'_N
$$

(18)

where $\Delta^0_N$ is any particular solution satisfying Eq. (17) (and such that $S + \Delta^0_N$ satisfies the structural constraints), and $\Delta'_N$ is subject only to the conditions that $S + \Delta'_N$ must satisfy the structural constraints and any linearly realized global symmetries, and

$$
(S, \Delta'_N) = 0.
$$

(19)

We may write the infinite $N$-th order terms in $\Gamma$ as

$$
\Gamma_{N,\infty} = \Delta'_N,_{\infty} - (S, F_{N,\infty}) + X_{N,\infty}
$$

(20)

where $X_N$ consists of terms from loop graphs, as well as from the term $\Delta^0_N$ and various terms in $\Gamma$ that involve $\Delta_M$ and $F_M$ for $M < N$. For instance, for $N = 2$ Eq. (10) gives

$$
X_2 = \Delta^0_2 + 2(F_1, \Delta_1) + (F_1, (F_1, S)) + \text{two loop terms involving only } S
$$

+ one loop terms involving $S$, $\Delta_1$ and $F_1$.

the right-hand-side $R_N$ of the equation for $\Delta_N$ does satisfy the condition $(S, R_N) = 0$, but we cannot find solutions of the equation $(S, \Delta_N) = R_N$ for arbitrary $R_N$ satisfying $(S, R_N) = 0$ unless the cohomology (known as $H^1(S|d)$, where $d$ denotes the exterior derivative) of the antibracket operation $X \mapsto (S, X)$ on the local functionals $X$ of ghost number $+1$ is trivial, which is not generally the case. (The condition $H^1(S|d) = 0$ would also rule out anomalies, but it is not a necessary condition for the theory to be anomaly free. Even for $H^1(S|d) \neq 0$, anomalies can cancel among different fermion multiplets, as is the case in the standard electroweak theory.) Fortunately, we are not trying to solve the equations $(S, \Delta_N) = R_N$ for arbitrary $R_N$ satisfying $(S, R_N) = 0$, but only for the particular functionals that appear on the right-hand-side of equations (17). The existence of such solutions is guaranteed by the assumption that the structural constraints allow the master equation to be solved for all values of $\bar{h}$. 

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For our purposes the only thing we need to know about $X_N$ is that it does not involve $\Delta'_N$ or $F_N$, and that it is invariant under any linearly realized global symmetries of $S$. It follows from Eqs. (15), (19), and (20) that

$$ (S, X_{N,\infty}) = 0 . $$

(21)

Hence the hypothesized cohomology theorem would allow us to write $X_N$ in the form (16):

$$ X_{N,\infty} = G_N + (S, H_N) , $$

(22)

where $G_N$ is a local functional for which $S + G_N$ satisfies the same structural constraints as $S$, and $H_N$ is a local fermionic functional, with both $G_N$ and $H_N$ invariant under the same linearly realized global symmetries as $S$. Since $\Delta'_N$ and $F_N$ are local functionals that can be varied independently of $X_N$, subject only to the conditions that they are invariant under linearly realized global symmetries, that $S + \Delta'_N$ satisfies the same structure constraints, and that $(S, \Delta'_N) = 0$, they can be chosen so that

$$ \Delta'_{N,\infty} = -G_N , \quad F_{N,\infty} = H_N . $$

(23)

According to Eq. (20), this eliminates the infinities in the quantum effective action to order $N$. Continuing this process allows a step-by-step construction of a counterterm $\Delta S$ and canonical transformation generator $F$ that render the quantum effective action finite to all orders.

4. Cohomology Theorems

The previous section shows how to use cohomology theorems to prove
the renormalizability of various ‘nonrenormalizable’ gauge theories. As an example of such a cohomology theorem, we note that Barnich, Brandt, and Henneaux have recently shown that if \( S \) is the action of a *semisimple* Yang-Mills theory, or of gravitation, or both together, which of course has ghost number zero and is linear in antifields, then the most general local functional \( X \) of ghost number zero that satisfies the condition \( (S, X) = 0 \) may be written as a local gauge-invariant functional \( G[\phi] \) of the ‘classical’ (gauge and matter) fields alone, so that in our language \( S + G[\phi] \) satisfies the structural constraints, plus a term of the form \( (S, H) \). Then by the reasoning of the previous section, we may eliminate all infinities in the quantum effective action by adjusting the counterterms in \( S_0 - S \) to cancel \( G[\phi] \), and performing a suitable canonical transformation on the fields and antifields to cancel \( (S, H) \).

Gauge theories with \( U(1) \) factors require special consideration. Reference 9 shows that in this case the most general local functional \( X \) of ghost number zero that satisfies the condition \( (S, X) = 0 \) may be written as a local gauge-invariant functional \( G[\phi] \) of the ‘classical’ fields alone, plus a term of the form \( (S, H) \), plus a term of the form\(^\dagger\dagger\)

\[
\int A^\mu(x) j^\mu(x) \, d^4x + \text{terms linear in } \phi^*_r ,
\]  

(24)

where \( j^\mu(x) \) is the gauge-invariant current associated with any symmetry of \(^\dagger\dagger\) There are additional complications\(^9\) in theories with certain exotic couplings between matter and gauge fields. We will not go into this here, because such theories do not seem to be of physical interest.
the action, and $A^\mu(x)$ is the $U(1)$ gauge field (supposing for simplicity that there is only one.) If $j_\mu(x)$ is the same current to which $A^\mu(x)$ is coupled in the bare action, then a term like (24) can be compensated by a renormalization of the field $A^\mu(x)$ and a corresponding renormalization of the antifield $A^*_\mu(x)$, which is one example of the canonical transformations discussed in Step A of the previous section.

On the other hand, if the action respects a global symmetry in addition to the $U(1)$ gauge symmetry, then $j^\mu(x)$ can be the current associated with that global symmetry, and in this case the cohomology includes terms whose antifield-independent part is only gauge-invariant ‘on-shell,’ that is, when the field equations are satisfied. Thus if infinite terms of the form (24) actually appeared in the quantum effective action, with $j^\mu(x)$ a conserved current other than that to which $A_\mu(x)$ was originally coupled, then the structural constraint we used for semisimple gauge theories, that the bare action has the form (4) with $I[\phi]$ off-shell invariant under a prescribed transformation $\delta \phi^r \rightarrow \phi^r + \epsilon^A C^r_A[\phi]$, would not lead to a renormalizable theory. In this case we would have to use the weaker structural constraint of type (b) discussed in Section 2, that the action is of the form (4), with the transformation functions $C^A_r[\phi]$ specified only as to their number and structure constants (in this case zero). The counterterms in the bare action would then only be constrained by the condition that they are linear in antifields, do not invalidate the master equation, and do not change the structure constants,
which in this case are zero. Thus such counterterms could be used to cancel infinite terms in the quantum effective action of the form (24).

It does not seem that infinities of the form (24), with \( j^\mu(x) \) a conserved current other than that to which \( A_\mu(x) \) was originally coupled, actually appear in the quantum effective action. We have not checked this by direct calculation, but such infinite terms would represent a change in the mixture of fermion currents to which long-wave photons couple, and this is prohibited by the Ward soft-photon theorem. It is not necessary for us to settle this question, because we have shown that any infinities of form (24) are cancelled by renormalization of the parameters in the \( U(1) \) gauge transformation, but this seems to be a case where the candidate divergences presented by cohomology theorems are not actually divergent.

An even clearer case of this sort is presented by theories containing a set of free \( U(1) \) gauge fields \( A^b_\mu(x) \). The cohomology of the antibracket operator also includes the terms

\[
f_{abc} \int d^x \left( F^{\nu \mu \alpha} A^{b \mu} A^{\alpha \nu} + 2 A^a_\mu A^b_\mu \omega^c + \omega^a_\mu \omega^b_\omega^c \right).
\] (25)

\[\text{As already noted in Section 2, the antifield-independent term } I[\phi] + \Delta I[\phi] \text{ is not required by these structural constraints and the master equation to be invariant under the original gauge transformations } \phi^r \rightarrow \phi^r + e^A C^r_A, \text{ but only under the modified gauge transformations } \phi^r \rightarrow \phi^r + e^A \left( C^r_A + \Delta C^r_A \right), \text{ so that}
\]

\[
\left( \delta \Delta I[\phi] / \delta \phi^r \right) C^r_A = - \left( \delta \left( I[\phi] + \Delta I[\phi] \right) / \delta \phi^r \right) \Delta C^r_A
\]

which only requires that \( \Delta I[\phi] \) should be invariant under the original gauge transformation \( \phi^r \rightarrow \phi^r + e^A C^r_A \) when the field equations are satisfied.

\[\text{We are grateful to F. Brandt for suggesting this to us.}\]
where $f_{abc}$ are totally antisymmetric constants. If these corresponded to actual divergences we would have to weaken the structural constraints so that not even the structure constants were prescribed in advance, leaving open the possibility that the fields $A^a_{\mu}(x)$ transform under a non-Abelian gauge group. But here it is quite clear that the terms in Eq. (25) are not produced by radiative corrections; no radiative corrections can give interactions to a field that does not interact to begin with.

A recent cohomology theorem of Brandt, Troost, and Van Proeyen\textsuperscript{10} shows that it is also necessary to weaken the structural constraints in dealing with first-quantized string theories — that is, with gravitation coupled to scalar matter in two dimensions. If the Liouville field is explicitly introduced the analysis of ref. 17 shows that the cohomology of $S$ contains terms corresponding to a change in the action of its local symmetries, though not of their algebra, so here one should impose a structural constraint of type (b). Analogous comments apply to the spinning string.\textsuperscript{18}

The possibility of weakening the structural constraints may become useful in applications to other theories. It is important to find out whether supergravity and general superstring theories are renormalizable in the modern sense, and for this purpose we need to know the cohomology generated by the bare action of these theories.

**Acknowledgments** We are grateful for helpful conversations with C. Becchi, F. Brandt, D. Buchholz, M. Henneaux, and J. Pons.
Appendix

We wish to prove that if

$$
\Phi^* = \frac{\delta \Psi(\Phi)}{\delta \Phi} n + K_n \, ,
$$

(26)

then canonically transformed variables $\Phi^n$ and $\Phi'^* n$ satisfy a relation of the same form

$$
\Phi'^* = \frac{\delta \Psi'(\Phi', K')}{\delta \Phi'^n} n + K_n \, ,
$$

(27)

though generally with a different (and $K$-dependent) fermionic functional $\Psi' \neq \Psi$. It is only necessary to show that this is true for infinitesimal canonical transformations, which are of the form

$$
\Phi'^n = \Phi^n + (F, \Phi^n) = \Phi^n - (\delta F/\delta \Phi)_{\Phi^* = \delta \Psi / \delta \Phi} + K \, ,
$$

(28)

$$
\Phi'^* = \Phi^* + (F, \Phi^* n) = \Phi^* + (\delta F/\delta \Phi)_{\Phi^* = \delta \Psi / \delta \Phi + K} \, ,
$$

(29)

where $F[\Phi, \Phi^*]$ is an infinitesimal fermionic functional. Continuity then implies that the same will be true for finite canonical transformations, in at least a finite region around the unit transformation.

To prove Eq. (26), we note that Eqs. (25) and (28) yield

$$
\Phi'^* = \delta \Psi / \delta \Phi^n + (\delta F/\delta \Phi)_{\Phi^* = \delta \Psi / \delta \Phi + K} + K_n \, .
$$

(30)

The derivative of $\Psi$ with respect to $\Phi$ may be expressed in terms of its derivative with respect to $\Phi'$, using Eq. (27) to write

$$
\frac{\delta L \Phi'^n}{\delta \Phi'^m} = \delta^n_m - \frac{\delta L}{\delta \Phi'^m} \left( \frac{\delta F}{\delta \Phi'^n} \right)_{\Phi^* = \delta \Psi / \delta \Phi + K} \, .
$$

(31)
Using this in Eq. (29) and keeping only terms of first order in $F$ gives

$$
\Phi_{nm}' = \delta \Psi_{nm} \frac{\delta F}{\delta \Phi^m} \left( \frac{\delta F}{\delta \Phi^m} \right)_{\Phi^* = \delta \Psi / \delta \Phi + K} + \delta \Phi^* = \delta \Psi / \delta \Phi + K
$$

To first order in $F$ this has the same form as the desired result (26), with

$$
\Psi' = \Psi - \delta \Psi \left( \frac{\delta F}{\delta \Phi^m} \right)_{\Phi^* = \delta \Psi / \delta \Phi + K} + (F)_{\Phi^* = \delta \Psi / \delta \Phi + K}.
$$

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