The Problem of Classical Limit in Quantum Cosmology: the Effective Action Language

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Abstract

The tool of functional averaging over some “large” diffeomorphisms is used to describe quantum systems with constraints, in particular quantum cosmology, in the language of quantum Effective Action. Simple toy models demonstrate a supposedly general phenomenon: the presence of a constraint results in “quantum repel” from the classical mass shell.

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1 Introduction

At this memorial Quantum Gravity Conference dedicated to Moisei Alexandrovich Markov, I think we must also remember some others who unfortunately passed away: Andrei Sakharov, Yuri Golfand, Yakov Zeldovich, and Igor Tamm whose centenary is being celebrated these days. Professor Stelle has already presented here some reminiscences on the political troubles concerning Sakharov and Golfand. I would like just to remind the audience that after highest political tops of the Former USSR had reacted to the strongest international pressure and restored Yuri Golfand in the Lebedev Institute, he, from 1980 and almost 10 years to the moment of his departure to Israel, worked in the laboratory headed by Moisei Alexandrovich Markov.

Classical limit in quantum cosmology becomes a problem because of the Wheeler–De Witt (WDW) and shift constraints:

\[ \hat{H}\ket{\Psi} = 0, \]
\[ \hat{H}_i\ket{\Psi} = 0, \]

(1.1)
(1.2)

An enormous number of papers on the topic come to a more or less unique conclusion: the construction of time and of the classical space-time is not possible from the pure wave function of the Universe; the system must be open, some external environment is necessary to provide decoherence, i.e. a wave function reduction from the time-independent WDW state \( \ket{\Psi} \) to the states localized in time. For the quantum-field theorist this just means that gauge freedom responsible for constraints (1.1), (1.2) is broken extrinsically. Another way to eliminate the problem is to use unitary gauge (see e.g. review [1]), i.e. to nominate one of the quantum coordinates (e.g. scale factor of the Universe, or some scalar field) to be a nonquantized classical clock. One can also use Mensky channels [2], i.e. to make a substitution (symbolically):

\[ \delta(H) = \int e^{iNH} dN \Rightarrow \int e^{-\gamma N^2} e^{iNH} dN. \]

(1.3)

An artificial weight factor for the Lagrange multiplier \( N \) in (1.3), nonquantized clocks, ‘environment’ — in all these approaches the question about the classical limit in quantum cosmology is in a sense erased ‘by hand’. Here we consider the gauge freedom of general coordinate transformations (and also the global ‘residual gauge freedom’ — see Sec. 2) seriously; it will be shown that in the Effective Action (EA) language, the presence of constraints results in the ‘quantum repel’ from the mass shell, i.e. ”repel” from the solutions of the classical dynamical equations

\[ \frac{\delta S}{\delta \phi^A} = 0, \]

(1.4)
where $S$ is the classical Action. Only one-loop corrections to EA will be considered:

$$\Gamma(\bar{\phi}) = S(\bar{\phi}) + \Gamma^{(1)}(\bar{\phi}, \frac{\delta S}{\delta \bar{\phi}^A}).$$  \hspace{1cm} (1.5)

$\bar{\phi}$ are the background fields. Quantum repel from the mass shell means that because of the constraints the one-loop term $\Gamma^{(1)}$ possesses a specific nonlocal dependence on $\frac{\delta S}{\delta \bar{\phi}^A}$ and is nonanalytic or divergent on the solutions of Eq. (1.4). ‘Bad’ behaviour of the one-loop term near $\bar{\phi}$ means the divergence of quantum dispersion and signals that this background is not a classical field. New mass shell (and in particular a regularized vacuum $\bar{\phi}_0$) is defined as a solution of the equation

$$\frac{\delta \Gamma}{\delta \bar{\phi}^A} = 0$$  \hspace{1cm} (1.6)

in full analogy with the standard procedure, e.g. in the Coleman–Weinberg model [3]. The ‘external current’ $\frac{\delta S}{\delta \bar{\phi}_0}$ may be found self-consistently from (1.5) and (1.6). (In this way dynamically generated mass of the Goldstone boson may be found in the Glashow-Higgs model [3]).

2 Constraints and functional integration

It is well known that restriction of the Hilbert space to some subspace results in nontrivial kinematics and dynamics of the physical observables and also results in problems with the classical limit [5]–[7]. E.g., if we postulate that only even states $\psi(x) = \psi(-x)$ of the most popular one-dimensional oscillator are permitted, then not $x$ but $|x|$ becomes a relevant physical observable and contrary to the conventional oscillator there is no classical limit near $x = 0$ [3]. The same is true for the dynamics of the radial coordinate of a free particle constrained to the $S$-state of its Hilbert space [8]. Van der Monde determinant and the phenomenon of quantum ‘level repel’ repeat the story for the unitary matrices averaged over angles [9].

Constraints – functional integration over Lagrange multipliers – gauge theory and gauge fixing – Faddeev–Popov ghosts – Generating Functional and the Effective Action: this well known logical chain puts the following questions:

1. How to construct a theory whose physical predictions do not depend on the gauge fixing?

2. What is left of a constraint after gauge-fixing?

As an answer to the first question we use here the Vilkovisky–DeWitt unique effective Action where relativistic Landau–DeWitt, the so called ‘natural’, gauge is used (see Sec. 3).
Let us discuss the second question? If the gauge freedom is over-fixed (as in RHS of (1.3)) then physics of the constraint is lost. If we conventionally postulate that functional integration is done over fields (including gauge diffeomorphisms) which are zero at infinity (at the boundary) then again the constraint is lost in the Green functions Generating functional and in EA; it must be imposed in the asymptotic Hilbert space as a supplementary condition. Instead of it one may generalize the definition of the functional integral so as to include into it the integration over some ‘global’ gauge transformations, nonzero at the infinity. We shall go this way.

This approach is not a new one. In quantum cosmology Teitelboim [10] and Halliwell [11] showed that WDW constraint (1.1) follows from the integration over a residual gauge mode $N = \text{const}$ (which corresponds to time reparametrization $t \rightarrow t + \text{const} \cdot t$) i.e. over Faddeev–Popov zero (‘residual gauge’) in the $\delta(N)$ gauge fixing. The supplementary integration over global gauge transformations is equivalent to the requirement of wave function invariance under ‘large’ diffeomorphisms and physically means that some charges are zero – and these are constraints. The residual gauge contribution to the functional integral is responsible for the specific IR peculiarities of EA at the mass shell (1.4); their regularization by external currents follows from the general formula (3.22) below (Cf. [13]).

## 3 Vilkovisky–De Witt Effective Action (VDWEA)

Conventional EA is defined through generating functional $W(j)$ [14]:

$$\Gamma(\bar{\phi}) = W(j) - \bar{\phi}^A j_A ,$$  \hspace{1cm} (3.1)

where $\bar{\phi}^A = \frac{\delta W}{\delta j^A}$, $A$ is a condensed De Witt index (continuous and discrete), and

$$e^{iW(j)} = \int D\varphi e^{iS(\varphi) + i\bar{\phi}^A j_A} .$$ \hspace{1cm} (3.2)

However this EA suffers from two ambiguities: (1) it depends on the local fields redefinition [15]; (2) in gauge theories EA depends (away from the mass shell) on the gauge fixing term.

The VDWEA approach [16, 17] (it is also applied to gravity in [18] and to scalar QED in [19]) heals the first disease by considering functional space $\varphi^A$ as a Riemannian space with an interval:

$$d\ell^2 = d\varphi^A G_{AB}(\varphi) d\varphi^B .$$ \hspace{1cm} (3.3)

In particular, for the gravitational field ($\varphi^A = g_{\mu\nu}$)

$$G_{AB} = \sqrt{-g} G^{\alpha\beta\gamma\delta} \delta(x - x') = \frac{1}{2} \sqrt{-g} (g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma} - cg^{\alpha\beta} g^{\gamma\delta}) \delta(x - x')$$ \hspace{1cm} (3.4)
(\(c\) is an arbitrary constant).

Now instead of the ordinary functional derivatives, functionally covariant derivatives are used, e.g.

\[
S_{AB} \rightarrow S_{\gamma AB} = S_{AB} - \left\{ \frac{C}{\gamma AB} \right\} S_{\gamma C}
\]

(3.5)

where Cristoffel symbols are built from the metric (3.3). Hence loops do not depend on the choice of coordinate in the functional space, i.e. on the fields reparametrization.

In gauge theories, the generators \(K^A_\gamma(\varphi)\) of the gauge transformations \(\xi^\gamma\)

\[
\delta \varphi = K^A_\gamma(\varphi)\xi^\gamma \equiv \eta^A_{(\xi)}
\]

(3.6)

(parametrized by the fields \(\xi^\gamma\)) must be Killing vectors of the metric (3.3). In particular in gravity we have

\[
\delta \varphi^{(g)} = \delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu} = (-g_{\mu\gamma}\nabla_\nu - g_{\nu\gamma}\nabla_\mu)\xi^\gamma = K^{(\mu\nu)}_\gamma(\xi) \equiv \eta^{(\mu\nu)}_{(\xi)}.
\]

(3.7)

At every ‘world’ point \(\bar{\varphi}^A\) of the functional space we can take differentials \(\eta^A\) of the field:

\[
\varphi^A = \bar{\varphi}^A + \eta^A
\]

(3.8)

in the direction normal to the gauge orbit:

\[
K^A_\gamma G_{AB}\eta^B \equiv (K_\gamma \eta) = 0.
\]

(3.9)

This is the Landau–DeWitt, or the so called “natural”, gauge condition: e.g. \(A^\mu,_{\mu} = 0\) in QED, and

\[
\chi^\gamma(\eta) \equiv (h_{\gamma\mu}^{\mu} - \frac{c}{2}\delta^{\mu}_{\gamma}h^\nu_{\mu})_{\mu} = 0
\]

(3.10)

for free gravity \((g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu})\) with the functional metric (3.4). The substitution \(\eta^A = \eta^A_{(\xi)}\) from (3.6) to (3.9) gives a Faddeev–Popov differential operator

\[
Q_{\gamma\beta} = K^A_\gamma G_{AB}K^B_{\beta}
\]

(3.11)

and equation for the “residual” gauge transformations \(\xi^\beta_0\):

\[
Q_{\gamma\beta}\xi^\beta_0 = 0;
\]

(3.12)

in free electrodynamics (3.12) is \(\Box \xi_0 = 0\) (\(\Box\) is a D’Alambertian), and in free gravity it is (from (3.7), (3.10)):

\[
(g_{\mu\nu} \Box + \nabla_\nu \nabla_\mu - c\nabla_\mu \nabla_\nu)\xi^\mu_0 = 0.
\]

(3.13)

In the most simplified formulation, the VDWEA remedy against the second ambiguity of EA mentioned at the beginning of this Section is: “Use ‘natural’ gauge-fixing (3.9)”. Actually
this is true only for one-loop terms; special Vilkovisky connection [17] must be introduced to calculate unique VDWEA in higher orders in \( \hbar \). But in this paper we are interested only in the one-loop term \( \Gamma^{(1)} \). Thus

\[
e^{i\Gamma^{(1)}} = \lim_{\alpha \to \infty} \int e^{iS^{(2)}} \det \hat{Q} M \left[ D\eta^A \right]. \tag{3.14}
\]

Here \( S^{(2)} \) is a sum of a second functionally covariant variation of the classical Action \( S \) and the gauge-fixing term:

\[
S^{(2)}(\eta) \equiv \frac{1}{2} \eta^A \tilde{F}_{AB} \eta^B = \frac{1}{2} \eta^A S_{;AB} \eta^B + \frac{\alpha}{2} (K_\gamma \eta) f^{\gamma\delta} (K_\delta \eta); \tag{3.15}
\]

\( f^{\gamma\delta} \) is some arbitrary matrix in the space of gauge indices; the measure reads

\[
M = \left[ \det(\alpha f^{\gamma\delta}) \right]^{1/2}; \tag{3.16}
\]

the other symbols are defined in (3.3), (3.8), and (3.9). Finally we have VDWEA to the one-loop term:

\[
\Gamma(\bar{\phi}) = S + \Gamma^{(1)} = S + \frac{1}{2} \lim_{\alpha \to \infty} \left\{ \log \det(DABG_{BC}) + \log M \right\} + \log \det \hat{Q}, \tag{3.17}
\]

where

\[
D_{AB} = (\tilde{F}_{AB})^{-1} \tag{3.18}
\]

and the differential operator \( \tilde{F}_{AB} \) is defined in (3.15); \( G_{AB}, \tilde{Q}, \) and \( M \) are defined in (3.3), (3.11) and (3.16).

The identify

\[
S_{;B} K^B_{\gamma} \xi^\gamma = 0 \tag{3.19}
\]

represents gauge-invariance of the Action \( S \). Its covariant functional differentiation gives

\[
S_{;AB} K^B_{\gamma} \xi^\gamma = -S_{,B} K^B_{\gamma;A} \xi^\gamma. \tag{3.20}
\]

Thus for the ‘residual gauge’ field variations

\[
\eta_0^A = K^A_{\gamma} \xi_0^\gamma \tag{3.21}
\]

(where \( \xi_0^\gamma \) are solutions to (3.12)) we have from (3.15), with account of (3.9) and (3.20):

\[
\tilde{F}_{AB} \eta_0^B = -S_{,B} K^B_{\gamma;A} \xi_0^\gamma, \tag{3.22}
\]

hence \( \eta_0^A \) are zeros of the differential operator \( \tilde{F} \) on the mass shell of the background field, i.e. when \( S_{,B} \equiv \delta S/\delta \bar{\phi}^B = 0 \). Zero modes (3.21) are conventionally excluded from the definition of \( D = \tilde{F}^{-1} \) in (3.17) (i.e. from the functional integration in (3.14)) because they
are nonvanishing all over the space-time including the infinity. Our hypothesis: they must be taken into account if we want to preserve constraints’ physics in the functional integral language.

**Note.** In General Relativity (GR), contrary to the vector gauge theories, the RHS of the identity (3.19) is not zero but is a surface integral: under general coordinate transformation $x^\mu \to x^\nu + \xi^\mu(x)$, we have

$$
\delta S = \delta \left( \int \mathcal{L} \sqrt{-g} \, dx \right) = \int (\mathcal{L} \sqrt{-g} \xi^\mu)_{,\mu} \, dx .
$$

(3.23)

Thus $S^{(2)}(\eta_0) = \frac{1}{2} \eta_0^A \tilde{F}_{AB} \eta_0^B$ (with $\eta_0$ taken from (3.21)) is zero on the mass shell only for those solutions to Eq. (1.4) which render gravitational Lagrangian $\mathcal{L} = 0$ at the boundary. Nevertheless (3.22) is valid in GR as well as in other gauge theories.

### 4 Simple quantum-mechanical example

Before turning to cosmology, let us consider the simplest illustrative theory: scalar QED reduced to quantum mechanics. Its Action and the functional space interval (3.3):

$$
S = \int \left[ \frac{1}{2} \dot{\rho}^2 + \frac{1}{2} \rho^2 (\dot{x} - ey)^2 - V(\rho) \right] \, dt
$$

(4.1)

$$
d\ell^2 = \left[ d\rho^2 + \rho^2 dx^2 - dy^2 \right] dt
$$

(4.2)

are the reduced version of the Glashow–Higgs model (the scalar field $\varphi = \rho e^{ix}$; $x$ is the Goldstone field; $A_\mu = \{A_0, 0, 0, 0\}$; $A_0 \equiv y(t)$; $e$ is the electric charge, and the dot symbolizes $d/dt$; in (4.1) and (4.2) all quantities are normalized to be dimensionless). $V(\rho)$ is a standard Higgs potential

$$
V = \frac{m^2}{8\rho_0^2} (\rho^2 - \rho_0^2)^2 .
$$

(4.3)

($m$ is the mass of the $\rho$-field at the extremum of $V$). We shall quantize $x(t), y(t)$ and not $\rho(t)$, and consider VDWEA correction to the potential (4.3). To calculate the effective potential

$$
V_{\text{eff}} = -\frac{1}{\theta} \Gamma = V + V^{(1)}
$$

(4.4)

(with $\theta$ being the time interval) we conventionally take a background

$$
\bar{\rho}(t) = \rho = \text{const}, \quad \bar{x}(t) = \bar{y}(t) = 0 .
$$

(4.5)

Now $x, y$ may be considered as small variations

$$
\eta^A = \{x; y\} ,
$$

(4.6)
their gauge transformations

\[ \delta x = K^{(x)} \xi = e \xi ; \quad \delta y = K^{(y)} \xi = \dot{\xi} \tag{4.7} \]

leave (4.1) invariant. The natural gauge (3.9) and the Faddeev–Popov equation (3.12) take the form (we used the metric (4.2):

\[ \frac{\dot{d}}{dt} G_{yy} \cdot y + e G_{xx} \cdot x = - \frac{\dot{d}}{dt} G_{yy} y + e G_{xx} \cdot x = \dot{y} + e \rho^2 x = 0 , \tag{4.8} \]

\[ Q \xi_0 = \xi_0 + e^2 \rho^2 \xi_0 = 0 . \tag{4.9} \]

Because of the dependence of the \( G_{xx} \) component of the metric (4.2) on \( \rho \), \( S_{\alpha \beta} \) in (3.15) includes the mass term of the “Goldstone boson” \( x(t) \) (see (3.5)):

\[ \left\{ \begin{array}{l} \rho \\ \beta \beta \end{array} \right\} S_{\rho} = - \rho \frac{dV}{d\rho} \equiv - \rho^2 \mu^2 , \tag{4.10} \]

which, as is well known, is zero on the mass shell, i.e. at the extremum of the potential (4.3). From (4.1), (4.8), (4.10) we have for this model the Action (3.15):

\[ S^{(2)}(\eta) = \int \left\{ \frac{1}{2} \rho^2 (\dot{x} - e y)^2 - \frac{1}{2} \rho^2 \mu^2 x^2 + \frac{\alpha}{2} (y + e \rho^2 x)^2 \right\} dt \equiv \frac{1}{2} \eta^A \bar{F}_{AB} \eta^B . \tag{4.11} \]

Performing the Fourier transform we arrive at the following expression for the determinant of the differential operator \( \bar{F} \) in (4.11) in the \( \{ x, y \} \) linear space:

\[ \det F(p) = \alpha Q^2 - \mu^2 (\alpha p^2 + e^2 \rho^2) \equiv \alpha (p^2 - m_1^2)(p^2 - m_2^2) \tag{4.12} \]

where

\[ Q(p) = p^2 - e^2 \rho^2 \tag{4.13} \]

is the Fourier transform of the Faddeev–Popov operator (4.9):

\[ m_{1,2}^2 = e^2 \rho^2 + \frac{1}{2} \mu^2 \pm \frac{1}{2} \sqrt{\mu^4 + 4e^2 \mu^2 \rho^2 (1 + \alpha^{-1})} . \tag{4.14} \]

Substitution of (4.12), (4.13), and \( \prod \alpha^{1/2} \) (3.16) into (3.17) gives the one-loop term of the EA. To calculate it we first take its derivative (see dependence on \( \mu^2 \) in (4.12)):

\[ \frac{d\Gamma^{(1)}}{d\mu^2} = \frac{\theta i}{4 \pi} \int \frac{p^2 + e^2 \rho^2 \alpha^{-1}}{(p^2 - m_1^2)(p^2 - m_2^2) - i \varepsilon} dp = - \frac{\theta}{4 (m_1 - m_2)} \left( 1 + \frac{e^2 \rho^2 \alpha^{-1}}{m_1 m_2} \right) . \tag{4.15} \]
Zeroes of the denominator in the integrand in (4.15) are located in the upper half-plane at \( p = m_1 \) and \( p = -m_2 \). Because of this fact (4.15) is proportional to

\[
(Dc_{m_1} - Dc_{m_2}) \bigg|_{t=0}
\]

(4.16)

\((Dc, Dc)\) are causal and anticausal Green functions), that is why (4.15) is divergent when \( m_1 \to m_2 \), i.e. at the extremum of \( V(\rho) \) where \( \mu = 0 \). From (4.15), (4.4) we finally have (for \( \alpha = \infty \)):

\[
V_{\text{eff}}(\rho) = V(\rho) - \int \frac{d\mu^2}{4(m_1 - m_2)}.
\]

(4.17)

For \( \mu \ll \rho \) this gives, with the account of (4.17), (4.14):

\[
V_{\text{eff}}(\rho) \approx V(\rho) - \frac{1}{2} \sqrt{\frac{1}{\rho}} \frac{dV}{d\rho}.
\]

(4.18)

(The ‘charge’ \( e \) does not enter the answer in this approximation. In [4] we considered Glashow–Higgs model in 4 dimensions. There, the Goldstone boson mass \( \mu \) at the extremum of \( V_{\text{eff}}(\rho) \) depends on \( e, \rho_0, m \). Formula (4.18) is the main result of this Section and also illustrates the main idea of the article: quantum repel from the mass shell. Indeed the minimum \( V_{\text{eff}} \) is shifted away from the minimum of the potential (4.3).

Let us contemplate a bit on this result. Substitution \( \rho = \text{const} \) in (4.1) makes this model the well known [20] simplest gauge theory generated by the constraint

\[
\hat{x}\psi = -i \frac{\partial}{\partial x}|\psi\rangle = 0.
\]

(4.19)

Eq. (4.19) means that the scalar field \( \varphi = \rho e^{ix} \) is in the S-state with regard to the phase \( x \). There is a standard “philosophical” question addressed to all theories with a spontaneous breakdown of symmetry: “Who selects a definite phase of the order parameter in the ‘broken’ vacuum state?” Bogolyubov’s quasi-averages are the most popular answer, but they suppose the temporary (before \( V^{(3)} \to \infty \)) introduction of the external field; this violates the constraint (4.19) and prepares a quantum state with a definite phase \( x_0 \). Here we postulate that the system is permanently in the S-state (4.19) and we come to the conclusion that the minimum \( \rho = \rho_0 \) of the potential (4.3) cannot be realized in this case. The use of a noncausal contour in the longitudinal sector (in the complex \( p \)-plain in (4.15)) is crucial; supposedly it is a counterpart of the constraint (4.19) in the EA approach to the theory (4.1).

Is this physically adequate? To our mind this question is perhaps not too adequate itself. Actually with the choice of the gauge-fixing and of the way of averaging over ‘large’ diffeomorphisms we do change the effective quantum dynamics and in particular one of its the most important physical predictions – the effective ‘classical’ dynamics, i.e. renormalized mass
shell, determined by the equation \((1.6)\). Thus postulating one or another way of averaging over global gauge modes in the functional integral \((3.14)\) we define the theory.

Our observation is that the presence of a constraint demands some sort of global averaging.

5 VDWEA language for the cosmological minisuperspace model

Earlier we investigated the VDWEA approach to the relativistic particle \([4]\). Now we consider the standard flat cosmological model in \((n + 1)\) dimensions with space-time metric

\[
ds^2 = e^{2\nu} dt^2 - e^{2\beta} d\Omega^{(n)}.
\]

For this model, the reduced Einstein Action with a \(\Lambda\)-term and the reduced functional space metric \((3.4)\) are of the form \((L = V^{(n)}/2\kappa, V^{(n)}\) is the space volume, and \(\kappa\) is the gravitational constant in \(n + 1\) dimensions; in fact, \(L\) is an arbitrary length parameter of the reduced ‘flat’ minisuperspace model \((5.2)\):

\[
S = \int \sqrt{-g} \mathcal{L}^{(0)} dt = L \cdot \int e^{\nu+n\beta} \left[-n(n-1)e^{-2\nu}\dot{\beta}^2 - 2\Lambda\right] dt,
\]

\[
dt^2 = L^{-1} \int e^{\nu+n\beta}[(2-c)d\nu^2 - 2cnd\nu d\beta - n(cn-2)d\beta^2] dt.
\]

The classical solution is the De-Sitter universe; in the proper time \((\nu = 0)\):

\[
\beta_{cl} = \lambda_0 t, \quad \lambda_0^2 = \frac{2\Lambda}{n(n-1)}.
\]

We calculate one-loop VDWEA term \((3.14)\) on the background:

\[
\bar{\nu} = 0, \quad \bar{\beta} = \lambda t.
\]

\(\lambda - \lambda_0 \neq 0\) is a measure of departure from the mass shell. On the background \((5.5)\) we have the sources:

\[
S_\nu = Le^{n\lambda}n(n-1)(\lambda^2 - \lambda_0^2), \quad S_\beta = nS_\nu.
\]

A reparametrization of time \(t \rightarrow t + \xi(t)\) generates gauge variations which on the background \((5.3)\) read

\[
\delta \nu \equiv K^\nu \xi = \dot{\nu} \xi + \dot{\xi}, \quad \delta \beta \equiv K^\beta \xi = \dot{\beta} \xi = \lambda \xi.
\]

Knowledge of the functional Killing vectors \((5.7)\) and of the metric \((5.3)\) allows us to write down the Landau–De Witt gauge condition \((5.3)\)

\[
\chi \equiv (2-c)\eta^\nu - cn\eta^\beta + 2n\lambda(\eta^\nu - \eta^\beta) = 0,
\]
and Eq. (3.12) for Faddeev–Popov zero modes:

$$\hat{Q}\xi_0 = (2 - c) \ddot{\xi}_0 + \lambda n(2 - c)\dot{\xi}_0 - 2\lambda^2 n\xi_0 = 0.$$  

(5.9)

Here,

$$\eta^A = \{\eta^\nu; \eta^\beta\} , \quad \eta^\nu \equiv d\nu , \quad \eta^\beta \equiv d\beta .$$  

(5.10)

Of course one may get formulae (5.8), (5.9) for the space-time (5.1) directly from the general formulae (3.4), (3.10) and (3.13).

Omitting elementary calculations we present the final expression for this model for $S^{(2)}$ from (3.15):

$$S^{(2)} = L \int e^{n\lambda t} \left[ -n(n-1)(\bar{\eta}^\beta - \lambda \eta^\nu)^2 - \frac{n}{n-1} E(\eta^\nu - \eta^\beta)^2 + \frac{\alpha}{2} \chi^2 \right] dt$$

$$- L \int \frac{d}{dt} \left[ n^2(n-1)\lambda e^{n\lambda t}(\eta^\beta)^2 \right] dt ,$$

(5.11)

where the gauge-fixing function $\chi$ is defined in (5.8), and

$$E \equiv \frac{n(n-1)^2}{cn + c - 2} (\lambda^2 - \lambda_0^2) .$$

(5.12)

Note. It is easily seen that the Hessian $(\partial^2 L / \partial \dot{q}^i \partial \dot{q}^k)$ of the Action $S^{(2)}$ (5.11) is proportional to the local functional metric (5.3) if:

$$c = 1 , \quad \alpha = 1 ,$$

(5.13)

($\alpha = 1$ is the so called Feynman gauge fixing). This Canonical quantization self-consistent condition is valid on the proper time background ($\bar{\nu} = 0$ in (5.1)); and it is rather general [17]. Following Vilkovisky, we put $c = 1$ in the functional gravitational metric (3.4). But we leave $\alpha$ arbitrary and in the end we put $\alpha \to \infty$, which gives the Landau–De Witt gauge fixing (5.8).

To pursue further calculations, the following substitution to (5.3), (5.8), (5.11) is useful:

$$\eta^\beta = e^{-\frac{n\lambda t}{2}} x , \quad \eta^\nu = e^{-\frac{n\lambda t}{2}} (y + nx) .$$

(5.14)

This diagonalizes functional metric (5.3):

$$d\ell^2 = L^{-1} \int [-2n(n-1)x^2 + y^2] dt ,$$

(5.15)

makes the Faddeev–Popov differential operator (5.3) self-conjugate,

$$\tilde{Q} = \frac{d^2}{dt^2} - \frac{1}{4} n(n+8)\lambda^2 ,$$

(5.16)
and results in the following expression for $S^{(2)}$ (5.11) (we discarded surface terms, and put $c = 1$ everywhere):

$$
S^{(2)} = L \int \left[ -n(n-1)\dot{x}^2 + \frac{\alpha}{2}y^2 + 2n(n-1)\lambda (\dot{xy} + \alpha \dot{x}y) + \frac{1}{2}x^i m_{ik} x^k \right] dt
\equiv \int \frac{1}{2} x^i \bar{F}_{ik} x^k dt,
$$

(5.17)

$x^i = \{x^1; x^2\} \equiv \{x; y\}$. We shall not write down expressions for the elements of the ‘mass’ matrix $m_{ik}$, but present the final answer in the momentum representation for the combination of determinants which, according to general formula (3.17), determine $\Gamma^{(1)} \ (i, k = \{1, 2\})$:

$$
\mathcal{I}(p) \equiv \frac{\det \bar{F}_{ik}}{\det G_{ik} \cdot \det (\alpha \delta_{ik})(\det \bar{Q})^2} = \frac{(p + \omega_1^2)(p + \omega_2^2)}{(p + \omega_{F-P}^2)^2},
$$

(5.18)

$\bar{F}_{ik}$ is given by variations of $S^{(2)}$ in (5.17):

$$
\frac{\delta S^{(2)}}{\delta x^i} = \bar{F}_{ik} x^k,
$$

(5.19)

for $G_{ik}$ see (5.15), and for $\bar{Q}$, (5.16);

$$
\begin{align*}
\omega_{1,2}^2 &= \omega_0^2 + 3 \frac{n+2}{n-1} E \pm \sqrt{\frac{1}{4} \frac{n+7}{n-1} E^2 + 2nE\lambda_0^2}, \\
\omega_{F-P}^2 &= \omega_0^2 + \frac{1}{4} \frac{n+8}{n-1} E, \\
\omega_0^2 &= \frac{1}{4} n(n+8)\lambda_0^2; \quad E = n(n-1)(\lambda^2 - \lambda_0^2).
\end{align*}

(5.20)

For the one-loop term of the effective Lagrangian (defined in (5.27) below) we have finally:

$$
\mathcal{L}^{(1)} = \frac{1}{2\pi} \int \frac{d\log \mathcal{I}}{dE} \left[ \int_C \frac{d \log \mathcal{I}}{dE} dp \right] dE.
$$

(5.21)

The crucial point is the choice of the contour which selects two out of the four poles of $\mathcal{I}^{-1}$ (5.18): $p = i\omega_1$ and $p = -i\omega_2$ (see discussion in Sec. 4). With the choice of the noncausal pole at $p = -i\omega_2$ we in a sense take into account the global Faddeev–Popov zeroes in the functional integral. With this choice of the contour in the $p$-plane and with the account of the formula (see (5.18), (5.22))

$$
\frac{d \log \mathcal{I}}{dE} = \frac{Ap^2 + BE + C\omega_0^2}{(p^2 + \omega_1^2)(p^2 + \omega_2^2)} - \frac{2D}{(p^2 + \omega_{F-P}^2)},
$$

$$
A = \frac{3n+2}{2n-1}; \quad B = \frac{5n^2+12n+64}{8(n-1)^2}; \quad C = \frac{3n^2+14n+64}{2(n+8)(n-1)}; \quad D = \frac{n+8}{4(n-1)}
$$

(5.22)
we have the following result of the momentum integration in (5.21):

\[
\int_C \frac{d \log I}{dE} dp = \frac{\pi}{\omega_1 - \omega_2} \left( A - \frac{BE + C\omega_0^2}{\omega_1\omega_2} \right) - \frac{2\pi D}{\omega_{F-P}}
\]  

(5.23)

(for \(\omega_1, \omega_2, \omega_0, \omega_{F-P}, E, \) see (5.20), and for \(A, B, C, D\) see (5.22)). The final integration over \(E\) in (5.21) is not elementary. We present the answer for the one-loop term (5.21) of the effective lagrangian for small \(E\):

\[
L^{(1)} = \sqrt{\frac{8}{n+8}} E, \quad E = n(n-1)(\lambda^2 - \lambda_0^2) \ll \lambda_0^2.
\]  

(5.24)

(We normalized \(L^{(1)}\) to zero at the mass shell \(E = 0\).) To see the physics behind this result let us remember that according to (5.3) the background parameter \(\lambda\) is the derivative of \(\bar{\beta}\). One may also change the background value of \(\bar{\nu}\) from zero to any constant; this corresponds to the time reparametrization \(t \to \text{const} \cdot t\) and according to (5.4) does not violate the crucial for our calculations condition \(\dot{\bar{\beta}} = \text{const}\). To write down gauge-invariant EA, the following substitution in the expression for \(\Gamma^{(1)}\) must be performed:

\[
\lambda = \dot{\bar{\beta}} \rightarrow e^{-\bar{\nu}} \dot{\bar{\beta}}; \quad dt \rightarrow e^{\bar{\nu}} dt.
\]  

(5.25)

Hence from the definition of \(E\) in (5.24) and \(\lambda_0^2\) in (5.4):

\[
E = n(n-1)e^{-2\bar{\nu}} \dot{\bar{\beta}}^2 - 2\Lambda.
\]  

(5.26)

The Effective Action (3.17) is here a time integral of the sum of the classical Lagrangian \(L^{(0)}\) (see (5.2)) and the one-loop term \(L^{(1)}\) (5.24), where \(E\) is taken in the form (5.26). Finally:

\[
\Gamma(\bar{\nu}, \bar{\beta}) = \int \sqrt{-g} (L^{(0)} + L^{(1)}) dt
\]

\[
= L \int e^{\bar{\nu} + n\bar{\beta}} \left\{ -n(n-1)e^{-2\bar{\nu}} \dot{\bar{\beta}}^2 - 2\Lambda \right\} + L^{-1} \left( \frac{8}{n+8} \right)^{1/2} E^{1/2} dt.
\]  

(5.27)

From (5.27) and (5.26) renormalized constraint equation

\[
\frac{\partial \Gamma}{\partial \bar{\nu}} = 0.
\]  

(5.28)

(cf. (1.6)) reads:

\[
E - 2\frac{L}{\Lambda} \left( \frac{8}{n+8} \right)^{1/2} \frac{1}{\sqrt{E}} = 0.
\]  

(5.29)

We see the “repel” from the classical constraint \(E = 0\). Solution of (5-29)

\[
\bar{E} = \left[ \frac{32}{n+8} \frac{\Lambda^2}{L^2} \right]^{1/3}
\]  

(5.30)

is the renormalized “vacuum” value of \(E\); \(E \ll \Lambda\), i.e. small \(E\) approximation (5.24) requires \(\Lambda \cdot L^2 \gg 1\).
6 Conclusion

More work and better understanding are necessary in the future. It would be interesting to consider a more realistic theory instead of toy models. Our statement: in the EA language, the presence of a constraint shows up in the quantum repel from the mass shell; this is demonstrated in (1.18) and (3.27). Concrete formulae for this phenomenon evidently depend on the choice of the gauge-fixing condition, which is not a surprise. Constraints are intimately connected with the gauge sector; all effects studied in this report are of longitudinal nature. However this does not mean that the results are unphysical; it is well known after all that in the relativistic gauges of nonabelian theories the very separation of the longitudinal and transverse sectors is problematic [21]. A choice of the gauge fixing means physically the choice of gauge-invariant observables. If gauge transformations are zero at infinity, they do not influence asymptotic observables. But global, ‘large’, gauge transformations cannot help influence physical predictions of the theory. And even if we, following the VDWEA approach, postulate Landau-De Witt gauge, the way of averaging over residual gauge modes in essential. We already discussed it in Sec. 4.

Let us sketch here the way of averaging alternative to the choice of the noncausal contour in (4.13) or (5.21). We may define functional integral in (3.14) as a product of the ordinary functional integration over fields vanishing at infinity and the integration over residual gauge modes (3.21). Because of (3.20) this will give logarithmic divergency of the EA when \( S_B \to 0 \). We shall not dwell on this approach here, but in order to illustrate the idea we present symbolically the expected EA for gravity:

\[
\Gamma(\bar{g}) = S_E + \text{const} \cdot \log \left\{ \det \left[ \int V_{ij}^{(\mu \nu)} (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \sqrt{-g} \ d^4 x \right] \right\} \tag{6.1}
\]

where \( S_E \) is the Einstein Action, \( V_{ij}^{(\mu \nu)} \) are some standard tensors constructed from the functional Cristoffel symbols, gauge Killings \( K_\beta^{(\mu \nu)} \), and combinations \( \xi_0^{\alpha} \xi_0^{\beta} \) of all solutions to Eq. (3.13); variations of the space-time metric \( \eta_{\xi_0}^{(\mu \nu)} \) (3.7) are normalized to unity with the functional metric (3.4). It is easily seen that dynamical equations (1.6) given by the variations of (6.1) are rather Machian: they forbid empty Einstein spaces which obey \( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 0 \). Although the action (6.1) differs drastically from the conventional GR dynamics, there are forcible grounds to conclude that it does not come into contradictions with the local observations. We shall touch this problem elsewhere.

The question arises: what is the ‘empty’ or ‘nonempty’ space? Why do we define external sources as a variation of the classical Action \( \delta S/\delta \varphi^A \) and not of the EA \( \delta \Gamma/\delta \varphi^A \)? A possible answer is that quantum corrections considered in this report are obligatory nonlocal (as in (6.1)) (they just look as local in (5.27) because we considered homogeneous background).
Thus it is physically natural to define external sources as nonzero variations of the local action $S$ (let $S$ include all quantum ultraviolet renormalizations), whereas $\delta \Gamma^{(\text{non local})}/\delta \varphi^A = 0$ (1.6) is a self-consistency condition to determine external currents.

Describing the Universe with a modified Einstein equations was in a trend of thought of Academician Markov. The integral formulation of Einstein equations [22, 23]

$$g_{ik}(x) = \kappa \int \mathcal{D}_{ik}^\alpha \delta(y) T_{\alpha\beta}(y) \sqrt{-g} \, d^4 y$$  \hspace{1cm} (6.2)

was also a topic of his interest. This field is intimately connected with the ideas of the present report. Let us consider instead of (6.2) the integral equation for a small “residual” gauge variations (3.7) of the metric (here $\xi = \xi_0$ is the solution of (3.13)):

$$\delta^{(\xi)} g_{ik} = \kappa \int \mathcal{D}^{\alpha\beta}_{ik} \delta^{(\xi)} T_{\alpha\beta} \sqrt{-g} \, d^4 y,$$ \hspace{1cm} (6.3)

where $\kappa \delta^{(\xi)} T_{\alpha\beta} = \bar{F}_{\alpha\beta} \delta^{(\xi)} g_{\gamma\delta}$ are the gauge variations of the external current, $\bar{F}$ is the VDW differential operator defined in (3.15), $\mathcal{D} = \bar{F}^{-1}$. Eq. (6.3) puts rather strong demands upon the background metric $\bar{g}^{ik}$; as well as (6.2), it forbids empty space-times. (6.3) means that residual gauge variations of the metric are totally created by the corresponding variations of the sources, there are no ‘free’ $\delta g_{ik}^{(\xi)}$ and hence no Faddeev–Popov zero modes in the functional integral (3.14). We may postulate that only the backgrounds, where (6.3) is valid, are realized in Nature. Supposedly quantum functional averaging over residual gauge modes considered in this report is a dynamical way to the selection rule (6.3). In [23] Mal’tsev and Markov proposed a scalar version of the integral form (6.2). It would be interesting to investigate for this simplified model the quantum approach of this report and its connection with the infinitesimal integral form (6.3).

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