Decoding algorithms of monotone codes and azinv codes and their unified view

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Abstract—This paper investigates linear-time decoding algorithms for two classes of error-correcting codes. One of the classes is monotone codes which are known as single deletion codes. The other is azinv codes which are known as single balanced adjacent deletion codes. As results, this paper proposes generalizations of Levenshtein’s decoding algorithm for Levenshtein’s single deletion codes. This paper points out that it is possible to unify our new two decoding algorithms.

I. INTRODUCTION

Insertion errors and deletion errors are considered to be synchronization errors over communication channels and storage channels such as DNA-data based storages [1], [2], racetrack memories [3], [4], or Bit Patterned Media [5], [6]. The study of deletion error-correcting codes started with Levenshtein’s work [7], where he proved that Varshamov-Tenengolts (VT) codes are capable of correcting single insertions or deletions. In Levenshtein’s proof, he showed beautiful decoding algorithm to correct single deletions. This paper provides two generalizations of Levenshtein’s decoding algorithms. One corrects single deletions and reversals for monotone codes, the other corrects single balanced adjacent deletions and single balanced adjacent reversals for azinv codes. A single Balanced Adjacent Deletion (BAD) error is a double deletion error that deletes two different consecutive binary symbols, i.e., 01 or 10. A single Balanced Adjacent Reversal (BAR) error is a swap error that reverses two different consecutive binary symbols.

For applications, the computational cost of decoding algorithms is preferable to be polynomial. One of remarkable aspects of our algorithms is the computational cost. The costs are only linear to their input, i.e., the length of received word. Another remarkable aspect is the following. While these classes and their correctable error are different, our algorithms can be allowed to be unified.

This paper is organized as follows. In Section II we first introduce monotone codes and provide a single deletion/reversal error-correcting algorithm of the monotone codes. In Section III we first introduce azinv codes and provide a single BAD/BAR error-correcting algorithm of the azinv codes. In Section IV we provide a unified view of the algorithms of the monotone codes and the azinv codes.

II. MONOTONE CODE AND ITS DECODING ALGORITHM

Throughout this paper, \( \mathbb{B} \) denotes the binary set \( \{0, 1\} \). For a positive integer \( n \), \( [n] \) denotes \( \{1, 2, \cdots, n\} \).

In this section, we introduce monotone codes and provide an algorithm to make the monotone codes single deletion/reversal error-correctable (Algorithm 1). Errors treated in this section are deletion errors and reversal errors, which are defined below.

**Definition II.1** (Deletion and Reversal). Let \( n \) be a positive integer and \( i \in [n] \). Define a map \( \text{del}_i : \mathbb{B}^n \rightarrow \mathbb{B}^{n-1} \) as

\[
\text{del}_i(x_1x_2\cdots x_n) := x_1\cdots x_{i-1}x_i+1\cdots x_n.
\]

We call the map \( \text{del}_i \) deletion. Note that \( \mathbb{B}^0 := \{\varepsilon\} \), where \( \varepsilon \) is the empty word.

Define a map \( \text{rev}_i : \mathbb{B}^n \rightarrow \mathbb{B}^n \) as

\[
\text{rev}_i(x_1x_2\cdots x_n) := x_1\cdots x_{i-1}\overline{x}_ix_{i+1}\cdots x_n
\]

with \( \overline{0} = 1 \) and \( \overline{1} = 0 \). We call the map \( \text{rev}_i \) reversal.

The following codes, monotone codes, are known as single deletion error-correcting codes [8]. However, no decoding algorithm has been studied.

**Definition II.2** (Monotone code [8]). Let \( n \) and \( m \) be positive integers, \( a \) an integer, and \( k = (k_1, k_2, \cdots, k_n) \) a positive monotonic increasing integer sequence with \( m > k_1 \). Define a set \( M_{a,m,k}(n) \) as

\[
M_{a,m,k}(n) := \{x \in \mathbb{B}^n \mid \rho_k(x) \equiv a \pmod{m}\},
\]

where

\[
\rho_k(x) := \sum_{i=1}^{\lvert x \rvert} k_ix_i
\]

and \( \lvert x \rvert \) denotes the length of \( x \). \( \rho_k(x) \) is defined only for \( k \) with \( \lvert k \rvert \geq \lvert x \rvert \). We call \( M_{a,m,k}(n) \) a monotone code.

**Remark II.3.** If \( k = (1, 2, \cdots, n) \), the monotone code is called a Levenshtein code. If \( k = (1, 2, \cdots, n) \) and \( m = n + 1 \), the monotone code is called a VT code. Levenshtein [7] proved that Levenshtein codes are single deletion error-correcting codes with \( m > n \). He also proved that Levenshtein...
codes are single deletion/reversal error-correcting codes with \( m \geq 2n \).

The following is one of the main contributions of this paper.

**Theorem II.4.** Let \( M_{a,m,k}(n) \) be a monotone code and \( x \) be a codeword of \( M_{a,m,k}(n) \). Let \( \text{Dec}_M \) denote Algorithm 1.

1. \( \text{Dec}_M(x) = x \).
2. Assume \( k_n < m \). For any single deletion \( \text{del} \), \( \text{Dec}_M \circ \text{del}(x) = x \).
3. If \( k_n < m \), any monotone code \( M_{a,m,k}(n) \) is a single deletion error-correcting code with Algorithm 1 as a decoding algorithm.
4. Assume \( 2k_n \leq m \). For any single reversal \( \text{rev} \), \( \text{Dec}_M \circ \text{rev}(x) = x \).
5. If \( 2k_n \leq m \), any monotone code \( M_{a,m,k}(n) \) is a single deletion/reversal error-correcting code with Algorithm 1 as a decoding algorithm. Here \( \circ \) denotes the map composition.

**Proof.**
1. It is trivial from the steps 3 and 4 of Algorithm 1.
2. A proof is provided in subsection II-B. 3. It is a corollary of 1. and 2. 4. A proof is provided in subsection II-C. 5. It is a corollary of 3. and 4. \( \square \)

A. Decoding algorithm for single deletion/reversal errors

In this subsection, we provide notation in Algorithm 1 and the details of Algorithm 1.

For an integer \( n \geq 1 \), a positive integer \( i \in [n] \), a positive monotonic increasing integer sequence \( k = (k_1, k_2, \cdots , k_n) \), and a binary sequence \( y = y_1 y_2 \cdots y_{n-1} \in \mathbb{B}^{n-1} \), define maps as

\[
L^{(0)}_{k}(i, y) := \begin{cases} 
\rho_i \sum_{j=i}^{n-1} j y_{j} (k_{j+1} - k_j) & (i \neq 1), \\
0 & (i = 1).
\end{cases}
\]

\[
L^{(1)}_{k}(i, y) := \begin{cases} 
\rho_i \sum_{j=i}^{n-1} j y_{j}(k_{j+1} - k_j) & (i \neq 1), \\
0 & (i = 1).
\end{cases}
\]

\[
R^{(0)}_{k}(i, y) := \begin{cases} 
\rho_i \sum_{j=i}^{n-1} j y_{j} (k_{j+1} - k_j) & (i \neq n), \\
0 & (i = n).
\end{cases}
\]

\[
R^{(1)}_{k}(i, y) := \begin{cases} 
\rho_i \sum_{j=i}^{n-1} j y_{j} (k_{j+1} - k_j) & (i \neq n), \\
0 & (i = n).
\end{cases}
\]

We denote \( R^{(1)}_{k}(1, y) \) by \( w_t^k(y) \). Note that \( w_t^k(y) \) is coincided with the Hamming weight of \( y \) if \( k = (1, 2, \cdots , n) \). We omit \( k \) from the notations, if \( k = (1, 2, \cdots , n) \).

For \( b \in \mathbb{B} \), \( i \in [n] \), and \( \text{ins}_{b,y}(y) \), \( L^{(0)}(i, y) \) (resp. \( L^{(1)}(i, y) \)) equals to the number of 0 (resp. 1) to the left of inserted position \( i \), \( R^{(0)}(i, y) \) (resp. \( R^{(1)}(i, y) \)) equals to the number of 0 (resp. 1) to the right of inserted position \( i \).

The map defined below is known as an inverse operation of deletion and is used in Algorithm 1.

**Definition II.5.** For a positive integer \( i \in [n+1] \) and a non-empty binary sequence \( b \), define a map \( \text{ins}_{i,b} : \mathbb{B}^n \rightarrow \mathbb{B}^{n+|b|} \) as

\[
\text{ins}_{i,b}(x_1 x_2 \cdots x_n) := x_1 \cdots x_{i-1} b x_i \cdots x_n.
\]

We call the map \( \text{ins}_{i,b} \) insertion.

**Algorithm 1 Decoding algorithm for single deletion/reversal errors**

1: Input: \( a \in \mathbb{Z}_{\geq 0} \), \( n \) and \( m \in \mathbb{Z}_{\geq 1} \), \( y \in \bigcup_{t \geq 0} \mathbb{B}^t \), and \( k = (k_1, k_2, \cdots , k_n) \in \mathbb{Z}^n \).

Output: \( z \in \mathbb{B}^n \) or a symbol ?.

2: Compute the length of \( y \), say \( |y| \).
3: if \( |y| = n \) then
4: go to 10.
5: else if \( |y| = n - 1 \) then
6: go to 21.
7: else
8: \( z := ?. \) Go to 35.
9: end if
10: Compute \( r := \min \{ s \in \mathbb{Z}_{\geq 0} \mid s \equiv a - \rho_k(y) \ (\text{mod} \ m) \} \).
11: if \( r = 0 \) then
12: \( z := y \). Go to 35.
13: else
14: Compute \( p := \min \{ j \in [n] \mid k_j = \min \{ r, m - r \} \} \).
15: if \( \text{rev}_p(y) \in M_{a,m,k}(n) \) then
16: \( z := \text{rev}_p(y) \). Go to 35.
17: else
18: \( z := ?. \) Go to 35.
19: end if
20: end if
21: Compute \( r := \min \{ s \in \mathbb{Z}_{\geq 0} \mid s \equiv a - \rho_k(y) \ (\text{mod} \ m) \} \).
22: Compute \( w := w_t^k(y) \).
23: if \( r \leq w \) then
24: Compute \( p := \max \{ j \in [n] \mid L^{(1)}_{k}(j, y) = r \} \).
25: \( b := 0 \).
26: else
27: Compute \( p := \min \{ j \in [n] \mid L^{(0)}_{k}(j, y) = w - k_1 \} \).
28: \( b := 1 \).
29: end if
30: if \( \text{ins}_{p,b}(y) \in M_{a,m,k}(n) \) then
31: \( z := \text{ins}_{p,b}(y) \).
32: else
33: \( z := ?. \)
34: end if
35: Output \( z \).

Note that the steps 21st - 34th of Algorithm 1 are coincided with Levenshtein’s decoding algorithm for single deletion errors if \( k = (1, 2, \cdots , n) \).

Before we move to proofs for Theorem II.4, we provide examples of Algorithm 1 for each error.
Example II.6 (Decoding for a single deletion error). Let \( n = 4, a = 0, m = 9, k = (1, 3, 6, 8), \) and \( y = 101. \) Note that the monotone code has four words,
\[
M_{0,9,k}(4) = \{0000, 1001, 0110, 1111\}.
\]
- Since \( |y| = 3, \) we go to step 6 and then to step 21.
- Algorithm 1 computes \( r \) and \( w. \) We obtain \( r = 2 \) and \( w = 4. \)
- Since \( r \leq w, \)
  \[
p = \max\{j \in [4] \mid R^{(1)}(j, y) = 2\}
  = 3,
\]
  \[
b = 0.
\]
- The output \( \text{Dec}_M(y) = \text{ins}_{p,b}(y) \) is \( \text{ins}_{3,0}(101) = 1001. \)

Example II.7 (Decoding for a single reversal error). Let \( n = 6, a = 0, m = 20, k = (1, 2, 3, 8, 9, 10), \) and \( y = 111110. \) Note that the monotone code has five words,
\[
M_{0,20,k}(6) = \{000000, 110110, 001110, 100011, 010101\}.
\]
- Since \( |y| = 6, \) we go to step 4 and then to step 10.
- Algorithm 1 computes \( r, \) then we obtain \( r = 17. \)
- Since \( r \neq 0, \)
  \[
p = \min\{3, 17\}
  = 3.
\]
- The output \( \text{Dec}_M(y) = \text{rev}_{p}(y) \) is \( \text{rev}_{3}(111110) = 110110. \)

B. Proof for single deletion error-correction

To prove 2 of Theorem II.4, we introduce the following four Lemmas II.9, II.10, II.11, and II.12. From now on, till the end of this subsection, we assume the following.

**Hypothesis II.8.** A binary sequence \( x \) is a codeword of \( M_{a,m,k}(n). \) Set \( y := \text{del}_i(x) \) for a fixed \( i. \) \( r \) is the value at the step 21 of Algorithm 1.

**Lemma II.9.**
\[
k_i = k_i + L_k^{(0)}(i, y) + L_k^{(1)}(i, y).
\]

**Proof.** It follows from the definitions of \( L_k^{(0)}(i, y) \) and \( L_k^{(1)}(i, y). \)

(R.H.S.) \[
k_i + \sum_{j=1}^{i-1} y_j(k_{j+1} - k_j) + \sum_{j=1}^{i-1} y_j(k_{j+1} - k_j)
  = k_i + \sum_{j=1}^{i-1} (y_j + y_j)(k_{j+1} - k_j)
  = k_i + \sum_{j=1}^{i-1} (k_{j+1} - k_j)
  = k_i.
\]

**Lemma II.10.** \( \text{wt}_k(y) = L_k^{(1)}(i, y) + R_k^{(1)}(i, y). \)

**Proof.** It follows from the definitions of \( \text{wt}_k(y), L_k^{(1)}(i, y), \) and \( R_k^{(1)}(i, y). \)

(R.H.S.) \[
\sum_{j=1}^{i-1} y_j(k_{j+1} - k_j) + \sum_{j=1}^{n-1} y_j(k_{j+1} - k_j)
  = \sum_{j=1}^{n-1} y_j(k_{j+1} - k_j) = \text{wt}_k(y).
\]

**Lemma II.11.** The following four inequalities hold.

\[
0 \leq R_k^{(1)}(i, y) \leq \text{wt}_k(y) \leq k_1 + \text{wt}_k(y) + L_k^{(0)}(i, y) \leq k_n. \tag{1-4}
\]

**Proof.** The inequality (1) follows from that \( k \) is a positive monotonic increasing integer sequence. The inequality (2) follows from Lemma II.10 and \( L_k^{(1)}(i, y) \geq 0. \) The inequality (3) follows from \( k_1 > 0 \) and \( L_k^{(0)}(i, y) \geq 0. \)

We show the inequality (4). From Lemma II.9 and Lemma II.10 the equation on the first line below follows.

\[
k_1 + \text{wt}_k(y) + L_k^{(0)}(i, y) = k_1 + R_k^{(1)}(i, y)
  = k_1 + \sum_{j=1}^{n-1} y_j(k_{j+1} - k_j)
  \leq k_1 + \sum_{j=1}^{n-1} (k_{j+1} - k_j)
  = k_n.
\]

**Lemma II.12.**
\[
r = \begin{cases} 
R_k^{(1)}(i, y) & (x_i = 0), \\
1 + \text{wt}_k(y) + L_k^{(0)}(i, y) & (x_i = 1).
\end{cases}
\]

**Proof.** The value of \( r \) is obtained by using Lemma II.9, Lemma II.10 and Lemma II.11. It follows from the definitions of \( r \) and \( a. \)

\[
r \equiv -a \rho_k(y) \pmod{m}
  = a - \sum_{i=1}^{n-1} k_i y_i
  \equiv \sum_{i=1}^{n-1} k_i x_i - \sum_{i=1}^{n-1} k_i y_i \pmod{m}.
\]

By the assumption \( y := \text{del}_i(x), \) the following holds.
\[
x = \text{ins}_{x_i}(y) = y_1 \cdots y_{i-1} x_i y_i \cdots y_{n-1}.
\]
Therefore,
\[
\begin{align*}
\sum_{i=1}^{n} k_{i} x_{i} - \sum_{i=1}^{n-1} k_{i} y_{i} &= k_{1} y_{1} + \ldots + k_{i-1} y_{i-1} + k_{i} x_{i} + k_{i+1} y_{i+1} + \ldots + k_{n} y_{n-1} - k_{1} y_{1} - \ldots - k_{i-1} y_{i-1} + k_{i} y_{i} + \ldots + k_{n} y_{n-1} \\
&= k_{i+1} x_{i+1} + \ldots + k_{n} y_{n-1}.
\end{align*}
\]

On the other hand, the following inequalities hold by Lemma II.9.

Thus, \( r > w \) holds. Assume that \( r = w \). Then, Lemma II.12 implies \( r = k_{1} + w t_{k}(y) + L^{(0)}_{k}(i, y) \), which contradicts to

\[
r = P^{(1)}_{k}(i, y).
\]

Therefore, \( x_{i} = 0 \).

Finally, we show that \( \text{ins}_{p, 0}(y) = x \). We showed the deleted symbol \( x_{i} \) is equal to 0. Since \( x_{i} = 0 \) and \( y = \text{del}_{i}(x) \), \( x = \text{ins}_{i, 0}(y) \) holds. Therefore, it suffices to prove that \( \text{ins}_{p, 0}(y) = \text{ins}_{i, 0}(y) \). Furthermore, we will show

\[
0 = y_{i} = y_{i+1} = \ldots = y_{p-1}.
\]

Since \( r = R^{(1)}_{k}(i, y) \), then

\[
i \in \{ j \in [n] \mid R^{(1)}_{k}(j, y) = r \}
\]

holds. Since

\[
p \in \{ j \in [n] \mid R^{(1)}_{k}(j, y) = r \},
\]

then

\[
r = R^{(1)}_{k}(i, y) = R^{(1)}_{k}(p, y)
\]

holds. Therefore,

\[
\sum_{j=i}^{n-1} y_{j}(k_{j+1} - k_{j}) = \sum_{j=p}^{n-1} y_{j}(k_{j+1} - k_{j})
\]

holds. In the case of \( r \leq w \), \( i \leq p \) follows from the definition of \( p \). Since \( i \leq p \), we have

\[
0 = \sum_{j=i}^{n-1} y_{j}(k_{j+1} - k_{j}) - \sum_{j=p}^{n-1} y_{j}(k_{j+1} - k_{j}) = \sum_{j=i}^{p-1} y_{j}(k_{j+1} - k_{j}).
\]

Since \( (k_{j+1} - k_{j}) > 0 \), we have

\[
0 = y_{i} = y_{i+1} = \ldots = y_{p-1}.
\]

By a similar argument, we can prove in the remaining case \( r > w \).

**proof of 2 of Theorem II.4.** Let us focus on the step 23 of Algorithm 1. The case of \( r \leq w \) and the case of \( r > w \) are shown separately.

In the case of \( r \leq w \), we have \( \text{Dec}_{M}(y) = \text{ins}_{p, 0}(y) \). We show \( \text{ins}_{p, 0}(y) = x \). First, we show that \( r = R^{(1)}_{k}(i, y) \) holds. This is shown by contradiction. From Lemma II.12, \( r = k_{1} + w t_{k}(y) + L^{(0)}_{k}(i, y) \) holds. Assume that \( r = k_{1} + w t_{k}(y) + L^{(0)}_{k}(i, y) \) holds. Lemma II.11 implies \( r > w = w t_{k}(y) \), which contradicts to \( r \leq w \). Thus, \( r = R^{(1)}_{k}(i, y) \) holds.

**C. Proof for single reversal error-correction**

To prove 4 of Theorem II.4, we introduce the following Lemma II.14. From now on, till the end of this subsection, we assume the following.

**Hypothesis II.13.** A binary sequence \( x \) is a codeword of \( M_{a, m, k}(n) \) with \( m \geq 2k_{n} \). Set \( y := \text{rev}_{i}(x) \) for a fixed \( i \). \( r \) is the value at the step 10 of Algorithm 1.

**Lemma II.14.**

\[
r = \begin{cases} 
k_{i} & (y_{i} = 0), 
\quad m - k_{i} & (y_{i} = 1)
\end{cases}
\]
and $r \neq 0$.

**Proof.** It follows from the definitions of $r$ and $a$.

$$r \equiv a - \rho_k(y) \pmod{m}$$

$$= a - \sum_{i=1}^{n} k_i y_i$$

$$\equiv \sum_{i=1}^{n} k_i x_i - \sum_{i=1}^{n} k_i y_i \pmod{m}.$$

By the assumption $y = \text{rev}_i(x)$, the following holds.

$$x = \text{rev}_i(y) = y_1 \cdots y_{i-1} y_{i+1} \cdots y_n.$$

Therefore, we have

$$\sum_{i=1}^{n} k_i x_i - \sum_{i=1}^{n} k_i y_i$$

$$= k_1 y_1 + \cdots + k_{i-1} y_{i-1} + k_i y_i + k_{i+1} y_{i+1} + \cdots + k_n y_n$$

$$- k_1 y_1 + \cdots + k_{i-1} y_{i-1} + k_i y_i + k_{i+1} y_{i+1} + \cdots + k_n y_n$$

$$= k_i (y_i - y_i)$$

$$= \begin{cases} k_i & (y_i = 0), \\ -k_i & (y_i = 1) \end{cases}$$

$$\equiv \begin{cases} k_i & (y_i = 0), \\ m - k_i & (y_i = 1) \end{cases}.$$

We show that the equality holds in the cases of $y_i = 0$ and $y_i = 1$. Since the sequence $k$ is a positive monotonic increasing integer sequence with $m \geq 2k_n$, we have $0 < k_i \leq k_n < m$ and $0 < m - k_n \leq m - k_i < m$. Therefore,

$$r = \begin{cases} k_i & (y_i = 0), \\ m - k_i & (y_i = 1) \end{cases}$$

and $r \neq 0$. \hfill $\square$

**proof of 4 of Theorem II.4.** Lemma II.14 implies $r \neq 0$. Therefore, Dec$_M(y) \neq y$. Let us focus on the step 14 of Algorithm 1. Lemma II.14 implies $r = k_i$ or $r = m - k_i$. Whichever $r = k_i$ or $r = m - k_i$,

$$\min\{r, m - r\} = \min\{k_i, m - k_i\}$$

holds, since $m \geq 2k_i$. Furthermore, for distinct indices $j_1$ and $j_2 \in [n]$, $k_{j_1} \neq k_{j_2}$ holds, since $k$ is a positive monotonic increasing integer sequence. Therefore, we have

$$p = \min\{j \in [n] \mid k_j = \min\{r, m - r\}\}$$

$$= i$$

holds. Thus,

$$\text{rev}_p(y) = \text{rev}_i(y)$$

$$= x$$

$$\in M_{a, m, k}(n).$$

Therefore, Dec$_M(y) = \text{rev}_p(y) = x$. 

**III. AZINV CODE AND ITS DECODING ALGORITHM**

In this section, we provide an algorithm to make azinv codes single BAD/BAR error-correctable (Algorithm 2). Errors treated in this section are BAD errors and BAR errors, which are defined below.

**Definition III.1** (BAD and BAR). For an integer $n \geq 2$ and $i \in [n-1]$, define a partial map $\text{BD}_i : \B^n \to \B^{n-2}$ as

$$\text{BD}_i(x_1 x_2 \cdots x_n) := x_1 \cdots x_{i-1} x_{i+2} \cdots x_n$$

only for $x$ with $x_i \neq x_{i+1}$. We call the partial map $\text{BD}_i$ balanced adjacent deletion (BAD).

Define a partial map $\text{BR}_i : \B^n \to \B^n$ as

$$\text{BR}_i(x_1 x_2 \cdots x_n) := x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_n$$

only for $x$ with $x_i \neq x_{i+1}$. We call the partial map $\text{BR}_i$ balanced adjacent reversal (BAR).

The following codes, azinv codes, are known as single BAD error-correcting codes [9]. However, no decoding algorithm has been studied.

**Definition III.2** (Azinv code [9]). For integers $n \geq 2$ and $m \geq 2$ and an integer $a$, define a set $A_{a, m}(n)$ as

$$A_{a, m}(n) := \{ x \in \B^n \mid \tau(x) = a \pmod{m}, x \neq 0, 1 \}$$

with $m \geq n$, where $0$ (resp. $1$) is the all zero (resp. one) word, the function $\tau$ is the composition of the function inv below and the permutation $\sigma^{-1}$ below i.e., $\tau := \text{inv} \circ \sigma^{-1}$.

The function inv is a map from a binary word to a non-negative integer and is defined as

$$\text{inv}(x_1 x_2 \cdots x_n) := \# \{ (i, j) \mid 1 \leq i < j \leq n, x_i > x_j \}.$$

The value inv$(x)$ is called the inversion number of $x$.

The permutation $\sigma$ is defined as

$$\sigma(x_1 x_2 \cdots x_n) := \begin{cases} x_1 x_n x_2 x_{n-1} x_3 \cdots x_{n+1} x_{n+2} x_n x_{n+2} \cdots x_1 (n : \text{even}), \\ x_1 x_n x_2 x_{n-1} x_3 \cdots x_{n+1} x_1 x_{n+2} x_{n+1} \cdots x_1 (n : \text{odd}) \end{cases}$$

We call $A_{a, m}(n)$ an azinv code.

The following is one of the main contributions of this paper.

**Theorem III.3.** Let $A_{a, m}(n)$ be an azinv code and $x$ be a codeword of $A_{a, m}(n)$. Let Dec$_A$ denote Algorithm 2.

1. Dec$_A(x) = x$.

2. Assume $n \leq m$. For any single BAD del$_{BA}$, Dec$_A \circ$ del$_{BA}(x) = x$.

3. If $n \leq m$, any azinv code $A_{a, m}(n)$ is a single BAD error-correcting code with Algorithm 2 as a decoding algorithm.

4. Assume $2(n-1) \leq m$. For any single BAR rev$_{BA}$, Dec$_A \circ$ rev$_{BA}(x) = x$.

5. If $2(n-1) \leq m$, any monotone code $A_{a, m}(n)$ is a single BAD/BAR error-correcting code with Algorithm 2 as a decoding algorithm. Here $\circ$ denotes the map composition.

**Proof.** 1. It is trivial from the steps 3 and 4 of Algorithm 2.

2. A proof is provided in subsection III-B. It is a corollary.
of 1. and 2. 4. A proof is provided in subsection III-C. 5. It is a corollary of 3. and 4.

A. Decoding algorithm for single BAD errors

In this subsection, we provide the details of Algorithm 2. Curiously, Algorithm 2 is similar to Algorithm 1 for correcting single deletion/reversal errors for monotone codes.

We introduce the following notation. For a positive integer \( n \) and a binary sequence \( y = y_1 y_2 \cdots y_n \), define \( \tilde{y} \) as

\[
\tilde{y} := \begin{cases} y_1 y_2 y_3 \cdots y_{n-1} y_n & (n : \text{even}), \\
y_1 y_2 y_3 \cdots y_{n-1} & (\text{otherwise}).
\end{cases}
\]

For the \( i \)th entry of \( y \), say \( y_i \), define \( \tilde{y}_i \) as

\[
\tilde{y}_i := \begin{cases} y_i & (i : \text{even}), \\
y_i & (\text{otherwise}).
\end{cases}
\]

For integers \( i \) and \( j \), define \([i, j]\) as \([i, j] := \{k \in \mathbb{Z} | i \leq k \leq j\}\). We denote the subsequence of \( y \) in the range \([i, j]\), by \( y_{[i, j]} \). I.e., \( y_{[i, j]} := y_i y_{i+1} \cdots y_j \). By using the range notation, the permutation \( \sigma^{-1} \) can be written in the following form.

**Remark III.4.** \( \sigma^{-1}(y_1 y_2 \cdots y_n) = y_1 \sigma^{-1}(y_{[3, n]}) y_2 \).

Before we move to proofs for Theorem III.3, we provide examples of Algorithm 2 for each error.

**Example III.5** (Decoding for a single BAD error). \( \text{Let } n = 5, a = 0, m = 5, \text{ and } y = 101. \text{ Note that the azinv code has six words,} \\
C_{0,5}(5) = \{010000, 010100, 010101, 010111, 011110, 101110, 10001\}. \\
\text{Since } |y| = 3, \text{ we go to step 6 and then to step 21.} \\
\text{Algorithm 2 computes } r \text{ and } w, \text{ then we get } r = 3 \text{ and } w = 3. \\
\text{Since } r \leq w, \text{ then } \\
\quad p = \max\{j \in [4] \mid L^{(1)}(j, \tilde{y}) = 3\} \\
\quad = 4. \\
\text{Since } p = 4 \text{ is even, then } b = 10. \\
\text{The output } \text{Dec}_A(y) = \text{ins}_{p, b}(y) \text{ is} \\
\qquad \text{BR}_1(101) = 10110.
\)

**Example III.6** (Decoding for a single BAR error). \( \text{Let } n = 6, a = 0, m = 10, \text{ and } y = 100000. \text{ Note that the azinv code has five words,} \\
C_{0,10}(6) = \{010000, 010100, 010101, 010111, 011111\}. \\
\text{Since } |y| = 6, \text{ we go to step 4 and then to step 10.} \\
\text{Algorithm 2 computes } r, \text{ then } r = 5. \\
\text{Since } r \neq 0, \\
\quad p = 6 - \min\{5, 5\} \\
\quad = 1. \\
\text{The output } \text{Dec}_A(y) = \text{rev}_{BA_p}(y) \text{ is} \\
\qquad \text{BR}_1(100000) = 010000.
\)

**Algorithm 2** Decoding algorithm for single BAD/BAR errors

\begin{verbatim}
1: Input: \( a \in \mathbb{Z}_{\geq 0}, n \text{ and } m \in \mathbb{Z}_{\geq 2}, \text{ and } y \in \bigcup_{t \geq 0} \mathbb{B}^t. \) \\
2: Compute the length of \( y \), say \(|y|\). \\
3: if \(|y| = n\) then \\
4: Go to 10. \\
5: else if \(|y| = n - 2\) then \\
6: Go to 21. \\
7: else \\
8: \( z := ? \). Go to 43. \\
9: end if \\
10: Compute \( r := \min\{s \in \mathbb{Z}_{\geq 0} \mid s \equiv a - \tau(y) \pmod{m}\} \). \\
11: if \( r = 0\) then \\
12: \( z := y \). Go to 43. \\
13: else \\
14: Compute \( p := n - \min\{r, m - r\} \). \\
15: if rev_{BA_p}(y) \in A_{a,m}(n) \) then \\
16: \( z := \text{rev}_{BA_p}(y) \). Go to 43. \\
17: else \\
18: \( z := ? \). Go to 43. \\
19: end if \\
20: end if \\
21: Compute \( r := \min\{s \in \mathbb{Z}_{\geq 0} \mid s \equiv a - \tau(y) \pmod{m}\} \). \\
22: Compute \( w := \text{wt}(\tilde{y}) \). \\
23: if \( r \leq w\) then \\
24: Compute \( p := \max\{j \in [n - 1] \mid L^{(1)}(j, \tilde{y}) = r\} \). \\
25: if \( p\) even then \\
26: \( b := 10. \) \\
27: else \\
28: \( b := 01. \) \\
29: end if \\
30: else \\
31: Compute \( p := \min\{j \in [n - 1] \mid R^{(0)}(j, \tilde{y}) = r - w - 1\} \). \\
32: if \( p\) even then \\
33: \( b := 01. \) \\
34: else \\
35: \( b := 10. \) \\
36: end if \\
37: end if \\
38: if \text{ins}_{p, b}(y) \in A_{a,m}(n) \) then \\
39: \( z := \text{ins}_{p, b}(y) \). \\
40: else \\
41: \( z := ? \). \\
42: end if \\
43: Output \( z \).
\end{verbatim}
B. Proof for single BAD error-correction

To prove 2 of Theorem III.3, we introduce the following three Lemmas III.8, III.9 and III.10. From now on, till the end of this subsection, we assume the following.

**Hypothesis III.7.** A binary sequence \( x \) is a codeword of \( A_{n,n}(n) \). Set \( y := \text{del}_{BA_i}(x) \) for a fixed \( i \). \( r \) is the value at the step 21 of Algorithm 2.

**Lemma III.8.** The following four inequalities hold.

\[
0 \leq L^{(1)}(i, y) \tag{1}
\]

\[
\leq \text{wt}(y) \tag{2}
\]

\[
< 1 + \text{wt}(y) + R^{(0)}(i, y) \tag{3}
\]

\[
< n. \tag{4}
\]

**Proof.** The inequality (1) follows from the definition of \( L^{(1)}(i, y) \). The inequality (2) follows from Lemma III.10 and \( R^{(1)}(i, y) \geq 0 \). The inequality (3) follows from \( R^{(0)}(i, y) \geq 0 \). We show the inequality (4).

\[
1 + \text{wt}(y) + R^{(0)}(i, y) = 1 + R^{(1)}(1, y) + R^{(0)}(1, y) \leq 1 + R^{(1)}(1, y) + R^{(0)}(1, y) = n - 1 < n.
\]

\( \square \)

**Lemma III.9.** Either \( \bar{x}, \bar{x}_{i+1} = 0 \) or \( \bar{x}, \bar{x}_{i+1} = 11 \) holds.

**Proof.** By the assumption \( y := \text{del}_{BA_i}(x) \), \( x_i \neq x_{i+1} \) holds. Thus, either \( x_i x_{i+1} = 01 \) or \( x_i x_{i+1} = 10 \) holds. Therefore, whichever \( i \) is odd or even, either \( \bar{x}, \bar{x}_{i+1} = 0 \) or \( \bar{x}, \bar{x}_{i+1} = 11 \) holds.

\( \square \)

**Lemma III.10.**

\[
r = \begin{cases} 
L^{(1)}(i, \bar{y}) & (\bar{x}, \bar{x}_{i+1} = 00), \\
1 + \text{wt}(\bar{y}) + R^{(0)}(i, \bar{y}) & (\bar{x}, \bar{x}_{i+1} = 11). 
\end{cases}
\]

**Proof.** It follows from the definitions of \( r \) and \( a \).

\[
r = a - \tau(y) \pmod{m}
\]

\[
= a - \text{inv}(\sigma^{-1}(y))
\]

\[
\equiv \text{inv}(\sigma^{-1}(x)) - \text{inv}(\sigma^{-1}(y)) \pmod{m}
\]

By the assumption \( y = \text{BAD}_{i}(x) \), the following holds.

\[
x = \text{ins}_{i,x_{i+1}}(y) = y_1 y_2 \cdots y_{i-1} x_i x_{i+1} y_i y_{i+1} \cdots y_{n-2}.
\]

Therefore, we have

\[
\text{inv}(\sigma^{-1}(x)) - \text{inv}(\sigma^{-1}(y))
\]

\[
= \text{inv}(\sigma^{-1}(\text{ins}_{i,x_{i+1}}(y))) - \text{inv}(\sigma^{-1}(y))
\]

\[
= \text{inv}(\sigma^{-1}(y_1 y_2 \cdots y_{i-1} x_i x_{i+1} y_i y_{i+1} \cdots y_{n-2}))
\]

\[
- \text{inv}(\sigma^{-1}(y_1 y_2 \cdots y_{i-1} y_i y_{i+1} \cdots y_{n-2}))
\]
III.8. On the other hand, the following inequalities hold by Lemma

This implies

holds, similarly to the case of Monotone codes.

Next, we show that the deleted symbols \( x,x_i+1 \) satisfy \( \tilde{x}\tilde{x}_{i+1} = 0 \). By the assumption \( y = \text{del}_{BA_i}(x) \), we have \( x = \text{ins}_{i,x_i+1}(y) \).

Therefore, the case of

Algorithm 2. The case of

0 = \tilde{y}_i = \tilde{y}_{i+1} = \cdots = \tilde{y}_{p-1}.

Since \( r = L^{(1)}(i, \tilde{y}) \), then

holds. Since

\( p \in \{ j \in [n-1] \mid L^{(1)}(j, \tilde{y}) = r \} \),

then

holds. Therefore, we have

By a similar argument, we can prove in the remaining case

which contradicts to

Therefore, \( \tilde{x}\tilde{x}_{i+1} = 0 \).

Finally, we show that \( \text{ins}_{p,b}(y) = x \). We showed that the deleted symbols \( x,x_i+1 \) satisfy \( \tilde{x}\tilde{x}_{i+1} = 0 \). By the assumption \( y = \text{del}_{BA}(x) \), we have \( x = \text{ins}_{i,x_i+1}(y) \).

Therefore, it suffices to prove that \( \text{ins}_{p,b}(y) = \text{ins}_{i,x_i+1}(y) \).

Furthermore we will show

holds. Therefore,

hence

This implies \( r = L^{(1)}(i, \tilde{y}) \).}

On the other hand, the following inequalities hold by Lemma

\[ 0 \leq L^{(1)}(i, \tilde{y}) < n < m. \]

By the definition of \( r \),

This implies \( r = L^{(1)}(i, \tilde{y}) \).

Case \( \tilde{x}\tilde{x}_{i+1} = 11 \): We have shown

On the other hand, the following inequalities hold by Lemma

\[ 0 \leq 1 + \text{wt}(\tilde{y}) + R^{(0)}(i, \tilde{y}) \leq n < m. \]

By the definition of \( r \),

This implies \( r = 1 + \text{wt}(\tilde{y}) + R^{(0)}(i, \tilde{y}) \). Therefore, Lemma

\[ \text{proof of 2 of Theorem III.3. Let us focus on the step 23 of} \]

Algorithm 2. The case of \( r \leq w \) and the case of \( r > w \) are shown separately.

In the case of \( r \leq w \), we have \( \text{Dec}_A(y) = \text{ins}_{p,b}(y) \). We show \( \text{ins}_{p,b}(y) = x \). First, we can show that \( r = L^{(1)}(i, \tilde{y}) \)

holds, similarly to the case of Monotone codes.

Next, we show that the deleted symbols \( x,x_i+1 \) satisfy \( \tilde{x}\tilde{x}_{i+1} = 0 \). This is shown by contradiction. Since \( x,x_i+1 \in \{0,1\} \), either \( \tilde{x}\tilde{x}_{i+1} = 0 \) or \( \tilde{x}\tilde{x}_{i+1} = 11 \) holds. Assume that \( \tilde{x}\tilde{x}_{i+1} = 11 \) holds. Lemma

implies

where \( \text{ins}_{p,b}(y) \) and \( \text{ins}_{i,x_i+1}(y) \) are obtained recursively by Algorithm 2.

C. Proof for single BAR error-correction

To prove 4 of Theorem III.3, we introduce the following two Lemmas [III.12] and [III.13]. From now on, till the end of this subsection, we assume the following.

Hypothesis III.11. A binary sequence \( x \) is a codeword of \( A_{a,m}(n) \) with \( m \geq 2(n-1) \). Let \( y := \text{rev}_{BA_i}(x) \) for a fixed \( i \). \( r \) is the value at the step 10 of Algorithm 2.

Lemma III.12. Either \( \tilde{y}_i\tilde{y}_{i+1} = 00 \) or \( \tilde{y}_i\tilde{y}_{i+1} = 11 \) holds.

Proof. By the assumption \( y = \text{rev}_{BA}(x) \), \( x_i \neq x_{i+1} \) holds. Hence, \( y_i \neq y_{i+1} \) holds. Thus, either \( y_iy_{i+1} = 01 \) or \( y_iy_{i+1} = 10 \) holds. Therefore, whichever \( i \) is odd or even, either \( \tilde{y}_i\tilde{y}_{i+1} = 00 \) or \( \tilde{y}_i\tilde{y}_{i+1} = 11 \) holds. □
Lemma III.13.
\[ r = \begin{cases} n - i & (\tilde{y}_i\tilde{y}_{i+1} = 00), \\ m - (n - i) & (\tilde{y}_i\tilde{y}_{i+1} = 11) \end{cases} \]
and \( r \neq 0. \)

Proof. It follows from the definitions of \( r \) and \( a. \)
\[
r = a - \tau(y) \pmod{m} \\
= a - \text{inv}(\sigma^{-1}(y)) \\
= \text{inv}(\sigma^{-1}(x)) - \text{inv}(\sigma^{-1}(y)) \pmod{m}
\]
By the assumption \( y = \text{rev}_{BA_1}(x), \) the following holds.
\[
x = \text{rev}_{BA_1}(y) = y_1y_2 \cdots y_{i-1}y_{i+1}y_{i} \cdots y_n.
\]
Therefore, we have
\[
\text{inv}(\sigma^{-1}(x)) - \text{inv}(\sigma^{-1}(y)) \\
= \text{inv}(\text{rev}_{BA_1}(y)) - \text{inv}(\sigma^{-1}(y)) \\
= \text{inv}(\text{rev}_{BA_1}(y_1y_2 \cdots y_{i-1}y_{i+1} \cdots y_n)) \\
= \text{inv}(y_1y_3 \cdots y_{i-1}\sigma^{-1}(y_{i+2,n})y_{i+1}y_{i-1}y_{i+2}y_2) \\
- \text{inv}(y_1y_3 \cdots y_{i-1}\sigma^{-1}(y_{i+2,n})y_{i+1}y_{i-1}y_{i+2}y_2) \\
= \begin{cases} 
(i: \text{odd}, \tilde{y}_i\tilde{y}_{i+1} = 01 \text{ or } i: \text{even}, \tilde{y}_i\tilde{y}_{i+1} = 10), \\
(n - i) \\
(i: \text{odd}, \tilde{y}_i\tilde{y}_{i+1} = 10 \text{ or } i: \text{even}, \tilde{y}_i\tilde{y}_{i+1} = 01) \\
n - i \\
m - (n - i) \end{cases}
\]
We show that the equality holds in the case of \( \tilde{y}_i\tilde{y}_{i+1} = 00 \) and \( \tilde{y}_i\tilde{y}_{i+1} = 11. \) Since \( i \in [n-1], \) we have \( 1 \leq n - i \leq n - 1 \leq m. \) Then, \( 0 < m - (n - i) \leq m - 1 < m. \) Therefore,
\[
r = \begin{cases} n - i & (\tilde{y}_i\tilde{y}_{i+1} = 00), \\ m - (n - i) & (\tilde{y}_i\tilde{y}_{i+1} = 11) \end{cases}
\]
and \( r \neq 0. \)

Proof of Theorem III.3. Lemma III.13 implies \( r \neq 0. \) Therefore, \( \text{Dec}(y) \neq y. \) Let us focus on the step 14 of Algorithm 2. Lemma III.13 implies \( r = n - i \) or \( r = m - (n - i). \) Whichever \( r = n - i \) or \( r = m - (n - i), \)
\[
\min\{r, m - r\} = \min\{n - i, m - (n - i)\}
\]
holds, since \( m \geq 2n. \) Then,
\[
p = n - \min\{r, m - r\} \\
= n - (n - i) \\
= i
\]
holds. Thus,
\[
\text{rev}_{BA_1}(y) = \text{rev}_{BA_1}(y) = x \\
\in A_{a,m}(n).
\]
Therefore, \( \text{Dec}(y) = \text{rev}_{BA_1}(y) = x. \)

IV. ANALYSIS AND UNIFICATION OF THE ALGORITHMS

In this section, we provide a unified view of our proposed algorithms. After that, we prove that the algorithms can be computed in linear time in the code-length.

Flowchart 1 is the unified representation of the parts of Algorithm 1 and Algorithm 2, that is, the steps 21st - 34th of Algorithm 1 and the steps 21st - 42th of Algorithm 2.

![Flowchart 1](image)

Fig. 1. Flowchart 1 for deletion error-correction

The following table summarizes the variables and the functions in Flowchart 1.

| VARIABLES AND FUNCTIONS IN FLOWCHART 1 |
|---|
| \( a, m, y, k \) |
| \( r = \text{remainder}(a, m, y, k) \) |
| \( w = \text{weight}(y, k) \) |
| \( p = \text{position}(r, y, k) \) |
| \( b = \text{sequence}(p) \) |
| \( \text{inserted}(p, b, y) \) |
| \( \text{condition of } a \) |
| \( \in \mathbb{Z}_{\geq 0} \) |
| \( \geq k_a + 1 \) |
| \( \in \mathbb{B}^{a-1} \) |
| \( (1, 2, \ldots, n) \) |
| \( \in \mathbb{B}^{a-2} \) |
| \( (a - \tau(y)) \% m \) |
| \( \text{weight}(y, k) \) |
| \( \max J_{1,1} \) |
| \( \min J_{2,2} \) |
| \( \text{sequence}(p) \) |
| 0 |
| 01(p: odd) |
| \( \in_{p,b}(y) \) |
| \( \text{inserted}(p, b, y) \) |
| \( \text{condition of } m \) |
| \( \in \mathbb{Z}_{\geq 0} \) |
| \( \geq n \) |
| \( \in \mathbb{B}^{a-2} \) |
| \( \text{condition of } k \) |
| \( \in \mathbb{B}^{a-1} \) |
| \( (k_1, k_2, \ldots, k_a) \) |
| \( \text{position}(1, y, k) \) |
| \( \text{position}(2, y, k) \) |
| \( \text{min} J_{2,1} \) |
Here, \((k_1, k_2, \cdots, k_n)\) is a positive monotonic increasing integer sequence and \(J_{1,1}, J_{1,2}, J_{2,1}\) and \(J_{2,2}\) are defined as follows.

\[
J_{1,1} := \{ j \in [n] \mid R_k^{(1)}(j, y) = r \},
J_{1,2} := \{ j \in [n - 1] \mid L_k^{(1)}(j, \bar{y}) = r \},
J_{2,1} := \{ j \in [n] \mid L_k^{(0)}(j, y) = r - w - k_1 \},
J_{2,2} := \{ j \in [n - 1] \mid R_k^{(0)}(j, \bar{y}) = r - w - 1 \}.
\]

For integers \(a\) and \(b\), \(a \% b\) denotes the remainder of \(a\) divided by \(b\).

Flowchart 2 is the unified representation of the parts of Algorithm 1 and Algorithm 2, that is, the steps 10th - 20th of Algorithm 1 and Algorithm 2.

\[
\text{Input: } a, m, y, k \\
\text{Compute: } r = \text{remainder}(a, m, y, k) \\
\text{Compare: } r = 0 \\
\text{yes} \\
\text{Compute: } p = \text{position}(r, m, y) \\
\text{Output: } y \\
\text{no} \\
\text{Compute: } p = \text{reversed}(p, y) \\
\text{Output: } \text{reversed}(p, y)
\]

![Fig. 2. Flowchart 2 for reversal-correction](image)

The following table summarizes the variables and the functions in Flowchart 2, where \((k_1, k_2, \cdots, k_n)\) is a positive monotonic increasing integer sequence and \(J\) is defined as follows.

\[
J := \{ j \in [n] \mid k_j = \min\{r, m - r\} \}.
\]

| the condition of \(a\) | Monotone codes |
|-------------------------|----------------|
| \( \in \mathbb{Z}_{>0} \) | \( \in \mathbb{Z}_{>0} \) |
| \( \geq 2k_n \) | \( \geq 2(n - 1) \) |
| \( \in \mathbb{B}^n \) | \( \in \mathbb{B}^n \) |
| \( (1, 2, \cdots, n) \) | \( (a - \rho_k(y))\%m \) |
| \( \min J \) | \( \min J \) |
| \( \text{rem}(y) \) | \( \text{rem}(y) \) |
| \( \text{rev}_p(y) \) | \( \text{rev}_{\text{BAD}}(p, y) \) |
| \( \text{remainder}(a, m, y, k) \) | \( \text{position}(r, y, k) \) |
| \( \text{reversed}(p, y) \) | \( \text{reversed}(p, y) \) |

**TABLE II**

VARIABLES AND FUNCTIONS IN FLOWCHART 2

We can compute \(\text{inv}(y)\) in linear time in the length of \(y\) by using the following Theorem IV.2.

**Theorem IV.1.** Set \(s := (n, n - 1, \cdots, 2, 1)\). For \( y \in \mathbb{B}^n \),

\[
\text{inv}(y) = \rho_s(y) - \left( \frac{\text{wt}(y) + 1}{2} \right).
\]

**Proof.** Since \( y \in \mathbb{B}^n \),

\[
\text{inv}(y) = \#\{ (i, j) \mid 1 \leq i < j \leq n, y_i > y_j \}
= \#\{ (i, j) \mid 1 \leq i < j \leq n, (y_i, y_j) = (1, 0) \}
\]

holds. Set \( I := \{ i \in [n] \mid y_i = 1 \} \). Then, we have

\[
\#\{ (i, j) \mid 1 \leq i < j \leq n, (y_i, y_j) = (1, 0) \}
= \sum_{i \in I} \#\{ j \in [n] \mid i < j, y_j = 0 \}
= \sum_{i \in I} \left( (n - i) - \#\{ j \in [n] \mid i < j, y_j = 1 \} \right)
= \sum_{i \in I} \left( (n - i + 1) - (1 + \#\{ j \in [n] \mid i < j, y_j = 1 \}) \right)
= \sum_{i \in I} (n - i + 1) - \sum_{i \in I} \#\{ j \in [n] \mid i \leq j, y_j = 1 \}
= \sum_{i \in I} y_i(n - i + 1) - \#I(\#I + 1)
= \rho_s(y) - \left( \frac{\text{wt}(y) + 1}{2} \right).
\]

**Theorem IV.2.** For each function in Flowcharts 1 or 2, its computational cost is \(O(|y|)\), where \(|y|\) is the length of \(y\).

**Proof.** Since the definitions of \(\text{sequence1}(p)\), \(\text{sequence2}(p)\), and \(\text{reversed}(p, y)\), they can be computed in constant time.

Since we only need to use "For loop" once, position1\((r, y, k)\), position2\((r, y, k)\), position\((r, y, k)\), inserted\((p, b, y)\), and \(\bar{y}\) can be computed in linear time in the length of \(y\).

Since the inner product, \(\text{inv}(y)\), and \(\bar{y}\) can be computed in linear time, \(\text{remainder}(a, m, y, k)\) and \(\text{weight}(y, k)\) can be computed in linear time in the length of \(y\).

The following is a corollary of Theorem IV.2.

**Corollary IV.3.** Algorithm 1 and Algorithm 2 are linear time algorithms in the code-length.

V. CONCLUSION

In this paper, we provided the single deletion/reversal error-correcting algorithm for monotone codes and the single BAD/BAR error-correcting algorithm for azinv codes. Constructions of these codes are different. However, algorithms for these codes and the proofs of Theorem II.4 and Theorem III.3 correspond to each other.

In Section IV we provided the unification of these decoding algorithms for monotone codes and azinv codes. The respective deletion error-correcting algorithms for monotone codes and azinv codes are represented by the same flowchart, and the respective reversal error-correcting algorithms for monotone codes and azinv codes are represented by the same flowchart. We also showed that these algorithms are linear-time algorithms.

As a future work, we will consider decoding algorithms for single insertion errors for monotone codes and azinv codes. Furthermore, we will consider decoding algorithms for other deletion/reversal errors. Monotone codes are defined by \(\rho_k(x)\) and azinv codes are defined by \(\tau(x)\). By replacing one of these functions with the other, we will create new codes that
are capable of the other deletion error-correcting and the other reversal error-correcting. These error-correcting algorithms are expected to have the same flowcharts as the ones for monotone codes and azinv codes.

Moreover, since monotone codes can freely take a positive monotonic increasing integer sequence \( k \), it is expected to be able to add the other property to monotone codes in addition to the single deletion/reversal error-correctable property. For example, it is known to be able to add properties of being two-deletion error-correctable \([10]\) and easy to encode \([8]\). Monotone codes are generalized by introducing parameter \( k \) into Levenshtein codes. In the same way, the generalization with parameter \( k \) in azinv codes can be considered. The function \( \text{inv}(x) \) used to define azinv codes has the property of \([4,1]\). We can generalize azinv codes by taking a positive monotonic decreasing integer sequence as \( s \) in \([4,1]\). The generalized azinv codes are expected to be able to add properties of being two-BAD error-correctable and easy to encode. In addition to these properties, there are some other similar properties in Levenshtein codes and azinv codes, such as optimality and convergence \([9]\). We would like to discuss these topics in a future work for further development of our research.

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REFERENCES

[1] Tilo Buschmann and Leonid V Bystrykh. Levenshtein error-correcting barcodes for multiplexed dna sequencing. \textit{BMC bioinformatics}, 14(1):272, 2013.
[2] Ryan Gabrys, Eitan Yaakobi, and Olgica Milenkovic. Codes in the damerau distance for deletion and adjacent transposition correction. \textit{IEEE Transactions on Information Theory}, 64(4):2550–2570, 2017.
[3] Yeow Meng Chee, Han Mao Kiah, Alexander Vardy, Eitan Yaakobi, et al. Coding for racetrack memories. \textit{IEEE Transactions on Information Theory}, 64(11):7094–7112, 2018.
[4] Jin Sima and Jehoshua Bruck. Correcting deletions in multiple-heads racetrack memories. In \textit{2019 IEEE International Symposium on Information Theory (ISIT)}, pages 1367–1371. IEEE, 2019.
[5] Masato Inoue and Haruhiko Kaneko. Deletion/insertion/reversal error correcting codes for bit-patterned media recording. In \textit{2011 IEEE International Symposium on Defect and Fault Tolerance in VLSI and Nanotechnology Systems}, pages 286–293. IEEE, 2011.
[6] Anantha Raman Krishnan and Bane Vasic. Coding for correcting insertions and deletions in bit-patterned media recording. In \textit{2011 IEEE Global Telecommunications Conference-GLOBECOM 2011}, pages 1–5. IEEE, 2011.
[7] V.I. Levenshtein. Binary codes capable of correcting deletions, insertions, and reversals. \textit{Soviet physics doklady}, 10(8):707–710, 1966.
[8] Manabu Hagiwara. On ordered syndromes for multi insertion/deletion error-correcting codes. In \textit{Information Theory (ISIT), 2016 IEEE International Symposium on}, pages 625–629. IEEE, 2016.
[9] Manabu Hagiwara. Perfect codes for single balanced adjacent deletions. In \textit{Information Theory (ISIT), 2017 IEEE International Symposium on}, pages 1938–1942. IEEE, 2017.
[10] Albertus SJ Helberg and Hendrik C Ferreira. On multiple insertion/deletion correcting codes. \textit{IEEE Transactions on Information Theory}, 48(1):305–308, 2002.