ORBITAL EVOLUTION OF A MASSIVE BLACK HOLE PAIR BY DYNAMICAL FRICTION

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We investigate the evolution of a massive black hole pair under the action of dynamical friction by a uniform background of light stars with isotropic velocity distribution. In our scenario, the primary black hole $M_1$ sits, at rest, in the center of the spherical star distribution and the secondary less massive companion $M_2$ moves along bound orbits determined by the background gravitational field. $M_2$ loses energy and angular momentum by dynamical friction, on a time scale longer than the orbital period. The uniform star core has total mass $M_c$ and radius $r_c$, and the following inequality $M_c > M_1 > M_2$ holds.

In this paper, we investigate mostly analytically the secular evolution of the orbital parameters, and find that angular momentum ($J$) and energy ($E$) are lost so as to cause the increase of the eccentricity $e$ with time, during the orbital decay of $M_2$.

In the region of the core where the motion of $M_2$ is determined by the mean field generated of the uniform stellar distribution, $E$ and $J$ are lost exponentially on a time scale $\sim \tau_{DF}$ determined by the properties of the ambient stars. The rise of $e$ establishes instead on a longer time $\sim \tau_{DF} (r_c/r_A)^2$ increasing as the apocenter distance $r_A$ decreases.

With the progressive decay of the orbit, $M_2$ enters the region $r < r_B \sim (M_1/M_c)^{1/3} r_c$, where the gravitational field of the primary black hole dominates, but the star background maintains still uniform (to first approximation). Inside $r_B$, a key parameter of the calculation is the ratio between the black hole velocity $v$ and the stellar dispersion velocity $\sigma$. (i) If $v < \sigma$, energy and angular momentum are lost exponentially on a time scale $\tau_{DF}$. The growth of $e$ occurs instead on a time $\tau_e$ longer than $\tau_{DF}$ by a factor $\sim [\sigma/v]^2$. Therefore, $e$ rises weakly during orbital decay. $\tau_e$ is also found to be a function of $e$ and increases as $e \rightarrow 1$. (ii) In the opposite limit, i.e., when $v > \sigma$, the evolution of $E$ and $J$ is close to a power-law and establishes on a time scale $\sim [v/\sigma]^3 \tau_{DF}$. The eccentricity grows on a time $\tau_e$ comparable to this scale. Along an evolutionary path, $e$ increases significantly: This rise leads the pericenter distance to diminish exponentially, in this limit. $\tau_e$ is a function of $e$ and decreases as $e \rightarrow 1$. This limit ($v > \sigma$) is attained close to the cusp radius $r_{\text{cusp}} \sim (M_1/M_c) r_c$, i.e., the distance below which the stellar distribution is affected by the gravitational field of $M_1$. Below $r_{\text{cusp}}$ our description is invalid, and we terminate our analysis.

Energy losses by gravitational wave emission become comparable to those by dynamical friction at a critical distance that depends on the ratio $M_1/M_c$: Consistency with the model assumptions implies $M_1 \ll M_c$. The braking index $n$ is calculated in this transition region: A measurable deviation from the value of $11/3$ correspond-
ing to pure gravitational wave losses provides ideally an indirect way for probing the ambient medium.
1. INTRODUCTION

Galaxies in their cores may contain massive binary black holes. This hypothesis follows mainly from two considerations. (i) Most galaxies harbor a central black hole, relic of an earlier active phase (Dressler 1989; Bland-Hawthorn, Wilson & Tully 1991). This assertion is plausible if the AGN phenomenon is the result of short-lived phases of activity that have involved most of the galaxies in the past (Rees 1990). (ii) Galaxies interact and merge. Evidence comes from the appearance of unusual morphological features interpreted as clear signature of a tidal interaction between companion galaxies or of a recent merger (see Barnes & Hernquist 1992 for a complete review). If galaxies participate a hierarchical process of clustering, large galaxies cannibalize the small ones and in these systems a black hole binary with unequal masses may form (Begelman, Blandford & Rees 1980, BBR hereafter; Roos 1981; Gaskell 1985). Recently, a claim was put forward for the observational evidence of a massive binary system in the nucleus of 1928+738 (Roos, Kaastra & Hummel 1993).

It is believed that the nuclei of two interacting galaxies sink, during a merger, toward the center of the common potential well by dynamical friction: a gravitationally bound pair of black holes may thus form, embedded in core stars (BBR; Valtaoja, Valtonen & Byrd 1989; Governato, Colpi & Maraschi 1993). The frictional deceleration on the pair itself progressively tightens the system and the two black holes eventually bind. When their distance becomes comparable to the mean star separation, single interactions with stars on a time less than the orbital period come into play and energy is transferred to the light particle. This process however may lead to a depletion of background stars that are driven into loss cone orbits via a sling-shot mechanism (BBR; Roos 1981, 1988). The decay of the binary may be relented or even inhibited at separations where the energy loss by emission of gravitational waves would be weak to drive the binary to coalescence within the Hubble time. Mechanisms for replenishing the loss cone are therefore invoked for triggering further evolution (BBR; Ross 1988; Polnarev & Rees 1994).

Recent numerical $N$-body simulations on the dynamical evolution of black hole binaries in the violently relaxed core of a merged galaxy (Fukushige, Ebisuzaky & Makino 1992a; Mikkola & Valtonen 1992; Makino et al. 1993), have indicated that the collective effect of dynamical friction leads to a “runaway” increase of the eccentricity of the binary, and to a steep drop of the pericenter distance. Losses by gravitational wave emission would thus intervene early in the life-time of the binary that evolves rapidly (within $10^9$ yrs) toward final coalescence. Proposing an approximate extension of the Chandrasekhar formula for dynamical friction in a non-uniform anisotropic
medium, Polnarev & Rees (1994; PR hereafter) studied numerically and analytically
the evolution of a black hole pair in a typical galaxy. They find that in the core
regions where sufficiently large density gradients develop the eccentricity decreases
and velocity anisotropies tend to stabilize this tendency.

The study of the black hole binary evolution is important since a correct deter-
mination of the life-time and occurrence rate of coalescence (Fukushige, Ebisuzaky
& Makino 1992b) is crucial for assessing the relevance of black hole binaries as can-
didate sources for the detection of gravitational waves (Thorne & Braginsky 1976;
Thorne 1992). These sources could indeed be detected by Doppler experiments by
spacecraft tracking (Estabrook & Wahlquist 1975) and long interferometer in space
as LISA (Danzmann et al. 1993) and SAGITTARIUS (Hellings et al. 1993).

Moving from these considerations, we here re-explore the orbital evolution of a
black hole pair, using mainly an analytical technique. In our model, a massive black
hole $M_1$ sits, at rest, in the center of a uniform galactic core with total mass $M_c,$
and radius $r_c$ (Binney & Tremaine 1987). A comparatively low mass black hole $M_2$
moves along bound orbits spiraling inward under the action of dynamical friction by
the field stars. We consider the relevant case when $M_2 < M_1 < M_c$ and introduce
the dimensionless parameter $\eta \equiv M_1/M_c < 1.$

Along the course of dynamical evolution, the frictional deceleration on $M_2$ can
be considered as small perturbation relative to the “background” gravitational field,
and therefore causes only secular changes in the orbital motion. In our scenario,$M_2$ enters the core region of radius $r_c$ along bound orbits as it experiences the mean
gravitational field generated by the uniform stellar distribution (see PR for detail). $M_2$
eventually bind to the massive central black hole, at a distance $r_B \sim \eta^{1/3} r_c.$ Below
this separation, $M_2$ is still surrounded by the homogeneous sea of field stars.
The influence of the central black hole on the stellar distribution becomes important
at a smaller separation $r_{cusp} \sim r_c \eta.$ Below $r_{cusp}$ steep stellar density gradients and
anisotropies in the velocity field appear, and our treatment ceases to be valid.

In this paper we determine the evolution of the orbital parameters of $M_2$ in the
two relevant regions: (a) the “outer” zone, in the interval $r_B < r < r_c,$ where the
mean gravitational field is generated by the core stars and (ii) the “inner” zone
$r_{cusp} < r < r_B$ where the gravitational field of the massive central black hole affects
the motion of $M_2.$

The plan of the paper is as follows. In §2 the expression for dynamical friction
is reviewed and key parameters of the model are introduced. In §3 we outline
the basic equations for the energy and angular momentum losses. We solve these equa-
tions using Gauss’ perturbation theory, both in the “outer” and “inner” core regions.
Characteristic time scales are derived as a function of the eccentricity. A criterion
for the growth of $e$ is also presented and applied to our analysis. In §4 we describe
possible evolutionary paths the binary experiences, within the inner region of the
galactic core. In §5 we review formulae on the emission of gravitational waves: the coordinated effect of dynamical friction and emission of gravity waves is then discussed. The deviation, due to dynamical friction, of the braking index $n$ from the canonical value of $11/3$ is calculated. §6 contains our conclusions.

2. THE MODEL

The frictional deceleration experienced by the secondary BH in a homogeneous background of stars with isotropic gaussian velocity field is

$$\dot{v}_{DF} = -\chi g(x) \hat{v},$$

(2.1)

where $\hat{v}$ is a unit vector in the direction of $v$, the velocity of the BH relative to the background.

In equation (2.1), the BH velocity is expressed in unit of the one-dimensional dispersion velocity ($\sqrt{2} \sigma$) of the stars

$$x \equiv \frac{v}{\sqrt{2} \sigma}$$

(2.2)

and the function $g(x)$ reads

$$g(x) = \frac{1}{x^2} \left[ \text{erf}(x) - \frac{2}{\sqrt{\pi}} xe^{-x^2} \right]$$

(2.3)

(Chandrasekhar 1943; Binney & Tremaine 1987).

The coefficient $\chi$ contains key parameters of the background stars

$$\chi = 2\pi \ln \Lambda \frac{G^2 m n_0 (M_2 + m)}{\sigma^2}$$

$$\sim 10^{-8} \left( \ln \frac{\Lambda}{10} \right) \left( \frac{M_2}{10^6 M_\odot} \right) \left( \frac{m}{1 M_\odot} \right) \left( \frac{300 \text{ km s}^{-1}}{\sigma} \right)^2 \text{ cm s}^{-2},$$

(2.4)

where $m \ll M_2$ is the mass of a typical individual star, $n_0 \sim M_c/m r_c^3$ the number density of stars and $\ln \Lambda = \ln(b_{\text{max}}/b_{\text{min}})$ the so called Coulomb logarithm (Binney & Tremaine 1987).

The dissipative force (eq. [2.1]) parallel to the velocity vector $\mathbf{v}$ clearly acts so as to decelerate the secondary BH causing the progressive decay of the binary separation.
Equation (2.1) takes a simple asymptotic expression in two relevant limits. In the first limit, i.e., for $v \ll \sigma$ (i.e., $x \ll 1$),

$$\dot{v}_{DF} = -\frac{4}{3\sqrt{\pi}}\chi (x - \frac{3}{5}x^3 + o(x^3)) \dot{v}.$$  \hfill (2.5)

This regime may characterize the early phase of the BH evolution when orbiting regions where field stars move with dispersion velocity $\sigma$ higher than $v$ (BBR).

In the second limit, i.e, for $v \gg \sigma$ (i.e., for $x \gg 1$), the deceleration takes the form

$$\dot{v}_{DF} \simeq -\frac{1}{x^2} \dot{v}.$$  \hfill (2.6)

The transition between the two asymptotic regimes occurs when $x \simeq 1$, corresponding to a value of the mean separation between the BHs $r_\simeq \eta r_c$.

Simulations have shown that equation (2.1) often provides an accurate description of the deceleration experienced by a single object. In our model for the binary evolution, the drag acts only on the secondary black hole since the primary is at rest relative to the medium. If the primary were to move a correction to equation (2.1) would be of need (Bekenstein & Zamir 1990; see however Gould 1993). A source of uncertainty in equation (2.1) is the evaluation of the Coulomb logarithm, and in particular the estimate of the maximum impact parameter $b_{\text{max}}$ whose value is predicted from numerical simulations to be in the range $N^{-1/3} r_c$ (Smith 1992) and $r_c$ (Farouki & Salpeter 1982): We have chosen as reference value $\ln \Lambda = 10$ (obtained considering $b_{\text{max}} = r_c$; Binney & Tremaine 1987). The deviation related to a different choice of $b_{\text{max}}$ (i.e., a fractional error of less that 30%) falls anyway within the intrinsic spread of $\ln \Lambda$ related to the typical parameters of galaxy models.

Equation (2.1) is considered valid down to the limiting radius $r_{\text{cusp}} \simeq \eta r_c$. At this distance the influence of the primary BH on the stellar distribution is important (Frank & Rees 1976; Binney & Tremaine 1987) and different mathematical tools are of need to follow the subsequent evolution (PR; see Maoz 1993 for a new approach to DF).

2.2 The Secondary Black Hole Orbital Motion

(a) The “outer” region: In absence of dissipative forces the secondary BH undergoes a periodic motion. In the region $r_B < r < r_c$, this motion is determined by the gravitational potential $U(r)$ generated by the uniform and isotropic distribution of stars. According to the Poisson equation

$$U(r) = \frac{1}{2} \Omega^2 r^2,$$  \hfill (2.7)
where $\Omega$ is related to the density of stars $\rho = m n_0$ through the following relation

$$\Omega^2 = \frac{4\pi}{3} G \rho. \quad (2.8)$$

The secondary BH undergoes a periodic motion along elliptic orbits: In cartesian coordinates, with origin in the center of the star distribution, the motion is described by

$$x = r_A \cos(\Omega t), \quad (2.9)$$
$$y = r_P \sin(\Omega t), \quad (2.10)$$

where $r_A$ and $r_P$ are the apocenter and pericenter distances, respectively.

We can express the two constants of motion, i.e., the total energy $E$ and angular momentum $J$ in the following form

$$r_A^2 + r_P^2 = 2 \frac{E}{M_2 \Omega^2}, \quad (2.11)$$
$$r_A r_P = \frac{J}{M_2 \Omega}, \quad (2.12)$$

and introduce the eccentricity

$$e = \frac{r_A - r_P}{r_A + r_P}. \quad (2.13)$$

For the analysis of the motion of the secondary BH driven by DF it is of importance to estimate the magnitude of the deceleration $\dot{v}_{DF}$ and compare it with the acceleration generated by the gravitational potential $\dot{v}_H = \Omega^2 r$. In the limit $x \ll 1$, we have

$$\frac{\dot{v}_{DF}}{\dot{v}_H} = \frac{4}{\sqrt{3}} G M_2 \ln \Lambda \sqrt{G \rho \sigma^{-3}}$$
$$= 5 \times 10^{-3} \left( \frac{M_2}{10^6 M_\odot} \right) \left( \frac{M_c}{2 \times 10^8 M_\odot} \right)^{1/2} \left( \frac{100 \text{ pc}}{r_c} \right)^{3/2} \left( \frac{300 \text{ km s}^{-1}}{\sigma} \right)^3 \quad (2.14)$$

The weakness of the frictional drag enable us to treat DF as a *perturbing* force.

For the harmonic potential, we consider only the case $x \ll 1$. From the virial theorem and from equation (2.8) this condition is equivalent to

$$\frac{r_A}{r_c} < 1 \quad (2.15)$$

which is obviously satisfied. The opposite limit ($x \gg 1$), equivalent to $r_P/r_c > 1$, corresponds to a motion confined in regions that are external to the galaxy core. This limit is therefore inconsistent.
(b) The “inner” region: The secondary BH becomes eventually bound to the primary BH. This occurs at a separation

\[ r_B \sim \eta^{1/3} r_c \sim 37 \left( \frac{M_1}{10^8 M_\odot} \frac{2 \times 10^9 M_\odot}{M_c} \right)^{1/3} \left( \frac{r_c}{100 \text{ pc}} \right) \text{ pc} \]  

(2.16)

for typical parameters of a galactic core (BBR). At radii \( r < r_B \), the potential of the central BH dominates over the mean potential generated by the homogeneous see of stars. At \( r_B \) the BH binary forms and the secondary BH moves along Keplerian orbits. In the region \( r_{\text{cusp}} < r < r_B \), the binary is surrounded by ambient stars having a uniform density and dispersion velocity \( \sigma \) larger than \( v \). This is the reason why only below the cusp radius

\[ r_{\text{cusp}} \sim \eta r_c \sim 1 \left( \frac{M_1}{10^8 M_\odot} \frac{2 \times 10^9 M_\odot}{M_c} \right) \left( \frac{r_c}{100 \text{ pc}} \right) \text{ pc} \]  

(2.17)

the stellar distribution is affected by the central BH.

The total newtonian energy of the binary

\[ E = -\frac{GM\mu}{2a} \]  

(2.18)

and angular momentum \( J \) of magnitude

\[ J^2 = GM\mu^2a(1 - e^2) \]  

(2.19)

are constants of motion (in eq. [2.18] and [2.19] \( M = M_1 + M_2 \sim M_1 \) and \( \mu \sim M_2 \) is the reduced mass). Given \( E \) and \( J \) the semimajor axis \( a \) and the eccentricity \( e \) are uniquely determined.

Analogously to the previous case, we estimate the magnitude of the frictional drag \( \dot{v}_{\text{DF}} \) relative to the gravitational acceleration, \( \dot{v}_K = GM/a^2 \). For \( x \ll 1 \) we have

\[ \frac{\dot{v}_{\text{DF}}}{\dot{v}_K} = \frac{4}{3\pi^{1/2}} \frac{\chi a^{3/2}}{\sigma (GM)^{1/2}} \]

\[ \sim 2.2 \times 10^{-3} \left( \frac{a}{10 \text{ pc}} \right)^{3/2} \left( \frac{300 \text{ km s}^{-1}}{\sigma} \right)^3 \left( \frac{10^8 M_\odot}{M} \right)^2 \]  

(2.20)

and for \( x \gg 1 \)

\[ \frac{\dot{v}_{\text{DF}}}{\dot{v}_K} = \frac{2\chi a^3}{(GM)^2} \]

\[ \sim 2.3 \times 10^{-8} \left( \frac{a}{0.1 \text{ pc}} \right)^3 \left( \frac{10^8 M_\odot}{M} \right)^2 \]  

(2.21)
Even in this case DF is weak compared to gravity. Therefore, DF can be considered as a perturbing force that controls only the evolution of the BH on a time scale longer than a orbital period. In both potentials, the motion of the secondary BH is almost periodic. On the secular scale fixed by DF, the mean motion varies, the system evolving toward states that are progressively more bound.

3. THE LONG TERM EVOLUTION OF THE SECONDARY BH

3.1 Long Term Variation of $E$ and $J$

In this section we derive the long term evolution of $E$ and $J$. In presence of DF, energy is not conserved and it is dissipated at a rate

$$\dot{E} = \mu \mathbf{v}_{DF} \cdot \mathbf{v} = -\mu \chi g(x) v. \quad (3.1)$$

Similarly, the equation for the angular momentum loss is

$$\dot{J} = \mu (r \times \dot{\mathbf{v}}_{DF}) = -\chi g(x) J v. \quad (3.2)$$

$\dot{E}$ and $\dot{J}$ represent the instantaneous values of the time derivatives for $E$ and $J$. Since DF is a weak perturbing force, it is meaningful to calculate $\langle \dot{E} \rangle$ and $\langle \dot{J} \rangle$, i.e., the average on the orbital period. This mean is defined for any orbital parameter $S$ (such as $E, J, r_P, r_A, a$ and $e$) as

$$\langle \dot{S} \rangle \equiv \frac{1}{P} \int_0^P \dot{S} dt, \quad (3.3)$$

where $P$ is the orbital period. Below, the results are presented separately in the two asymptotic limits of equation (2.1) and for the two expressions of the gravitational potential acting on the secondary BH: the calculations are fully analytical.

(a) The Harmonic Potential

When the gravitational potential of the star distribution is dominant ($r_B < r < r_c$), the losses can be evaluated for $x \ll 1$ retaining only the first term in the expansion of $g(x)$. This yields:

$$\langle \dot{E} \rangle = -\frac{4\chi}{3\sqrt{2\pi} \sigma} E \quad (3.4)$$
It is relevant to notice that in this limit the decay is exponential as DF is proportional to \( \langle v \rangle \) in this limit (see eqs. [2.5], [3.1] and [3.2]). These losses are also to lowest order independent of the eccentricity \( e \).

As indicated by equations (3.4) and (3.5), the magnitude of the decay time is primarily determined by the values of \( \chi \) and \( \sigma \). We are therefore led to introduce a characteristic time scale for DF

\[
\tau_{\text{DF}} \equiv \frac{\sqrt{2\sigma}}{\chi}
\]

\[
\approx 10^7 \left( \frac{\sigma}{300 \text{ km s}^{-1}} \right)^3 \left( \frac{10^6 M_\odot}{M_2} \right) \left( \frac{2 \times 10^3 \text{ pc}^{-3}}{n_0} \right) \text{ yrs}.
\]

(3.6)

In terms of \( \tau_{\text{DF}} \) the decay time scales for \( E \) and \( J \) are equal and read:

\[
\tau_{H}^E = \tau_{H}^J = \frac{3 \pi^{1/2}}{4} \tau_{\text{DF}}.
\]

(3.7)

(b) The Keplerian Potential

Below \( r_B \) the gravitational potential of the massive primary BH dominates and the motion of the secondary is nearly Keplerian.

(i) The limit \( x \ll 1 \)

We consider the case of \( x \ll 1 \) and, for simplicity, retain again the first term in the expansion of \( g(x) \). In this case the losses of energy and angular momentum read

\[
\langle \dot{E} \rangle = -\frac{8\chi}{3(2\pi)^{1/2}\sigma} E
\]

(3.8)

\[
\langle \dot{J} \rangle = -\frac{4\chi}{3(2\pi)^{1/2}\sigma} J
\]

(3.9)

(the details are presented in Appendix A and Appendix B. In the calculations \( x \) is a function of the orbital phase and the asymptotic limit is meant to be valid over the complete orbit). It is important to notice that even in this case the losses are exponential and independent of the eccentricity. According to the previous equations
the corresponding time scales are

\[
\tau_E^E = \frac{1}{2} \tau_J^J = \frac{3 \pi^{1/2}}{8} \tau_{\text{DF}}. \tag{3.10}
\]

(ii) The limit \( x \gg 1 \)

With the progressive loss of energy to the ambient stars, the orbital velocity of the secondary BH may increase above \( \sigma \) (see §4 for a discussion). For \( x \gg 1 \), the evolution equations, after considerable manipulations, can be cast into the following form

\[
\langle \dot{E} \rangle = -\frac{\chi \sigma^2 \mu^{3/2}}{2^{1/2} \pi} \frac{1}{E} \frac{1}{K(e)}
\]

\[
= -\frac{\chi \sigma^2 \mu}{\pi} \left( \frac{a}{GM} \right)^{1/2} K(e), \tag{3.11}
\]

\[
\langle \dot{J} \rangle = -\frac{\chi \sigma^2 \mu}{\pi} \frac{a^2}{GM} Z(e), \tag{3.12}
\]

where in \( K(e) \), and \( Z(e) \) we confine the dependence on \( e \) (the details are presented in Appendix B). Equations (3.11) and (3.12) show that the time evolution of \( E \) and \( J \) is not exponential in this limit but follows approximately a power–law.

These equations contain the relevant time scales \( \tau_E^E \) and \( \tau_J^J \) that depend, in this limit, on \( e \) and on a new quantity

\[
x_{\text{cir}} \equiv \left( \frac{G M}{2 \sigma^2 a} \right)^{1/2}, \tag{3.13}
\]

representing the velocity the BH would have as if it were to move along a circular orbit; this velocity is expressed in units of the dispersion velocity of the background stars. In \( x_{\text{cir}} \) we mainly retain the dependence of the time scale on the energy \( E \) (see eq. [2.18]).

The resulting time scales are

\[
\tau_E^E \sim x_{\text{cir}}^3 \tau_{\text{DF}} \frac{1}{K(e)}
\]

\[
\sim 5 \times 10^8 \left( \frac{x_{\text{cir}}}{3} \right)^3 \frac{1}{K(e)} \text{ yrs}, \tag{3.14}
\]
\[ \tau_j^J \sim x_{\text{cir}}^3 \tau_{\text{DF}} \frac{(1 - e^2)^{1/2}}{Z(e)} \]

\[ \sim 5 \times 10^8 \left( \frac{x_{\text{cir}}}{3} \right)^3 \frac{(1 - e^2)^{1/2}}{Z(e)} \text{ yrs}. \] (3.15)

It is of interest to notice that, in this regime, the time scales of orbital decay \( \tau_E^J \) and \( \tau_J^J \) are longer than the “reference” time scale \( \tau_{\text{DF}} \) (i.e., the scale for \( x \ll 1 \)), and the dilution factor is of \( \sim x^3 \), as DF on \( \propto v^{-2} \) (equations [2.6] and [3.1]). Considering further their dependence on \( e \) through \( K \) and \( Z \), both time scales are found to become progressively shorter with increasing eccentricity. In Figure 1, we show \( \tau_E^J \) and \( \tau_J^J \) versus \( e \) and note the decrease of the time scales as \( e \to 1 \). As an example for \( J \), \( \tau_J^J (e = 0.9) \sim 0.1 \tau_J^J (e = 0) \).

The release of energy by the decaying binary is a source of heating for the background. For our choice of parameters, however, the relative enhancement of the dispersion velocity is negligible, being \( \Delta \sigma/\sigma \sim G M_1 M_2/\eta r_c/N m \sigma^2 \sim M_2/M_c \ll 1 \).

### 3.2 The Criterion for the Growth of the Eccentricity

In the decay of the orbit, energy and angular momentum losses to the field stars may lead either to the growth or the decrease in the eccentricity of the system, the effect having important consequences on the life-time of the binary. We here formulate two simple and general criterions for the growth of \( e \), the first applies to the motion in the harmonic potential and the second to the motion in the Keplerian potential.

(a) The Harmonic Potential

From equation (2.13) we can express the eccentricity in the following form:

\[ e^2 = \left( \frac{r_A - r_P}{r_A + r_P} \right)^2 \]

\[ = \left( \frac{E}{\Omega} - J \right) \left( \frac{E}{\Omega} + J \right)^{-1}. \] (3.16)

The time derivative of the eccentricity therefore reads

\[ \frac{de^2}{dt} = \frac{2}{\Omega} (\dot{E}J - E \dot{J}) \left[ \frac{E}{\Omega} + J \right]^{-2}. \] (3.17)

It is obvious that the sign of \( \dot{e} \) depends on the competition between the energy loss and the angular momentum loss, i.e., on the sign of \( \dot{E}J - E \dot{J} \). We are therefore led to introduce a general criterion for the growth of \( e \) using equation (3.17). If the condition

\[ \langle \dot{E}J - E \dot{J} \rangle > 0 \] (3.18)
is satisfied along the BH orbital evolution, the eccentricity will rise.

Substituting into equation (3.18) the expressions of the energy and angular momentum losses (eqs.[3.4] and [3.5]) it is straightforward to verify that $\langle \dot{e} \rangle = 0$. The eccentricity remains thus frozen to the value set at the outset of the BH evolution. This holds if only the lowest order in $x$ is retained in equation (2.5). If we include the next leading term in the expansion of $g(x)$ we obtain

$$
\langle \dot{E}J - E\dot{J} \rangle = \frac{\chi}{20\sqrt{2\pi}\sigma^3} J\Omega^2 \left( r_A^2 - r_P^2 \right)^2.
$$

(3.19)

The r.h.s. of equation (3.19) is definite positive. We thus find that the eccentricity increases during the first stage of the orbital motion of the secondary BH.

We can also calculate the rate of increase of $e$

$$
\langle \dot{e} \rangle = \frac{1}{5} \frac{\chi}{\sqrt{2\pi}\sigma^3} \Omega^2 \left( r_A^2 - r_P^2 \right)
$$

(3.20)

and the associated time scale

$$
\tau_H = 5\sqrt{2\pi}\tau_{DF} \frac{r_c^2}{(r_A + r_P)^2}
$$

$$
\sim 6 \times 10^7 \left( \frac{r_c}{100\text{pc}} \right)^2 \left( \frac{70\text{pc}}{r} \right)^2 \text{yrs.}
$$

(3.21)

Note that $\tau_H$ exceeds $\tau_{DF}$ by a factor $\sim (r_c/r_A)^2$: This indicates that the rise of $e$ is not severe in this evolutionary phase as it occurs on a time scale longer than the time for energy and angular momentum losses.

(b) The Keplerian Potential

In the region $r_{cusp} < r < r_B$, the orbital motion of the secondary BH is described by the equations (2.18) and (2.19). They relate $e$ to $J$ and $E$ to give

$$
J^2 = -\mu^3 \frac{G^2 M^2}{2E} (1 - e^2).
$$

(3.22)

We can thus derive the expression for the time derivative of the eccentricity

$$
\dot{e} = -\frac{1 - e^2}{e} \left( \frac{\dot{J}}{J} + \frac{1}{2} \frac{\dot{E}}{E} \right).
$$

(3.23)

Similar to the previous case, we can introduce a general criterion for the growth of the $e$. Using equation [3.23] the condition is

$$
\frac{\langle \dot{E} \rangle}{\langle J \rangle} < -\frac{2E}{J}.
$$

(3.24)
Below, we apply this criterion of growth using the results of § 3.1.

(i) The limit \( x \ll 1 \)

Equations (3.8) and (3.9) indicate that the eccentricity is constant along the motion (i.e., \( \langle \dot{e} \rangle = 0 \)) since \( \langle \dot{E} \rangle / \langle J \rangle = -2E/J \) sharply. This result holds if only the lowest order in \( x \) is retained in equation (2.5). If we include the next leading term in the expansion of \( g(x) \), the long term changes of the energy and angular momentum are found to be characterized by a ratio

\[
\frac{\langle \dot{E} \rangle}{\langle J \rangle} \simeq -\frac{2E}{J} \times \left[ 1 - \frac{3}{20\pi\sigma^2} \frac{GM}{a} (1 - e^2)^{-1/2} I_2 \right]
\times \left[ 1 - \frac{3}{20\pi\sigma^2} \frac{GM}{a} (1 - e^2)^{1/2} I_1 \right]^{-1},
\]

(3.25)

that depends now explicitly on \( e \) (the integrals \( I_1 \) and \( I_2 \) are given in Appendix B). From equation (3.18), the criterion for the “growth” of \( e \) is fulfilled if

\[
1 < \frac{1}{1 - e^2} \frac{I_2(e)}{I_1(e)} = \frac{4 - 3(1 - e^2)^{1/2}}{(1 - e^2)^{1/2}}.
\]

(3.26)

This inequality holds for \( e^2 > 0 \), and equation (3.26) is thus satisfied \( \forall e \in [0, 1] \). In the limit \( x \ll 1 \) we thus find that DF on the secondary BH acts so as to increase the eccentricity, during orbit decay. The time scale for the growth of \( e \) can be estimated considering the evolution equation for the eccentricity.

Following Bertotti and Farinella (1990), we compute the secular change of \( e \) (see Appendix C). After some calculations, we are able to find a simple analytical expression

\[
\langle \dot{e} \rangle = \frac{8\chi}{5(2\pi)^{1/2}\sigma^3} \frac{GM}{a} \frac{(1 - e^2)^{1/2}}{e} \left[ 1 - (1 - e^2)^{1/2} \right].
\]

(3.27)

Note that \( \langle \dot{e} \rangle \) is positive definite for any value of \( e \), and the characteristic time of growth for \( e \) is

\[
\tau_{\langle \dot{e} \rangle} \sim \frac{e}{\langle \dot{e} \rangle} = \frac{5\pi^{1/2}}{8} \frac{\tau_{\text{DF}}}{x_{\text{cir}}^2} \frac{e^2}{2(1 - e^2)^{1/2} \left[ 1 - (1 - e^2)^{1/2} \right]^{\text{yrs}}} \\
\sim 7 \times 10^8 \left( \frac{0.3}{\chi_{\text{cir}}} \right)^2 \frac{e^2}{2(1 - e^2)^{1/2} \left[ 1 - (1 - e^2)^{1/2} \right]^{\text{yrs}}}.
\]

(3.28)

The behavior of \( \tau_{\langle \dot{e} \rangle} \) is reported in Figure 2. \( \tau_{\langle \dot{e} \rangle} \) is found to increase as \( e \to 1 \) but the deviation is significant only for extreme values of \( e \), very close to unity. This time scale is longer than the scale on which energy and angular momentum are lost (eq.
[3.10]), and the dilution factor is \( \sim 1/x_{\text{cir}}^2 \). Hence, only a weak rise of \( e \) is expected to take place, in the early phase of binary decay.

(ii) The limit \( x \gg 1 \)

In this limit, the ratio \( \langle \dot{E} \rangle / \langle \dot{J} \rangle \) reads

\[
\frac{\langle \dot{E} \rangle}{\langle \dot{J} \rangle} = \left( \frac{GM}{a^3} \right)^{1/2} \frac{\mathcal{K}(e)}{Z(e)} = -\frac{2E}{J} \left[ \frac{(1-e^2)^{1/2}}{Z(e)} \right].
\]

(3.29)

Considering the dependence of \( \mathcal{K}(e) \) and \( Z(e) \) (drown in Figure B1 and B2 of Appendix B) we find that criterion (3.24) is satisfied \( \forall e \). As a result, \( e \) increases since angular momentum is lost by DF more effectively than energy, as for the opposite limit.

In particular, at small eccentricities \( (e \ll 1) \) the ratio \( \langle \dot{E} \rangle / \langle \dot{J} \rangle \) takes a simple form that we derive in Appendix B:

\[
\frac{\langle \dot{E} \rangle}{\langle \dot{J} \rangle} \simeq -\frac{2E}{J} (1 - 3e^2) < -\frac{2E}{J}.
\]

(3.30)

From equations (C.1) and (2.6), the long term change of \( e \) is governed by the following equation (for the details see Appendix C)

\[
\langle \dot{e} \rangle = \frac{2\chi \sigma^2}{\pi} \left( \frac{a}{GM} \right)^{3/2} \mathcal{W}(e),
\]

(3.31)

where \( \mathcal{W}(e) \) is a positive function given in Appendix C.

\( \mathcal{W}(e) \) takes a very simple form in the case \( e \ll 1 \)

\[
\mathcal{W}(e) = \frac{3}{2} e \left[ 1 + \frac{3}{8} e^2 + o(e^2) \right].
\]

(3.32)

The associated time scale is

\[
\tau_e \sim x_{\text{cir}}^3 \tau_{\text{DF}} \frac{e}{\mathcal{W}(e)}
\]

\[
\sim 4 \times 10^8 \left( \frac{x_{\text{cir}}}{3} \right)^3 \frac{e}{\mathcal{W}(e)} \text{ yrs}.
\]

(3.33)

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The behaviour of \( \tau_\text{e} \) is shown in Figure 3. At small eccentrics the time scale is

\[
\tau_\text{e} \sim \frac{1}{3} x_{\text{cir}}^3 \tau_{\text{DF}} \left[ 1 - \frac{3}{8} e^2 + o(e^2) \right].
\]

Contrary to the case for \( x \ll 1 \), the evolution time is found to decrease with increasing \( e \): Thus, eccentric orbits become more eccentric on a progressively shorter time scale: We may refer to this effect as “runaway” increase of \( e \).

(iii) \( x \sim 1 \)

We have studied so far the long term evolution of \( E, J \) and \( e \) in the two asymptotic limits of \( x \). To complete our analysis we explore numerically the evolution equations (3.1) and (3.2) using the complete expression of \( g(x) \).

The long term evolution of \( E \) and \( J \) when \( \langle x \rangle \simeq 1 \) is found to be intermediate the two bracketing limits. Accordingly, we find that the eccentricity increases during orbit decay since angular momentum is always lost more effectively than energy. Whereas in the limit \( \langle x \rangle \gg 1 \) the losses at pericenter are weaker than at apocenter (eq.[2.6]) the opposite holds when \( \langle x \rangle \ll 1 \) (eq.[2.5]). Nonetheless also in the last case, \( \langle \dot{e} \rangle \) maintains the same sign. Pervious arguments based on comparisons between losses at different orbital phases were used to derive conclusive statements for the growth of \( e \); the above finding thus shows the weakness of that approach.

We summarize the results of numerical integrations in Figure 4, where \( \langle \dot{E} \rangle/\langle \dot{J} \rangle \) in units of \( -2E/J \) is shown as a function of \( e \), for selected values of \( x_{\text{cir}} \). It is remarkable that regardless the specific value of \( x \), this ratio maintains below unity over the complete domain \([0,1]\[, and becomes smaller with either increasing eccentricity (close to unity) or increasing \( x \). This is in accordance with the behavior of \( \tau_\text{e} \) (eqs. [3.28], and [3.33]).

4. POSSIBLE EVOLUTIONARY PATHS FOR THE BH BINARY

In §3 we have shown that the BH eccentricity increases slowly in the region \( r_B < r < r_c \) where the gravitational field of the core dominates. As a consequence, about \( r_B, e \) has a value close to the one acquired about \( r_c \). The orbital parameters thus keep memory of the previous BH evolution in regions \( r > r_c \). The action of DF at radii \( r > r_c \) where the density of the stellar distribution decreases nearly as \( r^{-2} \) have been investigated numerically in PR. In this work it is shown that the BH reaches \( r_c \) with
a small eccentricity: It follows that the BH trajectory is “prepared” to have \( e \ll 1 \) at \( r_B \).

Starting from these considerations, in this section we explore the long term evolution of the secondary BH and in particular the evolution of the eccentricity \( e(E,J) \) in the “inner” core region where the “instability” that drives a bound orbit to nearly radial infall \( (e \rightarrow 1) \) may set in.

Consistency with the model assumptions of a uniform star background requires the distance of closest approach not to drop below \( r_{\text{cusp}} \), around \( M_1 \) (i.e., around the focus of the ellipse). In an equivalent way, the apocenter should not exceed \( r_B \) for the motion to be consistent with the Keplerian potential. These constraints lead to the following inequalities (for \( a \) and \( e \))

\[
a (1 + e) < \eta^{1/3} r_c \quad \quad (4.1)
\]

\[
a (1 - e)(1 + e) \frac{e}{e} > \eta r_c \quad \quad (4.2)
\]

For sufficiently high values \( e \) and binding energy \( -E \), the BH can probe the region within the cusp where density gradients are present and where \( \sigma \approx v \); this region is not accessible to our model.

### 4.1 Selected Paths

We begin the numerical integration of the evolution equations (3.1-3.2) at a distance \( a_i = 20 \text{pc} \) (for which \( \langle x_i \rangle \approx 0.3 \)) and choose a value of \( \eta \approx 0.05 \) (following BBR). Along the course of the BH evolution \( \sigma \) is kept constant (see §3.1).

The eccentricity is treated as a free parameter; small initial values are preferred being consistent with the previous BH history (PR). In the evolution equations, the drag by the ambient stars is computed using the complete expression of \( g(x) \) (eq.[2.3]). The means (eq. [3.3]) are evaluated numerically and a test on the accuracy of the calculation is performed using the analytical results obtained in the asymptotic limits.

Figure 5 shows possible paths for \( e \) as a function of the semimajor axis \( a \) which can be considered in place of the time coordinate. It appears that, independent of the initial value of the eccentricity \( e_i \), the growth of \( e \) with decreasing \( a \) is not severe as long as the BH velocity does not exceed the dispersion velocity of the ambient stars (i.e., until \( \langle x \rangle < 1 \)). At separations where \( \langle x \rangle \sim 1 \), the value of \( e \) still depends on \( e_i \) and the binary keeps memory of the initial condition. The “runaway growth” of \( e \) occurs only as the mean value of \( \langle x \rangle \) increases just above unity. In Figure 5, the function \( e(a) \) is tracked also for \( a < r_{\text{cusp}} \) (dashed line). In this case the pericenter distance \( r_P \) falls within the cusp radius, and the above inequalities (eqs. [4.1] and
[4.2]) are violated. The path below \( r_{\text{cusp}} \) is shown just to indicate how rapid would be the growth of \( e \) if the BH were to move in a uniform medium even below \( r_{\text{cusp}} \).

We now consider an evolutionary path \( a(t) \) having \( a_i = 20 \) pc, and a small \( e_i = 0.03 \), to follow the evolution of the pericenter \( r_p(t) \equiv a (1 - e) \) and apocenter \( r_A(t) \equiv a (1 + e) \) distances with time. According to the analysis of §3 we find that in the “inner” core region where the BH velocity \( \langle v \rangle < \sigma \), energy and angular momentum are lost to the background stars exponentially, on a time scale \( \tau_{\text{DF}} \sim 10^7 \) yrs. In this phase, the semimajor axis decreases exponentially but \( e \) does not vary appreciably having a longer time scale of growth \( (\propto \tau_{\text{DF}}/\langle x \rangle)^2 \); cfr. eq. [3.28]): Accordingly \( r_p(t) \sim r_A(t) \sim a(t) \). The binary evolves rapidly through this early phase, until \( \langle v \rangle \sim \sigma \). The dynamical evolution will proceed from this moment onward on a different time scale \( \sim \langle x \rangle^3 \tau_{\text{DF}} \sim 10^8 \) yrs, since DF \( (\propto v^{-2}) \) weakens as the binary hardens. The increase of the stellar density (by a factor \( \sim 2.5 \) for a density profile \( \sim r^{-7/4} \); see Frank & Rees 1976) would however partly compensate this delay in the BH evolution.

In summary, our idealized study shows that we can distinguish two evolutionary phases for the BH “binary”: (1) an early phase, lasting a few \( 10^7 \) yrs, in which the inspiral of the two BHs occurs along bound orbits of slowly increasing eccentricity (the binary is “nearly stable” to the rise of \( e \)). (2) a subsequent phase, not described by our model, in which the BH probes the cusp region at pericenter, and the uniform core at apocenter. The non-local nature of DF prevents us to give a correct estimate of the evolution of the eccentricity, during the BH inspiral. According to recent studies, the eccentricity would remain nearly constant in the region below the cusp where stars can still be treated as a continuum medium (see PR). Where collisions with single stars come into play, \( e \) can instead increase again, as indicated in numerical experiments by Roos (1981) and Hills (1983). Hence, the evolution of \( e \) depends sensitively on the detail of the interaction with stars in the innermost region of the galactic core, before losses by gravitational waves drive the binary to final plunge.

5. DYNAMICAL FRICTION AND GRAVITATIONAL WAVES

In the previous sections we have shown that in a homogeneous isotropic background of stars, DF acts so as to bind the binary while progressively rising \( e \). Eventually, energy and angular momentum losses by gravitational wave (GW hereafter) emission will determine the terminal evolution of the binary. These losses depend sensitively on \( e \): For the same energy \( E \), a binary with \( e \sim 0.9 \) suffers a loss a factor 1000 larger.
than for a binary with $e \sim 0$ (Peters & Mathews 1963). Due to this dependence, the transition distance $a_t$ at which the energy loss by DF equals the power emitted via GW radiation in a binary is a function of $e$. $a_t(e)$ can be estimated setting $\langle \dot{E}_{\text{DF}} \rangle \simeq \langle \dot{E}_{\text{GW}} \rangle$ where

$$\langle \dot{E}_{\text{GW}} \rangle = \frac{32 G^4 M_5^5}{5 c^5} q^2 (1 + q) f(e)$$

is the energy loss by GWs (Peters & Mathews 1963); $q \equiv M_2/M_1$ is the mass ratio and the function $f(e)$ reads

$$f(e) = 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \left(1 - e^2\right)^{7/2}. \quad (5.2)$$

Considering equations (5.1) and (3.11), the equality in the energy losses leads to

$$a_t(e) = \left(\frac{16}{5} \frac{G^{5/2}}{c^5 \ln \Lambda n_o m}\right)^{2/11} \left[\frac{f(e)}{K(e)}\right]^{1/2} \left[\frac{1}{2} \frac{M_7}{r_{11/4} c} \right]^{1/2} \left(1 + q\right)^{7/4}. \quad (5.3)$$

Note that $a_t(e)$ depends on $e$ through $f(e)$ and $K(e)$ and increases with $e$.

The expression of $a_t$ is consistent with our treatment of DF only for $a_t(e) \geq \eta r_c$. This inequality translates into a limit on the parameter $\eta$ as a function of $e$, $q$ and of the core parameters:

$$\eta < \left(\frac{16}{5} \frac{G^{5/2}}{c^5 \ln \Lambda n_o m}\right)^{1/2} \left[\frac{f(e)}{K(e)}\right]^{1/2} \left[\frac{M_7}{r_{11/4} c} \right]^{1/2} \left(1 + q\right)^{7/4}. \quad (5.4)$$

$\eta$ depends sensitively on $r_c$ and increases with increasing $e$. However, for typical parameters of a galaxy core, we find that $\eta \simeq 10^{-7}$. In a real scenario DF is effective also at smaller radii and the actual value of the transition radius depends sensitively on the dissipative mechanisms for energy and angular momentum transfer by the ambient stars (see BBR).

### 5.1 Braking index

The simultaneous presence of the dissipative forces of DF and GW reaction close to $a_t$ leads, in this idealized scenario, to the equation for the evolution of the semimajor axis $a$

$$\dot{a} = -\frac{64 G^3 M_2^2 \mu}{5 c^5 a^3} f(e) - a \frac{\chi \sigma^2 \mu^{3/2} K(e)}{2^{1/2} \pi} \frac{E^{3/2}}{E^{3/2}} \quad (5.5)$$

derived using equation (5.1), and (3.11) for $x \gg 1$. 20
This evolution equation can formally be written as

\[ \dot{a} = -\frac{\alpha}{a^3} \left( 1 + \frac{\gamma}{\alpha} a^{11/2} \right), \]  

(5.6)

where

\[ \alpha = \frac{64}{5} \frac{G^3}{c^5} M^2 \mu f(e) \]  

(5.7)

and

\[ \gamma = \frac{2 \chi \sigma^2}{\pi (GM)^{3/2}} \mathcal{K}(e). \]  

(5.8)

With use of Kepler’s third law, we can relate the evolution of \( a(t) \) to the frequency \( f(t) = 1/P \) of orbital motion of the system:

\[ f = \frac{1}{2\pi} \left( \frac{GM}{a^3} \right)^{1/2}. \]  

(5.9)

Due to orbit decay, we further have from (5.8) and (5.9)

\[ \dot{f} = \frac{3}{2\pi} \sqrt{GM} \left( \frac{\alpha}{a^{11/2}} + \gamma \right), \]  

(5.10)

\[ \ddot{f} = \frac{33}{4\pi} \sqrt{GM} \frac{1}{a^{13/2}} \left( \frac{\alpha}{a^3} + \gamma a^{5/2} \right). \]  

(5.11)

We can thus introduce the braking index \( n \)

\[ n = \frac{f \ddot{f}}{\dot{f}^2}, \]  

(5.12)

that takes the following form

\[ n = \frac{11}{3} \left[ 1 + \frac{\gamma}{\alpha} a^{11/2} \right]^{-1}. \]

\[ = \frac{11}{3} \left[ 1 + \left( \frac{a}{a_t(e)} \right)^{11/2} \right]^{-1}, \]  

(5.13)

where \( n_{GW} = 11/3 \) is the value of the index in absence of DF.

In principle, a measurable deviation from 11/3 would provide an indirect way for testing the ambient star medium. In Figure 6 the braking index \( n \) (eq. [5.13]) is shown as a function of the BH binary separation for various \( e \). It appears that a large deviation from 11/3 occurs close to \( a_t \), where the two forces are competing. Above \( a_t \) the braking index is quite small since DF is the cause of the deceleration of the orbit. Below \( a_t \) on the contrary the index is close to \( n_{GW} \), since GW losses dominate.
Recently, Giampieri (1993) considered the deviation from the value $n_{GW}$ induced by mass transfer. The correction term in that case has a sign that depends on the mass difference of the two BHs and on the details of mass transfer. The calculation of $n$ is here proposed as example for suggesting a way to probe the environment of a BH binary. The validity of such approach depends however on the relevance of the process considered in the region where an outgoing GW signal is detectable. For the case of DF, the GW signal about $a_t$ would be still very weak and of extremely low-frequency. The possibility of revealing GWs about $a_t$ is thus quite remote. On the contrary, mass transfer processes occurring close to the gravitational radius of the BHs in the binary, could appear in the measure of $n$ as deviation from the pure gravitational effect.

At low enough BH separations the energy loss to the field stars via single encounters lasting less than a period $P$, might produce a sudden rise in the GW frequency $f$, with a $\Delta f/f \sim m/M$ for an impact parameter $\sim a$. For smaller impact parameters, of the order of the gravitational radius $R_G(M_2) = 2GM_2/c^2$, the “glitch” might have a magnitude $\Delta f/f$ as large as $\sim (m/M)a/R_G(M_2)$. The potential observability of these phenomena would be of considerable importance.

6. CONCLUSIONS

We have explored the long term evolution of a black hole pair under the action of dynamical friction by a stationary homogeneous and isotropic background of light particles. The result applies to the case of a BH pair in the center of a galactic nucleus, in regions where the stellar distribution can be approximated as homogeneous.

Future work should focus on the study of DF in a nonuniform anisotropic star background with the aim at exploring the subsequent binary evolution (BBR; PR).

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APPENDIX A

In this Section we outline the analytical method used to calculate $\langle \dot{S} \rangle$ in the case of the Keplerian potential. We proceed along three steps:

(1) First we specify the instantaneous motion of the secondary BH, described by the velocity vector $v$ and the separation vector $r$. Since DF is a weak perturbing force, the motion is determined by the driving acceleration force of the BH primary (see § 2). $v$ and $r$ are expressed in term of the orbital phase $\psi$: They are defined uniquely by the instantaneous values of $E, J, a$ and $e$. For $r \equiv r/\hat{r}$ we have

$$r = \frac{a(1 - e^2)}{1 + e \cos \psi}.$$

(A.1)

The velocity is decomposed along the radial and tangential directions to yield the following expressions:

$$v_r = \left[ \frac{GM}{a(1 - e^2)} \right]^{1/2} e \sin \psi,$$

(A.2)

$$v_t = \frac{J}{\mu r} \left[ \frac{GM}{a(1 - e^2)} \right]^{1/2} (1 + e \cos \psi).$$

(A.3)

Furthermore,

$$v^2 = \frac{GM}{a(1 - e^2)} (1 + 2e \cos \psi + e^2).$$

(A.4)

(2) Secondly, the perturbing force $\dot{v}_{\text{DF}}$ is decomposed along three orthonormal directions defined by the BHs separation vector $r$, the velocity $v$ and $J$. These components are indicated with $R$ acting along $r$, $W$ along $J$, and $T$ in the orbital plane. The three components of the frictional deceleration given by equation (2.1) are accordingly to Bertotti & Farinella (1990) (BF hereafter)

$$R = \dot{v}_{\text{DF}} \cdot \frac{r}{r}$$

$$= -\chi g(x) \frac{v_r}{v}$$

$$= -\chi g(x) \frac{e \sin \psi}{(1 + 2e \cos \psi + e^2)^{1/2}},$$

(A.5)
\[ T = \mathbf{v}_{DF} \cdot \frac{\mathbf{J} \times \mathbf{r}}{Jr} \]
\[ = -\mu \frac{\chi g(x) v^2 r}{v} \]
\[ = -\chi g(x) \frac{1 + e \cos \psi}{(1 + 2e \cos \psi + e^2)^{1/2}}, \tag{A.6} \]

\[ W = \mathbf{v}_{DF} \cdot \frac{\mathbf{J}}{J} \]
\[ = 0. \tag{A.7} \]

In particular, for \( x \ll 1 \) we have:

\[ R = -\frac{4\chi}{3(2\pi)^{1/2}\sigma} \left[ \frac{GM}{a(1 - e^2)} \right]^{1/2} e \sin \psi \]
\[ + \frac{2\chi}{5(2\pi)^{1/2}\sigma^3} \left[ \frac{GM}{a(1 - e^2)} \right]^{3/2} e \sin \psi (1 + 2e \cos \psi + e^2) \]
\[ + o(x^3), \tag{A.8} \]

\[ T = -\frac{4\chi}{3\sqrt{2\pi}\sigma} \left[ \frac{GM}{a(1 - e^2)} \right]^{1/2} (1 + e \cos \psi) \]
\[ + \frac{2\chi}{5(2\pi)^{1/2}\sigma^3} \left[ \frac{GM}{a(1 - e^2)} \right]^{3/2} (1 + e \cos \psi)(1 + 2e \cos \psi + e^2) \]
\[ + o(x^3), \tag{A.9} \]

and for \( x \gg 1 \):

\[ R = -2\chi \sigma^2 \frac{a(1 - e^2)}{GM} \frac{e \sin \psi}{(1 + 2e \cos \psi + e^2)^{3/2}} \tag{A.10} \]

\[ T = -2\chi \sigma^2 \frac{a(1 - e^2)}{GM} \frac{1 + e \cos \psi}{(1 + 2e \cos \psi + e^2)^{3/2}}, \tag{A.11} \]

(3) As third step, we express \( \dot{E}, \dot{J} \) and \( \dot{e} \) in terms of \( R \) and \( T \). Following closely BF we have

\[ \dot{E} = \mu \mathbf{v}_{DF} \cdot \mathbf{v} \]
\[ = \frac{GM}{J} \left[ Re \sin \psi + T(1 + e \cos \psi) \right], \tag{A.12} \]
\[ \dot{J} = \mu (r \times \dot{v}_{DF}) \cdot \frac{J}{J} = \mu r T, \quad (A.13) \]

and

\[ \dot{e} = \frac{P(1 - e^2)^{1/2}}{2\pi a} \left[ R\sin \psi + T \frac{e \cos^2 \psi + 2 \cos \psi + e}{1 + e \cos \psi} \right], \quad (A.14) \]

with \( P \) the Keplerian period. The derivatives written above are functions of the orbital phase \( \psi \). In Appendix B we carry out their mean, over the orbital motion.
APPENDIX B

In this Appendix we sketch the derivation of the equations for the long term evolution of $E$ and $J$. The calculations involve the explicit integration of equations (A.12) and (A.13) over the orbital phase $\psi$. Below we indicate the main steps.

Along a Keplerian orbit, the phase $\psi$ is related to the time coordinate by the relation

$$dt = \frac{P}{2\pi}(1 - e^2)^{3/2} \frac{1}{(1 + e \cos \psi)^2} d\psi.$$  \hspace{1cm} (B.1)

This equation is used to transform the time integral (3.3) to an integral over $\psi$. The mean variation of energy $E$ thus reads:

$$\langle \dot{E} \rangle = -\frac{\mu X}{2\pi} \left( \frac{GM}{a} \right)^{1/2} (1 - e^2) \int_0^{2\pi} g(x(\psi)) \frac{(1 + 2e \cos \psi + e^2)^{1/2}}{(1 + e \cos \psi)^2} d\psi; \hspace{1cm} (B.2)$$

equivalently for $J$:

$$\langle \dot{J} \rangle = -\frac{\mu X}{2\pi} a(1 - e^2)^{5/2} \int_0^{2\pi} g(x(\psi)) \frac{1}{(1 + 2e \cos \psi + e^2)^{1/2}(1 + e \cos \psi)^2} d\psi. \hspace{1cm} (B.3)$$

In the two separate limits $x \ll 1$ and $x \gg 1$, the above integrals can be evaluated analytically. For arbitrary $x(\psi)$, the calculation is carried out numerically and Figure 4 summarizes the results of the integration. The analytical expressions in the two asymptotic limits were used as a test on numerical accuracy.

(i) $x \ll 1$

In the calculation that follows, the frictional drag $g(x)$ is given by equation (2.5). According to the expression of $g(x)$, we distinguish in $\langle \dot{E} \rangle$ the contribution associated to the first leading term ($\propto x$) in the expansion of $g(x)$ (indicated with label 1) and the subsequent term ($\propto x^3$) (indicated with label 3). The frictional drag $g(x)$ is known function of $\psi$ through $x$ given by equations (2.3) and (A.4). The resulting integral will formally read:

$$\langle \dot{E} \rangle = \langle \dot{E} \rangle_1 + \langle \dot{E} \rangle_3 + o(x^3). \hspace{1cm} (B.4)$$

As for the energy, the variation of $J$ splits in two terms that will be evaluated separately

$$\langle \dot{J} \rangle = \langle \dot{J} \rangle_1 + \langle \dot{J} \rangle_3 + o(x^3). \hspace{1cm} (B.5)$$
From equations (B.2), (B.3) and (2.5) we find:

\[ \langle \dot{E} \rangle_1 = \frac{2\chi}{3\pi(2\pi)^{1/2}a}(1 - e^2)^{1/2}I_1(e), \]  

(\text{B.6})

\[ \langle \dot{E} \rangle_3 = \frac{\chi\mu}{5\pi(2\pi)^{1/2}a^3}\left(\frac{GM}{a}\right)^2 \frac{I_2(e)}{(1 - e^2)^{1/2}}, \]  

(\text{B.7})

\[ \langle \dot{J} \rangle_1 = -\frac{2\chi\mu}{3\pi(2\pi)^{1/2}a}(GMa)^{1/2}(1 - e^2)^2I_0(e), \]  

(\text{B.8})

\[ \langle \dot{J} \rangle_3 = \frac{\chi\mu}{5\pi(2\pi)^{1/2}a^3}\left(\frac{G^3M^3}{a}\right)^{1/2} \frac{(1 - e^2)I_1(e)}{2}, \]  

(\text{B.9})

where

\[ I_n(e) = \int_0^{2\pi} \frac{(1 + 2e \cos \psi + e^2)^n}{(1 + e \cos \psi)^2}d\psi, \]  

(\text{B.10})

After some straightforward algebra we have:

\[ I_0(e) = \frac{2\pi}{(1 - e^2)^{3/2}}, \]  

(\text{B.11})

\[ I_1(e) = \frac{2\pi}{(1 - e^2)^{1/2}}, \]  

(\text{B.12})

\[ I_2(e) = 2\pi \left[ 4 - 3(1 - e^2)^{1/2} \right], \]  

(\text{B.13})

Substituting the values of \( I_0, I_1 \) and \( I_2 \) in (\text{B.6}), (\text{B.7}), (\text{B.8}) and (\text{B.9}) we obtain equation (3.8) for \( \langle \dot{E} \rangle_1 \), equation (3.9) for \( \langle \dot{J} \rangle_1 \) and

\[ \langle \dot{E} \rangle_3 = \frac{2\chi\mu}{5(2\pi)^{1/2}a^3}\left(\frac{GM}{a}\right)^2 \frac{4 - 3(1 - e^2)^{1/2}}{(1 - e^2)^{1/2}}, \]  

(\text{B.14})

\[ \langle \dot{J} \rangle_3 = \frac{2\chi\mu}{5(2\pi)^{1/2}a^3}\left(\frac{G^3M^3}{a}\right)^{1/2} \frac{(1 - e^2)^{1/2}}{2}, \]  

(\text{B.15})

respectively.

(ii) \( x \gg 1 \)

In the limit \( x \gg 1 \), equation (\text{B.2}) becomes:

\[ \langle \dot{E} \rangle = -\frac{\mu \chi \sigma^2}{\pi} \left(\frac{a}{GM}\right)^{1/2} K(e), \]  

(\text{B.16})
where
\[ K(e) = (1 - e^2)^2 \frac{1}{2\pi} I_{-1/2}(e). \] (B.17)

We can perform the integration analytically, expanding (B.17) about \( e \sim 0 \); it is found
\[ K(e) = \left[ 1 + \frac{3}{4} e^2 + \frac{21}{64} e^4 + o(e^4) \right]. \] (B.18)

The behaviour of \( K(e) \) is given in Figure B1.

Analogously for \( \langle \dot{J} \rangle \), from equation (B.3) we find:
\[ \langle \dot{J} \rangle = -\frac{\mu \chi}{\pi} \frac{a^2}{GM} Z(e), \] (B.19)

where
\[ Z(e) = (1 - e^2)^{7/2} \frac{1}{2\pi} I_{-3/2}(e). \] (B.20)

Expanding equation (B.20) for \( e \ll 1 \) we have:
\[ Z(e) = \left[ 1 + \frac{13}{4} e^2 + \frac{157}{64} e^4 + o(e^4) \right]. \] (B.21)

The behaviour of \( Z(e) \) is given in Figure B2.

APPENDIX C

Now we derive the formula of the growth of the eccentricity. Substituting in equation (A.14) the equations (A.5) and (A.6), and subsequently averaging we find:
\[ \langle \dot{e} \rangle = -\frac{\chi}{\pi} (1 - e^2)^2 \left( \frac{a}{GM} \right)^{1/2} \int_0^{2\pi} g[x(\psi)] \frac{e + \cos \psi}{(1 + 2e \cos \psi + e^2)^{1/2}(1 + e \cos \psi)^2} d\psi. \] (C.1)

(i) \( x \ll 1 \)

Retaining, as in the case of \( E \) and \( J \), terms to order \( x^3 \) in \( g[x(\psi)] \), we write \( \langle \dot{e} \rangle \) in the form:
\[ \langle \dot{e} \rangle = \langle \dot{e} \rangle_1 + \langle \dot{e} \rangle_3 + o(x^3). \] (C.2)
Using the asymptotical expression of $g(x)$, we obtain:

\[
\langle \dot{e} \rangle_1 = -\frac{4\chi}{3\pi(2\pi)^{1/2}\sigma}(1 - e^2)^{3/2}Q_0(e), \quad (C.3)
\]

\[
\langle \dot{e} \rangle_3 = \frac{2\chi}{5\pi\sigma^3(2\pi)^{1/2}}\left(\frac{GM}{a}\right)(1 - e^2)^{1/2}Q_1(e), \quad (C.4)
\]

where

\[
Q_n(e) = \int_0^{2\pi} \frac{(e + \cos \psi)(1 + 2e\cos \psi + e^2)^n}{(1 + e\cos \psi)^2} d\psi. \quad (C.5)
\]

After some straightforward algebra we obtain:

\[
Q_0(e) = 0 \quad (C.6)
\]

and

\[
Q_1(e) = \frac{4\pi}{e}\left[1 - (1 - e^2)^{1/2}\right]. \quad (C.7)
\]

(ii) $x \gg 1$

Substituting equation (2.6) into (C.1) we obtain:

\[
\langle \dot{e} \rangle = 2\chi\frac{\sigma^2}{\pi}\left(\frac{a}{GM}\right)^{3/2}W(e), \quad (C.8)
\]

where

\[
W(e) = -(1 - e^2)^3\frac{1}{2\pi}Q_{-3/2}(e). \quad (C.9)
\]

Expanding $W(e)$ for $e \ll 1$ we find:

\[
W(e) = \frac{3}{2}e\left[1 + \frac{3}{8}e^2 + o(e^2)\right]. \quad (C.10)
\]

The behaviour of $W(e)$ is given in Figure C1.
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FIGURE CAPTIONS

**Figure 1.** Time scale $\tau_> \ (\text{in units of } 10^8 \ \text{yrs})$ describing the orbital decay of the secondary BH in the Keplerian potential of the primary, as a function of $e$ for $x \gg 1$. Top curve denotes the time scale for energy loss $\tau^E_>$ (eq. [3.14]); bottom curve denotes the time scale for angular momentum loss $\tau^J_>$ (eq. [3.15]).

**Figure 2.** Time scale for the growth of the eccentricity $e$, $\tau^e_<$ (in units of $10^8\text{yrs}$) as a function of $e$, for $x \ll 1$. The secondary BH is in the Keplerian potential of $M_1$.

**Figure 3.** Time scale for the growth of the eccentricity $e$, $\tau^e_<$ (in units of $10^8\text{yrs}$) as a function of $e$, for $x \gg 1$. The secondary BH is in the Keplerian potential of $M_1$.

**Figure 4.** $\langle \dot{E} \rangle / \langle J \rangle$ in units of $-2E/J$ as a function of $e$ for selected values of $x_{\text{cir}}$ (0.1, 0.5, 1, 10) summarizing the results of the eccentricity increase for the orbital evolution of the secondary BH in the Keplerian potential of $M_1$. Dots indicate the analytical expression of $\langle \dot{E} \rangle / \langle J \rangle$ as given by eq. [3.25] (top curve) and eq. [3.29] (bottom curve).

**Figure 5.** The eccentricity evolution for paths with initial semimajor axis $a_i = 20$ pc. The curves give $e$ as a function of $a$ (in pc) for initial values of the eccentricity $e_i = 0.05, 0.03, 0.01$ starting from the top respectively. The dashed lines indicate evolutionary paths with pericenter distance $r_p < r_{\text{cusp}}$.

**Figure 6.** Braking index $n$ as a function of the semimajor axis $a$ in unit of $a_i(e = 0)$, as defined by eq. (5.13) for selected values of $e = 0, 0.4, 0.7$ starting from left, respectively.

**Figure B1.** $K$ versus $e$, as defined by eq. (B.17): dots denote the analytical expansion (eq. [B.18]).

**Figure B2.** $Z$ versus $e$, as defined by eq. (B.20): dots denote the analytical expansion (eq. [B.21]).

**Figure C1.** $W$ versus $e$, as defined by eq. (C.9): dots denote the analytical expansion (eq. [C.10]).