Abstract—In this work we establish a simple yet effective strategy, based on optimal transport theory, for enabling a group of robots to accomplish complex tasks, such as shape formation and assembly. We demonstrate the feasibility of this approach and rigorously prove collision avoidance and convergence properties of the proposed algorithms.

Index Terms—Path planning, multi-agent systems, optimal transport, intermittent diffusion

I. INTRODUCTION

Motion planning for multi-robot systems has drawn significant attention in recent years due to the emergence of a number of new application scenarios, e.g., [1], [2]. Compared to single robot systems, multi-robot systems have many benefits, including spatial distribution, efficiency and robustness at completing a task due to division of labor, localization, information-sharing, redundancy, and potentially lower cost. On the other hand, motion planning for multi-robot systems must address significant challenges, such as collisions, deadlock due to the presence of local minima in the multi-objective functions from which the controllers are derived, and uncertainty introduced from the environment and stochastic effects in the system [3]. Computationally, the path planning problem can be NP-hard and not solvable in polynomial time even for some two dimensional cases [4]. Furthermore, all of these difficulties are exacerbated when the robots are limited in capabilities, for example, short range communications. In this paper we propose a robust method for multi-robot motion planning that enables groups of simple robots to avoid both collision and congestion without costing significant resources.

There is a vast literature for path planning that spans widely known methods, including graph based approaches such as A*, D*, or D*-lite, [5], [6], [7], [8], [9], [10], [11], [12], randomized algorithms such as Probabilistic Road Maps (PRM) [13], [14], [15], [16], and recently, tree-search algorithms, including Rapidly-exploring Random Tree (RRT) [17], [18], [19], [20], [21]. These methods find trajectories, often optimal ones, by generating feasible paths defined by nodes on a lattice or random tree that characterizes the space of possible configurations.

Artificial potential field (APF) methods, proposed in [22], formulate the shape-formation problem as a problem of minimizing a potential composed of an attractive field, based on the desired shape, and a repelling field based on obstacles. Designed originally for single-robot trajectories [23], [24], these theories and methods have been extended and improved upon over the past several decades, including the addition of simulated annealing and an extension to dynamic environments [25], [26], [27], [28], [29]. APF methods handle higher dimensional problems efficiently, however, a potential limitation of APF is the presence of local minima caused by the repelling forces of obstacles, leading to potential deadlocks.

Much progress has been made in adapting existing methods to cooperative path planning problems for relatively small groups of robots [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42] or the design of cooperative motion strategies without explicit preplanning of optimal paths [43], [44], [45], [35].

In recent work inspired by statistical physics, rigorous error estimates have been obtained between partial differential equations (PDEs) that model the swarm dynamics and the target distribution, enabling desired coverage performance [46], [47]. In addition, there are also other stochastic methods for path planning and control [48], [49], [50].

In this paper, we advocate designing path planning methods for multi-robot systems by using the theory of optimal transport [51], [52] that has been successfully applied to problems in optics, econometrics, and computer graphics [53], [54], [55], [56], [57], [58], to name a few. In this paper, we produce a new method for...
multi-robot path planning that controls the group dynamics using carefully designed potentials and stochasticity. Connecting the ideas of optimal transport to large multi-robot systems, we show that desired convergence can be guaranteed. Furthermore, our approach overcomes the problem of local minima and deadlocks by employing intermittent diffusion, as in the method of evolving junctions (MEJ) [59], [60].

The proposed method differs from similar applications of optimal transport to robot path-planning [61], [62] in that the resulting Stochastic Differential Equations (SDEs) can be solved using robust numerical methods and executed in real time. Furthermore, although our method shares a lot of similarities with APF, there are key differences. In APF methods, the repelling fields from obstacles affect the potential everywhere in the domain, while in the proposed method, each robot is viewed as a dynamically moving obstacle to the other robots. As a result, its repelling effect is restricted to a small, local region. More importantly, intermittent random perturbations are added in order to avoid deadlock.

The paper is organized as follows. In §II we formulate the continuous problem in terms of a system of SDEs. The discretized problem is described in §III. In §IV we provide numerical simulations of the shape formation problem for different shapes and different size groups. We provide theoretical guarantees for global convergence of the system and collision avoidance, both in the continuous and discrete settings in §V.

II. MODEL SETUP

Suppose is a set of spatial locations that form a desired shape, and consider the trajectories of robots given by

where is a curve in describing the position of the th robot at time . The objective is to produce a path from an initial state to a final state such that while avoiding collisions, meaning for all and all . Our strategy is to design a modified gradient flow whose solution preserves the path for all robots.

In order to do so, we first introduce a shape function that attains its minimal value if . A convenient choice, among many candidates, is the distance function defined by,

where is the Euclidean norm: . Obviously, is a non-negative function achieving its minimum only when . Figure illustrates the level-sets of corresponding to two different target shapes.

We also introduce a penalty function that takes a large value when exhibits undesirable behavior. In multi-robot systems, one of the main objectives is to ensure that the trajectories are collision free, meaning the pairwise distances , for all . For example, we can select the penalty term as the following smooth, “repelling” function that peaks when the pairwise distances are small,

The function has the following properties

For example: , here, the constant is related to the sensing radius of each robot, and the constant is calibrated to achieve desirable dynamics, see [13] for details. Further constraints on the system, such as obstacle avoidance, can also be easily included. To simplify the presentation, we do not consider obstacle avoidance in this paper.

Combining the shape function with the penalty function , we obtain an energy functional

Then the trajectories of the robots are primarily generated by the gradient flow that minimizes , i.e.

\[
\frac{dX^i(t)}{dt} = - (\nabla \Psi(X(t)))
\]
Following it, the robots get to the desired shape when $F(X) = 0$, while minimizing $G(X)$ helps to spread out their locations in addition to avoid collisions.

However, the path generated by such a simple gradient flow may suffer a well known shortcoming that the solution can stuck at local minimizers. To overcome this limitation, we borrow ideas from intermittent diffusion [63], a stochastic strategy developed for global optimization. More precisely, we intermittently add random perturbations to (7), leading to the following SDEs,

$$
dY^i(t) = -\nabla \Psi(Y(t)) dt + \sigma(t) dW(t), \quad t > 0, \quad Y(0) = Y_0, $$  

where $W(t)$ is the standard Brownian motion in $\mathbb{R}^2$ and $\sigma(t)$ is a piecewise constant function alternating between zero and a positive value, i.e.

$$\sigma(t) = \begin{cases} 
0 & \text{if } t \in [S_k, T_k] \\
\sigma_k & \text{if } t \in [T_{k-1}, S_k]. 
\end{cases}$$

Here we partition $[0, T]$ as $\cup_{k=1}^{K} ([T_{k-1}, T_k])$ with $T_0 = 0$, $T_N = T$ and $S_k \in [T_{k-1}, T_k]$.

We want to highlight that the random perturbations are added to the gradient flow to avoid trajectories being trapped at local minimizers. Therefore, the constant $\sigma_k$ doesn’t have to be small. This is different from the choice used in simulated annealing, in which the corresponding coefficient, also called temperature, must go to zero asymptotically. The effectiveness of random perturbations can be verified by numerical experiments and comes with guarantees based on optimal transport theory. More precisely, the solution of (8) converges to the global minimizer in the probability sense according to the so called Gibbs distribution

$$\rho^*(X) = P^{-1} \exp(-2\Psi(X)/\sigma^2),$$

where $P = \int_{\mathbb{R}} \exp(-2\Psi(X)/\sigma^2) dx$.

In fact, the Gibbs distribution is an invariant measure of the system (8), and $\rho^*(X)$ takes the largest value when $\Psi(X)$ reaches its global minimum. Further details of the theory are provided in [64].

It is important to emphasize that the random portion of the solution $Y(t)$ when $t \in [T_{k-1}, S_k]$ is not used as the trajectories for the robots due to inefficient jittering motions. Instead, $Y(t)$ is only computed virtually to create the vector $Y(S_k)$, denoted as $\hat{Y}$ in the rest of the paper, of intermediate positions to move the robots to. Once this position $\hat{Y}$ is computed, we define another objective function

$$\hat{F}(X) = \frac{1}{N} \sum_{i=1}^{N} \|X^i - \hat{Y}^i\|^2.$$  

Using it together with $G(X)$, we create another gradient flow

$$\frac{dX^i(t)}{dt} = -\left(\nabla \left(\hat{F}(X(t)) + G(X(t))\right)\right)^i.$$  

In the end, the path $X(t)$ of the robots is generated by alternating between two gradient flows (7) and (13). The implementation of the method is given in the next section.

III. Implementation

The gradient flows and the SDEs presented in the previous section must be solved numerically when calculating...
the path. We employ the simple Euler scheme to do so in this paper. More precisely, we compute
\[ X^n_{n+1} = X^n_n - \Delta t (\nabla \Psi (X^n_n)), \tag{14} \]
where \( \Delta t \) is the step size, \( \Psi (X) \) takes \( F(X) + G(X) \) for \( \Psi \) and \( \hat{F}(X) + G(X) \) for \( \Psi \) respectively. The SDEs \( \Psi \) is discretized as
\[ Y^n_{n+1} = Y^n_n - \Delta t (\nabla \Psi (Y^n_n)) + \xi_n \sqrt{\Delta t}, \tag{15} \]
where \( \xi_n \in \mathbb{R}^2 \) is a normally distributed random vector generated at each iteration.

As mentioned in the previous section, the path is generated by alternating between (7) and (13). This is implemented by repeating a 2-step strategy. In the first step, the robots are moved, using (13), toward temporary destinations computed by a simulation of (15). After the temporary locations are reached, the second step has the robots follow (7) toward the desired shape. The robots then repeat the two steps until reaching the desired shape. Details are presented in Algorithm 1, and the computed trajectories that are influenced by both the desired shape and random noise. This procedure is performed offline to generate collision free path for each robot that converges to the desired shape. Before doing so, we present a few numerical experiments to illustrate the performance in the next section.

IV. NUMERICAL RESULTS

In our numerical experiments, we confine the robots in a square domain given by \( \Omega = [-M, M] \times [-M, M] \).

Algorithm 1: Intermittent diffusion based motion-planning

1: **Initialization:** Given a feasible initial configuration \( X_0 \) in a computational domain \( \Omega = [-M, M]^2 \), and a tolerance \( \epsilon > 0 \). Pick ID parameters \( (\alpha, \beta) \), a small number \( \tau > 0 \) and a positive integer \( s_{\text{max}} \) as the maximum iteration number in step 3. Set \( X_{\text{opt}} = X_0, n = 0 \).
2: **Virtual diffusion:** If \( \Psi (X_{\text{opt}}) > \epsilon \), set \( k = k + 1, m = 0 \). Generate two random positive numbers \( d, t \in (0, 1) \) and set \( \sigma = \alpha d \) and \( V = \beta t \). Define \( Y^0 = X_{\text{opt}} \), and perform the following simulation for \( m \Delta t \leq V \):
\[ Y^t_{m+1} = Y^t_m - \left( \frac{1}{N} \nabla \mu (Y^t_m) + (\nabla G(Y^t_m)) \right) \Delta t + \sigma \xi^t_m \sqrt{\Delta t}, \tag{16} \]
in which \( \xi^t_m \) is a random vector following normal distribution. Record the final locations \( \hat{Y} = Y_m \).
3: **Gradient descent toward \( \hat{Y} \):** For \( 1 \leq i \leq N \), define
\[ \hat{F}_i(X^i) = \frac{1}{N} \| X^i - \hat{Y} \|^2. \]
Set \( s_0 = n \). Compute the following iterations until \( \max_i \| X^i_{n+1} - X^i_n \| < \tau \) or \( n > s_0 + s_{\text{max}} \).
\[ X^i_{n+1} = X^i_n - \left( \nabla \hat{F}_i(X^i_n) + (\nabla G(X^i_n)) \right) \Delta t. \]
If \( X^i_{n+1} \notin \Omega \), set \( X^i_{n+1} = X^i_{n+1} - 2 \text{sgn}(X^i_{n+1}) \mod (\| X^i_{n+1} \|_\infty, M) \).
4: **Gradient descent toward \( \Gamma \):** Calculate the following iterations until \( \| X^i_{n+1} - X^i_n \| < \tau \):
\[ X^i_{n+1} = X^i_n - (\nabla F_i(X^n_n) + (\nabla G(X^n_n)) \right) \Delta t. \]
If \( X^i_{n+1} \notin \Omega \), set \( X^i_{n+1} = X^i_{n+1} - 2 \text{sgn}(X^i_{n+1}) \mod (\| X^i_{n+1} \|_\infty, M) \).
If \( \Psi (X^i_n) < \Psi (X_{\text{opt}}) \), set \( X_{\text{opt}} = X^i_n \).
5: Repeat steps 2,3, and 4 until \( \Psi (X) < \epsilon \).

| Symbol | Description | Value |
|--------|-------------|-------|
| \( G_0 \) | Repelling function amplitude | 0.01 |
| \( R \) | Robot sensor radius | 10r |
| \( \Delta t \) | Time step | 0.1r |
| \( \alpha \) | ID Diffusion scale | r |
| \( \beta \) | ID Time scale | 10 |
| \( M \) | Computational domain size | 6 |
Fig. 2. The blue arrow indicates the direction $\nabla G$ and the red arrow indicates $\nabla F$ at the beginning of the iterations and midway through the iterations.

We make the assumptions that each robot has knowledge of its location $X^i$, the gradient of the shape function $(\nabla F_i(X))^i$, and a sensing radius $R$, meaning that a robot can only detect other robots if they are within a circular region centered at $X^i$ with radius $R$. This $R$ is also the parameter we use in $G(X)$:

$$G(X) = G_0 \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} \cot(\pi/2(||X^i - X^j||^2)/R^2).$$

We evaluate the success of the algorithm by determining if the robots are in the desired region, distributed uniformly, and if the nearest neighbors difference is minimized.

The numerical tests are performed on two shapes. The first shape consisting of points in the set $\Gamma_1$, corresponds to a handwritten letter ‘Q’. In this case, the closed loop feature poses difficulties. The second shape consisting of points in the set $\Gamma_2$, is a Chinese character, pronounced as ‘JIE’, with multiple complicated strokes and two disconnected components. The initial positions for the robots are either clustered at a corner (demonstrated for shape $\Gamma_1$) or randomly distributed in the domain (demonstrated for shape $\Gamma_2$). The time evolution, shown in two cases in Figure 3, indicates that the robot trajectories driven by our proposed algorithm drive the robots to the desired shapes without suffering from congestion or getting stuck at local minimizers.

To test the scalability of our algorithm, we varied the size of the robot radius (resulting in different values of $N$). The choice of $N$ is based on a-priori knowledge that there is a global minimum with $N$ robots positioned entirely in the desired shape, determined by trial and error.

From the experiments, we observe that the faster convergence occurs with a random initial configuration that minimizes congestion from the start and provides the robot group immediate access to all sides of the target shape. When robots are initialized in a cluster near one end of the domain, they risk stagnating near the corner of the shape and missing entire sections of the shape unless intermittent diffusion becomes active.

We compared our method to a standard gradient descent with potential $\Psi$. From the energy plots shown in Figure 4, it is clear that gradient descent alone leaves robots stuck in local minima. After about 2000 iterations, the congestion caused by the gradient descent iterations is not resolved. Furthermore, the energy decays at a much slower rate than in the iterations produced by the proposed algorithm.

| $r$ | $N$ | $\Psi(X_{ID})$ | $\Psi(X_{GD})$ |
|-----|-----|----------------|----------------|
| .1  | 50  | 0.02537        | 0.02808        |
| .05 | 150 | 0.02905        | 0.02931        |
| .01 | 1,000 | 0.00066       | 0.00223        |
| .1  | 200 | 0.12891        | 0.13273        |
| .05 | 400 | 0.05964        | 0.06545        |
| .01 | 3,000 | 0.00702      | 0.01648        |

TABLE II
Final objective function value for both shapes and varied $r$, starting from a random initialization.
V. MATHEMATICAL UNDERPINNINGS

In this section, we justify theoretically that the generated path using the proposed method can achieve the desired shape while maintaining collision free motions. We start with the collision free property first.

Our model determines the trajectories of the robots based on two different gradient flows, (7) and (13) respectively. In both cases, the energy functional $\Psi(X)$ consists of a potential $F(X)$ (or $\hat{F}(X)$) that attracts the robots to the destinations, and the repelling function $G(X)$ that keeps them away from each other. In our theoretical study, it suffices to consider a general potential $F$ that is differentiable, bounded, and has minimizers only at the desired regions. In this general setting, the governing equation for the path is still given by the gradient flow presented in (7). So we show that the trajectories are collision free in both continuous and discrete cases.

A. Continuous time collision avoidance

Let us recall that the location of the $i$th robot is given by $X^i$, and the set of admissible robot coordinates is $\mathcal{X}$, where

$$\mathcal{X} = \{ (X^1, \ldots, X^N) \in \Omega \mid \inf_{i \neq j} \| X^i - X^j \| > r \}.$$  

We note that the repelling function $G(X)$ satisfies (2), for a function $\varphi$ satisfying (3) - (5), which implies

$$\text{1) } \liminf_{i,j \neq i} \| X^i - X^j \| \to 0 \quad G(X) = +\infty$$

$$\text{2) } \limsup_{i,j \neq i} \| X^i - X^j \| \to R \quad G(X) = 0.$$  

Therefore, $G(X)$ can be arbitrarily large if the distance between a single pair of robots is small enough. Let $m > r$ be the smallest allowable distance between two robots, and define the constant

$$E_m = G_0 \varphi(m).$$  

This implies that we have must have $G(X) > E_m$, if there is at least one pair of robots that have pairwise distance less than $m$.

Then we have the following theorem.

Theorem 5.1: For any trajectory $X(t) = (X^1(t), \ldots, X^N(t)), N > 1$ of (7) with an admissible
initial condition \(X_0 \in \mathcal{X}\) satisfying \(E_0 = \Psi(X_0) < E_m\), the inequality
\[
\inf_{i,j \neq i} \|X^i(t) - X^j(t)\|^2 \geq r^2 \tag{17}
\]
holds for all \(t > 0\).

**Proof 1:** The function \(\Psi(X(t))\) is non-increasing along the solution of (7) since it satisfies
\[
\frac{d}{dt} \langle \Psi(X) \rangle = \sum_{i=1}^{N} (\nabla \Psi(X)_i) dt \frac{dX^i}{dt} = -\sum_{i=1}^{N} \|\nabla \Psi(X)_i\|^2 < 0,
\]
and \(F\) is non-negative by construction.

Assume there is a time \(t^* > 0\) such that \(\|X^i(t^*) - X^j(t^*)\|^2 < r^2\) for some \(i, j \neq i\), then
\[
\Psi(X(t^*)) = F(X(t^*)) + G(X(t^*)) \\
\geq G_0 \varphi(m) = E_m > E_0.
\]
This is a contradiction, because \(\Psi(X(t))\) is non-increasing, so we must have \(\Psi(X(t^*)) \leq E_0\).

**B. Discrete time collision avoidance**

Equations (7) and (13) are solved in discrete time using the iterations
\[
X^i_{n+1} = X^i_n - (\nabla \Psi(X_n)_i) \Delta t, \tag{18}
\]
where \(X_n \simeq X(t_n)\) and \(t_n = n \Delta t\). It is known that the Euler scheme converges to the continuous solution, if \(\nabla \Psi\) is \(L\)-Lipschitz continuous in space. This ensures no collision in the discrete case when the step size is small enough. In the next theorem, we present such a result, and prove it by using a standard argument from [64].

**Theorem 5.2:** Suppose \(\Psi \in C^1(\mathcal{X})\) is a positive function that is bounded below and \(\nabla \Psi\) is \(L\)-Lipschitz continuous in space. Then, if \(\Delta t \leq \frac{1}{L}\), one step of the gradient method \(13\) will not increase the objective function \(\Psi\), that is \(\Psi(X_{n+1}) \leq \Psi(X_n)\).

**Proof 2:** Denote the Euclidean inner product by
\[
\langle X, Z \rangle = \sum_{i=1}^{N} X^i \cdot Z^i \tag{19}
\]
For \(X, Z \in \mathcal{X}\), we can express \(\Psi(Z) - \Psi(X)\) by
\[
\Psi(Z) - \Psi(X) = \int_{0}^{1} \langle \nabla \Psi(X + \tau(Z - X)), Z - X \rangle d\tau \nonumber \]
\[
= 2 \int_{0}^{1} (\nabla \Psi(X), Z - X) d\tau + \int_{0}^{1} (\nabla \Psi(X + \tau(Z - X)) - \nabla \Psi(X), Z - X) d\tau.
\]
This results in
\[
\Psi(Z) - \Psi(X) = \int_{0}^{1} \langle \nabla \Psi(X + \tau(Z - X)), Z - X \rangle d\tau \nonumber \]
\[
\leq \int_{0}^{1} \sum_{i=1}^{N} \|\nabla \Psi(X + \tau(Z - X))_i\| \sum_{i=1}^{N} \|Z^i - X^i\| d\tau \\
\leq L \Delta t^2 \sum_{i=1}^{N} \|\nabla \Psi(X_n)_i\| \sum_{i=1}^{N} \|Z^i - X^i\|^2 d\tau \\
= \Psi(X_n) - \Delta t (1 - \frac{L \Delta t}{2}) \sum_{i=1}^{N} \|\nabla \Psi(X_n)_i\|^2.
\]
Therefore \(\Psi(X_{n+1}) \leq \Psi(X_n)\) if \(\Delta t \leq \frac{2}{L}\).

**Corollary 1:** The discrete trajectory \(X_n\) computed by (14) satisfies
\[
\inf_{i,j \neq i} \|X^i_n - X^j_n\|^2 \geq r^2, \tag{19}
\]
for all \(n \geq 0\), provided \(E_0 = \Psi(X_0) < E_m\).

The proof of this corollary follows directly from the proof of Theorem 5.1.

**C. Convergence to the global minima**

As described in our model, the goal of introducing (13) is to move the robots to the intermediate locations generated by the SDEs (8). Therefore, the convergence of the trajectories to the desired shape means that the solutions of (8) march to the global minima of \(\Phi(x)\), which is guaranteed by the theory of optimal transport. More precisely, the idea of combining (7) and (13) comes from the intermittent diffusion. Together, the dynamics can be equivalently described by a uniform formula given in (8), in which (7) is performed when \(\sigma = 0\), and (13) reaches the same spatial locations as (8) when \(\sigma\) is not zero. Therefore the final locations of \(X(t)\) are determined by the asymptotic distributions of (8).
The probability density function \( \rho(X,t) \) of the stochastic process \( X(t) \) from \([8]\) evolves according to the so-called Fokker-Planck equation

\[
\frac{\partial \rho(X,t)}{\partial t} = \nabla \cdot (\rho(X,t) \nabla \Psi(X)) + \frac{1}{2} \sigma^2 \Delta \rho(X,t). \tag{20}
\]

which is a transport equation when \( \sigma = 0 \), and a diffusion equation when \( \sigma > 0 \). In the diffusion case, the asymptotic solution, also called equilibrium or station solution, is the well-known Gibbs distribution defined in \([10]\). The Gibbs distribution suggests that \( X(t) \) has arbitrary large probability when settled in the global minima of \( \Psi \), for sufficiently small \( \sigma \).

By the convergence theory of intermittent diffusion, it can be shown that for \( \sigma \) taken in a discontinuous manner as in \([10]\), the convergence to the global minima is ensured according to the following theorem.

**Theorem 5.3:** Suppose \( \Psi(x) \) has a global minimum attained on a set \( Q \) of positive Lebesgue measure, and let \( U \) be a neighborhood of \( Q \). Then for any \( \eta > 0 \) there exists a \( T^* > 0 \), \( \sigma_0 > 0 \) and \( K_0 > 0 \) such that if \( T_i - S_i > T^* \), \( \sigma_i < \sigma_0 \) for \( 1 \leq i \leq K \) and \( K > K_0 \),

\[
P(X_{opt} \in U) \geq 1 - \eta.
\]

where \( P \) is the probability.

**VI. Conclusions and Future Work**

We present a path planning strategy for a large group of robots to accomplish shape formation, one of the fundamental tasks in many applications that employ multi-robot systems. Typical challenges include how to avoid collisions and deadlocks in motion planning and how to achieve the desirable shape with assurance. Those challenges become more significant for large groups of robots and for robots with low functionality. In our method, we calculate the individual robot trajectories by alternating two gradient flows that involve an attractive potential, a repelling function and a process of intermittent diffusion. The potential attracts robots to form the targeted shape, while the repelling function is designed to ensure collision-free motions. The intermittent diffusion, originally a stochastic approach but here realized by deterministic means, overcomes situations with deadlocks. Our strategy is inspired by recent developments in the theory of optimal transport which in turn provides the basis for theoretical guarantees of collision avoidance and global convergence. Numerical experiments confirm that the proposed algorithm is simple, yet effective in achieving desired objectives.

The presentation here in the two-dimensional setting can be extended to higher dimensions with straight forward adaptations. The proposed strategy can also be adapted to accommodate inhomogeneous multi-robot systems, in which robots may have different functionalities. In this scenario, the differences among robots must be reflected throughout the selections of the potential functions, including both \( F(x) \) and \( G(x) \). On the technical side, this may not be easy to accomplish and it is in our future plan for further investigation.

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