A Curry-Howard Correspondence for the Minimal Fragment of Łukasiewicz Logic

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Abstract
In this paper we introduce a term calculus $B$ which adds to the affine $\lambda$-calculus with pairing a new construct allowing for a restricted form of contraction. We obtain a Curry-Howard correspondence between $B$ and the sub-structural logical system which we call “minimal Łukasiewicz logic”, also known in the literature as the logic of hoops (a generalisation of MV-algebras). This logic lies strictly in between affine minimal logic and standard minimal logic. We prove that $B$ is strongly normalising and has the Church-Rosser property. We also give examples of terms in $B$ corresponding to some important derivations from our work and the literature. Finally, we discuss the relation between normalisation in $B$ and cut-elimination for a Gentzen-style formulation of minimal Łukasiewicz logic.

1 Introduction

We are interested in the proof theory of Łukasiewicz logic and subsystems thereof. In this context, designing proof systems with nice dynamical properties – cut-elimination or normalisation – has proved to be a hard problem. Systems with the cut-elimination property have been successfully obtained via an extension of Gentzen’s sequent calculus by means of hypersequents [13]. But this approach depends on the pre-linearity axiom

$$(A \Rightarrow B) \lor (B \Rightarrow A)$$

a principle that is not intuitionistically acceptable. As far as we are aware, in the quite extensive literature on fragments of Łukasiewicz logic that are compatible with intuitionistic or minimal logic, such as the logic of GBL algebras [11] and the logic of hoops [5, 7], no normalising or cut-free proof systems are to be found.

Benton et al. [4] gave a term calculus for intuitionistic linear logic. In this paper, we follow their approach and extend the simply-typed affine $\lambda$-calculus with a construct that captures propositional minimal Łukasiewicz logic. We prove this extended system $B$ preserves types, is strongly normalising and has the Church-Rosser property. We give examples of terms in $B$ corresponding to

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some important derivations from our work and from the literature on hoops and
GBL algebras.

1.1 Fragments of Lukasiewicz logic

The standard Hilbert-style axiomatisation of (classical) Lukasiewicz logic \( \text{LL}_c \) may be found in [9]. It has \textit{modus ponens} as its only inference rule and its
axioms comprise the axioms of basic logic:

\begin{align*}
\text{(B1)} & \quad (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C \\
\text{(B2)} & \quad A \otimes B \Rightarrow A \\
\text{(B3)} & \quad A \Rightarrow B \Rightarrow B \otimes A \\
\text{(B4)} & \quad A \otimes (A \Rightarrow B) \Rightarrow B \otimes (B \Rightarrow A) \\
\text{(B5a)} & \quad (A \otimes B \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C \\
\text{(B5b)} & \quad (A \Rightarrow B \Rightarrow C) \Rightarrow A \otimes B \Rightarrow C \\
\text{(B6)} & \quad ((A \Rightarrow B) \Rightarrow C) \Rightarrow ((B \Rightarrow A) \Rightarrow C) \Rightarrow C \\
\text{(B7)} & \quad \bot \Rightarrow A
\end{align*}

together with the axiom of \textit{double negation elimination}.

\[(\text{DNE}) \quad \neg
\neg A \Rightarrow A\]

where \( \neg A \) is defined as \( A \Rightarrow \bot \). If from these we drop the axioms that are
not valid in minimal logic \([16]\), i.e. (B6), (B7) and (DNE), we are left with
the fragment (B1)–(B5), which we will call \textit{minimal Lukasiewicz logic} \( \text{LL}_m \).
If we extend \( \text{LL}_m \) with the ex-falso-quadlibet axiom (B7), we obtain what we
have called \textit{intuitionistic Lukasiewicz logic} \( \text{LL}_i \). The logics \( \text{LL}_m \) and \( \text{LL}_i \) can
be faithfully characterised algebraically using the classes of algebraic structures
originally due to Büchi and Owen and known as hoops and bounded hoops,
respectively (see \([1, 2, 5, 7]\)). Hoops are reducts of commutative GBL algebras,
whose equational theory has been shown to be PSPACE-complete \([5]\). As the
equational theory of GBL algebras is a conservative extension of that of hoops,
it follows that the decision problems for \( \text{LL}_m \) and \( \text{LL}_i \) are also PSPACE-
complete\(^1\). If we omit (B4) from \( \text{LL}_m \), we obtain the minimal \((\Rightarrow, \otimes)-\)fragment
of affine logic \( \text{AL}_m \), which we will just refer to as affine logic in this paper. For
more details about the proof theory of these systems, including double negation
translations from \( \text{LL}_c \) into \( \text{LL}_i \) and \( \text{LL}_m \), see \([3]\).

Our goal in this paper is to devise a typed term calculus whose inhabited
types comprise the formulas provable in \( \text{LL}_m \) with \( \otimes \) and \( \Rightarrow \) viewed as the
product and function type constructors.

2 The \( B \) Calculus

Let us start by introducing the term language of the \( B \) calculus. Since we
want terms to be unambiguous representations of proofs, we give a Church-style
calculus of typed terms, rather than a Curry-style type-assignment system.

\(^1\)We are indebted to the late Franco Montagna who pointed this out to us back in 2014.
2.1 Term language

Types are formed from type variables $P_1, P_2, \ldots$ using the binary operators $\Rightarrow$ and $\otimes$. We use $P$ to range over type variables and $A, B, C, D$ will range over arbitrary types. The terms of the $B$ calculus are obtained inductively starting from typed variables ($x^A, y^B, \ldots$) via the following constructs:

- $\lambda$-abstraction and term application
  - $\lambda x^A.t$ is a term when $t$ is a term
  - $st$ is a term when $s$ and $t$ are terms

- constructor and destructor for pairs
  - $s \otimes t$ is a term when $s$ and $t$ are terms
  - let $t$ be $x^A \otimes y^B$ in $r$ is a term when $r$ and $t$ are terms, $x^A$ and $y^B$ are distinct variables

- the break constructor:
  - break $t$ as $x^A, y^B$ in $r$ is a term when $r$ and $t$ are terms, $x^A$ and $y^B$ are distinct variables.

Our typing rules will imply that the variables $x^A$ and $y^B$ in break $t$ as $x^A, y^B$ in $r$ denote a higher-order function and a function respectively. From now on, we will therefore generally use letters like $\varphi$ and $f$ for these variables instead of $x$ and $y$. This is illustrated in the following definition of the set $\text{FV}(t)$ of free variables of a term $t$:

- $\text{FV}(x^A) = \{x^A\}$
- $\text{FV}(\lambda x^A.t) = \text{FV}(t) \setminus \{x^A\}$
- $\text{FV}(st) = \text{FV}(s) \cup \text{FV}(t)$
- $\text{FV}(s \otimes t) = \text{FV}(s) \cup \text{FV}(t)$
- $\text{FV}(\text{let } s \text{ be } x^A \otimes y^B \text{ in } r) = (\text{FV}(t) \setminus \{x^A, y^B\}) \cup \text{FV}(s)$
- $\text{FV}(\text{break } s \text{ as } \varphi^A, f^B \text{ in } t) = (\text{FV}(t) \setminus \{\varphi^A, f^B\}) \cup \text{FV}(s)$

2.2 Type system

The typing rules for $B$ are given in Figure 1. In the sequents $\Gamma \vdash t : A$ used in the rules, the context $\Gamma$ is a finite function mapping (untyped) variable names to types, $t$ is a $B$ term and $A$ is a type. In the rule [BRK], we use the following abbreviation:

- $K_B A \equiv (A \Rightarrow B) \Rightarrow B$
- $S_B A \equiv A \Rightarrow B$.

The rules with two premises are subject to the side condition that the two contexts $\Gamma$ and $\Delta$ must be compatible, i.e. $\Gamma, \Delta$ must also be a finite function

\footnote{It is noteworthy that for each formula $B$, the mappings $A \Rightarrow K_B A$ and $A \Rightarrow S_B A$ can both be equipped with a monad structure in the simply typed $\lambda$-calculus, with $A \Rightarrow K_B A$ being the well-known continuation monad.}
mapping (untyped) variable names to types. This implies that typable terms are affine in the sense that in any subterm each free variable appears exactly once. However, as we will see, with the new rule \( \text{BRK} \) we populate many types that are uninhabited in the affine simply-typed \( \lambda \)-calculus with pairing.

We say a term \( t \) is typable with type \( A \) and write \( t : A \) if the rules of \( B \) allow us to infer a sequent of the form \( \Gamma \vdash t : A \), where \( \Gamma \) consists of the mappings \( x : B \) for each \( x^B \in \text{FV}(t) \). If \( t : A \), then \( A \) is uniquely determined by \( t \). If one presents \( B \) as a Curry-style type assignment system, then the Milner-Hindley principal type algorithm \[10\] extends easily to \( B \) allowing us to find a most general type assignment for a given term. From now on we will adopt the usual conversion of omitting type superscripts that can be inferred from the context.

### 2.3 Correspondence between LLm and \( B \)

In this section we show that types inhabited by closed terms in \( B \) are precisely the provable formulas of \( \text{LLm} \). To see that any formula provable in the Hilbert-style system \( \text{LLm} \) is (when viewed as a type) inhabited by a closed term of \( B \), the main work is showing that the axioms of \( \text{LLm} \) are inhabited:

**Theorem 1** If a formula \( A \) is provable in \( \text{LLm} \) then there exists a closed \( B \) term \( t \) which is typable with type \( A \).

**Proof:** The special case of \( \Rightarrow \text{E} \) where the contexts \( \Gamma \) and \( \Delta \) are empty provides us with *modus ponens*. Hence it is enough to show that we have closed terms whose types are precisely the axioms (B1)–(B5). We indeed have:

- (B1) \( \lambda f \lambda g x. g(f(x)) : (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C \)
- (B2) \( \lambda v. \text{let } v \text{ be } x \otimes y \text{ in } x : A \otimes B \Rightarrow A \)
- (B3) \( \lambda v. \text{let } v \text{ be } x \otimes y \text{ in } y \otimes x : A \otimes B \Rightarrow B \otimes A \)
- (B4) \( \lambda v. \text{let } v \text{ be } x \otimes f \text{ in } (\text{break } x \text{ as } \varphi, g \text{ in } \varphi(f) \otimes g) : A \otimes (A \Rightarrow B) \Rightarrow B \otimes (B \Rightarrow A) \)
- (B5a) \( \lambda f \lambda x y. f(x,y) : (A \otimes B \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C \)
- (B5b) \( \lambda y \lambda a. \text{let } a \text{ be } x \otimes y \text{ in } g(x)(y) : (A \Rightarrow B \Rightarrow C) \Rightarrow A \otimes B \Rightarrow C \)

For the converse, to see that any type inhabited by a closed term in \( B \) is a formula that is provable in the Hilbert-style system \( \text{LLm} \) requires a little
\[(\lambda x.t) s \xrightarrow{s{\text{-}conv}} t[s/x]\]

\[
\text{let } t \otimes u \text{ be } x \otimes y \text{ in } s \xrightarrow{l{\text{-}conv}} s[t/x, u/y]\]

\[
\text{break } t \text{ as } \varphi, f \text{ in } s \xrightarrow{b{\text{-}conv}} s[(\lambda p.tp)/\varphi, (\lambda t)/f] \quad \left\{ \begin{array}{l}
\varphi \notin \text{FV}(s) \\
\forall f \notin \text{FV}(s) \\
\forall \text{FV}(t) = \emptyset
\end{array} \right. \]

Figure 2: Standard conversions

ingenious: we show that the logical principle embodied in the rule [BRK] is derivable from the axiom (B4).

**Theorem 2** If \(\Gamma \vdash t : A\) in \(B\), where \(\Gamma \equiv x_1 : B_1, \ldots, x_n : B_n\), then \(B_1 \Rightarrow \ldots \Rightarrow B_n \Rightarrow A\) is provable in \(\text{LL}_m\).

**Proof:** If we erase the terms from the typing rules \([\text{ASM}], [\Rightarrow l], [\Rightarrow E], [\otimes I]\) and \([\otimes E]\), the resulting logical inference rules give a standard natural deduction presentation of affine logic, which is well known to be equivalent to the Hilbert-style system \(\text{AL}_m\). So we have only to show that the following rule, obtained by erasing the terms from [BRK], is derivable in \(\text{LL}_m = \text{AL}_m + (B4)\):

\[
\Gamma \vdash A \quad \Delta, K_B A, S_B A \vdash C \\
\Gamma, \Delta \vdash C
\]

That in turn, follows once we can show that \(A \Rightarrow K_B A \otimes S_B A\) is derivable in \(\text{LL}_m\). We start by noting that \(\text{AL}_m\) proves \(\vdash A \Rightarrow (A \Rightarrow B) \Rightarrow B\) i.e., \(\vdash A \Rightarrow K_B A\). Hence \(\text{AL}_m\) proves \(\vdash A \Rightarrow (A \otimes (A \Rightarrow K_B A))\). But then, using axiom (B4) to transform \(A \otimes (A \Rightarrow K_B A)\), we have that \(\text{LL}_m\) proves \(\vdash A \Rightarrow (K_B A \otimes (K_B A \Rightarrow A))\). As \(\text{AL}_m\) also proves \(\vdash B \Rightarrow (A \Rightarrow B) \Rightarrow B\), i.e., \(\vdash B \Rightarrow K_B A\). \(\text{LL}_m\) proves \(\vdash A \Rightarrow K_B A \otimes S_B A\). I.e., in \(\text{LL}_m\), \(A\) is logically stronger than the conjunction of \(K_B A\) and \(S_B A\), which justifies the inference given by [BRK].

In the above proof we have weakened \(A \Rightarrow (K_B A \otimes (K_B A \Rightarrow A))\) to \(A \Rightarrow (K_B A \otimes S_B A)\). Since \(A\) is strictly stronger than \(K_B A \otimes S_B A\) in general, the inference rule given by [BRK] is not invertible. An alternative invertible version of the rule [BRK] is

\[
\Gamma \vdash A \quad \Delta, K_B A, K_B A \Rightarrow A \vdash C \\
\Gamma, \Delta \vdash C
\]

However, the simpler rule is adequate for present purposes.

### 2.4 Conversion Rules

We now equip our term language with type preserving conversion rules. The conversion rules we propose are shown in Figures 2 and 3. The first two standard conversions are the usual conversions for the affine simply-typed \(\lambda\)-calculus with
(let t be \(x \otimes y\) in u) \(\Rightarrow\) (let t be \(x \otimes y\) in u)

let (let t be \(x \otimes y\) in u) be \(v \otimes w\) in s \(\Rightarrow\) let t be \(x \otimes y\) in (let u be \(v \otimes w\) in s) 

\((x, y \notin \text{FV}(s))\)

(break t as \(\varphi, f\) in u) \(\Rightarrow\) break t as \(\varphi, f\) in (u)

let (break t as \(\varphi, f\) in u) be \(x \otimes y\) in s \(\Rightarrow\) break t as \(\varphi, f\) in (let u be \(x \otimes y\) in s) 

\((\varphi, f \notin \text{FV}(s))\)

Figure 3: Permuting conversions

\[
g : B \Rightarrow A \vdash g : B \Rightarrow A \quad \varphi : K_B A \vdash \varphi : K_B A \quad [\Rightarrow \text{E}]
\]
\[
g : B \Rightarrow A, \varphi : K_B A \vdash \varphi(g) : B \quad f : B \Rightarrow A \vdash f : B \Rightarrow A \quad [\otimes \text{I}]
\]
\[
x : A \vdash x : A
\]
\[
\varphi : K_B A, f : B \Rightarrow A \vdash \lambda g. \varphi(g) \otimes f : (A \Rightarrow B) \Rightarrow B \otimes (B \Rightarrow A) \quad [\Rightarrow \text{I}]
\]
\[
\vdash \lambda x. \text{break} x \ as \ \varphi, f \ in \lambda g. \varphi(g) \otimes f : (A \Rightarrow B) \Rightarrow B \otimes (B \Rightarrow A) \quad \text{[BRK]}
\]

Figure 4: Sample derivation in \(B\)

pairing. The third standard conversion shows how to reduce terms involving the new constructor \(\text{break}\). The permuting conversions show how to move an occurrence of \(\text{break}\) or \(\text{let}\) up one level in the term structure. We write \(t \Rightarrow t'\) when \(t'\) can be obtained from \(t\) by applying one of the conversion of Figures 2 and 3 to a single sub-term of \(t\). We write \(\Rightarrow^*\) for the reflexive-transitive closure of \(\Rightarrow\).

3 Example Derivations

Before investigating the theory of the conversions introduced above, we will first look at some examples of type derivations and some examples of term normalisation using the conversions. For the examples we will use some important \(\text{LL}_m\) provable formulas from our work and the literature.

3.1 The divisibility axiom

Consider an example of a type whose corresponding formula is not provable in affine logic, but which is provable in \(\text{LL}_m\), namely

\[
A \Rightarrow (A \Rightarrow B) \Rightarrow (B \otimes (B \Rightarrow A))
\]

This is essentially axiom (B4), and is normally referred to as the divisibility axiom. It is well-known that the divisibility axiom characterises Lukasiewicz logic over affine logic [1].
We can build a term (which can be seen as a proof) having the above type in the calculus $B$ as follows: Given $x : A$ we can break it into $\varphi : A \Rightarrow B$ and $f : B \Rightarrow A$. Using these and $g : A \Rightarrow B$ we can build a term of type $B \otimes (B \Rightarrow A)$ as $\varphi(g) \otimes f$. The full type derivation for the inhabitant

$$t \equiv \lambda x.\text{break } x \text{ as } \varphi, f \text{ in } \lambda g.\varphi(g) \otimes f$$

of $A \Rightarrow (A \Rightarrow B) \Rightarrow B \otimes (B \Rightarrow A)$ is shown in Figure 4.

Clearly $t$ is in normal form, i.e., no conversion rules apply to it. However, using $t$ we can construct the term,

$$u \equiv \lambda x' \lambda g'.\text{let } t(x')(g') \text{ be } m \otimes n \text{ in } m$$

of type $A \Rightarrow (A \Rightarrow B) \Rightarrow B$, a type that is already inhabited in minimal affine logic. Hence, it is reasonable to ask whether the conversions in $B$ will reduce this term to a term without the $\text{break}$ constructor. This is indeed the case:

First we have:

$$u \xrightarrow{\beta-conv} \lambda x' \lambda g'.(\lambda p. p(x'))(g')$$

$$\xrightarrow{l-conv} \lambda x' \lambda g'.\text{break } x' \text{ as } \varphi, f \text{ in } (\lambda g'. \varphi(g') \otimes f \text{ be } m \otimes n \text{ in } m)$$

Now, in the term $v$, the variable $f$ no longer appears free in the body of the $\text{break}$ term, so the side-conditions of $(\text{b-conv})$ hold and we may continue as follow to get the normal form for $u$, which does not involve $\text{break}$.

$$v \xrightarrow{\text{b-conv}} \lambda x' \lambda g'.(\lambda p. p(x'))(g')$$

In this case we were able to reduce a term with a minimal affine type into a term without $\text{break}$ sub-terms, but this is not possible in general. The new $\text{break}$ constructor will give rise to new proofs of affine minimal logic theorems. For instance, we have the following normal form proof of identity $A \Rightarrow A$:

$$\lambda x^A.\text{break } x^A \text{ as } \varphi^{K,A}, f^{S,A} \text{ in } \varphi(f).$$

Nevertheless, when this is applied to a closed term $s$ of type $A$, we are able to reduce $(\lambda x.\text{break } x \text{ as } \varphi, f \text{ in } \varphi(f))(s)$ to $s$:

$$\text{break } s \text{ as } \varphi, f \text{ in } \varphi(f) \xrightarrow{\text{b-conv}} (\lambda p.p(s))(\lambda s)$$

$$\xrightarrow{\beta-conv} (\lambda s)(s)$$

$$\xrightarrow{\beta-conv} s$$

so that the new term also behaves like the identity function.

### 3.2 Axiom L

Consider another formula which is provable in basic logic (without using pre-linearity) but is not provable in affine logic, namely the axiom $L$ of \cite{5}:

$$(B \Rightarrow A) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow B)$$

7
Assuming $\Delta(B \Rightarrow A) \Rightarrow (A \Rightarrow B)$ and $x^A$, we can break $x$ as $\varphi^{K_B A}$ and $f^{B \Rightarrow A}$ and construct a term of type $B$ as $\varphi(\Delta(f))$. Discharging the two assumptions, in our system we obtain:

$$\vdash \lambda \Delta \lambda x. \text{break } x \text{ as } \varphi, f \text{ in } \varphi(\Delta(f)) : ((B \Rightarrow A) \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$$

If we take $A \equiv B$, and $\Delta(g) = g$, the term above reduces to

$$\vdash \lambda x. \text{break } x \text{ as } \varphi, f \text{ in } \varphi(f) : A \Rightarrow A$$

which, as we have seen in the previous sub-section, is in normal form and behaves as the identity function on each closed term $t^A$.

### 3.3 A homomorphism property

Ferreirim [8] proved an algebraic result (in the algebra of hoops) suggesting that the following formula should be provable in $\mathbb{LLm}$:

$$(A \Rightarrow A \otimes A) \Rightarrow (A \Rightarrow B \otimes C) \Rightarrow ((A \Rightarrow B) \otimes (A \Rightarrow C))$$

Her proof used model-theoretic methods and proved validity of the formula for a restricted class of algebras. This constitutes the main lemma in the proof that the mapping $X \mapsto A \Rightarrow X$ is a hoop homomorphism for idempotent elements $A$. With the assistance of the Otter system [12] and Veroff’s method of proof sketches [18], Veroff and Spinks [17] found a syntactic proof of the theorem in full generality. An indirect proof of the general result using algebraic methods is given in [2]. Here we present a term of $B$ with the above type.

Assuming $\alpha : A \Rightarrow A \otimes A$ and $h : A \Rightarrow B \otimes C$ we build a term of type $(A \Rightarrow B) \otimes (A \Rightarrow C)$. This term will be built using

$$\varphi : K_{A \Rightarrow B}(A \Rightarrow B \otimes C) \quad f : S_{A \Rightarrow B}(A \Rightarrow B \otimes C)$$

which we will obtain by breaking $h : A \Rightarrow B \otimes C$.

We will define a series of terms $t_1[\varphi], t_2[x, f], \ldots, t_9[h, \alpha]$ where we have listed the free-variables of each term in the brackets. The final term $t_9[h, \alpha]$ will satisfy

$$t_9[h, \alpha] : (A \Rightarrow B) \otimes (A \Rightarrow C)$$

so that the term $\lambda \alpha \lambda h. t_9[h, \alpha]$ will witness the provability of our homomorphism property.

Let $\pi_0 : B \otimes C \rightarrow B$ and $\pi_1 : (A \Rightarrow B) \otimes (A \Rightarrow C) \rightarrow A \Rightarrow C$ be two closed terms of the indicated types (such terms are easy to define in $B$). Now, we begin the definition of the $t_i$’s:

- $t_1[\varphi] \equiv \varphi(\lambda m A \Rightarrow B \otimes C \lambda x^A. \pi_0(m x))$
- $t_2[x^A, f] \equiv \lambda j^{A \Rightarrow B}. \text{let } f j x \text{ be } x' \otimes y' \text{ in } (\lambda x' \otimes (\lambda x' y'))$

so that

- $t_1[\varphi] : A \Rightarrow B$
- $t_2[x^A, f] : (A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \otimes (A \Rightarrow C))$
Let \(Y \equiv (A \Rightarrow B) \Rightarrow ((A \Rightarrow B) \otimes (A \Rightarrow C))\). Using \(t_2[x,f]\) we build
\[
t_3[x^A, f, p^{Y \Rightarrow (A \Rightarrow C)}] \equiv p(t_2[x,f])
\]
so \(t_3[x^A, f, p^{Y \Rightarrow (A \Rightarrow C)}]: A \Rightarrow C\). Then, using \(\alpha: A \Rightarrow A \otimes A\), we can get a term of type \(K_{A \Rightarrow C}Y\) as
\[
t_4[\alpha, f] \equiv \lambda p^{Y \Rightarrow (A \Rightarrow C)} \lambda y^A.\text{let } \alpha y \text{ be } y_0 \otimes y_1 \text{ in } t_3[y_0, f, p](y_1)
\]
So, in summary, we have built two terms
- \(t_1[\varphi]: A \Rightarrow B\)
- \(t_4[\alpha, f]: (Y \Rightarrow (A \Rightarrow C)) \Rightarrow A \Rightarrow C\)

Now we use \(t_1[\varphi]\) to build the term
\[
t_5[\varphi] \equiv \lambda q^Y.q(t_4[\varphi]) : Y \Rightarrow (A \Rightarrow B) \otimes (A \Rightarrow C)
\]
We then break \(t_5[\varphi]: Z\) into
\[
\psi: K_{Y \Rightarrow (A \Rightarrow C)}Z \quad g: S_{Y \Rightarrow (A \Rightarrow C)}Z
\]
Defining \(t_6: Z \Rightarrow (Y \Rightarrow (A \Rightarrow C))\) as the closed term
\[
t_6 \equiv \lambda u^Z \lambda v^Y.\pi_1(u \nu)
\]
we have that \(\psi(t_6): Y \Rightarrow (A \Rightarrow C)\). Finally, using \(g: (Y \Rightarrow (A \Rightarrow C)) \Rightarrow Z\) and
\[
t_7 \equiv \lambda u^{A \Rightarrow C} \lambda u^{A \Rightarrow B}.u \otimes v : (A \Rightarrow C) \Rightarrow Y
\]
we build
\[
t_8[g] \equiv \lambda i^{A \Rightarrow C}.\text{break}_{Y}i \text{ as } \eta, k \text{ in } g(k)(\eta(t_7))
\]
of type \((A \Rightarrow C) \Rightarrow ((A \Rightarrow B) \otimes (A \Rightarrow C))\). The final term \(t_9[h, \alpha]: (A \Rightarrow B) \otimes (A \Rightarrow C)\) can then be built as
\[
t_9[h, \alpha] \equiv \text{break } h \text{ as } \varphi, f \text{ in } (\text{break } t_5[\varphi] \text{ as } \psi, g \text{ in } t_8[g](t_4[\alpha, f](\psi(t_6))))
\]

4 Properties of the Calculus

In this section we prove three important properties of the calculus \(B\): the subject reduction property (conversions preserve types), strong normalisation and the Church-Rosser property.

4.1 Subject reduction

We now demonstrate that the \(B\) conversions proposed in Figures \(\text{[2]}\) and \(\text{[3]}\) all preserve types.

**Theorem 3** If \(t: A\) and \(t \leadsto t'\) then \(t': A\).
**Proof:** It is sufficient to consider each of the conversions applied to the top level of the term \( t \). This is clearly the case for the standard \( \beta \)-conv and \( l \)-conv. Assume we have a type derivation for \( \text{break } t \text{ as } \varphi, f \text{ in } s \). Consider first the case when \( t \) is closed, namely

\[
\begin{align*}
\Gamma &\vdash \lambda p.p(t) : K_B A \\
&\vdash \lambda_t t : S_B A \\
\Delta &\vdash \text{break } t \text{ as } \varphi, f \text{ in } s : C
\end{align*}
\]

with the two sub-derivations \( \pi_1 \) and \( \pi_2 \). The conversion \( b \)-conv in this case corresponds to the following proof transformation

\[
\begin{align*}
\Gamma &\vdash \lambda p.p(t) : K_B A \\
&\vdash \lambda_t t : S_B A \\
\Delta &\vdash s[\lambda p.p(t)/\varphi, \lambda_t t/f] : C
\end{align*}
\]

If \( t \) is not closed, but either \( \varphi \) or \( f \) is not free in \( s \), the argument is similar but an extra context \( \Gamma \) might be present in the derivation \( \Gamma \vdash t : A \) but since this derivation only needs to be used once this does not invalidate the proof transformation.

Each of the permuting conversions needs to be checked as well, but this is an easy exercise, e.g. for \( \text{ap-b-conv} \) we are transforming the derivation

\[
\begin{align*}
\Gamma &\vdash \text{break } t \text{ as } \varphi, f \text{ in } u : C \Rightarrow D \\
\Theta &\vdash s : C
\end{align*}
\]

into

\[
\begin{align*}
\Gamma, \Delta &\vdash \text{break } t \text{ as } \varphi, f \text{ in } u : C \Rightarrow D \\
\Theta &\vdash s : C
\end{align*}
\]

by moving \([\Rightarrow E]\) above the application of \([\text{BRK}]\)

\[\Box\]

### 4.2 Strong normalisation

Let us now prove that the system \( B \) is strongly normalising.

**Definition 1** Let us call a \( l \)-conv or \( b \)-conv conversion in which the variables being substituted do not actually appear free in the term \( s \) a silent conversion.

We first prove that the set of permuting conversions together with the silent \( l \)-conv and \( b \)-conv conversions is strongly normalising:
Lemma 4 There is no infinite sequence of terms \((t_i)_{i \in \mathbb{N}}\) such that each \(t_{i+1}\) is obtained from \(t_i\) by means of a permuting conversion or a silent \(l\text{-conv}\) or \(b\text{-conv}\) conversion.

Proof: The silent conversions make the resulting term strictly smaller than the original one. In a \textit{let} or a \textit{break} term

\[
\text{let } t \text{ be } x \otimes y \text{ in } s \quad \text{break } t \text{ as } \varphi, f \text{ in } s
\]

let us call \(t\) the \textit{first} argument, and \(s\) the \textit{second} argument. The permuting conversions \(\text{ap-l-conv}\) and \(\text{ap-b-conv}\) reduce the type complexity of the second argument, e.g. in

\[
(\text{let } t \text{ be } x \otimes y \text{ in } s) \ u
\]

the term \(s\) will have some type \(A \Rightarrow B\), but after the \(\text{ap-l-conv}\) conversion we have

\[
\text{let } t \text{ be } x \otimes y \text{ in } s \ u
\]

where the second argument \(s \ u\) has type \(B\). Finally, the permuting conversions \(\text{l-l-conv}\) and \(\text{l-b-conv}\) reduce the size of the first argument for the \textit{let} or \textit{break} expressions. If we take the product of these three measures with a lexicographical ordering we obtain a measure which decreases (on a well-founded ordering) after each of these conversions.

Our proof of strong normalisation will make use of the following translation of \(\mathcal{B}\) terms into terms in the simply typed \(\lambda\)-calculus with pairing, which we will denote by \(\Lambda^\otimes\).

Definition 2 Define a translation of \(\mathcal{B}\) terms into \(\Lambda^\otimes\) terms inductively as:

\[
(\lambda x.t)^* = \lambda x.t^*
\]

\[
(s \otimes t)^* = s^* \otimes t^*
\]

\[
(\text{let } s \text{ be } x \otimes y \text{ in } u)^* = u^*[\pi_0(s^*)/x][\pi_1(s^*)]/y
\]

\[
(break s \text{ as } \varphi, f \text{ in } u)^* = u^*[k_0(s^*)]/\varphi][k_1(s^*)]/f
\]

where \(\pi_0, \pi_1\) are the \(\Lambda^\otimes\) projections, and \(k_0 = \lambda x \lambda p.p \ x\) and \(k_1 = \lambda x \lambda _. x\).

First, it is easy to prove by structural induction on the term \(s\) that the translation \(s \mapsto s^*\) commutes with substitution:

Lemma 5 \((s[t/x])^* = s^*[t^*/x]\)

Using the lemma above we can state precisely how the translation of \(\mathcal{B}\) terms maps to a translation of conversions:

Lemma 6 We have that:

(i) If \(t \rightsquigarrow t'\) via a non-silent standard conversion in \(\mathcal{B}\) then \(t^* \rightsquigarrow (t')^*\) in one or more standard conversions in \(\Lambda^\otimes\).
(ii) If \( t \leadsto t' \) via a silent standard conversion or a permuting conversion in \( \mathcal{B} \) then \( t^* = (t')^* \).

**Theorem 7** \( \mathcal{B} \) is strongly normalising.

**Proof:** Suppose that there was an infinite sequence \((t_i)_{i \in \mathbb{N}}\) of \( \mathcal{B} \) terms such that for each \( i \) we have that \( t_{i+1} \) is obtained from \( t_i \) by one of the \( \mathcal{B} \) conversions (standard or permuting). By Lemma 6 we would then obtain a sequence of \( \Lambda^\odot \)-terms \( (t^*_i)_{i \in \mathbb{N}} \) where for each \( i \), either

- \( t^*_{i+1} \) is obtained from \( t^*_i \) via one or more \( \Lambda^\odot \) conversions, or
- \( t^*_{i+1} = t^*_i \)

Since \( \Lambda^\odot \) is strongly normalising, we know that from some number \( N \) and all \( i \geq N \) we must have that \( t^*_i = t^*_{i+1} \). But this would mean that in the original sequence, we have an infinite chain of permuting conversions or silent standard conversions, contradicting Lemma 4. \( \blacksquare \)

### 4.3 Church-Rosser property

**Theorem 8** \( \mathcal{B} \) has the Church-Rosser property.

**Proof:** Since we have strong normalisation for \( \mathcal{B} \), by Newman’s lemma, it is enough to prove the weak Church-Rosser property, i.e., that if \( w \leadsto w' \) and \( w \leadsto w'' \) then there is a \( w''' \) such that \( w' \leadsto^* w''' \) and \( w'' \leadsto^* w''' \). The proof is by induction on the structure of \( w \) and is fairly standard, so we will only sketch it here. See [14, Theorem 6.3.9] for an example of this kind of proof. One checks that for all substitutions \( \sigma \) and all terms \( s \) and \( s' \)

- if \( s \leadsto s' \) then \( s[\sigma] \leadsto s'[\sigma] \), and
- for all total functions \( f \subseteq \leadsto^* \) we have \( s[\sigma] \leadsto^* s[\sigma'] \), where \( \sigma' = f \circ \sigma \).

These facts deal with the only tricky case in the proof for the simply-typed \( \lambda \)-calculus, when \( w \) is the \( \beta \)-redex \((\lambda x.s)u, w' = s[u/x] \) and \( w'' \) is either \((\lambda x.s')u \) or \((\lambda x.s')u' \). The proof now reduces to an analysis of the critical pairs: i.e., the reducts \( w' \) and \( w'' \) of a term \( w \) in which two redexes overlap in such a way that carrying out either conversion affects the structure required by the other conversion. Inspection of the conversions shows that there are no critical pairs involving \( \beta \)-conv, but critical pairs do arise for the following pairs of conversions:

| l-conv v. ap-l-conv | l-conv v. l-l-conv | b-conv v. ap-b-conv | b-conv v. l-b-conv | ap-l-conv v. l-l-conv | ap-l-conv v. l-b-conv | l-l-conv v. l-l-conv |

In the first four types of critical pair, the conversion to \( w' \) (say) eliminates both the redexes while \( w'' \) still has a redex of the same type as was used to reduce
w to w'. E.g., consider the critical pair of the form l-conv v. ap-l-conv. If we set:

\[
\begin{align*}
w &= (\text{let } t \otimes u \text{ be } x \otimes y \text{ in } s) r \\
w' &= (s[t/x, u/y]) r \\
w'' &= \text{let } t \otimes u \text{ be } x \otimes y \text{ in } (s r)
\end{align*}
\]

then \( w \xrightarrow{\text{l-conv}} w' \) and \( w \xrightarrow{\text{ap-l-conv}} w''. \) So taking \( w''' = w' \), we have that \( w' \xrightarrow{\ast} w''' \) (trivially) and that \( w'' \xrightarrow{\text{l-conv}} w''' \).

In the remaining three types of critical pair, both \( w' \) and \( w'' \) require further conversion. E.g., consider ap-l-conv v. 1-b-conv. If we set:

\[
\begin{align*}
w &= (\text{let } (\text{break } t \text{ as } \phi, f \text{ in } u) \text{ be } x \otimes y \text{ in } s) r \\
w' &= \text{let } (\text{break } t \text{ as } \phi, f \text{ in } u) \text{ be } x \otimes y \text{ in } (s r) \\
w'' &= (\text{break } t \text{ as } \phi, f \text{ in } (\text{let } u \text{ be } x \otimes y \text{ in } s)) r
\end{align*}
\]

then \( w \xrightarrow{\text{ap-l-conv}} w' \) and \( w \xrightarrow{\text{1-b-conv}} w'' \). But then putting

\[
w''' = \text{break } t \text{ as } \phi, f \text{ in } (\text{let } u \text{ be } x \otimes y \text{ in } (s r))
\]

we have:

\[
\begin{align*}
w' \xrightarrow{\text{1-b-conv}} \text{break } t \text{ as } \phi, f \text{ in } (\text{let } u \text{ be } x \otimes y \text{ in } (s, r)) &= w'''
\end{align*}
\]

and

\[
\begin{align*}
w'' \xrightarrow{\text{ap-b-conv}} \text{break } t \text{ as } \phi, f \text{ in } ((\text{let } u \text{ be } x \otimes y \text{ in } s)r)
\end{align*}
\]

\[
\begin{align*}
w'' \xrightarrow{\text{ap-l-conv}} \text{break } t \text{ as } \phi, f \text{ in } (\text{let } u \text{ be } x \otimes y \text{ in } (s, r)) &= w'''
\end{align*}
\]

The treatment of the other types of critical pair is similar.

Although the Church-Rosser property is not difficult to prove, it was quite tricky to find a suitable system of conversions. One of our earlier attempts included the following conversion

\[
\text{break}(\text{let } t \text{ be } x \otimes y \text{ in } u) \text{ as } \varphi, f \text{ in } s \xrightarrow{\text{b-l-conv}} \text{let } t \text{ be } x \otimes y \text{ in } (\text{break } u \text{ as } \varphi, f \text{ in } s)
\]

If we put:

\[
w = \text{break } \text{let } t \text{ be } x \otimes y \text{ in } u \text{ as } \phi, f \text{ in } s
\]

then we find (assuming \( \text{FV}(t) = \text{FV}(u) = \emptyset \)) that:

\[
\begin{align*}
w \xrightarrow{\text{b-l-conv}} s[\lambda p. p(\text{let } t \text{ be } x \otimes y \text{ in } u)/\phi, \lambda _-\text{let } t \text{ be } x \otimes y \text{ in } u/f]
\end{align*}
\]

and

\[
\begin{align*}
w \xrightarrow{\text{b-l-conv}} \text{let } t \text{ be } x \otimes y \text{ in } (\text{break } u \text{ as } \phi, f \text{ in } s)
\end{align*}
\]

\[
\begin{align*}
w \xrightarrow{\text{b-l-conv}} \text{let } t \text{ be } x \otimes y \text{ in } s[\lambda p. p u/\phi, \lambda _-u/f]
\end{align*}
\]

So \( w \) would have two distinct normal forms if we admitted \( \text{b-l-conv} \).
We have so far discussed the dynamics of $\mathbf{LL_m}$ natural deduction proofs via normalisation, using our Curry-Howard correspondence. We can also look at these results using a Gentzen-style calculus with left and right rules, and look into cut elimination. Such a system for $\mathbf{LL_m}$ is given in Figure 5.

**Theorem 9** The provable sequents of the Gentzen system of Figure 5 are the same as those of $\mathbf{LL_m}$.  

**Proof:** The break rule of Figure 5 matches precisely the break rule of our natural deduction system $\mathcal{B}$, which we have already shown to coincide with $\mathbf{LL_m}$. It is standard to show that the left and right rules are inter-derivable with the introduction and elimination rules of the natural deduction system.

**Theorem 10** The cut rule $\text{[CUT]}$ is eliminable from the proof system of Figure 5.

**Proof:** For each left and right rule, let us call the formula which is being introduced the major formula. When the application of cut involves two major formulas, then such cut can be replaced by cuts of smaller complexity, as in the standard cut elimination procedure. In all other cases, when the cut formula $A$ is not major, the cut can be pushed up the proof tree. This can also be done with the new break rule $\text{[BRK]}$. When one of the premises of the cut rule is an axiom the cut rule can be eliminated.

The reader might have noticed, however, that $\text{[BRK]}$ has a very similar flavour to $\text{[CUT]}$. But it follows directly from the theorem above that the rule $\text{[BRK]}$ is not derivable from $\text{[CUT]}$, since $\text{[CUT]}$ is eliminable but $\text{[BRK]}$ is not. There are, however, some particular instance of $\text{[BRK]}$ which are indeed derivable from $\text{[CUT]}$.

**Theorem 11** When $\Gamma = \emptyset$ then $\text{[BRK]}$ is derivable from $\text{[CUT]}$. 

---

Figure 5: Sequent calculus for $\mathbf{LL_m}$

\[
\Gamma, A \vdash A \quad \text{[ASM]}
\]

\[
\frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C} \quad \text{[CUT]}
\]

\[
\frac{\Gamma \vdash A \quad \Delta, K_B A, S_B A \vdash C}{\Gamma, \Delta \vdash C} \quad \text{[BRK]}
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \quad \text{[⇒R]}
\]

\[
\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \Rightarrow B \vdash C} \quad \text{[⇒L]}
\]

\[
\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \quad \text{[⊗R]}
\]

\[
\frac{\Gamma, A, B \vdash u : C}{\Gamma, A \otimes B \vdash C} \quad \text{[⊗L]}
\]
Proof: We can derive $[BRK]$ as follows:

\[
\frac{\vdash A}{\vdash B \Rightarrow A} \quad \frac{\vdash A}{\vdash K_B A} \quad \frac{\Delta, B \Rightarrow A \vdash C}{\Delta, K_B A, S_B A \vdash C} \quad [CUT]
\]

where the double lines indicate one or more steps.

**Theorem 12** When $K_B A$ or $S_B A$ is a superfluous assumption in proving $\Delta, K_B A, S_B A \vdash C$ then $[BRK]$ is derivable from $[CUT]$. 

Proof: Consider the case when $K_B A$ is not needed, so that we actually have $\Delta, S_B A \vdash C$. Such instance of $[BRK]$ can be derived from $[CUT]$ as:

\[
\frac{\vdash A}{\vdash B \Rightarrow A} \quad \frac{\vdash A}{\Delta, B \Rightarrow A \vdash C} \quad [CUT]
\]

The case where $S_B A$ is superfluous can be treated in the same way. ■

These last two theorems justify our side conditions for the conversion rule $\text{(b-conv)}$ from Section 2.4. The context $\Gamma$ being empty corresponds to the term $t$ being closed $(\text{FV}(t) = \emptyset)$, whereas $K_B A$ or $B \Rightarrow A$ begin superfluous assumptions correspond to $\varphi \notin \text{FV}(s)$ or $f \notin \text{FV}(s)$. From a Gentzen-style point of view, these are the cases where we can always replace a break rule by a standard cut rule.

### 6 Concluding Remarks

Strong normalisation for the standard simply-typed $\lambda$-calculus with pairing is well-known. Strong normalisation for the affine fragment of that calculus follows or can be proved more directly when one observes that the conversions always reduce the size of an affine term. Troelstra [15] proved strong normalisation for a variant of the term calculus for intuitionistic linear logic proposed by Benton et al. [4]. In that calculus it is the exponential operator $!$ that makes the normalisation result tricky, since the usual introduction rule for $!$ also acts as an elimination rule. Similarly, in $B$, the rule for $\text{break}$ complicates the normalisation proof. In both cases, the desire to control contraction is the source of the difficulty.

The decision problem for classical Lukasiewicz logic is known to be co-NP-complete while the decision problem for minimal Lukasiewicz logic can be shown to reduce to the decision problem for the equational theory of commutative GBL-algebras, which is known to be PSPACE-complete [3]. In both cases, the known decision procedures are based on semantic methods and no effective proof search methods are known. The present work is motivated by a desire either to find such algorithms or to understand why they cannot exist. It seems highly unlikely that a logic with a PSPACE-complete decision problem could admit an analytic inference system. However from the strong normalisation property, one can hope to derive effective bounds on the size of a deduction and the formulas in it and so, perhaps, find some weak form of the sub-formula property that could enable a proof-theoretic decision procedure.
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