Magnetic properties of confined holographic QCD

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Abstract. We investigate the Sakai-Sugimoto model at nonzero baryon chemical potential in a background magnetic field in the confined phase where chiral symmetry is broken. The D8-brane Chern-Simons term holographically encodes the axial anomaly and generates a gradient of the $\eta'$ meson, which carries a non-vanishing baryon charge. Above a critical value of the chemical potential, there is a second-order phase transition to a mixed phase which includes also ordinary baryonic matter. However, at fixed baryon charge density, the matter is purely $\eta'$-gradient above a critical magnetic field.

1. Introduction
The behavior of QCD under external conditions is an interesting and physically relevant problem. At high temperature the ground state is believed to be a deconfined quark-gluon plasma, and at high density it is believed to be a color-superconductor. The former is relevant for the understanding of the physics at RHIC, and the latter may be relevant for the physics of dense stellar objects such as neutron stars. Background electromagnetic fields provide another kind of external condition. Their effect on the QCD ground state is possibly relevant for magnetars, neutron stars with very large magnetic fields. However, it is often the case that in physically relevant regimes, QCD is strongly coupled, and we cannot reliably use perturbation theory.

An alternative approach to strongly-coupled gauge theories has emerged in recent years from string theory via holographic gauge/gravity duality. This approach is well-poised to address questions related to external conditions since these translate simply to boundary conditions on internal fields in the bulk.

Due to asymptotic freedom, the holographic dual of QCD must go beyond the supergravity approximation. Nevertheless, it may be useful to study string theory in a background where the low-lying excitations resemble those of QCD and in which one can consistently study the supergravity limit. This does not give QCD, but these kinds of models share many of its low-energy properties. The closest so far to QCD is the Sakai-Sugimoto model [1]. This model consists of $N_c$ D4-branes wrapping a circle with anti-periodic boundary conditions for the fermions, $N_f$ D8-branes at a point on the circle, and $N_f$ anti-D8-branes at another point on the circle. The low-lying open string excitations are precisely those of $SU(N_c)$ Yang-Mills theory with $N_f$ flavors of massless quarks. The holographic limit corresponds to $N_c \rightarrow \infty$, and $N_f$ is kept finite, so that the D8-branes are treated as probes in the near-horizon background of the D4-branes. In this limit, the background is capped off in the IR, corresponding to confinement, and the D8-branes and anti-D8-branes connect, corresponding to chiral symmetry breaking.
This model has also been studied in various external conditions, including nonzero temperature [2], nonzero baryon density [3, 4], and background electromagnetic fields [5, 6, 7, 8, 9, 10, 11], and it exhibits many properties that are expected of QCD.

Background magnetic fields are particularly interesting in that they may be physically relevant in neutron stars, where they can reach values of about 10^{15} Gauss. On the theory side, background magnetic fields have some interesting effects on the QCD ground state. One effect is the catalysis of chiral symmetry breaking by a strong magnetic field [12]. The effect of a background magnetic field in the Sakai-Sugimoto model was studied in [5, 6], where the catalysis of chiral symmetry breaking was demonstrated explicitly.

Here we will be interested in the effects of a background magnetic field at nonzero baryon chemical potential. This question was recently studied in the the low-energy effective field theory [13, 14, 15], where it was shown that the axial anomaly leads to interesting effects. In the confined phase, the anomaly leads to a non-trivial pion gradient and an associated baryon charge density [15],

\[ \nabla \pi^0 = \frac{e}{4\pi^2 f_\pi} \mu_B B, \quad d = \frac{e}{4\pi^2 f_\pi} B \cdot \nabla \pi^0. \]  

(1)

We will present here the results of [8] which show that similar effects occur in the one-flavor Sakai-Sugimoto model. In this model, the anomaly is encoded in the Chern-Simons term of the D8-brane action. The model does not include a true electromagnetic field, but we can mimic the effect of a non-dynamical background electromagnetic field using the non-normalizable mode of the D8-brane worldvolume gauge field. This field is actually dual to the baryon current in the gauge theory but is equal in the one-flavor case to the electric current. We will therefore use the same bulk gauge field, but different components, to describe both the baryon chemical potential and the background magnetic field. We will show that these source a third, normalizable component of the gauge field via the Chern-Simons term. In the low-temperature confining background this field has a nonzero boundary value, which is interpreted as the gradient of the \( U(1)_A \) pseudo-scalar meson, \( i.e. \) the \( \eta' \). This also leads to a baryon number charge density. For small magnetic fields, our result agrees with (1) adapted to the \( U(1)_A \) sector. Furthermore, we will show that there is a phase transition at a critical value of the chemical potential to a mixed phase of ordinary baryonic matter and pseudo-scalar gradient matter. In the mixed phase, the relative proportion of ordinary baryonic matter at fixed chemical potential decreases with the magnetic field.

2. Review of finite density HQCD

We first present an abbreviated review of the Sakai-Sugimoto model with nonzero baryon density and with background electromagnetic fields. For more detailed background see, for example, [1, 2] for the model and finite temperature phase structure, [3, 4] for nonzero baryon density, and [5, 6, 7, 8, 11] for electromagnetic fields.

The Euclidean metric dual to the confining phase is given by

\[ ds^2 = u^{3/2} \left( (dx_4^E)^2 + dx^2 + f(u)dx_4^2 \right) + u^{-3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right) \]  

(2)

where \( x_4 \) is a compact coordinate with periodicity \( 2\pi R_4 \), and

\[ f(u) = 1 - \frac{u^3_{KK}}{u^3}, \quad u_{KK} = \frac{4R^2}{9R_4^2}. \]  

(3)

See also [9] and [10], which present some related and overlapping results.
The curvature radius of the space is \( R = (\pi g_s N_c)^{1/3}/\sqrt{\alpha'} \).

The antipodal embedding of the D8-brane in this background has a U shape that satisfies \( x'_4(u) = 0 \), with the tip at \( u_{KK} \). The D8-brane worldvolume theory contains a gauge field, which contains both a vector and an axial part depending on the parity under exchanging the two halves of the embedding,

\[
a_M(x^\mu, u) = a_M^V(x^\mu, u) + a_M^A(x^\mu, u) .
\]

The physical gauge field is \( A_M = a_M R/(2\pi\alpha') \). There is a discrete spectrum of normalizable radial modes corresponding to various low-spin mesons. In particular, the zero mode of \( a_u^A \) is identified with the massless pseudo-scalar corresponding to the Goldstone boson of the broken chiral symmetry. For a single flavor, this is the \( \eta' \). Working in the gauge \( a_u = 0 \) \cite{1}, the pseudo-scalar reappears in the zero mode of \( a_u^A \),

\[
a_u^A(x^\mu, u) = \partial_\mu \phi(x^\mu) \psi_0(u) + \text{higher modes} ,
\]

where the zero-mode wavefunction \( \psi_0(u) \) yields a normalizable field strength. The physical pseudo-scalar field is \( \eta'(X) = f_\pi \phi(x) R^2/(2\pi\alpha') \), where \( f_\pi^2 = N_c u_{KK}^3/(2\pi\alpha') \).

By contrast, the asymptotic solution for the vector component of the gauge field at large \( u \) is

\[
a_0^V(u) \approx \mu - \frac{2}{3} \frac{d}{u^{3/2}} ,
\]

where the zero mode \( \mu \) is \( u \)-independent, and therefore non-normalizable, and is identified with the baryon chemical potential. Varying the action with respect to \( \mu \) allows us to identify the constant \( d \) as the baryon charge density. The physical chemical potential is \( \mu_B = \mu R/(2\pi\alpha') \), and the physical baryon charge density is \( D = d(2\pi\alpha'N_c/R) \).

In the absence of sources, the only solution is a constant: \( a_0^V(u) = \mu \). A second solution becomes possible when one includes sources at the tip corresponding to D4-branes wrapped on the \( S^4 \), which are precisely the baryons of the model. A single baryon carries \( N_c \) units of baryon charge and has mass \( m_4 = \frac{1}{2} N_c u_{KK} \), which is the mass of a wrapped D4-brane located at \( u = u_{KK} \). The physical D4-brane mass and density are given by \( M_4 = m_4 R/(2\pi\alpha') \) and \( N_4 = n_4(2\pi\alpha'N_c/R) \). The baryon charge density is then just \( d = N_c n_4 \). In the presence of baryon sources, the gauge field at the tip is

\[
a_0^V(u_{KK}) = \frac{m_4}{N_c} ,
\]

which implies that the solution exists only for \( \mu > m_4/N_c \). This has the obvious phenomenological interpretation that, at low temperature, baryons can only appear when the chemical potential is high enough to produce them. Furthermore, above this point, this “nuclear matter” solution dominates over the vacuum solution. In other words, nuclear matter forms as soon as it can, and there is a phase transition at \( \mu_c = m_4/N_c \). This nuclear matter phase transition is marginally second-order, which is different from the expected first-order transition in QCD. However, the result is reasonable since we have ignored baryon interactions.

3. Magnetic properties of the confined phase

To mimic the effect of a background magnetic field, we turn on a background value for the zero mode of a spatial component of the vector gauge field,

\[
a_3^V(x_2, u) = h x_2 ,
\]
The physical magnetic field is $H = h/(2\pi\alpha')$. Since $a_1^V(u) \neq 0$, the five-dimensional CS term, which comes from the D8-brane CS coupling to $F_3$, will source the field $a_1^A(u)$. As we saw in the previous section, the boundary value of this field corresponds to the (constant) gradient of the pseudo-scalar (in the $x^1$ direction in this case),

$$a_1^A(\infty) = \nabla \varphi.$$  \hfill (9)

This corresponds to a field, rather than an external parameter, in the gauge theory since the zero mode of the axial field is normalizable. Its value is therefore determined by extremizing the action. Furthermore, since $a_1^A$ is an axial field, it must vanish at the tip,

$$a_1^A(u_{KK}) = 0.$$  \hfill (10)

The D8-brane action with all the relevant fields is $S_{DBI} + S_{CS}$, where

$$S_{DBI} = \mathcal{N} \int_{u_{KK}}^{\infty} du \frac{u^{5/2}}{f(u)} \sqrt{\left( \frac{1}{f(u)} - (a_0^V(u))^2 + (a_1^A(u))^2 \right) \left( 1 + \frac{h^2}{u^2} \right)}$$

$$S_{CS} = -\mathcal{N} \int_{u_{KK}}^{\infty} du \left( \partial_2 a_3^V a_0^V(u) a_1^A(u) - \partial_2 a_3^V a_0^V(u) a_1^A(u) - a_3^V \partial_2 a_0^V a_1^A + a_3^V \partial_2 a_0^V a_1^A \right).$$

To deal with the last two terms in $S_{CS}$ which give contributions at the boundary, we add to the action the following boundary terms (as in [8]):

$$\Delta S = \frac{\mathcal{N}}{2} \int \left\{ \partial_2 \left( a_3^V a_0^V \right) + \partial_0 \left( a_2^V a_1^A \right) + \partial_0 \left( a_3 \partial_2 a_1^A a_0^V + a_2 \partial_2 a_1^A a_0^V \right) \right\}.$$  \hfill (12)

The combined action simplifies to

$$S_{CS} + \Delta S = \frac{3\mathcal{N}}{2} \int \left( h a_0^V a_1^A + c a_1^A \right).$$  \hfill (13)

The corresponding integrated equations of motion are given by

$$\frac{\sqrt{u^3 + h^2 u^2 a_0^V(u)}}{\sqrt{f(u)} - (a_0^V(u))^2 + (a_1^A(u))^2} = 3h a_1^A(u) + \mathcal{N} c_n$$

$$\frac{\sqrt{u^3 + h^2 u^2 a_1^A(u)}}{\sqrt{f(u)} - (a_0^V(u))^2 + (a_1^A(u))^2} = 3h a_0^V(u) + c,$$  \hfill (14) \hfill (15)

where the constant of integration in (14) has been identified with the density of D4-brane sources, and the constant of integration in (15) will be determined below.

The baryon charge and currents are defined by

$$J^\mu(x) = \frac{\delta S_{EOM}}{\delta A_\mu(x, u = \infty)} = \lim_{u \to \infty} \left( \frac{\delta S_{EOM}}{\delta \partial_u A_\mu(x)} \right),$$

which gives the (dimensionless) baryon charge density

$$d = \mathcal{N} c_n + \frac{3}{2} h a_1^A(\infty) = \mathcal{N} c_n + \frac{3}{2} h \nabla \varphi.$$  \hfill (17)
The origin of the second term can be understood as an additional D4-brane charge inside the D8-brane, which is due to the orthogonal worldvolume field strengths in the \((x^2, x^3)\) and \((u, x^1)\) directions. We can likewise get the (dimensionless) axial current density,

\[
j_A = c + \frac{3}{2} h a_0 V(\infty) = c + \frac{3}{2} h \mu.
\]  

(18)

We still need to extremize the action with respect to \(\nabla \phi\), which has the effect of setting \(j_A = 0\), and therefore \(c = -\frac{3}{2} h \mu\).

We can simplify the equations of motion considerably by introducing a new coordinate

\[
y = \int_{u_{KK}}^{u} \frac{3h du}{\sqrt{f(u)} \sqrt{u^5 \left(1 + \frac{h^2}{c^2}\right) + (N_c n_4)^2 - \left(\frac{3}{2} h \mu\right)^2 - 6h \kappa}}
\]  

(19)

where \(\kappa\) is

\[
\kappa = \begin{cases} 
-\frac{3}{2} h (\nabla \phi)^2 & \text{sourceless case} \\
-\frac{3}{2} h \frac{m_4}{N_c} \left(\mu - \frac{m_4}{N_c}\right) & \text{sourced case}.
\end{cases}
\]  

(20)

With some algebra, (14) and (15) can be written as

\[
a_0^V(y) = a_1^A(y) + \frac{N_c n_4}{3h},
\]  

(21)

\[
a_1^A(y) = a_0^V(y) - \frac{\mu}{2},
\]  

(22)

where the derivative is with respect to \(y\). Let us now analyze the two types of solutions.

### 3.1. Pseudo-scalar gradient phase

In the absence of sources, \(n_4 = 0\), all the baryon charge density comes from the pseudo-scalar gradient:

\[
d = \frac{3}{2} h \nabla \phi.
\]  

(23)

The solution to (21) and (22) is given in this case by

\[
a_0^V(y) = \frac{\mu}{2} \left(\cosh y \cosh y_{\infty} + 1\right)
\]  

(24)

\[
a_1^A(y) = \frac{\mu}{2} \frac{\sinh y}{\cosh y_{\infty}},
\]  

(25)

where \(y_{\infty} \equiv y(u \to \infty)\) can be determined numerically. The pseudo-scalar gradient is then

\[
\nabla \phi = \frac{\mu}{2} \tanh y_{\infty}.
\]  

(26)

The numerical results for \(\nabla \phi\) as a function of \(\mu\) and \(h\) are presented in Figure 1. For \(h \ll 1\), \(i.e.\) sub-string scale magnetic fields, the behavior is linear, and the pseudo-scalar gradient is approximately given by

\[
\nabla \phi \approx \frac{\pi}{2 a_{KK}^{3/2}} \mu h.
\]  

(27)
In terms of the physical quantities this is

\[ \nabla \eta' \approx \frac{N_c}{8\pi f_\pi} \mu_B H \]  

(28)

for small magnetic fields. This agrees, with appropriate change in normalization,² with the one-flavor version of the result (1) from [15].

3.2. Mixed phase

At sufficiently large chemical potential, a solution with D4-brane sources is also possible. In this phase, both the baryons and the pseudo-scalar gradient contribute to the baryon charge density.

Using the boundary conditions at the tip \( a_0^V(y=0) = m_4/N_c \) and \( a_1^A(y=0) = 0 \), the solution to the equations of motion (21) and (22) is now given by

\[
\begin{align*}
a_0^V(y) &= \left( \frac{m_4}{N_c} - \frac{\mu}{2} \right) \cosh y + \frac{N_c n_4}{3h} \sinh y + \frac{\mu}{2} \\
a_1^A(y) &= \left( \frac{m_4}{N_c} - \frac{\mu}{2} \right) \sinh y + \frac{N_c n_4}{3h} (\cosh y - 1).
\end{align*}
\]  

(29)

(30)

The boundary conditions at infinity then determine the gradient and D4-brane density implicitly in terms of \( \mu \) and \( h \),

\[
\begin{align*}
\nabla \varphi &= \cosh y_\infty \left( 1 - \frac{m_4}{N_c} \right) \\
N_c n_4 &= \frac{3h\mu + \frac{3}{2} h \left( \mu - \frac{2m_4}{N_c} \right) \cosh y_\infty}{\sinh y_\infty}.
\end{align*}
\]  

(31)

(32)

The critical value of the chemical potential corresponds to the point at which the actions of the \( \nabla \varphi \) and mixed phases are equal. But, it also coincides, as it did when \( h = 0 \), with the minimal value of the chemical potential to create a baryon. The phase diagrams in the \((\mu, h)\) and \((d, h)\) planes are shown in Figure 2. The critical chemical potential increases from its \( h = 0 \) value \( m_4/N_c \) to \( 2m_4/N_c \) as \( h \to \infty \). For a given \( h \), there is a marginally second-order phase transition from the \( \nabla \varphi \) phase to the mixed phase at \( \mu_c(h) \), which generalizes the ordinary nuclear matter transition at \( h = 0 \). We also see that, for a fixed total baryon charge density, the pseudo-scalar gradient phase dominates above a critical magnetic field.

² The relative factor of \( N_c/2 \) is understood as follows. First, the baryon charge of a quark in [15] is \( 1/N_c \) so \( \mu_B^{\text{here}} = N_c \mu_B^{\text{here}} \). Second, the CS term coupling the baryonic \( U(1)_V \) gauge field to the \( \sigma_3 \)-component of the \( SU(2) \) gauge field has a factor of 2 relative to the purely abelian CS term once the boundary term at spatial infinity is omitted.
Figure 2. Phase diagram in the (a) canonical and (b) grand canonical ensemble.

The total baryon charge density is given by
\[
d = N_c n_4 + \frac{3}{2} h \nabla \varphi = \frac{3h}{2} \left( \mu - \frac{m_4}{N_c} \right) \frac{\cosh y_\infty + 1}{\sinh y_\infty}. \tag{33}
\]

Figure 3 shows the fraction, obtained numerically, of that charge carried by baryons. We see that the relative proportion of baryons at fixed \( h \) increases with \( \mu \). In the limit of large \( \mu \), the system is almost entirely baryonic nuclear matter. On the other hand, at fixed \( \mu \) the proportion of baryons decreases with \( h \). For \( m_4/N_c < \mu < 2m_4/N_c \), the proportion of baryons vanishes at the critical magnetic field \( h_c(\mu) \) shown in Figure 2, where the \( \nabla \varphi \) phase takes over.

Figure 3. The baryon fraction \( n_4 N_c / d \), as a function of \( \mu \) for fixed \( h = 1 \) and as a function of \( h \) for fixed \( \mu = 3m_4/N_c \), both with \( u_{KK} = 1 \).

3.3. Magnetization

The state described by either the pseudo-scalar gradient phase or the mixed phase responds to the external magnetic field by becoming magnetized. The magnetization \( M \) in the grand canonical or canonical ensemble is
\[
M(\mu, h) = -\partial \Omega(\mu, h) / \partial h, \quad M(d, h) = -\partial F(d, h) / \partial h,
\]
where the grand potential is \( \Omega(\mu, h) = S[a_0(u), a_1(u)]_{EOM} \) and the free energy is \( F = \Omega + \mu d \).

We focus here on the matter contribution to the magnetization. The magnetic properties of the vacuum were studied in [5, 6]. We will therefore subtract from the quantities above the formally divergent contribution of the vacuum, which gives a finite result that represents the corresponding contribution of just the matter. The numerical results for the magnetizations in the pseudo-scalar gradient and mixed phases are presented in Figure 4. For small \( h \), the response is linear, but for \( h \sim O(1) \) the non-linear effect of the DBI action becomes pronounced. We find that the matter is paramagnetic in both phases.
4. Conclusions
We have explored the properties of holographic QCD at finite density in a background magnetic field and find the system has a rich phenomenology. In the holographic dual, these axial anomaly-induced properties are induced by the Chern-Simon term on the D8-brane. In the confined phase, turning on a magnetic field induces a gradient for the pseudo-scalar $\eta'$ field. This gradient carries baryon charge, and at large enough magnetic fields, it is the dominant phase. That is, if we start at zero field with some baryons, as we increase the field those baryons will start being replaced by a gradient of the $\eta'$ field, eventually disappearing altogether.

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