Classification of $E_0$–Semigroups by Product Systems

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Abstract

In these notes we tie up some loose ends in the theory of $E_0$–semigroups and their classification by product systems of Hilbert modules. We explain how the notion of cocycle conjugacy must be modified in order to see how product systems classify $E_0$–semigroups. Actually, we will find two notions of cocycle conjugacy (which for Hilbert spaces coincide) that lead to classification up to isomorphism of product systems and up to Morita equivalence of product systems, respectively. (In between there is also a classification up to generalized isomorphism of product systems.) Together with the existence results of $E_0$–semigroups for product systems (here for the first time also for the $W^*$–case) this concludes the classification theory. As a by-product we now show also in the $W^*$–case that every faithful product system admits a faithful normal nondegenerate representation.

Apart from these new results, we provide also general versions of results known for Hilbert modules with unit vectors. In this context it is indispensable to review the notions of Morita equivalent product systems and Morita equivalent Hilbert modules, adding some generalities that have not yet been mentioned elsewhere. In any case, we underline the outstanding role played by Morita equivalence in the relation between $E_0$–semigroups and product systems. As usual with Morita equivalence, the most satisfying form of the results we find for von Neumann algebras. Some of the $C^*$–versions of the results will depend on countability assumptions.

Altogether, we have now a complete the theory of the classification of normal $E_0$–semigroups on $\mathcal{B}(E)$ by product systems of von Neumann correspondences. We have the same theory for the classification of strict $E_0$–semigroups by product systems of $C^*$–correspondences under countability hypotheses. In both cases, we apply our theory to prove that a Markov semigroup admits a Hudson–Parthasarathy dilation if and only if it is spatial. Mainly in the appendices we provide the necessary results of a beginning theory of strong type I product systems and of strongly continuous product systems.

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1 Introduction

The emergence of tensor product systems of bimodules over a unital ring $\mathcal{A}$ from endomorphism semigroup on $\mathcal{A}$ is, actually, a quite simple issue. The situation gets more interesting, when $\mathcal{A}$ is the algebra of all endomorphisms of right module $E$ over another ring $\mathcal{B}$. The $\mathcal{A}$–$\mathcal{B}$–module $E$ (with the natural left action of $\mathcal{A}$) plays, then, the role of a Morita equivalence from $\mathcal{A}$ to $\mathcal{B}$. This allows to “transform” the product system of $\mathcal{A}$–bimodules into a product system of $\mathcal{B}$–bimodules; confer Section 7 and Footnote [d] in Section 11.

It is natural to ask how endomorphism semigroups are classified by their product system. In these notes we give a conclusive answer for unital endomorphism semigroups in the case of arbitrary von Neumann (or $W^*$) modules $E$ (with and without continuity conditions). In the case of $C^*$–modules, the results partly depend on countability hypotheses.

Let us describe these algebraic ideas in more detail. By $\mathbb{S}$ we denote one of the additive semigroups $\mathbb{R}_+ = [0, \infty)$ or $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. An $E_0$–semigroup is a semigroup $\vartheta = (\vartheta_t)_{t \in \mathbb{S}}$ of unital endomorphisms $\vartheta_t$ of a unital $*$–algebra $\mathcal{A}$, fulfilling $\vartheta_0 = id_{\mathcal{A}}$. Every $E_0$–semigroup gives rise to a product system of bimodules $E_t$ over $\mathcal{A}$ under tensor product over $\mathcal{A}$ in the following way: Simply put $E_t = _t \mathcal{A}$, that is, the right module $\mathcal{A}$ with left action $a, x_t := \vartheta_1(a)x_t$ of $\mathcal{A}$ via $\vartheta_t$. Denote by $\otimes$ the tensor product over $\mathcal{A}$. For every $s, t \in \mathbb{S}$ we define an
isomorphism \( u_{s,t} : E_s \odot E_t \to E_{s+t} \) of bimodules by \( x_s \odot y_t \mapsto \vartheta_t(x_s)y_t \) and these isomorphisms iterate associatively. Moreover, \( E_0 = \mathcal{A} \), the **trivial** \( \mathcal{A} \)--bimodule, and for \( s = 0 \) and \( t = 0 \) the isomorphisms \( u_{0,0} \) and \( u_{s,0} \) reduce to left and right action of elements of \( \mathcal{A} \) on \( E_t \) and \( E_s \), respectively. (So far, this works even if \( \mathcal{A} \) is just a unital ring, not necessarily a \( * \)--algebra.) If \( \mathcal{A} \) is a \( C^* \) or a von Neumann algebra, then each \( E_t \) is also a Hilbert \( \mathcal{A} \)--module with inner product \( \langle x_t, y_t \rangle := x_t^*y_t \). In fact, \( E_t \) with its bimodule structure is a correspondence over \( \mathcal{A} \) and the \( u_{s,t} \) are also isometric for the tensor product of correspondences.

We see that the family \( E^\odot = (E_t)_{t \in \mathbb{S}} \) forms a tensor product system in the sense of Bhat and Skeide [BS00]. We call such a product system a **one-dimensional** product system, because all right \( \mathcal{A} \)--modules \( E_t \) are one-dimensional. Every one-dimensional product system of \( \mathcal{A} \)--correspondences arises in that way from an \( E_0 \)--semigroup on \( \mathcal{A} \). The **trivial** product system is that one-dimensional product system where also the left action on each \( \mathcal{A} \) is the trivial one and where the product system operation is just multiplication in \( \mathcal{A} \). The product system of an \( E_0 \)--semigroup \( \vartheta \) on \( \mathcal{A} \) is isomorphic to the trivial one if and only if \( \vartheta \) is a semigroup of inner automorphisms. All these statements are easy exercises; see [BS00] Remark 7.10 and the discussion preceding it.

If \( \mathcal{A} = \mathcal{B}^a(E) \) is the \( (C^* \) or von Neumann) algebra of all adjointable mappings on a Hilbert \( \mathcal{B} \)--module \( E \), and if all \( \vartheta_t \) are sufficiently continuous (strict in the \( C^* \)--case and normal in the von Neumann case), then the situation gets more interesting. The representation theory of \( \mathcal{B}^a(E) \) asserts that for each \( \vartheta_t \) there is a **multiplicity correspondence** \( E_t \) over \( \mathcal{B} \) and an identification \( E = E \odot E_t \) in such a way that \( \vartheta_t(a) = a \odot \text{id} \), for all \( a \in \mathcal{B}^a(E) \). Moreover, the \( E_t \) compose associatively as \( E_s \odot E_t = E_{s+t} \) under tensor product (of \( C^* \) or of von Neumann correspondences over \( \mathcal{B} \)), that is, they form a product system \( E^\odot \).

The representation theory, *cum grano salis*, may be viewed as an operation of **Morita equivalence** from the correspondence \( \vartheta, \mathcal{B}^a(E) \) over \( \mathcal{B}^a(E) \) to the correspondence \( E_t \) over \( \mathcal{B} \), where the Hilbert \( \mathcal{B} \)--module \( E \) plays, again *cum grano salis*, the role of the Morita equivalence from \( \mathcal{B}^a(E) \) to \( \mathcal{B} \); see Footnote [4] in Section 11. In this sense, that is *cum grano salis*, the product system \( E^\odot \) of correspondences over \( \mathcal{B} \) is **Morita equivalent** to the one-dimensional product system \( (\vartheta_t, \mathcal{B}^a(E))_{t \in \mathbb{S}} \) of correspondences over \( \mathcal{B}^a(E) \). (If \( E \) is a full Hilbert \( \mathcal{B} \)--module, then it is a Morita equivalence from the compacts \( \mathcal{K}(E) \) to \( \mathcal{B} \). This is what *cum grano salis* is referring to. In the von Neumann case the statements the statements are exact.)

There are several natural questions about the correspondence between \( E_0 \)--semigroups and product systems. The first question is which product systems occur as the product system of an \( E_0 \)--semigroup. In other words, what are necessary and sufficient conditions a product system must satisfy in order that it comes from an \( E_0 \)--semigroup. The second question is to what extent the product system of an \( E_0 \)--semigroup determines the \( E_0 \)--semigroup. In other words, what is the classification of \( E_0 \)--semigroups induced by the isomorphism classes among all product
systems that stem from $E_0$–semigroups.

Arveson, who initiated the modern theory of product system in [Arv89] with the first concise definition of product systems of Hilbert spaces (Arveson systems in the sequel), answered both questions for the case when $E = H$ is an infinite-dimensional separable Hilbert space: Every Arveson system is the product system of an $E_0$–semigroup on $\mathcal{B}(H)$ [Arv90]. Two $E_0$–semigroups have isomorphic Arveson systems if and only if they are unitary cocycle conjugate [Arv89].

The first result, existence of an $E_0$–semigroup for every Arveson system, has been reproved in Liebscher [Lie09] (preprint 2003; adding also that the $E_0$–semigroup may be chosen pure), in Skeide [Ske06a] (with a short and elementary proof), and in Arveson [Arv06] (leading to an $E_0$–semigroup unitarily equivalent to that in [Ske06a]; see [Ske06c]). It has been generalized to Hilbert modules over unital $C^*$–algebras in Skeide [Ske07a] and to modules over possibly nonunital $C^*$–algebras in Skeide [Ske09b]. In these notes we also add the case of von Neumann algebras (Theorem 12.4). The case of von Neumann algebras (different from the preceding case only if continuity in time is required) requires a certain amount of technical discussion and is, therefore, proved in Appendix B.1.

The proof of the second result, classification up to cocycle conjugacy, is quite simple for Hilbert spaces. But, it relies on two facts: First, there is the hidden assumption that all Hilbert spaces on whose operator algebras the $E_0$–semigroups act are infinite-dimensional and separable. In other words, they are all isomorphic. Second, all automorphisms of $\mathcal{B}(H)$ are inner. In fact, by the same proof as for Hilbert spaces, one also shows for Hilbert modules $E$ that two $E_0$–semigroups on $\mathcal{B}^a(E)$ have isomorphic product systems if and only if the semigroups are unitary cocycle equivalent [Ske02] Theorem 2.4]. (By this, we mean that one $E_0$–semigroup occurs as a unitary cocycle perturbation of the other; see Section 5.) Moreover, if $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are isomorphic, so that $E$ and $F$ are Morita equivalent in the sense of Skeide [Ske09d], then an $E_0$–semigroup on $\mathcal{B}^a(E)$ and an $E_0$–semigroup on $\mathcal{B}^a(F)$ have Morita equivalent product systems if and only if the semigroups are unitary cocycle conjugate [Ske09d, Corollary 5.11]. For this, $E$ and $F$ need not even be modules over the same algebra. However, there do exist pairs of $E_0$–semigroups with isomorphic product systems that act on nonisomorphic operator algebras $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ for Hilbert $\mathcal{B}$–modules $E$ and $F$. We also note that two $E_0$–semigroups acting on the same $\mathcal{B}^a(E)$ that are unitary cocycle conjugate (that is, a conjugate of one $E_0$–semigroup via an automorphism of $\mathcal{B}^a(E)$ is unitary cocycle equivalent to the other), need not be unitary cocycle equivalent. This is possible, because not all automorphisms of $\mathcal{B}^a(E)$ are inner. (Note that all endomorphisms and isomorphisms are required strict in the $C^*$–case and normal in the von Neumann case.)

For Hilbert modules, the question which classification of $E_0$–semigroups (acting on not nec-
essarily isomorphic $\mathcal{B}^a(E)$s is induced by classifying their product systems up to isomorphism, so far, has been open. Here is where these notes start. We answer the question for $C^*$—modules that are “sufficiently separable” (we need Kasparov’s stabilization theorem), and for von Neumann modules in full generality. We also answer the question when the classification is up to Morita equivalence of product systems. As an application we show that Markov semigroups admit so-called Hudson-Parthasarathy dilations if (and in a sense only if) they are spatial. In order to complete the classification of $E_0$—semigroup by product systems, we also provide a proof of existence of an $E_0$—semigroup for a strongly continuous product system of von Neumann correspondences. Definition and result, though announced several times, appear for the first time in these notes. As a side result, possible because all technical prerequisites had to be made available for the other proof, we also add existence of faithful nondegenerate normal representations (that is, right dilations) for such product systems. (Note that for existence of an $E_0$—semigroup the product system is necessarily strongly full, and that for existence of a right dilation the product systems is necessarily faithful. These properties are dual to each other in the sense of commutants of von Neumann correspondences.)

The solution of the classification problem is, in it its simplicity, a bit of a surprise: $E_0$—semigroups are classified up to stable cocycle conjugacy. That is, in order that two $E_0$—semigroups have isomorphic or Morita equivalent product systems, suitable amplifications of the $E_0$—semigroups must be cocycle conjugate in a suitable sense. The reason for this fact is that all full countably generated Hilbert $\mathcal{B}$—modules $E$ have isomorphic multiples $E^\infty$. More precisely, $E^\infty \cong B^\infty$, the column space over $\mathcal{B}$. Since amplification of an $E_0$—semigroup does not change its product system, it follows that we may examine the amplified $E_0$—semigroups, which now act on the same algebra $\mathcal{B}^a(\mathcal{B}^\infty)$, for unitary cocycle equivalence. Inner conjugacy (that is, conjugacy via an inner automorphism of $\mathcal{B}^a(\mathcal{B}^\infty)$) of one of the two $E_0$—semigroups $\vartheta$ and $\vartheta'$ acting on $\mathcal{B}^a(\mathcal{B}^\infty)$ does not change unitary cocycle equivalence among them; see Proposition 5.5. But, conjugacy via a non-inner automorphism possibly does! The classification of $E_0$—semigroups on $\mathcal{B}^a(\mathcal{B}^\infty)$ up to unitary cocycle equivalence is reflected by the isomorphism classes of the product systems. If we weaken to classification up to unitary cocycle equivalence after conjugation with an arbitrary (strict) automorphism of $\mathcal{B}^a(\mathcal{B}^\infty)$, then we get Morita equivalence of product systems; this is strictly weaker (see Example 7.4). The question for Morita equivalence of product systems, however, makes sense also for product systems of correspondences over different $C^*$—algebras $\mathcal{B}$ and $\mathcal{C}$. A necessary condition is, of course, that $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent, say, via a Morita equivalence $M$ from $\mathcal{B}$ to $\mathcal{C}$. However, in this case the Hilbert modules $\mathcal{B}^\infty$ ans $\mathcal{C}^\infty$ are Morita equivalent in the sense that $\mathcal{B}^\infty \odot M \cong \mathcal{C}^\infty$. Therefore, the operators algebras $\mathcal{B}^a(\mathcal{B}^\infty)$ and $\mathcal{B}^a(\mathcal{C}^\infty)$ are strictly isomorphic so that the question for unitary cocycle conjugacy makes sense. As a result, for countably generated Hilbert modules we get classification of $E_0$—semigroups up stable unitary cocycle conjugacy by Morita equivalence classes.
of their product systems. A third classification, *ternary equivalence*, emerges when we restrict Morita equivalence to isomorphic $C^*$–algebras so that $B^a(B)$ and $B^a(C)$ become *ternary isomorphic*; this notion lies strictly in between inner isomorphism and strict isomorphism. All this also holds in the for von Neumann case, now without any countability restriction (and generally simpler proofs).

**Remark.** Alevras [Ale04] investigated $E_0$–semigroups on type II$_1$ factors, and classified them up to unitary cocycle conjugacy by their product systems. (Also the notion of *stable* unitary cocycle conjugacy is mentioned there, but in the end not necessary in the classification.) It should be noted, however, that the product system constructed from an $E_0$–semigroup on a type II$_1$ factor $B$ in [Ale04] is the commutant of our one-dimensional product system as discussed in the beginning of this introduction. (We explain some facts about commutants of product systems and their relation in Appendix B.2. Our approach has the advantage that it gives “good” classification results also in the $C^*$–case, while the approach via intertwiner space is *a priori* restricted to von Neumann algebras.) This product system coincides with the product system of $B$–correspondence when we identify $B^a(B) = B$. And for $E_0$–semigroups on $B^a(E)$ we had the same result (classification up to unitary cocycle conjugacy, though without technical conditions like continuity or measurability) in [Ske02, Theorem 2.4]. And the commutant of product systems preserves both isomorphisms and Morita equivalences. We also mention that the product system in [Ale04] is shown to have a trivial bundle structure in terms of Borel bundles. (This depends on the choice to consider only type II$_1$ factors and on the assumption of a separable pre-dual.) A more general definition of *weakly measurable* product system has been proposed by Muhly and Solel [MS07b]. (A version of countably generated *measurable* product systems of $C^*$–correspondences has been introduced by Hirshberg [Hir04]. In [Hir05b] he showed the universal property of representations to extends uniquely to representations of the associated *spectral* $C^*$–algebra. As mentioned above, this direction is not so directly related to $E_0$–semigroup. For a given representation there is an $E_0$–semigroup on a certain von Neumann algebra; see Fact [B.20][2] in Appendix B.2. But this $E_0$–semigroup does, in general, not allow to reconstruct the product system of $C^*$–correspondences.) In these notes, when we take into consideration technical conditions, these will be continuity conditions. This has the advantage that in the von Neumann case no countability requirements are necessary.

These notes are organized as follows. Morita equivalence is crucial. In Section 2 we give a definition of Morita equivalence (Definition 2.1) that, in our opinion runs more smoothly than others, and we prove (Theorem 2.7) that it is equivalent to one of the usual definitions. We also review the results from Muhly, Skeide and Solel [MSS06] about strict representations of $B^a(E)$ and from Skeide [Ske09d] about strict isomorphisms.
In Section 3 we explain the basic idea based on the well-known result (in the form of Lance \cite[Proposition 7.4]{Lan95}) that asserts that for a full countably generated Hilbert module $E$ over a $\sigma$–unital $C^*$–algebra the multiple $E^\omega$ is isomorphic to the column space $B^\omega$. In particular, all such modules $E$ are stably isomorphic. A version for modules over different algebras gives rise to the notion of stable Morita equivalence for Hilbert modules (Definition 3.4) with the basic Theorem 3.5 about when stable Morita equivalence happens. In Section 4 we discuss ternary isomorphisms and derive analogue results for that subclass of Morita equivalences among Hilbert modules.

Starting from Section 5 we come to cocycle conjugacy of $E_0$–semigroups. While for algebras isomorphic to $\mathcal{B}(H)$ there is essentially only one notion of cocycle conjugacy, for general $*$–algebras this is not so. In Section 5 we discuss cocycle conjugacy of $E_0$–semigroups in the completely algebraic setting from the beginning of this introduction. In Section 6 we switch to $E_0$–semigroups on $B^\omega(E)$. We explain how the product system of an $E_0$–semigroup is defined, and how it gives back the $E_0$–semigroup in terms of a left dilation. We explain what it means that two $E_0$–semigroups acting on the same $B^\omega(E)$ have isomorphic product systems (Proposition 6.3 and Theorem 6.5). In Theorem 6.7 we give a necessary and sufficient criterion for that a family of $E_0$–semigroups acting on different full Hilbert $\mathcal{B}$–modules all have the same product system. This theorem is, however, not in terms of cocycle conjugacy and, therefore, gives only a partial solution to our classification problem. Section 7 classifies $E_0$–semigroup acting on strictly isomorphic $B^\omega(E)$ and $B^\omega(F)$ in terms of Morita equivalence of product systems (Theorem 7.3 and discusses the restriction to the special case of ternary equivalence. In Section 8 we apply the amplification idea from Section 3 to classify (under suitable countability assumptions) all $E_0$–semigroups in terms of stable cocycle conjugacy either by isomorphism classes, or by Morita equivalence classes, or by ternary equivalence classes of their product systems (Theorem 8.3). Section 9 finally, takes into account questions of continuity (strong continuity on the $E_0$–semigroup side and continuity on the product systems side). Provided we have a countably generated product system $E^\omega$, only with continuity conditions and for unital $C^*$–algebras we are able to guarantee that the $E_0$–semigroup constructed for $E^\omega$ in Skeide \cite{Ske07a} acts on a $B^\omega(E)$ where $E$ is countably generated. Under these hypotheses we obtain one-to-one classification (Theorem 9.4) of $E_0$–semigroups by product systems with continuity conditions. (The case of not necessarily unital $C^*$–algebras is resolved in Skeide \cite{Ske09b}.)

One of the basic problems of quantum probability is to obtain dilation of Markov semigroup to a unitary cocycle perturbation of a noise, a so-called Hudson-Parthasarathy dilation. In Section 10 we apply our classification result to prove existence of Hudson-Parthasarathy dilations for spatial Markov semigroups acting on a unital separable $C^*$–algebra (Theorem 10.12). A Markov semigroup is spatial if it dominates a so-called elementary CP-semigroup. Without continuity conditions, this property is always fulfilled. However, if the elementary CP-semigroup is
required continuous, then the algebraic domination condition is so strong that, in the $C^*$–case, spatial Markov semigroups are always uniformly continuous. (The von Neumann case is more interesting; see below.) However, the condition to be spatial for that a Markov semigroup admits a Hudson-Parthasarathy dilation is, in a sense, also necessary.

In the remainder we tackle the von Neumann case and resolve it, unlike the $C^*$–case where countability hypotheses are required, in full generality. In Section 11 we obtain the analogue results about algebraic classification of Sections 2–8 and also parts of Section 9. In Section 12 we obtain the analogue results in the continuous case. Since continuity in that case means strong continuity in the von Neumann sense (which is much weaker a condition), considerable work has to be done. In particular, for the first time we give a concise definition of strongly continuous product system of von Neumann correspondences (Definition 12.1) and we prove that every such (strongly full) product system is the strongly continuous product system of a strongly continuous normal $E_0$–semigroup (Theorem 12.4). We obtain results in full analogy with those of Section 9 and without any countability requirement. Section 13 finally deals with Hudson-Parthasarathy dilations of spatial Markov semigroups on von Neumann algebras. Once more, the classification up to stable cocycle conjugacy allows to prove that a Markov semigroup on a von Neumann algebra admits a Hudson-Parthasarathy dilation if (and, in a sense, only if) it is spatial (Theorem 13.3). It is here in the von Neumann case, where this result shows its full power. Indeed, while in the $C^*$–case the spatial Markov semigroups is limited to a subset of the uniformly continuous Markov semigroups, in the von Neumann case most strongly operator continuous Markov semigroups are spatial; see the introduction of Section 13.

These notes, clearly, aim at fixing the open problem in the classification of $E_0$–semigroups on possibly different operator algebras $B^o(E)$. But for having a complete solution of the classification problem (at least, in the von Neumann case without countability assumptions), we also need the existence result: Every suitably full and suitably continuous product system comes form a suitably continuous $E_0$–semigroup. While the $C^*$–case is known, the von Neumann case is a new result of these notes. However, even without that these notes are already quite long, and the existence result, though very important inside the whole theory, is not among main scopes of these notes. Therefore, we postpone its proof to Appendix B. Appendices A and B may be considered as the beginning of a more systematic theory of strongly continuous product systems. But, the theory is developed only as far as necessary for our applications: Appendix A deals with strong type I product systems (systems generated by their strongly continuous units), and develops what is necessary to prove the results about spatial Markov semigroups. Appendix A also fills a gap in the proofs of [BS00 Theorems 10.2 and 12.1] (see acknowledgments), generalizing them considerably (Theorems A.1 and A.4). As Corollary A.6 we reprove the result due to Markiewicz and Shalit [MS07a] that every weakly operator continuous CP-semigroup on a von Neumann algebra is strongly operator continuous. We also prove that spatial Markov semi-
groups on a von Neumann algebra have spatial GNS-systems (Theorem A.14) contrasting the negative statement in the $C^*$-case (see Bhat, Liebscher, and Skeide [BLS10]), and we prove that elementary CP-semigroups have trivial GNS-systems (Lemma A.13). Appendix B provides the proof of existence of $E_0$-semigroups in the von Neumann case (Theorem 12.4), utilizing also some results from Appendix A. In Appendix B.2 we prove the new result that every faithful strongly continuous product systems admits a strongly continuous nondegenerate normal representations by operators on a Hilbert space. This result is connected rather indirectly with the classification of product system. (See the remark in this introduction.) However, it is important in the theory of product systems, and the technical tools developed in both appendices allow easily the necessary modification to the proof of the $C^*$-case in Skeide [Ske06e]. Therefore, we include it here. As applications we provide a different proof of a result by Arveson and Kishimoto [AK92] about embeddability into an automorphism group, and a result about existence of elementary dilations of strongly continuous CP-semigroups on von Neumann algebras.

For the orientation of the reader, we mention that several sections contain more detailed introductions to special aspects that could be dealt with only rather superficially in the present general introduction: Section 10 (dilations, noises, and Hudson-Parthasarathy dilations); Section 11 (von Neumann and $W^*$-modules); Appendix B.2 (representations and commutants of product systems). Section 2 (Morita equivalence and representations of $\mathcal{B}(E)$) and Section 5 (algebraic cocycle conjugacy) are introductory in their own right.

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2 Morita equivalence and representations

The relation between $E_0$–semigroups and product systems goes via the representation theory of $\mathcal{B}^a(E)$ for a Hilbert $\mathcal{B}$–module $E$. The representation theory has been discussed first in Skeide [Ske02] in the case when $E$ has a unit vector $\xi$ (that is, $\langle \xi, \xi \rangle = 1 \in \mathcal{B}$) and in Muhly, Skeide and Solel [MSS06] for the general case. In particular the approach in [MSS06], a slight extension of Rieffel’s [Rie74a] discussion of the representations of the imprimitivity algebra (that is, the finite-rank operators; see below) underlines the role played by Morita equivalence. We use this section to introduce some notation and to review the relation between Morita equivalence and the theory of strict representations of $\mathcal{B}^a(E)$. The definition of Morita equivalence we use here is different from standard definitions. Although it is probably folklore that it is equivalent to standard definitions, we do not know any reference. Therefore, we include a proof.

Let $E$ be Hilbert module over a $C^*$–algebra $\mathcal{B}$. We say, $E$ is full if the range ideal $\mathcal{B}_E := \text{span}(E, E)$ in $\mathcal{B}$ coincides with $\mathcal{B}$. By $\mathcal{B}^a(E)$ we denote the algebra of all adjointable operators on $E$. Often, we consider an element $x \in E$ as mapping $x : b \mapsto xb$ from $\mathcal{B}$ to $E$. The adjoint of that mapping is $x^* : y \mapsto \langle x, y \rangle$. The linear hull of the rank-one operators $xy^*$ is the algebra $\mathcal{F}(E)$ of finite-rank operators. The completion $\mathcal{K}(E)$ of $\mathcal{F}(E)$ is the algebra of compact operators. We use similar notations for operators between Hilbert $\mathcal{B}$–modules $E$ and $F$.

A correspondence from a $C^*$–algebra $\mathcal{A}$ to a $C^*$–algebra $\mathcal{B}$ (or $\mathcal{A}$–$\mathcal{B}$–correspondence) is a Hilbert $\mathcal{B}$–module $E$ with a left action of $\mathcal{A}$ that defines a nondegenerate representation of $\mathcal{A}$ by adjointable operators on $E$. A correspondence is faithful if the left action defines a faithful representation. The (internal) tensor product of an $\mathcal{A}$–$\mathcal{B}$–correspondence $E$ and a $\mathcal{B}$–$\mathcal{C}$–correspondence $F$ is the unique (up to isomorphism) $\mathcal{A}$–$\mathcal{C}$–correspondence $E \otimes F$ generated by elementary tensors $x \otimes y$ with inner products $\langle x \otimes y, x' \otimes y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$.

Every $C^*$–algebra $\mathcal{B}$ is a $\mathcal{B}$–$\mathcal{B}$–correspondence with the natural bimodule operation and inner product $(b, b') := b^* b'$. We refer to $\mathcal{B}$ with this structure as the identity $\mathcal{B}$–correspondence. For every $\mathcal{A}$–$\mathcal{B}$–correspondence $E$ we will always identify both correspondences $E \otimes \mathcal{B}$ and $\mathcal{A} \otimes E$ with $E$ via the canonical identifications $x \otimes b \mapsto xb$ and $a \otimes x \mapsto ax$, respectively. (Note that the second identification is possible only, because we require the left action to be nondegenerate. Nondegeneracy of the right action is automatic.)

2.1 Definition. A Morita equivalence from $\mathcal{A}$ to $\mathcal{B}$ is an $\mathcal{A}$–$\mathcal{B}$–correspondence $M$ for which there exists a $\mathcal{B}$–$\mathcal{A}$–correspondence $N$ such that

$$N \otimes M \cong \mathcal{B}, \quad M \otimes N \cong \mathcal{A},$$

as correspondences over $\mathcal{B}$ and over $\mathcal{A}$, respectively. We call $N$ an inverse of $M$ under tensor product.
Following Rieffel [Rie74b], two \( C^* \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \) are **strongly Morita equivalent** if there exists an \( \mathcal{A}-\mathcal{B} \)-Morita equivalence. We use nowadays convention and speak just of **Morita equivalent** \( C^* \)-algebras.

We observe that a Morita equivalence is necessarily faithful and full. (If \( M \) is not full, then \( N \odot M \) is not full, too, and if \( M \) is not faithful, then \( M \odot N \) is not faithful, too.)

**2.2 Proposition.**

1. The correspondence \( N \) in (2.1) is unique up to (unique, in a sense) isomorphism.

2. Morita equivalence of \( C^* \)-algebras is an equivalence relation.

**Proof.**

1.) Suppose \( N' \) is another \( \mathcal{B}-\mathcal{A} \)-correspondence fulfilling (2.1). Then \( N \cong \mathcal{B} \odot N \cong N' \odot M \cong N \cong N' \cong N' \odot \mathcal{A} \cong N' \). (Taking into account the canonical identifications discussed in front of Definition 2.1, the isomorphism is unique, once the isomorphisms \( \mathcal{B} \cong N' \odot M \) and \( \mathcal{A} \cong M \odot N \) in (2.1) are fixed.)

2.) \( \mathcal{B} \) is a \( \mathcal{B}-\mathcal{B} \)-Morita equivalence (with \( N = \mathcal{B} \)). So, Morita equivalence is reflexive.

If \( M \) is an \( \mathcal{A}-\mathcal{B} \)-Morita equivalence, then \( N \) is a \( \mathcal{B}-\mathcal{A} \)-Morita equivalence. So, Morita equivalence is symmetric.

If \( M_1 \) is an \( \mathcal{A}-\mathcal{B} \)-Morita equivalence (with inverse \( N_1 \), say) and if \( M_2 \) is a \( \mathcal{B}-\mathcal{C} \)-Morita equivalence (with inverse \( N_2 \), say), then, obviously, \( M_1 \odot M_2 \) is an \( \mathcal{A}-\mathcal{C} \)-Morita equivalence with inverse \( N_2 \odot N_1 \). So, Morita equivalence is transitive. ■

**2.3 Proposition.** The identifications in (2.1) can be chosen such that diagrams

\[
\begin{array}{ccc}
\mathcal{A} \odot M & \longrightarrow & M \\
M \odot N \odot M & \longrightarrow & M \odot \mathcal{B} \\
\mathcal{B} \odot N & \longrightarrow & N \odot \mathcal{A} \\
N \odot M \odot N & \longrightarrow & N \odot M \\
\end{array}
\]

commute.

**Proof.** Fix two isomorphisms (that is, bilinear unitaries) \( u: N \odot M \to \mathcal{B} \) and \( v: M \odot N \to \mathcal{A} \). To begin with suppose that the left diagram commutes, that is, \( v(m \odot n)m' = mu(n \odot m') \) for all \( m, m' \in M \) and all \( n \in N \). Since \( M \) is faithful, the right diagram commutes if and only if it commutes also when tensored with \( M \) from the right. Evaluating the left hand path on an elementary tensor \( n \odot m \odot n' \odot m' \) we find

\[
n \odot m \odot n' \odot m' \longmapsto u(n \odot m)n' \odot m'.
\]

Evaluating the right hand path we find

\[
n \odot m \odot n' \odot m' \longmapsto nu(m \odot n') \odot m' = n \odot v(m \odot n')m' = n \odot mu(n' \odot m').
\]
Applying the isomorphism $u$ to both elements, by bilinearity we find

$$u(u(n \odot m)n' \odot m') = u(n \odot m)u(n' \odot m') = u(n \odot mu(n' \odot m')).$$

In conclusion: If the left diagram commutes then so does the right diagram. By symmetry, of course, also the converse statement is true.

Now suppose that the left diagram does not necessarily commute. Then, still, the map

$$w: M = M \odot \mathcal{B} \xrightarrow{id_M \odot u'} M \odot N \odot M \xrightarrow{\psi \odot id_M} \mathcal{A} \odot M = M$$

defines an automorphism $w$ of $M$ that sends $mu(n \odot m')$ to $v(m \odot n)m'$. If we replace $v$ with $v' := v(w^* \odot id_{\mathcal{B}})$, then the automorphism $w': mu(n \odot m') \mapsto v'(m \odot n)m'$ of $M$ corresponding to the new pair $u, v'$ satisfies

$$w'((wm)u(n \odot m')) = v'((wm) \odot n)m' = v(m \odot n)m' = w(mu(n \odot m')) = (wm)u(n \odot m').$$

Therefore, $w'$ is the identity. Equivalently, for the pair $u, v'$ the left and, therefore, both diagrams commute. ■

### 2.4 Convention.

After this proposition we shall always assume that the diagrams commute. This allows us to identify $\mathcal{B}$ with $N \odot M$ and $\mathcal{A}$ with $M \odot N$ without having to worry about brackets in tensor products.

The following example is basic for everything about Morita equivalence.

### 2.5 Example.

Every Hilbert $\mathcal{B}$–module $E$ may be viewed as Morita equivalence from $\mathcal{K}(E)$ to $\mathcal{B}_E$. In fact, the space $E^* = \{x^*: x \in E\}$ becomes a correspondence from $\mathcal{B}_E$ to $\mathcal{K}(E)$ if we define the inner product $\langle x^*, y^* \rangle := xy^* \in \mathcal{K}(E)$ and the bimodule operation $bx^*a := (a^*xb^*)^*$. Clearly, $E^* \odot E = \mathcal{B}_E$ (via $x^* \odot y = \langle x, y \rangle$), and $E \odot E^* = \mathcal{K}(E)$ (via $x \odot y^* = xy^*$). Moreover, since

$$(x \odot y^*) \odot z = (xy^*)z = x(y, z) = x \odot (y^* \odot z),$$

these identifications also satisfy Convention[2.4]

### 2.6 Corollary.

$\mathcal{K}(E^*) = \mathcal{B}_E$ and $E^{**} = E$ as correspondence from $\mathcal{K}(E)$ to $\mathcal{K}(E^*) = \mathcal{B}_E$.

**Proof.** Since $E$ is a full Hilbert $\mathcal{B}_E$–module, the left action of $\mathcal{B}_E$ on $E^*$ is faithful. It follows that $\langle x, y \rangle \mapsto x^*y^{**}$ defines an injective homomorphism from $\mathcal{B}_E$ onto $\mathcal{K}(E^*)$. ■

The following result makes the connection with the definition of Morita equivalence in Lance [Lan95 Chapter 7]. It also also shows that Example 2.5 captures, in a sense, the most general situation of Morita equivalence.
2.7 Theorem. An $\mathcal{A}$–$\mathcal{B}$–correspondence $M$ is a Morita equivalence if and only if $M$ is full and the left action defines an isomorphism $\mathcal{A} \to \mathcal{K}(M)$.

Proof. We know already that for being a Morita equivalence, $M$ must be full and faithful. So the only question is whether or not the injection $\mathcal{A} \to \mathcal{B}^a(M)$ is onto $\mathcal{K}(M)$.

By Example 2.5, $M$ with the canonical action of $\mathcal{K}(M)$ is a Morita equivalence from $\mathcal{K}(M)$ to $\mathcal{B}$. So, if the left action of $\mathcal{A}$ on $M$ defines an isomorphism $\alpha : \mathcal{A} \to \mathcal{K}(M)$, then we turn the $\mathcal{B}$–$\mathcal{K}(M)$–correspondence $M^*$ into a $\mathcal{B}$–$\mathcal{A}$–correspondence $N$ with inner product $\langle m^*, m'' \rangle_N := \alpha^{-1}(mm'')$ and right action $m'a = m'(\alpha(a))$. Clearly, $N$ is an inverse under tensor product.

Conversely, suppose that $M$ is a Morita equivalence with inverse $N$, say. The idea is to establish a map $u : N \to M^*$ such that $\langle n, n' \rangle \mapsto \langle un, un' \rangle$ extends as an isomorphism $\alpha : \mathcal{A} \to \mathcal{K}(M)$. A look at how to resolve $N = \mathcal{B} \cap N = M^* \cap M \cap N$ in the other direction to give $M^*$ by “bringing somehow $M \cap N$ under the $*$”, reveals

$$u : \langle m, m' \rangle n \mapsto ((m' \cap n)^* m)^*$$

as the only reasonable attempt. We do not worry, at that point, about whether $u$ is well-defined. (See, however, Remark 2.9 below.) What we wish to show is that the map

$$\alpha : \langle \langle m_1, m'_1 \rangle n_1, \langle m_2, m'_2 \rangle n_2 \rangle \mapsto \langle ((m'_1 \cap n_1)^* m_1)((m'_2 \cap n_2)^* m_2)^* $$

(the right-hand is, $\langle u((m_1, m'_1)n_1), u((m_2, m'_2)n_2) \rangle$, once $u$ showed to be well-defined) is nothing but the canonical homomorphism $\mathcal{A} \to \mathcal{B}^a(M)$ when applied to $a := \langle \langle m_1, m'_1 \rangle n_1, \langle m_2, m'_2 \rangle n_2 \rangle \in \mathcal{A}$. (From this everything follows: Well-definedness, because the canonical homomorphism is well-defined. Injectivity, because $M$ is faithful. Surjectivity onto $\mathcal{K}(M)$, because we obtain a dense subset of the rank-one operators.) To achieve our goal we calculate the matrix element $\langle \tilde{m}_1, a\tilde{m}_2 \rangle$ and convince ourselves that it coincides with the corresponding matrix element of the operator on the right-hand side. We find

$$\langle \tilde{m}_1, a\tilde{m}_2 \rangle = \langle m_1, m'_1 \rangle n_1 \cap \tilde{m}_1, \langle m_2, m'_2 \rangle n_2 \cap \tilde{m}_2 \rangle = \langle m_1, m'_1 \rangle n_1 \cap \tilde{m}_1, \langle m_2, m'_2 \rangle n_2 \cap \tilde{m}_2 \rangle$$

and

$$\langle \tilde{m}_1, ((m'_1 \cap n_1)^* m_1)((m'_2 \cap n_2)^* m_2)^* \rangle \tilde{m}_2 \rangle = \langle \langle m'_1 \cap n_1)^* m_1, (m'_2 \cap n_2)^* m_2 \rangle \tilde{m}_2 \rangle = \langle m_1, m'_1 \rangle (m_2, m'_2) \tilde{m}_2 \rangle = \langle m_1, m'_1 \rangle (m_2, m'_2) \tilde{m}_2 \rangle$$

where in the step from the second to the last line we applied the convention following Proposition 2.3.
2.8 Remark. Theorem 2.7 is most probably folklore. But we do not know any reference. In fact, we used the statement of Theorem 2.7 in the proof of [MSS06, Corollary 1.11]. Since that result is too important for these notes, we decided to include a formal proof of Theorem 2.7 and give also a formal proof of [MSS06, Corollary 1.11]; see Corollary 2.13 below.

2.9 Remark. The map $u$ in the proof, actually, is what we called a ternary isomorphism in Abbaspour and Skeide [AS07] and the Hilbert modules $N$ and $M^*$ are ternary isomorphic. (We come back to this in Section 4, see, in particular, Remark 4.3.) The preceding proof is inspired very much by [AS07, Theorem 2.1], which asserts that a ternary homomorphism between full Hilbert modules induces a homomorphisms between the corresponding $C^*$–algebras. However, it would have been (notationally) more complicated to prove that $u$ is a ternary homomorphism and that the isomorphism $A \to \mathcal{K}(M)$ induced by [AS07, Theorem 2.1], is just the canonical map. Here, by calculating matrix elements, we take advantage of the fact that both $A$ and $K(M)$ are represented faithfully as operators on $M$.

2.10 Corollary. Let $M$ be a full Hilbert $B$–module. Suppose we can turn $M$ into $A_i$–$B$–Morita equivalences $iM$ ($i = 1, 2$) via nondegenerate homomorphisms $A_i \to B^*(M)$ (necessarily faithful and onto $\mathcal{K}(M)$ by Theorem 2.7). Then there exists a unique isomorphism $\alpha: A_1 \to A_2$ fulfilling

$$\alpha(a_1)m = a_1m$$

for all $a_1 \in A_1$ and $m \in M$ (first, considered an element in $2M$ and, then, considered an element in $1M$). Moreover, the inverses $N_i$ of $iM$ are ternary isomorphic (via a ternary isomorphism inducing $\alpha$ as explained in Remark 2.9).

The following representation theory of $\mathcal{K}(E)$ is a simple consequence of $E \odot E^* = \mathcal{K}(E)$. The uniqueness statement follows from $E^* \odot E = B_E$.

2.11 Corollary. Let $E$ be a Hilbert $B$–module. Suppose $F$ is a correspondence from $\mathcal{K}(E)$ to $C$ (that is, a nondegenerate representation of $\mathcal{K}(E)$ by adjointable operators on the Hilbert $C$–module $F$). Then

$$F = \mathcal{K}(E) \odot F = E \odot E^* \odot F = E \odot \tilde{F}$$

as $\mathcal{K}(E)$–$C$–correspondences, where we defined the $B$–$C$–correspondence $\tilde{F} := E^* \odot F$. (The identifications are the canonical ones.)

Moreover, $\tilde{F}$ is also a $B_E$–$C$–correspondence and as such it the unique (up to isomorphism) $B_E$–$C$–correspondence for which $F \cong E \odot \tilde{F}$ (as $\mathcal{K}(E)$–$C$–correspondences).

So far, this has been discussed already by Rieffel [Rie74a]. Actually, Rieffel discussed representations of the pre–$C^*$–algebra $\mathcal{F}(E)$ on a Hilbert space. The extension to Hilbert $C$–modules as representation spaces is marginal. The observation that the representation of $\mathcal{F}(E)$ extends
uniquely not only to $\mathcal{K}(E)$ but even to $\mathcal{B}^a(E)$ is key! In fact, if we have a nondegenerate representation of the ideal $\mathcal{I}(E)$ in $\mathcal{B}^a(E)$ on $F$, then $f y \mapsto (af)y$ ($f \in \mathcal{I}(E), y \in F$) for $a \in \mathcal{B}^a(E)$ induces a representation of the $C^*$-algebra $\mathcal{B}^a(E)$ on the dense pre-Hilbert $C^*$-module $\operatorname{span} \mathcal{I}(E)F$ of $F$. Such a representation is by bounded operators, automatically. (Note that for this it is not even necessary to require that the representation of $\mathcal{I}(E)$ is by bounded operators.)

We see that representations of $\mathcal{B}^a(E)$ behave well as soon as the action of $\mathcal{I}(E)$ or, equivalently, of $\mathcal{K}(E)$ is already nondegenerate, so that the representation module becomes a correspondence with left action of $\mathcal{K}(E)$. But that condition, for representations, is equivalent to strictness on bounded subsets. We do not worry to give the usual definition of strictness. Instead, we define immediately that a homomorphism $\vartheta: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is strict if $\operatorname{span} \mathcal{K}(E)F = \vartheta(1)F$ (reminding the reader that this is equivalent to usual strict continuity on bounded subsets). For a unital homomorphism this means just that $F$ may be viewed as a correspondence with left action of $\mathcal{K}(E)$ via restriction of $\vartheta$.

Fixing what the canonical identifications in Corollary 2.11 do, we obtain the representation theorem [MSS06, Theorem 1.4].

2.12 Theorem. Let $E$ be a Hilbert $\mathcal{B}$–module, let $F$ be a Hilbert $\mathcal{C}$–module and let $\vartheta: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ be a strict unital homomorphism. (In other words, $F$ is a correspondence from $\mathcal{B}^a(E)$ to $\mathcal{C}$ with strict left action and, thus, also a correspondence from $\mathcal{K}(E)$ to $\mathcal{C}$.) Then $F_\vartheta := E^* \odot F$ is a correspondence from $\mathcal{B}$ to $\mathcal{C}$ and the formula

$$u(x_1 \odot (x_2^* \odot y)) := \vartheta(x_1x_2^*)y$$

defines a unitary

$$u: E \odot F_\vartheta \longrightarrow F$$

such that

$$\vartheta(a) = u(a \odot \operatorname{id}_{F_\vartheta})u^*.$$  

2.13 Corollary. A full Hilbert $\mathcal{B}$–module $E$ and a full Hilbert $\mathcal{C}$–module $F$ have strictly isomorphic operator algebras (the isomorphism and its inverse are strict mappings) if and only if there is a Morita equivalence $M$ from $\mathcal{B}$ to $\mathcal{C}$ such that $F \cong E \odot M$.

Proof. Taking into account also Theorem 2.7, this is the proof of [MSS06, Corollary 1.11]: The two correspondences $F_\vartheta$ and $E_{\vartheta^{-1}}$ of a bistrict isomorphism $\vartheta: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ have tensor products that induce the identity. By the uniqueness result in Corollary 2.11 $F_\vartheta$ and $E_{\vartheta^{-1}}$ are inverses under tensor product. □

2.14 Corollary [MSS06, Remark 1.13]. An isomorphism $\vartheta: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is bistrict if and only if both $\vartheta$ and $\vartheta^{-1}$ take the compacts into (and, therefore, onto) the compacts.
2.15 Definition [Ske09d, Definition 5.7]. A Hilbert $\mathcal{B}$–module $E$ and a Hilbert $\mathcal{C}$–module $F$ are **Morita equivalent** if there is a Morita equivalence $M$ from $\mathcal{B}$ to $\mathcal{C}$ such that $E \odot M \cong F$ (or $E \cong F \odot M^*$).

With this definition Corollary 2.13 may be rephrased as follows.

2.16 Corollary. Two full Hilbert modules have strictly isomorphic operator algebras if and only if they are Morita equivalent.

2.17 Observation. If, in the notation of Corollary 2.13 $E$ and $F$ are not necessarily full, then strict isomorphism of $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ does not necessarily imply that $\mathcal{B}$ and $\mathcal{C}$ Morita equivalent. (Only $B_E$ and $C_F$ are Morita equivalent. For instance, if $\mathcal{B}$ is a commutative $C^*$–algebra and $C$ an ideal in $\mathcal{B}$ not isomorphic to $\mathcal{B}$, then $E = C$ considered as Hilbert $\mathcal{B}$–module and the Hilbert $\mathcal{C}$–module $F = C$ have the same compact operators. So, $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are strictly isomorphic. But, $\mathcal{B}$ and $\mathcal{C}$ are not Morita equivalent, because commutative $C^*$–algebras are Morita equivalent if and only if they are isomorphic.)

However, if $E$ and $F$ are Morita equivalent via $M$, say, then still $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are strictly isomorphic and the Morita equivalence from $B_E$ to $C_F$ inducing that isomorphism is simply $B_E \odot M \odot C_F = \text{span} B_E MC_F$.

2.18 Remark. Anoussis and Todorov [AT05] show that for separable $C^*$–algebras and countably generated Hilbert modules every isomorphism $\mathcal{B}^a(E) \to \mathcal{B}^a(F)$ takes the compacts onto the compacts; see, once more, [MSS06, Remark 1.13]. Therefore, in such a situation the requirement strict is unnecessary.

3 Stable Morita equivalence for Hilbert modules

Let $\mathbb{K}$ denote an infinite-dimensional separable Hilbert space and denote $\mathcal{K} := \mathcal{K}(\mathbb{K})$. Two $C^*$–algebras $\mathcal{A}$ and $\mathcal{B}$ are **stably isomorphic** if $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{B} \otimes \mathcal{K}$. A $C^*$–algebra is **$\sigma$–unital** if it has a countable approximate unit. The main result of Brown, Green and Rieffel [BGR77] asserts that two $\sigma$–unital $C^*$–algebras $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic if and only if they are Morita equivalent.

The proof of the forward direction is simple and works for arbitrary $C^*$–algebras. Indeed, for a Hilbert $\mathcal{B}$–module we denote by $E \otimes \mathbb{K}$ the **external tensor product** (the Hilbert $\mathcal{B}$–module obtained by completion from the algebraic tensor product $E \otimes \mathbb{K}$ with the obvious inner product). One easily checks that $\mathcal{K}(E \otimes \mathbb{K}) = \mathcal{K}(E) \otimes \mathbb{K}$. In particular, if we put $\mathbb{K}_B := B \otimes \mathbb{K}$, then $\mathcal{K}(\mathbb{K}_B) = \mathcal{B} \otimes \mathbb{K}$. So $\mathbb{K}_B$ is a Morita equivalence from $\mathcal{B} \otimes \mathbb{K}$ to $\mathcal{B}$, and if $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent.
In the version of the proof of the backward direction as presented in Lance \cite[Chapter 7]{Lan95}, the following result is key.

### 3.1 Proposition \cite[Proposition 7.4]{Lan95}.

1. Suppose $E$ is a full Hilbert module over a $\sigma$–unital $C^*$–algebra $\mathcal{B}$. Then $E \otimes \mathbb{K}$ has a direct summand $\mathcal{B}$.

2. Suppose $E$ is a countably generated Hilbert $\mathcal{B}$–module that has $\mathcal{B}$ as a direct summand. Then $E \otimes \mathbb{K} \cong \mathbb{K}_B$.

So, if $E$ is a countably generated full Hilbert module over a $\sigma$–unital $C^*$–algebra $\mathcal{B}$, then $E \otimes \mathbb{K} \cong \mathbb{K}_B$.

### 3.2 Remark.

Part 1 has a much simpler proof when $\mathcal{B}$ is unital. In fact, in that case $\mathbb{K}$ may be replaced by a suitable finite-dimensional Hilbert space; see \cite[Lemma 3.2]{Ske09d}.

### 3.3 Remark.

The proof of Part 2 relies on Kasparov’s stabilization theorem \cite{Kas80}. In fact, if $E = \mathcal{B} \oplus F$, then $E \otimes \mathbb{K} \cong \mathbb{K}_B \oplus (F \otimes \mathbb{K})$. Since with $E$ also $F \otimes \mathbb{K}$ is countably generated, the stabilization theorem asserts $\mathbb{K}_B \oplus (F \otimes \mathbb{K}) \cong \mathbb{K}_B$.

### 3.4 Definition. Let $E$ and $F$ denote Hilbert modules.

1. $E$ and $F$ are **stably Morita equivalent** if $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are Morita equivalent.

2. $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are **stably strictly isomorphic** if $\mathcal{B}^a(E \otimes \mathbb{K})$ and $\mathcal{B}^a(F \otimes \mathbb{K})$ are strictly isomorphic.

By Corollary \cite[2.14]{2.14}, $\mathcal{B}^a(E \otimes \mathbb{K})$ and $\mathcal{B}^a(F \otimes \mathbb{K})$ are strictly isomorphic if (and only if) $\mathcal{K}(E \otimes \mathbb{K}) = \mathcal{K}(E) \otimes \mathbb{K}$ and $\mathcal{K}(F \otimes \mathbb{K}) = \mathcal{K}(F) \otimes \mathbb{K}$ are isomorphic, that is, if and only $\mathcal{K}(E)$ and $\mathcal{K}(F)$ are stably isomorphic.

### 3.5 Theorem. Let $E$ and $F$ denote full Hilbert modules over $C^*$–algebras $\mathcal{B}$ and $\mathcal{C}$, respectively.

1. $E$ and $F$ are stably Morita equivalent if and only if $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are stably strictly isomorphic. Either condition implies that $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent.

2. Suppose $E$ and $F$ are countably generated and $\mathcal{B}$ and $\mathcal{C}$ are $\sigma$–unital. Then the following conditions are all equivalent:

   (i) $E$ and $F$ are stably Morita equivalent.

   (ii) $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are stably strictly isomorphic.

   (iii) $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent.
(iv) $B$ and $C$ are stably isomorphic.

Proof. Part 1 is Corollary 2.16 and equivalence of (2i) and (2ii) is 1 restricted to the special case. Equivalence of (2iii) and (2iv) is [BGR77]. Clearly, (2i) $\Rightarrow$ (2iii) directly from the definition, while (2iv) $\Rightarrow$ (2i) follows from Proposition 3.1 and the observation that $K_B$ and $K_C$ are Morita equivalent if $B$ and $C$ are, via the same Morita equivalence. ■

4 Ternary isomorphisms

The isomorphisms in the category of Hilbert $B$–modules are the unitaries, that is, the inner product preserving surjections. If $u: E \to F$ is a unitary, then conjugation $u \cdot u^*: \mathcal{B}^e(E) \to \mathcal{B}^e(F)$ defines a strict isomorphism. If $E$ and $F$ are isomorphic, we say $B^e(E)$ and $B^e(F)$ are inner conjugate.

In the sequel, we shall say that strictly isomorphic $\mathcal{B}^e(E)$ and $\mathcal{B}^e(F)$ are strictly conjugate. We know that a full Hilbert $B$–module $E$ and a full Hilbert $C$–module $F$ have strictly isomorphic operator algebras if and only if the modules are Morita equivalent. Isomorphic Hilbert $B$–modules are Morita equivalent via the identity correspondence $\mathcal{B}$. But, Morita equivalent full Hilbert $B$–modules need not be isomorphic.

4.1 Example. Let $B := \begin{pmatrix} 0 & 0 \\ 0 & M_2 \end{pmatrix} \subset M_3$ an let $E := \begin{pmatrix} 0 & C^2 \\ C^2 & 0 \end{pmatrix} \subset M_3$ be the $B$–correspondence obtained by restricting the operations of the identity $M_3$–correspondence $M_3$ to the subsets $E$ and $B$. Then $E$ is a Morita equivalence. From $E \circ E = B$ it follows that $E$ and $B$ are Morita equivalent as Hilbert $B$–modules. Of course, they are not isomorphic. In fact, their dimensions as complex vector spaces differ, so that there is not even a linear bijection between them. Also, $E$ has no unit vector, but $B$, of course, has one.

In between isomorphism of Hilbert $B$–modules and Morita equivalence of Hilbert modules there is another equivalence relation, based on ternary isomorphisms. A ternary homomorphism from a Hilbert $B$–module $E$ to a Hilbert $C$–module $F$ is a map $u: E \to F$ (a priori neither linear nor bounded) that satisfies

$$u(x(y, z)) = (ux)\langle uy, uz \rangle$$

for all $x, y, z \in E$. A ternary unitary is a bijective ternary homomorphism. Clearly, if $u$ is a ternary unitary, then so is $u^{-1}$. If there is a ternary unitary from $E$ to $F$, we say $E$ and $F$ are ternary isomorphic.

Ternary homomorphisms have the advantage that they do not refer in any way to the $C^*$–algebras over which the modules are modules. (In fact, we may turn the class of all Hilbert
modules, without fixing an algebra, into a category by choosing as morphisms the ternary homomorphisms.) The following notion takes into account the algebras more explicitly. A \textbf{generalized isometry} from a Hilbert $\mathcal{B}$–module $E$ to a Hilbert $\mathcal{C}$–module $F$ is a map $u: E \to F$ (a priori neither linear nor bounded) such that there exists a homomorphism $\varphi: \mathcal{B} \to \mathcal{C}$ fulfilling

$$\langle ux, uy \rangle = \varphi(\langle x, y \rangle) \quad (4.1)$$

for all $x, y \in E$. Once the homomorphism $\varphi$ is fixed, we shall also speak of a $\varphi$–\textit{isometry} $u$; see Skeide \cite{Ske06d}.

The connection between ternary homomorphisms and generalized isometries is made by the following result.

\textbf{4.2 Theorem \cite{AS07} Theorem 2.1.} For a map $u$ from a full Hilbert $\mathcal{B}$–module $E$ to a Hilbert $\mathcal{C}$–module $F$ the following statements are equivalent:

1. $u$ is a generalized isometry.

2. $u$ is a ternary homomorphism.

\textbf{4.3 Remark.} Of course, the homomorphism $\varphi$ turning a ternary homomorphism into a generalized isometry is the unique homomorphism satisfying (4.1). This is essentially what we used in the proof of Theorem \cite{AS07} 2.7. As mentioned in Remark \cite{AS07} 2.9 the map $u$ in that proof is a ternary homomorphism. Just that it was easier in the particular case to establish $u$ as a generalized isometry. \cite{AS07} Theorem 2.1 now assures that $u$ is, indeed, a ternary homomorphism and, therefore, a ternary unitary.

Recognizing a ternary homomorphism as $\varphi$–isometry has more consequences. For instance, there is the notion of $\varphi$–adjointable operators with all results but also with all problems known from the usual adjointable operators; see \cite{Ske06d} Observation 1.9.

\textbf{4.4 Observation.} Clearly, a generalized isometry is linear and contractive (even completely contractive; see Theorem \cite{AS07} 4.5). Therefore, so is a ternary homomorphism.

Note that if $u: E \to F$ is a ternary homomorphism, then $u^*: x^* \mapsto (ux)^*$ is a ternary homomorphism from $E^*$ to $F^*$. The following theorem now follows easily from \cite{AS07} Theorem 2.1. We omit the proof.

\textbf{4.5 Theorem.} For a map $u$ from a Hilbert $\mathcal{B}$–module $E$ to a Hilbert $\mathcal{C}$–module $F$ the following statements are equivalent:

1. $u$ is a ternary homomorphism.
2. \( u \) extends as a (unique!) homomorphism between the \textbf{reduced linking algebras}

\[
\Phi_u: \begin{pmatrix} B_E & E^* \\ E & \mathcal{X}(E) \end{pmatrix} \rightarrow \begin{pmatrix} C_F & F^* \\ F & \mathcal{X}(F) \end{pmatrix}
\]

respecting the corners.

In either case, \( \Phi_u \) is injective if and only if \( u \) is, and \( \Phi_u \) is surjective if and only if \( u \) is.

Let \( u: E \rightarrow F \) be a ternary unitary. Then the conjugation \( u \cdot u^{-1} : a \mapsto uau^{-1} \) is, clearly, multiplicative. Note also

\[
(ux)(uy,(uau^{-1})uz) = (ux)(uy,uz) = u(x(y,az))
\]

\[
= u(x(a^*y,z)) = (ux)(ua^*y,uz) = (ux)((ua^*u^{-1})uy,uz)
\]

for all \( x, y, z \in E \), so that \( u \cdot u^{-1} \) is a \( * \)-map. In other words, conjugation with \( u \) still defines an isomorphism from \( \mathcal{B}^a(E) \) to \( \mathcal{B}^a(F) \). We call an isomorphisms obtained by conjugation with a ternary unitary a \textbf{ternary inner isomorphism} and we call \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \) \textbf{ternary conjugate}.

The restriction of a ternary inner isomorphisms induced by \( u \) to \( \mathcal{K}(E) \) is precisely the restriction of \( \Phi_u \) to \( \mathcal{K}(E) \) and, therefore, an isomorphism onto \( \mathcal{K}(F) \). It follows that ternary inner isomorphisms are bistrict. The Morita equivalence from \( \mathcal{B}_E \) to \( \mathcal{C}_F \) of such a ternary inner isomorphism is simply \( \varphi \mathcal{C}_F \), where \( \varphi \) is the restriction of \( \Phi_u \) to an isomorphisms from \( \mathcal{B}_E \) to \( \mathcal{C}_F \).

4.6 \textbf{Remark.} In Skeide [Ske06d] we have analyzed the group of ternary inner automorphisms of \( \mathcal{B}^a(E) \) and how it is reflected in the Picard group of \( \mathcal{B} \). The \textbf{Picard group} of \( \mathcal{B} \) is the group of isomorphism classes of Morita equivalences over \( \mathcal{B} \) under tensor product; see [BGR77]. It contains the (opposite of the) group of the automorphisms of \( \mathcal{B} \) modulo the multiplier inner automorphisms. (Multiplier inner automorphisms are obtained by conjugation with a unitary in the multiplier algebra. They are called generalized inner automorphisms in [BGR77, Ske06d].) We now prefer to follow the modern terminology in Blackadar [Bla06].) One main point of [Ske06d] is, very roughly, that there are full Hilbert \( \mathcal{B} \)-modules \( E \), such that not every automorphism of \( \mathcal{B} \) occurs as the automorphism \( \varphi \) induced by a ternary unitary on \( u \) on \( E \). In other words, not all automorphisms of \( \mathcal{B} \) extend to automorphisms of the linking algebra of \( E \). Equivalently, not for every automorphism \( \varphi \) of \( \mathcal{B} \) the Hilbert \( \mathcal{B} \)-modules \( E \) and \( E \odot \varphi \mathcal{B} \) are isomorphic.

4.7 \textbf{Definition.} Let \( E \) and \( F \) denote Hilbert modules.

1. \( E \) and \( F \) are \textbf{stably ternary isomorphic} if \( E \otimes \mathbb{K} \) and \( F \otimes \mathbb{K} \) are ternary isomorphic.

2. \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \) are \textbf{stably ternary conjugate} if \( \mathcal{B}^a(E \otimes \mathbb{K}) \) and \( \mathcal{B}^a(F \otimes \mathbb{K}) \) are ternary conjugate.
By definition of ternary conjugate, the two properties are equivalent.

4.8 Theorem. Let $E$ and $F$ denote full countably generated Hilbert modules over $\sigma$–unital $C^*$–algebras $B$ and $C$, respectively. Then either of the conditions in Definition 4.7 holds if and only if $B$ and $C$ are isomorphic.

**Proof.** $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are full Hilbert modules over $B$ and $C$, respectively. Suppose the first condition of Definition 4.7 holds. Then $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are ternary isomorphic, so that $B$ and $C$ are isomorphic. (This does not depend on countability hypotheses.) Suppose, on the other hand, $B$ and $C$ are isomorphic via an isomorphism $\varphi$, say. Then $E_\varphi := E \odot \varphi C$ is a full countably generated Hilbert $C^*$–module ternary isomorphic to $E$ via $x \odot c \mapsto x\varphi^{-1}(c)$. By Proposition 3.1 the Hilbert $C$–modules $E_\varphi \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are isomorphic, so that $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are ternary isomorphic. ■

5 Ccocycle conjugacy of $E_0$–semigroups

In this section we discuss several notions of cocycle in an algebraic context. (We use $C^*$–algebras just for convenience. General unital $*$–algebras, like in the beginning of the introduction, would do as well.) We put particular emphasis on the fact that, unlike the case $\mathcal{B}(H)$ where all cocycles are unitarily implemented, here the character of the automorphisms forming the cocycles may vary. In the end, also in these note we shall concentrate on unitary cocycles. (See, however, Remark 7.6.) But, this is our choice, and this choice does the job of resolving our classification problem. The differences here lie in what conjugacies we will allow.

Let $\vartheta$ and $\theta$ denote unital endomorphisms of unital $C^*$–algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. (For nonunital algebras one would replace unital with nondegenerate in the sense that $\vartheta(\mathcal{A})\mathcal{A}$ should be total in $\mathcal{A}$. We do not tackle these problems. Though, interesting phenomena may happen, worth of a separate investigation.) $\vartheta$ and $\theta$ are conjugate if there exists an isomorphism $\alpha: \mathcal{A} \rightarrow \mathcal{B}$ such that $\alpha \odot \vartheta = \theta \odot \alpha$. If $\vartheta$ and $\theta$ are conjugate, then for every $n \in \mathbb{N}_0$, the members $\vartheta_n := \vartheta^n$ and $\theta_n := \theta^n$ of the $E_0$–semigroups generated by $\vartheta$ and $\theta$, respectively, are conjugate by the same isomorphism $\alpha$. In general, we say two $E_0$–semigroups $\vartheta$ and $\theta$ are conjugate if there is an isomorphism $\alpha$ such that $\alpha \odot \vartheta_t = \theta_t \odot \alpha$ for all $t \in \mathbb{S}$. Of course, conjugacy of $E_0$–semigroups is an equivalence relation.

If $\vartheta$ and $\theta$ are two unital endomorphisms of the same unital $C^*$–algebra $\mathcal{A}$, then we may ask whether there is an automorphism $\beta$ of $\mathcal{A}$, such that $\theta = \beta \odot \vartheta$. In this case, we may not expect that $\theta_n = (\beta \odot \vartheta)^n$ would be equal to $\beta \odot \theta_n$. In the (rare) case when $\beta$ and $\vartheta$ commute, we find $\theta_n = \beta^n \odot \vartheta_n$. In general, we may not even expect that there exist automorphisms $\beta_n$ such that $\theta_n = \beta_n \odot \vartheta_n$. 

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5.1 Example. Let $\mathcal{A} = \mathbb{C}^3$, the diagonal subalgebra of $M_3$. Define the one-sided shift $\vartheta_{[n]} := \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$ and the cyclic permutation $\beta_{[n]} := \begin{bmatrix} n \\ 0 \\ 0 \end{bmatrix}$. Put $\vartheta := \beta \circ \vartheta$, so that $\vartheta_{[n]} = \beta_{[n]}$. Then $\vartheta^2_{[n]} = \beta_{[n]}$, while $\vartheta^3_{[n]} = \beta_{[n]}$. There is no automorphism $\beta_2$ such that $\beta_2_{[n]} = \beta_{[n]}$.

5.2 Definition. We say two $E_0$–semigroups $\vartheta$ and $\theta$ on $\mathcal{A}$ are cocycle equivalent if there exist automorphisms $\beta_i$ of $\mathcal{A}$ such that $\vartheta_i = \beta_i \circ \theta$. If, vice versa, $\theta$ is an $E_0$–semigroup and $\beta = (\beta_i)_{i \in \mathbb{S}}$ is a family of automorphisms such that $\vartheta_i^\beta := \beta_i \circ \theta_i$ defines an $E_0$–semigroup $\vartheta^\beta$, then we say $\beta$ is a cocycle on $\mathcal{A}$ with respect to $\vartheta$.

Clearly, if $\beta$ is a cocycle with respect to $\vartheta$, then $\beta^{-1} = (\beta_i^{-1})_{i \in \mathbb{S}}$ is a cocycle with respect to $\theta$. So, cocycle equivalence is symmetric. Clearly, it reflexive and transitive. In other words, cocycle equivalence is an equivalence relation.

The reader might ask, why we used the name cocycle equivalent instead of the more common cocycle conjugate. The reason is that in minute we will define the second term in a different way, which is closer to what is known as cocycle conjugate.

Cocycle equivalence is a notion that involves two semigroups of endomorphisms on the same algebra. A relation that allows to compare (semigroups of) endomorphisms on different algebras is conjugacy. Before, we can investigate two semigroups of endomorphisms on different algebras for cocycle equivalence, we must transport one of them to the other algebra via a conjugacy.

5.3 Definition. Let $\vartheta$ and $\theta$ denote $E_0$–semigroups on unital $C^*$–algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. We say $\vartheta$ and $\theta$ are cocycle conjugate if there exists an isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that the conjugate $E_0$–semigroup $\vartheta^\alpha := (\alpha \circ \vartheta_i \circ \alpha^{-1})_{i \in \mathbb{S}}$ on $\mathcal{B}$ and $\theta$ are cocycle equivalent.

If $\alpha$ satisfies additional conditions, then we will indicate these in front of the word conjugate. (For instance, if $\alpha$ is an inner isomorphism, we will say $\vartheta$ and $\theta$ are cocycle inner conjugate.)

If the cocycle satisfies additional conditions, then we will indicate these in front of the word cocycle. (For instance, if $\beta$ consists of inner automorphisms we will say $\vartheta$ and $\theta$ are inner cocycle conjugate.)

Also for two $E_0$–semigroups $\vartheta$ and $\theta$ on the same unital $C^*$–algebra $\mathcal{A}$ we may ask, whether they are cocycle conjugate. Of course, cocycle equivalent $E_0$–semigroups are cocycle (inner) conjugate via $\alpha = \text{id}_{\mathcal{A}}$. But the converse need not be true.

5.4 Example. Let $\mathcal{A} = \mathbb{C}^2$, the diagonal subalgebra of $M_2$. Define the one-sided shift $\vartheta_{[n]} := \begin{bmatrix} n \\ 0 \end{bmatrix}$ and the flip automorphism $\alpha_{[n]} := \begin{bmatrix} 0 \\ n \end{bmatrix}$. Then $\vartheta$ and $\theta := \vartheta^n = \alpha \circ \vartheta \circ \alpha^{-1} : \begin{bmatrix} 0 \\ n \end{bmatrix} \mapsto \begin{bmatrix} n \\ 0 \end{bmatrix}$ and, therefore, the whole semigroups $\vartheta^n$ and $\theta^n$ generated by them are conjugate. A fortiori these two semigroups
are cocycle conjugate via $\alpha$ by the identity cocycle $\beta_{\alpha} = \text{id}_{\mathcal{A}}$. But, no automorphism $\beta$ can recover $\theta$ as $\beta \circ \theta$. So, these semigroups are not cocycle equivalent.

**5.5 Proposition.** Two $E_0$–semigroups $\vartheta$ and $\theta$ on $\mathcal{A}$ are cocycle equivalent if and only if they are cocycle inner conjugate.

**Proof.** The forward implication being clear, suppose $\alpha = u \bullet u^*$ is an inner automorphism (for some unitary $u \in \mathcal{A}$) and $\beta = (\beta_t)_{t \in \mathbb{S}}$ a family of automorphisms $\beta_t$ of $\mathcal{A}$ such that $\beta_t \circ \theta^\alpha_t = \theta_t$. That is, $$\theta_t(a) = \beta_t(u \theta_t(u^*) u^*) = \beta_t((u \theta_t(u^*) ) \theta_t(a) (u \theta_t(u^*))^*).$$ In other words, $\theta_t = \beta_t \circ \theta_t$ for the automorphism $\beta_t := \beta_t \circ ((u \theta_t(u^*)) \bullet (u \theta_t(u^*))^*)$.

Since $\mathcal{A} = \mathcal{B}(H)$ ($H$ some Hilbert space) has only inner automorphisms, the notions of cocycle equivalence and cocycle conjugacy for $E_0$–semigroups $\mathcal{B}(H)$ coincide. But for $\mathcal{A} = \mathcal{B}^\omega(E)$, of course, this is not so. In fact, Example 5.4 gives a counterexample via $E := \mathcal{A} = \mathcal{B}^\omega(E)$.

Among the inner cocycles $\beta$ with respect to $\vartheta$, a particularly important class consists of those cocycles that are generated as $\beta_t = u_t \bullet u_t^*$ where $u = (u_t)_{t \in \mathbb{S}}$ is a family of unitaries in $\mathcal{A}$ fulfilling $$u_0 = 1, \quad \text{and} \quad u_{s+t} = u_t \theta_s(u_s)$$ for all $s, t \in \mathbb{S}$. Such a family is called a unitary left cocycle in $\mathcal{A}$ with respect to $\vartheta$ (or simply a left cocycle if the $u_t$ are not necessarily unitary). It is easy to check that every unitary left cocycle implements a cocycle $\beta^\vartheta$ via $\beta^\vartheta_t := u_t \bullet u_t^*$. We will say $\beta^\vartheta$ is a unitary cocycle on $\mathcal{A}$, and we will denote $\vartheta^\beta := \theta^\beta$.

**5.6 Definition.** Two $E_0$–semigroups are unitary cocycle conjugate (equivalent) if the conjugacy (the equivalence) can be implemented by a unitary left cocycle.

**5.7 Example.** Suppose two $E_0$–semigroups $\vartheta$ and $\theta$ on $\mathcal{A}$ are inner conjugate via a unitary $u \in \mathcal{A}$. It is easy to check that $u \theta_t(u^*)$ is a unitary left cocycle with respect to $\vartheta$ and that $\theta = \vartheta^u$.

In other words, inner conjugate $E_0$–semigroups on $\mathcal{A}$ are unitary cocycle equivalent. More generally, if the cocycle $\beta = \beta^\vartheta$ in the proof of Proposition 5.5 is implemented by a unitary left cocycle $v$ with respect to $\vartheta^u$, then $\beta' = \beta^\vartheta'$ where $\vartheta'$ is the unitary left cocycle with respect to $\vartheta$ defined by $\vartheta'_t := v_t u \theta_t(u^*)$. Indeed, from $v_{s+t} = v_s u \theta_s(u^*) u_t u^*$ on easily verifies that $$v_{s+t} = v_{s+t} u \theta_{s+t}(u^*) = v_s u \theta_s(u^*) u_t u^* \theta_{s+t}(u^*) = v_s u \theta_s(u^*) \theta_t(u^*) (v_t u \theta_t(u^*)) = v'_s \theta_s(v'_t).$$

**5.8 Corollary.** Two $E_0$–semigroups $\vartheta$ and $\theta$ on $\mathcal{A}$ are unitary cocycle equivalent if and only if they are unitary cocycle inner conjugate.
It is easy to check that also unitary cocycle conjugacy or equivalence are equivalence relations. It is unitary cocycle conjugacy that, usually, corresponds to cocycle conjugacy in literature; see, for instance, Takesaki [Tak03, Definition X.1.5]. But be aware that this notion refers rather to the context of groups, not so much to semigroups.

We do not tackle the questions whether every inner cocycle is implemented by a unitary left cocycle, or to what extent the cocycle $\beta$ is nonunique. We just mention the following easy to prove fact. (Recall that a left cocycle $u$ with respect to $\vartheta$ is local if $u_t \vartheta_t(a) = \vartheta_t(a) u_t$ for all $t \in \mathbb{S}, a \in A$. Every local left cocycle is also a right cocycle. Therefore, we usually say just local cocycle.)

5.9 Proposition. Two unitary left cocycles $u$ and $v$ implement the same inner cocycle $\beta$ if and only if the elements $v^*_t u_t$ form a local cocycle.

6 $E_0$–Semigroups, product systems, and unitary cocycles

In this section we, finally, explain how the representation theory of $B^a(E)$ gives rise to the construction of a product system from an $E_0$–semigroup on $B^a(E)$. We show that the $E_0$–semigroups acting on a fixed $B^a(E)$ are classified by their product systems up to unitary cocycle equivalence. We also give a criterion when $E_0$–semigroups acting on possibly different $B^a(E)$, varying the (full) Hilbert module $E$ but over a fixed $C^*$–algebra have the same product system. This criterion does, however, not involve cocycle conjugacy and is, therefore, not the criterion we really want.

Let $E$, $F$, and $G$ denote a Hilbert $B$–, a Hilbert $C$–, and a Hilbert $D$–module, respectively. Suppose $\vartheta: B^a(E) \to B^a(F)$ and $\theta: B^a(F) \to B^a(G)$ are unital strict homomorphisms. Then the multiplicity correspondences $F_\vartheta$ and $G_\theta$ compose contravariantly as tensor product by the isomorphism

$$F_\vartheta \circ G_\theta \ni (x^* \odot_\vartheta y) \odot (y^* \odot_\theta z) \mapsto x \odot_{\vartheta \circ \theta} \theta(y y^*) z \in G_{\theta \circ \vartheta}. \quad (6.1)$$

Moreover, under iterations these isomorphisms compose associatively; see [MSS06, Theorem 1.14].

It follows that every equality between compositions of unital strict homomorphisms is reflected by an isomorphism of the corresponding tensor products of the multiplicity correspondences in the reverse order. If $E$ is a Hilbert $B$–module and if $\vartheta$ is a strict $E_0$–semigroup on $B^a(E)$, then the semigroup property $\vartheta_t \circ \vartheta_s = \vartheta_{s+t}$ gives rise to isomorphisms

$$u_{s,t}: E_s \odot E_t \longrightarrow E_{s+t}$$
of the multiplicity $\mathcal{B}$–correspondences $E_t := E_{\theta_t}, t > 0$. (We shall abbreviate $x^* \circ_{\theta_t} y := x^* \circ_t y$.)

The “multiplication”

$$((x^* \circ_s x^'), (y^* \circ_t y')) \mapsto (x^* \circ_x x')(y^* \circ_t y') := u_{s,t}((x^* \circ_s x') \circ (y^* \circ_t y')) = x^* \circ_{s,t} \theta_t(x'y^*)y'$$

is associative. If $E$ is full, then everything also extends to $t = 0$ with $E_0 = E^* \circ E = \mathcal{B}$, and $u_{0,t}$ and $u_{t,0}$ are just the canonical left and right action, respectively, of $E_0 = \mathcal{B}$. If $E$ is not full, then we put $E_0 := \mathcal{B}$ by hand and the canonical actions $u_{0,t}$ and $u_{t,0}$ extend uniquely the above identifications from $E_1 = E^* \circ E = E^* \circ_0 E$ to $E = E_0$.

A family $E^\circ = (E_t)_{t \in \mathbb{S}}$ of correspondences over $\mathcal{B}$ with associative identifications $u_{s,t}$ and the conditions on $E_0, u_{0,t}, u_{t,0}$ at zero is what has been called a product system in Bhat and Skeide [BS00]. We call $E^\circ$ constructed as above from $\theta$ the product system associated with the strict $E_0$–semigroup $\theta$.

6.1 Convention. In the sequel, we restrict our attention to full Hilbert modules.

For full Hilbert modules the multiplicity correspondence of a unital strict homomorphism is unique (up to isomorphism) and the condition $E_0 = \mathcal{B}$ is automatic. As far as we are dealing with the connection between $E_0$–semigroups on $\mathcal{B}^a(E)$ and product systems associated with them, it is natural to restrict to full Hilbert $\mathcal{B}$–modules, as $\mathcal{B}$ can always be replaced with $\mathcal{B}_E$. When we take also into account continuity questions, then $E_0 := \mathcal{B} = \mathcal{B}_E$ is forced. (Observe that $\mathcal{B}_{E_t} = \mathcal{B}_E$. Therefore, $E^\circ$ will never have continuous sections reaching every point of $E_0$ unless $E_0 = \mathcal{B}$.)

Note that the product system associated with a strict $E_0$–semigroup on $\mathcal{B}^a(E)$ for a full Hilbert $\mathcal{B}$–module $E$ must be full in the sense that $E_t$ is full for each $t \in \mathbb{S}$. Note, too, that the $E_0$–semigroup consists of faithful endomorphisms if and only if the associated product system is faithful in the sense that all $E_t$ have a faithful left action.

Remember that the multiplicity correspondence $E_t$ of $\theta_t$ is related with $\theta_t$ via a unitary $\nu_t: E \circ E_t \to E$ such that $\theta_t(a) = \nu_t(a \circ \text{id}_t)\nu_t^*$. Moreover, the “multiplication” $(x, y_t) \mapsto xy_t := \nu_t(x \circ y_t)$ iterates associatively with the product system multiplication, and $\nu_t$ is the canonical identification. A family $\nu = (\nu_t)_{t \in \mathbb{S}}$ fulfilling these properties is what we started calling a left dilation of the full product system $E^\circ$ to the full Hilbert module $E$ in [Ske06a] (for the Hilbert space case) and in [Ske07a, Ske06e]. (For nonfull $E^\circ$ the term left dilation is not defined.)

\textsuperscript{[a]}If $E$ is not necessarily full, then we speak of a left quasi dilation. This is an interesting concept, too. But it has no nice relation with $E_0$–semigroups. ($E$ may be very well $[0,1]$.) In these notes we are interested only in the relation between $E_0$–semigroups and product systems. There is also a relation of product systems with $E$–semigroups, that is, semigroups of not necessarily unital endomorphisms. In that case the $\nu_t$ need not be unitary but just isometric, and we speak of semidilations. There is also the concept of right dilation [Ske06e] of faithful product systems, which is practically synonymous with faithful nondegenerate representation of a product system; see [Ske09d].
$E^\circ$ is the product system associated with a strict $E_0$–semigroup and the left dilation arises in the prescribed way, then we refer to it as the **standard dilation of $E^\circ$**.

For every left dilation of a product system $E^\circ$ to $E$, by $\vartheta_t^\circ(a) := v_t(a \circ \id)\vartheta_t^*$ we define a strict $E_0$–semigroup $\vartheta$ on $\mathcal{B}^\circ(E)$. We say a strict $E_0$–semigroup $\vartheta$ is **associated** with a full product system $E^\circ$ if it can be obtained as $\vartheta = \vartheta^\circ$ for some left dilation $\vartheta$ of $E^\circ$. Of course, every strict $E_0$–semigroup is associated with its associated product system via the standard dilation. But, this need not be the only left dilation that gives back the $E_0$–semigroup. We now investigate the possibilities in the slightly more general situation when two left dilations of two product systems induce conjugate $E_0$–semigroups.

**6.2 Definition.** A **morphism** between two product systems $E^\circ$ and $F^\circ$ of $\mathcal{B}$–correspondences is a family $w^\circ = (w_t)_{t \in S}$ of bilinear adjointable maps $w_t : E_t \to F_t$ such that $w_t(x_s)w_t(y_t) = w_{s+t}(x,y)$ and $w_0 = \id_B$. An **isomorphism** is a morphism that consists of unitaries. Of course, the inverse of an isomorphism is an isomorphism.

The following proposition is very important. It summarizes answers to all questions about product systems associated via left dilations with $E_0$–semigroups, when these $E_0$–semigroups are inner conjugate.

**6.3 Proposition.** Let $E$ and $E'$ be full Hilbert $\mathcal{B}$–modules. Let $u \in \mathcal{B}^\circ(E,E')$ be a unitary and define the inner isomorphism $\alpha := u \bullet u^*$. Suppose $v, v'$ are left dilations of product systems $E^\circ, E'^\circ$ to $E$ and $E'$, respectively, such that $(\vartheta^v)^* = \vartheta^v$. Then there is a unique isomorphism $w^\circ$ from $E^\circ$ to $E'^\circ$ such that

$$u(xy_t) = (ux)(w_t y_t) \quad (6.2)$$

for all $t \in S, x \in E, y_t \in E_t$. (That means $w_t = v_t'(u \circ w_t)$.) In particular:

1. If $u$ is the identity of $E = E'$, so that $\vartheta^v = \vartheta^v'$, then $w^\circ$ is the unique isomorphism satisfying $xy_t = x(w_t y_t)$.

2. If, in the situation of (1), $E^\circ$ is the product system associated with $\vartheta^v$ and $v'$ its standard dilation, then $w^\circ$ is the unique isomorphism satisfying

$$w_t((x,y)z_t) = x^* \circ_t (yz_t).$$

3. Let $\vartheta$ be a strict $E_0$–semigroup on $\mathcal{B}^\circ(E)$. If $E^\circ$ and $E'^\circ$ are the product systems associated with $\vartheta$ and $\vartheta' := \vartheta^\circ$, respectively, and if $v$ and $v'$ are their respective standard dilations, then $w^\circ$ is the unique isomorphism determined by

$$w_t(x^* \circ_t y) = (ux)^* \circ_t (uy).$$

Also this concept, for Hilbert modules, is not directly related to $E_0$–semigroups, while, for von Neumann modules, it parallels Arveson’s approach to Arveson systems. We come back to right dilations in Appendix B.2.
Proof. Regarding uniqueness, suppose that \( w_t \) and \( w'_t \) are bilinear adjointable maps satisfying 
\[
(ux)(w_t y_t) = (ux)(w'_t y_t).
\]
Then \( (ux)(w_t y_t - w'_t y_t) = u'(u \otimes id)(x \otimes (w_t y_t - w'_t y_t)) = 0 \) for all \( x \in E \). Since \( E \) is full and \( B \) acts nondegenerately, one easily verifies that, in a tensor product, \( x \otimes y = 0 \) for all \( x \in E \) implies \( y = 0 \). In other words, \( w_t y_t - w'_t y_t = 0 \) for all \( y_t \in E_t \) or \( w_t = w'_t \).

By associativity of left dilations and uniqueness it also follows that maps \( w_t \) fulfilling (6.2) for all \( t \in \mathcal{S} \), form a morphism. Indeed, we find
\[
(ux)w_{t+\iota}(y_t z_t) = u(xy_t z_t) = (u(xy_t))(w_t z_t) = (ux)(w_t y_t)(w_t z_t)
\]
for all \( x \in E \), or \( w_{t+\iota}(y_t z_t) = (w_t y_t)(w_t z_t) \). Once more, by fullness of \( E \) and \( E' \), and by nondegeneracy of \( \mathcal{B} \), the maps \( w_t \) must be isometries.

It follows that maps \( w_t \) satisfying (6.2) if they exist, then they are uniquely determined and form an isometric morphism. It remains to establish mappings \( w_t \) with total range that satisfy (6.2).

Note that (6.2) determines the \( w_t \) only indirectly, while the properties stated in 2 and 3 can in either special case be used as a direct definition. We just observe that it is enough to prove the cases 2 and 3 separately to prove also the general statement. (The general situation can be decomposed into an isomorphism of \( E^0 \) and the product system associated with \( \theta^0 \), an isomorphism between the product systems associated with \( \theta \) and with \( \theta^u \), and an isomorphism between the product system associated with \( \theta^0 \) and \( E'^{0,0} \).) Instead, is simply the restriction of the general statement to the special case. So, it only remains to prove 2 and 3.

To prove 2 we observe that the stated \( w_t \) are isometric (and, therefore, well-defined), bilinear and surjective. Since
\[
z(w_t((x, x')y_t)) = z(x^0 \otimes_t (x'y_t)) = \theta_t(zx^0)(x'y_t) = \theta_t^0((zx^0)(x'y_t)) = z((x, x')y_t),
\]
the \( w_t \) fulfill (6.2).

Similarly, to prove 3 we observe that the stated \( w_t \) are isometric (and, therefore, well-defined), bilinear and surjective. Since
\[
(ux)(w_t y^0 \otimes_t z) = (ux)(uy^0) \otimes_t (uz) = \theta_t^0((ux)(uy^0)uz = u\theta_t^0(xy^0)z = u(x(y^0 \otimes_t z),
\]
the \( w_t \) fulfill (6.2).

\[\blacktriangleright\]

6.4 Definition. In the situation of Proposition 6.3 we say the pairs \( (v, E^0) \) and \( (v', E'^0) \) are conjugate, and in the particular situation of Number 1 we say they are equivalent.

Roughly, speaking two left dilations (of not necessarily the same product system) are conjugate (equivalent) if they induce inner conjugate (the same) \( E_0 \)-semigroup(s). In either case, the two product systems are necessarily isomorphic. The isomorphism is uniquely determined by (6.2), once the unitary between the dilation spaces is fixed.
We see that associating different product systems with the same $E_0$–semigroup means establishing a unique isomorphism between the product systems that behaves well with respect to the left dilations providing the association. This remains even true if the two $E_0$–semigroups live on different but isomorphic $E$. But what happens if we have two ways $v$ and $v'$ to associate the same product system $E^\circ$ with the same $E_0$–semigroup $\vartheta$ on $B^a(E)$?

In general, an endomorphism $w^\circ$ of $E^\circ$ induces a family $u_t := v_t(id_E \circ w_t)v_t^*$ of elements in $B^a(E)$ that form a local cocycle for $\vartheta = \vartheta^w$. If $w_t$ is the automorphism that fulfills (6.2), that is, that fulfills $v_t(x \otimes y_t) = v'_t(x \otimes w_t y_t)$, then we find $v'_t = v_t(id_E \circ w_t^*) = u_t^* v_t$. In other words, $v$ and $v'$ are related by the local cocycle $u^*$ for $\vartheta = \vartheta^v$ and $\vartheta^{u^*} = \vartheta^v = \vartheta^w = \vartheta$.

We have seen that two product systems associated with the same $E_0$–semigroup or with inner conjugate $E_0$–semigroups are isomorphic in an essentially unique way, and we have seen the relation between two ways of associating the same product system with the same $E_0$–semigroup. The next natural question is which $E_0$–semigroups acting on a fixed $B^a(E)$ have the same or (equivalently, by the preceding discussion) isomorphic product systems. In other words, how are the $E_0$–semigroups that act on a fixed $B^a(E)$ classified by their product systems. This is a generalization of the result [Arv89, Corollary of Definition 3.20] for Hilbert spaces and of [Ske02, Theorem 2.4] for Hilbert modules with a unit vector.

**6.5 Theorem.** Let $\vartheta$ and $\vartheta'$ be two strict $E_0$–semigroups on $B^a(E)$ ($E$ a full Hilbert $B$–module). Then their associated product systems $E^\circ$ and $E'^\circ$ are isomorphic if and only if $\vartheta$ and $\vartheta'$ are unitary cocycle equivalent.

**Proof.** Denote by $v, v'$ the standard dilations of $E^\circ, E'^\circ$.

Suppose $w^\circ$ is a morphism from $E^\circ$ to $E'^\circ$. Then $u_t := v'_t(id_E \circ w_t)v_t^*$ defines a left cocycle with respect to $\vartheta$. The cocycle $u$ is unitary if and only if $w^\circ$ is an isomorphism. We find

$$u_t \vartheta_t(a) u_t^* = v'_t(id_E \circ w_t)v_t^* v_t(a \circ id_t)u_t^* v_t(id_E \circ w_t)v_t^* = v'_t(a \circ id_t)v_t^* = \vartheta'_t(a),$$

so that $\vartheta$ and $\vartheta'$ are cocycle equivalent.

Conversely, suppose $u$ is a unitary left cocycle such that $\vartheta^u = \vartheta'$. By $u_t \vartheta_t(a) = \vartheta'_t(a)u_t$, we see that $u_t$ is an isomorphism between the $B^a(E)$–$B$–correspondences $\vartheta_t E$ and $\vartheta'_t E$. It follows that $w_t := id_E \circ u_t$ defines a bilinear unitary from $E_t = E^\circ \circ_t E$ to $E'_t = E^\circ \circ'_t E$. We find

$$w_t(x^* \otimes x')w_t(y^* \otimes y') = (x^* \otimes u_t x')(y^* \otimes u_t y') = x^* \otimes u_t \vartheta_t(u_t x'y^*) u_t y' = x^* \vartheta'_t(u_t x'y^*) u_t y' = x^* \vartheta'_t u_t \vartheta_t(u_t x'y^*) y' = (x^* \otimes u_t \vartheta_t(x'y^*)) y' = (x^* \otimes u_t \vartheta_t(y^* \otimes y')) y' = w_{s+t}(x^* \circ_{s+t} \vartheta_t(x'y^*)) y',$$

so that the $w_t$ form a morphism. ■

**6.6 Corollary.** Let $E$ and $E'$ be isomorphic full Hilbert $B$–modules and suppose $\vartheta$ and $\vartheta'$ are two strict $E_0$–semigroups on $B^a(E)$ and $B^a(E')$, respectively. Then their associated product systems $E^\circ$ and $E'^\circ$ are isomorphic if and only if $\vartheta$ and $\vartheta'$ are unitary cocycle inner conjugate.
**Proof.** Fix a unitary \( u \in \mathcal{B}^a(E, E') \) and define the isomorphism \( \alpha := u \cdot u^* \). Then by Proposition 6.3, \( \vartheta \) and \( \vartheta^\alpha \) have the same product system. The statement now follows by applying the theorem to \( \vartheta^\alpha \) and \( \vartheta \). \( \blacksquare \)

Before analyzing in the following sections the relation between product systems of \( E_0 \)-semigroups acting on not necessarily inner conjugate \( \mathcal{B}^a(E) \)s in terms of more general notions of cocycle conjugacy, we close this section by giving a general result that does not involve cocycles. The result, well known for (separable) Hilbert spaces, provides a necessary and sufficient criterion for that all members of a family of \( E_0 \)-semigroups have isomorphic product systems.

**6.7 Theorem.** Let \( E^i \) \((i \in I)\) be a family of full Hilbert \( \mathcal{B} \)-modules and suppose that for each \( i \in I \) we have a strict \( E_0 \)-semigroup \( \vartheta^i \) on \( \mathcal{B}^a(E^i) \). Denote \( E := \bigoplus_{i \in I} E^i \).

Then the \( \vartheta^i \) have mutually isomorphic product systems if and only if there exists a strict \( E_0 \)-semigroup \( \vartheta \) on \( \mathcal{B}^a(E) \) such that \( \vartheta \uparrow \mathcal{B}^a(E^i) = \vartheta^i \) for all \( i \in I \). In the affirmative case, the product system of \( \vartheta \) is in the same isomorphism class as those of the \( \vartheta^i \).

**Proof.** Denote by \( p^i \in \mathcal{B}^a(E) \) the projection onto \( E^i \). We observe that an \( E_0 \)-semigroup \( \vartheta \) on \( \mathcal{B}^a(E) \) leaves all \( \mathcal{B}^a(E^i) \) invariant if and only if \( \vartheta_i(p^i) = p^i \) for all \( i \in I, t \in S \).

Suppose all \( \vartheta^i \) have isomorphic product systems. By Proposition 6.3 we may fix one product system \( E^\ominus \) in this isomorphism class, and left dilations \( v^i \) of \( E^\ominus \) to \( E^i \) such that \( \vartheta^i = \vartheta v^i \) for all \( i \in I \). Then \( v_i := \bigoplus_{i \in I} v^i \) defines a left dilation of \( E^\ominus \) to \( E \). Clearly, the \( E_0 \)-semigroup \( \vartheta := \vartheta v \) leaves all \( \mathcal{B}^a(E^i) \) invariant, and the restriction of \( \vartheta \) to \( \mathcal{B}^a(E^i) \) is \( \vartheta^i \).

On the contrary, suppose that \( \vartheta \) is a strict \( E_0 \)-semigroup on \( \mathcal{B}^a(E) \) that leaves each \( \mathcal{B}^a(E^i) \) invariant. Suppose \( E^\ominus \) is a product system and \( v \) is a left dilation of \( E^\ominus \) to \( E \) such that \( \vartheta v = \vartheta \). Since \( \vartheta_i(p^i) = p^i \), it follows that \( v^i := p^i v \uparrow (E^i \cap E_i) \) defines a left dilation of \( E^\ominus \) to \( E^i \). Clearly, \( \vartheta^i \) is just the restriction \( \vartheta^i \) of \( \vartheta \) to \( \mathcal{B}^a(E^i) \). Therefore, again by Proposition 6.3 the product system of \( \vartheta^i \) is isomorphic to \( E^\ominus \). \( \blacksquare \)

**6.8 Remark.** The problem in the preceding proof is somewhat similar to showing that a functor between two categories of Hilbert modules is uniquely determined by what it does to a single full object. (On a single object, Theorem 2.12 tells us that the functor is given by tensoring with a multiplicity correspondence. This is crucial for the proof of Blecher’s Eilenberg-Watts theorem for Hilbert modules [Ble97] from [MSS06, Section 2].) In fact, here we are concerned with a semigroup of endofunctors (for each \( t \in S \) induced by tensoring with the multiplicity correspondence \( E_i \)) leaving all objects fixed (that is, acting only on the morphisms) of the mini-category that has only the (full) objects \( E, E^i \) \((i \in I)\). However, thanks to the simple structure (only full objects which are fixed by the functor) the direct proof we gave here is considerably simpler than reducing the statement to [MSS06] Section 2.)
7 Conjugate $E_0$–Semigroups and Morita equivalent product systems

In the preceding section we showed that $E_0$–semigroups on the same $\mathcal{B}^a(E)$ have the same product system (up to isomorphism) if and only if they are unitary cocycle equivalent. This remains true if we replace unitary cocycle equivalence with unitary cocycle inner conjugacy, even between different $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$, as long as $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ are inner conjugate, that is, as long as $E$ and $F$ are isomorphic.

In this section we deal with the question what happens with $E_0$–semigroups on two conjugate $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ under a strict conjugacy provided by an arbitrary strict isomorphism $\alpha: \mathcal{B}^a(E) \rightarrow \mathcal{B}^a(F)$ where $E$ is a Hilbert $\mathcal{B}$–module and $F$ is a Hilbert $\mathcal{C}$–module. Following Convention 6.1, we shall assume that $E$ and $F$ are full.

By Corollary 2.13 there is a Morita equivalence $M$ from $\mathcal{B}$ to $\mathcal{C}$ such that $F \cong E \circ M$ and $E \cong F \circ M^*$, so that $E$ and $F$ are Morita equivalent. Moreover, $\alpha$ is the homomorphism implemented by the isomorphism $F \cong E \circ M$ and $\alpha^{-1}$ is the homomorphism implemented by the isomorphism $E \cong F \circ M^*$. Here, we wish to be more specific than making statements just up to isomorphism.

We fix $M := E^* \circ_{\alpha} F$ and $N := F^* \circ_{\alpha^{-1}} E$ with identifications according to Theorem 2.12. For the isomorphisms in (2.1) we choose (6.1), that is,

$$\begin{align*}
N \circ M &\ni (y^* \circ_{\alpha^{-1}} x^*) \circ (x^* \circ_{\alpha} y' ) \mapsto y^* \circ_{\text{id}_C} \alpha(x'x^*)y' = \langle y, \alpha(x'x^*)y' \rangle \in C, \\
M \circ N &\ni (x^* \circ_{\alpha} y') \circ (y^* \circ_{\alpha^{-1}} x^*) \mapsto x^* \circ_{\text{id}_C} \alpha^{-1}(y'y^*)x' = \langle x, \alpha^{-1}(y'y^*)x' \rangle \in B.
\end{align*}$$

By [MSS06, Theorem 1.14], the identifications according to (6.1) compose associatively. That is, we are in the situation required in Convention 2.4. We easily check that for every $m := x^* \circ_{\alpha} y \in M$ the element $n := y^* \circ_{\alpha^{-1}} x \in N$ allows to recover $m^* \in M^*$ as $m^* \mapsto n \circ m' \in N \circ M = C$. (We leave it as an instructive exercise to verify that the map $n \mapsto m^*$ is the map $u$ used in the proof of Theorem 2.7.)

Morita equivalence of correspondences has been defined by Muhly and Solel [MS00]. Recall the following version for product systems from [Ske09d].

7.1 Definition. Let $E^\otimes$ be a product system of $\mathcal{B}$–correspondences and let $M$ be a Morita equivalence from $\mathcal{B}$ to $\mathcal{C}$. Then the $M$–transformed product system of $E^\otimes$ is the product system $M^* \circ E^\otimes \circ M$ with $(M^* \circ E^\otimes \circ M)_t := M^* \circ E_t \circ M$ and identifications

$$M^* \circ E_{s+t} \circ M \cong M^* \circ E_s \circ E_t \circ M = (M^* \circ E_s \circ M) \circ (M^* \circ E_t \circ M).$$

Clearly, $E^\otimes \mapsto M^* \circ E^\otimes \circ M$ and $w^\otimes \mapsto \text{id}_{M^*} \circ w^\otimes \circ \text{id}_M$ define an equivalence between the category of product systems of $\mathcal{B}$–correspondences and the category of product systems of $\mathcal{C}$–correspondences.
Two product systems $E^\otimes$ and $F^\otimes$ are Morita equivalent if there exists a Morita equivalence $M$ such that $M^* \odot E^\otimes \odot M$ and $F^\otimes$ are isomorphic. Clearly, Morita equivalence of product systems is an equivalence relation.

Putting all these identifications together and taking into account, once more, the associativity result [MSS06, Theorem 1.14] for the identifications according to (6.1), we immediately read off the following result.

7.2 Proposition. Let $E$ be a full Hilbert $\mathcal{B}$–module, let $\vartheta$ be a strict $E_0$–semigroup on $\mathcal{B}^a(E)$, and denote by $E^\otimes$ the product system associated with $\vartheta$. Suppose $F$ is a full Hilbert $\mathcal{C}$–module with a strict isomorphism $\alpha: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$, and denote by $M$ the associated Morita $\mathcal{B}$–$\mathcal{C}$–equivalence (as discussed before).

Then the product system associated with $\vartheta^\alpha$ is isomorphic to $M^* \odot E^\otimes \odot M$. In particular, the product systems of $\vartheta$ and of $\vartheta^\alpha$ are Morita equivalent.

As a simple corollary, we obtain the main classification result for $E_0$–semigroup acting on strictly conjugate operator algebras.

7.3 Theorem. Let $\vartheta$ and $\vartheta$ be strict $E_0$–semigroups on $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$, respectively. Then $\vartheta$ and $\vartheta$ are unitary cocycle strictly conjugate if and only if there exists a strict isomorphism $\alpha: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ and their associated product systems are Morita equivalent via the same Morita equivalence inducing $\alpha$.

7.4 Example. Note that for $F = E$ the notion of unitary cocycle strict conjugacy is strictly wider than the notion of unitary cocycle equivalence. We may suspect this, because the Picard group of $\mathcal{B}$, in general, consists of more than multiplier inner automorphisms. But $M$ being a nontrivial Morita equivalence over $\mathcal{B}$ such that $F = E \odot M$ does not yet guarantee that $M^* \odot E^\otimes \odot M$ is not isomorphic to $E^\otimes$. (For example, take the trivial product system $\mathcal{B}^\otimes$. But also the time ordered product systems $I^{\otimes}(F)$ do not change under a transformation $M^* \odot \bullet \odot M$, whenever $M^* \odot F \odot M \equiv F$.)

But we may obtain a concrete example in the following way. Let $F$ be a correspondence over $\mathcal{B}$ and $M$ a Morita equivalence over $\mathcal{B}$ such that $M^* \odot F \odot M \neq F$. (Example 4.1 helps. Indeed, we choose $M = E$ from that example, and $F = \mathbb{C}$, the $\mathbb{C}$–component of $\mathcal{B} = \mathbb{C} \oplus M_2$. We easily check that $M^* \odot F \odot M = M_2 \neq F$.) In that case, also the time ordered product systems $\Pi^{\otimes}(F)$ and $\Pi^{\otimes}(M^* \odot F \odot M) = M^* \odot \Pi^{\otimes}(F) \odot M$ (see [BS00, LS01]) cannot be isomorphic, because the index $F$ of the time ordered product system $\Pi^{\otimes}(F)$ is an isomorphism invariant; see [Ske06]. Now $\Pi^{\otimes}(F)$ is the product system of the CCR-flow on $\mathcal{B}^a(\Pi(F))$ and $\Pi^{\otimes}(M^* \odot F \odot M)$ is the product system of the CCR-flow on $\mathcal{B}^a(\Pi(M^* \odot F \odot M))$. If $F$ and $M$ (like in the example) are countably generated, then $\Pi(F)$ and $\Pi(M^* \odot F \odot M)$ are stably isomorphic. So, the respective amplifications
of the CCR-flows are unitary cocycle strictly conjugate. But they are not unitary cocycle inner conjugate, because the have non-isomorphic product systems. (Note that the discussion about stable conjugacy and amplification of $E_0$–semigroups anticipates some arguments from Section 8; see there for details.)

We briefly specialize to the case, when the conjugacy of $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$ can be chosen ternary. In that case, $M = \phi C$ where $\phi: \mathcal{B} \to C$ is an isomorphism. One easily verifies that $M^* = \phi^{-1}B$ and that $M^* \circ E_i \circ M$ can be identified with $E_i$ via $b \odot x \odot c \mapsto bx_i\phi^{-1}(c)$ where, however, the inner product is $\langle x_t, y_t \rangle_C := \phi(\langle x_t, y_t \rangle)$ and the left action is $cx_t := \phi^{-1}(c)x_t$.

We call **ternary equivalent** two product systems that are Morita equivalent via a Morita equivalence $M$ that induces a ternary isomorphism. Theorem 7.3 remains true replacing ‘strictly conjugate’ with ‘ternary conjugate’ everywhere. Also Example 7.4 remains valid in either direction:

**7.5 Remark.** The notion of unitary cocycle ternary conjugacy lies strictly in between unitary cocycle strict conjugacy and (where it applies) unitary strict cocycle equivalence. This follows from Example 7.4 from existence of non inner ternary isomorphisms, and from the observation that either composition of an isomorphism $\alpha$ with a ternary (an inner) isomorphism is ternary (inner) if and only if $\alpha$ is ternary (inner).

**7.6 Remark.** We think the potential of the translation of equations between homomorphisms into equations between multiplicity correspondences, as discussed in the beginning of Section 6, is by far not yet exhausted. It would be an interesting exercise to do the computations of Section 6 in these terms. We did not do it in that way, because we do not gain simpler identifications, but rather a considerable complication concerning abstraction. A question where it appears unavoidable to proceed in that way is, what happens if we pass from unitary cocycles to arbitrary cocycles implementing the equivalence of $E_0$–semigroups on the same $\mathcal{B}^a(E)$. Already for ternary unitary cocycles we do not know the answer. (The main problem is that it is completely unclear what $\vartheta_s(u_t)$ might be for a ternary unitary $u$; see [Ske06a, Remark 3.8] and [AS07, Section 4].) As for the present notes, we do not need an answer to this question. So, we do not tackle the problem here.

**8 Stable unitary cocycle (inner) conjugacy of $E_0$–semigroups**

In the two preceding sections we established the main results about classification of $E_0$–semigroups in terms of product systems in the situation where the involved $E_0$–semigroups act on the same $\mathcal{B}^a(E)$ or at least on strictly (or ternary) isomorphic operator algebras. (The only exception is Theorem 6.7 which is, however, not in terms of cocycle conjugacy.) On the same $\mathcal{B}^a(E)$ we
found classification of $E_0$–semigroups up to unitary cocycle equivalence by product systems up to isomorphism. For conjugate $\mathcal{B}(E)$ and $\mathcal{B}(F)$ we found classification of $E_0$–semigroups up to unitary cocycle strict (ternary) conjugacy by product systems up to Morita (ternary) equivalence. We also showed that on the intersection of their domains, in general, the notions are all different. Only for Hilbert spaces the difference disappears.

But the question when two $E_0$–semigroups have isomorphic or Morita equivalent product systems, has a meaning also if the $E_0$–semigroups act on operator algebras of a priori unrelated Hilbert modules $E$ and $F$. In the present section we use the results from Section 3 (and Section 4) combining them with Sections 6 and 7 to answer this question under the (reasonable) countability conditions of Section 3. (The von Neumann case, without such separability conditions, will be discussed in Section 11.)

The problem is that before we can apply the results of Sections 6 and 7 in order to compare $E_0$–semigroups on $\mathcal{B}(E)$ and $\mathcal{B}(F)$ in terms of their product systems, we must be able to compare $\mathcal{B}(E)$ and $\mathcal{B}(F)$ in terms of strict conjugacy. If $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are not strictly conjugate then, maybe, $\mathcal{B}(E \otimes \mathbb{K})$ and $\mathcal{B}(F \otimes \mathbb{K})$ are. As soon as this is the case, we can apply Sections 6 and 7 to the amplified $E_0$–semigroups. The following proposition, which is a simple corollary of Theorem 6.7, taking into account that $E \otimes \mathbb{K} = \bigoplus_{n \in \mathbb{N}} E$, shows that passing to the stable versions does not change the product systems.

8.1 Proposition. Suppose $\vartheta$ is a strict $E_0$–semigroup on $\mathcal{B}(E)$ for some full Hilbert $\mathcal{B}$–module $E$. Then $\vartheta$ and its amplification $\vartheta^\mathbb{K} := \vartheta \otimes \text{id}_\mathbb{K}$ to $\mathcal{B}(E \otimes \mathbb{K})$ have the same product system.

8.2 Definition. Two strict $E_0$–semigroup are stably cocycle conjugate (equivalent) if their amplifications are cocycle conjugate (equivalent). We use all supplements (like unitary, inner, ternary, etc.) in the same way as before.

A version of stable cocycle conjugacy for von Neumann algebras has been considered by Alevras [Ale04]; see the remark in the introduction.

The following theorem merely collects most of the results of Sections 3, 4, 6 and 7.

8.3 Theorem. 1. If $\vartheta$ is a strict $E_0$–semigroup on $\mathcal{B}(E)$ for a full countably generated Hilbert module $E$ over a $\sigma$–unital $C^*$–algebra $\mathcal{B}$, then the amplification $\vartheta^\mathbb{K}$ is inner conjugate to an $E_0$–semigroup $\vartheta^\mathcal{B}$ on $\mathcal{B}(\mathbb{K} \otimes \mathcal{B})$.

2. Let $\vartheta$ and $\theta$ be strict $E_0$–semigroups on $\mathcal{B}(E)$ and $\mathcal{B}(F)$, respectively, where $E$ and $F$ are full countably generated Hilbert modules over a $\sigma$–unital $C^*$–algebra $\mathcal{B}$. Then the following conditions are equivalent:

(i) $\vartheta$ and $\theta$ are stably unitary cocycle inner conjugate.

(ii) $\vartheta^\mathcal{B}$ and $\theta^\mathcal{B}$ are unitary cocycle equivalent.
(iii) $\theta$ and $\vartheta$ have isomorphic product systems.

3. Let $\vartheta$ and $\theta$ be strict $E_0$–semigroups on $\mathcal{B}^a(E)$ and $\mathcal{B}^a(F)$, respectively, where $E$ and $F$ are full countably generated Hilbert modules over $\sigma$–unital $C^*$–algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Then $\vartheta$ and $\theta$ are stably unitary cocycle strictly (ternary) conjugate if and only if they have Morita (ternary) equivalent product systems.

**Proof.**

1. By Proposition 3.1, $E \otimes \mathbb{K} \cong \mathbb{K}_B$. Choose an isomorphism $u$. Then, conjugation of $\vartheta^\mathbb{K}$ with $\alpha := u \bullet u^*$ gives the conjugate semigroup on $\mathcal{B}^a(\mathbb{K}_B)$.

2. By definition, 2i is equivalent to that $\vartheta^\mathbb{K}$ and $\theta^\mathbb{K}$ are unitary cocycle inner conjugate. By Part 1, this is the same as 2ii, and by Theorem 6.5 this is equivalent to 2iii.

3. By Theorem 3.5 (Theorem 4.8), either condition means that $\mathcal{B}$ and $\mathcal{C}$ must be Morita equivalent (isomorphic). (Otherwise, none of the conditions can be satisfied.) So, if one of the conditions is satisfied, then there exists a Morita equivalence $M$ from $\mathcal{B}$ to $\mathcal{C}$. Since $\mathcal{K}(M) = \mathcal{B}$ is $\sigma$–unital, by [Lan95, Proposition 6.7] $M$ is countably generated, and with $M$ and $E$ also the Hilbert $\mathcal{C}$–module $E \otimes M$ is countably generated. We are now ready to apply Part 2 replacing $\vartheta$ with the conjugate $E_0$–semigroup on $\mathcal{B}^a(E \otimes M)$. The specialization to the ternary case (where $M = \varphi C$ for some isomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{C}$) is obvious. ■

9 About continuity

So far, we have answered completely the question, to what extent strict $E_0$–semigroups acting on operator algebras on countably generated Hilbert modules over $\sigma$–unital $C^*$–algebras are classified by their product systems (up to isomorphism, up to ternary equivalence, or up to Morita equivalence of the product systems). The answer is: Up to a suitable notion of stable unitary cocycle conjugacy. (The variation is just in the adjective preceding the word *conjugacy*.) For a complete treatment, there remains the problem to indicate which product systems can arise as product systems of $E_0$–semigroups.

In this section we recall the known results about existence of $E_0$–semigroups for product systems. We will see that in order that the constructed $E_0$–semigroups live on spaces that are compatible with the countability assumptions (originating in Section 3), we can no longer avoid to introduce technical constraints both on the $E_0$–semigroup side (strong continuity) and on the product system side (continuity and countability hypothesis).

First of all, recall that the product system of an $E_0$–semigroup is always full. Recall, too, that in these notes we deal only with the case of *unital* $C^*$–algebras. The case of not necessarily unital $C^*$–algebras (with similar results, though technically more involved) is done in Skeide [Ske09b]. For $\mathcal{S} = \mathbb{N}_0$, by [Ske09a, Theorem 7.6] we obtain an $E_0$–semigroup that acts on the operators on a Hilbert module $E$. It is easy to check that $E$ is countably generated if and
only if \( E_1 \) is countably generated as a (right) Hilbert module. As sketched only very briefly in [Ske07a], in the case \( S = \mathbb{R}_+ \) but without continuity conditions the algebraic part of the construction in the Hilbert space case in [Ske06a] (making also use of the result of [Ske09d]) generalizes easily to Hilbert modules. But the \( E_0 \)–semigroup obtained acts on the operators on a Hilbert module which is definitely not countably generated. Without continuity conditions, there is no construction known that would lead to a countably generated Hilbert module.

This negative statement ends the discussion of the continuous time case \( S = \mathbb{R}_+ \) in the purely algebraic situation.

Speaking about Hilbert modules, there remains the case \( S = \mathbb{R}_+ \) with continuity conditions. (The case of von Neumann modules will be discussed in Section 12.) To deal with that case, we have to repeat to some extent what these conditions are, and how the results from [Ske07a] allow to prove, as a new result, that suitable countability conditions on the continuous product system are preserved under the construction of an \( E_0 \)–semigroup.

Recall that an \( E_0 \)–semigroup \( \vartheta \) on \( B^a(E) \) is strongly continuous if \( t \mapsto \vartheta_t(a)x \) is continuous for all \( a \in B^a(E) \) and \( x \in E \). Obviously, the amplification \( \vartheta^X \) is strongly continuous if and only if \( \vartheta \) is strongly continuous. A family \( u = (u_t)_{t \in \mathbb{R}_+} \) of elements \( u_t \in B^a(E) \) (that is, in particular, a left cocycle) is strongly continuous if \( t \mapsto u_tx \) is continuous for all \( x \in E \).

Following the definitions in [Ske03b, Ske07a], a continuous product system is a product system \( E^\otimes = (E_t)_{t \in \mathbb{R}_+} \) with a family \( i_t : E_t \rightarrow \widehat{E} \) of isometric embeddings of the \( B \)–correspondences \( E_t \) into a common Hilbert \( B \)–module \( \widehat{E} \) as right modules (there is no left action on \( \widehat{E} \)), fulfilling the following conditions: Denote by

\[
\text{CS}_i(E^\otimes) = \left\{ (x_t)_{t \in \mathbb{R}_+} : x_t \in E_t, \ t \mapsto i_t x_t \text{ is continuous} \right\}
\]

the set of continuous sections of \( E^\otimes \) (with respect to the embeddings \( i_t \)). Then, firstly,

\[
\left\{ x_s : (x_t)_{t \in \mathbb{R}_+} \in \text{CS}_i(E^\otimes) \right\} = E_s
\]

for all \( s \in \mathbb{R}_+ \) (that is, \( E^\otimes \) has sufficiently many continuous sections), and, secondly,

\[
(s,t) \mapsto i_{s+t}(x_t,y_t)
\]

is continuous for all \( (x_t)_{t \in \mathbb{R}_+}, (y_t)_{t \in \mathbb{R}_+} \in \text{CS}_i(E^\otimes) \) (that is, the ‘product’ of continuous sections is continuous). A morphism between continuous product systems is continuous if it sends continuous sections to continuous sections. An isomorphism of continuous product systems is a continuous isomorphism. In [Ske09b, Theorem 2.2] we showed that a continuous isomorphism has continuous inverse. So, an isomorphism provides a bijection between the sets of continuous sections.
If $E$ is a full Hilbert module over a unital $C^*$–algebra $B$, and if $\theta$ is a strongly continuous strict $E_0$–semigroup on $B^a(E)$, then (generalizing on Skeide [Ske03b]) we define a continuous structure on the associated product system $E^\otimes$ in the following way:

Pass to the strongly continuous strict $E_0$–semigroup $\theta := \theta^K$ on $B^a(F)$ for $F := E \otimes \mathbb{K}$ and associated product system $F^\otimes$. We know from Proposition 8.1 that $\bar{\theta}$ and $\theta$ have the same product system. But we wish to have more specific identifications. Obviously,

$$F_i = (E \otimes \mathbb{K})^\tau (\phi, \phi) \otimes ((\theta, E) \otimes \mathbb{K}) = (E^\tau \otimes \theta) \otimes ((\mathbb{K} \otimes \mathbb{K}) = E_i \otimes \mathbb{C} = E,$$

everywhere the canonical identifications and the natural action of $\mathcal{K}$ on $\mathbb{K}$. (We leave it as an intriguing exercise for the reader to show that the concrete prescription

$$(x \otimes f)^\tau \otimes_i (y \otimes g) \mapsto (x \otimes f)(y, g)$$
suggested by the preceding identifications, define bilinear unitaries $F_i \rightarrow E_i$ that form an isomorphism $F^\otimes \rightarrow E^\otimes$. The inverse is determined by $x^* \otimes_i y \mapsto (x \otimes x)^\tau \otimes_i (y \otimes y)$ where $y$ can be any unit vector in $\mathbb{K}$.)

Even if $E$ has no unit vector, by [Ske09d, Lemma 3.2] $F$ has one, $\zeta$ say. That is, we are ready for the construction of the product system of $\theta$ following the first construction in [Ske02] (imitating Bhat’s construction [Bha96] in the Hilbert space case) based on existence of a unit vector: For every $t \in \mathbb{R}_+$ define the Hilbert $B$–submodule $\bar{F}_i := \theta_t(\zeta \zeta^*)F$ of $F$. Turn it into a correspondence over $B$ by setting $b\bar{y}_t = \theta_t(b\zeta \zeta^*)\bar{y}_t$. Then $\bar{F}^\otimes = (\bar{F}_i)_{t \in \mathbb{R}_+}$ is a product system via $\bar{y}_t \otimes \bar{z}_t \mapsto \theta_t(\bar{y}_t \zeta \zeta^*)\bar{z}_t$ and $y \otimes z_i \mapsto \theta_t(y \zeta \zeta^*)z_t$ defines a left dilation of $\bar{F}^\otimes$ to $F$ giving back $\theta$. Clearly, $F^\otimes \cong \bar{F}^\otimes$, and it is easy to verify that $y^* \otimes_i z \mapsto \theta_t(y \zeta \zeta^*)z$ defines the isomorphism. (See the old version [Ske04b, p.5] for details.) We now define the isometric embedding

$$j_t: y^* \otimes_i z \mapsto \theta_t(y \zeta \zeta^*)z \in F$$
of $F_i$ into $F$. It is easy to prove that this equips $F^\otimes$ with a continuous structure. (See [Ske03b, Ske07b] for details. For instance, for every $y_t \in F_t \subset F$ the continuous section $(\theta_t(\zeta \zeta^*)y_t)_{t \in \mathbb{R}_+}$ meets $y_t$ for $s = t$.) By composing the isomorphism $E_i \rightarrow F_i$ with $j_t$, we define isometric embeddings $i_t: E_t \rightarrow F$, turning $E^\otimes$ into a continuous product system isomorphic to $F^\otimes$. It is $E^\otimes$ with this continuous structure we have in mind if we speak about the continuous product system associated with $\theta$.

It is noteworthy that the continuous structure does not depend on the choice of the unit vector $\zeta \in F$. In fact, if $\zeta' \in F$ is another unit vector, then the isomorphism $\theta_t(\zeta \zeta')$ from the product system $\bar{F}^\otimes$ constructed from $\zeta$ to the product system $\bar{F}^\otimes$ constructed from $\zeta'$ (see [Ske02] for details), clearly, sends continuous sections to continuous sections, and so does its inverse $\theta_t(\zeta \zeta'^*)$. Even if $E$ has already a unit vector, $\xi$ say, and we started the construction from $\theta$ and that $\xi$, obtaining embeddings $E_t \rightarrow \theta_t(\xi \xi^*) \subset E$, the continuous structure would be the
same. In fact, we may choose $\zeta = \xi \otimes x$ for a unit vector $x \in \mathbb{K}$. If, then, we identify $E$ with the subspace $E \otimes x$ of $F$, then the embeddings $E_t \to \theta_t(\xi x^*)E \subset E \to E \otimes x \subset F$ and $E_t \to F_t \to \theta_t(\xi x^*)F \subset F$ are the same.

In conclusion, the continuous structure of the product system associated with $\theta$ is determined uniquely by the preceding construction and is isomorphic to the continuous structure defined in [Ske03b] in presence of a unit vector in $E$.

We show that the classification in Theorem 6.5 of $E_0$–semigroups of a fixed $\mathcal{B}^a(E)$ behaves well with respect to continuity.

9.1 Theorem. Let $\theta$ and $\theta'$ be two strongly continuous strict $E_0$–semigroups on $\mathcal{B}^a(E)$ ($E$ a full Hilbert $\mathcal{B}$–module). Then their associated continuous product systems $E^\circ$ and $E'^\circ$ are isomorphic if and only if $\theta$ and $\theta'$ are unitary cocycle equivalent via a strongly continuous unitary left cocycle.

Proof. If $\theta$ and $\theta'$ are not unitary cocycle equivalent, then by Theorem 6.5, $E^\circ$ and $E'^\circ$ are not even algebraically isomorphic. So, let us assume that $\theta$ and $\theta'$ are unitary cocycle equivalent and denote by $u$ and $w^\circ$ the unitary left cocycle and the isomorphism, respectively, related by the formulae in the proof of Theorem 6.5. Then $v := u \otimes \text{id}_E$ is a unitary left cocycle in $\mathcal{B}^a(F)$ providing cocycle equivalence of $\theta$ and $\theta'$. Clearly, $v$ is strongly continuous if and only if $u$ is. Moreover, the isomorphism related to $v$ is the same $w^\circ$.

We shall show that $u \otimes \text{id}_E$ is strongly continuous if and only if $w^\circ$ is continuous. To that goal we switch to the product systems $\tilde{F}^\circ$ and $\tilde{F}'^\circ$, where continuity of sections is defined. So let $\tilde{y}_t = \theta_t(\xi x^*)\tilde{y}_t$ be a section of $\tilde{F}^\circ$. Note that $v_t \tilde{y}_t = v_t \theta_t(\xi x^*) \tilde{y}_t = \theta'_t(\xi x^*) v_t \tilde{y}_t \in \tilde{F}_t$, so that $v_t \tilde{y}_t$ is a section of $\tilde{F}'^\circ$. One easily verifies that this is precisely what $w^\circ$ does in the picture $\tilde{F}^\circ \to \tilde{F}'^\circ$.

So, if $v$ is strongly continuous, then with the section $\tilde{y}$ also the section $v\tilde{y}$ is continuous, and vice versa. (Recall that the strong topology on unitaries coincides with the $*$–strong topology.) That is, $w^\circ$ is continuous.

On the other hand, suppose $w^\circ$ is continuous. Choose $y \in F$ and a continuous section $\tilde{z}$ of $\tilde{F}^\circ$. Observe that with $\tilde{z}$ also the function $t \mapsto \theta_t(y \xi^*)\tilde{z}_t$ is continuous. Likewise, this holds for $\theta'$ and continuous sections of $\tilde{F}'^\circ$. So,

$$
\| (v_t - v_s) \theta_t(y \xi^*) \tilde{z}_t \| \leq \| v_t \theta_t(y \xi^*) \tilde{z}_t - v_s \theta_s(y \xi^*) \tilde{z}_s \| + \| v_s \theta_s(y \xi^*) \tilde{z}_s - \theta_t(y \xi^*) \tilde{z}_t \|
$$

is small for $s$ close to $t$, because $v_t \tilde{z}_t$ is a continuous section of $\tilde{F}^\circ$. Since the set of all $\theta_t(y \xi^*) \tilde{z}_t$ is total in $F$, it follows that $v$ is strongly continuous. □

The basic result of [Ske07a] asserts that every full continuous product system of correspondences over a unital $C^*$–algebra is (isomorphic to) the continuous product system associated
with a strongly continuous strict $E_0$–semigroup on some $\mathcal{B}^\alpha(E)$. (See, however, Remark 9.6.) Theorem 9.1 tells us that all strongly continuous strict $E_0$–semigroups on that $\mathcal{B}^\alpha(E)$ are determined by the isomorphism class of their continuous product system up to continuous unitary cocycle equivalence. If, in order to complete the classification of strongly continuous strict $E_0$–semigroups by continuous product systems, we wish to apply Theorem 8.3 (in particular, the equivalence of (2) and (3)), then we have to analyze to what extent we can guarantee that the $E_0$–semigroup constructed in [Ske07a] lives on a $\mathcal{B}^\alpha(E)$ where $E$ satisfies the necessary countability hypotheses. Since $\mathcal{B}$ is unital, that is, in particular, $\sigma$–unital, we have only to worry about whether $E$ is countably generated.

Recall that starting from a strongly continuous $E_0$–semigroup $\vartheta$ on $\mathcal{B}^\alpha(E)$, the module $\hat{E}$ is simply $F = E \otimes \mathbb{K}$. It seems, therefore, reasonable to require for the opposite direction that $\hat{E}$ is countably generated. But, in general, that would not even guarantee that the submodules $i_t E_t$ of $\hat{E}$ are countably generated. (Submodules of countably generated Hilbert modules need not be countably generated. Think of the $\mathcal{B}(H)$–module $\mathcal{K}(H)$ of the singly generated $\mathcal{B}(H)$–module $\mathcal{B}(H)$ for nonseparable $H$.) In addition, having a look at the construction in [Ske07a], the question whether the constructed $E$ is countably generated reduces to the question whether the direct integral $\int_0^1 E_\alpha d\alpha$ (defined in [Ske07a]) is countably generated. It is a submodule of the countably generated $L^2([0,1], \hat{E})$. However, once more submodules of countably generated Hilbert modules need not be countably generated.

The problem disappears if we take into account that the submodules $\theta_t(\zeta \zeta^*) F$ of $F$ are the range of a projection. Indeed, if $E$ is countably generated, then so is $F$. Take a countable generating subset $S$ of $F$. Then the countable set of sections $\{ (\theta_t(\zeta \zeta^*) y)_t : y \in S \}$ shows that $E^\odot$ is countably generated in the following sense. (A proof goes like that of [Ske09b, Theorem 2.2].)

9.2 Definition. A continuous product system $E^\odot$ is countably generated if it admits a countable subset of $CS_i(E^\odot)$ that is total in the locally uniform topology of $CS_i(E^\odot)$.

9.3 Theorem. If $E^\odot$ is a countably generated continuous product system of correspondences over a unital $C^*$–algebra $\mathcal{B}$ then there exist a countably generated full Hilbert $\mathcal{B}$–module $E$ and a strongly continuous strict $E_0$–semigroup on $\mathcal{B}^\alpha(E)$ such that $E^\odot$ is isomorphic to the continuous product systems associated with $\vartheta$.

Proof. The continuous sections restricted to $[0,1]$ take an inner product $\langle x, y \rangle = \int_0^1 \langle x_\alpha, y_\alpha \rangle d\alpha$ and $\int_0^1 E_\alpha d\alpha$ is defined as the norm completion. A countable set of sections generating the restriction to $[0,1]$ in the uniform topology, is a fortiori generating for the $L^2$ topology.

The classification theorem follows immediately.
9.4 Theorem. Let $\mathcal{B}$ be a unital $C^*$–algebra. Then there is a one-to-one correspondence between equivalence classes (up to stable unitary cocycle inner conjugacy with strongly continuous unitary cocycles) of strongly continuous strict $E_0$–semigroups acting on the operators of countably generated full Hilbert $\mathcal{B}$–modules and isomorphism classes of countably generated continuous product systems of correspondences over $\mathcal{B}$.

We dispense with stating the obvious variants for unitary cocycle strict or ternary conjugacy following from Theorem 8.3(3).

9.5 Remark. The only problem is to make $M^* \otimes E \otimes M$ inherit a continuous structure from $E^\otimes$, and to show that $M \otimes (M^* \otimes E^\otimes \otimes M) \otimes M^*$ gives back the original structure. The simplest way, is to start with a strongly continuous $E_0$–semigroup on some $\mathcal{B}^a(E)$ that has product system $E^\otimes$, and then to equip $M^* \otimes E^\otimes \otimes M$ with the continuous structure that emerges from the conjugate $E_0$–semigroup on $\mathcal{B}^a(E \otimes M)$. This assures that the structures are compatible, and that iterating we get back the original one. To proceed in this way has also the advantage that the same sort of argument works in the von Neumann case.

A direct way to equip $M^* \otimes E^\otimes \otimes M$ with a continuous structure, without making reference to $E_0$–semigroups, is the following. Observe that, by [Ske09d, Lemma 3.2], there exists $n \in \mathbb{N}$ such that $M^n$ has a unit vector, $\mu$ say. Then $m^* \otimes x_i \otimes m' \mapsto \mu \otimes m^* \otimes x_i \otimes m'$ defines an isometry $M^* \otimes E_i \otimes M \to M^n \otimes M^* \otimes E_i \otimes M = (E_i \otimes M)^n$. Combining this with $i \otimes \text{id}_M$ on each component, we get isometries $k_i: M^* \otimes E_i \otimes M \to (\hat{E} \otimes M)^n$. It is not difficult to see that the $k_i$ turn $M^* \otimes E^\otimes \otimes M$ into a continuous product system. (Approximate everything locally uniformly by sections that are finite linear combinations of sections $(m^* \otimes y_i \otimes m')_{i \in \mathbb{R}_+}$ with $y \in CS_i(E^\otimes)$ and $m, m' \in M$.) It is also not difficult to see that the double iteration of this procedure gives a new continuous structure on $E^\otimes$ where all section from $CS_i(E^\otimes)$ are continuous. By [Ske09b, Theorem 2.2], the new and the old continuous structure coincide. (The von Neumann case works in a similar manner, but the cardinality $n \in \mathbb{N}$ must be allowed to be an arbitrary cardinal number.)

This ends the classification of $E_0$–semigroups by product systems in the case of Hilbert $C^*$–modules (under the manifest countability assumptions, of course). In the next section we apply the results to characterize the Markov semigroups that admit a special sort of dilation as the spatial ones. After that, the rest of the paper is devoted to discuss all the results for von Neumann algebras and modules. Apart from the absence of countability assumptions, in particular, the result about dilations is much more powerful, because there are much more interesting spatial Markov semigroups on von Neumann algebras.

9.6 Remark. We mention that the proof of [Ske07a, Proposition 4.9], which asserts that the $E_0$–semigroup constructed in [Ske07a] for a continuous product system $E^\otimes$, gives back the same continuous structure on $E^\otimes$ we started with, has a gap. [Ske09b, Theorem 2.2] does not
only fix that gap, but provides a considerably more general statement on the bundle structure of a continuous product system.

10 Hudson-Parthasarathy dilations of spatial Markov semigroups

Markov semigroups are models for irreversible evolutions both of classical and of quantum systems. Dilation means to understand an irreversible evolution as projection from a reversible evolution of a big system onto the small subsystem via a conditional expectation. Noises are models for big systems in which the small system is unperturbed, that is, dilations of the trivial evolution of the small system or, yet in another way, a big physical system in interaction picture with the interaction switched off. Finding a dilation of the Markov semigroup that is a cocycle perturbation of a noise, means establishing a so-called quantum Feynman-Kac formula. If the perturbation is by a unitary left cocycle, then we speak of a Hudson-Parthasarathy dilation.

Noises come along with filtrations of subalgebras that are conditionally independent in the conditional expectation for some notion of quantum independence. That is why they are called noises. They also are direct quantum generalizations of noises in the sense of Tsirelson [Tsi98, Tsi03]. In Skeide [Ske04a] we showed that every noise on $\mathcal{B}(E)$ comes along with a conditionally monotone independent filtration. In Skeide [Ske09c, Ske10b] we show that every such monotone noise may be “blown up” to a free noise by making a relation with free product systems. See the survey Skeide [Ske09a] for more details.

Spatiality is a property that exists on the level of $E_0$–semigroups, on the level of product systems, and on the level of CP-semigroups. If spatiality is present on one level, then it is present also on the other levels. Noises are spatial and, therefore, so are the product systems and all Markov semigroup that can be dilated to noises. Once more, see [Ske09a] for more details.

The scope of this section is to show by the means developed in the preceding sections that a Markov semigroup admits a Hudson-Parthasarathy dilation if and only if it is spatial. The key point is that, starting from the spatial Markov semigroup, we will construct two $E_0$–semigroups having the same product system. One is another sort of dilation, a so-called weak dilation, while the other is a noise. So far, it was unclear how to compare these two $E_0$–semigroups. But now, with the results obtained in the preceding sections, we know that (under countability conditions) their amplifications will act on the operator algebras of isomorphic Hilbert modules, so that there is a unitary left cocycle sending one amplification to the other. The only thing is to adjust the identification of the modules in such a way that they behave nicely in terms of the dilation. In [Ske08b] we performed that program for Markov semigroups on $\mathcal{B}(H)$, where all
the necessary classification results had already been known long before.

We start by explaining the necessary terms used in this introduction.

A **CP-semigroup** is a semigroup $T = (T_t)_{t \in S}$ of completely positive maps $T_t$ on a $C^*$-algebra $\mathcal{B}$. In the sequel, we fix a unital $C^*$-algebra $\mathcal{B}$. A **Markov semigroup** is a CP-semigroup $T$ where all $T_t$ are unital.

Suppose $(E, \vartheta, \xi)$ is a triple consisting of a Hilbert $\mathcal{B}$–module, a strict $E_0$–semigroup $\vartheta$ on $\mathcal{B}^a(E)$, and a unit vector $\xi \in E$. Then by [Ske02, Proposition 3.1], the family of maps $T_t: b \mapsto \langle \xi, \vartheta_t(\xi b \xi^*) \xi \rangle$ defines a CP-semigroup on $\mathcal{B}$ (which is unital automatically) if and only if the projection $\xi \xi^*$ is increasing for $\vartheta$, that is if and only if $\vartheta_t(\xi \xi^*) = \xi \xi^*$ for all $t \in \mathbb{R}$. In this case, $(E, \vartheta, \xi)$ is a weak dilation of $T$ in the sense of [BS00], that is, with the embedding $i: b \mapsto \xi b \xi^*$ and the vector expectation $\nu: a \mapsto \langle \xi, a \xi \rangle$ the diagram

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{B}^a(E) & \xrightarrow{\vartheta_t} & \mathcal{B}^a(E)
\end{array}
$$

commutes for all $t \in \mathbb{R}$. A weak dilation is primary if $\vartheta_t(\xi \xi^*)$ converges strongly to $\text{id}_{E}$ for $t \to \infty$. If the diagram commutes with $i$ replaced by an arbitrary embedding, then we say just $(E, \vartheta, \xi, i)$ is a dilation. A dilation $(E, \vartheta, \xi, i)$ is unital if $i$ is unital. It is reversible if $\vartheta$ consists of automorphisms. Note that whatever the dilation is, putting $t = 0$ it follows that $i$ is injective and that $i \circ \nu$ is a conditional expectation onto $i(\mathcal{B})$. In the case of a unital dilation, this means that $i$ turns $E$ into a faithful correspondence over $\mathcal{B}$. If we wish to think of $E$ as a correspondence in that way, we will identify $\mathcal{B}$ as a unital subalgebra of $\mathcal{B}^a(E)$ and write, slightly abusing notation, $(E, \vartheta, \xi, \text{id}_{\mathcal{B}})$ for the unital dilation.

By a noise over a unital $C^*$–algebra $\mathcal{B}$ we understand a triple $(E, S, \omega)$ consisting of a (necessarily faithful) correspondence $E$ over $\mathcal{B}$, an $E_0$–semigroup $S$ on $\mathcal{B}^a(E)$ (referred to as time shift), and a unit vector $\omega \in E$ (referred to as vacuum), such that:

1. $S$ leaves $\mathcal{B} \subset \mathcal{B}^a(E)$ invariant, that is, $S_t(b) = b$ for all $t \in \mathbb{R}_+, b \in \mathcal{B}$.

2. $(E, S, \omega, \text{id}_{\mathcal{B}})$ is a unital dilation.

3. $(E, S, \omega)$ is a weak dilation.

By $\langle \omega, b \omega \rangle = b$ for all $b \in \mathcal{B}$. Calculating the norm of $b \omega - \omega b$, it follows that $b \omega = \omega b$. By $(E, S, \omega, \text{id}_{\mathcal{B}})$ is a unital dilation of the trivial semigroup on $\mathcal{B}$. By [1] the projection $p := \omega \omega^*$ is increasing. From

$$
\omega b \omega^* = \omega b \omega \omega^*
$$

it follows that $(E, S, \omega)$ is a weak dilation of the trivial semigroup.
10.1 Remark. This definition of noise is more or less from Skeide [Ske06]. In the scalar case (that is, \( \mathcal{B} = \mathbb{C} \)) it corresponds to noises in the sense of Tsirelson [Tsi98] [Tsi03]. A reversible noise is closely related to a Bernoulli shift in the sense of Hellmich, Köstler and Kümmerer [HKK04].

A noise is inner and strongly continuous and strict if the time shift \( \mathcal{S} \) is inner and strongly continuous and strict, respectively. We use similar conventions for other properties of \( \mathcal{S} \), or of the weak dilation \( (E, \mathcal{S}, \omega) \). For instance, a reversible noise is a noise where \( \mathcal{S} \) consists of automorphisms. But, since there are, in general, noninner automorphism semigroups, a reversible noise need not be inner. An inner noise is vacuum preserving if the implementing unitary semigroup \( u \) can be chosen such that \( u_t \omega = \omega (= u'_t \omega) \) for all \( t \in \mathbb{R}_+ \).

10.2 Proposition. An inner noise fulfills \( \mathcal{S}_t(\omega \omega^*) = \omega \omega^* \) for all \( t \in \mathbb{R} \). Moreover, the elements \( \langle \omega, u_t \omega \rangle \) form a unitary semigroup in the center of \( \mathcal{B} \) such that the unitary semigroup \( u'_t := u_t (u_t \omega, \omega) \) implements an inner noise that is vacuum preserving.

Proof. For that \( \mathcal{S}_t(\omega \omega^*) = (u_t \omega)(u_t \omega)^* \geq \omega \omega^* \), it is necessary and sufficient that \( \langle u_t \omega, \omega \rangle \) is an isometry. For that \( \mathcal{S} \) leaves \( \mathcal{B} \) invariant, it is necessary and sufficient, that all \( u_t \) commute with all \( b \in \mathcal{B} \). It follows that also \( u_t \omega \) commutes with all \( b \in \mathcal{B} \). Therefore, \( \langle u_t \omega, \omega \rangle \) is in the center of \( \mathcal{B} \). But an isometry in a commutative algebra is a unitary. From this, also the inverse inequality \( (u_t \omega)(u_t \omega)^* \leq \omega \omega^* \) follows, so that \( (u_t \omega)(u_t \omega)^* = \omega \omega^* \).

Observe that \( u_{t+\tau} \omega = (u_t \omega)(u_{t+\tau} \omega)^*(u_t \omega) = \omega \omega^*(u_t \omega) = \omega(\omega, u_t \omega) \). Applying \( (u_{t+\tau} \omega)(u_{t+\tau} \omega)^* = (u_t \omega)(u_t \omega)^* \) to \( u_{t+\tau} \omega \), we find

\[
\begin{align*}
  u_{t+\tau} \omega &= (u_t \omega)(u_t \omega, u_{t+\tau} \omega) = (u_t \omega)(\omega, u_{t+\tau} \omega) = \omega(\omega, u_t \omega)(\omega, u_{t+\tau} \omega).
\end{align*}
\]

Multiplying with \( \omega^* \) from the left, we see that the \( \langle \omega, u_t \omega \rangle \) form a unitary semigroup in the center of \( \mathcal{B} \). The rest is obvious. 

10.3 Observation. The time shift \( \mathcal{S}_t \), differs from the modified time shift \( \mathcal{S}'_t := u'_t \cdot u_t^* \) by conjugation with the unitary semigroup \( \langle u_t \omega, \omega \rangle \) in \( \mathcal{B} \subset \mathcal{B}^a(E) \). The center of \( \mathcal{B} \) is isomorphic to the center of \( \mathcal{B}^a(E) \) (acting on \( E \) by multiplication from the right), but it need not coincide with the center of \( \mathcal{B}^a(E) \supset \mathcal{B} \supset C(\mathcal{B}) \) (that is, acting from the left). So \( \mathcal{S}' \) is, in general, different from \( \mathcal{S} \). But, since \( \mathcal{S} \) leaves \( \mathcal{B} \subset \mathcal{B}^a(E) \) invariant, the unitaries \( \langle u_t \omega, \omega \rangle \) form a (generally, nonlocal!) cocycle for \( \mathcal{S} \).

10.4 Example. If \( \mathcal{B} \) has trivial center, for instance if \( \mathcal{B} = \mathcal{B}(G) \) for some Hilbert space \( G \), then the cocycle \( \langle u_t \omega, \omega \rangle \) is local and does not change \( \mathcal{S} \). But, for nontrivial center of \( \mathcal{B} \), there is no reason why the left action of \( \mathcal{B} \) should map the center of \( \mathcal{B} \) into the center of \( \mathcal{B}^a(E) \). The latter is isomorphic to the center of \( \mathcal{B} \), where the center of \( \mathcal{B} \) acts by right multiplication; see
Ske01, Theorem 4.2.18]. For instance, let $B \subset \mathcal{B}(G)$ a von Neumann algebra with nontrivial center $C(B)$. Put $E = G \overrightarrow{\otimes} B \subset \mathcal{B}(G, G \otimes G)$ (exterior tensor product of von Neumann modules; see [Ske01, Section 4.3]). Then $\mathcal{B}^u(E) = \mathcal{B}(G) \overrightarrow{\otimes} B \subset \mathcal{B}(G \otimes G)$ and the center of $\mathcal{B}^u(E)$ is $\mathrm{id}_G \otimes C(B)$. We turn $E$ into a correspondence over $B$ by letting act $B$ on the factor $G$ of $E$. Clearly, conjugation with the left action of a unitary semigroup in $C(B)$ defines an automorphism semigroup leaving invariant the left action of $B$, but not $\mathcal{B}(G) \otimes \mathrm{id}_G \subset \mathcal{B}^u(E)$.

10.5 Definition. Let $T$ be a Markov semigroup on a unital $C^*$–algebra $B$. A Hudson-Parthasarathy dilation of $T$ is a noise $(E, S, \omega)$ together with a unitary left cocycle $u$ with respect to $S$, such that $(E, S^u, \omega, \mathrm{id})$ becomes an (automatically unital) dilation of $T$. We shall often write $(E, S^u, \omega)$ for a Hudson-Parthasarathy dilation.

A Hudson-Parthasarathy dilation is inner, vacuum preserving, and so forth, if the underlying noise is inner, vacuum preserving, and so forth. We will say the Hudson-Parthasarathy dilation is strongly continuous if both the time shift $S$ and the cocycle $u$ are strongly continuous.

A Hudson-Parthasarathy dilation of $T$ is weak if $(E, S^u, \omega)$ is also a weak dilation (by (10.1) necessarily of the same Markov semigroup $T$).

Note that a Hudson-Parthasarathy dilation cannot be inner and weak at the same time. But we will see in Theorem 10.13 that every weak Hudson-Parthasarathy dilation arises as the restriction from an inner one.

10.6 Remark. The name Hudson-Parthasarathy dilation refers to the seminal work of Hudson and Parthasarathy [HP84a]. Perturbations of Markov semigroup by cocycles have been introduced by Accardi [Acc78] under the name of quantum Feynman-Kac formula. Hudson and Parthasarathy constructed such a dilation for the first time for a uniformly continuous Markov semigroup on $\mathcal{B}(H)$ with a Lindblad generator of finite degree of freedom. The construction is with the help of their quantum stochastic calculus developed precisely for that purpose. Quantum stochastic calculus has been generalized to allow to find dilations of Markov semigroups with arbitrary Lindblad generator (Hudson and Parthasarathy [HP84b]), unbounded versions (Chebotarev and Fagnola [CF98]), and arbitrary von Neumann algebras (Goswami and Sinha [GS99]). While the cited works all deal with $\mathcal{B}(H)$ and more general von Neumann algebras, the quantum stochastic calculus in Skeide [Ske00b] deals completely within the $C^*$–framework (and generalizes easily to von Neumann algebras).

We said that our results about classification of $E_0$–semigroups up to stable cocycle conjugacy will allow to establish existence of the unitary cocycle of the Hudson-Parthasarathy dilation. As these results depend on continuity conditions, we switch immediately to sufficiently continuous Markov semigroups. For semigroups on unital $C^*$–algebras, apart from the uniform topology, there is only the strong topology. A semigroup $T$ of bounded linear maps on $B$ is
**strongly continuous** if \( t \mapsto T_t(b) \) is continuous for all \( b \in \mathcal{B} \). We will see in a minute that, under reasonable continuity conditions, the \( C^* \)-case automatically restricts to Markov semigroups with bounded generators. (The von Neumann case is more interesting; see Section [13]. On the other hand, the von Neumann case does not have some pathologies of the \( C^* \)-case. Knowing the \( C^* \)-case will help appreciating better the von Neumann case.) To understand this, we investigate better the product systems of the involved dilations.

Suppose \( (E, \theta, \xi) \) is a strongly continuous weak dilation of an (automatically strongly continuous) Markov semigroup \( T \). Then the projection \( \xi \xi^* \) is increasing. If we construct the product system of \( \theta \) with the unit vector construction (as described for \( \theta \) on \( \mathcal{B}^\omega(E \otimes \mathbb{K}) \) on the pages preceding Theorem [9.1]), then the \( E_t = \theta_t(\xi \xi^*)E \ni \xi \) become an increasing family of subsets of \( E \), all containing \( \xi \). It is easy to check that the family \( \xi^\circ = (\xi_t)_{t \in \mathbb{S}} \) with \( \xi_t := \xi \in E_t \) form a unit, that is, \( \xi_t \xi_i = \xi_{i+t} \) and \( \xi_0 = 1 \). The unit \( \xi^\circ \) is even continuous in that it is among the continuous sections of \( E^\circ \). (After all, under the embedding into \( E \) the section \( \xi^\circ \) is constant.) Obiously, we recover \( T \) from the unit \( \xi^\circ \) as \( T_t = \langle \xi_t, \bullet \xi_t \rangle \).

Now suppose, further, that \( \theta = S_t^\omega \) is the cocycle perturbation of a strongly continuous noise \( (E, S, \omega = \xi) \). (In other words, suppose that \( T \) admits a strongly continuous weak Hudson-Parthasarathy dilation.) Then the continuous product system of \( S \) is also (isomorphic to) \( E^\circ \). Since the noise is a weak dilation of the trivial CP-semigroup, its product system contains a continuous unit \( \omega^\circ = (\omega_t)_{t \in \mathbb{S}} \) such that \( b = \langle \omega_t, b \omega_t \rangle \) for all \( b \in \mathcal{B}, t \in \mathbb{S} \). One easily concludes that \( b \omega_t = \omega_t b \), that is, the unit is central. Moreover, the unit \( \omega^\circ \) is (like \( \xi^\circ \)) unital, in the sense that all \( \omega_t \) are unit vectors. In Skeide [Ske06f], we introduced spatial product systems as pairs \( (E^\circ, \omega^\circ) \) consisting of a product system \( E^\circ \) and a central unital reference unit \( \omega^\circ \). We agree here to say a spatial product system is continuous if \( E^\circ \) is a continuous product system and if the reference unit \( \omega^\circ \) is among its continuous sections.

We just proved:

**10.7 Proposition.** If a Markov semigroup \( T \) admits a strongly continuous weak Hudson-Parthasarathy dilation, then there is a continuous spatial product system \( (E^\circ, \omega^\circ) \) with a continuous unit \( \xi^\circ \) such that \( T_t = \langle \xi_t, \bullet \xi_t \rangle \).

The statement that for every CP-semigroup \( T \) there exists a product system \( E^\circ \) with a unit \( \xi^\circ \) such that \( T_t = \langle \xi_t, \bullet \xi_t \rangle \), is not new. In fact, by a GNS-type construction, Bhat and Skeide [BS00, Section 4] construct a product system \( E^\tau^\circ \) with a suitable unit \( \xi^\circ \), the GNS-system of \( T \). The GNS-system is minimal in the sense that there is no proper subsystem containing the unit \( \xi^\circ \), and the pair \( (E^\tau^\circ, \xi^\circ) \) is determined by these properties up to unit preserving isomorphism.

---

[9]: There is no unit defined for nonunital \( \mathcal{B} \). The condition \( \xi_0 = 1 \) reflects that all our semigroups are actually monoids. In contexts with continuity, dropping the condition at \( t = 0 \) would mean to speak about semigroups that are continuous only for \( t > 0 \). It is well known that continuity at \( t = 0 \) is often not automatic.
The point about Proposition 10.7 is that the GNS-system of that Markov semigroup embeds continuously into a continuous spatial product system. After recalling the definition of a spatial strongly continuous Markov semigroup and a theorem from Bhat, Liebscher and Skeide [BLS10], this is equivalent to that \( T \) is spatial.

10.8 Definition. (From [BLS10] modeled after Arveson [Arv97].) A unit for a strongly continuous CP-semigroup \( T \) on a unital C*-algebra \( B \) is a continuous semigroup \( c = (c_t)_{t \in \mathbb{S}} \) of elements in \( B \) such that \( T_t \) dominates the CP-map \( b \mapsto c_t^* b c_t \) for all \( t \in \mathbb{S} \) (that is, for all \( t \in \mathbb{S} \) the map \( T_t - c_t^* \bullet c_t \) is a CP-map). \( T \) is spatial if it admits units.

CP-semigroups of the form \( b \mapsto c_t^* b c_t \) are also called elementary CP-semigroups. Continuity requirements for an elementary CP-semigroup refer to continuity of the implementing semigroup \( c = (c_t)_{t \in \mathbb{R}_+} \).

10.9 Theorem [BLS10]. Let \( T \) be a strongly continuous CP-semigroup on a unital C*-algebra. Then the following conditions are equivalent:

1. \( T \) is spatial.

2. The GNS-system of \( T \) embeds into a continuous spatial product system in such a way that the unit \( \xi^\circ \) giving back \( T \) is among the continuous sections.

3. The generator \( L \) of \( T \) has Christensen-Evans form, that is, \( L(b) = L_0(b) + b\beta + \beta^* b \) for a CP-map \( L_0 \) on \( B \) and an element \( \beta \) of \( B \).

10.10 Remark. By 3 it follows, in particular, that a spatial CP-semigroup is uniformly continuous. It also follows by [Ske06f, Theorem 6.3] that the spatial product system into which the GNS-system embeds can be chosen to be a product system of time ordered Fock modules. Simply take the system generated by the two units \( \omega^\circ \) and \( \xi^\circ \). But, the example in [BLS10] shows that the GNS-system alone (that is, the subsystem generated by \( \xi^\circ \)) need not be spatial.

We sketch, very briefly, the proof from [BLS10]. To show that a CP-semigroup fulfilling 2 is spatial, observe that \( c_t := \langle \omega_t, \xi_t \rangle \) defines a semigroup in \( B \). From \( T_t - c_t^* \bullet c_t = \langle \xi_t, q \bullet \xi_t \rangle \) for the bilinear projection \( q := id_t - \omega_t \omega_t^* \) in \( B^a(E_t) \) we see that \( T_t \) dominates \( c_t^* \bullet c_t \). Moreover, since \( b \mapsto \langle \omega_t, b\xi_t \rangle = bc_t \) is strongly continuous, \( c_t = 1c_t = \langle \omega_t, 1\xi_t \rangle \) is norm continuous. So, \( T \) is spatial. It is also easy to check that as soon as \( L \) is bounded, the generator of \( c \) provides a suitable \( \beta \) as required for the Christensen-Evans form in 3, see [BBL04, Lemma 5.1.1]. Not so obvious is to see that \( L \) is, indeed, bounded. (See [BLS10] for the reduction to [Ske03b, Theorem 7.7].) For the opposite direction, observe that if \( T \) is a spatial CP-semigroup with unit \( c \), say, then the maps

\[
\begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  b_{11} & b_{12}c_t \\
  c_t^* b_{21} & T_t(b_{22})
\end{pmatrix}
\]
define a CP-semigroup on \( M_2(\mathcal{B}) \). Interpreting this in terms of so-called CPD-semigroups, by [BBLS04] there is a product system \( E^\circ \) with a unit \( \xi^\circ \) such that \( \langle \xi^t, \xi^s \rangle = T_t \) and a central unital unit \( \omega^\circ \) such that \( \langle \omega_t, \xi^s \rangle = e \). It is easy to check that this product system is continuous with \( \xi^\circ \) and \( \omega^\circ \) among the continuous sections. (The simplest way is to refer to Theorem A.1 in Appendix A.1.) We refer to \((E^\circ, \xi^\circ, \omega^\circ)\) as the spatial continuous product system of \( T \) associated with the unit \( e \).

Returning to our problem, Proposition \([10.7]\) together with Theorem \([10.9]\) tells us that we must seek among the spatial Markov semigroups if we wish that they admit a weak Hudson-Parthasarathy dilation. We now wish to show that every spatial Markov semigroup admits such a dilation.

By Theorem \([10.9]\) we may start with a continuous product system \( E^\circ \) that has at least two unital units among its continuous sections. One is \( \xi^\circ \) and generates \( T \) as \( T_t = \langle \xi^t, \xi^s \rangle \), the other the central unital reference unit \( \omega^\circ \). Already for Arveson systems it was known since [Arv89] that a unital unit allows easily to construct an \( E_0 \)-semigroup. The construction for Hilbert modules is from [BS00, Section 5]: Take a product system \( E^\circ \) and a unital unit \( \xi^\circ \). Embed \( E_t \) into \( E_{s+t} \) as \( \xi_t E_t \). The family of these embeddings forms an inductive system with inductive limit \( E^\xi \). The factorization \( E_t \otimes E_s = E_{s+t} \), under the limit, survives as \( E^\xi \otimes E_t = E^\xi_t \). In other words, we obtain a left dilation of \( E^\circ \) to \( E^\xi \), inducing a strict \( E_0 \)-semigroup \( \theta^\xi \) on \( \mathcal{B}^0(E^\xi) \). Moreover, \( E^\xi \) contains a unit vector \( \zeta \) (the image of the elements \( \zeta_t \in E_t \subset E^\xi \)) that factorizes as \( \zeta = \zeta^\xi_t \) under the left dilation, and \( (E^\xi, \theta^\xi, \zeta) \) is a weak dilation of the Markov semigroup \( T^\xi \) defined by \( T^\xi_t = \langle \zeta_t, \xi^\xi_t \rangle \); see [BS00] [Ske02] for details. By [Ske03b, Theorem 7.5], if \( E^\circ \) is continuous and if \( \xi^\circ \) is among its continuous sections, then \( \theta^\xi \) is strongly continuous and the continuous structure on \( E^\circ \) derived from \( \theta^\xi \) coincides with the original one. In particular, the continuous structure does not depend on the choice of \( \xi^\circ \). [c]

Constructing \( E^\omega \) and \( S := \theta^\omega \) from \( \omega^\circ \), we obtain a weak dilation \( (E^\omega, S, \omega) \) of the trivial semigroup. Since \( \omega^\circ \) is central, the left action of \( \mathcal{B} \) on \( E_t \) survives the inductive limit (\( b \omega_t x_t = \omega_t bx_t \)). So, \( E^\omega \) with that left action becomes a correspondence over \( \mathcal{B} \), and the unit vector \( \omega \) fulfills \( b \omega = \omega b \). Once more, by \([10.1]\) and since \( \omega \omega^t \) is increasing for \( S \), we see that \( (E^\omega, S, \omega) \) is a strongly continuous noise. Moreover, \( (E^\xi, \theta := \theta^\xi, \xi) \) provides us with a strongly continuous weak dilation of \( T \), sharing the product system with that noise.

The strategy is, like in Theorem \([8.3]\) and its continuous version Theorem \([9.4]\) to amplify the two \( E_0 \)-semigroup, appealing to that the modules \( E^\omega \) and \( E^\xi \) are stably isomorphic, so that

[c]In the proof of [Ske03b, Theorem 7.5] (with the unit denoted \( \xi^\circ \) instead of \( \xi^\circ \)), in proving that \( \theta^\xi \) is strongly continuous, we were negligent regarding left continuity. However, in the case of the proof of [Ske03b, Theorem 7.5], the omission is marginal. (To complete that proof, simply test strong left continuity at \( t \), by checking on the total set of vectors \( xy_t \) for some continuous section \( y \), taking also into account that \( y_t \approx y_{t-\epsilon} \xi^\xi_t \approx \xi^\xi_{t-\epsilon} y_t \). The rest of the proof is okay.)
there will be a unitary cocycle. But for that, we must be sure that both modules are countably generated. Also, if we wish that the dilations are related somehow, then we have to make sure that also the amplified semigroups can be turned into a noise and a weak dilation of $T$, respectively, that are related in the sense of Hudson-Parthasarathy dilation.

We first look at the hypothesis to be countably generated.

10.11 Proposition. Let $\mathcal{B}$ be a separable $C^*$–algebra and let $E^\otimes$ be a continuous product system of $\mathcal{B}$–correspondences. Suppose, further, that there is a countable set $S \subset CS_i(E^\otimes)$ of units that generate $E^\otimes$.

Then all $E_t$ are separable. Moreover, if $\zeta^\otimes \in CS_i(E^\otimes)$ is a unital unit, then also the inductive limit $E_\zeta^\otimes$ is separable.

Proof. One just has to observe that for each $t > 0$ the set

\[
\left\{ b_n \xi_n \cdots b_1 \xi_1 b_0 : n \in \mathbb{N}, b_i \in \mathcal{B}, \xi_i \in S, t_i > 0, t_1 + \ldots + t_n = t \right\}
\]

is total in $E_t$. (Every product subsystem of $E^\otimes$ containing the units in $S$, must contain these elements. On the other hand, it is easy to check that the closed linear spans form a product subsystem; see [BBLS04, Proposition 4.2.6]. Since $E^\otimes$ is generated by $S$, the subsystem must coincide with $E^\otimes$. ) Now, by continuity, the standard argument applies that in (10.2) the points $t_i$ can be restricted to the rational numbers and, of course, the elements $b_i$ can be restricted to a countable total subset of $\mathcal{B}$, without changing totality of the set. This subset is, then, a countable union of countable sets and, therefore, a countable subset of (10.2). So, $E_t$ is separable.

Moreover, the inductive limit of the $E_t$ over $t \in \mathbb{R}_+$ is increasing. It coincides, therefore, with the inductive limit of $E_n$ over $n \in \mathbb{N}_0$. So, also $E_\zeta^\otimes$, as countable inductive limit over separable spaces, is separable.

If $\mathcal{B}$ is separable, we see that both $E^\omega$ and $E^\xi$ are countably generated. So, in principle, we could now apply Theorem 9.4. But now we really have to worry about how to choose the identifications of the amplified modules $E^\omega \otimes \mathbb{K}$ and $E^\xi \otimes \mathbb{K}$ in order that they behave nice with respect to the dilations carried by the original modules.

Let us start by observing that the inductive limit $E_\zeta^\otimes$ obtained from a continuous unital unit $\zeta^\otimes$ has the unit vector $\zeta$. In other words, the submodule $\zeta \mathcal{B}$ is a direct summand of $E_\zeta^\otimes$. So, from the two parts of the proof that the amplification $E^\xi \otimes \mathbb{K}$ is isomorphic to $\mathbb{K}_\mathcal{B}$, namely, Parts 1 and 2 of Proposition 3.1, we need only the second part. However, instead of applying Proposition 3.1, we, first, take away a piece. Like in the discussion in Section 9, we choose a unit vector $\kappa \in \mathbb{K}$. Then $\zeta \kappa := \zeta \otimes \kappa$ is a unit vector in $E_\zeta^\otimes \otimes \mathbb{K}$. Moreover, $(E^\xi \otimes \mathbb{K}, (\theta^\xi)\kappa, \zeta_\kappa)$ remains a weak dilation of $T^\xi$, sharing all the properties of the weak dilation $(E_\zeta^\otimes, \theta^\zeta, \zeta)$ (except for that it is no longer primary). In particular, we know that the product system is the same.
Now fix an isometry $v$ from $|\kappa|_1$ onto $\mathbb{K}$. Then

\[
E^\xi \otimes \mathbb{K} = (\zeta B \otimes \mathbb{K}) \oplus (|\xi|_1 \otimes \mathbb{K}) = (\zeta B \otimes \kappa) \oplus (\zeta B \otimes |\kappa|_1) \oplus (|\xi|_1 \otimes \mathbb{K}) \\
\cong (\zeta B \otimes \kappa) \oplus (\zeta B \otimes \mathbb{K}) \oplus (|\xi|_1 \otimes \mathbb{K}) = \xi \kappa B \oplus (E^\xi \otimes \mathbb{K}),
\]

where from the first to the second line we applied the isomorphism $id_{\xi B} \otimes v$ to the middle summand. Applying Proposition 3.1(2) to the second summand of the last term, we obtain

\[
E^\xi \otimes \mathbb{K} \cong \xi \kappa B \oplus \mathbb{K}.B.
\]

If we now do the same for $E^\omega$ and $E^\xi$ we obtain

\[
E^\omega \otimes \mathbb{K} \cong \omega \kappa B \oplus \mathbb{K}.B \cong \xi \kappa B \oplus \mathbb{K} \cong E^\xi \otimes \mathbb{K}.
\]

It is crucial to observe that this isomorphism identifies the distinguished unit vectors $\omega_\kappa$ and $\xi_\kappa$. More precisely, we just have shown that there exists a unitary $u : E^\xi \otimes \mathbb{K} \rightarrow E^\omega \otimes \mathbb{K}$ such that $u\xi_\kappa = \omega_\kappa$.

By Theorem 6.5 there exists a strongly continuous unitary cocycle $u_t$ with respect to $S^\mathbb{K}$ that fulfills

\[
u \theta^\mathbb{K}_t (u^* au^*) = u_t S^\mathbb{K}_t (a) u^*_t
\]

for all $a \in B^a(E^\omega \otimes \mathbb{K})$. We find

\[
T_t(b) = \xi^*_\kappa \theta_1^\mathbb{K} (\xi_\kappa b \xi^*_\kappa) \xi_\kappa = \xi^*_\kappa u^* u \theta^\mathbb{K}_t (u^* u_\xi \xi_\kappa b \xi^*_\kappa u^* u) u^* u \xi_\kappa
\]

\[
= \omega_\kappa^* u^* \theta_1^\mathbb{K} (u^* \omega_\kappa^* b \omega_\kappa^* u) u^* \omega_\kappa = \omega_\kappa^* u^* S^\mathbb{K}_1 (\omega_\kappa^* \omega_\kappa^*) u^* \omega_\kappa,
\]

so that $u_t S^\mathbb{K}_1 (\bullet) u^*_t$ with the unit vector $\omega_\kappa$ is a weak dilation of $T$. In particular, the projection $\omega_\kappa \omega_\kappa^*$ must be increasing, that is, $u_t S^\mathbb{K}_1 (\omega_\kappa \omega_\kappa^*) u^*_t \omega_\kappa = \omega_\kappa \omega_\kappa^*$ or $u_t S^\mathbb{K}_1 (\omega_\kappa \omega_\kappa^*) u^*_t \omega_\kappa = \omega_\kappa$. Now, recall that also $\omega_\kappa$ fulfills (10.1). It follows that

\[
T_t(b) = \omega_\kappa^* u^* S^\mathbb{K}_1 (\omega_\kappa^* \omega_\kappa^*) u_t S^\mathbb{K}_1 (b) u^*_t u^* S^\mathbb{K}_1 (\omega_\kappa \omega_\kappa^*) u^*_t \omega_\kappa = \omega_\kappa^* u^* S^\mathbb{K}_1 (b) u^*_t \omega_\kappa.
\]

In other words, the cocycle perturbation of the noise $(E^\omega \otimes \mathbb{K}, S^\mathbb{K}, \omega_\kappa)$ by the cocycle $u_t$ is a weak Hudson-Parthasarathy dilation of $T$.

We collect what we have proved so far in the following characterization of Markov semigroups admitting weak Hudson-Parthasarathy dilations.

10.12 Theorem. Let $B$ be a separable unital $C^*$–algebra and let $T$ be a Markov semigroup on $B$. Then $T$ admits a strongly continuous strict weak Hudson-Parthasarathy dilation if and only if $T$ is spatial.
By our construction, the correspondence on which the noise acts can be chosen to be isomorphic to $\mathbb{K}_g$ as right module. This does not at all mean that $\mathbb{K}_g$ would carry the canonical left action of $\mathcal{B}$ that acts on each summand $\mathcal{B}$ in $\mathbb{K}_g$ just by multiplication from the left. Also, due to the amplification procedure, the weak dilation of $T$ coming shipped with the weak Hudson-Parthasarathy dilation, in our construction will never be the minimal one. (There is a similarity of these facts to what happens in Goswami and Sinha [GS99]. There a Hudson-Parthasarathy dilation is constructed on $\mathcal{B} \otimes \mathbb{K}$ where $\mathbb{K}$ is identified with a symmetric Fock space with the help of a quantum stochastic calculus. We mention, however, that the left action there is not even unital. Our construction improves this aspect.)

Apart from the mentioned problems with minimality, we can even say the following: There exist spatial Markov semigroups whose minimal weak dilation does not arise from a weak Hudson-Parthasarathy dilation. In fact, whenever the GNS-system of the spatial CP-semigroup is nonspatial, then a weak dilation obtained from a weak Hudson-Parthasarathy dilation is not minimal, because the product system of the dilation is spatial and, therefore, too big. An example is the counter example studied in [BLS10].

We close this long section on spatial CP-semigroups with the following result on inner Hudson-Parthasarathy dilations.

**10.13 Theorem.** For every strongly continuous strict weak Hudson-Parthasarathy dilation $(E, S^a, \omega)$ there exists a strongly continuous inner vacuum preserving Hudson-Parthasarathy dilation $(E', S'^u, \omega')$ of the same Markov semigroup that “contains” $(E, S^u, \omega)$ in the following sense:

1. There is a strict unital representation of $\mathcal{B}^a(E)$ on $E'$ that allows to identify $\mathcal{B}^a(E)$ as a unital subalgebra of $\mathcal{B}^a(E')$.
2. $S'$ leaves $\mathcal{B}^a(E) \subset \mathcal{B}^a(E')$ invariant.
3. $u'_t = u_t \in \mathcal{B}^a(E) \subset \mathcal{B}^a(E')$.

This is the result of [Ske07b] applied to the noise $(E, S, \omega)$ ornamented by the embedding of the cocycle $u$ into the bigger $\mathcal{B}^a(E')$. The algebraic properties are checked easily in the construction of [Ske07b]. Continuity, a matter completely neglected in [Ske07b], follows very similarly as many other proofs of continuity like, for instance, continuity of $(E^\epsilon, \partial^\epsilon, \zeta)$. We do not give any detail.

**11 Von Neumann case: Algebraic classification**

For the balance of these notes (Appendix A.1 being the only exception) we discuss the analogues for von Neumann or $W^*$-algebras (respectively, modules and correspondences) of the
statements we obtained so far for $C^*$–algebras and modules.

The algebraic part of the classification gets even simpler. This is mainly for two reasons. Firstly, a Morita equivalence $M$ from $\mathcal{A}$ to $\mathcal{B}$ relates the $C^*$–algebra $\mathcal{A}$ to the compact operators $\mathcal{K}(M)$ on $M$. This obscures somehow that the representation theory of $\mathcal{B}^a(E)$ is actually an operation of Morita equivalence, in that the statements that have interpretations in terms of Morita equivalence must be extended from $\mathcal{K}(E)$ to $\mathcal{B}^a(E)$ via strictness. In the von Neumann case this obstruction disappears, and the representations theory becomes pure Morita equivalence. Secondly, the stable versions of isomorphism results, in the $C^*$–case, depend on Kasparov’s stabilization theorem and, therefore, on countability assumptions. (The hypothesis of $\sigma$–unitality of $\mathcal{B}$, on the other hand, does not play a role, because $\mathcal{B}$ is even assumed unital.) Also this obstruction disappears in the von Neumann case. (A small price to be paid is that now the dimension of the space with which a module must be stabilized depends on the module. But we are more than happy to pay that price, because we are payed off by getting a general theory without countability limitations.)

On the other hand, the weaker topologies of von Neumann objects require more work. Also this work is, however, payed off by much wider applicability of the results. (For instance, the results about Hudson-Parthasarathy dilations now apply to the large class of spatial Markov semigroups on a von Neumann algebra, which need no longer be uniformly continuous as in the $C^*$–case.) For the first time, we give a concise definition of strongly continuous product system. (We explain that we consider this definition only a “working definition”, because it does not behave sufficiently “nice” with respect to the commutant of von Neumann correspondences. But it is enough for our purposes here.) Parts of the results for continuous product systems generalize more or less directly to strongly continuous versions. Other parts do not.

In the present section we repeat what we need to know about von Neumann modules. We specify the versions for von Neumann module of the results of Sections 2–8 (and, actually, a part of Section 9) that go through without any further complication. This is the algebraic part of the classification. Actually, that part simplifies for von Neumann modules. In Section 12 we deal with the analogues for the strong topology of the continuity results in Section 9. In particular, we give a “working definition” of strongly continuous product systems. Here, most proofs go similar to the continuous $C^*$–case or, at least, the necessary modifications are more or less obvious. An exception is the proof of existence of an $E_0$–semigroup for every strongly full strongly continuous product system; a main result of these notes. Although, the strategy, in principle, is the same as for the $C^*$–case, the technical differences are so substantial that we decided to discuss the proof in Appendix B. (On the other hand, this gives us the occasion also to include in Appendix B.2 the existence result for nondegenerate representations of a faithful strongly continuous product system; a result that is not that directly related to the main purpose of these notes: Classification of $E_0$–semigroups by product systems.) In Section 13
finally, we prove the results of Section 10 about Hudson-Parthasarathy dilations for spatial
Markov semigroups in the case of von Neumann algebras. In that section the differences become
most substantial. It is necessary to fix in Appendix A.2 a gap in the proof of [BS00, Theorem
12.1] on dilations of Markov semigroups on von Neumann algebras. But, Appendix A.2 does
more. Apart from presenting a more general version of [BS00, Theorem 12.1], a result from
[MS07a] is reproved as a corollary. We also prove the fundamental result that the GNS-system
of spatial Markov semigroups is spatial in the von Neumann case. This means a considerable
simplification of the $C^*$–case, where we have only embedding into a spatial product system.
Appendix A.2 contains, thus, considerable parts of a beginning theory of strong type I product
systems of von Neumann correspondences (that is, strongly continuous product systems of von
Neumann correspondences that are generated by their strongly continuous units).

The heart of Morita equivalence of $C^*$–algebras is Example 2.5 together with Corollary
2.6: What the inner products of elements of $E$ generate in norm coincides with the compact
operators on $E^*$. All the rest is writing down suitable isomorphisms of certain $C^*$–algebras
with $\mathcal{K}(E)$ or with $\mathcal{B}_E$. For utilizing the relation between $\mathcal{K}(E)$ and $\mathcal{B}_E^\sigma(E)$ in the representation
theory, we had to work. In particular, we had to require that the representations are strict.

For von Neumann or $W^*$–algebras and modules, once accepted the premise that all reason-
able mappings between them be normal (or $\sigma$–weak), everything is simpler. The range ideal $\mathcal{B}_E$
of a von Neumann (or $W^*$–) module over a von Neumann (or $W^*$–) algebra $\mathcal{B}$ will be replaced
by its strong (or $\sigma$–weak) closure $\mathcal{B}_E^\sigma$. It coincides with the von Neumann (or $W^*$–) algebra
$\mathcal{B}_E^\sigma(E^*)$. The list of results or proofs involving Morita equivalence, where the proofs of the von
Neumann version runs considerably more smoothly than that of the $C^*$–version (or where the
$C^*$–version even fails), is still getting longer. We resist the temptation to give such a list and
refer the reader to [Rie74b, Ske09d, Ske06b].

In order to avoid the notorious distinction between von Neumann and $W^*$–modules, we
have to make a decision. Although they form equivalent categories, von Neumann modules
[Ske00a, Ske06b] are technically simpler. In fact, many proofs of results about $W^*$–modules
run best, after transforming the modules into von Neumann modules by choosing a faith-
ful normal representation of the involved $W^*$–algebras. Other proofs do not even possess
intrinsic $W^*$–versions. However, the notion of $W^*$–modules is more wide-spread. In order
not to make these notes any longer, we will formulate our statements for $W^*$–algebras and
$W^*$–modules. To choose a representation plays only a small role in Section 12 and in Appendix
B. There we will simply choose the standard representation. Anyway, we shall assume that
the reader knows $W^*$–modules (that is, self-dual (pre-)Hilbert modules over a $W^*$–algebra),
$W^*$–correspondences (that is, a $W^*$–module with a nondegenerate and suitably normal left ac-
tion by another $W^*$–algebra), and the tensor product of $W^*$–correspondences $E$ and $F$, denoted
by $E \tilde{\otimes} F$, (that is, the unique minimal self-dual extension of $E \odot F$). Every subset $S$ of a $W^*$-module $E$ generates a $W^*$-submodule of $E$ that coincides with the orthogonal bicomplement of $S$ in $E$. An (algebraic) submodule of $E$ is called strongly total if its bicomplement is $E$. (It is dense in $E$ in the natural $\sigma$-weak topology of $E$, respectively, in the strong topology when considered as von Neumann module.) A $W^*$-module $E$ over $B$ is strongly full if $B_E \simeq B$.

Let us start with the notions and results of Section 2.

11.1 Definition. A $W^*$-correspondence $M$ from $A$ to $B$ is a Morita $W^*$-equivalence if there exists a $W^*$-correspondence $N$ from $B$ to $A$ such that

$$N \tilde{\otimes} M \simeq B, \quad M \tilde{\otimes} N \simeq A,$$

as $W^*$-correspondences over $B$ and over $A$, respectively. Also here, we call $N$ an inverse of $M$ under tensor product.

Following Rieffel [Rie74b], two $W^*$-algebras $A$ and $B$ are Morita equivalent if there exists a Morita $W^*$-equivalence from $A$ to $B$. Also here, a Morita equivalence is necessarily faithful and strongly full. Obviously, a strongly full $W^*$-module over $B$ is a Morita equivalence from $B\sigma(E)$ to $B$.

The results that follow in Section 2 (including their proofs) remain true until Theorem 2.12 if we replace everywhere $C^*$-algebras, modules, and correspondences with $W^*$-algebras, modules, and correspondences, if we replace range ideals with their strong closures, full with strongly full, $\mathcal{K}(E)$ with $B\sigma(E)$, and if we replace strict maps with normal maps. In particular, we have the compatibility result for tensor products in Proposition 2.3, we have the characterization of Morita equivalences by Theorem 2.7, and we have the theory of, now, normal representations of $B\sigma(E)$ in Theorem 2.12 (see [MSS06, Theorem 1.16]).

Isomorphisms between von Neumann algebras are normal, automatically. Therefore, in the modified version of Corollary 2.13 we may leave out the word ‘normal’ (that, according to our rules, has substituted the word ‘strict’). The same is true for Corollary 2.16 once we stated the following:

11.2 Definition. A $W^*$-module $E$ over $B$ and a $W^*$-module $F$ over $C$ are Morita equivalent if there is a Morita $W^*$-equivalence $M$ from $B$ to $C$ such that $E \tilde{\otimes} M \simeq F$ (or $F \tilde{\otimes} M^* \simeq E$).

Corollary 2.14 remains true, independently, in its original formulation. Apart from that the stated isomorphism has no choice but being normal, the corollary states a criterion for when it is also strict (which may happen or not).

Section 3 has to be overworked considerably. The stabilization results in the von Neumann context are more general, but they depend on the choice of a sufficiently big cardinal number $\kappa$. 52
Their proofs are completely different and less sophisticated than their $C^*$–counterparts. We take them mainly from [Ske09d], where it is also pointed out that these facts resemble statements from the representation theory of von Neumann algebras.

Given a cardinal number $n$, by $C^n$ we denote the canonical Hilbert space of dimension $n$. We denote $M_n := B(C^n)$. By $E^n := E \otimes^n C^n$ we mean the von Neumann or $W^*$–version of the exterior tensor product; see [Ske01, Section 4.3] for details and for the facts we are using in the sequel. We have $\mathcal{B}(E \otimes^n C^n) = \mathcal{B}(E) \otimes^n M_n$ (tensor product of von Neumann algebras). For an infinite cardinal number $n$, we say $W^*$–algebras $A$ and $B$ are $n$–stably isomorphic if $A \otimes^n M_n$ and $B \otimes^n M_n$ are isomorphic. We say $A$ and $B$ are stably isomorphic if they are $n$–stably isomorphic for some infinite cardinal number $n$. Since $B^n$ is a Morita $W^*$–equivalence from $B \otimes^n M_n$ to $B$ it follows that $W^*$–algebras $A$ and $B$ are Morita equivalent if they are stably isomorphic. By [Ske09d, Corollary 9.4], also the converse is true.

Proposition 3.1 gets the following shape.

11.3 Proposition. Suppose $E$ is a strongly full $W^*$–module over $B$. Then there exists a cardinal number $n$ such that:

1. $E^n$ has a direct summand $B$.

2. $E^n \cong B^n$.

Part 1 is [Ske09d, Lemma 4.2]. Part 2 is stated and proved in front of [Ske09d, Corollary 4.3]. Of course, 2 implies 1. But, like in the proof of Proposition 3.1, Part 2 is proved using Part 1.

It may be noted that $n$ cannot always be chosen to be the smallest cardinality of a subset that generates $E$ as a $W^*$–module:

11.4 Example. Let $H$ be a nonseparable Hilbert space and choose a nonzero vector $h \in H$. Then the strongly full $W^*$–module $H^*$ over $B = \mathcal{B}(H)$ is generated by the single element $h^*$. But no cardinality $n$ strictly smaller than $\dim H$ makes $H^* \otimes^n$ isomorphic to $B^n$. In fact, $H^* \otimes^n = \mathcal{B}(H, C^n)$ does not contain a single copy of $B$, because it contains only operators of “rank” not greater than $n$. In particular, it does not contain any unit vector.

11.5 Definition. Let $n$ be an infinite cardinal number. Two $W^*$–modules $E$ and $F$ are $n$–stably Morita equivalent if $E^n$ and $F^n$ are Morita equivalent. They are stably Morita equivalent (as $W^*$–modules) if they are $l$–stably Morita equivalent for some infinite cardinal number $l$.

11.6 Observation. Suppose $E^l$ and $F^l$ are Morita equivalent for some arbitrary (also finite) cardinal numbers $t$ and $l$. Then $E$ and $F$ are $n$–stably Morita equivalent for every infinite cardinal number $n \geq \max(t, l)$. (Simply choose isomorphisms $C^t \otimes C^n \cong C^n \cong C^l \otimes C^n$.) The same is true, of course, for stable isomorphisms of $W^*$–algebras.
The analogue of Theorem 3.5 reads as follows.

11.7 Theorem. Let $E$ and $F$ denote strongly full $W^*$–modules over $W^*$–algebras $B$ and $C$, respectively. Then the following conditions are equivalent:

1. $E$ and $F$ are stably Morita equivalent.
2. $B^\alpha(E)$ and $B^\alpha(F)$ are Morita equivalent.
3. $B$ and $C$ are Morita equivalent.
4. $B$ and $C$ are stably isomorphic.

Proof. Since $B^\alpha(E)$ and $B$ are Morita equivalent (similarly, for $B^\alpha(F)$ and $C$) and since Morita equivalence is an equivalence relation, [2] and [3] are equivalent. Equivalence of [3] and [4] is [Ske09d, Corollary 9.4]. Of course, [1] implies [3] and if [3] holds, then by Proposition 11.3[2] also [1] holds, so that also [1] and [3] are equivalent. ■

To save space we do not spend much time on ternary isomorphisms, because everything is quite obvious. We mention only one thing, which facilitates to understand why everything is obvious. A ternary homomorphism between $W^*$–modules extends to a normal homomorphism between their linking algebras if and only if it is $\sigma$–weak. For that this happens, it is already sufficient that the restriction of the extension to the corner $B^a_E$ or to the corner $B^\alpha(E)$ is normal. With this observation, everything in Section 4 goes through for the obvious modifications. In particular, the $W^*$–version of Theorem 4.8 asserts that strongly full $W^*$–modules are stably ternary isomorphic if and only if they are modules over isomorphic $W^*$–algebras.

Section 5 of course, remains unchanged, as it is completely on algebras without any modules or topologies.

With the same global substitutions as for Section 2 also Section 6 remains essentially unchanged. Only in Theorem 5.7 we have to replace the direct sum with the $W^*$–module direct sum. The same is true for Section 7 and cum grano salis also for Section 8[d] Cum grano salis

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[d]We dispense with giving a formal $W^*$–version of Definition 7.1 as the changes belong to our list of changes. But we like to mention that Morita equivalence of correspondences (as introduced by Muhly and Solel [MS00]) in the $W^*$–case has a particularly nice interpretation in terms of our representation theory when applied to a unital normal endomorphism $\theta$ of $B^\alpha(E)$. In fact, $gB^\alpha(E)$ is a $W^*$–correspondence over $B^\alpha(E)$ with left action via $\theta$. If $E$ is strongly full, then $E$ is a Morita $W^*$–equivalence from $B^\alpha(E)$ to $B$. The multiplicity correspondence of the endomorphism $\theta$ is nothing but $E_{\theta} = E^*\hat{\circ} gB^\alpha(E)\hat{\circ} E$, the “conjugate” of $gB^\alpha(E)$ with the Morita equivalence $E^*$. For a normal $E^\alpha$–semigroup $\theta$ on $B^\alpha(E)$, we see that the product system of $\theta$ is simply conjugate to the one-dimensional product system of $\theta$ (see the introduction) via the Morita equivalence $E^*$. In the proof of [Ske09d, Theorem 5.12] and its corollary we used the statement that a strongly full product system of $W^*$–correspondences is the product system of an $E^\alpha$–semigroup if and only if it arises in the described way by conjugation from a one-dimensional product system, that is, if and only if it is Morita equivalent to a one-dimensional product system.
for Section 8 means that stably, of course, has to be replaced with the version where stably means \( n \)-stably for some infinite cardinal number \( n \). We only reformulate the main results of Sections 7 and 8.

11.8 Theorem. Let \( \vartheta \) and \( \theta \) be normal \( E_0 \)-semigroups on \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \), respectively, and suppose that \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \) are isomorphic. Then \( \vartheta \) and \( \theta \) are unitary cocycle conjugate if and only if their associated product systems are Morita equivalent via the same Morita \( W^* \)-equivalence inducing the isomorphism of \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \).

If \( n \) is a cardinal number and if \( \vartheta \) is an \( E_0 \)-semigroup on \( \mathcal{B}^a(E) \), denote by \( \vartheta^n \) the amplification of \( \vartheta \) to \( \mathcal{B}^a(E^n) \).

11.9 Theorem. 1. Let \( \vartheta \) be a normal \( E_0 \)-semigroup on \( \mathcal{B}^a(E) \) for a strongly full \( W^* \)-module \( E \) over \( \mathcal{B} \). Then there exists a cardinal number \( n \) such that the amplification \( \vartheta^n \) is inner conjugate to an \( E_0 \)-semigroup \( \vartheta^{\mathcal{B}} \) on \( \mathcal{B}^a(\mathcal{B}^n) \).

2. Let \( \vartheta \) and \( \theta \) be normal \( E_0 \)-semigroups on \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \), respectively, where \( E \) and \( F \) are strongly full \( W^* \)-modules over \( \mathcal{B} \). Then the following conditions are equivalent:
   (i) \( \vartheta \) and \( \theta \) are stably unitary cocycle inner conjugate.
   (ii) There exists a cardinal number \( n \) such that \( \vartheta^{\mathcal{B}} \) and \( \theta^{\mathcal{B}} \) are unitary cocycle equivalent.
   (iii) \( \vartheta \) and \( \theta \) have isomorphic product systems.

3. Let \( \vartheta \) and \( \theta \) be normal \( E_0 \)-semigroups on \( \mathcal{B}^a(E) \) and \( \mathcal{B}^a(F) \), respectively, where \( E \) and \( F \) are strongly full \( W^* \)-modules over \( \mathcal{B} \) and \( \mathcal{C} \), respectively. Then \( \vartheta \) and \( \theta \) are stably unitary cocycle (ternary) conjugate if and only if they have Morita (ternary) equivalent product systems.

We explain briefly in which sense a part of the results of Section 9 are available already now in the \( W^* \)-context. Essentially, we mean all results that are algebraic without continuity conditions. The reason why we can allow this, is that Proposition 11.3, the analogue Proposition 3.1, now, does no longer depend on countability conditions. (Remember: The main reason, why in Section 9 we had to restrict to the continuous case, was precisely to guarantee these countability conditions.)

One of the main results of [Ske09d] asserts that every discrete product system \( (E_n)_{n \in \mathbb{N}_0} \) of strongly full \( W^* \)-correspondences is the product system of a discrete \( E_0 \)-semigroup. This completed the classification for the case of the discrete semigroup \( \mathbb{S} = \mathbb{N}_0 \). But, for the continuous case \( \mathbb{S} = \mathbb{R}_+ \) it also means that for every strongly full product system \( E^\circ = (E_t)_{t \in \mathbb{R}_+} \) of \( W^* \)-correspondences we can find a left dilation of the discrete subsystem \( (E_n)_{n \in \mathbb{N}_0} \) to a strongly full \( W^* \)-module \( \hat{E} \). Such a left dilation of the discrete subsystem is precisely the main input
for the construction in \[\text{Ske06a}\] of an \(E_0\)–semigroup for every Arveson system. We mentioned already in \[\text{Ske06a}\] that the construction works without any problem if all the direct integrals are with respect to the counting measure. So, if we define \(\int_a^b E_\alpha \, d\alpha := \bigoplus_{\alpha \in [a,b)} E_\alpha\) and if we put \(E := \tilde{E} \circ \int_0^1 E_\alpha \, d\alpha\), then the following formula from \[\text{Ske06a}\]
\[
E \circ E_t = \tilde{E} \circ \left( \int_0^1 E_\alpha \, d\alpha \right) \circ E_t = \tilde{E} \circ \int_t^{t+t} E_\alpha \, d\alpha
\]
\[
\approx \left( \tilde{E} \circ E_n \circ \int_{t-n}^t E_\alpha \, d\alpha \right) \oplus \left( \tilde{E} \circ E_{n+1} \circ \int_0^{t-n} E_\alpha \, d\alpha \right)
\]
\[
\approx \left( \tilde{E} \circ \int_{t-n}^t E_\alpha \, d\alpha \right) \oplus \left( \tilde{E} \circ \int_0^{t-n} E_\alpha \, d\alpha \right) = E
\]
\[\text{(11.1)}\]
suggests an isomorphism \(v_t : E \tilde{\circ} E_t \to E\) for every \(t \in \mathbb{R}_+\). By \[\text{Ske06a, Proposition 3.1}\], these maps \(v_t\) form a left dilation of \(E \tilde{\circ}\) to \(E\). Just that, by construction, the induced \(E_0\)–semigroup is definitely not continuous with time in any reasonable topology. Nevertheless, we can formulate the classification theorem for the classification of algebraic normal \(E_0\)–semigroups by algebraic product systems.

11.10 Theorem. 1. Let \(\mathcal{B}\) denote a \(W^*\)–algebra. Then there is a one-to-one correspondence between equivalence classes (up to stable unitary cocycle inner conjugacy) of normal \(E_0\)–semigroups acting on the algebras of operators on strongly full \(W^*\)–modules over \(\mathcal{B}\) and isomorphism classes of strongly full product systems of \(W^*\)–correspondences over \(\mathcal{B}\).

2. There is a one-to-one correspondence between equivalence classes (up to stable unitary cocycle conjugacy) of normal \(E_0\)–semigroups acting the algebras of operators on strongly full \(W^*\)–modules and Morita equivalence classes of strongly full product systems of \(W^*\)–correspondences.

12 Von Neumann case: Topological classification

We now come to strongly continuous \(E_0\)–semigroups in the \(W^*\)–case\[\text{[e]}\], a property that has to be reflected by a property of the associated product system of \(W^*\)–correspondences. This brings us

\[\text{[e]}\]Once for all, when we speak about strongly continuous semigroups on a \(W^*\)–algebra \(\mathcal{B}\), what we have in mind is the point-strong topology of a subalgebra \(\mathcal{B}\) of some \(\mathcal{B}(H)\): A semigroup \(T\) on \(\mathcal{B}\) is strongly continuous if \(t \mapsto T(t)b\) is continuous for all \(b \in \mathcal{B}, h \in H\). That is, we are rather thinking of \(\mathcal{B}\) as a von Neumann algebra. The strong topology depends on the representation, continuity results do not. Usually, continuity for semigroups on \(W^*\)–algebras is formulated in terms of the weak* topology induced by the pre-dual. But, a result by Markiewicz and Shalit \[\text{MS07a}\] asserts that this implies strong continuity for every representation of \(\mathcal{B}\) as von Neumann algebra on a Hilbert space \(H\). We reprove this result in Corollary \[\text{A.6}\].
to the problem that we have to give a concise definition of strongly continuous product system. Already in Skeide [Ske03b] we indicated briefly how this can be done, following the procedure in the \( C^* \)-case. This is what we will do here in order to be able to work. But we do not hide the fact that we think this definition should be considered as a preliminary working definition. The reason is as follows: In the von Neumann way to see things, von Neumann correspondences come shipped with a commutant generalizing the commutant of a von Neumann algebra; see [Ske03a, Ske06b]. The same is true for whole product systems; see [Ske03b, Ske09d, MS07b]. In [MS07b], Muhly and Solel introduced a weakly measurable version of product systems and showed (under separability assumptions, and using their independent way [MS04] to look at the commutant) that also the commutant system has a measurable structure by reducing it to Effros’ analogue result [Eff65] for fields of von Neumann algebras. It is not difficult to see that the definition we will use here, is manifestly asymmetric under commutant. Our scope in [Ske10a], among others, will be to provide a notion of strongly continuous product system that is compatible with the commutant. Therefore, we would like to consider the definition used here as preliminary.

In the sequel, we will assume that the \( \mathcal{W}^- \)-algebra \( \mathcal{B} \) is acting in standard representation on the Hilbert space \( G \). (Actually, we could have taken just an arbitrary faithful normal nondegenerate representation, and we would be in the context of von Neumann algebras and modules. Nothing in the following discussion really refers to properties of the standard representation.) If we have another \( \mathcal{W}^- \)-algebra \( \mathcal{C} \), then we will denote the Hilbert space of its standard representation by \( K \). Following the suggestion in [Ske03b], we define as follows:

12.1 Definition. Suppose \( E^\circ \) is a product system of \( \mathcal{W}^- \)-correspondences over \( \mathcal{B} \). Suppose further that \( i_t \) is a family of isometric embeddings of \( E_t \) into a fixed \( \mathcal{W}^- \)-module \( \hat{E} \) over \( \mathcal{B} \), and denote by

\[
CS^i_s(E^\circ) := \left\{ (x_t)_{t \in \mathbb{R}^+} : x_t \in E_t, t \mapsto i_t x_t \odot g \in \hat{H} := \hat{E} \odot G \text{ is continuous for all } g \in G \right\}
\]

the set of strongly continuous sections (with respect to the embedding \( i \)). We say \( E^\circ \) is a strongly continuous product system if

\[
\left\{ x_s : (x_t)_{t \in \mathbb{R}^+} \in CS^i_s(E^\circ) \right\} = E_s
\]

for all \( s \in \mathbb{R}^+ \), and if the function

\[
(s, t) \mapsto i_{s+t}(x_t y_t) \odot g \in \hat{H}
\]

is continuous for all \( (x_t)_{t \in \mathbb{R}^+}, (y_t)_{t \in \mathbb{R}^+} \in CS^i_s(E^\circ) \) and for all \( g \in G \).

A morphism between strongly continuous product systems is continuous if it sends strongly continuous sections to strongly continuous sections. By Theorem [B.5] a continuous isomorphism has a continuous inverse.
12.2 Remark. As said in the beginning of this section, we consider this a working definition. We do not know if the condition that each \( x_s \in E_s \) is the value \( x_s = y_s \) of a strongly continuous section \( y \) can be weakened to strong totality of such \( y_s \) in \( E_s \). (In the \( C^* \)-version this is so.) But the definition is justified by two facts: First, every product system of a strongly continuous \( E_0 \)-semigroup admits such a strongly continuous structure. (This will be explained immediately.) Second, every (strongly full) strongly continuous product system can be obtained in that way from an \( E_0 \)-semigroup. (The proof of the latter fact is postponed to Appendix B.1.)

If \( E \) is a \( W^* \)-module over \( \mathcal{B} \), then we turn \( \mathcal{B}^a(E) \) into a von Neumann algebra by embedding \( \mathcal{B}^a(E) \otimes \text{id}_G \) into \( \mathcal{B}(H) \), where \( H := E \otimes G \). Like in the \( C^* \)-case, if we have a normal \( E_0 \)-semigroup \( \vartheta \) acting on \( \mathcal{B}^a(E) \), then it is strongly continuous (with respect to the strong topology of \( \mathcal{B}(H) \)) if and only if each amplification \( \vartheta^n \) to \( \mathcal{B}^a(E^n) \) is strongly continuous (with respect to the strong topology of \( \mathcal{B}(H^n) \)). If \( E \) is strongly full, then by [Ske09d, Lemma 4.2] \( F := E^n \) has a unit vector \( \zeta \) as soon as the cardinal number \( n \) is big enough. Like in the \( C^* \)-case, we may use that unit vector to construct embeddings \( i_t : E_t \to F \). It is easy to show (similar to the \( C^* \)-case) that these embeddings equip \( E^\varnothing \) with a strongly continuous structure, and that this strongly continuous structure does not depend neither on the choice of cardinal number \( n \) nor on the choice of \( \zeta \). In particular, if \( E \) has already a unit vector \( \xi \), then \( n = 1 \) is among the admissible cardinal numbers and the strongly continuous structure derived from that \( \xi \) coincides with all others.

Once more, if \( u \) is a unitary cocycle for \( \vartheta \), then it is strongly continuous (in \( \mathcal{B}(H) \)) if and only if the induced automorphism of the associated product system \( E^\varnothing \) is strongly continuous. (The proof is exactly like that of Theorem 9.1 for the \( C^* \)-case, except that we have to tensor everywhere in the estimates with an element \( g \in G \).) We find:

12.3 Theorem. Let \( E \) be a strongly full \( W^* \)-module and suppose \( \vartheta \) and \( \vartheta' \) are two strongly continuous normal \( E_0 \)-semigroups on \( \mathcal{B}^a(E) \). Then the following are equivalent:

1. \( \vartheta \) and \( \vartheta' \) are unitary cocycle equivalent via a strongly continuous cocycle.

2. The strongly continuous product systems associated with \( \vartheta \) and \( \vartheta' \) are isomorphic.

This is the classification of strongly continuous normal \( E_0 \)-semigroups acting all on the same \( \mathcal{B}^a(E) \). Of course, also Theorem 11.9 remains true if we simply add everywhere strongly continuous, since (as pointed out before) amplification is compatible with strong continuity. To save space, we do not repeat it.

What is missing to obtain the strongly continuous analogue also for Theorem 11.10 is the following existence result:
12.4 Theorem. Every strongly full strongly continuous product system of $W^*$–correspondences is isomorphic to the strongly continuous product system associated with a normal strongly continuous $E_0$–semigroup acting on the algebra of all adjointable operators of a strongly full $W^*$–module.

We prove this theorem in Appendix B.1. It follows immediately that the following strongly continuous version of Theorem 11.10 holds, too, in all of its parts. (Two strongly continuous product systems $E^\otimes$ and $F^\otimes$ are in the same continuous Morita equivalence class if the isomorphism between $F^\otimes$ and the product system $M^* \otimes E^\otimes \otimes M$ Morita equivalent to $E^\otimes$ can be chosen continuous. See also Remark 9.5 for how the strongly continuous structure of $M^* \otimes E^\otimes \otimes M$ can be defined.)

12.5 Theorem. 1. Let $B$ denote a $W^*$–algebra. Then there is a one-to-one correspondence between equivalence classes (up to stable unitary cocycle inner conjugacy by a strongly continuous cocycle) of normal strongly continuous $E_0$–semigroups acting on the algebras of operators on strongly full $W^*$–modules over $B$ and continuous isomorphism classes of strongly full strongly continuous product systems of $W^*$–correspondences over $B$.

2. There is a one-to-one correspondence between equivalence classes (up to stable unitary cocycle conjugacy by a strongly continuous cocycle) of normal strongly continuous $E_0$–semigroups acting the algebras of operators on strongly full $W^*$–modules and continuous Morita equivalence classes of strongly full product systems of $W^*$–correspondences.

13 Von Neumann case: Spatial Markov semigroups

The discussion of spatial Markov semigroups on a $W^*$–algebra $B$ and their Hudson-Parthasarathy dilations, apart from the weaker topologies, is very similar to the $C^*$–case. (The weaker topologies require different proofs, which we discuss in Appendix A.2.) We even have the simplification that spatial Markov semigroups, here, turn out to have a spatial product system; see Theorem [A.14] (In the $C^*$–case, we had only embedding into a spatial product system.) Thanks to the weaker topology, the results are applicable to a much wider (thus, more interesting) class of Markov semigroups. In fact, in the case $B = \mathcal{B}(G)$ (for some Hilbert space $G$), we do not know examples of nonspatial Markov semigroups, except for nonspatial $E_0$–semigroups or Markov semigroups that arise as tensor products with a nonspatial $E_0$–semigroup. On the other hand, from Fagnola, Liebscher, and Skeide [PLS10, Ske05a] we know that the Brownian and Ornstein-Uhlenbeck semigroups on the commutative von Neumann algebra $L^\omega(\mathbb{R})$ have nonspatial product systems.

We would like to mention that the discussion of the case $\mathcal{B}(G)$ in Skeide [Ske08b], actually,
was inspired by the preparation of Section 10 and the present section. But, while in [Ske08b] we used mainly well-known results about spatial $E_0$–semigroups and spatial Arveson systems (that is, formulated with measurability conditions rather than continuity conditions), here we present a completely new treatment adapted to our notions of strong continuity.

While in Section 10 we could build on the results on continuous units in continuous product systems from [Ske03b], in this section we have to develop the strongly continuous analogues. There are results, like the following theorem, that can be proved simply by tensoring the vectors in $\hat{E}$ occurring in the estimates in [Ske03b], with a fixed vector $g$ in the representation space $G$ of the standard representation of $\mathcal{B}$.

13.1 Theorem. Let $E^\circ$ be a strongly continuous (with respect to embeddings $i_i: E_i \to \hat{E}$, say) product system of $W^*$–correspondences over $\mathcal{B}$ and suppose that $\xi^\circ \in CS_i^1(E^\circ)$ is unital unit among the strongly continuous sections.

Then the normal $E_0$–semigroup $\vartheta$ on $\mathcal{B}^{\circ}(\hat{E})$ is strongly continuous, and the strongly continuous structure induced on $E^\circ$ by $\vartheta^\circ$ via the unit vector $\xi$ coincides with the original one. In particular, the induced strongly continuous structure does not depend on the choice of the strongly continuous unital unit $\xi^\circ$.

Proof. Except for the modifications stated in front of the theorem, the proof goes exactly like the corresponding proof of [Ske03b, Theorem 7.5], including the add-on mentioned in Footnote [c].

And there are results like the construction of a strongly continuous weak dilation from a product system $E^\circ$ (so far, without a strongly continuous structure) and a unital strongly continuous unit $\xi^\circ$ that strongly generates $E^\circ$ (that is, the smallest product subsystem of $W^*$–correspondences containing $\xi^\circ$ is $E^\circ$ itself). Like in Section 10 from product system and strongly generating unit we construct the triple $(E^\circ, \vartheta^\circ, \xi)$ which is a weak dilation, the so-called unique minimal dilation, of the strongly continuous normal Markov semigroup $T^\circ := \langle \xi, \cdot \xi \rangle$ on $\mathcal{B}$. By [BS00, Theorem 12.1], every strongly continuous normal Markov semigroup $T$ on a $W^*$–algebra arises in that way from its GNS-system $E^\circ$ and a strongly continuous unit $\xi^\circ$. But [BS00, Theorem 12.1] also claims that the minimal dilation is strongly continuous. (Muhly and Solel showed it is, but only under the hypothesis that $\mathcal{B}$ has separable pre-dual.) Since the proof in [BS00] contained a gap, we give a complete proof of a slightly more general statement, Theorem A.4 in the Appendix A.2. The strongly continuous $E_0$–semigroup of the minimal dilation of $T$ equips the GNS-system, $E^\circ$, with a strongly continuous structure and, of course, the unit $\xi^\circ$ (being the constant element $\xi \in E^\xi$) is among the strongly continuous sections.

If a (necessarily strongly continuous and normal) Markov semigroup $T$ admits a strongly continuous normal weak Hudson-Parthasarathy dilation $(E, S^\circ, \omega)$ (by strongly continuous we
mean that also the cocycle \( u \) is strongly continuous in \( \mathcal{B}(H) \), then the product system of the dilation contains two unital units among its strongly continuous sections: One is the unit \( \xi^\circ \) that gives back \( T \) as \( T^\xi \), because \( S^u \) is a weak dilation, and the other is the unital reference unit \( \omega^\circ \) of the noise \( S \). (As in Section 10 it suffices just to observe that the strongly continuous product systems of \( S^u \) and of \( S \) are continuously isomorphic. The former contains \( \xi^\circ \) and the latter contains \( \omega^\circ \) as strongly continuous sections. So a section that is strongly continuous for one product system has an image in the other that is also strongly continuous.) We find the analogue of Proposition 10.7: There is a \textit{spatial} strongly continuous product system (that is, a strongly continuous product system that is spatial with a strongly continuous reference unit \( \omega^\circ \)) with a strongly continuous unit \( \xi^\circ \) such that \( T = T^\xi \).

If we define the semigroup \( c_t := \langle \omega_t, \xi_t \rangle \), then \( T \) is spatial in the sense of the following definition:

13.2 Definition. A \textbf{unit} for a strongly continuous normal CP-semigroup \( T \) on a \( W^\ast \)-algebra \( \mathcal{B} \) is a strongly continuous semigroup \( c = (c_t)_{t \in \mathbb{S}} \) of elements in \( \mathcal{B} \) such that \( T_t \) dominates the CP-map \( b \mapsto c_t^* b c_t \) for all \( t \in \mathbb{S} \). We say \( T \) is \textit{spatial} if it admits units.

For the backwards direction we are done as soon as we are able to find for every spatial Markov semigroup a strongly continuous spatial product system (with reference unit \( \omega^\circ \), say) and a strongly continuous unit \( \xi^\circ \) such that \( T = T^\xi \). This is done in Appendix A.2 in a way that is much more satisfactory than the \( C^\ast \)-case. In fact, Theorem A.14 asserts that a Markov semigroup on a \( W^\ast \)-algebra is spatial if and only if its GNS-system is spatial (including all requirements about strong continuity).

Once we have these ingredients, the construction of a Hudson-Parthasarathy dilation goes exactly as in Section 10. Just that now there are no countability assumptions. The price to be paid is that now we have to choose our amplifications \textit{big enough} when establishing stable cocycle conjugacy — a small price, of course. We do not give more details on these steps because they, really, are completely analogue to Section 10 and all necessary compatibility results regarding the strong topologies have been mentioned.

13.3 Theorem. Let \( \mathcal{B} \) be a \( W^\ast \)-algebra and let \( T \) be a strongly continuous normal Markov semigroup on \( \mathcal{B} \). Then \( T \) admits a strongly continuous normal weak Hudson-Parthasarathy dilation if and only if \( T \) is spatial. Like in Theorem 10.13 this dilation may be obtained as the restriction of a strongly continuous inner normal Hudson-Parthasarathy dilation.

13.4 Remark. It is easy to show that the construction preserves countability assumptions: If \( \mathcal{B} \) has separable pre-dual, then both the minimal dilation and the Hudson-Parthasarathy dilation of \( T \) act on a \( \mathcal{B}^0(E) \) with separable pre-dual.
Appendix A: Strong type I product systems

A strong type I system is a product system $E^\otimes$ that is generated by a unital strongly continuous set of units $S$. Unital means that $S$ contains at least one unital unit $\xi^\otimes$. The topology in which the units generate the product system depend on whether we are speaking about unital $C^*$–algebras and modules or about $W^*$–algebras and modules. Also strongly continuous means two different things: In the $C^*$–case we mean that a family of maps $T_t: \mathcal{A} \to \mathcal{B}$ is strongly continuous if $t \mapsto T_t(a)$ is continuous in $\mathcal{B}$ for all $a \in \mathcal{A}$. The occurring product systems will be continuous product systems. In the $W^*$–case we refer to the strong operator topology, assuming that the occurring $W^*$–algebras are represented in standard representation. (For the results any faithful normal representation would do.) So, $t \mapsto T_t(a)g$ is continuous in $G$ for all $a \in \mathcal{A}$ and $g \in G$. The occurring product systems will be strongly continuous product systems.

This appendix serves two purposes.

The first purpose is to show that every strong type I system comes along with a unique continuous structure making $S$ a set of (strongly) continuous sections. (Theorem A.1 in the $C^*$–case and Theorem A.4 in the $W^*$–case.) The idea is to use the unital unit $\xi^\otimes$ to construct an $E_0$–semigroup $\theta$ on $\mathcal{B}^\omega(\mathcal{E})$ that is strongly continuous and whose associated product system is $E^\otimes$. This equips $E^\otimes$ with a (strongly) continuous structure. (Note that [Ske03b, Theorem 7.5] (Theorem 13.1) start from a product system that has already a (strongly) continuous structure and from the assumption that $\xi^\otimes$ is among the (strongly) continuous sections.) By [Ske03b, Theorem 7.5] and Theorem 13.1 such a structure, if it exists, is unique. In particular, once we show that $S$ is a subset of the (strongly) continuous sections of this structure, we know that the structure does not depend on the choice of the unital unit $\xi^\otimes \in S$.

We thank Orr Shalit for having pointed out that the proofs of strong continuity of $\theta$ in [BS00, Theorems 10.2, 12.1] contain a gap. In fact, in either case only right continuity is shown. But in either case the situation is not among those where right continuity would imply also left continuity. (Strong or weak operator topology on $\mathcal{B}^\omega(\mathcal{E})$ or $\mathcal{B}(\mathcal{H})$ are too weak to achieve this immediately from standard theorems! This error is surprisingly frequent in literature and led Markiewicz and Shalit to prove, once for all, in [MS07a] that weak operator continuity of a normal CP-semigroup on a von Neumann algebra implies strong operator continuity.) In this appendix we fix this gap in systematic way for the most general situation. We also reprove the result of [MS07a] in a different way as Corollary A.6.

The second purpose is to provide, in the $W^*$–case, some (new) basic classification results for strong type I systems. (Corollary A.8, Lemma A.13 and Theorem A.14. Also some more technical results about strongly continuous sets of units, which have nice interpretations in terms of CPD-semigroups. CPD-semigroup is a notion from [BBL04] which we avoid to discuss here for reasons of space, but which would be part of a systematic study of strong type
I systems.) This part is limited to the $W^*$–case, because we know from Bhat, Liebscher and Skeide [BLS10] that these results fail in the $C^*$–case.

A.1: The $C^*$–case

Already from Arveson [Arv89] we know that is is easy to construct for a product system an $E_0$–semigroup, provided that $E^\otimes$ has a unital unit $\xi^\otimes$. There are several versions for Hilbert modules in [BS00, Ske02, BBLS04].

In Section 10 (the paragraph ending with Footnote [c]) we have described one possibility that results in a weak dilation $(E^\xi, \theta^\xi, \xi)$ of the Markov semigroup $T^\xi$ defined by $T^\xi_t = \langle \xi_t, \bullet \xi_t \rangle$. By [Ske03b, Theorem 7.5], if $E^\otimes$ is continuous and if $\xi^\otimes$ is among its continuous sections, then $\theta^\xi$ is strongly continuous and the continuous structure on $E^\otimes$ derived from $\theta^\xi$ coincides with the original one; see, once more, Footnote [c]. In particular, the continuous structure induced by $\theta^\xi$ does not depend on the choice of $\xi^\otimes$. As pointed out in the introduction to this appendix, we will start with a product system that is generated by a strongly continuous set of units $S \ni \xi^\otimes$ and that has not yet a continuous structure. Instead, it will be our scope to show that under these assumptions $\theta^\xi$ is strongly continuous. This $\theta^\xi$ can, then, be used to induce a continuous structure on $E^\otimes$.

A.1 Theorem. Suppose $S$ is a set of units for the product system $E^\otimes$ that generates $E^\otimes$, and suppose $\xi^\otimes \in S$ is unital. If for all $\xi^\otimes, \xi^\otimes \in S$ and $b \in B$ the map $t \mapsto \langle \xi_t, b\xi_t' \rangle$ is continuous, then the $E_0$–semigroup $\theta^\xi$ is strongly continuous. Moreover, the continuous structure induced on $E^\otimes$ by $\theta^\xi$ is the unique one making all $\xi^\otimes \in S$ continuous sections.

Proof. We explained already that if $\theta^\xi$ is strongly continuous, then by [Ske03b, Theorem 7.5] the continuous structure derived from it is the unique one, making $\xi^\otimes$ a continuous section. So it remains to show that $\theta^\xi$ is strongly continuous, and that $S$ is a subset of the set of continuous sections of this structure.

It is easy to see that the product subsystem $E^S\otimes$ generated by $S$ is formed by the spaces

$$E^S_t := \overline{\text{span}}\{b_n\xi^\otimes_{t_n} \ldots b_1\xi^\otimes_{t_1} b_0 : n \in \mathbb{N}, b_i \in B, \xi^\otimes \in S, t_i > 0, t_1 + \ldots + t_n = t\}$$

for $t > 0$ and, of course, $E^S_0 = E_0 = B$. Since $S$ is assumed generating, we have $E^S_t = E_t$. The inductive limit $E^\xi$ is, therefore, spanned by elements of the form

$$x = \xi b_n\xi^\otimes_{t_n} \ldots b_1\xi^\otimes_{t_1} b_0. $$

(No condition on $t_1 + \ldots + t_n$, here!) As in [BS00] one easily shows that the semigroup of time-shift operators $s_t : y \mapsto y\xi_t$ on $E^\xi \ni y$ is strongly continuous at $t = 0$. (Since all $\xi_t$ are contractions, it is sufficient to show this on the total set of elements of the form $x$. Inserting the
the concrete form of \(x\) and calculating \(\langle x\xi_e - x, x\xi_e - x \rangle\), one sees that this goes to 0 for \(\varepsilon \to 0\). Since that part of the proof in \[BS00\] is okay, and because we discuss in all detail the strong version in the \(W^*\)-case in Appendix A.2, we do not give more details, here.) It follows that the semigroup \(s_t\) is right-continuous everywhere. (It is even continuous, but we do not need this fact.) Like in \[BS00\], we compute

\[
\vartheta_t(a)x - ax = \vartheta_t(a)x - \vartheta_t(a)x\xi_t + \vartheta_t(a)x\xi_t - ax = \vartheta_t(a)(x - x\xi_t) + ((ax)\xi_t - (ax)),
\]

and conclude that \(\vartheta_t(a)\) is strongly right-continuous. (So far, this is the old proof.) Additionally, we observe that nothing changes in the proof if we replace the semigroup \(s_t\) with with the semigroup \(\vartheta_t : x \mapsto x\xi_t\) for any other unit \(\xi^0\) in \(S\).

To add left continuity, fix \(t > 0\) and fix an element of the form \(x\). Without changing \(x\), we may assume that: First, \(t_1 + \ldots + t_n > t\). (Otherwise, choose \(t_{n+1}\) sufficiently big, and \(b_{n+1} = 1\).) Second, there is \(m < n\) such that \(t_1 + \ldots + t_m = t\). (Otherwise, split that \(\xi^m_t\) into a product \(\xi^m_{t_1} \xi^m_{t_2} \xi^m_{t_3}\) such that \(t_1 + \ldots + t_m + s = t\) and insert a 1 in between.) Choose \(a \in \mathcal{B}^\varepsilon(E)\) and 0 < \(\varepsilon < \min\{t_1, \ldots, t_n\}\) (so that \(\varepsilon \leq t\), too). Then

\[
\vartheta_t(a)x = (a(\xi b_n\xi^m_{t_n} \ldots b_{m+1}\xi^m_{t_{m+1}} b_m))\xi^m_{t_{m+1}} \ldots b_1\xi^m_1 b_0 = (a(\xi b_n\xi^m_{t_n} \ldots b_{m+1}\xi^m_{t_{m+1}} b_m))\xi^m_{t_{m+1}} \ldots b_1\xi^m_1 b_0
\]

and

\[
\vartheta_{t-\varepsilon}(a)x = (a(\xi b_n\xi^m_{t_n} \ldots b_{m+1}\xi^m_{t_{m+1}} b_m))\xi^m_{t_{m+1}} \ldots b_1\xi^m_1 b_0.
\]

We observe that \(\xi^m_{t_{m+1}} \ldots b_1\xi^m_1 b_0\) is bounded uniformly in \(\varepsilon\), say, by \(M\). If we abbreviate \(y := \xi b_n\xi^m_{t_n} \ldots b_{m+1}\xi^m_{t_{m+1}} b_m\), we find

\[
\|\vartheta_{t-\varepsilon}(a)x - \vartheta_t(a)x\| \\
\leq M \|a(\xi b_n\xi^m_{t_n} \ldots b_{m+1}\xi^m_{t_{m+1}} b_m)\xi^m_{t_{m+1}} \ldots b_1\xi^m_1 b_0 - (a(\xi b_n\xi^m_{t_n} \ldots b_{m+1}\xi^m_{t_{m+1}} b_m))\xi^m_{t_{m+1}} \ldots b_1\xi^m_1 b_0\| = M \|a(\xi^0\varepsilon y - \xi^0\varepsilon (ay))\|,
\]

which goes to 0 for \(\varepsilon \to 0\).

We conclude that \(\vartheta\) is strongly right-continuous and strongly left-continuous and, therefore, strongly continuous.

To show that \(\xi^0 \in S\) is among the continuous sections, we have to compute the norm of \(\xi^0 - \xi^s\). For \(t \geq s\) this is \(\xi^0 - \xi^s = (\xi^0 - \xi^s)\xi^s\). Since \(\xi^s\) is bounded uniformly on any compact interval, continuity of \(\xi^\circ\) follows.

A.2: The \(W^*\)-case

The preceding part of the appendix fixes only a gap in the proof of \[BS00\] Theorem 10.2], providing also a slight generalization. (The product system of \[BS00\] Theorem 10.2] is generated by a single strongly continuous unit.) The present part, apart from fixing an analogue
gap in the proof of [BS00, Theorem 12.1], may also be considered as a start-up for the
to the theory of strongly continuous product systems of $W^*$–correspondences that are generated by their
strongly continuous units (strong type I). It is not exhaustive, and derives only those results that
we need for Section [13]. A systematic discussion would require to give a formal definition of
CPD-semigroups. It is not among the scopes of these notes.

Recall that from now on the $W^*$–algebras are assumed to be faithfully represented (for
convenience) in standard representation. “Strong” refers to the strong operator topology.

The following lemma is the generalization of Accardi and Mohari [AM96, Lemma 3.2] to
$W^*$–algebras $\mathcal{B}$ with not necessarily separable pre-dual. (The word “net” in their proof, should
be replaced by “sequence”. And even then, it seems that a restriction to bounded subsets is
still necessary. A similar result, which deals only with convergence but not with continuity, is
[MS02, Lemma 4.1(b)].) By the $\sigma$–weak topology on $\mathbb{R}_+ \times \mathcal{B}$ we mean the the product topology
of the usual topology on $\mathbb{R}_+$ and the $\sigma$–weak topology on $\mathcal{B}$.

A.2 Lemma. Let $T$ be a $\sigma$–weakly continuous one-parameter semigroup on an $W^*$–algebra $\mathcal{B}$,
and fix an arbitrary bounded subset $B$ of $\mathcal{B}$. Then the map $(t, b) \mapsto T_t(b)$ is a continuous map
$\mathbb{R}_+ \times B \rightarrow B$ for the $\sigma$–weak topology on either side.

Proof. $T$ being $\sigma$–weakly continuous, means that the pre-dual semigroup $T_\ast$ on the pre-dual
$\mathcal{B}_\ast$ of $\mathcal{B}$ is weakly, hence, strongly continuous. Therefore, $T$, like $T_\ast$, is bounded by a family of
constants $(Me^{\gamma t})_{t \in \mathbb{R}_+}$ for suitable positive numbers $M, \gamma$. We shall assume that $\gamma = 0$ passing, if
necessary, to the semigroup $T$ rescaled by $e^{-\gamma t}$. Denote by $(L_\ast, D(L_\ast))$ the generator of $T_\ast$, and
choose an element $\varphi \in D(L_\ast)$. Then

$$ (T_\ast)_\gamma(\varphi) = (T_\ast)_t(\varphi) + \int_0^t (T_\ast)_s(L_\ast(\varphi)) \, ds $$

(in norm of $\mathcal{B}_\ast$). Let $((t_\lambda, b_\lambda))_{\lambda \in \Lambda}$ be a net converging $\sigma$–weakly in $\mathbb{R}_+ \times B$ to $(t, b)$, that is, $t_\lambda \rightarrow t$
and $\varphi(b_\lambda) \rightarrow \varphi(b)$ for every $\varphi \in \mathcal{B}_\ast$. For $\varphi \in D(L_\ast)$ we find

$$ |\varphi(T_{t_\lambda}(b_\lambda) - T_t(b))| = \left| [(T_\ast)_t(\varphi)](b_\lambda) - [(T_\ast)_t(\varphi)](b) \right| $$

$$ \leq \left| [(T_\ast)_t(\varphi)][b_\lambda - b] \right| + \int_0^t \left| [(T_\ast)_s(L_\ast(\varphi))](b_\lambda) \right| \, ds $$

The first summand converges to 0. The second summand converges to $M \cdot 0 \cdot |L_\ast(\varphi)](b) = 0$, too. Now, since an arbitrary $\varphi \in \mathcal{B}_\ast$ is the norm limit of elements in $D(L_\ast)$ and since the $b_\lambda$ are
bounded uniformly in $\lambda$, it follows that $|\varphi(T_{t_\lambda}(b_\lambda) - T_t(b))| \rightarrow 0$ for all $\varphi \in \mathcal{B}_\ast$. \hfill \blacksquare

We are now ready to prove a generalization of [BS00, Theorem 12.1], fixing also the gap in
the proof of that theorem.
A.3 Definition. A product system of $W^*$– (or von Neumann) correspondences is $\sigma$–weak/strong/(weak) type I if it is generated by a $\sigma$–weakly/strongly/(weakly) continuous set $S$ of units, that is, for every $\xi^0, \xi^0 \in S$ the semigroup $(\langle \xi_i, \xi^i \rangle)_{t \in \mathbb{R}^+}$ is $\sigma$–weakly/strongly/(weakly) continuous. A strongly continuous product system is strong type I if the generating set of strongly continuous units can be chosen from the strongly continuous sections.

A.4 Theorem. Let $E^\circ$ be a product system of $W^*$–correspondences over a $W^*$–algebra $\mathcal{B}$ of $\sigma$–weak type I with generating set $S$ of $\sigma$–weakly continuous units. Furthermore, suppose that $\xi^0 \in S$ is a unital unit. Denote by \( \overline{E^\xi} \) the strong closure of $E^\xi$, the normal extension of the $E_0$–semigroup on $\mathcal{B}^a(E^\xi)$ to $\mathcal{B}^a(\overline{E^\xi})$ and the unit vector $\xi$ as constructed in Section 10. Then $\overline{\theta^\xi}$ is strongly continuous.

Proof. Recall that the elements of the form

$$\xi b_{n_1} \xi_{t_1} \ldots b_{i}^1 \xi_{t_i} b_0 \quad (n \in \mathbb{N}, t_i > 0, \xi^0 \in S, b_i \in \mathcal{B})$$  \hspace{1cm} (A.1)

are total in $E^\xi$. Thus, they are strongly total in $\overline{E^\xi}$.

(i) Let $x = \xi b_{n_1} \xi_{t_1} \ldots b_i^1 \xi_{t_i} b_0$. We will show that the map

$$t \mapsto x\xi_i$$

is strongly continuous. Put $\xi^{0\circ} := \xi^0$ and define the $\sigma$–weakly continuous semigroups $T^{i,j}$ and $T^j$ ($0 \leq i, j \leq n$) by setting $T^{i,j}_t := \langle \xi^{i,j}, \xi^i \rangle$ and $T^j := T^{i,i}$. Observe that

$$t \mapsto \|x\xi_i \odot g\|^2 = \langle g, T^0_t (b_0^1 T^1_{t_1} (b_1^1 \ldots T^n_{t_n} (b_n^1 b_n) \ldots b_1) b_0) g \rangle$$

is continuous. It remains to show that

$$t \mapsto \langle x\xi_i \odot g, x\xi_i \odot g \rangle$$

depends continuously on $t$ in a neighbourhood of $s$. To see that this is so, we observe that in either case, $t \geq s$ and $t \leq s$, we find $m \in \mathbb{N}; s_j > 0; 0 \leq i, k \leq n; c_j, c_j' \in \mathcal{B}$ such that the right-hand side becomes

$$\langle g, T^0_{s_j} c_1^1 \ldots T^0_{s_m} c_m^1 (c_m^1 c_m') \ldots c_1^1 g \rangle$$

with $s_j$ depending jointly continuously on $t$ and $s$. (Simply factor in $x\xi_s$ and in $x\xi_i$ the pieces $\xi_{t_i}$ of the units into products of smaller pieces, so that the involved time points in both coincide and the inner product can be calculated; see the proof of [BS00, Theorem 4.8] for a similar argument.) By induction, Lemma A.2 tells us that this depends continuously on $t$ in either case.

(ii) Let $x, y, z \in E^\xi$ have the form in (A.1), put $a := yz^*$, so that $ax = y(z, x)$ also has the form in (A.1), and choose $g \in G$. Recall that $\theta_t(a) x \xi_i = (ax) \xi_i$. Like in [BS00], we compute

$$\theta_t(a) x \odot g - ax \odot g = \theta_t(a) x \odot g - \theta_t(a) x \xi_i \odot g + \theta_t(a) x \xi_i \odot g - ax \odot g$$

$$= \theta_t(a) (x \odot g - x \xi_i \odot g) + ((ax) \xi_i \odot g - (ax) \odot g).$$
Since \( x \) and \( ax \) have the form in \((\text{A}.1)\), by (i) this converges to 0 for \( t \to 0 \). By boundedness of \( t \mapsto \vartheta_t(a) \), this shows that \( \vartheta_t(a) \) is strongly continuous at 0 at least for all \( a \) of the given form.

(iii) Let \( x_1, y_1, z_1, x_2, y_2, z_2 \in E^\xi \) have the form in \((\text{A}.1)\), and choose \( g_1, g_2 \in G \). Fix an arbitrary \( a \in B^\omega(E^\xi) \). Observe that \( y_1 x_1^* a x_2 y_2^* = y_1 (x_1, ax_2)y_2^* \) is an operator of the form dealt with in (ii). Observe also that the elements \( xy^*z \odot g := x(y, z) \odot g \) still form a total subset. Also here \( xy^*z \xi_t \odot g = \vartheta_t(xy^*)z \xi_t \odot g \). We compute

\[
\langle x_1 y_1^* z_1 \odot g_1, \vartheta_t(a) x_2 y_2^* z_2 \odot g_2 \rangle - \langle x_1 y_1^* z_1 \odot g_1, a x_2 y_2^* z_2 \odot g_2 \rangle \\
= \langle x_1 y_1^* z_1 \odot g_1, \vartheta_t(a) x_2 y_2^* z_2 \odot g_2 \rangle - \langle x_1 y_1^* z_1 \xi_t \odot g_1, \vartheta_t(a) x_2 y_2^* z_2 \xi_t \odot g_2 \rangle \\
+ \langle \vartheta_t(x_1 y_1^*) z_1 \xi_t \odot g_1, \vartheta_t(a) \vartheta_t(x_2 y_2^*) z_2 \xi_t \odot g_2 \rangle - \langle x_1 y_1^* z_1 \odot g_1, a x_2 y_2^* z_2 \odot g_2 \rangle \\
= \langle x_1 y_1^* z_1 v \odot g_1 - x_1 y_1^* z_1 \xi_t \odot g_1, \vartheta_t(a) x_2 y_2^* z_2 \odot g_2 \rangle \\
+ \langle x_1 y_1^* z_1 \xi_t \odot g_1, \vartheta_t(a) (x_2 y_2^* z_2 \odot g_2 - x_2 y_2^* z_2 \xi_t \odot g_2) \rangle \\
+ \langle z_1 \xi_t \xi_t \odot g_1 - z_1 \odot g_1, (y_1 x_1^* a x_2 y_2^*) z_2 \xi_t \odot g_2 \rangle \\
+ \langle z_1 \xi_t \odot g_1, (y_1^* a x_2 y_2^*) (z_2 \xi_t \odot g_2 - z_2 \odot g_2) \rangle.
\]

This converges to 0 for \( t \to 0 \). By boundedness of \( t \mapsto \vartheta_t(a) \), this shows that \( \vartheta_t(a) \) is \( \sigma \)--weakly continuous at 0. This means, the pre-dual semigroup of \( \vartheta \) is weakly continuous at 0 and, therefore, by [HP57] Theorem 10.2.3+Corollary, it is strongly continuous everywhere. In other words, \( \vartheta \) is \( \sigma \)--weakly continuous.

(iv) A \( \sigma \)--weakly continuous semigroup of endomorphisms is strongly continuous. Indeed,

\[
\|\vartheta_s(a)h - \vartheta_s(a)h\|^2 = \langle h, \vartheta_s(a^*a)h \rangle - \langle \vartheta_s(a)h, \vartheta_s(a)h \rangle - \langle \vartheta_s(a)h, \vartheta_s(a)h \rangle + \langle h, \vartheta_s(a^*a)h \rangle.
\]

For fixed \( s \) and \( t \to s \) this converges to 0. ■

A.5 Remark. If the unit \( \xi^\odot \) is only contractive, the statement remains true for the \( E \)--semigroup with pre-assigned product system constructed in Skeide [Ske08a]. We do not give details, as in these notes we are only interested in Markov semigroups. But the version of the following corollary for non-Markov semigroups depends on the contractive case.

We reprove the following result from Markiewicz and Shalit [MS07a].

A.6 Corollary. A weakly continuous contractive CP-semigroup on a von Neumann algebra is strongly continuous.

Proof. Since the semigroup is bounded, weak continuity implies \( \sigma \)--weak continuity. If we apply the theorem to the single strongly generating unit \( \xi^\odot \) of the GNS-system, we find that the minimal dilation is strongly continuous. It follows that \( T_s = \langle \xi_t^\odot \bullet \xi_s^\odot \rangle = \langle \xi_t^\odot, \vartheta_t^\odot (\xi_t^\odot \bullet \xi_s^\odot) \xi_t^\odot \rangle \) is strongly continuous in the strong topology of the standard representation and, therefore, \( \sigma \)--strongly continuous in any other. ■
A.7 Observation. It is even sufficient that $T$ is weakly continuous at $t = 0$. Indeed, $(T$ weakly continuous at $0) \Rightarrow (T$ $\sigma$–weakly continuous at $0) \Rightarrow (T$, strongly continuous at $0) \Rightarrow (T$, strongly continuous everywhere) $\Rightarrow (T$ $\sigma$–weakly (a fortiori weakly) continuous everywhere).

A.8 Corollary. A weak type I product system of $W^*$–correspondences is strong type I.

Proof. This is equivalent to the statement that for every finite subset $S’ = \{s_1, \ldots, s_n\}$ of $S$ the CP-semigroup $(b_{ij})_{i,j} \mapsto \langle \xi_i, \xi_j \rangle_{i,j}$ on $M_n(\mathcal{B})$ is strongly continuous.

Henceforth, we speak only of strong type I systems.

Let us fix a unital (or contractive) unit from the generating set of a strong type I system. Theorem A.4 (or Remark A.5) tells us that the $E_0$–semigroup constructed from it is strongly continuous, so that the product system inherits a strongly continuous structure, clearly having $\xi^\circ$ among the strongly continuous sections. It is important to know if the other units in $S$ are strongly continuous sections, too.

The next result gives general criteria to check if a unit is strongly continuous. The only-if-direction of the first part is a strong version of a considerable improvement of $3 \Rightarrow 1$ in [BBLS04, Lemma 4.4.11]. Also the other statements, true also in the situation of [BBLS04, Lemma 4.4.11] (for continuous units in continuous product systems), are new.

A.9 Lemma. Let $\xi^\circ$ by a strongly continuous unital unit in a strongly continuous product system $E^\circ$.

1. Suppose $\zeta^\circ$ is another unit in $E^\circ$. Then $\zeta^\circ$ is a strongly continuous section if and only if the functions $t \mapsto \langle \xi_t, \zeta_t \rangle$, $t \mapsto \langle \zeta_t, \xi_t \rangle$, and $t \mapsto \langle \zeta_t, \zeta_t \rangle$ are strongly continuous at $t = 0$ (and, therefore, everywhere).

2. If $\zeta^\circ$ is a strongly continuous unit in $E^\circ$, and if $\zeta^{\circ^\circ}$ is another unit such that the functions $t \mapsto \langle \zeta_t, \zeta_t \rangle$, $t \mapsto \langle \zeta_t, \xi_t \rangle$, and $t \mapsto \langle \xi_t, \xi_t \rangle$ are strongly continuous at $t = 0$, then $\zeta^{\circ^\circ}$ is a strongly continuous section.

A.10 Corollary. All units in the generating set $S$ of a strong type I product system are strongly continuous sections. Therefore, by Theorem 13.1, the strongly continuous structure does not depend on the choice of $\xi^\circ$.

A.11 Corollary. If $\zeta^\circ$ is a strongly continuous unit in the strongly continuous subsystem generated by $\zeta^\circ$, then $\zeta^{\circ^\circ}$ is a strongly continuous unit for $E^\circ$, too.

Proof of Lemma A.9. Define the CP-semigroup $S_i = \langle \xi_i, \bullet \xi_i \rangle$ generated by $\xi^\circ$. If $\xi^\circ$ is a strongly continuous section, then $S$ is weakly continuous at $t = 0$ and, hence, by Observation
strongly continuous everywhere. In particular, \( t \mapsto \langle \zeta, \zeta \rangle = S_t(1) \) is strongly continuous. The same argument applies to the mixed inner products. This is the if-direction of Part \[1\].

If we assume strong continuity of the inner products everywhere, then the proof of the only-if-direction in Part \[1\] is very similar to the proof of [BBLS04, Lemma 4.4.11]. But the strong version requires a refined argument. It is this refinement that allows to show the improvement that, actually, strong continuity (continuity in the case of [BBLS04, Lemma 4.4.11]) at \( t = 0 \) is sufficient.

So, let now \( \xi \) be a unit satisfying the continuity condition on the inner products. We first show that \( S \) is strongly continuous. Indeed, since

\[
S_{\varepsilon}(b) = \langle (\xi_{\varepsilon} - \xi_{\varepsilon}) + \xi_{\varepsilon}, b((\xi_{\varepsilon} - \xi_{\varepsilon}) + \xi_{\varepsilon}) \rangle \\
= \langle \xi_{\varepsilon} - \xi_{\varepsilon}, b(\xi_{\varepsilon} - \xi_{\varepsilon}) \rangle + \langle \xi_{\varepsilon}, b(\xi_{\varepsilon} - \xi_{\varepsilon}) \rangle + \langle \xi_{\varepsilon}, b\xi_{\varepsilon} \rangle + T_{\varepsilon}(b),
\]

since, by assumption,

\[
|\xi_{\varepsilon} - \xi_{\varepsilon}|^2 = (\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1) - (\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1) - (\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1) + (\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1)
\]

(A.2)

goesto0 strongly, and since \( T \) is strongly continuous, it follows that \( S_{\varepsilon}(b) - b = (S_{\varepsilon}(b) - T_{\varepsilon}(b)) + (T_{\varepsilon}(b) - b) \) goes to zero at least weakly. From this, by Observation A.7 strong continuity of \( S \) everywhere follows.

Recall that a section \( x \) of \( E \) is strongly continuous if and only if the function \( t \mapsto \xi x_t \odot g \) is continuous for all \( g \in G \). From \( \xi x_{t+\varepsilon} - \xi x_t = \xi (\xi_{\varepsilon} - \xi_{\varepsilon}) \xi_t \) (see [BBLS04]) and (A.2), it follows that

\[
|\xi_{t+\varepsilon} - \xi_t|^2 = S_t(\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1) - S_t(\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1) - S_t(\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1) + S_t(\langle \xi_{\varepsilon}, \xi_{\varepsilon} \rangle - 1).
\]

Of course, this implies strong right continuity of \( t \mapsto \xi x_t \). Substituting \( t > 0 \) with \( t - \varepsilon \), an appropriate application of Lemma A.2 shows also left strong continuity.

Part 2 follows by applying Part 1 and (A.2) to \( \langle \zeta', \zeta \rangle = \langle \zeta', \zeta_t - \zeta_t \rangle + \langle \zeta_t', \zeta_t \rangle \).

A.12 Remark. Also here the lemma and its corollaries remain true if the unit \( \xi \) is just contractive.

We close by showing that for \( W^* \)-algebras no spatial extension of the GNS-system is required. This makes the proof of Theorem 13.3 independent of the construction of the spatial extension (involving CPD-semigroups and their GNS-systems).

We start with a lemma on product systems of elementary CP-semigroups.

A.13 Lemma. Let \( c = (c_t)_{t \in \mathbb{R}_+} \) be a strongly continuous semigroup in the \( W^* \)-algebra \( \mathcal{B} \). Then the product system of the elementary CP-semigroup \( c_t' \bullet c_t \) is the trivial one with generating unit \( c_0 = (c_t)_{t \in \mathbb{R}_+} \).
Proof. Effectively, the trivial product system contains the unit $c^\odot$ and $T_i^c = \langle c_i, \bullet c_i \rangle = c_i^* \bullet c_i$. So the only thing to be shown is that $c^\odot$ generates the whole trivial product system and not only a subsystem. Denote by $E^\odot$ the product subsystem generated by $c^\odot$ and denote by $q_i$ the unique central projection such that $E_i = q_i \mathcal{B}$. It follows that

$$q_{s+t} \mathcal{B} = E_{s+t} = E_s \ominus E_t = q_s \mathcal{B} \ominus q_t \mathcal{B} = q_s q_t \mathcal{B},$$

or $q_{s+t} = q_s q_t$. The only semigroups of projections are constant for $t > 0$. (Indeed, for $s \geq t$ from the semigroup property it follows $q_{s+t} = q_s q_t^2 = q_s^2 q_t = q_s$. Therefore, for arbitrary $s, t > 0$ and sufficiently big $n$, we get $q_s q_t = q_n q_t = q_{ns} q_t = q_s$. By symmetry, $q_s = q_s q_t = q_t$.) Since $c_i$ approaches 1 strongly and since $E_i$ contains $c_i$, the only possibility for that constant $q_i$ is $q_i = 1$. So, $E_i = \mathcal{B}$ and the product system is the trivial one. ■

A.14 Theorem. Let $T$ be a strongly continuous normal Markov semigroup on a $W^*$–algebra $\mathcal{B}$. Then $T$ is spatial if and only if (the strong closure of) its GNS-system is spatial.

Proof. The backwards direction we know already. So let us assume that $T$ dominates the elementary CP-semigroup $c_i^* \bullet c_i$ for some strongly continuous semigroup $c$ in $\mathcal{B}$. By [BS00] Theorem 14.3] and its proof, for every CP-semigroup $S$ dominated by $T$ there exists a unique contractive positive endomorphism $w^\odot$ of the GNS-system $E^\odot$ of $T$ such that the unit $\xi^\odot := \sqrt{w\xi} = (\sqrt{w_i} \xi_i)_{i \in \mathbb{R}_+}$ generates $S$. If $S$ is elementary, then by Lemma[A.13] the subsystem generated by that unit is the trivial one. So, the only thing that remains to be shown, is that the unit $\xi^\odot$ is strongly continuous, because in that case, by Lemma[A.9](2), the unital central unit $(1)_{i \in \mathbb{R}_+}$ of that subsystem is strongly continuous also in $E^\odot$.

The map $t \mapsto \langle \xi_i, \xi_i \rangle = c_i^* c_i$ is strongly continuous. Observe that $1 - \langle \xi_i, \xi_i \rangle = \langle \xi_i, (1 - \sqrt{w_i}) \xi_i \rangle$. From

$$0 \leq \langle \xi_i, (1 - \sqrt{w_i}) \xi_i \rangle \leq \langle \xi_i, (1 - \sqrt{w_i})(1 + \sqrt{w_i}) \xi_i \rangle = \langle \xi_i, (1 - w_i) \xi_i \rangle = \langle \xi_i, \xi_i \rangle - \langle \xi_i, \xi_i \rangle,$$

it follows that also the map $t \mapsto \langle \xi_i, \xi_i \rangle = \langle \xi_i, \xi_i \rangle$ is strongly continuous at $t = 0$, as required by Lemma[A.9](1). ■

A.15 Remark. Apart from being crucial for the proof of existence of a Hudson-Parthasarathy dilation for a spatial Markov semigroup, this result is also important for the classification of strong type I systems. A slight modification asserts that a strongly continuous strong type I system is spatial if and only if the CPD-semigroup generated by the generating set $S$ is spatial in the sense of Skeide [Ske08c].
Appendix B: $E_0$–Semigroups and representations for strongly continuous product systems

The principal scope of this appendix is to prove Theorem [12.4]: Every strongly full strongly continuous product system is the product system associated with a norm strongly continuous $E_0$–semigroup on some $B^\sigma(E)$. However, since the existence of a nondegenerate faithful representation of a faithful product system is a closely related problem (dual to Theorem [12.4] in the sense of commutant of von Neumann correspondences in Skeide [Ske03a]), we include this result here. As corollaries we reprove a result by Arveson and Kishimoto [AK92], and we show that faithful strongly continuous product systems have a strongly continuous commutant system.

According to our convention in these notes, $\mathcal{B}$ is a $W^*$–algebra acting in standard representation on the Hilbert space $\mathcal{G}$. (As repeated several times, nothing really depends on that hypothesis. We just want to fix a faithful normal representation up to spatial isomorphism, and the standard representation is a canonical choice.) For the balance of this appendix, $E\otimes_t$ is a product system of $W^*$–correspondences over $\mathcal{B}$ that is strongly continuous with respect to a family of isometric embeddings $i_t: E_t \to \hat{E}$ in the sense of Definition [12.1]. We also use the other notations introduced there.

We start by proving some properties that hold for all strongly continuous product system, strongly full or not, faithful left action or not. First of all, we note that the embeddings $i_t: E_t \to \hat{E}$ give rise to embeddings $H_t := E_t \otimes G \to \hat{H} := \hat{E} \otimes G$, also denoted by $i_t$, defined by $x_t \otimes g \mapsto (i_t x_t) \otimes g$. We, therefore, may speak about continuous sections $h = (h_t)_{t \in \mathbb{R}_+}$ of $E^\sigma \otimes G$, in the sense that $t \mapsto i_t h_t$ is continuous. We denote the set of all continuous sections of $E^\sigma \otimes G$ by $CS_{i}(E^\sigma \otimes G)$. By definition, whenever $x, y \in CS_{i}^1(E^\sigma)$ and $g \in G$, then the functions $t \mapsto i_t x_t \otimes g$ and $(s, t) \mapsto i_{s+t}(x_t y_t) \otimes g$ are continuous.

**B.1 Corollary.** If $y \in CS_{i}^1(E^\sigma)$ and $g \in G$, then for every $b \in \mathcal{B}$ the function $t \mapsto i_t(b y_t) \otimes g$ is continuous.

**Proof.** Choose a strongly continuous section $x$ such that $x_0 = b$, and in the continuous function $(s, t) \mapsto i_{s+t}(x_t y_t) \otimes g$ put $s = 0$. ■

The following lemma is just the Dini theorem for nets. Later on, we will apply its corollary to the functionals $\langle y_t \otimes g, \bullet y_t \otimes g \rangle$ on $\mathcal{B}$.

**B.2 Lemma.** Let $(\phi_t)_{t \in \mathbb{R}_+}$ be a family of normal positive linear functionals on $\mathcal{B}$ such that $t \mapsto \phi_t(c)$ is continuous for all $c \in \mathcal{B}$. Suppose the net $(c_{A})_{A \in \Lambda}$ in $\mathcal{B}$ increases to $c \in \mathcal{B}$. Then for
every $0 \leq a < b < \infty$ and every $\varepsilon > 0$ there exists a $\lambda_0 \in \Lambda$ such that
\[ \varphi_\alpha(c - c_\lambda) < \varepsilon. \]
for all $\alpha \in [a, b]$ and all $\lambda \geq \lambda_0$.

**B.3 Corollary.** The positive linear functional on $\mathcal{B}$ defined by $c \mapsto \int_a^b \varphi_\alpha(c) \, d\alpha$ is normal.

**Proof of Lemma B.2** This is a standard application of compactness of the interval $[a, b]$, like many others that follow in this appendix. For each $\beta \in [a, b]$ choose $\lambda_\beta \in \Lambda$ such that $\varphi_\beta(c - c_\lambda) < \varepsilon$ for all $\lambda \geq \lambda_\beta$. Define $I_\beta$ to be the largest subinterval of $[a, b]$ such that $\varphi_\alpha(c - c_\lambda) < \varepsilon$ for all $\alpha \in I_\beta$. Since $c_\lambda$ increases to $c$, we get $\varphi_\alpha(c - c_\lambda) < \varepsilon$ for all $\alpha \in I_\beta$ and all $\lambda \geq \lambda_\beta$. Every $I_\beta$ is open in $[a, b]$ and contains at least $\beta$. Therefore, the family of all $I_\beta$ forms an open cover of the compact interval $[a, b]$. So, we may choose $\beta_1, \ldots, \beta_m$ such that the union over $I_{\beta_i}$ is $[a, b]$. Since every $\alpha \in [a, b]$ is contained in at least one of the intervals $I_{\beta_i}$, it follows that $\lambda_0 = \max_{i=1,\ldots,m} \lambda_i$ does the job. ■

The following density result is analogue to [Ske06c, Proposition 2.6]. The proof applies *a fortiori* also to the situation in [Ske06c, Proposition 2.6], and simplifies its proof quite a bit.

**B.4 Proposition.** Every continuous section $h \in \text{CS}_i(E^\odot \odot G)$ may be approximated locally uniformly by elements in $\text{span} \, \text{CS}_i^1(E^\odot) \odot G$. Moreover:

1. For every $k_i \in H_i$ we can find a continuous section $h \in \text{CS}_i(E^\odot \odot G)$ such that $h_i = k_i$.

2. For every pair $x \in \text{CS}_i^1(E^\odot)$ and $h \in \text{CS}_i(E^\odot \odot G)$ of sections the function
\[ (s, t) \mapsto i_{s+t}(x_i h_t) \]
is continuous.

**Proof.** Once we have the density statement, (1) is a standard result about continuous fields of Banach spaces (proved, for instance, like [Ske03b, Proposition 7.9]), and (2) follows by three epsilons, approximating $h \in \text{CS}_i(E^\odot \odot G)$ with an element in $\text{span} \, \text{CS}_i^1(E^\odot) \odot G$ on a suitably big interval. So, let us prove the density statement.

Let $h \in \text{CS}_i(E^\odot \odot G)$ and choose $0 \leq a < b < \infty$ and $\varepsilon > 0$. By Definition [1241] for every $\beta \in [a, b]$ there exists a section $h_\beta^\delta$ in $\text{span} \, \text{CS}_i^1(E^\odot) \odot G$ such that $\|h_\beta - h_\beta^\delta\| < \varepsilon$. For every $\beta$ define $I_\beta$ to be the largest interval such that $\|h_\alpha - h_\beta^\delta\| < \varepsilon$ for all $\alpha \in I_\beta$. Every $I_\beta$ is open in $[a, b]$ and contains at least $\beta$. Therefore, the family of all $I_\beta$ forms an open cover of the compact interval $[a, b]$. So, we may choose $\beta_1, \ldots, \beta_m$ such that the union over $I_{\beta_i}$ is $[a, b]$. By
standard theorems about partitions of unity there exist continuous functions \( \varphi_i \) on \([a, b]\) with the following properties:

\[
0 \leq \varphi_i \leq 1, \quad \varphi_i \upharpoonright I_{p_i}^c = 0, \quad \sum_{i=1}^m \varphi_i = 1.
\]

From these properties, one easily verifies that \( \|h_\alpha - \sum_{i=1}^m \varphi_i(\alpha)h_{\alpha i}^\delta\| < \varepsilon \) for all \( \alpha \in [a, b] \). This shows that \( \sum_{i=1}^m \varphi_i h_{\alpha i}^\delta \in \text{span } CS_\delta(E^\circ) \otimes G \) approximates \( h \) uniformly up to \( \varepsilon \) on the interval \([a, b]\). ■

**B.5 Theorem.** Let \( i_i: E_i \to E^i \) and \( k_i: E_i \to E^k \) be two strongly continuous structures on the product system \( E^\circ = (E_i)_{i \in \mathbb{R}_+} \). If the identity morphism is a strongly continuous isomorphism from \( E^\circ \) with respect to the embeddings \( (i_i) \) to \( E^\circ \) with respect to the embeddings \( (k_i) \), then it is a strongly continuous isomorphism also for the other direction.

**Proof.** For continuous product systems and continuous isomorphisms, this is [Ske09b, Theorem 2.2]. In the proof of that theorem we also mentioned that the theorem, actually, is a statement that is valid for every continuous field of Banach spaces that takes it continuous structure from a family of isometric embeddings into a fixed Banach space. Making use of Proposition B.4, we will apply [Ske09b, Theorem 2.2] to the continuous field of Hilbert spaces \( (H_i)_{i \in \mathbb{R}_+} \) with respect to the two embeddings.

What we have to show is that if \( x \otimes g \in CS_\delta(E^\circ \otimes G) \) for every \( g \in G \) implies \( x \otimes g \in CS_\delta(E^\circ \otimes G) \) for every \( g \in G \), then \( x \otimes g \in CS_\delta(E^\circ \otimes G) \) for every \( g \in G \). What we have from [Ske09b, Theorem 2.2] is the statement that if \( h \in CS_\delta(E^\circ \otimes G) \) implies \( h \in CS_\delta(E^\circ \otimes G) \), then \( h \in CS_\delta(E^\circ \otimes G) \) implies \( h \in CS_\delta(E^\circ \otimes G) \). It is, therefore, sufficient to show that if \( x \otimes g \in CS_\delta(E^\circ \otimes G) \) for every \( g \in G \) implies \( x \otimes g \in CS_\delta(E^\circ \otimes G) \) for every \( g \in G \), then \( h \in CS_\delta(E^\circ \otimes G) \) implies \( h \in CS_\delta(E^\circ \otimes G) \). But, this last statement follows if we approximate \( h \) locally uniformly in the by sections in span \( C^i(E^\circ) \otimes G \subset C^i_k(E^\circ) \) and take into account, like in the proof of [Ske09b, Theorem 2.2], that the locally uniform approximation does not depend on the choice of the continuous structure \( i \) or \( k \). ■

**B.1: \( E_0 \)-Semigroups**

After these general properties of strongly continuous product systems, we now come to the proof of Theorem 12.4. For this part of this appendix we shall assume that \( E^\circ \) is strongly full.

We mentioned already in Section 11 that the algebraic part of the proof in Skeide [Ske06a] for the Hilbert space case allows to construct an \( E_0 \)-semigroup that is definitely not continuous and acts on a \( B^\circ(E) \) whose pre-dual cannot be separable. The \( E_0 \)-semigroup is constructed in terms of a left dilation, and the construction, using the identifications in (11.1),
involves the choice of a left dilation of the discrete subsystem \((E_n)_{n \in \mathbb{N}_0}\) of \(E^\circ\). Existence of a left dilation of \((E_n)_{n \in \mathbb{N}_0}\) is granted by Skeide [Ske09d Theorem 6.3].

Like in Skeide [Ske07a], where the \(C^\ast\)-case is treated, also here it is convenient to adapt Arveson’s construction in [Arv06] of our \(E_0\)-semigroup in [Ske06a], rather than using our construction directly. (We refer to [Ske07a] for motivation.) This construction is based on the choice of a unit vector \(\xi_1 \in E_1\). It is a feature of continuous product systems that every \(E_t\) contains a unit vector; see [Ske07a Lemma 3.2]. Although we dare to conjecture that the same is also true for strongly continuous product systems of \(W^\ast\)-correspondences that are strongly full (it is difficult to imagine a counter example), we could not yet prove it. (Note that [Ske07a, Lemma 3.2] is about every continuous product system of \(C^\ast\)-correspondences over a unital \(C^\ast\)-algebra \(B\), which, therefore, are full automatically. If we drop strong fullness, then the statement for product systems of \(W^\ast\)-correspondence is surely false: Consider the [Ske04b Example 4.13 for a non-unital \(C^\ast\)-algebra and close it strongly.]

One basic idea in the construction of the left dilation of the discrete subsystem \((E_n)_{n \in \mathbb{N}_0}\) in [Ske09d] was that even if the strongly full \(W^\ast\)-correspondence \(E_1\) over \(B\) has no unit vector, then \(\overline{M_n(E_1)}\) has one; see [Ske09d, Proposition 6.2]. Here \(n\) is a sufficiently big cardinal number and \(\overline{M_n(E_1)} = E_1 \hat{\otimes} B(\mathbb{C}^n)\) is the (spatial) external tensor product of \(W^\ast\)-modules, which is a \(W^\ast\)-module over \(B\) \(\hat{\otimes}^\ast B(\mathbb{C}^n) = \overline{M_n(B)}\). The elements of \(\overline{M_n(E_1)}\) are understood best as \(E_1\)-valued matrices, and the \(W^\ast\)-module operations are the natural matrix operations; see [Ske09d Section 6] for details. One easily verifies that \(\overline{B^0(M_n(E_1))} = B^0(E_1) \hat{\otimes}^\ast B(\mathbb{C}^n) = \overline{M_n(B^0(E_1))}\).

We follow the construction in Skeide [Ske09b] where we proved the result of [Ske07a] for not necessarily unital \(C^\ast\)-algebras. We also refer to [Ske09b] for a detailed motivation and an explanation why what follows is the proper generalization of Arveson’s idea [Arv06]. Technically, the whole proof of the \(W^\ast\)-case here, is very similar to [Ske09b] (involving also technical results from [Ske07a] and [Ske09a]). For reasons of space we dispense with giving full proofs of these technical results, and often refer to either identical or at least very similar proofs in the cited papers.

So, let us fix a unit vector \(\Xi_1 \in \overline{M_n(E_1)}\). To facilitate notation we fix a set \(S\) of cardinality \(#S = n\) and denote the elements of \(E^\alpha_n\) as \(X_\alpha = (X^j_\alpha)_{j \in S}\). Also \(\Xi_1\) is given by the matrix \((\Xi_1)_{ss'}\) \(ss'\). Like a vector \(\xi_1 \in E_1\) can act on \(x_\alpha \in E_\alpha\) as \(\xi_1 x_\alpha \in E_{\alpha+1}\), the vector \(\Xi_1\) can act on \(X_\alpha \in E^\alpha_n\) as \(\Xi_1 X_\alpha := (\sum_{s' \in S} (\Xi_1)_{ss'} X^s_\alpha)_{s \in S} \in E^\alpha_{n+1}\).

Next, we define the direct integrals we need. The family of embeddings \(i_t : E_t \rightarrow \hat{E}\) gives rise to embeddings \(i^t_n : E^t_n \rightarrow \hat{E}^n\). Every section \(X = (X_t)_{t \in \mathbb{R}_\alpha}\) with \(X_t \in E^t_n\) gives rise to a function \(t \mapsto X(t) := i^t_n X_t\) with values in \(\hat{E}^n\). We denote by

\[
CS^t_n(i_n^t(E^\circ)) = \left\{ X : t \mapsto X(t) \text{ is strongly continuous} \right\}
\]
the set of all sections that are strongly continuous. Let \( 0 \leq a < b < \infty \). By \( \int_a^b E_a^n \, da \) we understand the self-dual extension of the pre-Hilbert \( \mathcal{B} \)–module that consists of continuous sections \( X \in CS^+_i(E^\mathcal{B}) \) restricted to \([a, b)\) with inner product
\[
\langle X, Y \rangle_{[a,b]} := \int_a^b \langle X_\alpha, Y_\alpha \rangle \, d\alpha = \int_a^b \langle X(\alpha), Y(\alpha) \rangle \, d\alpha.
\]
By an application of the principle of uniform boundedness, all strongly continuous sections are bounded on the compact interval \([a, b)\). Therefore, the integral exists as a Riemann integral in the weak operator topology of \( \mathcal{B}(G) \).

**B.6 Observation.** Some care is in place in calculating inner products of arbitrary elements \( X \) and \( Y \), when thinking as an “integral” over “sections”. Even when the bundle is trivial, these elements need no longer have an interpretation as sections with values in the fibers. [Ske01, Example 4.3.13] shows that this already fails under norm closure.

**B.7 Proposition.** \( \int_a^b E_a^n \, da \) contains as a pre-Hilbert submodule the space \( s\mathfrak{R}_{(a,b)} \) of restrictions to \([a, b)\) of those sections \( X \) for which \( t \mapsto X(t) \) has strong left limits everywhere and is strongly right continuous with a finite number of jumps in \([a, b)\).

**Proof.** The proof is more or less like that of [Ske07a, Proposition 4.2] for right continuous (not only strongly continuous) functions with left limits (not only strong limits) in each point. Just that now the approximation must be done for each of the functions \( t \mapsto X(t) \circ g \) separately. (A linear subspace \( C \) of \( s\mathfrak{R}_{(a,b)} \) is strongly dense if for each \( g \in G \), each \( \varepsilon > 0 \), and each \( X \in s\mathfrak{R}_{(a,b)} \) the function \( t \mapsto X(t) \circ g \) can be approximated up to \( \varepsilon \) by a function \( t \mapsto Y_t \circ g \) with \( Y_t \in C \). By the proof [Ske07a, Proposition 4.2], this is true for \( C \) being the space of strongly continuous sections restricted to \([a, b)\), because the approximation of the jumps is done in a ways that does not depend on \( g \). Better: It is uniform in \( \|g\| \leq 1 \).

**B.8 Proposition.** \( \int_a^b E_a^n \, da \) is strongly full.

**Proof.** Let \( b = \langle x_a, x_a \rangle \in \mathcal{B} \) for some \( x_a \in E_a \). These elements are strongly total, so it is sufficient if we strongly approximate each such \( b \). By definition, there exists a section \( y \in CS^+_i(E^\mathcal{B}) \) such that \( y_a = x_a \). It is not difficult to show that for the strongly right continuous sections \( y^\lambda := (\frac{\lambda E_{a+y(\lambda a)}(a)}{\lambda})_{a \in \mathbb{R}} \) (\( \lambda > 0 \)), the expression \( \langle y^\lambda, y^\lambda \rangle_{(a,b)} \) converges weakly to \( b \) for \( \lambda \to 0 \). So the the span of all \( \langle y^\lambda, y^\lambda \rangle \) is weakly, hence, strongly dense in \( \mathcal{B} \).

Let \( s\mathfrak{S} \) denote the right \( \mathcal{B} \)–module of all sections \( X \) that are locally \( s\mathfrak{R} \), that is, for every \( 0 \leq a < b < \infty \) the restriction of \( X \) to \([a, b)\) is in \( s\mathfrak{R}_{[a,b)} \), and which are stable with respect to the unit vector \( \Xi_1 \), that is, there exists an \( a_0 > 0 \) such that
\[
X_{a+1} = \Xi_1 X_a
\]

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for all $\alpha \geq \alpha_0$. By $sN$ we denote the subspace of all sections in $sS$ which are eventually 0, that is, of all sections $X \in sS$ for which there exists an $\alpha_0 \geq 0$ such that $X_\alpha = 0$ for all $\alpha \geq \alpha_0$. A straightforward verification shows that

$$\langle X, Y \rangle := \lim_{m \to \infty} \int_{m}^{m+1} \langle X(\alpha), Y(\alpha) \rangle d\alpha$$

defines a semiinner product on $sS$ and that $\langle X, X \rangle = 0$ if and only if $X \in sN$. Actually, we have

$$\langle X, Y \rangle = \int_{T}^{T+1} \langle X(\alpha), Y(\alpha) \rangle d\alpha$$

for all sufficiently large $T > 0$; see [Arv06, Lemma 2.1]. So, $sS/sN$ becomes a pre-Hilbert module with inner product $\langle X + sN, Y + sN \rangle := \langle X, Y \rangle$. By $E$ we denote its self-dual extension.

**B.9 Proposition.** For every section $X$ and every $\alpha_0 \geq 0$ define the section $X^{\alpha_0}$ as

$$X^{\alpha_0}_\alpha := \begin{cases} 0 & \alpha < \alpha_0 \\ \Xi^i X_{\alpha-n} & \alpha \in [\alpha_0 + n, \alpha_0 + n + 1), n \in \mathbb{N}_0. \end{cases}$$

If $X$ is in $CS_{i}^{n,t}(E^\circ)$, then $X^{\alpha_0}$ is in $sS$. Moreover, the set $\{X^{\alpha_0} + sN: X \in CS_{i}^{n,t}(E^\circ), \alpha_0 \geq 0\}$ is a strongly dense submodule of $E$.

**Proof.** The proof is like that of [Ske07a, Proposition 4.3]. It only takes a moments thought to convince oneself that the approximating sequence in that proof may replaced without problem by a strongly approximating net. □

After these preparations it is completely plain to see that for every $t \in \mathbb{R}_+$ the map $X \otimes y_t \mapsto Xy_t$, where

$$(Xy_t)_\alpha = \begin{cases} X_{\alpha-t}y_t & \alpha \geq t, \\ 0 & \text{else}, \end{cases}$$

and where $X_{\alpha}y_t = (X_{\alpha}y_t)_{\alpha \in S}$, defines an isometry $v_t: E \otimes E \to E$, and that these isometries iterate associatively with the product system structure.

**B.10 Proposition.** Each $v_t$ is surjective.

**Proof.** By Proposition [B.9] it is sufficient to approximate every section of the form $X^{\alpha_0}$ with $X \in CS_{i}^{n,t}(E^\circ), \alpha_0 \geq 0$ in the (semi-)inner product of $sS$ by finite sums of sections of the form $Y_{Z_t}$ for $Y \in sS, z_t \in E_t$. As what the section does on the finite interval $[0, t]$ is not important for the inner product, we may even assume that $\alpha_0 \geq t$. And as in the proof of Proposition [B.9] the approximation can be done by approximating $X$ in $sR_{(\alpha_0,\alpha_0+1)}$ and then extending the restriction to $[\alpha_0, \alpha_0 + 1)$ stably to the whole axis. (This stable extension is the main reason why we worry
to introduce the subspace of strongly right continuous sections.) On \( \mathfrak{sR}_{(\alpha_0, \alpha_0 + 1)} \), however, the approximation may be done for each direct summand in \( E \) separately. (The maps \( v_t \) decompose into the components of \( E^\alpha \).) In other words, it is sufficient to prove the density statement only on the interval \( [\alpha_0, \alpha_0 + 1) \) and for \( n = 1 \).

The continuous version of Proposition [B.10] for \( n = 1 \) is done in [Ske07a, Proposition 4.6]. Like in the proof of Proposition [B.4] for the strongly continuous version we have to modify the proof of [Ske07a, Proposition 4.6]. As modification is a bit too much for just referring to the continuous version, we give full detail.

Let \( \alpha_0 \geq t \) and let \( x \) be a strongly continuous section and choose \( g \in G \). We will approximate the continuous section \( \alpha \mapsto x_\alpha \odot g \) uniformly on \( [\alpha_0, \alpha_0 + 1) \) (and, therefore, in \( L^2 \)) by finite sums over sections of the form \( \alpha \mapsto y_{\alpha - t} \odot g \). Choose \( \varepsilon > 0 \). For every \( \beta \in [\alpha_0, \alpha_0 + 1) \) choose \( n^\beta \in \mathbb{N} \), \( y^\beta_k \in E_{\beta - i} \), \( z^\beta_k \in E_i \) such that \( \| (x_\beta - \sum_{k=1}^{n^\beta} y^\beta_k z^\beta_k) \odot g \| < \varepsilon \). Choose continuous sections \( \tilde{y}^\beta_k = (\tilde{y}^\beta_{k, i}, a \in \mathbb{R}) \in CS_i^s(E^\alpha) \) such that \( \tilde{y}^\beta_k(\beta - i) = y^\beta_k \). For every \( \beta \) chose the maximal interval \( I_\beta \subset [\alpha_0, \alpha_0 + 1) \) such that \( \| (x_{\beta} - \sum_{k=1}^{n^\beta} \tilde{y}^\beta_{k, a - i} \odot g) \| < \varepsilon \) for all \( \alpha \in I_\beta \). Like in the other proofs, \( I_\beta \) is open in \( [\alpha_0, \alpha_0 + 1) \) and contains at least \( \beta \). So, we may choose finitely many \( \beta_1, \ldots, \beta_m \in [\alpha_0, \alpha_0 + 1) \) such that the union of all \( I_{\beta_i} \) is \( [\alpha_0, \alpha_0 + 1) \). By taking away from \( I_{\beta_i} \), everything that is already contained in \( I_{\beta_1} \cup \ldots \cup I_{\beta_m} \), we define a finite partition \( I \) of \( [\alpha_0, \alpha_0 + 1) \). Taking away the point \( \alpha_0 + 1 \) and adjusting the endpoints of the \( I \), suitably, we may assume that all \( I \) are right open. Denote by \( \| \cdot \|_I \) the indicator function of \( I_t \). Then, restriction of the piecewise continuous section

\[
\alpha \mapsto \begin{cases} 0 & \alpha < t \\ \sum_{i=1}^{m} \sum_{k=1}^{n_i} (\tilde{y}^\beta_{k, a - i} \odot g I_i(\alpha)) & \alpha \geq t \end{cases}
\]

to \( [\alpha_0, \alpha_0 + 1) \) is in \( \mathfrak{sR}_{(\alpha_0, \alpha_0 + 1)} \) and approximates \( \alpha \mapsto x_\alpha \odot g \) uniformly on \( [\alpha_0, \alpha_0 + 1) \) up to \( \varepsilon \).

So, the \( v_t \) form a left dilation of \( E^\alpha \) to \( E \). This left dilation is strongly continuous in the following sense.

**B.11 Proposition.** For every \( X \in E \), every strongly continuous section \( y \in CS_i^s(E^\alpha) \), and every \( g \in G \) the function \( t \mapsto X y_t \odot g \) is continuous.

**Proof.** In the proof of the analogue [Ske07a, Proposition 4.7] for continuous product systems, we made use of the fact that, there, \( X \) could be approximated in norm by a section of the form \( X^m \), so that it was sufficient to show the statement of [Ske07a, Proposition 4.7] only for such sections. Also here proving the statement first for sections \( X^m \) will be an important step in the proof. However, Proposition [B.9] guarantees only strong approximation of \( X \), and the argument that this is sufficient differs considerably from the proof in [Ske07a].

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So, suppose for a moment we had proved that \( t \mapsto X^{α_0}y_t \circ g \) is continuous for every \( X \in C_γ^α(\mathbb{E}^0) \), every \( α_0 \geq 0 \), every \( y \in CS_γ^α(\mathbb{E}^0) \), and every \( g \in G \). In order to prove that \( t \mapsto Y_{z_t} \circ g' \) is continuous for every \( Y \in E \), every \( z \in CS_γ^α(\mathbb{E}^0) \), and every \( g' \in G \), it is sufficient to show that \( t \mapsto \|Y_{z_t} \circ g'\| \) is continuous, and that \( t \mapsto Y_{z_t} \circ g' \) is weakly continuous. Since that function is bounded uniformly on finite intervals, it is even sufficient to check weak continuity on a total subset of \( H \), only. Continuity of the norm follows from Corollary B.1. To see weak continuity at a fixed point \( t \), we observe that the elements \( X^{α_0}y_t \circ g \) form a total subset of \( H \). We compute

\[
\langle X^{α_0}y_t \circ g, Y_{z_t} \circ g' - Y_{z_t} \circ g' \rangle
\]

\[
= \langle X^{α_0}y_t \circ g - X^{α_0}y_t \circ g, Y_{z_t} \circ g' \rangle + \langle X^{α_0}y_t \circ g, Y_{z_t} \circ g' \rangle - \langle X^{α_0}y_t \circ g, Y_{z_t} \circ g' \rangle.
\]

For \( s \to t \) the first summand goes to 0 by strong continuity of \( t \mapsto X^{α_0}y_t \circ g \). Like for continuity of the norm, also the difference of the last two summands goes to 0 by Corollary B.1.

It remains, therefore, to show continuity of \( t \mapsto X^{α_0}y_t \circ g \) for \( X^{α_0} \) for \( X \in CS_γ^{α+n}(\mathbb{E}^0) \) and \( α_0 \geq 0 \). To calculate \( \|X^{α_0}y_t - X^{α_0}y_{s}\| \circ g \|^2 \) we have to integrate over \( α \) the values of \( \|X^{α_0}y_t - X^{α_0}y_{s}\| \circ g \|^2 \) for \( α \) in some unit interval such that \( α - t \) and \( α - s \) are not smaller than \( α_0 \). So

\[
\|X^{α_0}y_t - X^{α_0}y_{s}\| \circ g \|^2 = \int_{d}^{d+1} \left\|X^{α_0}_{α+t}y_t - X^{α_0}_{α+t}y_{s}\right\|^2 dα
\]

(B.1)

for all \( d \geq α_0 \). The function \( (α, t) \mapsto X^{α_0}_{α}y_t \circ g \) is uniformly continuous on each of the intervals \([α_0 + n, α_0 + n + 1] \times [a, b]\) and it is bounded on each \( \mathbb{R}_+ \times [a, b] \). We fix a \( t \). Choose \( d \geq α_0 \) such that \( d + t = n + α_0 \) for a suitable \( n \in \mathbb{N}_0 \). Then \( α + t \) in the integral in (B.1) goes over the interval \([α_0 + n, α_0 + n + 1]\). Choose \( ε \in (0, \frac{1}{2}) \). Then in

\[
\|X^{α_0}y_t - X^{α_0}y_{s}\| \circ g \|^2 = \int_{d}^{d+\epsilon} + \int_{d+\epsilon}^{d+\epsilon+1} + \int_{d+\epsilon-1}^{d+1} \left\|X^{α_0}_{α+t}y_t - X^{α_0}_{α+t}y_{s}\right\|^2 dα
\]

the first and the last integral are bounded by \( ε \) times a constant which can be chosen independent of \( s \) as long as \( s \) varies in a bounded set. If we choose \( s \in (t-\epsilon, t+\epsilon) \), in the middle integral also \( α + s \) is in the same interval \([α_0 + n, α_0 + n + 1]\), so that both \( X^{α_0}_{α+t}y_t \) and \( X^{α_0}_{α+t}y_{s} \) depend uniformly continuously on \( (α, s) \) in \((d + \epsilon, d + 1 - \epsilon) \times (t-\epsilon, t+\epsilon) \). In particular, if \( s \) is sufficiently close to \( t \), then both \( X^{α_0}_{α+t}y_t \circ g \) and \( X^{α_0}_{α+t}y_{s} \circ g \) are close to their common limit \( X^{α_0}_{α+t}y_t \circ g \) uniformly in \( α \). It follows that the middle integral goes to 0 for \( s \to t \). Sending also \( ε \to 0 \), the proposition is proved.

\[ \text{B.12 Corollary. The } E_0\text{-semigroup } \vartheta^t \text{ is strongly continuous.} \]

\[ \text{Proof. Fix } a \in B^α(a). \text{ Since } \vartheta^t_t(a) \text{ is bounded uniformly in } t \text{ by } \|a\|, \text{ we may check strong continuity at } t \text{ on the total subset of elements of the form } Xy_t \circ g \text{ (} X \in E, y \in CS_γ^α(\mathbb{E}^0), g \in G \).}\]
We find

\[(\vartheta^s(a) - \vartheta^t(a))Xy_t \circ g = (\vartheta^s(a)(Xy_t \circ g) - \vartheta^t(a)(Xy_t \circ g)) + (\vartheta^s(a)(Xy_t \circ g) - \vartheta^t(a)(Xy_t \circ g)) = \vartheta^s(a)(Xy_t \circ g - Xy_s \circ g) + ((aX)y_s \circ g - (aX)y_t \circ g).\]

By Proposition [B.11] both expressions are small, whenever \(s\) is sufficiently close to \(t\).

**B.13 Corollary.** The continuous structure induced by the \(E_0\)-semigroup \(\vartheta^e\) coincides with the continuous structure of \(E^0\).

**Proof.** By amplifying if necessary, we may assume that \(E\) has a unit vector \(\xi\). Each section \(y\) corresponds to the section \((\xi y_t)_{t \in \mathbb{R}_+}\) with respect to the new embedding of \(E^0\) into \(E\) induced by \(\vartheta^e\). By Proposition [B.11] (which, clearly, remains valid also under amplification of \(\vartheta^e\)), if \(y\) is strongly continuous, then so is \(t \mapsto \xi y_t\). By Theorem [B.5] the two continuous structures coincide.

This ends the proof of Theorem [12.4].

**B.14 Observation.** Note that the proofs of the two preceding corollaries do not depend on the concrete form of the left dilation. We, therefore, showed the following more general statement: If \(v_t\) is a left dilation of a strongly continuous product system \(E^0\) that is strongly continuous in the sense of Proposition [B.11], then the induced \(E_0\)-semigroup \(\vartheta^e\) is strongly continuous and the strongly continuous structure induced by that \(E_0\)-semigroup upon \(E^0\) coincides with the original one.

**B.15 Remark.** We do not need countability hypothesis. But of course, like in the \(C^*\)-case, if countability hypothesis are fulfilled, then the constructions \(E_0\)-semigroup \(\leftrightarrow\) product system in either direction preserve these. If \(B\) and \(B^a(E)\) have separable pre-duals, then the product system \(E^0\) of a strongly continuous \(E_0\)-semigroup on \(B^a(E)\) is countably generated in the sense that there exists a countable subset of \(C^*_f(E^0)\) that generates \(C^*_f(E^0)\) by locally uniform strong limits. Conversely, if \(B\) has separable pre-dual and if \(E^0\) is countably generated, then \(E\) and, therefore, also \(B^a(E)\) has separable pre-dual.

**B.2: Nondegenerate representations**

Since the introduction of Arveson systems in [Arv89], representations of product systems have been recognized as an important concept. For product systems of correspondences the definition is due to Muhly and Solel [MS02]. A concept equivalent to faithful and nondegenerate representation is that of right dilation, introduced in Skeide [Ske06a, Ske07a, Ske06c]. The naming,
left and right dilation underlines a deep symmetry between the concepts. In fact, the commu-
tant of von Neumann correspondences introduced in Skeide [Ske03a] (and, independently, in
Muhly and Solel [MS04] under the name of $\sigma$–dual) turns a left dilation of a product system
into a right dilation of its commutant system, and vice versa; see Skeide [Ske09d, Theorem 9.9]
(preprint 2004).

While on the algebraic level, the correspondence between left and right dilations is perfect,
this situation changes, when we take into consideration continuity. Under the hypothesis that all
occurring $W^*$–algebras and $W^*$–modules have separable pre-dual, Muhly and Solel [MS07b]
have proved that the duality is perfect for weakly measurable product systems, by reducing it
to a result by Effros [Eff65]: Measurable fields of von Neumann algebras have measurable
commutant fields.

Since we do not know a similar result for strongly continuous fields of von Neumann alge-
bras, we cannot imitate the reduction from [MS07b]. Instead, we have to prove from scratch
existence of strongly continuous right dilations under a condition, faithfulness of all left ac-
tions, which under commutant corresponds to strong fullness on the commutant side. In the
case of strongly continuous product systems that are both strongly full and faithful, we get sym-
metry: Also the commutant system possesses a strongly continuous structure. Note, however,
that this result relies heavily on the “semigroup structure” encoded by the product system; a
product system is a monoid of correspondences. In fact, we shall use that also a strongly con-
tinuous right dilation is related with a certain $E_0$–semigroup and that the product system of that
$E_0$–semigroup is the commutant of the one for which we construct a right dilation. If that right
dilation is strongly continuous, then so is the $E_0$–semigroup. Therefore, the commutant system
inherits a strongly continuous structure.

In the sequel, we discuss only $W^*$–versions. $C^*$–Versions have been discussed in [Ske09d,
Ske06e, Ske08a]. So, $E^\circ$ is a product system of $W^*$–correspondences over the $W^*$–algebra $B$.

B.16 Definition [MS02]. A representation of $E^\circ$ on a Hilbert space $H$ is a family $\eta = (\eta_t)_{t \in \mathbb{R}_+}$
of maps $\eta_t : E_t \rightarrow B(H)$ that fulfills

$$\eta_t(x_t)\eta_s(y_s) = \eta_{t+s}(x_t y_s), \quad \eta_t(x_t)^*\eta_s(y_t) = \eta_0(\langle x_t, y_t \rangle).$$

A representation is normal (faithful) if $\eta_0$ is normal (faithful). A representation is nondegen-
erate if each $\eta_t$ acts nondegenerately on $H$ (that is, $\overline{\text{span}} \eta_0(E_t)H = H$).

Note that $\eta_t$ is a ternary homomorphism into the Hilbert $B(H)$–module $B(H)$. Therefore,$\eta_t$ is linear and completely contractive. $\eta_0$ is a representation. (Simply, put $t = s = 0$.) In
particular, it makes sense to speak of $\eta_0$ being normal (that is, order continuous). If $\eta$ is faithful,
then each $\eta_t$ is faithful. If $\eta$ is normal, then each $\eta_t$ is $\sigma$–weak; see the remark following
[MS02, Lemma 2.16]. Note, too, that the pair $(\eta_t, \eta_0)$ is what is called a representation of the single
correspondence $E_t$. 80
Nondegenerate representations have been called essential in Arveson [Arv89] and Hirshberg [Hir05a]. Muhly and Solel [MS02] defined as a covariant representation a family that fulfills only \( \eta_i(x_i)\eta_i(y_i) = \eta_{i+\lambda}(x_i,y_i) \) plus the requirement that \( \eta_0 \) is a representation (which is no longer automatic). A covariant representation also fulfilling \( \eta_i(x_i)\eta_i(y_i) = \eta_0((x_i, y_i)) \), is called isometric, while a covariant representation fulfilling the nondegeneracy condition, is called fully coisometric. Also, covariant representations are usually required to be completely contractive. In the isometric case (that is, in particular, for our representations), this is automatic.

**B.17 Remark.** Nondegenerate representations have been called essential in Arveson [Arv89] and Hirshberg [Hir05a]. Muhly and Solel [MS02] defined as a covariant representation a family of bilinear unitaries \( w_i : E_i \otimes H \rightarrow H \), such that the product \((x_i, h) \mapsto x_i h := w_i(x_i \otimes h)\) iterates associatively with the product system structure.

Like left dilations to \( E \) require that \( E \) is strongly full, right dilations to \( H \) require that \( H \) is faithful. If either condition is missing, then we speak of left and right quasi dilations.

Note that existence of a right dilation \( E^\circ \) implies that \( E^\circ \) is faithful in the sense that each \( E_i \) has faithful left action.

Without the obvious proof, we state that faithful nondegenerate representations and right dilations are equivalent concepts.

**B.18 Definition [Ske06a,Ske06e].** A right dilation of \( E^\circ \) to a faithful \( W^\ast \)–correspondence \( H \) from \( B \) to \( \mathbb{C} \) (that is, a Hilbert space with a nondegenerate normal faithful left action of \( B \)) is a family of bilinear unitaries \( w_i : E_i \otimes H \rightarrow H \), such that the product \((x_i, h) \mapsto x_i h := w_i(x_i \otimes h)\) iterates associatively with the product system structure.

**B.19 Proposition.** The relation \( \eta_i(x_i)h = w_i(x_i \otimes h) \) establishes a one-to-one correspondence between faithful nondegenerate normal representations \( \eta \) of \( E^\circ \) on \( H \) and right dilations \( w \) of \( E^\circ \) to \( H \).

**B.20 Facts.** We collect some facts about the relation between right dilations of product systems of \( W^\ast \)–correspondences, \( E_0 \)–semigroups and commutants of product systems.

1. If \( w_i \) is a right dilation of \( E^\circ \) to \( H \), then \( \vartheta^w_i(a) := w_i(id_i \otimes a)w_i^\ast \) defines an \( E_0 \)–semigroup on \( B^{bil}(H) \), the von Neumann algebra of bilinear operators on \( H \); see [MS02].

2. Conversely, if \( H \) is a faithful \( W^\ast \)–correspondence from \( B \) to \( \mathbb{C} \) (that is, a Hilbert space with a faithful, nondegenerate, and normal representation of \( B \)), and if \( \vartheta \) is a normal \( E_0 \)–semigroup on \( B^{bil}(H) \), then

\[
E_i := \{x_i \in B(H) : \vartheta_i(a)x_i = x_i a (a \in B^{bil}(H))\}
\]

is a correspondence over \( B \) with left and right action given by the left action of \( B \) on \( H \), and with \( \langle x_i, y_i \rangle \) being that unique element in \( B \) that acts on \( H \) like \( x_i^t y_i \in B^{bil}(H)' \cong B \). Moreover, \( E^\circ = (E_i)_{i \in \mathbb{R}_+} \) becomes a product system with \( u_{i,s}(x_i \otimes y_i) := x_i y_{i,s} \), and \( w_i(x_i \otimes h) := x_i h \) defines a right dilation such that \( \vartheta^w = \vartheta \).
This product system has been constructed in [Ske03a]. In the case $\mathcal{B} = \mathbb{C}$ (so that, $\mathcal{B}^{bil}(H) = \mathcal{B}(H)$), we recover Arveson’s construction [Arv89] of an Arveson system for an $E_0$–semigroup on $\mathcal{B}(H)$.

3. Considering also $G$ as correspondence from $\mathcal{B}$ to $\mathbb{C}$, the space $E' := \mathcal{B}^{bil}(G, H)$ is a $W^*$–module over $\mathcal{B}'$, acting nondegenerately on $G$ in the sense that $E'G$ is total in $H$; see Rieffel [Ric74b, Proposition 6.10], Muhly and Solel [MS02, Lemma 2.10], and Skeide [Ske03a, Ske05b]. Moreover, $\mathcal{B}'(E') = \mathcal{B}^{bil}(H)$ and $E'$ is strongly full. So, the $E_0$–semigroup $\vartheta$ on this algebra also has a product system $E'^{\otimes}$ of $W^*$–correspondences over $\mathcal{B}'$.

This product system $E'^{\otimes}$ has been introduced in [Ske03a] as the commutant system of $E^{\otimes}$. In [Ske03a] it is also explained that the two product systems used in [BS00] and in [MS02] to construct in two different ways the unique minimal weak dilation of a Markov semigroup on $\mathcal{B}'$, actually are commutants of each other.

**B.21 Remark.** We emphasize that everything about the commutant works for an arbitrary faithful nondegenerate normal representation of $\mathcal{B}$ on $G$. The commutant depends on that representation (up to Morita equivalence), but the results do not. We repeat that we introduced our convention to choose the standard representation only for the sake of uniqueness.

**B.22 Observation.** We mentioned in the beginning of Appendix B.2 that the commutant also takes left dilations to right dilations and vice versa. (In fact, the $\mathcal{B}–\mathbb{C}$–correspondence $H$ under commutant goes precisely to the $\mathbb{C}–\mathcal{B}'$–correspondence $E'$ (note: $\mathbb{C}' = \mathbb{C}$!) and vice versa. Also, strong fullness on one side corresponds to faithfulness on the other. Moreover, the commutant functor is anti-multiplicative for the tensor product. See [Ske09d, Section 9].) So, as far as existence of a algebraic right dilations is concerned, we are done, because we know about existence of left dilations.

However, the transition from left to right dilations (and back) via the commutant functor involves the construction of intertwiner spaces. But we do not know theorems asserting that bundles of intertwiner spaces have enough sufficiently continuous sections. (With this in mind, it is even more noteworthy, that Arveson, actually, proved in [Arv89, Lemma 2.3] that the intertwiner spaces for an $E_0$–semigroup on $\mathcal{B}(H)$ ($H$ separable!) do have enough continuous sections. We pose as an open problem, to see if that proof has a chance to be generalized.) This bad behavior is the reason, why we have to do right dilations separately, instead of inferring them from left dilations.

The proof of existence of a strongly continuous right dilation for a faithful strongly continuous product system of $W^*$–correspondences is, a bit surprisingly, much more similar to that for the $C^*$–case in [Ske06c], than the proof for left dilations in Appendix B.1 is to that in [Ske07a]. This mainly so, because already in the $C^*$–case we had to deal with sections in $CS_j(E^\otimes \otimes G)$,
and the few places where substituting $C_i(E^\circ)$ with $C'_i(E^\circ)$ causes some differences have already been dealt with in Appendix B.1.

In [Ske06e], the basic ingredient for the construction of a right dilation was granted by [Ske06e] Theorem 1.2. In the introduction of [Ske06e] we spent some time to explain why this theorem corresponds to the existence of a unit vector in $M_n(E_1')^\infty$ in the construction of a left dilation. Here, we just ask the reader to keep in mind the following consideration based on the duality via commutant: If in the construction of a left dilation of $E^\circ$ a unit vector in (a suitable multiple of) $E_1$ plays a crucial role, then in the construction of a right dilation this role should be played by a unit vector in (a suitable multiple of) the commutant of $E_1$, $E_1'$. This explains, roughly speaking, why we get as main ingredient an element in an intertwiner space and not in $E_1$ itself.

**B.23 Theorem.** There exists a cardinality $n$ such that $\mathcal{B}^{bil}(G^n, E_1 \otimes G^n)$ admits an isometry $\Xi'_i$.

**B.24 Remark.** Note that $\mathcal{B}^{bil}(G^n, E_1 \otimes G^n)$ is nothing but $M_n(E_1')^\infty$.

**Proof of Theorem B.23** In principle, this is the $W^*$–analogue of [Ske06e] Theorem 1.2, and proved exactly in the same way from existence of a nondegenerate faithful representation of a correspondence $E_1$. Existence of that representation is granted, in the $W^*$–case, by [Ske09d] Theorem 8.2, and [Ske09d] Observation 8.5] tells us that we may even choose that representation in the desired form. (Actually, our proof of the $C^*$–case in [Ske09d] Theorem 8.3], providing the generalization of Hirshberg’s original result [Hir05a] to the non-full case, is by reduction to the von $W^*$–case [Ske09d] Theorem 8.2].

Note that $E_1 \otimes G^n$ is nothing but $H^n_1$. In particular, $E_1 \otimes G^n = H^n_1$. By standard results on tensor products, $\Xi'_i$ gives rise to an operator $(u_{i+1} \otimes \text{id}_G)(\text{id}_i \otimes \Xi'_i) \in \mathcal{B}^{bil}(E_1 \otimes G^n, E_{i+1} \otimes G^n) = \mathcal{B}^{bil}(H^n_i, H^n_{i+1})$. By abuse of notation, we shall denote this operator by $\Xi'_i$, too. With this notation, we obtain the suggestive commutation relation $\Xi'_i x_i = x_i \Xi'_i$.

In order to avoid conflicts with the Hilbert space $G$, we shall denote elements in $G^n$ by $g = (g_s)_{s \in S}$. We apply the same convention to other Hilbert spaces like $H^n$ and $H/n$.

It now is really important to note that results like Proposition B.4 do not depend on the precise form of the representation of $\mathcal{B}$ on $G$. Therefore, Proposition B.4 remains valid if we replace $G$ with $G^n$ (and the canonical representation), and $CS_i(E^\circ \otimes G)$ with $CS^\circ_i(E^\circ \otimes G) := CS_i(E^\circ \otimes G^n)$, the space of continuous sections $h : t \mapsto h_t \in H^n_1$.

For a section $h$ of $E^\circ \otimes G^n$ we shall denote $h(t) := i_{\alpha} h_t$. Let $0 \leq a < b < \infty$. By $\int_a^b H^n_\alpha d\alpha$ we understand the norm completion of the pre-Hilbert space that consists of continuous sections $h \in CS^\circ_i(E^\circ \otimes G)$ restricted to $[a, b)$ with inner product

$$\langle h, h' \rangle_{[a, b]} := \int_a^b \langle h_\alpha, h'_\alpha \rangle d\alpha = \int_a^b \langle h(\alpha), h'(\alpha) \rangle d\alpha.$$
As in [Ske07, Proposition 4.2] or Proposition [B.7] we show:

**B.25 Proposition.** \[ \int_{a}^{b} H_{n}^{\alpha} d\alpha \]

contains the space \( R_{\{a,b\}} \) of restrictions to \([a,b)\) of those sections \( h \) for which \( t \mapsto h(t) \) is right continuous with finite jumps (by this we mean, in particular, that there exists a left limit) in finitely many points of \([a,b)\), and bounded on \([a,b)\), as a pre-Hilbert subspace.

In Appendix B.1 we considered sections that were stable under multiplication with \( \Xi_{1} \). Here we have to multiply with \( \Xi_{1}' \). Let \( S \) denote the subspace of all sections \( h = (h_{t})_{t \in \mathbb{R}_{+}} \) of \( E^{\otimes} \otimes G^{n} \) which are **locally** \( R \), that is, for every \( 0 \leq a < b < \infty \) the restriction of \( h \) to \([a,b)\) is in \( R_{\{a,b\}} \), and which are **stable** with respect to the isometry \( \Xi_{1}' \), that is, there exists an \( \alpha_{0} \geq 0 \) such that

\[
\Xi_{1}' h_{\alpha} = h_{\alpha+1} \quad (B.2)
\]

holds for all \( \alpha \geq \alpha_{0} \). By \( N \) we denote the subspace of all sections in \( S \) which are eventually 0, that is, of all sections \( h \in S \) for which there exists an \( \alpha_{0} \geq 0 \) such that \( h_{\alpha} = 0 \) for all \( \alpha \geq \alpha_{0} \). A straightforward verification shows that

\[
\langle h, h' \rangle := \lim_{m \to \infty} \int_{m}^{m+1} \langle h(\alpha), h'(\alpha) \rangle d\alpha
\]

defines a semiinner product on \( S \) and that \( \langle h, h \rangle = 0 \) if and only if \( h \in N \). Actually, we have

\[
\langle h, h' \rangle = \int_{T}^{T+1} \langle h(\alpha), h'(\alpha) \rangle d\alpha
\]

for all sufficiently large \( T > 0 \); see [Arv06, Lemma 2.1]. So, \( S/N \) becomes a pre-Hilbert space with inner product \( \langle h + N, h' + N \rangle := \langle h, h' \rangle \). By \( H \) we denote its completion.

**B.26 Proposition.** If \( h \) is in \( CS_{i}^{n}(E^{\otimes} \otimes G) \), then the shifted section

\[
\begin{cases}
0 & t < 1 \\
\Xi_{i} h_{t-1} & t \geq 1
\end{cases}
\]

is continuous for \( t \geq 1 \).

**Proof.** (Like our proof of Proposition [B.4] also this proof is a simplification compared with the proof of [Ske06, Corollary 2.4].) By Proposition [B.4] the elements in \( \text{span} CS_{i}(E^{\otimes}) \otimes G^{n} \) approximate \( t \mapsto h_{t} \) locally uniformly. So, it is enough to show the statement for sections of the form \( t \mapsto x_{t} \otimes g \) (\( x \in CS_{i}(E^{\otimes}), g \in G^{n} \)). Again by Proposition [B.4] there is a section \( h \in CS_{i}^{n}(E^{\otimes} \otimes G) \) such that \( h_{1} = \Xi_{i} g \). Once more, by Proposition [B.4] the map

\[
(t, s) \mapsto (i_{t+s} u_{t,s} \otimes \text{id}_{G^{n}}) x_{t} h_{s}
\]

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is continuous. This holds a fortiori if we fix $s = 1$. ■

From Proposition \[\text{B.26}\] we easily deduce the following analogue of \[\text{[Ske07a, Proposition 4.3]}\] or Proposition \[\text{B.9}\].

**B.27 Corollary.** For every section $h$ and every $\alpha_0 \geq 0$ define the section $h^{\alpha_0}$ as

$$h^{\alpha_0}_\alpha := \begin{cases} 0 & \alpha < \alpha_0 \\ \zeta^{\alpha_0}_\alpha h_{\alpha-\alpha} & \alpha \in [\alpha_0 + n, \alpha_0 + n + 1), n \in \mathbb{N}. \end{cases}$$

If $h$ is in $CS_j(E^G \otimes G)$, then $h^{\alpha_0}$ is in $S$. Moreover, the set $\{h^{\alpha_0} + N : h \in CS_j(E^G \otimes G), \alpha_0 \geq 0\}$ is a dense subspace of $H$.

Observe that on $H$ we have a canonical representation of $\mathcal{B}$ that acts simply pointwise on sections. By a simple application of continuity, we see that this representation is faithful, because $E^G$ is faithful. Of course, $1 \in \mathcal{B}$ acts as identity, so that the representation is nondegenerate.

**B.28 Proposition.** The canonical action of $\mathcal{B}$ on $H$ is normal. (Equivalently, $\int_a^b E^G_\alpha \, d\alpha$ is a $W^*$–correspondence!)

**Proof.** This follows from strong density in Proposition \[\text{B.9}\] by an application of Corollary \[\text{B.3}\]. (Note how important it is that Proposition \[\text{B.9}\] guaranties density and not just totality.) ■

It is now completely plain to see that for every $t \in \mathbb{R}_+$ the map $x_t \odot h \mapsto x_t h$, where

$$(x_t h)_\alpha = \begin{cases} x_t h_{\alpha-t} & \alpha \geq t, \\ 0 & \text{else}, \end{cases}$$

defines an isometry $w_t : E_t \odot H \to H$, and that these isometries iterate associatively as required for a right dilation.

**B.29 Proposition.** Each $w_t$ is surjective.

**Proof.** The proof goes presicely like that of Proposition \[\text{B.10}\], just that in all expressions that concern $z_t$ and $y$ the order must be inverted. ■

**B.30 Proposition.** The $w_t$ are strongly continuous in the sense that for every section $x \in CS_j(E^G)$ and every $h \in H$ the function $t \mapsto x_t h$ is continuous.

**Proof.** The proof goes like that of Proposition \[\text{B.11}\]. (Expressions like $X^{\alpha_0} y_t$ or $X^{\alpha_0}_n y_t$ must be replaced with expressions like $y_t \odot x^{\alpha_0}$ or $y_t x^{\alpha_0}_n$. Actually, the proof here is quite a bit simpler,
because now the preparation in the first two paragraphs of that proof is no longer necessary, and we may immediately start showing the continuity statement only for section \( x^{(n)} \). ■

We summarize:

**B.31 Theorem.** Let \( E^\circ \) be a faithful strongly continuous product system of \( W^\ast \)–correspondences. Then \( E^\circ \) admits a strongly continuous right dilation.

**B.32 Remark.** Also here the construction preserves countability hypotheses: If \( \mathcal{B} \) has separable pre-dual and if \( E^\circ \) is countably generated, then \( H \) is separable.

We mentioned already the following consequence of Theorem [B.31]

**B.33 Theorem.** If \( E^\circ \) is a faithful strongly continuous product system of \( W^\ast \)–correspondences over \( \mathcal{B} \), then its commutant system \( E'^\circ \) is the product system of the the strongly continuous \( E_0 \)–semigroup \( \vartheta^w \) on \( \mathcal{B}^\circ(E') = \mathcal{B}^{bdl}(H) \). Therefore, \( E'^\circ \) possesses a strongly continuous structure.

**B.34 Theorem.** The commutant is a duality between strongly full and faithful strongly continuous product systems of correspondences over \( \mathcal{B} \) and strongly full and faithful strongly continuous product systems of correspondences over \( \mathcal{B}' \).

**Proof.** (Sketch.) A detailed proof must show that under bicommutant also the strongly continuous structure is recovered. This includes a proof of the fact that all constructions go through, also if we replace the standard representation of \( \mathcal{B} \) by a more general faithful normal nondegenerate representation on \( G \) (and, therefore, \( \mathcal{B}' \cong \mathcal{B}^{op} \) by a possibly different commutant). The proof that the continuous structure does not depend on that choice invokes Theorem [B.5]. We do not give details. ■

**B.35 Remark.** The theorem holds, in particular, for spatial strongly continuous product systems, which are strongly full and faithful, automatically, because they contain the trivial product system. But, of course, this can be seen much more easily, since in this case both \( E^\circ \) and \( E'^\circ \) contain a (central) unital unit. (In an interpretation as von Neumann correspondences, \( E_i \) and \( E'_i \) are subsets of \( \mathcal{B}(G, H_i) \) having the same space of central elements \( E_i \cap E'_i \). The reference unit \( \omega^\circ \) of \( E^\circ \) is, in fact, also a unit for \( E'^\circ \). Yet in other words: The commutant of a spatial product system is spatial, and this duality includes the continuous structure.)

Precisely as the \( C^\ast \)–case in [Ske06e, Theorem 3.1], we provide a completely different proof of the following result due to Arveson and Kishimoto [AK92].
B.36 Theorem. Let $E$ be a strongly full $W^*$–module over a $W^*$–algebra $B$ (for instance, let $E = B$ itself!) and let $\vartheta$ be a faithful strongly continuous normal $E_0$–semigroup on $B^a(E)$ (for instance $B$ itself if $E = B$). Then there exists a faithful $W^*$–correspondence $K$ from $B^a(E)$ to $\mathbb{C}$ with strict left action (that is, a Hilbert space with a faithful nondegenerate normal representation of $B^a(E)$) and a strongly continuous unitary group $u$ on $K$ such that $\vartheta_t(a)k = u_t au_t^*k$ for all $a \in B^a(E), t \in \mathbb{R}_+, k \in K$.

An elementary dilation of a CP-semigroup $T$ on $B$ is a $C^*$–algebra $A$ with an embedding $\varphi: B \rightarrow A$ and a semigroup $c = (c_t)_{t \in \mathbb{R}_+}$ of elements in $A$ such that

$$\varphi \circ T(b) = c_t^* \varphi(b) c_t$$

for all $b \in B$ and $t \in \mathbb{R}_+$. A CP-semigroup $T$ is semifaithful if it GNS-correspondence is faithful. (From this, it follows that an arbitrary weak dilation $\vartheta$ of $T$ is faithful.) Putting together Theorems B.33 and A.4, we get the following $W^*$–analogue of [Ske06e, Theorem 3.4]. (Note that the strong continuity condition here is referring to the strong operator topology, and is much weaker than the one in the $C^*$–case.)

B.37 Theorem. Every semifaithful strongly continuous normal CP-semigroup on a $W^*$–algebra admits a (strongly continuous) elementary dilation to some $B(H)$ with normal embedding $B \rightarrow B(H)$.

References

[Acc78] L. Accardi, On the quantum Feynman-Kac formula, Rendiconti Sem. Mat. e Fis. di Milano 48 (1978), 135–179.

[AK92] W. Arveson and A. Kishimoto, A note on extensions of semigroups of $*$–endomorphisms, Proc. Amer. Math. Soc. 116 (1992), 169–774.

[Ale04] A. Alevras, One parameter semigroups of endomorphisms of factors of type $II_1$, J. Operator Theory 51 (2004), 161–179.

[AM96] L. Accardi and A. Mohari, On the structure of classical and quantum flows, J. Funct. Anal. 135 (1996), 421–455.

[Arv89] W. Arveson, Continuous analogues of Fock space, Mem. Amer. Math. Soc., no. 409, American Mathematical Society, 1989.

[Arv90] , Continuous analogues of Fock space IV: essential states, Acta Math. 164 (1990), 265–300.

[Arv97] , The index of a quantum dynamical semigroup, J. Funct. Anal. 146 (1997), 557–588.
[Arv06] ______, On the existence of $E_0$–semigroups, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 315–320.

[AS07] G. Abbaspour and M. Skeide, Generators of dynamical systems on Hilbert modules, Commun. Stoch. Anal. 1 (2007), 193–207, (arXiv: math.OA/0611097).

[AT05] M. Anoussis and I. Todorov, Compact operators on Hilbert modules, Proc. Amer. Math. Soc. 133 (2005), 257–261.

[BBLS04] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide, Type I product systems of Hilbert modules, J. Funct. Anal. 212 (2004), 121–181, (Preprint, Cottbus 2001).

[BGR77] L.G. Brown, P. Green, and M.A. Rieffel, Stable isomorphism and strong Morita equivalence of $C^*$–algebras, Pacific J. Math. 71 (1977), 649–663.

[Bha96] B.V.R. Bhat, An index theory for quantum dynamical semigroups, Trans. Amer. Math. Soc. 348 (1996), 561–583.

[Bla06] B. Blackadar, Operator algebras, Encyclopaedia of Matematical Sciences, no. 122 (number III in the subseries Operator Algebras and Non-Commutative Geometry), Springer, 2006.

[Ble97] D.P. Blecher, A new approach to Hilbert $C^*$–modules, Math. Ann. 307 (1997), 253–290.

[BLS10] B.V.R. Bhat, V. Liebscher, and M. Skeide, Subsystems of Fock need not be Fock: Spatial CP-semigroups, 2010, Proc. Amer. Math. Soc. electronically Feb 2010, (arXiv: 0804.2169v2).

[BS00] B.V.R. Bhat and M. Skeide, Tensor product systems of Hilbert modules and dilations of completely positive semigroups, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000), 519–575, (Rome, Volterra-Preprint 1999/0370).

[CF98] A.M. Chebotarev and F. Fagnola, Sufficient conditions for conservativity of minimal quantum dynamical semigroups, J. Funct. Anal. 153 (1998), 382–404.

[Eff65] E.G. Effros, The Borel structure of von Neumann algebras on a separable Hilbert space, Pac. J. Math. 15 (1965), 1153–1164.

[FLS10] F. Fagnola, V. Liebscher, and M. Skeide, Product systems of Ornstein-Uhlenbeck processes (tentative title), Preprint, in preparation, 2010.

[GS99] D. Goswami and K.B. Sinha, Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra, Commun. Math. Phys. 205 (1999), 377–403.

[Hir04] I. Hirshberg, $C^*$–Algebras of Hilbert module product systems, J. Reine Angew. Math. 570 (2004), 131–142.

[Hir05a] ______, Essential representations of $C^*$–correspondences, Int. J. Math. 16 (2005), 765–775.

[Hir05b] ______, On the universal property of Pimsner-Toeplitz $C^*$–algebras and their continuous analogues, J. Funct. Anal. 219 (2005), 21–33.
[HKK04] J. Hellmich, C. Köstler, and B. Kümmerer, *Noncommutative continuous Bernoulli sifts*, Preprint, arXiv: math.OA/0411565, 2004.

[HP57] E. Hille and R.S. Phillips, *Functional analysis and semi-groups*, American Mathematical Society, 1957.

[HP84a] R.L. Hudson and K.R. Parthasarathy, *Quantum Ito’s formula and stochastic evolutions*, Commun. Math. Phys. 93 (1984), 301–323.

[HP84b] , *Stochastic dilations of uniformly continuous completely positive semi-groups*, Acta Appl. Math. 2 (1984), 353–378.

[Kas80] G.G. Kasparov, *Hilbert C∗–modules, theorems of Stinespring & Voiculescu*, J. Operator Theory 4 (1980), 133–150.

[Lan95] E.C. Lance, *Hilbert C∗–modules*, Cambridge University Press, 1995.

[Lie09] V. Liebscher, *Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces*, Mem. Amer. Math. Soc., no. 930, American Mathematical Society, 2009, (arXiv: math.PR/0306365).

[LS01] V. Liebscher and M. Skeide, *Units for the time ordered Fock module*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 4 (2001), 545–551, (Rome, Volterra-Preprint 2000/01).

[MS00] P.S. Muhly and B. Solel, *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc. 81 (2000), 113–168.

[MS02] , *Quantum Markov processes (correspondences and dilations)*, Int. J. Math. 51 (2002), 863–906, (arXiv: math.OA/0203193).

[MS04] , *Hardy algebras, W∗–correspondences and interpolation theory*, Math. Ann. 330 (2004), 353–415, (arXiv: math.OA/0308088).

[MS07a] D. Markiewicz and O.M. Shalit, *Continuity of CP-semigroups in the point-strong operator topology*, Preprint, arXiv: 0711.0111v1, 2007, To appear in J. Operator Theory.

[MS07b] P.S. Muhly and B. Solel, *Quantum Markov semigroups (product systems and subordination)*, Int. J. Math. 18 (2007), 633–669, (arXiv: math.OA/0510653).

[MSS06] P.S. Muhly, M. Skeide, and B. Solel, *Representations of B∗(E)*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 47–66, (arXiv: math.OA/0410607).

[Rie74a] M.A. Rieffel, *Induced representations of C∗–algebras*, Adv. Math. 13 (1974), 176–257.

[Rie74b] , *Morita equivalence for C∗–algebras and W∗–algebras*, J. Pure Appl. Algebra 5 (1974), 51–96.

[Ske00a] M. Skeide, *Generalized matrix C∗–algebras and representations of Hilbert modules*, Mathematical Proceedings of the Royal Irish Academy 100A (2000), 11–38, (Cottbus, Reihe Mathematik 1997/M-13).
[Ske00b] ______, Quantum stochastic calculus on full Fock modules, J. Funct. Anal. 173 (2000), 401–452, (Rome, Volterra-Preprint 1999/0374).

[Ske01] ______, Hilbert modules and applications in quantum probability, Habilitationsschrift, Cottbus, 2001, Available at http://www.math.tu-cottbus.de/INSTITUT/lswas/skeide.html.

[Ske02] ______, Dilations, product systems and weak dilations, Math. Notes 71 (2002), 914–923.

[Ske03a] ______, Commutants of von Neumann modules, representations of $\mathcal{B}(E)$ and other topics related to product systems of Hilbert modules, Advances in quantum dynamics (G.L. Price, B.M. Baker, P.E.T. Jorgensen, and P.S. Muhly, eds.), Contemporary Mathematics, no. 335, American Mathematical Society, 2003, (Preprint, Cottbus 2002, arXiv: math.OA/0308231), pp. 253–262.

[Ske03b] ______, Dilation theory and continuous tensor product systems of Hilbert modules, Quantum Probability and Infinite Dimensional Analysis (W. Freudenberg, ed.), Quantum Probability and White Noise Analysis, no. XV, World Scientific, 2003, (Preprint, Cottbus 2001), pp. 215–242.

[Ske04a] ______, Independence and product systems, Recent developments in stochastic analysis and related topics (S. Albeverio, Z.-M. Ma, and M. Röckner, eds.), World Scientific, 2004, (arXiv: math.OA/0308245), pp. 420–438.

[Ske04b] ______, Unit vectors, Morita equivalence and endomorphisms, Preprint, arXiv: math.OA/0412231v4 (Version 4), 2004.

[Ske05a] ______, Lévy processes and tensor product systems of Hilbert modules, Quantum Probability and Infinite Dimensional Analysis — From Foundations to Applications (M. Schürmann and U. Franz, eds.), Quantum Probability and White Noise Analysis, no. XVIII, World Scientific, 2005, pp. 492–503.

[Ske05b] ______, Von Neumann modules, intertwiners and self-duality, J. Operator Theory 54 (2005), 119–124, (arXiv: math.OA/0308230).

[Ske06a] ______, A simple proof of the fundamental theorem about Arveson systems, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 305–314, (arXiv: math.OA/0602014).

[Ske06b] ______, Commutants of von Neumann correspondences and duality of Eilenberg-Watts theorems by Rieffel and by Blecher, Quantum probability (M. Bozejko, W. Młotkowski, and J. Wysoczanski, eds.), Banach Center Publications, vol. 73, Polish Academy of Sciences — Institute of Mathematics, 2006, (arXiv: math.OA/0502241), pp. 391–408.

[Ske06c] ______, Existence of $E_0$–semigroups for Arveson systems: Making two proofs into one, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 373–378, (arXiv: math.OA/0605480).

[Ske06d] ______, Generalized unitaries and the Picard group, Proc. Ind. Ac. Sc. (Math Sc.) 116 (2006), 429–442, (arXiv: math.OA/0511661).
[Ske06e] _____, Nondegenerate representations of continuous product systems, Preprint, arXiv: math.OA/0607362, To appear in J. Operator Theory, 2006.

[Ske06f] _____, The index of (white) noises and their product systems, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 617–655, (Rome, Volterra-Preprint 2001/0458, arXiv: math.OA/0601228).

[Ske07a] _____, $E_0$–semigroups for continuous product systems, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 10 (2007), 381–395, (arXiv: math.OA/0607132).

[Ske07b] _____, Spatial $E_0$–semigroups are restrictions of inner automorphismgroups, Quantum Probability and Infinite Dimensional Analysis — Proceedings of the 26th Conference (L. Accardi, W. Freudenberg, and M. Schürmann, eds.), Quantum Probability and White Noise Analysis, no. XX, World Scientific, 2007, (arXiv: math.OA/0509323), pp. 348–355.

[Ske08a] _____, Isometric dilations of representations of product systems via commutants, Int. J. Math. 19 (2008), 521–539, (arXiv: math.OA/0602459).

[Ske08b] _____, Spatial Markov semigroups admit Hudson-Parthasarathy dilations, Preprint, arXiv: 0809.3538v1, 2008.

[Ske08c] _____, The Powers sum of spatial CPD-semigroups and CP-semigroups, Preprint, arXiv: 0812.0077v1, 2008, To appear in Banach Center Publications.

[Ske09a] _____, An introduction, Mini-workshop: Product systems and independence in quantum dynamics (B.V.R. Bhat, U. Franz, and M. Skeide, eds.), Oberwolfach Reports, no. 09/2009, Mathematisches Forschungsinstitut Oberwolfach, 2009, pp. 497–505.

[Ske09b] _____, $E_0$–Semigroups for continuous product systems: The nonunital case, Banach J. Math. Anal. 3 (2009), 16–27, (arXiv: 0901.1754v1).

[Ske09c] _____, Free product systems, Mini-workshop: Product systems and independence in quantum dynamics (B.V.R. Bhat, U. Franz, and M. Skeide, eds.), Oberwolfach Reports, no. 09/2009, Mathematisches Forschungsinstitut Oberwolfach, 2009, pp. 528–530.

[Ske09d] _____, Unit vectors, Morita equivalence and endomorphisms, Publ. Res. Inst. Math. Sci. 45 (2009), 475–518, (arXiv: math.OA/0412231v5 (Version 5)).

[Ske10a] _____, Dilations of product sytems and commutants of von Neumann modules (tentative title), Preprint, in preparation, 2010.

[Ske10b] _____, Free product systems generated by spatial tensor product systems, Preprint, in preparation, 2010.

[Tak03] M. Takesaki, Theory of operator algebras II, Encyclopaedia of Matematical Sciences, no. 125 (number VI in the subseries Operator Algebras and Non-Commutative Geometry), Springer, 2003.

[Tsi98] B. Tsirelson, Unitary Brownian motions are linearizable, Preprint, arXiv: math.PR/9806112, 1998.

[Tsi03] _____, Scaling limit, noise, stability, Preprint, arXiv: math.PR/0301237, 2003.