Combinatorial matrices

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Abstract. In this paper, we explore the algebraic structure of the set of combinatorial matrices, which is characterized as the $n \times n$ matrices induced by the linear combination of the identity matrix of order $n$ and the $n \times n$ matrix of 1's. We also explore some of its matrix properties such as its diagonalizability, its determinant, and its inverse. Lastly, we give a generalization on the error-correcting capability of twisted centralizer codes obtained by fixing a combinatorial matrix.

1. Introduction
Let $I_n$ be the identity matrix of order $n$ and $J_n$ be the square matrix of order $n$ whose entries are all equal to 1. A combinatorial matrix $A$, as defined in [1], is a square matrix of the form

$$A := xJ_n + yI_n = \begin{bmatrix} x+y & x & \cdots & x \\ x & x+y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x+y \end{bmatrix}.$$

In this paper, we denote the set of all combinatorial matrices over a field $\mathbb{F}$ by $(CM)_n$, the set of all matrices over $\mathbb{F}$ as $M_n(\mathbb{F})$ and the set of all invertible matrices as $G_n(\mathbb{F})$. We’ll use the conventional notations to denote groups, rings, and vector spaces.

The paper is organized as follows. In section 2, we’ll give the algebraic structures induced by combinatorial matrices as well as some of its properties. Lastly, section 3 will focus on the application of combinatorial matrix in coding theory, particularly by giving a generalization of a result in [4] about the error-correcting capability of the twisted centralizer code obtained from fixing any combinatorial matrix over a finite field $\mathbb{F}$.
2. Algebraic Structure and Some Properties of \((CM)_n\)

This section gives an in-depth discussion of the algebraic structure of the set of combinatorial matrices over \(\mathbb{F} = \mathbb{R}\), where \(\mathbb{R}\) is the set of all real numbers. We first start by showing that \((CM)_n\) forms a group over the binary operation \(\oplus\) and is in fact abelian. Note that in this section, we denote \(\oplus\) as the usual matrix addition.

**Theorem 2.1**
The ordered pair \(\langle (CM)_n, \oplus \rangle\) forms an abelian group.

**Proof**
Let \(A, B, C \in (CM)_n\). Specifically, let \(A = x_0 J_n \oplus y_0 I_n\), \(B = x_1 J_n \oplus y_1 I_n\), and \(C = x_2 J_n \oplus y_2 I_n\) for some \(x_i, y_i \in \mathbb{F}\), \(i = 1, 2, 3\). Then \(A \oplus B = (x_0 + x_1) J_n \oplus (y_0 + y_1) I_n\) must also be in \((CM)_n\) hence the set is closed. The commutativity and associativity of \(\oplus\) in \((CM)_n\) follows trivially from the additive group of \(\mathbb{F}\). The null combinatorial matrix is the identity element in \((CM)_n\). Trivially, for any \(A \in (CM)_n\), the inverse of \(A\) is given by \(-A := (-x_0) J_n \oplus (-y_0) I_n\). Hence, \(\langle (CM)_n, \oplus \rangle\) is an abelian group.

Let \( (CM)_n^+ := \{ x J_n \oplus y I_n \in (CM)_n \mid (x, y) \neq (i, 0), (i, -in) \forall i \in \mathbb{F} - \{0\} \} \). The next theorem shows that \((CM)_n^+\) forms a subset of \(G_n(\mathbb{F})\). The theorem also gives us the closed form for the determinant of any combinatorial matrix \(A\) in \((CM)_n^+\).

**Theorem 2.2**
\((CM)_n^+\) forms a subset of \(G_n(\mathbb{F})\).

**Proof**
Recall that \(G_n(\mathbb{F}) := \{ B \in M_n(\mathbb{F}) \mid det(B) \neq 0 \}\). Note that \((CM)_n^+\) is a non-empty set because \(I_n = 0 J_n \oplus I_n \in (CM)_n^+\). We need to show that \((CM)_n^+ \subset G_n(\mathbb{F})\). Let \(A := x J_n \oplus y I_n\) be in \((CM)_n\). Taking \(det(A)\), we have

\[
\begin{vmatrix}
  x + y & x & x & \cdots & x \\
  x & x + y & x & \cdots & x \\
  x & x & x + y & \cdots & x \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x & x & x & \cdots & x + y \\
\end{vmatrix}
\]

We then subtract column 1 from columns 2 to \(n\). From [2] and [3], this will have no effect on the value of the determinant of \(A\). Similarly after subtracting column 1 from columns 2 to \(n\), adding rows 2 to \(n\) to row 1 will deal no effect on the value of the determinant of \(A\). Hence,
Note that in the determinant of the triangular matrix above. Therefore, \( \det(A) = y^{n-1}(xn+y) \). Note that \( \det(A) \neq 0 \) for \( A \in (CM)^n \) because of our choice of \( y \). Therefore, \( A \in G_n(\mathbb{F}) \) and \( (CM)^n \subset G_n(\mathbb{F}) \). ■

**Corollary 2.3**

The spectrum of any \( A \in (CM)^n \) is given by the set \( S := \{xn+y,y\} \), with multiplicities of 1 and \( n-1 \) respectively.

**Proof**

The \( n-1 \) row operations and \( n-1 \) column operations done in matrix \( A \) in Theorem 2.2 could be written as a product of elementary matrices respectively. Denote by \( R_i \) the elementary matrix corresponding to subtracting row \( i \) to the \( i-th \) row of \( A \) and \( C_j \) the elementary matrix corresponding to adding column \( j \) to the \( j-th \) column of \( A \). It can be noticed that \( R_i^{-1} = C_j \) whenever \( i = j \). Hence, letting a diagonal matrix \( D := \text{diag}(xn+y,y,y,\ldots,y) \) with \( n-1 \) \( y \)'s, the row and column operations done to combinatorial matrix \( A \) could mathematically be expressed as \( (R_{n-1} \circ R_{n-2} \circ \cdots \circ R_2 \circ R_1) \circ A \circ (C_1 \circ C_2 \circ \cdots \circ C_{n-2} \circ C_{n-1}) = D \), therefore \( PAP^{-1} = D \) which could further be expressed as \( A = P^{-1}DP \) where \( P = (R_{n-1})(R_{n-2})\cdots(R_2)(R_1) \). Hence, the only possible eigenvalues for \( A \) are \( xn+y \) and \( y \), with multiplicities of 1 and \( n-1 \) respectively. ■

Let \( \circ \) denote matrix multiplication. Intuitively, we can conclude that all \( A \in (CM)^n \) have inverses based from Theorem 2.2. The next theorem shows that \( (CM)^n \) forms an abelian group with respect to the binary operation \( \circ \). Moreover, since we are restricting \( (x,y) \neq (i,0),(i,-in) \forall i \in \mathbb{F} - \{0\} \), the theorem below also gives us the closed form for \( A^{-1} \) for all \( A \in (CM)^n \).

**Theorem 2.4**

The ordered pair \( ((CM)^n,\circ) \) forms an abelian group.

**Proof**

It is trivial that \( I_n \) serves as the identity element of \( (CM)^n \) with respect to \( \circ \) since \( A \circ I_n = A \) for any \( A \in (CM)^n \). Hence, we only need to show that \( (CM)^n \) is closed under \( \circ \), \( \circ \) is associative in \( (CM)^n \), and that for any \( A \in (CM)^n \), \( A^{-1} \in (CM)^n \). Let \( A, B, C \in (CM)^n \subset (CM)^n \). We use the same notation for the entries of matrices \( A \), \( B \), \( C \) as in Theorem 2.1. Notice that by taking \( x_3 = nx_0x_1 + y_0x_1 + x_0y_1 \) and \( y_3 = y_0x_1 \), then \( A \circ B := x_3J_n \oplus y_3I_n \in (CM)^n \), hence the set is closed. Note that \( B \circ C = x_4J_n \oplus y_4I_n \) where \( x_4 = nx_1x_2 + y_1x_2 + x_1y_2 \) and \( y_4 = y_1y_2 \). Hence, it can be seen
that \((A \odot B) \odot C = A \odot (B \odot C)\). Therefore \(\odot\) is associative in \((CM)_n^*\). Take \(A \in (CM)_n^*\). Notice that \(J_n^2 = nJ_n\). Hence,

\[
(y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) = y_0^2I_n \oplus (-nx_0^2)J_n
\]

\[
\Leftrightarrow (y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) \oplus nx_0^2J_n = y_0^2I_n
\]

\[
\Leftrightarrow (y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) \oplus nx_0I_n \oplus x_0J_n = y_0^2I_n
\]

\[
\Leftrightarrow (y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) \oplus nx_0I_n \oplus x_0J_n = y_0^2I_n
\]

\[
\Leftrightarrow (y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) \oplus nx_0I_n \oplus x_0J_n = y_0^2I_n
\]

\[
\Leftrightarrow (y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) \oplus nx_0I_n \oplus x_0J_n = y_0^2I_n
\]

\[
\Leftrightarrow (y_0I_n \oplus x_0J_n) \odot (y_0I_n \oplus (-x_0)J_n) \oplus nx_0I_n \oplus x_0J_n = y_0^2I_n
\]

\[
\Leftrightarrow A \odot (y_0^{-1}I_n \oplus (-x_0)(y_0(x_0n + y_0))^{-1})J_n = I_n.
\]

We found a closed form for \(A^{-1}\), which is also in \((CM)_n^*\). Therefore, \((CM)_n^*, \odot\) forms a group. The proof for the group being abelian is easy and is therefore left as an exercise for the reader.  

**Corollary 2.5**

The set \((CM)_n\) forms a commutative monoid with respect to the binary operation \(\odot\).

**Proof**

Let \(\{iJ_n\}\) be the set of scalar multiples of \(J_n\) for all \(i \in \mathbb{R} \setminus \{0\}\). Note that \((CM)_n\) includes the set \(\{iJ_n\}\). Say, \(i = 1\). Then we have \(J_n \in \{iJ_n\}\) which is singular because its RREF contains a row of zeroes. All other conditions for an abelian group are satisfied by Theorem 2.4.  

**Corollary 2.6**

The triple \(\langle (CM)_n, \oplus, \odot \rangle\) forms a commutative matrix ring.

**Proof**

Theorem 2.1 and Corollary 2.5 showed that the two ring properties are satisfied by the triple \(\langle (CM)_n, \oplus, \odot \rangle\). We only need to show that the distributive property of multiplication over addition holds. Let \(A, B, C \in (CM)_n\) be the same matrices in Theorem 2.1. Note that

\[
(A \oplus B) \odot C = [(nx_2 + y_2)(x_0 + x_i) + x_2(y_0 + y_i)]J_n \odot y_2(y_0 + y_i)J_n = A \odot C \oplus B \odot C.
\]

Similarly, \(A \odot (B \oplus C) = A \odot B \oplus B \odot C\). Hence, left and right distributive property of matrix multiplication over matrix addition holds. Note that the commutativity of \(\odot\) in \((CM)_n\) follows from Corollary 2.5. Therefore, \(\langle (CM)_n, \oplus, \odot \rangle\) forms a commutative matrix ring.  

**Corollary 2.7**

The triple \(\langle (CM)_n^*, \oplus, \odot \rangle\) forms a matrix field.

**Proof**

It is trivial that \(\langle (CM)_n^*, \oplus, \odot \rangle\) forms an abelian group. From Theorem 2.4, it must be noted that \(\odot\) is both associative and commutative in \((CM)_n^*\). The proof for distributive property of matrix
multiplication over matrix addition is similar to that of Corollary 2.6. Because of our choice of \( y \) for \((CM)^*_n\), note that \( I_n \) also serves as its unity. Hence, \( (CM)^*_n, \oplus, \odot \) is a commutative matrix ring with unity. From Theorem 2.4, there exists a matrix \( A^{-1} \in (CM)^*_n \) such that \( A \odot A^{-1} = I_n, \forall A \in (CM)^*_n \). Hence, \( (CM)^*_n, \oplus, \odot \) is a matrix field. ■

### 3. Application of Combinatorial Matrices

This section presents the application of the results of this paper particularly in coding theory. In 2017, a new family of linear codes was established in [4] by fixing an arbitrary matrix \( A \in \mathbb{F}_q^m \) for some finite field \( q \) and collecting all matrices \( B \in M_n(\mathbb{F}_q) \) such that \( AB = aBA \). We denote this particular collection of such \( B \)'s as \( C(A,a) \), read as the centralizer of \( A \), twisted by \( a \). Note that in this section, usual notations for matrix addition, matrix multiplication, and scalar multiplication will be used.

**Definition**

The centralizer of \( A \), twisted by \( a \), is defined as the set

\[
C(A,a) := \{ B \in M_n(\mathbb{F}_q) \mid AB = aBA \}
\]

We view \( C(A,a) \) as a code whose codewords are the column vectors obtained by concatenating each column of \( B \in C(A,a) \). We denote each column vector obtained from \( B \in C(A,a) \) as \([ B ]\). We do not distinguish twisted centralizer codes to the set \( C(A,a) \) hence we use the same notation for such code. Clearly, by construction of its codewords, it has parameters \([n^2,k,d]\). It was shown in [4] that by fixing the arbitrary matrix \( J_n + I_n \), \( C(J_n + I_n, a) \) attains a minimum distance equivalent to \( n^2 \). The next theorem generalizes the result of [4] on the error-correcting capability of \( C(A,a) \) whenever we fix any \( A = xJ_n + yI_n \in M_n(\mathbb{F}_q) \) for any ordered pair \((x, y) \neq (0,0), \forall x, y \in \mathbb{F}_q \). Note that in this section, \( \{ J_n \} \) denotes the additive group of \( J_n \).

**Theorem 3.1**

Suppose \( \text{char}(\mathbb{F}_q) \mid xn + y \). If \( a \neq 0,1 \), then the twisted centralizer code \( C(xJ_n + yI_n, a) \) is equal to \( \{ J_n \} \) and has parameters \([n^2,1,n^2]\).

**Proof**

We begin by noting that \( \text{char}(\mathbb{F}_q) = p \), where \( p \) is a prime number, and note that \( p \mid xn + y \). By Corollary 2.3, it must be noted that \( A = xJ_n + yI_n \) must have an eigenvalue given by the set \( S := \{0,y\} \), with multiplicities \( 1 \) and \( n-1 \) respectively. Hence, \( A \) must be equivalent to a certain diagonal matrix \( D := (0,y,y,...,y,y) \) with \( n-1 \) \( y \)'s. Let \( E_{11} \) be the \( n \times n \) matrix whose \( (1,1) \) entry is \( 1 \) and all other entries are \( 0 \). It can be seen that \( D = yI_n - yE_{11} \). Let \( B \in C(D,a) \). Then

\[
DB = aBD
\]

\[
(yI_n - yE_{11})B = aB(yI_n - yE_{11})
\]

\[
y(1-a)B = yE_{11}B - ayBE_{11}.
\]
We write matrices $E_{11}$ and $B$ as matrices of column and row vectors, specifically,

$$
E_{11} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
e_1 & \ddots & \ddots & \ddots & 0 \\
1 & 0 & \ddots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
- & - & e_1 & - & - \\
0 & 0 & \cdots & 0 & 0 \\
: & : & \ddots & : & : \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
v_1 & v_2 & \cdots & v_{n-1} & v_n
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2 \\
: \\
u_{n-1} \\
u_n
\end{bmatrix}
$$

where $e_1$ is the element of the standard basis of $\mathbb{R}^n$ whose 1st entry is 1 and the rest are zeroes, and $v_i, u_i$ are column vectors and row vectors of length $n$ respectively for $i = 1, 2, \ldots, n-1, n$. Hence, it can be seen that

$$
E_{11}B = \begin{bmatrix}
- & - & u_1 & - & - \\
0 & 0 & \cdots & 0 & 0 \\
: & : & \ddots & : & : \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
$$

and

$$
BE_{11} = \begin{bmatrix}
v_1 & \ddots & \ddots & \ddots & \\
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots \\
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0
\end{bmatrix}
$$

We compare the entries of $y(1-a)B$ and $yE_{11}B - ayBE_{11}$. Recall that we have the assumption that $a \neq 0, 1$. Let $T = yE_{11}B - ayBE_{11}$. Let $t_{ij}, b_{ij}$ denote the $(i, j)$ -th entry of $T$ and $B$, respectively. Note that the first entries of $v_i$ and $u_i$ must be the same, and is equivalent to $b_{11}$. Hence it can be seen that $t_{11}$ always agrees with $y(1-a)b_{11}$. Let $(u_i)_j$ denote the $j$ -th entry of row vector $u_i$, $j = 1, 2, \ldots, n-1, n$. For $j > 1$, we actually have $t_{ij} = y(u_i)_j = yb_{ij}$. Thus, $t_{1j} = y(1-a)b_{1j}$ if and only if $b_{ij} = 0$. Hence, $u_i$ must be a multiple of $e_1$. Let $(v_i)_i$ denote the $i$ -th entry of column vector $v_i$, $i = 1, 2, \ldots, n-1, n$. For $i > 1$, we have $t_{ii} = -ay(v_i)_i = -y(b_{ii}$ and hence $t_{ii} = y(1-a)b_{ii}$ if and only if $b_{ii} = 0$. Thus $v_i$ must also be a multiple of $e_1$. Of course, for $i, j > 1$, we have
t_y = 0 = b_y. Therefore, B must be a multiple of \( E_{11} \), i.e., \( C(D, a) = (E_1) \). Thus, \( \dim \{ C(D, a) \} = 1 \).

Note that \( A \) and \( D \) are similar matrices thus for every \( B \in C(D, a) \), \( DB - aBD = 0 \). implies that \( A(PBP^{-1}) = a(PBP^{-1})A \). Hence, every element in \( C(A, a) \) are the conjugates of each \( B \in C(D, a) \) and therefore, \( C(A, a) \) must also be 1-dimensional. Notice that \( J_n A = A J_n = (xn + y)J_n = 0 \). Thus, \( C(A, a) = \langle J_n \rangle \). Therefore, every non-zero element of \( C(A, a) \) is of the form \( bJ_n \) for \( b \neq 0 \). Hence, \( C(A, a) \) has minimum distance equal to \( n^2 \). Therefore, \( C(A, a) \) has parameters \([n^2, 1, n^2] \), as desired.

The result above implies that fixing any combinatorial matrix \( A \) with the condition that the characteristic of \( \mathbb{F}_q \) divides \( xn + y \) will always give us a code whose elements are \( \{bJ_n\}, \forall b \in \mathbb{F}_q \), and will always give us an maximum distance separable (MDS) code. Recall that MDS codes are known to attain the Singleton bound hence are good error-correcting codes. However, the result above also implies that the twisted centralizer code generated by \( A \) will have a relatively low information rate which is only at \((n^2)^{-1}\).

4. Conclusions
In this paper, it was known that \((CM)_n \) induces a group over matrix addition. However, it only forms a monoidal structure over matrix multiplication since it contains the set \( \{iJ_n\} \). Moreover, the set itself forms a commutative matrix ring with unity and is also a vector space over \( \mathbb{F} = \mathbb{R} \). We also give the closed form for the determinant of any combinatorial matrix as well as the closed form for the inverse of a combinatorial matrix \( A = xJ_n + yJ_n \) where \( (x, y) \neq (i, 0), (i, -i), \forall i \in \mathbb{F} - \{0\} \). Since a combinatorial matrix is a symmetric matrix, then it is straightforward that it is diagonalizable. It was found out that the only possible eigenvalues for a combinatorial matrix is given by the set \( S := \{xn + y, y\} \), with \( xn + y \) and \( y \) having multiplicities of 1 and \( n-1 \) respectively. Lastly, this paper improved the result in [4] on the twisted centralizer code obtained from fixing a combinatorial matrix \( A \). In particular, it was found out that any \( A \in (CM)_n \subset M_n(\mathbb{F}_q) \) gives us an MSD code, but with a relatively very low information rate.

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References
[1] Definition: Combinatorial Matrix. (2018, February 25). Retrieved from https://proofwiki.org/wiki/Definition:Combinatorial\_Matrix
[2] Knuth, D. E. (1997). The Art of Computer Programming, Vol. 1: Fundamental Algorithms. Reading, Mass.: Addison-Wesley. ISBN: 0201896834 9780201896831
[3] Humphreys, J. F. (1996). A course in group theory. Oxford University Press, Oxford ; New York
[4] Adel Alahmadi, S.P. Glasby, Cheryl E. Praeger, Patrick Solé, Bahattin Yildiz, Twisted centralizer codes, Linear Algebra and its Applications, Volume 524, 2017, Pages 235-249, ISSN 0024-3795, https://doi.org/10.1016/j.laa.2017.03.011.
[5] Symmetric matrix. Encyclopedia of Mathematics. (2018, December 1). Retrieved from http://www.encyclopediaofmath.org/index.php?title=Symmetric\_matrix&oldid=43512