On the local well–posedness of the two component $b$-family of equations

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Abstract
In this paper we consider the two component $b$-family of equations on $\mathbb{R}$. We write the equations on a Sobolev type diffeomorphism group. As an application of this formulation we show that the dependence on the initial data is nowhere locally uniformly continuous. In particular it is nowhere locally Lipschitz and nowhere locally Hölder continuous.

Keywords Two component $b$-family of equations · Nonuniform dependence · Diffeomorphism group

Mathematics Subject Classification 35Q35

1 Introduction

The initial value problem for the two component $b$-family of equations on $\mathbb{R}$ is given by

$$
\begin{align*}
    u_t - u_{txx} + (b + 1) uu_x &= bu_x u_{xx} + uu_{xxx} + \rho \rho_x, \\
    \rho_t + (\rho u)_x &= 0, \\
    u(t = 0) &= u_0, \\
    \rho(t = 0) &= \rho_0,
\end{align*}
$$

(1)

where $b \in \mathbb{R}$ is a parameter. The case $b = 2$ and $\rho \equiv 0$ in (1) corresponds to the Camassa-Holm equation, which was introduced in [1] as an integrable PDE modeling shallow water, where $u(t, x)$ is the horizontal velocity of the fluid at $x \in \mathbb{R}$ and time $t$. The case $b = 3$ and $\rho \equiv 0$ in (1) corresponds to the Degasperis-Procesi
equation, another integrable PDE, which was introduced in [4]. The case \( b \in \mathbb{R} \) and \( \rho \equiv 0 \) in (1) is the so called Holm-Staley \( b \)-family of equations, a model for shallow water. In [5,6] this shallow water model was studied with regards to the changes in the behaviour of solitary wave solutions for different \( b \) values. In [13] it was shown that in the Holm-Staley \( b \)-family of equations only the values \( b = 2, 3 \) lead to integrable PDEs. In search for possible integrable generalizations of the \( b \)-family of equations in the case \( b = 2, 3 \) people coupled the equation with the additional variable \( \rho \) to get something like (1)–see e.g. [2,14]. This is the same approach as in [10] for the KdV equation. On the other hand one can derive (1) as a shallow water model where \( u(t, x) \) is the horizontal velocity of the fluid and \( \rho(t, x) \) is the horizontal deviation of the surface from equilibrium – see e.g. [3] for the derivation of this in the case of the two component Camassa-Holm equation.

Let us summarize some known results regarding the local well posedness of (1). Using Kato’s semigroup theory it was shown in [11] that (1) is locally well-posed in \((u, \rho) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \ s \geq 2\). Later this was improved to \((u, \rho) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \ s > 3/2, \) in [11], where local well posedness in a range of Besov spaces was established. As a by-product of our diffeomorphism group formulation we will get the same local well-posedness result in Sobolev spaces as in [11], i.e.

**Theorem 11** Let \( s > 3/2 \). For every \((u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\) there is a \( T > 0 \) s.t. there is a unique pair

\[(u, \rho) \in C([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})),\]

satisfying (1). For \( T > 0 \) we denote by \( U_T \subset H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) the set of initial values \((u_0, \rho_0)\) for which the solution to (1) exists longer than time \( T \). Then the time \( T \) solution map

\[\Phi_T : U_T \to H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \quad (u_0, \rho_0) \mapsto (u(T), \rho(T))\]

is continuous. Here \((u(T), \rho(T))\) is the time \( T \) value of the solution \((u, \rho)\) corresponding to the initial value \((u_0, \rho_0)\).

A natural question is how regular the solution map \(\Phi_T\) is, e.g. whether \(\Phi_T\) is \(C^1\) or at least locally Lipschitz. In [12] it was shown that there is a bounded set in \(H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \ s > 5/2, \) on which \(\Phi_T\) is not uniformly continuous. We will improve this both w.r.t. \(s\) and w.r.t. to the non uniformity. Our main result reads as

**Theorem 12** Let \( s > 3/2 \) and \( T > 0 \). Then

\[\Phi_T : U_T \subset H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \]

\[(u_0, \rho_0) \mapsto \Phi_T((u_0, \rho_0)) = (u(T), \rho(T)),\]

is nowhere locally uniformly continuous.
Theorem 12 tells us that $\Phi_T$ fails to be uniformly continuous on any ball $B \subset U_T$ regardless of how small the ball is. In particular $\Phi_T$ is nowhere locally Lipschitz, nowhere locally Hölder continuous and hence nowhere $C^1$.

Our strategy to prove Theorem 12 is similar to the procedure in [9] and consists of two steps. In a first step we will write (1) in Lagrangian coordinates as an equation on a diffeomorphism group, i.e. we consider the flow map $\varphi$ of $u$

$$\varphi_t(t, x) = u(t, \varphi(t, x)), \quad \varphi(0, x) = x,$$

and write (1) in terms of $\varphi$. The second equation in (1) reads as

$$\frac{d}{dt} \left( \varphi_x \cdot \rho \circ \varphi \right) = 0$$

or

$$\rho(t) = \left( \frac{\rho_0}{\varphi_x(t)} \right) \circ \varphi(t)^{-1}.$$  

In a second step we will use this composite expression for a “moving hump” argument to produce non uniformity.

2 Lagrangian formulation

The goal of this section is to write the equations (1) in terms of the flow map of $u$, i.e. in terms of $\varphi$ given by

$$\varphi_t(t) = u(t) \circ \varphi(t), \quad t \geq 0, \quad \varphi(0) = \text{id}.$$  

Here $\text{id} : \mathbb{R} \to \mathbb{R}, \ x \mapsto x$, is the identity map. We introduced in [7] the diffeomorphism group $D^s$ based on Sobolev spaces. This space will be the configuration space for $\varphi$. More precisely for $s > 3/2$ we define

$$D^s(\mathbb{R}) := \{ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi - \text{id} \in H^s(\mathbb{R}), \ det(d_x \varphi) > 0 \ \forall x \in \mathbb{R} \}.$$  

Here $H^s(\mathbb{R})$ is the Sobolev space of order $s$, i.e.

$$H^s(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) \mid \| f \|_{H^s} := \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \ d\xi \right)^{1/2} < \infty \},$$  

where $\hat{f}$ is the Fourier transform of $f$. By the Sobolev Imbedding Theorem the function space $D^s(\mathbb{R})$, $s > 3/2$, consists of $C^1$ diffeomorphisms. By the imbedding

$$D^s(\mathbb{R}) \to H^s(\mathbb{R}), \quad \varphi \mapsto \varphi - \text{id}.$$
we can identify $\mathcal{D}^s(\mathbb{R})$ with an open subset of $H^s(\mathbb{R})$, thus we get a differential structure on $\mathcal{D}^s(\mathbb{R})$. In [7] it was shown that the space $\mathcal{D}^s(\mathbb{R}), s > 3/2$, is a topological group under composition. That $\mathcal{D}^s(\mathbb{R})$ is the right space follows from a result in [8], that says that for every $u \in C([0, T]; H^s(\mathbb{R})), T > 0$, there is a unique $\varphi \in C^1([0, T]; \mathcal{D}^s(\mathbb{R}))$ satisfying

$$\varphi_t(t) = u(t) \circ \varphi(t), \ t \in [0, T], \ \varphi(0) = \text{id.}$$

To get a Lagrangian formulation of (1) we write the first equation in (1) in non local form

$$u_t + uu_x = (1 - \partial^2_x)^{-1}(-bu_x + (b - 3)u_xu_{xx} + \rho \rho_x) \quad (2)$$

Now let $\varphi$ be the flow map of $u$. By differentiating $\varphi = u \circ \varphi$ w.r.t. $t$ we get

$$\varphi_{tt} = (u_t + uu_x) \circ \varphi.$$

Using (2) we get

$$\varphi_{tt} = \left((1 - \partial^2_x)^{-1}(-bu_x + (b - 3)u_xu_{xx} + \rho \rho_x)\right) \circ \varphi.$$

As noted in the introduction we have

$$\rho = \left(\frac{\rho_0}{\varphi_x}\right) \circ \varphi^{-1}.$$

If we use this expression for $\rho$ and $u = \varphi_t \circ \varphi^{-1}$ we get

$$\varphi_{tt} = \left((1 - \partial^2_x)^{-1}\left(-b\varphi_t \circ \varphi^{-1} \cdot \partial_x (\varphi_t \circ \varphi^{-1}) + (b - 3)\partial_x (\varphi_t \circ \varphi^{-1}) \partial^2_x (\varphi_t \circ \varphi^{-1})\right)\right) \circ \varphi =: F(\varphi, \varphi_t, \rho_0).$$

We’ve proved in [9] that the expressions appearing in $F(\varphi, \varphi_t, \rho_0)$ are analytic in $(\varphi, \varphi_t, \rho_0)$. More precisely

$$\mathcal{D}^s(\mathbb{R}) \times H^s(\mathbb{R}) \to H^s(\mathbb{R}),$$

$$(\varphi, \varphi_t) \mapsto \left((1 - \partial^2_x)^{-1}\left(\partial_x (\varphi_t \circ \varphi^{-1}) \partial^2_x (\varphi_t \circ \varphi^{-1})\right)\right) \circ \varphi,$$

and

$$\mathcal{D}^s(\mathbb{R}) \times H^s(\mathbb{R}) \to H^s(\mathbb{R}),$$

$$(\varphi, \varphi_t) \mapsto \left((1 - \partial^2_x)^{-1}\left(\partial^2_x (\varphi_t \circ \varphi^{-1})\right)\right) \circ \varphi.$$
and
\[ D^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to H^s(\mathbb{R}), \]
\[ (\varphi, \rho_0) \mapsto \left( (1 - \partial_x^2)^{-1} \left( \left( \frac{\rho_0}{\varphi_x} \right) \circ \varphi^{-1} \cdot \partial_x \left( \left( \frac{\rho_0}{\varphi_x} \circ \varphi^{-1} \right) \right) \right) \right) \circ \varphi, \]
are analytic maps. Consult [9] for detailed computations. Let us just point out here that the mechanism underlying analyticity is that conjugation with \( \varphi^{-1} \) in combination with \( \partial_x \)
gives an expression involving only derivatives and multiplication. So we get

**Proposition 21** Let \( s > 3/2 \). Then the map
\[ D^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to H^s(\mathbb{R}), \quad (\varphi, \varphi_t, \rho_0) \mapsto F(\varphi, \varphi_t, \rho_0) \]
is analytic.

By applying the Picard-Lindelöf Theorem we get local in time existence for solutions of (1).

**Lemma 22** Let \( s > 3/2 \). Then for every initial value \( (u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) there is \( T > 0 \) and
\[ (u, \rho) \in C([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})), \]
satisfying
\[ u_t + uu_x = (1 - \partial_x^2)^{-1} (-buu_x + (b - 3)u_xu_{xx} + \rho \rho_x), \]
\[ \rho_t + (\rho u)_x = 0, \]
\[ u(0) = u_0, \quad \rho(0) = \rho_0 \]
on \([0, T] \).

**Proof** Consider the analytic second order ODE on \( D^s(\mathbb{R}) \)
\[ \varphi_{tt} = F(\varphi, \varphi_t, \rho_0), \quad \varphi(0) = \text{id}, \quad \varphi_t(0) = u_0. \] (4)

By Picard-Lindelöf there is \( T > 0 \) and a solution \( \varphi \in D^s(\mathbb{R}) \) to (4) on \([0, T]\). For \( t \in [0, T] \) we define
\[ u(t) := \varphi_t(t) \circ \varphi(t)^{-1}, \quad \rho(t) := \left( \frac{\rho_0}{\varphi_x(t)} \right) \circ \varphi(t)^{-1}. \]
From the regularity properties of composition established in [7] we know
\[(u, \rho) \in C([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))\]
and from the derivation above we see that \((u, \rho)\) solves (3). \square

By the Picard-Lindelöf Theorem we also get uniqueness.

**Lemma 23** Let \(s > 3/2\) and \(T > 0\). Suppose that 
\[(u, \rho), (\tilde{u}, \tilde{\rho}) \in C([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))\]
are two solutions to (3). Then \((u, \rho) = (\tilde{u}, \tilde{\rho})\) on \([0, T]\).

**Proof** We know by [8] that there is a unique \(\varphi \in C^1([0, T]; D^s(\mathbb{R}))\) s.t.
\[\varphi_t(t) = u(t) \circ \varphi(t), \ t \in [0, T], \ \varphi(0) = \text{id}.\]
Taking the derivative in \(\varphi_t = u \circ \varphi\) w.r.t. \(t\) we get pointwise the identity
\[\varphi_{tt} = (u_t + uu_x) \circ \varphi.\]
By using (3) we get pointwise
\[\varphi_{tt} = (1 - \partial_x^2)^{-1}(-buu_x + (b - 3)u_x u_{xx} + \rho \rho_x) \circ \varphi.\]
The right hand side is in \(C([0, T]; H^s(\mathbb{R}))\). This means that \(\varphi \in C^2([0, T]; D^s(\mathbb{R}))\) and it solves (4). The solution \((\tilde{u}, \tilde{\rho})\) generates in a similar fashion a \(\tilde{\varphi} \in C^2([0, T]; D^s(\mathbb{R}))\) solving (4). By Picard-Lindelöf we get \(\varphi = \tilde{\varphi}\) on \([0, T]\). Therefore we have \(u = \tilde{u}\) on \([0, T]\). This proves uniqueness. \square

Combining Lemma 22 and 23 we can prove Theorem 11.

**Proof** (Proof of Theorem 11) By solving (4) as in Lemma 22 we get for \(T > 0\) an open set \(U_T \subset H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\) as in the statement of the theorem. The solution map is given by
\[\Phi_T : U_T \rightarrow H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}),\]
\[(u_0, \rho_0) \mapsto (\varphi_T(T) \circ \varphi(T)^{-1}, \left(\frac{\rho_0}{\varphi_x(T)}\right) \circ \varphi(T)^{-1}),\]
where \(\varphi = \varphi(\cdot; u_0, \rho_0) \in C^2([0, T]; D^s(\mathbb{R}))\) is the solution of (4). We get from the regularity of the composition that \(\Phi_T\) is continuous. Together with the uniqueness of Lemma 23 this finishes the proof. \square
3 Nonuniform dependence

In this section we will prove Theorem 12. Throughout this section we assume $s > 3/2$. Note that (1) admits the scale invariance

$$u_\lambda(t, x) := \lambda u(\lambda t, x), \quad \rho_\lambda(t, x) := \lambda \rho(\lambda t, x), \quad \lambda > 0,$$

in the sense that $(u_\lambda, \rho_\lambda)$ is a solution to (1) whenever $(u, \rho)$ is. This scaling shows that $U_T \subset H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ is star shaped w.r.t. $(0, 0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$. Let us denote $U = U_T |_{T_1}$ and $\Phi = \Phi_{T_1}$. The scaling $(u_\lambda, \rho_\lambda)$ implies for $T > 0$ and $(u_0, \rho_0) \in U_T$ that $(Tu_0, T\rho_0) \in U$ and for the solution map that

$$\Phi_T((u_0, \rho_0)) = (u(T), \rho(T)) = (\frac{1}{T}u_T(1), \frac{1}{T}\rho_T(1)) = \frac{1}{T}\Phi(Tu_0, T\rho_0) \quad (5)$$

for $(u_0, \rho_0) \in U_T$. Using (5) Theorem (12) will follow from

**Proposition 31** The map

$$\Phi : U \subset H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$$

is nowhere uniformly continuous.

Let us also introduce the time $T = 1$ solution map in Lagrangian coordinates, i.e.

$$\Psi : U \subset H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to D^s(\mathbb{R}), (u_0, \rho_0) \mapsto \varphi(1; u_0, \rho_0),$$

where $\varphi(1; u_0, \rho_0)$ is the time $T = 1$ value of the solution to (4) with initial values $(u_0, \rho_0)$. We know by analytic dependence on the initial data that $\Psi$ is an analytic map. Moreover a simple computation shows that

$$\varphi(t; u_0, \rho_0) = \Psi(t(u_0, \rho_0)). \quad (6)$$

Before we prove Proposition 31 we need the following technical lemma about the map $\Psi$.

**Lemma 32** There is a dense subset $S \subset U$ consisting of smooth compactly supported $(u_\bullet, \rho_\bullet)$ s.t. for every $(u_\bullet, \rho_\bullet) \in S$ there is $w_\bullet = (w_1, 0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ and $a_\bullet \in \mathbb{R}$ with $\text{dist}(a_\bullet, \text{supp } \rho_\bullet) \geq 2$ satisfying

$$\left( d(u_\bullet, \rho_\bullet) \Psi(w) \right)(a_\bullet) \neq 0.$$ 

Here $\text{supp } \rho_\bullet \subset \mathbb{R}$ is the support of $\rho_\bullet$, $\text{dist}(a_\bullet, \text{supp } \rho_\bullet)$ is the distance of $a_\bullet$ to the support of $\rho_\bullet$ and $d(u_\bullet, \rho_\bullet) \Psi$ is the differential of $\Psi$ at $(u_\bullet, \rho_\bullet)$, i.e.

$$d(u_\bullet, \rho_\bullet) \Psi : H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to H^s(\mathbb{R}),$$

$$v = (v_1, v_2) \mapsto d(u_0, \rho_0) \Psi(v) = \lim_{t \to 0} \frac{\Psi(u_\bullet + tv_1, \rho_\bullet + tv_2) - \Psi(u_\bullet, \rho_\bullet)}{t}.$$
Proof For \( v = (v_1, v_2) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) we get from (6)

\[
d_{(0,0)} \Psi(v) = \frac{d}{dt} \bigg|_{t=0} \Psi(tv) = \frac{d}{dt} \bigg|_{t=0} \varphi(t; \ v_1, v_2) = v_1.
\]

We know that \( C^\infty_c(\mathbb{R}) \times C^\infty_c(\mathbb{R}) \cap U \) is dense in \( U \). Let \( (u_\bullet, \rho_\bullet) \in C^\infty_c(\mathbb{R}) \times C^\infty_c(\mathbb{R}) \cap U \) be arbitrary. We take \( a_\bullet \in \mathbb{R} \) with \( \text{dist}(a_\bullet, \text{supp} \rho_\bullet) \geq 2 \) and choose \( w_\bullet = (w_1, 0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) s.t. \( w_1(a_\bullet) \neq 0 \). Now consider the analytic function

\[
[0, 1] \to \mathbb{R}, \quad t \mapsto (d_{(u_\bullet, \rho_\bullet)} \Psi(w_\bullet))(a_\bullet),
\]

which for \( t = 0 \) is equal to \( w_1(a_\bullet) \neq 0 \). Since the function is analytic this means that there is a sequence \( t_n \uparrow 1 \) s.t.

\[
(d_{t_n(u_\bullet, \rho_\bullet)} \Psi(w_\bullet))(a_\bullet) \neq 0, \quad n \geq 1.
\]

So we can put \( \{t_n(u_\bullet, \rho_\bullet) \mid n \geq 1 \} \) into \( S \). By doing this for all \( (u_\bullet, \rho_\bullet) \in C^\infty_c(\mathbb{R}) \times C^\infty_c(\mathbb{R}) \cap U \) we get a dense \( S \subset U \) with the desired properties.

Now we can prove Proposition 31.

Proof (Proof of Proposition 31) Let \( S \subset U \) be as in Lemma 32 and \( (u_\bullet, \rho_\bullet) \in S \) with corresponding \( w_\bullet = (w_1, 0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) and \( a_\bullet \in \mathbb{R} \) satisfying \( \text{dist}(a_\bullet, \text{supp} \rho_\bullet) \geq 2 \) and \( (d_{(u_\bullet, \rho_\bullet)} \Psi(w_\bullet))(a_\bullet) \neq 0 \). We fix \( m > 0 \) s.t.

\[
\| (d_{(u_\bullet, \rho_\bullet)} \Psi(w_\bullet))(a_\bullet) \| > m \| w_1 \|_{H^s}.
\]

We will determine in successive steps \( R_\ast > 0 \) with \( B_{R_\ast}((u_\bullet, \rho_\bullet)) \subset U \), where

\[
B_{R_\ast}((u_\bullet, \rho_\bullet)) = \{(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \mid \max \{ \| u_0 - u_\bullet \|_{H^s}, \| \rho_0 - \rho_\bullet \|_{H^{s-1}} \} < R_\ast \},
\]

s.t. \( \Phi|_{B_R((u_\bullet, \rho_\bullet))} \) is not uniformly continuous for all \( 0 < R \leq R_\ast \).

The solution map \( \Phi = (\Phi_1, \Phi_2) \) has two components, a \( u \) and a \( \rho \) part. It is sufficient to prove the non uniform dependence for the \( \rho \) part

\[
\Phi_2(u_0, \rho_0) = \left( \frac{\rho_0}{\partial_x \Psi(u_0, \rho_0)} \right) \circ \Psi(u_0, \rho_0)^{-1}, \quad (u_0, \rho_0) \in U.
\]

We first choose \( R_1 > 0 \) with \( B_{R_1}((u_\bullet, \rho_\bullet)) \subset U \) and

\[
\| \Psi(u_0, \rho_0) - \text{id} \|_{H^s}, \| \Psi(u_0, \rho_0)^{-1} - \text{id} \|_{H^s} < C_1
\]

for all \( (u_0, \rho_0) \in B_{R_1}((u_\bullet, \rho_\bullet)) \) and for some \( C_1 > 0 \). This is clearly possible due to the continuity of \( \Psi \). The map

\[
H^{s-1}(\mathbb{R}) \times D^s(\mathbb{R}) \to H^{s-1}(\mathbb{R}), \quad (f, \varphi) \mapsto \left( \frac{f}{\varphi_x} \right) \circ \varphi^{-1}
\]
is continuous as we know from [7]. By the uniform boundedness principle there is
\[ 0 < R_2 \leq R_1 \] and \( C_2 > 0 \) s.t.
\[ \frac{1}{C_2} \| f \|_{H^{s-1}} \leq \left\| \left( \frac{f}{\partial_x \Psi(u_0, \rho_0)} \right) \circ \Psi(u_0, \rho_0)^{-1} \right\|_{H^{s-1}} \leq C_2 \| f \|_{H^{s-1}}, \quad (8) \]
for all \( f \in H^{s-1}(\mathbb{R}) \) and for all \((u_0, \rho_0) \in B_{R_2}((u_*, \rho_*))\). By Taylor’s Theorem we have
\[
\Psi(u_{\ast} + v, \rho_{\ast} + w) = \\
\Psi(u_{\ast} , \rho_{\ast}) + d(u_{\ast} , \rho_{\ast}) \Psi(v, w) + \int_0^1 (1 - s) d^2(u_{\ast} + sv, \rho_{\ast} + sw) \Psi((v, w), (v, w)) \, ds.
\]
We will need estimates for the second order differential \( d^2 \Psi \). Since \( \Psi \) is smooth there is
\[ 0 < R_3 \leq R_2 \] and \( C_3 > 0 \) s.t.
\[ \| d^2(u_0, \rho_0) \Psi((v, w), (\tilde{v}, \tilde{w})) \|_{H^s} \leq C_3 (\| v \|_{H^s} + \| w \|_{H^{s-1}})(\| \tilde{v} \|_{H^s} + \| \tilde{w} \|_{H^{s-1}}) \quad (9) \]
and
\[ \| d^2(u_0, \rho_0) \Psi(v, w)^2 - d^2(\tilde{u}_0, \tilde{\rho}_0) \Psi(v, w)^2 \|_{H^s} \leq C_3 (\| u_0 - \tilde{u}_0 \|_{H^s} + \| \rho_0 - \tilde{\rho}_0 \|_{H^{s-1}})(\| v \|_{H^s} + \| w \|_{H^{s-1}})^2 \quad (10) \]
for all \((u_0, \rho_0), (\tilde{u}_0, \tilde{\rho}_0) \in B_{R_3}((u_*, \rho_*))\) and for all \((v, w), (\tilde{v}, \tilde{w}) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\). Here we used \((u, v)^2 = ((u, v), (u, v))\). By the smoothness of \( \Psi \) there is
\[ 0 < R_4 \leq R_3 \] and \( C_4 > 0 \) s.t.
\[ \| \Psi(u_0, \rho_0) - \Psi(\tilde{u}_0, \tilde{\rho}_0) \|_{H^s} \leq C_4 (\| u_0 - \tilde{u}_0 \|_{H^s} + \| \rho_0 - \tilde{\rho}_0 \|_{H^{s-1}}) \quad (11) \]
for all \((u_0, \rho_0), (\tilde{u}_0, \tilde{\rho}_0) \in B_{R_4}((u_*, \rho_*))\). We also fix a constant \( C_5 > 0 \) for the Sobolev imbedding
\[ \| f \|_{C^1} \leq C_5 \| f \|_{H^s}, \quad \forall f \in H^s(\mathbb{R}). \quad (12) \]
After all this choices we set \( 0 < R_s \leq R_5 \) in such a way that we have
\[ \max\{C_3 C_5 R_s^2 / 16, C_3 C_5 R_s / 2\} < m/4. \quad (13) \]
By \( (7) \) and the Sobolev imbedding \( (12) \) there is \( L > 0 \) s.t.
\[ \frac{1}{L} |x - y| \leq |\Psi(u_0, \rho_0)(x) - \Psi(u_0, \rho_0)(y)| \leq L |x - y|, \quad \forall x, y \in \mathbb{R}, \quad (14) \]
for all \((u_0, \rho_0) \in B_{R_s}((u_*, \rho_*))\).
Let \( 0 < R \leq R_s \). Our strategy is to construct two sequences of initial data
\[(u_0^{(n)}, \rho_0^{(n)})_{n \geq 1}, (\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)})_{n \geq 1} \subset B_R((u_\bullet, \rho_\bullet)) \text{ s.t.} \]
\[
\lim_{n \to \infty} (\|u_0^{(n)} - \tilde{u}_0^{(n)}\|_{H^s} + \|\rho_0^{(n)} - \tilde{\rho}_0^{(n)}\|_{H^{s-1}}) = 0
\]
whereas
\[
\limsup_{n \to \infty} \|\Phi_2(u_0^{(n)}, \rho_0^{(n)}) - \Phi_2(\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)})\|_{H^{s-1}} > 0.
\]
This would show that \(\Phi|_{B_R((u_\bullet, \rho_\bullet))}\) is not uniformly continuous.

We define 
\[r_n := \frac{m}{8n}\|w_1\|_{H^s}\]
and a sequence \((\rho_n)_{n \geq 1} \subset C_c^\infty(\mathbb{R})\) with \(\|\rho_n\|_{H^{s-1}} = R/4\) and
\[\text{supp } \rho_n \subset [a_\bullet - \frac{1}{L}r_n, a_\bullet + \frac{1}{L}r_n].\]
With this we define the pair of initial data as
\[u_0^{(n)} = u_\bullet, \quad \rho_0^{(n)} = \rho_\bullet + \rho_n, \quad \tilde{u}_0^{(n)} = u_\bullet + \frac{1}{n}w_1, \quad \tilde{\rho}_0^{(n)} = \rho_\bullet + \rho_n.\]
There is \(N \geq 1\) s.t.
\[(u_0^{(n)}, \rho_0^{(n)}), (\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)}) \in B_R((u_\bullet, \rho_\bullet)), \quad \forall n \geq N,
\]
and \(\text{supp } \rho_n \subset [a_\bullet - 1, a_\bullet + 1]\) for \(n \geq N\). By construction we have
\[
\lim_{n \to \infty} (\|u_0^{(n)} - \tilde{u}_0^{(n)}\|_{H^s} + \|\rho_0^{(n)} - \tilde{\rho}_0^{(n)}\|_{H^{s-1}}) = 0.
\]
In order to make the notation easier we introduce for \(n \geq N\)
\[\varphi_n = \Psi(u_0^{(n)}, \rho_0^{(n)}), \quad \tilde{\varphi}_n = \Psi(\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)}).
\]
Thus we get for \(n \geq N\)
\[\Phi_2(u_0^{(n)}, \rho_0^{(n)}) = \left(\frac{\rho_0^{(n)}}{\partial_x \varphi_n}\right) \circ \varphi_n^{-1}, \quad \Phi_2(\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)}) = \left(\frac{\tilde{\rho}_0^{(n)}}{\partial_x \tilde{\varphi}_n}\right) \circ \tilde{\varphi}_n^{-1}.
\]
For the supports we get
\[\text{supp } \left(\frac{\rho_0^{(n)}}{\partial_x \varphi_n}\right) \circ \varphi_n^{-1} = \varphi_n(\text{supp } \rho_\bullet) \cup \varphi_n(\text{supp } \rho_n)\]
and
\[ \text{supp} \left( \frac{\tilde{\rho}_0^{(n)}}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} = \varphi_n(\text{supp } \rho \bullet) \cup \tilde{\varphi}_n(\text{supp } \rho_n). \]

Both are disjoint unions. But we can say more. We introduce the sets

\[ A_n = \varphi_n(\text{supp } \rho \bullet), \quad B_n = \varphi_n(\text{supp } \rho_n), \quad C_n = \tilde{\varphi}_n(\text{supp } \rho \bullet), \quad D_n = \tilde{\varphi}_n(\text{supp } \rho_n), \]

for \( n \geq N \). By (14) the sets \( A_n \) and \( B_n \) are separated by a distance not less than \( \delta := 1/L > 0 \). As

\[ \| u_0^{(n)} - \tilde{u}_0^{(n)} \|_{H^s} + \| \rho_0^{(n)} - \tilde{\rho}_0^{(n)} \|_{H^{s-1}} \xrightarrow{n \to \infty} 0 \]

we get by (11) and (12) that for \( n \) large enough \( A_n \cup C_n \) and \( B_n \cup D_n \) are separated by at least \( \delta/2 \). Thus there is \( C_\delta > 0 \) s.t.

\[ \| \Phi_2(u_0^{(n)}, \rho_0^{(n)}) - \Phi_2(\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)}) \|_{H^{s-1}} = \]

\[ C_\delta \left( \left\| \left( \frac{\rho_\bullet}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} - \left( \frac{\tilde{\rho}_0^{(n)}}{\partial_x \tilde{\varphi}_n} \right) \circ \tilde{\varphi}_n^{-1} \right\|_{H^{s-1}} \right) + \]

\[ C_\delta \left( \left\| \left( \frac{\rho_n}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} - \left( \frac{\rho_\bullet}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} \right\|_{H^{s-1}} \right) \geq \]

\[ C_\delta \left( \left\| \left( \frac{\rho_n}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} - \left( \frac{\rho_n}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} \right\|_{H^{s-1}} \right) \]

for \( n \) large. So it is sufficient to show

\[ \limsup_{n \to \infty} \left( \left\| \left( \frac{\rho_n}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} - \left( \frac{\rho_n}{\partial_x \varphi_n} \right) \circ \varphi_n^{-1} \right\|_{H^{s-1}} \right) > 0. \]

We will get this by showing that \( B_n \) and \( D_n \) are disjoint as well. For that we look at the “center” of \( B_n \) resp. \( D_n \), more precisely at \( \varphi_n(a_\bullet) \) resp. \( \tilde{\varphi}_n(a_\bullet) \) By Taylor’s Theorem we have for \( \varphi_n = \Psi(u_\bullet, \rho_\bullet + \rho_n) \)

\[ \varphi_n = \Psi(u_\bullet, \rho_\bullet) + d(u_\bullet, \rho_\bullet) \Psi(0, \rho_n) + \int_0^1 (1-s)d^2(u_\bullet, \rho_\bullet + s\rho_n) \Psi(0, \rho_n)^2 ds \]
and similarly for $\tilde{\varphi}_n = \Psi(u_\bullet + \frac{1}{n} w_1, \rho_\bullet + \rho_n)$

$$
\tilde{\varphi}_n = \Psi(u_\bullet, \rho_\bullet) + d_{(u_\bullet, \rho_\bullet)}(\frac{1}{n} w_1, \rho_n) + \int_0^1 (1 - s) d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi(\frac{1}{n} w_1, \rho_n)^2 \, ds
$$

$$
= \Psi(u_\bullet, \rho_\bullet) + d_{(u_\bullet, \rho_\bullet)}(\frac{1}{n} w_1, \rho_n) + \int_0^1 (1 - s) d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi(0, \rho_n)^2 \, ds
$$

$$
+ 2 \int_0^1 (1 - s) d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi\left(\frac{1}{n} w_1, 0\right), (0, \rho_n) \right) \, ds
$$

$$
+ \int_0^1 (1 - s) d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi(\frac{1}{n} w_1, 0)^2 \, ds.
$$

Thus

$$
\tilde{\varphi}_n - \varphi_n = d_{(u_\bullet, \rho_\bullet)}(\frac{1}{n} w_1, 0)
$$

$$
+ \int_0^1 (1 - s) \left( d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi - d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi \right) (0, \rho_n)^2 \, ds
$$

$$
+ 2 \int_0^1 (1 - s) d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi\left(\frac{1}{n} w_1, 0\right), (0, \rho_n) \right) \, ds
$$

$$
+ \int_0^1 (1 - s) d^2_{(u_\bullet + s \frac{1}{n} w_1, \rho_\bullet + s \rho_n)} \Psi(\frac{1}{n} w_1, 0)^2 \, ds.
$$

Using (10) and (12) we get

$$
\|R_1\|_\infty \leq C_5 \|R_1\|_{H^s} \leq C_5 C_3 \|\frac{1}{n} w_1\|_{H^s} \|\rho_n\|_{H^{s-1}}^2 = C_5 C_5 \frac{R^2}{16 n} \|\frac{1}{n} w_1\|_{H^s}.
$$

Using (9) and (12) we get

$$
\|R_2\|_\infty \leq C_5 \|R_2\|_{H^s} \leq 2 C_5 C_3 \|\frac{1}{n} w_1\|_{H^s} \|\rho_n\|_{H^{s-1}} = 2 C_5 C_5 \frac{R}{4 n} \|\frac{1}{n} w_1\|_{H^s}
$$

and

$$
\|R_3\|_\infty \leq C_5 \|R_3\|_{H^s} \leq C_5 C_3 \|\frac{1}{n} w_1\|_{H^s}^2 = C_5 C_5 \frac{1}{n^2} \|\frac{1}{n} w_1\|_{H^s}^2.
$$

By (13) we get for $n$ large

$$
\|R_1\|_\infty + \|R_2\|_\infty + \|R_3\|_\infty < \frac{m}{2n} \|\frac{1}{n} w_1\|_{H^s}.
$$
Therefore
\[ |\tilde{\phi}_n(a_\bullet) - \phi_n(a_\bullet)| \geq \left| \left( d(u_\bullet, \rho_\bullet) \Psi \left( \frac{1}{n} w_1, 0 \right) \right)(a_\bullet) \right| - \| R_1 \|_{\infty} - \| R_2 \|_{\infty} - \| R_3 \|_{\infty} \]
\[ \geq \frac{1}{n} \left| \left( d(u_\bullet, \rho_\bullet) \Psi (w_\bullet) \right)(a_\bullet) \right| - \frac{m}{2n} \| w_1 \|_{H^s} = \frac{m}{2n} \| w_1 \|_{H^s} \]
for \( n \) large. Since \( \text{supp} \rho_n \subset [a_\bullet - \frac{1}{T} r_n, a_\bullet + \frac{1}{T} r_n] \) we get by (14)
\[ \text{supp} \left( \frac{\rho_n^{(n)}}{\partial_x} \right) \circ \tilde{\phi}_n^{-1} \subset [\phi_n(a_\bullet) - r_n, \phi_n(a_\bullet) + r_n] \]
and
\[ \text{supp} \left( \frac{\rho_n^{(n)}}{\partial_x} \right) \circ \phi_n^{-1} \subset [\phi_n(a_\bullet) - r_n, \phi_n(a_\bullet) + r_n] \].

By our choice \( r_n = \frac{m}{8n} \| w_1 \|_{H^s} \) and the fact the the centers of the intervals have a
distance of at least \( \frac{m}{2} \| w_1 \|_{H^s} - 1 \), the supports are in such a way apart that we can
separate the expressions as we did in [9]. To be precise there is \( \tilde{C} > 0 \) s.t.
\[ \left\| \left( \frac{\rho_n}{\partial_x} \right) \circ \phi_n^{-1} - \left( \frac{\rho_n}{\partial_x} \right) \circ \tilde{\phi}_n^{-1} \right\|_{H^{s-1}} \geq \tilde{C} \left( \left\| \left( \frac{\rho_n}{\partial_x} \right) \circ \phi_n^{-1} \right\|_{H^{s-1}} + \left\| \left( \frac{\rho_n}{\partial_x} \right) \circ \tilde{\phi}_n^{-1} \right\|_{H^{s-1}} \right) \]
for large \( n \). Using this and (8) we can estimate
\[ \limsup_{n \to \infty} \left( \left\| \left( \frac{\rho_n}{\partial_x} \right) \circ \phi_n^{-1} - \left( \frac{\rho_n}{\partial_x} \right) \circ \tilde{\phi}_n^{-1} \right\|_{H^{s-1}} \right) \geq \frac{2\tilde{C}}{C_2} \| \rho_n \|_{H^{s-1}} = \frac{2\tilde{C}}{C_2} R/4 > 0. \]

In summary, we constructed for an arbitrary \( R \in (0, R_\bullet) \) a pair of sequences
\( (u_0^{(n)}, \rho_0^{(n)})_{n \geq N}, (\tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)})_{n \geq N} \in B_R((u_\bullet, \rho_\bullet)) \)
with
\[ \lim_{n \to \infty} \left( \| u_0^{(n)} - \tilde{u}_0^{(n)} \|_{H^s} + \| \rho_0^{(n)} - \tilde{\rho}_0^{(n)} \|_{H^{s-1}} \right) = 0 \]
whereas
\[ \limsup_{n \to \infty} \| \Phi \left( u_0^{(n)}, \rho_0^{(n)} \right) - \Phi \left( \tilde{u}_0^{(n)}, \tilde{\rho}_0^{(n)} \right) \|_{H^s \times H^{s-1}} > 0. \]
This shows that $\Phi_{B_R((u_0, \rho_0))}$ is not uniformly continuous. This finishes the proof. □

Using Proposition 31 we can prove Theorem 12.

**Proof** (Proof of Theorem 12) By (5) we have for $T > 0$

$$\Phi_T : U_T \to H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \quad (u_0, \rho_0) \mapsto \Phi_T(u_0, \rho_0) = \frac{1}{T} \Phi(Tu_0, T\rho_0).$$

Thus Proposition 31 shows that $\Phi_T$ is nowhere uniformly continuous. □

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