The torsion-free part of the Ziegler spectrum of orders over Dedekind domains

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We study the \( R \)-torsion-free part of the Ziegler spectrum of an order \( \Lambda \) over a Dedekind domain \( R \). We underline and comment on the role of lattices over \( \Lambda \). We describe the torsion-free part of the spectrum when \( \Lambda \) is of finite lattice representation type.

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1 Introduction

In his posthumous paper [14], Puninski made substantial progress in the description of the Cohen-Macaulay part of the Ziegler spectrum over Cohen-Macaulay rings. Puninski also raised a similar question for torsion-free \( \Lambda \)-modules containing \( \text{Latt}_\Lambda \), or also as direct summands (still over \( R \)) of \( R^n \) for some positive integer \( n \).

We are interested in (say, right) modules over such an order \( \Lambda \), in particular in \( R \)-torsion-free \( \Lambda \)-modules. Recall that a \( \Lambda \)-module \( M \) is \( R \)-torsion-free if for all \( 0 \neq m \in M \) and \( 0 \neq r \in R \) we have \( mr \neq 0 \). Finitely generated \( R \)-torsion-free \( \Lambda \)-modules are known as \( \Lambda \)-lattices. Over a Dedekind domain \( R \), \( \Lambda \)-lattices can be equivalently introduced as \( \Lambda \)-modules which are finitely generated and projective over \( R \), or also as direct summands (still over \( R \)) of \( R^n \) for some positive integer \( n \).

Let us fix some further notation. For \( \Lambda \) an order over a Dedekind domain \( R \), \( \text{Tf}_\Lambda \) is the category of all \( R \)-torsion-free (right) \( \Lambda \)-modules and \( \text{Latt}_\Lambda \) is the category of (right) \( \Lambda \)-lattices. Moreover \( L_\Lambda \) is the first order language of \( \Lambda \)-modules, and in this language \( T^L_\Lambda \) is the first order theory of \( R \)-torsion-free \( \Lambda \)-modules, so of \( \text{Tf}_\Lambda \).

Note that \( \text{Tf}_\Lambda \) is the smallest definable subcategory of the category of all \( \Lambda \)-modules containing \( \text{Latt}_\Lambda \).

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For every positive integer $n$, $pp_n^\Lambda$ denotes the lattice of $pp$-formulas with $n$ free variables of $L_\Lambda$ (warning: here the word lattice has a different meaning, that of an ordered structure, cf. below). In detail, $pp_n^\Lambda$ is the quotient set of these $pp$-formulas with respect to the logical equivalence relation (in the theory of $\Lambda$-modules). The lattice structure is given by the partial order relation determined by logical implication (modulo the same theory). Then meet corresponds to the conjunction of $pp$-formulas, and join to their sum +. If one identifies $pp$-formulas in $n$ variables equivalent in the first order theory of some given $\Lambda$-module $M$, then one forms another lattice $pp_n^\Lambda(M)$ — a quotient lattice of $pp_n^\Lambda$. The same can be done starting from a class of $\Lambda$-modules instead of a single $M$. For instance, one builds in this way $pp_n^\Lambda(T_{\Lambda\Lambda})$.

We shall denote the binary relation in these lattices by $\leq$ (with the usual meaning for $<$). When necessary, a subscript will specify to which lattice we refer. For instance, we write $\leq_{pp_n^\Lambda(M)}$ when dealing with $pp$-formulas in 1 free variable with respect to the first order theory of a module $M$. Likewise $[\ ]$ will denote a closed interval in a lattice, with possible use of subscripts to say which lattice we deal with, as before. Similar conventions will regard open or half closed intervals.

Throughout we shall use boldface, e.g., $\mathbf{x}$, $\mathbf{m}$, to denote tuples (of variables, elements respectively).

Note that, as $\Lambda$ is Noetherian, every finitely generated $\Lambda$-module is finitely presented. Therefore the $pp$-type of an $n$-tuple of a $\Lambda$-lattice is generated in $pp_n^\Lambda$ by a $pp$-formula $\varphi$ [13, 1.2.11]. Moreover, all $pp$-formulas $\varphi$ have a free realisation, i.e., for every $pp$-formula $\varphi$ there is a finitely presented $\Lambda$-module $M$ and a tuple $\mathbf{m}$ of elements of $M$ such that the $pp$-type of $\mathbf{m}$ in $M$ is generated by $\varphi$ [13, 1.2.14].

Let us come back to illustrate the aim of the paper. As said, we consider $R$-torsion-free modules over an order $\Lambda$ over a Dedekind domain $R$. Let $Zg_\Lambda$ denote the whole (right) Ziegler spectrum of $\Lambda$, i.e., the topological space of (isomorphism classes of) indecomposable pure injective $\Lambda$-modules. A basis of open sets of the topology is given by $[\varphi/\psi] = \{N \in Zg_\Lambda : \varphi \wedge \psi \leq_{pp_n^\Lambda(N)} \varphi\}$ where $\varphi$ and $\psi$ range over $pp_n^\Lambda$. We are interested in the subset of $Zg_\Lambda$ formed by $R$-torsion-free indecomposable pure injective $\Lambda$-modules. Notice that this is a closed set, as the complement of the union of $(xr = 0/x = 0)$ where $r$ ranges over the non-zero elements of $R$. Let $Zg_\Lambda^{\text{lf}}$ denote it. Observe that since $Zg_\Lambda$ is compact and $Zg_\Lambda^{\text{lf}}$ is a closed subset of $Zg_\Lambda$, $Zg_\Lambda^{\text{lf}}$ is also compact.

One may wonder what is the role of $\Lambda$-lattices in this framework, e.g., if indecomposable $\Lambda$-lattices are pure injective, so points of $Zg_\Lambda^{\text{lf}}$. We cannot expect that in general, but we shall see in the next section that the answer is positive at least over complete discrete valuation domains. Apart from this, we shall also discuss the relevance of lattices in the $R$-torsion-free part of the spectrum, just as [11] did over group rings.

Here is a more detailed plan of this paper. In § 2, we prove some first results on lattices, and above all that, when $R$ is a complete discrete valuation domain and $\Lambda$ is separable, they are isolated points of $Zg_\Lambda^{\text{lf}}$, dense in the whole space $Zg_\Lambda$. In § 3, we provide a description of the torsion-free part of the Ziegler spectrum of an order $\Lambda$ over a Dedekind domain $R$ in a semisimple $Q$-algebra, extending that over group rings in [11]. We also investigate the $m$-dimension of $pp_n^\Lambda(T_{\Lambda\Lambda})$ in that section. Applications of the (classical) Maranda theorem to our setting will be treated in § 4. The final § 5 considers orders of finite lattice representation type and provides a complete description of their Ziegler spectrum, confirming a conjecture of Puninski. As an application, it is shown that the theory of $Z$-torsion-free $ZG$-modules, with $G$ a cyclic group of order $p$ or $p^2$ for some prime $p$, is decidable, which positively answers a question in [10].

We assume some familiarity with model theory of modules, as treated in [12, 13, 19].

Finally let us call again the reader’s attention to the fact that, as this introduction already witnesses, the word lattice denotes in this paper two different concepts: lattice as a module, and as a partially ordered set. Indeed the same is true of order, that can be meant in the usual sense but also as a ring. We hope this coincidence will not cause any misunderstanding and the meaning of any occurrence of lattice or order will always be clear.

2 The role of lattices

We mainly devote this section to some first results on lattices. We keep $R$, $\Lambda$ etc. in agreement with the introduction. Since $R$ is hereditary and Noetherian, $\Lambda$-lattices are closed under submodules. On the contrary, quotients of lattices need not be lattices. Since $\Lambda$ is Noetherian and $\Lambda$-lattices are finitely generated $\Lambda$-modules, $\Lambda$-lattices are Noetherian. Therefore they cannot contain a strictly increasing chain of submodules $M_1 \subseteq M_1 \oplus M_2 \subseteq M_1 \oplus M_2 \oplus M_3 \subseteq \ldots$ and hence must decompose as a finite direct sum of indecomposable modules (cf. [2, 2.2] for details). Thus we have the following fact.
Fact 2.1 Every lattice $M$ over an $R$-order $\Lambda$ decomposes as a finite direct sum of indecomposable lattices.

This decomposition may not be unique. In other words the category of lattices over an order may not be Krull-Schmidt (cf. [5, §36] for examples). But this is true over complete discrete valuation domains [5, 30.6].

Proposition 2.2 Let $R$ be a complete discrete valuation domain and $\Lambda$ be an $R$-order. Then every $\Lambda$-lattice $L$ is pure injective and the set of indecomposable $\Lambda$-lattices is dense in $Zg_{\Lambda}^f$.

Proof. Since $R$ is a discrete valuation domain and $L$ is finitely generated and torsion-free over $R$, as an $R$-module, $L$ is isomorphic to $R^n$. Since $R$ is complete, it is linearly compact as a module over itself, cf. [13, §4.2.2] for the definition of linear compactness. Since the class of linearly compact modules is closed under extensions [13, 4.2.10], $R^n$ is linearly compact. Since all pp-definable subgroups of $L$ as a $\Lambda$-module are $R$-submodules, $L$ is algebraically compact over $\Lambda$.

If $M$ is an $R$-torsion-free $\Lambda$-module, then it is a direct limit of its finitely generated submodules, which are lattices. Then $M$ is in the closure of these lattices.

When $Q\Lambda$ is separable and $R$ is complete, the category of lattices has almost split sequences (cf. [17], [18], and [1, 2.1]). We shall use this to show that every indecomposable $\Lambda$-lattice is isolated in $Zg_{\Lambda}^f$.

The following result may have its own interest and indeed will be used also later. Let $\varphi \in pp_{\Lambda}^f$ and let $(M, m)$, $m \in M^n$, be a free realisation of $\varphi$. Look at the pp-type of $m + Tor M$ in $M / Tor M$ and take a pp-formula $\overline{\varphi} \in pp_{\Lambda}^f$ generating this pp-type.

Lemma 2.3 The map $\varphi \mapsto \overline{\varphi}$ defines a $+\text{-semi-lattice}$ homomorphism from $pp_{\Lambda}^f$ to $pp_{\Lambda}^f$, such that $\overline{\varphi} \leq \varphi$ for every $\varphi$. Moreover, if $N$ is an $R$-torsion-free $\Lambda$-module then $\varphi(N) = \overline{\varphi}(N)$. The partially ordered set $\{ \overline{\varphi} \mid \varphi \in pp_{\Lambda}^f \}$ is isomorphic to $pp_{\Lambda}(Tf_{\Lambda})$ and hence is a lattice.

Proof. Let $(M, m)$ be a free realisation of $\varphi \in pp_{\Lambda}^f$. Since there is a homomorphism from $M$ to $M / Tor M$ sending $m$ to $m + Tor M$, we have $\overline{\varphi} \leq \varphi$. Suppose that $\psi \leq \varphi$ in $pp_{\Lambda}^f$. Let $(N, n)$ be a free realisation of $\psi$. Since $\psi \leq \varphi$ there is a homomorphism $f : M \rightarrow N$ with $f(m) = n$. Let $\overline{f} : M / Tor M \rightarrow N / Tor N$ be the homomorphism induced by $f$. Then $\overline{f}(m + Tor M) = n + Tor N$. Thus $\overline{\varphi} \leq \overline{\psi}$. This also shows that the map sending $\varphi$ to $\overline{\varphi}$ is well-defined.

We now just have to observe that for all $\varphi, \psi \in pp_{\Lambda}^f$, $\overline{\varphi + \psi} = \overline{\varphi} + \overline{\psi}$. This is true because if $(M, m), (N, n)$ are free realisations of $\varphi, \psi$ respectively, then, cf. [13, 1.2.27], $(M \oplus N, m + n)$ is a free realisation of $\varphi + \psi$ and $Tor(M \oplus N) = Tor M \oplus Tor N$.

Now suppose that $N$ is an $R$-torsion-free $\Lambda$-module. Suppose $n \in \varphi(N)$. There exists $f : M \rightarrow N$ such that $f(m) = n$. Since $N$ is $R$-torsion-free, $Tor M \subseteq ker f$. Thus the homomorphism $\overline{f} : M / Tor M \rightarrow N$ induced by $f$ satisfies $\overline{f}(m + Tor M) = n$. Hence $n \in \overline{\varphi}(N)$.

We now provide a detailed proof that the indecomposable $\Lambda$-lattices are isolated in $Zg_{\Lambda}^f$, when $R$ is complete and $Q\Lambda$ is separable, following that of the analogous result for Artin algebras.

Lemma 2.4 Let $R$ be a complete discrete valuation domain with field of fractions $Q$. $\Lambda$ be an $R$-order in a finite dimensional separable $Q$-algebra $A$. Then all indecomposable $\Lambda$-lattices are isolated in $Zg_{\Lambda}^f$. by a $Tf_{\Lambda}$-minimal pair.

Proof. As said, the category of $\Lambda$-lattices has left almost split morphisms (cf., e.g., [1, 2.1]). That is, for all indecomposable lattices $N$ there exists a homomorphism of lattices $f : N \rightarrow E$ such that $f$ is a non-split monomorphism and for any $\Lambda$-homomorphism of lattices $h : N \rightarrow X$ which is not a split monomorphism, there exists $\lambda \in Hom(E, X)$ such that $\lambda f = h$.

Pick $n \in N$ a generating tuple for $N$. Let $\varphi$ generate the pp-type of $n$ and $\psi$ generate the pp-type of $f(n)$. That $N \in (\varphi / \psi)$ follows exactly as in [13, 5.3.31].

Now we take any $\sigma \in pp_{\Lambda}^f$, and we claim that if $\sigma <_{Tf_{\Lambda}} \varphi$, then $\sigma \leq_{Tf_{\Lambda}} \psi$. Since we are working modulo the theory of $R$-torsion-free $\Lambda$-modules, we may replace $\sigma$ by $\overline{\sigma}$ (cf. Lemma 2.3). Moreover, since $\varphi = \overline{\varphi}, \overline{\sigma} < \varphi$. Let $M \in Latt_{\Lambda}$ and $m \in M$ be such that $(M, m)$ is a free realisation of $\overline{\sigma}$. Thus there is a homomorphism $h : N \rightarrow M$ such that $h(n) = m$. As $\overline{\sigma} < \varphi, h$ is non-split, and so, there exists $\lambda \in Hom(E, X)$ such that $\lambda f = h$. Then $\sigma \leq \psi$.

Thus $\sigma \leq_{Tf_{\Lambda}} \psi$.

Therefore $\varphi / \psi$ is a $Tf_{\Lambda}$-minimal pair. Hence $(\varphi / \psi)$ isolates $N$ in $Zg_{\Lambda}^f$. □
Note that if \( R \) is a complete discrete valuation domain and \( \text{Latt}_\Lambda \) has left almost split morphisms then the above result holds for \( \text{Zg}_\Lambda^{\rm{tf}} \) with the same proof.

Next let us deal with the closed points of \( \text{Zg}_\Lambda^{\rm{tf}} \). We start by proving an auxiliary result.

**Lemma 2.5** For all \( N \in \text{TF}_\Lambda \) and \( \varphi \in \text{pp}_\Lambda^1 \), \( Q\varphi(N) = \varphi(QN) \).

**Proof.** Let \( \varphi(x) \equiv \exists y \, (x, y) T_\varphi = 0 \) where \( T_\varphi \) is a matrix of a suitable size with entries in \( \Lambda \). Since \( N \) is a submodule of \( QN \), \( \varphi(N) \subseteq \varphi(QN) \). All pp-definable subsets of \( QN \) are \( Q \)-vector subspaces, so \( \varphi(QN) \subseteq \varphi(QN) \).

Now suppose that \( m \in \varphi(QN) \). There exists \( \ell = (\ell_1, \ldots, \ell_k) \in QN \) such that \( (m, \ell) T_\varphi = 0 \). Let \( c \in R \) be such that \( mc, \ell_1c, \ldots, \ell_kc \in N \). Then \( (mc, \ell c) T_\varphi = 0 \). So \( mc \in \varphi(N) \). Thus \( m \in \varphi(QN) \). Therefore \( \varphi(QN) = \varphi(QN) \). \( \square \)

When \( R \) is a complete discrete valuation domain, we are now able to describe the closure of a \( \Lambda \)-lattice.

**Proposition 2.6** Let \( R \) be a complete discrete valuation domain and \( \Lambda \) an order over \( R \). If \( N \) is an indecomposable \( \Lambda \)-lattice and \( M \) is in the (Ziegler) closure of \( N \) but is not equal to \( N \), then \( M \) is a direct summand of \( QN \). In particular, \( M \) is a closed point and \( \text{pp}_\Lambda^1(M) \) is of finite length.

**Proof.** Let \( M \) be in the closure of \( N \) but not equal to \( N \). Suppose that \( M \in \varphi(\psi) \). We aim to show that \( \psi(QN) \supseteq \varphi(QN) \). Since \( M \in \varphi(\psi) \) and \( M \) is in the closure of \( N \), \( N \in \varphi(\psi) \). If \( \psi(QN) \) were finite length as an \( R \)-module then the interval \( \langle \psi, \psi \rangle \subseteq \text{pp}_\Lambda^1(N) \) would be finite length. Let \( \psi = \{ \psi_0 \geq \psi_1 \geq \ldots \geq \psi_n + 1 \} \). Then \( \psi(\psi) = \bigcup_{i=0}^{n} \langle \psi_i, \psi_{i+1} \rangle \). Since \( \langle \psi_i, \psi_{i+1} \rangle \) is simple, by [19, 7.10], \( \langle \psi, \psi \rangle \) is a matrix of a suitable size with entries in \( \phi \).

Since \( R \) is Noetherian and \( N \) is finitely generated as an \( R \)-module, \( \varphi(N)/\psi(N) \) is finitely generated as an \( R \)-module. Thus, since \( R \) is a principal ideal domain [9, Ch. III, 7.3], \( \psi(N)/\psi(N) \) is isomorphic to \( R^n \oplus T \) where \( T \) is a finitely generated torsion \( R \)-module. Since \( \psi(N)/\psi(N) \) is infinite length as an \( R \)-module, \( n \geq 1 \). Thus there exists \( m \in \psi(N) \) such that \( \psi m^\ell \notin \psi(N) \) for all \( \ell \in \mathbb{N} \) where \( \pi \) denotes a generator of the maximal ideal of \( R \). By Lemma 2.5, \( m \in \varphi(QN) \) and \( m \notin \psi(QN) \).

Since \( N \) is a lattice, \( QN \) is finite dimensional. Let \( L_1, \ldots, L_m \) be the indecomposable summands of \( QN \). If \( \varphi(QN) \supseteq \psi(QN) \) then \( L_i \in \varphi(\psi) \) for some \( 1 \leq i \leq m \). Thus if \( M \in \varphi(\psi) \) then \( L_i \in \varphi(\psi) \) for some \( 1 \leq i \leq m \). So \( M \) is in the closure of \( \{ L_1, \ldots, L_m \} \). Since each \( L_i \) is a closed point, \( M \) is in the closure of \( \{ L_1, \ldots, L_m \} \) if and only if \( M = L_i \) for some \( 1 \leq i \leq m \). \( \square \)

Recall that the support of a \( \Lambda \)-module \( M \), \( \text{Supp}(M) \), is the intersection of \( \text{Zg}_\Lambda \) with the definable subcategory generated by \( M \), i.e., \( \text{Supp}(M) \) is the set of indecomposable pure injective \( \Lambda \)-modules \( N \) such that \( \text{pp} \)-pairs \( \varphi, \psi(M) = \psi(M) \) implies \( \varphi(N) = \psi(N) \). In particular, if \( M \in \text{Zg}_\Lambda \) then \( \text{Supp}(M) \) is the closure of \( M \) in \( \text{Zg}_\Lambda \). For \( M \in \text{Zg}_\Lambda \), we shall write \( \text{cl}(M) \) for the closure of \( M \) in \( \text{Zg}_\Lambda \).

**Lemma 2.7** If \( N \in \text{Zg}_\Lambda^{\text{tf}} \) and \( S \in \text{Supp}(QN) \) then \( S \) is in the closure of \( N \). In particular, if \( N \in \text{Zg}_\Lambda^{\text{tf}} \) and \( S \in \text{Zg}_\Lambda^{\text{tf}} \) is a direct summand of \( QN \), then \( S \) is in the closure of \( N \).

**Proof.** By Lemma 2.5, \( QN \) is in the definable subcategory generated by \( N \). So \( \text{Supp}(QN) \subseteq \text{Supp}(N) \). Therefore, if \( S \in \text{Supp}(QN) \) then \( S \in \text{Supp}(N) \) and hence \( S \in \text{cl}(N) \). Finally, if \( S \in \text{Zg}_\Lambda \) is a direct summand of \( QN \) then \( S \in \text{cl}(N) \) since definable subcategories are closed under direct summands. \( \square \)

**Corollary 2.8** If \( N \in \text{Zg}_\Lambda^{\text{tf}} \) is a closed point then \( N \in \text{Zg}_\Lambda \) and \( N \) is a closed point in \( \text{Zg}_\Lambda \).

**Proof.** By Lemma 2.7, if \( N \in \text{Zg}_\Lambda^{\text{tf}} \) is a closed point then \( \text{Supp}(QN) = \text{Supp}(N) \). So, in particular \( N \) is in the definable subcategory generated by \( QN \) and hence \( R \)-divisible. Therefore \( N \in \text{Zg}_\Lambda \). \( \square \)

We include in this section some further useful remarks. For every \( R \)-module \( M \), let \( \text{Sub}_R(M) \) be the lattice of \( R \)-submodules of \( M \). For the definition of the \( \text{m} \)-dimension of a modular lattice, cf. [13, § 7.2].

**Lemma 2.9** Let \( R \) be a Dedekind domain and \( M \) a torsion-free finitely generated module over \( R \). Then \( \text{Sub}_R(M) \) has \( \text{m} \)-dimension 1.

**Proof.** Since \( M \) is torsion-free, \( R \) is a submodule of \( M \) and hence \( \text{Sub}_R(M) \) is not of finite length. Thus \( \text{m} \)-dim(\( \text{Sub}_R(M) \)) \( \neq 0 \). Since \( M \) is finitely generated and torsion-free, \( M \) is a direct summand of \( R^n \) for some...
positive integer \( n \). It follows that, if \( \text{m-dim}(\text{Sub}_R(R^n)) \leq 1 \) for all \( n \in \mathbb{N} \), then \( \text{m-dim}(\text{Sub}_R(M)) \leq 1 \) and hence \( \text{m-dim}(\text{Sub}_R(M)) = 1 \).

On the other hand, since \( R^n \) can be filtered as a finite chain of submodules with quotients isomorphic to \( R \), \( \text{m-dim}(\text{Sub}_R(R^n)) = \text{m-dim}(\text{Sub}_R(R)) \).

Let \( J \) be a non-zero ideal of \( R \). Then \( R/J \) is of finite length. Thus the interval \([J, R]\) in \( \text{Sub}_R(R) \) is of finite length. So \( \text{m-dim}(\text{Sub}_R(R)) = 1 \).

\[ \text{Corollary 2.10} \quad \text{Let } R \text{ be a Dedekind domain, } \Lambda \text{ an order over } R \text{ and } M \text{ a } \Lambda\text{-lattice. Then } \text{pp}^1_\Lambda(M) \text{ has m-dimension } I. \]

Proof. Since all pp-definable subgroups of \( M \) are \( R\)-submodules, \( \text{pp}^1_\Lambda(M) \) is a sublattice of \( \text{Sub}_R(M) \). Thus \( \text{m-dim}(\text{pp}^1_\Lambda(M)) \leq \text{m-dim}(\text{Sub}_R(M)) = 1 \). Since \( M \) is \( R\)-torsion-free and not \( R\)-divisible, there exist \( c \in R \setminus \{0\} \) and \( m \in M \) such that \( m \not\in Mc \). Since \( M \) is \( R\)-torsion-free, \( mc \not\in Mc+1 \). Thus \( Mc+1 \subseteq Mc \) for all \( n \in \mathbb{N} \). Thus \( \text{pp}^1_\Lambda(M) \) is not of finite length, whence its m-dimension cannot be 0.

\[ \Box \]

\[ \text{3 The torsion-free part of the Ziegler spectrum} \]

In this section we extend the main results of [11], about the torsion-free part of the Ziegler spectrum of a group ring \( RG \), with \( R \) a Dedekind domain of characteristic 0 and \( G \) a finite group, to arbitrary orders \( \Lambda \) over a Dedekind domain \( R \) in a semisimple \( Q\)-algebra \( A = QA \). We also investigate the m-dimension of \( \text{pp}^1_\Lambda(Tf_\Lambda) \) and the Cantor-Bendixson rank of \( Zg^{tf}_\Lambda \) in both this framework and the more general setting where \( A \) is not assumed to be semisimple.

If \( P \) is a prime ideal of \( R \) then we write \( \Lambda_P \) for the central localisation of \( \Lambda \) at \( P \) and \( \widetilde{\Lambda}_P \) for its completion at \( P \). Note that \( \Lambda_P \) is an \( R_P\)-order in \( \Lambda \) and \( \widetilde{\Lambda}_P \) is an \( R_P\)-order.

We start in the general setting. The endomorphism ring of every indecomposable pure injective module \( N \), \( \text{End}(N) \), is local. Let \( P(N) \) denote its maximal ideal. When \( N \) is \( R\)-torsion-free, \( R \) embeds in a natural way into \( \text{End}(N) \). Moreover \( P(N) \cap R \) is a prime ideal of \( R \). Thus, every indecomposable pure injective \( \Lambda\)-module is a module over \( \Lambda_P \) for some prime and even maximal ideal \( P \) of \( R \). The homomorphism \( \Lambda \rightarrow \Lambda_P \) is an epimorphism and hence restriction of scalars induces an embedding of \( Zg^{tf}_{\Lambda_P} \) into \( Zg^{tf}_\Lambda \) whose image is a closed subset. This embedding restricts to an embedding of \( Zg^{tf}_{\Lambda_P} \) into \( Zg^{tf}_{\Lambda_P} \) and again the image is a closed subset. Identifying \( Zg^{tf}_{\Lambda_P} \) with the set of \( N \in Zg^{tf}_\Lambda \) such that \( P(N) \cap R \subseteq P \), we may write

\[ Zg^{tf}_{\Lambda_P} = \bigcup_{P} Zg^{tf}_{\Lambda_P} \]

where \( P \) ranges over maximal ideals of \( R \). Since \( R \) has Krull dimension 1, if \( P, P' \) are distinct maximal ideals of \( R \) then \( Zg^{tf}_{\Lambda_P} \cap Zg^{tf}_{\Lambda_P'} = Zg^{tf}_\Lambda \).

This description of the space is not particularly useful for computing the Cantor-Bendixson rank of \( Zg^{tf}_{\Lambda_P} \) because if \( T \) is a topological space, \( X \) is a closed subset of \( T \) and \( p \in X \) then the Cantor-Bendixson rank of \( p \) as a point in \( X \) may be strictly less than the Cantor-Bendixson rank of \( p \) as a point in \( T \). Thus we now work to give a more useful description.

Since \( R \) is Noetherian, every (maximal) ideal \( P \) of \( R \) is finitely generated, whence there is a pp-formula of \( L_R \) defining in any \( R\)-module \( M \) just \( MP \); if \( r = (r_1, \ldots, r_\ell) \) is a generating tuple of \( P \), it suffices to take \( \exists y_1 \ldots \exists y_\ell \ (x = y_1 r_1 + \cdots + y_\ell r_\ell) \). Let \( P | x \) denote this formula. For instance, when \( R \) is a discrete valuation domain and \( \pi \) is a generator of its maximal ideal \( P \), then the formula is \( \pi | x \), i.e., \( \exists y (x = \pi y) \).

If \( N \in Zg^{tf}_{\Lambda_P} \) and \( P \not\in P' \) then \( NP = N \) since some element of \( P \) is not in \( P' \), i.e., some element of \( P \) acts invertibly on \( N \).

Now suppose that \( N \in Zg^{tf}_{\Lambda_P} \) and \( N \not\in (x = x/P|x) \). Let \( (r_1, \ldots, r_\ell) \) still denote a tuple of generators of \( P \). Since \( R_P \) is a valuation domain, there exists \( 1 \leq j \leq \ell \) such that \( r_j \in r_j R_P \) for all \( 1 \leq i \leq \ell \). Put \( r = r_j \) and for all \( i \) write \( r_i = r c_i/a_i \) with \( c_i, a_i \in R \) and \( a_i \not\in P \). Then \( r c_i = r_i a_i \) for all \( i \). Set \( a = \prod_{1 \leq k \leq \ell} a_k \). Then \( a \in R/P \).

Multiply the \( i \)th equation above by \( \prod_{1 \leq k \leq \ell} a_k \) and get for every \( i \) a new equation \( r b_i = r_i a \) for a suitable \( b_i \in R \).

Now, for all \( i \), \( Nr_i a \subseteq Nr_i \). Since \( a \not\in P \) and \( N \in Zg^{tf}_{\Lambda_P} \), \( a \) acts invertibly on \( N \) and hence \( Nr_i = Nr_i a \subseteq Nr_i \).

Therefore \( NP = Nr_i \). Hence the fact that \( NP = N \) implies that \( r \) acts invertibly on \( N \). So \( P \cap N \subseteq P \).
Therefore \( P(N) \cap R \), as a prime ideal of \( R \), coincides with 0. So \( N \) is \( R \)-divisible, i.e., \( N \in Zg_\Lambda \). Thus we have shown that

\[
Zg_\Lambda^{\Omega} = \bigcup_P ((x = x/P | x) \cap Zg_\Lambda^{\Omega}) \cup Zg_\Lambda
\]

where \( P \) ranges over maximal ideals of \( R \).

As promised, we now generalise the main results of [11] to orders in semisimple algebras. A large part of the proof is the same as over group rings, but adaptions are sometimes necessary.

**Theorem 3.1** Let \( R \) be a Dedekind domain with field of fractions \( Q \), and \( \Lambda \) an \( R \)-order in a semisimple \( Q \)-algebra \( A \). If \( N \in Zg_\Lambda^{\Omega} \), then either \( N \) is a simple \( A \)-module, or there is some maximal ideal \( P \) of \( R \) such that \( N \in Zg_{\hat{\Lambda}_P}^{\Omega} \) and \( N \) is \( \hat{P} \)-reduced. Moreover, if \( N \in Zg_{\hat{\Lambda}_P}^{\Omega} \) is \( \hat{R} \)-reduced then \( N \in Zg_\Lambda^{\Omega} \).

Here \( N \) being \( \hat{R} \)-reduced means that \( \bigcap_{i=0}^\infty N P^i = 0 \). Recall that \( \hat{\Lambda}_P \) is an order over \( \hat{R}_P \) in \( \hat{A} = \hat{Q} \otimes Q A \). Where \( \hat{Q} \) denotes the field of fractions of \( \hat{R}_P \).

**Proof.** We follow the proof of [11, Theorem 2.1]. For simplicity we divide our argument in several steps. Let \( N \) be an indecomposable pure injective \( R \)-module.

**Step 1:** For some maximal ideal \( P \) of \( R \), \( N \) is a module over \( \Lambda_P \). This step has already been covered in the discussion preceding this theorem. Let \( \pi \) denote a generator of the maximal ideal \( PR_P \) of \( \hat{R}_P \).

**Step 2:** Any \( \Lambda_P \)-module divisible and torsion-free over \( \hat{R}_P \) is injective over \( \Lambda_P \). The proof is the same as [11, Claim 2, proof of 2.1].

**Step 3:** \( N \), as an \( \hat{R}_P \)-torsion-free module over \( \Lambda_P \), decomposes over \( \Lambda_P \) as \( N' \oplus N'' \) where \( N'' \) is \( \hat{R}_P \)-divisible (hence an \( A \)-module) and \( N' \) is \( \hat{R}_P \)-reduced, i.e., \( \bigcap_{i=0}^\infty N' P^i = \bigcap_{i=0}^\infty N'' \pi^i = 0 \).

To prove this claim, first we put \( N'' := \{ m \in N | \pi^n|m \text{ for all } n \in \mathbb{N} \} \). Take \( n \in \mathbb{N} \), \( m, m' \in N'' \) and \( r \in \Lambda \). Since \( m, m' \in N'' \) there exists \( a, a' \in N \) such that \( m = a \pi^n \) and \( m' = a' \pi^n \). Thus \( mr + m'r = a \pi^n \pi r + a' \pi^n = (ar + a') \pi^n \) because \( \pi \) is central. Thus \( N'' \) is a submodule of \( N \). Since \( N'' \) is \( \hat{R}_P \)-divisible by definition, it is injective by Step 2 and thus a direct summand of \( N \).

Let \( N' \) be a complement of \( N'' \) in \( N \). If \( m \in \bigcap_{i=0}^\infty N' \pi^i \) then \( \pi^n|m \) for all \( n \in \mathbb{N} \) and thus \( m \in N'' \). So \( m = 0 \).

This concludes Step 3.

As \( N \) is indecomposable, either (a) \( N \) is an \( \Lambda \)-module or (b) \( N \) is \( \hat{R}_P \)-reduced. In the former case \( N \) must be a simple \( \Lambda \)-module. So let us turn to (b). We assume from now on that \( N \) is \( \hat{R}_P \)-reduced.

**Step 4:** Every \( \hat{R}_P \)-reduced pure injective \( \Lambda_P \)-module \( M \) can be equipped with a \( \hat{\Lambda}_P \)-module structure, and \( M \) remains pure injective over \( \hat{\Lambda}_P \).

To see this, one proceeds exactly as in [11, proof of Theorem 2.1, pp. 1128-1129]. Suppose \( r \in \hat{\Lambda}_P \) and \( m \in M \). For each \( i \in \mathbb{N} \), let \( r_i \in \Lambda_P \) satisfy \( \pi^i | r - r_i \). For each \( i \in \mathbb{N} \), look at the equation \( x - mr_i = y_i \pi^i \). When \( i \) ranges over \( \mathbb{N} \), this set of equations is finitely solvable and so, since \( M \) is pure injective, solvable in \( M \). Let \( n, n' \in \mathbb{M} \) be such that \( \pi^i | n - mr_i \) and \( \pi^i | n' - mr_i \) for all \( i \). Then \( n - n' \in \bigcap_{i=0}^\infty M \pi^i = 0 \).

Define \( mr \) to be the unique element \( n \in M \) such that \( \pi^i | n - mr_i \) for all \( i \). Note that this definition of \( mr \) does not depend on the particular choice of \( r_i \) above. If \( r, s \in \hat{\Lambda}_P \) then \( mr \) is the unique element \( m_1 \in M \) such that \( \pi^i | m_1 - mr_i \) and \( ms \) is the unique element \( n_2 \in M \) such that \( \pi^i | m_2 - ms_i \). Thus \( \pi^i | m_1 + m_2 - m(r_i + s_i) \) for all \( i \). Hence \( m_1 + m_2 = m(r + s) \).

That \( M \) is pure injective as a \( \hat{\Lambda}_P \)-module is a consequence of [11, Lemma §2, p. 1129].

Conversely, independently of the assumption that \( N \) is \( \hat{R}_P \)-reduced, every pure injective \( \hat{\Lambda}_P \)-module \( N \) remains pure injective after restricting it over \( \Lambda_P \). This is simply because \( \Lambda_P \) is a subring of its \( P \)-adic completion. For the same reason any decomposable module over \( \hat{\Lambda}_P \) is decomposable over \( \Lambda_P \). On the other hand the following holds.

**Step 5:** If \( N \) is an \( \hat{R}_P \)-reduced indecomposable pure injective \( \hat{\Lambda}_P \)-module, then \( N \) is indecomposable as a \( \Lambda \)-module. This is explained in [11, Remarks 1, §2, p. 1130].

The above theorem has shown that, when \( A \) is semisimple, \( \hat{R}_P \)-reduced \( R \)-torsion-free indecomposable pure injective modules are the same over \( \Lambda_P \) and over \( \hat{\Lambda}_P \). Moreover, the \( \hat{R}_P \)-reduced \( R \)-torsion-free indecomposable pure injective modules are exactly those modules in the open set \( \{ x = x/P | x \} \). For this reason, we shall sometimes write \( Zg_{\hat{\Lambda}_P}^{\Omega-\text{red}} \) for this open set.
Note that any module \( N \in \mathbb{Z}G^f_{\Lambda} \) that can be regarded as a \( \Lambda_P \)-module but does not belong to \( (x = x/P \mid x) \), i.e., satisfies \( N = PN \), is an \( A \)-module. Furthermore any two different sets \( (x = x/P \mid x) \cap \mathbb{Z}G^f_{\Lambda} \) are disjoint from each other. So we get the following corollary to Theorem 3.1.

**Corollary 3.2** With the same hypothesis as 3.1, \( \mathbb{Z}G^f_{\Lambda} \) is the disjoint union of \( \mathbb{Z}G_{\Lambda} \) and of the various \( \mathbb{Z}G^f_{\Lambda_P} \) where \( P \) ranges over maximal ideals of \( R \).

Now let us deal with the topology. Notice that for every maximal ideal \( P \) of \( R \) the embedding of \( \Lambda \) into \( \hat{\Lambda}_P \) induces an inclusion of the \( R_P \)-reduced part of \( (x = x/P \mid x) \cap \mathbb{Z}G^f_{\Lambda} = \mathbb{Z}G^f_{\Lambda_P} \) into \( \mathbb{Z}G^f_{\Lambda} \). By the same argument given in [11, Theorem 2.2] over group rings, this inclusion is homeomorphic:

**Theorem 3.3** Suppose that \( A = Q \Lambda \) and \( \hat{\Lambda} = \hat{Q} \Lambda \) (where \( \hat{Q} \) is the field of fractions of \( \hat{R}_P \)) are semisimple. For every maximal ideal \( P \) of \( R \), \( (x = x/P \mid x) \cap \mathbb{Z}G^f_{\Lambda} = \mathbb{Z}G^f_{\Lambda_P} \) has the same topology whether viewed as a subspace of \( ZG^f_{\Lambda_P} \) or of \( ZG^f_{\Lambda} \).

Here we give a different proof of a slightly stronger claim. First of all, observe that every pp-formula \( \alpha \) of \( L_{\Lambda_P}, \alpha \equiv \exists y (xS = yT) \), with \( S, T \) matrices of suitable sizes with entries in \( \Lambda_P = R_P \Lambda \), can be translated into a pp-formula \( \alpha' \) of \( \Lambda_P \) which is equivalent to \( \alpha \) in all \( R \)-torsion-free \( \Lambda_P \)-modules. To build \( \alpha' \), calculate the product \( r \) of all multiplicative inverses of scalars of \( R \) occurring in the entries of \( S \) and \( T \). Then \( r \in R \setminus P \), in particular \( r \neq 0 \). Now multiply the previous scalars by \( r \) and get \( \alpha' \) as required, as \( \exists y (xS = yT) \). In fact the entries of \( rS \) and \( rT \) are in \( \Lambda \). The torsion-free condition ensures the equivalence to \( \alpha \). That is, for every \( R \)-torsion-free \( \Lambda_P \)-module \( M \) and \( m, n \) in \( M, r(mS - nT) = 0 \) if and only if \( mS - nT = 0 \).

Thus we have to compare \( \Lambda_P \) and \( \hat{\Lambda}_P \). We may now assume that \( R \) is a discrete valuation domain and \( \pi \) is a generator of its maximal ideal \( P \); \( \pi \) is still the field of fraction of \( R \), \( A \) a finite dimensional \( Q \)-algebra, \( \Lambda \) an order over \( R \) in \( A \), \( \hat{\Lambda} \) its \( \pi \)-adic completion. We also assume both \( A \) and \( \hat{\Lambda} \) semisimple, which is true, in particular when \( A \) is separable. Under these conditions the following propositions hold.

**Proposition 3.4** Suppose that \( R \) is a discrete valuation domain whose maximal ideal is generated by \( \pi \) and that both \( A \) and \( \hat{\Lambda} \) are semisimple. The closed intervals \( [\pi \mid x, x = x] \) and \( [\pi \mid x, x = x] \) are isomorphic as lattices. Moreover the Ziegler open sets \( (x = x/\pi \mid x) \) in \( \mathbb{Z}G^f_{\Lambda} \) and \( (x = x/\pi \mid x) \) in \( \mathbb{Z}G^f_{\hat{\Lambda}} \) are homeomorphic.

**Proof.** For \( 1 \leq j \leq n \) and \( 1 \leq k \leq \ell \), let \( s_j, r_{jk} \in \hat{\Lambda} \), and let \( \phi \) be the pp-formula

\[
\exists y_1 \exists y_2 \ldots \exists y_{\ell} \bigwedge_{j=1}^{n} (xs_j + \sum_{k=1}^{\ell} y_k r_{jk} = 0).
\]

Further suppose that \( \pi | x \leq \phi \).

For \( i \in \mathbb{N} \), let \( s_j^i, r_{jk}^i \in \hat{\Lambda} \) be such that \( \pi^i \mid s_j^i = s_j \) and \( \pi^i \mid r_{jk}^i = r_{jk} \). For each \( i \in \mathbb{N} \), let \( \phi_i \) be the pp-formula

\[
\exists y_1 \exists y_2 \ldots \exists y_{\ell} \bigwedge_{j=1}^{n} \pi^i | (xs_j^i + \sum_{k=1}^{\ell} y_k r_{jk}^i).
\]

Clearly, \( \phi \leq \phi_i \) and \( \phi_i \geq \phi_{i+1} \) for each \( i \in \mathbb{N} \).

We now show that for all indecomposable pure injective \( \hat{\Lambda} \)-modules \( N \), \( \bigcap_{i \in \mathbb{N}} \phi_i(N) = \phi(N) \). Since \( \hat{\Lambda} \) is semisimple, every indecomposable pure injective \( \hat{R} \)-torsion-free \( \hat{\Lambda} \)-module \( N \) is either \( R_P \)-reduced or \( \hat{R} \)-divisible. In the latter case, since \( \phi_i \geq \phi \geq \pi | x \), \( \phi(N) = \phi(N) = N \). Hence assume that \( N \) is reduced. Suppose that \( m \in \phi_i(N) \) for all \( i \in \mathbb{N} \). Then the infinite system of linear equations

\[
ms_j^i + \sum_{k=1}^{\ell} y_k r_{jk}^i = z_j^i \pi^i
\]

where \( i \in \mathbb{N} \), \( 1 \leq j \leq n \) and \( 1 \leq k \leq \ell \), is finitely solvable. Consequently, since \( N \) is pure injective, it is solvable say with \( y_k = a_k \in \mathbb{N} \). So for each \( 1 \leq j \leq n \), \( ms_j + \sum_{k=1}^{\ell} a_k r_{jk}^i \in N \pi^i \) for all \( i \in \mathbb{N} \). Thus, since \( N \) is reduced, \( ms_j + \sum_{k=1}^{\ell} a_k r_{jk}^i = 0 \). Thus \( m \in \phi(N) \).

We now show that there exists an \( i \in \mathbb{N} \) such that \( \phi_i = \phi \). Suppose that \( \phi < \phi_i \) for all \( i \in \mathbb{N} \) with respect to the theory of \( \hat{R} \)-torsion-free \( \hat{\Lambda} \)-modules. Let \( \bar{F} \) be the filter generated by \( \{ \phi_i \mid i \in \mathbb{N} \} \) and \( I \) be the ideal generated
by \( \varphi \). Since \( F \cap I \) is empty, by [12, 433], we can construct an irreducible pp-type \( p \) such that \( \varphi_i \in p \) for all \( i \in \mathbb{N} \) and \( \varphi \notin p \). Now, \( N(p) \), the pure injective hull of \( p \), has an element \( m \) such that \( m \in \varphi(N(p)) \) for all \( i \in \mathbb{N} \) but \( m \notin \varphi(N(p)) \). This contradicts the fact that \( \varphi(N) = \bigcap_{i \in \mathbb{N}} \varphi_i(N) \) for all indecomposable pure injective \( R \)-torsion-free \( \Lambda \)-modules \( N \). Thus \( \varphi_i = \varphi \) for some \( i \in \mathbb{N} \).

Restriction of scalars gives a lattice homomorphism from \( pp^1_\Lambda(T_f \Lambda) \) to \( pp^1_\Lambda(T_f \Lambda) \). We have shown that the restriction of this map to \( [\tau | x, x = \bar{x}]_{T_f(\Lambda)} \) has image \( [\tau | x, x = \bar{x}]_{T_f(\Lambda)} \). Suppose that \( \varphi \geq \psi \geq \pi \) in \( pp^1_\Lambda(T_f \Lambda) \). There exists an \( R \)-torsion-free indecomposable pure injective \( \Lambda \)-module \( N \) such that \( \varphi(N) \supseteq \psi(N) \). Since \( \varphi \neq \psi \), this implies that \( N \) is \( R \)-reduced and hence \( N \) can be equipped with the structure of a \( \Lambda \)-module. Hence \( \varphi > \psi \) as pp-formulas in \( pp^1_\Lambda(T_f \Lambda) \). Thus we have shown that \( [\tau | x, x = \bar{x}]_{T_f(\Lambda)} \) and \( [\tau | x, x = \bar{x}]_{T_f(\Lambda)} \) are isomorphic as lattices.

Since both \( A \) and \( \hat{A} \) are semisimple, the open sets \( (x = x/\pi | x) \subseteq Zg^\mu_\Lambda \) and \( (x = x/\pi | x) \subseteq Zg^\mu_\Lambda \) contain exactly the reduced indecomposable pure injective modules. Thus restriction of scalars gives a bijection from \( (x = x/\pi | x) \subseteq Zg^\mu_\Lambda \) to \( (x = x/\pi | x) \subseteq Zg^\mu_\Lambda \). Since the sets of the form \( (\varphi/\psi) \) for \( \pi \leq \psi < \varphi \) in \( pp^1_\Lambda(T_f \Lambda) \) (respectively \( \pi \leq \varphi < \psi \) in \( pp^1_\Lambda(T_f \Lambda) \)) both have \( \varphi \) as a pp-pair and each \( x/\pi = \varphi \) has Cantor-Bendixson rank.

Having concluded the proof of Theorem 3.3, we now give some consequences of what we have shown so far for the Cantor-Bendixson rank of \( Zg^\mu_\Lambda \) and the m-dimension of \( pp^1_\Lambda(T_f \Lambda) \).

**Remark 3.5** Let \( X \) be a topological space and \( U_i \) for \( i \in I \) be open sets. Then \( X \) has Cantor-Bendixson rank if and only if \( U_i \) has Cantor-Bendixson rank for all \( i \in I \) and \( X \setminus \bigcup_{i \in I} U_i \) has Cantor-Bendixson rank.

**Proof.** Suppose \( U_i \) has Cantor-Bendixson rank for all \( i \in I \) and \( X \setminus \bigcup_{i \in I} U_i \) has Cantor-Bendixson rank. Let \( \alpha \) be an ordinal greater than the Cantor-Bendixson rank of any \( U_i \). Then \( X^{(\alpha)} \subseteq X \setminus \bigcup_{i \in I} U_i \). So \( X^{(\alpha)} \) has Cantor-Bendixson rank and hence \( X \) has Cantor-Bendixson rank.

The following proposition now follows from the above remark using Theorem 3.3 & Corollary 3.2.

**Proposition 3.6** Let \( A \) be semisimple. Then \( Zg^\mu_\Lambda \) has Cantor-Bendixson rank if and only if \( Zg^\mu_\Lambda \) and each \( Zg^\mu_\Lambda \), with \( P \) a maximal ideal of \( R \), has Cantor-Bendixson rank.

Next let us deal with the m-dimension of \( pp^1_\Lambda(T_f \Lambda) \). Recall the connection between m-dimension and Cantor-Bendixson rank [13, 5.3.60], at least under the isolation condition. The latter requires that for every closed subset \( C \) of \( Zg^\mu_\Lambda \) (indeed, in our case, of \( Zg^\mu_\Lambda \)) and every isolated point \( N \) of \( C \), there is a pp-pair \( \varphi/\psi \) which is minimal such that \( (\varphi/\psi) \cap C = N \) (cf. [13, 5.3.2]).

We first prove a general statement which we were not able to find elsewhere in the literature.

**Lemma 3.7** Let \( S \) be an arbitrary ring, \( X \) a closed subset of \( Zg^\mu_\Lambda \) and \( \{\varphi_i/\psi_i | i \in I\} \) a set of pp-pairs. Then \( pp^1_\Lambda(X) \) has m-dimension if and only if \( \{\varphi_i/\psi_i | x\} \) has m-dimension for all \( i \in I \) and \( pp^1_\Lambda(X \setminus \bigcup_{i \in I}(\varphi_i/\psi_i)) \) has m-dimension.

**Proof.** If \( pp^1_\Lambda(X) \) has m-dimension then \( pp^1_\Lambda(X \setminus \bigcup_{i \in I}(\varphi_i/\psi_i)) \) has m-dimension since it is a quotient of \( pp^1_\Lambda(X) \) and for all \( i \in I \), \( \{\varphi_i/\psi_i | x\} \) has m-dimension because \( \{\varphi_i/\psi_i | x\} \) is a sublattice of \( pp^1_\Lambda(X) \) for some \( n \in \mathbb{N} \).

Now suppose that \( \{\varphi_i/\psi_i | x\} \) has m-dimension for all \( i \in I \) and \( pp^1_\Lambda(X \setminus \bigcup_{i \in I}(\varphi_i/\psi_i)) \) has m-dimension. We first show that for all \( N \in X, \) \( N \) has an \( N \)-minimal pair, i.e., \( pp^1_\Lambda(N) \) has a simple interval.

Firstly, suppose that \( N \notin (\varphi_i/\psi_i) \) for all \( i \in I \). Then \( cl(N) \subseteq X \setminus \bigcup_{i \in I}(\varphi_i/\psi_i) \). Thus \( pp^1_\Lambda(cl(N)) = pp^1_\Lambda(N) \) has m-dimension and hence \( N \) has an \( N \)-minimal pair. Now suppose that \( N \in (\varphi_i/\psi_i) \) for some \( i \in I \). Since \( \{\varphi_i/\psi_i | x\} \) has m-dimension, so does \( \{\varphi_i/\psi_i | N\} \). Thus \( \{\varphi_i/\psi_i | N\} \) contains a simple interval. So \( N \) has an \( N \)-minimal pair.

Therefore, by [13, 5.3.16], the isolation condition holds for \( X \). So, by [13, 5.3.60], \( X \) has Cantor-Bendixson rank if and only if \( pp^1_\Lambda(X) \) has m-dimension. We now show that all points in \( X \) have Cantor-Bendixson rank.

Let \( \alpha > m-dim(\{\varphi_i/\psi_i | x\}) \) for all \( i \in I \). Then, by [13, 5.3.59], \( \{\varphi_i(M) | i \} \) for all \( M \in X^{(\alpha)} \). Therefore \( X \setminus \bigcup_{i \in I}(\varphi_i/\psi_i) \subseteq X^{(\alpha)} \). Since \( pp^1_\Lambda(X \setminus \bigcup_{i \in I}(\varphi_i/\psi_i)) \) has m-dimension, so does \( pp^1_\Lambda(X^{(\alpha)}) \). So \( X^{(\alpha)} \) has Cantor-Bendixson rank and hence, \( X \) also has Cantor-Bendixson rank.

**Corollary 3.8** The lattice \( pp^1_\Lambda(T_f \Lambda) \) has m-dimension if and only if, for all maximal ideals \( P \), \( \{P | x, x = x \} \) has m-dimension and \( pp^1_{\hat{Q}_\Lambda} \) has m-dimension.
\textbf{Proof.} Note that $Zg^i_{\Lambda}\left(\bigcup_{\rho}(x = x/P \mid x)\right)$ just consists of the $R$-divisible modules, i.e., of modules over $Q \otimes \Lambda$. \hfill \qedsymbol

The following is an easy consequence of the beginning of the proof of 3.7.

\textbf{Remark 3.9} $m\text{-dim}(\text{pp}^1_{\Lambda}(Tf_{\Lambda})) \geq m\text{-dim}([P \mid x, x = x]_{Tf_{\Lambda}})$ for every maximal ideal $P$ of $R$, and $m\text{-dim}(\text{pp}^p_{\Lambda}(Tf_{\Lambda})) \geq m\text{-dim}([P \mid x, x = x]_{Tf_{\Lambda}})$.

Specialising to the case where $A = QA$ is semisimple we get the following.

\textbf{Lemma 3.10} Suppose that $A = QA$ is semisimple and $[P \mid x, x = x]_{Tf_{\Lambda}}$ has $m\text{-dimension}$ for every maximal ideal $P$ of $R$. Let $\alpha$ be the least ordinal such that $\alpha > m\text{-dim}([P \mid x, x = x]_{Tf_{\Lambda}})$ for all maximal ideals $P$ of $R$. Then $m\text{-dim}(\text{pp}^1_{\Lambda}(Tf_{\Lambda})) = \alpha$.

\textbf{Proof.} Since $A$ is semisimple, $\text{pp}^1_{\Lambda}$ has $m\text{-dimension}$ zero. So by Corollary 3.8, $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ has $m\text{-dimension}$. Thus the $m\text{-dimension}$ of $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ is equal to the Cantor-Bendixson rank of $Zg^i_{\Lambda}$.

To improve readability, let $X := Zg^i_{\Lambda}$. Note that $X^{(\alpha)} \cap (x = x/P \mid x) = \emptyset$ for all maximal ideals $P$ of $R$. Thus $Z_{QA} \supseteq X^{(\alpha)}$. Since $A$ is semisimple, all points in $Z_{QA}$ are isolated. Thus $X^{(\alpha+1)} = \emptyset$. Thus we just need to show that $X^{(\alpha)} \neq \emptyset$. We deal with the cases where $\alpha$ is a successor ordinal and $\alpha$ is a limit ordinal separately.

Suppose $\alpha = \beta + 1$. There exists a maximal ideal $P$ of $R$ such that $\beta = m\text{-dim}([P \mid x, x = x])$. Thus $X^{(\beta)} \cap (x = x/P \mid x) \neq \emptyset$. Take $N \in X^{(\beta)} \cap (x = x/P \mid x)$ and let $L$ be an indecomposable direct summand of $QN$. By Lemma 2.7, $L$ is in the closure of $N$. Thus $L \in X^{(\alpha)}$.

Now suppose $\alpha$ is a limit ordinal. For all $\beta < \alpha$, there exists a maximal ideal $P$ of $R$ such that $(x = x/P \mid x) \cap X^{(\beta)}$ is non-empty. Thus $X^{(\beta)} \neq \emptyset$ for all $\beta < \alpha$. Since $Zg^i_{\Lambda}$ is compact, $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$ is non-empty. \hfill \qedsymbol

\textbf{Corollary 3.11} Suppose that $A = QA$ is semisimple and $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ has $m\text{-dimension}$ for every maximal ideal $P$ of $R$. Then the $m\text{-dimension}$ of $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ is equal to the supremum of the $m\text{-dimension}$s of $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ where $P$ ranges over maximal ideals of $R$. Moreover, if $A = QA$ is separable, the $m\text{-dimension}$ of $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ is equal to the supremum of the $m\text{-dimension}$s of $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ where $P$ ranges over maximal ideals of $R$.

We shall come back to $m\text{-dimension}$ and Cantor-Bendixson rank after dealing with orders of finite lattice representation type. We shall show, in Proposition 5.1, that if $\Lambda$ has finite lattice representation type then the $m\text{-dimension}$ of $\text{pp}^1_{\Lambda}(Tf_{\Lambda})$ is 1. This will allow us, in Corollary 5.2, to improve the above corollary when $QA$ is separable.

\section{Applications of Maranda’s Theorem}

Let us open a short parenthesis on Maranda’s Theorem. Following [5, 30.12], we assume throughout this section that $R$ is a discrete valuation domain, $\pi$ is a generator of its maximal ideal $P$, $Q$ is the field of fractions of $R$, $\Lambda$ is a finite dimensional \textit{separable} $Q$-algebra and $\Lambda$ is an $R$-order in $A$.

We shall deal with the quotient ring $\Lambda/\pi^k \Lambda$, often abbreviated as $\Lambda_k$, for every positive integer $k$. Similarly, for every $\Lambda$-module $M$, $M_k$ will denote the quotient module $M/\pi^k M$.

There is a non-negative integer, and hence a minimal non-negative integer $k_0$ such that $\pi^{k_0} \text{Ext}^1(L, N) = 0$ for all $\Lambda$-modules $L$ and $N$ (cf. [5, 30.13, § 30A]). For instance, when $\Lambda = RG$ for some finite group $G$, then $k_0$ is the largest integer such that $|G| \in \pi^{k_0} R$. Note that, since $\Lambda$ is Noetherian, $\text{Ext}^1(L, -)$ is a finitely presented functor from $\Lambda$-modules to abelian groups (cf. [13, 10.2.35]). Hence $\pi^{k_0} \text{Ext}^1(L, -)$ is also a finitely presented functor. As $Tf_{\Lambda}$ is the smallest definable subcategory of $\Lambda$-modules containing all $\Lambda$-lattices, it follows that $\pi^{k_0} \text{Ext}^1(L, N) = 0$ for every $\Lambda$-lattice $L$ and $N \in Tf_{\Lambda}$.

Maranda’s Theorem [5, 30.14] says that, under the previous assumptions, if $k$ is any integer $> k_0$ then any two $\Lambda$-lattices $M, N$ are isomorphic over $\Lambda$ if and only if $M_k$ and $N_k$ are isomorphic over $\Lambda_k$. Moreover, when $R$ is complete, even decomposability lifts from $M_k$ to $M$ [5, Theorem 30.19].

A generalisation of these results to pure injective $\Lambda$-modules is given in a parallel paper [6, 3.4, 3.5], where the following is shown.

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Theorem 4.1 Let $N$, $N'$ be $R$-reduced $R$-torsion-free pure injective $\Lambda$-modules. If $N_k \simeq N'_k$ for some integer $k > k_0$, then $N \simeq N'$. Moreover, if $N$ is indecomposable over $\Lambda$, then the same is true of $N_k$ over $\Lambda_k$ for every $k > k_0$. On the other hand, let us also mention, from [6, §4]:

Example 4.2 There exists a module $N$ over $\mathbb{Z}_2 C(2)^2$ such that $N$ is torsion-free and reduced over $\mathbb{Z}_2$, $N_k$ is pure injective for all positive integers $k$ but $N$ is not pure-injective.

This example is produced as part of more general theory describing the pure-injective hulls of direct limits up rays of irreducible morphisms in tubes in the Auslander-Reiten quivers of categories of lattices of orders over a complete discrete valuation domains under the hypotheses of Maranda’s theorem. It is an instance of [6, 4.9] and this particular example is discussed in detail at the end of [6, §4].

We propose here some applications of the classical Maranda Theorem to our framework. If $f : M \to N$ is a morphism of $\Lambda$-modules, then we shall write $\text{f}\text{'}$ for the induced homomorphism from $M_k$ to $N_k$ (a positive integer). The following lemma is implicit in the proof of Maranda’s Theorem in [5, 30.14].

Lemma 4.3 Let $L$ be a $\Lambda$-lattice and $N$ an $R$-torsion-free $\Lambda$-module. If $k > k_0$ then for all $g \in \text{Hom}_\Lambda(L, N_k)$ there exists $f \in \text{Hom}_\Lambda(L, N)$ such that for all $m \in L$, $\pi^{k-k_0} + \Lambda \pi^k \mid (f(m) + \pi^k M) - g(m + \pi^k M)$. 

Definition 4.4 Assume $k > k_0$. Let $\varphi$ be a pp-formula of $L_\Lambda$ in $l$ free variables $x = (x_1, \ldots, x_l)$. Suppose that $(M, m)$ is a free realisation of $\varphi$, where $M$ is $R$-torsion-free, so a $\Lambda$-lattice, and that $\varphi \geq \bigwedge_{i=1}^{l} \pi^{k-k_0} | x_i$. Then let $\varphi_k$ be the generator of the pp-type of $m + \pi^k M \in M_k$.

Thus $\varphi_k$ is a pp-formula of $L_\Lambda$ with $\ell$ free variables. Note that $\varphi_k$ is well defined. In fact, let $(M, m)$ and $(N, n)$ be free realisations of $\varphi$, with both $M$ and $N$ $\Lambda$-lattices. Then there exist $\Lambda$-module morphisms $f : M \to N$ and $g : N \to M$ such that $f(m) = n$ and $g(n) = m$. The homomorphisms $\text{f}\text{'} : M_k \to N_k$ and $\text{g}\text{'} : N_k \to M_k$ induced by $f$ and $g$ respectively are such that $\text{f}\text{'}(m + \pi^k M) = n + \pi^k N$ and $\text{g}\text{'}(n + \pi^k N) = m + \pi^k M$. Thus the pp-type of $n + \pi^k N$ in $N_k$ is equal to the pp-type of $m + \pi^k M$ in $M_k$, which guarantees that the above pp-formula $\varphi_k$ is well defined, as said.

Lemma 4.5 Let $k > k_0$. Suppose that $\varphi \in \text{pp}_\Lambda(N)$ is freely realised in a $\Lambda$-lattice and $\varphi \geq \bigwedge_{i=1}^{\ell} \pi^{k-k_0} | x_i$. Then, for all $R$-torsion-free $\Lambda$-modules $N$ and $\ell$-tuples $n \in N$, $n \in \varphi(N)$ if and only if $n + \pi^k N \in \varphi_k(N_k)$.

Proof. Let $M$ be a $\Lambda$-lattice and suppose that $\varphi$ is freely realised by $m \in M$. Then, by definition, $\varphi_k$ is freely realised by $m + \pi^k M \in M_k$.

Let $N$ be an $R$-torsion-free $\Lambda$-module. If $n \in \varphi(N)$ then there exists a morphism $f : M \to N$ sending $m$ to $n$. Consequently $\text{f}\text{'} : M_k \to N_k$ sends $m + \pi^k M$ to $n + \pi^k N$. Therefore $n + \pi^k N \in \varphi_k(N_k)$.

Now suppose that $n + \pi^k N \in \varphi_k(N_k)$. Then there exists a morphism $f : M_k \to N_k$ sending $m + \pi^k M$ to $n + \pi^k N$. By Lemma 4.3, there exists $g \in \text{Hom}_\Lambda(M, N)$ such that $g(m_i) - n_i \in \pi^{\ell-k_0}N$ for all $i = 1, \ldots, \ell$. Thus $n$ satisfies the pp-formula $\bigwedge_{i=1}^{\ell} \pi^{k-k_0} | x_i \vee \varphi(x)$. Since $k - k_0 \geq 1$ and $\varphi \geq \bigwedge_{i=1}^{\ell} \pi^{k-k_0} | x_i, n \in \varphi(N)$.

Given $\varphi \in \text{pp}_\Lambda(N)$, we define $\varphi^* \in \text{pp}_\Lambda(N)$ such that for all $N \in \text{Mod-}\Lambda$, $m \in \varphi^*(N)$ if and only if $m + \pi^k N \in \varphi(N)$. Suppose $\varphi \equiv \exists y \, (xy)T = 0$ where $T$ is an appropriately sized matrix with entries from $\Lambda_k$. Further suppose that each entry of $T$ has the form $T_{ij} := t_{ij} + \Lambda \pi^k$ where $t_{ij} \in \Lambda$. Let $T^\ast$ be the matrix with entries $t^\ast_{ij}$ and let $\varphi^* \equiv \exists y \, (xy)T^{\ast}$. A quick computation shows that for all $N \in \text{Mod-}\Lambda$, $m \in \varphi^*(N)$ if and only if $m + \pi^k N \in \varphi(N_k)$ as required. Using this property of $\varphi^*$, one can check that for all $N \in \text{Mod-}\Lambda$, the map which sends $\varphi(N_k) \in \text{pp}_\Lambda(N_k)$ to $\varphi^*(N) \in \text{pp}_\Lambda(N)$ is a lattice homomorphism. Note that $(\pi^{k-k_0} + \Lambda \pi^k | x)$ is equivalent to $\pi^{k-k_0} | x$.

The next proposition applies to arbitrary $R$-torsion-free $\Lambda$-modules. In its statement we write $\pi^{k-k_0} | x$ to mean the pp-$n$-formula $\bigwedge_{i=1}^{n} \pi^{k-k_0} | x_i$. Recall that, for every pp-formula $\varphi(x) \in \text{pp}_\Lambda(N)$, $\varphi(x)$ is the pp-formula associated to $\varphi(x)$ defined just before Lemma 2.3.

Proposition 4.6 Let $k > k_0$. The map from the closed interval $[\pi^{k-k_0} | x, x = x]$ of $\text{pp}_\Lambda(N)$ to the interval $[\pi^{k-k_0} | x, x = x]$ of $\text{pp}_\Lambda(N)$ which sends any $\varphi \in [\pi^{k-k_0} | x, x = x]$ to $\varphi_k \in [\pi^{k-k_0} | x, x = x]$ induces a lattice isomorphism from $[\pi^{k-k_0} | x, x = x]_N$ to $[\pi^{k-k_0} + \pi^k \Lambda | x, x = x]_{N_k}$ for all $N \in \text{TI}_\Lambda$. In particular, this lattice
isomorphism is inverse to the lattice homomorphism which sends $\psi(N_k) \in [\pi^{k-k_0} + \pi^k \Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]_{N_k}$ to $\psi^*(N) \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]_N$ for all $N \in \text{TF}_\Lambda$.

**Proof.** First note that $\pi^{k-k_0} | \mathbf{x}$ is freely realised by the n-tuple from $\Lambda$ with all entries $\pi^{k-k_0}$. Thus $\pi^{k-k_0} | \mathbf{x} \leq \varphi$ implies $\pi^{k-k_0} | \mathbf{x} \leq \varphi$. So $\varphi_k$ is defined. Suppose $\pi^{k-k_0} | \mathbf{x} \leq \varphi$. Then $(\pi^{k-k_0}, \ldots, \pi^{k-k_0}) \in \varphi(\Lambda) = \varphi(\Lambda)$. So $(\pi^{k-k_0} + \Lambda \pi^k, \ldots, \pi^{k-k_0} + \Lambda \pi^k) \in \varphi_k(\Lambda_k)$. Hence $\pi^{k-k_0} + \Lambda \pi^k | \mathbf{x} \leq \varphi_k$.

Recall that $(\pi^{k-k_0} + \Lambda \pi^k)\mathbf{x}$ is equivalent to $\pi^{k-k_0} | \mathbf{x}$. So $\psi(N_k) \mapsto \psi^*(N)$ defines a lattice homomorphism from $[\pi^{k-k_0} + \pi^k \Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]_{N_k}$ to $[\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]_N$ for all $N \in \text{TF}_\Lambda$. Since, when it exists, the setwise inverse of a lattice homomorphism is a lattice isomorphism, it is therefore enough to show that for all $N \in \text{TF}_\Lambda$, $\varphi \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ and $\psi \in [\pi^{k-k_0} + \pi^k \Lambda | \mathbf{x}, \mathbf{x} = \mathbf{x}]$ $(\varphi_k)^*(N) = \varphi(N)$ and $(\psi^*)_k(N_k) = \psi(N_k)$. But this follows from Lemma 4.5 and the property of $\psi^*$ described just before this proposition. □

### 5 Finite lattice representation type

In this final section we recover our largest setting and we deal with a Dedekind domain $R$ which is not a field, with its field of fractions $Q$ and with an $R$-order $\Lambda$ in a finite dimensional $Q$-algebra $A$.

Recall that $\Lambda$ is said to be of *finite lattice representation type* if it has only finitely many non-isomorphic indecomposable lattices. Our aim is to obtain a complete description of $Z\text{g}^\Lambda_d$ when $\Lambda$ is of finite lattice representation type.

But let us first concern ourselves with the $m$-dimension of $\text{pp}^1(\text{TF}_\Lambda)$ under the finite lattice representation type hypothesis. The simplest example of the following proposition is when $R$ is a Dedekind domain and we view $R$ as an order over itself. Then all indecomposable $R$-lattices are isomorphic to $R$ and $\text{TF}_R$ is the definable subcategory generated by $R$ and hence $\text{pp}^1(\text{TF}_R) \cong \text{pp}_R(R) \cong \text{Sub}_R(R)$. So, although, by [13, 5.2.6], the $m$-dimension of $\text{pp}^1_R$ is 2, the $m$-dimension of $\text{pp}^1_R(\text{TF}_R)$ is 1 by Lemma 2.9.

**Proposition 5.1** Let $\Lambda$ be an order over a Dedekind domain $R$. If $\Lambda$ is of finite lattice representation type then $m\text{-dim}(\text{pp}^1(\text{TF}_\Lambda)) = 1$.

**Proof.** Let $L_1, \ldots, L_n$ be a complete list of indecomposable $\Lambda$-lattices up to isomorphism. By Fact 2.1 and Proposition 2.2, the canonical surjection from $\text{pp}^1(\text{TF}_\Lambda)$ to $\text{pp}^1(\bigoplus_{i=1}^m L_i)$ is an isomorphism. Since $\bigoplus_{i=1}^n L_i$ is a $\Lambda$-lattice, by Corollary 2.10, $\text{pp}^1(\bigoplus_{i=1}^m L_i)$ has $m$-dimension 1. □

Here is a first consequence of this proposition. Recall that an $R$-order $\Lambda$ is **maximal** if, for all $R$-orders $\Lambda' \subseteq \Lambda$, $\Lambda \subseteq \Lambda'$ implies $\Lambda = \Lambda'$. Let $S(\Lambda)$ be the set of maximal ideals $P$ of $R$ such that $\Lambda_P$ is not maximal. If $\Lambda \varphi$ is separable then $S(\Lambda)$ is finite, cf. [5, § 31, p. 642]. Moreover, $\Lambda_P$ is maximal if and only if $\Lambda_P$ is maximal [16, 11.5]. By [16, 18.1], if $\Lambda_P$ is maximal then $\Lambda_P$ is hereditary. So, by [16, 10.6], all $\Lambda_P$-lattices are projective. Finally, since that category of $\Lambda_P$-lattices is Krull-Schmidt, there are only finitely many indecomposable projective lattices and hence $\Lambda_P$ is of finite lattice representation type.

In the following corollary we do not assume that $\Lambda$ is of finite lattice representation type. It is intended as a slight improvement on Corollary 3.11 when $\Lambda \varphi$ is separable. In particular, it means that in order to calculate the $m$-dimension of $\text{pp}^1(\text{TF}_\Lambda)$ we only need to calculate the $m$-dimension of $\text{pp}^1(\text{TF}_{\Lambda_P})$ for $\Lambda_P$ in the finite subset $S(\Lambda)$.

**Corollary 5.2** Suppose that $\Lambda \varphi$ is separable. If $S(\Lambda)$ is non-empty then the $m$-dimension of $\text{pp}^1(\text{TF}_\Lambda)$ is equal to the supremum of the $m$-dimensions of $\text{pp}^1(\text{TF}_{\Lambda_P})$ where $P$ ranges over the maximal ideals in $S(\Lambda)$. If $S(\Lambda)$ is empty, then the $m$-dimension of $\text{pp}^1(\text{TF}_\Lambda)$ is 1.

**Proof.** If $P \notin S(\Lambda)$ then $\Lambda_P$ is of finite lattice representation type. So, by Proposition 5.1, if $P \notin S(\Lambda)$ then $\text{pp}^1(\text{TF}_{\Lambda_P})$ has $m$-dimension 1. For any maximal ideal $P$ of $R$, the $m$-dimension of $\text{pp}^1(\text{TF}_{\Lambda_P})$ is greater than or equal to 1. Applying Corollary 3.11 now gives both the required statements. □

The following is a further consequence of Proposition 5.1: if $\Lambda$ is of finite lattice representation type, then $Z\text{g}^\Lambda_d$ has the isolation property and so its Cantor-Bendixson rank is equal to $m\text{-dim}(\text{pp}^1(\text{TF}_\Lambda))$, viz. it is 1 (cf.
Proposition 5.3 Let \( R \) be a complete discrete valuation domain and \( \Lambda \) an order over \( R \). If \( Z_{gf_{\Lambda}} \) has Cantor-Bendixson rank 1 then \( \Lambda \) is of finite lattice representation type.

Proof. Note that if a topological space \( X \) has Cantor-Bendixson rank 1 then all its points are either isolated or closed. This is because if \( p \in X \) is not isolated then \( p \in X^{(1)} \). Since \( X \) has Cantor-Bendixson rank 1, all points in \( X^{(1)} \) are isolated in \( X^{(1)} \). Therefore, all points in \( X^{(1)} \) are closed in \( X^{(1)} \) and hence also in \( X \).

Let \( \pi \) generate the maximal ideal of \( R \). If \( N \in (x = x/\pi \mid x) \) then \( N \) is not \( \pi \)-divisible and hence by Corollary 2.8 is not a closed point. Therefore \( (x = x/\pi \mid x) \) contains only isolated points. Since \( (x = x/\pi \mid x) \) is compact, it must be finite. Since all indecomposable \( \Lambda \)-lattices are pure injective and not \( \pi \)-divisible, \( \Lambda \) is of finite lattice representation type. \( \square \)

Next we provide the description of \( Z_{gf_{\Lambda}} \) when \( \Lambda \) is an order over a complete discrete valuation domain.

Proposition 5.4 Suppose that \( R \) is a complete discrete valuation domain with field of fractions \( Q \). \( \Lambda \) is an order over \( R \), \( A = Q\Lambda \) is a semisimple \( Q \)-algebra and \( \Lambda \) is of finite lattice representation type. Then the set \( Z_{gf_{\Lambda}} \) consists exactly of

1. finitely many indecomposable lattices over \( \Lambda \),
2. finitely many simple \( A \)-modules.

The simple modules over \( \Lambda \) are closed points. If \( N \) is an indecomposable lattice then a simple \( A \)-module \( M \) is in the closure of \( N \) if and only if \( M \) is a direct summand of \( QN \).

Proof. By Proposition 2.2, the set of indecomposable \( \Lambda \)-lattices is dense in \( Z_{gf_{\Lambda}} \). Thus if \( N_1, \ldots, N_n \) are the indecomposable lattices over \( \Lambda \), then \( Z_{gf_{\Lambda}} \) is equal to the closure of \( \{N_1, \ldots, N_n\} \) and, since closures commute with finite unions, \( Z_{gf_{\Lambda}} \) is equal to the union of the closures of the \( N_i \) with \( i = 1, \ldots, n \).

In Proposition 2.6, we showed that if \( N \) is a lattice and \( M \) is in the closure of \( N \) but not equal to \( N \), then \( M \) is a closed point. By Corollary 2.8, any closed point is an \( A \)-module. Thus we have shown that \( Z_{gf_{\Lambda}} \) has exactly the points stated in the proposition.

The description of the topology follows from Lemmas 2.7 & 2.4. \( \square \)

Corollary 5.5 Assume \( R \) and \( \Lambda \) as before, hence in particular \( \Lambda \) is of finite lattice representation type. Let \( p \) be a non-finitely generated indecomposable pp-type in the theory \( T_{\Lambda}^0 \) of \( R \)-torsion-free \( \Lambda \)-modules. Then \( p \) contains all divisibility formulas \( \pi^k \mid x \) for \( k \) a positive integer.

Proof. Any element of a simple \( A \)-module realises all these formulas. \( \square \)

Note that Herzog and Puninskaya verified a similar result for torsion-free modules over 1-dimensional commutative Noetherian local complete domains; cf. [7, Theorem 6.6].

Now let us come back to a Dedekind domain \( R \) and to an \( R \)-order \( \Lambda \) in a separable \( Q \)-algebra \( A \). The following hold:

1. there are only finitely many maximal ideals \( P \) of \( R \) such that \( \hat{\Lambda}_P \) is not a maximal order [5, Exercise 7, § 4, p. 99];
2. if \( \hat{\Lambda}_P \) is maximal, then it is of finite lattice representation type, and hence the topology of \( Z_{gf_{\hat{\Lambda}_P}} \) is that described in Proposition 5.1 [5, Proposition 33.1].

Recall that this topology is the same both in \( Z_{gf_{\hat{\Lambda}_P}} \) and \( Z_{gf_{\hat{\Lambda}_P}} \), when restricted to its \( R \)-reduced part (Theorem 3.3). By the proof of [5, 33.2], if \( \Lambda \) is of finite lattice representation type then each non-maximal \( \Lambda \) is. On this basis it is easy to deduce:

Theorem 5.6 Let \( R \) be a Dedekind domain and \( \Lambda \) an \( R \)-order in a separable \( Q \)-algebra \( A \). Assume \( \Lambda \) is of finite lattice representation type. Then the Cantor-Bendixson rank of \( Z_{gf_{\Lambda}} \) is 1, and
1. the isolated points are the indecomposable \( \hat{N}_P \)-lattices, where \( P \) ranges over maximal ideals of \( R \),
2. the points of Cantor-Bendixson rank 1 are the simple \( A \)-modules.

Let us give some examples illustrating the previous results. The first, over a complete discrete valuation domain \( R \), was proposed by Puninski. Indeed it is one of the last suggestions he left to us. So we like to mention it as a tribute to his memory.

**Example 5.7** Let \( R \) be as just said, \( \pi \) be a generator of its maximal ideal. Let \( \Lambda = ( R \pi \pi R R \pi R ) \) (cf. [5, § 37, p. 779]). Also, let \( e_1, e_2 \) denote for simplicity the idempotents
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
respectively. It is well known that \( \Lambda \) has finite lattice representation type. In fact \( \Lambda \) is Gorenstein, i.e., projective modules \( P_1 = e_1 \Lambda = ( R R 0 ) \) (so basically \( ( R, R ) \)) and \( P_2 = e_2 \Lambda = ( 0 \pi R R 0 ) \) (hence \( ( \pi^2, R ) \)) are injective (in the category of lattices).

The only remaining indecomposable lattice is \( P = ( \pi R, R ) \) (note that \( ( \pi R R 0 0 ) \) and \( ( 0 \pi R R 0 ) \) are isomorphic as \( \Lambda \)-modules).

Hence a description of \( Zg/\Lambda \) follows from Proposition 5.4. Anyway let us follow Puninski’s approach for the reasons we said.

First of all, note that, being Gorenstein, \( \Lambda \) admits a unique overorder \( \Lambda' = ( R R R ) \) which is hereditary; and \( P \) is defined over \( \Lambda' \), i.e., \( \Lambda' \) is the ring of definable scalars of \( P \). Furthermore the following is the AR-quiver of \( \Lambda \)

![AR-quiver](image.png)

where \( \pi \) denotes the multiplication by \( \pi \). From that we can see irreducible morphisms in the category of lattices and the unique almost split sequence:

\[
0 \rightarrow P \xrightarrow{(i, \pi)} P_1 \bigoplus P_2 \xrightarrow{(\pi, i)} P \rightarrow 0,
\]

where \( i \) denotes inclusion. In detail the two intermediate morphisms act as follows:

1. for all \( a, b \in R \), \((i, \pi)\) maps \((\pi a, b)\) to \(((\pi a, b), (\pi^2 a, \pi b))\),
2. for all \( a', b', c', d' \in R \), \((\pi, i)\) sends \(((a', b'), (\pi^2 c', d'))\) to \((\pi a' - \pi^2 c', \pi b' - d')\).

Let \( N \) be an indecomposable \( R \)-torsion-free pure injective \( \Lambda \)-module. First suppose that there exists \( 0 \neq n \in Ne_1 \). Hence look at pointed indecomposable lattices \((M, m)\) such that \( m \in Me_1 \). Up to equivalence (of types realised by \( m \)) here is a complete list of them:

1. \((P_1, (\pi^k, 0)), k \geq 0, \)
2. \((P_2, (\pi^\ell, 0)), \ell \geq 2, \)
3. \((P, (\pi^m, 0)), m \geq 1, \)

Furthermore the following is the pattern of the module \((P_1, 0)\), i.e., the poset of morphisms from \( P_1 \) to indecomposable lattices (cf. [14, § 4] for a definition). Here we use an “exponential” notation: e.g., \((P_1, k)\) abbreviates \((P_1, (\pi^k, 0))\).
We easily derive that there is a unique (critical over zero) indecomposable non-finitely generated type $p$ in the interval $[x = 0, e_2 | x]$ in $\mathfrak{p}_A(T_{\mathfrak{p}})$. Furthermore $p$ is realised by $(1, 0) \in (Q, Q) = S$, the simple module over $A = M_2(Q)$. Thus in this case $N \cong P_1, P$ or $N \cong S$.

It remains to consider the case when there exists $0 \neq n \in Ne_2$. Again look at indecomposable pointed lattices $(M, m)$ where $m \in Me_2$. They form the following pattern, where $(P_1, k)$ now abbreviates $(P_1, (0, \pi^k))$, etc.

Then there is a unique indecomposable non-finitely generated type $q$ in the interval $[x = 0, e_2 | x]$ which is realised as $(0, 1) \in (Q, Q) = S$. Again we conclude that $N$ is isomorphic to $P_1, P$ or $S$. This completes the description of the $R$-torsion-free part of the Ziegler spectrum of our ring.

**Example 5.8** The second example concerns an integral group ring $\mathbb{Z}C(p)$ with $p$ a prime. This is of finite lattice representation type (cf. [5, 33.6]), whence Theorem 5.6 applies. Here is an explicit description of the torsion-free part of the Ziegler spectrum. Let $g$ denote a generator of $C(p)$ and $\zeta_p$ a primitive $p$th root of 1 in $\mathbb{C}$, so a root of the cyclotomic polynomial $\Phi_p(t) = t^{p-1} + t^{p-2} + \cdots + t + 1 \in \mathbb{Z}[t]$. Incidentally, let $\Phi_1(t) = t - 1$, whence $t^p - 1 = \Phi_1(t) \cdot \Phi_p(t)$. Also, let $e_1 = \frac{1}{p} \Phi_p(g)$ and $e_2 = 1 - e_1$ be the primitive idempotents of the algebra $\mathbb{Q}C(p)$. Thus the points of $\mathbb{Z}_p C(p)$ are the following:

First of all, three isolated lattices over $\Lambda = \mathbb{Z}_p C(p)$, i.e., $\Lambda e_1, e_2 = \mathbb{Z}_p(\zeta_p)$ and $\mathbb{Z}_p C(p)$. In the first two cases $g$ acts as the identity and the multiplication by $\zeta_p$, respectively. See [4, 3.5(a)]. Next, for every prime $q \neq p$, two more isolated points, $\mathbb{Z}_q$ and $\mathbb{Z}_q(\zeta_p)$ respectively, as now $\mathbb{Z}_q C(p)$ is their direct sum. Finally, two more points of Cantor-Bendixson rank 1, $Q$ and $Q(\zeta_p)$, as $Q C(p)$ is again their direct sum.

The topology is also easy to describe: (a) $\Lambda e_1 = \mathbb{Z}_p$, $\Lambda e_2 = \mathbb{Z}_p(\zeta_p)$ and $\Lambda = \mathbb{Z}_p C(p)$ are the only points in the basic open set $(x = x/p | x)$, and indeed can be isolated, and separated from each other, as follows: $\Lambda$ by $(\sum_{j < \ell} g^j | x/p | x)$ where $\sum_{j < \ell} g^j = pe_1$ (cf. [4, §2, end of p. 57]) and $\Lambda e_1$, $\Lambda e_2$ by $(e_1 | x/p e_1 | x), (e_2 | x/(1 - \zeta_p) e_2 | x)$, respectively (cf. [4, proof of 3.3(a)]). Note that, properly speaking, $e_1$ and $e_2$ are not in $\mathbb{Z}_p C(p)$.

However, as we are working in a $\mathbb{Z}$-torsion-free framework we can use here the simple trick of multiplying every involved scalar by $p$ and expressing the previous open sets as $(pe_1 | x/p^2 e_1 | x)$ and $(pe_2 | x/p(1 - \zeta_p) e_2 | x)$.
(b) For every prime \( q \neq p \), \( \mathbb{Z}_q \) and \( \mathbb{Z}_q(\zeta_p) \) are isolated from the other points by \( (x = x/q \mid x) \) and indeed separated from each other, and hence isolated at all, by \( ((1 + g + \cdots + g^{p^s - 1}) \mid x/q \mid x) \) and \( ((1 - g) \mid x/q \mid x) \) respectively.

(c) The points \( \mathbb{Q} \) and \( \mathbb{Q}(\zeta_p) \), i.e., \( \mathbb{Q} \Lambda e_i \) and \( \mathbb{Q} \Lambda e_2 \), are the points of Cantor-Bendixson rank 1 and at this level can be separated from each other, e.g., by \( (x(1 - e_1) = 0 \mid x = 0) \) and \( (x(1 - e_2) = 0 \mid x = 0) \), respectively.

Example 5.9 Finally let us deal with the integral group ring \( \mathbb{Z} C(p^2) \) with \( p \) a prime. This is again a \( \mathbb{Z} \)-order of finite lattice representation type. A description of \( \mathbb{Z} g_{C(p^2)} \), both points and topology, can be extracted from the classification of lattices over \( \mathbb{Z}_p C(p^2) \) given in [5, § 34C, p. 730] in terms of extension groups, or in [8] in terms of pullbacks, or also in [4, § 4]. We follow this third approach. Let \( g \) still denote a generator of the group \( C(p^2) \), \( e_1, e_2, e_3 \) be the primitive idempotents of the algebra \( \mathbb{Q} C(p^2) \). Thus

\[
e_1 = \frac{1}{p^2} \sum_{j < p^2} g^j = \frac{1}{p^2} \Phi_p(g) \Phi_p(g^p),
\]

\[
e_2 = \frac{1}{p^2} (p - \Phi_p(g)) \Phi_p(g^p), \quad e_3 = \frac{1}{p} (p - \Phi_p(g^p))
\]

where \( \Phi_p(t^p) = \Phi_{p^2}(t) \) is the cyclotomic polynomial of order \( p^2 \). Then the points of \( \mathbb{Z} g_{C(p^2)} \) are the following.

Let us start this time from simple \( \mathbb{Q} C(p^2) \)-modules, i.e., from points of Cantor-Bendixson rank 1. They are \( \mathbb{Q}, \mathbb{Q}(\zeta_p) \) and \( \mathbb{Q}(\zeta_p) \) where \( \zeta_p \) and \( \zeta_p^2 \) are primitive roots of 1 of order \( p \), \( p^2 \) respectively.

When \( q \) is a prime different from \( p \), \( \mathbb{Z}_q C(p^2) \)-lattices admit a similar description. Hence let us focus on \( \Lambda = \mathbb{Z}_p C(p^2) \). Indecomposable lattices are now \( 4p + 1 \). The first four are \( \Lambda \) itself and the \( \Lambda e_i, i = 1, 2, 3 \), i.e., \( \mathbb{Z}_p C(p^2) \) and then \( \mathbb{Z}_p, \mathbb{Z}_p(\zeta_p) \) and \( \mathbb{Z}_p(\zeta_p^2) \). In the three last cases \( g \) acts as the multiplication by \( 1, \zeta_p \) and \( \zeta_p^2 \) respectively. The remaining \( 4p - 3 \) correspond via the representation equivalence described in [4, §§ 3 & 4] to the indecomposable objects in the category of finite dimensional \( \mathbb{Z}/p\mathbb{Z} \)-representations of the directed Dynkin diagram \( D_{2p} \). These can be viewed as tuples \((W, (W^i)_{0 < i < p}, h, n_1, n_2, W_1, W_2)\), where

1. \( W \) is a vector space over \( \mathbb{Z}/p\mathbb{Z} \).
2. \( W_0 = W_{0i}, W_1, W_2 \) are subspaces of \( W \) such that the sum of any two of them gives the whole \( W \).
3. the \( W^i \) form an increasing chain of subspaces \( W_0 = W^1_{0i} \subseteq W^2_{0i} \subseteq W^2_{0i} \subseteq \cdots \subseteq W^p_{0i} \subseteq W^{p-1}_{0i} \subseteq W^{p-1}_{0i} = W \).

Moreover Butler’s functor uniformly defines in a first order way every such representation in the associated lattice via quotients of pp-subgroups. Then it suffices for our purposes to list these indecomposable representations of \( D_{2p} \). In fact they recursively determine the corresponding lattices as a sort of ID card. As said, they are \( 4p - 3 \). In all of them the dimension of \( W \) is either 1 or 2. In the former case, i.e., in dimension 1, we meet

a) three points where \( W_0 = W \) (and hence \( W^i_{0i} = W \) for every \( h \) and \( s \)) and the pair \((W_1, W_2)\) is one among \((W, 0), (0, W)\) and \((W, W)\).

b) \( 2p - 3 \) points where \( W_1 = W_2 = W, W_0 = 0 \) and the \( W^i_{0i} \) are constantly 0 before some \( s \) and \( h \) (\( h = 1 \) when \( s = 1 \)) and then become equal to \( W \).

In the 2-dimensional case, we find \( 2p - 3 \) additional points in which \( W_0, W_1, W_2 \) are 1-dimensional subspaces such that \( W \) is the sum of any two of them (and so the intersection of any two of them is 0), and the \( W^i_{0i} \) equal \( W_0 \) before some \( s \) and \( h \) (\( h = 1 \) when \( s = 1 \)) and then coincide with \( W \).

Next let us see the topology, so how to separate isolated points from each other. The case of primes \( q \neq p \) can be handled as for \( C(p) \), with slight complications. For instance the open sets isolating \( \mathbb{Z}_q, \mathbb{Z}_q(\zeta_p) \) and \( \mathbb{Z}_q(\zeta_p^2) \) are now \( ((1 + g + \cdots + g^{p^s - 1}) \mid x \wedge (1 + g^p + \cdots + g^{p^{s - 1}}) \mid x/q \mid x), ((1 - g) \mid x \wedge (1 + g^p + \cdots + g^{p^{s - 1}}) \mid x/q \mid x), ((1 - g) \mid x \wedge (1 + g + \cdots + g^{p^{s - 1}}) \mid x/q \mid x) \) respectively. Also the analysis of simple \( \mathbb{Q} C(p^2)\)-modules is similar to that of \( C(p) \).

Hence let us deal with \( q = p \) and with indecomposable lattices over \( \Lambda = \mathbb{Z}_p C(p^2) \), those in \((x = x/p \mid x) \). The way to isolate \( \Lambda \) and the \( \Lambda e_i, i = 1, 2, 3 \), is the same as for \( C(p) \), by \((\sum_{j < p^2} g^j \mid x/p \mid x), (e_1 \mid x/pe_1 \mid x), (e_2 \mid x/(1 - \zeta_p)e_2 \mid x), (e_3 \mid x/(1 - \zeta_p^2)e_3 \mid x) \) respectively.

The further \( 4p - 3 \) points are those in the open set \((p^2 \sum_{1 \leq i \leq 5} e_i \mid x/p^2 \mid x) \) (cf. the construction in [4, § 3]). To separate them from each other, we can look at the associated representations of \( D_{2p} \) as abelian structures in their

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own language, because these representations are uniformly pp-definable without parameters in the corresponding
lattices. Let us write for simplicity $x \in W_0, x \in W_1$ and so on to denote the formulas admitting this interpretation
in any given representation. Thus

a) the first three points are isolated by $(x \in W_0 \land x \in W_1 / x \in W_2), (x \in W_0 \land x \in W_2 / x \in W_1)$ and $(x \in W_1 \land x \in W_2 / x \in W_0 / x = 0)$,

b) the following $2p - 3$ are isolated by $(x \in W_1 \land x \in W_2 / x \in W_{01}^{x+1} / x \in W_{02})$ or $(x \in W_1 \land x \in W_2 / x \in W_{02} / x \in W_{01})$ for the right $s$.

Similarly, the last $2p - 3$ are isolated by $(x \in W_{01}^{x+1} / x \in W_{02} + (x \in W_1 \land x \in W_2))$ or $(x \in W_{02} / x \in W_{01} + (x \in W_1 \land x \in W_2))$ for the right $s$.

On this basis, one easily deduces the following:

**Theorem 5.10** For every prime $p$, the first order theories of both $\mathbb{Z}$-torsion-free $\mathbb{ZC}(p)$-modules and $\mathbb{Z}$-
torsion-free $\mathbb{ZC}(p^2)$-modules are decidable.

In fact, the descriptions of the torsion-free part of the Ziegler spectrum of $\mathbb{ZC}(p)$ and $\mathbb{ZC}(p^2)$ fit with the
conditions of the decidability criterion in [19, Theorem 9.4] and [12, Theorem 17.12]. In fact both an effective
list of indecomposable pure injective modules $N$ (possibly through the related representations of $D_{2p^r}$) and, for
every such module, an effective list of basic open sets $(\varphi/\psi)$ around $N$ were already provided. Furthermore
straightforward calculations recursively determine $\varphi(N)/\psi(N)$ from $N, \varphi$ and $\psi$.

Notice that Theorem 5.10 positively solves expectations in the final lines of [10].

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