COUNTING STRONGLY-CONNECTED, SPARSELY EDGED DIRECTED GRAPHS.

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Abstract. A sharp asymptotic formula for the number of strongly connected digraphs on \(n\) labelled vertices with \(m\) arcs, under a condition \(m−n → \infty\), \(m = O(n)\), is obtained; this solves a problem posed by Wright back in 1977. Our formula is a counterpart of a classic asymptotic formula, due to Bender, Canfield and McKay, for the total number of connected undirected graphs on \(n\) vertices with \(m\) edges. A key ingredient of their proof was a recurrence equation for the connected graphs count due to Wright. No analogue of Wright’s recurrence seems to exist for digraphs. In a previous paper with Nick Wormald we rederived the BCM formula via counting two-connected graphs among the graphs of minimum degree 2, at least. In this paper, using a similar embedding for directed graphs, we find an asymptotic formula, which includes an explicit error term, for the fraction of strongly-connected digraphs with parameters \(m\) and \(n\) among all such digraphs with positive in/out-degrees.

0. Introduction. In a pioneering paper [24], Wright described two potent approaches to calculating \(c(n, n + k)\), the number of connected labeled graphs with \(n\) vertices and \(n + k\) edges. One was based on a classic exponential identity expressing a bivariate generating function of all labeled graphs through that of the connected graphs. This identity lent itself to a recursive procedure well tailored for computing the exponential generating functions \(W_k(x) = \sum_n x^n c(n, n + k)/n!\), and finding the asymptotic formulas for \(c(n, n + k)\) for small \(k\), going beyond a formula for the first nontrivial \(c(n, n + 1)\), due to Bagaev [1]. By representing a connected graph as a connected 2-core with a forest of trees sprouting from the core vertices, Wright found an alternative way of computing \(W_k(x)\) for small \(k\), and established a remarkable expression of \(W_k(x)\) through the exponential generating function of the rooted trees for all \(k\). Neither of these approaches could be used to get asymptotics for \(k = k(n) \to \infty\).

In [25] Wright described a quadratic recurrence equation for the numbers \(c(n, n + k)\), and the related recurrence for the exponential generating function \(W_k(x)\). The recurrence relates to the last, \((n + k)\)-th, step of an edge-insertion process that begins at an empty graph on \([n]\); at the terminal step a connected graph with \(n + k\) edges is born either if the penultimate graph is already connected and has \(n + k − 1\) edges, or if it consists of two components with parameters \((n_i, n_i + k_i)\).
with \( n_1 + n_2 = n \), \( k_1 + k_2 + 1 = k \), and the \((n+k)\)-th edge joins the two components. Wright used the recurrence to obtain a sharp asymptotic formula for \( c(n, n+k) \) with \( k = o(n^{1/3}) \) as \( n \to \infty \), a difficult result with many ramifications.

Bollobás [6], [7] discovered that a leading factor in Wright’s formula [25] was an upper bound for \( c(n, n+k) \) far beyond \( k = o(n^{1/3}) \), and used his bound to identify sharply—for the first time—a transition window in the Erdős–Rényi random edge-insertion graph process, [9]. Wright’s results and enumerational insight were also a key ingredient in the subsequent studies of random graphs, Luczak [14], Luczak et al [16], Janson et al [12], Flajolet et al [10], Daudé and Ravelomanana [8], Pittel and Yeum [22], Stepanov [23], to name just a few.

In a monumental paper [3], Bender et al managed to use Wright’s quadratic recurrence to derive an extension of Wright’s asymptotic formula for \( k = O(n \ln n) \), i.e. all the way till the number of edges is so large that \( c(n, n+k) \) is asymptotic to the number of all graphs with parameters \( n \) and \( n+k \). Later with Nick Wormald we found [20], [21] a more combinatorial, and less technical way to rederive the Bender-Canfield-McKay formula. It was based on Wright’s decomposition device. First we counted the connected 2-cores among all the graphs with minimum degree 2, at least. After this step, enumeration of the connected graphs was reduced to enumeration of the cores with a forest of trees attached to the core vertices. More recently van der Hofstadt and Spencer [11] found yet another way to obtain the BCM formula. At the heart of their approach is a breadth-first search for connected components in the Bernoulli-type random graph \( G(n, P(\text{edge}) = p) \). Interestingly, this algorithm had already been used by Karp [13] to obtain a sharp estimate of the largest strongly-connected subgraph size of the random digraph with \( P(\text{arc}) = c/n \) in a supercritical phase \( c > 1 \).

Wright also wrote a companion paper [26] on determination of \( g(n, n+k) \), the total number of strongly-connected digraphs with \( n+k \) arcs. He was able to find a qualitative description of how \( g(n, n+k) \) depends on \( n \), \( k \), and to find explicit formulas for \( k = 0, 1, 2 \). We gather that he did not find a recurrence for \( g(n, n+k) \) that might have been used to study the case \( k = k(n) \to \infty \). (We are not aware of any such recurrence either.)

Aside from Bender et al [4] who found an elegant way to use their formula from [3] for enumeration of weakly-connected digraphs, the problem of asymptotic behavior of \( g(n, n+k) \) has remained open.

Our goal in this paper is to obtain a sharp asymptotic formula for \( g(n, n+k) \), with essentially best remainder term—a directed counterpart of the BCM formula—by using a proper version of the embedding device.

Consider the digraphs on \( n \) labelled vertices, with \( m \) arcs. Let \( g(n, m) \) stand for the total number of strongly connected digraphs, with parameters \( n \) and \( m \); thus \( m > n \) necessarily. Suppose \( r := m - n \to \infty \) and \( m = O(n) \). We will show
that

\[ g(n, m) = \frac{m!}{2\pi n \text{Var}[Y]} \frac{(e^\lambda - 1)^{2n}}{\lambda^{2m}} \times \left( \frac{1 - \frac{\lambda}{e^\lambda - 1}}{1 - \frac{\lambda}{e^\lambda (e^\lambda - 1)}} \right)^2 \exp \left( -\frac{m}{n} \frac{\lambda^2}{2} \right) \times \left( 1 + O\left( r^{-1} \ln^2 r + r^\varepsilon n^{1/2+\gamma} \right) \right), \]

for any fixed \( \varepsilon > 0, \gamma \in (0, 1/2) \). Here \( \lambda e^\lambda / (e^\lambda - 1) = m/n \), and \( Y \) is a positive Poisson variable with mean \( m/n \), so that \( \text{Var}[Y] = (m/n)(\lambda - r/n) \). In particular, if \( r = O(n^{1/2-\sigma}) \), \( \sigma > 0 \), then

\[ g(n, m) = \frac{m!}{6\pi e n} \left( \frac{n}{2r} \right)^{2r} \left( 1 + O(r^{-1} \ln^2 r) \right). \]

Neglecting \( \ln^2 r \), the error term would be of order \( 1/r \), which we believe is the correct order of the remainder term. Inevitably our proofs run parallel to the argument in [20]. There are quite a few unexpected challenges though, as the graph component notion morphs into two, harder-to-handle, dual notions of a sink-set and a source-set, the subsets of vertices with no arcs going outside, and no arcs coming from outside, respectively.

**Note.** The author is well aware of a research started earlier by Xavier Pérez and Nick Wormald. According to Wormald, (private communication), they expect a full argument to be less technical since their aim is a cruder version of (1), i.e. an explicit leading factor times \( 1 + o(1) \).

The author plans to use the results and the insights of this paper for a sharp analysis of a phase transition window for the directed counterpart of the Erdős-Rényi graph process.

The rest of the paper is organized as follows. In Section 1 we state an asymptotic estimate for the number of digraphs with given in/out degrees (Theorem 1.1), analogous to those by Bender and Canfield [2], Bollobás [5], McKay [17], and McKay and Wormald [18] for graphs. In a proof sketch we introduce a random matching scheme similar to a random pairing introduced by Bollobás [5] for graphs. We use this estimate to prove an asymptotic formula for the number of digraphs with constrained in/out degrees (Theorem 1.2), and to formulate auxiliary bounds we use later. In Section 2, (Theorem 2.1) we prove a bound for the number of those digraphs without isolated cycles. In Section 3 we use Theorem 2.1 and the bounds from Section 1 to prove (Theorem 3.1) a sharp \( O(n^{-1}) \) bound for the fraction of digraphs with a complex “sink-set” (“source-set”) having less than half of all arcs, but without a simple sink-set (source-set). This implies that the fraction of strongly-connected digraphs differs from the fraction of digraphs without simple sink/source-sets by at most \( O(n^{-1}) \). Finally, in Section 4 we determine a sharp estimate for the latter fraction, which turns out to be \( \Theta((m - n)/n) \gg n^{-1} \) (Theorem 4.2), and this result yields the formula (1).
1. Enumerating the digraphs with restricted in/out-degrees.

In this Section we provide a set of estimates, both crude and sharp, for the counts of all digraphs with (most of) their in/out-degrees being positive.

**Theorem 1.1.** Let \( \delta_1, \ldots, \delta_n \geq 0 \) and \( \Delta_1, \ldots, \Delta_n \geq 0 \) be such that

\[
\sum_i \delta_i = \sum_i \Delta_i = m.
\]

where \( m \geq n \). Introduce \( g(\delta, \Delta) \), the total number of simple digraphs with in-degrees \( \delta_i \) and out-degrees \( \delta_i \). If \( D := \max\{\max_i \delta_i, \max_j \Delta_j\} = o(m^{1/4}) \), then

\[
g(\delta, \Delta) = m! \left( \prod_{i \in [n]} \frac{1}{\delta_i! \Delta_i!} \right) F(\delta, \Delta),
\]

where the “fudge factor”, always 1 at most, is given by

\[
F(\delta, \Delta) = \exp \left( -\frac{1}{m} \sum_i \delta_i \Delta_i - \frac{1}{2m^2} \sum_i (\delta_i)_2 \sum_j (\Delta_j)_2 + O(D^4/m) \right);
\]

\((d)_2 := d(d-1))\).

A combinatorial core of the proof is a random matching scheme, similar to a random pairing model for undirected graphs introduced by Bollobás [5] in his probabilistic proof of the Bender-Canfield [2] formula for the number of graphs with given, bounded, degrees. Consider two copies \([m]_1\) and \([m]_2\) of the set \([m]\), together with the partitions \([m]_1 = \cup_i I_i\), \([m]_2 = \cup_j O_j\), \(|I_i| = \delta_i\), \(|O_j| = \Delta_j\). Each of \(m!\) bijections \( \pi : [m]_2 \to [m]_1 \) determines a directed multigraph \( G(\pi) : i \to j \) is an arc if \( \pi(\nu) = \mu \) for some \( \nu \in O_j \) and \( \mu \in I_i \). Notice that, each simple digraph \( G \) with in-degrees \( \delta_i \) and out-degrees \( \Delta_j \), corresponds to exactly \( \prod_i (\delta_i! \Delta_i!) \) bijections \( \pi \). We call those bijections digraph-induced. Therefore \( F(\delta, \Delta) \) defined by (2) is the probability that the bijection \( \pi \) chosen uniformly at random from among all \( m! \) bijections is digraph-induced, i.e. there is no \( i \) such that \( \pi(\nu) = \mu \) for \( \nu \in O_i \), \( \mu \in I_i \), and there is no \( (i, j) \) such that \( \pi(\nu_t) = \mu_t \), \( t = 1, 2 \), \( (\nu_1 \neq \nu_2, \mu_1 \neq \mu_2) \), and \( \nu_t \in O_j \), \( \mu_t \in O_i \).

Since \( F(\delta, \Delta) \leq 1 \), we see that

\[
g(\delta, \Delta) \leq m! \left( \prod_{i \in [n]} \frac{1}{\delta_i! \Delta_i!} \right)
\]

always.

So the task is to show that this probability \( F(\delta, \Delta) \leq 1 \) is given by (1.3). The sums in the exponent in (1.3) are asymptotic estimates for the expected number of loops and number of double arcs in \( G(\pi) \). Our claim is a digraph analogue of McKay’s far-reaching extension [17] (McKay and Wormald [18]), based on Bollobás’ pairing model, of the Bender-Canfield formula for the number of undirected graphs.
with a given degree sequence \(d_1, \ldots, d_n\), and \(\max_i d_i = o(M^{1/4})\). We omit the proof as it is very similar to those in [17] and [18].

**Note 1.1.** Notice that the RHS of (1.4) is not necessarily an integer, but

\[
h(\delta, \Delta) := (m!)^2 \left( \prod_{i \in [n]} \frac{1}{\delta_i! \Delta_i!} \right) = \binom{m}{\delta} \binom{m}{\Delta}
\]

most certainly is! In fact, \(h(\delta, \Delta)\) counts the total number of \(m\)-long ordered sequences of insertions of single arcs into an initially empty digraph that result in a multi-digraph with in-degrees \(\delta\) and out-degrees \(\Delta\).

This formula follows from a bijection between the set of those sequences and the set of pairs of \(m\)-long words in the alphabet of \(n\) letters \(1, \ldots, n\), such that the first (second, resp.) word has \(\delta_i\) (\(\Delta_i\), resp.) letters \(i\). Observe also that, given \(\mu = \{\mu_{j,i}\}_{j,i \in [n]}\) with \(\|\mu\| = m\), the number \(h(\mu)\) of the sequences, that result in the multigraph with exactly \(\mu_{j,i}\) arcs \(j \rightarrow i\), is obviously the multinomial coefficient

\[(1.5) \quad h(\mu) = \binom{m}{\mu}.
\]

On the other hand, \(m! g(\delta, \Delta)\) is the total number of the \(m\)-long sequences that result in a simple digraph with in-degrees \(\delta\) and out-degrees \(\Delta\). We call these special sequences graphic. Thus \(F(\delta, \Delta)\) is the fraction of graphic sequences among all \(m\)-long insertion sequences leading to a multi-digraph with the prescribed in/out-degrees.

McKay’s formula was used by Pittel and Wormald [19] to derive a sharp estimate of \(C_k(n, m)\), the total number of graphs on \([n]\) with \(m = O(n \ln n)\) edges, and with minimum vertex degree \(k\), at least, \((k \geq 1)\). We use Theorem 1.1 to derive an asymptotic formula for \(C_{1,1}(n, m)\), the total number of digraphs on \([n]\) with \(\min_i \delta_i \geq 1\), \(\min_j \Delta_j \geq 1\). In fact we will need a slightly more general set-up.

Given \(n_1 \leq n\), let \(n = (n_1, n)\), and let \(C_{1,1}(n, m)\) denote the total number of digraphs on \([n]\), with \(m\) arcs, such that

\[(1.6) \quad \delta_i \geq 1, (i \in [n]); \quad \Delta_j \geq 1, (j \in [n_1]); \quad \Delta_j \equiv 0, (j > n_1).
\]

By (1.4),

\[
C_{1,1}(n, m) \leq m! \sum_{\delta, \Delta \text{ meet (1.6)}} \prod_{i \in [n]} \frac{1}{\delta_i! \Delta_i!}
= m! \left[x^m x_1^m\right] \sum_{\delta, \Delta \text{ meet (1.6)}} \prod_{i \in [n]} \frac{x^\delta_i x_1^\Delta_i}{\delta_i! \Delta_i!}
= m! \left[x^m \right] f(x)^n \left[x_1^m \right] f(x_1)^{n_1}, \quad f(y) := e^y - 1.
\]

So, for all \(x > 0, x_1 > 0\),

\[(1.7) \quad C_{1,1}(n, m) \leq m! \frac{f(x)^n}{x^m} \frac{f(x_1)^{n_1}}{x_1^m},
\]
and the best values of $x$ and $x_1$ are the minimum points of the first fraction and the second fraction, i.e. $\lambda$ and $\lambda_1$, the roots of

\begin{equation}
\frac{\lambda e^\lambda}{e^\lambda - 1} = \frac{m}{n}, \quad \frac{\lambda_1 e^{\lambda_1}}{e^{\lambda_1} - 1} = \frac{m}{n_1}.
\end{equation}

Using the Cauchy integral formula

\[ [y^a] f(y)^b = \frac{1}{2\pi i} \oint_{|z| = \rho} \frac{f(z)^b}{z^{a+1}} \, dz, \]

and an inequality

\[ |e^z - 1| \leq (e^{|z|} - 1) \exp\left(\frac{-|z| - \text{Re} \, z}{2}\right), \]

one can show easily that, in fact,

\[ [y^a] f(y)^b \leq b \frac{1}{(y b)^{1/2}} \frac{f(y)^b}{y^a}, \quad \forall \, y > 0. \]

(Here and elsewhere $A \leq_b B$ means that $A = O(B)$ uniformly over all parameters that determine the values of $A$ and $B$.) So (1.7) can be strengthened to

\begin{equation}
C_{1,1}(n, m) \leq b \frac{m!}{(1 + n x)^{1/2} x^m} \cdot \frac{f(x)^n}{(1 + n x_1)^{1/2} x_1^{n_1}},
\end{equation}

for all $x, x_1 > 0$. In particular,

\begin{equation}
C_{1,1}(n, m) \leq b \frac{m!}{(1 + n x)^2 x^m}, \quad \forall \, x > 0.
\end{equation}

The next theorem gives an asymptotically sharp formula for $C_{1,1}(n, m)$.

**Theorem 1.2.** Let $n_1 = \Theta(n)$, $r := m - n \to \infty$, $m = O(n)$. Introduce $Y$ and $Y_1$, two positive Poissons, with parameters $\lambda$ and $\lambda_1$, i.e.

\[ P(Y = j) = \frac{\lambda^j / j!}{f(\lambda)}, \quad P(Y = j) = \frac{\lambda_1^j / j!}{f(\lambda_1)}; \quad (j \geq 1). \]

Then

\begin{equation}
C_{1,1}(n, m) = (1 + O(r^{-1} + n^{-1/2} r^\varepsilon)) \frac{m! f(\lambda)^n f(\lambda_1)^{n_1}}{(\lambda \lambda_1)^m}
\end{equation}

\[ \times \frac{e^{-n}}{2\pi (n \text{Var}[Y])^{1/2} (n_1 \text{Var}[Y_1])^{1/2}}, \]

for every $\varepsilon > 0$, where

\begin{equation}
\eta = \frac{m}{n} + \frac{\lambda_1}{2}.
\end{equation}
The idea of the proof is that, introducing the independent copies $Y^1, \ldots, Y^n$ of $Y$ and $Y_1^1, \ldots, Y_1^{m_1}$ and denoting $Y = (Y^1, \ldots, Y^n)$, $Y_1 = (Y_1^1, \ldots, Y_1^{m_1}, n - n_1$ zeroes), we can rewrite (1.2) in the following, suggestive, way:

$$C_{1,1}(n, m) = m! \frac{f(\lambda)^n f(\lambda_1)^{m_1}}{\lambda^m \lambda_1^{m_1}} \cdot [x^m x_1^{m_1}] \sum_{\delta, \Delta \text{ meet } (1.6)} F(\delta, \Delta) \prod_{i \in [n]} \frac{(\lambda x)^{\delta_i}/\delta_i!}{f(\lambda)} \cdot \frac{(\lambda_1 x_1)^{\Delta_i}/\Delta_i!}{f(\lambda)}.$$

(1.13)

$$= m! \frac{f(\lambda)^n f(\lambda_1)^{m_1}}{\lambda^m \lambda_1^{m_1}} \mathbb{E} \left[ F(Y, Y_1) 1_{\{\|Y\| = \|Y_1\| = m\}} \right].$$

(1.14)

$$= m! \frac{f(\lambda)^n f(\lambda_1)^{m_1}}{\lambda^m \lambda_1^{m_1}} \mathbb{E} \left[ F(Y, Y_1) \mid \|Y\| = \|Y_1\| = m \right].$$

Here

(1.15)

$$\mathbb{E} \left[ F(Y, Y_1) 1_{\{\|Y\| = \|Y_1\| = m\}} \right] = \mathbb{E} \left[ F(Z, Z_1) \right],$$

where, denoting $i = (i_1, \ldots, i_n)$, $j = (j_1, \ldots, j_n)$,

(1.16)

$$P(Z = i) = \frac{\prod_{i=1}^{n} 1/i_{i_1}! \prod_{j=1}^{n} 1/j_{j_1}!}{\sum_{j: \|j\| = m} \prod_{i=1}^{n} 1/i_{i_1}! \prod_{j=1}^{n} 1/j_{j_1}!}, \quad (\|i\| = m),$$

and, denoting $i = (i_1, \ldots, i_{n_1})$, $j = (j_1, \ldots, j_{n_1})$, \n
(1.17)

$$P(Z_1 = i) = \frac{\prod_{i=1}^{n_1} 1/i_{i_1}! \prod_{j=1}^{n_1} 1/j_{j_1}!}{\sum_{j: \|j\| = m} \prod_{i=1}^{n_1} 1/i_{i_1}! \prod_{j=1}^{n_1} 1/j_{j_1}!}, \quad (\|i\| = m).$$

The next step is to prove two similar cases of a local limit theorem

$$P(\|Y\| = m) = \frac{1 + O(r^{-1})}{(2\pi n Var[Y])^{1/2}},$$

(1.18)

$$P(\|Y_1\| = m) = \frac{1 + O(r^{-1})}{(2\pi n Var[Y_1])^{1/2}},$$

with the remainder term $O(r^{-1})$.

The last step is to prove that on the event $\{|Y| = |Y_1| = m\}$, in probability $F(Y, Y_1)$ is within $1 + o(1)$ factor from the RHS of (1.3), where $\delta_i$, $(\delta_i)_2$, $(i \in [n])$, are replaced with $E[Y]$ and $E[(Y_2)|Y]$, and $\Delta_i$, $(\Delta_i)_2$, $(i \in [n_1])$, are replaced with $E[Y_1]$ and $E[(Y_1)_2]$. We omit the technical details as the argument is a natural modification of the proof of a corresponding formula in [19] for the total number of undirected graphs of mindegree $k \geq 1$, at least. We mention only that in (1.11) $r^\epsilon$ comes from an observation that, with sufficiently high probability, $\max\{\max_i Y_i, \max_j Y_1^j\} < r^\epsilon$, and that $r^{-1}$ comes from the remainder term in (1.18). $\Box$

**Note 1.2.** Implicit in the theorem 1.2 and its proof is the following: for $r \to \infty$, $m = O(n)$,

(1.19)

$$[x^n](e^x - 1)^n = \frac{(e^\lambda - 1)^n}{\lambda^{2m}} \cdot \frac{1 + O(r^{-1})}{(2\pi n Var[Y])^{1/2}}.$$
Corollary 1.3. Let \( r \to \infty, m = O(n) \). Then

\[
C_{1,1}(n, m) = (1 + O(r^{-1} + n^{-1/2}r^\varepsilon)) \ m! \frac{f(\lambda)^{2n}}{\lambda^{2m}} \cdot \frac{e^{-\eta}}{2\pi n \text{Var}[Y]}
\]

\[(1.20)\]

\[\eta = \eta(n, m) := \frac{m}{n} + \frac{\lambda^2}{2},\]

for every \( \varepsilon > 0 \).

Since \( \text{Var}[Y] = \Theta(\lambda) \), \( \text{Var}[Y_1] = \Theta(\lambda_1) \), the bound (1.10) for \( C_{1,1}(n, m) \) with \( x = \lambda, x_1 = \lambda_1 \) contains all the key factors in the sharp estimate (1.15), except for \( e^{-\eta} \).

Note 1.3 Recalling the note 1.1, \( m! C_{1,1}(n, m) \) is the total number of \( m \)-long insertion sequences \( s \) resulting in a digraph with non-zero in/out-degrees. Let \( h_{1,1}(n, m) \) denote the total number of those sequences resulting in multi-digraphs with non-zero in/out-degrees. Dropping the fudge factor \( F \), we have a counterpart of (1.13)-(1.15):

\[
h_{1,1}(n, m) = (m!)^2 \frac{f(\lambda)^{2n}}{\lambda^{2m}} \times P(||Y|| = m) \ P(||Y_1|| = m).
\]

Consequently,

\[
\frac{m! C_{1,1}(n, m)}{h_{1,1}(n, m)} = \text{E}[F(Z, Z_1)],
\]

where \( Z, Z_1 \) are defined in (1.16)-(1.17), with \( n_1 = n \) this time. Now \( \text{E}[F(Z, Z_1)] \) is asymptotically equivalent to \( e^{-\eta} \), which is positive if \( r = O(n) \). Thus, for \( r = O(n) \) the graphic \( m \)-long sequences constitute an asymptotically positive fraction among all \( m \)-long sequences of arc insertions. Moreover, from the proof of the theorem 1.2 it follows that, for \( r = O(n) \),

\[
|\{s : |F(\delta(s), \Delta(s)) - e^{-\eta}| \leq r^\varepsilon n^{-1/2+\gamma}\}| \geq h_{1,1}(n, m) \ (1 - O(e^{-r^c n^{2\gamma}})),
\]

for all \( \varepsilon > 0, \gamma \in (0, 1/2) \), and some absolute constant \( c > 0 \).

Section 2. Counting digraphs with restricted in/out-degrees that have no isolated cycles.

Recall that, given \( n_1 \leq n \), \( C_{1,1}(n, m) \) denotes the total number of digraphs with in-degrees of all vertices and outdegrees of the vertices in \([n_1]\) each being 1, at least. Introduce \( C_{1,1}^0(n, m) \), the total number of these digraphs without isolated cycles. How much smaller is \( C_{1,1}^0(n, m) \) compared with \( C_{1,1}(n, m) \)?
Theorem 2.1. If \((n = (n_1, n_2), m)\) meet the conditions of Theorem 1.2, and \(r_1 \equiv m - n_1 = O(r)\), then

\[
C_{1,1}^0(n, m) \leq b \frac{r}{n} C_{1,1}(n, m), \quad r = m - n.
\]

Proof of Theorem 2.1. Obviously, we may and will assume that \(r = o(n)\). If so, in (1.10) \(\eta = O(1)\), see (1.11). Suppose a digraph is chosen uniformly at random from among all \(C_{1,1}(n, m)\) digraphs. Let \(X\) denote the total number of isolated cycles in the random digraph. We need to show that \(P(X = 0) = O(r/n)\).

Since the exponential generating function of directed cycles is

\[
\sum_{\ell \geq 2} \frac{x^\ell}{\ell} = \ln(1 - x)^{-1} - x,
\]

the binomial moments of \(X\) are given by

\[
E \left[ \binom{X}{k} \right] = \frac{1}{k!} \sum_{a < n_1} \binom{n_1}{a} \frac{C_{1,1}(n(a), m - a)}{C_{1,1}(n, m)} \cdot a! [x^a] \left( \sum_{\ell \geq 2} \frac{x^\ell}{\ell} \right)^k
\]

\[
= \frac{1}{k!} \sum_{a < n_1} R_{n, m}(a) [x^a] \left( \sum_{\ell \geq 2} \frac{x^\ell}{\ell} \right)^k,
\]

\[
R_{n, m}(a) := \frac{n_1! C_{1,1}(n(a), m - a)}{(n_1 - a)! C_{1,1}(n, m)}, \quad (n(a) = (n_1 - a, n_2)).
\]

Explanation: The binomial moment is the expected number of \(k\)-long unordered tuples of isolated cycles. To generate those tuples in all digraphs in question we choose a vertex set of a generic cardinality \(a\) from \(n_1\) in \(\binom{n_1}{a}\) ways, then form an unordered family of \(k\) disjoint cycles on these \(a\) vertices, in \((a!/k!)[x^a] \left( \sum_{\ell \geq 2} \frac{x^\ell}{\ell} \right)^k\) ways, and finally select an admissible complementary digraph on the remaining \(n - a\) vertices with \(m - a\) arcs, in \(C_{1,1}(n(a), m - a)\) ways. Furthermore, an admissible \(a\) also satisfies \(m - a \leq (n - a)^2\), which implies that \(n - a \geq \sqrt{r}\).

Consequently, by using an inversion formula

\[
P(X = j) = \sum_{k \geq j} (-1)^{k-j} \binom{k}{j} E \left[ \binom{X}{k} \right],
\]

we obtain

\[
P(X = 0) = \sum_{k \geq 0} (-1)^k E \left[ \binom{X}{k} \right] = \sum_{a} R_{n, m}(a) [x^a] h(x),
\]

where

\[
h(x) = \exp \left(- \sum_{\ell \geq 2} \frac{x^\ell}{\ell} \right) = (1 - x)e^x.
\]
So
\[ [x^a]h(x) = \frac{1}{a!} - \frac{1}{(a - 1)!}, \quad a \geq 1, \quad [x^0]h(x) = 1. \]

Let us find a sharp asymptotic formula for the sum in (2.2). Using (1.10) for \( C_{1,1}(n, m) \) and (1.8), with \( x = \lambda, \ x_1 = \lambda_1 \), for \( C_{1,1}(n(a), m - a) \), we bound
\[
R_{n,m}(a) \leq \frac{\Gamma(n_1)}{(m)_a} \cdot \left( \frac{\lambda \lambda_1}{f(\lambda)f(\lambda_1)} \right)^a \cdot \sqrt{\frac{nn_1}{(n - a)(n_1 - a)}} \\
\leq \frac{n}{n_1 - a} \sigma^a.
\]

Here, by (1.7),
\[
\sigma = \frac{n_1}{m} \cdot \frac{\lambda \lambda_1}{f(\lambda)f(\lambda_1)} = \frac{\lambda e^{-\lambda_1}}{e^\lambda - 1} < 1.
\]

Introduce
\[ A_n = 3 \left[ \frac{\ln(n/r)}{\ln \ln(n/r)} \right]. \]

Consider \( a \geq A_n \):
\[
\left| \sum_{a \geq A_n} R_{n,m}(a) [x^a]h(x) \right| \leq \sum_{n_1/2 \leq a \leq n_1} \frac{1}{a!} + \frac{n}{[n_1/2]!} \\
\leq \frac{1}{A_n!} + \frac{n}{[n_1/2]!} \\
\leq b \left( \frac{r}{n} \right)^2.
\]

For \( 1 \leq a \leq A_n \) we need a sharp estimate \( R_{n,m}(a) \), within a factor \( 1 + O(r/n) \). Observe upfront that the asymptotic estimate (1.10) used separately for \( C_{1,1}(n, m) \) and \( C_{1,1}(n(a), m - a) \) would not work, as the remainder term \( O(r^{-1} + n^{-1/2 + \varepsilon}) \) is too big. (We already confronted a similar issue in [20]; the hurdle is higher this time, as in [20] we were content to have an error term of order \( o(r/n)^{1/2} \), rather than \( O(r/n) \). For the next theorem, at a similar point we will even need \( o(r/n) \).)

Instead of (1.10), we use the exact formulas (1.12)-(1.13) for both \( C_{1,1}(n, m) \) and \( C_{1,1}(n(a), m - a) \), using \( \lambda \) and \( \lambda_1 \), the roots of (1.7), for both \( C_{1,1}(n, m) \) and \( C_{1,1}(n(a), m - a) \). So
\[
R_{n,m}(a) = \frac{\Gamma(n_1)}{(m)_a} \cdot \frac{\lambda \lambda_1}{f(\lambda)f(\lambda_1)}^a \cdot \frac{E_{n(a), m - a}}{E_{n,m}} \cdot \frac{P(\sum_{i=1}^{n-a} Y^i = m - a)}{P(\sum_{i=1}^n Y^i = m)} \cdot \frac{P(\sum_{j=1}^{n_1-a} Y_1^j = m - a)}{P(\sum_{j=1}^{n_1} Y_1^j = m)}.
\]
where \( E_{n,m} = E_{n(a),m-a} \big|_{a=0} \), (see (1.14)-(1.16))

\[
(2.6) \quad E_{n(a),m-a} = E \left[ 1_{B(a)} \exp \left( - \frac{1}{m-a} \sum_{i=1}^{n_1-a} Z^i(a)Z^i_1(a) \right. \right.
\]
\[
\left. \left. - \frac{1}{2(m-a)2} \sum_{i \notin (n_1-a,n_1]} (Z^i(a))_2 \sum_{j=1}^{n_1-a} (Z^j_1(a))_2 \right. \right.
\]
\[
\left. \left. + O(m^{-1} \max_{i,j}(Z^i(a) + Z^j_1(a)))^4) \right] \right) + O(P(B(a)^c)), \]

and

\[
B(a) := \{ \max_{i,j}(Z^i(a) + Z^j_1(a)) \leq \omega \},
\]

with \( \omega = o(m^{-1/4}) \) to be specified shortly. Our notation emphasizes dependence of \( Z \)'s on \( a \): for instance, \( Z^i(a), \ldots, Z^{n-a}(a) \) are occupancy numbers in the uniformly random allocation of \( m-a \) distinguishable balls among \( n-a \) boxes, subject to the condition “no box is empty”.

First, let us dispense with the ratios of local probabilities. In [20] the following estimate was proved. Let \( Y^1, \ldots, Y^n \) be independent Poissons distributed on \( \{2, 3, \ldots\} \), such that \( E[Y^j] = 2m/n \). Then, for \( a = o(n) \),

\[
P(\sum_{i=1}^{n-a} Y^i = 2(m-a)) \quad \frac{P(\sum_{i=1}^{n-a} Y^i = 2(m-a))}{P(\sum_{i=1}^{n-a} Y^i = 2m)} = 1 + O(an^{-1} + a^2rn^{-2}), \quad r := 2(m-n).
\]

No real changes are needed to show that for our Poissons \( Y^1, \ldots, Y^n \) (distributed on \( \{1, 2, \ldots\} \)),

\[
P(\sum_{i=1}^{n-a} Y^i = m-a) \quad \frac{P(\sum_{i=1}^{n-a} Y^i = m-a)}{P(\sum_{i=1}^{n-a} Y^i = m)} = 1 + O(an^{-1} + a^2rn^{-2}), \quad r := m-n,
\]

and likewise

\[
P(\sum_{j=1}^{n_1-a} Y^j_1 = m-a) \quad \frac{P(\sum_{j=1}^{n_1-a} Y^j_1 = m-a)}{P(\sum_{j=1}^{n_1-a} Y^j_1 = m)} = 1 + O(an^{-1} + a^2r_1n^{-2}), \quad r_1 = m-n_1.
\]

Since \( r_1 = O(r) \) we conclude that

\[
(2.8) \quad \frac{P(\sum_{i=1}^{n-a} Y^i = m-a)}{P(\sum_{i=1}^{n-a} Y^i = m)} \cdot \frac{P(\sum_{j=1}^{n_1-a} Y^j_1 = m-a)}{P(\sum_{j=1}^{n_1-a} Y^j_1 = m)} = 1 + O(an^{-1} + a^2rn^{-2}).
\]

Next we focus on \( E_{n(a),m-a}/E_{n,m} \). Let us look at \( \max_{i \notin (n_1-a,n_1]} Z^i(a) \). For \( j \geq 1 \),
by (1.15),

\[ P(Z(a) = j) = \frac{1}{j!} \sum_{i>0: ||i||=m-a-j} \prod_{s=2}^{n-a} 1/i_s! \prod_{i=1}^{n-a} 1/j_i! , \]

\[ = \frac{1}{j!} \left[ x^{m-a-j} (e^x - 1)^{n-a-1} \right] \]

\[ \leq b \frac{1}{j!} \frac{(e^\lambda - 1)^{n-a-1} / [\lambda^{m-a-j} \sqrt{(n-a-1)\lambda}] }{(e^\lambda - 1)^{n-a} / [\lambda^{m-a} \sqrt{(n-a)\lambda}] } \]

\[ \leq b \frac{1}{j!} \frac{\lambda^j}{e^\lambda - 1} . \]

As \( \lambda \leq 2r/n \), we see then that \( P(Z(a) > j) \leq b (2r/n)^j \). Therefore, picking \( \varepsilon > 0 \),

\[ P(\max Z^i(a) > r^\varepsilon) \leq n \left( \frac{2r}{n} \right)^{r^\varepsilon} \]

\[ \leq b \frac{r}{n} \exp [\ln n - r^\varepsilon \ln(n/(2r))] \]

\[ \leq b \frac{r}{n} \left( \frac{r}{n} \right)^{r^\varepsilon/2} , \]

(2.9)

since, for \( r \to \infty \) and \( r = o(n) \),

\[ r^\varepsilon \ln(n/r) \gg \ln n. \]

And on the event \{max_i Z^i(a) \leq r^\varepsilon \}, we have

\[ \frac{\max_i Z^i(a)^4}{m} \leq b \frac{r^{4\varepsilon}}{n} = \frac{r}{n} r^{4\varepsilon-1} \ll \frac{r}{n} , \]

(2.10)

provided that \( \varepsilon < 1/4 \). Obviously the counterparts of (2.9), (2.10) hold for \( \max_j Z^j(a) \) as well.

So, setting \( \omega = r^\varepsilon \) in (2.6),

\[ E_{n(a),m-a} = O\left( (r/n)^{r^\varepsilon/2 + 1} \right) + (1 + O(r^{4\varepsilon}/n)) \]

\[ \times \mathbb{E} \left[ \exp \left( -\frac{1}{m-a} \sum_{i=1}^{n-a} Z^i(a) Z^i_1(a) \right. \right. \]

\[ \left. \left. - \frac{1}{2(m-a)^2} \sum_{i \notin \{n_1-a,n_1\}} (Z^i(a))^2 \sum_{j=1}^{n_1-a} (Z^j_1(a))^2 \right) \right] , \]

and, by \( \mathbb{E}[e^U] \geq e^{\mathbb{E}[U]} \), the last expected value is bounded below by

\[ \exp \left( -\frac{n_1-a}{m-a} \mathbb{E}[Z(a)] \mathbb{E}[Z_1(a)] - \frac{(n-a)(n_1-a)}{2(m-a)^2} \mathbb{E}[(Z(a))^2] \mathbb{E}[(Z_1(a))^2] \right) , \]
which is bounded away from 0. Therefore

\[ E_{n(a),m-a} = (1 + O((r/n)^{r^*/2+1} + r^{4\epsilon}/n)) E_{n(a),m-a}^*, \]

\[ E_{n(a),m-a}^* := E \left[ \exp \left( -\frac{1}{m-a} \sum_{i=1}^{n_1-a} Z_i(a) Z_i^*(a) \right) \right. \]

\[ \left. - \frac{1}{2(m-a)^2} \sum_{i \notin (n_1-a,n_1]} (Z_i(a))^2 \sum_{j=1}^{n_1-a} (Z_j^*(a))^2 \right], \]

It remains to consider \( E_{n(a),m-a}^*/E_{n(0),m}^* \). Notice that, conditioned on the event

\[ A = \{ Z_i(0) \equiv 1, i \in (n_1 - a, n_1] \} \cap \{ Z_1^*(0) \equiv 1, i \in (n_1 - a, n_1] \}, \]

\( \{ Z_i(0), Z_1^*(0); i \notin (n_1 - a, n_1], j \leq n_1 - a \} \) has the same distribution as \( \{ Z_i(a), Z_1^*(a); i \notin (n_1 - a, n_1], j \leq n_1 - a \} \), and

\[ P(A) \geq 1 - a (P(Z(0) > 1) + P(Z_1(0) > 0)) = 1 - O(a\lambda) = 1 - O(ar/n). \]

Therefore

\[ E_{n(a),m-a}^* = (1 + O(ar/n)) E_{n(a),m-a}^*. \]

\[ E_{n(a),m-a}^{**} := E \left[ \exp \left( -\frac{1}{m-a} \sum_{i=1}^{n_1-a} Z_i(0) Z_i^*(0) \right) \right. \]

\[ - \frac{1}{2(m-a)^2} \sum_{i \notin (n_1-a,n_1]} (Z_i^*(0))^2 \sum_{j=1}^{n_1-a} (Z_j^*(0))^2 \right]. \]

The contribution to the expectation \( E_{n(a),m-a}^{**} \) from the random outcomes with

\[ \max_{i,j}(Z_i(0) + Z_j^*(0)) > r^\epsilon \]

is of order

\[ P(\max_{i,j}(Z_i(0) + Z_j^*(0)) > r^\epsilon) \leq (r/n)^{r^*/2+1}. \]

If \( \max_{i,j}(Z_i(0) + Z_j^*(0)) \leq r^\epsilon \), then the difference between the random exponents in (2.11) for \( a = 0 \) and \( a > 0 \) is, by simple algebra, of order \( ar^{4\epsilon}/n \). So \( E_{n(a),m-a}^{**} \) is within the multiplicative factor

\[ (1 + O((r/n)^{r^*/2+1})) \left[ \exp(O(r^{4\epsilon}/n)) \right]^{a} \]

away from \( E_{n(0),m}^{*} \). Collecting the pieces we obtain

\[ \frac{E_{n(a),m-a}}{E_{n,m}} = (1 + O(ar/n)) \left[ \exp(O(r^{4\epsilon}/n)) \right]^{a}, \]

uniformly for \( 1 \leq a \leq A_n \).
Combining (2.5), (2.8) and (2.12), we conclude: uniformly for \(1 \leq a \leq A_n\),

\[
R_{n,m}(a) = \frac{\binom{n}{a}}{\binom{m}{a}} \left( \frac{\lambda_1}{f(\lambda)f(\lambda_1)} e^{O(r^{4\varepsilon}/n)} \right)^a (1 + O(ar/n)),
\]

\[
= \left( \frac{\lambda_1}{(m/n_1)f(\lambda)f(\lambda_1)} e^{O(r^{4\varepsilon}/n)} \right)^a \cdot \exp \left( -\frac{a^2}{2n_1} + \frac{a^2}{2m} \right) (1 + O(ar/n))
\]

\[
= [\sigma e^{O(r^{4\varepsilon}/n)}]^a (1 + O(ar/n)),
\]

see (2.3) for the definition of \(\sigma\). Since \(\sigma = 1 - \Theta(r/n)\), and \(\varepsilon < 1/4\),

\[
\sigma e^{O(r^{4\varepsilon}/n)} = 1 - \Theta(r/n).
\]

Therefore, invoking the inversion formula (2.2) and (2.4), and using

\[
\frac{1}{a!} - \frac{1}{(a-1)!} \leq 0, \quad \forall a \geq 1,
\]

we see that \(P(X = 0)\) is bounded above and below by

\[
1 + \sum_{a=1}^{A_n} \left( \frac{1}{a!} - \frac{1}{(a-1)!} \right) [\sigma e^{O(r^{4\varepsilon}/n)}]^a (1 + O(ar/n)) + O((r/n)^2)
\]

\[
= (1 - \sigma e^{O(r^{4\varepsilon}/n)}) e^\sigma + O \left[ \frac{r}{n} \sum_{a \geq 1} a \left( \frac{1}{(a-1)!} - \frac{1}{a!} \right) \right] + O((r/n)^2)
\]

\[
= (1 - \sigma e^{O(r^{4\varepsilon}/n)}) e^{\sigma e^{O(r^{4\varepsilon}/n)}} + O(r/n) = O(r/n).
\]

This completes the proof of Theorem 2.1. \(\square\)

**Note 2.1.** Using Theorem 2.1 we will prove in the next section 3, Theorem 3.1, that it is quite unlikely that the random digraph with non-zero in/out-degrees has no isolated cycles and no sink/source-set with fewer than \(m/2\) arcs. In Section 4, we will use the proof of Theorem 2.1 as a rough template for proving a genuinely sharp asymptotic estimate for the probability of non-existence of “simple sink/source-sets” in the random digraph with non-zero in/out-degrees. This estimate coupled with Theorem 3.1 will deliver an asymptotic fraction of strongly-connected digraphs among all such digraphs.

### 3. Bounding the number of the digraphs without simple sink-sets and small complex sink-sets.

Let \(G_{1,1}(n,m)\) denote a digraph on \([n]\) which is chosen uniformly at random from all such digraphs with \(m\) arcs and with the smallest in-degree and the smallest out-degree both at least 1. We call \(S \subseteq [n]\) a sink-set (source-set resp.) if there is no arc \(i \to j\) (\(j \to i\) resp.) for \(i \in S, j \notin S\). A digraph is strongly-connected if it has no proper sink-sets and no proper source-sets. We call a sink-set (source-set resp.) \(S\) simple, if all the out-degrees (in-degrees) of \(G_S\), the subdigraph induced by \(S\), are equal 1, and complex otherwise. Our first step is to the following result.
Theorem 3.1. Suppose $r := m - n \to \infty$ and $r = O(n)$. Let $A_{n,m}$ denote the event “$G_{1,1}(n, m)$ has a complex sink-set containing at most $m/2$ arcs, and has no simple sink-set”. Then $P(A_{n,m}) = O(n^{-1})$.

Proof of Theorem 3.1. We will focus on a core case $r = o(n)$, and at the end of the proof we will briefly discuss how to handle $r = \Theta(n)$.

Given $\nu \in [3, n)$, $\nu < \mu \leq m_0$, let $X_{\nu, \mu}$ denote the total number partitions $[n] = A \cup B$, $(|A| = \nu)$, such that (1) $A$ is a minimal complex sink-set with $\mu$ induced arcs, and (2) neither $G_A$ nor $G_B$ contain a simple sink-set or source set. (Minimal means that $A$ does not contain a smaller complex sink-set.) Let $\varepsilon > 0$ be fixed. Set

$$
(3.1) \quad \nu_0 = \varepsilon \left( \frac{mn^2}{r^2} \right)^{1/3} \sim \varepsilon \frac{n}{2r^{2/3}},
$$

clearly $\nu_0 = o(n)$. Define

$$
(3.2) \quad X = \sum_{\nu \leq \nu_0, \mu \leq m/2} X_{\nu, \mu} + \sum_{\nu > \nu_0, \mu \leq m/2} X_{\nu, \mu}.
$$

In both sums $\nu$ and $\mu$ are subject to an additional condition, $\mu - \nu \leq r$: as we shall see shortly, $X_{\nu, \mu} = 0$ otherwise. The theorem will be proven when we show that $E[X] = O(n^{-1})$.

For this we will use the following bound:

$$
(3.3) \quad E[X_{\nu, \mu}] \leq b r^\frac{r}{n} E_{\nu, \mu},
$$

$$
E_{\nu, \mu} := (x + 1) \frac{(n)}{\binom{m}{\mu}} \cdot \frac{\nu^{\mu - \nu}}{\mu (\mu - \nu)!} \cdot \frac{(e^x - 1)^\nu}{x^{\mu}} \cdot \frac{\lambda^{2\mu} \epsilon^{\nu \lambda}}{(e^\lambda - 1)^{2\nu}},
$$

uniformly for all $\nu < \mu$, $m - \mu \geq c m$, $c \in (0, 1)$ being fixed, and $x > 0$.

To prove (3.3), let us bound the total number of partitions $(A, B)$ in question in all the digraphs with parameters $n$, $m$, of minimum in-degree and minimum out-degree 1 at least. A set $A$ of cardinality $\nu$ can be chosen in $\binom{n}{\nu}$ ways. By symmetry, we may consider $A = [\nu]$, and we will use $[n - \nu]$ to denote $B = \{ \nu + 1, \ldots, n \}$. Let us bound the total number of $G_A$’s. Introduce $R$, the set consisting of 1 and all vertices $j \in [n]$ reachable from $i$ by directed paths. $R \subseteq A$ as $A$ is a sink-set. Since $R$ is a sink-set itself, $R = A$ by minimality of $A$. Thus there exists a directed tree $T$ rooted at 1 that spans $A$, with arcs oriented away from the vertex 1. This spanning tree is such that the out-degree of 1 equals its the out-degree of 1 in $G_A$ itself. Given

$$
(3.4) \quad d_1 \geq 1, \quad d_2, \ldots, d_\nu \geq 0, \quad (d_1 - 1) + d_2 + \cdots + d_\nu = \nu - 2,
$$

there are

$$
\frac{(\nu - 2)!}{(d_1 - 1)! \prod_{i \geq 2} d_i!}
$$
rooted trees $T$ with outdegrees $d_1, \ldots, d_\nu$. The digraph $G_A$ is a disjoint union of $T$ and a complementary digraph $H$ on $[\nu]$. $\delta_i, \Delta_i$, $(i \in [\nu])$, respectively the in-degrees and the out-degrees of $H$, must meet the constraints

\begin{align}
\sum_i \delta_i &= \sum_i \Delta_i = \mu - \nu + 1, \\
(3.6) \quad d_1 &\geq 1, \delta_1 \geq 1, \Delta_1 = 0; \quad \Delta_i + d_i \geq 1, (i \geq 2).
\end{align}

(All in-degrees in $H$, except the root 1, are positive, whence the single condition $\delta_1 \geq 1$ ensures that $T \cup H$ has all its in-degrees positive. The condition $\Delta_1 + d_1 \geq 1$, redundant for the root 1, ensures that $T \cup H$ has all its out-degrees positive as well.) The total number of $H$’s with the in-degrees $\delta_i$ and the out-degrees $\Delta_i$ is $(\mu - \nu + 1)! \prod_i (\delta_i! \Delta_i!)$, at most. Hence the total number of $G_A$’s, with $A = [\nu]$, is on the order of

$$(\nu - 2)! (\mu - \nu + 1)! \sum_{d, \delta, \Delta} \prod_i \frac{1}{(d_i - 1)!(\delta_i! \Delta_i)!},$$

with $d, \delta, \Delta$ meeting the constraints (3.4)-(3.6). Using the generating functions,

\begin{align}
&\sum_{d, \delta, \Delta} \prod_i \frac{1}{(d_i - 1)!(\delta_i! \Delta_i)!} \\
&= [x^{\nu - 2} y^{\mu - \nu + 1} z^{\mu - \nu + 1}] \sum_{d, \delta, \Delta} \prod_i \frac{x^{d_i - 1} y^{\delta_i} z^{\Delta_i}}{(d_i - 1)!(\delta_i! \Delta_i)!} \\
&= [y^{\mu - \nu + 1}] (e^y - 1) e^{(\nu - 1)y} \\
&\times [x^{\nu - 2} z^{\mu - \nu + 1}] \left( \sum_{d \geq 1, \Delta = 0} \frac{x^{d - 1} z^{\Delta}}{d! \Delta!} \right) \cdot \left( \sum_{d \geq 1, \Delta \geq 1} \frac{x^{d} z^{\Delta}}{d! \Delta!} \right) \\
&= [y^{\mu - \nu + 1}] (e^y - 1) e^{(\nu - 1)y} \times [x^{\nu - 2} z^{\mu - \nu + 1}] e^x (e^x - 1)^{\nu - 1}.
\end{align}

Here

$$[y^{\mu - \nu + 1}] (e^y - 1) e^{(\nu - 1)y} \leq [y^{\mu - \nu + 1}] y e^y = [y^{\mu - \nu}] e^y = \frac{\nu^{\mu - \nu}}{(\mu - \nu)!},$$

and

$$[x^{\nu - 2} z^{\mu - \nu + 1}] e^x (e^x - 1)^{\nu - 1} \leq [x^{\nu - 2} z^{\mu - \nu + 1}] e^x (e^x - 1)^{\nu - 1} = \frac{(\mu - 1)!}{(\nu - 2)! (\mu - \nu + 1)!} [x^{\mu - 1}] e^x (e^x - 1)^{\nu - 1},$$

where

$$[x^{\mu - 1}] e^x (e^x - 1)^{\nu - 1} \leq \frac{e^x (e^x - 1)^{\nu - 1}}{x^{\mu - 1}}, \quad (\forall x > 0),$$

$$\leq (x + 1) \frac{(e^x - 1)^\nu}{x^\mu}. $$
Collecting the pieces, we bound the total number of $G_A$'s with $|A| = \nu$ by

$$
(3.7) \quad \binom{n}{\nu} (\nu - 2)! (\mu - \nu + 1)! \sum_{d, \delta, \Delta} \prod_{i} \frac{1}{(d_i - 1)_{(i=1)}! \delta_i! \Delta_i!} \leq (x + 1) \binom{n}{\nu} \nu^{\nu - \nu} (\mu - 1)! \left( e^x - 1 \right)^\nu \nu^\mu.
$$

Turn to the complementary subgraphs $G_B$, $B = [\nu]^c := [n - \nu]$. Let $\delta_i, \Delta_i, i \in [n - \nu]$, denote in-degrees and out-degrees of $G_{[n-\nu]}$, and let $\partial_i$ denote the total number of arcs emanating from $i \in [n - \nu]$ and ending at a vertex in $[\nu]$. Obviously

$$
(3.8) \quad \delta_i \geq 1; \quad \partial_i + \Delta_i \geq 1, \quad i \in [n - \nu],
$$

$$
(3.9) \quad \sum_{i \in [n - \nu]} \delta_i = \sum_{i \in [n - \nu]} \Delta_i = m - \nu - \partial.
$$

Let us bound the total number of the admissible $G_{[n-\nu]}$ for given $\partial_i, i \in [n - \nu]$. The second condition in (3.8) simplifies to

$$
\Delta_i \geq 1, \quad i \in I(\sigma) := \{ i \in [n - \nu] : \partial_i = 0 \}.
$$

Clearly,

$$
|I(\sigma)| \geq n - \nu - \partial, \quad \partial := \sum_{i \in [n - \nu]} \partial_i.
$$

By the first condition in (3.8), $n - \nu \leq m - \nu - \partial$; in particular, $\mu - \nu \leq m - n = r$. Actually $n - \nu < m - \mu - \partial$, since otherwise all $\delta_i = 1$ and $G_{[n-\nu]}$ would contain an isolated cycle, a special case of a simple sink-set. So $\partial \leq r - (\mu - \nu) < r$, and

$$
|\{ n - \nu \} \setminus I(\sigma)| \leq r \ll n - \nu,
$$

i.e. out-degrees $\Delta_i$ cannot be zero for all but at most $r$ specified vertices. Let $C_{\sigma}(m - \mu - \partial, n - \nu)$ denote the total number of digraphs on $[n - \nu]$ with $m - \mu - \partial$ arcs such that

$$
(3.10) \quad \delta_i \geq 1, \quad (i \in [n - \nu]); \quad \Delta_i \geq 1, \quad (i \in I(\sigma)).
$$

Further,

$$
C_{\sigma}(m - \mu - \partial, n - \nu) \leq (m - \mu - \partial)! \sum_{\delta, \Delta \text{ meet (3.8),(3.9)}} \prod_{i \in [n - \nu]} \frac{1}{\delta_i! \Delta_i!}.
$$

And, by the theorem 2.1, the total number of $G_{[n-\nu]}$ is on the order of

$$
\frac{m - \mu - (n - \nu - \partial)}{n - \nu} C_{\sigma}(m - \mu - \partial, n - \nu) \leq \frac{r}{n} C_{\sigma}(m - \mu - \partial, n - \nu).
$$

Also, the total number of ways to choose in $[\nu]$ the partners of the vertices in $[n - \nu]$ is

$$
\prod_{i \in [n - \nu]} \binom{\nu}{\partial_i} \leq \nu^\partial \prod_{i \in [n - \nu]} \frac{1}{\partial_i!}.
$$
Therefore, for a given \( \partial \), the total number of the subdigraphs that complement \( G[\nu] \) is on the order of

\[
\frac{r}{n} \nu^\partial (m - \mu - \partial)! \sum_{\delta, \Delta \text{ meet (3.8), (3.9)}} \prod_{i \in [n-\nu]} \frac{1}{\partial_i! \delta_i!} \\
= \frac{r}{n} \nu^\partial (m - \mu - \partial)! [u^\partial v^{m-\mu-\partial}] \left( \sum_{i+j \geq 1} \frac{u^i v^j}{i! j!} \right) [w^{m-\mu-\partial}] \left( \sum_{k \geq 1} \frac{w^k}{k!} \right) \\
= \frac{r}{n} \nu^\partial (m - \mu - \partial)! [u^\partial v^{m-\mu-\partial}] (e^{u+v} - 1)^{n-\nu} [w^{m-\mu-\partial}] (e^w - 1)^{n-\nu} \\
= \frac{r}{n} \nu^\partial (m - \mu - \partial)! \left( \frac{m-\mu}{\partial} \right) [u^{m-\mu}] (e^u - 1)^{n-\nu} [w^{m-\mu-\partial}] (e^w - 1)^{n-\nu} \\
\leq b \frac{r}{n} (m - \mu)! \nu^\partial ((n - \nu)u)^{-1/2} \left( \frac{e^u - 1}{u^{m-\mu}} \right) ((n - \nu)w)^{-1/2} \left( \frac{e^w - 1}{w^{m-\mu-\partial}} \right),
\]

for all \( u > 0 \), \( w > 0 \).

Setting \( u = w = \lambda \), and summing for \( \partial \geq 0 \), we obtain an overall bound for the count of complementary subdigraphs:

\[
(3.11) \quad \frac{r}{n} (n\lambda)^{-1} (m - \mu)! \left( \sum_{\partial \geq 0} \frac{(\nu \lambda)^\partial}{\partial!} \right) \frac{(e^\lambda - 1)^{2(n-\nu)}}{\lambda^{2(m-\mu)}} = \frac{r}{n} (n\lambda)^{-1} (m - \mu)! \frac{(e^\lambda - 1)^{2(n-\nu)} e^{\nu \lambda}}{\lambda^{2(m-\mu)}}.
\]

(We have used \( n - \nu = \Theta(n) \).)

The total number of the partitions \((A, B)\) is bounded above by the product of (3.7), (3.11) and \( \binom{n}{\nu} \). Dividing this product by the total number of the digraphs in question, we arrive at the bound (3.3).

To get the most out of (3.3), we will use \( x = x(\nu, \mu) \), the minimum point of

\[
h(\nu, \mu, x) = 2\nu \ln(e^x - 1) - 2\mu \ln x,
\]

i.e. the root of

\[
(3.12) \quad h_x(\nu, \mu, x) = 2\nu \frac{e^x}{e^x - 1} - \frac{2\mu}{x} = 0.
\]

Considering \( \nu \) and \( \mu \) as continuously varying, we compute the partial derivatives

\[
x_\nu = -\frac{\mu}{\nu^2} g(x), \quad x_\mu = \frac{1}{\nu} g(x),
\]

\[
g(x) = \frac{(e^x - 1)^2}{e^x (e^x - 1 - x)^2}.
\]

Since \( g(0+) = 2 \) and \( g(+\infty) = 1 \), it follows that \( x(\nu, \mu) = \Theta((\mu - \nu)/\nu) \), uniformly for all \( 0 < \nu < \mu \). In particular, in (3.3) the factor \((x + 1)/\mu\) is \( O(1/\nu) \); so we replace \( x + 1 \) with \( 1/\nu \). (It is easy to show also that \( x(\nu, \mu) > 2(\mu - \nu)/\mu \).)
Consider $\nu \leq \nu_0$ and $\nu < \mu \leq m/2$. By (3.3) and (3.13),

$$
\frac{E_{\nu,\mu}}{E_{\nu,\mu-1}} \leq \lambda^2 \frac{\mu}{m - \mu} \frac{\nu}{\mu - \nu} \exp[h(\nu,\mu, x(\nu,\mu)) - h(\nu, \mu - 1, x(\nu, \mu - 1))]
$$

($\tilde{\mu} \in [\mu - 1, \mu]$, $\tilde{x} = x(\tilde{\mu}, \nu)$)

$$
= \lambda^2 \frac{\mu}{m - \mu} \frac{\nu}{\mu - \nu} \exp[h_{\nu}(\nu, \tilde{\mu}, \tilde{x}) + h_{x}(\nu, \tilde{\mu}, \tilde{x})x_{\nu}(\nu, \tilde{\mu})]
$$

$$
= \lambda^2 \frac{\mu}{m - \mu} \frac{\nu}{\mu - \nu} \exp(-\ln \tilde{x}) \leq \lambda^2 \frac{\mu}{m - \mu} \frac{\nu}{\mu - \nu - 1} \exp(-\ln \tilde{x})
$$

(3.14) \quad \leq_b \lambda^2 \frac{\mu}{m - \mu} x^{-2(\nu, \mu - 1)};

($x(\nu, \mu)$ increases with $\mu$, $\frac{\nu - x}{\nu} = \phi(x(\nu, \mu))$) where

$$
\phi(x) := \frac{xe^x}{e^x - 1} - 1 = \Theta(x),
$$

uniformly for $x > 0$.) Let us show that the last expression decreases with $\mu$. Using the formula for $x_{\mu}$ in (3.13) for $\mu - 1$ instead of $\mu$, we compute

$$
d \frac{d}{d\mu} \left( \frac{\mu}{m - \mu} x^{-2(\nu, \mu - 1)} \right)
$$

$$
= \frac{m}{(m - \mu)^2 x^2} - \frac{2}{(m - \mu)} \frac{\mu}{\nu} \cdot \frac{(e^x - 1)^2}{x^3 e^x (e^x - 1 - x)} \quad (x := x(\nu, \mu - 1))
$$

$$
\leq \frac{m}{(m - \mu)^2 x^2} - \frac{2}{(m - \mu)} \frac{e^x}{e^x - 1} \cdot \frac{(e^x - 1)^2}{x^3 e^x (e^x - 1 - x)}
$$

$$
= \frac{1}{x^2(m - \mu)^2} \left[ m - 2(m - \mu) \frac{e^x - 1}{e^x - 1 - x} \right] < 0, \quad (!)
$$

as $\mu \leq m/2$. Thus indeed the RHS in (3.14) decreases with $\mu$. At $\mu = \nu + 2$, for $\nu \leq \nu_0 = \varepsilon(mn^2/r^2)^{1/3}$, this RHS is

$$
\lambda^\nu \frac{\nu + 2}{m - (\nu + 2)} x^{-2(\nu, \nu + 1)} \sim \lambda^\nu \frac{\nu}{m} \left( \frac{2}{\nu} \right)^{-2} (1 + O(1/\nu))
$$

$$
= \lambda^\nu \frac{\nu}{4m} (1 + O(1/\nu)) = \varepsilon^3 \frac{4\nu^2/n^2}{4m} \frac{mn^2}{r^2} (1 + O(1/\nu)) < 2\varepsilon^3,
$$

if $\nu$ exceeds a large enough $\nu^*$. The same bound holds trivially for $\nu \leq \nu^*$, as $x(\nu, \nu + 1) \geq 2/(\nu + 1)$. Hence

$$
\frac{E_{\nu,\mu}}{E_{\nu,\mu-1}} \leq_b 2\varepsilon^3 \leq \frac{1}{2}, \quad (\nu \leq \nu_0, \, \nu + 2 \leq \mu \leq m/2),
$$

provided that $\varepsilon > 0$ is chosen sufficiently small. Consequently, denoting $x = x(\nu, \nu + 1)$ and using

$$
\frac{(n)}{(\nu + 1)} \leq_b \frac{\nu}{m} \left( \frac{n}{m} \right)^\nu, \quad \frac{n}{m} = \frac{\lambda e^\lambda}{e^\lambda - 1}, \quad \frac{e^x - 1}{x} = e^x \frac{\nu}{\nu + 1} = e^{2/\nu} (1 + O(1/\nu))
$$

COUNTING DIGRAPHS 19
we bound
\[
\frac{r}{n} \sum_{\nu < \mu} \nu^{-1} E_{\nu,\mu} \leq \frac{r}{n} \sum_{\nu \leq n_0} \nu^{-1} E_{\nu,\nu+1} \left( \sum_{j \geq 1} (2/3)^{j-1} \right) \\
\leq b \frac{r}{n} \sum_{\nu \leq n_0} \left( \frac{n}{\nu + 1} \right) \nu^2 \cdot \frac{\lambda^2}{\nu x} \left[ \frac{(e^x - 1)^2}{x^2} \cdot \frac{\lambda^2 e^\lambda}{(e^\lambda - 1)^2} \right] \nu \\
\leq b \frac{r \lambda^2}{n m} \sum_{\nu \geq 1} \nu^2 \left( \frac{\lambda}{e^\lambda - 1} \right)^\nu \leq b \frac{r \lambda^2}{n m} \left( 1 - \frac{\lambda}{e^\lambda - 1} \right)^{-3} \\
= O((r/n)(m\lambda)^{-1}) = O(n^{-1}).
\]

Thus
\[
\Sigma_1 := \frac{r}{n} \sum_{\nu \leq n_0, \mu \leq m/2} \frac{1}{\nu} E_{\nu,\mu} = O(n^{-1}).
\]

It remains to bound
\[
\Sigma_2 := \frac{r}{n} \sum_{\nu > n_0, \mu \leq m/2} \tilde{E}_{\nu,\mu}.
\]

Since \( k! = \Theta(k^{1/2}(k/e)^k) \), for \( E_{\nu,\mu} \) in (3.3) we have
\[
E_{\nu,\mu} \leq b \frac{x + 1}{\mu} \exp(H(\nu,\mu,x)) \leq b \frac{1}{\nu_0} \exp(H(\nu,\mu,x)),
\]

where
\[
H_{n,m}(\nu,\mu,x) = H(\nu,\mu,x) := n \ln n - \nu \ln \nu - (n - \nu) \ln(n - \nu) - m \ln m + \mu \ln \mu + (m - \mu) \ln(m - \mu)
\]
\[
\quad + \nu \left( \ln(e^x - 1) - 2 \ln(e^\lambda - 1) + \lambda \right)
\]
\[
\quad + (\mu - \nu) \ln \nu - (\mu - \nu) \ln \frac{\mu - \nu}{e}
\]
\[
\quad + \mu(2 \ln \lambda - \ln x).
\]

We will use \( H_\nu, H_\mu, H_x \) to denote the partial derivatives of \( H(\nu,\mu,x) \). Like before, given \( \nu < \mu \), we choose \( x = x(\nu,\mu) \), the root of (3.12), or equivalently the root of \( H_x(\nu,\mu,x) = 0 \). Then
\[
\frac{\partial H(\nu,\mu,x(\nu,\mu))}{\partial \nu} = H_\nu(\nu,\mu,x(\nu,\mu))
\]
\[
= - \ln \nu + \ln(n - \nu) + \ln \frac{e^{x(\nu,\mu)} - 1}{(e^\lambda - 1)^2} + \lambda
\]
\[
+ \frac{\mu - \nu}{\nu} - \ln \nu + \ln(\mu - \nu).
\]
By (3.13), \(x(\nu, \mu)\) strictly decreases with \(\nu\), and then so does \(\partial H(\nu, \mu, x(\nu, \mu))/\partial \nu\). That is, as a function of \(\nu\), \(H(\nu, \mu, x(\nu, \mu))\) is convex. In fact, using \(x_{\nu}(\nu, \mu) < 0\) and \(\mu - \nu \leq r\), we have

\[
\frac{\partial^2 H(\nu, \mu, x(\nu, \mu))}{\partial \nu^2} = \frac{\partial H_{\nu}(\nu, \mu, x(\nu, \mu))}{\partial \nu} \\
= -\frac{1}{\nu} - \frac{1}{n - \nu} + \frac{e^x(\nu, \mu) x_{\nu}(\nu, \mu)}{e^x(\nu, \mu) - 1} - \frac{\mu}{\nu^2} - \frac{1}{\mu - \nu} \\
\leq -\frac{1}{r}.
\]  

Since \(x(\nu, \mu) \downarrow 0\) as \(\nu \uparrow \mu\), and \(x(\nu, \mu) \uparrow \infty\) as \(\nu \downarrow 0\), there exists a unique \(\nu(\mu)\), the root of \(\partial H(\nu, \mu, x(\nu, \mu))/\partial \nu = 0\), at which \(H(\nu, \mu, x(\nu, \mu))\) attains its absolute maximum \(f(\mu) := H(\nu(\mu), \mu, x(\nu(\mu), \mu))\). Observe immediately that the equation \(H_{\nu}(\nu, \mu, x(\nu, \mu)) = 0\), with \(H_{\nu}(\nu, \mu, x(\nu, \mu))\) given by (3.18), and (3.17) allow us to obtain a much simpler expression for \(f(\mu)\), namely

\[
f(\mu) = -n \ln(1 - \nu/n) + \mu \ln(\mu) + (m - \mu) \ln(1 - \mu/n) - m \ln m \\
+ \mu \ln \frac{\chi^2 \nu}{x(\mu - \nu)}.
\]  

Let us have a close look at \(\nu(\mu)\) and \(x(\nu(\mu), \mu)\).

Observe first that, by (3.13),

\[
\frac{dx(\mu, \nu(\mu))}{d\mu} = x_{\mu} + x_{\nu} \nu_{\mu} = \frac{g(x)}{\nu} \left(1 - \frac{\mu}{\nu} \nu_{\mu}\right).
\]

Differentiating \(\partial H(\nu, \mu, x(\nu, \mu))/\partial \nu = 0\) with respect to \(\mu\), using (3.13) and solving for \(\nu_{\mu}\), we obtain

\[
\nu_{\mu} = \frac{(e^x - 1)^{-1} g(x)/\nu}{2\nu^{-1} + (n - \nu)^{-1} + (\mu - \nu)^{-1} + \mu \nu^{-2} + (e^x - 1)^{-1} g(x) \mu \nu^{-2}}.
\]

So \(\nu_{\mu}(\mu) \in (0, \nu(\mu)/\mu)\), i.e. \(\nu(\mu)\) and, by (30), \(x(\nu(\mu), \mu)\) both strictly increase with \(\mu\).

Turning the tables around, introduce the functions \(\nu(x)\) and \(\mu(x)\) determined by two equations, (3.12) and \(H_{\nu}(\nu, \mu, x) = 0\), (see (3.18) for \(H_{\nu}(\nu, \mu, x)\)):

\[
\nu = \nu(x) := n \frac{e^x - 1}{(e^x - 1)^2 \phi(x) e^{\phi(x)}} \frac{e^x - 1}{(e^x - 1)^2 \phi(x) e^{\phi(x)}}, \quad \mu = \mu(x) := \nu(x) \frac{e^x}{e^x - 1},
\]

where \(\phi(x) = xe^x/(e^x - 1) - 1\). Since \(\nu(\mu)\), \(x(\nu(\mu), \mu)\) are strictly increasing, then so are \(\nu(x)\) and \(\mu(x)\).

Using the parameterization \(\nu(x), \mu(x)\), let us evaluate sharply \(F(x) := f(\mu(x)) = H(\nu(x), \mu(x), x)\) for \(x \leq c \lambda\), with \(c > 0\) to be specified shortly. After some
algebra, (3.22) becomes
\[ \nu(x) = n \frac{0.5(x/\lambda)^2}{1 + 0.5(x/\lambda)^2} (1 + \theta_1 + O(\lambda^2)) , \]
(3.23)
\[ \mu(x) = m \frac{0.5(x/\lambda)^2}{1 + 0.5(x/\lambda)^2} (1 + \theta_2 + O(\lambda^2)) , \]
\[ \theta_1 = \frac{7x/6 - \lambda}{1 + 0.5(x/\lambda)^2} , \quad \theta_2 = \theta_1 + \frac{x - \lambda}{2} ; \]

(we have used \( m/n = 1 + \lambda/2 + O(\lambda^2) \)). In particular, \( \mu(x) \) is of order \( m(x/\lambda)^2 \) exactly. Plugging these expressions into (3.20) we obtain, after massive simplifications,
\[ F(x) = -r G(x/\lambda) + O(mx^2) , \]
(3.24)
\[ G(z) := \ln(1 + 0.5z^2) - \frac{z^3}{1 + 0.5z^2} + \frac{z^2}{1 + 0.5z^2} , \]

(\( r = m - n \)). \( G(z) > 0 \) for \( z \in (0, z_0) \), \( z_0 = 1.772 \), and \( G(z) \sim 1.5 z^2 \) as \( z \downarrow 0 \).

For \( x_0 := \lambda z_0 \), it follows from (3.23) that
\[ m_1 := \mu(x_0) = m \frac{z_0^2}{1 + z_0^2} (1 + O(\lambda)) > 0.61 m . \]

Returning to the parameterization \( \nu(\mu), x(\nu(\mu), \mu) \), we see that \( x(\nu(\mu), \mu) \leq x_0 \) for \( \mu \leq m_0 = 0.61 m \), and
\[ f(\mu) = F(x(\nu(\mu), \mu)) \leq -c_2 r \left( \frac{x(\nu(\mu), \mu)}{\lambda} \right)^2 + O(mx^2) \]
(3.25)
\[ = -c_3 r \frac{\mu}{m} + O(\mu\lambda^2) \leq -c_4 r \frac{\mu}{m} , \]
as \( \lambda = O(r/m) \).

With these preparations out of the way, turn directly to \( \Sigma_2 \). Given \( \mu \in (\nu_0 + 1, m_1] \), \((n_0 = \Theta(n/\mu^{2/3}))\), by (3.16), (3.19) and (3.25),
\[ \sum_{\nu \in (n_0, \mu)} \sum_{\nu \geq \mu - r} E_{\nu, \mu} \leq b \nu_0^{-1} \exp \left( -c_4 r \frac{\mu}{m} \right) \times \sum_{\nu \in (n_0, \mu)} \exp \left( -c_1 \frac{(\nu - \nu(\mu))^2}{r} \right) \]
\[ \leq b \nu_0^{-1} \exp \left( -c_4 r \frac{\mu}{m} \right) \int_0^{\infty} e^{-c_1 z^2/r} \, dz \]
\[ \leq b \nu_0^{-1} r^{7/6} \exp \left( -c_4 r \frac{\mu}{m} \right) . \]

Therefore
\[ \frac{r}{n} \sum_{\mu \in (n_0, m_1]} \sum_{\nu \in (n_0, \mu)} E_{\nu, \mu} \leq b \frac{r^{13/6}}{n^2} \sum_{\nu > n_0} \exp \left( -c_4 r \frac{\mu}{m} \right) \]
(3.26)
\[ \leq b \frac{r^{13/6}}{n^2} \frac{\exp(-c_4 r n_0/m)}{1 - e^{-c_4 r/m}} \]
\[ \leq b \frac{r^{7/6}}{n} e^{-c_4 r^{1/3}} = O(1/n) . \]
Combining (3.26) and (3.15), we conclude that $E[X] = O(n^{-1})$. This completes the proof of Theorem 3.1 for the case $r = o(n)$. Since $r/n \to 0$ however slowly, we actually established our claim for $r \to \infty$ such that $r \leq \varepsilon n$, where $\varepsilon > 0$ is sufficiently small.

Turn to $r = \Theta(n)$. Up to and including the equations (3.22) the proof is basically the same. Furthermore, for $x \leq x_0$, $x_0$ being small enough, $F_{n,m}(x) = H(\nu(x), \mu(x), x) = -n\Theta(x^2)$. The challenge is to show that $F_{n,m}(x) = -\Theta(n)$ for all $x \geq x_0$, such that $\mu(x) \leq m/2$. This is definitely so for $r \leq \varepsilon n$. Let

$$c^* = \sup \{c \geq \varepsilon : F_{n,n+c\varepsilon}(x) < 0 \text{ if } x > x_0 \text{ and } \mu(x) \leq m/2\}.$$ 

If $c^* < \infty$, then for $m = n + c^*n$ there exists $x^* > x_0$ such that $\mu(x^*) \leq m/2$ and $F_{n,m}(x^*) = 0$, $F'_{n,m}(x^*) = 0$. Equivalently, we must have $f_{n,m}(\mu) = 0$, $f'_{n,m}(\mu) = 0$ for some $0 \leq \mu \leq m/2$, with $f_{n,m}(\mu) = f(\mu)$ defined by (3.20).

Since $H_x(\mu, \nu, x(\nu, \mu)) = 0$, $H_x(\nu(\mu), \mu, x(\nu, \mu)) = 0$, by (3.17) we have

$$f'_{n,m}(\mu) = H_\mu(\mu, \nu(\mu), x(\nu(\mu), \mu)) = \ln \left(\frac{\mu}{m - \mu} - \frac{\nu}{\mu - \nu} \frac{\lambda^2}{x}\right) = 0,$$ 

so that, denoting $x = x(\nu(\mu), \mu)$,

$$\mu = \frac{m}{1 + \frac{x}{x\phi(x)}}, \quad \phi(x) = \frac{xe^x}{e^x - 1} - 1.$$

Since $\mu \leq m/2$, it must be true that $\lambda^2/x\phi(x) \geq 1$, so that $\lambda > \phi(x)$, as $\phi(x) < x$ for $x > 0$. Furthermore, combining (3.27) and (3.20), we obtain

$$f_{n,m}(\mu) = -n \ln(1 - \nu(\mu)/n) + m \ln(1 - \mu/m) = 0. \quad (!)$$

Combining (3.28)-(3.29) with (3.22), and denoting $\psi(x) = xe^x/(e^x - 1)$, we see that there must exist a solution $(\lambda, x)$ of the following system of two equations:

$$1 + \frac{e^x - 1}{(e^x - 1)^2} \phi(x)e^{\phi(x)} = \left(1 + \frac{x\phi(x)}{\lambda^2}\right)\psi(\lambda),$$

$$\psi(\lambda) \frac{1}{\psi(x)} \frac{\lambda^2}{1 + \frac{x}{x\phi(x)}} = \frac{e^x - 1}{(e^x - 1)^2} \phi(x)e^{\phi(x)}.$$  

Our task is to show that assuming existence of such a solution $(\lambda, x)$ we get a contradiction.

Since $\psi(\lambda) > 1$, the first equation in (3.30) implies that

$$1 + \frac{e^x - 1}{(e^x - 1)^2} \phi(x)e^{\phi(x)} > 1 + \frac{x\phi(x)}{\lambda^2} \psi(\lambda),$$  

or, using the definition of $\psi(\cdot)$,

$$\frac{e^x - 1}{x} e^{\phi(x)} > \frac{e^\lambda - 1}{\lambda} e^\lambda.$$ 

As $\phi(x) < \lambda$, and $(e^y - 1)/y$ is increasing, it follows that $x > \lambda$. By (3.31), the second equation yields an inequality

$$\frac{\psi(x)}{\psi(\lambda)} \frac{1}{1 + \frac{x^2}{\lambda^2 \phi(x)}} > \frac{x \phi(x)}{\lambda^2} \frac{\psi(\lambda)}{\psi(\lambda)}.$$

or

$$\psi(x) \left(1 + \frac{x^2}{\phi(x)}\right) < \frac{x^2}{\phi(x)} + \psi(\lambda),$$

or

$$\frac{\lambda^2}{x \phi(x)} (\psi(x) - 1) < \psi(\lambda) - \psi(x).$$

This is impossible since $\psi(x) > 1$ and $\psi(\lambda) < \psi(x)$.

Thus, for $r = \Theta(n)$, $F_{n,m}(x) = -n \Theta(x^2)$ for $x \leq x_0$, and $F_{n,m}(x) = -\Theta(n)$, for all $x \geq x_0$ such that $\mu(x) \leq m/2$.

With this property established, the bound for $\Sigma_2$ can be proved in essentially the same way as for $r = o(n)$. □

**Note 3.2.** If a digraph with positive in/out degrees has a sink-set (a source-set) $T$ with $|T| < n$, then there is a source-set (a sink-set) $T \subseteq T^c$. (If, say, $T$ is a source-set, then, for a vertex $v \in T$, the set consisting of $v$ and all vertices reachable from $v$ is a sink-set in $T^c$.) Therefore if a digraph of this sort is not strongly-connected and has no simple sink/source-sets, it must have either a complex sink-set or a complex source-set with at most $m/2$ induced arcs. By Theorem 3.1, the probability of this event is $O(n^{-1})$. Thus, it remains to find a sharp estimate for the probability of no simple sink/source-sets and to check that this probability far exceeds $n^{-1}$.

### 4. Counting the digraphs without simple sink/source-sets.

A subgraph induced by a simple sink-set (source-set resp.) is a disjoint union of cycles. So our task in this section is to find a sharp asymptotic formula for the probability that $G_{1,1}(n,m)$ has no induced cycles $C$ such that there are no arcs $i \to j$ $(j \to i$ resp.) with $i \in C$ and $j \notin C$.

Consider first the corresponding probability for the random multi-digraph $MG_{1,1}(n,m)$ induced by the $m$-long insertion sequence chosen uniformly at random among all $h_{1,1}(n,m)$ such sequences. (See Note 1.2 and (1.1) for the definition of $h_{1,1}(n,m)$.) Let $X = X_n$ denote the total length of the cycles in all simple sink-sets and source-sets.
Lemma 4.1.

\[ E[z^X] = \frac{(ml)^2}{h_{1,1}(n, m)} \sum_a \frac{(n)_a}{(m)_a} \]

\[ \times [(x_1x_2)^{m-a}y^a] \left( H(x, y, z) \prod_{i=1}^{2}(e^{x_i} - 1)^{n-a} \right), \]

where

\[ H(x, y, z) = \frac{1 - yz}{1 - y} \prod_{i=1}^{2} \frac{1 - ye^{x_i}}{1 - yze^{x_i}}. \]

In particular,

\[ P(X = 0) = E[0^X] = \frac{(ml)^2}{h_{1,1}(n, m)} \sum_a \frac{(n)_a}{(m)_a} \]

\[ \times [(x_1x_2)^{m-a}y^a] \left( (1 - y)^{-1} \prod_{i=1}^{2}(1 - ye^{x_i})(e^{x_i} - 1)^{n-a} \right). \]

Proof of Lemma 4.1. Let us first derive a formula for \( E[(X_k)] \). \( (X_k) \) is the total number of unordered \( k \) tuples of vertices belonging to the cycles of simple sink/source-sets. By symmetry, \( E[(X_k)] \) equals \( \binom{n}{k} \) times the probability that the cycles contain vertices from \( [k] \). To evaluate this probability we need to count the number of \( m \)-long insertion sequences that result in multi-digraphs in which the set \( [k] \) belongs to those cycles. We find it a bit easier to turn things around. First, we evaluate the number of the insertion sequences having several such (disjoint) cycles induced by a subset of vertices. Second, we multiply this number by the total number of ways to select a set of \( k \) points contained in these cycles, under a constraint that every cycle is represented in such a set. Third, we sum these products over all those collections of disjoint sets. Finally we divide the sum by \( h_{1,1}(n, m) \), the total number of the insertion sequences.

Let \( a \) denote a (generic) cardinality of a chosen set \( A \) of vertices; there are \( \binom{n}{a} \) such sets. Let \( a_{1,1}, \ldots, a_{1,\ell_1}, (a_{2,1}, \ldots, a_{2,\ell_2}, \text{ resp.}) \) denote the lengths of sink (source, resp.) cycles to be formed out of the \( a \) vertices. As loops allowed, \( a_{i,j} \geq 1 \) and

\[ \sum_{i=1}^{2} \sum_{j=1}^{\ell_i} a_{i,j} = a. \]

The total number of ways to form \( \ell_1 + \ell_2 \) directed cycles is

\[ \frac{a!}{\ell_1!\ell_2!} \prod_{i,j} \frac{(a_{i,j} - 1)!}{a_{i,j}!} = \frac{a!}{\ell_1!\ell_2!} \prod_{i,j} \frac{1}{a_{i,j}}. \]
The total number of ways to select \( k \) points from these cycles subject to the representation constraint is

\[
\sum_{k_{i,j} \geq 1: \|k\|=k} \prod_{i,j} \left( \binom{a_{i,j}}{k_{i,j}} \right) = [w^k] \prod_{i,j} \left( \sum_{\kappa \geq 1} \binom{a_{i,j}}{\kappa} w^{\kappa} \right) = [w^k] \prod_{i,j} \left( (1 + w)^{a_{i,j}} - 1 \right).
\]

(4.6)

The product of the counts in (4.5) and (4.6) is

\[
\frac{a!}{\ell_1! \ell_2!} \prod_{i,j} \left( \frac{(1 + w)^{a_{i,j}} - 1}{a_{i,j}} \right).
\]

(4.7)

Given these cycles \( C_{1,1}, \ldots, C_{1,\ell_1}, C_{2,1}, \ldots, C_{2,\ell_2} \), let \( \delta_i, 1 \leq i \leq \ell_1 \), denote the number of arcs \( u \rightarrow v \) with \( u \notin C_{1,i} \) and \( v \in C_{1,i} \), and let \( \Delta_i, 1 \leq i \leq \ell_2 \), denote the number of arcs \( u \rightarrow v \) with \( u \in C_{2,i} \) and \( v \notin C_{2,i} \). Since \( C_{1,i} \) is a sink cycle, there are no arcs from \( C_{1,i} \) to its complement; likewise, there are no arcs leading to \( C_{2,i} \) from its complement. To avoid overcounting we consider an isolated cycle only as a source cycle. Thus we have only the constraint \( \hat{\delta}_i \geq 1, 1 \leq i \leq \ell_1 \). For \( i \notin A \), let \( \delta_i \geq 1, \Delta_i \geq 1 \) denote its in-degree and its out-degree. Clearly

\[
\sum_{i=1}^{\ell_1} \hat{\delta}_i + \sum_{i \notin A} \delta_i = m - a,
\]

\[
\sum_{i=1}^{\ell_2} \hat{\Delta}_i + \sum_{i \notin A} \Delta_i = m - a.
\]

(4.8)

By (1.5), the total number of the \( (m - a) \)-long insertion sequences that result in a multi-digraph on the vertex set \( \{1, \ldots, \ell_1, 1, \ldots, \ell_1 + \ell_2, [n] \setminus A\} \), with the in-degrees \( \hat{\delta}_1, \ldots, \hat{\delta}_{\ell_1}, \delta_i, (i \in [n] \setminus A) \), and the out-degrees \( \hat{\Delta}_1, \ldots, \hat{\Delta}_{\ell_2}, \Delta_i, (i \in [n] \setminus A) \), is

\[
((m - a)!)^2 \prod_{i=1}^{\ell_1} \frac{1}{\delta_i!} \prod_{i=1}^{\ell_2} \frac{1}{\Delta_i!} \prod_{i \in [n] \setminus A} \frac{1}{\delta_i! \Delta_i!}.
\]

We need to multiply this count by

\[
\prod_{i=1}^{\ell_1} a_{1,i}^{\hat{\delta}_i} \prod_{i=1}^{\ell_2} a_{2,i}^{\hat{\Delta}_i},
\]

since, say, for each of \( \hat{\delta}_i \) arcs ending at the “aggregated” vertex, that represents the cycle \( C_{1,i} \), there are \( a_{1,i} = |C_{1,i}| \) ways to specify the arc end. There is another factor still missing, namely \( (m)_{m-a} \), which counts the total number of ways to select those \( (m - a) \) positions in the whole \( m \)-long insertion sequence when a newly added arc belongs to that multi-digraph, rather than to one of the cycles. By (4.8), and the constraints on \( \hat{\delta}_i, \delta_i, \Delta_i \), the resulting product equals

\[
m!(m-a)! \left[ x_1^{m-a} \right] (e^{x_1} - 1)^{n-a} \prod_{i=1}^{\ell_1} (e^{a_{1,i}x_1} - 1)
\]

(4.9)

\[
\times \left[ x_2^{m-a} \right] (e^{x_2} - 1)^{n-a} \prod_{i=1}^{\ell_2} e^{a_{2,i}x_2}.
\]
Further, the product of the factors in (4.7) and (4.9) dependent on \(a_{1,i}, a_{2,i}\), equals

\[
\frac{1}{\ell_1!} \prod_{i=1}^{\ell_1} \left( e^{a_{1,i}x_1} - 1 \right) \frac{1}{a_{1,i} \left( 1 + w \right)^{a_{1,i} - 1}}
\]

\[
\times \frac{1}{\ell_2!} \prod_{i=1}^{\ell_2} e^{a_{2,i}x_1} \frac{\left( 1 + w \right)^{a_{2,i} - 1}}{a_{1,i}}.
\]

(4.10)

Summing the products (4.10) over \(a_{i,j} \geq 2\) meeting the constraint (4.4), and then over all \(\ell_1, \ell_2\) we get

\[
[y^a] \sum_{\ell_1 \geq 0} \frac{1}{\ell_1!} \left( \sum_{b \geq 1} y^b \frac{e^{bx_1} - 1}{b} \left( 1 + w \right)^{b - 1} \right)^{\ell_1}
\]

\[
\times \sum_{\ell_2 \geq 0} \frac{1}{\ell_2!} \left( \sum_{b \geq 1} y^b e^{bx_2} \frac{\left( 1 + w \right)^{b - 1}}{b} \right)^{\ell_2}
\]

\[
= [y^a] \exp \left[ \sum_{b \geq 1} y^b \frac{e^{bx_1} - 1}{b} \left( 1 + w \right)^{b - 1} + \sum_{b \geq 1} y^b e^{bx_2} \frac{\left( 1 + w \right)^{b - 1}}{b} \right]
\]

\[
= [y^a] H(x, y, 1 + w),
\]

see (4.2) for the definition of \(H(x, y, z)\).

Collecting all the pieces, we write

\[
E \left[ \binom{X}{k} \right] = \frac{(m!)^2}{h(n, m)} [w^k] \sum_a \frac{(n)_a}{(m)_a} [x_1 x_2]^{m-a} H(x, y, 1 + w) \prod_{i=1}^{2} (e^{x_i} - 1)^{n-a}.
\]

Consequently

\[
E[(1 + w)^X] = \sum_k w^k E \left[ \binom{X}{k} \right]
\]

\[
= \frac{(m!)^2}{h(n, m)} \sum_a \frac{(n)_a}{(m)_a} [(x_1 x_2)^{m-a}] \left( H(x, y, 1 + w) \prod_{i=1}^{2} (e^{x_i} - 1)^{n-a} \right),
\]

which is equivalent to (4.1). □

Armed with this lemma, we obtain an asymptotic formula for \(P(X = 0)\).

**Theorem 4.2.** For the random multi-digraph \(MG_{1,1}(n, m)\), with \(r = m - n \to \infty\), \(m = O(n)\),

\[
P(X = 0) = \frac{\left( 1 - \frac{\lambda}{e^\lambda - 1} \right)^2}{1 - \frac{\lambda}{e^\lambda (e^\lambda - 1)}} \left( 1 + O(r^{-1} \ln^2 r) \right),
\]

(4.12)
with $\lambda$ determined by $\lambda e^\lambda/(e^\lambda - 1) = m/n$.

**Proof of Theorem 4.2.** In the formula (4.3) for $P(X = 0)$,

$$(1 - y)^{-1} \prod_{i=1}^2 (1 - ye^{x_i}) = 1 + \prod_{i=1}^2 (e^{x_i} - 1) \sum_{a \neq 0,1} y^a - [1 + \sum_{i=1}^2 (e^{x_i} - 1)] y$$

and

$$h_{1,1}(n, m) = (m!)^2 ([x^m](e^x - 1)^n)^2.$$  

So (4.3) becomes

$$P(X = 0) = 1 + \sum_{1 \leq a < n} \frac{(n)_a}{(m)_a} Q^2(a, a - 1) - \frac{n}{m} (Q(1, 1) + Q(1, 0))^2,$$

where

$$Q(a, b) := \frac{[x^{m-a}](e^x - 1)^{n-b}}{[x^m](e^x - 1)^n}.$$  

As in the proof of the theorem 2.1, for $a, b < n$,

$$Q(a, b) \leq_b \frac{\lambda^a}{(e^\lambda - 1)^b} \sqrt{\frac{n}{n - b}}, \quad \frac{(n)_a}{(m)_a} \leq \left(\frac{n}{m}\right)^a = \left(\frac{e^\lambda - 1}{\lambda e^\lambda}\right)^a,$$

and consequently, setting

$$A_n := \left\lceil \frac{2 \ln r}{\lambda} \right\rceil = \Theta(r^{-1}n \ln r),$$

after simple estimates we have

$$\sum_{a \geq A_n} \frac{(n)_a}{(m)_a} Q^2(a, a - 1) \leq_b e^{-A_n \lambda} \min\{1, \lambda\} \leq r^{-2} \min\{1, \lambda\}.$$  

(We compare the series tail with $\min\{1, \lambda\}$ because the latter is the order of the RHS in (4.12).)

Turn to $a < A_n$. Let us have a close look at $[x^{m-a}](e^x - 1)^{n+1-a}$, the numerator in the fraction $Q(a, a - 1)$. Introduce $\tilde{\lambda}$, satisfying

$$\frac{xe^x}{e^x - 1} = \frac{m}{n+1},$$

and the independent positive Poissons $\tilde{Y}^i = Y^i(\tilde{\lambda})$. Using (2.7) we have

$$[x^{m-a}](e^x - 1)^{n+1-a} = \frac{(e^\tilde{\lambda} - 1)^{n+1-a}}{\lambda^{m-a}} \left(\sum_{i=1}^{n+1-a} \tilde{Y}^i = m-a\right) = \frac{(e^\tilde{\lambda} - 1)^{n+1-a}}{\lambda^{m-a}} \left(\sum_{i=1}^{n+1} \tilde{Y}^i = m\right) \left(1 + O(an^{-1} + a^2rn^{-2})\right)$$
A bit of calculus, based on (3.13), shows that \( \tilde{\lambda} = \lambda + O(n^{-1}) \). Consequently

\[
\frac{(e^\lambda - 1)^{n+1-a}}{\lambda^{m-a}} = \frac{(e^\lambda - 1)^{n+1-a}}{\lambda^{m-a}} (1 + O(r^{-1} + n^{-1}a)),
\]

and

\[
P \left( \sum_{i=1}^{n+1} \tilde{Y}^i = m \right) = \frac{1 + O(r^{-1})}{(2\pi(n+1)\text{Var}[\tilde{Y}])^{1/2}} = (1 + O(r^{-1}))P \left( \sum_{i=1}^{n} Y^i = m \right).
\]

Therefore

\[
Q(a, a - 1) = \frac{1 + O(r^{-1} + an^{-1} + a^2rn^{-2})}{(e^\lambda - 1)^{a-1}} \frac{\Lambda^a}{(e^\lambda - 1)^{a-1}}
\]

(4.15)

\[
= \left( 1 + O(r^{-1} \ln^2 r) \right) \frac{\lambda^a}{(e^\lambda - 1)^{a-1}},
\]

uniformly for \( a \leq A_n \).

Also, uniformly for \( a < A_n \),

\[
\frac{(n)_a}{(m)_a} = \left( \frac{n}{m} \right)^a (1 + O(a^2rn^{-2})) = \left( \frac{n}{m} \right)^a (1 + O(r^{-1} \ln^2 r)).
\]

(4.16)

Using (4.15)-(4.16), \( m/n = \lambda e^\lambda/(e^\lambda - 1) \), and denoting \( \sigma = \lambda/[e^\lambda(e^\lambda - 1)] \), we compute:

\[
\sum_{1 \leq a < A_n} \frac{(n)_a}{(m)_a} Q^2(a, a - 1) = (e^\lambda - 1)^2 \left( \sum_{1 \leq a < A_n} \sigma^a \right) (1 + O(r^{-1} \ln^2 r))
\]

\[
= \frac{\sigma(e^\lambda - 1)^2}{1 - \sigma} \left[ 1 + O(\sigma A_n) + O(r^{-1} \ln^2 r) \right]
\]

\[
= \frac{\sigma(e^\lambda - 1)^2}{1 - \sigma} (1 + O(r^{-1} \ln^2 r)),
\]

as

\[
\sigma A_n \leq e^{-1}\sigma A_n \leq e^{-\lambda A_n} \leq r^{-1}.
\]

For the last, negative, term in (4.13) we need to be more precise. Picking a simple contour \( L \) enclosing the origin, by Cauchy integral formula we have

\[
[x^m] (e^x - 1)^n = \frac{1}{2\pi i} \oint_L \frac{(e^z - 1)^n}{z^m+1} \, dz
\]

\[
= \frac{1}{2\pi i} \cdot \frac{n}{m} \oint_L \frac{(e^z - 1)^{n-1}e^z}{z^m} \, dz
\]

\[
= \frac{n}{m} \left\{ [x^{m-1}] (e^x - 1)^n + [x^{m-1}] (e^x - 1)^{n-1} \right\}.
\]

So, recalling the definition of \( Q(\cdot, \cdot) \),

\[
Q(1, 1) + Q(1, 0) = \frac{m}{n}. \quad (!)
\]
Combining (4.14), (4.16) and (4.17), we easily transform (4.13) into (4.12).

Turn now to the random digraph $G_{1,1}(n,m)$. Let $P_{n,m}$ denote the probability that $G_{1,1}(n,m)$ has no simple sink-sets and source sets. The theorem 4.2 provides the asymptotic estimate for $P_{n,m} = P(X = 0)$, the corresponding probability for the random multigraph $MG_{1,1}(n,m)$. We can write

$$\mathcal{P}_{n,m} = \frac{|S^*|}{h_{1,1}(n,m)};$$

here $S^*$ is the set of all $m$-long sequences of arc-insertions that result in a multi-digraph without the simple sink/source sets. Denoting by $\delta(s)$ and $\Delta(s)$ the in/out degrees of a multi-digraph corresponding to a generic sequence $s$, we see that the total number of the graphic sequences $s$ leading to a digraph without simple sink/source sets is

$$h^G_{1,1}(n,m) := \sum_{s \in S^*} F(\delta(s), \Delta(s)).$$

By (1.22), and $\mathcal{P}_{n,m} = \Theta(r/n)$,

$$h^G_{1,1}(n,m) = (e^{-\eta} + O(r^\varepsilon n^{-1/2+\gamma}))|S^*| + O(h_{1,1}(n,m)e^{-r^\varepsilon n^{2\gamma}})$$

$$= e^{-\eta} \mathcal{P}_{n,m} h_{1,1}(n,m) \left(1 + O(r^\varepsilon n^{-1/2+\gamma}) + O(ne^{-r^\varepsilon n^{2\gamma}})\right)$$

$$= e^{-\eta} \mathcal{P}_{n,m} h_{1,1}(n,m) \left(1 + O(r^\varepsilon n^{-1/2+\gamma})\right).$$

Therefore

$$P_{n,m} = \frac{h^G_{1,1}(n,m)}{h_{1,1}(n,m)} = e^{-\eta} \mathcal{P}_{n,m} \left(1 + O(r^\varepsilon n^{-1/2+\gamma})\right),$$

for $\varepsilon > 0$ and $\gamma \in (0,1/2)$.

Combining (4.19), (4.12) and recalling the formula (1.12) for $\eta$, we have proved the following.

**Theorem 4.3.** For $r \to \infty$ and $m = O(n)$,

$$P_{n,m} = \left(1 - \frac{\lambda}{e^\lambda - 1}\right)^2 \frac{\lambda}{1 - \frac{\lambda}{e^\lambda - 1}} \exp \left(-\frac{m}{n} \frac{\lambda^2}{2}\right)$$

$$\times \left(1 + O(r^\varepsilon \ln^2 r + r^\varepsilon n^{-1/2+\gamma})\right),$$

for $\varepsilon > 0$ and $\gamma \in (0,1/2)$.

By (4.20), $P_{n,m} = \Theta(\lambda) = \Theta(r/n) \gg n^{-1}$. In light of Note 3.2, (4.20) combined with Theorem 1.2 complete the proof of our claim (1) in the introduction.

**Acknowledgement.** Back in 2001, during my memorable visit to Nick Wormald at University of Melbourne, we spent most of our time together working on the
alternative derivation of the Bender-Canfield-McKay formula for the count of connected sparse graphs, the remarkable formula that struck us by how sharp it was. It occurred to us then that the embedding idea might also work for counting the strongly-connected directed graphs. It was immediately clear though that an intrinsically harder notion of strong connectivity presented a difficult new challenge. In Summer of 2009 Nick informed me that together with Xavier Pérez they had made a further progress toward obtaining a fully-proved asymptotic formula for the counts of strongly-connected digraphs. In particular, this formula would yield a directed counterpart of the BCM formula, but without an explicit estimate of an error term. Reenergized by the news, I set up to see if I could obtain a sharp asymptotic formula, with a remainder term qualitatively matching that of the BCM formula. The approach in this paper is naturally very different from the one used by Pérez and Wormald. However the paper would not be possible without the joint work with Nick on the BCM formula, and our initial attacks on the directed case. I owe to Nick my debt of gratitude.

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