A Semigroup Proof of the Bounded Degree Case of S.B. Rao’s Conjecture on Degree Sequences and a Bipartite Analogue

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Abstract

S.B. Rao conjectured in 1971 that graphic degree sequences are well quasi ordered by a relation $\preceq$ defined in terms of the induced subgraph relation $\subseteq$. In 2008, M. Chudnovsky and P. Seymour proved this long standing Rao’s Conjecture by giving structure theorems for graphic degree sequences.

In this paper, we prove and use a variant of Dickson’s Lemma from commutative semigroup theory to give a short proof of the bounded degree case of Rao’s Conjecture that is independent of the Chudnovsky-Seymour structure theory. In fact, we affirmatively answer two questions of N. Robertson, the first of which implies the bounded degree case of Rao’s Conjecture.

1. Introduction

Let $G$ be a finite, simple graph and let $D(G) = (d_1, \ldots, d_n)$ be its list of vertex degrees listed in decreasing order. The sequence $D(G)$ is known as the degree sequence of $G$, and $G$ is said to realize $D$. A sequence $(d_1, \ldots, d_n)$ of nonnegative integers is said to be a graphic degree sequence if it is realized by some graph. Given graphic degree sequences $D_1$ and $D_2$, we define $D_1 \preceq D_2$ to mean there is $G_1$ realizing $D_1$ and $G_2$ realizing $D_2$ such that $G_1 \subseteq G_2$, where $\subseteq$ is the induced subgraph relation. The reader may check that $\preceq$ is a transitive relation.
relation on degree sequences. For other basic graph theoretic definitions, we refer the reader to [4].

We recall that a quasi order \((Q, \leq)\) is a reflexive, transitive relation \(\leq\) on a class \(Q\). A quasi order \((Q, \leq)\) is said to be a well quasi order if \(Q\) contains no infinite decreasing sequence and no infinite antichain. Equivalently, \((Q, \leq)\) is a well quasi order if for every infinite sequence \(q_1, q_2, \ldots\) in \(Q\) there are positive integers \(i < j\) such that \(q_i \leq q_j\).

With these definitions, we may state Rao’s Conjecture, posed in 1971 by S.B. Rao and finally proved in 2008 by M. Chudnovsky and P. Seymour.

**Theorem 1.** *Degree sequences of finite graphs are well quasi ordered by \(\preceq\).*

Independently, N. Robertson had asked if graphic degree sequences of bounded degree can be realized as disjoint unions of graphs with bounded sized components, noticing that an affirmative answer would imply the bounded degree case of [1]. Motivated by this question, he further asked for a bipartite analogue. Namely, Robertson asked if degree sequences of bipartite graphs of bounded degree can be realized as disjoint unions of bipartite graphs with bounded sized components.

In this work, we use a variant of Dickson’s Lemma from commutative semigroup theory to prove a more general fact that yields affirmative answers to both of Robertson’s questions as corollaries. In particular, we obtain a new proof of the bounded degree case of [1] that does not depend on the Chudnovsky-Seymour structure theory. This proof abstracts a proof given in the author’s doctoral thesis [1].

While our proof has the disadvantage of only going through for bounded degree, it is fairly short. Moreover, our proof is no longer restricted to graphs and goes through equally well for partial orders, hypergraphs, or any class of structured sets at all for which nonnegative integers can be assigned to each point in a way that respect disjoint union and such that regular elements exist. In particular, even for graphs, these nonnegative integers need no longer represent the degree. This is worth noting since some of the most commonly used tools
for degree sequences, such as switchings \[6\] and the Erdős-Gallai inequalities \[5\], have no known counterparts in this more general setting.

2. The Semigroup Lemma

A commutative semigroup is a set \( S \) together with an associative, commutative binary operation \( + \). We need not assume existence of an identity element. For basic facts and terminology, we refer the reader to \[7\], but our presentation is self contained. Given a semigroup \((S,+)\) and subsets \(Y \subseteq X\) of \(S\), we say that \(Y\) generates \(X\) if every \(x\) in \(X\) can be written as \(y_1 + y_2 + \cdots + y_n\) for some points \(y_1, \ldots, y_n\) of \(Y\). We say that \(X\) is finitely generated if some finite subset \(Y\) of \(X\) generates \(X\).

We now work exclusively with the free commutative semigroup \(\mathbb{N}^k\), where we assume \(k\) is fixed throughout. Given \(x = (x_1, \ldots, x_k)\) in \(\mathbb{N}^k\), the support \(\text{supp}(x)\) is defined as the set of \(i\) such that \(x_i > 0\).

**Definition 1.** Let \(X\) be a subset of \(\mathbb{N}^k\). We say that \(X\) is grounded if for all \(i\) in \(\{1, \ldots, k\}\) there is \(x\) in \(X\) with \(\text{supp}(x) = \{i\}\).

Let \((x_1, \ldots, x_k), (y_1, \ldots, y_k),\) and \((t_1, \ldots, t_k)\) be elements in \(\mathbb{N}^k\). We say that

\[
(x_1, \ldots, x_k) \equiv (y_1, \ldots, y_k) \mod (t_1, \ldots, t_k)
\]

if \(x_i \equiv y_i \mod t_i\) for each \(i\).

**Lemma 1.** Every grounded subset of \((\mathbb{N}^k,+)\) is finitely generated.

**Proof.** Fix a grounded set \(X\). Then for each \(i\) in \(\{1, \ldots, k\}\), we may choose an element of the form \((0, \ldots, 0, t_i, 0, \ldots, 0)\) in \(X\), where \(t_i > 0\) occurs in position \(i\). Without loss of generality, we may choose the minimal such \(t_i\) for each \(i\). Note that equivalence modulo \((t_1, \ldots, t_k)\) is an equivalence relation \(\sim\) on \(\mathbb{N}^k\) with only finitely many equivalence classes.

The partial order \((\mathbb{N}, \leq)\) with the usual ordering of the natural numbers is obviously a well quasi order. Since the product of finitely many well quasi orders
is a well quasi order, we see that \((\mathbb{N}^k, \leq)\) is well quasi ordered when considered as a product order. In particular, every antichain in \((\mathbb{N}^k, \leq)\) is finite.

Given a nonempty \(\sim\) class \(C\), the (possibly empty) set \(M_C\) of \((\mathbb{N}^k, \leq)\) minimal elements of \(C\) is an antichain in \((\mathbb{N}^k, \leq)\) and therefore finite by the previous paragraph. Let \(Y\) be the union over all \(\sim\) classes \(C\) of the sets \(M_C\). Then \(Y\) is the finite union of finite sets and so is finite. It is thus enough to show \(Y' = Y \cup \{(0, \ldots, 0)\}\) generates \(X\).

Choose \(x = (x_1, \ldots, x_k)\) in \(X\). If \(x = (0, \ldots, 0)\) then \(x\) is in \(Y'\) and we are done. Assume not. Let \(C\) be the \(\sim\) class of \(x\). Since \((\mathbb{N}^k, \leq)\) is well founded, \(C\) contains a \((\mathbb{N}^k, \leq)\) minimal element \((m_1, \ldots, m_k)\) such that \((m_1, \ldots, m_k) \leq (x_1, \ldots, x_k)\). Then \(m_i \leq x_i\) for each \(i\). Since \((m_1, \ldots, m_k) \sim (x_1, \ldots, x_k)\) by hypothesis, we see that for each \(i\), the equation \(x_i - m_i = c_i t_i\) holds for some nonnegative integer \(c_i\). Therefore

\[
(x_1, \ldots, x_k) = (m_1, \ldots, m_k) + \sum_{i=1}^{k} c_i (0, \ldots, 0, t_i, 0, \ldots, 0).
\]

We know \((m_1, \ldots, m_k)\) is in \(Y'\) by hypothesis. It is easy to see that \((0, \ldots, 0, t_i, 0, \ldots, 0)\) is a minimal nonzero element in its \(\sim\) class. Therefore \((0, \ldots, 0, t_i, 0, \ldots, 0)\) is in \(Y'\) as well, by which we see \(Y'\) generates \((x_1, \ldots, x_k)\). As \((x_1, \ldots, x_k)\) in \(X\) was chosen arbitrarily, we see that \(Y'\) generates \(X\) as claimed.

### 3. Structured Sets

Our main theorem will apply equally to the class of finite graphs and the class of finite, bipartite graphs, the class of finite posets, and so on, so we need a general way to speak of all these classes of objects. As numerous mathematical objects are defined as a set together with some structure on it, which could be a binary operation, a relation, a set of subsets, and so on, rather than try to define some generalized structure on a set that includes all these things, we prefer to consider the structure as nothing more than a label. For instance, a partial order \((P, \leq)\) would be considered a set \(P\) together with label \(\leq\). For us then, a structured set is simply a set together with a label.
The one operation we need on our structured sets is that of coproduct, which will correspond to $+$ in the semigroup. For our purposes, we simply consider $\coprod$ as an arbitrary associative, commutative binary operation on a class of structured sets. We make this and the previous paragraph precise in the following definition.

**Definition 2.** A structured set class is a class $U$ of ordered pairs together with an associative, commutative binary operation $\coprod : U \times U \to U$ called coproduct such that $P$ is a finite set for each ordered pair $(P,T)$ in $U$. We call members of $U$ structured sets.

When no confusion arises, we sometimes say $P$ instead of $(P,T)$.

Note that in the above definition, $\coprod$ is not a function in the sense of being a set of ordered pairs. The binary operation $\coprod$ is, in natural cases, a proper class of ordered pairs as $U$ is. While this fact is worth noting, it creates no problems, and we do not concern ourselves with such foundational issues here. We use proper classes freely and without comment.

Also note that since $\coprod$ is both associative and commutative, we could consider $\coprod$ itself as a semigroup whose domain is a proper class. Though formally correct, we do not take this point of view, as we find it is a greater aid to intuition to think of $\coprod$ as a coproduct of structures than as addition in an abelian semigroup.

We have generalized the notion of a class of finite graphs to the notion of a structured set class. We now need to generalize the notion of the degree sequence of a graph to this new setting. In fact, using degree sequences for graphs would not allow us to use the semigroup lemma as even graphs of bounded degree may have arbitrarily long degree sequences. The degree sequences of graphs of degree at most $k$ are not, therefore, contained in $\mathbb{N}^r$ for any $r$.

The solution is to instead use what we define as the regularity sequence of a graph. Given a finite graph $G$, its regularity sequence is the unique sequence $R_G$ such that

$$R_G(i) = |\{v : v \in G \text{ and } d_G(v) = i\}|$$

for each natural number $i$. It is simple to check that the degree sequence and
regularity sequence of a graph each uniquely determine the other. Instead of
generalizing the notion of degree sequence to structured set classes, we generalize
the notion of regularity sequences to structured set classes.

**Definition 3.** Let $U$ be a structured set class. A structured set function for
the class $U$ is a function $F$ whose range is a subset of $\mathbb{N}$ and whose domain
consists of all triples $(P, T, x)$ such that $(P, T)$ is a structured set in $U$ and $x$ is
in $P$.

**Definition 4.** Let $U$ be a structured set class. Let $F$ be a structured set
function for $U$ and let $(P, T)$ in $U$ be a structured set. The $F$-regularity sequence
$R_{F,P}$ of $(P, T)$ is defined by letting

$$R_{F,P}(i) = |\{v : v \in P \text{ and } F(P, T, v) = i\}|$$

for each natural number $i$.

Note that a regularity sequence is, in particular, a sequence. We may there-
fore add regularity sequences.

**Definition 5.** Let $U$ be a structured set class. A structured set function $F$ for
$U$ is called additive if for all structured sets $P$ and $Q$, we have

$$R_{F,P \sqcup Q} = R_{F,P} + R_{F,Q}.$$  

The reader may check that if $U$ is the class of finite graphs, considered as
structured sets by letting the edge set $E(G)$ be the label of the finite vertex set
$V(G)$ and $\sqcup$ the disjoint union of graphs, then the structured set function $F$
taking a triple $(V(G), E(G), v)$ to $d_G(v)$ is additive. Additivity of this $F$ simply
states that the number of vertices of degree $i$ in the disjoint union of $G$ and $H$
is the number of vertices of degree $i$ in $G$ plus the number of vertices of degree
$i$ in $H$.

The following definition generalizes to our new setting the notion of two
graphs having the same regularity sequence.

**Definition 6.** Let $U$ be a structured set class. Let $F$ be a structured set
function for $U$. Let $P$ and $Q$ be structured sets. We say that $P$ and $Q$ are
$F$-equivalent if $R_{F,P} = R_{F,Q}$.

**Definition 7.** Let $U$ be a structured set class and $F$ be a structured set function
for $U$. We say that $U$ is $F$-finitely representable if there is some finite subset $Z$
of $U$ such that every structured set in $U$ is $F$-equivalent to some structured set of the form 
\[ \prod_{i=1}^{n} P_i, \]
with each $P_i$ in $Z$.

The following definition abstracts the notion of the class of finite graphs with all degrees at most $k$.

**Definition 8.** Let $U$ be a structured set class, $F$ a structured set function for $U$, and $k$ a nonnegative integer. Then $U_{F,k}$ denotes the class of structured sets $(P, T)$ such that $R_{F,P}(i) = 0$ for all integers $i > k$.

We now make a definition that will be used as a hypothesis to ensure a subset of a semigroup is grounded.

**Definition 9.** Let $U$ be a structured set class and $F$ a structured set function for $U$. Then $F$ is said to have regulars if for all nonnegative integers $i$ there is a structured set $P$ such that $R_{F,P}$ has support \( \{i\} \).

Note that for the structured set class $U$ of finite graphs and the structured set function $F$ taking each vertex to its degree, $F$ has regulars since there are $k$-regular graphs for each nonnegative integer $k$.

### 4. The Main Theorems

We now state our main theorem.

**Theorem 2.** Let $U$ be a structured set class, $k$ a nonnegative integer, and $F$ an additive structured set function for $U$ that has regulars. Then $U_{F,k}$ is $F$-finitely representable.

**Proof.** By definition of $U_{F,k}$, we know that for each structured set $P$ in $U_{F,k}$ and $i > k$ that $R_{F,P}(i) = 0$. We may therefore think of the regularity sequence $R_{F,P}$ as the finite sequence $R_{F,P}(0), \ldots, R_{F,P}(k)$ of length $k + 1$, which we consider as an element of the additive semigroup $(\mathbb{N}^{k+1}, +)$. Let $X$ be the set of points in $(\mathbb{N}^{k+1}, +)$ corresponding to regularity sequences of structured sets in $U_{F,k}$. 

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Since $F$ has regulars, we see that $X$ is grounded. By 11 we know that $X$ is finitely generated. Let $Y$ be a finite generating set. Each member $y$ of $Y$ is in particular a member of $X$, and therefore there is a structured set $P$ in $U_{F,k}$ that has $y$ as its regularity sequence. We may thus choose a finite set $Z$ of structured sets in $U_{F,k}$ such that each regularity sequence $y$ in $Y$ is the regularity sequence of some structured set in $Z$.

Now take an arbitrary structured set $P$ in $U_{F,k}$. We know its regularity sequence $x$ is in $X$ by definition. Therefore
\[ x = a_1y_1 + \ldots + a_ny_n \]
for some $n \geq 1$, nonnegative integers $a_i$, and members $y_i$ of $Y$. Therefore $R_{F,P} = R_{F,Q}$, where $Q$ is the structured set
\[ \prod_{i=1}^{n} a_iQ_i, \]
where $Q_i$ is a structured set in $Z$ with regularity sequence $y_i$ and $a_iQ_i$ denotes the structured set
\[ \prod_{j=1}^{a_i} Q_i. \]

We therefore see there is a finite set $Z$ of structured sets in $U_{F,k}$ such that each structured set in $U_{F,k}$ is $F$-equivalent to a coproduct of structured sets in $Z$. By definition of $F$-finite representability, this completes the proof.

We now apply this theorem to answer Robertson’s original questions. Though Robertson asked if graphic degree sequences of bounded degree may be realized with bounded sized components, we note this is equivalent to asking if graphic degree sequences of bounded degree may be realized as disjoint unions of graphs from a fixed finite set, and similarly for the bipartite analogue. We find this reformulation somewhat more convenient as then 2 more directly applies.

**Corollary 1.** Degree sequences of finite, bipartite graphs with bounded degree can be realized as disjoint unions of bipartite graphs from a fixed finite set.
Proof. Let $U$ be the class of finite, bipartite graphs $G = (V(G), E(G))$, considered as a structured set class by choosing label $E(G)$ for the set $V(G)$, $\coprod$ representing disjoint union, and $F$ the additive structured set function for $U$ taking $(V(G), E(G), v)$ to $d_G(v)$. To show that $F$ has regulars, simply note that $K_{j,j}$ is a $j$-regular bipartite graph for each nonnegative $j$. We thus see by 2 that regularity sequences of finite, bipartite graphs with bounded degree can be realized as disjoint unions of bipartite graphs from a fixed finite set $Z$. It is trivial that the same is thus true for degree sequences.

Corollary 2. Degree sequences of finite graphs with bounded degree can be realized as disjoint unions of graphs from a fixed finite set.

Proof. The proof is exactly as in that of the previous corollary except we let $U$ be the structured set class of finite graphs.

It is worth noting that neither 1 nor 2 is stronger than the other, as both the hypotheses and the conclusions of 1 are stronger than that of 2. We now give the simple proof that 2 implies the bounded degree case of Rao’s Conjecture.

Corollary 3. Fix $k$. Degree sequences of finite graphs with degrees at most $k$ are well quasi ordered by $\preceq$.

Proof. By 2 there is a finite set $Z$ of finite graphs with degrees at most $k$ such that every graphic degree sequence with degrees at most $k$ can be realized as a disjoint union of graphs in $Z$. Since degree sequences and regularity sequences contain the same information, we may consider $\preceq$ as a relation on regularity sequences, and we again think of regularity sequences as points in $(\mathbb{N}^{k+1}, +)$.

We note that given points $x$ and $x'$ in $\mathbb{N}^{k+1}$, if $x \leq x'$ in the product order $(\mathbb{N}^{k+1}, \leq)$ then $x \preceq x'$. This implies every $\preceq$ antichain is a $\leq$ antichain. We have previously noted that all $\leq$ antichains are finite, which implies all $\preceq$ antichains are finite, thus completing the proof.
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