Zero Modes of Matter Fields on Scalar Flat Thick Branes

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Abstract

Zero modes of various matters with spin 0, 1 and 1/2 on a class of scalar flat thick branes are discussed in this paper. We show that scalar field with spin 0 is localized on all thick branes without additional condition, while spin 1 vector field is not localized. In addition, for spin 1/2 fermionic field, the zero mode is localized on the branes under certain conditions.

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I. INTRODUCTION

In the 1920’s Kaluza and Klein came up with the Compactification theory. In their original work it was shown that if we start with a theory of general relativity in 5-spacetime dimensions and then curl up one of the dimensions into a circle we end up with a 4-dimensional theory of general relativity plus electromagnetism. More recently, at least since about 1980, the Kaluza-Klein theory has been revived in attempts to derive supergravity theories. Nowadays it has been acknowledged that our four dimensional world is a three-brane embedded in a higher dimensional space-time with non-factorizable warped geometry. It means that we are free from the moduli stabilization problem in the sense that the internal manifold is noncompact and does not need to be compactified to the Planck scale any more.

The most popular model in this field is the so-called RS model which consists of a flat brane embedded in an AdS5 bulk. The particular idea of the model was the inclusion of tension of the brane, describing the energy density per volume in three-dimensional space. The researchers focused on models with only one extra dimension, but the basic concept can be generalized to an arbitrary higher-dimensional spacetime. Positive result suggests Randall-Sundrum extra dimensions are correct, and we could extend RS brane worlds in principle. According to the model, spin 0 field is localized on a brane with positive tension, spin 1/2 and 3/2 fields are localized not on a brane with positive tension but on a brane with negative tension.

The key ingredient for realizing the brane world idea is the localization of various bulk fields on a brane by a natural mechanism. Gravity is known to be the unique interaction having universal coupling with all matter fields. Considering RS model in the thin brane limit, thick brane scenarios based on gravity coupled to scalars have been constructed recently. With these models, one can obtain branes naturally without introducing them by hand in the action of the theory. And some work has been made to generalize the models which based on gravity coupled to scalars. Then it is of great interest to extend the investigation to the localization of various matters on the thick branes.

The organization of the paper is as follows: The first part of this paper will review the thick brane worlds arising from scalar fields, then we will discuss in detail the localization of various matters with spin 0, 1 and 1/2. One may ponder in this section whether all the
fields can be localized on various thick branes. Finally, in Section 4, we summarize what have been achieved in the paper, and give some further discussions.

II. REVIEW OF THICK BRANE WORLDS ARISING FROM SCALAR FIELDS

The model that we investigate is described by a theory of five-dimensional gravity coupled to scalar fields governed by the following action

\[ S = \int d^4x dy \sqrt{|g|} \left( \frac{1}{4} R + \mathcal{L}(\phi, \partial_i \phi) \right), \]  

(1)

where \( \phi \) stands for a real scalar field and we are using \( 4\pi G = 1 \). These theories with one or many scalar fields can simulate true five-dimensional supergravity theories under certain consistent truncations. The line element \( ds_5^2 \) of the five-dimensional space-time can be written as

\[ ds_5^2 = g_{MN} dx^M dx^N = e^{2A} ds_4^2 + dy^2, \]  

(2)

where \( M, N = 0, 1, ..., 4 \). Also, \( ds_4^2 \) represents the line element of the four-dimensional space-time, which can have the form

\[ ds_4^2 = -dt^2 + e^{2\sqrt{\Lambda} t} (dx_1^2 + dx_2^2 + dx_3^2), \]  

(3)

\[ ds_4^2 = e^{-2\sqrt{\Lambda} x_3} (-dt^2 + dx_1^2 + dx_2^2) + dx_3^2, \]  

(4)

for dS and AdS geometry, respectively. Here \( e^{2A} \) is the warp factor and \( \Lambda \) represents the cosmological constant of the four-dimensional space-time; the limit \( \Lambda \to 0 \) leads to the line element

\[ ds_5^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \]  

(5)

where \( \eta_{\mu\nu} \) describes Minkowski geometry. The scalar field dynamics is governed by the Lagrangian density

\[ \mathcal{L} = -\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V, \]  

(6)

where \( V = V(\phi) \) represents the potential, which specifies the model to be considered.

The standard scenario is to suppose that both \( A \) and \( \phi \) are static, depending only on the extra dimension, that is we can set \( A = A(y) \) and \( \phi = \phi(y) \). The example for flat branes\(^{17}\) is given by

\[ V = \frac{1}{6} \phi^{2-2p} \left[ 3 \left( \phi^\frac{2}{p} - 1 \right)^2 - 8p^2 \phi^2 \left( \frac{1}{2p-1} - \frac{\phi^\frac{2}{p}}{2p+1} \right)^2 \right], \]
where \( p \) is odd integer. The case \( p = 1 \) is special, and reproduces the \( \phi^4 \) model in flat spacetime. This model was first introduced in \([18]\), and it was recently considered within the braneworld context in \([12]\). The scalar field and the warp factor are given by

\[
\phi = \tanh^p \left( \frac{y}{p} \right),
\]

\[
A = -\frac{1}{32p+1} \tanh^{2p} \left( \frac{y}{p} \right) + \frac{4}{3} \frac{p^2}{1-4p^2} \left[ \ln \cosh \left( \frac{y}{p} \right) - \sum_{n=1}^{p-1} \frac{1}{2n} \tanh^{2n} \left( \frac{y}{p} \right) \right].
\]

The energy density of the scalar matter is

\[
T_{00}(y) \propto -3e^{2A} \left( 2A^2 + A'' \right).
\]

For \( p = 1 \), this function has two negative minima and a positive maximum at \( y = 0 \), and finally it vanishes asymptotically. The fact can be compared with Randall-Sundrum thin brane case, where one of the branes has a positive brane tension meanwhile the second brane has a negative one \([5, 6, 19]\). For the case of \( p > 1 \), however, this function has two negative minima and two positive maxima at \( y \neq 0 \), and finally it vanishes also asymptotically. Note that, for this case, the energy density at \( y = 0 \) is zero.

### III. LOCALIZATION OF VARIOUS MATTER FIELDS

In this section, we focus attention on the problems of localization of various bulk fields with spin 0, 1 and 1/2. It is necessary to point out that the solutions given in previous section remain valid even in the presence of bulk fields since various bulk fields considered below make little contribution to the bulk energy. Now we discuss the situation of various matter fields in detail.

#### A. Spin 0 scalar field

For a real scalar field on the thick branes, we first investigate how to localize it in the backgrounds \([4, 8]\). Let us consider the action of a massless real scalar coupled to gravity

\[
S_0 = -\frac{1}{2} \int d^5x \sqrt{-g} \ g^{MN} \partial_M \Phi \partial_N \Phi,
\]
from which the equation of motion can be derived

\[
\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N \Phi) = 0. \tag{11}
\]

For simplicity, we define \( P(y) = e^{2A(y)} \). Then the background metric is determined by

\[
ds^2_5 = P(y) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{12}
\]

and the equation of motion (11) becomes

\[
P \eta^{\mu\nu} \partial_\mu \partial_\nu \Phi + \partial_y (P^2 \partial_y \Phi) = 0. \tag{13}
\]

Making the following decomposition

\[
\Phi(x, y) = \phi(x) \chi(y) = \phi(x) \sum_m \chi_m(y), \tag{14}
\]

and demanding \( \phi(x) \) satisfies the massless 4-dimensional Klein-Gordon equation

\[
\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) = 0,
\]

we obtain the zero-mass constant solution of Eq. (13), i.e. \( \chi(y) = \chi_0 = \text{constant} \).

Now we show that this constant mode is localized on the thick branes arising from scalar kink (7). The condition for having localized 4-dimensional scalar field is that \( \chi(y) = \chi_0 \) is normalizable, which is equivalent to the condition that the "coupling" constant is nonvanishing. Substituting the zero mode into the starting action (10), we get

\[
S_0^{(0)} = -\frac{1}{2} \chi_0^2 \int_{-\infty}^{\infty} dy P \int d^4x \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{15}
\]

Hence, in order to localize 4-dimensional scalar field on the thick branes one requires that the integral \( I_0 \), which is defined by

\[
I_0 = \int_{-\infty}^{\infty} dy P,
\]

should be finite. Noting that \( \tanh^{2n}(y/p) \leq 1 \) for arbitrary finite \( y \) and positive \( n \), we have

\[
I_0 = \int_{-\infty}^{\infty} dy P < \int_{-\infty}^{\infty} dy P_\infty \tag{17}
\]

is finite for arbitrary positive odd \( p \), where

\[
P_\infty = \left( \cosh \frac{y}{p} \right)^{-\frac{p^2}{2p+2} \left( \frac{p}{2p+2} \sum_{n=1}^{p-1} \frac{1}{n-1} \right)}.
\]
For examples, we have

\[ I_0 < \frac{e^{-2/9} \sqrt{\pi} \Gamma(4/9)}{\Gamma(17/18)} = 1.5\sqrt{\pi} \quad \text{for } p = 1, \]

\[ I_0 < \frac{3e^{8/35} \sqrt{\pi} \Gamma(12/35)}{\Gamma(59/70)} = 8.8\sqrt{\pi} \quad \text{for } p = 3. \]

From above analysis, we conclude that the spin 0 scalar field can be localized on thick branes without additional conditions.

**B. Spin 1 vector field**

Next we turn our attention to spin 1 vector field. Now the action of \( U(1) \) vector field is considered as the following:

\[ S_1 = -\frac{1}{4} \int d^5 x \sqrt{-g} g^{MN} g^{RS} F_{MR} F_{NS}, \tag{18} \]

where \( F_{MN} = \partial_M A_N - \partial_N A_M \). And the equation motion can be obtained easily by using this action, i.e.

\[ \frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} g^{RS} F_{NS}) = 0. \tag{19} \]

From the background geometry (12), this equation is reduced to

\[ \eta^{\mu
u} \partial_\mu F_{\nu y} = 0, \tag{20} \]

\[ \partial^\mu F_{\mu
u} + \partial_y (PF_{y\nu}) = 0. \tag{21} \]

By decomposing the vector field as

\[ A_\mu(x, y) = a_\mu(x) \sum_m \rho_m(y), \quad A_y(x, y) = a_y(x) \sum_m \rho_m(y), \]

it is easy to see that there is the constant solution \( \rho_m(y) = \rho_0 = \text{constant} \) and \( a_y = \text{constant} \) if we use \( \partial^\mu f_{\mu\nu} = 0 \) with the definition of \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \).

Now let us substitute this constant solution into the action (18) in order to see whether the solution is a normalizable one or not. It turns out that the action is reduced to

\[ S_1^{(0)} = -\frac{1}{4} \rho_0^2 \int_{-\infty}^{\infty} dy \int d^4 x \eta^{\mu\nu} \eta^{\alpha\beta} f_{\mu\alpha} f_{\nu\beta}. \tag{22} \]
Here we define $I_1$ by

$$I_1 = \int_{-\infty}^{\infty} dy,$$  \hspace{1cm} (23)

then the condition of having localized 4-dimensional vector field on the branes requires $I_1$ to be finite. Obviously, the integral is divergent, which shows that the vector field can not be localized on the thick brane. This conclusion can be compared with the work of others. It was shown in the Randall-Sundrum model in $AdS_5$ space that spin 1 vector field is not localized neither on a brane with positive tension nor on a brane with negative tension so the Dvali-Shifman mechanism \cite{20} must be considered for the vector field localization \cite{21}.

### C. Spin 1/2 fermionic field

For spin 1/2 fermionic field, it is well known that fermions can not be localized in Randall-Sundrum branes\cite{6}. Melfo et al. studied the localization of fermions on various different scalar thick branes recently\cite{22, 23, 24}, and had shown that only one massless chiral mode is localized in double walls and branes interpolating between different $AdS_5$ spacetimes whenever the wall thickness is keep finite, while Chiral fermionic modes can not be localized in $dS_4$ walls embedded in a $M_5$ spacetime. It is therefore of interest to investigate the possibility that fermion confinement, being directly dependent on the scalar field solution and not only on the spacetime metric, can be affected by the internal structure on the thick brane. We can shown that, for the case of no Yukawa coupling, there is no bound states for both left and right chiral fermions. Hence, for the massless chiral fermion localization, there must be some kind of Yukawa coupling. In this subsection localization of a spin 1/2 fermionic field on the thick branes will be investigated.

The discussions begin with the Dirac action of a massless spin 1/2 fermion coupled to gravity and scalar

$$S_{1/2} = \int d^5x \sqrt{-g} (\bar{\Psi} \Gamma^M D_M \Psi + \lambda \bar{\Psi} \phi \Psi),$$  \hspace{1cm} (24)

from which the equation of motion is given by

$$(\Gamma^M (\partial_M + \omega_M) + \lambda \phi) \Psi = 0,$$  \hspace{1cm} (25)

where $\omega_M = \frac{1}{4} \omega^\mathcal{M}, \mathcal{N} \gamma_\mathcal{M} \gamma_\mathcal{N}$ is the spin connection with $\mathcal{M}, \mathcal{N}, \cdots$ denoting the local Lorentz indices, $\Gamma^M$ and $\gamma^\mathcal{M}$ are the curved gamma matrices and the flat gamma ones respectively, and
have the relations $\Gamma^M = e^M_M \gamma^M$ with $e^M_M$ being the vielbein. The non-vanishing components of $\omega_M$ are

$$\omega_\mu = \frac{P'}{4\sqrt{P}} \gamma^\mu \gamma_y.$$  \hspace{1cm} (26)

And the Dirac equation (25) then becomes

$$\left \{ P^{-\frac{1}{2}} \gamma^\mu \partial_\mu + \gamma^y \partial_y + P' P'^{-1} \gamma^y + \lambda \phi \right \} \Psi = 0,$$  \hspace{1cm} (27)

where $\gamma^\mu \partial_\mu$ is the Dirac operator on the brane. We are now ready to study the above Dirac equation for 5-dimensional fluctuations, and write it in terms of 4-dimensional effective fields. From the equation of motion (27), we will search for the solutions of the chiral decomposition

$$\Psi_\pm = \psi_\pm (x) \alpha_\pm (y),$$  \hspace{1cm} (28)

where $\psi_\pm (x)$ satisfies the massless 4-dimensional Dirac equation. The equation for the chiral modes becomes

$$\left \{ \gamma^\mu \partial_\mu + P^\frac{1}{2} \left ( \gamma^y \partial_y + P' P'^{-1} \gamma^y + \lambda \phi \right ) \right \} \Psi_\pm = 0.$$  \hspace{1cm} (29)

Setting $\gamma^y \Psi_\pm = \pm \Psi_\pm$, namely assume $\Psi_+$ is left chiral fermion $\gamma^y \Psi_+ = \Psi_+$, $\Psi_-$ is right chiral fermion $\gamma^y \Psi_- = -\Psi_-$, the new form of above equation can be obtained. For zero modes, corresponding to 4D massless fermions, we have $\gamma^\mu \partial_\mu \psi_\pm (x) = 0$, and

$$\left ( \pm \partial_y \pm \frac{P'}{P} + \lambda \phi \right ) \alpha_\pm (y) = 0.$$  \hspace{1cm} (30)

So the solutions of zero modes are

$$\Psi_\pm = c_0 \exp \left \{ - \int^y \left ( \frac{P'(\bar{y})}{P(\bar{y})} \pm \lambda \phi(\bar{y}) \right ) d\bar{y} \right \} \psi_\pm (x).$$  \hspace{1cm} (31)

Let us analyze whether this zero mode can be localized on the branes under certain conditions. Substituting (31) into the Dirac action (24), we obtain

$$S^{(0)}_{1/2} = I_{1/2} \int d^4 x \bar{\psi}_\pm \gamma^\mu \partial_\mu \psi_\pm,$$  \hspace{1cm} (32)

where $I_{1/2}$ is defined as

$$I_{1/2} = c_0^2 \int_{-\infty}^{\infty} dy P^{-\frac{1}{2}} e^{-2 \int^y \pm \lambda \phi(y) dy}.$$  \hspace{1cm} (33)
In order to localize spin 1/2 fermion in this framework, the integral (33) should be finite. Using the configuration (7, 8) and \( P(y) = e^{2A(y)} \), the requirement of convergence for integral is easily discussed. For example, for \( p = 1 \), substituting \( P = \cosh^{-\frac{2}{9}} y \exp(-\frac{2}{9} \tanh^2 y) \) and \( \phi = \tanh y \) into (33), we get

\[
I_{1/2} = c_0^2 \int_{-\infty}^{\infty} dy \cosh^{\frac{4}{9} - 2(\pm \lambda)} y \exp \left( \frac{1}{9} \tanh^2 y \right). \tag{34}
\]

Noting that for \( \lambda > \frac{2}{9} \), we have

\[
\cosh^{\left(\frac{4}{9} - 2\lambda\right)} y \big|_{y \to \pm \infty} = 0, \quad \exp \left( \frac{1}{9} \tanh^2 y \right) \big|_{y \to \pm \infty} = e^{\frac{1}{9}}.
\]

Now one can check that the integral (34) is finite when \( \lambda > \frac{2}{9} \) for left chiral fermion. For right chiral fermion, the localization condition is \( \lambda < -\frac{2}{9} \). For \( p = 3 \), using the similar method, we can also get the localization condition for left chiral fermion is \( \lambda > \frac{2}{35} \).

Based on above analysis, the requirement of convergence for integral is studied, and the discussions end with the conclusion that the spin 1/2 field can be localized with some conditions satisfied.

**IV. SUMMARY**

From the viewpoint of field theory, the possibility of localizing various matter fields on thick branes has been investigated in this paper. The analytical study shows that spin 0 field is localized on thick branes without additional condition. But the result of spin 1 field is the same as the case of RS model in \( AdS_5 \) space, in which vector field is not localized neither on a brane with positive tension nor on a brane with negative tension so the Dvali-Shifman mechanism should be considered for the vector field localization. Since localization of fermions on branes requires other interactions but gravity, we show that massless chiral fermionic modes can be localized on the thick branes, coupled through a Yukawa term. There are still some other backgrounds, for example, vortex background [25, 26, 27], could be considered besides gauge field and supergravity [28, 29, 30]. The topological vortex coupled to fermions may lead to chiral fermion zero modes [31]. Usually the number of the zero modes coincides with the topological number, that is, with the magnetic flux of the vortex. Under these backgrounds, more extensive work can be carried out in the future.
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