Approximations with deep neural networks in Sobolev time-space

Ahmed Abdeljawad* and Philipp Grohs*, †

ahmed.abdeljawad@ricam.oeaw.ac.at, philipp.grohs@univie.ac.at

*Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Linz, Austria.
†Faculty of Mathematics, University of Vienna, Vienna, Austria.

Solutions of evolution equation generally lies in certain Bochner-Sobolev spaces, in which the solution may have regularity and integrability properties for the time variable that can be different for the space variables. Therefore, in this paper, we develop a framework shows that deep neural networks can approximate Sobolev-regular functions with respect to Bochner-Sobolev spaces. In our work we use the so-called Rectified Cubic Unit (ReCU) as an activation function in our networks, which allows us to deduce approximation results of the neural networks while avoiding issues caused by the non regularity of the most commonly used Rectified Linear Unit (ReLU) activation function.

1. Introduction

In recent years, methods from deep learning have been applied to the numerical solution of partial differential equations with impressive results [2, 3, 5, 8, 10, 13, 16, 17, 20, 21, 24, 26, 27, 34]. One key component of this success lies in the expressive power of neural networks, which constitute a parametrizes class of functions constructed by iterative compositions of affine mappings and pointwise application of a nonlinear activation function. Neural networks have been demonstrated to be at least on par with most known approximation methods, including (hp) finite elements, wavelts, shearlets, in terms of their approximation power, see for example [4, 9, 25, 28, 29, 31, 33]. In these works it is shown that functions belonging to certain smoothness classes can be approximated by neural networks at a complexity corresponding to the optimal approximation rate as dictated by the metric entropy of the smoothness class, where the approximation error is typically measured in an isotropic Sobolev norm. However, if one considers the problem of approximating solutions to time dependent partial differential equations, the natural norm in which the error is measured are typically of a different form coming from a space time Sobolev space [1, 7, 11, 23, 32]. Motivated by this fact we consider in this paper the approximation of functions in space time Sobolev spaces. Our main result Theorem 4.1 shows that, similar to functions in isotropic Sobolev spaces, these functions can be efficiently approximated by neural networks, also when the approximation error is measured with respect to a space time Sobolev norm. Our result is constructive and similar in spirit to [19] in the sense that our explicit constructions of
approximants emulate a local polynomial approximant. In contrast to [18, 19, 30, 37] where the so-called ReLU function \( x \mapsto \max\{0, x\} \) is chosen as activation function, our results hold for the so-called ReCU function \( x \mapsto \max\{0, x\}^{\alpha} \), the main reason being that the latter is continuously differentiable.

### 1.1. Outline

This paper is organized as follows. In Section 2 we provide definitions and properties of the Sobolev time-space. All the proofs of the result in Section 2 can be found in Appendix A. We start Section 3 by introducing the mathematical definition of deep neural networks, moreover we show some of its properties. Finally, in Section 4 we prove the main result of our paper. That is, Theorem 4.1, where we show that deep neural networks can approximate certain function in Sobolev time-space, thus we get information about regularity and approximation rate. For the approximation and estimation results, we always work with ReCU activation function.

### 1.2. Notations

Throughout the paper, the following notation is used: The sets of natural numbers and real numbers are denoted by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. Furthermore, \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \), denotes the set of non-negative integers.

If \( x \in \mathbb{R} \), then we write \( \lfloor x \rfloor := \min\{k \in \mathbb{Z} : k \geq x\} \) where \( \mathbb{Z} \) is the set of integers.

If \( d \in \mathbb{N} \) and \( \| \cdot \| \) is a norm on \( \mathbb{R}^d \), then we denote for \( x \in \mathbb{R}^d \) and \( r > 0 \) by \( B_r(x) \) the open ball around \( x \) in \( \mathbb{R}^d \) with radius \( r \), where the distance is measured in \( \| \cdot \| \). By \( |x| \) we denote the Euclidean norm of \( x \) and by \( |x|_{\ell^\infty} \) the maximum norm. Moreover, throughout this paper \( \| \cdot \|_{\ell^0} \) be referred to as the counting norm that return the total number of non-zero elements in a given vector. Strictly speaking, \( \ell^0 \)-norm is not actually a norm in the mathematical sense.

We endow \( \mathbb{R}^d \) with the standard topology and for \( A \subset \mathbb{R}^d \) we denote by \( \bar{A} \) the closure of \( A \) and by \( \partial A \) the boundary of \( A \). The diameter of a non-empty set \( A \subset \mathbb{R}^d \) is always taken with respect to the euclidean distance, i.e. \( \text{diam} A := \text{diam}_{|x|} A := \sup_{x,y \in A} |x - y| \).

Note that if \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d \) is a multi-index, then \( \alpha! = \alpha_1! \cdots \alpha_d! \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_d \). Let \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), and \( u \in C^\alpha(\Omega) \), and let \( \alpha \in \mathbb{N}^d \) be a multi-index such that \( |\alpha| \leq n \), then we denote

\[
D^\alpha u = \frac{\partial^{\alpha_1} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad D_{x_j} u = \frac{\partial u}{\partial x_j}, \quad \text{where } 1 \leq j \leq d.
\]

Let \( \Omega \subset \mathbb{R}^d \), \( 1 \leq p \leq \infty \), then \( L^p(\Omega, \mathbb{R}) = L^p(\Omega) \) denotes the Lebesgue space. Moreover, if \( X \) is a Banach space, then the space \( L^p(\Omega, X) \) is called vector-valued Lebesgue space or Bochner space, defined as the space of all measurable functions such that \( \|f\|_X \in L^p(\Omega, \mathbb{R}) \) and the norm on this space will be defined via \( \|f\|_{L^p(\Omega, X)} := \|\|f\|_X\|_{L^p(\Omega, \mathbb{R})} \).

If \( d_1, d_2, d_3 \in \mathbb{N} \) and \( A \in \mathbb{R}^{d_1}, B \in \mathbb{R}^{d_2}, \) then we use the common block matrix notation and write for the horizontal concatenation of \( A \) and \( B \)
\[
\begin{bmatrix}
  A & B \\
\end{bmatrix} \in \mathbb{R}^{d_1,d_2+d_3}.
\]

A similar notation is used for the vertical concatenation of \( A \in \mathbb{R}^{d_1,d_2} \) and \( B \in \mathbb{R}^{d_1,d_2} \).

We define the Rectified Power Unit (RePU) as follows
\[
\rho_s(x) = \begin{cases} 
  x^s, & x \geq 0, \\
  0, & x < 0,
\end{cases}, \quad s \in \mathbb{N}_0.
\] (1.1)

Note that \( \rho_0 \) is the binary step function while \( \rho_1 \) is the commonly used Rectified Linear Unit (ReLU) function. We call \( \rho_2, \rho_3 \) Rectified Quadratic Unit (ReQU) and Rectified Cubic Unit (ReCU), respectively.

\section{Sobolev time-space definition and properties}

In the current section we review some properties of mixed Sobolev spaces and extend some. Moreover we show that Bramble-Hilbert lemma is valid in our setting. More details about the proofs can be found in the Appendix.

\textbf{Definition 2.1 (Sobolev space).} Assume that \( \Omega \) is an open subset of \( \mathbb{R}^d \), and let \( n \in \mathbb{N}, \ 1 \leq p \leq \infty \). The Sobolev space \( W^{n,p}(\Omega) \) consists of functions \( u \in L^p(\Omega) \) such that for every multi-index \( \alpha \) with \( |\alpha| \leq k \), \( D^\alpha u \) exists and \( D^\alpha u \in L^p(\Omega) \). Thus
\[
W^{n,p}(\Omega) := \left\{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq n \right\}.
\]

Furthermore, for \( f \in W^{n,p}(\Omega) \) and \( 1 \leq p < \infty \), we define the norm
\[
\|f\|_{W^{n,p}(\Omega)} := \left( \sum_{0 \leq |\alpha| \leq n} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}
\]
and
\[
\|f\|_{W^{n,\infty}(\Omega)} := \max_{0 \leq |\alpha| \leq n} \|D^\alpha f\|_{L^\infty(\Omega)}.
\]

\textbf{Definition 2.2 (Sobolev time-space).} Let \( 1 \leq p, q \leq \infty, m, n \in \mathbb{N}, I \subset \subset \mathbb{R}, \) and \( \Omega \subset \subset \mathbb{R}^d \).

Let \( W^{n,p}_{m,q}(I, \Omega) \) defined as follows
\[
W^{n,p}_{m,q}(I, \Omega) = \left\{ f \in L^q(I, W^{n,p}(\Omega)) : \partial^k_t f \in L^q(I, W^{n,p}(\Omega)) \text{ for all } k \leq m \right\}
\]
such that
\[
\|f\|_{W^{n,p}_{m,q}(I, \Omega)} = \sum_{k \leq m} \|\partial^k_t f\|_{L^q(I,W^{n,p}(\Omega))}
\]
when \( 1 \leq p, q < \infty \), with the obvious modifications when \( p = \infty \) and/or \( q = \infty \).

Note that if \( n = m = 0 \), then \( W^{0,p}_{0,q}(I, \Omega) = L^q(I, L^p(\Omega)) \). Hence, we shall write \( L^q_t L^p_x(I \times \Omega) := W^{0,p}_{0,q}(I, \Omega) \), where \( L^q_t \) and \( L^p_x \) stand for the Lebesgue integral with respect to \( t \in I \) and \( x \in \Omega \), respectively.

Next we introduce the Sobolev time-space semi-norms in order to simplify the notations in the proofs came in the sequel.
**Definition 2.3** (Sobolev time-space semi-norm). Let \( I \subset \mathbb{R}, \Omega \subset \mathbb{R}^d \). For \( n,k \in \mathbb{N}_0 \) with \( \ell \leq k, m \leq n, m \in \mathbb{N} \) and \( 1 \leq p,q \leq \infty \), we define for \( f \in W^{n,p}_{k,q}(I,\Omega) \) the Sobolev time-space semi-norm

\[
|f|_{W^{n,p}_{k,q}(I,\Omega)} := \left( \sum_{|\alpha|=m} \left\| D^{\alpha}_{x} D^\ell_{t} f \right\|_{L^p(I \times \Omega)}^p \right)^{1/p}
\]

for \( 1 \leq p,q < \infty \) and

\[
|f|_{W^{n,\infty}_{k,q}(I,\Omega)} := \max_{|\alpha|=m} \left\| D^{\alpha}_{x} D^\ell_{t} f \right\|_{L^\infty(I \times \Omega)},
\]

with the obvious modification when \( q = \infty \), \( 1 \leq p < \infty \) and when \( p = q = \infty \).

**Definition 2.4.** Let \( I \subset \mathbb{R}, \Omega \subset \mathbb{R}^d \) and \( m \in \mathbb{N} \). Then the Taylor polynomial of order \( m \) evaluated at \((\tau,\xi)\in I \times \Omega\) is given by

\[
T^{m}_{\tau,\xi} u(t,x) = \sum_{k+|\alpha|<m} \frac{1}{\alpha!k!} D^{\alpha}_{x} D^k_{t} u(\tau,\xi)(x-\xi)^\alpha(t-\tau)^k,
\]

where \( \alpha \) is the \( d \)-tuple of nonnegative integers and \( k \in \mathbb{N}_0 \).

**Definition 2.5** (averaged Taylor polynomial). Let \( I \subset \mathbb{R}, \Omega \subset \mathbb{R}^d \), \( k,n \in \mathbb{N}_0 \), \( m \in \mathbb{N} \), such that \( k+n \in \{0, \ldots, m-1\}, 1 \leq p,q \leq \infty \) and \( u \in W^{n,p}_{k,q}(I \times \Omega) \), and let \((t_0,x_0) \in I \times \Omega \), \( r > 0 \) such that for the ball \( B := \{(t,x) \in I \times \Omega \text{ such that } |t-t_0|+|x-x_0| < r \} \) it holds that \( B \subset \subset \Omega \). The corresponding Taylor polynomial of order \( m \) of \( u \) averaged over \( B \) is defined for \((t,x) \in I \times \Omega \) as

\[
Q^m u(t,x) := \int_B T^{m}_{\tau,\xi} u(t,x) \phi(\tau,\xi) d\xi d\tau,
\]

where \( T^{m}_{\tau,\xi} u \) is the Taylor polynomial of order \( m \) defined in Definition 2.4 and \( \phi \) is an arbitrary cut-off function supported in \( B \), with \( \phi(t,x) \geq 0 \) for all \((t,x) \in \mathbb{R} \times \mathbb{R}^d \), \( \supp \phi = B \) and \( \int_\mathbb{R} \int_\mathbb{R}^d \phi(t,x) dx dt = 1 \).

A cut-off function as used in the previous definition always exists. A possible choice is

\[
\phi(t,x) = \begin{cases} e^{-\left(1-\frac{|t-t_0|}{r}\right)^2-\left(1-\frac{|x-x_0|}{r}\right)^2}, & \text{if } |t-t_0|+|x-x_0| < r \\ 0, & \text{else} \end{cases}
\]

normalized by \( \int_\mathbb{R} \int_\mathbb{R}^d \phi(t,x) dx dt = 1 \).

**Proposition 2.1.** Let \( C_{t,x} \) denotes the convex hull of \( \{(t,x)\} \cup B \). Then, the remainder \( R^m u := u - Q^m u \) satisfies

\[
R^m u(t,x) = \sum_{|\alpha|+k=m} \int_{C_{t,x}} K_{\alpha,k}(t,T,x,\Xi) D^{\alpha}_{x} D^k_{t} u(T,\Xi) d\Xi dT
\]

where \( \Xi = x + s(\xi-x) \), \( T = t + s(\tau-t) \), \( K_{\alpha,k}(t,T,x;\Xi) = \Delta_{x,\Xi}^{-1}(x-\Xi)^\alpha(t-T)^k K(t,T,x,\Xi) \)

and

\[
|K(t,T,x,\Xi)| \leq C (1+(|x-x_0|+|t-t_0|)/r)^d+1 (|\Xi - x| + |T-t|)^{-d-1}.
\]

**Proof.** The proof can be found in Appendix A.1. \( \square \)
Next we recall some geometric definitions needed for the control of the non-degeneracy of a given family of subdivisions of a domain $\Omega$ through the so-called chunkiness parameter. More details can be found in the discussion after [6, Definition 10.5.1].

**Definition 2.6.** Let $\Omega, B \subset \subset \mathbb{R}^d$, then $\Omega$ is star-shaped with respect to $B$ if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of $\Omega$.

**Definition 2.7.** Let $\Omega \subset \subset \mathbb{R}^d$ have diameter $diam(\Omega) > 0$ and is star-shaped with respect to a ball $B$. Let $\mathcal{R} = \{r > 0 : \Omega$ is star-shaped with respect to a ball of radius $r\}$. If $\mathcal{R} \neq \emptyset$, then $r_{\text{max}}^* = \sup \mathcal{R}$ and the chunkiness parameter of $\Omega$ is defined by

\[ \gamma = \frac{diam(\Omega)}{r_{\text{max}}^*} \]

**Lemma 2.1** (Bramble-Hilbert). Let $I \subset \subset \mathbb{R}$, $\Omega \subset \subset \mathbb{R}^d$, $t_0 \in I$, $x_0 \in \Omega$ and $r > 0$ such that $I \times \Omega$ is star-shaped with respect to $B := B_r(t_0, x_0)$, and $r > (1/2)r_{\text{max}}^*$. Moreover, let $k, m, n \in \mathbb{N}$, such that $k + n \in \{0, 1, \ldots, m\}$, $1 \leq p, q \leq \infty$ and denote by $\gamma$ the chunkiness parameter of $I \times \Omega$. Then there exists a constant $C = C(m, d, \gamma) > 0$ such that for all $u \in W^m_{m,q}(I, \Omega)$

\[ |u - Q^m u|_{W^m_{m,q}(I, \Omega)} \leq C h^{m-k-n} \|u\|_{W^{m,k}_{m,q}(I, \Omega)} \]

where $Q^m u$ denotes the Taylor polynomial of order $m$ of $u$ averaged over $B$ and $h = diam(I \times \Omega)$.

**Proof.** A proof can be found in Appendix [A.2].

**Lemma 2.2.** Let $I \subset \subset \mathbb{R}$, $\Omega \subset \subset \mathbb{R}^d$, $k, n \in \mathbb{N}$, $1 \leq p \leq \infty$ and $u \in W^{k+n-1,p}_{k+n-1,p}(I, \Omega)$, and let $(t_0, x_0) \in I \times \Omega$, $r > 0$, $R \geq 1$ such that for the ball $B := B_{r, [1]}((t_0, x_0))$ it holds that $B \subset \subset I \times \Omega$ and $B \subset B_{R, \|\cdot\|_{L^\infty}}(0)$. Then the Taylor polynomial of order $n + k$ of $u$ averaged over $B$ can be written as

\[ Q^{n+k} u(t, x) = \sum_{|\alpha| + \kappa \leq k + n - 1} c_{\alpha, \kappa} t^\alpha x^\kappa \]

for $(t, x) \in I \times \Omega$.

Moreover, there exists a constant $c = c(k, n, d, R) > 0$ such that the coefficients $c_{\alpha, \kappa}$ are bounded with $|c_{\alpha, \kappa}| \leq c R^{-(d+1)p} \|u\|_{W^{k+n-1,p}_{k+n-1,p}(I, \Omega)}$ for all $\alpha, \kappa$ with $|\alpha| + \kappa \leq k + n - 1$.

**Proof.** A detailed proof can be found in Appendix [A.3].

We need the following lemma to estimate the semi-norm of a product of weakly differentiable functions on the mixed Sobolev space.

**Lemma 2.3.** Let $1 \leq p, q \leq \infty$, and $I \subset \subset \mathbb{R}$, $\Omega \subset \subset \mathbb{R}^d$, $f \in W^{1,\infty}_{1,\infty}(I, \Omega)$, and $g \in W^{1,p}_{1,\infty}(I, \Omega)$, then $fg \in W^{1,q}_{1,q}(I, \Omega)$ and there exists a constant $C_1, C_2 > 0$ depend on $d$ and $p$ such that

\[
\begin{align*}
|fg|_{W^{1,q}_{1,q}(I,\Omega)} &\leq C_1 \left( |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|g\|_{L^q_tL^\infty_x(I \times \Omega)} + \|f\|_{L^\infty_tL^q_x(\Omega)} \|g\|_{W^{1,q}_{1,q}(I,\Omega)} \right), \\
|fg|_{W^{0,q}_{1,q}(I,\Omega)} &\leq |f|_{W^{0,\infty}_{1,\infty}(I,\Omega)} \|g\|_{L^q_tL^\infty_x(I \times \Omega)} + \|f\|_{L^\infty_tL^q_x(\Omega)} \|g\|_{W^{0,q}_{1,q}(I,\Omega)} , \\
|fg|_{W^{1,q}_{1,\infty}(I,\Omega)} &\leq C_2 \left( |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|L^q_tL^\infty_x(I \times \Omega) + \|f\|_{W^{0,\infty}_{1,\infty}(I,\Omega)} \|g\|_{W^{1,q}_{1,q}(I,\Omega)} \right) + |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|g\|_{W^{0,q}_{1,q}(I,\Omega)} + \|f\|_{L^\infty_tL^q_x(\Omega)} \|g\|_{W^{1,q}_{1,q}(I,\Omega)} \right), \\
\end{align*}
\]

For $p = \infty$, we have $C_1 = C_2 = 1$. 


Proof. A proof can be found in Appendix A.1.

The following corollary establishes a chain rule estimate for $W_{1,\infty}^1$.

Lemma 2.4. Let $p_i, m_i \in \mathbb{N}$, for $i = 1, 2, n, k \in \{0, 1\}$ such that $n + k = 1, p_1 + p_2 = p$, and $m_1 + m_2 = m$ and let $\Omega_i \subset \mathbb{R}^{p_i}$, $\Theta_i \subset \mathbb{R}^{m_i}$ with $i = 1, 2$, be open, bounded, and convex. Then, there is a constant $C = C(p, m) > 0$ with the following property:

If $f \in \mathcal{F} = \mathcal{F}(\Omega_1, \Omega_2)$ and $g \in \mathcal{G} = \mathcal{G}(\Theta_1, \Theta_2)$ are Lipschitz continuous functions such that $\text{ran}(f) \subset \Theta_1 \times \Theta_2$, then $g \circ f \in \mathcal{F}$ and we have

$$|g \circ f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} \leq C |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)} |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)}.$$

Moreover, if $n = k = 1$ there exists $C' = C'(p, m) > 0$, such that

$$|g \circ f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} \leq C' \max \left( |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)}, |f|^2_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)} \right).$$

Proof. The proof of Lemma 2.4 can be found in Appendix A.3.

3. Mathematical theory of neural networks

Deep neural networks have been shown to perform well on classification or regression tasks, that is supervised learning problems.

Here we introduce the basic mathematical theory of neural networks that will be used during this paper.

Definition 3.1. Let $d, L \in \mathbb{N}$. A neural network $\Phi$ with input dimension $d$ and $L$ layers is a sequence of matrix-vector tuples

$$\Phi = ((A_1, b_1), (A_2, b_2), \ldots, (A_L, b_L))$$

where $N_0 = d$ and $N_1, \ldots, N_L \in \mathbb{N}$, and where each $A_\ell$ is an $N_\ell \times N_{\ell-1}$ matrix, and $b_\ell \in \mathbb{R}^{N_\ell}$. If $\Phi$ is a neural network as above, and if $\rho : \mathbb{R} \to \mathbb{R}$ is arbitrary, then we define the associated realization of $\Phi$ with activation function $\rho$ as the map $R_\rho(\Phi) : \mathbb{R}^d \to \mathbb{R}^{N_L}$ such that

$$R_\rho(\Phi)(x) = x_L$$

where $x_L$ results from the following scheme:

$$x_0 := x$$
$$x_\ell := \rho(A_\ell x_{\ell-1} + b_\ell), \quad \text{for } \ell = 1, \ldots, L - 1$$
$$x_L := A_L x_{L-1} + b_L$$

where $\rho$ acts componentwise, i.e., for a given vector $y \in \mathbb{R}^m$, $\rho(y) = [\rho(y_1), \ldots, \rho(y_m)]$.

We call $N(\Phi) := d + \sum_{\ell=1}^L N_\ell$ the number of neurons of the network $\Phi$, while $L(\Phi) := L$ denotes the number of layers of $\Phi$. Moreover, $M(\Phi) := \sum_{\ell=1}^L (\|A_\ell\|_{\ell} + \|b_\ell\|_{\ell})$ denotes the total number of nonzero entries of all $A_\ell, b_\ell$, which we call the number of weights of $\Phi$. Finally, we refer to $N_L$ as the dimension of the output layer of $\Phi$, or simply as the output dimension of $\Phi$. We shall also sometimes refer to $A(\Phi) := (N_0, \ldots, N_L) \in \mathbb{N}^{L+1}$ as the architecture of $\Phi$. 
When dealing with neural networks, usually one has to fix a specific architecture (see Definition 3.2) e.g., fully-connected feedforward neural networks where information in such architecture flows in one direction from input to output layer (via hidden nodes if any), that is they do not form any circles or loopbacks. More details about different architecture can be found in e.g., [22, 35].

**Definition 3.2.** Let \( d, L \in \mathbb{N} \), a neural network architecture \( \mathcal{A} \) with input dimension \( d \) and \( L \) layers is a sequence of matrix-vector tuples

\[
\mathcal{A} = ((A_1, b_1), (A_2, b_2), \ldots, (A_L, b_L))
\]

such that \( N_0 = d \) and \( N_1, \ldots, N_L \in \mathbb{N} \), where each \( A_l \) is an \( N_l \times \sum_{k=0}^{l-1} N_k \) matrix, and \( b_l \) a vector of length \( N_l \) with elements in \( \{0, 1\} \). We call \( N(\mathcal{A}) := d + \sum_{j=1}^{L} N_j \) the number of neurons of the architecture \( \mathcal{A} \), \( L(\mathcal{A}) = L \) the number of layers and \( M(\mathcal{A}) := \sum_{j=1}^{L} (\|A_j\|_0 + \|b_j\|_0) \) Moreover, \( N_L \) denotes the dimension of the output layer of \( \mathcal{A} \). We say that a neural network \( \Phi = ((A_1', b_1'), (A_2', b_2'), \ldots, (A_L', b_L')) \) with input dimension \( d \) and \( L \) layers has architecture \( \mathcal{A} \) if the followings are satisfied

(i) \( N_l(\Phi) = N_l \) for all \( l = 1, \ldots, L \),

(ii) \( [A'_l]_{i,j} \neq 0 \) implies \( [A_l]_{i,j} 
eq 0 \) such that \( l = 1, \ldots, L \) where \( i = 1, \ldots, N_l \) and \( j = 1, \ldots, \sum_{k=0}^{l-1} N_k \).

Throughout the paper, we consider the Rectified Cubic Unit (ReCU) activation function, which is defined as follows:

\[
\rho_3 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \max(0, x^3).
\]  

(3.1)

To construct new neural networks from existing ones, we will frequently need to concatenate networks or put them in parallel. Most of the following results are well-known, see for example [30]. We first define the concatenation of networks.

**Definition 3.3.** Let \( L_1, L_2 \in \mathbb{N} \), and let

\[
\Phi^1 = ((A^1_{1}, b^1_{1}), \ldots, (A^1_{L_1}, b^1_{L_1})) \quad \Phi^2 = ((A^2_{1}, b^2_{1}), \ldots, (A^2_{L_2}, b^2_{L_2}))
\]

be two neural networks such that the input layer of \( \Phi^1 \) has the same dimension as the output layer of \( \Phi^2 \). Then, \( \Phi^1 \bullet \Phi^2 \) denotes the following \( L_1 + L_2 - 1 \) layer network:

\[
\Phi^1 \bullet \Phi^2 := ((A^1_{1}, b^1_{1}), \ldots, (A^2_{L_2-1}, b^2_{L_2-1}), \ldots, (A^1_{L_1}, A^2_{L_1} b^2_{L_2} + b^1_{L_1})), (A^2_{1}, b^2_{1}), \ldots, (A^2_{L_2}, b^2_{L_2}))
\]

We call \( \Phi^1 \bullet \Phi^2 \) the concatenation of \( \Phi^1 \) and \( \Phi^2 \).

**Lemma 3.1.** Let \( \Phi^1 \) and \( \Phi^2 \) be two neural networks where the input layer of \( \Phi^1 \) has the same dimension as the output layer of \( \Phi^2 \), then

\[
R_{\rho_3} (\Phi^1 \bullet \Phi^2) = R_{\rho_3} (\Phi^1) \circ R_{\rho_3} (\Phi^2).
\]

(3.2)

*Proof.* Equality in (3.2) is immediate and follows from the previous Definition 3.3. \( \square \)

Next we show that small neural networks are capable of emulating the identity.
Lemma 3.2. Let $\rho_3$ be the ReCU, $\Omega_r \doteq \prod_{j=1}^d [-r_j, r_j]$, where $r_j > 0$, let $d \in \mathbb{N}$, and define two layers neural network $\Phi^d_{d,r} \doteq ((A_1, b_1), (A_2, b_2))$ with

$$A_1 \doteq \left( \begin{array}{cc} \text{Id}_{R^d} & \text{Id}_{R^d} \\ \text{Id}_{R^d} & -\text{Id}_{R^d} \end{array} \right), \quad b_1 \doteq \left( \begin{array}{c} r_1 + 2 \\ \vdots \\ r_d + 2 \\ r_1 \\ \vdots \\ r_d \\ r_1 + 2 \\ \vdots \\ r_d + 2 \end{array} \right),$$

$$A_2 \doteq 1/24 \left( \begin{array}{cc} \text{diag}(1/(r_1 + 1),\ldots,1/(r_d + 1)) & \text{diag}(1/(r_1 + 1),\ldots,1/(r_d + 1)) \\ \text{diag}(1/(r_1 + 1),\ldots,1/(r_d + 1)) & -\text{diag}(1/(r_1 + 1),\ldots,1/(r_d + 1)) \\ -\text{diag}(1/(r_1 + 1),\ldots,1/(r_d + 1)) & \text{diag}(1/(r_1 + 1),\ldots,1/(r_d + 1)) \end{array} \right)^t, \quad b_2 \doteq 0.$$ 

Then, the realization $R_{\rho_3}(\Phi^d_{d,r}) = \text{Id}_{\Omega_r}$.

Proof. The proof of the lemma follows from the following identity

$$x = \frac{1}{24(r + 1)}\left(\rho_3(x + r + 2) + \rho_3(-x + r) - \rho_3(-x + r + 2)\right),$$

for any $x \in [-r, r]$ where $r > 0$. The extension to general domain is straightforward, thus the details are left for the reader. \hfill \Box

Remark 3.1. In view of Definition 3.3, we can bound the number of layers, neurons and weights as follows

$$L(\Phi^1 \bullet \Phi^2) = L_1 + L_2 - 1 \leq L_1 + L_2,$$

$$N(\Phi^1 \bullet \Phi^2) = N_1 + N_2 - N_0 \leq N_1 + N_2,$$

$$M(\Phi^1 \bullet \Phi^2) \leq M_1 + M_2 + M_1 M_2.$$

In the current paper we need another operation between networks, which is the parallelization. That is, one can put two networks of same length in parallel as next definition shows.

Definition 3.4. Let $L \in \mathbb{N}$ and let $\Phi^1 = ((A_1^1, b_1^1), \ldots, (A_L^1, b_L^1))$ and $\Phi^2 = ((A_1^2, b_1^2), \ldots, (A_L^2, b_L^2))$ be two neural networks with $L$ layers and with $d$-dimensional input. We define

$$P(\Phi^1, \Phi^2) \doteq \left( \begin{array}{c} (\tilde{A}_1, \tilde{b}_1), \ldots, (\tilde{A}_L, \tilde{b}_L) \end{array} \right)$$

where

$$\tilde{A}_1 \doteq \left( \begin{array}{c} A_1^1 \\ A_2^1 \end{array} \right), \quad \tilde{b}_1 \doteq \left( \begin{array}{c} b_1^1 \\ b_1^2 \end{array} \right) \quad \text{and} \quad \tilde{A}_\ell \doteq \left( \begin{array}{cc} A_\ell^1 & 0 \\ 0 & A_\ell^2 \end{array} \right), \quad \tilde{b}_\ell \doteq \left( \begin{array}{c} b_\ell^1 \\ b_\ell^2 \end{array} \right) \quad \text{for } 1 < \ell \leq L.$$ 

Then, $P(\Phi^1, \Phi^2)$ is a neural network with $d$-dimensional input and $L$ layers, called the parallelization of $\Phi^1$ and $\Phi^2$. 

8
Lemma 3.3. Let $L, d \in \mathbb{N}$, $\Phi^1$ and $\Phi^2$ be two neural networks with $L$ layers and with $d$-dimensional input. Then, $M(P(\Phi^1, \Phi^2)) = M(\Phi^1) + M(\Phi^2)$, and

$$R_{\rho_3}(P(\Phi^1, \Phi^2))(x) = (R_{\rho_3}(\Phi^1)(x), R_{\rho_3}(\Phi^2)(x)),$$  for any $x \in \mathbb{R}^d$.

Proof. The proof is straightforward and therefore is left for the reader. \qed

4. Approximations with deep ReCU neural networks in mixed Sobolev space

We are interested in approximating functions in subsets of the Sobolev space $W^{n,p}_{k,q}((0,1),(0,1)^d)$ with realizations of neural networks. For this we define the set:

$$\mathcal{U}_{k,q,n,p,d,B} := \left\{ u \in W^{n,p}_{k,q}((0,1),(0,1)^d) : \|u\|_{W^{n,p}_{k,q}((0,1),(0,1)^d)} \leq B \right\} \tag{4.1}$$

Next, we construct a partition of unity that can be defined as a product of piecewise linear functions, such that each factor of the product can be realized by a neural network.

Lemma 4.1. For any $d, N \in \mathbb{N}$ there exists a collection of functions

$$\Psi = \{ \phi_\mu : \mu \in \{0, \ldots, N\}^{d+1} \}$$

with $\phi_\mu : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ for all $\mu \in \{0, \ldots, N\}^{d+1}$ with the following properties:

(i) $0 \leq \phi_\mu(t,x) \leq 1$ for every $\phi_\mu \in \Psi$ and every $(t,x) \in \mathbb{R} \times \mathbb{R}^d$;

(ii) $\sum_{\phi_\mu \in \Psi} \phi_\mu(t,x) = 1$ for every $(t,x) \in [0,1] \times [0,1]^d$;

(iii) $\text{supp} \phi_\mu \subset B_{\frac{1}{\sqrt{N}},\|\cdot\|_{C_0}}(\frac{d}{N})$ for every $\phi_\mu \in \Psi$;

(iv) there exists a constant $c \geq 1$ such that $\|\phi_\mu\|_{W^{k,\infty}_{1,\infty}(\mathbb{R}^d)} \leq (c \cdot N)^{n+k}$ for $k, n \in \{0,1\}$;

(v) there exist absolute constants $C, c \geq 1$ such that for each $\phi_\mu \in \Psi$ there is a neural network $\Phi_\mu$ with $d+1$-dimensional input and $d+1$-dimensional output, with at most three layers, $C(d+1)$ nonzero weights and neurons, that satisfies

$$\prod_{l=0}^{d} R_{\rho_3}(\Phi_\mu)(x_l) = \prod_{l=0}^{d} [R_{\rho_3}(\Phi_\mu)]_l(t,x) = \phi_\mu(t,x), \ \text{where} \ x_0 = t$$

and $\|[R_{\rho_3}(\Phi_\mu)]_l\|_{W^{k,\infty}_{1,\infty}((0,1),(0,1)^d)} \leq (cN)^{n+k}$ for all $l = 0, \ldots, d$ such that $k, n \in \{0,1\}$.

Proof. As in [37], we define the functions

$$\psi : \mathbb{R} \to \mathbb{R}, \ \ \psi(x) := \begin{cases}
54 + 81x + \frac{81}{2}x^2 + \frac{87}{4}x^3, & x \in [-2, -\frac{5}{3}], \\
\frac{17}{2} - \frac{63}{2}x - 27x^2 - \frac{27}{4}x^3, & x \in [-\frac{5}{3}, -1], \\
\frac{1}{4}(20 + 36x + 54x^2 + 27x^3), & x \in [-1, -\frac{2}{3}], \\
1, & x \in (-\frac{2}{3}, \frac{2}{3}), \\
\frac{1}{4}(20 - 36x + 54x^2 - 27x^3), & x \in \left[\frac{2}{3}, 1\right], \\
\frac{17}{2} + \frac{63}{2}x - 27x^2 + \frac{27}{4}x^3, & x \in [1, \frac{5}{3}], \\
54 - 81x + \frac{87}{2}x^2 - \frac{87}{4}x^3, & x \in \left[\frac{5}{3}, 2\right], \\
0, & x \in \mathbb{R} \setminus (-2, 2).
\end{cases}$$
The function \( \phi_\mu : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is a product of scaled and shifted versions of \( \psi \). Concretely, we set

\[
\phi_\mu(t,x) := \psi \left( 3N \left( t - \frac{\mu_0}{N} \right) \right) \times \prod_{l=1}^{d} \psi \left( 3N \left( x_l - \frac{\mu_l}{N} \right) \right),
\]

(4.2)

for \( \mu = (\mu_0, \mu_1, \ldots, \mu_d) \in \{0, \ldots, N\}^{d+1} \). Then, (i), (ii) and (iii) follow easily from the definition.

To show (iv), note that \( \| \phi_\mu \|_{L^\infty_t L^\infty_x} \leq 1 \) follows already from (i). We need to show that the claim holds for \( \frac{\partial}{\partial x_l} \phi_\mu(t,x), \frac{\partial}{\partial t} \phi_\mu(t,x) \) and \( \frac{\partial}{\partial x_l} \frac{\partial}{\partial t} \phi_\mu(t,x) \). For this, let \( l \in \{1, \ldots, d\} \) \( t \in \mathbb{R} \), and \( x \in \mathbb{R}^d \), then, using the fact that \( |\psi'(x)| \leq \frac{9}{2} \) for any \( x \in \mathbb{R} \), we get

\[
\left| \frac{\partial}{\partial x_l} \phi_\mu(x) \right| = \left| \psi \left( 3N \left( t - \frac{\mu_0}{N} \right) \right) \prod_{i=1, i \neq l}^{d} \psi \left( 3N \left( x_i - \frac{\mu_i}{N} \right) \right) \right| \left| \psi' \left( 3N \left( x_l - \frac{\mu_l}{N} \right) \right) \right| 3N
\leq cN,
\]

\[
\left| \frac{\partial}{\partial t} \phi_\mu(x) \right| = \left| \psi \left( 3N \left( t - \frac{\mu_0}{N} \right) \right) \prod_{i=1}^{d} \psi \left( 3N \left( x_i - \frac{\mu_i}{N} \right) \right) \right| \left| \psi' \left( 3N \left( t - \frac{\mu_0}{N} \right) \right) \right| 3N
\leq cN,
\]

\[
\left| \frac{\partial}{\partial x_l} \frac{\partial}{\partial t} \phi_\mu(x) \right| = \left| \psi \left( 3N \left( t - \frac{\mu_0}{N} \right) \right) \prod_{i=1, i \neq l}^{d} \psi \left( 3N \left( x_i - \frac{\mu_i}{N} \right) \right) \right|
\times \left| \psi' \left( 3N \left( t - \frac{\mu_0}{N} \right) \right) \right| 3N \left| \psi' \left( 3N \left( x_l - \frac{\mu_l}{N} \right) \right) \right| 3N
\leq (cN)^2,
\]

where \( c > \frac{27}{2} \) is a suitable constant. It follows that \( \| \phi_\mu \|_{W^{n,\infty}_{t,x}} \leq (cN)^{n+k} \).

The proof of (v), is given by constructing a network \( \Phi_\psi \) that realizes the function \( \psi \). Thus,
let

\[
A_1 := \frac{3}{2} \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix},
\quad b_1 := \frac{1}{2} \begin{bmatrix}
6 \\
3 \\
2 \\
-3 \\
-6
\end{bmatrix},
\quad A_2 := \frac{2}{3} \begin{bmatrix}
0 & -2 & 2 & -1 & -1 & 2 & -2 & 1
\end{bmatrix},
\quad b_2 := 0,
\]

and \( \Phi_\psi := ((A_1, b_1), (A_2, b_2)). \) Then \( \Phi_\psi \) is a two-layer network with one-dimensional input and one-dimensional output, with 24 nonzero weights and 10 neurons such that

\[
R_{\rho_3}(\Phi_\psi)(x) = \psi(x) \quad \text{for all} \quad x \in \mathbb{R}.
\]

The remaining part of the proof is similar to \cite{[19]} Lemma C.3(v)]. The details are left to the reader.

Next we show that any function in the space \( W^{m,p}_{m,p}((0, 1), (0, 1)^d) \), can be approximated by a sum of localized polynomial of degree at most \( m-1 \). This makes Lemma \ref{lemma:4.2} one of the main ingredients in our strategy to prove the main result in Theorem \ref{thm:4.1}.

**Lemma 4.2.** Let \( d, N \in \mathbb{N} \), \( n, k \in \{0, 1\} \), and \( m \in \mathbb{N} \) such that \( m \geq n+k+1, 1 \leq p, q \leq \infty \) and \( \Psi = \Psi(d+1, N) = \{ \phi_\mu : \mu \in \{0, \ldots, N\}^{d+1} \} \) be the partition of unity from Lemma \ref{lemma:4.1}.

Then there is a constant \( C = C(d+1, m, p) > 0 \) such that for any \( u \in W^{m,p}_{m,p}((0, 1), (0, 1)^d) \), there exist polynomials \( p_{u,\mu}(t, x) = \sum_{|\alpha|+\kappa \leq m-1} c_{\mu, \kappa, \alpha} t^\kappa x^\alpha \) for \( \mu \in \{0, \ldots, d\}^{d+1} \) with the following properties:

Let \( u_N := \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_{u,\mu} \), then the operator

\[
T_N : W^{m,p}_{m,p}((0, 1), (0, 1)^d) \rightarrow W^{m,p}_{m,p}((0, 1), (0, 1)^d)
\]

with \( T_N u = u - u_N \) is linear and bounded with

\[
\|T_N u\|_{W^{m,p}_{m,p}((0, 1), (0, 1)^d)} \leq C \left( \frac{1}{N} \right)^{m-n-1} \|u\|_{W^{m,p}_{m,p}((0, 1), (0, 1)^d)}.
\]

Furthermore, there is a constant \( c = c(m, d+1) > 0 \) such that for any \( u \in W^{m,p}_{m,p}((0, 1), (0, 1)^d) \) the coefficients of the polynomials \( p_{u,\mu} \) satisfy

\[
|c_{\mu, \kappa, \alpha}| \leq cN^{(d+1)/p} \|U\|_{W^{m,p}_{m,p}(\Omega_{m,N})}
\]

for all \( \kappa \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^d \) with \( |\alpha| + \kappa \leq m - 1 \) and \( \mu \in \{0, \ldots, N\}^{d+1} \), where \( \Omega_{m,N} := B_\mathbb{R}^d \|\cdot\|_{\infty} \left( \frac{1}{N} \right) \) and \( U \in W^{m,p}_{m,p}(\mathbb{R}, \mathbb{R}^d) \) is an extension of \( u \).

**Proof.** We proceed in a similar way as the proof of \cite{[19]} Theorem 1. Thus, we need the notion of the extension operator on the anisotropic Sobolev space cf. \cite{[30]}, which is a generalization of Stein theorem for the extension operator to anisotropic Sobolev spaces. That is, we can define the operator \( E : W^{m,p}_{k,q}((0, 1), (0, 1)^d) \rightarrow W^{m,p}_{k,q}(\mathbb{R}, \mathbb{R}^d) \) as the extension operator.
Moreover, we use approximation properties of averaged Taylor polynomials (see Bramble-Hilbert Lemma [2.1]) to derive local estimates and then combine them using a partition of unity to obtain a global estimate. Following similar approach as in [19]

**Step 1 (Averaged Taylor polynomials):** Let $U := Eu$ and $\mu \in \{0, \ldots , N\}^{d+1}$, we set

$$\Omega_{\mu,N} := B_{\mu,N}^{\mu} \left( \frac{\mu}{N} \right) \quad \text{and} \quad B_{\mu,N} := B_{\mu,N}^{\mu} \left( \frac{\mu}{N} \right),$$

and $p_\mu = p_{n,\mu}$ stands for the Taylor polynomial of order $m$ of $U$ averaged over $B_{\mu,N}$ (cf. Definition [2.3]). It follows from Lemma 2.2 that we can write $p_\mu = \sum |\alpha|+\kappa \leq m-1 c_{\mu,\kappa,\alpha} \frac{d\kappa}{\alpha}$, such that for $c' = c'(m, d + 1) > 0$, we have

$$|c_{\mu,\kappa,\alpha}| \leq c' \left( \frac{3}{4N} \right)^{-(d+1)/p} \|U\|_{W^m_{m,p} (\Omega_{\mu,N})} \leq c'' \sqrt{N}^{(d+1)/p} \|U\|_{W^m_{m,p} (\Omega_{\mu,N})},$$

where $c''$ is a nonnegative constant depends on $m$ and $d + 1$.

**Step 2 (Local estimates in $\|\|_{W^m_{m,p}, k, n \in \{0, 1\}}$: To check that the conditions of the Bramble-Hilbert Lemma [2.1] are fulfilled, note that $B_{\mu,N} \subset \subset \Omega_{\mu,N}$. Furthermore, $B_{\mu,N}$ is a ball in $\Omega_{\mu,N}$ such that $\Omega_{\mu,N}$ is star-shaped with respect to $B_{\mu,N}$. Moreover, $\text{diam}_{\mu,N} = \left( S \sqrt{d+1} \right)$ where $S = 2/N$, $r_{\mu,N}^\ast (\Omega_{\mu,N}) = 1/N$ and, $r_{\mu,N}^\ast (B_{\mu,N}) > \frac{1}{2} r_{\mu,N}^\ast (\Omega_{\mu,N})$. Finally, the chunkiness parameter of $\Omega_{\mu,N}

$$\gamma(\Omega_{\mu,N}) = \text{diam}(\Omega_{\mu,N}) \cdot \frac{1}{r_{\mu,N}^\ast (\Omega_{\mu,N})} = \frac{2 \sqrt{d+1}}{N} \cdot N = 2 \sqrt{d+1}. \quad (4.3)$$

Applying the Bramble-Hilbert Lemma [2.1] yields for each $\mu \in \{0, \ldots , N\}^{d+1}$ the local estimate

$$\|U - p_\mu\|_{L^p_\mu L^p_\mu (\Omega_{\mu,N})} \leq C \left( \frac{2 \sqrt{d+1}}{N} \right)^m \|U\|_{W^m_{m,p} (\Omega_{\mu,N})} \leq \tilde{C} \left( \frac{1}{N} \right)^m \|U\|_{W^m_{m,p} (\Omega_{\mu,N})}, \quad (4.4)$$

where $C$ depends on $m$ and $d$ (see Lemma [2.1]), since the chunkiness parameter of $\Omega_{\mu,N}$ is a constant depending only on $d$ (see [4.3]) and $\tilde{C} = \tilde{C}(m, d) > 0$. Similarly, we get

$$|U - p_\mu|_{W^{1,p}_{0,\mu} (\Omega_{\mu,N})} \leq c_1 \left( \frac{1}{N} \right)^{m-1} \|U\|_{W^m_{m,p} (\Omega_{\mu,N})},$$

$$|U - p_\mu|_{W^{0,p}_{1,\mu} (\Omega_{\mu,N})} \leq c_2 \left( \frac{1}{N} \right)^{m-1} \|U\|_{W^m_{m,p} (\Omega_{\mu,N})}, \quad (4.5)$$

$$|U - p_\mu|_{W^{1,p}_{1,\mu} (\Omega_{\mu,N})} \leq c_3 \left( \frac{1}{N} \right)^{m-2} \|U\|_{W^m_{m,p} (\Omega_{\mu,N})},$$

where $c_i$ is a suitable constant depends on $m$ and $d$ for $i \in \{1, 2, 3\}$.

Combining Lemma 4.1 inequalities (4.4) and (4.5) using the cut-off functions from the partition of unity, we get

$$\|\phi_\mu (U - p_\mu)\|_{L^p_\mu L^p_\mu (\Omega_{\mu,N})} \leq \|\phi_\mu\|_{L^p_\mu L^p_\mu (\Omega_{\mu,N})} \cdot \|U - p_\mu\|_{L^p_\mu L^p_\mu (\Omega_{\mu,N})} \leq \tilde{C} \left( \frac{1}{N} \right)^m \|U\|_{W^m_{m,p} (\Omega_{\mu,N})}. \quad (4.6)$$
Next we use the product inequality for weak derivatives from Lemma 2.3. Under this consideration, there are constants $C_1, C_2$ depend on $d$ and $p$ such that

$$|\phi_\mu (U - p_\mu)|_{W_0^{1,p} (\Omega , N)} \leq C_1 \left( |\phi_\mu|_{W_0^{1,\infty} (\Omega , N)} \|U - p_\mu\|_{L^1_t L^p_x (\Omega , N)} + |\phi_\mu|_{W_0^{1,\infty} (\Omega , N)} \|U - p_\mu\|_{W_0^{1,p} (\Omega , N)} \right)$$

$$\leq C_1 \cdot c N \cdot \tilde{C} \left( \frac{1}{N} \right)^m m \|U\|_{W^{m,p}_{m,p} (\Omega , N)} + C_1 \cdot c_1 \left( \frac{1}{N} \right)^{m-1} m \|U\|_{W^{m,p}_{m,p} (\Omega , N)}$$

$$\leq c_4 \left( \frac{1}{N} \right)^{m-1} m \|U\|_{W^{m,p}_{m,p} (\Omega , N)}$$

(4.7)

where $c_4 = c_4(m, d, p) > 0$, such that the first part of the second inequality follows from Lemma 4.1(iv), with (4.4) and the second part from Lemma 4.1(iv), together with (4.5).

In a similar way we get the following results

$$|\phi_\mu (U - p_\mu)|_{W^{0,p}_{1,q} (\Omega , N)} \leq |\phi_\mu|_{W_0^{1,\infty} (\Omega , N)} \|U - p_\mu\|_{L^1_t L^p_x (\Omega , N)} + |\phi_\mu|_{W_0^{1,\infty} (\Omega , N)} \|U - p_\mu\|_{W_0^{1,p} (\Omega , N)}$$

$$\leq c N \cdot \tilde{C} \left( \frac{1}{N} \right)^m m \|U\|_{W^{m,p}_{m,p} (\Omega , N)} + c_2 \left( \frac{1}{N} \right)^{m-1} m \|U\|_{W^{m,p}_{m,p} (\Omega , N)}$$

$$\leq c_5 \left( \frac{1}{N} \right)^{m-1} m \|U\|_{W^{m,p}_{m,p} (\Omega , N)}$$

(4.8)

Now it easily follows from (4.6), (4.7), (4.8) and (4.9) that

$$\|\phi_\mu (U - p_\mu)\|_{W^{1,p}_{1,q} (\Omega , N)} \leq C \left( \frac{1}{N} \right)^{m-2} m \|U\|_{W^{m,p}_{m,p} (\Omega , N)}$$

(4.9)

for some constant $C = C(m, d, p) > 0$.

**Step 3 (Global estimate in $||\cdot||_{W^{n,p}_{k,q}}$, $k, n \in \{0, 1\}$):** To derive the global estimate, we start by noting that with property (ii) from Lemma 4.1 we have

$$U(t, x) = \sum_{\mu \in \{0, \ldots , N\}^{d+1}} \phi_\mu (t, x) U(t, x), \text{ for a.e. } (t, x) \in (0, 1) \times (0, 1)^d.$$  

(4.11)
Let \( k, n \in \{0, 1\} \), we have

\[
\|u - \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_\mu \|_{W_{k,q}^n((0,1),(0,1)^d)} = \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu (U - p_\mu) \|_{W_{k,q}^n((0,1),(0,1)^d)} \\
\leq \sum_{\tilde{\mu} \in \{0, \ldots, N\}^{d+1}} \| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu (U - p_\mu) \|_{W_{k,q}^n(\Omega_{\tilde{\mu}})} \tag{4.12}
\]

where in the first step we use the fact that \( U \) is an extension of \( u \) on \((0,1) \times (0,1)^d\), the last step follows from the fact that \((0,1) \times (0,1)^d \subset \bigcup_{\tilde{\mu} \in \{0, \ldots, N\}^{d+1}} \Omega_{\tilde{\mu}}\). Consequently, for each \( \tilde{\mu} \in \{0, \ldots, N\}^{d+1} \), we get

\[
\| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu (U - p_\mu) \|_{W_{k,q}^n(\Omega_{\tilde{\mu}})} \leq \sum_{\mu \in \{0, \ldots, N\}^{d+1}, \|\mu - \tilde{\mu}\|_\infty \leq 1} \| \phi_\mu (U - p_\mu) \|_{W_{k,q}^n(\Omega_{\mu})} \\
\leq \sum_{\mu \in \{0, \ldots, N\}^{d+1}, \|\mu - \tilde{\mu}\|_\infty \leq 1} \| \phi_\mu (U - p_\mu) \|_{W_{k,q}^n(\Omega_{\mu})} \\
\leq C \left( \frac{1}{N} \right)^{m-n-k} \sum_{\mu \in \{0, \ldots, N\}^{d+1}, \|\mu - \tilde{\mu}\|_\infty \leq 1} \| U \|_{W_{m,p}^n(\Omega_{\mu})} \tag{4.13}
\]

where first and second steps follow from the support property \([\text{iii}]\) from Lemma 4.1, third step follows from \((4.6), (4.7), (4.8)\) and from \((4.9)\) for \((k = n = 0), (k = 0, n = 1), (k = 1, n = 0),\) and for \((k = n = 1)\) respectively. Here \( C > 0 \) depends on \(m, d\) and \(p\).

Using the fact that \( u_N := \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_\mu \), \((4.12)\) and with Equation \((4.13)\), we get the following bound

\[
\|u - u_N\|_{W_{k,q}^n((0,1),(0,1)^d)} \leq \sum_{\tilde{\mu} \in \{0, \ldots, N\}^d} C \left( \frac{1}{N} \right)^{m-n-k} \sum_{\mu \in \{0, \ldots, N\}^{d+1}, \|\mu - \tilde{\mu}\|_\infty \leq 1} \| U \|_{W_{m,p}^n(\Omega_{\mu})} \\
\leq C \left( \frac{1}{N} \right)^{m-n-k} \sum_{\mu \in \{0, \ldots, N\}^{d+1}, \|\mu - \tilde{\mu}\|_\infty \leq 1} \| U \|_{W_{m,p}^n(\Omega_{\tilde{\mu}})} \\
\leq C \left( \frac{1}{N} \right)^{m-n-k} 3^d \sum_{\tilde{\mu} \in \{0, \ldots, N\}^d} \| U \|_{W_{m,p}^n(\Omega_{\tilde{\mu}})} \\
\leq C \left( \frac{1}{N} \right)^{m-n-k} 3^d 2^d \| U \|_{W_{m,p}^n(\bigcup_{\tilde{\mu} \in \{0, \ldots, N\}^{d+1}} \Omega_{\tilde{\mu}})}
\]

where the last two steps follow from the definition of \( \Omega_{\tilde{\mu}} \). Thus, we have

\[
\|u - u_N\|_{W_{k,q}^n((0,1),(0,1)^d)} \leq C_7 \left( \frac{1}{N} \right)^{m-n-k} \| U \|_{W_{m,p}^n(\Omega_{\tilde{\mu}})} \leq \tilde{C}_7 \left( \frac{1}{N} \right)^{m-n-k} \| U \|_{W_{m,p}^n(\Omega_{\tilde{\mu}})}
\]

for \( k, n \in \{0, 1\} \), where the extension operator continuity was used in the first and second step. Here \( C_7 \) and \( \tilde{C}_7 \) are positive constants depend on \(m, d\) and \(p\). \(\square\)
Remark 4.1. The function \( f(x) = x^2 \) can be represented by ReCU neural network in a compact interval. Indeed let \( r > 0 \),
\[
A_1 := \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b_1 := r \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad A_2 := \frac{1}{6r} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \quad b_2 := -\frac{r^2}{3}
\]
and \( \Phi_{x,r} := ((A_1, b_1), (A_2, b_2)) \). Then \( \Phi_{x,r} \) is a two-layer network with one-dimensional input and one-dimensional output, with 7 nonzero weights and 4 neurons such that
\[
R_{\rho_3}(\Phi_{x,r})(x) = x^2 \quad \text{for any } x \in [-r, r].
\]

Remark 4.2. The product \( tx \) can be represented by two-layer ReCU network with two-dimensional input and one-dimensional output, 16 nonzero weights and 7 neurons. Indeed, let \( \times_r = ((A_1, b_1), (A_2, b_2)) \) where
\[
A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b_1 = 2r \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad A_2 = \frac{1}{48r} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad b_2 = 0.
\]

Hence, \( R_{\rho_3}(\times_r)(t, x) = tx \) such that \( t, x \in [-r, r] \) and \( r > 0 \). Moreover, if \( n, k \in \{0, 1\} \), then
\[
|R_{\rho_3}(\times_r)|_{W_{l,\infty}^{n,\infty}((-r,r),(-r,r))} = r^{2-k-n}.
\]

Using (\ref{remark41}) from Lemma 4.1, a localized (mixed) monomial \( \phi_\mu(t, x) t^\kappa x^\alpha \) can be expressed by the product of the output components of a network \( \Phi_{(\mu,\alpha,n)} \) as follows:
\[
\phi_\mu(t, x) t^\kappa x^\alpha = \prod_{l=1}^m [R_{\rho_3}(\Phi_{(\mu,\alpha,n)})]_l(t, x).
\]

In the following lemma we show that the localized monomials (\ref{remark41}) can be approximated by ReCU neural networks, using the fact that \( R_{\rho_3}(\Phi_{x,1}) = x^2 \) on \( (0, 1) \) cf. Remark 4.1.

Lemma 4.3. Let \( d, \mu, K \in \mathbb{N} \) and \( N \geq 1 \) be arbitrary. Then there is a constant \( C = C(\mu) > 0 \) such that the following holds:

For any \( \epsilon \in (0, 1/2) \), and any neural network \( \Phi \) with \( (d+1) \)-dimensional input and \( m \)-dimensional output where \( m \leq \mu \), and with number of layers, neurons and weights all bounded by \( K \), such that
\[
\|[R_{\rho_3}(\Phi)]_l\|_{W_{k,\infty}^{n,\infty}((0,1),(0,1)^d)} \leq N^{k+n} \quad \text{for} \quad n, k \in \{0, 1\} \quad \text{and} \quad l = 1, \ldots, m
\]
there exists a neural network \( \Psi_{\epsilon,\Phi} \) with \( (d+1) \)-dimensional input and one-dimensional output, and with number of layers, neurons and weights all bounded by \( CK \), such that
\[
\|R_{\rho_3}(\Psi_{\epsilon,\Phi}) - \prod_{l=1}^m [R_{\rho_3}(\Phi)]_l\|_{W_{k,\infty}^{n,\infty}((0,1),(0,1)^d)} \leq cN^{k+n} \epsilon
\]
for \( n, k \in \{0, 1\} \) and some constant \( c = c(d+1, \mu, k, n) \). Moreover, for \( t \in (0, 1), x \in (0, 1)^d \), we have
\[
R_{\rho_3}(\Psi_{\epsilon,\Phi})(t, x) = 0 \quad \text{if} \quad \prod_{l=1}^m [R_{\rho_3}(\Phi)]_l(t, x) = 0.
\]
Proof. We show the proof by induction over $\mu \in \mathbb{N}$. Moreover, we will make sure that the constant $c$ in (4.14) can be written as $c = \mu^{2-k-n}c_1^{k+n}$, where $c_1$ depends on the dimension $d+1$ and $\mu$. Furthermore, we show that the first $L(\Phi) - 1$ layers of $\Psi_{\epsilon, \Phi}$ and $\Phi$ coincide, and that

$$|R_{\rho_3}(\Psi_{\epsilon, \Phi})|_{W^{n,\infty}_{k,\infty}((0,1),(0,1)^d)} \leq C_1 N^{n+k},$$

(4.18)

where $n, k \in \{0, 1\}$, such that $n + k = 1$ or $n = k = 1$, $N \geq 1$ and $C_1$ depends on $d + 1$ and $\mu$. The first case in the induction is fulfilled obviously when $\mu = 1$ we can choose $\Psi_{\epsilon, \Phi} = \Phi$ and the claim holds for any $\epsilon \in (0, 1/2)$.

Now we show the second case of the induction, that is, let the claim holds for constants $\tilde{c}_0, \tilde{c}_1$ and $\tilde{\epsilon}$ depending on $\mu + 1$, possibly different from the constants $c_0, c_1$ and $C_1$ from Step 1 respectively.

We denote by $\Phi_{\mu}$ the neural network with $d + 1$-dimensional input and $\mu$-dimensional output which results from $\Phi$ by removing the last output neuron and corresponding weights. In detail, we write

$$A_L = \begin{bmatrix} A_L^{(1,\mu)} \\ a_L^{(\mu+1)} \end{bmatrix} \quad \text{and} \quad b_L = \begin{bmatrix} b_L^{(1,\mu)} \\ b_L^{(\mu+1)} \end{bmatrix},$$

where $A_L^{(1,\mu)}$ is a $\mu \times \sum_{k=1}^{L-1} N_k$ matrix and $a_L^{(\mu+1)}$ is a $1 \times \sum_{k=1}^{L-1} N_k$ vector, and $b_L^{(1,\mu)} \in \mathbb{R}^\mu$ and $b_L^{(\mu+1)} \in \mathbb{R}$. Now we set

$$\Phi_{\mu} := ((A_1, b_1), (A_2, b_2), \ldots, (A_{L-1}, b_{L-1}), (A_L^{(1,\mu)}, b_L^{(1,\mu)})).$$

Using the induction hypothesis and the constants $c_0, c_1$ and $C_1$ from Step 1 we get that there is a neural network $\Psi_{\epsilon, \Phi_{\mu}} = ((A_1', b_1'), (A_2', b_2'), \ldots, (A_{L}', b_{L}'))$ with $d + 1$-dimensional
input and one-dimensional output, and at most $Kc_0$ layers, neurons and weights such that

$$\|R_{\rho_3}(\Psi_{\epsilon,\Phi}) - \prod_{l=1}^{\mu} [R_{\rho_3}(\Phi_{\mu})]t\|_{W_{l_1}^{n,\infty}((0,1),(0,1)^d)} \leq \mu^{2-k-n} c_1^{k+n} N^{k+n}\epsilon$$  \hspace{1cm} (4.19)

for $n, k \in \{0, 1\}$. Moreover,

$$R_{\rho_3}(\Psi_{\epsilon,\Phi}) (t, x) = 0 \text{ if } \prod_{l=1}^{\mu} [R_{\rho_3}(\Phi_{\mu})]t(t, x) = 0,$$  \hspace{1cm} (4.20)

for any $t \in (0, 1)$ and $x \in (0, 1)^d$. Furthermore, we can assume that $\|R_{\rho_3}(\Psi_{\epsilon,\Phi})|_{W_{l_1}^{1,\infty}((0,1),(0,1)^d)} \leq C_1 N^2$, and that the first $L(\Phi) - 1$ layers of $\Psi_{\epsilon,\Phi}$ and $\Phi_{\mu}$ coincide and, thus, also the first $L(\Phi) - 1$ layers of $\Psi_{\epsilon,\Phi}$ and $\Phi$, i.e. $A_l = A_{l'}$ for $l = 1, \ldots, L(\Phi) - 1$.

Now, we add the formerly removed neuron with corresponding weights back to the last layer of $\Psi_{\epsilon,\Phi}$. For the resulting network

$$\tilde{\Psi}_{\epsilon,\Phi} := \left( (A_1', b_1'), (A_2', b_2'), \ldots, (A_{L'}', b_{L'}'), \begin{bmatrix} A_{L'}' \mu+1 \\ a_{L'} \end{bmatrix} \begin{bmatrix} 0 \\ \sum_{l=1}^{L} A_l' \\ b_{L'}' \mu+1 \end{bmatrix} \right)$$

it holds that the first $L - 1$ layers of $\tilde{\Psi}_{\epsilon,\Phi}$ and $\Phi$ coincide, and $\tilde{\Psi}_{\epsilon,\Phi}$ is a neural network with two-dimensional output. Note that

$$\| \left[ R_{\rho_3}(\tilde{\Psi}_{\epsilon,\Phi}) \right]_1 \|_{L^\infty L^\infty((0,1),(0,1)^d)} = \| R_{\rho_3}(\Psi_{\epsilon,\Phi}) \|_{L^\infty L^\infty((0,1),(0,1)^d)}$$

$$\leq \| R_{\rho_3}(\Psi_{\epsilon,\Phi}) - \prod_{l=1}^{\mu} [R_{\rho_3}(\Phi_{\mu})]t\|_{L^\infty L^\infty((0,1),(0,1)^d)} + \| \prod_{l=1}^{\mu} [R_{\rho_3}(\Phi_{\mu})]t\|_{L^\infty L^\infty((0,1),(0,1)^d)}$$

$$\leq \mu^2 \epsilon + 1 < \mu^2 + 1,$$

where we used \((4.19)\) for $n = k = 0$, \((4.15)\) and the properties of the partition of unity. Additionally, we have

$$\| \left[ R_{\rho_3}(\tilde{\Psi}_{\epsilon,\Phi}) \right]_2 \|_{L^\infty L^\infty((0,1),(0,1)^d)} = \| R_{\rho_3}(\tilde{\Phi}) \|_{L^\infty L^\infty((0,1),(0,1)^d)} \leq 1.$$

Now, we denote by $\times_r$ the network from Remark \[4.2\] with $r = \mu^2 + 1$ such that for any $\epsilon \in (0, 1/2)$, we have

$$\| R_{\rho_3}(\times_r)(t, x) - t x \|_{W_1^{1,\infty}((-\mu^2 + 1, \mu^2 + 1), (-\mu^2 + 1, \mu^2 + 1))} < \epsilon$$

previous estimate holds true since, in Remark \[4.2\], $R_{\rho_3}(\times_r)$ present exactly the product $tx$ in $W_1^{1,\infty}$. Moreover, we define

$$\Psi_{\epsilon,\Phi} := \times_r \cdot \tilde{\Psi}_{\epsilon,\Phi}.$$  

Consequently, combining the induction hypothesis with Remark \[4.2\] and Remark \[3.1\], $\Psi_{\epsilon,\Phi}$ has $d + 1$-dimensional input, one-dimensional output and at most $K' + Kc_0 + K'Kc_0 \leq KC$ layers, number of neurons and weights, where $K' = 16$ is the constant from Remark \[4.2\] and $C = C(\mu) > 0$ is a suitable constant. Moreover, the first $L - 1$ layers of $\Psi_{\epsilon,\Phi}$ and $\Phi$ coincide.
and for \( n, k \in \{0, 1\} \) the following approximation holds

\[
\left| R_{p_3}(\Psi_{e, \Phi}) - \prod_{l=1}^{\mu+1} [R_{p_3}(\Phi)]_l \right|_{W^{n,\infty}_k((0,1),(0,1)^d)} 
= \left| R_{p_3}(\times_r) \circ R_{p_3}(\hat{\Psi}_{e, \Phi}) - [R_{p_3}(\Phi)]_{\mu+1} \cdot \prod_{l=1}^{\mu} [R_{p_3}(\Phi)]_l \right|_{W^{n,\infty}_k((0,1),(0,1)^d)} 
\leq \left| R_{p_3}(\times_r) \circ (R_{p_3}(\Psi_{e, \Phi}), [R_{p_3}(\Phi)]_{\mu+1}) - R_{p_3}(\Psi_{e, \Phi}) \cdot [R_{p_3}(\Phi)]_{\mu+1} \right|_{W^{n,\infty}_k((0,1),(0,1)^d)} 
+ \left| [R_{p_3}(\Phi)]_{\mu+1} \cdot \left( R_{p_3}(\Psi_{e, \Phi}) - \prod_{l=1}^{\mu} [R_{p_3}(\Phi)]_l \right) \right|_{W^{n,\infty}_k((0,1),(0,1)^d)}.
\] (4.21)

Let \( n = k = 0 \), then, for the first term in inequality (4.21), using Remark 4.2 we obtain

\[
\| R_{p_3}(\times_r) \circ (R_{p_3}(\Psi_{e, \Phi}), [R_{p_3}(\Phi)]_{\mu+1}) - R_{p_3}(\Psi_{e, \Phi}) \cdot [R_{p_3}(\Phi)]_{\mu+1} \|_{L^p_\infty L^p_\infty((0,1),(0,1)^d)} 
\leq \| R_{p_3}(\times_r)(t, x) - t \cdot x \|_{L^p_\infty L^p_\infty((-(\mu^2+1), \mu^2+1),(-(\mu^2+1), \mu^2+1))} \leq \epsilon.
\] (4.22)

Next, for \( n, k \in \{0, 1\} \) such that \( n + k = 1 \) and apply the chain rule from Lemma 2.4 to (4.22). For this, let \( \hat{C} = \hat{C}(d+1) \) be the constant from Lemma 2.4 (for \( p = d+1 \) and \( m = 2 \)). Using the induction hypothesis together with the fact that \( \| [R_{p_3}(\Phi)]_{\mu+1} \|_{W^{n,\infty}_k((0,1)^d)} \leq N^{n+k} \), we get

\[
\left| R_{p_3}(\times_r) \circ (R_{p_3}(\Psi_{e, \Phi}), [R_{p_3}(\Phi)]_{\mu+1}) - R_{p_3}(\Psi_{e, \Phi}) \cdot [R_{p_3}(\Phi)]_{\mu+1} \right|_{W^{n,\infty}_k((0,1),(0,1)^d)} 
\leq \hat{C} \cdot \max\{C_1, N, N\} = \hat{C} \epsilon N,
\] (4.24)

where \( C_1 = C_1'(d+1) > 0 \). Now, in similar way, we treat the case where \( n = k = 1 \) for the same quantity in the previous inequality. In view of the second result of Lemma 2.4 for some constant \( C' = C''(d+1) > 0 \), we get

\[
\left| R_{p_3}(\times_r) \circ (R_{p_3}(\Psi_{e, \Phi}), [R_{p_3}(\Phi)]_{\mu+1}) - R_{p_3}(\Psi_{e, \Phi}) \cdot [R_{p_3}(\Phi)]_{\mu+1} \right|_{W^{1,\infty}_k((0,1),(0,1)^d)} 
\leq C' \cdot \max\left( \left| R_{p_3}(\times_r)(t, x) - t \cdot x \right|_{W^{1,\infty}_k((-(\mu^2+1), \mu^2+1),(-(\mu^2+1), \mu^2+1))} \right|_{W^{1,\infty}_k((0,1),(0,1)^d)} \right),
\]

\[
\left| R_{p_3}(\times_r)(t, x) - t \cdot x \right|_{W^{1,\infty}_k((-(\mu^2+1), \mu^2+1),(-(\mu^2+1), \mu^2+1))} \right|_{W^{1,\infty}_k((0,1),(0,1)^d)} \right),
\]

\[
\leq C' \cdot \max\{C_1, N^2, N^2\} + \epsilon \max\{C_2, N^2, N^2\} = C_2 N^2 \epsilon,
\] (4.25)

where \( C_2 = C_2'(d+1) > 0 \).

It remains to estimate the second term of (4.21) for \( n = k = 0 \). Thus, under the induction hypothesis (for \( n = k = 0 \)) and get

\[
\| [R_{p_3}(\Phi)]_{\mu+1} \cdot \left( R_{p_3}(\Psi_{e, \Phi}) - \prod_{l=1}^{\mu} [R_{p_3}(\Phi)]_l \right) \|_{L^\infty((0,1)^d)} 
\leq \| [R_{p_3}(\Phi)]_{\mu+1} \|_{L^\infty((0,1)^d)} \cdot \| R_{p_3}(\Psi_{e, \Phi}) - \prod_{l=1}^{\mu} [R_{p_3}(\Phi)]_l \|_{L^\infty((0,1)^d)} \leq 1 \cdot \mu^2 \epsilon.
\] (4.26)
where we used the induction hypothesis for $n + k = 1$ can be obtained by applying the product rule from Lemma 2.3 together with $\| [R_{p_3}(\Phi)]_{\mu+1} \|_{L^\infty} \leq 1$, indeed

$$
\left[ R_{p_3}(\Phi) \right]_{\mu+1} \cdot \left( R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right)
\leq \left[ R_{p_3}(\Phi) \right]_{\mu+1} \left[ W_{1,\infty}^{0,\infty,\infty}((0,1),(0,1)^d) \right] \cdot \left[ R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right]
+ \left[ R_{p_3}(\Phi) \right]_{\mu+1} \left[ L^\infty((0,1),(0,1)^d) \right] \cdot \left[ R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right]
\leq N \cdot \mu^2 \epsilon + 1 \cdot \mu c_1 N \epsilon = c_1' N \epsilon,
$$

(4.27)

where we used the induction hypothesis for $k + n = 1$, and $c_1' = c_1'(d + 1, \mu) > 0$.

The last case is $n = k = 1$ can be concluded, in a similar way as the previous case, by applying the product rule from Lemma 2.3 together with $\| [R_{p_3}(\Phi)]_{\mu+1} \|_{L^\infty} \leq 1$, indeed

$$
\left[ R_{p_3}(\Phi) \right]_{\mu+1} \cdot \left( R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right)
\leq \left[ R_{p_3}(\Phi) \right]_{\mu+1} \left[ W_{1,\infty}^{1,\infty}((0,1),(0,1)^d) \right] \cdot \left[ R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right]
+ \left[ R_{p_3}(\Phi) \right]_{\mu+1} \left[ W_{1,\infty}^{0,\infty}((0,1),(0,1)^d) \right] \cdot \left[ R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right]
+ \left[ R_{p_3}(\Phi) \right]_{\mu+1} \left[ L^\infty((0,1),(0,1)^d) \right] \cdot \left[ R_{p_3}(\Psi_{\epsilon,\Phi}) - \mu \prod_{l=1}^{\mu} \left[ R_{p_3}(\Phi) \right]_l \right]
\leq N^2 \cdot \mu^2 \epsilon + N \cdot \mu c_1 N \epsilon + N \cdot \mu c_1 N \epsilon + 1 \cdot c_1^2 N^2 \epsilon = c_2' N^2 \epsilon,
$$

(4.28)

where we used the induction hypothesis for $k + n = 1$, $n = k = 1$, and $c_2' = c_2'(d + 1, \mu) > 0$.

Then, a combination of (4.21) with (4.23), (4.26) and (4.26) yields

$$
\| R_{p_3}(\Phi_{\epsilon}) - \mu \prod_{l=1}^{\mu+1} \left[ R_{p_3}(\Phi) \right]_l \|_{L^\infty((0,1),(0,1)^d)} \leq \epsilon + \mu^2 \cdot \epsilon = (\mu^2 + 1) \cdot \epsilon.
$$

(4.29)

Similarly a combination of (4.21) with (4.24) and (4.27), for $n, k \in \{0, 1\}$ such that $n + k = 1$, we get

$$
\left| R_{p_3}(\Phi_{\epsilon}) - \mu \prod_{l=1}^{\mu+1} \left[ R_{p_3}(\Phi) \right]_l \right| \leq (C_1' + c_1') \cdot N \cdot \epsilon = c_1'' N \epsilon,
$$

where $c_1'' = c_1'(d + 1, \mu) > 0$. 

19
Moreover, for the case where \( n = k = 1 \) we combine (4.21) with (4.25) and (4.28), we get
\[
\left| R_{\rho_3}(\Psi, \phi) - \sum_{l=1}^{\mu+1} [R_{\rho_3}(\Phi)]_l \right|_{W^{1,\infty}_{k,\infty}((0,1),(0,1)^d)} \leq (C_2' + c_2') \cdot N^2 \cdot \epsilon = c_2'' N^2 \epsilon,
\]
where \( c_2'' = c_2'(d + 1, \mu) > 0 \). In view of the three previous estimates, we get
\[
\| R_{\rho_3}(\Psi, \phi) - \sum_{l=1}^{\mu+1} [R_{\rho_3}(\Phi)]_l \|_{W^{n,\infty}_{k,\infty}((0,1),(0,1)^d)} \leq c''_{1} N^{n+k} \epsilon,
\]
for a suitable constant \( c''_{1} = c''_1(d + 1, \mu, k, n) > 0 \).

Finally, we show (4.17) for \( \mu + 1 \), this is follow by similar argument as in [10, Lemma C.5], for the sake of completeness we show it. Thus, let \( [R_{\rho_3}(\Phi)]_l(t, x) = 0 \) for some \( l \in \{1, \ldots, \mu + 1\} \), \( t \in (0, 1) \) and \( x \in (0, 1)^d \). In the case where \( l \leq \mu \), (4.21) implies that
\[
[R_{\rho_3}(\Psi, \phi)]_1(t, x) = R_{\rho_3}(\Psi, \phi)(t, x) = 0.
\]
Moreover, if \( l = \mu + 1 \), then
\[
[R_{\rho_3}(\Psi, \phi)]_2(t, x) = [R_{\rho_3}(\Phi)]_{\mu+1}(t, x) = 0.
\]

Hence, by application of Remark 4.2, we have
\[
R_{\rho_3}(\Psi, \phi)(x) = R_{\rho_3}(\times_r) \left( [R_{\rho_3}(\Psi, \phi)]_1(x), [R_{\rho_3}(\Psi, \phi)]_2(x) \right) = 0.
\]

Before concluding the proof, we need to show (4.18). If \( n + k = 1 \), we use Lemma 2.4 Remark 4.2 and similar argument as in (4.24), then we get
\[
|R_{\rho_3}(\Psi, \phi)|_{W^{n,\infty}_{k,\infty}((0,1),(0,1)^d)} = \left| R_{\rho_3}(\times_r) \circ R_{\rho_3}(\Psi, \phi) \right|_{W^{n,\infty}_{k,\infty}((0,1),(0,1)^d)} \leq \hat{C} \cdot |R_{\rho_3}(\Psi, \phi)|_{W^{n,\infty}_{k,\infty}((-\mu^2+1),\mu^2+1),(-\mu^2+1),\mu^2+1))} \cdot \left| R_{\rho_3}(\Psi, \phi) \right|_{W^{n,\infty}_{k,\infty}((0,1),(0,1)^{d};\mathbb{R})} \leq \hat{C} \cdot (\mu^2 + 1) \cdot \max \{ C_1 N, N \} = C_1'' N,
\]
where \( C_1'' = C_1''(d + 1, \mu) > 0 \) is a suitable constant.

If \( n = k = 1 \), in view of Lemma 2.4 Remark 4.2 and similar argument as in (4.25), we have
\[
|R_{\rho_3}(\Psi, \phi)|_{W^{1,\infty}_{1,\infty}((0,1),(0,1)^d)} = \left| R_{\rho_3}(\times_r) \circ R_{\rho_3}(\Psi, \phi) \right|_{W^{1,\infty}_{1,\infty}((0,1),(0,1)^d)} \leq C' \cdot \max \left( |R_{\rho_3}(\times_r)|_{W^{1,\infty}_{1,\infty}((-\mu^2+1),\mu^2+1),(-\mu^2+1),\mu^2+1))} |R_{\rho_3}(\Psi, \phi)|_{W^{1,\infty}_{1,\infty}((0,1),(0,1)^{d};\mathbb{R})}^2 \right) \leq C' \cdot (\mu^2 + 1) \cdot \max \{ C_1 N^2, N^2 \} \cdot \max \{ C_1^2 N^2, N^2 \} \leq C_2'' N^2,
\]
where \( C_2'' = C_2''(d + 1, \mu) > 0 \) is a suitable constant.

To conclude, we take the maximum of the constants derived in Step 1 and Step 2.
Next result is the final step toward the main theorem of our paper. Mainly, we show an upper bound error in Sobolev time-space for a sum of localized polynomials with deep neural network. Thus, we get approximation and regularity information about the network.

**Lemma 4.4.** Let $d, m, N \in \mathbb{N}, 1 \leq p, q \leq \infty$, such that $m \geq n + k + 1$, $n, k \in \{0, 1\}$ and $\Psi = \{\phi_\mu : \mu \in \{0, \ldots, N\}^{d+1}\}$ be the partition of unity from Lemma 4.1. Then, there are constants $C_1 = C_1(m,d+1, n, k) > 0$ and $C_2 = C_2(m,d+1), C_3 = C_3(m,d+1) > 0$ with the following properties: For any $\epsilon \in (0,1/2)$, there is a neural network architecture $A_i = A_i(d+1, m, N, \epsilon)$ with $d + 1$-dimensional input and one-dimensional output, at most $C_2$ layers and $C_3(N + 1)^{2(d+1)}$ neurons and weights such that the following holds: Let $u \in W_{m,p}^\ast((0,1), (0,1)^d)$ and $p_{\mu,p}(t, x) = \sum_{\kappa+|\alpha| \leq m-1} c_{\mu,\kappa,\alpha} t^\kappa x^\alpha$ for $\mu \in \{0, \ldots, N\}^{d+1}$ be the polynomials from Lemma 4.1 then there is a neural network $\Phi_{\mu,\kappa,\alpha}$ that has architecture $A_i$, such that

$$
\left\| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_{\mu} p_{\mu,\kappa,\alpha} - R_{p_3} (\Phi_{\mu,\kappa,\alpha}) \right\|_{W_{m,p}^{\ast}((0,1), (0,1)^d)} \leq C_1 N^{n+k+1} \epsilon \left\| u \right\|_{W_{m,p}^\ast((0,1), (0,1)^d)} \quad (4.31)
$$

**Proof.** We divide the proof in three steps.

**Step 1 (Approximating localized monomials $\phi_{\mu}(t, x) t^\kappa x^\alpha$):** Let $\kappa + |\alpha| \leq m - 1$ and $\mu \in \{0, \ldots, N\}^{d+1}$. Since we can get $x$ out of the ReLU realization of $\phi_x$ for any $x \in [0, 1]$, where $\phi_x = ((A_1, b_1), (A_2, b_2))$ and

$$
A_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad A_2 = \frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad b_2 = 0.
$$

Here $\phi_x$ is a two layer ReLU network with one-dimensional input and one-dimensional output, 8 nonzero weights and 5 neurons. Then, we can construct a neural network $\Phi_{\mu,\kappa,\alpha}$ with $d + 1$-dimensional input and $\kappa + |\alpha|$-dimensional output, with two layer and at most $8(m - 1)$ nonzero weights and $d + 1 + 4(m - 1)$ neurons such that

$$
t^\kappa x^\alpha = \prod_{l=1}^{\kappa+|\alpha|} [R_{p_3} (\Phi_{\mu,\kappa,\alpha})]_l (t, x) \quad \text{for any } (t, x) \in (0, 1) \times (0, 1)^d
$$

and

$$
\left\| [R_{p_3} (\Phi_{\mu,\kappa,\alpha})]_l \right\|_{W_{m,p}^\ast((0,1), (0,1)^d)} \leq 2 \quad \text{for any } l = 1, \ldots, \kappa + |\alpha| \quad \text{and} \quad n, k \in \{0, 1\}. \quad (4.32)
$$

From Lemma 4.1 we use the neural network $\Phi_{\mu,\kappa,\alpha}$ and the constants $C, c \geq 1$ to define the network $\Phi_{\mu,\kappa,\alpha}$ as the parallelization of $\Phi_{\mu,\kappa,\alpha}$ (see Definition 3.4), that is,

$$
\Phi_{\mu,\kappa,\alpha} := P(\Phi_{\mu,\kappa,\alpha}).
$$

Then $\Phi_{\mu,\kappa,\alpha}$ has at most $3 \leq K_0$ layers, $C(d + 1) + 8(m - 1) \leq K_0$ nonzero weights, and $C(d + 1) + 4(m - 1) \leq K_0$ neurons for a suitable constant $K_0 = K_0(m,d + 1) \in \mathbb{N}$, and

$$
\prod_{l=1}^{\kappa+|\alpha|+d+1} [R_{p_3} (\Phi_{\mu,\kappa,\alpha})]_l (t, x) = \phi_{\mu,\kappa,\alpha}(t, x) t^\kappa x^\alpha \quad \text{for all } (t, x) \in (0, 1) \times (0, 1)^d.
$$

Moreover, as a consequence of Lemma 4.1, together with (4.32), we have

$$
\left\| [R_{p_3} (\Phi_{\mu,\kappa,\alpha})]_l \right\|_{W_{m,p}^\ast((0,1), (0,1)^d)} \leq (cN)^{k+n} \quad \text{for any } l = 1, \ldots, \kappa + |\alpha| + d + 1 \quad \text{and} \quad k, n \in \{0, 1\}.
$$
Let $\Psi, \Phi_{\kappa, \alpha}$ be the neural network from Lemma 4.3 (with $\Phi_{\kappa, \alpha}$ instead of $\Phi$, $\mu = m + d \in \mathbb{N}$, $K = K_0 \in \mathbb{N}$ and $cN$ instead of $N$) for $\mu \in \{0, \ldots, N\}^{d+1}$ and $\alpha \in \mathbb{N}_0^d$, $\kappa + |\alpha| \leq m - 1$. There exists a constant $C_1 = C_1(m, d + 1) \geq 1$ such that $\Psi, \Phi_{\kappa, \alpha}$ has at most $C_1$ layers, number of neurons, and weights. Moreover, for a constant $C = C_1 + 1 + \frac{1}{2}$ and define the matrix $A_{\text{sum}} \in \mathbb{R}^{1 \times M}$ by

$$A_{\text{sum}} := [c_{\mu, \kappa, \alpha} : \mu \in \{0, \ldots, N\}^{d+1}, \alpha \in \mathbb{N}_0^d, \kappa + |\alpha| \leq m - 1].$$

and the neural network $\Phi_{\text{sum}} := ((A_{\text{sum}}, 0))$. Finally, we set

$$\Phi_{P, \epsilon} := \Phi_{\text{sum}} \bullet P (\Psi, \Phi_{\kappa, \alpha} : \mu \in \{0, \ldots, N\}^{d+1}, \alpha \in \mathbb{N}_0^d, \kappa + |\alpha| \leq m - 1).$$

Then, there are constants $C_2 = C_2(m, d + 1), C_3 = C_3(m, d + 1) > 0$ such that $\Phi_{P, \epsilon}$ is a neural network with $d + 1$-dimensional input and one-dimensional output, with at most $1 + C_1 \leq C_2$ layers, $M + 2C_1 \leq 2M^2C_1 \leq C_3(N + 1)^2(d+1)$ nonzero weights and $M + MC_1 \leq 2MC_1 \leq C_3(N + 1)^d(d+1)$ neurons, and

$$R_{p_3} (\Phi_{P, \epsilon}) = \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \sum_{\kappa + |\alpha| \leq m - 1} c_{\mu, \kappa, \alpha} R_{p_3} (\Psi, \Phi_{\kappa, \alpha}).$$

Moreover, the network $\Phi_{P, \epsilon}$ depends only on $p_{u, \mu}$ (and thus on $u$) via the coefficients $c_{\mu, \kappa, \alpha}$. Now, it is easy to see that there exists a neural network architecture $A_\epsilon = A_\epsilon(d + 1, m, N, \epsilon)$ with $L (A_\epsilon) \leq C_2$ layers and number of neurons and weights bounded by $C_3(N + 1)^{2(d+1)}$ such that $\Phi_{P, \epsilon}$ has architecture $A_\epsilon$ for every choice of coefficients $c_{\mu, \kappa, \alpha}$, and hence for every choice of $u$.

**Step 3 (Estimating the approximation error in $\| \cdot \|_{W_{k, \epsilon}}$)** Let

$$\Omega_{\mu, N} := B_{\frac{1}{N}} \| \cdot \|_{\infty} \left( \frac{\mu}{N} \right).$$
such that $\mu \in \{0, \ldots, N\}^{d+1}$. Moreover, for $n,k \in \{0,1\}$

\[
\left\| \sum_{\mu \in \{0,\ldots,N\}^{d+1}} \phi_{\mu}(t,x)p_{n,\mu}(t,x) - R_{p_3}(\Phi_{P,\mu})(t,x) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})} = \left\| \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} c_{\mu,\kappa,\alpha} \left( \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})}
\]

\[
\leq \sum_{\mu \in \{0,\ldots,N\}^{d+1}} \left\| \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} c_{\mu,\kappa,\alpha} \left( \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})}
\]

last inequality holds true since $(0,1)^{d+1} \subset \bigcup_{\tilde{\mu} \in \{0,\ldots,N\}^{d+1}} \Omega_{\tilde{\mu},N}$. Therefore, using Lemma 4.12 such that $\tilde{\mu} \in \{0, \ldots, N\}^{d+1}$, we get

\[
\left\| \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} c_{\mu,\kappa,\alpha} \left( \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})} \leq c_1 N^{d+1/q} \left\| U \right\|_{W_{m,p}^{m,p}(\Omega_{\mu,N})} \left\| \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})}(4.35)
\]

where $U$ is an extension of $u$ and $c_1 = c_1(m,d+1) > 0$ is a constant. Next, note that

\[
\left\| \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})} \leq \lambda \left( \Omega_{\tilde{\mu},N} \cap (0,1)^{d+1} \right) \left\| U \right\|_{W_{m,p}^{m,p}(\Omega_{\mu,N})} \left\| \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})} \leq c_3 \left( \frac{1}{N} \right)^{1/q + d/p} \left\| \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})}(4.36)
\]

where $\lambda$ denotes the Lebesgue measure and $c_3 = c_3(d+1,p,q) > 0$ is a constant. A direct combination of (4.36) with the last estimate, such that $N^{1/p - 1/q} \leq N$, yields

\[
\left\| \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} c_{m,\alpha} \left( \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})} \leq c_4 N \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} \sum_{\mu \in \{0,\ldots,N\}^{d+1}+1} \left\| U \right\|_{W_{m,p}^{m,p}(\Omega_{\mu,N})} \left\| \phi_{\mu}(t,x)t^\kappa x^\alpha - R_{p_3}(\Psi_{\epsilon,\Phi_{\mu,\alpha}})(t,x) \right\|_{W_{k,q}^{n,p}(\Omega_{\tilde{\mu},N})}(4.37)
\]
where, $c_4 = c_4(m, d+1, p) > 0$ is a constant, Lemma 4.1 and (4.34) give the second step, while (4.33) conclude the last step. Now, using that $\{\kappa \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^d, \kappa + |\alpha| \leq m - 1\} \leq (m - 1)^{d+1}$ shows that

$$\sum_{\mu \in \{0, \ldots, N\}^{d+1}} \sum_{|\mu|_{\infty} \leq 1, \kappa + |\alpha| \leq m - 1} \|U\|_{W_{m,p}^p(\Omega_{\mu,N})} \leq (m - 1)^{d+1} \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \|U\|_{W_{m,p}^p(\Omega_{\mu,N})}. \quad (4.38)$$

Combining (4.37) with (4.38) and plugging the result in (4.35) finally yields

$$\left\| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu(t, x)p_{u,\mu}(t, x) - R_{\rho_3}(\Phi_{\epsilon})(t, x) \right\|_{W_{n,q}^p((0,1),(0,1)^d)} \leq c_4 \epsilon (m - 1)^{d+1} N^{n+k+1} \epsilon \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \sum_{|\mu|_{\infty} \leq 1} \|U\|_{W_{m,p}^p(\Omega_{\mu,N})} \leq c_5 N^{n+k+1} \epsilon \|u\|_{W_{m,p}^p((0,1),(0,1)^d)},$$

where the last step is the same as Step 3 of the proof of Lemma 4.2 and $c_5 > 0$ depends only on $m, d + 1, p, n$ and $k$. \hfill \Box

Theorem 4.1 is the main result in our paper, here we show that any function in $U_{m,p,m,p,d,B}$ can be approximated by neural network with ReCU activation function.

**Theorem 4.1.** Let $d, m \in \mathbb{N}$, $n, k \in \{0, 1\}$ such that $m \geq n + k + 1$, $1 \leq p, q \leq \infty$, $B > 0$. Then there exists a constant $c = c(m, d + 1, p, B, n, k) > 0$ with the following properties: For any $\epsilon \in (0, 1/2)$, there is a neural network architecture $A_\epsilon = A_\epsilon(d + 1, m, p, B, \epsilon)$ with $d + 1$-dimensional input and one-dimensional output such that for any $u \in U_{m,p,m,p,d,B}$ (defined in 4.1), there is a neural network $\Phi_\epsilon^u$ that has architecture $A_\epsilon$ such that

1. $L(A_\epsilon) \leq c$;
2. $M(A_\epsilon) \leq c \cdot \epsilon^{-\frac{d+1}{m-n-k}}$;
3. $N(A_\epsilon) \leq c \cdot \epsilon^{-\frac{d+1}{m-n-k}}$;

and

$$\|u - R_{\rho_3}(\Phi_\epsilon^u)\|_{W_{n,q}^p((0,1),(0,1)^d)} \leq \sqrt{\epsilon}.$$

**Proof of Theorem 4.1.** The idea of the proof is simple, indeed, we need to approximate the function $u$ by a sum of localized polynomials and then approximate the sum by a neural network. We start by setting

$$N := \left\lceil \left( \frac{\epsilon^k}{2CB} \right)^{-1/(m-n-k)} \right\rceil \quad (4.39)$$

where $C = C(d + 1, m, p) > 0$ is the constant from Lemma 4.2. Without loss of generality, we may assume that $CB \geq 1$. Moreover Lemma 4.2 yields that if $\Psi = \{\phi_\mu : \mu \in \{0, \ldots, N\}^{d+1}\}$ is the partition of unity from Lemma 4.1 then there exist polynomials $p_{u,\mu}$ where

$$p_{u,\mu}(t, x) = \sum_{\kappa + |\alpha| \leq m - 1} c_{\mu,\kappa,\alpha} t^\kappa x^\alpha \quad \text{for } \mu \in \{0, \ldots, N\}^{d+1}.$$
such that
\[
\left\| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_{u,\mu} \right\|_{W_{k,q}^{n,p}(0,1),(0,1)^d)} \leq CB \left( \frac{1}{N} \right)^{m-n-k}
\]
\[
\leq CB \frac{\varepsilon'}{2CB} = \varepsilon' \quad (4.40)
\]

For the second step, let \( C_1 = C_1(m, d + 1, p, n, k) > 0 \), \( C_2 = C_2(m, d + 1) > 0 \) and \( C_3 = C_3(m, d+1) > 0 \) be the constants from Lemma \( \textbf{[1.4]} \) and \( \Phi_{p,e'} \) be the neural network given in the same lemma (independent of the function \( u \)) with \( \frac{\varepsilon'}{2CB} \left( \frac{\varepsilon'}{2CB} \right)^{-1/(m-n-k)} + 1 \) \(-n-k-1\) instead of \( \varepsilon \) in \( \textbf{(4.31)} \). The neural network \( \Phi_{p,e'} \) has \( d + 1 \)-dimensional input and one-dimensional output, at most \( C_2 \) layers and
\[
C_3(N+1)^{2(d+1)} \leq C_3 \left( \frac{\varepsilon'}{2CB} \right)^{-1/(m-n-k)} + 2)^{2(d+1)} \leq C_3 3^{2(d+1)} \left( \frac{\varepsilon'}{2CB} \right)^{1/(m-n-k)} \leq C'' \varepsilon' \frac{2^{2(d+1)}}{m-n-k}
\]

nonzero weights and neurons, such that \( C'' = C''(m, d + 1, p, B, n, k) \) is a positive constant, where in the first inequality we used the fact that \( \frac{2CB \varepsilon'}{e'} \geq 1 \). Thus, for the statement in the theorem, we choose \( c = \max(C_2, C'') \). Furthermore, we have
\[
\left\| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_{u,\mu} - R_{p3} (\Phi_{p,e'}) \right\|_{W_{k,q}^{n,p}(0,1),(0,1)^d)} \leq C_1 BN^{n+k+1} \varepsilon' \frac{\varepsilon'}{2C_1 B} \left( \frac{\varepsilon'}{2CB} \right)^{1/(m-n-k)} + 1 \)
\[
\leq \varepsilon' \frac{\varepsilon'}{2} \quad (4.41)
\]

Using the triangle inequality and Eqs. \( \textbf{(1.40)} \) and \( \textbf{(1.41)} \), we finally obtain
\[
\left\| u - R_{p3} (\Phi_{p,e'}) \right\|_{W_{k,q}^{n,p}(0,1),(0,1)^d)} \leq \left\| u - \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_{u,\mu} \right\|_{W_{k,q}^{n,p}(0,1),(0,1)^d)} + \left\| \sum_{\mu \in \{0, \ldots, N\}^{d+1}} \phi_\mu p_{u,\mu} - R_{p3} (\Phi_{p,e'}) \right\|_{W_{k,q}^{n,p}(0,1),(0,1)^d)} \leq \frac{\varepsilon'}{2} + \frac{\varepsilon'}{2} = \varepsilon'
\]
which concludes the proof for \( \varepsilon' = \sqrt{\varepsilon} \). \( \square \)

References

[1] W. Arendt, D. Dier, and M.K. Fijavž. Diffusion in networks with time-dependent transmission conditions. Appl. Math. Optim., 69(2), pp.315-336, 2014.
[2] C. Beck, S. Becker, P. Grohs, N. Jaafari, and A. Jentzen. Solving stochastic differential equations and Kolmogorov equations by means of deep learning. *arXiv preprint arXiv:1806.00421*, 2018.

[3] J. Berg and K. Nystrom. A unified deep artificial neural network approach to partial differential equations in complex geometries. Neurocomputing 317 (2018), 28–41.

[4] H. Bölcskei, P. Grohs, G. Kutyniok, and P. Petersen. Optimal approximation with sparsely connected deep neural networks. SIAM Journal on Mathematics of Data Science, 1(1):8–45, 2019.

[5] J. Berner, P. Grohs and A. Jentzen. Analysis of the Generalization Error: Empirical Risk Minimization over Deep Artificial Neural Networks Overcomes the Curse of Dimensionality in the Numerical Approximation of Black–Scholes Partial Differential Equations. SIAM Journal on Mathematics of Data Science 2, 3 (2020), 631–657.

[6] S. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer Science+Business Media, New York, third edition, 2008.

[7] T. Cazenave and M. Scialom. A Schrödinger equation with time-oscillating nonlinearity. Rev. Mat. Complut., 23(2), pp.321-339, 2010.

[8] Y. Chen and J. W. Wan. Deep neural network framework based on backward stochastic differential equations for pricing and hedging American options in high dimensions. *arXiv:1909.11532* (2019).

[9] C. K. Chui, S.-B. Lin, and D.-X. Zhou. Construction of neural networks for realization of localized deep learning. Frontiers in Applied Mathematics and Statistics, 4, 2018.

[10] D. Elbrächter, P. Grohs, A. Jentzen, and C. Schwab. DNN expression rate analysis of high-dimensional PDEs: Application to option pricing. *arXiv preprint arXiv:1809.07669*, 2018.

[11] C. L. Fefferman, D. S. McCormick, J. C. Robinson and J. L. Rodrigo. Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces. Archive for Rational Mechanics and Analysis, 223(2), pp.677-691, 2017.

[12] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224 of *A Series of Comprehensive Studies in Mathematics*. Springer-Verlag, Berlin, second edition, 1998.

[13] L. Gonon, P. Grohs, A. Jentzen, D. Kofler and D. Šiška. Uniform error estimates for artificial neural network approximations for heat equations. *arXiv:1911.09647* (2019).

[14] W. R. Grey, Inclusions Among Mixed-Norm Lebesgue Spaces, PHD thesis, https://ir.lib.uwo.ca/cgi/viewcontent.cgi?article=4270&context=etd

[15] P. Grohs and L. Herrmann. Deep neural network approximation for high-dimensional elliptic PDEs with boundary conditions. *arXiv:2007.05384* (2020).

[16] P. Grohs, F. Hornung, A. Jentzen and P. Zimmermann. Space-time error estimates for deep neural network approximations for differential equations. *arXiv:1908.03333* (2019).
[17] P. Grohs, D. Perekrestenko, D. Elbrächter, and H. Bölcskei. Deep neural network approximation theory. arXiv preprint arXiv:1901.02220, 2019.

[18] I. Gühring. Error bounds for approximations with deep ReLU neural networks in general norms. Master thesis. https://www.math.tu-berlin.de/fileadmin/i26_fg-kutyniok/G%C3%BChring/publications/master_thesis.pdf, 2018. [Online; accessed 20-February-2019].

[19] I. Gühring, G. Kutyniok and P. Petersen, Error bounds for approximations with deep ReLU neural networks in $W^{s,p}$ norms. Analysis and Applications, 1-57, doi:10.1142/S0219530519410021

[20] J. Hana, A. Jentzenb and W. E Solving high-dimensional partial differential equations using deep learning. Proceedings of the National Academy of Sciences, 115 (34):8505-8510, 2018.

[21] M. Hutzenthaler, A. Jentzen, T. Kruse and T. A. Nguyen A proof that rectified deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear heat equations. SN Partial Differ. Equ. Appl. 1, 10 (2020).

[22] A. Khan, A. Sohail, U. Zahoora, A. S. Qureshi . A survey of the recent architectures of deep convolutional neural networks. Artif Intell Rev (2020). https://doi.org/10.1007/s10462-020-09825-6

[23] H.O. Kreiss and J. Lorenz. Initial-boundary value problems and the Navier-Stokes equations. Society for Industrial and Applied Mathematics, 2004.

[24] G. Kutyniok, P. Petersen, M. Raslan and R. Schneider. A theoretical analysis of deep neural networks and parametric pdes. arXiv:1904.00377 (2019).

[25] S. Liang and R. Srikant. Why deep neural networks for function approximation? In Proc. of ICLR 2017, pages 1 – 17, 2017.

[26] K. O. Lye, S. Mishra and D. Ray. Deep learning observables in computational fluid dynamics. arXiv:1903.03040 (2019).

[27] M. Magill, F. Qureshi and H. W. de Haan. Neural networks trained to solve differential equations learn general representations. In Advances in Neural Information Processing Systems (2018), pp. 4071–4081.

[28] H. N. Mhaskar. Approximation properties of a multilayered feedforward artificial neural network. Adv. Comput. Math., 1(1):61–80, 1993.

[29] J. Opschoor, P. Petersen, and C. Schwab. Deep ReLU networks and high-order finite element methods. SAM, ETH Zürich, 2019.

[30] P. Petersen and F. Voigtländer. Optimal approximation of piecewise smooth functions using deep ReLU neural networks. Neural Networks, 108:296–330, 2018.

[31] D. Rolnick and M. Tegmark. The power of deeper networks for expressing natural functions. In International Conference on Learning Representations, 2018.

[32] F Rothe. Global solutions of reaction-diffusion systems. Springer, 2006.
A. Proof of the results in Section 2

A.1. Proof of Proposition 2.1

The $m^{th}$-order remainder term is given by $R_m^u(t, x) = u(t, x) - Q^m u(t, x)$.

$$u(t, x) = \sum_{|\alpha|+k<m} \frac{1}{\alpha!k!} D_x^\alpha D_t^k u(\tau, \xi)(x - \xi)^\alpha(t - \tau)^k$$

$$+ \sum_{|\alpha|+k=m} (x - \xi)^\alpha(t - \tau)^k \int_0^1 \frac{m}{\alpha!k!} s^{m-1} D_x^\alpha D_t^k u(t + s(\tau - t), x + s(\xi - x)) ds.$$

Using the previous equality and the properties of a cut-off function, we get

$$R_m^u(t, x) = \int_B u(t, x) \phi(\tau, \xi) dy - \int_B T_{\tau, \xi}^m u(t, x) \phi(\tau, \xi) d\tau$$

$$= \int_B \left[ u(t, x) - T_{\tau, \xi}^m u(t, x) \right] \phi(\tau, \xi) d\tau$$

$$= \int_B \phi(\tau, \xi) m \sum_{|\alpha|=m} (x - \xi)^\alpha(t - \tau)^k$$

$$\times \int_0^1 \frac{s^{m-1}}{\alpha!k!} D_x^\alpha D_t^k u(t + s(\tau - t), x + s(\xi - x)) ds d\tau.$$

We make a change of variables from the $(\tau, \xi, s)$-space to the $(T, \Xi, s)$-space, where $\Xi = x + s(\xi - x)$, $T = t + s(\tau - t)$. Then, we have

$$ds d\xi d\tau = s^{-(d+1)} ds d\Xi dT.$$

The domain of integration in the $(\tau, \xi, s)$-space is $B \times (0, 1]$ and the corresponding domain in the $(T, \Xi, s)$-space is the set

$$A = \left\{ (T, \Xi, s) : s \in (0, 1], \left| \frac{1}{s}(\Xi - x) + x - x_0 \right| + \left| \frac{1}{s}(T - t) + t - t_0 \right| < r \right\}.$$
If we define
\[ (T, \Xi, s) \in A \] implies that \[ \frac{|\Xi - x| + |T - t|}{|x - x_0| + |t - t_0| + r} < s. \]
Moreover, for \(|\alpha| + k = m\), we have
\[ (x - \xi)^\alpha (t - \tau)^k = s^{-m}(x - \Xi)^\alpha (t - T)^k. \] (A.2)

Letting \(\chi_A\) be the characteristic function of \(A\), from (A.1) and (A.2) we obtain
\[
R^m u(t, x) = \sum_{|\alpha| + k = m} \int_{C_{t,x}} \frac{1}{\alpha!k!} D_x^\alpha D_t^k u(T, \Xi)(x - \Xi)^\alpha (t - T)^k \\
\times \int_0^1 \phi(t + \frac{1}{s}(T - t), x + \frac{1}{s}(z - x)) \chi_A(T, \Xi, s) s^{-d-2} ds \, d\Xi dT
\]

The projection of \(A\) onto the \((T, \Xi)\)-space is \(C_{t,x}\). Therefore, by Fubini’s Theorem,
\[
R^m u(t, x) = m \sum_{|\alpha| + k = m} \int_{C_{t,x}} K_{\alpha,k}(t, T; x, \Xi) D_x^\alpha D_t^k u(T, \Xi) d\Xi dT
\]
if we define
\[
K(t, T; x, \Xi) = \int_0^1 \phi(t + \frac{1}{s}(T - t), x + \frac{1}{s}(z - x)) \chi_A(T, \Xi, s) s^{-d-2} ds
\]
and
\[
K_{\alpha,k}(t, T; x, \Xi) = \frac{1}{\alpha!k!} (x - \Xi)^\alpha (t - T)^k K(t, T; x, \Xi).
\]

It remains to prove estimate (2.3) for \(K(t, T; x, \Xi)\). Thus, let \(y = \frac{|\Xi - x| + |T - t|}{|x - x_0| + |t - t_0| + r}\).
Then,
\[
|K(t, T; x, \Xi)| = \left| \int_0^1 \chi_A(T, \Xi, s) \phi(t + \frac{1}{s}(T - t), x + \frac{1}{s}(\Xi - x)) s^{-d-2} ds \right|
\leq \int_0^1 \left| \phi(t + \frac{1}{s}(T - t), x + \frac{1}{s}(\Xi - x)) \right| s^{-d-2} ds
\leq \|\phi\|_{L^\infty_x L^\infty_s(B)} \frac{s^{-d-1}}{d+1} \leq \frac{1}{d+1} \|\phi\|_{L^\infty_x L^\infty_s(B)} y^{-d-1}
= \frac{1}{d+1} \|\phi\|_{L^\infty_x L^\infty_s(B)} (|x - x_0| + |t - t_0| + r)^{d+1} (|\Xi - x| + |T - t|)^{-d-1}
\leq Cr^{-d-1} (|x - x_0| + |t - t_0| + r)^{d+1} (|\Xi - x| + |T - t|)^{-d-1}
= C (1 + (|x - x_0| + |t - t_0|) / r)^{d+1} (|\Xi - x| + |T - t|)^{-d-1}.\]
A.2. Proof of Lemma \textbf{2.1}

We need the next result for the proof of Lemma \textbf{2.1}.

**Lemma A.1.** Let $I \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^d$ be open and bounded, $k, n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, such that $n + k = m$, $1 \leq p, q \leq \infty$, $u \in L^p_I L^q_x(I \times \Omega)$, and let

$$g(t, x) = \int_{I \times \Omega} \frac{|\xi - x|^n |\tau - t|^k}{(|\xi - x| + |\tau - t|)^{d+1}} u(\tau, \xi) \, d\xi d\tau.$$ 

Then,

$$\|g\|_{L_t^p L_x^q(I \times \Omega)} \leq C_{m,k,n} h^{m(p+q)} \|u\|_{L_t^p L_x^q(I \times \Omega)},$$

where $h = \text{diam}(I \times \Omega)$. \hfill (A.3)

**Proof.** First we assume that $1 < p, q < \infty$. Then, using Hölder’s inequality with $\frac{1}{p} + \frac{1}{q'} = 1$, we get

$$\|g\|_{L_t^p L_x^q(I \times \Omega)}^q = \int_I \left( \int_{I \times \Omega} \left( \int_{I \times \Omega} \frac{|\xi - x|^n |\tau - t|^k}{(|\xi - x| + |\tau - t|)^{d+1}} u(\tau, \xi) \, d\xi d\tau \right)^p \right)^{q/p} \, dx \, dt$$

$$\leq \int_I \left( \int_{I \times \Omega} \left( \int_{I \times \Omega} \frac{|\xi - x|^n |\tau - t|^k}{(|\xi - x| + |\tau - t|)^{d+1}} |u(\tau, \xi)|^p \, d\xi d\tau \right)^{\frac{q}{p}} \right)^{\frac{1}{q'}} \, dx \, dt$$

$$\leq \int_I \left( \int_{I \times \Omega} \frac{|\xi - x|^n |\tau - t|^k}{(|\xi - x| + |\tau - t|)^{d+1}} |u(\tau, \xi)|^p \, d\xi d\tau \right)^{\frac{q}{p}} \, dx \, dt$$

$$\leq C_{d,m} \text{diam}(I \times \Omega)^{mp/q'} \int_I \left( \int_{I \times \Omega} \frac{|\xi - x|^n |\tau - t|^k}{(|\xi - x| + |\tau - t|)^{d+1}} |u(\tau, \xi)|^p \, d\xi d\tau \right)^{q/p} \, dx$$

$$\leq C_{d,m} \text{diam}(I \times \Omega)^{m(1+p'/q')} \int_I \left( \int_{I \times \Omega} |\tau - t|^k |u(\tau, \xi)|^p \, d\xi d\tau \right)^{q/p} \, dx$$

$$\leq C_{d,m} \text{diam}(I \times \Omega)^{m(1+p'/q')} \|u\|_{L_t^p L_x^q(I \times \Omega)}^{q/p} = C_{d,m} \text{diam}(I \times \Omega)^{m(p+q)} \|u\|_{L_t^p L_x^q(I \times \Omega)}.$$ 

The cases where $p, q \in \{1, \infty\}$ are straightforward and therefore left to the reader.

Proof of Lemma \textbf{2.1} In a similar way as in \textbf{[6] Lemma 4.3.8}, we prove Lemma \textbf{2.1} using \textbf{[6] Proposition 4.1.9} and the fact that $Q^m$ is a polynomial in both $t$ and $x$ of order less than $m$. Let $\text{diam}(I \times \Omega) = 1$, for $k = n = 0$, using Lemma \textbf{A.1} we get

$$\|u - Q^m u\|_{L_t^p L_x^q(I \times \Omega)} = \|R^m u\|_{L_t^p L_x^q(I \times \Omega)}$$

$$\leq m \sum_{|\alpha| + \kappa = m} \left\| \int_{I \times \Omega} K_{\alpha,\kappa}(t, x, \xi) D_x^\alpha D_t^\kappa u(\tau, \xi) \, d\xi d\tau \right\|_{L_t^p L_x^q(I \times \Omega)}$$

$$\leq C_{d,m} (1 + r^{-1})^{d+1} \sum_{|\alpha| + \kappa = m} \left\| \int_{I \times \Omega} \frac{|\xi - x|^{|\alpha|} |T - t|^\kappa}{(|\xi - x| + |T - t|)^{d+1}} D_x^\alpha D_t^\kappa u(\tau, \xi) \, d\xi d\tau \right\|_{L_t^p L_x^q(I \times \Omega)}$$

$$\leq C_{d,m} \|u\|_{W^{m,p}_{m,p}(I \times \Omega)}.$$ 

For $0 < k + n \leq m$,
A.3. Proof of Lemma \ref{lem:2.2}

The first part of the proof of this lemma follows closely the chain of arguments in \cite[Equations (4.1.5) - (4.1.8)]{6} and the Binomial theorem. We write for $\kappa \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$

$$(t - \tau)^{\kappa}(x - y)^{\alpha} = \sum_{\nu + \mu = \kappa, \gamma + \beta = \alpha} a_{(\nu, \mu, \gamma, \beta)} t^\nu x^\gamma y^\beta,$$

where $a_{(\nu, \mu, \gamma, \beta)} \in \mathbb{R}$ are suitable constants with

$$|a_{(\nu, \mu, \gamma, \beta)}| \leq \frac{(\nu + \mu)! (\gamma + \beta)!}{\nu! \gamma! \beta!} \quad (A.4)$$

in multi-index notation. Then, combining Equation \eqref{eq:2.1} and \eqref{eq:2.2} yields

$$Q^{k+n}u(x) = \sum_{|\alpha|+\kappa \leq k+n-1} \sum_{\gamma + \beta = \alpha} \frac{1}{\kappa! \alpha!} a_{(\nu, \mu, \gamma, \beta)} t^\nu x^\gamma \int_B D_x^\alpha D_t^\beta u(\tau, y) t^{\mu} y^{\beta} \phi(\tau, y) dy d\tau$$

$$= \sum_{|\gamma|+\nu \leq k+n-1} t^{\nu} x^\gamma \sum_{|\gamma + \beta| = k+n} \frac{1}{(\nu + \mu)! (\gamma + \beta)!} a_{(\nu, \mu, \gamma, \beta)} \int_B D_x^{\gamma + \beta} D_t^{\nu + \mu} u(\tau, y) t^{\mu} y^{\beta} \phi(\tau, y) dy d\tau. \quad (A.5)$$

For the second part, note that

$$\left| \int_B D_x^{\gamma + \beta} D_t^{\nu + \mu} u(\tau, y) t^{\mu} y^{\beta} \phi(\tau, y) dy d\tau \right| \leq \int_B |D_x^{\gamma + \beta} D_t^{\nu + \mu} u(\tau, y)| t^{\mu} y^{\beta} |\phi(\tau, y)| dy d\tau$$

$$\leq R^{\beta+\mu} \|u\|_{W^{k+n-1,p}(B)} \|\phi\|_{L^q_x L^\infty_t(B)}, \quad (A.6)$$

where we used the fact that $B \subset B_{R,\|\cdot\|_{L^\infty}}(0)$ and the Hölder’s inequality with $1/p = 1 - 1/q$. Next, since $\phi \in L^1_t L^2_x(B) \cap L^\infty_t L^2_x(B)$ and $\|\phi\|_{L^1_t L^2_x} = 1$, using the Mixed interpolative Hölder’s inequality cf. \cite{14}, we get

$$\|\phi\|_{L^q_x L^\infty_t} \leq \|\phi\|_{L^1_t L^2_x}^{1/q} \|\phi\|_{L^\infty_t L^\infty_x}^{1-1/q} = \|\phi\|_{L^1_t L^2_x}^{1/p} \|\phi\|_{L^\infty_t L^\infty_x}^{1/q}.$$

Combining the last estimate with Equation \eqref{eq:4.3} yields

$$\left| \int_B D_x^{\gamma + \beta} D_t^{\nu + \mu} u(\tau, y) t^{\mu} y^{\beta} \phi(\tau, y) dy d\tau \right| \leq R^{\kappa+n-1} \|u\|_{W^{k+n-1,p}(\Omega)} \|\phi\|^{1/p}_{L^1_t L^2_x} \|\phi\|^{1/q}_{L^\infty_t L^\infty_x} \leq cR^{\kappa+n-1} \tau^{(d+1)/p} \|u\|_{W^{k+n-1,p}(I, \Omega)}, \quad (A.7)$$

For a general domain $\Omega$, we define $\Theta = \{x/h \text{ such that } x \in \Omega\}$, using similar argument to the previous calculus, we conclude the result. The details are left to the reader.
where the second step follows from \( \| \phi \|_{L^\infty} \leq c r^{-(d+1)} \) for some constant \( c = c(d) > 0 \) (see Section 4.1]. To estimate the absolute value of the coefficients \( c_{\gamma, \nu} \) (defined in Equation A.3), we have
\[
|c_{\gamma, \nu}| \leq \sum_{|\gamma + \beta + \nu| \leq k + n - 1} \frac{1}{(\gamma + \beta)! (\nu + \mu)!} \left| a_{(\nu, \mu, \gamma, \beta)} \right| \int_B D_x^{\gamma + \beta} D_t^\nu g(\tau, y) \phi(\tau, y) dy d\tau
\]
\[
\leq c R^{k + n - 1} r^{-(d+1)/p} \sum_{|\gamma + \beta + \nu| \leq k + n - 1} \frac{1}{\gamma! \beta! \nu! \mu!}
\]
\[
= c' r^{-(d+1)/p} \| u \|_{W^{k+n-1,p}(I, \Omega)},
\]
where \( c' = c'(k, n, d, R) > 0 \) is a constant.

### A.4. Proof of Lemma 2.3

From the given assumptions on \( f \) and \( g \) it is clear that \( fg \in L^q_t L^p_x(I \times \Omega) \). Moreover, \( f_g, (D_t f) g + f(D_t g), (D_x f) g + f(D_x g), (D_x D_t f) g + (D_t f) (D_x g) + (D_x f) (D_t g) + f(D_x D_t g) \in L^q_t L^p_x(I \times \Omega) \) so that the product formula in [12] Chapter 7.3 yields that for the weak derivatives of \( fg \) it holds
\[
D_t (fg) = (D_t f) g + f(D_t g), \quad D_{x_i} (fg) = (D_{x_i} f) g + f(D_{x_i} g)
\]
and
\[
D_{x_i} D_t (fg) = (D_{x_i} D_t f) g + (D_t f) (D_{x_i} g) + (D_{x_i} f) (D_t g) + f(D_{x_i} D_t g)
\]
for \( i = 1, 2, \ldots, d \). Thus, we have
\[
\| fg \|_{W^{1, p}_{q}(I, \Omega)} = \| fg \|_{L^q_t L^p_x(I \times \Omega)} = \left( \sum_{i=1}^d \| (D_{x_i} f) g + f(D_{x_i} g) \|_{L^p_x(\Omega)} \right)^{1/p} \leq \sum_{i=1}^d \| (D_{x_i} f) g + f(D_{x_i} g) \|_{L^p_x(\Omega)}^{1/p}
\]
\[
\leq \sum_{i=1}^d \| (D_{x_i} f) g \|_{L^p_x(I \times \Omega)} + \| f(D_{x_i} g) \|_{L^p_x(I \times \Omega)},
\]
where
\[
\| (D_{x_i} f) g \|_{L^p_x(I \times \Omega)} \leq \| D_{x_i} f \|_{L^\infty_t L^\infty_x(I \times \Omega)} \| g \|_{L^p_x(I \times \Omega)} \leq |f|_{W^{1, \infty}_{q}(I, \Omega)} \| g \|_{L^p_x(I \times \Omega)}
\]
\[
\| f(D_{x_i} g) \|_{L^p_x(I \times \Omega)} \leq \| f \|_{L^\infty_t L^\infty_x(I \times \Omega)} \| D_{x_i} g \|_{L^p_x(I \times \Omega)}.
\]
Thus,
\[
\| fg \|_{W^{1, p}_{q}(I, \Omega)} \leq d \| f \|_{W^{1, \infty}_{q}(I, \Omega)} \| g \|_{L^p_x(I \times \Omega)} + \| f \|_{L^\infty_t L^\infty_x(I \times \Omega)} \sum_{i=1}^d \| D_{x_i} g \|_{L^p_x(I \times \Omega)}
\]
\[
\leq C \left( \| f \|_{W^{1, \infty}_{q}(I, \Omega)} \| g \|_{L^p_x(I \times \Omega)} + \| f \|_{L^\infty_t L^\infty_x(I \times \Omega)} \| g \|_{W^{1, p}_{q}(I, \Omega)} \right).
\]
where $C > 0$ depends on $d$ and $p$.

Moreover,

$$|fg|_{W^{1,p}_{1,q}(I,\Omega)} = |fg|_{W^{1,p}_{1,q}(I,\Omega)} = \|(D_t f)g + f(D_t g)\|_{L^p_t L^q_x(I \times \Omega)}$$

such that

$$\|(D_t f)g\|_{L^p_t L^q_x(I \times \Omega)} \leq \|(D_t f)\|_{L^p_t L^q_x(I \times \Omega)} \|g\|_{L^q_x(I \times \Omega)} \leq |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|g\|_{L^q_x(I \times \Omega)}$$

$$\|f(D_t g)\|_{L^p_t L^q_x(I \times \Omega)} \leq \|f\|_{L^p_t L^q_x(I \times \Omega)} \|D_t g\|_{L^p_t L^q_x(I \times \Omega)} \leq \|f\|_{L^p_t L^q_x(I \times \Omega)} |g|_{W^{1,p}_{1,q}(I,\Omega)}.$$ 

Then, we get

$$|fg|_{W^{1,p}_{1,q}(I,\Omega)} \leq |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|g\|_{L^q_x(I \times \Omega)} + \|f\|_{L^p_t L^q_x(I \times \Omega)} |g|_{W^{1,p}_{1,q}(I,\Omega)}.$$ 

For the mixed derivatives, we have

$$|fg|_{W^{1,p}_{1,q}(I,\Omega)} = \|(D_t f)g + f(D_t g)\|_{L^p_t L^q_x(I \times \Omega)}$$

$$= \left\| \left( \sum_{i=1}^d (D_x D_t f)g + (D_t f)(D_x g) + (D_x f)(D_t g) + f(D_x D_t g) \right) \right\|_{L^p_t L^q_x(I \times \Omega)}^{1/p}$$

$$\leq \sum_{i=1}^d \|(D_x D_t f)g\|_{L^p_t L^q_x(I \times \Omega)} + \|(D_t f)(D_x g)\|_{L^p_t L^q_x(I \times \Omega)}$$

Note that we have

$$\|(D_x D_t f)g\|_{L^p_t L^q_x(I \times \Omega)} \leq \|D_x D_t f\|_{L^p_t L^q_x(I \times \Omega)} \|g\|_{L^q_x(I \times \Omega)} \leq |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|g\|_{L^q_x(I \times \Omega)}$$

$$\|(D_t f)(D_x g)\|_{L^p_t L^q_x(I \times \Omega)} \leq \|D_t f\|_{L^p_t L^q_x(I \times \Omega)} \|D_x g\|_{L^q_x(I \times \Omega)} \leq |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|D_x g\|_{L^q_x(I \times \Omega)}$$

$$\|(D_x f)(D_t g)\|_{L^p_t L^q_x(I \times \Omega)} \leq \|D_x f\|_{L^p_t L^q_x(I \times \Omega)} \|D_t g\|_{L^q_x(I \times \Omega)} \leq |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|D_t g\|_{L^q_x(I \times \Omega)}$$

Finally, we get

$$|fg|_{W^{1,p}_{1,q}(I,\Omega)} \leq C \left( |f|_{W^{1,\infty}_{1,\infty}(I,\Omega)} \|g\|_{L^q_x(I \times \Omega)} + |f|_{W^{0,\infty}_{1,\infty}(I,\Omega)} |g|_{W^{1,p}_{1,q}(I,\Omega)} \right)$$

$$+ |f|_{W^{0,\infty}_{1,\infty}(I,\Omega)} |g|_{W^{0,p}_{1,q}(I,\Omega)} + \|f\|_{L^p_t L^q_x(I \times \Omega)} \|g\|_{W^{1,p}_{1,q}(I,\Omega)}$$

where $C = C(d, p) > 0$ is a constant.
A.5. Proof of Lemma 2.4

We start by the case \( n + k = 1 \). Let \( \nabla = (\nabla_1, \nabla_2) \), where \( \nabla_1 \) and \( \nabla_2 \) are the gradient with respect to the first and second block of variables respectively. Moreover, we set for \( j = 1, \ldots, m \)

\[
L_j := \|\nabla f_j\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} \quad \text{and} \quad L_f := (L_1, \ldots, L_m).
\]

Using similar ideas from the proof of [19, Corollary B.5], we conclude that \( f_j \) is \( L_j \)-Lipschitz and therefore \( f \) is \( |L_f| \)-Lipschitz. Similarly, \( g \) is \( L_g \)-Lipschitz, where \( L_g := \|\nabla g\|_{L^\infty L^p(\Theta_1 \times \Theta_2)} \). Furthermore, \( g \circ f \in W_{k,\infty}^n(\Omega_1, \Omega_2) \), and \( \|\nabla (g \circ f)\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} \leq |L_f| \cdot L_g \). Thus, we have

\[
|g \circ f|_{W_{k,\infty}^n(\Omega_1, \Omega_2)} \leq \|\nabla (g \circ f)\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} \leq |L_f| \cdot L_g
\]

\[
\leq m|L_f|\|\nabla \circ g\|_{W_{k,\infty}^n(\Theta_1, \Theta_2)}
\]

\[
\leq p m^2 |f|_{W_{k,\infty}^n(\Omega_1, \Omega_2)} \cdot |g|_{W_{k,\infty}^n(\Theta_1, \Theta_2)},
\]

where we use the estimate of the \( \ell^2 \) norm with the \( \ell^\infty \) norm on \( \mathbb{R}^m \) and the fact that if \( f \in W_{k,\infty}^n(\Omega_1, \Omega_2) \) and that \( \dim(\Omega_i) = p_i \) for \( i = 1, 2 \), then we have the following observation

\[
|f|_{W_{k,\infty}^n(\Omega_1, \Omega_2)} \leq \|\nabla f\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} \leq \sqrt{p_1 p_2} |f|_{W_{k,\infty}^n(\Omega_1, \Omega_2)} \leq p |f|_{W_{k,\infty}^n(\Omega_1, \Omega_2)}.
\]

If \( n = k = 1 \), we denote by \( D_1 \) the derivative with respect to the first block of variables, then for \( |\alpha| = 1 \) we have \( D_1^\alpha (g \circ f) = \sum_{j=1}^m \partial_1 f_j ((\partial_j g) \circ f) \). Since \( \dim(\Omega_i) = p_1 \), we get

\[
|g \circ f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} \leq \max_{1 \leq i \leq p_1} \sum_{j=1}^m |\partial_1 f_j (\partial_j g) \circ f|_{W_{0,\infty}^1(\Omega_1, \Omega_2)}
\]

\[
\leq \max_{1 \leq i \leq p_1} \max_{1 \leq j \leq p_1} \left( \sum_{j=1}^m \|\partial_1 \partial_j f_j (\partial_j g) \circ f\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} + \|\partial_1 f_j \partial_1 ((\partial_j g) \circ f)\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} \right)
\]

\[
\leq \sum_{j=1}^m |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} \|\partial_j g\|_{L^\infty L^p(\Omega_1 \times \Omega_2)} + |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} \|((\partial_j g) \circ f)\|_{W_{0,\infty}^1(\Omega_1, \Omega_2)}
\]

\[
\leq \sum_{j=1}^m |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} \|\partial_j g\|_{L^\infty L^p(\Theta_1 \times \Theta_2)} + p m^2 |f|_{W_{0,\infty}^1(\Omega_1, \Omega_2)} |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |\partial_j g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)}
\]

\[
\leq \sum_{j=1}^m |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)} + p m^2 |f|_{W_{0,\infty}^1(\Omega_1, \Omega_2)} |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)}
\]

\[
\leq m |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)} + p m^3 |f|_{W_{0,\infty}^1(\Omega_1, \Omega_2)} |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)}
\]

\[
\leq 2 \max \left( m |f|_{W_{1,\infty}^1(\Omega_1, \Omega_2)} |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)}, p m^3 |f|_{W_{0,\infty}^1(\Omega_1, \Omega_2)}^2 |g|_{W_{1,\infty}^1(\Theta_1, \Theta_2)} \right)
\]