A COUNTEREXAMPLE IN THE THEORY OF D-SPACES

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Abstract. Assuming ♦, we construct a $T_2$ example of a hereditarily Lindelöf space of size $\omega_1$ which is not a $D$-space. The example has the property that all finite powers are also Lindelöf.

1. Introduction

The following notion is due to van Douwen, first studied with Pfeffer in [3].

Definition 1.1. A $T_1$ space $X$ is said to be a $D$-space if for each open neighbourhood assignment $\{U_x : x \in X\}$ there is a closed and discrete subset $D \subseteq X$ such that $\{U_x : x \in D\}$ covers the space.

The question whether every regular Lindelöf space is $D$ has been attributed to van Douwen [8]. Moreover, van Douwen and Pfeffer pointed out that

"No satisfactory example of a space which is not a $D$-space is known, where by satisfactory example we mean an example having a covering property at least as strong as metacompactness or subparacompactness."

Indeed, the lack of satisfactory examples of $D$-spaces satisfying some interesting covering properties continues and there has been quite a bit of activity in the area in the last decades (see the surveys [5] and [8] for other related results and open problems). Whether regular Lindelöf spaces are $D$-spaces was listed as Problem 14 in Hrušák and Moore’s list of 20 open problems in set-theoretic topology [10], and there are no consistency results in either direction even for hereditarily Lindelöf spaces. The question whether Lindelöf implies $D$ for the class of $T_1$ spaces was also open and explicitly asked in [6] and more recently in [1].
In this note, assuming ♦, we construct an example of a hereditarily Lindelöf $T_2$ space that is not a $D$-space. The example also has the property that every finite power is Lindelöf, but we do not know if it can be made regular.

The article is structured as follows; in Section 2 we gather a few general facts and definitions, and in Section 3 we present the construction. In Section 4 we make some remarks and prove further properties of our construction. Finally, in Section 5 we state a few open problems.

2. Preliminaries

Delicate use of elementary submodels play crucial role in our arguments. We do not intend to give a precise introduction to this powerful tool since elementary submodels are widely used in topology nowadays; let us refer to [4]. However, we present here a few easy facts and a lemma which could serve as a warm-up exercise for the readers less involved in the use of elementary submodels.

Let $H(\vartheta)$ denote the sets which have transitive closure of size less than $\vartheta$ for some cardinal $\vartheta$. The following facts will be used regularly without explicitly referring to them.

Fact 2.1. Suppose that $M \prec H(\vartheta)$ for some cardinal $\vartheta$ and $M$ is countable.

(a) If $F \in M$ and there is $F \in F \setminus M$ then $F$ is uncountable.
(b) If $B \in M$ and $B$ is countable then $B \subseteq M$.

The next lemma is well-known, nonetheless we present a proof.

Lemma 2.2. Let $\mathcal{F} \subseteq [\omega_1]^\omega$ and suppose that $M$ is a countable elementary submodel of $H(\vartheta)$ for some cardinal $\vartheta$ such that $\mathcal{F} \in M$. If there is an $F \in \mathcal{F}$ such that $F \notin M$ then there is an uncountable $\Delta$-system $\mathcal{G} \subseteq \mathcal{F}$ in $M$ with kernel $F \cap M$.

Moreover, if $\psi(x,\ldots)$ is any formula with parameters from $M$ and $\psi(F,\ldots)$ holds then $\mathcal{G}$ can be chosen in such a way that $\psi(G,\ldots)$ holds for every $G \in \mathcal{G}$, as well.

Proof. Suppose that $\mathcal{F}, M, F \in \mathcal{F} \setminus M$ and $\psi$ is as above. Let $D = F \cap M$ and let $\mathcal{F}_0 = \{G \in \mathcal{F} : D \subseteq G \text{ and } \psi(G,\ldots)\}$. Clearly, $F \in \mathcal{F}_0 \in M$ and $F \notin M$ thus $\mathcal{F}_0$ is uncountable. Moreover, $F \cap \alpha = D$ for all $\alpha$ in a tail of $M \cap \omega_1$; that is, $\exists G \in \mathcal{F}_0 : G \cap \alpha = D$ and this holds in $M$ as well, by elementary. Thus

$$M \models \exists \beta < \omega_1 \forall \alpha \in (\beta, \omega_1) \exists G \in \mathcal{F}_0 : G \cap \alpha = D.$$
Thus this holds in $H(\vartheta)$ as well, by elementary. Hence we can select inductively an uncountable $\Delta$-system from $\mathcal{F}_0$. Using elementary again, there is such a $\Delta$-system in $M$ too.

\[ \square \]

For any set-theoretic notion, including background on ♦, see [11].

There are different conventions in general topology whether to add regularity to the definition of a Lindelöf space. In this article, any topological space $X$ is said to be *Lindelöf* iff every open cover has a countable subcover; that is, no separation is assumed.

Finally, we need a few other definitions.

**Definition 2.3.** A collection $\mathcal{U}$ of subsets of a space $X$ is called an $\omega$-cover if for every finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$.

**Definition 2.4.** A collection $\mathcal{N}$ of subsets of a space $X$ is called a local $\pi$-network at the point $x$ if for each open neighbourhood $U$ of $x$ in $X$, there is an $N \in \mathcal{N}$ such that $N \subseteq U$ (it is not required that the sets in $\mathcal{N}$ be open, nor that they contain the point $x$).

3. The Construction

We construct a topology by constructing a sequence $\{U_\gamma : \gamma < \omega_1\}$ of subsets of $\omega_1$ such that $\gamma \in U_\gamma$ for every $\gamma \in \omega_1$. The example will be obtained by first taking the family $\{U_\gamma : \gamma < \omega_1\}$ as a subbasis for a topology on $\omega_1$ and then refining it with a Hausdorff topology of countable weight.

The following lemma will be used to prove the Lindelöf property.

**Lemma 3.1.** Consider a topology on $\omega_1$ generated by a family $\{U_\gamma : \gamma < \omega_1\}$ as a subbase; sets of the form $U_F = \bigcap \{U_\gamma : \gamma \in F\}$ for $F \in [\omega_1]^\omega$ form a base. If for every uncountable family $B \subseteq [\omega_1]^\omega$ of pairwise disjoint sets there is a countable $B' \subseteq B$ such that

$$|\omega_1 \setminus \bigcup \{U_F : F \in B'\}| \leq \omega$$

then the topology is hereditarily Lindelöf.

**Proof.** Fix an open family $\mathcal{U}$; we may assume that $\mathcal{U} = \{U_F : F \in \mathcal{G}\}$ for some $\mathcal{G} \subseteq [\omega_1]^\omega$. Let $M$ be a countably elementary submodel of $H(\vartheta)$ for some sufficiently large $\vartheta$ such that $\{U_\gamma : \gamma \in \omega_1\}, \mathcal{G} \in M$.

It suffices to prove that $\bigcup \mathcal{U}$ is covered by the countable family $\mathcal{V} = \{U_F : F \in \mathcal{G} \cap M\}$. $\mathcal{V}$ clearly covers $(\bigcup \mathcal{U}) \cap M$ thus we consider an arbitrary $\alpha \in (\bigcup \mathcal{U}) \setminus M$. There is $G_0 \in \mathcal{G}$ such that $\alpha \in U_{G_0}$; let $F = G_0 \cap M$. If $F = G_0$ then $\mathcal{V}$ covers $\alpha$ thus we are done.
Otherwise there is an uncountable $\Delta$-system $D \subseteq G$ in $M$ with kernel $F$ by Lemma 2.2. Consider the uncountable, pairwise disjoint family $B = \{G \setminus F : G \in D\}$; by our hypothesis there is a countable $B' \subseteq B$ such that

$$|\omega_1 \setminus \bigcup \{U_H : H \in B'\}| \leq \omega.$$  

$B'$ can be chosen in $M$ since $B \in M$; note that $B' \subseteq M$. Thus the countable set of points not covered also lie in $M$. Therefore, there is $G \setminus F \in B'$ such that $\alpha \in U_{G \setminus F}$. Hence $\alpha \in U_{G \setminus F}$ and $U_G = U_{G \setminus F} \cap U_F$. This completes the proof of the lemma.  

Let us define now the topology which will be used to ensure the Hausdorff property.

**Definition 3.2.** Define a topology on $[\mathbb{R}]^{<\omega}$ as follows. Let $Q \subseteq \mathbb{R}$ be a Euclidean open set and let $Q^* = \{H \in [\mathbb{R}]^{<\omega} : H \subseteq Q\}$. Sets of the form $Q^*$ define a $\rho$ topology on $[\mathbb{R}]^{<\omega}$.

The proof of the following claim is straightforward.

**Claim 3.3.**

1. $([\mathbb{R}]^{<\omega}, \rho)$ is of countable weight,
2. any family $X \subseteq [\mathbb{R}]^{<\omega}$ of pairwise disjoint nonempty sets forms a Hausdorff subspace of $([\mathbb{R}]^{<\omega}, \rho)$.

Let us fix the countable base $\mathcal{W} = \{Q : Q$ is a disjoint union of finitely many intervals with rational endpoints} for $([\mathbb{R}]^{<\omega}, \rho)$.

For the remainder of this section we suppose $\diamondsuit$; thus $2^\omega = \omega_1$ and we can fix an enumeration

- $\{C_\alpha\}_{\alpha < \omega_1} = [\omega_1]^{<\omega}$ such that $C_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$.

Also, there is a $\diamondsuit$-sequence $\{B_\gamma\}_{\gamma < \omega_1}$ on $[\omega_1]^{<\omega}$; that is,

- for every uncountable $B \subseteq [\omega_1]^{<\omega}$ there are stationary many $\beta \in \omega_1$ such that $B \cap [\beta]^{<\omega} = B_\beta$.

The next theorem is the key to our main result; we encourage the reader to first skip the quite technical proof of Theorem 3.4 and go to Corollary 3.8 to see how our main result is deduced. In particular, $\text{IH}(3)$ assures the space is not a $D$-space and $\text{IH}(4)$ makes the space hereditarily Lindelöf.

**Theorem 3.4.** There exist $\{U_\gamma^\alpha\}_{\gamma \leq \alpha}$ and $\varphi_\alpha : (\alpha + 1) \to [\mathbb{R}]^{<\omega}$ for $\alpha < \omega_1$ with the following properties:

1. $U_\gamma^\alpha \subseteq \alpha + 1$ and $U_\alpha^\alpha = \alpha + 1$ for every $\gamma \leq \alpha < \omega_1$, and the family $\varphi_\alpha[\alpha + 1]$ is pairwise disjoint for every $\alpha < \omega_1$.
2. $U_\gamma^\alpha = U_\gamma^\alpha \cap (\alpha + 1)$ and $\varphi_\alpha = \varphi_\alpha \upharpoonright (\alpha + 1)$ for all $\gamma \leq \alpha \leq \alpha_0$. 


Let \( \tau_\alpha \) denote the topology generated by the sets
\[
\{ U_\gamma^\alpha : \gamma \leq \alpha \} \cup \{ \varphi^{-1}_\alpha(W) : W \in \mathcal{W} \}
\]
as a subbase. Let \( U_\beta^\alpha = \bigcap \{ U_\gamma^\alpha : \gamma \in F \} \) for \( F \in [\alpha+1]^{<\omega} \).

**IH(3)** If \( C_\alpha \) is \( \tau_\alpha \) closed discrete then \( \bigcup \{ U_\alpha^\gamma : \gamma \in C_\alpha \} \neq \alpha + 1 \).

**IH(4)** Let \( T_\alpha = \{ \beta \leq \alpha : B_\beta \) is a pairwise disjoint family of finite subsets of \( \beta \) and there is a countable elementary submodel \( M \prec H(\vartheta) \) for some sufficiently large \( \vartheta \) such that (i)-(v) holds from below \}.

(i) \( M \cap \omega_1 = \beta \),
(ii) \( \mathcal{W}, \{ B_\gamma \}_{\gamma < \omega_1} \in M \),
(iii) there is a function \( \varphi \in M \) such that \( \varphi \upharpoonright \beta = \varphi_\beta \upharpoonright \beta \),
(iv) there is an uncountable \( B \in M \) such that \( M \cap B = B_\beta \),
and
(v) there is a \( \{ V_\gamma \}_{\gamma < \omega_1} \in M \) such that \( V_\gamma \cap \beta = U_\gamma^\beta \cap \beta \) for all \( \gamma < \beta \).

Then
(a) if \( \beta \in T_\alpha \) then \( B_\beta \) is a local \( \pi \)-network at \( \beta \) in \( \tau_\alpha \),
(b) if \( \beta \in T_\alpha \cap \alpha \) then for every \( V \in \tau_\alpha \) with \( \beta \in V \) the family
\[ \{ U_\gamma^\alpha : F \in B_\beta, F \subseteq V \} \]
is an \( \omega \)-cover of \( (\beta, \alpha) \).

**Proof.** We prove by induction on \( \alpha < \omega_1 \) with inductional hypotheses **IH(1)-IH(4)!** Suppose we constructed \( \{ U_\gamma^\beta \}_{\gamma \leq \beta} \) for \( \beta < \alpha \). Let
\[
U_\gamma^{<\alpha} = \bigcup \{ U_\gamma^\beta : \gamma \leq \beta < \alpha \} \text{ and } \varphi_{<\alpha} = \bigcup \{ \varphi_\beta : \beta < \alpha \}.
\]

Let \( \tau^-_\alpha \) denote the topology on \( \alpha \) generated by the sets
\[
\{ U_\gamma^{<\alpha} : \gamma < \omega_1 \} \cup \{ \varphi^{-1}_{<\alpha}(W) : W \in \mathcal{W} \}
\]
as a subbase. Let \( U_F^{<\alpha} = \bigcap \{ U_\gamma^{<\alpha} : \gamma \in F \} \) for \( F \in [\alpha]^{<\omega} \). Note that if \( \beta \in T_\alpha \cap \alpha \) then \( \beta \in T_\alpha \) for every \( \alpha' \in (\beta, \alpha) \); hence \( \mathbf{N}(i) \) and \( \mathbf{N}(ii) \) below holds by **IH(4):**

\( \mathbf{N}(i) \) for every \( V \in \tau^-_\alpha \) with \( \beta \in V \) the family
\[ \{ U_F^{<\alpha} : F \in B_\beta, F \subseteq V \} \]
is an \( \omega \)-cover of \( (\beta, \alpha) \);
\( \mathbf{N}(ii) \) \( B_\beta \) is a local \( \pi \)-network at \( \beta \) in \( \tau^-_\alpha \).

Therefore, it suffices to define \( U_\alpha^\alpha = \alpha + 1 \) and
\[
U_\gamma^\alpha = \begin{cases} U_\gamma^{<\alpha} & \text{or} \\ U_\gamma^{<\alpha} \cup \{ \alpha \} \end{cases}
\]
and \( \varphi_\alpha \upharpoonright \alpha = \varphi_{<\alpha}, \varphi_\alpha(\alpha) = x_\alpha \in [\mathbb{R}]^{<\omega} \) such that
D(i) $x_\alpha$ is disjoint from $x_\beta = \varphi_\alpha(\beta)$ for all $\beta < \alpha$,

D(ii) if $\beta \in T_\alpha \cap \alpha$ then for every $\beta \in V \in \tau_\alpha$ the family
$$\{ U_\alpha^F : F \in B_\beta, F \subseteq V \}$$
is an $\omega$-cover of $(\beta, \alpha)$,

D(iii) if $C_\alpha$ is $\tau^-_\alpha$ closed discrete then $\alpha \notin U_\alpha^\gamma$ for all $\gamma \in C_\alpha$,

D(iv) if $\alpha \in T_\alpha$ then $B_\alpha$ is a local $\pi$-network at $\alpha$ in $\tau_\alpha$.

**Case I.** $T_\alpha \cap \alpha = \emptyset$

Let $U_\alpha^\alpha = \alpha + 1$, $U_\gamma^\alpha = U_\gamma^\gamma$ for $\gamma < \alpha$. We proceed differently according to whether $\alpha \notin T_\alpha$ or $\alpha \in T_\alpha$.

**Subcase A.** $\alpha \notin T_\alpha$

Pick any $x_\alpha \in [\mathbb{R}]^{<\omega}$ disjoint from $x_\beta$ for all $\beta < \alpha$. Clearly, D(i)-D(iv) are satisfied.

**Subcase B.** $\alpha \in T_\alpha$

It is clear that D(ii) and D(iii) are satisfied. Let $M$ be a countable elementary submodel of $H(\vartheta)$ for some sufficiently large $\vartheta$ showing that $\alpha \in T_\alpha$. To find the appropriate $x_\alpha \in [\mathbb{R}]^{<\omega}$ we need that $B_\alpha$ is a local $\pi$-network at $\alpha$ in $\tau_\alpha$. Since $\{ \varphi^{-1}_\alpha(W) : W \in \mathcal{W}, x_\alpha \in W \}$ will be a base at $\alpha$ in $\tau_\alpha$ we need that for all $W \in \mathcal{W}$ such that $x_\alpha \in W$ there is an $F \in B_\alpha$ such that $\varphi[F] \subseteq W$. Since $\varphi[F] \subseteq W$ iff $\bigcup \varphi[F] \in W$, we need to find an accumulation point of the finite sets $\{ \bigcup \varphi[F] : F \in B_\alpha \}$. We prove the following which will suffice:

**Claim 3.5.** There is an $x_\alpha \in [\mathbb{R}]^{<\omega}$ such that $x_\alpha \cap x_\beta = \emptyset$ for all $\beta < \alpha$ and for all $W \in \mathcal{W}$ such that $x_\alpha \in W$ we have
$$M \models |\{ F \in B : \bigcup \varphi[F] \in W \}| > \omega.$$

Indeed, D(i) is satisfied. Let us check D(iv); clearly,
$$M \models \exists F \in B : \bigcup \varphi[F] \in W$$
for every $W \in \mathcal{W}$ such that $x_\alpha \in W$. Hence there is $F \in B \cap M = B_\alpha$ such that $\bigcup \varphi[F] = \bigcup \varphi_\alpha[F] \in W$, that is $\varphi_\alpha[F] \subseteq W$. Thus $B_\alpha$ is a local $\pi$-network at $\alpha$ in $\tau_\alpha$; that is, D(iv) is satisfied.
Proof of Claim 3.5. Since $M | B > \omega$ there is $\tilde{B} \in [B]^\omega \cap M$ and $k \in \omega, \{n_i : i < k\} \subseteq \omega$ such that $|F| = k$ for all $F \in \tilde{B}$ and if $F = \{\gamma_i : i < k\}$ then $|\varphi(\gamma_i)| = n_i$ for all $i < k$.

Let $s = \sum_{i<k} n_i$. Now consider the pairwise disjoint $s$-element subsets $\{\cup \varphi[F] : F \in \tilde{B}\}$ in $R$. Clearly $M \models (\text{there are uncountably many pairwise disjoint } x \in [R]^s \text{ such that } |\{F \in \tilde{B} : \cup \varphi[F] \in W\}| > \omega \text{ for every } W \in W \text{ with } x \in W)$. Hence, there is $x_\alpha \in [R]^s$ disjoint from $x_\beta$ for all $\beta < \alpha$ such that $|\{F \in \tilde{B} : \cup \varphi[F] \in W\}| > \omega \text{ for all } W \in W \text{ with } x_\alpha \in W$. Thus $M \models |\{F \in \tilde{B} : \cup \varphi[F] \in W\}| > \omega$

which we wanted to prove. \hfill \Box

Case II. $T_\alpha \cap \alpha \neq \emptyset$

Let $T_\alpha \cap \alpha = \{\beta_n : n \in \omega\}$ and $\{G_n : n \in \omega\} \subseteq [\alpha]^{<\omega}$ such that for all $\beta \in T_\alpha \cap \alpha$ and $G \subseteq (\beta, \alpha)$ there are infinitely many $n \in \omega$ such that $\beta = \beta_n$ and $G = G_n$. Let $\{V_k(\beta) : k < \omega\}$ denote a decreasing neighbourhood base for the point $\beta \in T_\alpha \cap \alpha$ in $\tau^-_{\alpha}$. Note that $\{V_n(\beta_n) : n \in \omega, \beta_n = \beta\}$ is a base for $\beta \in T_\alpha \cap \alpha$.

Subcase A. $\alpha \notin T_\alpha$

We need the following claim:

Claim 3.6. There is $F_n \in B_{\beta_n}$ for $n \in \omega$ such that

A(i) $F_n \subseteq V_n(\beta_n)$,
A(ii) $G_n \subseteq U^{\leq \alpha}_{F_n}$,
A(iii) $F_n \cap C_\alpha = \emptyset$ if $C_\alpha$ is $\tau^-_{\alpha}$ closed discrete,

Proof. There is $V \in \tau^-_{\alpha}$ such that $\beta_n \in V \subseteq V_n(\beta_n)$ and if $C_\alpha$ is closed discrete, then $C_\alpha \cap V \subseteq \{\beta_n\}$. The family

$$\{U^{\leq \alpha}_F : F \in B_{\beta_n}, F \subseteq V\}$$

is an $\omega$-cover of $(\beta_n, \alpha)$ by N(ii), thus there is $F_n \in B_{\beta_n}$ such that $F_n \subseteq V$ and $G_n \subseteq U^{\leq \alpha}_{F_n}$.

\hfill \Box
Let $U^\alpha_\gamma = \alpha + 1$ and for $\gamma < \alpha$ let

$$U^\alpha_\gamma = \begin{cases} U^\alpha_\gamma & \text{if } \gamma \notin \cup \{F_n : n \in \omega\}, \\ U^\alpha_\gamma \cup \{\alpha\} & \text{if } \gamma \in \cup \{F_n : n \in \omega\}. \end{cases}$$

Pick any $x_\alpha \in [\mathbb{R}]^{<\omega}$ disjoint from $x_\beta$ for all $\beta < \alpha$.

$D(i)$, $D(iii)$, and $D(iv)$ are trivially satisfied. Let us check $D(ii)$; fix $\beta \in T_{\alpha} \cap \alpha$, any neighbourhood $V \in \tau_{\alpha}$ such that $\beta \in V$, and a finite subset $G \subseteq (\beta, \alpha)$. We show that there is an $F \in B_{\beta}$, such that $U^\alpha_F$ covers $G \cup \{\alpha\}$ and $F \subseteq V$. There is $n \in \omega$ such that $\beta_n = \beta$, $G_n = G$, and $V_n(\beta_n) \subseteq V$; then $F_n \in B_{\beta_n}$ and $F = F_n$ does the job by $A(i)$, $A(ii)$ and the fact that $\alpha \in U^\alpha_{F_n}$.

**Subcase B. $\alpha \in T_{\alpha}$**

Let $M$ be a countable elementary submodel of $H(\varnothing)$ for some sufficiently large $\varnothing$ showing that $\alpha \in T_{\alpha}$. Since $M \models |B| > \omega$ there is $\bar{B} \in [\bar{B}]^{\omega_1} \cap M$ and $k \in \omega$, $\{n_i : i < k\} \subseteq \omega$ such that $|F| = k$ for all $F \in \bar{B}$ and if $F = \{\gamma_i : i < k\}$ then $|\varphi(\gamma_i)| = n_i$ for all $i < k$. Let $s = \sum_{i<k} n_i$. Enumerate $\alpha$ as $\{\alpha_n : n \in \omega\}$.

**Claim 3.7.** There are $F_n \in B_{\beta_n}$ and $W_n \in \mathcal{W}$ for $n \in \omega$ such that

- $B(i)$ $F_n \subseteq V_n(\beta_n)$,
- $B(ii)$ $G_n \subseteq U^\alpha_{F_n}$,
- $B(iii)$ $F_n \cap C_\alpha = \emptyset$ if $C_\alpha$ is $\tau_\alpha$ closed discrete,
- $B(iv)$ $W_n = (\cup \{Q_{n,i} : i < s\})^*$ is a basic open set of the topology $\rho$ corresponding to $s$ many disjoint rational intervals $\{Q_{n,i} : i < s\}$ of diameter less than $\frac{1}{n}$,
- $B(v)$ $Q_{n+1,i} \subseteq Q_{n,i}$ for every $i < s$ in the Euclidean topology,
- $B(vi)$ $\varphi(\alpha_n)$ is disjoint from $\cup \{Q_{n,i} : i < s\}$,
- $B(vii)$ and finally

$$M \models |\{F \in \bar{B} : F \subseteq \cap \{V_{F_k} : k \leq n\} \text{ and } \cup \varphi[F] \in W_n\}| > \omega.$$  

**Proof.** We construct $F_n$ and $W_n$ by induction on $n \in \omega$. Suppose we constructed $F_k$ and $W_k$ for $k < n$ such that the hypotheses $B(i)$-$B(vii)$ above are satisfied.

Let $D = \{F \in \bar{B} : F \subseteq \cap \{V_{F_k} : k < n\} \text{ and } \cup \varphi[F] \in W_{n-1}\}$ if $n > 0$ and $D = \bar{B}$ if $n = 0$; then $M \models |D| > \omega$. Just as in Claim 3.5

$$M \models \text{(there are uncountably many pairwise disjoint } x \in \mathbb{R}^s \cap W_{n-1} \text{ such that } |\{F \in D : \cup \varphi[F] \in W\}| > \omega \text{ for every } W \in \mathcal{W} \text{ with } x \in W).$$
Choose \( x \in \mathbb{R}^s \cap W_{n-1} \) such that \( |\{ F \in D : \cup_{\varphi[F] \in W} \} | > \omega \) for every \( W \in \mathcal{W} \) with \( x \in W \) and \( x \cap \varphi(\alpha_n) = \emptyset \). Let \( x = \{ x_i : i < s \} \) and choose \( W_n = (\cup \{ Q_{n,i} : i < s \})^* \) such that

- \( Q_{n,i} \) is a rational interval of diameter less than \( \frac{1}{n} \) for every \( i < s \),
- \( x_i \in Q_{n,i} \) for every \( i < s \),
- \( Q_{n,i} \subseteq Q_{n-1,i} \) for every \( i < s \) in the Euclidean topology (if \( n > 0 \)),
- \( \cup \{ Q_{n,i} : i < s \} \cap \varphi(\alpha_n) = \emptyset \).

Let \( D' = \{ F \in \tilde{B} : F \subseteq \cap \{ V_{F_k} : k < n \}, \cup_{\varphi[F]} \in W_n \text{ and } \beta_n < \min F \} \); clearly, \( M \models |D'| > \omega \). Let \( V \in \tau_{\alpha} \) such that \( \beta_n \in V \subseteq V_n(\beta_n) \) and \( V \cap C_{\alpha} \subseteq \{ \beta_n \} \) if \( C_{\alpha} \) is \( \tau_{\alpha} \) closed discrete. Applying \( N(i) \) to \( F \cup G_n \) for \( F \in D' \cap M \) gives us that there is \( F_n(F) \in B_{\beta_n} \) such that \( F_n(F) \subseteq V \) and

\[
U_{F_n(F)}^\alpha \text{ covers } F \cup G_n
\]

and hence \( V_{F_n(F)} \) covers \( F \cup G_n \). Thus

\[
M \models (\text{for every } F \in D' \text{ there is } F_n(F) \in B_{\beta_n} \text{ such that } F_n(F) \subseteq V \text{ and } V_{F_n(F)} \text{ covers } F \cup G_n.).
\]

Finally, note that \( M \models |B_{\beta_n}| \leq \omega \); thus

\[
M \models (\text{there is } F_n \in B_{\beta_n} \text{ such that } F_n \subseteq V \text{ and } V_{F_n} \text{ covers } F \cup G_n \text{ for uncountably many } F \in D').
\]

It is now easily checked that \( F_n \) and \( W_n \) satisfies properties \( \mathbf{B}(i) - \mathbf{B}(vii) \). \( \square \)

Let \( U^\alpha_\gamma = \alpha + 1 \) and for \( \gamma < \alpha \) let

\[
U^\alpha_\gamma = \begin{cases} U^\gamma_{\beta} & \text{if } \gamma \notin \cup F_n : n \in \omega \},
\end{cases}

U^\alpha_\gamma \cup \{ \alpha \} \text{ if } \gamma \in \cup \{ F_n : n \in \omega \}.
\]

Let \( x_\alpha \in [\mathbb{R}]^\omega \) be the unique \( s \)-element subset of \( \mathbb{R} \) in the intersection \( \cap \{ \cup Q_{n,i} : i < s \} ; n \in \omega \} \); existence and uniqueness follows from \( \mathbf{B}(iv) \) and \( \mathbf{B}(v) \), and \( x_\alpha \) is disjoint from \( x_\beta \) for all \( \beta < \alpha \) by \( \mathbf{B}(vi) \). Note that

\[
\cap \{ U^\alpha_{F_k} : k \leq n \} \cap \varphi^{-1}(W_n) : n \in \omega \}
\]

is a base for the point \( \alpha \) in \( \tau_\alpha \).

\( \mathbf{D}(i) \) is satisfied by \( \mathbf{B}(vi) \) and the fact that \( x_\alpha \subseteq \cup \{ Q_{n,i} : i < s \} \).

Let us check \( \mathbf{D}(ii) \); fix \( \beta \in \tau_\alpha \cap \alpha \), any neighbourhood \( V \in \tau_\alpha \) such that \( \beta \in V \), and a finite subset \( G \subseteq (\beta, \alpha) \). We show that there is an \( F \in B_\beta \), such that \( U^\alpha_F \) covers \( G \cup \{ \alpha \} \) and \( F \subseteq V \). There is \( n \in \omega \) such
that $\beta_n = \beta$, $G_n = G$, and $V_n(\beta_n) \subseteq V$; $F_n \in B_{\beta_n}$ does the job by B(i), B(ii) and the fact that $\alpha \in U_{F_n}^\alpha$.

D(iii) is satisfied by B(iii) and the definition of $U_n^\alpha$.

Finally, let us check D(iv); it suffices to show that for every $n \in \omega$ there is $F \in B_\alpha$ such

$$F \subseteq \bigcap \{U_{F_k}^\alpha : k \leq n\} \cap \varphi_\alpha^{-1}(W_n).$$

Condition B(vii) gives us this, using the observation that $\varphi_\alpha[F] \subseteq W$ iff $\bigcup \varphi_\alpha[F] \in W$ for any $F \in B_\alpha$ and $W \in \mathcal{W}$.

By all means, this completes the proof of the theorem.

□

Now we are ready to deduce our main result.

**Corollary 3.8.** Suppose that $\{U_\gamma^\alpha\}_{\gamma \leq \alpha}$ and $\varphi_\alpha : (\alpha + 1) \to [\mathbb{R}]^{<\omega}$ for $\alpha < \omega_1$ are as in Theorem 3.4 and let $U_\gamma = \bigcup \{U_\gamma^\alpha : \gamma \leq \alpha < \omega_1\}$ for $\gamma < \omega_1$ and $\varphi = \bigcup \{\varphi_\alpha : \alpha < \omega_1\}$. Let $\tau$ denote the topology on $\omega_1$ generated by the sets

$$\{U_\gamma : \gamma < \omega_1\} \cup \{\varphi^{-1}(W) : W \in \mathcal{W}\}$$

as a subbase.

The space $(\omega_1, \tau)$ is hereditarily Lindelöf, Hausdorff but not a D-space. Also, $(\omega_1, \tau)$ has countable $\Psi$-weight.

**Proof.** First, we show that $(\omega_1, \tau)$ is hereditarily Lindelöf and Hausdorff. We need the following observation.

**Claim 3.9.** A Hausdorff topology of countable weight $\tau_{sc}$ refined by a hereditarily Lindelöf topology $\tau_{hl}$ on some set $X$ is again a hereditarily Lindelöf, Hausdorff topology on $X$.

**Proof.** Let $\tau_{ref}$ denote the topology generated by $\tau_{sc} \cup \tau_{hl}$ as a subbase; that is, $\tau_{ref}$ is the common refinement of $\tau_{sc}$ and $\tau_{hl}$. $\tau_{ref}$ is clearly Hausdorff, we prove that for any open family $\mathcal{U} \subseteq \tau_{ref}$ there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$. We can suppose that

$$\mathcal{U} = \{U_i \cap V_j^i : i \in \omega, j \in I_i\}$$

where $\{U_i : i \in \omega\} \subseteq \tau_{sc}$ and $\{V_j^i : i \in \omega, j \in I_i\} \subseteq \tau_{hl}$ for some index sets $\{I_i : i \in \omega\}$. For every $i \in \omega$ there is a countable $J_i \subseteq I_i$ such that

$$U_i \cap \bigcup \{V_j^i : j \in I_i\} = U_i \cap \bigcup \{V_j^i : j \in J_i\}$$

by the hereditarily Lindelöfness of $\tau_{hl}$. Thus

$$\bigcup \mathcal{U} = \bigcup \{U_i \cap V_j^i : i \in \omega, j \in J_i\}$$
which completes the proof. □

Therefore, it suffices to prove that the topology generated by \( \{ U_\gamma : \gamma < \omega_1 \} \) as a subbase on \( \omega_1 \) is hereditarily Lindelöf. Lemma 3.1 and the proposition below gives us this result. Let \( U_F = \bigcap \{ U_\gamma : \gamma \in F \} \) for \( F \in [\omega_1]^{<\omega} \).

**Proposition 3.10.** For any uncountable family of pairwise disjoint sets \( B \subseteq [\omega_1]^{<\omega} \), there is a countable \( B' \subseteq B \) such that \( \bigcup \{ U_F : F \in B' \} \) is a cover, moreover an \( \omega \)-cover of a tail of \( \omega_1 \).

**Proof.** Fix some uncountable family \( B \subseteq [\omega_1]^{<\omega} \) of pairwise disjoint sets. There is an \( M \prec H(\vartheta) \) for some sufficiently large \( \vartheta \) such that \( B, \varphi, \{ U_\gamma : \gamma < \omega_1 \}, \{ B_\gamma : \gamma < \omega_1 \}, W \in M \) and \( M \cap \omega_1 = \beta \) and \( B \cap M = B \cap [\beta]^{<\omega} = B_\beta \).

We claim that \( \bigcup \{ U_F : F \in B' \} \) is an \( \omega \)-cover of \( \omega_1 \setminus (\beta + 1) \) for the countable \( B' = B_\beta \). Indeed, fix some finite \( K \subseteq \omega_1 \setminus (\beta + 1) \) and let \( \alpha \in \omega_1 \setminus (\beta + 1) \) such that \( K \subseteq \alpha \). Then \( \beta \in T_\alpha \) ensured by the model \( M \), and hence there is some \( F \in B_\beta = B' \) such that \( K \subseteq U_F \subseteq U_{F_{\alpha}} \subseteq U_F \) by IH(4).

Now we prove that \( (\omega_1, \tau) \) is not a \( D \)-space. Consider the neighbourhood assignment \( \gamma \mapsto U_\gamma \); we show that \( \bigcup \{ U_\gamma : \gamma \in C \} \neq \omega_1 \) for every closed discrete \( C \subseteq \omega_1 \). Since \( (\omega_1, \tau) \) is Lindelöf, \( |C| \leq \omega \) and hence there is \( \alpha < \omega_1 \) such that \( C_\alpha = C \). It suffices to note that \( C_\alpha \) is \( \tau_\alpha \) closed discrete if \( \tau \) closed discrete; indeed, then \( \bigcup \{ U_\gamma : \gamma \in C_\alpha \} \neq \alpha + 1 \) by IH(3).

Finally, \( (\omega_1, \tau) \) has countable \( \Psi \)-weight since \( \tau \) is a refinement of a Hausdorff topology which is of countable weight. □

4. Further properties

In [12] the authors asked the following:

**Problem 4.1 ([12], Problem 4.6).** Suppose that a space \( X \) has the property that for every open neighbourhood assignment \( \{ U_x : x \in X \} \) there is a second countable subspace \( Y \) of \( X \) such that \( \bigcup \{ U_x : x \in Y \} = X \) (dually second countable, in short). Is \( X \) a \( D \)-space?

Our construction answers this question in the negative.

**Proposition 4.2.** The space \( X \) constructed in Corollary 3.8 is dually second countable, however not a \( D \)-space.
Proof. The space $X$ has the property that every countable subspace is second countable; indeed, the subspace topology on $\alpha \in \omega_1$ is generated by the sets $U_\beta \cap \alpha$ for $\beta < \alpha$ and $\{\varphi^{-1}(W) : W \in \mathcal{W}\}$, using the notations of the previous section. Therefore, by the Lindelöf property, for every open neighbourhood assignment there is a countable and hence second countable subspace whose neighbourhoods cover the space. \(\square\)

Our aim now is to prove that the space constructed in Corollary 3.8 has the property that all its finite powers are Lindelöf. Indeed, by a theorem of Gerlits and Nagy [9], a space has all finite powers Lindelöf if and only if the space is an $(\varepsilon)$-space, i.e., every $\omega$-cover has a countable $\omega$-subcover.

Let us call our space from Corollary 3.8 $X$, and now establish the following theorem:

**Theorem 4.3.** Every subspace of $X$ is an $(\varepsilon)$-space.

**Proof.** First, let us prove the following analogue of Lemma 3.1.

**Lemma 4.4.** Consider a topology on $\omega_1$ generated by a family $\{U_\gamma : \gamma < \omega_1\}$ as a subbase. If for every uncountable family $B \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets there is a countable $B' \subseteq B$ such that $\{U_F : F \in B'\}$ is an $\omega$-cover of a tail of $\omega_1$ then the topology is a hereditarily $(\varepsilon)$-space.

**Proof.** Fix $Y \subseteq X$ and an $\omega$-cover $\mathcal{U}$ of $Y$; we can suppose that $\mathcal{U} = \{\cup\{U_{F_i} : i < m\} : \{F_i : i < m\} \in \mathbb{F}\}$ for some $\mathbb{F} \subseteq [\omega_1]^{<\omega}$. Let $M$ be a countably elementary submodel of $H(\vartheta)$ for some sufficiently large $\vartheta$ such that $\{U_\gamma : \gamma \in \omega_1\}, \mathbb{F} \in M$. It suffices to prove the following.

**Claim 4.5.** $M \cap \mathcal{U}$ is a countable $\omega$-cover of $Y$.

**Proof.** Let $K \in [Y]^{<\omega}$ and let $L = K \cap M$. Clearly, $M \cap \mathcal{U}$ covers $K$ if $K = L \subseteq M$; thus, we can suppose that $K \not= L$ and hence $K \not\subseteq M$. There is some $\{F_i : i < m\} \in \mathbb{F}$ such that $K \subseteq \cup\{U_{F_i} : i < m\}$. Let $D_i = F_i \cap M$ for $i < m$ and we can suppose that there is some $n \leq m$ such that $F_i \not= D_i$ for $i < n$ and $F_i = D_i$ for $n \leq i < m$. It follows from Lemma 2.2 that there is an uncountable sequence $\{\{F_\alpha^i : i < m\} : \alpha < \omega_1\} \subseteq \mathbb{F}$ in $M$ such that

1. $\{F_\alpha^i : \alpha < \omega_1\}$ is an uncountable $\Delta$-system with kernel $D_i$ for every $i < n$,
2. $F_\alpha^i = F_i$ for all $\alpha < \omega_1$ and $\alpha < i < m$,
3. $\beta \in U_{F_i}$ iff $\beta \in U_{F_\alpha^i}$ for every $\beta \in L$ and $\alpha < \omega_1, i < m$.\[\]
The uncountable family \( \{ F^\alpha \setminus D_i : \alpha < \omega_1 \} \) is pairwise disjoint for every \( i < n \). Hence if we let \( F^\alpha = \bigcup_{i<n} (F^\alpha \setminus D_i) \) for \( \alpha < \omega_1 \) then there is \( \Theta \subseteq [\omega_1]^\omega \cap M \) such that \( B = \{ F^\alpha : \alpha \in \Theta \} \) is pairwise disjoint as well. By our hypothesis and elementary of \( \beta \in U \) also lie in \( \omega \leq n \) and \( \beta \in U \) implies that \( \beta \in \omega \). Hence if we let \( \beta \in U \). We claim that Proposition 3.10 implies that \( X \) is a hereditarily (\( \varepsilon \))-space. Indeed, our topology \( \tau \) on \( \omega_1 \) is generated by the sets
\[
\{ U'_\delta : \delta < \omega_1 \} = \{ U_\gamma : \gamma < \omega_1 \} \cup \{ \varphi^{-1}(W) : W \in W \}
\]
as a subbase. Suppose that \( B \subseteq [\omega_1]^\omega \) is an uncountable family of pairwise disjoint sets; since \( W \) is countable, there is \( B_0 \in [B]^\omega \) such that \( U'_\delta \in \{ U_\gamma : \gamma \in \omega_1 \} \) for \( \delta \in B_0 \). Thus by Proposition 3.10, there is some countable \( B' \subseteq B_0 \) such that \( \{ U'_F : F \in B' \} \) is an \( \omega \)-cover of a tail of \( \omega_1 \). Hence the assumption of Lemma 4.4 holds for \( X \), thus \( X \) is a hereditarily (\( \varepsilon \))-space.

We claim that Proposition 3.10 and Lemma 4.4 implies that \( X \) is a hereditarily (\( \varepsilon \))-space. Indeed, our topology \( \tau \) on \( \omega_1 \) is generated by the sets
\[
\{ U'_\delta : \delta < \omega_1 \} = \{ U_\gamma : \gamma < \omega_1 \} \cup \{ \varphi^{-1}(W) : W \in W \}
\]
as a subbase. Suppose that \( B \subseteq [\omega_1]^\omega \) is an uncountable family of pairwise disjoint sets; since \( W \) is countable, there is \( B_0 \in [B]^\omega \) such that \( U'_\delta \in \{ U_\gamma : \gamma \in \omega_1 \} \) for \( \delta \in B_0 \). Thus by Proposition 3.10, there is some countable \( B' \subseteq B_0 \) such that \( \{ U'_F : F \in B' \} \) is an \( \omega \)-cover of a tail of \( \omega_1 \). Hence the assumption of Lemma 4.4 holds for \( X \), thus \( X \) is a hereditarily (\( \varepsilon \))-space.

A well known weakening of \( \diamond \) is Ostaszewski’s \( \clubsuit \), which is known to be consistent with \( \omega_1 < 2^{\omega} \). We remark that \( \clubsuit \) is not enough to construct a space of size \( \omega_1 \) which is Hausdorff and Lindelöf but not a \( D \)-space.

**Claim 4.6.** It is consistent that \( \clubsuit \) holds, \( 2^{\omega} \) is arbitrarily large, and every \( T_1 \) Lindelöf space of size less than \( 2^{\omega} \) is a \( D \)-space.

**Proof.** It is known that \( T_1 \) Lindelöf spaces of size less than the dominating number \( \mathfrak{d} \) are Menger, and L. Aurichi proved that every Menger space is a \( D \)-space [2]. Thus, it suffices to show that there is a model of ZFC where \( \clubsuit \) holds, \( 2^{\omega} \) is arbitrarily large, and \( \mathfrak{d} = 2^{\omega} \). I. Juhász proved in an unpublished note that it is consistent that \( \clubsuit \) holds, \( 2^{\omega} \) is arbitrarily large, and Martin’s Axiom holds for countable posets; for a proof see [4]. It is easy to see that Martin’s Axiom for countable posets imply \( \mathfrak{d} = 2^{\omega} \).
Let us state some questions concerning Theorem 4.3. We do not know whether the analogue of Claim 3.9 holds for hereditarily \((\varepsilon,\varepsilon_0)\)-spaces.

**Question 5.1.** Suppose that \(\tau\) and \(\sigma\) are second countable and hereditarily \((\varepsilon,\varepsilon_0)\)-space topologies respectively on some set \(X\). Is the topology generated by \(\tau \cup \sigma\) a hereditarily \((\varepsilon,\varepsilon_0)\)-space again?

We do not know if being a hereditarily \((\varepsilon,\varepsilon_0)\)-space implies the hereditarily Lindelöfness of finite powers.

**Question 5.2.** Suppose that a space \(X\) is a hereditarily \((\varepsilon,\varepsilon_0)\)-space. Is \(X^n\) hereditarily Lindelöf for all \(n \in \omega\)?

The following might be easier, nonetheless seems to be open.

**Question 5.3.** Suppose that a space \(X\) has the property that \(A^2\) is Lindelöf for all \(A \subseteq X\). Is \(X^2\) hereditarily Lindelöf?

We mention two other versions of the question above.

**Question 5.4.** (i) Suppose that a space \(X\) has the property that \(A \times B\) is Lindelöf for all \(A, B \subseteq X\). Is \(X^2\) hereditarily Lindelöf?

(ii) Suppose that the spaces \(X, Y\) have the property that that \(A \times B\) is Lindelöf for all \(A \subseteq X\) and \(B \subseteq Y\). Is \(X \times Y\) hereditarily Lindelöf?

Of course, the main interest is in obtaining a regular counterexample to van Douwen’s question. We conjecture that one should be able to modify our construction in such a way that the sets \(\{U_\gamma : \gamma \in \omega_1\} \cup \{\omega_1 \setminus U_\gamma : \gamma \in \omega_1\}\) generate a 0-dimensional, \(T_1\) topology that is not a \(D\)-space and has some additional interesting covering properties. E.g.,

**Question 5.5.** Can we modify the construction to obtain a 0-dimensional \(T_1\) (hence regular) Lindelöf non \(D\)-space?

Finally, let us finish with a more general question. We believe that our construction can be modified so that its finite powers are hereditarily Lindelöf, thus its \(\omega\)th power as well.

**Question 5.6.** Suppose that a regular space \(X\) has the property that \(X^\omega\) is hereditarily Lindelöf. Is \(X\) a \(D\)-space?
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