Unsteady vibration model of the euler-bernoulli beam taking into account diffusion

A V Zemskov1,2 and D V Tarlakovskii2,1

1 Moscow Aviation Institute (National Research University), Volokolamskoe highway, 4, Moscow, Russia
2 Research Institute of Mechanics, Lomonosov Moscow State University, Michurinskiy pr., 1, Moscow, Russia

E-mail: azemskov1975@mail.ru

Abstract. In this paper we study the unsteady vibration problem for the Euler-Bernoulli beam taking mass transfer into account. To formulate the problem, we use the coupled elastic diffusion model for a homogeneous isotropic multicomponent continuum. Equations of motion and mass transfer are written in the variational form of the Hamilton functional. Using the necessary condition for the Hamilton functional stationarity, we obtain the unsteady plane bending model for an elastodiffusive beam. To solve this problem, we use the integral Laplace transform with respect to time and expansion in Fourier series with respect to spatial coordinate.

Introduction
In mechanics, a calculation of structures and their individual elements, which working under conditions of various intensive physical loads, often leads to the consideration of a unsteady coupled fields problems. As a special case, to the problems of mechanodiffusion.

Among the modern publications devoted to this problem can be noted [1-15]. It is important to say, that in the above-mentioned works the mechanodiffusion problems are solved mainly in stationary formulations [4, 7, 8]. As a rule, unsteady problems are solved numerically. In this case, either difference schemes are used, or the transformants of unknown functions are found with help of the Laplace and Fourier integral transforms. But the inversion of the integral Laplace transform is performed numerically [9-12, 14, 15].

In this paper, we consider the unsteady vibrations problem of the Euler-Bernoulli beam with taking into account masstransfer effect. We also propose a method for constructing a solution based on the use of the integral Laplace transform and eigenfunction expansion. At present, there are no publications on the analytical solution to this problem.

1. Problem formulation
The Euler-Bernoulli beam unsteady vibrations are investigated in this paper. The scheme of applied forces and bending moments, as well as axes orientation in a rectangular Cartesian coordinate system, are presented in Fig. 1.

For the mathematical problem formulation we use the coupled elastic diffusion medium model in a rectangular Cartesian coordinate system, which has the next form [1-18]
\[ \dot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i, \quad \eta^{(q)}(t) = -\frac{\partial J^{(q)}}{\partial x_i} + Y^{(q)}(q=1,N). \]  

**Figure 1.** Figure to the statement of the problem.

where \( \sigma_{ij} \) and \( J_i^{(q)} \) are the stress tensor components and the diffusion flux vector respectively, which are defined as follows:

\[ \sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l}, \quad J_i^{(q)} = -D_i^{(q)} \frac{\partial \eta^{(q)}}{\partial x_i} + \Lambda^{(q)} \frac{\partial^2 u_k}{\partial x_j \partial x_i}. \]

Here the dots denote the time derivative. All quantities in (1) and (2) are dimensionless. For them the following notation is used:

\[ x_i = \frac{x_i}{l}, \quad u_i = \frac{u_i}{l}, \quad \tau = \frac{Ct}{l}, \quad C_{ijkl} = \frac{C_{ijkl}}{C_{1111}}, \quad \alpha_{ij} = \frac{\alpha_{ij}^{(q)}}{C_{1111}}, \quad D_i^{(q)} = \frac{D_i^{(q)}}{C^2}, \quad \Lambda^{(q)} = \frac{\Lambda^{(q)}}{\rho R T_0 C}, \quad F_i = \frac{F_i}{C_{1111}}, \quad Y^{(q)} = \frac{Y^{(q)}}{C}. \]

where \( t \) is time; \( x_i \) are rectangular Cartesian coordinates; \( \rho \) is the medium density; \( u_i \) are displacement vector components; \( C_{ijkl} \) are elastic constant tensor components; \( T_0 \) is initial temperature; \( c \) is speed of light; \( D_i^{(q)} \) are the self-diffusion coefficients; \( R \) is the universal gas constant; \( m^{(q)} \) is the molar mass; \( \eta^{(q)} = n^{(q)} - n_0^{(q)} \) is increment of concentration of \( q \)-th component in the \( N \)-component medium; \( n_0^{(q)} \) and \( n^{(q)} \) are the initial and actual concentrations of the medium; \( \alpha_{ij}^{(q)} \) are coefficients characterizing the medium volumetric changes due to diffusion; \( l \) is beam length; \( F_i \) and \( Y^{(q)} \) are mechanical and diffusive bulk perturbations.

The formulation of the problem is completed by the initial and boundary conditions. 

**The initial conditions are:**

\[ u_i \big|_{t=0} = u_{i0}, \quad \frac{\partial u_i}{\partial t} \big|_{t=0} = v_{i0}, \quad \eta^{(q)} \big|_{t=0} = \eta_0^{(q)}, \quad q=1,N. \] 

Here \( u_{i0}, v_{i0}, \eta_0^{(q)} \) are given functions of spatial coordinates. Further in the paper we assume that \( t_0 = 0, u_{i0} = 0, v_{i0} = 0, \eta_0^{(q)} = 0. \)
Boundary conditions (domain $G$ is bounded; $n_i$ are components of the outer normal unit vector to
$\partial G$ , $\partial G = \Pi_u \cup \Pi_\sigma = \Pi_h \cup \Pi_f$):

$$u_{\mid \Pi_u} = U_i , \quad \sigma_{\mid \Pi_u} = P_i , \quad \eta_i^{(0)}_{\mid \Pi_h} = N_i^{(0)} , \quad J_i^{(0)}_{\mid \Pi_f} = I_i^{(0)} \left( \tau > 0 , q = \overline{1 , N} \right).$$

(5)

The quantities on the boundary conditions right sides are surface kinematic $U_i$ , $N^{(0)}$ and dynamic $P_i$ , $I_i^{(0)}$ perturbations.

To construct the beam bending equations, a transition to the problem variational formulation (1) –
(5) is used. According to the Hamilton's variational principle, the relations (1) – (5) can be regarded as
a condition for the stationarity of a certain functional $H \left( u_i , \eta_i^{(0)} \right)$, the whose variation is written thus:

$$\delta H = \int_{t_1}^{t_2} \int_G \left( \frac{\partial \sigma}{\partial x} - F \right) \delta u_i dG + \sum_{q=1}^{N} \int_{t_1}^{t_2} \int_{\Pi_f} \left( \frac{\partial J_q^{(0)}}{\partial \chi} - Y^{(0)} \right) \delta n_i^{(0)} dG +$$

$$+ \sum_{q=1}^{N} \int_{t_1}^{t_2} \int_{\Pi_f} \left( J_q^{(0)} - I_q^{(0)} \right) \delta \eta_i^{(0)} dSd\tau.$$

(6)

Further we will assume that:

1) The problem solution domain is cylinder $G = \Omega \times [0,1]$, where $\Omega$ - area occupied by the
beam cross-section. Cross-section boundary is $\Gamma = \partial \Omega = \gamma_1 (x_i) \cup \gamma_2 (x_i)$ (Fig. 1).

2) Beam surface is $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_h$, where $\Pi_0$ is the end surface corresponding to $x_i = 0$ , $\Pi_1$ is
the end surface corresponding to $x_i = 1$ , $\Pi_h$ is the lateral surface. It is assumed that the lateral surface
is free of mechanical loads and mass transfer through the lateral surface is absent.

3) The beam material is a homogeneous isotropic medium

$$C_{ij} = \lambda \delta_{ij} + \mu (\delta_{ij} \delta_{\alpha \beta} + \delta_{\alpha \beta} \delta_{ij}), \quad \Lambda_{ij}^{(0)} = \Lambda_q^{(0)} = \alpha_q ^{(0)}, \quad D_{ij}^{(0)} = D_q^{(0)}.$$  

(7)

Here $\lambda$, $\mu$ are Lamé coefficients, $\delta_{ij}$ is the Kronecker symbol.

4) The bending of beam is considered in plane $x_i x_2$. Then $u_i = u_0 (x_i, x_2, \tau), k = 1, 2$ , $u_3 = 0$,
$\varepsilon_{ij} = 0$. Mass transfer occurs also in the plane $x_i x_2$, i.e. $\eta_i^{(0)} = \eta_i^{(0)} (x_i, x_2, \tau)$.

5) Transverse deflections are considered small. Then the linearization of the unknown quantities
with respect to the variable $x_i$ will have the form:

$$u_i (x_i, x_2, \tau) = u_i (x_i, \tau) + x_2 \chi (x_i, \tau), \quad u_i (x_i, x_2, \tau) = v_i (x_i, \tau) + x_2 \psi (x_i, \tau).$$

$$\eta_i^{(0)} (x_i, x_2, \tau) = N_i (x_i, \tau) + x_2 H_i (x_i, \tau).$$

(8)

6) The cross-sections after deformation remain normal to the neutral line of the beam (Euler
Bernoulli's beam theory). Also we will assume that there are no deformations along the $x_2$ axis.

Then [19] (the prime denotes the derivative with respect to the variable $x_i$):

$$\psi (x_i, \tau) = 0, \quad \chi (x_i, \tau) = - v_i (x_i, \tau), \quad u_i = u_i - x_2 v_i', \quad u_3 = v, \quad u_3 = 0.$$

(9)

Substituting the equations (2), (7) - (9) into (6) we get:

$$\delta H = \int_{t_1}^{t_2} \int_{\Omega} \left[ F \left( \overline{u} - u^* + \sum_{q=1}^{N} \alpha_q N^* \right) - \left( J_q \left( v^* - v'^* \right) - \sum_{q=1}^{N} \alpha_q H_q^* \right) \right] \delta v' d\Omega +$$

$$+ \sum_{q=1}^{N} \int_{t_1}^{t_2} \int_{\Pi_f} \left( J_q \left( H_q - D_q H_q^* - \Lambda_q v'^* \right) \right) \delta H_q' + \sum_{q=1}^{N} \int_{t_1}^{t_2} \int_{\Pi_f} \left( J_q \left( H_q + \Lambda_q v'^* \right) \right) \delta N_q' d\Omega.$$
We introduce the following notation:

1) \[ \int \frac{dx_1}{D} = F \] is the cross-sectional area,
2) \[ \int \frac{x_2^2}{D} dx_3 = J_3 \] is moment of inertia of the beam section relative to the axis \( O_{x_3} \),
3) \[ \int \frac{F}{D} dx_2 dx_3 = p \] is the linearly distributed axial load,
4) \[-\int \frac{F_1}{D} dx_2 dx_3 = m \] is the linearly distributed moment,
5) \[-\int \frac{F_2}{D} dx_2 dx_3 = q \] is the linearly distributed transverse load,
6) \[ \int_{D} \gamma^{(q)} dx_3 = y^{(q)} \] is the linear density of bulk mass transfer sources,
7) \[ \int_{D} \gamma^{(q)} x dx_3 = z^{(q)} \] is the linear density of density of bulk mass transfer sources.

Thus, the following boundary-value problems are the necessary condition for the stationarity of the Hamilton functional under the conditions (6) – (9):

- problems with longitudinal deformations of the beam

\[ \ddot{u} = u' - \sum_{q=1}^{N} \alpha_q N_q' + \frac{N}{F}, \quad \ddot{N}_q = D_q N_q' - \Lambda_q u'' + \frac{y^{(q)}}{F}; \]

\[ \left( u' - \sum_{q=1}^{N} \alpha_q N_q \right) \bigg|_{t=0} = \frac{N_0}{F}, \quad \left( u' - \sum_{q=1}^{N} \alpha_q N_q \right) \bigg|_{t=1} = \frac{N_1}{F}; \] (10)

\[ \left( -D_q N_q' + \Lambda_q u'' \right) \bigg|_{t=0} = \frac{\Gamma^{(q)}_0}{F}, \quad \left( -D_q N_q' + \Lambda_q u'' \right) \bigg|_{t=1} = \frac{\Gamma^{(q)}_1}{F}; \] (11)

- problems with beam deflections

\[ \frac{J_3}{F} \dddot{v} = \frac{J_3}{F} \left( y' + \sum_{j=1}^{N} \alpha_j H_j' \right) + q' + \frac{q}{F} + \dot{H}_q = D_q H_q' + \Lambda_q v'' + \frac{z^{(q)}}{J_3}; \]

\[ \left( y' + \sum_{j=1}^{N} \alpha_j H_j \right) \bigg|_{t=0} = \frac{M_0}{J_3}, \quad \left( y' + \sum_{j=1}^{N} \alpha_j H_j \right) \bigg|_{t=1} = \frac{M_1}{J_3}; \] (13)
\[
\left( \psi'' + \sum_{j=1}^{N} \alpha_{j} H_{j} - \psi \right)_{t=0}^{J_{3}} = \frac{Q_{0} - m_{0}}{J_{3}},
\]
\[
\left( \psi'' + \sum_{j=1}^{N} \alpha_{j} H_{j} - \psi \right)_{t=1}^{J_{3}} = \frac{Q_{1} - m_{1}}{J_{3}},
\]
\[
\left( D_{q} H_{q} + \Lambda_{q} \psi'' \right)_{t=0}^{J_{3}} = -\frac{\Omega_{q}^{(0)}}{J_{3}}, \quad \left( D_{q} H_{q} + \Lambda_{q} \psi'' \right)_{t=1}^{J_{3}} = -\frac{\Omega_{q}^{(1)}}{J_{3}},
\]
where
\[
N_{0} = N(0), \quad \frac{N_{1}}{N_{0}} = N(1), \quad M_{0} = M(0), \quad M_{1} = M(1), \quad Q_{0} = Q(0), \quad Q_{1} = Q(1),
\]
\[
\Gamma_{1}^{(0)} = \Gamma^{(0)}(0), \quad \Gamma_{1}^{(1)} = \Gamma^{(0)}(1), \quad \Omega_{0}^{(0)} = \Omega^{(0)}(0), \quad \Omega_{1}^{(0)} = \Omega^{(0)}(1), \quad m_{0} = m(0), \quad m_{1} = m(1).
\]

In accordance with the variational principle of Lagrange, the boundary conditions (10), (11), (13) – (15) combined with the kinematic boundary conditions:
\[
\begin{align*}
&u_{t=0} = U_{0}, \quad u_{t=1} = U_{1}, \quad N_{q=0} = N_{q0}, \quad N_{q=1} = N_{q1}, \\
&v_{t=0} = V_{0}, \quad v_{t=1} = V_{1}, \quad v_{q=0} = V_{0}', \quad v_{q=1} = V_{1}', \\
&H_{t=0} = H_{q0}, \quad H_{t=1} = H_{q1}.
\end{align*}
\]

2. Method of solution
We will consider the problems (12), (13) and (16). The solutions of the problem are represented in the form [16, 17] (symbol \(\ast\) notes the time convolution):
\[
\begin{align*}
&u(x, \tau) = \sum_{k=1}^{N} \int G_{ik}(x, \tau) \ast f_{ik}(\tau) + G_{ik}(1-x, \tau) \ast f_{ik}(\tau) + \int_{0}^{1} G_{ik}(x, \xi, \tau) \ast f_{ik}(\xi, \tau) d\xi, \\
&\eta_{q}(x, \tau) = \sum_{k=1}^{N} \int G_{q+1,k}(x, \tau) \ast f_{q+1,k}(\tau) + G_{q+1,k}(1-x, \tau) \ast f_{q+1,k}(\tau) + \int_{0}^{1} G_{q+1,k}(x, \xi, \tau) \ast f_{q+1,k}(\xi, \tau) d\xi.
\end{align*}
\]
Here \(x = x_{j}; \quad F_{k}(x, \tau), \quad k = 1, N\) are body forces entering into equations (12); \(f_{ik}(\tau)\) are surface perturbations entering into boundary conditions (13); \(G_{mk}\) are the surface Green's functions that satisfy equations
\[
\frac{J_{k}}{F} \tilde{G}_{k} - \tilde{G}_{k} = \frac{J_{3}}{F} \left( G_{k}^{w} + \sum_{j=1}^{N} \alpha_{j} G_{j+1,k}^{w} \right), \quad \tilde{G}_{q+1,k} = D_{q} G_{q+1,k}^{w} + \Lambda_{q} G_{q+1,k}^{w},
\]
and boundary conditions:
\[
\begin{align*}
&\left( \tilde{G}_{k}^{w} + \sum_{j=1}^{N} \alpha_{j} G_{j+1,k}^{w} \right)_{t=0}^{J_{3}} = \delta_{q} \delta(\tau), \quad \left( \tilde{G}_{k}^{w} + \sum_{j=1}^{N} \alpha_{j} G_{j+1,k}^{w} \right)_{t=1}^{J_{3}} = 0, \\
&G_{q+1,k}^{w} = \delta_{q+1,k} \delta(\tau), \quad G_{q+1,k}^{w} = 0.
\end{align*}
\]
\(G_{mk}\) are the bulk Green's functions that satisfy equations
\[
\begin{align*}
&\frac{J_{k}}{F} \tilde{G}_{k} - \tilde{G}_{k} = \frac{J_{3}}{F} \left( \tilde{G}_{k}^{w} + \sum_{j=1}^{N} \alpha_{j} \tilde{G}_{j+1,k}^{w} \right) + \delta_{q} \delta(\tau), \\
&\tilde{G}_{q+1,k} = D_{q} \tilde{G}_{q+1,k} + \Lambda_{q} \tilde{G}_{q+1,k} + \delta_{q+1,k} \delta(\tau),
\end{align*}
\]
and homogeneous boundary conditions corresponding to (19).
We consider the problem of finding the surface Green's functions \(G_{mk}\). Applying the Laplace transformation with respect to time to (18) and (19) and expanding the Green's functions into Fourier series, we obtain \((m,k = 1, N+1)\):
\[
\left[\frac{J_{k}^{2} + 1 + J_{k}}{F}\right]G_{ik}^{ls} - \frac{J_{k}^{2} + 1}{F}\sum_{j=1}^{N} \alpha_{j} G_{jik}^{ls} = -2 \frac{J_{k}}{F} \lambda_{n} \delta_{ik},
\]
\[-\Delta_{q} \lambda_{n}^{s} G_{ik}^{ls} + (s + D_{q} \lambda_{n}^{2}) G_{ik}^{ls} = -2 \Delta_{q} \lambda_{n} \delta_{ik} + 2 D_{q} \lambda_{n} \delta_{q+1,k} - 2 \Delta_{q} \lambda_{n} \sum_{j=1}^{N} \alpha_{j} \delta_{j+1,k},
\]
\[G_{ik}^{ls}(x,s) = \sum_{n=1}^{N} G_{ik}^{ls}(\lambda_{n},s) \sin \lambda_{n} x, \quad G_{ik}^{ls}(\lambda_{n},s) = 2 \int_{0}^{1} G_{ik}^{ls}(x,s) \sin \lambda_{n} x \, dx, \quad \lambda_{n} = \pi n.
\]
Solution of the system has the form \((q,p=1,N,k=1,N+1)\):
\[G_{ik}^{ls}(\lambda_{n},s) = \frac{P_{ik}(\lambda_{n},s)}{P(\lambda_{n},s)}, \quad G_{q+1,k}^{ls}(\lambda_{n},s) = \frac{Q_{q}(\lambda_{n},s)}{Q(\lambda_{n},s)} + 2 \left[\frac{D_{q} \delta_{q+1,k} - \Delta_{q} \alpha_{q} (\delta_{q+1,k} + \delta_{p+1,k}) - \Delta_{q} \delta_{ik}}{s + D_{q} \lambda_{n}^{2}}\right] \lambda_{n}, \quad \lambda_{n} = \pi n.
\]
where
\[P(\lambda_{n},s) = \left[\lambda_{n}^{2} + \frac{F}{J_{s}}\right]^{2} + \lambda_{n}^{2} \sum_{j=1}^{N} \alpha_{j} \Lambda_{j} \Pi_{j}(s,\lambda_{n}),
\]
\[Q_{q}(\lambda_{n},s) = (s + D_{q} \lambda_{n}^{2}) P(\lambda_{n},s), \quad \Pi_{j}(s,\lambda_{n}) = \sum_{j=1}^{N} (s + D_{q} \lambda_{n}^{2}), \quad \Pi_{q}(s,\lambda_{n}) = \prod_{j=1}^{N} (s + D_{q} \lambda_{n}^{2}).
\]
\[P_{ik}(\lambda_{n},s) = \sum_{j=1}^{N} A_{ik}^{(j)}(\lambda_{n}) e^{\delta_{j,k} s \lambda_{n}}, \quad A_{ik}^{(j)}(\lambda_{n}) = \frac{P_{ik}(\lambda_{n},s)}{P(\lambda_{n},s)}, \quad A_{q+1,k}^{(j)}(\lambda_{n}) = \frac{Q_{q}(\lambda_{n},s)}{Q(\lambda_{n},s)} + 2 \left[\frac{D_{q} \delta_{q+1,k} - \Delta_{q} \alpha_{q} (\delta_{q+1,k} + \delta_{p+1,k}) - \Delta_{q} \delta_{ik}}{s + D_{q} \lambda_{n}^{2}}\right] \lambda_{n} e^{-D_{q} \lambda_{n}^{2} s}.
\]

3. Example

We assume in the boundary conditions (13), (16) that \((H(\tau) \text{ is Heaviside function)}\):
\[\frac{M(x)}{J_{s}} = H(\tau), \quad \frac{M_{x}}{J_{s}} = H(\tau), \quad H_{q0}(\tau) = 0, \quad H_{q1}(\tau) = 0.
\]

Then, according to (17) in the absence case of volume perturbations, we have
\[u(x,\tau) = 2 \sum_{n=1}^{N} \frac{\lambda_{n}^{2} \cos \lambda_{n}}{2} \sum_{j=1}^{N} A_{1j}^{(j)}(\lambda_{n}) e^{\delta_{j,k} s \lambda_{n}} - \frac{1}{s_{1}(\lambda_{n})},
\]
\[\eta_{u}(x,\tau) = 2 \sum_{n=1}^{N} \frac{\lambda_{n}^{2} \cos \lambda_{n}}{2} \sum_{j=1}^{N} A_{1j}^{(j)}(\lambda_{n}) e^{\delta_{j,k} s \lambda_{n}} - \frac{1}{s_{1}(\lambda_{n})}.
\]
We take for the calculation example the one-component $N=1$ medium is aluminum, with the following characteristics (the subscript $q$ denoting the component number, for brevity we omit it) [21]:

\[
\begin{align*}
\lambda &= 6.93 \times 10^0 \frac{N}{m^2}, \\
\mu &= 2.56 \times 10^0 \frac{N}{m^2}, \\
T_0 &= 800 K, \\
\rho &= 2700 \frac{kg}{m^3}, \\
\alpha &= 4.2 \times 10^5 \frac{J}{mol K}, \\
D &= 7.73 \times 10^{-14} \frac{m^2}{s}, \\
L &= 1 m, \\
F &= 5.00 \times 10^{-3} m^2, \\
J &= 4.17 \times 10^{-5} m^4.
\end{align*}
\]

The calculation results are shown in Figures 2, 3.

Figure 4 demonstrates a comparison of the obtained solution with the purely elastic problem solution.

**Conclusion**

The unsteady vibrations model of the Euler Bernoulli beam is constructed. The algorithm is proposed for finding the surface Green's functions, based on the representation of solutions in the Fourier series form. The method verification shown by comparing of the obtained results with the known solutions of the classical problems about bending of beams. The solutions obtained by the algorithm can be used to refine and verify the numerical solutions of more complex unsteady vibration problem for the beam taking mass transfer into account.

![Figure 2. Beam deflections $u(x, \tau)$](image1)

![Figure 3. Concentration increment $\eta(x, \tau)$](image2)
Figure 4. The solid line is the solution of the problem (13), (14). The dashed line is the elastic problem solution, $x=0.5$.

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