Quasicharacters, recoupling calculus and costratifications of lattice quantum gauge theory

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Abstract

We study the algebra $R$ of $G$-invariant representative functions over the $N$-fold Cartesian product of copies of a compact Lie group $G$ modulo the action of conjugation by the diagonal subgroup. Using the representation theory of $G$ on the Hilbert space $\mathcal{H} = L^2(G^N)^G$, we construct a subset of $G$-invariant representative functions which, by standard theorems, span $\mathcal{H}$ and thus generate $R$. The elements of this basis will be referred to as quasicharacters. For $N = 1$, they coincide with the ordinary irreducible group characters of $G$. The form of the quasicharacters depends on the choice of a certain unitary $G$-representation isomorphism, or reduction scheme, for every $\lambda \in \hat{G}^N$, where $\hat{G}$ denotes the set of isomorphism classes of irreps of $G$. We determine the multiplication law of $R$ in terms of the quasicharacters with structure constants. Next, we use the one-to-one correspondence between complete bracketing schemes for the reduction of multiple tensor products of $G$-representations and rooted binary trees. This provides a link to the recoupling theory for $G$-representations. In particular, via this link, the choice of the reduction scheme acquires an interpretation in terms of binary trees. Using these tools, we prove that the structure constants of the algebra $R$ are given by a certain type of recoupling coefficients of $G$-representations. For these recouplings we derive a reduction law in terms of a product over primitive elements of 9j symbol type. The latter may be further expressed in terms of sums over products of Clebsch-Gordan coefficients of $G$. For $G = SU(2)$, everything boils down to combinatorics of angular momentum theory.

In the final part of the paper, we present one application: the construction of the Hilbert space costratification of (finite) lattice quantum gauge theory. Here, we build on previous work [8,9], where we have implemented the classical gauge orbit strata on the quantum level within a suitable holomorphic picture. In this picture, each element $\tau$ of the classical stratification corresponds to the zero locus of a finite subset $\{p_i\}$ of the algebra $R$ of $G$-invariant representative functions on
$G^N$. Viewing the invariants as multiplication operators $\hat{p}_i$ on the Hilbert space $\mathcal{H}$, the union of their images defines a subspace of $\mathcal{H}$ whose orthogonal complement $\mathcal{H}_\tau$ is the element of the costratification corresponding to $\tau$. In [9], we applied the above described theory to construct the subspaces $\mathcal{H}_\tau$ for the case $G = \text{SU}(2)$. In the present paper, we start to analyze the case $G = \text{SU}(3)$.

We note that the methods developed in this paper may be useful in the study of virtually all quantum models with polynomial constraints related to some symmetry.

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1 Introduction

This paper builds on previous work [8,9], where we have developed tools for the implementation and the study of the classical gauge orbit stratification in lattice quantum gauge theory with gauge group $G = SU(2)$. Our work is part of a program which aims at constructing a non-perturbative quantum theory of gauge fields in the Hamiltonian framework, with special emphasis on the role of non-generic gauge orbit types. The starting point is a classical finite-dimensional Hamiltonian system with symmetries, obtained by lattice approximation of the gauge model under consideration. The quantum theory is obtained via canonical quantization. It is best described in the language of $C^*$-algebras with a field algebra which (for a pure gauge theory) may be identified with the algebra of compact operators on the Hilbert space of square-integrable functions over the product $G^N$ of a number of copies of the gauge group manifold $G$. Correspondingly, the observable algebra is obtained via gauge symmetry reduction. We refer to [17,21,22,30] for details. For the construction of the thermodynamical limit, see [10,11]. In the present paper, we extend our tools to the case of an arbitrary compact Lie group $G$.

Let us recall that for a nonabelian gauge group the action of the group of local gauge transformations has more than one orbit type. Accordingly, the reduced phase space obtained by symplectic reduction is a stratified symplectic space [27,30,32]. The stratification is given by the orbit type strata. It consists of an open and dense principal stratum, and several secondary strata. Each of these strata is invariant under the dynamics with respect to any invariant Hamiltonian. For case studies we refer to [4,5,7]. To study the influence of the classical gauge orbit type stratification on quantum level, we combine the concept of costratification of the quantum Hilbert space as developed by Huebschmann [15] with a localization concept taken from the theory of coherent states. Loosely speaking, a costratification is given by a family of closed subspaces, one for each stratum. The closed subspace associated with a certain classical stratum consists of the wave functions which are optimally localized at that stratum, in the sense that they are orthogonal to all states vanishing at that stratum. The vanishing condition can be given sense in the framework of holomorphic quantization, where wave functions are true functions and not just classes of functions. In [16] we have constructed this costratification for a toy model with gauge group $SU(2)$ on a single lattice plaquette. As physical effects, we have found a nontrivial overlap between distant strata and, for a certain range of the coupling, a very large overlap between the ground state of the lattice Hamiltonian and one of the two secondary strata.

Every classical gauge orbit stratum may be characterized by a set of polynomial relations. Within the above mentioned holomorphic picture, the latter may be implemented on quantum level as follows. In this picture, each element $\tau$ of the stratification corresponds to the zero locus of a finite subset $\{p_1, \ldots, p_\ell\}$ of the algebra $R$ of $G$-invariant representative functions on $G^N_C$, where $G_C$ denotes the complexification of $G$. Viewing the invariants $p_i$ as multiplication operators $\hat{p}_i$ on the physical Hilbert space $\mathcal{H} = L^2(G^N)^G$, the union of their images defines a subspace of $\mathcal{H}$ whose orthogonal complement $\mathcal{H}_\tau$ is, by definition, the element of the costratification corresponding to $\tau$. 

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Thus, to construct $\mathcal{H}_\tau$, one has to determine the images of the $\hat{p}_i$ explicitly.

To accomplish that goal, one needs deeper insight into the structure of the algebra $\mathcal{R}$. To achieve that we proceed as follows. Using the representation theory of $G$ on the Hilbert space $\mathcal{H}$, we construct a subset of $G$-invariant representative functions which, by standard theorems, span $\mathcal{H}$ and thus generate $\mathcal{R}$. The elements of this basis will be referred to as quasicharacters. For $N = 1$, they coincide with the ordinary irreducible group characters of $G$. The form of the quasicharacters depends on the choice of a certain unitary $G$-representation isomorphism, or reduction scheme, for every $\Lambda \in \hat{G}^N$, where $\hat{G}$ denotes the set of isomorphism classes of irreps of $G$. We determine the multiplication law of $\mathcal{R}$ in terms of the quasicharacters. Next, we use the one-to-one correspondence between complete bracketing schemes for the reduction of multiple tensor products of $G$-representations and rooted binary trees. This provides a link to the recoupling theory for $G$-representations. In particular, via this link, the choice of the reduction scheme acquires an interpretation in terms of binary trees. Using these tools, we prove that the structure constants of the algebra $\mathcal{R}$ are given by a certain type of recoupling coefficients of $G$-representations. For these recouplings we derive a reduction law in terms of a product over primitive elements of $9j$ symbol type. The latter may be further expressed in terms of sums over products of Clebsch-Gordan coefficients of $G$. For $G = SU(2)$, everything boils down to combinatorics of angular momentum theory.

Given these tools, one may work through the following programme:

1. Express the functions $\hat{p}_i$ in terms of the quasicharacters.

2. Use the multiplication law for the quasicharacters and the fact that they constitute an orthonormal basis in $\mathcal{H}$, to construct a basis in the vanishing subspace $\mathcal{V}_\tau$. The elements of this basis are linear combinations of the quasicharacters with coefficients built from products of the multiplicative structure constants of the quasicharacters with the expansion coefficients of the functions $\hat{p}_i$.

3. Obtain the subspace $\mathcal{H}_\tau$ by taking the orthogonal complement of $\mathcal{V}_\tau$ in $\mathcal{H}$. This yields $\mathcal{H}_\tau$ in terms of a family of vectors (again expanded with respect to the quasicharacters) whose components are determined by a system of linear equations with coefficients given by the above mentioned products of the structure constants with the expansion coefficients of the functions $\hat{p}_i$.

By this procedure, the construction of the costratification boils down to a numerical problem in linear algebra. In the final part of the paper, we present a sample calculation for the gauge group SU(3). This opens the door for a systematic discussion of models with this gauge group.

Finally, we note that the methods developed in this paper may be useful in the study of virtually all quantum models with polynomial constraints related to some symmetry. For this reason we present the methods first, without reference to lattice gauge theory.
2 Quasicharacters

In this section, we recall the construction of an orthonormal basis in $L^2(G^N)^G$ and use it to analyze the multiplication law in the commutative algebra of $G$-invariant representative functions on $G^N$, see [9].

Let $\mathcal{R}(G^N)$ denote the commutative algebra of representative functions on $G^N$ and let $\mathcal{R} := \mathcal{R}(G^N)^G$ be the subalgebra of $G$-invariant elements. Since $G^N_{\mathbb{C}}$ is the complexification of the compact Lie group $G^N$, the proposition and Theorem 3 in Section 8.7.2 of [29] imply that $\mathcal{R}(G^N)$ coincides with the coordinate ring of $G^N_{\mathbb{C}}$, viewed as a complex affine variety, and that $\mathcal{R}(G^N)$ coincides with the algebra of representative functions on $G^N_{\mathbb{C}}$. As a consequence, $\mathcal{R}$ coincides with the algebra of $G$-invariant representative functions on $G^N_{\mathbb{C}}$, where the relation is given by restriction and analytic continuation, respectively.

Let $\hat{\mathcal{G}}$ denote the set of isomorphism classes of finite-dimensional irreps of $G$. Given a finite-dimensional unitary representation $(H, D)$ of $G$, let $C(G)_D \subset \mathcal{R}(G)$ denote the subspace of representative functions of $D$ and let $\chi_D \in C(G)_D$ be the character of $D$, defined by $\chi_D(a) := \text{tr}(D(a))$. The same notation will be used for the Lie group $G^N$. Below, all representations are assumed to be continuous and unitary without further notice. The elements of $\hat{\mathcal{G}}$ will be labelled by the corresponding highest weights $\lambda$ relative to some chosen Cartan subalgebra and some chosen dominant Weyl chamber.

Assume that for every $\lambda \in \hat{\mathcal{G}}$ a concrete unitary irrep $(H_{\lambda}, D_{\lambda})$ of highest weight $\lambda$ in the Hilbert space $H_{\lambda}$ has been chosen. Given $\lambda = (\lambda_1, \ldots, \lambda_N) \in \hat{\mathcal{G}}^N$, we define a representation $(H_\lambda, D_\lambda)$ of $G^N$ by

$$H_\lambda = \bigotimes_{i=1}^N H_{\lambda_i}, \quad D_\lambda(\underline{a}) = \bigotimes_{i=1}^N D_{\lambda_i}(a_i),$$

(2.1)

where $\underline{a} = (a_1, a_2, \ldots, a_N)$. This representation is irreducible and we have

$$C(G^N)_{D_\lambda} \cong \bigotimes_{i=1}^N C(G)_{D_{\lambda_i}},$$

isometrically with respect to the $L^2$-norms. Using this, together with the Peter-Weyl theorem for $G$, we obtain that $\bigoplus_{\lambda \in \hat{\mathcal{G}}^N} C(G^N)_{D_\lambda}$ is dense in $L^2(G^N, d^N\alpha)$, where $d\alpha$ denotes the normalized Haar measure on $G$. Since

$$\bigoplus_{\lambda \in \hat{\mathcal{G}}^N} C(G^N)_{D_\lambda} \subset \bigoplus_{D \in \hat{\mathcal{G}}^N} C(G^N)_D,$$

this implies

**Lemma 2.1.** Every irreducible representation of $G^N$ is equivalent to a product representation $(H_\lambda, D_\lambda)$ with $\lambda \in \hat{\mathcal{G}}^N$. If $(H_\lambda, D_\lambda)$ and $(H_{\lambda'}, D_{\lambda'})$ are isomorphic, then $\lambda = \lambda'$.

---

[1] The subspace spanned by all matrix coefficients $\langle \zeta, D(\cdot)v \rangle$ with $v \in H$ and $\zeta \in H^*$. 

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Given $\lambda \in \hat{G}$, let $D^\lambda$ denote the representation of $G$ on $H_\lambda$ defined by

$$D^\lambda (a) := D^\lambda (a, \ldots, a).$$  \hfill (2.2)

This representation will be referred to as the diagonal representation induced by $D^\lambda$. It is reducible and has the isotypical decomposition

$$H_\lambda = \bigoplus_{\lambda \in \hat{G}} H_{\lambda,\lambda}$$

into uniquely determined subspaces $H_{\lambda,\lambda}$. Recall that these subspaces may be obtained as the images of the orthogonal projectors

$$P_\lambda := \dim(H_\lambda) \int_G \chi_{D^\lambda} D^\lambda (a) da$$  \hfill (2.3)

on $H_\lambda$. These projectors commute with one another and with $D^\lambda$. If an isotypical subspace $H_{\lambda,\lambda}$ is reducible, we can further decompose it in a non-unique way into irreducible subspaces of isomorphism type $\lambda$. Let $m_\lambda(\lambda)$ denote the number of these irreducible subspaces (the multiplicity of $D^\lambda$ in $D^\lambda_d$) and let $\hat{G}_\lambda$ denote the subset of $\hat{G}$ consisting of the highest weights $\lambda$ such that $m_\lambda(\lambda) > 0$. In this way, we obtain a unitary $G$-representation isomorphism

$$H_\lambda \cong \bigoplus_{\lambda \in \hat{G}_{\lambda}} m_\lambda(\lambda) \bigoplus_{l=1}^{m_\lambda(\lambda)} H_{\lambda,\lambda}. \hfill (2.4)$$

Let us assume that we have chosen such an isomorphism for every $\lambda \in \hat{G}_\lambda$ and for every $n = 2, 3, \ldots$. Composing this isomorphism with the natural projections and injections of the direct sum, we obtain projections and injections

$$p_{\lambda,\lambda,l} : H_\lambda \to H_{\lambda,l}, \quad i_{\lambda,\lambda,l} : H_{\lambda,l} \to H_\lambda, \quad \lambda \in \hat{G}_{\lambda}, \quad l = 1, \ldots, m_\lambda(\lambda). \hfill (2.5)$$

We have

$$\sum_{l,l'} i_{\lambda,\lambda,l} \circ p_{\lambda,\lambda,l'} = 1_{H_\lambda}, \quad p_{\lambda,\lambda,l} \circ i_{\lambda,\lambda,l'} = \delta_{l,l'} \delta^{\lambda,l}_{\lambda} 1_{H_\lambda}.$$

For every $\lambda \in \hat{G}_\lambda$ and every $l, l' = 1, \ldots, m_\lambda(\lambda)$, we define an operator on $H_{\lambda,l}$,

$$(A_{\lambda,l})_{l'} := \frac{1}{\sqrt{\dim(H_\lambda)}} i_{\lambda,\lambda,l} \circ p_{\lambda,\lambda,l'}, \hfill (2.6)$$

which is a $G$-representation endomorphism of $D^\lambda_d$ and a $G$-invariant function $(\chi_{\lambda,\lambda})_{l'}$ on $G^N$ by

$$(\chi_{\lambda,\lambda})_{l'} (a) := \sqrt{\dim(H_\lambda)} \operatorname{tr}(D^\lambda(a)(A_{\lambda,l})_{l'}). \hfill (2.7)$$

In the sequel, the functions $(\chi_{\lambda,\lambda})_{l'}$ will be referred to as quasicharacters.
Proposition 2.2. The family of functions
\[
\left\{ (\chi_{\underline{\lambda}})^l_{\underline{l}} : \underline{\lambda} \in \hat{G}^N, \; l \in \hat{G}_\lambda, \; l, l' = 1, \ldots, m_{\underline{\lambda}}(\lambda) \right\}
\]
constitutes an orthonormal basis in \(L^2(G^N)^G\).

Proof. See Proposition 3.7 in [9].

By analytic continuation, the irreps \(D^\lambda\) of \(G\) induce irreps \(D^\underline{\lambda}\) of \(G_C\), the irreps \(D^\underline{\lambda}\) of \(G^N\) induce irreps \(D^\underline{\lambda}\) of \(G_C^N\), and the functions \((\chi_{\underline{\lambda}}^C)^l_{\underline{l}}\) on \(G^N\) induce holomorphic functions \((\chi_{\underline{\lambda}}^C)^l_{\underline{l}}\) on \(G_C^N\). Then, (2.1), (2.2) and (2.7) hold with \(D^\lambda, D^\underline{\lambda}\) and \((\chi_{\underline{\lambda}}^C)^l_{\underline{l}}\) replaced by, respectively, \(D^\underline{\lambda}, D^\underline{\lambda}\) and \((\chi_{\underline{\lambda}}^C)^l_{\underline{l}}\).

Corollary 2.3. The family of functions
\[
\left\{ (\chi_{\underline{\lambda}}^C)^l_{\underline{l}} : \underline{\lambda} \in \hat{G}^N, \; l \in \hat{G}_\lambda, \; l, l' = 1, \ldots, m_{\underline{\lambda}}(\lambda) \right\}
\]
constitutes an orthogonal basis in \(H\). The norms are
\[
\| (\chi_{\underline{\lambda}}^C)^l_{\underline{l}} \|^2 = \prod_{r=1}^N C_{\lambda^r}, \quad C_{\lambda^r} = (\hbar \pi)^{\dim(G)/2} e^{\frac{\hbar}{2} |\lambda^r|^2}, \tag{2.8}
\]
where \(\rho\) denotes half the sum of the positive roots. The expansion coefficients of \(f \in H\) with respect to this basis are given by the scalar products \(\langle (\chi_{\underline{\lambda}}^C)^l_{\underline{l}} f \rangle_{\hat{G}^N} \) in \(L^2(G^N)^G\).

Proof. See Corollary 3.8 in [9].

It follows that the quasicharacters span the algebra \(\mathfrak{R}(G_N)\). Hence, to study the multiplicative structure of this algebra, it suffices to find the multiplication law for quasicharacters. To get rid of dimension factors, we will pass to the modified quasicharacters
\[
(\tilde{\chi}_{\underline{\lambda}}^C)^l_{\underline{l}} := \sqrt{\frac{\dim(H_{\lambda})}{\dim(H_{\underline{\lambda}})}} (\chi_{\underline{\lambda}}^C)^l_{\underline{l}}.
\]
We assume that a unitary \(G\)-representation isomorphism (2.4) has been chosen for every \(\underline{\lambda} \in \hat{G}^N\) and every \(N\). Writing
\[
(\tilde{\chi}_{\underline{\lambda}_1}^C)^l_{\underline{l}_1}(\underline{a}) (\tilde{\chi}_{\underline{\lambda}_2}^C)^l_{\underline{l}_2}(\underline{a}) = \sqrt{\dim(H_{\lambda_1}) \dim(H_{\lambda_2})} \tr \left( \left((A_{\underline{\lambda}_1}^\lambda)^l_{\underline{l}_1} \times (A_{\underline{\lambda}_2}^\lambda)^l_{\underline{l}_2}\right) \circ \left(D^\underline{\lambda}(\underline{a}) \times D^\underline{\lambda}(\underline{a})\right) \right), \tag{2.9}
\]
we see that in order to expand the product \((\tilde{\chi}_{\underline{\lambda}_1}^C)^l_{\underline{l}_1} \times (\tilde{\chi}_{\underline{\lambda}_2}^C)^l_{\underline{l}_2}\) in terms of the basis functions \((\tilde{\chi}_{\underline{\lambda}}^C)^l_{\underline{l}}\), a reasonable strategy is to decompose the \(G^N\)-representation \(D^\underline{\lambda} \times D^\underline{\lambda}\) into \(G^N\)-irreps \(\underline{\lambda}\) and then relate these \(G^N\)-irreps to the basis functions using the chosen \(G\)-representation isomorphisms (2.4). To implement this, we define two different
decompositions of the diagonal representation $D^{\lambda}_{d} \otimes D^{\lambda}_{d}$ into irreps. The first one is adapted to the tensor product on the right hand side of (2.9). It is defined by the projections

$$H_{\lambda_{1}} \otimes H_{\lambda_{2}} \xrightarrow{p^{\lambda_{1},\lambda_{1};l_{1}} \otimes p^{\lambda_{2},\lambda_{2};l_{2}}} H_{\lambda_{1}} \otimes H_{\lambda_{2}} \xrightarrow{p^{(\lambda_{1},\lambda_{2})},\lambda,k} H_{\lambda},$$

(2.10)

where $\lambda_{i} \in \hat{G}^{N}$, $l_{i} = 1, \ldots, m_{\lambda_{i}}(\lambda_{i})$ and $\lambda \in \hat{G}^{2}_{(\lambda_{1},\lambda_{2})}$, $k = 1, \ldots, m_{(\lambda_{1},\lambda_{2})}(\lambda)$. The second decomposition is adapted to the definition of the basis functions $\left(\tilde{\chi}_{\Delta}^{\lambda}\right)^{i}_{l}$. It is defined by the projections

$$H_{\lambda_{1}} \otimes H_{\lambda_{2}} \otimes_{i=1}^{N} p^{(\lambda^{i}_{1},\lambda^{i}_{2})},\lambda^{i},k^{i}) \xrightarrow{\otimes_{i=1}^{N} i^{(\lambda^{i}_{1},\lambda^{i}_{2})},\lambda^{i},k^{i})} H_{\lambda} \xrightarrow{\otimes_{i=1}^{N} i^{(\lambda^{i}_{1},\lambda^{i}_{2})},\lambda^{i},k^{i})} H_{\lambda},$$

(2.11)

where $\Delta = (\lambda^{1}, \ldots, \lambda^{N})$ with $\lambda^{i} \in \hat{G}^{2}_{(\lambda^{i}_{1},\lambda^{i}_{2})}$, $k^{i} = 1, \ldots, m_{(\lambda^{i}_{1},\lambda^{i}_{2})}(\lambda^{i})$ and $\lambda \in \hat{G}^{N}_{\Delta}$, $l = 1, \ldots, m_{\lambda}(\lambda)$. Composition of the injection corresponding to (2.11) with the projection (2.10) yields a unitary representation isomorphism of irreps $H_{\lambda}$. Hence, Schur’s lemma implies that

$$p^{(\lambda_{1},\lambda_{2}),\lambda^{i},k^{i})} \circ \left(\otimes_{i=1}^{N} i^{(\lambda^{i}_{1},\lambda^{i}_{2})},\lambda^{i},k^{i})\right) \circ i^{(\lambda_{1},\lambda_{2}),\lambda^{i},k^{i})} = \delta_{\lambda}^{\lambda} U^{(\lambda_{1},\lambda_{1};l_{1}) \otimes (\lambda_{2},\lambda_{2};l_{2})^{k}} id_{H_{\lambda}}$$

(2.12)

with certain coefficients $U^{(\lambda_{1},\lambda_{1};l_{1}) \otimes (\lambda_{2},\lambda_{2};l_{2})^{k}}$. Here, we have denoted $k = (k^{1}, \ldots, k^{N})$. Now, the multiplication law may be expressed in terms of these coefficients.

**Theorem 2.4.** In terms of the basis functions, the multiplication in $\mathcal{R}$ is given by

$$\left(\tilde{\chi}_{\Delta_{1}}^{\lambda_{1}}\right)^{i}_{l_{1}} \cdot \left(\tilde{\chi}_{\Delta_{2}}^{\lambda_{2}}\right)^{j}_{l_{2}} = \sum_{\lambda} \sum_{l_{i}=1}^{m_{\lambda_{i}}(\lambda_{i})} \sum_{l_{i}=1}^{m_{\lambda_{i}}(\lambda_{i})} \sum_{k=1}^{m_{(\lambda_{1},\lambda_{2})}(\lambda)} \sum_{k=1}^{m_{(\lambda_{1},\lambda_{2})}(\lambda)} \sum_{k=1}^{m_{(\lambda_{1},\lambda_{2})}(\lambda)} U^{(\lambda_{1},\lambda_{1};l_{1}) \otimes (\lambda_{2},\lambda_{2};l_{2})^{k}} \cdot \left(U^{(\lambda_{1},\lambda_{1};l_{1}) \otimes (\lambda_{2},\lambda_{2};l_{2})^{k}}ight)^{*} \left(\tilde{\chi}_{\Delta_{2}^{\lambda}}\right)^{i}_{l_{1}}.$$  

The same formula holds true for the basis functions $\left(\tilde{\chi}_{\Delta_{2}^{\lambda}}\right)^{i}_{l_{1}}$ on $G^{N}$.  

**Proof.** See Proposition 3.10 in [9].

**Remark 2.5.** Note that the coefficients $U$ in Theorem 2.4 depend on the unitary $G$-representation isomorphisms (2.11). In the next section, we will see that they are given by appropriate recoupling coefficients.  

**Remark 2.6.** For completeness, we also recall the case $N = 1$. Here, the quasicharacters are ordinary characters

$$\chi^{\lambda}_{\Delta}(a) = \text{tr} \left(D^{\lambda}(a)\right), \quad \lambda \in \hat{G},$$
and the multiplication law can be obtained directly from the Clebsch-Gordan series,

\[ D^{\lambda_1} \otimes D^{\lambda_2} = \bigoplus_{\lambda \in \hat{G}} m_{(\lambda_1, \lambda_2)}(\lambda) \, D^\lambda. \]

Indeed, we have

\[ \text{tr} \left( D^{\lambda_1}(a) \right) \, \text{tr} \left( D^{\lambda_2}(a) \right) = \text{tr} \left( D^{\lambda_1}(a) \otimes D^{\lambda_2}(a) \right) = \sum_{\lambda \in \hat{G}} m_{(\lambda_1, \lambda_2)}(\lambda) \, \text{tr} \left( D^\lambda(a) \right) \]

and hence

\[ \chi_{\lambda_1} \cdot \chi_{\lambda_2} = \sum_{\lambda \in \hat{G}} m_{(\lambda_1, \lambda_2)}(\lambda) \, \chi_{\lambda}. \]  \hspace{1cm} (2.13)

This generalizes to

\[ \chi_{\lambda_1} \cdot \cdots \cdot \chi_{\lambda_r} = \sum_{\lambda \in \hat{G}} m_{(\lambda_1, \ldots, \lambda_r)}(\lambda) \, \chi_{\lambda}, \]  \hspace{1cm} (2.14)

where \( m_{(\lambda_1, \ldots, \lambda_r)}(\lambda) \) denotes the multiplicity of the irrep \( D^\lambda \) in \( D^{\lambda_1} \otimes \cdots \otimes D^{\lambda_r}. \)

\[ \blacksquare \]

3 Recoupling calculus

As observed in the preceding section, to fix concrete basis functions \( (\chi_{\lambda})_i^j \), we have to fix the unitary \( G \)-representation isomorphisms \( (2.4) \) entering their definition. As a consequence, we obtain concrete formulae for the unitary operators in the multiplication law of the algebra \( \mathcal{R} \), expressed in terms of \( G \)-recoupling coefficients. This relates the algebra structure to the combinatorics of recoupling theory for \( G \)-representations, see \cite{2,3,18,19,24,33}.

3.1 Reduction schemes, binary trees, and recoupling coefficients

Given an element \( \lambda \in \hat{G} \), recall that \( (H_{\lambda}, D^\lambda) \) denotes the standard irrep of highest weight \( \lambda \). Let \( \hat{u}(\lambda) \) denote the weight system of this representation and let \( u(\lambda) \) denote the set of pairs \( \mu = (\hat{\mu}, \hat{\mu}) \), where \( \hat{\mu} \in \hat{u}(\lambda) \) and \( \hat{\mu} \) is a multiplicity counter for \( \hat{\mu} \). We refer to the pairs \( \mu \) as weight labels. All results carry over to arbitrary orthonormal bases by interpreting \( \mu \) as a label without an inner structure and by ignoring statements about weights. The representation space \( H_{\lambda} \) is spanned by an orthonormal weight vector basis \( \{|\lambda \mu\rangle : \mu \in u(\lambda)\} \), where \( |\lambda \mu\rangle \) denotes the normalized common eigenvector of the image under \( D^\lambda \) of the Cartan subalgebra chosen in the Lie algebra \( \mathfrak{g} \) of \( G \) corresponding to the eigenvalue functional \( i\hat{\mu} \) and the multiplicity counter \( \hat{\mu} \). Here, \( i \) denotes the imaginary unit. The matrix elements of \( D^\lambda(a), \, a \in G \), in that basis are

\[ D^\lambda_{\mu \mu'}(a) = \langle \lambda \mu | D^\lambda(a) | \lambda \mu' \rangle, \quad \mu, \mu' \in u(\lambda). \]

Accordingly, the matrix elements of

\[ D^\lambda(a) = D^{\lambda_1}(a_1) \otimes D^{\lambda_2}(a_2) \otimes \cdots \otimes D^{\lambda_N}(a_N) \]
with respect to the tensor product weight basis
\[
|\lambda_\mu\rangle = |\lambda^n\rangle_1 \otimes \cdots \otimes |\lambda^n\rangle_N, \quad \mu^n \in \mathfrak{w}(\lambda^n), \quad n = 1, \ldots, N,
\]
are given by
\[
\langle \lambda_\mu | D^{\lambda}_{\mu}(a) | \lambda_\mu' \rangle = D^{\lambda_1}_{\mu_1}(a_1) \cdots D^{\lambda_N}_{\mu_N}(a_N).
\]
We write \( D^{\lambda}_{\mu} \) for the induced diagonal representation of \( G \) and \( \mu \in \mathfrak{w}(\lambda) \) for the condition that \( \mu^n \in \mathfrak{w}(\lambda^n) \) for all \( n = 1, \ldots, N \). Moreover, we put \( \Sigma(\mu) := \sum_{n=1}^N \mu^n \) (sum of linear functionals).

Recall that for highest weights \( \lambda^1, \lambda^2 \in \hat{G} \), it may happen that in the decomposition of \( H_{(\lambda_1,\lambda_2)} = H_{\lambda_1} \otimes H_{\lambda_2} \) into irreps, given by the Clebsch-Gordan series
\[
H_{(\lambda_1,\lambda_2)} \cong \bigoplus_{\lambda \in \hat{G}} m_{(\lambda_1,\lambda_2)}(\lambda) H_{\lambda}, \quad (3.1)
\]
the multiplicity \( m_{(\lambda_1,\lambda_2)}(\lambda) > 1 \). We assume that in each such case, a concrete orthogonal decomposition of each isotypical subspace \( H_{(\lambda_1,\lambda_2),\lambda} \) into \( m_{(\lambda_1,\lambda_2)}(\lambda) \) irreducible subspaces has been chosen. By identifying each of them with \( H_{\lambda} \), we obtain a unitary representation isomorphism
\[
H_{(\lambda_1,\lambda_2),\lambda} \cong \bigoplus_{k=1}^{m_{(\lambda_1,\lambda_2)}(\lambda)} H_{\lambda} \quad (3.2)
\]
for each isotypical subspace. Let us introduce the bracket sets
\[
\langle \lambda^1, \lambda^2 \rangle := \{ \lambda \in \hat{G} : m_{(\lambda_1,\lambda_2)}(\lambda) \neq 0 \},
\]
\[
\langle \lambda^1, \lambda^2 \rangle := \{ (\lambda, k) \in \hat{G} \times \mathbb{N} : \lambda \in \langle \lambda^1, \lambda^2 \rangle, \quad k = 1, \ldots, m_{(\lambda_1,\lambda_2)}(\lambda) \}.
\]
The integer \( k \) plays the role of a multiplicity counter. For the first bracket set, there is an iterated version like \( \langle \lambda^1, \langle \lambda^2, \lambda^3 \rangle \rangle \) defined by taking the union of \( \langle \lambda^1, \lambda \rangle \) over \( \lambda \in \langle \lambda^1, \lambda^2 \rangle, \quad k = 1, \ldots, m_{(\lambda_1,\lambda_2)}(\lambda) \). Note that this bracketing is associative,
\[
\langle \lambda^1, \langle \lambda^2, \lambda^3 \rangle \rangle = \langle \langle \lambda^1, \lambda^2 \rangle, \lambda^3 \rangle,
\]
and so for any given set \( \lambda \) of highest weights we may write \( \langle \lambda \rangle \). The elements \( \lambda \) of this set label the (uniquely determined) isotypical subspaces \( H_{\lambda,\lambda} \) of \( (H_{\lambda}, D^{\lambda}_{\mu}) \). Concrete irreducible subspaces for the diagonal \( G \)-representation \( (H_{\lambda}, D^{\lambda}_{\mu}) \) can be obtained by choosing a reduction scheme for \( N \)-fold tensor products of \( G \)-irreps, which breaks the reduction into iterated reductions of twofold tensor products. Such reduction schemes are enumerated combinatorially by specifying complete bracketing schemes, or equivalently, by rooted binary trees [2, 3, 18, 19, 24].

Remark 3.1. A rooted tree is an undirected connected graph which does not contain cycles and which has a distinguished vertex, called the root. The parent of a vertex \( x \)
is the vertex connected to \( x \) on the unique path to the root. Any vertex on that path is called an ascendant of \( x \). A child of a vertex \( x \) is a vertex of which \( x \) is the parent. A descendant of \( x \) is a vertex of which \( x \) is an ascendant. We write \( x < y \) if \( x \) is an ascendant of \( y \) (or equivalently \( y \) is a descendant of \( x \)). The number of edges attached to a vertex is called the valence of that vertex. A vertex of valence 1 is called a leaf. All other vertices are called nodes. A rooted binary tree is a rooted tree whose root has valence 2 and whose other nodes have valence 3. For convenience, we will view the root as a node of valence 3, with an additional pendant root edge. The nodes different from the root are called internal.

A labelling of a rooted binary tree \( T \) is an assignment \( \alpha : x \mapsto \alpha_x \) of a label to every vertex of \( T \). The pair \( (T, \alpha) \) is called a labelled, rooted binary tree.

It is convenient to view a rooted binary tree as being embedded in the plane. Then, the child vertices of every node can be ordered from left to right. Conversely, an ordering of the child vertices of every node defines a unique planar embedding. Given such an ordering, or equivalently an embedding in the plane, one speaks of an ordered tree. In particular, in such a tree, the leaves are ordered and thus their labels are given by a sequence. ♦

**Example 3.2.** Consider the specific case of the reduction of a tensor product of \( N = 5 \) irreps according to the bracketing

\[
(\big(H_{\lambda^1} \otimes H_{\lambda^3}\big) \otimes \big(H_{\lambda^4} \otimes (H_{\lambda^5} \otimes H_{\lambda^5})\big)),
\]

as shown in Figure 1. The rooted binary tree corresponding to this reduction scheme is as follows. There are 5 leaves representing the tensor factors, 3 internal nodes, representing intermediate stages of the reduction procedure, and the root representing the final irreducible subspaces. The leaves are labelled by the highest weights \( \lambda^1, \ldots, \lambda^5 \) of the respective irreducible representations. The 3 internal nodes are labelled by \((\lambda^{12}, k^{12}), (\lambda^{45}, k^{45}), (\lambda^{345}, k^{345})\) enumerating the admissible weight values occurring in the pairwise tensor products indicated by their child nodes and the root is labelled by \((\lambda^{12345}, k^{12345}) \equiv (\lambda, k)\). We have \((\lambda^{12}, k^{12}) \in \langle \langle \lambda^1, \lambda^2 \rangle \rangle, (\lambda^{45}, k^{45}) \in \langle \langle \lambda^4, \lambda^5 \rangle \rangle, (\lambda^{345}, k^{345}) \in \langle \langle \lambda^3, \lambda^{45} \rangle \rangle, \text{ and } (\lambda, k) \in \langle \langle \lambda^{12}, \lambda^{345} \rangle \rangle.

Figure 1: A reduction scheme for a tensor product of \( N = 5 \) irreps.
Example 3.3 (Standard coupling). Given $N$ highest weights $\lambda_1, \ldots, \lambda_N$, we may start with decomposing $H_{\lambda_1} \otimes H_{\lambda_2}$ into the unique irreducible subspaces $(H_{\lambda_1} \otimes H_{\lambda_2})_{(\lambda_{12}, k_{12})}$ with $(\lambda_{12}, k_{12}) \in \langle \langle \lambda_1, \lambda_2 \rangle \rangle$. Then, we decompose the invariant subspaces $(H_{\lambda_1} \otimes H_{\lambda_2})_{(\lambda_{12}, k_{12})} \otimes H_{\lambda_3} \subset H_{\lambda_1} \otimes H_{\lambda_2} \otimes H_{\lambda_3}$ into the unique irreducible subspaces $(\cdots ((H_{\lambda_1} \otimes H_{\lambda_2})_{(\lambda_{12}, k_{12})} \otimes H_{\lambda_3})_{(\lambda_{123}, k_{123})} \cdots \otimes H_{\lambda_N})_{(\lambda_{1\ldots N}, k_{1\ldots N})}$, where $(\lambda_{1\ldots n}, k_{1\ldots n}) \in \langle \langle \lambda_{1\ldots n-1}, \lambda_n \rangle \rangle$ for $n = 3, 4, \ldots, N$. We may number the nodes of $T$ by assigning the number $n$ to the parent of leaf $n$, where $n = 2, \ldots, N$. Then, $x_2, \ldots, x_{N-1}$ are the internal nodes and $x_N$ is the root. Accordingly, we may write $(\lambda_{1\ldots n}, k_{1\ldots n}) = (\lambda_{1\ldots n-1}, \lambda_n)$ for $n = 2, \ldots, N-1$ and $(\lambda, k) = (\lambda_{1\ldots N}, k_{1\ldots N})$. The corresponding bracketing is $(\cdots ((\cdot, \cdot), \cdot, \cdot))$ and the corresponding coupling tree is the caterpillar tree shown in Figure 2 also referred to as the standard coupling tree.

In general, for an $N$-fold tensor product of $G$-irreps, the reduction schemes (bracketings) correspond 1-1 to rooted binary trees with $N$ leaves, referred to as coupling trees. Such a tree has $N-2$ internal nodes. The leaves of a coupling tree correspond to the irreps entering the tensor product and the root corresponds to an irreducible subspace of the tensor product. The root will usually be denoted by $r$.

A given labelling $\alpha$ of a coupling tree $T$ assigns to every leaf $y$ of $T$ a highest weight $\alpha^y = \lambda^y \in \hat{G}$ and to every node $x$ of $T$ a pair $\alpha^x = (\lambda^x, k^x) \in \hat{G} \times \mathbb{N}$. We say that $\alpha$ is admissible if $\alpha^x \in \langle \langle \lambda^{x'}, \lambda^{x''} \rangle \rangle$ for every node $x$ of $T$, where $x'$ and $x''$ denote the child vertices of $x$. In what follows, we will assume all labellings to be admissible without explicitly stating that. By forgetting about the multiplicity counters, $\alpha$ induces a labelling of $T$ by highest weights, referred to as the highest weight labelling underlying $\alpha$. By forgetting about the highest weights, $\alpha$ induces a labelling of $T$ by multiplicity...
counters, referred to as the multiplicity counter labelling underlying \( \alpha \). Given a coupling tree \( T \) with leaf labelling \( \Delta \), let \( \mathcal{L}^T(\Delta) \) denote the set of admissible labellings of \( T \) having \( \Delta \) as their leaf labelling. Given, in addition, a highest weight \( \lambda \in \langle \Delta \rangle \), let \( \mathcal{L}^T(\Delta, \lambda) \) denote the set of labellings of \( T \) having \( \Delta \) as their leaf labelling and \( \lambda \) as the highest weight label of the root. These subsets establish a disjoint decomposition of the totality of all (admissible) labellings of \( T \). We will say that two given labellings of \( T \) are combinable if they belong to the same such subset, that is, if they share the same leaf labelling and the same highest weight of the root. To be combinable is an equivalence relation.

Remark 3.4. Each internal node \( x \) is the root of a unique subtree made up by \( x \) and all its descendants. This subtree represents a reduction scheme for a tensor product of \( l \) \( G \)-irreps, where \( l \) is the number of leaves in the subtree. In case the leaves are numbered by \( 1, \ldots, N \), the subtrees associated with the nodes are in one-to-one correspondence with the subsets \( S \subseteq \{1, \ldots, N\} \) made up by their leaves, and one may use these subsets to name the nodes, as in Examples 3.2 and 3.3. Thus, the leaf labelling reads \( \lambda = (\lambda_1, \ldots, \lambda_N) \) and the node labels read \( \alpha_S = (\lambda_S, k_S) \). The subset \( S = \{1, \ldots, N\} \) corresponds to the root, so that \( \alpha^{\{1,\ldots,N\}} = (\lambda^{\{1,\ldots,N\}}, k^{\{1,\ldots,N\}}) = (\lambda, k) \). Admissibility ensures that

\[ \lambda S \in \langle \lambda^n : n \in S \rangle. \]

The subsets \( S \) form a hierarchy, that is, for any two members \( S \) and \( S' \), either \( S \cap S' = \emptyset \), \( S \cap S' = S \), or \( S \cap S' = S' \).

Now, for given leaf labelling \( \Delta \), the diagonal representation \( (H_\Delta, D_\Delta) \) decomposes into uniquely determined isotypical subspaces \( H_{\Delta, \lambda} \) labelled by an admissible highest weight label \( \lambda \in \langle \Delta \rangle \) of the root. According to the coupling tree \( T \) chosen, the isotypical subspaces \( H_{\Delta, \lambda} \) can be further decomposed in a non-unique way into irreducible subspaces \( H_\alpha \) enumerated by the tree labellings \( \alpha \in \mathcal{L}^T(\Delta, \lambda) \). Thus,

\[ H_\Delta \cong \bigoplus_{\lambda \in \langle \Delta \rangle} \bigoplus_{\alpha \in \mathcal{L}^T(\Delta, \lambda)} H_\alpha \]

as \( G \)-representations, and it is the labels \( \alpha^x \) of the internal nodes \( x \) and the multiplicity counter \( k \) of the root that are in 1-1 correspondence with the irreducible subspaces obtained by means of the reduction scheme related to \( T \). Accordingly, the multiplicity of the irrep \( (H_\lambda, D_\lambda) \) of highest weight \( \lambda \) in the diagonal representation on \( H_\Delta \) is

\[ m_\lambda(\lambda) = |\mathcal{L}^T(\Delta, \lambda)|, \]

and the isomorphism (2.4) reads

\[ H_\Delta \cong \bigoplus_{\lambda \in \langle \Delta \rangle} \bigoplus_{\alpha \in \mathcal{L}^T(\Delta, \lambda)} H_\lambda. \]

That is, the latter is obtained by identifying each irreducible subspace \( H_\alpha \) with a copy of \( H_\lambda \). By Schur’s Lemma, each of these identifications is unique up to a phase. In what follows, we assume that a phase has been chosen. To summarize, the invisible choices
made in the definition of the isomorphism (2.4) via $T$ amount to these phases and to the choice of a unitary $G$-representation isomorphism (3.2) for every pair $(\lambda_1, \lambda_2) \in \hat{G} \times \hat{G}$ (affecting the definition of the subspaces $H^T_{\alpha}$. Note that we may absorb the direct sum over the final highest weights $\lambda$ into the sum over the labellings and thus write

$$H_{\Lambda} \cong \bigoplus_{\lambda \in \mathcal{L}(\Lambda)} H_{\lambda}. \quad (3.4)$$

We use the $\alpha$ as labels for the copies of $H_{\lambda}$ in the direct sum on the right hand side. This has the following consequences.

Firstly, the projections and injections (2.5) obtained by composing this isomorphism with the natural projections and injections of the direct sum read

$$H_{\Lambda} \xrightarrow{p^T_{\alpha}} H_{\lambda}, \quad H_{\lambda} \xrightarrow{i^T_{\alpha}} H_{\Lambda}.$$

By construction, each $p^T_{\alpha}$ is obtained by composing the elementary projections associated with twofold tensor products according to $T$. Thus, $p^T_{\alpha}$ is uniquely determined by the choice of a reduction tree $T$, a labelling $\alpha$, and the choice of an isomorphism (3.2) for every combination of $\lambda_1, \lambda_2 \in \hat{G}$ and every $\lambda \in \langle \lambda_1, \lambda_2 \rangle$. An analogous statement holds true for $i^T_{\alpha}$.

Secondly, the quasicharacters read $\chi^C(T)_{\alpha'}^{\alpha'}$ and the representation homomorphisms entering their definition read $A(T)_{\alpha'}^{\alpha'}$, for any combinable $\alpha, \alpha'$. Using the injections $i^T_{\alpha}$ and the normalized weight bases $\{|\lambda\mu\rangle : \mu \in \mathfrak{w}(\lambda)\}$ chosen in $H_{\Lambda}$ for every $\lambda \in \hat{G}$, we can define elements of $\mathcal{H}_{\alpha'}$ by

$$|T; \alpha, \mu\rangle := i^T_{\alpha}(|\lambda\mu\rangle), \quad \alpha \in \mathcal{L}(\Lambda), \quad \lambda \in \langle \Lambda \rangle \lambda \in \langle \Lambda \rangle, \quad \mu \in \mathfrak{w}(\lambda). \quad (3.5)$$

These elements form an orthonormal basis in $\mathcal{H}_{\alpha'}$.

$$\langle T; \alpha, \mu | T; \alpha', \mu' \rangle = \delta_{\alpha\alpha'} \delta_{\mu\mu'},$$

and they are common eigenvectors of the Cartan subalgebra with eigenvalue functional $i\hat{\mu}$. For fixed $\alpha$, the elements $|T; \alpha, \mu\rangle, \mu \in \mathfrak{w}(\lambda)$, form an orthonormal weight basis in the irreducible subspace $H^T_{\alpha}$. Using (2.6) and the relation $p^T_{\alpha'} \circ i^T_{\alpha'} = \delta_{\alpha\alpha'} \text{id}_{H_{\lambda}}$ holding for $\alpha, \alpha' \in \mathcal{L}(\Lambda)$, we compute

$$A(T)_{\alpha'}^{\alpha'}(|T; \alpha'', \mu\rangle) = \frac{1}{\sqrt{\dim(H_{\lambda})}} i^T_{\alpha} \circ p^T_{\alpha'}(|T; \alpha'', \mu\rangle) = \frac{\delta_{\alpha\alpha''}}{\sqrt{\dim(H_{\lambda})}} |T; \alpha, \mu\rangle$$

for all $\alpha, \alpha', \alpha'' \in \mathcal{L}(\Lambda)$. This implies

$$A(T)_{\alpha'}^{\alpha'} = \frac{1}{\sqrt{\dim(H_{\lambda})}} \sum_{\mu \in \mathfrak{w}(\lambda)} |T; \alpha, \mu\rangle \langle T; \alpha', \mu|, \quad \alpha, \alpha' \in \mathcal{L}(\Lambda), \quad (3.6)$$

and

$$\hat{\chi}^C(T)_{\alpha'}^{\alpha'}(\chi) = \sum_{\mu \in \mathfrak{w}(\lambda)} \langle T; \alpha, \mu | D(\chi) | T; \alpha, \mu\rangle, \quad \alpha, \alpha' \in \mathcal{L}(\Lambda). \quad (3.7)$$
It will turn out that the key to unravelling the algebraic structure of the invariants in terms of the quasicharacters is the dependence on the coupling tree and the transformation law between different such trees. For \( i = 1, 2 \), let \( T_i \) be a coupling tree, \( \lambda_i \) a leaf labelling of \( T_i \), \( \lambda_i \in \langle \lambda_i \rangle \) and \( \alpha_i \in \mathcal{L}^{T_i}(\lambda_i, \lambda_i) \). If \( \lambda_2 \) is obtained from \( \lambda_1 \) by a permutation \( \sigma \) of the entries, there exists a unitary transformation \( \Pi : H_{\lambda_1} \rightarrow H_{\lambda_2} \), permuting the tensor factors according to \( \sigma \), that is,
\[
\Pi|\lambda_1, \mu \rangle = |\lambda_2, \sigma(\mu) \rangle, \quad \mu \in \mathfrak{u}(\lambda_1).
\]
Then, by orthogonality of isotypical subspaces, the overlap of the basis vectors \(|T_1; \alpha_1, \mu_1\rangle\) and \(|T_2; \alpha_2, \mu_2\rangle\) vanishes unless \( \lambda_1 = \lambda_2 \). In that case, \( p^\alpha_1 \circ \Pi \circ \delta^\alpha_1 \) is a unitary representation automorphism of the unitary irrep \((H_{\lambda_1}, D^{\lambda_1})\) and hence it is given by multiplication by a complex factor of modulus one. Denoting this factor by \( R(T_2|T_1)^{\alpha_2}_{\alpha_1} \), we obtain
\[
\langle T_2; \alpha_2, \mu_2|\Pi|T_1; \alpha_1, \mu_1 \rangle = \delta_{\lambda_1, \lambda_2} \delta_{\mu_1, \mu_2} R(T_2|T_1)^{\alpha_2}_{\alpha_1}.
\]
We will refer to \( R(T_2|T_1)^{\alpha_2}_{\alpha_1} \) as a recoupling coefficient. In case \( G = SU(2) \), the recoupling coefficients coincide with the angular momentum recoupling coefficients (Racah coefficients) \( [3] \).

Remark 3.5. Explicitly, we have
\[
R(T_2|T_1)^{\alpha_2}_{\alpha_1} = \langle T_2; \alpha_2, \mu|\Pi|T_1; \alpha_1, \mu \rangle,
\]
individually of \( \mu \). Note that (3.9) yields the symmetry
\[
R(T_2|T_1)^{\alpha_2}_{\alpha_1} = \left( R(T_1|T_2)^{\alpha_1}_{\alpha_2} \right)^*.
\]
Consider a further coupling tree \( T_3 \) with labelling \( \alpha_3 \) whose leaf labelling \( \lambda_3 \) is obtained from \( \lambda_2 \) by some further permutation. Denote the respective induced unitary transformation permuting the tensor factors by \( \Pi_{12} : H_{\lambda_2} \rightarrow H_{\lambda_3} \) and \( \Pi_{23} : H_{\lambda_3} \rightarrow H_{\lambda_3} \). Then, \( \Pi_{23} \circ \Pi_{12} \) permutes the tensor factors according to the composite permutation. Assume further that \( \alpha_1 \) and \( \alpha_3 \) assign the same weight label \( \lambda \) to the root. Then, by inserting an appropriate unit into (3.9), with index 2 replaced by 3 and \( \Pi \) replaced by \( \Pi_{23} \circ \Pi_{12} \), we obtain the cycle relation
\[
R(T_3|T_1)^{\alpha_3}_{\alpha_1} = \sum_{\alpha_2 \in \mathcal{L}^{T_2}(\lambda_2, \lambda)} R(T_3|T_2)^{\alpha_3}_{\alpha_2} R(T_2|T_1)^{\alpha_2}_{\alpha_1}.
\]

\( \diamond \)

Proposition 3.6 (Change of Coupling Tree). Let \( T_1 \) and \( T_2 \) be coupling trees with the same number of leaves, let \( \lambda \) be a leaf labelling for both of them and let \( \lambda \in \langle \lambda \rangle \). Then, for labellings \( \alpha_1, \alpha_1' \in \mathcal{L}^{T_1}(\lambda, \lambda) \) and \( \alpha_2, \alpha_2' \in \mathcal{L}^{T_2}(\lambda, \lambda) \), the transformation rule between quasicharacters coupled according to \( T_1 \) and \( T_2 \) is
\[
\hat{\chi}_\lambda^{C}(T_1)^{\alpha_1'}_{\alpha_1} = \sum_{\alpha_2, \alpha_2' \in \mathcal{L}^{T_2}(\lambda, \lambda)} R(T_1|T_2)^{\alpha_1'}_{\alpha_2'} \hat{\chi}_\lambda^{C}(T_2)^{\alpha_2'}_{\alpha_2} R(T_2|T_1)^{\alpha_2}_{\alpha_1}.
\]
Proof. Inserting complete sets of coupled states for the coupling tree \( T_2 \),

\[
1 = \sum_{\alpha_2 \in \mathcal{C}^{T_2}(\lambda)} \sum_{\mu_2} \langle T_2; \alpha_2, \mu_2 \rangle \langle T_2; \alpha_2, \mu_2 \rangle ,
\]

into (3.7), we obtain

\[
\hat{\chi}^C(T_1)|_{\alpha_1}(a) = \sum_{\mu} \sum_{\alpha_2, \alpha_2' \in \mathcal{C}^{T_2}(\lambda)} \sum_{\mu_2, \mu_2'} \langle T_1; \alpha_1', \mu \rangle \langle T_2; \alpha_2', \mu_2' \rangle \cdot \langle T_2; \alpha_2', \mu_2' \rangle D_{\lambda, \mu}^{\lambda_1}(a) \langle T_2; \alpha_2, \mu_2 \rangle \langle T_2; \alpha_2, \mu_2 \rangle \langle T_1; \alpha_1, \mu \rangle .
\]

Application of (3.8) now yields the assertion. \( \Box \)

3.2 The multiplication law for quasicharacters

In this section, we study the multiplicative structure of the algebra \( \mathcal{R} \), that is, we analyze the general multiplication law given by Theorem 2.4 in terms of the recoupling calculus developed above. To start with, we define two operations on trees.

Definition 3.7 (Operations on coupling trees). Let \( T_1, T_2 \) and \( T \) be coupling trees with, respectively, \( N_1, N_2 \) and \( N \) leaves.

Tree join: The join \( T_1 \cdot T_2 \) is the coupling tree with \( N_1 + N_2 \) leaves formed by glueing the pendant root edges of \( T_1 \) and \( T_2 \) to make a new root and add to this new root a pendant edge.

Leaf duplication: The tree \( T' \) is the coupling tree with 2N leaves formed by replacing each leaf of \( T \) by the root of a copy of the root node binary tree with two leaves (a cherry), thereby identifying the leaf edge with the pendant root edge of that copy.

See Figure 3 for specific examples of tree join and leaf duplication.

For \( i = 1, 2 \), let \( r_i \) denote the root of \( T_i \) and assume that admissible labellings \( \alpha_i \) of \( T_i \) with underlying highest weight labelling \( x \to \lambda_i^\infty \) are given.

In case of the operation of join, one can obtain an admissible labelling of \( T_1 \cdot T_2 \) by retaining the labellings \( \alpha_1 \) for \( T_1 \) and \( \alpha_2 \) for \( T_2 \), and by assigning to the new root some value \( (\lambda, k) \in \langle (\lambda_1^\infty, 1), (\lambda_2^\infty, 2) \rangle \). The labelling of \( T_1 \cdot T_2 \) so arising will be denoted by \( [\alpha_1 \cdot \alpha_2, (\lambda, k)] \). In addition, given leaf labellings \( \Lambda_i \) of \( T_i \), let \( \Lambda_1 \cdot \Lambda_2 \) denote the leaf labelling of \( T_1 \cdot T_2 \) obtained by retaining the leaf labellings \( \Lambda_1 \) for \( T_1 \) and \( \Lambda_2 \) for \( T_2 \). Thus, if \( \Lambda_i \) is the leaf labelling belonging to \( \alpha_i \), then \( \Lambda_1 \cdot \Lambda_2 \) is the leaf labelling belonging to \( [\alpha_1 \cdot \alpha_2, (\lambda, k)] \). Having drawn \( T_1 \) and \( T_2 \) in a plane, with \( T_1 \) to the left and \( T_2 \) to the right, then the leaf labelling \( \Lambda_1 \cdot \Lambda_2 \) is given by

\[
\Lambda_1 \cdot \Lambda_2 = (\lambda_1^1, \ldots, \lambda_1^{N_1}, \lambda_2^1, \ldots, \lambda_2^{N_2}) \quad (3.12)
\]

and the corresponding tensor product by \( H_{\lambda_1^1} \otimes \cdots \otimes H_{\lambda_1^{N_1}} \otimes H_{\lambda_2^1} \otimes \cdots \otimes H_{\lambda_2^{N_2}} \). If \( N_1 = N_2 = N \), there corresponds a representation of \( G^N \),

\[
D_{\Lambda_1 \cdot \Lambda_2}(a) = D_{\Lambda_1}(a) \otimes D_{\Lambda_2}(a) = D_{\lambda_1^1}(a_1) \otimes \cdots \otimes D_{\lambda_1^{N_1}}(a_N) \otimes D_{\lambda_2^1}(a_1) \otimes \cdots \otimes D_{\lambda_2^{N_2}}(a_N)
\]

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Figure 3: Join $T_1 \cdot T_2$ of two standard trees $T_i$ with leaves $y_i^1, \ldots, y_i^{N_i}$ and root $r_i$, $i = 1, 2$ (left) and leaf duplication $T^\lor$ of a standard tree $T$ with leaves $y^1, \ldots, y^N$ and root $r$ (right).

In case of the operation of leaf duplication, the situation is different. Here, given labellings $\alpha_1$ of $T_1$ and $\alpha_2$ of $T_2$, their only parts which can be assigned are their leaf labelings $\lambda_1$ and $\lambda_2$, respectively. They define a leaf labelling $\lambda_1 \ast \lambda_2$ of $T^\lor$ by assigning $\lambda_y^i$ to the $i$-th child vertex of the cherry replacing leaf $y$ of $T$. As the nodes of $T^\lor$ correspond 1-1 to the vertices of $T$, every node labelling of $T^\lor$ splits into a labelling of $T$ and an assignment $k_y$ of a multiplicity counter $k^y$ to every leaf $y$ of $T$. Conversely, assume that we are given a further labelling $\alpha_3$ of $T$ with leaf labelling $\lambda_3$ and an assignment $k_3$ of a positive integer to every leaf of $T$. If $(\lambda_y^3, k^3_y) \in \langle \langle \lambda^1_y, \lambda^2_y \rangle \rangle$ for all leaves $y$ of $T$, then the node labelling of $T^\lor$ arising from $\alpha_3$ and $k_3$ is compatible with the leaf labelling $\lambda_1 \ast \lambda_2$, and so the two combine to an admissible labelling of $T^\lor$. This labelling will be denoted by $[\alpha_1 \ast \alpha_2, \alpha_3, k_3]$. Finally, having drawn $T^\lor$ in a plane, with the first child vertex of every cherry to the left and the second one to the right, the corresponding sequence of leaf labels is given by

$$\lambda_1 \ast \lambda_2 = (\lambda_1^1, \lambda_2^1, \ldots, \lambda_1^N, \lambda_2^N) \tag{3.13}$$

and the corresponding tensor product by $H_{\lambda_1^1} \otimes H_{\lambda_2^1} \otimes \cdots \otimes H_{\lambda_1^N} \otimes H_{\lambda_2^N}$. There corresponds a representation of $G^N$,

$$D^{\Delta \lambda^1 \lambda^2}(a) = D^{\lambda_1^1}(a_1) \otimes D^{\lambda_2^1}(a_1) \otimes \cdots \otimes D^{\lambda_1^N}(a_N) \otimes D^{\lambda_2^N}(a_N).$$

**Remark 3.8.** Leaf duplication is a special case of (rooted) tree composition, $T_1 \ast T_2$, defined as the tree with $N_1N_2$ leaves formed by replacing each leaf of $T_1$ by the root of a copy of $T_2$, thereby identifying the leaf edge with the pendant root edge of that copy. Thus, $T^\lor = T \ast \lor$. Given $N_2$ leaf labellings $\lambda_1(1), \ldots, \lambda_1(N_2)$ of $T_1$, one defines the leaf labelling of $T_1 \ast T_2$ by

$$\lambda_1(1) \ast \cdots \ast \lambda_1(N_2) := (\lambda_1(1), \ldots, \lambda_1(N_2), \ldots, \lambda_1(1), \ldots, \lambda_1(N_2)).$$
It is immediate that tree join and tree composition are subject to the ’distributive law’

\((T_1 \cdot T_2) \ast T_3 = (T_1 \ast T_3) \cdot (T_2 \ast T_3)\).  

\(\checkmark\)

In the sequel we adapt the notations \(\lambda_1 \cdot \lambda_2\) and \(\lambda_1 \ast \lambda_2\) given by (3.12) and (3.13) to weight labels to obtain weight labelings \(\mu_1 \cdot \mu_2\) and \(\mu_1 \ast \mu_2\) of the leaves of \(T_1 \cdot T_2\) and \(T\), respectively. There correspond tensor product states \(|\lambda_1 \cdot \lambda_2 \mu_1 \cdot \mu_2\rangle\) and \(|\lambda_1 \ast \lambda_2 \mu_1 \ast \mu_2\rangle\), respectively. In the case where \(T_1, T_2\) and \(T\) have the same number \(N\) of leaves, there exists a permutation operator \(\Pi\) such that

\(|\lambda_1 \cdot \lambda_2 \mu_1 \cdot \mu_2\rangle = \Pi |\lambda_1 \ast \lambda_2 \mu_1 \ast \mu_2\rangle, \quad \Pi^{-1} D_{\lambda_1 \cdot \lambda_2}(q) \Pi = D_{\lambda_1 \ast \lambda_2}(q)\). (3.14)

With these preliminaries, we are able to express the multiplication law for quasicharacters in terms of the recoupling coefficients. For that purpose, let labellings \(\alpha_1, \alpha_2, \alpha_3\) of \(T\), an assignment \(k\) of a positive integer to every leaf of \(T\) and a positive integer \(k\) be given. Let \(x \mapsto \lambda_x^r\) be the highest weight labelling underlying \(\alpha_i\). We write \((\alpha_3, \lambda_x^r, k) \in \langle \alpha_1, \alpha_2 \rangle\) if \((\lambda_x^r, k) \in \langle \lambda_1^r, \lambda_2^r \rangle\) for all leaves \(y\) of \(T\) and \((\lambda_y^r, k) \in \langle \lambda_3^r, \lambda_3^r \rangle\), where \(r\) denotes the root of \(T\). We define

\[ R(T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, k} := \begin{cases} R(T^\ast T^\ast T)_{\alpha_1, \alpha_2, \lambda_x^r, k}^{\alpha_1, \alpha_2, \lambda_x^r, k} & (\alpha_3, \lambda_x^r, k) \in \langle \alpha_1, \alpha_2 \rangle, \\ 0 & \text{otherwise} \end{cases} \]

and the adjoint

\[ R(T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, k} := \begin{cases} R(T^\ast T^\ast T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, \lambda_x^r, k} & (\alpha_3, \lambda_x^r, k) \in \langle \alpha_1, \alpha_2 \rangle, \\ 0 & \text{otherwise} \end{cases} \]

By (3.11),

\[ R(T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, k} = (R(T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, k})^*. \]

It turns out that for our choice of the isomorphism (2.14), the coefficients \(U\) in Formula (2.12) are given by the recoupling coefficients \(R(T)\):

**Lemma 3.9.** For \(i = 1, 2, 3\), let \(\lambda_i\) be a leaf labelling of \(T\), let \(\lambda_i \in \langle \lambda_i \rangle\) and let \(\alpha_i \in \mathcal{L}^T(\lambda_i, \lambda_i)\). Let \(k\) be an assignment of a positive integer to every leaf of \(T\), let \(k\) be a positive integer and assume that \((\alpha_3, \lambda_x^r, k) \in \langle \alpha_1, \alpha_2 \rangle\). Then,

\[ R(T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, k} = U_{\lambda_1 \cdot \lambda_2, \lambda_3, l_1, l_2, l_3, k}^{\lambda_1 \cdot \lambda_2, \lambda_3, l_1, l_2, l_3, k}, \]

where \(l_i\) is the positive integer corresponding to \(\alpha_i\) relative to some chosen enumeration of the elements of \(\mathcal{L}^T(\lambda_i, \lambda_i)\), \(i = 1, 2, 3\).

**Proof.** For any \(\mu \in \mathcal{U}(\lambda_3)\), we have

\[ R(T)_{\alpha_3, \lambda_x^r, k}^{\alpha_1, \alpha_2, k} = R(T^\ast T^\ast T)_{\alpha_1, \alpha_2, \lambda_x^r, k}^{\alpha_1, \alpha_2, \lambda_x^r, k} \]

\[ = \langle T^\ast T; [\alpha_1 \cdot \alpha_2, (\lambda_3, k)], \mu \rangle \cdot \Pi | T^\ast T; [\alpha_1 \ast \alpha_2, \lambda_3, k], \mu \rangle. \]

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where \( \Pi \) denotes the unitary transformation given by (3.14). We observe that

\[
\Pi \circ i^{T'}_{[\alpha_1, \alpha_2, \alpha_3, \lambda]} = \left( \otimes_{i=1}^{N} i^{T'}_{(\lambda'_1, \lambda'_2, \lambda'_3, k_i)} \right) \circ i^{T'}_{\lambda, \lambda'}.\]

and that

\[
p^{\alpha_1, \alpha_2, \alpha_3, \lambda}_{[\alpha_1, \alpha_2, \alpha_3, \lambda]} = p^{(\lambda_1, \lambda_2), \lambda', k} \circ \left( p^{\lambda_1, \lambda_1', j_1} \otimes p^{\lambda_2, \lambda_2', j_2} \right).\]

Hence, the assertion follows from (2.12). \( \square \)

Using this lemma, we immediately obtain the following corollary to Theorem 2.4.

**Corollary 3.10 (Multiplication law for quasicharacters).** For a given coupling tree \( T \), the product of quasicharacters is given by

\[
\hat{\chi}^C(T)_{\alpha_1} \chi^C(T)_{\alpha_2} = \sum_{\alpha, \alpha'} R(T)_{\alpha \alpha'}^\alpha R(T)_{\alpha_1 \alpha_2, \lambda} \chi^C(T)_{\alpha'},
\]

where the sum is over all combinable labellings \( \alpha, \alpha' \) of \( T \), all assignments \( k \) of a positive integer to every leaf of \( T \) and all positive integers \( k \) such that \( (\alpha, k, k, k) \in \langle \alpha_1, \alpha_2 \rangle \).

For a direct proof of this corollary using the Clebsch-Gordan calculus see the appendix.

**Remark 3.11.**

1. As a result, the structure constants of the algebra \( \mathcal{R} \) with respect to the basis \( \{ \chi^C(T)_{\alpha} \} \) are given by \( \sum_k R(T)_{\alpha}^\alpha R(T)_{\alpha_1 \alpha_2, \lambda} \) with free summation indices \( \alpha_3 \) and \( \alpha'_3 \).

2. For a threefold product, we obtain

\[
\hat{\chi}^C(T)_{\alpha_1} \chi^C(T)_{\alpha_2} \chi^C(T)_{\alpha_3} = \sum_{\beta_2, \beta_3} \sum_{\lambda_3} R(T)_{\alpha_1 \alpha_2, \beta_2} R(T)_{\beta_2 \beta_3, \lambda_3} \hat{\chi}^C(T)_{\beta_3} R(T)_{\beta_3 k_3} R(T)_{\alpha_1 \alpha_2, k_2}.
\]

This generalizes in an obvious way to \( n \)-fold products. \( \diamond \)

**Remark 3.12.** The above formula generalizes directly to the pointwise multiplication \( \hat{\chi}(T_1) \cdot \hat{\chi}(T_2) \) of modified quasicharacters for two different coupling trees \( T_1 \) and \( T_2 \) having the same number of leaves. The result is expressible in terms of the \( \hat{\chi}(T_3) \)-basis provided by an arbitrary third tree \( T_3 \) with the same number of leaves, with \( T' \cdot T \) replaced by \( T_1 \cdot T_2 \), and \( T' \) replaced by \( T_3' \),

\[
\hat{\chi}^C(T_1)_{\alpha_1} \chi^C(T_2)_{\alpha_2} = \sum_{\lambda_3} \sum_{\alpha_3, \alpha'_3} R(T_1 \cdot T_2 | T_3') R(T_3 | T_1 \cdot T_2) \hat{\chi}^C(T_3)_{\alpha_3} R(T_3 | T_1 \cdot T_2) \chi^C(T_3)_{\alpha_3} R(T_3 | T_1 \cdot T_2).
\]
where the sum is over all combinable labellings $\alpha_3$, $\alpha'_3$ of $T_3$, all assignments $k_3$ of a positive integer to every leaf of $T_3$ and all positive integers $k_3$ such that $(\alpha_3, k_3) \in \langle \alpha_1, \alpha_2 \rangle$ and $(\lambda_3, k) \in \langle \lambda_1, \lambda_2 \rangle$. In this case, the recoupling coefficients measure the overlap of two $2N$-leaf coupling trees, and so each of them entails $2N - 2$ internal labels; including the final coupling to $\lambda_3$ and adding the $2N$ leaf labels $\Delta_1, \Delta_2$, they are thus of $(6N - 3)j$-type, reflecting the count for structure constants arising from expanding the pointwise product of two $\chi$'s into a sum, $3(2N - 1) = 6N - 3$. The structure constants reflect the commutativity of the pointwise product because of their symmetry (under interchange of $\alpha_1$ with $\alpha_2$ and $\alpha'_3$ with $\alpha'_2$). Associativity on the other hand is not manifest, but is clearly a concomitant of the tens or-categorial origins of the recoupling calculus itself; there appears moreover to be a functorial association between the pointwise algebraic product, and the above combinatorial tree operations. In this light, it might be expected that the compound structure constants derived above should be replaced by single, but more elaborated, recouplings involving higher degree tree operations such as $(\langle \cdot \cdot \cdot \rangle, \cdot \cdot \cdot )$, and their $n$-ary generalizations.

\[ \phi \]

### 3.3 Composite Clebsch-Gordan coefficients

In this subsection, we analyze the combinatorics of angular momentum theory phrased in terms of Clebsch-Gordan coefficients for the case of a compact group $G$. Using this calculus, we will be able to reduce the recoupling coefficients arising in our quasicharacter manipulations to products of more basic coefficients (see \[\text{3.3}\] below).

Let $T$ be a coupling tree, $\Lambda$ a leaf labelling and $\lambda \in \langle \Lambda \rangle$. For $\alpha \in L^T(\Lambda, \lambda)$, $\mu \in w(\lambda)$ and $\mu \in w(\lambda)$, we define

\[ C(T)^{\alpha}_{\mu} := \langle \Lambda | T; \alpha, \mu \rangle. \]

Then, for $\mu \in w(\lambda)$, we have

\[ |T; \alpha, \mu \rangle = \sum_{\mu \in w(\Lambda)} C(T)^{\alpha}_{\mu} |\Lambda \mu\rangle. \tag{3.15} \]

Since $|\Lambda \mu\rangle$ and $|T; \alpha, \mu \rangle$ are common eigenvectors of a commutative algebra of skew-adjoint operators with eigenvalue functionals $\Sigma(\hat{\mu})$ and $i\Sigma(\hat{\mu})$, respectively, we have $C(T)^{\alpha}_{\mu} = 0$ unless $\Sigma(\hat{\mu}) = \hat{\mu}$. Hence, the sum restricts automatically to $\mu$ satisfying $\Sigma(\hat{\mu}) = \hat{\mu}$. In case $N = 2$, the only coupling tree is the cherry $\nu$, with labelling assigning $\lambda^1$ and $\lambda^2$ to the leaves and $(\lambda, k) \in \langle \lambda^1, \lambda^2 \rangle$ to the root. For $\mu^n \in w(\lambda^n)$ and $\mu \in w(\lambda)$, the coefficients

\[ C^{\lambda^1, \lambda^2, \lambda, k}_{\mu^1, \mu^2, \mu} := C(\nu)^{(\lambda^1, \lambda^2, \lambda, k)}_{(\mu^1, \mu^2, \mu)} \]

are the analogues of the ordinary Clebsch-Gordan coefficients in the case $G = SU(2)$ and hence will be referred to as the ordinary Clebsch-Gordan coefficients for $G$. These coefficients can be chosen to be real [1]. Occasionally, we will make use of this. The coefficients $C(T)^{\alpha}_{\mu}$ will turn out to be products of ordinary Clebsch-Gordan coefficients for $G$. Therefore, they will be referred to as composite Clebsch-Gordan coefficients for
G. Quasicharacters and recoupling coefficients can be expressed in terms of composite Clebsch-Gordan coefficients as follows.

**Proposition 3.13.**

1. For every coupling tree $T$, leaf labelling $\lambda \in \langle \lambda \rangle$ and $\alpha, \alpha' \in \mathcal{L}^T(\lambda, \lambda)$, one has

$$
\chi^G(T)_{\alpha}^{\alpha'}(\mu) = \sum_{\mu \in \mathcal{w}(\lambda)} \sum_{\mu' \in \mathcal{w}(\lambda)} C(T)_{\mu \mu'}^{\alpha} \left( C(T)_{\mu' \mu}^{\alpha'} \right)^* D_{\mu_1}^{\alpha_1} \cdots D_{\mu_N}^{\alpha_N}(a_1) \cdots D_{\mu_N}^{\alpha_N}(a_N).
$$

2. For $i = 1, 2$, let $T_i$ be a coupling tree, $\lambda_i \in \langle \lambda_i \rangle$ and $\alpha_i \in \mathcal{L}^T(\lambda_i, \lambda_i)$. If $\lambda_i = \sigma(\lambda_i)$ for some permutation $\sigma$ and $\lambda_2 = \lambda_1$, then

$$
R(T_2|T_1)_{\alpha_1}^{\alpha_2} = \sum_{\mu \in \mathcal{w}(\lambda_1)} \left( C(T_2)_{\sigma(\mu) \mu}^{\alpha_2} \right)^* C(T_1)_{\mu \mu}^{\alpha_1}
$$

for any $\mu \in \mathcal{w}(\lambda_1)$.

In both situations, the sum over $\mu$ (and $\mu'$) restricts automatically to contributions where $\Sigma(\mu) = \hat{\mu}$.

**Proof.** Point 1 follows by plugging (3.15) into (3.7). Point 2 follows by plugging (3.15) into the definition of $R$ and using that $\langle \lambda_2, \mu_2, \Pi \rangle = \delta_{\mu, \sigma(\mu)}$, where $\Pi$ denotes the unitary transformation permuting the tensor factors according to $\sigma$.

Let us introduce the following terminology. Given a coupling tree $T$, a leaf labelling $\lambda$ and $\mu \in \mathcal{w}(\lambda)$, for every vertex $x$ of $T$, let $\hat{\mu}^x$ be the sum of $\hat{\mu}^y$ over all leaves $y$ among the descendants of $x$. We refer to the assignment $x \mapsto \hat{\mu}^x$ as the weight labelling of $T$ generated by $\mu$. Given, in addition, $\alpha \in \mathcal{L}^T(\lambda)$ with underlying highest weight labelling $x \mapsto \lambda^x$, we say that $\mu$ and $\alpha$ are compatible if $\hat{\mu}^x \in \mathcal{w}(\lambda^x)$ for all nodes $x$ (for the leaves this holds true by construction).

**Lemma 3.14** (Product formula for composite Clebsch-Gordan coefficients). Let $T$ be a coupling tree, $\lambda$ a leaf labelling of $T$, $\lambda \in \langle \lambda \rangle$, $\alpha \in \mathcal{L}^T(\lambda, \lambda)$, $\mu \in \mathcal{w}(\lambda)$ and $\mu \in \mathcal{w}(\lambda)$. Let $x \mapsto \lambda^x$ be the highest weight labelling and $x \mapsto k^x$ the multiplicity counter labelling underlying $\alpha$. Let $x \mapsto \hat{\mu}^x$ be the weight labelling generated by $\mu$. Then, $C(T)_{\mu \mu}^{\alpha} = 0$ unless $\mu$ and $\alpha$ are compatible. In that case,

$$
C(T)_{\mu \mu}^{\alpha} = \sum_{\hat{\mu} \mu} \prod_{x \text{ node of } T} \left( C(\mu', \mu, \lambda^x, k^x, (\hat{\mu}^x, \hat{\mu}^y)) \right), \quad (3.16)
$$

where $x', x''$ denote the child vertices of $x$ and where the sum runs over all assignments $x \mapsto \hat{\mu}^x = 1, \ldots, m_{\lambda^x}(\hat{\mu}^x)$ of a weight multiplicity counter to every internal node $x$. 

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Proof. The proof is by induction on the number $N$ of leaves. For $N = 2$, the assertion holds by definition of the ordinary Clebsch-Gordan coefficients. Thus, assume that it holds for $N$ and that $T$ has $N + 1$ leaves. Let $r$ be the root of $T$. For clarity, we write $\lambda^r$ for $\lambda$. The child nodes $r'$ and $r''$ of $r$ are the roots of subtrees $S'$ and $S''$, respectively. Accordingly, $\lambda^r$ splits into leaf labellings $\lambda'$ of $S'$ and $\lambda''$ of $S''$, $\mu^r$ splits into $\mu^r \in \mathcal{U}(\lambda')$ and $\mu'' \in \mathcal{U}(\lambda'')$, and $\alpha$ splits into labellings $\alpha'$ of $S'$, $\alpha''$ of $S''$ and $\alpha^r = (\lambda^r, k^r)$ of the root. For convenience, let us introduce the notation $\tilde{\alpha}$ for the induced labelling of the cherry $\lor$ made up by the nodes $r$, $r'$ and $r''$. Thus, $\tilde{\alpha}$ consists of the leaf labels $\lambda^r$, $\lambda'$ and the root label $\alpha^r$.

Since $H_{\lambda^r} = H_{\lambda'} \otimes H_{\lambda''}$ under the bracketing associated with $T$, we may expand $|T; \alpha, \mu\rangle$ with respect to the orthonormal basis vectors $|S'; \beta', \nu\rangle \otimes |S''; \beta'', \nu''\rangle$. To compute the expansion coefficients, we observe that the diagram

\[
\begin{array}{cc}
H_{\lambda'} \otimes H_{\lambda''} & = \rightarrow \rightarrow \\
H_{\lambda'} \otimes H_{\lambda''} & \downarrow \downarrow \\
|T; \alpha, \mu\rangle & \rightarrow \rightarrow \\
|T; \alpha, \mu\rangle & \downarrow \downarrow \\
\end{array}
\]

commutes, by construction of the projections involved. Thus, for the corresponding injections we have

\[
\tilde{i}_{\alpha}^T = (\tilde{i}_{\alpha'}^T \otimes \tilde{i}_{\alpha''}^T) \circ \tilde{i}_T^\lambda.
\]

Hence,

\[
|T; \alpha, \mu\rangle = \sum_{\mu' \in \mathcal{U}(\lambda')} \sum_{\mu'' \in \mathcal{U}(\lambda'')} C^{\lambda'}_{\mu', \mu''} \langle \lambda^r | \mu' \rangle \langle \lambda^r | \mu'' \rangle.
\]

Using this, we find

\[
C(T)^{\alpha}_{\mu, \mu'} = \langle \lambda^r | T; \alpha, \mu\rangle = \sum_{\mu' \in \mathcal{U}(\lambda')} \sum_{\mu'' \in \mathcal{U}(\lambda'')} C^{\lambda'}_{\mu', \mu''} \langle \lambda^r | \mu' \rangle \langle \lambda'' | \mu'' \rangle.
\]

We have $\langle \lambda' | \mu' | S'; \alpha', \mu'\rangle = 0$ unless $\hat{\mu}' = \Sigma(\hat{\mu}')$ and $\langle \lambda' | \mu' | S''; \alpha'', \mu''\rangle = 0$ unless $\hat{\mu}'' = \Sigma(\hat{\mu}'')$. Moreover, $\Sigma(\hat{\mu}') = \hat{\mu}'$ and $\Sigma(\hat{\mu}'') = \hat{\mu}''$. It follows that $C(T)^{\alpha}_{\mu, \mu'}$ vanishes unless $\hat{\mu}' \in \hat{\mu}(\lambda')$ and $\hat{\mu}'' \in \hat{\mu}(\lambda'')$. In that case,

\[
C(T)^{\alpha}_{\mu, \mu'} = \sum_{\hat{\mu}' = 1}^{m_{\lambda'}(\hat{\mu}')} \sum_{\hat{\mu}'' = 1}^{m_{\lambda''}(\hat{\mu}'')} C_{\hat{\mu}', \hat{\mu}''}^{\lambda'} C_{\hat{\mu}', \hat{\mu}''}^{\lambda''} C(S')^{\alpha'}_{\mu', \hat{\mu}'}, C(S'')^{\alpha''}_{\mu'', \hat{\mu}''}.
\]
3.4 Reduction of recoupling coefficients

In this subsection, we are going to express the recoupling coefficients $R(T)$ appearing in the multiplication law for quasicharacters in terms of the elementary recoupling coefficients $R(\vee)$ for a cherry $\vee$. The latter correspond to the recoupling between $\vee \cdot \vee$ and $\vee^\vee$, see Figure 4. One may organize the parameters entering this coefficient into a matrix symbol as follows. For $i = 1, 2, 3$, let $\alpha_i$ be a labelling of $\vee$ assigning $\lambda^1_i$ and $\lambda^2_i$ to the leaves and $(\lambda_i, k_i)$ to the root. Given an assignment $k = (k^1, k^2)$ of positive integers to the leaves of $\vee$ and a positive integer $k$, one defines

$$
\begin{pmatrix}
\lambda^1_1 & \lambda^2_1 & \lambda_1 & k^1 \\
\lambda^1_2 & \lambda^2_2 & \lambda_2 & k^2 \\
\lambda^1_3 & \lambda^2_3 & \lambda_3 & k^3 \\
\end{pmatrix} := R(\vee)^{\alpha_1 \alpha_2, k}_{\alpha_3,k}.
$$

In case $G = SU(2)$, where the multiplicity labels may be omitted, these matrix symbols reduce to the ordinary Wigner $9j$ symbols, up to a dimension factor. Therefore, we will refer to them as $9\lambda$ symbols. By construction, the entries #1, \ldots, #4 of any full row satisfy $(#3, #4) \in \langle \langle #1, #2 \rangle \rangle$, because the $\alpha_i$ are admissible labellings of $\vee$. We refer to this statement as the full row property. We may extend the definition of the $9\lambda$ symbol to arbitrary weights in the upper left $(3 \times 3)$-block and arbitrary positive integers as the remaining entries by putting it to 0 whenever the full row property does not hold. Moreover, by definition of the recoupling coefficient on the right hand side, the $9\lambda$ symbol vanishes unless $(\alpha_3, k) \in \langle \langle \alpha_1, \alpha_2 \rangle \rangle$. Thus, it vanishes unless the entries #1, \ldots, #4 of any full column satisfy $(#3, #4) \in \langle \langle #1, #2 \rangle \rangle$. We refer to this statement as the full column property.
Remark 3.15. By point 2 of Proposition 3.13 and (3.16), one finds the following expression of the 9λ symbol in terms of ordinary Clebsch-Gordan coefficients,

\[
\begin{pmatrix}
\lambda_1^1 & \lambda_1^2 & \lambda_1^3 \\
\lambda_2^1 & \lambda_2^2 & \lambda_2^3 \\
\lambda_3^1 & \lambda_3^2 & \lambda_3^3 \\
k_1 & k_2 & k_3
\end{pmatrix}
= \sum_{\mu_1, \mu_2, \mu_3 = 1}^{\lambda} m_{\lambda_1}^1(\mu_1) m_{\lambda_2}^1(\mu_2) m_{\lambda_3}^1(\mu_3) 
\sum_{\mu_1, \mu_2 = 1}^{\lambda} (C_{\mu_1}^{\lambda_1} C_{\mu_2}^{\lambda_2} C_{\mu_3}^{\lambda_3}) 
\sum_{\mu_3 = 1}^{\lambda} \left( C_{\mu_1}^{\lambda_1} C_{\mu_2}^{\lambda_2} C_{\mu_3}^{\lambda_3} \right) 
\cdot \left( C_{\mu_1}^{\lambda_1} C_{\mu_2}^{\lambda_2} C_{\mu_3}^{\lambda_3} \right) 
\cdot \left( C_{\mu_1}^{\lambda_1} C_{\mu_2}^{\lambda_2} C_{\mu_3}^{\lambda_3} \right)
\]
Proof. I In the proof, we use the shorthand notations
\[ T_{ij} = T_i \cdot T_j , \quad T_{ijkl} = (T_i \cdot T_j) \cdot (T_k \cdot T_l) . \]
Let \( \sigma_{13} \) and \( \sigma_{24} \) be the permutations turning \( \Lambda_1 \) into \( \Lambda_3 \) and \( \Lambda_2 \) into \( \Lambda_4 \), respectively. They define a permutation \( \sigma \) turning \( \Lambda_1, \Lambda_2 \) into \( \Lambda_3, \Lambda_4 \). By point \( 2 \) of Proposition \( 3.13 \), for any \( \mu \in \mathfrak{w}(\lambda) \),
\[
R(T_{12}|T_{34})_{[\sigma_{13}, \alpha_2, (\lambda, k_{12})]} = \sum_{\sigma \in \mathfrak{w}(\lambda)} \left( C(T_{12})_{[\sigma_{13}, \alpha_2, (\lambda, k_{12})]} \right)^* C(T_{34})_{[\sigma_{24}, (\lambda, k_{34})]} . \tag{3.18}
\]

Clearly, \( \mu_1 \) and \( \alpha_i \) are compatible for \( i = 1, 2 \) iff \( \mu_1 \cdot \alpha_i \) and \( \alpha_1 \cdot \alpha_2, (\lambda, k_{12}) \) are compatible. If this condition is violated, then Proposition \( 3.14 \) yields that \( C(T_{12})_{[\alpha_1, \alpha_2, (\lambda, k_{12})]} = 0 \). Hence, for every nonzero contribution to the right hand side of \( 3.18 \), compatibility holds true. Then, Proposition \( 3.14 \) implies that \( C(T_{12})_{[\alpha_1, \alpha_2, (\lambda, k_{12})]} \) and \( C(T_i)_{[\alpha_i, (\Sigma(\mu_i), \mu_i)]} \) for any \( \mu_i = 1, \ldots, m_{\lambda_i}(\Sigma(\mu_i)) \), \( i = 1, 2 \), can be decomposed according to \( 3.10 \). It follows that
\[
C(T_{12})_{[\alpha_1, \alpha_2, (\lambda, k_{12})]} = \sum_{\mu_1} m_{\lambda_1}(\mu_1) m_{\lambda_2}(\mu_2) C(T_1)_{[\alpha_1, \mu_1]} C(T_2)_{[\alpha_2, \mu_2]} C(\lambda_1, \lambda_2, k_{12}) ,
\]
where we have denoted \( \mu_i := \Sigma(\mu_i) \) and \( \mu_i := (\mu_i, \mu_i) \). Since \( \Sigma(\mu_i) = (\sigma_{13}(\mu_i)) \) and \( \Sigma(\mu_i) = (\sigma_{24}(\mu_i)) \), an analogous argument yields that for every nonzero contribution to the right hand side of \( 3.18 \), we have \( \mu_1 \in \mathfrak{w}(\lambda_3) \) and \( \mu_2 \in \mathfrak{w}(\lambda_4) \), and
\[
C(T_{34})_{[\sigma_{24}, (\lambda, k_{34})]} = \sum_{\mu_3} m_{\lambda_3}(\mu_3) m_{\lambda_4}(\mu_4) C(T_3)_{[\sigma_{13}(\mu_3), \mu_3]} C(T_4)_{[\sigma_{24}(\mu_4), \mu_4]} C(\lambda_3, \lambda_4, k_{34}) ,
\]
where we have denoted \( \mu_3 := (\mu_1, \mu_3) \) and \( \mu_4 := (\mu_2, \mu_4) \). Grouping the factors and decomposing the sum accordingly, we obtain
\[
R(T_{12}|T_{34})_{[\alpha_1, \alpha_2, (\lambda, k_{12})]} = \sum_{\mu_1, \mu_2} m_{\lambda_1}(\mu_1) m_{\lambda_2}(\mu_2) \sum_{\mu_3, \mu_4} m_{\lambda_3}(\mu_3) m_{\lambda_4}(\mu_4) \left( \sum_{\mu \in \mathfrak{w}(\lambda)} \left( C(T_1)_{[\alpha_1, \mu_1]} \right)^* C(T_3)_{[\sigma_{13}(\mu_3), \mu_3]} \right) \cdot \left( \sum_{\mu \in \mathfrak{w}(\lambda_2)} \left( C(T_2)_{[\alpha_2, \mu_2]} \right)^* C(T_4)_{[\sigma_{24}(\mu_4), \mu_4]} \right) (\lambda_1)_{\lambda_2} \lambda_3 k_{12} (\lambda_3)_{\lambda_4} k_{34} \tag{3.19}
\]
Using \( \sigma_{13}(\mu_1) \) and \( \Pi_{13} \), where \( \Pi_{13} \) is the unitary transformation permuting the tensor according to \( \sigma_{13} \), and \( 3.8 \), we find
\[
\sum_{\mu_1} \left( C(T_1)_{[\alpha_1, \mu_1]} \right)^* C(T_3)_{[\sigma_{13}(\mu_1), \mu_3]} = \sum_{\mu \in \mathfrak{w}(\lambda_1)} \langle T_1; \alpha_1, \mu_1 | \Lambda_1, \mu_1 \rangle \langle \Lambda_1, \mu_1 | \Pi_{13} | T_3; \alpha_3, \mu_1 \rangle
\]
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= \langle T_1; \alpha_1, \mu_1 | \Pi_{13} | T_3; \alpha_3, \mu_3 \rangle \\
= \delta_{\lambda_1 \lambda_3} \delta_{\mu_1 \mu_3} R(T_1 | T_3)_{\alpha_1}^{\alpha_3}.

By analogy, the second expression in parentheses equals \( \delta_{\lambda_2 \lambda_4} \delta_{\mu_2 \mu_4} R(T_2 | T_4)_{\alpha_2}^{\alpha_4} \). Then, by unitarity of the isomorphisms \((3.2)\) chosen, the remaining sums yield

\[
\sum_{\mu_2 \in w(\lambda_1)} \sum_{\mu_2 \in w(\lambda_2)} \left( C_{\lambda_1 \lambda_2, \lambda} \right)^* C_{\lambda_1 \lambda_2, \lambda} = \delta_{k_{12} k_{34}}.
\]

2 We proceed by analogy. First, we use point 2 of Proposition 3.13 to rewrite the recoupling coefficient under consideration as

\[
\sum_{\mu_1 \in w(\lambda_1)} \cdots \sum_{\mu_4 \in w(\lambda_4)} \left( C(T_{1324})_{\mu_1 \mu_5, \mu_2 \mu_4} \right)^* C(T_{1324})_{\mu_1 \mu_5, \mu_2 \mu_4} = \delta_{\lambda_{12} \lambda_{34}},
\]

for some chosen \( \mu \in w(\lambda) \). For \( i = 1, 2, \ldots, 4 \), put \( \hat{\mu}_i := \Sigma(\hat{\mu}_i) \). Then, we define cherry labellings \( \alpha_{12}, \alpha_{34}, \alpha \) by, respectively,

\[
\lambda_1 \lambda_2 \quad \lambda_3 \lambda_4 \quad \lambda_1 \lambda_3 \lambda_4 \lambda_2 \lambda_4
\]

and the multiplicity pair \( k := (k_{13}, k_{24}) \), check compatibility and use (3.16) to factorize

\[
C(T_{1324})_{\mu_1 \mu_5, \mu_2 \mu_4} = \prod_{i=1}^{4} \left( C(T_i)_{\mu_i, \mu_i} \right)^* C(T_i)_{\mu_i, \mu_i},
\]

and

\[
C(T_{1324})_{\mu_1 \mu_5, \mu_2 \mu_4} = \prod_{i=1}^{4} \left( C(T_i)_{\mu_i, \mu_i} \right)^* C(T_i)_{\mu_i, \mu_i}.
\]

see Figure 5. Here, we have denoted \( \mu_i := (\hat{\mu}_i, \hat{\mu}_i) \), \( \mu_i' := (\hat{\mu}_i, \hat{\mu}_i) \) and \( \underline{\mu}_j := (\mu_j, \mu_j) \), \( \underline{\mu}_{ij} := (\mu_i', \mu_j') \), \( i, j = 1, \ldots, 4 \). By grouping the factors and decomposing the sum, we find that (3.20) equals

\[
\sum_{\hat{\mu}_1 \in w(\lambda_1)} \sum_{\hat{\mu}_1 \hat{\mu}_1'} \cdots \sum_{\hat{\mu}_4 \in w(\lambda_4)} \sum_{\hat{\mu}_4 \hat{\mu}_4'} \left( \prod_{i=1}^{4} \left( \sum_{\mu \in w(\lambda)} \left( C(T_i)_{\mu_i, \mu_i} \right)^* C(T_i)_{\mu_i, \mu_i} \right) \right)
\]
The sum is over all assignments of an integer \( x \) where \( x \equiv 3 \pmod{16}. \) The root labels a positive integer. Then, (3.21) boils down to

\[
\left( C(\mathcal{V} \cdot \mathcal{V})^{[\alpha_{12} \cdot \alpha_{34}, (\lambda, k)]} \right)_{\bar{\mu}_1, \bar{\mu}_3, \mu} \cdot \left( C(\mathcal{V} \cdot \mathcal{V})^{[\alpha_{34} \cdot \alpha, k]} \right)_{\bar{\mu}_3, \bar{\mu}_4, \mu}.
\]

(3.21) boils down to

\[
\left( \prod_{i=1}^{4} \delta_{\alpha_i, \alpha'_i} \right) \sum_{\mu_1 \in \Lambda(\lambda_1)} \ldots \sum_{\mu_4 \in \Lambda(\lambda_4)} \left( C(\mathcal{V} \cdot \mathcal{V})^{[\alpha_{12} \cdot \alpha_{34}, (\lambda, k)]} \right)_{\bar{\mu}_1, \bar{\mu}_3, \mu} \cdot \left( C(\mathcal{V} \cdot \mathcal{V})^{[\alpha_{34} \cdot \alpha, k]} \right)_{\bar{\mu}_3, \bar{\mu}_4, \mu}
\]

\[
= \left( \prod_{i=1}^{4} \delta_{\alpha_i, \alpha'_i} \right) R(\mathcal{V})^{[\alpha_{12} \cdot \alpha_{34}, k]},
\]

where we have used by point 2 of Proposition 3.13.

**Theorem 3.17.** Let \( T \) be a coupling tree. For \( i = 1, 2, 3, \) let \( \alpha_i \) be a labelling of \( T \) with underlying highest weight labelling \( x \mapsto \lambda^x_1 \) and underlying multiplicity counter labelling \( x \mapsto k^x_1 \). Let \( k \) be an assignment of a positive integer to every leaf of \( T \) and let \( k \) be a positive integer. Then,

\[
R(T)^{\alpha_{12}, k} = \sum_{x \text{ node of } T} \prod_{x' \text{ child} \ x} \begin{pmatrix} \lambda^x_1 & \lambda^{x'}_1 & \lambda^{x''}_1 & k^x_1 \\ \lambda^x_2 & \lambda^{x'}_2 & \lambda^{x''}_2 & k^x_2 \\ \lambda^x_3 & \lambda^{x'}_3 & \lambda^{x''}_3 & k^x_3 \\ k^{x'} & k^{x''} & k^x & k^x \end{pmatrix},
\]

where \( x' \) and \( x'' \) denote the child vertices of \( x \) and \( k^x = k \) in case \( x \) is the root of \( T \). The sum is over all assignments of an integer \( k^x = 1, \ldots, m(\lambda^x_1, \lambda^x_2)(\lambda^x_3) \) to every internal node \( x \) of \( T \).
We have to show that

Consequently, see Figure 6, and follows from the observation that, otherwise, we have either

The proof. Let \( \mathbf{r} \) denote the root of \( T \). By definition of \( R(T)_{\alpha_1 \alpha_2, k}^{\alpha_1 \alpha_2} \), we have to show that the right hand side of the asserted formula vanishes unless

and that, in that case, it coincides with \( R(T \cdot T|T^\triangledown)_{\alpha_1 \alpha_2, (\lambda^r, k)}^{\alpha_1 \alpha_2} \). The first statement follows from the observation that, otherwise, we have either \((\lambda^r, k) \notin \langle \lambda^y, k \rangle\) or there is a leaf \( y \) of \( T \) such that \( \langle \lambda^r, k \rangle \notin \langle \lambda^y, k \rangle \). In the first case, the factor contributed by \( \mathbf{r} \) to any of the summands on the right hand side is a 9\( \lambda \) symbol with a column having the entries \( \lambda^1, \lambda^2, \lambda^3, k \). Therefore, this factor vanishes by the full column property. In the second case, the factor contributed by the parent of \( y \) to any of the summands on the right hand side vanishes by the full column property as well.

Thus, in what follows, we may assume that (3.22) holds. The main step in the proof will consist in splitting \( R(T \cdot T|T^\triangledown)_{\alpha_1 \alpha_2, (\lambda^r, k)}^{\alpha_1 \alpha_2} \) into a product of analogous coefficients with \( T \) replaced by subtrees as follows. The child nodes \( \mathbf{r}' \) and \( \mathbf{r}'' \) of \( \mathbf{r} \) are the roots of subtrees \( S' \) and \( S'' \), respectively. By restriction, for \( i = 1, 2, 3 \), \( \alpha_i \) induces labellings \( \alpha_i' \) of \( S' \) and \( \alpha_i'' \) of \( S'' \), and \( \lambda_i \) splits into leaf labellings \( \lambda_i' \) of \( S' \) and \( \lambda_i'' \) of \( S'' \). Accordingly, \( k \) splits into \( k' \) and \( k'' \). We have

\[
T = S' \cdot S'', \quad \alpha_i = [\alpha_i', \alpha_i''] \quad (i = 1, 2, 3)
\]

Consequently,

\[
T \cdot T = (S' \cdot S'') \cdot (S' \cdot S'') = \left[\alpha_1 \cdot \alpha_2, (\lambda^r, k)\right] = \left[[\alpha_1', \alpha_1''] \cdot \alpha_2' \cdot \alpha_2'', \alpha_2', \alpha_2''\right], (\lambda^r, k) \right), (3.23)
\]

see Figure 6 and

\[
T^\triangledown = S'^\triangledown \cdot S''^\triangledown, \quad \left[\alpha_1 \ast \alpha_2, \alpha_3, k\right] = \left[[\alpha_1', \alpha_2', \alpha_3, k] \cdot \alpha_2'', \alpha_3, k''\right], (\alpha_1', \alpha_2', \alpha_3, k''\right), \alpha_3'\right]. (3.24)
\]
Consider the intermediate rooted binary tree
\[ \tilde{T} := (S' \cdot S') \cdot (S'' \cdot S'') \] (3.25)

with leaf labelling \( \tilde{\lambda} := (\tilde{\lambda}_1' \cdot \tilde{\lambda}_2') \cdot (\tilde{\lambda}_1'' \cdot \tilde{\lambda}_2'') \). By (3.11), we have the decomposition
\[
R(T \cdot T|T')_{[\alpha_1 \cdot \alpha_2, (\tilde{\lambda}_1, k)]} = \sum_{\tilde{\alpha} \in E(T)} R(T \cdot T|\tilde{T})_{[\alpha_1 \cdot \alpha_2, (\tilde{\lambda}_1, k)]} R(T|T')_{[\alpha_1 \cdot \alpha_2, (\tilde{\lambda}_1, k)]}. \tag{3.26}
\]

According to the decomposition (3.25), the labelling \( \tilde{\alpha} \) of \( \tilde{T} \) induces labellings \( \tilde{\alpha}_i' \) of the \( i \)-th copy of \( S' \) and \( \tilde{\alpha}_i'' \) of the \( i \)-th copy of \( S'' \). Denoting the roots of \( S' \cdot S' \) and \( S'' \cdot S'' \) by \( \tilde{r}' \) and \( \tilde{r}'' \), respectively, we can write
\[
\tilde{\alpha} = [[\tilde{\alpha}_1', \tilde{\alpha}_2', \tilde{\alpha}^{\tilde{r}'}], [\tilde{\alpha}_1'', \tilde{\alpha}_2'', \tilde{\alpha}^{\tilde{r}''}], (\tilde{\lambda}_3, \tilde{k}^{\tilde{r}})], \tag{3.27}
\]
where \( \tilde{k}^{\tilde{r}} \) is the multiplicity counter assigned by \( \tilde{\alpha} \) to \( r \). See Figure 6. In view of (3.27) and (3.24), we can apply point 2 of Lemma 3.16 by identifying \( T_1 \) with the first copy of \( S' \), \( T_2 \) with the first copy of \( S'' \), \( T_3 \) with the second copy of \( S' \), and \( T_4 \) with the second copy of \( S'' \) (as shown in Figures 5 and 6) to get
\[
R(T \cdot T|T')_{[\alpha_1 \cdot \alpha_2, (\tilde{\lambda}_1, k)]} = \left( \prod_{i=1,2} \delta_{\alpha_i', \tilde{\alpha}_i'} \delta_{\tilde{\alpha}_i', \tilde{\alpha}_i''} \right) \left( \begin{array}{c}
\lambda_1' \quad \lambda_1'' \\
\lambda_2' \quad \lambda_2'' \\
\lambda_3' \quad \lambda_3'' \\
\tilde{k}^{\tilde{r}} \\
k
\end{array} \right). \tag{3.28}
\]

Here, we have written out \( \tilde{\alpha}^{\tilde{r}'} \) as \( (\lambda_1', \lambda_2', \lambda_3', \tilde{k}^{\tilde{r}}) \) and \( \tilde{\alpha}^{\tilde{r}''} \) as \( (\lambda_1'', \lambda_2'', \lambda_3'', \tilde{k}^{\tilde{r}''}) \). In view of (3.27) and (3.24), we can apply point 1 of Lemma 3.16 to \( T_1 = S' \cdot S' \), \( T_2 = S'' \cdot S'' \), \( T_3 = S'' \cdot S' \) and \( T_4 = S'' \cdot S'' \) to get
\[
R(\tilde{T}|T')_{[\alpha_1 \cdot \alpha_2, (\tilde{\lambda}_1, k)]} = \delta_{\lambda_1', \tilde{\lambda}_1'} \delta_{\lambda_1'', \tilde{\lambda}_1''} \delta_{\tilde{k}^{\tilde{r}'}, \tilde{k}^{\tilde{r}''}} R(S' \cdot S'|S'')_{[\alpha_1', \alpha_2', \tilde{\alpha}^{\tilde{r}'}]} R(S'' \cdot S'|S'')_{[\alpha_1'', \alpha_2'', \tilde{\alpha}^{\tilde{r}''}]} \tag{3.29}
\]

Here, we have used that \( (\lambda_1')' = \lambda_1' \) and \( (\lambda_1'')' = \lambda_1'' \). Substituting (3.28) and (3.29) into (3.26), renaming \( \kappa^{\tilde{r}'} = \tilde{k}^{\tilde{r}} \), \( \kappa^{\tilde{r}''} = \tilde{k}^{\tilde{r}''} \) and taking the sum over \( \tilde{\alpha} \), we obtain
\[
R(T \cdot T|T')_{[\alpha_1 \cdot \alpha_2, (\tilde{\lambda}_1, k)]} = \sum_{\kappa^{\tilde{r}'} = 1}^{m_{\lambda_1', \lambda_1''}(\lambda_3', \tilde{k}^{\tilde{r}'})} \sum_{\kappa^{\tilde{r}''} = 1}^{m_{\lambda_1', \lambda_1''}(\lambda_3', \tilde{k}^{\tilde{r}'})} \left( \begin{array}{c}
\lambda_1' \quad \lambda_1'' \\
\lambda_2' \quad \lambda_2'' \\
\lambda_3' \quad \lambda_3'' \\
\kappa^{\tilde{r}'} \quad \kappa^{\tilde{r}''} \\
k
\end{array} \right) \cdot R(S' \cdot S'|S'')_{[\alpha_1', \alpha_2', (\lambda_3', \tilde{k}^{\tilde{r}'})]} R(S'' \cdot S'|S'')_{[\alpha_1'', \alpha_2'', (\lambda_3'', \tilde{k}^{\tilde{r}''})]} \tag{3.29}
\]

Now, we can iterate this formula by replacing \( T \) by \( S', S'' \), thereby replacing successively \( r \) by the root \( x \) of a subtree and \( k \) by \( \kappa^{x'} \), until we arrive at the situation where one of the child nodes of \( x, x' \) say, is a leaf of \( T \). In that case, the subtree \( S' \) with root \( x' \) is
the trivial coupling tree consisting of $x'$ alone, so that both $S^I \cdot S'$ and $S'^I$ are cherries, and we have $\alpha'_I = \lambda'_I$ and $k'' = k'$, so that $[\alpha'_I \cdot \alpha'_2, (\lambda'_3, \kappa''_3)]$ and $[\alpha'_I \cdot \alpha'_2, \alpha''_I, k'_2]$ differ only in their multiplicity counters $\kappa''_3$, $k''_2$. Thus, the recoupling coefficient is in fact a coupling coefficient, so that unitarity of the representation isomorphisms (3.2) implies

$$R(S^I \cdot S'|S''^I)_{[\alpha'_I \cdot \alpha'_2, (\lambda'_3, \kappa''_3)]} = \delta_{\kappa''_3, k''_2}.$$  

Summation over $\kappa''_3$ then yields that in the $9\lambda$ symbols, $\kappa''_3$ gets replaced by $k''_2$, with the remaining summation being over those $\kappa''_3$ where $x$ is an internal node. This completes the proof.

Example 3.18 (Structure constants for standard coupling). Consider the standard tree (caterpillar tree) of $N$ leaves and let labellings $\alpha_i$ with leaf labellings $\lambda_i$, $i = 1, 2, 3$, be given. As in Example 3.3 we number the nodes of $T$ by assigning, for $n = 2, \ldots, N$, the number $n$ to the parent of the $n$-th leaf, see Figure 2. Then, the nodes numbered $2, \ldots, N - 1$ are internal and node number $N$ is the root. Accordingly, the labels of the internal nodes are $\alpha_i = (\lambda_i^n, k_i^n)$ and the multiplicity counters one has to sum over are $k_i^n = 1, \ldots, m_i (\lambda_i^n, \lambda_i^n)$ for $n = 2, \ldots, N - 1$. In this notation, the formula given in Theorem 3.17 reads

$$R(T)^{\alpha_1 \alpha_2 \alpha_3}_{\lambda_1 \lambda_2 \lambda_3} = \sum_{k_i^n, \ldots, k_i^{n-1}} \left( \begin{array}{ccc} \lambda_1^n & \lambda_2^n & \lambda_3^n \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} \end{array} \right) \left( \begin{array}{c} k_1^n \\ k_2^n \\ k_3^n \end{array} \right) \prod_{n=2}^{N-1} \left( \begin{array}{ccc} \lambda_1^n & \lambda_2^n & \lambda_3^n \\ \lambda_1^{n+1} & \lambda_2^{n+1} & \lambda_3^{n+1} \end{array} \right) \left( \begin{array}{c} k_1^{n+1} \\ k_2^{n+1} \\ k_3^{n+1} \end{array} \right),$$

where $k_i^n = k$ and where the summation range is given above. The subtree $S'$ used in the proof of that theorem consists of the leaves $1, \ldots, N - 1$ and the nodes $2, \ldots, N - 1$ and has the shape of the standard tree on $N - 1$ leaves. The subtree $S''$, on the other hand, consists of the leaf $N$ only. The join $T \cdot T$ and the intermediate coupling tree $\tilde{T}$, together with the subtrees $S'$ and $S''$, are shown in Figure 7.

3.5 The special case $G = SU(2)$

In the case of $G = SU(2)$, the highest weights $\lambda$ of irreps correspond to spins $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The elements of the weight system of the highest weight corresponding to spin $j$ are nondegenerate, and correspond to spin projections $m = -j, -j + 1, \ldots, j$. Therefore, we write $j$ for $\lambda$, $j$ for $\mu$ for $m$ and $m$ for $\mu$, and we omit the weight multiplicity counters. Thus, $(\overline{H}_j, D^j)$ is the standard $SU(2)$-irrep of spin $j$, spanned by the orthonormal ladder basis $\{|jm\} : m = -j, -j + 1, \ldots, j\}$ which is unique up to a phase. Let us denote

$$d_j := \dim(H_j) = 2j + 1, \quad d_{\frac{j}{2}} := \dim(H_{\frac{j}{2}}) = 2(j_1 + \cdots + j_N) + N.$$

In contrast to the general case, in the decomposition of the tensor product of two irreps into irreps the multiplicities are equal to 1. Thus, we need only one type of bracket,
Figure 7: The tree join $T \cdot T$, the subtrees $S'$ and $S''$, and the intermediate coupling tree $\tilde{T}$ constructed in the proof of Theorem 3.17 for the standard coupling tree $T$. Here, $r$ is the root of $T$ and $r'$, $r''$ are its child nodes.

$\langle \cdot, \cdot \rangle$, given by

$$\langle j^1, j^2 \rangle = \{ |j^1 - j^2|, |j^1 - j^2| + 1, |j^1 - j^2| + 2, \ldots, j^1 + j^2 \},$$

for any two spins $j^1, j^2$, which is known as the triangle rule. The unitary representation isomorphism (3.1) reads

$$H_{j_1} \otimes H_{j_2} \rightarrow \bigoplus_{j \in \langle j_1, j_2 \rangle} H_{j}.$$

Accordingly, the bracketing schemes and their description in terms of coupling trees simplify: a labelling $\alpha$ of a coupling tree $T$ assigns to every vertex $x$ of $T$ a spin $j_x$. It is admissible if for every node $x$ one has $j_x \in \langle j_x', j_x'' \rangle$, where $x'$ and $x''$ denote the child nodes of $x$. The modified quasicharacters read

$$\hat{\chi}_{C}(T)_{\alpha}'(a) = \sum_{m=-j}^{j} \langle T; \alpha', m | D \hat{\xi}(a) | T; \alpha, m \rangle, \quad \alpha, \alpha' \in L^T(j, j). \quad (3.30)$$

Computation of their norms (2.8) yields

$$\left\| \hat{\chi}_{C}(T)_{\alpha}'(a) \right\|^2 = (\hbar\beta)^{N/2} \frac{d_j}{d_2} \frac{1}{2} e^{- \hbar \beta^2 (d_1^2 + \cdots + d_N^2)}, \quad (3.31)$$

where $\beta$ is a scaling factor for the invariant scalar product $\langle \cdot | \cdot \rangle$ on $g$ defined by

$$\langle X|Y \rangle = -\frac{1}{2\beta^2} \text{tr}(XY).$$

Formula (3.8) giving the recoupling coefficients reads

$$\langle T_2; \alpha_2, m_2 | P | T_1; \alpha_1, m_1 \rangle = \delta_{j_1 j_2} \delta_{m_1 m_2} R(T_2|T_1)^{\alpha_2}_{\alpha_1}. \quad (3.32)$$
The latter are the recoupling coefficients (Racah coefficients) of angular momentum theory \([3]\). Explicitly,

\[
R(T_2|T_1)_{\alpha_1}^{\alpha_2} = \langle T_2; \alpha_2, m|\Pi|T_1; \alpha_1, m \rangle, \tag{3.33}
\]

independently of \(m\). The recoupling coefficients are real. Hence, we have the symmetry relation

\[
R(T_2|T_1)_{\alpha_2}^{\alpha_1} = R(T_1|T_2)_{\alpha_2}^{\alpha_1}. \tag{3.34}
\]

The labelling of join and leaf duplication simplifies as follows. For \(i = 1, 2, 3\), let \(\tilde{j}_i\) be a leaf labelling of \(T_i\), let \(j_i \in \langle \tilde{j}_i \rangle\) and let \(\alpha_i \in L^T(\tilde{j}_i, j_i)\). For \(T \cdot T\), the label of the new root reduces to \(j \in \langle j_1, j_2 \rangle\). Hence, labellings of \(T \cdot T\) read \([\alpha_1, \alpha_2, j]\). For \(T^\vee\), node labellings compatible with the leaf labelling \(\tilde{j}_1 \odot \tilde{j}_2\) are given by labellings \(\alpha_3\) of \(T\) satisfying the condition that for all leaves \(y\) one has

\[
j_3^y \in \langle j_1^y, j_2^y \rangle,
\]

that is, \(j_1^y, j_2^y\) and \(j_3^y\) satisfy the triangle condition. The corresponding labellings of \(T^\vee\) read \([\alpha_1 \ast \alpha_2, \alpha_3]\). If, in addition, also \(j_1, j_2\) and \(j_3\) satisfy the triangle condition, we will say that \(\alpha_1, \alpha_2\) and \(\alpha_3\) satisfy the triangle condition. Then, the definitions of the recoupling coefficient \(R(T)\) and of its transpose read

\[
R(T)^{\alpha_1 \alpha_2}_{\alpha_3} := \begin{cases} R(T \cdot T^\vee)^{[\alpha_1 \ast \alpha_2, j_3]}_{[\alpha_1^{\ast \alpha_2}, \alpha_3]} & | \alpha_1, \alpha_2, \alpha_3\ \text{satisfy the triangle condition}, \\ 0 & | \text{otherwise} \end{cases}
\]

and

\[
R(T)^{\alpha_3}_{\alpha_1 \alpha_2} := \begin{cases} R(T^\vee \cdot T)^{[\alpha_1^{\ast \alpha_2}, \alpha_3]}_{[\alpha_1 \ast \alpha_2, j_3]} & | \alpha_1, \alpha_2, \alpha_3\ \text{satisfy the triangle condition}, \\ 0 & | \text{otherwise}, \end{cases}
\]

respectively. Since the recoupling coefficients can be chosen to be real, we have

\[
R(T)^{\alpha_1 \alpha_2}_{\alpha_3} = R(T)^{\alpha_3}_{\alpha_1 \alpha_2}.
\]

In this notation, the multiplication law for quasicharacters given by Corollary \([3.10]\) reads

\[
\hat{\chi}(T)^{[\alpha_1]}_{\alpha_1} \hat{\chi}(T)^{[\alpha_2]}_{\alpha_2} = \sum_{\alpha_3, \alpha_3'} R(T)^{[\alpha_1]}_{\alpha_1} R(T)^{[\alpha_2]}_{\alpha_2} \hat{\chi}(T)^{[\alpha_3']}_{\alpha_3} R(T)^{[\alpha_3]}_{\alpha_3'}, \tag{3.35}
\]

where the sum is over all combinable labellings \(\alpha_3, \alpha_3'\) of \(T\) such that \(\alpha_1, \alpha_2\) and \(\alpha_3\) satisfy the triangle condition.

Finally, we observe that the combinatorics based upon Clebsch-Gordan coefficients boils down to the combinatorics of angular momentum theory. According to Lemma \([3.11]\) for a given leaf labelling \(\tilde{j}\), \(j \in \langle \tilde{j} \rangle\) and \(\alpha \in L^T(\tilde{j}, j)\), the composite Clebsch-Gordan coefficient

\[
C(T)^{[\alpha]}_{m,m} = \langle j m|T; \alpha, m \rangle
\]

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vanishes unless $|m^x| \leq j^x$ for every node $x$, where $m^x$ denotes the sum of $m^y$ over all leaves $y$ descending from $x$. In that case,

$$C(T)^\alpha_{m^x;m^y} = \prod_{\text{nodes } x} C_{m^x;m^x'}, j^x$$

(3.36)

where $x'$, $x''$ denote the child vertices of $x$. Here, $C_{m_1,m_2,m_3}^{j_1,j_2,j_3}$ are the ordinary Clebsch-Gordan coefficients of angular momentum theory. It follows that the composite Clebsch-Gordan coefficients are real. The formula expressing the recoupling coefficients in terms of composite Clebsch-Gordan coefficients given in point 2 of Proposition 3.13 reads

$$R(T_2|T_1)^{\alpha_2}_{\alpha_1} = \sum_m C(T_2;\alpha_2,\sigma(m)) C(T_1;\alpha_1,m),$$

(3.37)

where $\sigma$ is the permutation turning $j_1$ into $j_2$ and the sum is over all spin projections $m$ of $j$, summing to $m$ for some fixed spin projection $m$ of $j$.

Finally, the decomposition of recoupling coefficients into a product of primitive 9 symbols given by Theorem 3.17 boils down to the decomposition of Racah-recoupling coefficients in terms of 9j symbols of angular momentum theory. In more detail, the 9 symbols become $(3 \times 3)$-arrays,

$$\begin{pmatrix}
  j_1^1 & j_1^2 & j_1^3 \\
  j_2^1 & j_2^2 & j_2^3 \\
  j_3^1 & j_3^2 & j_3^3
\end{pmatrix} := \begin{cases}
  R(V)^{\alpha_1\alpha_2}_{\alpha_3} & | j_i \in \langle j_1^i, j_2^i \rangle \text{ for all } i = 1, 2, 3, \\
  0 & | \text{ otherwise.}
\end{cases}$$

Here, $\alpha_i$ assigns $j_i^1$, $j_i^2$ to the leaves of $\vee$ and $j_i$ to its root. By definition of $R(V)^{\alpha_1\alpha_2}_{\alpha_3}$, these symbols vanish unless all rows satisfy the triangle condition. They are related to Wigner’s 9j symbols via a dimension factor.

By Theorem 3.17 for every coupling tree $T$ and all labellings $\alpha_1$, $\alpha_2$, $\alpha_3$, we have

$$R(T)^{\alpha_1\alpha_2}_{\alpha_3} = \prod_{x \text{ node of } T} \begin{pmatrix}
  j_1^x & j_1''^x & j_1^x' \\
  j_2^x & j_2''^x & j_2^x' \\
  j_3^x & j_3''^x & j_3^x'
\end{pmatrix},$$

(3.38)

where $x'$ and $x''$ denote the child vertices of $x$. For the standard coupling tree, this decomposition formula was already found in [9].

### 4 Application: quantum lattice gauge theory

In this section, we are going to use the quasicharacters studied above to characterize the strata of the reduced phase space of classical Hamiltonian lattice gauge theory in terms of relations, and to construct the costratification of the Hilbert space of quantum lattice gauge theory.
4.1 The classical picture

Let us briefly recall the classical picture of (finite) lattice gauge theory with compact gauge group $G$ in the Hamiltonian approach. For details, see [21–23]. Let $\Lambda$ be a finite regular three-dimensional spatial lattice and let $\Lambda^0$, $\Lambda^1$ and $\Lambda^2$ denote, respectively, the sets of lattice sites, lattice links and lattice plaquettes. For the links and plaquettes, let there be chosen an arbitrary orientation. Gauge potentials (the variables) are approximated by their parallel transporters along links and gauge transformations (the symmetries) are approximated by their values at the lattice sites. Thus, the classical configuration space is the space $G^{\Lambda^1}$ of mappings $\Lambda^1 \to G$, the classical symmetry group is the group $G^{\Lambda^0}$ of mappings $\Lambda^0 \to G$ with pointwise multiplication and the action of $g \in G^{\Lambda^0}$ on $a \in G^{\Lambda^1}$ is given by

\[
(g \cdot a)(\ell) := g(x)a(\ell)g(y)^{-1},
\]

where $(x, y) = \ell \in \Lambda^1$ with $x$ and $y$ the source and target of $\ell$, respectively. The classical phase space is given by the associated Hamiltonian $G$-manifold and the reduced classical phase space is obtained from that by symplectic reduction, as developed in [27, 31, 32]. Dynamics is ruled by the classical counterpart of the Kogut-Susskind lattice Hamiltonian [23].

In gauge theory, it is convenient to perform symplectic reduction by stages. First, one factorizes with respect to the free action of pointed gauge transformations. Here, given a lattice site $x_0$, it is easy to see that the normal subgroup

\[
\{g \in G^{\Lambda^0} : g(x_0) = 1\},
\]

where $1$ denotes the unit element of $G$, acts freely on $G^{\Lambda^1}$. Hence, by regular symplectic reduction, one may pass to the quotient manifold carrying the action of the residual group of local gauge transformations which is naturally isomorphic to $G$. By choosing a maximal tree $T$ in the graph $\Lambda^1$ and imposing the tree gauge, one finds that the subset

\[
\{a \in G^{\Lambda^1} : a(\ell) = 1 \text{ for all } \ell \in T\} \subset G^{\Lambda^1}
\]

intersects every orbit of the subgroup (4.2) exactly once. Hence, one can identify the quotient manifold with the direct product $G^N$ of $N$ copies of $G$. Then, the action of the residual gauge group turns into the action of $G$ by diagonal conjugation,

\[
g \cdot (a_1, \ldots, a_N) = (ga_1g^{-1}, \ldots, ga_Ng^{-1}).
\]

Details are given for example in [9]. Here, $N$ denotes the number of off-tree links of $T$. If $G$ is nonabelian, this action has non-trivial orbit types and, thus, in that case the second step in symplectic reduction is singular.

We denote $Q = G^N$ and recall that the $G$-action on $Q$ naturally lifts to a symplectic action on the (partially) reduced phase space $T^*Q$. The latter admits the momentum mapping

\[
\mu : T^*Q \to \mathfrak{g}^*, \quad \mu(p)(A) := p(A_\ast),
\]

where $\mathfrak{g}^*$ denotes the dual lie algebra of $\mathfrak{g}$.
where \( g \) is the Lie algebra of \( G \), \( A \in g \) and \( A_\ast \) denotes the Killing vector field defined by \( A \). An easy calculation shows that under the global trivialization

\[
T^*G^N \cong G^N \times g^N
\]

induced by left-invariant vector fields and an invariant scalar product on \( g \), the lifted action is given by diagonal conjugation,

\[
g \cdot (a_1, \ldots, a_N, A_1, \ldots, A_N) = (ga_1g^{-1}, \ldots, ga_Ng^{-1}, \text{Ad}(g)A_1, \ldots, \text{Ad}(g)A_N)
\]

and the associated momentum mapping is given by

\[
\mu(a_1, \ldots, a_N, A_1, \ldots, A_N) = \sum_{i=1}^N \text{Ad}(a_i)A_i - A_i,
\]

see e.g. [31, §10.7]. The fully reduced phase space \( P \) is obtained from \( T^*G^N \) by singular symplectic reduction at \( \mu = 0 \). The latter condition is the lattice approximation to the Gauß law constraint. Thus, as a topological space, \( P = \mu^{-1}(0)/G \). By the theory of singular symplectic reduction [27][32], it is endowed with the structure of a stratified symplectic space, where the strata are given by the connected components of the orbit type subsets. It can be easily shown that the action of \( G \) on \( \mu^{-1}(0) \) has the same orbit types as that on \( Q \).

### 4.2 Kähler quantization

The quantum theory of the above model is constructed via canonical quantization. We refer to [10][21][22] for the details, including the discussion of the field algebra and the observable algebra of the quantum system. It turns out that the quantum model so obtained may be, equivalently, derived by Kähler quantization and, in this context, the classical gauge orbit stratification may be implemented at the quantum level. As regular reduction commutes with quantization [22 Thm. 5.2], we may start with the partially reduced phase space \( T^*G^N \). The latter is endowed with a natural Kähler structure as follows. Let \( g_C \) denote the complexification of \( g \) and let \( G_C \) denote the complexification of \( G \). By restriction, the exponential mapping \( \exp : g_C \to G_C \) induces a diffeomorphism

\[
G \times g \to G_C, \quad (a, A) \mapsto a \exp(iA),
\]

which is equivariant with respect to the action of \( G \) on \( G \times g \) by

\[
g \cdot (a, A) := (ga^{-1}, \text{Ad}(g)A)
\]

and the action of \( G \) on \( G_C \) by conjugation. For \( G = SU(n) \), this diffeomorphism amounts to the inverse of the polar decomposition. By applying this diffeomorphism to each copy, we obtain a diffeomorphism

\[
G^N \times g^N \to G_C^N, \quad (a_1, \ldots, a_N, A_1, \ldots, A_N) \mapsto (a_1 \exp(iA_1), \ldots, a_N \exp(iA_N)).
\]
By composing the latter with the global trivialization (4.4), we obtain a diffeomorphism
\[ T^*G^N \rightarrow G^N_C \] (4.8)
which, due to (4.5), is equivariant with respect to the lifted action of \( G \) on \( T^*G^N \) and the action of \( G \) on \( G^N_C \) by diagonal conjugation. Via this diffeomorphism, the complex structure of \( G^N_C \) and the symplectic structure of \( T^*G^N \) combine to a Kähler structure.

Now, half-form Kähler quantization on \( G^N_C \) yields the Hilbert space
\[ HL^2(G^N_C, d\nu) \]
of holomorphic functions on \( G^N_C \) which are square-integrable with respect to the measure
\[ d\nu = e^{-\kappa/\hbar} \eta \varepsilon, \] (4.9)
where \( \kappa \) is the Kähler potential on \( G^N_C \), \( \eta \) is the half-form correction and \( \varepsilon \) is the Liouville measure on \( T^*G^N \). Details are given in [16], based on results of Hall [13].

Reduction of the residual \( G \)-symmetry on quantum level then yields the closed sub-
\[ \mathcal{H} = HL^2(G^N_C, d\nu)^G \]
space of \( G \)-invariants as the Hilbert space of the reduced system. Alternatively, the Hilbert space \( HL^2(G^N_C, d\nu) \) is obtained via the Segal-Bargmann transformation for compact Lie groups [12]. In more detail, the Segal-Bargmann transformation
\[ \Phi : L^2(G^N) \rightarrow HL^2(G^N_C, d\nu) \]
is a unitary isomorphism, which restricts to a unitary isomorphism of the subspaces of invariants. Thus, it is the mapping \( \Phi \) which constitutes the link between ordinary canonical quantization and Kähler quantization used here.

### 4.3 The orbit type costratification

Combining the concept of costratification as developed by Huebschmann [15] with a localization concept taken from the theory of coherent states, we define the subspaces associated with the orbit type strata of \( P \) to be the orthogonal complements of the subspaces of functions vanishing on those strata. To accomplish this goal, one first has to clarify how to interpret elements of \( \mathcal{H} \) as functions on \( P \). In the case \( N = 1 \) discussed in [16] and [14], this is obvious by observing that \( P \cong T_C/W \), where \( T \) is a maximal torus in \( G \) and \( W \) the corresponding Weyl group, and using the isomorphism
\[ HL^2(G_C, d\nu)^G \cong HL^2(T_C, d\nu_T)^W \] [16] §3.1. Here, the measure \( d\nu_T \) is obtained from \( d\nu \) by integration over the orbits of the action by conjugation in \( G_C \), thus yielding an analogue of Weyl’s integration formula for \( HL^2(G^N_C, d\nu) \). For the argument in the general case, we refer to [8, 9]. There, we have shown that the elements of \( \mathcal{H} \) may be viewed as continuous functions on the categorial quotient \( G^N_C/G_C \) in the sense of geometric invariant theory [26]. Next, via [14,8], the momentum mapping may be viewed
as a mapping $\mu : G_C^N \to \mathfrak{g}^*$ and, thus, $\mathcal{P}$ may be viewed as the quotient of $\mu^{-1}(0) \subset G_C^N$ by the action of $G$. In this language, $\mu^{-1}(0)$ turns out to be a Kempf-Ness set \[20\].

Using this fact, one can prove that the natural inclusion mapping $\mu^{-1}(0) \to G_C^N$ induces a homeomorphism

$$\mathcal{P} \to G_C^N // G_C.$$  \hspace{1cm} (4.10)

Via this homeomorphism, elements of $\mathcal{H}$ may be viewed as continuous functions on $\mathcal{P}$. By virtue of this interpretation, to a given orbit type stratum $\mathcal{P}_\tau \subset \mathcal{P}$, there corresponds the closed subspace

$$\mathcal{V}_\tau := \{ \psi \in \mathcal{H} : \psi|_{\mathcal{P}_\tau} = 0 \}.$$  \hspace{1cm} (4.11)

We define the subspace $\mathcal{H}_\tau$ associated with $\mathcal{P}_\tau$ to be the orthogonal complement of $\mathcal{V}_\tau$ in $\mathcal{H}$. Then, we have the orthogonal decomposition

$$\mathcal{H}_\tau \oplus \mathcal{V}_\tau = \mathcal{H}.$$  \hspace{1cm} (4.12)

Remark 4.1. Since holomorphic functions are continuous, one has

$$\mathcal{V}_\tau = \{ \psi \in \mathcal{H} : \psi|_{\mathcal{P}_\tau} = 0 \}.$$  \hspace{1cm} (4.13)

This implies the following. Firstly, since the principal stratum is dense in $\mathcal{P}$, the subspace associated with that stratum coincides with $\mathcal{H}$. Thus, in the discussion of the orbit type subspaces below, the principal stratum need not be treated separately. Secondly, since the strata satisfy the frontier condition, which means that $\mathcal{P}_\sigma \cap \mathcal{P}_\tau \neq \emptyset$ implies $\mathcal{P}_\sigma \subset \mathcal{P}_\tau$, it follows that if $\mathcal{P}_\sigma \cap \mathcal{P}_\tau \neq \emptyset$, then $\mathcal{V}_\tau \subset \mathcal{V}_\sigma$ and hence $\mathcal{H}_\sigma \subset \mathcal{H}_\tau$. Thus, the family of orthogonal projections

$$\mathcal{H}_\tau \to \mathcal{H}_\sigma$$

makes the family of closed subspaces $\mathcal{H}_\tau$ into a costratification in the sense of Huebschmann \[15\].

In order to analyse the condition $\psi|_{\mathcal{P}_\tau} = 0$, it is convenient to work with those subsets of $G_C^N$ which under the natural projection $G_C^N \to G_C^N // G_C$ and the homeomorphism \[4.10\] correspond to the orbit type strata of $\mathcal{P}$. For a given orbit type stratum $\mathcal{P}_\tau$, denote this subset by $(G_C^N)_\tau$. That is, $(G_C^N)_\tau$ consists of the elements $a$ of $G_C^N$ whose orbit closure equivalence class belongs to the image of $\mathcal{P}_\tau$ under the homeomorphism \[4.10\]. In other words, $a \in (G_C^N)_\tau$ iff it is orbit closure equivalent (o.c.e.) to some element of $\mu^{-1}(0)$ whose $G$-orbit belongs to $\mathcal{P}_\tau$. Thus, \[4.11\] reads

$$\mathcal{V}_\tau = \{ \psi \in \mathcal{H} : \psi|_{(G_C^N)_\tau} = 0 \}.$$  \hspace{1cm} (4.13)

Next, we explain how to construct $\mathcal{V}_\tau$ and $\mathcal{H}_\tau$ using defining relations for the orbit type strata $\mathcal{P}_\tau$. Recall from Section 2 the algebra $\mathcal{R}(G^N)$ of representative functions on $G^N$ and its subalgebra $\mathcal{R} := \mathcal{R}(G^N)^G$ of $G$-invariant elements, called quasicharacters. Since $G_C^N$ is the complexification of the compact Lie group $G^N$, Theorem 3 and the preceding proposition in Section 8.7.2 of \[29\] imply that $\mathcal{R}(G^N)$ coincides with the coordinate ring of $G_C^N$, viewed as a complex affine variety, and that $\mathcal{R}(G^N)$ coincides with the
algebra of representative functions on $G_N^\mathbb{C}$, where the relation is given by restriction and analytic continuation. In this sense, $\mathcal{R}$ may be identified with the algebra of $G$-invariant representative functions on $G_N^\mathbb{C}$.

Recall that an ideal $I \subset \mathcal{R}$ is called a radical ideal if for all $f \in \mathcal{R}$ satisfying $f^n \in I$ for some $n$ one has $f \in I$. Moreover, given a subset $R \subset \mathcal{R}$, one defines the zero locus of $R$ by

$$\{g \in G_N^\mathbb{C} : f(g) = 0 \text{ for all } f \in R\} \subset G_N^\mathbb{C}.$$ It coincides with the zero locus of the ideal in $\mathcal{R}$ generated by $R$.

**Proposition 4.2.** Let $\mathcal{P}_\tau$ be an orbit type stratum and let $R_\tau$ be a subset of $\mathcal{R}$ satisfying

1. The zero locus of $R_\tau$ coincides with the (topological) closure of $(G_N^\mathbb{C})_\tau$,
2. The ideal generated by $R_\tau$ in $\mathcal{R}$ is a radical ideal.

Then, $\mathcal{V}_\tau$ is obtained by intersecting $\mathcal{H}$ with the ideal generated by $R_\tau$ in the algebra $\text{Hol}(G_N^\mathbb{C})^G$ of $G$-invariant holomorphic functions on $G_N^\mathbb{C}$.

**Proof.** See [8]. □

By Hilbert’s Basissatz, finite subsets $R_\tau \subset \mathcal{R}$ satisfying conditions 1 and 2 of Proposition 4.2 exist. Given $R_\tau$, Proposition 4.2 implies the following explicit characterization of the subspaces $\mathcal{V}_\tau$ and $\mathcal{H}_\tau$ in terms of multiplication operators. For $f \in \mathcal{R}$, let $\hat{f} : \mathcal{H} \to \mathcal{H}$ denote the operator of multiplication by $f$.

**Corollary 4.3.** Let $\mathcal{P}_\tau$ be an orbit type stratum and let $R_\tau = \{p_1, \ldots, p_c\}$ be a finite subset of $\mathcal{R}$ satisfying conditions 1 and 2 of Proposition 4.2. Then,

$$\mathcal{V}_\tau = \text{im}(\hat{p}_1) + \cdots + \text{im}(\hat{p}_c), \quad \mathcal{H}_\tau = \ker(\hat{p}_1^\dagger) \cap \cdots \cap \ker(\hat{p}_c^\dagger).$$ □

In what follows, we will refer to conditions 1 and 2 of Proposition 4.2 as the zero locus condition and the radical ideal condition, respectively.

## 4.4 The costratification for the gauge group SU(2)

This case has been fully elaborated in [8], [9]. Here, we briefly recall the results.

For $G = \text{SU}(2)$, we have $G_\mathbb{C} = \text{SL}(2, \mathbb{C})$. Denote by $Z$ the center of $G$, by $\mathbb{T} = \text{U}(1)$ and $\mathbb{T}_\mathbb{C}$ the maximal toral subgroups of $G$ and $G_\mathbb{C}$, respectively, and by $\mathfrak{t}$ and $\mathfrak{t}_\mathbb{C}$ their Lie algebras. The orbit type strata of the momentum level set $\mu^{-1}(0) \subset G_N^\mathbb{C} \times \mathfrak{g}^\mathbb{C}$ with respect to the $G$-action are as follows. The case $N = 1$ is very simple, see [10]. So, let us assume $N \geq 2$.

1. **(G)** There exist $2^N$ strata of orbit type $G$, each of which consists of a single point representing the (trivial) orbit of an element of $Z^N \times \{0\}^N$. Since such an element is of the form $(\gamma_1 \mathbb{1}, \ldots, \gamma_N \mathbb{1}, 0, \ldots, 0)$ for some sequence of signs $\underline{\gamma} = (\gamma_1, \ldots, \gamma_N)$, we denote the corresponding stratum by $\mathcal{P}_{\underline{\gamma}}$. 

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(T) There is one stratum of orbit type represented by $\mathbb{T}$. It will be denoted by $\mathcal{P}_\mathbb{T}$.

Every element of this stratum is conjugate to an element of the subset

$$(\mathbb{T}^N \times \mathfrak{t}^N) \setminus (Z^N \times \{0\}^N) \subset \mu^{-1}(0).$$

(Z) The principal stratum is of orbit type $Z$.

As a consequence, the costratification for $G = SU(2)$ consists of the Hilbert subspaces

$$\mathcal{H}_\gamma, \quad \mathcal{H}_\mathbb{T}, \quad \mathcal{H}_Z = \mathcal{H},$$

together with their orthogonal projectors. To construct them, it is helpful to pass from the strata $\mathcal{P}_\tau$ to the corresponding subsets $(G^N_N)_{\tau} \subset G^N_C$. It suffices to do this for every sequence of signs $\gamma = (\gamma_1, \ldots, \gamma_N)$ and for $\mathbb{T}$. The result is as follows [8].

**Theorem 4.4.** Let $\mathfrak{a} \in G^N_C$. Then,

1. $\mathfrak{a} \in (G^N_C)_{\gamma}$ iff $\mathfrak{a}$ is o.c.e. to $(\gamma_1 \mathbb{1}, \ldots, \gamma_N \mathbb{1})$,

2. $\mathfrak{a} \in (G^N_C)_{\mathbb{T}}$ iff $\mathfrak{a}$ is o.c.e. to an element of $\mathbb{T}^N \setminus Z^N$.

To proceed, one may look for finite subsets $R_\tau$ of the algebra $\mathcal{R}$ having the orbit type subsets $(G^N_N)_{\tau}$ as their zero loci and satisfying the radical ideal condition. Then, general arguments [8] ensure that

$$\mathcal{V}_\tau = \sum_{\mathfrak{p} \in R_\tau} \text{im}(\hat{\mathfrak{p}}).$$

(4.13)

In the case at hand, we define the following $G$-invariant representative functions:

$$p^T_{r,s}(\mathfrak{a}) := \text{tr} \left( [a_r, a_s]^2 \right), \quad 1 \leq r < s \leq N,$$

$$p^T_{r,s,t}(\mathfrak{a}) := \text{tr} \left( [a_r, a_s]a_t \right), \quad 1 \leq r < s < t \leq N.$$

The following was shown in [8, 9].

**Theorem 4.5.** The closure $(G^N_C)_{\mathbb{T}}$ is the set of common zeros of the $G$-invariant representative functions

$$p^T_{r,s}, \quad 1 \leq r < s \leq N, \quad p^T_{r,s,t}, \quad 1 \leq r < s < t \leq N.$$ 

(4.14)

The ideal in $\mathcal{R}$ generated by these functions is a radical ideal.

**Theorem 4.6.** The subset $(G^N_C)_{\gamma} \subset G^N_C$ is the set of common zeros of the $G_C$-invariant functions $p^T_{r,s}$ with $1 \leq r < s \leq N$, $p^T_{r,s,t}$ with $1 \leq r < s < t \leq N$ and

$$\tilde{p}^\gamma_{r}(\mathfrak{a}) := \text{tr}(a_r) - 2\gamma_r, \quad 1 \leq r \leq N, \quad \tilde{p}^\gamma_{r,s}(\mathfrak{a}) := \text{tr}(a_r a_s) - 2\gamma_r \gamma_s, \quad 1 \leq r < s \leq N.$$

The ideal in $\mathcal{R}$ generated by these functions is a radical ideal.
For the vanishing subspaces \( \mathcal{V}_2 \), instead of (4.13), there is a much simpler characterization. Let \( 1_\gamma := (\gamma_1 \mathbb{1}, \ldots, \gamma_N \mathbb{1}) \) and let 1 denote the constant function with value 1 in \( \mathcal{G}_C^N \). Then,
\[
\mathcal{V}_2 = \{ \psi - \psi(1_\gamma) \mathbb{1} : \psi \in \mathcal{H} \}.
\]
Indeed, the right hand side is certainly contained in the left hand side. If \( \psi \in \mathcal{V}_2 \), then \( \psi(1_\gamma) = 0 \) and hence \( \psi = \psi(1_\gamma) \mathbb{1} \), so that \( \psi \) is contained in the right hand side. Thus, (4.15) holds true, indeed. It follows that \( \mathcal{V}_2 \) has codimension 1, because the constant function 1 obviously spans a complement of the right hand side. Hence, \( \mathcal{H}_2 \) has dimension 1 and is thus spanned by a normalized vector \( \psi_\gamma \) that is unique up to a phase. Given an orthonormal basis \( \{ \psi_i : i \in I \} \) of \( \mathcal{H} \) which contains a constant function \( \psi_0 \) (eg. the basis built from quasicharacters), we find that
\[
\psi_\gamma = \sum_{i \in I} \frac{\psi_i(1_\gamma) \psi_i}{\| \psi_i(1_\gamma) \|_2^2}.
\]
Indeed, \( \psi_\gamma \) is orthogonal to \( \mathcal{V}_2 \) for any \( i' \in I \),
\[
\langle \sum_{i \in I} \psi_i(1_\gamma) \psi_i' \rho - \psi_i'(1_\gamma) | 1 \rangle = \psi_i'(1_\gamma) - \psi_i(1_\gamma) \rangle \sum_{i \in I} \psi_i(1_\gamma) \langle \psi_i | 1 \rangle = 0,
\]
because 1 = \( \| 1 \| \psi_0 \) and hence
\[
\langle \psi_i | 1 \rangle = \frac{\delta_{i,0}}{\psi_0(1_\gamma)}.
\]
Let us calculate (4.16) for the normalized quaicharacters as basis elements. Let \( j \in \hat{G}^N \), \( j \in \langle j \rangle \) and \( \alpha, \alpha' \in \mathcal{L}^T (j, j) \). Since
\[
D_j^n (\gamma_\alpha \mathbb{1}) = \gamma_\alpha^{2j^n} D_j (\mathbb{1}), \quad n = 1, \ldots, N, \quad \hat{\chi}_C^T (\alpha') (\mathbb{1}, \ldots, \mathbb{1}) = d_j,
\]
we find
\[
\hat{\chi}_C^T (\alpha'^\prime) (1_\gamma) = \gamma_1^{2j_1^1} \cdots \gamma_N^{2j_N^N} d_j.
\]
By (3.31), then
\[
\left\| \hat{\chi}_C^T (\alpha') \right\|_2 \left( \frac{1_\gamma}{\left\| \hat{\chi}_C^T (\alpha') \right\|_2} \right)^\alpha (1_\gamma) = (h\pi)^{-3N/2} e^{-\hbar^2(d_j^1 + \cdots + d_j^N)} \gamma_1^{2j_1^1} \cdots \gamma_N^{2j_N^N} d_j \hat{\chi}_C^T (\alpha')
\]
and
\[
\left\| \hat{\chi}_C^T (\alpha') \right\|_2^2 (1_\gamma) = (h\pi)^{-3N/2} e^{-\hbar^2(d_j^1 + \cdots + d_j^N)} d_j d_j.
\]
Since \( \sum_{j \in \langle j \rangle} \sum_{\alpha, \alpha' \in \mathcal{L}^T (j, j)} d_j = d_j \), the latter implies that for the denominator in (4.16) we obtain
\[
(h\pi)^{-3N/2} \sum_{\langle j \rangle \in \hat{G}^N} e^{-\hbar^2(d_j^1 + \cdots + d_j^N)} d_j^2 = (h\pi)^{-3N/2} \Delta^N,
\]
where
\[
\Delta = \sum_{j \in \hat{G}} e^{-\hbar \beta^2 d_j^2} d_j^2 = \sum_{k=1}^{\infty} e^{-\hbar \beta^2 k^2} k^2 = \frac{1}{2} e^{-\hbar \beta^2} \theta_3'(e^{-\hbar \beta^2})
\]
with the theta-constant
\[
\theta_3(z) = \sum_{k=-\infty}^{\infty} e^{-\hbar \beta^2 k^2}.
\]
Thus, we have proved the following.

**Proposition 4.7.** For every \( \gamma \), the subspace \( \mathcal{H}_\gamma \) is spanned by
\[
\psi_\gamma = \frac{2e^{N\hbar \beta^2}}{\theta_3'(e^{-\hbar \beta^2})^N} \sum_{j \in \hat{G}^N} \gamma_1^{2j_1} \cdots \gamma_N^{2j_N} d_j^2 e^{-\hbar \beta^2 (d_{j_1}^2 + \cdots + d_{j_N}^2)} \sum_{j \in \langle \hat{j} \rangle} \sum_{\alpha, \alpha' \in \mathcal{L}(T)} \chi_C(T)^{\alpha'} \alpha.
\]
□

For \( N = 1 \), this reproduces the corresponding result of \[16\]. It remains to construct the subspace \( \mathcal{H}_T \) associated with the stratum \( T \). As already outlined in the introduction, this can be done as follows.

1. Expand the functions \( p_{r_1}^T \) and \( p_{r_2}^T \) with respect to the basis of quasicharacters \[9, Lemma 4.7\].

2. Use the multiplication law for the quasicharacters and the fact that they constitute a basis in \( \mathcal{H} \) to construct a basis in the vanishing subspace \( \mathcal{V}_T \). The elements of this basis turn out to be linear combinations of quasicharacters with coefficients built from products of the structure constants of the quasicharacter basis with the expansion coefficients of the functions \( p_{r_1}^T \) and \( p_{r_2}^T \) \[9, Thm. 4.8\].

3. Determine the subspace \( \mathcal{H}_T \) by taking the orthogonal complement of \( \mathcal{V}_T \) in \( \mathcal{H} \). This yields \( \mathcal{H}_T \) in terms of a family of vectors whose expansion coefficients with respect to the quasicharacter basis are determined by a system of linear equations with coefficients given by the above mentioned products of the structure constants with the expansion coefficients of the functions \( p_{r_1}^T \) and \( p_{r_2}^T \) \[9, Cor. 4.7\].

This analysis will be done elsewhere. By the decomposition formula (3.38), the coefficients entering the linear equations to be analyzed under point 3 may be reduced to the combinatorics of \( 9j \) symbols. For the latter there exist symbolic calculators \[25\]. Thus, the construction of the costratification is reduced to a problem in linear algebra.

### 4.5 Towards \( G = SU(3) \)

#### 4.5.1 The orbit type stratification

We start with deriving the orbit types. For that purpose, we define the following subgroups of \( G = SU(3) \):
\[
Z := \text{center}, \quad T := \{ \text{diag} \left( z_1, z_2, z_1^{-1} z_2^{-1} \right) : z_1, z_2 \in U(1) \}.
\]
\[ U^1 := \{ \text{diag} (z, z, z^{-2}) : z \in U(1) \} , \quad U^2 := \left\{ \begin{bmatrix} \det(a)^{-1} & 0 & 0 \\ 0 & 0 & a \end{bmatrix} : a \in U(2) \right\} . \]

Denote the corresponding Lie subalgebras of \( g = u(3) \) by
\[ \mathfrak{z} := \{0\} , \quad \mathfrak{t} := \{ \text{diag} (ix_1, ix_2, -i(x_1 + x_2)) : x_1, x_2 \in \mathbb{R} \} , \]
\[ \mathfrak{u}^1 := \{ \text{diag} (ix, ix, -2ix) : x \in \mathbb{R} \} , \quad \mathfrak{u}^2 := \left\{ \begin{bmatrix} -\text{tr}(A) & 0 & 0 \\ 0 & 0 & A \end{bmatrix} : A \in \mathfrak{u}(2) \right\} . \]

There correspond subgroups \( Z_C, U^1_C, T_C \) and \( U^2_C \) of \( G_C = \text{SL}(3, \mathbb{C}) \) and Lie subalgebras \( \mathfrak{z}_C, \mathfrak{u}^1_C, \mathfrak{t}_C \) and \( \mathfrak{u}^2_C \) of \( \mathfrak{g}_C = \mathfrak{sl}(3, \mathbb{C}) \), defined accordingly with \( z, z_1, z_2 \in \mathbb{C}^* \), \( a \in \text{SL}(2, \mathbb{C}) \), \( x \in \mathbb{C} \) and \( A \in \mathfrak{gl}(2, \mathbb{C}) \).

The following lemma lists the orbit types \( \tau \) together with the closed orbit type subsets \( (G^N_C)_\tau \subset G^N_C \).

**Lemma 4.8.**

Case \( N = 1 \): There exists one principal orbit type, represented by \( T \), and two secondary orbit types, represented by \( U^2 \) and \( G \). They are characterized as follows. For \( a \in G_C \),

\[ (U^2) \ a \in (G^N_C)_{U^2} \text{ iff it is o.c.e. to some element of } U^1_C , \]
\[ (G) \ a \in (G^N_C)_G \text{ iff it is o.c.e. to some element of } Z_C . \]

Case \( N > 1 \): There exists one principal orbit type, \( Z \), and four secondary orbit types, represented by \( U^1, T, U^2 \) and \( G \). They are characterized as follows. For \( a \in G^N_C \),

\[ (U^1) \ a \in (G^N_C)_{U^1} \text{ iff it is o.c.e. to some element of } (U^1_C)^N , \]
\[ (T) \ a \in (G^N_C)_T \text{ iff it is o.c.e. to some element of } T^N_C , \]
\[ (U^2) \ a \in (G^N_C)_{U^2} \text{ iff it is o.c.e. to some element of } (U^1_C)^N , \]
\[ (G) \ a \in (G^N_C)_G \text{ iff it is o.c.e. to some element of } Z^N_C . \]

One has \( (G^N_C)_G = (G^N_C)_G \).

**Proof.** First, we determine the orbit types of the action of \( G \) on \( G^N \times \mathfrak{g}^N \). This has been done in [4] via a common eigenspace analysis. The result is as follows. Let \( (u, X) \in G^N \times \mathfrak{g}^N \). By a common eigenspace of a set of operators we mean a maximal subspace on which all these operators act via multiplication by a complex number.

Z If \( u_1, \ldots, u_N \) and \( X_1, \ldots, X_N \) do not admit a common eigenvector, the stabilizer of \( (u, X) \) is \( Z \).
$U^1$ If $u_1, \ldots, u_N$ and $X_1, \ldots, X_N$ have exactly one common eigenspace and if this eigenspace has dimension 1, the stabilizer of $(u, X)$ is conjugate to $U^1$.

$T$ If $u_1, \ldots, u_N$ and $X_1, \ldots, X_N$ have three common eigenspaces, the stabilizer of $(u, X)$ is conjugate to $T$.

$U^2$ If $u_1, \ldots, u_N$ and $X_1, \ldots, X_N$ have a common eigenspace of dimension 2, the stabilizer of $(u, X)$ is conjugate to $U^2$.

$G$ If $u_1, \ldots, u_N$ and $X_1, \ldots, X_N$ have a common eigenspace of dimension 3, and thus are all proportional to $I$, the stabilizer of $(u, X)$ is $G$.

Accordingly, the orbit types of the action of $G$ on $G^N \times g^N$ are $Z, U^1, T, U^2$ and $G$, and they may be characterized as follows.

1. $(u, X)$ has orbit type $G$ iff it belongs to $Z^N \times 1^N$.
2. $(u, X)$ has orbit type $U^2$ iff it is conjugate to $(U^1)^N \times (u^1)^N$ but does not belong to $Z^N \times 3^N$.
3. $(u, X)$ has orbit type $T$ iff it is conjugate to an element of $T^N \times t^N$ but not to an element of $(U^1)^N \times (u^1)^N$.
4. $(u, X)$ has orbit type $U^1$ iff it is conjugate to an element of $(U^2)^N \times (u^2)^N$ but not to an element of $T^N \times t^N$.
5. $(u, X)$ has orbit type $Z$ iff it is not conjugate to an element of $(U^2)^N \times (u^2)^N$.

In particular, the orbit types are totally ordered and $Z$ is the principal orbit type on all of $G^N \times g^N$. Now, we intersect with $\mu^{-1}(0)$. For $N = 1$, the condition $\mu(u, X) = 0$ requires $u$ and $X$ to commute. This is equivalent to having a common eigenbasis and hence to being conjugate to an element of $T \times t$. Therefore, all elements of orbit types $T, U^2$ or $G$ belong to $\mu^{-1}(0)$ and they exhaust this subset. In particular, in the case $N = 1$, $T$ is the principal orbit type. For $N > 1$, it is still true that the orbit type subsets of types $T, U^2$ and $G$ are contained in $\mu^{-1}(0)$. In addition, $\mu^{-1}(0)$ intersects all orbit type subsets: it suffices to find $u_1, u_2, X_1, X_2$ such that

$$\operatorname{Ad}_{u_1} X_1 - X_1 + \operatorname{Ad}_{u_2} X_2 - X_2 = 0$$

and

$$Z_G(u_1) \cap Z_G(u_2) \cap Z_G(X_1) \cap Z_G(X_2) = Z.$$ 

By putting $X_1 = X_2 = 0$, the first condition is satisfied automatically and the second one reads $Z_G(u_1) \cap Z_G(u_2) = Z$. The latter holds true e.g. whenever $u_1$ is diagonal with pairwise distinct entries and $u_2$ is not diagonal.

Now, let $a \in G_C^N$ be given. It is o.c.e. to some $b$ whose polar decomposition $(u_a, X)$ belongs to $\mu^{-1}(0)$. It has been shown in §3 of in [8] that $a$ belongs to $(G_C^N)_+ \iff (u_a, X)$
is invariant under a subgroup of \( G \) representing \( \tau \). Thus, for example, \( \mathbf{a} \in (G_N^N)_{U^1} \) iff \((u, X)\) is invariant under a subgroup of \( G \) conjugate to \( U^1 \). By the above analysis, this is equivalent to \((u, X)\) being conjugate to an element of \((U^2)^N \times (u^2)^N\). Since the polar mapping turns \( U^2 \times u^2 \) into \( U^2_\mathbb{C} \), the latter is equivalent to \( \mathbf{b} \) being conjugate to, and hence to \( \mathbf{a} \) being o.c.e. to, an element of \((U^2_\mathbb{C})^N\). The argument for the other types is analogous.

To find the strata, we have to pass to the connected components of the orbit type subsets. Dimension counting yields

**Lemma 4.9.** The subset \((G_N^N)_\tau\) is connected for \( \tau = Z, U^1, \mathbb{T}, U^2 \). The subset \((G_N^N)_G\) consists of isolated points, each corresponding to the orbit of an element of \( Z^N \).

Denote the cyclic subgroup of \( U(1) \) of order 3 (made up by the third roots of unity) by \( \mathbb{Z}_3 \). Then,

\[
Z^N = \{(\gamma_1, \ldots, \gamma_N) : \gamma_1, \ldots, \gamma_N \in \mathbb{Z}_3^U\}.
\]

Consequently, the subset \((G_N^N)_\tau\) corresponds to a stratum of the reduced phase space for \( \tau = Z, U^1, \mathbb{T}, U^2 \), whereas \((G_N^N)_G\) splits into the connected components \((G_N^N)_{G, \gamma}\), where \( \gamma = (\gamma_1, \ldots, \gamma_N) \in (\mathbb{Z}_3^U)^N \), consisting of the elements \( \mathbf{a} \) of \( G_N^N \) which are o.c.e. to \((\gamma_1, \ldots, \gamma_N)\). Each \((G_N^N)_{G, \gamma}\) corresponds to an isolated point stratum of the reduced phase space.

Lemmas 4.8 and 4.9 together with the obvious inclusion relations

\[
(G_N^N)_{G, \tau} \subset (G_N^N)_{U^2} \subset (G_N^N)_\mathbb{T} \subset (G_N^N)_{U^1} \subset (G_N^N)_Z = G_N^N,
\]

imply

**Theorem 4.10.** The stratification of \( G_N^N \) by orbit type connected components is given for \( N = 1 \) by the Hasse diagram

![Hasse diagram for N=1]

and for \( N > 1 \) by the Hasse diagram

![Hasse diagram for N>1]

(with inclusion of closures from left to right).
4.5.2 The point strata

Let \( \gamma \in (\mathbb{Z}^3)^N \) and consider the stratum \( (G^N)_{G, \gamma} \), consisting of the single point

\[ 1_\gamma := (\gamma_1 1, \ldots, \gamma_N 1) \].

As done already for the case \( G = SU(2) \), the corresponding subspace \( H_{G, \gamma} \) can be constructed directly. The vanishing subspace \( V_{G, \gamma} \) is given by (4.15), the subspace \( H_{G, \gamma} \) has dimension one and it is spanned by the single vector (4.16). We evaluate this formula for the orthonormal basis of normalized quasicharacters. For that purpose, we need some representation theoretic basics about \( SL(3, \mathbb{C}) \). The highest weights correspond to pairs \((n, m)\) of nonnegative integers, each representing twice the spin of an appropriate \( SU(2) \) subgroup. For example, \((1, 0)\) corresponds to the identical representation, \((0, 1)\) to the contragredient of the identical representation (the representation induced on the dual space) and \((1, 1)\) to the adjoint representation. We write \( \lambda_{n,m} \) for the highest weight corresponding to \((n, m)\), \( D_{n,m} \) for the corresponding irrep and \( d_{n,m} \) for its dimension.

Recall that

\[ d_{n,m} = \frac{1}{2} (n + 1)(m + 1)(n + m + 2) \].

Given sequences \( n, m \) of length \( N \) of nonnegative integers we write

\[ \gamma_{n,m} := (\lambda_{n_1,m_1}, \ldots, \lambda_{n_N,m_N}), \quad d_{n,m} := \prod_{i=1}^N d_{n_i,m_i} \].

Thus, the modified quasicharacters are enumerated by \( \lambda_{n,m} \in \hat{G}^N \), \( \gamma_{n,m} \in \langle \lambda_{n,m} \rangle \) and \( \alpha, \alpha' \in L_T(\gamma_{n,m}, \lambda_{n,m}) \). Let us compute their norms. By (2.8),

\[ \| \hat{\chi}_{C}(T)_\alpha \|_2^2 = \frac{d_{n,m}}{d_{n,m}} (h\pi)^{4N} e^h \sum_{i=1}^N |\lambda_{n_i,m_i} + \rho|^2. \]

To compute the exponent, let \( \alpha_1 \) and \( \alpha_2 \) be simple roots of \( sl(3, \mathbb{C}) \). Then,

\[ \rho = \alpha_1 + \alpha_2, \quad \lambda_{n,m} = \frac{2n + m}{3} \alpha_1 + \frac{n + 2m}{3} \alpha_2 \]

and hence

\[ |\lambda_{n,m} + \rho|^2 = \frac{(n + 1)^2 + (m + 1)^2 + (n + 1)(m + 1)}{3} |\alpha_1|^2, \]

where we have used that \( |\alpha_1| = |\alpha_2| \) and \( \frac{2(\alpha_1, \alpha_2)}{|\alpha_1|^2} = -1 \) (Cartan matrix). Since \( sl(3, \mathbb{C}) \) is simple, the invariant scalar product in \( su(3) \) can be written in the form

\[ \langle X, Y \rangle = -\frac{1}{2\beta^2} \text{tr}(XY) \]

with a scaling parameter \( \beta > 0 \). Then,

\[ |\alpha_1|^2 = 4\beta^2, \]

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so that, altogether, we obtain

$$
\| \hat{\chi}^C(T)_{\alpha'} \|^2 = \frac{d_{n,m}}{d_{n,m}} (\hbar \pi)^{4N} e^{2\hbar \beta^2 \zeta_{n,m}}
$$

(4.17)

with

$$
\zeta_{n,m} := \frac{2}{3} \sum_{i=1}^{N} \left( (n_i + 1)^2 + (m_i + 1)^2 + (n_i + 1)(m_i + 1) \right).
$$

Next, we evaluate the modified quasicharacters at $1_{\gamma}$. Since $D^{n_i,m_i}$ is a subrepresentation of $(D^{1,0})^{\otimes m_i} \otimes (D^{0,1})^{\otimes m_i}$, and since $D^{1,0}(\gamma a) = \gamma D^{1,0}(a)$ and $D^{0,1}(\gamma a) = \gamma D^{0,1}(a)$, we have

$$
D^{n_i,m_i}(\gamma a 1) = \gamma^{n_i-m_i} D^{n_i,m_i}(1).
$$

Moreover,

$$
\hat{\chi}^C(T)_{\alpha'}(1, \ldots, 1) = d_{n,m}.
$$

Hence,

$$
\hat{\chi}^C(T)_{\alpha'}(1_{\gamma}) = \gamma_{1}^{n_1-m_1} \cdots \gamma_{N}^{n_N-m_N} d_{n,m}.
$$

It follows that

$$
\frac{\hat{\chi}^C(T)_{\alpha'}}{\| \hat{\chi}^C(T)_{\alpha'} \|} (1_{\gamma}) \frac{\hat{\chi}^C(T)_{\alpha'}}{\| \hat{\chi}^C(T)_{\alpha'} \|} = (\hbar \pi)^{-4N} e^{-2\hbar \beta^2 \zeta_{n,m}} \gamma_{1}^{n_1-m_1} \cdots \gamma_{N}^{n_N-m_N} d_{n,m} \hat{\chi}^C(T)_{\alpha'}
$$

and

$$
\left| \frac{\hat{\chi}^C(T)_{\alpha'}}{\| \hat{\chi}^C(T)_{\alpha'} \|} (1_{\gamma}) \right|^2 = (\hbar \pi)^{-4N} e^{-2\hbar \beta^2 \zeta_{n,m}} d_{n,m} d_{n,m}.
$$

Since $\sum_{\lambda_{n,m} \in (\pi_{1,0})^{\otimes m}} \sum_{\alpha, \alpha' \in \mathcal{L}^T(\lambda_{n,m}, \lambda_{n,m})} d_{n,m} = d_{n,m}$ for the denominator in (4.16) we obtain

$$
(\hbar \pi)^{-4N} \sum_{n,m=0}^{\infty} e^{-2\hbar \beta^2 \zeta_{n,m}} d_{n,m}^2 = (\hbar \pi)^{-4N} \Delta
$$

where

$$
\Delta = \sum_{n,m=0}^{\infty} e^{-2\hbar \beta^2 \zeta_{n,m}} d_{n,m}^2 = \sum_{k,l=1}^{\infty} e^{-\frac{\hbar^2 \beta^2}{2} (k^2 + l^2 + kl)} k^2 l^2 (k + l)^2.
$$

Thus, we have proved the following.

**Proposition 4.11.** For every $\gamma \in (\pi_{1,0})^{\otimes m}$, the subspace $\mathcal{H}_{1,0}$ is spanned by

$$
\psi_{n,m} = e^{-2\hbar \beta^2 \zeta_{n,m}} \prod_{i=1}^{N} j_{i}^{n_i-m_i} \sum_{\lambda_{n,m} \in (\pi_{1,0})^{\otimes m}} \sum_{\alpha, \alpha' \in \mathcal{L}^T(\lambda_{n,m}, \lambda_{n,m})} \hat{\chi}^C(T)_{\alpha'}. 
$$
4.5.3 The other strata

In case \( N = 1 \), according to Theorem 4.10, there is one further secondary stratum, \( U^2 \). Since here the quasicharacters boil down to the ordinary characters, the multiplication law is given by (2.13), and no recoupling coefficients appear. Therefore, this case will be discussed in a separate paper.

In case \( N > 1 \), Theorem 4.10 states that there remain three secondary strata, \( U^2 \), \( T \), and \( U^1 \). It turns out that for each of them, the description in terms of relations is quite elaborated and beyond the scope of the present paper. Therefore, let us just illustrate the method here. Let us consider the stratum \( T \) for \( N = 2 \) and let us pick a single one of the defining relations,

\[
\text{tr}([a_1, a_2]a_1a_2) = 0,
\]

say. Clearly, this holds for any element of \( T^2 \) and hence for any element of \( \overline{(G^N)}_T \). Put

\[
t_1(a_1, a_2) := \text{tr}((a_1a_2)^2), \quad t_2(a_1, a_2) := \text{tr}(a_1^2a_2^2), \quad p := t_1 - t_2,
\]

so that the relation under consideration is

\[
p = 0.
\]

We compute the matrix elements of the corresponding multiplication operator \( \hat{p} \) with respect to the orthonormal basis of normalized complex quasicharacters. In case \( N = 2 \), the only reduction tree is the cherry, and a labelling is given by two leaf labels, \( \lambda_{n_1,m_1} \) and \( \lambda_{n_2,m_2} \), and a root label, \( (\lambda_{n,m}, k) \in \langle \langle \lambda_{n_1,m_1}, \lambda_{n_2,m_2} \rangle \rangle \). Thus, the elements of

\[
\mathcal{E}^\vee ((\lambda_{n_1,m_1}, \lambda_{n_2,m_2}), \lambda_{n,m})
\]

are enumerated by the multiplicity counter \( k \). Therefore, the quasicharacters are labelled by the triple \( (\lambda_{n_1,m_1}, \lambda_{n_2,m_2}, \lambda_{n,m}) \) and two multiplicity counters \( k', k \). For simplicity, we will write

\[
(\hat{x}C^\vee (\lambda_{n_1,m_1,n_2,m_2}, \lambda_{n,m}))(k') := \hat{x}C^\vee (\langle \langle \lambda_{n_1,m_1}, \lambda_{n_2,m_2}, \lambda_{n,m} \rangle \rangle). \quad
\]

In addition, we will omit the brackets and the multiplicity counters \( k, k' \) in case the multiplicity is 1. By (4.17),

\[
\left\| (\hat{x}C^\vee (\lambda_{n_1,m_1,n_2,m_2}, \lambda_{n,m}))(k') \right\| = (\hbar \pi)^4 e^{\hbar \beta^2 (\zeta_{n_1,m_1} + \zeta_{n_2,m_2})} \sqrt{\frac{d_{n,m}}{d_{n_1,m_1} d_{n_2,m_2}}}. \quad (4.18)
\]

According to [6], the Clebsch-Gordan series for twofold tensor products reads

\[
D^{n,m} \otimes D^{n',m'} = \bigoplus_{i=0}^{\min(n,m')} \bigoplus_{j=0}^{\min(n',m)} \left( D^{n+i-j, m+m'-i-j} \right)
\]

\[
\otimes \bigoplus_{k=1}^{\min(n-i,n'-j)} \bigoplus_{l=0}^{\min(n-i,n'-j)} \left( D^{n+i-j-2k, m+m'-i-j+k} \right).
\]

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The normalized real quasicharacters. According to point 1 of Proposition 3.13, real. We have

$$\sum_{k=1}^{\min(n-j,m'-1)} D^{n+n'-i-j+k,m+m'-i-j-2k}. \quad (4.19)$$

For the ordinary Clebsch-Gordan coefficients, we introduce the shorthand notation

$$C_{\mu_1,\mu_2,\mu}^{m_1,m_2,m,k} := C_{\mu_1,\mu_2,\mu}^{\lambda_1,m_1,\lambda_2,m_2,\lambda,n,m,k},$$

with the convention that $k$ can be omitted whenever the multiplicity is 1. For the computation of these coefficients, see [1].

We start by expressing $p$ in terms of the modified quasicharacters. For $t_1$, we find

$$t_1(a_1, a_2) = \sum_{\mu_1,\mu_2 \in \mu} D^{1,0}_{\mu_1,\mu_2}(a_1) D^{1,0}_{\mu_2,\mu_3}(a_2) D^{1,0}_{\mu_3,\mu_4}(a_1) D^{1,0}_{\mu_4,\mu_1}(a_2)$$

$$= \sum_{\mu_1,\mu_2 \in \mu} \langle \langle \lambda_1,0,\mu_1 | \mu_4 \rangle \rangle \langle \langle \lambda_1,0,\mu_2 | \mu_4 \rangle \rangle \langle \langle \lambda_1,0,\mu_3 | \mu_4 \rangle \rangle \langle \langle \lambda_1,0,\mu_4 | \mu_4 \rangle \rangle$$

$$= \sum_{\mu_1,\mu_2 \in \mu} \langle \langle \lambda_1,0,\mu_1 | \mu_4 \rangle \rangle \langle \langle \lambda_1,0,\mu_2 | \mu_4 \rangle \rangle \langle \langle \lambda_1,0,\mu_3 | \mu_4 \rangle \rangle \langle \langle \lambda_1,0,\mu_4 | \mu_4 \rangle \rangle$$

where we have used that the ordinary Clebsch-Gordan coefficients can be chosen to be real. We have

$$C_{\mu_2,\mu_1,\mu}'^{10,0} = -C_{\mu_1,\mu_2,\mu}^{10,0}, \quad C_{\mu_2,\mu_1,\mu}^{10,20} = C_{\mu_1,\mu_2,\mu}^{10,20}.$$

Thus, by orthogonality,

$$\sum_{\mu_1 \in \mu} C_{\mu_1,\mu_3,\mu_1}'^{10,0} C_{\mu_2,\mu_4,\mu_2}'^{10,0} C_{\mu_2,\mu_4,\mu_2}'^{10,0} C_{\mu_3,\mu_3,\mu_1}'^{10,0} \delta_{\mu,\mu'} \delta_{\nu,\nu'} = \delta_{\mu,\mu'} \delta_{\nu,\nu'},$$

whereas the mixed terms vanish. Hence,

$$t_1(a_1, a_2) = \sum_{\mu,\nu \in \mu} D^{2,0}_{\mu,\mu}(a_1) D^{2,0}_{\nu,\mu}(a_2) - \sum_{\mu,\nu \in \mu} D^{0,1}_{\mu,\mu}(a_1) D^{0,1}_{\nu,\mu}(a_2). \quad (4.20)$$

To express $t_1$ in terms of modified complex quasicharacters, we compute the expansion coefficients with respect to the basis of complex quasicharacters. According to Corollary 2.3, it suffices to compute the scalar products of the restriction of $t_1$ to $G^2$ with the normalized real quasicharacters. According to point \[\text{II}\] of Proposition 3.13,

$$(\lambda_1 m_1, \lambda_2 m_2, \lambda n, m)_{k'}^{k} (a_1, a_2) = \sqrt{d_{n,m}} \sum_{\mu_1,\mu'_1 \in \mu(\lambda_1, m_1)} \sum_{\mu,\mu' \in \mu(\lambda_2, m_2)} C_{\mu_1,\mu_2,\mu}^{m_1,m_2,m,n,k}$$
Using
\[
\langle D^{n_1,m_1}_{\mu'_1,\nu'_1} | D^{n_2,m_2}_{\mu'_2,\nu'_2} \rangle_{L^2(G^2)} = \frac{1}{d_{n_1,m_1}} \delta_{n_1,n_2} \delta_{m_1,m_2} \delta_{\mu_1,\mu_2} \delta_{\nu_1,\nu_2},
\]
we obtain
\[
\langle (\chi_{n_1,n_2,m,n,m})^{k'}_{k} \mid t_1 \rangle_{G^2} = \frac{\delta_{n_1,2} \delta_{m_1,0} \delta_{n_2,2} \delta_{m_2,0}}{d_{2,0} \sqrt{d_{n,m}}} \sum_{\mu,\mu',\nu \in \mathbf{w}(\lambda_{n,m})} C^{20,20,0,0,\nu}_{\mu,\mu',\nu} C^{20,20,0,0,\nu}_{\mu',\mu,\nu'} - \frac{\delta_{n_1,0} \delta_{m_1,1} \delta_{n_2,0} \delta_{m_2,1}}{d_{0,1} \sqrt{d_{n,m}}} \sum_{\mu,\mu',\nu \in \mathbf{w}(\lambda_{n,m})} C^{01,01,0,1,\nu}_{\mu,\mu',\nu} C^{01,01,0,1,\nu}_{\mu',\mu,\nu'}.
\]

By (4.19), we have
\[
D^{2,0} \otimes D^{2,0} = D^{4,0} \oplus D^{2,1} \oplus D^{0,2}, \quad D^{0,1} \otimes D^{0,1} = D^{0,2} \oplus D^{1,0}, \quad (4.21)
\]
Hence, \( k, k' = 1 \) and the relevant values of \((n, m)\) are \((4, 0), (2, 1)\) and \((0, 2)\) for the first sum and \((1, 0)\) for the second sum. Using
\[
C^{20,20,40}_{\mu',\mu,\nu} = C^{20,20,40}_{\mu,\mu',\nu}, \quad C^{20,20,21}_{\mu',\mu,\nu} = -C^{20,20,21}_{\mu,\mu',\nu}, \quad C^{20,20,02}_{\mu',\mu,\nu} = C^{20,20,02}_{\mu,\mu',\nu},
\]
and taking the sums over weights, we finally arrive at
\[
t_1 = \hat{\chi}^{\mathbb{C}}_{20,20,40} - \hat{\chi}^{\mathbb{C}}_{20,20,21} + \hat{\chi}^{\mathbb{C}}_{01,01,02} - \hat{\chi}^{\mathbb{C}}_{01,01,10}.
\]
A similar calculation leads to
\[
t_2(a_1, a_2) = \sum_{\mu_i \in \mathbf{w}(\lambda_{n,m})} \sum_{\nu_i \in \mathbf{w}(\lambda_{n,m})} \sum_{\nu'_i \in \mathbf{w}(\lambda_{n,m})} C^{10,10,10,\lambda_1}_{\mu_1,\mu_2,\nu_1,\nu'_1} C^{10,10,10,\lambda_2}_{\mu_2,\mu_3,\nu'_1,\nu'_2} C^{10,10,10,\lambda_2}_{\mu_3,\mu_4,\nu'_2} C^{10,10,10,\lambda_1}_{\mu_4,\mu_1,\nu'_2} \sum_{\nu'_1 \in \mathbf{w}(\lambda_{n,m})} \sum_{\nu'_2 \in \mathbf{w}(\lambda_{n,m})} d_{\lambda_1,\nu_1,\nu'_1} D^{\lambda_1}_{\nu_1,\nu'_1}(a_1) D^{\lambda_2}_{\nu_2,\nu'_2}(a_2)
\]
and
\[
\langle (\chi_{n_1,n_2,m,n,m})^{k'}_{k} \mid t_2 \rangle_{G^2} = \sum_{\lambda_i = \lambda_2,0,\lambda_{0,1}} \frac{\delta_{\lambda_1,\lambda_{n_1,m_1}} \delta_{\lambda_2,\lambda_{n_2,m_2}}}{d_{\lambda_1} d_{\lambda_2} d_{n,m}} \sum_{\mu_i \in \mathbf{w}(\lambda_{n,m})} \sum_{\nu_i \in \mathbf{w}(\lambda_{n,m})} \sum_{\nu'_i \in \mathbf{w}(\lambda_{n,m})} C^{10,10,10,10,10,10,\lambda_1}_{\mu_1,\mu_2,\nu_1,\nu'_1} C^{10,10,10,10,10,10,\lambda_2}_{\mu_2,\mu_3,\nu'_1,\nu'_2} C^{10,10,10,10,10,10,\lambda_1}_{\mu_3,\mu_4,\nu'_2} C^{10,10,10,10,10,10,\lambda_2}_{\mu_4,\mu_1,\nu'_2}.
\]
Here, in addition to (4.21), the Clebsch-Gordan series \( D^{2,0} \otimes D^{0,1} = D^{2,1} \oplus D^{1,0} \) occurs, showing that, still, \( k, k' = 1 \) throughout. Evaluation of the sums using the online calculator [1] yields
\[
t_2 = \hat{\chi}^{\mathbb{C}}_{20,20,40} - \frac{1}{2} \hat{\chi}^{\mathbb{C}}_{20,20,21} - \frac{1}{2} \hat{\chi}^{\mathbb{C}}_{01,01,21} + \frac{1}{2} \hat{\chi}^{\mathbb{C}}_{01,01,10} + \frac{1}{2} \hat{\chi}^{\mathbb{C}}_{01,01,10} + \frac{1}{2} \hat{\chi}^{\mathbb{C}}_{01,01,10}.
\]
so that, altogether,

\[ p = -\hat{\chi}_{20,20,21}^C + \frac{3}{2} \hat{\chi}_{20,20,02}^C + \frac{1}{2} \left( \hat{\chi}_{20,01,21}^C + \hat{\chi}_{01,20,21}^C \right) - \frac{1}{2} \left( \hat{\chi}_{20,01,10}^C + \hat{\chi}_{01,20,10}^C \right) - \frac{3}{2} \hat{\chi}_{01,01,21}^C + \frac{3}{2} \hat{\chi}_{01,01,10}^C. \]  \hspace{1cm} (4.22)

The matrix elements of the corresponding multiplication operator \( \hat{p} \) with respect to the orthonormal basis of normalized complex quasicharacters are given by

\[ \hat{p}_{r_1s_1,r_2s_2,r,s,l,l'}^{n_1m_1,n_2m_2,nm,k'k} = \sqrt{C_{r_1s_1,r_2s_2,r,s,l,l'}} \left( \hat{\chi}_{n_1m_1,n_2m_2,nm}^C \right)_k^{k'}, \]

where the coefficients \( C \) are defined by

\[ p \cdot \left( \hat{\chi}_{r_1s_1,r_2s_2,r,s}^C \right)_l = \sum_{n_1m_1,n_2m_2,nm,k'k} C_{r_1s_1,r_2s_2,r,s,l,l'}^{n_1m_1,n_2m_2,nm,k'k} \left( \hat{\chi}_{n_1m_1,n_2m_2,nm}^C \right)_k^{k'}. \]

To illustrate the computation of these coefficients, we restrict attention to the case where \( (r_1, s_1) = (r, s) = (0, 1) \) and \( (r_2, s_2) = (0, 0) \), thus deriving a specific row of the matrix of \( \hat{p} \). First, we compute \( C \). As an example, consider the contribution of the term \( \hat{\chi}_{20,20,21}^C \) in (4.22). According to Corollary 3.10,

\[ \hat{\chi}_{20,20,21}^C \cdot \hat{\chi}_{01,00,01}^C = \sum R(\mathcal{V})_{(n_1m_1,n_2m_2,nm,l,l')} (\hat{\chi}_{n_1m_1,n_2m_2,nm}^C)^l_{l'} R(\mathcal{V})_{(20,20,21),(01,00,01),k}, \]

where the sum is taken over \( n_1, m_1, n_2, m_2, n, m, k, k \) satisfying

\[ (\lambda_{n_1, m_1}, k^1) \in \langle \langle \lambda_{2,0}, \lambda_{0,1} \rangle \rangle, \quad (\lambda_{n_2, m_2}, k^2) \in \langle \langle \lambda_{2,0}, \lambda_{0,0} \rangle \rangle, \quad (\lambda_{n, m}, l) \in \langle \langle \lambda_{2,1}, \lambda_{0,1} \rangle \rangle, \quad (\lambda_{n, m}, l') \in \langle \langle \lambda_{n_1, m_1}, \lambda_{n_2, m_2} \rangle \rangle. \]

Clearly, the second condition yields \( (n_2, m_2) = (2, 0) \) with \( k^2 = 1 \). Since

\[ D^{2,0} \otimes D^{0,1} = D^{2,1} \oplus D^{1,0}, \]

the first condition implies that \( (n_1, m_1) = (2, 1) \) or \( (1, 0) \) with \( k^1 = 1 \). Since

\[ D^{2,1} \otimes D^{0,1} = D^{2,2} \oplus D^{3,0} \oplus D^{1,1}, \]

the third condition implies \( (n, m) = (2, 2), (3, 0) \) or \( (1, 1) \) with \( k = 1 \). Then, due to

\[ D^{2,1} \otimes D^{2,0} = D^{4,1} \oplus D^{2,2} \oplus D^{3,0} \oplus D^{0,3} \oplus D^{1,1}, \quad D^{1,0} \otimes D^{2,0} = D^{3,0} \oplus D^{1,1}, \]

the last condition implies that \( l = l' = 1 \) and that \( (n_1m_1, n_2m_2, nm) \) can take the values

\[ (21, 20, 22), \quad (21, 20, 30), \quad (21, 20, 11), \quad (10, 20, 30), \quad (10, 20, 11). \]
Since in addition $R(\vee)$ is real and thus coincides with its transpose, we obtain
\[
\hat{\chi}^{20,20,21} \cdot \hat{\chi}^{01,00,01} = \sum_{(n_1 m_1, nm)} W^2_{n_1 m_1, nm} \chi^{C}_{n_1 m_1, 20, nm},
\] (4.23)
where the sum runs over the tuples
\[
(n_1 m_1, nm) = (21, 22), \ (21, 30), \ (21, 11), \ (10, 30), \ (10, 11).
\]

Here, according to Theorem 3.17, we have expressed $R(\vee)$ in terms of the 9$\lambda$ symbols
\[
W_{n_1 m_1, nm} = \begin{pmatrix}
\lambda_{2,0} & \lambda_{2,0} & \lambda_{2,1} \\
\lambda_{0,1} & \lambda_{0,0} & \lambda_{0,1} \\
\lambda_{1,0} & \lambda_{2,0} & \lambda_{n,m}
\end{pmatrix}.
\]

According to Remark 3.15, the 9$\lambda$ symbols can be expressed, in turn, in terms of ordinary Clebsch-Gordan coefficients. Since the latter are real and since
\[
C^\lambda_{\mu_1, \mu_2, \mu_3} = \delta_{\mu_1, \mu_3},
\]
we obtain
\[
W_{n_1 m_1, nm} = \sum_{\mu_1^*, \mu_2^*, \mu_3^*} C^{20,20,21}_{\mu_1^*, \mu_2^*, \mu_3^*} C^{21,01, nm}_{\mu_1^*, \mu_2^*, \mu_3^*} C^{20,01, n_1 m_1}_{\mu_1^*, \mu_2^*, \mu_3^*} C^{n_1 m_1, 20, nm}_{\mu_1^*, \mu_2^*, \mu_3^*},
\]
for any given $\mu_3^* \in \mathcal{W}(\lambda_{n,m})$. The ranges of the summation variables are
\[
\mu_1^* \in \mathcal{W}(\lambda_{2,0}), \quad \mu_2^* \in \mathcal{W}(\lambda_{2,1}), \quad \mu_3^* \in \mathcal{W}(\lambda_{n,1}), \quad \mu_2^* \in \mathcal{W}(\lambda_{0,1}).
\]

Using the online calculator [1], we find
\[
W_{21, 22} = 1, \quad W_{21, 30} = -\frac{\sqrt{3}}{2}, \quad W_{21, 11} = -\frac{\sqrt{6}}{4}, \quad W_{10, 30} = \frac{1}{2}, \quad W_{10, 11} = \frac{\sqrt{10}}{4}.
\]
Thus,
\[
\hat{\chi}^{C}_{20,20,21} \cdot \hat{\chi}^{C}_{01,00,01} = \hat{\chi}^{C}_{20,20,22} - \frac{\sqrt{3}}{2} \hat{\chi}^{C}_{21,20,30} - \frac{\sqrt{6}}{4} \hat{\chi}^{C}_{21,20,11} + \frac{1}{2} \hat{\chi}^{C}_{10,20,30} + \frac{\sqrt{10}}{4} \hat{\chi}^{C}_{10,20,11}.
\]

The corresponding contribution to $C$ can be obtained from the right hand side by replacing the modified quasicharacters by appropriate Kronecker deltas. By analogy, we determine the contributions of the other terms in (4.22). Putting all of this together, and computing the norm ratios using (4.18), we finally arrive at the following formula for the elements of the row of $\hat{p}$ under consideration:
\[
(\hat{p})^{n_1 m_1, n_2 m_2, nm, k'k}_{01,00,01}
\]

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\begin{align*}
&= -\frac{3}{\sqrt{30}} z^{11} \delta_{\ell}(21,20,22) - \frac{1}{4} z^{11} \delta_{\ell}(21,20,30) + \frac{1}{2} z^{11} \delta_{\ell}(21,20,03) + \frac{3}{8\sqrt{5}} z^{11} \delta_{\ell}(21,20,11) \\
&+ \frac{3}{2\sqrt{15}} z^{8} \delta_{\ell}(21,01,22) + \frac{1}{\sqrt{6}} z^{8} \delta_{\ell}(21,01,30) - \frac{7}{12\sqrt{10}} z^{8} \delta_{\ell}(21,01,11) + \frac{\sqrt{3}}{4} z^{8} \delta_{\ell}(02,20,22) \\
&- \frac{1}{12\sqrt{2}} z^{8} \delta_{\ell}(02,20,11) - \frac{1}{12} z^{8} \delta_{\ell}(02,20,00) + \frac{\sqrt{5}}{12} z^{5} \delta_{\ell}(10,20,30) + \frac{1}{24} z^{5} \delta_{\ell}(10,20,11) \\
&- \frac{\sqrt{5}}{2} z^{5} \delta_{\ell}(02,01,03) + \frac{1}{4} z^{5} \delta_{\ell}(02,01,11) - \frac{7}{12\sqrt{2}} z^{2} \delta_{\ell}(10,01,11) + \frac{1}{6} z^{2} \delta_{\ell}(10,01,00) .
\end{align*}

Here,
\[ z = e^{4\hbar \beta^2/3} . \]

and the double bracket \( () \) in the Kronecker delta symbols is a shorthand notation for \((n_1 m_1, n_2 m_2, nm)\).

5 Outlook

For future work, the following tasks will be interesting.

1. There is a deep relation between rooted binary trees and trivalent graphs \([2,33]\). It would be interesting to study whether our methods may be reformulated in terms of trivalent graph theory.

2. For a systematic investigation of \( G = SU(3) \), it seems reasonable to start with analyzing the strata subspaces and their orthoprojectors for \( N = 2 \), the simplest situation where all the strata are present.

3. In \([9]\) we have shown for the case \( G = SU(2) \) that the above tools are also useful for the analysis of the spectral problem of the quantum Hamiltonian. It will be interesting to extend this study to \( G = SU(3) \), starting from the simplest toy model provided by \( N = 1 \).

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Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.
A A direct proof of Corollary 3.10

Here, we prove this corollary by means of the composite Clebsch-Gordan coefficients.

Let $T$ be a coupling tree. For simplicity, we assume that the leaves of $T$ are numbered $1, \ldots, N$. For $i = 1, 2, 3$, let $\lambda_i$ be a leaf labelling of $T$, let $\lambda_i \in T(\lambda_i, \alpha_i)$ and let $\alpha_i \in \mathcal{L}_T(\lambda_i, \alpha_i)$. Let $k$ be an assignment of a positive integer to every leaf of $T$ and let $\hat{k}$ be a positive integer. Assume that $(\alpha_3, k, k) \in \langle \alpha_1, \alpha_2 \rangle$.

Under the identification $H_{\lambda_1} \otimes H_{\lambda_2} = H_{\lambda_3}$, the vectors $[T \cdot T; [\alpha_1 \cdot \alpha_2, (\lambda, k)], \mu]$, where $(\lambda, k) \in \langle \lambda_1, \lambda_2 \rangle$ and $\mu \in \omega(\lambda)$, can be expanded with respect to the tensor product vectors $[T; \alpha_1, \mu_1] \otimes [T; \alpha_2, \mu_2]$, where $\mu_i \in \omega(\lambda_i)$, and vice versa:

$$[T \cdot T; [\alpha_1 \cdot \alpha_2, (\lambda, k)], \mu] = \sum_{\mu_1 \in \omega(\lambda_1)} \sum_{\mu_2 \in \omega(\lambda_2)} C^{\lambda_1, \lambda_2, \lambda, k}_{\mu_1, \mu_2, \mu} [T; \alpha_1, \mu_1] \otimes [T; \alpha_2, \mu_2],$$

$$[T; \alpha_1, \mu_1] \otimes [T; \alpha_2, \mu_2] = \sum_{(\lambda, k)} \sum_{\hat{\mu}_1 = 1}^{m_1(\hat{\mu})} \sum_{\hat{\mu}_2 = 1}^{m_2(\hat{\mu})} (C^{\lambda_1, \lambda_2, \lambda, k}_{\mu_1, \mu_2, \mu})^* [T \cdot T; [\alpha_1 \cdot \alpha_2, (\lambda, k)], \mu],$$

(A–1)

where $\hat{\mu}_1 := \hat{\mu}_1 + \hat{\mu}_2$ and $\mu := (\hat{\mu}, \hat{\mu})$ and where the sum is over $(\lambda, k) \in \langle \lambda_1, \lambda_2 \rangle$ such that $\omega(\lambda)$ contains $\hat{\mu}$. Similarly, under the identification $H_{\lambda_1} \otimes H_{\lambda_2} = \bigotimes_{n=1}^N (H_{\lambda_1} \otimes H_{\lambda_2})$, the vectors $[T^\Lambda; [\alpha_1 \cdot \alpha_2, \alpha_3, k], \mu]$, where $\mu \in \omega(\lambda_3)$, can be expanded with respect to the vectors

$$[\Lambda; (\lambda_1^1, \lambda_2^1, (\lambda_3^1, k^1)), \mu^1] \otimes \cdots \otimes [\Lambda; (\lambda_1^N, \lambda_2^N, (\lambda_3^N, k^N)), \mu^N],$$

where $\mu^n \in \omega(\lambda_3^n)$ for all $n$. Here, $(\lambda_1^n, \lambda_2^n, (\lambda_3^n, k^n))$ stands for the labelling of $\Lambda$ assigning $\lambda_1^n, \lambda_2^n$ to the leaves and $(\lambda_3^n, k^n)$ to the root. To find the expansion coefficients, we first expand with respect to the tensor product basis,

$$[T^\Lambda; [\alpha_1 \cdot \alpha_2, \alpha_3], \mu] = \sum_{\mu \in \omega(\lambda_1)} \sum_{\mu \in \omega(\lambda_2)} C(T^\Lambda)_{[\alpha_1 \cdot \alpha_2, \alpha_3]}^{[\alpha_1 \cdot \alpha_2, \alpha_3]} [\Lambda; \alpha_1^1, \alpha_2^1, \mu_1^1, \mu_2^1].$$

(A–2)

In view of (A.13), we may factorize the composite Clebsch-Gordan coefficient as

$$C(T^\Lambda)_{[\alpha_1 \cdot \alpha_2, \alpha_3]}^{[\alpha_1 \cdot \alpha_2, \alpha_3]} = \prod_{\hat{\mu}_3 = 1}^{m_3(\hat{\mu})} \prod_{\hat{\mu}_3 = 1}^{m_3(\hat{\mu})} C(T)_{[\alpha_1^1, \alpha_2^1]}^{[\alpha_1^1, \alpha_2^1]} \prod_{n=1}^{N} C_{\mu_1^n, \mu_2^n, \mu_3^n}^{\lambda_1^n, \lambda_2^n, \lambda_3^n},$$

where $\hat{\mu}_3^n := \hat{\mu}_1^n + \hat{\mu}_2^n$ and $\mu_3^n := (\hat{\mu}_3^n, \hat{\mu}_3^n)$. Decomposing the sum accordingly, from (A–2) we obtain

$$[T^\Lambda; [\alpha_1 \cdot \alpha_2, \alpha_3], \mu]$$

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\[
\sum_{\mu_3 \in w(\lambda_3)} C(T)^{\alpha_3, \mu_3}_{\mu_1, \mu_2} \left( \prod_{n=1}^{N} \left( \sum_{\mu_1 \in w(\lambda_1^n)} \sum_{\mu_2 \in w(\lambda_2^n)} \sum_{\mu_3 \in w(\lambda_3^n)} \lambda_1^n \mu_1^n \otimes \lambda_2^n \mu_2^n \right) \right).
\]

Since
\[
\sum_{\mu_1 \in w(\lambda_1^n)} \sum_{\mu_2 \in w(\lambda_2^n)} \lambda_1^n \mu_1^n \otimes \lambda_2^n \mu_2^n = |\lambda_1^n \mu_1^n \otimes \lambda_2^n \mu_2^n\rangle = |\chi_1^n \chi_2^n\rangle,
\]
we may rewrite
\[
|T^{\nu}; [\alpha_1 \ast \alpha_2, \alpha_3], \mu\rangle = \sum_{\mu_3 \in w(\lambda_3)} C(T)^{\alpha_3, \mu_3}_{\mu_1, \mu_2} |\chi_1^n \chi_2^n \chi_3^n\rangle \otimes \cdots \otimes |\lambda_1^n \lambda_2^n \lambda_3^n\rangle.
\]

Now, we use (3.7), (A–1) and (3.14) to calculate
\[
\chi^{\mu_1, \mu_2}_{\alpha_1, \alpha_2}(q) \cdot \chi^{\mu_1, \mu_2}_{\alpha_1, \alpha_2}(q)
\]
\[
= \sum_{\mu_1 \in w(\lambda_1)} \sum_{\mu_2 \in w(\lambda_2)} \langle T; \alpha_1', \mu_1 \mid D^{\Lambda_1}(\alpha_1) \mid T; \alpha_1, \mu_1 \rangle \langle T; \alpha_2', \mu_2 \mid D^{\Lambda_2}(\alpha_2) \mid T; \alpha_2, \mu_2 \rangle
\]
\[
= \sum_{\mu_1 \in w(\lambda_1)} \sum_{\mu_2 \in w(\lambda_2)} \langle \langle T; \alpha_1', \mu_1 \otimes T; \alpha_2', \mu_2 \mid D^{\Lambda_1}(\alpha_1) \mid T; \alpha_1, \mu_1 \otimes T; \alpha_2, \mu_2 \rangle \rangle
\]
\[
= \sum_{\mu_1 \in w(\lambda_1)} \sum_{\mu_2 \in w(\lambda_2)} \langle T; [\alpha_1', \alpha_2', (\lambda', k')] \mid (\mu_1 + \mu_2, \mu') \mid D^{\Lambda_1}(\alpha_1) \mid T; [\alpha_1 \cdot \alpha_2, (\lambda, k)] \rangle \langle T; [\alpha_1', \alpha_2', (\lambda', k')] \mid (\mu_1 + \mu_2, \mu') \rangle
\]
\[
= \sum_{\lambda, k, k'} \sum_{\mu_1 \in w(\lambda)} \langle T; [\alpha_1 \cdot \alpha_2, (\lambda, k)] \mid (\mu_1 + \mu_2, \mu') \rangle \Pi D^{\Lambda_1}(\alpha_1) \Pi^{-1} \langle T; [\alpha_1 \cdot \alpha_2, (\lambda, k)] \mid (\mu_1 + \mu_2, \mu') \rangle
\]
\[
= \sum_{\lambda, k, k'} \langle T; [\alpha_1 \cdot \alpha_2, (\lambda, k)] \mid (\mu_1 \cdot (\mu_1 + \mu_2), \mu') \rangle \Pi D^{\Lambda_1}(\alpha_1) \Pi^{-1} \langle T; [\alpha_1 \cdot \alpha_2, (\lambda, k)] \mid (\mu_1 + \mu_2, \mu) \rangle
\]
where the sum is over \( \lambda \in \{\lambda_1, \lambda_2\} \) and \( k = 1, \ldots, m(\lambda_1, \lambda_2)(\lambda) \). In the last step, we have used
\[
\sum_{\mu_1 \in w(\lambda_1)} \sum_{\mu_2 \in w(\lambda_2)} C^{\lambda_1, \lambda_2, \lambda', k'}_{\mu_1, \mu_2, \mu'} (C^{\lambda_1, \lambda_2, \lambda, k}_{\mu_1, \mu_2, \mu})^* = \delta_{\lambda \lambda'} \delta_{k k'} \delta_{\mu \mu'}.
\]

Inserting the unit operator
\[
1 = \sum_{\alpha_2, \alpha_3, \lambda} |T^{\nu}; [\alpha_1 \ast \alpha_2, \alpha_3, \lambda], \mu_3 \rangle \langle T^{\nu}; [\alpha_1 \ast \alpha_2, \alpha_3, \lambda] \mid \mu_3 |
\]

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twice and using the relations

\[ \langle T \cdot T; [\alpha_1' \cdot \alpha_2, (\lambda, k)], \mu \mid \Pi \rangle T^\alpha_1 \alpha_2, k; \mu' \rangle = \delta_{\lambda, \lambda_3} \delta_{\mu \cdot \mu'} R(T)^{\alpha_1' \alpha_2' k \cdot \mu} \alpha_3' \alpha_3 \alpha_2 k, \]
\[ \langle T^\alpha \mid [\alpha_1 \cdot \alpha_2, (\lambda, k)], \mu \mid \Pi \rangle T^\alpha_1 \alpha_2, k; \mu' \rangle = \delta_{\lambda, \lambda_3} \delta_{\mu \cdot \mu'} R(T)^{\alpha_3' \cdot \mu} \alpha_1 \alpha_2, k \]

we obtain

\[ \hat{\chi}(T)^{\alpha_1}_\alpha (a) \hat{\chi}(T)^{\alpha_2}_\alpha (a) = \sum_{\alpha_3, \alpha'_3} \sum_{\hat{\alpha}'_3, \hat{\alpha}_3} \sum_{k} R(T)^{\alpha'_1 \alpha'_2, k} \] \[ \alpha_1 \alpha_2, k \cdot \mu \rangle D_{\hat{\alpha}'_3 \hat{\alpha}_3 \hat{\alpha}_2} (a) \] \[ \cdot \left( \sum_{\mu} \langle T^\alpha; [\alpha_1' \cdot \alpha_2, (\lambda, k)], \mu \mid D_{\hat{\alpha}'_3 \hat{\alpha}_3 \hat{\alpha}_2} (a) \rangle \right), \]

where \( \alpha_3 \) and \( \alpha'_3 \) assign the same highest weight \( \lambda_3 \) to the root and \( \mu \in \mathfrak{u}(\lambda_3) \). We plug in \((3.3)\) and use unitarity of the isomorphisms \((3.2)\), as well as \((3.15)\) and \((3.7)\), to rewrite the sum over \( \mu \) as

\[ \sum_{\mu} \langle T^\alpha; [\alpha_1' \cdot \alpha_2, (\lambda, k)], \mu \mid D_{\hat{\alpha}'_3 \hat{\alpha}_3 \hat{\alpha}_2} (a) \rangle \]
\[ = \sum_{\mu} \sum_{\mu \in \mathfrak{u}(\lambda_3) \cdot \mu' \in \mathfrak{u}(\lambda_3)} \left( C(T)^{\alpha'_2 \hat{\alpha}_2, \mu} \cdot C(T)^{\alpha_3 \hat{\alpha}_3 \hat{\alpha}_2} \right) \]
\[ \cdot \left( \prod_{n=1}^{N} \langle \lambda n, \lambda m, (\lambda k, n), \mu \rangle, D_{\hat{\alpha}_3} (a) \mid D_{\hat{\alpha}_3} (a) \rangle \right) \]
\[ = \delta_{\lambda, \lambda_3} \delta_{\mu \cdot \mu'} \sum_{\mu} \sum_{\mu \in \mathfrak{u}(\lambda_3) \cdot \mu' \in \mathfrak{u}(\lambda_3)} \left( C(T)^{\alpha'_2 \hat{\alpha}_2, \mu} \cdot C(T)^{\alpha_3 \hat{\alpha}_3 \hat{\alpha}_2} \right) \]
\[ \cdot \langle \lambda n, \lambda m, (\lambda k, n), \mu \rangle, D_{\hat{\alpha}_3} (a) \mid T; \alpha_3, \mu \rangle \]
\[ = \delta_{\lambda, \lambda_3} \delta_{\mu \cdot \mu'} \hat{\chi}(T)^{\alpha_3}_\alpha (a). \]

Plugging this in, we finally obtain the assertion.

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