NON-AUTONOMOUS STOCHASTIC EVOLUTION EQUATIONS WITH NONLINEAR NOISE AND NONLOCAL CONDITIONS GOVERNED BY NONCOMPACT EVOLUTION FAMILIES

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Abstract. In this paper, we investigate the non-autonomous stochastic evolution equations of parabolic type with nonlinear noise and nonlocal initial conditions in Hilbert spaces, where the operators in linear part depend on time and generate an noncompact evolution family. New existence result of mild solutions is established under some weaker growth and measure of noncompactness conditions on nonlinear functions and nonlocal term. The discussions are based on Sadovskii’s fixed-point theorem as well as the theory of evolution family. At last, as a sample of application, the obtained abstract result is applied to a class of non-autonomous stochastic partial differential equations of parabolic type with nonlocal initial conditions. The result obtained in this paper is a supplement to the existing literature and essentially extends some existing results in this area.

1. Introduction. In this paper, we investigate the existence of mild solutions for the following non-autonomous stochastic evolution equations (NSEE) of parabolic type with nonlinear noise and nonlocal initial conditions in the real separable Hilbert space $\mathbb{H}$

$$
\begin{cases}
  du(t) = [A(t)u(t) + f(t, u(t))]dt + g(t, u(t))dW(t), \quad t \in J, \\
  u(0) = H(u),
\end{cases}
$$

where $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $D(A(t))$ for every $t \in J$, $J = [0, a]$, $a > 0$ is a constant, the state $u(\cdot)$ takes values in the real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot, \cdot)_K$ and norm $\| \cdot \|_K$. Assume that $\{W(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We are also employing the same notation $\| \cdot \|$ for the norm of $\mathcal{L}(\mathbb{K}, \mathbb{H})$, which denotes the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. We denote by $\mathcal{L}(\mathbb{H}) = \mathcal{L}(\mathbb{H}, \mathbb{H})$ and $L^2(\Omega, \mathbb{H})$ the collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables. $f : J \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ and $g : J \times L^2(\Omega, \mathbb{H}) \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous nonlinear functions, and $H$ is an appropriate

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continuous function mapping from some space of functions so that it constitutes a nonlocal condition to be specified later.

The nonlocal problem was motivated by physical problems. Indeed, it is demonstrated that the nonlocal condition can be more useful than the classical initial condition to describe some physical phenomena [5], [7], [13], [18], [19], [23], [28]. Since semilinear stochastic evolution equations are abstract formulations for many problems arising in the domain of engineering technology, biology and economic system etc., stochastic evolution equations have attracted increasing attention in recent years and the existence, uniqueness and asymptotic behavior of mild solutions to stochastic evolution equations have been considered by many authors. For more details about stochastic evolution equations, please see [15], [16], [25], [26], [29] and the references therein.

In recent years, stochastic evolution equations with nonlocal initial conditions have also been investigated extensively and some interesting results have been obtained. In 2012, Cui et al. [14] studied the existence results of mild solutions for a class of stochastic integro-differential evolution equations with nonlocal initial conditions in Hilbert spaces assuming that the nonlocal item is only continuous but without imposing some compactness and convexity. Later, Chen and Li [8] obtained the existence of $\alpha$-mild solutions for a class of fractional stochastic integro-differential evolution equations with nonlocal initial conditions in a real separable Hilbert space by using a new strategy which relies on the compactness of the operator semigroup, Schauder fixed point theorem and approximating techniques in 2015. In 2016, by using the concept of $\alpha$-order fractional solution operator and $\alpha$-resolvent family combined with fractional calculations, Schauder fixed point theorem and stochastic analysis theory, Chen et al. [9] obtained the existence of mild solutions for a class of fractional stochastic evolution equations with nonlocal initial conditions under the situation that the nonlinear term satisfies some appropriate growth conditions and the $\alpha$-order fractional solution operator is compact. In 2017, by establishing a sufficient condition for judging the relative compactness of a class of abstract continuous family of functions on infinite intervals, Chen et al. [6] obtained the global existence, uniqueness and asymptotic stability of mild solutions for a class of semilinear evolution equations with nonlocal initial conditions on infinite interval by using stochastic analysis theory, analytic semigroup theory, relevant fixed point theory and the well known Gronwall-Bellman type inequality.

We notice that among the previous researches, most of researchers focus on the case that the differential operators in the main parts are independent of time $t$, which means that the problems under consideration are autonomous. However, when treating some parabolic evolution equations, it is usually assumed that the partial differential operators depend on time $t$ on account of this class of operators appears frequently in the applications, for the details please see [1], [2], [3], [10], [11], [12], [20], [24] and [30]. Therefore, it is interesting and significant to investigate stochastic non-autonomous evolution equations with nonlocal initial conditions, i.e., the differential operators in the main parts of the considered problems are dependent of time $t$. Motivated by the above mentioned aspects, in this paper, we will investigate the existence of mild solutions for the non-autonomous stochastic evolution equations of parabolic type with nonlocal initial conditions (1).

We point out that the work of this paper is the following two wedges: on the one hand, to the best of the author’s knowledge, all the existing articles used various methods to study autonomous stochastic evolution equations, i.e., the differential operators in the main parts of the considered problems are independent of time $t$, but for the case that the corresponding differential operators in the main parts are dependent of time...
t, we have not seen the relevant papers to study non-autonomous stochastic evolution equations with nonlocal initial conditions. In order to fill this gap, we are concerned with the existence of mild solutions for NSEE (1) in this paper; on the other hand, we notice that non-autonomous evolution equations have been extensively studied in recent years using various fixed point theorems when the corresponding evolution family is compact, see for example [3, 20, 24, 30], this is very convenient to the equations with compact resolvent. But for the case that the corresponding evolution family is noncompact, we have not seen the relevant papers to study non-autonomous stochastic evolution equations of parabolic type with nonlocal initial conditions. Therefore, inspired by the previous works, we are devoted to studying the existence of mild solution for NSEE (1) under the situation that the corresponding evolution family is noncompact in this paper.

2. Preliminaries. We begin with this section by giving some notations. Through out this paper, denote \( J = [0, a] \), where \( a > 0 \) is a constant. Let \( \mathbb{H} \) and \( \mathbb{K} \) be two real separable Hilbert spaces, we denote by \( (\cdot, \cdot) \) and \( (\cdot, \cdot)_\mathbb{K} \) their inner products, and by \( \| \cdot \| \) and \( \| \cdot \|_\mathbb{K} \) their vector norms, respectively. We denote by \( \mathcal{L}(\mathbb{H}) \) the Banach space of all linear and bounded operators on \( \mathbb{H} \) endowed with the topology defined by operator norm. Let \( L^1(J, \mathbb{H}) \) be the Banach space of all \( \mathbb{H} \)-valued Bochner integrable functions defined on \( J \) with the norm \( \| u \|_{L^1} = \int_0^a \| u(t) \| dt \). In this paper, we assume that \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) is a complete filtered probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets of \( \mathcal{F} \). Let \( \{e_k, k \in \mathbb{N}\} \) be a complete orthonormal basis of \( \mathbb{K} \). Suppose that \( \{\mathcal{W}(t) : t \geq 0\} \) is a cylindrical \( \mathbb{K} \)-valued Wiener process defined on the probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) with a finite trace nuclear covariance operator \( Q \geq 0 \), denote \( \text{Tr}(Q) = \sum_{k=1}^\infty \lambda_k = \lambda < \infty \), which satisfies that \( Q e_k = \lambda_k e_k, k \in \mathbb{N} \). Let \( \{\mathcal{W}_k(t), k \in \mathbb{N}\} \) be a sequence of one-dimensional standard Wiener processes mutually independent on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) such that

\[
\mathcal{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \mathcal{W}_k(t) e_k.
\]

We further assume that \( \mathcal{F}_t = \sigma(\mathcal{W}(s), 0 \leq s \leq t) \) is the \( \sigma \)-algebra generated by \( \mathcal{W} \) and \( \mathcal{F}_a = \mathcal{F} \). For \( \varphi, \psi \in \mathcal{L}(\mathbb{K}, \mathbb{H}) \), we define \( (\varphi, \psi) = \text{Tr}(\varphi Q \psi^*) \), where \( \psi^* \) is the adjoint of the operator \( \psi \). Clearly, for any bounded operator \( \psi \in \mathcal{L}(\mathbb{K}, \mathbb{H}) \),

\[
\| \psi \|^2_Q = \text{Tr}(\psi Q \psi^*) = \sum_{k=1}^{\infty} \| \sqrt{\lambda_k} \psi e_k \|^2.
\]

If \( \| \psi \|^2_Q < \infty \), then \( \psi \) is called a \( Q \)-Hilbert-Schmidt operator.

The collection of all strongly-measurable, square-integrable \( \mathbb{H} \)-valued random variables, denoted \( L^2(\Omega, \mathbb{H}) \), which is a Banach space equipped with the norm \( \| u(\cdot) \|_{L^2} = (\mathbb{E}\| u(\cdot, \mathcal{W}) \|^2)^{\frac{1}{2}} \), where the expectation \( \mathbb{E} \) is defined by \( \mathbb{E}u = \int_{\Omega} u(\mathcal{W}) d\mathbb{P} \). An important subspace of \( L^2(\Omega, \mathbb{H}) \) is given by

\[
L^2_0(\Omega, \mathbb{H}) = \{u \in L^2(\Omega, \mathbb{H}) \mid u \text{ is } \mathcal{F}_0 \text{-measurable}\}.
\]

We denote by \( C(J, L^2(\Omega, \mathbb{H})) \) the space of all continuous \( \mathcal{F}_t \)-adapted measurable processes from \( J \) to \( L^2(\Omega, \mathbb{H}) \) satisfying \( \sup_{t \in J} \mathbb{E}\| u(t) \|^2 < \infty \). Then it is easy to see that
$C(J, L^2(\Omega, \mathbb{H}))$ is a Banach space endowed with the supnorm
\[ \|u\|_C = \left( \sup_{t \in J} \mathbb{E}\|u(t)\|^2 \right)^{\frac{1}{2}}. \]

For any constant $r > 0$, let
\[ B_r = \{ u \in C(J, L^2(\Omega, \mathbb{H})) : \|u\|_C^2 \leq r \}. \]
Clearly, $B_r$ is a bounded closed convex set in $C(J, L^2(\Omega, \mathbb{H}))$.

By [15, Proposition 2.8], we have the following result which will be used throughout this paper.

**Lemma 2.1.** If $g : J \times L^2(\Omega, \mathbb{H}) \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ is continuous and $u \in C(J, L^2(\Omega, \mathbb{H}))$, then
\[ \mathbb{E}\left\| \int_0^a g(t, u(t))dW(t) \right\|^2 \leq \text{Tr}(Q) \int_0^a \mathbb{E}\|g(t, u(t))\|^2 dt. \]

Throughout the paper, we assume that $\{A(t) : 0 \leq t \leq a\}$ is a family of closed and densely defined operator on Hilbert space $\mathbb{H}$, which satisfies the following well-known “Acquistapace-Terreni condition”.

(A1) There exist constants $\lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi)$ and $M_1 \geq 1$ such that $\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0 I)$ and for all $\lambda \in \Sigma_\theta \cup \{0\}$ and $t \in J$,
\[ \|R(\lambda, A(t) - \lambda_0 I)\|_{\mathcal{L}(\mathbb{H})} \leq \frac{M_1}{1 + |\lambda|}; \]

(A2) There exist constants $M_2 > 0$ and $\theta, \beta \in (0, 1]$ with $\theta + \beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $0 \leq s \leq t \leq a$,
\[ \|(A(t) - \lambda_0 I)R(\lambda, A(t) - \lambda_0 I)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \frac{M_2|t - s|^\theta}{|\lambda|^\beta}, \]
where
\[ \Sigma_\theta = \{ \lambda \in \mathbb{C} \setminus \{0\} : \arg|\lambda| \leq \theta \}. \]

Conditions (A1) and (A2), which are initiated by Acquistapace and Terreni in [2] and Acquistapace in [1] for $\lambda_0 = 0$, are well understood and widely used in the literature. Under the above conditions (A1) and (A2), the family $\{A(t) : 0 \leq t \leq a\}$ generates a unique linear evolution system, or called linear evolution family, $\{U(t, s) : 0 \leq s \leq t \leq a\}$. Furthermore, by an obvious rescaling from [1, Theorem 2.3] and [2, Theorem 2.1] combined with the Acquistapace and Terreni conditions (A1) and (A2) one gets the following properties for the family of linear operator $\{U(t, s) : 0 \leq s \leq t \leq a\}$.

**Lemma 2.2.** The family of the linear operator $\{U(t, s) : 0 \leq s \leq t \leq a\}$ satisfies the following properties:

(i) $U(t, r)U(r, s) = U(t, s)$, $U(t, t) = I$ for $0 \leq s \leq r \leq t \leq a$;

(ii) The map $(t, s) \mapsto U(t, s)x$ is continuous for all $x \in \mathbb{H}$ and $0 \leq s \leq t \leq a$;

(iii) $U(\cdot, s) \in C^1((s, \infty), \mathcal{L}(\mathbb{H}))$, $\frac{\partial U(t, s)}{\partial t} = A(t)U(t, s)$ for $t > s$, and $\|A^k(t)U(t, s)\| \leq M(t - s)^{-k}$ for $0 < s < t \leq 1$ and $k = 0, 1$;

(iv) $\frac{\partial U(t, s)}{\partial s} = -U(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in D(A(s))$.

From the property (iii) we know that
\[ \|U(t, s)\|_{\mathcal{L}(\mathbb{H})} \leq M \quad \text{for} \quad 0 \leq s \leq t \leq a. \]

In (2) and property (iii), $M > 0$ is a constant.
Definition 2.3. ([27]) An evolution family \{U(t, s) : 0 \leq s \leq t \leq a\} is said to be equicontinuous if the function \( t \mapsto U(t, s) \) is continuous by operator norm for \( t \in (s, +\infty) \).

Remark 1. If the domain \( D(A(t)) = D \) of \( \{A(t) : 0 \leq t \leq a\} \) is dense in \( \mathbb{H} \) and independent of \( t \), then the evolution family \( \{U(t, s) : 0 \leq s \leq t \leq a\} \) is equicontinuous when the conditions (AT1) and (AT2) are satisfied.

Definition 2.4. An \( \mathcal{F}_t \)-adapted stochastic process \( u : J \to \mathbb{H} \) is called a mild solution of NSEE (1) if \( u(t) \in \mathbb{H} \) has càdlàg paths on \( t \in J \) almost surely and for each \( t \in J \), \( u(t) \) \( \mathbb{P} \)-almost surely satisfies the integral equation

\[
  u(t) = U(t, 0)H(u) + \int_0^t U(t, s)f(s, u(s))ds + \int_0^t U(t, s)g(s, u(s))d\mathbb{W}(s).
\]

Next, we introduce some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 2.5. ([4, 17]) The Kuratowski measure of noncompactness \( \alpha(\cdot) \) defined on bounded set \( S \) of Banach space \( E \) is

\[
  \alpha(S) := \inf\{\delta > 0 : S = \bigcup_{i=1}^m S_i \text{ and } \text{diam}(S_i) \leq \delta \text{ for } i = 1, 2, \ldots, m\}.
\]

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.6. ([4, 17]) Let \( E \) be a Banach space and \( S, U \in E \) be bounded. The following properties are satisfied:

(i) \( \alpha(S) = 0 \) if and only if \( \overline{S} \) is compact, where \( \overline{S} \) means the closure hull of \( S \);

(ii) \( \alpha(S) = \alpha(\overline{S}) = \alpha(\text{conv} S) \), where \( \text{conv} S \) means the convex hull of \( S \);

(iii) \( \alpha(\kappa S) = |\kappa|\alpha(S) \) for any \( \kappa \in \mathbb{R} \);

(iv) \( S \cup U \) implies \( \alpha(S) \leq \alpha(U) \);

(v) \( \alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\} \);

(vi) \( \alpha(S + U) \leq \alpha(S) + \alpha(U) \), where \( S + U = \{x + y : x \in S, y \in U\} \);

(vii) If the map \( Q : \mathcal{D}(Q) \subset E \to X \) is Lipschitz continuous with constant \( k \), then \( \alpha(Q(V)) \leq k\alpha(V) \) for any bounded subset \( V \subset \mathcal{D}(Q) \), where \( X \) is another Banach space.

In order to introduce the useful lemmas which will be used in our argument, we denote by \( E \) a Banach space and \( C(J, E) \) the Banach space of all continuous \( E \)-valued functions on interval \( J \) in the sequel of this section. We use \( \overline{\pi}(\cdot) \) and \( \overline{\pi}_C(\cdot) \) to denote the Kuratowski measure of noncompactness on the bounded set of \( E \) and \( C(J, E) \), respectively. For any \( D \subset C(J, E) \) and \( t \in J \), set \( D(t) = \{u(t) : u \in D\} \) then \( D(t) \subset E \).

If \( D \subset C(J, E) \) is bounded, then \( D(t) \) is bounded in \( E \) and \( \overline{\pi}(D(t)) \leq \overline{\pi}_C(D) \). For more details about the properties of the Kuratowski measure of noncompactness, we refer to the monographs [4] and [17].

Lemma 2.7. ([4]) Let \( D \subset C(J, E) \) be bounded and equicontinuous. Then \( \overline{\pi}(D(t)) \) is continuous on \( J \), and \( \overline{\pi}_C(D) = \max_{t \in J} \overline{\pi}(D(t)) \).

Lemma 2.8. ([7]) Let \( E \) be a Banach space, and let \( D \subset E \) be bounded. Then there exists a countable set \( D_0 \subset D \), such that \( \overline{\pi}(D) \leq 2\overline{\pi}(D_0) \).

Lemma 2.9. ([21]) Let \( E \) be a Banach space. If \( D = \{u_n\}_{n=1}^\infty \subset C(J, E) \) is a countable set and there exists a function \( m \in L^1(J, \mathbb{R}^+) \) such that for every \( n \in \mathbb{N} \)

\[
  \|u_n(t)\| \leq m(t), \quad \text{a.e. } t \in J.
\]
Proof. Consider the operator \( Q : S \to E \) is called to be condensing if for every bounded set \( D \subset S \),
\[
\nu(Q(D)) < \nu(D).
\]

Definition 2.10. ([17]) Let \( S \) be a nonempty subset of \( E \). A continuous mapping \( Q : S \to E \) is called to be condensing if for every bounded set \( D \subset S \),
\[
\nu(Q(D)) < \nu(D).
\]

Lemma 2.11. (Sadovskii’s fixed point theorem [17]) Let \( E \) be a Banach space. Assume that \( D \subset E \) is a bounded closed and convex set on \( E \) and \( Q : D \to D \) is a condensing operator. Then \( Q \) has at least one fixed point in \( D \).

In the sequel of this paper, we denote by \( \alpha(\cdot) \) and \( \alpha_C(\cdot) \) the Kuratowski measure of noncompactness on the bounded set of \( L^2(\Omega, \mathbb{H}) \) and \( C(J, L^2(\Omega, \mathbb{H})) \), respectively.

3. Main results. In this section, we will state and prove the main result in this paper.

Theorem 3.1. Assume that the evolution family \( \{U(t,s) : 0 \leq s \leq t \leq a\} \) generated by \( \{A(t) : 0 \leq t \leq a\} \) is equicontinuous, the nonlinear functions \( f : J \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H}) \), \( g : J \times L^2(\Omega, \mathbb{H}) \to \mathcal{L}(\mathbb{K}, \mathbb{H}) \) and nonlocal function \( H : C(J, L^2(\Omega, \mathbb{H})) \to L^2(\Omega, \mathbb{H}) \) are continuous. If the following assumptions

\( (P1) \) For some \( r > 0 \), there exist positive constants \( \rho_1, \rho_2 \) and functions \( \varphi_r, \psi_r \in L(J, \mathbb{R}_+) \) such that for all \( u \in L^2(\Omega, \mathbb{H}) \) satisfying \( \nu(u(t)) \leq r \) and a.e. \( t \in J \),
\[
E||f(t, u)||^2 \leq \varphi_r(t) \quad \text{and} \quad \liminf_{r \to +\infty} \frac{||\varphi_r||_{L(J, \mathbb{R}_+)}}{r} := \rho_1 < +\infty,
\]
\[
E||g(t, u)||^2 \leq \psi_r(t) \quad \text{and} \quad \liminf_{r \to +\infty} \frac{||\psi_r||_{L(J, \mathbb{R}_+)}}{r} := \rho_2 < +\infty,
\]

\( (P2) \) There exist a constant \( \rho_3 > 0 \) and a nondecreasing continuous function \( \Psi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for some \( r > 0 \) and all \( u \in B_r \) \( (B_r \text{ is given in Preliminaries}) \),
\[
E||H(u)||^2 \leq \Psi(r) \quad \text{and} \quad \liminf_{r \to +\infty} \frac{\Psi(r)}{r} := \rho_3 < +\infty,
\]

\( (P3) \) There exist positive constants \( L_f, L_g \) and \( L_H \) satisfying
\[
2(ML_H + 2aML_f + \sqrt{2a Tr(Q)}ML_g) < 1,
\]
such that for any \( t \in J \), bounded and countable set \( D \subset \mathbb{H} \) and \( \mathcal{D} \subset C(J, L^2(\Omega, \mathbb{H})) \)
\[
\alpha(f(t, D)) \leq L_f \alpha(D), \quad \alpha(g(t, D)) \leq L_g \alpha(D), \quad \alpha(H(D)) \leq L_H \alpha_C(D),
\]
hold, then NSEE (1) has at least one mild solution on \( J \) provided that
\[
3M^2(a \rho_1 + \rho_3 + Tr(Q) \rho_2) < 1. \tag{3}
\]
Proof. Consider the operator \( Q : C(J, L^2(\Omega, \mathbb{H})) \to C(J, L^2(\Omega, \mathbb{H})) \) defined by
\[
(Qu)(t) = U(t, 0)H(u) + \int_0^t U(t, s)f(s, u(s))ds + \int_0^t U(t, s)g(s, u(s))d\mathcal{W}(s), \quad t \in J. \tag{4}
\]
From Definition 2.4, it is easy to see that the mild solution of NSEE (1) on \( J \) is equivalent to the fixed point of the operator \( Q \) defined by (4). In what follows, we will prove that the operator \( Q \) has at least one fixed point by applying the famous Sadovskii’s fixed point theorem.
Therefore, we prove that there exists a positive constant R such that the operator Q defined by (4) maps the set $B_R$ into itself. If this is not true, then there would exist $t_r \in J$ and $u_r \in B_r$ such that $E\langle (Q u_r)(t_r) \rangle > r$ for each $r > 0$. By Lemma 2.1, Hölder’s inequality, (2), (4) and the assumptions (P1) and (P2), we get that

$$r < E\langle (Q u_r)(t_r) \rangle \leq 3M^2 E\|H(u)\|^2 + 3M^2 t_r \int_0^{t_r} E\|f(s, u_r(s))\|^2 ds$$

$$+ 3 \text{Tr}(Q) M^2 \int_0^{t_r} E\|g(s, u_r(s))\|^2 ds$$

$$\leq 3M^2 \Psi(r) + 3M^2 a \|\varphi_r\|_{L(J, R^+)} + 3 \text{Tr}(Q) M^2 \|\psi_r\|_{L(J, R^+)}.$$  \hspace{1cm} (5)

Dividing both side of (5) by $r$ and taking the lower limit as $r \to +\infty$, combined with the assumption (3) we get that

$$1 \leq 3M^2 (a \rho_1 + \rho_3 + \text{Tr}(Q) \rho_2) < 1,$$

which is a contradiction. Therefore, we have proved that $Q : B_R \to B_R$.

Secondly, we prove that the operator $Q : B_R \to B_R$ is continuous. To this end, let the sequence $\{u_n\}_{n=1}^{\infty} \subset B_R$ such that $\lim_{n \to +\infty} u_n = u$ in $B_R$. By the continuity of the nonlinear functions $f$, $g$ and nonlocal function $H$, we know that when $n \to +\infty$

$$E\|f(s, u_n(s)) - f(s, u(s))\|^2 \to 0 \quad \text{a.e. } s \in J,$$  \hspace{1cm} (6)

$$E\|g(s, u_n(s)) - g(s, u(s))\|^2 \to 0 \quad \text{a.e. } s \in J$$  \hspace{1cm} (7)

and

$$E\|H(u_n) - H(u)\|^2 \to 0.$$  \hspace{1cm} (8)

Furthermore, from the assumption (P1), we get that for a.e. $s \in J$,

$$E\|f(s, u_n(s)) - f(s, u(s))\|^2 \leq 2E\|f(s, u_n(s))\|^2 + 2E\|f(s, u(s))\|^2 \leq 4\varphi_R(s)$$  \hspace{1cm} (9)

and

$$E\|g(s, u_n(s)) - g(s, u(s))\|^2 \leq 2E\|g(s, u_n(s))\|^2 + 2E\|g(s, u(s))\|^2 \leq 4\psi_R(s).$$  \hspace{1cm} (10)

Using the fact that the functions $s \to 4\varphi_R(s)$ and $s \to 4\psi_R(s)$ are Lebesgue integrable for a.e. $s \in [0, t]$ and every $t \in J$, combined with Lemma 2.1, (2), (4), (6)-(10) and the Lebesgue dominated convergence theorem, we know that

$$E\| (Qu_n)(t) - (Qu)(t) \|^2 \leq 3M^2 E\|H(u_n) - H(u)\|^2$$

$$+ 3M^2 \int_0^t E\|f(s, u_n(s)) - f(s, u(s))\|^2 ds$$

$$+ 3 \text{Tr}(Q) M^2 \int_0^t E\|g(s, u_n(s)) - g(s, u(s))\|^2 ds$$

$$\to 0 \quad \text{as } n \to \infty,$$

which means that

$$\| (Qu_n) - (Qu) \|_C = \left( \sup_{t \in J} E\| (Qu_n)(t) - (Qu)(t) \|^2 \right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$  

Therefore, $Q : B_R \to B_R$ is a continuous operator.
Now, we are in the position to demonstrate that \( \{Qu : u \in B_R \} \) is a family of equicontinuous functions in \( C(J, L^2(\Omega, \mathbb{H})) \). For any \( u \in B_R \) and \( 0 \leq t_1 < t_2 \leq a \), by Lemma 2.1, (4) and the assumption (P1), we have

\[
E \| (Qu)(t_2) - (Qu)(t_1) \|^2 \leq 5E \| U(t_2, 0)H(u) - U(t_1, 0)H(u) \|^2 + 5E \int_0^{t_1} \| U(t_2, s) - U(t_1, s) \| f(s, u(s))ds + 5E \int_{t_1}^{t_2} U(t_2, s)f(s, u(s))ds + 5E \int_0^{t_1} U(t_2, s)g(s, u(s))dW(s) + 5E \int_{t_1}^{t_2} U(t_2, s)g(s, u(s))dW(s)
\]

\[
\leq 5 \| U(t_2, 0) - U(t_1, 0) \| H(u) + 5t_1 \| U(t_2, s) - U(t_1, s) \| \varphi_R(s)ds + 5M^2(t_2 - t_1) \| \varphi_R(s)ds + 5Tr(Q) \int_0^{t_1} \| U(t_2, s) - U(t_1, s) \| \psi_R(s)ds + 5Tr(Q)M^2 \int_{t_1}^{t_2} \psi_R(s)ds
\]

\[
:= I_1 + I_2 + I_3 + I_4 + I_5.
\]

In order to prove that \( E \| (Qu)(t_2) - (Qu)(t_1) \|^2 \to 0 \) as \( t_2 - t_1 \to 0 \), we only need to check \( I_i \to 0 \) independently of \( u \in B_R \) when \( t_2 - t_1 \to 0 \) for \( i = 1, 2, \cdots, 5 \).

For \( I_1 \), by the fact that linear evolution family \( \{U(t, s) : 0 \leq s \leq t \leq a \} \) is strongly continuous one can easily obtain that \( I_1 \to 0 \) as \( t_2 - t_1 \to 0 \).

For \( t_1 = 0, 0 < t_2 \leq a \), it is easy to see that \( I_2 = I_4 = 0 \). For \( 0 < t_1 < a \) and arbitrary \( 0 < \delta < t_1 \), by Lemma 2.2, Definition 2.3, the assumption (P1), (2) and the arbitrariness of \( \delta \), we get that

\[
I_2 \leq 5t_1 \int_0^{t_1-\delta} \| U(t_2, s) - U(t_1, s) \| ^2 \varphi_R(s)ds + 5t_1 \int_{t_1-\delta}^{t_1} \| U(t_2, s) - U(t_1, s) \| ^2 \varphi_R(s)ds \leq \sup_{t_1} \| U(t_2, s) - U(t_1, s) \|^2 \cdot 5t_1 \int_0^{t_1-\delta} \varphi_R(s)ds + 10M^2t_1 \int_{t_1-\delta}^{t_1} \varphi_R(s)ds \to 0 \text{ as } t_2 - t_1 \to 0 \text{ and } \delta \to 0,
\]

and

\[
I_4 \leq 5Tr(Q) \int_0^{t_1-\delta} \| U(t_2, s) - U(t_1, s) \|^2 \psi_R(s)ds + 5Tr(Q) \int_{t_1-\delta}^{t_1} \| U(t_2, s) - U(t_1, s) \|^2 \psi_R(s)ds
\]
Therefore, by Lemma 2.9, (2), (4), (11), (13) and the assumption (P3), we get that
\[ I \leq \sup_{s \in [0,t_1-\delta]} \|U(t_2,s) - U(t_1,s)\|_{L^2(\mathbb{H})}^2 : 5T(Q) \int_0^{t_1-\delta} \psi_R(s) \, ds + 10M^2T(Q) \int_{t_1-\delta}^{t_1} \psi_R(s) \, ds \]
\[ \rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0 \quad \text{and} \quad \delta \rightarrow 0. \]
For \( I_3 \) and \( I_5 \), by the assumption (P1), we get that
\[ I_3 = 5M^2(t_2 - t_1) \int_{t_1}^{t_2} \varphi_R(s) \, ds \rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0, \]
and
\[ I_5 = 5M^2T(Q) \int_{t_1}^{t_2} \psi_R(s) \, ds \rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0. \]
As a result, we have proved that \( \mathbb{E}[(Q(u(t_2)) - (Q(u))(t_1)]^2 \rightarrow 0 \) independently of \( u \in B_R \) as \( t_2 - t_1 \rightarrow 0 \), which means that the operator \( Q : B_R \rightarrow B_R \) is equicontinuous.

In what follows, we will prove that \( \mathbb{Q} : B_R \rightarrow B_R \) is a condensing operator. For any \( D \subset B_R \), by Lemma 2.8, there exists a countable set \( D_0 = \{ u_n \} \subset D \) such that
\[ \alpha_C(\mathbb{Q}(D)) \leq 2\alpha_C(\mathbb{Q}(D_0)). \quad (11) \]
Since \( \mathbb{Q}(D_0) \subset \mathbb{Q}(B_R) \) is equicontinuous, we get from Lemma 2.7 that
\[ \alpha_C(\mathbb{Q}(D_0)) = \max_{t \in J} \alpha(\mathbb{Q}(D_0)(t)). \quad (12) \]
For any \( u_1, u_2 \in D_0 \), we know from Lemma 2.1 that
\[ \mathbb{E}\left\| \int_0^t U(t,s)g(s,u_1(s)) \, d\mathbb{W}(s) - \int_0^t U(t,s)g(s,u_2(s)) \, d\mathbb{W}(s) \right\|^2 \]
\[ = \mathbb{E}\left\| \int_0^t U(t,s)[g(s,u_1(s)) - g(s,u_2(s))] \, d\mathbb{W}(s) \right\|^2 \]
\[ \leq M^2T(Q) \int_0^t \mathbb{E}\|g(s,u_1(s)) - g(s,u_2(s))\|^2 \, ds. \]
The above inequality combined with Lemma 2.6 (vii), Lemma 2.9 and the fact that the norm of Banach space \( L^2(\Omega, \mathbb{H}) \) is \( \|u(\cdot)\|_{L^2} = (\mathbb{E}||u(\cdot, \mathbb{W})||^2)^{\frac{1}{2}} \), one gets that
\[ \alpha \left( \int_0^t U(t,s)g(s,D_0(s)) \, d\mathbb{W}(s) \right) \leq M \left( 2T(Q) \int_0^t \alpha(\mathbb{Q}(D_0(s)))^2 \, ds \right)^{\frac{1}{2}}. \quad (13) \]
Therefore, by Lemma 2.9, (2), (4), (11), (13) and the assumption (P3), we get that
\[ \alpha(\mathbb{Q}(D_0)(t)) \leq \alpha(\{U(t,0)H(u_n)\}) + \alpha \left( \left\{ \int_0^t U(t,s)f(s,u_n(s)) \, ds \right\} \right) \]
\[ + \alpha \left( \left\{ \int_0^t U(t-s)g(s,u_n(s)) \, d\mathbb{W}(s) \right\} \right) \]
\[ \leq MLH\alpha_C(D_0) + 2M \int_0^t \alpha \left( \{f(s,u_n(s))\} \right) \, ds \]
Hence, from (11), (12), (14) and the assumption (P3), we get that
\[ \alpha_C(Q(D)) \leq 2 \left( M L_H + 2 a ML_f + \sqrt{2 a \text{Tr}(Q) ML_g} \right) \alpha_C(D) < \alpha_C(D). \] (14)

Therefore, \( Q : B_R \to B_R \) is a condensing operator. It follows from Lemma 2.11 that the operator \( Q \) has at least one fixed point \( u \in B_R \), which is just a mild solution of NSEE (1). This completes the proof of Theorem 3.1.

\[ \square \]

**Remark 2.** Theorem 3.1 can be applied to a class of nonautonomous stochastic partial differential equations of evolution type with nonlocal initial conditions, in which the corresponding evolution family is not compact. Therefore, Theorem 3.1 in this paper is supplement to the papers [8], [9], [6] and [14]. This distinguishes the present paper from earlier works on stochastic evolution equations with nonlocal initial conditions.

**Remark 3.** As the readers can see, Theorem 3.1 extends the studying of the papers [24] and [30], which investigate the solvability of non-autonomous parabolic evolution equations with nonlocal initial conditions, to the case of stochastic non-autonomous evolution equations with nonlocal initial conditions and noncompact evolution family.

### 4. An application.

In order to illustrate the applicability of our main result, we consider the following non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= A(x,t,D)u(x,t) + \frac{\cos(\pi t)}{10 + |u(x,t)|} + \frac{\sin(x,t,u(x,t))dW(t)}{10e^{at}}, \quad x \in \Theta, \ t \in J, \\
D^\alpha u(x,t) &= 0, \quad (x,t) \in \partial\Theta \times J, \ |\alpha| \leq n, \\
u(x,0) &= \int_0^x K(s)ds \frac{1}{1 + |u(x,s)|}, \quad x \in \Theta,
\end{aligned}
\] (15)

where \( \Theta \subset \mathbb{R}^N (N \geq 1) \) is a bounded domain with smooth boundary \( \partial\Theta \), \( n \in \mathbb{Z}^+ \), \( J = [0,a] \) is a constant, \( W(t) \) denotes a one-dimensional standard cylindrical Wiener process with a finite trace nuclear covariance operator \( Q \geq 0 \) defined on a filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \):

\[ A(x,t,D)u = \sum_{|\alpha| \leq 2n} a_\alpha(x,t)D^\alpha u \]

are uniformly strongly elliptic operators in \( \Theta \), i.e., there exist a constant \( C_1 > 0 \) such that for every \( x \in \Theta, \ t \in J \) and \( \xi \in \mathbb{R}^N \),

\[ (-1)^n \text{Re} \sum_{|\alpha| = 2n} a_\alpha(x,t)\xi^\alpha \geq C_1 \xi^{2n}; \]
the coefficients $a_\alpha(\cdot, t) \in C^{2n}(\overline{\Theta})$ for $t \in J$ and $a_\alpha(\cdot, t): J \rightarrow \mathbb{R}$ are uniformly Hölder continuous, i.e., there exist constants $C_2 > 0$ and $0 < \eta \leq 1$ such that for every $x \in \overline{\Theta}$, $t, s \in J$ and $|\alpha| \leq 2n$,

$$|a_\alpha(x, t) - a_\alpha(x, s)| \leq C_2|t - s|^\eta;$$

$K \in L(J, \mathbb{R}^+)$

Let $H = L^2(\Theta, \mathbb{R})$. Then $H$ is a Hilbert space with the norm $\| \cdot \|_2$ and inner product $(\cdot, \cdot)$. Consider the operator $A(t)$ on $H$ defined by

$$A(t)u(x) = A(x, t, D)u(x), \quad D(A(t)) = H^{2n}(\Theta) \cap H^6_0(\Theta).$$

It follows from [27, Lemma 6.1 in Chapter 7] that there are constants $\theta \in (\frac{\pi}{2}, \pi)$ and $M_1 \geq 0$ such that $A(t)$ satisfy the condition $(\text{AT}_1)$. Furthermore, by again [27, Lemma 6.1 in Chapter 7] together with Hölder continuity of coefficients $a_\alpha(x, t)$ one know that there exist constants $M_2 > 0$ and $\vartheta, \beta \in (0, 1]$ with $\vartheta + \beta > 1$ such that for all $\lambda \in \Sigma_a$ and $0 \leq s \leq t \leq a$, the condition $(\text{AT}_2)$ is satisfied. Therefore, the family $\{A(t): 0 \leq t \leq a\}$ generates an equicontinuous evolution family $\{U(t, s): 0 \leq s \leq t \leq a\}$. Denote

$$M := \sup_{0 \leq s \leq t \leq a} \|U(t, s)\|_{L(H)}.$$

For every $t \in J$, denote

$$u(t) = u(\cdot, t), \quad f(t, u(t)) = \frac{\cos(\pi t)}{10 + |u(\cdot, t)|},$$

$$g(t, u(t)) = \frac{\sin(x, t, u(x, t))}{10e^t}, \quad H(u) = \int_0^a K(s)ds$$

Then the non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions (15) can be rewritten into the abstract form of NSEE (1) in $L^2(\Theta, \mathbb{R})$.

**Theorem 4.1.** Assume that $M(a + 5\sqrt{2a Tr(Q)}) < 25$. Then the non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions (15) has at least one mild solution $u \in C(J, L^2(\Omega, L^2(\Theta, \mathbb{R})))$.

**Proof.** From the definition of nonlinear functions $f, g$ and nonlocal function $H$, we can easily to verify that the assumptions (P1) and (P2) hold with

$$\varphi_r(t) = \frac{\text{mes}(\Theta) \cos^2(\pi t)}{100}, \quad \psi_r(t) = \frac{\text{mes}(\Theta)e^{-2t}}{100},$$

$$\Psi(r) = \text{mes}(\Theta)(\int_0^a K(s)ds)^2, \quad \rho_1 = \rho_2 = \rho_3 = 0. \quad (16)$$

From (16) one can easily to verify that the condition (3) is satisfied. Furthermore, from the definition of nonlinear functions $f$ and $g$, we know that $f(t, u)$ and $g(t, u)$ is Lipschitz continuous about the variable $u$ with Lipschitz constants $k_f = \frac{1}{100}$ and $k_g = \frac{1}{10}$, respectively. In addition, From [23] and some basic analysis we know that the nonlocal term $H$ is a compact operator. Therefore, by Lemma 2.6 and the assumption $M(a + 5\sqrt{2a Tr(Q)}) < 25$ we know that the assumption (P3) is satisfied with constants

$$L_f = \frac{1}{100}, \quad L_g = \frac{1}{10}, \quad L_H = 0.$$ 

Therefore, all the assumptions of Theorem 3.1 are satisfied. Hence, the non-autonomous stochastic partial differential equation of parabolic type with nonlocal initial conditions (15) has at least one mild solution $u \in C(J, L^2(\Omega, L^2(\Theta, \mathbb{R})))$ due to Theorem 3.1. This completes the proof of Theorem 4.1.
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