AN EXPLICIT VAN DER CORPUT ESTIMATE FOR $\zeta(1/2 + it)$

GHAITH A. HIARY

Abstract. An explicit estimate for the Riemann zeta function on the critical line is derived using the van der Corput method. An explicit van der Corput lemma is presented.

1. Introduction

The Riemann zeta function is defined for $s = \sigma + it$ by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \sigma > 1.$$  \hfill (1)

It can be analytically continued everywhere except for a simple pole at $s = 1$. A well-known problem in number theory is to bound the growth rate of zeta on the critical line $\sigma = 1/2$. This problem has led to deep ideas in the theory of exponential sums. In particular, the method of exponent pairs (see [6], [14, page 116]), and the Bombieri-Iwaniec method [1, 2].

Since $|\zeta(\sigma + it)| = |\zeta(\sigma - it)|$, we can suppose that $t \geq 0$. Starting with the series (1), it follows by the Euler-Maclaurin formula that $\zeta(1/2 + it) \ll t^{1/2}$ for large enough $t$. This can be improved substantially by appealing to the Riemann–Siegel formula, which gives $\zeta(1/2 + it) \ll t^{1/4}$. The Weyl-Hardy-Littlewood method (see [14, section 5.3]) detects a certain amount of cancellation in the main sum of the Riemann–Siegel formula, further improving the bound to $\zeta(1/2 + it) \ll t^{1/6} \log^{3/2} t$.

The van der Corput method (e.g. [16, 15]) removes an extra $\sqrt{\log t}$ factor, which sharpens the estimate to

$$\zeta(1/2 + it) \ll t^{1/6} \log t.$$  \hfill (2)

The exponent $1/6$ in estimate (2) is hard to improve. The sharpest result so far is due to Huxley [9], who proved that $\zeta(1/2 + it) \ll t^{32/205} \log^{\gamma} t$ for some constant $\gamma$. Note $32/205 = 0.15609 \ldots$ (See also the recent result in [8].) Assuming the Riemann hypothesis, one can show that $\zeta(1/2 + it) \ll \exp(A \log t / \log \log t)$ for some constant $A$; see [14, §14.4]. Hence, $\zeta(1/2 + it)$ is conjectured to grow slower than any fixed power of $t$.

In this article, we obtain an explicit bound of the van der Corput type (2). That is, we compute constants $C_1$ and $C_2$ such that $|\zeta(1/2 + it)| \leq C_1 t^{1/6} \log t$ for $t \geq C_2$. This is part of our work in progress about subconvexity bounds for zeta. We are also motivated by the following computational considerations where the simplicity of an explicit van der Corput estimate is particularly attractive.

2010 Mathematics Subject Classification. Primary 11Y05.

Key words and phrases. Van der Corput estimate, exponential sums, Riemann zeta function.

Preparation of this material is partially supported by the National Science Foundation under agreements No. DMS-1406190.
Specifically, in numerical tests of the growth rate of \( \zeta(1/2 + it) \), it is necessary to have explicit bounds. For one usually cannot distinguish a small power of \( t \) from a logarithm or a subexponential factor unless \( t \) is prohibitively large. An explicit bound offers an unconditional measure against which one can compare large values of \( |\zeta(1/2 + it)| \) found by special numerical searches, such as those in [7].

Another motivation comes from the algorithms in [8] which employ a Taylor expansion to express zeta as a sum of low degree exponential sums and their derivatives. The size of the remainder term in these algorithms is bounded by the highest order derivative used and the maximal size of a certain subdivision of the main sum of zeta. In this context, an improved and simple method to obtain an explicit bound is of value as it enables reducing the number of derivatives needed to guarantee a given error tolerance. This in turn will improve the running time appreciably.

Explicit bounds for zeta were obtained by Cheng and Graham [4], and recently by Platt and Trudgian [12] who proved that

\[
|\zeta(1/2 + it)| \leq 0.732t^{1/6}\log t, \quad (t \geq 2).
\]

We improve the leading constant in (3) by 14%.

**Theorem 1.1.** If \( 0 \leq t \leq 3 \), then \( |\zeta(1/2 + it)| \leq 1.461 \). If \( t \geq 3 \), then

\[
|\zeta(1/2 + it)| \leq 0.63t^{1/6}\log t.
\]

A numerical computation reveals that

\[
\min_{t \geq 3} \frac{4(t/(2\pi))^{1/4} - 2.08}{t^{1/6}\log t} > 0.541.
\]

Our proof of Theorem 1.1 uses a subdivision of the Riemann–Siegel main sum different than [4] [12], giving rise to a simpler optimization problem. In particular, we divide the main sum into short pieces of length \( \approx t^{1/3} \) (which appears to be a new, and natural, subdivision; c.f. Weyl’s method in [14] Section 5.3) which divides the main sum into pieces of length \( \lesssim t^{1/6} \). This subdivision enables better control of the oscillations in each piece via Lemma 1.2 since the range of \( f'''(x) \) for us will be more restricted.

**Lemma 1.2.** Let \( f(x) \) be a real-valued function with three continuous derivatives on \([N + 1, N + L]\). Suppose there are \( W > 1 \) and \( \lambda \geq 1 \) such that \( \frac{1}{W} \leq |f'''(x)| \leq \frac{1}{W} \) for \( N + 1 \leq x \leq N + L \). If \( \eta > 0 \), then

\[
\left| \sum_{n=N+1}^{N+L} e^{2\pi if(n)} \right|^2 \leq (LW^{-1/3} + \eta)(\alpha L + \beta W^{2/3}),
\]

where

\[
\alpha := \alpha(W, \lambda, \eta) = \frac{1}{\eta} + \frac{64\lambda}{75}\sqrt{\eta + W^{-1/3}} + \frac{\lambda\eta}{W^{1/3}} + \frac{\lambda}{W^{2/3}},
\]

\[
\beta := \beta(W, \eta) = \frac{64}{15}\frac{1}{\sqrt{\eta}} + \frac{3}{W^{1/3}}.
\]
Remarks. Lemma 1.2 is proved in [3]. This lemma is an explicit version of the process \( AB \) in the method of exponent pairs. The condition \( W > 1 \) in the lemma can be relaxed to \( W > 2/\pi \). The number \( \eta \) will be chosen of size \( \approx 1 \) in our application, Theorem 1.1.

The proof of Theorem 1.1 completely overlooks cancellation among the \( \approx t^{1/6} \) pieces where Lemma 1.2 is applied. This is the main source of inefficiency in our proof. As far as we know, there is no definitive mechanism to take advantage of this cancellation. Instead, one works with longer pieces, and tries to prove cancellation within each one.

Of course, the bound (4) is asymptotically far from the truth. And even for moderately large values of \( t \), this bound is still probably a substantial overestimate.

Evidence for this comes from computations by J. W. Bober and the author, some of which are summarized in [7]. In these computations, several hundred large values of zeta were recorded by computing \( \zeta(1/2 + it) \) at certain special points. The largest value found this way was

\[
|\zeta(1/2 + iT_0)| = 16244.86526\ldots,
\]

where \( T_0 = 39246764589894309155251169284104.050622\ldots \), so \( T_0 \approx 3.9 \times 10^{31} \). (To our knowledge, (8) is the largest value of zeta computed so far.) In comparison, the bound (4) gives \( |\zeta(1/2 + iT_0)| \leq 8448744 \), which is about 521 times larger than (8). Alternatively, one can use the van der Corput bound (20) directly. This gives a bound that is about 507 times larger than our computed value. It should be stressed, though, that (8) was found by searching a thin set of special points, not by an exhaustive search, and so only provides a lower bound for \( \max_{t \in [0,T_0]} |\zeta(1/2+it)| \).

2. Proof of Theorem 1.1

For the remainder of the paper, we set

\[
c_0 := 0.63, \quad t_0 := 9.3 \times 10^7.
\]

Proof. We consider \( t \) over three ranges. In the range \( 0 \leq t \leq 200 \), we use the computational bound from Lemma 2.2. In the intermediate range \( 200 \leq t \leq t_0 \) we use the Riemann–Siegel–Lehman bound supplied by Lemma 2.3. And in the last range \( t \geq t_0 \) we use the van der Corput bound supplied by Lemma 2.4.

To handle the intermediate and last ranges of \( t \), we rely on Lemma 2.1, which is consequence of the Riemann–Siegel formula. This lemma requires \( t \geq 200 \), which is the reason we treat the initial range \( t \in [0, 200] \) separately.

Lemma 2.1. If \( t \geq 200 \) and \( n_1 = \lfloor \sqrt{t/(2\pi)} \rfloor \), then

\[
|\zeta(1/2 + it)| \leq 2 \sum_{n=1}^{n_1} n^{-1/2+it} + \mathcal{R}(t).
\]

where \( \mathcal{R}(t) := 1.48t^{-1/4} + 0.127t^{-3/4} \).

Proof. The lemma gives a small improvement on [12, Lemma 3]. The improvement is that the inequality for \( \mathcal{R}(t) \) is tighter if \( t \geq 200 \). The lemma is stated separately for emphasis, as it is an essential first step in all that follows.
follows from the inequality

$$\beta < n$$

where $n_1 = \lceil \sqrt{t}/2\pi \rceil$, $|R_0(t)| < 0.127t^{-3/4}$ for $t \geq 200$, and

$$\beta = \max_{0 \leq \phi \leq 1} \left| \frac{\cos \frac{3}{2}(\phi^2 + 3/4)}{\cos \pi \phi} \right|.$$ 

By [5] page 65, we have $\beta < 0.93$ (more precisely, $\beta = \cos(\pi/8)$). The lemma now follows from the inequality $\beta(2\pi)^{1/4} < 1.48$, and using

$$\sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \leq \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{it \log n - i\theta(t)}}{\sqrt{n}} + \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{i\theta(t) - it \log n}}{\sqrt{n}}$$

(13)

we apply the triangle inequality to the Riemann–Siegel formula in [5, page 9] to obtain

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \right| + \beta \left( \frac{2\pi}{t} \right)^{1/4} + |R_0(t)|,$$

where $n_1 = \lceil \sqrt{t}/2\pi \rceil$, $|R_0(t)| < 0.127t^{-3/4}$ for $t \geq 200$, and

$$\beta = \max_{0 \leq \phi \leq 1} \left| \frac{\cos \frac{3}{2}(\phi^2 + 3/4)}{\cos \pi \phi} \right|.$$ 

By [5] page 65, we have $\beta < 0.93$ (more precisely, $\beta = \cos(\pi/8)$). The lemma now follows from the inequality $\beta(2\pi)^{1/4} < 1.48$, and using

$$\sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \leq \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{it \log n - i\theta(t)}}{\sqrt{n}} + \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{i\theta(t) - it \log n}}{\sqrt{n}}$$

(13)

we apply the triangle inequality to the Riemann–Siegel formula in [5, page 9] to obtain

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \right| + \beta \left( \frac{2\pi}{t} \right)^{1/4} + |R_0(t)|,$$

where $n_1 = \lceil \sqrt{t}/2\pi \rceil$, $|R_0(t)| < 0.127t^{-3/4}$ for $t \geq 200$, and

$$\beta = \max_{0 \leq \phi \leq 1} \left| \frac{\cos \frac{3}{2}(\phi^2 + 3/4)}{\cos \pi \phi} \right|.$$ 

By [5] page 65, we have $\beta < 0.93$ (more precisely, $\beta = \cos(\pi/8)$). The lemma now follows from the inequality $\beta(2\pi)^{1/4} < 1.48$, and using

$$\sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \leq \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{it \log n - i\theta(t)}}{\sqrt{n}} + \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{i\theta(t) - it \log n}}{\sqrt{n}}$$

(13)

we apply the triangle inequality to the Riemann–Siegel formula in [5, page 9] to obtain

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \right| + \beta \left( \frac{2\pi}{t} \right)^{1/4} + |R_0(t)|,$$

where $n_1 = \lceil \sqrt{t}/2\pi \rceil$, $|R_0(t)| < 0.127t^{-3/4}$ for $t \geq 200$, and

$$\beta = \max_{0 \leq \phi \leq 1} \left| \frac{\cos \frac{3}{2}(\phi^2 + 3/4)}{\cos \pi \phi} \right|.$$ 

By [5] page 65, we have $\beta < 0.93$ (more precisely, $\beta = \cos(\pi/8)$). The lemma now follows from the inequality $\beta(2\pi)^{1/4} < 1.48$, and using

$$\sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \leq \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{it \log n - i\theta(t)}}{\sqrt{n}} + \frac{1}{2} \sum_{n=1}^{n_1} \frac{e^{i\theta(t) - it \log n}}{\sqrt{n}}$$

(13)

we apply the triangle inequality to the Riemann–Siegel formula in [5, page 9] to obtain

$$|\zeta(1/2 + it)| \leq 2 \left| \sum_{n=1}^{n_1} \frac{\cos(t \log n - \theta(t))}{\sqrt{n}} \right| + \beta \left( \frac{2\pi}{t} \right)^{1/4} + |R_0(t)|,$$
modifying some of its parameters. Specifically, we used the Euler-Maclaurin formula with a main sum of \([60(t + 1)]\) terms, which is much longer than before. We kept a single correction term, and also used a finer partition with \(Q = 2^{14}\). The longer main sum for \(0 \leq t \leq 3\) ensured that the error in the Euler-Maclaurin formula was sufficiently small. Given this, we were able to verify the claimed bound. 

Lemma 2.3 (Riemann–Siegel–Lehman bound). If \(t \geq 200\), then

\[
|\zeta(1/2 + it)| \leq \frac{4t^{1/4}}{(2\pi)^{1/4}} - 2.08.
\]

In particular, for \(200 \leq t \leq t_0\), we have \(|\zeta(1/2 + it)| \leq c_0 t^{1/6} \log t\).

Proof. This lemma gives a small improvement on the Lehman bound in [10, Lemma 2]. The improvement is in the term \(2.08\), which is significant if \(t\) is not too large.

We start with the bound furnished by the Riemann–Siegel Lemma 2.1. To bound the main sum there, we employ the estimate

\[
2 \left[ \sum_{n=1}^{n_1} \frac{e^{it \log n}}{\sqrt{n}} \right] \leq 2 \left( \sum_{n=1}^{5} \frac{1}{\sqrt{n}} + 2 \int_{5}^{n_1} \frac{1}{\sqrt{x}} \, dx \right)
= 2 \left( \sum_{n=1}^{5} \frac{1}{\sqrt{n}} + 4\sqrt{n_1} - 4\sqrt{5}. \right)
\]

(17)

Note that \(n_1 = [\sqrt{t/(2\pi)}] \geq 5\) for \(t \geq 200\), so the integration step in (17) makes sense. Also, we have \(2 \sum_{n=1}^{5} n^{-1/2} - 4\sqrt{5} < -2.48\). To bound the remainder term \(\mathcal{R}(t)\) in the Riemann–Siegel Lemma 2.1, we employ the estimate

\[
\mathcal{R}(t) \leq 1.48(200)^{-1/4} + 0.127(200)^{-3/4} < 0.4, \quad (t \geq 200).
\]

The first part of the lemma now follows on substituting (17) and (18) back into (10), and noting that \(\sqrt{n_1} \leq (t/2\pi)^{1/4}\).

To prove the second part of the lemma, we use Mathematica to verify that there is no solution to the equation

\[
\frac{4t^{1/4}}{(2\pi)^{1/4}} - 2.08 = c_0 t^{1/6} \log t, \quad (200 \leq t \leq t_0),
\]

and that the l.h.s of the equation is smaller than the r.h.s. at \(t = 200\), and therefore throughout the range \(t \in [200, t_0]\). 

Lemma 2.4 (Van der Corput bound). If \(t \geq t_0\), then

\[
|\zeta(1/2 + it)| \leq a_1 t^{1/6} \log t + a_2 t^{1/6} + a_3
\]

where \(a_1 := 0.6058490462530\), \(a_2 := 0.5743984045897\), and \(a_3 := -2.884626766806\). In particular, for \(t \geq t_0\), we have \(|\zeta(1/2 + it)| \leq c_0 t^{1/6} \log t\).

Proof. We plan to divide the main sum in the Riemann–Siegel Lemma 2.1 into pieces of length \(\approx t^{1/3}\), then apply the van der Corput Lemma 1.2 to each piece. To this end, let \(K = [t^{1/3}]\) and \(R = [n_1/K]\), where, as before, \(n_1 = \lfloor \sqrt{t/2\pi} \rfloor\). Here, \(K\) is the length of each piece (except possibly the last one, which can be shorter), and \(R + 1\) is the total number of pieces.
The remainder term $R(t) := 1.48t^{-1/4} + 0.127t^{-3/4}$ in the Riemann–Siegel Lemma \ref{LS} satisfies $R \leq R(t_0)$ for $t \geq t_0$. Thus, carrying out the aforementioned subdivision, and using the triangle inequality, we obtain

$$\left| \zeta(1/2 + it) \right| \leq 2 \sum_{n=1}^{r_0K - 1} \frac{1}{\sqrt{n}} + 2 \sum_{r = r_0}^{R-1} \left| \sum_{n = rK}^{(r+1)K-1} e^{it \log n} \right|$$

(21)

$$+ 2 \left| \sum_{n=1}^{r_1} e^{it \log n} \right| + R(t_0).$$

Here, we trivially estimated the part of the main sum with $n < r_0 K$, where $r_0$ is a positive integer to be chosen later. Also, we used $R > 0$, which is due to $t \geq t_0$.

To bound the first sum in (21), we note that $r_0 K \geq \lceil r_0 t_1/3 \rceil$. So, proceeding as in (17), we obtain

$$2 \sum_{n=1}^{r_0K - 1} \frac{1}{\sqrt{n}} \leq 4 \sqrt{r_0 K} + \mathcal{I}(r_0, t_0),$$

(22)

where

$$\mathcal{I}(r_0, t_0) := 2 \sum_{n=1}^{\lceil r_0 t_1/3 \rceil - 1} \frac{1}{\sqrt{n}} - 4 \sqrt{\lceil r_0 t_1/3 \rceil} - 1.$$  

To bound the remaining sums in (21), we use partial summation \cite{LL} (5.2.1). Put together, if we let

$$S := 2 \sum_{r = r_0}^{R} \frac{1}{\sqrt{rK}} \max_{\Delta \leq K} \left| \sum_{k=0}^{\Delta-1} e^{it \log(rK+k)} \right|,$$

(24)

then we obtain

$$\left| \zeta(1/2 + it) \right| \leq S + 4 \sqrt{r_0 K} + \mathcal{I}(r_0, t_0) + R(t_0).$$

In order to bound the inner sum of $S$, we employ the van der Corput Lemma \ref{VC}.

Setting

$$f(x) := \frac{t}{2\pi} \log(rK + x), \quad (0 \leq x \leq \Delta - 1),$$

(26)

then

$$f'''(x) = \frac{t}{\pi(rK + x)^3}, \quad (0 \leq x \leq \Delta - 1).$$

(27)

So, on defining

$$W := W_r = \frac{\pi(r + 1)^3 K^3}{t}, \quad \lambda := \lambda_r = \frac{(r + 1)^3}{r^3},$$

(28)

and noting that $\Delta \leq K$, we obtain

$$\frac{1}{W} \leq |f'''(y)| \leq \frac{\lambda}{W}, \quad (0 \leq y \leq K).$$

(29)

We apply Lemma \ref{VC} with $L = K$, $\alpha_r := \alpha(W_r, \lambda_r, \eta)$, $\beta_r := \beta(W_r, \eta)$, and with $\eta > 0$ to be chosen later. This yields

$$S \leq 2 \sum_{r = r_0}^{R} \frac{1}{\sqrt{rK}} \sqrt{\alpha_r K^2 W_r^{-1/3} + \eta \alpha_r K + \beta_r KW_r^{1/3} + \eta \beta_r W_r^{2/3}}.$$  

(30)
We factor out $K^2 W_r^{-1/3} = \frac{K^{1/3}}{\pi^{1/3} (r+1)}$ from under the square-root. This gives

\[ S \leq \frac{2^{1/6}}{\pi^{1/6}} \sum_{r=r_0}^R \sqrt{\frac{B_r}{r(r+1)}}, \]

where

\[ B_r := \alpha_r + \frac{\eta \alpha_r W_r^{1/3}}{K} + \frac{\beta_r W_r^{2/3}}{K} + \frac{\eta \beta_r W_r}{K^2}. \]

Since $K < t^{1/3} + 1$ and $R \leq t^{1/6}/\sqrt{2\pi}$, we have

\[ W_r \leq \pi(R + 1)^3(1 + t^{-1/3})^3 \leq \frac{\sqrt{t_0^3}}{2^{3/2} \sqrt{\pi}}, \quad \rho := \left(1 + \frac{1}{R}\right) \left(1 + \frac{1}{t^{1/3}}\right). \]

In addition, $K \geq t^{1/3}$, so put together we obtain

\[ B_r \leq \alpha_r + \frac{\eta \alpha_r \rho}{\sqrt{2\pi^{1/6} t^{1/3}}} + \frac{\beta_r \rho^2}{2\pi^{1/3}} + \frac{\eta \beta_r \rho^3}{2^{3/2} \sqrt{\pi} t^{1/6}}. \]

At this point, we make several observations. First, $W_r$ is monotonically increasing in $r$, and $\lambda_r$ is monotonically decreasing in $r$. Hence, by the definitions of $\alpha_r$ and $\beta_r$, we see that they monotonically decrease with $r$. So $\alpha_r \leq \alpha_{r_0}$ and $\beta_r \leq \beta_{r_0}$ for $r \geq r_0$. Second, we fix $\eta = \eta_0 := (75/64)^2/3$, which is to help balance the leading two terms in the formula for $\alpha_r$ in (3). Third, if we define

\[ R_0 := \left\lfloor \frac{\sqrt{t_0/(2\pi)} - 1}{t_0^{1/3}} + 1 \right\rfloor, \]

then $R \geq R_0$, and therefore

\[ \rho \leq \rho_0 := \left(1 + \frac{1}{R_0}\right) \left(1 + \frac{1}{t_0^{1/3}}\right). \]

Assembling these estimates into (31), and using

\[ \sum_{r=r_0}^R \frac{1}{\sqrt{r(r+1)}} \leq \log \frac{R}{R_0 - 1}, \]

together with the inequality $t \geq t_0$ to bound the denominators in (34) from below, we obtain

\[ S \leq \frac{2^{1/6}}{\pi^{1/6}} \sqrt{\frac{\alpha_{r_0}}{\sqrt{2\pi^{1/6} t_0^{1/6}}}} + \frac{\eta \alpha_{r_0} \rho_0}{\sqrt{2\pi^{1/6} t_0^{1/6}}} + \frac{\beta_{r_0} \rho_0^2}{2\pi^{1/3}} + \frac{\eta \beta_{r_0} \rho_0^3}{2^{3/2} \sqrt{\pi} t_0^{1/6}} \log \frac{R}{R_0 - 1}. \]

We substitute (38) back into (35), use the inequality $W_{r_0} \geq \pi(r_0 + 1)^3$ to simplify the bounds for $\alpha_{r_0}$ and $\beta_{r_0}$, and choose $r_0 = 5$ which is suggested by numerical experimentation. Then, we combine the resulting expression with the bounds

\[ 4\sqrt{r_0 K} \leq 4\sqrt{r_0 (1 + t_0^{-1/3}) t_0^{1/6}}, \quad R \leq t^{1/6}/\sqrt{2\pi}, \]

and numerically evaluate the resulting constants using Mathematica. On completion, this yields the first part of the lemma.

To prove the second part of the lemma, denote the r.h.s. of (20) by $(*)$, and consider the equation $(*) = c_0 t^{1/6} \log t$. Using Mathematica, we find that there is no solution $t \geq t_0$ to this equation, and that $(*)$ is smaller than the $c_0 t^{1/6} \log t$ at $t = t_0$, and therefore for all $t \geq t_0$. □
Remarks. The reason we restrict \( t \geq t_0 \) in Lemma 2.4 is that the Riemann–Siegel–Lehman bound from Lemma 2.3 is tighter for \( t < t_0 \). So we may as well take \( t \geq t_0 \), which gives marginally better constants.

3. Proof of Lemma 1.2

Proof. We use the Weyl-van der Corput Lemma in [4, Lemma 5], but in the more precise form presented at the bottom of page 1273. Also, we incorporate a refinement pointed out by Platt and Trudgian in [12, Lemma 2] that allows writing the leading term in (40) as \( L + M - 1 \) instead of \( L + M \). Put together, if \( M \) is a positive integer, then

\[
\left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 \leq (L + M - 1) \left( \frac{L}{M} + \frac{2}{M} \sum_{m=1}^{M} \left(1 - \frac{m}{M}\right) |S'_m(L)| \right),
\]

where

\[
S'_m(L) := \sum_{r=N+1}^{N+L-m} e^{2\pi i f(r+m) - f(r)}.
\]

Henceforth, we may assume that \( m < L \) and \( L > 1 \). Otherwise, the sum \( S'_m(L) \) is empty, and so it does not contribute to the upper bound in (40).

Let \( g(x) := f(x + m) - f(x) \), where \( N + 1 \leq x \leq N + L - m \). Then, \( g''(x) = f''(x + m) - f''(x) \). Therefore, by the mean-value theorem, \( g''(x) = m f'''(\xi^*) \) for some \( \xi^* \in (x, x + m) \). Since we assumed that \( 1 \leq m < L \) and \( L > 1 \) then \((x, x + m) \subset [N + 1, N + L] \). So, given our bound on \( f''' \), we deduce that

\[
m \leq \frac{|g''(x)|}{\lambda W}, \quad (N + 1 \leq x \leq N + L - m).
\]

Applying the van der Corput Lemma in [4, Lemma 3] to \( S'_m(L) \) thus yields

\[
|S'_m(L)| \leq \frac{8\lambda L \sqrt{m/W}}{5} + \frac{3\lambda L m}{W} + \frac{8 \sqrt{W/m}}{5} + 3.
\]

We substitute (43) into (40), then execute the summation over \( m \) using the estimates that appear after [4, Lemma 7]: namely,

\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M}\right) \sqrt{m} \leq \frac{4M^{3/2}}{15}, \quad \sum_{m=1}^{M} \left(1 - \frac{m}{M}\right) \frac{1}{\sqrt{m}} \leq \frac{4\sqrt{M}}{3},
\]

as well as the Euler-Maclaurin summation estimates

\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M}\right) \leq \frac{M}{2}, \quad \sum_{m=1}^{M} \left(1 - \frac{m}{M}\right) m \leq \frac{M^2}{6}.
\]

From this, we conclude that

\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M}\right) |S'_m(L)| \leq \frac{32\lambda L M^{3/2}}{75 \sqrt{W}} + \frac{\lambda L M^2}{2W} + \frac{32 \sqrt{W M}}{15} + \frac{3 M}{2}.
\]

In particular, substituting (43) back into (40), we obtain

\[
\left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 \leq (L + M - 1) \left( \frac{L}{M} + \frac{64\lambda L}{75 \sqrt{W}} + \frac{\lambda L M}{W} + \frac{64}{15} \sqrt{W} + 3 \right).
\]
We choose $M = \lceil \eta W^{1/3} \rceil$ for some free parameter $\eta > 0$ that can be optimized (usually, $\eta$ will be around 1). This choice is in order to balance the first two terms on the r.h.s. as they typically dominate in our application. So, now, appealing to the inequality $\eta W^{1/3} \leq M \leq \eta W^{1/3} + 1$, we deduce that

$$\left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 \leq (L + \eta W^{1/3}) \left( \frac{L}{\eta W^{1/3}} + \frac{64\lambda L}{75} \eta + W^{-1/3} \right) + \frac{\lambda L (\eta W^{1/3} + 1)}{W} + \frac{1}{15} \sqrt{\frac{W}{\eta W^{1/3}}} + 3W^{-1/3},$$

(47)

We factor out $LW^{-1/3}$ from the first three terms in the second bracket, and $W^{1/3}$ from the last two terms. This gives

$$\left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 \leq (L + \eta W^{1/3}) \left( \frac{1}{\eta} + \frac{64\lambda}{75} \sqrt{\eta} + W^{-1/3} \right) + \frac{\lambda (\eta + W^{-1/3})}{W^{1/3}} + (L + \eta W^{1/3})W^{1/3} \left( \frac{64}{15\sqrt{\eta}} + 3W^{-1/3} \right).$$

Last, the lemma follows on recalling the definitions of $\alpha$ and $\beta$ in (7). \qed

**References**

1. E. Bombieri and H. Iwaniec, *On the order of $\zeta(\frac{1}{2} + it)$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13** (1986), no. 3, 449–472. MR 881101 (88i:11054a)
2. J. Bourgain, *Decoupling, exponential sums and the Riemann zeta function*, arXiv:1408.5794 (2014).
3. Yuanyou F. Cheng and Sidney W. Graham, *Explicit estimates for the Riemann zeta function*, Rocky Mountain J. Math. **34** (2004), no. 4, 1261–1280. MR 2095256 (2005f:11179)
4. W. Gabcke, *Neue herleitung und explicite restabschätzung der Riemann-Siegel-formel*. Ph.D. thesis, Göttingen, 1979.
5. S. W. Graham and G. Kolesnik, *van der Corput’s method of exponential sums*, London Mathematical Society Lecture Note Series, vol. 126, Cambridge University Press, Cambridge, 1991. MR 1145488 (92k:11082)
6. G. A. Hiary, [https://people.math.osu.edu/hiary.1/fastmethods.html](https://people.math.osu.edu/hiary.1/fastmethods.html)
7. G. A. Hiary, *Fast methods to compute the Riemann zeta function*, Ann. of Math. (2) **174** (2011), no. 2, 891–946. MR 2831110 (2012g:11154)
8. M. N. Huxley, *Exponential sums and the Riemann zeta function. V*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 1–41. MR 2107036 (2005h:11180)
9. R. Sherman Lehman, *On the distribution of zeros of the Riemann zeta-function*, Proc. London Math. Soc. (3) **20** (1970), 303–320. MR 0258708 (41 #3414)
10. A. M. Odlyzko and A. Schönhage, *Fast algorithms for multiple evaluations of the Riemann zeta function*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 797–809. MR 961614 (89j:11083)
11. D. J. Platt and T. S. Trudgian, *An improved explicit bound on $|\zeta(\frac{1}{2} + it)|$*, J. Number Theory **147** (2015), 842–851. MR 3276357
12. Michael Rubinstein, *Computational methods and experiments in analytic number theory*, Recent perspectives in random matrix theory and number theory, London Math. Soc. Lecture Note Ser., vol. 322, Cambridge Univ. Press, Cambridge, 2005, pp. 425–506. MR 2166470 (2006d:11153)
13. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press Oxford University Press, New York, 1986. Edited and with a preface by D. R. Heath-Brown. MR 882550 (88c:11049)
14. J. G. van der Corput, *Neue zahlentheoretische abschätzungen*, M. Z. **29** (1929), 397–426.
16. J. G. van der Corput and J.F. Koksma, *Sur l’ordre de grandeur de la fonction $\zeta(s)$ de Riemann dans la bande critique*, Annales de Toulouse (3) 22 (1930), 1–39.

**Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210.**

*E-mail address: hiaryg@gmail.com*