FBM: A Flexible Random Walk Based Generative Model for Social Network

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Abstract: This paper studies the static and dynamic characteristics of the real social networks as well as their proposed generative models, among which the Butterfly Model [1] is useful while not flexible enough to generate the social networks with the expected power-law exponent. And a novel Flexible Butterfly Model (FBM) is proposed based on the Butterfly Model and combined with the Monte Carlo method, and a Bayesian Graph Model for the training of the FBM Model is built in order to learn parameters from real social networks. Experiments have shown that the FBM model can adjust the law power exponent of the generated social network effectively by the introduced parameters. Meanwhile, the FBM model also maintains the vast majority of important characteristics that the Butterfly model has.

Keywords: Generative model, graph model, random walk, social network.

1. INTRODUCTION

In recent years, virtual social networks, such as Facebook, Twitter etc, have emerged in large numbers, and have impacted human society in various aspects profoundly along with the penetrating applications of the internet. As the extension of complex network in human society, the social network are the relationships among individuals or groups in the activities of human society, for example: friend relationships of individuals, business relationships of companies, or cooperation relationships of authors.

With social entity represented by node and social relationship represented by edge, social network can be described as a graph, and it has some prominent static and dynamic characteristics. How to establish the reasonable generative model to simulate the formative procedure of the social network? The answers are significant to explain the formative mechanism of the social network, forecast the trend and detect the abnormal behaviors in the social networks.

Finding the characteristics and patterns of social networks is the foundation of constructing generative model, and a method informally named “observations and imitations” is widely adopted by the researchers of social network: Firstly, some characteristics and metrics are established via observations on the data of real social networks, then the patterns of them are sought out, further more possible explanations for the discovered patterns are provided. After that, to set up reasonable generative model is required to mimic the formative procedure of the social network according to the explanations, and sequentially generate the synthetic social networks complying with those patterns.

Currently, studies on generative model of social networks are concentrated in two areas: one is to find new characteristics, metrics and patterns, and the other is to build new generative models. Researchers have found many interesting characteristics and metrics, and most of them are based on two original properties on the graph: degrees and distances of nodes. With the accumulation of data and the renovation of methods, researchers are constantly discovering new interesting characteristics and metrics as well as static or dynamic patterns.

Analyzing characteristics and constructing generative model of social networks, lay the groundwork for numerous applied researches, have penetrated into many application fields: It contributes to identifying the reputation of the webpage and thereby promoting the performance of search engine [2], and is helpful to determinate the importance of customers in the virtual market and priority of recommender in social recommender system [3]; It can help to forecast the trend of social networks and can benefit social network operators to plan the scale and time of purchasing new hardware and storage devices; It can be engaged to detect abnormalities in the social network so as to find and prevent the attack and abuse of social network such as spam links [4] etc; It can be used to generate data for scientific studies with the similar characteristics to the real social network, which can not only protect privacies in the real data, but also can cheaply produce the experimental data with the different characteristics and scales by adjusting parameters of generative model, and then, reduce the costs of obtaining the data; It is also possible to generate the small-scale samples of the large-scale social networks to accelerate the experiments, or extrapolate the small-scale social networks to predict the trend in the further, etc. Therefore, studies on the characteristics of the social network and generative model are hot areas in the social network research.
The main contributions of this paper are: (1) we conclude the faults that power law exponent generated by Butterfly Model holds fixed value, and by introducing parameters, propose a new FBM model with adjustable power law exponent. (2) a new random number generator function is derived to adjust the power law exponent of the FBM model taking advantage of the Monte Carlo method. (3) we build a Bayesian graph model that can be easily learned by Gibbs Sampling method with the purpose of training FBM mode.

The contents of this paper are organized as follows: section 2 introduces some found static and dynamic characteristics and their generative models; section 3 analyses the defects of Butterfly Model; section 4 deduces a novel FBM model on basis of Butterfly Model and proposed a Bayesian graph model for its training; section 5 designs several experiments to verify the FBM, and the last section draws a conclusion for this paper.

2. RELATED WORKS

In this section, we will first introduce some discovered static and dynamic characteristics, metrics and patterns, and main proposed generative models.

2.1. Researches Related to Characteristics of Social Network

The studies on the characteristics and metrics mainly focus on three aspects: the power-laws characteristics that the nodes and edges satisfy, network diameters, and community structure. By means of different metrics, researchers have discovered various significant power-laws characteristics that edges and nodes meet, variation laws of network diameter, and community structures in the social network.

The characteristics of power-laws related to nodes and edges include: Degree of nodes follows a power-law distribution [5], the discrete probability distributions of degree $f(d)$ follows $f(d) \propto d^{-\gamma}$, $\gamma > 0$; Density obeys a power-law distribution [6], that is, at any time $t$, the number of total edges $Et$ and the number of total nodes $Nt$ in the social network meet $Et \mu Nt \alpha > 0$; The weight of the network follows the power-law distribution [1], namely at any time $t$, the sum of weights on all edges $Wt$ and the number of all edges $Et$ comply with $Wt \sim Et^\beta$, $\beta > 1$; Weight of any individual node $N$ follows the power-law distribution [1], the sum of weights on the edges connected to the node $Wn$ and degree of the node $dn$ submit to $Wn \sim dn^\theta$, $\theta > 1$; The number of triangles follows the power-law distribution [7], the number of triangles existing in the social network $\Delta$ and the number of nodes involved in the triangles $f(\Delta)$ are obedient to $f(\Delta) \sim \Delta^\sigma$, $\sigma < 0$; The eigenvalues of the adjacency matrix representing the social network follow the power-laws amazingly [8], more than that, the maximum of eigenvalues follows the power-law distribution too [9], at any time $t$, the maximum of eigenvalues $\lambda t$ and the number of all edges $Et$ observe $\lambda t \mu Et^\delta$, $\delta < 0.5$.

Apart from this, researches also find some meaningful characteristics in the growth of the social network: with the scale increasing, social networks emerge the small-world phenomenon [10, 11], gradual shrink of the network diameter [12], and clear community structures [13, 14]; the number of nodes in the second and the third connected components maintain relatively invariance [1], similar to the network traffic, newly arrived edges or weights which exhibit the property of self-similar in different timescales [15], have the burstiness rather than uniform peak, etc.

2.2. Researches Related to Generative Models

The role of the generative model is to simulate and generate the social network with some special characteristics. In order to establish reasonable generative models, substantial studies are made to find out the formative mechanism of the social networks. The procedure of exploring new characteristics and the procedure of establishing new generative models have moved forward alternately, and the first try of the generative model named as ER model [16] is made by Erdos and Renyi, while the ER model is ideal, explanatory and not practical; In 1999, Albert and Barabasi put forward the BA model [17], which explained the formative mechanism of power-law as a Yule process, and the BA model can partly simulate the power-law characteristics in the social network, as well as the small-world phenomenon, but there are strict restrictions, nevertheless, the BA model laid the foundation for later researches. To generate community structure, based on the BA model, Kleinberg et al proposed the Copy Model [18], which mimic a newcomer’s behavior when he join an unfamiliar society: when a new node arrives, it randomly selects a node in the social network to connect, and then, the chosen node continues to copy his friends (relationships of neighbors) to the new node with certain probability. Copy Model can not only product the community structure, but also simulate more power-law characteristics in social networks. Subsequently, Jure et al found density power-law and shrink of the network diameter, and consider that the Self-Organized Criticality [19] existed in the system account for their findings, thus put forward the Forestfire model [6]; Because the power laws could also derive from the self-similar, the researchers studied the formative procedures of various self-similar phenomena in the natural [20, 21], and on these basis, proposed R-MAT model [22], Kronecker graph model [23], RTG model [24].

Different from others, McGlohon, Kang et al propose the Butterfly model and Community Connect model [25] based on random walk. Besides the different perspective, two model are the same, and they can simulate the majority of the found characteristics and patterns in the social network, but there are some obvious shortcomings that the models can only generate social network with fixed power-law exponents, and lack effective methods to fit the models according to real data. Even so, they are the rarely models near to the state-of-the-art at present. This paper analyzes the lack of the two models, combined with Monte Carlo method, and proposes a novel Flexible Butterfly Model and its Bayesian probabilistic graphical model for the training.
3. DEFECT ANALYSIS OF THE BUTTERFLY MODEL

3.1. The Procedure Description of the Butterfly Model

The Butterfly Model contains three parameters\( (P_{\text{link}}, P_{\text{step}}, P_{\text{host}})\), and can be simply described as follows:

\[ P_1 \] When a node \( \text{node}_{\text{newer}} \) wants to join the social network, it’s assigned with a probability \( P_{\text{step}} \) following a specific distribution, and then it transfers to the process \( P_2 \).

\[ P_2 \] A new walk process is restarted with probability \( P_{\text{host}} \), if the restart is successful, a node \( \text{node}_{\text{host}} \) in the social network as the starting point for the new walk process is uniformly selected, then it transfers to the process \( P_3 \). If the restart is unsuccessful, the restart is failed and the formative processes are terminated.

\[ P_3 \] The \( \text{node}_{\text{host}} \) is taken as the current node \( \text{node}_{\text{current}} \), and moves one step forward with the probability \( P_{\text{step}} \) repeatedly until the end of this walk process, then it transfers to the process \( P_2 \).

The sketch of Butterfly Model can be shown as the Fig. (1).

3.2. The Probability Analysis of Butterfly Model

In term of the above process, some conclusions can be drawn: the length of a walk process \( L \) follows a geometric distribution \( L = l \sim \text{Geom}(l, 1 - P_{\text{step}}) \), the expectation of \( L \) is \( E(L) = 1 / (1 - P_{\text{step}}) \), the time of all walk processes \( H \) follows a geometric distribution \( H = h \sim \text{Geom}(h+1, 1 - P_{\text{host}}) \), \( \text{Geom} \) denotes a negative binomial distribution, and degree of the new node \( \text{node}_{\text{newer}} \) follows \( D = d \sim \text{NBin}(d, h, 1 - P_{\text{step}}) \cdot \text{Geom}(h+1, 1 - P_{\text{host}}) \cdot P_{\text{link}} \), and approximately follows \( P(D = d) = h \cdot E(L) = h / (1 - P_{\text{step}}) \). The Butterfly Model in detailed procedures is shown as Fig. (2).

3.3. Defects and Improvements of the Butterfly Model

Here, defects of the Butterfly model will be analyzed, and it starts from deducing the quantitative relation between the expectation of \( H \) is \( E(H) = 1 / (1 - P_{\text{host}}) \), total length of all walk processes \( LA \) follows \( P(LA = la) = \text{NBin} \) \( (la, h, 1 - P_{\text{step}}) \cdot \text{Geom}(h+1, 1 - P_{\text{host}}) \), \( \text{NBin} \) denotes a negative binomial distribution, and degree of the new node \( \text{node}_{\text{newer}} \) follows \( P(D = d) = \text{NBin}(d, h, 1 - P_{\text{step}}) \cdot \text{Geom}(h+1, 1 - P_{\text{host}}) \cdot P_{\text{link}} \), and approximately follows \( P(D = d) = h \cdot E(L) = h / (1 - P_{\text{step}}) \). Evidently, let \( P_{\text{link}} \) be constant, given \( P_{\text{host}} \) and \( P_{\text{step}} \) following the same distribution, \( D \) and \( L \) will comply with the same power-law, and if \( P_{\text{host}} \) is assigned by \( E(P_{\text{step}}) \), \( D \) and \( L \) will obey the near power-laws. The Butterfly Model in detailed procedures is shown as Fig. (2).
the distribution of degree D and length of a walk process L: for the convenience of deducing, \( P_{\text{step}} \) is relabeled by \( X \), the length of a walk process \( L \) is relabeled by \( Y \), in the Butterfly Model, \( P_{\text{step}} \) follows the uniform distribution \( f_{\text{step}} \sim \text{uniform}(0,1) \), and \( Y = E(L) - 1 = 1/(1-P_{\text{step}}) - 1 = x/(1-x) \). Now let \( y = g(x) = \frac{x}{1-x} \), hence \( x = g^{-1}(y) = h(y) = \frac{y}{y+1} \), and according to the formula of probability density on function of random variables, we have formula (1):

\[
f_y(y) = f_x(h(y)) \cdot \left| \frac{dh(y)}{dy} \right| = (y+1)^{-2}
\]

Therefore, the Butterfly model can only generate social network with power-law exponent near -2. Multiple edges and the number of restarts are very small part, therefore, the distribution of LA can be approximately given by the distribution of \( L \), and the above conclusion is also applicable to the repeatedly walk processes in the Butterfly Model.

Obviously, the power-law exponent generated by the Butterfly Model is not flexible enough to meet the needs of the social networks with mutative power-law exponents. In order to generate social network with flexible power-law exponents, authors have introduced two new parameters to the Butterfly model, and then have used the Monte Carlo inverse function method to adjust the probability distribution of \( X \) and \( Y \), thus, have adjusted the power-law exponents of the Butterfly model, and then have used the Monte Carlo inverse function method to adjust the probability distribution of \( X \) and \( Y \), thus, have adjusted the power-law exponents of the model. The improved Butterfly Model, named as Flexible Butterfly Model (FBM), can learn parameters from the real data through building a Bayesians graph model, hence appears more practical.

4. FLEXIBLE BUTTERY MODEL

4.1. Description of the Flexible Butterfly Model

Main difference between FBM model and Butterfly model exists in \( P1 \), in which the new model is no longer subjected to uniform distribution but a deduced special distribution. FBM model may change the generated power-law exponent by changing the parameters of the special distribution, besides that, \( P2 \) and \( P3 \) are the same.

Assume that \( Y \) wants to follow the target distribution: \( f_Y(y) = b \cdot (y+1)^{b-2} \), \( 1 < y < \infty \), \( a \) and \( b \) are the introduced parameters, \( a \) is the power-law exponents, \( b \) is magnification factor related to scale of social network(number of nodes in the social network), in the light of the Monte Carlo inverse function method, let \( x = h(y) = \frac{y}{y+1} \) as primitive function, let \( y = h^{-1}(x) = g(x) = \frac{x}{1-x} \) as inverse function, so that \( X \) need to follows:

\[
f_X(x) = f_Y(g(x)) \cdot |g(x)| = b \cdot (1-x)^{b-2}
\]

To generate the above distribution of \( X \), we turn to another random variables \( R \) following uniform distribution \( f_R \sim \text{uniform}(0,1) \). Similarly in line with the Monte Carlo inverse function method, taking \( x = \phi(r) \) as primitive function and \( r = \phi^{-1}(x) \) as inverse function, we have

\[
f_X(x) = f_R(\phi^{-1}(x)) \cdot \left| \frac{d}{dx} \phi^{-1}(x) \right| = b \cdot (1-x)^{b-2} = \frac{d}{dx} \phi^{-1}(x),
\]

then it can be deduced that:

\[
\phi(r) = 1 - \left( \frac{a-1}{b} \right) \frac{1}{r^{b-1}} \quad a > 1, b \geq a - 1
\]

As recalled that \( L \) is relabeled as \( Y \) as well as \( L \) and \( D \) will obey the near power-laws, the graph model of the \( D \)'s target distribution can be shown in Fig. (3), and it means that new model can change the distribution of \( P_{\text{step}} \) by changing parameter \( a \) and hence change the power-law exponents of \( D \). Conversely, the introduced parameters could be trained from degrees distribution through sampling methods, such as MCMC with Gibbs Sampling [26], on graph model. From above derivation, it is not difficult that: let \( a = 2.0, b = 1.0 \), then \( P_{\text{step}} \) obey the uniform distribution between \((0, 1)\), and the FBM model is equivalent to Butterfly Model. According to the above derivations and explanations, we propose the algorithm description of the FBM model as shown in Table 1.

Fig. (3). Graph model of the degree’s target distribution.

4.2. Training of the FBM Model

The FBM model takes the modifying of the distribution \( P_{\text{step}} \) as the entry point, by introducing parameter \( a \) and \( b \), and have achieved the regulation of the power-law exponent. Previously, in order to simplify the discussion, we set \( P_{\text{link}} \) and \( P_{\text{knot}} \) in new model as the empirical values in the original model, respectively, 0.3 and 0.5. Now if training the FBM model, \( P_{\text{knot}} + P_{\text{link}} \) together with the introduced parameters \( a, b \), four parameters must be learned, so it needs to establish a complete training model. Based on Fig. (3), we established the Bayesian graph model to train the FBM, and describe the Bayesian graph model via a pseudo-code similar to OpenBUGS [27] syntax, and it is shown in Table 2.
**Table 1. Algorithm description of Flexible Butterfly Model.**

| P1: `getPStepByMonteCarlo(node_newer, a, b)` |
|---|
| **input:** node_newer with uninitialized `p_step` and `a`, `b` is parameters to adapt the exponents of the power lawer  
**output:** node_newer with initialized `p_step` |
| `r ← Uniform(0,1)`  
If `a > 1.0` and `b ≥ a - 1` then   
`node_newer.p_step ← 1 - \((a-1.0)b^{-\frac{1}{a}}\)`  
Else   
`node_newer.p_step ← 1`  
End |

**Table 2. Bayesian graph model of the FBM.**

| Init: `{N = N_0}` |
|---|
| **Data:** `{d[i] = degrees of real social networks}` |
| **Model:**  
- `a`: Normal(1.0, 10^(-5))  
- `b`: Normal(2.0, 10^(-5))  
- `p_host`: Normal(0.3, 10^(-4))  
- `p_host`: Normal(0.5, 10^(-4))  
- `h`: Geom(1 - `p_host`)  
**for** `{i in 1 : N}`  
- `r[i]`: Uniform(0,1.0)  
- `p_step[i] ← \(\Theta_0 \text{ pow}((a, 1) r[i] / b, (a, 1))\)`  
- `la[i]`: Binomial(1 - `p_step[i]`, `h`)  
- `d[i] ← la[i]`  
| }

The Bayesian graphical model consists of three parts: part **Init** expresses initialization of model, part **Data** lists the observed data in the model, which is mainly the random sequence of degrees in the social network, and part **Model** presents the distributions of random variables and dependencies among the random variables in the model. The Bayesian model takes the degrees of nodes `D` as observed variable, takes other variables as the latent random variable, takes `p_host`, `p_step`, `a`, and `b` as the parameters to learn, assumes that the parameters are subjected to the normal distribution with very small variance, sets the mean of `p_host`, `p_step` as the empirical values of original model, and sets the mean of `a`, `b` as 1.0 and 2.0 respectively. `h` is the random variables representing the number of restart, and follows the geometric distribution with `1 - p_host`, `N` is number of sampling, `r` obeys the uniform distribution between 0 and 1, `p_step` follows the special distribution defined by `\phi()`, `la` follow the negative binomial distribution with `1 - p_step`, and `h = 1`. The training of the Bayesian graph model can use MCMC with Gibbs sampling, calculate the posterior distribution of the parameters, and treat posterior mean value of the normal distribution as the learned values of the parameters. The learned parameters can be employed for FBM to generate the social network with the power-law exponents similar to the real social network. Due to limited space, MCMC with Gibbs sampling on the Bayesian graph model and related tools are no longer to discuss here.

**5. Design of Experiments**

The main purposes of the experiment are to verify the effectiveness of adjusting the power-law exponent in the FBM model, and verify whether the FBM model keep all the main characteristics of Butterfly Mode. Therefore the experimental data generated by the FBM model.
5.1. Validation of Adjustment on Power-Law Exponent

The experiments verify the effectiveness of the adjusting the power-law exponent which is estimated by 1-5% normal interval estimates in statistics. Firstly, the experiments assign respectively five different values in (1.6-2.0) to \( a \), and treat \( b \) as the constant and maintain the value of 1.0, so as to each pair of \( a \) and \( b \) is used to generate 20 social networks with 10000 node repeatedly. Experiments statistics the degree frequency of nodes, and take the degree as \( x \) axis and degree frequency as the \( y \) axis so as to draw scatter plot under the loglog coordinates. In order to measure the power-law exponent, experiments fit straight lines on the scatter plot by the least squares method, and treat the fitted slope \( e \) as the estimated the power-law exponent. Experiments suppose that \( e \) follows normal distribution. With the estimated values of power-law exponent denoted by \( e_{1}-e_{20} \) for each pair of \( a, b \), the 1-5% confidence interval and the mean value of the random variable \( e \) are calculated. Experiments take the fifth group as the reference, where the FBM model is equivalent to the Butterfly model by letting parameters \( a=2.0, b=1.0 \).

The results of experiment are shown as Table 3 and Fig. (4).

| GID | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| \( a \) | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 |
| \( e \) | 1-5% 1.42-1.59 | 1.58-1.69 | 1.65-1.80 | 1.81-1.95 | 1.87-2.01 |
| mean | 1.51 | 1.64 | 1.73 | 1.88 | 1.94 |

5.2. Social Network Characteristics of the FBM Model

Similar to the Butterfly Model in [1], this paper does not verify all the social network characteristics the FBM gener-
ated, but verifies the four major dynamic characteristics: power law of density, power law of weight, shrinking diameter phenomenon in giant connected component (GCC) and constant size of next large connected component (NLCC).

Fig. (6) shows the temporal variations of dynamic characteristics in a FBM-generated social network assigning $a=2.0, b=1.0$.

Fig. (6) shows that the density and weight of social networks have approximately linear growth with the nodes under loglog coordinates; and NLCC integrates into the GCC after growing to a certain extent, and its nodes are almost constant; network diameter of GCC increases with time, and begins to shrink after the peak. Therefore, the FBM model is not only able to adjust the power-law exponent, but also maintains the major dynamic social networking characteristics that Butterfly Model has.

6. CONCLUSION

This paper analyzes the defect of Butterfly model, and has proposed a novel FBM model based on Butterfly Model by introducing new adjustable parameters. The experiments show that the FBM model has overcome the shortcoming of Butterfly Model, power-law exponent of which can only be near to fixed value, and the introduced parameters may adjust the power-law exponent of generated social networks, and it makes FBM able to simulate social networks with different power-law exponents; The experiments also show that the FBM model maintained some important social network characteristics, such that the power-law of density and weight, network diameter shrinks in the largest connected component(GCC), and nodes of next large connected component(NLCC) remains relatively constant.

In order to learn required parameters from the sequence data of the real social, this article has also established a Bayesian graph model for the FBM model, and it makes the FBM model more practical because of its capability of fitting real social networks. Evaluating and validating the performance of Bayesian graph model will the further works of this paper.

CONFLICT OF INTEREST

The authors confirm that this article content has no conflict of interest.

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