The universality of renormalization group limit cycle behavior is illustrated with a simple discrete Hamiltonian model. A non-perturbative renormalization group equation for the model is soluble analytically at criticality and exhibits one marginal operator (made necessary by the limit cycle) and an infinite set of irrelevant operators. Relevant operators are absent. The model exhibits an infinite series of bound state energy eigenvalues. This infinite series approaches an exact geometric series as the eigenvalues approach zero - also a consequence of the limit cycle. Wegner’s eigenvalues for irrelevant operators are calculated generically for all choices of parameters in the model. We show that Wegner’s eigenvalues are independent of location on the limit cycle, in contrast with Wegner’s operators themselves, which vary depending on their location on the limit cycle. An example is then used to illustrate numerically how one can tune the initial Hamiltonian to eliminate the first two irrelevant operators. After tuning, the Hamiltonian’s bound state eigenvalues converge much more quickly than otherwise to an exact geometric series.

I. INTRODUCTION

The renormalization group has been known to be able to produce (in principle) a limit cycle for a long time [1]. A limit cycle is an alternative to a fixed point, although a limit cycle necessarily implies the existence of a fixed point for a discrete version of the renormalization group transformation: see below. Recently, the two of us have described a discrete, analytically soluble Hamiltonian matrix that can be analyzed using a renormalization group procedure. The result, after an initial analysis, was a remarkably simple form of limit cycle involving only one coupling constant [2]. Our model can also lead to chaotic renormalization group behavior [3]. Here, we provide a more complete analysis of the same model. The more complete analysis leads to renormalization group transformations involving either a finite set of coupling constants (with 2, 3, 4, or more couplings, as one wishes) or in the limit to a transformation in a space of functions of one variable. The discrete coupling constants become identified with the coefficients of the function’s expansion about the origin of its argument. The focus of the more complete analysis is universality and the nature of marginal and irrelevant operators (in Wegner’s sense [4]) in the model in the presence of the limit cycle. The possibility for chaotic behavior is disregarded here.

The limit cycle picture of the model may be helpful in qualitative understanding of more complex systems. In particular it is already known that a three-body Hamiltonian with short range forces exhibits a limit cycle when there is a two-body bound state at threshold [5]. Systems that might be understood approximately through the three-body limit cycle example include nuclear [6], and atomic [6] three-body systems with short-range interactions or many-body systems with similar interactions that have not been studied yet [6, 7, 8]. In the exact critical limit of the three-body case there is a three-body bound state spectrum that has an infinite sequence of states converging to zero energy as a geometric series; the infinite sequence of bound states was noticed [9, 10] well before an underlying renormalization group limit cycle was identified.

Unfortunately, physical three-body systems known to date do not have bound states precisely at threshold, the requirement for criticality and an infinite sequence of bound states. This makes it necessary to understand corrections to critical behavior in the three-body case. Obtaining these corrections is a very complex undertaking that is far from complete. One purpose of this paper is to provide an analysis of corrections to the limit cycle behavior due to irrelevant operators in a much simpler model to analyze than is the three-body Hamiltonian already under investigation [11].

A second purpose of our paper is to lay a basis for searches across quantum condensed matter physics more broadly for evidence of limit cycle behavior. We will stress those aspects of limit cycle behavior that are or could be universal, and therefore present if examples of limit cycles are found in systems other than the three-body case already under study. One very distinctive phenomenon to be on the lookout for is the presence of geometric series of bound state energies (discrete energy levels) converging toward zero. One may have to tune a real system by applying external fields and stresses or introducing impurities in order to approach criticality. Therefore, our discussion of corrections to criticality
leads to an example of how one can tune a finite model Hamiltonian to achieve rapid approach of its eigenvalues to their limiting behavior for low energies.

Despite the complexity of a non-perturbative limit cycle as opposed to a simpler fixed point, the model discussed here is largely soluble analytically, and even when numerical procedures are needed, they demand relatively little in the way of computer time or complex computer programs. To be more specific, matrix elements of the model Hamiltonian studied here, \( H_{mn} = \langle m | H | n \rangle \), are

\[
H_{mn}(g_N, h_N) = (E_m E_n)^{1/2} ( \delta_{mn} - g_N - i h_N s_{mn} ) ,
\]

where \( m \) and \( n \) are integers. For \( m = n, \delta_{mn} = 1 \) and \( s_{mn} = 0 \). For \( m \neq n, \delta_{mn} = 0 \) and \( s_{mn} = (m-n)/|m-n| \). The numbers \( E_m \) are \( b^n \) with \( b > 1 \), are eigenvalues of the operator \( H_0 \), which is defined in the same basis to have matrix elements \( \langle m | H_0 | n \rangle = H_{mn}(0,0) \). The basis states are normalized, \( \langle m | n \rangle = \delta_{mn} \). The energies \( E_n \) of the basis states are limited by a cutoff \( \Lambda = b^{N+1} \), so that \( m, n \leq N \). The indices can be also limited from below by a large negative integer \( M \), in order to make the Hamiltonian matrix finite. But the lower bound \( M \) will be set equal to \(-\infty\) in most of the discussion that follows.

Hamiltonians defined by Eq. (1) have a general ultraviolet logarithmically divergent structure, with both real and imaginary parts contributing to the divergence. Therefore, the coupling constants \( g_N \) and \( h_N \) are expected to depend on the cutoff \( N = \ln \Lambda / \ln b \) if the physical content of the theory, e.g., the energy spectrum, is to be independent of the cutoff. In fact, \( g_N \) exhibits asymptotic freedom as a function of \( \Lambda \) if \( h_N = 0 \). When \( h_N \neq 0, g_N \) exhibits instead the limit cycle behavior (or chaos) [2]. The universality of the limit cycle behavior is studied here.

Section II introduces the renormalization group (RG) equation in the model and sets the stage for next sections. Section III describes the limit cycle and a related fixed point. Section IV discusses the spectrum of the Hamiltonian and explains how the cycle is related to a geometric sequence of binding energies (discrete levels) that approach zero. Then Section V discusses universality and Wegner’s marginal and leading irrelevant operators in a linearized approximation, with non-leading operators treated in Section VI. The RG analysis enables us then to show details in Section VII of how one can tune a Hamiltonian so that the limit cycle structure becomes rapidly visible in the spectrum. Section VII includes a numerical example with first two leading irrelevant operators removed through the tuning so that the remaining irrelevant corrections to the limit cycle disappear at the rate given by factor 1/512 per cycle, for \( b = 2 \). Section VIII describes generic properties of limit cycles and Section IX briefly concludes the paper.

II. RENORMALIZATION GROUP

The eigenvalue problem with \( H \) given by Eq. (1),

\[
\sum_{n=-\infty}^{N} H_{mn} \psi_n = E \psi_m ,
\]

can be solved for \( \psi_m, m \leq N \), assuming that one knows \( E \), by using Gaussian elimination. One solves for \( \psi_N \) in terms of all other components, \( \psi_n \) with \( n < N \). Then, one expresses \( \psi_{N-1} \) in terms of components \( \psi_n \) with \( n < N - 1 \), and so on. After the first \( p \) such steps, for negligibly small \( E \), one is able to recognize the existence of limit cycles in the coupling constants \( g \) and \( h \) of period \( p \) or less, when they occur.

The eigenstate components are re-written as \( \psi_n = b^{-n/2} \phi_n \) for all \( n \leq N \), and one defines

\[
\sigma_N = \sum_{n=-\infty}^{N} \phi_n , \tag{3}
\]

\[
\pi_{Nm} = \sum_{n=-\infty}^{N} s_{mn} \phi_n , \tag{4}
\]

The eigenvalue condition produces then a set of equations,

\[
\left( 1 - \frac{E}{E_m} \right) \phi_m - g_N \sigma_N - i h_N \pi_{Nm} = 0 , \tag{5}
\]

for all integers \( m \leq N \). For \( m = N, \pi_{Nm} = \delta_{N-1} \) and,

\[
(1 - g_N - \epsilon_N) \phi_N = (g_N + i h_N) \delta_{N-1} , \tag{6}
\]

where \( \epsilon_N = E/E_N \). Equation (6) gives \( \phi_N \) in terms of all \( \phi_m \) with \( m < N \) contained in \( \sigma_{N-1} \). This result is then inserted into all the remaining equations with \( m \leq N - 1 \), leading to the new set of equations with the highest energy component removed,

\[
(1 - \epsilon_m) \phi_m - g_N \sigma_{N-1} - i h_N \pi_{N-1 m} + (-g_N + i h_N) \frac{1}{1 - g_N - \epsilon_N} (g_N + i h_N) \delta_{N-1} = 0 ,
\]

or

\[
(1 - \epsilon_m) \phi_m - \left( g_N + \frac{g_N^2 + h_N^2}{1 - g_N - \epsilon_N} \right) \delta_{N-1} - i h_N \pi_{N-1 m} = 0 . \tag{8}
\]

These equations appear to represent a new eigenvalue problem for a Hamiltonian matrix with elements,

\[
H_{mn}(g_{N-1}, h_{N-1}) = (E_m E_n)^{1/2} ( \delta_{mn} - g_{N-1} - i h_{N-1} s_{mn} ) , \tag{9}
\]
where
\[ g_{N-1} = g_N + \frac{g_N^2 + h^2}{1 - g_N - \epsilon_N}, \]  
(10)
\[ h_{N-1} = h_N + \epsilon, \]  
(11)

the new cutoff is \( \Lambda_{N-1} = b^{N-1} = \Lambda/b \). When \( N \) is reduced from \( N-1 \) to \( N-2, \epsilon_N \) goes over to \( \epsilon_{N-1} \), and so on, along with the cutoff being reduced by successive powers of \( b \). The coupling constant \( g_{N-p} \) obtained after \( p \) such steps depends on \( g_{N-p+1} \) and \( \epsilon_{N-p+1} \).

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Besides the sequence of coupling constants, the RG recursion formula can be used to find the eigenvalues \( E \). A procedure to do so can be used to show the connection between the universal RG limit-cycle behavior and the energy spectrum. In principle, one specifies an energy \( E \) and computes the sequence of couplings \( g_n \) for that value of \( E \) from the recursion
\[ g_{n-1} = g_n + \frac{g_n^2 + h^2}{1 - g_n - E/E_n}, \]  
(13)
starting from \( n = N \) and iterating all the way down to \( n = M+1 \), whereby one arrives at a \( g_M \) that must satisfy the condition
\[ 1 - g_M - E/E_M = 0, \]  
(14)
if \( E \) is an eigenvalue. Although one does not know in advance what \( E \) to pick, there exist simple search routines to find out values of \( E \) that make the expression \( 1 - g_M - E/E_M \) equal to zero. The solution to the recursion formula of Eq. (13) will be developed in the next section. We will also employ later the reverse relation to Eq. (13),
\[ g_{n+1} = g_n - \frac{g_n^2 + h^2}{1 + g_n - E/E_{n+1}}, \]  
(15)

III. LIMIT CYCLE AND RESULTING FIXED POINTS

Equation (13) can be rewritten as,
\[ \frac{g_{n-1}}{h} = \frac{(g_n/h) + h/(1 - \epsilon_n)}{1 - (g_n/h)[h/(1 - \epsilon_n)]}, \]  
(16)
to exhibit the structure of a trigonometric identity,
\[ \tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \]  
(17)
Therefore, one can introduce the angles \( \alpha_n \), and \( \beta_n \),
\[ \alpha_n = \arctan \frac{g_n}{h}, \]  
(18)
\[ \beta_n = \arctan \frac{h}{1 - \epsilon_n}, \]  
(19)
and obtain a simplified equation,
\[ \alpha_{n-1} = \alpha_n + \beta_n. \]  
(20)
After \( p \) steps, one obtains
\[ \alpha_{n-p} = \alpha_n + \gamma(E, n, n - p + 1), \]  
(21)
where
\[ \gamma(E, k, l) = \beta_k + \ldots + \beta_l. \]  
(22)
It is clear that the coupling constant returns to its value after the \( p \) steps, if
\[ \gamma(E, n, n - p + 1) = \pi. \]  
(23)
Although this is unlikely for arbitrary \( n \) and given \( E \), a significant simplification occurs in the recursion when \( E \to 0 \), or, equivalently, when \( \epsilon_n \) is so small that it can be neglected. In these circumstances,
\[ \beta_n = \arctan h + \frac{h}{1 + h^2} \frac{E}{b^n} + \frac{h}{(1 + h^2) b^n} \left( \frac{E}{b^n} \right)^2 \]  
\[ + \frac{h(1 - h^2/3)}{(1 + h^2)^3} \left( \frac{E}{b^n} \right)^3 + O(E^4), \]  
(24)
where \( O(E^4) \) denotes terms order \( E^4 \) and higher. The simplification occurs because all angles \( \beta_n \) become equal in the limit \( E \to 0 \),
\[ \beta_n = \arctan h, \]  
(25)
\[ \equiv \beta, \]  
(26)
and
\[ g_{n-p} = h \tan (\alpha_n + p\beta) \]  
\[ = g_n, \]  
(27)
if \( \beta = \pi/p \). Thus, one obtains a cyclic behavior of \( g_n \) with period \( p \), if
\[ h = \tan (\pi/p) \]  
(28)
with an integer \( p \), and if there exist eigenvalues equivalent to zero. The latter condition corresponds to the existence of two-body bound state at threshold in the three-body dynamics that was mentioned in the introduction.

Equation (28) for \( h \) shows how the cycle emerges for integer \( p \geq 3 \), \( p = 2 \) requires infinite \( h \), and \( p = 1 \) is equivalent to \( p = \infty \) and \( h = 0 \). In that case, the cycle is infinite and one has asymptotic freedom instead of a finite cycle. For rational values of \( p \) in Eq. (28), say, \( p = p_1/p_2 \) with \( p_1 \) and \( p_2 \) both integers, the sequence of coupling constants goes over a number of twists over the full period \( p_2 \geq 3 \). For irrational values of \( p \) in Eq. (28), chaotic behavior appears. We limit further discussion to integer \( p \geq 3 \) in Eq. (28). The sequence \( g_n, g_{n-1}, \ldots, g_{n-p+1} \), repeats itself for as long as \( E \) is equivalent to 0.
Equation (27) has an important implication that the recursion formula must simplify when one computes $g_{n-p}$ directly from $g_n$, because if $E = 0$, $g_{n-p}$ must equal $g_n$ regardless of the value of $n$. This result suggests that one should study the relationship between functions $f_n(\epsilon_n) = g_n(\epsilon_n)/h$ and $f_{n-p}(\epsilon_{n-p}) = g_{n-p}(\epsilon_{n-p})/h$ when $E \neq 0$. Note that $\epsilon_{n-p} = r^{p} \epsilon_n$, where
\[ r = b^{p}, \quad (29) \]
is the energy scale-factor associated with moving down in energy from $E_n$ to $E_{n-p}$, which means over the entire cycle. But in order to derive the recursion that connects $f_n(x)$ with $f_{n-p}(r x)$, one needs to introduce $p - 1$ intermediate functions.

Namely, we introduce functions $f_{n-k}(x)$ such that $f_{n-k}(\epsilon_{n-k}) = g_{n-k}(\epsilon_{n-k})/h$, with $\epsilon_{n-k} = b^{k} \epsilon_n$. In this notation, the recursion formula (10) reads,
\[ f_{n-1}(x_{n-1}) = \frac{f_n(x_n) + z_1(x_n)}{1 - f_n(x_n) z_1(x_n)}, \quad (30) \]
where $x_{n-1} = b x_n$ and $z_1(x) = h/(1 - x)$. The RG transformation over two steps gives
\[ f_{n-2}(x_{n-2}) = \frac{f_n(x_n) + z_2(x_n)}{1 - f_n(x_n) z_2(x_n)}, \quad (31) \]
where
\[ z_2(x) = \frac{z_1(x) + z_1(b x)}{1 - z_1(x) z_1(b x)}. \quad (32) \]
Proceeding by induction, suppose that after $k$ steps one has
\[ f_{n-k}(x_{n-k}) = \frac{f_n(x_n) + z_k(x_n)}{1 - f_n(x_n) z_k(x_n)}, \quad (33) \]
and one performs one more step according to Eq. (10). The result is,
\[ f_{n-k-1}(x_{n-k-1}) = \frac{f_{n-k}(x_{n-k}) + z_1(x_{n-k})}{1 - f_{n-k}(x_{n-k}) z_1(x_{n-k})}. \quad (34) \]
We substitute here the assumed expression (33) for $f_{n-k}(x_{n-k})$, and obtain,
\[ f_{n-k-1}(x_{n-k-1}) = \frac{f_n(x_n) + z_{k+1}(x_n)}{1 - f_n(x_n) z_{k+1}(x_n)}, \quad (35) \]
where
\[ z_{k+1}(x) = \frac{z_k(x) + z_1(b^k x)}{1 - z_k(x) z_1(b^k x)}. \quad (36) \]
This sequence of intermediate steps implies that after $p$ of them,
\[ f_{n-p}(x_{n-p}) = \frac{f_n(x_n) + z_p(x_n)}{1 - f_n(x_n) z_p(x_n)}, \quad (37) \]
where $z_p(x)$ is a function that can be calculated starting with $z_1(x) = h/(1 - x)$ and using recursion given in Eq. (30). The point is that one must have
\[ z_p(0) = 0 \quad (38) \]
for the identity
\[ f_{n-p}(0) = f_n(0) \quad (39) \]
to hold with arbitrary values of $f_n(0)$, which was observed before as a condition for the cycle to close in the case $E = 0$ after $p$ steps, Eq. (27). This implies that the Taylor series expansion for $z_p(x)$ around $x = 0$, starts with a term linear in $x$.

To give an example, we take $p = 3$, the shortest possible period. We focus on this example because when we discuss tuning to criticality for matrices smaller than $100 \times 100$, the shortest period will allow us to display approach to the limit cycle spectrum most clearly. We have,
\[ f_{n-3}(b^3 x) = \frac{f_n(x) + z_3(x)}{1 - f_n(x) z_3(x)}, \quad (40) \]
where,
\[ z_3(x) = \frac{a_0 + a_1 + a_2 - a_0 a_1 a_2}{1 - (a_0 a_1 + a_1 a_2 + a_2 a_0)}, \quad (41) \]
and $a_j = h/(1 - b^j x)$. For $b = 2$, one obtains
\[ z_3(x) = \frac{(7/4)x(1-x)}{1 + x^3 - (7/4)x(1+x)}, \quad (42) \]
which has an expansion,
\[ \frac{z_3(x)}{h} = (7/4)x + (21/16)x^2 + (343/64)x^3 + O(x^4), \quad (43) \]
that starts with a term order $x$, as expected. For $x = 0$, i.e., for $E = 0$, one obtains a fixed point $g_{n-3k} = g_n = g^*_n$ for arbitrary integer values of $k$. There are three such fixed-modulo-cycle points in the cycle of period 3, $g_{n-3k} = g^*_n$, $g_{n-1-3k} = g^*_{n-1}$, and $g_{n-2-3k} = g^*_{n-2}$. The value of $n$ does not matter.

**IV. THE LIMIT CYCLE**

In this section, we show that the limit cycle leads to a spectrum in the form of multiple exact geometric series with eigenvalues in each series separated by the factor $r$ of Eq. (28) in the limit $N \to +\infty$ and $M \to -\infty$. One of these series describes bound states and is negative. We prove this by showing that for fixed $E$, $M$ near enough to $-\infty$, and $N$ near enough to $+\infty$, the sequence of coupling
constants \(g_m(E)\) approaches, for \(m\) very negative but still well above \(M\),

\[
g_m(E) = -E (1 - 1/b)/b^m. \tag{44}\]

The above result can be obtained from Eq. (15) assuming the eigenvalue condition \(g_M(E) = 1 - E/E_M\), which means

\[
g_M(E) = 1 - E/b^M. \tag{45}\]

The second term is huge in comparison to \(1 + h^2\) for finite \(E\) and \(M \to -\infty\). Then Eq. (15) gives,

\[
g_{M+1}(E) = g_M(E) - \frac{g_M^2(E) + h^2}{1 + g_M(E) - E/b^{M+1}}. \tag{46}\]

By inspection, \(g_{M+1}(E) = g_M(E)/(1 + b)\) plus corrections of order 1 or \(h^2\). If \(g_{M-n+1}(E) = -g_M(E)/\alpha_{n-1}\), then, neglecting terms order 1 and \(h^2\) again, \(g_{M+n}(E) = -g_M(E)/(\alpha_{n-1} + b^n)\), which implies that \(\alpha_n = 1 + b + \ldots + b^n\) and for finite \(m = M + n\) one obtains

\[
g_{M+n}(E) = -E (1 - 1/b)/b^{M+n}, \tag{47}\]

as promised. But the existence of the limit cycle means that in the limit of large positive \(n\) the coupling constants return to the same value after \(p\) steps.

Let \(g_N = g^*(E)\) in the limit as \(N \to \infty\) through steps starting with a given finite \(m = M + n\) with large positive \(n\) and \(M \to -\infty\). Then Eq. (15) says that with \(k = 1\) one has \(g_{m+p}(rE) = g_m(E)\). But by translational invariance in \(m\) by \(p\), which is valid in the limit cycle, this equation continues to hold for all larger values \(k\). In the limit of large positive \(k\), \(g_{m+k}(rE) = g_{m+(k-1)p}(E) = g^*(E)\). This means \(rE\) is also an eigenvalue for the same \(g^*\) as \(E\) is. The same argument works for all values of \(m\) within one cycle, leading to as many series as elements in the cycle, which equals \(p\).

It is necessary to actually construct \(p\) eigenvalues not separated by the factor \(r\) using Newton’s method to determine the full set of \(p\) distinct geometric series. Our numerical experience is that for any combination of \(b\) and \(p\) that we have tried, there is precisely one negative eigenvalue sequence and \(p - 1\) positive eigenvalue sequences. Besides the numerical experience with \(1 < b < 10\), one can also consider very large \(b\). In this case the eigenvalues are \(E_n(1 - g_n)\), which translates, using Eq. (15), into \(E_n(1 - h \tan \alpha_n)\), so that one might expect more negative eigenvalue series, one for every \(g_n > 1\) in one cycle. But we observe here that with \(h = \tan \beta\) a coupling constant \(g_0\) can be greater than 1 when \(\pi/2 - \beta < \alpha_0 < \pi/2\), and this can only occur for one \(n\) value in a cycle. This implies that there is only one negative eigenvalue series for all choices of \(b, g^*,\) and \(p\), i.e., including the large \(b\) limit. The apparent discrepancy between this result and Eq. (17) disappears when one observes that Eq. (17) is valid for \(|E| \gg E_{M+n}\).
V. MARGINAL AND IRRELEVANT OPERATORS

In this section we present an analysis of the two-variable renormalization group, in which $f_{n-kp}(x)$ of Section III is represented by a linear approximation in $x$ for any $k$,

$$f_{n-kp}(x_{n-kp}) = \frac{g_n^*}{h} + f_{n-kp}^{(1)} x_{n-kp}, \quad (56)$$

neglecting terms of order $x_{n-kp}^2$ or higher. The subscript $n$ that numbers distinct points in the cycle will be omitted to simplify notation wherever possible.

The RG equations for $g^*$ and $f^{(1)}$ for a jump of $p$ steps are,

$$g^* = g^*, \quad (57)$$

$$r f_{n-p}^{(1)} = f_n^{(1)} + (1 + g^2 / h^2) z_p^{(1)}/(r - 1), \quad (58)$$

where $z_p^{(1)}$ is the coefficient of $x_n$ in the expansion of $z_p(x_n)$ in powers of $x_n$. The first step in the analysis is to solve Eq. 59 for a Hamiltonian with $g_n$ given and fixed, which means that $f_n^{(1)}$ is zero. For $N - n$ large, and a multiple of $p$, $f_n^{(1)}$ is fixed at a value which ensures that $f_{n-p}^{(1)}$ is the same as $f_n^{(1)}$, namely

$$f_n^{(1)} = f^{(1)*} = \frac{(1 + g^2 / h^2) z_p^{(1)} / (r - 1),}{(59)}$$

which is valid for $n$ of the form $N - kp$ with $k$ large enough. The solution leading from the boundary condition $f_N^{(1)} = 0$ to the fixed point is,

$$f_{N-kp}^{(1)} = z_p^{(1)} (1 + g^2 / h^2)$$

$$\times (r^{-1} + r^{-2} + \ldots + r^{-(k-1)}). \quad (60)$$

Our discussion of the fixed point and operators near the fixed point uses the variable $x$ without any reference to $n$ or $k$, and the function $f(x)$ is assumed to have the generic form,

$$f(x) = f_0 + f_1 x. \quad (62)$$

The fixed point condition is,

$$f(rx) = f(x) + (1 + f_0^2 / h^2) z_p^{(1)} x. \quad (63)$$

Or, written in terms of a transformation $R$,

$$f(x) = R[f]$$

$$= f(x/r) + (1 + f_0^2 / h^2) z_p^{(1)} x/r. \quad (64)$$

This has already been solved to give a one parameter family of fixed points,

$$f^*(g^*, x) = f_0^*(g^*) + f_1^*(g^*) x. \quad (65)$$

where

$$f_0^*(g^*) = g^*/h, \quad (66)$$

$$f_1^*(g^*) = f^{(1)*}(g^*). \quad (67)$$

Operators are obtained by linearizing the transformation $R$ about a fixed point. We write,

$$f(x) = f^*(g^*, x) + df(g^*, x), \quad (68)$$

and treat $df$ as small compared to $f$. Then,

$$R[f + df] = f(x/r) + df(x/r) + z_p^{(1)}$$

$$\times (1 + f_0^2 / h^2 + (2 f_0 d h_0 / h^2) x/r). \quad (69)$$

Wegner’s operators are extracted from an eigenvalue condition,

$$L(df) = w df(x), \quad (70)$$

where $w$ is an eigenvalue and $L$ is the linearized form of $R$. Assuming a linear function $df$,

$$df(x) = c_0 + c_1 x, \quad (71)$$

one obtains two equations that must be satisfied simultaneously,

$$c_0 = w c_0, \quad (72)$$

$$\frac{1}{r} c_1 + 2 z_p^{(1)} g^*/r c_0 = w c_1. \quad (73)$$

The eigenvalue problem has two solutions, one corresponding to a marginal operator with $w_0 = 1$, and one corresponding to an irrelevant operator with $w_1 = r^{-1} = 1/b^p$, which is less than 1. The marginal operator has $c_0$ arbitrary (let it be 1) but it has a $c_1$ value also, which is determined given $c_0$. The irrelevant operator has $c_0$ equal to zero and an arbitrary value for $c_1$. Critical exponents, $\lambda$ for $l$-th operator, can be read from Wegner’s eigenvalues $R$ using relation,

$$w_l = r^\lambda. \quad (74)$$

Clearly, $\lambda_0$ is 0 for the marginal operator. This operator is just the derivative of the fixed point function with respect to $g^*$. $\lambda_1$ is $-1$ for the irrelevant operator. This result is obtained here generically for all choices of parameters in the model.

The link from operators to the behavior of eigenvalues is following. The marginal operator causes the entire eigenvalue spectrum to shift while preserving the fact that successive eigenvalues have a ratio of $r$ as long as eigenvalues are far from either the ultraviolet cutoff $b^N$ or the infrared cutoff $b^M$. But as one approaches either the ultraviolet or infrared cutoff limits, in the energy, the eigenvalue ratios change in ways that can only be determined numerically, except for the case that $g^*$ is cycled continuously through all real values from $g^*$ to $\infty$, then from $-\infty$ up to $g^*$ again. This complete cycle can
be shown to be equivalent to the renormalization group transformation through $p$ steps already discussed.

The leading irrelevant operator, if present (as it is if one solves the Hamiltonian of this article for finite $N$), causes the ratios of eigenvalues to approach the number $r$ as the eigenvalue itself decreases from its maximum near $E_N$ at a rate given by $w_1 = 1/r$ for each reduction of the eigenvalue by $r$. This phenomenon occurs through the same recursion features that we used in Section IV to show the existence of geometric sequences of eigenvalues, except that one has to look for corrections due to finite $N$. If the leading irrelevant operator is tuned out (see Section VII), the next operator causes the rate to become $1/r^2$. It will be shown in the next section that Wegner’s irrelevant operators of successive orders have eigenvalues given by successive powers of $1/r$, which suggests that the elimination of the next-to-leading irrelevant operator should accelerate the rate to $1/r^3$. Numerical example of this behavior is shown in Section VII.

VI. HIGHER-ORDER CONTRIBUTIONS

In order to illustrate numerically in Section VII how the tuning of a finite initial Hamiltonian to criticality can be achieved, we need to discuss the behavior of the higher-order terms in $f(x)$, i.e., the terms beyond $f_0$ and $f_1$ introduced in Eq. (72). Our numerical example in Section VII uses $N = 17$ and $M = -51$, and we choose the smallest possible cycle period, $p = 3$, for the largest possible number of cycles to fit within the bounds of these $M$ and $N$. Accordingly, we focus in this section on the case $p = 3$ and discuss the higher-order terms in functions $f_{n−3k}(x)$, $f_{n−3k−1}(x)$, and $f_{n−3k−2}(x)$, with arbitrary $n$ and $k = \tan \pi/3 = \sqrt{3}$.

In principle, the choice of $g*_{N}$ is not important, but in order to have the limit cycle universality pattern described in terms of integers or ratios of small integers, it is useful to aim at $g^* = -7$ at some value of $n < N$, see Table I and Appendix A.

| $n$  | $h f^0_{n}$ | $h f^*_n$ |
|------|-------------|-----------|
| $n$  | $-7$        | $13$      |
| $n - 1$ | -$1/2$   | $13/16$  |
| $n - 2$ | $5/3$      | $13/9$   |

For some $n$, the fixed point Eq. (40) can be written in the form,

$$f(8x) = f(x) + z_3 [1 + f(8x) f(x)],$$

which can be used to generate a power series solution for the function $f(x)$. The constant term, which we denoted $g^*/h$, is arbitrary, but given $g^*$, the remaining terms are determined. We can write

$$h f(x) = h f(x, g^*)$$

$$= g^* + c_1(g^*) x + c_2(g^*) x^2 + c_3(g^*) x^3 + ..., \quad (76)$$

and the solution for the case $g^* = -7$ is,

$$h f(x, -7) = -7 + 13x - (65/3)x^2 + (221/6)x^3 + O(x^4). \quad (77)$$

This result is extended to arbitrary values of $b$, $p > 3$, and $g^*$, in Appendix B.

Wegner’s operators are infinitesimal functions $df(x, g^*)$ which satisfy Eq. (40) when added to the fixed-point solution for $f^*(x, g^*)$. Namely,

$$\frac{f(x, g^*) + df(x, g^*) + z_3}{1 - z_3 [f(x, g^*) + df(x, g^*)]} - f(8x, g^*) = w df(8x, g^*), \quad (78)$$

valid to first order in $df$, with $w$ being Wegner’s eigenvalue. Each eigenoperator begins with a different power of $x$, and the leading power of $x$ determines $w$, namely $w_j$ is $1/8^j$ for the operator that begins with the power $x^j$, where $j$ can be 1, 2, 3, etc. Each operator then has coefficients for all higher powers of $x$, just as $f(x, g^*)$ does. Key details of the supporting argument are given in Appendix C for arbitrary $b$, $g^*$, and integers $p \geq 3$.

Extracting the linear terms, the precise equation for determining higher-order terms in $df$ is the following, cf. Eq. (79).

$$w df(8x, g^*) = df(x, g^*) + z_3 \left[ df(x, g^*) f(8x, g^*) \right] + f(x, g^*) df(8x, g^*). \quad (79)$$

VII. TUNING TO A CYCLE

We show below how to tune parameters so that a finite Hamiltonian can exhibit universal features of a limit cycle in its spectrum. We consider the case of $p = 3$ and we quote numerical results for $b = 2$.

Our numerical studies began with alteration of the matrix element $H_{NN}$ and attempts to use this element to obtain a fixed point value for the leading marginal operator already in two iterations. The idea was to obtain the approach to scaling with the rate of 1/64 instead of only 1/8 per cycle. Then, alterations of four matrix elements with largest subscripts were studied, and it was found that by changing only these 4 matrix elements in the Hamiltonian of Eq. (11), one can also make the next-to-leading irrelevant operator vanish, which leads to extremely fast approach to the geometric scaling of the eigenvalues, clearly visible within a small number of cycles.

We look at mappings from parameters in the initial Hamiltonian matrix with additional couplings in the elements $H_{NN}$, $H_{N-1N}$, $H_{N-1N}$, $H_{NN-1}$, to the coupling
$g_{N-2}$ and its expansion for small $E$. Denote the relevant matrix elements as,

$$H_{NN} = E_N t,$$

$$H_{N-1N} = \sqrt{E_N E_{N-1}} (u + iv),$$

$$H_{NN-1} = \sqrt{E_N E_{N-1}} (u - iv),$$

$$H_{N-1,N-1} = E_{N-1} z.$$

Then, $g_{N-2}$ can be written as the ratio of two determinants. The numerator is the determinant of a 3×3 matrix that can be written with overall energy factors $E_N$, etc., removed. Then, the 3×3 matrix is,

$$
\begin{bmatrix}
t - x/b^2 & u - iv & -g - ih \\
u + iv & z - x/b & -g - ih \\
-g + ih & -g + ih & -g
\end{bmatrix}
$$

where $g$ is $g_N$ and $x$ is $E/E_{N-2}$. The denominator is the determinant of the upper left $2 \times 2$ matrix within this 3×3 matrix. The result is that

$$g_{N-2} = g - (g^2 + h^2)$$

$$\times \frac{2u - (t + z) + x(b + 1)/b^2}{tz - (u^2 + v^2) - x(tb + z)/b^2 + x^2/b^2}.$$  \hspace{1cm} (85)

It is convenient to set $t = u = z$, leaving $v$ and $z$ as free parameters. Then, with $h = \sqrt{3},$

$$g_{N-2} = g + x (g^2 + 3)$$

$$\frac{b + 1}{2b^2v^2}$$

$$- x^2 (g^2 + 3) z \left( \frac{b + 1}{2b^2v^2} \right)^2 + O(x^3).$$ \hspace{1cm} (86)

The fixed point value for $g_{N-2}$, through order $x^2$, can also be worked out, and gives

$$g_{N-2} = g + x (g^2 + 3)$$

$$\frac{1}{4(b - 1)}$$

$$+ x^2 (g^2 + 3) \frac{g + (b - 1)/(b + 1)}{16(b - 1)^2} + O(x^3).$$ \hspace{1cm} (87)

From matching these two formulae, one obtains

$$v^2 = 4(b^2 - 1)/b^2 = 3,$$ \hspace{1cm} (88)

and,

$$z = -g - (b - 1)/(b + 1) = 20/3,$$ \hspace{1cm} (89)

with the numerical values valid for $g = -7$ and $b = 2$. Using $t = u = z = 20/3$, $v = \sqrt{3}$, and $g_{N} = -7$, we find that

$$g_{N-2}(x) = -7 + 13x/(1 + 5x/3 - x^2/24)$$

$$= -7 + 13 x - \frac{65}{3} x^2 + o(x^3),$$ \hspace{1cm} (90)

in agreement with Eq. \[(90)\] , and indeed the eigenvalues converge very fast towards a fixed ratio, as illustrated by Table \[(10)\] . There is a negative eigenvalue sequence separated by a factor rapidly approaching 8 and a positive eigenvalue sequence whose ratios rapidly approach an alternating sequence of two ratios both near the square root of 8. As a numerical check, we have verified that the sum of these eigenvalues is to twelve decimal places the trace of the original Hamiltonian. The eigenvalues in Table \[(10)\] are given for $N = 17$ and $M = -51$ (low enough to have no impact on these numbers: smaller eigenvalues are omitted). The number $\sim 400$ in the right column turns into the original Hamiltonian. The ratios for the positive eigenvalues can be compared with $b^{p/r^{(p-1)}} = \sqrt{3} \approx 2.82842712474619$, looking at the average value $(2.82833219599 + 2.828521033011)/2 = 2.82842712605$, and at the product $2.82833219599 \times 2.828521033011 \sim 7.999999999999491$.

\begin{table}[h]
| $E$          | ratio  | $w_{3-1}$ |
|--------------|--------|-----------|
| 0.115359460521 | 3.28333219599 | 2.828333219599 |
| 0.326274994287 | 3.28333219599 | 2.828333219599 |
| 0.922875684170 | 3.28333219599 | 2.828333219599 |
| 2.610199550997 | 3.28333219599 | 2.828333219599 |
| 7.383005473558 | 3.28333219599 | 2.828333219599 |
| 20.881596407900 | 3.28333219599 | 2.828333219599 |
| 59.064043787554 | 3.28333219599 | 2.828333219599 |
| 167.052797170935 | 3.283332200314 | 2.8283332200314 |
| 472.51235312001 | 3.283333591344 | 2.828333591344 |
| 1336.422560688180 | 3.28333591344 | 2.828333591344 |
| 3780.109315479000 | 3.28333591344 | 2.828333591344 |
| 10692.223891890000 | 3.283347649430 | 2.828347649430 |
| 30306.851247105530 | 3.283347649430 | 2.828347649430 |
| 91327.456629399810 | 3.014326102394 | 2.828333219599 |
| 184274.370928830000 | 19.975092247797 | 2.828333219599 |
\end{table}
VIII. GENERIC PROPERTIES OF LIMIT CYCLES

The model Hamiltonian discussed in this paper has an extraordinarily simple solution despite its exhibiting a limit cycle. The initial coupling $g_N$ is given as an explicit analytic expression as a function of an energy eigenvalue $E$ that involves nothing worse than a convergent infinite sum of arc tangents. The renormalization group transformation for the model can be written in terms of one, two, or an arbitrary number of coupling parameters. Analysis of the transformation with $j$ couplings yields exact results for the limit cycle itself and for the critical exponents of the $j$ leading operators, including a marginal operator and a succession of irrelevant operators with integer exponents.

Obviously, this is too simple to all carry over to more general renormalization group limit cycles. What does carry over to the more general case? First of all, we cannot guarantee that a renormalization group limit cycle will lead to any bound states. But if there are bound states close to threshold, then these bound states must form an infinite geometric series converging on zero. The reason for this infinite geometric series is the existence of a fixed point Hamiltonian for a change of scale $r$ that corresponds to a transformation around the complete limit cycle ($r = b^p$ in the discrete model). The proof of this statement begins with the fact that if the fixed point Hamiltonian with cutoff $\Lambda$ has an energy eigenvalue $E$ much less than $\Lambda$, then the Hamiltonian after the renormalization group transformation has cutoff $\Lambda/r$, yet exactly the same form. Therefore the Hamiltonian after the transformation must have an eigenvalue $E/r$. But the definition of the transformation is that it preserves eigenvalues, which means that the Hamiltonian before the transformation must have $E/r$ as an eigenvalue too, as well as its original eigenvalue $E$. Then by following the same reasoning over again, one can establish that the complete geometric series $E, E/r, E/r^2$, etc., must all be eigenvalues for the fixed point Hamiltonian.

The second observation to make is that marginal and irrelevant operators vary at different locations on the limit cycle, but the critical exponents of these operators do not vary. In the model, the operators have the form of a set of coefficients $c_1(g_n^*)$, $c_2(g_n^*)$, etc., that depend on the values $g_n^*$ that make up the limit cycle. The critical exponents are the integers $-1$, $-2$, etc., and do not depend on the location $n$ on the cycle. Let us us then consider the general case. Suppose there is a non-linear transformation $R_b$ that forms the basic (this is why we introduce the subscript $b$) renormalization group transformation that when iterated $p$ times, $R_bR_b \ldots R_b$, produces the transformation $R$ defined earlier that corresponds to a complete circuit of the cycle, which we could denote here by $R_r$. Suppose then that one has a limit cycle of Hamiltonians $H'_n$ of period $r = b^p$. Suppose that $dH'_n$ is one of Wegner’s operators for this limit cycle. We now apply the renormalization group transformation $R_r$ to $H'_n + dH_n$ $k$ times, where $k$ is arbitrarily large. Suppose that the critical exponent for $dH_n$ were an exponent $\lambda_n$ dependent on $n$, i.e., we would have different exponents for different $n$s. Then there would exist two ways to apply the iterated transformation that would give two different results, while on the other hand they would have to give the same result in the case of limit cycle. The first way is to replace $R_b$ iterated $kp$ times by $R_r$ iterated $k$ times, in which case $H'_n + dH_n$ becomes $H'^*_n + r^{\lambda_n}dH_n$. The second way is to apply $R_b$ to $H'^*_n + dH_n$, then apply the transformation $R$ for $k - 1$ times, then apply $R_b$ for $p - 1$ times. The only source of $k$ dependence is the $k - 1$ applications of $R_r$, but it is applied to $H'^*_n + dH_n - 1$ which means that $dH$ is multiplied by $r^{(k-1)\lambda_n-1}$. If $\lambda_n - 1$ is different from $\lambda_n$, there is an immediate contradiction in the $k$-dependence of the two different approaches. The same argument works for all values of $n$ within the cycle.

The third observation is that the model has a marginal operator, that is, an operator with a critical exponent of zero. The presence of a marginal operator is guaranteed if one has a renormalization group in differential form so that the limit cycle must be continuous, and position on the cycle is labeled by a continuous parameter, just as $g^*$ is continuous in the model. In this case, the marginal operator is obtained by differentiating the limit cycle Hamiltonians themselves with respect to the parameter, and it is easily established that the critical exponent for this operator must be zero. But when the renormalization group is discrete, and the limit cycle is a finite set of points rather than a continuous curve in the space of Hamiltonians, we do not know whether or not a marginal operator has to occur. In addition, while our model has no relevant operators, this need not be true for other limit cycles, and indeed, the limit cycle for the three-body case has a relevant operator associated with infinitesimal changes in the strength of the two-body potential in the model. When the relevant operator is absent, there is a two-body bound state exactly at threshold, and inclusion of the relevant operator causes this threshold bound state either to acquire a non-zero binding energy or to blend into the two-body scattering continuum.

Finally, we showed that parameters in an initial Hamiltonian could be tuned to eliminate the two leading irrelevant operators and thereby ensure a very rapid approach to a pure geometric series of the bound state eigenvalues for the Hamiltonian. We cannot guarantee that such tuning will be possible in other examples of renormalization group limit cycles, such as the limit cycles for the three-body Hamiltonians $A, E, B, C, D, T$. The problem is that there is a non-linear mapping from parameters in an initial Hamiltonian to coefficients of Wegner’s operators, and with non-linear mappings, there is no assurance that any given set of output parameters can be obtained from the mapping. We were lucky to find an easily identified solution for the tuning problem for the starting Hamiltonian that we used. In other cases, one has to make careful studies to establish if a desired option for tuning
The fixed point of this transformation over the cycle is given by
\[ c_n^* = u_n / (1 - v_n) , \]  
and the \( p - 1 \) values in between can be calculated for given \( g_n^* \) at one value of \( n \) using Eq. (A3).

**APPENDIX B: DIRECT ENERGY EXPANSION IN THE MODEL**

Including corrections due to \( E \neq 0 \) in Eq. (24), and inserting them in Eq. (21), one obtains for \( p \) \( \arctan \) \( h = \pi \) that,
\[ \frac{g_{n-p}}{h} = \tan [\alpha_n + e_1 E + e_2 E^2 + e_3 E^3 + O(E^4)] , \]  
where \( g_n \) and \( \alpha_n \) are functions of \( E \), and
\[ e_1 = \frac{h}{1 + h^2} b_1 \]  
\[ e_2 = \frac{h}{(1 + h^2)^2} b_2 \]  
\[ e_3 = \frac{h}{(1 + h^2)^3} \left( 1 - \frac{h^2}{3} \right) b_3 + \frac{b_2}{b_3} \]  
with \( b_k = (b^{kp} - 1)/(b^{kp}) \). Then, the coupling constant dependence on \( E \) obeys the recursion,
\[ g_{n-p} = g_n + \tilde{c}_1 E + \tilde{c}_2 E^2 + \tilde{c}_3 E^3 + O(E^4) , \]  
\[ \tilde{c}_1 = \frac{g_n^2 + h^2}{1 + h^2} b_1 \]  
\[ \tilde{c}_2 = \frac{g_n^2 + h^2}{(1 + h^2)^2} \left[ b_2 + g_n b_n^2 \right] \frac{1}{b_3 n} \]  
\[ \tilde{c}_3 = \frac{g_n^2 + h^2}{(1 + h^2)^3} \left[ 1 - \frac{h^2}{3} \right] b_3 + 2 g_n b_2 b_1 \]  
\[ + \left( g_n^2 + h^2 / 3 b_n^3 \right) \frac{1}{b_3 n} \]  
Assuming that \( f_n(x) = g_{n-p}(rE) \) and \( g_n(E) = f_n(x) \) with \( x = E/E_n \), and expanding Eq. (B5) in a series of powers of \( x \), one arrives at (the subscript \( n \) is dropped, but there are \( p \) distinct functions with \( p \) different values of \( g^* \), forming a cycle),
\[ f(x) = g^* + c_1 x + c_2 x^2 + c_3 x^3 + O(x^4) , \]  
with the coefficients,
\[ c_1 = \frac{g^* + h^2}{1 + h^2} b_1 \]  
\[ c_2 = \frac{g^* + h^2}{(1 + h^2)^2} \left[ g^* \left( \frac{b_1}{r - 1} \right)^2 + \frac{b_2}{r^2 - 1} \right] , \]  
\[ c_3 = \frac{g^* + h^2}{(1 + h^2)^3} \left[ 1 - \frac{h^2}{3} \right] b_3 + 2 g_n b_2 b_1 \]  
\[ + \left( g_n^2 + h^2 / 3 b_n^3 \right) \frac{1}{b_3 n} , \]  
Assuming that \( f_n(x) = g_{n-p}(rE) \) and \( g_n(E) = f_n(x) \) with \( x = E/E_n \), and expanding Eq. (B5) in a series of powers of \( x \), one arrives at (the subscript \( n \) is dropped, but there are \( p \) distinct functions with \( p \) different values of \( g^* \), forming a cycle),
\[ f(x) = g^* + c_1 x + c_2 x^2 + c_3 x^3 + O(x^4) , \]  
with the coefficients,
\[ c_3^* = \frac{g^{*2} + h^2}{(1 + h^2)^3} \left[ (g^{*2} + h^2/3) \left( \frac{b_1}{r - 1} \right)^3 + 2g^* \frac{b_1}{r - 1} \frac{b_2}{r^2 - 1} + \frac{(1 - h^2/3)b_2}{r^3 - 1} \right] \] (B12)

These expressions provide fixed-modulo-cycle values of 3 additional coupling constants for arbitrary choices of \( g^* \), \( h = \tan (\pi/p) \), and \( b \), with integer \( p > 3 \), in addition to the results obtained in Section V. The \( p \) different constants \( g^* \) in the limit cycle, \( g_n^*, g_{n-1}^*, \ldots , g_{n-p+1}^* \), are determined by a choice of any one of them with an arbitrarily selected \( n \).

**APPENDIX C: EIGENVALUES** \( w_j \) FOR \( j > 1 \)

Equation (21) implies
\[ \alpha_{n-p}(x) = \alpha_n(x) + \gamma(x), \] (C1)
where
\[ \gamma(x) = \sum_{k=0}^{p-1} \arctan \frac{h}{1 - b^k x}, \] (C2)
see Eq. (22). One has \( \gamma(0) = \pi \), and the fixed point relation can be written as,
\[ \alpha^*(x,g^*) = \alpha^*(x,g^*) + \gamma(x) - \gamma(0). \] (C3)

The coefficients of the Taylor series for the function \( \alpha(x,g^*) \) around zero,
\[ \alpha(x,g^*) = \tau_0(g^*) + \sum_{m=1}^{\infty} \tau_m(g^*)x^m/m!, \] (C4)
have fixed point values given by \( \tau_m(g^*) = \tau_m^*(g^*) \), with
\[ \tau_m^*(g^*) = \gamma_m/(r^m - 1), \] (C5)
where \( \gamma_m \) are given by the expansion of the known function \( \gamma(x) \),
\[ \gamma(x) = \pi + \sum_{m=1}^{\infty} \gamma_m x^m/m!. \] (C6)

Suppose that
\[ \alpha(x,g^*) = \alpha^*(x,g^*) + \delta_j x^j, \] (C7)
with infinitesimal \( \delta_j \) and \( j > 1 \). The RG transformation over the cycle gives,
\[ w_j \delta_j = r^{-j} \delta_j, \] (C8)
and a whole series \( df_j(x,g^*) \) of powers of \( x \) greater than or equal to \( j \) is generated, by Wegner’s eigenvalue condition,
\[ w_j df_j(x,g^*) = df_j(x,g^*) + z_p(x)[df_j(x,g^*) f(x,g^*) + f(x,g^*) w_j df_j(x,g^*)], \] (C9)
with \( f(x,g^*) = h \tan [\alpha^*(x,g^*)] \).

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