Existence of symplectic structures on torus bundles over surfaces

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Abstract

Let $E$ be the total space of a locally trivial torus bundle over the surface $\Sigma_g$ of genus $g > 1$. Using the Seiberg–Witten theory and spectral sequences we prove that $E$ carries a symplectic structure if and only if the homology class of the fiber $[T^2]$ is nonzero in $H_2(E, \mathbb{R})$.

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Contents

1 Introduction 1

2 $\text{Spin}^c$–structures and SW invariants 6
   2.1 Classification and pulling back of $\text{Spin}^c$–structures . . . . . . 6
   2.2 Seiberg–Witten invariants ........................................... 8

3 Principal torus bundles 11
   3.1 $n$ odd ................................................................. 16
   3.2 $n$ even ................................................................. 17

4 General torus bundles 19
   4.1 First case .............................................................. 26
   4.2 Second case ........................................................... 26
   4.3 A characterization of the property $[T^2] \neq 0$ ................. 27

1 Introduction

Let $E$ be a closed 4–manifold. The problem whether $E$ admits a symplectic structure is in general very difficult. There are, however, some theorems on

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existence of symplectic forms, for example Thurston’s construction \cite{TH1}. Recalling that $\Sigma_g$ denotes the surface of genus $g$, his theorem can be stated as follows:

**Theorem 1.1 [Thurston]** Let $E \xrightarrow{\pi} \Sigma_g$ be a surface $\Sigma_h$–bundle over a surface $\Sigma_g$. If the homology class of the fiber is nonzero in $H_2(E, \mathbb{R})$, then $E$ admits a $\pi$–compatible (i.e. compatible with the bundle structure) symplectic form.

Consider a locally trivial surface bundle over a surface. If the fiber is different from a torus, Thurston’s construction gives a symplectic form on $E$. If the fiber is the torus, the situation is more complicated. If the base is the sphere $S^2$, then all torus bundles are principal and only the trivial bundle admits a symplectic structure, as for nontrivial bundles we have $b_2 = 0$ (by $b_2$ we mean the dimension of the second cohomology group $\dim H^2(\ast, \mathbb{R})$). If the base is the torus, then Geiges \cite{G} showed that the total space always supports a symplectic form. The case when the base is a surface of genus $g > 1$ was still unsolved in general.

The principal motivations for us were relations between the space of all symplectic forms and the subspace of invariant symplectic forms, assuming $M$ is endowed with a free circle action. In \cite{HW} we were concerned with the space $S_{inv}$ of such invariant symplectic forms on 4–manifolds. Here we study torus fibration from viewpoint of existence of free circle action preserving fibers. This vehicle recognizes these bundles which admit symplectic structure. That’s why in section 4 we determine necessary and sufficient condition for existence of such action (Lemma 4.5). We treat the two cases (i.e. these admitting and not admitting such actions) separately. In the case where there is no such action we use the Leray–Serre spectral sequences in order to prove 4.9. In the other case we use relevant facts from \cite{HW} and results from section 3 to achieve our goals.

Another motivation for us was the following (see \cite{B2}):

**Conjecture 1.2 (Baldridge)** Let $E$ be a closed oriented 4–manifold admitting a symplectic structure and equipped with a free action of the circle $S^1$. Then the quotient $E/S^1$ fibers over $S^1$.

Conjecture 1.2 is a generalization of Taubes’ Conjecture, which we quote below.

**Conjecture 1.3 (Taubes)** Let $M$ be a closed oriented 3–manifold. Then $S^1 \times M$ admits a symplectic structure if and only if $M$ fibers over $S^1$.

Note that Taubes’ Conjecture 1.3 is equivalent to the existence of invariant symplectic form under the action of $S^1$ on the first factor of $S^1 \times M^3$.

Total spaces of torus bundles equipped with free circle action preserving fibers over surfaces form a natural class to check the conjecture 1.2. We
prove it for this class in Section 3 by showing that whenever $E$ is a torus bundle over $\Sigma_g$ with genus greater than 1 (here and in the sequel $\Sigma_g$ will denote the oriented surface of genus $g$), which admits a free circle action preserving fibers and a symplectic form, then $E/S^1 \cong \Sigma_g \times S^1$.

The main result of this paper is the following

**Theorem 4.9** The total space $E$ of the fibration $T^2 \to E \overset{\pi}{\to} \Sigma_g$ ($g > 1$) is symplectic if and only if the homology class $[T^2]$ represented by the fiber is nonzero in $H_2(E, \mathbb{R})$.

We should recall that there exists a topological characterization of 4–dimensional symplectic manifolds ([Go]). They can be characterized by existence of the hyperpencil. This characterization, however, does not seem to be applicable in the case of our interests, i.e. torus bundles over the surface $\Sigma_g$ of genus $g > 1$. The reason is that there is no evident proof of nonexistence of the hyperpencil on these torus bundles which do not admit any symplectic structure.

The main tools used in this paper are the Seiberg–Witten invariants and spectral sequences.

Recently Etgü proved in [E] that a principal torus bundle $E$ over $\Sigma_g$ admits no symplectic form if $g > 1$ provided that $E$ is a product of the circle $S^1$ and a three–dimensional manifold which is a nontrivial $S^1$ bundle over $\Sigma_g$ ($g > 1$). His result verifies positively Taubes’ Conjecture [3] for manifold $M$ which are $S^1$–principal bundles over $\Sigma_g$ ($g > 1$). In the first part of this paper we prove a theorem which extends his results. Before stating the theorem let us recall basic definitions concerning principal torus bundles and the Euler class of such bundles.

Each $T^2$–principal bundle is obtained by pulling back the universal bundle $ET^2 \to BT^2$. Since $BT^2$ is homotopy equivalent to $BS^1 \times BS^1$, therefore $T^2$–principal bundles over a surface $\Sigma_g$ are classified by a pair $(m, n)$ of two integers. This pair is also called the Euler class of the bundle. The Euler class has yet another interpretation. Take the product $\Sigma_g \times T^2$ and choose any disc $D^2 \hookrightarrow \Sigma_g$. Remove the counterimage $\pi^{-1}(D^2)$ and glue it back via the identification map $T^2 \times \partial D^2 \to T^2 \times \partial D^2$ given by the formula $((x_1, x_2), \theta) \mapsto ((x_1 + \frac{m\theta}{2\pi}, x_2 + \frac{n\theta}{2\pi}), \theta)$. The resulting bundle has the Euler class equal to $(m, n)$.

In Section 3 we consider principal $T^2$–bundles over surfaces and prove the following:

**Theorem 3.1** Let $T^2 \to E \overset{\pi}{\to} \Sigma_g$ be a principal fibration over the surface $\Sigma_g$, where $g > 1$. Let $(m, n)$ be the Euler class of the fibration. If $mn \neq 0$, then $E$ is not a symplectic manifold.
Theorem 3.1 is proved in two steps. In the first we prove, following Etgü \[E\], that $0 \in H^2(E, \mathbb{Z})$ can be the only basic class. In the second we write the formula counting the SW–invariant of $0 \in H^2(E, \mathbb{Z})$ and justify that $sw^4_E(0)$ is always even, thus the conclusion comes from Taubes’ theorem.

Theorem 3.1 generalizes directly to the case of torus bundles which are circle bundles over total spaces of nontrivial circle bundles over surfaces $\Sigma_g$, $g > 1$. The only difference is that the formula for $sw^4_E(0)$ slightly changes, but the proof remains the same.

The second part is devoted to arbitrary (locally trivial and orientable) torus bundles over $\Sigma_g$, $g > 1$. In this part we prove Theorem 4.9. In fact our proofs give a method to decide whether a given bundle satisfy the condition of Theorem 4.9 (i.e. when $[T^2] \neq 0$ in $H^2(E, \mathbb{Z})$). This theorem completes the classification of surface bundles over surfaces admitting symplectic structures. Theorem 4.9 is equivalent to the fact that the total space $E$ of the fibration admits a symplectic structure if and only if it supports a $\pi$–compatible symplectic form (\[TH1\] \[MS\]). Obviously, the only nontrivial implication is that whenever $E$ supports a symplectic structure, it has a $\pi$–compatible symplectic form.

We describe now this paper in some details. In section 3 we are mainly concerned with principal $T^2$–bundles over $\Sigma_g$ ($g > 1$).

We start with an observation that only $0 \in H^2(E, \mathbb{Z})$ can be basic class. In order to prove this we use similar reasoning as in \[E\]. Next we write the formula for SW–invariant of $0 \in H^2(E, \mathbb{Z})$ using \[B1\] \[B2\] \[MOY\]. In order to use Taubes’ theorem \[2.2\] we prove that the formula always gives an even number. In Section 2 are gathered preliminary notions and facts used in the proof of Theorem 4.9. Further details of the proof are contained in Section 3.

In section 4 we deal with general torus bundles over surfaces and we finish the proof of the Theorem 4.9. We consider two cases:

1. bundles which does not admit any free circle action preserving fibers,
2. bundles which support such action.

A key idea here is that the property that $E$ admits a free circle action preserving fibers can be described in terms of monodromy (Proposition 4.4). This allows us to prove a useful trichotomy (Lemma 4.5) which gives a connection between the Betti number $b_1(E)$ and existence of such actions for flat fibrations. We next prove that if $E$ does not admit such actions, then we have $[T^2] \neq 0$ and $E$ is symplectic (Theorem 4.3). If $E$ supports such actions, then we prove that $E$ is symplectic if and only if $E/S^1 \cong \Sigma_g \times S^1$. 

4
and $E$ is not a principal torus bundle over $\Sigma_g$. We also prove Conjecture 1.2 in the case where $E/S^1$ is a circle bundle over $\Sigma_g$, $g > 1$.

To prove Theorem 4.9 we also apply the Leray–Serre spectral sequence of the fibration $T^2 \to E$ to establish two facts. The first states that $E^{\infty}_{11}$ depends only on monodromy and not on the Euler class of the bundle. The second (Lemma 4.6) gives two conditions $E^{\infty}_{02} \cong \mathbb{Z}$, and $E^{\infty}_{20} \cong \mathbb{Z}$, which both are equivalent to the fact that $[T^2] \neq 0$.

These facts give answer in Case 1. To see that consider first a flat bundle $E$. Corollary 4.2 gives that $T^2 \neq 0$, which yields that $b_2(E) = 2 + \text{rank}(E^{\infty}_{11})$.

Consider next the bundle $E'$ which has the same monodromy, but possibly different Euler from $E$. Due to Lemma 4.5 we have then $b_2(E) = b_2(E')$. Since $\text{rank}(E^{\infty}_{11}) = \text{rank}(E'^{\infty}_{11})$ we have $b_2(E') = 2 + \text{rank}(E'^{\infty}_{11})$.

This implies (Lemma 4.5) that $[T^2] \neq 0$ and means that all bundles in case 1 admit a $\pi$–compatible symplectic form.

Our approach to Case 2 is different. From section 3 we know that if $E$ is symplectic, then $E/S^1 \cong \Sigma_g \times S^1$. Using results form [HW] we prove that if $E$ is not a principal fibration (i.e. $E$ has nontrivial monodromy), then $E$ admits a symplectic from. This is done in the following manner.

Recall that in [HW] we gave necessary and sufficient conditions for existence of invariant symplectic forms on four–dimensional manifolds $E$ equipped in a free circle action. Following Bouyakoub ([Bo]) we define the subspace

$$L = \{ \alpha \in H^1(E/S^1, \mathbb{R}) \mid \alpha \cup c_1(\xi) = 0 \},$$

where $\xi$ is the principal fibration $S^1 \to E \xrightarrow{\xi} E/S^1$. Then $E$ supports an invariant symplectic form if and only if $L$ possesses a cohomology class which has a closed and nondegenerate representative.

For $E/S^1 \cong \Sigma_g \times S^1$ we obtain that only the classes which are pullbacks from $\Sigma_g$ do not have such representatives. But for these pullback classes we have

$$L = \{ \pi_1^*a \mid a \in H^1(\Sigma_g, \mathbb{R}) \},$$

(where $pr_1 : \Sigma_g \times S^1 \to \Sigma_g$ is the projection) which means that the bundle $E$ is principal with the Euler class ($\ast$, 0). For all other cases we have that $E$ admits an invariant symplectic structure and $[T^2] \neq 0$ (see 4.16).

In 4.3 we decide which torus bundles $E$ have $[T^2] = 0$ in terms of monodromy and the Euler class.
Finally we give a couple of applications of Lemma 4.10 which concerns averaging of $\pi$–compatible symplectic forms. They are given in Remarks 4.11 and 4.12.

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2 $Spin^c$–structures and SW invariants

In this section we recall some definition and theorems on $Spin^c$–structures and SW invariants that we will be using. In this context we also collect facts concerning circle bundles over 3–dimensional manifolds.

2.1 Classification and pulling back of $Spin^c$–structures

The set of $Spin^c$–structures on 4–dimensional closed, oriented manifold $E$ is classified by $H^2(E, \mathbb{Z})$, but no such correspondence can be chosen canonical. Indeed, given two $Spin^c$–structures $\xi_1, \xi_2$ on $E$ their difference $\delta(\xi_1, \xi_2)$ is a well–defined element of $H^2(E, \mathbb{Z})$. To see this consider the following diagram.

\[
\begin{array}{c}
BS^1 \to BS^1 \\
E' \to BSpin^c(4) \\
E \to BSO(4)
\end{array}
\]

In this diagram $f$ is the classifying map for $TE$ and $f_2$ is the mapping classifying $Spin^c$–structure $\xi_2$. Furthermore, bundle $\Xi = (BS^1, E', E)$ over $E$ is the pullback of $BS^1$–bundle over $BSO(4)$. Since $S^1$ is abelian we have that $BS^1$ is an H–group and therefore $\Xi$ can be consider as a principal $BS^1$–bundle over $E$. Moreover, section $\sigma_2$ of $\Xi$ corresponding to $f_2$ gives a trivialization of this bundle. This trivialization gives a bijection between $Spin^c$–structures on $E$ and homotopy classes $[E, BS^1]$ of section of $\Xi$, which are naturally isomorphic to $H^2(E, \mathbb{Z})$. Once we have fixed $\xi_2$ we define

$$\delta(\xi_1, \xi_2) := \xi_1 - \xi_2 \in H^2(E, \mathbb{Z})$$

as element corresponding to $\xi_1$. This element is well–defined regardless of which $Spin^c$–structure we use to trivialize $\Xi$.

Here and subsequently the set of all $Spin^c$–structures on $E$ will be denoted by $S(E)$. Recall ([1A3]) we have a map

$$c_1 : S(E) \to H^2(X, \mathbb{Z})$$
which assigns to a $\text{Spin}^c$–structure $\xi \in S(E)$ its first Chern class. Observe that assignment

$$\xi_1 - \xi_2 \mapsto c_1(\xi_1) - c_1(\xi_2)$$

yields a mapping

$$H^2(E, \mathbb{Z}) \ni x \mapsto 2x \in H^2(E, \mathbb{Z}).$$

This is a direct consequence of the following fact.

**Proposition 2.1** Under the above assumptions

$$c_1(\xi_1) - c_1(\xi_2) = 2\delta(\xi_1, \xi_2).$$

For the proof of Proposition 2.1 the reader is referred to [FM].

Proposition 2.1 gives that if $H^2(E, \mathbb{Z})$ is $\mathbb{Z}_2$–torsion free, then the first Chern class associated to a $\text{Spin}^c$–structure gives an injection of $S(E)$ in $H^2(E, \mathbb{Z})$. However, in presence of $\mathbb{Z}_2$–torsion there is still a natural choice of unique element $a$ satisfying $c_1(\xi_1) - c_1(\xi_2) = 2a$, since we have $c_1(\xi_1) - c_1(\xi_2) = 2\delta(\xi_1, \xi_2)$. This assignment also classifies $\text{Spin}^c$–structures with equal first Chern classes of their determinants line bundles. They are classified by $A \subset H^2(E, \mathbb{Z})$, where

$$A = \{a \in H^2(E, \mathbb{Z}) \mid 2a = 0\}. \quad (2.1)$$

Analogous facts hold in 3–dimensional case. For more details see [GS], chapter 10.

Our task is to write an explicit formula for $sw^4_E(0)$ and prove that it always gives even number. We will to this by applying Baldridge’s formula 2.3. In order to apply this formula we need to establish the set of these $\text{Spin}^c$–structures $\xi$ such that $c_1(\pi^*(\xi)) = 0$.

We know (12) that if $\pi : E \to M$ is a principal $S^1$–bundle and $\xi$ is a $\text{Spin}^c$–structure on $M$, then $c_1(\pi^*(\xi)) = \pi^*(c_1(\xi))$. The other pulled back $\text{Spin}^c$–structures are obtained by adding of classes $\pi^*(a)$ for $a \in H^2(M, \mathbb{Z})$.

We will determine the elements $a \in H^2(M, \mathbb{Z})$, for which $\pi^*(a) = 0$. To do this apply the Gysin sequence to get the following exact sequence

$$\cdots \to H^0(M, \mathbb{Z}) \xrightarrow{\cup \chi} H^2(M, \mathbb{Z}) \xrightarrow{\pi^*} H^2(E, \mathbb{Z}) \to H^1(M, \mathbb{Z}) \to \cdots,$$

where $\chi$ denotes the Euler class of the bundle $\pi : E \to M$. It yields that kernel of the mapping $\pi^* : H^2(M, \mathbb{Z}) \to H^2(E, \mathbb{Z})$ is equal to the subgroup generated by the Euler class $\chi$.

This simple fact allows us to determine all $\text{Spin}^c$–structures on $M$ whose pullback has vanishing first Chern class. They can be described by the
condition that their first Chern classes belong to subgroup \( \langle \chi \rangle < H^2(M, \mathbb{Z}) \) generated by the Euler class of the fibration.

Next we apply these results to case where \( M \) is a nontrivial principal \( S^1 \)-bundle \( \eta' \) over \( \Sigma_g, \ (g > 1) \) whose Euler class \( e \) equals \( n[\Sigma_g] \) for some \( n \neq 0 \).

It is relatively easy to compute \( H^2(M, \mathbb{Z}) \). We start with recalling the formula for the group \( H_1(M, \mathbb{Z}) \). We thus get

\[
H_1(M, \mathbb{Z}) \cong \langle b_1, b_2, \ldots, b_{2g}, x \mid nx \rangle \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_n, \tag{2.2}
\]

where \( x \) denotes the homology class of the fiber and \( b_1, b_2, \ldots, b_{2g} \) the generators of the base. Here and subsequently \( \langle \cdots \mid \cdots \rangle \) will denote the abelian group given by generators and relations.

By Poincare duality we have \( PD : H_1(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z}) \). We also have (see \[BT\], Ch. 1, Sec. 6) that \( PD(x) = e \). This yields that \( H^2(M, \mathbb{Z}) \) decomposes as \( T \oplus \mathbb{Z}^{2g} \), where \( T \) denotes finite group generated by \( e \).

This yields that for \( n \) odd group \( H^2(M, \mathbb{Z}) \) does not have 2–torsion and the first Chern class gives an injection of \( Spin^c \)–structures in \( \langle e \rangle < H^2(M, \mathbb{Z}) \). For \( n \) even there are two elements of order two in \( H^2(M, \mathbb{Z}) \), therefore there are precisely two \( Spin^c \)–structures corresponding to the same first Chern class. Similarly, these first Chern classes belong to \( \langle e \rangle \).

This will determine all \( Spin^c \)–structure \( \xi \) on \( M \) such that \( c_1(\pi^*(\xi)) = 0 \). To do this let us introduce the following definition.

For \( m \in \mathbb{Z}, n \in \mathbb{N} \) such that \( mn \neq 0 \) define

\[
A_{m,n} = \{ x \in \mathbb{Z}_n \mid (\exists k \in \mathbb{Z}) \ x \equiv km(\text{mod } n) \},
\]

where \( (\mathbb{Z}_n, +) \) denotes the additive group \( \{0, \ldots, n-1\} \) modulo \( n \). Note that \( A_{m,n} \) is a subgroup of \( \mathbb{Z}_n \) generated by element \( m(\text{mod } n) \).

We have that for \( n \in \mathbb{Z} \) the set of first Chern classes of these \( Spin^c \)–structures \( \xi \) such that \( c_1(\pi^*(\xi)) = 0 \) corresponds to \( A_{m,|n|} < T \cong \mathbb{Z}_n \).

### 2.2 Seiberg–Witten invariants

In this section we review facts and theorems on for the Seiberg–Witten invariants on three and four–dimensional manifolds which we will be using.

A powerful tool to deal with the problem of existence of symplectic structures is the Seiberg–Witten theory. In 1994 Taubes (\[TA1\],\[TA2\]) proved the following strong theorem which yields obstruction to existence of symplectic forms. Namely, for any symplectic form there exists a compatible almost complex structure \( J \), for which we can in turn canonically associate a \( Spin^c \)–structure \( \xi \) (more details can be found in \[LM\],\[MS\]). This \( Spin^c \)–structure has a Seiberg–Witten invariant \( SW(\xi) \in \mathbb{Z} \).
Theorem 2.2 [Taubes] Let $X$ be a closed 4–manifold with $b^+_2 > 1$ and a symplectic form $\omega$. Then there is a canonical $Spin^c$–structure $\xi$ on $X$ such that $SW^4_X(\xi) = \pm 1$ and $\alpha = c_1(\det(\xi))$ is the canonical line bundle $K$ of $(X, \omega)$. Moreover, if $sw^4_X(\alpha) \neq 0$ (i.e. $\alpha$ is a basic class), then

$$|\alpha \cdot [\omega]| \leq |c_1(K) \cdot [\omega]|$$

and $\alpha \cdot [\omega] = 0$ if and only if $\alpha = 0$; $|\alpha \cdot [\omega]| = |c_1(K) \cdot [\omega]|$ if and only if $\alpha = \pm c_1(K)$.

The theorem 2.2 can be applied to decide whether $E$ is symplectic in the following way. If all nonzero values of SW invariants for $Spin^c$–structures are different from $\pm 1$, then $E$ admits no symplectic structure. These $Spin^c$–structures (or equivalently elements of $H^2(X, \mathbb{Z})$) with nonvanishing SW–invariants are called basic. For example, $\#3\mathbb{CP}^2$ admits no symplectic structure although it possesses almost complex structure and positive second Betti number.

Let us now recall the notion of the Seiberg–Witten polynomial. As we said in Section 2.1 there are two ways of associating $Spin^c$–structures with $H^2(X, \mathbb{Z})$. The first uses the fibration $BS^1 \to BSpin^c(4) \to BSO(4)$ induced from sequence of homomorphisms $S^1 \to Spin^c(4) \to SO(4)$. In this case the set $\mathcal{S}(X)$ of all $Spin^c$–structures is in bijective correspondence with $H^2(X, \mathbb{Z})$, but no such correspondence is natural without choosing first a distinguished element in $\mathcal{S}(X)$. However, there is the first Chern class map

$$c_1: \mathcal{S}(X) \to H^2(X, \mathbb{Z})$$

which assigns to a $Spin^c$–structure $\xi$ its canonical class $K$. This class is induced by the canonical homomorphism $Spin^c(4) \to U(1)$. This gives another map of $\mathcal{S}(X)$ to $H^2(X, \mathbb{Z})$. This mapping is injective if there is no 2–torsion in $H^2(X, \mathbb{Z})$ (see [LM], Appendix D).

When $X$ is compact, oriented 4–manifold with $b^+_2 > 1$, then the Seiberg–Witten invariants define, via the map $c_1$, a map

$$sw^4_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$$

which is defined up to $\pm 1$ without any additional choices. That is,

$$sw^4_X(z) = \sum_{\{s \mid c(s) = z\}} SW(s),$$

where $SW(s)$ denotes the value of the Seiberg–Witten invariant on the class $s \in \mathcal{S}(X)$. 

9
Note that if there is 2–torsion in $H^2(X, \mathbb{Z})$, then for $a \in H^2(X, \mathbb{Z})$ we can calculate $sw^4_X(a)$ by the same formula.

It proved useful to package the map $sw^4_X$ in a manner which will now be described. We start with the group ring $\mathbb{Z}H^2(X, \mathbb{Z})$, the free $\mathbb{Z}$ module generated by the elements of $H^2(X, \mathbb{Z})$. Then the invariant $sw^4_X$, in the $b_2^+ > 1$ case, can be written as an element in $\mathbb{Z}H^2(X, \mathbb{Z})$, called the Seiberg–Witten polynomial of $X$:

$$SW^4_X = \sum_z sw^4_X(z)z.$$  \hspace{1cm} (2.5)

This formula means that for given $z \in H^2(X, \mathbb{Z})$ its SW invariant is given by $sw^4_X(z)$.

More details on Seiberg–Witten invariants can be found in [1A3].

Our proof of nonexistence of symplectic structures on principal torus fibrations $E \to \Sigma_g$, for $g > 1$ and the Euler class $(m, n)$ satisfying $mn \neq 0$ begins with the observation that we have a decomposition of $E$ as circle bundle $\eta$ over three–dimensional manifold $M \cong E/S^1$, which itself is a nontrivial circle bundle $\eta'$ over $\Sigma_g$. This helps us to prove that $0 \in H^2(E, \mathbb{Z})$ can be the only basic class and to write the formula for $SW(0)$. For the first task we follow Etgii ([E]). Namely, from the second part of Taubes’ Theorem 2.2 we know that a basic class cannot be a nonzero torsion element. Furthermore, by the results cited below we also get that any basic class must be torsion since it is included in pullback of group $T \subset H^2(M, \mathbb{Z})$, where $T$ is generated by the first Chern class $c_1(\eta')$.

**Theorem 2.3 (Baldridge [B2])** Let $E$ be a closed oriented 4–manifold with $b_2^+ > 1$ and a free circle action. Then the orbit space $M^3$ is a smooth 3–manifold and suppose that $\chi \in H^2(M, \mathbb{Z})$ is the Euler class of the circle action on $E$. If $\xi$ is Spin$^c$ structure on $E$ with $SW^3_E(\xi) \neq 0$, then $\xi = \pi^*(\xi_0)$ for some Spin$^c$ structure on $M$ and

$$SW^3_E(\xi) = \sum_{\xi' \equiv \xi_0 \text{ mod } \chi} SW^3_M(\xi').$$  \hspace{1cm} (2.6)

**Theorem 2.4 (Mrowka, Ozsváth, Yu [MOY])** Let $M$ be a $S^1$ principal bundle over a surface $B$ of genus $g \geq 1$ with Euler class $n\lambda$, where $\lambda$ is the (positive) generator of $H^2(B, \mathbb{Z})$. If $n \neq 0$, then all basic classes of $M$ are in $\{k\pi^*(\lambda) | 0 \leq k \leq |n| - 1\}$, where $\pi$ is the projection. Moreover, we have

$$sw^3_M(k \cdot \pi^*(\lambda)) = \sum_{s \equiv k \text{ (mod } n)} sw^3_{S^1 \times B}(s \cdot pr_2^*(\lambda)),$$  \hspace{1cm} (2.7)

where $pr_2$ is the projection $S^1 \times B \to B$. 

10
The Seiberg–Witten polynomial of 3–manifold $M$ is defined similarly as in the case of 4–manifolds (see the formula 2.5). For other necessary background on Seiberg–Witten theory concerning 3–dimensional manifolds consult [MOY, B2].

Theorem 2.4 together with 2.7 gives an explicit formula for the SW–invariant of $M$ ([B1]).

**Theorem 2.5** (Balridge [B1]) Let $\pi : M \to \Sigma_g$ be the total space of a circle bundle over a genus $g$ surface. Assume $c_1(M) = n\lambda$ where $\lambda \in H^2(\Sigma_g, \mathbb{Z})$ is the generator and $n$ is an even number $n = 2l \neq 0$, then the Seiberg–Witten polynomial of $M$ is

$$SW^3_M = \sum_{i=0}^{n-1} sw^3_M(t)^i =$$

$$= \text{sign}(n) \sum_{i=0}^{\frac{|l|-1}{2}} \sum_{k=-(2g-2)}^{2g-2} (-1)^{(g-1)+i+k|l|} \left( \frac{2g-2}{(g-1) + i + k|l|} \right) t^{2i}, \tag{2.8}$$

where $t = \exp(\pi^*(\lambda))$ and the binomial coefficient $\left( \frac{p}{q} \right) = 0$ for $q < 0$ and $q > p$. For the formula where $n$ is odd, replace $l$ by $n$ and $t^{2i}$ by $t^i$.

The Formula 2.8 should be understood that the Seiberg–Witten invariant of the class $k\lambda$ equals to coefficient at $t^k$ of the Seiberg–Witten polynomial $SW^3_M$. Observe also that we use the variable $t = \exp(\pi^*(\lambda))$ in order to keep multiplicative notation of the Seiberg–Witten polynomial $SW^3_M$.

Formula 2.6 allows us to write down an explicit expression which computes $sw^4_E(0)$ (recall that only $0 \in H^2(E, \mathbb{Z})$ can be basic). We prove that under our assumptions this formula always gives even number. It yields that $E$ is not symplectic due to Taubes’ theorem 2.2.

### 3 Principal torus bundles

In this section we consider principal $T^2$–bundles over surfaces $\Sigma_g$ of genus $g > 1$ as well as $S^1$ bundles over 3–manifolds which are $S^1$ bundles over $\Sigma_g$ ($g > 1$). Our main result of this Section is the following.

**Theorem 3.1** Let $T^2 \to E \xrightarrow{\pi} \Sigma_g$ be a principal fibration over the surface $\Sigma_g$, where $g > 1$. Let $(m, n)$ be the Euler class of the fibration. If $mn \neq 0$, then $E$ is not a symplectic manifold.

Assume we have a principal torus fibration $E \to \Sigma_g$ with the Euler class equal to $(m, n)$ (for a complete definition of the Euler class see the beginning of section 4). We will prove that if $mn \neq 0$, then $E$ does not
support any symplectic structure. The case where \( mn = 0 \) was settled by Etgü in [E]. Namely, he proved (Lemma 3.8 in [E]) that a nontrivial \( S^1 \)-bundle \( M^3 \rightarrow \Sigma_g \) \((g > 1)\) cannot fiber over \( S^1 \).

This fact can be proven by another argument using a Thurston theorem [TH2]. Recall (2.2) that for a bundle with the Euler class \( e = n[\Sigma_g] \), \( n \neq 0 \), we have \( H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_n \). This easily yields that the homology group \( H_2(M, \mathbb{Z}) \cong H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g} \) has a base which consists of embedded tori. To see this take any set of embedded circles \( S_i \) representing \( b_1, b_2, \ldots, b_{2g} \), and consider their counterimages \((\pi')^{-1}(S_i)\). It follows that the Thurston norm vanishes identically on \( H_2(M, \mathbb{Z}) \). Now if \( M \) fibers over the circle we have that the fiber must be the sphere or the torus ([TH2]). Since we have \( 4 \leq 2g = b_1(M) \leq 3 \) the claim follows.

To finish the proof of Taubes’ Conjecture he proves using the SW–theory that a nontrivial circle bundle over \( \Sigma_g \times S^1 \) whose Euler class is \( n\pi_2^*\Sigma \), does not support any symplectic structure. Obviously this bundle is the principal \( T^2 \)-bundle over \( \Sigma \) with the Euler class equal to \((n, 0)\).

Assume we have a principal torus fibration \( E \rightarrow \Sigma_g \) with the Euler class equal to \((m, n)\) (for a complete definition of the Euler class see the beginning of section 4). The total space \( E \) can be decomposed as a principal \( S^1 \)-bundle over 3–manifold \( M \) which is itself a \( S^1 \)-principal bundle over \( \Sigma_g \). This decomposition goes as follows. We first take a \( S^1 \) principal bundle \( S^1 
rightarrow M \rightarrow \Sigma_g \) with the Euler class equal to \( n\lambda \), where \( \lambda \in H^2(\Sigma_g, \mathbb{Z}) \) is the orientation class. Next we take a \( S^1 \) principal bundle \( S^1 \rightarrow X \rightarrow M \) with the Euler class equal to \( m \cdot \pi^*\lambda \). We want to prove that the total space \( X \) of the last fibration is diffeomorphic to \( E \). This procedure of decomposing is an example of some more general pattern.

Assume we are given a \( G \)-principal bundle \( \xi : G \rightarrow E \rightarrow Y \) over some CW–complex \( Y \) and a normal subgroup \( H < G \) where the inclusion is denoted by \( i : H \hookrightarrow G \). We can write \( \pi = E \rightarrow E/H \rightarrow Y \), where \( \pi' \) is a \( G/H \)-principal bundle over \( Y \) and \( \xi'' \) is a \( H \)-principal bundle over \( E/H \). Our goal is to find relations between classifying maps for these three bundles \( \xi, \xi' \) and \( \xi'' \).

Consider the following diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & BG \\
& \searrow & \downarrow \text{Br} \\
& & B(G/H)
\end{array}
\]

where \( r : G \rightarrow G/H \) is the quotient map and \( f \) is the classifying map for the bundle \( \xi \). We obviously get that \( d \) is the classifying map for the \( G/H \)-principal bundle \( \xi' \) which was obtained from \( \xi \) by dividing the total space \( E \) by the free right \( H \)-action.
To proceed consider another diagram.

\[
\begin{array}{ccc}
E & \longrightarrow & EG \\
\pi & \downarrow & \downarrow \\
Y & \underset{f}{\longrightarrow} & BG
\end{array}
\]

Let us divide the total spaces \(E\) and \(EG\) by the free right \(H\)–action. From the commutative diagram

\[
\begin{array}{ccc}
E/H & \underset{\tilde{f}}{\longrightarrow} & EG/H \cong BH \\
\downarrow & & \downarrow Bi \\
Y & \underset{f}{\longrightarrow} & BG
\end{array}
\]

we get that \(\tilde{f} : E/H \to BH\) is the classifying map for the principal \(H\)–bundle \(\xi''\).

In our case we have \(G = T^2\) and \(H = S^1\). We also have that \(d^* : H^2(BS^1) \to H^2(\Sigma_g)\) is given by \(d^*(c_1) = n[\Sigma_g]\). Furthermore, the classifying map \(\tilde{f} : E/S^1 \to BS^1\) is given by \(\tilde{f}^*(c_1) = m\pi^*[\Sigma_g]\). This can be seen by considering diagram

\[
\begin{array}{ccc}
E/S^1 & \underset{\tilde{f}}{\longrightarrow} & ET^2/S^1 \cong BS^1 \\
\downarrow & & \downarrow Bi \\
\Sigma_g & \underset{f}{\longrightarrow} & BT^2
\end{array}
\]

In order to use Taubes’ Theorem 2.2 we need to show that \(b_2^+(E) > 1\). This follows from the vanishing of the signature \(\sigma(E)\) and the fact that \(b_2(E) > 3\).

For the proof of the first fact see \([K]\). The second is then quite clear. We know that \(b_1(E) \geq 2g\) (see Formula 4.3). The Euler characteristics \(\chi(E) = 0\), which gives \(b_2(E) = 2b_1(E) - 2 \geq 4g - 2 \geq 6\), thus \(b_2^+ \geq 3\).

We will check that only \(0 \in H^2(E, \mathbb{Z})\) can be basic. The idea here is to modify slightly the argument in \([E]\). First notice that a basic class cannot be a nonzero torsion class. This is a direct consequence of the second part of Taubes’ theorem \(2.2\). To see this recall that if \(\alpha\) is basic and \(\alpha \cdot [\omega] = 0\), then \(\alpha = 0\).

We also have that basic class of \(E\) cannot be non–torsion. To justify this notice that all nonzero Seiberg–Witten invariants \(SW^3_M\) are included in a group \(T\) which was defined as

\[
T = \{k \cdot \pi^*\lambda \mid k \in \mathbb{Z}\}.
\]
This fact follows directly from the Theorem 2.4. Due to Theorem 2.3 we get that basic classes of $E$ are included in the set $\pi^*T$. This gives our claim since $T$ is a finite group.

Theorem 2.5 also gives an explicit formula to count the Seiberg–Witten invariant of the zero class. Seiberg–Witten invariants of $M$ are given as coefficients of the Seiberg–Witten polynomial exhibited in a very convenient form in Theorem 2.5. Take any $\text{Spin}^c$–structure $\xi_0$ on $M$ such that $c_1(\pi^*(\xi_0)) = 0$. We know (see Section 2.1) that these $\text{Spin}^c$–structures are characterized by the condition that $c_1(\xi_0) \in A_{m,|n|} < T$. For each such structure $\xi_0$ consider the set

$$\{ \xi \mid \xi \equiv \xi_0 \pmod{\chi} \}$$

corresponding to $\{c_1(\xi_0)\} + A_{2m,|n|} < T$ via the first Chern class. Formula for $sw^4_E(0)$ has now the form

$$sw^4_E(0) = \sum_{\{\xi \mid 2\chi([c_1(\xi) - c_1(\xi_0)]) \} \{\xi_0 \mid \chi(c_1(\xi_0)) \}} \sum SW^3_M(\xi).$$

(3.2)

This means that

$$sw^4_E(0) = \sum_{\{j \in \mathbb{Z}_n \mid (i-j) \in A_{2m,|n|}\}} \left( \sum_{i \in A_{m,|n|}} sw^3_M(t^i) \right).$$

(3.3)

If we denote

$$A = \{ i \mid 0 \leq i \leq |n| - 1, \gcd(|m|, |n|) \mid i \}$$

and

$$A' = \{ i \mid 0 \leq i \leq |n| - 1, \gcd(|m|, |n|) \mid i, 2 \mid i \},$$

the formula 3.3 can be explicitly given by

$$sw^4_E(0) = \frac{\text{sign}(n) |n|}{\gcd(|m|, |n|)} \cdot \left( \sum_{i \in A} \sum_{k=-(2g-2)}^{2g-2} (-1)^{(g-1)+i+k|n|} \left( \left( \begin{array}{c} 2g-2 \\ (g-1) + i + k|n| \end{array} \right) \right) \right).$$

(3.4)

if $n$ is odd, and

$$sw^4_E(0) =$$

14
\[ = \text{sign}(n) \frac{|n|}{x \gcd(|m|, |n|)}. \]

\[ . \left( \sum_{i \in A', k = -(2g-2)} \sum_{i \in A, k = -(2g-2)} (-1)^{(g-1)+i+k|\frac{m}{n}|} \left( \frac{2g - 2}{(g - 1) + i + k|\frac{m}{n}|} \right) \right), \quad (3.5) \]

if \( n \) is even, where \( x \) equals 1 or 2 depending on whether \( \gcd(|2m|, |n|) = \gcd(|m|, |n|) \) or \( \gcd(|2m|, |n|) = 2 \gcd(|m|, |n|) \).

For \( n \) odd this can be proven in the following way. Since \( \gcd(2m, |n|) = \gcd(m, |n|) \) we have \( \{e_1(\xi_0)\} \cup A_{2m, |n|} = A_{m, |n|} \subset T \). Thus for fixed \( i \in A_{m, |n|} \) the sums in (3.3) are all equal and they are counted \( \frac{|n|}{2 \gcd(|m|, |n|)} \) times. Furthermore, for fixed \( i \) these sums are obviously equal to

\[ \text{sign}(n) \sum_{i \in A} \sum_{k = -(2g-2)} (-1)^{(g-1)+i+k|\frac{m}{n}|} \left( \frac{2g - 2}{(g - 1) + i + k|\frac{m}{n}|} \right). \]

For \( n \) even the reasoning is similar. Observe also that if \( \gcd(|2m|, |n|) = 2 \gcd(|m|, |n|) \), then \( \frac{|n|}{2 \gcd(|m|, |n|)} \) is an integer number.

Therefore it is enough to examine parity of the following sums:

\[ \sum_{i \in A} \sum_{k = -(2g-2)} (-1)^{(g-1)+i+k|\frac{m}{n}|} \left( \frac{2g - 2}{(g - 1) + i + k|\frac{m}{n}|} \right), \quad (3.6) \]

if \( n \) is odd, and

\[ \sum_{i \in A'} \sum_{k = -(2g-2)} (-1)^{(g-1)+i+k|\frac{m}{n}|} \left( \frac{2g - 2}{(g - 1) + i + k|\frac{m}{n}|} \right), \quad (3.7) \]

if \( n \) is even.

The formulas (3.6) and (3.7) seemed to be complicated. Using the Mathematica program I checked that for \( m, n = -20, \ldots, -1, 1, \ldots, 20 \) and \( g = 2, \ldots, 20 \) (which is approximately 30000 cases) the formulas give even numbers. This convinced me that the sums should be even.

We will show that the sums are always even. Taubes’ theorem will tell us then that \( E \) carries no symplectic structure.

In order to continue, we will consider two cases.
3.1 $n$ odd

We analyze the parity of the sum $\text{(3.6)}$ and $\text{(3.7)}$ therefore we omit the powers of $(-1)$ in these formulas.

Summands of the sum

$$\sum_{i \in A} \sum_{k = -(2g-2)}^{2g-2} \left( \frac{2g-2}{(g-1) + i + k|n|} \right)$$

are in bijective correspondence with the set

$$B = \left\{ \left( \frac{2g-2}{r} \right) \mid r \in C \right\},$$

where

$$C = \{ (g-1) - (2g-2)|n|, \ldots, (g-1) - (2g-2)|n| + u \gcd(|m|, |n|), \ldots, (g-1) + (2g-1)|n| - \gcd(|m|, |n|) \}$$

and $u$ is a natural number. To see this just change the order of summing to get

$$\sum_{k = -(2g-2)}^{2g-2} \sum_{i \in A} \left( \frac{2g-2}{(g-1) + i + k|n|} \right)$$

and expand this sum. Thus for given $k \in \{ -(2g-2), \ldots, 2g-2 \}$ and $i \in A$ we obtain elements $(g-1) + k|n|, \ldots, (g-1) + (k+1)|n| - \gcd(|m|, |n|)$ of the set $C$. Since

$$(g-1) - (2g-2)|n| < 0,$$

$$(g-1) + (2g-1)|n| - \gcd(|m|, |n|) > 2g-2$$

and

$$\left( \frac{2g-2}{g-1} \right) \in B,$$

all nonzero members of $B$ may be represented as a sum of pairs

$$\left\{ \left( \frac{2g-2}{r} \right), \left( \frac{2g-2}{2g-2 - r} \right) \mid r < g-1, r \in C \right\}$$

and a single element

$$\left( \frac{2g-2}{g-1} \right).$$

It suffices to notice that $\left( \frac{2g-2}{g-1} \right)$ is an even number provided that $g > 1$, since

$$\left( \frac{2g-2}{g-1} \right) = 2 \left( \frac{2g-3}{g-1} \right).$$
3.2 \( n \) even

The case when \( n \) is even and \( m \) is odd is proved in exactly the same way as before and will be omitted. Therefore without loss of generality we can assume that \( m \) is also even.

In this case we have \( 2 \mid \gcd(|m|,|n|) \), so \( A = A' \) and therefore we shall examine the parity of the sum

\[
\sum_{i \in A} \sum_{k = -(2g-2)}^{2g-2} \left( \frac{2g-2}{(g-1) + i + k|\frac{n}{2}|} \right),
\]

where \( A \) and \( A' \) are defined as in the previous case. Let us divide the sum into two summands:

\[
\sum_{i \in A} \sum_{k \in F} \left( \frac{2g-2}{(g-1) + i + k|\frac{n}{2}|} \right)
\]

and

\[
\sum_{i \in A} \sum_{k \in F'} \left( \frac{2g-2}{(g-1) + i + k|\frac{n}{2}|} \right),
\]

where \( F = \{-(2g-2), -(2g-4), \ldots, 2g-4, 2g-2\} \) and \( F' = \{-(2g-3), -(2g-5), \ldots, 2g-5, 2g-3\} \). Thus the first sum may be expressed as

\[
\sum_{i \in A} \sum_{k = -(g-1)}^{g-1} \left( \frac{2g-2}{(g-1) + i + k|n|} \right).
\]

By the same reasoning as in the case where \( n \) is odd we deduce that this sum is even (observe that \( (g-1) + (-g-1)|n| < 0 \) and \( (g-1) + (g+1)|n| > 0 \) since \( n \) is a nonzero even number). As for the second sum, similarly as in the previous case, we notice that elements of this sum are in bijective correspondence with the set

\[
B' = \left\{ \left( \frac{2g-2}{r} \right) \mid r \in C' \right\},
\]

where

\[
C' = \{(g-1) - (2g-3)|\frac{n}{2}|, \ldots, (g-1) - (2g-3)|\frac{n}{2}| + u \gcd(|m|,|n|), \ldots,
\]

\[
(g-1) + (2g-1)|\frac{n}{2}| - u \gcd(|m|,|n|)\}
\]

and \( u \) is a natural number. Observe that

\[
(g-1) - (2g-3)|\frac{n}{2}| \leq 0,
\]

17
\[(g - 1) + (2g - 1)\frac{n}{2} - \gcd(|m|, |n|) \geq 2g - 2,\]

and that all nonzero members of \(B'\) are distributed symmetrically with respect to \(\left(\begin{array}{c}
\frac{2g - 2}{g - 1}
\end{array}\right)\). To be more precise: if

\[y = (g - 1) + k\frac{n}{2} + x \gcd(|m|, |n|) \in C',\]

where \(0 \leq x < \frac{|n|}{\gcd(|m|, |n|)}\), \(k \in F'\) and \(0 \leq y \leq g - 1\), then

\[z = (g - 1) - (k + 2)\frac{n}{2} + \left(\frac{|n|}{\gcd(|m|, |n|)} - x\right) \gcd(|m|, |n|) \in C',\]

\(k \in F'\) and \(z \geq g - 1\) as well, since we have \(z + y = 2g - 2\).

QED

The theorem we have just proved can be slightly generalized. Namely, we do not need the Euler class to be included in \(T\) (see 3.1). Assume then that \(\chi \not\in T\). Observe that the same argument as before yields that only zero class can be basic. The difference is that the formula for \(\text{sw}_4(E)(0)\) slightly changes. To be more precise, we have that \(\langle \chi \rangle \cap T = \{0\}\) and therefore \(\{\xi \mid 2\chi \mid (c_1(\xi_0) - c_1(\xi))\} = \{\xi_0\}\) for every \(\text{Spin}^c\)-structure \(\xi_0\) such that \(\pi^*(c_1(\xi_0)) = 0\). Thus Formulas 3.5 and 3.4 have the following forms.

\[\text{sw}_4(E)(0) = \]

\[= \text{sign}(n) \sum_{i \in A} \sum_{k = -(2g - 2)}^{2g - 2} (-1)^{(g - 1) + i + k|n|} \left(\begin{array}{c}
2g - 2
\end{array}\right) \left(\begin{array}{c}
g - 1 + i + k|n|
\end{array}\right), \quad (3.8)\]

if \(n\) is odd, and

\[\text{sw}_4(E)(0) = \]

\[= \text{sign}(n) \sum_{i \in A'} \sum_{k = -(2g - 2)}^{2g - 2} (-1)^{(g - 1) + i + k|\frac{n}{2}|} \left(\begin{array}{c}
2g - 2
\end{array}\right) \left(\begin{array}{c}
g - 1 + i + k|\frac{n}{2}|
\end{array}\right), \quad (3.9)\]

if \(n\) is even.

These are precisely Formulas 3.7 and 3.6. We already proved they represent even numbers provided that \(g > 1\) and \(n \neq 0\).

**Corollary 3.2** If \(T^2 \to E \to \Sigma_g\) is a nontrivial principal fibration, then homology class of the fiber \([T^2] = 0\) in \(H_2(E, \mathbb{R})\).
Remark 3.3 Note that in this section we proved our Conjecture 1.2 for torus bundles equipped with free circle action preserving fibers. Recall that the Conjecture 1.2 states that if $E$ is a closed 4–manifold equipped with a free circle action admitting a symplectic structure, then the quotient $E/S^1$ fibers over $S^1$. Indeed, in our case we have that $E/S^1$ is a principal circle bundle $\xi$ over surface $\Sigma_g$. If this surface is $S^2$ or $T^2$, the Conjecture obviously hold. if the genus $g$ is greater than 1, then $\xi$ must be trivial (recall that if $n \neq 0$, then $E$ is not symplectic). In the next section we settle the question which $S^1$ bundles over $\Sigma_g \times S^1$ admit symplectic structures. □

4 General torus bundles

In this section we discuss the general case of $T^2$ bundles over surfaces $\Sigma_g$ of genus $g > 1$.

Each such bundle is, up to isomorphism, uniquely determined by the monodromy homomorphism $\pi_1 \Sigma_g \to \pi_0 Diff^+ T^2 \cong SL(2, \mathbb{Z})$, which reveals the structure of the bundle over 1–skeleton of the base, and the Euler class. We can assume that monodromy of such a bundle are linear automorphisms of $T^2$, thus they are given by $A_i \in SL(2, \mathbb{Z})$.

The Euler class, as an obstruction for a cross section of the bundle, is a pair of integer numbers $(m, n)$, which may be defined as follows. Let $f : \Sigma_g \to BDiff^+ T^2$ be the classifying map for the bundle and $B\pi : BDiff^+ T^2 \to BSL(2, \mathbb{Z})$ is the map associated to the natural map $\pi : Diff^+ T^2 \to \pi_0 Diff^+ T^2 \cong SL(2, \mathbb{Z})$, then we can define the mapping $\rho$ by the formula $\rho = (\pi_1)_*(B\pi \circ f) : \pi_1(\Sigma_g) \to \pi_1(BSL(2, \mathbb{Z})) = SL(2, \mathbb{Z})$. This mapping is a representation $\pi_1(\Sigma_g) \to SL(2, \mathbb{Z})$ which also determines monodromy associated with $E$.

We call the bundle flat if its structural group can be reduced to a discrete group. In case of torus bundle it means that the Euler class of the bundle is equal to $(0, 0)$ and the structural group reduces to $SL(2, \mathbb{Z})$.

A realization of a given Euler class $(m, n)$ for a flat bundle can be described in the following way. Take a flat bundle and choose any disc $D^2 \hookrightarrow \Sigma_g$. Remove the counterimage $\pi^{-1}(D^2)$ and glue it back via the identification map $T^2 \times \partial D^2 \to T^2 \times \partial D^2$ given by the formula $((x_1, x_2), \theta) \mapsto ((x_1 + \frac{m\theta}{2\pi}, x_2 + \frac{n\theta}{2\pi}), \theta)$.

We begin with the following lemma.
Lemma 4.1 Any flat $T^2$ bundle over $\Sigma_g$ supports a $\pi$–compatible symplectic form.

Proof. The total space $E$ of the bundle can be described as the quotient by the following action of $\pi_1(\Sigma_g)$ on $\mathbb{H}^2 \times T^2$. First take representation $\rho : \pi_1(\Sigma_g) \to SL_2\mathbb{Z}$ induced by monodromy associated with $E$. Define next

$$g(x, y) = (xg^{-1}, \rho(g)y), \quad (4.1)$$

where on the first coordinate we have the natural action of $\pi_1(\Sigma_g)$ on the universal covering of $\Sigma_g$. To finish notice that the product symplectic form on $\mathbb{H}^2 \times T^2$ is invariant with respect to $4.1$. QED

An equivalent formulation of the Lemma 4.1 is given by the following corollary.

Corollary 4.2 For any flat $T^2$–bundle over $\Sigma_g$ we have that the homology class of the fiber $[T^2]$ is nonzero in $H_2(E, \mathbb{R})$.

The equivalence between Lemma 4.1 and Corollary 4.2 is due to Thurston [TH1]:

Theorem 4.3 Let $E \xrightarrow{\pi} \Sigma_1$ be a surface $\Sigma_2$–bundle over surface $\Sigma_1$. If the homology class of the fiber is nonzero in $H_2(E, \mathbb{R})$, then $E$ admits a $\pi$–compatible symplectic form.

Recall now the following facts which we shall need in the sequel.

If $T^2 \to E \xrightarrow{\pi} \Sigma_g$ is a flat bundle, then

$$H_1(E) \cong \langle b_1, b_2, \cdots, b_{2g}, x_1, x_2 \mid A_i x_1 - x_1, A_i x_2 - x_2, i = 1, 2, \cdots, 2g \rangle. \quad (4.2)$$

The generators $x_1, x_2$ give the basis for $H_1(T^2, \mathbb{Z})$ and $b_1, b_2, \cdots, b_{2g}$ are the standard basis for $H_1(\Sigma_g, \mathbb{Z})$.

If the bundle $T^2 \to E \xrightarrow{\pi} \Sigma_g$ is not flat, we have

$$H_1(E) \cong \langle b_1, b_2, \cdots, b_{2g}, x_1, x_2 \mid$$

$$mx_1 + nx_2, A_i x_1 - x_1, A_i x_2 - x_2, i = 1, 2, \cdots, 2g \rangle, \quad (4.3)$$

where the pair $(m, n) \in \mathbb{Z}^{\oplus 2}$ denotes the Euler class. Note that the pair $(m, n)$ is determined by the basis $x_1, x_2$ of $H_1(F)$. 20
To proceed we make an observation that we will use occurs here is that the question whether $E$ admits preserving fibers action can be answer in terms of the monodromy. Namely, existence of such action is equivalent to the existence of $x \in H^1(T^2, \mathbb{Z})$ preserved by the monodromy. This is the content of the next proposition.

**Proposition 4.4** The total space $E$ of a fibration $T^2 \to E \xrightarrow{\pi} \Sigma_g$ supports a free circle action preserving fibers if and only if there exists a nonzero element $z \in \mathbb{Z}^2$ such that $A_i z = z$ for $i = 1, \ldots, 2g$.

**Proof.** Assume that there exists a common eigenvector $z \in \mathbb{Z}^2$ with eigenvalue 1 for all monodromy. We can assume that the bundle $E$ is isomorphic with the bundle whose all monodromy $A_i \in SL(2, \mathbb{Z})$. We can also assume that $z$ is primitive, i.e. $\gcd(a, b) = 1$, where $z = (a, b)$. If $\{z, u\}$ is a basis of $H_1(T^2, \mathbb{Z})$, than all monodromy have the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. We have then a free circle action preserving fibers on $E$ given by the subgroup corresponding to $z$.

Assume now that $E$ admits such circle action. Any effective circle action on $T^2$ is determined, up to isomorphism, by the homology class of the orbit. As a result we can choose a basis $\{z, u\} \in H_1(T^2, \mathbb{Z})$ for which all monodromy matrices $A_i$ have the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. QED

The Proposition 4.4 allows us to prove the following trichotomy.

**Lemma 4.5** Let $T^2 \to E \xrightarrow{\pi} \Sigma_g$ be a flat fibration. Then

1. $E$ is trivial $\iff b_1(E) = 2g + 2$,
2. $E$ is nontrivial and admits a free circle action preserving fibers $\iff b_1(E) = 2g + 1$,
3. $E$ does not admit any free circle action preserving fibers $\iff b_1(E) = 2g$.

**Proof.** The first equivalence follows from cohomological computation and is obvious.

Assume that $b_1(E) = 2g + 1$. We will prove that $E$ supports a free circle action preserving fibers. Our assumption yields that the group $S < \mathbb{Z}^2$ defined as

$$S = \langle A_i x_1 - x_1, A_i x_2 - x_2, i = 1, 2, \cdots, 2g \rangle,$$

where $\langle \ldots \rangle$ denotes the smallest subgroup generated by given relations, is isomorphic to $\mathbb{Z}$. To proceed take any basis $\{z, u\} \in \mathbb{Z}^2$ such that $S < \mathbb{Z}z$. Thus we obtain $A_i z - z = k_i z$ for some integers $k_i \in \mathbb{Z}$. In the basis $\{z, u\}$
the matrices $A_i$ have the form \(
abla A_i = k^i + 1 \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \). Additionally we cannot have $k^i + 1 = -1$ since $A_i$ would have the form \(\begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix} \) and $b_1(E) = 2g$.

Finally we have that $A_i$ are given by \(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \). But this and Proposition 4.4 means that $E$ admits a free circle action preserving fibers.

Assume now that $E$ is nontrivial and admits such circle action. Due to Proposition 4.4 we have that in some base \(\{z, u\} \in \mathbb{Z}^2\) the monodromy matrices have the form \(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \). This yields that $b_1(E) = 2g + 1$.

The third equivalence follows from the first two equivalences. \(\text{QED}\)

At this point we would like to analyze the Leray–Serre spectral sequence for the fibration $T^2 \rightarrow E \rightarrow \Sigma_g$. We need this to establish two facts concerning our bundles. First is to justify that $E_{11}^\infty$ does not depend on the Euler class, but only on monodromy. Second is to prove a lemma which states that the homology class $[T^2]$ is nonzero in $H_2(E, \mathbb{R})$ if and only if any of the conditions: $E_{02}^\infty \cong \mathbb{Z}$, or $E_{20}^\infty \cong \mathbb{Z}$, hold.

To prove the first fact we follow Geiges (C). The $E^2$–page of the Leray–Serre spectral sequence is given by

\[ E^2_{pq} = H_p(\Sigma_g, \mathcal{H}_q(T^2)), \]

where $\mathcal{H}$ denotes the system of local coefficients, and the spectral sequence converges to $H_* (E, \mathbb{Z})$.

For our purposes it is enough to consider

\[ E_{11}^\infty = E_{11}^2 = H_1(\Sigma_g, \mathcal{H}_1(T^2)). \]

Using the standard action of $\pi_1(\Sigma_g)$ on the universal cover we get the following free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Pi$, where $\Pi \cong \pi_1(\Sigma_g)$.

\[ 0 \rightarrow \mathbb{Z}\Pi \rightarrow \bigoplus_{i=1}^{2g} \mathbb{Z}\Pi \rightarrow \mathbb{Z}\Pi \rightarrow \mathbb{Z} \rightarrow 0. \]  
(4.5)

Recall that if $\mathcal{R}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Pi$ and $V$ is a $Q$–module, then

\[ H_* (Q, V) = H_* (\mathcal{R} \otimes_{\mathbb{Z}\Pi} V) \]

by definition.

Tensoring the above resolution with $\mathbb{Z} \oplus \mathbb{Z}$ over $\mathbb{Z}\Pi$ gives

\[ 0 \rightarrow \mathbb{Z}^{\oplus 2} \rightarrow \bigoplus_{i=1}^{2g} \mathbb{Z}^{\oplus 4g}. p^* \rightarrow \mathbb{Z}^{\oplus 2} \rightarrow \mathbb{Z}^{\oplus 2} \rightarrow 0. \]  
(4.6)
Then $E_{11}^2 = ker(p_*)/im(i_*)$.

This implies that $E_{11}^2 = E_{11}^\infty$ depends only on monodromy and not on the Euler class. Indeed, if we keep the monodromy and change the Euler class then the sequence $\text{4.6}$ remains the same.

Next we establish the second fact, which will shall be using.

**Lemma 4.6** Let $T^2 \rightarrow E \xrightarrow{\pi} \Sigma_g$ be an orientable fibration where $g > 1$. Then the condition $[T^2] \neq 0$ is equivalent to each of the following two conditions: $E_{02}^\infty \cong \mathbb{Z}$, or $E_{20}^\infty \cong \mathbb{Z}$.

**Proof.** In the $E^2$–page of the spectral sequence for the fibration we have $H_0(\Sigma_g, H_2(F)) = E_{02}^2 \cong \mathbb{Z}$ and $H_2(\Sigma_g, H_0(F)) = E_{20}^2 \cong \mathbb{Z}$ since the local coefficients systems for $H_0(T^2, \mathbb{Z})$ and $H_2(T^2, \mathbb{Z})$ are both trivial. It follows that the summand $E_{02}^\infty$ has the following interpretation: $[\Sigma_g] \neq 0 \Leftrightarrow E_{02}^\infty \cong \mathbb{Z}$. This can be easily seen by noticing that (see for example [Sp], ch.9, sec.3, p. 482) the homomorphism $i_* : H_2(T^2, \mathbb{Z}) \rightarrow H_2(E, \mathbb{Z})$ is the composition

$H_2(T^2, \mathbb{Z}) \cong H_0(\Sigma_g, H_2(T^2, \mathbb{Z})) \cong E_{02}^2 \rightarrow E_{02}^\infty = F_0 H_2(E, \mathbb{Z}) \subset H_2(E, \mathbb{Z}), \quad (4.7)$

where $F_i$ is the filtration (see also [S] and [TH1]).

Similarly the summand $E_{20}^\infty$ has an analogous interpretation: $[T^2] \neq 0 \Leftrightarrow E_{20}^\infty \cong \mathbb{Z}$. This can proven the same way as before. The local coefficients systems for $H_0(T^2, \mathbb{Z})$ is trivial, so ([Sp], ch.9, sec.3, p. 483) $\pi_*$ is the composition

$H_2(E, \mathbb{Z}) = F_2(H_2(E, \mathbb{Z})) \rightarrow E_{20}^\infty \rightarrow E_{20}^2 \cong H_2(\Sigma_g, \mathbb{Z}), \quad (4.8)$

The isomorphism $E_{02}^\infty \cong \mathbb{Z}$ is equivalent to the fact that $\pi_* : H_2(E) \rightarrow H_2(\Sigma_g)$ is rationally onto. This in turn means that $\pi^*[\Sigma_g] \neq 0$, where $[\Sigma_g]$ is the orientation class. Since $PD(\pi^*[\Sigma_g]) = [T^2]$ in cohomology with real coefficients the assertion is proven.

QED

To continue we review some relevant facts from [Bo, HW, MD, MT, TH1, TH2].

We denote by $M$ a compact oriented smooth manifold with a smooth free action of $S^1$ and $N \cong M/S^1$. The space of invariant symplectic forms consistent with the given orientation will be denoted by $S_{inv}$.

Let $\pi : M \rightarrow N$ denote the principal $S^1$–fibration given by the action. By $\alpha$ we will denote the nondegenerate (=nowhere vanishing) closed 1–form $\alpha$ on $N$ satisfying

$\pi^* \alpha = \iota_X \omega,$

where $X$ is the infinitesimal generator of the action.
Lemma 4.7 [Bo] If $\omega \in S_{\text{inv}}$, then

$$[\alpha] \cup c_1(\pi) = 0. \quad (4.9)$$

Proof. Take any connection form $\eta \in \Omega^1(M, \mathbb{R})$. By Chern – Weyl there exists a closed 2–form $c_1 \in H^2(N, \mathbb{R})$ such that $\pi^*c_1 = d\eta$ and $c_1$ represents $c_1(\pi) \in H^2(N^3, \mathbb{R})$. There also exists a unique 2–form $\beta' \in \Omega^2(N^3, \mathbb{R})$ such that $\omega - \eta \wedge \iota_X \omega = \pi^*\beta'$. By differentiating both sides of the last equation we get $d\eta \wedge \iota_X \omega = \pi^*d\beta'$ and this implies the lemma. \hfill \text{QED}

From now on we assume that $\dim M = 4$.

Define the subspace $L \subset H^1(N^3, \mathbb{R})$ by

$$L = \{ \alpha \in H^1(N^3, \mathbb{R}) \mid \alpha \cup c_1(\pi) = 0 \}. \quad (4.10)$$

Let us also recall the notion of Thurston norm [TH2] (see also [MT]) for a 3–manifold. If $N^3$ is a compact, connected and oriented manifold without boundary then for any compact oriented n–component surface $S = S_1 \sqcup \cdots \sqcup S_n$ embedded in $N$ define

$$\chi_-(S) = \sum_{\chi(S_i) < 0} |\chi(S_i)|. \quad (4.11)$$

The Thurston norm on $H_2(N^3, \mathbb{Z})$ and, by the Poincare duality, on $H^1(N^3, \mathbb{Z})$ is given by

$$\|\phi\|_T = \inf \{ \chi_-(S) \mid [S] = \phi \}. \quad (4.12)$$

The Thurston norm can be extended linearly to $H^1(N^3, \mathbb{R})$. Let $B_T = \{ \phi : \|\phi\|_T \leq 1 \}$ denote the unit ball in the Thurston norm. It is a (possibly noncompact) polyhedron in $H^1(N^3, \mathbb{R})$. Suppose $\phi' \in H^1(N^3, \mathbb{Z})$ is represented by a fibration $N^3 \to S^1$. Then $\phi'$ is contained in the open cone $\mathbb{R}_+ \cdot F$ over a top – dimensional face $F$ of the Thurston norm ball $B_T$. In this case we say $F$ is a fibered face of the Thurston norm ball. The Thurston norm can be also defined if boundary of $N^3$ is a union of tori.

Assume now that $N^3$ fibers over the circle, $b_1 = m$ and it is not $S^2 \times S^1$. In this case we have that $L \cap \mathbb{R}_+ \cdot F$ has a homotopy type of a point, or a sphere $S^{m-1}$ if $c_1 = 0$ in $H^2(N, \mathbb{R})$, and $S^{m-2}$ when $c_1 \neq 0$ provided that Thurston norm vanishes identically. If $N \cong S^2 \times S^1$, then total space of any circle fibration over $N$ happens to be symplectic if and only if $c_1 = 0$ and is diffeomorphic $S^2 \times T^2$; furthermore we have that $L \cap \mathbb{R}_+ \cdot F \cong \mathbb{R}\{0\}$ has a homotopy type of a two–point set.
Given a closed nondegenerate 1–form on \( N \) satisfying (4.9) it is relatively easy to construct an invariant symplectic form on \( M \) such that \( \iota_X \omega = \pi^* \alpha \). The construction follows from a formula [Bo] in the spirit of the inflation trick of Thurston [TH1] and McDuff [MD]. It consists in enlarging the form along the foliation determined by \( \ker \alpha \).

For a given form \( \alpha \),

\[
\omega = \eta \wedge \pi^* \alpha + \pi^* (K \beta + \phi)
\]  

is an invariant symplectic form if \( \eta \in \Omega^1(M^4, \mathbb{R}) \) is a connection form, \( \beta \) is a closed 2–form on \( N^3 \) such that \( \alpha \wedge \beta \) is a volume form on \( N^3 \), \( d\phi = -c_1 \wedge \alpha \) and \( K \) is sufficiently large real number. Obviously \( \omega \) satisfies \( \pi^* \alpha = \iota_X \omega \).

Existence of \( \beta \) is well–known ([Pl, Su]).

**Lemma 4.8** Let \( N^n \) be a closed and oriented manifold. Assume that a closed and nondegenerate 1–form \( \alpha \) on \( N \) is given. Then there is a closed \((n-1)\)–form \( \beta \) such that \( \alpha \wedge \beta \) is a volume form on \( M \). Equivalently, \( \beta \) is nondegenerate on leaves of the foliation defined by \( \ker \alpha \).

We are able to state and prove the main theorem of this section.

**Theorem 4.9** The total space \( E \) of the fibration \( T^2 \to E \to \Sigma_g \ (g > 1) \) is symplectic if and only if \([T^2] \neq 0 \) in \( H_2(E, \mathbb{R}) \).

**Proof.** Note that due to Theorem 4.3 it is enough to prove that if \( E \) is symplectic, then \([T^2] \neq 0 \). In the sequel we shall need also the fact that \( b_2(E) = 2b_1(E) - 2 \). This is a direct consequence of the fact that the Euler characteristics \( \chi(E) \) vanishes. We will divide the proof into two parts:

1. bundles which does not admit a free circle action preserving fibers,
2. bundles which support such action.

The first part will be proved by using spectral sequences. We begin with flat \( E \) bundle to obtain \( b_2(E) = 2 + \text{rank}(E_1^\infty) \). We change the Euler class keeping the monodromy fixed to get \( E' \). Since \( b_2(E) = b_2(E') \) and \( \text{rank}(E_1^\infty) = \text{rank}(E_1'^\infty) \) we have \( b_2(E') = 2 + \text{rank}(E_1'^\infty) \). But this means that all bundles in this category have \( \pi \)–compatible symplectic forms.

The second part is proved in different way. Using section 3 we know that if \( E \) symplectic, then \( E/S^1 \) is the product \( \Sigma_g \times S^1 \). Furthermore, \( E \) cannot be nontrivial principal \( T^2 \)–fibration over \( \Sigma_g \) in order to be symplectic ([E]). However, it means that \( L \neq A \) ([HW]) and \( E \) admits an invariant symplectic form. We also get that for these bundles we have \([T^2] \neq 0 \).
4.1 First case

In the Leray–Serre spectral sequence for the fibration $T^2 \to E \xrightarrow{\pi} \Sigma_g$ in the $E^\infty$–page we have

$$b_2(E) = \text{rank}(E_{20}^\infty) + \text{rank}(E_{11}^\infty) + \text{rank}(E_{02}^\infty). \quad (4.14)$$

In addition to that, we know from Corollary 4.2 that $[T^2]$ is nonzero in $H_2(E, \mathbb{Z})$ since the bundle is flat.

Finally lemma 4.6 simplifies the formula 4.14 to

$$b_2(E) = \text{rank}(E_{11}^\infty) + 2.$$

Note also the summand $E_{11}^\infty$ does not depend on the Euler class, but only on monodromy.

Consider now the bundle $E'$ with the same monodromy as in the bundle $E$ but nonzero Euler class. Lemma 4.5 yields

$$b_1(E') = 2g = b_1(E) \quad \text{and} \quad b_2(E') = 2b_1(E') - 2 = 2b_1(E) - 2 = b_2(E).$$

In both cases $\text{rank}(E_{11}^\infty)$ is the same, so $b_2(E') = 2 + \text{rank}(E_{11}^\infty).$ But the latter means (Lemma 4.6) that $[T^2] \neq 0.$

4.2 Second case

We have here the total space $E$ equipped with a free circle action preserving fibers. Furthermore, the quotient space $E/S^1$ is a principal $S^1$–bundle over $\Sigma_g.$ From section 3 we know that if $E$ is symplectic, then the latter must be trivial and $E/S^1 \cong \Sigma_g \times S^1.$ Therefore we shall restrict our considerations to bundles of the form

$$\xi : S^1 \to E \to \Sigma_g \times S^1. \quad (4.15)$$

Note that by [E] we can also exclude principal nontrivial $T^2$ fibrations over $\Sigma_g$ from our consideration.

We shall describe the Thurston norm on $\Sigma_g \times S^1.$ From [TH2] we know that $||[\Sigma_g]||_T = |\chi(\Sigma_g)| = 2g - 2.$ Furthermore, if we denote the bundle tangent to $\Sigma_g$ by $\tau$ and its Euler class by $\chi(\tau)$ we have that the equality $||a||_T = |\chi(\tau) \cdot a|$ holds for all $a \in H_2(\Sigma_g \times S^1, \mathbb{R})$ in some neighborhood of the ray through $[\Sigma_g].$ In addition to that we know that second homology classes of the form $b_i \times S^1, \ i = 1, \ldots, 2g$ can be represented by embedded tori, where $b_i \to \Sigma_g$ are embedded circles representing the base of $H_1(\Sigma_g, \mathbb{Z}).$ It follows that the Thurston norm vanishes identically on the subspace generated by classes of the form $b_i \times S^1, \ i = 1, \ldots, 2g.$ These data completely describe the Thurston norm on $\Sigma_g \times S^1.$
When we pass to first real cohomology via Poincare duality we can rewrite this data as \( \|k[S^1] + b\|_T = |k|(2g - 2) \), where \( b \) is any cohomology class coming from \( \Sigma_g \). Thus all cohomology classes, except of those which are pullbacks from \( \Sigma_g \), have a closed and nondegenerate representative.

Denote then by \( A \subset H^1(\Sigma_g \times S^1, \mathbb{R}) \) a codimension 1 subspace generated by all these classes which are pullbacks from \( \Sigma_g \). This subspace consists of classes on which Thurston norm vanishes identically.

Recall that there exists an invariant symplectic form if and only if \( L \neq A \) (\cite{HW}). Thus \( L \) (see \( 4.10 \)) coincides with \( A \) if and only if \( \xi \) is a nontrivial \( T^2 \)-principal bundle over \( \Sigma_g \). This bundle does not have any symplectic form by \( \cite{E} \). Recall that there exists an invariant symplectic form on \( E \) provided that \( L \neq A \) (see \( 4.13 \)). In all these cases \( E \) supports an invariant symplectic form.

As we show below, whenever such \( E \) supports an invariant symplectic form, the homology class \( [T^2] \) is nonzero in \( H_2(E, \mathbb{R}) \).

Formula \( 4.13 \) gives the following application: if we have any embedded circle \( S^1 \) in \( N \) such that \( \int_{S^1} \alpha > 0 \), then

\[
\int_{\pi^{-1}(S^1)} \omega = 2\pi \int_{S^1} \alpha > 0. \tag{4.16}
\]

Now, since \( L \) does not coincide with \( A \) we have that for any nondegenerate 1–form \( \alpha \) such that \( [\alpha] \in L \) the inequality \( \int_{S'} \alpha > 0 \) holds, where \( S' \) is the circle \( S^1 \hookrightarrow S^1 \times \Sigma_g \) which is the first factor of the product \( S^1 \times \Sigma_g \). But this in turn means that \( [T^2] \neq 0 \).

### 4.3 A characterization of the property \([T^2] \neq 0\)

In this section we give a complete characterization of the bundles with nonzero homology class of the fiber in terms of monodromy and the Euler class. We also give an alternative proof of the fact that if \( E \) has a free circle action preserving fibers, then the relation \( mx_1 + nx_2 \) belongs to the group generated by the homology class of the orbit if and only if \( [T^2] \neq 0 \).

We start with bundles which do not admit a free circle action preserving fibers (i.e. \( \not\exists \ z \ A_i z = z \), \( i = 1, \ldots, 2g \)). From Lemma \( (4.5) \) and subsection \( 4.1 \) we know that for these bundles we always have \( [T^2] \neq 0 \). By Proposition \( 4.4 \) the condition of admitting such action can be described in terms of monodromy.

Consider now bundles which admit such free circle action. If such bundle is flat, then \( [T^2] \neq 0 \) (see Corollary \( 4.2 \)). Any other bundle \( E' \) can be obtained by changing the Euler class and keeping the monodromy fixed.

In the sequel we use notation of Lemma \( (4.5) \). From the homology computations \( 4.2, 4.3 \) we see that the first Betti number can decrease by one.
Recall that the subspace
\[ S = \langle A_i x_1 - x_1, A_i x_2 - x_2, i = 1, 2, \ldots, 2g \rangle \]
has dimensional one over \( \mathbb{Z} \) provided that \( E \) is flat and nontrivial (Lemma 4.5). Take also as before any basis \( \{ z, u \} \in \mathbb{Z}^2 \) for which all the monodromy matrices have the form \( \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \). Obviously we have that \( S < \mathbb{Z} z \).

If the extra relations \( mx_1 + nx_2 \in \mathbb{Z} z \), then \( b_1(E) = b_1(E') \) and \( b_2(E) = b_2(E') \). Following subsection 4.1 we get that \( [T^2] \neq 0 \) in this case.

Otherwise we have \( mx_1 + nx_2 \notin \mathbb{Z} z \). It yields \( b_1(E') = b_1(E) - 1 \) and \( b_2(E') = b_2(E) - 2 \). But this means that \( \text{rank}(E'_{02}) = \text{rank}(E'_{20}) = 0 \). However the last condition implies that \( [T^2] = 0 \) (see Lemma 4.6).

These computations are concordant with our previous results. To explain this consider a flat bundle \( T^2 \to E \xrightarrow{\pi} \Sigma_g \) with monodromy of the form \( \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \) in some basis \( \{ z, u \} \).

This bundle is a \( S^1 \) bundle \( \xi \) over the product \( S^1 \times \Sigma_g \). This can be proven as follows. Each flat \( T^2 \)-bundle over \( \Sigma_g \) has an obvious section, which assigns the neutral element group \( 1 \in T^2 \) to any point in base. This section defines another section \( \sigma : \Sigma_g \to E/S^1 \), but since \( E/S^1 \) is a principal \( S^1 \)-bundle over \( \Sigma_g \), it must be trivial. As in 4.15 denote this bundle by \( \xi \):

\[ \xi : S^1 \to E \to \Sigma_g \times S^1. \]

Using the Poincare duality it is easy to describe the first Chern class of this bundle. Namely, it is equal to union of disjoint circles representing the standard base \( b_1, \ldots, b_{2g} \in H_1(\Sigma_g, \mathbb{Z}) \) with multiplicities equal accordingly to \( k_i \).

We next change the Euler class to obtain another bundle \( E' \). If we have \( mu + nv \in \mathbb{Z} u \), then the section \( \sigma \) is still well defined as a mapping from \( \Sigma_g \) to \( T^2/[z] \). The latter means that the new bundle \( E' \) can be interpreted as another \( S^1 \)-bundle \( \xi' \) over the product \( \Sigma_g \times S^1 : \]

\[ \xi' : S^1 \to E \to \Sigma_g \times S^1. \]

The difference between \( \xi \) and \( \xi' \) can be described by use of the first Chern class. To be more precise: we have \( c_1(\xi) - c_1(\xi') \in \mathbb{Z}[S^1] \), where this circle is the second factor in \( \Sigma_g \times S^1 \).

As we mentioned earlier, the argument concerning spectral sequences gives that if \( mu + nv \in \mathbb{Z} u \), then \( [T^2] \neq 0 \). It follows (Theorem 4.3) that \( E' \) supports a \( \pi \)-compatible symplectic form. This form may be chosen to be invariant, as the next lemma shows.
Lemma 4.10 If $E$ is equipped with a free circle action included in fibers and $[T^2] \neq 0$, then $E$ has an invariant $\pi$–compatible symplectic form.

Proof. It follows (\cite{TH1}) that $E$ supports a symplectic structure $\omega$ compatible with the fibration. The symplectic form $\omega$ can written down as

$$\omega = \omega_F + K \pi^* \omega_B,$$

where $\omega_B$ is the symplectic structure on the base $\Sigma_g$, $\omega_F$ is a closed 2–form nondegenerate on the fibers and $K$ is a big enough real number. Observe that $\omega_F$ can be chosen invariant since we can average this form with respect to the action. Finally, we can choose $K$ so big that the form $\omega$ is symplectic.

QED

Remark 4.11 For torus bundles satisfying conditions that $[T^2] \neq 0$ and $E$ is equipped with a free circle action preserving fibers Lemma \ref{lem:4.10} gives a simpler argument that $b_1(E) \geq 2g + 1$ (cf. Lemma \ref{lem:4.5}). To see this denote by $i : T^2 \subset E$ the inclusion of the fiber in the total space. Next, take any basis $\{z, u\}$ for $H_1(T^2, \mathbb{Z})$, where $i_* z$ denotes the homology class of the orbit, and the invariant $\pi$–compatible symplectic form $\omega$. Thus we obtain

$$0 \neq \int_u i^*(\iota_Z \omega) = \int_{i_* u} \iota_Z \omega,$$

where $Z$ denotes the infinitesimal generator of the action. This equality yields that $i_* u \neq 0$ in $H_1(E, \mathbb{R})$.

Using this approach we can also provide another proof of Corollary \ref{cor:4.2} (i.e. for every nontrivial principal torus fibration over $\Sigma_g$ we have $[T^2] = 0$). Assume the opposite and take any $\pi$–compatible symplectic form $\omega$. Similarly as Lemma \ref{lem:4.10} we can assume that $\omega$ is averaged with respect to principal torus action. Furthermore we have $\int_{i_* u} \iota_Z \omega \neq 0$ and $\int_{i_* z} \iota_U \omega \neq 0$, where $\{z, u\}$ is some basis for $H_1(T^2, \mathbb{Z})$ and $Z, U$ are corresponding infinitesimal generators. However this means that $b_1(E) = 2g + 2$ (see formula \ref{eq:4.3}) which contradicts Lemma \ref{lem:4.5}.

Remark 4.12 Averaging of forms can also be used to give a simple proof of the following fact. Let $\xi : T^2 \to E \to M$ be a principal torus bundle over some manifold $M$. Then $[T^2] \neq 0$ in $H_2(E, \mathbb{R})$ if and only if $\xi$ trivial. To see this assume $[T^2] \neq 0$. A slight generalization of the Thurston theorem \ref{thm:4.3} yields existence of some closed two form $\beta$ on $E$ which is the volume form on each fiber. Average the form $\beta$ with respect to the torus action to get a closed invariant 2–form which is also the volume form on each fiber. Now the same argument as in Remark \ref{rem:4.11} gives that $u, v \neq 0$ in $H_1(E, \mathbb{R})$, where $\{u, v\}$ is some basis for $H_1(F, \mathbb{R})$. This yields $b_1(E) = b_1(M) + 2$ which means that the bundle $\xi$ is trivial.

□

29
In this way we are able to answer which $T^2$ bundles over $\Sigma_g$ has the property that $[T^2] \neq 0$. They are precisely:

1. $E = T^2 \times \Sigma_g$,
2. bundles which do not support any free circle action included in fiber,
3. bundles which support some free circle action included in fiber and the Euler class is the multiple of the orbit class $[S^1] \in H_1(E, \mathbb{Z})$.

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31