GENERALIZED INTRANSITIVE DICE II:
PARTITION CONSTRUCTIONS

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ABSTRACT. A generalized \( N \)-sided die is a random variable \( D \) on a sample space of \( N \) equally likely outcomes taking values in the set of positive integers. We say of independent \( N \) sided dice \( D_i, D_j \) that \( D_i \) beats \( D_j \), written \( D_i \rightarrow D_j \), if \( \text{Prob}(D_i > D_j) > \frac{1}{2} \). A collection of dice \( \{D_i : i = 1, \ldots, n\} \) models a tournament on the set \([n] = \{1, 2, \ldots, n\}\), i.e. a complete digraph with \( n \) vertices, when \( D_i \rightarrow D_j \) if and only if \( i \rightarrow j \) in the tournament. By using \( n \)-fold partitions of the set \([Nn]\) with each set of size \( N \) we can model an arbitrary tournament on \([n]\). A bound on the required size of \( N \) is obtained by examples with \( N = 3^{n-2} \).

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1. INTRODUCTION: TOURNAMENTS USING DICE

A generalized die is a cube with each face labeled with a positive number. The possibility of repeated labels is allowed. On the standard die each of numbers 1, 2, \ldots, 6 occurs once. We identify the die with the random variable of the outcome of a roll with each face equally likely. With a pair of dice we assume that the rolls are independent.

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Of two dice $D_1$ and $D_2$ we say that $D_1$ beats $D_2$ (written $D_1 \rightarrow D_2$) if the probability that $D_1 > D_2$ is greater than $\frac{1}{2}$ where $D_1$ and $D_2$ are the independent outcomes of the rolls of the dice.

Of course, with standard dice this does not happen. If $D_1$ and $D_2$ are standard, then $P(D_1 > D_2) = \frac{15}{36}$.

If we use the labels

$$A_1 = \{3\},$$

$$A_2 = \{2\},$$

$$A_3 = \{1\}.$$ 

and repeat each label six times to get 6-sided dice, then clearly always $D_1 \rightarrow D_2, D_3$ and $D_2 \rightarrow D_3$.

However, there exist examples of nontransitive dice, or intransitive dice, three dice $D_1, D_2, D_3$ such that $D_1 \rightarrow D_2, D_2 \rightarrow D_3$, and $D_3 \rightarrow D_1$. For example, if we let

$$A_1 = \{3, 5, 7\},$$

$$A_2 = \{2, 4, 9\},$$

$$A_3 = \{1, 6, 8\}.$$ 

and repeat each label twice to get 6-sided dice, $\{D_1, D_2, D_3\}$, then $P(D_i > D_{i+1}) = \frac{5}{9}$ for $i = 1, 2, 3$ (counting mod 3).

A digraph $R$ on a set $I$ of size (= cardinality) $|I|$ is a set of ordered pairs $(i, j)$ of distinct elements $i, j \in I$ such that at most one of the pairs $(i, j), (j, i)$ lies in $R$. The digraph is called a tournament when exactly one of the pairs $(i, j), (j, i)$ lies in $R$. The name arises because $R$ models the outcomes of a round-robin tournament where every pair of players competes once with $i$ beating $j$, written $i \rightarrow j$, if $(i, j) \in R$. Alternatively, we can think of $I$ as a list of strategies or actions so that $i \rightarrow j$ when $i$ wins against $j$. The output set $R(i) = \{j : i \rightarrow j\}$ consists of the elements of $I$ which are beaten by $i$.

Up to relabeling, there are two tournaments of size 3: the ordering $\{(1, 2), (2, 3), (1, 3)\}$ and the 3-cycle $\{(1, 2), (2, 3), (3, 1)\}$. The above examples show that each can be mimicked by using dice.

The 3-cycle models the game Rock-Paper-Scissors. In general, we will call a tournament $R$ on $I$ a game when its size $|I|$ is odd and for each $i$, $|R(i)| = \frac{1}{2}(|I| - 1)$. That is, each strategy beats exactly half of its competing strategies and is beaten by the other half. Clearly, the 3-cycle is (up to isomorphism) the only game of size 3.

With size 5 there is also a game which is unique up to isomorphism. On the television show The Big Bang Theory this game was described as Rock-Paper-Scissors-Lizard-Spock. It can also be modeled using
generalized dice:
\[
A = \{1, 6, 10, 22, 24, 30\}, \\
B = \{7, 12, 13, 15, 19, 27\}, \\
C = \{3, 4, 17, 18, 23, 28\}, \\
D = \{2, 9, 11, 16, 26, 29\}, \\
E = \{5, 8, 14, 20, 21, 25\},
\]
(1.3) which satisfy, with probability 19/36 in each case,
\[
(1.4) \quad A \rightarrow C, E; \quad B \rightarrow A, D; \quad C \rightarrow B, D; \quad D \rightarrow A, E; \quad E \rightarrow B, C.
\]

We would like to similarly mimic an arbitrary tournament. However, as the size of the tournament grows we will require larger dice, dice with more than 6 “faces”. On a sample space of \( N \) equally likely outcomes, which we will call the faces, an \( N \) sided die is a random variable taking positive integer values. Again a pair of competing \( N \)-sided dice are assumed independent. See, for example, [3].

An \( N \) sided die is called proper when the values are all in \([N] = \{1, \ldots, N\}\) and the sum of the values is \( \frac{(N+1)N}{2} \), or, equivalently, the expected value is \( \frac{N+1}{2} \) which is the same as that of the standard \( N \) sided die which takes on each value of \([N]\) once.

In [2] it is shown in various ways that an arbitrary tournament can be modeled by proper \( N \) sided dice.

A convenient way of constructing examples is by the use of partitions.

A partition \( \mathcal{A} \) of a set \( I \) is a collection of disjoint subsets with union \( I \). We call it a regular partition when the cardinalities of the elements of \( \mathcal{A} \) are all the same. From now on we will assume that our partitions are regular so that \( \mathcal{A} = \{A_1, \ldots, A_n\} \) is an \( n \) partition of \([Nn] = \{1, \ldots, Nn\}\) when it is a partition of \([N]\) with \( |A_i| = N \) for \( i = 1, \ldots n \). There are \( (Nn)!/(N!)^n \) \( n \) partitions of \([Nn]\).

For a finite subset \( A \subset \mathbb{N} \) we denote by \( \sigma(A) \) the sum of the elements of \( A \). If \( \mathcal{A} = \{A_1, \ldots A_n\} \) is an \( n \) partition of \([Nn]\), we call the partition proper when the sums \( \sigma(A_i) \) are all equal and so
\[
(1.5) \quad \sigma(A_i) = \frac{N(Nn + 1)}{2} \quad \text{for} \quad i \in [n].
\]

Equivalently, for each \( i \), the expected value of a random element of \( A_i \) is \( \frac{Nn+1}{2} \).

For an \( n \) partition \( \mathcal{A} \) on \([Nn]\) we define the digraph
\[
(1.6) \quad R[\mathcal{A}] = \{(i, j) \in [n] \times [n] : |\{(a, b) \in A_i \times A_j : a > b\}| > N^2/2\}.
\]
That is, \((i, j) \in R[A]\) or \(A_i \to A_j\) if it is more likely that a randomly chosen element of \(A_i\) is greater than a randomly chosen element of \(A_j\) rather than the reverse.

If \(N\) is odd, then \(R[A]\) is necessarily a tournament on \([n]\). That is, for every pair \(i, j \in [n]\) with \(i \neq j\) either \(A_i \to A_j\) or \(A_j \to A_i\). Note that for \(i = j\), \(|\{(a, b) \in A_i \times A_i : a > b\}| = N(N - 1)/2\).

We can use the partition to label the faces of \(n\) different \(N\) sided dice, with distinct values selected from \([Nn]\). If \(D_i\) is the random variable associated with the die labeled with values from \(A_i\), then \(A_i \to A_j\) exactly when \(D_i \to D_j\) in the previous sense. If \(A\) is a proper \(n\) partition of \([Nn]\) then repeating each value \(n\) times, we obtain \(n\) proper \(Nn\) sided dice \(\{D_i : i \in [n]\}\) with \(D_i \to D_j\) if and only if \(A_i \to A_j\).

Example (1.2) is a proper 3 partition of \([9]\), and Example (1.3) is a proper 5 partition of 30.

We will say that an \(n\) partition \(A\) of \([Nn]\) models a tournament \(R\) on \([n]\) when \(R[A] = R\).

For \(N\) large enough we can model any tournament on \([n]\) by using an \(n\) partition on \([Nn]\). In \([2]\) the following is proved.

**Theorem 1.1.** If \(R\) is a tournament on \([n]\), then there is a positive integer \(M\) such that for every integer \(N \geq M\), there exists an \(n\) partition of \([Nn]\) \(A = \{A_1, \ldots, A_n\}\) such that for \(i, j \in [n]\), \(A_i \to A_j\) if and only if \(i \to j\) in \(R\). That is, \(R = R[A]\).

However the proof of this theorem and the related results in \([2]\) are all rather non-constructive. If we let \(N_n\) be the smallest positive integer \(N\) such that every tournament on \([n]\) can be modeled by an \(n\) partition on \([Nn]\), then the results of \([2]\) do not provide a bound on the size \(N_n\).

In Section 3 we provide an explicit construction which will yield such a bound. In Theorem 3.14 below we will show the following

**Theorem 1.2.** If \(R\) is a tournament on \([n]\) with \(n \geq 2\), then there exists an \(n\) partition of \([3n-2n]\) \(A = \{A_1, \ldots, A_n\}\) such that \(R = R[A]\). In particular, \(N_n \leq 3^{n-2}\).

The bound is probably very crude. Furthermore, the examples constructed are not necessarily proper. On the other hand, we will show that for arbitrary positive \(n\), there is a game of size \(2n + 1\) which can be modeled by a proper \(2n + 1\) partition of \([3(2n + 1)]\).

Notice that \(N_n\) does tend to infinity with \(n\). To see this, recall that the number of \(n\) partitions of \([Nn]\) is \(P_n = (Nn)!/(N!)^n\). Since
\[ \int_1^{n+1} \ln(x) \, dx > \ln(n!) > \int_1^n \ln(x) \, dx, \]

(1.7)

\[ \ln(P_n) \leq \left[ (Nn + 1)(\ln(Nn + 1) - 1) + 1 \right] - n[N(\ln(N) - 1) + 1] \]
\[ \quad = (Nn + 1)(\ln(n + (1/N)) + \ln(N) - n. \]

On the other hand, the number of tournaments of size \( n \) is \( T_n = 2^{n(n-1)/2} \) and so \( \ln(T_n) = \ln(2)2^{n(n-1)/2} \). Because \( n^2 \) grows faster than \( n \ln(n) \) it follows that \( N_n \) cannot remain bounded as \( n \) tends to infinity.

## 2. Tournaments and Games

All the sets we consider are assumed to be finite.

A **digraph** on a nonempty set \( I \) is a subset \( R \subseteq I \times I \) such that \( R \cap R^{-1} = \emptyset \) with \( R^{-1} = \{ (j, i) : (i, j) \in R \} \). In particular, \( R \) is disjoint from the diagonal \( \Delta = \{ (i, i) : i \in I \} \). We write \( i \rightarrow j \) for \( (i, j) \in R \).

For a vertex \( i \), i.e. \( i \in I \), the *output set* \( R(i) = \{ j : (i, j) \in R \} \) and so \( R^{-1}(i) = \{ j : (j, i) \in R \} \) is the *input set*. If \( J \subseteq I \), then the restriction of \( R \) to \( J \) is \( R|_J = R \cap (J \times J) \).

We use \( |I| \) to denote the cardinality of a set \( I \). Notice that if \( I \) is the singleton \( \{ u \} \), then \( R = \emptyset \) is the only digraph on \( I \). We call this the *trivial* digraph and denote it \( \emptyset[u] \).

Given a map \( \rho : I \rightarrow J \) we let \( \bar{\rho} \) denote the product map \( \rho \times \rho : I \times I \rightarrow J \times J \).

**Definition 2.1.** Let \( R \) and \( S \) be digraphs on \( I \) and \( J \), respectively. A *morphism* \( \rho : R \rightarrow S \) is a map \( \rho : I \rightarrow J \) such that \( (\bar{\rho})^{-1}(S) = R \setminus (\bar{\rho})^{-1}(\Delta_J) \). That is, for \( i_1, i_2 \in I \) with \( \rho(i_1) \neq \rho(i_2) \) \( \rho(i_1) \rightarrow \rho(i_2) \) if and only if \( i_1 \rightarrow i_2 \).

Clearly, if \( \rho \) is a bijective morphism then \( \rho^{-1} \) is a morphism and so \( \rho \) is an *isomorphism*. Two digraphs are isomorphic when each can be obtained from the other by relabeling the vertices.

An *automorphism* of \( R \) is an isomorphism with \( R = S \).

If \( R \) is a digraph on \( I \) and \( \pi \) is a permutation of \( I \), then we let \( \pi R \) be the digraph on \( I \) given by

\[ \pi R = \pi(R) = \{ (\pi(i), \pi(j)) : (i, j) \in R \}. \]

Clearly, if \( R \) and \( S \) are digraphs on \( I \), then \( \rho : R \rightarrow S \) is an isomorphism if and only if the map \( \rho \) on \( I \) is bijective, i.e. a permutation, and \( S = \rho R \).
An \( R \) path \([i_0, \ldots, i_n]\) from \( i_0 \) to \( i_n \) (or simply a path when \( R \) is understood) is a sequence of elements of \( I \) with \( n \geq 1 \) such that \((i_k, i_{k+1}) \in R\) for \( k = 0, \ldots, n-1 \). The length of the path is \( n \). It is a closed path when \( i_n = i_0 \). An \( n \) cycle, denoted \( \langle i_1, \ldots, i_n \rangle \), is a closed path \([i_n, i_1, \ldots, i_n]\) such that the vertices \( i_1, \ldots, i_n \) are distinct. A path spans \( I \) when every \( i \in I \) occurs on the path. A spanning cycle is called a Hamiltonian cycle for \( R \).

A digraph \( R \) is called strongly connected, or just strong, if for every pair \( i, j \) of distinct elements of \( I \) there is a path from \( i \) to \( j \). It follows that if \(|I| > 1\), then for any \( i \in I \) there is a path beginning and ending at \( i \). We may eliminate any repeated vertices \( j_k = j_\ell \) with \( 0 < k < \ell \) by removing the portion of the path \( j_k, j_{k+1}, \ldots, j_{\ell-1} \) and renumbering. This shows that if \( R \) is strong and nontrivial, there is a cycle through each vertex. The trivial digraph on a singleton is strong vacuously.

A subset \( J \subset I \) is invariant if \( i \in J \) implies that the output set \( R(i) \) is contained in \( J \), or, equivalently, if any path which begins in \( J \) remains in \( J \). It is clear that \( R \) is strong if and only if it does not contain any proper invariant subset.

A digraph \( R \) is called a tournament when \( R \cup R^{-1} = (I \times I) \setminus \Delta \). Thus, \( R \) is a tournament on \( I \) when for each pair of distinct elements \( i, j \in I \) either \((i, j)\) or \((j, i)\) lies in \( R \) but not both. Clearly, if \( R \) is a tournament on \( I \) and \( J \subset I \), then the restriction \( R|J \) is a tournament on \( J \). Harary and Moser provide a nice exposition of tournaments in [5].

**Proposition 2.2.** If \( R \) is a strong tournament on \( I \) with \(|I| = p > 1\) and \( i \in I \), then for every \( \ell \) with \( 3 \leq \ell \leq p \) there exists a \( \ell \)-cycle in \( R \) passing through \( i \). In particular, \( R \) is a strong, nontrivial tournament if and only if it admits a Hamiltonian cycle.

**Proof.** See Moon, [6] Theorem 3 for a proof of this result which is a sharpening of Harary and Moser [5] Theorem 7. See also [1] Proposition 1.2.

\[ \square \]

For \( S' \) and \( S \) tournaments on \( J', J \), respectively, with \( J' \) and \( J \) disjoint, the domination product is the tournament \( S' \triangleright S \) on \( J' \cup J \) defined by:

\[
S' \triangleright S = S' \cup (J' \times J) \cup S,
\]

so that \( J \) is a proper invariant subset for \( S' \triangleright S \).
Conversely, if \( J \) is a proper invariant subset for a tournament \( R \) on \( I \) and \( J' = I \setminus J \), then
\[
R = (R|J') \triangleright (R|J).
\]

Let \( R \) be a nontrivial tournament on \( I \), \( v \in I \) and \( J = I \setminus \{v\} \). The vertex \( v \) is called a **maximum** when it satisfies the following equivalent conditions
\[
\begin{align*}
&v \to u \text{ for all } u \neq v \text{ in } I. \\
&R(v) = J \\
&R^{-1}(v) = \emptyset. \\
&R = (\emptyset[v]) \triangleright (R|J).
\end{align*}
\]

Similarly, vertex \( v \) is a **minimum** when it satisfies the following equivalent conditions
\[
\begin{align*}
&u \to v \text{ for all } u \neq v \text{ in } I. \\
&R^{-1}(v) = J \\
&R(v) = \emptyset. \\
&R = (R|J) \triangleright (\emptyset[v]).
\end{align*}
\]

**Proposition 2.3.** Let \( R \) be a nontrivial tournament on \( I \).

(a) The tournament \( R \) is not strong if and only if it is a domination product, i.e. \( R = S' \triangleright S \) for some tournaments \( S \) and \( S' \).

(b) If \( v \in I \) with \( J = I \setminus \{v\} \) and \( R|J \) is strong, then exactly one of the following is true.

\[
\begin{align*}
&(i) \text{ The vertex } v \text{ is a maximum vertex for } R. \\
&(ii) \text{ The vertex } v \text{ is a minimum vertex for } R. \\
&(iii) \text{ The tournament } R \text{ is strong.}
\end{align*}
\]

**Proof.** (a) This follows from (2.3) and the remarks before it.

(b) Because \( R|J \) is strong, there is a Hamiltonian cycle \( \langle i_0, \ldots, i_p \rangle \) for \( R|J \). If \( R^{-1}(v) \neq \emptyset \) we can renumber and so assume \( i_0 \in R^{-1}(v) \). Let \( k \) be the maximum integer such that \( i_q \in R^{-1}(v) \) for \( 0 \leq q \leq k \). If also \( R(v) \neq \emptyset \) then \( k < p \) and so \( i_{k+1} \in R(v) \). Thus, \( \langle i_0, \ldots, i_k, v, i_{k+1}, \ldots, i_p \rangle \) is a Hamiltonian cycle for \( R \) and so \( R \) is strong.

\( \square \)

For a positive integer \( k \), a digraph \( R \) is called **\( k \)-regular** when both the input set and the output set of of every vertex have cardinality \( k \). That is, \( |R(i)| = |R^{-1}(i)| = k \) for all \( i \in I \). A digraph which is \( k \)-regular for some \( k \) is called **regular**. If a tournament on \( I \) is \( k \)-regular, then \( |I| = 2k + 1 \). We will call a regular tournament a **game** because such a tournament generalizes the Rock-Paper-Scissors game. Such games
are described in [11]. In particular, it is demonstrated there that up to isomorphism there is a unique game of size 5.

Of special interest are the group games described in Section 3 of [11].

Let \( \mathbb{Z}_{2n+1} \) denote the additive group of integers mod \( 2n + 1 \) with congruence classes labeled by \( 0, 1, \ldots, 2n \). Call \( A \subseteq \mathbb{Z}_{2n+1} \) a game subset if \( A \cap -A = \emptyset \) and \( \mathbb{Z}_{2n+1} = \{0\} \cup A \cup -A \) where \( -A = \{-a : a \in A\} \). In particular, \( |A| = n \). The set \( \mathbb{Z}_{2n+1} \setminus \{0\} \) is decomposed by the \( n \) pairs \( \{\{a, -a\} : a \in \mathbb{Z}_{2n+1} \setminus \{0\}\} \) and a game subset is obtained by choosing one element from each pair. In particular, there are \( 2^n \) game subsets. For example \( [n] = \{1, 2, \ldots, n\} \) is a game subset.

For any game subset \( A \) define the associated game \( R[A] \) on \( \mathbb{Z}_{2n+1} \) by
\[
(2.4) \quad R[A] = \{(i, j) : j - i \in A\}.
\]
Since \( R[A](i) = i + A, R[A]^{-1}(i) = i - A \) it follows that \( R[A] \) is a regular tournament, i.e. a game. Furthermore, the translation map \( \ell_j \) on \( \mathbb{Z}_{2n+1} \) defined by \( \ell_j(i) = j + i \) is an automorphism of \( R[A] \) for each \( j \in \mathbb{Z}_{2n+1} \).

It follows that for every positive integer \( n \) there is a game of size \( 2n + 1 \). Another way of seeing this is by induction using the following construction.

Let \( R \) be a tournament on \( I \). With \( J \subseteq I \), let \( J' = I \setminus J \). For \( u, v \) distinct vertices not in \( I \), define the tournament \( R^+ \), called the extension of \( R \) via \( J \) and \( u \to v \), by
\[
(2.5) \quad R^+|I = R, \quad R^+(u) = \{v\} \cup J', \quad R^+(v) = J,
\]
so that \( (R^+)^{-1}(u) = J \) and \( (R^+)^{-1}(v) = \{u\} \cup J' \).

If \( R \) is a game with \( |I| = 2n - 1 \) and \( |J| = n \), then the extension \( R^+ \) is a game of size \( 2n + 1 \).

We conclude the section with the definition of the lexicographic product following the definition in [7] and [8] for graphs and in [11] for tournaments, see also [11] Section 6.

For \( R, S \) digraphs on \( I, J \), respectively, the lexicographic product \( R \times S \) is a digraph on \( I \times J \). For \( u, v \in I \times J \) we define \( u \to v \) when
\[
(2.6) \quad u_1 \to v_1 \text{ in } R, \quad \text{or} \quad u_1 = v_1 \text{ and } u_2 \to v_2 \text{ in } S.
\]

It is easy to check that \( R \times S \) is a tournament (or a game) if both \( R \) and \( S \) are tournaments (resp. both are games).
Partition Constructions

Recall that we defined for an \( n \) partition \( A \) of \([Nn]\) the digraph
\[
R[A] = \{(i, j) \in [n] \times [n] : \{(a, b) \in A_i \times A_j : a > b\} > N^2/2\}.
\]
If \( R[A] \) is a tournament, e.g. if \( N \) is odd, then by permuting \([n]\) or, equivalently, by relabelling the elements of \( A \), we can obtain every tournament isomorphic to \( R[A] \) as the tournament of an \( n \) partition of \([Nn]\).

For two disjoint sets \( A, B \subset \mathbb{N} \) we define
\[
Q(A, B) = 2 \cdot |\{(a, b) \in A \times B : a > b\}| - |A| \cdot |B|.
\]
Clearly,
\[
Q(A, B) \equiv |A| \cdot |B| \mod 2.
\]
We will write \( A \rightarrow B \) when \( Q(A, B) > 0 \). Observe that
\[
\frac{Q(A, B)}{2|A| \cdot |B|} = P(a > b) - \frac{1}{2}
\]
where \( a \) and \( b \) are chosen randomly from \( A \) and \( B \), respectively. In particular, if \( A = \{A_1, \ldots, A_n\} \) is an \( n \) partition of \([Nn]\), then for the digraph \( R[A] \) on \([n]\)
\[
i \rightarrow j \iff Q(A_i, A_j) > 0,
\]

**Lemma 3.1.** For disjoint sets \( A, B \subset \mathbb{N} \)
\[
Q(A, B) = |\{(a, b) \in A \times B : a > b\}| - |\{(a, b) \in A \times B : b > a\}|,
\]
\[
Q(B, A) = -Q(A, B).
\]

**Proof.** Clearly, \(|\{(a, b) \in A \times B : a > b\}| + |\{(a, b) \in A \times B : b > a\}| = |A| \cdot |B| \) since \( A \) and \( B \) are disjoint. From this and (3.2) (3.7) is obvious. Subtract from the equation \( 2|\{(a, b) \in A \times B : a > b\}| = |A| \cdot |B| + Q(A, B) \), to get (3.6).

\[\square\]

If \( |A| \) and \( |B| \) are odd, then by (3.3) \( Q(A, B) \) is odd and so cannot equal zero.

On the other hand, if \( A = \{a_1 < a_2\}, B = \{b_1 < b_2\} \subset \mathbb{N} \), then
\[
\text{Case}(i) \quad a_1 < b_1 < b_2 < a_2 \quad \implies \quad Q(A, B) = 2 \cdot 2 - 4 = 0,
\]
\[
\text{Case}(ii) \quad b_1 < a_1 < b_2 < a_2 \quad \implies \quad Q(A, B) = 2 \cdot 3 - 4 = 2.
\]
We call Case (i) a pair inclusion which we will write as $B \hookrightarrow A$ and Case (ii) a pair overlap with $A$ higher which we will write as $A \rightarrow B$.

We write $B < A$ when $b < a$ for all $(a, b) \in A \times B$. In that case, $Q(A, B) = |A| \cdot |B|$.

**Lemma 3.2.** Let $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ be pairwise disjoint finite subsets of $\mathbb{N}$. With $A = \bigcup_{i=1}^{k} A_i$, $B = \bigcup_{j=1}^{\ell} B_j$

\begin{equation}
Q(A, B) = \sum_{(i, j) \in [k] \times [\ell]} Q(A_i, B_j).
\end{equation}

If $k = \ell$, $|A_i| \cdot |B_j| = |A_j| \cdot |B_i|$ for $i, j = 1, \ldots, k$ and $A_1 \cup B_1 < A_2 \cup B_2, \ldots < A_k \cup B_k$, then

\begin{equation}
Q(A, B) = \sum_{i \in [k]} Q(A_i, B_i).
\end{equation}

**Proof.** Every $(a, b) \in A \times B$ is in a unique $A_i \times B_j$. Furthermore, $|A| \cdot |B| = \sum_{(i, j) \in [k] \times [\ell]} |A_i| \cdot |B_j|$. These imply (3.9).

If $A_i \cup B_i < A_j \cup B_j$ with $|A_i| \cdot |B_j| = |A_j| \cdot |B_i|$ then $Q(A_j, B_i) = -Q(A_i, B_j) = |A_j| \cdot |B_i|$. So (3.10) follows from (3.9).

\[\square\]

For a triple $A = \{a_1 < a_2 < a_3\}$ we write $A^- = \{a_1, a_2\}$ and $A^+ = \{a_2, a_3\}$.

**Lemma 3.3.** Assume $A = \{a_1 < a_2 < a_3\}$ and $B = \{b_1 < b_2 < b_3\}$ are disjoint subsets of $\mathbb{N}$.

(a) If $A^-$ and $B^-$ are inclusion pairs, then $a_3 > b_3 > a_2$ implies $Q(A, B) = 1$. If $A^+$ and $B^+$ are inclusion pairs, then $b_2 > a_1 > b_1$ implies $Q(A, B) = 1$.

(b) If $A^- \rightarrow B^-$, then $a_3 > b_3 > a_2$ implies $Q(A, B) = 3$ and $b_3 > a_3 > b_2$ implies $Q(A, B) = 1$. If $A^+ \rightarrow B^+$, then $b_2 > a_1 > b_1$ implies $Q(A, B) = 3$ and $a_2 > b_1 > a_1$ implies $Q(A, B) = 1$. In particular, in all of these cases $Q(A, B) > 0$.

**Proof.** These are obvious from (3.10) and (3.8) or direct computation.

\[\square\]

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an $n$ partition of $[Nn]$, and $M$ be a positive integer. Define $\mathcal{A}^* = \{A_1^*, \ldots, A_n^*\}$ and $\mathcal{A}^{(M)} = \{A_1^{(M)}, \ldots, A_n^{(M)}\}$, $n$
partitions of \([Nn]\) and of \([MNn]\), respectively, by
\[
A^*_i = \{Nn - a + 1 : a \in A_i\},
\]
\[
A^{(M)}_i = \{Nn(q-1) + a : a \in A_i, q \in [M]\},
\]
for \(i \in [n]\).

It is easy to check that, using (3.10) for the latter, that for \(i, j \in [n]\)
\[
Q(A^*_i, A^*_j) = -Q(A_i, A_j),
\]
\[
Q(A^{(M)}_i, A^{(M)}_j) = M \cdot Q(A_i, A_j).
\]

So we see that
\[
R[A^*] = R[\mathcal{A}]^{-1}, \quad R[A^{(M)}] = R[\mathcal{A}].
\]

Furthermore, for \(i \in [n]\)
\[
\sigma(A^*_i) = (Nn + 1)N - \sigma(A_i),
\]
\[
\sigma(A^{(M)}_i) = \frac{Nn \cdot M(M-1) \cdot N}{2} + M \cdot \sigma(A_i).
\]

It follows that if \(\mathcal{A}\) is proper, then \(\mathcal{A}^*\) and \(\mathcal{A}^{(M)}\) are proper.

**Theorem 3.4.** If \(\mathcal{A} = \{A_1, \ldots, A_n\}\) is an \(n\)-partition of \([Nn]\), then there exists \(\mathcal{A}'' = \{A'_1, \ldots, A''_n\}\) an \(n\)-partition of \([(N+2)n]\) with \(R[\mathcal{A}''] = R[\mathcal{A}]\).

**Proof.** Define \(C_i = \{Nn+i, Nn+2n-i+1\}\) for \(i \in [n]\). If \(j > i \in [n]\), then \(C_j \subset C_i\). So \((3.8)\) implies \(Q(C_i, C_j) = 0\) for distinct \(i, j \in [n]\). Let \(A''_i = A_i \cup C_i\). Since \(A_i < C_j\) for all \(i, j \in [n]\), \((3.10)\) implies that \(Q(A''_i, A''_j) = Q(A_i, A_j) + Q(C_i, C_j) = Q(A_i, A_j)\) for distinct \(i, j \in [n]\).

\[
\Box
\]

**Definition 3.5.** For \(\mathcal{A} = \{A_1, \ldots, A_n\}\) an \(n\)-partition of \([Nn]\) and \(\mathcal{B} = \{B_1, \ldots, B_m\}\) an \(m\)-partition of \([Nm]\), the domination product \(\mathcal{B} \triangleright \mathcal{A} = \{A_1, \ldots, A_{m+n}\}\) is the \(m+n\)-partition of \([N(m+n)]\) which extends \(\mathcal{A}\) by
\[
A_{n+p} = \{Nn + j : j \in B_p\}\quad \text{for } p \in [m].
\]

This is the partition version of the tournament construction given in \((2.2)\). To be precise:
Theorem 3.6. For $A = \{A_1, \ldots, A_n\}$ an $n$ partition of $[Nn]$ and $B = \{B_1, \ldots, B_m\}$ an $m$ partition of $[Nm]$, let $R[B]'$ be the tournament on $[n + 1, n + m] = \{n + 1, \ldots, n + m\}$ so that $n + i \rightarrow n + j$ in $R[B]'$ if and only if $i \rightarrow j$ in $R[B]$. Thus, $R[A]$ and $R[B]'$ are tournaments on disjoint sets. Furthermore, $R[B \triangleright A] = R[B]' \triangleright R[A]$.

Proof. For $p, q \in [m]$, clearly, $Q(A_{n+p}, A_{n+q}) = Q(B_p, B_q)$ and for $p \in [m], q \in [n] Q(A_{n+p}, A_q) = N^2$ since $A_{n+p} > A_q$.

\[ \Box \]

Definition 3.7. For $A \subset [M]$ and $B \subset \mathbb{N}$, not necessarily disjoint, the lexicographic product $B \prec A$ is defined by

\begin{equation}
B \prec A = \{M(b - 1) + a : b \in B, a \in A\}.
\end{equation}

The name is adopted because if $a_1, a_2 \in [M]$ then $|a_1 - a_2| < M$ and so for $b_1, b_2 \in \mathbb{N}$ it follows that

\begin{equation}
(M(b_1 - 1) + a_1) > (M(b_2 - 1) + a_2) \iff \begin{cases} b_1 > b_2 \\ b_1 = b_2 \text{ and } a_1 > a_2. \end{cases}
\end{equation}

In particular, we see that $|B \prec A| = |B| \cdot |A|$. It is easy to check that

\begin{equation}
\sigma(B \prec A) = M \cdot |A| \cdot (\sigma(B) - |B|) + |B| \cdot \sigma(A).
\end{equation}

Notice that the definition requires that we specify $M$.

Lemma 3.8. (a) If $A_1, A_2 \subset [M]$, and $B_1, B_2$ are disjoint subsets of $\mathbb{N}$, then $B_1 \prec A_1$ and $B_2 \prec A_2$ are disjoint with

\begin{equation}
Q(B_1 \prec A_1, B_2 \prec A_2) = Q(B_1, B_2) \cdot |A_1| \cdot |A_2|.
\end{equation}

(b) If $A_1, A_2$ are disjoint subsets of $[M]$, and $B$ is a subset of $\mathbb{N}$, then $B \prec A_1$ and $B \prec A_2$ are disjoint with

\begin{equation}
Q(B \prec A_1, B \prec A_2) = Q(A_1, A_2) \cdot |B|.
\end{equation}

Proof. (a) Since $B_1$ and $B_2$ are disjoint, $(M(b_1 - 1) + a_1) > (M(b_2 - 1) + a_2)$ if and only if $b_1 > b_2$. Hence, (3.19) easily follows from (3.6).

(b) Write $B \prec A_p = \bigcup \{b \mid A_p : b \in B\}$ for $p = 1, 2$. From (3.10) it follows that

\begin{equation}
Q(B \prec A_1, B \prec A_2) = \sum_{b \in B} Q(\{b\} \prec A_1, \{b\} \prec A_2).
\end{equation}
Clearly, \( Q(\{b\} \times A_1, \{b\} \times A_2) = Q(A_1, A_2) \) for every \( b \in B \) and so (3.20) follows.

\[ \square \]

**Definition 3.9.** For \( \mathcal{A} = \{A_1, \ldots, A_n\} \) an \( n \) partition of \([Nn]\) and \( \mathcal{B} = \{B_1, \ldots, B_k\} \) a \( k \) partition of \([Kk]\), the lexicographic product 
\( \mathcal{B} \ltimes \mathcal{A} = \{C_1, \ldots, C_{kn}\} \) is the \( kn \) partition of \([KNkn]\) defined by
\[
(3.22) \quad C_{n(i-1)+j} = B_i \ltimes A_j \quad \text{with} \quad M = Nn.
\]

From Lemma 3.8, we obtain.

**Lemma 3.10.** For \( \mathcal{B} \ltimes \mathcal{A} = \{C_1, \ldots, C_{kn}\} \), we have
\[
(3.23) \quad Q(C_{n(i_1-1)+j_1}, C_{n(i_2-1)+j_2}) = Q(B_{i_1}, B_{i_2}) \cdot N^2 \quad \text{when} \quad i_1 \neq i_2,
\]
\[
(3.24) \quad Q(C_{n(i_1-1)+j_1}, C_{n(i_2-1)+j_2}) = Q(A_{j_1}, A_{j_2}) \cdot K \quad \text{when} \quad j_1 \neq j_2.
\]

From this computation we immediately obtain the following theorem. Note that the lexicographic product of proper partitions is proper by (3.18).

**Theorem 3.11.** For \( \mathcal{A} \) an \( n \) partition of \([Nn]\) and \( \mathcal{B} \) a \( k \) partition of 
\([Kk]\), the \( kn \) partition of \([NK^{kn}]\) \( \mathcal{B} \ltimes \mathcal{A} \) satisfies \( R[\mathcal{B} \ltimes \mathcal{A}] = R[\mathcal{B}] \ltimes R[\mathcal{A}] \). If \( \mathcal{A} \) and \( \mathcal{B} \) are proper, then \( \mathcal{B} \ltimes \mathcal{A} \) is proper.

By adapting this construction we now show how to extend the digraph of a partition, by inserting an additional vertex.

Let \( \mathcal{A} = \{A_1, \ldots, A_n\} \) be an \( n \) partition of \([Nn]\) and let \( J \subset [n] \) with \( J' = [n] \setminus J \). We will show how to obtain an \( n+1 \) partition whose tournament on \([n+1]\) restricts to \( R[\mathcal{A}] \) on \([n]\) and with \( n+1 \rightarrow j \) if and only if \( j \in J \).

First, we consider the extreme cases where \( J = [n] \) or \( J = \emptyset \), i.e. where \( n+1 \) is a maximum or minimum vertex.

Let \( \mathcal{O}_N = \{[N]\} \) which we regard as a 1 partition of \([N] = [N \cdot 1] \). Thus, \( R[\mathcal{O}_N] = \emptyset[[1]] \), the trivial tournament on the singleton \([1]\).

By Theorem 3.6 \( \mathcal{C} = \mathcal{O}_N \triangleright \mathcal{A} \) is an \( n+1 \) partition of \([N(n+1)]\) which extends \( \mathcal{A} \) and which has \( A_{n+1} \rightarrow A_p \) for all \( p \in [n] \). This is the case with \( J = [n] \).

On the other hand, if \( \mathcal{C}' = \mathcal{A} \triangleright \mathcal{O}_N = \{C'_1, \ldots, C'_{n+1}\} \) and we permute, defining \( C_p = C'_{p+1} \) for \( p \in [n] \) and \( C'_{n+1} = C'_1 \), then for \( \mathcal{E} = \{C_1, \ldots, C_n, C_{n+1}\} \), \( R[\mathcal{E}] \) restricts to \( R[\mathcal{A}] \) on \([n]\) and \( C_p \rightarrow C_{n+1} \) for all \( p \in [n] \). This is the case with \( J = \emptyset \).
For the remaining construction we will use $\mathcal{B} = \{B_1, B_2, B_3\}$ with

\[
\begin{align*}
B_1 &= \{2, 4, 6\}, \\
B_2 &= \{2, 3, 8\}, \\
B_3 &= \{1, 5, 7\},
\end{align*}
\]

(3.25)

so that $B_1$ and $B_2$ are disjoint from $B_3$, but $B_1 \cap B_2 = \{2\}$. Clearly, $B_1^+ \hookrightarrow B_2^+$ and $B_3^+ \rightarrow B_1^+$.

Using Lemma 3.3 or computing directly we see that

\[
Q(B_2, B_3) = Q(B_3, B_1) = 2 \cdot 5 - 9 = 1.
\]

(3.26)

Now let $J \subset [n]$ with $J$ and $J' = [n] \setminus J$ nonempty. Define the sets $C_1, \ldots, C_{n+1} \subset [8Nn]$ using a variation of the lexicographic order construction.

With $M = Nn$ define

\[
\begin{align*}
C_i &= B_1 \ltimes A_i \quad \text{for } i \in J, \\
C_i &= B_2 \ltimes A_i \quad \text{for } i \in J', \\
C_{n+1} &= B_3 \ltimes [N].
\end{align*}
\]

(3.27)

**Lemma 3.12.** The sets $C_1, \ldots, C_{n+1}$ are pairwise disjoint with $|C_i| = 3N$ for $i \in [n+1]$ and satisfying

\[
\begin{align*}
Q(C_i, C_j) &= 3 \cdot Q(A_i, A_j) \text{ when } i, j \in J \text{ or } i, j \in J' \quad \text{(3.28)} \\
Q(C_i, C_j) &= Q(A_i, A_j) \text{ when } i \in J \text{ and } j \in J' \quad \text{(3.29)} \\
Q(C_i, C_{n+1}) &= N^2 \text{ when } i \in J' \quad \text{(3.30)} \\
Q(C_{n+1}, C_i) &= N^2 \text{ when } i \in J. \quad \text{(3.31)}
\end{align*}
\]

**Proof.** Because the sets $B_1$ and $B_2$ are combined with different sets in $\mathcal{A}$, it is clear that $C_1, \ldots, C_{n+1}$ are pairwise disjoint.

Equation (3.28) follows from (3.20).

Equations (3.30) and (3.31) follow from (3.19).

For (3.29) we write $B_0 = \{2\}$, so that $B_1 = B_0 \cup B_1^+$ and $B_2 = B_0 \cup B_2^+$. Since $B_1^+ \hookrightarrow B_2^+$ and $B_0 \oplus B_1^+, B_2^+$ it follows that

\[
Q(B_1^+, B_2^+) = 0,
\]

(3.32)

\[
Q(B_1^+, B_0) = Q(B_2^+, B_0) = 2.
\]

(3.33)

So for distinct $i, j \in [n]$, (3.19) implies

\[
Q(B_1^+ \ltimes A_i, B_0 \ltimes A_j) = 0,
\]

(3.33)

\[
Q(B_1^+ \ltimes A_i, B_0 \ltimes A_j) = Q(B_2^+ \ltimes A_i, B_0 \ltimes A_j) = 2N^2.
\]
From (3.20) we have
\[ Q(B_0 \times A_i, B_0 \times A_j) = Q(A_i, A_j). \]
Because \( B_1 \times A_i = (B_0 \times A_i) \cup (B_1^+ \times A_i) \) and \( B_2 \times A_j = (B_0 \times A_j) \cup (B_2^+ \times A_j) \), (3.9), (3.33) and (3.34) imply (3.29).

\[ \square \]

Now assume that \( \mathcal{C} = \{C_1, \ldots, C_k\} \) is a collection of pairwise disjoint subsets of \( \mathbb{N} \) with \( |C_i| = K \) for \( i \in [k] \). We obtain a \( k \)-partition of \( [Kk] \) \( \mathcal{D} = \{D_1, \ldots, D_k\} \) by packing \( \mathcal{C} \) as follows. Let \( m_1, \ldots, m_{Kk} \) number in order the points of \( \bigcup_{p=1}^{k} C_p \) so that \( m_i > m_j \) if and only if \( i > j \).

Define
\[ D_p = \{i : m_i \in C_p\} \quad \text{for} \quad p \in [k]. \]
Since the ordering of the elements is preserved by the renumbering it follows that
\[ Q(D_p, D_q) = Q(C_p, C_q) \quad \text{for} \quad p, q \in [k]. \]

**Theorem 3.13.** Let \( \mathcal{A} = \{A_1, \ldots, A_n\} \) be an \( n \) partition of \( [Nn] \) with \( R[\mathcal{A}] \) a tournament on \( [n] \), e.g. with \( N \) odd. Assume that \( R \) is a tournament on \( [n+1] \) with restriction \( R|[n] \) equal to \( R[\mathcal{A}] \). There exists \( \mathcal{D} = \{D_1, \ldots, D_{n+1}\} \) an \( n+1 \) partition of \( [3N(n+1)] \) with \( R[\mathcal{D}] = R \).

**Proof.** Case 1 (Either \( R(n+1) = [n] \) or \( R^{-1}(n+1) = [n] \)).

If \( R(n+1) = [n] \), then we begin with \( \mathcal{C} = \emptyset_N \triangleright \mathcal{A} \) and let \( \mathcal{D} = \mathcal{C}^{(3)} \), see (3.11). This is an \( n+1 \) partition of \( [3N(n+1)] \) and by (3.12) \( R[\mathcal{D}] = R \).

Similarly if \( R^{-1}(n+1) = [n] \), then we begin with \( \mathcal{C} \) the renumbering given above of \( \mathcal{C}' = \mathcal{A} \triangleright \emptyset_N \) and let \( \mathcal{D} = \mathcal{C}^{(3)} \).

Case 2 (\( R(n+1) = J \subset [n] \) with \( J \) and \( J' = [n] \setminus J \) nonempty).

We apply the construction of (3.27) to the partition \( \mathcal{A} \) yielding the disjoint sets \( \{C_1, \ldots, C_{n+1}\} \). From Lemma 3.12 we see that \( i \rightarrow j \) in \( R \) if and only if \( Q(C_i, C_j) > 0 \).

We obtain \( \mathcal{D} \) by packing \( \mathcal{C} = \{C_1, \ldots, C_{n+1}\} \) to obtain an \( n+1 \) partition of \( [3N(n+1)] \).

\[ \square \]

From this follows our main result and, in particular, Theorem 1.2.
Theorem 3.14. For $n \geq 2$, if $R$ is a tournament on $[n]$ and $N$ is any odd integer with $N \geq 3^{n-2}$ or an even integer with $N \geq 2 \cdot 3^{n-2}$, then there exists an $n$ partition $A$ of $[Nn]$ with $R[A] = R$.

Proof. We use induction on $n$ to prove that $R$ can be modeled by $A$ an $n$ partition of $[3^{n-2} \cdot n]$.

With $n = 2$, $\{\{2\}, \{1\}\}$ is a 2 partition of $1 \cdot 2 = 3^{n-2} \cdot n$. Reversing the two elements we obtain the other tournament on $[2]$.

For the inductive step, we apply Theorem 3.13.

By using $A^{(2)}$ we obtain an $n$ partition on $[2 \cdot 3^{n-2}n]$ which models $R$.

If $N = 3^{n-2} + 2m$ or $N = 2 \cdot 3^{n-2} + 2m$, then the result follows by induction on $m$, using Theorem 3.14 for the inductive step.

\[\square\]

We illustrate Theorem 3.13 by beginning with $A = \{A_1, A_2\} = \{\{2\}, \{1\}\}$ and using $J = \{1\}$. Thus, $N = 1$ and $n = 2$.

\[
\begin{align*}
C_1 &= B_1 \circ A_1 = \{2 + 2, 6 + 2, 10 + 2\} = \{4, 8, 12\}, \\
C_2 &= B_2 \circ A_2 = \{2 + 1, 4 + 1, 14 + 1\} = \{3, 5, 15\}, \\
C_3 &= B_3 \circ [1] = \{0 + 1, 8 + 1, 12 + 1\} = \{1, 9, 13\}.
\end{align*}
\]

Packing we obtain

\[
\begin{align*}
D_1 &= \{3, 5, 7\}, \\
D_2 &= \{2, 4, 9\}, \\
D_3 &= \{1, 6, 8\}.
\end{align*}
\]

This is example (1.2) with which we began.

On the other hand, the exponential growth in Corollary 3.14 provides what is probably only a crude upper bound. For example, it shows that tournaments on $[5]$ can be modeled using 5 partitions on $[27 \cdot 5] = [135]$. The example (1.3) models the Rock-Paper-Scissors-Lizard-Spock tournament as a 5 partition of $[30]$ and we will see in the next section that we can do even better.

We can mimic for partitions the extension construction of (2.5) by using the following list.
Define $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ with
\[
\begin{align*}
B_0 &= \{10\}, \\
B_1 &= \{2, 7, 10\}, \\
B_2 &= \{4, 5, 10\}, \\
B_3 &= \{3, 8, 9\}, \\
B_4 &= \{1, 6, 11\}.
\end{align*}
\]
(3.39)

Clearly: $B_2^- \hookrightarrow B_1^-, B_3^-, B_4^-, B_3^- \rightarrow B_1^-, B_4^-$ and $B_1^- \rightarrow B_4^-$. From Lemma 3.3 we have
\[
Q(B_2, B_3) = Q(B_3, B_4) = Q(B_4, B_2) = 1,
\]
\[
Q(B_3, B_1) = Q(B_1, B_4) = 1,
\]
\[
Q(B_1^-, B_2^-) = 0,
\]
(3.40)
\[
Q(B_0, B_1^-) = Q(B_0, B_2^-) = 2.
\]

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an $n$ partition of $[Nn]$ and let $J \subset [n]$ with $J' = [n] \setminus J$ with neither empty. With $M = Nn$ define
\[
\begin{align*}
C_i &= B_1 \times A_i \quad \text{for } i \in J', \\
C_j &= B_2 \times A_j \quad \text{for } j \in J,
\end{align*}
\]
(3.41)
\[
\begin{align*}
\bar{U} &= B_3 \times [N], \\
\bar{V} &= B_4 \times [N].
\end{align*}
\]

**Lemma 3.15.** The sets $C_1, \ldots, C_n, \bar{U}, \bar{V}$ are pairwise disjoint with $|\bar{U}| = |\bar{V}| = |C_i| = 3N$ for $i \in [n]$ and they satisfy
\[
Q(C_i, C_j) = 3 \cdot Q(A_i, A_j) \quad \text{when } i, j \in J \text{ or } i \in J', j \in J
\]
(3.42)
\[
Q(C_i, C_j) = Q(A_i, A_j) \quad \text{when } j \in J, i \in J'
\]
(3.43)
\[
Q(C_j, \bar{U}) = Q(\bar{U}, \bar{V}) = Q(\bar{U}, C_i) = N^2 \quad \text{when } j \in J, i \in J'
\]
(3.44)
\[
Q(\bar{V}, C_j) = Q(C_i, \bar{V}) = N^2 \quad \text{when } j \in J, i \in J'.
\]
(3.45)

**Proof.** Using Lemmas 3.2 and 3.8 with (3.40), the proof proceeds just as in Lemma 3.12. \qed
Theorem 3.16. Let $A = \{A_1, \ldots, A_n\}$ be an $n$ partition of $[Nn]$ with $R[A]$ a tournament on $[n]$, and let $J \subset [n]$ and $J' = [n] \setminus J$ with neither empty. There exists $D = \{D_1, \ldots, D_n, U, V\}$ an $n+2$ partition of $[3N(n+2)]$ with $R[D] = R^+$ the extension of $R$ via $J$ and $U \rightarrow V$.

Proof. Use $C = \{C_1, \ldots, C_n, \bar{U}, \bar{V}\}$ from Lemma 3.15 and then pack to obtain $D$. □

4. Examples

We saw at the end of Section 1 that for any $N$ there exist, for $n$ sufficiently large, tournaments $R$ on $[n]$ which cannot be modeled using an $n$ partition of $[Nn]$.

Proposition 4.1. For $N$ a positive integer, let $R$ be a tournament on $[n]$ which cannot be modeled using an $n$ partition of $[Nn]$. If $n$ is the minimum size of such an unobtainable tournament, then $R$ is a strong tournament.

Proof. Assume that every tournament of size $k < n$ can be modeled using a $k$ partition of $[Nk]$. If $R$ is a tournament on $[n]$ which is not strong, then by Proposition 2.3 it can be written as the domination product of two tournaments of smaller size. So by Theorem 3.6 it can be modeled as the domination production of some $k_1$ partition of $[Nk_1]$ and a $k_2$ partition of $[Nk_2]$ with $k_1, k_2$ positive integers such that $k_1 + k_2 = n$. Hence, $R$ can be modeled by an $n$ partition of $[Nn]$.

In this section we will consider examples of tournaments on $[n]$ which can be modeled by using $n$ partitions of $[3n]$, i.e. with $N = 3$.

For $\pi$ a permutation of $[n]$ and $A = \{A_1, \ldots, A_n\}$ an $n$ partition of $[Nn]$, define $A^\pi = \{A^\pi_1, \ldots, A^\pi_n\}$ by

\begin{equation}
A^\pi_i = A_{\pi^{-1}i},
\end{equation}

for $i \in [n]$, so that $Q(A^\pi_i, A^\pi_j) = Q(A_{\pi^{-1}i}, A_{\pi^{-1}j})$ from which it follows that $i \rightarrow j$ in $R[A^\pi]$ if and only if $\pi^{-1}i \rightarrow \pi^{-1}j$ in $R[A]$. Thus,

\begin{equation}
R[A^\pi] = \pi R[A].
\end{equation}

It follows that if $R[A]$ is isomorphic to a tournament $R$ on $[n]$, then $R[A^\pi] = R$ for some permutation $\pi$ of $[n]$. 
On the other hand, if \( \pi \) is a permutation of \( [Nn] \) we can define \( \pi(A) = \{ \pi(A_1), \ldots, \pi(A_n) \} \) with \( \pi(A_p) = \{ \pi(j) : j \in A_p \} \), the image of \( A_p \) by the map \( \pi \). If we start with \( \mathcal{A} \) an arbitrary \( n \) partition of \( [Nn] \), then by varying the permutation \( \pi \) we can obtain any \( n \) partition of \( [Nn] \).

For \( k = 1, \ldots, M - 1 \) call the transposition \( (k, k + 1) \) on \( [M] \) a simple transposition.

**Lemma 4.2.** With \( M \geq 2 \), every permutation on \( [M] \) is a product of simple transpositions.

*Proof.* For a permutation \( \pi \) we show that a product of simple transpositions applied to the sequence \( \pi(1), \ldots, \pi(M) \) transforms the sequence to the identity sequence \( 1, \ldots, M \).

We first use induction on \( k \) with \( \pi(M) = M - k \), to obtain a sequence of simple transpositions after which the sequence terminates at \( M \).

If \( k = 0 \), then no transpositions are necessary.

If \( 0 < k \leq M - 1 \), apply the transposition \( (M - k, M - (k - 1)) \) after which the sequence terminates at \( M - (k - 1) \). Now use the inductive hypothesis to transform so that the sequence terminates at \( M \).

For a permutation \( \pi \) we show by induction on \( M \) that a product of simple transpositions applied to the sequence \( \pi(1), \ldots, \pi(M) \) transforms the sequence to \( 1, \ldots, M \). This is trivial for \( M = 2 \). Now assume \( M > 2 \).

By our first result we may assume that \( \pi(M) = M \). Now \( \pi(1), \ldots, \pi(M - 1) \) is a permutation of \( [M - 1] \) and so by inductive hypothesis, it can be transformed to the identity by a sequence of simple transpositions on \( [M - 1] \).

\[\square\]

Begin with an arbitrary \( n \) partition of \( [Nn] \), \( \mathcal{A} = \{A_1, \ldots, A_n\} \) and \( k \in [Nn - 1] \). Assume that \( k \in A_{p_1}, k + 1 \in A_{p_2} \) with \( p_1 \neq p_2 \). If \( \pi \) is the simple transposition \( (k, k + 1) \), then

\[
\begin{align*}
\pi(A_{p_1}) &= (A_{p_1} \setminus \{k\}) \cup \{k + 1\}, \\
\pi(A_{p_2}) &= (A_{p_2} \setminus \{k + 1\}) \cup \{k\}, \\
\pi(A_p) &= A_p \quad \text{for } p \neq p_1, p_2.
\end{align*}
\]
For \( j \in [Nn] \) with \( j \neq k, k + 1 \) it is clear that \( k > j \) if and only if \( k + 1 > j \). So it follows that
\[
Q(\pi(A_p), \pi(A_q)) = Q(A_p, A_q) \quad \text{if} \quad \{p, q\} \neq \{p_1, p_2\},
\]
\[
Q(\pi(A_{p_1}), \pi(A_{p_2})) = Q(A_{p_1}, A_{p_2}) + 2.
\]

Thus, either \( R[\pi(A)] = R[A] \) or the only possible changes are
(i) the reversal of the edge from \( A_{p_2} \) to \( A_{p_1} \), which occurs if \( Q(A_{p_2}, A_{p_1}) = 1 \),
(ii) the elimination of the edge from \( A_{p_2} \) to \( A_{p_1} \), which occurs if \( Q(A_{p_2}, A_{p_1}) = 2 \),
(iii) the introduction of a new edge \( A_{p_1} \) to \( A_{p_2} \), which occurs if \( Q(A_{p_2}, A_{p_1}) = 0 \).

Notice that if \( N \) is odd, then by (3.3), cases (ii) and (iii) cannot occur.

Call this operation a simple switch. From Lemma 1.2 it follows that we can get from any \( n \) partition of \([Nn]\) to any other by a sequence of simple switches.

If \( A = \{A_1, \ldots, A_n\} \) is an \( n \) partition of \([3n]\) we write \( A_i = \{a^i_1 < a^i_2 < a^i_3\} \) calling \( a^i_s \) the level \( s \) element for \( s = 1, 2, 3 \). We call \( A \) a stratified partition if
\[
\{a^i_s : i \in [n]\} = \{n(s - 1) + j : j \in [n]\}, \quad \text{for} \quad s = 1, 2, 3.
\]
That is, the level one elements are \([1, \ldots, n]\) and the level two elements are \([n+1, \ldots, 2n]\) so that the level three elements are \([2n+1, \ldots, 3n]\).

**Theorem 4.3.** If \( B = \{B_1, \ldots, B_n\} \) is an \( n \) partition of \([3n]\), then there exists \( A = \{A_1, \ldots, A_n\} \) a stratified \( n \) partition of \([3n]\) with \( R[A] = R[B] \).

**Proof.** As before label \( B_i = \{b^i_1 < b^i_2 < b^i_3\} \). Notice that for \( i, j \in [n] \)
\[
b^i_2 > b^j_3 \quad \text{or} \quad b^i_1 > b^j_3 \quad \implies \quad Q(A_i, A_j) \geq 2 \cdot 6 - 9 = 3,
\]
\[
b^i_1 > b^j_2 \quad \implies \quad Q(A_i, A_j) \geq 2 \cdot 6 - 9 = 3.
\]

We first prove that by using a sequence of simple switches we can obtain \( \{b^i_3 : i \in [n]\} = \{2n + j : j \in [n]\} \). If we let \( m_3 = \min\{b^i_3 : i \in [n]\} \) then this is equivalent to \( m_3 = 2n + 1 \). Observe that always \( m_3 \leq 2n + 1 \) since \( \{b^i_3 : i \in [n]\} \) consists of \( n \) distinct integers with maximum \( 3n \). We use induction assuming that \( m_3 = 2n + 1 - r \). If \( r = 0 \) then there is nothing to prove.

Assume that \( r \geq 1 \). Among the \( n + r \) numbers in the interval \([2n + 1 - r, 3n]\) some are level two elements and perhaps some are even from level one. Let \( k + 1 \) be the smallest such so that each of the numbers in the interval \([2n + 1 - r, k] \) is from level three. In particular, \( k = b^j_3 \).
with \( k + 1 = b^j_i \) for \( s = 2 \) or \( 1 \). If we do a simple switch of \( k \) with \( k + 1 \), to obtain \( \bar{B}_i, \bar{B}_j \) then \( k = \bar{b}^i_s \) and \( k + 1 = \bar{b}^j_i \). Furthermore, (4.4) implies that \( Q(\bar{B}_i, \bar{B}_j) = Q(B_i, B_j) - 2 \) which is still positive by (4.6). As this is the only change in the \( Q \) values it follows that \( R[\bar{B}] = R[B] \). We continue down, switching until finally, we obtain \( B \) with \( 2n + 1 - r = b^i_p \) and with \( m_3 = 2n + 1 - r + 1 = 2n + 1 - (r - 1) \). Now apply the induction hypothesis.

Now assuming that \( B \) satisfies \( \{ b^i_j : i \in [n] \} = \{ 2n + j : j \in [n] \} \), we let \( m_2 = \min \{ b^i_j : i \in [n] \} = n + 1 - r \). If \( r = 0 \), then \( \{ b^i_j : i \in [n] \} = \{ n + j : j \in [n] \} \) and so \( \{ b^i_j : i \in [n] \} = \{ j : j \in [n] \} \). As above, \( m_2 \leq n + 1 \) and we are done if \( r = 0 \).

Assume that \( r \geq 1 \). Among the \( n + r \) numbers in the interval \( [n + 1 - r, 2n] \) some are from level one. Let \( k + 1 \) be the smallest such so that each of the numbers in \( [n + 1 - r, k] \) is from level two. In particular, \( k = b^i_2 \) with \( k + 1 = b^i_1 \). As before we do a sequence of switches to get \( \bar{B} \) with \( b^i_1 = n + 1 - r \) and with \( m_2 = n + 1 - r + 1 = n + 1 - (r - 1) \). Applying the induction hypothesis we arrive at \( A \).

The main result of this section is the observation that for arbitrary \( n \) there is a game of size \( 2n + 1 \) which can be modeled using a \( 2n + 1 \) partition of \( [3(2n + 1)] \), that is, with \( N = 3 \).

**Theorem 4.4.** Let \( R \) be the group game on \( Z_{2n+1} \) with game subset \( [n] \). That is, for \( p, q \in Z_{2n+1} \) \( p \to q \) if and only if \( q - p \text{ (mod } 2n+1) \) lies in \( [n] \). There exists \( \mathcal{A} = \{ A_0, A_1, \ldots, A_{2n} \} \) a proper, stratified \( 2n + 1 \) partition of \( [3(2n + 1)] \) with \( R[\mathcal{A}] = R \). That is,

\[
A_p \to A_q \iff q - p \in [n] \text{ mod } 2n + 1.
\]

**Proof.** Notice that for convenience of the algebraic description we are labelling the elements of the partition by \( Z_{2n+1} = \{ 0, 1, \ldots, 2n \} \) rather than by \( [2n + 1] = \{ 1, 2, \ldots, 2n + 1 \} \).

Define \( \mathcal{A} = \{ A_0, A_1, \ldots, A_{2n} \} \) the stratified \( 2n+1 \) partition of \( [3(2n+1)] \) by (with \( j = 1, \ldots n \))

\[
\begin{align*}
A_0 &= \{ 2n + 1, 3n + 2, 4n + 3 \}, \\
A_j &= \{ 2n + 1 - j, 3n + 2 - j, 4n + 3 + 2j \}, \\
A_n &= \{ n + 1, 2n + 2, 6n + 3 \}, \\
A_{n+1} &= \{ n, 4n + 2, 4n + 4 \}, \\
A_{n+j} &= \{ n + 1 - j, 4n + 3 - j, 4n + 2 + 2j \}, \\
A_{2n} &= \{ 1, 3n + 3, 6n + 2 \},
\end{align*}
\]

\( \square \)
Note first that $\sigma(A_p) = 9n + 6$ for all $p$ and so the partition is proper. Observe that for $p < q \in [0, n]

\begin{equation}
(4.9) \quad a_1^q < a_1^p < a_2^q < a_2^p.
\end{equation}

Thus, $A_p^- \rightarrow A_q^-$. Similarly, for $p < q \in [1, n]

\begin{equation}
(4.10) \quad a_1^{n+q} < a_1^{n+p} < a_2^{n+q} < a_2^{n+p}.
\end{equation}

So $A_{n+p}^- \rightarrow A_{n+q}^-$. From (3.8) Case (ii) and (3.10) it follows that

\begin{align*}
Q(A_p, A_q) &= 1 \quad \text{for } p < q \in [0, n], \\
Q(A_{n+p}, A_{n+q}) &= 1 \quad \text{for } p < q \in [n].
\end{align*}

(4.11)

Next note that

\begin{equation}
(4.12) \quad a_1^{n+p} < a_1^q < a_2^q < a_2^{n+p}
\end{equation}

for $p \in [n], q \in [0, n]$. Thus, each $A_q^- \rightarrow A_{n+p}^-$. Furthermore,

\begin{align*}
& a_1^j > a_1^{n+j} > a_3^{n+p} \quad \text{for } 1 \leq p \leq j, \\
& a_3^{n+j} > a_3^{j-1} > a_3^q \quad \text{for } 0 \leq q \leq j-1.
\end{align*}

(4.13)

From (3.8) Case (ii) and (3.10) it follows that, for $j \in [n]

\begin{align*}
Q(A_j, A_{n+p}) &= 1 \quad \text{for } 1 \leq p \leq j, \\
Q(A_{n+j}, A_q) &= 1 \quad \text{for } 0 \leq q \leq j-1.
\end{align*}

(4.14)

The relations of (4.7) follow from (4.11) and (4.14).

\[ \Box \]

In particular, with $n = 2$ we obtain the unique game of size 5 via

\begin{align*}
A_0 &= \{5, 8, 11\}, \\
A_1 &= \{4, 7, 13\}, \\
A_2 &= \{3, 6, 15\}, \\
A_3 &= \{2, 10, 12\}, \\
A_4 &= \{1, 9, 14\}, \\
\end{align*}

(4.15)
Even among proper, stratified partitions this representation is not unique. For example we can also obtain the game of size 5 via

\[
\begin{align*}
A_0 &= \{5, 7, 12\}, \\
A_1 &= \{4, 6, 14\}, \\
A_2 &= \{3, 10, 11\}, \\
A_3 &= \{2, 9, 13\}, \\
A_4 &= \{1, 8, 15\},
\end{align*}
\]

(4.16)

Using these and suitable simple switches one can show that every tournament on \([n]\) with \(n \leq 5\) can be modeled using an \(n\) partition of \([3n]\). By Proposition [4.1] one need only consider strong tournaments with \(2 < n \leq 5\).

There are three isomorphism classes of games of size seven, labeled Type I, II and III in Section 10 of [1]. All three can be obtained using 7 partitions of \([21]\). The Type I game is given by Theorem [4.4] with \(n = 3\):  

\[
\begin{align*}
A_0 &= \{7, 11, 15\}, \\
A_1 &= \{6, 10, 17\}, \\
A_2 &= \{5, 9, 19\}, \\
A_3 &= \{4, 8, 21\}, \\
A_4 &= \{3, 14, 16\}, \\
A_5 &= \{2, 13, 18\}, \\
A_6 &= \{1, 12, 20\}.
\end{align*}
\]

(4.17)

The game of Type II can be obtained via the following proper, stratified 7 partition of 21:  

\[
\begin{align*}
A_0 &= \{7, 9, 17\}, \\
A_1 &= \{6, 12, 15\}, \\
A_2 &= \{5, 8, 20\}, \\
A_3 &= \{4, 11, 18\}, \\
A_4 &= \{3, 14, 16\}, \\
A_5 &= \{2, 10, 21\}, \\
A_6 &= \{1, 13, 19\}.
\end{align*}
\]

(4.18)
\[ A_0 \rightarrow A_1, A_2, A_4; \quad A_1 \rightarrow A_2, A_3, A_5; \]
\[ A_2 \rightarrow A_3, A_4, A_6; \quad A_3 \rightarrow A_0, A_4, A_5; \]
\[ A_4 \rightarrow A_1, A_5, A_6; \quad A_5 \rightarrow A_0, A_2, A_6; \]
\[ A_6 \rightarrow A_0, A_1, A_3. \]

Each with a \( Q \) value of 1. This is group game on \( \mathbb{Z}_7 \) with game subset \( \{1, 2, 4\} \).

By doing a 1, 2 simple switch we reverse the arrow \( A_5 \rightarrow A_6 \). By doing a 10, 11 simple switch we reverse the arrow \( A_3 \rightarrow A_5 \). By doing a 18, 19 simple switch we reverse the arrow \( A_6 \rightarrow A_3 \). Together these three simple switches reverse the 3-cycle \( (A_3, A_5, A_6) \). This yields a Type III game which is not a group game. The result is still a stratified, proper partition.

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