Buckling Analyses of Functionally Graded Graphene Nanoplatelets Reinforced Nonlocal Piezoelectric Microplate

H M Lu, W Zhang and J J Mao

College of Mechanical Engineering, Beijing University of Technology, Beijing 100124, P. R. China
Email: luhaoming@emails.bjut.edu.cn; sandyzhang0@yahoo.com; jiajia.mao@bjut.edu.cn

Abstract. This paper analyses the buckling character of a multi-layered functionally graded graphene nanoplatelets reinforced nonlocal piezoelectric (FG-GRNP) microplate. The FG-GRNP microplate is acted by the in-plane axial force and external potential. Graphene nanoplatelets are assumed to dispersing parallel in each layer of the FG-GRNP microplate, but it is distributed graded along the thickness direction. The rule of mixture and Halpin-Tsai parallel model are used to calculate the properties of the FG-GRNP microplate. The governing equations of buckling behaviors of the FG-GRNP microplate are derived by nonlocal elastic theory, minimum potential energy principle and first-order shear deformation plate theory. The differential quadrature (DQ) method is adopted to solve the eigenvalue equations. The effects of the axial forces, piezoelectric multiple, nonlocal parameter, external voltage, characteristics of graphene nanoplatelets on the critical buckling load of the FG-GRNP microplate are studied.

Keywords. Graphene nanoplatelets, critical buckling load, small-size effect, DQ method.

1. Introduction
With the rapid development of the material science, graphene has been widely concerned by scholars due to its excellent electrical and mechanical properties [1]. Graphene can enhance the performances of polymer composite material [2, 3] with its excellent two-dimensional structure and properties. Comparing with pure epoxy, the graphene nanoplatelets Young’s modulus reinforce composite grow up 131% by adding 0.1% volume fraction of graphene nanoplatelets [2]. It is also recorded that the storage modulus, Young’s modulus and stress at break of GF-PVDF composite material separately grow up 124%, 121% and 97% by adding 0.75% volume fraction of graphene nanoplatelets [3]. In addition, the positive piezoelectric effect of graphene has been found by Xu’s team [4]. Many theoretical and experimental researches indicated that graphene polymer micro-composites are widely applied to the fields of micro-electro-mechanical and aerospace [5].

In recent years, functionally graded materials (FGMs) have got extensive attention in the academic community as the smooth and continuous manner of FGMs can greatly improve the damage resistance capacities and deformation resistance [6] hence strengthen the dynamic stability of structures [7]. Zhang’s team has systematically studied on the nonlinear dynamic behaviors of functionally graded material structures, containing the FGM sandwich doubly-curved shallow shell [8], FGM truncated conical shell [9], imperfect FGM conical panel [10].

Due to the force-electro coupling effect of the graphene nanoplatelets [4], graphene is beneficial for the development of the piezoelectric structures. Small-size effect is an important factor for the micro- and nano- structures. However, the classical elasticity theory cannot describe the small-size effect of...
micro- and nano-structures. Many high-order continuum theories are produced, such as nonlocal elastic theory [11]. Liew and his co-writers concluded the prospects and challenges of the nonlocal elasticity theory for graphene model [12]. In addition, Sahmani and his team [13] took the nonlocal effect into consideration to research the instability and nonlinear bending of the functionally graded graphene nanoplatelets reinforced porous micro/nano-structures.

This paper combines the nonlocal elastic theory, first order shear deformation plate theory and DQ method to analyse the buckling behaviors of the multi-layer FG-GRNP microplate. Graphene nanoplatelets distribute parallel and homogeneously in each layer, but they present graded distribution along the thickness of the novel microplate. The rule of mixture and Halpin-Tsai parallel model are employed to calculate the effective material properties of the FG-GRNP microplate. The effects of the axial forces, piezoelectric multiple, nonlocal parameter, external voltage, characteristics of graphene nanoplatelets on the critical buckling load of the FG-GRNP microplate are studied.

2. Theoretical Derivation

The multilayer FG-GRNP microplate (a) with length $a_M$, width $b_M$ and thickness $h_M$ is applied to the loading of uniform distributed in-plane compression force $P_x$ and $P_y$ (b) and the external potential, as shown in figure 1.

Graphene nanoplatelets present three graded distributions of FG-GRNP microplate, U-pattern, X-pattern, O-pattern, which are shown in figure 2. In the U-pattern, graphene nanoplatelets are evenly distributed in each layer of FG-GRNP microplate. In addition, the bottom and top layers of the X-pattern are darker than the middle layers, which mean graphene nanoplatelets volume graphene of both sides is bigger than middle part. On the contrary, the middle layers have the darker colour for O-pattern FG-GRNP microplate. Graphene volume fractions are different in the three distributions. The total graphene nanoplatelets volume fraction of the three different FG-GRNP microplates are suppose in the same value $V_G$. $V_n$ is the volume fraction of graphene nanoplatelets in the $n$th sub-layer of a certain FG-GRNP microplate. Halpin-Tsai parallel model and rule of mixture are applied to estimate Young’s modulus, Poisson’s ratio, mass density, piezoelectric parameter and dielectric parameter of the $n$th sub-layer of the FG-GRNP microplate [14].

Figure 1. Model diagram of FG-GRNP microplate.

Figure 2. GNPLs distribution patterns U-pattern, X-pattern and O-pattern.
For X-pattern

\[ V_n = \left( \frac{N}{2} + 1 - n \right) V', \text{ when } n \leq \frac{N}{2}; \quad V_n = \left( n - \frac{N}{2} \right) V', \text{ when } n \geq \frac{N}{2} \]  

(1.1)

for O-pattern

\[ V_n = n V', \text{ when } n \leq \frac{N}{2}; \quad V_n = (N + 1 - n) V', \text{ when } n \geq \frac{N}{2} \]  

(1.2)

and for U-pattern

\[ V_n = V_G, \quad (n = 1, 2, \cdots, N) \]  

(1.3)

where

\[ V' = \frac{2}{1 + \frac{N}{2}} V_G \]  

(1.4)

The displacement fields of an arbitrary point \( u(x, y, z) \), \( u_z(x, y, z) \) and \( u_{zz}(x, y, z) \) along the \( x \), \( y \) and \( z \) directions can be expressed by first-order shear deformation theory. According to the Eringen’s nonlocal elasticity theory [11], the nonlocal constitutive relations for the \( n \)th sub-layer of FG-GRNP microplate can be expressed as

\[
\begin{align*}
\mathbf{\sigma}_{(x)} &= \mathbf{Q}_{(1)} \mathbf{Q}_{(2)} \mathbf{Q}_{(3)} \mathbf{Q}_{(4)} \mathbf{Q}_{(5)} \mathbf{Q}_{(6)} \mathbf{Q}_{(7)} \mathbf{Q}_{(8)} \mathbf{Q}_{(9)} \\
\mathbf{e}_{(x)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

(2)

where

\[
\mathbf{Q}_{(n)} = \frac{V G}{1 - V_n^2}, \quad \mathbf{Q}_{(1)} = \frac{E_1}{1 - V_n^2}, \quad \mathbf{Q}_{(2)} = \frac{E_2}{1 - V_n^2}, \quad \mathbf{Q}_{(3)} = \frac{Q_{11}}{2h}, \quad \mathbf{Q}_{(4)} = \frac{Q_{22}}{2h}, \quad \mathbf{Q}_{(5)} = \frac{Q_{33}}{2h}, \quad \mathbf{Q}_{(6)} = \frac{Q_{12}}{2h}, \quad \mathbf{Q}_{(7)} = \frac{Q_{13}}{2h}, \quad \mathbf{Q}_{(8)} = \frac{Q_{23}}{2h}, \quad \mathbf{Q}_{(9)} = \frac{Q_{66}}{2h}
\]

(3)

where \( \nabla \times \) is the Laplace operator and \( e_0a_0 \) is the scale coefficient of micro-scale reaction.

The distribution of external potential \( \phi \) for FG-GRNP microplate is based on Maxwell equation and it is supposed as combination of a cosine and linear forms [15]:

\[
\hat{\phi}(x, y, z) = -\cos(\beta z) \phi(x, y) + \frac{2zV_0}{h}
\]

(4)

Then the electric fields of the \( n \)th sub-layer for the FG-GRNP microplate can be expressed as

\[
E_x = -\frac{\partial \hat{\phi}}{\partial x}, \quad E_y = -\frac{\partial \hat{\phi}}{\partial y}, \quad E_z = -\frac{\partial \hat{\phi}}{\partial z}
\]

(5)

The equation of strain energy \( \Pi \) and the work done by uniform distribution in-plane compression forces are same as [14]. Hence, the buckling government equations of the FG-GRNP microplate are expressed by

\[
A_1 \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x^2} \right) + A_2 \left( \frac{\partial^2 V}{\partial y^2} + \frac{\partial W}{\partial y} \frac{\partial^2 W}{\partial y^2} \right) + A_6 \left( \frac{\partial^2 U}{\partial x^2} \frac{\partial^2 V}{\partial y^2} + \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial y} + \frac{\partial W}{\partial y} \frac{\partial^2 W}{\partial y \partial x} \right) = 0
\]

(6.1)
\[
A_4 \left( \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial y^2} \right) + A_{14} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial V}{\partial x} \frac{\partial^2 V}{\partial y^2} \right) + A_{44} \left( \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial V}{\partial y} \frac{\partial^2 V}{\partial x} + \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial y^2} \right) = 0 \quad (6.2)
\]

\[
-L_{nl} \left[ (N^e_{\phi} - P_h) \frac{\partial^2 W}{\partial x^2} + (N^e_{\phi} - P_h) \frac{\partial^2 W}{\partial y^2} \right] + A_{44} \left( \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial \phi}{\partial x} \right) + A_{44} \left( \frac{\partial^2 W}{\partial x \partial y} + \frac{\partial \phi}{\partial y} \right) - K \left( E_{15} \frac{\partial^2 \phi}{\partial x^2} + E_{24} \frac{\partial^2 \phi}{\partial y^2} \right) = 0 \quad (6.3)
\]

\[
D_{11} \frac{\partial^2 \phi}{\partial x^2} + D_{12} \frac{\partial^2 \phi}{\partial x \partial y} + D_{13} \left( \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi_x}{\partial y^2} \right) + E_{13} \frac{\partial \phi}{\partial x} - A_{14} \left( \frac{\partial W}{\partial x} + \frac{\partial \phi}{\partial y} \right) + K E_{15} \frac{\partial \phi}{\partial x} = 0 \quad (6.4)
\]

\[
D_{11} \frac{\partial^2 \phi}{\partial x^2} + D_{22} \frac{\partial^2 \phi}{\partial y^2} + D_{13} \left( \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2} \right) + E_{13} \frac{\partial \phi}{\partial x} + E_{13} \frac{\partial \phi}{\partial y} + X_{11} \frac{\partial^2 \phi}{\partial x^2} + X_{22} \frac{\partial^2 \phi}{\partial y^2} - X_3 \phi = 0 \quad (6.6)
\]

where

\[
L_{nl} = 1 - \left( \varepsilon_{i0} a_0 \right)^2 \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \quad (7.1)
\]

\[
\left[ \begin{array}{c}
N^e_{\phi} \\
N^e_{\phi}
\end{array} \right] = \left[ \begin{array}{c}
\frac{-2V_h}{h} \sum_{i=1}^{N_x^i} e_{i1} d z_i \\
\frac{-2V_h}{h} \sum_{i=1}^{N_x^i} e_{i2} d z_i
\end{array} \right] \quad (7.2)
\]

The dimensionless parameters are introduced as follows:

\[
\left( \begin{array}{c}
\xi \\
\eta
\end{array} \right) = \left( \begin{array}{c}
\frac{x}{a_m} \\
\frac{y}{b_m}
\end{array} \right) \cdot \left( \begin{array}{c}
\mu \\
\nu \frac{W}{h}
\end{array} \right), \quad \eta = \frac{a_m}{b_m}, \quad \lambda = \frac{a_m}{b_m}, \quad \bar{A}_i = \frac{A_{10}}{A_{10}}, \quad \bar{D}_{10} = \frac{D_0}{A_{10} h_M}, \quad \{ I_i, I_1 \} = \left( \begin{array}{c}
I_i \\
I_{10}
\end{array} \right)
\]

\[
\begin{align*}
I_{10} &= \rho_a h_M, \\
\mu &= \frac{e_0 a_0 d_m}{a_m}, \\
\bar{A}_{110} &= \frac{E_M}{1 - \nu^2} h_M, \\
P^* &= \frac{P_z}{P_i}, \\
k_p &= \frac{\phi}{\phi_0}, \\
\phi_0 &= \frac{\bar{A}_{110}}{X_{11}}, \\
\{ N_{\phi}, N_{\phi} \} &= \left( \begin{array}{c}
N_{\phi} \\
N_{\phi}
\end{array} \right)
\end{align*}
\]

\[
\left\{ X_{11}, X_{22}, X_{33} \right\} = \left\{ \frac{X_{11} \phi_{11}}{A_{110} h_M}, \frac{X_{22} \phi_{22}}{A_{110} h_M}, \frac{X_{33} \phi_{33}}{A_{110} h_M} \right\}, \quad \{ E_{11}, E_{22}, E_{33} \} = \left\{ \frac{E_{11} \phi_{11}}{A_{110} h_M}, \frac{E_{22} \phi_{22}}{A_{110} h_M}, \frac{E_{33} \phi_{33}}{A_{110} h_M} \right\}
\]

\[
(8)
\]

The buckling governing equations (6) of FG-GRNP microplate can be further rewritten. The FG-GRNP microplate is clamped at two opposite edges, and the other two opposite edges are simply supported, named as CSCS.

3. Solution Method
In order to study the buckling of FG-GRNP microplate, DQ method [15] is adopted to discretize the dimensionless equation and solve the numerical results. According to DQ method, the nonlinear algebraic equations can be expressed as vector form

\[
d = \left\{ \begin{array}{c}
\nu_{ij} \\
\nu_{ij}
\end{array} \right\}, \left\{ \begin{array}{c}
\nu_{ij} \\
\phi_{ij}
\end{array} \right\}, \left\{ \begin{array}{c}
\phi_{ij} \\
\phi_{ij}
\end{array} \right\}, \left\{ \begin{array}{c}
\phi_{ij} \\
\phi_{ij}
\end{array} \right\}, \left\{ \begin{array}{c}
\phi_{ij} \\
\phi_{ij}
\end{array} \right\}, i = 1, 2, \cdots, N_1, \quad j = 1, 2, \cdots, N_2
\]

\[
(9)
\]

Hence, the discretized governing equation for buckling performances of the GR-FG micro-plate can be expressed in the matrix form:

\[
(K_L + K_p)_{6N \times 6N} d_{6N \times 1} = 0
\]

\[
(10)
\]

where $K_L$ is the constant coefficient, $K_p$ is the coefficient matrices. Equation (10) can be applied to study the effect of the axial forces, piezoelectric multiple, nonlocal parameter, external voltage, characteristics of graphene nanoplatelets on the critical buckling load.
4. Numerical Results and Discussion

In this part, the dimensionless critical buckling load of the FG-GRNP microplate under uniform distribution in-plane compression forces and external potential are studied. Furthermore, the effects of the different distribution patterns, axial force ratio, nonlocal parameter, graphene volume fraction, external voltage, length-to-thickness ratio and piezoelectric multiple on the buckling of the FG-GRNP microplate are revealed through figures and tabulations.

Unless otherwise specified, the parameters of FG-GRNP microplate are as follows [14]:

\[ a_w = 100 \, \mu m, \quad b_w = 100 \, \mu m, \quad h_w = 5 \, \mu m, \quad \mu = 0.02, \quad V_G = 0.2V, \quad a_G = 2.5 \, \mu m, \quad b_G = 1.5 \, \mu m, \quad h_G = h_m, \quad V_G = 0.5\%, \]
\[ \alpha = 100 \times 10^3, \quad E_G = 1010 \, GPa, \quad \nu_G = 0.186, \quad \rho_G = 1062.5 \, kg/m^3, \quad E_M = 1.44 \, GPa, \quad \nu_M = 0.29, \quad \rho_M = 1920 \, kg/m^3 \]

where \( a_G, \ b_G, \ h_G, \ a_M, \ b_M, \ h_M, \) respectively represent the length, width and thickness of graphene nanoplatelet and FG-GRNP microplate [14]. The piezoelectric multiple and volume fraction of graphene nanoplatelets are respectively \( \alpha \) and \( V_G \). \( \mu, \ a_G/h_G \) and \( V_0 \) separately represent the nonlocal coefficient, the length-to-thickness ratio of graphene and external voltage. Besides, \( E, \nu \) and \( \rho \) are respectively elastic properties of the graphene nanoplatelets and PVDF [14]. The results are compared with Mao and Zhang [14].

Figures 3 and 4 manifest the effect of the axial force ratio \( k_p \), nonlocal parameter \( \mu \) and volume fraction of graphene nanoplatelets \( V_G \) on the critical buckling load \( P_{cr} \) of FG-GRNP microplate. As expected, the dimensionless critical buckling load \( P_{cr} \) decreases along with the growth of axial force ratio \( k_p \). It is because the critical buckling load \( P_{cr} \) of FG-GRNP microplate under simple support boundary condition is less than clamped boundary condition. Therefore, the increase of the force on the weak side of FG-GRNP microplate can lead to the decrease of the critical buckling load. In addition, the critical buckling load \( P_{cr} \) of FG-GRNP microplate decreases with nonlocal parameter growth, mainly due to the growth of nonlocal parameter can cause the system stiffness to decrease. Moreover, the growth of graphene nanoplatelets volume fraction \( V_G \) can cause the growth of critical buckling load \( P_{cr} \) of FG-GRNP microplate, because the graphene nanoplatelets can strengthen matrix material very well.

**Figure 3.** The effect of the axial force ratio and nonlocal parameter on the critical buckling load.

**Figure 4.** The effect of the graphene volume fraction on the critical buckling load.

Figure 5 gives the effects of piezoelectric multiple \( \alpha \) on critical buckling load \( P_{cr} \) of the FG-GRNP microplate. As expected, the critical buckling load \( P_{cr} \) rises along with the piezoelectric multiple \( \alpha \). This is mainly due to the piezoelectric property of graphene nanoplatelets is better than matrix material. It can be seen that with the growth of piezoelectric multiple, the capacity of FG-GRNP microplate to resist buckling load is significantly enhanced.

In addition, figure 6 presents the effects of the length-to-thickness ratio of graphene nanoplatelets on the dimensionless critical buckling load \( P_{cr} \). Obviously, it can be seen that the dimensionless critical buckling load \( P_{cr} \) grows with the length-to-thickness ratio of graphene nanoplatelets. Mainly
due to the growth of graphene nanoplatelets length-to-thickness ratio can lead to the growth of contact area between PVDF and graphene nanoplatelets.

![Figure 5](image1.png)

**Figure 5.** The effect of piezoelectric multiple on critical buckling load.

![Figure 6](image2.png)

**Figure 6.** The effect of the length-to-thickness ratio on the critical buckling load.

5. Conclusion

This paper studies the buckling of FG-GRNP microplate subjected to in-plane forces and external voltage under the CSCS boundary condition. The Halpin-Tsai parallel model is adopted to calculate Young’s modulus and the mixing law is applied to estimate other material properties of the novel micro-composite plate. The first-order shear deformation theory, nonlocal elastic theory and minimum potential energy principle are employed to acquire governing equations of the buckling of the novel plate. X-pattern, U-pattern, O-pattern distribution forms are considered to study the effects of the axial forces, piezoelectric multiple, nonlocal parameter, external voltage, characteristics of graphene nanoplatelets on critical buckling load of FG-GRNP microplate. It is found that the axial force ratio, nonlocal parameter, graphene volume fraction, external voltage, length-to-thickness ratio of graphene nanoplatelets and piezoelectric multiple have important effects on the buckling of FG-GRNP microplate. Regardless of the small-scale effect might lead into errors for studying the buckling of FG-GRNP microplate. The rising axial force ratio, nonlocal parameter or reducing piezoelectric multiple and the length-to-thickness ratio of graphene can speed up the buckling process. As the excellent electrical and mechanical properties of graphene nanoplatelets, it can strengthen the PVDF matrix effectively. In addition, X-pattern is the most useful distribution to enlarge buckling stability area of FG-GRNP microplate.

References

[1] Novoselov K S, Geim A K, S. Morozov V, Jiang D, Zhang Y, Dubonos S V, Grigorieva I V and Firsov A A 2004 Electric field effect in atomically thin carbon films Science 306 666-9

[2] Zaman I, Phan T T, Kuan H C, Meng Q S, La L T B, Luong L, Youssf O and Ma J 2011 Epoxy/graphene platelets nanocomposites with two levels of interface strength Polymer 52 1603-11

[3] Chen J H, Jang C, Xiao S D, Ishigami M and Fuhrer M S 2008 Intrinsic and extrinsic performance limits of graphene devices on SiO$_2$ J. Nature Nanotechnology 3 206-9

[4] Xu K, Wang K, Zhao W, et al. 2015 The positive piezoconductive effect in graphene Nature Communications 6 8119

[5] Maity N, Mandal A and Nandi A K 2016 Hierarchical nanostructured polyaniline functionalized graphene/poly(vinylidene fluoride) composites for improved dielectric performances Polymer 103 83-97

[6] Suresh S and Mortensen A 1998 Fundamentals of Functionally Graded Materials The Institute of Materials
[7] Liu J, Ke L L, Wang Y S, Yang J and Alam F 2012 Thermoelastic frictional contact of functionally graded materials with arbitrarily varying properties International Journal of Mechanical Sciences 63 86-98

[8] Liu Y Z, Hao Y X, Zhang W, Chen J and Li S B 2015 Nonlinear dynamics of initially imperfect functionally graded circular cylindrical shell under complex loads Journal of Sound and Vibration 348 294-328

[9] Liang F, Yang X D, Bao R D and Zhang W 2016 Frequency analysis of functionally graded curved pipes conveying fluid J. Advances in Materials Science and Engineering 2016 1-9

[10] Chen H Y, Wang A W, Hao Y X and Zhang W 2017 Free vibration of FGM sandwich doubly-curved shallow shell based on a new shear deformation theory with stretching effects Composite Structures 179 50-60

[11] Eringen A C 2002 Nonlocal Continuum Field Theories Springer Science & Business Media

[12] Liew K M, Zhang Y and Zhang L W 2017 Nonlocal elasticity theory for graphene modeling and simulation: Prospects and challenges Journal of Modeling in Mechanics and Materials 1 20160159

[13] Sahmani S, Aghdam M M and Rabczuk T 2018 Nonlinear bending of functionally graded porous micro/nano-beams reinforced with graphene platelets based upon nonlocal strain gradient theory Composite Structures 186 68-78

[14] Mao J J and Zhang W 2019 Buckling and post-buckling analyses of functionally graded graphene reinforced piezoelectric plate subjected to electric potential and axial forces Composite Structures 216 392-405

[15] Liu C, Ke L L, Yang J, Kitipornchai S and Wang Y S 2018 Nonlinear vibration of piezoelectric nanoplates using nonlocal mindlin plate theory Mechanics of Advanced Materials and Structures 25 1252-64