On the 4-Adic Complexity of Quaternary Sequences with Ideal Autocorrelation

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Abstract

In this paper, we determine the 4-adic complexity of the balanced quaternary sequences of period $2p$ and $2(2^n - 1)$ with ideal autocorrelation defined by Kim et al. (ISIT, pp. 282-285, 2009) and Jang et al. (ISIT, pp. 278-281, 2009), respectively. Our results show that the 4-adic complexity of the quaternary sequences defined in these two papers is large enough to resist the attack of the rational approximation algorithm.

Index Terms

4-adic complexity, balance, ideal autocorrelation, quaternary sequences, the rational approximation algorithm

I. INTRODUCTION

With the development of correlation attack and algebraic attack, it is becoming the main trend to use the nonlinear feedback shift register sequences with pseudorandom property as the driving sequences in stream cipher design. The feedback with carry shift register (FCSR) proposed by [6] and [7] is a kind of generator which can produce nonlinear sequences quickly.

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Balanced binary and quaternary sequences with good autocorrelation play important roles in communication and cryptography systems. The $d$-adic complexity $\Phi_d(s)$ measures the smallest length of FCSR which generates the sequence $s$ over $\mathbb{Z}/(d)$. Sequences over $\mathbb{Z}/(d)$ with low $d$-adic complexity are susceptible to decoding by the rational approximation algorithm, see [6], [8-9]. Particularly, a quaternary sequence $s$ can be decoded by the rational approximation algorithm with $6\Phi_4(s) + 16$ consecutive bits. Hence, the 4-adic complexity $\Phi_4(s)$ of a safe sequence $s$ with period $N$ should exceed $\frac{N - 16}{6}$. There are numerous results about the 2-adic complexity of binary sequences with good autocorrelation, see [2-3], [11-14], for example. However, the 4-adic complexity of quaternary sequences with good autocorrelation has not been studied so fully and there are few quaternary sequences with good autocorrelation whose 4-adic complexity is known, see [10]. This may pose risk to communication and cryptography systems.

In this paper, we determine the 4-adic complexity of the balanced quaternary sequences of even period $2p$ and $2(2^n - 1)$ with ideal autocorrelation defined in [5] and [4], respectively. Our results show that the 4-adic complexity of the quaternary sequences with period $2p$ and $2(2^n - 1)$ defined in these two papers is larger than $\frac{2p - 16}{6}$ and $\frac{2(2^n - 1) - 16}{6}$ respectively. Hence they are safe enough to resist the attack of the rational approximation algorithm.

II. Preliminaries

In the application of communication and cryptography, balanced sequences with good autocorrelation property are preferred.

For a sequence $g = (g_0, g_1, \ldots, g_{N-1})$ over $\mathbb{Z}/(d)$ with period $N$, it is said to be balanced if $|A_i - A_j| \leq 1$ for any pair of $i, j$ with $0 \leq i \neq j \leq N - 1$, where

$$A_k = \{t | g_t = k, 0 \leq t < N\}, \ k = 0, 1, \ldots, d - 1.$$

The autocorrelation function of a sequence $s = (s_0, s_1, \ldots, s_{N-1})$ over $\mathbb{Z}/(d)$ with period $N$ is defined by

$$C_s(\tau) = \sum_{i=0}^{N-1} \zeta_d^{s_i - s_{i+\tau}}, \quad 0 \leq \tau < N,$$

where $\zeta_d$ is a complex $d$-th primitive root of unity.

The maximal out-of-phase autocorrelation magnitude should be as small as possible and the number of the occurrences of the maximal out-of-phase autocorrelation magnitude should be minimized. A sequence with the possible minimum value of the maximal out-of-phase autocorrelation magnitude and the minimum number of occurrences of the maximal out-of-phase autocorrelation magnitude is said to have the ideal autocorrelation property.
For a binary sequence $s$ with period $N$, it is well known that if
\[ C_s(\tau) = -1 \quad \text{for all } 0 < \tau < N, \tag{1} \]
then $s$ is an ideal autocorrelation sequence.

The autocorrelation distribution of a quaternary sequence $s$ of even period $N$ with ideal autocorrelation and balance property is given by
\[
C_s(\tau) = \begin{cases} 
N, & \text{1 times,} \\
0, & \text{\(\frac{N}{2}\) times,} \\
-2, & \text{\(\frac{N}{2}\) times.}
\end{cases}
\]
in [5].

By using the Legendre sequences and the Gray mapping, two classes of balanced quaternary sequences of even period $2p$ with ideal autocorrelation were constructed in [5]. Balanced quaternary sequences of period $2(2^n - 1)$ with ideal autocorrelation were constructed in [4] by using the binary sequences of period $2^n - 1$ with ideal autocorrelation and the Gray mapping.

For an odd prime $p$, let $QR$ and $QNR$ be the set of quadratic residues and quadratic non-residues in the set $\mathbb{Z}_p^* = \mathbb{Z}/(p) \setminus \{0\} = \{1, 2, \ldots, p-1\}$, respectively. Two classes of Legendre sequences $b$ and $c$ of period $p$ are defined by
\[
b_t = \begin{cases} 
0, & \text{for } t = 0 \\
0, & \text{for } t \in QR \\
1, & \text{for } t \in QNR
\end{cases}
\]
\[
c_t = \begin{cases} 
1, & \text{for } t = 0 \\
0, & \text{for } t \in QR \\
1, & \text{for } t \in QNR
\end{cases}
\]
respectively.

The Gray mapping $\phi$ is defined by
\[
\phi(0,0) = 0, \ \phi(0,1) = 1, \ \phi(1,1) = 2, \ \phi(1,0) = 3.
\]

According to the definition of the Gray mapping, we can get
\[
\phi(a,e) = 2a - a(e - 1) - (a - 1)e \tag{2}
\]
where $0 \leq a \leq 1$ and $0 \leq e \leq 1$. 
The following two classes of quaternary sequences $g^1$ and $g^2$ of even period $2p$ defined by using the Gray mapping and the Legendre sequences were shown to have ideal autocorrelation and balance property in [5].

**Definition 1.** ([5]) For an odd prime $p$ with $p \equiv 1 \pmod{4}$, let $s^0$ and $s^1$ be two binary sequences of the same period $2p$ defined by

\[
s^0_t = \begin{cases} b_t, & \text{for } t \equiv 0 \pmod{2} \\ c_t, & \text{for } t \equiv 1 \pmod{2} \end{cases}
\]

\[
s^1_t = \begin{cases} b_t, & \text{for } t \equiv 0 \pmod{2} \\ 1 - c_t, & \text{for } t \equiv 1 \pmod{2} \end{cases}
\]

The quaternary sequence $g^1$ of period $2p$ is defined by $g^1_t = \phi(s^0_t, s^1_t)$.

**Definition 2.** ([5]) For an odd prime $p$ with $p \equiv 3 \pmod{4}$, let $s^2$ and $s^3$ be two binary sequences of the same period $2p$ defined by

\[
s^2_t = \begin{cases} b_t, & \text{for } 0 \leq t < p \\ b_t, & \text{for } p \leq t < 2p \end{cases}
\]

\[
s^3_t = \begin{cases} c_t, & \text{for } t \equiv 0 \pmod{2} \\ 1 - c_t, & \text{for } t \equiv 1 \pmod{2} \end{cases}
\]

The quaternary sequence $g^2$ of period $2p$ is defined by $g^2_t = \phi(s^2_t, s^3_t)$.

Let $Z_{2^n-1} = \mathbb{Z}/(2^n - 1) = \{0, 1, 2, \ldots, 2^n - 2\}$. Assume that $s$ is a binary sequence of period $2^n - 1$ with ideal autocorrelation. Let $D_0$ be the characteristic set of $s$ defined by

\[D_0 = \{t | s_t = 1, 0 \leq t \leq 2^n - 2\}\]

and $\overline{D}_0 = Z_{2^n-1} \setminus D_0$. By the Chinese remainder theorem, we have the isomorphism

\[
\phi : Z_{2^n-1} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^n-1}, h \mapsto (h \mod 2, h \mod 2^n - 1).
\]

The following class of quaternary sequences $g^3$ of even period $2(2^n - 1)$ defined by using the Gray mapping and the ideal autocorrelation sequences with period $2^n - 1$ were shown to have ideal autocorrelation and balance property in [4].

**Definition 3.** ([4]) Let $s$ be binary sequence of period $2^n - 1$ with ideal autocorrelation and $D_0$ a characteristic set of $s$. Let $g^3$ be the quaternary sequence defined by

\[g^3_t = \phi(u_t, v_t),\]
where \( u \) and \( v \) are the binary sequences of period \( 2^{n+1} - 2 \) defined by
\[
\begin{align*}
u_t &= \begin{cases} 
1, & \text{if } t \in \{0, 1\} \times D_0 \\
0, & \text{if } t \in \{0, 1\} \times \overline{D}_0
\end{cases} \\
v_t &= \begin{cases} 
1, & \text{if } t \in \{0\} \times D_0 \cup \{1\} \times \overline{D}_0 \\
0, & \text{if } t \in \{0\} \times \overline{D}_0 \cup \{1\} \times D_0.
\end{cases}
\end{align*}
\]

The definition about the 4-adic complexity of quaternary sequences with period \( N \) is defined as follows.

**Definition 4. ([15], [9])** For a quaternary sequence \( s = (s_0, s_1, \ldots, s_{N-1}) \) with period \( N \), let \( S(4) = \sum_{i=0}^{N-1} s_i4^i \).

The 4-adic complexity \( \Phi_4(s) \) is defined by
\[
\log_4 \left( \frac{4^{N-1}}{\gcd(4^{N-1}, S(4))} \right),
\]
where \( \gcd(a, b) \) denotes the greatest common divisor of \( a \) and \( b \). (The exact value of the smallest length of FCSR which generates the quaternary sequence is \( \lceil \log_4 \left( \frac{(4^N - 1)}{\gcd(4^N - 1, S(4)) + 1} \right) \rceil \).

According to Definition 4, determining the 4-adic complexity of quaternary sequences is equivalent to determining \( \gcd(4^{N-1}, S(4)) \).

**III. MAIN RESULT**

In this section, we study the 4-adic complexity of the quaternary sequences of period \( 2p \) and \( 2(2^n - 1) \) with ideal autocorrelation in Section III.

For \( i \in \mathbb{Z}_p^* \), the Legendre symbol \((i/p)\) is defined by
\[
\left( \frac{i}{p} \right) = \begin{cases} 
1, & \text{if } i \in \text{QR} \\
-1, & \text{otherwise}.
\end{cases}
\]
The following four lemmas are useful in the sequel.

**Lemma 1. ([11], Theorem 7.3)** If \( s \) is a periodic binary sequence of odd period \( 2^n - 1 \) with ideal autocorrelation, then the number of nonzero bits in one period of \( s \) is \( 2^{n-1} \).

The proof of the lemma is similar to that of Lemma 2(1) in [14]. For the completeness of the paper, we give a simple proof.

**Lemma 2.** Let \( p \) be an odd prime. Then
\[
\left( \sum_{i=1}^{p-1} \left( \frac{i}{p} \right) 4^i \right)^2 \equiv - \left( \frac{-1}{p} \right) \frac{4^p - 1}{3} + \left( \frac{-1}{p} \right) p \pmod{4^p - 1}.
\]

**Proof.** Since
\[
\left( \sum_{i=1}^{p-1} \left( \frac{i}{p} \right) 4^i \right)^2 = \sum_{i=1}^{p-1} \left( \frac{i}{p} \right) 4^i \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) 4^j
\]

Proof. From Lemma 4.

For an odd prime \( p \), we have
\[
\sum_{t=1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t} + \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} \equiv \begin{cases} 
2 \sum_{t=1}^{p-1} \left( \frac{t}{p} \right) 4^t & \text{(mod } 4^p - 1) \\
0 & \text{(mod } 4^p + 1) \end{cases}
\]

Proof. From
\[
\sum_{t=1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t} \equiv \sum_{t=0}^{p-1} \left( \frac{t}{p} \right) 4^t \equiv \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} \text{ (mod } 4^p - 1) \]

we get
\[
\sum_{t=1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t} + \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} \equiv 2 \sum_{t=1}^{p-1} \left( \frac{t}{p} \right) 4^t \text{ (mod } 4^p - 1). \]

Since
\[
\sum_{t=1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t} \equiv - \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} \text{ (mod } 4^p + 1) \]
Theorem 5. For the quaternary sequence \( g^1 \) and then the result follows from

\[
\sum_{t=0}^{p-1} \left( \frac{2t}{p} \right) 4^{2t} = \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} \quad (\text{mod} \ 4^p + 1)
\]

then the rest result follows from

\[
\sum_{t=1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t} = \sum_{t=1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t+1} + \sum_{t=\frac{p+1}{2}+1}^{p-1} \left( \frac{2t}{p} \right) 4^{2t}
\]

and

\[
\sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} = \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} + \sum_{t=\frac{p+1}{2}+2}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1}.
\]

\[ \Box \]

Now we study the 4-adic complexity of the quaternary sequence \( g^1 \) in Definition [7].

**Theorem 5.** For the quaternary sequence \( g^1 \) in Definition [7] we have

\[
\Phi_4(g^1) = \begin{cases} 
\log_4 \frac{4^{2r-1}}{3}, & \text{if } 5|(p+2) \\
\log_4 \frac{4^{2r-1}}{3}, & \text{else.}
\end{cases}
\]

**Proof.** (i) Firstly, we prove

\[
\gcd(g^1(4), 4^p - 1) = 3.
\]

Let the symbols be the same as before. Then we get

\[
g^1(4) = \sum_{t=0}^{2p-1} \phi(s_t, r_t) 4^t
\]

\[
= \sum_{t=0}^{p-1} \phi(b_{2t+1}, 1-c_{2t+1}) 4^{2t+1} + \sum_{t=0}^{p-1} \phi(c_{2t+1}, 1-c_{2t+1}) 4^{2t+1}
\]

\[
= \sum_{t=0}^{p-1} (2b_{2t} - b_{2t} - (b_{2t} - 1)) 4^{2t} + \sum_{t=0}^{p-1} [2c_{2t+1} - c_{2t+1}(-c_{2t+1})
\]

\[
- (c_{2t+1} - 1)(1-c_{2t+1})] 4^{2t+1} \quad \text{(by (2))}
\]

\[
= \sum_{t=0}^{p-1} (2b_{2t} - (b_{2t})^2 + b_{2t} - (b_{2t})^2 + b_{2t}) 4^{2t} + \sum_{t=0}^{p-1} [2c_{2t+1} + (c_{2t+1})^2 + 1 - 2c_{2t+1} + (c_{2t+1})^2] 4^{2t+1}
\]

\[
= \sum_{t=0}^{p-1} 2b_{2t} 4^{2t} + \sum_{t=0}^{p-1} (2c_{2t+1} + 1) 4^{2t+1} \quad \text{(since } a^2 = a \ (0 \leq a \leq 1))
\]

\[
= 2b_0 4^0 + \sum_{t=1}^{p-1} 2b_{2t} 4^{2t} + (2c_{p+1} + 1) 4^p + \sum_{t=0}^{p-1} (2c_{2t+1} + 1) 4^{2t+1}
\]

\[
= \sum_{t=1}^{p-1} 2b_{2t} 4^{2t} + 3 \cdot 4^p + \sum_{t=0}^{p-1} (2c_{2t+1} + 1) 4^{2t+1} \quad \text{(since } b_0 = 0, c_0 = c_p = 1)\]
\[\begin{align*}
&= \sum_{t=0}^{p-1} 4^{2^t+1} + 3 \cdot 4^p + \sum_{t=1}^{p-1} 2b_2 4^{2^t} + \sum_{t=1}^{p-1} 2c_{2^t+1} 4^{2^t+1} \\
&= 4 \sum_{t=0}^{p-1} 4^{2^t} - 4^p + 3 \cdot 4^p + 2 \sum_{t=1}^{p-1} 1 - \left(\frac{2}{p}\right) 4^{2^t} + 2 \sum_{t=0}^{p-1} 1 - \left(\frac{2^{t+1}}{p}\right) 4^{2^t+1} \\
&= \begin{cases}
9 \sum_{t=0}^{p-1} 4^{2^t} - 2 \sum_{t=1}^{p-1} \left(\frac{t}{p}\right) 4^t \pmod{4^p - 1} \\
9 \sum_{t=0}^{p-1} 4^{2^t} - 2 \pmod{4^p + 1}.
\end{cases}
\]
(by Lemma 4)

Since \(3 \mid (4^p - 1)\), then from (3) we know
\[g^1(4) \equiv \sum_{t=1}^{p-1} \left(\frac{t}{p}\right) 4^t \equiv 0 \pmod{3} \quad \text{(since } \sum_{t=1}^{p-1} \left(\frac{t}{p}\right) = 0 ).\]

It then follows that \(3 \mid \gcd(g^1(4), 4^p - 1)\). If \(9 \mid (4^p - 1)\), then from \(4^3 \equiv 1 \pmod{9}\) and \(4^p \equiv 1 \pmod{9}\), we get \(p = 3\) which contradicts with \(p \equiv 1 \pmod{4}\). Therefore \(9 \nmid \gcd(g^1(4), 4^p - 1)\).

Assume that \(d_1\) is a prime divisor of \(\gcd(g^1(4), 4^p - 1)\) such that \(d_1 \neq 3\). By (3) we get
\[g^1(4) \equiv -2 \sum_{t=1}^{p-1} \left(\frac{t}{p}\right) 4^t \pmod{d_1}.
\]

Then we have \(d_1 \mid \left(\sum_{t=1}^{p-1} \left(\frac{t}{p}\right) 4^t\right)^2\). Combining with Lemma 2 we have \(d_1 \mid p\) which implies \(d_1 = p\). Hence, we have \(4^p \equiv 1 \pmod{p}\). By Fermat’s little Theorem, we get \(4^{p-1} \equiv 1 \pmod{p}\). Then we have \(p \mid (p-1)\) which is a contradiction. Hence, we know \(d_1 = 1\).

Therefore we get
\[\gcd(g^1(4), 4^p - 1) = 3.\]

(ii) Next, we prove
\[\gcd(g^1(4), 4^p + 1) = \begin{cases}
5, & \text{if } 5 \mid (p + 2) \\
1, & \text{else.}
\end{cases}\]

By (3) we have
\[g^1(4) \equiv -p - 2 \pmod{5}.
\]

Then we get \(5 \mid \gcd(g^1(4), 4^p + 1)\) only when \(5 \mid (p + 2)\).

Assume that \(5 \mid (p + 2)\) and \(25 \mid (4^p + 1)\), then by Lemma 3 we get \(p = 5\) which contradicts with \(5 \mid (p + 2)\). It then follows that \(25 \nmid \gcd(g^1(4), 4^p + 1)\).
Let \(d_2\) be a divisor of \(\gcd(g^1(4), 4^p + 1)\) such that \(5 \mid d_2\). Then from (3) we have \(g^1(4) \equiv -2 \pmod{d_2}\). Thus \(d_2 \mid 2\) which implies \(d_2 = 1\). Therefore

\[
\gcd(g^1(4), 4^p + 1) = \begin{cases} 
5, & \text{if } 5|(p+2) \\
1, & \text{else.}
\end{cases}
\]  

(5)

Combining with (4-5) and the definition of the 4-adic complexity, the result is proven. \(\square\)

The 4-adic complexity of the sequence \(g^2\) in Definition 2 is given by the following theorem.

**Theorem 6.** For the quaternary sequence \(g^2\) in Definition 2, we have

\[
\Phi_4(g^2) = \begin{cases} 
\log_4 \frac{4^{2p-1}}{5}, & \text{if } 5|(p-2) \\
\log_4 (4^{2p} - 1), & \text{else.}
\end{cases}
\]

**Proof.** (i) Firstly, we determine the exact value of \(\gcd(g^2(4), 4^p - 1)\).

Assume that the symbols are the same as before. Then we have

\[
g^2(4) = \sum_{t=0}^{2p-1} \phi(s^2_t, s^3_t)4^t
\]

\[
= \sum_{t=0}^{p-1} \phi(b_{2t}, c_{2t})4^{2t} + \sum_{t=0}^{p-1} \phi(b_{2t+1}, 1 - c_{2t+1})4^{2t+1}
\]

\[
= \sum_{t=0}^{p-1} (2b_{2t} - b_{2t}c_{2t} - (b_{2t+1} - 1)(1 - c_{2t+1})4^{2t+1}) + \sum_{t=0}^{p-1} [2b_{2t+1} + b_{2t+1}c_{2t+1}]
\]

\[
= (2 \times 0 - 0 \times (1 - 1) - (0 - 1) \times 1) \times 4^0 + \sum_{t=0}^{p-1} (2b_{2t} - b_{2t}c_{2t} + b_{2t} - b_{2t}c_{2t} + c_{2t})4^{2t}
\]

\[
+ [2b_p + b_p c_p - (b_p - 1)(1 - c_p)]4^p + \sum_{t=0}^{p-1} [2b_{2t+1} + b_{2t+1}c_{2t+1} + (b_{2t+1})^2 + 1 - 2b_{2t+1}]4^{2t+1}
\]

\[
= 1 + \sum_{t=1}^{p-1} 2b_{2t}4^{2t} + \sum_{t=0}^{p-1} (2b_{2t+1} + 1)4^{2t+1} \quad \text{(since } a^2 = a \text{ (0} \leq a \leq 1))
\]

\[
= 1 + \sum_{t=1}^{p-1} 2b_{2t}4^{2t} + 4 \sum_{t=0}^{p-1} 4^{2t} - 4^{2\frac{p-1}{2}} + 1 + \sum_{t=0}^{p-1} 2b_{2t+1}4^{2t+1}
\]

\[
= 2 \sum_{t=1}^{p-1} \left( \frac{3}{p} \right) 4^{2t} + 2 \sum_{t=0}^{p-1} \left( \frac{2t+1}{p} \right) 4^{2t+1} + 1 - 4^p + 4 \sum_{t=0}^{p-1} 4^{2t}
\]

\[
\equiv \begin{cases} 
9 \sum_{t=0}^{p-1} 4^t - 2 \sum_{t=1}^{p-1} \left( \frac{t}{p} \right) 4^t - 2 \pmod{4^p - 1} \\
9 \sum_{t=0}^{p-1} 4^t + 2 \pmod{4^p + 1}.
\end{cases}
\]  

(by Lemma 4) 

(6)
Since \(3|(4^p-1)\), then by (6) we know
\[
g^2(4) \equiv 1 \pmod{3}.
\]
Then we get \(3 \nmid \gcd(g^2(4), 4^p - 1)\).

Let \(d_3\) be a prime divisor of \(\gcd(g^2(4), 4^p - 1)\). From (6) and \(d_3 \neq 3\) we have
\[
g^2(4) \equiv -2 \left( \sum_{t=1}^{p-1} \left( \frac{t}{p} \right) 4' + 1 \right) \pmod{d_3}.
\]

Then by Lemma \(\text{[2]}\) and \(p \equiv 3 \pmod{4}\), we have
\[
1 \equiv \left( \sum_{t=1}^{p-1} \left( \frac{t}{p} \right) 4' \right)^2 \equiv -p + \frac{4^p - 1}{3} \equiv -p \pmod{d_3}.
\]
Hence, we have \(d_3|(p + 1)\). From \(4^{d_3 - 1} \equiv 1 \pmod{d_3}\) and \(4^p \equiv 1 \pmod{d_3}\) we get \(p|(d_3 - 1)\) which is a contradiction. Therefore
\[
\gcd(g^2(4), 4^p - 1) = 1. \tag{7}
\]

(ii) Now we determine \(\gcd(g^2(4), 4^p + 1)\).

Since \(5|(4^p + 1)\), then by (6) we get
\[
g^2(4) \equiv - \sum_{t=0}^{p-1} 4^t + 2 \equiv -p + 2 \pmod{5}. \tag{5}
\]
Hence we get \(5|\gcd(g^2(4), 4^p + 1)\) only when \(5|(p - 2)\).

Assume that \(5|(p - 2)\) and \(25|(4^p + 1)\), then from Lemma \(\text{[3]}\) we have \(p = 5\) which contradicts with \(5|(p - 2)\). It then follows that \(25 \nmid \gcd(g^2(4), 4^p + 1)\). Assume that \(d_4\) is a prime divisor of \(g^2(4)\) and \(4^p + 1\) such that \(d_4 \neq 5\). Then by (6) we get
\[
g^2(4) \equiv 2 \pmod{d_4}
\]
which implies \(d_4 = 1\). Hence we have
\[
\gcd(g^2(4), 4^p + 1) = \begin{cases} 
5, & \text{if } 5|(p - 2) \\
1, & \text{otherwise.}
\end{cases} \tag{8}
\]
Combining with (7-8) and the definition of the 4-adic complexity, the result is proven.

The 4-adic complexity of the sequence \(g^3\) with period \(2^{n+1} - 2\) in Definition \(\text{[3]}\) is given as follows.

**Theorem 7.** For the quaternary sequence \(g^3\) with period \(2^{n+1} - 2\) in Definition \(\text{[3]}\) we have
\[
\Phi(g^3) = \log_4 5 \cdot (4^{2^n-1} - 1).
\]

**Proof.** With the symbols the same as before, we have
\[
g^3(4) = \sum_{t=0}^{2^{n+1}-3} \phi(u_t, v_t)4^t
\]
\(\frac{2^{n+1}-3}{2|} 4^t + 2 \cdot \sum_{t=0}^{2^{n+1}-3} 4^t + \sum_{t=0}^{2^{n+1}-3} 3 \cdot 4^t = \frac{2^{n+1}-3}{2|} + \sum_{t=0}^{2^{n+1}-3} 2 \cdot 4^t + \sum_{t=0}^{2^{n+1}-3} 3 \cdot 4^t = \frac{2^{n+1}-3}{2|} + 2 \cdot \sum_{t=0}^{2^{n+1}-3} 4^t + \sum_{t=0}^{2^{n+1}-3} 3 \cdot 4^t \)

\(= \sum_{t=0}^{2^{n+1}-3} (1-s_t)4^t + 2 \cdot \sum_{t=0}^{2^{n+1}-3} s_t \cdot 4^t + 3 \cdot \sum_{t=0}^{2^{n+1}-3} s_t \cdot 4^t \)

\(= \sum_{t=0}^{2^{n+1}-3} 4^t + 2 \sum_{t=0}^{2^{n+1}-3} s_t \cdot 4^t \)

\(= \sum_{t=0}^{2^{n-2}} 4^{2^t+1} + 2 \sum_{t=0}^{2^{n-2}} s_t \cdot 4^t \cdot (1 + 4^{2^t-1}) \)

\(= 4 \cdot \frac{4^{2^{n-1}} + 1}{5} \cdot \frac{4^{2^{n-1}} - 1}{3} + 2 \sum_{t=0}^{2^{n-2}} s_t \cdot 4^t \cdot (1 + 4^{2^{n-1}-1}) \).

(i) Firstly, we determine \(\gcd(g^3(4), 4^{2^{n-1}} + 1)\).

By (10) and \(\frac{4^{2^{n-1}-1}}{3} = \sum_{t=0}^{2^{n-2}} 4^t \equiv 1 \pmod{5}\), we have

\[ g^3(4) \equiv 4 \cdot \frac{4^{2^{n-1}} + 1}{5} \pmod{4^{2^{n-1}} + 1}. \]

It then follows that

\[ \gcd(g^3(4), 4^{2^{n-1}} + 1) = \gcd(\frac{4^{2^{n-1}} + 1}{5}, 4^{2^{n-1}} + 1) = \frac{4^{2^{n-1}} + 1}{5}. \]  (11)

(ii) Secondly, we determine \(\gcd(g^3(4), 4^{2^{n-1}} - 1)\). By (10) and \(\frac{4^{2^{n-1}-1}}{3} = \sum_{t=0}^{2^{n-2}} (-4)^t \equiv 1 \pmod{3}\), we have

\[ g^3(4) \equiv 4 \cdot \sum_{t=0}^{2^{n-2}} s_t 4^t + \frac{4^{2^{n-1}} - 1}{3} \pmod{4^{2^{n-1}} - 1}. \]  (12)

Since \(s_t \in \{0, 1\}\), we have \(s_t = \frac{1 - (-1)^n}{2}\). Then from (12), we get

\[ g^3(4) \equiv 4 \cdot \sum_{t=0}^{2^{n-2}} \frac{1 - (-1)^n}{2} \cdot 4^t + \frac{4^{2^{n-1}} - 1}{3} \]

\[ \equiv 2 \cdot \frac{4^{2^{n-1}} - 1}{3} - 2 \sum_{t=0}^{2^{n-2}} (-1)^s_t \cdot 4^t + \frac{4^{2^{n-1}} - 1}{3} \]

\[ \equiv -2 \sum_{t=0}^{2^{n-2}} (-1)^{s_t} \cdot 4^t \pmod{4^{2^{n-1}} - 1}. \]

Hence, we have

\[ \gcd(g^3(4), 4^{2^{n-1}} - 1) = \gcd(- \sum_{t=0}^{2^{n-2}} (-1)^{s_t} 4^t, 4^{2^{n-1}} - 1). \]  (13)
Let $r$ be a prime divisor of $\gcd(g^3(4), 4^{2^n} - 1)$. Then from (13), we get

$$0 \equiv g^3(4) \equiv \sum_{i=0}^{2^{n-2}} (-1)^{s_i} \cdot 4^i \quad (\text{mod } r).$$

It then follows that

$$0 \equiv \sum_{i,j=0}^{2^n-2} (-1)^{s_i + 4j} \cdot 4^{i-j}$$

$$\equiv \sum_{i=0}^{2^n-2} 4^i \sum_{j=0}^{2^n-2} (-1)^{s_i + sj} \cdot 4^j \quad (\text{mod } r)$$

Then from the fact that $s$ is a binary sequence of period $2^n - 1$ with ideal autocorrelation and (11), we get

$$0 \equiv 2^n - 1 + \sum_{f=1}^{2^n-2} (-1)4^f \equiv 2^n - \frac{4^{2^n-1} - 1}{3} \quad (\text{mod } r). \quad (14)$$

By (9), we know $g^3(4) \equiv \sum_{t=0}^{2^n-2} s_t + 2^n - 1 \quad (\text{mod } 3).$ Since $s$ is a binary sequence of period $2^n - 1$ with ideal autocorrelation, then from Lemma (11), we have $\sum_{t=0}^{2^n-2} s_t = 2^{n-1}$. It then follows that $g^3(4) \equiv 2^{n-1} + 2^n - 1 \equiv 2 \quad (\text{mod } 3)$. Therefore we get $3 \nmid g^3(4)$. It then follows that $r \neq 3$ which implies

$$r \mid \frac{4^{2^n-1} - 1}{3}.$$ 

By (14) we get $0 \equiv 2^n \quad (\text{mod } r)$ which is a contradiction. Hence, we have

$$\gcd(g^3(4), 4^{2^n-1} - 1) = 1. \quad (15)$$

Combining with (11), (13) and the definition of the 4-adic complexity, the result is proven. □

We give several examples to demonstrate our main results.

**Example 1.** For $p = 5 \equiv 1 \quad (\text{mod } 4)$, we have $\mathbb{F}_2^5 = \langle 2 \rangle$ and $b_0 = b_5 = 0, c_0 = c_5 = 1, b_1 = b_6 = 0, c_1 = c_6 = 0, b_2 = b_7 = 1, c_2 = c_7 = 1, b_3 = b_8 = 1, c_3 = c_8 = 1, b_4 = b_9 = 0, c_4 = c_9 = 0$. Then according to the definition of the sequence $g^1$, we get $g^1 = (0, 1, 2, 3, 0, 3, 0, 3, 2, 1)$. Then we have

$$\gcd(g^1(4), 4^{10} - 1)$$

$$= \gcd(1 \times 4 + 2 \times 4^2 + 3 \times 4^3 + 3 \times 4^5 + 3 \times 4^7 + 2 \times 4^8 + 4^9, 4^{10} - 1)$$

$$= 3$$

which implies $\Phi_4(g^1) = \log_4 \frac{4^{10} - 1}{3}$. This result is consistent with Theorem 5.
Example 2. For \( p = 13 \equiv 1 \pmod{4} \), we have \( \mathbb{F}_{13} = \langle 2 \rangle \). According to the definition of the sequence \( g^1 \), we have \( g^1 = (0,1,2,1,0,3,2,3,2,1,0,3,0,3,0,1,2,3,2,3,0,1,2,1) \). Then we have
\[
\gcd(g^1(4),4^{26} - 1)
= \gcd(1 \times 4 + 2 \times 4^2 + 4^3 + 3 \times 4^5 + 2 \times 4^6 + 3 \times 4^7 + 2 \times 4^8 + 4^9 + 3 \times 4^{11} + 3 \times 4^{13} \\
+ 3 \times 4^{15} + 4^{17} + 2 \times 4^{18} + 3 \times 4^{19} + 2 \times 4^{20} + 3 \times 4^{21} + 4^{23} + 2 \times 4^{24} + 4^{25} + 4^{26} - 1) 
= 15
\]
which implies \( \Phi_4(g^1) = \log_4 \frac{4^{26} - 1}{15} \). This result is consistent with Theorem 5.

Example 3. For \( p = 3 \equiv 3 \pmod{4} \), we have \( \mathbb{F}_3 = \langle 2 \rangle \) and \( b_0 = b_3 = 0, c_0 = c_3 = 1, b_1 = b_4 = 0, c_1 = c_4 = 0, b_2 = b_5 = 1, c_2 = c_5 = 1 \). According to the definition of the sequence \( g^2 \), we have \( g^2 = (1,1,2,0,0,3) \). Then we get
\[
\gcd(g^2(4),4^6 - 1)
= \gcd(1 + 1 \times 4 + 2 \times 4^2 + 3 \times 4^5,4^6 - 1) 
= 1
\]
which implies \( \Phi_4(g^2) = \log_4 (4^6 - 1) \). This result is consistent with Theorem 6.

Example 4. For \( p = 7 \equiv 3 \pmod{4} \), according to the definition of the sequence \( g^2 \), we have \( g^2 = (1,1,0,3,0,3,2,0,0,1,2,1,2,3) \). Then we get
\[
\gcd(g^2(4),4^{14} - 1)
= \gcd(1 + 1 \times 4 + 3 \times 4^3 + 3 \times 4^5 + 2 \times 4^6 + 4^9 + 2 \times 4^{10} + 4^{11} + 2 \times 4^{12} + 3 \times 4^{13},4^{14} - 1) 
= 5
\]
which implies \( \Phi_4(g^2) = \log_4 \frac{4^{14} - 1}{5} \). This result is consistent with Theorem 7.

Example 5. For \( n = 4 \), we have the binary m-sequence \( s = (0,0,0,1,0,0,1,1,0,1,0,1,1,1) \) of period 15, according to the definition of the sequence \( g^3 \), we have
\[
g^3 = (0,1,0,3,0,1,2,3,0,3,0,3,2,3,2,1,0,1,2,1,0,3,2,1,2,1,2,3,2,3) .
\]
Then we get
\[
\gcd(g^3(4),4^{30} - 1)
= \gcd(4 + 3 \times 4^3 + 4^5 + 2 \times 4^6 + 3 \times 4^7 + 3 \times 4^9 + 3 \times 4^{11} + 2 \times 4^{12} + 3 \times 4^{13} + 2 \times 4^{14} + 4^{15} + 4^{17}
\]
\[+ 2 \times 4^{18} + 4^{19} + 3 \times 4^{21} + 2 \times 4^{22} + 4^{23} + 2 \times 4^{24} + 4^{25} + 2 \times 4^{26} + 3 \times 4^{27} + 2 \times 4^{28} + 3 \times 4^{29} + 4^{30} - 1\]

\[= 214748365\]

\[= \frac{4^{15} + 1}{5}\]

which implies \(\Phi_4(3^5) = \log_4(5 \times (4^{15} - 1))\). This result is consistent with Theorem 7.

**Remark 1.** For a sequence \(s\) with period \(N\), the 4-adic complexity \(\Phi_4(s)\) should exceed \(\frac{N-16}{6}\) to resist the rational approximation algorithm. Theorems 5-7 show that the 4-adic complexity of the balanced quaternary sequences of period \(2^p\) and \(2(2^n-1)\) with ideal autocorrelation defined in [5] and [4] is larger than \(\frac{2^p-16}{6}\) and \(\frac{2(2^n-1)-16}{6}\) respectively.

**IV. Conclusion**

In this paper, we study the 4-adic complexity of the balanced quaternary sequences with ideal autocorrelation constructed in [5] and [4], respectively. It turns out that the balanced quaternary sequences with ideal autocorrelation constructed in these two papers are safe enough to resist the attack of the rational approximation algorithm. It would be interesting to investigate the 4-adic complexity of more quaternary sequences with good autocorrelation and balance property.

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