A variational inequality of Kirchhoff-type in $\mathbb{R}^N$

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Abstract

In this paper, we investigate the existence of nontrivial radial solutions for a kind of variational inequalities in $\mathbb{R}^N$. Our main technique is the non-smooth critical point theory, based on the Szulkin-type functionals.

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1 Introduction

Variational inequalities describe a lot of phenomena in the real world and have a wide range of applications in physics, mechanics, engineering etc.; see, for example, [1–3, 5–7, 9, 10, 12–14, 18]. This paper is concerned with a kind of variational inequalities in $\mathbb{R}^N$, the aim is to prove the existence of infinite radial solutions under suitable conditions.

Let $H^1_{0}(\mathbb{R}^N)$ be the Sobolev space of $O(N)$ invariant functions (see the definition in Sect. 3), and $B$ be a closed convex set in $H^1_{0}(\mathbb{R}^N)$ with $0 \in B$. Our problem, denoted by $(Q)$, is to find $u \in B$ such that

$$
\left( a + b \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right) \left( \int_{\mathbb{R}^N} \nabla u \cdot \nabla (v - u) \, dx \right) + \int_{\mathbb{R}^N} u(v - u) \, dx - \int_{\mathbb{R}^N} g(x, u)(v - u) \, dx \geq 0,
$$

for all $v \in B$,

where $a, b > 0, N \geq 2$ and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

This problem is related to the obstacle problems, extensively studied due to the physical applications (see [15, 17]).

It is well known that the variational inequality is discussed in different ways in the case of regional bounded and unbounded. In [4], on the bounded interval $(0, 1)$, a class of variational inequalities of Kirchhoff-type is discussed by applying the non-smooth critical point theory based on Szulkin functionals [16]. In [11], the authors study a kind of variational inequality defined on $(0, \infty)$. Motivated by the above work, in this paper we want to study the radial solutions of the problem $(Q)$ by using two kinds of theorem in [16]. Our research scope is an extension of some problems studied by [4] and [11]. Since the domain is unbounded and the continuous embedding $H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is not compact. We consider the symmetric method of the action of a group, similar to [8], to overcome this difficulty.
Meanwhile, suppose the function $g$ satisfies:

$(g_1)$ $\lim_{|u|\to 0} \frac{g(x,u)}{|u|} = 0$ uniformly for $x \in \mathbb{R}^N$.

$(g_2)$ For $1 < p < 2^* - 1$ and there exists $c > 0$ such that

$$
|g(x,u)| \leq c(1 + |u|^p), \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R},
$$

where

$$
2^* - 1 = \begin{cases} 
\frac{N+2}{N-2}, & N \geq 3, \\
+\infty, & N = 1, 2.
\end{cases}
$$

$(g_3)$ There is a constant $\mu > 4$ such that

$$
ug(x,u) \geq \mu G(x,u) = \int_0^u g(x,s) \, ds, \quad \text{for all } x \in \mathbb{R}^N, \text{ and } u \in \mathbb{R}^N.
$$

$(g_4)$ $\lim_{|u|\to +\infty} \frac{G(x,u)}{|u|^p} \to +\infty$ uniformly for all $x \in \mathbb{R}^N$.

$(g_5)$ $g(x,u) = g(zx,u)$ for any $z \in O(N)$ and $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

$(g_6)$ $g(x,u) = -g(x,-u)$ for any $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

We state the main result of this paper.

**Theorem 1.1** If assumptions $(g_1)$–$(g_5)$ hold, then the problem $(Q)$ has a nontrivial radial solution in $B$. Furthermore, if the condition $(g_6)$ holds, then the problem $(Q)$ has infinitely many pairs of nontrivial radial solutions in $B$.

The structure of the paper is as follows. In Sect. 2, we review some preliminaries. Section 3 gives the proof of our main result.

### 2 Szulkin-type functionals

Let $X$ be a real Banach space and denote by $X^*$ its dual. Let $T = \Phi + \psi$ with $\Phi \in C^1(X, \mathbb{R})$ and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous. Then $T = \Phi + \psi$ is a Szulkin-type functional. A point $u \in X$ is called critical if $\psi(u) \neq +\infty$ and

$$
\Phi'(u)(v - u) + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X,
$$

or equivalently

$$
0 \in \Phi'(u) + \partial \psi(u) \quad \text{in } X^*,
$$

where $\partial \psi(u)$ is called the subdifferential of $\psi$ at $u$.

**Definition 2.1** ([16]) The functional $T = \Phi + \psi$ fulfills the $(PS)$ condition at level $c \in \mathbb{R}$; it can be written as $(PSZ)$, if every sequence $\{u_n\} \subset X$ such that $\lim_{n \to \infty} T(u_n) = c$ and

$$
\langle \Phi'(u_n)(v - u_n), v - \psi(u_n) \rangle_X + \psi(v) - \psi(u_n) \geq \varepsilon_n \|v - u_n\| \quad \text{for all } v \in X,
$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.
Lemma 2.2 ([16], Mountain pass theorem) Suppose that $T = \Phi + \psi : X \to \mathbb{R} \cup \{+\infty\}$ be a Szulkin-type functional and that

(i) $T(0) = 0$ and there exist $\alpha, \rho > 0$ such that $T(u) \geq \alpha$ for all $\|u\| = \rho$;

(ii) $T(e) \leq 0$ for some $e \in X$ with $\|e\| > \rho$.

If $T$ satisfies the $(PSZ)_c$-condition, then $T$ has a critical value $c \geq \alpha$ which may be characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} T(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}$.

Lemma 2.3 ([16], Corollary 4.8) Suppose that $T = \Phi + \psi : E \to \mathbb{R} \cup \{+\infty\}$ is an even Szulkin-type functional and satisfies the $(PSZ)_c$-condition with $T(0) = 0$. If $E = X \oplus Y$, where $X$ is a finite dimensional, and assume also that

$(A_1)$ there are constants $\alpha, \rho > 0$ such that $T|_{\beta F_{\rho} \cap Y} \geq \alpha$;

$(A_2)$ for any positive integer $k$, there is $k$-dimensional subspace $E_k \subset E$, such that $T(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in E_k$.

Then $T$ has infinitely many pairs of nontrivial critical points, where $F_{\rho} = \{u \in E : \|u\| < \rho\}$.

3 The proof of the main result

Let

$$H := H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

be the Sobolev space with inner product and corresponding norm

$$\langle u, v \rangle := \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \, dx, \quad \|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx\right)^{\frac{1}{2}}.$$

Denote by $\| \cdot \|_p$ the norm of $L^p(\mathbb{R}^N)$, i.e. $\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{\frac{1}{p}}$.

Let $O(N)$ is an orthogonal transformation group on $\mathbb{R}^N$. We have that

$$E = H^1_{O(N)}(\mathbb{R}^N) := \{u \in H \mid zu(x) := u(\varepsilon^{-1}x) = u(x), \forall z \in O(N)\}$$

is a subspace of $H^1(\mathbb{R}^N)$, and it is invariant. We note that the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is compact when $s \in (2, 2^*)$ by Corollary 1.26 of [19]. Define the functional $\Phi : E \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} b \|u\|^4 - \Psi(u),$$

where $\Psi(u) := \int_{\mathbb{R}^N} G(x, u) \, dx$, and the indicator function of the set $B$ as follows:

$$\psi_B(u) := \begin{cases} 0, & \text{if } u \in B, \\ +\infty, & \text{otherwise.} \end{cases}$$

The function $\psi_B(u)$ is convex, proper, even, and lower semicontinuous. In order to show that $T = \Phi + \psi_B$ is a Szulkin-type functional, we need the following proposition.
**Proposition 3.1** Every critical point \( u \in E \) of \( T = \Phi + \psi_B \) is a solution of (Q).

**Proof** Since \( u \in E \) of \( T = \Phi + \psi_B \) is a critical point, we have

\[
\Phi'(u)(v - u) + \psi_B(v) - \psi_B(u) \geq 0 \quad \text{for all } v \in E.
\]

It is clear that \( u \) belongs to \( B \). If not, we get \( \psi_B = +\infty \), and in the inequality above, setting \( v = 0 \in B \) we get a contradiction. We fix \( v \in B \). Since

\[
\Phi'(u)(v - u) = \left( a + b\|u\|^2 \right) \left( \int_{\mathbb{R}^N} \nabla u \nabla (v - u) \, dx + \int_{\mathbb{R}^N} (v - u) \, dx \right)
\]

\[
- \int_{\mathbb{R}^N} g(x, u(x))(v - u) \, dx \geq 0,
\]

\( u \) is a solution of (Q). \( \square \)

**Proposition 3.2** Suppose that \( g \) satisfies the conditions \((g_1)\) and \((g_2)\) and \( \langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} g(x, u)v \, dx \), then \( \Phi \in C^1(E, \mathbb{R}) \),

\[
\langle \Phi'(u), v \rangle = \left( a + b \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right) \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \, dx - \langle \Psi'(u), v \rangle.
\]

**Proof** By (3.1), we only need to prove that

\[
\Psi \in C^1(H, \mathbb{R}), \quad \langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} g(x, u)v \, dx, \quad \forall u, v \in H.
\]

Thus, we divide the whole proof into the following two steps.

**Step 1.** We verify that \( \Psi \) is a Gateaux derivative.

For small enough \( \varepsilon > 0 \), using \((g_1)\) and \((g_2)\), there is a positive constant \( \varepsilon \) depend on \( \varepsilon \) such that

\[
|g(x, u)| \leq \varepsilon |u| + c(\varepsilon)|u|^p
\]

(3.2)

for every \((x, u) \in \mathbb{R}^N \times \mathbb{R} \). For any \( u(x), v(x) \in H \) and \( 0 < |t| < 1 \), according to (3.2) and using the mean value theorem, there exists \( \theta \in (0, 1) \) such that

\[
\frac{|G(x, u + tv) - G(x, u)|}{|t|} = |g(x, u + \theta tv)v|
\]

\[
\leq \varepsilon |u||v| + \varepsilon |v|^2 + c(\varepsilon)(|u + \theta tv|)^p|v|
\]

\[
\leq \varepsilon |u||v| + \varepsilon |v|^2 + 2^p c(\varepsilon)(|u|^p|v| + |v|^{p+1}).
\]

By the Hölder inequality, it follows that

\[
h := \varepsilon |u||v| + \varepsilon |v|^2 + 2^p c(\varepsilon)(|u|^p|v| + |v|^{p+1}) \in L^1(\mathbb{R}^N).
\]

So, by the Lebesgue dominated convergence theorem, we have

\[
\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} g(x, u)v \, dx.
\]
Step 2. We show that $\Psi'(\cdot): H \to H^*$ is continuous.

Suppose that $u_n \to u$ in $H$. Since the imbedding $H \hookrightarrow L^s(\mathbb{R}^N)$ ($2 \leq s \leq 2^*$) is continuous, we see that, for each $s \in [2, 2^*]$, there is a constant $\eta_s > 0$ such that

$$\|w\|_s \leq \eta_s \|w\|, \quad \forall w \in H^1(\mathbb{R}^N), \quad u_n \to u \quad \text{in} \ L^s(\mathbb{R}^N).$$

Note that

$$\left\| \Psi'(u_n) - \Psi'(u) \right\| = \sup_{\|v\| \leq 1} \left| \int_{\mathbb{R}^N} (g(x, u_n) - g(x, u))v \, dx \right|$$

$$\leq \sup_{\|v\| \leq 1} \int_{\mathbb{R}^N} \left| (g(x, u_n) - g(x, u)) \right| |v| \, dx.$$

According to the Hölder inequality, and Theorem A.4 in [19], we have

$$\sup_{\|v\| \leq 1} \int_{\mathbb{R}^N} \left| (g(x, u_n) - g(x, u)) \right| |v| \, dx \to 0$$
as $n \to \infty$. So, we obtain $\|\Psi'(u_n) - \Psi'(u)\| \to 0$, and thus the claim is proven. Consequently, $T = \Phi + \psi_B$ is a Szulkin-type functional.

It follows from $(g_5)$ that $T$ is $O(N)$-invariant, i.e. for all $(z, u) \in O(N) \times H$, $T(u) = T(zu)$, and the action of the group $O(N)$ on $H$ is isometric, i.e. for all $(z, u) \in O(N) \times H$, $\|u\| = \|zu\|$. Furthermore, because of Lemma 2.2 and Theorem 1.28 of [19], we notice that $u$ is a critical point of $T|_E$ if and only if it is a critical point of $T$ in $H$. We will use the symmetric mountain pass theorem to obtain the critical points of the functional $T|_E$.

**Proposition 3.3** If the continuous function $f$ fulfills $(g_3)$ and $(g_4)$, then $T = \Phi + \psi_B$ fulfills $(PSZ)_c$-condition for every $c \in \mathbb{R}$.

**Proof** Fix $c \in \mathbb{R}$. Set $\{u_n\} \subset E$ such that

$$T(u_n) = \Phi(u_n) + \psi_B(u_n) \to c, \quad (3.3)$$

$$\Phi'(u_n)(v - u_n) + \psi_B(v) - \psi_B(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in E, \quad (3.4)$$

where $\varepsilon_n \to 0$ in $[0, \infty)$. According to (3.3), obviously, we notice that the sequence $\{u_n\} \subset B$. Setting $v = 2u_n$ in (3.4) we have

$$\Phi'(u_n)(u_n) \geq -\varepsilon_n \|u_n\|.$$ 

Thus

$$a \|u_n\|^2 + b \|u_n\|^4 - \int_{\mathbb{R}^N} G(x, u_n(x))u_n(x) \, dx \geq -\varepsilon_n \|u_n\|. \quad (3.5)$$

On the basis of (3.3), for large enough $n \in N$, we get

$$c + 1 \geq \frac{1}{2} a \|u_n\|^2 + \frac{1}{4} b \|u_n\|^4 - \int_{\mathbb{R}^N} G(x, u_n) \, dx. \quad (3.6)$$
Multiply both sides of inequality (3.5) by \( \mu^{-1} \), adding it to another inequality (3.6), and applying the condition (g4). When \( n \in N \) is sufficiently large, we have

\[
c + 1 + \frac{1}{\mu} \| u_n \| \geq a \left(\frac{1}{2} - \frac{1}{\mu}\right) \| u_n \|^2 + b \left(\frac{1}{4} - \frac{1}{\mu}\right) \| u_n \|^4
\]

\[
- \int_{\mathbb{R}^N} \left( G(x, u_n(x)) - \frac{1}{\mu} g(x, u_n(x)) u_n(x) \right) dx
\]

\[
= a \left(\frac{1}{2} - \frac{1}{\mu}\right) \| u_n \|^2 + b \left(\frac{1}{4} - \frac{1}{\mu}\right) \| u_n \|^4
\]

\[
- \frac{1}{\mu} \int_{\mathbb{R}^N} (\mu G(x, u_n(x)) - g(x, u_n(x)) u_n(x)) dx
\]

\[
\geq a \left(\frac{1}{2} - \frac{1}{\mu}\right) \| u_n \|^2 + b \left(\frac{1}{4} - \frac{1}{\mu}\right) \| u_n \|^4.
\]

Since \( \mu > 4 \), the sequence \{\( u_n \)\} is bounded in \( B \). Then there exists a subsequence converging weakly in \( E \). According to the compactness embedding \( E \hookrightarrow L^s(\mathbb{R}^N) \). Without loss of generality, assume

\[
u_n \rightharpoonup u \quad \text{in} \quad E; \tag{3.7}
\]

\[
u_n \rightharpoonup u \quad \text{in} \quad L^s(\mathbb{R}^N), s \in (2, 2^*). \tag{3.8}
\]

By observing that \( B \) is weakly closed, we get \( u \in B \). Let again \( v = u \) in (3.4), we have

\[
(a + b \| u_n \|^2) \langle u_n, u - u_n \rangle_E + \int_{\mathbb{R}^N} g(x, u_n(x)) (u_n(x) - u(x)) dx \geq -\varepsilon_n \| u - u_n \|. \tag{3.9}
\]

We use

\[
(a + b \| u_n \|^2) \| u - u_n \|^2 = (a + b \| u_n \|^2) \langle u - u_n, u - u_n \rangle_E. \tag{3.10}
\]

So, for large enough \( n \) and any \( \varepsilon > 0 \), it follows from (3.9) and (3.10) that

\[
(a + b \| u_n \|^2) \langle u - u_n \rangle_E
\]

\[
\leq (a + b \| u_n \|^2) \langle u, u - u_n \rangle_E + \int_{\mathbb{R}^N} g(x, u_n)(u_n - u) dx + \varepsilon_n \| u - u_n \|
\]

\[
\leq (a + b \| u_n \|^2) \langle u, u - u_n \rangle_E + \int_{\mathbb{R}^N} (\varepsilon |u_n| + c(\varepsilon) |u_n|^p) |u - u_n| dx + \varepsilon_n \| u - u_n \|
\]

\[
\leq (a + b \| u_n \|^2) \langle u, u - u_n \rangle_E + \varepsilon c_1 + c(\varepsilon) \| u_n - u \|_{p+1} \| u_n \|^p_{p+1} + \varepsilon_n \| u - u_n \|
\]

\[
\leq (a + b \| u_n \|^2) \langle u, u - u_n \rangle_E + \varepsilon c_1 + c_2(\varepsilon) \| u_n - u \|_{p+1} + \varepsilon_n \| u - u_n \|,
\]

where the constants \( c_1 \) and \( c_2 \) are independent of \( n \) and \( \varepsilon \). By (3.7) and the fact that \{\( u_n \)\}
is bounded in \( E \), we obtain

\[
\lim_n (a + b\|u_n\|^2)(u, u - u_n)_E = 0.
\]

Taking into account (3.8), \( \|u_n - u\|_{p+1} \to 0 \). Setting \( \epsilon_n \to 0^+ \), then we have proved that

\[
(a + b\|u_n\|^2)\|u - u_n\|^2 \to 0.
\]

Consequently, we get \( u_n \to u \) in \( E \). This means that the proof of this conclusion has been completed. \( \square \)

Now we give the proof of Theorem 1.1.

Proof By (3.2), for any \( 0 < \epsilon < \frac{a}{\eta_2^2} \) (\( \eta_2 \) is continuous imbedding constant \( E \hookrightarrow L^2(\mathbb{R}^N) \)), we obtain

\[
|G(x, u)| \leq \int_0^1 |g(x, tu)u| \, dt \leq \frac{\epsilon}{2} |u|^2 + \frac{c(\epsilon)}{p + 1} |u|^{p+1}, \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{R}.
\]

The condition (\( g_4 \)) implies \( p > 4 \). Therefore, for small enough \( \rho > 0 \), we have

\[
T(u) \geq \frac{1}{2} a|u|^2 + \frac{1}{4} b|u|^4 - \frac{\epsilon}{2} |u|^2 - \frac{c(\epsilon)}{p + 1} |u|^{p+1}
\]

\[
\geq \frac{1}{2} \left( a - \eta_2^2 \epsilon \right) |u|^2 + \frac{1}{4} b|u|^4 - \frac{c(\epsilon)}{p + 1} |u|^{p+1}
\]

\[
\geq \frac{1}{4} \left( a - \eta_2^2 \epsilon \right) |u|^2 + \frac{1}{4} b|u|^4,
\]

for all \( u \in F_\rho \). Thus,

\[
T|_{F_\rho} \geq \frac{1}{4} \left( a - \eta_2^2 \epsilon \right) \rho^2 + \frac{1}{4} b \rho^4 := \alpha > 0.
\]

Let \( \{e_i\} \) be a complete normal orthogonal basis of \( E \). Take \( X = \text{span}\{e_1, e_2, \ldots, e_n\} \) and \( Y = X^\perp \). Then \( E = X \oplus Y \). Thus,

\[
T|_{\overline{F}_\rho \cap Y} \geq \alpha > 0.
\]

For every finite dimensional subspace \( \overline{E} \subset E \), there exists \( k \in N^+ \) such that \( \overline{E} \subset E_k \). Due to the equivalence of all norms in a finite dimensional space, for some positive constant \( c_4 \) we have

\[
\|u\|_k \geq c_4 \|u\|, \quad \text{for all } u \in E_k.
\]

According to the conditions (\( g_1 \)), (\( g_2 \)), and (\( g_4 \)), we note that, for \( D > \frac{b}{c_4^2} \), there exists a positive constant \( C(D) \) such that

\[
G(x, u) \geq D|u|^4 - C(D)|u|^2, \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{R}.
\]
So, fixing \( u_0 \in B \setminus \{0\} \subset E_k \), and taking \( u = su_0(s > 0) \), we get

\[
T(su_0) \leq \frac{1}{2} a s^2 \|u_0\|^2 + \frac{1}{4} b s^4 \|u_0\|^4 - Ds^4 \|u_0\|^4 + C(D)s^2 \|u_0\|^2 \\
\leq \frac{1}{2} a s^2 \|u_0\|^2 - \left( Dc^4 - \frac{1}{4} b s^4 \right) \|u_0\|^4 + C(D)s^2 \|u_0\|^2.
\]

Obviously, we have \( T(su_0) \to -\infty \) as \( s \to +\infty \). Therefore, we take \( s (e = su_0) \) large enough such that \( \|e\| > \rho \) and \( T(e) < 0 \).

By Proposition 3.3, we know that \( T \) satisfies the (PSZ)\(_c\)-condition \( (c \in \mathbb{R}) \), and \( T(0) = 0 \). So \( T \) has a critical value according to Lemma 2.2. We remark that the critical point \( u_1 \in E \) associated to the critical value \( \eta \) is nontrivial due to \( T(u_1) = \eta > 0 = T(0) \). From Proposition 3.1, we notice that \( u_1 \in B \) and it is a radial solution of \((Q)\).

If the condition \( (g_6) \) holds, then \( T \) is even. Similar to the previous discussion, we see that all conditions of Lemma 2.3 are satisfied. Therefore, the second conclusion of Theorem 1.1 is obtained.

\[\Box\]

**Example 3.4** For \( n = 1, 2, 3, \ldots \), considering \( g(x, u) = u^{2n+1}|u|^{2n+1} \), it is satisfied with all assumptions of Theorem 1.1.

**4 Conclusion**

In this article, the existence of nontrivial radial solutions to problem \((Q)\) is established by using the variational methods under suitable conditions. We consider a variational inequality of Kirchhoff-type in \( \mathbb{R}^N \), which improves the previous results. In order to overcome new difficulties, we need to adopt symmetric method of the action of a group in our paper.

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