Beating effects in cubic Schrödinger systems and growth of Sobolev norms

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Abstract

We consider the following coupled cubic Schrödinger equations

\[
\begin{aligned}
    i\partial_t u + \partial_x^2 u &= \varepsilon^2 |v|^2 u, \\
    i\partial_t v + \partial_x^2 v &= \varepsilon^2 |u|^2 v.
\end{aligned}
\]

We prove that there exists a beating effect, i.e. an energy exchange between different modes. This construction may be transported to the linear time-dependent Schrödinger equation: we build solutions such that their Sobolev norms grow logarithmically. All of these results are stated for large but finite times.

Mathematics Subject Classification: 37K45, 35Q55, 35B34, 35B35

1. Introduction

1.1. General introduction

Denote the circle by $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and let $\varepsilon > 0$ be a small parameter. In this paper we are concerned with the following cubic coupled nonlinear Schrödinger equations

\[
\begin{aligned}
    i\partial_t u + \partial_x^2 u &= \varepsilon^2 |v|^2 u, \\
    i\partial_t v + \partial_x^2 v &= \varepsilon^2 |u|^2 v, \\
    u(0, x) &= u_0(x), \\
    v(0, x) &= v_0(x).
\end{aligned}
\]  

We present some solutions of this system which are close to the solutions of a finite dimensional nonlinear system for long times. We stress that these solutions are not obtained
by perturbations of the associated linear system. Thanks to the nonlinearity, we may produce a beating effect, i.e. a transfer of energy between two different modes, something which is not possible in the linear case. The solutions of the initial system are then found thanks to a resonant Birkhoff normal form and approximation arguments, and they enjoy the same beating properties as those of the reduced system. This phenomenon relies heavily on the presence of resonances. Actually, Bambusi and Grébert \cite{2} showed that, in the nonresonant setting (e.g. adding a typical potential in each equation of (1.1)), the dynamics stay close to linear for long times (see the introduction of \cite{8}).

This new example depends on a principle that was already used in \cite{9} and \cite{8}: we make it explicit in Section 2, where the conditions for applying our method to different resonant Hamiltonian PDEs are enumerated.

The control of Sobolev norms in Hamiltonian PDEs has a long story, both in the nonlinear and the linear time-dependant setting. Concerning nonlinear equations, one of the most outstanding results is due to \cite{5} for the cubic two-dimensional NLS, recently completed by \cite{10}, where there is a construction of specific solutions which exhibit a polynomial growth of Sobolev norms for large finite times.

In the linear setting, Bourgain \cite{3} proves a polynomial bound of the Sobolev norm of the solution \( u \) of

\[
i \partial_t u + \partial_x^2 u + V(t, x) u = 0, \quad (t, x) \in \mathbb{R} \times S^1,
\]

where \( V(t, x) \) is a bounded (real analytic) potential. Moreover, when the potential is quasi-periodic in time he obtains in \cite{4} a logarithmic bound. This last result has been enhanced by Delort \cite{6} and Wang \cite{12}, who finds a logarithmic bound for bounded potentials. As a by-product of our work, we recover a result of Bourgain \cite{4}, who showed that these logarithmic bounds are optimal in the case of analytic potentials (see section 4). Note that it is possible to obtain a higher order (but still logarithmic) growth when considering potentials in Gevrey classes (as in Fang–Zhang \cite{7}), or even a sub-polynomial growth in the case of \( C^\infty \) potentials.

### 1.2. Beating effect in the system (1.1)

Our first result concerns the dynamics of (1.1).

**Theorem 1.1.** For all \( 0 < \gamma < 1/2 \), there exist \( 0 < T_\gamma < C |\ln \gamma| \), a \( 2T_\gamma \)-periodic function \( K_\gamma: \mathbb{R} \mapsto [0, 1] \) which satisfies \( K_\gamma(0) = \gamma \) and \( K_\gamma(T_\gamma) = 1 - \gamma \), and there exists \( \varepsilon_0 < \gamma^2 \) so that for \( p, q \in \mathbb{Z} \) and \( 0 < \varepsilon < \varepsilon_0 \), there exists a solution to (1.1) satisfying, for all \( |t| \leq \varepsilon^{-5/2} \),

\[
\begin{align*}
(u(t, x) &= u_p(t)e^{ipx} + u_q(t)e^{iqx} + \varepsilon^{1/2} r_u(t, x), \\
v(t, x) &= v_p(t)e^{ipx} + v_q(t)e^{iqx} + \varepsilon^{1/2} r_v(t, x),
\end{align*}
\]

with

\[
\begin{align*}
|u_q(t)|^2 &= |v_p(t)|^2 = K_\gamma(\varepsilon^2 t) \\
|u_p(t)|^2 &= |v_q(t)|^2 = 1 - K_\gamma(\varepsilon^2 t),
\end{align*}
\]

and where \( r_u \) and \( r_v \) are:

- smooth in time and analytic in space on \([-\varepsilon^{-5/2}, \varepsilon^{-5/2}] \times S^1 \).

- for \( r = r_u, r_v \) the Fourier coefficients \( \hat{r}_j(t) \) of \( r(t) \) satisfy, for some \( \rho > 0 \),

\[
\sup_{|t| \leq \varepsilon^{-5/2}} |\hat{r}_j(t)| \leq Ce^{-\rho|j|},
\]

uniformly in \( \varepsilon > 0 \) and \( p, q \in \mathbb{Z} \).
This statement shows an exchange of energy between the modes \( p \) and \( q \): the modes \( u_q \) and \( v_p \) grow from \( \gamma \) to \( 1 - \gamma \) in time \( t = \varepsilon^{-2} T \). Considering the larger time scales \( \varepsilon^{-5/2} \), we obtain a periodic phenomenon which we will call a beating effect.

Of course the solutions satisfy the three conservation laws: the mass, the momentum and the energy are constant quantities.

- Conservation of mass: \( \int |u|^2 + \int |v|^2 \)
  \[ |u_q|^2 + |u_p|^2 = cst. \tag{1.3} \]
- Conservation of momentum: \( \text{Im} \int \bar{u}_q \partial_t u + \text{Im} \int \bar{v}_q \partial_t v \)
  \[ q|u_q|^2 + p|u_p|^2 + q|v_q|^2 + p|v_p|^2 = cst. \tag{1.4} \]
- Conservation of energy: \( \int (|\partial_t u|^2 + |\partial_t v|^2) + \varepsilon^2 \int |u|^2|v|^2 \)
  \[ q^2|u_q|^2 + p^2|u_p|^2 + q^2|v_q|^2 + p^2|v_p|^2 = cst. \tag{1.5} \]

On the other hand, the solutions given by theorem 1.1 satisfy, for \( 0 \leq t \leq \varepsilon^{-5/2} \) and \( s \geq 0 \),
\[ \|u(t, \cdot)\|_{H^s}^2 = (q^{2s} - p^{2s}) K_\varepsilon (\varepsilon^2 t) + p^{2s} + O(\varepsilon). \tag{1.6} \]

In particular, this norm does not remain constant in time, which is a true nonlinear effect. However, the sum \( \|u(t, \cdot)\|_{H^s}^2 + \|v(t, \cdot)\|_{H^s}^2 \) remains almost constant and thus (1.6) cannot be interpreted as a norm inflation. Nevertheless, this effect will be used in the linear case (see theorem 1.3).

**Remark 1.2.** For the defocusing–focusing system
\[
\begin{align*}
\bar{\partial}_t u + \partial_x^2 u &= \varepsilon^2 |v|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{S}^1, \\
\bar{\partial}_t v + \partial_x^2 v &= -\varepsilon^2 |u|^2 v, \\
u(0, x) &= u_0(x), \\
v(0, x) &= v_0(x),
\end{align*}
\tag{1.7}
\]
one can show a beating effect only for the case \( q = -p \) (see also remark 3.3).

**1.3. Growth of Sobolev norms in linear Schrödinger equations**

Consider a solution \((u, v)\) of the equation (1.1) and set \( V(t, x) = -\varepsilon^2 |v(t, x)|^2 \). Then the first line in (1.1) can be interpreted as a linear Schrödinger equation with a time-dependent potential
\[ \bar{\partial}_t u + \partial_x^2 u + V(t, x) u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{S}^1. \tag{1.8} \]

For \( \alpha \geq 1 \) we define the Gevrey class \( G_\alpha(\mathbb{S}) \) as the set of functions \( f \in C^\infty(\mathbb{S}) \) satisfying, for some \( A > 0 \) and \( C > 0 \):
\[ \sup_{x \in \mathbb{S}^1} |f^{(n)}(x)| \leq CA^n(n!)^\alpha, \quad \forall n \in \mathbb{N}. \]

In the periodic setting, an equivalent formulation is available (see [11]): a function \( f \in C^\infty(\mathbb{S}^1) \) is in \( G_\alpha(\mathbb{S}^1) \) if, for some \( K > 0 \) and \( B > 0 \), we have for any \( j \in \mathbb{Z} \),
\[ |\hat{f}_j| \leq Ke^{-|j|^{1/\alpha}}. \tag{1.9} \]

where \((\hat{f}_j)_{j \in \mathbb{Z}}\) denote the Fourier coefficients of \( f \). We then define a semi-norm \( \|f\|_{G_\alpha} \) as the best constant \( K \) in (1.9) (see [11] for more details).

The beating phenomenon then leads to the growth of Sobolev norms (for finite but arbitrary large times) for some solutions of this equation. Obviously, since \( V \) is a real potential, the \( L^2 \) norm of any solution of (1.8) is constant. However, we are able to prove...
Theorem 1.3. Fix $s > 0$ and $\alpha \geq 1$. There exist a sequence of real potentials $V_q(t, x)$, a sequence of initial conditions $(u_0^q)$ and a sequence of times $T_q \to +\infty$ as $q \to +\infty$ such that

- The potentials are smooth in time, real analytic in space and uniformly bounded in Gevrey classes:
  \[ \forall q \in \mathbb{Z}, \forall t \in [0, T_q], \quad \|V_q(t, .)\|_{G^\alpha} \leq C_\alpha, \]
- The corresponding solutions to the Cauchy problem $u^q(t, .)$ are real analytic in space for $t \in [0, T_q]$, 
- There exists a constant $C_{\alpha, s}$ depending only on $\alpha$ and $s$ such that
  \[ \|u^q(T_q)\|_{H^s} \geq C_{\alpha, s} (\ln T_q)^{\kappa s} \|u_0^q\|_{H^s}, \]
where $\kappa$ is a constant independent of $s, u_0$.

Bourgain also showed in [4] that $\|u\|_{H^s} \leq C(\ln t)^{\eta/2} \|u_0\|_{H^s}$ when $V$ is analytic and quasi-periodic in time. This has been extended by Fang and Zhang [7]. If $V$ is $C^\infty$ but not supposed to be quasi-periodic in time, Bourgain [3] proves the bound $\|u\|_{H^s} \leq C_\eta t^{\eta/2} \|u_0\|_{H^s}$, for all $\eta > 0$. See also the nice generalization by Delort [6].

1.4. Plan of the paper

We describe in section 2 the normal form method used to extract from the infinite dimensional Hamiltonian system a nonlinear finite dimensional and integrable system which will drive our solutions. The solutions of this small system are then studied in section 3. The proof of theorem 1.1 then relies on the control of the other terms in the initial Hamiltonian system. That is the topic of section 4. Finally, in section 5 we prove theorem 1.3.

2. The normal form

2.1. Principle of the result

In this section, we formalize the principle already used in [9] and [8], in order to follow, for arbitrary long times, solutions of an integrable model equation. The system has to be Hamiltonian: let $H$ be the Hamilton function describing its dynamics on some Hilbert phase space. We assume that $H$ is smooth in the neighbourhood of the origin, and that its Taylor expansion is given by

\[ H = N + Z^{res}_p + Z^{nr}_p + R_{p+1}, \]

where

- $N$ is a homogeneous polynomial of order 2, coming from the linear part of the system.
- It usually gathers the linear actions, i.e. the first integrals of the linearized system at the origin, which may easily be written in action-angle coordinates thanks to a Fourier transform, for instance. We take the following form for clarity:
  \[ N = \sum_{j \in \mathbb{Z}} \lambda_j I_j, \]
where the \( \lambda_j \) are the eigenvalues of the linearization of the system at 0. We suppose that \( \lambda_j \) grows polynomially with \( j \): \( \lambda_j \sim |j|^r \), with \( r > 1 \).

– For \( p \) an even integer, \( Z_p^{\text{res}} + Z_p^{\text{nr}} \) is the next nonzero term in the Taylor expansion of \( H \). It is a homogeneous polynomial of degree \( p \). We distinguish between resonant and nonresonant terms in the sense of Birkhoff normal forms: a monomial \( M \) of degree \( p \) is called resonant if it commutes with \( N \), i.e. \( \{ M, N \} = 0 \). In the contrary case, a nonresonant monomial may be removed by one step of the Birkhoff normal form (see proposition 2.1): we suppose that a Birkhoff normal form is available for the system in a ball \( B \) centred at the origin.

– \( R_{p+1} \) is an analytic Hamiltonian which vanishes at the origin up to order \( p + 1 \).

In order to observe some beating effect, we have to focus on the resonant part. Suppose that we may decompose

\[
N + Z_p^{\text{res}} = H^{\square} + N^{\text{ext}} + Z_{p,1} + Z_{p,2} + Z_{p,>2},
\]

where

– \( H^{\square} \), defining the reduced Hamiltonian system, depends only on a finite number of variables (indexed by \( j \in A \)), called the internal modes,

– \( N^{\text{ext}} \) contains all the monomials of \( N \) depending on the external modes, i.e. the variables indexed by \( j \notin A \),

– \( Z_{p,1} \) gathers monomials involving exactly one external mode,

– \( Z_{p,2} \) gathers monomials involving exactly two external modes,

– \( Z_{p,>2} \) gathers monomials involving at least three external modes.

We may now write the principle already used in [9] and [8], brought to light again in this paper. This brings together the assumptions needed to exhibit beating phenomena for Hamiltonian PDEs using our method.

**Principle.** If the following assumptions are fulfilled:

– \( H^{\square} \) defines a completely integrable Hamiltonian system,

– \( t \mapsto (q(t), p(t)) \) is a solution of the reduced system satisfying that, for every \( t \), \( (q(t), p(t)) \) stays in the ball \( B \),

– \( Z_{p,1} = 0 \), i.e. resonances cannot light on one single outer mode,

– \( \{ N^{\text{ext}}, Z_{p,2} \} = 0 \), i.e. \( Z_{p,2} \) does not affect the external modes.

– There exists a strictly convex combination of the \( (I_j)_{j \in A} \) denoted by \( I \) such that \( \{ I, H^{\square} \} = 0 \).

Then there exists solutions of the system governed by \( H \) which follow \( (q(t), p(t)) \) for long times, i.e. their projection on the reduced phase space stays close to \( (q(t), p(t)) \) and the difference between the solution and its projection stays small, for long times.

The beating effect is then obtained when we are able to construct a periodic solution \( t \mapsto (q(t), p(t)) \) of the reduced system. Note that the 2D cubic NLS equation enters in this setting when considering ‘small squares’ of indices, e.g. \( I = \{(0,0), (1,0), (0,1), (1,1)\} \) in the Fourier modes decomposition. We do not write the details.
2.2. Hamiltonian formulation and Birkhoff normal form

To apply a normal form procedure it is convenient to transform the original system where the nonlinear term is small into a system where the solutions are small. Namely, by an obvious change of variable, (1.1) is equivalent to the system

\[
\begin{cases}
  i \partial_t u + \partial_x^2 u = |v|^2 u, \\
  i \partial_t v + \partial_x^2 v = |u|^2 v,
\end{cases}
\]

\[u(0, x) = \varepsilon u_0(x), \quad v(0, x) = \varepsilon v_0(x).\]  

(2.1)

Denote by

\[H = \int |\partial_x u|^2 + |\partial_x v|^2 + \int |u|^2 |v|^2 \, dx,\]

the Hamiltonian of (2.1) with the symplectic structure \(d\omega = d\omega + d\theta\). In other words, (2.1) is equivalent to

\[
\begin{cases}
  \dot{u} = -i \frac{\delta H}{\delta u}, \\
  \dot{v} = -i \frac{\delta H}{\delta v}.
\end{cases}
\]

(2.2)

Let us expand \(u, \bar{u}, v\) and \(\bar{v}\) in Fourier modes:

\[
u(x) = \sum_{j \in \mathbb{Z}} \alpha_j e^{ijx}, \quad \bar{u}(x) = \sum_{j \in \mathbb{Z}} \alpha_j e^{-ijx},
\]

\[
\begin{aligned}
  v(x) &= \sum_{j \in \mathbb{Z}} \beta_j e^{ijx}, \\
  \bar{v}(x) &= \sum_{j \in \mathbb{Z}} \beta_j e^{-ijx}.
\end{aligned}
\]

We define

\[
P(\alpha, \beta) = \int_{S^1} |u(x)|^2 |v(x)|^2 \, dx = \sum_{j, l \in \mathbb{Z}, M(j,l) = 0} \alpha_j \bar{\alpha}_l \beta_j \beta_{\ell},
\]

where \(M(j, \ell) = j_1 - j_2 + \ell_1 - \ell_2\) denotes the momentum of the multi-index \((\alpha, \beta) \in \mathbb{Z}^4\).

In this Fourier setting the equation (2.2) reads as an infinite Hamiltonian system

\[
\begin{cases}
  i \dot{\alpha}_j = j^2 \alpha_j + \frac{\partial P}{\partial \alpha_j} = \frac{\partial H}{\partial \alpha_j}, \\
  i \dot{\beta}_j = j^2 \beta_j + \frac{\partial P}{\partial \beta_j} = \frac{\partial H}{\partial \beta_j},
\end{cases}
\]

\[j \in \mathbb{Z},\]  

(2.3)

For \(\rho > 0\), we consider the following phase space

\[\mathcal{F}_\rho = \{(\alpha, \beta) \in (\ell^1(\mathbb{Z}))^4, \text{s.t.} \| (\alpha, \beta) \|_\rho := \sum_{j \in \mathbb{Z}} e^{\rho |j|} (|\alpha_j| + |\beta_j|) < \infty \},\]

which we endow with the canonical symplectic structure \(-i \sum_j (d\alpha_j \wedge d\bar{\alpha}_j + d\beta_j \wedge d\bar{\beta}_j)\).

According to this structure, the Poisson bracket between two functions \(f\) and \(g\) of \((\alpha, \bar{\alpha}, \beta, \bar{\beta})\) is defined by

\[
\{ f, g \} = -i \sum_{j \in \mathbb{Z}} \left[ \frac{\partial f}{\partial \alpha_j} \frac{\partial g}{\partial \bar{\alpha}_j} - \frac{\partial f}{\partial \bar{\alpha}_j} \frac{\partial g}{\partial \alpha_j} + \left( \frac{\partial f}{\partial \beta_j} \frac{\partial g}{\partial \bar{\beta}_j} - \frac{\partial f}{\partial \bar{\beta}_j} \frac{\partial g}{\partial \beta_j} \right) \right].
\]
It is convenient to work in the symplectic polar coordinates $(\alpha_j = \sqrt{I_j} e^{i\theta_j}, \beta_j = \sqrt{J_j} e^{i\phi_j})_{j \in \mathbb{Z}}$. Since we have $d\alpha \wedge d\overline{\alpha} = i d\theta \wedge dI$ and $d\beta \wedge d\overline{\beta} = i d\phi \wedge dJ$ the system (2.2) is equivalent to

$$
\begin{cases}
\dot{\theta}_j = -\frac{\partial H}{\partial I_j}, & j \in \mathbb{Z}, \\
\dot{I}_j = \frac{\partial H}{\partial \theta_j}, & j \in \mathbb{Z}, \\
\dot{\phi}_j = -\frac{\partial H}{\partial J_j}, & \dot{J}_j = \frac{\partial H}{\partial \phi_j}, & j \in \mathbb{Z}.
\end{cases}
$$

We denote by $B_\rho(r)$ the ball of radius $r$ centred at the origin in $\mathcal{F}_\rho$, and introduce the resonant set

$$
\mathcal{R} = \{(j_1, j_2, \ell_1, \ell_2) \in \mathbb{Z}^4 \mid j_1 - j_2 + \ell_1 - \ell_2 = 0 \text{ and } j_1^2 - j_2^2 + \ell_1^2 - \ell_2^2 = 0\}.
$$

**Proposition 2.1.** There exists a canonical change of variable $\tau$ from $B_\rho(\varepsilon)$ into $B_\rho(2\varepsilon)$ with $\varepsilon$ small enough such that

$$
\overline{H} := H \circ \tau = N + Z_4 + R_6,
$$

where

(i) $N$ is the term $N(I) = \sum_{j \in \mathbb{Z}} j^2 (I_j + J_j)$,

(ii) $Z_4$ is the homogeneous polynomial of degree 4

$$
Z_4 = \sum_{\mathcal{R}} \alpha_{j_1} \alpha_{j_2} \beta_{\ell_1} \beta_{\ell_2}.
$$

In particular, $Z_4$ is made of resonant monomials: it satisfies $\{Z_4, N\} = 0$.

(iii) $R_6$ is the remainder of order 6, i.e. a Hamiltonian satisfying $\|X_{R_6}(z)\|_\rho \leq C\|z\|_\rho^5$ for $z = (\alpha, \beta, \overline{\alpha}, \overline{\beta}) \in B_\rho(\varepsilon)$.

(iv) $\tau$ is close to the identity: there exist a constant $C_\rho$ such that $\|\tau(z) - z\|_\rho \leq C_\rho\|z\|_\rho^2$ for all $z \in B_\rho(\varepsilon)$.

The proof is similar to the proof of proposition 2.1 in [9]: we essentially use that if $\mathcal{M}(j, \ell) = 0$ and $(j, \ell) \not\in \mathcal{R}$ then $|j_1^2 - j_2^2 + \ell_1^2 - \ell_2^2| \geq 1$, i.e. there are no small divisors involved.

For the construction of a more general Birkhoff Normal Form, see [2]. By abuse of notation, in the proposition and in the sequel, the new variables $(\alpha', \beta') = \tau^{-1}(\alpha, \beta)$ are still denoted by $(\alpha, \beta)$.

### 2.3. Description of the resonant normal form

In this subsection we study the resonant part of the normal form given by proposition 2.1. Denote

$$
I = \sum_{n \in \mathbb{Z}} |\alpha_n|^2, \quad J = \sum_{n \in \mathbb{Z}} |\beta_n|^2, \quad S = \sum_{n \in \mathbb{Z}} \alpha_n \beta_{-n}.
$$

**Proposition 2.2.** The polynomial $Z_4$ reads:

$$
Z_4 = IJ + |S|^2 - \sum_{n \in \mathbb{Z}} |\alpha_n|^2 |\beta_n|^2.
$$

**Proof.** By an elementary computation, we know that $(j_1, j_2, \ell_1, \ell_2) \in \mathcal{R}$ iff $(j_1, \ell_1) = (j_2, \ell_2)$ and the result follows. $\square$
3. The reduced model

We want to describe the dynamics of the Hamiltonian system obtained by reducing (2.3) to the space
\[ \mathcal{J}(p, q) := \{ (\alpha, \beta) \in \mathcal{F}_p \mid \alpha_j = \beta_j = \bar{\beta}_j = 0 \text{ when } j \neq p, q \}, \]
and we denote the reduced Hamiltonian by \( \hat{H} \), i.e.
\[ \hat{H} = H \big|_{\mathcal{J}(p, q)}. \]
After calculation we obtain
\[ \hat{H} = p^2(I_p + J_p) + q^2(I_q + J_q) + (I_p + I_q)(J_p + J_q) + (\alpha_p \alpha_q \beta_p \beta_q + \bar{\alpha}_p \bar{\alpha}_q \bar{\beta}_p \bar{\beta}_q) \]
\[ = p^2(I_p + J_p) + q^2(I_q + J_q) + (I_p + I_q)(J_p + J_q) + 2(I_p I_q J_p J_q)^{1/2} \cos(\psi_0), \]
with \( \psi_0 = \theta_q - \theta_p + \phi_p - \phi_q \).

The Hamiltonian system associated with \( \hat{H} \) is defined on the phase space \( \mathbb{T}^4 \times \mathbb{R}^4 \ni (\theta_p, \theta_q, \phi_p, \phi_q; I_p, I_q, J_p, J_q) \) by
\[
\begin{align*}
\dot{\theta}_j &= -\frac{\partial \hat{H}}{\partial I_j}, & \dot{I}_j &= \frac{\partial \hat{H}}{\partial \theta_j}, & j = p, q, \\
\dot{\phi}_j &= -\frac{\partial \hat{H}}{\partial J_j}, & \dot{J}_j &= \frac{\partial \hat{H}}{\partial \phi_j}, & j = p, q.
\end{align*}
\] (3.1)

Since the Hamiltonian \( \hat{H} \) only depends on one angle (\( \psi_0 \)), the system (3.1) is completely integrable (this is also a consequence of the invariance properties recalled in (1.3)–(1.5)).

**Lemma 3.1.** The system (3.1) is completely integrable. Moreover, the change of variables
\[
\begin{align*}
K_1 &= I_q + I_p, & K_2 &= J_q + J_p, & K_3 &= I_q + J_q, & K_0 &= I_q, \\
\psi_1 &= \theta_p, & \psi_2 &= \phi_p, & \psi_2 &= \phi_p - \phi_q, & \psi_0 &= \theta_q - \theta_p + \phi_p - \phi_q
\end{align*}
\] (3.2)

is symplectic: \( dI \wedge d\theta + dJ \wedge d\phi = dK \wedge d\psi \).

**Proof.** It is straightforward to check that
\[ K_1 = I_q + I_p, \quad K_2 = J_q + J_p \quad \text{and} \quad K_3 = I_q + J_q, \]
are constants of motion. Furthermore we verify
\[ \{ K_1, \hat{H} \} = \{ K_2, \hat{H} \} = \{ K_3, \hat{H} \} = 0, \]
as well as
\[ \{ K_1, K_2 \} = \{ K_2, K_3 \} = \{ K_3, K_1 \} = 0. \]
Moreover the previous quantities are independent. So \( \hat{H} \) admits four integrals of motions that are independent and in involution and thus \( \hat{H} \) is completely integrable. \( \square \)

In the new coordinates, the Hamiltonian \( \hat{H} \) reads
\[ \hat{H} = \hat{H}(\psi_0, K_0, K_1, K_2, K_3) \]
\[ = p^2(K_1 + K_2 - K_3) + q^2K_3 + K_1K_2 \]
\[ + 2[K(K - K)(K_2 - K_3 + K)(K_1 - K)]^{1/2} \cos(\psi_0). \] (3.3)

We set \( K_1 = K_2 \Rightarrow K_3 = \varepsilon^2 \), and we denote
\[ \hat{H}_0(\psi_0, K_0) := \hat{H}(\psi_0, K_0, \varepsilon^2, \varepsilon^2, \varepsilon^2) = \varepsilon^4(p^2 + q^2) + \varepsilon^4 + 2K_0(\varepsilon^2 - K_0) \cos(\psi_0). \]
The evolution of \((\psi_0, K_0)\) is given by
\[
\dot{\psi}_0 = -\frac{\partial \hat{H}_0}{\partial K_0}, \quad \dot{K}_0 = \frac{\partial \hat{H}_0}{\partial \psi_0}.
\]

Then, we make the change of unknown
\[
\psi_0(t) = \psi(\varepsilon^2 t) \quad \text{and} \quad K_0(t) = \varepsilon^2 K(\varepsilon^2 t).
\]

An elementary computation shows that the evolution of \((\psi, K)\) is given by
\[
\begin{aligned}
\dot{\psi} &= -2(1 - 2K) \cos \psi = -\frac{\partial H_\varepsilon}{\partial K}, \\
\dot{K} &= -2K(1 - K) \sin \psi = \frac{\partial H_\varepsilon}{\partial \psi},
\end{aligned}
\]

where
\[
H_\varepsilon = H_\varepsilon(\psi, K) = 2\varepsilon^2 (q^2 - p^2)K + 2K(1 - K) \cos \psi.
\]

The dynamical system \((3.5)\) is a pendulum whose phase portrait is drawn in figure 1 and we easily deduce that

**Lemma 3.2.** Let \(\gamma > 0\) be arbitrary small, then the dynamical system \((3.5)\) admits a periodic orbit \(\Gamma_\varepsilon := \{(\psi_\varepsilon(t), K_\varepsilon(t)) \mid t \in \mathbb{R}\}\) of period \(2T_\varepsilon\) satisfying \((\psi_\varepsilon(0), K_\varepsilon(0)) = (0, \gamma)\) and \((\psi_\varepsilon(T_\varepsilon), K_\varepsilon(T_\varepsilon)) = (0, 1 - \gamma)\).

**Remark 3.3.** For system \((1.7)\), we can perform a similar analysis, and we get the reduced Hamiltonian
\[
H_\varepsilon = H_\varepsilon(\psi, K) = \frac{2}{\varepsilon^2} (q^2 - p^2)K + 2K(1 - K) \cos \psi.
\]

This Hamiltonian is of pendulum type for \(\varepsilon\) arbitrary small iff \(q = \pm p\), and only with this choice are we able to prove the same result as for \((1.1)\).
4. Proof of theorem 1.1

Consider the Hamiltonian \( \mathcal{H} \) given by (2.4), which is a function of \((\alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j)_{j \in \mathbb{Z}}\). We want to prove that, for a good choice of initial datum, the solution of the Hamiltonian system governed by \( \mathcal{H} \) remains close to the solution of the reduced system governed by \( \mathcal{H} \) (or \( \mathcal{H}^{\ast} \)).

We make the linear change of variables given by lemma 3.1. Then \( \mathcal{H} \) induces the system

\[
\begin{align*}
\dot{\psi}_j &= -\frac{\partial \mathcal{H}}{\partial K_j}, \\
\dot{K}_j &= \frac{\partial \mathcal{H}}{\partial \psi_j},
\end{align*}
\]

\( j \in \mathbb{Z}. \) (4.1)

Denote \( A = \{ p, q \} \) and \( \mathcal{L} = \mathbb{Z} \setminus \{ p, q \} \). By propositions 2.1 and 2.2 we have \( \mathcal{H} = \hat{\mathcal{H}} + R_I + Z_{4,2} + Z_{4,4} + R_0. \)

Proof. We first remark that by the preservation of the \( L^2 \) norm in each NLS equation, we have

\[
\sum_{j \in \mathbb{Z}} I_j(t) = \sum_{j \in \mathbb{Z}} I_j(0) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} J_j(t) = \sum_{j \in \mathbb{Z}} J_j(0) \quad \text{for all} \ t \in \mathbb{R},
\]

where the \( K_j \)'s are defined by (3.2).

Lemma 4.1. Assume that

\[
\alpha_j(0), \beta_j(0) = O(\epsilon), \ \forall j \in A \quad \text{and} \quad \alpha_j(0), \beta_j(0) = O(\epsilon^2), \ \forall j \in \mathcal{L}. \quad (4.2)
\]

Then for all \( 0 \leq t \leq C \epsilon^{-5/2} \),

\[
I_j(t), J_j(t) = O(\epsilon^4) \quad \text{when} \ j \in \mathcal{L},
\]

and

\[
\begin{align*}
K_1(t) &= K_1(0) + O(\epsilon^6)t, \\
K_2(t) &= K_2(0) + O(\epsilon^6)t, \\
K_3(t) &= K_3(0) + O(\epsilon^6)t,
\end{align*}
\]

(4.4-4.6)

where the \( K_j \)'s are defined by (3.2).

Proof. We first remark that by the preservation of the \( L^2 \) norm in each NLS equation, we have

and therefore by using (4.2)
\[
I_n(t) = O(\varepsilon^2), \ J_n(t) = O(\varepsilon^2)
\]
for all \( n \in \mathbb{Z} \) and for all \( t \in \mathbb{R} \).

On the other hand by propositions 2.1 and 2.2, we have for \( n \in \mathbb{Z} \)
\[
\dot{I}_n = \{I_n, \mathcal{H}\} = \{I_n, Z_4\} + \{I_n, R_6\},
\]
(4.7)
and the same for \( J_n \).

\begin{itemize}
  \item To prove (4.3), we compute
    \[
    \{I_n, Z_4\} = -i(\alpha S - \alpha S S),
    \]
(4.8)
and
\[
\{J_n, Z_4\} = i(\alpha S - \alpha S S).
\]
Then by (4.7), if we denote by \( L_n = I_n + J_n \) we get
\[
\dot{L}_n = \{L_n, R_6\}.
\]
Furthermore, for \( n \neq p, q \) all the monomials appearing in \( \{L_n, R_6\} \) are of order 6 and contain at least one mode in \( L \). Therefore as soon as (4.3) remains valid, we have
\[
\dot{L}_n(t) = O(\varepsilon^2 + 5)\]
and thus \( |L_n(t)| = O(\varepsilon^4) + t O(\varepsilon^5) \). We then conclude by a classical bootstrap argument that (4.3) holds true for \( t \leq C \varepsilon^{-5/2} \).
  \item It remains to prove (4.4)–(4.6). We denote by
\[
Z_4^b = (I_p + I_q)(J_p + J_q) + (\alpha_p \bar{\alpha}_q \bar{\beta}_p \beta_q + \bar{\alpha}_p \alpha_q \beta_p \bar{\beta}_q),
\]
the fourth order part of the model Hamiltonian. From (4.8), we get
\[
\{I_p, Z_4^b\} = -i(\alpha_p \bar{\beta}_p \alpha_q \bar{\beta}_q - \alpha_p \bar{\beta}_p \alpha_q \bar{\beta}_q) = -\{J_q, Z_4^b\} = \{J_q, Z_4^b\},
\]
thus \( \{K_1, Z_4^b\} = \{I_p + I_q, Z_4^b\} = 0 \). Similarly, \( \{K_2, Z_4^b\} = 0 \) and \( \{K_3, Z_4^b\} = 0 \).
Therefore, by using (4.7) we deduce that for all \( j \in \{1, 2, 3\} \)
\[
\dot{K}_j = \{K_j, Z_4, R_6\} + \{K_j, R_6\},
\]
(4.9)
Then we use that each monomial of \( Z_4, R_6 \) contains at least two terms with indices \( j \in L \). Therefore, as soon as (4.2) holds, \( |\{K_j, Z_4\}| \leq C \varepsilon^6 \). Furthermore \( |\{K_j, R_6\}| \leq C \varepsilon^6 \). Therefore, by (4.9),
\[
K_j(t) = K_j(0) + t O(\varepsilon^6).
\]
\end{itemize}

From now, we fix the initial conditions
\[
K_1(0) = \varepsilon^2, \quad K_2(0) = \varepsilon^2, \quad K_3(0) = \varepsilon^2,
\]
and \( |\alpha_j(0)|, |\bar{\alpha_j}(0)|, |\beta_j(0)|, |\bar{\beta}_j(0)| \leq C \varepsilon^2 \) for \( j \neq p, q \).
(4.10)

Let \( \mathcal{H} \) be given by (2.4). Then according to the result of lemma 4.1, which says that for a suitably long time we remain close to the regime of section 3, we hope that we can write \( \mathcal{H} = \mathcal{H}_0 + R \), where \( R \) is an error term which remains small for times \( 0 \leq t \leq \varepsilon^{-5/2} \).

Let \( \psi(t) \) be the solution of (4.11). Then we state
\[
\psi(t) = \psi(\varepsilon^2 t) \quad \text{and} \quad K_0(t) = \varepsilon^2 K(\varepsilon^2 t),
\]
(4.11)
and we work with the scaled time variable \( \tau = \varepsilon^2 t \). Then we can state
Proposition 4.2. Consider the solution of (4.1) with the initial conditions (4.10). Then \((\psi, K)\) defined by (4.11) satisfies, for \(0 \leq \tau \leq \varepsilon^{-1}\),

\[
\begin{align*}
\dot{\psi} &= -\frac{\partial H}{\partial K} + O(\varepsilon^2) \\
\dot{K} &= \frac{\partial H}{\partial \psi} + O(\varepsilon^2),
\end{align*}
\]

where \(H\) is the Hamiltonian

\[H = 2K(1 - K)\cos\psi.\]

Proof. First recall that \(\hat{H} = \hat{H}(\psi_0, K_0, K_1, K_2, K_3)\) is the reduced Hamiltonian given by (3.3). By propositions 2.1 and 2.2 we have

\[\overline{H} = \hat{H} + R_I + Z_{4,2} + Z_{4,4} + R_6.\]  

(4.13)

Thanks to the Taylor formula there is \(Q\) so that

\[\hat{H}(\psi_0, K_0, K_1, K_2, K_3) = \hat{H}(\psi_0, K_0, \varepsilon^2, \varepsilon^2, \varepsilon^2) + Q = \hat{H}_0 + Q.\]

(4.14)

Thus, by (4.13) and (4.14) we have \(\overline{H} = \hat{H}_0 + R\) with

\[R = Q + R_I + Z_{4,2} + Z_{4,4} + R_6.\]

Then, the change of variables (4.11) we obtain

\[
\begin{align*}
\dot{\psi}(\tau) &= -\frac{1}{\varepsilon^4} \frac{\partial R}{\partial K}(\psi(\tau), \varepsilon^2 K(\tau), \ldots) \\
\dot{K}(\tau) &= \frac{1}{\varepsilon^4} \frac{\partial R}{\partial \psi}(\psi(\tau), \varepsilon^2 K(\tau), \ldots).
\end{align*}
\]

Now write \(\overline{H} = \hat{H}_0 + R\) and observe that \(\hat{H}_0(\psi, \varepsilon^2 K) = C_\varepsilon + \varepsilon^4 H_\varepsilon(\psi, K)\). As a consequence, \((\psi, K)\) satisfies

\[
\begin{align*}
\dot{\psi} &= -\frac{\partial H_\varepsilon}{\partial K} - \frac{1}{\varepsilon^4} \frac{\partial R(\psi, \varepsilon^2 K, \ldots)}{\partial K} \\
\dot{K} &= \frac{\partial H_\varepsilon}{\partial \psi} + \frac{1}{\varepsilon^4} \frac{\partial R(\psi, \varepsilon^2 K, \ldots)}{\partial \psi}.\n\end{align*}
\]

Thus it remains to estimate \(\frac{\partial R}{\partial \psi}(\psi, \varepsilon^2 K, \ldots)\) and \(\frac{\partial R}{\partial K}(\psi, \varepsilon^2 K, \ldots)\). Notice that \(\psi\) and \(K\) are dimensionless variables. Thus, if \(P\) is a polynomial involving \(k\) internal modes, \((\alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j)_{j \in A}\), and \(\ell\) external modes, \((\alpha_j, \bar{\alpha}_j, \beta_j, \bar{\beta}_j)_{j \in \ell}\), we have, by using lemma 4.1,

\[
\begin{align*}
\frac{\partial}{\partial \psi} P(\psi, \varepsilon^2 K, \ldots) &= O(\varepsilon^{k+2\ell}) \\
\frac{\partial}{\partial K} P(\psi, \varepsilon^2 K, \ldots) &= O(\varepsilon^{k+2\ell}).
\end{align*}
\]
As $R_i$ contains only monomials involving at least one external action $(I_k)_{k=4}$ we get
\[
\partial_\psi R_i(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6), \quad \partial_K R_i(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6),
\]
\[
\partial_\psi Z_{d,2}(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6), \quad \partial_K Z_{d,2}(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6),
\]
\[
\partial_\psi Z_{d,4}(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6), \quad \partial_K Z_{d,4}(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6),
\]
\[
\partial_\psi R_6(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6), \quad \partial_K R_6(\psi, \varepsilon^2 K, \ldots) = O(\varepsilon^6).
\]
On the other hand, by construction $Q$ reads $P_1 \Delta K_1 + P_2 \Delta K_2 + P_3 \Delta K_3$ where $P_1$, $P_2$ and $P_3$ are polynomials of order 1 in $K_0$, $K_1$, $K_2$, $K_3$ and $\varepsilon^2$ while $\Delta K_j$ denotes the variation of $K_j$: $\Delta K_j = K_j - K_j(0)$. Using again lemma 4.1, we check that for $0 \leq \tau \leq \varepsilon^{-1}$
\[
\partial_\psi Q = O(\varepsilon^2), \quad \partial_K Q = O(\varepsilon^2),
\]
hence the result.

We now consider the solution $(\psi_\gamma, K_\gamma)$ of (3.5), which is issued from the initial condition $(\psi_\gamma, K_\gamma)(0) = (0, \gamma)$ for some $\gamma$ such that $\varepsilon^{1/2} \ll \gamma \ll 1$ and we compare it with the solution $(\psi, K)$ of (4.12) issued from the same initial datum:

**Lemma 4.3.** For all $0 \leq \tau \leq \varepsilon^{-1/2}$ we have
\[
(\psi, K)(\tau) = (\psi_\gamma, K_\gamma)(\tau) + O(\varepsilon^2)\tau. \tag{4.15}
\]

**Proof.** Consider system (3.5) and the open domain $U = (-\pi, \pi) \times (0, 1)$. By the Arnold Theorem (see [1, p 113], see also [8, lemma 4.3]), this Hamiltonian system admits action-angle coordinates $(L, \alpha) = \Phi(\psi, K)$ defined on $U \setminus \{(0, 0)\}$ by a $C^1$ symplectic map $\Phi$ satisfying that uniformly on any compact $U \subset U \setminus \{(0, 0)\}$:
\[
\|d\Phi\| \leq C, \quad \|d\Phi^{-1}\| \leq C.
\]
Then we obtain that for $0 \leq \tau \leq \varepsilon^{-1/2}$
\[
\frac{d}{d\tau} (L, \alpha) = \frac{d}{d\tau} \Phi(\psi, K) = d\Phi(\psi, K) \cdot (\dot{\psi}, \dot{K})
\]
\[
= d\Phi(\psi, K) \cdot \left( \frac{\partial H_\alpha}{\partial \psi}, \frac{\partial H_\alpha}{\partial K} \right) + O(\varepsilon^2)
\]
\[
= \dot{\psi} \frac{\partial H_\alpha}{\partial \psi} + \dot{K} \frac{\partial H_\alpha}{\partial K} + O(\varepsilon^2)
\]
\[
= (0, -\frac{\partial H_\alpha}{\partial L}) + O(\varepsilon^2).
\]
Therefore there exists $L_\ast \in \mathbb{R}$ so that $L(\tau) = L_\ast + O(\varepsilon^2)\tau$ and if we define $\omega_\ast = -\frac{\partial H_\alpha}{\partial L}(L_\ast)$, we obtain $\alpha(\tau) = \omega_\ast \tau + O(\varepsilon^2)\tau$. Notice that by construction $\Phi(\psi(\gamma), K(\gamma))(\tau) = (L_\ast, \omega_\ast \tau)$ for all $\tau \in \mathbb{R}$. Next, as $d\Phi^{-1}$ is bounded, we get
\[
(\psi, K)(\tau) = \Phi^{-1}(L(\tau), \alpha(\tau)) = \Phi^{-1}(L_\ast, \omega_\ast \tau) + O(\varepsilon^2)\tau + O(\varepsilon^2)\tau^2
\]
\[
= (\psi_\gamma, K_\gamma)(\tau) + O(\varepsilon^2)(\tau + \tau^2).
\]
With the choice $\varepsilon^{1/2} \leq \gamma$, the remainder term in (4.15) is so that $|O(\varepsilon^2)\tau + O(\varepsilon^2)\tau^2| \ll \gamma \leq K_\gamma$ for $\tau \leq \varepsilon^{-1/2}$.

**Proof of theorem 1.1.** As a consequence of lemma 4.3, the solution of (4.1), with initial datum (4.10) and $(\psi_0, K_0)(0) = (0, \varepsilon^2 \gamma)$, satisfies for $0 \leq t \leq \varepsilon^{-5/2}$
\[
K_0(t) = \varepsilon^2 K_\gamma(\varepsilon^2 t) + O(\varepsilon^6 t) + O(\varepsilon^8 t^2)
\]
\[
\psi_0(t) = \psi_\gamma(\varepsilon^2 t) + O(\varepsilon^4 t) + O(\varepsilon^6 t^2),
\]
and with the condition $\varepsilon^{1/2} \leq \gamma$ we obtain (1.2).
We now compute the period $2T_{\gamma}$. From the expression of the Hamiltonian $H_*$, we infer

$$K = 2K(1 - K) \sqrt{1 - \frac{h^2}{(2K(1 - K))^2}},$$

where $h = 2\gamma(1 - \gamma) > 0$. Thanks to the symmetries of $H_*$, $T_{\gamma}/2$ is the travel time for the solution between $(0, \gamma)$ and $(-\cos^{-1}(2h), 1/2)$. In the interval $[0, T_{\gamma}/2]$, the function $K$ is strictly increasing, hence invertible, so we can write the time $t$ as a function of $K$, and this implies that

$$T_{\gamma} = 2 \int_{0}^{1} \frac{dK}{\sqrt{(2K(1 - K))^2 - h^2}}. \quad (4.16)$$

Next, we estimate $T_{\gamma}$. It is easy to check that there exists $C > 0$ such that for all $\gamma < K < \frac{1}{2}$, we have

$$K^2(1 - K)^2 - \gamma^2(1 - \gamma)^2 \geq C(K^2 - \gamma^2).$$

Hence by integration in (4.16) we deduce that there exists $C > 0$ such that

$$T_{\gamma} \leq -C \ln \gamma. \quad (4.17)$$

5. Proof of theorem 1.3

Fix $\alpha \geq 1$ and consider system (1.1). We fix the set of internal modes: $p = 0, q \gg 1$ and

$$s := e^{-\frac{1}{q}t^{1/\alpha}}.$$

Then we consider the first equation in (1.1) as a linear time-dependent Schrödinger equation

$$i\partial_t u + \partial_x^2 u + V_q(t, x)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{S}^1,$$

with potential $V_q = -e^{-q|x|} |v|^2$.

The regularity properties of the solutions given in theorem 1.1 imply that, on $[0, e^{-5/2} \times \mathbb{S}^1$ the function $V_q$ is smooth in time and real analytic in space. In order to construct the sequence of initial conditions announced in theorem 1.3, we have to ensure uniform bounds w.r.t. the integer $q \in \mathbb{N}$ in the Gevrey class $\mathcal{G}_0(\mathbb{S}^1)$, given in (1.9). We have

$$V_q(t, x) = e^{-q|x|} |v_0(t) + v_q(t)e^{i\alpha x} + e^{i\gamma q} r_\gamma(t, x)|^2,$$

where $r_\gamma(t, \cdot)$ is an analytic function, whose norm is uniformly bounded with respect to $|t| \leq e^{-5/2}$ and $q \in \mathbb{N}$. We then compute the Fourier coefficients $\hat{V}_j$ of $V_q(t, \cdot)$.

The dominant coefficients are labelled by the indices $0, q$ and $-q$: for them, we have

$$|\hat{V}_0| \leq e^{-q|x|} (|v_0|^2 + |v_q|^2 + e^{-q|x|} R),$$

$$|\hat{V}_q| \leq e^{-q|x|} (|v_0||v_q| + e^{-q|x|} R'),$$

$$|\hat{V}_{-q}| \leq e^{-q|x|} (|v_0||v_q| + e^{-q|x|} R''),$$

where $R, R'$, $R''$ are uniformly bounded w.r.t. $q \in \mathbb{N}$ and $0 \leq t \leq e^{-5/2}$. Since $v_0$ and $v_q$ stay (in modulus) between 0 and 1, estimate (1.9) is obtained for these indices.

The coefficients $\hat{V}_j$ for $j \neq -q, 0, q$ decay much faster in general: using the analyticity of $r_\gamma(t, \cdot)$, and the fact that for every $q \in \mathbb{N}$ and $\ell \in \mathbb{Z}$

$$\frac{5}{4}|q|^{1/u} + \rho|\ell - q| \geq c|\ell|^{1/u},$$
we obtain
\[ |\hat{V}_j| \leq C e^{-c|j|^{1/\alpha}}. \]

Once again, this estimate is uniform w.r.t. \( q \in \mathbb{N} \) and \( 0 \leq t \leq \varepsilon^{-5/2} \).

Choose initial conditions so that \((\psi(0), K(0)) = (0, \gamma)\) for some \( 0 < \gamma \ll 1 \). In order to apply the result of theorem 1.1 we must have \( T_\gamma \leq \varepsilon^{-1/2} \). Therefore we impose
\[ T_q := \frac{T_\gamma}{\varepsilon^2} < \frac{1}{\varepsilon^{5/2}}, \]
which leads to \( e^{-1/2q^{1/\alpha}} < C |\ln \gamma| \) since \( T_\gamma \leq C |\ln \gamma| \). We fix the \( H^s\)-norm of the initial condition with the choice \( \gamma = q^{-2s} \) (observe that for \( q \gg 1, \varepsilon < \gamma^2 \), so that we are in the conditions of application of theorem 1.1), then the previous constraint becomes \( e^{-1/2q^{1/\alpha}} < \frac{C}{2q^{1/\alpha}} \), hence is satisfied for \( q \) large enough. From (4.17), we get
\[ T_q \leq C |\ln \gamma| \varepsilon^{-2} = 2s C e^{q^{1/\alpha}} \ln q. \]

Now, the growth rate of \( \|u\|_{H^s} \) between \( t = 0 \) and \( t = T_q \) is bounded from below by \( C' \gamma^{-1/2} \), where \( C' \) is independent of \( s \). So we have, by (1.6),
\[ \frac{\|u(T_q)\|_{H^s}}{\|u(0)\|_{H^s}} \geq C' \gamma^{-1/2} = C' q^s \geq \frac{C'}{(1 + 2as)^{as}} (\ln T_q)^{as}. \]

Note that the constant \( C_{a,s} := \frac{C'}{(1 + 2as)^{as}} \) goes to 0 as \( s \) or \( \alpha \) goes to infinity. \( \square \)

**Remark 5.1.** If we choose a different \( \varepsilon \), as for example \( \varepsilon = \exp(-((\ln q)^{1+\kappa})) \), with \( \kappa > 0 \), we have that for all \( s > 0 \)
\[ \forall q \in \mathbb{N}, \forall t \in [0, T_q], \|V_q(t, \cdot)\|_{H^s} \leq C_{s,a}, \]
and we obtain the growth
\[ \frac{\|u(T_q)\|_{H^s}}{\|u(0)\|_{H^s}} \geq C \exp \left(s((\ln T_q)^{1/(1+\kappa)})\right), \]
that is, a sub-polynomial growth.

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