IMPROVED LOCAL SMOOTHING ESTIMATES FOR THE FRACTIONAL SCHRÖDINGER OPERATOR

CHUANWEI GAO, CHANGXING MIAO, AND JIQIANG ZHENG

Abstract. In this paper, we consider local smoothing estimates for the fractional Schrödinger operator $e^{it(-\Delta)^{\alpha/2}}$ with $\alpha > 1$. Using the $k$-broad “norm” estimates of Guth-Hickman-Iliopoulou [8], we improve the previously best-known results of local smoothing estimates of [6, 10].

1. INTRODUCTION

Let $u$ be the solution for the Cauchy problem of the fractional Schrödinger equation

$$\begin{cases}
  i\partial_t u + (-\Delta)^{\alpha/2} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n \\
  u(0, x) = f(x),
\end{cases}$$

where $\alpha > 1$ and $f$ is a Schwartz function. The solution $u$ is expressed by

$$u(x, t) = e^{it(-\Delta)^{\alpha/2}} f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^{\alpha/2})} \hat{f}(\xi) d\xi.$$  

We are concerned with $L^p$-regularity for the solution $u$. For the fixed time $t$, Fefferman and Stein [5], Miyachi [9] showed the following optimal $L^p$ estimate:

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L^p(\mathbb{R}^n)} \leq C_{t, p} \|f\|_{L^{s_{\alpha, p}}(\mathbb{R}^n)}, \quad s_{\alpha, p} := \alpha n \left(\frac{1}{2} - \frac{1}{p}\right), \quad 1 < p < \infty,$$

where the constant $C_{t, p}$ is locally bounded.

This estimate trivially leads to the following spacetime estimate

$$(\int_1^2 \|e^{it(-\Delta)^{\alpha/2}} f\|_{L^p(\mathbb{R}^n)} dt)^{1/p} \lesssim \|f\|_{L^{s_{\alpha, p}}(\mathbb{R}^n)}.$$  

As one can see, compared with (1.3), (1.4) does not gain any profits from taking an average over time. In contrast with the fixed time estimate, a natural question appears: can one gain some regularities by considering the spacetime integral? More precisely, is there an $\varepsilon > 0$ such that

$$(\int_1^2 \|e^{it(-\Delta)^{\alpha/2}} f\|_{L^p(\mathbb{R}^n)} dt)^{1/p} \lesssim \|f\|_{L^{s_{\alpha, p} - \varepsilon}(\mathbb{R}^n)}?$$  

Taking the example in [10] into account, it seems natural to formulate the following local smoothing conjecture for the fractional Schrödinger operator.

**Conjecture 1.1** (Local smoothing for the fractional Schrödinger operator). Let $\alpha > 1, p > 2 + \frac{2}{n}$ and $s \geq \alpha n \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2}{p}$. Then

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L^p(\mathbb{R}^n)} \leq C_{p, s} \|f\|_{L^p(\mathbb{R}^n)}.$$  

When $\alpha = 2$, which corresponds to the Schrödinger operator, Rogers [11] proposed this conjecture, and showed that it could be deduced from the restriction conjecture. To be more precise, for $q > 2 + \frac{2}{n}$ with $p' = \frac{2n}{n+2}$, the adjoint restriction estimate

$$\|e^{it\Delta} f\|_{L^q(\mathbb{R}^{n+1})} \leq \|\hat{f}\|_{L^p(\mathbb{R}^n)}$$

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will imply that
\[
\|e^{it\Delta}f\|_{L^s_t(L^r_x \times [1,2])} \leq C_{q,s}\|f\|_{L^2_x}^{1/2}, \quad s > 2n\left(\frac{1}{q} - \frac{1}{r}\right) - \frac{2}{q}.
\] (1.8)

The proof of the above implication relies deeply on the structure of the phase function and the “completing of square” trick. Roughly speaking, we may explicitly write \(e^{it\Delta}f\) to be
\[
e^{it\Delta}f(x) = \frac{1}{(4\pi it)^\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} f(y)dy.
\] (1.9)

Squaring out \(|x-y|^2\), we obtain
\[
|e^{it\Delta}f(x)| = \left|\frac{c_n}{2t} e^{-\frac{|x|^2}{4t}} f\left(\frac{c_n x}{t}\right)\right|.
\]

This equality and (1.2) enable us to express \(e^{it\Delta}f\) freely in terms of spatial or frequency variables. After some appropriate reductions and making use of the pseudo-conformal change of variables, one can obtain (1.8) by (1.7). The above approach is, unfortunately, unavailable for the general fractional Schrödinger operators. Using the bilinear method, Rogers and Seeger [10] established the sharp local smoothing results for \(p > 2 + \frac{1}{n+1}\). Away from the endpoint regularity, their results were further improved by Guo-Roo-Yung in [6] by means of the Bourgain-Guth [3] iteration argument. In this paper, motivated by the seminal work of Guth [7], up to the endpoint regularity, we further refresh the range of \(p\) of [6, 10] by means of weakened versions of the multilinear restriction estimates of Bennett-Carbery-Tao [1], the so-called \(k\)-broad “norm” estimates.

**Theorem 1.2.** Let \(\alpha > 1\), \(n \geq 1\) and \(s > s_{\alpha,p} = \frac{\alpha}{p}\) with
\[
p > \begin{cases} 2\frac{2n+4}{3n+5}, & \text{for } n \text{ even}, \\ 2\frac{2n+5}{3n+6}, & \text{for } n \text{ odd}. \end{cases}
\] (1.10)

Then
\[
\|e^{it(-\Delta)_{\alpha}}f\|_{L^p_t(L^r_x \times [1,2])} \leq C\|f\|_{L^2_x}.
\] (1.11)

**Remark 1.3.** We recover the sharp local smoothing results for \(n = 1\) and improve the previously best-known results in [6, 10] for \(n \geq 3\). In particular, when \(n = 1\), \(\alpha = 2\), the above result follows from the known restriction theorem in \(\mathbb{R}^2\) and Rogers’s implication.

The crucial observation is that away from the origin, the phase function \(|\xi|^{\alpha}\) with \(\alpha > 1\) always has non-vanishing Gaussian curvature. This fact facilitates us to incorporate them all into a class of elliptic phase functions. A prototypical example for such class is the Schrödinger propagator, for which local smoothing estimates have extensive applications in various aspects. The Strichartz estimate, among other things, plays a critical role in the study of the semilinear Schrödinger equations.

The purpose of this paper is to explore to what extent the current available methods and tools for studying the Fourier restriction can be applied to the local smoothing problems. This paper is organized as follows. In Section 2, we will provide some preliminaries and reductions. In Section 3, we will prove Theorem 1.2. In the appendix, we will show some probable tractable approaches toward further improvement.

**Notations.** For nonnegative quantities \(X\) and \(Y\), we will write \(X \lesssim Y\) to denote the inequality \(X \leq CY\) for some \(C > 0\). If \(X \lesssim Y \lesssim X\), we will write \(X \sim Y\). Dependence of implicit constants on the spatial dimensions or integral exponents such as \(p\) will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, \(X \lesssim_u Y\) indicates \(X \leq CY\) for some \(C = C(u)\). We write \(A(R) \leq \text{RapDec}(R)B\) to mean that for any power \(\beta\), there is a constant \(C_\beta\) such that
\[
|A(R)| \leq C_\beta R^{-\beta}B \quad \text{for all } R \geq 1.
\]
For a spacetime slab $\mathbb{R}^n \times I$, we write $L^s_x L^q_t(\mathbb{R}^n \times I)$ for the Banach space of functions $u : \mathbb{R}^n \times I \to \mathbb{C}$ equipped with the norm

$$||u||_{L^s_x L^q_t(\mathbb{R}^n \times I)} := \left( \int_{\mathbb{R}^n} ||u(x, \cdot)||_{L^q_t(I)}^r dx \right)^{\frac{1}{r}},$$

with the usual adjustments when $q$ or $r$ is infinity. When $q = r$, we abbreviate $L^s_x L^q_t = L^s_{x,t}$. We will also often abbreviate $\|f\|_{L^s_x L^q_t(\mathbb{R}^n)}$ to $\|f\|_{L^s_r}$. For $1 \leq r \leq \infty$, we use $r'$ to denote the dual exponent to $r$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. Throughout the paper, $\chi_E$ is the characteristic function of the set $E$. We usually denote by $B^n_0(a)$ a ball in $\mathbb{R}^n$ with center $a$ and radius $r$. We will also denote by $B^n_R$ a ball of radius $R$ and arbitrary center in $\mathbb{R}^n$. Denote by $A(r) := B^n_0(0) \setminus B^n_{r/2}(0)$.

We define $w_{B^n_R(x_0)}$ to be a nonnegative weight function adapted to the ball $B^n_R(x_0)$ such that

$$w_{B^n_R(x_0)}(x) \lesssim (1 + R^{-1}|x - x_0|)^{-M},$$

for some large constant $M \in \mathbb{N}$.

We define the Fourier transform on $\mathbb{R}^n$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx := \mathcal{F}f(\xi).$$

and the inverse Fourier transform by

$$\hat{g}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi := (\mathcal{F}^{-1}g)(x).$$

These help us to define the fractional differentiation operators $|\nabla|^s$ and $(\nabla)^s$ for $s \in \mathbb{R}$ via

$$|\nabla|^s f(x) := \mathcal{F}^{-1} \left\{ |\xi|^s \hat{f}(\xi) \right\}(x) \quad \text{and} \quad (\nabla)^s f(x) := \mathcal{F}^{-1} \left\{ (1 + |\xi|^2)^s \hat{f}(\xi) \right\}(x).$$

In this manner, we define the Sobolev norm of the space $L^p_0(\mathbb{R}^n)$ by

$$\|f\|_{L^p_0(\mathbb{R}^n)} := \|(|\nabla|^\alpha f\|_{L^p(\mathbb{R}^n)}).$$

Let $\varphi$ be a radial bump function supported on the ball $|\xi| \leq 2$ and equal to 1 on the ball $|\xi| \leq 1$. For $N \in 2\mathbb{Z}$, we define the Littlewood–Paley projection operators by

$$\widetilde{P_{\leq N}}f(\xi) := \varphi(\xi/N)\hat{f}(\xi),$$

$$\widetilde{P_{> N}}f(\xi) := (1 - \varphi(\xi/N))\hat{f}(\xi),$$

$$\widetilde{P_{N}}f(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi).$$

2. PRELIMINARIES

Define the pseudo-differential operator $P$ by

$$Pf(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi) d\xi,$$

where the symbol $p(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies

$$|\partial^\alpha_x \partial^\beta_\xi p(x, \xi)| \leq_{\alpha, \beta} \frac{1}{(1 + |\xi|)^{1+|\beta|}}, \quad \forall \alpha, \beta \in \mathbb{N}^n.$$  

It is well known that the pseudo-differential operator $P$ satisfies the following pseudo-locality property

$$\int_{|x - x_0| \leq 1} |Pf(x)|^2 dx \lesssim_M \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1 + |x - x_0|)^M} dx, \quad \text{for} \ M \geq 0. \quad (2.1)$$

One may refer to [12, Chapter VI] for details. Roughly speaking, the main contribution of $Pf$ in the unit ball about $x_0$ comes from the values of $f(x)$ for $x$ near that ball, in view of the rapidly decaying term $(1 + |x - x_0|)^{-M}$. One may justify (2.1) through integration by parts. In particular, one has

$$\chi_{B^n_R(x_0)} \cdot Pf(x) = P(\chi_{B^n_R(x_0)}f)(x) + \text{RapDec}(r)\|f\|_{L^r},$$
Lemma 2.2. Assume holds \( \psi \) obtained by the following simple observation

\[
\text{Without loss of generality, we may assume that}
\]

\[
\text{Proof.}
\]

We decompose \( e^{it\phi(D)} \) given by

\[
e^{it\phi(D)} f(x) := \int e^{i(x \cdot \xi + t\phi(\xi))} \hat{f}(\xi) d\xi,
\]

where the function \( \phi \) belongs to a class of elliptic phase functions.

**Definition 2.1 (Elliptic phase functions).** For a given \( n \)-tuple consisting of \( n \) dyadic numbers \( A = (A_1, \cdots, A_n) \), we say the smooth function \( \phi \) is of elliptic type \( E_A \), if \( \text{supp} \phi \subset B^\varepsilon(0) \) and satisfies the following conditions

- \( \phi(0) = 0, \nabla \phi(0) = 0. \)
- \( \phi(0) = 0, \nabla \phi(0) = 0. \)

\( \text{Let } \phi \) be a nonnegative smooth function on \( \mathbb{R}^n \) such that

\[
\text{supp } \hat{\psi} \subset B^\varepsilon(0), \sum_{\ell \in \mathbb{Z}^n} \psi(x - \ell) \equiv 1, \quad \forall \, x \in \mathbb{R}^n.
\]

Define \( \psi_t(x) := \psi(R^{-2}x - \ell) \) and \( f_t = \psi_t f \).

**Lemma 2.2.** Assume \( \phi \in E_1 = (1, \cdots, 1) \) and \( \text{supp } \hat{f} \subset B^\varepsilon(0) \). Then, for any \( \varepsilon > 0 \), there holds

\[
|e^{it\phi(D)} f(x)| \lesssim \varepsilon |e^{it\phi(D)}(\Psi_{B_{R^2+\varepsilon}(x_0)} f)(x)| + \text{RapDec}(R) \sum_{|\ell| > R^\varepsilon} \|f_t|(-x_0)|^2\|_{L^p(\supp \psi_{\varepsilon}(x_0))},
\]

for \( (x, t) \in B^\varepsilon_{R^2}(x_0) \times [-R^2, R^2] \), \( 1 < p < \infty \), where

\[
\Psi_{B^\varepsilon_{R^2+\varepsilon}(x_0)}(x) := \sum_{|\ell| \leq R^\varepsilon} \psi(R^{-2}(x - x_0) - \ell).
\]

**Proof.** Without loss of generality, we may assume that \( x_0 = 0 \). The general cases can be obtained by the following simple observation

\[
e^{it\phi(D)} f(x_0) = (e^{it\phi(D)} f(\cdot + x_0))(0).
\]

We rewrite \( e^{it\phi(D)} f \) by (2.2)

\[
e^{it\phi(D)} f(x) = \sum_{\ell \in \mathbb{Z}^n} \int_{\mathbb{R}^n} e^{i(x \cdot y - \xi \cdot \ell + t\phi(\xi))} \eta(\xi) f_t(y) dy d\xi,
\]

where \( \eta(\xi) \in C^\varepsilon(B^\varepsilon(0)) \) with \( \eta(\xi) = 1 \) when \( \xi \in B^\varepsilon(0) \). The associated kernel \( K_t(\cdot) \) of the operator \( e^{it\phi(D)} \eta(-i\nabla) \) is

\[
K_t(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t\phi(\xi))} \eta(\xi) d\xi.
\]

Note that \( |\ell| \leq R^2 \), by stationary phase argument, we obtain

\[
|K_t(x)| \leq C \chi_{|x| \leq C R^2} + C_M \chi_{|x| > C R^2} \frac{\chi_{|x| > C R^2}}{(1 + |x|)^M}.
\]

We decompose \( e^{it\phi(D)} f \) into two parts

\[
e^{it\phi(D)} f(x) = \sum_{|\ell| \leq R^\varepsilon} e^{it\phi(D)} f_t(x) + \sum_{|\ell| > R^\varepsilon} e^{it\phi(D)} f_t(x)
\]

\[
= e^{it\phi(D)}(\Psi_{B^\varepsilon_{R^2+\varepsilon}(0)} f)(x) + \sum_{|\ell| > R^\varepsilon} e^{it\phi(D)} f_t(x).
\]
Now we turn to estimate the second term of the right-hand side of (2.6). By Hölder’s inequality, we have
\[
\left| \sum_{|\ell|>R^e} e^{it\phi(D)} f_\ell(x) \right| \leq \sum_{|\ell|>R^e} \left| K_\ell(x-y) f_\ell(y) dy \right|
\leq \sum_{|\ell|>R^e} \left( \int_{\mathbb{R}^n} |K_\ell(x-y)|^{\frac{2}{p'}} |\psi(y)|^{\frac{2}{p'}} |f(y)|^{\frac{2}{p'}} |K_\ell(x-y)|^{\frac{2}{p'}} dy \right)^{\frac{1}{p'}}
\leq \left( \int_{\mathbb{R}^n} |K_\ell(x-y)|^{\frac{2}{p'}} |f(y)|^{\frac{2}{p'}} |K_\ell(x-y)|^{\frac{2}{p'}} dy \right)^{\frac{1}{p'}}.
\]
For \((x, t) \in B_{R^2}^n(0) \times [-R^2, R^2]\), using the rapidly decaying property of \(K_\ell\) and \(\psi\), we have
\[
|K_\ell(x-y)\psi(y)| \lesssim_M \frac{R^{-eM}}{\left(1 + |R^{-2}y - \ell|\right)^M}, \quad |\ell| > R^e, \forall x \in B_{R^2}^n(0), y \in \mathbb{R}^n,
\]
and
\[
|K_\ell(x-y)| \lesssim_M \frac{1}{\left(1 + \frac{|y|}{R^e}\right)^{M/2}}, \quad \forall x \in B_{R^2}^n(0), y \in \mathbb{R}^n.
\]
Hence,
\[
\left| \sum_{|\ell|>R^e} e^{it\phi(D)} f_\ell(x) \right| \lesssim_M R^{-eM + \frac{1}{p'}} \sum_{|\ell|>R^e} \| f \|_{L^p(\mathcal{W}_n_R^e(0))}^{\frac{1}{p'}} \| \psi f \|_{L^p(\mathcal{W}_n_R^e(0))}^{\frac{1}{p'}}.
\]
Therefore, we complete the proof.

As a direct consequence of Lemma 2.2, we immediately obtain the relation between local and global estimates in the spatial space.

**Corollary 2.3.** Let \(\phi \in E_1, s \in \mathbb{R}, 2 < p < \infty\) and \(I\) be an interval with \(I \subset [-R^2, R^2]\). Suppose that \(\text{supp} \; \hat{f} \subset B_1^n(0)\) and
\[
\| e^{it\phi(D)} f \|_{L^p_x(B_{R^2}^n \times I)} \leq CR^s \| f \|_{L^p},
\]
then, \(\forall \varepsilon > 0\), there holds
\[
\| e^{it\phi(D)} f \|_{L^p_x(B_{R^2}^n \times I)} \lesssim \varepsilon R^{s+\varepsilon} \| f \|_{L^p}.
\]

**Proof.** Let \(\{B_{R^2}^n(x_k)\}_{k \in \mathbb{Z}^n}\) with \(x_k = kR^2\) be a family of finitely overlapping balls which cover \(\mathbb{R}^n\). Thus
\[
\| e^{it\phi(D)} f \|_{L^p_x(B_{R^2}^n \times I)}^p \leq \sum_k \| e^{it\phi(D)} f \|_{L^p_x(B_{R^2}^n(x_k) \times I)}^p.
\]
Using Lemma 2.2, we get
\[
\| e^{it\phi(D)} f \|_{L^p_x(B_{R^2}^n(x_k) \times I)}^p \lesssim \| e^{it\phi(D)} (\Phi_{B_{R^2+1/10^n}^n(x_k)} f) \|_{L^p_x(B_{R^2}^n(x_k) \times I)}^p + \text{RapDec}(R) \sum_{|\ell|>R^e} \| f |\psi f | \|_{L^p(\mathcal{W}_n_R^e(0))}^p\|_{L^p(\mathcal{W}_n_R^e(0))}^p.
\]
We take the summation with respect to \(k\), and obtain
\[
\sum_k \| e^{it\phi(D)} f \|_{L^p_x(B_{R^2}^n(x_k) \times I)}^p \lesssim \sum_k \left( \left( \left| e^{it\phi(D)} (\Phi_{B_{R^2+1/10^n}^n(x_k)} f) \|_{L^p_x(B_{R^2}^n(x_k) \times I)} \right)^p + \text{RapDec}(R) \sum_{|\ell|>R^e} \| f |\psi f | \|_{L^p(\mathcal{W}_n_R^e(0))}^p \right)^p.
\]
It follows from (2.7)\(^1\) and the bounded overlapping property of the balls \(\{B^n_R(x_k)\}_k\) that
\[
\sum_k \left( \|e^{it\phi(D)}(\Psi^{B^n_R(x_k)}f)\|_{L^p_x(B^n_R(x_k) \times I)} \right)^p \lesssim R^{sp+p}\|f\|_{L^p}^p.
\]
As for the error term, using Minkowski’s inequality and the separation property of \(x_k\), we obtain
\[
\text{RapDec}(R) \sum_k \left( \sum_{|\xi| > R} \|f|\hat{\psi}(\cdot - x_k)\|_{L^p_w(w_{B^n_R(x_k)})} \right)^p \lesssim \text{RapDec}(R)\|f\|_{L^p}^p.
\]
Thus we complete the proof of Corollary 2.3. \(\square\)

For later use, we also need the following lemma concerning the eigenvalues of the Hessian matrix of a radially symmetric function. One can carry out the approach in [4] to obtain the following Lemma.

**Lemma 2.4.** Let \(\phi = \phi(|x|)\) be a radially symmetric \(C^2\) function on \(R^n \setminus \{0\}, n \geq 2\). Then the determinant of the Hessian matrix is
\[
\det \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{n \times n} = \left( \frac{\phi'(r)}{r} \right)^{n-1} \phi''(r).
\]
Furthermore, the eigenvalues of the Hessian matrix are
\[
\frac{\phi'(r)}{r}, \quad \phi''(r),
\]
\(\text{(n-1)-fold}\)

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Bourgain-Guth [3] have developed a strategy to convert \(k\)-linear into linear inequalities in the context of the Fourier extension operators. In [7], Guth observed that full power of the \(k\)-linear inequality could be replaced by a certain weakened version of the multilinear estimate for the Fourier extension operators known as \(k\)-broad “norm” estimates. Following the approach developed by Guth in [7], we shall divide \(e^{it(-\Delta)^{\alpha/2}}f\) into narrow and broad parts in the frequency space, and one part is around a neighborhood of \((k-1)\)-dimensional subspace, another comes from its outside. We estimate the contribution of the first part through the decoupling theorem and an induction on scales argument, and then use the \(k\)-broad “norm” estimates to handle the broad part. In this process, we should take advantage of the pseudo-locality property of the fractional Schrödinger operators. It is worth noting that, for \(\alpha > 1\), we can incorporate the phase functions \(|\xi|^\alpha\) into the class of “elliptic phase functions”.

In order to prove Theorem 1.2, we will use the following \(k\)-broad “norm” estimates of Guth-Hickman-Iliopoulo\( [8]\). For some \(0 < \varepsilon < 1\), let \(1 \ll K \ll R^\varepsilon\). We assume that \(\theta, \tau\) are balls in \(R^n\) of radius \(R^{-1}\) and \(K^{-1}\), respectively. Correspondingly, we define \(G(\theta)\) and \(G(\tau)\) to be the set of unit normal vectors as follows:
\[
G(\theta) := \left\{ \frac{1}{\sqrt{1 + |\nabla \phi|^2}}(-\nabla \phi(\xi), 1) : \xi \in \theta \right\}, \quad G(\tau) := \bigcup_{\theta \subset \tau} G(\theta).
\]
Let \(V \subset \mathbb{R}^{n+1}\) be a \((k-1)\)-dimensional subspace. We denote by \(\text{Ang}(G(\tau), V)\) the smallest angle between the non-zero vectors \(v \in V\) and \(v' \in G(\tau)\).

Define
\[
f_{\tau} := F^{-1}(f\chi_{\tau}).
\]

\(^1\)It should be noted that the Fourier support condition of \(f\) may not be satisfied. However, it can be easily fixed by dividing \(B^0_1(0)\) into several smaller balls and considering the estimate on each of these smaller balls.
For each ball $B_{K^2}^{n+1} \subset B_{R_0}^{n} \times [-R^2, R^2]$, define
\[
\mu_\phi(B_{K^2}^{n+1}) := \min_{V_1, \ldots, V_L} \max_{\tau \not\in V_t} \left( \int_{B_{K^2}^{n+1}} |e^{it\phi(D)} f_\tau|^p \, dx \, dt \right),
\]
where $\tau \not\in V_t$ means that for all $1 \leq \ell \leq L$, $\text{Ang}(G(\tau), V_t) > K^{-1}$.

Let $\{B_{K^2}^{n+1}\}$ be a collection of finitely overlapping balls which form a cover of $B_{R_0}^{n} \times [-R^2, R^2]$. In this setting, we define the $k$-brood “norm” by
\[
\left\| e^{it\phi(D)} f \right\|_{\text{BL}_k^{p, L}(B_{R_0}^{n} \times [-R^2, R^2])} := \sum_{B_{K^2}^{n+1} \subset B_{R_0}^{n} \times [-R^2, R^2]} \mu_\phi(B_{K^2}^{n+1}).
\]

**Theorem 3.1** ([8]). Let $2 \leq k \leq n + 1$ and $\phi \in E_A$. There exists a large constant $L$ such that
\[
\left\| e^{it\phi(D)} f \right\|_{\text{BL}_k^{p, L}(B_{R_0}^{n} \times [-R^2, R^2])} \lesssim_{A, \epsilon, L} R^{2n(\frac{1}{2} - \frac{k}{p}) + \epsilon} \left\| f \right\|_{L^p(B^n)}
\]
for all $\epsilon > 0$ and $p \geq 2(n + k + 1)/(n + k - 1)$.

As a direct consequence of Theorem 3.1, we obtain

**Corollary 3.2.** Let $2 \leq k \leq n + 1$, $\epsilon > 0$ and $\phi \in E_A$. There is a large constant $L$ such that
\[
\left\| e^{it\phi(D)} f \right\|_{\text{BL}_k^{p, L}(B_{R_0}^{n} \times [-R^2, R^2])} \lesssim_{A, \epsilon, L} R^{2n(\frac{1}{2} - \frac{k}{p}) + \epsilon} \left\| f \right\|_{L^p(B^n)},
\]
for all $f$ with $\supp \hat{f} \subset B_{R_0}^{n}(0)$ and $p \geq 2(n + k + 1)/(n + k - 1)$.

**Proof.** It follows from (2.3) that for $(x, t) \in B_{R_0}^{n} \times [-R^2, R^2]$
\[
e^{it\phi(D)} f(x) = \int_{\mathbb{R}^n} e^{it\xi} \eta(\xi) \hat{f}(\xi) \, d\xi = \int_{\mathbb{R}^n} e^{ix \xi} \eta(\xi) (\widehat{\Psi_{B_{R_0}^{n+1}(0)}} \hat{f})(\xi) \, d\xi + \text{RapDec}(R) \left\| f \right\|_{L^p},
\]
where $\eta(\xi)$ is the same as in (2.4). By Theorem 3.1 and Hölder’s inequality, we obtain the desired estimate. \(\square\)

We also need the following decoupling theorem due to Bourgain-Demeter [2].

**Theorem 3.3** (Decoupling theorem). Let $\phi \in E_A$, then
\[
\left\| \sum_{\tau} e^{it\phi(D)} f_\tau \right\|_{L^p(B_{R_0}^{n+1})} \lesssim_{A, \delta} R^{n(\frac{1}{2} - \frac{1}{p}) + \delta} \left( \sum_{\tau} \left\| e^{it\phi(D)} f_\tau \right\|_{L^p(B_{R_0}^{n+1})} \right)^{\frac{1}{p}},
\]
for $2 \leq p \leq \frac{2(n+2)}{n}$ and $\delta > 0$.

As a consequence of Theorem 3.3, we have

**Lemma 3.4.** Let $\phi \in E_A$ and $V \subset \mathbb{R}^{n+1}$ be a $(k-1)$-dimensional linear subspace, then
\[
\left\| \sum_{\tau \in V} e^{it\phi(D)} f_\tau \right\|_{L^p(B_{R_0}^{n+1})} \lesssim_{A, \delta} R^{n(\frac{k-2}{2} - \frac{1}{p}) + \delta} \left( \sum_{\tau \in V} \left\| e^{it\phi(D)} f_\tau \right\|_{L^p(B_{R_0}^{n+1})} \right)^{\frac{1}{p}},
\]
for $2 \leq p \leq \frac{2k}{k-2}$ and $\delta > 0$. Here the sum is taken over all caps $\tau$ for which $\text{Ang}(G(\tau), V) \leq K^{-1}$.

For the proof of Lemma 3.4, one may refer to [7] for the details.

**Parabolic rescaling.** For given $\phi \in E_A$, we denote $Q_A(R)$ to be the optimal constant such that
\[
\left\| e^{it\phi(D)} f \right\|_{L^p_{t, x}(B_{R_0}^{n+1} \times [-R^2, R^2])} \leq Q_A(R) R^{2n(\frac{1}{2} - \frac{1}{p})} \left\| f \right\|_{L^p(\mathbb{R}^n)}, \quad \supp \hat{f} \subset B_{R_0}^{n}(0).
\]

The parabolic rescaling transformation establishes the bridge among the estimates at different scales, which enables us to use an induction on scales argument. We utilize the pseudo-locality property of the propagator $e^{it\phi(D)}$ to establish a parabolic rescaling in our setting.
Lemma 3.5 (Parabolic rescaling). Suppose that $\phi \in E_1$, and $\tau \subset \mathbb{R}^n$ is a ball with radius $K^{-1}$, then for $0 < \varepsilon \ll 1$, we have

$$
\left\| e^{it\phi(D)} f_\tau \right\|_{L^p(B^n_{R^2} \times [-R^2, R^2])} \leq C(\varepsilon)K^{-2n\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{\varepsilon}{2}}Q_1\left(\frac{R}{K}\right)R^{2n\left(\frac{1}{2} - \frac{1}{p}\right) + \varepsilon} \left\| f_\tau \right\|_{L^p + \text{RapDec}(R)} \left\| f \right\|_{L^p}.
$$

(3.5)

Proof. Suppose that supp $\hat{f}_\tau \subset B^n_{K^{-1}}(\xi)$, and denote $\tilde{\phi}(\xi)$ by

$$
\tilde{\phi}(\xi) := \phi(\xi) - \phi(\xi_{\tau}) - \nabla \phi(\xi_{\tau}) \cdot \xi,
$$

then

$$
\left| e^{it\phi(D)} f_\tau(x) \right| = \left| \int_{B^n_{K^{-1}}(\xi)} e^{i(x \cdot \xi + t\tilde{\phi}(\xi))} \hat{f}_\tau(\xi) d\xi \right|
= \left| \int_{B^n_{K^{-1}}(\xi)} e^{i(x \cdot \xi + t\tilde{\phi}(\xi))} \hat{f}_\tau(\xi) d\xi \right|.
$$

(3.6)

Under an invertible map $\Phi: (x, t) \to (y, s)$, i.e.

$$
x + t\nabla \phi(\xi_{\tau}) \to y, \quad t \to s,
$$

(3.6) can be reduced to dealing with

$$
\left| \int_{B^n_{K^{-1}}(\xi)} e^{i(y \cdot \xi + s\tilde{\phi}(\xi))} \hat{f}_\tau(\xi) d\xi \right|.
$$

(3.7)

By the change of variables

$$
\xi \to K^{-1} \xi + \xi_{\tau},
$$

(3.7) is further reduced to estimating

$$
\left| K^{-n} \int_{B^n_{1}(0)} e^{i(K^{-1}y \cdot \xi + K^{-2}s\tilde{\phi}(\xi))} \hat{f}_\tau(\xi) d\xi \right|,
$$

where $\tilde{f}_\tau(\cdot) = e^{-iK\xi_{\tau}}K^n f_\tau(K \cdot)$ and

$$
\tilde{\phi}_\tau(\xi) = K^2(\phi(\xi + K^{-1}\xi_{\tau}) - \phi(\xi_{\tau}) - K^{-1}\nabla \phi(\xi_{\tau}) \cdot \xi).
$$

Thus, we have

$$
\left\| e^{it\phi(D)} f_\tau \right\|_{L^p(B^n_{R^2} \times [-R^2, R^2])} \lesssim K^{-np}\left\| (e^{iku^{2}s\tilde{\phi}_\tau(D)} \hat{f}_\tau(K^{-1})) \right\|_{L^p(\Phi(B^n_{R^2} \times [-R^2, R^2]))}.
$$

(3.8)

After the change of variables: $y \to K\tilde{x}, s \to K^2\tilde{s}$, we denote by $\tilde{\Phi}(B^n_{R^2} \times [-R^2, R^2])$ the transformed region from $\Phi(B^n_{R^2} \times [-R^2, R^2])$. Note that $\tilde{\Phi}(B^n_{R^2} \times [-R^2, R^2])$ can be contained in a cylinder of the type $B^n_{C,R^2/K^4} \times [-C R^2/K^2, C R^2/K^2]$. Thus we may construct a class of cylinders $\{B_\gamma\}$, such that

$$
\tilde{\Phi}(B^n_{R^2} \times [-R^2, R^2]) \subset \bigcup_\gamma B_\gamma, \quad B_\gamma =: B^n_{C,R^2/K^4}(c_\gamma) \times [-C R^2/K^2, C R^2/K^2].
$$

Define

$$
\tilde{f}_{\gamma, \tau} := \Psi_{B^n_{(R/K)^{2\gamma + 1/n}}(c_\gamma)} \tilde{f}_\tau,
$$

where $\Psi$ is the function introduced in Lemma 2.2.

In order to perform the induction on scales argument, we should verify that the phase function $\tilde{\phi}_\tau$ belongs to the elliptic class $E_1$. Obviously, $\tilde{\phi}_\tau(0) = 0$, and the Hessian of $\tilde{\phi}_\tau$ is

$$
\left( \frac{\partial^2 \tilde{\phi}}{\partial \xi_i \partial \xi_j} \right)_{n \times n}(K^{-1} \xi + \xi_{\tau}).
Noting that $\phi \in E_1$ and $K^{-1}\xi + \xi_\tau \in \text{supp } \phi$ for $\xi \in B_1^n(0)$, we have $\tilde{\phi}_\tau \in E_1$. Hence, using Lemma 2.2, we have
\[
K^{-np}\|e^{itK^{-2}\tilde{\phi}_\tau(D)}\tilde{f}(K^{-1})\|_{L^p(B_1^n(0))}^p \lesssim K^{-(n + \frac{2p}{p - 1})p} \sum_\gamma \|e^{it\tilde{\phi}_\tau(D)}\tilde{f}\|_{L^p(B_1^n)}^p
\]
\[
\lesssim \epsilon K^{-(n + \frac{2p}{p - 1})p} \sum_\gamma \|e^{it\tilde{\phi}_\tau(D)}\tilde{f}\|_{L^p(B_1^n)}^p \lesssim \epsilon^{2p} \left(\frac{R}{K}\right)^{2p(\frac{1}{p} - \frac{1}{2})} \left(\frac{R}{K}\right)^{\frac{1}{2}} \sum_\gamma \|f\|_{L^p(B_1^n)}^p + \text{RapDec}(R) \|f\|_{L^p}^p
\]
\[
\lesssim \epsilon K^{-(n + \frac{2p}{p - 1})p} \left(\frac{R}{K}\right)^{2p(\frac{1}{p} - \frac{1}{2})} \left(\frac{R}{K}\right)^{\frac{1}{2}} \sum_\gamma \|f\|_{L^p(B_1^n)}^p + \text{RapDec}(R) \|f\|_{L^p}^p.
\]
This inequality together with (3.8) yields (3.5).

We come back to prove Theorem 1.2. We first claim that (1.11) can be reduced to showing for $R \geq 1$,
\[
\|e^{it(-\Delta)\frac{p}{p - 1}} f\|_{L^p_{x,t}(B^{n}_{R^2} \times [-R^2, R^2])} \lesssim_{\alpha, \varepsilon} R^{2n(\frac{1}{p} - \frac{1}{2}) + \varepsilon} \|f\|_{L^p(R^n)}, \quad \text{supp } \tilde{f} \subseteq B_1^n(0),
\]
(3.9)
where $p$ is as in (1.10).

Indeed, by Littlewood-Paley decomposition,
\[
e^{it(-\Delta)^{\frac{p}{2}}} f = e^{it(-\Delta)^{\frac{p}{2}}} P_1 f + \sum_{N > 1} e^{it(-\Delta)^{\frac{p}{2}}} P_N f,
\]
and using the fixed-time estimate (1.3), we easily conclude
\[
\|e^{it(-\Delta)^{\frac{p}{2}}} P_1 f\|_{L^p_{x,t}(R^n \times [1, 2])} \lesssim \|P_1 f\|_{L^p_{x,t}(R^n)} \lesssim \|f\|_{L^p(R^n)},
\]
(3.10)
Now we come to estimate $e^{it(-\Delta)^{\frac{p}{2}}} P_N f$ with $N > 1$. For $R \geq 1$, by Corollary 2.3, (3.9), we have
\[
\|e^{it(-\Delta)^{\frac{p}{2}}} g\|_{L^p_{x,t}(R^n \times [1, 2])} \lesssim_{\alpha, \varepsilon} R^{2n(\frac{1}{p} - \frac{1}{2}) + \varepsilon} \|g\|_{L^p(R^n)}, \quad \text{supp } \tilde{g} \subseteq A(1).
\]
Therefore, we obtain
\[
\|e^{it(-\Delta)^{\frac{p}{2}}} P_N f\|_{L^p_{x,t}(R^n \times [1, 2])} \lesssim_{\alpha, \varepsilon} R^{2n(\frac{1}{p} - \frac{1}{2}) + \varepsilon} N^{-\frac{p}{2n}} \|P_N f\|_{L^p(R^n)}.
\]
Setting $R = \sqrt{2} N^{\alpha/2}$, we have
\[
\|e^{it(-\Delta)^{\frac{p}{2}}} P_N f\|_{L^p_{x,t}(R^n \times [1, 2])} \lesssim_{\alpha, \varepsilon} N^{\alpha n(\frac{1}{p} - \frac{1}{2})} \|P_N f\|_{L^p(R^n)}.
\]
(3.11)
This estimate together with (3.10) implies that for $s > s_{\alpha, p} - \frac{\alpha}{p}$,
\[
\|e^{it(-\Delta)^{\frac{p}{2}}} f\|_{L^p_{x,t}(R^n \times [1, 2])} \leq \|e^{it(-\Delta)^{\frac{p}{2}}} P_1 f\|_{L^p_{x,t}(R^n \times [1, 2])} + \sum_{N > 1} \|e^{it(-\Delta)^{\frac{p}{2}}} P_N f\|_{L^p_{x,t}(R^n \times [1, 2])}
\]
\[
\lesssim_{\alpha, \varepsilon} f\|_{L^p(R^n)} + \sum_{N > 1} N^{\alpha n(\frac{1}{p} - \frac{1}{2})} \|P_N f\|_{L^p(R^n)}
\]
\[
\lesssim_{\alpha, \varepsilon} f\|_{L^p(R^n)},
\]
and so we verify the above claim. This proves Theorem 1.2 under the assumption of (3.9).

It remains to prove (3.9). We first observe that (3.9) can be deduced from
\[
\|e^{it\phi(D)} f\|_{L^p_{x,t}(B^{n}_{R^2} \times [-R^2, R^2])} \lesssim \epsilon \left(\frac{R}{K}\right)^{2p(\frac{1}{p} - \frac{1}{2}) + \varepsilon} \|f\|_{L^p(R^n)} , \quad \text{supp } \tilde{f} \subseteq B_1^n(0),
\]
(3.12)
where the phase function $\phi(\xi) \in E_1$. 

Note that for $\alpha > 1, \xi \neq 0$, by Lemma 2.4, one easily sees that the eigenvalues of the Hessian matrix of $|\xi|^\alpha$ are
\[ \alpha^2 \sum_{(n-1)\text{-fold}} |\xi|^{\alpha-2}, \quad \alpha(\alpha - 1)|\xi|^{\alpha-2}. \]
Obviously, the phase function $|\xi|^\alpha, \xi \in A(1)$ may not belong to the $E_1$. This problem can be fixed by decomposing $A(1)$ into a series of sufficiently small pieces and making appropriate affine transformations.

Now we show that (3.12) can be deduced from the following proposition.

**Proposition 3.6.** Let $\phi \in E_1$. Suppose that
\[ \|e^{it\phi(D)}f\|_{L^p(\mathbb{R}^n \setminus [-r, r]^2)} \lesssim K \varepsilon R^{2n(\frac{1}{2} - \frac{1}{p}) + \varepsilon}\|f\|_{L^p}, \]
for all $K \geq 1, \varepsilon > 0$ and
\[ \frac{2n - k + 4}{2n - k + 2} < p \leq \frac{2k}{k - 2}, \]
then
\[ \|e^{it\phi(D)}f\|_{L^p(\mathbb{R}^n \setminus [-r, r]^2)} \lesssim \varepsilon R^{2n(\frac{1}{2} - \frac{1}{p}) + \varepsilon}\|f\|_{L^p}. \]

**Proof of (3.12).** By Corollary 3.2 and Proposition 3.6, we obtain (3.12) if
\[ p > \min_{2 \leq k \leq n+1} \max \left\{ \frac{2n + k + 1}{n + k - 1}, \frac{2n - k + 4}{2n - k + 2} \right\}. \]
In particular, if we choose
\[ k = \begin{cases} \frac{n+3}{2}, & n \text{ is odd}, \\ \frac{n+4}{2}, & n \text{ is even}, \end{cases} \]
then we obtain the optimal range as in (1.10).

**The proof of Proposition 3.6.** Let $r > 0$, for the sake of convenience, we denote $C_{n+1}^k$ to be the cylinder $B_{2}^n \times [-r, r]$. For a given ball $B_{2}^{n+1} \subset C_{n+1}^k$, assume that a choice of $(k-1)$-dimensional subspaces $V_1 \ldots V_L$, which achieves the minimum in the definition of the $k$-broad “norm”, we obtain
\[ \int_{B_{2}^{n+1}} |e^{it\phi(D)}f(x)|^p \, dx \, dt \lesssim K^{O(1)} \max_{\tau \notin V_1} \int_{B_{2}^{n+1}} |e^{it\phi(D)}f_\tau(x)|^p \, dx \, dt + \sum_{\ell=1}^{L} \int_{B_{2}^{n+1}} \left| \sum_{\tau \in V_\ell} e^{it\phi(D)}f_\tau(x) \right|^p \, dx \, dt. \]
Summing over the balls $\{B_{2}^{n+1}\}$ yields
\[ \int_{C_{n+1}^k} |e^{it\phi(D)}f(x)|^p \, dx \, dt \lesssim K^{O(1)} \sum_{B_{2}^{n+1} \subset C_{n+1}^k} \min_{V_1 \ldots V_L} \max_{\tau \notin V_1} \int_{B_{2}^{n+1}} |e^{it\phi(D)}f_\tau(x)|^p \, dx \, dt + \sum_{\ell=1}^{L} \int_{B_{2}^{n+1}} \left| \sum_{\tau \in V_\ell} e^{it\phi(D)}f_\tau(x) \right|^p \, dx \, dt. \] \quad (3.13)
Now we use (3.2) to estimate the contribution of the first term in the right-hand side of (3.13). Let $\varepsilon > 0$ be determined later. Using Corollary 3.2, we have
\[ \sum_{B_{2}^{n+1} \subset C_{n+1}^k} \min_{V_1 \ldots V_L} \max_{\tau \notin V_1} \int_{B_{2}^{n+1}} |e^{it\phi(D)}f_\tau(x)|^p \, dx \, dt \lesssim C(\varepsilon, L, K) R^{2np(\frac{1}{2} - \frac{1}{p}) + \varepsilon} \|f\|_{L^p}^p. \quad (3.14) \]
Now we use Lemma 3.4 and parabolic rescaling as in Lemma 3.5 to estimate the contribution of the second term in the right-hand side of (3.13). Let $\delta > 0$ to be chosen later. It follows from Lemma 3.4 that
\[ \sum_{\ell=1}^{L} \int_{B_{2}^{n+1}} \left| \sum_{\tau \in V_\ell} e^{it\phi(D)}f_\tau(x) \right|^p \, dx \, dt \lesssim C(\delta, L) K^{\delta} K^{(k-2)(\frac{1}{2} - \frac{1}{p})} \sum_{\tau} \int_{B_{2}^{n+1}} |e^{it\phi(D)}f_\tau(x)|^p \, dx \, dt. \]
Summing over $B^{n+1}_{K^2}$ in both sides of the above inequality, we obtain

$$\sum_{B^{n+1}_{K^2}} \sum_{\ell=1}^L \int_{B^{n+1}_{K^2}} \left| \sum_{\tau \in V_{\ell}} e^{it\phi(D)} f_{\tau}(x) \right|^p \, dx \, dt \leq C(L, K) R^{\delta} K^{(k-2)(\frac{4}{p} - \frac{1}{2})} \sum_{\tau} \int_{\mathbb{R}^{n+1}} w_{C^{n+1}} \left| e^{it\phi(D)} f_{\tau}(x) \right|^p \, dx \, dt. \tag{3.15}$$

Using the rapidly decaying property of the weight function, we have

$$\int_{\mathbb{R}^{n+1}} w_{C^{n+1}} \left| e^{it\phi(D)} f_{\tau}(x) \right|^p \, dx \, dt \leq \int_{C^{n+1}} \left| e^{it\phi(D)} f_{\tau}(x) \right|^p \, dx \, dt + \text{RapDec}(R) \| f \|_p^p.$$

Choosing $\varepsilon_1 > 0$, we obtain by Lemma 3.5

$$\int_{C^{n+1}} \left| e^{it\phi(D)} f_{\tau}(x) \right|^p \, dx \, dt \leq \varepsilon_1 K^{-2n(\frac{4}{p} - \frac{1}{2}) p + 2 - \epsilon_1} Q_1^p \left( \frac{R^{1+\delta}}{K} \right) R^{2(1+\delta)n p (\frac{4}{p} - \frac{1}{2}) + \epsilon_1} \| f \|_p^p + \text{RapDec}(R) \| f \|_p^p. \tag{3.16}$$

Summing over $\tau$ and noting that

$$\sum_{\tau} \| f_{\tau} \|_p^p \leq C \| f \|_{L^p}, \quad \text{for } 2 \leq p \leq \infty,$$

we have

$$\sum \int_{\mathbb{R}^{n+1}} w_{C^{n+1}} \left| e^{it\phi(D)} f_{\tau}(x) \right|^p \, dx \, dt \leq \varepsilon_1 K^{-2n(\frac{4}{p} - \frac{1}{2}) p + 2 - \epsilon_1} Q_1^p \left( \frac{R^{1+\delta}}{K} \right) R^{2(1+\delta)n p (\frac{4}{p} - \frac{1}{2}) + \epsilon_1} \| f \|_p^p + \text{RapDec}(R) \| f \|_p^p. \tag{3.16}$$

Collecting the estimates (3.14)-(3.16) and inserting them into (3.13), we obtain

$$\int_{C^{n+1}} \left| e^{it\phi(D)} f(x) \right|^p \, dx \, dt \leq C(\varepsilon, L, K) R^{2n p (\frac{4}{p} - \frac{1}{2}) + \epsilon/2} \| f \|_{L^p}^p + C(\delta, \varepsilon_1, L) K^{\delta} R^{2(1+\delta)n p (\frac{4}{p} - \frac{1}{2}) + \epsilon_1} K^{-\epsilon(p,k,n) - \epsilon_1} Q_1^p \left( \frac{R^{1+\delta}}{K} \right) \| f \|_{L^p}^p,$$

where

$$\epsilon(p,k,n) := -(k-2) \left( \frac{1}{p} - \frac{1}{2} \right) p + 2n \left( \frac{1}{2} - \frac{1}{p} \right) p - 2 > 0, \quad p > \frac{2n - k + 4}{2n - k + 2}.$$

Therefore by the definition of $Q_1(R)$, we have

$$Q_1^p(R) \leq C(\varepsilon, L, K) R^{\hat{\delta} p} + C(\delta, \varepsilon_1, L) K^{\hat{\delta} p} R^{2 \delta n p (\frac{4}{p} - \frac{1}{2}) + \epsilon_1} K^{-\epsilon(p,k,n) - \epsilon_1} Q_1^p \left( \frac{R^{1+\hat{\delta}}}{K} \right).$$

Fix $p > \frac{2n - k + 4}{2n - k + 2}$, let $K = K_0 R^{\hat{\delta}}$ with $K_0 > 0$ being a large constant to be chosen later, and

$$\delta = \varepsilon_1/2, \quad \hat{\delta} = \frac{2 \varepsilon_1 (1 + np(1/2 - 1/p))}{\epsilon(p,k,n) + \varepsilon_1/2}, \quad 0 < \varepsilon_1 < \frac{2 \epsilon(p,k,n)}{4 + 4np(\frac{1}{2} - \frac{1}{p})} - \epsilon$$

such that the resulting power of $R$ is negative and $0 < \hat{\delta} < \varepsilon$.

Recall that, in the process of estimating the broad part, the constant $C(\varepsilon, L, K)$ grows at most polynomially with respect to $K$, by choosing $1 \ll K_0$ such that

$$C(\delta, \varepsilon_1, L) K_0^{\delta - \epsilon(p,k,n) - \varepsilon_1} < \frac{1}{2},$$

it follows

$$Q_1(R) \lesssim \varepsilon R^\varepsilon.$$

Thus we finish the proof of Proposition 3.6. \qed
Further tractable approach. The result in Theorem 1.2 relies on the following sharp $k$-broad “norm” estimates in [8]: Let $p \geq \frac{2(n+k+1)}{n+k-1}$, supp $f(\xi) \subset A(1)$, then

$$
\|e^{it(-\Delta)^{\frac{\alpha}{2}}}f\|_{BL^p_{x,L}(B^m_{\infty} \times [-R^2, R^2])} \lesssim_{\alpha, \varepsilon} R^\alpha \|\hat{f}\|_{L^p}, \quad \alpha > 1. \tag{4.1}
$$

Using the pseudo-locality property of the operator $e^{it(-\Delta)^{\frac{\alpha}{2}}}$ with $\alpha > 1$ and Hölder’s inequality, we have

$$
\|e^{it(-\Delta)^{\frac{\alpha}{2}}}f\|_{BL^p_{x,L}(B^m_{\infty} \times [-R^2, R^2])} \lesssim_{\alpha, \varepsilon} R^{2n(\frac{1}{p} - \frac{1}{2}) + \varepsilon} \|f\|_{L^p(R^n)}, \quad \alpha > 1. \tag{4.2}
$$

To improve the result in Theorem 1.2, we expect to establish (4.2) directly for some $p < \frac{2(n+k+1)}{n+k-1}$.

Remark on the local smoothing of the half-wave operator. There are some troubles in generalizing the above method to handle the local smoothing estimates of the half-wave operator $e^{it\sqrt{-\Delta}}$, which is of great interest. For the restriction problem of the cone operator, using the Lorentz transformation, we can reduce to considering

$$
\int_{\mathbb{R}^n} e^{i(x_1^2 + \cdots + x_n^2 + x_{n+1}^2 + \frac{x_{n+1}^2}{2x_n})} \eta(\xi) \hat{f}(\xi) d\xi,
$$

of which the structure is well suited for the rescaling argument. But, for the local smoothing problem, we can’t establish the corresponding parabolic rescaling lemma in this setting. In fact, the relationship between $e^{it\sqrt{-\Delta}}f$ and

$$
\int_{\mathbb{R}^n} e^{i(x_1^2 + \cdots + x_n^2 + x_{n+1}^2 + \frac{x_{n+1}^2}{2x_n})} \eta(\xi) \hat{f}(\xi) d\xi
$$

is uncertain for us.

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