Can Anyon statistics explain high temperature superconductivity?

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Abstract
In this paper, we find a reasonable explanation of high temperature superconductivity phenomena using Anyon statistics.

1 Introduction

Superconductivity is a phenomena occurring in certain materials when cooled below some characteristic temperature called a critical temperature. In a superconductor, the resistance drops suddenly to zero when the material is cooled below its critical temperature. This phenomena was first discovered by the dutch physicist Heike Kamerlingh Onnes, which received the Nobel prize of physics in 1913 for discovering this phenomena and the related work regarding liquefaction of helium [1]. There is another phenomena related to superconductor called (Meissner effect), which is the expulsion of a magnetic field from a superconductor during its transition to the superconducting state. This phenomena was discovered in 1933 by Walther Meissner and Robert Ochsenfeld [2]. Although the microscopic theory of superconductivity was developed by Bardeen, Cooper and Schrieffer (BCS theory) in 1957 [3], other descriptions of the phenomena of the superconductivity were developed [4-6].

Whereas ordinary superconductors usually have transition temperatures below 30 Kelvin, It is well known that some materials behave as superconductors at unusually high critical temperatures and this materials called high temperature superconductors or (HTS). The first HTS was discovered in 1986 by Georg Bednorz and K. Alex Muller, who were awarded the 1987 Nobel Prize in Physics [7].

Although the standard BCS theory (weak-coupling electron–phonon interaction) can explain all known low temperature superconductors and even some high temperature superconductors, some of high temperature superconductors cannot be explained by BCS theory and some physicist believe that they obey a strong interaction.
In this paper, we find that if we use a unified statistics [8] instead Fermi-Dirac statistics, then we can find a reasonable explanation of all high temperature superconductivity phenomena using a standard BCS theory.

2 The model

We follow the derivation of Hamiltonian model using the mean field approximation and Bogoliubov unitary transformation as in the reference [9], where we can write the following relation,

\[ |\Delta| = |V| \sum_k \frac{|\Delta|}{2\sqrt{(\epsilon(k) - \mu)^2 + \Delta^2}} [f(-E_K) - f(E_K)], \]  

(1)

where \( f(E_K) \) determined from the unified statistics as follow [8],

\[ \left( \frac{1}{p f(E_k)} + 1 \right)^{1-p} \left( \frac{1}{p f(E_k)} - 1 \right)^p = e^{\frac{\sqrt{(\epsilon(k) - \mu)^2 + \Delta^2}}{k_B T}}. \]  

(2)

3 BCS Theory ( \( p \rightarrow 1 \))

For \( p \rightarrow 1 \), we recover the usual BCS Theory.

When \( p \rightarrow 1 \), we have the following usual form of the Fermi function \( f(E_K) \),

\[ f(E_K) = \frac{1}{\sqrt{(\epsilon - \mu)^2 + \Delta^2}}, \]  

(3)

so, we have,

\[ 1 = |V| \sum_k \int \frac{1}{2\sqrt{(\epsilon(k) - \mu)^2 + \Delta^2}} \left[ \frac{1}{1 + e^{\frac{-\sqrt{(\epsilon(k) - \mu)^2 + \Delta^2}}{k_B T}}} - \frac{1}{1 + e^{\frac{\sqrt{(\epsilon(k) - \mu)^2 + \Delta^2}}{k_B T}}} \right]. \]  

(4)

Replacing the summation by integration over the energy states, we have,

\[ 1 = |V| \int_{\mu - \epsilon_D}^{\mu + \epsilon_D} \frac{N(0) de}{2\sqrt{(\epsilon - \mu)^2 + \Delta^2}} \left[ \frac{1}{1 + e^{\frac{-\sqrt{(\epsilon - \mu)^2 + \Delta^2}}{k_B T}}} - \frac{1}{1 + e^{\frac{\sqrt{(\epsilon - \mu)^2 + \Delta^2}}{k_B T}}} \right]. \]  

(5)
Let $\zeta = \epsilon - \mu$, then we have the master equation for BCS theory as,

$$
\frac{1}{|V| N(0)} = \int_0^{\epsilon_D} \frac{d\zeta}{\sqrt{\zeta^2 + \Delta^2}} \left[ \frac{1}{1 + e^{-\sqrt{\zeta^2 + \Delta^2}/k_BT}} - \frac{1}{1 + e^{\sqrt{\zeta^2 + \Delta^2}/k_BT}} \right].
$$

(Determining $T_C$.)

At $\Delta = 0$, we have $T = T_C$, so we can write,

$$
\frac{1}{|V| N(0)} = \int_0^{\epsilon_D} \frac{d\zeta}{\sqrt{\zeta^2}} \left[ \frac{1}{1 + e^{-\zeta/k_BT_C}} - \frac{1}{1 + e^{\zeta/k_BT_C}} \right].
$$

Define $\frac{\zeta}{k_BT_C} = x$, we can write,

$$
\frac{1}{|V| N(0)} = \int_0^{\epsilon_D} \frac{dx}{x} \phi(x),
$$

where $\phi(x) = \left[ \frac{1}{1 + e^{-x}} - \frac{1}{1 + e^x} \right]$.

Using integration by parts we have,

$$
\int_0^{\epsilon_D} \frac{dx}{x} \phi(x) = \left[ \phi(x) \ln x \right]_{0}^{\epsilon_D} k_BT_C - \int_0^{\epsilon_D} \frac{dx}{x} \frac{d\phi(x)}{dx} \ln x dx.
$$

For a weak coupling limit, i.e. $\{\frac{1}{|V|N_0} \gg 1, k_BT_C \ll \epsilon_D, |\Delta_0| \ll k_BT_C\}$, we have,

$$
[\phi(x) \ln x]_{0}^{\epsilon_D} k_BT_C = \ln \frac{\epsilon_D}{k_BT_C},
$$

$$
\int_0^{\epsilon_D} \frac{d\phi(x)}{dx} \ln x dx \approx \int_0^{\infty} \frac{d\phi(x)}{dx} \ln x dx \approx -0.12563.
$$

So, we have

$$
\frac{1}{|V| N(0)} = \ln \left( 1, 134 \frac{\epsilon_D}{k_BT_C} \right).
$$

We can rewrite the last equation to obtain the well known formula for critical temperature for BCS Theory as,
\[ T_C = \frac{1.134 \epsilon_D}{K_B} e^{|V| N(0)}. \] (13)

**Determine \( \Delta_0 \).**

At \( T = 0 \), we have \( \Delta = \Delta_0 \), so from the equation we have,

\[ \frac{1}{|V| N(0)} = \int_0^{\epsilon_D} \frac{d\zeta}{\sqrt{\zeta^2 + \Delta_0^2}} = \sinh^{-1}\left( \frac{\epsilon_D}{\Delta_0} \right), \] (14)

so we have,

\[ \frac{\epsilon_D}{\Delta_0} = \sinh\left( \frac{1}{|V| N(0)} \right). \] (15)

Again, for a weak coupling limit, we have \( \sinh\left( \frac{1}{|V| N(0)} \right) \approx \frac{1}{2} e^{\frac{1}{|V| N(0)}} \), so we can write the BCS formula of \( \Delta_0 \) as follow,

\[ \frac{\epsilon_D}{\Delta_0} = \frac{1}{2} e^{\frac{1}{|V| N(0)}}. \] (16)

From equations (13) and (16), we arrive at the universal BCS formula,

\[ \Delta_0 = 1.76 K_B T_C. \] (17)

Using equations (16) and (17) in the master equation (6), we can plot the universal curve \( y = y(x) \), where \( x = \frac{T}{T_C}, y = \frac{\Delta}{\Delta_0} \) as follow,

\[ \frac{1}{|V| N(0)} = \int_0^{\epsilon_D} \frac{d\zeta}{y \sqrt{\zeta^2 + \Delta_0^2}} \left( \frac{1}{1 + e^{\frac{1.76 y \sqrt{\zeta^2 + 1}}{x}}} - \frac{1}{1 + e^{\frac{1.76 \sqrt{\zeta^2 + 1}}{x}}} \right). \] (18)

### 4 Anyon superconductivity

From equations (1) and (2), we can write the general form of the master equation for any value of \( p \) as follow,

\[ \frac{1}{|V| N(0)} = \int_0^{\epsilon_D} \frac{d\zeta}{\sqrt{\zeta^2 + \Delta^2}} \left[ \phi\left( \frac{1}{r(\zeta)} \right) - \phi(r(\zeta)) \right], \] (19)
where,
\[ r(\zeta) = e^{\sqrt{\zeta^2 + \Delta^2}/K_B T}, \]  
(20)

and \( \phi(r) \) determined from the following equation,
\[ \left( \frac{1}{1 - p\phi(r)} + 1 \right)^{1-p} \left( \frac{1}{p\phi(r)} - 1 \right)^p = r. \]  
(21)

4.1 Determine \( T_C \) for any value of \( p \)

At \( \Delta = 0 \), we have \( T = T_C \) and then from equations (19-21) we can write,
\[ \frac{1}{|V| N(0)} = \frac{\epsilon_D}{K_B T_C} \int_0 dx x \left[ \phi \left( \frac{1}{r(x)} \right) - \phi(r(x)) \right], \]  
(22)

\[ r(x) = e^x, \]  
(23)

and again, \( \phi(r) \) determined from the equation (21).

4.2 Determine \( \Delta_0 \) for any value of \( p \)

At \( T = 0 \), we have \( \Delta = \Delta_0 \) and from equation (20), we have \( r = \infty \), so equation (21) takes the form,
\[ \left( \frac{1}{1 - p\phi(r)} + 1 \right)^{1-p} \left( \frac{1}{p\phi(r)} - 1 \right)^p = \infty. \]  
(24)

Then for any value of \( p \), we have
\[ \phi(r) = 0 \text{ at } T = 0. \]  
(25)

Similarly, we can evaluate \( \phi \left( \frac{1}{r} \right) \) by solving the following equation,
\[ \left( \frac{1}{1 - p\phi \left( \frac{1}{r} \right)} + 1 \right)^{1-p} \left( \frac{1}{p\phi \left( \frac{1}{r} \right)} - 1 \right)^p = \frac{1}{r}. \]  
(26)

So at \( T = 0 \), We have \( r = \infty, \frac{1}{r} = 0 \) and then equation (26) takes the form,
\[ \left( \frac{1}{1 - p\phi \left( \frac{1}{r} \right)} + 1 \right)^{1-p} \left( \frac{1}{p\phi \left( \frac{1}{r} \right)} - 1 \right)^p = 0 \]  
(27)
The solution of (26) (for positive $\phi(\frac{1}{r})$) takes the form,

$$\phi(\frac{1}{r}) = \frac{1}{p} \text{ at } T = 0.$$  \hspace{1cm} (28)

Using (25) and (28) in the master equation (19), we have at $T = 0$ the following equation,

$$\frac{p}{|V| N(0)} = \frac{\epsilon_D}{\sqrt{\frac{1}{2} + \Delta_0^2}} = \sinh^{-1} \frac{\epsilon_D}{\Delta_0},$$  \hspace{1cm} (29)

or in the equivalent form,

$$\epsilon_D = \Delta_0 \sinh \frac{p}{|V| N(0)}.$$  \hspace{1cm} (30)

At weak coupling limit, from the equation (30), we have the following universal relation for $\Delta_0$ at any value of $p$ as,

$$\Delta_0 = 2 \epsilon_D e^{-\frac{p}{|V| N(0)}}.$$  \hspace{1cm} (31)

In the next section, we will study some special cases of Anyon superconductivity for some special values of $p$.

5 Special cases of Anyon superconductivity

5.1 $p = \frac{1}{2}$ Anyon

From the unified statistics (equation(2)), we can find the formula of $f(E_k)$ as follow,

$$f(E_k) = \frac{2}{2 \sqrt{(\epsilon - \mu)^2 + \Delta^2} \frac{1}{k_B T} \frac{1}{2}}.$$ \hspace{1cm} (32)

So, after some calculations, we arrive at the master equation for the case $p = \frac{1}{2}$ as follow,
\[
\frac{1}{2} |V| N(0) = \int_0^{\epsilon_D} \frac{d\zeta}{\sqrt{\zeta^2 + \Delta^2}} \left[ \frac{1}{1 + e^{-\frac{2\sqrt{\zeta^2 + \Delta^2}}{k_B T}}} - \frac{1}{1 + e^{-\frac{2\sqrt{\zeta^2 + \Delta^2}}{k_B T}}} \right].
\]

(33)

Determine \( T_C \)

At \( \Delta = 0 \), we have \( T = T_C \), so from equations (32,33) we can write,

\[
\frac{1}{2} |V| N(0) = \int_0^{\epsilon_D} \frac{d\zeta}{k_B T_C} \left[ \frac{1}{1 + e^{-\frac{2\sqrt{\zeta^2 + \Delta^2}}{k_B T}}} - \frac{1}{1 + e^{-\frac{2\sqrt{\zeta^2 + \Delta^2}}{k_B T}}} \right].
\]

(34)

Define \( \phi(x) = \left[ \frac{1}{(1 + e^{-2x})^2} - \frac{1}{(1 + e^{2x})^2} \right] \), we can calculate the last integral as,

\[
\int_0^{\epsilon_D} \frac{dx}{K_B T_C} \phi(x) \approx \left[ \phi(x) \ln x \right]_0^{K_B T_C} = \int_0^\infty dx \frac{d\phi(x)}{x} \ln x = \ln \frac{\epsilon_D}{K_B T_C} + 0.4228 = \ln \frac{1.535\epsilon_D}{K_B T_C}.
\]

(35)

So, we can write the equation (34) as,

\[
\frac{1}{2} |V| N(0) = \ln \frac{1.535\epsilon_D}{K_B T_C},
\]

(36)

or on the equivalent form,

\[
T_C = \frac{1.535\epsilon_D}{K_B} e^{\frac{-1}{2|V| N(0)}}.
\]

(37)

Comparing equation (37), the critical temperature for the Anyon case \( (P = \frac{1}{2}) \) and equation (13), the critical temperature for BCS theory (at the same value of Debye energy) we have,
\[
\frac{T_{C_p=\frac{1}{2}}}{T_{C_{BCS}}} = 1.35 e^{\frac{1}{2} |V| N(0)}.
\] (38)

So, for some weak coupling limit, say \( \frac{1}{|V| N(0)} = 5 \), we have
\[
T_{C_p=\frac{1}{2}} \approx 16,446 T_{C_{BCS}}.
\] (39)

So, if the critical temperature in the standard BSC is equal to 1 Kelvin (with \( \frac{1}{|V| N(0)} = 5 \)), then using a unified statistics with \( P = \frac{1}{2} \), the critical temperature jumps to 16,446 Kelvin.

Notice that although the weak coupling is the reason of why we cannot explain some HTS using the standard BCS theory, in the Anyonic statistics the critical temperature is increased whenever the coupling limit is decreased!!

As an example, consider that we have more weak interaction (say \( \frac{1}{|V| N(0)} = 10 \)), then we have,
\[
T_{C_p=\frac{1}{2}} \approx 201 T_{C_{BCS}}!!
\] (40)

So at the same condition, if the critical temperature in the standard BSC is equal to 1 Kelvin then for a weak coupling \( \frac{1}{|V| N(0)} = 5 \) we see that the critical temperature jumps to 16,446 Kelvin and for more weak coupling \( \frac{1}{|V| N(0)} = 10 \) the critical temperature jumps to 201 Kelvin!

**Determine \( \Delta_0 \)**

At \( T = 0 \), we have \( \Delta = \Delta_0 \), so from the equation (29) we have,
\[
\frac{1}{2 |V| N(0)} = \int_0^{\epsilon_D} \frac{d\zeta}{\sqrt{\zeta^2 + \Delta_0^2}} = \sinh^{-1} \left( \frac{\epsilon_D}{\Delta_0} \right),
\] (41)

so we have,
\[
-1 = 2 \epsilon_D e^{\frac{1}{2} |V| N(0)}.
\] (42)

From (24) and (28), we have
\[
\Delta_0 = 1.3 K_B T_C.
\] (43)

Finally, we can plot the universal curve \( y = y(x) \), where \( x = \frac{T}{T_C}, y = \frac{\Delta}{\Delta_0} \) for \( p = \frac{1}{2} \) as follow,
\[
\frac{1}{2|V|N(0)} = \int_0^y \frac{dz}{\sqrt{z^2 + 1}} \left[ \frac{1}{1 + e^{-2.6\sqrt{z^2 + 1}/x}} \right]^{1/2} \left[ \frac{1}{1 + e^{2.6\sqrt{z^2 + 1}/x}} \right]^{1/2}.
\]

\section*{5.2 $p = \frac{1}{3}$ Anyon superconductor}

For $p = \frac{1}{3}$, we can determine $\phi(r)$ from equation (21) as,

\begin{equation}
\phi(r) = \frac{3}{(2r^3 + 2\sqrt{r^6 + r^3} + 1)^3 + (2r^3 + 2\sqrt{r^6 + r^3} + 1)^3 - 1}.
\end{equation}

From equations (22-23) and after some calculations, we can find the formula of $T_C$ for the Anyon $p = \frac{1}{3}$ as,

\begin{equation}
T_C = \frac{32.627}{K_B} \epsilon_D e^{3|V|N(0)}.
\end{equation}

Also, from the general equation (31) we have,

\begin{equation}
\Delta_0 = 2\epsilon_D e^{3|V|N(0)}.
\end{equation}

\textbf{Conclusion 1} In this paper, we calculated the critical temperature of some metals using the standard BCS theory but assuming that the particles inside this metals obey Anyonic statistics instead Fermi-Dirac statistics. We find that the lower of the interaction we have, the greater of the critical temperature which may be a reasonable explanation of some high temperature superconductors which the standard BCS theory cannot explain it.

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\section*{6 References}

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