NONEXISTENCE OF SOLUTIONS OF CERTAIN SEMILINEAR HEAT EQUATIONS

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Abstract. We consider a semilinear heat equation involving a forcing term which depends only upon the space variable. Existence of a local mild solution is proved through an application of the Banach fixed point theorem. An upper bound for the blow-up time of local solutions is also provided. With the help of carefully defined test functions, we prove nonexistence results for global weak solutions. This leads to lower bounds for a possible critical Fujita-type exponent.

1. Introduction

Let $X = (X_1, \ldots, X_m)$ be a system of $C^\infty$ real vector fields on $\mathbb{R}^n$. We study the self-adjoint operator

$$\Delta_X = -\sum_{k=1}^m X^*_k X_k,$$

where $X^*_k = -X_k - \text{div} X_k$. The operator $\Delta_X$ is well-defined on

$$D(\Delta_X) = \{ u \in H^1_{X,0}(\mathbb{R}^n) : \Delta_X u \in L^2(\mathbb{R}^n) \},$$

where $H^1_{X,0}(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : X_i u \in L^2(\mathbb{R}^n), 1 \leq i \leq m \}$ and $H^1_{X,0}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in $H^1_{X,0}(\mathbb{R}^n)$.

For $p > 1$ and $n > 2$ (except for Theorem 4.1, where $n = 2$), we consider the problem of finding conditions for the non-existence of solutions for

$$\begin{cases}
  u_t(t, x) - \Delta_X u(t, x) = |u(t, x)|^p + f(x), & (t, x) \in (0, T) \times \mathbb{R}^n \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^n.
\end{cases} \tag{1.1}$$

Note that the complexity in finding the existence of a solution for such a problem lies in the fact that whereas a time dependent solution $u$ is sought, the function $u_t(t, x) - \Delta_X u(t, x) - |u(t, x)|^p$ is required be independent of time. The simplest way around is to look for a stationary solution, i.e. a solution which is independent of time. When one does this, the term $u_t(t, x)$ looses its value in the problem.

As already mentioned, this paper focuses solely on describing some conditions under which a global solution for the problem (1.1) does not exists. A notion which becomes important for this goal is the notion of critical exponent, which is the least real number $p_c$ such that for every $p > p_c$ the existence of a global solution is guaranteed for some $f$ and $u_0$; and a global solution never exists for $p \leq p_c$.

Literature is vast for similar problems which originated from the seminal work [10] in which Fujita assumed $f$ to be zero and established that the critical exponent is $\frac{n+2}{n}$ when $X_k = \partial_k$ and $k = 1, \ldots, n$. For non-zero $f$, a system similar to (1.1) was considered by Zhang [20] on Riemannian manifold $M^n$ with possibly non-negative Ricci curvature. There $\frac{\alpha}{\alpha-2}$ for some $\alpha$ was proved to be the critical exponent, where $\alpha \leq n$ in general and $\alpha = n$ when $M^n = \mathbb{R}^n$. Bandle

2020 Mathematics Subject Classification. 35B33, 35B44.

Key words and phrases. Fixed point theorem, blow-up, local solution, global solution, weak solution, Grushin-type PDO, Engel-type PDOs.
et al. proved in [1, Theorem 2.1] that if one replaces $\Delta_X$ with $\Delta$ in (1.1), then the critical exponent is $\frac{n}{n-2}$. As $\frac{n}{n-2} > \frac{n+2}{n}$, it shows that one may choose $p$ from a bigger interval $(1, \frac{n}{n-2}]$ and still the nonexistence of a global solution is guaranteed when a suitable non-linearity $f$ is introduced. Another interesting phenomenon observed during similar studies is that a solution exists for some finite time and then it blows-up in the supremum norm, hindering the global existence. Finding the time at which this blow-up occurs is another challenging question in the field. Ever since [10] was published, many existence, nonexistence and blow-up results have been obtained not just on $\mathbb{R}^n$ but on general topological groups as well, which include the works on Lie groups—see [8, 18, 11, 12, 17, 16, 4, 14, 2] and references therein.

Although $\Delta_X$ is well-studied with an extra Hörmander condition (H) on $X$ which is described below, we do not impose any such restriction. However, this condition is met by a class of operators we study, especially in Section 4. Note that the counterpart of $\Delta_X$ on bounded domains was recently studied in [9], along with the Hörmander condition.

(H): There exists a natural number $r$ such that the vector fields $X_1, \ldots, X_m$ together with their $r$ commutators span the tangent space at each point of $\mathbb{R}^n$.

Different systems $X$ are considered in this paper. Along with a general local existence result, lower bound for a possible critical exponents is provided. Our techniques and results are inspired by [8, 12]. It must be observed that in [8] the group structure of the Heisenberg group $\mathbb{H}^n$ does not play any serious role and the most useful tool turns out to be the dilation structure on the group. Let us briefly summarize the layout and main results of this paper.

1. As an application of the Banach fixed point theorem, generalizing [8, Theorem 3.2], in Theorem 2.2 we prove the existence and blow-up of a local in time mild solution.

2. In Section 3 we assume that for $X_k = \sum_{i=1}^{n} a_{k,i} \frac{\partial}{\partial x_i}$ with $1 \leq k \leq m$, the functions $a_{k,i}$ and $\frac{\partial a_{k,i}}{\partial x_j}$ are bounded. This holds, for example, for sine and cosine functions. If $\int_{\mathbb{R}^n} f(x) dx > 0$, then it is proved through Theorem 3.2 and Theorem 3.3 that a lower bound for the possible critical exponent is $\frac{n}{n-1}$ which follows Theorem 3.4 in which it is shown that for a special class $X$ this lower bound increases to $\frac{n}{n-2}$. Making choices of suitable test functions is the crucial part of the proofs. This section is conclude by providing an upper bound on the time of blow-up. In Remark 3.6 we mention how one may get rid of the non-negativity assumption imposed on the initial condition $u_0$ in our theorems.

3. The Remark 3.6 is well justified through the proofs of Theorem 4.1 and Theorem 4.2. In Section 4 the considered $X$ give rise to the so-called Grushin-type and Engel-type partial differential operator. We refer the reader to [5, 3] and references therein for details on the significance of these operators. A noteworthy information about these operators is that they do not give rise to a stratified Lie group structure on the base spaces under consideration.

Throughout the paper, we make a major assumption of existence if a non-negative function $h_t(x,y)$ satisfying $\int_{\mathbb{R}^n} h_t(x,y) dy \leq 1$ and making $\{e^{-t\Delta X}\}_{t \geq 0}$ a semigroup where

$$e^{-t\Delta X} w(x) = \int_{\mathbb{R}^n} h_t(x,y) w(y) dy$$

for every $x \in \mathbb{R}^n$ whenever $w \in C_0(\mathbb{R}^n)$. As a consequence, we have

$$\|e^{-t\Delta X} w(x)\| \leq \int_{\mathbb{R}^n} h_t(x,y) \|w(y)\| dy \leq \|w\|_{\text{sup}}$$
for every \( x \in \mathbb{R}^n \) and \( t > 0 \). This property of the semigroup \( \{e^{-t\Delta}x\}_{t \geq 0} \) will be crucial to the proof of Theorem 2.2. Note that the so-called heat-kernel satisfies these properties and its existence is guaranteed in many general cases, for example, [15, Theorem 10.1] provides the existence on compact manifolds with boundary and [19, Chapter 4] gives existence on nilpotent Lie groups. On \( \mathbb{R}^n \), existence of heat-kernel for \( \Delta_X - \frac{\partial}{\partial s} \) is mentioned in [13] and in [6] global estimates for heat kernel are given for the operators we study in Section 4.

2. Local existence

Let us fix some notations before proving the local existence of weak solutions of (1.1). We denote the space of all continuous functions on \( \mathbb{R}^n \) which vanish at infinity by \( C_0(\mathbb{R}^n) \). For a Banach space \( (Y, \| \cdot \|) \), the space of all \( Y \)-valued continuous and vanishing at infinity functions on a locally compact Hausdorff topological space \( S \) is denoted by \( C_0(S, Y) \). This space is a Banach space when norm of \( w \in C_0(S, Y) \) is given by

\[
\|w\|_{sup} = \sup\{\|w(s)\| : s \in S\}.
\]

Similarly, \( C_c(S, Y) \) will denote the normed space of compactly supported \( Y \)-valued continuous functions on \( S \). We set \( x = (x_1, x_2, \ldots, x_n) \) and similarly \( x' = (x'_1, x'_2, \ldots, x'_n) \).

**Definition 2.1.** (Mild solution) A local mild solution of (1.1) is a function \( u \in C([0, T], C_0(\mathbb{R}^n)) \) such that

\[
 u(t) = e^{-t\Delta}u(0) + \int_0^t e^{-(t-s)\Delta}(|u(s)|^p + f)ds,
\]

for any \( t \in [0, T) \). The function \( u \) is called a global mild solution of (1.1) if \( T = +\infty \).

**Theorem 2.2.** Let \( u_0, f \in C_0(\mathbb{R}^n) \). Then the following holds.

1. There exists a time \( T \in (0, \infty) \) such that (1.1) has a unique mild solution \( u \).
2. The solution in (1) can be uniquely extended to a maximal interval \([0, T_{\text{max}})\). If \( T_{\text{max}} \) is a real number, then supremum norm of \( u(t) \in C_0(\mathbb{R}^n) \) approaches infinity as \( t \) tends to \( T_{\text{max}} \).
3. If \( u_0, f \in L^q(\mathbb{R}^n) \) where \( q \in [1, \infty] \), then

\[
 u \in C([0, T_{\text{max}}], C_0(\mathbb{R}^n)) \cap C([0, T_{\text{max}}], L^q(\mathbb{R}^n)).
\]

**Proof.** 1. Let us define a metric space

\[
 \Theta = \{u \in C([0, T], C_0(\mathbb{R}^n)) : ||u||_{sup} \leq 2\delta(u_0, f), u(0) = u_0\}
\]

where \( \delta(u_0, f) = \max\{||u_0||_{sup}, ||f||_{sup}\} \). The distance between two elements of \( \Theta \) is given by the supremum norm of their difference. However, for convenience, we denote this distance between \( u, v \in \Theta \) by \( ||u - v||_\Theta \). As convergence in supremum norm implies pointwise convergence, \( || \cdot ||_\Theta \) defines a metric on \( \Theta \) making it a complete metric space. Define \( \Phi : \Theta \to \Theta \) such that

\[
 \Phi(v)(t) = e^{-t\Delta}v(0) + \int_0^t e^{-(t-s)\Delta}(|v(s)|^p + f)ds.
\]

We prove that for an appropriate \( T \), \( \Phi \) is a well-defined contraction and then use the Banach fixed point theorem for complete metric spaces to obtain a unique function \( u \in \Theta \) such that \( \Phi(u) = u \) which will be the required solution.

(i) Well-defined: We need to prove that \( \Phi(v) \in \Theta \) whenever \( v \in \Theta \). In other words, we must prove that for \( v \in \Theta \) we have \( \Phi(v) \in C([0, T], C_0(\mathbb{R}^n)) \), \( ||\Phi(v)|| \leq 2\delta(u_0, f) \) and \( \Phi(v)(0) = u_0 \). It is clear that \( \Phi(v) \) is a continuous function as it is composition of several continuous functions.
Also, $\Phi(v(0)) = v(0) = u_0$. As $e^{-t\Delta}w(\cdot) = \int_{\mathbb{R}^n} h_{t, y}w(y)dy$ on $\mathbb{R}^n$ for every $t > 0$, we have

$$
\|\Phi(v)(t)\| = \left\|e^{-t\Delta}v(0) + \int_0^t e^{-(t-s)\Delta} (|v(s)|^p + f)ds \right\|
$$

$$
\leq \|e^{-t\Delta}v(0)\| + \int_0^t \|e^{-(t-s)\Delta} (|v(s)|^p + f)\|ds
$$

Thus,

$$
\|\Phi(v)\|_{\text{sup}} \leq \|v(0)\|_{\text{sup}} + T\|v(s)\|_{\text{sup}} + T\|f\|_{\text{sup}} \leq (1 + T)\delta(u_0, f) + T2^p\delta(u_0, f)^p.
$$

Finally, one chooses $T$ small enough to obtain

$$
\|\Phi(v)\|_{\text{sup}} \leq (1 + T + 2^{p-1}T\delta(u_0, f)^{p-1})\delta(u_0, f) \leq 2\delta(u_0, f).
$$

Next we prove that for every $t \in [0, T]$, we have $\Phi(v)(t) \in C_0(\mathbb{R}^n)$. Let $1 > \epsilon > 0$ be given. For any fixed $t \in [0, T]$, as $f$ and $v(t)$ are vanishing at infinity, there exists a compact set $K_{t, \epsilon}$ such that for $x \in \mathbb{R}^n \setminus K_{t, \epsilon}$ both $|f(x)|$ and $|v(t)(x)|$ are less than $\epsilon/2$. Using continuity of $v$, choose a neighbourhood $V_t$ of $t \in [0, T]$ such that $|v(s)(x)| \leq \epsilon$ for every $x \in \mathbb{R}^n \setminus K_{t, \epsilon}$ and $s \in V_t$. Since $[0, T]$ is compact, there exists $t_1, t_2, \ldots, t_l \in [0, T]$ such that $[0, T] = \bigcup_{i=1}^l V_{t_i}$. Then for $x \in \mathbb{R}^n \setminus \bigcup_{i=1}^l K_{t_i, \epsilon}$, we have

$$
|\Phi(v)(t)(x)| = \left|e^{-t\Delta}v(0)(x) + \int_0^t e^{-(t-s)\Delta} (|v(s)(x)|^p + f(x))ds \right|
$$

$$
\leq \epsilon + Te^p + \int_0^t \left|e^{-(t-s)\Delta} f(x)\right|ds
$$

$$
\leq \epsilon + Te^p + T\epsilon \leq (1 + 2T)\epsilon.
$$

As $\bigcup_{i=1}^l K_{t_i, \epsilon}$ is compact, we have established that $\Phi(v)(t) \in C_0(\mathbb{R}^n)$ for every $t \in [0, T]$. (ii) Contraction: Let $u, v \in \Theta$. As $u(0) = v(0)$, for some constant $C(p)$ depending upon $p$ we have

$$
\|\Phi(u)(t) - \Phi(v)(t)\| = \left\|e^{-t\Delta}u(0) - e^{-t\Delta}v(0) + \int_0^t e^{-(t-s)\Delta} (|u(s)|^p - |v(s)|^p)ds \right\|
$$

$$
= \left\|\int_0^t e^{-(t-s)\Delta} (|u(s)|^p - |v(s)|^p)ds \right\|
$$

$$
\leq C(p) \int_0^t \|e^{-(t-s)\Delta} |u(s) - v(s)|^p + |v(s)|^{p-1}\|ds
$$

$$
\leq C(p)2^p\delta(u_0, f)^{p-1} \int_0^t \|e^{-(t-s)\Delta} |u(s) - v(s)|\|ds
$$

$$
\leq C(p)2^p\delta(u_0, f)^{p-1}T\|u - v\|_{\Theta}.
$$

One can now choose $T$ small enough to obtain that $\Phi$ is a contraction on $[0, T]$. 2. Existence of $[0, T_{\text{max}})$ comes from the uniqueness part of (1). Set $u$ to be this unique solution. Let us now assume that $T_{\text{max}}$ is finite and on the contrary, there exists $K > 0$ such that $\|u(t)\|_{\Theta} \leq K$ for every $0 \leq t < T_{\text{max}}$. By demonstrating that there exists a mild solution of $(1.1)$ in $[0, \tau)$ where $\tau > T_{\text{max}}$ we will contradict the maximality of $T_{\text{max}}$.

Fix $t^* \in (\frac{T_{\text{max}}}{2}, T_{\text{max}})$, $T < T_{\text{max}}$ and define a complete metric space

$$
\Theta_1 = \{v \in C([0, T], C_0(\mathbb{R}^n)) : \|v\| \leq 2\delta(K, f), v(0) = u_t\},
$$

where $u_t$ is the unique solution for $[0, t^*]$ with $u(t^*) = v(0)$. We define the operator $\Phi_1(v)$ similar to $\Phi$ and prove that $\Phi_1$ is a contraction on $\Theta_1$. Let $\tau > T_{\text{max}}$ be such that $\Phi(v)(\tau) = v(\tau)$. By the uniqueness of $u$, we have $\Phi(v)(\tau) = u(\tau)$. This contradicts the maximality of $T_{\text{max}}$. Therefore, $T_{\text{max}}$ is infinite.
where $\delta(K, f) = \max\{K, \|f\|_{\sup}\}$ and norm is denoted by $\|\cdot\|_{\Theta_1}$. Then just as in (1), the operator

$$\Phi_1(v) = e^{-t\Delta_x}u_{t*} + \int_0^t e^{-(t-s)\Delta_x}(|v(s)|^p + f)ds,$$

is a well-defined contraction. From the Banach fixed point theorem, there exists a fixed point $v$ of $\Theta_1$.

One must also notice that no matter which $t^*$ we start with, the choice of $T$ such that a fixed point of $\Phi_1$ exists in $\Theta_1$ does not change. This is because for $\Phi_1$ to be a contraction from $\Theta_1$ to itself, we need $T$ small enough so that

$$\|\Phi_1(v)\|_{\sup} \leq (1 + T + 2^{p-1}\delta(K, f)^{p-1})\delta(u_0, f) \leq 2\delta(K, f)$$

and

$$\|\Phi_1(u)(t) - \Phi_1(v)(t)\| \leq C(p)2^p\delta(K, f)^{p-1}T\|u - v\|_{\Theta_1}$$

as well as none of these bounds depend upon $t^*$. In view of this, choose $t^*$ such that $T + t^* > T_{\max}$. Since $u(t^*) = v_0$, it can be easily verified that

$$\pi(t) = \begin{cases} u(t), & t \in [0, t^*], \\ v(t - t^*), & t \in [t^*, t^* + T], \end{cases} \quad (2.1)$$

defines a solution for (1), contradicting the maximality of $T_{\max}$.

(3) As $C_0(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$, the result holds trivially for $q = \infty$. Let $q \in [1, \infty)$ and consider the complete metric space $\Theta_2$ which is nothing but the intersection of

$$\left\{ v \in C([0, T_{\max}], C_0(\mathbb{R}^n)) : \|v\|_{\sup} \leq 2\delta(u_0, f), v(0) = u_0 \right\}$$

and

$$\left\{ v \in C([0, T_{\max}], L^q(\mathbb{R}^n)) : \|v\|_{\sup} \leq 2\delta(u_0, f) \right\},$$

where $\delta(u_0, f) = \max\{\|u_0\|_{\sup}, \|f\|_{\sup}\}$, $\delta_q(u_0, f) = \max\{\|u_0\|_{L^q(\mathbb{R}^n)}, \|f\|_{L^q(\mathbb{R}^n)}\}$ and norm is given by

$$\|v\|_{\Theta_2} = \|v\|_{C([0, T_{\max}], C_0(\mathbb{R}^n))} + \|v\|_{C([0, T_{\max}], L^q(\mathbb{R}^n))}.$$ 

We prove that if $v \in C([0, T_{\max}], L^q(\mathbb{R}^n))$, then for every $t \in [0, T_{\max})$ we have $\Phi(v)(t) \in L^q(\mathbb{R}^n)$. Rest follows from the proof of (1).

Recall that

$$\Phi(v)(t) = e^{-t\Delta_x}v(0) + \int_0^t e^{-(t-s)\Delta_x}(|v(s)|^p + f)ds.$$ 

Note that $e^{-t\Delta_x}$ is an operator from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ since, by definition, $e^{-t\Delta_x}(v) = v * h_t$ is convolution of $v \in L^q(\mathbb{R}^n)$ with $h_t \in L^1(\mathbb{R}^n)$. As $u_0 \in L^q(\mathbb{R}^n)$ we obtain $e^{-t\Delta_x}v(0) \in L^q(\mathbb{R}^n)$. Moreover, as $v(s) \in C_0(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ we have $|v(s)|^p \leq \|v(s)^{p-1}\|_{\sup}|v(s)|$. Using $f \in L^q(\mathbb{R}^n)$ gives $|v(s)|^p + f \in L^q(\mathbb{R}^n)$. Application of the operator $e^{-t\Delta_x}$ then gives an element of $L^q(\mathbb{R}^n)$. Being linear combination of elements of $L^q(\mathbb{R}^n)$ the function $\Phi(v)(t)$ belongs to $L^q(\mathbb{R}^n)$. \hfill \Box

3. Global nonexistence

**Definition 3.1.** Let $u_0 \in L^1_{loc}(\mathbb{R}^n)$. Then $u \in L^p_{loc}((0, T), L^p_{loc}(\mathbb{R}^n))$ is called a local in time weak solution of (1.7) if for every non-negative test function $\psi \in C^1_\infty([0, T); \mathbb{R}^n)$ we have

$$\int_0^T \int_{\mathbb{R}^n} |u|^p\psi + \int_{\mathbb{R}^n} u_0\psi(0, x) + \int_0^T \int_{\mathbb{R}^n} f\psi + \int_0^T \int_{\mathbb{R}^n} u\psi_t + \int_0^T \int_{\mathbb{R}^n} u\Delta_x\psi = 0. \quad (3.1)$$

When $T = \infty$, such $u$ is called global in time weak solution.
Throughout this paper we fix $\frac{1}{p} + \frac{1}{p'} = 1$. We keep $C, C_1$, and $C_2$ to denote the constants, their values may keep on changing from line to line. Using the self-adjointness of the operator $\Delta X$, it can be proved using standard techniques that when $u_0 \in C_0(\mathbb{R}^n)$, every local in time mild solution belonging to the class $C([0,T], L^\infty(\mathbb{R}^n))$ is a local in time weak solution as well.

In Theorem 3.2 we provide a lower bound for the critical exponent as $\frac{n}{n-1}$. It can be seen from Theorem 3.4 this lower bound is not sharp.

**Theorem 3.2.** Let $p < \frac{n}{n-1}$ and $u_0 \in L^1_{loc}(\mathbb{R}^n)$ be non-negative. If $\int_{\mathbb{R}^n} f(x) dx > 0$ and the functions $a_{k,i}$ and $\frac{\partial a_{k,i}}{\partial x_j}$ are bounded for every $1 \leq k \leq m$ and $1 \leq i,j \leq n$, then (1.1) does not admit global in time weak solution.

**Proof.** Let us suppose on the contrary that $u$ is a global in time weak solution of (1.1). Then for every test function $\psi$,

$$\int_0^T \int_{\mathbb{R}^n} |u|^p \psi + \int_{\mathbb{R}^n} u_0 \psi(0,x) + \int_0^T \int_{\mathbb{R}^n} f \psi + \int_0^T \int_{\mathbb{R}^n} u \psi_t + \int_0^T \int_{\mathbb{R}^n} u \Delta X \psi = 0.$$ 

As both $\psi$ and $u_0$ are non-negative, we obtain

$$\int_0^T \int_{\mathbb{R}^n} f \psi + \int_0^T \int_{\mathbb{R}^n} |u|^p \psi \leq - \int_0^T \int_{\mathbb{R}^n} u \psi_t - \int_0^T \int_{\mathbb{R}^n} u \Delta X \psi$$

$$\leq \int_0^T \int_{\mathbb{R}^n} |u||\psi_t| + |u||\Delta X \psi|.$$  

(3.2)

Set

$$P_x := \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{T}.$$ 

Let us define a test function (see Definition 3.1) with separated variables as

$$\psi(t,x) = \Phi(P_x)\Phi(2t/T),$$

where

$$\Phi(z) = \begin{cases} 1, & z \in [0,1], \\ \gamma, & z \in [1,2], \\ 0, & z \in (2,\infty), \end{cases}$$

is a function in $C^2_c(\mathbb{R}^+)$ satisfying the estimates

$$\int_{1 \leq ||x|| \leq 2} \left| \frac{\Phi'(||x||) + \Phi''(||x||)}{\Phi(||x||)} \right|^{\frac{1}{p-1}} \Phi(||x||)^p < \infty$$

and

$$\int_0^2 \left| \frac{\Phi(t)^p}{\Phi(t)} \right|^{\frac{1}{p-1}} dt < \infty.$$ 

Note that

$$\text{supp}(\psi) = \left\{ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq \frac{2t}{T} \leq 2, 0 \leq P_x \leq 2 \right\}.$$ 

Now,

$$\psi_t(x,t) = \frac{\partial}{\partial t}(\Phi(P_x)\Phi(2t/T)) = \frac{2t}{T}\Phi(P_x)\Phi'(2t/T)$$

implying that

$$\text{supp}(\psi_t) = \left\{ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 1 \leq 2t/T \leq 2, 0 \leq P_x \leq 2 \right\}.$$
Using inequality 3.2 we now obtain
\[
\Delta_X \psi(t, x) = \Phi(2t/T) \Delta_X \Phi(P_x)
\]
\[
= \Phi(2t/T) \sum_{k=1}^{m} (X_k^2 \Phi(P_x) + (\text{div}X_k)X_k \Phi(P_x))
\]
\[
= \Phi(2t/T) \sum_{k=1}^{m} \left( \Phi''(P_x)(X_k(P_x))^2 + \Phi'(P_x)X_k^2(P_x) \right)
\]
\[
+ \left( \sum_{i=1}^{n} \frac{\partial a_{k,i}}{\partial x_i} \right) \left( \sum_{j=1}^{n} a_{k,j} \frac{\partial}{\partial x_j} \Phi(P_x) \right)
\]
implies that
\[
\text{supp}(\Delta_X \psi) = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq 2t/T \leq 2, 1 \leq P_x \leq 2\}.
\]
As \( \Phi \) is decreasing in \([1, 2]\), \( \psi \) is zero whenever \( \psi_t \) or \( \Delta_X \psi \) is zero. Hence we can write \( |u||\psi_t| \) as \( |u||\psi_t|\rightarrow |\psi_t| \) and \( |u||\Delta_X \psi| \) as \( |u||\psi_t|\rightarrow |\Delta_X \psi| \). Applying \( \frac{p}{2} \)-Young’s inequality, we obtain
\[
\int_0^T \int_{\mathbb{R}^n} |u||\psi_t| \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} |u|^{\frac{p}{2}} |\psi_t|^{\frac{p}{2}} \int_0^T \int_{\mathbb{R}^n} |\psi_t|^{\frac{p}{2}} |\psi_t|^{\frac{p}{2}}
\]
and
\[
\int_0^T \int_{\mathbb{R}^n} |u||\Delta_X \psi| \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} |u|^{\frac{p}{2}} |\Delta_X \psi|^{\frac{p}{2}} \int_0^T \int_{\mathbb{R}^n} |\Delta_X \psi|^{\frac{p}{2}} |\psi_t|^{\frac{p}{2}}.
\]
Using inequality 3.2 we now obtain
\[
\int_0^T \int_{\mathbb{R}^n} f\psi \leq \frac{1}{p'(\frac{p}{2})^{p'-1}} \left( \int_0^T \int_{\mathbb{R}^n} |\psi_t|^{\frac{p}{2}} |\Delta_X \psi|^{\frac{p}{2}} + |\psi|^{\frac{p}{2}} |\psi_t|^{\frac{p}{2}} \right).
\]
In the support of \( \Delta_X \psi \), we have \( |x_i| \leq \sqrt{2T} \) for every \( 1 \leq i \leq n \). Using the assumptions on \( a_{k,i} \) and the fact that \( \Phi \) is twice continuously differentiable, one can choose
\[
M = \max \left\{ ||a_{k,i}(x)||_{\text{sup}}, ||a_{k,i}^2(x)||_{\text{sup}}, \left\| \frac{\partial a_{k,j}(x)}{\partial x_i} \right\|_{\text{sup}} \right\}_{1 \leq k \leq m; 1 \leq i, j \leq n}
\]
Thus, for large \( T \) there exists a constant \( K > 0 \) such that for every \((t, x) \in \text{supp}(\Delta_X \psi)\) we have
\[
|\Delta_X \psi(t, x)| \leq |\Phi(2t/T)| \sum_{k=1}^{m} \left| \Phi''(P_x) \left( \sum_{j=1}^{n} a_{k,j} \left| \frac{2x_j}{T} \right| \right)^2 \right|
\]
\[ + |\Phi'(P_x)| \left( \sum_{j=1}^{n} a_{k,j}^2 \frac{2}{T} + \sum_{j=1}^{n} \frac{2|x_j|}{T} \left( \sum_{i=1}^{n} |a_{k,i}| \left| \frac{\partial a_{k,j}}{\partial x_i} \right| \right) \right) \]
\[ + \left( \sum_{i=1}^{n} \left| \frac{\partial a_{k,j}}{\partial x_i} \right| \right) \left( \sum_{j=1}^{n} |a_{k,j}| |\Phi'(P_x)| \frac{2|x_j|}{T} \right) \]
\[ \leq \frac{K|\Phi(2t/T)| |\Phi'(P_x) + \Phi''(P_x)|}{\sqrt{T}}. \]

With change of variables $t \leftrightarrow \frac{2t}{T}$ and $x_i \leftrightarrow \frac{x_i}{\sqrt{T}}$ for every $1 \leq i \leq n$, we have

\[ \int_0^T \int_{\mathbb{R}^n} |\psi|^{p-1} |\Delta_X \psi|^{\frac{p}{p-1}} = \int_{\supp(\Delta_X \psi)} |\psi|^{p-1} |\Delta_X \psi|^{\frac{p}{p-1}} \]
\[ \leq \int_0^T \int_{1 \leq P_x \leq 2} |\Phi(P_x)| \Phi(\frac{2t}{T})^{\frac{1}{p-1}} \left( \frac{K|\Phi(\frac{2t}{T})| |\Phi'(P_x) + \Phi''(P_x)|}{\sqrt{T}} \right)^{\frac{p}{p-1}} \]
\[ = \left( \frac{K}{\sqrt{T}} \right)^{\frac{p}{p-1}} \int_0^T \int_{1 \leq P_x \leq 2} |\Phi(P_x)|^{\frac{1}{p-1}} |(\Phi' + \Phi'')(P_x)|^{\frac{1}{p-1}} |\Phi(\frac{2t}{T})| \]
\[ = \left( \frac{K}{\sqrt{T}} \right)^{\frac{p}{p-1}} T^{\frac{p}{2}} \int_0^2 \int_{1 \leq \|x\| \leq 2} \frac{|(\Phi' + \Phi'')(\|x\|)|}{\Phi(\|x\|)}^{\frac{1}{p-1}} |\Phi(t)| \]
\[ \leq CT^{\frac{n}{2}+1} \frac{p}{2(p-1)}. \]

Similarly,

\[ \int_0^T \int_{\mathbb{R}^n} |\psi|^{\frac{1}{p-1}} |\psi_t|^{\frac{1}{p-1}} = \int_{\supp(\psi_t)} |\Phi(P_x)| \Phi(\frac{2t}{T})^{\frac{1}{p-1}} \left| \frac{2}{T} \Phi(P_x) \Phi'(\frac{2t}{T}) \right|^{\frac{p}{p-1}} \]
\[ = \left( \frac{2}{T} \right)^{\frac{p}{p-1}} \int_{\supp(\psi_t)} |\Phi(P_x)| \left| \frac{\Phi'(\frac{2t}{T})^p}{\Phi(\frac{2t}{T})} \right|^{\frac{1}{p-1}} \]
\[ = \left( \frac{2}{T} \right)^{\frac{p}{p-1}} T^{\frac{p}{2}} \int_1^2 \int_{0 \leq \|x\| \leq 2} \left| \Phi(\|x\|) \right| \left| \frac{\Phi'(t)^p}{\Phi(t)} \right|^{\frac{1}{p-1}} \leq CT^{\frac{n}{2}+1} \frac{p}{2(p-1)}. \]

As,

\[ \int_0^T \int_{\mathbb{R}^n} f \psi = \int_0^T \Phi \left( \frac{2t}{T} \right) \int_{\mathbb{R}^n} f \Phi(P_x) = \frac{T}{2} \int_0^2 \Phi(t') \int_{\mathbb{R}^n} f \Phi(P_x) \geq T \int_{\mathbb{R}^n} f \Phi(P_x) \]
we obtain

\[ T \int_{\mathbb{R}^n} f \Phi(P_x) \leq \int_0^T \int_{\mathbb{R}^n} f \psi \leq CT^{\frac{n}{2}+1} \frac{p}{2(p-1)}. \]

By dominated convergence theorem, $\int_{\mathbb{R}^n} f \Phi \to \int_{\mathbb{R}^n} f$ as $T \to \infty$. Taking $T \to \infty$ and using the fact that $\frac{n}{2} < \frac{p}{2(p-1)}$ gives $\int_{\mathbb{R}^n} f \leq 0$. This contradicts the hypothesis that $\int_{\mathbb{R}^n} f(x)dx > 0$. So, no such $u$ exists and the statement is proved. \( \square \)

**Theorem 3.3.** Let $p = \frac{n}{n-1}$ and $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$ be non-negative. If $\int_{\mathbb{R}^n} f(x)dx > 0$ and the functions $a_{k,i}$ and $\frac{\partial a_{k,j}}{\partial x_i}$ are bounded for every $1 \leq k \leq m$ and $1 \leq i, j \leq n$, then \( \text{(1.1)} \) does not admit global in time weak solution.
Proof. Let us suppose on the contrary that $u$ is a global in time weak solution for (1.1). For any $0 < R < \infty$, set

$$P_x := \frac{\ln(\|x\|/\sqrt{R})}{\ln(\sqrt{R})}$$

and define a test function (see Definition 3.1) with separated variables as

$$\psi(t,x) = \Phi(P_x)\Phi(t/T),$$

where

$$\Phi(z) = \begin{cases} 
1, & z \in (-\infty, 0], \\
\chi_z, & z \in [0, 1], \\
0, & z \in (1, \infty),
\end{cases}$$

is a function in $C^2(\mathbb{R})$ satisfying the estimates

$$\int_{0 \leq r \leq 1} \left| \frac{(\Phi' + \Phi'')(r)}{\Phi(r)} \right|^{\frac{1}{p-1}} r^{n-2} dr < \infty$$

and

$$\int_0^1 \left| \frac{\Phi(t)^p}{\Phi(t)} \right|^{\frac{1}{p-1}} dt < \infty.$$  

Note that

$$\text{supp}(\psi) \subseteq \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq t \leq T, -\infty < P_x \leq 1\}.$$  

As $-\infty < P_x \leq 1 \iff 0 < \|x\| \leq R$, the set $\text{supp}(\psi)$ is compact. The fact that

$$\psi_t(x,t) = \frac{\partial}{\partial t} (\Phi(P_x)\Phi(t/T)) = \frac{1}{T}\Phi(P_x)\Phi'(t/T)$$

implies

$$\text{supp}(\psi_t) \subseteq \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq t \leq T, -\infty < P_x \leq 1\}$$

and for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

$$\Delta_X \psi(t,x) = \Phi(t/T)\Delta_X \Phi(P_x)$$

$$= \Phi(t/T) \sum_{k=1}^m \left( X^2_k \Phi(P_x) + (\text{div}X_k)X_k \Phi(P_x) \right)$$

$$= \Phi(t/T) \sum_{k=1}^m \left( \Phi''(P_x)(X_k(P_x))^2 + \Phi'(P_x)X^2_k(P_x) \right)$$

$$+ \left( \sum_{i=1}^n \frac{\partial a_{k,i}}{\partial x_i} \right) \left( \sum_{j=1}^n a_{k,j} \frac{\partial}{\partial x_j} \Phi(P_x) \right)$$

$$+ \Phi(P_x) \left( \sum_{j=1}^n \frac{\partial P_x}{\partial x_j} \right)^2$$

$$+ \Phi'(P_x) \left( \sum_{i,j=1}^n a_{k,i}a_{k,j} \frac{\partial^2 P_x}{\partial x_i \partial x_j} + \sum_{j=1}^n \frac{\partial P_x}{\partial x_j} \left( \sum_{i=1}^n a_{k,i} \frac{\partial a_{k,j}}{\partial x_i} \right) \right)$$

$$+ \left( \sum_{i=1}^n \frac{\partial a_{k,i}}{\partial x_i} \right) \left( \sum_{j=1}^n a_{k,j} \Phi'(P_x) \frac{\partial P_x}{\partial x_j} \right).$$
implies that

\[\text{supp}(\Delta_X(\psi)) \subseteq \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq P_x \leq 1, 0 \leq t \leq T\}.\]

For large \(R\) we have \(\ln(\sqrt{R}) \geq 0\). Thus, \(0 \leq \frac{\ln(|x|)}{\ln(\sqrt{R})} \leq 1 \iff 0 \leq \ln\left(\frac{|x|}{\sqrt{R}}\right) \leq 0 \iff \exp(0) \leq \exp\left(\ln\left(\frac{|x|}{\sqrt{R}}\right)\right) \iff 1 \leq \frac{|x|}{\sqrt{R}} \leq \sqrt{R} \iff \sqrt{R} \leq \|x\| \leq R\). Hence,

\[\text{supp}(\Delta_X(\psi)) \subseteq \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \sqrt{R} \leq \|x\| \leq R, 0 \leq t \leq T\}.

As in Theorem 3.2,

\[
\int_0^T \int_{\mathbb{R}^n} f \psi \leq \frac{1}{p'(\frac{R}{2})p'-1}\left(\int_0^T \int_{\mathbb{R}^n} \left|\psi\right|^{-\frac{p}{p'-1}}\left|\Delta_X\psi\right|^{\frac{p}{p'-1}} + \left|\psi\right|^{-\frac{1}{p'-1}}\left|\psi_x\right|^{\frac{p}{p'-1}}\right).
\]

Using the assumptions on \(a_{k,i}\) and the fact that \(\Phi\) is twice continuously differentiable, one can choose

\[M = \max \left\{\|a_{k,i}(x)\|_{\text{sup}}, \|a_{k,i}(x)\|_{\text{sup}}, \left\|\frac{\partial a_{k,j}(x)}{\partial x_i}\right\|_{\text{sup}} : 1 \leq k \leq m, 1 \leq i,j \leq n\right\}.
\]

Note that \(|x_i| \leq \|x\|\) and \(\sqrt{R} \leq |x_i| \leq R\), in the support of \(\Delta_X\psi\), for every \(1 \leq i \leq n\). Thus, there exists a constant \(C > 0\) such that for every \((t, x) \in \text{supp}(\Delta_X\psi)\) we have

\[
\left|\Delta_X\psi(t, x)\right| \leq C|\Phi(t/T)| \sum_{k=1}^m \left(\left|\Phi''(P_x)\right| \left(\frac{\|x\|}{\ln(\sqrt{R})\|x\|}\right)^2 + \left|\Phi'(P_x)\right| \left(\sum_{i=1}^n \left|a_{k,j}a_{k,i}\right| \left(\frac{\delta_{ij}}{\|x\|} + \frac{2\|x\|^2}{\|x\|^4}\right)\right)\right.
\]

\[
+ \sum_{j=1}^n \frac{\|x\|}{\ln(\sqrt{R})\|x\|^2} \left(\sum_{i=1}^n \left|a_{k,i}\right| \left|\frac{\partial a_{k,j}}{\partial x_i}\right|\right)
\]

\[
+ \left(\sum_{i=1}^n \left|\frac{\partial a_{k,i}}{\partial x_i}\right|\right) \left(\sum_{j=1}^n \left|a_{k,j}\right| \left|\Phi'(P_x)\right| \left(\frac{\|x\|}{\ln(\sqrt{R})\|x\|^2}\right)\right)\right)
\]

\[
\leq \frac{C|\Phi(t/T)\Phi'(P_x) + \Phi''(P_x)|}{\ln \sqrt{R}\|x\|} \leq \frac{C|\Phi(t/T)\Phi'(P_x) + \Phi''(P_x)|}{\ln \sqrt{R}\sqrt{R}}.
\]

Here we have used the fact that 1-norm is equivalent to 2-norm on \(\mathbb{R}^n\).
Now, with change of the variables $x \leftrightarrow \frac{x}{\sqrt{R}}$ and $t \leftrightarrow \frac{t}{T}$ we obtain

\[
\int_0^T \int_{\mathbb{R}^n} |\psi|^{-\frac{1}{p-1}} |\Delta_X \psi|^{\frac{p}{p-1}} = \int_{\text{supp}(\Delta_X(\psi))} |\psi|^{-\frac{1}{p-1}} |\Delta_X \psi|^{\frac{p}{p-1}}
\]

\[
\leq \int_{\text{supp}(\Delta_X(\psi))} \Phi(P_x) \Phi \left( \frac{t}{T} \right) \left| \int_1^\infty \left( \frac{\Phi'(t)}{\Phi(t)} \right) |(\Phi' + \Phi'')(P_x)|^{\frac{1}{p-1}} \right| \frac{1}{p-1} \left( \frac{\ln(1/r)}{\ln(1/R)} \right) \frac{1}{p-1} r^{n-2} dr
\]

\[
= C \left( \frac{1}{\ln(1/R)} \right)^{\frac{p}{p-1}} T \int_0^1 \left| \Phi(t) \right| \int_{1 \leq r \leq 1} \left| \frac{(\Phi' + \Phi'')(P_x)}{\Phi(t)} \right|^{\frac{1}{p-1}} \left( \frac{\ln(1/r)}{\ln(1/R)} \right) \left( \frac{\ln(1/r)}{\ln(1/R)} \right) \frac{1}{p-1} r^{n-2} dr
\]

\[
\leq CT \ln \left( \frac{1}{R} \right)^{-n-1}.
\]

Where, we have used the fact that $\frac{p}{p-1} = n$. Similarly,

\[
\int_0^T \int_{\mathbb{R}^n} |\psi|^{-\frac{1}{p-1}} |\psi_t|^{\frac{p}{p-1}} \leq \int_{\text{supp}(\psi_t)} \left| \Phi(P_x) \Phi(t/T) \left| \frac{1}{T} \right| \Phi(P_x) \Phi(t/T) \right|^{\frac{p}{p-1}}
\]

\[
\leq \left( \frac{1}{T} \right)^{\frac{p}{p-1}} \int_{\text{supp}(\psi_t)} \left| \Phi(P_x) \left| \Phi(t/T) \right| \right|^{\frac{p}{p-1}} \left| \Phi'(t/T) \right|^{\frac{p}{p-1}}
\]

\[
\leq \left( \frac{1}{T} \right)^{\frac{p}{p-1}} T R^{\frac{p}{2}} \int_0^1 \int_{1 \leq \|x\| \leq \sqrt{R}} \left| \Phi \left( \frac{\ln(\|x\|)}{\ln(\sqrt{R})} \right) \right| \left| \Phi'(t) \right|^{\frac{1}{p-1}}
\]

\[
= \left( \frac{1}{T} \right)^{\frac{p}{p-1}} T R^{\frac{p}{2}} \int_1^{\sqrt{R}} \left| \Phi \left( \frac{\ln(\|x\|)}{\ln(\sqrt{R})} \right) \right| \left| \Phi'(t) \right|^{\frac{1}{p-1}}
\]

\[
= CT^{-n+1} R^{\frac{p}{2}} \int_1^{\sqrt{R}} \left| \Phi \left( \frac{\ln(\|x\|)}{\ln(\sqrt{R})} \right) \right| dx
\]

\[
= CT^{-n+1} R^{\frac{p}{2}} \int_1^{\sqrt{R}} \left| \Phi \left( \frac{\ln(r)}{\ln(\sqrt{R})} \right) \right| r^{n-1} dr
\]

\[
\leq CT^{-n+1} R^{\frac{p}{2}} \frac{1}{\ln(\sqrt{R})} \int_{0 \leq r \leq 1} \left| \Phi(r) \right| r^{n-2} dr
\]
the computations which we describe below. Set

\[ u_0 \in L^1_{\text{loc}}(\mathbb{R}^n) \]

Proof. Let \( u_0 \in L^1_{\text{loc}}(\mathbb{R}^n) \) be non-negative. If \( p < \frac{n}{n-2} \) and \( \int_{\mathbb{R}^n} f(x)dx > 0 \), then (1.1) does not admit a global in time weak solution.

One may now conclude that \( \frac{n}{n-1} \) serves as a lower bound for the possible critical exponent. In the following theorem we assume that all the functions \( a_{k,i} \) are constant. For an example of such vector fields, consider \( X_1 = \frac{\partial}{\partial x_1} \) and \( X_2 = 2 \frac{\partial}{\partial x_2} \) on \( \mathbb{R}^2 \). Then \( \Delta_X = X_1^2 + X_2^2 = \frac{\partial^2}{\partial x_1^2} + 4 \frac{\partial^2}{\partial x_2^2} \).

For such an \( X \) it is shown that there is a better lower bound \( \frac{n}{n-2} \). This covers, in particular, the well-known case of Laplacian \( \Delta \) where the critical exponent (famously known as the Fujita exponent) is \( \frac{n}{n-2} \). Thus, Theorem 3.4, which generalizes [1, Theorem 2.1 (a)], demonstrates that the technique used to prove Theorem 3.2 and Theorem 3.3 for obtaining a lower bound of a possible critical exponent may have actually led us to the actual critical exponent for (1.1), in some cases at least. In other words, we expect \( \frac{n}{n-1} \) to be the critical exponent for several classes of vector fields.

**Theorem 3.4.** Assume that for every \( 1 \leq k \leq m \) and \( 1 \leq i \leq n \) the function \( a_{k,i} \) is constant and let \( u_0 \in L^1_{\text{loc}}(\mathbb{R}^n) \) be non-negative. If \( p < \frac{n}{n-2} \) and \( \int_{\mathbb{R}^n} f(x)dx > 0 \), then (1.1) does not admit a global in time weak solution.

**Proof.** The proof is similar to that of Theorem 3.2 apart from some necessary modifications in the computations which we describe below. Set \( P_x \), \( \psi \) and \( \Phi \) as in Theorem 3.2.

Then

\[
\Delta_X \psi(t, x) = \Phi(2t/T) \Delta_X \Phi(P_x)
\]

\[
= \Phi(2t/T) \sum_{k=1}^{m} X_k^2 \Phi(P_x)
\]

\[
= \Phi(2t/T) \sum_{k=1}^{m} \left( \Phi''(P_x) (X_k(P_x))^2 + \Phi'(P_x) X_k^2(P_x) \right)
\]

\[
= \Phi(2t/T) \sum_{k=1}^{m} \left( \Phi''(P_x) \left( \sum_{j=1}^{n} a_{k,j} \frac{\partial P_x}{\partial x_j} \right)^2 \right)
\]

\[
+ \Phi'(P_x) \left( \sum_{k,i,j=1}^{n} a_{k,i} a_{k,j} \frac{\partial^2 P_x}{\partial x_i \partial x_j} + \sum_{j=1}^{n} \frac{\partial P_x}{\partial x_j} \left( \sum_{i=1}^{n} a_{k,j} \frac{\partial a_{k,i}}{\partial x_i} \right) \right)
\]

\[
= \Phi(2t/T) \sum_{k=1}^{m} \left( \Phi''(P_x) \left( \sum_{j=1}^{n} a_{k,j} \frac{2x_j}{T} \right)^2 \right) + \Phi'(P_x) \left( \sum_{j=1}^{n} a_{k,j} \frac{2x_j}{T} \right)
\]

gives

\[
\text{supp}(\Delta_X(\psi)) \subseteq \{(t, x) : 1 \leq P_x \leq 2, 0 \leq 2t/T \leq 2 \}. 
\]
To prove the theorem by contradiction, let us suppose that \( u \) is a global in time weak solution. As in Theorem 3.2, we obtain

\[
\int_0^T \int_{\mathbb{R}^n} f' \leq \frac{1}{p'(\frac{2}{p})^{p-1}} \left( \int_0^T \int_{\mathbb{R}^n} |\psi|^{\frac{1}{p-1}} |\Delta_X \psi|^{\frac{p}{p-1}} + |\psi|^{\frac{1}{p-1}} |\psi|^{\frac{p}{p-1}} \right).
\]

In the support of \( \Delta_X \psi \), we have \( |x_i| \leq \sqrt{T} \) for every \( 1 \leq i \leq n \). Using the assumptions on \( a_{k,i} \) and the fact that \( \Phi \) is twice continuously differentiable, choose

\[
M = \sup_{(t,x) \in (0,\infty) \times \mathbb{R}^n} \{ |a_{k,i}(x)|, |a_{k,i}(x)|^2 \}_{1 \leq k \leq m; 1 \leq i, j \leq n}.
\]

Thus, for large \( T \) there exists a constant \( K > 0 \) such that

\[
|\Delta_X \psi| \leq |\Phi(2t/T)| \sum_{k=1}^m \left( |\Phi''(P_x)| \left( \sum_{j=1}^n |a_{k,j}| \left| \frac{2x_j}{T} \right| \right)^2 + |\Phi'(P_x)| \left( \sum_{j=1}^n a_{k,j}^2 \right) \right)
\]

\[
\leq K |\Phi(2t/T)| (\Phi'(P_x) + \Phi''(P_x)) \sum_{k=1}^m \left( \left( \sum_{j=1}^n \left| \frac{2\sqrt{T}}{T} \right| \right)^2 + \sum_{j=1}^n \right)
\]

\[
\leq \frac{K |\Phi(2t/T)| (\Phi'(P_x) + \Phi''(P_x))}{T}.
\]

Now,

\[
\int_0^T \int_{\mathbb{R}^n} |\psi|^{\frac{1}{p-1}} |\Delta_X \psi|^{\frac{p}{p-1}} = \int_{\text{supp}(\Delta_X(\psi))} |\psi|^{\frac{1}{p-1}} |\Delta_X \psi|^{\frac{p}{p-1}}
\]

\[
\leq \int_{\text{supp}(\Delta_X(\psi))} \left| \Phi(P_x) \Phi(2t/T) \right|^{\frac{1}{p-1}} T \left( \frac{K |\Phi(2t/T)| (\Phi' + \Phi'')(P_x)}{T} \right)^{\frac{p}{p-1}}
\]

\[
\leq \frac{\left( \frac{K}{T} \right)^{\frac{p-1}{2}} T^{\frac{p-1}{2}}}{2} \int \int_{1 \leq \|x\| \leq 2} \left| \frac{(\Phi' + \Phi'')^p(\|x\|)}{\Phi(\|x\|)} \right|^{\frac{1}{p-1}} \Phi(t)
\]

\[
\leq CT^{\frac{p}{2}+1-\frac{p}{p-1}}.
\]

Similarly,

\[
\int_0^T \int_{\mathbb{R}^n} |\psi|^{\frac{1}{p-1}} |\psi|^{\frac{p}{p-1}} = \int_{\text{supp}(\psi)} |\psi|^{\frac{1}{p-1}} |\psi|^{\frac{p}{p-1}}
\]

\[
\leq \int_{\text{supp}(\psi)} \left| \Phi(P) \Phi(2t/T) \right|^{\frac{1}{p-1}} \left| \frac{2}{T} \Phi(P) \Phi'(2t/T) \right|
\]

\[
\leq \left( \frac{2}{T} \right)^{\frac{2}{p-1}} \int_{\text{supp}(\psi)} \left| \Phi(P) \right| \left| \Phi(2t/T) \right|^{\frac{p}{p-1}} \left| \Phi'(2t/T) \right|^{\frac{p}{p-1}}
\]

\[
\leq \left( \frac{2}{T} \right)^{\frac{p-1}{2}} T^{\frac{p-1}{2}} \int \int_{0 \leq \|x\| \leq 2} \left| \Phi(\|x\|) \right| \left| \Phi(t) \right|^{\frac{1}{p-1}} \left| \Phi'(t) \right|^{\frac{p}{p-1}}.
\]

Moreover,

\[
\int_0^T \int_{\mathbb{R}^n} f' \psi = \int_0^T \Phi(2t/T) \int_{\mathbb{R}^n} f \Phi(P_x) \geq T \int_{\mathbb{R}^n} f \Phi(P_x).
\]
Consequently, we obtain
\[ T \int_{\mathbb{R}^n} f \Phi(P_x) \leq \int_0^T \int_{\mathbb{R}^n} f \psi \leq C(T^{\frac{n}{2} + 1 - \frac{p}{p-1}} + T^{\frac{n}{2} + 1 - \frac{p}{p-1}}). \]

Using the fact that \( \frac{n}{2} < \frac{p}{p-1} \), taking \( T \to \infty \) and applying the Lebesgue dominated convergence theorem we obtain \( \int_{\mathbb{R}^n} f \Phi \to \int_{\mathbb{R}^n} f \leq 0 \). This contradicts the hypothesis that \( \int_{\mathbb{R}^n} f > 0 \). So, no such \( u \) exists and the statement is proved.

The following theorem provides an estimate for the upper bound on the blow-up time of local in time weak solution to \( (1.1) \).

**Theorem 3.5.** Let \( p < \frac{n}{n-1} \). Assume that for \( f \in L^1(\mathbb{R}^n) \) there exists an \( \epsilon > 0 \) such that \( f(x) \geq \epsilon \| x \|^{-\lambda} \) whenever \( \| x \| \geq 1 \) and \( \lambda \in (0, \frac{p}{p-1}) \). Let \( T_\epsilon \) denote a time such that a weak solution exists in \([0, T_\epsilon)\). For every \( 1 \leq k \leq m \) and \( 1 \leq i, j \leq n \), if the functions \( \alpha_{k,i} \) and \( \frac{\partial \alpha_{k,i}}{\partial x_i} \) are bounded then there exists a positive constant \( C \) such that \( T_\epsilon \leq C\epsilon^{1/(\lambda - \frac{p}{2(p-1)})} \).

**Proof.** For the test function as used in the proof of Theorem 3.2, we observed that
\[ \int_{\mathbb{R}^n} f \Phi(P_x) \leq CT_\epsilon^{\frac{n}{2} - \frac{p}{2(p-1)}}. \]

For an arbitrary \( T_0 < T_\epsilon \), changing the variable \( x \to x' := \frac{x}{\sqrt{T_\epsilon}} \) implies that \( \| x \| \geq 1 \) if and only if \( \| x' \| \geq \frac{1}{\sqrt{T_\epsilon}} \). Thus,
\[ C T_\epsilon^{\frac{n}{2} - \frac{p}{2(p-1)}} \geq \int_{\mathbb{R}^n} f(x) \Phi(P_x) dx \geq \int_{\| x \| \geq 1} f(x) \Phi(P_x) dx \geq \epsilon \int_{\| x \| \geq 1} \| x \|^{-\lambda} \Phi(P_x) dx \]
\[ = \epsilon T_\epsilon^{-\frac{\lambda}{2}} \int_{\| x \| \geq 1} \left\| \frac{x}{\sqrt{T_\epsilon}} \right\|^{-\lambda} \Phi(P_x) dx = \epsilon T_\epsilon^{\frac{n-\lambda}{2}} \int_{\| x' \| \geq \frac{1}{\sqrt{T_\epsilon}}} \| x' \|^{-\lambda} \Phi(P_x) dx' \]
\[ \geq \epsilon T_\epsilon^{\frac{n-\lambda}{2}} \int_{\| x' \| \geq \frac{1}{\sqrt{T_\epsilon}}} \| x' \|^{-\lambda} \Phi(\| x' \|^2) dx' = \epsilon C_2 T_\epsilon^{\frac{n-\lambda}{2}}. \]

As \( \frac{n-\lambda}{p-1} > \lambda \), we obtain
\[ \epsilon \leq \frac{1}{C_1} T_\epsilon^{\frac{n}{2} - \frac{p}{2(p-1)} - \frac{\lambda}{p-1}} = \frac{1}{C_1} T_\epsilon^{\frac{p}{2(p-1)} + \frac{1}{2}} = \frac{1}{C_1} T_\epsilon^{\frac{1}{2}} = \frac{1}{C_1} \]
\[ \text{and hence } T_\epsilon^{\frac{p}{2(p-1)} - \frac{\lambda}{p-1}} \leq \frac{1}{C_1 \epsilon} \text{ which implies } T_\epsilon \leq \left( \frac{1}{C_1 \epsilon} \right)^{\frac{p}{2(p-1)} + \frac{1}{2}} = C \epsilon^{2/(\lambda - \frac{p}{2(p-1)}}. \]

**Remark 3.6.** With a minor modification in the test functions used to obtain results in this section, the non-negativity assumption on \( u_0 \) may be removed. The only use of non-negativity of \( u_0 \) in the proofs was to get rid of the term \( \int_{\mathbb{R}^n} u_0 \psi(0, x) \) in equation 3.1 so that we have inequality 3.2. However, this can be achieved by considering a test function \( \psi \) which vanishes at time \( t = 0 \).

In particular, in Theorem 3.2, instead of the test function \( \psi(t, x) = \Phi(P_x)\Phi(t/T) \), one may use \( \psi(t, x) = \Phi(P_x)\Phi_1(t/T) \) where \( \Phi_1 \in C_0^\infty(\mathbb{R}^n) \) is an appropriate function supported in \((0, 1)\).

It will be demonstrated through the following section that similar steps then lead to the desired conclusion.

4. On certain non-invariant vector fields

We now study more general vector fields and take upon the delicate task of constructing test functions which will help us find conditions under which \((1.1)\) does not admit a global in time
weak solution. What is special about these vector fields is that although they fall within the class of vector fields being considered in this paper (with \( \text{div} X_k = 0 \)), they do not give rise to a stratified Lie group structure. For this reason, our results generalize the subcritical case considered in [8] in the sense that the group structure on the domain is not contributing to our proofs. The reader is referred to [7], Examples 1 & 6.6, for more detailed account on the Hörmander sum of squares of vector fields discussed in this section.

**Theorem 4.1.** Let \( p < \frac{k+2}{k} \), \( X_1 = \frac{\partial}{\partial x_1} \) and \( X_2 = x_1^k \frac{\partial}{\partial x_2} \) on \( \mathbb{R}^2 \). If \( \int_{\mathbb{R}^2} f(x)dx > 0 \), then (1.1) does not admit global in time weak solution.

**Proof.** Let us suppose on the contrary that \( u \) is a global in time weak solution. Then for every test function \( \psi \),

\[
\int_0^T \int_{\mathbb{R}^2} |u|^p \psi + \int_{\mathbb{R}^2} u_0 \psi(0, x) + \int_0^T \int_{\mathbb{R}^2} f \psi + \int_0^T \int_{\mathbb{R}^2} u \psi_t + \int_0^T \int_{\mathbb{R}^2} u \Delta X \psi = 0.
\]

For a test function \( \psi \) which is zero at time \( t = 0 \), we obtain

\[
\int_0^T \int_{\mathbb{R}^2} f \psi + \int_0^T \int_{\mathbb{R}^2} |u|^p \psi = - \int_0^T \int_{\mathbb{R}^2} u \psi_t - \int_0^T \int_{\mathbb{R}^2} u \Delta X \psi \\
\leq \int_0^T \int_{\mathbb{R}^n} |u||\psi_t| + |u||\Delta X \psi|.
\]

(4.1)

Let

\[
d(x) = x_1^{2k+2} + x_2^2
\]

and

\[
f(a) = \frac{a}{T^{k+1}}, \text{ for } a \in \mathbb{R}.
\]

Define a test function with separated variables as

\[
\psi_T(t, x) = \Phi_1(t/T) \Phi_2((f \circ d)(x)),
\]

where \( 0 \leq \Phi_1 \in C^\infty(\mathbb{R}) \) with \( \text{supp}(\Phi_1) \subseteq (0, 1) \) satisfies the estimate

\[
\int_0^1 |\Phi_1(t)|^{-\frac{p}{p-1}} |\Phi_1'(t)|^{-\frac{p}{p-1}} < \infty
\]

and

\[
\Phi_2(z) = \begin{cases} 
1, & z \in [0, 1], \\
\sqrt{z}, & z \in [1, 2], \\
0, & z \in (2, \infty),
\end{cases}
\]

is a function in \( C_0^2(\mathbb{R}^+ \right) \) satisfying the estimate

\[
\int_{0 \leq P'(x) \leq 1} \left| \frac{(\Phi_2')^p(P'(x))}{\Phi_2(P'(x))} \right|^{\frac{1}{p-1}} < \infty
\]

where \( P'(x) := (x_1')^{2k+2} + (x_2')^2 \) with \( x_1' := \frac{x_1}{\sqrt{T}} \) and \( x_2' := \frac{x_2}{\sqrt{T}} \). Note that

\[
\text{supp}(\psi_T) \subseteq \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2 : 0 \leq \frac{t}{T} \leq 1, 0 \leq (f \circ d)(x) \leq 2 \right\}.
\]

Now

\[
\Delta X \psi_T(t, x) = \Phi_1(t/T) \Delta X ((\Phi_2 \circ f) \circ d)(x)
\]

and with \( \Psi := \Phi_2 \circ f \) we obtain

\[
\Delta X ((\Phi_2 \circ f) \circ d) = \Delta X (\Psi \circ d)
\]
Now, for $\Delta X\psi_T(x) = \sum_{k=1}^{2} (X_k^2(\Psi \circ d))
= \sum_{k=1}^{2} (\Psi''(d)(X_k(d))^2 + \Psi'(d)X_k^2(d))$
implies that

$$\text{supp}(\Delta_X \psi_T) = \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2 : 0 \leq \frac{t}{T} \leq 1 \leq (f \circ d)(x) \leq 2 \right\}.$$ 

Note that

$$\Psi''(d) = (\Phi_2 \circ f)''(d) = \frac{(\Phi_2'(f)(d))'}{T^{k+1}}.$$ 

Since

$$X_1(d(x)) = (2k + 2)x_1^{2k+1}, \quad X_2(d(x)) = 2x_1^k x_2,$$
$$X_1^2(d(x)) = (2k + 2)(2k + 1)x_1^{2k} \quad \text{and} \quad X_2^2(d(x)) = 2x_1^{2k};$$
we obtain

$$\Delta_X((\Phi_2 \circ f) \circ d) = \sum_{k=1}^{2} (\Psi''(d)(X_k(d))^2 + \Psi'(d)X_k^2(d))$$
$$= \frac{(\Phi_2'(f)(d))'}{T^{k+1}} \left( ((2k + 2)x_1^{2k+1})^2 + (2x_1^k x_2)^2 \right)$$
$$+ \left( \frac{(\Phi_2'(f)(d))'}{T^{k+1}} \right) ((2k + 2)(2k + 1)x_1^{2k} + 2x_1^{2k}).$$

For $x \in \text{supp}(\Delta_X \psi_T)$ we have $|x_1| \leq C\sqrt{T}$ and $|x_2| \leq C T^{\frac{k+1}{2}}$ and hence

$$|\Delta_X((\Phi_2 \circ f) \circ d)| \leq C \left( \Phi_2'(f \circ d)(x) + \Phi_2''((f \circ d)(x)) \right).$$

Now

$$(\psi_T)_t(t, x) = \frac{\partial}{\partial t} \left( \Phi_1 \left( \frac{t}{T} \right) \Phi_2((f \circ d)(x)) \right) = \frac{\Phi_1'(t/T)\Phi_2((f \circ d)(x))}{T}$$
implies that

$$\text{supp}((\psi_T)_t) \subseteq \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2 : 0 \leq \frac{t}{T} \leq 1, 0 \leq (f \circ d)(x) \leq 2 \right\}$$
and

$$|(\psi_T)_t| \leq \frac{C}{T}.$$ 

As in Theorem 3.2, one obtains

$$\int_{0}^{T} \int_{\mathbb{R}^2} f \psi_T \leq C \left( \int_{0}^{T} \int_{\mathbb{R}^2} |\psi_T|^{\frac{1}{p-1}}|\Delta_X \psi_T|^{\frac{p}{p-1}} + |\psi_T|^{\frac{1}{p-1}}|((\psi_T)_t)|^{\frac{p}{p-1}} \right).$$

Now, for $P(x) := (f \circ d)(x)$ and $S := \text{supp}(\Delta_X \psi_T)$, changing the variable $x \leftrightarrow x'$ and $t \leftrightarrow t' := \frac{1}{T}$ gives

$$\int_{0}^{T} \int_{\mathbb{R}^2} |\psi_T|^{\frac{1}{p-1}}|\Delta_X \psi_T|^{\frac{p}{p-1}} = \int_{S} |\psi_T|^{\frac{1}{p-1}}|\Delta_X \psi_T|^{\frac{p}{p-1}}$$
By dominated convergence theorem,

\[ \int_S \bigg| \Phi_2(P(x)) \Phi_1 \left( \frac{t}{T} \right) \bigg|^{-\frac{1}{p-1}} \left( C(\Phi_2 + \Phi_2')(P(x)) \right)^{\frac{1}{p-1}} \]

\[ = \left( \frac{C}{T} \right)^{\frac{p}{p-1}} \int_S \left| \Phi_2(P(x)) \Phi_1 \left( \frac{t}{T} \right) \right|^{-\frac{1}{p-1}} \left( \Phi_2 + \Phi_2'(P(x)) \right)^{\frac{1}{p-1}} \]

\[ = C \left( \frac{1}{T} \right)^{\frac{p}{p-1}} T^{\frac{k+2}{2}} \int_0^1 \int_{0 \leq P(x) \leq 1} \left| \Phi_2(P(x)) \right| \left| \Phi_1(t') \right|^{\frac{1}{p-1}} \left| \Phi_1(t) \right|^{\frac{p}{p-1}} \]

Similarly, for \( S_1 := \text{supp}(\psi) \) we have

\[ \int_0^T \int_{\mathbb{R}^n} |\psi_T|^\frac{1}{p-1} |\psi_T'|^{\frac{p}{p-1}} = \int_{S_1} |\psi_T|^\frac{1}{p-1} |\psi_T'|^{\frac{p}{p-1}} \]

\[ = \int_{S_1} \left| \Phi_2(P(x)) \Phi_1 \left( \frac{t}{T} \right) \right|^{\frac{p}{p-1}} \left| \frac{1}{T} \Phi_2(P(x)) \Phi_1' \left( \frac{t}{T} \right) \right|^{\frac{p}{p-1}} \]

\[ = \left( \frac{1}{T} \right)^{\frac{p}{p-1}} \int_{S_1} \left| \Phi_2(P(x)) \right| \left| \Phi_1(t'/T) \right|^{\frac{1}{p-1}} \left| \Phi_1(t/T) \right|^{\frac{p}{p-1}} \]

\[ = \left( \frac{1}{T} \right)^{\frac{p}{p-1}} T^{\frac{k+2}{2}} \int_0^1 \int_{0 \leq P(x) \leq 1} \left| \Phi_2(P'(x)) \right| \left| \Phi_1(t') \right|^{\frac{1}{p-1}} \left| \Phi_1(t) \right|^{\frac{p}{p-1}} \]

\[ \leq C \left( \frac{1}{T} \right)^{\frac{p}{p-1}} T^{\frac{k+2}{2}}. \]

Since

\[ \int_0^T \int_{\mathbb{R}^2} f \psi_T = \int_0^T \int_{\mathbb{R}^2} \Phi_1 \left( \frac{t}{T} \right) \int_{\mathbb{R}^2} \Phi_2(P(x)) \int_0^1 \int_{\mathbb{R}^2} \Phi_2(P(x)) = CT \int_{\mathbb{R}^2} \Phi_2(P(x)), \]

we obtain

\[ \int_{\mathbb{R}^2} \Phi_2(P(x)) \leq CT^{-1} \int_0^T \int_{\mathbb{R}^2} f \psi_T \leq CT^{\frac{k+2}{2}} - \frac{p}{p-1}. \]

By dominated convergence theorem, \( \int_{\mathbb{R}^2} \Phi_2 \to \int_{\mathbb{R}^2} f \) as \( T \to \infty \). Using the fact that \( p < \frac{k+2}{k} \) and taking \( T \to \infty \) we obtain \( \int_{\mathbb{R}^2} f \leq 0 \). This contradicts the hypothesis that \( \int_{\mathbb{R}^2} f > 0 \). So, no such \( u \) exists and the statement is proved.

**Theorem 4.2.** Let \( p < \frac{2^n-1}{2^n-3} \), \( X_1 = \frac{\partial}{\partial x_1} \) and \( X_2 = x_1 \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial x_2} + \ldots + x_n^2 \frac{\partial}{\partial x_n} \) on \( \mathbb{R}^n \). If \( \int_{\mathbb{R}^n} f(x)dx > 0 \), then (1.1) does not admit a global in time weak solution.

**Proof.** Let us suppose on the contrary that \( u \) is a global in time weak solution. As in Theorem 4.1, for a test function \( \psi \) which is zero at time \( t = 0 \), we obtain

\[ \int_0^T \int_{\mathbb{R}^n} f \psi + \int_0^T \int_{\mathbb{R}^n} |u|^p \psi = - \int_0^T \int_{\mathbb{R}^n} u \psi_t - \int_0^T \int_{\mathbb{R}^n} u \Delta x \psi \]

\[ \leq \int_0^T \int_{\mathbb{R}^n} |u| |\psi_t| + |u| |\Delta x \psi|. \]
Let
\[ d(x) = x_1^{2n} + x_2^{2n-1} + x_3^{2n-2} + \cdots + x_n^2 \]
and
\[ f(a) = \frac{a}{T^{2n-1}}, \text{ for every } a \in \mathbb{R}. \]
Define a test function with separated variables as
\[ \psi_T(t, x) = \Phi_1(t/T)\Phi_2((f \circ d)(x)), \]
where \( 0 \leq \Phi_1 \in C^\infty(\mathbb{R}) \) with \( \text{supp}(\Phi_1) \subseteq (0, 1) \) satisfies the estimate
\[ \int_0^1 |\Phi_1(t)|^{\frac{1}{p-1}}|\Phi_1'(t)|^{\frac{1}{p-1}} \, dt < \infty \]
and
\[ \Phi_2(z) = \begin{cases} 1, & z \in [0, 1], \\ \zeta, & z \in [1, 2], \\ 0, & z \in (2, \infty), \end{cases} \]
is a function in \( C^2_0(\mathbb{R}^+) \) satisfying the estimate
\[ \int_{0 \leq P'(x) \leq 1} \left| \frac{\Phi_2' + \Phi_2'' P'(x)}{\Phi_2(P'(x))} \right|^{\frac{1}{p-1}} \, dx < \infty \]
where \( P'(x) := (x_1')^{2n} + (x_2')^{2n-1} + (x_3')^{2n-2} + \cdots + (x_n')^2 \) with \( x_i' := \frac{x_i}{\sqrt{T^{2n-1}}} \) for every \( 1 \leq i \leq n. \)
Note that
\[ \text{supp}(\psi_T) \subseteq \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq \frac{t}{T} \leq 1, 0 \leq P(x) \leq 2 \right\}. \]
Now
\[ \Delta_X \psi_T(t, x) = \Phi_1(t/T)\Delta_X((\Phi_2 \circ f) \circ d)(x) \]
and for \( \Psi := \Phi_2 \circ f \) we obtain
\[ \Delta_X((\Phi_2 \circ f) \circ d) = \Delta_X(\Psi \circ d) \]
\[ = \sum_{k=1}^2 (X_k^2(\Psi \circ d)) \]
\[ = \sum_{k=1}^2 (\Psi''(d)(X_k(d))^2 + \Psi'(d)X_k^2(d)) \]
implying that
\[ \text{supp}(\Delta_X \psi_T) = \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq \frac{t}{T} \leq 1 \leq (f \circ d)(x) \leq 2 \right\}. \]
Note that
\[ \Psi''(d) = (\Phi_2 \circ f)'(d)' = \frac{(\Phi_2'(f)(d))'}{T^{2n-1}} = \frac{\Phi_2''(f)(d)}{T^{2n}}. \]
Therefore, for \( x \in \text{supp}(\Delta_X \psi_T) \) we have \( x_i \leq C \sqrt{T^{2n-1}} \) for every \( 1 \leq i \leq n. \) Since
\[ X_1(d(x)) = 2^n x_1^{2n-1}, \quad X_2(d(x)) = x_1 2^{n-1} x_2^{2n-1} + 2^{n-2} x_2 x_3^{2n-2} + \cdots + 2 x_1^{n-1} x_n, \]
\[ X_1^2(d(x)) = 2^n(2^{n-1} - 1)x_1^{2n-2} \]
and
\[ X_2^2(d(x)) = x_1^{2n-1}(2^{n-1} - 1)x_2^{2n-2} + 2^{n-2}(2^{n-2} - 1)x_1^4 x_3^{2n-2} + \cdots + 2 x_1^{2n-2}, \]
we have
\[ \Delta_X((\Phi_2 \circ f) \circ d) = \sum_{k=1}^{2} (\Psi''(d)(X_k(d))^2 + \Psi'(d)X_k^2(d)) \]
\[ = \frac{\Phi_2''(f)(d)}{T^{2n+1}} ((2^n x_1^{2n-1})^2 + (x_1 2^{n-1} x_2^{2n-1} - 1 + \cdots + 2x_1^{n-1} x_n)^2) \]
\[ + \left( \frac{\Phi_2'(f)(d)}{T^{2n}} \right) (2^n (2^n - 1) x_1^{2n-2} + x_1^{2n-1} (2^n - 1) x_2^{2n-1} + 2^n - 1) x_1^{2n-2} + \cdots + 2^{2n-2} \]
and hence
\[ |\Delta_X((\Phi_2 \circ f) \circ d)| \leq C |(\Phi_2' + \Phi_2''(f)(d))| \frac{T}{T} \]
Also,
\[ (\psi_T)(t, x) = \frac{\partial}{\partial t} \left( \Phi_1 \left( \frac{t}{T} \right) \Phi_2((f \circ d)(x)) \right) = \frac{\Phi_1'(t/T) \Phi_2((f \circ d)(x))}{T} \]
implies that
\[ \text{supp}((\psi_T)_t) \subseteq \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : 0 \leq \frac{t}{T} \leq 1, 0 \leq (f \circ d)(x) \leq 2 \right\} \]
and
\[ |(\psi_T)_t| \leq \frac{C}{T}. \]
As in Theorem 3.2 one obtains
\[ \int_0^T \int_{\mathbb{R}^n} f(\psi_T) \leq C \int_0^T \int_{\mathbb{R}^n} |\psi_T| \frac{1}{p-1} |\Delta_X \psi_T| \frac{p}{p-1} + |\psi_T| \frac{1}{p-1} |(\psi_T)_t| \frac{1}{p-1}. \]
Now, for \( t' = \frac{T}{2}, P(x) := (f \circ d)(x) \) and \( S := \text{supp}(\Delta_X \psi_T) \), we have
\[ \int_0^T \int_{\mathbb{R}^n} |\psi_T| \frac{1}{p-1} |\Delta_X \psi| \frac{p}{p-1} = \int_S |\psi_T| \frac{1}{p-1} |\Delta_X \psi| \frac{p}{p-1} \]
\[ \leq \int_S |\Phi_2(P(x))| \frac{1}{p-1} \left( \frac{C |(\Phi_2' + \Phi_2''(f)(d))|}{T} \right) \frac{p}{p-1} |\Phi_1(t/T)| \]
\[ = C \left( \frac{1}{T} \right) \frac{p}{p-1} \sqrt{T^{1+2^2+\cdots+2^{n-1}}} T \int_{0 \leq P'(x) \leq 2} \left| \frac{(\Phi_2 + \Phi_2'')P'(x)}{\Phi_2(P'(x))} \right| \frac{1}{p-1} \]
\[ = C \left( \frac{1}{T} \right) \frac{p}{p-1} \sqrt{T^{2^{n-1}}}. \]
Similarly, for \( S_1 := \text{supp}((\psi_T)_t) \), we have
\[ \int_0^T \int_{\mathbb{R}^n} |\psi_T| \frac{1}{p-1} |(\psi_T)_t| \frac{p}{p-1} = \int_{S_1} |\psi_T| \frac{1}{p-1} |(\psi_T)_t| \frac{p}{p-1} \]
\[ = \int_{S_1} |\Phi_2(P(x))\Phi_1(t/T)| \frac{1}{p-1} \left| \frac{1}{T} \Phi_2(P(x)) \Phi_1(t/T) \right| \frac{p}{p-1} \]
\[
\frac{1}{T} \int_{S_1} |\Phi_2(P(x))| |\Phi(t/T)|^{\frac{1}{p-1}} |\Phi'(t/T)|^{\frac{1}{p-1}} \\
= \left( \frac{2}{T} \right)^{\frac{n^2-3}{2}} T \int_0^1 \int_{0 \leq P'(x) \leq 2} |\Phi_2(P'(x))| |\Phi_1(t')|^{\frac{1}{p-1}} |\Phi'_1(t')|^{\frac{1}{p-1}}.
\]

As,
\[
\int_0^T \int_{\mathbb{R}^n} f \psi_T = \int_0^T \Phi_1(t/T) \int_{\mathbb{R}^n} f \Phi_2(P(x)) = T \int_0^1 \Phi_1(t') \int_{\mathbb{R}^n} f \Phi_2(P(x)) = CT \int_{\mathbb{R}^n} f \Phi_2(P(x))
\]
we obtain
\[
\int_{\mathbb{R}^n} f \Phi_2(P(x)) \leq CT^{-1} \int_0^T \int_{\mathbb{R}^n} f \psi_T \leq CT^{\frac{n^2-1}{2}} - \frac{p}{p-1}.
\]

By dominated convergence theorem, \(\int_{\mathbb{R}^n} f \Phi_2 \rightarrow \int_{\mathbb{R}^n} f\) as \(T \rightarrow \infty\). Using the fact that \(p < \frac{2n-1}{2n-3}\) and taking \(T \rightarrow \infty\) we obtain \(\int_{\mathbb{R}^n} f \leq 0\). This contradicts the hypothesis that \(\int_{\mathbb{R}^n} f > 0\). So, no such \(u\) exists and the statement is proved. \(\square\)

**Acknowledgment**

The authors grateful to anonymous referees who pointed out several inaccuracies in an earlier version of this paper.

**Funding**

This research was funded by Nazarbayev University under Collaborative Research Program Grant 20122022CRP1601.

**Author contribution**

The research is a result of continuous discussions between the two authors. Every part of it is contributed by both of them.

**Conflict of interest**

The authors declare that they have no conflict of interest.

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