Markov Approximations of the Evolution of Quantum Systems

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Abstract—The convergence in probability of a sequence of iterations of independent random quantum dynamical semigroups to a Markov process describing the evolution of an open quantum system is studied. The statistical properties of the dynamics of open quantum systems with random generators of Markovian evolution are described in terms of the law of large numbers for operator-valued random processes. For compositions of independent random semigroups of completely positive operators, the convergence of mean values to a semigroup described by the Gorini–Kossakowski–Sudarshan–Lindblad equation is established. Moreover, a sequence of random operator-valued functions with values in the set of operators without the infinite divisibility property is shown to converge in probability to an operator-valued function with values in the set of infinitely divisible operators.

Keywords: random linear operator, random operator-valued function, operator-valued random process, law of large numbers, open quantum system, Markovian process, Gorini–Kossakowski–Sudarshan–Lindblad equation

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Below, we discuss the convergence in probability of iterations of independent random quantum dynamical semigroups to a Markov process describing the evolution of an open quantum system. In doing this, we apply one-parameter families of completely positive mappings of the algebra of bounded linear operators into itself that satisfy the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation [1–3]. Additionally, we examine the asymptotic properties of a sequence of compositions of independent random completely positive transformations of the algebra \(\mathcal{L}(H)\) of linear operators acting in a finite-dimensional complex Hilbert space \(H\). The results presented in this paper are obtained using the approach to the general theory of random semigroups developed in [4, 5].

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Statistical properties of the dynamics of open quantum systems with random generators of Markovian evolution were considered in [6, 7]. In the approach proposed in this paper, such properties are described in terms of the law of large numbers for operator-valued random processes. We discuss random variables with values in the space of strongly continuous mappings of the real half-line to the cone of completely positive operators of the Banach space \(\mathcal{L}(\mathcal{L}(H))\). In what follows, we use the terminology and notation from [8]. Given a locally convex space \(E\), the set of linear continuous mapping from \(E\) to \(E\) is denoted by the symbol \(\mathcal{L}(E)\) (rather than by \(\mathcal{L}(E, E)\)). Specifically, the linear space of linear continuous mappings from \(E\) to \(\mathcal{L}(E)\) is denoted by \(\mathcal{L}(E)\).

For random completely positive semigroups, we establish the convergence of mean values to a semigroup of mappings described by the GKSL equation. Moreover, a sequence of operator-valued functions with values in the set of operators without the infinite divisibility property is shown to converge to an operator-valued function with values in the set of infinitely divisible operators.

Note that a similar problem, namely, the convergence in distribution of a sequence of products of independent random matrices was studied in [9]. However, the problem formulation and the results presented below differ considerably from the results of [9]. Specifically, Theorem 7 in [9] presents a law of large numbers stating the convergence in distribution
of a sequence of products of random matrices to a
deterministic limit matrix, while the law of large
numbers in this paper establishes the convergence in prob-
ability of a sequence of random compositions to a limit
semigroup and allows us to estimate the probability of deviation in the form of the Chebyshev inequality [4].

Note that the conditions we impose on random semigroups differ significantly from the conditions used in [4, 10]. Moreover, we systematically apply the combinatorial approach proposed in [11].

1. QUANTUM DYNAMICAL SEMIGROUPS

Throughout this paper, $H = \mathbb{C}^d$, where $d \in \mathbb{N}$. Since $H$ is a finite-dimensional space, it follows that any of two norms in each of the tensor products discussed below are equivalent and that these tensor products are complete with respect to the topologies defined by each of these norms.

In what follows, let $\mathcal{L}(H)$ be the Banach space of linear operators in $H$ equipped with the standard operator norm, and let $\mathcal{L}(\mathcal{L}(H))$ be the Banach space of linear mappings of $\mathcal{L}(H)$ into itself equipped with the standard operator norm. Recall that an element $A \in \mathcal{L}(\mathcal{L}(H))$ is called positive if $A(X) \geq 0$ for any $X \in \mathcal{L}(H)$ such that $X \geq 0$.

An element $A \in \mathcal{L}(\mathcal{L}(H))$ is called a completely positive mapping of the space $\mathcal{L}(H)$ into itself (see [12]) if the linear operator $1d \otimes A$ acting in the algebra $\mathcal{M}_d \otimes \mathcal{L}(H)$ according to the rule

$$1d \otimes A(M \otimes X) = M \otimes A(X) \quad \forall M \in \mathcal{M}_d,$$

is positive in the operator algebra $\mathcal{M}_d \otimes \mathcal{L}(H)$, where $\mathcal{M}_d$ is the algebra of $d \times d$ matrices over the field of complex numbers.

**Theorem [13].** If $A \in \mathcal{L}(\mathcal{L}(H))$, then $A$ is completely positive if and only if there exists a set of operators

$$V_\alpha \in \mathcal{L}(H), \quad a = 1, 2, \ldots, d^2$$

such that

$$A(X) = \sum_{a=1}^{d^2} V_\alpha^* X V_\alpha.$$

The following lemmas are proved by direct verification.

**Lemma 1.** The sum of two completely positive mappings is a completely positive mapping.

**Lemma 2.** The composition of two completely positive mappings is a completely positive mapping.

Recall that a quantum channel in a space of observables is defined as a linear completely positive mapping of the space $\mathcal{L}(H)$ into itself that preserves the identity operator $I \in \mathcal{L}(H)$. A one-parameter continuous semigroup of quantum channels in the algebra of linear operators of $\mathcal{L}(H)$ is called a quantum dynamical semigroup.

**Theorem** (Gorini–Kossakowski–Sudarshan–Lindblad theorem) [14, 15]. The generator $\mathcal{L}$ of any one-parameter uniformly continuous semigroup $W(t), t \in \mathbb{R}_+$, of completely positive mappings of the space $\mathcal{L}(H)$ into itself is defined by the equality

$$\mathcal{L}(X) = \sum_{a} L_a^* XL_a - XK - K^*X,$$

where $K = \frac{1}{2} \sum_{a} L_a^* L_a + iH$, $\{L_a\}$ is a set of at most $d^2 - 1$ operators from the algebra $\mathcal{M}_d$ and $H = H^* \in \mathcal{M}_d$.

2. RANDOM SEMIGROUPS

The generator of a quantum dynamical semigroup $\mathcal{L}$ is random if the operators $L_a$ and $H$ in (1) are random variables with values in the matrix algebra $\mathcal{M}_d$. A random variable with values in the space of linear mappings of $\mathcal{L}(H)$ into itself is defined as a measurable mapping of the probability space $(\Omega, \mathcal{F}, P)$ to $\mathcal{L}(\mathcal{L}(H))$ equipped with the weak operator topology (in the case of a finite-dimensional space $H$, coincides with the operator norm topology). Compositions of random orthogonal transformations of finite-dimensional Euclidean spaces were studied in [17].

The task we address is to investigate the properties of compositions of independent random processes with values in the cone $\mathcal{Q}(\mathcal{L}(H))_{CP}$ of completely positive mappings of $\mathcal{L}(H)$.

The set of uniformly continuous quantum dynamical semigroups acting in the Banach algebra $\mathcal{L}(H)$ is denoted by $\mathcal{Q}(\mathcal{L}(H))$. By the Hille–Yosida and Lindblad theorems, there is a bijection between the set $\mathcal{Q}(\mathcal{L}(H))$ equipped with the topology of uniform convergence on each interval in the space of linear operators and the set $\mathcal{Q}(\mathcal{L}(H))$ of generators of continuous quantum dynamical semigroups equipped with the operator norm topology. Note that, in our case, uniform continuity follows from continuity.

Let $\Phi_t, t \geq 0$, be a random quantum dynamical semigroup in the algebra $\mathcal{L}(H)$ defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e., $(\Phi_t)_{t \geq 0}: \omega \rightarrow (\Phi_t(\omega))_{t \geq 0}$ is an $\mathcal{F}$-measurable mapping with values in $\mathcal{Q}(\mathcal{L}(H))$. The mean value of $\Phi_t, t \geq 0$, is its expectation $M[\Phi_t]_{t \geq 0}$ defined by the equality

$$M[\Phi_t] = \int \Phi_t(\omega) P(d\omega), \quad t \geq 0.$$

If $\Phi_t, t \geq 0$, is a random quantum dynamical semigroup in $\mathcal{L}(H)$, then its expectation is a one-parameter family of completely positive mappings, which may
not be a one-parameter semigroup. However, for a large class of random quantum dynamical semigroups, \( M(\{\Phi_t\}) \), \( t \geq 0 \), is Chernoff equivalent to a semigroup with an averaged generator. Recall that an operator-valued function \( F: [0, +\infty) \to L(X) \) is called Chernoff equivalent to the linear transformation semigroup \( U(t), t \geq 0 \), on the space \( X \) if, for each \( x \in X \) and each 
\[
T > 0, \text{ it is true that } \lim_{n \to \infty} \sup_{t \in [0, T]} \left| \left( F \left( \frac{t}{n} \right)^n - U(t)x \right) \right| = 0.
\]

The Chernoff theorem (see [16]) provides sufficient conditions for the Chernoff equivalence of the expectation \( M(\{\Phi_t\}) \), \( t \geq 0 \), and the semigroup \( \exp(M(\Phi(0)) \cdot t \geq 0 \).

**Theorem 1.** Let \( \Phi_t, t \geq 0 \), be a random uniformly continuous quantum dynamical semigroup in the algebra \( L(\mathcal{L}(H)) \), and let \( G \) be its random generator taking values in a ball of some radius \( \rho > 0 \) in the space \( L(\mathcal{L}(H)) \). Then, if \( \bar{G} = \int G \rho(\omega) d\omega \), then the expectation \( M(\{\Phi_t\}) \), \( t \geq 0 \), is Chernoff equivalent to the semigroup \( \exp(\bar{G}) \), \( t \geq 0 \).

It suffices to check the conditions of the Chernoff theorem. Since the operator-valued functions \( \Phi_{t,\omega} \), \( t \geq 0 \), are continuous for each \( \omega \in \Omega \), satisfy the estimate \( \left\| \Phi_{t,\omega} \right\|_{L(\mathcal{L}(H))} \leq \exp(\rho t), t \geq 0 \), take values in the cone \( \left( \mathcal{L}(\mathcal{L}(H)) \right)_{\rho > 0} \), preserve the identity operator, and satisfy the condition \( \Phi_{0,\omega} = 1d \), we conclude that all these properties are possessed by the operator-valued function \( F(t) = \int \Phi_{t,\omega} \rho(\omega) d\omega, t \geq 0 \). For each \( \omega \in \Omega \), we have \( \Phi_{t,\omega} = 1d + tG_{\omega} + r(t, \omega) \), where \( \left\| r(t, \omega) \right\|_{\mathcal{L}(X)} \leq C_1^2 \exp(\rho t) \), \( \rho = \sup_{t \in \Omega} \left\| G_{\omega} \right\|_{\mathcal{L}(X)} \). Therefore, \( F(t) = 1d + t\bar{G} + r(t) \), where \( \left\| r(t) \right\|_{\mathcal{L}(X)} \leq C_1^2 \exp(\rho t) \), so there exists a derivative \( F'(0) = \bar{G} \in \mathcal{L}(\mathcal{L}(H)) \), and \( \left\| F'(0) \right\|_{\mathcal{L}(\mathcal{L}(H))} \leq \rho \). Hence, all the conditions of the Chernoff theorem are satisfied, which proves Theorem 1.

### 3. LIMIT THEOREMS

Let \( A \) be a random variable with values in the Banach space \( L = \mathcal{L}(\mathcal{L}(H)) \) of linear operators acting in the Banach algebra \( \mathcal{L}(H) \), and let \( A \) be defined as a weakly measurable mapping of the probability space \( (\Omega, \mathcal{F}, P) \) to \( L \). In the case of a finite-dimensional space \( H \), the weak measurability of \( A: \Omega \to L \) (i.e., the measurability of functions \( \langle g, A \rangle \) for any \( f \in \mathcal{L}(H) \), \( g \in (\mathcal{L}(H))^* \)) is equivalent to the measurability of this mapping to \( L \) and implies the measurability of the real-valued random variable \( \left\| A \right\|_L \).

**Theorem 2.** Let \( U \) be a random quantum dynamical semigroup acting in the space \( \mathcal{L}(H) \). Suppose that the random generator \( A \) of \( U \) takes values in a ball of the Banach space \( \mathcal{L}(\mathcal{L}(H)) \). Let \( \{U_n\} \) be a sequence of independent identically distributed random semigroups such that the distribution of each of them coincides with the distribution of \( U \). Then the sequence 
\[
W_n(t) = U_n \left( \frac{t}{n} \right) \cdots U_n \left( \frac{1}{n} \right), \quad t \geq 0, \quad n \in \mathbb{N},
\]
of compositions of random semigroups converges in probability to the one-parameter semigroup \( U(t) = \exp(A t) \), \( t \geq 0 \): for any \( X \in \mathcal{L}(H) \), \( T > 0 \), and \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} P \left( \sup_{t \in [0, T]} \left\| W_n(t) - U(t)X \right\|_{\mathcal{L}(H)} > \varepsilon \right) = 0.
\]

**Proof.** The random variable \( A \), which takes values in a ball of some radius \( \rho > 0 \) in the Banach space \( L \), has the expectation \( M(A) = \bar{A} \in L \).

For each \( t \geq 0 \), the operator \( \exp(A t) \) is a random variable with values in the ball of radius \( \rho > 0 \) in the Banach space \( L \). Therefore, the function \( F: \mathbb{R}_+ \to L \) is defined by the relation \( F(t) = M(\exp(A t)) = \int \exp(A t) d\rho(\omega) \). By Theorem 1 [10], the function \( F \) is a continuous mapping from \( \mathbb{R}_+ \) to \( L \) in the strong operator topology (since \( L \) is finite-dimensional, the mapping is also continuous in the norm topology).

By the Lagrange mean value theorem (see [8, Subsection 4.6.4]), for each \( \omega \in \Omega \) and each \( t \geq 0 \), we have the estimates
\[
\left\| \exp(A t) - I \right\|_L \leq t \rho \exp(\rho t).
\]

Theorem 1 implies that \( \frac{d}{dt} F(t) \big|_{t=0} = \bar{A} \).

The analogue of the law of large numbers for the composition of random semigroups (2) is established using an operator counterpart of the Chebyshev inequality [4, 11], to derive which the operator-valued variance of a random operator is introduced. Since the spaces \( H \) and \( \mathcal{L}(H) \) are finite-dimensional, we equip \( \mathcal{L}(H) \) with a norm equivalent to \( \left\| \cdot \right\|_{\mathcal{L}(H)} \), namely, with the norm \( \left\| \cdot \right\|_{\mathcal{L}(H)} \) of the space of Hilbert–Schmidt operators, which turns \( \mathcal{L}(H) \) into a Hilbert space. The symbol \( \mathcal{L}_2(H) \) denotes the linear space \( \mathcal{L}(H) \) equipped with the Hilbert–Schmidt norm. Then \( (W_n(t)) \in \mathcal{L}_2(H) \) and \( (W_n(t)) \in \mathcal{L}_2(H) \) for each \( n \in \mathbb{N} \) and each \( t \geq 0 \). Since \( \mathcal{L}_2(H) \) is a Hilbert space, for compositions of random operators with values in the space \( \mathcal{L}_2(H) \) equipped with the strong operator topology, we apply the approach proposed in [4, 11]. The variance of the random operator-valued function (2) is defined as...
\( D(W_n(t)) = M[(W_n(t) - MW_n(t))^n(W_n(t) - MW_n(t))] \),
\[ t \geq 0, \quad n \in \mathbb{N}. \]

To estimate the variance of the random composition \( W_n \), we introduce the following random deviation from the expectation. For each \( k \in \mathbb{N} \), let \( V_k(t) = U_k(t) - F(t), \quad t \geq 0 \). Then \( M[V_k(t)] = 0 \) \( \forall k \in \mathbb{N} \), \( \forall t \in \mathbb{R} \), and, in view of (3), for each \( t \geq 0 \) with probability 1 we have the estimate
\[ \|V_k(t)\|_* \leq \|U_k(t) - I\|_* + \|F(t) - I\|_* \leq 2\rho \exp(p). \] (4)

By virtue of (2), for each \( n \in \mathbb{N} \), it is true that
\[ W_n(t) = \left( F\left( \frac{t}{n} \right) \right)^n + \sum_{j=1}^{n-1} \left( F\left( \frac{t}{n} \right) \right)^{n-j} V_j(t) \left( F\left( \frac{t}{n} \right) \right)^{k-j} + \ldots + V_n(t) \circ \ldots \circ V_1(t), \quad t \geq 0; \]
\[ W_n^*(t)W_n(t) = \left( F^*\left( \frac{t}{n} \right) \right)^n \]
\[ + \sum_{j=1}^{n-1} \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} V_j(t) \left( F^*\left( \frac{t}{n} \right) \right)^{k-j} + \ldots + V_n^*(t) \circ \ldots \circ V_1^*(t). \] (5)

The product of \( 2n \) binomials in (5) contains \( 2^{2n} \) different terms. Since the expectation of any of the random variables \( V_j(t) \) is zero, a nonzero contribution to the expectation of the sum of \( 2^{2n} \) terms is made only by those terms in which the set of factors from \( V_j \) coincides with the set of factors from \( V_k^* \). Therefore, for each \( t \geq 0 \),
\[ MW_n^*(t)W_n(t) = \left( F^*\left( \frac{t}{n} \right) \right)^n \left( F\left( \frac{t}{n} \right) \right)^n \]
\[ + \sum_{j=1}^{n-1} M \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} V_j^*(t) \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} \]
\[ \times \left( F\left( \frac{t}{n} \right) \right)^{k-j} V_k(t) \left( F\left( \frac{t}{n} \right) \right)^{n-k} + \ldots + MV_n^*(t) \circ \ldots \circ V_1^*(t). \]

Consequently, the variance of the operator-valued random variable \( W_n(t) \) can be expressed as
\[ D[W_n(t)] = MW_n^*(t)W_n(t) - MW_n^*(t)MW_n(t) \]
\[ = \sum_{j=1}^{n} \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} V_j^*(t) \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} \]
\[ \times \left( F\left( \frac{t}{n} \right) \right)^{k-j} V_k(t) \left( F\left( \frac{t}{n} \right) \right)^{n-k} + \ldots + MV_n^*(t) \circ \ldots \circ V_1^*(t). \] (6)

Since \( \left\| \frac{t}{n} \right\|_\rho \leq \exp(p) \), it follows from estimate (4) that
\[ \left\| \sum_{j=1}^{n} \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} V_j^*(t) \left( F^*\left( \frac{t}{n} \right) \right)^{n-j} \]
\[ \times \left( F\left( \frac{t}{n} \right) \right)^{k-j} V_k(t) \left( F\left( \frac{t}{n} \right) \right)^{n-k} \]
\[ \leq C_n e^{2\nu \left( \frac{2\rho}{n^2} \right)^2}. \]

Estimating the remaining terms in formula (6) in the same manner, by virtue of (3) and (4), we obtain the estimate
\[ \|D[W_n(t)]\|_\rho \leq C_n e^{2\nu \left( \frac{2\rho}{n^2} \right)^2} \]
\[ + C_n e^{2\nu \left( \frac{2\rho}{n^2} \right) + \ldots + C_n e^{2\nu (\frac{2\rho}{n^2})^{2n}} \}
\[ = e^{2\nu \left( \frac{2\rho}{n^2} \right)^2 - 1}. \]

According to the Taylor formula, there exists a number \( s \in (0, 1) \) such that
\[ \|D[W_n(t)]\|_\rho \leq \frac{t^2 \rho^2}{n^2} e^{2\nu \left( \frac{2\rho}{n^2} \right)^2} \]
\[ \leq \frac{(2\rho)^2}{n} e^{2\nu \left( \frac{2\rho}{n^2} \right)^2}. \]

Therefore, for every \( T > 0 \), there exists a number
\[ C = C(T, \rho) > 0 \] such that \( \sup_{n \in [0, T]} \|D[W_n(t)]\|_\rho \leq C_n \) for all \( n \in \mathbb{N} \). Then, according to the Chebyshev inequality, for operator-valued random variables (see Lemma 1 in [4]), for any \( T > 0 \) and \( X \in \mathcal{F}(H) \), it is true that
Since the random generators are independent, we have \( M(W_n(t)) = \left( F\left( \frac{t^n}{n} \right) \right)^n \). By Theorem 1,
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left\| M(W_n(t)) - U(t) \right\|_{L_2(H)} = 0
\]
for any \( X \in L_2(H) \) and \( T > 0 \). Since the norms of \( L_2(H) \) and \( L_2^2(H) \) are equivalent, inequality (7) proves Theorem 2.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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