Uniqueness of Asymptotically Conical Gradient Shrinking Solitons in \( G_2 \)-Laplacian Flow.

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Joint work with  
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**Definition**

A **$G_2$-structure** on a 7-manifold $M$ is a 3-form $\varphi \in \Omega^3(M)$ such that for each $x \in M$, there is an isomorphism $\iota : T_x M \to \mathbb{R}^7$ such that $\iota^* \varphi_0 = \varphi_x$, where $\varphi_0$ is the 3-form on $\mathbb{R}^7 = \text{Im}(\Theta)$ given by

$$\varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle.$$ 

- **closed** $G_2$-structure: $d\varphi = 0$; **coclosed** $G_2$-structure: $d^* \varphi = 0$.
- **torsion-free** $G_2$-structure: $d\varphi = 0$ and $d^* \varphi = 0$.
- The **torsion** of a $G_2$-structure is a two-tensor $T_{ij}$ satisfying

$$\nabla \varphi = T_{ik} \ast_{\varphi} \varphi_{mjk}. \,$$

- If $\varphi$ is closed, then $T_{ij}$ is a two-form satisfying $T \wedge \ast \varphi = 0$. Also,

$$R = -|T|^2 \quad \text{and} \quad \nabla T = Rm \ast \varphi + T^2 \ast \varphi.$$
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- **torsion-free** $G_2$-structure: $d\varphi = 0$ and $d^*\varphi = 0$.
- The torsion of a $G_2$-structure is a two-tensor $T_{ij}$ satisfying
  $$\nabla \varphi = T_{i}^{\ m}(\star \varphi \varphi)_{mjk}.$$
- If $\varphi$ is closed, then $T_{ij}$ is a two-form satisfying $T \wedge \star \varphi \varphi = 0$. Also, $R = -|T|^2$ and $\nabla T = Rm \star \varphi + T^2 \star \varphi$. 
A $G_2$-structure gives an orientation and a metric $g_\varphi$ on $M$ satisfying

$$g_\varphi(u, v)\text{Vol}_{g_\varphi} = \frac{1}{6} \iota_u \varphi \wedge \iota_v \varphi \wedge \varphi.$$ 

**Definition**

If $M$ is compact and $\varphi_0$ is closed, then the Hitchin volume functional, $\mathcal{H} : [\varphi_0]_+ \to \mathbb{R}$, is defined for $\varphi \in [\varphi_0]_+$ by

$$\mathcal{H}(\varphi) := \frac{1}{7} \int_M \varphi \wedge *\varphi \varphi = \text{Vol}(M, g_\varphi).$$

Critical points of $\mathcal{H}$ are exactly the torsion-free $G_2$-structures, and these are local maxima of $\mathcal{H}$. 
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For a compact $M$ with a closed $G_2$-structure $\varphi_0$,

**Proposition**

The upward gradient flow of $\mathcal{H}$ on $[\varphi_0]_+$ satisfies

$$\frac{\partial}{\partial t} \varphi = \Delta \varphi.$$

This is the **Laplacian flow** on a compact $M$.

For any geometric flow of closed $G_2$-structures, $\pi_1\left(\frac{\partial \varphi}{\partial t}\right) = 3f_0 \varphi$ and

$$\pi_1\left(\frac{\partial}{\partial t}(\star \varphi \varphi)\right) = 4f_0 \star \varphi \varphi. \text{ For } \varphi(t) = \varphi(0) + d\eta(t),$$

$$\frac{\partial}{\partial t} \mathcal{H}(\varphi) = \int_M f_0 \varphi \land \star \varphi \varphi = \frac{1}{3} \left\langle \frac{\partial \varphi}{\partial t}, \varphi \right\rangle_{L^2} = \frac{1}{3} \left\langle d \frac{\partial \eta}{\partial t}, \varphi \right\rangle_{L^2} = \frac{1}{3} \left\langle \frac{\partial \eta}{\partial t}, d^* \varphi \right\rangle_{L^2}.$$

The upward gradient flow is $\frac{\partial \eta}{\partial t} = d^* \varphi$, and thus $\frac{\partial}{\partial t} \varphi = dd^* \varphi = \Delta \varphi$. 

**Introduction to Laplacian Flow**

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**Definition**

For a closed $G_2$-structure $\varphi_0$ on a (possibly) noncompact $M$, the **Laplacian flow** of $\varphi_0$ is a solution $\varphi = \varphi(t)$ to

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\begin{align*}
\frac{\partial \varphi}{\partial t} &= \Delta \varphi, \\
d\varphi &= 0,
\end{align*}
\]

for $t \in [0, \epsilon)$ with initial condition $\varphi(0) = \varphi_0$.

- Fixed points of the flow are exactly the torsion-free $G_2$-structures.
- The flow may help construct $G_2$-holonomy metrics. All previous constructions involve perturbing closed $G_2$-structures with small torsion to exactly torsion-free ones. (Joyce, Kovalev, Karigiannis, Corti-Haskins-Nordström-Pacini)
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The metric $g = g_\varphi(t)$ evolves by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - \frac{2}{3} |T|^2 g_{ij} - 4 T^k_i T_{kj},$$

where $T_{ij}$ is the torsion tensor.

- Bryant–Xu: Short-time existence and uniqueness of solutions to Laplacian flow on compact $M$.
- If the flow exists for $t \in [0, \infty)$, it need not converge to a torsion-free $G_2$-structure as $t \to \infty$.
- Lotay–Wei: Torsion-free $G_2$-structures are stable under Laplacian flow.
- The flow can have singularities in finite time, i.e. $|Rm|_{g_t} \to \infty$ for $t \to T < \infty$. No compact examples with finite-time singularities are presently known.
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The context of other geometric flows

Ricci flow:

\[ \frac{\partial g_{ij}}{\partial t} = -2R_{ij}. \]

- Hamilton: Under volume-preserving Ricci flow, any positive Ricci curvature metric on a compact 3-manifold converges to a spherical space form as \( t \to \infty \). In higher dimensions, similar results hold under positive curvature assumptions which are preserved by the flow. (Huisken, Böhm-Wilking, Brendle-Schoen)

- Without any curvature restrictions, Ricci flow on 3-manifolds has finite-time singularities. Classifying finite-time singularities is central to topological results. Thurston’s Geometrization Conjecture guides us, e.g. neckpinches modeled on the self-similarly shrinking \( S^2 \times \mathbb{R} \) implement connected sum decompositions of the manifold. (Perelman)
An **ancient solution** to a geometric flow is a solution which exists for \( t \in (-\infty, T] \). These are **singularity models** for the flow.

We study the behavior of the flow around singularities by parabolically rescaling the flow and taking limits using a compactness theory to find ancient solutions.

- In Ricci flow, \((M, g(t))\) is rescaled by choosing a sequence of spacetime points \((x_k, t_k) \in M \times \mathbb{R}\) and \(C_k = |\text{Rm}(x_k, t_k)| > 0\) and considering the sequence of flows \(g_k(t) = C_k g(t_k + C_k^{-1}t)\). The compactness theory requires bounded curvature and injectivity radius estimates (noncollapsing: lower bounds on volume ratio).

- In Laplacian flow, there is a compactness theory due to Lotay-Wei. Just as in Ricci flow, compactness requires bounded curvature and injectivity radius estimates. Chen showed that Laplacian flow is noncollapsing under a bound on scalar curvature, \(R = -|T|^2\).
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**Solitons in Laplacian flow**

Solitons are self-similar solutions to a geometric flow, and understanding them is the first step to understanding ancient solutions.

**Definition**

A *Laplacian soliton* on a 7-manifold is a closed $G_2$-structure $\varphi$ together with a vector field $X$ and $\lambda \in \mathbb{R}$ satisfying

\[
\begin{cases}
  d\varphi_0 = 0 \\
  \Delta\varphi_0 = \lambda\varphi_0 + \mathcal{L}_X\varphi_0
\end{cases}
\]

Let $c(t) = (1 + \frac{2}{3} \lambda t)^{3/2}$, and let $X(t) = c(t)^{-2/3} X$. Then, $\varphi(t) = c(t)\phi_t^*\varphi_0$ is a solution to Laplacian flow, and it is self-similar. Note that $c'(t) = \lambda c(t)^{1/3}$.

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\frac{\partial \varphi}{\partial t} = c'(t)\phi_t^*\varphi_0 + c(t)\phi_t^*(\mathcal{L}_X(t)\varphi_0) = c(t)^{1/3}\phi_t^*(\lambda\varphi_0 + \mathcal{L}_X\varphi_0)
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Facts about Laplacian Solitons

For a Laplacian soliton $\varphi$ with constant $\lambda \in \mathbb{R}$,

$\lambda < 0$: **shrinking soliton** ("shrinker"), exists for $t \in (-\infty, 0)$, models finite-time singularities.

$\lambda = 0$: **steady soliton**, exists for $t \in (-\infty, \infty)$, models infinite-time singularities and degenerate cases of finite-time singularities.

$\lambda > 0$: **expanding soliton** ("expander"), exists for $t \in (0, \infty)$, models singularity resolution.

- Solitons with $X$ a gradient vector field, $X = \nabla f$, are special and are known as **gradient solitons**.

- Solitons on a compact manifold are either torsion-free or expanding. Shrinking solitons must be noncompact.

To understand finite-time singularities, we need to understand the asymptotics of shrinkers. Hence, our interest in AC shrinkers.
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Examples of Laplacian Solitons

- Fowdar’s inhomogeneous gradient shrinker with metric given by
  \[ g(t) = 2^{-1/2} e^u(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + 2e^{2u}(dx_5^2 + dx_6^2) + du^2. \]

- Lauret’s non-gradient shrinking homogeneous soliton on a solvmanifold.

- Steady and expanding homogeneous solitons (Lauret, Fino–Raffero, Fernández–Fino–Manero, etc.)

- Ball’s inhomogeneous steady gradient solitons on \( \mathbb{R} \times N \), where \( N \) is some admissible 6-manifolds, e.g. a particular \( T^2 \)-bundle over a hyperkähler 4-manifold.

- Haskins–Nordström: cohomogeneity-one AC gradient shrinkers, steadies, and expanders. Also, a steady soliton with exponential volume growth.
R. Bryant, Some remarks on G2-structures, arXiv:0305124.

M. Haskins and J. Nordström. Cohomogeneity-one solitons in Laplacian flow: local, smoothly-closing and steady solitons. arXiv:2112.09095 (2021).

J. D. Lotay. “Geometric Flows of G2 Structures.” Lectures and Surveys on G2-Manifolds and Related Topics. Springer, New York, NY, 2020. 113-140.

J. D. Lotay and Y. Wei, Laplacian flow for closed G2-structures: Shi-type estimates, uniqueness, and compactness, Geom. Funct. Anal 27 (2017), 165–233.

S. Karigiannis. “Introduction to G2 Geometry.” Lectures and Surveys on G2-manifolds and related topics. Springer, New York, NY, 2020. 3-50.