Some topics concerning analysis on metric spaces and semigroups of operators

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Classical analysis on Euclidean spaces (as in [100, 103])

Fix a positive integer \( n \), and consider the Euclidean space \( \mathbb{R}^n \) equipped with the standard distance function \( |x - y| \) and Lebesgue measure. If \( f(x) \) is a locally-integrable function on \( \mathbb{R}^n \), then the Hardy–Littlewood maximal function \( f^*(x) \) associated to \( f \) is defined by

\[
f^*(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),
\]

where the supremum is taken over all open balls \( B \) in \( \mathbb{R}^n \) which contain \( x \). The supremum may be \( +\infty \), so that \( f^* \) is actually a (nonnegative) extended real-valued function. This is sometimes referred to as the uncentered maximal function, and there are variants defined in terms of balls centered at \( x \), or using cubes instead of balls. A nice feature of \( f^*(x) \) is that it is upper semicontinuous, which is to say that

\[
\{ x \in \mathbb{R}^n : f^*(x) > t \}
\]

is an open subset of \( \mathbb{R}^n \) for each positive real number \( t \). Indeed, if \( f^*(x) > t \) for some \( x \), then there is a ball \( B \) containing \( x \) such that

\[
\frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y) > t,
\]

and it follows that \( f^*(z) > t \) for all \( z \) in \( B \).

Clearly the supremum of \( f^* \) is less than or equal to the \( L^\infty \) norm of \( f \). A famous weak type \((1, 1)\) result says that

\[
|\{ x \in M : f^*(x) > t \}| \leq C(n) t^{-1} \| f \|_1,
\]
for some constant \( C(n) > 0 \) that depends only on the dimension \( n \) and all functions \( f \), where \( |E| \) denotes the Lebesgue measure of a set \( E \) and \( \|f\|_1 \) is the usual \( L^1 \) norm of \( f \). In particular, \( f^* \) is finite almost everywhere in this case. For \( p > 1 \) there is a strong type result, which means that

\[
\|f^*\|_p \leq C(n, p) \|f\|_p
\]

for some constant \( C(n, p) \) which depends only on \( n \) and \( p \), and where \( \|f\|_p \) denotes the usual \( L^p \) norm of \( f \). This can in fact be derived from the preceding estimates for \( p = 1, \infty \) through a general interpolation result.

One might be interested in other kinds of averages of \( f \), such as those given by integrating \( f \) against the Poisson kernel or the Gauss–Weierstrass kernel. These are exactly the quantities which arise in the extensions of \( f \) to the upper half space \( \mathbb{R}^n \times (0, \infty) \) which are harmonic or satisfy the heat equation (and which satisfy additional mild growth conditions to avoid modest ambiguities). Fortunately, these averages can be estimated in terms of averages over balls in a simple way, so that the corresponding maximal functions are bounded in terms of \( f^* \). Thus the inequalities above for \( f^* \) provide basic results about the boundary behavior of solutions to the Laplace and heat equations on \( \mathbb{R}^n \times (0, \infty) \).

Some other interesting operators are the singular integral operators

\[
R_j(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) \, dy, \\
1 \leq j \leq n, \text{ and }
\]

\[
I_t(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n+t}} f(y) \, dy, \\
t \in \mathbb{R}, t \neq 0.
\]

Some care is involved in taking the principal values, especially in the second case. For \( I_t \), different ways of defining the principal values will even lead to different answers, but the difference is rather mild (a multiple of the identity operator).

One can show that these operators are bounded on \( L^2 \) using special structure related to \( p = 2 \), i.e., Fourier transform and Hilbert space methods. This can be extended to boundedness on \( L^p \) when \( 1 < p < \infty \) and the weak type \((1,1)\) property for \( p = 1 \) using well-known techniques in harmonic analysis. For \( p = \infty \) there are estimates in terms of BMO, as a substitute for \( L^\infty \) bounds which do not work. Similar results apply to numerous other operators of similar type.
Spaces of homogeneous type [20, 21]

A space of homogeneous type can be described as a triple \((M, d(x, y), \mu)\), where \(M\) is a nonempty set, \(d(x, y)\) is a metric on \(M\) (and thus is a symmetric nonnegative real-valued function on \(M \times M\) which vanishes exactly when \(x = y\) and satisfies the triangle inequality), and \(\mu\) is a doubling measure on \(M\). The latter means that \(\mu\) is a nonnegative Borel measure which assigns positive finite measure to open balls in \(M\), and for which there is a constant \(C > 0\) such that

\[
\mu(B(x, 2r)) \leq C \mu(B(x, r))
\]

(8)

for every ball \(B(x, r)\) in \(M\). Of course (8) implies that \(\mu\) assigns positive finite measure to every open ball in \(M\) as soon as this holds for a single such ball. One might also ask that \(M\) be complete, in the sense that Cauchy sequences converge, or that open subsets of \(M\) be realizable as countable unions of compact sets. Basic examples of spaces of homogeneous type are given by Euclidean spaces with the standard metric and Lebesgue measure. Reasonably-smooth domains or manifolds are also included in this notion.

It can be convenient to allow \(d(x, y)\) to be a quasimetric instead of a metric, which means that a positive constant factor is allowed on the right side of the triangle inequality, and the notion of a space of homogeneous type is often formulated in this manner. As in [79], there are always metrics not too far from quasimetrics, so that for many purposes one might as well restrict to metrics.

The Hardy–Littlewood maximal function \(f^*\) associated to a locally-integrable function \(f\) can be defined on a space of homogeneous type in the same manner as on Euclidean spaces. A basic fact is that the weak type \((1, 1)\) estimate extends to this general setting. The supremum of \(f^*\) is still bounded by the \(L^\infty\) norm of \(f\), and \(L^p\) estimates for \(1 < p < \infty\) follow from the \(p = 1, \infty\) estimates through general interpolation arguments, as before.

Another basic result is that one has “Calderón–Zygmund inequalities” for singular integral operators analogous to those on \(\mathbb{R}^n\). That is, one can start with a linear operator \(T\) which is bounded on \(L^2\), or some other fixed \(L^{p_1}\), and which is associated to a kernel that satisfies suitable size and smoothness conditions, and derive boundedness on \(L^p\) for all \(1 < p < \infty\) and a weak-type inequality for \(p = 1\). One can also get BMO estimates for \(p = \infty\), estimates on Hardy spaces as an alternative to the weak-type inequality for \(p = 1\) as well as allowing for some \(p < 1\), etc. The compatibility between the metric and the measure given by the doubling condition is quite remarkable.
Let us mention two classes of examples of spaces of homogeneous type which were examined on their own before the general notion. In the first case, which was studied by my colleague Frank Jones [65], one takes $\mathbb{R}^n \times \mathbb{R}$ with the distance between two points $(x, s), (y, t)$ defined to be

$$|x - y| + |s - t|^{1/2},$$

where $|x - y|, |s - t|$ denote the usual distances in $\mathbb{R}^n, \mathbb{R}$, respectively. Sometimes other expressions are used for essentially the same geometry; a key point is that the distance behaves well under the non-isotropic dilations

$$r > 0, \quad (x, t) \mapsto (rx, r^2t),$$

for $r > 0$, just as the ordinary metric on $\mathbb{R}^n$ behaves well under the dilations $x \mapsto rx$. Of course the metric is also invariant under translations on $\mathbb{R}^n \times \mathbb{R}$, and is compatible with the usual topology. For the measure one still uses Lebesgue measure. The measure of a ball of radius $\rho$ is a constant multiple of $\rho^{n+2}$, and the doubling condition is satisfied. In this case the singular integral theory can be applied to operators related to the heat operator, whereas the standard geometry on $\mathbb{R}^n$ fits with operators related to the Laplacian. Note that there is a kind of tricky point here, in which the $t$ parameter is included in the underlying space.

A second basic situation corresponds to the unit sphere in $\mathbb{C}^n$, which, for $n \geq 2$, has a non-Euclidean geometry which is adapted to several complex variables, holomorphic functions on the unit ball in $\mathbb{C}^n$, etc. Just as in the previous case, one can still use ordinary Lebesgue measure on the sphere, and this measure is doubling with respect to the non-Euclidean geometry. (For that matter, it is also doubling with respect to the usual Euclidean geometry.) The Hardy–Littlewood maximal function with respect to the non-Euclidean geometry is closely connected to maximal functions and limits for holomorphic functions in the ball along certain “admissible” regions, just as the classical maximal function is connected to nontangential maximal functions for holomorphic functions in one complex variable or harmonic functions in several real variables. A fundamental singular integral operator in this situation is the Szegö projection, which is the orthogonal projection from $L^2$ of the unit sphere onto the subspace of functions which are boundary values of holomorphic functions on the ball. This operator is bounded on $L^2$ with norm 1 by definition, and its kernel can be computed explicitly. With respect to the non-Euclidean geometry, the kernel satisfies the appropriate
size and smoothness conditions, so that the operator is in fact bounded on $L^p$, $1 < p < \infty$, and so on. See \[68, 69, 70, 71, 72, 74, 85, 101, 102\].

**Semigroups of operators**

In another direction, suppose that $\mathcal{B}$ is a Banach space, and that $\{T_t\}_{t \geq 0}$ is a semigroup of bounded operators on $\mathcal{B}$. Specifically, assume that $T_0$ is the identity operator $I$, that the operator norm of $T_t$ is bounded by some constant $k$ for $0 \leq t \leq 1$, that $T_{s+t} = T_s \circ T_t$ for all $s, t \geq 0$, and that $\lim_{t \to 0} T_t(f) = f$ for all $f$ in $\mathcal{B}$. Of course the semigroup property together with the uniform bound for the operator norm of the $T_t$’s for $0 \leq t \leq 1$ implies an exponentially-increasing bound for the operator norm of $T_t$ for all $t$’s.

There is a remarkable amount of mathematics around this kind of situation. In fact, this is just the beginning; one can add relatively-simple hypotheses which occur in numerous settings and which add quite a bit more structure. As a basic distinction, one might think of $\{T_t\}_{t \geq 0}$ as being a semigroup of unitary transformations on a Hilbert space, or a semigroup of invertible linear mappings on a Banach space more generally, as is associated to solutions of a wave equation, or one might think of $\{T_t\}_{t \geq 0}$ as defining a diffusion, as is associated to solutions of a heat equation.

Here we shall mostly focus on the second type of situation. We assume now that we have a measure space $M$ with a positive measure $\mu$, and we take for our Banach space $\mathcal{B}$ the Hilbert space $L^2(M, \mu)$. We ask too that each $T_t$ be self-adjoint and positivity-preserving, which means that for each nonnegative function $f$ on $M$, $T_t(f)$ is also a nonnegative function on $M$ for every $t \geq 0$. Each of these conditions is significant in its own right, and part of the beauty of the subject arises from the interplay between them.

Let us also ask that the $T_t$’s extend to bounded operators on $L^p(M, \mu)$ for each $1 \leq p \leq \infty$, and in fact that the $T_t$’s are contractions on all $L^p$, which is to say that the operator norms are all less than or equal to 1. If $1$ denotes the function on $M$ which is identically equal to 1, then we ask that $T_t(1) = 1$ for all $t$. These conditions are satisfied by the semigroups associated to the heat kernel and Poisson kernel on $\mathbb{R}^n$, for instance.

A famous result of Stein states that the maximal function inequalities

$$\| \sup_{t>0} |T_t(f)| \|_p \leq A_p \| f \|_p$$

(12)
hold for $1 < p \leq \infty$, i.e., for some constant $A_p$ and all functions $f$ in $L^p$. In this setting there is also a “singular integral operator” theory, for operators which are functions of the generator of the semigroup. Boundedness on $L^2$ for these operators is easily determined through the spectral representation. Some general conditions for boundedness on $L^p$ are described in [99], and of course more precise information depends on the particular situation.

The significance of self-adjointness is illustrated by the example where $T_t$ is defined on functions on $\mathbb{R}$ to be translation by $t$. This semigroup of operators is positivity-preserving and preserves all $L^p$ norms, but the maximal inequality fails completely for $p < \infty$. In this case it is natural to consider averages of $T_t f$ and suprema of the averages, as in ergodic theory, and as for semigroups associated to measure-preserving transformations on the underlying measure space more generally.

**Semigroups and geometry**

There are very interesting combinations of the spaces of homogeneous type and semigroups of operators pictures, involving bounds for kernels of semigroups, and $L^p$ mapping properties of operators related to the semigroup. See [3, 4, 25, 26, 34, 35], for instance. Another perspective has recently been studied in [48], with the following set-up. One assumes again that $(M, d(x, y))$ is a metric space, that $\mu$ is a positive Borel measure on it, and that $T_t$ is a symmetric contraction semigroup of linear operators as before. Now one asks in addition that for $t > 0$ the operator $T_t$ is defined by a nonnegative kernel $k_t(x, y)$, so that

\[
T_t(f)(x) = \int_M k_t(x, y) f(y) d\mu(y).
\]

For the kernel $k_t(x, y)$ one considers upper and lower bounds of the form

\[
\frac{1}{t^{\alpha/\beta}} \phi_1\left(\frac{d(x, y)^{\beta}}{t}\right) \leq k_t(x, y) \leq \frac{1}{t^{\alpha/\beta}} \phi_2\left(\frac{d(x, y)^{\beta}}{t}\right).
\]

Here $\alpha, \beta$ are positive constants, and $\phi_1(u), \phi_2(u)$ are monotone decreasing positive functions on $[0, \infty)$, with $\phi_1(u_1) > 0$ for some $u_1 > 0$ and $\phi_2(u)$ normally asked to satisfy decay conditions.

The parameter $\alpha$ is related to volume growth in $M$, and this is discussed in [48]. The connection between $\beta$ and the geometry of $M$ is also treated in [48]. For the standard heat semigroup on Euclidean spaces, $\beta$ is always equal
to 2. There are heat semigroups associated to subelliptic operators in place of the ordinary Laplacian which also satisfy these conditions with $\beta = 2$, with respect to an associated metric. A basic version of this arises for the unit sphere in $\mathbb{C}^n$, $n \geq 2$, and non-Euclidean geometry on it, as indicated earlier. There are a number of fractals such as Sierpinski gaskets and carpets and semigroups on them which satisfy the conditions above with various values of $\beta$. Compare with [6, 7, 8, 9, 10, 37, 67, 75].

Without the semigroup property, there are well-known fairly simple constructions of approximations to the identity on spaces of homogeneous type with nice properties. The semigroup property of course imposes very strong restrictions. For that matter, commutativity of the operators in the family is a substantial condition.

Let us note that decay conditions on $\phi_2(u)$ above can be quite significant. A very nice contraction semigroup on $\mathbb{R}^n$ is given by the Poisson kernel, and for this kernel the decay is not very fast. Modest decay conditions for the kernel are adequate for a number of applications, even if they are not sufficient for other results, as in [18].

**Analysis on fractals like Sierpinski gaskets and carpets**

Of course decay conditions for the kernel of a semigroup are closely connected to locality conditions for the generator of the semigroup. In the classical cases on Euclidean spaces or tori for periodic functions, etc., the heat kernel has fast decay and is generated by the Laplace operator, while the Poisson kernel does not have very fast decay and is generated by a constant multiple of the square root of the Laplace operator, which is not a local operator.

The fast decay for the kernels of the semigroups on Sierpinski gaskets and carpets mentioned before reflects the fact that the generators are nice local operators, versions of “Laplacians” for these fractals. The fractal structures play a role and are reflected in the parameter $\beta$, but still there are nice operators which are like differential operators.

For example, there are remarkable results concerning elliptic and parabolic Harnack inequalities for these operators. See [6, 7, 8, 9, 67].
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