Stability of Nonlinear Regime-switching Jump Diffusions

Zhixin Yang,* G. Yin†

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Abstract

Motivated by networked systems, stochastic control, optimization, and a wide variety of applications, this work is devoted to systems of switching jump diffusions. Treating such nonlinear systems, we focus on stability issues. First asymptotic stability in the large is obtained. Then the study on exponential p-stability is carried out. Connection between almost surely exponential stability and exponential p-stability is exploited. Also presented are smooth-dependence on the initial data. Using the smooth-dependence, necessary conditions for exponential p-stability are derived. Then criteria for asymptotic stability in distribution are provided. A couple of examples are given to illustrate our results.

Key Words. jump diffusion, switching process, stability in the large, smooth dependence on initial data, stability in distribution.

Brief Title. Stability of Jump Diffusions with Switching
1 Introduction

Randomly varying switching systems have drawn increasing attention recently, especially in the fields of control and optimization. This is largely owing to their ability to model complex systems, which can be used in a wide range of applications in consensus controls, distributed computing, autonomous or semi-autonomous vehicles, multi-agent systems, tele-medicine, smart grids, and financial engineering etc. One of the common features of the many systems mentioned above is that they may be represented as networked systems. In a typical networked system, different nodes are connected through a communication link described by a network topology or configuration. Most work to date dealt with networked systems with fixed topology. Nevertheless, data routing, packet aggregations, channel uncertainties, and switching links to different network hubs demand the consideration of topology changes in a networked system. Thus fixed topology becomes inadequate and random environment and uncertainty must be taken into consideration. For example, in the original formulation of consensus problems [21, 24, 25], one dealt with a fixed configuration or topology, whereas consideration of randomly varying topologies leads to switching diffusion processes [30].

Facing the demands and pressing needs, this paper considers systems that are formulated as regime-switching jump diffusions. Because many systems in networked systems are in operation for very long time, their asymptotic behavior, namely, stability is of crucial importance; see [1, 15] and references therein for related work. Due to the involvement of multiple stochastic processes, care must be taken to treat the stability issues, which is the objective of the current paper.

One of the main features of the underlying systems we consider here is the coexistence of dynamics described by solutions of differential equations and discrete events whose values belong to a finite set; see [7, 9] and references therein. The usual formulation in the traditional dynamic system setup described by differential or difference equations alone becomes unsuitable. A class of models naturally replacing the traditional setup is a process with two components in which one of them delineates the dynamics that may be represented as a solution of a differential equation and the other portraits the discrete event movements (see [2] for an example in finance application). To take into consideration of possible inclusion of the Poisson type of random processes, we consider jump diffusion processes with random switching.

In recent years, switching stochastic systems have received much attention; see [17, 18, 34] and references therein for a systematic treatment on Markov modulated switching diffusions; see also [35] for stability of switching diffusions with delays. In addition, switching diffusion with continuous dependence on initial data were treated in [33]. Concerning jump diffusions, we refer the reader to [14, 19, 26] for the study on such properties as ergodicity and stability. Switching jump diffusions with state dependent switching have also been examined in [29, 31, 32] etc., in which stability in probability, asymptotic stability in probability, and almost surely exponential stability were dealt with. Our aims in this paper are to establish a number of results on different modes of stability that have not been studied for switching jump diffusions to date to the best of our knowledge. We begin with asymptotic stability in the large, proceed to exponential $p$-stability and obtain smooth dependence on the initial data. As a nice application of the smooth dependence on the initial
data, we derive necessary conditions for $p$-stability, which can be viewed as a Lyapunov converse theorem. The aforementioned results all begin with an equilibrium point of the switching jump diffusion. In absence of information of the equilibrium, an appropriate notion of stability is stability in distribution. Under simple conditions, we obtain sufficient conditions for asymptotic stability in distribution.

The rest of the paper is arranged as follows. We begin with the precise description of the system in Section 2. Section 3 concentrates on asymptotic stability in the large. Section 4 proceeds to the study on exponential $p$-stability. Section 5 furthers our investigation by examining the smooth-dependence on the initial data, and Section 6 presents criteria for asymptotic stability in distribution. Section 7 presents a few examples to demonstrate our results. Finally, the paper is concluded in section 8 with further remarks.

2 Formulation

This section presents the formulation of the problem. We begin with certain notation needed together with a number of definitions. We use $z^t$ to denote the transpose of $z \in \mathbb{R}^{l_1 \times l_2}$ with $l_i \geq 1$, and $\mathbb{R}^{r \times 1}$ is simply written as $\mathbb{R}^r$. Denote $1 = (1, 1, \ldots, 1)^t \in \mathbb{R}^r$ that is a column vector with all entries being 1. For a matrix $A$, its trace norm is denoted by $|A| = \sqrt{\text{tr}(A^T A)}$. Let $(X(t), \alpha(t))$ be a two-component Markov process in which $X(\cdot)$ takes values in $\mathbb{R}^r$ and $\alpha(\cdot)$ is a switching process taking values in a finite set $\mathcal{M} = \{1, 2, 3, \ldots, m\}$. Let $\Gamma$ be a subset of $\mathbb{R}^r - \{0\}$ that is the range space of the impulsive jumps. For any subset $B$ in $\Gamma$, $N(t, B)$ counts the number of impulses on $[0, t]$ with values in $B$; $b(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r$, $\sigma(\cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r \times \mathbb{R}^d$, and $g(\cdot, \cdot, \cdot) : \mathbb{R}^r \times \mathcal{M} \times \Gamma \mapsto \mathbb{R}^r$ are suitable functions whose precise conditions will be given later. Consider the dynamic system given by

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t) + dJ(t),$$
$$J(t) = \int_0^t \int_\Gamma g(X(s^-), \alpha(s^-), \gamma)N(ds, d\gamma),$$
$$X(0) = x, \alpha(0) = \alpha,$$

where the switching process $\alpha(\cdot)$ obeys the transition rule

$$P\{\alpha(t + \Delta t) = j|\alpha(t) = i, X(s), \alpha(s), s \leq t\} = q_{ij}(X(t))\Delta t + o(\Delta t), \text{ for } i \neq j,$$

$w(t)$ is a $d$-dimensional standard Brownian motion, and $N(\cdot, \cdot)$ is a Poisson measure such that the jump process $N(\cdot, \cdot)$ is independent of the Brownian motion $w(\cdot)$. Equation (2.1) can be written as integral form:

$$X(t) = x + \int_0^t b(X(s), \alpha(s))ds + \int_0^t \sigma(X(s), \alpha(s))dw(s) + \int_0^t \int_\Gamma g(X(s^-), \alpha(s^-), \gamma)N(ds, d\gamma).$$

Here we have used a setup similar to [31]. When we wish to emphasize the initial data dependence in the sequel, we write the process as $(X^{x,\alpha}(t), \alpha^{x,\alpha}(t))$. Note that although the two-component process $(X(t), \alpha(t))$ is Markov, $\alpha(t)$ generally is not a Markov chain due to the dependence of the state $x$ in the generator. The transition rule indicates that $\alpha(t)$ depends on the jump diffusion.
component. Thus the setup we consider is more general than that of considered in the literature, whereas in the past work it was often assumed that \( \alpha(t) \) itself is a Markov chain and \( w(t) \) and \( \alpha(t) \) are independent.

For future use, we define a compensated or centered Poisson measure as

\[
\tilde{N}(t, B) = N(t, B) - \lambda t \pi(B) \text{ for } B \subset \Gamma,
\]

where \( 0 < \lambda < \infty \) is known as the jump rate and \( \pi(\cdot) \) is the jump distribution (a probability measure). With this centered Poisson measure, we can rewrite \( J(t) \) as

\[
J(t) = \int_0^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \tilde{N}(ds, d\gamma) + \lambda \int_0^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \pi(d\gamma)ds,
\]

which is the sum of a martingale and an absolute continuous process provided certain conditions are satisfied for the function \( g(\cdot) \).

Note that the evolution of the discrete component \( \alpha(\cdot) \) can be represented by a stochastic integral with respect to a Poisson measure (e.g., [12]). For \( x \in \mathbb{R}^r \) and \( i, j \in \mathcal{M} \) with \( j \neq i \), let \( \Delta_{ij}(x) \) be the consecutive (with respect to the lexicographic ordering on \( \mathcal{M} \times \mathcal{M} \)), left-closed, right-open intervals of the real line, each having length \( q_{ij}(x) \). Define a function \( h : \mathbb{R}^r \times \mathcal{M} \times \mathbb{R} \mapsto \mathbb{R} \) by

\[
h(x, i, z) = \sum_{j=1}^m (j - i) I_{z \in \Delta_{ij}(x)}.
\]

That is, with the partition \( \{ \Delta_{ij}(x) : i, j \in \mathcal{M} \} \) used and for each \( i \in \mathcal{M} \), if \( z \in \Delta_{ij}(x) \), \( h(x, i, z) = j - i \); otherwise \( h(x, i, z) = 0 \). Then we may write the switching process as a stochastic integral

\[
d\alpha(t) = \int_{\mathbb{R}} h(X(t), \alpha(t^-), z) N_1(dt, dz),
\]

where \( N_1(dt, dz) \) is a Poisson random measure with intensity \( dt \times \tilde{m}(dz) \), and \( \tilde{m}(\cdot) \) is the Lebesgue measure on \( \mathbb{R} \). The Poisson random measure \( N_1(\cdot, \cdot) \) is independent of the Brownian motion \( w(\cdot) \) and the Poisson measure \( N(\cdot, \cdot) \). For subsequent use, we define another centered Poisson measure as

\[
\mu(dt, dz) = N_1(dt, dz) - dt \times \tilde{m}(dz).
\]

The generator \( G \) associated with the process \( (X(t), \alpha(t)) \) is defined as follows: For each \( i \in \mathcal{M} \), and for any twice continuously differentiable function \( f(\cdot, i) \),

\[
Gf(x, \cdot)(i) = \mathcal{L} f(x, \cdot)(i) + \lambda \int_{\Gamma} [f(x + g(x, i, \gamma), i) - f(x, i)] \pi(d\gamma),
\]

where \( \mathcal{L} \) is the operator for a switching diffusion process given by

\[
\mathcal{L} f(x, \cdot)(i) = \frac{1}{2} \sum_{k,l=1}^r a_{kl}(x, i) \frac{\partial^2 f(x, i)}{\partial x_k \partial x_l} + \sum_{k=1}^r b_k(x, i) \frac{\partial f(x, i)}{\partial x_k} + Q(x)f(x, \cdot)(i)
\]

\[
= \frac{1}{2} \text{tr}(a(x, i)H f(x, i)) + b'(x, i) \nabla f(x, i) + Q(x)f(x, \cdot)(i),
\]

\( i \in \mathcal{M} \).
whenever there is no confusion. By virtue of the generalized Itô’s formula, we have that

\[ b(x, i) = \sigma(x, i) \sigma'(x, i), \] and \( Hf(t, i) \) denote the gradient and Hessian matrix of \( f(t, i) \), respectively, and \( Q(x) = (q_{ij}(x)) \) is an \( m \times m \) matrix depending on \( x \) satisfying the q-property, namely, (i) \( q_{ij}(x) \) is Borel measurable and uniformly bounded for all \( i, j \in \mathcal{M} \) and \( x \in \mathbb{R}^r \);

(ii) \( q_{ij}(x) \geq 0 \) for all \( x \in \mathbb{R}^r \) and \( i \neq j \);

(iii) \( q_{ii}(x) = -\sum_{j \neq i} q_{ij}(x) \) for all \( x \in \mathbb{R}^r \) and \( i \in \mathcal{M} \). Denote

\[ Q(x)f(x, \cdot)(i) = \sum_{j \in \mathcal{M}} q_{ij}(x)f(x, j) = \sum_{j \neq i} q_{ij}(x)(f(x, j) - f(x, i)), i \in \mathcal{M}. \]

In what follows, we often write \( \mathcal{L}f(x, \cdot)(i) \) as \( \mathcal{L}f(x, i) \) and \( \mathcal{G}f(x, \cdot)(i) \) as \( \mathcal{G}f(x, i) \) for convenience whenever there is no confusion. By virtue of the generalized Itô’s formula, we have that

\[ f(X(t), \alpha(t)) - f(x, \alpha) - \int_0^t \mathcal{G}f(X(s), \alpha(s))ds \] is a martingale.

To proceed, we need the following assumptions.

(A1) The functions \( b(\cdot, i), \sigma(\cdot, i) \), and \( g(\cdot, i, \gamma) \) satisfy \( b(0, i) = 0, \sigma(0, i) = 0, \) and \( g(0, i, \gamma) = 0 \) for each \( i \in \mathcal{M}; \) \( \sigma(x, i) \) vanishes only at \( x = 0 \) for each \( i \in \mathcal{M} \).

(A2) There exists a positive constant \( K_0 \) such that for each \( i \in \mathcal{M}, x, y \in \mathbb{R}^r \) and \( \gamma \in \Gamma \),

\[ |b(x, i) - b(y, i)| + |\sigma(x, i) - \sigma(y, i)| \leq K_0|x - y|, \]

\[ |g(x, i, \gamma) - g(y, i, \gamma)| \leq K_0|x - y|. \]

(A3) There exists \( g^*(i) \) satisfying

\[ |g(x, i, \gamma)| \leq g^*(i)|x| \] for each \( x \in \mathbb{R}^r, i \in \mathcal{M}, \) and each \( \gamma \in \Gamma \).

We elaborate on the conditions briefly. Condition (A1) indicates that 0 is an equilibrium point; (A2) is a Lipschitz condition on the functions. It together with the equilibrium point 0 implies that the functions grow at most linearly. Several of our results to follow are concerned with equilibrium point of the switching jump diffusions. To proceed, we make the following definitions by adopting the terminologies of [32].

**Definition 2.1** The equilibrium point \( x = 0 \) of system (2.1) and (2.2) is said to be

(i) **stable in probability**, if for any \( \varepsilon > 0 \) and any \( \alpha \in \mathcal{M} \), \( \lim_{x \to 0} \sup_{t \geq 0} P\{|X^{x,\alpha}_t| > \varepsilon\} = 0 \); and \( x = 0 \) is said to be **unstable in probability** if it is not stable in probability.

(ii) **asymptotically stable in probability**, if it is stable in probability and \( \lim_{x \to 0} \lim_{t \to \infty} P\{X^{x,\alpha}_t = 0\} = 1 \), for any \( \alpha \in \mathcal{M} \);

(iii) **asymptotically stable in the large**, if it is stable in probability and \( \lim_{t \to \infty} P\{X^{x,\alpha}_t = 0\} = 1 \), for any \( (x, \alpha) \in \mathbb{R}^r \times \mathcal{M} \);
(iv) exponential p-stable, if for some positive constants $K$ and $k$, $E|X^{x,\alpha}(t)|^p \leq K|x|^{p} e^{-kt}$, for any $(x, \alpha) \in \mathbb{R}^r \times \mathcal{M}$;

(v) almost surely exponential stable, if for any $(x, \alpha) \in \mathbb{R}^r \times \mathcal{M}$, $\limsup_{t \to \infty} \frac{1}{t} \ln(|X^{x,\alpha}(t)|) < 0$ w.p.1.

As a preparation, we first recall a lemma, which indicates that the equilibrium $(0, \alpha)$ is inaccessible in that starting with any $x \neq 0$, the system will not reach the origin with probability one. The proof of this lemma can be found in [31, Lemma 2.10].

**Lemma 2.2** $P\{X^{x,\alpha}(t) \neq 0, t \geq 0\} = 1$, for any $x \neq 0$ and $\alpha \in \mathcal{M}$.

### 3 Asymptotic Stability in the Large

To proceed, we first recall two lemmas. The detailed proof can be found in [31].

**Lemma 3.1** Let $D \subset \mathbb{R}^r$ is a neighborhood of 0. Suppose that for each $i \in \mathcal{M}$, there exists a nonnegative Lyapunov function $V(\cdot, i) : D \to \mathbb{R}$ such that

(i) $V(\cdot, i)$ is continuous in $D$ and vanishes only at $x = 0$;

(ii) $V(\cdot, i)$ is twice continuously differentiable in $D - \{0\}$ and satisfies $G V(x, i) \leq 0$ for all $x \in D - \{0\}$.

Then the equilibrium point $x = 0$ is stable in probability.

Define

$$\tau_{\rho,\varsigma} := \inf\{t \geq 0 : |X(t)| = \rho \text{ or } |X(t)| = \varsigma\},$$

for any $0 < \rho < \varsigma$ and any $(x, \alpha) \in \mathbb{R}^r \times \mathcal{M}$ with $\rho < |x| < \varsigma$.

**Lemma 3.2** Assume that the conditions of Lemma 3.1 hold, and that for any sufficiently small $0 < \rho < \varsigma$ and any $(x, \alpha) \in \mathbb{R}^r \times \mathcal{M}$ with $\rho < |x| < \varsigma$, $P\{\tau_{\rho,\varsigma} < \infty\} = 1$. Then the equilibrium point $x = 0$ is asymptotically stable in probability.

**Theorem 3.3** Assume that the conditions of Lemma 3.2 hold, and that $V_\varsigma := \inf_{\|x\| \geq \varsigma, i \in \mathcal{M}} V(x, i) \to \infty$ as $\varsigma \to \infty$. Then the equilibrium point $x = 0$ is asymptotically stable in the large.

**Proof.** For each $i \in \mathcal{M}$, for any $\varepsilon > 0$ and $(x, \alpha) \in \mathbb{R}^r \times \mathcal{M}$, there exists a $\varsigma > |x|$ large enough such that $\inf_{\|X\| \geq \varsigma, i \in \mathcal{M}} V(X, i) \geq 2V(x, \alpha)/\varepsilon$.

Let $\tau_\varsigma$ be the stopping time $\tau_\varsigma := \inf\{t \geq 0 : |X(t)| \geq \varsigma\}$ and $t_\varsigma = \tau_\varsigma \land t$. Then it follows from Dynkin’s formula that

$$EV(X(t_\varsigma), \alpha(t_\varsigma)) - V(x, \alpha) = E \int_0^{t_\varsigma} G V(X(u), \alpha(u)) du \leq 0.$$
Consequently, $EV(X(t), \alpha(t)) \leq V(x, \alpha)$. Then we have

$$E[V(X(\tau_\varsigma), \alpha(\tau_\varsigma))I_{\{\tau_\varsigma < t\}}] \leq V(x, \alpha).$$

Hence, $\frac{2V(x, \alpha)}{\epsilon}P(\tau_\varsigma < t) \leq V(x, \alpha)$. So $P(\tau_\varsigma < t) \leq \epsilon/2$. Let $t \to \infty$, $P(\tau_\varsigma < \infty) \leq \epsilon/2$. Then it follows from Lemma 3.2 that, for any $\rho > 0$ with $\rho < |x| < \varsigma$ we have

$$1 = P(\tau_{\rho, \varsigma} < \infty) \leq P(\tau_{\rho} < \infty) + P(\tau_\varsigma < \infty),$$

in which $\tau_\rho$ is the stopping time $\tau_\rho := \inf\{t \geq 0 : |X(t)| \leq \rho\}$, where $\tau_{\rho, \varsigma}$ was defined in (3.1). Consequently, $P(\tau_{\rho} < \infty) \geq 1 - \epsilon/2$. This implies that $P\{\inf_{t \geq 0} |X(t)| \leq \rho\} \geq 1 - \epsilon/2$. Since $\rho > 0$ can be arbitrarily small, $P\{\inf_{t \geq 0} |X(t)| = 0\} \geq 1 - \epsilon/2$.

Now we can follow the same techniques in [32, Lemma 7.6] and obtain $P\{\lim_{t \to \infty} X(t) = 0\} \geq 1 - \epsilon/2$. That is, the equilibrium point $x = 0$ is asymptotically stable in the large as desired. \qed

For application, it is important to be able to handle linearized systems. Similar to [31], we pose the following condition.

(A4) For each $i \in \mathcal{M}$, there exist $b(i), \sigma_l(i) \in \mathbb{R}^{n \times r}$ for $l = 1, 2, \ldots, d$, and a generator of a continuous-time Markov chain $\hat{Q} = (\hat{q}_{ij})$ with the corresponding Markov chain denoted by $\hat{\alpha}(t)$ such that as $x \to 0$,

$$b(x, i) = b(i)x + o(|x|),$$

$$\sigma(x, i) = (\sigma_1(i)x, \sigma_2(i)x, \ldots, \sigma_d(i)x) + o(|x|),$$

$$Q(x) = \hat{Q} + o(1).$$

Moreover, $\hat{Q}$ is irreducible.

Assumption (A4) indicates that near the origin, the coefficients are locally linear. By choosing a Lyapunov function properly, we have the same sufficient condition for asymptotically stable in the large as that of asymptotically stable in probability. The result is provided below, and the proof is omitted. The method involved is similar to [31, Theorem 3.5].

**Corollary 3.4** Under assumptions (A1)-(A4), the equilibrium point $x = 0$ of the system given by (2.1) and (2.2) is asymptotically stable in the large if

$$\sum_{i \in \mathcal{M}} \mu_i \left( \Lambda_{\text{max}}(b(i) + b'(i)) + \frac{1}{2} \Lambda_{\text{max}} \left( \sum_{l=1}^{d} \sigma_l(i)\sigma_l(i) \right) + \lambda g^*(i) \right) < 0.$$

In which $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^{1 \times m}$ is the stationary distribution of $\hat{\alpha}(t)$ and $\Lambda_{\text{max}}(A)$ denotes the largest eigenvalue of the symmetric part of $A$.  

7
4 Exponential $p$-stability

In this section, we give a sufficient condition for exponential $p$-stability. To proceed, we first recall a lemma, which indicates that the process $X(t) = (X(t), \alpha(t))$ has no finite explosion time, also known as regular. The proof of this lemma can be found in [31, Lemma 2.8].

**Lemma 4.1** Under assumptions (A1)-(A3), the switching jump diffusion $(X(t), \alpha(t))$ is regular.

**Theorem 4.2** Let $D \subset \mathbb{R}^r$ be a neighborhood of 0. Assume that the conditions of Lemma 4.1 hold and assume that for each $i \in \mathcal{M}$, there exists a nonnegative Lyapunov function $V(\cdot, i) : D \mapsto \mathbb{R}$ such that $V(\cdot, i)$ is twice continuously differentiable in $D - \{0\}$, and satisfies the following conditions:

\[
  k_1|x|^p \leq V(x, i) \leq k_2|x|^p, \quad x \in D, \\
  \mathcal{G}V(x, i) \leq -kV(x, i) \quad \text{for all } x \in D - \{0\},
\]

for some positive constants $k_1, k_2$ and $k$. Then the equilibrium point $x = 0$ is exponential $p$-stable.

**Proof.** Consider a sequence of stopping times $\{\tau_n\}$ defined by $\tau_n := \inf\{t \geq 0 : |X(t)| \geq n\}$ and let $t_n = t \land \tau_n$. By virtue of Dynkin’s formula and (4.2), since $kV(X(s), \alpha(s)) + \mathcal{G}V(X(s), \alpha(s)) \leq 0$, so

\[
  E[e^{k(t_n)}V(X(t_n), \alpha(t_n))] = V(x, \alpha) + E \int_0^{t_n} [e^{ks}(kV(X(s), \alpha(s)) + \mathcal{G}V(X(s), \alpha(s)))]ds \\
  \leq V(x, \alpha).
\]

Let $n \to \infty$, by Fatou’s Lemma and Lemma 4.1, we have $E[e^{kt}V(X(t), \alpha(t))] \leq V(x, \alpha)$. Hence,

\[
e^{kt}E(k_1|x|^p) \leq e^{kt}EV(X(t), \alpha(t)) \leq V(x, \alpha) \leq k_2|x|^p.
\]

Then we obtain $E|X(t)|^p \leq K|x|^p e^{-kt}$. The theorem is thus proved. \qed

**Remark 4.3** In the above and hereafter, $K$ is used as a generic positive constant, whose value may be different in different appearances. Under the conditions of Theorem 4.2, we can also obtain the result of almost surely exponential stability by similar argument in [10, Theorem 5.8.1].

**Theorem 4.4** Under assumptions (A1)-(A3), exponential $p$-stability implies almost surely exponential stability.

**Proof.** Because

\[
  X(t) = x + \int_0^t b(X(s), \alpha(s))ds + \int_0^t \sigma(X(s), \alpha(s))dw(s) + \int_0^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma)N(ds, d\gamma),
\]

we have for any $p \geq 2$ that

\[
  |X(t)|^p \leq 4^{p-1}||x||^p + \left|\int_0^t b(X(s), \alpha(s))ds \right|^p + \left|\int_0^t \sigma(X(s), \alpha(s))dw(s) \right|^p \\
  + \left|\int_0^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma)N(ds, d\gamma) \right|^p.
\]
For any $t$, there exists such an $n$ that $t \in [n-1, n]$ and the following inequality holds

$$E \left[ \sup_{n-1 \leq t \leq n} |X(t)|^p \right]$$

\[ \leq 4^{p-1} E|X(n-1)|^p + 4^{p-1} E \left( \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t b(X(s), \alpha(s)) \, ds \right|^p \right) \]

\[ + 4^{p-1} E \left( \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \sigma(X(s), \alpha(s)) \, dw(s) \right|^p \right) \]

\[ + 4^{p-1} E \left( \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, N(ds, d\gamma) \right|^p \right). \]

(4.3)

By (A1) together with the Hölder inequality and the martingale inequality,

$$E \left( \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t b(X(s), \alpha(s)) \, ds \right|^p \right) \leq K \int_{n-1}^n E|X(s)|^p \, ds,$$

$$E \left[ \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \sigma(X(s), \alpha(s)) \, dw(s) \right|^p \right] \leq K \int_{n-1}^n E|X(s)|^p \, ds.$$

For the Poisson jump part,

$$E \left[ \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, N(ds, d\gamma) \right|^p \right]$$

\[ = E \left[ \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, \tilde{N}(ds, d\gamma) \right|^p \right] \]

\[ + \lambda \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, \pi(d\gamma) \, ds \]

\[ \leq 2^{p-1} E \left[ \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, \tilde{N}(ds, d\gamma) \right|^p \right] \]

\[ + 2^{p-1} E \left[ \lambda \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, \pi(d\gamma) \, ds \right|^p \right]. \]

(4.4)

Using Hölder inequality and assumptions (A1)-(A3) for the last term of (4.4), detailed computation leads to

$$E \left[ \sup_{n-1 \leq t \leq n} \left| \int_{n-1}^t \int_{\Gamma} g(X(s^-), \alpha(s^-), \gamma) \, \pi(d\gamma) \, ds \right|^p \right]$$

\[ \leq K E \left[ \int_{n-1}^n \left| X(s^-) \right|^p \, \pi(d\gamma) \, ds \right] \]

\[ \leq K \int_{n-1}^n E|X(s^-)|^p \, ds. \]
Now let us handle the martingale part in (4.4). By Hölder inequality, assumptions (A1)-(A3), and properties of stochastic integral with respect to a Poisson measure, we have

\[
E \left[ \sup_{n-1 \leq t \leq n} \left| \int_{t_1}^{t_2} g(X(s^-), \alpha(s^-), \gamma) \tilde{N}(ds, d\gamma) \right|^p \right] \\
\leq KE \left( \int_{t_1}^{t_2} \left| g(X(s^-), \alpha(s^-), \gamma) \right|^2 ds \pi(d\gamma) \right)^{p/2} \\
\leq KE \left( \int_{t_1}^{t_2} \left| X(s^-) \right|^2 ds \pi(d\gamma) \right)^{p/2} \\
\leq KE \left( \int_{t_1}^{t_2} \left| X(s^-) \right|^2 ds \right)^{p/2} \\
\leq K \int_{t_1}^{t_2} E|X(s^-)|^p ds.
\]

Given that \( X(t) \) is exponential \( p \)-stable, we have

\[
E|X(t)|^p \leq K|\gamma|^p e^{-\kappa t} \text{ for all } t \geq 0.
\]

Substituting the above bounds to (4.3), careful calculations lead to

\[
E \left[ \sup_{n-1 \leq t \leq n} |X(t)|^p \right] \leq Ke^{-\kappa(n-1)}.
\]

For any \( 1 \leq p < 2 \), we have

\[
E \left[ \sup_{n-1 \leq t \leq n} |X(t)|^p \right] \leq \left( E( \sup_{n-1 \leq t \leq n} |X(t)|^{2p}) \right)^{1/2} \leq Ke^{-\kappa(n-1)}.
\]

Finally, for any \( 0 < p < 1 \), we have

\[
|X(t)|^p = |X(t)|^p I_{\{X(t) \geq 1\}} + |X(t)|^p I_{\{X(t) < 1\}} \leq 1 + |X(t)|^{1+p}.
\]

Therefore,

\[
E \left( \sup_{n-1 \leq t \leq n} |X(t)|^p \right) \leq 1 + E( \sup_{n-1 \leq t \leq n} |X(t)|^{1+p}) \leq Ke^{-\kappa(n-1)}.
\]

Note that in the above \( K \) and \( \kappa \) have different values in different appearances. Now let \( \varepsilon \in (0, \kappa) \) be arbitrary, then it follows that

\[
P \left\{ \sup_{n-1 \leq t \leq n} |X(t)|^p > e^{-(\kappa-\varepsilon)(n-1)} \right\} \\
\leq e^{(\kappa-\varepsilon)(n-1)} E \left( \sup_{n-1 \leq t \leq n} |X(t)|^p \right) \\
\leq Ke^{-\varepsilon(n-1)}.
\]

Since \( \sum_{n=1}^{\infty} K \exp(-\varepsilon(n-1)) < \infty \), by Borel-Cantelli lemma, we have

\[
\sup_{n-1 \leq t \leq n} |X(t)|^p \leq e^{-(\kappa-\varepsilon)(n-1)} \text{ a.s.} \quad (4.5)
\]
for all but finitely many \( n \). Hence, there exists such an \( n_0 \) that whenever \( n \geq n_0 \), (4.5) holds a.s. So,

\[
\frac{1}{t} \ln |X(t)| = \frac{1}{pt} \ln(|X(t)|^p) \leq -\frac{(\kappa - \varepsilon)(n - 1)}{p(n - 1)} < 0 \quad \text{a.s.}
\]  

(4.6)

Taking \( \lim \sup \) in (4.6) leads to almost surely exponential stability. Thus, the proof is completed.

\[ \square \]

5 Smooth-Dependence on Initial Data

One of the important properties of a diffusion processes is the continuous and smooth dependence on the initial data. This property is preserved for the switching diffusion processes with state-dependent switching; however much work is needed. In what follows, we show that this property is also preserved for the switching jump diffusion processes. The results are stated for multi-dimensional cases, whereas the proofs are carried out for a one-dimensional process for notational simplicity. Let \((X(t), \alpha(t))\) denote the switching jump process with initial condition \((x, \alpha)\) and \((\tilde{X}(t), \tilde{\alpha}(t))\) be the process starting from \((\tilde{x}, \alpha)\), let \( \Delta \neq 0 \) be small and denote \( \tilde{x} = x + \Delta \) in the sequel.

**Lemma 5.1** Under conditions (A1)-(A3), we have for \( 0 \leq t \leq T \) and any positive constant \( \iota \), \( E|X(t)|^\iota \leq |x|^\iota e^{\kappa t} \leq C \), for \( x \neq 0 \), \( \alpha \in \mathcal{M} \), where \( \kappa = \kappa(\iota, K_0, m, g^\iota(i)) \) and \( C = C(\kappa, T) \).

**Proof.** For each \( \iota \in \mathcal{M} \) and \( x \neq 0 \), define \( V(x, \iota) = |x|^\iota \) for any \( \iota \in \mathbb{R}_+ - \{0\} \). Then for any \( \Delta > 0 \) and \( |x| > \Delta \),

\[
\mathcal{G}|x|^\iota = \iota|x|^{\iota - 2}x'b(x, \iota) + \lambda \int_\Gamma (|x + g(x, \iota, \gamma)|^\iota - |x|^\iota)\pi(d\gamma) \\
+ \frac{1}{2} \text{tr}[\sigma(x, \iota)\sigma'(x, \iota)|x|^{\iota - 4}(|x|^2I + (\iota - 2)x'x')].
\]

Since 0 is an equilibrium point, Cauchy-Schwartz inequality implies \( |x' b(x, \iota)| \leq |x| |b(x, \iota)| \leq K_0 |x|^2 \),

\[
\text{tr}(\sigma \sigma') = |\sigma|^2 \leq K_0 |\sigma|^2, \\
\text{tr}(\sigma' x'x') = x' \sigma' x \leq |x|^2 |\sigma|^2 \leq K_0 |x|^4.
\]

Therefore, we have

\[
|\mathcal{G}|x|^\iota| \leq K_0 \iota |x|^\iota + \frac{1}{2} K_0 \iota |x|^{\iota - 2} (|x|^2 + (\iota - 2)|x|^2) \\
+ \lambda |x|^\iota (1 + g^\iota(i)^\iota - 1) \leq \kappa |x|^\iota.
\]

Define the stopping time \( \tau_\Delta := \inf\{t \geq 0, |X(t)| \leq \Delta\} \). Then by the generalized Itô lemma, we obtain

\[
E|X(\tau_\Delta \wedge t)|^\iota = |x|^\iota + E \int_0^{\tau_\Delta \wedge t} \mathcal{G}|X(u)|^\iota du \\
\leq |x|^\iota + \kappa E \int_0^{\tau_\Delta \wedge t} |X(u)|^\iota du \\
\leq |x|^\iota + \kappa E \int_0^{\tau} |X(u \wedge \tau_\Delta)|^\iota du.
\]
By Gronwall’s inequality, it follows that

\[ E|X(\tau_t \land t)|^t \leq |x|^t e^{\epsilon t}. \]

Letting \( \Delta \to 0 \), by virtue of non-zero property of \( X(t) \) shown in Lemma 2.2, we have

\[ E|X(t)|^t \leq |x|^t e^{\epsilon t}. \]

For \( 0 \leq t \leq T \), we further have

\[ E|X(t)|^t \leq |x|^t e^{\epsilon t} \leq |x|^t e^{\epsilon T} = C. \]

Thus, the proof is completed.

**Theorem 5.2** Under the conditions of Lemma 5.1, define

\[ \phi^\Delta(t) = \frac{1}{\Delta} \int_0^t [b(\bar{X}(s), \bar{\alpha}(s)) - b(\bar{X}(s), \alpha(s))] ds 
+ \frac{1}{\Delta} \int_0^t [\sigma(\bar{X}(s), \bar{\alpha}(s)) - \sigma(\bar{X}(s), \alpha(s))] dw(s) 
+ \frac{1}{\Delta} \int_0^t \int_\Gamma [g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma)] N(ds, d\gamma). \tag{5.1} \]

Then we have \( \lim_{\Delta \to 0} E \sup_{0 \leq t \leq T} |\phi^\Delta(t)|^2 = 0. \)

**Proof.** It can be verified that

\[ E \sup_{0 \leq t \leq T} |\phi^\Delta(t)|^2 = \frac{K}{\Delta^2} E \int_0^T |b(\bar{X}(s), \bar{\alpha}(s)) - b(\bar{X}(s), \alpha(s))|^2 ds 
+ \frac{K}{\Delta^2} E \sup_{0 \leq t \leq T} \int_0^t |\sigma(\bar{X}(s), \bar{\alpha}(s)) - \sigma(\bar{X}(s), \alpha(s))| dw(s)^2 
+ \frac{K}{\Delta^2} E \int_0^T \int_\Gamma |g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma)|^2 ds\pi(d\gamma) 
+ \frac{K}{\Delta^2} E \sup_{0 \leq t \leq T} \int_0^t \int_\Gamma [g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma)] N(ds, d\gamma)^2. \tag{5.2} \]

Let us first consider the next to the last line of (5.2). By choosing \( \eta = \Delta^{-\gamma_0} \) with \( \gamma_0 > 2 \) and partition the interval \([0, T]\) by \( \eta \) we obtain

\[ E \int_0^T \int_{[\frac{t}{\Delta}]^{-1}} |g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma)|^2 ds\pi(d\gamma) 
= E \sum_{k=0}^{\lfloor T \Delta^{-\gamma} \rfloor} \int_{k\eta}^{k^{\eta}+\eta} \int_\Gamma |g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma)|^2 ds\pi(d\gamma) 
= KE \sum_{k=0}^{\lfloor T \Delta^{-\gamma} \rfloor} \left[ \int_{k\eta}^{k^{\eta}+\eta} \int_\Gamma |g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(k\eta), \bar{\alpha}(s^-), \gamma)|^2 ds\pi(d\gamma) 
+ \int_{k\eta}^{k^{\eta}+\eta} \int_\Gamma |g(\bar{X}(s^-), \bar{\alpha}(s^-), \gamma) - g(\bar{X}(k\eta), \alpha(s^-), \gamma)|^2 ds\pi(d\gamma) 
+ \int_{k\eta}^{k^{\eta}+\eta} \int_\Gamma |g(\bar{X}(s^-), \alpha(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma)|^2 ds\pi(d\gamma) \right]. \tag{5.3} \]
For the third line of (5.3), we have the following bound by virtue of (A2) and \[13, \text{Theorem 3.7.1},\]

\[
E \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} \left| g(\bar{X}(s^-), \bar{a}(s^-), \gamma) - g(\bar{X}(k\eta), \bar{a}(s^-), \gamma) \right|^2 ds \pi(d\gamma)
\leq KE \int_{k\eta}^{k\eta+\eta} E \left| \bar{X}(s^-) - \bar{X}(k\eta) \right|^2 ds
\leq K \int_{k\eta}^{k\eta+\eta} (s - k\eta) ds \leq K\eta^2.
\]

Recall that $K$ is a generic positive constant, whose values may be different for different appearances.

We can derive the upper bound for the last line of (5.3) similarly,

\[
E \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} \left| g(\bar{X}(k\eta), \alpha(s^-), \gamma) - g(\bar{X}(s^-), \alpha(s^-), \gamma) \right|^2 ds \pi(d\gamma) \leq O(\eta^2).
\]

To treat the term on the next to the last line of (5.3), note that

\[
E \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} |g(\bar{X}(k\eta), \bar{a}(s^-), \gamma) - g(\bar{X}(k\eta), \alpha(s^-), \gamma)|^2 ds \pi(d\gamma)
\leq KE \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} |g(\bar{X}(k\eta), \bar{a}(s^-), \gamma) - g(\bar{X}(k\eta), \bar{a}(k\eta), \gamma)|^2 ds \pi(d\gamma)
+ KE \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} |g(\bar{X}(k\eta), \bar{a}(k\eta), \gamma) - g(\bar{X}(k\eta), \alpha(s^-), \gamma)|^2 ds \pi(d\gamma).
\]

For the term on the second line of (5.5) and $k = 0, 1, \cdots, \left\lfloor \frac{T}{\eta} \right\rfloor - 1$,

\[
E \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} |g(\bar{X}(k\eta), \bar{a}(s^-), \gamma) - g(\bar{X}(k\eta), \bar{a}(k\eta), \gamma)|^2 ds \pi(d\gamma)
= E \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} |g(\bar{X}(k\eta), \bar{a}(s^-), \gamma) - g(\bar{X}(k\eta), \bar{a}(k\eta), \gamma)|^2 I_{\{\bar{a}(s^-) \neq \bar{a}(k\eta)\}} ds \pi(d\gamma)

\leq KE \sum_{i \in \mathcal{M}, j \neq i} \int_{k\eta}^{k\eta+\eta} \left[ 1 + |\bar{X}(k\eta)| \right]^2 I_{\{\bar{a}(k\eta) = i\}} \times E \left[ I_{\{\bar{a}(s) = j\}} \right] |\bar{X}(k\eta), \bar{a}(k\eta) = i| ds
\leq K \int_{k\eta}^{k\eta+\eta} O(\eta) ds \leq K\eta^2.
\]

In the above, we employed the fact that the time of jump of $X(t)$ does not coincide with that of switching part $\alpha(t)$ in \[29, \text{Proposition 2.2.}\] Also, Lemma 5.1 and boundedness of $Q(x)$ are involved. Now let us deal with the last line of (5.5) by using the basic coupling techniques \[4, \text{p. 11}]. Consider the measure

\[
\Lambda((x, j), (\bar{x}, i)) = |x - \bar{x}| + d(j, i), \quad \text{where} \quad d(j, i) = \begin{cases} 0 & \text{if } j = i, \\ 1 & \text{if } j \neq i. \end{cases}
\]
Let \((\alpha(t), \tilde{\alpha}(t))\) be a random process with a finite state space \(M \times M\) such that

\[
P[(\alpha(t+h), \tilde{\alpha}(t+h)) = (j, i) | (\alpha(t), \tilde{\alpha}(t)) = (k, l), (X(t), \tilde{X}(t)) = (x, \tilde{x})] = \begin{cases} 
    \tilde{q}(k, l)(j, i)(x, \tilde{x})h + o(h), & \text{if } (k, l) \neq (j, i), \\
    1 + \tilde{q}(k, l)(j, i)(x, \tilde{x})h + o(h), & \text{if } (k, l) = (j, i), 
\end{cases}
\]

where \(h \to 0\), and the matrix \((\tilde{q}(k, l)(j, i)(x, \tilde{x}))\) is the basic coupling of matrices \(Q(x) = (q_{kl}(x))\) and \(Q(\tilde{x}) = (q_{kl}(\tilde{x}))\) satisfying

\[
\tilde{Q}(x, \tilde{x})\tilde{f}(k, l) = \sum_{(j, i) \in M \times M} \tilde{q}(k, l)(j, i)(x, \tilde{x})(\tilde{f}(j, i) - \tilde{f}(k, l))
\]

for any function \(\tilde{f}(\cdot, \cdot)\) defined on \(M \times M\). Then we have

\[
E[I_{\{\alpha(s) = j\}}|\alpha(kn) = i_1, \tilde{\alpha}(kn) = i, X(kn) = x, \tilde{X}(kn) = \tilde{x}] = \sum_{i \in M} E[I_{\{\alpha(s) = j\}}I_{\{\tilde{\alpha}(kn) = i\}}|\alpha(kn) = i_1, \tilde{\alpha}(kn) = i, X(kn) = x, \tilde{X}(kn) = \tilde{x}]
\]

(5.8)

Therefore, for \(k = 1, \ldots, \left\lfloor \frac{T}{\eta} \right\rfloor - 1\), we have

\[
E \int_{kn}^{kn+\eta} \int_{\Gamma} \left| g(\tilde{X}(kn), \tilde{\alpha}(kn), \gamma) - g(\tilde{X}(kn), \alpha(s^-), \gamma) \right|^2 ds d\pi(d\gamma)
\]

\[
= E \sum_{i \in M} \sum_{j \neq i} \int_{kn}^{kn+\eta} \int_{\Gamma} \left| g(\tilde{X}(kn), i, \gamma) - g(\tilde{X}(kn), j, \gamma) \right|^2 I_{\{\alpha(s^-) = j\}} I_{\{\tilde{\alpha}(kn) = i\}} ds d\pi(d\gamma)
\]

\[
\leq KE \sum_{i, i_1 \in M} \sum_{j \neq i} \int_{kn}^{kn+\eta} [1 + |\tilde{X}(kn)|^2] I_{\{\tilde{\alpha}(kn) = i, \alpha(kn) = i_1\}} \times E[I_{\{\alpha(s) = j\}}|\alpha(kn) = i_1, \tilde{\alpha}(kn) = i, X(kn) = x, \tilde{X}(kn) = \tilde{x}] ds = O(\eta^2).
\]

(5.9)

For \(k = 0\), recall that \(\alpha(0) = \tilde{\alpha}(0) = \alpha, X(0) = x\) and \(\tilde{X}(0) = \tilde{x}\), we have

\[
E \int_0^\eta \int_{\Gamma_\eta} \left| g(\tilde{X}(0), \tilde{\alpha}(0), \gamma) - g(\tilde{X}(0), \alpha(s), \gamma) \right|^2 ds d\pi(d\gamma)
\]

\[
= E \int_0^\eta \int_{\Gamma_\eta} \sum_{j \neq \alpha} \left| g(\tilde{x}, \alpha, \gamma) - g(\tilde{x}, j, \gamma) \right|^2 I_{\{\alpha(s) = j\}} ds d\pi(d\gamma)
\]

\[
\leq K \sum_{j \neq \alpha} \int_0^\eta [1 + \tilde{x}^2] E[I_{\{\alpha(s) = j\}}|\alpha(0) = \alpha, \tilde{X}(0) = \tilde{x}] ds
\]

\[
\leq K \int_0^\eta \sum_{j \neq \alpha} \left| q_{\alpha j}(\tilde{x}) s + o(s) \right| ds \leq K \eta^2.
\]

(5.10)
Thus, for $k = 0, 1, \cdots, \lfloor \frac{T}{\eta} \rfloor - 1$,

$$E \int_{k\eta}^{k\eta+\eta} \int_{\Gamma} |g(\tilde{X}(k\eta), \tilde{\alpha}(k\eta), \gamma) - g(\tilde{X}(k\eta), \alpha(s^-), \gamma)|^2 ds d\pi(d\gamma) \leq K\eta^2. \quad (5.11)$$

Now we can obtain

$$E \int_0^T \int_{\Gamma} |g(\tilde{X}(s^-), \tilde{\alpha}(s^-), \gamma) - g(\tilde{X}(s^-), \alpha(s^-), \gamma)|^2 ds d\pi(d\gamma) \leq \sum_{k=0}^{\lfloor \frac{T}{\eta} \rfloor - 1} K\eta^2 \leq K\eta.$$

Likewise, we also obtain the bound for the martingale part

$$E \sup_{0 \leq t \leq T} | \int_0^t \int_{\Gamma} [g(\tilde{X}(s^-), \tilde{\alpha}(s^-), \gamma) - g(\tilde{X}(s^-), \alpha(s^-), \gamma)] N(ds, d\gamma) |^2 \leq K\eta.$$

For the drift and diffusion parts involved, the argument in [33, Lemma 4.3] leads to

$$E \int_0^T |(b(\tilde{X}(s), \tilde{\alpha}(s)) - b(\tilde{X}(s), \alpha(s)))|^2 ds \leq K\eta,$$

$$E \sup_{0 \leq t \leq T} | \int_0^t [\sigma(\tilde{X}(s), \tilde{\alpha}(s)) - \sigma(\tilde{X}(s), \alpha(s))] dw(s) |^2 \leq K\eta.$$

Therefore, we obtain

$$E \sup_{0 \leq t \leq T} | \phi^\Delta(t) |^2 \leq K\frac{\eta}{\Delta^2} = K\Delta^{\gamma_0 - 2} \to 0 \text{ as } \Delta \to 0. \quad (5.12)$$

This concludes the proof.

**Lemma 5.3** Under the conditions of Theorem 5.2, $E[ \sup_{0 \leq t \leq T} | \tilde{X}^{\tilde{x},\alpha}(t) - X^{x,\alpha}(t) |^2 ] \leq C|\tilde{x} - x|^2$, where the constant $C$ satisfies $C = C(K_0, T)$.

**Proof.** Let $T > 0$ be fixed and recall that $\Delta = \tilde{x} - x$, then we have $\tilde{X}^{\tilde{x},\alpha}(t) - X^{x,\alpha}(t) = \Delta + A(t) + B(t)$, where

$$A(t) = \int_0^t [b(\tilde{X}(s), \tilde{\alpha}(s)) - b(\tilde{X}(s), \alpha(s))] ds$$

$$+ \int_0^t [\sigma(\tilde{X}(s), \tilde{\alpha}(s)) - \sigma(\tilde{X}(s), \alpha(s))] dw(s)$$

$$+ \int_0^t \int_{\Gamma} [g(\tilde{X}(s^-), \tilde{\alpha}(s^-), \gamma) - g(\tilde{X}(s^-), \alpha(s^-), \gamma)] N(ds, d\gamma)$$

$$= \Delta \phi^\Delta(t), \quad (5.13)$$

$$B(t) = \int_0^t [b(\tilde{X}(s), \alpha(s)) - b(X(s), \alpha(s))] ds$$

$$+ \int_0^t [\sigma(\tilde{X}(s), \alpha(s)) - \sigma(X(s), \alpha(s))] dw(s)$$

$$+ \int_0^t \int_{\Gamma} [g(\tilde{X}(s^-), \alpha(s^-), \gamma) - g(X(s^-), \alpha(s^-), \gamma)] N(ds, d\gamma). \quad (5.14)$$
Hence
\[
\sup_{t \in [0,T]} |\tilde{X}_{\bar{x},\alpha}(t) - X_{x,\alpha}(t)|^2 \leq 3\Delta^2 + 3 \sup_{t \in [0,T]} |A(t)|^2 + 3 \sup_{t \in [0,T]} |B(t)|^2.
\]

It follows from (5.12) that
\[
E[\sup_{t \in [0,T]} |A(t)|^2] \leq \Delta^2 E[\sup_{t \in [0,T]} |\phi(t)|^2] \leq K\Delta \sigma_0 = o(\Delta^2).
\]

By the Hölder inequality and the Lipschitz continuity, we have
\[
E[\sup_{t \in [0,T]} \left| \int_0^t [b(\tilde{X}(s), \alpha(s)) - b(X(s), \alpha(s))]ds \right|^2] \leq K \int_0^T E|\tilde{X}(s) - X(s)|^2 ds
\]
and
\[
E[\sup_{t \in [0,T]} \left| \int_0^T \int_{\Gamma} [g(\tilde{X}(s^-), \alpha(s^-), \gamma) - g(X(s^-), \alpha(s^-), \gamma)]ds\pi(ds) |^2] \leq K \int_0^T E|\tilde{X}(s^-) - X(s^-)|^2 ds.
\]

Then the basic properties of stochastic integrals (w.r.t. \( w(\cdot) \) and \( \tilde{N}(\cdot) \)) together with the Lipschitz continuity lead to
\[
E[\sup_{t \in [0,T]} \left| \int_0^t [\sigma(\tilde{X}(s), \alpha(s)) - \sigma(X(s), \alpha(s))]dw(s) \right|^2] \leq K \int_0^T E|\tilde{X}(s) - X(s)|^2 ds
\]
and
\[
E[\sup_{t \in [0,T]} \left| \int_0^T \int_{\Gamma} [g(\tilde{X}(s^-), \alpha(s^-), \gamma) - g(X(s^-), \alpha(s^-), \gamma)] \tilde{N}(ds, d\gamma) \right|^2] \leq K \int_0^T E|\tilde{X}(s^-) - X(s^-)|^2 ds.
\]

So,
\[
E[\sup_{t \in [0,T]} |\tilde{X}_{\bar{x},\alpha}(t) - X_{x,\alpha}(t)|^2] \leq 3\Delta^2 + K \int_0^T E[\sup_{u \in [0,T]} |\tilde{X}(u) - X(u)|^2] du + o(\Delta^2). \tag{5.15}
\]

Now, by Gronwall’s inequality
\[
E[\sup_{t \in [0,T]} |\tilde{X}_{\bar{x},\alpha}(t) - X_{x,\alpha}(t)|^2] \leq 3\Delta^2 \exp(KT) + o(\Delta^2) \leq K|\bar{x} - x|^2.
\]

Thus, we have completed the proof. \(\square\)

Let us introduce some notations to proceed. Recall that a vector \( \beta = (\beta_1, \beta_2, \ldots, \beta_r) \) with nonnegative integer component is referred to as a multi-index. Put \( |\beta| = \beta_1 + \beta_2 + \cdots + \beta_r \), we define \( D_\beta^2 \) as
\[
D_\beta^2 = \frac{\partial^\beta}{\partial x^\beta} = \frac{\partial|\beta|}{\partial x_1^{\beta_1} \cdots \partial x_r^{\beta_r}}.
\]

Recall that \( \Delta = \bar{x} - x \) and define
\[
Z^{\Delta}(t) = \frac{\tilde{X}_{\bar{x},\alpha}(t) - X_{x,\alpha}(t)}{\Delta}. \tag{5.16}
\]
Then we have the following expression:

\[ Z^\Delta(t) = 1 + \phi^\Delta(t) + \frac{1}{\Delta} \int_0^t \left[ b(\tilde{X}(s), \alpha(s)) - b(X(s), \alpha(s)) \right] ds \]

\[ + \frac{1}{\Delta} \int_0^t \left[ \sigma(\tilde{X}(s), \alpha(s)) - \sigma(X(s), \alpha(s)) \right] dw(s) \]

\[ + \frac{1}{\Delta} \int_0^t \left[ g(\tilde{X}(s), \alpha(s^-), \gamma) - g(X(s^-), \alpha(s^-), \gamma) \right] N(ds, d\gamma), \] (5.17)

where \( \phi^\Delta(t) \) is defined in (5.1).

**Lemma 5.4** Under the conditions of Theorem 5.3, assume that for each \( i \in \mathcal{M} \), \( b(\cdot, i), \sigma(\cdot, i) \) and \( g(\cdot, i, \gamma) \) have continuously partial derivatives with respect to the variable \( x \) up to the second order and that

\[ |D^\beta_x b(x, i)| + |D^\beta_x \sigma(x, i)| + |D^\beta_x g(x, i, \gamma)| \leq K(1 + |x|^\rho), \]

where \( K \) and \( \rho \) are positive constants and \( \beta \) is a multi-index with \( |eta| \leq 2 \). Then \( X^{x,\alpha}(t) \) is twice continuously differentiable in mean square with respect to \( x \).

**Proof.** Given the definition of \( Z^\Delta(t) \) above and Theorem 5.2, we just need to consider the last three terms of (5.17). First, note that

\[ \frac{1}{\Delta} \int_0^t \int_\Gamma \left[ g(\tilde{X}(s^-), \alpha(s^-), \gamma) - g(X(s^-), \alpha(s^-), \gamma) \right] ds \pi(ds, d\gamma) \]

\[ = \frac{1}{\Delta} \int_\Gamma \int_0^t \int_0^1 \frac{d}{d\nu} g(X(s^-) + \nu(\tilde{X}(s^-) - X(s^-)), \alpha(s^-), \gamma) d\nu ds \pi(ds, d\gamma) \]

\[ = \int_\Gamma \int_0^t \left[ \int_0^1 g_x(X(s^-) + \nu(\tilde{X}(s^-) - X(s^-)), \alpha(s^-), \gamma) d\nu \right] Z^\Delta(s^-) ds \pi(ds, d\gamma), \] (5.18)

where \( g_x(\cdot) \) denotes the partial derivative of \( g(\cdot, i, \gamma) \) with respect to \( x \). It follows from Lemma 5.3 that for any \( s \in [0, T] \), \( \tilde{X}(s^-) - X(s^-) \rightarrow 0 \) in probability as \( \Delta \rightarrow 0 \). This implies that

\[ \int_0^1 g_x(X(s^-) + \nu(\tilde{X}(s^-) - X(s^-)), \alpha(s^-), \gamma) d\nu \rightarrow g_x(X(s^-), \alpha(s^-), \gamma) \] (5.19)

in probability as \( \Delta \rightarrow 0 \). Therefore, we have

\[ \frac{1}{\Delta} \int_0^t \int_\Gamma \left[ g(\tilde{X}(s^-), \alpha(s^-), \gamma) - g(X(s^-), \alpha(s^-), \gamma) \right] ds \pi(ds, d\gamma) \]

\[ \rightarrow \int_0^t \int_\Gamma g_x(X(s^-), \alpha(s^-), \gamma) Z^\Delta(s^-) ds \pi(ds, d\gamma). \] (5.20)

Similarly, we have

\[ \frac{1}{\Delta} \int_0^t [b(\tilde{X}(s), \alpha(s)) - b(X(s), \alpha(s))] ds \rightarrow \int_0^t b_x(X(s), \alpha(s)) Z^\Delta(s) ds \] (5.21)

in probability as \( \Delta \rightarrow 0 \) and

\[ \frac{1}{\Delta} \int_0^t [\sigma(\tilde{X}(s), \alpha(s)) - \sigma(X(s), \alpha(s))] dw(s) \rightarrow \int_0^t \sigma_x(X(s), \alpha(s)) Z^\Delta(s) dw(s) \] (5.22)

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in probability as $\Delta \to 0$. $b_x(\cdot)$ and $\sigma_x(\cdot)$ denote the partial derivative of $b(\cdot, i)$ and $\sigma(\cdot, i)$ with respect to $x$, respectively. Recall the definition of $Z^\Delta(t)$ in equation (5.16), Theorem 5.2, (5.19)-(5.22), and [6, Theorem 5.5.2] yield

$$E|Z^\Delta(t) - \zeta(t)|^2 \to 0 \text{ as } \Delta \to 0. \quad (5.23)$$

where

$$\zeta(t) = 1 + \int_0^t b_x(X(s), \alpha(s))\zeta(s)ds + \int_0^t \sigma_x(X(s), \alpha(s))\zeta(s)dw(s) + \int_0^t \int_\Gamma g_x(X(s), \alpha(s), \gamma)\zeta(s)N(ds, d\gamma)$$

and $\zeta(t) = \zeta_x^\alpha(t)$ is mean square continuous with respect to $x$. Therefore, $\frac{\partial}{\partial x} X^{x, \alpha}(t)$ exists in the mean square sense and $\zeta(t) = \frac{\partial}{\partial x} X^{x, \alpha}(t)$. Likewise, we can show $\frac{\partial^2}{\partial x^2} X^{x, \alpha}(t)$ exists in the mean square sense and is mean square continuous with respect to $x$. \hfill \Box

**Lemma 5.5** **Under the assumptions of Lemma 5.4, we have**

$$\sup_{t \in [0, T]} E|\zeta(t)|^2 \leq K = K(x, \bar{x}, T, K_0) < \infty.$$

**Proof.** For any $t \in [0, T]$, $E|\zeta(t)|^2 \leq 2E|\zeta(t) - Z^\Delta(t)|^2 + 2E|Z^\Delta(t)|^2$. By (5.23), it suffices to consider the last term above. In fact,

$$E|Z^\Delta(t)|^2 \leq K + 5E|\phi^\Delta(t)|^2 + 5E\frac{1}{\Delta} \int_0^t |b(\tilde{X}(u), \alpha(u)) - b(X(u), \alpha(u))|du|^2$$

$$+ 5E\frac{1}{\Delta} \int_0^t |\sigma(\tilde{X}(u), \alpha(u)) - \sigma(X(u), \alpha(u))|du|^2$$

$$+ 5E\frac{1}{\Delta} \int_0^t \int_\Gamma |g(\tilde{X}(u^-), \alpha(u^-), \gamma) - g(X(u^-), \alpha(u^-), \gamma)|N(du, d\gamma)|^2,$$

so

$$E|Z^\Delta(t)|^2 \leq K + 5t \frac{1}{|\Delta|^2} E \int_0^t |b(\tilde{X}(u), \alpha(u)) - b(X(u), \alpha(u))|^2 du$$

$$+ 5 \frac{1}{|\Delta|^2} E \int_0^t |\sigma(\tilde{X}(u), \alpha(u)) - \sigma(X(u), \alpha(u))|^2 du$$

$$+ 5t \frac{1}{|\Delta|^2} E \int_0^t \int_\Gamma |g(\tilde{X}(u^-), \alpha(u^-), \gamma) - g(X(u^-), \alpha(u^-), \gamma)|^2 d\gamma|du|$$

$$\leq K + 5K_0(T + 1) \frac{1}{|\Delta|^2} E \int_0^t |\tilde{X}(u) - X(u)|^2 du$$

$$+ 5K_0(T + 1) \frac{1}{|\Delta|^2} E \int_0^t |\tilde{X}(u^-) - X(u^-)|^2 du \leq K = K(x, \bar{x}, T, K_0).$$

Hence the proof is completed. \hfill \Box

**Lemma 5.6** **Assume the conditions of Lemma 5.5 hold. Then the function $E|X^{x, \alpha}(t)|^p$ is twice continuously differentiable with respect to the variable $x$, except possibly at $x = 0$.**
Proof. In what follows, let \( u(t, x, \alpha) = E[\phi(X(t), \alpha(t))] = E|X_{x, \alpha}(t)|^p \), then

\[
\frac{u(t, \tilde{x}, \alpha) - u(t, x, \alpha)}{\Delta} = \frac{1}{\Delta} E[|\tilde{X}(t)|^p - |X(t)|^p] \\
= \frac{1}{\Delta} E \int_0^1 \frac{d}{dv}[X(t) + v(\tilde{X}(t) - X(t))]^p dv \\
= E[Z^\Delta(t) \int_0^1 |X(t) + v(\tilde{X}(t) - X(t))|^p dv],
\]

where \(| \cdot |^p_x\) denotes the partial derivative of \( \phi(\cdot, i) = | \cdot |^p_x\) with respect to \( x \). Consider

\[
\frac{1}{\Delta} E[|\tilde{X}(t)|^p - |X(t)|^p] - E[|X(t)|^p_x \varsigma(t)] \\
\leq |E \int_0^1 [X(t) + v(\tilde{X}(t) - X(t))|^p_x dvZ^\Delta(t)] - E|X(t)|^p_x \varsigma(t)| \\
\leq E \int_0^1 | [X(t) + v(\tilde{X}(t) - X(t))|^p_x dv - |X(t)|^p_x|^2 Z^\Delta(t)] + E||X(t)|^p_x [Z^\Delta(t) - \varsigma(t)]|.
\]

For the second part of last line of (5.26), by Cauchy-Schwartz inequality, we obtain

\[
E ||X(t)|^p_x [Z^\Delta(t) - \varsigma(t)]| \leq E^{1/2} |X(t)|_2^p E^{1/2} [Z^\Delta(t) - \varsigma(t)]^2 \\
\leq KE^{1/2} [Z^\Delta(t) - \varsigma(t)]^2 \rightarrow 0 \text{ as } \Delta \rightarrow 0.
\]

Here we used Lemma 5.1 and (5.23). Similarly, we can show the first term of last line of (5.26) goes to 0 as \( \Delta \rightarrow 0 \). Thus \( E|X_{x, \alpha}(t)|^p \) is differentiable with respect to the variable \( x \). Likewise, we can also see it is twice continuously differentiable with respect to the variable \( x \). As a nice application of the smooth dependence on the initial data, we obtain a Lyapunov converse theorem, namely, necessary conditions for exponential \( p \) stability.

**Theorem 5.7** Assume that the conditions of Lemma 5.6 hold and that the equilibrium point 0 is exponentially \( p \)-stable. Then for each \( i \in M \), there exists a function \( V(\cdot, i) \in C^2(\mathbb{R}^r : \mathbb{R}_+) \) such that

\[
k_1|x|^p \leq V(x, i) \leq k_2|x|^p \quad x \in D, \\
GV(x, i) \leq -k_3|x|^p \quad \text{for all } x \in D - \{0\}, \\
\left| \frac{\partial V}{\partial x_j}(x, i) \right| \leq k_4|x|^{p-1}, \\
\left| \frac{\partial^2 V}{\partial x_j \partial x_l}(x, i) \right| \leq k|x|^{p-2},
\]

for all \( 1 \leq j, l \leq r, x \in D - \{0\} \), and for some positive constants \( k_1, k_2, k_3 \) and \( k_4 \), where \( D \) is a neighborhood of 0.

**Proof.** For each \( i \in M \), consider the function

\[
V(x, i) = \int_0^T E|X^{x, i}(u)|^p du.
\]

It follows from Lemma 5.6, \( V(x, i) \) is twice continuously differentiable with respect to \( x \) except possibly at 0. The equilibrium point 0 is exponential \( p \)-stable, therefore there is a \( \kappa > 0 \) such that

\[
V(x, i) = \int_0^T E|X^{x, i}(u)|^p du \leq \int_0^T K|x|^p e^{-\kappa u} du \leq k_2|x|^p.
\]
For the function \(|x|^p\), we have \(|G|x|^p| \leq K|x|^p\) for some positive real number \(K\). An application of generalized Itô’s formula leads to

\[
E|X^{x,i}(T)|^p - |x|^p = E \int_0^T G|X^{x,i}(u)|^p du \geq -KE \int_0^T |X^{x,i}(u)|^p du = -KV(x,i).
\]

Again recall that equilibrium point \(x = 0\) is exponential \(p\)-stable, we can choose \(T\) such that \(E|X^{x,i}(T)|^p \leq \frac{1}{2}|x|^p\), and therefore, we have \(V(x,i) \geq \frac{|x|^p}{2K} = k_1|x|^p\). Notice that

\[
GV(x,i) = \int_0^T GE|X^{x,i}(u)|^p du.
\]

Let \(u(t,x,i) = E|X^{x,i}(t)|^p\), by the similar argument in step 1 and step 2 of [32, Theorem 7.10], we obtain

\[
GV(x,i) = \int_0^T GE|X^{x,i}(u)|^p du = u(T,x,i) - u(0,x,i)
\]

\[
= E|X^{x,i}(T)|^p - E|X^{x,i}(0)|^p = E|X^{x,i}(T)|^p - |x|^p
\]

\[
\leq -\frac{1}{2}|x|^p = -k_3|x|^p.
\]

Note that

\[
\frac{\partial E|X^{x,i}(t)|^p}{\partial x_j} = pE|X^{x,i}(t)|^{p-1} \text{sgn}(X^{x,i}(t)) \frac{\partial X^{x,i}(t)}{\partial x_j},
\]

so

\[
\left|\frac{\partial E|X^{x,i}(t)|^p}{\partial x_j}\right| = pE \left(|X^{x,i}(t)|^{p-1} \left|\frac{\partial X^{x,i}(t)}{\partial x_j}\right|\right)
\]

\[
\leq pE \frac{1}{2} |X^{x,i}(t)|^{2p-2} E \frac{1}{2} \left|\frac{\partial X^{x,i}(t)}{\partial x_j}\right|^2
\]

\[
\leq K(|x|^{2p-2} e^{-\kappa t})^{\frac{1}{2}} = K|x|^{p-1} e^{-\kappa t/2}.
\]

For the last line above, we used the Lemma 5.1 and Lemma 5.5. Consequently, we have

\[
\left|\frac{\partial V(x,i)}{\partial x_j}\right| = \left|\int_0^T \frac{\partial}{\partial x_j} E|X^{x,i}(u)|^p du\right| \leq \int_0^T K|x|^{p-1} e^{-\kappa t/2} du \leq k_4|x|^{p-1}.
\]

We can have estimate of the second derivative of \(V(x,i)\) by similar argument, the theorem is thus proved.

\[
\square
\]

6 Asymptotic Stability in Distribution

For practical systems, frequently, we do not have information regarding the equilibria of the systems. Nevertheless, the systems still possesses certain kind of stability properties. Thus it is necessary to extend our definition to consider the so-called the asymptotic stability in distribution. Here, the assumptions \(b(0,i) = 0\) and \(\sigma(0,i) = 0\) for each item \(i \in \mathcal{M}\) are not needed. That is, the system under consideration may have no equilibrium point at all. To proceed, let us first give two definitions.

**Definition 6.1** The system given by (2.1) and (2.2) is asymptotically stable in distribution if, there exists such a probability measure \(\nu(\cdot \times \cdot)\) on \(\mathbb{R}^r \times \mathcal{M}\) that the transition probability \(p(t,x,\alpha,dy \times \{i\})\) of \((X(t),\alpha(t))\) converges weakly to \(\nu(dy \times \{i\})\) as \(t \to \infty\) for every \((x,\alpha) \in \mathbb{R}^r \times \mathcal{M}\).
Definition 6.2  The definitions of (P1) and (P2) are as follows.

- The switching jump diffusion process given by (2.1) and (2.2) is said to have property (P1) if, for any $(x, \alpha) \in \mathbb{R}^r \times \mathcal{M}$ and any $\varepsilon > 0$, there exists a constant $R > 0$ such that
  
  $$P\{|X^{x,\alpha}(t)| \geq R\} < \varepsilon, \text{ for any } t \geq 0.$$ 

- The switching jump diffusion process given by (2.1) and (2.2) is said to have property (P2) if, for any $\varepsilon > 0$ and any compact subset $\hat{\mathcal{C}}$ of $\mathbb{R}^r$, there exists a $T = T(\varepsilon, \hat{\mathcal{C}}) > 0$ such that
  
  $$P(|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)| \leq \varepsilon) \to 1 \text{ as } t \to \infty,$$

  whenever $(x_0, y_0, i_0) \in \hat{\mathcal{C}} \times \hat{\mathcal{C}} \times \mathcal{M}$. 

In this section, we first establish asymptotic stability in distribution of the process $(X(t), \alpha(t))$ in which $\alpha(t)$ is a Markov chain that is independent of the Brownian motion, which is referred as Markov switching jump diffusions. Then we further extend the results to state-dependent switching process.

6.1 Markov Switching Jump Diffusions

Throughout this section, $\alpha(t)$ is a Markov chain independent of the Brownian motion. We first establish a result on stability in distribution.

Proposition 6.3 Suppose that (A2) is satisfied, that $b(\cdot, i), \sigma(\cdot, i)$, and $g(\cdot, i, \gamma)$ grow at most linearly for each $i \in \mathcal{M}$ and $\gamma \in \Gamma$, that conditions (P1) and (P2) hold, and that the generator of the Markov chain $Q$ is irreducible. Then the switching jump diffusion process $(X(t), \alpha(t))$ is stable in distribution.

Proof. We note that [34, Theorem 3.1] in fact works not only for Markov switching diffusion processes but also for more general Markov processes. In our current setup, $(X(t), \alpha(t))$ is a Markov process. So we can use essentially the same steps as in the aforementioned reference to show the process is stable in distribution. The verbatim argument is omitted. \(\square\)

Our next task is to find sufficient conditions that ensure conditions (P1) and (P2) are in force. The result is stated in the next theorem.

Theorem 6.4 Assume that for each $i \in \mathcal{M}$, there exists function $V(\cdot, i) \in C^2(\mathbb{R}^r : \mathbb{R}_+)$ satisfying the following two conditions: There exists a positive real number $\beta$ such that

\begin{align*}
\mathcal{G}V(x, i) &\leq -\beta V(x, i), \quad (6.1) \\
V_R := \inf_{|x| \geq R \atop i \in \mathcal{M}} V(x, i) &\to \infty \text{ as } R \to \infty. \quad (6.2)
\end{align*}

Then (P1) and (P2) hold.
Proof. Let us first verify (P1). Define the stopping time

$$
\tau_R := \inf\{t \geq 0 : |X(t)| \geq R\}.
$$

Consider $V(x,i)e^{\beta t}$ and let $t_R = \tau_R \wedge t$. By virtue of Dynkin’s formula, we have

$$
E_{x,\alpha}[V(X(t_R), \alpha(t_R))e^{\beta R}] - V(x, \alpha) = E_{x,\alpha}\int_0^{t_R} e^{\beta s} \mathcal{G}V(X(s), \alpha(s))\,ds
+ \beta E_{x,\alpha}\int_0^{t_R} e^{\beta s}V(X(s), \alpha(s))\,ds,
$$

where $E_{x,\alpha}$ denotes the expectation with $X(0) = x$ and $\alpha(0) = \alpha$.

Hence, by virtue of (6.1), $E_{x,\alpha}V(X(t_R), \alpha(t_R)) \leq V(x, \alpha)e^{-\beta R}$. We further have

$$
V_R P\{\tau_R \leq t\} \leq E_{x,\alpha}[V(X(\tau_R), \alpha(\tau_R))I_{[\tau_R \leq t]}] \leq V(x, \alpha)e^{-\beta \tau_R}.
$$

Note that $\tau_R \leq t$ if and only if $\sup_{0 \leq u \leq t} |X(u)| \geq R$. Therefore, it follows that

$$
P\{\sup_{0 \leq u \leq t} |X_{x,\alpha}(u)| \geq R\} \leq \frac{V(x, \alpha)e^{-\beta \tau_R}}{V_R} \leq \frac{V(x, \alpha)}{V_R}.
$$

Then upon using (6.2), $P\{|X_{x,\alpha}(t)| \geq R\} \to 0$ as $R \to \infty$, for all $t \geq 0$. To guarantee (P2) hold, similar technique is involved here. But now we need to consider the difference between two solutions of equation (2.1) starting from different initial values in compact set $\bar{C}$. Namely, $(x, \alpha)$ and $(y, \alpha)$.

$$
X_{x,\alpha}(t) - X_{y,\alpha}(t)
= x - y + \int_0^t [b(X_{x,\alpha}(s), \alpha(s)) - b(X_{y,\alpha}(s), \alpha(s))]\,ds
+ \int_0^t [\sigma(X_{x,\alpha}(s), \alpha(s)) - \sigma(X_{y,\alpha}(s), \alpha(s))]\,dw(s)
+ \int_0^t \int_{\Gamma} [g(X_{x,\alpha}(s^-), \alpha(s^-), \gamma) - g(X_{y,\alpha}(s^-), \alpha(s^-), \gamma)]\,N(ds, d\gamma).
$$

Let $Z_{x,y,\alpha}(t) = X_{x,\alpha}(t) - X_{y,\alpha}(t)$, so $Z(0) = z = x - y$. Then

$$
dZ_{x,y,\alpha}(t) = [b(X_{x,\alpha}(t), \alpha(t)) - b(X_{y,\alpha}(t), \alpha(t))]\,dt
+ [\sigma(X_{x,\alpha}(t), \alpha(t)) - \sigma(X_{y,\alpha}(t), \alpha(t))]\,dw(t)
+ \int_{\Gamma} [g(X_{x,\alpha}(t^-), \alpha(t^-), \gamma) - g(X_{y,\alpha}(t^-), \alpha(t^-), \gamma)]\,N(dt, d\gamma).
$$

Define a stopping time $\tau_\varepsilon := \inf\{t \geq 0, |X_{x,\alpha}(t) - X_{y,\alpha}(t)| \geq \varepsilon\}$ and let $t_\varepsilon = \tau_\varepsilon \wedge t$. Then we have

$$
E_{z,\alpha}V(Z(t_\varepsilon), \alpha(t_\varepsilon)) - V(z, \alpha) = E_{z,\alpha}\int_0^{t_\varepsilon} \mathcal{G}V(Z(s), \alpha(s))\,ds
\leq -\beta \int_0^{t_\varepsilon} E_{z,\alpha}V(Z(s), \alpha(s))\,ds.
$$

Given $s \leq \tau_\varepsilon \wedge t$, we have $s \wedge \tau_\varepsilon = s$. As a result,

$$
E_{z,\alpha}V(Z(t \wedge \tau_\varepsilon), \alpha(t \wedge \tau_\varepsilon)) - V(z, \alpha) \leq -\beta \int_0^{t} E_{z,\alpha}V(Z(s \wedge \tau_\varepsilon), \alpha(s \wedge \tau_\varepsilon))\,ds.
$$
By applying Gronwall’s inequality, we obtain

$$ E_{x,\alpha} V(Z(\tau_\varepsilon \wedge t), \alpha(\tau_\varepsilon \wedge t)) \leq V(z, \alpha) e^{-\beta t}. $$

Hence,

$$ V_\varepsilon P(\tau_\varepsilon \leq t) \leq E_{x,\alpha} [V(Z(\tau_\varepsilon), \alpha(\tau_\varepsilon)) I_{\{\tau_\varepsilon \leq t\}}] \leq V(z, \alpha) e^{-\beta t}, $$

in which $V_\varepsilon = \inf \{ V(z, i), z \in \mathbb{R}^n \setminus B_\varepsilon, i \in \mathcal{M} \}$ and $B_\varepsilon = \{ z \in \tilde{\mathcal{C}}, |z| < \varepsilon \}$, so $V_\varepsilon > 0$. Note that $\tau_\varepsilon \leq t$ if and only if $\sup_{0 \leq u \leq t} |Z(u)| \geq \varepsilon$. Therefore, it follows that $P\{ \sup_{0 \leq u \leq t} |Z(u)| \geq \varepsilon \} \leq \frac{V(z, \alpha) e^{-\beta t}}{V_\varepsilon}$, so $P(|Z(t)| \geq \varepsilon) \to 0$ as $t \to \infty$. That is, $P(|X^{x,\alpha}(t) - X^{y,\alpha}(t)| \leq \varepsilon) \to 1$ as $t \to \infty$. Thus, the proof is concluded. $\square$

### 6.2 State-Dependent Case

Now, let us consider the case when the generator of the discrete component $\alpha(t)$ is $x$-dependent. In this case, the switching part is no longer a Markov chain. Because of the interplays between $\alpha(t)$ and $X(t)$, we need more complex notations. We use the same notations and technique as that of [3]. Switching diffusions were treated in [3], whereas we deal with switching jump diffusions. Define

$$ \tilde{X}(t) = \left[ X'(t) I_{\{\alpha(t)=1\}}, X'(t) I_{\{\alpha(t)=2\}}, \ldots, X'(t) I_{\{\alpha(t)=m\}} \right]', $$

$$ S = \bigcup_{i \in \mathcal{M}} 0_{r(i-1)} \times \mathbb{R}^r \times 0_{r(m-i)}, $$

(6.3)

Here and in the sequel $0_{k \times k_2}$ is a $k \times k_2$ zero matrix, $0_k$ denotes the $k$-dimensional zero column vector. It is seen that $S \subseteq \mathbb{R}^{mr}$ and $\tilde{X}(t)$ is an $S$-valued process. For $i \in \mathcal{M}$, $x \in \mathbb{R}^r$, define

$$ \check{x}^i = 0_{r(i-1)} \times x \times 0_{r(m-i)} \in S, $$

$$ \check{\Xi} = \bigcup_{i,j \in \mathcal{M}} 0_{r(i-1)} \times \mathbb{R}^r \times 0_{r(j-i-1)} \times \mathbb{R}^r \times 0_{r(m-j)}. $$

(6.4)

Then $\check{\Xi} \subseteq \mathbb{R}^{mr}$ and $\tilde{X}^{x_0,i_0}(t) - \tilde{X}^{y_0,j_0}(t)$ is a $\check{\Xi} \cup S$-valued process. For $x, y \in \mathbb{R}^r, i, j \in \mathcal{M}$,

$$ \check{x}^i - \check{y}^j \begin{cases} [0'_{r(i-1)}, x'-y', 0'_{r(m-i)}] \in S \text{ for } i = j, \\ [0'_{r(i-1)}, x', 0'_{r(j-i-1)}, -y', 0'_{r(m-j)}] \in \check{\Xi} \text{ for } i < j, \\ [0'_{r(j-1)}, -y', 0'_{r(i-j-1)}, x', 0'_{r(m-j)}] \in \check{\Xi} \text{ for } i > j. \end{cases} $$

Similar to the conditions we mentioned in the previous part, under the condition (P1) and (P2'), we can obtain stability in distribution similar to the approach in [3]. Now let us give condition (P2').

**Definition 6.5** The switching jump diffusion given by (2.1) and (2.2) is said to satisfy condition (P2') if, for any $\varepsilon > 0$ and any compact subset $\tilde{\mathcal{C}}$ of $\mathbb{R}^r$, there exists a $T = T(\varepsilon, \tilde{\mathcal{C}}) > 0$ such that

$$ E[|\tilde{X}^{x_0,i_0}(t) - \tilde{X}^{y_0,j_0}(t)|] < \varepsilon \text{ for all } t \geq T, $$

whenever $(x_0, i_0, y_0, j_0) \in \tilde{\mathcal{C}} \times \mathcal{M} \times \tilde{\mathcal{C}} \times \mathcal{M}$. 23
We can obtain (P2) from (P2'). To continue, we focus on obtaining sufficient conditions for conditions (P1) and (P2'). From [3, Theorem 3.8] we can see these two properties imply asymptotic stability in distribution. So it is necessary to establish sufficient criteria for the two properties. To proceed, we need to introduce the following notations.

The generator $\tilde{G}$ associated with the process $\tilde{x}^i - \tilde{y}^j$ is defined as follows: For each $i, j \in \mathcal{M}$, and for any twice continuously differentiable function $f$,

$$\tilde{G} f(\tilde{x}^i - \tilde{y}^j) = \tilde{L} f(\tilde{x}^i - \tilde{y}^j) + \lambda \int_{\Gamma} [f(\tilde{x}^i + \tilde{y}(x, i, \gamma) - \tilde{y}^j - \tilde{y}(y, j, \gamma)) - f(\tilde{x}^i - \tilde{y}^j)] \pi(du),$$

where $\tilde{L}$ is the operator for a switching diffusion process given by

$$\tilde{L} f(\tilde{x}^i - \tilde{y}^j) = \frac{1}{2} \text{tr}(\tilde{a}(\tilde{x}^i, \tilde{y}^j) H f(\tilde{x}^i - \tilde{y}^j)) + (\tilde{b}(x, i) - \tilde{b}(y, j))' \nabla f(\tilde{x}^i - \tilde{y}^j)
+ \sum_{k=1}^{m} q_{ik}(x) f(\tilde{x}^i - \tilde{y}^j) + \sum_{k=1}^{m} q_{jk}(x) f(\tilde{x}^i - \tilde{y}^j)
+ \sum_{k=1}^{m} \sum_{l=1}^{m} \tilde{m}(\Delta_{ik}(x) \cap \Delta_{jl}(y))
\times [f(\tilde{x}^i - \tilde{y}^j) - f(\tilde{x}^i - \tilde{y}^j) - f(\tilde{x}^i - \tilde{y}^j) + f(\tilde{x}^i - \tilde{y}^j)],$$

in which

$$\tilde{b}(x, i) = \left[ 0_{r(1-i),d}, b'(x, i), 0_{r(m-i)}' \right],$$

$$\tilde{\sigma}(x, i) = \left[ 0_{r(1-i) \times d}, \sigma'(x, i), 0_{r(m-i) \times d} \right],$$

$$\tilde{y}(x, i, \gamma) = \left[ 0_{r(1-i)}, g'(x, i, \gamma), 0_{r(m-i)} \right],$$

$$\tilde{a}(\tilde{x}^i, \tilde{y}^j) = (\tilde{\sigma}(x, i) - \tilde{\sigma}(y, j)) \times (\tilde{\sigma}(x, i) - \tilde{\sigma}(y, j))'.$$

Recall that $\Delta_{ik}(x)$ are the intervals having length $q_{ik}(x)$; $\tilde{m}$ is the Lebesgue measure on $\mathbb{R}$ such that $dt \times \tilde{m}(dz)$ is the density of Poisson measure with which we can represent the discrete component $\alpha(t)$ by a stochastic integral as mentioned in Section 2.

**Theorem 6.6** Assume the conditions of Theorem 6.4 hold and assume that for each $i, j \in \mathcal{M}$, there exists a Lyapunov function $V(z) = z'z \in C^2(\mathbb{R}^{mr} : \mathbb{R})$ satisfying the following condition: There exists a positive real number $\beta$ such that

$$\tilde{G} V(\tilde{x}^i - \tilde{y}^j) \leq -\beta V(\tilde{x}^i - \tilde{y}^j),$$

then (P1) and (P2') hold.

**Proof.** We need only verify (P2'). Let $\tilde{C}$ be any compact subset of $\mathbb{R}^r$, and fix any $x_0, y_0 \in \tilde{C}$, $i_0, j_0 \in \mathcal{M}$. Define

$$\zeta_N = \inf \{ t \geq 0, |\tilde{X}^{x_0,i_0}(t) - \tilde{X}^{y_0,j_0}(t)| > N \},$$

$$\zeta_R = \inf \{ t \geq 0, |\tilde{X}^{x_0,i_0}(t)|^2 + |\tilde{X}^{y_0,j_0}(t)|^2 > R \}.$$

Let $\zeta = \zeta_N \wedge \zeta_R$.

By virtue of the generalized Itô formula, we have

$$E|\tilde{X}^{x_0,i_0}(t \wedge \zeta) - \tilde{X}^{y_0,j_0}(t \wedge \zeta)|^2 = |\tilde{x}_0^{i_0} - \tilde{y}_0^{j_0}|^2 + \int_0^{t \wedge \zeta} E \tilde{G} |\tilde{X}^{x_0,i_0}(u) - \tilde{X}^{y_0,j_0}(u)|^2 du.$$
Given the fact that for \( u \leq t \land \zeta \), we have \( u \land \zeta = u \). As a result,

\[
E[\tilde{X}^{x_0, i_0}(t \land \zeta) - \tilde{X}^{y_0, j_0}(t \land \zeta)]^2 = |\tilde{x}_0^i - \tilde{y}_0^j|^2 + \int_0^t E[\tilde{G}] [\tilde{X}^{x_0, i_0}(u \land \zeta) - \tilde{X}^{y_0, j_0}(u \land \zeta)]^2 du.
\]

Then

\[
\frac{dE}{dt} [\tilde{X}^{x_0, i_0}(t \land \zeta) - \tilde{X}^{y_0, j_0}(t \land \zeta)]^2 = E[\tilde{G}] [\tilde{X}^{x_0, i_0}(t \land \zeta) - \tilde{X}^{y_0, j_0}(t \land \zeta)]^2
\]
\[ \leq -\beta E[\tilde{X}^{x_0, i_0}(t \land \zeta) - \tilde{X}^{y_0, j_0}(t \land \zeta)]^2.
\]

Solving the differential inequality above leads to

\[
E[\tilde{X}^{x_0, i_0}(t \land \zeta) - \tilde{X}^{y_0, j_0}(t \land \zeta)]^2 \leq e^{-\beta t} |\tilde{x}_0^i - \tilde{y}_0^j|^2.
\]

Let \( N \to \infty, R \to \infty \), we obtain

\[
E[\tilde{X}^{x_0, i_0}(t) - \tilde{X}^{y_0, j_0}(t)]^2 \leq e^{-\beta t} |\tilde{x}_0^i - \tilde{y}_0^j|^2.
\]

Condition (P2') is thus verified. \( \square \)

7 Examples

This section provides two simple examples. The purposes of these examples are to demonstrate our results.

Example 7.1 Consider a Markov switching jump diffusion \((X(t), \alpha(t))\) given by (2.1) and (2.2) with \( X(t) \in \mathbb{R}^1 \), \( \lambda = 1/8 \), \( g(x, \alpha, \gamma) = x \), \( w(t) \) is a one-dimensional standard Brownian motion, \( \alpha(t) \in \mathcal{M} = \{1, 2\} \) with

\[
Q = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}, \quad b(x, 1) = \frac{x \sin x}{8}, \quad b(x, 2) = \frac{x \cos x}{2}, \quad \sigma(x, 1) = \frac{3x}{2}, \quad \sigma(x, 2) = \frac{x}{2}.
\]

Then by choosing Lyapunov function \( V = x^2 e^{-4t} \), we can verify that the equilibrium \( x = 0 \) is \( p \)-exponential stable with \( p = 2 \). It is almost surely exponentially stable according to Theorem 4.4 as also confirmed in [31, Example 6.2]. The following figure plots the trajectories of the switched system.

Example 7.2 Consider a switching jump diffusion \((X(t), \alpha(t))\) given by (2.1) and (2.2) with \( X(t) \in \mathbb{R}^1 \), \( \lambda = 1 \), \( g(x, \alpha, \gamma) = x \), \( w(t) \) is one-dimensional standard Brownian motion, \( \alpha(t) \in \mathcal{M} = \{1, 2, 3\} \) with

\[
Q(x) = \begin{pmatrix} -3 - |\cos x| + \sin^2 x \cos x & 1 + |\cos x| & 2 - \sin^2 x \cos x \\ 1 & -\frac{x^2}{1+x^2} & \frac{x^2}{1+x^2} \\ 2 - \sin x \cos x & 1 - \frac{|x|}{1+|x|} \cos x & -3 + \sin x \cos x + \frac{|x|}{1+|x|} \cos x \end{pmatrix},
\]

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\begin{align*}
  b(x, 1) &= x + \sin x, \quad b(x, 2) = 2x + x \sin x \cos x, \quad b(x, 3) = 3x + \sin^2 x, \\
  \sigma(x, 1) &= x + x \sin x, \quad \sigma(x, 2) = 3x + x \cos x \sin x, \quad \sigma(x, 3) = x + \frac{x}{1 + x} \sin x, \\
  g(x, i, \gamma) &= x \quad \text{for } i = 1, 2, 3.
\end{align*}

For the \(Q(x)\) given above, \(\hat{Q} = \begin{pmatrix} -4 & 2 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}\),

\(b(1) = 2, \ b(2) = 2, \ b(3) = 3, \ \sigma(1) = 1, \ \sigma(2) = 3, \ \sigma(3) = 1.\)

Corresponding to \(\hat{Q}\), the stationary distribution of the associated Markov chain is given by \(\xi = (\frac{3}{13}, \frac{8}{13}, \frac{2}{13})\). There are three associating jump diffusions that interact and switch back and forth. They are given by

\begin{align*}
  X(t) &= x + \int_0^t (X(s) + \sin X(s))ds + \int_0^t (X(s) + X(s) \sin X(s))dw(s) \\
  &\quad + \int_0^t \int_\Gamma X(s^-)N(ds, d\gamma), \\
  X(t) &= x + \int_0^t (2X(s) + X(s) \sin X(s) \cos X(s))ds \\
  &\quad + \int_0^t (3X(s) + X(s) \cos X(s) \sin X(s))dw(s) + \int_0^t \int_\Gamma X(s^-)N(ds, d\gamma), \\
  X(t) &= x + \int_0^t (3X(s) + \sin^2 X(s))ds + \int_0^t (X(s) + \frac{X(s)}{1 + X(s)} \sin X(s))dw(s) \\
  &\quad + \int_0^t \int_\Gamma X(s^-)N(ds, d\gamma).
\end{align*}

From [31, Example 6.1], the first and the third jump diffusion are unstable in probability and therefore, not asymptotically stable in the large. In view of [26] and Theorem 3.3, the second jump diffusion is asymptotically stable in the large. Using Corollary 3.4, we obtain that the switching jump diffusion is asymptotically stable in the large. The following plot provides a sample path of the regime-switching jump diffusion.
8 Further Remarks

This work focused on stability of regime switching jump diffusions. Under simple conditions, we derived sufficient conditions for asymptotic stability in the large and asymptotic stability in distribution. We also provided necessary and sufficient conditions for exponential stability. The connection between exponential stability and almost surely exponential stability was studied. Smooth dependence on the initial data was demonstrated as well. Future research efforts can be directed to the study of positive recurrence and ergodicity of regime-switching jump diffusions, which was coined as weak stability in [27] for diffusion processes. Treating stability with non-Lipschitz coefficients is of great importance. Numerical methods will be welcomed since the nonlinear systems rarely have closed-form solutions. All of these deserve more thoughts and further considerations.

References

[1] G. Badowski and G. Yin, Stability of hybrid dynamic systems containing singularly perturbed random processes, *IEEE Trans. Automatic Control*, 47 (2002), 2021–2032.

[2] G. Barone-Adesi and R. Whaley, Efficient analytic approximation of American option values, *J. Finance*, 42 (1987), 301–320.

[3] G.K. Basak, A. Bisi and M.K. Ghosh, Stability of degenerate diffusions with state-dependent switching, *J. Math. Anal. Appl.*, 240 (1999), 219–248.

[4] M.-F. Chen, *From Markov Chains to Non-equilibrium Particle Systems*, 2nd ed., World Scientific, Singapore, 2004.

[5] M.-F. Chen, *Coupling methods for multidimensional diffusion processes*, The Ann. Probab., Vol.17. (1989), 151–177.

[6] A. Friedman, *Stochastic Differential Equations and Applications*, Vol I, Academic Press, New York, 1975.

[7] J.P. Hespanha, A model for stochastic hybrid systems with application to communication networks, *Nonlinear Anal.*, 62 (2005), 1353–1383.
A.M. Il’in, R.Z. Khasminskii, and G. Yin, Asymptotic expansion of solutions of integro-differential equations for transition densities of singularly perturbed switching diffusions: rapid switchings, *J. Math. Anal. Appl.*, **238** (1999), 516–539.

S. Kar and J.M.F. Moura, Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise, *IEEE Trans. Signal Processing*, **57** (2009), no.1, 355–369.

R.Z. Khasminskii, *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, Alphen aan den Rijn, Netherlands, 1980.

R.Z. Khasminskii, C. Zhu, and G. Yin, Stability of regime-switching diffusions, *Stochastic Process Appl.*, **117** (2007), 1037–1051.

A.V. Skorohod, *Asymptotic Methods in the Theory of Stochastic Differential Equations*, Amer. Math. Soc., Providence, RI, 1989.

H.J. Kushner, *Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems*, Birhkhäuser, Boston, 1990.

C.W. Li, Z. Dong, and R. Situ, Almost sure stability of linear stochastic differential equations with jump, *Probab. Theory Related Fields*, **123** (2002), 121–155.

Y. Liu, K. Passino, and M.M. Polycarpou, Stability analysis of M-dimensional asynchronous swarms with a fixed communication topology, *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 76-95, Jan. 2003.

X. Mao, *Stochastic Differential Equations and Applications*, 2nd Ed., Horwood, Chichester, UK, 2007.

X. Mao, Stability of stochastic differential equations with Markovian switching, *Stochastic Processes Appl.*, **79** (1999), 45–67.

X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, UK, 2006.

J.L. Menaldi and M. Robin, Invariant measure for diffusions with jumps, *Appl. Math. Optim.*, **40** (1999), 105–140.

R. Olfati-Saber and R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Automatic Control*, **49** (2004), 1520–1533.

C.W. Reynolds, Flocks, herds, and schools: a distributed behavioral model, *Computer Graphics*, **21** (4): 25-34, July 1987.

B. Sklar, *Digital Communications: Fundamentals and Applications*, 2nd Ed., Prentice Hall, 2001.

A.V. Sviishchuk and Yu.I. Kazmerchuk, Stability of stochastic delay equations of Itô form with jumps and Markovian switchings, and their applications in finance, *Theory Probab. Math. Statist.*, **64** (2002), 167–178.

J. Toner and Y. Tu, Flocks, herds, and schools: A quantitative theory of flocking, *Physical Review E*, **58** (4): 4828–4858, October, 1998.

T. Viseck, A. Czirook, E. Ben-Jacob, O. Cohen, and I. Shochet, Novel type of phase transition in a system of self-derived particles, *Physical Review Letters*, **75** (6): 1226–1229, August, 1995.

I.S. Wee, Stability for multidimensional jump-diffusion processes, *Stochastic Process. Appl.*, **80** (1999), 193–299.

W.M. Wonham, Liapunov criteria for weak stochastic stability, *J. Differential Eqs.*, **2** (1966), 195–207.

S. Wu and Y. Zeng, The term structure of interest rates under regime shifts and jumps, *Economics Lett.*, **93** (2006), 215–221.

F. Xi, Asymptotic properties of jump-diffusion processes with state-depedent switching, *Stochastic Process. Appl.*, **119** (2009), 2198–2221.

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[30] G. Yin, Y. Sun, and L.Y. Wang, Asymptotic properties of consensus-type algorithms for networked systems with regime-switching topologies, *Automatica*, **47** (2011) 1366–1378.

[31] G. Yin and F. Xi, Stability of regime-switching jump diffusions, *SIAM J. Control Optim.*, **48** (2010), 4525–4549.

[32] G. Yin and C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York, 2010.

[33] G. Yin and C. Zhu, Properties of solutions of stochastic differential equations with continuous-state-dependent switching, *J. Differential Eqs.*, **249** (2010), 2409–2439.

[34] C. Yuan and X. Mao, Asymptotic stability in distribution of stochastic differential equations with Markovian switching, *Stochastic Process. Appl.*, **103** (2003), 277–291.

[35] C. Yuan and X. Mao, Stability of stochastic delay hybrid systems with jumps, *Eur. J. Control*, **16** (2010), no.6,595-608.