Adjoint representation of the graded Lie algebra \(osp(2/1; \mathbb{C})\) and its exponentiation

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(Dated: August 1, 2003)

We construct explicitly the grade star Hermitian adjoint representation of \(osp(2/1; \mathbb{C})\) graded Lie algebra. Its proper Lie subalgebra, the even part of the graded Lie algebra \(osp(2/1; \mathbb{C})\), is given by \(su(2)\) compact Lie algebra. The Baker-Campbell-Hausdorff formula is considered and reality conditions for the Grassman-odd transformation parameters, which multiply the pair of odd generators of the graded Lie algebra, are clarified.

PACS numbers: 11.10.Ef, 11.15.-q
Keywords: graded Lie algebra, Grassman variables, Baker-Campbell-Hausdorff formula.

I. INTRODUCTION

The number of gauge bosons of a gauge theory is given by the number of generators of the underlying compact Lie algebra. The requirement of being a compact Lie algebra stems from the reality of the action functional, which follows from the properties of the Fermat principle in optics and the Feynman path integral. This leads to physically admissible dynamics of a system with such an action functional. In the case of a Yang-Mills theory one utilizes properties of compact Lie algebras to ensure the real property of the action functional. In this paper we explore how one can utilize a \(\mathbb{Z}_2\)-graded extension of the compact Lie algebra \(su(2)\) for the purposes of defining a meaningful gauge theory of the Yang-Mills type (see [1, 2]). In particular, the matrices of the graded Lie algebra generators in the adjoint representation and the corresponding super-Killing form are explicitly calculated. By exponentiating the graded algebra, we make preparations for treating gauge transformations based on the graded Lie algebra and study some properties of one-parameter families of transformations of such a supergroup.

II. GRADED LIE ALGEBRA \(osp(2/1; \mathbb{C})\)

The algebra \(osp(2/1; \mathbb{C})\) is a graded extension of \(su(2)\) algebra by a pair of odd generators, \(\tau_A\), which anticommute with one another and commute with the three even generators, \(T_a\), of \(su(2)\). It is customary to assign a degree, \(\deg T_a\), to the even (\(\deg T_a = 0\)) and odd (\(\deg \tau_A = 1\)) generators. We use the square brackets to denote the commutator and the curly ones to denote the anticommutator. The defining relations have the form, [1, 3, 4]:

\[
[T_a, T_b] = i \varepsilon_{abc} T_c, \quad [T_a, \tau_A] = \frac{i}{2} (\sigma^a)_A^B \tau_B, \quad \{\tau_A, \tau_B\} = \frac{i}{2} (\sigma^a)_{AB} T_a.
\]

(2.1)

Summation is assumed over all repeated indices. Lower-case Roman indices from the beginning of the alphabet run from 1 to 3; upper-case Roman indices run over 1 and 2; \(\delta_{ab} = \delta^{ab}\) (\(\delta_{ab} = \delta_{ba}\)), \(\varepsilon_{abc} (\varepsilon_{123} = \varepsilon^{123} = 1)\) and \(\epsilon_{AB}\) (\(\epsilon_{12} = \epsilon^{12} = 1\)) are the three dimensional identity matrix and the Levi-Civita totally antisymmetric symbols in three and two dimensions, respectively; the matrices \((\sigma_a)_A^B \ (\sigma^a)_{AB} = \delta^{ab}(\sigma_b)_A^C C_{CB}\) are just the usual Pauli matrices:

\[
(\sigma^a)_A^B = (\sigma_a)_A^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^a)_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We use the Levi-Civita symbols in two dimensions to raise and lower upper-case Roman indices paying attention to their antisymmetric properties:

\[
\Sigma = ||\epsilon_{AB}|| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = ||\epsilon_{AB}|| = -\Sigma^{-1}.
\]

Note that, as concerned to these indices, we are working with two-component spinors and adopt conventions of the book [2]. We shall follow those conventions as more suitable for our purposes when complex conjugation of spinor and Grassman quantities is involved.

In the adjoint representation the matrices \(T_a\) and \(\tau_A\) can be written as follows (solid lines are drawn to emphasize their block structure):

\[
T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

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Let us denote $T_4 = \tau_1$, $T_5 = \tau_2$ and employ lowercase Greek indices from the beginning of the alphabet ($\alpha, \beta$, etc.) to run over the whole set, $T_{\alpha}$, of the generators of $osp(2|1; \mathbb{C})$. We then find that the non-degenerate super-Killing form, $B(T_{\alpha}, T_{\beta})$, is given by

$$B(T_{\alpha}, T_{\beta}) = \frac{2}{3} \text{str}(T_{\alpha} T_{\beta}) = \left( \frac{\delta_{\alpha \beta}}{0 + i \epsilon_{AB}} \right),$$

(2.3)

where the supertrace operation is adopted from [11] pp. 18-19, 42. It is a linear operation, which in a certain way inherits all the properties of the ordinary trace.

It turns out that all of the generators are grade star Hermitian (the operation is denoted by $^\dagger$): on the even generators the operation coincides with ordinary Hermitian conjugation ($^\dagger$) while the odd ones obey more complicated relations. Following the papers [12, 13], we shall call them the grade star Hermiticity conditions: $\tau_\pm = \pm \tau_\mp (\tau_\pm = \tau_1 \pm i \tau_2)$.

Let us consider complex-valued matrices divided into blocks according to the scheme (see (2.2) and (2.3)):

$$M_{\text{even}} = \begin{pmatrix} A & 0 \cr 0 & D \end{pmatrix} \quad \text{and} \quad M_{\text{odd}} = \begin{pmatrix} 0 & B \cr C & 0 \end{pmatrix},$$

where $B$ and $C$ are $2 \times 3$ rectangular blocks and $A$ and $D$ are $3 \times 2$ and $2 \times 2$ square blocks, respectively. On these matrices the supertrace operation is defined by $\text{str} M_{\text{even}} = \text{tr} A - \text{tr} D$ and $\text{str} M_{\text{odd}} = 0$ (here “tr” denotes the ordinary trace), while the grade star Hermiticity condition reads

$$M_{\text{even}}^\dagger = \begin{pmatrix} A^\dagger & 0 \\ 0 & D^\dagger \end{pmatrix} \quad \text{and} \quad M_{\text{odd}}^\dagger = \begin{pmatrix} 0 & -C^\dagger \\ B^\dagger & 0 \end{pmatrix}.$$

We shall also use multiplication of algebra generators by scalars. Such an operation must take into account that Grassman-odd scalars anticommute with the odd algebra generators while commute with complex numbers and the even algebra generators, [11]. The following construction possesses all of these properties. Let $a$ be a scalar and $\deg a$ be its degree (0 or 1 depending on whether it is Grassman-even or Grassman-odd, respectively). Then multiplication by $a$ is defined as follows:

$$a M_{\text{odd}} = \begin{pmatrix} a & 0 \\ 0 & (-1)^{\deg a} \end{pmatrix} \begin{pmatrix} A & 0 \cr 0 & D \end{pmatrix} = (-1)^{\deg a} \begin{pmatrix} 0 & B \cr C & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & (-1)^{\deg a} \end{pmatrix} = (-1)^{\deg a} a M_{\text{odd}},$$

$$a M_{\text{even}} = \begin{pmatrix} a & 0 \\ 0 & (-1)^{\deg a} \end{pmatrix} \begin{pmatrix} A & 0 \cr 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \cr 0 & D \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & (-1)^{\deg a} \end{pmatrix} = M_{\text{even}} a,$$

where $\xi^{(A \theta B)} = 1/2(\xi A \theta B + \xi B \theta A)$ and $\xi^{[A \theta B]} = 1/2(\xi A \theta B - \xi B \theta A)$ are convenient shorthand notations. This result was obtained using anticommutator for odd generators in definition (2.1). Using a fundamental fact of spinor algebra, $\epsilon_{AB} \epsilon_{CD} + \epsilon_{AC} \epsilon_{DB} + \epsilon_{AD} \epsilon_{BC} = 0$, one can calculate

$$\xi^{[A \theta B]} = \frac{1}{2} \xi C \theta^C e^{AB}.$$

From symmetry of $(\sigma^a)_{AB}$ in the uppercase indices, it then follows that

$$[\xi^A \tau_A, \theta^B \tau_B] = -\frac{i}{2} \xi^A \theta^B (\sigma^a)_{AB} T_a = -\frac{i}{2} \xi^A \theta^B (\sigma^a)_{AB} T_a \quad (3.1)$$

and, in particular, the commutator $[\theta^A \tau_A, \theta^B \tau_B]$ vanishes identically. One can also calculate

$$[\kappa^a \tau_a, \varepsilon^b \tau_b] = i \kappa^a \varepsilon^b \varepsilon_{abc} T_c,$$

$$[\varepsilon^a \tau_a, \theta^B \tau_B] = \theta^B \tau_B,$$

(3.2)
where \( \hat{\theta}^B = 1/2 \varepsilon^a (\sigma_3)_A B \) is again a Grassman-odd transformation parameter.

Group elements are obtained by exponentiating the algebra

\[
U(\varepsilon, \theta) = \exp[i (\varepsilon^a T_a + \theta^A T_A)]
\]

and the Baker-Campbell-Hausdorff formula,

\[
\exp(M) \exp(N) = \exp(M + N + \frac{1}{2} [M, N] + \ldots),
\]

may be applied to determine motion in the parameter space under a (left) multiplication with a group element \( U(\kappa, \xi) \):

\[
U(\varepsilon', \theta') = U(\kappa, \xi) U(\varepsilon, \theta).
\]

Substituting (3.3) and using (3.1), we obtain after some algebra

\[
\begin{align*}
\varepsilon'^a &= \varepsilon^a + \kappa^a + \frac{1}{2} \varepsilon_b \kappa^c \varepsilon^{bca} + \frac{1}{4} \varepsilon^A \theta^B (\sigma^a)_{AB} + \ldots \\
\theta'^A &= \theta^A + \xi^A + \frac{i}{4} (\kappa_b \theta^b - \varepsilon_b \kappa^B) (\sigma^a)_{AB} A + \ldots
\end{align*}
\]

(3.4)

Here the dots denote the remaining contribution from linear combinations of \( k \)-fold \( (k > 2, k \in \mathbb{Z}) \) commutators of \( M \) and \( N \) to the Baker-Campbell-Hausdorff formula.

The last term in the first equation of system (3.4) needs to be investigated in more detail. First, let us calculate that

\[
2[\xi^A, \theta^B] (\sigma^a)_{AB} = \xi_A \epsilon^{AB} (\sigma^a)_{AB} C \theta_C - \theta_A \epsilon^{AB} (\sigma^a)_{AB} C \xi_C = \xi^T \Sigma \sigma^a \theta - \theta^T \Sigma \sigma^a \xi,
\]

(3.5)

where we employed some self-evident matrix notations. Comparing the result (3.5) and a description of \( su(2) \)-spinors of 3D Euclidean space in the book [11, p. 48], one immediately realizes that the last term of the first equation in system (3.4) is, in general, a complex vector of 3D Euclidean space, e.g. it transforms like a vector under \( SO(3) \) transformations. Second, the representation (3.3) tells us that components of this vector vanish if \( \xi = \theta = 0 \) as required by a property of a one-parameter subgroup of transformations (3.6). Finally, this vector also has all components equal to zero if \( \xi = -\theta \). This shows that the inverse of the group element \( U(\varepsilon, \theta) \) has the form

\[
U^{-1}(\varepsilon, \theta) = \exp[-i (\varepsilon^a T_a + \theta^A T_A)].
\]

(3.6)

If one intends, as we actually do, to treat \( \varepsilon^a, \kappa^a \), etc. as real-valued transformation parameters, then it is necessary to impose some conditions on the \( su(2) \)-spinors \( \xi_A, \theta_A, \) etc. in order to ensure that (3.5) will be a real 3D Euclidean vector. Such a condition must be compatible with transformation properties of the corresponding space of \( su(2) \)-spinors, \( \xi_A \), and take into account that its members are also Grassman-odd quantities. In fact, this condition should involve a passage from an \( su(2) \)-spinor to its conjugate and, thus, rely on the definition of an anti-involution in the space of spinors (see, e.g. [11, p. 100]). Let us observe first that for a Grassman algebra on one generator the last term in the first relation in (3.4) vanishes identically. This is a somewhat trivial situation. The non-trivial one arises when all \( su(2) \)-spinors under consideration take values in a Grassman algebra on two odd generators, \( \varepsilon_1 \) and \( \varepsilon_2 \); \( \varepsilon_1^2 = \varepsilon_2^2 = 0, \varepsilon_1 \varepsilon_2 = -\varepsilon_2 \varepsilon_1 \) (see, e.g. [11, p. 7]). It is hoped that there would be no misunderstanding due to the fact that we use the same kernel letter to denote vector components \( \varepsilon^a \) and the two odd generating elements of the Grassman algebra. We shall employ lowercase Roman indices from the middle of the alphabet running over 1 and 2 to enumerate the decompositions of various quantities in the corresponding basis of the Grassman algebra. Decomposing \( \xi_A \) and \( \theta_A \) into this basis one obtains

\[
\xi_A = \xi_A \varepsilon_1 \varepsilon_i \quad \text{and} \quad \theta_B = \theta_B \varepsilon_j \varepsilon_j,
\]

where \( \xi_A \) and \( \theta_B \) are ordinary, i.e. commuting, \( su(2) \)-spinors of 3D Euclidean space, and summation over repeated indices is assumed. In this case we can write

\[
\frac{1}{2} (\xi^A \theta^B - \theta^A \xi^B) (\sigma^a)_{AB} = \varepsilon_1 \varepsilon_2 (\xi^T \Sigma \sigma^a \theta - \theta^T \Sigma \sigma^a \xi).
\]

(3.7)

Now we shall impose some additional conditions on \( su(2) \)-spinors \( \xi_A, \theta_A, \) etc. to ensure that (3.4) gives a real Grassman-even 3D Euclidean vector. One way of doing so in a manner preserving all the spinor transformations properties is to define

\[
\xi_A = i C_A B^r \xi_B^r, \quad \theta_A = i C_A B^r \theta_B^r, \quad \text{etc.,}
\]

(3.8)

where the ‘charge conjugation’ matrix \( C \) \( (C C^r = -I) \) is given by

\[
C = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).
\]

In (3.8) a bar over the spinors in the left-hand sides of the relations and primes over the indices denote complex conjugation. We again adhere to Penrose notations when spinors are concerned. The charge conjugation matrix, \( C \), \( B^r \), is responsible for invariant preservation of spinor properties (for details see, e.g. the review article [12, pp. 108 – 109], where this object is denoted by \( \Pi \); also compare with the treatment in [11, p. 100]). As seen from the relation (3.8) in this case each Grassman-odd \( su(2) \)-spinor \( \xi_A, \theta_A, \) etc. is defined by a single ordinary (Grassman-even) \( su(2) \)-spinor. For the sake of notations denoting, respectively,

\[
\xi_A = \eta_A \quad \text{and} \quad \theta_B = \theta_B,
\]

we write

\[
\begin{align*}
\varepsilon^a &= \xi^T \Sigma \sigma^a \theta - \theta^T \Sigma \sigma^a \xi = i (\eta^T C \Sigma \sigma^a \theta - \theta^T C \Sigma \sigma^a \eta).
\end{align*}
\]

(3.9)

On comparison with [11, p. 50], one can check that \( \varepsilon^a \) is indeed a real 3D Euclidean vector. In components it reads:

\[
\begin{align*}
x^1 &= i (\eta_1 \varepsilon_2 - \varepsilon_2 \eta_1 + \eta_2 \varepsilon_1 - \varepsilon_1 \eta_2), \\
x^2 &= i (\eta_2 \varepsilon_1 + \varepsilon_1 \eta_2 - \varepsilon_2 \eta_1 - \eta_1 \varepsilon_2), \\
x^3 &= i (\eta_1 \varepsilon_1 - \varepsilon_1 \eta_1 + \eta_2 \varepsilon_2 - \varepsilon_2 \eta_2).
\end{align*}
\]

(3.10)

These are obviously real quantities and the vector \( \varepsilon^a \) vanishes if and only if \( \eta_A \equiv \pm \theta_A \) as required.
IV. DISCUSSION

To the best of the author’s knowledge the graded extension of $u(1)$ Lie algebra for the first time was considered by Berezin and Kac in \[10\]. However, they didn’t build the adjoint representation, neither they calculated the super-Killing form for a product of two one-parameter families of transformations. From the current standpoint, the last task could be achieved in an analogous manner to that of presented in Sec. III. The only necessary remark here is that one would use complex Euclidean spinors in one dimension, i.e. complex numbers, and conditions of $\mathfrak{ps}$-type. In the present paper we treated the next simplest case: the graded extension of $su(2)$ Lie algebra.

For a reader’s convenience we present below a table with some first members of the two major series of graded Lie algebras (cf., \[13\]). Their respective proper Lie subalgebras and the numbers of even, $m$, and odd, $n$, generators are shown, \[3, 6\]. We are concerned here only with compact choice for the proper Lie subalgebras of the graded Lie algebras:

| Low-dimensional graded Lie algebras of two main series | proper Lie subalgebra | $u(1)$ | $u(1)@su(2)$ | $u(1)@su(2)@su(2)$ | $u(1)@su(3)$ |
|------------------------------------------------------|-----------------------|--------|----------------|------------------|----------------|
| $spl(p/q; \mathbb{C})$: $p, q > 0$; $m = p^2 + q^2 - 1, n = 2pq$ | proper Lie subalgebra | $u(1)$ | $u(1)@su(2)$ | $u(1)@su(2)@su(2)$ | $u(1)@su(3)$ |
| $n$ – dim. even subspace | 1 | 4 | 7 | 9 |
| $n$ – dim. odd subspace | 2 | 4 | 6 | 6 |

| $osp(p/q; \mathbb{C})$: $p \geq 1$, even $q > 0$; $m = \frac{p(p - 1)}{2} + \frac{q(q + 1)}{2}, n = pq$ | proper Lie subalgebra | $su(2)$ | $u(1)@su(2)$ | $su(2)@su(2)$ | $su(2)@su(2)@su(2)$ |
|------------------------------------------------------|-----------------------|--------|----------------|------------------|----------------|
| $n$ – dim. even subspace | 3 | 4 | 6 | 9 |
| $n$ – dim. odd subspace | 2 | 4 | 6 | 8 |

Acknowledgements

I am grateful to Dr. T.S. Tsou for interest in this work and to Prof. Yu.P. Stepanovsky for numerous helpful discussions. I would also like to acknowledge an inspiring criticism from Dr. V. Pidstrigach at an early stage of this development.

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