On the Connectivity of Unions of Random Graphs

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Abstract

Graph-theoretic tools and techniques have seen wide use in the multi-agent systems literature, and the unpredictable nature of some multi-agent communications has been successfully modeled using random communication graphs. Across both network control and network optimization, a common assumption is that the union of agents’ communication graphs is connected across any finite interval of some prescribed length, and some convergence results explicitly depend upon this length. Despite the prevalence of this assumption and the prevalence of random graphs in studying multi-agent systems, to the best of our knowledge, there has not been a study dedicated to determining how many random graphs must be in a union before it is connected. To address this point, this paper solves two related problems. The first bounds the number of random graphs required in a union before its expected algebraic connectivity exceeds the minimum needed for connectedness. The second bounds the probability that a union of random graphs is connected. The random graph model used is the Erdős-Rényi model, and, in solving these problems, we also bound the expectation and variance of the algebraic connectivity of unions of such graphs. Numerical results for several use cases are given to supplement the theoretical developments made.

I. Introduction

Multi-agent systems have been studied in a number of applications, including sensor networks [10], robotics [35], communications [8], and smart power grids [6]. Across these applications, the agents in a network and their associated communications are often abstractly represented as graphs [26]. In general, graph-theoretic methods in multi-agent systems represent each agent as a node in a graph and each communication link as an edge, and multi-agent coordination algorithms have been developed for both static and time-varying graphs [26, Chapter 1.4].

Time-varying random graphs in particular have been used to model communications which are unreliable and intermittent due to interference and poor channel quality [26, Chapter 5], and such graphs have seen use in a number of multi-agent settings. For example, distributed agreement problems over random graphs are studied in [37] and [19], while optimization over random graphs was explored in [25]. The work in [43] provides a means to modify random graphs to make them robust to network failures, and [24] discusses general properties of random graphs as they pertain to multi-agent systems. A broad survey of graph-theoretic results for control can be found in [26], and well-known graph-theoretic results in optimization include [3], [29], [44].

When time-varying graphs (random or not) are used, a common assumption is that the unions of these graphs are connected over intervals of some finite length, i.e., the graph containing all edges present over time is itself a

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connected graph. A partial sampling of works using this assumption (or a related variant) includes [3], [7], [14], [20], [22], [27]–[31], [33], [34], [36], [39], [40], [44]. In addition, some works derive convergence rates or other results that explicitly use the length of such intervals, including [3], [7], [14], [28], [29], [31], [36], [39], [40]. In applying these results, one may wish to determine the time needed for the system to attain a connected union graph. To the best of our knowledge, no study has been undertaken that addresses this problem for unions of random graphs, despite their frequent use in multi-agent systems.

Owing to the success of Erdős-Rényi graphs in modeling some time-varying multi-agent communications [26, Chapter 5], we consider unions of random graphs generated by the Erdős-Rényi model and examine the connectedness of such unions. In particular, this paper solves two problems: lower-bounding the number of random graphs required in a union before one may expect it to be connected (in a precise sense to be defined in Section II), and lower-bounding the probability that a union of random graphs is connected.

Our results use spectral properties of the first four (matrix-valued) moments of the Laplacian of a union of random graphs. The eigenvalues of these moments are used to bound the expected value of the Laplacian’s second-smallest eigenvalue, called the algebraic connectivity [15] of the underlying union graph. This bound in turn enables a lower bound on the number of graphs needed in a union to have its algebraic connectivity reach a specified expected value, and also enables a lower bound on the probability of the algebraic connectivity exceeding some given threshold.

The results presented rely heavily upon the spectral properties of random graphs’ Laplacians, which are random matrices. A common approach to analyzing the spectra of random matrices is to let the dimension of the matrix grow arbitrarily large [11], [16], [21], [42], and the work in [9] considers similar asymptotic results focused on Laplacians of random graphs. For random graphs specifically, a common approach is to derive results in which the size of the graph grows arbitrarily large, and doing so enables results that hold for almost all graphs [5]. While there is clear theoretical appeal to such results, our focus on multi-agent systems leads us to consider non-asymptotic results precisely because such systems are typically comprised by a fixed number of agents. Our results are therefore stated for graphs of fixed (but unspecified) size.

In addition, while some work on random graphs considers edge probabilities that bear some known relationship to the number of nodes in a graph [23], [38], we do not do so here. Our use of random graphs to model multi-agent communications is inspired by applications in which poor channel quality, interference, and other factors make communications unreliable. In such cases, the probability of a communication link being active may not bear any known relationship to the size of the network. We therefore proceed with edge probabilities and network sizes that are fixed and not assumed to be related.

The rest of the paper is organized as follows. Section II reviews the required elements of graph theory and gives formal statements for the two problems that are the focus of this paper. Then, Section III computes moments of random graph Laplacians and certain spectral properties of these moments to enable the results of Section IV. Section IV then presents the main results of the paper and solves the problems stated in Section II. Next, Section V presents numerical solutions to several instantiations of the problems studied. Finally, Section VI provides concluding
remarks and future directions for extending this work.

II. REVIEW OF GRAPH THEORY AND PROBLEM STATEMENTS

In this section, we review the required elements of graph theory. We begin with basic definitions, including the definition of algebraic connectivity, and then review the Erdős-Rényi model for random graphs; throughout this paper, all uses of the phrase “random graphs” refer to Erdős-Rényi graphs. Then we formally state the two problems solved in this paper. Below, we use the notation \([m] := \{1, \ldots, m\}\) for any \(m \in \mathbb{N}\).

A. Basic Graph Theory

A graph is defined over a set of nodes, denoted \(V\), and describes connections between these nodes in the form of edges, which are contained in an edge set \(E\). Formally, for \(n \in \mathbb{N}\), the elements of \(V\) are indexed over \([n]\). The set of edges in the graph is a subset \(E \subseteq V \times V\), where a pair \((i, j) \in E\) if nodes \(i\) and \(j\) share a connection, and \((i, j) \notin E\) if they do not. This paper considers graphs which are undirected, meaning an edge \((i, j)\) is not distinguished from an edge \((j, i)\), and simple, so that \((i, i) \notin E\) for all \(i\). A graph \(G\) is then defined as the pair \(G = (V, E)\). One main focus of this paper is on connected graphs, which we define now.

Definition 1: (E.g., \([17]\)) A graph \(G\) is called connected if, for all \(i \in [n]\) and \(j \in [n], i \neq j\), there is a sequence of edges one can traverse from node \(i\) to node \(j\), i.e., there is a sequence of indices \(\{i_\ell\}_{\ell=1}^k\) and nodes \(\{v_p\}_{p=1}^k\) such that \(E\) contains all of the edges
\[
(i, v_{i_1}), (v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{k-1}}, v_{i_k}), (v_{i_k}, j).
\]

\(\triangle\)

The results of this paper are developed in terms of graph Laplacians, which we define now. First, the adjacency matrix \(A(G) \in \mathbb{R}^{n \times n}\) associated with the graph \(G\) is defined element-wise as
\[
a_{ij} = \begin{cases} 1 & (i, j) \in E \\ 0 & \text{otherwise}, \end{cases}
\]
where \(a_{ij}\) is the \(i^{th}\), \(j^{th}\) element of \(A(G)\). When there is no ambiguity, we will simply denote \(A(G)\) by \(A\). Because we consider undirected graphs, \(A\) is symmetric by definition.

Next, the degree matrix \(D(G) \in \mathbb{R}^{n \times n}\) associated with a graph \(G\) is a diagonal matrix whose entries count the number of edges connecting to a node. Using \(d_i\) to denote the degree of node \(i\), we find
\[
d_i = \sum_{\substack{j=1 \ j \neq i}}^n a_{ij} = |\{j \mid (i, j) \in E\}|,
\]
where $|\cdot|$ denotes the cardinality of a set. Then the degree matrix associated with a graph $G$ is

$$D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$$

which we will denote $D$ when $G$ is clear from context. Clearly $D$ is also symmetric by definition.

The Laplacian of a graph $G$ is then defined as

$$L(G) = D(G) - A(G),$$

which will be written simply as $L$ when $G$ is unambiguous. The results of this paper rely in particular on spectral properties of $L$. Letting $\lambda_k(\cdot)$ denote the $k^{th}$ smallest eigenvalue of a matrix, it is known that $\lambda_1(L) = 0$ for all graph Laplacians [26], and thus we have

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L). \quad (1)$$

The value of $\lambda_2(L)$ is central to the work in this paper and some other works in graph theory, and it gives rise to the following definition.

**Definition 2:** (From [15]) The algebraic connectivity of a graph $G$ is the second smallest eigenvalue of its Laplacian, $\lambda_2(L)$, and $G$ is connected if and only if $\lambda_2(L) > 0$. $\triangle$

This paper is dedicated to studying the statistical properties of $\lambda_2$ for unions of random graphs. Toward doing so, we now review the necessary elements of the theory of random graphs.

### B. Random Graphs

Several well-known random graph models exist in the literature [12], [41], and Erdős-Rényi graphs in particular have been successfully used in the multi-agent systems literature. Erdős-Rényi graphs can model, for example, unreliable, intermittent and time-varying communications in multi-agent networks [26], and we therefore consider the Erdős-Rényi model in this paper. Under this model, a graph on $n$ vertices contains each admissible edge with some fixed edge probability $p \in (0, 1)$. Therefore, for each $i \in [n]$ and $j \in [n]$ with $i \neq j$, an Erdős-Rényi graph satisfies

$$\mathbb{P}[(i, j) \in E] = p \quad \text{and} \quad \mathbb{P}[(i, j) \notin E] = 1 - p. \quad (2)$$

Equivalently, based on Equation (2) one finds

$$\mathbb{E}[a_{ij} = 1] = p \quad \text{and} \quad \mathbb{E}[a_{ij} = 0] = 1 - p,$$

i.e., that $a_{ij}$ is a Bernoulli random variable for $i \neq j$.

We denote the sample space of all Erdős-Rényi graphs on $n$ nodes with edge probability $p$ by $\mathcal{G}(n, p)$, and we denote the set of Laplacians of all such graphs by $\mathcal{L}(n, p)$. One approach to spectral graph theory commonly used in the literature is to let $n \to \infty$ [4]. The value of doing so is that one may draw conclusions that hold for *almost all* graphs in a rigorous way. In the study of multi-agent systems, however, one is often focused on networks with a
fixed number of agents that is not well approximated by letting \( n \) become arbitrarily large. Accordingly, we develop our results in terms of an arbitrary but fixed value of \( n \).

In addition, some well-known results in the graph theory literature assume that \( p \) has some known relationship to \( n \) [4], or else that the number of edges in a random graph has some relationship to the number of nodes in the graph [13]. While the theoretical utility of these relationships is certainly clear from those works, this relationship will often not hold in multi-agent systems where communications are unpredictable because these communications are affected by a wide variety of external factors. We therefore proceed with a value of \( p \in (0, 1) \) that is not assumed to have any relationship to the value of \( n \).

In the study of multi-agent systems, it is also common for algorithms and results to be stated in terms of unions of graphs, which we define now.

**Definition 3:** For a collection of graphs \( \{G_k = (V, E_k)\}_{k=1}^N \) defined on the same node set \( V \), the union of these graphs, denoted \( U_N \), is defined as
\[
U_N := \bigcup_{k=1}^N G_k = (V, \bigcup_{k=1}^N E_k),
\]
i.e., the union graph \( U_N \) contains all edges in all \( N \) graphs that comprise the union. \( \triangle \)

### C. Problem Statement

A common requirement in some multi-agent systems is that the communication graphs in a network form a connected union graph over intervals of some fixed length. To help determine when this occurs, we formulate and solve two related problems in this paper. The first concerns when a union graph has expected algebraic connectivity above some threshold.

**Problem 1:** Find \( N \in \mathbb{N} \) such that
\[
\mathbb{E}[\lambda_2(U_N)] \geq \lambda_{\text{min}},
\]
where \( \lambda_{\text{min}} \) is the minimum algebraic connectivity of a connected graph, \( U_N \) is given by
\[
U_N = \bigcup_{k=1}^N G_k,
\]
and \( G_k \in \mathcal{G}(n, p) \) for all \( k \in [N] \). \( \diamond \)

The second problem we solve concerns the probability with which a union graph has algebraic connectivity exceeding the minimum among connected graphs.

**Problem 2:** Given \( N \in \mathbb{N} \), lower bound the value of
\[
\mathbb{P} \left[ \lambda_2(U_N) \geq \lambda_{\text{min}} \right],
\]
where \( \lambda_{\text{min}} \) is the minimum algebraic connectivity of a connected graph, and where \( U_N \) is defined as it is in Problem 1. \( \diamond \)

Section III next provides theoretical developments that enable the solutions to these problems in Section IV.
III. MOMENTS AND SPECTRA OF RANDOM GRAPH LAPLACIANS

Towards solving Problems 1 and 2, this section computes the first four moments of a random graph’s Laplacian. The values of these moments will be used below to compute the expectation and variance of the algebraic connectivity of random graphs and their unions, and these results later enable solutions to Problems 1 and 2.

A. Moments of Random Graph Laplacians

We begin by stating a lemma that will be used below to compute eigenvalues of moments of $L$.

**Lemma 1:** Let $I$ be the $n \times n$ identity matrix, and let $J$ be the $n \times n$ matrix whose entries are all 1. Then the matrix

$$M := (\alpha - \beta)I + \beta J = \begin{pmatrix} \alpha & \beta & \beta & \cdots & \beta \\ \beta & \alpha & \beta & \cdots & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \beta & \cdots & \alpha \end{pmatrix}$$

has $\alpha + (n-1)\beta$ as an eigenvalue with multiplicity one and $\alpha - \beta$ as an eigenvalue with multiplicity $n-1$.

*Proof:* See Lemma 1 in [18].

We now present the first four moments of a random graph Laplacian $L \in \mathcal{L}(n, p)$.

**Lemma 2:** Let $G \in \mathcal{G}(n, p)$ have Laplacian $L$. Then

$$\mathbb{E}[L] = p(nI - J)$$
$$\mathbb{E}[L^2] = [(n-2)p^2 + 2p](nI - J)$$
$$\mathbb{E}[L^3] = [(n-2)(n-4)p^3 + 6(n-2)p^2 + 4p](nI - J)$$
$$\mathbb{E}[L^4] = [(n-7)(n-3)(n-2)p^4 + 6(2n-7)(n-2)p^3 + 25(n-2)p^2 + 8p](nI - J),$$

where $I$ is the $n \times n$ identity matrix and $J$ is the $n \times n$ matrix of ones.

*Proof:* See Lemma 2 in [18].

Next, we present a lemma showing the equivalence between the expected spectrum of powers of $L$ and the spectrum of the corresponding moments of $L$. The purpose of this lemma is to enable the use of Lemma 2 in computing the expected eigenvalues for a random graph Laplacian $L \in \mathcal{L}(n, p)$.

Before doing so, we draw an important distinction between the eigenvalues $\{\ell_i\}_{i=1}^n$ studied in this section and the eigenvalues $\{\lambda_i\}_{i=1}^n$ in Section II. The eigenvalues $\{\ell_i\}_{i=1}^n$ comprise an unordered collection and are simply the eigenvalues of a random graph’s Laplacian; as a result, each $\ell_i$ is itself a random variable. In the setting of random graphs, $\lambda_i$ is then the $i^{th}$ order statistic over these random variables, i.e., the $i^{th}$ smallest value realized by any of the random variables in the collection $\{\ell_i\}_{i=1}^n$. 
More concretely, using the convention that $\ell_1 = 0$ is always the zero eigenvalue of a random graph’s Laplacian (which is guaranteed to exist by Equation (1)), the algebraic connectivity of a random graph is defined as

$$\lambda_2 = \min_{2 \leq i \leq n} \ell_i.$$ 

This section characterizes each $\ell_i$, and Section IV uses these results to characterize $\lambda_2$. Towards doing so, we have the following lemma.

**Lemma 3:** Let $L$ be the Laplacian of a random graph $G \in \mathcal{G}(n, p)$, and let $\text{eig}(M)$ denote the set of eigenvalues of a matrix $M \in \mathbb{R}^{n \times n}$. Then, for $k \in [4]$,

$$\text{eig}(\mathbb{E}[L^k]) = \mathbb{E}[\text{eig}(L^k)].$$

**Proof:** The expected value of any diagonal element of a graph Laplacian takes the form

$$\mathbb{E}[L_{ii}] = \mathbb{E} \left[ \sum_{j=1}^{n} a_{ij} \right] = (n - 1)p$$

because the random variables $a_{ij}$ are independent Bernoulli random variables which take value 1 with probability $p$. Summing these diagonal entries, we find the expected trace of $L$ to be

$$\mathbb{E}[\text{trace}(L)] = \sum_{i=1}^{n} \mathbb{E}[L_{ii}] = n(n - 1)p. \quad (3)$$

Denote the eigenvalues of a matrix $L \in \mathcal{L}(n, p)$ by $\ell_i$, $1 \leq i \leq n$. Due to the fact that all off-diagonal elements of $L$ are i.i.d. random variables, and that the diagonal elements are simply sums of these variables, the non-zero eigenvalues of $L$ are equal in expectation. That is, apart from the guaranteed zero eigenvalue $\ell_1 = 0$, all other eigenvalues have equal expectation, precisely because all off-diagonal entries of $L$ take the same form and because all diagonal entries do as well. By Equation (3) and the definition of the trace of a matrix, we then find

$$\sum_{i=2}^{n} \mathbb{E}[\ell_i] = n(n - 1)p,$$

giving

$$\mathbb{E}[\ell_i] = np. \quad (4)$$

As for $\text{eig}(\mathbb{E}[L])$, we note that

$$\mathbb{E}[L] = p(nI - J),$$

which by Lemma 1 has eigenvalues

$$\ell_1 = 0 \text{ and } \ell_i = np \text{ for } k \in \{2, \ldots, n\}. \quad (5)$$

Comparing Equations (4) and (5), we find that

$$\mathbb{E}[\text{eig}(L)] = \text{eig}(\mathbb{E}[L]).$$

By the same reasoning, one can repeatedly exploit the symmetries of $L$ and its powers to obtain the same result for $L^k$. ■
In words, Lemma 3 says that the expected spectrum of $L^k$ is equal to the spectrum of the expectation of $L^k$. It was shown in Lemma 2 that $\mathbb{E}[L^k]$ takes a simple form for $k \in [4]$, and thus Lemma 3 simplifies the process of computing the expected eigenvalues of $L^k$.

Using Lemmas 2 and 3 we can compute moments of the eigenvalues of $L$. The eigenvalues of $L$ are denoted by $\ell_i$, and these are random variables because they are functions of the entries of $L$, which in turn are either Bernoulli random variables (for off-diagonal entries) or sums of Bernoulli random variables (for diagonal entries). We begin with the smallest eigenvalue of $L$.

**Lemma 4**: Let $L \in \mathcal{L}(n, p)$. Then $\ell_1 = 0$ is in $\mathbb{E}[\text{eig}(L)]$, and $\ell_1^k = \ell_1 = 0$ is in $\mathbb{E}[\text{eig}(L^k)]$ for $k \in \{2, 3, 4\}$.

**Proof**: Let $\mathbb{I}$ denote the vector in \( \mathbb{R}^n \) whose entries are all 1. Then we see that

\[
(nI - J)\mathbb{I} = 0. \]

Then 0 is an eigenvalue of $nI - J$. Because $\mathbb{E}[L]$ in Lemma 2 is a scalar multiple of $nI - J$, 0 is an eigenvalue of $\mathbb{E}[L]$. By Lemma 3, 0 is then also the expected value of an eigenvalue of $L$. Because 0 is an eigenvalue of $nI - J$, it is also an eigenvalue of $(nI - J)^k$ for $k \in \{2, 3, 4\}$, and repeating the preceding argument for these values of $k$ completes the lemma. ■

Having established the expectation of $\ell_1$ in Lemma 4 we now compute the first four moments of all other $\ell_i$’s.

**Theorem 1**: Let $G \in \mathcal{G}(n, p)$ and let $\ell_i$ denote the $i^{th}$ eigenvalue of its Laplacian. For all $i \in [n] \backslash \{1\}$,

\[
\begin{align*}
\mathbb{E}[\ell_i] &= np \\
\mathbb{E}[\ell_i^2] &= n(n - 2)p^2 + 2np \\
\mathbb{E}[\ell_i^3] &= n(n - 2)(n - 4)p^3 + 6n(n - 2)p^2 + 4np \\
\mathbb{E}[\ell_i^4] &= n(n - 7)(n - 3)(n - 2)p^4 + 6n(2n - 7)(n - 2)p^3 + 25n(n - 2)p^2 + 8np.
\end{align*}
\]

**Proof**: Applying Lemma 4 to Lemma 2 gives the above quantities as eigenvalues of $\mathbb{E}[L]$, and Lemma 3 establishes that these eigenvalues are moments of eigenvalues of $L$. ■

Using Theorem 1 we have the following corollary which computes the variances of $\ell_i$ and $\ell_i^2$, and these variances will be applied in the next section to bound certain properties of $\lambda_2$.

**Corollary 1**: Let $G \in \mathcal{G}(n, p)$ and let $\ell_i$ denote the $i^{th}$ eigenvalue of its Laplacian. Then for $i \in [n] \backslash \{1\}$,

\[
\text{Var}[\ell_i] = 2npq
\]

and

\[
\text{Var}[\ell_i^2] = n(n - 7)(n - 3)(n - 2)p^4 + 6n(2n - 7)(n - 2)p^3 + 25n(n - 2)p^2 + 8np - (n(n - 2)p^2 + 2np)^2.
\]

**Proof**: By definition,

\[
\text{Var}[\ell_i] = \mathbb{E}[\ell_i^2] - \mathbb{E}[\ell_i]^2 \quad \text{and} \quad \text{Var}[\ell_i^2] = \mathbb{E}[\ell_i^4] - \mathbb{E}[\ell_i^2]^2,
\]

and the result follows using the results of Theorem 1. ■
Having characterized certain statistical properties of the collection \(\{\ell_i\}_{i=1}^n\), the next section translates these properties into bounds on statistical properties of \(\lambda_2\).

### IV. Algebraic Connectivity of Unions of Random Graphs

This section translates the bounds on \(\ell_i\) derived in Section III for single random graphs into bounds on \(\lambda_2\) for unions of random graphs. First, we show that a union of random graphs can itself be represented as a random graph with a different edge probability. Second, we present results that bound the expectation of order statistics in terms of the expectations and variances of the underlying collection of random variables. Third, we present our solutions to Problems 1 and 2. In this section, we use the notation \(q = 1 - p\).

#### A. Unions of Random Graphs are Random Graphs

Section III derived results for single random graphs, and we show now that these results are easily adapted to unions of random graphs because such unions are themselves equivalent to single random graphs with a different edge probability.

**Lemma 5:** Let \(\mathcal{U}_N(n,p)\) denote the set of all unions of \(N\) random graphs on \(n\) nodes with edge probability \(p\), i.e.,

\[
\mathcal{U}_N(n,p) := \left\{ \bigcup_{i=1}^N G_i \mid G_i \in \mathcal{G}(n,p) \right\}.
\]

Then

\[
\mathcal{U}_N(n,p) = \mathcal{G}(n, 1 - (1 - p)^N).
\]

**Proof:** Consider some \(G \in \mathcal{U}_N(n,p)\). Fix any admissible node indices \(i\) and \(j\). Then an edge is absent between \(i\) and \(j\) only if it is absent in all \(N\) graphs that comprise \(G\). That is, an edge between \(i\) and \(j\) is absent in \(G\) with probability \(q^N\). Then that edge is present with probability \(1 - q^N = 1 - (1 - p)^N\). \(\blacksquare\)

With Lemma 5, results pertaining to individual random graphs can be applied to unions of such graphs with only minor modifications.

#### B. Expectation of Order Statistics

It was noted in Section III that the algebraic connectivity of a random graph is the first order statistic over the non-zero eigenvalues of that random graph’s Laplacian, i.e.,

\[
\lambda_2 = \min_{2 \leq i \leq n} \ell_i.
\]

Thus, while the expected value of each \(\ell_i\) is known, the expected value of \(\lambda_2\) is not. To apply what is known about \(\ell_i\) to \(\lambda_2\), we state the following lemma from [1] which bounds the expectation of order statistics in terms of properties of the underlying collection of random variables.
**Lemma 6:** Let $X_1, X_2, \ldots, X_m$ be jointly distributed with common mean $\mu$ and variance $\sigma^2$. Then the $k^{th}$ order statistic of this collection, denoted $X_{k:m}$, has expectation bounded according to

$$\mu - \sigma \sqrt{\frac{m-k}{k}} \leq \mathbb{E}[X_{k:m}] \leq \mu + \sigma \sqrt{\frac{k-1}{m-k+1}}.$$  

**Proof:** See Equation (4) in [1].

The bound in Lemma 6 is shown in [1] to be tight when the underlying random variables have identical means and variances. Using Lemma 6, we now bound the expected value of $\lambda_2$ for a single random graph.

**Lemma 7:** Let $G \in G(n, p)$. Its algebraic connectivity, $\lambda_2$, has expectation bounded according to

$$\max\{np - \sqrt{2n(n-2)pq}, 0\} \leq \mathbb{E}[\lambda_2] \leq np,$$

where $q := 1 - p$.

**Proof:** This follows from Equation (6) and using Lemma 6 with $m = n - 1$, $\mu$ from Theorem 1, and $\text{Var}[\ell_i]$ from Corollary 1. The non-negativity of the left-hand side of Equation (7) follows from the non-negativity of all eigenvalues of all $L \in \mathcal{L}(n, p)$, stated in Equation (1).

It is possible that the left-hand side of Equation (7) is zero for some values of $p$. In particular, a straightforward calculation shows that the left-hand side of Equation (7) is only positive when

$$p > \frac{2n - 4}{3n - 4}$$

and for $p$ outside this range, Lemma 7 does not provide a lower bound on $\lambda_2$ beyond its non-negativity (which can be inferred from the non-negativity of each $\ell_i$). However, despite this limitation, Lemma 7 will be instrumental in solving Problem 1 below. Before doing so, we now bound the variance of $\lambda_2$ by following an argument similar to that in Lemma 7.

**Lemma 8:** Let $G \in G(n, p)$. Its algebraic connectivity, $\lambda_2$, has variance bounded according to

$$\text{Var}[\lambda_2] \leq n(n-2)p^2 + 2np - (np - \sqrt{2n(n-2)pq})^2$$

and

$$\text{Var}[\lambda_2] \geq n(n-2)p^2 + 2np - \sigma[\ell_i^2] \sqrt{n-2 - n^2 p^2},$$

where

$$\sigma[\ell_i^2] = \left( n(n-7)(n-3)(n-2)p^4 + 6n(2n-7)(n-2)p^3 + 25n(n-2)p^2 + 8np - (n(n-2)p^2 + 2np)^2 \right)^{1/2}$$

as in Corollary 1.

**Proof:** Using Lemma 6 and Theorem 1 we find that

$$n(n-2)p^2 + 2np - \sigma[\ell_i^2] \sqrt{n-2} \leq \mathbb{E}[\lambda_2^2] \leq n(n-2)p^2 + 2np.$$  

Using that $\text{Var}[\lambda_2] = \mathbb{E}[\lambda_2^2] - \mathbb{E}[\lambda_2]^2$ and the bounds on $\mathbb{E}[\lambda_2]$ from Lemma 7 the result follows.

To assess connectivity of random graphs, the final result needed is a lower bound on the algebraic connectivity of connected graphs. It is known that the connected graph with least algebraic connectivity is a line graph [15], and we present this value below.
Lemma 9: The minimum algebraic connectivity attained by a connected graph on \( n \) nodes is that of a line graph, equal to
\[
\lambda_{\text{min}} = 2 \left(1 - \cos \frac{\pi}{n}\right)
\].

**Proof:** See Proposition 1.12 in [2].

While Definition 2 says that a graph is connected if and only if \( \lambda_2 > 0 \), Lemma 9 shows that there is a minimum value of \( \lambda_2 \) attained by any connected graph. Definition 2 still holds because any graph with \( \lambda_2 > 0 \) will also have \( \lambda_2 \geq \lambda_{\text{min}} \). However, when computing \( \mathbb{E}[\lambda_2] \) for a random graph, it is possible to have \( 0 < \mathbb{E}[\lambda_2] < \lambda_{\text{min}} \), in which case \( \mathbb{E}[\lambda_2] \) is not large enough to imply connectivity of the underlying graph. Thus, while an actual graph has \( \lambda_2 \geq \lambda_{\text{min}} \) whenever \( \lambda_2 > 0 \), an “expected graph” may not. Accordingly, our solutions to Problems 1 and 2 use \( \lambda_{\text{min}} \) as the desired lower bound on \( \lambda_2 \), with the knowledge that doing so is sufficient for connectivity.

C. Solutions to Problems 1 and 2

We now present the main results of the paper: solutions to Problems 1 and 2. We begin with Problem 1 and provide a lower bound on the number of random graphs needed in a union before its expected algebraic connectivity is bounded below by the minimum among all connected graphs, as determined in Lemma 9.

**Theorem 2:** (Solution to Problem 1) The expected algebraic connectivity of a union of \( N \) random graphs is bounded below by the minimum for connected graphs if
\[
N \geq N_{\text{min}} := \frac{1}{\log q} \log \left(\frac{4n^2 + 4n \cos \frac{\pi}{n} - \tau(n) - 8n}{6n^2 - 8n}\right),
\]
where
\[
\tau(n) := \left(16n^2(n - 2) \left(1 - \cos \frac{\pi}{n}\right) + 32n(2 - n) \left(1 - \cos \frac{\pi}{n}\right)^2 + 4n^2(n - 2)^2\right)^{1/2}.
\]

**Proof:** From Lemma 9 the minimum algebraic connectivity attained by any connected graph is \( \lambda_{\text{min}} = 2 \left(1 - \cos \frac{\pi}{n}\right) \). By Lemma 7 we find that
\[
np - \sqrt{2n(n - 2)p\hat{q}} \leq \mathbb{E}[\lambda_2],
\]
where \( \lambda_2 \) is the algebraic connectivity for a union of \( N \) random graphs.

Using Lemma 5 and replacing \( p \) by \( \hat{p} := 1 - (1 - p)^N \) in Equation 8 gives
\[
n\hat{p} - \sqrt{2n(n - 2)\hat{p}\hat{q}} \leq \mathbb{E}[\lambda_2],
\]
where \( \hat{q} = 1 - \hat{p} \). To lower-bound \( \mathbb{E}[\lambda_2] \) by \( \lambda_{\text{min}} \), it is sufficient for
\[
\lambda_{\text{min}} \leq n\hat{p} - \sqrt{2n(n - 2)\hat{p}\hat{q}},
\]
or, rearranging terms, it is sufficient for
\[
2n(n - 2)\hat{p}\hat{q} \leq n^2\hat{p}^2 - 2n\hat{p}\lambda_{\text{line}} + \lambda_{\text{line}}^2.
\]
Replacing \( \hat{p} \) by \( 1 - \hat{q} \) gives a quadratic inequality for \( \hat{q} \), which can be solved for \( \hat{q} \) using the quadratic equation. Then, expanding \( \hat{q} = (1 - p)^N \) and solving for \( N \) gives the desired result.
Section V gives numerical lower bounds on $N$ generated by Theorem 2 for a range of values of $n$ and $p$. We emphasize that Theorem 2 holds for any fixed values of $n$ and $p$, without requiring any relationship between them.

In Theorem 2, it can also be shown that $\tau(n)$ is dominated by $2n^2$ for large $n$, and therefore the argument of the second $\log$ is dominated by the $2n^2$ term in the numerator and the $6n^2$ term in the denominator, resulting in this term limiting to $\log(1/3) = -\log(3)$. Therefore, as $n$ becomes large, the lower bound on $N_{\min}$ in Theorem 2 approaches a limiting value, namely, for large $n$,

$$N_{\min} \approx \frac{\log(3)}{\log q}. \tag{9}$$

Having solved Problem 1 we now focus on Problem 2. Toward solving Problem 2 we now state the Paley-Zygmund inequality in the form in which we use it below.

Lemma 10: (Paley-Zygmund inequality) Let $Z$ be a non-negative random variable with $\text{Var}[Z] < \infty$ and let $\theta \in [0, 1]$. Then

$$P(z > \theta \mathbb{E}[Z]) \geq (1 - \theta)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}.$$  

Proof: See [32].

Theorem 3: (Solution to Problem 2) The probability that the algebraic connectivity of a union of $N \geq N_{\min}$ random graphs is at least the minimum algebraic connectivity of a connected graph is

$$P[\lambda_2(U_N) \geq \lambda_{\min}] \geq \left(1 - 2 \left(1 - \cos \frac{\pi}{n}\right) \right)^2 \frac{\left(n\hat{p} - \sqrt{2n(n-2)\hat{p}\hat{q}}\right)^2}{n(n-2)\hat{p}^2 + 2n\hat{p}},$$

where $\hat{p} = 1 - (1 - p)^N$ and $\hat{q} = 1 - \hat{p}$.

Proof: From Lemma 7 we see that

$$n\hat{p} - \sqrt{2n(n-2)\hat{p}\hat{q}} \leq \mathbb{E}[\lambda_2],$$

while from Lemma 8 we find that

$$\mathbb{E}[\lambda_2^2] \leq n(n-2)\hat{p}^2 + 2n\hat{p}.$$ 

Substituting these bounds into Lemma 10 and setting $\theta = \lambda_{\min}/\mathbb{E}[\lambda_2]$ completes the proof.

The condition that $N \geq N_{\min}$ in Theorem 3 is enforced so that $\theta = \lambda_{\min}/\mathbb{E}[\lambda_2]$ can be used in the Paley-Zygmund inequality. Of course, this condition can be eliminated and a different form of probabilistic bound can be derived in place of Theorem 3. As stated, Theorem 3 gives a probabilistic bound that is a function only of $N$, the number of random graphs in a union, because $n$ and $p$ are fixed. Together, Theorems 2 and 3 characterize any union of Erdős-Rényi graphs and help determine the number of graphs required to attain connectivity in such unions.

In the next section, we give numerical results derived from Theorems 2 and 3 for values of $n$ and $p$ across several orders of magnitude.

V. NUMERICAL RESULTS

In this section, we simulate the results of Theorems 2 and 3 to provide numerical solutions to Problems 1 and 2 for select values of $n$ and $p$. 
A. Numerical Results for Problem 1

We now present results using Theorem 2 by providing values of \( N_{\text{min}} \) for a range of values of \( n \) and \( p \), representing the solutions to Problem 1 under different conditions.

Table I gives the value of \( N_{\text{min}} \) (rounded up) as determined by Theorem 2 for \( n \) ranging from 10 to 100,000 and \( p \) ranging from 0.00001 to 0.1. The values of \( N_{\text{min}} \) shown in the table are the numbers of graphs from \( G(n, p) \) needed in a union before the algebraic connectivity of the union has expectation bounded below by that of a line graph (which has least algebraic connectivity among all connected graphs).

| \( p \)     | 0.00001 | 0.0001  | 0.001   | 0.01    | 0.1     |
|------------|---------|---------|---------|---------|---------|
| \( n \)    | 10      | 100     | 1,000   | 10,000  | 100,000 |
| 117,846   | 110,539 | 109,928 | 109,868 | 109,862 |
| 11,785    | 11,054  | 10,993  | 10,987  | 10,986  |
| 1,178     | 1,105   | 1,099   | 1,099   | 1,099   |
| 118       | 110     | 110     | 110     | 110     |
| 12        | 11      | 11      | 11      | 11      |

TABLE I
VALUES OF \( N_{\text{min}} \), AS DETERMINED BY THEOREM 2

Throughout these values, it can be seen that, for each fixed value of \( n \), an order of magnitude increase in \( p \) corresponds well to an order of magnitude decrease in the number of graphs required for connectivity of a union. In addition, for a fixed value of \( p \), increasing \( n \) causes a decrease in the lower bound on \( N_{\text{min}} \). This means that, as graphs become larger, fewer total graphs are required in a union to make it connected. This occurs because a graph on \( n \) nodes has \( \frac{n(n-1)}{2} \) possible edges and, because a larger graph has more possible edges, larger graphs have more possible ways to attain connectivity, resulting in fewer required in a union to make it connected. The limiting behavior of Theorem 2 seen in Equation (9) can also be seen in Table I where the lower bounds on \( N_{\text{min}} \) appear to saturate when \( p \) is held fixed and \( n \) is increased. For example, for \( p = 0.00001 \), we have

\[
-\frac{\log(3)}{\log q} = 109,861,
\]

which agrees closely with the values of \( N_{\text{min}} \) for \( p = 0.00001 \) and \( n \geq 1,000 \) seen in Table I.

B. Numerical Results for Problem 2

We now present numerical results from Theorem 3. In particular, for \( n = 50 \) nodes and \( p \in \{0.05, 0.10, 0.15, 0.20, 0.25\} \), we present lower bounds on the probability of a union of \( N = 50 \) random graphs being connected. While Theorem 2 concerns the expected algebraic connectivity of a union of random graphs, Theorem 3 concerns the probability of the algebraic connectivity itself exceeding that of a line graph (which is the minimum among all connected graphs). Below, Table II gives numerical values for the probability of a particular union of \( N = 50 \) random graphs having algebraic connectivity bounded below by that of a line graph.
In Table II we see that the probability of having a connected union increases rapidly as p increases, and that this probability approaches 1 even when p is far from 1. Next, in Table III we present results that have fixed values of n = 50 and p = 0.10, but changing values of N.

| n  | N  | 25  | 50  | 75  | 100 | 125 |
|----|----|-----|-----|-----|-----|-----|
| 50 | 0.377 | 0.810 | 0.947 | 0.986 | 0.996 |

TABLE III
A lower bound on the probability of N graphs from G(50, 0.1) being connected, as determined by Theorem 3.

Similar to what was seen in Table II these results show that the probability of a union being connected increases rapidly with N. To further illustrate this trend, a union of N = 250 graphs with n = 50 and p = 0.1 is connected with probability at least 0.9998 according to Theorem 3.

The numerical results in this section indicate that the results in Theorems 2 and 3 solve Problems 1 and 2 in a manner which readily provides numerical results. The three parameters n, p, and N can vary dramatically across problem formulations, though the results obtained here apply to broad ranges across all three of these parameters, allowing for these results to be used in a wide range of applications.

VI. Conclusion

We presented results that determine the number of random graphs required for their union to attain some lower bound on its expected algebraic connectivity, and results that lower-bound the probability with which a union of random graphs is connected. In multi-agent systems, a common assumption is that agents’ communication graphs have connected unions over time, and these results can be used to enforce this assumption.

Future work includes extending our results to the cases of different edge probabilities and of time-varying edge probabilities. Another direction for future work concerns reformulating these results for directed graphs, including with time-varying probabilities that are direction-dependent. The directed case breaks the symmetries used in this paper and would therefore likely require different techniques to derive bounds of the same form derived here.

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