Iteration scheme for initial value problem for PDEs: Existence, convergence and comparison

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Abstract Results about existence and uniqueness of solutions of initial value problem for certain types of partial differential equations are recalled as well as iterative scheme and an error estimate for approximate solutions obtained using this scheme. Several numerical examples are presented to demonstrate how the proposed iterative scheme can be applied, with emphasis given to verifying assumptions of using the scheme. Comparison to other recently presented results is done in this respect.

1 Introduction

The significance of partial differential equations is growing in the field of mathematical analysis recently. Researchers are looking for new methods which are computationally efficient and simple in application to find exact or approximate solutions of partial differential equations.

Many techniques including variational iteration method, homotopy analysis method, homotopy perturbation method and differential transformation method have been recently used for solving particular problems. Unfortunately, most of the above mentioned methods need some analytical preparation before using some computer software. Most of the algorithms require software which is able to perform symbolical integration.

Following direction of current research, we found a lot of papers whose content was unsatisfactory. Some authors do not verify conditions that justify using of the particular method. Some authors even do not mention any assumptions under
2 Preliminaries

Let $\Omega$ be a compact subset of $\mathbb{R}^k$. Denote $J = [-\delta, \delta] \times \Omega$, where $\delta > 0$ will be specified later, then $J$ is a compact subset of $\mathbb{R}^{k+1}$. Let $u(t,x) = u(t,x_1,\ldots,x_k)$ be a real function of $k+1$ variables defined on $J$. We introduce the following operators: $\nabla = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k})$ and $D = (\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k})$. We deal with partial differential equations

$$\frac{\partial^n}{\partial t^n} u(t,x) = F(t,x,u,\nabla u,\ldots,\nabla^m u) \quad \text{for } m < n$$

(1)

and

$$\frac{\partial^n}{\partial t^n} u(t,x) = F(t,x,u,\nabla u,\ldots,\nabla^{m-1} u, D\nabla^{m-1} u, D^2\nabla^{m-1} u,\ldots, D^{m-(n-1)}\nabla^{m-n-1} u), m \geq n.$$ (2)

In both cases, left-hand side of the equation contains only the highest derivative with respect to $t$. We do not consider equations where the order of partial derivatives with respect to $t$ is $n$ or higher on the righthand side, including mixed derivatives.

When convenient, we will use multiindex notation as well:

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_k^{\alpha_k}},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. 
Denote $N = \max\{m, n\}$. We consider equation (1) or (2) with the set of initial conditions

$$u(0, x) = c_1(x), \quad \frac{\partial}{\partial t} u(0, x) = c_2(x), \quad \vdots \quad \frac{\partial^{n-1}}{\partial t^{n-1}} u(0, x) = c_n(x),$$

where initial functions $c_i(x), i = 1, \ldots, n$ are taken from space $C^N(\Omega, \mathbb{R})$. It means that we are looking for classical solutions.

For the purpose of clarity, we emphasize that our formulation covers for instance heat, wave, Burger, Boussinesq or Korteweg-de Vries (KdV) equations.

Obviously, $F : J \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}^{(k+1)^2} \times \cdots \times \mathbb{R}^{(k+1)^m} \rightarrow \mathbb{R}$ if $m < n$, and $F : J \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}^{(k+1)^2} \times \cdots \times \mathbb{R}^{(k+1)^n-1} \times \mathbb{R}^{(k+1)^{n-1}} \times \cdots \times \mathbb{R}^{k^{m-n+1}(k+1)^{n-1}} \rightarrow \mathbb{R}$ if $m \geq n$. Denote

$$K_1 = \frac{(k+1)^{m+1} - 1}{k} \quad \text{for } m < n$$

and

$$K_2 = \frac{(k+1)^{n-1} - 1}{k} + (k+1)^{n-1} \frac{k^{m-n+2} - 1}{k-1} \quad \text{for } m \geq n.\quad (5)$$

Then, if we consider $u$ as dependent variable, we see that $F$ is a function of $k + 1 + K_1$ variables in case $m < n$ or $k + 1 + K_2$ variables in case $m \geq n$.

Denote

$$u_0(t, x) = \sum_{i=1}^{n} c_i(x) t^{i-1} \frac{1}{(i-1)!} = \sum_{i=1}^{n} \left( \frac{\partial^{i-1}}{\partial t^{i-1}} u(0, x) \right) t^{i-1} \frac{1}{(i-1)!}.\quad (6)$$

Then $u_0 \in C^N(J, \mathbb{R})$.

We suppose that $F$ is Lipschitz continuous in last $K_1$ ($m < n$) or $K_2$ ($m \geq n$) variables, i.e. $F$ satisfies condition

$$|F(t, x, y_1, \ldots, y_{K_l}) - F(t, x, z_1, \ldots, z_{K_l})| \leq L \left( \sum_{i=1}^{K_l} |y_i - z_i| \right), \quad \text{for } l = 1 \text{ or } 2,\quad (7)$$

on a compact set which is defined as follows: There is $R \in \mathbb{R}, R > 0$ such that (7) holds on

$$J \times \prod_{\alpha_0 + |\alpha| \leq m} \overline{[c_{\alpha_0, \alpha}, d_{\alpha_0, \alpha}]},\quad (8)$$

where
\[c_{\alpha_0, \alpha} = \min_{(t,x) \in J} \left( \frac{\partial^{\alpha_0 + |\alpha|}}{\partial t^{\alpha_0} \partial x^\alpha} u_0(t,x) \right) - R, \quad d_{\alpha_0, \alpha} = \max_{(t,x) \in J} \left( \frac{\partial^{\alpha_0 + |\alpha|}}{\partial t^{\alpha_0} \partial x^\alpha} u_0(t,x) \right) + R,\]

and \(\alpha_0 < n\) in all cases.

Since \(F\) is continuous on compact set, \(|F|\) attains its maximal value on this set, denote it \(M\). Then we put

\[\delta = \left( \frac{R \cdot (n-1)!}{M} \right)^{1/n} \tag{10}\]

Finally, define the following operator

\[Tu(t,x) = u_0(t,x) + \int_0^t (t-\xi)^{n-1} \frac{1}{(n-1)!} F(\xi, x, u(\xi, x), \nabla u(\xi, x), \ldots) d\xi, \tag{11}\]

where the function \(F\) has either \(k+1+K_1\) or \(k+1+K_2\) arguments and the last \(K_1\) or \(K_2\) arguments involve dependent variable \(u\).

3 Overview of results

This section is devoted to recalling the results presented in [2]. We would like to emphasize that all results have local character.

**Theorem 1.** Let the condition (7) hold. Then problem consisting of equation (1) or (2) and initial conditions (3) has a unique local solution on \((-\delta, \delta) \times \Omega\), where \(\delta\) is defined by (10).

**Theorem 2.** Assume that condition (7) holds. Then iterative scheme \(u_p = Tu_{p-1}\), \(p \geq 1\) with initial approximation \(u_0\) defined by (6), where \(T\) is defined by (11), converges to unique local solution \(u(t,x)\) of problem (1), (3), respective (2), (3). Moreover, we have the following error estimate for this scheme:

\[\|u(t,x) - u_p(t,x)\|_{C^N} \leq \frac{R \cdot \gamma^p}{1 - \gamma} \tag{12}\]

on \((-\delta_1, \delta_1) \times \Omega\), where \(\delta_1\) is chosen such that operator \(T\) is a contraction, \(\gamma = \frac{\delta_1}{(n-1)!}\) and constants \(L\) and \(R\) are defined by (7) and (8).

**Corollary 1.** Let condition (7) be valid and suppose that \(F\) can be written as \(G + g:\)

\[F(t,x,z(t,x),\nabla z(t,x),...) = G(t,x,z(t,x),\nabla z(t,x),...) + g(t,x).\]  

Then we may choose initial approximation
\[ \bar{u}_0 = u_0 + \int_0^t \left( \frac{(t - \xi)^{n-1}}{(n-1)!} - g(\xi, x) \right) d\xi \]
\[ = \sum_{i=1}^{n} \left( \frac{\partial^{i-1}}{\partial t^{i-1}}u(0,x) \right) \frac{t^{i-1}}{(i-1)!} + \int_0^t \left( \frac{(t - \xi)^{n-1}}{(n-1)!} - g(\xi, x) \right) d\xi. \] (15)

4 Comparison

Now we are prepared to compare our results to other recently published results. We would like to point out that formulation and verification of conditions sufficient to use particular algorithms is important.

Example 1. The first example to compare was published in [3] in 2010 as Example 1. Consider the following third-order nonlinear PDE
\[ u_t + 6u^2u_x + u_{xxx} = 0 \] (16)
with initial function \( u(0, x) = \sqrt{c} \text{sech}(k + \sqrt{c}x), \) where \( c \geq 0 \) and \( k \) is a constant.

The authors of paper [3] claim that their “variational iteration formula” converges “for some constants \( \gamma_i = \alpha_i + m_iT < 1 \)” However, no verification of validity of such condition is done in the examples.

On the other hand, to apply our iteration scheme, we need to verify that condition (7) is valid on some connected compact set of the form (8). In our terms, \( F(t, x, y_1, y_2, y_3) = -6y_1^2y_2 - y_3 \). Again, this function has continuous partial derivatives of all orders in \( \mathbb{R}^3 \), hence the same argument as in Example 1 shows that our iteration scheme is applicable in a neighbourhood of origin. Indeed, calculating first few terms, we get the following sequence:
Calculating a few more terms reveals that all terms of order 2 and higher are vanishing one by one in each step, thus the sequence converges to exact solution \( u(t) = 1 + t \) near the origin. This also is the case in [4], if the author tried to calculate the sequence of partial sums. However, the verification led to the false conclusion which prevented him to try it.

**Example 3.** The third example was published in [5] in 1997 as Example 4. It is a first order nonlinear PDE

\[
\frac{u_t}{1 + t} = x^2 - \frac{1}{4} u_x^2 \tag{18}
\]

with initial function \( u(0, x) = 0 \).

In this case, since the cited paper does not contain any assumptions, conditions, theorems or proofs, no verification was done in the examples.

The same argument as in previous two examples (continuous partial derivatives of \( F \)) allows us to apply our iteration scheme to find local approximation of unique solution of the problem. The first few approximations are

\[
\begin{align*}
  u_1(t,x) & = x^2 t, \\
  u_2(t,x) & = x^2 (t - \frac{1}{3} t^3), \\
  u_3(t,x) & = x^2 (t - \frac{1}{3} t^3 + \frac{2}{15} t^5 + o(t^5)), \\
  u_4(t,x) & = x^2 (t - \frac{1}{3} t^3 + \frac{2}{15} t^5 + \frac{17}{315} t^7 + o(t^7)).
\end{align*}
\]

Careful examination of this sequence leads to a guess that it converges to the function \( u(t,x) = x^2 \tanh t \). Indeed, it is not difficult to verify that this function is an exact solution of the considered problem. According to Theorem 1 this solution is unique, hence our sequence of approximations indeed converges to this unique solution.

**5 Conclusion**

We formulated an existence and uniqueness result for initial value problem for certain classes of PDEs in this paper. Using this result, we derived an iteration scheme, investigated its convergence and compared our results to other recently published results. It turned out that our results are less restrictive than the compared results. A
subject of further investigation is to develop similar approach for systems of PDEs and for other types of problems.

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References

1. Rebenda, J., Šmarda, Z.: A new iterative method for linear and nonlinear partial differential equations. In: Proceedings of the conference ICNAAM 2014, AIP Conf. Proc. 1648, 810002 (2015) doi: 10.1063/1.4913011
2. Rebenda, J., Šmarda, Z.: Convergence analysis of an iterative scheme for solving initial value problem for multidimensional partial differential equations. Comput. Math. Appl. 70, 1772–1780 (2015) doi: 10.1016/j.camwa.2015.07.018
3. Sweilam, N.H., Khader, M.M.: On the convergence of variational iteration method for nonlinear coupled system of partial differential equations. Int. J. Comput. Math. 87 (5), 1120–1130 (2010) doi: 10.1080/00207160903124959
4. Odibat, Z.M.: A study on the convergence of variational iteration method. Mathematical and Computer Modelling 51, 1181–1192 (2010)
5. He, J.: A new approach to nonlinear partial differential equations. Commun. Nonlin. Sci. Numer. Simulat. 2, 230–235 (1997)