The Constrained Blaschke Functional for Spherical Curves

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Abstract

We study critical trajectories in the sphere for the Blaschke variational problem with length constraint. For every Lagrange multiplier encoding the conservation of the length during the variation, we show the existence of infinitely many closed trajectories which depend on a pair of relative prime natural numbers. A geometric description of these numbers and the relation with the shape of the corresponding critical trajectories is also given.

Keywords: Blaschke Functional, Closed Trajectories, Critical Curves.

Mathematics Subject Classification: 53A04, 49Q10.

1 Introduction

Functionals on curves depending on the curvatures are ubiquitous in differential geometry, mathematical physics and biomathematics. Their study dates back to the days of the Bernoulli family and Euler. Indeed, the problem of determining the bending deformation of rods was first formulated by J. Bernoulli in 1691 ([41]) and the possible qualitative types for untwisted plane configurations were completely described by L. Euler ([21]), although some particular cases were already known to J. Bernoulli ([7]).

More generally, in a letter to L. Euler in 1738 ([58]), D. Bernoulli proposed to investigate the extrema of the functionals

$$\mathcal{E}_p : \gamma \mapsto \int_{\gamma} \kappa^p$$

where $\kappa$ is the curvature of the curve $\gamma$.

The case $p = 2$ corresponds with the classical Euler-Bernoulli elastic curves, which have received a considerable interest in the last decades because of their applications in the following research topics: Willmore surfaces ([40, 54, 61]), constrained Willmore surfaces ([10, 57]), the Canham-Helfrich-Evans model for lipid bilayers ([15, 16, 22, 33, 35, 59]), and surfaces with spherical curvature lines ([17]), among others. Another source of interest is due to the interrelationships with integrable flows of curves governed by the mKdV hierarchy ([18, 26, 27, 38, 46, 50]). In fact, (constrained) elastic curves do evolve maintaining their shape under the first non-trivial Goldstein-Petrič flow. From an analytical viewpoint this amounts to say that the curvatures of the extrema are traveling wave solutions of the mKdV equation. A similar relation between elastic curves and the nonlinear Schr"odinger equation can...
be obtained applying the Hasimoto transformation \([31, 32, 37]\). This phenomenon occurs in other contexts such as Lorentzian, centro-affine, equi-affine, projective and pseudo-conformal geometries \([12, 13, 14, 19, 20, 45, 47, 48, 49, 55, 60]\).

For natural values of \(p > 2\), this variational problem has been considered in \([2]\). Critical curves for this problem have been used to construct Willmore-Chen submanifolds in spaces with Riemannian and pseudo-Riemannian warped product metrics \([1, 6]\) and they have been applied to analyze conformal tensions in string theories \([5]\). Quite unexpectedly, in the case of spherical curves, the only closed critical trajectories are geodesics. However, as shown in \([43, 44]\), for suitable rational values of \(p \in (0, 1)\), there exist infinitely many closed critical spherical curves. These curves arise in the theory of biconservative hypersurfaces as the generating curves of rotational ones \([43, 44]\).

The case \(p = 1/3\) corresponds with the equi-affine length for convex curves, investigated by W. Blaschke \([8\), Vol II, 1923\). After the seminal paper \([11]\), equi-affine geometry of convex curves has been consistently used in recent studies on human curvilinear 2-dimensional drawing movements and recognition for non-rigid planar shapes (see for instance \([23, 56]\) and the literature therein). Critical curves of the equi-affine arc-length functional are parabolas \([8, 30]\). Replacing the arc-element with the equi-affine arc-element and \(\kappa\) with the equi-affine curvature and putting \(p = 1\), the corresponding variational problem is Liouville-integrable and has countably many closed critical curves \([60]\). Putting \(p = 1/2\), one obtains the affine arc-length functional \([60]\). The ellipses are the only simple closed critical curves. Due to the rather complicated nature of the Euler-Lagrange equation \([60]\), not much is known about the existence of other closed (non-simple) critical curves.

The case \(p = 1/2\) in the Bernoulli’s list was again considered by W. Blaschke \([8\), Vol I, 1921\), who explicitly obtained the curvatures of the critical curves in terms of their arc-length parameter. He showed that plane critical curves are catenaries. The existence of countably many closed critical curves in the unit-sphere \(S^2\) was proved in \([2, 4]\). Recently, critical curves for an extension of this case in Riemannian and Lorentzian 3-space forms have been characterized as the profile curves of invariant constant mean curvature surfaces \([3]\).

The present paper deals with the constrained Blaschke variational problem

\[
B_\lambda : \gamma \mapsto \int_\gamma (\sqrt{\kappa} + \lambda),
\]

where \(\kappa\) is the geodesic curvature of the curve \(\gamma\) and \(\lambda \in \mathbb{R}\) is a Lagrange multiplier encoding the conservation of the length during the variation. The functional \(B_\lambda\) is defined (perhaps not everywhere) either on convex plane curves in 2-dimensional Riemannian manifolds or on curves with no inflection points in a Riemannian or pseudo-Riemannian manifold of dimension greater than two (in the latter case the curve is either time-like or space-like). Although in the constrained case there exist critical curves in \(\mathbb{R}^2\) with periodic curvature, it is quite easy to see that none of them are closed. The family of critical curves for \(B_\lambda\) in \(\mathbb{R}^2\) was geometrically described in \([32]\).

Motivated by the above mentioned results of Arroyo, Garay, Mencía and Pámpano about the existence of closed spherical trajectories of the unconstrained case \([2, 3, 4]\), this paper aims to investigate closed critical curves of the constrained Blaschke functional for spherical curves. For the sake of brevity, a critical curve \(\gamma\) for \(B_\lambda\) with non-constant periodic curvature is said to be a \(B\)-curve. If, in addition, the curve \(\gamma\) is periodic, we say that the curve is a \(B\)-string. One of the main differences between the unconstrained and constrained cases is due to the fact that the curvatures of unconstrained \(B\)-curves can be written in terms of trigonometric functions while in the unconstrained case one must deal with elliptic integrals involving the square root of a quartic polynomial. Or, from a
more geometric viewpoint: the phase portraits of unconstrained B-curves are singular rational curves while, in the constrained case they are, in general, singular elliptic curves. This explains why the study of the constrained variational problem is technically more challenging than the unconstrained one.

Our approach can be applied to the study of critical curves in the hyperbolic plane. However, in this case, the situation is more complicated since the momenta of the critical curves can be diagonalizable over $\mathbb{R}$. It is likely that the corresponding curves are never closed. Nevertheless, critical curves whose momenta are diagonalizable over $\mathbb{C}$ can be treated more or less in analogy with the spherical case. Consequently, the existence of countably many closed critical trajectories is expected. We plan to consider the constrained Blaschke variational problem in the hyperbolic plane in a future research project.

We next state the main results of the paper.

**Theorem 1.1** Let $\gamma : I \subset \mathbb{R} \rightarrow S^2$ be a critical curve for $B_\lambda$ with non-constant curvature defined on its maximal domain $I \subset \mathbb{R}$. Then, $I = \mathbb{R}$ and $\gamma$ is a B-curve which can be parameterized in terms of its arc-length parameter $s \in \mathbb{R}$, up to rigid motions, as

$$\gamma(s) \equiv \gamma_\xi(s) = \frac{1}{2\xi \mu} \left(1, -\sqrt{4\xi^2 \mu^2 - 1} \cos \theta(s), \sqrt{4\xi^2 \mu^2 - 1} \sin \theta(s)\right),$$

where

$$\theta(s) := 2\xi \int_0^s \frac{\mu^2 (\mu + 2\lambda)}{1 - 4\xi^2 \mu^2} \, ds,$$

and $\mu \equiv \mu(s) = \sqrt{\kappa(s)}$ is a solution of

$$\mu^2 = -\mu^2 \left(\mu^4 + 4\lambda \mu^3 + 4 \left[\lambda^2 - \xi^2\right] \mu^2 + 1\right),$$

for suitable constant $\xi > 0$.

We will say that two B-curves are *equivalent* if there is a rigid motion taking one into another. This theorem implies that $\gamma_\xi$, and so all the curves in the equivalence class of $\gamma_\xi$, are B-strings if and only if

$$\Psi_\lambda(\xi) := 2\xi \int_0^\omega \frac{\mu^2 (\mu + 2\lambda)}{1 - 4\xi^2 \mu^2} \, ds,$$

is a rational multiple of $2\pi$. Here, $\omega$ denotes the least period of $\mu$. After regularizing the function $\Psi_\lambda$, for fixed $\lambda \in \mathbb{R}$, this rational multiple can be identified with a rational number $q \in \mathbb{Q}$ of the type $q = m/n$ for relative prime natural numbers $m < n$. We say that $q$ is the *characteristic number* of the B-string $\gamma_\xi \equiv \gamma_{m,n}$. The number $n$ is said to be the *wave number*.

Our second main result deals with the existence of B-strings.

**Theorem 1.2** For every $\lambda \in \mathbb{R}$ there exists an interval $I_\lambda \subset (0, 1/2)$ such that for any $q = m/n \in I_\lambda$, the equivalence class of $\gamma_{m,n}$ is a B-string with multiplier $\lambda$ and characteristic number $q$. 

The proof of this theorem is based on the analysis of the complete elliptic integral $\Psi_\lambda$ and its asymptotic behavior. The numerical experiments strongly support the ansatz that $\Psi_\lambda$ is a strictly decreasing function of $\xi$. The validity of the ansatz would imply that for each pair of relative prime
natural numbers \((m,n)\) such that \(m/n \in I_{\lambda}\) there exists a unique equivalence class of B-strings with multiplier \(\lambda\) and characteristic number \(q = m/n\). Moreover, this would also show that all B-strings have self-intersections and wave number \(n \geq 3\).

Finally, the third result is about basic geometric features of B-strings.

**Theorem 1.3** Let \(\gamma_{\xi}, \xi > 0\), be a suitable representative of an equivalence class of B-strings with multiplier \(\lambda\) and characteristic number \(q = m/n\). Then, the following conclusions hold true:

1. The trajectory of \(\gamma_{\xi}\) is invariant by the group generated by rotation of \(2\pi/n\) around the \(Ox\)-axis and it is contained in a region of the upper hemisphere \(S^2_+ = \{(x,y,z) \in S^2 | x > 0\}\) bounded by two horizontal planes.

2. If \(4\lambda\xi + 1 \neq 0\), then \(\gamma_{\xi}\) does not intersect the \(Ox\)-axis. Moreover:
   (a) If \(4\lambda\xi + 1 > 0\), \(n - m\) is the linking number with the \(Ox\)-axis (equipped with the upward orientation) and \(\gamma_{\xi}\) possesses, exactly, \(n(n - m - 1)\) ordinary double points.
   (b) If \(4\lambda\xi + 1 < 0\), \(-m\) is the linking number with the \(Ox\)-axis (equipped with the upward orientation) and \(\gamma_{\xi}\) possesses, at least, \(nm\) points of self-intersection.

3. If \(4\lambda\xi + 1 = 0\) (necessarily, \(\lambda < 0\)), then \(\gamma_{\xi}\) intersects the \(Ox\)-axis \(n\) times and the moving point \(\gamma_{\xi}(s)\) travels counter-clockwise around the \(Ox\)-axis (equipped with the upward orientation). In this case \(n - m\) is the turning number of the plane projection of \(\gamma_{\xi}\) to the plane \(x = 0\).

In Figure 1 we show three equivalence classes of B-strings with three-fold symmetry for different values of the Lagrange multiplier \(\lambda\) and for suitable characteristic numbers \(q = m/n\). These cases cover all the possible options for the sign of \(4\lambda\xi + 1\) discussed in Theorem 1.3. More examples will be discussed in detail in Section 5.

![Figure 1: Equivalence classes of B-strings with three-fold symmetry with different multipliers \(\lambda\) and characteristic numbers \(q = m/n\). From left to right: \(4\lambda\xi + 1 < 0\), \(4\lambda\xi + 1 = 0\) and \(4\lambda\xi + 1 > 0\). For each of them we show the corresponding parameters \((\lambda, m, n)\).](image)

The material of this paper is organized into four sections. In Section 2 we write the Euler-Lagrange equation and its associated conservation law. Subsequently, we consider the monodromy map and we formulate the closure condition for a B-curve in terms of the monodromy. In Section 3 we prove
the three main theorems (which are stated in a more technical form than the one presented in this introduction) and we briefly discuss the Hopf tori arising from B-strings. In Section 4 we focus on the theoretical aspects, analyzing the rich “hidden” geometry surrounding the Blaschke variational problem. Following Griffith’s approach to calculus of variations ([25, 28, 30, 34]) we construct the phase space of $B_\lambda$, a 5-dimensional submanifold $M$ of $T^*\text{[SO}(3)]$. The restriction on $M$ of the symplectic form of $T^*\text{[SO}(3)]$ is an exact 2-form $d\zeta \in \Omega^2(M)$ of maximal rank, invariant by the action of $\text{SO}(3)$. The relevant fact is that the characteristic curves of $d\zeta$ are canonical lifts of $B$-curves with multiplier $\lambda$. We then compute the moment map of $d\zeta$ for the $\text{SO}(3)$-action ([51]) and we reformulate Theorem 1.1 in terms of the Marsden-Weinstein reduced spaces. We quote [24, 29, 36, 39, 51] as our basic references on the moment map of hamiltonian actions and the Marsden-Weinstein reduction. Finally, Section 5 is devoted to the discussion of explicit examples which illustrate the theoretical properties shown in previous sections.

The graphics, symbolic computations and numerical evaluations have been performed with the software Mathematica 13. Complete elliptic integrals involving the square root of a quartic polynomial do appear several times. We used the monograph [9] as our reference on this technical topic. The bibliographic references, while relatively consistent, do not completely reflect the vast literature devoted to functionals depending on curvatures and their interrelations with symplectic geometry and applied mathematics. It is the result of a very partial selection, aimed at the specific themes considered in this work.

2 Critical Curves

Let $(x, y, z)$ be the standard coordinates of the Euclidean space $\mathbb{R}^3$. The round 2-sphere (of radius one) can be understood as the hyperquadric

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

endowed with the induced metric of $\mathbb{R}^3$.

Let $\gamma : I \subset \mathbb{R} \rightarrow S^2$ be a smooth immersed (spherical) curve parameterized by the arc-length $s \in I$. Denote by $T(s) := \dot{\gamma}(s)$ the unit tangent vector field along the curve $\gamma(s)$, where the upper dot represents the derivative with respect to the arc-length parameter, and define the unit normal vector field $N(s)$ along $\gamma(s)$ to be the counter-clockwise rotation of $T(s)$ through an angle $\pi/2$ in the tangent bundle of $S^2$. In this setting, the (signed) geodesic curvature $\kappa(s)$ of $\gamma(s)$ is defined by the Frenet-Serret equation

$$\nabla_T T(s) = \kappa(s) N(s),$$

where $\nabla$ denotes the Levi-Civita connection on $S^2$. We will say that a curve is convex if its curvature is strictly positive everywhere, i.e., if $\kappa(s) > 0$ for all $s \in I$. For convex curves we introduce the following geometric invariant

$$\mu(s) := \sqrt{\kappa(s)}.$$ 

It follows from the Fundamental Theorem for Plane Curves that the geometric invariant $\mu(s)$ completely determines the (convex) curve, up to rigid motions.

Let $C^\infty_\text{conv}(\mathbb{R}, S^2)$ be the space of smooth immersed convex curves $\gamma : \mathbb{R} \rightarrow S^2$ parameterized by the arc-length $s \in \mathbb{R}$ and let $\lambda \in \mathbb{R}$ be a constant. The constrained Blaschke functional with Lagrange
multiplier \( \lambda \) is defined by

\[
B_\lambda : \gamma \in C^\infty_*(\mathbb{R}, \mathbb{S}^2) \mapsto \int_0^{L_\gamma} (\mu(s) + \lambda) \, ds,
\]
where \( L_\gamma \) stands for the length of \( \gamma \).

Using a standard formula for the variational derivative of functionals depending on the curvature \( \kappa \) (see for instance \([2, 3, 10, 53]\)) we obtain the Euler-Lagrange equation associated to \( B_\lambda \), which is

\[
\frac{d^2}{ds^2} \left( \frac{1}{\mu} \right) - \frac{1}{\mu} \left( \mu^4 - 1 \right) - 2\lambda \mu^2 = 0.
\]

**Remark 2.1** For every \( \lambda \in \mathbb{R} \) there exists a unique circle critical for \( B_\lambda \). Its constant invariant \( \mu \) is the (unique) positive solution of

\[
\mu^4 + 2\lambda \mu^3 - 1 = 0,
\]
which we denote by \( \eta_\lambda \). Consequently, \( \eta : \lambda \in \mathbb{R} \mapsto \eta_\lambda \in \mathbb{R}^+ \) is a real-analytic function.

From now on we assume that the invariant \( \mu(s) \) associated to \( \gamma(s) \) is non-constant. Then (2) admits a first integral from standard arguments (see \([2, 3, 10, 53]\) again for details). We describe this conservation law in the following result.

**Proposition 2.2** Let \( \mu(s) \) be a non-constant function solution of (2). Then \( \mu(s) \) satisfies the first order differential equation

\[
\dot{\mu}^2(s) = -\mu^2(s) Q(\mu(s)),
\]
where \( Q \) is the quartic polynomial defined by

\[
Q(t) := t^4 + 4\lambda t^3 + 4 \left( \lambda^2 - \xi^2 \right) t^2 + 1,
\]
and \( \xi > 0 \) is a constant of integration.

From Proposition 2.2 it follows that, for fixed \( \lambda \in \mathbb{R} \), solutions of the Euler-Lagrange equation (2) belong to a two parameter family. Nevertheless, by translating the origin of the arc-length parameter \( s \) if necessary, we can assume that non-constant solutions \( \mu(s) \) belong to a one parameter family, since the constant of integration arising from integrating (4) may be assumed to be zero. Consequently, the one parameter family of solutions depends on the constant of integration \( \xi > 0 \), whose physical meaning will be clarified in Section 4.

A **B-curve** (with multiplier \( \lambda \)) is an arc-length parameterized convex curve \( \gamma : \mathbb{R} \rightarrow \mathbb{S}^2 \) with non-constant positive periodic curvature satisfying (4). Two B-curves \( \gamma \) and \( \tilde{\gamma} \) are said to be equivalent if there exists \( A \in SO(3) \) and \( c \in \mathbb{R} \) such that \( \tilde{\gamma}(s) = A \cdot \gamma(s + c) \), i.e., if there exists an isometry transforming one into another and a translation of the arc-length parameter. We denote the equivalence class of \( \gamma \) by \( [\gamma] \) and the set of the equivalence classes of B-curves with multiplier \( \lambda \) by \( \mathcal{M}_\lambda \).

In the following result we show that any non-constant solution \( \mu(s) \) of (4) is periodic and so the associated curve is a B-curve.

**Proposition 2.3** Let \( \gamma : I \subset \mathbb{R} \rightarrow \mathbb{S}^2 \) be an arc-length parameterized convex curve with non-constant invariant \( \mu(s) \) satisfying (4), and assume that \( I \) is the maximal domain of definition of \( \gamma \). Then, \( I = \mathbb{R} \) and \( \gamma \) is a B-curve.
Proof. Since the differential equation (4) is satisfied, the polynomial (5) must be negative for some \( t > 0 \).

We first observe that the limit when \( t \to \infty \) of \( Q(t) \) is \( \infty \), while \( Q(0) = 1 > 0 \). Regardless of the values of \( \lambda \) and \( \lambda^2 - \xi^2 \), \( Q(t) \) has either zero or two changes of signs among its coefficients. It then follows from Descartes’ rule of signs that \( Q(t) \) has either zero, one (double) or two (distinct) positive roots. The case of zero positive roots can be discarded since from above limits one would conclude that \( Q(t) > 0 \) for all \( t > 0 \). The case of the double root corresponds to a circle, which has been studied in Remark 2.1. Therefore, it only remains the case of two distinct positive roots.

In this case, applying the standard square root method of algebraic geometry we get that the algebraic curve

\[
y^2 = -x^2 Q(x)
\]

is closed. Moreover, the curve \((\mu(s), \dot{\mu}(s))\), which is a bounded integral curve of a smooth vector field, is included in the trace of this algebraic curve because \( \mu(s) \) is a solution of (4). We then conclude from the Poincaré-Bendixson Theorem that \( \mu(s) \) is a periodic function and so its maximal domain of definition is \( \mathbb{R} \). Consequently, the associated curve is a B-curve. \( \square \)

This proposition proves that, up to equivalence, there exists a one parameter family of B-curves, depending on the constant of integration \( \xi > 0 \). It turns out that this constant of integration \( \xi > 0 \) can be described in terms of the Lagrange multiplier \( \lambda \) and the largest root of the polynomial \( Q(\mu) \). Indeed, as shown above, this polynomial has two positive roots, which we denote by \( e_1 > e_2 > 0 \). Then, it follows from \( Q(e_1) = 0 \) that

\[
\xi = \frac{1}{2e_1} \sqrt{1 + (e_1(e_1 + 2\lambda))^2}.
\] (6)

From this relation we can assume that B-curves depend on \( \lambda \) and \( e_1 \) and we can consider these parameters as the fundamental ones.

The set of equivalence classes of B-curves is in one-to-one correspondence with the plane domain \( P = \{(\lambda, e_1) \in \mathbb{R}^2 \mid e_1 > \eta_\lambda\} \), where \( \eta_\lambda \) was defined in Remark 2.1 as the only positive solution of \( \mu^4 + 2\lambda\mu^3 - 1 = 0 \).

**Theorem 2.4** For every \( \lambda \in \mathbb{R} \), the map \( e_1 \in (\eta_\lambda, \infty) \longrightarrow [\gamma_{\lambda,e_1}] \in \mathcal{M}_\lambda \) is bijective.

*Proof.* The proof is a straightforward consequence of the existence and uniqueness of solutions for ordinary differential equations. For each \( \lambda \in \mathbb{R} \) and \( e_1 \in (\eta_\lambda, \infty) \) fixed, we get a unique \( \xi > 0 \) from (6). Note that for \( \lambda \in \mathbb{R} \) fixed the relation (6) between \( \xi \) and \( e_1 \) is bijective. Then, there exists a unique, up to translation of the arc-length parameter, solution of (4), denoted by \( \mu_{\lambda,e_1}(s) \). This geometric invariant, from the Fundamental Theorem for Plane Curves, uniquely determines a convex spherical curve \( \gamma_{\lambda,e_1} \), up to rigid motions. In conclusion, the curve \( \gamma_{\lambda,e_1} \) is unique, up to equivalence. \( \square \)

In order to treat the B-curves, it is convenient to fix a suitable representative for each equivalence class. Let \((\lambda, e_1) \in P \) and \( \mu \equiv \mu_{\lambda,e_1} \) be the unique solution of (4) such that \( \mu_{\lambda,e_1}(\omega_{\lambda,e_1}/2) = e_1 \), where \( \omega_{\lambda,e_1} \) denotes the least period of \( \mu_{\lambda,e_1} \). Consider \( \gamma_{\lambda,e_1} \) to be the unique B-curve with curvature \( \mu_{\lambda,e_1}^2 \).
such that
\[
\begin{align*}
\gamma_{\lambda,e_1} \left( \frac{\omega_{\lambda,e_1}}{2} \right) &= \left( \frac{1}{\sqrt{1 + (e_1 + 2\lambda)^2}}, \frac{-e_1 (e_1 + 2\lambda)}{\sqrt{1 + (e_1 + 2\lambda)^2}}, 0 \right), \\
\gamma_{\lambda,e_1} \left( \frac{\omega_{\lambda,e_1}}{2} \right) &= (0, 0, -1)
\end{align*}
\]

We call \( \gamma_{\lambda,e_1} \), obtained as above, the standard B-curve with parameters \( \lambda \) and \( e_1 \).

**Remark 2.5** It is not restrictive to focus exclusively on standard B-curves and, hence, from now on we implicitly assume that the B-curves in consideration are in their standard form.

We finish this section by formulating the closure condition for a B-curve in terms of the monodromy map. For a B-curve \( \gamma_{\lambda,e_1} \) we define the (spherical) Frenet frame field along \( \gamma_{\lambda,e_1} \) as the map \( \mathcal{F}_{\lambda,e_1} \equiv (\gamma_{\lambda,e_1}, \dot{\gamma}_{\lambda,e_1}, \gamma_{\lambda,e_1} \times \dot{\gamma}_{\lambda,e_1}) : \mathbb{R} \rightarrow SO(3) \) where \( \times \) denotes the usual vector cross product of \( \mathbb{R}^3 \). Then, the map \( m : (\lambda, e_1) \in \mathcal{P} \mapsto \mathcal{F}_{\lambda,e_1}(\omega_{\lambda,e_1}) \cdot [\mathcal{F}_{\lambda,e_1}(\omega_{\lambda,e_1})]^{-1} \in SO(3) \) is the monodromy of the Blaschke variational problem. The order of this map is essential to describe the closure condition of \( \gamma_{\lambda,e_1} \).

**Theorem 2.6** The monodromy \( m \) is a continuous function of \( \lambda \) and \( e_1 \). Moreover, \( \gamma_{\lambda,e_1} \) is a B-string if and only if \( m_{\lambda,e_1} \) has finite order.

**Proof.** Let \( (\lambda, e_1) \in \mathcal{P} \) and consider a B-curve \( \gamma_{\lambda,e_1} \) (in its standard form). Since \( \mu_{\lambda,e_1} \) is a solution of (3), it also satisfies the second order Euler-Lagrange equation (2), together with the initial conditions \( \mu_{\lambda,e_1}(\omega_{\lambda,e_1}/2) = e_1 \) and \( \mu_{\lambda,e_1}(0) = 0 \). Therefore, \( \mu_{\lambda,e_1}(s) \) is a real-analytic function of \( s \in \mathbb{R} \) and \( (\lambda, e_1) \in \mathcal{P} \), and so is \( \kappa_{\lambda,e_1}(s) = \sqrt{\mu_{\lambda,e_1}(s)} \).

On the other hand, the (spherical) Frenet frame \( \mathcal{F}_{\lambda,e_1} \) satisfies
\[
\begin{align*}
\dot{\mathcal{F}}_{\lambda,e_1} &= \mathcal{F}_{\lambda,e_1} \cdot \mathcal{K}_{\lambda,e_1} \\
\mathcal{F}_{\lambda,e_1}(\omega_{\lambda,e_1}/2) &= \mathcal{F}_{\lambda,e_1},
\end{align*}
\]
where
\[
\mathcal{K}_{\lambda,e_1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\kappa_{\lambda,e_1} \\ 0 & \kappa_{\lambda,e_1} & 0 \end{pmatrix}
\]
and \( \mathcal{F}_{\lambda,e_1} = ([E_1]_{\lambda,e_1}, [E_2]_{\lambda,e_1}, [E_3]_{\lambda,e_1}) \) for
\[
\begin{align*}
(E_1)_{\lambda,e_1} &= \left( \frac{1}{\sqrt{1 + (e_1 + 2\lambda)^2}}, \frac{-e_1 (e_1 + 2\lambda)}{\sqrt{1 + (e_1 + 2\lambda)^2}}, 0 \right), \\
(E_2)_{\lambda,e_1} &= (0, 0, -1) \\
(E_3)_{\lambda,e_1} &= (E_1)_{\lambda,e_1} \times (E_2)_{\lambda,e_1}
\end{align*}
\]
Consequently, \( \mathcal{F}_{\lambda,e_1}(s) \) is also a real-analytic function of \( s \in \mathbb{R} \) and \( (\lambda, e_1) \in \mathcal{P} \).

Note that the least period of \( \mu, \omega \), is a standard complete elliptic integral which can be solved in terms of complete elliptic integrals of the first and third kind (see [3], 257.12 and 259.04). It then follows from the properties of these complete elliptic integrals that \( \omega : (\lambda, e_1) \in \mathcal{P} \mapsto \omega_{\lambda,e_1} \in \mathbb{R} \) is a continuous function which is real-analytic on the Zariski-open set \( \mathcal{P} \), i.e., the complement of the zero
locus of the non-constant real-analytic function \((\lambda, e_1) \in P \mapsto (e_1 + [e_2]_{\lambda,e_1})^3 - 4e_1^3e_2^3_{\lambda,e_1} \). Thus, \(m\) is also a continuous function and real-analytic on \(\hat{P}\).

Finally, since \(\mu_{\lambda,e_1}\) is periodic with least period \(\omega_{\lambda,e_1}\), we conclude from (2) that for every \(k \in \mathbb{Z}\),

\[
\mathcal{F}_{\lambda,e_1}(s + k \omega_{\lambda,e_1}) = m_{\lambda,e_1}^k \cdot \mathcal{F}_{\lambda,e_1}(s).
\]

This finishes the proof. \(\square\)

**Remark 2.7** Observe that the order of the monodromy \(m_{\lambda,e_1}\) is, precisely, the wave number \(n\) of \(\gamma_{\lambda,e_1}\).

### 3 Integrability by Quadratures and Existence of B-Strings

In this section we give the parameterization of B-curves in terms of just one quadrature and prove the main theorems of the paper.

We begin by defining a curve in the plane domain

\[
P = \{(\lambda, e_1) \in \mathbb{R}^2 | e_1 > \eta_{\lambda}\}.
\]

The **exceptional locus** is the smooth curve \(P_* \subset P\) defined by the equation

\[
4\lambda^2 e_1^3 + 8\lambda^3 e_1^2 - e_1 + 2\lambda = 0.
\]

It is easy to check that this exceptional curve is contained in \(\{\lambda, e_1) \in P | \lambda < 0\}\). We say that the parameters \((\lambda, e_1)\) are **exceptional** if they belong to \(P_*\). Moreover, for any \(\lambda < 0\), the cubic equation (7) has a unique positive root \(u_{\lambda}\), which may be explicitly computed. The function \(u : \lambda \in \mathbb{R}^- \mapsto u_{\lambda} \in \mathbb{R}^+\) is real-analytic and \(P_*\) is the graph of \(u\) (see Figure 5). For convenience, if \(\lambda \geq 0\) we will define \(u_{\lambda} = \infty\).

We next introduce some functions which will play an essential role on the parameterization of B-curves. Let \(\sigma : (s, \lambda, e_1) \in \mathbb{R} \times P \mapsto \sigma_{\lambda,e_1}(s) \in \mathbb{Z}_2\) be defined by

\[
\sigma_{\lambda,e_1}(s) := \begin{cases} 
1, & \text{if } s \in [2k\omega_{\lambda,e_1}, (2k+1)\omega_{\lambda,e_1}), k \in \mathbb{Z} \\
(-1)^{\chi(\lambda,e_1)}, & \text{if } s \in [(2k+1)\omega_{\lambda,e_1}, 2(k+1)\omega_{\lambda,e_1}), k \in \mathbb{Z},
\end{cases}
\]

where \(\omega_{\lambda,e_1}\) is the least period of \(\mu_{\lambda,e_1}\) and \(\chi : P \mapsto \mathbb{Z}_2\) is the indicator function of \(P_*\), i.e., \(\chi\) is zero everywhere but at the points \((\lambda, e_1) \in P_*\) in which case \(\chi(\lambda, e_1) = 1\). We then define the **angular function**

\[
\theta_{\lambda,e_1}(s) := 2\xi_{\lambda,e_1} \int_{\omega_{\lambda,e_1}}^{s} \frac{\mu_{\lambda,e_1}^2(t)(\mu_{\lambda,e_1}(t) + 2\lambda)}{1 - 4\xi_{\lambda,e_1}^2 \mu_{\lambda,e_1}^2(t)} dt
\]

and the **radial** and **height functions**, respectively,

\[
\rho_{\lambda,e_1}(s) := \frac{\sigma_{\lambda,e_1}(s) \sqrt{4\xi_{\lambda,e_1}^2 \mu_{\lambda,e_1}^2(s) - 1}}{2\xi_{\lambda,e_1} \mu_{\lambda,e_1}(s)},
\]

\[
h_{\lambda,e_1}(s) := \frac{1}{2\xi_{\lambda,e_1} \mu_{\lambda,e_1}(s)}.
\]

From these definitions, some basic features of these functions can be deduced:

1. The height functions are periodic (with least period \(\omega_{\lambda,e_1}\)) and even. They have a minimum at \(s = \omega_{\lambda,e_1}/2\) and a maximum at \(s = \omega_{\lambda,e_1}\). (See Figure 2)
2. The radial functions have two different qualitative behaviors depending on whether \( e_1 = u_\lambda \) or not:

(a) If \( e_1 \neq u_\lambda \), then \( \rho_{\lambda,e_1} \) is periodic (with least period \( \omega_{\lambda,e_1} \)) and positive. (See Figure 3, Left.)

(b) If \( e_1 = u_\lambda \), then \( \rho_{\lambda,e_1} \) is periodic (with least period \( 2\omega_{\lambda,e_1} \)) and \( \rho_{\lambda,e_1}(s+\omega_{\lambda,e_1}) = -\rho_{\lambda,e_1}(s) \). It has two zeros in the interval \( [\omega_{\lambda,e_1}/2,\omega_{\lambda,e_1}/2+2\omega_{\lambda,e_1}] \), precisely, at \( s = \omega_{\lambda,e_1} \) and \( s = 2\omega_{\lambda,e_1} \). (See Figure 3, Right.)

Figure 3: The graphs of the radial functions \( \rho_{\lambda,e_1} \) for: \( \lambda = -1.1 \) and \( u_\lambda < e_1 \simeq 4.59 \) (left); and, \( \lambda = -0.27 \) and \( e_1 = u_\lambda \simeq 2.34 \) (right). In the left figure we take the portion of the graph on the interval \( [\omega_{\lambda,e_1}/2,\omega_{\lambda,e_1}/2+\omega_{\lambda,e_1}] \), while in the right one we are showing the graph on the interval \( [\omega_{\lambda,e_1}/2,\omega_{\lambda,e_1}/2+2\omega_{\lambda,e_1}] \).

3. The angular functions are arithmetic quasi-periodic (with quasi-period \( \omega_{\lambda,e_1} \)) and odd. In the interval \( [\omega_{\lambda,e_1}/2,\omega_{\lambda,e_1}/2+\omega_{\lambda,e_1}] \), \( \theta_{\lambda,e_1} \) possesses an inflection point at \( s = \omega_{\lambda,e_1} \) with \( \theta_{\lambda,e_1}(\omega_{\lambda,e_1}/2) = \theta(\omega_{\lambda,e_1})/2 \). However, their qualitative behavior depends on whether \( e_1 > u_\lambda \) or \( e_1 \leq u_\lambda \):

(a) If \( e_1 > u_\lambda \), in the same interval, \( \theta_{\lambda,e_1} \) has exactly two critical points, an absolute minimum somewhere between \( s \in (\omega_{\lambda,e_1}/2,\omega_{\lambda,e_1}) \) and an absolute maximum in \( (\omega_{\lambda,e_1},\omega_{\lambda,e_1}/2+\omega_{\lambda,e_1}) \).

These functions are increasing from the minimum to the maximum and they tend to \( \infty \) as \( s \to \infty \). (See Figure 4, Left.)

(b) If \( e_1 \leq u_\lambda \), then \( \theta_{\lambda,e_1} \) is strictly decreasing and it tends to \( -\infty \) as \( s \to \infty \). (See Figure 4, Right.)

We have now all the necessary information to prove the parameterization of B-curves (the following result corresponds to Theorem 1.1 in the Introduction).
Figure 4: The graphs of the angular functions $\theta_{\lambda,e_1}$ for: $\lambda = -1.1$ and $u_\lambda < e_1 \simeq 4.59$ (left); and, $\lambda = -0.27$ and $e_1 = u_\lambda \simeq 2.34$ (right). The black segment represents the jump of the quasi-periodic function. If the length of the segment is a rational multiple of $2\pi$, the critical curve is periodic (see Theorem 3.3). In this pictures we show the portion of the graph on the interval $[\omega_{\lambda,e_1}/2, \omega_{\lambda,e_1}/2 + 2\omega_{\lambda,e_1}]$.

Theorem 3.1 Let $(\lambda, e_1) \in \mathcal{P} = \{(\lambda, e_1) \in \mathbb{R}^2 \mid e_1 > \eta_\lambda\}$ where $\eta_\lambda$ is defined in Remark 3.1 and $\mu$ be a solution of the Cauchy problem

\[
\begin{align*}
\dot{\mu}^2 &= -\mu^2 Q(\mu) \\
\mu(\omega/2) &= e_1,
\end{align*}
\]

where $Q$ is the quartic polynomial defined in (5) and $\omega$ is the least period of $\mu$. Then, $\mu \equiv \mu_{\lambda,e_1}$ is a positive periodic function. Moreover, define the curve

$$
\gamma_{\lambda,e_1}(s) = (h_{\lambda,e_1}(s), -\rho_{\lambda,e_1}(s) \cos \theta_{\lambda,e_1}(s), \rho_{\lambda,e_1}(s) \sin \theta_{\lambda,e_1}(s))
$$

where $h_{\lambda,e_1}$, $\rho_{\lambda,e_1}$ and $\theta_{\lambda,e_1}$ are the height, radial and angular functions introduced above, (6)–(10). Then, $\gamma_{\lambda,e_1}$ is the B-curve with parameters $(\lambda, e_1)$, in its standard form.

Proof. The first part of the statement was already proven in previous section. It remains to prove that $\gamma_{\lambda,e_1}$ is the B-curve with parameters $(\lambda, e_1)$. For simplicity, we will omit the subscripts $\lambda$ and $e_1$ throughout this proof.

We first note that for every $(\lambda, e_1) \in \mathcal{P}$ the function

$$
\frac{\mu(s) + 2\lambda}{1 - 4\xi^2\mu^2(s)}
$$

is real-analytic. In fact, if $(\lambda, e_1)$ is not exceptional, then $1 - 4\xi^2 \mu^2(s) \leq 1 - 4\xi^2 e_1^2 < 0$. Note that $(\lambda, e_1)$ is exceptional if and only if $\xi = -1/(4\lambda)$ and $\lambda = -e_2/2 < 0$. Thus, in this case

$$
\frac{\mu(s) + 2\lambda}{1 - 4\xi^2\mu^2(s)} = 4\lambda^2 \frac{\mu(s) + 2\lambda}{4\lambda^2 - \mu^2(s)} = -4\lambda^2 \frac{1}{\mu(s) - 2\lambda} = -\frac{4\lambda^2}{\mu(s) + e_2} < 0.
$$

This implies that the angular function $\theta(s)$ is real-analytic. By construction, so is the height function $h_s$.

We next distinguish between the cases where $(\lambda, e_1) \in \mathcal{P}$ is exceptional or not.

Suppose first that $(\lambda, e_1) \in \mathcal{P}$ is not exceptional. Then, as shown above $4\xi^2 \mu^2 - 1 > 0$ holds and, hence, $\gamma(s)$ is a real-analytic spherical curve. Moreover, since $\mu(\omega/2) = e_1$, we have

$$
\gamma\left(\frac{\omega}{2}\right) = (h\left(\frac{\omega}{2}\right), -\rho\left(\frac{\omega}{2}\right), 0) = \frac{1}{2\xi e_1} \left(1, -\sqrt{4\xi^2 e_1^2 - 1}, 0\right).
$$
Computing the derivative we get
\[ \dot{\gamma}(s) \cdot \ddot{\gamma}(s) = -\frac{4\lambda^2 \mu^2 + 4\lambda \dot{\mu} + \mu^2}{\mu^2 - 4\xi^2 \mu^4} = 1, \]
and \( \ddot{\gamma}(\omega/2) = (0, 0, -1) \). Here we have used the conservation law \(4\) to simplify above expression. Hence, \( \gamma \) is an arc-length parameterized spherical curve. Then, computing the second order derivative of \( \gamma \), and using the Euler-Lagrange equation \(2\) and the conservation law \(4\) to simplify the expression, we obtain that
\[ \dddot{\gamma}(s) \cdot \dddot{\gamma}(s) = \mu^4(s) + 1, \]
from which we conclude that the geodesic curvature of \( \gamma \) is either \( \mu^2 \) or \( -\mu^2 \). In what follows, we will discard the second case. From above computations we also deduce that
\[ \gamma \left( \frac{\omega}{2} \right) \times \dot{\gamma} \left( \frac{\omega}{2} \right) = \left( \frac{e_1 (e_1 + 2\lambda)}{\sqrt{1 + (e_1^2 + 2\lambda)^2}}, \frac{1}{\sqrt{1 + (e_1^2 + 2\lambda)^2}}, 0 \right). \]
This implies
\[ \kappa \left( \frac{\omega}{2} \right) = \frac{\ddot{\gamma} \left( \frac{\omega}{2} \right) \cdot \left( \frac{e_1 (e_1 + 2\lambda)}{\sqrt{1 + (e_1^2 + 2\lambda)^2}}, \frac{1}{\sqrt{1 + (e_1^2 + 2\lambda)^2}}, 0 \right)}{\gamma \left( \frac{\omega}{2} \right) \times \dot{\gamma} \left( \frac{\omega}{2} \right)}, \]
A straightforward computation yields
\[ \dddot{\gamma} \left( \frac{\omega}{2} \right) = \left( \frac{e_1^2 + 2\lambda e_1^3 - 1}{1 + (e_1^2 + 2\lambda)^2}, \frac{2e_1 (e_1 + \lambda)}{1 + (e_1^2 + 2\lambda)^2}, 0 \right), \]
and, hence, \( \kappa(\omega/2) = e_1^2 = \mu^2(\omega/2) \). That is \( \kappa = \mu^2 \) and \( \gamma \) is an arc-length parameterized curve with curvature \( \mu^2 \) and the same initial conditions of the (standard) B-curve. Consequently, both of them coincide.

Finally, we treat the case where \( (\lambda, e_1) \in \mathcal{P} \) is exceptional. In this case \( 4\xi^2 \mu^2 - 1 \) vanishes on \( \omega \mathbb{Z} \) and is positive on \( \mathbb{R} \setminus \omega \mathbb{Z} \). By definition,
\[ \gamma(s) = \begin{cases} (h(s), -\tilde{\rho}(s) \cos \theta(s), \tilde{\rho}(s) \sin \theta(s)), & \text{if } s \in [2k \omega, (2k + 1) \omega), \ k \in \mathbb{Z} \\ (h(s), \tilde{\rho}(s) \cos \theta(s), -\tilde{\rho}(s) \sin \theta(s)), & \text{if } s \in [(2k + 1)\omega, (2k + 1)\omega), \ k \in \mathbb{Z} \end{cases}, \]
where \( \tilde{\rho} = \sqrt{4\xi^2 \mu^2 - 1}/(2\xi \mu) \).

It is then clear that \( \gamma \) is continuous and real-analytic away from the discrete set \( \omega \mathbb{Z} \). On the set \( \mathbb{R} \setminus \omega \mathbb{Z} \) we can argue as in the non-exceptional case to prove that \( \gamma \) is an arc-length parameterized spherical curve with \( \kappa = \mu^2 \) and such that
\[ \left\{ \begin{array}{l} \gamma \left( \frac{\omega}{2} \right) = \frac{1}{2\xi e_1} \left( 1, -\sqrt{4\xi^2 e_1^2 - 1}, 0 \right). \\ \gamma \left( \frac{\omega}{2} \right) = (0, 0, -1) \end{array} \right. \]
We next prove that \( \gamma \) is of class \( C^1 \) on \( \omega \mathbb{Z} \). By construction, \( \gamma(s + k \omega) = R_k \cdot \gamma(s) \), for any \( k \in \mathbb{Z} \), where \( R \) is a rotation around the \( Ox \)-axis. Hence, it suffices to show that \( \gamma \) is of class \( C^1 \) at \( s = \omega \). Note that the first component of \( \gamma \) is everywhere real-analytic. Since \( \theta \) is a real-analytic function, the second and third components of \( \gamma \) are of class \( C^1 \) at \( s = \omega \) if and only if the function
\[ f(s) = \begin{cases} \sqrt{4\xi^2 \mu^2(s)} - 1, & s \in [0, \omega) \\ -\sqrt{4\xi^2 \mu^2(s)} - 1, & s \in [\omega, 2\omega) \end{cases}, \]
is of class $\mathcal{C}^1$ at $s = \omega$. Recall that $\mu(\omega) = e_2$ and that, in the exceptional case $\xi = 1/(2e_2)$. Thus, $f$ is continuous at $s = \omega$. Moreover, on the left of $\omega$ we have

$$\dot{\mu} = \mu \sqrt{-(\mu - e_1)(\mu - e_2)(\mu - e_3)(\mu - e_4)},$$

where $e_1 > e_2 > 0$, $e_3$ and $e_4$ are the roots of the polynomial $Q$, (3). Therefore,

$$f = \frac{\mu \dot{\mu}}{e_2 \sqrt{\mu^2 - e_2^2}} = \frac{\mu^2 \sqrt{-(\mu - e_1)(\mu - e_3)(\mu - e_4)}}{e_2 \sqrt{\mu^2 + e_2^2}}.$$

While, on the right of $\omega$,

$$\dot{\mu} = -\mu \sqrt{-(\mu - e_1)(\mu - e_2)(\mu - e_3)(\mu - e_4)},$$

and so

$$\dot{f} = -\frac{\mu \dot{\mu}}{e_2 \sqrt{\mu^2 - e_2^2}} = \frac{\mu^2 \sqrt{-(\mu - e_1)(\mu - e_3)(\mu - e_4)}}{e_2 \sqrt{\mu^2 + e_2^2}}.$$

This implies that

$$\lim_{s \to \omega^-} f(s) = \lim_{s \to \omega^+} f(s) = \frac{\sqrt{(e_1 - e_2)(e_2 - e_3)(e_2 - e_4)}}{\sqrt{2}}.$$

Consequently, $\gamma$ is an arc-length parameterized spherical curve of class $\mathcal{C}^1$. Let $\tilde{\gamma}$ be the standard B-curve with parameters $\lambda$ and $e_1$. Then, both $\gamma$ and $\tilde{\gamma}$ are arc-length parameterized spherical curves with the same curvature and the same initial conditions, $\gamma(\omega/2) = \tilde{\gamma}(\omega/2)$ and $\dot{\gamma}(\omega/2) = \dot{\tilde{\gamma}}(\omega/2)$. Since $\tilde{\gamma}$ is real-analytic and $\gamma$ is of class $\mathcal{C}^1$ and real-analytic on $\mathbb{R} \setminus \omega \mathbb{Z}$, we may conclude that $\gamma = \tilde{\gamma}$.

More precisely, let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be the (spherical) Frenet frame fields along $\gamma$ and $\tilde{\gamma}$, respectively. By construction, $\mathcal{F}$ is continuous and real-analytic on $\mathbb{R} \setminus \omega \mathbb{Z}$, while $\tilde{\mathcal{F}}$ is real-analytic everywhere. Then there exists a continuous map $A : \mathbb{R} \to SO(3)$ such that $\tilde{\mathcal{F}} = A \mathcal{F}$. On $\mathbb{R} \setminus \omega \mathbb{Z}$, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are both solutions of the linear system

$$\dot{X} = X \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\mu^2 \\ 0 & \mu^2 & 0 \end{pmatrix}.$$

Thus, $A$ is constant on $\mathbb{R} \setminus \omega \mathbb{Z}$. By continuity, one concludes that $A$ is constant everywhere. Moreover, since $\mathcal{F}(\omega/2) = \tilde{\mathcal{F}}(\omega/2)$, we obtain that $A = \text{Id}_{3x3}$ is the identity. This proves that $\gamma = \tilde{\gamma}$. \hfill $\square$

**Remark 3.2** A B-curve $\gamma_{\lambda,e_1}$ passes through the pole $(1,0,0)$ of the unit-sphere $\mathbb{S}^2$ if and only if the parameters $(\lambda, e_1) \in \mathcal{P}$ are exceptional, i.e., if $(\lambda, e_1) \in \mathcal{P}^\ast$ or, equivalently, if $e_1 = u_\lambda$. Moreover, the B-curve passes through the pole, precisely, whenever $s \in \omega \mathbb{Z}$.

The jump of the B-curve $\gamma$ with parameters $(\lambda, e_1) \in \mathcal{P}$ is defined by

$$\Psi(\lambda, e_1) \equiv \Psi_\lambda(e_1) := \theta_{\lambda,e_1}(\omega_{\lambda,e_1}) - \theta_{\lambda,e_1}(0).$$

The curve is periodic if and only if $\Psi_\lambda(e_1)$ is a rational multiple of $2\pi$. (See Figure 4.)

The jump function can be seen as a period map in the following way. Let $SO(2) \subset SO(3)$ be the stabilizer of the vector $\vec{t} = (1,0,0)$. From Theorem 3.1 it follows that the monodromy map $m$
is $SO(2)$-valued. Then, since $\mathcal{P}$ is contractible and $m$ is continuous, there exist continuous functions $\Phi: \mathcal{P} \to \mathbb{R}$ such that

$$
m_{\lambda, e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi_{\lambda, e_1} & -\sin \Phi_{\lambda, e_1} \\ 0 & \sin \Phi_{\lambda, e_1} & \cos \Phi_{\lambda, e_1} \end{pmatrix}.
$$

We say that $\Phi$ is a period map. The period map is unique mod $\mathbb{Z}$ and $\Phi_{\lambda, e_1} \equiv \Psi_{\lambda}(e_1) \pmod{1}$. Keeping in mind that $\omega$ is real-analytic on the Zariski-open set $\hat{\mathcal{P}}$, that $\xi$ and the integrand are real-analytic on $\mathcal{P}$, we deduce that the period maps are real-analytic on $\hat{\mathcal{P}}$.

We now introduce the strictly increasing real-analytic function $p: \mathbb{R} \to (-1, -1/\sqrt{3})$ given by

$$p(\lambda) := -\sqrt{\frac{1 + \eta_{\lambda}^4}{3 + \eta_{\lambda}^4}} = -\sqrt{\frac{1 - \lambda \eta_{\lambda}^3}{2 - \lambda \eta_{\lambda}^3}},$$

where $\eta_{\lambda}$ is the (unique) positive solution of (3). This function will arise when analyzing the asymptotic behavior of $\Psi_{\lambda}(e_1)$.

We next prove the second main result of the paper, which shows the existence of periodic B-curves, i.e., B-strings (this result corresponds to Theorem 1.2 in the Introduction).

**Theorem 3.3** Let $\lambda \in \mathbb{R}$. Then, there exists an unbounded countable set $\Delta_{\lambda} \subset (\eta_{\lambda}, \infty)$ such that a B-curve $\gamma$ with parameters $\lambda$ and $e_1$ is periodic if and only if $e_1 \in \Delta_{\lambda}$. In other words, for every $q \in (1 + p(\lambda), 1/2) \cap \mathbb{Q}$ there exists a B-string $\gamma$ with Lagrange multiplier $\lambda$ and monodromy

$$m_q \equiv m_{\lambda, e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi q) & -\sin(2\pi q) \\ 0 & \sin(2\pi q) & \cos(2\pi q) \end{pmatrix}.$$
We begin with the limit when $e_1 \to \eta^+_{\lambda}$. In this case, for every $\lambda \in \mathbb{R}$ we have

$$\lim_{e_1 \to \eta^+_{\lambda}} \Psi(e_1) = p(\lambda),$$

where $p(\lambda)$ is, precisely, the function introduced in (12). On the other hand, we have the following limits when $e_1 \to \infty$, depending on the sign of $\lambda$,

$$\lim_{e_1 \to \infty} \Psi(e_1) = -\frac{1}{2},$$

if $\lambda \geq 0$, and

$$\lim_{e_1 \to \infty} \Psi(e_1) = \frac{1}{2},$$

if $\lambda < 0$.

Finally, let $\Phi_\pm : \mathcal{P} \to \mathbb{R}$ be the period maps such that $\Phi_-|_{\mathcal{P}_-} = \Psi|_{\mathcal{P}_-}$ and $\Phi_+|_{\mathcal{P}_+} = \Psi|_{\mathcal{P}_+}$. Then, there exists a $k \in \mathbb{Z}$ such that $\Phi_- = \Phi_+ + k$. From above limits, $k = -1$ and $\{\Phi_-(\lambda, e_1) | e_1 > \eta_{\lambda}\}$ contains the open interval $(-1/2, p(\lambda)) \equiv (1 + p(\lambda), 1/2) \pmod{1}$. This concludes the proof. \quad \Box

As mentioned in the previous proof, the function $\Psi_\lambda$ has a jump discontinuity on $\mathcal{P}_*$, i.e., when $e_1 = u_\lambda$. Consequently, it is convenient to regularize this function in order to work with a continuous function. Let us define the regularized jump function $\hat{\Psi} : \mathcal{P} \to \mathbb{R}$ by

$$\hat{\Psi}(\lambda, e_1) \equiv \hat{\Psi}_\lambda(e_1) = \begin{cases} 
\Psi_\lambda(e_1) \pmod{1}, & e_1 \neq u_\lambda \\
\Psi_\lambda(e_1) + 1/2, & e_1 = u_\lambda.
\end{cases}$$

In terms of $\hat{\Psi}_\lambda$, a B-curve $\gamma$ with parameters $(\lambda, e_1) \in \mathcal{P}$ is closed, i.e., a B-string, if and only if $\hat{\Psi}_\lambda(e_1) = 2\pi q$, where $q \in \mathbb{Q}$ is a rational number of the type $q = m/n$ for relative prime natural numbers $m < n$. The rational number $q$ is the characteristic number of the B-string, while the natural number $n$ is its wave number.

From Theorem 3.3 we conclude that for any pair $(m, n)$ of relative prime natural numbers and $\lambda \in \mathbb{R}$ satisfying

$$2(1 + p(\lambda)) n < 2m < n$$

there exists a B-string with multiplier $\lambda \in \mathbb{R}$ and characteristic number $q = m/n$. 

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Remark 3.4 Numerical experiments strongly support the ansatz that the regularized jump function \( \hat{\Psi}_\lambda \) is a real-analytic period map for the Blaschke variational problem. In addition, for every \( \lambda \in \mathbb{R} \) the function \( \hat{\Psi}_\lambda : e^1 \in (\eta_\lambda, \infty) \mapsto (1 + p(\lambda), 1/2) \) is a strictly increasing diffeomorphism. This experimental fact would lead to the following conclusions (which are stronger than those of Theorem 3.3):

1. For every \( \lambda \in \mathbb{R} \) and \( q \in (1 + p(\lambda), 1/2) \cap \mathbb{Q} \) there exists a unique B-string \( \gamma \) with multiplier \( \lambda \) such that \( e^1 = \hat{\Psi}_\lambda^{-1}(q) \). In particular, B-strings \( \gamma \) with multiplier \( \lambda \) are in one-to-one correspondence with the countable set \( (1 + p(\lambda), 1/2) \cap \mathbb{Q} \).

2. The wave number of \( \gamma \equiv \gamma_q \) is \( n \geq 3 \), and hence B-strings are not embedded (see Lemma 3.5 for the number of self-intersection points).

In the last part of this section we will prove Theorem 1.3. We first show a technical lemma regarding the number of points of self-intersection.

Lemma 3.5 Let \( \gamma \) be a B-string with multiplier \( \lambda \) and characteristic number \( q = m/n \). Assume also that \( e^1 \neq u_\lambda \). Then,

1. The points \( \gamma(\omega/2 + k\omega) \) and \( \gamma(\omega + k\omega) \), \( k = 0, ..., n - 1 \) are simple.

2. If \( e^1 < u_\lambda \), then \( \gamma \) possesses, exactly, \( n(n - m - 1) \) ordinary double points.

3. If \( e^1 > u_\lambda \), then \( \gamma \) possesses, at least, \( nm \) points of self-intersection. The angular function has a unique absolute maximum at \( s_\ast \in [\omega/2, \omega] \) such that \( \theta(s_\ast) > \theta(\omega) = \pi m/n \), and we have three different scenarios:

   (a) Case \( \theta(s_\ast) < \pi(n + m)/n \). In this case the B-string has exactly \( nm \) ordinary double points.

   (b) Case \( \pi(m + k)/n < \theta(s_\ast) < \pi(m + k + 1)/n \) for some \( k \in \mathbb{N} \). In this case the B-string has exactly \( n(m + 2k) \) points of self-intersection.

   (c) Case \( \pi(m + k)/n < \theta(s_\ast) = \pi(m + k + 1)/n \) for some \( k \in \mathbb{N} \cup \{0\} \). In this case the B-string has exactly \( n(m + 2k + 1) \) points of self-intersection.

Proof. We begin by proving that the points \( \gamma(\omega/2 + k\omega) \) and \( \gamma(\omega + k\omega) \), \( k = 0, ..., n - 1 \) are simple. Since \( \gamma(s + \omega) = R_{2\pi m/n} \gamma(s) \), where \( R \) is a rotation around the \( Ox \)-axis (see previous Theorems 3.1 and 3.3) it suffices to show that \( \gamma(\omega/2) \) and \( \gamma(\omega) \) are simple. Consider \( \gamma(\omega/2) \). We exhibit that, if \( s_\ast \in [\omega/2, \omega/2 + n\omega] \) satisfies \( \gamma(s_\ast) = \gamma(\omega/2) \), then \( s_\ast = \omega/2 \). The radial function reaches its maximum at \( \omega/2 + h\omega \), \( h \in \mathbb{Z} \). Hence \( s_\ast = \omega/2 + p\omega \), for some integer \( p = 0, ..., n - 1 \). In addition, \( \theta(s_\ast) = \theta(\omega/2) = 0(\text{mod} \ 2\pi) \). Then, \( \theta(s_\ast) = 2k\pi, \ k \in \mathbb{Z} \). On the other hand, \( \theta(s_\ast) = \theta(\omega/2 + p\omega) = 2\pi mp/n \). This implies \( mp = kn \). Since \( m \) and \( n \) are relatively prime, we have \( p = hn, \ h \in \mathbb{Z} \). Therefore, \( s_\ast = \omega/2 + hn\omega \). But \( \omega/2 \leq s_\ast < \omega/2 + n\omega \). Thus, \( h = 0 \) and \( s_\ast = \omega/2 \). Next, consider \( \gamma(\omega) \). The radial function reaches its minimum at \( h\omega, \ h \in \mathbb{Z} \). Thus, if \( s_\ast \in [\omega/2, \omega/2 + n\omega] \) satisfies \( \gamma(s_\ast) = \gamma(\omega) \), then \( s_\ast = p\omega \), for some integer \( p = 0, ..., n \). In addition, \( \theta(s_\ast) \equiv \theta(\omega) = 2\pi m/n \), \( \text{mod} \ 2\pi \mathbb{Z} \), ie \( \theta(s_\ast) = 2\pi(m/n + k), \ k \in \mathbb{Z} \). On the other hand, \( \theta(s_\ast) = \theta(p\omega) = 2\pi pm/n \). Then, \( m(p - 1) = nk \). Thus, \( p - 1 = hn \), \( h \in \mathbb{Z} \). Since \( 1 \leq p \leq n \), the only option is \( p = 1 \). This proves that \( s_\ast = \omega \).
To prove the other assertions, let $k_1$ and $k_2$ be the integers such that
\[ k_1n + k_2m = 1, \]
whose existence is guaranteed because $m$ and $n$ are relative prime.

Consider first that $e_1 < u_{\lambda}$. The angular function is strictly decreasing on $(\omega/2, \omega)$ and $\theta(\omega/2) = 0$ while $\theta(\omega) = -\pi(n - m)/n$. Thus, for every $j = 1, \ldots, n - m - 1$ there exists a unique $s_j \in (\omega/2, \omega)$ such that $\theta(s_j) = -\pi j/n$. We are going to prove that, for every $k = 0, \ldots, n - 1$, $\gamma(s_j - k\omega)$ is a multiple point. For this purpose, consider
\[ \hat{s}_j = -s_j - (k + jk_2)\omega. \]
The height and radial functions are even and periodic, with period $\omega$. Then, $\rho(s_j) = \rho(\hat{s}_j)$ and $h(s_j) = h(\hat{s}_j)$. Taking into account that $\theta$ is an odd function and using (13) we have
\[
\theta(\hat{s}_j) = \theta(s_j) - 2\pi (k + jk_2) \left( \frac{m}{n} \right) = \pi j/n - 2\pi \frac{j - k_1}{n} - 2\pi \frac{k m}{n} \]
This implies that $\gamma(s_j) = \gamma(\hat{s}_j)$. It remains to prove that $s_j \not\equiv \hat{s}_j \pmod{n\omega}$. By contradiction, suppose $s_j \equiv \hat{s}_j \pmod{n\omega}$, then there exists a $p \in \mathbb{Z}$ such that $2s_j = p\omega$, but this is impossible because $s_j \in (\omega/2, \omega)$. We next prove that the points $\gamma(s_j - k\omega)$, $1 \leq j \leq n - m - 1$ and $0 \leq k \leq n - 1$ are distinct. Let $0 \leq \hat{k} < k \leq n - 1$, then
\[
\theta(s_j - k\omega) = -\pi j/n - 2\pi \frac{km}{n} = \theta(s_j - \hat{k}\omega) - 2\pi \frac{m(k - \hat{k})}{n}.
\]
Since $m$ and $n$ are relative prime, $m(k - \hat{k})/n$ is an integer number if and only if $k - \hat{k}$ is an integer multiple of $n$. On the other hand, it is clear that $1 \leq k - \hat{k} < n - 1$. Hence, $m(k - \hat{k})/n \notin \mathbb{Z}$ and so $\theta(s_j - k\omega) \neq \theta(s_j - \hat{k}\omega) \pmod{2\pi}$. This implies that $\gamma(s_j - k\omega) \neq \gamma(s_j - \hat{k}\omega)$.

Suppose now that $1 \leq j < \hat{j} \leq n - m - 1$, then $\theta(s_j) = -\pi j/n > -\pi \hat{j}/n = \theta(s_{\hat{j}})$. Since $\theta$ is strictly decreasing, we have $\omega/2 < s_j < s_{\hat{j}} < \omega$. On the other hand, $\rho$ is also strictly decreasing on $(\omega/2, \omega)$, so $\rho(s_j - k\omega) \neq \rho(s_{\hat{j}}) = \rho(s_j - \hat{k}\omega)$, for every $k, \hat{k} = 0, \ldots, n - 1$.

In conclusion, $\gamma(s_j - k\omega) \neq \gamma(s_{\hat{j}} - \hat{k}\omega)$, for every $k, \hat{k} = 0, \ldots, n - 1$, and it then follows that the points $\gamma(s_j - k\omega)$, $1 \leq j \leq n - m - 1$, $0 \leq k \leq n - 1$ are distinct. This proves that $\gamma$ possesses, at least, $n(n - m - 1)$ points of self-intersection.

We finally conclude from the first assertion and from the fact that the angular function is strictly decreasing that the points constructed above are the only multiple points of $\gamma$.

In what follows, we analyze the case $e_1 > u_{\lambda}$. Denote by $\hat{s}, \tilde{s} \in (\omega/2, \omega)$ the absolute minimum and absolute maximum, respectively, of $\theta$ on the closed interval $[\omega/2, \omega]$. Then, for every $j = 1, \ldots, m$ there exists a unique $s_j \in (\hat{s}, \tilde{s})$ such that $\theta(s_j) = \pi j/n$. Then, arguing as in the case $e_1 < u_{\lambda}$, we prove that $\gamma(s_j - k\omega)$, $j = 1, \ldots, m$ and $k = 0, \ldots, n - 1$ are distinct multiple points of $\gamma$. However, contrary to the case $e_1 < u_{\lambda}$, these may not be the only multiple points of $\gamma$ since the angular function has a unique absolute maximum at $s_* \in [\omega/2, \omega]$ such that $\theta(s_*) > \theta(\omega) = \pi m/n$, and we have three different scenarios:
(a) Case $\theta(s_\ast) < \pi(m+1)/n$. In this case the B-string has exactly $nm$ points of self-intersection. These points are the ones constructed above. (See Figure 12 of Subsection 5.4.)

(b) Case $\pi(m+k)/n < \theta(s_\ast) < \pi(m+k+1)/n$. In this case the B-string has exactly $n(m+2k)$ points of self-intersection. All of them are ordinary double points. The first $nm$ multiple points are the ones constructed in the proof, while the others arise as the two solutions of the equations $\theta(s) = \pi(m+p)/n$ with $p = 1, \ldots, k$. (See Figure 16 of Subsection 5.6.)

(c) Case $\pi(m+k)/n < \theta(s_\ast) = \pi(m+k+1)/n$. In this case the B-string has exactly $n(m+2k)$ points of self-intersection. The first $nm$ multiple points are the ones constructed in the proof, while the remaining $n$ points are points of tangential self-intersection. More precisely, these points are, exactly, $\gamma(s_\ast + p\omega)$, $p = 0, \ldots, n-1$. (See Figure 14 of Subsection 5.5.)

This finishes the proof. □

A similar argument may be used to obtain the analogue result for the case $e_1 = u\lambda$, which we state in the following remark.

**Remark 3.6** Let $\gamma$ be a B-string with multiplier $\lambda$ and characteristic number $q = m/n$. Assume that $e_1 = u\lambda$ and let $k$ be the largest natural number relatively prime with $n$ and such that $2k < n$. Then, $\gamma$ has a multiple point of multiplicity $n$ and $n(k-m)$ ordinary double points (see Figures 8 and 10 of Subsections 5.2 and 5.3, respectively).

We are now in the right position to prove Theorem 1.3, which we state here in a more technical form.

**Theorem 3.7** Let $\gamma$ be a B-string with parameters $(\lambda, e_1) \in P$ and characteristic number $q = m/n$. Then, the following conclusions hold true:

1. The trajectory of $\gamma$ is invariant by the group generated by the rotation $2\pi/n$ around the Ox-axis and $\gamma$ is contained in the spherical region bounded by the planes $x = 1/(2\xi e_1)$ and $x = 1/(2\xi e_2)$.

2. If $e_1 \neq u\lambda$, then $\gamma$ does not intersect the Ox-axis. Moreover:

   (a) If $e_1 < u\lambda$, $n - m$ is the linking number with the Ox-axis (equipped with the upward orientation) and $\gamma$ possesses, exactly, $n(n-m-1)$ ordinary double points.

   (b) If $e_1 > u\lambda$, $-m$ is the linking number with the Ox-axis (equipped with the upward orientation) and $\gamma$ possesses, at least, $nm$ points of self-intersection.

3. If $e_1 = u\lambda$, then $\gamma$ intersects the Ox-axis $n$ times and the moving point $\gamma(s)$ travels counterclockwise around the Ox-axis (equipped with the upward orientation). Moreover, $n - m$ is the turning number of the plane projection of $\gamma$ to the plane $x = 0$.

**Proof.** The first assertion is trivial. In fact, the stabilizer of the trajectory of a B-string is the subgroup generated by the monodromy, that is the group generated by the rotation of an angle $2\pi/n$ around the Ox-axis. Moreover, from Theorem 3.1 it follows that

$$\gamma(s) = (h(s), -\rho(s)\cos\theta(s), \rho(s)\sin\theta(s)),$$
where \( h(s) \) is defined in (10). As customary in our proofs, we are avoiding to write the subscripts \( \lambda \) and \( e_1 \), for simplicity in the expressions. Since \( h(s) > 0 \), the trajectory of the curve is clearly contained in \( \mathbb{S}^2 \{ (x, y, z) \in \mathbb{S}^2 | x > 0 \} \). Similarly, since \( e_2 \leq \mu(s) \leq e_1 \), from the definition of \( h(s) \), one also concludes that the trajectory of the curve lies in the spherical region bounded by the planes of the statement.

Suppose now that \( e_1 \neq u_\lambda \). In this case, the radial function \( \rho \) defined in (9) is strictly positive and so the linking number \( \text{Lk}(\gamma, Ox) \) is the winding number of \( \widehat{\gamma} = (-\rho \cos \theta, \rho \sin \theta) \) around the origin, that is, the degree of the circle map

\[
s \in \mathbb{R} \setminus n\omega\mathbb{Z} \cong \mathbb{S}^1 \longmapsto (-\cos \theta, \sin \theta) \in \mathbb{S}^1,
\]

which is equal to

\[
-\frac{n}{2\pi} \left( \theta \left[ \frac{\omega}{2} + \omega \right] - \theta \left[ \frac{\omega}{2} \right] \right) = -m.
\]

If \( e_1 > u_\lambda \), we have \( \theta(\omega/2 + \omega) > \theta(\omega/2) \) and, if \( e_1 < u_\lambda \), \( \theta(\omega/2 + \omega) < \theta(\omega/2) \). This proves the second statement (the number of points of self-intersection follows from Lemma 3.3).

Finally, suppose that \( e_1 = u_\lambda \), then \( \gamma(s) \) passes through the pole \((1, 0, 0)\) if and only if \( s \) is a zero of the radial function \( \rho \). In other words, if and only if, \( s \equiv 0 \pmod{\omega} \). This implies that \( \gamma \) crosses the \( Ox \)-axis when \( s = 0, \ldots, (n - 1)\omega \), i.e., \( n \) times.

Recall that in this exceptional case

\[
\Psi(e_1) = \theta(\omega) - \theta(0) = 2\pi \left( q - \frac{1}{2} \right) = 2\pi \left( \frac{m}{n} - \frac{1}{2} \right),
\]

where \( q = m/n \) is the characteristic number of \( \gamma \), and we denote by \( \hat{m}/\hat{n} = q - 1/2 \). The plane projection of \( \gamma \) is the “polar” curve

\[
\widehat{\gamma}(s) = \rho(s) (-\cos \theta(s), \sin \theta(s)).
\]

The radial and angular functions satisfy \( \theta(s + \omega) = \theta(s) = 2\pi \hat{m}/\hat{n} \) and \( \rho(s + \omega) = -\rho(s) \). Thus, if \( \hat{n} = 2k \) and \( k \) is an odd integer, the least period of \( \widehat{\gamma}, \omega_{\#} \), is \( k\omega \), while if \( \hat{n} = 2k \) and \( k \) is even, the least period of \( \widehat{\gamma}, \omega_{\#} \), is \( n\omega \). Similarly, if \( \hat{n} \) is odd, \( \omega_{\#} = 2\hat{n}\omega \). Let \( \tilde{T} \) and \( \tilde{N} \) be the unit tangent and unit normal vector fields along \( \widehat{\gamma} \). Then

\[
\tilde{N} \cdot d\tilde{T} = -\frac{2\rho^2 \dot{\theta} + \rho^2 \ddot{\rho} - \rho \ddot{\rho} \dot{\theta} + \rho \dot{\rho} \ddot{\theta}}{\dot{\rho}^2 + \rho^2 \dot{\theta}^2} ds.
\]

This implies that

\[
\frac{1}{2\pi} \int \tilde{N} \cdot d\tilde{T} = \frac{1}{2\pi} \left( f(s) - \theta(s) \right) + c, \tag{14}
\]

where \( f \) is the continuous determination of \(-\arctan(\rho \dot{\theta}/\dot{\rho})\) such that \( f(0) = 0 \). Considering the properties of the radial and angular functions, \( \arctan(\rho \dot{\theta}/\dot{\rho}) \) possesses jump discontinuities at the points \( p_k = \omega/2 + k\omega, k \in \mathbb{Z} \), and is real-analytic elsewhere. At the points of discontinuity we have

\[
\lim_{s \to p_k^+} \arctan \frac{\rho \dot{\theta}}{\dot{\rho}} = \frac{\pi}{2},
\]

\[
\lim_{s \to p_k^+} \arctan \frac{\rho \dot{\theta}}{\dot{\rho}} = -\frac{\pi}{2}.
\]

Consequently, \( f \) is a quasi-periodic function with quasi-period \( \omega \) such that \( f(\omega) - f(0) = \pi \). Then, we deduce the following properties from (14):
1. If \( \hat{n} = 2k \) and \( k \) is an odd integer, then \( 2m = \hat{m} + k \), \( n = k \) and \((\omega_{\gamma} = k\omega)\)
\[
\frac{1}{2\pi} \int_{0}^{k\omega} \tilde{N} \cdot d\tilde{T} = -\frac{\hat{m}}{2} + \frac{k}{2} = n - m.
\]
2. If \( \hat{n} = 2k \) and \( k \) is an even integer, then \( m = \hat{m} + k \), \( n = \hat{n} \) and \((\omega_{\gamma} = \hat{n}\omega)\)
\[
\frac{1}{2\pi} \int_{0}^{\hat{n}\omega} \tilde{N} \cdot d\tilde{T} = \frac{\hat{n}}{2} - \hat{m} = n - m.
\]
3. If \( \hat{n} \) is an odd integer, \( m = 2\hat{m} + \hat{n} \), \( n = 2\hat{n} \) and \((\omega_{\gamma} = 2\hat{n}\omega)\)
\[
\frac{1}{2\pi} \int_{0}^{2\hat{n}\omega} \tilde{N} \cdot d\tilde{T} = \hat{n} - 2\hat{m} = n - m.
\]
This proves that the total curvature, i.e., the turning number of \( \tilde{\gamma} \) is \( n - m \), as claimed. \( \square \)

**Remark 3.8** Theorem 3.7 implies that the geometry of a B-string is encoded by its projection onto an annular region (disc-type region if the B-string intersects the Ox-axis) of the oriented plane through the origin and orthogonal to the symmetry axis. The multiple points of the B-string are projected onto the multiple points of the projected plane curve and, vice-versa, each multiple point of the projection gives rise to a multiple point of the string.

If \( e_1 \leq u_\lambda \), the projection is counter-clockwise oriented, while if \( e_1 > u_\lambda \) is clockwise oriented. Its symmetry group is the same as the one of the B-string. If \( e_1 \neq u_\lambda \), the linking number of the string is the homotopy class of the projection, viewed as a plane curve of the annular region.

We finish this section showing a possible application of the existence of B-strings.

Let \( X \) be an oriented, connected and compact surface without boundary. A smooth, generically one-to-one, immersion \( f : X \rightarrow S^3 \) of \( X \) into the 3-dimensional sphere of radius one \( S^3 \) is said to be weakly convex if its Gaussian curvature is non-negative and its mean curvature \( H \) (with respect to the orientation of \( X \)) is strictly positive.

Denote by \( C^\infty_*(X,S^3) \) the space of weakly convex immersions. We then say that \( f \) is a (constrained) B-surface with multiplier \( \lambda \in \mathbb{R} \), if it is a critical point of the functional
\[
B_\lambda : f \in C^\infty_*(X,S^3) \mapsto \int_{X} \left( \sqrt{\pi I} + \lambda \right) dA_f .\quad (15)
\]
This functional is the 2-dimensional analogue of the constrained Blaschke functional \( \Gamma \) for curves, and so we employ here the same notation for both of them.

Given a periodic arc-length parameterized curve \( \gamma : \mathbb{R} \rightarrow S^2 \) of length \( L_\gamma \) we consider the (spherical) Frenet frame field \( F : \mathbb{R} \rightarrow SO(3) \) along \( \gamma \), and we denote by \( \tilde{F} : \mathbb{R} \rightarrow SU(2) \) the lift of \( F \) to \( SU(2) \), the universal covering of \( SO(3) \). Identifying \( SU(2) \) with \( S^3 \), \( \tilde{F} \) originates a lift \( \tilde{\gamma} : \mathbb{R} \rightarrow S^3 \) of \( \gamma \) to the total space of the Clifford fibration \( S^3 \rightarrow S^2 \), called the spinor lift of \( \gamma \). We point out here that the Clifford fibration is sometimes referred as to the (standard) Hopf fibration.

The spinor lift of \( \gamma \) is unique up to the sign. Since, in our case, \( \kappa > 0 \), i.e., the curve will be convex, the spinor lifts are transversal to the contact distribution of \( S^3 \) and to the fibers of the Clifford fibration as well. The least period of \( \tilde{\gamma} \) can be either \( L_\gamma \) or \( 2L_\gamma \). We say that \( \gamma \) has Clifford spin 1 or 1/2, respectively, and we denote it by \( s_\gamma \).

Let \( \gamma \) be a B-string with characteristic number \( q = m/n \). Then:
1. If \( m + n \) is even, \( s_\gamma = 1 \); while, if \( m + n \) is odd, \( s_\gamma = 1/2 \).

2. If \( s_\gamma = 1/2 \), \( \hat{\gamma} \) is a negative torus knot of type \((n - m, -[n + m])\).

3. If \( s_\gamma = 1 \), \( \hat{\gamma} \) is a negative torus knot of type \([n - m]/2, -[n + m]/2\).

Consequently, a B-string with fixed Lagrange multiplier can be characterized by the Clifford spin and the knot type of its spinor lift.

The Hopf torus based on \( \gamma \) is the, generically one-to-one, immersion

\[
f_\gamma : (s, \varphi) \in \mathbb{R}^2 \setminus \mathbb{Z}_\gamma \mapsto e^{i\varphi}\hat{\gamma}(s) \in S^3,
\]

where \( \mathbb{Z}_\gamma \) is the lattice spanned by \((s_\gamma L_\gamma, 2\pi s_\gamma)\). If the curvature of \( \gamma \) is positive, \( f_\gamma \) is weakly convex. Moreover, by the Principle of Symmetric Criticality \((\text{PSC})\), the Hopf tori based on B-strings are B-surfaces. In this setting, Theorem 3.3 implies the existence of countably many B-surfaces of genus one with Lagrange multiplier \( \lambda \), for every \( \lambda \in \mathbb{R} \).

In view of Theorem 3.7 and of the ansatz on the monotonicity of the (regularized) jump function \( \hat{\Psi}_\lambda \), the Hopf torus based on a B-string with characteristic number \( q = m/n \) and \( e_1 \leq u_\lambda \) possesses \( n(n - m - 1) \) disjoint circles of ordinary double points; whereas, if \( e_1 > u_\lambda \), the number of disjoint circles of multiple points is, at least, \( nm \). See Section 5 for several illustrations.

4 Theoretical Aspects

In this section we comment on the theoretical aspects behind Theorems 3.1, 3.3 and 3.7 and on the rich “hidden” geometry surrounding the Blaschke variational problem, which is typical of variational problems related to non-commutative completely integrable Hamiltonian contact systems \((\mathbb{24, 28, 29, 36, 51})\).

4.1 The Phase Space

Let

\[
\Omega := \begin{pmatrix}
0 & -\bar{\omega}_0^1 & -\bar{\omega}_0^2 \\
\bar{\omega}_0^1 & 0 & -\bar{\omega}_0^2 \\
\bar{\omega}_0^2 & \bar{\omega}_0^1 & 0
\end{pmatrix},
\]

be the Maurer-Cartan form of \( SO(3) \). Then, \( \{\bar{\omega}_0^0, \bar{\omega}_0^2, \bar{\omega}_0^1\} \) is a basis for the space of left-invariant 1-forms. Using the algorithmic procedure illustrated in \(\mathbb{28, 90}\), the momentum space of the functional \( B_\lambda \) is the 5-dimensional submanifold \( M \) of \( T^*[SO(3)] \) defined by the embedding

\[
(A, \mu, \dot{\mu}) \in SO(3) \times \mathbb{R}^+ \times \mathbb{R} \mapsto \left(\frac{\mu}{2} + \lambda\right) \bar{\omega}_0^1|_A - \frac{\mu^2}{\mu^2} \bar{\omega}_0^2|_A + \frac{1}{2\mu} \bar{\omega}_0^1|_A \in T^*[SO(3)].
\]

The restriction of the Liouville form of \( T^*[SO(3)] \) to \( M \) is the 1-form

\[
\zeta = \left(\frac{\mu}{2} + \lambda\right) \bar{\omega}_0^1 - \frac{\mu^2}{\mu^2} \bar{\omega}_0^2 + \frac{1}{2\mu} \bar{\omega}_0^1.
\]
The 2-form $d\zeta$ has maximal rank and its characteristic line bundle is generated by the vector field

$$X = \partial_{\omega_0} + \mu^2 \partial_{\omega_1} + \dot{\mu} \partial_{\mu} + \left( \mu - 2\lambda \mu^4 - \mu^5 + 2\frac{\mu^2}{2} \right) \partial_{\dot{\mu}},$$

where $\left( \partial_{\omega_1}, \partial_{\omega_2}, \partial_{\omega_1}, \partial_{\mu}, \partial_{\dot{\mu}} \right)$ is the trivialization of $TM$ dual to the coframe $\left( \omega_0, \omega_1, \omega_2, d\mu, d\dot{\mu} \right)$.

The integral curves of $X$ are the canonical lifts of the B-curves (not necessarily in their standard form) with multiplier $\lambda$, that is curves of the following type

$$\Gamma : s \in \mathbb{R} \mapsto \left( (\gamma, \dot{\gamma}, \gamma \times \dot{\gamma}), \mu, \dot{\mu} \right) \in M,$$

where $\gamma$ is an arc-length parameterized B-curve with multiplier $\lambda$, curvature $\mu^2$ and where $\dot{\mu}$ is the derivative of $\mu$ with respect to the arc-length parameter. Thus, the problem has been reduced to the integration of the characteristic vector field of the $SO(3)$-invariant 2-form $d\zeta$.

### 4.2 The Moment Map

The moment map $\mathfrak{M} : M \rightarrow so(3)^*$ for the $SO(3)$-action on $(M, d\zeta)$ is the restriction of the moment map for the $SO(3)$-action on $T^*[SO(3)]$ equipped with its standard symplectic form. So, we get

$$\mathfrak{M} : (A, \mu, \dot{\mu}) \mapsto \operatorname{Ad}^*(A) \cdot H(\mu, \dot{\mu}) \in so(3)^*,$$

where

$$H(\mu, \dot{\mu}) = \left( \frac{\mu}{2} + \lambda \right) \omega_0 - \frac{\mu^2}{\mu^2} \omega_0^2 + \frac{1}{2\mu} \omega_1^2 \in so(3)^*.$$

Using the Killing form we identify $so(3)^*$ with $so(3)$, which is isomorphic to $\mathbb{R}^3$ equipped with the Lie algebra structure defined by the usual vector cross product. Modulo these identifications, the moment map can be written as

$$\mathfrak{M} : (A, \mu, \dot{\mu}) \in M \mapsto A \cdot \left( \frac{1}{2\mu}, -\frac{\mu}{2}, \frac{\mu}{2} + \lambda \right)^T \in \mathbb{R}^3.$$

By construction, $\mathfrak{M}$ is constant along the integral curves of $X$. This implies that if $\gamma$ is a B-curve with multiplier $\lambda$ and parameter $e_1$, then

$$\frac{1}{2\mu} \dot{\gamma} - \frac{\mu}{2} \dot{\gamma} + \left( \frac{\mu}{2} + \lambda \right) \gamma \times \dot{\gamma} = J,$$

is constant. Note that $||J|| = \xi$, where $\xi$ is the constant of integration defined in (6). The oriented line passing through the origin and parallel to $J$ is the axis of symmetry of $\gamma$. The element $J \in \mathbb{R}^3$ is the momentum of the B-curve. Identifying $\mathbb{R}^3$ with $so(3)$ and letting $J^*$ be the corresponding fundamental vector field of $S^2$, then $J^*$ is the sum of the (adapted) Killing vector fields along $\gamma$ which arise in the Lagrangian approach (10).

The momentum map has maximal rank at each point of $M$ and its image is the open set

$$\Upsilon_\lambda := \{ J \in \mathbb{R}^3 | ||J|| > \frac{1}{2\eta_\lambda} \sqrt{1 + (\eta_\lambda |\eta_\lambda + 2\lambda|)^2} \} \subset \mathbb{R}^3,$$

where $\eta_\lambda$ is defined in Remark 2.1.
If $\mathcal{J} \in \Upsilon_{\lambda}$, the quartic polynomial

$$Q_{\mathcal{J}}(t) = t^4 + 4\lambda t^3 + 4\left(\lambda - \|\mathcal{J}\|^2\right) t^2 + 1,$$

possesses exactly two distinct positive real roots, $e_1 > e_2 > 0$. The integral curves of $X$ with momentum $\mathcal{J}$ are, precisely, the canonical lifts of B-curves with multiplier $\lambda$ and parameter $e_1$.

The phase portrait $\mathcal{C}_{\mathcal{J}}^*$ of $\mathcal{J}$ is the connected component of the singular algebraic curve

$$C_{\mathcal{J}} := \{(x, y) \in \mathbb{R}^2 | y^2 = -x^2Q_{\mathcal{J}}(x)\} \subset \mathbb{R}^2,$$

contained in the half-plane $x > 0$. It turns out that $\mathcal{C}_{\mathcal{J}}^*$ is a smooth curve and $C_{\mathcal{J}} \otimes \mathbb{C}$ is the affine part of a singular algebraic curve of $\mathbb{CP}^2$. Such a curve is, in general, elliptic and it is rational if and only if either $\lambda = 0$ or $\lambda < 0$ and $(e_1 + e_2)^2 = 4e_1^2e_2^3$.

### 4.3 The Marsden-Weinstein Reduction and the Arnold Connection

Let $\mathcal{J} \in \Upsilon_{\lambda}$. The (Marsden-Weinstein) reduced space of $\mathcal{J}$ is the 2-dimensional torus $O_{\mathcal{J}} := \mathfrak{M}^{-1}(\mathcal{J}) \subset M$. This torus is invariant by the action of the stabilizer $S_{\mathcal{J}} := \{A \in SO(3) | A \cdot \mathcal{J} = \mathcal{J}\} \cong SO(2)$ and the vector field $X$ is tangent to $O_{\mathcal{J}}$.

The map $\pi_{\mathcal{J}} : (A, \mu, \dot{\mu}) \in O_{\mathcal{J}} \mapsto (\mu, \dot{\mu}) \in C_{\mathcal{J}}^*$ is $S_{\mathcal{J}}$-invariant and gives, on $O_{\mathcal{J}}$, the structure of a principal circle bundle. The line bundle of $T \tilde{O}_{\mathcal{J}}$ spanned by $X_{\mathcal{J}} := X|_{O_{\mathcal{J}}}$ defines a connection on the principal bundle $O_{\mathcal{J}} \mapsto \mathcal{C}_{\mathcal{J}}^*$.

### 4.4 Closure Conditions and Integrability by Quadratures

Let $e_1$ be the largest positive root of $Q_{\mathcal{J}}(x)$. Then, $(e_1, 0) \in C_{\mathcal{J}}^*$. Let $\Delta_{\mathcal{J}} \subset S_{\mathcal{J}}$ be the (discrete) holonomy group of the connection, with reference point $(e_1, 0)$. We have that $\Delta_{\mathcal{J}}$ is isomorphic to the monodromy of a B-curve with multiplier $\lambda$ and parameter $e_1$. From a theoretical point of view, the closure condition for a critical curve can be rephrased as follows.

**Remark 4.1** A B-curve with momentum $\mathcal{J}$ is periodic if and only if $\Delta_{\mathcal{J}}$ is finite.

Let $R = (R_1, R_2, R_3) \in SO(3)$ be a positively oriented orthogonal basis such that

$$\mathcal{J} = \frac{1}{2e_1}R_1 + \left(\frac{e_1}{2} + \lambda\right) R_3,$$

and $\tilde{O}_{\mathcal{J}} \subset O_{\mathcal{J}}$ be the holonomy bundle of the connection passing through $(R, e_1, 0)$. Then:

1. The map $\tilde{O}_{\mathcal{J}} \mapsto O_{\mathcal{J}}$ is a covering map with deck transformation group $\Delta_{\mathcal{J}}$.
2. If $\Delta_{\mathcal{J}}$ is finite, $\tilde{O}_{\mathcal{J}} \cong S^1$. Otherwise, $\tilde{O}_{\mathcal{J}} \cong \mathbb{R}$.
3. The set $\tilde{O}_{\mathcal{J}}$ is an integral curve of $X$ and, hence, $(A, \mu, \dot{\mu}) \in \tilde{O}_{\mathcal{J}} \mapsto A_1 \in S^2$ is a B-curve with multiplier $\lambda$ and momentum $\mathcal{J}$.

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We next explain how to find the parameterization of a B-curve given in Theorem 3.1. Let \((\lambda, e_1) \in \mathcal{P}\) and

\[
J \equiv J_{\lambda, e_1} = \xi \vec{i} = \frac{1}{2e_1} \sqrt{1 + (e_1 [e_1 + 2\lambda])^2} \vec{i}.
\]

Assume that \(\mu : \mathbb{R} \rightarrow \mathbb{R}^+\) is a (periodic) solution to the Cauchy problem

\[
\begin{align*}
\dot{\mu}^2 &= -\mu^2 Q_J(\mu) \\
\mu(\pi/2) &= e_1.
\end{align*}
\]

We denote by \(\varphi = (\mu, \dot{\mu}) : \mathbb{R} \rightarrow \mathbb{C}^* J\) the parameterization of the phase portrait. Choose any cross section of \(\varphi^*(O_J) \mapsto \mathbb{R}\). This amounts to find \(A : \mathbb{R} \rightarrow SO(3)\) satisfying

\[
A \cdot \left( \frac{1}{2\mu}, \frac{\dot{\mu}}{2\mu^2}, \frac{\mu^2 + \lambda}{2} \right)^T = (\xi, 0, 0).
\]

Such a map can be easily found by elementary linear algebra. For instance,

\[
A = \begin{pmatrix}
-\frac{\sqrt{2 e_1^2 \mu^2 - 1}}{2\xi \mu^2} & -\frac{\sqrt{2 e_1^2 \mu^2 - 1}}{2\xi \mu^2} & \frac{2\lambda + \mu}{2e_1^2} \\
0 & -\frac{\sqrt{2 e_1^2 \mu^2 - 1}}{2\xi \mu^2} & \frac{2\lambda + \mu}{2e_1^2} \\
0 & \frac{\sqrt{2 e_1^2 \mu^2 - 1}}{2\xi \mu^2} & -\frac{\mu^2 + \lambda}{2e_1^2}
\end{pmatrix}
\]

satisfies above equation. Clearly, any other cross section is of the form

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \Psi & -\sin \Psi \\
0 & \sin \Psi & \cos \Psi
\end{pmatrix} \cdot A.
\]

Parallel sections of the connection are solutions of

\[
\dot{B} = B \cdot \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -\mu^2 \\
0 & \mu^2 & 0
\end{pmatrix}.
\]

It follows from the expression of \(A\) and \(B\), that \(B\) is, precisely, a parallel section if and only if

\[
\Psi = 2\xi \int \frac{\mu^2(\mu + \lambda)}{1 - 4\xi^2 \mu^2} ds.
\]

Hence, the map \(s \mapsto (B(s), \mu(s), \dot{\mu}(s))\) is a parallel cross section and \(s \mapsto B_1(s)\) is the standard configuration of a B-curve with parameters \((\lambda, e_1)\) exhibited in Theorem 3.1.

We finish this section by reformulating Theorems 3.3 and 3.7 in terms of the momentum.

Let \(\lambda \in \mathbb{R}\) and \(\Sigma_\lambda := \text{Im}(\hat{\Psi}_\lambda) \cap \mathbb{Q}\) (which is a countable set containing \((1 + p(\lambda), 1/2) \cap \mathbb{Q}\)). The function \(\xi\) is a strictly increasing real-analytic diffeomorphism of \((\eta_\lambda, \infty)\) onto \((\hat{\eta}_\lambda, \infty)\) where

\[
\hat{\eta}_\lambda := \frac{1}{2\eta_\lambda} \sqrt{1 + (\eta_\lambda [\eta_\lambda + 2\lambda])^2}.
\]

Then, we may choose \(\xi\) as a fundamental parameter and express \(e_1\) and \(\hat{\Psi}_\lambda\) as functions of \(\xi \in (\hat{\eta}_\lambda, \infty)\).

Let \(\xi^*_\lambda \in (\hat{\eta}_\lambda, \infty\] be defined by \(u_\lambda = e_1(\xi^*_\lambda)\) (if \(\lambda < 0\), then \(\xi^*_\lambda = -1/(4\lambda)\), while if \(\lambda \geq 0\), we set \(\xi^*_\lambda = \infty\)). Note that there exist countably many \(\lambda < 0\) such that \(\hat{\Psi}_\lambda(\xi^*_\lambda) \in \Sigma_\lambda\).
Remark 4.2  With this notation the results of Theorems 3.3 and 3.7 can be reformulated as follows:

1. An arc-length parameterized curve $\gamma$ is a B-string with multiplier $\lambda$ if and only if
   \[ \frac{1}{2\mu} \gamma - \frac{\mu}{2\mu^2} \dot{\gamma} + \left( \frac{\mu}{2} + \lambda \right) \gamma \times \dot{\gamma} = J, \]
   is constant and $\hat{\Psi}_\lambda(\|J\|) \in \Sigma_\lambda$.

2. For every $J$ such that $\hat{\Psi}_\lambda(\|J\|) \in \Sigma_\lambda$, there exists a B-string with momentum $J$. Two B-strings are equivalent if and only if their momenta have the same length.

3. Let $\hat{\Psi}_\lambda(\|J\|) = m/n \in \Sigma_\lambda$. Then, the stabilizer of a B-string with momentum $J$ is generated by the rotation around $J$ of an angle $2\pi/n$. Moreover:
   (a) If $\|J\| < \xi_\lambda^\ast$, $n - m$ is the linking number of the B-string with the oriented axis $A_J := \{O + tJ | t \in \mathbb{R}\}$. In this case, the B-string possesses, exactly, $n(n - m - 1)$ ordinary double points.
   (b) If $\|J\| = \xi_\lambda^\ast$, the B-string turns counter-clockwise around $A_J$ and intersects this axis $n$ times. The turning number of the plane projection of the B-string to the oriented plane through the origin and orthogonal to $A_J$ is $n - m$.
   (c) If $\|J\| > \xi_\lambda^\ast$, $-m$ is the linking number of the B-string with the axis $A_J$. In this case, the B-string possesses, at least, $nm$ points of self-intersection. (See Lemma 3.5 for more details about the intersection points.)

Remark 4.3  Assuming the ansatz that $\hat{\Psi}_\lambda$ is strictly increasing, then $\Sigma_\lambda = (1 + p(\lambda), 1/2) \cap \mathbb{Q}$ and, for every $m/n \in \Sigma_\lambda$ there exists a unique equivalence class of B-strings with multiplier $\lambda$ such that $\hat{\Psi}_\lambda(\|J\|) = m/n$.

5 Examples

In this section we will consider several examples which illustrate all the theoretical findings of previous sections.

5.1 Case $\eta_\lambda < e_1 < u_\lambda$:

In this example we consider a B-string with multiplier $\lambda = 1.1$ and characteristic number $q = 2/5$ ($n = 5$ and $m = 2$). Since $\lambda > 0$ the B-string is of negative type (i.e., $e_1 < u_\lambda = \infty$).

According to Theorems 3.3 and 3.7, the B-string has a counter-clockwise five-fold symmetry; its linking number with the (upward oriented) $Ox$-axis is $n - m = 3$ (which coincides with the winding number and the turning number of the plane projection), and it possesses $n(n - m - 1) = 10$ ordinary double points. In Figure 6, we show the corresponding B-string, its plane projection and its associated phase portrait, where we illustrate these properties.

Note that since $n + m = 7$ is odd the Clifford spin of this B-string is $1/2$. In Figure 7, we show the stereographic projection of its associated Hopf torus, a piece of it (to observe the self-intersecting circles) and the spinor lift of the B-string, which is a negative torus knot of type $(n - m, -[n + m]) = (3, -7)$.
Figure 6: A B-string of negative type \((e_1 < u_\lambda)\) with multiplier \(\lambda = 1.1\) and characteristic number \(q = 2/5\), together with its plane projection and phase portrait.

Figure 7: Stereographic projection of the Hopf torus associated to the B-string of negative type \((e_1 < u_\lambda)\) with multiplier \(\lambda = 1.1\) and characteristic number \(q = 2/5\), a piece of it and the spinor lift of this B-string.

5.2 Case \(e_1 = u_\lambda\) (without ordinary double points):

In this example we consider an exceptional B-string with multiplier \(\lambda \simeq -0.11\) and characteristic number \(q = 4/9\) \((n = 9\) and \(m = 4\)).

Observe in Figure 8 that, as stated in Theorems 3.3 and 3.7, the B-string possesses a counterclockwise nine-fold symmetry, the turning number of the plane projection is \(n - m = 5\) and that it possesses a multiple point (at the pole \((1,0,0)\)) of multiplicity \(n = 9\). Observe that the largest natural number \(k\) relatively prime with \(n = 9\) and such that \(2k < n\) is \(k = 4\) and so, we conclude from Remark 3.6, that this B-string has \(n(k - m) = 0\) ordinary double points.

From \(n + m = 13\), we conclude that the Clifford spin of this B-string is \(1/2\). In Figure 9 we show the stereographic projections of: the associated Hopf torus, a part of this torus and the spinor lift of the B-string, which represents a negative torus knot of type \((n - m, -|n + m|) = (5, -13)\).

5.3 Case \(e_1 = u_\lambda\) (with \(n(k - m)\) ordinary double points):

In this example we consider an exceptional B-string with multiplier \(\lambda \simeq -0.45\) and characteristic number \(q = 2/9\) \((n = 9\) and \(m = 2\)).
Figure 8: An exceptional \((e_1 = u\lambda)\) B-string with multiplier \(\lambda \simeq -0.11\) and characteristic number \(q = 4/9\), together with its plane projection and phase portrait.

Figure 9: Stereographic projection of the Hopf torus associated to the exceptional \((e_1 = u\lambda)\) B-string with multiplier \(\lambda \simeq -0.11\) and characteristic number \(q = 4/9\), a piece of it and the spinor lift of this B-string.

In Figure 10 we can see that, as stated in Theorems 3.3 and 3.7, the B-string possesses a counter-clockwise nine-fold symmetry, the turning number of the plane projection is \(n - m = 7\) and that it possesses a multiple point (at the pole \((1,0,0)\)) of multiplicity \(n = 9\). Moreover, the B-string also possesses \(n(k - m) = 18\) ordinary double points, since in this case the largest natural number \(k\) relatively prime with \(n = 9\) and such that \(2k < n\) is \(k = 4\) (see Remark 3.6).

The Clifford spin of this B-string is \(1/2\) since \(n + m = 11\) is odd. In Figure 11 we show the stereographic projections of: the associated Hopf torus, a part of this torus and the spinor lift of the B-string, which represents a negative torus knot of type \((n - m, -|n + m|) = (7, -11)\).

5.4 Case \(e_1 > u\lambda\) (with \(nm\) ordinary double points):

In this example we consider a B-string of positive type with multiplier \(\lambda = -0.5\) and characteristic number \(q = 3/8\) \((n = 8\) and \(m = 3)\). By positive type, we mean that \(e_1 > u\lambda\) holds.

Observe in Figure 12 that, as stated in Theorems 3.3 and 3.7, the B-string possesses a clockwise eight-fold symmetry, the winding number (also, the turning number) of the plane projection is \(-m = -3\) and it possesses \(nm = 24\) ordinary double points.
Figure 10: An exceptional \( (e_1 = u \lambda) \) B-string with multiplier \( \lambda \simeq -0.45 \) and characteristic number \( q = 2/9 \), together with its plane projection and phase portrait.

Figure 11: Stereographic projection of the Hopf torus associated to the exceptional \( (e_1 = u \lambda) \) B-string with multiplier \( \lambda \simeq -0.45 \) and characteristic number \( q = 2/9 \), a piece of it and the spinor lift of this B-string.

Figure 12: A B-string of positive type \( (e_1 > u \lambda) \) with multiplier \( \lambda = -0.5 \) and characteristic number \( q = 3/8 \), together with its plane projection and phase portrait.

The Clifford spin of this B-string is 1/2 since \( n + m = 11 \) is odd. In Figure 13 we show the stereographic projections of: the associated Hopf torus, a part of this torus and the spinor lift of the B-string, which represents a negative torus knot of type \( (n - m, -[n + m]) = (5, -11) \).
Figure 13: Stereographic projection of the Hopf torus associated to the B-string of positive type $e_1 > u_\lambda$ with multiplier $\lambda = -0.5$ and characteristic number $q = 3/8$, a piece of it and the spinor lift of this B-string.

5.5 Case $e_1 > u_\lambda$ (with tangential double points):

In this example we consider a B-string of positive type with multiplier $\lambda = -0.5$ and characteristic number $q = 3/8$ ($n = 11$ and $m = 3$).

This B-string possesses a clockwise eleven-fold symmetry, the winding number (also, the turning number) of the plane projection is $-m = -3$ and it possesses $nm = 33$ ordinary double points and $n = 11$ points of tangential self-intersection.

Figure 14: A B-string of positive type $(e_1 > u_\lambda)$ with multiplier $\lambda = -0.5$ and characteristic number $q = 3/11$, together with its plane projection and phase portrait.

It follows from $n + m = 14$ that the Clifford spin of this B-string is 1. In Figure 15 we show the stereographic projections of: the associated Hopf torus, a part of this torus and the spinor lift of the B-string, which represents a negative torus knot of type $([n - m]/2, -[n + m]/2) = (4, -7)$.

5.6 Case $e_1 > u_\lambda$ (with $n(m + 2k)$ ordinary double points):

In this example we consider a B-string of positive type with multiplier $\lambda = -0.5$ and characteristic number $q = 2/9$ ($n = 9$ and $m = 2$).
Observe in Figure 16 that, as stated in Theorems 3.3 and 3.7, the B-string possesses a clockwise nine-fold symmetry, the winding number (also, the turning number) of the plane projection is $-m = -2$ and that it possesses $n(m + 2k) = 36$ ordinary double points, since $k = 1$.

The Clifford spin of this B-string is $\frac{1}{2}$. In Figure 17 we show the stereographic projections of: the associated Hopf torus, a part of this torus and the spinor lift of the B-string, which represents a negative torus knot of type $(n - m, -[n + m]) = (7, -11)$.

**Appendix A. The Complete Elliptic Integral $\Psi_\lambda$**

In this appendix we will decompose the integral $\Psi_\lambda$ and compute its limits as $e_1$ approaches the boundaries of its domain.

Using (4) to make a change of variable in the definition of $\Psi_\lambda$, (11), we have

\[
\Psi_\lambda(e_1) = 2\xi \int_0^\infty \frac{\mu^2 (\mu + 2\lambda)}{1 - 4\xi^2 \mu^2} \, ds = 4\xi \int_{e_2}^{e_1} \frac{\mu (\mu + 2\lambda)}{(1 - 4\xi^2 \mu^2)\sqrt{-(\mu - e_1)(\mu - e_2)(\mu - e_3)(\mu - e_4)}} \, d\mu,
\]
Figure 17: Stereographic projection of the Hopf torus associated to the B-string of positive type \((e_1 > u_\lambda)\) with multiplier \(\lambda = -0.5\) and characteristic number \(q = 2/9\), a piece of it and the spinor lift of this B-string.

where \(e_1 > e_2 > 0\) are the only two positive roots of the quartic polynomial \(Q\), \(\[5\]\). This integral can be solved in terms of complete elliptic integrals of the first and third kind.

Let \(K(\delta)\) and \(\Pi(\zeta, \delta)\) be the complete elliptic integrals of the first and third kind, respectively, defined as the meromorphic extensions of

\[
K(\delta) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - \delta \sin^2 t}}, \quad \Pi(\zeta, \delta) = \int_0^{\pi/2} \frac{dt}{(1 - \zeta \sin^2 t)\sqrt{1 - \delta \sin^2 t}}.
\]

The function \(K(\delta)\) has a branch cut discontinuity in the complex plane running from 1 to \(\infty\), while \(\Pi(\zeta, \delta)\) has branch cut discontinuities at \(\Im(\zeta) = 0\) and \(\Re(\zeta) > 1\), and at \(\Im(\delta) = 0\) and \(\Re(\delta) > 1\).

For simplicity, we denote by

\[
\alpha \equiv \alpha(\lambda, e_1) = \frac{e_2 - e_1}{e_2 - e_4}, \quad \beta \equiv \beta(\lambda, e_1) = \frac{2}{\sqrt{(e_1 - e_3)(e_2 - e_4)}}, \quad \delta \equiv \delta(\lambda, e_1) = \frac{(e_1 - e_2)(e_3 - e_4)}{(e_1 - e_3)(e_2 - e_4)},
\]

and

\[
\zeta_+ \equiv \zeta_+(\lambda, e_1) = -\frac{(e_1 - e_2)(-1 + 2e_4\xi)}{(e_2 - e_4)(-1 + 2e_1\xi)}, \quad \zeta_- \equiv \zeta_-(\lambda, e_1) = -\frac{(e_1 - e_2)(1 + 2e_4\xi)}{(e_2 - e_4)(1 + 2e_1\xi)}.
\]

We then have

\[
\Psi_\lambda = I + \chi II + III,
\]

where \(\chi\) is the indicator function of the exceptional locus \(P\), and,

\[
I = \frac{\beta}{4\pi\xi} \left(-2 + \alpha \left[\frac{-1 - 4\lambda\xi}{\zeta_+(-1 + 2e_1\xi)} + \frac{1 - 4\lambda\xi}{\zeta_-(1 + 2e_1\xi)}\right]\right) K(\delta), \quad (16)
\]

\[
II = \frac{\beta(\alpha - \zeta_+)(1 + 4\lambda\xi)}{4\pi\zeta_+\zeta_-(-1 + 2e_1)} \Pi(\zeta_+, \delta), \quad (17)
\]

\[
III = \frac{\beta(\alpha - \zeta_+)(-1 + 4\lambda\xi)}{4\pi\zeta_+\zeta_-(-1 + 2e_1)} \Pi(\zeta_-, \delta). \quad (18)
\]

These formulas follow from three standard elliptic integrals. The first one (cf. 340.01 and 341.03 of [4]) is

\[
\int_0^{K(\delta)} \frac{1 - \sin^2(u, \delta)}{1 - b \sin^2(u, \delta)} du = \frac{a}{b} K(\delta) - \frac{a - b}{b} \Pi(b, \delta),
\]

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where $\text{sn}(u, \delta)$ is the Jacobi’s $\text{sn}$-function. The second elliptic integral (cf. 257 and 259 of [9]) is

$$
\int_{e_2}^{e_1} \frac{d\mu}{\sqrt{-(\mu-e_1)(\mu-e_2)(\mu-e_3)(\mu-e_4)}} = \beta K(\delta),
$$

where $\beta$ and $\delta$ are as above. The third relevant elliptic integral (cf. 257.39 and 259.04 of [9]) is

$$
\int_{e_2}^{e_1} \frac{d\mu}{(p-\mu)\sqrt{-(\mu-e_1)(\mu-e_2)(\mu-e_3)(\mu-e_4)}} = \frac{\beta}{p-e_1} \int_0^K (1 - \alpha \text{sn}^2(u, \delta)) \, du
$$

$$
= \frac{\beta}{p-e_1} \left( \frac{\alpha}{\zeta} K(\delta) - \frac{\alpha - \zeta}{\zeta} \Pi(\zeta, \delta) \right),
$$

where $p \neq e_1$, $\alpha$, $\beta$ and $\delta$ are as above, and

$$
\zeta = \frac{(p-e_4)(e_1-e_2)}{(e_1-p)(e_2-e_4)}.
$$

We begin proving that $\Psi_\lambda \to p(\lambda)$ when $e_1$ approaches $\eta_\lambda$ from the right. By construction we have that $e_2 \to \eta_\lambda$ too, and, hence, it follows that

$$
\begin{align*}
\lim_{e_1 \to \eta_\lambda} e_3(\lambda, e_1) &= \frac{-1 + \sqrt{1 - \eta^4_\lambda}}{\eta^2_\lambda}, \\
\lim_{e_1 \to \eta_\lambda} e_4(\lambda, e_1) &= \frac{-1 - \sqrt{1 - \eta^4_\lambda}}{\eta^2_\lambda},
\end{align*}
$$

and

$$
\lambda = \frac{1 - \eta^4_\lambda}{2\eta^3_\lambda}.
$$

We now see that the coefficients in (16)-(18) tend to, respectively,

$$
\lim_{e_1 \to \eta_\lambda^+} \alpha(\lambda, e_1) = 0, \quad \lim_{e_1 \to \eta_\lambda^-} \beta(\lambda, e_1) = \frac{2\eta_\lambda}{\sqrt{3 + \eta^2_\lambda}}, \quad \lim_{e_1 \to \eta_\lambda^-} \delta(\lambda, e_1) = 0,
$$

while

$$
\lim_{e_1 \to \eta_\lambda^+} \zeta(\lambda, e_1) = \frac{\sqrt{1 + \eta^4_\lambda}}{2\eta^3_\lambda}, \quad \lim_{e_1 \to \eta_\lambda^-} \zeta+(\lambda, e_1) = 0, \quad \lim_{e_1 \to \eta_\lambda^-} \zeta-(\lambda, e_1) = 0.
$$

Finally, recalling that $K(0) = \Pi(0, 0) = \pi/2$ and using above limits we conclude that

$$
\begin{align*}
\lim_{e_1 \to \eta_\lambda^+} I(\lambda, e_1) &= -\frac{\eta^4_\lambda}{\sqrt{1 + \eta^4_\lambda}}
\sqrt{3 + \eta^2_\lambda}, \\
\lim_{e_1 \to \eta_\lambda^-} II(\lambda, e_1) &= \frac{\eta^6_\lambda + 2\eta^3_\lambda \sqrt{1 + \eta^4_\lambda}}{2\sqrt{1 + \eta^2_\lambda} \sqrt{3 + \eta^2_\lambda} \left( \eta^2_\lambda - \sqrt{1 + \eta^4_\lambda} \right)}, \\
\lim_{e_1 \to \eta_\lambda^-} III(\lambda, e_1) &= \frac{\eta^6_\lambda - 2\eta^3_\lambda \sqrt{1 + \eta^4_\lambda}}{2\sqrt{1 + \eta^2_\lambda} \sqrt{3 + \eta^2_\lambda} \left( \eta^2_\lambda + \sqrt{1 + \eta^4_\lambda} \right)}.
\end{align*}
$$
which implies, after straightforward simplifications, that $\Psi_\lambda$ tends to $p(\lambda)$ when $e_1 \to \eta_\lambda^+$. 

In what follows, we prove the limit when $e_1 \to \infty$. This limit will depend on the sign of $\lambda$. More precisely, we will see that

$$\lim_{e_1 \to \infty} I(\lambda, e_1) = \lim_{e_1 \to \infty} III(\lambda, e_1) = 0,$$

and

$$\begin{cases} 
\lim_{e_1 \to \infty} II(\lambda, e_1) = -\frac{1}{2}, & \text{if } \lambda \geq 0 \\
\lim_{e_1 \to \infty} II(\lambda, e_1) = \frac{1}{2}, & \text{if } \lambda < 0.
\end{cases}$$

In order to prove these limits we observe that, as $e_1 \to \infty$, the following asymptotic estimates hold true:

$$e_2 \sim 1/e_1, \quad e_3 \sim -1/e_1, \quad e_4 \sim -e_1, \quad \xi \sim 1/(2e_1), \quad \beta \sim 2/e_1,$$

$$\delta \sim 1 - 4/e_1^2, \quad \alpha \sim (1 - e_1^2)/(1 + e_1^2), \quad \zeta_- \sim 1 - 4/e_1^2,$$

and

$$\begin{cases} 
\zeta_+ \sim 1 - \frac{4\lambda^2}{e_1^2}, & \text{if } \lambda \neq 0 \\
\zeta_+ \sim 1 - \frac{1}{e_1^2}, & \text{if } \lambda = 0.
\end{cases}$$

Moreover, recall that $K(\delta) \sim -\log(1 - \delta)/2$ as $\delta \to 1^-$ and so, in our case, we have

$$K(\delta(\lambda, e_1)) \sim -\frac{1}{2} \log \frac{4}{e_1^2},$$

as $e_1 \to \infty$. Combining this and above estimates we conclude that

$$I(\lambda, e_1) \sim -\frac{1}{\pi e_1^2} \log \frac{4}{e_1^2},$$

as $e_1 \to \infty$. This proves the first limit.

For the other limits we need some basic properties of the complete elliptic integral of the third kind $\Pi(\zeta, \delta)$. Let $\Lambda = \{ (\zeta, \delta) \in [0, 1] \times [0, 1) \mid \zeta \geq \delta \}$ and consider the function

$$f : (\zeta, \delta) \in \Lambda \mapsto \frac{2}{\pi} \sqrt{1 - \zeta} \sqrt{1 - \delta} \Pi(\zeta, \delta) \in \mathbb{R}.$$ 

This function $f$ is bounded below by $2/\pi$ and above by 1. In addition, $f(\zeta, 0) = 1$ for every $\zeta \in [0, 1)$ and $f(\zeta, \delta) \to 1$ when $\zeta \to 1^-$, for every value of $\delta \in [0, 1)$. Moreover, for every $\zeta \in [0, 1)$,

$$f(\zeta, \zeta) = \frac{2}{\pi} \sqrt{1 - \zeta} \left( \frac{\zeta}{\zeta - 1} \right)^E,$$

where $E$ is the complete elliptic integral of the second kind. From this we infer that

$$\lim_{\zeta \to 1^-} f(\zeta, \zeta) = \frac{2}{\pi}.$$

From these properties we deduce the following facts:
1. If \( \gamma : (a, \infty) \rightarrow \Lambda \) is a smooth curve such that \( \gamma(t) \to (1,1) \) when \( t \to \infty \) and \( \zeta = \delta \) is an asymptote of \( \gamma \) as \( t \to \infty \), then
\[
\lim_{t \to \infty} f(\gamma(t)) = \frac{2}{\pi}.
\]
2. If \( \tilde{\gamma} : (a, \infty) \rightarrow \Lambda \) is a smooth curve such that \( \tilde{\gamma}(t) \to (1,1) \) when \( t \to \infty \) and \( \zeta = 1 \) is an asymptote of \( \tilde{\gamma} \) as \( t \to \infty \), then
\[
\lim_{t \to \infty} f(\tilde{\gamma}(t)) = 1.
\]

Combining both things, it follows that as \( t \to \infty \),
\[
\begin{cases}
\Pi(\gamma(t)) \sim \frac{1}{\sqrt{1 - \gamma_1(t)\sqrt{1 - \gamma_2(t)}}} \\ 
\Pi(\tilde{\gamma}(t)) \sim \frac{\pi}{2\sqrt{1 - \tilde{\gamma}_1(t)\sqrt{1 - \tilde{\gamma}_2(t)}}}
\end{cases}
\]

In view of these properties, fix \( \tau_\lambda \) sufficiently large and consider the curves
\[
\begin{cases}
\gamma_\lambda : e_1 \in (\tau_\lambda, \infty) \rightarrow (\zeta_-(\lambda, e_1), \delta(\lambda, e_1)) \in \Lambda \\
\tilde{\gamma}_\lambda : e_1 \in (\tau_\lambda, \infty) \rightarrow (\zeta_+(\lambda, e_1), \delta(\lambda, e_1)) \in \Lambda
\end{cases}
\]

From above estimates when \( e_1 \to \infty \), we have the following asymptotic behavior for \( \gamma_\lambda \),
\[
\gamma_\lambda \sim \left(1 - \frac{4}{e_1^2}, 1 - \frac{4}{e_1^2}\right),
\]
while for \( \tilde{\gamma}_\lambda \), it depends on the value of \( \lambda \),
\[
\begin{cases}
\tilde{\gamma}_\lambda \sim \left(1 - \frac{4\lambda^2}{e_1^4}, 1 - \frac{4}{e_1^2}\right), & \text{if } \lambda \neq 0 \\
\tilde{\gamma}_0 \sim \left(1 - \frac{1}{e_1^2}, 1 - \frac{4}{e_1^2}\right), & \text{if } \lambda = 0
\end{cases}
\]

Thus, \( \gamma_\lambda \to (1, 1) \) and \( \zeta = \delta \) is an asymptote of \( \gamma_\lambda \) as \( e_1 \to \infty \). Similarly, \( \tilde{\gamma}_\lambda \to (1, 1) \) as \( e_1 \to \infty \).

Hence, it follows from above facts that
\[
\begin{cases}
\Pi((\zeta_-(\lambda, e_1), \delta(\lambda, e_1))) \sim \frac{e_1^2}{e_1^4 - 4} \\
\Pi((\zeta_+(\lambda \neq 0, e_1), \delta(\lambda \neq 0, e_1))) \sim \frac{\pi e_1^3}{8|\lambda|} \\
\Pi((\zeta_+(\lambda = 0, e_1), \delta(\lambda = 0, e_1))) \sim \frac{\pi e_1^4}{4}
\end{cases}
\]
when \( e_1 \to \infty \).

It is then clear, combining this and above estimates, that
\[
\begin{cases}
\lim_{e_1 \to \infty} III(\lambda, e_1) = 0 \\
\lim_{e_1 \to \infty} II(\lambda \geq 0, e_1) = -\frac{1}{2} \\
\lim_{e_1 \to \infty} II(\lambda < 0, e_1) = \frac{1}{2}
\end{cases}
\]
This completes the proof about the claimed limits for \( \Psi_\lambda \) when \( e_1 \to \infty \).
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