SHARP RESOLVENT ESTIMATES OUTSIDE OF THE UNIFORM BOUNDEDNESS RANGE

YEHYUN KWON AND SANGHYUK LEE

ABSTRACT. In this paper we are concerned with resolvent estimates for the Laplacian $\Delta$ in Euclidean spaces. Uniform resolvent estimates for $\Delta$ were shown by Kenig, Ruiz and Sogge [31] who established rather a complete description of the Lebesgue spaces allowing such uniform resolvent estimate. The estimates have a variety of applications, particularly, to proving uniform Sobolev and Carleman estimates. Recently, interest in such estimates was renewed in connection to the Carleman estimate related to inverse problems. Especially, on a compact Riemannian manifold $(M,g)$, an interesting new phenomenon was discovered by Dos Santos Ferreira, Kenig and Salo [15]. Precisely, the estimate
\[
\|(-\Delta - z)^{-1}f\|_{L^q(M)} \leq C\|f\|_{L^p(M)}
\]
holds with $C$ independent of $z \in \mathbb{C} \setminus [0, \infty)$ when $\text{Im} \sqrt{z} \geq \delta$ for $\delta > 0$. Later it was shown by Bourgain, Shao, Sogge and Yao [8] that the region of $z$ can not be extended to a wider range in general, and under the same assumption on the range of $z$, the range of $p, q$ was further extended by Shao and Yao [41]. However, even in the Euclidean spaces, the problem of obtaining sharp $L^p-L^q$ bounds depending on $z$ has not been considered in general framework which admits all possible $p,q$. In this paper, we present a complete picture of sharp $L^p-L^q$ resolvent estimates, which may depend on $z$. The resolvent estimates in Euclidean space seem to be expected to behave in a simpler way compared with those on manifolds. However, it turns out that, for some $p, q$, the estimates exhibit unexpected behavior which is similar to those on compact Riemannian manifolds. We also obtain the non-uniform sharp resolvent estimates for the fractional Laplacians and a new result for the Bochner–Riesz operators of negative index.

1. INTRODUCTION AND MAIN RESULTS

In this paper we are concerned with the resolvent estimate for the Laplacian which is of the form
\[
\|(-\Delta - z)^{-1}f\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall z \in \mathbb{C} \setminus [0, \infty).
\]
When $z = 0$ the estimate is simply the classical Hardy–Littlewood–Sobolev inequality. If $z \in (0, \infty)$ the left hand side can not be defined even as a distribution without additional assumption. Throughout this article we assume $z \in \mathbb{C} \setminus [0, \infty)$. The inequality (1.1) and its variants (especially, with $C$ independent of $z$) have applications to various related problems. Among them are uniform Sobolev estimates, unique continuation properties [31, 30], limiting absorption principles [20], absolute continuity of the spectrum of periodic Schrödinger operators [42] and eigenvalue bounds for Schrödinger operators with complex potentials [17, 18]. As just mentioned, (1.1) has been usually considered with $C$ independent of $z$ but the sharp bounds which are allowed to be dependent on $z$ are not studied in a general framework. The primary purpose of this paper is to provide complete characterization of the sharp $L^p-L^q$ bounds for the resolvent operators up to a multiplicative constant.

Uniform resolvent estimate. In their celebrated work [31] Kenig, Ruiz and Sogge showed that, for certain pairs of $p, q$, the constant $C$ in (1.1) can be chosen uniformly in $z \in \mathbb{C} \setminus [0, \infty)$. More precisely, for $d \geq 3$, it was shown that there is a uniform constant $C = C(p,q,d) > 0$ such that

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holds if and only if $1/p - 1/q = 2/d$ and $2d/3 < p < 2d/5$, or equivalently $(1/p, 1/q)$ lies on the open line segment whose endpoints are

$$A = A(d) := \left(\frac{d + 1}{2d}, \frac{d - 3}{2d}\right), \quad A' = A'(d) := \left(\frac{d + 3}{2d}, \frac{d - 1}{2d}\right), \quad d \geq 3.$$  

(See Figure 2). They used these estimates to show uniform Sobolev estimates for second order elliptic differential operators on the same range of $p, q$ (see [31 Theorem 2.2]). When $\left(\frac{1}{p}, \frac{1}{q}\right) = (d + 2/2d, d - 2/2d)$ the same estimate was also obtained by Kato and Yajima [32] pp. 493–494 by a different approach.

The result in [31] gives complete characterization of the range of $p, q$ which admits the uniform resolvent estimate. However, it is not difficult to see that if $C$ in (1.1) is allowed to be dependent on $z \in \mathbb{C} \setminus [0, \infty)$, there is a larger set of $p, q$ for which the estimate (1.1) holds. To be precise, for $z \in \mathbb{C} \setminus [0, \infty)$ let us set

$$\|(-\Delta - z)^{-1}\|_{p \to q} := \inf \left\{ C_z : \|(-\Delta - z)^{-1}f\|_{L^p(\mathbb{R}^d)} \leq C_z\|f\|_{L^q(\mathbb{R}^d)}, \forall f \in \mathcal{S}(\mathbb{R}^d) \right\},$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions on $\mathbb{R}^d$.

**Proposition 1.1.** Let $d \geq 2$, $1 < p, q \leq \infty$ and $z \in \mathbb{C} \setminus [0, \infty)$. Then $\|(-\Delta - z)^{-1}\|_{p \to q} < \infty$ if and only if $(1/p, 1/q) \in \mathcal{R}_0$ which is given by

$$\mathcal{R}_0 = \mathcal{R}_0(d) := \begin{cases} 
\{(x, y) : 0 \leq x, y \leq 1, \ 0 \leq x - y < 1\} & \text{if } d = 2, \\
\{(x, y) : 0 \leq x, y \leq 1, \ 0 \leq x - y \leq \frac{3}{2}\} \setminus \{(1, \frac{d - 2}{d})\} & \text{if } d \geq 3.
\end{cases}$$

In view of Proposition 1.1 it is natural to ask what is the sharp value of $\|(-\Delta - z)^{-1}\|_{p \to q}$ which depends on $z$. For some $p, q$ such estimate (modulo a constant multiplication) can be deduced by interpolation between estimates in [31] and the easy bound

$$\|(-\Delta - z)^{-1}f\|_2 \leq \frac{\|f\|_2}{\text{dist}(z, [0, \infty))},$$

which directly follows from the Fourier transform and Parseval’s identity. Some of related results can be found in [18]. Moreover, these estimates turn out to be sharp (see [15] Proposition 1.3 below). But, the sharp bound for $\|(-\Delta - z)^{-1}\|_{p \to q}$ with general $p, q$ can not be deduced from interpolation between previously known estimates. For the purpose we need to make use of $L^p$ theory of oscillatory integral operators of Carleson–Sjölin type under the additional elliptic condition ([11] [27] [60] [35] [23], also see Section 2.1 below).

**Boundedness of the associated multiplier operators.** To obtain the sharp resolvent estimates, it is convenient to consider bounds for the associated multiplier operators. Clearly,

$$\|(-\Delta - z)^{-1}\|_{p \to q} = \sup_{\|f\|_p \leq 1} \left\| \mathcal{F}^{-1}\left(\frac{\mathcal{F}f(z)}{|z|^2 - z}\right) \right\|_{L^q(\mathbb{R}^d)}, \quad \forall z \in \mathbb{C} \setminus [0, \infty).$$

Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms on $\mathbb{R}^d$, respectively. Since the multiplier $\|\xi|^2 - z\|^{-1}$ becomes singular as $z$ approaches to the set $[0, \infty)$ it is reasonable to expect that the bound $\|(-\Delta - z)^{-1}\|_{p \to q}$ gets worse as $\text{dist}(z, [0, \infty)) \to 0$. Thanks to homogeneity and scaling, we have that

$$\|(-\Delta - z)^{-1}\|_{p \to q} = |z|^{-1 + \frac{d}{p} - \frac{d}{q}} \left\| (-\Delta - \frac{2}{|z|})^{-1} \right\|_{p \to q}, \quad \forall z \in \mathbb{C} \setminus [0, \infty).$$

Thus we may assume that $|z| = 1$, $z \neq 1$ to get the sharp bounds for $\|(-\Delta - z)^{-1}\|_{p \to q}$. Indeed, when $d \geq 3$, it was shown in [31] that there is a uniform constant $C$, independent of $z$, such that

$$(1.6) \quad \|(-\Delta - z)^{-1}\|_{p \to q} \leq C, \quad \forall z \in S^1 \setminus \{1\} \setminus \{1\}.$$

The midpoint of the line segment $AA'$ in Figure 2.  

$S^1 := \{z \in \mathbb{C} : |z| = 1\}.$
if \((1/p, 1/q)\) lies in either the open line segment of which endpoints are \(A\) and \(A'\) (see (1.2) and Figure 2), or the line of duality \(1/p + 1/q = 1\) restricted to \(\frac{d+3}{2d+2} \leq \frac{1}{p} \leq \frac{d+2}{2d}\) (see [31, Lemma 2.2(b) and Theorem 2.3]). Later, Gutiérrez ([25, Theorem 6]) extended (1.6) to the optimal range of \(p,q\).

More precisely, she proved that the uniform bound (1.6) is true if \((1/p, 1/q)\) lies in the set

\[
R_1 = R_1(d) := \{(x, y) \in R_0(d) : \frac{2}{d+1} \leq x - y \leq \frac{2}{d}, x > \frac{d+1}{2d}, y < \frac{d-1}{2d}\}, \quad d \geq 3.
\]

This region is the closed trapezoid \(ABB'A'\) from which the closed line segments joining \(A,B\) and \(A',B'\) are removed (see Figure 2). She also established the \(L^p,L^q\)-analogues of (1.6) when \((1/p, 1/q)\) is either \(B\) or \(B'\), where

\[
B = B(d) := \left(\frac{d+1}{2d}, \frac{(d-1)^2}{2d(d+1)}\right), \quad B' = B'(d) := \left(\frac{d^2 + 4d - 1}{2d(d+1)}, \frac{d-1}{2d}\right).
\]

Failure of (1.6) for \((1/p, 1/q) \notin R_1\) has been actually known before in the studies of the Bochner–Riesz operators of negative orders (see Section 2.6). In fact, the necessity of the conditions \(\frac{1}{p} > \frac{d+1}{2d}\) and \(\frac{1}{q} < \frac{d-1}{2d}\) follow since (1.6) combined with (1.4) implies \(L^p-L^q\) boundedness of the restriction-extension operator on the sphere (see Theorem 2.14 and [31, pp. 341–342]), which is a constant multiple of the Bochner–Riesz operator (2.49) of order \(-1\). The other two conditions \(\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q}\) and \(\frac{1}{p} - \frac{1}{q} \leq \frac{2}{d}\) can be obtained by the Knapp type example (see Böjeson [4]) and a simple argument involved with the Littlewood–Paley projection (see Proof of Proposition 1.1), respectively.

When \(d = 2\), as far as the authors are aware, the corresponding results regarding the uniform resolvent estimate (1.6) are not explicitly stated anywhere else before, although the \(L^p-L^q\) mapping properties of the closely related Bochner–Riesz operators of negative order are well known (see e.g., [11, 12] and references therein). However, the method in [25] can be applied to obtain (1.6) provided that \((1/p, 1/q)\) is contained in the pentagon

\[
R_1(2) := \{(x, y) : 2/3 \leq x - y < 1, 3/4 < x \leq 1, 0 \leq y < 1/4\}.
\]

See Figure 1 and Remark 1.

**Conjecture regarding \(L^p-L^q\) resolvent estimate with \((1/p, 1/q) \in R_0 \setminus R_1\).** Having seen that we have the uniform bound (1.6) on the optimal range \(R_1\), we now proceed to investigate the (non-uniform) sharp bounds with \(p,q\) which lie outside of the uniform boundedness range. As
becomes clear later, the problem is closely related to sharp $L^p-L^q$ boundedness of the Bochner-Riesz operators of negative orders (see Section 2.0). The non-uniform bounds on the resolvents have been used to study eigenvalues of the Schrödinger operators with complex potentials (for example, see [13, 14]).

In order to state our results we introduce some notations which denote points and regions in the closed unit square $I^2 := \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$. For each $(x, y) \in I^2$ we set 
\[(x, y)' := (1 - y, 1 - x)\].

Similarly, for every subset $R$ of $I^2$ we define $R' \subset I^2$ by 
\[R' := \{(x, y) \in I^2 : (x, y)' \in R\}\].

**Definition 1.2.** For $X_1, \ldots, X_5 \in I^2$, we denote by $[X_1, \ldots, X_5]$ the convex hull of the points $X_1, \ldots, X_5$. In particular, if $X, Y \in I^2$, $[X, Y]$ denotes the closed line segment connecting $X$ and $Y$ in $I^2$. We also denote by $(X, Y)$ and $[X, Y]$ the open interval $[X, Y] \setminus \{X, Y\}$ and the half-open interval $[X, Y] \setminus \{Y\}$, respectively.

For every $d \geq 2$ and every $(1/p, 1/q) \in I^2$, define a nonnegative number 
\[(1.7) \quad \gamma_{p,q} = \gamma_{p,q}(d) := \max\left\{0, 1 - \frac{d+1}{2}(\frac{1}{p} - \frac{1}{q}), \frac{d+1}{2} - \frac{d}{p} - \frac{d}{q} - \frac{1}{2}\right\}\].

The definition of $\gamma_{p,q}$ naturally leads to division of $\{(x, y) \in I^2 : y \leq x\}$ into the four regions 
\[(1.8) \quad \mathcal{P} = \mathcal{P}(d) := \{(x, y) \in I^2 : x - y \geq \frac{2}{d+1}, x > \frac{d+1}{2d}, y < \frac{d-1}{2d}\}\],
\[(1.9) \quad \mathcal{T} = \mathcal{T}(d) := \{(x, y) \in I^2 : 0 \leq x - y < \frac{2}{d+1}, \frac{d-1}{d+1}(1 - x) \leq y \leq \frac{d+1}{d-1}(1 - x)\}\],
\[(1.10) \quad \mathcal{Q} = \mathcal{Q}(d) := \{(x, y) \in I^2 : y < x, x \leq \frac{d+1}{d+1}(1 - y), y \leq x \leq \frac{d+1}{2d}\}\].

We now observe that $R_i(d) = \mathcal{P}(d) \cap R_0(d)$. Setting $H := (\frac{1}{2}, \frac{1}{2})$ and $D = D(d) := (\frac{d-1}{2d}, \frac{d}{2d})$ we also define $R_2 = R_2(d)$ and $R_3 = R_3(d)$ by 
\[R_2 := \mathcal{T}(d) \setminus ([D, H] \cup [D', H]), \quad R_3 := \mathcal{Q}(d) \cap R_0(d)\].

See Figure 1 and Figure 2. Observe that the sets $R_i$ $(i = 1, 2, 3)$ and $R_3$ are mutually disjoint. Setting $E = E(d) := (\frac{d+1}{2d}, 0)$ we have that 
\[\left(\bigcup_{i=1}^{3} R_i\right) \cup R_3 = R_0 \setminus ([B, E] \cup [B', E'] \cup [D, H] \cup [D', H]),\]

and we also see that 
\[(1.11) \quad \gamma_{p,q} = \begin{cases} 0 & \text{if } (\frac{1}{p}, \frac{1}{q}) \in R_1, \\ 1 - \frac{d+1}{2}(\frac{1}{p} - \frac{1}{q}) & \text{if } (\frac{1}{p}, \frac{1}{q}) \in R_2, \\ \frac{d+1}{2} - \frac{d}{p} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in R_3, \\ \frac{d}{q} - \frac{d+1}{2} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in R_3. \end{cases}\]

In Section 5 we obtain the following lower bounds for $\|(-\Delta - z)^{-1}\|_{p \to q}$.

**Proposition 1.3.** Let $d \geq 2$. Suppose that $(1/p, 1/q) \in (\bigcup_{i=1}^{3} R_i) \cup R_3$. Then, for $z \in S^1 \setminus \{1\}$, 
\[(1.12) \quad \|(-\Delta - z)^{-1}\|_{p \to q} \geq \text{dist}(z, [0, \infty))^{-\gamma_{p,q}},\]

where the implicit constant is independent of $z \in S^1 \setminus \{1\}$.

\[\mathcal{P} = ([0, 1), E, B, B', E'] \cup ([E, B] \cup [E', B']), \quad \mathcal{T} = [B, D, D', B'] \setminus [B, B'], \quad \text{and} \quad \mathcal{Q} = ([0, 0), D, B, E] \setminus ([D, B] \cup [B, E]). \] See Figure 1 and Figure 2.
As mentioned in the above, when \((1/p,1/q) \in [B,E] \cup [B',E']\), sup\(_{z \in S^1(D)} \|(-\Delta - z)^{-1}\|_{p \to q} = \infty\). For \((1/p,1/q) \in [D,H] \cup [D',H']\), it is likely that by adapting Fefferman’s disproof of disk multiplier conjecture [16] one can show sup\(_{z \in S^1(D)} \|(-\Delta - z)^{-1}\|_{p \to q} = \infty\). However, for the other \(p,q\) with \((1/p,1/q) \in \bigcup_{i=2}^{3} \mathcal{R}_i \cup \mathcal{R}_3^c\) it seems to be natural to expect that the lower bound in (1.12) is also an upper bound.

For \(p,q\) with \((1/p,1/q) \in \bigcup_{i=2}^{3} \mathcal{R}_i \cup \mathcal{R}_3^c\) and \(z \in \mathbb{C} \setminus [0,\infty)\), let us set

\[ \kappa_{p,q}(z) = \kappa_{p,q,d}(z) := \frac{1}{|z|^{\gamma_p + \frac{1}{2}|p-q|}} \text{dist}(z, [0,\infty))^{-\gamma_p}. \]

Since dist\((|z|^{-1}z,[0,\infty)) = |z|^{-1} \text{dist}(z,[0,\infty))\), from Proposition 1.3 and (1.5) we conjecture the following which completely characterizes the resolvent estimates outside of the uniform boundedness range.

**Conjecture 1.** Let \(d \geq 2\) and \((1/p,1/q) \in \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3^c\). There exists an absolute constant \(C\), depending only on \(p,q\) and \(d\), such that, for \(z \in \mathbb{C} \setminus [0,\infty)\),

\[ C^{-1} \kappa_{p,q}(z) \leq \|(-\Delta - z)^{-1}\|_{p \to q} \leq C \kappa_{p,q}(z). \]

**Sharp \(L^p - L^q\) resolvent estimate with** \((1/p,1/q) \notin \mathcal{R}_1\). Our main result is that Conjecture 1 is true for most of cases of \(p,q\). For the statement of the result we introduce additional notations. Let \(p_o, q_o,\) and \(p_*\) be defined by

\[ \frac{1}{p_*} := \begin{cases} \frac{3(d-1)}{2(3d+1)}, & \text{if } d \text{ is odd} \\ \frac{3d-2}{2(3d+2)}, & \text{if } d \text{ is even} \end{cases}, \quad \left(\frac{1}{p_o}, \frac{1}{q_o}\right) := \begin{cases} \frac{1}{p_*}, \frac{1}{q_o}, & \text{if } d \text{ is odd} \\ \frac{1}{p_*}, \frac{1}{q_o}, & \text{if } d \text{ is even} \end{cases}. \]

We also set \(P_* = P_o(d) := (1/p_o,1/p_*), P_o := P_o(d) = (1/p_o,1/q_o)\). See Figure 3 and Figure 4 When \(d \geq 2\) we define \(\mathcal{R}_2 = \mathcal{R}_2(d)\) and \(\mathcal{R}_3 = \mathcal{R}_3(d)\) by

\[ \mathcal{R}_2 := [B,B',P_o,H,P_o] \setminus ([P_o,H] \cup [P'_o,H] \cup [B,B']) \quad \text{and} \quad \mathcal{R}_3 := \mathcal{R}_3 \setminus [D,P_o,P_*]. \]

If \(d = 2\), note that \(P_o = P_* = D = (1/4,1/4)\), and \(\mathcal{R}_i = \mathcal{R}_{2i}\) for \(i = 2,3\).

**Theorem 1.4.** Let \(z \in \mathbb{C} \setminus [0,\infty)\). If \(d = 2\), Conjecture 1 is true. If \(d \geq 3\), the conjectured estimate (1.13) is true whenever \((1/p,1/q) \in \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3^c\). Furthermore, when \(d \geq 2\), for \((1/p,1/q) \in \{B,B'\}\) the estimate \(||(-\Delta - z)^{-1}f||_{p,q} \leq C|z|^{-1 + \frac{1}{4p}}||f||_{p,1}\) holds, and for \((1/p,1/q) \in (B',E') \cap \mathcal{R}_0\) the estimate \(||(-\Delta - z)^{-1}f||_{p,p} \leq C|z|^{-1 + \frac{1}{4(1 - \frac{1}{d})}}||f||_p\) holds.
It is also possible to obtain similar results regarding the Laplace-Beltrami operator on compact manifolds \([\mathbb{S}^3]\). To prove the sharp resolvent estimates \((1.13)\), we dyadically decompose the multipliers \(\langle |\xi|^2 - z \rangle^{-1}\) by taking into account the region of \(\xi\) where the multiplier gets singular as \(\operatorname{Im} z \to 0\). Such idea is now classical in the context of the Bochner–Riesz conjecture (e.g. \([13, 25]\)).

It is important to obtain the optimal \(L^p - L^q\) bounds for each of the operators which are given by the dyadic decomposition. For the purpose we use the Carleson–Sjölin reduction \((11, 46)\), and combine this with Theorem 2.2 in Section 2.1 \((23)\) and bilinear estimate for the extension operator associated to the hypersurfaces of elliptic type \((19)\). For more details, see Section 2 \((\text{Corollary 2.12})\).

**Remark 1.** As mentioned in the above, the restricted weak type \((p,q)\) estimates with \((1/p,1/q) = B,B'\) when \(d \geq 3\) were shown in \([25]\). In Section 4 we provide a different proof of those restricted weak type estimates for \(d \geq 2\), together with the weak type \((p,q)\) estimates when \((1/p,1/q)\) is in the half open line segment \([B',B'] \cap \mathcal{R}_0\) (see Figure 3 and Figure 4). This upgrades the endpoint case of uniform Sobolev estimate in \([40]\) from the restricted weak type \((p,q)\) to the weak type \((p,q)\) for \((1/p,1/q) = A' = \left(\frac{d+3}{2d+2}, \frac{d-1}{2d}\right)\) when \(d \geq 4\). Also, for \(p, q\) satisfying \((1/p,1/q) \in \mathcal{R}_1(d)\), the uniform resolvent estimate \((1.6)\) follows by duality and interpolation. (For \(d = 2\) an additional simple argument involving frequency localization and Young’s inequality is necessary to cover the case \((1/p,1/q) \in \mathcal{R}_1(2)\).)

**Remark 2.** When \(d = 1\) it is also possible and much simpler to obtain the sharp resolvent estimates. For \(z \in \mathbb{C} \setminus [0, \infty)\) we write \((-d^2/dx^2 - z)^{-1} f(x) = G_z * f(x)\), where \(G_z(x) = \frac{i}{\sqrt{2\pi}} e^{i\sqrt{z}x}\) (see \([47, p. 203]\)). Since the kernel is bounded and integrable, Young’s inequality and \((1.5)\) yield

\[
\|(-d^2/dx^2 - z)^{-1}\|_{p \to q} \lesssim |z|^{-\frac{1}{2}} \left(\frac{1}{p} - \frac{1}{q}\right)^{-1} \operatorname{dist}(z, [0, \infty))^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}}, \quad \forall z \in \mathbb{C} \setminus [0, \infty)
\]

for all \(p, q\) such that \(1 \leq p \leq q \leq \infty\). Following the argument in Section 5.2 one can easily check that the estimates are sharp.

**Resolvent estimates on compact Riemannian manifolds.** Let \((M, g)\) be a \(d\)-dimensional compact Riemannian manifold without boundary. When \(d \geq 3\) Dos Santos Ferreira, Kenig, and Salo proved in \([15]\) that for any fixed \(\delta > 0\) the uniform estimate

\[
(1.15) \quad \|(-\Delta_g - z)^{-1} f\|_{L^{\frac{2d}{d+2}}(M)} \leq C\|f\|_{L^{\frac{2d}{d+1}}(M)}
\]

holds for all \(z \in \Xi_\delta := \{ z \in \mathbb{C} \setminus [0, \infty) : \operatorname{Im} \sqrt{z} \geq \delta\}\).\(^4\) Shortly afterwards, Bourgain, Shao, Sogge and Yao \([8]\) proved that if \(M\) is Zoll, then the region \(\Xi_\delta\) cannot be significantly improved by showing that

\[
(1.16) \quad \lim_{\lambda \to +\infty} \sup_{\tau \in [1, \lambda]} \|(-\Delta_g - (\tau^2 + i\varepsilon(\tau)\tau))^{-1}\|_{L^{\frac{2d}{d+2}}(M) \to L^{\frac{2d}{d+3}}(M)} = +\infty
\]

whenever \(\varepsilon(\tau) > 0\) for all \(\tau\), and \(\varepsilon(\tau) \to 0\) as \(\tau \to +\infty\). However, in some cases where the manifold has favorable geometry such as the flat torus or Riemannian manifolds with nonpositive sectional curvature, the range of \(z\) for \((1.15)\) can be extended (see \([8]\)). Shao and Yao \([41]\) proved the off-diagonal \(L^p(M) - L^q(M)\) estimate of \((1.15)\) for \(p, q\) satisfying \(1/p - 1/q = 2/d, p \leq \frac{2(d+1)}{d+3}\) and \(q \geq \frac{2d}{d+1}\), but it is not known whether this range of \(p, q\) is optimal even for \(p, q\) which satisfy \(1/p - 1/q = 2/d\). In \([19]\) Frank and Schimmer observed that the argument in \([15]\) can be applied to establish \(L^p(M) - L^q(M)\) analogue of \((1.15)\) when \(\frac{2d}{d+2} < p < \frac{2(d+1)}{d+3}\) and \(d \geq 2\). They also obtained the estimate

\[
\|(-\Delta_g - z)^{-1} f\|_{L^{\frac{2(d+1)}{d+3}}(M)} \leq C|z|^{-\frac{1}{d+1}}\|f\|_{L^{\frac{2(d+1)}{d+3}}(M)}
\]

with \(C\) independent of \(z \in \Xi_\delta\) by proving an off-diagonal restricted weak type bound for the parametrix constructed in \([15]\).

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\(^4\)Here we choose the branch of \(\sqrt{z}\), \(z \in \mathbb{C} \setminus [0, \infty)\), such that the imaginary part is positive. Note that \(\Xi_\delta = \{ z \in \mathbb{C} \setminus [0, \infty) : (\operatorname{Im} z)^2 \geq 4\delta^2(\operatorname{Re} z + \delta^2)\}\). In the complex plane this region excludes a neighborhood of the origin and a parabolic region opening to the right.
A remarkably interesting phenomena occurs when $(1/p, 1/q)$, the spectral parameters by $\|f\|$, holds for $z\in \mathbb{C} \setminus [0, \infty)$ while the uniform estimate $\Delta + \bar{V}$ on compact manifold holds only for $z\in \mathcal{X}_d$ (see Figure 5). Thus, we may reasonably expect that the bound $\|(-\Delta - z)^{-1}\|_{p\to q}$ behaves better on $\mathbb{R}^d$ than on compact manifolds. However, as is to be seen below, it is rather surprising that, for certain $p, q$, the bound for $\|(-\Delta - z)^{-1}\|_{p\to q}$ has a similar behavior with those on compact manifolds and the profile of the $z$-region where $\|(-\Delta - z)^{-1}\|_{p\to q}$ is uniformly bounded changes dramatically depending on the values of $p, q$.

For $p, q$ which satisfy $(1/p, 1/q) \in \mathcal{R}_1 \cup (\bigcup_{i=2}^\ell \bar{\mathcal{R}}_i) \cup \bar{\mathcal{R}}_3'$, and $\ell > 0$ we define the region $\mathcal{Z}_{p,q}(\ell)$ of spectral parameters by

$$\mathcal{Z}_{p,q}(\ell) := \{z \in \mathbb{C} \setminus [0, \infty) : \kappa_{p,q}(z) \leq \ell\}.$$ 

For simplicity, let us focus on the case $\ell = 1$, and describe roughly the typical shapes of $\mathcal{Z}_{p,q}(1)$. See Section 4.4 and Figure 7 for detailed description of $\mathcal{Z}_{p,q}(\ell)$ in terms of $p, q, d$, and $\ell$.

- If $d \geq 3$ and $(1/p, 1/q) \in (A, A')$, then $\mathcal{Z}_{p,q}(1) = \mathbb{C} \setminus [0, \infty)$ (see Figure 5b).
- If $(1/p, 1/q) \in \mathcal{R}_1$, and $1/p - 1/q < 2/d$, then $\mathcal{Z}_{p,q}(1)$ is given by removing the unit disk centered at zero from $\mathbb{C} \setminus [0, \infty)$ (see Figure 5a).
- If $(1/p, 1/q) \in \mathcal{R}_2$, then $\mathcal{Z}_{p,q}(1)$ basically have two different types. When $(p, q) \neq (2, 2)$, $\mathcal{Z}_{p,q}(1)$ is the complex plane minus a neighborhood of $[0, \infty)$ which shrinks along the positive real line as $\text{Re } z \to \infty$ (see Figure 5b). When $p = q = 2$, $\mathcal{Z}_{2,2}(1)$ is the complex plane from which the 1-neighborhood of $[0, \infty)$ is removed (see Figure 5c).

A remarkably interesting phenomena occurs when $(1/p, 1/q) \in \bar{\mathcal{R}}_3 \cup \bar{\mathcal{R}}_3'$. To describe this let us divide $\bar{\mathcal{R}}_3$ into the three subsets $\bar{\mathcal{R}}_{3,+}$, $\bar{\mathcal{R}}_{3,0}$, and $\bar{\mathcal{R}}_{3,-}$, given by

$$\bar{\mathcal{R}}_{3,\pm} := \{(x, y) \in \bar{\mathcal{R}}_3 : x + y - \frac{d-1}{d} > 0\}, \quad \bar{\mathcal{R}}_{3,0} := \{(x, y) \in \bar{\mathcal{R}}_3 : x + y - \frac{d-1}{d} = 0\}.$$ 

- If $(1/p, 1/q) \in \bar{\mathcal{R}}_{3,+} \cup \bar{\mathcal{R}}_{3,+}'$, $\mathcal{Z}_{p,q}(1)$ is similar type as in the case $(1/p, 1/q) \in \bar{\mathcal{R}}_2 \setminus \{H\}$ (see Figure 5b).
- If $(1/p, 1/q) \in \bar{\mathcal{R}}_{3,0} \cup \bar{\mathcal{R}}_{3,o}'$, we have $\mathcal{Z}_{p,q}(1) = \mathcal{Z}_{2,2}(1)$ (see Figure 5c).
- Let $(1/p, 1/q) \in \bar{\mathcal{R}}_{3,-} \cup \bar{\mathcal{R}}_{3,-}'$. If $1/p - 1/q < 2/d$, $\mathcal{Z}_{p,q}(1)$ is the complement (in $\mathbb{C}$) of a neighborhood of $[0, \infty)$ whose boundary becomes wider as $\text{Re } z$ gets large (see Figure 5d). If $1/p - 1/q = 2/d$, then $\mathcal{Z}_{p,q}(1) = \{z \in \mathbb{C} \setminus \{0\} : \text{Re } z \leq -d\}$.

**Location of the eigenvalues of $-\Delta + V$.** The sharp resolvent estimates (Theorem 1.4) can be used to specify the location of eigenvalues of non-self-adjoint Schrödinger operators $-\Delta + V$ acting
in $L^q(R^d)$, $1 \leq q \leq \infty$. As was shown in [17], [18], if $-\Delta + V$ acts in $L^2(R^d)$ one can use the Birman–Schwinger principle, but this is not the case when $-\Delta + V$ acts in $L^q(R^d)$, $q \neq 2$.

**Corollary 1.5.** Let $(1/p, 1/q) \in R_1 \cup (\bigcup_{i=2}^3 \tilde{R}_i) \cup \tilde{R}_d'$ and let $C > 0$ be the constant which appears in (1.13). Fix a positive number $\ell > 0$ (we choose $\ell \geq 1$ if $1/p - 1/q = 2/d$). Suppose that, for some $\ell \in (0, 1)$,

\[
(1.17) \quad \|V\|_{L^{\frac{pq}{p+q}}(R^d)} \leq t(\ell)^{-1}.
\]

If $E \in C \setminus [0, \infty)$ is an eigenvalue of $-\Delta + V$ acting in $L^q(R^d)$, then $E$ must lie in $C \setminus Z_{p,q}(\ell)$.

This is rather a direct consequence of Theorem 1.4. Let $u \in L^q(R^d)$ be an eigenfunction of $-\Delta + V$ with eigenvalue $E \in C \setminus [0, \infty)$. If $E$ were contained in $Z_{p,q}(\ell)$, Theorem 1.4 gives $\|(\Delta - E)^{-1}\|_{p\rightarrow q} \leq C\kappa_{p,q}(E) \leq Ct$. By Minkowski’s and Hölder’s inequalities, and (1.17) we have

\[
\|u\|_q \leq Ct\left(\|(-\Delta + V - E)u\|_p + \|Vu\|_p\right) \leq C\ell\|V\|_{L^{\frac{pq}{p+q}}} \|u\|_q \leq t\|u\|_q,
\]

which implies $u = 0$ since $t < 1$. This is contradiction, hence $E$ must be in $C \setminus Z_{p,q}(\ell)$.

**Remark 3.** It is possible to formulate a statement which is analogous to the observation in [18], p. 220, Remark (1)]. For example, if $(1/p, 1/q) \in \tilde{R}_2 \cup \tilde{R}_3_+ \cup \tilde{R}_d'$, then for a sequence of eigenvalues $\{E_j\}$ of $-\Delta + V$ acting in $L^q(R^d)$ such that $\text{Re} E_j \to \infty$ we have $\text{Im} E_j \to 0$ provided that $\|V\|_{L^{\frac{pq}{p+q}}}$ is small enough. However, it does not seem to be likely that this phenomenon continues to be true for $p, q$ satisfying $(1/p, 1/q) \in (\tilde{R}_{3,0} \cup \tilde{R}_{3,-}) \cup (\tilde{R}_{3,0} \cup \tilde{R}_{3,-})'$ and it would be interesting to ask whether there is a potential $V \in L^{\frac{pq}{p+q}}$ for which this kind of phenomenon fails.

**Remark 4.** If $1/p - 1/q = 2/d$, and (1.17) is satisfied with some $\ell \geq 1$ and $t \in (0, 1)$, then it follows from Corollary 1.5 that the Schrödinger operator $-\Delta + V$ acting in $L^q(R^d)$ does not have any eigenvalue of which real part is negative.
Sharp resolvent estimate for the fractional Laplacian. We also consider the sharp bound on \( \|((-\Delta)^s - z)^{-1}\|_{p \rightarrow q} \), that is to say, the \( L^p - L^q \) resolvent estimate for the fractional Laplacian \((-\Delta)^s\) which is defined by

\[
(-\Delta)^sf(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} |\xi|^s \hat{f}(\xi) d\xi.
\]

Uniform bounds on \( \|((-\Delta)^s - z)^{-1}\|_{p \rightarrow q} \) for \( p, q \) on certain range were obtained in Cuenin [14] and these bounds were used to study eigenvalues of the fractional Schrödinger operators with complex potentials. Later, uniform bounds up to the optimal range of \( p, q \) were obtained by Huang, Yao, and Zheng [29]. We also obtain the sharp bounds on \( \|((-\Delta)^s - z)^{-1}\|_{p \rightarrow q} \) for \( p, q \) which is not contained in the uniform bounded range. See Theorem 6.2.

Our method here is flexible and robust enough so that it is rather straightforward to extend our argument from the Laplacian to the fractional Laplacian. This allows us to obtain the sharp bounds on \(( -\Delta )^2 - z\) for \( s \in (0, d) \), which include the results for the resolvent of the Laplacian. Furthermore, Proposition 1.3, Theorem 1.4, and Corollary 1.5 can also be generalized in the context of the fractional Laplacian \((-\Delta)^s\), \( s > 0 \). There are also some new phenomena which do not appear in the study of the resolvent of the Laplacian. For example, if \( s \) is small, the profile of the spectral parameter region where uniform bound is allowed never takes the form such as in Figure 6b (see Section 6 for details). However, we postpone discussion regarding the resolvent of the fractional Laplacian until the last section to keep the presentation simpler.

Organization of this paper. In Section 2, we review some properties of hypersurfaces of elliptic type, and the \( L^p - L^q \) estimate for the Carleson–Sjölin type oscillatory integral operators. Then we obtain sharp estimates for the related multiplier operators of which frequency is localized. In Section 3, based on the results obtained in Section 2, we establish Proposition 2.3 which is the main ingredient for the proof of Theorem 1.4. In Section 4, we prove Theorem 1.4 and give descriptions in detail for various regions of spectral parameters \( Z_{p,q}(\ell) \) depending on \( p, q, d, \) and \( \ell \). In Section 5, the proof of Proposition 1.1 and Proposition 1.3 are given. In Section 6, we obtain the sharp resolvent estimates for the fractional Laplacian \((-\Delta)^s\), \( 0 < s < d \).

Notations. For positive numbers \( A \) and \( B \), \( A \lesssim B \) means that there is a constant \( C \) such that \( A \leq CB \). We write \( A \approx B \) if \( A \lesssim B \) and \( B \lesssim A \). Both \( \langle x, y \rangle \) and \( |x-y| \) denote the Euclidean inner product of \( x \) and \( y \). For a function \( f \) on \( \mathbb{R}^d \)

\[
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \mathcal{F}^{-1}f(\xi) = f^\prime(\xi) = (2\pi)^{-d} \hat{f}(-\xi)
\]

denote the Fourier and inverse Fourier transforms, respectively. We set \( D = -i(\partial/\partial x_1, \ldots, \partial/\partial x_d) \). For a bounded measurable function \( m \), \( m(D) \) denotes the Fourier multiplier operator \( (m\hat{f})^\vee \). For \( p, q \) \( \in [1, \infty] \) we define \( \|m(D)\|_{p \rightarrow q} := \sup_{\|f\|_{\leq 1}} \|m(D)f\|_q \). For any pair of subsets \( A, B \) of the Euclidean spaces or the complex plane, we write \( \text{dist}(A, B) := \inf\{|x-y| : x \in A, y \in B\} \). For any rectangle \( Q \) and a positive number \( a \), \( aQ \) is the rectangle whose side length is \( a \) times that of \( Q \) with same center as \( Q \). \( B_Q(c, r) \) is the open ball in \( \mathbb{R}^d \) centered at \( c \) with radius \( r \). If \( A \) is a set \( \chi_A \) is the characteristic function of \( A \). We denote by \( C^\infty_0(X) \) the class of smooth functions which are compactly supported in the set \( X \). Throughout this paper, we fix an even function \( \beta \in C^\infty_0(\mathbb{R}) \) which is supported in the interval \([-9/8, -3/8] \cup [3/8, 9/8] \) and satisfies \( \sum_{j=-\infty}^{\infty} \beta(2^{-j}t) = 1 \) whenever \( t \neq 0 \). We also set \( \beta_0 = 1 - \sum_{j \geq 0} \beta(2^{-j} \cdot) \in C^\infty_0((-3/4, 3/4)) \). For a variable \( x \) \( \in \mathbb{R}^d \) and a multi-index \( \alpha \in \mathbb{N}_0^d \) we sometimes write \( x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \) and \( \alpha = (\alpha', \alpha_d) \in \mathbb{N}_0^{d-1} \times \mathbb{N}_0 \).

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2. Estimates for localized frequency

In this section we prove basic estimates which play important roles in obtaining our main result.

2.1. Oscillatory integral operator of Carleson–Sjölin type. Let $\lambda \geq 1$, $a \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^{d-1})$, $\Phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$, and let $T_\lambda[\Phi, a]$ be the operator defined by

$$T_\lambda[\Phi, a]f(x) = \int_{\mathbb{R}^{d-1}} e^{i\lambda \Phi(x, u)} a(x, u) f(u) du, \quad (x, u) \in \mathbb{R}^d \times \mathbb{R}^{d-1}. $$

Suppose that, for every $(x, u) \in \text{ supp } a$,

$$\text{rank } (\partial_x \partial_u \Phi(x, u)) = d - 1. \quad (2.1)$$

We also assume that, for every $(x_0, u_0)$ $\in \text{ supp } a$, if $v \in S^{d-1}$ is the (unique up to sign) direction such that the function $u \rightarrow \langle v, \partial_x \Phi(x_0, u) \rangle$ has a critical point at $u = u_0$, then

$$\text{rank } (\partial_u^2 \langle v, \partial_x \Phi(x_0, u) \rangle)_{|u=u_0}) = d - 1. \quad (2.2)$$

The operator $T_\lambda[\Phi, a]$ with $\Phi$ satisfying $(2.1)$, $(2.2)$ on $\text{ supp } a$ is called the Carleson–Sjölin type oscillatory integral operator which originated from the work of Carleson and Sjölin [11] for the study of the two dimensional Bochner–Riesz problem (also, see [44, pp. 60–70], [38]). Hörmander [27] proved

$$\|T_\lambda[\Phi, a]f\|_{L^q(\mathbb{R}^2)} \lesssim \lambda^{-2/q} \|f\|_{L^p(\mathbb{R})} \quad \text{if } 4 < q \leq \infty \text{ and } 3/q \leq 1 - 1/p, \text{ and the range of } p, q \text{ for } (2.3) \text{ is optimal.} \quad (2.3)$$

Theorem 2.1. Suppose $\Phi$ satisfies $(2.1)$ and $(2.2)$ on $\text{ supp } a$. Then, for $1 \leq p, q \leq \infty$ satisfying $q \geq \frac{2(d+1)}{d-1}$ and $\frac{2d+1}{q} \leq (d - 1)(1 - \frac{1}{p})$, the following estimate holds:

$$\|T_\lambda[\Phi, a]f\|_{L^q(\mathbb{R}^d)} \lesssim \lambda^{-d/q} \|f\|_{L^p(\mathbb{R}^{d-1})}. \quad (2.4)$$

Bourgain [6] showed that the estimate $(2.4)$ under the conditions $(2.1)$ and $(2.2)$ generally fails if $q < \frac{2(d+1)}{d-1}$ when $d \geq 3$ is odd. However, in [30] one of the authors observed that in addition to $(2.1)$, $(2.2)$ if we assume that

$$\text{the surface } u \rightarrow \partial_x \Phi(x, u) \text{ has } d - 1 \text{ nonzero principal curvatures of the same sign} \quad (2.5)$$

then the range of $p, q$ for which $(2.4)$ holds can be enlarged to $q > \frac{2(d+1)}{d}$. For most recent developments see Bourgain and Guth [17] and Guth, Hickman and Iliopoulou [23]. These results are based on multilinear estimates due to Bennett, Carbery and Tao [3] and the method of polynomial partitioning due to Guth [22, 21]. We record here the recent sharp result due to Guth, Hickman and Iliopoulou [23].

Theorem 2.2. Let $d \geq 2$ and suppose $\Phi$ satisfies $(2.1)$, $(2.2)$ and $(2.5)$ on $\text{ supp } a$. Then, the estimate $(2.4)$ holds whenever $p = q > p_\ast$, for $p_\ast$ given in (1.14). This is sharp (up to endpoint) in the sense that there are examples of Carleson–Sjölin type operators $T_\lambda[\Phi, a]$ with phase functions satisfying all of $(2.1)$, $(2.2)$ and $(2.5)$ for which the estimate $(2.4)$ with $p = q$ fails whenever $p < p_\ast$.

Remark 5. The estimate $(2.4)$ with $p = q$ in Theorem 2.2 is uniform under small smooth perturbation of the phase $\Phi$ and the amplitude $a$. In fact, the estimate $(2.4)$ in [23] was obtained by running induction argument over a class of operators while the phase functions are properly normalized. See [23, Lemma 4.1, Definition 11.3]. Since small smooth perturbation of the phase functions are allowed within the class of operators, stability of the estimates follows.

\footnote{This is equivalent to saying that the matrix $\partial_u^2 \langle v, \partial_x \Phi \rangle$ is either positive or negative definite.}
2.2. Functions of elliptic type. Let \( N \in \mathbb{N} \) and \( \varepsilon > 0 \). Let us set \( I = [-1, 1] \). Following [50] and [37], we define \( \text{Ell}(N, \varepsilon) \) as the class of \( C^N \)-functions \( \psi : I^{d-1} \to \mathbb{R} \) satisfying

- \( \psi(0) = 0 \) and \( \nabla \psi(0) = 0 \);
- \( \text{Let } w(\xi') = \psi(\xi') - |\xi'|^2/2. \)

Then

\[
\sup_{\xi' \in I^{d-1}} \max_{0 \leq |\alpha| \leq N} |\partial^\alpha w(\xi')| \leq \epsilon.
\]

(2.6)

Typically \( N \) is chosen to be large and \( \varepsilon \) to be small. As was pointed out in [50], every convex smooth hypersurface with nonvanishing Gaussian curvature can be locally parametrized as graph of a function of elliptic type after a proper affine transformation.

For later use, we record here an approximate property of functions of elliptic type, which is an easy consequence of Taylor's theorem. Let \( H\psi \) denote the Hessian matrix of \( \psi \).

**Lemma 2.3.** Let \( N, \varepsilon \) be as above and \( 0 < \rho \leq 2^{-1} \). For \( \psi \in \text{Ell}(N, \varepsilon) \) and \( c \in (2^{-1}I)^{d-1} \) set

\[
\psi_{c, \rho}(\xi') = \rho^{-2} (\psi(c - \rho \xi') + \psi(c) - \rho \nabla \psi(c) \cdot \xi').
\]

Then, we have

\[
\sup_{(\xi', c) \in I^{d-1} \times (2^{-1}I)^{d-1}} |\partial^\alpha_\xi \left( \psi_{c, \rho}(\xi') - \frac{1}{2} \langle H\psi(c)\xi', \xi' \rangle \right)| \leq \left\{
\begin{array}{ll}
\frac{(d-1)|\alpha|}{4} \rho \varepsilon, & \text{if } 0 \leq |\alpha| \leq 2, \\
\rho^{3-|\alpha|} 3! \varepsilon, & \text{if } 3 \leq |\alpha| \leq N.
\end{array}
\right.
\]

(2.7)

Moreover, there is a constant \( c \), depending only on \( d \), such that if \( \psi \in \text{Ell}(N, \varepsilon) \), then, for all \( c \in (2^{-1}I)^{d-1} \) and \( 0 < \rho \leq 2^{-1} \), \( \psi_{c, \rho} \in \text{Ell}(N, \varepsilon) \).

**Proof.** Clearly \( \psi_{c, \rho}(0) = 0 \) and \( \nabla \psi_{c, \rho}(0) = 0 \). Let \( \xi'_1 := \sum_{k=1}^{d-1} |\xi_k| \). By Taylor's theorem we have for \( (\xi', c) \in I^{d-1} \times (2^{-1}I)^{d-1} \) that

\[
|\psi_{c, \rho}(\xi') - \frac{1}{2} \langle H\psi(c)\xi', \xi' \rangle| \leq \frac{|\xi'|^3}{3!} \sup_{0 \leq t \leq 1} |\rho(\partial^3 \psi)(\rho t \xi' + c)| \leq \frac{|\xi'|^3}{3!} \rho \varepsilon.
\]

The second inequality follows from (2.6) since \( \rho t \xi' + c \in I^{d-1} \) whenever \( (\xi', c) \in I^{d-1} \times (2^{-1}I)^{d-1} \), \( 0 < \rho \leq 2^{-1} \) and \( 0 \leq t \leq 1 \). Similarly, using Taylor’s theorem and (2.6) we also have

\[
|\partial^\alpha_\xi \left( \psi_{c, \rho}(\xi') - \frac{1}{2} \langle H\psi(c)\xi', \xi' \rangle \right)| \leq \left\{
\begin{array}{ll}
\frac{\rho^{|\alpha|}}{24!} |\xi'|^2, & \text{if } |\alpha| = 1, \\
\rho |\xi'|^2, & \text{if } |\alpha| = 2.
\end{array}
\right.
\]

If \( |\alpha| \geq 3 \), then \( \partial^\alpha_\xi \left( \psi_{c, \rho}(\xi') - \frac{1}{2} \langle H\psi(c)\xi', \xi' \rangle \right) = \rho^{3-|\alpha|} \partial^3 \psi(\rho \xi' + c) \). Hence we have (2.8). The second assertion follows immediately from (2.6), (2.8) and the comparison of \( \langle H\psi(c)\xi', \xi' \rangle \) with \( |\xi'|^2 \).

2.3. Estimates for the operator with localized frequency. To obtain the sharp bound (1.13), the case in which \( |\xi| \approx 1 \), \( \text{Re} \xi > 0 \), and \( |\text{Im} \xi| \ll 1 \) is most important (see Subsection 4.1 below). In this case, the corresponding Fourier multiplier carries most of its mass near the sphere \( S_z := \{ \xi \in \mathbb{R}^d : |\xi| = \sqrt{\text{Re} \xi} \} \), where \( \sqrt{\text{Re} \xi} \approx 1 \). Since \( S_z \) is compact and convex with nonvanishing curvature, using finite decomposition and affine transformations, we can regard \( S_z \) as a finite union of graphs of functions of elliptic type. Such operations do not have significant effect on the estimate (1.13) except for a minor change of the multiplicative constant \( C \).

Now, by a dyadic decomposition (away from the graph of a function \( \psi(\xi') \) of elliptic type) in the Fourier side, we need to obtain the sharp bounds for the multiplier operators of which Fourier transform is supported in a \( \delta \)-neighborhood of the surface \( \xi_d = \psi(\xi') \).

For \( \beta > 0 \), let us set

\[
\text{Mul}(N, b) = \{ m \in C^N(\mathbb{R}^d) : 2^{-1} \leq |m| \leq 2, |\partial^\alpha m| \leq b, 1 \leq |\alpha| \leq N \}.
\]
Let $0 \leq \delta \leq 1$, $\lambda \geq 1$ and $\delta \lambda \leq 1/10$, and let $\varphi \in C^\infty(\mathbb{R})$. For $\psi \in \text{Ell}(N, \epsilon)$ and $m \in \text{Mul}(N, b)$ we set

\begin{align}
\mathcal{M}_\delta(\xi) &:= \varphi\left(\frac{m(\xi)(\xi_d - \psi(\xi'))}{\delta}\right)\beta_0\left(\frac{m(\xi)(\xi_d - \psi(\xi'))}{\delta}\right)\chi_0(\xi), \\
\mathcal{M}_{\delta,\lambda}(\xi) &:= \varphi\left(\frac{m(\xi)(\xi_d - \psi(\xi'))}{\delta}\right)\beta\left(\frac{m(\xi)(\xi_d - \psi(\xi'))}{\delta \lambda}\right)\chi_0(\xi),
\end{align}

where $\chi_0 \in C^\infty(2^{-1}\mathbb{I}^d)$. By the additional $m$ we may perturb the multipliers $\mathcal{M}_\delta$ and $\mathcal{M}_{\delta,\lambda}$, so this allows us to handle other classes of operators which are given by multipliers with similar structure.

The following provide sharp estimates for $\mathcal{M}_\delta(D)$ and $\mathcal{M}_{\delta,\lambda}(D)$ and these are most important ingredients in proving Theorem 1.4.

**Proposition 2.4.** Let $b > 0$ and suppose that, for $k \geq 0$,

\begin{align}
\left|\left(\frac{d}{dt}\right)^k \varphi(t)\right| &\leq C_b t^{k-1}, \quad |t| \geq 1. \tag{2.11}
\end{align}

Then, for $p, q$ satisfying $\frac{1}{q} = \frac{d-1}{q_0}(1 - \frac{1}{q_0})$ and $q_0 < q \leq \frac{2(d+1)}{d-1}$, there exist $N$ and $\epsilon > 0$ such that the following hold uniformly provided that $\psi \in \text{Ell}(N, \epsilon)$ and $m \in \text{Mul}(N, b)$:

\begin{align}
\|\mathcal{M}_\delta(D)f\|_{L^q(\mathbb{R}^d)} &\leq C\delta^{-\frac{d-2}{2}}\|f\|_{L^p(\mathbb{R}^d)}, \tag{2.12} \\
\|\mathcal{M}_{\delta,\lambda}(D)f\|_{L^q(\mathbb{R}^d)} &\leq C\lambda^{-1}\delta \lambda^{-\frac{d-2}{2}}\|f\|_{L^p(\mathbb{R}^d)}. \tag{2.13}
\end{align}

Here the constant $C$ may depend on $b$, $d$, $p$, $q$, $N$, $\epsilon$, $\varphi$ and $\chi_0$, but is independent of $\delta$, $\lambda$, $m$, $\psi$ and $f$.

Remark 6. Similar estimates were obtained in [12 Proposition 2.4]. However, there are differences which need to be mentioned. Firstly, the function $\varphi \in S(\mathbb{R})$ in [12 Proposition 2.4] is assumed to have the special cancellation property $\text{supp } \hat{\varphi} = \{ t \in \mathbb{R} : |t| \approx 1 \}$ which was crucial in obtaining the sharp estimate, whereas we do not need such extra assumption in Proposition 2.4. This is necessary for the proof of Theorem 1.4. Unlike [12] the associated multipliers $\langle |\xi|^2 - z \rangle^{-1}$ are not homogeneous, so we can not decompose them in such a nice way as in [12 Lemma 2.1] (also, see [30] Section 2.1). Secondly, we allow smooth perturbation of $m$ in (2.9) and (2.10) with $m$ satisfying $|m| \approx 1$. Lastly, the estimates (2.12) and (2.13) hold on a wider range of $p, q$ than that of the estimate in [12 Proposition 2.4].

We postpone the proof of Proposition 2.4 until the next section. For the rest of this section we present results which will be used for the proof of Proposition 2.4.

2.4. $L^p$ boundedness of multiplier operators. In this subsection, we obtain the sharp $L^p$ estimates for the multiplier operators $\mathcal{M}_\delta(D)$ and $\mathcal{M}_{\delta,\lambda}(D)$ which are consequences of Theorem 2.2. We work with $\mathcal{M}_{\delta,\lambda}(D)$ only, since the same argument also works for $\mathcal{M}_\delta(D)$. In what follows all of the $L^p$ estimates are uniform in $\psi \in \text{Ell}(N, \epsilon)$, $m \in \text{Mul}(N, b)$, provided that $N$ is sufficiently large and $\epsilon$ is sufficiently small.

**Proposition 2.5.** Let $b$, $\chi_0$, $\delta$, $\lambda$ and $\varphi$ be as in Proposition 2.4 and suppose that $p_0 < p \leq \infty$.

Then there exist a large $N > 0$, a small $\epsilon > 0$ and a constant $C > 0$ such that

\begin{align}
\|\mathcal{M}_\delta(D)f\|_{L^p(\mathbb{R}^d)} &\leq C\delta^{-\frac{d-2}{2}}\|f\|_{L^p(\mathbb{R}^d)}, \tag{2.14} \\
\|\mathcal{M}_{\delta,\lambda}(D)f\|_{L^p(\mathbb{R}^d)} &\leq C\lambda^{-1}\delta \lambda^{-\frac{d-2}{2}}\|f\|_{L^p(\mathbb{R}^d)},
\end{align}

where the constants $C$ are independent of $\delta$, $\lambda$, $\psi \in \text{Ell}(N, \epsilon)$ and $m \in \text{Mul}(N, b)$.

We will achieve this by making use of Theorem 2.2. For this purpose we need to compute the kernel $K_{\delta,\lambda}$ of the operator $\mathcal{M}_{\delta,\lambda}(D)$. 


Remark 7. To begin with, we readjust the cutoff functions \( \chi_0 \) in (2.10) of which role is not so significant for the overall estimates. We may regard \( \chi_0 \tilde{f} \) as if it is \( \tilde{f} \) (note that \( \| \mathcal{F}^{-1}(\chi_0 \tilde{f}) \|_p \lesssim \| \tilde{f} \|_p \) if \( \chi_0 \in C^\infty_0 \)). We may also introduce a new cutoff function \( \chi_1 \) whenever \( \chi_0 \chi_1 = \chi_0 \) and replace \( \tilde{f} \) with \( \chi_1 \tilde{f} \). By decomposing (with a suitable partition of unity) \( \chi_0 \) into finitely many cutoff functions with smaller support (of diameter \( \lesssim \varepsilon_0 \)) we may assume \( \chi_0 \) is supported in a small neighborhood near the surface \( \xi_d = \psi(\xi') \). Otherwise, the contribution is negligible. In fact, the associated kernel has a bounded \( L^1 \)-norm as can be seen easily by a straightforward kernel estimate. Let \( \xi_0 = (c, \psi(c)) \) and suppose \( \chi_0 \) is supported in \( B(\xi_0, \varepsilon_0) \) for a fixed \( \varepsilon_0 \). Then, for \( 0 < \rho \leq 2^{-1} \), we may use the harmless affine transform

\[
\xi \to L_{c,p}(\xi) = (\rho \xi' + c, \rho^2 \xi_d + \psi(c) + \rho \nabla \psi(c) \cdot \xi')
\]
to write

\[
\mathfrak{m}_{\delta,\lambda}(L_{c,p}(\xi)) = \varphi \left( \frac{m(L_{c,p}(\xi_d - \psi_{c,\rho}(\xi')))}{\rho^{-2} \delta} \right) \beta \left( \frac{m(L_{c,p}(\xi_d - \psi_{c,\rho}(\xi')))}{\rho^{-2} \delta} \right) \chi_0(L_{c,p}(\xi)).
\]

By Lemma 2.3 \( \psi_{c,\rho} \in \text{Ell}(N, \varepsilon) \). Also, \( m \circ L_{c,\rho} \in \text{Mul}(N, Cb) \) for some \( C > 0 \). Thus we may regard this as the same multiplier given by (2.10) by simply replacing \( \rho^{-2} \delta \), \( m \circ L_{c,\rho} \), and \( \psi_{c,\rho} \) with \( \delta, m, \) and \( \psi \), respectively. Hence taking \( \varepsilon_0 \) small enough and \( \rho = \varepsilon_0 2^7 \), after a simple manipulation (discarding the part of multiplier which is away from the surface) we may assume the cutoff function takes the form

\[
\chi_0(\xi) = \chi(\xi') \chi_0(\xi_d - \psi(\xi'))
\]
and \( \chi \in C^\infty_0(B_{d-1}(0, 2^{-7})) \) and \( \chi_0 \in C^\infty_0((-2^{-7}, 2^{-7})) \).

By Remark 7 and change of variables (\( (\xi', \xi_d) \to (\xi', \xi_d + \psi(\xi')) \)) we may write

\[
K_{\delta,\lambda}(x) := \mathcal{F}^{-1}(\mathfrak{m}_{\delta,\lambda})(x) = \frac{1}{2\pi} \int e^{ix \cdot \xi_d} \chi_0(\xi_d) I_{\psi}(x; \xi_d) \, d\xi_d,
\]
where

\[
I_{\psi}(x; \xi_d) = \frac{1}{2\pi} \int e^{i(x' \cdot \xi' + x_d \psi(\xi'))} \varphi \left( \frac{m(x')}{\delta} \right) \beta \left( \frac{m(x')}{\delta} \right) \chi(\xi') \, d\xi',
\]
and \( \tilde{m}(\xi) = m(x', \xi_d + \psi(\xi')) \) still enjoys the same property as \( m \) in Proposition 2.4 that is to say, \( m \in \text{Mul}(N, Cb) \) for some \( C > 0 \). For simplicity we put

\[
A_{\delta,\lambda}(\xi) := \varphi \left( \frac{m(\xi)}{\delta} \right) \beta \left( \frac{m(\xi)}{\delta} \right) \chi(\xi').
\]

Let us collect some bounds for the functions \( A_{\delta,\lambda} \) and their differentials which will be useful later to show the uniformity of the constant \( C \) in \( m, \delta, \lambda \) and \( \psi \) in Proposition 2.5.

Lemma 2.6. Let \( 0 < \delta \leq \delta \lambda \leq 1 \), \( b > 0 \) and let \( \psi \in \text{Ell}(N, \varepsilon) \), \( m \in \text{Mul}(N, b) \). Then, for every \( (d - 1) \)-dimensional multi-index \( \vartheta \in \mathbb{N}^{d-1}_0 \) with \( |\vartheta| \leq N \), we have

\[
\sup_{\xi} |\partial_{\xi}^\vartheta A_{\delta,\lambda}(\xi)| \leq C_{\delta} \lambda^{-1}
\]
uniformly in \( \delta, \lambda, \psi \in \text{Ell}(N, \varepsilon) \) and \( m \in \text{Mul}(N, b) \). More generally, for every \( d \)-dimensional multi-index \( \alpha = (\alpha', \alpha_d) \in \mathbb{N}^{d-1}_0 \times N \) and every \( \vartheta \in \mathbb{N}^{d-1}_0 \) such that \( |\alpha| + |\vartheta| \leq N \), we have

\[
\sup_{\xi} |\partial_{\xi}^\alpha \partial_{\xi_d}^\vartheta (\xi_d) \partial_{\xi}^\vartheta A_{\delta,\lambda}(\xi)| \leq C_{\alpha,\delta} \lambda^{-1} (\delta \lambda)^{-\alpha_d + \ell}
\]
with \( C_{\alpha,\delta} \) independent of \( \delta, \lambda, m, \) and \( \psi \).

Proof. For every \( k \in \mathbb{N}_0 \) note that \( \beta^{(k)} \left( \frac{\tilde{m}(\xi_d)}{\delta} \right) \neq 0 \) only if \( |\xi_d| \approx \delta \lambda \) since \( |\tilde{m}| \approx 1 \). Hence for every \( \vartheta \in \mathbb{N}^{d-1}_0 \) with \( 0 \leq |\vartheta| \leq N \) it is easy to see that \( \text{supp} \partial_{\xi}^\vartheta A_{\delta,\lambda} \) is contained in the set \( \{ \xi : |\xi_d| \approx \delta \lambda \} \) and that

\[
\sup_{\xi} \left| \partial_{\xi}^\vartheta \left( \beta \left( \frac{\tilde{m}(\xi_d)}{\delta} \right) \right) \right| \leq C_{\delta}
\]
with $C_\phi$ independent of $\delta, \lambda, m$ and $\psi$. Also, for $0 \leq |\vartheta| \leq N$,
\[
\sup_{\xi} |\partial^\vartheta_\xi \left( \varphi \left( \frac{\tilde{m}(\xi)\xi_d}{\delta} \right) \right)| \lesssim \sum_{k=0}^{\lfloor |\vartheta| \rfloor} \sup_{\xi} |\varphi^{(k)} \left( \frac{\tilde{m}(\xi)\xi_d}{\delta} \right) \left( \frac{\xi_d}{\delta} \right)^k|,
\]
where the implicit constant is independent of $\delta, \lambda, m$ and $\psi$. Since $|\tilde{m}| \approx 1$ and $|\xi_d| \approx \lambda \delta$ on $\text{supp} A_{\delta, \lambda}$, by (2.11) we see that
\[
(2.22) \quad \sup_{\xi'} \left| \partial^\vartheta_{\xi'} \left( \varphi \left( \frac{\tilde{m}(\xi')\xi'_d}{\delta} \right) \right) \right| \leq C_\phi \lambda^{-1}.
\]
By combining (2.21) and (2.22) it is easy to see (2.19).

For the proof of (2.20) we first consider the case $\vartheta' = \vartheta = 0$. Note that $\partial^\vartheta_{\xi_d} ((\xi_d)^\vartheta A_{\delta, \lambda})$ is given by a linear combination of $(\xi_d)^{\ell-n} \partial^\vartheta_{\xi_d} A_{\delta, \lambda}$ and $\partial^\vartheta_{\xi_d} A_{\delta, \lambda}$ is also a linear combination of
\[
\delta^{-\nu} (\lambda \delta)^{-\mu} \chi_{\mu, \nu}(\xi) \varphi^{(\nu)} \left( \frac{\tilde{m}(\xi)\xi_d}{\delta} \right) \beta^{(\mu)} \left( \frac{\tilde{m}(\xi)\xi_d}{\delta} \right), \quad \mu + \nu \leq \alpha_d - n.
\]
Here $\chi_{\mu, \nu}$ is a smooth function with bounded derivatives. Since $|\xi_d| \approx \lambda \delta$ on $\text{supp} A_{\delta, \lambda}$, we deduce the desired bound (2.20) by (2.11). For the general cases one can routinely repeat the same argument keeping in mind that $\partial^\vartheta_{\xi'}$ or $\partial^\vartheta_{\xi_d}$ behaves almost similarly as in (2.19) on $\text{supp} A_{\delta, \lambda}$. So, we omit the detail.

We now obtain the asymptotic for the function $I_\psi (\cdot, \xi_d)$. Since $\psi \in \text{Ell}(N, \epsilon)$, $\nabla \psi(0) = 0$ and $\nabla \psi(\xi') = \xi' + O(\epsilon)$ for $\xi' \in I^{d-1}$. Thus, by the inverse function theorem we see that there exist neighborhoods $U, V$ of the origin and a unique diffeomorphism $g : U \to V$ such that $g(0) = 0$ and
\[
(2.23) \quad \nabla g + \nabla \psi(g(\xi')) = 0.
\]
If we take $\epsilon$ sufficiently small, we may assume that $U \supset B_{d-1}(0, 1/2)$. In fact, $(g(\xi'), \psi \circ g(\xi'))$ is the unique point on the graph $G(\psi) := \{(\xi', \psi(\xi')) : \xi' \in \text{supp} \chi \}$ at which the normal vector is parallel to $(t', 1)$. We denote by $K(\xi)$ the Gaussian curvature of the surface $G(\psi)$ at point $\xi = (\xi', \psi(\xi'))$ and by $Jg$ the Jacobian matrix of the diffeomorphism $g$. Direct differentiation of the equation (2.23) gives
\[
(2.24) \quad ((H \psi) \circ g) \cdot Jg = -I_{d-1}.
\]

**Lemma 2.7.** Let $0 \leq \delta \leq \delta \lambda \leq 1$. Suppose that $N$ (resp., $\epsilon$) is large (resp., small) enough so that for every $\psi \in \text{Ell}(N, \epsilon)$, the aforementioned diffeomorphism $g : U \supset B_{d-1}(0, 1/2) \to V$ exists. Then the following hold.

1. If $|x_d| \geq 1/2$ and $2^j|x'| \leq |x_d|$, then for every $M \in \mathbb{N}$ satisfying $2M \leq N$ we have
\[
(2.25) \quad I_\psi (x; \tau) = (2\pi)^{-\frac{d-1}{2}} |x_d|^{-\frac{d-1}{2}} e^{-\frac{(x'-g(x_d'))^2}{4|x_d'|}} \left( 1 + \sum_{j=0}^{M-1} D_j A_{\delta, \lambda}(\xi', \tau) \right) \left( \frac{|x_d|}{2} \right)^{-j} + E_{\delta, \lambda, M}(x; \tau),
\]
where $D_0 A_{\delta, \lambda} = A_{\delta, \lambda}$ and for each $j \geq 1$ $D_j$ is a differential operator in $\xi'$ of order $2j$ whose coefficients vary smoothly depending on $(\partial^\alpha \psi) \circ g \left( \frac{x'}{x_d} \right)$, $2 \leq |\alpha| \leq 2j + 2$. For $E_{\delta, \lambda, M}(x; \tau)$ we have the estimate
\[
(2.26) \quad |E_{\delta, \lambda, M}(x; \tau)| \leq C |x_d|^{-M} \sum_{|\alpha| \leq 2M} \sup_{(\xi', \tau')} |\partial^\alpha \psi \circ g \left( \frac{x'}{x_d} \right)| \leq C |x_d|^{-M} \lambda^{-1}
\]
with $C'$ independent of $\delta, \lambda, m$ and $\psi$.

2. On the other hand, if $2^j|x'| \geq |x_d|$ or $|x_d| \leq 2$, then for every $0 \leq M \leq N$ there exists a constant $C_M$, independent of $\delta, \lambda, m$ and $\psi$, such that
\[
(2.27) \quad |I_\psi (x; \tau)| \leq C_M \lambda^{-1} (1 + |x|)^{-M}.
\]
Proof. The asymptotic expansion (2.25) in (I) is a consequence of the stationary phase method. For its proof we refer the reader to [28, Theorems 7.7.5 and 7.7.6]. In (2.26) the uniformity of $C'$ in $\delta, \lambda, m$ and $\psi$ follows from Lemma 2.6.

For the second statement (II) we use integration by parts. Since $\text{supp} \chi \subset B_{d-1}(0,2^{-7})$ and $|\nabla \psi(\xi')| \leq (1+c\epsilon)|\xi'|$, it is easy to observe that, if $c$ is sufficiently small and $2^6|x'| \geq |x_d|$, $|\nabla \psi(\xi') + x_d \psi(\xi')| \geq |x'| (1 - 2^6|\nabla \psi(\xi')|) \gtrsim |x'|$. And if $|x_d| \leq 2$ the same estimate also holds with $|x'| \geq 1$. Hence (2.27) follows from integration by parts in $\xi'$ together with (2.19) in Lemma 2.6. $\square$

Now we prove (2.14) by combining Theorem 2.2 and Lemma 2.7.

**Proof of (2.14).** Let $\tilde{\chi}$ be a smooth function on $\mathbb{R}$ supported in the interval $(-2^{-5}, 2^{-5})$ and equal to 1 on $(-2^{-6}, 2^{-6})$. We break the kernel $K_{\delta\lambda}$ as follows:

$$K_{\delta\lambda}(x) = K_{\delta\lambda,0}(x) + \sum_{l=1}^{\infty} K_{\delta\lambda,l}(x),$$

where

$$K_{\delta\lambda,l}(x) = \tilde{\chi}\left(\frac{|x'|}{x_d}\right) \beta(2^{-l}x_d) K_{\delta\lambda}(x)$$

for $l \in \mathbb{N}$. So, the function $K_{\delta\lambda,0}$ is supported on the set $R := \{x : 2^6|x'| \geq |x_d|\} \cup \{x : |x_d| \leq 2\}$, and it follows from (2.27) (II) that

$$|K_{\delta\lambda,0}(x)| \leq C_M \lambda^{-1}(1 + |x|)^{-M}$$

for any $0 \leq M \leq N$, uniformly in $\delta, \lambda$. Since $\|K_{\delta\lambda,0}\|_{L^p} \lesssim \lambda^{-1}$, the operator $f \to K_{\delta\lambda,0} * f$ admits much better estimate than (2.14), since $p > p_{\ast} \geq \frac{2d}{d-1}$ and $\lambda \lesssim 1$. Therefore it suffices to prove that

$$\sum_{l=1}^{\infty} \|K_{\delta\lambda,l} * f\|_{L^p} \leq C \lambda^{-1}(\delta \lambda)^{\frac{d}{2} - \frac{d+1}{2}} \|f\|_{L^p}. \tag{2.28}$$

To show this we need the asymptotic (2.25) for $I_{\psi}(\cdot, \xi_d)$, which is to be combined with (2.17). Fixing $M \leq 2N$ large enough, it is enough to handle the finite summation in (2.25) since the contribution from the error term $E_{d, M}$ in (2.26) is at most $\lambda^{-1}$. In fact, since $(K_{\delta\lambda} - K_{\delta\lambda,0})(x) \neq 0$ only if $|x'| \gtrsim |x_d|$, if we set $K_{\text{err}}(x) = \frac{1}{x_d} \int e^{ix_d\xi_d} \chi_0(\xi) E_{d, M}(x, \xi_d) d\xi_d$, it follows from (2.26) that $\|K_{\text{err}}\|_{L^p} \lesssim \lambda^{-1}$. Thus, the contribution from the first term in (2.25) is most significant and it suffices to prove (2.28) by replacing $K_{\delta\lambda,l}$ with

$$\tilde{K}_{\delta\lambda,l}(x) = \tilde{\chi}\left(\frac{|x'|}{x_d}\right) \beta(2^{-l}x_d) |x_d|^{-\frac{d+1}{2}} \int e^{ix_d\xi_d} \chi_0(\xi) E_{d, M}(g(\frac{x'}{x_d}), \psi \circ g(\frac{x'}{x_d})) \left|K\left(g\left(\frac{x'}{x_d}\right), \psi \circ g\left(\frac{x'}{x_d}\right)\right)\right|^{-\frac{1}{2}}$$

$$\times \int e^{ix_d\xi_d} \chi_0(\xi) d\xi_d,$$

for $l \in \mathbb{N}$. The contributions from the other terms given by replacing $A_{\delta\lambda}$ with $D_{\delta\lambda} A_{\delta\lambda}$ can be handled similarly. In fact, since $D_{\delta\lambda}$ are only involved with derivatives in $\xi'$, by making use of Remark 8 below, we may repeat the same argument for those terms but they give even better bounds because of the additional decay factor $|x_d|^{-1/2}$. Therefore, for (2.28) we need only to show that

$$\sum_{l=1}^{\infty} \|\tilde{K}_{\delta\lambda,l} * f\|_{L^p} \leq C \lambda^{-1}(\delta \lambda)^{\frac{d}{2} - \frac{d+1}{2}} \|f\|_{L^p}. \tag{2.29}$$

By scaling, the $L^p - L^p$ norm of the convolution operator $f \to K \ast f$ is equal to that of $f \to L^d K(L \cdot) \ast f$ for any $L > 0$. Thus for (2.29) we are reduced to showing that for a large enough $M > 0$

$$\|2^d\tilde{K}_{\delta\lambda,l}(2^d\cdot) * f\|_{L^p} \lesssim_M 2^{\frac{d+1}{2} - \frac{d}{2}}(1 + 2^d \delta \lambda)^{-M} \|f\|_{L^p}. \tag{2.30}$$

And if $|x_d| \leq 2$ the same estimate also holds with $|x'| \geq 1$. Hence (2.27) follows from integration by parts in $\xi'$ together with (2.19) in Lemma 2.6. $\square$
Summation (2.31) over \( l \geq \log_2(\frac{1}{\delta}) \) and \( l < \log_2(\frac{1}{\delta}) \), separately yield (2.30). Indeed,
\[
\sum_{2^l \geq (\delta\lambda)^{-1}} \| \tilde{K}_{\delta, \lambda, l} \ast f \|_p \lesssim \delta(\delta\lambda)^{-M} \| f \|_p \sum_{2^l \geq (\delta\lambda)^{-1}} 2^l \left( \frac{M}{p} - \frac{d}{p} \right) \lesssim \lambda^{-1}(\delta\lambda)^{\frac{d}{p} - \frac{d-1}{p}} \| f \|_p
\]
by choosing \( M > \frac{d+1}{2} - \frac{d}{p} \). On the other hand, since \( \frac{d+1}{2} > \frac{d}{p} \),
\[
\sum_{2^l \geq (\delta\lambda)^{-1}} \| \tilde{K}_{\delta, \lambda, l} \ast f \|_p \lesssim \delta(\delta\lambda)^{-1} \sum_{2^l \geq (\delta\lambda)^{-1}} 2^l \left( \frac{d+1}{2} - \frac{d}{p} \right) \lesssim \lambda^{-1}(\delta\lambda)^{\frac{d}{p} - \frac{d-1}{p}} \| f \|_p.
\]
Combining these two estimates we get (2.30).

We now turn to the proof of (2.31). Since the kernel \( K_{\delta, \lambda, l}(2^l x) \) is supported in the set \( \{ x : |x'| < 2^{-1}, 3/8 \leq |x| \leq 9/8 \} \), it is enough to show the local estimate
\[
\| 2^l \tilde{K}_{\delta, \lambda, l}(2^l \cdot) \ast f \|_{L^p(B_d(x_o, \epsilon))} \leq C_M \delta(1 + 2^l \delta\lambda)^{-M} \| f \|_{L^p(B_d(x_o, 4))}
\]
with \( C_M \) independent of \( l, \delta, \lambda \) and \( x_o \in \mathbb{R}^d \). Estimate (2.31) follows directly from (2.32) by integrating with respect to the \( x_o \)-variable and using Fubini’s theorem. The rest of this section is devoted to proof of (2.32). Clearly, we may assume that \( x_o = 0 \) by translation.

Let us set \( \tilde{\gamma}(t) = |t|^{-\frac{d+1}{2}} \beta(t) \) and fix a function \( \beta_o \in C_0^\infty([-2, -2^{-2}] \cup [2^{-2}, 2]) \) such that \( \tilde{\beta} \beta_o = \tilde{\beta} \). We also set
\[
a_{\delta, \lambda, l}(x) := \tilde{\chi} \left( \frac{|x'|}{x_d} \right) \beta_o(x_d) \left| K \left( \frac{x'}{x_d}, \psi \circ g \left( \frac{x'}{x_d} \right) \right) \right|^{-\frac{1}{2}} \int e^{i2^lx_d\sigma} \chi_o(\sigma) \lambda \left( \frac{x'}{x_d}, \sigma \right) d\sigma.
\]
and
\[
\Phi(x, y) := (x' - y') \circ g \left( \frac{x'}{x_d} \right) + (x_d - y_d)(\psi \circ g \left( \frac{x'}{x_d} \right)).
\]
Let \( \eta \) be a nonnegative smooth function \( \eta \in C_0^\infty(B_o(0, 2)) \) whose value is equal to 1 on the unit ball \( B_o(0, 1) \). Freezing \( y_d \) we put
\[
(2.33) \quad \Phi^{y_d}(x, y') := \Phi(x, y', y_d), \quad a^{y_d}_{\delta, \lambda, l}(x, y') := \eta(x) a_{\delta, \lambda, l}(x - (y', y_d)).
\]
Then from (2.29) and the choice of \( \beta \) it is clear that, for \( x \in B_d(0, 1) \),
\[
(2.34) \quad \int \tilde{\gamma}(x_d - y_d) \left( T_{2^l} [\Phi^{y_d}, a^{y_d}_{\delta, \lambda, l}] f (\cdot, y_d) \right)(x) dy_d.
\]
Next, we show that the phase \( \Phi^{y_d} \) in (2.34) satisfies the Carleson–Sjölin condition (2.1) and (2.2) and the elliptic condition (2.5) uniformly in \( \psi \in \mathcal{E} \left[ N, \epsilon \right] \) and \( y_d \in [-4, 4] \) on the set
\[
S_{y_d} := \{(x, y') \in \mathbb{R}^d \times \mathbb{R}^{d-1} : |x| \leq 2, |x' - y'| \leq 2^{-3}, 2^{-2} \leq |x_d - y_d| \leq 2 \}.
\]
Let us write \( g = (g_1, \cdots, g_{d-1}) \). Differentiating (2.34) directly and then using (2.23) it is easy to see that
\[
\partial_x \Phi^{y_d}(x, y') = g \left( \frac{x' - y'}{x_d - y_d} \right), \quad \partial_{x_d} \Phi^{y_d}(x, y') = \psi \circ g \left( \frac{x' - y'}{x_d - y_d} \right)
\]
Differentiating these equations with respect to \( y' \) the rank condition (2.1) can be easily verified by (2.24). For \( v = (v', v_d) = (v_1, \cdots, v_{d-1}, v_d) \in \mathbb{R}^d \) we see that
\[
(2.35) \quad \partial_{y'} \langle v, \partial_x \Phi^{y_d}(x, y') \rangle = \frac{-1}{x_d - y_d} \left( v' + v_d (\nabla \psi) \circ g \left( \frac{x' - y'}{x_d - y_d} \right) \right) J g \left( \frac{x' - y'}{x_d - y_d} \right)
\]
Hence, for fixed \( y_d \in [-4, 4] \) and \( (x, y'_c) \in S_{y_d} \), the unique (up to sign) direction \( v \) in (2.2) can be chosen as
\[
v = \frac{w}{|w|}, \quad w = \left( - (\nabla \psi) \circ g \left( \frac{x' - y'_c}{x_d - y_d} \right), \frac{1}{x_d - y_d} \right) = \left( \frac{x' - y'_c}{x_d - y_d}, 1 \right),
\]
where the second equality holds because of (2.23). By a straightforward computation we see that
\[
\partial_{y'}^2 \langle v, \partial_x \Phi^{y_d}(x, y') \rangle |_{y' = y'_c} = \frac{1}{(x_d - y_d)^2|w|^2 (H \psi) \circ g \left( \frac{x' - y'_c}{x_d - y_d} \right) (J g)^2 \left( \frac{x' - y'_c}{x_d - y_d} \right)}.
\]
Since $-I_2 = ((H^*)^g)^{-1}$ is close to the identity matrix $I_2$ (see (2.24) and (2.8)), we see that the nondegeneracy (2.2) and the ellipticity (2.3) hold whenever $(x, y, y_d) \in \bigcup \{y_d \leq 4\} S_{y_d} \times \{y_d\}$.

For the moment, let $a \in C_0^\infty (S_{y_d})$ for $y_d \in [-4, 4]$. We apply Theorem 2.2 to the operator $T_{a^j} [\Phi^{yd}, a]$, which gives

$$\left\| T_{a^j} [\Phi^{yd}, a] h \right\|_{L^p(R^d)} \lesssim \rho^{-d/p} \left\| h \right\|_{L^p(R^{d-1})}$$
for $p > p_*$. The bound is uniform not only for $y_d \in [-4, 4]$ but also $\psi \in Ell(N, \epsilon)$ (see Remark 3).

To get estimate for $T_{a^j} [\Phi^{yd}, a^d_{\delta, \lambda, l}]$ in (2.35), we need to replace $a$ in (2.37) with $a^{yd}$ for the purpose we need the following which is a modification of [48, Lemma 2.1]. In fact, for the proof of Lemma 2.8 one only need to expend $a_1$ into the Fourier series.

**Lemma 2.8** (48). Let $a_0, a_1 \in C_0^\infty (H \times H)$ such that $a_0 a_1 = a_1$. Suppose \( \| T_{p} [\Phi, a_0] h \|_q \leq L \| h \|_p \) and suppose \( |\partial^\alpha a_1| \lesssim B \) for $|\alpha| \leq d$. Then, there is a constant $C$ independent of $\Phi$, and $a_1$ such that \( \| T_{p} [\Phi, a_1] h \|_q \leq CBL \| h \|_p \).

**Lemma 2.9.** Let $0 < \delta \leq \delta \lambda \leq 1$, $b > 0$ and let $\epsilon > 0$ be small enough. Then for every $M \geq 0$ and every multi-index $\alpha$ such that $|\alpha| \leq N - M$ there exists a constant $C_{\alpha, M}$ independent of $\delta$, $\lambda$, $l$, and $(\psi, m) \in Ell(N, \epsilon)$ such that

$$|\partial^\alpha a_{\delta, \lambda, l}(x)| \leq C_{\alpha, M} \delta (1 + 2\delta \lambda)^{-M}.$$

Now, by combining the estimate (2.37), Lemma 2.8 and Lemma 2.9 we obtain the estimate for $T_{a^j} [\Phi^{yd}, a^d_{\delta, \lambda, l}]$. Indeed, observe that in (2.33) and (2.34) the amplitude $\eta(x_a, x_a - (y', y_d))$ is nonzero only if $|x| \leq 2$, $3/8 \leq |x_a - y_d| \leq 9/8$ and $|x' - y'| \leq 2^{-4}$. Taking a smooth function $a$ such that $supp a \subset \bigcup \{y_d \leq 4\} S_{y_d}$ and $a a^d_{\delta, \lambda, l} = a^d_{\delta, \lambda, l}$ for all $y_d \in [-4, 4]$, we may apply Lemma 2.8 and Lemma 2.9 to get

$$\left\| T_{a} [\Phi^{yd}, a^d_{\delta, \lambda, l}] h \right\|_{L^p(R^d)} \leq C_M 2^{-d/p} \delta (1 + 2\delta \lambda)^{-M} \left\| h \right\|_{L^p(R^{d-1})}.$$

We now recall (2.35) and use Minkowski’s inequality to obtain

$$\left\| 2^{l} \overline{K}_{\delta, \lambda, l} (2^l \cdot) \ast f \right\|_{L^p(B(0,1))} \leq 2^{l(\frac{d+1}{2} - \frac{1}{p})} \left( \int_{|x| \leq 2} \left( \int_{-4}^{4} |\tilde{\beta}(x_a - y_d) (T_{a} [\Phi^{yd}, a^d_{\delta, \lambda, l}] f(\cdot, y_d))(x)| d y_d \right)^p dx \right)^{\frac{1}{p}}$$

$$\leq 2^{l(\frac{d+1}{2} - \frac{1}{p})} \left( \int_{-4}^{4} \left( \int_{\bigcup \{y_d \leq 4\}} \left| (T_{a^j} [\Phi^{yd}, a^d_{\delta, \lambda, l}] f(\cdot, y_d))(x) \right|^p d y_d \right)^{\frac{1}{p}} dx \right)^{\frac{1}{p}}$$

Finally using (2.39) which is followed by integration in $y_d$ gives the desired estimate (2.32). To complete proof of Proposition 2.5 it remains to show Lemma 2.9. 

**Proof of Lemma 2.9** Let us set

$$\mathcal{I}_{\delta, \lambda, l}(x) := \int e^{i2^l x_a} \tilde{x} \chi(g(x_a, y_d, \tau)) d \tau.$$

Since the term $\tilde{x}(\frac{x'}{x_d}) \beta_\lambda g(x_a, \tau)$ has bounded derivatives of any order it is sufficient to show that for $2^l |x'| \leq |x_d| \approx 1$

$$|\partial^\alpha \mathcal{I}_{\delta, \lambda, l}(x)| \leq C_{\alpha, M} \delta (1 + 2\delta \lambda)^{-M}.$$

Let us first consider the case $|\alpha| = 0$. By integration by parts

$$\mathcal{I}_{\delta, \lambda, l}(x) = \left( \frac{-1}{2} i 2^l x_d \right)^M \int e^{i2^l x_a} \left( \frac{d}{d \tau} \right)^M \chi(\tau) A_{\delta, \lambda} \left( g(x_a, x_d, \tau) \right) d \tau.$$

Since $0 < \delta \lambda \leq 1$, recalling (2.18), (2.11) and using Lemma 2.6 (2.20) with $|\alpha'| = |\partial^\ell| = 0$, we get

$$\left| \left( \frac{d}{d \tau} \right)^M \chi(\tau) A_{\delta, \lambda} \left( g(x_a, x_d, \tau) \right) \right| \leq C_M \lambda^{-1} (\delta \lambda)^{-M}$$
for $M \leq N$. Thus we obtain the desired bound (2.40) when $|\alpha| = 0$. 

\[ \square \]
Next we turn to proof of (2.40) for the case $|\alpha| \geq 1$. We observe that the case $\alpha_d = 0$ can be handled similarly as before in the case $|\alpha| = 0$ by making use of Lemma 2.6 (2.20) in [2.20] with $\ell = 0$ since the derivative $\partial_x^\alpha = \partial_x^{\alpha'}$ produces additional terms given by $(\partial_x^{\alpha'} A_{\delta,\lambda})(g(x_d, \tau)), |\alpha'| \leq |\alpha|$. However, if $\partial_{x,d}$ is involved we need to be additionally careful. Note that

$$
\partial_{x,d} I_{\delta,\lambda, l}(x) = i 2 \int e^{i x_d \tau} \chi_\delta(\tau) \tau A_{\delta,\lambda}(g(x_d, \tau)) d\tau
- \frac{1}{x_d} \int e^{i x_d \tau} \chi_\delta(\tau)(\nabla \tau A_{\delta,\lambda})^k(g(x_d, \tau)) \cdot j_k(x_d) \cdot x_d d\tau.
$$

For the first term, using Lemma 2.6 (2.20) in [2.20] and repeating the same argument as before in the case $|\alpha| = 0$, we see that it is bounded by $C_{\alpha,M} \delta(1 + 2^d \delta)^{-M+1}$. For the second term we use (2.19) to see that this is bounded by $C_{\alpha,M} \delta(1 + 2^d \delta)^{-M}$. Then we may repeat the same argument for general $\partial_x^\alpha$ to get

$$
|\partial_{x,d}^\alpha \partial_x^{\alpha'} I_{\delta,\lambda, l}(x)| \leq C_{\alpha,M} \delta(1 + 2^d \delta)^{-M+\alpha_d}
$$

for any $M + |\alpha| \leq N$.

Remark 8. It is not difficult to see that the same estimate for $\alpha_{\delta,\lambda, l}$ remains valid even if we replace $A_{\delta,\lambda}$ in (2.33) with $D_j A_{\delta,\lambda}$ which appears in (2.25). This is due to Lemma 2.6 and the fact that $D_j$ is given by derivatives in $\xi'$, thus the above argument also works.

2.5. Bilinear estimates for multiplier operators. In this section we obtain bilinear $L^2 \times L^2 \to L^{q/2}$ estimates for the multiplier operators $M_{\delta}(D)$ and $M_{\delta,\lambda}(D)$ when $q > 2(d+2)/d$. For this let us first recall the bilinear estimate for the extension operators given by elliptic surfaces which is due to Tao [49].

Theorem 2.10 [49]. Let $q > \frac{2(d+2)}{d}, a_0 \in (2^{-5}, 1/2]$. Then there exist $N, \epsilon > 0$ and $C = C(d,q,a_0) > 0$ such that

$$
\left\| \prod_{k=1,2} \int_{\mathbf{R}^{d-1}} h_k(\xi') e^{i(x' - x_d + x_d \psi(\xi'))} d\xi' \right\|_{L^{q/2}(\mathbf{R}^d)} \leq C \prod_{k=1,2} \|h_k\|_{L^2([-1,1]^{d-1})}
$$

for all $\psi \in \mathcal{E}(N, \epsilon)$ and all $h_1, h_2 \in L^2(\mathbf{R}^{d-1})$ satisfying $\text{dist}(\text{supp } h_1, \text{supp } h_2) \geq a_0$.

From Theorem 2.10 we deduce the following bilinear estimate. We follow the proof of [35] Lemma 2.4 (also, see [37] Lemma 3.1).

Corollary 11. Let $q, a_0, N, \epsilon$ and $\psi$ be as in Theorem 2.10 and let $\delta, \lambda, b, m$ and $M_{\delta,\lambda}$ be given as in Proposition 2.4. Suppose that

$$
(\xi', \xi_d) \in \text{supp } \hat{f}_1, (\xi', \xi_d) \in \text{supp } \hat{f}_2 \implies |\xi' - \xi' | \geq a_0.
$$

Then there is a constant $C$, independent of $\delta, \lambda$ and $\psi$, such that, for $f_1, f_2 \in L^2(\mathbf{R}^d)$ satisfying

$$
(2.42)
$$

$$
(2.43)
$$

The same estimate holds for $M_{\delta}(D)$ with $\delta^{-1}$ replaced by $\delta$ in (2.43).

Proof. Recalling (2.18) and changing variables $(\xi', \xi_d) \to (\xi', \xi_d + \psi(\xi'))$, we see that for $k = 1, 2$,

$$
|M_{\delta,\lambda}(D) f_k(x)| \lesssim \int \left| \int e^{i(x' - \xi_d + x_d \psi(\xi'))} A_{\delta,\lambda}(\xi) \hat{f}_k(\xi', \xi_d + \psi(\xi')) d\xi' \right| \left| \tilde{\beta}(\xi_d) \right| d\xi_d,
$$

where $\tilde{\beta} \in C_0^\infty([-5, -1/5] \cup [1/5, 5])$ satisfies $\tilde{\beta} = 1$ on the $\xi_d$-support of $\beta(\tilde{m}(\xi) \xi_d)$. Freezing $\xi_d$ we apply the bilinear estimate (2.41) to $h_k(\xi') = A_{\delta,\lambda}(\xi', \xi_d) \hat{f}_k(\xi', \xi_d + \psi(\xi'))$, $k = 1, 2.$
By the condition \((2.42)\), \(\text{dist}(\text{supp} \, h_1, \text{supp} \, h_2) \geq a_0\). Thus from Theorem \((2.10)\) and Minkowski’s inequality we see that the left side of \((2.43)\) is bounded by

\[
\int \int \prod_{k=1,2} \int e^{i(x'_k - a_k \psi(x_k))} A_{\delta,\lambda}(x'_k, \tau_k) \tilde{f}_k(x'_k, \tau_k + \psi(x_k)) \, dx'_k \, d\tau_k \leq C \int \int \prod_{k=1,2} \left\| A_{\delta,\lambda}(x'_k, \tau_k) \tilde{f}_k(x'_k, \tau_k + \psi(x_k)) \right\|_{L^2(\mathbb{R}^{d-1}; d\xi)} \beta(\frac{\tau_k}{\delta}) \beta(\frac{\tau_k}{\lambda}) \, d\tau_1 \, d\tau_2,
\]

\((2.44)\)

where \(C\) is independent of \(\delta, \lambda\) and \(\psi\). Since \(\bar{m} \in \text{Mul}(N,Cb)\) for some \(C > 0\), from \((2.19)\) in Lemma \(2.6\) we note that \(|A_{\delta,\lambda}| \lesssim \lambda^{-1}\). By the Cauchy–Schwarz inequality and the change of variables \(\tau_k \rightarrow \tau_k - \psi(\cdot)\), we see \((2.44)\) is bounded by

\[
C \bar{m}^\delta \left( \prod_{k=1,2} \int \lambda^{-2} |\tilde{f}_k(\xi)|^2 d\xi \right)^{1/2}.
\]

The inequality \((2.43)\) follows from Parseval’s identity. The estimate for \(\mathfrak{R}(D)\) can be obtained in exactly the same way. \(\square\)

Before closing this subsection, we state a result which is necessary to prove Proposition \((2.4)\) in the next section. Trivially, by Hölder’s inequality and Proposition \((2.5)\) it follows that

\[
\prod_{k=1,2} \left\| \mathfrak{R}(D) \tilde{f}_k \right\|_{L^{2/\delta}(\mathbb{R}^d)} \leq C \lambda^{-2(\delta) \frac{d}{2} - d + 1} \prod_{k=1,2} \left\| f_k \right\|_{L^\delta(\mathbb{R}^d)}
\]

\((2.45)\)

whenever \(p_* < q \leq \infty\). Under the additional transversality condition \((2.42)\) we have \((2.43)\). Since \((2.45)\) holds regardless of \((2.43)\), we may interpolate this with \((2.43)\) while assuming \((2.42)\). This yields the following.

**Corollary 2.12.** Let \(0 < \delta < \lambda \leq 1\), \(b > 0\), \(a_0 \in (0,1/2]\) and suppose \(2 \leq p \leq q \leq \infty\) and

\[
\frac{1}{q} - \frac{d}{2(d + 2)} < \frac{d/[2(d + 2)] - 1/p_*}{1/2 - 1/p_*} \left( \frac{1}{p} - \frac{1}{2} \right).
\]

Then, there exist large \(N\), small \(\epsilon\), and \(C > 0\) such that

\[
\prod_{k=1,2} \left\| \mathfrak{R}(D) \tilde{f}_k \right\|_{L^{\delta^*}(\mathbb{R}^d)} \leq C \delta^{1 - d + \frac{2d}{p}} \prod_{k=1,2} \left\| f_k \right\|_{L^{\delta}(\mathbb{R}^d)},
\]

\((2.47)\)

\[
\prod_{k=1,2} \left\| \mathfrak{R}(D) \tilde{f}_k \right\|_{L^{\delta^*(\mathbb{R}^d)}} \leq C \lambda^{-2(\delta) \frac{1}{2} - d + \frac{2d}{p}} \prod_{k=1,2} \left\| f_k \right\|_{L^{\delta}(\mathbb{R}^d)},
\]

\((2.48)\)

for \(\psi \in \text{Ell}(N,\epsilon)\), \(m \in \text{Mul}(N,b)\), and \(f_1\) and \(f_2\) satisfying the separation \((2.42)\).

**2.6. Bochner–Riesz operator of negative order.** If \((1/p,1/q) \in \{B,B'\} \cup \{A',B'\}\), then the (restricted) weak type estimates stated in Theorem \((1.4)\) can be obtained as consequences of the well-known estimates for the restriction-extension operator \(f \rightarrow F^{-1}(\hat{f}d\sigma)\) which is defined by

\[
F^{-1}(\hat{f}d\sigma)(x) = \frac{1}{(2\pi)^d} \int_{S^{d-1}} \hat{f}(\theta) e^{ix \cdot \theta} d\sigma(\theta),
\]

where \(d\sigma\) is the surface measure on the unit sphere \(S^{d-1}\). In fact, this is a special case of order \(-1\) of the classical Bochner–Riesz operator

\[
(2.49)
\]

which is defined by analytic continuation when \(\alpha \leq -1\). Here \(\Gamma\) is the gamma function. For \(d \geq 2\) and \(\alpha \in (0, \frac{d+1}{2}]\) let us set

\[
P_\alpha(d) := \left( \frac{d-1}{2d} + \frac{\alpha}{d} \right) \left( \frac{d-1}{2d} + \frac{\alpha}{d} \right), \quad Q_\alpha(d) := \left( \frac{d-1}{2d} + \frac{\alpha}{d} \right) \left( \frac{d-1}{2d} + \frac{\alpha}{d} \right).
\]

\([6] When \(d = 2\) this is \(1/q < 1/4\). When \(d \geq 3\) this is equivalent to saying that \((1/p,1/q)\) lies strictly below the line passing through the points \(P_\alpha\) and \(P_{\alpha/2}\). See Figure 3 and Figure 4. \)
This problem was studied by several authors \[12, 4, 10, 1, 24\]. The complete characterization of the restricted weak type estimates for the sphere, which we need later. Recalling \(1.8\), we note that \(R\) holds if and only if 
\[
\text{(1)} \quad \text{if and only if } \frac{d+1}{2} \leq \frac{2\alpha}{d+1}, \quad x > \frac{d-1}{2d} + \frac{\alpha}{d}, \quad y < \frac{d+1}{2d} - \frac{\alpha}{d}.
\]

The following has been conjectured.

**Conjecture 2.** Let \(d \geq 2\) and \(0 < \alpha \leq \frac{d+1}{2}\). \(R^{-\alpha}\) is bounded from \(L^p(\mathbb{R}^d)\) to \(L^q(\mathbb{R}^d)\) if and only if \((1/p, 1/q) \in \mathcal{P}_\alpha(d)\).

This problem was studied by several authors \[12, 4, 10, 1, 24\]. The complete characterization of the necessity part is due to Börjeson \[4\]. Estimates for \(R^{-\alpha}\) with \(\alpha > 1/2\) and \((1/p, 1/q) \in \mathcal{P}_\alpha(d)\) were obtained by Sogge \[43\]. Partial results regarding the critical estimate with \((1/p, 1/q) \in (Q_\alpha(d), Q'_\alpha(d))\) were obtained by Bak, McMichael and Oberlin \[2\]. When \(d = 2\), the conjecture was solved by Bak \[1\]. The restricted weak type estimates at \(Q_\alpha(d)\) and \(Q'_\alpha(d)\) were proven by Gutiérrez \[24\] for \(\alpha > 0\) when \(d \geq 2\), and for \(\alpha > 1/2\) when \(d \geq 3\). The conjecture was verified by Cho, Kim, Lee and Shim \[12\] for \((d-2)(d+1)/2 < \alpha\) and weaker endpoint estimates were also obtained.

From Proposition \[2.4\] and typical dyadic decomposition we can improve the current state of the boundedness of \(R^{-\alpha}\).

**Theorem 2.13.** If \(d \geq 3\) and \(\alpha > \frac{d+1}{2}(\frac{1}{p_0} - \frac{1}{q})\) (that is to say, \(\alpha > \frac{(d+1)(d-1)}{2(d^2+d-2)}\) if \(d\) is odd, and \(\alpha > \frac{(d+1)(d-2)}{2(d^2-d)}\) if \(d\) is even), then Conjecture 2 is true. Moreover, \(R^{-\alpha}\) is of restricted weak type \((p, q)\) when \((1/p, 1/q) \in Q'_\alpha(d)\), and of weak type \((p, q)\) if \((1/p, 1/q) \in (Q_\alpha(d), P'_\alpha(d))\).

**Proof.** By Proposition \[2.4\] we may replace the condition \(\frac{2d+4}{d} < q\) in \[12\, Proposition 2.4\] with \(q_0 < q\). Now the rest of the proof is identical with that of \[12, Theorem 1.1\]. \(\square\)

Especially, when \(\alpha = 1\), the result gives the following characterization of \(L^p-L^q\) boundedness for the restriction-extension operator, which we need later. Recalling \(1.8\), we note that \(P_1 = \mathcal{P}\).  

**Theorem 2.14** (Restriction-extension estimates for the sphere). Let \(d \geq 2\). The estimate
\[
(2.51) \quad \left\| \int_{S^{d-1}} \tilde{f}(\theta) e^{it\cdot \theta} d\sigma(\theta) \right\|_{L^q(\mathbb{R}^d)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}
\]
holds if and only if \((1/p, 1/q) \in \mathcal{P}\). Furthermore, for the critical \(p, q\) such that \((1/p, 1/q) = (B, B')\), the restricted weak type estimate holds instead of \(2.51\). If \((1/p, 1/q) \in (B', E')\), the weak type estimate holds (see Figure 3 and Figure 4).

Finally, we record here the following real interpolation technique (see \[5, 9, 34\]), which will be needed several times in the succeeding sections. Here \(\| \cdot \|_{r,s}\) denotes the norm of the Lorentz space \(L^{r,s}\).

**Lemma 2.15** \(\quad 34\). Let \(\epsilon_1, \epsilon_2 > 0\), \(1 \leq p_1, p_2, \epsilon_1 < \infty\), \(1 \leq i \leq k\), \(1 \leq q_1, q_2 < \infty\). For every \(j \in \mathbb{Z}\) let \(T_j\) be \(k\)-linear operators satisfying \(\| T_j(f_1, \cdots, f_k) \|_{q_1} \leq M_1 2^{-\epsilon_1 j} \prod_{i=1}^k \| f_i \|_{p_i}\) and \(\| T_j(f_1, \cdots, f_k) \|_{q_2} \leq M_2 2^{-\epsilon_2 j} \prod_{i=1}^k \| f_i \|_{p_i}\). Then, for \(\theta, q, p_i\) defined by \(\theta = \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}, \frac{q}{q_1} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}\) and \(\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}\), the following hold:

(I) \(\| \sum_j T_j(f_1, \cdots, f_k) \|_{q, \infty} \leq CM_0^\theta M_1^{1-\theta} \prod_{i=1}^k \| f_i \|_{p_i, 1}\),

(II) \(\| \sum_j T_j(f_1, \cdots, f_k) \|_q \leq CM_0^\theta M_1^{1-\theta} \prod_{i=1}^k \| f_i \|_{p_i, 1}\) if \(q_1 = q_2 = q\),

(III) \(\| \sum_j T_j(f_1, \cdots, f_k) \|_{q, \infty} \leq CM_0^\theta M_1^{1-\theta} \prod_{i=1}^k \| f_i \|_{p'_i}\) if \(p'_1 = p'_2 = p'\) for every \(i\).
3. Proof of Proposition 2.4

In order to deduce the linear estimates (2.12) and (2.13) from the bilinear estimates in Corollary 2.12 we basically follow the strategy in [35, 12] with some modifications. As before, we may only prove (2.13). The estimate (2.12) can be obtained by the same argument.

Let us put 2.12 we basically follow the strategy in [35, 12] with some modifications. As before, we may only deduce the linear estimates (2.12) and (2.13) from the bilinear estimates in Corollary 2.12. Let us put for the moment, we prove Proposition 2.4.

For every integer \( j \geq 0 \) let \( D(j) \) be the collection of the closed dyadic cubes of size \( 2^{-j} \) in \( Q \), that is,

\[
D(j) := \left\{ \prod_{k=1}^{d-1} [n_k 2^{-j}, (n_k + 1) 2^{-j}] : n_k \in \mathbb{Z}, -2^j \leq n_k \leq 2^j - 1 \right\}.
\]

For convenience let us denote by \( Q_k^j \) the members of \( D(j) \).

For every \( j \geq 1 \) we define a relation \( \sim \) on the dyadic cubes contained in \( D(j) \) as follows. For \( Q_{k_1}^j, Q_{k_2}^j \in D(j) \) we write \( Q_{k_1}^j \sim Q_{k_2}^j \) if \( Q_{k_1}^j \cap Q_{k_2}^j = \emptyset \), but there are parent cubes in \( D(j-1) \) which have nonempty intersection. It is easy to see that \( 2^{-j} \leq \text{dist}(Q_{k_1}^j, Q_{k_2}^j) \lesssim 2^{-j} \) if \( k_1 \neq k_2 \).

By a kind of Whitney decomposition of \( Q \times Q \) away from its diagonal \( \Lambda_Q = \{(\xi', \xi') : \xi' \in Q\} \),

\[
Q \times Q \setminus \Lambda_Q = \bigcup_{j \geq 1} \bigcup_{Q_{k_1}^j \sim Q_{k_2}^j} Q_{k_1}^j \times Q_{k_2}^j,
\]

hence

\[
\sum_{j \geq 1} \sum_{Q_{k_1}^j \sim Q_{k_2}^j} \chi_{Q_{k_1}^j} \chi_{Q_{k_2}^j} = 1
\]

almost everywhere in \( Q \times Q \) ([50, 35, 12]). For \( Q_k^j \in D(j) \) we define \( f_k^j \) by

\[
\hat{f}_k^j(\xi) = \chi_{Q_k^j}(\xi') \hat{f}(\xi).
\]

As mentioned before (Remark 7), with \( \chi_0 \) supported near the origin we may assume \( \hat{f} \) is supported in \( 2^{-5} I^d \). Then, by (3.1) we can write

\[
(\mathcal{M}_{\delta, \lambda}(D) f)^2 = \sum_{j \geq 6} T_j(f, f) := \sum_{j \geq 6} \sum_{Q_{k_1}^j \sim Q_{k_2}^j} \prod_{i=1,2} \mathcal{M}_{\delta, \lambda}(D) f_{k_i}^j.
\]

We now try to obtain sharp estimates for the bilinear operators \( \{T_j : j \geq 6\} \). We separately consider the cases \( 2^{2j} \lesssim 1/\delta \lambda \) and \( 2^{2j} \gtrsim 1/\delta \lambda \).

**Lemma 3.1.** Let \( p, q \) satisfy \( 2 \leq p < q \leq 4 \), (2.46) and suppose that \( 2^{2j} \delta \lambda < 1/10 \). Then, there are \( N \) and \( \epsilon \) which are independent of such \( p, q, j, \delta \) and \( \lambda \), such that

\[
\|T_j(f_1, f_2)\|_{q/2} \leq C 2^{j(3d+1)/(d+1) - (d+1)(1-\frac{1}{2})} 2^{-2j(\delta \lambda)^{1-d+\frac{2d}{d-1}}} \|f_1\|_p \|f_2\|_p
\]

for \( \psi \in \text{Ell}(N, \epsilon) \) and \( m \in \text{Mul}(N, b) \). Here the constant \( C \) is independent of \( j, \delta, \lambda, n, \) and \( \psi \).

**Lemma 3.2.** Suppose \( 2 \leq p < q \leq 4 \), (2.4d) \( \Delta \leq q \leq 4 \) and \( 2^{2j} \delta \lambda \geq 1/10 \). Then there are \( N \) and \( \epsilon \), independent of such \( p, q, j, \delta \) and \( \lambda \), such that

\[
\|T_j(f_1, f_2)\|_{q/2} \leq C 2^{-2j(\delta \lambda)^{1-d+\frac{2d}{d-1}}} 2^{(\delta \lambda)^{2d+\frac{1}{2}}} \|f_1\|_p \|f_2\|_p.
\]

for \( \psi \in \text{Ell}(N, \epsilon) \) and \( m \in \text{Mul}(N, b) \). The constant \( C \) is independent of \( j, \delta, \lambda, n, \) and \( \psi \).

Assuming Lemma 3.1 and Lemma 3.2 for the moment, we prove Proposition 2.4.

**Proof of Proposition 2.4.** Choose \( N \) and \( \epsilon > 0 \) so that both Lemma 3.2 and Lemma 3.3 hold. For \( p, q \) such that \( \frac{d+1}{q} = (d-1)(1-\frac{1}{2}) \) and \( q_0 < q < \frac{2(d+1)}{d-1} \), applying (I) in Lemma 2.15 with \( k = 2 \)
to the estimate \((3.2)\), we get
\[
\left\| \sum_{2^{2i} \delta \lambda < \frac{1}{n}} T_j(f_1, f_2) \right\|_{q/2,\infty} \leq C \lambda^{-2}(\delta \lambda)^{1-d+\frac{2d}{p}} \prod_{i=1,2} \| f_i \|_{p,1}.
\]
On the other hand, when \(2 \leq p < q\) and \(\frac{2d}{d-1} \leq q \leq 4\), direct summation of \((3.3)\) over \(j\) with \(2^{2i} \delta \lambda \geq 1\) gives
\[
\left\| \sum_{2^{2i} \delta \lambda \geq \frac{1}{n}} T_j(f_1, f_2) \right\|_{q/2} \leq C \lambda^{-2}(\delta \lambda)^{(d+1)(\frac{d}{p} - \frac{1}{q})} \prod_{i=1,2} \| f_i \|_{p}.
\]
Combining \((3.4)\) and \((3.5)\) we obtain the following restricted weak type estimate
\[
\left\| (\mathcal{N}_{\delta, \lambda}(D)f)^2 \right\|_{q/2,\infty} \leq C \lambda^{-2}(\delta \lambda)^{1-d+\frac{2d}{p}} \| f \|_{p,1}
\]
for \(\psi \in \mathbf{Ell}(N, \epsilon)\) and \(m \in \mathbf{Mul}(N, b)\) whenever \(\frac{1}{q} = \frac{d-1}{d+1}(1 - \frac{1}{p})\) and \(\frac{d-1}{2(d+1)} < \frac{1}{q} < \frac{1}{g_\epsilon}\). On the same range of \(p, q\) we can upgrade the restricted weak type estimates \((3.6)\) to strong type bounds by using (real) interpolation between those estimates. This completes the proof of Proposition \(2.4\). \qed

Before we proceed to show Lemma \(3.1\) and Lemma \(3.2\), we recall the following lemma which is a slight modification of \([12, \text{Lemma 3.5}]\). Since the proof of \([12, \text{Lemma 3.5}]\) works without modification, we state it without proof.

**Lemma 3.3.** Let \(2 \leq p < q \leq 4\). Suppose that there is a constant \(L\), independent of all pairs \((Q_{k_1}', Q_{k_2}')\) with \(Q_{k_1}' \sim Q_{k_2}'\), such that
\[
\left\| \prod_{i=1,2} \mathcal{M}_{\delta, \lambda}(D)(f_i)^{\frac{1}{2}} \right\|_{q/2} \leq L \prod_{i=1,2} \| f_i \|_{p}.
\]
Then there is a constant \(C\) independent of \(j, \delta, \lambda\), such that
\[
\left\| T_j(f_1, f_2) \right\|_{q/2} \leq C L \| f_1 \|_p \| f_2 \|_p.
\]

**Proof of Lemma 3.3.** By Lemma 3.3, it is sufficient to show that, for \(Q_{k_1}' \sim Q_{k_2}'\) and \(p, q\) satisfying \(2 \leq p < q \leq 4\) and \((2.16)\),
\[
\left\| \prod_{i=1,2} \mathcal{M}_{\delta, \lambda}(D)(f_i)^{\frac{1}{2}} \right\|_{q/2} \leq C 2^{2d(\frac{4}{q} - (d-1)(1 - \frac{1}{p}))} \lambda^{-2(d+1)} \prod_{i=1,2} \| f_i \|_{p}
\]
with \(C\) independent of \(j, k_1, k_2, \delta, \lambda, \psi \in \mathbf{Ell}(N, \epsilon)\) and \(m \in \mathbf{Mul}(N, b)\). For given \(k_1, k_2\) with \(Q_{k_1}' \sim Q_{k_2}'\) let \(R(j, k_1, k_2)\) be the smallest closed \((d-1)\)-dimensional rectangle containing \(Q_{k_1}' \cup Q_{k_2}'\) and let \(c \in \mathbf{I}^{d-1}\) be the center of \(R(j, k_1, k_2)\) and set
\[
\rho := 2^{1-j}.
\]
Now we perform the change of variables \(\xi \to L_{c, \rho}(\xi)\) in the frequency side. For example, see \((2.15)\) and \((2.16)\). By setting
\[
\tilde{g}_i(\xi) = \rho^{d+1} \chi_{Q_{k_i}'}(\rho \xi_i + c) \widehat{f}_i(L_{c, \rho}(\xi)), \quad i = 1, 2,
\]
one can easily see that
\[
\mathcal{M}_{\delta, \lambda}(D)(f_i)^{\frac{1}{2}}(x) = \left( [\mathcal{M}_{\delta, \lambda} \circ L_{c, \rho}](D)g_i \right)(\rho x_i + \rho \xi_i) + \rho \xi_i \nabla \psi(c), \rho^2 x_i.
\]
Thus we have
\[
\left\| \prod_{i=1,2} \mathcal{M}_{\delta, \lambda}(D)(f_i)^{\frac{1}{2}} \right\|_{q/2} = \rho^{-\frac{2(d+1)}{q}} \left\| \prod_{i=1,2} \mathcal{M}_{\delta, \lambda} \circ L_{c, \rho}(D)g_i \right\|_{q/2}.
\]
We now notice that \(\tilde{g}_i\) is supported in \(\tilde{Q}_i \times \mathbf{I}\), where \(\tilde{Q}_i's\) are cubes in \(\mathbf{I}^{d-1}\) of sidelength \(1/2\), and \(\text{dist}(\tilde{Q}_1, \tilde{Q}_2) \geq 1/2\). We now recall \((2.16)\) and that \(\mathcal{M}_{\delta, \lambda} \circ L_{c, \rho}\) can be regarded as a multiplier \(\mathcal{M}_{\delta', \lambda}\) given by putting \(\delta' = \rho^{-2} \delta\), \(m = m \circ L_{c, \rho}\), and \(\psi = \psi_{c, \rho}\) (see Remark 7). Since \(\psi_{c, \rho} \in \mathbf{Ell}(N, \epsilon)\)
by Lemma 2.3 and \( m \circ L_{c, \rho} \in \text{Mul}(N, Cb) \) for some \( C > 0 \), we can apply Corollary 2.12 to the right hand side of (3.11) with \( \delta \) replaced by \( \rho^{-2} \delta \), to get
\[
\left\| \prod_{i=1,2} \mathfrak{M}_{\delta, \lambda}(D)(f_i^j)_{\kappa_i} \right\|_{q/2} \lesssim \rho^{-\frac{2(d+1)}{q}} \lambda^{-2} (\rho^{-2} \delta \lambda)^{1-d+\frac{q}{p}} \prod_{i=1,2} \|g_i\|_p.
\]

It is easy to see that \( \|g_i\|_p = \rho^{\frac{d+1}{p}} \|(f_i)_{\kappa_i}\|_p \). Hence, we get (3.9).

**Proof of Lemma 3.3.** In order to show (3.3) by Lemma 3.3 it is sufficient to show
\[
(3.12) \quad \left\| \prod_{i=1,2} \mathfrak{M}_{\delta, \lambda}(D)(f_i^j)_{\kappa_i} \right\|_{q/2} \lesssim \lambda^{-2} (2^{-j(d-1)} \delta \lambda)^{2(\frac{1}{2} - \frac{q}{p})} \prod_{i=1,2} \|f_i^j\|_p.
\]

Let \( \tilde{\chi} \in C_0^\infty(\mathbb{R}^{d-1}) \) be a smooth cutoff function supported in \( \mathbf{1}^{d-1} \) such that \( \tilde{\chi} = 1 \) on \( (2^{-1} \mathbf{1})^{d-1} \), and \( c^j_{k_i} \), the center of \( Q_{k_i}^j \). Then \( \tilde{\chi}Q_{k_i}^j(\xi') = \tilde{\chi}(2^j(\xi' - c^j_{k_i})) \) is supported in \( 2Q_{k_i}^j \) and equal to 1 on \( Q_{k_i}^j \). Thus \( \tilde{\chi}Q_{k_i}^j \chi_{k_i} = \tilde{\chi}_{k_i} \) and we may write \( \mathfrak{M}_{\delta, \lambda}(D)f_{k_i}^j = K_{k_i}^j \ast f_{k_i}^j \), where
\[
K_{k_i}^j(x) = (2\pi)^{-d} \int e^{i\xi \cdot x} \mathfrak{M}_{\delta, \lambda}(\xi) \tilde{\chi}Q_{k_i}^j(\xi')d\xi.
\]

For notational convenience let us set \( c = c^j_{k_i} \), \( \rho = 2^{-j} \). By changing variables \( \xi \to L_{c, \rho} \xi \) we have
\[
K_{k_i}^j(x) = (2\pi)^{-d} e^{i(x' \cdot \rho \psi(c) \cdot \psi(c))} \rho^{d+1} \int e^{i(\rho(x' + x_d \psi(c)) \cdot \xi' + \rho^2 x_d \psi(c))} \mathfrak{M}_{\delta, \lambda}(L_{c, \rho} \xi') \tilde{\chi}(\xi')d\xi.
\]

As before we regard \( \mathfrak{M}_{\delta, \lambda} \circ L_{c, \rho} \) as a multiplier \( \mathfrak{M}_{\delta, \lambda} \) given by \( \rho^{-2} \delta \to \delta, m \circ L_{c, \rho} \to m \), and \( \psi_{c, \rho} \to \psi \) (see Remark 7). From (2.16), (2.11) and Lemma 2.6 it easily follows that \( |\partial^p_\xi \mathfrak{M}_{\delta, \lambda}(L_{c, \rho} \xi)| \lesssim \lambda^{1-r} \) uniformly in \( \psi \in \text{Ell}(N, \epsilon) \) and \( m \in \text{Mul}(N, b) \) whenever \( |\alpha| \leq N \). Since \( (\mathfrak{M}_{\delta, \lambda} \circ L_{c, \rho}) \tilde{\chi} \) is supported in \( \mathbf{1}^d \), it is clear that \( \|F^{-1}((\mathfrak{M}_{\delta, \lambda} \circ L_{c, \rho}) \tilde{\chi})(x)\| \lesssim \lambda^{1-1} (1 + |x|)^{-M} \) for any \( M \leq N \). Thus \( \|F^{-1}((\mathfrak{M}_{\delta, \lambda} \circ L_{c, \rho}) \tilde{\chi})\|_q \lesssim \lambda^{-1} \), and trivially we also have \( \|K_{k_i}^j\|_\infty \lesssim \lambda^{-1} \rho^{-d-1} \lambda/2 \) for \( 1 \leq r \leq \infty \). Since \( p \leq q \), from Young’s inequality we see that
\[
\|\mathfrak{M}_{\delta, \lambda}(D)f_{k_i}^j\|_q \lesssim \lambda^{-1} (\rho^{d+1}\delta \lambda)^{\frac{q}{p}-\frac{1}{2}} \|f_{k_i}^j\|_p.
\]

Hence, by Hölder’s inequality we get the desired estimate (3.12). \( \square \)

4. Resolvent estimates: Proof of Theorem 1.4

4.1. Reduction. For \( z \in S^1 \setminus \{1\} \) let us set
\[
m(\xi, z) = (|\xi|^2 - z)^{-1}.
\]
For every multi-index \( \alpha \), it is easy to see that
\[
|\partial^\alpha_\xi m(\xi, z)| \leq C_\alpha \max\{||\alpha||_1, 1\} \frac{1}{||\xi|^2 - z||_{|\alpha|+1}^{1}},
\]
where the constant \( C_\alpha \) is independent of \( z \in S^1 \setminus \{1\} \). We decompose \( m(\xi, z) \) into singular and regular parts. Let us fix a small number \( \delta_0 > 0 \) and choose a function \( \rho_0 \in C_0^\infty(\mathbb{R}^d) \) such that \( \rho_0(\xi) = 1 \) if \( 1 - \delta_0 \leq |\xi| \leq 1 + \delta_0 \) and \( \rho_0(\xi) = 0 \) if \( |\xi| \leq 1 - 2\delta_0 \) or \( |\xi| \geq 1 + 2\delta_0 \). Setting
\[
\rho_1 := (1 - \rho_0) \chi_{B_\delta}(0, 1), \quad \rho_2 := (1 - \rho_0) \chi_{\mathbb{R}^d \setminus B_\delta}(0, 1); \quad m_j(\xi, z) := m(\xi, z) \rho_j(\xi), \quad j = 0, 1, 2,
\]
we have
\[
m(\xi, z) = \sum_{j=0}^2 m_j(\xi, z),
\]
Since both \( m_1 \) and \( m_2 \) are zero on the annulus \( 1 - \delta_0 \leq |\xi| \leq 1 + \delta_0 \) it is easy to check that 
\[
|||\xi|^2 - z| \geq 2\delta_0 \pm \delta_2^2 \text{ on supp } m_1 \cup \text{supp } m_2.
\]
From this and (4.1) it follows that \( m_1 \) and \( m_2 \) are uniformly bounded in \( C^\infty(\mathbb{R}^d) \) for all \( z \in S^1 \setminus \{1\} \). More precisely, for all \( z \in S^1 \setminus \{1\} \), we have
\[
|\partial^2_z m_1(\xi, z)| \leq C_{\alpha, \delta_0},
\]
\[
|\partial^2_z m_2(\xi, z)| \leq C_{\alpha, \delta_0} |\xi|^{-|\alpha| - 2}.
\]

Since \( \rho_1 \) is a compactly supported smooth function, \( m_1(D, z) \) are bounded from \( L^p(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \) for \( 1 \leq p \leq q \leq \infty \). Moreover, the bounds are independent of \( z \in S^1 \setminus \{1\} \) because of (4.2). On the other hand, the operators \( m_2(D, z) \) are uniformly bounded from \( L^p(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \) when \( \{1/p, 1/q\} \in \mathcal{R}_0(d) \), which can be seen in a similar manner as in the proof of Proposition 1.1 because of (4.3). Hence it remains to deal with the operators \( m_0(D, z) \).

Let \( \theta_0 \) be a small number and set \( \mathbb{S}^1(\theta_0) := \{ e^{i\theta} \in \mathbb{S}^1 : \theta \in [\theta_0, 2\pi - \theta_0]\} \). By (4.1) we have, for any \( \alpha \) and \( z \in S^1 \setminus \{1\} \), \( |\partial^\alpha_z m_0(\xi, z)| \leq C_{\alpha, \delta_0, \theta_0} \). Hence similar argument shows that \( m_0(D, z) \) are bounded from \( L^p(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \) uniformly in \( z \in \{1 \leq p \leq q \leq \infty \} \). As a result, we conclude that the uniform estimate
\[
|||(-\Delta - z)^{-1}||_{p \to q} \leq C, \quad \forall z \in \mathbb{S}^1(\theta_0)
\]
holds if \( \{1/p, 1/q\} \in \mathcal{R}_0(d) \).

For the rest of this section, we focus on obtaining sharp bounds for \( m_0(D, z) \) when \( 0 < |\text{Im } z| \ll \text{Re } z < 1 \), which is the main part of obtaining the estimate \( (1.13) \). By scaling \( \xi \to (\text{Re } z)^{1/2}\xi \) it is harmless to assume that \( z = 1 + \delta \) and \( 0 < \delta \ll \theta_0 \). Now we are reduced to showing that
\[
\left\| F^{-1} \left( \frac{\tilde{\chi}(|\xi|^2) \hat{f}(|\xi|^2)}{|\xi|^s - 1 - i\delta} \right) \right\|_q \leq |\delta|^{-\gamma_{p,q}} \|f\|_p
\]
where \( s = 2 \) and \( \tilde{\chi} \in C^\infty_0([1 - 2\delta_0, 1 + 2\delta_0]) \) for a small \( \delta_0 > 0 \).

The estimate (4.5) actually holds on a range which is wider than \( \mathcal{R}_0 \). All the required estimates for the proof of Theorem 1.4 are contained in the following, which complete the proof of Theorem 1.4. Though we need only to deal with the case \( s = 2 \), we prove (4.5) with \( s \neq 0 \) for later use.

**Proposition 4.1.** Let \( d \geq 2 \), \( s \neq 0 \), and \((1/p, 1/q) \neq (1,0)\). Suppose that \((1/p, 1/q) \in \mathcal{P} \cup \mathcal{R}_2 \cup (\mathcal{Q} \setminus \{P_*, P_0, D\}) \cup (\mathcal{Q} \setminus \{P_*, P_0, D\})' \). Then the estimate (4.5) is true provided that \( \delta_0 \) is small. If \((1/p, 1/q) \in \{B, B'\} \), then the restricted weak type \((p, q)\) estimate holds. If \((1/p, 1/q) \in \{B', E'\}\) the weak type \((p, q)\) estimate holds.

In what follows we consider the cases \((1/p, 1/q) \in \mathcal{P} \cup \{B, B'\} \cup (B', E') \setminus \{(0,1)\}\) and \((1/p, 1/q) \in \mathcal{R}_2 \cup (\mathcal{Q} \setminus \{P_*, P_0, D\}) \cup (\mathcal{Q} \setminus \{P_*, P_0, D\})' \) separately.

**4.2. Proof of Proposition 4.1** when \((1/p, 1/q) \in \mathcal{P} \cup \{B, B'\} \cup (B', E')\) and \((1/p, 1/q) \neq (1,0)\). It is enough to show the following:
\[
\left\| F^{-1} \left( \frac{\tilde{\chi}(|\xi|^2) \hat{f}(|\xi|^2)}{|\xi|^s - 1 - i\delta} \right) \right\|_{2d \gamma_{p,q}} \leq \|f\|_{2d \gamma_{p,q} + 1},
\]
\[
\left\| F^{-1} \left( \frac{\tilde{\chi}(|\xi|^2) \hat{f}(|\xi|^2)}{|\xi|^s - 1 - i\delta} \right) \right\|_{2d \gamma_{p,q}} \leq \|f\|_p, \quad 1 \leq p < \frac{2d(d+1)}{\delta^2 + 4d - 1}.
\]

The estimates in (4.6) are the weak type \((p, q)\) estimates for \((1/p, 1/q) \in \{B', E'\}\), and (4.6) is the restricted weak type \((p, q)\) estimate with \((1/p, 1/q) = B'\). By duality, (real) interpolation, and Young’s inequality (note that the multiplier has compact support), it is easy to see that the estimate (4.5) for \((1/p, 1/q) \in \mathcal{P} \setminus \{(0,1)\}\) follows from (4.6) and (4.7). Indeed, note that \( \gamma_{p,q} = 0 \) when \((1/p, 1/q) \in \{(1,0), E, B, B', E'\} \).
We prove (4.6) and (4.7) by making use of Theorem 2.14 as in [30, 31]. Both arguments to show (4.6) and (4.7) are not much different from each other except for using different estimates in Theorem 2.14.

**Proof of (4.6).** Let us fix \((1/p_0, 1/q_0) = B'\) and write

\[
\frac{\tilde{\chi}(\xi)}{|\xi|^s - 1 - i\delta} = \mathcal{R}(\xi) + i\mathcal{J}(\xi) := \frac{(|\xi|^s - 1)\tilde{\chi}(\xi)}{|\xi|^s - 1 + i\delta} + i \frac{\delta \tilde{\chi}(\xi)}{|\xi|^s - 1 + \delta^2}.
\]

Then (4.6) follows if we show that both the operators \(\mathcal{R}(D)\) and \(\mathcal{J}(D)\) are of restricted weak type \((p_0, q_0)\). The desired estimate for \(\mathcal{J}(D)\) is easier than that for \(\mathcal{R}(D)\). Writing in the spherical coordinates, application of Minkowski’s inequality and Theorem 2.14 gives

\[
\|\mathcal{J}(D)f\|_{q_0, \infty} \lesssim \int_{1-2\delta_0}^{1+2\delta_0} \frac{|\delta|}{\rho^s(\rho^s - 1)^2 + \delta^2} \left\| \int_{S^{j-1}} \hat{f}(\rho\theta)e^{i\rho \xi \cdot \theta} d\sigma(\theta) \right\|_{q_0, \infty} \, d\rho
\]

\[
\lesssim \|f\|_{p_0, 1} \int_{t^2 + \delta^2}^{t^2 + 2\delta^2} dt \lesssim \|f\|_{p_0, 1}.
\]

For the real part, we decompose the multiplier \(\mathcal{R}(\xi)\) as in [30, Section 4]. Let \(\phi \in S(\mathbb{R})\) be such that \(\text{supp } \tilde{\phi} \subset [-2, -1/2] \cup [1/2, 2] \cup \sum_{j = -\infty}^{\infty} 2^{-j} t \phi(2^{-j} t) = 1\) whenever \(t \in \mathbb{R} \setminus \{0\}\), and we set \(\tilde{\phi}(t) = t\phi(t)\). Let us define

\[
A_j(\xi) := \mathcal{R}(\xi) \tilde{\phi}(2^{-j}(|\xi|^s - 1)), \quad B_j(\xi) := \left( \mathcal{R}(\xi) - \frac{\tilde{\chi}(\xi)}{|\xi|^s - 1} \right) \tilde{\phi}(2^{-j}(|\xi|^s - 1)),
\]

\[
C_j(\xi) := \frac{\tilde{\chi}(\xi)}{|\xi|^s - 1} \tilde{\phi}(2^{-j}(|\xi|^s - 1))
\]

for each \(j \in \mathbb{Z}\), and break the multiplier into

\[\mathcal{R}(\xi) = \sum_{2^j < \delta} A_j(\xi) + \sum_{2^j \geq \delta} B_j(\xi) + \sum_{2^j \geq \delta} C_j(\xi).\]

Again, by using the spherical coordinate, Minkowski’s inequality, and Theorem 2.14 we see that

\[
\left\| \mathcal{F}^{-1} \left( \sum_{2^j < \delta} A_j(\xi) \tilde{\phi}(\xi) \right) \right\|_{q_0, \infty} \lesssim \|f\|_{p_0, 1} \sum_{2^j < \delta} \int_{1-2\delta_0}^{1+2\delta_0} \frac{|\rho^s - 1||\phi(2^{-j}(\rho^s + 1))|}{(\rho^s - 1)^2 + \delta^2} \, d\rho
\]

\[
\lesssim \|f\|_{p_0, 1} \int_{2^j - 2\delta_0}^{2^j + 2\delta_0} \frac{|\delta|}{\rho^s(\rho^s - 1)^2 + \delta^2} d\rho \lesssim \|f\|_{p_0, 1}
\]

since \(\sum_{2^j < \delta} |t\phi(2^{-j} t)| \lesssim \sum_{2^j < \delta} 2^j \lesssim |\delta|\). Similarly we have

\[
\left\| \mathcal{F}^{-1} \left( \sum_{2^j \geq \delta} B_j(\xi) \tilde{\phi}(\xi) \right) \right\|_{q_0, \infty} \lesssim \|f\|_{p_0, 1} \sum_{2^j \geq |\delta|} \int_{1-2\delta_0}^{1+2\delta_0} \frac{\delta^2 2^{-j}|\phi(2^{-j}(\rho^s + 1))|}{(\rho^s - 1)^2 + \delta^2} \, d\rho
\]

\[
\lesssim \|f\|_{p_0, 1} \int_{2^j - 2\delta_0}^{2^j + 2\delta_0} \frac{|\delta|}{\rho^s(\rho^s - 1)^2 + \delta^2} d\rho \lesssim \|f\|_{p_0, 1}.
\]

To estimate the multiplier operator given by \(C_j\) we need the following.

**Lemma 4.2.** Let \(s \neq 0\) and \(\lambda > 0\). Suppose \(\phi \in S(\mathbb{R})\) with \(\text{supp } \tilde{\phi} \subset [-2, -1/2] \cup [1/2, 2] \cup \sum_{j = -\infty}^{\infty} 2^{-j} t \phi(2^{-j} t) = 1\) whenever \(t \in \mathbb{R} \setminus \{0\}\), and set \(\phi \in C_0^\infty([1 - \delta_0, 1 + \delta_0])\) for some small \(\delta_0 > 0\).

\[
\left\| \mathcal{F}^{-1} \left( \phi(\lambda^{-1}(|\xi|^s - 1)) \tilde{\chi}(\xi) \tilde{\phi}(\xi) \right) \right\|_q \lesssim \lambda^{\frac{d+1}{d} - \frac{q}{p}} \|f\|_p,
\]

where \(\tilde{\chi} \in C_0^\infty([1 - \delta_0, 1 + \delta_0])\) for some small \(\delta_0 > 0\).

\[\text{[For a proof of existence of such } \phi \text{ we refer the reader to [30, Lemma 2.2]. Also, see [12, Lemma 2.1].} \]

\"
Assuming this lemma for the moment let us continue. Since \( C_j(\xi) = \tilde{\chi}(|\xi|)2^{-j}\phi(2^{-j}(|\xi|^s - 1)) \), by Lemma 4.2 we have
\[
\|C_j(D)f\|_{\sigma} = 2^{-j}\left\|F^{-1}\left(\phi(2^{-j}(|\xi|^s - 1))\tilde{\chi}(|\xi|)\hat{f}(\xi)\right)\right\|_{\sigma} \lesssim 2^{j(\frac{d+1}{2} - \frac{s}{2})}\|f\|_r
\]
for \( 2 \leq \sigma \leq \infty, \frac{1}{\sigma} \geq \frac{d+1}{2}(1 - \frac{1}{r}) \). Application of (I) in Lemma 2.15 yields
\[
\left\|F^{-1}\left(\sum_{2^j \geq |\delta|} C_j(\xi)\hat{f}(\xi)\right)\right\|_{q_0, \infty} \lesssim \|f\|_{p_{0}, 1}.
\]
Therefore, the proof of (4.6) is completed.

Proof of (4.7) for \((1/p, 1/q) \in (B', E')\). We may follow the same lines of argument as in the proof of (4.6) by replacing the \(L^{p_0 - 1} - L^{q_0, \infty}\) estimate for the restriction-extension operator with the \(L^p - L^{q, \infty}\) estimate for the same operator with \((1/p, 1/q) \in (B', E')\) in Theorem 2.14. The only difference occurs when we attempt to prove
\[
\left\|F^{-1}\left(\sum_{2^j \geq |\delta|} C_j(\xi)\hat{f}(\xi)\right)\right\|_{q, \infty} \lesssim \|f\|_p.
\]
However, this can be obtained again by (4.9) and using the last statement (III) in Lemma 2.15 since we can fix \(p\) while \(q\) is allowed to be chosen to satisfy the assumption in Lemma 2.15. This observation first appeared in Bak [11]. Also, see [12].

Now, we prove Lemma 4.2.

Proof of Lemma 4.2. We may assume that \(\lambda \leq 1/100\). Otherwise, for every \(M \geq 0\), the multiplier in (4.8) is smooth and uniformly bounded in \(C^{M}_0(\mathbb{R}^d)\), hence (4.8) is trivial. By interpolation and Young’s inequality, it is sufficient to show (4.8) for \((p, q) = \left(\frac{2(d+1)}{d+3}, 2\right)\), and for \((p, q) = (1, \infty)\). When \((p, q) = \left(\frac{2(d+1)}{d+3}, 2\right)\) using Parseval’s identity and the Stein–Tomas restriction theorem [15, 51] we have
\[
\left\|F^{-1}\left(\phi(\lambda^{-1}(|\xi|^s - 1))\tilde{\chi}(|\xi|)\hat{f}(\xi)\right)\right\|_2 \approx \left(\int |\phi(\lambda^{-1}(\rho^s - 1))|\tilde{\chi}(\rho)|\hat{f}(\rho)\rho\,d\rho\right)^{1/2} \lesssim \left(\int |\phi(\lambda^{-1}(\rho^s - 1))|^{2}\rho\,d\rho\right)^{1/2} \lesssim \lambda^2 \|f\|_{2(d+1)/(d+3)}.
\]
Thus, it remains to show (4.8) when \((p, q) = (1, \infty)\). The related kernel is given by
\[
K(x) = (2\pi)^{d-1} \int \int_{\mathbb{R}^{d-1}} e^{i(\rho x - \theta \lambda^{-1}(\rho^s - 1))}\chi(\rho)\hat{\phi}(\rho)\rho\,d\rho\,d\theta d\rho,
\]
where \(\chi(\rho) := \tilde{\chi}(\rho)\rho^{d-1}\), and it suffices to show that
\[
|K(x)| \lesssim \lambda^{\frac{d+1}{2}}.
\]
We separately consider the three cases \(|x| \leq \lambda^{-1}/100, |x| \geq 100\lambda^{-1}, \text{ and } |x| \approx \lambda^{-1}\). For the first case, since \(\text{supp} \tilde{\phi} \subset [-2, 1/2] \cup [1/2, 2]\), we have \(|\phi(\lambda^{-1}(\rho^s - 1))| \gtrsim \lambda^{-1}\). Hence integration by parts gives \(|K(x)| \lesssim \lambda^M\) for any \(M \geq 0\). For the rest of the cases, we recall \(\int_{\mathbb{R}^{d-1}} e^{\rho x} d\rho = c_d x^{d-2} J_{d-2}(|x|)\) and use the asymptotic expansion of the Bessel function \(J_\nu\) (39, 44). Thus we have
\[
K(x) = \sum_{j=0}^{M} \sum_{\pm} C_{j, \pm} x^{-\frac{d+1}{2} - j} \int e^{i(\rho x - \theta \lambda^{-1}(\rho^s - 1))} \chi_{j, \pm}(\rho) \hat{\phi}(\rho) d\rho + O(|x|^{-M - \frac{d+1}{2}})
\]
(4.11)
\[
= (2\pi)^{-1} \sum_{j=0}^{M} \sum_{\pm} C_{j, \pm} x^{-\frac{d+1}{2} - j} \int e^{\pm j\rho x} \chi_{j, \pm}(\rho) \hat{\phi}(\lambda^{-1}(\rho^s - 1)) d\rho + O(|x|^{-M - \frac{d+1}{2}}),
\]
(4.12)
for \(M \geq d\) and \(\chi_{j, \pm} \in C_{0}^{M}([-1 - \delta_0, 1 + \delta_0])\). When \(|x| \geq 100\lambda^{-1}\) we use (4.11). Since \(\hat{\phi}(\rho) \neq 0\) only if \(|\rho| \approx 1\), we have \(|\hat{\phi}(\lambda^{-1}(\rho^s - 1))| \gtrsim |\rho|\), so \(|K(x)| \lesssim |x|^{-M}\) for any \(M \geq 0\) by
integration by parts. When $|x| \approx \lambda^{-1}$ taking the absolute value of the integrands in (4.12) we get $|K(x)| \lesssim \lambda^{\frac{d+1}{2}}$. Therefore, (4.10) follows. □

4.3. Proof of Proposition 4.1 when $(1/p, 1/q) \in \bar{R}_0 \cup (Q \setminus [P_*, P_0, D]) \cup (Q \setminus [P_*, P_0, D])'$. Since we already have the estimates (4.6), (4.7), and (4.5) with $p = q = 2$, in view of interpolation and duality, it is sufficient to show (4.5) for $p, q$ satisfying $(1/p, 1/q) \in (P_0, B) \cup [0, 0), P_*)$.

For the purpose we may assume $|\delta|$ is small enough. Thus, by finite decomposition, rotation, and discarding harmless smooth part of the multiplier, we may assume that the multiplier is supported near $(0, \cdots, 0, -1) \in \mathbb{R}^d$. We write $\xi = (\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and

$$|\xi|^s = 1 - \frac{(\tau + \sqrt{1 - |\eta|^2})(\tau - \sqrt{1 - |\eta|^2})(|\tau|^2 + |\eta|^2)^{\frac{s}{2}} - 1}{\tau^2 + |\eta|^2 - 1}.$$ 

Let us set

$$\psi(\eta) = 1 - \sqrt{1 - |\eta|^2}, \quad m(\eta, \tau) = \frac{1}{s} \frac{(\tau + \psi(\eta) - 2)((\tau - 1)^2 + |\eta|^2)^{\frac{s}{2}} - 1}{(\tau - 1)^2 + |\eta|^2 - 1}.$$ 

It is easy to see that $m(\eta, \tau) = -1 + O(|\tau| + |\eta|^2)$. After change of variables $\tau \to \tau - 1$, we may further assume that the multiplier is of the form

$$\mathcal{M}_\delta(\eta, \tau) = \frac{\chi_0(\eta, \tau)}{s(\tau - \psi(\eta))m(\eta, \tau) - i\delta},$$

where $\chi_0$ is a smooth function supported on a small neighborhood of the origin in $\mathbb{R}^d$. By further harmless affine transformations (see (2.15) and Lemma 2.3), we may assume that $\psi \in Ell(N, \epsilon)$ for a large $N \geq 10d$ and a small $\epsilon > 0$, and $m \in \text{Mul}(N, b)$ for some $b > 0$, so that both Proposition 2.4 and Proposition 2.5 are valid. Thus $\mathcal{M}_\delta$ takes the form

$$\mathcal{M}_\delta(\eta, \tau) = \frac{1}{|\delta|} \varphi\left(\frac{m(\eta, \tau)(\tau - \psi(\eta))}{|\delta|}\right)\chi_0(\eta, \tau)$$

for $\varphi(t) = (st \pm i)^{-1}$, which clearly satisfies the condition (2.11). We break $\mathcal{M}_\delta$ as follows:

$$\mathcal{M}_\delta(\eta, \tau) = \frac{1}{|\delta|} \varphi\left(\frac{m(\eta, \tau)(\tau - \psi(\eta))}{|\delta|}\right)\beta_1\left(\frac{m(\eta, \tau)(\tau - \psi(\eta))}{|\delta|}\right)\chi_0(\eta, \tau)$$

$$+ \frac{1}{|\delta|} \sum_{j=1}^{\log_2\frac{1}{\delta}} \varphi\left(\frac{m(\eta, \tau)(\tau - \psi(\eta))}{2^j|\delta|}\right)\beta\left(\frac{m(\eta, \tau)(\tau - \psi(\eta))}{2^j|\delta|}\right)\chi_0(\eta, \tau).$$

The case $(1/p, 1/q) \in (P_0, B) \cup [(0, 0), P_*)$. In this case we note that $\gamma_{p, q} = \frac{d+1}{2} - \frac{d}{p}$, and that $(p, q)$ are the pairs given by $\frac{2d}{d+1} < p < p_0$ and $\frac{1}{q} = \frac{d+1}{2} - \frac{1}{p}$, or $p_0 < p = q < \infty$. We apply Proposition 2.4 with $\delta$ and $\lambda$ replaced with $|\delta|$ and $2^{-j-1}$, respectively, to each of the multiplier operators which are given by the functions on the right hand side of (4.13). This yields, for $2 \leq p < p_0$ and $\frac{1}{q} = \frac{d+1}{2} - \frac{1}{p}$,

$$|\mathcal{M}_\delta(D)|_{p \to q} \lesssim \frac{1}{|\delta|} \left( |\delta|^{\frac{d+1}{2} - \frac{d}{p}} + \sum_j 2^{-j} (2^j|\delta|)^{\frac{d+1}{2} - \frac{d}{p}} \right) \lesssim |\delta|^{\frac{d}{p} - \frac{d+1}{2}}.$$ 

Now interpolation between these estimate and (4.6) gives the desired estimate (4.5) for $p, q$ satisfying $(1/p, 1/q) \in (P_0, B)$. The remaining cases $(1/p, 1/q) \in [(0, 0), P_*)$ can be handled similarly by making use of Proposition 2.5. Repeating the same argument, we get $|\mathcal{M}_\delta(D)|_{p \to p} \lesssim |\delta|^{\frac{d}{p} - \frac{d+1}{2}}$ for $p_* < p \leq \infty$. □
4.4. Description of $Z_{p,q}(\ell)$. The case $(1/p, 1/q) \in \bar{\mathcal{R}}_3'$ can be deduced from the case $(1/p, 1/q) \in \bar{\mathcal{R}}_3$ by duality, hence we may consider the case $(1/p, 1/q) \in \mathcal{R}_1 \cup (\bigcup_{i=2}^3 \mathcal{R}_i)$ only. For $d \geq 2$ and $(1/p, 1/q) \in \mathcal{R}_1 \cup (\bigcup_{i=2}^3 \mathcal{R}_i)$, we set

$$\omega_{p,q} = \omega_{p,q}(d) := 1 - \frac{d}{2}(\frac{1}{p} - \frac{1}{q}),$$

which lies in $[0, 1]$. Since $\kappa_{p,q}(z) = |z|^{-\omega_{p,q}}(\text{dist}(z, [0, \infty])/|z|)^{-\gamma_{p,q}}$, we get

$$Z_{p,q}(\ell) = \{ z \in \mathbb{C} \setminus \{0\} : \text{Re} z \leq 0, \ell|z|^{\omega_{p,q}} \geq 1 \}$$

$$\cup \{ z \in \mathbb{C} \setminus [0, \infty) : \text{Re} z > 0, \ell|z|^{\gamma_{p,q}} \geq |z|^{-\omega_{p,q}} \}$$

for $\ell > 0$ and $(1/p, 1/q) \in \mathcal{R}_1 \cup (\bigcup_{i=2}^3 \mathcal{R}_i)$. The shape of $Z_{p,q}(\ell)$ is mainly determined by the value of $\gamma_{p,q}$ and $\gamma_{p,q} - \omega_{p,q}$. When $\omega_{p,q} > 0$, the value $\ell$ does not have particular role in determining the overall shape of $Z_{p,q}(\ell)$. However, if $\omega_{p,q} = 0$ the profile of $Z_{p,q}(\ell)$ depends not only on $p, q, d$, but also on $\ell$. In what follows we handle these two cases separately.

The case $\omega_{p,q} > 0$. We further subdivide this case into the cases $(1/p, 1/q) \in \mathcal{R}_1$, $(1/p, 1/q) \in \mathcal{R}_2$, and $(1/p, 1/q) \in \mathcal{R}_3$.

- $(1/p, 1/q) \in \mathcal{R}_1(2)$, or $(1/p, 1/q) \in \mathcal{R}_1(d) \setminus (A, A')$ if $d \geq 3$: Then $\omega_{p,q} \in (0, \frac{1}{d+1}]$ and $\gamma_{p,q} = 0$. Hence $Z_{p,q}(\ell) = \{ z \in \mathbb{C} \setminus [0, \infty) : |z| \geq \ell^{-1/\omega_{p,q}} \}$. See Figure 7a.

- $(1/p, 1/q) \in \mathcal{R}_2$: Then $\omega_{p,q} \in (\frac{1}{d+1}, 1]$, $\gamma_{p,q} = 1 - \frac{d+1}{d+1}(\frac{1}{p} - \frac{1}{q}) \in (0, 1]$, and $\gamma_{p,q} - \omega_{p,q} = -\frac{1}{(\frac{1}{p} - \frac{1}{q})} \leq 0$. If $(1/p, 1/q) \in \mathcal{R}_2 \setminus \{H\}$, since $\gamma_{p,q} - \omega_{p,q} < 0$, $Z_{p,q}(\ell)$ is the complement of a neighborhood of $[0, \infty)$ which shrinks along the positive real line as $\text{Re} z \to \infty$. See Figure 7b, Figure 7c, Figure 7d, and Figure 7e. Also, $Z_{2,2}(\ell) = \{ z : \text{Re} z \leq 0, |z| \geq 1/\ell \} \cup \{ z : \text{Re} z > 0, |\text{Im} z| \geq \gamma_{p,q} \}$. See Figure 7f.

- $(1/p, 1/q) \in \mathcal{R}_3$: In this case $\omega_{p,q} \in (0, 1]$, $\gamma_{p,q} = \frac{d+1}{d} - \frac{d}{p} > 0$, and $\gamma_{p,q} - \omega_{p,q} = \frac{1}{2}(\frac{1}{p} - \frac{1}{q})$. So, we divide $\mathcal{R}_3$ into the three sets $\mathcal{R}_{3,+}$, $\mathcal{R}_{3,0}$, and $\mathcal{R}_{3,-}$.

\[\text{Figure 7.} \quad Z_{p,q}(1) ((a) \to (b) \to (c) \to (d) \to (e) \to (f)) \text{ as } 1/p \text{ decreases while } (\frac{1}{p}, \frac{1}{q}) \in [H, (\frac{d+2}{2d}, \frac{d-2}{2d})].\]
In this section we obtain lower bounds for \( \| \omega \|_{p,q} \). The case doing this we provide proof of Proposition 1.1 which is simpler.

1. Let \( (1/p, 1/q) \in (A, A') \). Since \( \gamma_{p,q} - \omega_{p,q} < 0 \), \( Z_{p,q}(\ell) \) is the complement of a neighborhood of \([0, \infty)\) which shrinks along positive real line as \( \Re z \to \infty \). See Figure 8c.

2. Let \( (1/p, 1/q) \in \mathcal{R}_{3,0} \). Since \( \gamma_{p,q} - \omega_{p,q} = 0 \), \( Z_{p,q}(\ell) \) is the complement of the \( \ell^{-1}/\gamma_{p,q} \)-neighborhood of \([0, \infty)\). See Figure 8d.

3. Let \( (1/p, 1/q) \in \mathcal{R}_{3,-} \). In this case \( \gamma_{p,q} - \omega_{p,q} > 0 \). Hence \( Z_{p,q}(\ell) \) is the complement of a neighborhood of \([0, \infty)\) whose boundary asymptotically satisfies \( |\Im z| \approx (\Re z)^{1-\omega_{p,q}/\gamma_{p,q}} \) when \( \Re z \) is large. See Figure 8a, Figure 8b, and Figure 8c.

The case \( \omega_{p,q} = 0 \). In this case the shape of \( Z_{p,q}(\ell) \) depends on the value of \( \ell \) as well.

- Let \( (1/p, 1/q) \in (A, A') \). Since \( \omega_{p,q} = \gamma_{p,q} = 0 \), we have \( Z_{p,q}(\ell) = \emptyset \) if \( \ell < 1 \), and \( Z_{p,q}(\ell) = \mathbb{C} \setminus [0, \infty) \) if \( \ell \geq 1 \).
- Let \( d \geq 4 \) and \( (1/p, 1/q) \in ((2/d, 0), A) \). In this case \( \omega_{p,q} = 0 \) and \( \gamma_{p,q} = \frac{d+1}{2} - \frac{d}{p} \in (0, \frac{d-3}{2}) \).
- If \( \ell = 1 \), there is a rigid dichotomy of \( Z_{p,q}(1) \) between the case of uniform bound and the other case; \( Z_{p,q}(1) = \mathbb{C} \setminus [0, \infty) \) if \( (1/p, 1/q) \in (A, A') \), but \( Z_{p,q}(1) = \{ z : \Re z \leq 0 \} \setminus \{ 0 \} \) if \( (1/p, 1/q) \notin [A, A'] \).
- When \( \ell > 1 \), there is also a kind of dichotomy although it is not so rigid as in the former case with \( \ell = 1 \). Indeed, if \( \ell > 1 \) and \( (1/p, 1/q) \in (A, A') \), then \( Z_{p,q}(\ell) = \mathbb{C} \setminus [0, \infty) \). Otherwise, \( Z_{p,q}(\ell) \) is the complement (in \( \mathbb{C} \)) of a (planar) cone of which axis is the positive real line \([0, \infty)\), and apex is the origin. It is interesting to note that, as \( (1/p, 1/q) \) moves from (near) \( A \) to (near) \((2/d, 0)\) along the line \( 1/p - 1/q = 2/d \), the apex angle gets larger from 0 to \( 2 \arctan \sqrt{\frac{1}{2d} - 1} \). See Figure 9.

5. LOWER BOUNDS FOR \( \|(-\Delta - z)^{-1}\|_{p,q} \): PROPOSITION 1.3

In this section we obtain lower bounds for \( \|(-\Delta - z)^{-1}\|_{p,q} \), which prove Proposition 1.3. Before doing this we provide proof of Proposition 1.1 which is simpler.
Hence \( \beta \) is clear that we take limit \( \lim_{p,q} \leq \frac{d+1}{d} \) whenever \( 1 \leq p,q < \infty \). The first condition is obvious since the multiplier operator is translation invariant \((26)\). For the second condition we notice that, for large \( j \), the estimate \((5.1)\) remains valid with \( C \) independent of \( j \) if \( \hat{f}(\xi) \) is replaced with \( \beta(2^{-j}\xi))\hat{f}(\xi) \). Then re-scaling the estimate gives

\[
\left\| \mathcal{F}^{-1}\left( \frac{\beta(2^{-j}|\xi|)}{|\xi|^2 - z} \hat{f}(\xi) \right) \right\|_q \leq C \| f \|_p
\]

for \( f \in \mathcal{S}(\mathbb{R}^d) \). Now fix a nonzero Schwartz function \( f \) such that \( \| \mathcal{F}^{-1}(|\xi|^2 \beta(|\xi|)\hat{f}(\xi)) \|_q > 0 \). If we take limit \( j \to +\infty \) the left side of \((5.4)\) converges to \( \| \mathcal{F}^{-1}(|\xi|^2 \beta(|\xi|)\hat{f}(\xi)) \|_q \), but the quantity

5.1. **Proof of Proposition 1.1.** We first show the sufficiency part. By \((1.4)\) it is sufficient to show the estimate

\[
(5.1) \quad \left\| \mathcal{F}^{-1}\left( \frac{\beta(2^{-j}|\xi|)}{|\xi|^2 - z} \right) \hat{f}(\xi) \right\|_q \leq C \| f \|_p
\]

holds whenever \( (1/p, 1/q) \in \mathcal{R}_0 \). Since \( z \neq 0 \), thanks to the scaling property \((1.5)\) we may assume that \( z \in S^1 \setminus \{1\} \). Let us break the multiplier

\[
(\beta(2^{-j}|\xi|)(|\xi|^2 - z)^{-1} - \beta_0(|\xi||\xi|^2 - z)^{-1} + \sum_{j \geq 0} \beta(2^{-j}|\xi|)(|\xi|^2 - z)^{-1}.
\]

It is clear that \( \beta_0(|\xi||\xi|^2 - z)^{-1} \) is smooth and compactly supported in the open ball \( B_d(0,3/4) \). Hence

\[
(5.2) \quad \left\| \mathcal{F}^{-1}\left( \beta_0(|\xi||\xi|^2 - z)^{-1} \right) \hat{f}(\xi) \right\|_q \leq C \| f \|_p
\]

for \( p,q \) satisfying \( 1 \leq \frac{p}{q} < \infty \). By scaling it is easy to see \( \| \mathcal{F}^{-1}(\beta(2^{-j}|\xi|)/|\xi|^2 - z) \|_r \leq 2^{d} \leq 2^{d-\frac{d}{2}} \) for \( 1 \leq r \leq \infty \). Thus, Young’s inequality gives

\[
\left\| \mathcal{F}^{-1}\left( \frac{\beta(2^{-j}|\xi|)}{|\xi|^2 - z} \right) \right\|_q \leq 2^{d} \leq 2^{d-\frac{d}{2}} \| f \|_p
\]

whenever \( 1 \leq \frac{p}{q} < \infty \). Thus summation along \( j \) and combining the resulting estimate with \((5.2)\) give \((5.1)\) all desired estimates except the case \( \frac{1}{p} - \frac{1}{q} = \frac{2}{d} \). In order to obtain the estimate \((5.1)\) for \( p,q \) with \( \frac{1}{p} - \frac{1}{q} = \frac{2}{d} \) in the case \( d \geq 3 \), we use \((11)\) in Lemma 2.15 with \( k = 1 \) to get

\[
\left\| \sum_{j \geq 1} \mathcal{F}^{-1}\left( \frac{\beta(2^{-j}|\xi|)}{|\xi|^2 - z} \right) \hat{f}(\xi) \right\|_{q,\infty} \leq \| f \|_p
\]

provided that \( 1 \leq \frac{p}{q} < \infty \) and \( \frac{1}{p} - \frac{1}{q} = \frac{2}{d} \). Real interpolation between those estimates together with \((5.2)\) gives the desired estimate \((5.1)\) for all \( p,q \) satisfying \( (1/p, 1/q) \in \mathcal{R}_0 \).

Now we consider the necessity part of Proposition 1.1. In the case \( d \geq 3 \) we need to show \((5.1)\) holds only if

\[
(5.3) \quad \frac{1}{p} - \frac{1}{q} \geq 0, \quad \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d} \quad (p,q) \neq \left(1, \frac{d}{d-2}\right), \quad (p,q) \neq \left(\frac{d}{2}, \infty\right).
\]

The first condition is obvious since the multiplier operator is translation invariant \((26)\). For the second condition we notice that, for large \( j \), the estimate \((5.1)\) remains valid with \( C \) independent of \( j \) if \( \hat{f}(\xi) \) is replaced with \( \beta(2^{-j}|\xi|)\hat{f}(\xi) \). Then re-scaling the estimate gives

\[
(5.4) \quad \left\| \mathcal{F}^{-1}\left( \frac{\beta(|\xi|)}{|\xi|^2 - 2^{-2j}z} \right) \hat{f}(\xi) \right\|_q \leq C \| f \|_p
\]

for \( f \in \mathcal{S}(\mathbb{R}^d) \). Now fix a nonzero Schwartz function \( f \) such that \( \| \mathcal{F}^{-1}(\beta(|\xi|)\hat{f}(\xi)) \|_q > 0 \). If we take limit \( j \to +\infty \) the left side of \((5.4)\) converges to \( \| \mathcal{F}^{-1}(\beta(|\xi|)\hat{f}(\xi)) \|_q \), but the quantity
Let us choose a \( \phi \) more straightforward manner. We need to prove that for any fixed \( z \) we still need to show the failure of (5.1) with \( 1 \) we prove this in a slightly more general form.

By scaling this also implies, for all \( \epsilon > 0 \),
\[
\left\| \mathcal{F}^{-1}\left( (1 - \beta_0(\epsilon|\xi|)|\xi|^2 - z)^{-1} \hat{f}(\xi) \right) \right\|_{\frac{d}{2}} \leq C \|f\|_1.
\]

Letting \( \epsilon \to 0 \) gives \( \left\| \mathcal{F}^{-1}(\xi|^{-2} \hat{f}(\xi)) \right\|_{\frac{d}{2}} \leq C \|f\|_1 \) which is obviously not true. Therefore we conclude the estimate (5.1) can not be true with \( p = 1 \) and \( q = d/(d-2) \).

When \( d = 2 \) the above argument works for \( p, q \) satisfying \( 1/p - 1/q < 2/d \), but not for \( p, q \) with \( 1/p - 1/q = 2/d \), that is, \( p = 1, q = \infty \) because Mikhlin’s theorem does not hold with \( q = \infty \). So we still need to show the failure of (5.1) with \( p = 1, q = \infty \). But the failure can be shown in a more straightforward manner. We need to prove that for any fixed \( z \in \mathbb{C} \setminus [0, \infty) \) there does not exist a constant \( C > 0 \) such that
\[
\left\| (-\Delta - z)^{-1} f \right\|_{L^\infty(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)}, \quad \forall f \in L^1(\mathbb{R}^2).
\]

To show this let us assume (5.5) and recall from [47] p. 202 that
\[
(-\Delta - z)^{-1} f(x) = (2\pi)^{-1} \int_{\mathbb{R}^2} K_0(\sqrt{-z}|x-y|) f(y) dy,
\]
where \( K_\nu(w) \) is the modified Bessel function of the second kind (see [39, 47]). It is well-known ([39, p. 252]) that
\[
\lim_{w \to 0} \frac{K_0(w)}{w} = 1.
\]

Let us choose a \( \phi \in C^\infty_0(B_2(0,1)) \) such that \( \phi \geq 0 \) and \( \|\phi\|_1 = 1 \), and set \( \phi_\epsilon(x) = e^{-\epsilon^2 \phi(\epsilon^{-1} x)} \), \( \epsilon > 0 \). Testing (5.5) with \( f = \phi_\epsilon \) and letting \( \epsilon \to 0^+ \) yield \( \sup_{x \in \mathbb{R}^2} |K_0(2\pi|x|)| \lesssim 1 \). This contradicts the asymptotic (5.6), thus we conclude (5.5) fails. \( \square \)

5.2. Proof of Proposition 1.3. For a bounded function \( m(\xi) \) which is symmetric under reflection \( \xi \to -\xi \) it is easy to see that the \( L^p-L^q \) norms of the operators \( m(D) \) and \( \overline{m}(D) \) are equal, hence we have
\[
\|(m \pm \overline{m})(D)\|_{p \to q} \leq 2 \|m(D)\|_{p \to q}.
\]

Therefore, to prove the lower bounds (1.12), we may work with the imaginary part
\[
\text{Im} \left( \frac{1}{\|\xi|^2 - z} \right) = \frac{1}{2i} \left( \frac{1}{\|\xi|^2 - z} - \frac{1}{\|\xi|^2 - \bar{z}} \right) = \frac{\text{Im} z}{(\|\xi|^2 - \text{Re} z)^2 + (\text{Im} z)^2}.
\]
The lower bounds in (1.12) is meaningful only when \( 0 < |\text{Im} z| \ll |\text{Re} z| < 1 \). Hence we only need to consider \( z = 1 + i \delta \) with \( 0 < 100\delta < 1 \) in this section. The lower bound \( \gamma_{p,q} = 1 \) (in the case of \( \gamma_{p,q} = 0 \)) is clear since the resolvent operators are nontrivial. Thus, recalling (1.17) it is sufficient to show that
\[
\left\| \mathcal{F}^{-1}\left( \frac{\delta \hat{f}(\xi)}{(\|\xi|^2 - 1)^2 + \delta^2} \right) \right\|_q \geq \max \left\{ \delta^{-1+\frac{d+1}{d+1}}(\frac{1}{\delta} - \frac{1}{\delta^2}), \delta^{-\frac{d+1}{2d+1}} - \frac{d}{4} \right\} \|f\|_p.
\]

for \( f \in L^p(\mathbb{R}^d) \). The other lower bound \( \delta^{-\frac{d+1}{d+1}} \) in (1.12) follows from the lower bound \( \delta^{-\frac{d+1}{2d+1}} - \frac{d}{4} \) by duality. Since we also consider the resolvent estimates for the fractional Laplacian in Section 6 we prove this in a slightly more general form.
Lemma 5.1. Let $1 \leq p, q \leq \infty$, and for $r, s > 0$ let $m_s^p(r) := \frac{\delta}{(r - s)^{d\mu} + \delta^s}$. Then, if $0 < \delta < c$ for a small $c > 0$,

\begin{align}
\|m_s^p(D)\|_{p \rightarrow q} & \gtrsim \delta^{-1 + \frac{d+1}{2} (\frac{1}{p} - \frac{1}{q})}, \\
\|m_s^p(D)\|_{p \rightarrow q} & \gtrsim \delta^{\frac{d+1}{2} - \frac{d}{q}},
\end{align}

where the implicit constants depend only on $p, q, s$, and $d$.

Proof of (5.7). Let $c, k$ be positive constants to be chosen later, depending only on $d$ and $s$. If $|\xi| \leq c\sqrt{\delta}$, $j = 1, \ldots, d - 1$ and $k\delta/4 \leq \xi_d - 1 \leq k\delta$, then

$$0 < |\xi|^n - 1 \leq (1 + ((d - 1)c^2 + 2k + 10^{-2}k^2)\delta)^{s/2} - 1$$

provided that $0 < 100\delta \leq 1$. Let us set $\mu(t) := (1 + t)^{s/2}$ and $M_s := \max\{|\mu''(t)| : 0 \leq t \leq 1\}$. Then, from Taylor’s theorem it follows that

$$0 < |\xi|^n - 1 \leq (1 + ((d - 1)c^2 + 2k + 10^{-2}k^2)\delta)^{s/2} - 1$$

if we choose $\delta$ small enough, that is to say, $0 < ((d - 1)c^2 + 2k + 10^{-2}k^2)\delta \leq \min\{1, s/M_s\}$. We now choose $c = 1/\sqrt{2(d - 1)s}$ and $k$ as the positive solution of the quadratic equation $2k + 10^{-2}k^2 = 1/2s$ so that, for $0 < \delta \leq \min\{s, s^2/M_s\}$,

$$0 < |\xi|^n - 1 \leq \delta.$$

Let us set $c_s := \min\{10^{-2}, s, s^2/M_s\}$, and choose $\phi, \psi \in C^\infty_0(\mathbb{R})$ such that supp $\phi \subset [-1, 1]$, $0 < \phi \leq 1$, $\phi = 1$ on $[-1/2, 1/2]$, supp $\psi \subset [1/4, 1]$, $0 \leq \psi \leq 1$ and $\psi = 1$ on $[1/2, 3/4]$. For every $\delta \in (0, c_s)$ we define $f_\delta \in \mathcal{S}(\mathbb{R}^d)$ by

$$f_\delta(\xi) = \psi(\frac{\xi_d - 1}{k\delta}) \prod_{j=1}^{d-1} \phi(\frac{\xi_j}{c\sqrt{\delta}}).$$

Since $|\int m_s^p(|\xi|)\hat{f}_\delta(\xi)e^{ix\cdot\xi}d\xi| = |\int m_s^p(|\xi|)\hat{f}_\delta(\xi)e^{ix\cdot\xi}d\xi|$, we have

$$|\int m_s^p(|\xi|)\hat{f}_\delta(\xi)e^{ix\cdot\xi}d\xi| \geq \left| \int m_s^p(|\xi|)\hat{f}_\delta(\xi)\cos(x \cdot \xi - x_d)d\xi \right| - \left| \int m_s^p(|\xi|)\hat{f}_\delta(\xi)\sin(x \cdot \xi - x_d)d\xi \right|.$$

On supp $\hat{f}_\delta$, $|\xi| \leq c\sqrt{\delta}$ for $j = 1, \ldots, d - 1$ and $k\delta/4 \leq \xi_d - 1 \leq k\delta$. Thus $m_s^p(|\xi|) \geq 1/2\delta$ whenever $\xi \in \text{supp} \hat{f}_\delta$ and $0 < \delta < c_s$. Also, if $\xi \in \text{supp} \hat{f}_\delta$ and

$$x \in A_\delta := \left\{ x \in \mathbb{R}^d : |x_j| \leq \frac{1}{200(d - 1)c\sqrt{\delta}}, j = 1, \ldots, d - 1, |x_d| \leq \frac{1}{200k\delta} \right\},$$

then $|x \cdot \xi - x_d| \leq 1/100$, hence

$$\left| \int \mathbb{R}^d m_s^p(|\xi|)\hat{f}_\delta(\xi)e^{ix\cdot\xi}d\xi \right| \geq \frac{1}{2\delta} \int \hat{f}_\delta(\xi) \left( 1 - \frac{1}{100} \right) d\xi - \frac{1}{100\delta} \int \hat{f}_\delta(\xi)d\xi \approx \delta^{s/2} ||\hat{f}_\delta||_1 \approx \delta^{s/2}.$$

Integration on the box $A_\delta$ yields

$$\left\| \int \mathbb{R}^d m_s^p(|\xi|)\hat{f}_\delta(\xi)e^{ix\cdot\xi}d\xi \right\|_q \gtrsim \delta^{\frac{s+1}{q}} |A_\delta|^{1/q} \approx \delta^{\frac{s+1}{q} - \frac{d+1}{q}}.$$

On the other hand it is easy to check that $||f_\delta||_p \approx \delta^{\frac{s+1}{q} - \frac{d+1}{q}}$. Thus we obtain (5.7). \qed

Proof of (5.8). Let $\phi$ be a non-negative smooth function on $\mathbb{R}$ such that supp $\phi \subset (1 - 2\varepsilon_0, 1 + 2\varepsilon_0)$ for some small $\varepsilon_0 > 0$ to be determined later depending on $s$. We take $f \in C^\infty_0(\mathbb{R}^d)$ so that $\hat{f}(\xi) = \phi(|\xi|)$ and set

$$Q(x) := \int_{\mathbb{R}^d} m_s^p(|\xi|)\hat{f}(\xi)e^{ix\cdot\xi}d\xi.$$

By the spherical coordinate we write

$$Q(x) = (2\pi)^\frac{d}{2} \int_{1-2\varepsilon_0}^{1+2\varepsilon_0} m_s^p(r) \phi(r)r^{d-1}|x|^{\frac{2s}{d} - 1} J_{\frac{2s}{d} - 1}(|rx|)dr,$$
where \( J_\nu \) denotes the Bessel function of order \( \nu \). It is well-known (see [46] p.338) that for \( \nu > -1/2 \)

\[
J_\nu(r) = \left( \frac{\pi r}{2} \right)^{-\frac{1}{2}} \cos \left( r - \frac{\pi \nu}{2} - \frac{\pi}{4} \right) + R_\nu(r), \quad r > 0,
\]

where \( R_\nu \) satisfies \(|R_\nu(r)| \leq c_r r^{-3/2}\) if \( r \geq 1 \).

By (5.9) with \( \nu = \frac{d-2}{2} \) and the formula \( \cos(u + v) = \cos u \cos v - \sin u \sin v \), we write \( Q \) as follows:

\[
Q(x) = Q_1(x) - Q_2(x) + Q_3(x),
\]

where

\[
Q_1(x) := 2(2\pi)^{\frac{d-1}{2}} |x|^{\frac{1-d}{2}} \cos \left( \frac{\pi(d-1)}{4} \right) \int_{1-2s}^{1+2s} m_\phi(r) \phi(r) r^{\frac{d-1}{2}} \cos \left( (r-1)|x| \right) dr,
\]

\[
Q_2(x) := 2(2\pi)^{\frac{d-1}{2}} |x|^{\frac{1-d}{2}} \sin \left( \frac{\pi(d-1)}{4} \right) \int_{1-2s}^{1+2s} m_\phi(r) \phi(r) r^{\frac{d-1}{2}} \sin \left( (r-1)|x| \right) dr,
\]

\[
Q_3(x) := (2\pi)^{\frac{d}{2}} \int_{1-2s}^{1+2s} m_\phi(r) \phi(r) r^{d-1} |x|^{\frac{2-d}{2}} R_{d-2}(r|x|) dr.
\]

We now split the domain of the integral \( \int_{1-2s}^{1+2s} m_\phi(r) \phi(r) r^{\frac{d-1}{2}} \cos ((r-1)|x|) dr \) of \( Q_1(x) \) into subintervals on which \(|r-1| \leq \delta\) and \(|r-1| \geq \delta\), respectively. To be precise let us set \( k(t) := \int_{|r| \leq \frac{1}{\sqrt{t+1}}} \frac{1}{r} dr \) and fix a large \( \lambda > 0 \) such that

\[
k(s\lambda) \geq 100(\pi - k(s\lambda)).
\]

Clearly, such \( \lambda \) exists since \( \int_{\frac{1}{\sqrt{t+1}}}^\infty \frac{1}{r} dr = \pi \). Let \( \mu \) be a small number so that \( \lambda \mu \leq 10^{-2} \), and let

\[
A' := \left\{ x \in \mathbb{R}^d : \frac{\mu}{4\delta} \leq |x| \leq \frac{\mu}{2\delta} \right\}.
\]

Put \( \psi(r) := r - s \), \( r > 0 \) and set \( M_\lambda := \max\{|\psi''(r)| : |r-1| \leq 1/2\} \). By Taylor’s theorem, \( |\psi(r) - s(r-1)| \leq M_\lambda (r-1)^2/2 \) when \(|r-1| \leq 1/2\). Thus, if \( |\psi(r)| \leq s\lambda \delta \), then

\[
|r-1| \leq \frac{M_\lambda}{2s} (r-1)^2 + \frac{M_\lambda \delta}{s} |r-1| \leq \lambda \delta, \quad \forall r \in \text{supp} \phi \cap [1/2, 3/2].
\]

Choosing \( \varepsilon_\circ := \min\{s/(2M_\lambda), 1/4\} \) we have \(|r-1| \leq 2\delta \) on \( \text{supp} \phi \) whenever \(|\psi(r)| \leq s\lambda \delta \). Therefore, if \( x \in A' \) and \(|\psi(r)| \leq s\lambda \delta \), then \(|(r-1)x| \leq \lambda \mu \leq 10^{-2} \), so \( \text{cos}((r-1)|x|) \geq 99/100 \) on \( \text{supp} \phi \).

Now we break the integral part of \( Q_1(x) \) as the following:

\[
I_1(x) + I_2(x) := \left( \int_{|\psi(r)| \leq s\lambda \delta} + \int_{|\psi(r)| > s\lambda \delta} \right) \frac{\delta}{\psi(r)^2 + \delta^2} \phi(r)^{\frac{d+1}{2}} \cos \left( (r-1)|x| \right) dr.
\]

If \( x \in A' \), by the above choice of \( \lambda \) and \( \varepsilon_\circ \), we have

\[
I_1(x) \geq \frac{99}{100} \int_{|\psi(r)| \leq s\lambda \delta} \frac{\delta}{\psi(r)^2 + \delta^2} \phi(r)^{\frac{d+1}{2}-s+1} \frac{d\psi(r)}{s}.
\]

Hence, we choose \( \phi \in C_\infty^\circ((1-2\varepsilon_\circ, 1+2\varepsilon_\circ)) \) such that \( \phi(r)^{\frac{d+1}{2}-s+1} = 1 \) if \(|r-1| \leq \varepsilon_\circ \), and \( 0 \leq \phi(r)^{\frac{d+1}{2}-s+1} \leq 2 \) for all \( r \). Thus it follows that, if \( x \in A' \) and \( \delta \leq \varepsilon_\circ/(2\lambda) \), then

\[
I_1(x) \geq \frac{99}{100s} \int_{|t| \leq s\lambda \delta} \frac{\delta}{t^2 + \delta^2} dt = \frac{99}{100s} k(s\lambda).
\]

On the other hand, by our choice of \( \phi \) and (5.11)

\[
|I_2(x)| \leq \int_{|\psi(r)| > s\lambda \delta} \frac{\delta}{\psi(r)^2 + \delta^2} \phi(r)^{\frac{d+1}{2}-s+1} \frac{d\psi(r)}{s} \leq \frac{2}{s} \left( \pi - k(s\lambda) \right) \leq \frac{k(s\lambda)}{50s}.
\]

Therefore we have, for \( x \in A' \) and \( 0 < \delta \leq \varepsilon_\circ/(2\lambda) \),

\[
I_1(x) + I_2(x) \geq I_1(x) - |I_2(x)| \geq \frac{97k(s\lambda)}{100s}.
\]
For each \( n \in \mathbb{N} \) let us set
\[
A_n := \left\{ x \in A' : \pi\left(2n + \frac{d-1}{4}\right) - \frac{1}{100} \leq |x| \leq \pi\left(2n + \frac{d-1}{4}\right) + \frac{1}{100} \right\},
\]
which is nonempty only if \( n \approx b^{-1} \). We also set
\[
A := \bigcup_{n \in \mathbb{N}} A_n.
\]
From now on suppose that \( x \in A \). It is clear that \( \cos(|x| - \pi(d-1)/4) \geq 99/100 \), and \( |\sin(|x| - \pi(d-1)/4)| \leq 1/100 \). Hence, when \( 0 < \delta \leq \varepsilon \delta/(2\lambda) \) it follows from (5.10) and (5.12) that
\[
Q_1(x) \geq 2(2\pi)^{\frac{d-1}{4}} |x|^{\frac{1-d}{2}} \cdot \frac{99}{100} \cdot \frac{97k(s\lambda)}{100s}.
\]
Similarly, we get
\[
|Q_2(x)| \leq 2(2\pi)^{\frac{d-1}{4}} |x|^{\frac{1-d}{2}} \cdot \frac{2}{100s} \cdot \frac{101k(s\lambda)}{100}.
\]
Combining the estimates for \( Q_1(x) \) and \( |Q_2(x)| \), we have
\[
Q_1(x) - Q_2(x) \geq |x|^{\frac{1-d}{2}} \approx \delta^{\frac{d-1}{4}},
\]
where the implicit constants depend only on \( d \) and \( s \). On the other hand, it is straightforward to see that \( Q_3(x) = O(\delta^{\frac{d-1}{4}}) \) as \( \delta \to 0^+ \). Thus, with sufficiently small \( \delta \), we have that
\[
Q(x) = Q_1(x) - Q_2(x) + Q_3(x) \geq \delta^{\frac{d-1}{4}}, \quad \forall x \in A.
\]
Observe that we can choose a constant \( c \in (0,1/10) \), independent of all small \( \delta > 0 \), such that \( |A| \geq c |B(0, \mu(2\delta)^{-1})| \). Therefore, we conclude that
\[
\|Q\|_{L^q(A)} \gtrsim \delta^{\frac{d-1}{4}}.
\]
for all sufficiently small \( \delta > 0 \), with the implicit constant depending only on \( d, s \). Meanwhile, \( f = F^{-1}(\phi(|\cdot|)) \in L^p(\mathbb{R}^d) \) for any \( p \in [1, \infty] \). Thus, this completes the proof of (5.8). \( \square \)

6. Sharp resolvent estimates for the fractional Laplace operators

The resolvent estimates for \((-\Delta)^{\frac{s}{2}}\) can be obtained by making use of the argument we have used for the resolvent estimates for \(-\Delta\) (the case \( s = 2 \)). In technical aspect there is not much difference, but it is worthwhile to record the result for the operator \((-\Delta)^{\frac{s}{2}} - z)^{-1}\). We shall be brief, but include the statements of results and sketch their proofs. In what follows we consider \( s \in (0, d) \) though generalization to \( s \geq d \) is also possible.

We begin with the following which can be shown by adapting the proof of Proposition 1.1 so we state it without proof.

**Proposition 6.1.** Let \( d \geq 2 \), \( 0 < s < d \), \( 1 \leq p, q \leq \infty \), and \( z \in \mathbb{C} \setminus [0, \infty) \). Then, \( \|((-\Delta)^{\frac{s}{2}} - z)^{-1}\|_{p \to q} < \infty \) if and only if \((1/p, 1/q) \in \mathcal{R}_d^s\) which is given by
\[
\mathcal{R}_d^s = \mathcal{R}_d^s(0) := \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1, \ 0 \leq x - y \leq s/d, \left(1, \frac{d-s}{d}, \left(\frac{s}{d}, 0\right)\right) \right\}.
\]

We introduce some notations which we need to state our results. For \( d \geq 2 \) and \( 0 < s < d \), let us define \( \mathcal{R}_i = \mathcal{R}_i^s(d), \mathcal{R}_2^s = \mathcal{R}_2^s(d), \) and \( \mathcal{R}_3^s = \mathcal{R}_3^s(d) \) by
\[
\mathcal{R}_1 := \mathcal{P}(d) \cap \mathcal{R}_0^s(d), \quad \mathcal{R}_2 := \mathcal{T}(d) \cap \mathcal{R}_0^s(d), \quad \mathcal{R}_3 := \mathcal{Q}(d) \cap \mathcal{R}_0^s(d).
\]
Note that if \( 0 < s < \frac{2d}{d+1} \), \( \mathcal{R}_1 = \emptyset \) (see Figure 10), and \( \mathcal{R}_2^s = \mathcal{R}_i \) for \( i = 0, 1, 2, 3 \). If \( 0 < s < 2 \), then \( \mathcal{R}_i^s = \mathcal{R}_i \cap \mathcal{R}_0^s \) for every \( i \). From the definition of \( \gamma_{p,q} \) (recall (1.7)), it follows that (1.11) holds with \( \mathcal{R}_i \) replaced by \( \mathcal{R}_i^s, i = 1, 2, 3 \).
As before, by scaling we have \( \|(−Δ)^{\frac{1}{2}} z \|_{p-q} \leq |z|^{-1 + \frac{2}{q} + \frac{1}{p} - \frac{1}{q}} \|(−Δ)^{\frac{1}{2}} - |z|^{-1} z \|_{p-q} \) for \( z \in \mathbb{C} \setminus [0, \infty) \), and using Lemma 5.1 we may repeat the argument in the proof of Proposition 1.3 to get
\[
\|(−Δ)^{\frac{1}{2}} z - z \|_{p-q} \gtrsim \text{dist}(z, [0, \infty))^{-\gamma_{p,q}}, \quad z \in \mathbb{S} \setminus \{1\}.
\]
Combining these two, we get, for \( z \in \mathbb{C} \setminus [0, \infty) \),
\[
(6.1) \quad \|(−Δ)^{\frac{1}{2}} z - z \|_{p-q} \gtrsim \kappa_{p,q}(z) := |z|^{-1 + \frac{2}{q} + \frac{1}{p} - \frac{1}{q}} \text{dist}(z, [0, \infty))^{-\gamma_{p,q}},
\]
and we may conjecture the following which is a natural extension of Conjecture 1

Conjecture 3. Let \( d \geq 2, \ 0 < s < d, \) and \((1/p, 1/q) \in (\bigcup_{j=1}^3 R_j^s) \cup R_3^s \). There exists an absolute constant \( C \), depending only on \( p, q, d \) and \( s \), such that, for \( z \in \mathbb{C} \setminus [0, \infty) \),
\[
(6.2) \quad C^{-1} \kappa_{p,q}(z) \leq \|(−Δ)^{\frac{1}{2}} z - z \|_{p-q} \leq C \kappa_{p,q}(z),
\]
When \( d \geq 3 \) let us set \( \tilde{R}_2^s := R_0^s(d) \cap \tilde{R}_2(d) \) and \( \tilde{R}_3^s := R_3^s(d) \setminus [D, P_o, P_s] \). We have the following.

Theorem 6.2. Let \( z \in \mathbb{C} \setminus [0, \infty) \). If \( d = 2 \), Conjecture 3 is true. If \( d \geq 3 \), the conjectured estimate (6.2) is true whenever \((1/p, 1/q) \in R_j^s \cup (\bigcup_{j=2}^3 \tilde{R}_j^s) \cup R_3^s \). Furthermore, if \( d \geq 2 \), for \( p, q, s \) satisfying \((1/p, 1/q) \in \{B, B'\} \) and \( \frac{2d}{d+1} \leq s < d \), we have \( \|(−Δ)^{\frac{1}{2}} z - z \|_{q, \infty} \lesssim |z|^{-1 + \frac{2}{q} + \frac{1}{p} - \frac{1}{q}} \|f\|_{p,1} \) and, for \( p, q, s \) satisfying \((1/p, 1/q) \in \{B', E'\} \cap R_0^s \) and \( \frac{2d}{d+1} < s < d \), we have \( \|(−Δ)^{\frac{1}{2}} z - z \|_{q, \infty} \lesssim |z|^{-1 + \frac{2}{q} + \frac{1}{p} - \frac{1}{q}} \|f\|_{p} \).

We remark that Theorem 1.4 is a special case of Theorem 6.2 when \( d \geq 3 \). If \( \frac{2d}{d+1} \leq s < d \) and \((1/p, 1/q) \in R_1^s(d) \), Theorem 6.2 covers the result by Huang, Yao, and Zheng [20, Theorem 1.4].

Proof of Theorem 6.2. We basically follow the argument in Section 4 with some modifications. For \( z \in \mathbb{S} \setminus \{1\} \) let us set
\[
m_s^{\ast}(\xi, z) = (|\xi|^{s} - z)^{-1}.
\]
Using the same functions \( \rho_0, \rho_1, \) and \( \rho_2 \) as in Section 4 we break \( m^{\ast} \) such that \( m_j^{\ast}(\xi, z) := m(\xi, z) \rho_j(\xi), \ j = 0, 1, 2, \) and \( m^{\ast}(\xi, z) = \sum_{j=0}^2 m_j^{\ast}(\xi, z) \). Since \( |\partial_\xi^a m_j^{\ast}(\xi, z)| \lesssim |\xi|^{-s-|\alpha|} \) and \( m_j^{\ast}(\xi, z) \) is supported away from the origin, by the standard argument (for example, the proof of sufficiency part of Proposition 1.1) we see that \( \|m_j^{\ast}(D, z) f\|_{q} \leq C \|f\|_{p} \) if \((1/p, 1/q) \in R_0^s \). Similarly, since \( m_0^{\ast} \) is supported in \( B(0, 1) \) and \( |\partial_\xi^a m_0^{\ast}(\xi, z)| \lesssim |\xi|^{s-|\alpha|} \), we see that \( \|m_0^{\ast}(D, z) f\|_{q} \leq C \|f\|_{p} \) if
$1 \leq p \leq q \leq \infty$. Thus, we need only to handle $m^q_\gamma(D, z)$. If $z \in S^1(\theta, \omega)$, $\partial^\gamma m^q_\gamma(\ell, z)$ is uniformly bounded. So, we may assume $z \notin S^1(\theta)$ and we are reduced to showing (4.15) when $(1/p, 1/q) \in \mathcal{R}_1^* \cup \left( \bigcup_{i=1}^3 \tilde{\mathcal{R}}_i^* \right) \cup \mathcal{R}_3^*$, and its (restricted) weak type variants when $(1/p, 1/q) \in [B', E'] \cap \mathcal{R}_0^*$. All of these estimates are contained in Proposition 4.1.

### 6.1. Region of spectral parameters for uniform estimate.

Let $p, q, d,$ and $s$ be as in Theorem 6.2 and let $\ell > 0$. Making use of Theorem 6.2, we can also describe the region

$$Z_{p,q}(\ell) := \{ z \in \mathbb{C} \setminus [0, \infty) : k_{p,q}^*(z) \leq \ell \}.$$

We consider three cases $0 < s < \frac{2d}{d+1}$, $s = \frac{2d}{d+1}$, and $\frac{2d}{d+1} < s < d$, separately. Also, by duality it is sufficient to consider $p, q$ satisfying $(1/p, 1/q) \in \mathcal{R}_1^* \cup \mathcal{R}_2^* \cup \mathcal{R}_3^*$. As before we set the homogeneity degree $\omega_{p,q}^s = \omega_{p,q}^s(d) := 1 - \frac{d}{s} \left( \frac{1}{p} - \frac{1}{q} \right)$, which is in $[0, 1]$ when $(1/p, 1/q) \in \mathcal{R}_0^*$, and note that

$$Z_{p,q}^s(\ell) = \{ z \in \mathbb{C} \setminus \{0\} : \text{Re} z \leq 0, \ell|z|^{-\omega_{p,q}^s} \geq 1 \}
\cup \{ z \in \mathbb{C} \setminus [0, \infty) : \text{Re} z > 0, \ell \text{Im} z|z|^{-\omega_{p,q}^s} \geq |z|^{-\gamma_{p,q}} \}.$$

Since $\omega_{p,q}^s = 1 = \gamma_{2,2}$ for any $s$ and $d$, regardless of the values of $d$ and $s$, $Z_{p,q}^s(\ell)$ is always the complement of the $\ell^{-1}$-neighborhood of $[0, \infty)$ (see Figure 6c or Figure 7). We also note that $Z_{p,q}^s(\ell) = \emptyset$ if $\omega_{p,q}^s = 0$ and $\ell < 1$. In what follows we disregard the case $p = q = 2$, and the case $\ell < 1$ whenever $\omega_{p,q}^s = 0$.

#### When $0 < s < \frac{2d}{d+1}$.

In this case, $\mathcal{R}_1^* = \emptyset$, and $\gamma_{p,q} > 0$ for all $p, q$ with $(1/p, 1/q) \in \mathcal{R}_2^* \cup \mathcal{R}_3^*$. If $\omega_{p,q}^s = 0$, $Z_{p,q}^s(1) = \{ z \in \mathbb{C} \setminus \{0\} : \text{Re} z \leq 0 \}$, and $Z_{p,q}^s(\ell)$ with $\ell > 1$ is a complement of a planar cone such as in Figure 9. If $\omega_{p,q}^s > 0$, one can easily check that $\gamma_{p,q} - \omega_{p,q}^s > 0$ for all $(1/p, 1/q) \in (\mathcal{R}_2^* \setminus \{H\}) \cup \mathcal{R}_3^*$. Thus, $Z_{p,q}^s(\ell)$ has profiles such as the regions in Figure 8a, Figure 8b, or Figure 8c.

#### When $s = \frac{2d}{d+1}$.

In this case, $\frac{1}{p} - \frac{1}{q} = \frac{2d}{d+1}$ and $\mathcal{R}_1^* = (B, B')$. If $\omega_{p,q}^s = 0$, $Z_{p,q}^s(\ell) = \mathbb{C} \setminus [0, \infty)$ for $(1/p, 1/q) \in \mathcal{R}_1^*$ and $\ell \geq 1$; $Z_{p,q}^s(1)$ is the left half-plane for $(1/p, 1/q) \in \mathcal{R}_2^*$. $Z_{p,q}^s(\ell)$ is a complement of a planar cone for $(1/p, 1/q) \in \mathcal{R}_3^*$ and $\ell > 1$ (see Figure 9); there is no $p, q$ satisfying $(1/p, 1/q) \in \mathcal{R}_2^*$. If $\omega_{p,q}^s > 0$, we consider the following two cases:

- $(1/p, 1/q) \in \mathcal{R}_2^* \setminus \{H\}$: In this case, $\gamma_{p,q} - \omega_{p,q}^s = 0$ since $s = \frac{2d}{d+1}$. Hence, $Z_{p,q}^s(\ell)$ is the complement of the $\ell^{-1/\gamma_{p,q}}$-neighborhood of $[0, \infty)$.

- $(1/p, 1/q) \in \mathcal{R}_3^*$: Then, $\gamma_{p,q} - \omega_{p,q}^s = -d_1^+ \left( \frac{1}{p} - \frac{d-1}{d+1} (1 - \frac{1}{q}) \right) > 0$. Thus, the profiles of $Z_{p,q}^s(\ell)$ take the forms of the regions in Figure 8a, Figure 8b, or Figure 8c.

#### When $\frac{2d}{d+1} < s < d$.

The classification of the profiles of $Z_{p,q}^s(\ell)$ is similar to that in Section 4.4 where $s = 2$. Again we consider the cases $\omega_{p,q}^s = 0$ and $\omega_{p,q}^s > 0$, separately. If $\omega_{p,q}^s = 0$, there are only two cases $(1/p, 1/q) \in \mathcal{R}_1^*$ and $(1/p, 1/q) \in \mathcal{R}_3^*$. For the first case $Z_{p,q}^s(\ell) = \mathbb{C} \setminus [0, \infty)$ when $\ell \geq 1$, and for the latter $Z_{p,q}^s(1)$ is the left half-plane and $Z_{p,q}^s(\ell)$ with $\ell > 1$ is the complement of a planar cone (Figure 9). If $\omega_{p,q}^s > 0$, we consider the following cases:

- $(1/p, 1/q) \in \mathcal{R}_1^*$: Since $\gamma_{p,q} = 0$ and $0 < \omega_{p,q}^s < \frac{d(d+1)-2d}{s(d+1)}$, $Z_{p,q}^s(\ell) = \{ z \in \mathbb{C} \setminus [0, \infty) : |z| \geq \ell^{-1/\omega_{p,q}^s} \}$.

- $(1/p, 1/q) \in \mathcal{R}_3^*$: Since $\gamma_{p,q} - \omega_{p,q}^s = \left( \frac{d-1}{d+1} \right) (\frac{1}{p} - \frac{1}{q}) < 0$, $Z_{p,q}^s(\ell)$ is the complement of a neighborhood of $[0, \infty)$ which shrinks along the positive real line as $\text{Re} z \to \infty$. 




\[
\begin{align*}
&\bullet \ (1/p, 1/q) \in \mathcal{R}^3_3; \text{ Note that } \gamma_{p,q} = \frac{d+1}{2} - \frac{d}{q} > 0, \text{ and } \gamma_{p,q} - \omega_{p,q}^* = \frac{d}{s} \left( \frac{1-s}{p} + \frac{s(d-1)}{2d} - \frac{1}{q} \right). \text{ We divide } \mathcal{R}^3_3 \text{ into } \mathcal{R}^3_{3,+}, \mathcal{R}^3_{3,0}, \text{ and } \mathcal{R}^3_{3,-} \text{ which are given by}
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}^3_{3,+} &= \left\{ (x, y) \in \mathcal{R}^3_3 : \pm \left( y - (1-s)x - \frac{s(d-1)}{2d} \right) > 0 \right\}, \\
\mathcal{R}^3_{3,0} &= \left\{ (x, y) \in \mathcal{R}^3_3 : y = (1-s)x + \frac{s(d-1)}{2d} \right\}, \\
\mathcal{R}^3_{3,-} &= \left\{ (x, y) \in \mathcal{R}^3_3 : \pm \left( y - (1-s)x - \frac{s(d-1)}{2d} \right) < 0 \right\}.
\end{align*}
\]

\[\dagger \ (1/p, 1/q) \in \mathcal{R}^3_{3,+}: \text{ Since } \gamma_{p,q} - \omega_{p,q}^* < 0, \text{ } Z_{p,q}(\ell) \text{ is the complement of a neighborhood of } [0, \infty) \text{ which shrinks along positive real line as } \text{Re} \ z \rightarrow \infty.\]

\[\dagger \ (1/p, 1/q) \in \mathcal{R}^3_{3,0}: \text{ Then } \gamma_{p,q} - \omega_{p,q}^* = 0 \text{ and } Z_{p,q}(\ell) \text{ is the complement of the } \ell^{-1/\gamma_{p,q}^*} \text{ neighborhood of } [0, \infty).\]

\[\dagger \ (1/p, 1/q) \in \mathcal{R}^3_{3,-}: \text{ Since } \gamma_{p,q} - \omega_{p,q}^* > 0, \text{ } Z_{p,q}(\ell) \text{ is the complement of a neighborhood of } [0, \infty) \text{ whose boundary asymptotically satisfies } | \text{Im} \ z | \approx (\text{Re} \ z)^{1-\omega_{p,q}^*/\gamma_{p,q}^*} \text{ when } \text{Re} \ z \text{ is large (Figure 8a, Figure 8b, and Figure 8c).}\]

6.2. Location of the eigenvalues of \((-\Delta)^{\frac{1}{2}} + V\). Finally, using Theorem (6.2) we can obtain the following which describes location of eigenvalues of the fractional operator \((-\Delta)^{\frac{1}{2}} + V\) acting in \(L^q(\mathbb{R}^d)\). The proof is similar to that of Corollary 1.5.

**Corollary 6.3.** Let \(d \geq 2, \ 0 < s < d, \ (1/p, 1/q) \in \mathcal{R}^3_3 \cup (\bigcup_{i=2}^{3} \mathcal{R}^3_i^*) \cup \mathcal{R}^3_{3,-}^*, \text{ and let } C > 0 \text{ be the constant which appears in (6.2). Fix a positive number } \ell > 0 (\text{we choose } \ell \geq 1 \text{ if } 1/p - 1/q = s/d). \text{ Suppose that } |V|_{L^p(\mathbb{R}^d)} \leq t(C\ell)^{-1} \text{ for some } t \in (0, 1). \text{ Then, if } E \in C \setminus [0, \infty) \text{ is an eigenvalue of } (-\Delta)^{\frac{1}{2}} + V \text{ acting in } L^q(\mathbb{R}^d), E \text{ must lie in } C \setminus Z_{p,q}(\ell).\]

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