Many Faces of Born-Infeld Theory

Sergei V. Ketov

Department of Physics, University of Kaiserslautern
Erwin-Strödinger Str., 67653 Kaiserslautern, Germany

Abstract

Born-Infeld theory is the non-linear generalization of Maxwell electrodynamics. It naturally arises as the low-energy effective action of open strings, and it is also part of the world-volume effective action of D-branes. The $N = 1$ and $N = 2$ supersymmetric generalizations of the Born-Infeld action are closely related to partial spontaneous breaking of rigid extended supersymmetry. We review some remarkable features of the Born-Infeld action and outline its supersymmetric generalizations in four dimensions. The non-abelian $N = 1$ supersymmetric extension of the Born-Infeld theory and its $N = 1$ supergravitational avatars are given in superspace.

1 Introduction

The Born-Infeld (BI) non-linear electrodynamics in Minkowski spacetime is defined by the Lagrangian

$$\mathcal{L}_{\text{BI}}(F) = \frac{1}{b^2} \left\{ 1 - \sqrt{\det(\eta_{\mu\nu} + bF_{\mu\nu})} \right\},$$

(1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu, \nu = 0, 1, 2, 3$, and $b$ is the dimensional coupling constant ($b = 1$ in what follows). The BI theory implies famous taming of Coulomb self-energy of a point-like electric charge, while it shares with the Maxwell theory electric-magnetic self-duality and physical propagation (no shock waves). In order to appreciate these highly non-trivial features, let's recall that a generic non-linear electrodynamics is defined by the field equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \cdot \vec{B} = 0,$$

$$\nabla \times \vec{H} = +\frac{\partial \vec{D}}{\partial t}, \quad \nabla \cdot \vec{D} = 0.$$ 

(2)

If there exists a Lagrangian $\mathcal{L}(\vec{E}, \vec{B})$, then we have

$$\vec{H} = -\frac{\partial \mathcal{L}}{\partial \vec{B}} \quad \text{and} \quad \vec{D} = +\frac{\partial \mathcal{L}}{\partial \vec{E}}.$$ 

(3)

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Lorentz invariance in four dimensions implies further restrictions,
\[ \mathcal{L} = \mathcal{L}(\alpha, \beta), \quad \text{where} \quad \alpha = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) \quad \text{and} \quad \beta = \vec{E} \cdot \vec{B}. \]  

(4)

The electric-magnetic self-duality of the non-linear electrodynamics (2) under rigid rotations,
\[ \vec{E} + i\vec{H} \rightarrow e^{i\theta}(\vec{E} + i\vec{H}) \quad \text{and} \quad \vec{D} + i\vec{B} \rightarrow e^{i\theta}(\vec{D} + i\vec{B}), \]  

together with eq. (3) gives rise to a highly non-trivial non-linear constraint [2],
\[ \vec{E} \cdot \vec{B} = \vec{D} \cdot \vec{H}. \]  

(6)

In the manifestly Lorentz-covariant setting with \( \mathcal{L}(F_{\mu\nu}) \), it is natural to deal with the equations of motion and the Bianchi identities having the same form,
\[ \partial^\nu \tilde{G}_{\mu\nu} = 0 \quad \text{and} \quad \partial^\nu \tilde{F}_{\mu\nu} = 0, \]  

(7)

respectively, where we have defined
\[ \tilde{G}_{\mu\nu}(F) = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} G^{\lambda\rho}(F) = 2 \frac{\partial \mathcal{L}(F)}{\partial F^{\mu\nu}}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}. \]  

(8)

Equation (6) then amounts to the non-linear constraint [2]
\[ G^{\mu\nu} \tilde{G}_{\mu\nu} + F^{\mu\nu} \tilde{F}_{\mu\nu} = 0. \]  

(9)

Causal propagation in a classical field theory follows from the dominant energy condition on the energy-momentum tensor \( T_{\mu\nu} \) (Hawking theorem) [3]
\[ T_{00} \geq T_{\mu\nu} \quad \text{for all} \quad \mu \quad \text{and} \quad \nu. \]  

(10)

It is straightforward (and very instructive) to verify that the BI theory (1) does satisfy both eqs. (9) and (10). The absence of shock waves means that the phase speed is phase-independent — it is also the truly non-perturbative feature of the BI theory!

The BI theory possesses even more magical properties, such as the built-in upper bound for the electro-magnetic field-strength and the existence of exact soliton-like solutions (called BIons) of finite total energy, \( \int d^3x T_{00} < \infty \) [5].

The existence of the maximal electromagnetic field strength is obvious from the form of the dual Hamiltonian density of the BI theory,
\[ \mathcal{H}_{\text{dual}} = 1 - \sqrt{1 - \vec{H}^2 - \vec{E}^2 + (\vec{H} \times \vec{E})^2}. \]  

(11)

In string theory, approaching the upper bound results in the breakdown of the BI theory due to a production of the open string massive states.
The BIon solution of electric charge $Q$ to the field equation

$$\nabla \cdot \vec{D} = 4\pi Q \delta(\vec{r})$$

is given by

$$\vec{D} = \frac{Q}{r^2} \hat{e}_r, \quad \vec{E} = \frac{\vec{D}}{\sqrt{1 + \vec{D}^2}} = \frac{Q}{\sqrt{r^4 + Q^2}} \hat{e}_r,$$

so that the electric field singularity of the Maxwell theory is not present in the BI theory, while the effective electric density $\rho_{\text{eff}}$ of a point-like electric charge $Q$,

$$\rho_{\text{eff}} = \frac{1}{4\pi} \nabla \cdot \vec{E},$$

has a finite non-vanishing radius (of order $\sqrt{b}$).

In string theory, when a constant Kalb-Ramond background $B_{\mu\nu}$ is turned on, $F_{\mu\nu} \to F_{\mu\nu} + B_{\mu\nu}$, the BI theory in the limit $b = 2\pi\alpha' \to 0$ appears to be equivalent to a non-commutative $U(1)$ gauge field theory in flat spacetime with $[x^\mu, x^\nu] = iB^{\mu\nu}$, via the Seiberg-Witten map. Note that Lorentz invariance is broken in this case. The BI Lagrangian (1) in Euclidean spacetime interpolates between the Maxwell Lagrangian $\frac{1}{4}F^2$ for small $F$, and the topological density $\frac{1}{4\pi} F\tilde{F}$ for large $F$, because of the identity

$$\sqrt{\det(F_{\mu\nu})} = \frac{1}{4} \left| F\tilde{F} \right|.$$ 

The non-trivial (Euclidean) BI Lagrangian in the $b \to 0$ limit reads

$$\frac{F^2}{|F\tilde{F}|},$$

where we have used the relations

$$\sqrt{\det(\varepsilon^{1/2} + F)} \to \sqrt{\varepsilon^2 + \frac{1}{4}F^2 + \frac{1}{16}(F\tilde{F})^2} \xrightarrow{\varepsilon \to 0} \frac{1}{4} \left| F\tilde{F} \right| + \varepsilon \frac{F^2}{|F\tilde{F}|}.$$ 

The first term on the r.h.s. of this equation is a total derivative, so that one arrives at eq. (16) after rescaling eq. (17) by a factor of $\varepsilon^{-1}$.

The BI Lagrangian (1) in Euclidean spacetime obeys the (BPS) bound

$$\mathcal{L}_{\text{BI}} = \sqrt{\left(1 + \frac{1}{4}F\tilde{F}\right)^2 + \frac{1}{4} \left( F - \tilde{F} \right)^2} - 1 \geq \frac{1}{4} \left| F\tilde{F} \right|$$

that is saturated at self-dual field configurations, $F = \tilde{F}$, like in the Maxwell case.
2 BI theory and rigid supersymmetry

The $N = 1$ supersymmetric extension of the abelian BI action in four spacetime dimensions is the Goldstone-Maxwell (GM) action associated with Partial (1/2) Spontaneous Supersymmetry Breaking (PSSB) $N = 2$ to $N = 1$, whose Goldstone fields belong to a Maxwell (vector) supermultiplet with respect to unbroken $N = 1$ supersymmetry [7, 8].

Manifest supersymmetry does not respect the standard determinantal form of the BI Lagrangian in eq. (1). Moreover, eq. (1) is not even the most elegant form of the BI theory! The complex bosonic variable, having the most natural $N = 1$ supersymmetric extension (with linearly realized $N = 1$ supersymmetry in superspace), is given by

$$
\omega = \alpha + i\beta, \quad \alpha = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \equiv \frac{1}{4}F^{2}, \quad \beta = \frac{1}{4}F^{\mu\nu}\tilde{F}_{\mu\nu} \equiv \frac{1}{4}\tilde{F}. \quad (19)
$$

The BI Lagrangian (1) can be rewritten to the form ($b = 1$)

$$
L_{\text{BI}}(\omega, \bar{\omega}) = 1 - \sqrt{1 + (\omega + \bar{\omega}) + \frac{1}{4}(\omega - \bar{\omega})^2}, \quad (20)
$$
or, equivalently,

$$
L_{\text{BI}}(\omega, \bar{\omega}) = L_{\text{free}} + L_{\text{int.}} \equiv -\frac{1}{2}(\omega + \bar{\omega}) + \omega\bar{\omega}Y(\omega, \bar{\omega}), \quad (21)
$$
whose structure function is given by

$$
Y(\omega, \bar{\omega}) \equiv \frac{1}{1 + \frac{1}{2}(\omega + \bar{\omega}) + \sqrt{1 + (\omega + \bar{\omega}) + \frac{1}{4}(\omega - \bar{\omega})^2}}. \quad (22)
$$

The remarkably compact form of the BI action [7, 8]

$$
L_{\text{BI}}(\omega, \bar{\omega}) = -\frac{1}{2}(\chi + \bar{\chi}) = -\text{Re} \chi \quad (23)
$$
arises as the iterative solution to the simple non-linear constraint

$$
\chi = -\frac{1}{2}\chi\bar{\chi} + \omega. \quad (24)
$$

This Non-Linear Sigma-Model (NLSM) form of the bosonic BI theory is quite natural from the viewpoint of PSSB [8]. Indeed, to spontaneously break any rigid symmetry, one may start with a free action that is invariant under the linearly realized symmetry, and then impose an invariant non-linear constraint that gives rise to the NLSM whose solutions break the symmetry.

The bosonic BI theory in the NLSM form is most convenient for a direct supersymmetrization in superspace. One simply replaces the abelian bosonic
field strength $F_{\mu\nu}$ by the abelian $N = 1$ chiral spinor superfield strength $W_\alpha$ obeying the standard off-shell $N = 1$ superspace Bianchi identities,
\[ D_\alpha W_\alpha = 0, \quad D_\alpha W^\alpha = 0, \quad \alpha = 1, 2. \quad (25) \]
In the chiral basis the $N = 1$ superfield $W_\alpha$ reads
\[
W_\alpha(x, \theta) = \psi_\alpha(x) + \theta^\beta [\sigma^{\mu\nu}]_\beta \alpha F_{\mu\nu}(x) + i \delta_\beta ^\alpha D_\alpha \tilde{\psi}_\beta(x), \quad (26)
\]
where $\psi_\alpha(x)$ is the fermionic superpartner (Goldstone fermion) of the BI vector field, and $D$ is the real auxiliary field. The $N = 1$ superextension of $\omega$ is simply given by $\frac{1}{2} D^2 W^2$, where $W^2 \equiv W_\alpha W^\alpha$ and $D^2 = D_\alpha D^\alpha$. The $N = 1$ manifestly supersymmetric abelian BI action \[7\] in the NLSM form reads
\[
S_{1BI} = \int d^4x d^2\theta X + \text{h.c.,} \quad (27)
\]
where the $N = 1$ chiral superfield Lagrangian $X$ obeys the non-linear constraint
\[
X = \frac{1}{2} X \bar{D}^2 \bar{X} + \frac{1}{2} W^\alpha W_\alpha. \quad (28)
\]
The iterative solution to eq. (28) gives rise to the superfield action \[7\]
\[
S_{1BI} = \frac{1}{2} \left( \int d^4x d^2\theta W^2 + \text{h.c.} \right) + \int d^4x d^2\theta \mathcal{Y}(\frac{1}{2} D^2 W^2, \frac{1}{2} \bar{D}^2 \bar{W}^2) W^2 \bar{W}^2 \quad (29)
\]
with the same structure function (22) as in the bosonic BI case.

The NLSM form (27) and (28) of the $N = 1$ BI action is also most useful in proving its invariance under the second (non-linearly realized and spontaneously broken) $N = 1$ supersymmetry with the rigid spinor parameter $\eta^\alpha$ \[7, 8\]$
\delta_2 X = \eta^\alpha W_\alpha, \quad \delta_2 W_\alpha = \eta_\alpha \left( 1 - \frac{1}{2} \bar{D}^2 \bar{X} \right) + i \bar{\eta}^\beta \partial_\beta ^\alpha X, \quad (30)$
and its $N = 1$ supersymmetric electric-magnetic self-duality as well. The latter amounts to a verification of the non-local constraint \[10\]
\[
\int d^4x d^2\theta (W^2 + M^2) = \int d^4x d^2\bar{\theta}(\bar{W}^2 + \bar{M}^2), \quad \text{where} \quad \frac{i}{2} M_\alpha = \frac{\delta S_{1BI}}{\delta W^\alpha}, \quad (31)
\]
which is just the N=1 supersymmetric extension of eq. (9).

The $N = 1$ supersymmetric BI action can be put into the $N = 1$ superconformal form by inserting a conformal compensator ($N = 1$ chiral superfield) $\Phi$ into the non-linear constraint (28) as follows \[11\]
\[
X = \frac{X}{2\Phi^2} \bar{D}^2 \left( \frac{\bar{X}}{\Phi^2} \right) + \frac{1}{2} W^\alpha W_\alpha. \quad (32)
\]
\[2\]The spontaneously broken supercharges do not exist, but the supercurrents do.
Equation (28) is recovered from eq. (32) in the gauge $\Phi = 1$. Varying the action (32) with respect to $\Phi$ yields $W_\alpha = 0$ and, hence, $\omega = 0$ that, in turn, implies $F^2 = F\bar{F} = 0$. Thus we arrive at the $N = 1$ superconformal extension of the ultra-BI (conformal) bosonic theory of Bialyncki-Birula [11].

An off-shell $N = 2$ supersymmetric Maxwell field strength also exists in $N = 2$ superspace, where it is given by the $N = 2$ restricted chiral superfield $\mathcal{W}$ subject to the $N = 2$ Bianchi identities [9]

$$\bar{D}_{i j} \mathcal{W} = 0 \quad \text{and} \quad D_{i j} \mathcal{W} = \bar{D}_{i j} \bar{\mathcal{W}} , \quad i,j = 1,2.$$  \hspace{1cm} (33)

We use the notation

$$D_{i j} = D_i^\alpha D_{j\alpha} , \quad D^4 = \prod_{i,\alpha} D_i^\alpha = \frac{1}{12} D_{i j} D^{i j} ,$$  \hspace{1cm} (34)

for the products of the $N = 2$ flat superspace supercovariant derivatives, and similarly for the $N = 2$ superspace anticommuting coordinates $\theta_i^\alpha$. In the chiral basis the $N = 2$ superfield $\mathcal{W}$ takes the form

$$\mathcal{W}(x, \theta) = \phi + \theta^i \psi_i^\alpha - \frac{1}{2} (\theta \cdot \bar{\tau} \theta) \bar{D} + \frac{i}{6} (\theta \sigma^{\mu \nu} \theta) F_{\mu \nu} - i (\theta^3 \cdot \partial \bar{\psi}) + \theta^4 \Box \bar{\phi} ,$$  \hspace{1cm} (35)

where $\phi(x)$ is the complex scalar, $\psi_i^\alpha(x)$ is the $SU(2)$ doublet of fermions, and $\bar{D}(x)$ is the auxiliary $SU(2)$ triplet.

The manifestly $N = 2$ supersymmetric extension of the bosonic BI theory in the form (21) is given by [12]

$$S_{2BI} = \frac{1}{2} \int d^4 x d^4 \theta \mathcal{W}^2 + \frac{1}{4} \int d^4 x d^8 \theta \mathcal{Y}(K, \bar{K}) \bar{\mathcal{W}}^2 \mathcal{W}^2 ,$$  \hspace{1cm} (36)

where we have introduced the $N = 2$ chiral superfield extension of the bosonic $\omega$ variable in eq. (19),

$$K = D^4 \mathcal{W}^2 ,$$  \hspace{1cm} (37)

and used the same structure function $\mathcal{Y}$ of eq. (22) as in the bosonic BI case.

The manifest $N = 2$ supersymmetric extension of the BI-NLSM in eqs. (23) and (24) was first proposed in ref. [13],

$$S_{2GM} = \frac{1}{4} \int d^4 x d^4 \theta \mathcal{X}^2 + \text{h.c.} ,$$  \hspace{1cm} (38)

where the $N = 2$ chiral superfield Lagrangian $\mathcal{X}$ obeys the non-linear $N = 2$ superfield constraint [3]

$$\mathcal{X} = \frac{1}{4} \mathcal{X} D^4 \mathcal{X} + \mathcal{W}^2 .$$  \hspace{1cm} (39)

Unlike the $N = 1$ BI actions (27) and (29), the $N = 2$ actions (36) and (38) merely coincide modulo terms with the spacetime derivatives of $\mathcal{W}$, i.e. [12]

$$S_{2GM} = S_{2BI} + O(\partial \mu \mathcal{W}) .$$  \hspace{1cm} (40)
The $N = 2$ supersymmetric extension of electric-magnetic self-duality in the $N = 2$ non-linear electrodynamics characterized by some action $S(W, \bar{W})$ amounts to the relation \[ \int d^4xd^4\bar{\theta}(\bar{W}^2 + \bar{M}^2) = \int d^4xd^4\theta(W^2 + M^2), \quad \frac{i}{4}M = \frac{\delta S}{\delta \bar{W}}, \] (41) which is the natural generalization of eq. (31). As was demonstrated in ref. [10], it is the action (38) that precisely obeys eq. (41). Hence, the higher-derivative terms present in the action (38) on the top of the action (36) should be taken seriously – e.g., they contribute to the effective worldvolume (static-gauge) action of a D3-brane propagating in six dimensions. The leading terms of the action (38) up to the 8th order in $(W, \bar{W})$ were computed in ref. [14],

\[
S_{2\text{GM}} = \frac{1}{4} \left( \int d^4xd^4\bar{\theta}W^2 + \text{h.c.} \right) + \frac{i}{4} \int d^4xd^8\theta \left\{ W^2\bar{W}^2 \left[ 2 + D^4W^2 + D^4\bar{W}^2 \right] \right.
- \frac{1}{6}W^3\Box\bar{W}^3 + \frac{1}{2}W^2\bar{W}^2 \left[ (D^4W^2 + \bar{D}^4\bar{W}^2)^2 + D^4W^2\bar{D}^4\bar{W}^2 \right] \\
- \frac{1}{6}D^4W^2\bar{W}^3\Box W^3 - \frac{1}{6}\bar{D}^4\bar{W}^2W^3\Box \bar{W}^3 + \frac{1}{288}W^4\Box^2\bar{W}^4 \right\} + \ldots ,
\] (42)

while they coincide with the ones obtained from the non-linear realization of the PSSB $N = 4$ to $N = 2$ in four spacetime dimensions [14].

The $N = 2$ supersymmetric extension of the BI action can, therefore, be interpreted as the most essential (if not full) part of the Goldstone-Maxwell action associated with the PSSB $N = 4$ to $N = 2$ in four spacetime dimensions [13, 14]. The transformation laws of extra (spontaneously broken) $N = 2$ supersymmetry have the form \[ \delta X = 2\Lambda W, \quad \delta W = \Lambda \left( 1 - \frac{1}{4}\bar{D}^4\bar{X} \right) + \ldots , \] (43)

where the dots stand for the higher-derivative terms, and the $N = 2$ rigid supersymmetry parameter is given by

\[ \Lambda(\theta) = \lambda + \theta^i\lambda^i + \theta^{ij}\lambda_{ij} . \] (44)

Here $\lambda$ is the complex parameter of two broken translations, $\lambda^i_a$ are the Grassmann parameters of two broken supersymmetries, and $\lambda_{ij}$ are the parameters of the spontaneously broken R-symmetry $SU(2)$. Accordingly, the complex scalar $\phi = P + iQ$ is the Goldstone scalar associated with two spontaneously broken translations, whereas $\psi^i_{\alpha}$ are two Goldstone spinors associated with two spontaneously broken supersymmetries.

Via a complicated field redefinition, the leading bosonic part of the $N = 2$ supersymmetric BI action can be rewritten to the form [13]

\[
S_{\text{brane}} = -\int d^4x\sqrt{-\text{det}(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu P\partial_\nu P + \partial_\mu Q\partial_\nu Q)},
\] (45)

which is characteristic to a D3-brane propagating in six spacetime dimensions.
3 Non-abelian $N = 1$ supersymmetric BI action

The simple structure of the $N = 1$ supersymmetric abelian BI action in eq. (27) dictated by the Gaussian non-linear superfield constraint (28) allows us to easily construct its non-abelian (NBI) generalization \[15\] which may be relevant for the effective description of the D3-brane clusters (i.e. the ‘spacetime-filling’ D3-branes on the top of each other).

The $N = 1$ Super-Yang-Mills (SYM) theory in $N = 1$ superspace is defined by the standard off-shell constraints:

$$\{\nabla_\alpha, \nabla_\beta\} = \{\bar{\nabla}_\alpha, \bar{\nabla}_\beta\} = 0$$

$$[\nabla_\alpha, \nabla_\beta] = 2i\epsilon_{\alpha\beta}\check{W}_\beta$$

$$[\bar{\nabla}_\alpha, \nabla_\beta] = 2i\epsilon_{\alpha\beta}\bar{\check{W}}_\beta$$

in terms of the $N = 1$ covariantly-chiral (Lie algebra-valued) gauge superfield strength $\hat{W}_\alpha = \hat{W}_a^\alpha t_a$ obeying the Bianchi identities \[14\]

$$\nabla_\alpha \hat{W}_\alpha = 0$$

$$\nabla_\alpha \hat{W}_\alpha = \nabla_\alpha \hat{W}_\alpha$$

The natural $N = 1$ supersymmetric NBI action is \[13\]

$$S_{\text{NBI}} = \int d^4x d^2\theta \ Tr\ \hat{\Phi} + h.c.$$  \hspace{1cm} (48)

whose $N = 1$ covariantly chiral Lagrangian $\hat{\Phi}$ is subject to the ‘minimal’ non-abelian generalization of the abelian non-linear constraint (28),

$$\hat{\Phi} = \frac{1}{2} \check{W}^{\alpha}\nabla_\alpha \hat{\Phi} + \frac{1}{4} \hat{W}^2$$  \hspace{1cm} (49)

The leading contribution to the NBI action (48) is the standard $N = 1$ SYM action in superspace,

$$S_{\text{SYM}} = \frac{1}{2} \int d^4x d^2\theta \ Tr\ \hat{W}^{2} + h.c.$$ \hspace{1cm} (50)

The next (quartic in the SYM field strength) correction in the Yang-Mills sector (in components) is given by the non-abelian Euler-Heisenberg term \[13\],

$$\frac{1}{4} \text{tr} \left( (F^2)^2 + (F\tilde{F})^2 \right)$$  \hspace{1cm} (51)

Our NBI action does not have ordering ambiguities that are well known in the bosonic NBI theory, because an iterative solution to the NBI constraint (49) also specifies the order of the non-abelian quantities.

The $N = 2$ supersymmetric NBI action \[15\] has similar structure.

\[3\]The Lie algebra generators $t_a$ obey the relations $[t_a, t_b] = f_{abc} t_c$ and $\text{tr}(t_a t_b) = -2\delta_{ab}$.
4 Born-Infeld supergravity

It is of interest to construct possible gravitational avatars of the BI action (see, e.g., ref. [17] for the earlier discussion without supersymmetry). Requiring local supersymmetry implies more constraints on the possible BI-type gravity actions that are to be non-linear in the spacetime curvature [18].

The $N = 1$ supergravity in four dimensions is most naturally described in curved superspace $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$, $m = 0, 1, 2, 3$ and $\mu = 1, 2$, where we now have to distinguish between curved ($M$) and flat ($A$) indices related by a supervielbein $E^M_A$ and its inverse $E_M^A$ with $E = \text{Ber}(E^M_A) \neq 0$ [19]. The supervielbein $E^M_A$ and a superconnection $\Omega_A$ are most conveniently described by (super) one-forms,

$$E_A = E^M_A(z)\partial_M \quad \text{and} \quad \Omega = dz^M \Omega_M(z) = E^A \Omega_A ,$$

where $\Omega_A$ take their values in the Lorentz algebra,

$$\Omega_A = \frac{1}{2} \Omega_A^{bc}(z) M_{bc} = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \Omega_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} ,$$

and $M_{bc} \sim (M_{\beta\gamma}, M_{\dot{\beta}\dot{\gamma}})$ are the Lorentz generators. The curved superspace covariant derivatives

$$D_A = \left( D_a, D_{\alpha}, \bar{D}_{\dot{\alpha}} \right) = E_A + \Omega_A$$

obey the algebra

$$[D_A, D_B] = T_{AB}^C D_C + R_{AB} ,$$

where the supertorsion $T_{AB}^C$ and the (Lorentz algebra-valued) supercurvature $R_{AB} = \frac{1}{2} R_{AB}^{cd} M_{cd} = R_{AB}^{\beta\gamma} M_{\beta\gamma} + R_{AB}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}$ have been introduced. We assume that the latter satisfy the standard (Wess-Zumino) off-shell superfield constraints [19] defining the minimal $N = 1$ supergravity in superspace. As a result of the constraints and the Bianchi identities, all the superfield components of the supertorsion and the supercurvature appear to be merely dependent upon three (constrained) supertorsion tensors: the complex (covariantly) chiral scalar superfield $R$, the real vector superfield $G_a$ and the complex (covariantly) chiral superfield $W_{\alpha\beta\gamma}$ that is totally symmetric with respect to its spinor indices [19].

The bosonic superfield $R$ has an auxiliary complex scalar $B$ as the leading component, while it also contains the spacetime scalar curvature as another bosonic field component. Similarly, the bosonic vector superfield $G_a$ has the spacetime Ricci curvature amongst its field components. The fermionic superfield $W_{\alpha\beta\gamma}$ has the gravitino field strength as its leading component, while it also contains the spacetime Weyl tensor $C_{\alpha\beta\gamma\delta}$ (totally symmetric on its spinor indices) as the fermionic field component.
The NBI superfield NLSM constraint (49) can be considered as the powerful tool converting any fundamental (input) chiral superfield Lagrangian \( \hat{W}^2 \) into the corresponding BI-type chiral Lagrangian \( \hat{\Phi} \). The supergravitational analogue of the (covariantly chiral) \( N = 1 \) SYM spinor superfield strength \( W^a_{\alpha\beta}\gamma \) is given by the super-Weyl curvature tensor \( W_{\alpha\beta\gamma}M^{\beta\gamma} \). This essentially amounts to replacing the Yang-Mills gauge group by the Lorentz group in the \( N = 1 \) NBI action. The standard Weyl supergravity action [19]

\[
S_{W} = \int d^4x d^2\theta \text{tr} W^2 + \text{h.c.}
\]  

(56)
is now easily extended to the new Born-Infeld-Weyl (BIW) supergravity action [18]

\[
S_{\text{BIW}} = \int d^4x d^2\theta \text{tr} F + \text{h.c.},
\]  

(57)
whose covariantly chiral (Lorentz algebra-valued) Lagrangian \( F \) is a solution to the non-linear superfield constraint

\[
F = \frac{1}{2} F(\hat{D}^2 - 4\mathcal{R})\hat{F} + W^2.
\]  

(58)
We have also introduced the chiral local density

\[
\mathcal{E} = -\frac{1}{4} \mathcal{R}^{-1}(\hat{D}^2 - 4\mathcal{R})E^{-1}
\]  

(59)
in eqs. (56) and (57).

The subleading correction to the Weyl supergravity action in the BIW theory (57) is given by

\[
S_{\text{BR}} = \frac{1}{2} \int d^4x d^2\theta E^{-1} W^2_{\alpha\beta\gamma} \hat{W}^2_{\alpha\beta\gamma},
\]  

(60)
whose purely bosonic (gravitational) part is proportional to the Bel-Robinson (BR) tensor squared [20],

\[
T_{mnpq} = R_{mstp}R_{n \cdot q \cdot t} + R_{msql}R_{n \cdot p \cdot s} - \frac{1}{2} g_{mn}R_{prst}R_{q \cdot r \cdot s},
\]  

(61)
In four dimensions the BR tensor (61) can be rewritten to the form

\[
T_{mnpq} = R_{mstp}R_{n \cdot q \cdot t} + \tilde{R}_{mstp}\tilde{R}_{n \cdot q \cdot t},
\]  

(62)
where \( \tilde{R}_{mstp} \) is the dual curvature. Moreover, the four-dimensional BR tensor is also known to be totally symmetric and pairwise traceless [20]. The BR tensor squared, \( T^2_{mnpq} \), can be considered as the gravitational analogue of the gauge Euler-Heisenberg term (51).

Unfortunately, even the leading terms of the BIW action have higher derivatives. Unlike the gauge theory that is quadratic in the field strength, the Einstein gravity action is linear in the curvature. The \( N = 1 \) Einstein supergravity action

\[
S_{SG} = -\frac{3}{\kappa^2} \int d^8z E^{-1}
\]  

(63)
does not contain the supercurvature at all. Here \( \kappa \) is the gravitational coupling constant of dimension of length.

In fact, any full superspace action containing a supercurvature (even linearly) gives rise to the terms that are non-linear in the component curvature. As an example, let’s consider the simplest invariant (BI-Einstein) action

\[
S_{\text{BI-E}} = \int d^8z E^{-1}(\Lambda + \mathcal{R}) + \text{h.c.}, \tag{64}
\]

where \( \Lambda \) is a non-vanishing constant. In components, eq. (64) gives rise to the following bosonic terms:

\[
S_{\text{bos.}} = -\frac{1}{9} \int d^4x \sqrt{-g}(R + \frac{1}{3} B\bar{B})(2\Lambda + B + \bar{B}) , \tag{65}
\]

where the auxiliary complex scalar field \( B \) is the leading component of \( \mathcal{R} \). The algebraic \( B \)-equation of motion has a solution,

\[
B = \bar{B} = -\frac{1}{3} \Lambda \pm \sqrt{\frac{1}{9} \Lambda^2 - R}. \tag{66}
\]

Being inserted back into the action (65), this yields

\[
S_{\text{bos.}} = -\frac{4}{27} \int d^4x \sqrt{-g} \left\{ \frac{4}{3} \Lambda R + (\frac{1}{9} \Lambda^2 - R) \left( \frac{1}{3} \Lambda \mp \sqrt{\frac{1}{9} \Lambda^2 - R} \right) \right\} . \tag{67}
\]

This action is already of the BI type, while it also implies taming of the space-time scalar curvature from above,

\[
R \leq \left( \frac{1}{3} \Lambda \right)^2 . \tag{68}
\]

Having chosen the upper sign (minus) choice in eq. (67), we can adjust the free parameter \( \Lambda \) as

\[
\Lambda = \left( \frac{3}{2\kappa} \right)^2 , \tag{69}
\]

so that the leading term (in the curvature) in the action (67) takes the standard (Einstein-Hilbert) form, \( -\frac{1}{2\kappa^2} R \).

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