Hamiltonian theory of gaps, masses and polarization in quantum Hall states: full disclosure.

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In two short papers I had described an extension, to all length scales, of the Hamiltonian theory of composite fermions (CF) that Murthy and I had developed for the infrared, and applied it to compute finite temperature quantities for quantum Hall fractions. I furnish details of the extended theory and apply it to Jain fractions \( \nu = p/(2ps + 1) \). The explicit operator description in terms of the CF allows one to answer quantitative and qualitative issues, some of which cannot even be posed otherwise. I compute activation gaps for several potentials, exhibit their particle hole symmetry, the profiles of charge density in states with a quasiparticles or hole, (all in closed form) and compare to results from trial wavefunctions and exact diagonalization. The Hartree-Fock approximation is used since much of the nonperturbative physics is built in at tree level. I compare the gaps to experiment and comment on the rough equality of normalized masses near half and quarter filling. I compute the critical fields at which the Hall system will jump from one quantized value of polarization to another, and the polarization and relaxation rates for half filling as a function of temperature and propose a Korringa like law. After providing some plausibility arguments, I explore the possibility of describing several magnetic phenomena in dirty systems with an effective potential, by extracting a free parameter describing the potential from one data point and then using it to predict all the others from that sample. This works to the accuracy typical of this theory (10 -20 percent). I explain why the CF behaves like free particle in some magnetic experiments when it is not, what exactly the CF is made of, what one means by its dipole moment, and how the comparison of theory to experiment must be modified to fit the peculiarities of the quantized Hall problem.

I. INTRODUCTION

There is a consensus among theorists and experimentalists that the concept of the Composite Fermion (CF) is very useful in understanding the fractional quantum Hall effect (FQHE). This concept allows us, among other things, to decide which fractions are robust (the Jain series), to generate very accurate trial wavefunctions and gaps, and to determine the allowed values of polarization when the spin is not fully polarized.

The aim of this paper is to describe, in detail, a Hamiltonian formulation which provides a comprehensive way to describe FQHE states qualitatively and quantitatively, at zero and nonzero temperatures. Recall that in the theory of superconductivity or Fermi liquids one always seeks a transformation relating the original electronic variables to those of the ultimate quasiparticles, these being the Cooper pairs and Landau’s quasiparticles respectively. In the Hamiltonian formalism used here one passes form a description in terms of electrons to one in terms of CF’s through a sequence of transformations. One ends with an operator description of the CF. Even though such a change of basis is approximate, it provides us with valuable insights, and occasionally, quantitative information that is unavailable in the wavefunction approach, such as ways of coupling to impurities and calculating unequal-time correlations.

Given that the FQHE has no small parameters, how was this passage possible, even approximately? The answer has two parts. Originally Murthy and I developed the transformation using a combination of the random phase approximation (RPA) and the infrared limit, and obtained the electron density operator in the CF basis at small \( q_l \), \( q \) being the momentum and \( l \) the magnetic length. The RPA kept operators at different \( q \)’s from mixing during the transformation. Given the density operator, the hamiltonian, which is just the interaction energy when electrons are restricted to the lowest Landau level (LLL), could be written down. For potentials that were soft at large \( q_l \), we could compute objects like activation gaps. I then extended the infrared results to all \( q_l \) by appealing to certain algebraic properties known to be true for the LLL projected problem in any basis. The extension consisted of taking the small \( q_l \) series for charge and the constraints, and exponentiating them. The resulting operators obeyed the desired algebras. Given that the extensions were not unique, but just minimal and consistent, it was very satisfying that they embodied, despite their questionable pedigree, much of what was known about the internal structure of the CF, illustrating once again that the tight constraints of the FQHE problem in the LLL can actually work in our favor. Two short papers described this extension and its application to finite temperatures, \( T > 0 \). Here I provide the promised details and additional insights gained in the meantime.

The topics covered here fall into two classes.

The first has to do with matters of principle. For ex-
ample, can CF be free or nearly so? Without going into any details, we can say no, since it takes two very different masses $m_a$ and $m_p$, to describe their polarization and activation phenomena, something impossible in a free theory. Yet certain polarization phenomena at $T = 0$ seem to be very accurately fit by free particles of mass $m_p$. The hamiltonian formalism not only allows one to compute these distinct masses, it also resolves the paradox posed above. Next, CF are supposed to derive their kinetic energy from electron-electron interactions. The present formalism provides an explicit expression for not only this kinetic energy but also the CF interactions, i.e., the full CF hamiltonian. Both kinetic and interaction terms have unusual functional forms, which are determined uniquely by the theory. Those attempting to fit data to traditional forms of energy should bear this in mind. Having a concrete hamiltonian also eliminates questions such as which of $m_a$ and $m_p$ should be used at $T > 0$. A Hartree-Fock calculation on the full CF hamiltonian gives the answer.

The second set of topics consists of application aimed at showing that it is possible to compute, by analytic means and often in closed form, numerous physical quantities pertaining to FQHE states. I consider here the computation of activation and polarization masses and gaps, charge profiles of quasiparticles and quasiholes, critical fields for magnetic transitions from one quantized value of polarization to another for the gapped states, all of which are $T = 0$ quantities. I also compute the polarization $P$ and relaxation rate $1/T_1$ at $T > 0$ for the gapless fractions.

Given the importance and utility of the CF idea, and the potential for misunderstandings, I have made every effort to make my arguments accessible to as wide an audience as possible, underscoring the various assumptions that go into the calculations, emphasizing not only the formalism but the physical picture that goes with it.

In Section II, I describe how one arrives at the hamiltonian in terms of CF, starting with electrons. This discussion will be brief, given that details have already been published. It does include recent insights on the internal structure of the CF, such as what it is made of and what exactly its dipole moment means. In Section III, the final equations are analyzed to gain familiarity with their main properties. The theory is then recast in a form that makes it more amenable to the Hartree Fock approximation. Readers not interested in looking under the hood, may begin with the equations listed towards the end of Section III, which form the basis for the subsequent calculations.

In Section IV, I calculate the activation gaps $\Delta_a$ for several fully polarized fractions within the HF approximation and compare to the results of Park, Meskini and Jain, (PMJ) based on trial wavefunctions. With one exception, all calculations will be carried out using the Zhang-Das Sarma (ZDS) potential

$$v(q) = \frac{2\pi e^2}{q} e^{q\lambda}$$

where $l$ is the magnetic length and $l\lambda = \Lambda$ was originally introduced to describe sample thickness, but employed here as a free parameter. I use this potential to illustrate the method, which is instantly adaptable to any other.

This present calculation differs from earlier work based on the infrared theory in that it yields finite results even for the coulomb case, $\lambda = 0$. The numbers agree to within $10 - 20\%$ (and occasionally better) for potentials that seem to describe real systems ($\lambda \simeq 1 - 2$). Similar results are found when I compare to the exact diagonalization results of Morf et al for a similar range of the parameter $b$ that enters their potential:

$$v(q) = \frac{2\pi e^2}{q} e^{(qbl)^2} Erf\ (qlb)$$

Why bother to reproduce numbers that are already known, to a lower accuracy? The point is that the present approach is fully analytic, makes the underlying physics very transparent, and furnishes an explicit operator description of the final quasiparticles, which permits a precise formulation of many question pertaining to them that would be otherwise nebulous. It is also worth mentioning that the closed expressions for physical quantities allows them to be computed in a few seconds on a PC. The reduced precision is a price we must pay in return.

I compare the theoretical activation gaps to the experiments of Du et al and Pan et al. In comparing the theory to experiments, one needs to decide how to handle disorder and LL mixing, which are suppressed in the PMJ and Morf at al computations. There does not exist at present an analytical theory for incorporating disorder. (There does exist numerical work demonstrating the effect of disorder, see for example Ref). My approach has been to take experimental points and fit them to the theory with the ZDS potential and ask what $\lambda$ is needed. This is done solely to get a feeling for its size and also compare it to the values computed for the pure system with no LL mixing, using say, the Local Density Approximation (LDA). It is not assumed that the ZDS potential actually describes the problem at hand. It simply taken as a reasonable variant of the coulomb potential with a free parameter that can parameterize sample thickness and illustrate the hamiltonian method. When results for polarization phenomena are compared to experiment, a more ambitious approach to $\lambda$ is undertaken.

I explore the question raised by Pan et al of how the normalized effective mass of CF near half filling compares with that near quarter filling. Is the rough equality, observed experimentally, in accord with theory (in the absence of disorder)? In general the answers depend on how the fractions are reached– by varying the density, the field or a combination of both. Typically these masses lie within a factor of two of each other and there appears to be no deep reason why they should be exactly equal, a point also made in Ref.

I provide the profiles of charge density in some gapped states with one quasiparticle or quasihole and compare to the unpublished work of Park and Jain based on trial wavefunctions. I explore the $\nu = 1/2$ case, especially the dipole moment and what it means, in some detail.
Since the CF Hamiltonian naturally separates into a free part $H_0$ and an interaction $H_I$, I explore the effect of turning off $H_I$ and find it can change the answer by as much as a factor of two.

I ask how well particle-hole symmetry works, i.e., to what extent gaps at $\nu = p/(2ps+1)$ in the fully polarized case equal those at $1 - \nu$ and find it works very well. I point out that this was not a foregone conclusion since the formulae for the two cases are quite different.

Section V is devoted to spin physics at $T = 0$, an area investigated in the past and more recently by Park and Jain in the wavefunction approach. In the absence of an overwhelming Zeeman term, one has to consider CF of both spins. The polarization of the ground state will be decided by a competition between ferromagnetism and antiferromagnetism. When the energy difference (not counting the Zeeman energy) between two ground states of different polarizations equals the corresponding Zeeman energy difference, a transition will take place. The transition can be driven, for example, if the density and field are varied together at fixed filling fraction, or by tilting the sample at fixed perpendicular field and density.

The critical fields $B^c$ at which these transitions happen are calculated. The calculations reveal a feature noticed by Park and Jain using trial wavefunctions, namely that they may be fit very well to free fermions with a constant polarization gap, $\Delta_p$. How do we reconcile this with the fact that activation gap $\Delta_a$ is substantially different from $\Delta_p$, and that turning off $H_I$ makes a sizeable difference to $\Delta_a$? I will show how two-dimensionality and rotational invariance can conspire to mimic free-field behavior for these polarization phenomena. For example, in the gapless case of $\nu = 1/2$, I will show that while the CF energies $E(k_{\pm F})$ of fermions on top of the spin up/down Fermi seas are not even quadratic functions of the corresponding momenta $k_{\pm F}$, (and have substantial $k_{\pm F}^3$ pieces), the energy cost of transferring a particle from the top of one sea to the top of the other (which is what determines the polarization) takes the free-field form $(k_{\pm F}^2 - k_{\pm F}^2)/2m_p$. A similar situation exists for the gapped fractions. These arguments should caution experimentalists and theorists against misinterpreting the free-field fit.

My results for the critical fields $B^c$ are then compared to the data of Kukushkin et al. In the case of magnetic phenomena I take a different approach to $\lambda$. First $\lambda$ at one transition is obtained by fitting to the observed $B^c$. Scaling laws then determine it for the other transitions, whose $B^c$ can be predicted to within 20%. In other words, it seems to be true for magnetic transitions in the samples considered, that the disordered system can be described by an effective translationally invariant potential. I describe a limit in which this result can be justified. However, I pursue this approach for all magnetic phenomena, even though at present I cannot provide similar arguments for all of them. I do so because it works to within the accuracy typical of this theory, a feature that needs to be understood.

Section VI considers magnetic phenomena of gapless states as a function of temperature $T$. The $T > 0$ physics is the first instance the present methods outperform complementary approaches based on exact diagonalization (limited to small systems) or trial wavefunctions (limited the ground state and very low excitations.) It has no finite size effects since one works in the thermodynamic limit all along. One need not agonize over whether $m_o$ or $m_p$ should be used in computing a $T > 0$ quantity such as polarizations: given a concrete Hamiltonian, a Hartree-Fock (HF) calculation gives the results. The HF works well because most of the right, nonperturbative physics is already built into the Hamiltonian.

A Hartree-Fock calculation gives the polarization $P$ and relaxation rate $1/T_1$ as a function of $T$ and the potential. These numbers are then compared to experiment, again by assuming that the real system can be described by an effective potential, fitting $\lambda$ at any one data point from each sample and explaining the rest of the data from that sample at other fields, tilts and temperatures. Into these calculations go the noncanonical, nonconstant, density of states peculiar to this Hamiltonian. These results are compared to experiments of Dementyev et al. who measured $P$ and $1/T_1$ at zero and a 38.3 ° tilt for a range of temperatures. They had pointed out that attempts to fit all four graphs with a single hamiltonian (with a mass $m$ and Stoner coupling $J$) led to four disjoint or disjoint set of values. On the other hand if the hamiltonian theory, with its peculiar functional form for $H$ is used, a good fit to all four graphs is possible with a single $\lambda = 1.75$.

On comparing to the polarization data of Melinte et al I find the predictions work for the untilted case but not at a tilt of 61 °. The reasons for this are discussed.

I provide an approximate expression for $1/T_1$ as a function of temperature. It has the Korringa form only in the critical case where the polarization saturates exactly at $T = 0$.

Conclusions follow in Section VII and end with a discussion of a procedure for comparison of theory to experiment that is tailor made for the Hall problem. Many details are relegated to the Appendix, which ends with a summary of symbols for which there does yet exist a uniform convention.

II. THE HAMILTONIAN FORMALISM

Let us begin by tracing the path from the Hamiltonian in terms of electronic coordinates to that in terms of CF, focusing on the spin-polarized case for fractions of the form

$$\nu = \frac{p}{2ps + 1}. \quad (3)$$

The results extend easily to $\nu = \frac{p}{2ps + 1}$. The treatment of old published steps, presented here for completeness, will be schematic.

Consider electrons of band mass $m$ and number density $n$, described by the following first quantized hamiltonian:

$$H_{el} = \sum_i \left( \frac{p_i + eA}{2m} \right)^2 + V \quad (4)$$
\begin{align*}
\eta &= \sum_i \frac{(\Pi_i)^2}{2m} + V \\
\eta &= \sum_i \frac{\eta_i^2}{2ml^2} + V \\
\eta &= \frac{1}{2} \mathbf{r} + l^2 \hat{z} \times \mathbf{p} = l^2 \hat{z} \times \Pi \\
l^2 &= \frac{1}{eB} \\
\nabla \times \mathbf{A} &= -eB
\end{align*}

where \( h = c = 1 \), \( \hat{z} \) the unit vector along the z-axis, \( l \) is the magnetic length, \( B \) is the applied field, \( V \) is the inter-electron potential, and \( \eta \) is the cyclotron coordinate, whose components are canonically conjugate:

\[ [\eta_x, \eta_y] = il^2. \]  

Thus the spectrum is given by Landau Levels (LL):

\[ E = \omega_0(n + 1/2) \]

\[ \omega_0 = eB/m. \]

In the lowest Landau level (LLL),

\[ \langle \eta \rangle_{LLL} = l. \]

There is a huge degeneracy of each LL due to the fact that the guiding center coordinate

\[ \mathbf{R} = \frac{1}{2} \mathbf{r} - l^2 \hat{z} \times \mathbf{p} \]

whose components obey

\[ [R_x, R_y] = -il^2 \]

does not enter \( H \). The conjugate pair \( (R_x, R_y) \) ranges over the entire sample, whose area is its phase space, and determines the degeneracy if one employs Bohr-Sommerfeld quantization with \( l^2 \) playing the role of \( \hbar \).

At the Jain fractions, the inverse filling fraction

\[ \nu^{-1} = \frac{eB}{2\pi n} = 2s + \frac{1}{p} \]

\[ = \text{flux quanta per electron} \]

\[ = \text{states in the LL per electron.} \]

If \( \nu \leq 1 \), there is enough room in the LLL to fit all the electrons in the noninteracting case. One expects that if the cyclotron energy \( \omega_0 \) is much larger than the interelectron potential, the ground state and low lying excitations will be formed out of states in the LLL.

Since

\[ \mathbf{r} = \mathbf{R} + \eta \]

a natural projection to the LLL is

\[ \mathcal{P} \Rightarrow \mathbf{r} \to \mathbf{R}. \]

After this projection the two commuting coordinates become canonically conjugate.

Given the huge degeneracy of the LLL for \( \nu < 1 \), the problem is the selection of a unique ground state. Laughlin blazed one trail, writing down inspired trial wavefunctions for \( \nu = 1/(2s+1) \). The other route is to try to start with the electronic hamiltonian and try to reach, through a sequence of approximations, the final quasiparticles, which in this work, will be the composite fermions.

For Laughlin fractions, where the wavefunction is extraordinarily compact and simple, one has the option of working with Composite Bosons (CB)

\[ \nu = 1/(2s+1), \]

which have considerable appeal of their own.

### A. What is a Composite Fermion?

What exactly is a CF composed of? I am grateful to G. Murthy for some very useful discussions of this issue. Laughlin showed (using arguments involving adiabatic introduction of a flux quantum) that at \( \nu = 1/(2s+1) \) the elementary excitations have a charge

\[ e^* = \frac{1}{2s+1}. \]

Consider the following state

\[ \psi_{\text{vortex}} = \prod_j (z - z_0) \prod_{i<j} (z_i - z_j)^{2s+1}. \]

The prefactor \( \prod_j (z - z_0) \), multiplying Laughlin’s ground state wave function (whose gaussian factor has been suppressed) is a vortex. Due to the zero at \( z_0 \), there is a charge deficit near that point, whose value, in electronic units, can be shown to be \(-1/(2s+1)\).

In CF theory the quasiparticle is believed to be an electron bound to 2s vortices. We shall see that while this is clearly so for the Laughlin series, the situation for the Jain series is more complex.

If we look at the Laughlin wavefunction we see a 2s+1-fold zero at each electron: one is due to the Pauli principle and the other 2s represent the zeros due to the captured vortices. There is no question of which vortex belongs to which electron since the vortices are sitting on the electrons. The charge of the electron plus 2s−fold vortex, i.e., CF charge is given by

\[ 1 - \frac{2s}{2s+1} = \frac{1}{2s+1} = e^*. \]

Sometimes the vortex is incorrectly used interchangeably with a flux tube,

\[ \text{flux tube} = \prod_j (z - z_0) \prod_{i<j} |z_i - z_0| \]

which has the phase of the vortex but not the zero. In other words the CF is described as an electron bound to 2s flux tubes. It was emphasized very early on by Halperin\cite{halperin1984} that for Laughlin fractions, electrons like to bind to vortices due to the coulomb attraction. This was also at the heart of Read’s work\cite{read1989}, which extended the
concept to \( \nu = 1/2 \) where the wavefunction is obtained by projection to the LLL (more on this shortly) which in turn causes the vortices to move off the electrons. In any event, electrons are not attracted to flux tubes, which are neither charged nor low energy excitations.

Consider the Jain wavefunctions at \( \nu = p/(2ps + 1) \):

\[
\Psi_{\text{Jain}} = \mathcal{P} \prod_{i < j} (z_i - z_j)^{2s} \chi_p(z, \bar{z}).
\]

The factor \( \chi_p(z, \bar{z}) \) describes \( p \)-filled CF Landau levels and the Jastrow factor \( J(2s) = \prod_{i < j} (z_i - z_j)^{2s} \) describes vortices sitting at the locations of the particles and \( \mathcal{P} \), the LLL projector, replaces the \( \bar{z} \)'s as per Eqs. (25):

\[
\mathcal{P} : \bar{z} \rightarrow 2l^2 \frac{\partial}{\partial z}.
\]

Let us first ignore \( \mathcal{P} \). Then there are indeed \( 2s \) zeros per particle in the Jastrow factor (located on the electrons) and one (not necessarily analytic) zero per particle in \( \chi_p(z, \bar{z}) \). (By zeros of the wavefunction, I always mean as a function of one coordinate, all others being held fixed.) The \( 2s \)-fold vortex has a charge

\[
e_v = -\frac{2ps}{(2ps + 1)}
\]

in electronic units, a result that can be deduced from just the Hall conductance and incompressibility of the state. If we add the vortex charge to that of the electron, we do indeed get

\[
e^* = \frac{1}{2ps + 1}.
\]

which according to Su, is the correct, quasiparticle charge at all gapped fractions. This is also confirmed by focusing experiments. So we may say on the basis of this unprojected wavefunction that the CF is the union of an electron and a \( 2s \)-fold vortex. Since the vortices sit right on the electrons, there is no confusion on who they are bound to and all moments vanish except the total charge.

A lot of this changes upon projection by \( \mathcal{P} \): \( \mathcal{P} \) acting on the Jastrow factors, moves the zeros away from the particles and many of them vanish, leaving \( 2s + 1/p \) per electron, (determined by the number of flux quanta per electron). Thus, after projection, vortices cannot be associated with the electrons in an unambiguous way. For example at 2/5, there are 2.5 zeros per electron. One sits on the electron due to the Pauli principle, leaving 1.5 non-Pauli zeros per electron, which will neither lie on the other electrons, nor be numerous enough to form two vortices per electron. Presumably, in states involving projection, where the wavefunction has a very complicated form due to the action of \( \mathcal{P} \), there is some nontrivial sharing of vortices between electrons. In the limiting case of \( \nu = 1/2 \) only one non-Pauli zero per electron will remain after projection. We shall return to \( \nu = 1/2 \) later.

These remarks do not imply that the CF approach to writing down trial wavefunctions based on electrons binding to vortices is in jeopardy. Thinking in terms of vortices still gives the unprojected wavefunction. This is all one needs, since the act of projection, while complicated, has a definite algorithm that is routinely carried out, and yields wavefunctions with their incredible overlaps with exact results. My message is only that in the end, if one looks at the projected wavefunction, (which is going to very complicated) one is not likely to find any simple correlation between electrons to vortices. These remarks apply to any projected wavefunction \( \Psi_{\text{LLL}} \).

It is quite remarkable that even though we cannot assign to each electron a \( 2s \)-fold vortex in \( \Psi_{\text{Jain}} \), \( e^* \) is still given by adding the electron’s charge to that of a charge \( e_v = -2ps/(2ps + 1) \) object. In other words, the charge we associate with the CF, being linked to the Hall conductance and incompressibility, is unaffected by the projection, although the notion of each electron having \( 2s \) vortices sitting on (or even near) it is no longer true. What then is this object that pairs with the electron and how is one to describe it theoretically?

The hamiltonian theory described here provides an answer. In this theory we enlarge the Hilbert space to include additional degrees of freedom, accompanied by an equal number of constraints. These new degrees of freedom (prevented from having any density fluctuations by constraints) will turn out to have charge \( e_v \) and pair with electrons and charge \( e \) to \( e^* \). I shall refer to them as vortices, for want of a better name, but in view of what was said above, they are not related in any simple way to zeros of \( \Psi_{\text{LLL}} \).

We now review the hamiltonian description, which, not surprisingly, is relies heavily on earlier work.

- Lopez and Fradkin took the first major step and attached \( 2s \) flux tubes by the singular gauge transformation of the wavefunction (due to Leinaas and Myrheim), from electrons to Chern-Simons fermions:

\[
\Psi_e = \prod_{i < j} \frac{(z_i - z_j)^{2s}}{|z_i - z_j|^{2s}} \Psi_{\text{CS}}
\]

and opened up the field theoretical description of the Jain states. This was applied to the gapless case \( \nu = 1/2 \) by Kalmeyer and Zhang, by Marston et al who used bosonization, and in a very exhaustive treatment by Halperin, Lee and Read (HLR).

- Murthy and I introduced collective coordinates, to describe long wavelength density fluctuations, as did Bohm and Pisarski in their treatment of plasmons. For every extra degree of freedom so introduced, there was a constraint on physical states, to
keep the problem same as before. The collective coordinates corresponded to oscillators at the cyclotron scale. Putting them in their ground states and projecting to the physical sector using constraints, was seen to produce the zeros that turned flux tubes into vortices, i.e., produced the Jastrow factors.

- To expose the low energy physics, Murthy and I introduced an additional unitary transformation that decoupled the oscillators and the fermions. This was however done approximately:

1: We worked at long distances. Thus if any quantity had an expansion in powers of $ql$, we kept just the leading term.
2. When the density operator was encountered in a product with other operators, we used the RPA:

$$\sum_j e^{i(q-k) \cdot r_j} \simeq n(2\pi)^2 \delta^2(q-k).$$

We made the first approximation so that we could introduce a small parameter $ql$ where there was none. The second ensured that the operators at small $q$ like $\rho(q)$, did not mix with those at high $q$ in the unitary transformation. These were the minimal assumptions we had to make before we could carry out the decoupling transformation. Despite these approximations a reasonable quantitative and qualitative description emerged.

The long-distance, low-energy theory Murthy and I derived was given by the following set of equations:

$$H = V = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \tilde{\rho}(q) \nu(q) \tilde{\rho}(-q)$$

$$\tilde{\rho}(q) = \sum_j e^{-iqr_j} \left[ 1 - \frac{il^2}{1+c} q \times \Pi^* j + \cdots \right]$$

$$\tilde{\chi}(q) = \sum_j e^{-iqr_j} \left[ 1 + \frac{il^2}{c(1+c)} q \times \Pi^* j + \cdots \right]$$

$$0 = \tilde{\chi}(q) |\text{Physical State}\rangle$$

$$\Pi^* = p + eA^*$$

$$A^* = \frac{A}{2ps + 1}$$

$$c^2 = \frac{2ps}{2ps + 1} = 2\nu/s$$

Note that physical states are to be annihilated by the constraints $\tilde{\chi}$. The magnetic moment of $e/2m$ on each particle, predicted by Simon, Stern and Halperin [1], that arises naturally here is not shown, and neither is the contributions to $H$ or the charge from the oscillators, which are frozen in their ground state. The kinetic energy of the fermions is quenched in the small $q$ sector if the number of oscillators equals the number of particles, i.e., $Q$, the largest oscillator momentum obeys $Q = \sqrt{4\pi n}$.

With this choice of $Q$, $H$ reduces to the electrostatic interaction between electrons written in the new basis.

While this formalism is good only for small $ql$, it can still be useful. For the Zhang-Das Sarma (ZDS) potential with $\lambda > 1$ we were able to calculate gaps that agreed well with the results of Park and Jain. In a collaboration with Park and Jain we also established some scaling relations between fractions at the same $p$ (number of filled CF Landau levels) but different $2s$ (number of vortices attached) that seemed to work very well.

- While the small $ql$ theory had its share of numerical successes, it had some disturbing conceptual problems. For example the constraints did not close to form an algebra and did not commute with the charge or the Hamiltonian built out of it, except to leading order in $q$. This implied that charge was not gauge invariant and the physical sector not defined: how could constraints at two different $q$’s annihilate the physical states but not their commutator? While gauge invariance can be implemented order by order in $q$, this is not so with respect to $q$ which is integrated over. (One can use $Q$ as a small parameter but some of the central physics gets modified.)

These problems were resolved in my minimal extension of these results to all $ql$ that is mathematically and physically attractive. Let us assume that Eqns. (32-39) represents the beginnings of two exponential series and adopt the following expressions for charge, constraint, and Hamiltonian:

$$\tilde{\rho}(q) = \sum_j \exp(-iq \cdot (r_j - \frac{l^2}{1+c} \mathbf{z} \times \Pi^* j))$$

$$\tilde{\chi}(q) = \sum_j \exp(-iq \cdot (r_j + \frac{l^2}{c(1+c)} \mathbf{z} \times \Pi^* j))$$

$$H = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \tilde{\rho}(q) \nu(q)e^{-(ql)^2/2} \tilde{\rho}(-q)$$

Note that $R_c$ and $R_v$ were fully determined by the two terms we did derive. The gaussian in Eqn. (42) will be explained shortly.

This is my final answer.

III. ANALYSIS OF THE CF HAMILTONIAN

Readers who either skipped the derivation or were troubled by the approximations, are invited to take Eqns. (32-39) as an effective field theory of CF for which the
author can provide a plausible lineage going back to the electronic hamiltonian.

To understand what these equations imply, let us begin with the coordinate appearing in the exponential in the expression for $\tilde{\rho}(q)$:

$$\mathbf{R}_e = \mathbf{r} - \frac{l^2}{(1+c)} \mathbf{z} \times \Pi^*.$$  \hspace{1cm} (43)

Its components obey

$$[R_{ex}, R_{ey}] = -il^2,$$  \hspace{1cm} (44)

the commutation rules of the guiding center of a unit charge object. This, together with the fact that it enters the electronic hamiltonian, tells us it describes, in the CF basis, the guiding center coordinates of the electron.

Next consider the coordinates appearing in $\tilde{\chi}(q)$

$$\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1+c)} \mathbf{z} \times \Pi^*$$  \hspace{1cm} (45)

$$[R_{vx}, R_{vy}] = il^2/c^2$$  \hspace{1cm} (46)

These describe, in the CF basis, the guiding center coordinates of a particle whose charge is $-c^2 = e_v = -(2ps)/(2gs + 1)$. It is exactly charge of the 2s-fold vortex. It will be seen to pair with the electron and reduce the charge down to $e^*$. For these reasons we shall refer to it as the vortex, although the nomenclature is not ideal. For one thing, we know that this object does not correspond to the 2s-fold zero of the LLL wavefunction, which does not generally have 2s non-Pauli zeros per electron. In addition, the assignment of physical meaning to objects that appear in an enlarged Hilbert space is at best schematic. Recall the oscillators, which when put in the ground state and projected to the physical sector, produced the Jastrow factor with its vortices. We could, for this reason, call them vortices. But we must not forget that prior to projection, neither the oscillators, nor their wavefunction had any meaning in electronic language, and that conversely, in the electronic Hilbert space, there were no independent degrees of freedom corresponding to vortices, which are really made up of electrons. The whole idea of going to an enlarged space, as in the case of Bohm and Pines, is to be able to handle, in intermediate stages, collective variables as canonical coordinates independent of electrons.

Finally

$$[\mathbf{R}_e, \mathbf{R}_v] = 0.$$  \hspace{1cm} (47)

Thus the four dimensional fermionic phase space has yielded two independent sets of canonical coordinates, $\mathbf{R}_e$ and $\mathbf{R}_v$. Now this is twice as many coordinates per particle as in the **electronic LLL problem**. But the constraints Eqn. (13) tell us the density formed out of $\mathbf{R}_e$ has no fluctuations, so that the number of independent coordinates matches the LLL. This is reminiscent of Bohm-Pines theory[14], where, once plasmons are introduced at small $q$’s, the fermions are not allowed collective density oscillations at these $q$’s.

Consider Eqn. (13 and 15). They show that a CF at $\mathbf{r}$ with kinetic momentum $\Pi^*$, is flanked by the electron and "vortex" within a distance of order $l^2\Pi^*$. (See Figure 1). Its total charge is their sum $e^* = 1/(2ps + 1)$. Its dipole moment (in the frame $\mathbf{r} = 0$) is $d^* = -l^2 \mathbf{z} \times \Pi^*$. Its size $l^2\Pi^* \approx l$ near the Fermi surface, making it a well defined object in this energy range. The vortices are bound to the electrons, since as their separation (proportional to $\Pi^*$) grows, so does the energy since $H$ will be seen to have terms that grow with $\Pi^*$. Thus we have managed to reach one of the goals of any theory of the FQHE: explain how CF get their kinetic energy from the electrostatic energy of the electrons. Indeed the entire CF hamiltonian is just the electrostatic interaction of electrons written in the CF basis. While these features are interesting, they are still heuristic. First, the discussions involving operators are semiclassical. Next, $\Pi^*$ is not a constant of motion except at $\nu = 1/2$ when it equals $p$. Finally the first quantized expressions do not include the effects antisymmetrization. However, as we proceed, we will find an operator expression of these ideas that is robust and survives in a second quantized theory fermions.

Given the commutation relations of $\mathbf{R}_e$ and $\mathbf{R}_v$, we easily deduce those of $\tilde{\rho}$ and $\tilde{\chi}$:

$$[\tilde{\rho}(q), \tilde{\rho}(q')] = 2i \sin \left[ \frac{(q \times q')^2}{2} \right] \tilde{\rho}(q + q')$$  \hspace{1cm} (48)

$$[\tilde{\chi}(q), \tilde{\chi}(q')] = -2i \sin \left[ \frac{(q \times q')^2}{2c^2} \right] \tilde{\chi}(q + q')$$  \hspace{1cm} (49)

$$[\tilde{\chi}(q), \tilde{\rho}(q')] = 0.$$  \hspace{1cm} (50)

One sees that $\tilde{\rho}$ is not only algebraically closed, but obeys the Girvin-MacDonald-Platzman GMP algebra[23] for magnetic translations. I will keep referring to it as the projected charge density, from which it differs by a

![Figure 1. Anatomy of the CF: note that a CF at a point $\mathbf{r}$ is flanked by the vortex and electron. They are bound by terms in $H$ that grow with $\Pi^*$.](image-url)
factor $e^{-(q\ell)^2/4}$. In the hamiltonian Eqn. (43), which is just the electrostatic interaction written in the CF basis, the factor $e^{-(q\ell)^2/2}$ takes this difference into account.

Note that $\bar{\rho}$ constitutes a specific realization or representation of the GMP algebra in terms of the final, canonical, CF variables, a feature that allows one to apply standard many-body methods to $H$.

Finally the constraint commutes with the projected charge and hence the low energy hamiltonian which is equivalent to ours for small $q\ell$. We should have these coordinates, could be derived in another theory, guided by the LLL algebras.

How does the small $q$ - RPA treatment manage to generate these coordinates $R_\nu$ and $R_\nu$ with their nice commutation relations? This is my current understanding. The first two terms in $\bar{\rho}$ and $\bar{\chi}$, which fully determine these coordinates, could be derived in another theory, equivalent to ours for small $q\ell$ but not all $q\ell$: the theory explored by Stern et al.[4]. In this theory $Q$, the upper cut-off for the oscillators is, assumed to be vanishingly small. This means that we can safely assume that every $q$ in the problem, including those integrated over, are small, being bounded by $Q$. Next, RPA becomes exact in this case since non RPA terms involve multiple $q$ integrals which are suppressed by higher powers of $Q$ in the unitary transformation. In such a derivation one gets exactly the same first two terms. These encode the charge and dipole moment of the CF, which characterize the CF in the infrared. While my extension to all $q$ is mathematically satisfactory, it need not be numerically exact down to arbitrarily small length scales. We should have been prepared for this since one cannot have at the same time a scheme that is analytically tractable and numerically exact, unless we are dealing with exact solutions. We managed to make the nonperturbative passage from electrons to CF by the exponentiation, which in turn was guided by the LLL algebras.

Recall that all proofs of nonzero static compressibility at $\nu = 1/2$ relied on a careful implementation of the constraints or gauge invariance[22-24]. We may now understand this as a follows: $\bar{\chi} = 0$ means that only the electrons in the CF respond to the static potential, exhibiting nonzero static compressibility of unit charge objects.

A. How to solve $H$?

Now we ask how we are to handle Eqns. (38-42). As shown in Appendix 2, $p$-filled LL’s and particle and hole excitations on top of it, are HF states of our $H$. (The proof relies on the rotational invariance of the hamiltonian.) One possibility is to ignore the constraints altogether and proceed with the HF approximation. This will however lead to the following fatal flaw: transition matrix elements of $\bar{\rho}$ will start out as $q$ and the structure factor $S(q)$ will go as $q^2$, in violation of Kohn’s theorem. (The the $q^2$ sum rule is saturated by the oscillators that were decoupled.)

We must therefore bring in the constraints and hope they will bail us out. A standard way to incorporate first class constraints is to introduce them into the path integral with a Lagrange multiplier and try to solve the theory in an approximation that respects the gauge symmetry i.e., the constraints. We will discuss it shortly, but in the present case I will use a solution that is essentially what Murthy and I used in the small $q$ theory[3]: replace $\bar{\rho}$ by the preferred combination for charge density

$$\bar{\rho'} = \bar{\rho} - c^2 f \bar{\chi}$$

where

$$f = e^{-q^2 \ell^2 / 8ps} \text{ Vortex Form Factor}$$

This combination is equivalent to $\bar{\rho}$ in the physical sector. The factor $e^{-q^2 \ell^2 / 8ps}$ (absent in the earlier work and unimportant for any fraction other than $1/3$) takes into account the fact that since the vortex and the electron have different magnetic lengths, to convert the vortex magnetic translation operator to the magnetic number density we need a different gaussian from the one we absorbed into $H$ in Eqn. (42). Note that $\bar{\rho'}$ is weakly gauge invariant:

$$[\bar{\chi}, \bar{\rho'}] \simeq \bar{\chi}.$$ (53)

Clearly so is the $H(\bar{\rho'})$ that I shall use. Weak gauge invariance is enough to keep physical and unphysical states from mixing.

Consider the series expansion of $\bar{\rho'}$:

$$\bar{\rho'} = \sum_j e^{-iq \cdot r_j} \left( \frac{1}{2ps + 1} - i \frac{q \times \Pi_j + 0 \cdot (q \times \Pi_j)^2}{2ps + 1} \right)$$

(54)

If we expand $e^{-iq \cdot r_j}$ to first order in $q$ we can verify that the term linear in $q$ contains only the guiding center coordinate of the CF ($r = \ell^2 z \times \Pi$) with no admixture of the cyclotron coordinate. Thus it does not contribute to the order $q$ transition matrix element. This is the unique multiple of the constraint we can add to $\bar{\rho}$, with this property.

With the constraint implemented this way, we are in compliance with Kohn’s theorem. But there is more. Consider the series Eqn. (54). The first term, proportional to CF density has the coefficient $e^*$. The next term has the dipole moment given by Read using wavefunction arguments. (This was done for $\nu = 1/2$ and is expected for the whole series, as suggested by Figure 1.) The vanishing of the third order term explains the success of the small $q$ theory.

It can also shown that if only terms linear in $q$ are kept in $\bar{\rho'}$, the algebra closes with

$$\sin(q \times q' \ell^2 / 2) \rightarrow q \times q' \ell^2 / 2$$

(55)

in the structure constant of the GMP algebra Eqn. (43). The significance of this is not known.

It is remarkable that a single guiding principle, Kohn’s theorem, leads to a combination with all these properties. Since the internal structure of the CF is built in at tree
level, we expect that vertex corrections (due to the constraints) must vanish as q → 0. We shall not refer to the constraint any further.

Recall the cautionary note about ascribing meaning to objects in an enlarged Hilbert space. Why should we give any significance to the terms in Eqn. (54)? After all, we could have added any multiple of the constraint χ to the physical sector. Since this led to a unique preferred combination (at small q), we assume that any corrections due to constraints will affect only the higher order moments.

In the operator approach there is no problem with how to assign this or that vortex to an electron. All one claims is that the density operator that obeys Kohn’s theorem in our HF calculation couples to any external potential with a charge e∗ and a dipole moment d∗ = l2 q × Π∗.

There are no problems with antisymmetrization among particles: we simply express this first quantized density operator in second quantized form with Fermi operators.

Let us return to the standard way of incorporating first class constraints in an approximation that respects the gauge symmetry i.e., the constraints. While this has not been done for general fractions, it has been done by Read 47 for bosons at ν = 1, which turn into fermions in zero field upon single vortex attachment a la Pasquier and Haldane. 48 (The lessons learnt from this exercise are directly applicable to us since the only difference is in the coupling constant of the gauge and matter fields.) Implementing the constraint in a conserving approximation leads to a propagating gauge field whose longitudinal part aL screens the charge fully, leaving behind dipoles of moment d∗ which then interact via the transverse gauge field aT. The propagator of aT, just as in HLR, favors the region iω ≈ q3. (The field aT, also produces mass divergences at the Fermi energy as well as HLR and restores compressibility.) However, away from the ultra-low frequency region, the answer is given by the correlation function of independent objects of charge e∗ = 0 and dipole moment d∗ = l2 q × p.

For gapped fractions (not too close to ν = 1/2) and/or at T > 0, a description in terms of independent particles with the right e∗ and d∗ is likewise expected to be a good approximation, likely for all ω, since the gap and/or T will cut-off the low frequency end where aT raises its head, and the major effects of aL are already encoded in e∗ and d∗. Rather than reach this description using the conserving approximation (which is very difficult away from ν = 1/2 due to LL structure), I use a scheme Murthy and I devised for small qL. (Away from ν = 1/2, aT could affect the statistics of the quasiparticles.)

For the benefit of the readers who just joined in, I display the equations will be used in the subsequent calculations:

\[ H^p = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \tilde{\rho}^p(q) \varphi(q) e^{-\left(iq\cdot r - \frac{q^2}{2} \cdot 2 \times \Pi^* \right)^2} \]

\[ \tilde{\rho}^p(q) = \tilde{\rho}(q) - c^2 e^{-q^2 l^2 / 8ps} \chi(q) \]

\[ \tilde{\rho}(q) = \sum_j \exp(-iq \cdot r_j - \frac{l^2}{1 + c} \cdot 2 \times \Pi^* \right)) \]

\[ \tilde{\chi}(q) = \sum_j \exp(-iq \cdot r_j + \frac{l^2}{c(1 + c)} \cdot 2 \times \Pi^* \right)) \]

\[ [H^p, \tilde{\chi}] \simeq \tilde{\chi} \]

\[ H^p = H_0 + H_f \]

Many leading long wavelength effects of the constraints have been built into \( H^p \); they will be ignored in the subsequent HF calculations, as will be the superscript on \( H^p \) since we shall always use this expression in terms of the preferred charge \( H^p \).

We separate \( H \) into free and interacting parts

\[ H_0 = \sum_{i} \int \frac{d^2q}{4\pi^2} \sin^2 \left( \frac{q \times k_i l^2}{2} \right) \tilde{\phi}(q) e^{-q^2 l^2} \]

If we expand the sin in a series and keep the lowest term, we get an expression quadratic in momentum from which we can define an effective mass 1/m∗. Thus we have managed to generate the CF kinetic energy, in operator form, in terms of the electron-electron interaction. But we have more. First, there are more powers of momentum in the kinetic energy. (We shall however see that only quartic term is important in HF.) Next, there is also \( H_f \), which can modify all the numbers. Thus 1/m∗ depends on the momentum and will be defined at the Fermi surface. For ν = 1 bosons, Haldane and Pasquier obtained the same \( H_0 \) by algebraic methods aimed at a direct LLL formalism. In the present case we have \( H \) for the entire Jain sequence. Away from \( \frac{1}{2} \) or \( \frac{1}{5} \), \( H \) is even more complicated, but of a very definite, known, functional form.

IV. THE ACTIVATION GAPS OF FULLY POLARIZED STATES

Here we use the Hamiltonian theory to compute activation gaps for fractions ν = p/(2ps + 1) in a field so strong that the system is fully polarized. We will probe the theory in the following ways:

- We will compare the gaps to those obtained by Park, Meskini and Jain (PMJ) using trial wavefunctions for the Zhang-Das Sarma (ZDS) potential
\[ v(q) = \frac{2\pi e^2}{q} e^{-q\lambda} \]  

(64)

for \( p = 1, 2, 3 \) and \( 4 \) and \( s = 1 \). These serve as a benchmark, at least for fractions not too close to \( \nu = 1/2 \).

- For the benefit of other users, a fit to the gaps in the experimentally significant region \( 1 \leq \lambda \leq 2 \) will be given for both \( s = 1 \) and \( s = 2 \). The gaps are also expressed in terms of an effective activation mass \( m_a \).
- We will compare the theory to the experiments of Du et al. and Pan et al.
- We will examine the charge density profiles in states with a quasiparticle or quasihole.
- The gaps are computed for a gaussian potential and compared to PMJ for \( p = 1, 2, 3, 4 \) and \( s = 1 \).
- The effect of turning off the interaction \( H_I \) in \( H = H_0 + H_I \), will be explored.
- We will compare the results to the exact diagonalization results of Morf et al.
- It will be seen how well particle-hole (PH) symmetry works. For example, is the activation gap for 2/5 same as that for 3/5 in the fully polarized case?

**A. Comparison of gaps in the HF approximation to PMJ**

We use the HF approximation. As shown in Appendix 2, \( p \)-filled CF LL and particle-hole excitations thereof are HF states of our \( H \). The HF ground state is given by CF filling \( p \) LL’s, which we will denote by \( |p\rangle \). I will use a boldface symbol such as \( p \) to label a Slater determinant with \( p \) occupied Landau Levels. Nonboldface symbols will label single particle states. Note also that the actual LL index \( n \) for the state labeled by \( p \) is \( n = p - 1 \) since the LLL has index \( n = 0 \).

The gap is defined by

\[ \Delta = \langle p + PH|H|p + PH \rangle - \langle p|H|p \rangle \]  

(65)

where \( PH \) stands for a widely separated particle-hole pair. Now that this is exactly how gaps are computed in PMJ’s wavefunction approach. There is however one big difference hidden in the notations. There too the hamiltonian is just the interaction, but *written in the electronic basis* (with \( \rho(q) = \sum \exp(\mathbf{q} \cdot \mathbf{r}_i) \)) while the states (which carry the same label) are these same wavefunction, multiplied by the Jastrow factor and then projected to the LLL. Projection leads to a very complicated expression for the wavefunctions. In the present approach we have tried to incorporate these effects by going in the reverse direction, from electrons to CF’s, and obtaining complicated expressions for the charge and other operators, but with simple expressions for the wavefunctions. While these operator expressions are unusual in form, they are still simple enough in appearance and amenable to exact analytical treatment, because of the approximations that went into the derivation. Thus does not expect the present results to match those of the Jain approach in their accuracy. This is indeed the case, unless the potential is fairly soft (\( A \) is larger than, say unity).

The aim of the present approach is to provide a 10–20% theory for soft potentials (which do seem relevant to experiment) in which the calculations can be performed analytically and the physics of the quasiparticles is transparent.

Rather than work with a widely separated particle-hole (PH) pair, I first find the energy in a state with just the particle and add to it the energy of a state with just the hole and subtract double the ground state energy. While the details are relegated to Appendix 4, here is the central idea.

One begins with the second quantized expression for the preferred charge operator \( \bar{\rho}(\mathbf{q}) \):

\[ \bar{\rho}(\mathbf{q}) = \sum_{m_z} d_{m_z}^d d_{m_z}^\dagger \rho_{m_z,m_z} \]  

(66)

where \( d_{m_z}^\dagger \) creates a particle in the state \( |m_z n_2 \rangle \) where \( m \) is the angular momentum and \( n \) is the LL index of CF in the weakened field \( A^* = A/(2ps + 1) \) with a magnetic length

\[ l^* = l\sqrt{2ps + 1}. \]  

(67)

The key ingredient in the HF calculation is the matrix element \( \rho_{m_z,m_z} \). It is shown in Appendix 1 that this matrix element factorizes:

\[ \rho_{m_z,m_z} = \rho_{m_z,m_z}^m \otimes \rho_{n_2,n_2}^0 \]  

(68)

The gaps depend only on \( \rho_{n_2,n_2}^0 \), the superscript on which will be generally dropped.

As shown in Appendix 4,

\[ \Delta_a(p, s, \lambda) = \int \frac{d^d q}{4\pi^2} \exp(-x/(2ps + 1))v(q)G(p, x, s) \]  

(69)

\[ x = \frac{g^2}{2} = \frac{g^2l^2(2ps + 1)}{2} \]  

(70)

\[ G(p, x, s) = G_0(p, x, s) + \Phi(p, x) \]  

(71)

\[ G_0(p, x, s) = c^2 e^{-x^2/2} \exp \left( \frac{-x}{(4ps(2ps + 1))} \right) \times (L_{p-1}(x\tau) - L_p(x\tau)) \]  

(72)

\[ \Phi(p, x) = \sum_{m=0}^{p-1} (R(p - 1, m, x) - R(p, m, x)) \]  

(73)

\[ R(a, b, x) = x^{a-b} \left( \frac{a!}{b!} \right) \left[ e^{-x^2/2} L_b^{a-b}(x^2) \right. \]  

\[ - e^{2a-b} \exp \left( -\frac{x}{4ps(2ps + 1)} \right) \]  

\[ L_b^{a-b}(x/c) \right] \]  

(74)

\[ c^2 = \frac{2ps}{2ps + 1} \]  

(75)

\[ \tau = (c - 1/c) \]  

(76)
where $L^a_m$ is the associated Laguerre polynomial. Often we will use the dimensionless gap $\delta$ defined by

$$\Delta_a = \frac{e^2}{\varepsilon l} \delta_a. \quad (77)$$

It is useful to know that

$$\frac{\Delta_a}{k_B} = \frac{e^2 \delta_a}{\varepsilon k_B} \simeq 50 \sqrt{B(T)} \delta_a^0 K \quad (78)$$

I shall use the values given in Table 1:

| $\frac{eB}{m_e k_B}$ | $1.34 B(T)^0 K$ |
|----------------------|-----------------|
| $\frac{e^2}{\varepsilon l k_B}$ | $50 \sqrt{B(T)}^0 K$ |
| $\frac{e}{\varepsilon l}$ | $.026 m_e \sqrt{B(T)}$ |

**Table 1** Approximate numbers used in this paper, with $k_B$ the Boltzmann’s constant, $B(T)$ the field in Tesla.

Figure 2 shows the gaps computed for 1/3, 2/5, 3/7 and 4/9 for the ZDS potential and compared to the work of PMJ in the region $0 \leq \lambda \leq 3$. The following features are noteworthy.

- At $\lambda = 0$, the coulomb case, the gaps are finite in contrast to the small $q$ theory. This is due to the gaussian factor $e^{-q^2/2}$ which was absent there. The slope of the graphs in the present theory is nonzero at this point. It is readily verified that $d\Delta/d\lambda$ at $\lambda = 0$ is the gap due to a delta-function potential, and should vanish for spinless fermions. It does happen for PMJ, whose electronic wave-function is explicitly antisymmetric. The present theory of CF does not give good answers for potentials as short ranged as the delta function. Indeed even the coulomb interaction is too singular, and the theory begins to work well only beyond $\lambda \simeq 1$.

- Beyond $\lambda \simeq 1$ the agreement is quite fair in general and very good for 2/5.

- The gaps do not vanish for any fraction and any finite $\lambda$.

**B. Activation masses**

I have computed gaps for many other fractions, including for $s = 2$, when four vortices are attached to form CF’s.

Rather than show more plots, I will now analyze the theory in terms of $m_a$ the activation mass defined by

$$\Delta_a = \frac{eB^*}{m_a} = \frac{eB}{(2ps + 1)m_a}. \quad (79)$$

Comparison to Eqn. (77) shows that

$$\frac{1}{m_a} = \frac{e^2}{\varepsilon} \delta_a (2ps + 1) \equiv \frac{e^2}{\varepsilon} C_a. \quad (80)$$

Thus

$$C_a = \delta_a (2ps + 1). \quad (81)$$

The significance of $C_a$ is that it approaches a limit as we approach $\nu = 1/2$ or 1/4, as first emphasized by HLR. What we will see now is that $C_a$ does indeed have a nice limit, but this limit depends on $\lambda$, a parameter that was set equal to zero (coulomb case) in HLR. We will focus on the $s = 1$ series, which converges to $\nu = 1/2$ as $p \to \infty$. To remind us of this fact a superscript 2 = 2 will be appended. Consider the Table 2, of gaps fitted as a function of $\lambda$ in the interval $1 \leq \lambda \leq 2$, although the fit will work for modest excursions on either side.

| $p$ | $\frac{\Delta_a^{(2)}/k_B}{50\sqrt{B(T)}\delta_a^{(2)}}$ | $\frac{\delta_a^{(2)}}{\lambda}$ | $C_a^{(2)}$ |
|-----|------------------------------------------------|-----------------|---------|
| 1   | 5.31 $\sqrt{B(T)/\lambda}$                       | .106/\lambda .32/\lambda |
| 2   | 2.08 $\sqrt{B(T)/\lambda}$                       | .042/\lambda .21/\lambda |
| 3   | 1.23 $\sqrt{B(T)/\lambda}$                       | .025/\lambda .17/\lambda |
| 4   | 0.87 $\sqrt{B(T)/\lambda}$                       | .017/\lambda .16/\lambda |

**Table 2** Activation gaps as a function of $\lambda$ for $1 \leq \lambda \leq 2$ according to the hamiltonian theory. Note the convergence of $C_a^{(2)}$ as $p \to \infty$. 

FIG. 2. Comparison of dimensionless activation gaps $\delta_a$ to the work of Park, Meskini and Jain et al for the fractions $1/3, 2/5, 3/7, 4/9$ ($p = 1, 2, 3, 4,$ and $s=1$) as a function of $\lambda$, the thickness parameter in the ZDS potential.
Based on Table (2), and a similar one for states near \( \nu = 1/4 \), I find that we may write, near these states,

\[
C_a^{(2)} = \frac{160}{\lambda} \quad (82)
\]

\[
C_a^{(4)} = \frac{148}{\lambda^{5/4}} \quad (83)
\]

where the expression, including the exponents \( (1, \frac{5}{4}) \) are approximate.

Consider the normalized mass defined by Pan et al [5]

\[
m_{a,nor} = \frac{m_a}{m_e \sqrt{B(T)}} \quad (84)
\]

where \( m_e \) is the electron mass and \( B(T) \) is the field in Tesla.

In terms of \( C \),

\[
m_{a,nor} = \frac{0.026}{C_a^{(2s)}} \quad (85)
\]

Combining Eqns. (82-85)

\[
m_{a,nor} = 0.163 \lambda \quad (s=1)
\]

\[
= 0.175 \lambda^{5/4} \quad (s=2)
\]

We find that the suitably scaled masses \( m_{a,nor} \) are comparable for \( s = 1 \) and \( s = 2 \) but not exactly equal. No fundamental arguments exist for their equality since the answer depends on the potential, parameterized by \( \lambda \).

C. Comparison to data

In comparing these gaps to experiments, I will limit myself to \( \nu \leq 1/2 \). States like \( 3/4 \) are related by particle-hole symmetry if full polarization is assumed, and states with \( \nu > 1 \) require assumptions about filled electronic LL's which I do not want to make.

Consider the experiments of Du et al [2], who have extensive data on activation gaps. Given that the experiments, unlike PMJ, have an unknown contribution from LL mixing and impurities, it is not clear how to apply the theory. There is no ab initio calculation that includes these effects. (There is however reliable evidence that LL mixing is a very small effect at the values of \( \lambda \) under consideration [4].) I will therefore compute gaps using the ZDS potential with \( \lambda \) as a free parameter, and ask what \( \lambda \) fits the data, just to get a feel for its size. The results are summarized in Table 3.

Table 3 Comparison of activation masses to Du et al, sample A, which has a density \( n = 1.12 \cdot 10^{11} \text{cm}^{-2} \). The last column gives the best fit to \( \lambda \).

| \( \nu \) | \( B(T) \) | \( \Delta_{exp}(^oK) \) | \( \Delta_{theo}(^oK) \) | \( \lambda \) |
|---|---|---|---|---|
| 1/3 | 13.9 | 8.2 | 5.3 \sqrt{B(T)/\lambda} | 2.4 |
| 2/5 | 11.6 | 3 | 2.08 \sqrt{B(T)/\lambda} | 2.4 |
| 3/7 | 10.8 | 2 | 1.23 \sqrt{B(T)/\lambda} | 2.0 |

Is it possible to describe the disordered sample by some effective \( \lambda \)? It does not seem likely, given these three data points: the line through them (plotted against \( B^* = B - 9.27 \text{ Tesla} \) has a negative intercept, while the pure system calculations give a gap that never vanishes for any finite \( \lambda \). Furthermore, the negative intercept is between \( 1^o - 2^oK \), while the gaps are at best \( 6 - 8^oK \). Thus the effects of disorder appear to be quite significant. Therefore no attempt will be made to find an effective \( \lambda \). If one day we get samples for which the disorder broadening is much smaller, we can attempt this. In the meantime, LDA and exact diagonalization calculations suggest that the answers for the pure system differs by roughly a factor of two from the data [3].

Consider now the results of Pan et al. Rather than attempt to fit their gaps to the theory, I consider the following issue they raise about the normalized activation mass

\[
m_{a,nor} = \frac{m_a}{m_e \sqrt{B(T)}}. \quad (88)
\]

They observe that these masses are in the range \( .25 - .35 \) near \( \nu = 1/2 \) and \( 1/4 \). How does this rough equality of masses of fermions with two and four vortices fit in the present theory? If we compare their results to Eqns. (84) we extract the range of values for \( \lambda \) listed in Table 4. They seem to be spread over a range that is more or less equal. The theoretical prediction for this case is sensitive to how exactly the two fractions are reached. For the present case, wherein \( n \) is fixed and \( B \) is altered, we expect \( \lambda^{(4)}/\lambda^{(2)} = \sqrt{2} \). I suspect that the experimental \( \lambda \)'s do not show this factor of \( \sqrt{2} \) because the effects of disorder are most pronounced near the gapless states.

Table 4 Given that normalized masses for 2 and 4 vortex attachment are in the range \( .25 - .35 \) the table asks what the corresponding values of \( \lambda \) are. Theory predicts that \( \lambda^{(4)} = \sqrt{2} \lambda^{(2)} s \).

| Theoretical value | \( \lambda \) implied by data |
|---|---|
| \( m_{a,nor} = 0.163\lambda^{(2)} \) (s = 1) | \( \lambda^{(2)} = 1.5 - 2 \) |
| \( m_{a,nor} = 0.175\lambda^{(4)5/4} \) (s = 2) | \( \lambda^{(4)} = 1.3 - 1.75 \) |

D. A study of particle and hole profiles.

Now we consider the charge densities in a state with either a widely separated particle-hole pair in one of the gapped fractions or just a CF at \( \nu = 1/2 \). The calculations are detailed in Appendix 3.

Figure 3 which shows the distribution of charge in units of the ambient density, for three gapped fractions with a widely separated particle and hole. To compare
to the unpublished work of Park and Jain who placed them at antipodal points of a sphere, I have done the same. The hole is placed at the north pole \( r = 0 \) and \( r \) is the distance along a great circle in units of \( l \). The solid lines refer to my results. The small horizontal lines indicate the values of the maxima and minima of Park and Jain results, while the small vertical lines show their locations. Three features are noteworthy.

First, the density is best predicted at 2/5. It is not surprising that the gap follows PMJ best in this case.

Next, in the case of 3/7, the particle and hole actually overlap and get entangled in the Monte Carlo work on a finite sphere. In the present infinite volume approach, we have no trouble distinguishing them apart since they were computed individually and superposed to draw the figure.

Finally, consider the case of 1/3, where one expects to find the best results for gaps (due to a large gap), but does not. Since there is just one filled CF level, if you make a hole at \( r = 0 \) the density must vanish. This feature found in Park and Jain, is absent in the present approximation where the function drops to about half the ambient density. This breakdown of the present picture at short distances is generic. Note also that while our unitary transformations attempt to go from electrons to CF while Jain’s wavefunction approach goes from CF to the electrons, the two are not of equal accuracy. Given the simplicity and tractability of the final operators in the present model (which cannot possibly carry the complexity of the projected Jain wavefunctions in electron coordinates) this is to be expected.

**E. A closer look at \( \nu = 1/2 \)**

At \( \nu = 1/2 \) the CF of momentum \( k \) is predicted to have a dipole moment \( \vec{r} \times \vec{k} \). Let us understand this statement. It stems from examining the trial state

\[
\Psi = \mathcal{P} \prod_{i<j} (z_i - z_j)^2 \text{Det} |e^{ik_i \cdot r_j}|. \tag{89}
\]

Before projection, the vortices are on the electrons. The CF charge is \( e^* \) but the dipole moment is zero. To project, one begins with

\[
e^{ik \cdot r} = \exp \frac{i}{2} (k \bar{z} + \bar{k} z) \tag{90}
\]

(where \( k = k_x + ik_y \)) recalls that \( \bar{z} \rightarrow 2 \bar{z} \partial / \partial z \), and sees that the derivatives in the exponential will move \( z \) to \( z + ik \). This suggests that the vortex will move off the particle by an amount \( ik l^2 \), which explains the origin of the dipole and its moment. This analysis in terms of just one particle corresponds to what was inferred from Figure 1 or the analysis of \( R_x \) and \( R_y \).

While this is true, it does mean that the spatial distribution of zeros in the LLL wavefunction (as a function of one coordinate, the others being frozen) will resemble the momenta that fill the Fermi sea. There are several reasons for this.

- Besides the one exponential we considered, there are other exponentials that move every other coordinate by the corresponding \( k \), so that in the end \( (z_i - z_j)^2 \rightarrow (z_i + il^2 k_i - z_j - il^2 k_j)^2 \). This dependence on the difference of momenta is in accord with the symmetry first noted by Haldane, and dubbed K-invariance by Stern *et al.*

- All of the above refers to just one term in the expansion of the determinant. We must still antisymmetrize over all possible pairing of momenta with coordinates.

- Many of these zeros will have to disappear upon projection, leaving just a total of two per particle. It is far from clear how the zeros will be distributed in the end. All we know is that one has to lie on the electron by the Pauli principle, and the other forms a single vortex, as in D.H. Lee’s picture. The latter must be at a distance \( 2kl^2 \) from the electron to preserve the dipole moment. Since the ”size” of
the CF at the Fermi energy, 2l, is close to the inter-
particle spacing of $2\sqrt{\pi l} \approx 3.5l$, there is definitely
going to be some ambiguity in pairing zeros with
electrons.

These complications notwithstanding, the dipole makes a clear appearance in the operator approach. It is
not tied to zeros of the wavefunctions and emerges as fol-
lo, if the expression for the preferred charge density $\bar{\rho}^p$, (which obeys Kohn’s theorem), is coupled to an external
potential, the second term in the series Eqn. (24) is pre-
cisely that of a dipole of strength $l^2 z \times k$. This operator
definition is unchanged by antisymmetrization (we sim-
ply write Eqn. (24) in terms of fermion operators in sec-
ond quantization) and crystallizes the notion of the dipole
inspired by an analysis of the trial wavefunction. The
dipoles also emerge in the high frequency density-density
response computed in the conserving approximations [46]
as stated earlier. The zeros of trial wavefunctions do
not seem to be the likely place to look for evidence of
dipoles.

I will now illustrate these points in a many-body state
with an extra CF. It however takes a bit of work. Suppose
we create a CF in a state of definite momentum $p$ above the
Fermi surface. If we evaluate $\bar{\rho}^p(r)$ in this state, we
get zero. This is because the CF has an equal amplitude
to be at all places and the dipolar density gets smeared out
completely. If we localize it at the origin, we use
all momenta, and hence all values of the dipole moment
which again averages to zero charge density. Consider
however the following state

$$|p_0\rangle = \int_{-\infty}^{\infty} dp_x dp_y |(p_x, p_y)\rangle |FS\rangle$$

(91)

where $|FS\rangle$ is the Fermi sea, and $p_0 > p_F$. This is a
superposition of states of all values of $p_x$, localizing the
CF at $x = 0$ but with fixed $p_y = p_0$. (As $p_0 > p_F$, all
these states lie on a line outside the FS.) The particle is
thus spread out completely in the $y$-direction. However,
its dipole moment is fixed at $l^2 p_0$ along the $x$-axis. A
calculation done in Appendix 3 yields the following result
in compliance with these expectations:

$$\rho(r) \approx \left[ \exp \left( -\frac{(x - \frac{1}{2} p_0 l^2)^2}{l^2} \right) - \exp \left( -(x + \frac{1}{2} p_0 l^2)^2 \right) \right]$$

(92)

depicted in Figure (4).

If one evaluates the dipole moment of this charge dis-
tribution, one finds it equals $p_0 l^2$. This is the sense in
which the dipole moment appears in the the operator
approach.

I conclude with one significant difference between $e^*$
and $d^*$. The CF charge $e^*$ is robust under projection,
while $d^*$ is not. In the unprojected wavefunction at
$\nu = 1/2$, the dipole moment of the CF is zero since the
vortices are on top of the electrons. The charge is zero as
well. Consider a system with an interaction comparable
to the cyclotron gap, such that the best wavefunction is
the above unprojected one. Based on this wavefunction
we would assign to the CF a charge $e^*$, but no dipole
moment. Consider now a problem where the projected
state is the best. Instead of applying $P$ in one shot,
imagine slowly reducing the non-LLL component of the
wavefunction to zero. Along the way, the zeros get ripped
off the electrons and some of them disappear. Through
it all $e^*$ is invariant, since it depends on the Hall conduc-
tance, which is constant. But the dipole moment, which
describes the internal structure of the CF, changes from
zero to some value in the LLL. This value seems to be
$l^2 z \times k$, in the wave function based arguments of Read,
in the operator series and the density-density response.
I do not know if the dipole moment is robust within the
LLL, but suspect it is, since Kohn’s theorem (which lim-
its the matrix element in the LLL) gives a unique answer
in the operator approach.

Some caution must be used in looking for this mo-
ment. Even if we know how the CF couples to an ex-
ternal potential, the response function will depend on the
Hamiltonian as well. In particular, we know that at
$\omega = 0$, $q \to 0$, the compressibility will not vanish as $q^2$
but as a constant, because of the special symmetries of
$H$, a point made by Halperin and Stern and discussed at
length in Refs. [37] and [38]. Only at high frequencies will
the dipoles behave classically.

![FIG. 4. The charge density of a CF showing its dipole moment. The CF is localized in $x$ but spread out uniformly in $y$. Luckily this does not smear out the dipole moment.](image1)

![FIG. 5. Comparison to the work of Park, Meskini and Jain et al for the gaussian potential $v(q) = 2\pi e^2 l e^{-q^2 l^2/2}$ for $p = 1, 2, 3, 4$.](image2)
F. Other potentials

Figure 6 shows a comparison to the PMJ results for a gaussian potential

\[ v(q) = 2\pi e^2 l e^{-q^2 l^2/2} \]  
(93)

Note that except for \( \nu = 1/3 \) the agreement is exceptional. This is the kind of potential for which the present theory works best.

On the other hand for a potential,

\[ v(r) = \frac{e^{-kr}}{r} \]  
(94)

the agreement is worse than for the coulomb case since this potential is just as bad as \( r \to 0 \) and does not give the large \( r \) values a chance. Likewise \( 1/r^2 \) fares worse than \( 1/r \).

From playing with these potentials and using the PMJ results as a benchmark we can thus learn when the present model can be trusted.

G. Effects of CF interactions

Figure 6 shows what happens to the gaps if the CF interactions are turned off. These correspond to the contribution from the \( G_0 \) term in Eqn. (71). Note that interactions seem less important for \( \nu = 1/4 \) and systematically get less important as \( p \) increases.

There is some freedom in defining the measure of interactions, which I took to mean the effect of \( H_I \) on \( \Delta_o \). But the effect of \( H_I \) is two fold: it renormalizes the self-energy of the particles and also mediates interactions between them. One can envisage a situation in which individual energies get strongly renormalized by \( H_I \) but the dressed particles are barely interacting, i.e., their energy barely varies with separation. In this case one could argue that CF are weakly interacting. Such a separation, between the quasiparticle and quasihole, was set to infinity in our gap calculations. This energy, as a function of separation, is contained in the \( q \)-dependence of the magnetoexciton spectrum. Since the variation is typically comparable to the gap, CF seem to be quite strongly interacting by this measure as well.

H. Comparison to the work of Morf et al

Morf et al. have calculated the activation gaps for 1/3, 2/5 and 3/7 by exact diagonalization of finite systems, paying great care in extrapolating to the thermodynamic limit. The potential they use is

\[ v(q) = \frac{2\pi e^2}{q} e^{(qlb)^2} \text{Erfc} (qlb) \]  
(95)

where \( b \) is the analog of \( \lambda \). Our numbers are compared in Figure 7.

The following features are worthy of note.

- The calculated gaps always lie above the exact diagonalization results for the two fractions shown (as well as for the 1/3 case, not shown). This result agrees with the general belief that HF always overestimates the gaps by neglecting fluctuations. Compare this to the case of PMJ where for 3/7 and 4/9 the calculated gaps were sometimes lower and sometimes higher.

The fact that the theory predicts gaps that always exceed the exact diagonalization results suggests the possibility that the problem may lie not in \( H^p \) but in the HF approximation. If \( H^p \) were solved by a more accurate method than HF, the agreement might have been better. This however merely remains a possibility till someone solves it by, say, exact diagonalization. This will be hard since the theory has constraints and is formulated in the full fermionic Hilbert space.

- The general agreement is worse for this potential than for the ZDS case. This is because at large \( q \) this potential goes as \( 1/q \) while the ZDS potential falls exponentially.

I. Particle-hole symmetry

For the fully polarized case one expects that within the LLL, \( \Delta_o \) for \( \nu \) and \( 1 - \nu \) will be equal. Let us focus on just the case \( s = 1 \), which corresponds to two-vortex attachment. The fractions related are now \( \nu \) and
\[ 1 - \nu = 1 - \frac{p}{2p + 1} = \frac{p + 1}{2(p + 1) - 1} \]  

(96)

Thus to find the gap at \( 1 - \nu \) corresponding to a certain \( \nu = p/(2p + 1) \) we must

- Replace \( p \) by \( p' = p + 1 \) and consider fractions of the form \( p'/2p' - 1 \). The flux quantum per CF is now \(-1/p'\) which means the double-vortices have overturned the applied field and changed its direction and \( p' \) levels are filled in this weakened and reversed field. In Eqs. (59-70) we must make the obvious changes, \( p \to p' = p + 1 \) and \((2p + 1) \to (2p' - 1)\), in all the denominators, and the not so obvious change \( \exp(-x/(4p(2p + 1))) \to \exp(x/(4p'(2p' - 1))) \) in the double-vortex form factor which reflects the fact that the vortex charge is bigger than that of the electron.

- If we are working at \( \Lambda = \lambda \neq 0 \), we have to keep \( \Lambda \) constant in the comparison. Thus we need to verify that

\[ \Delta(\nu, \lambda) = \Delta \left( 1 - \nu, \lambda \sqrt{\frac{\nu}{1 - \nu}} \right) \]  

(97)

or for the dimensionless gap \( \delta_a = \Delta_a/(e^2/\epsilon \ell) \),

\[ \delta_a(\nu, \lambda) = \sqrt{\frac{\nu}{1 - \nu}} \delta_a(1 - \nu, \lambda \sqrt{\frac{\nu}{1 - \nu}}) \]  

(98)

My results are summarized in Table 5.

| \( \nu \) | \( \delta_a(\nu, 0) \) | \( \delta_a(\nu, 0)/\nu \) | \( \delta_a(\nu, 1) \) | \( \delta_a(1, \nu) \sqrt{1 - \nu/\nu} \) |
|----------|-----------------|----------------|----------------|------------------|
| 1/3      | .213            | .203           | .106           | .101             |
| 2/5      | .097            | .098           | .042           | .043             |
| 3/7      | .063            | .065           | .025           | .026             |
| 4/9      | .047            | .048           | .017           | .018             |
| 5/11     | .038            | .039           | .013           | .014             |

Table 5 Particle-hole symmetry in the LLL for the polarized cases with \( \nu = p/(2p + 1) \) requires that the dimensionless gap \( \delta_a(\nu, \lambda) = \sqrt{\frac{\nu}{1 - \nu}} \delta_a(1 - \nu, \lambda \sqrt{\frac{\nu}{1 - \nu}}) \). The Table considers \( \lambda = 0 \) and 1 and indicates that the theoretical numbers obey this symmetry.

Note that particle-hole symmetry holds very well even at \( \lambda = 0 \) when the absolute value of the gap is not close to the benchmark value set by PMJ. If one looks at the expressions for the gap, one sees that this agreement is very nontrivial since the two problems are very different in the CF formalism: they have different number of filled LL, and the matrix elements involved are quite different.

### V. MAGNETIC TRANSITIONS AT \( T = 0 \).

Now we turn to the behavior of the spin of the system, which was assumed to be frozen along the applied field. This topic has explored within the CF approach rather extensively by Park and Jain. \[^{14}\]

The coupling of electron spin to the applied field is given a Zeeman term

\[ H_Z = -g \left( \frac{e}{2m_e} \right) \frac{S}{2} B \]  

(99)

where \( g = .44 \), \( m_e \) is the electron mass in free space, \( S \) is given by

\[ S = n P \]  

(100)

where \( n \) is the density and \( P \) is the polarization, to which each electron contributes \( \pm 1 \).

#### A. Magnetic transitions in gapped fractions

If \( H_Z \) dominates, we expect the system to be fully polarized \( (P = 1) \). As we lower \( H_Z \), we may expect \( P \) to drop. In CF theory for gapped fractions there is a discrete set of allowed values of \( P \). At \( \nu = p/(2p + 1) \), these correspond to states of the form \( \langle p - r, r \rangle \). These stand for many-body states in which \( p - r \) LL are occupied by up spins and \( r \) LL by down spins. In Jain’s approach, the actual wavefunction will be such a state times the Jastrow factor, followed by projection to the LLL. In the present approach, \( \langle p - r, r \rangle \) is literally the state, but the operators for charge and spin are obtained by canonical transformations. For the interested reader I mention that flux attachment and canonical transformations are same for both spins.

Since the uniform external field couples to the \( q = 0 \) component of the spin density which is unaffected by the canonical transformations, \( H_Z \) will have the same form in the final CF representation.

It is important to note that even though the states are labeled by free particles (in a weakened field \( A' \)), the problem is not really free: in Jain’s version the free states turn to highly correlated wavefunctions for electrons, and our case, the states may look free, but \( H \) is not.

In any event, the allowed values of polarization are given by

\[ P = \frac{p - 2r}{p}. \]  

(101)

Thus for example, when \( p = 4 \), the allowed values are \( P = 1, 5 \), and 0 corresponding to \( \langle 4, 0 \rangle, \langle 3, 1 \rangle \) and \( \langle 2, 2 \rangle \).

Our goal is to calculate the critical fields at which the system will jump from one value of \( r \) to the next as \( H_Z \) is varied. Let

\[ E(p - r, r) = \langle p - r, r | H | p - r, r \rangle \]  

(102)

where \( H \) does not contain the energy due to \( H_Z \). This will be case for the single-particle and ground state energies, with one exception which will be clearly pointed out. Since \( H_Z \) is diagonal in the HF states which have definite spin, its effects can be trivially incorporated.
The HF calculation of $E(p-r,r)$, detailed in the Appendix 5, gives

$$E(p-r,r) = \frac{n}{p} \sum_{q} \sum_{n_1=0}^{p-r-1} \langle n_1 | \rho(q) \rho(-q) | n_1 \rangle - \sum_{n_1=0}^{r-1} \sum_{n_2=0}^{p-r-1} | \rho_{n_1 n_2} |^2$$

where $\rho_{n_1 n_2}$ was introduced earlier and discussed in Appendix 1 and

$$\langle n | \rho(q) \rho(-q) | n \rangle = \sum_{n'=0}^{\infty} | \rho(q)_{nn'} |^2$$  \hspace{1cm} (103)

The critical field $B_c$ for the transition from $r$ to $r+1$ is given by:

$$E(p-r,r) - E(p-r-1,r+1) = g \frac{e B_c}{2 m_e} \frac{n}{p}$$  \hspace{1cm} (104)

where the right hand side denotes the Zeeman cost of flipping the $n/p$ spins in the LL that switched its spin. This discussion assumes that $B$ is perpendicular to the sample. If there is a tilt $\theta$, we write

$$E(p-r,r) - E(p-r-1,r+1) = g \frac{e B_c}{2 m_e \cos \theta} \frac{n}{p}$$  \hspace{1cm} (105)

When these $B_c$’s were calculated, I noticed the same remarkable regularity first noted by Park and Jain, namely that they could be fit by a theory of free fermions of mass $m_p$ (the polarization mass) that occupy LL with a gap $\Delta_p = e B^*/m_p$. In this case we would have

$$E(p-r,r) - E(p-r-1,r+1) = \frac{n(p-r-1)}{p} \Delta_p$$  \hspace{1cm} (106)

since $(n/p)$ spin-up fermions of energy $(p-r-1 + \frac{1}{2}) \Delta_p$ drop to the spin-down level with energy $(r + \frac{1}{2}) \Delta_p$.

Suppose we evaluate the left-hand-side of Eqn. (106) in the HF approximation to $H$ and define

$$\Delta_p(r)^{def} = \frac{p E(p-r,r) - E(p-r-1,r+1)}{p - 2r - 1}.$$  \hspace{1cm} (107)

Given that $H$ is not free, there is no reason why $\Delta_p(r)^{def}$ should be $r$-independent. But it is very nearly so. For example at $p = 6, \lambda = 1$,

$$\Delta_p(0,1,2)^{def} = \frac{e^2}{\xi_l} (0.000660, 0.000649, 0.000641)$$  \hspace{1cm} (108)

which describe $|6,0\rangle \rightarrow |5,1\rangle$, $|5,1\rangle \rightarrow |4,2\rangle$, and $|4,2\rangle \rightarrow |3,3\rangle$. This $r$-independence of the gaps was true for every fraction and every value of $\lambda$ I looked at. Yet I knew that $H$ was definitely not free since the activation gap $\Delta_a$ to make a widely separated particle-hole pair differs from $\Delta_p$ by factors like 2 or 4 (depending on $\lambda$) and turning off $H_I$ makes a substantial difference, as demonstrated earlier.

I will place this result in perspective shortly, after noting that it has a counterpart in the gapless case as well. But first I present the HF results for (the $r$-independent) $m_p$ and $\Delta_p = e B^*/m_p$. At and near $\nu = \frac{1}{3}$ and $\frac{2}{3}$, for $.75 < \lambda < 2$, $m_p$ may be approximated by

$$\frac{1}{m_p^{(2)}} = \frac{2}{\xi} C_p^{(2)}(\lambda)$$  \hspace{1cm} (109)

$$\frac{1}{m_p^{(4)}} = \frac{2}{\xi} C_p^{(4)}(\lambda)$$  \hspace{1cm} (110)

For fractions like 2/5, not too close to 1/2, I will use the actual $m_p$ in comparing to experiment.

The transition $(p-r,r) \rightarrow ||p-r-1,r+1)$ occurs when

$$g \frac{e}{2 m_e \cos \theta} B_c = (p - 2r - 1) \Delta_p.$$  \hspace{1cm} (111)

B. Magnetic transitions of gapless fractions

Let us now turn to the gapless fractions 1/2 and 1/4. The discrete labels $p - r$ and $r$ of the HF states that count the spin-up and down LL’s are now replaced by continuous variables $k_{\perp} \mu$ which label the Fermi momenta of the spin-up and down seas. These momenta are such that the total number of particles equals $n$:

$$k_{\perp}^2 + \mu^2 = 4 \pi n$$  \hspace{1cm} (112)

where $k_{\perp}$ denotes the Fermi momentum of a fully polarized sea.

In the gapped case there were several critical fields $B_c^*$, each corresponding to one more CF -LL flipping its spin, each describing one more jump in the allowed values of $P$. In the gapless case the situation is different. For very large Zeeman energy, the sea will be fully polarized. It will not be worth including even one fermion of the opposite spin since the Zeeman energy cost alone will exceed the Fermi energy of the polarized sea. As we lower the Zeeman term, we will reach a critical field at which it will be worth introducing one fermion of the other spin with zero kinetic energy. At this point the energy of a particle on top of the spin-up sea obeys

$$E_+(k_{\perp}^*) = g \frac{e}{2 m_e \cos \theta} B_{\perp}^{*}$$  \hspace{1cm} (113)

If we lower the Zeeman term further, the polarization will fall continuously and be determined by $E_+(k_{\perp}^*)$, the energies of the particles on top of these two seas according to

$$E_+(k_{\perp}^*) - E_-(k_{\perp}^*) = g \frac{e}{2 m_e \cos \theta} B_{\perp}^{*}$$  \hspace{1cm} (114)

since this equation states that the system is indifferent to the transfer of a particle from one sea to another, i.e., has minimized its energy with respect to polarization.
Using the fact that at these fraction when the mean magnetic field vanishes, we deal with a very simple expression:

\[ \tilde{\rho}(q) = \int \frac{d^2k}{4\pi^2} (-2i) \sin \left( \frac{q \cdot k^2}{2} \right) \tilde{d}_k \cdot d_k \]

(115)

it is easy to do a HF calculation and obtain

\[ E_\pm(k) = \frac{2}{\pi} \int \frac{d^2q}{\pi} \frac{\tilde{\rho}(q)}{\sin \left( \frac{q \cdot q^2}{2} \right)} \]

\[ = -4 \int \frac{d^2k'}{4\pi^2} \hat{\nu}(|k'|) \sin \left( \frac{k' \cdot k^2}{2} \right) \]

\[ = E_0 + E_I \]

where the Zeeman energy is not included, \( n_{\pm k}^{F} \) is the (step) Fermi function for the two species, \( E_0 \) and \( E_I \) represent single particle energy (due to what was called \( H_0 \) earlier) and the energy of interaction of this particle at the Fermi surface with those inside the sea, and

\[ \hat{\nu}(k) = v(k)e^{-k^2l^2/2}. \]

(116)

When this result is used to compute \( E_+(k_{+F}) - E_-(k_{-F}) \), I find once again that the numbers fit a free theory in the following sense. Imagine that CF were free and had a mass \( m_p \). We would then have

\[ E_+(k_{+F}) - E_-(k_{-F}) = \frac{k^2_{+F} - k^2_{-F}}{2m_p} \]

(117)

What I find is that the HF number for \( E(k_{+F}) - E(k_{-F}) \) may be fit to the above form with an \( m_p \) that is essentially constant as we vary \( k_{+F} \) i.e., the relative sizes of the up and down seas (which is analogous to an \( m_p \) that does not depend on the index \( r \) in the gapped case) and that this \( m_p \) matches smoothly with that defined for the nearby gapped fraction.

This result is surprising because we know the CF are not free from a variety of reasons. Indeed the HF energies do not have a quadratic dispersion: for example at \( \nu = \frac{1}{2} \) and \( \lambda = 1 \)

\[ \frac{E(k_{+F})}{(e^2/\lambda l)} = a \frac{k_{+F}^2}{k_F^2} + b \left( \frac{k_{+F}}{k_F} \right)^4 \]

(118)

where \( a = .075 \), \( b = -.030 \).

The proper interpretation of this free-field behaviour will be taken up shortly. For now I assume this feature of the results and define \( m_p \) by

\[ \frac{1}{m_p} = 2 \frac{E_+(k_{+F}) - E_-(k_{-F})}{k_{+F}^2 - k_{-F}^2} \]

(119)

As mentioned above, \( m_p \) for the gapless cases merges smoothly with the \( m_p \) for the nearby gapped states. The results are thus given by Eqns. (109)-(111) for the range \( .75 \leq \lambda \leq 2 \) for the ZDS potential.

C. Why the free-field behavior?

The fact that magnetic phenomena at \( T = 0 \) can be described (to excellent accuracy) by free fermions of mass \( m_p \) needs to be properly understood and interpreted. For example one must resist the thought that perhaps by some further change of variables one could take the present hamiltonian and convert it to a free one. This is because if there were really an underlying free theory, it would be able to predict an activation mass \( m_a \) and this would have to coincide with \( m_p \). We know within this theory, within Jain’s approach, or from experiment, that these masses differ by at least a factor of two.

I will now show that a single assumption about the form of the ground state energy, an assumption that is not equivalent to the free-field assumption or even to a quadratic dispersion relation in the gapless cases, will explain this behavior for gapped and gapless fractions. Consider \( E(S) \), the ground state energy as a function of \( S = nP \). By rotational invariance it must have only even powers of \( S \) in its series. Assume the series is dominated by the first two terms:

\[ E(S) = E(0) + \frac{\alpha}{2} S^2 \]

(120)

where \( \alpha \) is the inverse linear static susceptibility.

Consider first the gapless case. When \( dn \) particles go from spin-down to spin-up,

\[ dE = \alpha S dS = \alpha S (2dn) \]

(121)

\[ = \frac{\alpha k_{+F}^2 - k_{-F}^2}{4\pi} (2dn) \]

(122)

using the volumes of the Fermi seas. We see that \( dE \) has precisely the form of the kinetic energy difference of particles of mass \( m_p \) given by

\[ \frac{1}{m_p} = \frac{\alpha}{\pi} \]

(123)

Thus \( m_p \) is essentially the static susceptibility, which happens to have dimensions of mass in \( d = 2 \). The statement that \( m_p \) has no \( r \)-dependence in the gapped case or no spin dependence in the gapless case is the same as saying that the full nonlinear susceptibility does not depend on the spin \( S \), which in turn means \( E(S) \) is quadratic in \( S \).

Note that the free-field form of \( dE \) comes from \( E \propto S^2 \) and \( d = 2 \): in \( d = 3 \), we would have \( dE/dn \propto S \sim (k_{+F}^2 - k_{-F}^2) \) which no one would interpret as a difference of kinetic energies.

This general argument notwithstanding, it is worth explicitly considering the case in hand.

First, one must not think that \( E(S) \) being quadratic in \( S \) is equivalent to assuming that the single-particle HF energies are quadratic in momenta. Consider the HF energies quoted earlier

\[ \frac{E(k_{+F})}{(e^2/\lambda l)} = a \frac{k_{+F}^2}{k_F^2} + b \left( \frac{k_{+F}}{k_F} \right)^4 \]

(124)
The quartic terms miraculously drop out in the energy cost of transferring a particle from the top of the spin-down sea to the top of the spin-up sea:

\[
\frac{dE}{(e^2/\varepsilon)} = a\frac{k_{+F}^2 - k_{-F}^2}{k_F^2} + b\frac{k_{+F}^2 - k_{-F}^2}{k_F^2} + \frac{\alpha}{2}\left(\frac{n^2}{p^2}\right)(p - 2r + p - (2r + 2))(p - 2r + 2r + 2)
\]

Thus the \(k^4\) terms in \(E(k)\) are not the cause of the \(S^4\) term. However, a small \(k^6\) term in \(E(k)\), corresponds to small quartic terms in \(E(S)\).

To understand why the \(k^6\) term is so small, we turn to Eqn. (8) for \(H_0\). Expanding the \(\sin^2\) in a series, we find the \(k^2\) term is down by a factor of at least 15 (50) relative to the \(k^2\) term, at \(\lambda = 0\) (\(\lambda = 1\)), all the way up to \(k = k_F\). Presumably this feature (and its counterpart in the gapped case) persists in the HF approximation to \(H\) and keeps \(E(S)\) essentially quadratic, which in turn mimics free-field behavior.

To really drive home the point, consider a problem where particles are free and have a dispersion relation

\[
E(k) = ak^2 + bk^4.
\]

Let \(yn\) be density of spin-down particles and \((1 - y)n\) particles that of spin-up particles. Thus

\[
S = (1 - 2y)n
\]

The total energy, as a function of \(y\) is

\[
E(y) \approx \int_0^{\pi n y} dk^2(ak^2 + bk^4) + \int_0^{\pi n (1 - y)} dk^2(ak^2 + bk^4)
\]

Note that the cubic terms in \(y\) cancel. Since \(y\) is linearly related to \(S\), it follows \(E(S)\) is also quadratic in \(S\). However a \(k^6\) term would have led to \(S^4\) terms in \(E(S)\).

We wrap up this topic with one thought: even if CF are free or nearly so, there is no reason their kinetic energy should be quadratic in momentum. These particles owe their kinetic energy to electron-electron interactions, and given this fact, all we can say is that their energy must be an even function of \(k\), starting out as \(k^2\) at small \(k\). What constitutes small \(k\) is an open question that is answered unambiguously here: our expression of the energy has substantial \(k^4\) terms for momenta of interest.

Let us now turn to the gapped case and verify that \(E(S) = E(0) + \frac{a}{2}S^2\) implies that \(\Delta(r)\) will be \(r\)-independent. First note that

\[
S = nP = n(1 - \frac{2r}{p}) = \frac{n}{p}(p - 2r).
\]

It then follows that

\[
E(p - r, r) - E(p - r - 1, r + 1) = \frac{\alpha}{2}(S^2(p - r, r) - S^2(p - r - 1, r + 1))
\]

\[
= \frac{\alpha}{2}\left[\frac{n^2}{p^2}\right](p - 2r + p - (2r + 2))(p - 2r + 2r + 2)
\]

\[
= \frac{\alpha}{2}\left[\frac{n^2}{p^2}\right](2p - 4r + 2)
\]

\[
= \frac{n}{p}(p - 2r - 1)\Delta_p(r)_{def}
\]

We find that the \(r\)-dependence of \(\Delta_p(r)_{def}\) drops out and gives

\[
\Delta = \frac{2na}{p}.
\]

If we write

\[
\Delta_p = \frac{eB}{m_p} = \frac{2\pi n}{vm_p}\frac{1}{2ps + 1} = \frac{2\pi n}{m_p p}
\]

it implies, upon comparing to Eqn.(133) that, as in the gapless case

\[
1 = \frac{1}{m_p} = \frac{\alpha}{\pi}.
\]

Analogously to the gapless case, we can show that if the HF energies vary with the LL index \(n\) as \(E(n) = an + bn^2\), \(E(S)\) will be quadratic in \(S\). Thus the CF-LL’s do not have to be uniformly spaced for them to behave as if they were (with a spacing \(\Delta_p\)) in \(T = 0\) magnetic transitions.

## D. Effective potentials for dirty systems

Here I ask if it is possible that a ZDS potential with some effective \(\lambda\) can describe the dirty system. First of all, I realize this cannot be true with respect to all observables, if at all it is true for any. For example, if one were considering conductance, one knows the electron in a disordered potential will typically get localized whereas no ZDS interaction will predict this. As for transport gaps, the present day samples, with a disorder broadening of the same order as the gaps, again preclude this possibility. Magnetic transitions, on the other hand, are controlled by total energies and one may expect that disorder will have a rather innocuous effect and can be represented in an average way by some translationally invariant interaction. I will show below that at least in some limiting case this is a reasonable approximation. I will show that even in cases where I cannot provide a similar argument, it seem to work. This needs to be fully understood.
As seen above, the critical fields are controlled by ground state energies in states of different polarization. Let us write the ground state energy as

$$E_0 = \langle \Omega | H | \Omega \rangle$$  \hfill (137)

where $|\Omega\rangle$ is the ground state, and

$$H = \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \tilde{p}^p(q) v(q) e^{-q^2/2} \rho^p(-q).$$  \hfill (138)

Now add on a perturbation that couples the system to an external impurity potential $\Phi(q)$. To second order we find, upon disorder averaging,

$$E = E_0 + \int \frac{d^2 q}{(2\pi)^2} \tilde{p}^p(q) \langle \Omega | \tilde{p}^p(q)^2 | \Phi_0^2(q) \rangle \frac{E_0 - E_N}{2\Delta}$$ \hfill (139)

where $N$ is any intermediate state and $\Phi_0^2$ is the average of $\Phi(q)\Phi(-q)$ over realizations. We will now replace $E_N - E_0$ by $\Delta$, the smallest excitation gap produced by $\tilde{p}^p$. In our HF states, this is the gap to the next CF LL, ignoring the $q$ dependence of the magnetoexciton, or the roton minimum if one includes it. This replacement overestimates the second order contribution. Since matrix elements to more distant LL are accompanied by higher powers of $q$, the error will be small if the disorder potential (due to faraway impurities) has only long wavelength components. If we now use completeness, we see that the second order energy can be found by sandwiching a Hamiltonian, once again quadratic in $\tilde{p}^p$, but with an effective potential

$$v_{eff}(q) = v(q) - \frac{\Phi_0^2(q)}{\Delta}$$ \hfill (140)

While the second term, due to disorder, need not be of the original form, what is important is that it is translationally invariant and makes a negative definite contribution to $v_{eff}$. We could incorporate its effect by increasing $\lambda$ in the original ZDS term by a suitable amount. While such a replacement cannot reproduce all the effects of $v_{eff}$ in detail, let us note that the energies depend on the potential $v(q)$ primarily through the corresponding Haldane pseudopotential $V_1$. (This is for $2s = 2$. For $2s = 4$, it is $V_3$. In any event, one only dominates.) We may thus choose the effective $\lambda$, so that it reproduces the dominant $V$ corresponding to $v_{eff}$.

We see that at $T = 0$, in a state with a robust gap, to second order in the impurities, there is an effective $\lambda$, if we ignore higher CF LL’s. These are a lot of restrictions. Of these, only use of second order perturbation theory may be justifiable since the CF has a charge $e^+$ and not $e$, and correlations, which are taken into account from the outset, reduce the coupling to disorder by a factor $1/(2p + 1)^2$ at small $q$. The other restrictions cannot be justified at this point. I will however pursue the notion of an effective $\lambda$ for all fractions and at $T > 0$.

Specifically, $\lambda$ will be extracted from one data point and used to explain the rest of the data from that sample. If the other data points differ only in the temperature $T$, the same $\lambda$ will be used. If it differs in $\nu$ or $n$ or $\nu$, the following scaling argument will be used.

In a heterojunction, the donors of density $n$ produce a confining linear potential of slope that goes as $n$. If one considers a variational wavefunction of the Frank-Howard form $\psi(z) = A(w)z \exp(-z/w)$ in the transverse direction, then the optimal $w$ (to which $\lambda$ must be proportional), varies as $w \simeq n^{−3/4}$. Consequently $\lambda = \lambda/l$ varies as

$$\lambda \simeq n^{-1/3}B^{1/2} \simeq B^{1/6}n^{-1/3} \simeq n^{1/6}n^{-1/2}.$$ \hfill (141)

E. Comparison to data of Kukushkin et al.

Kukushkin et al. vary both $n$ and $B$ and drive the system through various transitions at $T = 0$ (by extrapolation). The field $B$ is always perpendicular to the sample. We will compare the Hamiltonian theory to these experiments by calculating the critical fields at which the $\nu = 1/2$ and $\nu = 1/4$ systems saturate ($P = 1$) and the gapped fractions undergo transitions from one quantized value of $P$ to the next.

Let us recall that as far as these transitions go, the systems behave like free fermions of mass $m_p$ which is independent of the index $r$ which labels how many LL’s have reversed their spins in the gapped case or in the gapless cases, the size of the up and down Fermi circles. At and near the gapless states $m_p$ may be fit by the expressions

$$\frac{1}{m_p^{(2)}} = \frac{c^2l}{\varepsilon} C^{(2)}_p(\lambda)$$ \hfill (142)

$$\frac{1}{m_p^{(4)}} = \frac{c^2l}{\varepsilon} C^{(4)}_p(\lambda)$$ \hfill (143)

where the superscripts on $C$ refer to the number of vortex attached.

I consider $B^c$’s at which the systems at $1/4, 2/5, 3/7, 4/9$, and $1/2$ lose full polarization ($r = 0$ for gapped cases, saturation for the gapless cases) and, for $4/9$, also the $r = 1$ transition, $|3, 1\rangle \rightarrow |2, 2\rangle$.

I fit $\lambda$ to the $\nu = 3/7$ transition $|3, 0\rangle \rightarrow |2, 1\rangle$ at $B^c = 4.5T$.

Two points need to be mentioned in connection with the experiment. First, each of these transitions seems to take place via a narrow intermediate step with a polarization half-way between the ones allowed by CF theory based on spatially homogeneous states. Murthy has suggested that these correspond to spatially inhomogeneous states. I use the center of these narrow steps as the transition points for comparison to the present theory. Secondly, I use the actual values for $C^{(2)}_p$ for these fractions, rather than the asymptotic value in Eqs. (142).

I then obtain $\lambda_{3/7} = 1.42$ on solving Eqn. (111) which takes the following specific form here:

$$\frac{e}{2m_p} \frac{B^c}{(e^2/\varepsilon l)} = \frac{2\Delta_p [(3, 0) \rightarrow (2, 1)]}{(e^2/\varepsilon l)} = \frac{2(0.0117)}{\lambda_{3/7}^{7/4}}$$ \hfill (144)
For transitions at other $B_\perp$ and $n$, I need the corresponding $\lambda$'s. One can argue, as per Eqn. (141) that

$$\lambda_\nu = \lambda_{3/7} \left[ \frac{B}{4.5} \right]^{1/6} \left[ \frac{3}{7} T \right]^{1/3} = .83 \frac{B^{1/6}}{\nu^{1/3}}. \quad (145)$$

Given $\lambda$ one finds $B^c$ using Eqn. (111) for gapped cases.

For the gapless cases, there are two equivalent approaches. First, at the critical field the Fermi energy is the self-consistent solution to

$$\int \frac{d^2q}{(2\pi)^2} \lambda_{\nu} = \lambda_{\nu} \nu B \frac{2\pi n}{m_p} = \frac{eB\nu}{m_p} \quad (147)$$

$$= \frac{e^2}{\sqrt{2} \Lambda_{7/4}} \nu = \frac{1}{2} \quad (148)$$

$$= \frac{e^2}{\sqrt{4\Lambda_{7/4}}} \nu = \frac{1}{4} \quad (149)$$

Using Eqn. (145) one solves for $B^c$ and obtains the values given in Table 6.

Equivalently we can write for the total ground state energy density $E^Z(S)$, (where the superscript indicates that the Zeeman energy is included),

$$E^Z(S) = \frac{\alpha}{2} S^2 - g \frac{e}{2m_e \cos \theta} B \perp S \quad (150)$$

where $\alpha = \pi/m_p$. This expression is minimized (for $P \leq 1$) to give $P$:

$$P = \frac{1.3 \sqrt{B \perp \Lambda_{7/4}}}{\cos \theta} \nu = \frac{1}{2}, \ B \ \text{in Tesla} \quad (151)$$

$$= \frac{1.9 \sqrt{B \perp \Lambda_{7/4}}}{\cos \theta} \nu = \frac{1}{4}, \ B \ \text{in Tesla} \quad (152)$$

Setting $P = 1$ gives the critical fields.

| $\nu$ | comment | $B^c$ (exp) | $B^c$ (theo) | $\nu B^c$ (exp) |
|-------|---------|-------------|-------------|---------------|
| 4/9   | $(3, 1) \rightarrow (2, 2)$ | 2.7 T | 1.6 T | 1.2 |
| 2/5   | $(2, 0) \rightarrow (1, 1)$ | 3 T | 2.65 T | 1.2 |
| 1/4   | saturation | 5.2 T | 4.4 T | 1.3 |
| 3/7   | $(3, 0) \rightarrow (2, 1)$ | 4.5 T | 4.5 T | 1.93 |
| 4/9   | $(4, 0) \rightarrow (3, 1)$ | 5.9 T | 5.9 T | 2.62 |
| 1/2   | saturation | 9.3 T | 11.8 T | 4.65 |

Table 6 Critical fields based on a fit at 3/7. The rows are ordered by the last column which measures density.

Note that in rows above (below) 3/7, where I fit $\lambda$, the predicted $B^c$'s are lower (higher) than the observed values, i.e., the actual $\lambda$'s are less (more) than what Eqn. (145) gives. This is consistent with the expectation that interactions will increase the effective thickness with increased density. If I fit to the 2/5 point, I obtain similar numbers, with the agreement worsening as we move off in density from 2/5. Thus 3/7 was chosen as the fitting point since its density was somewhere in the middle of all the densities considered.

An alternate approach is to attempt to calculate $\lambda \ aba\ initio$ using, for example, the Local Density Approximation (LDA) as done by of PMJ and numerous predecessors. This method will however not include the effect of disorder. It typically gives a $\lambda$ that is half as big. We cannot attribute the entire difference to disorder. For one thing, the HF approximation tends to inflate $\lambda$. Next LDA, as the name suggests, is an approximation. Lastly, LL mixing can account for some of the difference, though not much at these values of $\lambda$. It is possible, that due to all these, the effects of disorder are not that significant, for magnetic phenomena. The present strategy of determining $\lambda$ phenomenologically stands or falls depending on how well the fit to any one data point allows us to make predictions for other measurements made on the same sample. Table 4 suggests it is quite a useful point of view.

VI. PHYSICS AT NONZERO TEMPERATURES

$T > 0$.

So far we have seen the Hamiltonian theory may be used to compute quantities such as gaps, particle-hole profiles, critical fields for magnetic transitions and so on. All such quantities have been readily computed using trial wavefunctions, giving numbers that are superior to ours. The main idea of this paper so far has been to get a 10-20% theory in which we see the underlying physics as transparently as possible and to resolve questions such as why CF behave like particles on some occasions.

We turn to physics at finite $T$ where the method has fewer rivals. Exact diagonalization is limited to very small systems and trial wavefunctions typically cover the ground state and very low excitations. The Hamiltonian approach is able to yield, in the HF approximation, the polarization $P$ and the relaxation rate $1/T_1$ for the gapless states as a function of temperature. If $\lambda$ is treated as before (fit to one data point per sample) we will see it is possible to give a very satisfactory account of experiments up to about $1^6K$, which is of the order of the Fermi energy.

The HF energy of a particle including the Zeeman energy is the self-consistent solution to

$$E^Z_\parallel(k) =$$

$$\mp \frac{1}{2} \tilde{g} \left[ \frac{eB}{2m} \right] + 2 \int \frac{d^2q}{4\pi^2} \tilde{\nu}(q) \sin^2 \left[ \frac{k \times q^2}{2} \right]$$

$$- 4 \int \frac{d^2k'}{4\pi^2} n_\pm(|k'|) \tilde{\nu}(|k' - k|) \sin^2 \left[ \frac{k' \times k^2}{2} \right]$$

where the superscript on $E^Z_\parallel$ reminds us it is the total energy including the Zeeman part, the Fermi functions
depend on the energies $E_\pm(k)$ and the chemical potential $\mu$. At each $T$, one must choose a $\mu$, solve for $E_\pm(k)$ till a self-consistent answer with the right total particle density $n$ is obtained. From this one may obtain the polarization by taking the difference of up and down densities. As usual we use the ZDS potential for which

$$\hat{\rho}(q) = e^{-q^2/2} \frac{2\pi e^{-q\lambda}}{q}.$$  \hspace{1cm} (154)

The computation of $1/T_1$ is more involved. The question we ask is the following. The fermions are in a quantum well, with their density varying across the width. So the nuclear relaxation rate will be a function of position. Consider a nucleus at the center of the quantum well, (as well as the $x-y$ plane) where the density is the largest. Let us call this point the origin and let $1/T_1$ be the relaxation rate there. The theory predicts

$$\frac{1}{T_1} = 4\pi k_B T \left( \frac{K_{\max}^\mu}{n} \right)^2 \times \int_0^\infty dE \left( \frac{d\hat{F}(E)}{dE} \right) \rho_+(E)\rho_-(E) F(k_+,k_-)$$

$$F = e^{-(k_+^2+k_-^2)/2} I_0(k_+ k_- l^2)$$

$$\rho_\pm(E) = \int \frac{dk}{2\pi} \delta(E - E_\pm(k))$$

where $E_0$ is the lowest possible energy for up spin fermions, and $K_{\max}^\mu$ is the measured maximum Knight shift for the fraction $\nu = 1/2$ or $1/4$.

Here is a rough description of the derivation, the details of which may be found in the Appendix 6. Suppose for a moment we were dealing with electrons and not CF’s. The Knight shift at the chosen point, the origin, will be determined by the spin density there. The same parameter enters the $1/T_1$ calculation quadratically. This is why $K_{\max}^\nu$ enters the answer. The idea is that $K_{\max}^\nu$ is not calculated ab initio but taken from the same experiment. The density of states and Fermi factor are standard. The only new feature here is the presence of $F(k_+,k_-)$ which reflects the fact that the spin density has to be projected into the LLL when going to the CF basis. The effect of this factor (which is none other than the $e^{-q^2/2}$ which appeared on the projected charge density) is to suppress processes with momenta much larger than $1/l$, as these have no place within the LLL.

A. Comparison to experiment

We now compare to some experiments at $\nu = 1/2$ and $T > 0$. Consider first Dementiev et al. From their data point $P = .75$ for $B_{\perp} = 5.52 T$ at 300 mK I deduce

$$\lambda = 1.75.$$  \hspace{1cm} (158)

I have once again chosen instead to match my HF results with the above data point, (which gave $\lambda = 1.75$) and see to what extent a sole parameter $\lambda$, can describe $P$ and $1/T_1$ for the given sample at a given $B_{\perp}$, but various temperatures and tilts.

Since there does not exist a model, including disorder, that describes how $\lambda$ should vary with tilt I include no such variation.

Dementiev et al find $K_{1/3}^{\max} = 4.856 \cdot 10^{-7} \sigma K$, which is believed to describe a saturated system at $P = 1$. They
estimate that $K_{1/2}^{\max} = 0.953 K_1^{\max}$, which is what we need here. It is assumed that the nuclear wavefunction $u_E(0)$ (defined in Eqn. (222) Appendix 6) does not vary between $\nu = 1/3$ and $\nu = 1/2$. Given this information, $1/T_1$ follows.

The top and bottom halves of Figure 9 compare the HF calculation of $1/T_1$ and $P$ respectively, to the data. (The graphs for $1/T_1$ differ slightly from those in Ref.22 since the present calculation treats the spin of the CF more carefully. The $1/T_1$ graph at 5.52 $T$ appears a little jagged since it was computed at just six points which were then connected. This is not apparent in the tilted case since the points lie on a straight line.)

Dementyev et al. had pointed out that a two parameter fit (using a mass $m$ and interaction $J$) led to disjoint pairs of values for these curves. Given that $H$ is neither free nor of the standard form ($p^2/2m + U(x)$) this is to be expected. By contrast, a single $\lambda$ is able to describe the data here rather well since $H$ has the right functional form. Given how the theory fits the polarization data up to the Fermi energy of $\approx 1^o K$, it is clear that changing the data point used to fix $\lambda$ will be inconsequential.

If $P$ were computed from the LDA value $\lambda \approx 1$, it would be down by 15-50 % as $T$ drops from 1$^o$K to 0$^o$K. The present work establishes a phenomenological, non-trivial and nonobvious fact that a single $\lambda$ parameter, (like $g$ or $\varepsilon$) determined from one data point, can describe both $P$ and $1/T_1$ for the given sample under a variety of conditions. That the fitted $\lambda$ is larger than the LDA value makes sense, as both disorder and LL mixing will lower the gap and raise $\lambda$. As mentioned earlier, it is not clear exactly how strong the disorder corrections are since there is LL mixing and some intrinsic errors in the LDA.

Consider next sample M280 of Melinte et al. which had $P = .76$ at .06$^o$K and $B = B_- = 7.1 T$, from which I deduced $\lambda = 1.6$. Figure 10 compares my $T$ -dependence with data. The initial rise of $P$ with temperature was also seen by Chakroborthy and Pietlian.3

Note that agreement is quite poor for the tilted case. There are confusing aspects of both the theory and an experiment. In the experiment, one may ask why the polarization does not increase with increased tilt and hence increased Zeeman coupling. Of course this will happen if $P = 1$ to begin with, but it is not, it is clearly below unity, say 80%. On the other hand, the theory for tilted fields in not in good shape either. First of all, orbital effects have to be considered due to the tilt. The thickness parameter $\Lambda$ can be affected by it. As pointed out by Jungwirth in a private communication, once there is an in-plane component of $B$, the problem is no longer rotationally invariant. This means that our states are no longer HF states and get scattered into each other by the potential. Presumably these effects were negligible in the case of Dementyev et al. but not at nearly twice the tilt of 61°. No attempt is made here to take into account all the effects of the tilt. Instead I include just the increased Zeeman coupling and hope for the best.

For the benefit of others who measure $1/T_1$ at $\nu = 1/2$ in the future on similar samples, I give some very approximate formulae (to be used for zero or small tilts). From Figure 9, we note that in general, the graphs of $1/T_1$ become linear and parallel for temperatures above .3 $^o$K. In this region we can write

$$\frac{d(1/T_1)}{d\nu} \equiv 3 \frac{\bar{K}}{\bar{n}} \cdot 10^{-3} s^{-1} \langle 0 K \rangle^{-1} \quad \text{for } T > .3^o K$$

(159)

with $\bar{K}$ the Knight shift in KHz and

$$\bar{n} = \frac{n}{10^{10}/cm^2} \quad (160)$$

(In this approximate formula, I ignore the $\lambda$ dependence of Eqns. (155)-(157), and the distinction between the average and maximum Knight shift.)

The graphs do not generally obey the Korringa-like law because as $T \to 0$ they are sublinear (superlinear) for saturated (unsaturated) cases. Only the critical case with $P(0) \to 1$ as $T \to 0$ is linear. For $T > .3^o K$, (which in general must be replaced by either the energy gap or energy overlap between the up and down Fermi energies)

$$\frac{1}{T_1} = \left[ 3 T(\alpha K) + C \right] \left[ \frac{\bar{K}}{\bar{n}} \right]^2 \cdot 10^{-3} s^{-1} \quad (161)$$

$$C = 0 \quad \text{(critical)}$$

$$= 0 \quad \text{(unsaturated)}$$

$$= < 0 \quad \text{(saturated)} \quad (164)$$

For the critical case (only), we have Korringa law

$$\frac{1}{T_1} = \left[ 3 T(\alpha K) \right] \left[ \frac{\bar{K}}{\bar{n}} \right]^2 \cdot 10^{-3} s^{-1} (\alpha K)^{-1} \quad (165)$$

For Dementyev et al. $C \approx 1$. This value may be used as a first approximation, for example, if Melinte et al. measure $1/T_1$ on sample M280. For more accurate results they must solve Eqns. (155)-(157) with $\lambda = 1.6$.

The present formalism has been applied by G. Murthy to calculate the $T$ -dependence of polarization in the 1/3 and 2/5 states.
VII. CONCLUSION

This paper described in detail a formalism for describing FQHE states, in which the hamiltonian and various operators were expressed in terms of the quasiparticles, the Composite Fermions. It was shown that the formalism could be used to calculate a variety of quantities at zero and nonzero temperatures to some reasonable accuracy, and to resolve matters of principle (some of which could not even be posed otherwise), such as the internal structure of the CF, the reason it sometimes appears to be free when it is not, and how theory is to be compared to experiment in the quantum Hall problem.

First a review of the wavefunction approach to CF was given. It was seen that whereas in the Laughlin fractions, CF was evidently an electron bound to 2s vortices, the situation was more complicated for the Jain series due to the action of the LLL projector \( P \). This caused the vortices to move off the electrons and also got rid of many of them. It was not clear how the vortices paired off with the electrons, say at \( \nu = 2/5 \), when there were 1.5 (non-Pauli) vortices per electron in the projected state. Since the charge of the CF was the same (\( e^* = 1/(2ps + 1) \)) before and after projection, whatever paired with the electron still had the charge of 2s vortices, though it could not be associated with zeros of the wavefunction in any simple way. What is this object and how is one to incorporate it in the theory?

The hamiltonian theory was seen to provide the answer. In this theory we enlarged the Hilbert space to include additional degrees of freedom, accompanied by constraints. These new degrees of freedom (prevented from having any density fluctuations by constraints) turned out to have charge \( e_v = -2ps/(2ps + 1) \) and paired with electrons. They were referred to as vortices, for want of a better name, but in view of what was said above, they are not related in any simple way to zeros of \( \Psi_{LLL} \), the electronic wavefunction in the LLL.

To solve the theory with constraints, the HF approximation was used. Free particle and hole states of CF’s in a reduced field \( B' = B/(2ps + 1) \) were seen to be HF states. All matrix elements were evaluated in closed form and usually a single numerical integral gave the numbers. As for the constraints, the method proposed with Murthy was used: the charge density operator was written as a judicious combination of the transformed and LLL projected electronic charge density and the constraint, which is just the vortex charge density, the judicious combination being the unique one (as \( q \to 0 \)) that obeyed Kohn’s theorem. When this combination was expanded in \( q \), the monopole term was \( e^* \) and the next term corresponded to a dipole moment \( \mathbf{r} \times \mathbf{k} \). The operator approach thus gave a precise meaning to the CF dipole moment, in terms of how it coupled to an external potential. This coupling did not change when second quantized with Fermi fields. Since this procedure took into account the most important effects of the constraints (away from \( T = 0 \) and ultralow frequencies \( \omega \sim q^3 \)), constraints were neglected thereafter.

In Section IV activation gaps for fully polarized states were calculated for the Zhang Das Sarma (ZDS) potential as a function of the parameter \( \lambda \). Analytic expression were derived for all fractions of the form \( p/(2ps + 1) \) and compared to the Monte-Carlo work of PMJ based on trial wavefunctions for \( s = 1, p = 1, 2, 3, 4 \). It was found that the numbers were within 10-20% for \( \lambda > 1 \).

A comparison was made to experiments of Du et al. Rather than try to compute \( \lambda \) ab initio, it was fitted to the three gaps at 1/3, 2/5, 4/3/7 and was seen to take the values 2.4, 2.4, and 2.0.

It was found that normalized activation masses (scaled by \( 1/\sqrt{B} \)) were not too different for \( s = 1, s = 2 \), i.e., near \( \nu = 1/2 \) and \( \nu = 1/4 \) in reasonable accord with the experiments of Pan et al. It was pointed out that no deep reasons existed for their exact equality.

For the gaussian potential the numbers were in excellent agreement with PMJ except for \( \nu = 1/3 \). In was clear that although the extended theory was defined for all length scales, it gave good numbers only for soft potentials, ones that were smooth within a magnetic length.

A comparison to the exact diagonalization work of Morf et al was made for fractions 2/5 and 3/7. The HF results of this theory lay consistently above their numbers. The differences were somewhat larger than in the case of PMJ.

The hamiltonian formalism we can make precise the question of whether or not CF are interacting since \( H \) naturally separates into a free and interacting parts \( H_0 \) and \( H_I \). It was seen that turning off \( H_I \) made a sizeable difference to activation gaps.

The theory was used to compute the profiles of charge densities in states with a single particle or hole and compared to the unpublished work of Park and Jain. It was seen that while the salient features were reproduced with ups and downs at the right places, the amplitudes were not as pronounced. The best fraction was 2/5, and not surprisingly, this was also the one where the gaps came out best. The case \( \nu = 1/2 \) and the dipole moment of the CF were analyzed in some depth.

It was found that the activation gaps for the fully polarized states obeyed particle-hole symmetry to an excellent approximation. This was a nontrivial result since \( \nu \) naturally separates into a free and interacting parts \( H_0 \) and \( H_I \). It was seen that turning off \( H_I \) made a sizeable difference to activation gaps.

Polarization phenomena at \( T = 0 \) were the subject of Section V. By varying the density and field at fixed filling, or by placing the sample in a tilted field it is possible to increase the Zeeman coupling and drive the system though many magnetic transitions. For the fraction \( p/(2ps + 1) \), CF theory has states with \( p - r \) LL of spin up and \( r \) levels with spin down, \( r = 0 \) being the fully polarized case. Critical fields \( B_c \) at which the system would jump from one value of \( r \) to the next were computed. It was found, as Park and Jain did, that it was possible to fit all the numbers very well by assuming that CF were free and occupied LL with a polarization gap \( \Delta_p \) or a corresponding polarization mass \( m_p \). Yet we know CF cannot be free, given that the activation mass and gap are substantially different from these values. It was shown that rotational invariance and \( d = 2 \) conspired to
create this impression of free fermions. In particular, for the gapless case, while the individual HF energies of spin up and down fermions had sizeable $k^4$ terms in the formula for $E_\pm(k)$, the energy cost of transferring a fermion from the top of one sea to the top of the other equaled that of free fermions with a quadratic dispersion $k^2/2m_\nu$.

The theory was compared to the experiments of Kukushkin et al. who varied both $n$ and $B_z$ to drive the magnetic transitions. Some plausibility arguments were given for why an effective, translationally invariant potential might be able to describe magnetic phenomena under restricted conditions. It however used to describe all magnetic phenomena: an effective $\lambda$ was deduced from one data point (here one transition), and used to predict all the others, using scaling laws. The numbers agreed to within $10 - 20\%$. While the arguments for an effective potential description were trustworthy only in a limited region, the procedure seemed to work in a wider region, including gapless states and $T > 0$. This deserves to be understood.

The most important results of this theory, not limited by finite size, involve the computation of relaxation rates and polarizations as a function of temperature. These were discussed in Section VI. Finite $T$ HF equations were derive analytically and solved numerically to yield these quantities. They were compared to the work of Dementyev et al and Melinte et al. In the first case, the measured polarization at 300 mK for a perpendicular field of 5.52T was used to fix $\lambda = 1.75$. Using this value the polarization and relaxation rate $1/T_1$ were computed for a range of temperatures going up to $2^\circ$K, and $1^\circ$K respectively, where the latter is roughly the Fermi temperature. The agreement was very satisfactory. For a tilt of $38.3^\circ$, the agreement was again good till about $1^\circ$K. In their paper Dementyev et al had pointed out that fitting these four data sets to a model with a mass term $m$ and Stoner enhancement $J$ lead to four disjoint islands in the parameter space. In the present case a single $\lambda = 1.75$ seemed to describe all four sets quite well. It was claimed this was because the $H$ used was of the right functional form, with an unusual kinetic and interaction terms chock full of momenta and currents. For any one graph it may be mimicked by a hamiltonian of the $(n, J)$ form, but these numbers would then vary from set to set. In particular, to get the right relaxation rates, with just one parameter, $\lambda$, one needs the unusual dispersion relations (far from quadratic in momenta) and the corresponding unusual density of states that arise naturally here.

As for the polarization data of Melinte et al, there was reasonable agreement for the untitled sample, but not the tilted (by over $60^\circ$) case. The latter could be attributed to theory, which is not designed to handle such large tilts, or to the data which exhibit some unusual features discussed in the text, or both.

An approximate Korringa-like law for states that are just fully polarized at $T = 0$ and the slope of $1/T_1$ versus $T$ for the general case for $T > 3^\circ$K were provided to allow future experimenters (working on samples similar to that used by Dementyev et al) to make a quick and approximate comparison to this theory without having to solve the finite-T HF equations.

In summary it was shown that the hamiltonian theory of CF provides a comprehensive and analytical scheme for describing the low energy physics of the FQHE states, clarifying concepts regarding the internal structure of CF’s and how they are to be coupled to external potentials, and computing a variety of quantities at zero and nonzero temperatures to an accuracy of $10 - 20\%$ under typical conditions, and sometimes considerably better. The deviations could be due to the HF approximation or to the hamiltonian itself. In comparing to magnetic experiments, if one extracted a single parameter $\lambda$ that charaterized electron-electron interaction in a given sample from one data point, the theory gave a reasonable account of other data from that sample.

A. General philosophy

I conclude with some remarks that put the present approach in perspective and underscore the differences between the FQHE problem and others. Let us begin with the fact that in the FQHE restricted to the LLL, and in the absence of disorder, every physical quantity is a functional of the electrostatic interaction between electrons, since the kinetic energy is quenched. I will address the important question of disorder shortly. If this interaction is pure coulomb, it is a zero-parameter theory. If it is modeled by the ZDS potential as done here, all observables – $m_\sigma, m_\pi, B_\perp, P(T), 1/T_1$ – are functions of $\lambda$. Once a single data point is known, $\lambda$ may be determined, and from it, all others calculated (using scaling laws if needed). While I chose the ZDS potential to make this point, one could use another, say the one used by Morf et al. The the numbers in question are predominantly controlled by one of Haldane’s $V$’s, and these are different ways of varying it.

Why do we not do this all the time? Consider for example a Fermi liquid. Why do we bother with Landau’s $F$ functions? Why do we not start with the coulomb interaction and calculate everything with zero parameters? Or if we wanted to model it with a short range force, why don’t we employ the Hubbard model, calculate any one observable, fit it to data, extract the Hubbard $U$ and predict everything else? The reason is well known: we do not know how to go from the model to the physical quantities except in perturbation theory. Let us restate this in Landau’s language. Suppose we turn off $U$. The states, at least the low lying ones, are labeled by free field theory. We now turn on the interaction and the states and energies evolve to those of the interacting theory. We finally should end up with the quasiparticle basis, at least near the Fermi surface. Unfortunately this change of basis from free to interacting theory cannot be carried out in practice except perturbatively.

The same thing happens in S-matrix theory, which appears to have guided Landau’s intuition. The in and out states carry free particle labels and their dot product gives the elements of the $S$-matrix. They are obtained by
taking particles that are infinitely separated (and non-interacting) in the distant past or future and evolving them (forwards or backwards) in time to \( t = 0 \). Unfortunately this evolution too can be done only perturbatively.

In the FQHE problem the situation is a lot better, though it looks bad to start with. The usual idea of starting with free electrons and turning on interactions is doomed from the start since for \( \nu < 1 \), the free electrons do not have a unique ground state. But lurking in the background are again some free states or wavefunctions. These are the CF states with \( p \) filled LL and some low energy excitations. They get mapped to the electronic wavefunctions in Jain’s approach by a transformation that consists of attaching \( 2s \)-fold vortices with the Jastrow factor \( J(2s) \) and projecting with \( \mathcal{P} \). The resulting wavefunctions are known to be excellent, with nearly unit overlap with the results of exact diagonalization. If we know the electron-electron potential (in the absence of disorder) we can calculate anything related to the ground state and low energy excitations using these. In the hamiltonian approach we go the other way: from electrons to CF. We first do \( 2s \)-fold flux attachment by the CS transformations and this becomes vortex attachment following the additional transformations Murthy and I developed. The states not only carry free labels, they are simply free (in the HF calculations); it is the operators that become complicated. They are however simple enough to work with analytically, and lead to less accurate numbers than Jain’s approach.

All this is possible because of the one major difference compared to the Fermi liquid and the S-matrix problems: the transformation here is discrete and given by the attachment of \( 2s \) vortices (before projection, which leads to a complicated wavefunction, but is fully implementable). The CF know only about \( p \), the transformation about \( 2s \), and the electrons know about \( p \) and \( 2s \). It is this discreteness, (absent in the Fermi liquid or S-matrix problem), that eventually works in our favor, that sufficiently constrain the problem and lead us to the right answer. For example, in Jain’s approach analyticity and Fermi statistics fix the vortex number to be \( 2s \) and in the hamiltonian approach the same is true as well, and additionally, the LLL algebra pointed to the exponentiation which was the nonperturbative step that had to exist in any transformation linking electron to CF.

There is a mistaken belief that CF have to be free to be useful. Note that Jain is able to calculate a slew of quantities with his wavefunctions and to a lesser accuracy, so can the hamiltonian method. Neither method requires CF to be free. What is free about the CF is the label for the states. While this is true in S-matrix theory and in Landau theory, what is special here, as explained above, is that the transformation from the free states to the interacting counterparts is known to a great accuracy.

I believe this happy situation, of being able to go from the free to the interacting states in the FQHE, deserves further exploitation in comparing to experiment. If there were no disorder, what we should try to extract from the data is the electron-electron interaction, \( v(q) \), which is a c-number function that multiplies the operator quadratic in \( \rho^p \) and defines \( H \). By extracting \( v(q) \) I mean some parameter like \( \lambda \) of the ZDS potential or \( b \) in the Morf et al case. As mentioned earlier, the physics is dominated by one Haldane pseudopotential, \( V_1 \) for \( 2s = 2 \) and these parameters just control it. It does not seem fruitful to fit the complicated CF hamiltonian to a string of standard or nonstandard operators. The hamiltonian for the CF’s is not of the canonical form with a quadratic kinetic energy and some interaction terms of the usual (density-density) form. There is no reason such should be the case, given the internal structure of CF’s. Rather, the free and interacting parts of the CF hamiltonian form a monolith, of unusual functional form, fully determined by the density operator written in the CF basis. While it is possible to fit any one experiment with one set of standard interaction parameters, (like \( m \) and \( J \) that DeMentev ef al used) the fact that the functional form is wrong will manifest itself, as they found out, in the need for many disjoint sets of parameter for different measurements. (This is analogous to the fact that if we try to fit the kinetic energy of a relativistic particle as a function of its velocity to a nonrelativistic form, we will need a velocity dependent mass, while if we fit it to the correct form we will extract a fixed rest mass.) Likewise, if one extracts \( \lambda \) (or its counterpart for another potential) from the data, a single simple result is more likely to emerge. This \( \lambda \) will of course have the usual variance of 10–20% characteristic of this theory.

Now it is time to wake up and smell the disorder. Does it completely destroy the approach presented here? Is it possible that even the disordered system can be modeled by a pure potential with some effective \( \lambda \) or \( b \)? The answer is clearly negative if one wants to describe everything. For example no translationally invariant interaction will predict localization. As for activation gaps, the data suggest that disorder effects are quite strong: the disorder broadening (measured by the negative intercept of the gaps versus effective field \( B^* \)) is comparable to the transport gaps. As \( B^* \to 0 \), the pure system gaps never vanish while in experiments they do, implying no effective potential exists. If at some future date, we get samples with even lower disorder, we may be able describe these gaps with an effective potential, even for small \( B_{\text{eff}} \).

For the present, for magnetic transitions at \( T = 0 \), which depend only on total energies, I have tried to ask if an effective potential exists which can subsume the effects of disorder. I gave crude arguments that showed that in the limit of weak and smooth disorder one could get the disorder averaged ground state energy to second order by replacing \( v(q) \) with a \( v_{\text{eff}}(q) \). While \( v_{\text{eff}} \) did not have the same functional form as \( v \), it was always weaker, which meant the effective \( \lambda \) (which produced the corresponding dominant pseudopotential \( V \)) is larger. I have explored the possibility, of determining an effective \( \lambda \) from one data point and using it (combined with a scaling law if necessary) to predict the other data on that sample, for all magnetic phenomena. While the plausibility arguments were given for \( T = 0 \) and that too under a variety of restrictions, I have tried it for all magnetic phenomena. What I find that if one is prepared to work with a 10–
20% theory, such a program actually works. If such an effective potential is found from static or equal time data, it can be imported to the Hamiltonian scheme to do other things such as \( \omega \) dependent response functions and \( T > 0 \) physics.

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APPENDIX

1. Matrix elements

Many of the calculations performed in this paper deal with the preferred density \( \tilde{\rho}^p \). In second quantization we write it as

\[
\tilde{\rho}^p \left( \mathbf{q} \right) = \sum_{m_2 n_2; m_1 n_1} d^\dagger_{m_2 n_2} d_{m_1 n_1} \rho_{m_2 n_2; m_1 n_1}
\]

where \( d^\dagger_{m_2 n_2} \) creates a particle in the state \( |m_2 n_2\rangle \) where \( m \) is the angular momentum and \( n \) is the LL index. They are related to the CF cyclotron and guiding center coordinates, \( \mathbf{R} \) and \( \eta \) as follows. Let

\[
\mathbf{b} = \frac{\mathbf{R}_x - i \mathbf{R}_y}{\sqrt{2l^*}} \quad \mathbf{b}^\dagger = \frac{\mathbf{R}_x + i \mathbf{R}_y}{\sqrt{2l^*}}
\]

where \( l^* = l/\sqrt{1 - c^2} \) is the CF magnetic length. These obey the oscillator algebra

\[
\left[ \mathbf{b}, \mathbf{b}^\dagger \right] = 1 \quad (168)
\]

given

\[
\left[ \mathbf{R}_x, \mathbf{R}_y \right] = -i l^* \text{.} \quad (169)
\]

Similarly we define, in terms of the cyclotron coordinates,

\[
a = \frac{\eta_x + i \eta_y}{\sqrt{2l^*}} \quad a^\dagger = \frac{\eta_x - i \eta_y}{\sqrt{2l^*}}
\]

which obey the oscillator algebra

\[
\left[ a, a^\dagger \right] = 1 \quad (171)
\]

given

\[
\left[ \eta_x, \eta_y \right] = i l^* \text{.} \quad (172)
\]

The states \( |mn\rangle \) are just the tensor products

\[
|mn\rangle = \left( \frac{b^\dagger}{{\sqrt{m!}}} \right)^m \left( \frac{a^\dagger}{{\sqrt{n!}}} \right)^n |0\rangle
\]

where \( |0\rangle \) is annihilated by both \( a \) and \( b \).

I will now evaluate matrix elements of \( e^{-i q \mathbf{R}} \text{ and } e^{-i q \eta} \) and show that

\[
\langle m_2 | e^{-i q \mathbf{R}} | m_1 \rangle = \langle m_2 | e^{-i q \eta} \rangle |m_1\rangle \quad (173)
\]

\[
\langle m_2 | e^{-i q \mathbf{R}} \rangle = \sum_{m_1 = 0}^\infty \frac{m_2!}{m_1!} \langle m_2 | e^{-i q \eta} \rangle |m_1\rangle \quad (174)
\]

where \( x = q^2 l^*^2 / 2 \), \( q_z = q_x + i q_y \).

\( L \) is the associated Laguerre polynomial, and \( m_1 \geq m_2 \).

If \( m_1 < m_2 \) one may invoke the relation

\[
\langle m_2 | e^{-i q \mathbf{R}} | m_1 \rangle = \langle m_1 | e^{-i q \eta} | m_2 \rangle^* \quad (176)
\]

To establish Eqn. \( (174) \), consider the coherent states

\[
|z\rangle = e^{l^* z} |0\rangle = \sum_{m = 0}^\infty \langle m \rangle \frac{|m\rangle}{\sqrt{m!}} z^m \quad (177)
\]

with the inner product

\[
\langle \tilde{z} | z \rangle = e^{\tilde{z}^2 z^2} \quad (178)
\]

First we write from the definitions given above

\[
\langle \tilde{z} | e^{-i q \mathbf{R}} | z \rangle = \sum_{m_1 = 0}^\infty \sum_{m_2 = 0}^\infty \tilde{z}^{m_2} z^{m_1} \langle m_2 | e^{-i q \eta} \rangle |m_1\rangle \quad (179)
\]

\[
\equiv R(\tilde{z}, z, q) \quad (180)
\]

On the other hand

\[
\langle \tilde{z} | e^{-i q \mathbf{R}} | z \rangle = \langle \tilde{z} | \exp \left( \frac{-i l^*}{\sqrt{2}} (q_+ b^\dagger + q_- b) \right) | z \rangle \quad (181)
\]

\[
= \langle \tilde{z} - \frac{i l^*}{\sqrt{2}} q_+ z - \frac{i l^*}{\sqrt{2}} q_- (z) e^{q^2 l^*^2 / 4} \rangle \quad (182)
\]

\[
= \exp \left[ i \tilde{z} q_+ + q_+ z \right] e^{-q^2 l^*^2 / 4} \quad (183)
\]

\[
\equiv R(\tilde{z}, z, q) \quad (184)
\]

Comparing Eqns. \( (179-183) \) and matching powers of \( \tilde{z}^a z^b \), we obtain Eqn. \( (174) \) if we recall
$$L_{m_2}^{n_1-m_2}(x) = \sum_{t=0}^{m_2} \frac{m_1!}{(m_2-t)!(m_1-m_2+t)!} \frac{(-1)^t}{t!} x^t$$

(185)

To establish

$$\langle n_2 | e^{-i\mathbf{q} \cdot \mathbf{\eta}} | n_1 \rangle = \sqrt{\frac{2^l}{n_1!}} e^{-x/2} \left( \frac{-i\mathbf{q} \cdot \mathbf{l}^*}{\sqrt{2}} \right)^{n_1-n_2} L_{n_2}^{n_1-n_2}(x)$$

(186)

(again for \( n_1 \geq n_2 \)) we just need to remember that the commutation rules of the components of \( \mathbf{\eta} \) have a minus sign relative to those of \( \mathbf{R} \), which exchanges the roles of creation and destruction operators and hence \( q^+ \) and \( q^- \).

Now we consider matrix elements of \( \vec{\rho}, \vec{\chi}, \vec{\eta} \). As a first step, let us express the operators \( \mathbf{R}_c \) and \( \mathbf{R}_s \) in terms of CF guiding center and vortex coordinates \( \mathbf{R} \) and \( \mathbf{\eta} \). We have seen that in the CF representation

$$\mathbf{R}_c = \mathbf{r} - \frac{e}{c} \hat{\mathbf{z}} \times \mathbf{\Pi}^*$$

(187)

if we recall \( l^2 = l^s(1 - c^2) \).

It can similarly be shown that

$$\mathbf{\eta}_c = \mathbf{r} + \frac{e}{c} \hat{\mathbf{z}} \times \mathbf{\Pi}$$

(188)

Thus in first quantization

$$\vec{\rho} = \vec{\rho}^c - \frac{e^2}{c} \hat{\mathbf{z}} \times \mathbf{\Pi}$$

(189)

$$\vec{\rho} = \sum_i \exp(-i\mathbf{q} \cdot \mathbf{R}_i) \exp(-i\mathbf{q} \cdot \mathbf{\eta}_i c)$$

(190)

$$\vec{\chi} = \sum_i \exp(-i\mathbf{q} \cdot \mathbf{R}_i) \exp(-i\mathbf{q} \cdot \mathbf{\eta}_i c)$$

(191)

$$f = \exp \left( \frac{-q^2 l^2}{8 ps(2ps + 1)} \right)$$

(Vortex form factor)

(192)

Armed with Eqns. (174 and 186) we may finally write

$$\vec{\rho}_{m_2 n_2; m_1 n_1} =$$

$$\sqrt{\frac{m_1!}{m_2!}} e^{-x/2} \left( \frac{-i\mathbf{q} \cdot \mathbf{l}^*}{\sqrt{2}} \right)^{m_1-m_2} L_{m_2}^{m_1-m_2}(x)$$

$$\bigotimes \left[ \sqrt{\frac{n_2!}{n_1!}} \left( \frac{-i\mathbf{q} \cdot \mathbf{l}^*}{\sqrt{2}} \right)^{n_1-n_2} e^{-xc^2/2} L_{n_2}^{n_1-n_2}(xc^2) \right]$$

$$-c^2 \cdot f, \left( \frac{-i\mathbf{q} \cdot \mathbf{l}^*}{2c^2} \right)^{n_1-n_2} e^{-x/2c^2} L_{n_2}^{n_1-n_2}(x/c^2)$$

$$\equiv \rho_{m_2 n_2}^m \otimes \rho_{n_2 n_1}^n$$

Superscripts on \( \rho_{m_2 n_1}^m \) and \( \rho_{n_2 n_1}^n \), which will be apparent from the subscripts, will usually be suppressed.

2. Proof of Hartree-Fock nature of trial states

Consider

$$\langle f | H | i \rangle = \langle p | d_f H d_i | p \rangle$$

(193)

where \( | p \rangle \) stands for the (ground) state with \( p \)-filled LL, and \( i, f \) label single-particle excitations on top of this ground state. We want to show that this matrix element vanishes if \( i \neq f \), i.e., the hamiltonian does not mix these putative HF particle states. (This result was established for the small \( q \) theory by G. Murthy.) The proof, which relies on just the rotational invariance of the potential, applies with trivial modifications to the hole states, i.e., to

$$\langle p | d_f H d_i | p \rangle.$$  

(194)

This matrix element in question takes the schematic form

$$\langle f | H | i \rangle = \int_{\rho_1 l_2 \delta_1 l_3 \delta_1 \delta_4 l_4} \rho_{l_2}(q) \rho_{l_4}(-q)$$

(195)

where \( 1 \) stands for \( m_1 n_1 \) and so on, and \( \int_q \) stands for an integral over a rotationally invariant measure:

$$\int_q = \frac{1}{2} \int \frac{d^2 q}{4\pi^2} \rho(q)e^{-q^2 i / 2}.$$  

(196)

Now we use Wick’s theorem and perform pairwise contractions on the vacuum expectation value, bearing mind that

- We cannot contract the indices 1 and 2 or 3 and 4 since this will require that \( q = 0 \) at which point the measure (which contains the potential) vanishes.
- If we contract \( i \) and \( f \) we already have the desired result.

Here is a representative of the contractions we can get:

$$\rho_{l_2} \delta_{l_1}(1-n^F_1) \delta_{l_3}(1-n_4^F) \delta_{l_4}(1-n^F_4).$$

(197)

where \( n^F_1 \) is the Fermi function for the LL labeled by \( n_1 \)

$$n_1^F = \theta(p - 1 - n_1)$$

(198)

and so on. Since \( f = 1 \) and \( i = 4 \), the \((1-n^F_2)(1-n^F_4) = 1 \). The integrand assumes the form

$$\sum_{m_2 = 0}^\infty \sum_{n_2 = p}^\infty \rho_{l_2}(q) \rho_{l_4}(-q) =$$

$$\left[ \sum_{m_2 = 0}^\infty \rho_{m_2 l_2}(q) \rho_{m_2 l_4}(-q) \right] \left[ \sum_{n_2 = p}^\infty \rho_{n_2 l_2}(q) \rho_{n_2 l_4}(-q) \right]$$

$$= \delta_{m_2 n_2} \sum_{n_2} q^{n_2-n_f} F(|q|)$$

where we have also used the fact that \( e^{-i\mathbf{q} \cdot \mathbf{R}}, e^{-i\mathbf{q} \cdot \mathbf{R}} = I \) in doing the sum over \( m_2 \), and \( F(|q|) \) is some rotationally invariant function. It follows that every term in the sum over \( n_2 \) vanishes unless \( n_f = n_i \) due to the angular integral in \( q \).
3. Particle-hole profiles

Here we deal with both gapped and gapless states.

a. Gapped case

Let us create a particle in the \( p + 1 \)-th LL and ask what the corresponding charge density looks like. In \( q \) space this is given by

\[
\langle \tilde{\rho}^p(q) \rangle = \sum_{1,2} \langle d^\dagger_{\mu} d_{\mu} \rangle \rho_{12}(q) e^{-q^2 l^2 / 4} \quad (199)
\]

where \( \mu \) labels both \( n = p \) and \( m = 0 \) of the created particle, and 1 and 2 stand for the double labels summed over in the definition of \( \tilde{\rho}^p \).

Now we do the contractions bearing in mind that we should not contract 1 with 2 since this gives the background charge. This gives

\[
\langle \tilde{\rho}^p(q) \rangle = \sum_{1,2} \langle d^\dagger_{\mu} d_{\mu} \rangle \rho_{12} \quad (200)
\]

\[
= \sum_{1,2} \delta_{\mu 1} (1 - n^F_1) \delta_{\mu 2} (1 - n^F_2) \rho_{12} \quad (201)
\]

\[
= \rho_{\mu \mu} \quad (202)
\]

\[
= \langle e^{-iq \cdot R} \rangle_{00} \left( e^{-iq \cdot \eta c} - e^{i q \cdot \eta c} \right)_{pp} \quad (203)
\]

\[
= e^{-q^2 l^2 / 4} \left( e^{q^2 c^2 / 4 L_p} \left( \frac{q^2 l^2 c^2}{2} \right) \right) \quad (204)
\]

\[
- e^{-q^2 l^2 / 4} e^{-q^2 l^2 / 4 L_p} \left( \frac{q^2 l^2 c^2}{2} \right) \quad (205)
\]

It is straightforward to go to real space by Fourier transform. The figures depict the charge so obtained plus the background charge \( \nu / (2 \pi l^2) \) in units of the background charge. The result is

\[
\frac{\tilde{\rho}^p(r/l)}{(\nu / 2 \pi l^2)} = 1 + \frac{1}{\nu} \int_0^\infty y dy e^{-y^2 / 4} J_0(y r/l) F(y) \quad (206)
\]

where

\[
F(y) = e^{-y^2 (2p + 1) / 4} \left( e^{-y^2 c^2 (2p + 1) / 4 L_p} \left( \frac{y^2 c^2 (2p + 1)}{2} \right) \right)
\]

\[
- e^{-y^2 (2p + 1) / 4c^2 L_p} \left( \frac{y^2 (2p + 1)}{2c^2} \right) \]

The computation of the hole charge is analogous. To facilitate comparison to Park and Jain I have placed the particle and hole at antipodal points on a sphere of the same radius as the one they used.

b. The CF at \( \nu = 1/2 \)

As in the gapped case, we now place a fermion just above the Fermi sea. As explained in the main body of the text, we need a superposition of states of fixed \( p_y = p_0 \) and a sum over all values of \( p_x \) to localize the CF at \( x = 0 \). Since the dipole always has a nonzero \( x \)-component in this superposition, it does not get washed out.

For this case, with \( A^+ = 0 \), the LL expression for charge simplify greatly:

\[
\tilde{\rho}^p(q) = \int d^2 k / (4\pi l^2) \left( \frac{q \cdot k l^2}{2} \right) \]

\[
d^4 q \quad (207)
\]

In other words, the pair of labels \( (m, n) \) is replaced by a momentum vector \( q \). The density operator (for each particle)

\[
\tilde{\rho}^p(q) = e^{-i q \cdot r + q \cdot p / l^2} - e^{-i q \cdot r - q \cdot p / l^2} \quad (208)
\]

only connects states differing by \( q \). Carrying out the Wick contractions we get to the following result (dropping irrelevant constants)

\[
\langle \rho(r, y) \rangle \approx \int dp_x dq \ e^{-q^2 l^2 / 4} \left( e^{-i q_x p_0 l^2 / 2} + e^{i q_x p_0 l^2 / 2} \right)
\]

\[
\approx \exp \left[ \left[ \frac{x - p_0 l^2 / l^2}{2} \right]^2 \right] - [p_0 \rightarrow -p_0] \quad (209)
\]

Similar methods may be employed to calculate the structure factor \( S(q) \).

4. Activation gaps

Now we need to find the energy cost of producing a widely separated particle-hole (PH) pair. This will be done by evaluating

\[
\Delta_a = \langle p + p | H(p + P) + (p + H)| p + P \rangle - 2 \langle p | H | p \rangle
\]

\[
= \int_q E(P) + E(H). \quad (210)
\]

where \( P \) denotes a particle added to the state labeled \( \mu = (n = p, m = 0) \) and \( H \) denotes a state in which a hole has been made in the state \( \mu = (n = p - 1, m = 0) \). Let us consider

\[
E(P) = \langle p | d^\dagger_{\mu} d_{\mu} d^\dagger_{\nu} d_{\nu} | p \rangle \rho_{12} \rho_{34} \quad (212)
\]

In performing the contractions we

- Do not make any contractions within \( H \). This gets rid of \( E_0 = \langle p | H | p \rangle \), the ground state energy.
- Do not contract 1 with 2 or 3 with 4 since \( \nu(0) = 0 \).

We end up with

\[
\int_q \left[ \delta_{\mu 1} \delta_{\nu 2} \delta_{\mu 3} \delta_{\nu 4} (1 - n^F_1)(1 - n^F_2)(1 - n^F_3) \rho_{12} \rho_{34}
\]

\[
- \delta_{\mu 3} \delta_{\nu 2} \delta_{\mu 1} \delta_{\nu 4} (1 - n^F_3)(1 - n^F_2)(1 - n^F_4) \rho_{12} \rho_{34}
\]
Since $4 = \mu = 1$ in the first term, we can drop $(1 - n_{F}^{1})(1 - n_{F}^{2})$ and for similar reasons $(1 - n_{F}^{1})(1 - n_{F}^{2})$ in the second giving us

$$E(P) = \sum_{m_{2}=0}^{\infty} \sum_{n_{2}=0}^{\infty} \rho_{\mu 2}(q)\rho_{2\mu}(-q) - \sum_{m_{2}=0}^{\infty} \sum_{n_{2}=0}^{p-1} \rho_{\mu 2}(q)\rho_{\mu 2}(-q). \quad (214)$$

Since the sum over $m_{2}$ is unrestricted, we can use completeness and $e^{-i\mathbf{q}\cdot\mathbf{r}}e^{-i\mathbf{q}\cdot\mathbf{R}} = I$ to get rid of the $m$-index altogether. Thus we end up with

$$E(P) = \left(\sum_{n=p}^{\infty} |\rho_{pn}|^2 - \sum_{n=0}^{p-1} |\rho_{pn}|^2\right)$$

$$= \left[ \langle p|\rho(q)\rho(-q)|p\rangle - 2 \sum_{n=0}^{p-1} |\rho_{pn}|^2 \right] \quad (215)$$

A similar calculation for the hole state gives (upon dropping the ground state energy as usual)

$$E(H) = \left[ -\langle p-1|\rho(q)\rho(-q)|p-1\rangle + 2 \sum_{n=0}^{p-1} |\rho_{p-1,n}|^2 \right]$$

$$\text{where}$$

$$\langle n|\rho(q)\rho(-q)|n\rangle = \sum_{n'=0}^{\infty} |\rho(q)_{nn'}|^2. \quad (218)$$

Putting all the pieces together, and recalling the various matrix elements, we obtain Eqs. (69, 71).

5. Critical fields for magnetic transitions

We need to calculate

$$E(p - r, r) = \langle p - r, r|H| p - r, r \rangle \quad (219)$$

the energy in a state with $p - r$ spin-up LL’s and $r$ spin-down LL’s. Since the HF calculation for the spinless case is very similar, this treatment will be brief. We write

$$H = \sum_{1234} \int_{q} d_{1}d_{2}d_{3}d_{4}\rho_{12}\rho_{34} \quad (220)$$

with the understanding that a label like 1 stands for the triplet $(n_{1}, m_{1}, s_{1})$, $s$ being the spin. The matrix elements $\rho_{ij}$ are defined by

$$\rho_{ij} = \langle 1|e^{-i\mathbf{q}\cdot\mathbf{R}}(e^{-i\mathbf{q}\cdot\mathbf{r}}e^{-i\mathbf{q}\cdot\mathbf{r}})|2\rangle$$

$$= \rho_{m_{1}m_{2}} \otimes \rho_{n_{1}n_{2}} \otimes \delta_{s_{1}s_{2}}$$

and as a result

$$E(p - r, r) = \int_{q} \sum_{n_{1}n_{2}} n_{1}^{F}(s)(1 - n_{2}^{F}(s))|\rho_{n_{1}n_{2}}|^2 \sum_{m} \langle m|I|m\rangle \quad (221)$$

where we acknowledge the fact that the occupation factors $n_{F}^{1}$ and $n_{F}^{2}$ can depend on the spin. We have also used the fact that the sum over all values of $m$ is the degeneracy of each CF-LL, $n/p$. Carrying out the sums over $n_{1}$ and $n_{2}$, we obtain

$$E(p - r, r) = \frac{n}{p} \int_{q} \sum_{n_{1}=0}^{p-r-1} \langle n_{1}|\rho(q)\rho(-q)|n_{1}\rangle - \sum_{n_{1},n_{2}=0}^{p-r-1} |\rho_{n_{1}n_{2}}|^2$$

$$+ \sum_{n_{1}=0}^{r-1} \langle n_{1}|\rho(q)\rho(-q)|n_{1}\rangle - \sum_{n_{1},n_{2}=0}^{r-1} |\rho_{n_{1}n_{2}}|^2 \quad (214)$$

It is now straightforward to compute the critical field for the transition $|p - r, r\rangle \rightarrow |p - r - 1, r + 1\rangle$ by invoking

$$E(p - r, r) - E(p - r - 1, r + 1) = g \left[ \frac{e}{2m_{e}} \right] B\frac{n}{p} \quad (221)$$

6. Calculation of $1/T_{1}$

As explained in the main text, our strategy for computing $1/T_{1}$ will be to compute $K^{\text{max}}$, the Knight shift at the center of the well, in a fully polarized sample, in terms of an unknown nuclear matrix element squared $|u(0)|^{2}$, and then to express $1/T_{1}$ (which depends on $|u(0)|^{4}$) in terms of the measured value of $K^{\text{max}}$.

a. Calculation of Knight shift

In first quantization, the hyperfine interaction of electrons with a nucleus at the origin is

$$H_{hf} = \frac{8\pi}{3} \gamma_{n}\gamma_{e}\mathbf{S} \cdot \mathbf{I} \delta^{3}(0) \quad (222)$$

where $\gamma_{e}$ and $\gamma_{n}$ are gyromagnetic ratios of the nucleus and electron, $\mathbf{S} = \frac{1}{2}\sigma$ and $\mathbf{I}$ is the nuclear spin ($3/2$ in this case).

Our plan is to first express this operator in second-quantized form (in the electron basis) and then transform to the CF basis.

Consider now the quantum well, which we take to be infinite in the $x - y$ plane and of width $w$ in the $z$-direction. In the absence any nuclear potential the single-particle wavefunctions will be given by

$$\Phi_{(k,s)}(r, z, s) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{L^{2}}} \psi_{\perp}(z) \chi_{s} \quad (223)$$
where, $\chi_s$ is a spinor with $s = \pm 1/2$, $\mathbf{r}$, like $\mathbf{k}$ lies in the $x - y$ plane, $\psi_\perp(z)$ is a function like $\sqrt{2/w \sin(\pi z/w)}$. Note that in the $z$-direction there is essentially just one wavefunction, other excitations being too high in energy to interest us. We now turn on the nuclear potential, whose effect will be to modify the states to

$$
\Phi_{(k,s)}(r,z) = e^{ik \cdot r} u_{(k,s)}(r,z) \chi_s, \quad (224)
$$

where, by Bloch’s theorem, $u_{(k,s)}(r,z)$ is invariant under translations by the unit cell in the $x - y$ plane. The unit cell here extends into the $z$-direction from one end of the well to the other and may contain many nuclei. In the $z$-direction, $u_{(k,s)}(r,z)$ will vary and possibly vanish rapidly beyond the ends of the well. We will assume $u_{(k,s)}(r,z)$ is normalized to unity over the sample. (This means it contains all normalization factors like $1/\sqrt{L}$ that we may usually associate with the plane wave.) The only thing of importance is that the states are still labeled by $(k,s)$. As we shall see, detailed knowledge of $u_{(k,s)}(r,z)$ will not come into play.

In second quantization we define a spinor field operator

$$
\Psi(r,z) = \sum_{s,k} d_{k,s} e^{ik \cdot r} u_{(k,s)}(r,z) \chi_s \quad (225)
$$

(Note that this operator is good for low energy physics in which higher excited states in the $z$-direction are ignored. We could add these to the sum over states to obtain an operator that had Dirac delta function anti-commutation relations. However if we limit ourselves to the low energy sector as defined above, this is a waste.)

The spin density at the origin becomes

$$
S(0) = \sum_{s,k,s',k'} u_{(k,s)}(0) u_{(k',s')}(0) d_{k',s'}^{*} \frac{\sigma_{s's}}{2} d_{k,s} \quad (226)
$$

where

$$
u_{(k,s)}(0) = u_{(k,s)}(0,0,0). \quad (227)
$$

Now we make an assumption that is often made: $u_{(k,s)}(0)$ does not depend on $s$ or $k$. We will simply call it $u(0)$. Given this assumption

$$
S(0) = |u(0)|^2 \sum_{k,k',s,s'} d_{k',s'}^{*} \frac{\sigma_{s's}}{2} d_{k,s} \quad (228)
$$

$$
= |u(0)|^2 \sum_{k,q,s,s'} d_{k+q,s}^{*} \frac{\sigma_{s's}}{2} d_{k,s} \quad (229)
$$

$$
= |u(0)|^2 \sum_{q} S(q) \quad (230)
$$

where $S(q)$ is familiar expression for spin density at momentum $q$, the effect of the nuclear potential being encoded in $|u(0)|^2$. Note that $d_{k,s}^{*}$ still creates particles in states that are solutions to the nuclear and quantum well potential and not plane waves. However the commutation rules of the $d's$ are canonical and the spin density operator will obey the usual commutation rules

$$
[S^a(q), S^b(q')] = i\epsilon^{abc} S^c(q + q') \quad (231)
$$

When we go to the CF basis, we assume will see the same change as in the case of the charge density:

$$
\sum_{k} d_{k+q}^{*} d_{k} \rightarrow \sum_{k} d_{k+q}^{*} d_{k} e^{i\mathbf{q} \cdot \mathbf{k}/2} e^{-q^2 \mathbf{r}^2/4} \quad (232)
$$

where I have included the factor $e^{-q^2 \mathbf{r}^2/4}$ so as to work with the projected density and not the magnetic translation operator. Thus we write

$$
S(0)^{CF} = |u(0)|^2 \sum_{k,k',s,s'} d_{k',s'}^{*} \frac{\sigma_{s's}}{2} d_{k,s} \times e^{i\mathbf{k} \cdot \mathbf{r}/2} e^{-|\mathbf{k} - \mathbf{\gamma}^{2}/4} \quad (233)
$$

The factor $e^{i\mathbf{k} \cdot \mathbf{r}/2} e^{-|\mathbf{k} - \mathbf{\gamma}^{2}/4}$, which is just $e^{i\mathbf{q} \cdot \mathbf{r}} e^{-q^2 \mathbf{r}^2/4}$ ensures that the upper spin operators (without the $|u(0)|^2$) have the right commutation relations (Eqn. (28)) of Moon et al., among themselves and the projected charge density. (The same criterion was used by Murthy in Ref. 13 in the small $q$ limit.)

We are now ready to eliminate $|u(0)|^2$ in terms of a measurable quantity. The hyperfine interaction takes the form of an average field $\langle \mathbf{B} \rangle$ acting on the nuclei:

$$
H_{hf} = \gamma_s \langle \mathbf{B} \rangle \quad (234)
$$

$$
\langle \mathbf{B} \rangle = |u(0)|^2 \frac{8\pi}{3} e \sum_{i} \left( \sum_{k,s} d_{k,s}^{*} d_{k,s} \frac{1}{2} \frac{\sigma_{s's}}{2} \right) \quad (235)
$$

where I have used the fact that by symmetry, only $k = k'$ and $S_2$ can have mean values.

Assume we are in a fully polarized state. Then the Knight shift is readily found to be

$$
K_{max} = \frac{2}{3} \gamma_s \gamma_n N |u(0)|^2 \quad (236)
$$

where $N$ is the total number of particles. For future use we invert this to write

$$
|u(0)|^2 = \left[ \frac{3K_{max}}{2\gamma_s \gamma_n N} \right]. \quad (237)
$$

b. Calculation of $1/T_1$

Consider now the relaxation rate. By the standard procedure one arrives at the following expression for $1/T_1$:

$$
\frac{1}{T_1} = \frac{1}{2} \sum_{mn} W_m (E_m - E_n)^2 \quad (238)
$$

$$
W_m = 2\pi \sum_{F} \left( |<m| H_{hf} |n>|^2 \right) \delta(E_F - E_l) \quad (239)
$$
where the bar indicates a thermal average, \((m, n)\) label nuclear spin states, \((I, F)\) denote many-body fermionic states, the energy difference, \(E_m - E_n\), between nuclear spin states has been neglected in the energy conserving delta function, and \(H_{sf}\) is the hyperfine interaction, which now know how to write in the CF basis. Thus we have

\[
W_{mn} = 2\pi \left( \frac{8\pi e \gamma_n |u(0)|^2}{3} \right)^2 \times \sum_{F,J} \sum_\alpha (m|I_a|n)(F|S_{CF}^\alpha (0)|I)|^2 \delta(E_F - E_I) \tag{240}
\]

and we arrive at

\[
1/T_1 = \frac{1}{\pi} \left( \frac{8\pi e \gamma_n |u(0)|^2}{3} \right)^2 \times \int_{-\infty}^{\infty} dt \sum_{I \alpha \rho z} |(I|S_{CF}^\alpha (0)S_{CF}^\alpha (t)|I)|^2 \tag{241}
\]

where \(S(t)\) is the Heisenberg operator at time \(t\).

Here is a brief explanation of some steps leading to Eqn. (241). First I have used the above, \(\langle \text{bar} \rangle\) (m, n) states, the energy difference, \(E\), \(E_m - E_n\), where the bar indicates a thermal average, \(|u(0)|^2\) via Eqn. (237). Next we recall that

\[
\sum_{m} \sum_{n} \frac{\langle m|I_a|n\rangle \langle n|I_{\alpha'}|m\rangle (E_m - E_n)^2}{\sum_n E_n^2} = \delta_{\alpha \alpha'} \tag{242}
\]

where the transverse delta function \(\delta_{\alpha \alpha'}\) means that \(\alpha \neq \alpha'\) and then done some standard manipulations. To establish Eqn. (242), we need to invoke the following facts.

- The kets \(|m\rangle\) and \(|n\rangle\) are eigenstates of the nuclear hamiltonian \(A I_z\) where \(A\) is some constant.
- The factors \((E_m - E_n) I_a\) or \((E_m - E_n) I_{\alpha'}\) can be traded for commutators of some other \(I_3\) or \(I_{\alpha'}\) with \(A I_z\). The result is zero if either \(\alpha\) or \(\alpha'\) equals \(z\), which explains the transverse delta function that emerges.
- The previous step allows one to invoke completeness and reduce the numerator to the trace of the hamiltonian \((A I_z)^2\), which then cancels the denominator.

Next we recall that

\[
S_{CF}^\alpha (t) = \frac{1}{2} \sum_{k, k'} d^{\dagger}_{k,-s}(t) d_{k,s}(t) e^{i(k \cdot \k' t^2)/2 e^{i(k \cdot \k' t^2)/2}}
\]

\[
d_{k,s}(t) = d_{k,s}(0) e^{-i\mathcal{E}_s(k)t}
\]

(and similarly for \(S_{CF}^\alpha \)) in the HF approximation. In the above, \(\mathcal{E}_s(k)\) is the HF energy of a fermion of spin \(s\) and momentum \(k\).

To arrive at Eqns. (235, 237) in the text, we do the integral over \(t\) (obtaining a delta function), use the explicit expressions for spin operators, resort to standard HF factorization of the quartic operators and perform a standard change of variables in the measure, and finally eliminate \(|u(0)|^2\) via Eqn. (237).

7. Symbols

| Symbol | Significance |
|--------|-------------|
| \(\nu\) | = \(p/(2ps + 1)\)=filling fraction |
| \(p\) | Number of CF LL’s |
| \(2s\) | Number of vortices attached |
| \(c^2\) | 2ps/(2ps + 1) |
| \(B^*\) | Reduced field seen by CF = \(B/(2ps + 1)\) |
| \(A^*\) | Reduced potential by CF = \(A/(2ps + 1)\) |
| \(l\) | electron magnetic length |
| \(l^*\) | CF magnetic length = \(l/\sqrt{1-e^2}\) |
| \(\Pi^*\) | Velocity operator for CF |
| \(\bar{\rho}(q)\) | Electron density in CF basis, \(O(ql)\) |
| \(\bar{\chi}(q)\) | Constraint in CF basis, \(O(qt)\) |
| \(\bar{\rho}(q)\) | Electron density in CF basis, \(O(ql)\) |
| \(\bar{\chi}(q)\) | Constraint in CF basis, \(O(qt)\) |
| \(\bar{\rho}(q) - \bar{\rho}(q)^2\) | Vortex form factor \((e^{-q^2 l^2}/(8\pi l^2))\) |
| \(\lambda\) | Defined by \(v(q) = 2\pi e^2 q^2/(8\pi l^2)\) |
| \(\Delta_{a,p}\) | Activation or polarizarion gap |
| \(\delta\) | \(\Delta/(e^2/\ell)\) |
| \(1/m_{a,p}(2s)\) | Defined by \(\Delta_{a,p} = eB^*/(m_{a,p}(2s))\) |
| \(C_{a,p}^+(2s)\) | Defined by \(1/m_{a,p}(2s) = (e^2l/\varepsilon)C_{a,p}^+(2s)\) |
| \(m_{a,p}/(m_\varepsilon \sqrt{B/(\ell)})\) | (normalized mass) |
| \(m_\varepsilon\) | Electron mass in free space |
| \(\rho_p\) | Polarization |
| \(S\) | Number of spin up minus down CF |
| \(E(S)\) | Ground state energy density |
| \(g\) | \(-\text{factor of CF, taken to be .44}\) |
| \(p - r, r'\) | Many-body CF state with \(p - r\) spin LL’s and \(r\) spin-down LL’s. |
| \(E_\pm(k)\) | Hartree Fock energy for up/down spin |
| \(\theta\) | Tilt angle |
| \(\rho_{n_1, n_2}\) | One-particle Matrix element of \(\rho^p\) between LL \(n_1\) and \(n_2\) |

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