Generator of an abstract quantum walk

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Abstract

We give an explicit formula of the generator of an abstract Szegedy evolution operator in terms of the discriminant operator of the evolution. We also characterize the asymptotic behavior of a quantum walker through the spectral property of the discriminant operator by using the discrete analog of the RAGE theorem.

1 Introduction

Quantum walks (QWs) are one of the interesting topics which have overlaps to various kinds of study fields (see [3, 22, 24] and their references). While there are several opinions of the priority of QW, primitive forms of discrete-time QWs can been seen, for example, Feynman and Hibbs [8], Aharonov et al [1], and Watrous [25]. Gudder [9], Meyer [15], and Ambainis et al [2] introduced the current notion of discrete-time QWs, independently. The Szegedy walk, whose original form was introduced in [19], is one of well-investigated discrete-time QWs on graphs. This includes Grover walk [10, 25] and has been intensively studied from various perspectives (see, for example, [11, 12, 16, 20]). Recently, Higuchi et al [12] introduced an extended version of the Szegedy walk, the twisted Szegedy walk, and proved a spectral mapping theorem for the new walk on a finite graph. Using the theorem, they studied the spectral and asymptotic properties of the Grover walks on crystal lattices. In our previous paper [13], we studied an abstract evolution of the form

\[ U = S(2d_A d_A - 1). \]  

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Here $d_A$ is a coisometry from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ and $S$ is a unitary involution on $\mathcal{H}$:

$$d_A d_A^* = I_K, \quad S = S^* = S^{-1},$$

where $I_K$ is the identity operator on $\mathcal{K}$. Let $T = d_A S d_A^*$, called the discriminant of $U$, and $\varphi(x) = (x + x^{-1})/2$. Then the following spectral mapping theorem was proved:

$$\sigma(U) = \varphi^{-1}(\sigma(T)) \cup \{\pm 1\}^{M_+} \cup \{-1\}^{M_-},$$

$$\sigma_p(U) = \varphi^{-1}(\sigma_p(T)) \cup \{\pm 1\}^{M_+} \cup \{-1\}^{M_-},$$

where $M_\pm = \dim D_\pm$ and $D_\pm = \ker(T^2 - 1)^\perp$.

Let $G = (V,D)$ be a symmetric directed graph, i.e., an arc $e \in D$ if and only if the inverse arc $\bar{e} \in D$. The evolution $U^{(w,\theta)}$ of the twisted Szegedy walk on $G$ is of the form $U^{(w,\theta)} = S^{\theta}(2d_A^{(w)} - 1)$, where $d_A^{(w)} : \ell^2(D) \to \ell^2(V)$ is a boundary operator defined from a weight function $w : D \to \mathbb{C}$ and $S^{\theta}$ on $\ell^2(D)$ a (twisted) shift operator defined from a 1-form $\theta : D \to \mathbb{R}$. Because $U$ becomes $U^{(w,\theta)}$ with $d_A = d_A^{(w)}$ and $S = S^{(\theta)}$, the twisted Szegedy walk on any symmetric directed graph is an example of the abstract Szegedy walks.

In particular, the result of [12] was extended to infinite graphs other than crystal lattices. The evolution of the Grover walk on $G$ is given by $U^{(w,\theta)}$ with $w(e) = 1/\sqrt{\deg(o(e))}$ and $\theta(e) = 0$ ($e \in D$). In this case, the discriminant $T$ is unitarily equivalent to the transition operator $P_G$ of the symmetric random walk on $G$. This allows us to determine the the spectrum of $U^{(w,\theta)}$ from the spectrum of $P_G$ and the subspaces $D_\pm$.

In this paper, we continue the study of the abstract evolution $U$ defined by (1.1). In the case of a continuous-time QW, the time evolution is defined as $U(t) = e^{itH}$, where $H$ is the (negative) Hamiltonian (see [6] and [3] for details). By the Wiener theorem [26] and the RAGE theorem [18, 5, 7] (see also [17]), the asymptotic behavior of a quantum walker is deduced from the spectral properties of $H$. Motivated by the continuous case, we give an explicit formula of the generator $H$ such that $H$ is self-adjoint and $U^n = e^{inH}$. For the evolution $U$ defined by (1.1), we prove the following.

(1) The operators

$$d_+ = \frac{1}{\sqrt{2(1-T^2)}}(d_A - e^{-i\theta(T)}d_A S),$$

$$d_- = \frac{1}{\sqrt{2(1-T^2)}}(e^{-i\theta(T)}d_A - d_A S)$$

can be extended to unitary operators from $\text{Ran}(d_\pm^* d_\pm)$ to $\ker(T^2 - 1)^\perp$. 

2
Let $\vartheta : [-1, 1] \to [0, \pi]$ be the function defined by $\vartheta(\lambda) = \arccos \lambda$. Then, the generator $H$ of $U$ is expressed as

$$H = \vartheta(d^*_+ Td_+) \oplus (2\pi - \vartheta(d^*_+ Td_-)) \oplus 0 \oplus \pi$$

on $\mathcal{H} = \text{Ran}(d^*_+) \oplus \text{Ran}(d^*) \oplus \ker(U - 1) \oplus \ker(U + 1)$. Moreover,

$$\ker(U \mp 1) = d^*_A \ker(T \mp 1) \oplus D_{\pm}.$$

Let $\mathcal{H}_p(A)$ denote the direct sum of all eigenspaces of a self-adjoint operator $A$ and $\mathcal{H}_q(A) (\mp = c, ac, sc)$ the subspaces of continuity, absolute continuity, and singular continuity, respectively. As a direct consequence of (1) and (2), the spectral property of the generator $H$ (or the evolution $U$) is determined by the discriminant of $T$ and the subspaces $D_{\pm}$:

(3) Let $\mathcal{H}_p^T := \mathcal{H}_p(T) \cap \ker(T^2 - 1)$. Then,

$$\mathcal{H}_p(H) = d^*_+ \mathcal{H}_p^T \oplus d^-_+ \mathcal{H}_p^T \oplus (U^2 - 1),$$

$$\mathcal{H}_q(H) = d^*_+ \mathcal{H}_q(T) \oplus d^-_+ \mathcal{H}_q(T).$$

In what follows, we consider the long-time asymptotic behavior. We begin with a general setting. The setting allows us to introduce the notion of unitary equivalence among QWs and unify several concrete examples of QWs such as the Gudder-type QW and the Ambainis-type QW defined in [11]. Recently, Ohno proved that any space-homogeneous QWs on the line [9, 15, 2] are unitarily equivalent to abstract Szegedy walks. Given a unitary operator $U$ on a Hilbert space $\mathcal{H}$ and a direct sum decomposition $\mathcal{H} = \bigoplus_{x \in V} \mathcal{H}_x$, we can naturally introduce a directed graph $G_U$ with vertices $V$ and a probability distribution on $V$:

$$\nu_n(x) = \|P_x U^n \Psi_0\|^2 \quad (x \in V),$$

where $P_x$ is the orthogonal projection onto $\mathcal{H}_x$ and $\Psi_0 \in \mathcal{H}$ is a normalized vector. We interpret $\nu_n(x)$ as the finding probability of a quantum walker on $G_U$ and $\Psi_0$ as the initial state of the quantum walker. In this sense, we say that $U$ is an evolution of QW and write $(U, \{\mathcal{H}_x\}_{x \in V}) \in \mathcal{F}_{QW}$. In the case of the twisted Szegedy evolution $U^{(w, \theta)}$ on $G = (V, D)$, there is a natural decomposition $\ell^2(D) = \bigoplus_{x \in V} \mathcal{H}_x$ such that $(U^{(w, \theta)}, \{\mathcal{H}_x\}_{x \in V}) \in \mathcal{F}_{QW}$. Assuming $\dim \mathcal{H}_x < \infty (x \in V)$, we obtain the discrete analog of the RAGE theorem (see [17]):

(4) $\Psi_0 \in \mathcal{H}_c(H)$ if and only if $\lim_{N \to \infty} \sum_{n=0}^{N-1} \nu_n^\Psi_0(R) / N = 0$ for all finite set $R \subset V$.
(5) \( \Psi_0 \in \mathcal{H}_p(H) \) if and only if \( \lim_{m \to \infty} \sup_n \nu_n^{\Psi_0}(R_m) = 0 \) for any increasing sequence \( \{R_m\}_m \) of finite sets such that \( \bigcup_m R_m = V \).

In [12, Definition 6], the authors say that localization occurs if
\[
\limsup_{n \to \infty} \nu_n^{\Psi_0}(x) > 0 \quad \text{with some } x \in V.
\]

Let \( P_p(H) \) be the orthogonal projection onto \( \mathcal{H}_p(H) \). Assuming that \( \Psi_0 \in \mathcal{H}_{sc}(H) \), we observe form (5) the following assertion:

(6) Localization occurs if and only if \( \Psi_0 \) overlaps with \( \mathcal{H}_p(H) \), i.e., \( P_p(H)\Psi_0 \neq 0 \).

For the abstract Szegedy walk, we know the following from (3) and (6).

(7) Localization occurs for some initial state \( \Psi_0 \) if and only if \( \sigma_p(T) \neq \emptyset \) or \( D^\perp \neq \emptyset \).

(8) If \( T \) has a complete set of eigenstates, localization occurs for any initial state \( \Psi_0 \).

The remainder of this paper is organized as follows. Section 2 is devoted to the study of the abstract QW. In Section 2.1, we give the axiom of an abstract QW and some concrete examples. In Section 2.2, we discuss the relation between the generator of an abstract evolution and the long-time asymptotic behavior of a quantum walker. In particular, we prove (4), (5) and (6). Section 3 contains a brief review of the abstract Szegedy walk. We summarize the results from [13] without proofs. In Section 4, we state the main results of this paper and prove (7) and (8). Section 5 is devoted to the derivation of the generator of the abstract evolution. In Subsection 5.1, we present the rigorous definitions of the operators \( d_\pm \) and prove (1). In Subsection 5.2, we prove (2) and (3). In the appendix, we present the proofs of the discrete analog of the RAGE theorem and a relation between the initial state and localization.

2 Abstract quantum walks

In this section, we first propose QW defied by a unitary operator \( U \), where \( U \) is not assumed to be of the form \( U = S(2d_A^+d_A - 1) \) but is assumed to act on a Hilbert space written as a direct sum of Hilbert spaces \( \{\mathcal{H}_v\}_{v \in V} \). Then, as shown in the following subsection, \( U \) naturally defines a directed graph \( G_U = (V, D) \) and the probability of finding a quantum walker thereon. In addition, we see that the dynamics of a quantum walker is governed by the generator of the evolution \( U \).
2.1 Axiom of abstract quantum walks

Let $V$ be a countable set, $\{\mathcal{H}_v\}_{v \in V}$ a family of separable Hilbert spaces (possibly $\dim \mathcal{H}_v = \infty$) and $U$ a unitary on $\mathcal{H} = \bigoplus_{v \in V} \mathcal{H}_v$. We say that $(U, \{\mathcal{H}_v\}_{v \in V})$ is an evolution of QW and write $(U, \{\mathcal{H}_v\}_{v \in V}) \in \mathcal{F}_{QW}$. If there is no danger of confusion, we simply say that $U$ is an evolution of QW and write $U \in \mathcal{F}_{QW}$. We use $P_v$ to denote the projection from $\mathcal{H}$ onto $\mathcal{H}_v$ and define operators $U_{uv}: \mathcal{H}_v \to \mathcal{H}_u$ ($u, v \in V$) by

$$U_{uv} = P_u U P_v.$$

First, we introduce a graph associated with $U \in \mathcal{F}_{QW}$. We use $o(e)$ and $t(e)$ to denote the origin and terminal, respectively, of a directed edge $e$ of a graph.

**Definition 2.1.** The graph $G_U = (V_U, D_U)$ associated with an evolution $(U, \{\mathcal{H}_v\}_{v \in V}) \in \mathcal{F}_{QW}$ is a directed graph defined as follows:

1. The set $V_U$ of vertices of $G_U$ is given by $V_U = V$.

2. If $U_{uv} \neq 0$, there exists an arc $e \in D_U$ from $v$ to $u$.

Hereafter, we simply write $G_U = (V, D)$ when no confusion can arise. It is possible that depending on the choice of the separation $\{\mathcal{H}_v\}_{v \in V}$, there is no inverse arc of an arc $e \in D$, because it is not necessary that $U_{vu} \neq 0$ even if $U_{uv} \neq 0$.

**Example 2.1.** Let us consider the Hilbert space $\mathcal{H} = \mathbb{C}^3$. Let $\{\delta_1, \delta_2, \delta_3\}$ be the standard basis of $\mathcal{H}$ and

$$U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}$$

a unitary matrix on $\mathcal{H}$.

(i) Let $V = \{a, b\}$. We consider the separation $\{\mathcal{H}_a, \mathcal{H}_b\}$ of $\mathcal{H}$, where $\mathcal{H}_a = \text{Span}\{\delta_1\}$ and $\mathcal{H}_b = \text{Span}\{\delta_2, \delta_3\}$. By this separation, $U$ is decomposed as

$$U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}.$$

Hence, $G_U$ has an arc from $a$ to $b$ and its inverse arc. $G_U$ has loops at $a$ and $b$. 5
(ii) Let $V = \{a, b, c\}$ and consider the separation $\{H_v\}_{v \in V}$, where $H_a = \text{Span}\{\delta_1\}$, $H_b = \text{Span}\{\delta_2\}$, and $H_c = \text{Span}\{\delta_3\}$. $U$ is decomposed as

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}. $$

We observe that $U_{ba} = U_{ca} = 0$, whereas $U_{ab} \neq 0$ and $U_{ac} \neq 0$. Hence, $G_U$ has no inverse arcs of an arc from $b$ to $a$ and an arc from $c$ to $a$. $G_U$ has an arc from $b$ to $c$, its inverse arc, and a loop only at $a$.

In the following, we introduce an abstract QW on $G_U$.

**Axiom.** QW with an evolution $(U, \{H_v\}_{v \in V}) \in \mathcal{F}_{QW}$ is defined as follows:

1. The state of a quantum walker at time $n \in \mathbb{N}$ with the initial state $\Psi_0 \in \mathcal{H}$ ($\|\Psi_0\| = 1$) is given by $\Psi_n = U^n \Psi_0$.

2. The probability $\nu_n(x)$ of finding the quantum walker at vertex $x \in V$ at time $n \in \mathbb{N}$ is given by $\nu_n(x) = \|P_x \Psi_n\|^2$.

**Example 2.2.** The evolution of a typical QW on $\mathbb{Z}$ is of the form

$$U = \sum_{x \in \mathbb{Z}} (|x+1\rangle \langle x| \otimes Q + |x-1\rangle \langle x| \otimes P),$$

which converges in the strong operator topology. Here, $P, Q \in M_2(\mathbb{C})$ and the Hilbert space of states is given by $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. Noting that $\mathcal{H} = \bigoplus_{x \in \mathbb{Z}} \mathcal{H}_x$ with $\mathcal{H}_x = \text{Ran}(|x\rangle \langle x| \otimes \mathbb{I}_{\mathbb{C}^2}) \simeq \mathbb{C}^2$, we see that

$$U_{yx} = \begin{cases} |y\rangle \langle x| \otimes P, & y = x - 1, \\
|y\rangle \langle x| \otimes Q, & y = x + 1, \\
0, & \text{otherwise}. \end{cases} \quad (2.1)$$

We observe from Proposition 2.1 below that $U$ is unitary if and only if $P$ and $Q$ satisfy

$$PP^* + QQ^* = P^*P + Q^*Q = 1, \quad PQ^* = Q^*P = 0. \quad (2.2)$$

For example, if $P = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ and $P + Q$ is unitary, $P$ and $Q$ satisfy (2.2). Hence, $(U, \{\mathcal{H}_x\}_{x \in \mathbb{Z}}) \in \mathcal{F}_{QW}$ and the graph $G_U$ associated with $U$ is the symmetric directed graph of $\mathbb{Z}$. Because $P_x = |x\rangle \langle x| \otimes \mathbb{I}_{\mathbb{C}^2}$, we know that the probability of finding a quantum walker at vertex $x \in \mathbb{Z}$ at time $n \in \mathbb{N}$ with an initial state $\Psi_0 \in \mathcal{H}$ is $\nu_n(x) = \|\Psi_n(x)\|_{\mathbb{C}^2}^2$. For a deeper discussion of this QW, we refer the reader to [2, 3].
Proposition 2.1. Let $W$ be a bounded operator on $H = \bigoplus_{v \in V} H_v$ and $W_{uv} = P_u W P_v$ $(u,v \in V)$. The following are equivalent:

(i) $W$ is unitary.

(ii) $\sum_{x \in V} W_{ux}(W^*)_x = \sum_{x \in V} (W^*)_ux W_{xv} = \delta_{uv} P_v$ for all $u,v \in V$.

Proof. The operator equality $I = \sum_{v \in V} P_v$ and the equalities

$$(WW^*)_{uv} = \sum_{x \in V} W_{ux}(W^*)_x \quad \text{and} \quad (W^*W)_{uv} = \sum_{x \in V} (W^*)_ux W_{xv}$$

all hold in the strong convergence sense. Hence, (ii) is equivalent to $WW^* = W^*W = I_H$, which proves the proposition. \qed

Definition 2.2. $(U_1, \{H^{(1)}_{v_1}\}_{v_1 \in V_1}) \in \mathcal{F}_{\text{QW}}$ and $(U_2, \{H^{(2)}_{v_2}\}_{v_2 \in V_2}) \in \mathcal{F}_{\text{QW}}$ are unitarily equivalent, written $(U_1, \{H^{(1)}_{v_1}\}_{v_1 \in V_1}) \simeq (U_2, \{H^{(2)}_{v_2}\}_{v_2 \in V_2})$, if there exist a unital map $\mathcal{U} : \bigoplus_{v_1 \in V_1} H_{v_1} \to \bigoplus_{v_2 \in V_2} H_{v_2}$ and a bijection $\phi : V_1 \to V_2$ such that $\mathcal{U}H^{(1)}_{v_1} = H^{(2)}_{\phi(v_1)}$ and $\mathcal{U}U_1\mathcal{U}^{-1} = U_2$.

Let $(U_1, \{H^{(1)}_{v_1}\}_{v_1 \in V_1}) \in \mathcal{F}_{\text{QW}}$ and $(U_2, \{H^{(2)}_{v_2}\}_{v_2 \in V_2}) \in \mathcal{F}_{\text{QW}}$ be unitarily equivalent. The state $U_1^*\Psi_0^{(1)} \in \mathcal{H}_1 := \bigoplus_{v \in V_1} H_v^{(1)}$ of a quantum walker at time $n \in \mathbb{N}$ is identified with $U_2^*\Psi_0^{(2)} = \mathcal{U}(U_1^*\Psi_0^{(1)}) \in \mathcal{H}_2 := \bigoplus_{v \in V_2} H_v^{(2)}$, where $\Psi_0^{(2)} = \mathcal{U}\Psi_0^{(1)}$. Since $\mathcal{U}H^{(1)}_{v_1} = H^{(2)}_{\phi(v_1)}$, we have $P_{\phi(v_1)} = \mathcal{U}P_v\mathcal{U}^{-1}$.

Hence, the probability $\nu_n^{(1)}(x_1) := \|P_{x_1}\Psi_n^{(1)}\|^2$ of finding a quantum walker at vertex $x_1 \in V_1$ and at time $n \in \mathbb{N}$ is equal to $\nu_n^{(2)}(\phi(x_1)) := \|P_{\phi(x_1)}\Psi_n^{(2)}\|^2$.

We also know that the bijection $\phi : V_1 \to V_2$ is an isomorphism between the associated graphs $G_{U_1}$ and $G_{U_2}$.

Proposition 2.2. Let $W_1$ and $W_2$ be unitary operators on $H = \bigoplus_{v \in V} H_v$ and set $U = W_1 W_2$ and $\bar{U} = W_2 W_1$. Then,

$$(U, \{H_v\}) \simeq (\bar{U}, \{W_2 H_v\}) \simeq (U, \{W_1 H_v\}).$$

Proof. Let $\mathcal{U} = W_2$ and $\phi$ be an identity map on $V$. Then, $\mathcal{U}H_v = W_2 H_v$ and $\mathcal{U}U\mathcal{U}^{-1} = W_2(W_1W_2)W_2^{-1} = \bar{U}$. Hence, $(U, \{H_v\}) \simeq (\bar{U}, \{W_2 H_v\})$. Similarly, we know that $(U, \{W_2 H_v\}) \simeq (U, \{W_1 H_v\})$ if we take $\mathcal{U} = W_1^*$. \qed

Example 2.3 (Gudder and Ambainis type QWs). Here, we follow the notation of \[\text{[11]}\]. Let $S_\pi$ be a shift operator and $C = \oplus_{j \in V(G)} H_j$ a coin flip operator, where $\pi$ is a partition on the line digraph of a graph $G$ and $\{H_j\}$ is a sequence of unitary operators on $H_j$. Note that $CH_j = H_j$. We observe, from Proposition 2.2, that the Gudder type evolution $U^{(G)} = CS_\pi$ and the Ambainis type evolution $U^{(A)} = S_\pi C$ are unitarily equivalent and

$$(U^{(G)}, \{H_j\}) \simeq (U^{(A)}, \{H_j\}).$$
2.2 Generators

It is well known that for a unitary operator \( U \), there exists a unique self-adjoint operator \( H \) such that
\[
E_H([0,2\pi]) = I \quad \text{and} \quad U = e^{iH},
\]
where \( E_H \) is the spectral measure of \( H \). The state of a quantum walker at time \( n \in \mathbb{N} \) is represented as \( \Psi_n = e^{inH}\Psi_0 \) \((n \in \mathbb{N})\). In this sense, we define the generator of a unitary operator as follows:

**Definition 2.3.** A self-adjoint operator \( H \) is the generator of a unitary operator \( U \), if (2.3) holds.

Let \( H \) be the generator of an evolution \( (U, \{\mathcal{H}_v\}_{v \in V}) \in \mathcal{F}_{QW} \). Then, the probability \( \nu_n(x) \) of finding a quantum walker at vertex \( x \in V \) at time \( n \in \mathbb{N} \) is given by
\[
\nu_n(x) = \|P_x e^{inH}\Psi_0\|^2.
\]
We denote by \( \nu_n(R) \) the probability of finding a quantum walker in \( R \subset V \):
\[
\nu_n(R) = \sum_{x \in R} \nu_n(x).
\]
We denote \( \nu_n(x) \) (resp., \( \nu_n(R) \)) by \( \nu_n^{\Psi_0}(x) \) (resp., \( \nu_n^{\Psi_0}(R) \)) to emphasize the dependence on the initial state. The time average \( \bar{\nu}_n^{\Psi_0} \) of \( \nu_n \) and its infinite time limit \( \bar{\nu}_\infty^{\Psi_0} \), if it exists, are given by
\[
\bar{\nu}_n^{\Psi_0}(R) = \frac{1}{N} \sum_{n=0}^{N-1} \nu_n^{\Psi_0}(R) \quad \text{and} \quad \bar{\nu}_\infty^{\Psi_0}(R) = \lim_{N \to \infty} \bar{\nu}_n^{\Psi_0}(R).
\]

**Proposition 2.3.** Let \( H \) be the generator of an evolution \( (U, \{\mathcal{H}_v\}_{v \in V}) \in \mathcal{F}_{QW} \), and assume that \( \dim \mathcal{H}_v < \infty \) \((v \in V)\).

(i) \( \Psi_0 \in \mathcal{H}_c(H) \) if and only if \( \bar{\nu}_\infty^{\Psi_0}(R) = 0 \) for all finite sets \( R \subset V \).

(ii) \( \Psi_0 \in \mathcal{H}_p(H) \) if and only if \( \lim_{m \to \infty} \sup_n \nu_n^{\Psi_0}(R_m^c) = 0 \) for any increasing sequence \( \{R_m\}_m \) of finite sets such that \( \bigcup_m R_m = V \).

This proposition is the discrete analog of the RAGE theorem. The proof is standard, but we include it in the appendix for completeness.

In [12, Definition 6], the authors say that localization occurs if
\[
\limsup_{n \to \infty} \nu_n^{\Psi_0}(x) > 0 \quad \text{with some} \quad x \in V. \tag{2.4}
\]
As will be proved in the appendix, (2.4) holds if \( \lim_{m \to \infty} \sup_n \nu_n^{\Psi_0}(R_m) = 0 \) for some increasing sequence \( \{R_m\} \) such that \( \bigcup_m R_m = V \). Hence, localization occurs if \( \Psi_0 \in H_p(H) \).

Let \( P_s(H) \) be the orthogonal projection onto \( H_s(H) \) for \( s = p, ac \).

**Proposition 2.4.** Let \( H \) and \((U, \{H_v\}_{v \in V}) \in F_{QW} \) be as in Proposition 2.3. Suppose that \( \Psi_0 \in H_{sc}(H) \). Then, the following are equivalent:

(a) Localization occurs.

(b) \( \Psi_0 \) overlaps with \( H_p(H) \), i.e., \( P_p(H)\Psi_0 \neq 0 \).

**Proof.** By assumption, we can write \( \Psi_0 = \Psi^p + \Psi_{ac} \) with \( \Psi^s = P_s(H)\Psi_0 \) (\( s = p, ac \)). Because, by assumption, \( P_x \) is compact, \( P_x U^n \Psi_{ac} \) converges strongly to zero as \( n \to \infty \). Hence,

\[
|\nu_n^{\Psi_0}(x)^{1/2} - \nu_n^{\Psi^p}(x)^{1/2}| \leq \|P_x U^n \Psi_{ac}\| \to 0 \quad (n \to \infty).
\]

Assuming (a), we get \( \epsilon_0 := \limsup_{n \to \infty} \nu_n^{\Psi_0}(x) > 0 \) with some \( x \in V \). Because \( \nu_n^{\Psi_0}(x) \geq \nu_n^{\Psi^p}(x) - \epsilon_0/2 \) for sufficiently large \( n \),

\[
\|\Psi_{p}\|^2 \geq \nu_n^{\Psi_0}(x) - \epsilon_0/2.
\]

Taking the limit superior on both sides, we have (b). Conversely, we assume (b). Then, \( \Psi_p \neq 0 \). By the above argument, \( \epsilon_1 := \limsup_{n \to \infty} \nu_n^{\Psi^p}(x) > 0 \) with some \( x \in V \). Because \( \nu_n^{\Psi_0}(x) \geq \nu_n^{\Psi^p}(x) - \epsilon_1/2 \) for sufficiently large \( n \),

\[
\limsup_{n \to \infty} \nu_n^{\Psi_0}(x) \geq \epsilon_1/2 > 0.
\]

This proves (a). \( \square \)

### 3 Abstract Szegedy walk

In this section, we treat a specific class of abstract QWs, an extension of the Szegedy walks. Let us recall some notations and facts from [13]. Let \( H \) and \( K \) be complex Hilbert spaces. We assume that there exists a coisometry operator \( d_A : H \to K \), i.e., \( d_A \) is bounded and satisfies

\[
d_AD_A^* = I_K,
\]

where \( I_K \) is the identity operator on \( K \). By (3.1), \( d_A \) is a partial isometry and surjection, its adjoint \( d_A^* : K \to H \) is an isometry, and \( \Pi_A := d_A^* d_A \) is the projection onto \( A := \text{Ran}(d_A^* d_A) = d_A^* K \). We call the self-adjoint operator
C := 2d_A^*A - 1 on \mathcal{H} a coin operator, because we observe that C is a unitary involution and decomposed into

\[ C = I_A \oplus (-I_A^\perp) \quad \text{on} \quad \mathcal{H} = A \oplus A^\perp. \]

This also proves that A = ker(C - 1) and A^\perp = ker(C + 1).

Let S be a unitary involution on \mathcal{H}. We decompose S into S = I_S \oplus (-I_S^\perp) on \mathcal{H} = S \oplus S^\perp, where S = ker(S - 1) and S^\perp = ker(S + 1). Then d_B := d_AS is also a coisometry. Throughout this subsection, we fix d_A and S, and call them a boundary operator and a shift operator, respectively. In analogy with the twisted Szegedy walk (see Example 3.1 below), we define an abstract evolution U and its discriminant T as follows:

**Definition 3.1.** Let d_A, d_B, C, and S be as above.

1. The evolution associated with the boundary operator d_A and the shift operator S is defined by

   \[ U = SC. \]

2. The discriminant of U is defined by

   \[ T = d_A d_B^*. \]

We note that S, C, and U are unitary on \mathcal{H}. By definition, the discriminant T is a bounded self-adjoint operator on K with \|T\| \leq 1. Let \[ D^\perp_+ = A^\perp \cap S^\perp, \quad D^\perp_- = A^\perp \cap S. \] (3.2)

**Theorem 3.1** ([13]). Let \( M^\pm = \dim D^\perp \pm \).

1. \( \sigma(U) = \{ e^{i\xi} \mid \cos \xi \in \sigma(T), \xi \in [0, 2\pi) \} \cup \{ +1 \}^{M^+} \cup \{ -1 \}^{M^-}; \)

2. \( \sigma_p(U) = \{ e^{i\xi} \mid \cos \xi \in \sigma_p(T), \xi \in [0, 2\pi) \} \cup \{ +1 \}^{M^+} \cup \{ -1 \}^{M^-}, \)

where we use \( \{ \pm 1 \}^{M^\pm} \) to denote the multiplicity of \pm 1 and set \( \{ \pm 1 \}^{M^\pm} = \emptyset \) if \( M^\pm = 0. \)

**Example 3.1** (Twisted Szegedy walk [12]). Let \( G = (V, E) \) be a (possibly infinite) graph with the sets V of vertices and E of unoriented edges (possibly including multiple edges and loops). We consider that each edge \( e \in E \) with end vertices \( V(e) = \{ u, v \} \) has two orientations such that the origin of \( e \) is \( u \) or \( v \), and we denote the set of such oriented edges by \( D \). The inverse edge of \( e \in D \) is denoted by \( \bar{e} \), with the result that \( o(\bar{e}) = t(e) \) and \( t(\bar{e}) = o(e) \). Note that \( e \in D \) if and only if \( \bar{e} \in D \). Let \( \mathcal{H} = \ell^2(D) \) and \( \mathcal{K} = \ell^2(V) \). We define a boundary operator \( d_A^w : \mathcal{H} \to \mathcal{K} \) as follows. We call \( w : D \to \mathbb{C} \setminus \{ 0 \} \) a weight if it satisfies \( w(e) \neq 0 \) and

\[ \sum_{e : o(e) = v} |w(e)|^2 = 1 \quad \text{for all} \quad v \in V. \] (3.3)
For a weight \( w \) and all \( \psi \in \mathcal{H} \), \( d_A^{(w)} \psi \in \mathcal{K} \) is given by

\[
(d_A^{(w)} \psi)(v) = \sum_{e : o(e) = v} \psi(e) \overline{w(e)}, \quad v \in V.
\]

The adjoint \( d_A^{(w)^*} : \mathcal{K} \to \mathcal{H} \) of \( d_A^{(w)} \) is a coboundary operator and satisfies

\[
(d_A^{(w)^*} f)(e) = w(e) f(o(e)), \quad e \in D
\]

for all \( f \in \mathcal{K} \). We observe that \( d_A^{(w)} \) is a coisometry, i.e., \( d_A^{(w)} d_A^{(w)^*} = I_{\mathcal{K}} \), because, from (3.3),

\[
(d_A^{(w)} d_A^{(w)^*} f)(v) = \sum_{e : o(e) = v} (d_A^{(w)^*} f)(e) \overline{w(e)} = \sum_{e : o(e) = v} |w(e)|^2 f(o(e)) = f(v).
\]

The coin operator is defined by \( C^{(w)} = 2 d_A^{(w)^*} d_A^{(w)} - 1 \), and the (twisted) shift operator by \( (S^{(\theta)} \psi)(e) = e^{-i \theta(e)} \psi(e) \) \( (e \in D) \), where \( \theta : D \to \mathbb{R} \) is a 1-form and satisfies \( \theta(\bar{e}) = -\theta(e) \) \( (e \in D) \). It is easy to check that \( S^{(\theta)} \) is a unitary involution. The evolution of the twisted Szegedy walk associated with the weight \( w \) and the 1-form \( \theta \) is defined by \( U^{(w, \theta)} = S^{(\theta)} C^{(w)} \). The operators \( d_A^{(w)} \), \( C^{(w)} \), and \( S^{(\theta)} \) are examples of the abstract coisometry \( d_A \), coin operator \( C \), and shift operator \( S \), respectively. The discriminant of \( U^{(w, \theta)} \) is defined by \( T^{(w, \theta)} = d_A^{(w)} d_B^{(w, \theta)^*} \), where \( d_B^{(w, \theta)} = d_A^{(w)} S^{(\theta)} \). We now show that \( U^{(w, \theta)} \) is an evolution of QW. To this end, we set

\[
\mathcal{H}_v = \text{Span} \{ \delta_e | e \in D, o(e) = v \}, \quad (3.4)
\]

where \( \text{Span}A \) is the closure of the linear span of a set \( A \) and \( \delta_e \in \ell^2(D) \) is given by \( \delta_e(e) = 1 \) and \( \delta_e(f) = 0 \) \( (e \neq f) \). Then, we can decompose \( \mathcal{H} = \bigoplus_{v \in V} \mathcal{H}_v \). Thus, we know that \( (U^{(w, \theta)}, \{ \mathcal{H}_v \}_{v \in V}) \in \mathcal{F}_{\text{QW}} \). Observe that the orthogonal projection onto \( \mathcal{H}_v \) is given by

\[
P_v = \sum_{e \in D : o(e) = v} |e \rangle \langle e|,
\]

where \( |e \rangle \langle e| = \langle \delta_e , \cdot \rangle \delta_e \) is the orthogonal projection onto the one dimensional subspace \( \{ \alpha \delta_e | \alpha \in \mathbb{C} \} \). The probability \( \nu_n : V \to [0, 1] \) of finding a quantum walker at time \( n \) is

\[
\nu_n(x) = \sum_{e \in D : o(e) = x} |\langle \delta_e , \Psi_n \rangle|^2 = \sum_{e \in D : o(e) = x} |\Psi_n(e)|^2.
\]
Let $G_U$ be the associated graph of $U$. We observe that

$$U_{uv} = \sum_{e : o(e) = u, t(e) = v} \sum_{f : o(f) = v} w(e)w(f)(2 - \delta_{fe})e^{i\theta(e)}|e\rangle\langle f|$$

is non-zero if and only if there exists $e \in D$ such that $o(e) = u$ and $t(e) = v$. Hence, $G_U$ is identified with a subgraph of $G$. If $G$ has no multiple edges, $G_U \simeq G$.

**Remark 3.1.** Recently, Ohno proved that any space-homogeneous QW on $\mathbb{Z}$ such as the model in Example 2.2 is equivalent to an abstract QW associated with some boundary operator and shift operator. Even for an inhomogeneous case such as [14, 23], we can show that the evolution is associated with some boundary operator and shift operator.

**Remark 3.2.** We should remark that for a time-dependent abstract Szegedy walk $U_1 \to U_2 \to \cdots \to U_n$, the spectrum of $U_nU_{n-1} \cdots U_1$ cannot be described by the discriminant operators $T_nT_{n-1} \cdots T_1$ in general, since our analysis proposed here essentially works well when the time evolution is decomposed into two involution operators $U = E_2E_1$ [13]. Applying our abstract QW to the time-dependent QW effectively is an open problem.

In what follows, we introduce closed subspaces of $\mathcal{H}$ that play an important role in this paper:

$$\mathcal{D} = \overline{A + B}, \quad \mathcal{D}_0 = A \cap B, \quad \mathcal{D}_1 = \mathcal{D}_0^\perp \cap \mathcal{D}.$$

Here, we denote by $A$ and $B$ the subspaces $\text{Ran}(d_A^*d_A)$ and $\text{Ran}(d_B^*d_B)$, respectively. Clearly,

$$\mathcal{H} = \mathcal{D} \oplus \mathcal{D}^\perp = \mathcal{D}_1 \oplus \mathcal{D}_0 \oplus \mathcal{D}^\perp.$$

We state the basic properties of these subspaces without proof. For the proof, one can consult [13], where we used the notations $\mathcal{L}$, $\mathcal{L}_1$, and $\mathcal{L}_0$ with $\mathcal{D} = \overline{\mathcal{L}}$, $\mathcal{D}_1 = \overline{\mathcal{L}_1}$, and $\mathcal{D}_0 = \overline{\mathcal{L}}$.

**Proposition 3.1.** Let $U$ be as above and $T = d_Ad_B^*$ the discriminant of $U$. $U$ leaves $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_0$, and $\mathcal{D}^\perp$ invariant. Moreover, the following hold:

(i) $\mathcal{D}_0 = d_A^* \ker(T^2 - 1) = d_B^* \ker(T^2 - 1)$;

(ii) $\mathcal{D}_1 = d_A^* \ker(T^2 - 1)^\perp + d_B^* \ker(T^2 - 1)^\perp$.
(iii) \( \mathcal{D}^\perp = \ker(d_A) \cap \ker(d_B) \).

By Proposition 3.1, \( U \) is decomposed as
\[
U = U_{\mathcal{D}_1} \oplus U_{\mathcal{D}_0} \oplus U_{\mathcal{D}^\perp}.
\] (3.5)

Since \( \ker(T^2 - 1) = \ker(T - 1) \oplus \ker(T + 1) \), we know that
\[
\mathcal{D}_0 = \mathcal{D}_0^+ \oplus \mathcal{D}_0^-,
\]
where \( \mathcal{D}_0^\pm = d_A^* \ker(T \mp 1) \). We also have
\[
\mathcal{D}^\perp = \mathcal{D}_\perp^+ \oplus \mathcal{D}_\perp^-,
\]
where \( \mathcal{D}_\perp^\pm := \mathcal{D}^\perp \cap \ker(S \mp 1) \). By Proposition 3.1 (iii), we have \( \mathcal{D}_\perp \). The following is essentially proved in \( \text{[13]} \).

**Proposition 3.2.** Let \( M_\pm = \dim \mathcal{D}_\pm \).

1. \( \ker(U \mp 1) = \mathcal{D}_\pm^+ \oplus \mathcal{D}_\pm^\perp \) and \( \ker(U^2 - 1) = \mathcal{D}_1 \);
2. \( U_{\mathcal{D}_0} = I_{\mathcal{D}_0^+} \oplus (-I_{\mathcal{D}_0^-}) \) and \( U_{\mathcal{D}^\perp} = I_{\mathcal{D}_\perp^+} \oplus (-I_{\mathcal{D}_\perp^-}) \).

### 4 Main results

Let \( U = S(2d_A^*d_A - 1) \) be an evolution associated with a boundary operator \( d_A : \mathcal{H} \to \mathcal{K} \) and a shift operator \( S \) on \( \mathcal{H} \). As will be seen in Section 5, the operators
\[
d_+ = \frac{1}{\sqrt{2(1 - T^2)}}(d_A - e^{-i\vartheta(T)}d_B), \quad d_- = \frac{1}{\sqrt{2(1 - T^2)}}(e^{-i\vartheta(T)}d_A - d_B),
\]
can be extended to bounded operators, where \( T \) is the discriminant of \( U \) and \( \vartheta : [-1, 1] \to [0, \pi] \) is given by \( \vartheta(\lambda) = \arccos \lambda \).

**Theorem 4.1.** Let \( U, d_\pm \) and \( T \) be as above. Then, \( \mathcal{H} \) is decomposed as
\[
\mathcal{H} = \text{Ran}(d_A^*d_+) \oplus \text{Ran}(d_A^*d_-) \oplus \ker(U - 1) \oplus \ker(U + 1)\]
(4.1)
and the generator \( H \) of \( U \) is given by
\[
H = \vartheta(d_A^*Td_+) \oplus (2\pi - \vartheta(d_A^*Td_-)) \oplus 0 \oplus \pi,
\]
(4.2)
where
\[
\ker(U \mp 1) = d_A^* \ker(T \mp 1) \oplus \mathcal{D}_\pm^\perp.
\]
By this theorem, $U$ is expressed by
\[
U = e^{i\vartheta(d^*_+T)\psi_0} \oplus e^{-i\vartheta(d^-_+T)\psi_0} \oplus 1 \oplus (-1)
\] (4.3)
under the decomposition of (4.1). We consider the iteration of $U$, $\psi_0 \xrightarrow{U} \psi_1 \xrightarrow{U} \psi_2 \xrightarrow{U} \cdots$. From (4.3), we obtain the following temporal and spatial discrete analog of the wave equation.

**Corollary 4.2.** Let $\psi_0 \in \text{Ran}(d^*_+d_+)$ and $f_n = d_+\psi_n$. Then,
\[
\frac{1}{2} (f_{n+1} + f_{n-1}) = T f_n.
\]

Moreover, we obtain the following corollary, which is important for discriminating the localization of QW under the time evolution $U$.

**Corollary 4.3.** Let $U$, $d_\pm$, $T$ and $H$ be as in Theorem 4.1. Then,
\[
\mathcal{H}_p(H) = d^*_+ \mathcal{H}_p^T \oplus d^-_+ \mathcal{H}_p^T \oplus \ker(U^2 - 1),
\]
\[
\mathcal{H}_s(H) = d^*_+ \mathcal{H}_s^p(T) \oplus d^-_+ \mathcal{H}_s^p(T), \quad \sharp = c, ac, sc
\]
where $\mathcal{H}_p^T := \mathcal{H}_p(T) \cap \ker(T^2 - 1)^\perp$.

Combining Corollary 4.3 with Proposition 2.4 we have the following criterion for localization.

**Corollary 4.4.** Let $U = S(2d^*_Ad_A - 1)$ and $H$ be as in Theorem 4.1. Assume that there exists a family $\{\mathcal{H}_v\}_{v \in V}$ of Hilbert spaces such that $(U, \{\mathcal{H}_v\}_{v \in V}) \in \mathcal{F}_{QW}$ and $\text{dim}\mathcal{H}_v < \infty \ (v \in V)$. Then,

(1) Localization occurs for some initial state $\Psi_0$ if and only if $\sigma_p(T) \neq \emptyset$ or $D^\perp \neq \emptyset$.

(2) If $T$ has a complete set of eigenstates, localization occurs for any initial state $\Psi_0$.

**Proof.** In the case of (1), we know from Corollary 4.3 that $\sigma_p(H) \neq \emptyset$. By Proposition 2.4 we need only take the initial state $\Psi_0 \in \mathcal{H}_{\text{sc}}(H)^\perp$ overlapping with $\mathcal{H}_p(H)$. In the case of (2), $\sigma_{\text{sc}}(H) = \emptyset$ and any initial state $\Psi_0$ overlaps with $\mathcal{H}_p(H)$.

\[\square\]

5 **Generator of an evolution**

In this section, we prove Theorem 4.1 and Corollary 4.3. We begin with the precise definition of notations.
5.1 Definition and properties of $d_\pm$

Let $\vartheta : [-1, 1] \to [0, \pi]$ be a function defined by

$$\vartheta(\lambda) = \arccos \lambda, \quad \lambda \in [-1, 1].$$

Because $\sigma(T) \subseteq [-1, 1],

$$\cos \vartheta(T) = T, \quad \sin \vartheta(T) = \sqrt{1 - T^2}, \quad e^{\pm i\vartheta(T)} = T \pm i\sqrt{1 - T^2}.$$ 

Note that $\ker(T^2 - 1) = \ker \sqrt{1 - T^2}$ and $\ker(T^2 - 1)^\perp = \text{Ran} \sqrt{1 - T^2}$. We first define operators $d_\pm^*: \text{Ran}(T^2 - 1) \to D_1$ as follows: for $f \in \text{Ran}(T^2 - 1),

$$d_+^* f = (d_A^* - d_B^* e^{i\vartheta(T)}) \frac{1}{\sqrt{2(1 - T^2)}} f, \quad d_-^* f = (d_A^* - d_B^*) e^{i\vartheta(T)} \frac{1}{\sqrt{2(1 - T^2)}} f.$$ 

Because $\frac{1}{\sqrt{2(1 - T^2)}} f \in \text{Ran} \sqrt{1 - T^2}$ for all $f \in \text{Ran}(1 - T^2)$, we know that $d_\pm^* f \in D_1$.

**Lemma 5.1.** $d_\pm^*$ are isometries from $\text{Ran}(T^2 - 1)$ to $D_1$.

**Proof.** Because by direct calculation,

$$(d_A - e^{-i\vartheta(T)} d_B)(d_A^* - d_B^* e^{i\vartheta(T)}) = (2 - 2T \cos \vartheta(T)) = 2(1 - T^2)$$

it follows that for all $f \in \text{Ran}(T^2 - 1),

$$||d_\pm^* f||^2 = \left< \frac{1}{\sqrt{2(1 - T^2)}} f, (d_A - e^{-i\vartheta(T)} d_B)(d_A^* - d_B^* e^{i\vartheta(T)}) \frac{1}{\sqrt{2(1 - T^2)}} f \right> = ||f||^2.$$ 

This implies that $d_\pm^*$ is an isometry on $\text{Ran}(T^2 - 1)$. Noting that $(e^{-i\vartheta(T)} d_A - d_B)(d_A^* e^{i\vartheta(T)} - d_B^*) = 2(1 - T^2)$, we also know that $d_-^*$ is an isometry on $\text{Ran}(T^2 - 1)$.

From Lemma 5.1, $d_\pm^*$ have unique extensions, whose domains are $\text{Ran}(T^2 - 1)^\perp = \ker(T^2 - 1)^\perp$. We denote the extension by the same symbol, i.e., $d_\pm^*: \ker(T^2 - 1)^\perp \to D_1$ is given by

$$d_\pm^* f = \lim_{n \to \infty} d_\pm^* f_n, \quad f \in \ker(T^2 - 1)^\perp,$$

where $\{f_n\} \subset \text{Ran}(T^2 - 1)$ is an arbitrary sequence satisfying $\lim_n f_n = f$. Thus, we have the following:
Proposition 5.1. \(d_\pm^\dagger\) are isometries from \(\ker(T^2 - 1)\perp\) to \(D_1\).

We use \(D_1^\pm\) to denote the range of \(d_\pm^\dagger\):

\[D_1^\pm = d_\pm^\dagger \ker(T^2 - 1)\perp.\]

Lemma 5.2. \(D_1^\pm\) are closed subspaces of \(D_1\) and

\[D_1 = D_1^+ \oplus D_1^- .\]

Proof. Because \(d_\pm^\dagger\) is an isometry, it is clear that \(D_1^\pm\) is a closed subspace of \(D\). We first show that \(D_1^\pm\) are orthogonal to each other. Let \(\psi_\pm \in D_1^\pm\) and write it as \(\psi_\pm = \lim_{n \to \infty} d_\pm^\dagger f_\pm^n (f_\pm^n \in \text{Ran}(T^2 - 1)).\) It follows that

\[\langle \psi_+, \psi_- \rangle = \lim_{n \to \infty} \left\langle d_\dagger_+ f_+^n, d_-^\dagger f_-^n \right\rangle = \lim_{n \to \infty} \left( \frac{1}{\sqrt{2(1 - T^2)}} f_+^n, (d_A - e^{-i\vartheta(T)} d_B) (d_A^* e^{i\vartheta(T)} - d_B^*) \frac{1}{\sqrt{2(1 - T^2)}} f_-^n \right) = 0 ,\]

where in the last equality, we have used the fact that

\[(d_A - e^{-i\vartheta(T)} d_B)(d_A^* e^{i\vartheta(T)} - d_B^*) = 2 \cos \vartheta(T) - 2T = 0. \quad (5.1)\]

It remains to be shown that \(D_1 = D_1^+ \oplus D_1^-\). It suffices to show that \(d_A^\dagger \ker(T^2 - 1)^\perp + d_B^\dagger \ker(T^2 - 1) \subset D_1^+ \oplus D_1^-\). To this end, take a \(\psi \in d_A^\dagger \ker(T^2 - 1)^\perp + d_B^\dagger \ker(T^2 - 1)\). From [13], there exist unique vectors \(f, g \in \ker(T^2 - 1)^\perp\) such that \(\psi = d_A^\dagger f + d_B^\dagger g\).

We now take vectors \(f_n, g_n \in \text{Ran}\sqrt{1 - T^2}\) satisfying \(f = \lim_{n \to \infty} f_n\) and \(g = \lim_{n \to \infty} g_n\) and set

\[F_n = -\frac{1}{\sqrt{2i}} (e^{-i\vartheta(T)} f_n + g_n), \quad G_n = \frac{1}{\sqrt{2i}} (f_n + e^{-i\vartheta(T)} g_n).\]

Then, \(F_n, G_n \in \text{Ran}\sqrt{1 - T^2}\) and

\[f_n = \frac{1}{\sqrt{2(1 - T^2)} (F_n + e^{i\vartheta(T)} G_n), \quad g_n = -\frac{1}{\sqrt{2(1 - T^2)}} (e^{i\vartheta(T)} F_n + G_n).\]

By direct calculation,

\[d_\dagger_+ F_n + d_-^\dagger G_n = d_A^\dagger f_n + d_B^\dagger g_n.\]
Since the limits $F := \lim_{n \to \infty} F_n$ and $G := \lim_{n \to \infty} G_n$ exist and $F, G \in \ker(T^2 - 1)^\perp$, 

$$
\psi = \lim_{n \to \infty} (d_A f_n + d_B g_n) = \lim_{n \to \infty} (d_{+n} F_n + d_{-n} G_n) = d_{+} F + d_{-} G \in D_1^+ \oplus D_1^-.
$$

This completes the proof. \qed

Let $d_{\pm,1}$ be the adjoint of $d_{\pm} : \ker(T^2 - 1)^\perp \to D_1$. Then,

$$
d_{\pm,1} = (d_{\pm})^*, \quad d_{+1} = d_{+}.
$$

**Proposition 5.2.** On the entire $D_1$, 

$$
d_{+,1} = \frac{1}{\sqrt{2(1 - T^2)}}(d_A - e^{-i\vartheta(T)}d_B), \quad d_{-,1} = \frac{1}{\sqrt{2(1 - T^2)}}(e^{-i\vartheta(T)}d_A - d_B).
$$

Moreover,

(i) $d_{\pm,1} d_{\pm,1}^* = I_{\ker(T^2 - 1)^\perp}$, $d_{\pm,1} d_{\pm,1}^* = 0$.

(ii) $\Pi_{D_1^\pm} : = d_{\pm,1}^* d_{\pm,1}$ is the projection from $D_1$ onto $D_1^\pm$.

To prove this proposition, we use the following lemma:

**Lemma 5.3.** (i) $(d_A - e^{-i\vartheta(T)}d_B)(d_A^* e^{i\vartheta(T)} - d_B^*) = 0$.

(ii) $(e^{-i\vartheta(T)}d_A - d_B)(d_A^* e^{i\vartheta(T)} - d_B^*) = 0$.

(iii) $(d_A - e^{-i\vartheta(T)}d_B)(d_A^* - d_B^* e^{i\vartheta(T)}) = 2(1 - T^2)$.

(iv) $(e^{-i\vartheta(T)}d_A - d_B)(d_A^* e^{i\vartheta(T)} - d_B^*) = 2(1 - T^2)$.

**Proof.** (i) is proved in (5.1). (ii) is obtained from (i) by taking the adjoint. (iii) is also obtained from the adjoint of (iv). (iv) is proved by direct calculation:

$$(e^{-i\vartheta(T)}d_A - d_B)(d_A^* e^{i\vartheta(T)} - d_B^*) = 2 - 2T \cos \vartheta(T) = 2(1 - T^2).$$

\qed
Proof of Proposition 5.2. For all $F \in \ker(T^2 - 1)\dag$, there exists a sequence $\{F_n\} \subset \text{Ran}\sqrt{1-T^2}$ such that $F = \lim_{n \to \infty} F_n$. From (iii) and (iv) of Lemma 5.3

$$(d_A - e^{-i\theta(T)}d_B)d^\dag_+ F = \lim_{n \to \infty} \sqrt{2(1-T^2)}F_n$$

$$= \sqrt{2(1-T^2)}F \in \text{Ran}\sqrt{1-T^2}, \quad (5.2)$$

$$(e^{-i\theta(T)}d_A - d_B)d^\dag_- F = \lim_{n \to \infty} \sqrt{2(1-T^2)}F_n$$

$$= \sqrt{2(1-T^2)}F \in \text{Ran}\sqrt{1-T^2}. \quad (5.3)$$

In addition, from (i) and (ii) of Lemma 5.3

$$(d_A - e^{-i\theta(T)}d_B)d^\dag_+ F = 0,$$  

$$(e^{-i\theta(T)}d_A - d_B)d^\dag_- F = 0. \quad (5.4)$$

By (5.2), (5.3), (5.4) and (5.5), we know that the operators $o_+ := \frac{1}{\sqrt{2(1-T^2)}}(d_A - e^{-i\theta(T)}d_B)$ and $o_- := \frac{1}{\sqrt{2(1-T^2)}}(e^{-i\theta(T)}d_A - d_B)$ can be defined on the entire $\mathcal{D}_1$. To prove that $d_{\pm,1} = o_{\pm}$, it suffices to show that the adjoint of $o_{\pm}$ are $d^\dag_{\pm}$. For all $\psi \in \mathcal{D}_1$ and $f \in \ker(T^2 - 1)\dag$,

$$\langle f, o_+ \psi \rangle = \lim_{n \to \infty} \left\langle \frac{1}{\sqrt{2(1-T^2)}}f_n, (d_A - e^{-i\theta(T)}d_B)\psi \right\rangle$$

$$= \lim_{n \to \infty} \left\langle (d_A^*-d_B^*e^{i\theta(T)})\frac{1}{\sqrt{2(1-T^2)}}f_n, \psi \right\rangle = \langle d^\dag_+ f, \psi \rangle,$$

where $\{f_n\} \subset \text{Ran}(T^2 - 1)$ is a sequence such that $f \equiv \lim_{n \to \infty} f_n$. This means that $d^\dag_+$ is the adjoint of $o_+$. Hence, $d_{+,1} = o_+$. The same proof works for $d_{-,1} = o_-$. The former statement of the proposition is proved.

(i) is proved from Lemma 5.3. We prove (ii). To this end, we take $\psi_\pm \in \mathcal{D}^\dag_1$ and write it as $\psi_\pm = d^\dag_\pm F$ ($F \in \ker(T^2 - 1)\dag$). Combining (i) with $d^*_{\pm,1} = d^\dag_\pm$ yields the result that

$$\tilde{\Pi}_{\mathcal{D}^\dag_1}\psi_\pm = (d^*_{\pm,1}d^\dag_\pm)(d^\dag_\pm F) = d^*_{\pm} F = \psi_\pm.$$  

Hence, $\text{Ran}\tilde{\Pi}_{\mathcal{D}^\dag_1} = \mathcal{D}^\dag_1$. It remains to be proved that $\tilde{\Pi}_{\mathcal{D}^\dag_1}$ is a projection. It is clear, by definition, that $\tilde{\Pi}_{\mathcal{D}^\dag_1}$ is self-adjoint. By (i), $\tilde{\Pi}_{\mathcal{D}^\dag_1}^2 = d^*_{\pm,1}(d_{\pm,1}d^*_{\pm,1})d_{\pm,1} = \tilde{\Pi}_{\mathcal{D}^\dag_1}$, and we obtain the desired result. \qed

In what follows, we extend the domain $\mathcal{D}_1$ of $d_{\pm,1}$ to the entire space $\mathcal{H}$. We will denote the extension of $d_{\pm,1}$ by $d_\pm$. 18
Lemma 5.4. On $D_0 \oplus D^\perp$,

(i) $d_A - e^{-i\vartheta(T)}d_B = 0$;
(ii) $e^{-i\vartheta(T)}d_A - d_B = 0$.

Proof. Because by (iii) of Proposition 3.1, (i) and (ii) hold on $D^\perp$, we need to only establish them on $D_0$. Let $\psi_0 \in D_0$ and write it as $\psi_0 = d^*_Af_0$ ($f_0 \in \ker(T^2 - 1)$). Then,
\[
(d_A - e^{-i\vartheta(T)}d_B)\psi_0 = (1 - e^{-i\vartheta(T)}T)f_0 = i\sqrt{1 - T^2}e^{-i\vartheta(T)}f_0 = 0.
\]
Similarly,
\[
(e^{-i\vartheta(T)}d_A - d_B)\psi_0 = -i\sqrt{1 - T^2}f_0 = 0.
\]
By Lemma 5.4, operators $d_{\pm}: \mathcal{H} \rightarrow \mathcal{K}$ can be defined by
\[
d_{+} = \frac{1}{\sqrt{2(1 - T^2)}}(d_A - e^{-i\vartheta(T)}d_B), \quad d_{-} = \frac{1}{\sqrt{2(1 - T^2)}}(e^{-i\vartheta(T)}d_A - d_B)
\]
and
\[
d_{\pm} = d_{\pm,1}\Pi_{D_{1}^{\pm}}, \quad d^*_{\pm} = d^*_{\pm,1}\Pi_{\ker(T^2 - 1)^{\perp}},
\]
where $\Pi_{D_{1}^{\pm}}$ and $\Pi_{\ker(T^2 - 1)^{\perp}}$ are the projections onto $D_{1}^{\pm}$ and $\ker(T^2 - 1)^{\perp}$, respectively. From (5.6) and Proposition 5.2 we have the following:

Proposition 5.3. Let $d_{\pm}$ be defined as above.

(i) $\ker(d_{\pm}) = D_{1}^{\mp} \oplus D_0 \oplus D^{\perp}$ and $\text{Ran}(d_{\pm}) = \ker(T^2 - 1)^{\perp}$;
(ii) $D_{1}^{\pm} = d_{\pm,1}^{\ast}\Pi_{\ker(T^2 - 1)^{\perp}}$;
(iii) $d_{\pm}d_{\mp} = \Pi_{\ker(T^2 - 1)^{\perp}}$, $d_{\pm}d_{\mp}^{\ast} = 0$;
(iv) $d_{\pm}^{\ast}d_{\pm} = \Pi_{D_{1}^{\pm}}$. 

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5.2 Generator of $U$

By Proposition 3.2, the evolution $U = S(2d_A^*d_A - 1)$ associated with $d_A$ and $S$ is decomposed as

$$U = U_{D_1} \oplus I_{\ker(U - 1)} \oplus (-I_{\ker(U + 1)}),$$

(5.7)

where $D_1 = \ker(U^2 - 1)^\perp$ and $\ker(U \mp 1) = D_0^+ \oplus D_0^-$. We first prove the following representation of $U_{D_1}$:

**Theorem 5.1.** Let $U$ be as above. $U$ leaves $D_1^\mp$ invariant, and $U_{D_1}$ is decomposed as

$$U_{D_1} = e^{i\vartheta(d_A^*Td_+) + e^{-i\vartheta(d_A^*Td_-)}}$$

on $D_1 = D_1^+ \oplus D_1^-$. $U_{D_1}$ leaves $D_1^\mp$ invariant. Similarly, using $e^{-i\vartheta(T)} - 2T = -e^{-i\vartheta(T)}$, it follows that

$$U(\Pi_{D_1^+}\psi) = \lim_n U d_A^* F_n^+$$

$$= \lim_n U (d_A^* - d_B^*e^{i\vartheta(T)}) \frac{1}{\sqrt{2(1 - T^2)}} F_n^+$$

$$= \lim_n \left(d_A^* + d_B^*(e^{-i\vartheta(T)} - 2T)\right) \frac{1}{\sqrt{2(1 - T^2)}} e^{i\vartheta(T)} F_n^+,$$

(5.8)

where we have used the facts that $U d_A^* = d_B^*$ and $U d_B^* = 2d_A^*T - d_A^*$. Because $e^{-i\vartheta(T)} - 2T = -e^{i\vartheta(T)}$, it follows that

$$U(\Pi_{D_1^-}\psi) = \lim_n d_A^* e^{i\vartheta(T)} d_+^* F_n^+ = d_+^* e^{i\vartheta(T)} d_+^* \psi \in D_1^+,$$

(5.9)
Hence, the former half of the theorem follows. By (5.8) and (5.9), it follows that for all \( \psi \in D_1 \),

\[
U \psi = U(\Pi_{D_1^+} \psi) + U(\Pi_{D_1^-} \psi) = d_+^* e^{i\vartheta(T)} d_+ \psi + d_-^* e^{-i\vartheta(T)} d_- \psi.
\]

Because by Proposition 5.3, \( d_\pm : D_1^\perp \to \ker(T^2 - 1) \) is unitary,

\[
d_\pm e^{\pm i\vartheta(T)} d_\pm = e^{\pm i\vartheta(\vartheta^1_2)},
\]

which completes the proof.

\( \square \)

**Proof of Theorem 4.1.** Let \( H \) be defined by (4.2).

\[
e^{iH} = e^{i\vartheta(d_1^+ T_{d_1^+})} \oplus e^{i(2\pi - \vartheta(d_1^- T_{d_1^-}))} \oplus e^0 \oplus e^{i\varpi}
\]

\[
e^{i\vartheta(d_1^+ T_{d_1^+})} \oplus e^{-i\vartheta(d_1^- T_{d_1^-})} \oplus 1 \oplus (-1)
\]
one \( \mathcal{H} = D_1^+ \oplus D_1^- \oplus \ker(U - 1) \oplus \ker(U + 1) \). By (5.7), \( e^{iH} = U \). Because \( E_H([0, 2\pi]) = I \), we obtain the desired result.

\( \square \)

**Proof of Corollary 4.2.** Because \( \psi_0 \in \text{Ran}(d_+^* d_+^1) = D_1^+ \),

\[
f_n = d_+ U^n \psi_0 = e^{in\vartheta(T)} d_+ \psi_0.
\]

Hence,

\[
\frac{1}{2}(f_n + f_{n-1}) = \frac{e^{i\vartheta(T)} + e^{-i\vartheta(T)}}{2} e^{in\vartheta(T)} d_+ \psi_0 = T f_n.
\]

\( \square \)

**Proof of Corollary 4.3.** Let \( T_1 = T \Pi_{\ker(T^2 - 1) \perp} \). Because \( H \) has of the form (4.2), it follows that

\[
\sigma_p(H) = \{ \vartheta_+(\lambda) \mid \lambda \in \sigma_p(T_1) \} \cup \{ \vartheta_-^*(\lambda) \mid \lambda \in \sigma_p(T_1) \} \cup \{ 0, \pi \}.
\]

Here, we set \( \vartheta_+ = \vartheta \) and \( \vartheta_- = 2\pi - \vartheta \). It is clear that \( \ker(H) = \ker(U - 1) \) and \( \ker(H - \pi) = \ker(U + 1) \). Because \( d_\pm : D_1^\perp \to \ker(T^2 - 1) \) are unitary,

\[
\ker(H - \vartheta_\pm(\lambda)) = d_\pm^* \ker(T - \lambda).
\]

Hence,

\[
\mathcal{H}_p(H) = \left[ \bigoplus_{\lambda \in \sigma_p(T_1)} d_+^* \ker(T - \lambda) \right] \oplus \left[ \bigoplus_{\lambda \in \sigma_p(T_1)} d_-^* \ker(T - \lambda) \right] \oplus \ker(U^2 - 1)
\]

\[
= d_+^* \mathcal{H}_p(T_1) \oplus d_-^* \mathcal{H}_p(T_1) \oplus \ker(U^2 - 1).
\]

Because \( \mathcal{H}_p(T_1) = \mathcal{H}_p^T \), we obtain the former statement of the corollary. The latter follows from \( \mathcal{H}_p(T)^\perp = \mathcal{H}_c(T) = \mathcal{H}_{ac}(T) \oplus \mathcal{H}_{ac}(T) \) and the unitarity of \( d_\pm \).

\( \square \)
6 Conclusion

In this paper, we gave the explicit formula of the generator $H$ of the abstract Szegedy evolution operator $U = S(2d_A^*d_A - 1)$ in terms of the discriminant operator $T = d_A S d_A^*$. Using this formula, we characterized the spectral properties of $H$. By the discrete analog of the RAGE theorem, we also characterized the asymptotic properties of a quantum walker in terms of the generator $H$. In the case of the abstract Szegedy walk, we obtained the criteria for localization in terms of $T$ and the subspace $D^\perp$. In particular, for the Grover walk on a symmetric graph $G$, this implies that localization occurs for some initial state $\Psi_0$ only when the transition operator $P_G$ has an eigenvalue or $D^\perp \neq \emptyset$. In our future work, we will apply the theory developed in this paper to an inhomogeneous QW on $\mathbb{Z}$ such as [14, 23].

We also gave the axiom of the abstract discrete-time QWs, which includes many QWs. Given a unitary operator $U$ on a Hilbert space $\mathcal{H}$ and a decomposition $\mathcal{H} = \bigoplus_{x \in V} \mathcal{H}_x$, we can naturally define a directed graph $G_U$ with vertices $V$ and the finding probability of a quantum walker moving on $G_U$.

In forthcoming papers, we will treat the following problems:

(1) What kind of unitary operator $U$ has a boundary operator $d_A$ and a shift operator $S$ such that $U = S(2d_A^*d_A - 1)$?

(2) What is the graph $G_U$?

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A Appendix

A.1 Proof of Proposition 2.3

We present a proof of Proposition 2.3. Let $H$ be the generator of an evolution $(U, \{\mathcal{H}_v\}_{v \in V}) \in \mathcal{F}_{QW}$. Throughout this subsection, we assume that $\dim \mathcal{H}_v < \infty (v \in V)$. Let $\mathcal{H}_1$ be the set of vectors $\Psi_0 \in \mathcal{H}$ satisfying

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu_0(\Psi_n(R)) = 0$$
for any finite subset $R$ of $V$, and $\mathcal{H}_2$ the set of vectors $\Psi_0 \in \mathcal{H}$ satisfying

$$\lim_{m \to \infty} \sup_n \nu_n^0(R^c_m) = 0$$

for any sequence $\{R_m\}$ of finite subsets of $V$ such that $R_m \subset R_{m+1}$ and $V = \bigcup_m R_m$. Because $\nu_n^0 \Psi_0 + \beta \Phi_0(R) \leq 2 (|\alpha|^2 \nu_n^0(R) + |\beta|^2 \nu_n^0(R))$, we know that $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of $\mathcal{H}$. Let $P_R = \sum_{x \in R} P_x$ ($R \subset V$). Then,

$$\nu_n^0(R) = \|P_R e_{in} \Psi_0\|^2.$$ 

**Lemma A.1.** $\mathcal{H}_1 \perp \mathcal{H}_2$.

**Proof.** Let $\Psi_0 \in \mathcal{H}_1$ and $\Phi_0 \in \mathcal{H}_2$. Then, for all $R \subset V$,

$$|\langle \Psi_0, \Phi_0 \rangle| = \frac{1}{N} \sum_{n=0}^{N-1} |\langle \Psi_n, \Phi_n \rangle|$$

$$\leq \frac{1}{N} \sum_{n=0}^{N-1} |\langle P_R \Psi_n, P_R \Phi_n \rangle| + \frac{1}{N} \sum_{n=0}^{N-1} |\langle P_R^c \Psi_n, P_R^c \Phi_n \rangle|$$

$$\leq \| \Phi_0 \| \left( \frac{1}{N} \sum_{n=0}^{N-1} \| P_R \Psi_n \| \right) + \| \Psi_0 \| \left( \frac{1}{N} \sum_{n=0}^{N-1} \| P_R^c \Phi_n \| \right).$$

We first estimate the first term. By the Cauchy-Schwarz inequality,

$$\frac{1}{N} \sum_{n=0}^{N-1} \| P_R \Psi_n \| \leq \left( \frac{1}{N} \sum_{n=0}^{N-1} \| P_R \Psi_n \|^2 \right)^{1/2} = \bar{\nu}_N^{\Psi_0}(R)^{1/2}.$$ 

The second term is estimated as follows:

$$\frac{1}{N} \sum_{n=0}^{N-1} \| P_R^c \Phi_n \| \leq \sup_{n \geq 0} \| P_R^c \Phi_n \| = \sup_{n \geq 0} \nu_n^0(R^c)^{1/2}.$$ 

Combining these inequalities yields the result that

$$|\langle \Psi_0, \Phi_0 \rangle| \leq \| \Phi_0 \| \bar{\nu}_N^{\Psi_0}(R)^{1/2} + \| \Psi_0 \| \sup_{n \geq 0} \nu_n^0(R^c)^{1/2}. \quad (A.1)$$

Let $\epsilon > 0$ and $\{R_m\}_{m \geq 1}$ be a family of finite subsets of $V$ such that $R_m \subset R_{m+1}$ and $V = \bigcup_{m \geq 1} R_m$. Because $\Phi_0 \in \mathcal{H}_2$, there exists an $m_0 \in \mathbb{N}$ such that $\nu_n^0(R^c_m) < \epsilon^2 \| \Psi_0 \|^2$ ($m \geq m_0$). Because $\Psi_0 \in \mathcal{H}_1$, it follows from (A.1) that

$$\lim_{N \to \infty} |\langle \Psi_0, \Phi_0 \rangle| \leq \epsilon,$$

which completes the proof.

$\square$
Lemma A.2.  (i) $\mathcal{H}_c(H) \subset \mathcal{H}_1$;

(ii) $\mathcal{H}_p(H) \subset \mathcal{H}_2$.

Proof. Let $\Psi_0 \in \mathcal{H}_c(H)$. For any finite set $R$,

$$\bar{\nu}^\Psi_0(R) = \sum_{x \in R} \sum_{j=1}^{\dim \mathcal{H}_c} \bar{\nu}_N(\phi_{x,j}),$$

(A.2)

where $\{\phi_{x,j}\}$ is a complete orthonormal system of $\mathcal{H}_x$ and $\bar{\nu}_N(\phi) := \frac{1}{N} \sum_{n=0}^{N-1} |\langle \phi, e^{inH} \Psi_0 \rangle|^2$. Because, by assumption, the sum in (A.2) runs over a finite set, it suffices to show that $\lim_{N \to \infty} \bar{\nu}_N(\phi) = 0$. Let $x(t) = e^{it} \in H$ and $g_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} e^{inx}$. Then, $g_N(x) = \frac{1}{N(1-x^2)}$ if $x \neq 1$ and $g_N(1) = 1$. By the Fubini theorem,

$$\bar{\nu}_N(\phi) = \int_0^{2\pi} \int_0^{2\pi} g_N(x,0) d\lambda d\mu,$$

where $P_c(H)$ is the projection onto $\mathcal{H}_c(H)$. By the polarization identity, there exists $\{\psi_j\}_{j=1,2,3,4} \subset \mathcal{H}_c(H)$ such that

$$\bar{\nu}_N(\phi) \leq \text{const.} \sum_{j,k=1,2,3,4} \int_0^{2\pi} \int_0^{2\pi} |g_N(x,0)||E_H(\lambda)|^2 |\psi_j(\lambda)|^2 d\lambda d\mu.$$

Because $F_j := ||E_H(\cdot)\psi_j||^2$ is continuous,

$$\int \int_{\{x,y\}|x=y} dF_j(x) dF_k(y) \leq \int_0^{2\pi} dF_k(\mu) \int_0^{2\pi} dF_j(\lambda)$$

$$= \int_0^{2\pi} dF_k(\mu)(F_j(\mu + \epsilon) - F_j(\mu - \epsilon)) \to 0,$$

as $\epsilon \to 0$. Because $\sup_{|\omega| = 1} |g_N(\omega)| \leq 1$ and $\lim_{N \to \infty} g_N(x,\lambda - \mu) = 0$ ($x \neq 1$), we obtain $\lim_{N \to \infty} \bar{\nu}_N(\phi) = 0$ by the dominated convergence theorem. This completes the proof of (i).

Let $\Psi_0 \in \mathcal{H}_p(H)$. For any $\epsilon > 0$, there exist eigenvectors $\{\phi_j\}_{j=1}^M (M \in \mathbb{N})$ of $H$ such that $\|\Psi_0 - \sum_{j=1}^M \langle \phi_j, \Psi_0 \rangle \phi_j \| < \epsilon$. Let $\{R_m\}$ be a sequence of finite subsets of $V$ such that $R_m \subset R_{m+1}$ and $\bigcup_m R_m = V$. It follows that

$$\nu_n^\Psi_0(R_m^c)^{1/2} \leq \sum_{j=1}^M |\langle \phi_j, \Psi_0 \rangle||P_{R_m^c} \phi_j| + \epsilon,$$

which proves $\lim_{m \to \infty} \sup_n \nu_n^\Psi_0(R_m^c) = 0$. Hence we have (ii). \qed

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**Proof of Proposition 2.3.** Combining Lemmas A.1 and A.2 yields the result that
\[ H_2 \subset H_1^\perp \subset \mathcal{H}_p(H) \subset H_2, \quad H_1 \subset H_2^\perp \subset \mathcal{H}_c(H) \subset H_1, \]
which proves the proposition.

**A.2 Proof of Equation (2.4)**

In this subsection, we prove the following:

**Lemma A.3.** Let \((U, \{H_v\}_{v \in V}) \in \mathcal{F}_{QW}\) and \(\Psi_0 \in \mathcal{H}\) satisfy
\[
\lim_{m \to \infty} \sup_n \nu_n^{\Psi_0}(R_m^c) = 0
\]
for an increasing sequence \(\{R_m\}\) of finite subsets of \(V\). Then, (2.4) holds. In particular, (2.4) holds for all \(\Psi_0 \in \mathcal{H}_p(H)\).

**Proof.** By assumption, we know that for any \(\epsilon > 0\), there exists \(m_0 \in \mathbb{N}\) such that \(\sup_n \nu_n^{\Psi_0}(R_m^c) < \epsilon\). Hence,
\[
\limsup_{n \to \infty} \nu_n^{\Psi_0}(R_m) \geq 1 - \epsilon. \tag{A.3}
\]
If \(\limsup_n \nu_n^{\Psi_0}(x) = 0\) for any \(x \in R_{m_0}\), then
\[
\limsup_n \nu_n^{\Psi_0}(R_{m_0}) = \sum_{x \in R_{m_0}} \limsup_n \nu_n^{\Psi_0}(x) = 0,
\]
which contradicts (A.3). Therefore, (2.4) holds for some \(x \in R_{m_0}\).

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