Asymptotic Outage Probability Analysis for General Fixed-Gain Amplify-and-Forward Multihop Relay Systems

Justin P. Coon, Senior Member, IEEE, Yue Wang, Member, IEEE, and Gillian Huang, Student Member, IEEE

Abstract

In this paper, we present an analysis of the outage probability for fixed-gain amplify-and-forward (AF) multihop relay links operating in the high SNR regime. Our analysis exploits properties of Mellin transforms to derive an asymptotic approximation that is accurate even when the per-hop channel gains adhere to completely different fading models. The main result contained in the paper is a general expression for the outage probability, which is a functional of the Mellin transforms of the per-hop channel gains. Furthermore, we explicitly calculate the asymptotic outage probability for four different systems, whereby in each system the per-hop channels adhere to either a Nakagami-$m$, Weibull, Rician, or Hoyt fading profile, but where the distributional parameters may differ from hop to hop. This analysis leads to our second main result, which is a semi-general closed-form formula for the outage probability of general fixed-gain AF multihop systems. We exploit this formula to analyze an example scenario for a four-hop system where the per-hop channels follow the four aforementioned fading models, i.e., the first channel is Nakagami-$m$ fading, the second is Weibull fading, and so on. Finally, we provide simulation results to corroborate our analysis.

Index Terms

Amplify-and-forward relaying, semi-blind relaying, fading, outage, Mellin transforms.

The authors are with Toshiba Research Europe Ltd., Telecommunications Research Laboratory, 32 Queen Square, Bristol, BS1 4ND; tel: +44 (0)117 906 0700, fax: +44 (0)117 906 0701, email: {justin.yue.wang.gillian.huang}@toshiba-trel.com.
I. INTRODUCTION

Amplify-and-forward (AF) relay systems have received a lot of attention recently due to their ability to improve coverage and, thus, capacity in a geographical sense. To date, two main AF protocols have been focused on in the literature: variable-gain (a.k.a. channel state information (CSI) assisted) AF relaying and fixed-gain AF relaying (see, e.g., [1] and the references therein). While the former method yields good performance when CSI is available at the relay nodes, the latter technique is more suitable in simple systems where such information is not available, although the performance of the system often suffers. In particular, fixed-gain AF relaying may be a good choice in low-complexity systems, such as emerging energy and utility management applications (e.g., “smart grid” and water metering communication networks) as well as industrial wireless sensor networks [2]. Consequently, we focus on the fixed-gain protocol in this paper.

As with many wireless communication systems operating in fading environments, the end-to-end outage probability is an important metric that can be used to characterize the performance of a fixed-gain AF relaying system. Several results on this topic have been published. In [1], the authors derived a bound on the end-to-end SNR of a fixed-gain AF link, which was used to study the outage probability when each hop fades according to a Nakagami-\(m\) distribution. This analysis was adapted and extended in [3] for cases where the per-hop fading distributions are Nakagami-\(n\) (Rice) and Nakagami-\(q\) (Hoyt), and the outage probability was studied using Padé approximants. In [4], the authors derived tight closed-form bounds on the outage probability at asymptotically high SNR for the case where the underlying channel power probability density function (PDF) is nonzero at the origin, a condition that is valid for Rayleigh, Rician, and Hoyt fading, but excludes Nakagami-\(m\) (for \(m > 1\)) and Weibull fading. Other more recent performance analysis results for multihop relay networks focus on variable-gain AF relaying, particularly in Nakagami-\(m\) fading channels [6], [7].

There are two main drawbacks to results currently available on fixed-gain AF relaying, which

\(^1\)Symbol error probability is also an important metric that has been investigated for fixed-gain AF relaying (see, e.g., [3]–[5] and the references therein).
we aim to improve upon in this paper:

1) Many of the results found in the literature are given as lower bounds on the outage probability (see, e.g., [1], [3], [4]). To date, exact asymptotic results (not bounds) for general multihop systems have not been reported. Indeed, exact results for dual-hop systems operating in a class of Nakagami-\(m\) channels have only recently been published [8], [9].

2) Most analysis of multihop systems to date considers homogeneous fading, i.e., each hop fades according to the same distribution. Results for inhomogeneous systems appear to be limited to dual-hop links (see, e.g., [10], [11]), although this scenario is likely to be encountered frequently in practical multihop systems.

In this paper, we present a general framework for analyzing the outage probability of fixed-gain AF multihop relay systems operating in the high SNR regime. In contrast to outage performance analysis that currently exists in the literature, our approach does not rely on bounding the outage probability, but rather exploits properties of Mellin transforms to derive an asymptotic approximation – which is in the form of a functional of the Mellin transforms of the per-hop channel gain PDFs – that is accurate even when the per-hop channel gains adhere to completely different fading models. Furthermore, the nature of the functional allows us to apply the residue theorem from complex analysis to derive asymptotic approximations for the outage probability for specific fading models – including Nakagami-\(m\), Weibull, Rician, and Hoyt fading – in terms of elementary functions, which can be calculated easily in practice. Our contribution culminates in the introduction of a semi-general asymptotic formula for the outage probability of fixed-gain AF multihop systems.

The rest of the paper is organized as follows. In Section II, we define the AF system model. In Section III, the general framework for the asymptotic outage probability is detailed. This analysis draws heavily on properties of Mellin transforms (outlined in Appendix A for convenience). In Section IV, the analytical framework is applied to scenarios whereby the per-hop channels adhere to either a Nakagami-\(m\), Weibull, Rician, or Hoyt fading profile, or a combination thereof. This section concludes with the introduction of a semi-general asymptotic formula for the outage
probability and a brief discussion on the convergence of the asymptotics. Simulation results that corroborate our analysis are presented in Section V, and conclusions are drawn in Section VI.

II. SYSTEM MODEL

Consider a multihop network with a source node, a destination node, and \( N - 1 \) relay nodes in between where \( N \geq 2 \) (see Fig. [I]). Communication can only be achieved in a half-duplex manner between adjacent nodes. A data symbol \( d \) is conveyed to the first relay node. For simplicity and without loss of generality, we let \( E[|d|^2] = 1 \). This symbol is affected by flat fading in the transmission medium and additive Gaussian noise at the receiver (i.e., the relay node).

The received signal is then amplified by a fixed gain \( A_1^2 \), then conveyed to the next relay node and so on until the destination is reached. Denote the channel coefficient modelling the channel between the \((n - 1)\)th relay (or the source) and the \( n \)th relay (or the destination) by \( h_n \). Also, denote the additive noise at the \( n \)th relay node (or the destination) by \( v_n \), which is zero-mean complex Gaussian distributed with variance \( N_{0,n}/2 \) per dimension. Now, we can write the following input-output system equation \[ r = \left( \prod_{n=1}^{N} A_{n-1} h_n \right) d + \sum_{n=1}^{N} \left( \prod_{j=n+1}^{N} A_{j-1} h_j \right) v_n \] (1)

where \( A_0 = 1 \). Various amplification factors have been proposed in the literature, including \[ A_n = \frac{1}{\sqrt{E[|h_n|^2] + N_{0,n}}} \] \( n = 1, \ldots, N - 1 \). (2)

In order to maintain generality, however, we do not explicitly define \( A_n \) in what follows.

Since we will eventually be interested in the asymptotics of the outage probability, it is beneficial to define a reference parameter \( \bar{\gamma} \) that gives a notion of average SNR across the \( N \)-hop link. In this case, we let \( N_{0,n} = \rho_n/\bar{\gamma} \) for \( n = 1, \ldots, N \) where \( \{\rho_n\} \) are strictly positive (and finite) scaling factors and \( \rho_1 = 1 \) for convenience. In order to illustrate the physical definition of \( \bar{\gamma} \), we note that when (2) is adopted for the amplification factors, the average SNR for the \( n \)th hop is given by \( \bar{\gamma}_n = (E[|h_n|^2]/\rho_n) \bar{\gamma} \). We can now write the following expression for the
instantaneous end-to-end SNR:

$$\text{SNR} = \frac{\prod_{n=1}^{N} A_{n-1}^2 |h_n|^2}{\sum_{n=1}^{N} \rho_n \prod_{j=n+1}^{N} A_{j-1}^2 |h_j|^2 \bar{\gamma}}.$$  \hspace{1cm} (3)

III. Outage Probability

For the ease of exposition, we define the random variable $X_n = A_{n-1}^2 |h_n|^2$ in the following analysis. It follows from (3) that the outage probability of the fixed-gain AF multihop link can be written as

$$P_o = P(\text{SNR} < \gamma_{th})$$

$$= P \left( \prod_{n=1}^{N} X_n - \sum_{n=1}^{N-1} \sigma_n \prod_{j=n+1}^{N} X_j < \sigma_N \right)$$

$$= P \left( \left( \cdots \left( (X_1 - \sigma_1) X_2 - \sigma_2 \right) \cdots \right) X_N < \sigma_N \right)$$  \hspace{1cm} (4)

where we have defined $\sigma_n(\bar{\gamma}) = \rho_n \gamma_{th}/\bar{\gamma}$ for brevity. We further define the random variables

$$Z_n = W_n X_{n+1}, \quad n = 0, \ldots, N - 1$$  \hspace{1cm} (5)

and

$$W_n = W_{n-1} X_n - \sigma_n > 0, \quad n = 1, \ldots, N - 1$$  \hspace{1cm} (6)

with $W_0 \triangleq 1$. Note that $W_n$ is a conditional random variable in that it relates to the translation of $Z_{n-1}$ where it is given that $Z_{n-1} > \sigma_n$. Also, it is clear that $W_n$ and $X_{n+1}$ are statistically independent. Now we can apply this same conditioning on (4) recursively to obtain

$$P_o = P( Z_0 \leq \sigma_1 ) + P( Z_0 > \sigma_1 ) P \left( \left( \cdots (W_1 X_2 - \sigma_2) \cdots \right) X_N < \sigma_N \right)$$

$$= P( Z_0 \leq \sigma_1 ) + P( Z_0 > \sigma_1 ) \left( P( Z_1 \leq \sigma_2 ) + P( Z_1 > \sigma_2 ) \cdots \right. \times \left. \left( P( Z_{N-2} \leq \sigma_{N-1} ) + P( Z_{N-2} > \sigma_{N-1} ) P( Z_{N-1} < \sigma_N ) \right) \cdots \right)$$

$$= 1 - \prod_{n=1}^{N} F_{Z_{n-1}}(\sigma_n).$$  \hspace{1cm} (7)
This is a satisfyingly simple exact expression for the outage probability, although the calculation of the CCDFs of \( \{ Z_n \} \) remain. It turns out we can construct an elegant lemma using Mellin transforms (see Appendix A) that aids this calculation.

**Lemma 1:** The Mellin transform of \( f_{Z_n} \) can be approximated by

\[
M\left[ f_{Z_n}; s \right] \approx \frac{1}{\prod_{j=1}^{n} F_{Z_{j-1}}(\sigma_j)} \sum_{\ell_1=0}^{L_1-1} \cdots \sum_{\ell_n=0}^{L_n-1} (-1)^{\sum_{j=1}^{n} \ell_j} \times \frac{\Gamma(s) \prod_{j=1}^{n} \sigma_j^{\ell_j}}{\Gamma(s - \sum_{j} \ell_j) \prod_{j=1}^{n} \ell_j!} \prod_{j=1}^{n+1} M\left[ f_{X_j}; s - \sum_{k=j}^{n} \ell_k \right]
\]

where \( L_1, \ldots, L_n \geq 1 \) are integers that define the order of the approximation.

**Proof:** First, we note that \( M\left[ f_{Z_0}; s \right] = M\left[ f_{X_1}; s \right] \). Now, from (5) and property (32), we have that \( M\left[ f_{Z_{n+1}}; s \right] = M\left[ f_{W_{n+1}}; s \right] M\left[ f_{X_{n+2}}; s \right] \). It is easy to see from the definition of the random variable \( W_n \) that \( f_{W_{n+1}}(w) = f_{Z_n}(w + \sigma_{n+1}) / F_{Z_n}(\sigma_{n+1}) \) for \( w \geq 0 \). The proof follows from induction on \( n \), where properties (29) and (30) are applied in the inductive step.

We are now in the position to state our first main result in the form of the following proposition.

**Proposition 2:** The outage probability of an \( N \)-hop fixed-gain AF link is asymptotically given by

\[
P_o \sim 1 - \sum_{\ell_1=0}^{L_1-1} \cdots \sum_{\ell_{N-1}=0}^{L_{N-1}-1} (-1)^{\lambda_{N-1}} \prod_{n=1}^{N-1} \frac{1}{\ell_n!} \left( \frac{\rho_n}{\rho_N} \right)^{\ell_n} \times \frac{1}{2\pi i} \int_{-\kappa-i\infty}^{-\kappa+i\infty} \left( \frac{\rho_N \gamma \theta_n}{\bar{\gamma}} \right)^s \frac{\Gamma(\lambda_{N-1} - s)}{\Gamma(1 - s)} \prod_{n=1}^{N} M\left[ f_{X_n}; 1 + \lambda_{n-1} - s \right] ds
\]

as \( \bar{\gamma} \to \infty \) with \( \kappa > \lambda_{N-1} \), where \( \lambda_n = \sum_{k=1}^{n} \ell_k \) and \( f_{X_n} \) denotes the PDF of the channel power for the \( n \)th hop.

**Proof:** Starting from Lemma 1 we use (31) to calculate \( M\left[ F_{Z_{N-1}}; s \right] \), then take the inverse transform using (28) and substitute this into (7), resulting in a cancellation of the product \( \prod_{n=1}^{N-1} F_{Z_{n-1}}(\sigma_n) \). Eq. (9) follows from a change of variables in the contour integral.

\[\text{The notation } \sim \text{ denotes asymptotic equivalence in the relevant variable and limit, } \bar{\gamma} \to \infty \text{ in this case.}\]
Although this appears to be a rather complicated expression for $P_o$, we note that it can be computed as long as we know the Mellin transforms of the individual PDFs $f_{X_n}$, which can generally be calculated easily for many cases of interest. Thus, assuming we can compute these transforms, we can at least evaluate the outage probability bound numerically to a high degree of accuracy provided the transforms decay quickly as $|\text{Im}(s)| \to \infty$.

The Mellin transforms for the main fading distributions of interest are given in Table I. The key thing to notice from this table that makes the ensuing analysis uniform and tractable is that each PDF decays exponentially. In the transform domain, this results in the property that the transform of each fading distribution has poles at various points along the real axis. We will exploit this property in the next section to derive simple asymptotic expressions for $P_o$ under various fading conditions.

IV. ANALYSIS OF COMMON FADING DISTRIBUTIONS

We now apply the general analysis detailed above to a number of common fading distributions. For much of this section, we assume that all hops adhere to the same class of distribution (e.g., Nakagami-$m$) but where the channels for individual hops may vary in their distributional parameters (e.g., shape factors). We label this condition “homogeneity” in this context, and we later relax this restriction in order to analyze general multihop links as well as to show the power and versatility of adopting the proposed analytical framework. The distributions considered here (also listed in Table I) are derived from Nakagami-$m$, Weibull, Rician, and Hoyt fading. The analysis can be extended to other fading distributions using (9) and the techniques outlined below. The section concludes with our second major result, which is in the form of a semi-general closed-form formula for the outage probability for large $\bar{\gamma}$.

$^3$The transforms for the Rician and Hoyt distributions can be calculated by consulting standard tables of integrals and transforms (e.g., [14], [15]).
A. Nakagami-m Fading

If the per hop channels adhere to a Nakagami-m fading profile, the random variable \( X_n \) has a gamma density function with scale parameter \( \theta_n \) and shape parameter \( m_n \). The corresponding PDF and Mellin transform are given in Table I. By substituting this transform into (9) and applying the Mellin-Barnes integral definition of the Meijer G-function \([15]\) along with the functional relations \([15, 9.31.2]\) and \([15, 9.31.5]\), we can write the outage probability as

\[
P_{o,Nak} \sim 1 - \sum_{\ell_1, \ldots, \ell_{N-1}} (-1)^{\lambda_{N-1}} \xi_\ell G_{1,N+1}^{N+1,0} \left( \frac{\sigma_N}{\prod_{j=1}^{N} \theta_j} \right)_{\lambda_{N-1}, m_1, m_2 + \lambda_1, \ldots, m_N + \lambda_{N-1}}^{1}
\]

where

\[
\xi_\ell = \frac{1}{\Gamma(m_N)} \prod_{n=1}^{N-1} \frac{\theta_n^{\lambda_n}}{\ell_n! \Gamma(m_n)}
\]

with \( \ell = (\ell_1, \ldots, \ell_{N-1}) \) and \( \lambda_n = \sum_{k=1}^{n} \ell_k \) (defining \( \lambda_0 = 0 \)). It is understood that the summation in (10) is \((N-1)\)-fold, with the sum over the \( n \)th index \( \ell_n \) running from zero to \( L_n - 1 \). This expression can be evaluated easily using mathematical software tools such as Mathematica.

We may wish to consider the leading order expansion of \( P_{o,Nak} \) at high SNR, which would yield expressions for the diversity and coding gains of the multihop link. To do this, we first note that the Meijer G-function used here is defined as a Mellin-Barnes integral where the integration path goes from \(-i\infty\) to \(i\infty\) such that it separates the poles of the integrand \([16, \S16.17]\). It is straightforward to show that this integral converges in our application and that the integrand, given by

\[
I_{Nak}(s) = z(\bar{\gamma})^s \frac{\Gamma(\lambda_{N-1} - s) \prod_{n=1}^{N} \Gamma(m_n + \lambda_{n-1} - s)}{\Gamma(1 - s)}
\]

where \( z(\bar{\gamma}) = (\rho_N \gamma_{th} / \prod_n \theta_n) \bar{\gamma}^{-1} \), is well-behaved as \( s \to \infty \) in the right half \( s \)-plane. Moreover, \( I_{Nak}(s) \) has poles at \( s = m_n + \lambda_{n-1} + j \) for \( n = 0, \ldots, N - 1 \) and \( j = 0, 1, \ldots \), and at \( s = 0 \) when \( \lambda_{N-1} = 0 \). Thus, we can employ the residue theorem with the usual closing arc.

\[\text{See, e.g.,} [17] \text{for an introduction to complex analysis and the residue theorem.}\]
in the right half plane to evaluate the $G$-functions in (10) at high SNR, which leads to a more accessible and intuitive asymptotic expression for the outage probability given by

$$P_{o,Nak} \sim \frac{(\rho N \gamma_{th})^m}{\prod_n \theta_n^m} \sum_{\ell_1, \ldots, \ell_{N-1}} \sum_{r=1}^{\mu} (-1)^{\lambda_{N-1}} \xi_{\ell} \sum_{l=0}^{r-1} \frac{\mu^{(r-1-l)}(m)}{(r-1-l)!!} \prod_n \theta_n \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in L_r} \left(\frac{-1}{l}ight)^p (\log \rho N \gamma_{th})^{l-p} \frac{(\log \bar{\gamma})^p}{\bar{\gamma}^m} + o(\bar{\gamma}^{-m}) \quad (12)$$

where $m = \min \{m_n\}$, $\mu$ denotes the multiplicity of $m$, and

$$\nu(s) = (s - m)^r \frac{I_{Nak}(s)}{z^s}.$$

The sets $L_1, \ldots, L_\mu$ are disjoint sets of $(N-1)$-tuples $\{\ell_1, \ldots, \ell_{N-1}\}$ defined such that $I_{Nak}(s)$ has an $r$th order pole at $s = m$ when $\{\ell_1, \ldots, \ell_{N-1}\} \in L_r$. The details of the calculations that lead to this result (and a rigorous definition of the sets $L_1, \ldots, L_\mu$) are given in the appendix.

This expression can be evaluated easily for specific examples, but by retaining only the leading order term (i.e., $r = \mu$), we arrive at the following general asymptotic equivalence:

$$P_{o,Nak} \sim \psi_{Nak} (\log \bar{\gamma})^{\mu-1} \bar{\gamma}^{-m} \quad (13)$$

where

$$\psi_{Nak} = \frac{(\rho N \gamma_{th})^m}{\prod_n \theta_n^m} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in L_\mu} \sum_{\ell_1} (-1)^{\lambda_{N-1}+\mu-1} \xi_{\ell} \nu(m) \frac{\mu^\nu(m)}{(\mu-1)!} \quad (14)$$

is the coding gain of the link. For the case where each hop fades independently of others and all hops experience nonidentically shaped fading (i.e., $m_1 \neq \cdots \neq m_N$), $\mu = 1$ and we have

$$P_{o,Nak} \sim \frac{(\rho N \gamma_{th})^m}{\prod_n \theta_n^m} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in L_1} (-1)^{\lambda_{N-1}} \xi_{\ell} \nu(m) \bar{\gamma}^{-m} \quad (15)$$

The notation $g(\bar{\gamma}) = o(h(\bar{\gamma}))$ signifies that $\lim_{\bar{\gamma} \to \infty} g(\bar{\gamma}) / h(\bar{\gamma}) = 0$.

This definition of the coding gain is slightly different to the one typically used in system analysis. In fact, the standard definition of the coding gain (see, e.g., [18]) cannot be applied here since it is only valid when $P_{out}$ obeys a power law decay in $\bar{\gamma}$.
In the other extreme where all hops experience identically shaped fading (i.e., $m = m_1 = \cdots = m_N$), $\mu = N$ and we can write

$$P_{o,Nak} \sim \frac{(\rho N \gamma_{th})^m}{(N-1)!m \Gamma(m)^N \prod_n \hat{\theta}_n^m} \frac{(\log \bar{\gamma})^{N-1}}{\bar{\gamma}^m}.$$  

(16)

The expressions given above demonstrate the well-known fact that $m$th order diversity is achieved in these cases. However, the analysis is useful in illustrating the rate (with respect to SNR growth) at which $m$th order diversity is attained. By taking the (usual) definition of diversity to be

$$d = \lim_{\bar{\gamma} \to \infty} \frac{\log P_{o,Nak}(\bar{\gamma})}{-\log \bar{\gamma}},$$

(17)

we see from (13) that, for finite but large values of $\bar{\gamma}$,

$$d(\bar{\gamma}) = m - (\mu - 1) \frac{\log \log \bar{\gamma}}{\log \bar{\gamma}} + O\left(\frac{1}{\log \bar{\gamma}}\right).$$

(18)

Although this result points to slow convergence in terms of diversity (due to the $\log \log \bar{\gamma}/\log \bar{\gamma}$ term), it does not illustrate the full picture since the coding gain must be taken into account. Indeed, through our analysis, we have presented an accurate expression for the coding gain of a fixed-gain AF multihop link, which to the best of the authors’ knowledge has not yet been reported in the literature. The coding gain is, in general, a complicated expression, although it is straightforward to compute. However, we can draw some conclusions about the behavior of certain systems, such as those that experience identically shaped fading. For example, (16) points to the importance of having a well-designed destination receiver with a high-performance low-noise amplifier (LNA) when operating in the high SNR regime ($\rho N/\bar{\gamma} = N_{0,N}$ must be as low as possible).

Finally, it should be noted that the leading order results given here are not very accurate for low to mid-range $\bar{\gamma}$ when $\mu > 1$. This results from the fact that $\log \bar{\gamma}$ increases very slowly, which effectively means that all $(\log \bar{\gamma})^p \bar{\gamma}^{-m}$ terms are of roughly the same order for finite $\bar{\gamma}$ and, thus, should be included in the approximation. In such a case, it is best to use the general expansion given by (12).
B. Weibull Fading

If the channels adhere to a Weibull fading profile, the random variable $X_n$ has a Weibull density function. The corresponding PDF and Mellin transform are given in Table I. By substituting this transform into (9) and applying the definition of the Fox $H$-function [19] along with the functional relations [20, (2.8)] and [20, (2.11)], we can write the outage probability as

$$P_{o,Weibull} \sim 1 - \sum_{\ell_1, \ldots, \ell_{N-1}} (-1)^{\lambda_{N-1}} \varphi_{\ell} H_{1,N+1}^{N+1,0} \left( \frac{\sigma_N}{\prod_{j=1}^{N} \theta_j} \left( \lambda_{N-1}, 1 \right), \left( 1 + \frac{\lambda_n}{m_1}, \frac{1}{m_1} \right), \ldots, \left( 1 + \frac{\lambda_{N-1}}{m_N}, \frac{1}{m_N} \right) \right)$$

where

$$\varphi_{\ell} = \prod_{n=1}^{N-1} \frac{\beta_n^{\ell_n}}{\ell_n!} \left( \frac{\rho_n}{\rho_N} \right)^{\ell_n}.$$

As with the Nakagami-$m$ case, we can employ the residue theorem to obtain a simple asymptotic expression for $P_{o,Weibull}$ using elementary functions. Omitting the details (the methodology is the same as was described for Nakagami-$m$ fading in the appendix), it is possible to derive the following asymptotic expression:

$$P_{o,Weibull} \sim \frac{(\rho_N \gamma_{th})^m}{\prod_n \theta_n^m} \sum_{r=1}^{\mu} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_r} (-1)^{\lambda_{N-1}} \varphi_{\ell} \sum_{l=0}^{r-1} \omega^{(r-1-l)}(m) \frac{(r-1-l)!l!}{(r-1-l)!}$$

$$\times \sum_{p=0}^{l} \binom{l}{p} (-1)^p \left( \log \frac{\rho_N \gamma_{th}}{\prod_n \theta_n} \right)^{l-p} \frac{\left( \log \bar{\gamma} \right)^p}{\bar{\gamma}^m} + o(\bar{\gamma}^m)$$

where again $m = \min \{m_n\}$, $\mu$ denotes the multiplicity of $m$, and

$$\omega(s) = (s-m)^{r} \frac{\Gamma(\lambda_{N-1} - s) \prod_{n=1}^{N} \Gamma \left( 1 + \frac{\lambda_n - 1}{m_n} - s \right)}{\Gamma(1-s)}.$$

To leading order in $\bar{\gamma}$, we have the asymptotic equivalence

$$P_{o,Weibull} \sim \psi_{Weibull} \frac{(\log \bar{\gamma})^{\mu-1}}{\bar{\gamma}^m}$$

where

$$\psi_{Weibull} = \frac{(\rho_N \gamma_{th})^m}{\prod_n \theta_n^m} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_\mu} (-1)^{\lambda_{N-1}+\mu-1} \varphi_{\ell} \omega(m) \frac{(\mu-1)!}{(\mu-1)!}$$

is the coding gain. Note that this analysis points to similar diversity and coding gain behavior as was discussed for Nakagami-$m$ fading channels.
C. Rician and Hoyt Fading

No closed form expression of $P_o$ exists for the cases where all hops follow a Rician (or Hoyt) fading model. For the Rician case, we can employ the transform given in Table I along with Proposition 2 to obtain an expression for $P_{o,Rice}$ in terms of a contour integral, which can be evaluated numerically using, for example, Mathematica. The residue theorem can be applied to this integral to derive an asymptotic expression in the form of

$$P_{o,Rice} \sim \frac{\rho_N \gamma_{th} \prod_n (K_n + 1)}{\prod_n \theta_n} \sum_{r=1}^{N} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_r} (-1)^{\lambda_{N-1}} \zeta_\ell \sum_{l=0}^{r-1} \frac{\varrho^{(r-1-l)}(1)}{(r-1-l)!} \prod_{n=1}^{r-1} \left( -1 \right)^{l-p} \left( \log \bar{\gamma} \right)^p \frac{\gamma}{\bar{\gamma}} + o \left( \bar{\gamma}^{-1} \right) \quad (23)$$

where

$$\zeta_\ell = e^{-K_N} \prod_{n=1}^{N-1} e^{-K_n} \left( \frac{\rho_n}{\rho_N} \right)^{\epsilon_n} \left( \frac{\theta_{n+1}}{K_{n+1} + 1} \right)^{\lambda_n}$$

and

$$\varrho(s) = (s - 1)^r \frac{\Gamma (\lambda_{N-1} - s)}{\Gamma (1 - s)} \prod_{n=1}^{N} (1 + \lambda_{n-1} - s) \ {}_1 F_1 (1 + \lambda_{n-1} - s; 1; K_n).$$

To leading order in $\bar{\gamma}$, we have

$$P_{o,Rice} \sim \frac{\rho_N \gamma_{th} \prod_n (K_n + 1)}{(N-1)! \prod_n \theta_n e^{K_n}} \frac{(\log \bar{\gamma})^{N-1}}{\bar{\gamma}} \quad (24)$$

Similarly, for the case where all hops follow a Hoyt fading model, we can follow the same procedure to derive the asymptotic expression

$$P_{o,Hoyt} \sim \frac{\rho_N \gamma_{th} \prod_n \left( 1 + q_n^2 \right)^2}{\prod_n 4q_n^2 \theta_n} \sum_{r=1}^{N} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_r} (-1)^{\lambda_{N-1}} \chi_\ell \sum_{l=0}^{r-1} \frac{\eta^{(r-1-l)}(1)}{(r-1-l)!} \prod_{n=1}^{r-1} \left( -1 \right)^{l-p} \left( \log \bar{\gamma} \right)^p \frac{\gamma}{\bar{\gamma}} \quad (25)$$

where

$$\chi_\ell = \left( \frac{2q_N}{1 + q_N^2} \right)^{1 + 2\lambda_{N-1}} \prod_{n=1}^{N-1} \frac{\theta_{n+1}}{\epsilon_n} \left( \frac{\rho_n}{\rho_N} \right)^{\epsilon_n} \left( \frac{2q_n}{1 + q_n^2} \right)^{1 + 2\lambda_{n-1}}$$
and
\[ \eta(s) = (s - 1)^r \frac{\Gamma(\lambda_{N-1} - s)}{\Gamma(1 - s)} \prod_{n=1}^{N} \Gamma(1 + \lambda_{n-1} - s) \ _2F_1 \left( \frac{1 + \lambda_{n-1} - s}{2}, \frac{2 + \lambda_{n-1} - s}{2}; 1; \left( \frac{1 - q_n^2}{1 + q_n^2} \right)^2 \right). \]

To leading order in $\bar{\gamma}$, (25) reduces to
\[ P_{o,Hoyt} \sim \frac{\rho N \gamma_{th}}{(N - 1)!} \prod_{n=1}^{N} \frac{1 + q_n^2 (\log \bar{\gamma})^{N-1}}{2q_n \bar{\theta}_n} \bar{\gamma}. \tag{26} \]

### D. Semi-general Formula

It is clear that similarities exist between the expressions for $P_o$ given for the different fading distributions analyzed above. This observation leads to our second main result, which is in the form of a semi-general asymptotic formula for the outage probability of fixed-gain AF multihop links.

**Conjecture 3:** Consider an $N$-hop fixed-gain AF link, where the PDF of the channel power for each hop decays exponentially, and thus has a Mellin transform that is well-behaved at $|\text{Im}(s)| = \infty$. The outage probability of this link is given by
\[ P_o \sim A^m \sum_{r=1}^{\mu} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_r} (-1)^{\lambda_{N-1}} B_{\ell} \sum_{l=0}^{r-1} C_r^{(r-1-l)} (m) \frac{(\log \bar{\gamma})^{N-1}}{2^{l-1} l!} \times \sum_{p=0}^{l} \left( \frac{l}{p} \right) (-1)^p (\log A)^{l-p} \left( \frac{\log \bar{\gamma}}{\bar{\gamma}^m} \right)^p + o(\bar{\gamma}^{-m}) \tag{27} \]
as $\bar{\gamma} \to \infty$. In (27), the constants $A$ and $B_{\ell}$ are dependent upon the distributional parameters for the $N$ channels in the link with $B_{\ell}$ being dependent upon the indices in the vector $\ell = (\ell_1, \ldots, \ell_{N-1})$ as well; $m$ defines the minimum shape parameter of the $N$ channel power distributions with $\mu$ being the multiplicity of $m$ (e.g., $m = 1$ and $\mu = N$ if all hops are Rayleigh, Rician, or Hoyt); and $C_r(s) = (s - m)^r I(s) (\bar{\gamma}/A)^s$ with $I(s)$ being a simple form of the integrand in (9) where all terms independent of $s$ have been removed.

This result is stated as a conjecture since we make no attempt to rigorously define the properties that the channel power PDFs must have to make it a theorem. Indeed, only a few channel models are typically employed in practice. These models include those discussed in Sections IV-A through IV-C. We can easily apply the formula to those scenarios.
Additionally, this formula can be used to characterize the outage probability of inhomogeneous links. As a toy example, which is perhaps not often encountered in practice but serves as a useful illustration, consider a four-hop link where the per-hop channels are consecutively modelled as experiencing Nakagami-\(m\), Weibull, Rician, and Hoyt fading. In this case, it is straightforward to show that the semi-general formula given by (27) holds with the following definitions:

\[
A = \frac{\rho N \gamma_{th} (K_3 + 1) (1 + q_4^2)^2}{4 q_4^2 \prod_n \theta_n}
\]

\[
B_\ell = \frac{e^{-K_3}}{\Gamma (m_1) (K_3 + 1)^{\lambda_2}} \left( \frac{2 q_4}{1 + q_4^2} \right)^{1+2\lambda_3} \prod_{n=1}^{N-1} \frac{\theta_n^{\lambda_n}}{\ell_n!} \left( \frac{\rho_n}{\rho N} \right)^{\ell_n}
\]

\[
C_r (s) = (s - m)^r \frac{\Gamma (\lambda_{N-1} - s)}{\Gamma (1 - s)} \Gamma (m_1 - s) \Gamma \left( 1 + \frac{\lambda_1 - s}{m_2} \right) \Gamma (1 + \lambda_2 - s) \Gamma (1 + \lambda_3 - s)
\]

\[
\times {}_1 \! F_1 \left( 1 + \lambda_2 - s, 1; K_3 \right) {}_2 \! F_1 \left( \frac{1 + \lambda_3 - s}{2}, \frac{2 + \lambda_3 - s}{2}; 1; \left( \frac{1 - q_4^2}{1 + q_4^2} \right)^{2} \right)
\]

\[
m = \min \{ m_1, m_2, 1 \}.
\]

This example is discussed further in Section [V]

**E. Convergence of Asymptotics**

It is natural to determine the conditions under which the leading order expressions for \(P_o\) given in the preceding section serve as good approximations to the actual outage probability. First, we study the question: how large does \(\bar{\gamma}\) have to be before “asymptotic” becomes “approximate”? For nonidentical Nakagami-\(m\) and Weibull fading scenarios, convergence occurs quickly since the leading order expression monotonically decreases with increasing \(\bar{\gamma}\). For quasi-identical fading, as well as Rician and Hoyt fading channels, we note that the asymptotic bound for \(P_o\) given above has the form \(P_o \sim b (\log \bar{\gamma})^{\mu - 1} \bar{\gamma}^{-m}\) where \(b\) is independent of \(\bar{\gamma}\), \(\mu \geq 2\), and \(m \geq 1\). For \(\bar{\gamma} > 1\), it can be shown that this expression has a maximum at \(\bar{\gamma} = e^{(\mu - 1)/m}\), and is monotonically decreasing for \(\bar{\gamma} > e^{(\mu - 1)/m}\). Thus, it is necessary that \(\bar{\gamma} \gg e^{(\mu - 1)/m}\) for the asymptotic expressions given above to be reasonable approximations for the outage probability. For hardened channels (i.e., when \(m \gg 1\) for Nakagami-\(m\) and Weibull fading) or systems with only a few hops, this condition is satisfied easily in practice.
V. SIMULATION RESULTS AND DISCUSSION

In this section, we present numerical results obtained for homogeneous and inhomogeneous multihop links. For the homogeneous case, we simulated a number of scenarios in order to validate our analytical results. All four fading distributions mentioned above were considered, and different system parameters were chosen to illustrate the accuracy of our analytical framework as well as some interesting behavior of fixed-gain multihop links. For the inhomogeneous case, the four-hop example discussed in Section IV-D was studied. For all calculations, we assume only a first order asymptotic correction, i.e., $L_n = 2$ for $n = 1, \ldots, N - 1$.

First, we present results for the case where each hop fades according to a Nakagami-$m$ distribution. Fig. 2 through Fig. 4 illustrate the outage probability as a function of $\bar{\gamma}$ for different three-hop links. Simulations are plotting along with analytical results obtained through the Meijer $G$-function expression (10) as well as the leading order asymptotic result derived from the application of the residue theorem, which is given by (13) and (14). Furthermore, we compared our results to a lower bound on the probability of outage that was developed in [3]. This bound, which can be obtained through an application of a harmonic-geometric mean bound on the end-to-end SNR, has subsequently been applied and discussed in a number of works (e.g., [4]).

The accuracy of our analysis is clear from these examples. Moreover, although the asymptotic expression converges to simulation results for large $\bar{\gamma}$, we see that the Meijer $G$-function expression is a good approximation even at low and mid-range SNR. We also observe that the harmonic-geometric mean bound is loose in some cases, particularly when the inherent diversity in the channel decreases with each hop (i.e., $m_n$ decreases with increasing $n$). This behavior was also noted in [4].

Fig. 4 illustrates the effect that quasi-identical fading has on the convergence of the asymptotic expression. In this example, the first and third hops yield the minimum shape parameter ($m_1 = m_3 = 1$). This leads to a second order pole in the residue analysis, and thus $P_{o,Nak} \sim \psi_{Nak} (\log \bar{\gamma}) \bar{\gamma}^{-1}$. The logarithmic term delays convergence as discussed in Section IV-E and this is apparent in the figure in the plot of the leading order expansion. It is also
clear that the approximation can be made to be much more accurate for moderate SNR levels by refining the expansion through the inclusion of all correction terms of order $\Omega(\bar{\gamma}^{-1})$ (the curve labelled “Refined Asymptotic” in the figure).

Finally, we provide results for a five-hop system in Fig. 5. Here, we adopt the same shape factors that were used in [1, cf. Fig. 4 therein]; specifically, we have $m_1 = m_2 = 5, m_3 = m_4 = 2.5, \text{ and } m_5 = 1.5$. Again, there is excellent agreement between the simulation results and the Meijer $G$-function expression for $P_{o,Nak}$, and the asymptotic result converges around 13 dB. In contrast, the harmonic-geometric mean lower bound diverges.

In Fig. 6 we present results for homogeneous links where each hop experiences Weibull fading. Two, three, and four hop systems were considered where the shape parameters were chosen to be $m_1 = 1.5, m_2 = 2, m_3 = 2.5, \text{ and } m_4 = 1$. Again, the analytical results agree well with the simulations. Furthermore, we see how the addition of a fourth hop with less inherent diversity compared to the other hops affects performance. As previously mentioned, the hop with the minimum shape parameter dictates performance at high SNR, which is evident from the loss in diversity in the $N=4$ hop link in the figure.

Results for Rician and Hoyt channels are given in Fig. 7 and Fig. 8 respectively. The degradation in performance with increasing numbers of hops is apparent for both cases. Moreover, the loss in finite SNR diversity can also be observed, which results from the $\log \bar{\gamma}$ terms in the numerator of the high SNR expansion for $P_o$. The characterization of this behavior is beyond the scope of this paper, but is an interesting observation, nonetheless.

Finally, we present results for an inhomogeneous link in Fig. 9. In this example, the fading channel corresponding to the first hop adheres to a Nakagami-$m$ distribution with $m_1 = 2$. The second hop channel follows a Weibull profile with $m_2 = 1.5$. The channels related to the third and fourth hops follow Rician and Hoyt distributions, respectively, with $K_3 = 3$ and $q_4 = 3/4$. The scale parameter for the $n$th hop is given by $\theta_n = n/2$ and we have defined $\rho_n = 1 - (n - 1)/10$. In Fig. 9 the leading order asymptotic and the semi-general formula are

\footnote{The notation $f(x) = \Omega(g(x))$ implies $\exists x_0, k > 0$ such that $f(x) \geq k \cdot g(x)$ for $x > x_0$.}
plotted along with simulation results. Again, we see that although the graph points to convergence for the leading order expression in the asymptotic limit, the semi-general formula provides a much more accurate expression at low and mid-range SNR. Finally, it should be noted that this complex example cannot be studied with the theory that has been detailed in the literature to date, which exemplifies the versatility of our analytical framework.

VI. CONCLUSIONS

In this paper, we presented a novel, rigorous asymptotic analysis of the outage probability for fixed-gain AF multihop relay systems. Our analysis was general in nature, lending itself to application in a range of scenarios, including cases where the per hop fading processes adhere to completely different models and distributions. Specifically, we first provided a general asymptotic formula for the outage probability that is applicable to any system where the hops are statistically independent. We then provided analytical expressions for different fading distributions – namely Nakagami-$m$, Weibull, Rician, and Hoyt fading – and gave a brief discussion on the convergence of these formulae, which culminated in a semi-general closed-form formula for the outage probability at high SNR that can be applied to analyze homogeneous and inhomogeneous systems. Finally, we demonstrated through simulations that our theory is accurate, even at low to mid-range SNR in many cases of interest.

APPENDIX A

PROPERTIES OF MELLIN TRANSFORMS

The Mellin transform of a real-valued function $f(x)$ where $x \geq 0$ is defined by

$$M[f; s] = \int_0^\infty x^{s-1} f(x) \, dx$$

and its inverse is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M[f; s] \, ds \tag{28}$$
for $c > 0$. A useful identity exists for the Mellin transform of derivatives of $f$, namely that \cite{14} \ref{6.1 (10)}
\begin{equation}
M[f^{(\ell)}; s] = (-1)^{\ell} \frac{\Gamma(s)}{\Gamma(s-\ell)} M[f; s-\ell]
\end{equation}
provided the first $\ell$ derivatives exist and are well behaved. We can also employ Mellin transforms to approximate the average of a function (with respect to the kernel $x^{s-1}$) evaluated at a point close to, but to the right of $x$, \emph{i.e.}, $f (x + \epsilon)$ for $\epsilon > 0$. This is done by expanding $f$ near $x$, which yields
\begin{equation}
\int_{0}^{\infty} x^{s-1} f (x + \epsilon) \, dx = \int_{0}^{\infty} x^{s-1} \left( \sum_{\ell=0}^{L-1} \frac{\epsilon^{\ell}}{\ell!} f^{(\ell)} (x) + O (\epsilon^{L}) \right) \, dx \sim \sum_{\ell=0}^{L-1} \frac{\epsilon^{\ell}}{\ell!} M[f^{(\ell)}; s] + O (\epsilon^{L})
\end{equation}
again, provided the derivatives exist\footnote{The notation $\sim$ denotes asymptotic equivalence for $\epsilon \to 0$ in this case.}. Note that this is an asymptotic expansion; convergence of the series is not guaranteed in general.

If $X$ is a random variable and $f_X (x)$ denotes its density function, then $M[f_X; s] = E [x^{s-1}]$. If $f_X (x)$ is defined for $x \geq 0$, then we can define the complementary cumulative distribution function (CCDF)
\[ F_X (x) = \int_{x}^{\infty} f_X (t) \, dt. \]
Now we have the following identity, which is calculated using integration by parts and, in particular, holds for probability distributions with exponentially decaying tails\footnote{The identity is derived for the CCDF rather than the CDF in order to ensure the transform converges.}:
\begin{equation}
M[F_X; s] = s^{-1} M[f_X; s + 1].
\end{equation}
Finally, suppose $Z = \prod_{n=1}^{N} X_n$ where $\{X_n\}$ are statistically independent. Then it is easy to see that
\begin{equation}
M[f_Z; s] = \prod_{n=1}^{N} M[f_{X_n}; s].
\end{equation}
APPENDIX B
RESIDUE CALCULATIONS

It is instructive to outline some of the residue calculations that were made to obtain the results given in this paper. Most calculations follow similar reasoning. Consequently, we only include calculations for Nakagami-$m$ fading in this appendix.

Consider the function $I_{Nak} (s)$ defined in (11). Suppose there are $T \leq N$ unique shape parameters, where the $t$th parameter $\tilde{m}_t$ has multiplicity $\mu_t$ and $\sum_{t=1}^{T} \mu_t = N$. If $\lambda_{N-1} = 0$, $I_{Nak} (s)$ has a simple pole at $s = 0$ with residue $\text{res} (I_{Nak}, 0) = \prod_{n=1}^{N} \Gamma (m_n)$. Moreover, $I_{Nak}$ also has a pole at $s = s_{q,j} = m_q + \lambda_{q-1} + j$ for $q \in \{1, \ldots, N\}$ and $j \in \{0, 1, \ldots\}$, which is, in general, an $r$th order pole where $1 \leq r \leq N$. The residue of $I_{Nak}$ at this pole is given by

$$\text{res} (I_{Nak}, s_{q,j}) = -\frac{1}{(r-1)!} \lim_{s \to s_{q,j}} \frac{\partial^{r-1}}{\partial s^{r-1}} \left\{ \left( s - s_{q,j} \right)^r I_{Nak} (s) \right\}$$

$$= -\frac{1}{(r-1)!} \lim_{s \to s_{q,j}} \frac{\partial^{r-1}}{\partial s^{r-1}} \left\{ \nu_{q,j} (s) z^s \right\}$$

$$= -\frac{1}{(r-1)!} \sum_{l=0}^{r-1} \left( \begin{array}{c} r - 1 \\ l \end{array} \right) \nu_{q,j}^{(r-1-l)} (s_{q,j}) (\log z)^l z^{s_{q,j}}$$

(33)

where the third equality follows from the Leibniz rule of differentiation of products (with $\nu^{(n)} (a)$ being the $n$th derivative of $\nu$ evaluated at $a$) and $\nu_{q,j} (s) = (s - s_{q,j})^r I_{Nak} (s) z^{-s}$.

The residue theorem states that $P_o$ can be expressed as a series of residues of poles of $I_{Nak}$ [17]. Since $z \propto \bar{\gamma}^{-1}$, if we wish to construct an approximation to $P_o$ that is a function of the leading power of $z$ (as $\bar{\gamma}$ grows large), then we can ignore all residues for which $j > 0$. This leaves a finite summation of residues corresponding to different shape parameters. In fact, it is clear from (33) that out of these residues, those at poles relating to the smallest shape parameter in the set $\{m_1, \ldots, m_N\}$ dominate for large $\bar{\gamma}$. Thus, we only care about residues of the form

$$\text{res} (I_{Nak}, m_{\hat{q}} + \lambda_{\hat{q}-1}) = -\frac{1}{(r-1)!} \sum_{l=0}^{r-1} \left( \begin{array}{c} r - 1 \\ l \end{array} \right) \nu_{\hat{q},0}^{(r-1-l)} (m_{\hat{q}} + \lambda_{\hat{q}-1}) (\log z)^l z^{m_q + \lambda_{\hat{q}-1}}$$

where $\hat{q} \in \{1, \ldots, N\}$ such that $m_{\hat{q}} \leq m_n$ for all $n$, i.e., $m_{\hat{q}}$ is the smallest shape parameter, which has multiplicity $\tilde{\mu}$. In particular, we must determine conditions under which $\lambda_{\hat{q}-1} = 0$ since this yields the leading order in $z$. 

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Denote the ordered indices of the corresponding minimum shape parameters by \( \hat{q}_1, \ldots, \hat{q}_{\hat{\mu}} \).

Now, since \( \{\lambda_1, \ldots, \lambda_{N-1}\} \) are cumulative sums of the indices \( \{\ell_1, \ldots, \ell_{N-1}\} \), it follows that \( I_{\text{Nak}}(s) \) will have a \( \hat{\mu} \)th order pole at \( m_{\hat{q}} \) when \( \lambda_{\hat{q}_{\hat{\mu}}-1} = 0 \), a \( (\hat{\mu} - 1) \)th order pole at \( m_{\hat{q}} \) when \( \lambda_{\hat{q}_{\hat{\mu}}-1} = 0 \) but \( \lambda_{\hat{q}_{\hat{\mu}}-1} > 0 \), and so on. We wish to enumerate these instances through the indices \( \{\ell_1, \ldots, \ell_{N-1}\} \). To this end, it is possible to construct \( \hat{\mu} \) disjoint sets of \( (N-1) \)-tuples, which we denote \( \mathcal{L}_1, \ldots, \mathcal{L}_{\hat{\mu}} \), such that \( \mathcal{L}_r \) consists of all sets \( \{\ell_1, \ldots, \ell_{N-1}\} \) that satisfy the conditions \( \lambda_{\hat{q}_{r-1}} = 0 \) and \( \lambda_{\hat{q}_{r+1}} > 0 \), where the second condition is only necessary (and valid) for \( r \leq \hat{\mu} - 1 \). It follows that the order of the pole at \( s = m_{\hat{q}} \) is \( r \) if and only if \( \{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_r \).

This allows us to partition the expression of \( P_o \) into terms related to the order of the poles. Finally, we can substitute the leading order residues, summing over the appropriate sets of \( \ell \) indices, to write the following expression for \( P_o \)

\[
P_o \sim \frac{1}{\prod_n \Gamma(m_n)} \sum_{r=1}^{\hat{\mu}} \sum_{\{\ell_1, \ldots, \ell_{N-1}\} \in \mathcal{L}_r} (-1)^{\lambda_{N-1}} \xi_{\ell} \sum_{l=0}^{r-1} \frac{\nu(r-l)}{(r-1-l)! l!} (\log z)^{l} z^{m_{\hat{q}}}.
\]

Sustituting for \( z \) and applying the binomial theorem yields the result given in (12).

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Fig. 1. Multihop system diagram with one source transmitting to a destination via $N - 1$ relay nodes; no direct source-destination link exists.

Fig. 2. $P_{o,Nak}$ vs. $\bar{\gamma}$ for a fixed-gain AF multihop systems with Nakagami-$m$ fading channels ($N = 3$; $m_n = n$, $K_n = 2$ and $\theta_n = \rho_n = 1$ for all $n$).
Fig. 3.  $P_{o,Nak}$ vs. $\bar{\gamma}$ for a fixed-gain AF multihop systems with Nakagami-$m$ fading channels ($N = 3$; $m_n = N - n + 1$, $K_n = 2$, and $\theta_n = \rho_n = 1$ for all $n$).

**TABLE I**

Mellin transforms for various fading distributions.

| Distribution | Channel Power Density Function $f_X(x), x \geq 0$ | Mellin Transform $M \{f_X; s + 1\}$ |
|--------------|--------------------------------------------------|-------------------------------------|
| Nakagami-$m$ | $\theta^{-m} \Gamma (m)^{-1} x^{m-1} e^{-x/\theta}$ | $\theta^s \Gamma (s + m) / \Gamma (m)$ |
| Weibull      | $m \theta^{-m} x^{m-1} e^{-(x/\theta)^m}$          | $\theta^s (s/m + 1)$               |
| Rician       | $\theta^{-1} (K + 1) e^{-(K + \theta^{-1}(K+1)x)} I_0 \left( \frac{2K(K+1)x}{\theta} \right)$ | $e^{-K \left( \frac{\theta}{K+1} \right)^s} \Gamma (s + 1) \ 1_F \left( s + 1; 1; K \right)$ |
| Hoyt         | $\frac{1+q^2}{2q^2} e^{\frac{(1+q^2)^2}{4q^2} x} I_0 \left( \frac{1-q^4}{4q^2} x \right)$ | $\left( \frac{2q}{1+q^2} \right)^{2s+1} \theta^s \Gamma (s + 1) \ 2_F \left( \frac{1+1}{2}, \frac{3+1}{2}, 1; \left( \frac{1+q^2}{1+q^2} \right)^2 \right)$ |
Fig. 4. $P_{o, Nak}$ vs. $\bar{\gamma}$ for a fixed-gain AF multihop systems with Nakagami-$m$ fading channels ($N = 3$, $m_1 = m_3 = 1$, $m_2 = 2$, $\rho_1 = 1$, $\rho_2 = 1/3$, $\rho_3 = 5/3$; and $\theta_n = (N - n + 1)/2$ and $K_n = 2$ for all $n$).
Fig. 5. $P_{o,Nak}$ vs. $\bar{\gamma}$ for a fixed-gain AF multihop systems with Nakagami-$m$ fading channels ($N = 5$, $m_1 = m_2 = 5$, $m_3 = m_4 = 2.5$, $m_5 = 1.5$; and $K_n = 2$ and $\theta_n = \rho_n = 1$ for all $n$).
Fig. 6. $P_{o,W_{ei}}$ vs. $\bar{\gamma}$ for a fixed-gain AF multihop system with Weibull distributed channels ($N = 2, 3, 4$, $m_1 = 1.5$, $m_2 = 2$, $m_3 = 2.5$, $m_4 = 1$; and $K_n = 2$ and $\theta_n = \rho_n = 1$ for all $n$).
Fig. 7. $P_{o,Rice}$ vs. $\gamma$ for a fixed-gain AF multihop system with Rician distributed channels ($N = 2, 3, 4$, $K_1 = 1$, $K_2 = 3$, $K_3 = 5$, $K_4 = 0$; and $K_n = 2$ and $\theta_n = \rho_n = 1$ for all $n$).
Fig. 8. $P_{o,Hoyt}$ vs. $\bar{\gamma}$ for a fixed-gain AF multihop system with Hoyt distributed channels ($N = 2, 3, 4$, $q_1 = 3/4$, $q_2 = 1/2$, $q_3 = 1/3$, $q_4 = 1/4$; and $K_n = 2$ and $\theta_n = \rho_n = 1$ for all $n$).
Fig. 9. $P_o$ vs. $\bar{\gamma}$ for a four-hop inhomogeneous fixed-gain AF multihop system. The channels corresponding to the four hops follow a Nakagami-$m$, Weibull, Rician, and Hoyt distribution, respectively with the following parameters: $m_1 = 2$, $m_2 = 1.5$, $K_3 = 3$, $q_4 = 3/4$, $\theta_n = n/2$, and $\rho_n = 1 - (n - 1)/10$. 

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