General Mapping between Complex Spatial and Temporal Frequencies Using the Analytical Continuation

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Abstract—This paper introduces an analytical technique to map the eigen-space complex frequencies (temporal frequencies $\omega = \omega_r + j\omega_i$) to the complex propagation constant (spatial frequencies $\gamma = \alpha + j\beta$). The technique analytically finds the driven-mode solutions from the eigen-mode analysis of the associated one-dimensional (1-D) periodic structure unit cell. Since the approach is based on the analyticity of the physical function $\Omega(\gamma)$, it is not only valid for simple canonical problems but also in principle for any physical problem. We apply this general technique on different practical problems including an unbounded lossy medium, rectangular waveguide, periodic dielectric slabs and a series fed patch (SFP) leaky-wave antenna and validate the mapped solutions with those based on either closed-form analytical solutions or numerical finite-element method (FEM) solutions.

I. INTRODUCTION

PERIODIC structures have enormous applications in electromagnetics including diffraction gratings, metamaterials, metasurfaces [1], frequency selective surfaces [2], absorbing interfaces, phased arrays and travelling-wave [3] and leaky-wave antennas [4]. Wave propagation inside and scattering from a periodic structure can be explored by two different analysis, driven-mode and eigen-mode [5]. The driven-mode analysis provides the scattering (S)-parameters due to a given wave excitation while the eigen-mode analysis affords the dispersion diagram without any source.

One example of a 1-D periodic structure is a leaky-wave antenna (LWA) [5]. In the driven-mode analysis, we design a finite-size practical structure consisting of large enough number of unit-cells, define two terminals/ports and then excite them at some specified (real) frequency points $f_i$. This analysis results in the scattering (S-) parameters of the device (LWA) as function of (real) frequencies $f_i$. On the other hand, in the eigen-mode simulation we analyze a single unit-cell of the structure and define periodic boundary conditions (PBCs) with some specified phase difference between the boundaries. While there is no excitation, we look for proper field distributions/modes for which the boundary conditions are fully satisfied. This analysis results in the complex frequencies $\Omega = \omega_r + j\omega_i = 2\pi(f_r + jf_i)$ in terms of propagation phase constant $\beta$.

By the first analysis (driven-mode), we can find an important LWA parameter which is the complex propagation constant $(\gamma = \alpha + j\beta)$ directly from the S-parameters. However, this analysis is extremely time-consuming. On the other hand, the second analysis (eigen-mode analysis) has remarkably smaller computational domain (one unit-cell) and thus extremely more efficient than the first analysis. However, derivation of the complex propagation constants $\gamma$ from the complex frequencies $\Omega$ is not a straight forward task.

The idea of mapping between the $\gamma$ and $\Omega$ was originally introduced in [6] based on transmission line modeling and in [7] based on plane-wave propagation inside an unbounded lossy medium. Recently, the mapping for a canonical case of a dispersive waveguide has been addressed in [8]. In all these cases, the mappings between $\gamma$ and $\Omega$ (or the dispersion relations) are solely based on the wave propagation equations inside the media. Thus, they need the field equations $a$ priori and therefore are limited for a specific problem of interest and not general. In this paper, we propose a general mapping technique that does not depend on the wave equations. It relies on the analytic property of the function $\Omega(\beta)$ and can generally be applied on any 1-D periodic structure.

The organization of the paper is as follows. Section II poses the problem. It explains the concepts of complex frequencies and general complex mapping. Next, in Sec. III we propose the general mapping solution that is based on the analytical continuation theorem. Section IV provides numerical validations and illustrations for various examples. Finally, Sec. V provides discussion and conclusion.

II. PROBLEM STATEMENT

Let us consider a general one-dimensional (1-D) periodic structure shown in Fig. 1 where similar particles are arranged periodically along the $z$ axis with periodicity $L$. We first analyze a single unit cell of the structure by the eigen-mode analysis to find the dispersion diagrams; complex temporal frequency $\Omega = \omega_r + j\omega_i = 2\pi(f_r + jf_i)$ in terms of the purely imaginary propagation constant $\gamma = j\beta$. Then, the problem is to find the complex propagation constant $\gamma = \alpha + j\beta$ in terms of the real frequency $\Omega = \omega_r$.

We propose a general mapping solution from the $\Omega$-plane to the $\gamma$-plane that does not depend on the structure and can be applied in principle on any 1-D periodic structure.
A. Concept of Complex Frequency

First, let us assume an exponentially damping or growing cosine signal \( f(t) \), given by

\[
f(t) = e^{-bt} \cos(at),
\]
where \( a \) and \( b \) are real numbers. We can express \( f(t) \) in the following phasor form

\[
f(t) = \Re\{e^{-bt}e^{\pm jat}\} = \Re\{e^{j(\pm a + jb)t}\},
\]
that reveals the complex frequency of \( f(t) \) as \( \Omega = \pm a + jb \).

The \( f(t) \) is an oscillating function whose amplitude approaches zero if \( b > 0 \) or infinity if \( b < 0 \) as time increases. Therefore, the positiveness (negativeness) of the imaginary constant \( b \) shows the decay (growing) constant of the signal \( f(t) \).

Next, we explore the concept of a complex frequency in a wave propagation problem. Let us assume a relatively simple case: monochromatic plane wave with for example a z-directed electric field given by \( E_z(x,t) \) that propagates along \( x \) axis inside an unbounded lossy medium. The wave function \( \psi = E_z(x,t) \) satisfies the Helmholtz wave equation, given by

\[
\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{\mu \epsilon} \frac{\partial^2 \psi}{\partial x^2} + \frac{\sigma}{\epsilon} \frac{\partial \psi}{\partial t} = 0,
\]
where \( \epsilon \), \( \mu \) and \( \sigma \) are respectively the permittivity, permeability and conductivity of the medium. A solution for \( \psi \) is

\[
E_z(x,t) = e^{-\alpha x} \cos(\omega_0 t - \beta x),
\]

where \( \gamma = \alpha + j \beta \) is the complex propagation constant. Substituting \( \psi \) in \( \Omega \) results in

\[
\gamma(\omega_0) = \pm j \frac{\omega_0}{\nu} \sqrt{1 - j \frac{\sigma}{\omega_0 \epsilon}},
\]
where \( \nu = 1/\sqrt{\mu \epsilon} \) is the speed of the wave in the medium. Equation \( \Omega \) gives the complex propagation constant \( \gamma \), given the real frequency \( \omega_0 \) and the medium parameters \( \epsilon \), \( \mu \) and \( \sigma \).

Now, what is the complex frequency? By fixing the magnitude of the wave in space \( \alpha = 0 \) and letting only the phase variation \( \beta \neq 0 \), we look for a solution that has a complex frequency \( \Omega = \omega_0 + j\omega_1 \) given by

\[
E_z(x,t) = \Re\{e^{-j\beta x}e^{j\Omega t}\} = e^{-\omega_0 t} \cos(\omega_1 t - \beta x),
\]
and satisfies the wave equation \( \psi \). Substituting \( \psi \) in \( \Omega \) results in

\[
\beta = \pm \frac{\Omega}{\nu} \sqrt{1 - j \frac{\sigma}{\omega_0 \epsilon}},
\]
which is exactly Eq. \( \Omega \), except \( \gamma \) and \( \omega_0 \) are replaced by \( j\beta \) and \( \Omega \), respectively. Solving Eq. \( \Omega \) for \( \Omega \) yields

\[
\Omega(\beta) = \pm \sqrt{\nu^2 \beta^2 - \frac{\sigma^2}{4 \epsilon^2} + j \frac{\sigma}{2 \epsilon} \beta},
\]
Equation \( \Omega \) gives the complex frequency \( \Omega \), given the phase constant \( \beta \) and the medium parameters \( \epsilon \), \( \mu \) and \( \sigma \).

There are couple of important points in order.

1) The real part of the complex frequency \( \omega_1 = \pm \sqrt{\nu^2 \beta^2 - \sigma^2/4 \epsilon^2} \) is approximately \( \omega_1 \approx \pm \nu \beta \) for low loss media where \( \sigma/2 \epsilon \ll \nu \beta \).

2) The imaginary part of the complex frequency \( \omega_1 = \sigma/2 \epsilon \) is positive. Thus, the wave \( \psi \) decays as time increases.

3) According to Eq. \( \Omega \), the fields damps as it moves along the \( x \) axis. Likewise, according to Eq. \( \Omega \) the wave damps as time \( t \) passes.

B. Concept of General Mapping

In the last Sec. II-A we have been able to analytically find the complex frequency \( \Omega \) as a function of \( \beta \) by Eq. \( \Omega \) for a particular problem, wave in an unbounded lossy medium. The solution is based on the Helmholtz wave equation \( \psi \). Let us assume a general wave function given by

\[
\psi(x,t) = \Re\{e^{-\gamma x}e^{j\Omega t}\}.
\]

Inserting Eq. \( \psi \) into Eq. \( \psi \) results in the following complex dispersion relation

\[
\Omega^2 + \frac{1}{\mu \epsilon} \gamma^2 - j \frac{\sigma}{\epsilon} \Omega = 0,
\]
that provides mapping functions from the \( \Omega \)-plane into \( \gamma \)-plane and vise versa. However, we may not be able to use Eq. \( \Omega \) for a general problem. For example, in a rectangular waveguide with dimensions \( a \) and \( b \), the dispersion relation \( \Omega \) need to be replaced by

\[
-\gamma^2 + \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 = \left( \frac{\Omega}{\nu} \sqrt{1 - j \frac{\sigma}{\omega_0 \epsilon}} \right)^2,
\]
where every \( (m,n) \) combination specifies a possible mode for the TE_{mn} and TM_{mn} waves. Furthermore, according to Eq. \( \Omega \) we need an extra information about the wave mode number \( (m,n) \) to properly map \( \Omega \) to \( \gamma \) and vise versa.

In fact, to analyze a general one-dimensional (1-D) periodic problem (e.g., leakywave antenna), a closed-form dispersion relation such as Eqs. \( \Omega \) or \( \Omega \) is not accessible and we need to numerically solve the Maxwell’s equations with specified
boundary conditions. In addition, we do not know the wave mode number a priori. The only information is the complex frequency $\Omega$ whose real part is inside the frequency range of operation as a function of the phase propagation constant $\beta$. The procedure is as follows.

1) The Eigen-space/mode analysis of for example HFSS using the finite element method (FEM) provides the input data (complex frequency $\Omega$ in terms of $\beta$). Employing the periodic boundary condition (BC) on front and back sides of the unit cell with a given phase progression $\phi$, the eigen-space analysis provides $\Omega$ in terms of $\phi$ where $\phi = -\beta L$ and $L$ is periodicity.

2) Applying a general mapping technique, we obtain $\gamma(\omega_i)$ having the input $\Omega(\beta)$.

3) The mapped solution $\gamma(\omega_i)$ is validated either by a closed-form analytical solution or by the driven-mode analysis of HFSS.

III. GENERAL MAPPING USING THE ANALYTICAL CONTINUATION

The eigen-space analysis provides a map from $\gamma = j\beta$ to the complex $\Omega$. The question is what is the map of $\Omega = \omega_i$ in the $\gamma$-plane?

Let us for example consider the dispersion diagram of the $\text{TE}_{10}$ of a rectangular waveguide Eq. (11), filled with a lossy dielectric with relative permittivity $\epsilon_r = 2.2$ and conductivity $\sigma = 0.01$ [S/m] with $a = \lambda_g/2$ where $\lambda_g = \lambda_0/\sqrt{\epsilon_r}$ is the wavelength inside the dielectric medium and $\lambda_0 = 30$ [cm].

The eigen-space analysis provides the map of $\gamma = j\beta$ [solid line in Fig. 2(a)] that according to Eq. (11) is given by

$$\Omega(\beta) = \pm \sqrt{\nu^2 \left[ \beta^2 + \left( \frac{\pi}{a} \right)^2 \right] - \frac{\alpha^2}{4\epsilon_r} + j \frac{\sigma}{2\epsilon_r}}.$$  \hspace{1cm} (12)

[solid line in Fig. 2(b)]. Now, what is the map of $\Omega = \omega_i$ [dashed line in Fig. 2(b)]? According to the Eq. (11), it is given by

$$\gamma(\omega_i) = j \left( \frac{\omega_i}{\nu} \right)^2 \left( 1 - j \frac{\sigma}{\omega_i \epsilon_r} \right) - \left( \frac{\pi}{a} \right)^2,$$  \hspace{1cm} (13)

dashed line in Fig. 2(a).

In a general problem, we do not necessarily have access to an equation that relates $\gamma$ to $\Omega$ such as the one in Eqs. (10) or (11). We propose a general technique that is based on the analyticity of the function $\Omega(\beta)$ and gives the $\gamma(\omega_i)$ without knowing the $(\gamma, \Omega)$ equation such as Eqs. (10) or (11).

A function that is analytic in a domain $D$ is uniquely determined over $D$ by its values in a domain, or along a line segment, contained in $D$.

Let us assume that we have an analytic function $w = f(z)$ that maps $z = (x, y)$-plane to $w = (u, v)$-plane and for example, we have only data of $w$ along a line $z = jy$, given by

$$f(0, y) = (1 - y^2) + jy.$$  \hspace{1cm} (14)

According to the theorem, we can uniquely determine the function $f(z)$. Since $z = jy$, we replace $y$ by $-jz$ in Eq. (14) and obtain

$$f(z) = u + jv = 1 - (jz)^2 + z = z^2 + z + 1 = (x^2 - y^2 + x + 1) + j(2xy + y),$$  \hspace{1cm} (15)

which reduces to Eq. (14) if $z = (0, y)$. Because an analytic function is unique [9], Eq. (15) is the only analytic function that satisfies Eq. (14).

B. Proposed General Mapping Technique

Having the complex frequency $\Omega$ data points for different phase propagation constant $\beta$ values, the proposed technique to find the $\gamma(\omega_i)$ is as follows.

1) We write the complex angular frequency $\Omega$ as $\omega_i(\beta) + j\omega_i(\beta)$.

2) Since $\Omega$ is an analytic function and an analytic function has a Taylor series expansion, we fit a power series of $\beta$ to the $\omega_i(\beta)$ and $\omega_i(\beta)$ data. Therefore, we have

$$\Omega = \omega_i(\beta) + j\omega_i(\beta) = \sum_{m=0}^{M} A_m \beta^m + j \sum_{n=0}^{N} B_n \beta^n,$$  \hspace{1cm} (16)

3Wave solution inside a general 1-D periodic structure at a given (real) frequency is generally a superposition of different spatial modes with distinct amplitudes. The modes amplitudes not only depend on the frequency of operation but on the structure geometry and the excitation configuration. For example, to effectively excite the $\text{TE}_{10}$ mode inside a rectangular waveguide 1) the width $a = 2b$ is set twice the height $b$ and 2) a small coaxial probe is inserted in the middle of the waveguide and a quarter of a guided wavelength $\lambda_g$ from the short circuited end of the waveguide. Here, we make a practical assumption that only one dominant mode is propagating inside the 1-D periodic structure and effects of higher or lower modes are negligible.

4The MATLAB function polyfit(x,y,n) returns the coefficients for a polynomial of variable $x$ and degree $n$ that is a best fit (in a least-squares sense) for the data in $y$. 
where \((M,A_m)\) and \((N,B_n)\) are the best fitted-polynomial degrees and coefficients for the \(\omega_i\) and \(\omega_i\), respectively.

3) We replace the argument \(\beta\) by \(-j\gamma\) in the Taylor expansions of the \(\omega_i(\beta)\) and \(\omega_i(\beta)\). Therefore, Eq. (16) leads to the following complex dispersion relation

\[
\Omega = \sum_{m=0}^{M} A_m(-j\gamma)^m + j \sum_{n=0}^{N} B_n(-j\gamma)^n, \tag{17}
\]

which gives a relation between the complex \(\Omega\) and complex \(\gamma\) similar to Eqs. (10) or (11). Now, we can replace \(\Omega\) by \(\omega_i\) in Eq. (17) which results in

\[
\sum_{m=0}^{M} A_m(-j\gamma)^m + j \sum_{n=0}^{N} B_n(-j\gamma)^n - \omega_i = 0, \tag{18}
\]

which is a Max\((M,N)\) degree polynomial of \(\gamma\) and has Max\((M,N)\) roots. Finally, we look for \(\gamma\) that its imaginary part is close to \(\beta\).

IV. VALIDATIONS AND ILLUSTRATIONS

Let us illustrate results of the general mapping technique, proposed in Sec. III-B for 4 practical examples as follow and validate their results by either closed-form analytical solutions or results of the HFSS driven-mode analysis.

A. An unbounded lossy medium

The medium is a lossy dielectric with relative permittivity \(\epsilon_r = 2.2\) and conductivity \(\sigma = 0.01\) [S/m]. Given the complex frequency \(\Omega(\beta)\) data by Eq. (8), we would like to see if the proposed mapping technique is consistent with the analytical solution, given by

\[
\gamma(\omega_i) = \frac{\omega_i}{\nu} \sqrt{1 - j \frac{\sigma}{\omega_i \epsilon}}. \tag{19}
\]

Figures 3(a) and (b) respectively show the \(\omega_i\) and \(\omega_i\) data using Eq. (8) and their fitted curves/polyomials as functions of \(\beta\). The fitting polynomial of \(\omega_i\) has degree 1 and that of \(\omega_i\) has degree 0. Figures 3(c) and (d) respectively show the \(\alpha\) and \(\beta\) estimated by the mapping technique and the analytical formulation (19) where close agreements between the two are observed.

B. Dielectric-filled Rectangular Waveguide

We consider the rectangular waveguide example in Sec. III that is filled with a lossy dielectric with \(\epsilon_r = 2.2\) and \(\sigma = 0.01\) [S/m] and has \(\alpha = \lambda_0/2\sqrt{\epsilon_r}\) and \(\lambda_0 = 30\) [cm]. Given the complex frequency \(\Omega(\beta)\) data by Eq. (12), we would like to see if the proposed mapping technique is in agreement with the analytical solution, given by Eq. (13).

Figures 4(a) and (b) respectively show the \(\omega_i\) and \(\omega_i\) data using Eq. (12) and their fitted polynomials as functions of \(\beta\). The fitting polynomial of \(\omega_i\) has degree 4 and that of \(\omega_i\) has degree 0. Figures 3(c) and (d) respectively show the \(\alpha\) and \(\beta\) estimated by the mapping technique and the analytical formulation (13) where perfect agreements between the two are observed.

\(\gamma(\omega_i) = \frac{\omega_i}{\nu} \sqrt{1 - j \frac{\sigma}{\omega_i \epsilon}}. \tag{19}\)

\(\gamma(\omega_i) = \frac{\omega_i}{\nu} \sqrt{1 - j \frac{\sigma}{\omega_i \epsilon}}. \tag{19}\)

C. Periodic Dielectric Slab

Figure 5 shows a 1-D periodic structure consisting of lossy dielectric slabs with \(\epsilon_r = 4\) and \(\sigma = 0.01\) [S/m], the periodicity \(L = 3\) [cm] and thickness \(\ell = L/2\).

As shown in the Appendix A the dispersion relation for this structure is given by

\[
\text{det } \begin{pmatrix} e^{-jk\ell} & e^{jk\ell} & -e^{-jk_0\ell} & -e^{jk_0\ell} \\ e^{-jk\ell} & -e^{jk\ell} & -e^{-jk_0\ell} & \zeta e^{jk_0\ell} \\ e^{-\gamma L} & e^{-\gamma L} & -e^{-jk_0 L} & -\zeta e^{-jk_0 L} \\ e^{-\gamma L} & -e^{-\gamma L} & -
\end{pmatrix} = 0, \tag{20}
\]

where \(k = k_0/\sqrt{\epsilon_r}\), \(\epsilon_r = \epsilon_r - j\sigma/\Omega_0\), \(k_0 = \Omega/c\), \(c\) is the speed of light in vacuum and \(\zeta = 1/\sqrt{\epsilon_r}\).

Equation (20) provides mapping function from the \(\Omega\)-plane.
into $\gamma$-plane and vise versa. By setting $\gamma = j\beta$ in Eq. (20) and finding complex roots $\Omega$ for which the determinant is zero, we obtain the real and imaginary frequencies as functions of $\beta$. Figures 6(a) and (b) respectively show the $\omega_r$ and $\omega_i$ data and their fitted polynomials as functions of $\beta$. The polynomial of $\omega_r$ has degree 3 and that of $\omega_i$ has degree 4.

Next, we apply the proposed mapping technique and compare the results with the analytical ones. The analytical results are again calculated by Eq. (20) in which we set the $\Omega$ to be purely real $\omega_r$ and look for the complex roots $\gamma$ for which the determinant is zero. Figures 6(c) and (d) respectively show the $\alpha$ and $\beta$ estimated by the mapping technique and the analytical formulation based on (20) where again perfect agreements between the two are observed.

D. Leaky Wave Antenna

Let us now consider a series fed patch (SFP) leaky-wave antenna (LWA) problem shown in Fig. 7 for which no analytical solution exist and thus we use the FEM numerical analysis of HFSS. Figures 7(a) and 7(b) respectively show a unit cell design in the eigen-mode analysis and the entire LWA 19-cell structure with the microstrip transmission line ports.

The antenna center frequency is set $f_0 = 5.8 \text{ GHz}$ and the unit cell dimensions (see [5]) as shown in Fig. 7(a) are $L = 33.64 \text{ mm}$, $l_1 = L/2$, $w_1 = 20 \text{ mm}$ and $w_2 = 2 \text{ mm}$. The loss-less substrate has a relative permittivity of $\varepsilon_r = 2.2$ and a height of $h = 1.5 \text{ mm}$.

As shown in Fig. 7(a), we place the unit cell inside a polygonal cylinder (here, for example a 16-segment polygon) with assigned PBCs to the front and back faces and assigned surface impedance of $120\pi$ to the peripheral faces and a large enough radius, $R = 6 \text{ cm}$ in order to simulate radiation in to the free-space.

![Diagram of leaky wave antenna](image)

Fig. 7. Problem of the leaky wave antenna. HFSS (a) eigen-mode and (b) driven-mode analysis.

Figures 8(a) and (b) show the real and imaginary frequencies, $\omega_r$ and $\omega_i$ given by the HFSS eigen-mode analysis.

Figures 8(a) and (b) show the real and imaginary frequencies, $\omega_r$ and $\omega_i$ given by the HFSS eigen-mode analysis.

$^6$The radius $R \gg \lambda_0$ is large enough so that the field at the cylinder boundary can be approximated by a plane-wave. Here, $\lambda_0 \approx 5.2 \text{ cm}$ at $f_c = 5.8 \text{ GHz}$.

$^7$The HFSS eigen-space solver provides $\Omega$ in terms of phase difference $\phi$ between the front and back faces of the polygonal cylinder shown in Fig. 7(a). The phase constant is then given by $\beta = -\phi/L$. 
their corresponding fitted polynomials as functions of $\beta$. The polynomial of $\omega_1$ has degree 1 and that of $\omega_1$ has degree 5. We then apply the proposed mapping technique and compare the results with solutions of the driven-mode analysis of the HFSS$^8$. Figures 8(c) and (d) respectively show the $\alpha$ and $\beta$ estimated by the mapping technique and the HFSS driven-mode analysis where great agreements between the two are observed.

![Fig. 8. Solutions of the leaky wave antenna in Fig. 7. (a) $\omega_1$ (b) $\omega_1$ (c) $\alpha$ and (d) $\beta$](image)

V. DISCUSSION AND CONCLUSION

We introduced a new mapping methodology to map the complex frequencies to the complex propagation constants which only relies on the analyticity of the physical function $\Omega$ in terms of the phase constant $\beta$. The method in principle can be applied on any 1-D periodic structure.

APPENDIX A

DERIVATION OF EQUATION (20)

Let us assume that an $x$-polarized plane wave is propagating along the $z$ direction. Electromagnetic fields inside the slab ($E_x$, $H_y$) and inside the free-space ($E_{x0}$, $H_{y0}$) media are given in terms of forward and backward propagating waves with respectively $(A,B)$ and $(C,D)$ unknown coefficients, given by

$$
\begin{align*}
\frac{E_x}{H_y} &= A \left( \frac{1}{1/\eta} \right) e^{-jkz} + B \left( -\frac{1}{1/\eta} \right) e^{jkz}, \quad (A.1a) \\
\frac{E_{x0}}{H_{y0}} &= C \left( \frac{1}{1/\eta_0} \right) e^{-jk_0z} + D \left( -\frac{1}{1/\eta_0} \right) e^{jk_0z}, \quad (A.1b)
\end{align*}
$$

where $\eta = \eta_0/\sqrt{\mu_0}, \epsilon_\infty = \epsilon_1 - j\sigma/\Omega_\infty, \eta_0 = \sqrt{\mu_0/\epsilon_0}, k = k_0\sqrt{\epsilon_\infty}$, $k_0 = \Omega_\infty/c$ and $c$ is the speed of light in vacuum.

According to Fig. 8 we first apply the continuity boundary conditions ($E_x|_{z=L} = 0$ and ($H_y|_{z=L}$, given by

$$
A \left( \frac{1}{1/\eta} \right) e^{-jkL} + B \left( -\frac{1}{1/\eta} \right) e^{jkL} = 0
$$

and next the periodic boundary conditions $e^{-\gamma L}E_x|_{z=0} = E_{x0}|_{z=L}$ and $e^{-\gamma L}H_y|_{z=0} = H_{y0}|_{z=L}$, given by

$$
e^{-\gamma L} \begin{pmatrix} A \left( \frac{1}{1/\eta} \right) + B \left( -\frac{1}{1/\eta} \right) \end{pmatrix} = 0
$$

Therefore, the boundary conditions (A.2) yield the following matrix equation

$$
\begin{pmatrix} e^{-jkL} & e^{jkL} & -e^{-jk_0L} & -e^{jk_0L} \\
-e^{-jkL} & -e^{jkL} & \zeta e^{-jk_0L} & -\zeta e^{jk_0L} \\
e^{-\gamma L} & -e^{-\gamma L} & -e^{-jk_0L} & \zeta e^{jk_0L} \\
\gamma L & -\gamma L & -\zeta e^{-jk_0L} & \zeta e^{jk_0L} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

where $\zeta = \eta/\eta_0 = 1/\sqrt{\mu_0}$.

REFERENCES

[1] C. Caloz and T. Itoh, Electromagnetic metamaterials: transmission line theory and microwave applications. John Wiley & Sons, 2005.

[2] B. A. Munk, Frequency selective surfaces: theory and design. John Wiley & Sons, 2005.

[3] A. Hessel, “General characteristics of travelling-wave antennas,” in Antenna Theory, Part II, R. E. Collin and R. F. Zucker, Eds. New York: McGraw-Hill, 1969, ch. 19, pp. 151–258.

[4] T. Tamir, “Leaky-wave antennas,” in Antenna Theory, Part II, R. E. Collin and R. F. Zucker, Eds. New York: McGraw-Hill, 1969, ch. 20, pp. 259–297.

[5] S. Otto, A. Rennings, K. Solbach, and C. Caloz, “Transmission line modeling and asymptotic formulas for periodic leaky-wave antennas scanning through broadside,” IEEE transactions on antennas and propagation, vol. 59, no. 10, pp. 3695–3709, 2011.

[6] ———, “Complex frequency versus complex propagation constant modeling and q-balancing in periodic structures,” in 2012 IEEE/MTT-S International Microwave Symposium Digest. IEEE, 2012, pp. 1–3.

[7] W. Dyab, C. Caloz, and S. Otto, “Interpretation of complex frequencies in propagation problems,” in 2015 International Symposium on Antennas and Propagation (ISAP). IEEE, 2015, pp. 1–4.

[8] D. J. King and S. Gupta, “Relation between complex propagation constant and complex eigenmodes in lossy traveling-wave structures,” in 2019 IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting. IEEE, 2019, pp. 493–494.

[9] J. W. Brown and R. V. Churchill, Complex variables and applications. Boston: McGraw-Hill Higher Education, 2009.

[10] D. M. Pozar, Microwave Engineering. John Wiley & Sons, Inc., 2011.