On some classes of lazy cocycles and categorical structures *

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Abstract

We study some classes of lazy cocycles, called pure (respectively neat), together with their categorical counterparts, entwined (respectively strongly entwined) monoidal categories.

Introduction

Let $H$ be a Hopf algebra with bijective antipode. A left 2-cocycle $\sigma : H \otimes H \to k$ is called lazy if it satisfies the condition

$$\sigma(h_1, h'_1)h_2h'_2 = h_1h'_1\sigma(h_2, h'_2), \quad \forall h, h' \in H.$$ 

In dual form (and with a different name) lazy cocycles appear for instance in Majid’s book [10]; their most important property, the fact that they form a group (denoted now by $Z^2_L(H)$) appears in the paper of Chen [7]. Present terminology stems from [2], [6], inspired by the fact that a Doi twisting by a lazy cocycle does not modify $H$. Moreover, one may define lazy 2-coboundaries $B^2_L(H)$ and the second lazy cohomology group $H^2_L(H) = Z^2_L(H)/B^2_L(H)$, generalizing Sweedler’s second cohomology group of a cocommutative Hopf algebra (this is done by Schauenburg in [13]). Lazy cocycles have been studied systematically in [2], [5], [8], also in connection with Brauer groups of Hopf algebras, Bigalois groups, projective representations.

In this paper we study a certain class of lazy cocycles, satisfying the condition

$$\sigma(ab_1, c_1)\sigma^{-1}(b_2, c_2)\sigma(b_3, c_3d) = \sigma(b_1, c_1d)\sigma^{-1}(b_2, c_2)\sigma(ab_3, c_3),$$

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for all $a, b, c, d \in H$, called pure lazy cocycles. In dual form, they have been introduced in [14] as pure-braided structure. This purity condition has a topological meaning: pure lazy cocycles give rise to representations of pure braid groups and invariants for long knots, cf. [14]. A natural problem is whether it is possible to determine all pure lazy cocycles on a given Hopf algebra; this seems to be complicated even for “easy” Hopf algebras. This is why we have looked for a stronger condition than purity, and we were led to the following concept: a lazy cocycle is called neat if it satisfies the condition

$$\sigma(a, b_1)\sigma(b_2, c) = \sigma(b_1, c)\sigma(a, b_2),$$

for all $a, b, c \in H$. It turns out that a neat lazy cocycle is pure, and, using the description of lazy cocycles for Sweedler’s Hopf algebra $H_4$ from [2], it is quite easy to see that any lazy cocycle for $H_4$ is neat (hence also pure).

The categorical counterpart of pure lazy cocycles was introduced in [14] as a pure-braided category and independently in [4] as entwined category. These concepts look different but we prove here that they are equivalent (and provide another equivalent formulation). We introduce the categorical analogue of neat lazy cocycles, as a strongly entwined category, and prove that strongly entwined implies entwined. We show that there exists a canonical way to produce a strong twine starting from a D-structure (consisting of isomorphisms) in the sense of [3]. A natural question is to see what kind of algebraic properties pure and neat lazy cocycles have. It turns out that their algebraic properties are not too good (for instance they do not seem to form subgroups of $Z^2_k(H)$), but are also not so bad, for instance they have a good behaviour when extending to a Drinfeld double or a Radford biproduct (this extension property may also be regarded as a potential source of examples of pure and neat lazy cocycles).

1 Preliminaries

In this section we recall basic definitions and results and we fix notation to be used throughout the paper. All algebras, linear spaces, etc, will be over a base field $k$; unadorned $\otimes$ means $\otimes_k$. For a Hopf algebra $H$ with comultiplication $\Delta$ we use Sweedler’s sigma notation: $\Delta(h) = h_1 \otimes h_2$ or $\Delta(h) = h(1) \otimes h(2)$. Unless otherwise stated, $H$ will denote a Hopf algebra with bijective antipode $S$. For a linear map $\sigma : H \otimes H \rightarrow k$ we use either the notation $\sigma(h, h')$ or $\sigma(h \otimes h')$. For terminology concerning Hopf algebras and monoidal categories we refer to [9], [10], [11], [15].

A linear map $\sigma : H \otimes H \rightarrow k$ is called a left 2-cocycle if it satisfies the condition

$$\sigma(a_1, b_1)\sigma(a_2b_2, c) = \sigma(b_1, c_1)\sigma(a, b_2c_2),$$

(1.1)

for all $a, b, c \in H$, and it is called a right 2-cocycle if it satisfies the condition

$$\sigma(a_1b_1, c)\sigma(a_2, b_2) = \sigma(a, b_1c_1)\sigma(b_2, c_2).$$

(1.2)

Given a linear map $\sigma : H \otimes H \rightarrow k$, define a product $\cdot_\sigma$ on $H$ by

$$h \cdot_\sigma h' = \sigma(h_1, h'_1)h_2h'_2, \quad \forall h, h' \in H.$$

Then $\cdot_\sigma$ is associative if and only if $\sigma$ is a left 2-cocycle. If we define $\cdot_\sigma$ by

$$h \cdot_\sigma h' = h_1h'_1\sigma(h_2, h'_2), \quad \forall h, h' \in H,$$

then $\cdot_\sigma$ is associative if and only if $\sigma$ is a right 2-cocycle. In any of the two cases, $\sigma$ is normalized (i.e. $\sigma(1, h) = \sigma(h, 1) = \varepsilon(h)$ for all $h \in H$) if and only if $1_H$ is the unit for $\cdot_\sigma$. If $\sigma$ is a normalized
left (respectively right) 2-cocycle, we denote the algebra \((H, \gamma)\) by \(\sigma H\) (respectively \(H_\sigma\)). It is well-known that \(\sigma H\) (respectively \(H_\sigma\)) is a right (respectively left) \(H\)-comodule algebra via the comultiplication \(\Delta\) of \(H\). If \(\sigma : H \otimes H \to \mathbb{k}\) is normalized and convolution invertible, then \(\sigma\) is a left 2-cocycle if and only if \(\sigma^{-1}\) is a right 2-cocycle.

If \(\gamma : H \to \mathbb{k}\) is linear, normalized (i.e. \(\gamma(1) = 1\)) and convolution invertible, define

\[
D^1(\gamma) : H \otimes H \to \mathbb{k}, \quad D^1(\gamma)(h, h') = \gamma(h_1)\gamma(h'_1)\gamma^{-1}(h_2h'_2), \quad \forall h, h' \in H.
\]

Then \(D^1(\gamma)\) is a normalized and convolution invertible left 2-cocycle.

We recall from [2] some facts about lazy cocycles and lazy cohomology. The set \(\text{Reg}^1(H)\) (respectively \(\text{Reg}^2(H)\)) consisting of normalized and convolution invertible linear maps \(\gamma : H \to \mathbb{k}\) (respectively \(\sigma : H \otimes H \to \mathbb{k}\)), is a group with respect to the convolution product. An element \(\gamma \in \text{Reg}^1(H)\) is called lazy if

\[
\gamma(h_1)h_2 = h_1\gamma(h_2), \quad \forall h \in H. \tag{1.3}
\]

The set of lazy elements of \(\text{Reg}^1(H)\), denoted by \(\text{Reg}_L^1(H)\), is a central subgroup of \(\text{Reg}^1(H)\). An element \(\sigma \in \text{Reg}^2(H)\) is called lazy if

\[
\sigma(h_1, h'_1)h_2h'_2 = h_1h'_1\sigma(h_2, h'_2), \quad \forall h, h' \in H. \tag{1.4}
\]

The set of lazy elements of \(\text{Reg}^2(H)\), denoted by \(\text{Reg}_L^2(H)\), is a subgroup of \(\text{Reg}^2(H)\). We denote by \(Z^2(H)\) the set of left 2-cocycles on \(H\) and by \(Z_L^2(H)\) the set \(Z^2(H) \cap \text{Reg}_L^2(H)\) of normalized and convolution invertible lazy 2-cocycles. If \(\sigma \in Z_L^2(H)\), then the algebras \(\sigma H\) and \(H_\sigma\) coincide and will be denoted by \(H(\sigma)\); moreover, \(H(\sigma)\) is an \(H\)-bicomodule algebra via \(\Delta\).

It is well-known that in general the set \(Z^2(H)\) of left 2-cocycles is not closed under convolution. One of the main features of lazy 2-cocycles is that the set \(Z_L^2(H)\) is closed under convolution, and that the convolution inverse of an element \(\sigma \in Z_L^2(H)\) is again a lazy 2-cocycle, so \(Z_L^2(H)\) is a group under convolution. In particular, a lazy 2-cocycle is also a right 2-cocycle.

Consider now the map \(D^1 : \text{Reg}^1(H) \to \text{Reg}^2(H)\), \(D^1(\gamma)(h, h') = \gamma(h_1)\gamma(h'_1)\gamma^{-1}(h_2h'_2)\), for all \(h, h' \in H\). Then, by [2], the map \(D^1\) induces a group morphism \(\text{Reg}_L^1(H) \to Z_L^2(H)\), with image contained in the centre of \(Z_L^2(H)\); denote by \(B_L^2(H)\) this central subgroup \(D^1(\text{Reg}_L^1(H))\) of \(Z_L^2(H)\) (its elements are called lazy 2-coboundaries). Then define the second lazy cohomology group \(H_L^2(H) = Z_L^2(H)/B_L^2(H)\).

## 2 Pure-braided and entwined monoidal categories

We begin this section by recalling the following two concepts (all monoidal categories are assumed to be strict, with unit denoted by \(I\)).

**Definition 2.1** ([13]) Let \(\mathcal{C}\) be a monoidal category. A pure-braided structure of \(\mathcal{C}\) consists of two families of natural isomorphisms \(A_{U,V,W} : U \otimes V \otimes W \to U \otimes V \otimes W\) and \(B_{U,V,W} : U \otimes V \otimes W \to U \otimes V \otimes W\) such that:

\[
\begin{align*}
A_{U \otimes V, W, X} & = A_{U, V \otimes W, X}(id_U \otimes A_{V, W, X}), \tag{2.1} \\
A_{U, V \otimes W, X} & = (A_{U, V, W} \otimes id_X)A_{U, V \otimes W, X}, \tag{2.2} \\
B_{U \otimes V, W, X} & = (id_U \otimes B_{V, W, X})B_{U, V \otimes W, X}, \tag{2.3} \\
B_{U, V \otimes W, X} & = B_{U, V, W}(B_{U, V, W} \otimes id_X), \tag{2.4} \\
(A_{U, V, W} \otimes id_X)(id_U \otimes B_{V, W, X}) & = (id_U \otimes B_{V, W, X})(A_{U, V, W} \otimes id_X), \tag{2.5} \\
A_{U, I, V} & = B_{U, I, V}. \tag{2.6}
\end{align*}
\]

A category equipped with a pure-braided structure is called a pure-braided category.
Remark 2.2 The axioms \((2.1)\)\textemdash \((2.4)\) imply also the following relations:

\begin{align*}
A_{U,V} &= A_{U,V} = id_{U \otimes V}, \quad (2.7) \\
B_{U,V} &= B_{U,V} = id_{U \otimes V}. \quad (2.8)
\end{align*}

Definition 2.3 \((4)\) Let \(C\) be a monoidal category. A twine of \(C\) is a natural isomorphism \(D_{X,Y} : X \otimes Y \to X \otimes Y\) satisfying the following axioms:

\[
D_{I,X} = id_X,
\]

\[
(D_{X,Y} \otimes id_Z)D_{X \otimes Y,Z} = (id_X \otimes D_{Y,Z})D_{X,Y \otimes Z},
\]

\[
(D_{X,Y,Z} \otimes id_T)(id_X \otimes D_{Y,Z}^{-1} \otimes id_T)(id_X \otimes D_{Y,Z \otimes T})
\]

\[
= (id_X \otimes D_{Y,Z \otimes T})(id_X \otimes D_{Y,Z}^{-1} \otimes id_T)(D_{X,Y,Z} \otimes id_T).
\]

A category equipped with a twine is called an entwined category.

Remark 2.4 By \((4)\), if \((C, D)\) is an entwined category then \(D_{X,I} = D_{I,X} = id_X, \ \forall X \in C\).

Remark 2.5 If \(C\) is a monoidal category and \(D_{X,Y} : X \otimes Y \to X \otimes Y\) is a natural isomorphism, the naturality of \(D\) implies (for all \(X, Y, Z \in C\)):

\[
(D_{X,Y} \otimes id_Z)D_{X \otimes Y,Z} = (id_X \otimes D_{Y,Z})D_{X,Y \otimes Z},
\]

\[
(id_X \otimes D_{Y,Z})D_{X,Y \otimes Z} = D_{X,Y \otimes Z}(id_X \otimes D_{Y,Z}).
\]

We prove now that these two concepts are equivalent.

Proposition 2.6 Let \(C\) be a monoidal category.

a) If \((C, A, B)\) is a pure-braided category and we define \(D_{U,V} : U \otimes V \to U \otimes V\) by \(D_{U,V} := A_{U,I,V} = B_{U,I,V}\), then \(D_{U,V}\) is a natural isomorphism satisfying

\[
D_{I,X} = D_{X,I} = id_X, \quad (2.11)
\]

\[
(D_{X,Y} \otimes id_Z \otimes id_T)(id_X \otimes D_{Y,Z} \otimes id_T)(D_{X,Y,Z} \otimes id_T)
\]

\[
= (id_X \otimes id_Y \otimes D_{Z,T})(D_{X,Y,Z} \otimes id_T)(id_X \otimes D_{Y,Z \otimes T}). \quad (2.12)
\]

b) If \(D_{U,V} : U \otimes V \to U \otimes V\) is a natural isomorphism satisfying \((2.11)\) and \((2.12)\), then \((C, D)\) is an entwined category.

c) If \((C, D)\) is an entwined category and we define \(A_{X,Y,Z}, B_{X,Y,Z} : X \otimes Y \otimes Z \to X \otimes Y \otimes Z\) by

\[
A_{X,Y,Z} = D_{X,Y,Z}(id_X \otimes D_{Y,Z}^{-1}) = (D_{X,Y}^{-1} \otimes id_Z)D_{X,Y \otimes Z}, \quad (2.13)
\]

\[
B_{X,Y,Z} = (id_X \otimes D_{Y,Z}^{-1})D_{X \otimes Y,Z} = D_{X,Y \otimes Z}(D_{X,Y}^{-1} \otimes id_Z), \quad (2.14)
\]

then \((C, A, B)\) is a pure-braided category.

Proof. a) Define \(D_{U,V} := A_{U,I,V} = B_{U,I,V}\). By \((2.1)\) we have \(A_{U \otimes I,X} = A_{U \otimes I,L}(id_U \otimes A_{I,I,X})\), hence we obtain \(D_{I,X} = A_{I,I,X} = id_X\) and similarly \(D_{X,I} = id_X\). We prove that \(A_{U,V,X} = D_{U \otimes V,X}(id_U \otimes D_{V,X}^{-1});\) indeed, we have:

\[
D_{U \otimes V,X} = A_{U \otimes V,I,X} \quad (2.1) = A_{U,V,I,X}(id_U \otimes A_{V,I,X}) = A_{U,V,X}(id_U \otimes D_{V,X}),
\]
and similarly
\[ A_{U,V,X} = (D_{U,V}^{-1} \otimes id_X)D_{U,V \otimes X}, \]
\[ B_{U,V,X} = (id_U \otimes D_{V,X}^{-1})D_{U \otimes V,X}, \]
\[ B_{U,V,X} = D_{U,V \otimes X}(D_{U,V}^{-1} \otimes id_X). \]

Using these formulae we obtain:
\[
(A_{U,V,W} \otimes id_X)(id_U \otimes B_{V,W,X})
= (D_{U,V}^{-1} \otimes id_W \otimes id_X)(D_{U,V \otimes W} \otimes id_X)(id_U \otimes D_{V,W \otimes X}(id_U \otimes D_{V,W}^{-1} \otimes id_X),
\]
\[
(id_U \otimes B_{V,W,X})(A_{U,V,W} \otimes id_X)
= (id_U \otimes id_V \otimes D_{W,X}^{-1})(id_U \otimes D_{V,W \otimes X},(D_{U \otimes V,W} \otimes id_X)(id_U \otimes D_{V,W}^{-1} \otimes id_X).
\]

Now using (2.5) we get (2.12).
b) We take \( T = I \) in (2.12), obtaining
\[
(D_{X,Y} \otimes id_Z \otimes id_I)(id_X \otimes D_{Y \otimes Z,I})(D_{X \otimes Y,Z} \otimes id_I)
= (id_X \otimes id_Y \otimes D_{Z,I})(D_{X,Y \otimes Z} \otimes id_I)(id_X \otimes D_{Y,Z \otimes I}),
\]
which can be rewritten as
\[
(D_{X,Y} \otimes id_Z)D_{X,Y,Z} = D_{X,Y,Z}(id_X \otimes D_{Y,Z}).
\] (2.15)

Also, (2.12) implies
\[
(id_X \otimes id_Y \otimes D_{Z,T}^{-1})(id_X \otimes D_{Y \otimes Z,T})(D_{X \otimes Y,Z} \otimes id_T)
= (D_{X,Y}^{-1} \otimes id_Z \otimes id_T)(D_{X,Y \otimes Z} \otimes id_T)(id_X \otimes D_{Y,Z \otimes T}),
\]
and using (2.15) we obtain
\[
(id_X \otimes D_{Y,Z \otimes T})(id_X \otimes D_{Y,Z}^{-1} \otimes id_T)(D_{X \otimes Y,Z} \otimes id_T)
= (D_{X,Y,Z} \otimes id_T)(id_X \otimes D_{Y,Z}^{-1} \otimes id_T)(id_X \otimes D_{Y,Z \otimes T}).
\]

c) Define \( A \) and \( B \) by (2.13) and (2.14) respectively. We prove (2.1):
\[
A_{U \otimes V,W,X} = D_{U \otimes V \otimes W,X}(id_U \otimes V \otimes D_{W,X}^{-1}),
\]
\[
= D_{U \otimes V \otimes W,X}(id_U \otimes D_{V \otimes W,X}^{-1})(id_U \otimes D_{V \otimes W,X}^{-1})(id_U \otimes D_{W,X}^{-1})
= A_{U \otimes V \otimes W,X}(id_U \otimes A_{W,X}).
\]

Similarly we get (2.2), (2.3) and (2.4). From the definition we have \( A_{U,I,V} = D_{U,V} = B_{U,I,V}. \)
Finally, we prove (2.5):
\[
(A_{U,V,W} \otimes id_X)(id_U \otimes B_{V,W,X})
= (D_{U,V,W} \otimes id_X)(id_U \otimes D_{V,W}^{-1} \otimes id_X)(id_U \otimes D_{V,W \otimes X})(id_U \otimes D_{V,W}^{-1} \otimes id_X)
= (id_U \otimes D_{V,W \otimes X})(id_U \otimes D_{V,W}^{-1} \otimes id_X)(D_{U \otimes V,W \otimes X})(id_U \otimes D_{V,W}^{-1} \otimes id_X)
= (id_U \otimes B_{V,W,X})(A_{U,V,W} \otimes id_X),
\]
finishing the proof. \( \square \)
**Definition 2.7** Let $\mathcal{C}$ be a monoidal category and $T_{U,V} : U \otimes V \to U \otimes V$ a natural isomorphism. We say that $T$ is a strong twine (or $(\mathcal{C}, T)$ is strongly entwined) if:

$$T_{U} = id_{U},$$

$$T_{U,V} \otimes id_{W})T_{U \otimes V,W} = (id_{U} \otimes T_{V,W})T_{U,V \otimes W},$$

$$T_{U,V} \otimes id_{W})(id_{U} \otimes T_{V,W}) = (id_{U} \otimes T_{V,W})(T_{U,V} \otimes id_{W}).$$

**Proposition 2.8** If $(\mathcal{C}, T)$ is strongly entwined then $(\mathcal{C}, T)$ is entwined.

**Proof.** First we prove that

$$(T_{U,V} \otimes id_{W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X}) = (id_{U} \otimes T_{V \otimes W \otimes X})(T_{U,V} \otimes id_{W \otimes X}).$$

Indeed, we have:

$$(T_{U,V} \otimes id_{W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X})$$

$$= (T_{U,V} \otimes id_{W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X}^{-1} \otimes id_{X})(id_{U} \otimes T_{V \otimes W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X}^{-1} \otimes id_{X})$$

$$= (id_{U} \otimes T_{V \otimes W \otimes X}^{-1} \otimes id_{X})(T_{U,V} \otimes id_{W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X}^{-1} \otimes id_{X})$$

$$= (id_{U} \otimes T_{V \otimes W \otimes X}^{-1} \otimes id_{X})(id_{U} \otimes T_{V \otimes W \otimes X})(id_{U} \otimes T_{V \otimes W \otimes X}^{-1} \otimes id_{X})$$

and similarly

$$(T_{U,V} \otimes id_{X})(id_{U} \otimes T_{V \otimes W \otimes X}) = (id_{U} \otimes T_{V \otimes W \otimes X})(T_{U,V} \otimes id_{X}).$$

Now we compute:

$$(T_{U \otimes V,W} \otimes id_{X})(id_{U} \otimes T_{V \otimes W}^{-1} \otimes id_{X})(id_{U} \otimes T_{V,W \otimes X})$$

$$= (T_{U \otimes V,W} \otimes id_{X})(id_{U} \otimes T_{V,W}^{-1} \otimes id_{X})(id_{U} \otimes T_{V,W} \otimes id_{X})$$

$$= (id_{U} \otimes T_{V,W}^{-1} \otimes id_{X})(id_{U} \otimes T_{V,W} \otimes id_{X})(id_{U} \otimes T_{V,W} \otimes id_{X})$$

$$= (id_{U} \otimes T_{V,W} \otimes id_{X})(id_{U} \otimes T_{V,W} \otimes id_{X})$$

$$= (id_{U} \otimes T_{V,W} \otimes id_{X})(id_{U} \otimes T_{V,W} \otimes id_{X})$$

showing that $(\mathcal{C}, T)$ is an entwined category. \hfill $\square$

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Remark 2.9 Any monoidal category contains at least one strong twine: the trivial one.

The categorical analogue of the operator $D^1$ from the Preliminaries looks as follows (see [4]). If $\mathcal{C}$ is a monoidal category and $R_X : X \to X$ is a natural isomorphism in $\mathcal{C}$ such that $R_1 = \text{id}_I$, we denote $D^1(R)_{X,Y} := (R_X \otimes R_Y)R_{X \otimes Y}^{-1} = R_{X \otimes Y}^{-1}(R_X \otimes R_Y)$ as a morphism $X \otimes Y \to X \otimes Y$.

Definition 2.10 (3) Let $\mathcal{C}$ be a monoidal category. A $D$-structure on $\mathcal{C}$ consists of a family of natural morphisms $R_X : X \to X$ in $\mathcal{C}$, such that $R_I = \text{id}_I$ and (for all $X, Y, Z \in \mathcal{C}$):

$$(R_X \otimes \text{id}_Z)(\text{id}_X \otimes R_Y \otimes Z) = (\text{id}_X \otimes R_Y \otimes Z)(R_X \otimes \text{id}_Z). \quad (2.21)$$

This concept provides a method for constructing strong twines, as follows:

Proposition 2.11 Let $\mathcal{C}$ be a monoidal category and $R$ a $D$-structure on $\mathcal{C}$ such that all $R_X$ are isomorphisms. Then $D^1(R)$ is a strong twine on $\mathcal{C}$.

Proof. We only have to check (2.18). We compute:

$$(D^1(R)_{U,V} \otimes \text{id}_W)(\text{id}_U \otimes D^1(R)_{V,W})$$

$$= ((R_U \otimes R_V)R_{U \otimes V}^{-1} \otimes \text{id}_W)(\text{id}_U \otimes R_{V \otimes W}^{-1}(R_V \otimes R_W))$$

$$= (R_U \otimes R_V \otimes \text{id}_W)(R_{U \otimes V}^{-1} \otimes \text{id}_W)(\text{id}_U \otimes R_{V \otimes W}^{-1})$$

$$= (R_U \otimes (R_V \otimes \text{id}_W)R_{V \otimes W}^{-1})(id_U \otimes R_{V \otimes W})$$

$$= (R_U \otimes R_{V \otimes W}^{-1})(id_U \otimes R_{V \otimes W})$$

$$= (id_U \otimes R_{V \otimes W}^{-1})(R_U \otimes R_{V \otimes W})(R_{U \otimes V} \otimes \text{id}_W)$$

$$= (id_U \otimes D^1(R)_{V,W})(D^1(R)_{U,V} \otimes \text{id}_W),$$

finishing the proof. \qed

3 Pure and neat lazy cocycles

Definition 3.1 Let $\sigma \in \text{Reg}^2(\mathcal{H})$; we call $\sigma$ pure if it satisfies the condition:

$$\sigma(ab_1, c_1)\sigma^{-1}(b_2, c_2)\sigma(b_3, c_3d) = \sigma(b_1, c_1d)\sigma^{-1}(b_2, c_2)\sigma(ab_3, c_3), \quad (3.1)$$

for all $a, b, c \in H$. If $\sigma$ is moreover lazy we call it pure lazy and denote by $\text{Reg}^2_{PL}(\mathcal{H})$ the set of pure lazy elements. We also denote by $Z^2_{PL}(\mathcal{H})$ the set of pure lazy 2-cocycles.

Remark 3.2 The concept of pure lazy cocycle is dual to the concept of pure-braided structure in [4].

Example 3.3 If $r, s$ are two coquasiflatticy structures on $H$, then $r_{21} \ast s$ is a pure lazy 2-cocycle. The fact that it is a lazy 2-cocycle was noticed in [2], and the fact that it is pure is analogous to a remark due to Virelizier, see [4].
Definition 3.4 Let $\sigma \in \text{Reg}^2(H)$; we call $\sigma$ neat if it satisfies the condition:
\[\sigma(a,b_1)\sigma(b_2,c) = \sigma(b_1,c)\sigma(a,b_2),\] (3.2)
for all $a,b,c \in H$. If $\sigma$ is moreover lazy we call it neat lazy and denote by $\text{Reg}^2_{NL}(H)$ the set of neat lazy elements. We also denote by $Z^2_{NL}(H)$ the set of neat lazy 2-cocycles.

Remark 3.5 Relation (3.2) is a commutation condition. Namely, define the maps $\varphi, \psi : H \to H^*$, $\varphi(a)(b) = \sigma(a,b)$ and $\psi(a)(b) = \sigma(b,a)$. Then (3.2) holds if and only if $\varphi(a) * \psi(c) = \psi(c) * \varphi(a)$ in $H^*$, for all $a,c \in H$.

We have the following dictionary between lazy cocycles and categorical structures:

Proposition 3.6 Let $H$ be a Hopf algebra, $\sigma \in \text{Reg}^2(H)$, and consider $\mathcal{C} = \mathcal{M}^H$, the category of right $H$-comodules, with tensor product given by $(m \otimes n)(0) \otimes (m \otimes n)(1) = (m(0) \otimes n(0)) \otimes m(1)n(1)$. Define $T_{M,N}(m \otimes n) = m(0) \otimes n(0)\sigma(m(1),n(1))$. Then $\sigma$ is a pure (respectively neat) lazy 2-cocycle if and only if $T$ is a twine (respectively strong twine).

As a consequence of this and Proposition 2.8, we obtain:

Proposition 3.7 $Z^2_{NL}(H) \subseteq Z^2_{PL}(H)$, that is any neat lazy cocycle is pure.

Remark 3.8 A pure lazy cocycle of the type $r_2 \ast s$, with $r,s$ coquasitriangular structures on $H$, is not necessarily neat.

Example 3.9 Let $H_4$ be Sweedler’s 4-dimensional Hopf algebra. A description of $Z^2_2(H_4)$ was given in [2], Example 2.1. Using the formulae in [2], one can prove, by a direct computation, that any lazy 2-cocycle on $H_4$ is neat, hence we obtain $Z^2_{NL}(H_4) = Z^2_{PL}(H_4) = Z^2_{L}(H_4)$.

Proposition 3.10 (see [4], [14]) If $\gamma \in \text{Reg}^1(H)$ satisfies the condition
\[\gamma(a_1b_1)\gamma^{-1}(a_2b_2c_1)\gamma(b_3c_2)\gamma^{-1}(b_4c_3d_1)\gamma(c_4d_2) = \gamma(c_1d_1)\gamma^{-1}(b_1c_2d_2)\gamma(b_2c_3)\gamma^{-1}(a_1b_3c_4)\gamma(a_2b_4),\] (3.3)
for all $a,b,c,d \in H$, then $D^1(\gamma) \in Z^2_{PL}(H)$. An element $\gamma \in \text{Reg}^1(H)$ satisfying (3.3) is said to be pure. We denote by $\text{Reg}^1_{PL}(H)$ the set of pure lazy elements.

Definition 3.11 An element $\gamma \in \text{Reg}^1(H)$ satisfying the condition
\[\gamma(ab_1)\gamma(b_2c) = \gamma(b_1c)\gamma(ab_2),\] (3.4)
for all $a,b,c \in H$, is said to be neat. We denote the set of neat elements by $\text{Reg}^1_{N}(H)$ and the set of neat lazy elements by $\text{Reg}^1_{NL}(H)$.

Remark 3.12 A neat lazy element corresponds to a $D$-morphism in [3], except for the fact that a $D$-morphism is not required to be convolution invertible.

Proposition 3.13 $\text{Reg}^1_{NL}(H) \subseteq \text{Reg}^1_{PL}(H)$ and $D^1(\text{Reg}^1_{NL}(H)) \subseteq Z^2_{NL}(H)$.

Proof. Straightforward computation. □

Proposition 3.14 If $\gamma \in \text{Reg}^1(H)$ satisfies the condition
\[\gamma(ab_1)b_2 = \gamma(ab_2)b_1,\] (3.5)
for all $a,b \in H$, then $\gamma \in \text{Reg}^1_{NL}(H)$.

Proof. An element $\gamma$ satisfying (3.5) is automatically lazy and also satisfies (3.4). □
4 Extending pure and neat lazy cocycles to Drinfeld doubles and Radford biproducts

Let $H$ be a finite dimensional Hopf algebra. Recall that the Drinfeld double $D(H)$ is a quasitriangular Hopf algebra realized on the $k$-linear space $H^* \otimes H$; its coalgebra structure is $H^{*cop} \otimes H$ and the algebra structure is given by

$$(p \otimes h)(q \otimes l) = p(h_1 \rightarrow q \leftarrow S^{-1}(h_3)) \otimes h_2 l,$$

for all $p, q \in H^*$ and $h, l \in H$, where $\rightarrow$ and $\leftarrow$ are the left and right regular actions of $H$ on $H^*$ given by $(h \rightarrow p)(l) = p(hl)$ and $(p \leftarrow h)(l) = p(hl)$ for all $h, l \in H$ and $p \in H^*$. If $\sigma \in Z^2_H(H)$, define $\overline{\sigma} : D(H) \otimes D(H) \rightarrow k$ by

$$\overline{\sigma}(p \otimes h, q \otimes l) = p(1)q(S^{-1}(h_3)h_1)\sigma(h_2, l),$$

(4.1)

for all $p, q \in H^*$ and $h, l \in H$. Then, by [3], $\overline{\sigma} \in Z^2_H(D(H))$, and its convolution inverse is

$$\overline{\sigma}^{-1}(p \otimes h, q \otimes l) = p(1)q(S^{-1}(h_3)h_1)\sigma^{-1}(h_2, l).$$

(4.2)

Moreover, we have:

**Proposition 4.1** If $\sigma \in Z^2_{PL}(H)$ then $\overline{\sigma} \in Z^2_{PL}(D(H))$. If $\sigma \in Z^2_{NL}(H)$ then $\overline{\sigma} \in Z^2_{NL}(D(H))$.

**Proof.** Assume first that $\sigma \in Z^2_{PL}(H)$ and let $a, b, c, d \in H$ and $A, B, C, D \in H^*$; we prove (3.1) for $\overline{\sigma}$ and the elements $A \otimes a, B \otimes b, C \otimes c, D \otimes d$ in $D(H)$. We compute:

$$\overline{\sigma}((A \otimes a)(B \otimes b)_1, (C \otimes c)_1)\overline{\sigma}^{-1}((B \otimes b)_2, (C \otimes c)_2)\overline{\sigma}((B \otimes b)_3, (C \otimes c)_3(D \otimes d))$$

$$= \overline{\sigma}((A \otimes a)(B_3 \otimes b_1, C_3 \otimes c_1)\overline{\sigma}^{-1}(B_2 \otimes b_2, C_2 \otimes c_2)
\overline{\sigma}(B_1 \otimes b_3, (C_1 \otimes c_3)(D \otimes d))$$

$$= \overline{\sigma}(A(a_1 \rightarrow B_3 \leftarrow S^{-1}(a_3)) \otimes a_2 b_1, C_3 \otimes c_1)\overline{\sigma}^{-1}(B_2 \otimes b_2, C_2 \otimes c_2)
\overline{\sigma}(B_1 \otimes b_3, C_1(c_3 \rightarrow D \leftarrow S^{-1}(c_5)) \otimes c_4 d)$$

$$= A(1)B_3(S^{-1}(a_3 a_1)C_3(S^{-1}(a_2 b_1))a_2 b_1, c_1)\sigma(a_2, b_2), c_1)
B_3(1)C_2(S^{-1}(b_2, c_2))\sigma^{-1}(b_2, c_2)
B_1(1)C_1(S^{-1}(b_3, c_2)) \sigma(b_3, c_2, c_4 d)$$

$$= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_1))b_7 S^{-1}(b_6) b_4 S^{-1}(b_3)S^{-1}(a_4 a_2 b_1)
D(S^{-1}(c_5) S^{-1}(b_10) b_8 c_3) \sigma(b_3, c_2, c_4 d)
= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)
= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)$$

$$= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)$$

$$= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)$$

$$= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)$$

$$= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)$$

$$= A(1)B(S^{-1}(a_5 a_1)C(S^{-1}(b_7) S^{-1}(a_4) a_2 b_1)D(S^{-1}(b_6 c_5) b_4 c_3)
\sigma(a_3 b_2, c_1)\sigma^{-1}(b_5, c_4 d)$$

The result follows. \hfill $\Box$
and we see that the two terms are equal.

Assume now that \( \sigma \in Z_{NL}^2(H) \); we prove (3.2) for \( \mathfrak{B} \) and the elements \( A \otimes a, B \otimes b, C \otimes c \) in \( D(H) \). We compute:

\[
\mathfrak{B}(A \otimes a, (B \otimes b)_1)\mathfrak{B}((B \otimes b)_2, C \otimes c) \\
= \mathfrak{B}(A \otimes a, B_2 \otimes b_1)\mathfrak{B}(B_1 \otimes b_2, C \otimes c)
\]

(1.1) \quad = A(1)B_2(S^{-1}(a_3)a_1)\sigma(a_2, b_1)B_1(1)C(S^{-1}(b_2, b_3)b_2, b_1)\sigma(b_2, b_2, c)

(1.2) \quad = A(1)B(S^{-1}(a_3)a_1)C(S^{-1}(b_4)b_2, b_2)\sigma(a_2, b_1)\sigma(b_3, c)

(1.3) \quad = A(1)B(S^{-1}(a_3)b_1)C(S^{-1}(b_4)b_2c_2S^{-1}(c_1))\sigma(a_2, b_1)\sigma(b_3, c)

(1.4) \quad = A(1)B(S^{-1}(a_3)a_1)C(S^{-1}(b_4)b_3c_3S^{-1}(c_1))\sigma(a_2, b_1)\sigma(b_2, c_2)

(3.2) \quad = A(1)B(S^{-1}(a_3)a_1)C(c_3S^{-1}(c_1))\sigma(b_1, c_2)\sigma(a_2, b_2),

finishing the proof.

From a similar computation, the following result follows.

**Proposition 4.2** Let \( \gamma \in Reg^1_L(H) \) and define

\[ \mathfrak{T}: D(H) \to k, \quad \mathfrak{T}(p \otimes h) = p(1)\gamma(h), \quad \forall \ p \in H^*, \ h \in H. \]

If \( \gamma \in Reg^1_{PL}(H) \) then \( \mathfrak{T} \in Reg^1_{PL}(D(H)) \) and if \( \gamma \in Reg^1_{NL}(H) \) then \( \mathfrak{T} \in Reg^1_{NL}(D(H)). \)
We recall now from [12] the construction of the Radford biproduct. Let \( H \) be a bialgebra and \( B \) a vector space such that \((B, 1_B)\) is an algebra (with multiplication denoted by \( b \otimes c \mapsto bc \) for all \( b, c \in B \)) and \((B, \Delta_B, \varepsilon_B)\) is a coalgebra. The pair \((H, B)\) is called admissible if \( B \) is endowed with a left \( H \)-module structure (denoted by \( h \otimes b \mapsto h \cdot b \)) and with a left \( H \)-comodule structure (denoted by \( b \mapsto b^{(-1)} \otimes b^{(0)} \in H \otimes B \)) such that:

1. \( B \) is a left \( H \)-module algebra;
2. \( B \) is a left \( H \)-comodule algebra;
3. \( B \) is a left \( H \)-comodule coalgebra, that is, for all \( b \in B \):
   \[
   b_1^{(-1)} b_2^{(-1)} \otimes b_1^{(0)} \otimes b_2^{(0)} = b^{(-1)} \otimes (b^{(0)})_1 \otimes (b^{(0)})_2,
   \]
   \[
   b^{(-1)} \varepsilon_B(b^{(0)}) = \varepsilon_B(b) 1_H.
   \]
4. \( B \) is a left \( H \)-module coalgebra, that is, for all \( h \in H \) and \( b \in B \):
   \[
   \Delta_B(h \cdot b) = h_1 \cdot b_1 \otimes h_2 \cdot b_2,
   \]
   \[
   \varepsilon_B(h \cdot b) = \varepsilon_H(h) \varepsilon_B(b).
   \]
5. \( \varepsilon_B \) is an algebra map and \( \Delta_B(1_B) = 1_B \otimes 1_B \);
6. The following relations hold for all \( h \in H \) and \( b, c \in B \):
   \[
   \Delta_B(bc) = b_1^{(-1)} \cdot c_1 \otimes b_2^{(-1)} \cdot c_2,
   \]
   \[
   (h_1 \cdot b)^{(-1)} h_2 \otimes (h_1 \cdot b)^{(0)} = h_1 b^{(-1)} \otimes h_2 \cdot b^{(0)}.
   \]

If \((H, B)\) is an admissible pair, then we know from [12] that the smash product algebra structure and smash coproduct coalgebra structure on \( B \otimes H \) afford \( B \otimes H \) a bialgebra structure, denoted by \( B \times H \) and called the smash biproduct or Radford biproduct. Its comultiplication is given by

\[
\Delta(b \times h) = (b_1 \times b_2^{(-1)} h_1) \otimes (b_2^{(0)} \times h_2),
\]

for all \( b \in B \), \( h \in H \), and its counit is \( \varepsilon_B \otimes \varepsilon_H \). If \( H \) is a Hopf algebra with antipode \( S_H \) and \((H, B)\) is an admissible pair such that there exists \( S_B \in \text{Hom}(B, B) \) a convolution inverse for \( id_B \), then \( B \times H \) is a Hopf algebra with antipode

\[
S(b \times h) = (1 \times S_H(b^{(-1)} h))(S_B(b^{(0)}) \times 1),
\]

for all \( h \in H \), \( b \in B \). In this case, we will say that \((H, B)\) is a Hopf admissible pair. For a Hopf algebra \( H \), it is well-known (see [10], [11]) that \((H, B)\) being an admissible pair (respectively Hopf admissible pair) is equivalent to \( B \) being a bialgebra (respectively Hopf algebra) in the Yetter-Drinfeld category \( H \text{-YD} \).

Let \( \mathcal{C} \) be a braided monoidal category and \( B \) a Hopf algebra in \( \mathcal{C} \). Then, just as if \( B \) would be a usual Hopf algebra, one can define 2-cocycles, crossed products, Galois extensions, etc, for \( B \) in \( \mathcal{C} \), see for instance [14], [16]. Also, one can define lazy 2-cocycles, lazy 2 coboundaries and the second lazy cohomology group \( \bar{H}_2^L(B) = Z_2^L(B)/B_2^L(B) \), see [8]. We recall these concepts in the case when \( \mathcal{C} = H \text{-YD} \), the category of left Yetter-Drinfeld modules over a Hopf algebra \( H \), and \( B \) a Hopf algebra in \( H \text{-YD} \) (that is, \((H, B)\) is a Hopf admissible pair, so \( B \times H \) is a Hopf algebra). If \( M, N \in H \text{-YD} \), then \( M \otimes N \in H \text{-YD} \) with module structure \( h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n \) and comodule structure \( m \otimes n \mapsto m_{<\cdot>} n_{<\cdot>} \otimes (m_{<0>} \otimes n_{<0>}) \), where \( m \mapsto m_{<\cdot>} \otimes m_{<0>} \) and \( n \mapsto n_{<\cdot>} \otimes n_{<0>} \) are the comodule structures of \( M \) and \( N \), and the braiding is given by

\[
c_{M, N} : M \otimes N \to N \otimes M, \quad c_{M, N}(m \otimes n) = m_{<\cdot>} \cdot n \otimes m_{<0>}.
\]
Hence, the coalgebra structure of $B \otimes B$ in $\mathcal{H}YD$ is given by
\[
\Delta_{B \otimes B}(b \otimes b') = (id \otimes c_{B,B} \otimes id) \circ (\Delta_B \otimes \Delta_B)(b \otimes b')
\]
\[
= (b_1 \otimes b_2^{(-1)} \cdot b'_1 \otimes (b_2^{(0)} \otimes b'_2).
\]
So, if $\sigma, \tau : B \otimes B \to k$ are morphisms in $\mathcal{H}YD$, their convolution in $\mathcal{H}YD$ is given by:
\[
(\sigma * \tau)(b \otimes b') = \sigma(b_1 \otimes b_2^{(-1)} \cdot b'_1) \tau(b_2^{(0)} \otimes b'_2).
\]
(4.12)

Let $\sigma : B \otimes B \to k$ be a morphism in $\mathcal{H}YD$, that is, it satisfies the conditions:
\[
\sigma(h_1 \cdot b \otimes h_2 \cdot b') = \varepsilon(h) \sigma(b \otimes b'),
\]
\[
\sigma(b^{(0)} \otimes b'^{(0)})b^{(-1)}b'^{(-1)} = \sigma(b \otimes b')1_H,
\]
for all $h \in H$ and $b, b' \in B$. Then $\sigma$ is a lazy element if it satisfies the categorical laziness condition (for all $b, b' \in B$):
\[
\sigma(b_1 \otimes b_2^{(-1)} \cdot b'_1)b_2^{(0)}b'_2 = \sigma(b_2^{(0)} \otimes b'_2)b_1(b_2^{(-1)} \cdot b'_1).
\]
(4.15)

Let $\sigma : B \otimes B \to k$ be a normalized left 2-cocycle in $\mathcal{H}YD$, that is $\sigma$ is a normalized morphism in $\mathcal{H}YD$ satisfying the categorical left 2-cocycle condition
\[
\sigma(a_1 \otimes a_2^{(-1)} \cdot b_1)\sigma(a_2^{(0)}b_2 \otimes c) = \sigma(b_1 \otimes b_2^{(-1)} \cdot c_1)\sigma(a \otimes b_2^{(0)} c_2),
\]
(4.16)
for all $a, b, c \in B$. Then we can consider the crossed product $\sigma B = k\#\sigma B$ as in [16], which is an algebra in $\mathcal{H}YD$, and whose multiplication is:
\[
b \cdot b' = \sigma(b_1 \otimes b_2^{(-1)} \cdot b'_1)b_2^{(0)}b'_2.
\]
(4.17)

Since $\sigma B$ is an algebra in $\mathcal{H}YD$, it is in particular a left $H$-module algebra, so one can consider the smash product $\sigma B \# H$.

Let $\gamma : B \to k$ be a morphism in $\mathcal{H}YD$, that is
\[
\gamma(h \cdot b) = \varepsilon(h)\gamma(b),
\]
\[
\gamma(b^{(0)})b^{(-1)} = \gamma(b)1_H,
\]
(4.18) (4.19)
for all $h \in H$ and $b \in B$. If $\gamma$ is normalized and convolution invertible in $\mathcal{H}YD$, with convolution inverse $\gamma^{-1}$ in $\mathcal{H}YD$, the analogue of the operator $D^1$ is given in $\mathcal{H}YD$ by:
\[
D^1(\gamma)(b \otimes b') = \gamma(b_1)\gamma(b_2^{(-1)} \cdot b'_1)\gamma^{-1}(b_2^{(0)}b'_2)
\]
\[
= \gamma(b_1)\gamma(b_2^{(-1)} \cdot b'_1)\gamma^{-1}(b_2^{(0)}b'_2),
\]
(4.18) (4.19)
that is $D^1$ is given by the same formula as for ordinary Hopf algebras. For a morphism $\gamma : B \to k$ in $\mathcal{H}YD$, the laziness condition is identical to the usual one: $\gamma(b_1)b_2 = b_1\gamma(b_2)$ for all $b \in B$.

We recall also the following result from [8].

**Theorem 4.3** ([8]) Let $(H,B)$ be a Hopf admissible pair.

(i) For a normalized left 2-cocycle $\sigma : B \otimes B \to k$ in $\mathcal{H}YD$ define $\overline{\sigma} : (B \times H) \otimes (B \times H) \to k$,
\[
\overline{\sigma}(b \times h, b' \times h') = \sigma(b \otimes h \cdot b')\varepsilon(h').
\]
(4.20)

Then $\overline{\sigma}$ is a normalized left 2-cocycle on $B \times H$ and we have $\sigma B \# H = \overline{\sigma}(B \times H)$ as algebras. Moreover, $\overline{\sigma}$ is unique with this property.
(ii) If \( \sigma \) is convolution invertible in \( H^1_HYD \), then \( \overline{\sigma} \) is convolution invertible, with inverse
\[
\overline{\sigma}^{-1}(b \times h, b' \times h') = \sigma^{-1}(b \otimes h \cdot b') \varepsilon(h'),
\]
where \( \sigma^{-1} \) is the convolution inverse of \( \sigma \) in \( H^1_HYD \).

(iii) If \( \sigma \) is lazy in \( H^1_HYD \), then \( \overline{\sigma} \) is lazy.

(iv) If \( \sigma, \gamma : B \otimes B \to k \) are lazy 2-cocycles in \( H^1_HYD \), then \( \overline{\sigma} \ast \overline{\gamma} = \overline{\sigma \ast \gamma} \), hence the map \( \sigma \mapsto \overline{\sigma} \) is a group homomorphism from \( Z^2_L(B) \) to \( Z^2_L(B \times H) \).

(v) If \( \gamma : B \to k \) is a normalized and convolution invertible morphism in \( H^1_HYD \), define \( \overline{\gamma} : B \times H \to k \) by
\[
\overline{\gamma}(b \times h) = \gamma(b) \varepsilon(h).
\]

Then \( \overline{\gamma} \) is normalized and convolution invertible and \( D^1(\overline{\gamma}) = D^1(\gamma) \). If \( \gamma \) is lazy in \( H^1_HYD \), then \( \overline{\gamma} \) is also lazy.

(vi) If \( \sigma \) is a lazy 2-coboundary for \( B \) in \( H^1_HYD \), then \( \overline{\sigma} \) is a lazy 2-coboundary for \( B \times H \), so the group homomorphism \( Z^2_L(B) \to Z^2_L(B \times H) \), \( \sigma \mapsto \overline{\sigma} \), factorizes to a group homomorphism \( H^1_B(B) \to H^1_B(B \times H) \).

For a morphism \( \sigma : B \otimes B \to k \) in \( H^1_HYD \), we record the following useful formula
\[
\sigma(a \otimes h \cdot b) = \sigma(S^{-1}(h) \cdot a \otimes b),
\]
for all \( a, b \in B \) and \( h \in H \), which is obtained as follows:
\[
\begin{align*}
\sigma(a \otimes h \cdot b) &= \sigma(h_2 S^{-1}(h_1) \cdot a \otimes h_3 \cdot b) \\
&= \sigma(S^{-1}(h) \cdot a \otimes b). \tag{4.13}
\end{align*}
\]

As for the 2-cocycle condition and laziness condition, there exists a categorical analogue for the purity condition (3.1), which is obtained by appropriately introducing the braiding in (3.1); for \( C = H^1_HYD \), the condition which is obtained may be simplified using repeatedly the formulae (4.13), (4.14), (4.18), (4.14), so we arrive at the following concept:

**Definition 4.4** Let \((H, B)\) be a Hopf admissible pair and \( \sigma : B \otimes B \to k \) a normalized and convolution invertible morphism in \( H^1_HYD \), with convolution inverse \( \sigma^{-1} \) in \( H^1_HYD \). We call \( \sigma \) **pure** in \( H^1_HYD \) if it satisfies the condition (for all \( a, b, c, d \in B \)):
\[
\sigma(ab_1 \otimes b_2^{(-1)} \cdot c_1) \sigma^{-1}((b_2^{(0)})_1 \otimes (b_2^{(0)})_2^{(-1)} \cdot c_2) \sigma((b_2^{(0)})_2^{(0)} \otimes c_3 d) \\
= \sigma(b_1 \otimes [b_2^{(-1)} \cdot c_1][(b_2^{(0)})_2^{(0)}(b_2^{(0)})_2^{(-1)} \cdot c_3] \cdot d) \sigma^{-1}((b_2^{(0)})_1 \otimes (b_2^{(0)})_2^{(-1)} \cdot c_2) \\
\sigma(a(b_2^{(0)})_2^{(0)}(b_2^{(0)})_2^{(-1)} \cdot c_3^{(0)}).
\]

Similarly, we have the categorical analogue of the condition (3.2).

**Definition 4.5** Let \((H, B)\) be a Hopf admissible pair and \( \sigma : B \otimes B \to k \) a normalized and convolution invertible morphism in \( H^1_HYD \). We call \( \sigma \) **neat** in \( H^1_HYD \) if it satisfies the condition:
\[
\sigma(a \otimes b_1) \sigma(b_2 \otimes c) = \sigma(a^{(-1)} \cdot b_1 \otimes c) \sigma(a^{(0)} \otimes b_2), \quad \forall a, b, c \in B.
\]

\[
\sigma(a \otimes b_1) \sigma(b_2 \otimes c) = \sigma(a^{(-1)} \cdot b_1 \otimes c) \sigma(a^{(0)} \otimes b_2), \quad \forall a, b, c \in B. 
\]
Remark 4.6 It is not straightforward to prove that a neat lazy cocycle $\sigma : B \otimes B \to k$ in $H^kH \mathcal{YD}$ is pure in $H^kH \mathcal{YD}$. We will see an indirect proof below.

Motivated by Theorem 4.3 we prove the following result.

**Theorem 4.7** Let $(H, B)$ be a Hopf admissible pair and $\sigma : B \otimes B \to k$ pure (respectively neat) in $H^kH \mathcal{YD}$. If we define $\sigma : (B \times H) \otimes (B \times H) \to k$ by formula (4.20), then $\sigma$ is pure (respectively neat). In particular, if $\sigma$ is a pure (respectively neat) lazy cocycle in $H^kH \mathcal{YD}$, then $\sigma$ is a pure (respectively neat) lazy cocycle for $B \times H$.

**Proof.** Note first that $\sigma$ is convolution invertible, with convolution inverse given by (4.21) ($\sigma$ does not have to be a 2-cocycle for this). Now let $a, b, c, d \in B$ and $h, g, l, t \in H$ and assume that $\sigma$ is pure in $H^kH \mathcal{YD}$; we prove the purity condition (3.1) for $\sigma$ on $B \times H$, for the elements $a \times h, b \times g, c \times l, d \times t$. First we compute the right hand side of (3.1):

$$
\sigma((b \times g)_1, (c \times l)_1(d \times t))\sigma^{-1}((b \times g)_2, (c \times l)_2)\sigma((a \times h)(b \times g)_3, (c \times l)_3)
$$

$$
= \sigma(b_1 \times b_2(b_2^{(-1)}, g_1, c_1((c_2^{(-1)})_1 l_1 \cdot d) \times (c_2^{(-1)})_2 l_2))
$$

$$
\sigma^{-1}((b_2^{(0)}_1)_1 \times (b_2^{(0)}_2(b_2^{(-1)}), g_1, c_1((c_2^{(0)}_1)_1 l_3))
$$

$$
\sigma(a(h_1 \cdot b_2^{(0)}_2) \times h_2 g_3, (c_2^{(0)}_2)_2 \times l_4)
$$

$$
= \sigma(b_1 \otimes (b_2^{(-1)}_1 g_1 \cdot c_1((b_2^{(-1)}_2) \times 2 g_2 c_2^{(-1)}_2 c_3^{(-1)} l \cdot d))
$$

$$
\sigma^{-1}((b_2^{(0)}_1)_1 \otimes (b_2^{(0)}_2)_2(-1) g_3(c_2^{(-1)}))
$$

$$
\sigma(a(h_1 \cdot b_2^{(0)}_2)_2 \otimes h_2 g_4 \cdot (c_2^{(-1)}))\varepsilon(t)
$$

$$
= \sigma(b_1 \otimes [(b_2^{(-1)}_1 g_1 \cdot c_1)((b_2^{(-1)}_2) \times 2 g_2 c_2^{(-1)}_2 c_3^{(-1)} l \cdot d))
$$

$$
\sigma^{-1}((b_2^{(0)}_1)_1 \otimes (b_2^{(0)}_2)_2(-1) g_3(c_2^{(0)}))
$$

$$
\sigma(a(h_1 \cdot b_2^{(0)}_2)_2 \otimes h_2 g_4 \cdot (c_2^{(-1)}))\varepsilon(t)
$$

$$
= \sigma(b_1 \otimes [(b_2^{(-1)}_1 g_1 \cdot c_1)((b_2^{(-1)}_2) \times 2 g_2 c_2^{(-1)}_2 c_3^{(-1)} l \cdot d))
$$

$$
\sigma^{-1}((b_2^{(0)}_1)_1 \otimes (b_2^{(0)}_2)_2(-1) g_3(c_2^{(0)}))
$$

$$
\sigma(a(h_1 \cdot b_2^{(0)}_2)_2 \otimes h_2 g_4 \cdot (c_2^{(-1)}))\varepsilon(t)
$$

$$
= \sigma(b_1 \otimes [(b_2^{(-1)})(b_2^{(-1)}_1 g_1 \cdot c_1)((b_2^{(-1)}_2) \times 2 g_2 c_2^{(-1)}_2 c_3^{(-1)} l \cdot d))
$$

$$
\sigma^{-1}((b_2^{(0)}_1)_1 \otimes (b_2^{(0)}_2)_2(-1) g_3(c_2^{(0)}))
$$

$$
\sigma(a(h_1 \cdot b_2^{(0)}_2)_2 \otimes h_2 g_4 \cdot (c_2^{(-1)}))\varepsilon(t)
$$

Now we compute the left hand side of (3.1):

$$
\sigma((a \times h)(b \times g)_1, (c \times l)_1(d \times t))\sigma((b \times g)_2, (c \times l)_2)\sigma((b \times g)_3, (c \times l)_3(d \times t))
$$

$$
\sigma(a(h_1 \cdot b_1) \times h_2 b_2^{(-1)} b_1, c_1 \times c_2^{(-1)} l_1)
$$
\[ \mathbf{\sigma}^{-1}((b_2^{(0)})_1 \times (b_2^{(0)})_{2}^{-1}) g_2, (c_2^{(0)})_1 \times (c_2^{(0)})_{2}^{-1}) l_2, \]
\[ \mathbf{\sigma}((b_2^{(0)})_2) \times g_3, (c_2^{(0)})_{2}([g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes h_2 b_2^{-1} g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}((b_2^{(0)})_1 \otimes (b_2^{(0)})_{2}^{-1}) g_2 \cdot c_2) \]
\[ \mathbf{\sigma}((b_2^{(0)})_2) \otimes [g_3 \cdot c_3][g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes h_2 b_2^{-1} g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}((S^{-1}(h_4) h_3 \cdot b_2^{(0)})_1 \otimes (S^{-1}(h_4) h_3 \cdot b_2^{(0)})_{2}^{-1}) g_2 \cdot c_2) \]
\[ \mathbf{\sigma}((S^{-1}(h_4) h_3 \cdot b_2^{(0)})_2) \otimes [g_3 \cdot c_3][g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_2 \cdot b_2)^{-1} h_3 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}((h_2 \cdot b_2^{(0)})_1 \otimes (h_2 \cdot b_2^{(0)})_{2}^{-1} h_4 g_2 \cdot c_2) \]
\[ \mathbf{\sigma}(S^{-1}(h_5) \cdot ((h_2 \cdot b_2^{(0)})_2) \otimes [g_3 \cdot c_3][g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_2 \cdot b_2)^{-1} h_3 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}((h_2 \cdot b_2^{(0)})_1 \otimes (h_2 \cdot b_2^{(0)})_{2}^{-1} h_4 g_2 \cdot c_2) \]
\[ \mathbf{\sigma}(S^{-1}(h_5) \cdot ((h_2 \cdot b_2^{(0)})_2) \otimes [g_3 \cdot c_3][g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_1 \cdot b_2)^{-1} (h_2 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}(((h_1 \cdot b_2^{(0)})_1 \otimes ((h_1 \cdot b_2^{(0)})_{2}^{-1} (h_2 g_1 \cdot c_2) \]
\[ \mathbf{\sigma}(((h_1 \cdot b_2^{(0)})_2) \otimes [h_4 g_3 \cdot c_3][h_5 g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_1 \cdot b_2)^{-1} (h_2 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}(((h_1 \cdot b_2^{(0)})_1 \otimes ((h_1 \cdot b_2^{(0)})_{2}^{-1} (h_2 g_1 \cdot c_2) \]
\[ \mathbf{\sigma}(((h_1 \cdot b_2^{(0)})_2) \otimes [h_4 g_3 \cdot c_3][h_5 g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_1 \cdot b_2)^{-1} (h_2 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}(((h_1 \cdot b_2^{(0)})_1 \otimes ((h_1 \cdot b_2^{(0)})_{2}^{-1} (h_2 g_1 \cdot c_2) \]
\[ \mathbf{\sigma}(((h_1 \cdot b_2^{(0)})_2) \otimes [h_4 g_3 \cdot c_3][h_5 g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_1 \cdot b_2)^{-1} (h_2 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}(((h_1 \cdot b_2^{(0)})_1 \otimes ((h_1 \cdot b_2^{(0)})_{2}^{-1} (h_2 g_1 \cdot c_2) \]
\[ \mathbf{\sigma}(((h_1 \cdot b_2^{(0)})_2) \otimes [h_4 g_3 \cdot c_3][h_5 g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_1 \cdot b_2)^{-1} (h_2 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}(((h_1 \cdot b_2^{(0)})_1 \otimes ((h_1 \cdot b_2^{(0)})_{2}^{-1} (h_2 g_1 \cdot c_2) \]
\[ \mathbf{\sigma}(((h_1 \cdot b_2^{(0)})_2) \otimes [h_4 g_3 \cdot c_3][h_5 g_4 l \cdot d]) \varepsilon(t) \]
\[ = \mathbf{\sigma}(a(h_1 \cdot b_1) \otimes (h_1 \cdot b_2)^{-1} (h_2 g_1 \cdot c_1) \]
\[ \mathbf{\sigma}^{-1}(((h_1 \cdot b_2^{(0)})_1 \otimes ((h_1 \cdot b_2^{(0)})_{2}^{-1} (h_2 g_1 \cdot c_2) \]
\[ \mathbf{\sigma}(((h_1 \cdot b_2^{(0)})_2) \otimes [h_4 g_3 \cdot c_3][h_5 g_4 l \cdot d]) \varepsilon(t) \]
\[4.8\] \[4.9\] \[4.10\]

and we see that the two terms are equal. Assume now that \( \sigma \) is neat in \( H \)-YD; we prove (3.2) for \( \overline{\sigma} \) on \( B \times H \), for the elements \( a \times h \), \( b \times g \), \( c \times l \). We compute:

\[\overline{\sigma}(a \times h, (b \times g)_1)\overline{\sigma}((b \times g)_2, c \times l)\]

\[\begin{align*}
\overline{\sigma}(a \times h, (b \times g)_1)\overline{\sigma}((b \times g)_2, c \times l) &= \overline{\sigma}(a \times h, b_1 \times b_2^{(-1)}g_1)\overline{\sigma}(b_2^{(0)} \times g_2, c \times l) \\
&= \sigma(a \otimes h \cdot b_1)\sigma(b_2 \otimes g \cdot c)\epsilon(l) \\
&= \sigma(a \otimes h_1 \cdot b_1)\sigma(S^{-1}(h_3)h_2 \cdot b_2 \otimes g \cdot c)\epsilon(l) \\
&= \sigma(a \otimes (h_1 \cdot b_1))\sigma(S^{-1}(h_2) \cdot (h_1 \cdot b_2) \otimes g \cdot c)\epsilon(l) \\
&= \sigma(a \otimes (h_1 \cdot b_1))\sigma((h_1 \cdot b_2 \otimes h_2g \cdot c)\epsilon(l) \\
&= \sigma(a^{(-1)} \cdot (h_1 \cdot b_1) \otimes h_2g \cdot c)\sigma(a^{(0)} \otimes (h_1 \cdot b_2)\epsilon(l) \\
&= \sigma(a^{(-1)}h_1 \cdot b_1 \otimes h_3g \cdot c)\sigma(a^{(0)} \otimes h_2 \cdot b_2)\epsilon(l),
\end{align*}\]

\[\overline{\sigma}(b \times g)_1, c \times l)\overline{\sigma}(a \times h, (b \times g)_2)\]

\[\begin{align*}
\overline{\sigma}(b \times g)_1, c \times l)\overline{\sigma}(a \times h, (b \times g)_2) &= \overline{\sigma}(b_1 \times b_2^{(-1)}g_1, c \times l)\overline{\sigma}(a \times h, b_2^{(0)} \times g_2) \\
&= \sigma(b_1 \otimes b_2^{(-1)}g \cdot c)\sigma(a \otimes h \cdot b_2^{(0)})\epsilon(l) \\
&= \sigma(b_1 \otimes S(h_1)h_2b_2^{(-1)}g \cdot c)\sigma(a \otimes h_3 \cdot b_2^{(0)})\epsilon(l) \\
&= \sigma(b_1 \otimes S(h_1)(h_2 \cdot b_2)^{(1)}h_3g \cdot c)\sigma(a \otimes (h_2 \cdot b_2)\epsilon(l) \\
&= \sigma(b_1 \otimes S(h_1)S((a^{(-1)})_1)(a^{(-1)})_2(h_2 \cdot b_2)^{(1)}h_3g \cdot c) \\
&\sigma(a^{(0)} \otimes (h_2 \cdot b_2)^{(0)})\epsilon(l)
\end{align*}\]
If \((3.3)\) looks very complicated, so we treat only the analogue of \((3.4)\).

There exist also categorical analogues of the relations \((3.3)\) and \((3.4)\); the one corresponding to \((4.7)\) this proves that, if \(\sigma\) is neat (respectively neat) in \(H_2\) then \(\sigma\) is pure (respectively neat) in \(H_2\). Together with Proposition \(3.13\) and Theorems \(4.3\) and \(4.7\) this proves that, if \(\sigma\) is a neat lazy cocycle in \(H_2\), then \(\sigma\) is a pure lazy cocycle in \(H_2\).

**Remark 4.8** Let \((H, B)\) be a Hopf admissible pair and \(\sigma : B \otimes B \to k\) a normalized and convolution invertible morphism in \(H_2\), and define \(\overline{\sigma}\) by formula \((4.20)\). From the computation in the proof of Theorem \(4.7\) it follows that, conversely, if \(\overline{\sigma}\) is pure (respectively neat) on \(B \times H\), then \(\sigma\) is pure (respectively neat) in \(H_2\). Together with Proposition \(3.13\) and Theorems \(4.3\) and \(4.7\) this proves that, if \(\sigma\) is a neat lazy cocycle in \(H_2\), then \(\sigma\) is a pure lazy cocycle in \(H_2\).

**Definition 4.9** Let \((H, B)\) be a Hopf admissible pair and \(\gamma : B \to k\) a normalized and convolution invertible morphism in \(H_2\); we call \(\gamma\) neat in \(H_2\) if it satisfies the condition

\[
\gamma(ab_1)\gamma(b_2c) = \gamma(b_1(b_{2(1)} \cdot c))\gamma(ab_2^{(0)}), \quad \forall a, b, c \in B.
\]

**Proposition 4.10** If \(\gamma\) is neat in \(H_2\), then the map \(\overline{\gamma} : B \times H \to k\) given by \((4.22)\) is neat.

**Proof.** Let \(a, b, c \in B\) and \(h, g, l \in H\); we check \((3.4)\) for the elements \(a \times h, b \times g, c \times l\). We compute:

\[
\overline{\gamma}((a \times h)(b \times g)_{11})\overline{\gamma}((b \times g)_{22}(c \times l))
\]

\[
\overset{(4.9)}{=} \overline{\gamma}((a \times h)(b \times b_{2(1)}^{-1}g_1))\overline{\gamma}((b_{2(0)} \times g_2)(c \times l))
\]

\[
\overset{(4.22)}{=} \gamma(a \cdot h \cdot b_{11})\gamma(b_{2(0)} \cdot c)\varepsilon(l)
\]

\[
\overset{(4.5)}{=} \gamma(a \cdot h \cdot b_{11})\gamma((S^{-1}(b_{2(1)}^{-1})(b \times b_2)(g \cdot c))\varepsilon(l)
\]

\[
\overset{(4.15)}{=} \gamma(a \cdot h \cdot b_{11})\gamma((h_1 \cdot b_{22}(h_2 \times g \cdot c))\varepsilon(l)
\]

\[
\overset{(4.26)}{=} \gamma((h_1 \cdot b_{11})(h_1 \cdot b_{22}(h_2 \times g \cdot c))\gamma(a \cdot h \cdot b_{22}^{(0)})\varepsilon(l)
\]

\[
\overset{(4.5)}{=} \gamma((h_1 \cdot b_{11})(h_2 \times b_{22}(c))\gamma(a \cdot h \cdot b_{22}^{(0)})\varepsilon(l)
\]

\[
\overset{(4.8)}{=} \gamma((h_1 \cdot b_{11})(h_2 \times b_{22}(c))\gamma(a \cdot h \cdot b_{22}^{(0)})\varepsilon(l)
\]

\[
\overset{(4.15)}{=} \gamma((h_1 \cdot b_{11})(h_2 \times b_{22}(c))\gamma(a \cdot h \cdot b_{22}^{(0)})\varepsilon(l)
\]

and we see that the two terms are equal. □

**Remark 4.11** Combining Proposition \(3.13\), Theorem \(4.3\) (v) and Remark \(4.8\) we obtain: if \(\gamma : B \to k\) is neat lazy in \(H_2\) then \(D^1(\gamma)\) is neat lazy in \(H_2\).
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