QUANTUM COHOMOLOGY VIA D-MODULES

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Dedicated to Graeme Segal on the occasion of his 60th birthday

Abstract. We propose a new point of view on quantum cohomology, motivated by the work of Givental and Dubrovin, but closer to differential geometry than the existing approaches. The central object is a D-module which “quantizes” a commutative algebra associated to the (uncompactified) space of rational curves. Under appropriate conditions, we show that the associated flat connection may be gauged to the flat connection underlying quantum cohomology. This method clarifies the role of the Birkhoff factorization in the “mirror transformation”, and it gives a new algorithm (requiring construction of a Groebner basis and solution of a system of o.d.e.) for computation of the quantum product.

Quantum cohomology first arose in physics, and its (mathematically conjectural) properties were supported by physical intuition. A rigorous mathematical definition came later, based on deep properties of certain moduli spaces. We shall propose another point of view on quantum cohomology, closer in spirit to differential geometry.

The main ingredient in our approach is a flat connection, considered as a holonomic D-module (or maximally overdetermined system of p.d.e.). This object itself is not new: Givental’s “quantum cohomology D-module” is already well known ([Gi1]), and the associated flat connection appears in Dubrovin’s theory of Frobenius manifolds ([Du]). But, in the existing literature, the D-module plays a subservient role, being a consequence of the construction of the Gromov-Witten invariants and the quantum cohomology algebra. For us, the D-module will be the main object of interest.

We define a quantization of a (commutative) algebra $A$ to be a (non-commutative) D-module $M^h$ which satisfies certain properties. The quantum cohomology D-module is a particular kind of quantization, which arises in the following way. For a Kähler manifold $M$, we start with an algebra $A$ which is associated to the “raw data” consisting of the set of all rational curves in $M$. Then we construct (or assume the existence of) a quantization $M^h$. Next we transform $M^h$ into a new D-module $\hat{M}^h$ with certain properties. Finally, de-quantization (“semi-classical limit”) produces a commutative algebra $\hat{A}$, which (under appropriate conditions) turns out to be the quantum cohomology $QH^*M$. 

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Our scope will be very modest in this article: we consider only the “small” quantum cohomology algebra \( QH^*M \) of a manifold \( M \) whose ordinary cohomology algebra \( H^*M \) is generated by two-dimensional classes. But this case is sufficiently nontrivial to demonstrate that our method has something to offer, both conceptually and computationally. The most obvious conceptual benefit is that the usual moduli space \( M \) has been replaced by the D-module \( M^h \). As a first application we give an algorithm for computing the structure constants of the quantum cohomology algebra (3-point genus zero Gromov-Witten invariants), in the case of a Fano manifold. This involves a Gröbner basis calculation and a finite number of “quadratures”; it is quite different from previously known methods. A second application is a new interpretation of the “mirror coordinate transformation”. Impressively mysterious in its original context ([Gi3]–[Gi4], [LLY1]–[LLY3]), it arises here in a straightforward differential geometric fashion, reminiscent of the well known transformation to local Euclidean coordinates for a flat Riemannian manifold.

Here is a more detailed description of the organization of this paper. In §1 we review some facts concerning D-modules, mainly to establish notation. In §2 we recall the quantum cohomology algebra and the quantum product, again to set up notation. “Quantum cohomology algebra” refers to the isomorphism type of the algebra, while “quantum product” means the product operation on the vector space \( H^*M \), i.e. a way of multiplying ordinary cohomology classes.

Our point of view is introduced in §3: we start with an algebra \( A \) and construct from it both a “quantum cohomology algebra” and a “quantum product”. The method is conceptually straightforward. To a quantization \( M^h \) of \( A \) there corresponds a flat connection \( \nabla = d + \Omega^h \), where \( \Omega^h \) has a simple pole at \( h = 0 \). We may write \( \Omega^h = L^{-1}dL \) for some loop group-valued map \( L \). Replacing \( L \) by \( L_- \), where \( L = L_- L_+ \) is the Birkhoff factorization, we obtain \( \hat{\Omega}^h = L_-^{-1}dL_- \), and the connection \( d + \hat{\Omega}^h \) is the required connection. The map \( L \) is a (complicated) generating function for certain Gromov-Witten invariants but we shall not need it. Our main interest is the gauge transformation \( L_+ = Q_0 + O(h) \) which converts \( \Omega^h \) to \( \hat{\Omega}^h \). For the manifolds discussed here, \( A \) and \( M^h \) are known, and \( \Omega^h \) can be computed. If \( L_+ \) can be computed, then \( \hat{\Omega}^h \) (and the quantum cohomology algebra, together with its structure constants) can be computed too.

In §4 we discuss the case of Fano manifolds. Here it turns out that \( A = \hat{A} \), i.e. the “provisional” algebra is actually the “correct answer”. The gauge transformation \( L_+ \) has a special form but it is not trivial; indeed, its first term \( Q_0 \) tells us how to produce the quantum product. Thus all quantum products can be determined explicitly by our method from the relations of the quantum cohomology algebra (more precisely, from their quantizations). The following two families of manifolds are of special interest:

(1) Let \( M = G/B \), the full flag manifold of a complex semisimple Lie group \( G \). The quantum cohomology algebra was found originally by Givental and Kim ([Gi-Ki], [Ki]) and justified via the conventional moduli space theory. The first integrals of the quantum Toda lattice provide a quantization \( M^h \). It is known that the quantum product can be
described using quantum Schubert polynomials (see [FGP], [Ki-Ma] for the case \( G = GL_n \mathbb{C} \)); therefore, the theory of such polynomials is governed by our matrix \( Q_0 \). A more detailed treatment of flag manifolds from our point of view can be found in [Am-Gu].

(2) Let \( M \) be a Fano toric manifold. In this case a formula for the quantum cohomology was proposed by Batyrev ([Ba1]), but the subsequent proof of the correctness of the formula (see [Co-Ka], Chapter 11) depended on Givental’s mirror theorem from [Gi4]. The appropriate quantization is the generalized hypergeometric D-module of [GKZ] (whose relevance to mirror symmetry was already known; cf. [Ba2], [HLY]). Again, the matrix \( Q_0 \) produces the quantum product.

Beyond Fano manifolds there arises the interesting possibility that \( \hat{A} \) may be different from \( A \), and we discuss this in §5, primarily with toric manifolds in mind. Several authors have pointed out that the quantum cohomology algebra constructed by Batyrev in [Ba1] is generally the “wrong answer” for a non-Fano toric manifold. Our point of view resolves this apparent conflict, at least in the case of semi-positive toric manifolds: Batyrev’s algebra is \( A \), the “usual” quantum cohomology algebra is \( \hat{A} \), and the two are related via \( L_+ \). The gauge transformation \( L_+ \) contains more information than in the Fano case, namely a coordinate transformation. For toric manifolds this is Givental’s mirror transformation. It is a natural operation from the point of view of D-modules, but considerably less so from the point of view of the quantum cohomology algebra, where it seems miraculous ([Gi4], [Co-Ka]).

The results of this paper can probably be generalized in various directions. For manifolds whose ordinary cohomology is not generated by two-dimensional classes, one may work with the subalgebra generated by such classes, as is standard in discussions of mirror symmetry. For “big” quantum cohomology our methods may apply to some extent. Finally, there may well be more general algebras \( A \) to which our methods apply, i.e. algebras without any obvious connection to quantum cohomology theory.

This project began with a conviction that integrable systems methods could be used to rehabilitate Batyrev’s “incorrect” computations of quantum cohomology algebras of toric varieties in [Ba1]. It will be obvious to the experts that our framework owes much to the ideas of Givental ([Gi1]-[Gi4]) and Dubrovin ([Du]), and we gladly acknowledge these as our main sources of inspiration, though we would not have made much progress without the excellent treatments of quantum cohomology in [Co-Ka] and hypergeometric D-modules in [SST].

For background information on quantum cohomology we refer the reader to the books [Co-Ka], [Mn] and their references. In addition, survey articles related to the quantum differential equations include [BCPP], [Pa], [Gu2]. An introduction to loop group techniques in integrable systems can be found in the book [Gu1].

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§1 D-modules and flat connections

Let $K$ be an algebra of functions of the complex variables $q_1, \ldots, q_r$. (In practice we shall use the polynomial algebra $C[q_1, \ldots, q_r]$, or the field of rational functions or germs of holomorphic functions.) Depending on the context, we regard $q_i$ either as a formal variable or as a function $t \mapsto q_i = e^{t_i}$ where $t = (t_1, \ldots, t_r) \in C^r$. We introduce the notation $\partial_i = \frac{\partial}{\partial t_i} = q_i \frac{\partial}{\partial q_i}$, and define $D$ to be the algebra of differential operators generated by $\partial_1, \ldots, \partial_r$ with coefficients in $K$. Let $M = D/(D_1, \ldots, D_u)$ be a cyclic $D$-module (a left module over $D$, generated by the constant differential operator 1), where $(D_1, \ldots, D_u)$ means the left ideal generated by differential operators $D_1, \ldots, D_u$. In this section we shall assume that $M$ is free over $K$ of rank $s + 1$. For basic facts on D-modules we refer to [SST], [Ph], [Co].

The D-module $M$ is an algebraic version of the system of partial differential equations $D_1 f = \cdots = D_u f = 0$. Here, $f$ belongs to a given function space $F$, but $M$ is of course independent of $F$ (and this is its advantage). To say that $M$ has finite rank over $K$ is to say, roughly speaking, that the system is “maximally overdetermined”; in particular its solution space is finite dimensional. More precisely, the vector space $\text{Hom}_D(M, F)$ is called the solution space of $M$ with respect to the function space $F$, and this is isomorphic to the usual solution space $\{f \in F \mid D_1 f = \cdots = D_u f = 0\}$ of the system: to a solution $f$ there corresponds the D-module homomorphism $M \to F$ given by $P \mapsto Pf$ (for any $P \in D$). The solution space (in either sense) is a complex vector space of dimension $s + 1$.

We shall review briefly the relation between D-modules and flat connections. Let us choose differential operators $P_0, \ldots, P_s$ such that the equivalence classes $[P_0], \ldots, [P_s]$ form a $K$-module basis of $M$. (There is a standard way of doing this, by constructing first a Gröbner basis of the ideal $(D_1, \ldots, D_u)$, as explained in Section 1.4 of [SST].) Without loss of generality we may assume $P_0 = 1$. With respect to this basis we define matrices $\Omega_i = (\Omega^i_{kj})_{0 \leq k, j \leq s}$ by

$$[\partial_i P_j] = \sum_{k=0}^s \Omega^i_{kj}[P_k],$$

and we put $\Omega = \sum_{i=1}^r \Omega_i dt_i$, a 1-form with values in the space $\text{End}(C^{s+1})$ of complex $(s + 1) \times (s + 1)$ matrices. The formula $\nabla = d + \Omega$ defines a connection in the trivial vector bundle $C^r \times C^{s+1} \to C^r$, where $C^{s+1}$ is identified with the vector space spanned by $[P_0], \ldots, [P_s]$. Namely, $\nabla_{\partial_i}[P_j] = \sum_{k=0}^s \Omega^j_{kj}[P_k]$, and more generally for any section
\[ \sum_{j=0}^{s} y_j [P_j] \] of this bundle, \( \nabla_{\partial_i} (\sum_{j=0}^{s} y_j [P_j]) = \sum_{j=0}^{s} \partial_i y_j [P_j] + \sum_{j=0}^{s} y_j \nabla_{\partial_i} [P_j] \).

**Proposition 1.1.** The connection \( \nabla \) is flat.

**Proof.** By definition we have \( \nabla_{\partial_i} \nabla_{\partial_j} = \nabla_{\partial_j} \nabla_{\partial_i} \) (since \( \partial_i \partial_j = \partial_j \partial_i \)), so the curvature tensor of \( \nabla \) is zero. Alternatively, the zero curvature condition \( \partial \Omega + \Omega \wedge \Omega = 0 \) follows directly from computing both sides of the equation \( \partial_i \partial_j [P_k] = \partial_j \partial_i [P_k] \). \( \square \)

**Proposition 1.2.** We have an isomorphism of vector spaces

\[
\text{Hom}_D(M, F) \longrightarrow \{ \text{covariant constant sections of } \nabla^* \}, \quad f \mapsto \begin{pmatrix} P_0 f \\ \vdots \\ P_s f \end{pmatrix}
\]

where \( \nabla^* \) is the dual connection to \( \nabla \).

**Proof.** On the left hand side, \( f \) is regarded as the D-module homomorphism \( P \mapsto Pf \), whereas on the right hand side \( f \) is a solution of the system \( D_1 f = \cdots = D_u f = 0 \). The dual connection is defined by \( (\nabla^*_{\partial_i} [P_j]^* )[P_k] = -[P_j]^* (\nabla_{\partial_i} [P_k] ) \) where \([P_0]^*, \ldots, [P_s]^*\) is the dual basis to \([P_0], \ldots, [P_s]\). The column vector in the statement of the proposition refers to the section \( \sum_{j=0}^{s} (P_j f) [P_j]^* \). A section \( \sum_{j=0}^{s} y_j [P_j]^* \) is covariant constant if the following expression is zero for all \( k \):

\[
(\nabla^*_{\partial_i} \sum_{j=0}^{s} y_j [P_j]^* )[P_k] = (\sum_{j=0}^{s} \partial_i y_j [P_j]^* + \sum_{j=0}^{s} y_j \nabla^*_{\partial_i} [P_j]^* )[P_k]
\]

\[
= \partial_i y_k - \sum_{j=0}^{s} y_j ([P_j]^* \sum_{l=0}^{s} \Omega^i_{jk} [P_l])
\]

\[
= \partial_i y_k - \sum_{j=0}^{s} y_j \Omega^i_{jk}.
\]

For any \( f \in \text{Hom}_D(M, F) \), we have to verify that \( y_k = P_k f \) defines a covariant constant section. But this follows immediately from the formula \( [\partial_i P_k] = \sum_{k=0}^{s} \Omega^i_{jk} [P_j] \) defining \( \Omega \). The map in question is therefore a well defined, linear, map. To prove that it is an isomorphism, we observe that the kernel is zero (because \( P_0 f = f \)), and that \( \dim \text{Hom}_D(M, F) = s + 1 \) by assumption. \( \square \)

This generalizes the well known elementary construction of a system of first order o.d.e. equivalent to a higher order o.d.e. Here we construct the system \( \partial_i y_k = \sum_{j=0}^{s} y_j \Omega^i_{jk} \) of first order p.d.e. equivalent to the higher order system \( D_1 f = \cdots = D_u f = 0 \). Conversely,
given a flat connection (hence a system of first order p.d.e.), it is possible to construct a
cyclic D-module of finite rank over an appropriate algebra $K$ (hence a system of higher
order p.d.e.).

Since the dual connection $\nabla^* = d - \Omega^t$ is flat, there exist covariant constant sections
$H_0, \ldots, H_s$ which are linearly independent at each point of $\mathbb{C}^r$. Representing these sections
by column vectors, as above, let us introduce

$$H = \begin{pmatrix} H_0 & \cdots & H_s \end{pmatrix}$$

i.e. the “fundamental solution matrix” of the first order system. By definition we have
$\Omega^t = dHH^{-1}$. Up to multiplication on the right by a constant invertible matrix, this
equation determines $H$ uniquely. Equivalently, if $f_0, \ldots, f_s$ are a basis of solutions of the
higher order system $D_1f = \cdots = D_uf = 0$, and if $J = (f_0, \ldots, f_s)$ is regarded as a row
vector, then

$$H = \begin{pmatrix} -P_0J & - \\ \vdots & \vdots \\ -P_sJ & - \end{pmatrix}$$

satisfies $\Omega^t = dHH^{-1}$.

A standard technique is to study the transformation (symbol map) $\partial_i \mapsto b_i$ from the
non-commutative algebra $D$ to the commutative algebra $K$. A differential operator $P$ maps
to a polynomial $\tilde{P}$. The D-module $M = D/(D_1, \ldots, D_u)$ is transformed to a $K$-module
$\tilde{M} = K[b_1, \ldots, b_r]/(\tilde{D}_1, \ldots, \tilde{D}_u)$, and the associated flat connection $\nabla$ is transformed to
a connection $\tilde{\nabla}$, but the connection $\tilde{\nabla}$ is not in general flat. In more detail, we have
$\nabla = d + \sum_{i=1}^r \Omega_i dt_i$ where $\Omega_i$ is the matrix representing the action of the differential
operator $\partial_i$, and $\tilde{\nabla} = d + \sum_{i=0}^r \tilde{\Omega}_i dt_i$ where $\tilde{\Omega}_i$ is the matrix representing the action of the
operator $b_i$. As explained earlier, the fact that $\partial_i \partial_j = \partial_j \partial_i$ leads to the flatness condition
d$\Omega + \Omega \wedge \Omega = 0$. However, the condition $b_i b_j = b_j b_i$ says only that $\tilde{\Omega} \wedge \tilde{\Omega} = 0$. The exterior
derivative $d\tilde{\Omega}$ is not in general zero. This phenomenon is the key to our construction of
quantum cohomology in §3.

§2 The quantum cohomology D-module

In this section we shall review briefly the Dubrovin connection (or D-module) which
arises in the standard construction of quantum cohomology theory. We begin with a compacKähler manifold $M$ of (complex) dimension $n$, whose ordinary cohomology algebra
— with complex coefficients — is of the form

$$H^*M = \mathbb{C}[b_1, \ldots, b_r]/(R_1, \ldots, R_u)$$
where \(b_1, \ldots, b_r\) are additive generators of \(H^2 M\) and \(R_1, \ldots, R_u\) are certain relations (polynomials in \(b_1, \ldots, b_r\)). (As mentioned in the introduction, this assumption can be removed by studying the subalgebra generated by two-dimensional cohomology classes.) By general principles it follows that the (small) quantum cohomology algebra is of the form

\[
QH^* M = K[b_1, \ldots, b_r]/(R_1, \ldots, R_u)
\]

where \(K = \mathbb{C}[q_1, \ldots, q_r]\) and each \(R_i\) is a “\(q\)-deformation” of \(R_i\). (For certain \(M\), an extension or completion of \(K\) may be necessary here, but we shall assume in this section that \(M\) is not of this type.) As in §1, the variables \(q_1, \ldots, q_r\) here may be considered either as formal variables or as functions \(q_i: t = \sum_{j=1}^r t_j b_j \mapsto e^{t_i} \) on \(H^2 M\). With the latter convention, \(H^* M\) and \(QH^* M\) are isomorphic as vector spaces (but not, in general, as algebras), for each value of \(t\).

Quantum cohomology theory gives, in addition to \(QH^* M\), a quantum product operation on \(H^* M\). That is, for any \(x, y \in H^* M\), there is an element \(x \circ_t y \in H^* M\), which has the property \(x \circ_t y = x \cdot y + \text{terms involving } q_i = e^{t_i}, 1 \leq i \leq r, \text{ where } x \cdot y \text{ denotes the cup product. The relations } R_1, \ldots, R_u \text{ are those of the algebra } (H^* M, \cdot), \text{ while the relations } R_1, \ldots, R_u \text{ are those of the algebra } (H^* M, \circ_t). \text{ In particular this gives rise to an isomorphism of vector spaces } \delta: QH^* M \rightarrow H^* M \text{ which “evaluates” a polynomial using the quantum product.}

The Dubrovin connection is the (complex) connection \(\nabla = d + \frac{1}{h^*} \omega\) on the trivial bundle \(\mathbb{C}^r \times \mathbb{C}^{s+1} \rightarrow \mathbb{C}^r\) where \(\omega\) is the complex \(\text{End } \mathbb{C}^{s+1}\)-valued 1-form on \(\mathbb{C}^r\) defined by \(\omega_t(x)(y) = x \circ_t y\). Here \(h\) is a nonzero complex parameter, so in fact we have a family of connections.

**Theorem 2.1.** For any \(h\) the connection \(\nabla = d + \frac{1}{h^*} \omega\) is flat, i.e. \(d \omega = \omega \wedge \omega = 0\). □

A proof of this well known theorem and further explanation can be found in [Co-Ka] and the other references on quantum cohomology at the end of this paper.

§3 Reconstructing quantum cohomology

We begin with an abstract algebra of the form

\[
\mathcal{A} = K[b_1, \ldots, b_r]/(R_1, \ldots, R_u),
\]

where the relations \(R_1, \ldots, R_u\) are homogeneous with respect to a fixed assignment of degrees \(|b_i|, |q_j|\). We shall always choose \(|b_1| = \cdots = |b_r| = 2, \text{ but } |q_1|, \ldots, |q_r| \text{ (not necessarily non-negative) will be specified later. In addition we assume that } \mathcal{A} \text{ is a free } K\text{-module of rank } s+1. \text{ Finally, we assume that } \mathcal{A} \text{ is a deformation of an algebra}

\[
\mathcal{A}_0 = \mathbb{C}[b_1, \ldots, b_r]/(R_1, \ldots, R_u)
\]
in the sense that \( R_i|_{q=0} = R_i \) for \( i = 1, \ldots, u \) and \( \dim_{\mathbb{C}} \mathcal{A}_0 = s + 1 \).

Although it will play no role in this section, we should mention that the situation we have in mind is where \( \mathcal{A}_0 = H^*M \) for a compact connected Kähler manifold \( M \), and where \( \mathcal{A} \) is obtained by using the (uncompactified) space of rational curves in \( M \) to define structure constants in the “naive” way as in early papers in the physics literature. In our examples \( M \) will be a flag manifold \( G/B \) or a toric manifold, and we shall specify \( \mathcal{A} \) precisely when we discuss those cases.

Our main objective in this section will be to construct connections satisfying the property of Theorem 2.1. For this purpose, we introduce the ring \( D^h \) of differential operators generated by \( h\partial_1, \ldots, h\partial_r \) with coefficients in \( K[h] \), and we make the following fundamental definition:

**Definition 3.1.** A quantization of \( \mathcal{A} \) is a \( D \)-module \( \mathcal{M}^h = D^h/(D^h_1, \ldots, D^h_u) \) such that

1. \( \mathcal{M}^h \) is free over \( K[h] \) of rank \( s + 1 \),
2. \( \lim_{h \to 0} S(D^h_i) = R_i \), where \( S(D^h_i) \) is the result of replacing \( h\partial_1, \ldots, h\partial_r \) by \( b_1, \ldots, b_r \) in \( D^h_i \) (for \( i = 1, \ldots, u \)).

This notion depends on the specified generators and relations of \( \mathcal{A} \), of course. There is no guarantee that such a quantization exists, but it is sometimes possible to produce a quantization simply by replacing \( b_1, \ldots, b_r \) by \( h\partial_1, \ldots, h\partial_r \) in each \( R_i \). When this works, i.e. when the resulting \( D \)-module is free of rank \( s+1 \), we refer to it as the naive quantization.

Assume now that \( \mathcal{M}^h \) is a quantization of \( \mathcal{A} \). Then we may choose a \( K[h] \)-module basis \( [P_0], \ldots, [P_s] \) of \( \mathcal{M}^h \) such that \( [c_0 = \lim_{h \to 0} S(P_0)], \ldots, [c_s = \lim_{h \to 0} S(P_s)] \) is a \( K \)-module basis of \( \mathcal{A} \). We shall always do this by taking \( P_0, \ldots, P_s \) to be the “standard monomials” in \( h\partial_1, \ldots, h\partial_r \) with respect to a choice of Gröbner basis for the ideal \( (D^h_1, \ldots, D^h_u) \). For definiteness we use the graded reverse lexicographic monomial order in which \( \partial_1, \ldots, \partial_r \) are assigned weight one with \( \partial_1 > \cdots > \partial_r \). (Gröbner basis theory for this situation is explained in [SST]. Explicit computations may be carried out using the Ore algebra package of the software Maple, [Ma].) We define a connection form \( \Omega^h = \sum_{i=1}^r \Omega^h_i dt_i \) as follows:

**Notation.** For \( i = 1, \ldots, r \):

1. let \( \Omega^h_i \) denote “the matrix of the action of \( \partial_i \)” on the \( K[h] \)-module \( \mathcal{M}^h \), i.e. \( [\partial_i P_j] = \sum_{k=0}^s (\Omega^h_i)_{kj} [P_k] \);
2. let \( \omega_i \) denote the matrix of multiplication by \( b_i \) on the \( K \)-module \( \mathcal{A} \), i.e. \( [b_i c_j] = \sum_{k=0}^s (\omega_i)_{kj} [c_k] \).
It follows that $h\Omega^h$ is polynomial in $h$, so $\Omega^h$ is of the form

$$\Omega^h = \frac{1}{h}\omega + \theta^{(0)} + h\theta^{(1)} + \cdots + h^p\theta^{(p)},$$

where $\omega = \sum_{i=1}^r \omega_idt_i$, $\theta^{(0)}, \ldots, \theta^{(p)}$ are matrix-valued 1-forms, and $p$ is a non-negative integer which depends on the relations $\mathcal{R}_1, \ldots, \mathcal{R}_u$.

If $\theta^{(0)}, \ldots, \theta^{(p)}$ were all zero, then the connection $\nabla = d + \Omega^h$ (which is flat, by §1) would satisfy the condition of Theorem 2.1, and hence would be a candidate for the Dubrovin connection. It turns out that this situation can be achieved by making a suitable modification:

**Proposition 3.2.** Assume that $\Omega^h$ depends holomorphically on $q = (q_1, \ldots, q_r)$, for $q$ in some open subset $V$. Then, for any point $q_0$ in $V$, there is a neighbourhood $U_0$ of $q_0$ on which the connection $\nabla = d + \Omega^h$ is gauge equivalent to a connection $\hat{\nabla} = d + \hat{\Omega}^h$ with $\hat{\Omega}^h = \frac{1}{h}\hat{\omega}$, $\hat{\omega} = Q_0\omega Q_0^{-1}$, for some holomorphic map $Q_0 : U_0 \to GL(C^{s+1})$.

**Proof.** Since $d + \Omega^h$ is flat, we have $\Omega^h = L^{-1}dL$ for some $L : V \to AGL(C^{s+1})$. (In the notation of §1, $L = H^t$.) Here, $AGL(C^{s+1})$ is the (smooth) loop group of $GL(C^{s+1})$, i.e. the space of all (smooth) maps $S^1 \to GL(C^{s+1})$, where $S^1 = \{h \in C \mid |h| = 1\}$.

Let $L = L_-L_+$ be the Birkhoff factorization of $L$, where $L_+$ extends holomorphically to the disc $0 \leq |h| < 1$ and $L_-$ to the disc $1 < |h| \leq \infty$, and where $L_-|_{h=\infty} = I$. This factorization exists if and only if $L$ takes values in the “big cell” of the loop group. For any given point $q_0$ of $V$, we may choose $\gamma \in AGL(C^{s+1})$ so that $\gamma L(q_0)$ belongs to this big cell. Replacing $L$ by $\gamma L$, we obtain a factorization at $q_0$, and hence on a neighbourhood $U_0$ of this point. We may write

$$L_-(q, h) = I + h^{-1}A_1(q) + h^{-2}A_2(q) + \cdots$$

$$L_+(q, h) = Q_0(q)(I + hQ_1(q) + h^2Q_2(q) + \cdots)$$

for some $A_i, Q_j : U_0 \to GL(C^{s+1})$.

Now we employ a well known argument from the theory of integrable systems. The gauge transformation $L \mapsto \hat{L} = L(L_+)^{-1} = L_-$ transforms $\Omega^h = L^{-1}dL$ into $\hat{\Omega}^h = \hat{L}^{-1}d\hat{L} = L_-^{-1}dL_-$, and the Laurent expansion of the latter manifestly contains only negative powers of $h$. But we have the alternative expression

$$L_-^{-1}dL_- = (LL_+^{-1})^{-1}d(LL_+^{-1}) = L_+L^{-1}dLL_+^{-1} + L_+d(L_+^{-1})$$

$$= L_+\left(\frac{1}{h}\omega + \theta^{(0)} + h\theta^{(1)} + \cdots + h^p\theta^{(p)}\right)L_+^{-1} + L_+(dL_+^{-1}),$$

whose only negative power of $h$ occurs in the term $\frac{1}{h}Q_0\omega Q_0^{-1}$. It follows that $\hat{\Omega}^h = \frac{1}{h}Q_0\omega Q_0^{-1}$, as required. □
Another way to express this modification is to say that we replace the original basis $[P_0], \ldots, [P_s]$ of $M^h$ by a new basis $[\hat{P}_0], \ldots, [\hat{P}_s]$, where $\hat{P}_i = \sum_{j=0}^{s}(L_+)^{-1}_{ij}P_j$. Then $\hat{\Omega}_i^h$ is the matrix of the action of $\partial_i$ with respect to the basis $[\hat{P}_0], \ldots, [\hat{P}_s]$. At the same time, we replace the original basis $[c_0], \ldots, [c_s]$ of $\mathcal{A}$ by the new basis $[\hat{c}_0], \ldots, [\hat{c}_s]$, where $\hat{c}_i = \sum_{j=0}^{s}(Q_0^{-1})_{ji}c_j$; $\hat{\omega}_i$ is the matrix of multiplication by $[b_i]$ with respect to this new basis. In this description, the entries of $(L_+)^{-1}$ are assumed to lie in $K[h]$.

The modified connection $\hat{\nabla} = d + \hat{\Omega}^h$ will be the basic ingredient in our construction of a “quantum cohomology algebra” $\hat{\mathcal{A}}$ and a “quantum product operation”. The construction will be given here in a special case, the general case being postponed to §5. Namely, we assume that

\[ \hat{c}_0 = c_0 = 1 \text{ and } \hat{c}_i = c_i = b_i \text{ for } 1 \leq i \leq r, \]
and that $L_+|_{q=0} = I$ ($L_+$ is then determined uniquely). In this situation we simply define $\hat{\mathcal{A}} = \mathcal{A}$. The “quantum product operation” will be defined on $\mathcal{A}_0$, and for this it is convenient to introduce the following terminology.

**Notation.** For a polynomial $c$ in $b_1, \ldots, b_r, q_1, \ldots, q_r$, we denote the corresponding element of $\mathcal{A}$ — the equivalence class of $c \mod \mathcal{R}_1, \ldots, \mathcal{R}_u$ — by $[c]$. If $c$ is a polynomial in $b_1, \ldots, b_r$, we denote the corresponding element of $\mathcal{A}_0$ by $[[c]]$.

We define

\[ \delta : \mathcal{A} \to \mathcal{A}_0, \quad [\hat{c}_i] \mapsto [[\hat{c}_i]|_{q=0}] \quad (0 \leq i \leq r). \]

This is obviously an isomorphism of vector spaces if $q_1, \ldots, q_r$ are considered as functions (and if $q_1, \ldots, q_r$ are considered as formal variables, $\delta$ defines an isomorphism of $K$-modules $\mathcal{A} \to \mathcal{A}_0 \otimes K$). We introduce a “quantum product operation” $\circ_t$ on $\mathcal{A}_0$ as follows:

\[ x \circ_t y = \delta(\delta^{-1}(x)\delta^{-1}(y)). \]

(For a discussion of the relation between $\delta$ and $\circ_t$, see §1 of [Am-Gu].) It follows that the matrix of the operator $b_t \circ_t$ on $\mathcal{A}$, with respect to the basis $[\hat{c}_0], \ldots, [\hat{c}_s]$, is $\hat{\omega}_t$, and hence that the “Dubrovin connection” associated to $\circ_t$ is $d + \frac{1}{t}\hat{\omega}$. This is flat (since the gauge equivalent connection $d + \Omega^h$ is flat, by §1), and so it satisfies $d\hat{\omega} = \hat{\omega} \wedge \hat{\omega} = 0$.

We postpone to later sections a discussion of when our abstract quantum product coincides with the usual quantum product. For the moment we wish to emphasize that we have constructed a product with the expected properties, and that our construction involves a priori the following steps: (1) an algebraic (Gröbner basis) calculation to find $\Omega^h$; (2) solution of a system of ordinary differential equations to find $L$; (3) the factorization $L = L_- L_+$. Although steps (2) and (3) seem formidable in general, we shall see that they can sometimes be reduced to a straightforward algorithm.

We conclude this section by giving some general properties of $\Omega^h$. Let $M_i^h$ be the subspace of $M^h$ which is spanned (over $K[h]$) by the basis vectors $P_j$ of degree $i$ in
h∂₁, . . . , h∂ₙ. Then we have a decomposition $M^h = M^h_0 \oplus M^h_1 \oplus \cdots \oplus M^h_v$, with respect to which the $(\alpha, \beta)$-th block of the matrix $\Omega^h_\alpha$ will be denoted $(\Omega^h_\alpha)_{\alpha, \beta}$. We shall generally use Greek indices, separated by commas, in reference to block matrices.

**Proposition 3.3.** (1) For $\alpha \geq \beta + 2$ we have $(\Omega^h)_{\alpha, \beta} = 0$.

Assume that the generators $D^h_i$ are homogeneous in $h, q_1, \ldots, q_r, \partial_1, \ldots, \partial_r$, where: $h$ is assigned degree 2, $q_1, \ldots, q_r$ have their usual degrees, and $\partial_1, \ldots, \partial_r$ are assigned degree 0. Then:

(2a) Each nonzero entry of the block $(\Omega^h_\alpha)_{\alpha, \beta}$ has degree $2(\beta - \alpha)$.

Assume further that $L_+|_{q=0} = I$. Then:

(2b) Each nonzero entry of the block $(L_+^h)_{\alpha, \beta}$ has degree $2(\beta - \alpha)$. In particular each nonzero entry of $(Q^h_i)_{\alpha, \beta}$ has degree $2(\beta - \alpha - i)$.

**Proof.** (1) It follows from the division algorithm that the filtration of $M^h$ defined by $M^h_{(j)} = \bigoplus_{k=0}^j M^h_k$ satisfies $h\partial_i M^h_{(j)} \subseteq M^h_{(j+1)}$. (2a) This is immediate from the definition of $\Omega^h$ and the homogeneity of the $D^h_i$. (2b) The homogeneity property of $\Omega^h$ can be expressed as

$$\Omega^h(q_1, \ldots, q_r) = \text{diag}(\lambda^{2v}, \lambda^{2v-2}, \ldots, 1)^{-1} \Omega^h \lambda^2 (\lambda^{q_1}|q_1, \ldots, \lambda^{q_r}|q_r) \text{diag}(\lambda^{2v}, \lambda^{2v-2}, \ldots, 1)$$

where $\text{diag}(\lambda^{2v}, \lambda^{2v-2}, \ldots, 1)$ denotes a matrix in block diagonal form. We must show that the function $L_+$ satisfies the same condition. By the proof of Proposition 3.2, $L_+$ is determined uniquely by the differential equation

$$\frac{1}{h} Q_0 \omega Q_0^{-1} L_+ = L_+ \Omega^h - dL_+$$

and the condition $L_+|_{q=0} = I$. Therefore, it suffices to observe that

$$\text{diag}(\lambda^{2v}, \lambda^{2v-2}, \ldots, 1)^{-1} L_+ (\lambda^{q_1}|q_1, \ldots, \lambda^{q_r}|q_r, \lambda^2 h) \text{diag}(\lambda^{2v}, \lambda^{2v-2}, \ldots, 1)$$

satisfies the same conditions. □

§4 Fano manifolds

It is well known that a Fano manifold, by which we mean a Kähler manifold $M$ whose Kähler 2-form represents the first Chern class $c_1 M$ of the manifold, has particularly well behaved quantum cohomology. It is natural to begin by applying the theory of §3 in this case.
We start with a deformation \( A = K[b_1, \ldots, b_r]/(R_1, \ldots, R_u) \) of the cohomology algebra \( A_0 = H^* M = C[b_1, \ldots, b_r]/(R_1, \ldots, R_u) \). (A priori, \( A \) may or may not be isomorphic to the quantum cohomology algebra.) For \( G/B \) and toric manifolds, suitable algebras \( A \), and, most importantly, their quantizations \( M^h \), are already available “off the shelf”. Before looking at these in more detail, we shall point out some further properties of the connection form \( \Omega^h \) in the Fano case. A basic ingredient is the fact that, from the naive construction of \( A \) using rational curves, the degree of \( q_i \) satisfies \( |q_i| \geq 2 \). In the case of flag manifolds and Fano toric manifolds, this property leads to operators \( D_i^h \) of the form \( h^{[I]} \partial_I + \) lower order terms, where \( |I| \geq 2 \) and the lower order terms have coefficients in the polynomial algebra \( K[h] = C[q_1, \ldots, q_r, h] \); we shall say that such \( D_i^h \) are “regular”. It follows from this and the homogeneity property that the elements of the Gröbner basis are also regular, and hence that \( M^h \) is free over \( K[h] \). The matrices \( h\Omega^h \) will then have entries in \( K[h] \).

**Proposition 4.1.** Assume that \( \Omega^h_1, \ldots, \Omega^h_r \) are polynomial in \( q_1, \ldots, q_r \) with \( |q_1|, \ldots, |q_r| \geq 4 \). Then \( L_+ = Q_0(I + hQ_1 + h^2Q_2 + \ldots) \) satisfies:

1. \( Q_0 = \exp X \) where \( X_{\alpha,\beta} = 0 \) for \( \alpha \geq \beta - 1 \),
2. for \( i \geq 1 \), \( (Q_i)_{\alpha,\beta} = 0 \) for \( \alpha \geq \beta - i - 1 \).

In particular, \( Q_i = 0 \) for \( i \) sufficiently large, i.e. \( L_+ \) must be a polynomial in \( h \).

**Proof.** Since \( h\Omega^h = \omega + h\theta^{(0)} + h^2\theta^{(1)} + \cdots + h^{p+1}\theta^{(p)} \), it follows from the homogeneity and polynomiality properties that \( \theta^{(j)}_i \) satisfies \( (\theta^{(j)}_i)_{\alpha,\beta} = 0 \) for \( \alpha \geq \beta - j - 1 \). Hence \( \Omega^h \) takes values in the Lie algebra consisting of loops of the form \( \sum_{i \in \mathbb{Z}} h^i A_i \) whose coefficients satisfy the following conditions: \( (A_i)_{\alpha,\beta} = 0 \) for \( \alpha \geq \beta - i - 1 \) when \( i \geq 0 \), and \( (A_i)_{\alpha,\beta} = 0 \) for \( \alpha \geq \beta - i + 1 \) when \( i < 0 \). Hence \( L \) and \( L_- \) take values in the corresponding loop group. In particular \( (L_+)^{-1} dL_+ = \sum_{i \geq 0} h^i A_i \) where \( (A_i)_{\alpha,\beta} = 0 \) for \( \alpha \geq \beta - i - 1 \), from which the stated properties of \( L_+ \) follow. □

**Corollary 4.2.** With the assumptions of Proposition 4.1, we may assume that \( \hat{c}_0 = c_0 = 1 \) and \( \hat{c}_i = c_i = b_i \) for \( i = 1, \ldots, r \).

**Proof.** We can assume that \( P_0 = 1 \) and \( P_i = h\partial_i \) for \( 1 \leq i \leq r \) (as a nontrivial relation between \( h\partial_1, \ldots, h\partial_r \) would lead to a nontrivial relation between \( b_1, \ldots, b_r \)). Hence we may take \( c_0 = 1 \) and \( c_i = b_i \) for \( 1 \leq i \leq r \). Next, by (1) of Proposition 4.1, we have

\[
Q_0 = \begin{pmatrix}
1 & 0 & * \\
0 & I & * \\
[0] & [0] & [*]
\end{pmatrix}
\]

where \([*]\) denotes a submatrix and \([0]\) denotes a zero submatrix (where a submatrix may consist of several blocks). Thus, \( \hat{c}_i = c_i \) for \( i = 0, \ldots, r \). □
Let us now look at the two main families of examples in more detail (we postpone comments on the case where \( \deg q_i = 2 \) to the end of this section).

1. Full flag manifolds \( G/B \)

For the algebra \( \mathcal{A} \) we take the deformation of the ordinary cohomology algebra whose relations are the conserved quantities of the open one-dimensional Toda lattice. It may seem that we are “starting with the answer”, since this algebra has already been identified with the quantum cohomology of \( G/B \) in [Ki], but our point of view here is that this algebra exists naturally without reference to quantum cohomology. We have \( |q_i| = 4 \) for all \( i \).

To construct the D-module \( M^h \) we use the conserved quantities of the open one-dimensional quantum Toda lattice — see [Ki] and [Mr] for the precise definition. These are commuting differential operators which also have been studied independently of quantum cohomology theory. In particular, it follows from [Go-Wa] and the remarks at the beginning of this section that \( M^h \) is free over \( K[h] \) with rank equal to \( \dim H^*G/B \). This is a quantization of \( M^h \), and so our method produces a “quantum cohomology algebra” and a “quantum product operation”. Summarizing:

Theorem 4.3. The D-module \( M^h \) associated to the open one-dimensional quantum Toda lattice is a quantization (in the sense of Definition 3.1) of the algebra \( \mathcal{A} \) associated to the open one-dimensional Toda lattice. Hence we obtain a “quantum product” on \( H^*G/B \) which may be computed explicitly by the method explained above. \( \square \)

Using the fact ([Ki]) that \( M^h \) is known to be a quantization of the usual quantum cohomology algebra of \( G/B \), it can be shown (see [Am-Gu]) that \( L_+ \) can be chosen to satisfy \( L_+|_{q=0} = I \), and furthermore that our quantum product agrees with the usual quantum product. Computations for \( G = GL_n \mathbb{C} \) (\( n = 2, 3, 4 \)) are also given in [Am-Gu].

If the Schubert polynomial basis of \( H^*G/B \) is used instead of the monomial basis, then this procedure gives the so called quantum Schubert polynomials. Thus, \( Q_0 \) is essentially
the “quantization map” of [FGP] and [Ki-Ma] (for the case $G = GL_n \mathbb{C}$). This theory has been well studied, but our approach makes clear why such a rich structure can be expected, and in particular why the quantum products can be computed from surprisingly minimal assumptions about quantum cohomology.

Finally, we should point out that the role of D-modules in the approach of [Gi-Ki], [Ki] to the computation of the quantum cohomology algebra of $G/B$ (see also [Mr]) is quite different. The main step there is to show that the conserved quantities $D_i^h$ of the quantum Toda lattice imply relations $\lim_{h \to 0} S(D_i^h)$ of the quantum cohomology algebra (in the notation of Definition 3.1). This uses the special fact that the differential operators $D_i^h$ commute.

2. Fano toric manifolds with $|q_1|, \ldots, |q_r| \geq 4$

For the algebra $\mathcal{A}$ we take the “provisional” quantum cohomology algebra of Batyrev ([Ba1]). This exists for Fano and non-Fano toric manifolds alike. To construct a quantization we shall use the theory for generalized hypergeometric partial differential equations of Gelfand, Kapranov and Zelevinsky ([GKZ], [HLY], [SST], [Co-Ka]). This theory associates to a certain polytope a system of partial differential equations or D-module, which we refer to as a GKZ D-module. Now, by a well known construction (see [Od]), such a polytope gives rise to a toric variety $M$ with a line bundle. We shall use this to prove:

**Theorem 4.4.** Let $M$ be a Fano toric manifold. Then there is a GKZ D-module which is a quantization $M^h$ (in the sense of Definition 3.1) of Batyrev’s algebra $\mathcal{A}$. Hence we obtain a “quantum product” on $H^* M$ which may be computed explicitly by the method explained above.

**Proof.** We need a GKZ D-module $M^{GKZ}$ whose rank is equal to the dimension of the vector space $H^* M$. The construction of suitable differential operators (defining $M^h$) may then be carried out exactly as in Section 5.5 of [Co-Ka], and it is easy to see that these satisfy the conditions of Definition 3.1.

To obtain $M^{GKZ}$ we need a suitable polytope. It is known (see Lemma 2.20 of [Od] and Section 2 of [Ba2]) that, for a Fano toric manifold, there exists a reflexive polytope which gives rise to $M$ and has the following property: in the decomposition of the polytope given by taking the cones on the maximal faces with common vertex at the origin, each such cone has unit volume. Therefore, the volume of the polytope is the number of maximal faces, which (because the polytope is reflexive) is equal to the number of maximal cones in a fan defining the toric variety, and this in turn (by standard theory of toric varieties) is equal to the number of fixed points of the action of the torus on $M$. This number is equal to the Euler characteristic of $M$, and hence to $\dim H^* M$. On the other hand, it was proved in [GKZ], [SST] that the GKZ system in this situation is free, with rank equal to the volume of the polytope. We conclude that the rank of $M^{GKZ}$ (and hence of $M^h$) is equal to $\dim H^* M$. □
To compute our quantum product explicitly, the method of [Am-Gu] may be used, exactly as in the case $M = G/B$. To establish agreement with the usual quantum product, the method of [Am-Gu] applies if one uses the fact that the that the GKZ D-module quantizes the usual quantum cohomology algebra. This fact is known from very general arguments (essentially, the mirror theorem of Givental, as explained in Example 11.2.5.2 of [Co-Ka]). The simpler method used in [Ki] in the case $M = G/B$ cannot be used in the Fano toric case, because the GKZ differential operators do not in general commute.

We have assumed so far that $|q_i| \geq 4$ for all $i$. If some $|q_i| = 2$, the method of this section still applies, but in Proposition 4.1 we have

1. $Q_0 = \exp X$ where $X_{\alpha,\beta} = 0$ for $\alpha \geq \beta$,
2. for $i \geq 1$, $(Q_i)_{\alpha,\beta} = 0$ for $\alpha \geq \beta - i$.

In Corollary 4.2 we have $\hat{c}_0 = c_0 = 1$ and $c_i = b_i$ for $i = 1, \ldots, r$, but $\hat{c}_i$ will in general be of the form $b_i + \sum a_j q_j$ (summing over $j$ such that $|q_j| = 2$). A similar phenomenon occurs for non-Fano manifolds, which are the subject of the next section.

§5 Beyond Fano manifolds

Even for non-Fano manifolds, an algebra $\mathcal{A}$ and a quantization $M^h$ (with suitable coefficient algebra $K$) lead to a gauge transformation $L_+ = Q_0 + O(h)$ and a connection $d + \hat{\omega}$ with $d\hat{\omega} = \hat{\omega} \wedge \hat{\omega} = 0$. However, we do not necessarily have $\hat{c}_i = c_i = b_i$ for $i = 1, \ldots, r$, so we are not simply making a change of basis in the algebra $\mathcal{A}$. We shall see that an important new feature in the non-Fano case is the appearance of a coordinate transformation (“mirror transformation”).

Referring to the proof of Proposition 3.2, let us define

\[ \tilde{L}_-(q, h) = Q_0(q)(I + h^{-1}A_1(q) + h^{-2}A_2(q) + \ldots) \]
\[ \tilde{L}_+(q, h) = I + hQ_1(q) + h^2Q_2(q) + \ldots \]

i.e. we modify $L_-, L_+$ by moving the $Q_0$ factor from $L_+$ to $L_-$. In this case the proof shows that $\tilde{L}^{-1}_-d\tilde{L}_-$ is linear in $1/h$. Since the constant term of $\tilde{L}_+$ is the identity matrix, the gauge transformation by $\tilde{L}_+$ simply changes the basis of $\mathcal{A}$ as in §4. In principle, therefore, it suffices to study the case

\[ \hat{\Omega}^h = \frac{1}{h} \hat{\omega} + \theta. \]

Our first observation concerning this case is that the (usually complicated) computation of $L_+$ becomes easy. Namely, we have $L_+ = Q_0$ where $Q_0$ is a solution of $Q_0^{-1}dQ_0 = \theta$. Then $\hat{\Omega}^h = \frac{1}{h} \hat{\omega}$ where $\hat{\omega} = Q_0\omega Q_0^{-1}$. The next step depends on the form of $Q_0$. For example:
Proposition 5.1. Assume that $|q_1|, \ldots, |q_r| \geq 0$ and that $M^h = D^h/(D_1^h, \ldots, D_u^h)$ is a quantization of $\mathcal{A}$, where each $D_i^h$ is homogeneous in the sense of Proposition 3.3. Assume further that $\Omega^h = \frac{1}{h^2} \omega + \theta$ and that the zero-th order term of any second order element of a Gröbner basis of $(D_1^h, \ldots, D_u^h)$ is independent of $h$. Then the block structure of $Q_0$ has the form

$$Q_0 = \begin{pmatrix}
1 & 0 & [\ast] \\
0 & T & [\ast] \\
[0] & [0] & [\ast]
\end{pmatrix}$$

where $[\ast]$ denotes a submatrix and $[0]$ denotes a zero submatrix (where a submatrix may consist of several blocks). That is, $(Q_0)_{\alpha, \beta} = 0$ if $\alpha = 0$ or $1$, unless $(\alpha, \beta) = (0, 0)$ or $(\alpha, \beta) = (1, 1)$.

Proof. Since $Q_0^{-1} dQ_0 = \theta$, it suffices to prove that $\theta$ has the block form

$$
\begin{pmatrix}
0 & 0 & [\ast] \\
0 & * & [\ast] \\
[0] & [0] & [\ast]
\end{pmatrix},
$$

since the matrices of this type form a Lie algebra. From the definition of $\Omega^h$, the nonzero entries of $\theta$ arise from expressions of the form $h\partial_i P_j$ which contain terms with “excess $h$”, i.e. terms which still contain $h$ after replacing $h\partial_1, \ldots, h\partial_r$ by $b_1, \ldots, b_r$. Since $P_0 = 1$ we have $h\partial_i P_0 = h\partial_i$, and by the definition of quantization (cf. the proof of Corollary 4.2) the reduction of $h\partial_i$ modulo these generators is $h\partial_i$ itself. There are no excess $h$ here, so the first column of $\theta$ is zero. Regarding the second column of $\theta$, the third sub-matrix is zero by (2a) of Proposition 3.3: each block is homogeneous of negative degree and well defined at $q = 0$, hence polynomial in $q_1, \ldots, q_r$; but this contradicts the assumption $|q_1|, \ldots, |q_r| \geq 0$, unless that block is zero. By assumption, there are no excess $h$ in the zero-th order term of $h^2 \partial_i \partial_j$, so the first block is also zero, as required. □

Our second observation is that, while $T$ is not necessarily the identity matrix, it does have a special form:

Proposition 5.2. Assume that $M^h = D^h/(D_1^h, \ldots, D_u^h)$ is as in Proposition 5.1. Then the matrix $T$ is a Jacobian matrix, i.e. there exist new local coordinates $\hat{t}_1, \ldots, \hat{t}_r$ on the vector space $\mathbb{C}^r$ such that

$$T = \begin{pmatrix}
\partial_{\hat{t}_1} & \cdots & \partial_{\hat{t}_1} \\
\vdots & & \vdots \\
\partial_{\hat{t}_r} & \cdots & \partial_{\hat{t}_r}
\end{pmatrix}.$$

Proof. By definition, $\partial_i P_j = \sum_{k=0}^h (\Omega^h_i)_{kj} P_k$ mod $(D_1^h, \ldots, D_u^h)$. We have $h\partial_i P_j = h^2 \partial_i \partial_j$ for $1 \leq i, j \leq r$. Since $\partial_i \partial_j = \partial_j \partial_i$, it follows that $(\Omega^h_i)_{kj} = (\Omega^h_j)_{ki}$ for $1 \leq i, j \leq r$. In
particular this symmetry is valid for θ, and for the (1, 1) block of θ = Q_0^{-1}dQ_0, namely for $T^{-1}dT$. It is easy to verify that this implies that the operators $\hat{\partial}_i = \sum_{j=1}^r (T^{-1})_{ij}\partial_j$ ($i = 1, \ldots, r$) commute, hence define new local coordinates. □

Under the coordinate transformation $t \mapsto \hat{t}$, the differential operators $D^h_i$ transform to differential operators $\hat{D}^h_i$. Let $\hat{D}^h$ be the ring of differential operators analogous to $D^h$, using $\hat{\partial}_i = \partial/\partial \hat{t}_i = \hat{q}_i\partial/\partial \hat{q}_i$ instead of $\partial_i$. Then we obtain a new D-module $\hat{M}^h = \hat{D}^h/(\hat{D}^h_1, \ldots, \hat{D}^h_r)$ and a de-quantized commutative algebra $\hat{A}$. With respect to the basis of standard monomials in $h\hat{\partial}_1, \ldots, h\hat{\partial}_r$ we obtain a connection $\hat{d} + \hat{\Omega}^h$, and by construction $\hat{\Omega}^h$ has the property $\hat{T} = I$. We can now apply the procedure of §3 to $\hat{M}^h$. A gauge transformation produces a connection $\hat{d} + \hat{\Omega}^h$ with $\hat{\Omega}^h = \frac{1}{h^2}\hat{\omega}$, so we can define a “quantum product operation” on $A_0$.

This is our general procedure for reconstructing quantum cohomology: first we make a change of variable to obtain a connection of the kind discussed in §3, then we make a gauge transformation to obtain a connection with the properties of the Dubrovin connection. The first operation is natural from the point of view of the D-module $M^h$, but it does not in general preserve the isomorphism type of the associated algebra $A$. The second one preserves this isomorphism type, and just introduces the additional information needed to define a quantum product. We summarize this in the following theorem.

**Theorem 5.3.** Assume that $M^h = D^h/(D^h_1, \ldots, D^h_u)$ is a quantization of $A$, that the conditions of Proposition 5.1 hold, and that $\Omega^h = \frac{1}{h^2}\omega + \theta$. Then by a change of variable and a gauge transformation we obtain a connection form $\hat{\Omega}^h = \frac{1}{h^2}\hat{\omega}$ satisfying $\hat{d}\hat{\omega} = \hat{\omega} \wedge \hat{\omega} = 0$, and a “quantum product operation” on $A_0 = H^*M$. □

**Example 5.4:** The Hirzebruch surfaces $\Sigma_k = \mathbb{P}(O(0) \oplus O(-k))$, where $O(i)$ denotes the holomorphic line bundle on $\mathbb{C}P^1$ with first Chern class $i$, are Fano when $k = 0, 1$. We shall consider the first non-Fano case, $\Sigma_2$. The ordinary cohomology algebra is

$$A_0 = H^*\Sigma_2 = C[b_1, b_2]/(b_1^2, b_2(b_2 - 2b_1))$$

(in the notation of [Gu2], $b_1 = x_1$ and $b_2 = x_4$.) This is a complex vector space of dimension 4. Batyrev’s algebra ([Ba1]), obtained by consideration of rational curves in $\Sigma_2$, is in this case

$$A = C[b_1, b_2, q_1, q_2]/(b_1^2 - q_1(b_2 - 2b_1)^2, b_2(b_2 - 2b_1) - q_2).$$

It is a $C[q_1, q_2]$-module of rank 4. We have $|q_1| = 0$ and $|q_2| = 4$ here.

Consider the D-module $M^h = D^h/(D^h_1, D^h_2)$ where

$$D^h_1 = h^2\partial_1^2 - q_1h^2(\partial_2 - 2\partial_1)(\partial_2 - \partial_1 - 1), \quad D^h_2 = h^2\partial_2(\partial_2 - 2\partial_1) - q_2.$$
This can be derived from a GKZ D-module, as in the Fano case (see [Co-Ka], Section 5.5). It is a D-module which is free over $K[h]$ of rank 4, where $K$ is the field of rational functions, and therefore a quantization of $A$. (It is interesting to note that the “naive quantization”, obtained by using $D_1^h = h^2 \partial_1^2 - q_1 h^2 (\partial_2 - 2 \partial_1)^2$ and $D_2^h = h^2 \partial_2 (\partial_2 - 2 \partial_1) - q_2$, has rank 0, and is therefore not a valid quantization of $A$.) The Gröbner basis for the ideal $(D_1^h, D_2^h)$, turns out to be

$$
\begin{align*}
2h^2 \partial_1 \partial_2 - h^2 \partial_2^2 + q_2, \\
(4q_1 - 1)h^2 \partial_1^2 - q_1 h^2 \partial_2^2 + 2q_1 h^2 \partial_1 - q_1 h^2 \partial_2 + 2q_1 q_2, \\
h^3 \partial_2^3 + 2q_2 (4q_1 - 1)h \partial_1 - q_2 (4q_1 + 1)h \partial_2 -hq_2
\end{align*}
$$

The equivalence classes of the standard monomials $1, h \partial_2, h \partial_1, h^2 \partial_2^2$ (i.e. the monomials $(h \partial_1)^i (h \partial_2)^j$ not “divisible” by any of the leading terms, which are underlined) form a basis of $M^h$. With respect to this basis, the matrices $\Omega_i^h$ (of the action of $\partial_i$) are:

$$
\Omega_1^h = \frac{1}{h} \begin{pmatrix} 0 & -\frac{q_2}{2} & \frac{-2q_1 q_2}{4q_1 - 1} & 0 \\ 0 & 0 & \frac{h}{4q_1 - 1} & 2q_1 q_2 \\ 1 & 0 & -\frac{2q_1}{4q_1 - 1} & -q_2 (4q_1 - 1) \\ 0 & \frac{1}{2} & \frac{q_1}{4q_1 - 1} & 0 \end{pmatrix}, \quad \Omega_2^h = \frac{1}{h} \begin{pmatrix} 0 & 0 & \frac{-q_2}{2} & h q_2 \\ 1 & 0 & 0 & q_2 (4q_1 + 1) \\ 0 & 0 & 0 & -2q_2 (4q_1 - 1) \\ 0 & 1 & \frac{1}{2} & 0 \end{pmatrix}
$$

In particular we see that $\Omega_i^h$ is of the form $\frac{1}{h} \omega + \theta$ here.

The gauge transformation $L_+ = Q_0$ such that $Q_0^{-1} dQ_0 = \theta$ and $Q_0|_{q=0} = I$ is easily found. Its inverse is

$$
Q_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & -q_2 \\ 0 & 1 & \frac{1}{2} (1 - \sqrt{1 - 4q_1}) & 0 \\ 0 & 0 & \sqrt{1 - 4q_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

The coordinate transformation is determined by the central $2 \times 2$ block of $Q_0^{-1}$, i.e.

$$
\hat{\partial}_2 = \partial_2, \quad \hat{\partial}_1 = \frac{1}{2} (1 - \sqrt{1 - 4q_1}) \partial_2 + \sqrt{1 - 4q_1} \partial_1.
$$

Writing $\hat{q}_i = e^{\hat{i}}$, it is easy to deduce that $q_1 = \hat{q}_1/(1 + \hat{q}_1)^2$, $q_2 = \hat{q}_2 (1 + \hat{q}_1)$, if we impose the condition that the origin maps to the origin. Let us see what effect this transformation has on the D-module $M^h$. From

$$
\partial_2 = \hat{\partial}_2, \quad \partial_1 = -\frac{\hat{q}_1}{1 - \hat{q}_1} \hat{\partial}_2 + \frac{1 + \hat{q}_1}{1 - \hat{q}_1} \hat{\partial}_1
$$

we obtain

$$
\begin{align*}
D_1^h &= h^2 \hat{\partial}_1^2 - \hat{q}_1 \hat{q}_2 - \frac{\hat{q}_1}{1 - \hat{q}_1} (h^2 \hat{\partial}_2 (\hat{\partial}_2 - 2 \hat{\partial}_1) - \hat{q}_2 (1 - \hat{q}_1)) \quad (= \hat{D}_1^h, \text{ by definition}) \\
D_2^h &= \frac{1 + \hat{q}_1}{1 - \hat{q}_1} (h^2 \hat{\partial}_2 (\hat{\partial}_2 - 2 \hat{\partial}_1) - \hat{q}_2 (1 - \hat{q}_1)) \quad (= \hat{D}_2^h, \text{ by definition}).
\end{align*}
$$
These operators define an equivalent D-module $\hat{M}^h$, but the de-quantized algebra $\hat{A}$ is quite different from $A$:

$$\hat{A} = C[\hat{b}_1, \hat{b}_2, \hat{q}_1, \hat{q}_2]/(\hat{b}_1^2 - \hat{q}_1 \hat{q}_2, \hat{b}_2 \hat{b}_2 - 2\hat{b}_1 - \hat{q}_2(1 - \hat{q}_1)).$$

To obtain a “quantum product operation” on $H^*\Sigma_2$ we carry out the procedure of §3, but this time starting from $\hat{A}$. With respect to the standard monomials $1, h\hat{b}_2, h\hat{b}_1, h^2\hat{b}_2^2$, the matrices $\hat{\Omega}_i^h$ (of the action of $\hat{\partial}_i$) can be computed as

$$\hat{\Omega}_1^h = \frac{1}{h} \begin{pmatrix} 0 & -\hat{q}_2(1-\hat{q}_1)/2 & \hat{q}_1 \hat{q}_2 & h\hat{q}_1 \hat{q}_2 \\ 0 & 0 & 0 & 2\hat{q}_1 \hat{q}_2 \\ 1 & 0 & 0 & \hat{q}_2(1 - \hat{q}_1) \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad \hat{\Omega}_2^h = \frac{1}{h} \begin{pmatrix} 0 & 0 & -\hat{q}_2(1+\hat{q}_1)/2 & h\hat{q}_2(1+\hat{q}_1) \\ 1 & 0 & 0 & \hat{q}_2(1 + 3\hat{q}_1) \\ 0 & 0 & 0 & 2\hat{q}_2(1 - \hat{q}_1) \\ 0 & 1 & \frac{1}{2} & 0 \end{pmatrix}.$$  

We have $\hat{\Omega}^h = \frac{1}{h}\hat{\omega} + \hat{\theta}$. The inverse of the matrix $\hat{Q}_0$ such that $\hat{Q}_0^{-1}d\hat{Q}_0 = \hat{\theta}$ and $\hat{Q}_0|q=0 = I$ is

$$\hat{Q}_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\hat{q}_2(1 + \hat{q}_1) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

This converts $\hat{\Omega}^h$ to $\hat{\Omega}^h = \frac{1}{h}\hat{\omega}$, where $\hat{\omega} = \hat{Q}_0 \hat{\omega} \hat{Q}_0^{-1}$, and we have:

$$\hat{\omega}_1 = \begin{pmatrix} 0 & \hat{q}_1 \hat{q}_2 & \hat{q}_1 \hat{q}_2 & 0 \\ 0 & 0 & 0 & 2\hat{q}_1 \hat{q}_2 \\ 1 & 0 & 0 & -2\hat{q}_1 \hat{q}_2 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad \hat{\omega}_2 = \begin{pmatrix} 0 & (1 + \hat{q}_1)\hat{q}_2 & \hat{q}_1 \hat{q}_2 & 0 \\ 1 & 0 & 0 & 2\hat{q}_1 \hat{q}_2 \\ 0 & 0 & 0 & -2\hat{q}_2(-1 + \hat{q}_1) \\ 0 & 1 & \frac{1}{2} & 0 \end{pmatrix}.$$  

We obtain the following basic products: $\hat{b}_1 \circ_i \hat{b}_1 = \hat{q}_1 \hat{q}_2$, $\hat{b}_1 \circ_i \hat{b}_2 = \hat{b}_1 \hat{b}_2 + \hat{q}_1 \hat{q}_2$, $\hat{b}_2 \circ_i \hat{b}_2 = \hat{b}_2^2 + \hat{q}_2(1 + \hat{q}_1)$. These are in agreement with the observation made at the end of Chapter 11 of [Co-Ka] that the quantum products of $\Sigma_2$ can be deduced from those of $\Sigma_0 = CP^1 \times CP^1$, if one uses the symplectic invariance of Gromov-Witten invariants. Thus our product is indeed the usual quantum product. □

The coordinate transformation (“mirror transformation”) in this example was obtained in Example 11.2.5.2 of [Co-Ka], as a consequence of Givental’s “Toric Mirror Theorem”. It appeared originally, in a similar situation, in the Introduction to [Gi4]. In fact, as we shall discuss elsewhere, this example is typical of the case of a semi-positive toric manifold $M$, i.e. a toric manifold such that the evaluation of $c_1 M$ on any homology class represented by a rational curve is non-negative.

There are two ways to apply our theory in this situation. The first point of view is to assume that the quantum cohomology of $M$ is given by Givental’s mirror theorem, then use this to deduce that our quantum cohomology agrees with the usual quantum
cohomology. Alternatively, our construction of quantum cohomology can be used to prove a version of the mirror theorem.

To explain the latter, we need the explicit solution $J_{GKZ}$ of the GKZ system constructed in [GKZ], [St], [SST] (the function $I_\nu$ in (11.73) of [Co-Ka] with $\nu = 0$ and $t_0 = 0$). Let

$$L_{GKZ} = \begin{pmatrix} P_0 J_{GKZ} & \cdots & P_s J_{GKZ} \end{pmatrix}.$$  

This is the (transpose of the) fundamental solution of the first order system $(d - (\Omega^h)^t) L^t = 0$. (Since $P_0 = 1$ we have $P_0 J_{GKZ} = J_{GKZ}$, of course.) For a semi-positive toric manifold we may apply the method of this section to $L = L_{GKZ}$. We obtain a new first order system $(\hat{d} - (\hat{\Omega}^h)^t) \hat{L}^t = 0$, with fundamental solution of the form

$$\hat{L} = \begin{pmatrix} \hat{J} & \cdots \end{pmatrix}.$$  

The relation between $\hat{\Omega}^h$ and $\Omega^h$ is $\hat{\Omega}^h = G^{-1}(X \Omega^h)G - dGG^{-1}$, where $G$ is a gauge transformation and $X$ is the matrix function expressing the relation between the standard monomial bases of $\hat{M}^h$ and $M^h$. From our earlier descriptions of $G$ and $X$, it is obvious that the first rows of $\hat{\Omega}^h, \Omega^h$ (i.e. the first columns of $(\hat{\Omega}^h)^t, (\Omega^h)^t$) are unaffected by $G$ or $X$. Hence

$$\hat{J}(\hat{t}) = J_{GKZ}(t).$$  

This is (our version of) Givental’s toric mirror theorem. It expresses a relation between the structure constants (“Gromov-Witten invariants”) for our quantum product operation and the coefficients of the generalized hypergeometric series $J_{GKZ}$. Of course this is merely a reflection of the more fundamental underlying relation between $\hat{M}^h$ and $M^h$.

References

[Am-Gu] A. Amarzaya and M.A. Guest, Gromov-Witten invariants of flag manifolds, via D-modules, math.DG/0306374.

[Ba1] V. Batyrev, Quantum cohomology rings of toric manifolds, Astérisque (Journées de géométrie algébrique D’Orsay) 218 (1993), 9–34.

[Ba2] V.V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3 (1994), 495–535.

[BCPP] G. Bini, C. de Concini, M. Polito and C. Procesi, On the work of Givental relative to mirror symmetry, Appunti dei Corsi Tenuti da Docenti della Scuola, Scuola Normale Superiore, Pisa, 1998 (math.AG/9805097).

[Co] S.C. Coutinho, A Primer of Algebraic D-Modules, LMS Student Texts 33, Cambridge Univ. Press, 1995.

[Co-Ka] D.A. Cox and S. Katz, Mirror Symmetry and Algebraic Geometry, Math. Surveys and Monographs 68, Amer. Math. Soc., 1999 (see http://www.cs.amherst.edu/~dac/ms.html for corrections).

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[Du] B. Dubrovin, *Geometry of 2D topological field theories*, Lecture Notes in Math. 1620 (1996), 120–348.

[FGP] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. 10 (1997), 565–596.

[GKZ] I.M. Gelfand, A.V. Zelevinskii and M.M. Kapranov, *Hypergeometric functions and toral manifolds*, Funct. Anal. Appl. 23 (1989), 94–106 (with correction in Funct. Anal. Appl., 27, 1993, 295).

[Gi1] A.B. Givental, *Homological geometry I. Projective hypersurfaces*, Selecta Math. 1 (1995), 325–345.

[Gi2] A.B. Givental, *Homological geometry and mirror symmetry*, Proc. Int. Congress of Math. I, Zürich 1994 (S.D. Chatterji, ed.), Birkhäuser, 1995, pp. 472–480.

[Gi3] A.B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices 13 (1996), 1–63.

[Gi4] A. Givental, *A mirror theorem for toric complete intersections*, Topological Field Theory, Primitive Forms and Related Topics (M. Kashiwara et al, eds.), Progr. Math. 160, Birkhäuser, 1998, pp. 141–175.

[Gi-Ki] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Commun. Math. Phys. 168 (1995), 609–641.

[Go-Wa] R. Goodman and N. Wallach, *Classical and quantum mechanical systems of Toda-lattice type. III. Joint eigenfunctions of the quantized systems*, Commun. Math. Phys. 105 (1986), 473–509.

[Gu1] M.A. Guest, *Harmonic Maps, Loop Groups, and Integrable Systems*, LMS Student Texts 38, Cambridge Univ. Press, 1997.

[Gu2] M.A. Guest, *Introduction to homological geometry I,II*, math.DG/0104274, math.DG/0105032.

[HLY] S. Hosono, B.H. Lian and S.-T. Yau, *GKZ-generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces*, Commun. Math. Phys. 182 (1996), 535–577.

[Ki] B. Kim, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Ann. of Math. 149 (1999), 129–148.

[Ki-Ma] A.N. Kirillov and T. Maeno, *Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa-Intriligator formula*, Discrete Math. 217 (2000), 191–223.

[LLY1] B.H. Lian, K. Liu and S.-T. Yau, *Mirror principle I*, Asian J. Math. 1 (1997), 729–763.

[LLY2] B.H. Lian, K. Liu and S.-T. Yau, *Mirror principle II*, Asian J. Math. 3 (1999), 109–146.

[LLY3] B.H. Lian, K. Liu and S.-T. Yau, *Mirror principle III*, Asian J. Math. 3 (1999), 771–800.

[Ma] Maple 7, Waterloo Maple (1990–2001).

[Mn] Y.I. Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, Amer. Math. Soc. Colloquium Publications 47, Amer. Math. Soc., 1999.

[Mr] A.-L. Mare, *On the theorem of Kim concerning QH∗(G/B)*, Integrable Systems, Topology, and Physics (M.A. Guest et al, eds.), Contemp. Math. 309, Amer. Math. Soc., 2002, pp. 151–163.

[Od] T. Oda, *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*, Springer, 1988.

[Pa] R. Pandharipande, *Rational curves on hypersurfaces (after A. Givental)*, Séminaire Bourbaki 848, Astérisque 252 (1998), 307–340.

[Ph] F. Pham, *Singularités des systèmes différentiels de Gauss-Manin*, Progr. Math. 2, Birkhäuser, 1979.

[SST] M. Saito, B. Sturmfels and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Algorithms and Computation in Math. 6, Springer, 2000.

[St] J. Stienstra, *Resonant hypergeometric systems and mirror symmetry*, Integrable Systems and Algebraic Geometry (M.H. Saito et al, eds.), World Scientific, 1998, pp. 412–452.

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