The Flavor Group $\Delta(3n^2)$

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Abstract

The large neutrino mixing angles have generated interest in finite subgroups of $SU(3)$, as clues towards understanding the flavor structure of the Standard Model. In this work, we study the mathematical structure of the simplest non-Abelian subgroup, $\Delta(3n^2)$. Using simple mathematical techniques, we derive its conjugacy classes, character table, build its irreducible representations, their Kronecker products, and its invariants.
1 Introduction

At a time when the gauge structure of the Standard Model is well understood, the origin of the triplication of chiral families remains a mystery. Several authors tried early on to explain chiral family copies as manifestations of continuous family symmetries: $SU(2)$, $SO(3)$, and $SU(3)$. Unfortunately, these theories which generically introduce more parameters than need to be explained, have met with at best partial success.

Recent experimental clues have been added in the stew with the observation of neutrino oscillations where two large mixing angles have been measured. This fact has led to a renewal of interest in the exploration of finite flavor groups, where matrices with large angles appear naturally, as possible solutions to the flavor problem.

Many authors, inspired by their favorite entries in the MNSP matrix, have suggested specific finite groups, as holding the key to the flavor problem. One of their guiding principles has been to focus on those finite groups that are subgroups of a continuous flavor $SU(3)$. These were listed long ago, and studied in some details fifty years later.

However, there is no systematic approach to the study of the finite groups that may explain flavor triplication. It is the purpose of this paper to provide a first step in that direction by presenting simple mathematical techniques for studying the finite $SU(3)$ subgroups.

Three trivial finite $SU(3)$ subgroups are generated by $(3 \times 3)$ diagonal matrices of unit determinant

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha^m
\end{pmatrix}, \quad \begin{pmatrix}
\beta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \beta
\end{pmatrix}, \quad \begin{pmatrix}
\gamma & 0 & 0 \\
0 & \gamma & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with $\alpha^m = \beta^n = \gamma^p = 1$, generating the discrete group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_p$. A less trivial finite group is one where the three entries of these diagonal matrices are permuted into one another. In that case, there are only two independent diagonal matrices

\[
\begin{pmatrix}
\beta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \beta
\end{pmatrix}, \quad \begin{pmatrix}
\gamma & 0 & 0 \\
0 & \gamma & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where now $\beta^n = \gamma^n = 1$, generating $\mathbb{Z}_n \times \mathbb{Z}_n$. When their entries are permuted by $S_3$, the group of six permutations on the three diagonal entries, the finite group that ensues is called $\Delta(6n^2)$. In the simpler case when the diagonal entries are permuted only by the three cyclic permutations, we obtain its finite subgroup $\Delta(3n^2)$.

In this paper, we use $\Delta(3n^2)$ to display general techniques for the study of finite groups. We derive its class structure, character table, and irreducible representations.
Using the Kronecker products of its irreducible representations, we construct the Clebsch-Gordan coefficients, and list its quadratic and cubic invariants. Although most of these results can be found in the literature [12], we hope that our exposition will make the study of these finite groups accessible to particle physicists.

2 The Structure of $\Delta(3n^2)$

The group $\Delta(3n^2)$ is a non-Abelian finite subgroup of $SU(3)$ of order $3n^2$. It is isomorphic to the semidirect product of the cyclic group $Z_3$ with $(Z_n \times Z_n)$ [12],

$$\Delta(3n^2) \sim (Z_n \times Z_n) \rtimes Z_3 .$$

Finite groups are most easily defined by their presentation in terms of their generators. $\Delta(3n^2)$ has three generators, $a$ which generates $Z_3$ with $a^3 = 1$, $c$ and $d$ which generate $(Z_n \times Z_n)$, with $c^n = d^n = 1$ and $cd = dc$. $Z_3$ acts on $(Z_n \times Z_n)$ by similarity transformation (conjugation).

$$aca^{-1} = c^{-1}d^{-1} , \quad ada^{-1} = c .$$

(2.1)

Any of the $3n^2$ group elements $g \in \Delta(3n^2)$ can be written as a product of powers of $a$, $c$, and $d$

$$g = a^\alpha c^\gamma d^\delta ,$$

(2.2)

with $\alpha = 0, 1, 2$ and $\gamma, \delta = 0, 1, ..., n - 1$.

2.1 Conjugacy Classes

The structure of finite groups derives from its conjugacy classes. These are the set of group elements obtained from one another by similarity transformations. Clearly, the identity element $e$ of any finite group is always in a class by itself [12]. $1C_1(e) = \{ e \}$. From the presentation, we find that

$$cac^{-1} = ac^{-1}d , \quad dad^{-1} = ac^{-1}d^{-2} ,$$

(2.3)

indicating that conjugacy does not alter the power of $a$. This enables us to study the class structure in terms of $a$. There are three types of classes; those that contain:

1In the case of supersymmetry, renormalizability allows only the quadratic and trilinear terms in the superpotential.

2Different actions of $Z_3$ on $(Z_n \times Z_n)$ yield semidirect products which are not $SU(3)$ subgroups.

3The prefix before each class indicates the number of elements in the class.
• Elements without \( a \)

For fixed \( \rho \) and \( \sigma \), where \( \rho, \sigma = 0, 1, \ldots, n - 1 \) and \( (\rho, \sigma) \neq (0, 0) \), the conjugacy class of the element \( c^\rho d^\sigma \) can be obtained by the action

\[
g(c^\rho d^\sigma)g^{-1},
\]
leading to a class with at most three elements

\[
\{c^\rho d^\sigma, c^{-\rho+\sigma}d^{-\rho}, c^{-\sigma}d^{\rho-\sigma}\},
\]
obtained by a straightforward application of the presentation Eq. (2.1). The second element is obtained by conjugation with \( a^{-1} \). It is easy to see that whenever the conditions

\[
3\rho = \rho + \sigma = 0 \mod(n),
\]
are satisfied, the three elements in this class are the same, but this can occur only if \( n \) is a multiple of three. Thus we consider two possibilities:

(i) \( n \neq 3 \mathbb{Z} \). The three elements in each class are all different. Although \( \rho \) and \( \sigma \) can take on \((n^2 - 1)\) possible values, the triples \((\rho, \sigma), (-\rho+\sigma, -\rho), (-\sigma, \rho-\sigma)\) lead to the same class, leaving us with \((n^2 - 1)/3\) classes of three elements

\[
3 C_1^{(\rho, \sigma)} = \{c^\rho d^\sigma, c^{-\rho+\sigma}d^{-\rho}, c^{-\sigma}d^{\rho-\sigma}\}.
\]

(ii) \( n = 3 \mathbb{Z} \). Eq. (2.7) has two solutions, giving two classes with one element each. The remainder yields \((n^2 - 3)/3\) three-element classes

\[
1 C_1^{(\rho)} = \{c^\rho d^{-\rho}\}, \quad \rho = \frac{n}{3}, \frac{2n}{3},
\]
\[
3 C_1^{(\rho, \sigma)} = \{c^\rho d^\sigma, c^{-\rho+\sigma}d^{-\rho}, c^{-\sigma}d^{\rho-\sigma}\}, \quad (\rho, \sigma) \neq \left(\frac{n}{3}, \frac{2n}{3}\right), \left(\frac{2n}{3}, \frac{n}{3}\right).
\]

• Elements with \( a \)

Consider the element \( ac^\rho d^\sigma \), with \( \rho, \sigma = 0, 1, \ldots, n - 1 \). We discuss its conjugation by \( g_0 = a^\alpha c^\gamma d^\delta \) separately for all three values of \( \alpha = 0, 1, 2 \). The case \( \alpha = 0 \) yields, using Eq. (2.3),

\[
g_0(ac^\rho d^\sigma)g_0^{-1} = ac^{\rho-\gamma-\delta}d^{\sigma+\gamma-2\delta} = ac^{\rho+\sigma-y_0-3x_0}d^{y_0},
\]
written in terms of the new parameters \( y_0 \equiv \sigma + \gamma - 2\delta \) and \( x_0 \equiv \delta \), with \( x_0, y_0 = 0, 1, ..., n - 1 \). For \( \alpha = 1, 2 \) we get recursively from Eq. (2.11)

\[
g_{\alpha}(ac^\rho d^\sigma)g_{\alpha}^{-1} = \frac{aac^{\rho + \sigma - y_{\alpha - 1} - 3x_{\alpha - 1}}d^{y_{\alpha - 1}}a^{-1}}{aac^{\rho + \sigma - y_{\alpha - 3x_{\alpha}}d^{y_{\alpha}}}} ,
\]

(2.12)

where \( y_{\alpha} \equiv -\rho - \sigma + y_{\alpha - 1} + 3x_{\alpha - 1} \) and \( x_{\alpha} \equiv x_{\alpha - 1} - y_{\alpha} \), with \( x_{\alpha}, y_{\alpha} = 0, 1, ..., n - 1 \). The set of elements obtained by conjugation of \( ac^\rho d^\sigma \) by \( g_{\alpha} \) is therefore independent of \( \alpha \). The corresponding conjugacy class is given by the elements

\[
aac^{\rho - y - 3x}d^{y},
\]

(2.13)

with \( x, y = 0, 1, ..., n - 1 \). This shows that, for fixed \( \rho, \sigma, \) and \( y \), the exponent of \( c \) can only change in steps of three mod \( (n) \), leading us to two different cases:

(i) \( n \neq 3Z \). All exponents of \( c \) in Eq. (2.13) can be obtained just by varying \( x \).
To see this, note that as \( x \) takes its \( n \) possible values, so does \( 3x \) mod \( (n) \); otherwise, there would have to be an \( x' \neq x \), so that \( 3(x' - x) = 0 \) mod \( (n) \), which is impossible for integer \( x, x' \). We obtain only one conjugacy class of \( n^2 \) elements

\[
n^2 C_2 = \{ac^x d^y | x, y = 0, 1, ..., n - 1\}.
\]

(2.14)

(ii) \( n = 3Z \). In this case, only one third of all exponents can be obtained by varying \( x \). Thus we are led to three different conjugacy classes parameterized by \( \tau = 0, 1, 2 \):

\[
\frac{n^2}{3} C_2^{(\tau)} = \{ac^{\tau - y - 3x}d^y | x = 0, 1, ..., \frac{n-3}{3}; y = 0, 1, ..., n - 1\} .
\]

(2.15)

• Elements with \( a^2 \)

This case is similar to the one above. The elements obtained from \( a^2 c^\rho d^\sigma \) by the action of \( g_{\alpha} = a^\alpha c^\gamma d^\delta \) are given by

\[
g_{\alpha}(a^2 c^\rho d^\sigma)g_{\alpha}^{-1} = \frac{a^2 a^\rho c^{-2\gamma + \delta}d^{\sigma - \gamma - \delta}}{a^2 a^\rho c^{\rho + \sigma - y_{\alpha - 3x_{\alpha}}d^{y_{\alpha}}}},
\]

(2.16)

with \( y_{0} \equiv \sigma - \gamma - \delta \) and \( x_0 \equiv \gamma \) for \( \alpha = 0 \). In the case where \( \alpha = 1, 2 \) we find similar to Eq. (2.12)

\[
g_{\alpha}(a^2 c^\rho d^\sigma)g_{\alpha}^{-1} = \frac{a^2 a^\rho c^{\rho + \sigma - y_{\alpha - 3x_{\alpha}}d^{y_{\alpha}}a^{-1}}}{a^2 a^\rho c^{\rho + \sigma - y_{\alpha - 3x_{\alpha}}d^{y_{\alpha}}}} ,
\]

(2.17)
with \( y_\alpha \equiv -\rho - \sigma + y_{\alpha - 1} + 3x_{\alpha - 1} \) and \( x_\alpha \equiv x_{\alpha - 1} - y_\alpha \). The elements of the conjugacy class are thus given by

\[
a^2 e^{\rho + \sigma - 3x} d^y,
\]

with \( x, y = 0, 1, ..., n - 1 \), which is to be compared with Eq. (2.13). The analysis proceeds as before:

(i) \( n \neq 3 \mathbb{Z} \). Use the same argument that led to Eq. (2.14), except for another factor of \( a \), and obtain

\[
n^2 C_3' = \{ a^2 c^x d^y \mid x, y = 0, 1, ..., n - 1 \}.
\]

(ii) \( n = 3 \mathbb{Z} \). Here, from Eq. (2.18) the classes with elements containing \( a^2 \) are given by \( (\tau = 0, 1, 2) \)

\[
n^2 (\tau) C_3^{(\tau)} = \{ a^2 c^{\tau - y - 3x} d^y \mid x = 0, 1, ..., \frac{n - 3}{3}; y = 0, 1, ..., n - 1 \}.
\]

This completes the derivation of the class structure of the group \( \Delta(3n^2) \). Our results can be summarized as:

(i) \( n \neq 3 \mathbb{Z} \). Four types of classes

\[
1 C_1, \quad 3 C_1^{(\rho, \sigma)}, \quad n^2 C_2, \quad n^2 C_3,
\]

adding up to \( 1 + \frac{n^2 - 1}{3} + 1 + 1 \) distinct classes.

(ii) \( n = 3 \mathbb{Z} \). Five types of classes

\[
1 C_1, \quad 1 C_1^{(\rho)}, \quad 3 C_1^{(\rho, \sigma)}, \quad \frac{n^2}{3} C_2^{(\tau)}, \quad \frac{n^2}{3} C_3^{(\tau)},
\]

resulting in \( 1 + 2 + \frac{n^2 - 3}{3} + 3 + 3 \) different classes.

3 Irreducible Representations

Knowing the class structure, we construct the (unitary) irreducible representations of \( \Delta(3n^2) \). Thereafter, we can determine the character table by taking the trace of the corresponding matrices.

- One-dimensional Representations

In the one-dimensional representations, all generators commute with one another. Hence \( a \) is solely constrained by \( a^3 = 1 \), giving

\[
a = 1, \omega, \omega^2, \quad \text{with} \quad \omega \equiv e^{\frac{2\pi i}{3}}.
\]
From Eq. (2.1), we immediately see that
\[ d = c, \quad c^3 = 1. \quad (3.2) \]
This is compatible with \( c^n = d^n = 1 \) only if \( c = 1 \) or \( n \) is divisible by three. Hence the two cases:

(i) \( n \neq 3 \mathbb{Z} \). \( c \) and \( d \) are necessarily one, and there are only three singlets
\[ 1_r: \quad a = \omega^r, \quad c = d = 1, \quad \text{with} \quad r = 0, 1, 2. \quad (3.3) \]

(ii) \( n = 3 \mathbb{Z} \). \( c = 1, \omega, \omega^2 \), leading to nine one-dimensional representations
\[ 1_{r,s}: \quad a = \omega^r, \quad c = d = \omega^s, \quad \text{with} \quad r, s = 0, 1, 2. \quad (3.4) \]

**Three-dimensional Representations**

Using the method of induced representations, detailed in Appendix B it is easy to show the existence of irreducible three-dimensional representations. Since the generators \( c \) and \( d \) commute, they can be represented by diagonal \((3 \times 3)\) matrices with entries which are powers of the \( n \)-th root of unity \( \eta \equiv e^{2\pi i n} \). We find for any two integers \( k, l = 0, 1, \ldots, (n - 1) \)

\[ a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k-l} \end{pmatrix}, \quad d = \begin{pmatrix} \eta^{-k-l} & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}. \quad (3.5) \]

However, we must be careful not to count equivalent representations more than once. The easiest way to avoid overcounting is to note that the representations obtained by conjugation with \( a \) and \( a^2 \) are equivalent.

\[ c' \equiv ac a^{-1} = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k-l} & 0 \\ 0 & 0 & \eta^l \end{pmatrix}, \quad d' \equiv ada^{-1} = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k-l} \end{pmatrix}. \quad (3.6) \]

Therefore the representation labeled by the pair \((k, l)\) is equivalent to that labeled \((-k - l, k)\). Conjugation by \( a^2 \) yields another equivalent labeling. The same representation is thus labeled in three different ways
\[ \begin{pmatrix} k \\ l \end{pmatrix}, \quad \begin{pmatrix} -k - l \\ k \end{pmatrix}, \quad \begin{pmatrix} l \\ -k - l \end{pmatrix}. \quad (3.7) \]

\[ ^4 \text{excluding the values of } k \text{ and } l \text{ that lead to reducible representations, which happens if } k = l \text{ and } 3k = 0 \text{ mod}(n). \]
The matrix

\[
M \equiv \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}, \quad M^3 = 1,
\]

allows us to write these three pairs in the form

\[
M^p \begin{pmatrix} k \\ l \end{pmatrix}, \quad \text{with } p = 0, 1, 2.
\]

(3.9)

We have encountered a similar situation in labeling the classes \(C_1^{(\rho, \sigma)}\) by the pairs \((\rho, \sigma)\) [see above Eq. (2.8)]. There, we generated the three pairs of the triple by \((\rho, \sigma)M^p\), with \(p = 0, 1, 2\). When counting the number of classes with three elements we introduced a factor of 1/3 which takes care of the labeling ambiguity of \(C_1^{(\rho, \sigma)}\). To avoid the labeling ambiguity of the three-dimensional irreducible representations, it is useful to select a standard representative of the three pairs in Eq. (3.7). One possible choice is provided in Appendix A. Here we simply assume the existence of a mapping \(\widetilde{\cdot}\) that gives the standard representative:

\[
\begin{pmatrix} k \\ l \end{pmatrix} \mapsto \text{either } \begin{pmatrix} k \\ l \end{pmatrix}, \text{ or } \begin{pmatrix} -k - l \\ k \end{pmatrix}, \text{ or } \begin{pmatrix} l - k - l \\ -k - l \end{pmatrix},
\]

(3.10)

depending on the values of \(k\) and \(l\). This leads us to the following inequivalent and irreducible three-dimensional representations:

(i) \(n \neq 3 \mathbb{Z}\).

\[
3_{\widehat{(k,l)}}, \quad \text{with } (k, l) \neq (0, 0).
\]

(3.11)

Due to the mapping of Eq. (3.10) there are \(\frac{n^2 - 1}{3}\) different three-dimensional representations of this type. Under the implicit assumption that \((k, l)\) take only their standard values (see e.g. Appendix A) we can omit the \(\widetilde{\cdot}\) and simply write \(3_{(k, l)}\). Then the generators \(a, c, d\) are given as in Eq. (3.5).

(ii) \(n = 3 \mathbb{Z}\).

\[
3_{(k, l)}, \quad \text{with } (k, l) \neq (0, 0), \left(\frac{n}{3}, \frac{n}{3}\right), \left(\frac{2n}{3}, \frac{2n}{3}\right).
\]

(3.12)

This gives \(\frac{n^2 - 3}{3}\) three-dimensional representations. For notational ease, we will also write \(3_{(k, l)}\) instead of \(3_{\widehat{(k,l)}}\) in this case. In both cases, the complex conjugate representation is obtained by reversing \(k\) and \(l\) mod \(n\).

There are no other irreducible representations. This can be easily seen from the formula which relates the order of the group to the sum of the squared dimensions of its irreducible representations.
where \( d_i \) denotes the dimension of the irreducible representation \( i \). Knowing all one- and three-dimensional representations we can check Eq. (3.13) to see if there are more irreducible representations. For \( n \neq 3 \mathbb{Z} \) we have identified 3 one-dimensional and \( \frac{n^2-1}{3} \) three-dimensional representations. Plugging this into the right-hand side of Eq. (3.13), we obtain \( 3 \cdot 1^2 + \frac{n^2-1}{3} \cdot 3^2 = 3n^2 \). Similarly we get for the case where \( n = 3 \mathbb{Z} \) that \( 9 \cdot 1^2 + \frac{n^2-3}{3} \cdot 3^2 = 3n^2 \).

We can now deduce the \( \Delta(3n^2) \) character table by taking traces over the relevant matrices, for both \( n \neq 3 \mathbb{Z} \) and \( n = 3 \mathbb{Z} \). The results are displayed in Table 1 using the classes and representations just derived, including the restrictions on the parameters \((\rho, \sigma)\) and \((k, l)\).

Table 1: The character tables of \( \Delta(3n^2) \) for (a) \( n \neq 3 \mathbb{Z} \) and (b) \( n = 3 \mathbb{Z} \). Here \( \omega \equiv e^{2\pi i/3} \) and \( \eta \equiv e^{2\pi i/n} \). The sum runs over \( p = 0, 1, 2 \).

4 Kronecker Products

The construction of a theory that is invariant under a group of transformation requires the knowledge of its polynomial invariants. Simplest are the quadratic invariants, which occur whenever the Kronecker product of two representations contains the singlet representation. Given two irreps \( r \) and \( s \), their Kronecker product

\[
\mathbf{r} \otimes \mathbf{s} = \sum_\mathbf{t} d(\mathbf{r}, \mathbf{s}, \mathbf{t}) \mathbf{t},
\]

(4.1)
can be expressed as a sum of irreducible representations. The integer numerical factors 
\(d(r, s, t)\) can be calculated from the character table by means of the formula

\[
d(r, s, t) = \frac{1}{N} \sum_i n_i \cdot \chi_i^{[r]} \chi_i^{[s]} \overline{\chi_i^{[t]}},
\]  

(4.2)

where \(N\) is the order of the group; \(i\) labels a class of \(n_i\) elements and character \(\chi_i\). \(\overline{\chi}_i\) denotes the complex conjugate character. The sum over classes is rather intricate and, although the cases \(n \neq 3\mathbb{Z}\) and \(n = 3\mathbb{Z}\) are to some extent similar, we discuss them separately.

(i) \(n \neq 3\mathbb{Z}\).

The character table for this case is given in Table Ia. There are three \((r = 0, 1, 2)\) one-dimensional and \(n^2-1/3\) three-dimensional irreducible representations. The sum over each class is complicated by the redundant \((k, l)\) labeling, and its detailed calculation is left to Appendix C. Here we simply write the results

\[
\begin{align*}
1_r \otimes 1_r' &= 1_{r+r'} \\
1_r \otimes 3_{(k,l)} &= 3_{(k,l)} \\
3_{(k,l)} \otimes 3_{(k',l')} &= \delta_{(k')} \left(\frac{k'}{l'}\right) \left(-\frac{k}{l}\right) \left(1_0 + 1_1 + 1_2\right) + \\
&+ 3_{\left(\frac{k' + k}{l' + l}\right)} + 3_{\left(\frac{k' - k - l}{l' + k}\right)} + 3_{\left(\frac{k' + l}{l' - k - l}\right)}.
\end{align*}
\]

(ii) \(n = 3\mathbb{Z}\).

The character table in this case is given in Table Ib. We have nine \((r, s = 0, 1, 2)\) one-dimensional and \(n^2-3/3\) three-dimensional representations. Again the subtleties of evaluating the sum are relegated to Appendix C. The result is

\[
\begin{align*}
1_{r,s} \otimes 1_{r',s'} &= 1_{r+r'+s+s'} \\
1_{r,s} \otimes 3_{(k,l)} &= 3_{\left(\frac{k + sn/3}{l + sn/3}\right)} \\
3_{(k,l)} \otimes 3_{(k',l')} &= \sum_{s=0}^{2} \delta_{(k')} \left(\frac{k'}{l'}\right) \left(-\frac{k + sn/3}{l + sn/3}\right) \left(1_{0,s} + 1_{1,s} + 1_{2,s}\right) + \\
&+ 3_{\left(\frac{k' + k}{l' + l}\right)} + 3_{\left(\frac{k' - k - l}{l' + k}\right)} + 3_{\left(\frac{k' + l}{l' - k - l}\right)}.
\end{align*}
\]
5 Building Invariants

In the previous section we have decomposed the product of two three-dimensional representations in terms of irreducible representations. In this section we use these results to explicitly build invariants out of fields that transform as triplets. The corresponding Clebsch-Gordan coefficients are derived in Appendix D. Throughout this section we use the following notation: \( \vec{\varphi} \) denotes a \( 3_{(k,l)} \) and \( \vec{\varphi}' \) is a \( 3_{(k',l')} \). In terms of the component fields we have

\[
3_{(k,l)} = \vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \quad 3_{(k',l')} = \vec{\varphi}' = \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \\ \varphi'_3 \end{pmatrix},
\]

and the action of the generators in the triplet representation on these fields is given by the matrices in Eq. (3.5). We discuss separately the construction of invariants, depending on whether \( n \) is a multiple of three or not.

(i) \( n \neq 3 \mathbb{Z} \).

In order to obtain a product of \( \vec{\varphi} \) and \( \vec{\varphi}' \) which transforms as one of the three one-dimensional irreducible representations in Eq. (C.11) we need to have

\[
\begin{pmatrix} k' \\ l' \end{pmatrix} = \begin{pmatrix} -k \\ -l \end{pmatrix} = M^p \begin{pmatrix} -k \\ -l \end{pmatrix},
\]

for some value of \( p = 0, 1, 2 \). From the character Table Ia, we see that \( \vec{\varphi}' \) transforms as the complex conjugate representation of \( \vec{\varphi} \). The ambiguity of the \( \sim \) mapping yields three combinations that transform as the singlet \( 1_0 \),

\[
p = 0 : \quad \varphi'_1 \varphi_1 + \varphi'_2 \varphi_2 + \varphi'_3 \varphi_3,
\]

\[
p = 1 : \quad \varphi'_1 \varphi_2 + \varphi'_2 \varphi_3 + \varphi'_3 \varphi_1,
\]

\[
p = 2 : \quad \varphi'_1 \varphi_3 + \varphi'_2 \varphi_1 + \varphi'_3 \varphi_2.
\]

It is easy to verify the invariance of these expressions by acting the generating matrices of Eq. (3.5) on the fields.

For completeness, and mindful of the ambiguity Eq. (5.2), we choose the case \( p = 0 \), and represent the three different one-dimensional representations built out of \( \vec{\varphi}' \) and \( \vec{\varphi} \) as \( (r = 0, 1, 2) \)

\[
1_r : \quad \vec{\varphi} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & \omega \end{pmatrix}^r \vec{\varphi}', \quad \text{with} \quad \begin{pmatrix} k' \\ l' \end{pmatrix} = \begin{pmatrix} -k \\ -l \end{pmatrix}.
\]

These are helpful in the construction of cubic invariants out of two triplets and one field that transforms as a one-dimensional representation.
Now we turn to the construction of cubic invariants out of three triplets. For this we need to build triplets out of two triplets. To be specific, let us focus on the second $3(k'',l'')$ on the right-hand side of Eq. (C.11). Due to the mapping $\tilde{~}$, we have

$$
\begin{pmatrix}
k'' \\
l''
\end{pmatrix} = M_p \begin{pmatrix}
k' - k - l \\
l' + k
\end{pmatrix},
$$

(5.5)

where $p = 0, 1, \text{or } 2$. We find three ways to build the $3(k'',l'')$ out of the two triplets $\vec{\varphi}'$ and $\vec{\varphi}''$

$$
3(k'',l'') : \begin{pmatrix}
\varphi'_1 \varphi_2 \\
\varphi'_2 \varphi_3 \\
\varphi'_3 \varphi_1
\end{pmatrix}_{p=0}, \begin{pmatrix}
\varphi'_2 \varphi_3 \\
\varphi'_3 \varphi_1 \\
\varphi'_1 \varphi_2
\end{pmatrix}_{p=1}, \begin{pmatrix}
\varphi'_3 \varphi_1 \\
\varphi'_1 \varphi_2 \\
\varphi'_2 \varphi_3
\end{pmatrix}_{p=2}.
$$

(5.6)

These three triplets are related by a cyclic permutation of the components expressible by the action of $a$.

The cubic invariant is obtained by multiplying the appropriate triplet in Eq. (5.6) with the complex conjugate of a third triplet $\vec{\varphi}''$ which belongs to $3(k'',l'')$, as in Eq. (5.5).

Denoting the complex conjugate field by $\overline{\varphi}_i$, the cubic invariants are given by

$$
1_0 : \begin{pmatrix}
\bar{\varphi}''_1 \\
\bar{\varphi}''_2 \\
\bar{\varphi}''_3
\end{pmatrix} a^p \begin{pmatrix}
\varphi'_1 \varphi_2 \\
\varphi'_2 \varphi_3 \\
\varphi'_3 \varphi_1
\end{pmatrix}.
$$

(5.7)

This result depends on the value of $p$ in Eq. (5.5) and thus on the choice of the mapping $\tilde{~}$. Analogous to Eq. (5.4) we obtain all three singlets

$$
1_r : \begin{pmatrix}
\bar{\varphi}''_1 \\
\bar{\varphi}''_2 \\
\bar{\varphi}''_3
\end{pmatrix} a^p \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^{-1} & 0 \\
0 & 0 & \omega
\end{pmatrix}^r \begin{pmatrix}
\varphi'_1 \varphi_2 \\
\varphi'_2 \varphi_3 \\
\varphi'_3 \varphi_1
\end{pmatrix},
$$

(5.8)

with

$$
\begin{pmatrix}
k'' \\
l''
\end{pmatrix} = M_p \begin{pmatrix}
k' - k - l \\
l' + k
\end{pmatrix}.
$$

(5.9)

So far we have only considered the second triplet in Eq. (C.11). Alternatively, we can obtain one-dimensional representations from the first and the third $3(k'',l'')$ in Eq. (C.11) as well. Generalizing Eq. (5.5) to include these possibilities leads to

5The first $3(k'',l'')$ is absent in the case where $(k',l') = (-k,-l)$. 

12
\[
\begin{pmatrix}
  k'' \\
  l''
\end{pmatrix} = M_p \left[ \begin{pmatrix}
  k' \\
  l'
\end{pmatrix} + M_q \begin{pmatrix}
  k \\
  l
\end{pmatrix} \right].
\] (5.10)

\(q = 0\) corresponds to the first triplet of Eq. (C.11), \(q = 1\) to the second [hence it gives back Eq. (5.3)], and \(q = 2\) to the third. As above the value of \(p\) depends on the choice of the standard representatives. With this notation, the cubic combinations that transform as the one-dimensional representations are given, in a way similar to Eq. (5.8), by

\[
1_r : \bar{\varphi}'' \cdot a^p \begin{pmatrix}
  \varphi_1' & 0 & 0 \\
  0 & \omega^{-1} \varphi_2' & 0 \\
  0 & 0 & \omega^r \varphi_3'
\end{pmatrix} a^q \varphi.
\] (5.11)

The pairs \((k, l)\) and \((k', l')\) can take any (standard) value. \((k'', l'')\), which denotes the representation of \(\bar{\varphi}''\), is defined in Eq. (5.10). \(q\) is 0, 1, and 2, while the value of \(p\) is either 0 or 1 or 2, depending on the choice of the standard representatives. As mentioned before, the first triplet in Eq. (C.11) is absent in the case where \((k', l') = (-k, -l)\). Then the choice \(q = 0\) is forbidden in Eq. (5.11).

\((ii)\) \(n = 3 \cdot \mathbb{Z}\).

The decomposition of the product of two triplets in terms of irreducible representations depends on whether or not the parameters \(k, l, k', l'\) are multiples of \(n/3\). If they are, one obtains either nine or zero one-dimensional representations; if at least one of the four parameters is not a multiple of \(n/3\), we instead obtain three or zero one-dimensional representations (see Appendix C). We discuss each case separately.

\((a)\) \(k, l, k', l'\) are all multiples of \(n/3\).

As discussed below Eq. (C.22) the pairs \((k, l)\) and \((k', l')\) can take only two different values. Choosing \((0, \frac{n}{3})\) and \((0, \frac{-n}{3})\) as the standard representatives, we represent the nine different one-dimensional representations built out of \(\bar{\varphi}\) and \(\varphi\) as \((r, s = 0, 1, 2)\)

\[
1_{r,s} : \bar{\varphi} \cdot a^{r+s} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \omega^{-1} & 0 \\
  0 & 0 & \omega
\end{pmatrix}^r \varphi, \quad \text{with} \quad \begin{pmatrix}
  -k' \\
  -l'
\end{pmatrix} = \begin{pmatrix}
  k \\
  l
\end{pmatrix} = \begin{pmatrix}
  0 \\
  \pm n/3
\end{pmatrix}. \] (5.12)

Except for the additional factor of \(a^{r+s}\), this expression is identical to Eq. (5.4). Under the action of \(a/c, d\) it gets multiplied by \(\omega^r / \omega^s\), respectively; that is, it transforms as a \(1_{r,s}\).

Let us now turn to the construction of the three-dimensional representations out of two triplets \(\bar{\varphi}'\) and \(\bar{\varphi}\). If \((k', l') = (k, l) = (0, \pm n/3)\), we obtain three \(3_{(k'', l'')}\) representations.
where can be written in a more compact form, with \( q = 0, 1, 2 \) labeling the three possibilities,

\[
3_{(k'', l'')} : \begin{pmatrix}
\varphi_1' & 0 & 0 \\
0 & \varphi_2' & 0 \\
0 & 0 & \varphi_3'
\end{pmatrix}
\]

(5.14)

Since these new constructed triplets transform as \((k'', l'') = (0, \pm \frac{n}{3})\) representations, they constitute complex conjugate representations of the initial triplets \( \vec{\varphi} \) and \( \vec{\varphi} \).

The cubic invariants can be built by multiplying the three \((q = 0, 1, 2)\) triplets in Eq. (5.14) with a third triplet \( \vec{\varphi}'' \) transforming as \((k''', l'''') = (0, \pm \frac{n}{3})\). Using Eq. (5.12), we obtain the cubic combinations that transform as the one-dimensional representations:

\[
1_{r,s} : \vec{\varphi}'' \cdot a^{\pm s} a^q \begin{pmatrix}
\varphi_1' & 0 & 0 \\
0 & \varphi_2' & 0 \\
0 & 0 & \varphi_3'
\end{pmatrix} \begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{pmatrix}
\]

(5.15)

with

\[
\begin{pmatrix}
k'''
l'''
\end{pmatrix} = \begin{pmatrix}
k' \\
l'
\end{pmatrix} = \begin{pmatrix}
k \\
l
\end{pmatrix} = \begin{pmatrix}
0 \\
\pm n/3
\end{pmatrix}.
\]

(5.16)

The parameter \( q \) indicates that there are three different possibilities to build the one-dimensional representation \( 1_{r,s} \) from the product of \( \vec{\varphi}'' \), \( \vec{\varphi}' \), and \( \vec{\varphi} \).

(b) \( k, l, k', l' \) are not all multiples of \( n/3 \).

In order to obtain the three possible one-dimensional representations by multiplying two triplets \( \vec{\varphi}' \) and \( \vec{\varphi} \), we need to have

\[
\begin{pmatrix}
k' \\
l'
\end{pmatrix} = \begin{pmatrix}
-k + \frac{sn}{3} \\
-l + \frac{sn}{3}
\end{pmatrix} = M^p \begin{pmatrix}
-k \\
-l
\end{pmatrix} + \frac{sn}{3} \begin{pmatrix}
1 \\
1
\end{pmatrix}.
\]

(5.17)

The value of \( s \) determines which set of singlets we are constructing; for example, with \( s = 0 \) we get the three singlets \( 1_{0,0} \), \( 1_{1,0} \), and \( 1_{2,0} \). With Eq. (5.17), we can calculate the one-dimensional representations similar to Eq. (5.4).

---

\[\text{Note that we have specified the standard representatives explicitly. Therefore a parameter like } p \text{ in Eq. (5.11) is not needed.}\]
\[ \mathbf{1}_{r,s} : \quad \bar{\varphi} \cdot a^p \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & \omega \end{pmatrix}^r \bar{\varphi}. \]  

(5.18)

In this expression, \( r \) can take any of its three possible values, while \( p \) is determined in Eq. (5.17) by the values of \((k, l)\) and \( s \) as well as the mapping \( \sim \).

The triplets obtained from the product of \( \bar{\varphi} \) and \( \varphi \) are constructed analogous to Eq. (5.6). Multiplication with the appropriate triplet \( \bar{\varphi}'' \) yields the cubic combinations that transform as the one-dimensional representations

\[ \mathbf{1}_{r,s} : \quad \bar{\varphi}'' \cdot a^p \begin{pmatrix} \varphi_1' & 0 & 0 \\ 0 & \omega^{-r} \varphi_2' & 0 \\ 0 & 0 & \omega^s \varphi_3' \end{pmatrix} a^q \bar{\varphi}, \]  

(5.19)

with

\[ \begin{pmatrix} k'' \\ l'' \end{pmatrix} = M^p \left[ - \begin{pmatrix} k' \\ l' \end{pmatrix} - M^q \begin{pmatrix} k \\ l \end{pmatrix} \right] + \frac{2}{3 \eta} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]  

(5.20)

The pairs \((k, l)\) and \((k', l')\) can take any (standard) value. \( s \) and \( r \) determine what type of singlet we want to build. \( q \) can be 0, 1, and 2, while \( p \) is restricted to only one value [depending on the choice of the standard representative for \((k'', l'')\)]. In the case where the product of \( \bar{\varphi} \) and \( \varphi \) already includes singlets, i.e. where \( \begin{pmatrix} k' \\ l' \end{pmatrix} = M^p \begin{pmatrix} -k + s' \eta / 3 \\ -l + s' \eta / 3 \end{pmatrix} \) for some pair \( p', s' = 0, 1, 2 \), we must omit the choice \( q = p' \) in Eq. (5.19).

In summary, the products of two \( \Delta(3n^2) \) triplet fields yielding one-dimensional representations are given in Eqs. (5.4), (5.12), and (5.18). The products of three triplets which - as a product - transform as one-dimensional representations are stated in Eqs. (5.11), (5.15), and (5.19).

6 Outlook

In this article we have studied the class structure of the group \( \Delta(3n^2) \) and its irreducible representations. We have derived the Kronecker products and constructed the Clebsch-Gordan coefficients. In order to formulate a \( \Delta(3n^2) \)-invariant theory it is necessary to know how to build the (quadratic and cubic) invariants. Previous studies usually restrict themselves to building invariants in a specific way. For example, the authors of Refs. [8] only consider products of a triplet \( 3_{(k,l)} \) of \( \Delta(27) \) and its complex conjugate \( \bar{3}_{(k,l)} = 3_{(-k,-l)} \); also they do not include particles transforming as \( \mathbf{1}_{r,s} \) representations with \( r, s \neq 0 \).

Other models of flavor (e.g. [6]) adopt less trivial ways to build invariants. However, a systematic presentation of “what is possible” is still missing. Here we have tried to fill this gap. We hope that our study will be useful for model builders aiming at a thorough investigation of flavor models based on the group \( \Delta(3n^2) \). In future publication, we plan to apply the same techniques to \( \Delta(6n^2) \) and other finite groups such as \( PSL_2(7) \).
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Appendix

A Labeling the Three-Dimensional Representations

When labeling the three-dimensional representations of $\Delta(3n^2)$ by the subscripts $(k,l)$ there is a certain ambiguity involved. Taking, say, $(0,1)$ is equivalent to the pairs $(-1,0)$ and $(1,-1)$ [see Eq. (3.7)]. It is crucial for the discussion of the products of two representations that we list only one of the three equivalent representations. For this purpose the mapping $\tilde{\ }$ was introduced in Eq. (3.10). It remained unspecified in Section 3. In this appendix we wish to give one possible definition of $\tilde{\ }$ by picturing the full set of standard representatives for $(k,l)$ on the $n \times n$ grid. We have to distinguish between three cases: ($i_1$) $n = 3z + 1$ with $z \in \mathbb{Z}$, ($i_2$) $n = 3z + 2$, and ($ii$) $n = 3z$.

($i_1$) $n = 3z + 1$.

\[
\begin{array}{ccccccccc}
(k,l) & 0 & 1 & \cdots & z & \cdots & 2z & \cdots & 3z \\
0 & & & & & & & & \\
1 & & & & & & & & \\
z & & & & & & & & \\
2z & & & & & & & & \\
3z & & & & & & & & \\
\end{array}
\]

The dots indicate the chosen standard representatives for $(k,l)$. The diagram consists of three regions. First there are the representations with $k = 0$ and $l = 1, 2, \ldots, 3z$. This results in $3z$ representations. Note that $(k,l) = (0,0)$ does not label an irreducible representation. The second region is in the upper left corner. Commencing with the $z \times z$ square, we extend the area of representations to the right, reducing the value of the largest $k$ by one in each step: For $l = z + 1$ we have $k \leq z - 1$, then $l = z + 2$ requires $k \leq z - 2$, etc. Thus we obtain $z^2 + \frac{z(z-1)}{2}$ representations in the second region. The third region is obtained from the second by taking the negative values of the pairs $(k,l)$ in the second region and shifting these...
by \( n \). For instance, \((1,1)\) of the second region is transferred to \((-1,-1) = (3z, 3z)\).
Therefore the representations in the third region are the complex conjugates of those in the second region.
Altogether we obtain \(3z + 2z^2 + z(z - 1) = \frac{1}{3}[(9z^2 + 6z + 1) - 1] = \frac{n^2 - 1}{3}\) representations, a value which is required for \(n \neq 3 \mathbb{Z}\). It can be shown that all of these representations are inequivalent.

\((i_2)\) \( n = 3z + 2 \).

\[
\begin{array}{c|cccccccc}
(k,l) & 0 & 1 & \ldots & z & \ldots & 2z & \ldots & 3z & 3z+1 \\
\hline
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
z & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
2z & \cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
3z & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
3z + 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

In this case the table of standard representatives for \((k,l)\) is obtained similarly to the above. We have \(3z + 1\) pairs in the first region. Compared to \((i_1)\), the triangle on the right side of the second region is shifted one step to the right and the free column is filled up with \(z\) representations. This gives \(z^2 + \frac{(z+1)z}{2}\) pairs. The third region is again only the complex conjugate of the second one. Summing the number of all representations yields the required value of \(\frac{n^2 - 1}{3}\).

\((ii)\) \( n = 3z \).

\[
\begin{array}{c|cccccccc}
(k,l) & 0 & 1 & \ldots & z & \ldots & 2z & \ldots & 3z-1 \\
\hline
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
z & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
2z & \cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
3z - 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
Compared to the case \((i_1)\), the triangle of the second region is shifted by one step to the left. Because \((k, l) = (z, z)\) [and also \((2z, 2z)\)] does not give an irreducible representation, this point of the \(z \times z\) square has to be excluded. Counting all pairs of \((k, l)\) gives \(\frac{n^2 - 3}{3}\) different representations as required for the case where \(n = 3z\).

B The Method of Induced Representations

The method of induced representations is sometimes useful for building irreducible representations, although in most cases it yields reducible representations. Luckily, it allows us to build the \(\Delta(3n^2)\) triplet representations in a particularly simple way. Here we sketch the main ideas and apply them to our group.

Let \(\mathcal{H}\) be a subgroup of \(\mathcal{G}\) of index \(N\), (order of \(\mathcal{G}\) divided by that of \(\mathcal{H}\))

\[
\mathcal{G} \supset \mathcal{H}. \tag{B.1}
\]

Suppose we know a \(d\)-dimensional representation \(\mathbf{r}\) of \(\mathcal{H}\), that is for every \(h \in \mathcal{H}\),

\[
h : |i> \mapsto \mathcal{M}(h)^{[\mathbf{r}]}_{ji} |j>, \tag{B.2}
\]

acting on the \(d\)-dimensional Hilbert space \(\mathcal{H}\), spanned by \(|i>\), \(i = 1, 2, \ldots, d\). This enables us to construct the induced representation of \(\mathcal{G}\), generated by the representation of \(\mathcal{H}\). We begin by considering the coset made up of \(N\) points,

\[
(\mathcal{G}/\mathcal{H}) : \mathcal{H} \oplus g_1\mathcal{H} \oplus g_2\mathcal{H} \oplus \cdots \oplus g_{N-1}\mathcal{H}, \tag{B.3}
\]

where \(g_k (k = 1, \ldots, N - 1)\) are elements of \(\mathcal{G}\) not in \(\mathcal{H}\). Consider the set of elements

\[
\mathcal{G} \times \mathcal{H} : (g, |i>); \quad g \in \mathcal{G}, \quad |i> \in \mathcal{H}. \tag{B.4}
\]

In order to establish a one-to-one correspondence with the coset, we assume that whenever the \(\mathcal{G}\) group element is of the form

\[
g = g_kh_a , \tag{B.5}
\]

we make the identification

\[
(g_kh_a, |i>) = (g_k, \mathcal{M}(h_a)^{[\mathbf{r}]}_{ji} |j>), \tag{B.6}
\]

so that both \(g\) and \(gh\) are equivalent in the sense that they differ only by reshuffling \(\mathcal{H}\). In this way, we obtain \(N\) copies of \(\mathcal{H}\), the Hilbert space of the \(\mathbf{r}\) representation of \(\mathcal{H}\), one at each point of the coset, which we take to be
\[(g_0 \equiv e, \mathbf{H}), (g_1, \mathbf{H}), (g_2, \mathbf{H}), \cdots, (g_{N-1}, \mathbf{H})\]. \hfill (B.7)

The action of \(G\) on this set is simply group multiplication

\[g : (g_k, |i>) \mapsto (gg_k, |i>) \quad g \in G, \quad k = 0, 1, \cdots, N - 1. \hfill (B.8)\]

Suppose that \(g = h_a\), the coset decomposition tells us that

\[h_a g_k = g_l h_b, \hfill (B.9)\]

where \(g_l\) and \(h_b\) are uniquely determined. Hence,

\[h_a : (g_k, |i>) \mapsto (h_a g_k, |i>) = (g_l h_b, |i>) = (g_l, \mathcal{M}(h_b)]_{ji} |j> \hfill (B.10)\]

Similarly, when \(g = g_l\), the product \(g_l g_k\) is itself a group element and can be rewritten uniquely as

\[g_l g_k = g_m h_c, \hfill (B.11)\]

for some \(g_m\) and \(h_c\). It follows that

\[g_l : (g_k, |i>) \mapsto (g_l g_k, |i>) = (g_m h_c, |i>) = (g_m, \mathcal{M}(h_c)]_{ji} |j> \hfill (B.12)\]

We have shown that these \(N\) copies of the vector space are linearly mapped into one another under the action of \(G\). The action of \(G\) is thus represented by a \((dN \times dN)\) matrix, which is not necessarily irreducible.

We now apply this method to \(G = \Delta(3n^2)\) and take the subgroup to be \(H = (\mathbb{Z}_n \times \mathbb{Z}_n)\), of index \(N = 3\). Since the subgroup is Abelian, its representations are one-dimensional, with generators\footnote{The exponents of \(c\) and \(d\) are chosen so that they give the \((1,1)\)-entries of the \((3 \times 3)\) matrices \(c\) and \(d\) in Eq. 5. \(\eta\) is the \(n\)-th root of unity: \(\eta^n = 1\).}

\[c = \eta^l; \quad d = \eta^{-k-l}, \hfill (B.13)\]

acting on complex numbers \(z\). The coset contains just three points, and we consider the action of \(\Delta(3n^2)\) on

\[(e, z), \quad (a^2, z), \quad (a, z). \hfill (B.14)\]
Then
\[ a(e, z) = (a, z), \quad a(a^2, z) = (e, z), \quad a(a, z) = (a^2, z), \]
so that \( a \) is represented by the \((3 \times 3)\) permutation matrix
\[
a = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

Next, comes the action of \( c \)
\[
c(e, z) = (c, z) = (e, cz) = (e, \eta^l z) = \eta^l (e, z).
\]

Using
\[
ca^2 = a^2 c^{-1} d^{-1}; \quad ca = ad,
\]
derived from the presentation, we find
\[
c(a^2, z) = (ca^2, z) = (a^2 c^{-1} d^{-1}, z) = (a^2, \eta^{-l} \eta^{k+l} z) = \eta^k (a^2, z),
\]
\[
c(a, z) = (ca, z) = (ad, z) = (a, \eta^{-k-l} z) = \eta^{-k-l} (a, z).
\]
The action of \( d \) is derived in the same way, using \( da^2 = a^2 c, \) \( da = ac^{-1} d^{-1} \). As a result, both \( c \) and \( d \) are represented by the diagonal matrices
\[
c = \begin{pmatrix}
\eta^l & 0 & 0 \\
0 & \eta^k & 0 \\
0 & 0 & \eta^{-k-l}
\end{pmatrix},
\quad
d = \begin{pmatrix}
\eta^{-k-l} & 0 & 0 \\
0 & \eta^l & 0 \\
0 & 0 & \eta^k
\end{pmatrix},
\]
which completes the construction of the induced \((3 \times 3)\) representation.

C Product of Irreducible Representations: Details

Using Eq. (4.2) and the character Tables 1 we calculate the numerical factors \( d(r, s, t) \) in this appendix.

(i) \( n \neq 3 \cdot \mathbb{Z} \).
There are \( \frac{n^2-1}{3} \) three-dimensional irreducible representations. We need to exclude \((k, l) = (0, 0)\), and we implicitly assume that \((k, l)\) only take their standard values (see e.g. Appendix A). In addition, for the classes \( C_1^{(\rho, \sigma)} \), we need to exclude
\((\rho, \sigma) = (0,0)\). Recalling that summing over the remaining pairs \((\rho, \sigma)\) overcounts each class three times, we can write the sum over the classes \(C^{(\rho,\sigma)}_1\) formally as

\[
\sum_{C^{(\rho,\sigma)}_1} = \frac{1}{3} \left( \sum_{\rho,\sigma=0}^{n-1} - \sum_{\rho=\sigma=0} \right).
\]  

(C.1)

Noting that the characters of the class \(C_1\) are identical to the characters of \(C^{(\rho,\sigma)}_1\) with \((\rho, \sigma) = (0,0)\), we can write

\[
\chi^{[r]}_{C_1} \chi^{[s]}_{C_1} \chi^{[t]}_{C_1} + \sum_{C^{(\rho,\sigma)}_1} 3 \cdot \chi^{[r]}_{C^{(\rho,\sigma)}_1} \chi^{[s]}_{C^{(\rho,\sigma)}_1} \chi^{[t]}_{C^{(\rho,\sigma)}_1} = \sum_{\rho,\sigma=0}^{n-1} \chi^{[r]}_{C^{(\rho,\sigma)}_1} \chi^{[s]}_{C^{(\rho,\sigma)}_1} \chi^{[t]}_{C^{(\rho,\sigma)}_1}.
\]

(C.2)

Therefore, in Eq. (4.2), the sum over the classes \(C_1\) and \(C^{(\rho,\sigma)}_1\) can be neatly written as a sum over all \(n^2\) pairs of \((\rho, \sigma)\). With this observation, we now calculate the products \(1_r \otimes 1_{r'}\), \(1_r \otimes 3_{(k,l)}\), and \(3_{(k,l)} \otimes 3_{(k',l')}\) in turn.

- \(1_r \otimes 1_{r'}\).

\[
d(1_r, 1_{r'}, 1_{r''}) = \frac{1}{3n^2} \cdot \left[ \sum_{\rho,\sigma=0}^{n-1} 1 + n^2 \cdot \omega^{r+r'-r''} + n^2 \cdot \omega^{-(r+r'-r'')} \right]
= \frac{1}{3} \cdot \left[ 1 + \omega^{r+r'-r''} + \omega^{-(r+r'-r'')} \right]
= \delta^{(3)}_{r''-r+r'},
\]

(C.3)

\[
d(1_r, 1_{r'}, 3_{(k,l)}) = \frac{1}{3n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p=0}^{2} \eta^{-(\rho,\sigma)M^p_{(k,l)}} = 0.
\]

(C.4)

Due to the relation \(1 + \omega + \omega^{-1} = 0\), Eq. (C.3) is non-zero only if the exponent of \(\omega\) is zero mod (3). The superscript \((3)\) on the Kronecker delta indicates that we calculate modulo 3. Analogously, due to \(\sum_{p=0}^{n-1} \eta^p = 0\), Eq. (C.4) would be non-zero only if \((k,l) = (0,0)\). This choice however is forbidden. We thus have

\[
1_r \otimes 1_{r'} = 1_{r+r'}.
\]

(C.5)

- \(1_r \otimes 3_{(k,l)}\).

\[
d(1_r, 3_{(k,l)}, 1_{r'}) = \frac{1}{3n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p=0}^{2} \eta^{(\rho,\sigma)M^p_{(k,l)}} = 0.
\]

(C.6)
For a non-zero contribution we need \( M \) to be non-zero. In the first step of Eq. (C.7) we have absorbed \( M' \) into a redefinition of \((\rho,\sigma)\) and \(p\), leading to a factor of 3. The resulting expression only contributes a non-zero value if \( \begin{pmatrix} k' \\ \mu \end{pmatrix} = M' \begin{pmatrix} k \\ \ell \end{pmatrix} \). As \((k',\ell')\) and \((k,\ell)\) take their standard values, this condition is satisfied only for \( p = 0 \) and thus \((k',\ell') = (k,\ell)\). Hence we find the \( r \) independent equation

\[
1_r \otimes 3_{(k,\ell)} = 3_{(k,\ell)}.
\]  

For a non-zero contribution we need \( \begin{pmatrix} k' \\ \nu' \end{pmatrix} = M \begin{pmatrix} -k \\ -\nu \end{pmatrix} \). As \((k',\nu')\) takes its standard value, only one value of \( p' \) can satisfy this equation, namely the one that maps \((-k,-\nu)\) onto \((-k,-\nu')\).

Again, only one value of \( p' \) contributes to this sum. On the other hand, the three possibilities for \( p \) persist, so that we are left with three Kronecker deltas.
From Eqs. (C.9) and (C.10) we get

\[\begin{align*}
3_{(k,l)} \otimes 3_{(k',l')} &= \delta_{(k' - l')} \cdot (1_0 + 1 + 1_2) + \\
&+ 3_{(k' + l')} + 3_{(k' - l - l)} + 3_{(k + l - l)}.
\end{align*}\]  

(C.11)

Notice that if \((k', l') = \bar{(-k, -l)})\), then one of the three \(3\)'s is absent as there is no representation \(3_{(0,0)}\).

\((ii)\) \(n = 3\mathbb{Z}\).

In this case, \((k, l) = (0, 0), (\frac{n}{3}, \frac{n}{3}), (\frac{2n}{3}, \frac{2n}{3})\) result in reducible representations and have to be excluded. As above, \((k, l)\) are assumed to take their standard values only.

When summing over the classes we have similar to Eq. (C.1)

\[\frac{1}{3} \sum_{C_{1}^{(\rho,\sigma)}} = \frac{1}{3} \left( \sum_{\rho,\sigma=0}^{n-1} - \sum_{\rho=-\sigma=\frac{n}{3}, \frac{2n}{3}} - \sum_{\rho=0} \right).\]  

(C.12)

Noting that the characters of \(C_{1}/C_{1}^{(\rho)}\) are identical to the characters of \(C_{1}^{(\rho,\sigma)}\) with \((\rho, \sigma) = (0, 0)/C_{1}^{(\rho,\sigma)} = (\rho, -\rho), \rho = \frac{n}{3}, \frac{2n}{3}\), respectively, we can write the sum over the classes of type 1 in Eq. (4.12) as

\[\begin{align*}
\chi_{C_{1}}^{[r]} \chi_{C_{1}}^{[s]} \chi_{C_{1}}^{[t]} + \sum_{C_{1}^{(\rho)}} \chi_{C_{1}^{(\rho)}}^{[r]} \chi_{C_{1}^{(\rho)}}^{[s]} \chi_{C_{1}^{(\rho)}}^{[t]} + \sum_{C_{1}^{(\rho,\sigma)}} 3 \cdot \chi_{C_{1}^{(\rho,\sigma)}}^{[r]} \chi_{C_{1}^{(\rho,\sigma)}}^{[s]} \chi_{C_{1}^{(\rho,\sigma)}}^{[t]} \\
= \sum_{\rho,\sigma=0}^{n-1} \chi_{C_{1}^{(\rho,\sigma)}}^{[r]} \chi_{C_{1}^{(\rho,\sigma)}}^{[s]} \chi_{C_{1}^{(\rho,\sigma)}}^{[t]},
\end{align*}\]  

(C.13)

The following calculations of the products of two representations can be further simplified by observing that

\[\frac{n}{3} \cdot M^p \left(\begin{array}{c} 1 \\ 1 \end{array}\right) = \frac{n}{3} \cdot \left(\begin{array}{c} 1 \\ 1 \end{array}\right) \mod (n),\]  

(C.14)

independent of the value of \(p\).
• $1_{r,s} \otimes 1_{r',s'}$.

With the definitions of $S \equiv s + s' - s''$ and $R \equiv r + r' - r''$, we get

$$d(1_{r,s}, 1_{r',s'}, 1_{r'',s''}) = \frac{1}{3n^2} \left( \sum_{\rho,\sigma=0}^{n-1} \omega^{(\rho+\sigma)S} + \sum_{\tau=0}^{2} \frac{n^2}{3} \omega^{\tau S} (\omega^R + \omega^{-R}) \right)$$

$$= \frac{1}{3} \cdot \delta_{S,0} \cdot (1 + \omega^R + \omega^{-R})$$

$$= \delta_{S,0} \cdot \delta_{R,0}.$$  \hfill (C.15)

$$d(1_{r,s}, 1_{r',s'}, 3_{(k,l)}) = \frac{1}{3n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p=0}^{2} \omega^{(\rho+\sigma)(s+s')} \cdot \eta^{-(\rho,\sigma)M^p(k)}$$

$$= \frac{1}{3n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p=0}^{2} \eta^{(\rho,\sigma)\left[(s+s')\frac{\omega^{\rho}}{\omega^{\rho}}(1) - M^p(k)\right]}$$

$$= \frac{1}{n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \eta^{(\rho,\sigma)\left[(s+s')\frac{\omega^{\rho}}{\omega^{\rho}}(1) - (k)\right]} = 0. \hfill (C.16)$$

In the second step of Eq. (C.16) we have absorbed $M^p$ into a redefinition of $(\rho, \sigma)$ and made use of Eq. (C.14). As $(k,l)$ with $k = l = 0, \frac{n}{3}, \frac{2n}{3}$ does not label an irreducible representation, the sum over $\rho, \sigma$ yields zero. Combining the results of Eq. (C.15) and (C.16) we find

$$1_{r,s} \otimes 1_{r',s'} = 1_{r+r',s+s'}. \hfill (C.17)$$

• $1_{r,s} \otimes 3_{(k,l)}$.

Similar to Eq. (C.16) we get

$$d(1_{r,s}, 3_{(k,l)}, 1_{r',s'}) = \frac{1}{3n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p=0}^{2} \eta^{(\rho,\sigma)\left[(s-s')\frac{\omega^{\rho}}{\omega^{\rho}}(1) + M^p(k)\right]} = 0. \hfill (C.18)$$

$$d(1_{r,s}, 3_{(k,l)}, 3_{(k',l')}) = \frac{1}{3n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p,\rho'=0}^{p} \eta^{(\rho,\sigma)\left[\frac{\omega^{\rho}}{\omega^{\rho}}(1) + M^p(k) - M^{p'}(k')\right]}$$

$$= \frac{1}{n^2} \cdot \sum_{\rho,\sigma=0}^{n-1} \sum_{p=0}^{2} \eta^{(\rho,\sigma)\left[M^p\left(\frac{\omega^{\rho}}{\omega^{\rho}}(1) + (k)\right) - (k')\right]}$$

$$= \delta^{(k')_{l'}}. \hfill (C.19)$$

Again we have absorbed $M^{p'}$ into $(\rho, \sigma)$ and applied Eq. (C.14). As the pairs $(k',l')$ take standard values, only one $p$ contributes to the sum. Together, Eqs. (C.18) and (C.19) yield

$$1_{r,s} \otimes 3_{(k,l)} = 3_{(k+s/n, l+s/n)}. \hfill (C.20)$$
• $3\otimes 3$

Up to some sign changes, the coefficient $d(3, 3, 1)$ is calculated as in Eq. (C.14):

$$d(3, 3, 1) = \delta \left( \frac{k', l'}{-l + sn/3} \right).$$

The calculation of $d(3, 3, 3, 3)$ is completely identical to Eq. (C.10). We therefore have

$$3 \otimes 3 = \sum_{s=0}^{2} \delta \left( \frac{k'}{-l + sn/3} \right) \left( 1_{0,s} + 1_{1,s} + 1_{2,s} \right) +$$

$$+ 3 \left( \frac{k'}{l'} + 1 \right) + 3 \left( \frac{k'-l}{l'-k} \right).$$

Depending on the values of $(k, l)$ and $(k', l')$ we can have either nine, three, or zero singlets.

Let us first focus on the case where $(k, l, k', l')$ are all multiples of $\frac{n}{3}$. Adopting the conventions of Appendix A for choosing the standard representatives, $(k, l)$ and $(k', l')$ can take only two values: $(0, \frac{n}{3})$ and $(0, \frac{2n}{3})$. For $(k', l') = (-k, -l)$ we are left with nine singlets. If $(k', l') = (k, l)$, we have no singlets but three (identical) triplets $\bar{3}^{(2k, 2l)}$.

In all other cases, i.e. those cases where at least one of the four parameters $k, l, k', l'$ is not a multiple of $\frac{n}{3}$, it is possible to show that the Kronecker delta can be non-zero only for one value of $s$. Thus we obtain either three or zero singlets. In order to have singlets, there must be a pair $p', s = 0, 1, 2$ such that

$$\left( \frac{k'}{l'} \right) = M^{p'} \left( \frac{-k}{-l} \right) + \frac{sn}{3} \left( \frac{1}{1} \right).$$

Then, one of the three triplets would not exist. This can be seen by rewriting the subscripts of the triplets $3\otimes 3$ in Eq. (C.22) and inserting Eq. (C.23):

$$\left( \frac{k''}{l''} \right) = M^{p''} \left( \frac{k''}{l''} \right) + M^{p} \left( \frac{k}{l} \right)$$

$$= M^{p''} \left( M^{p'} \left( \frac{-k}{-l} \right) + \frac{sn}{3} \left( \frac{1}{1} \right) + M^{p} \left( \frac{k}{l} \right) \right)$$

$$= M^{p''} \left( M^{p'} - M^{p} \right) \left( \frac{k}{l} \right) + \frac{sn}{3} \left( \frac{1}{1} \right).$$

$p = 0/1/2$ correspond to the first/second/third triplet in Eq. (C.22). The matrix $M^{p''}$ takes care of the mapping $\sim$ onto the standard representatives. As the pairs $(k'', l'') = (0, 0), (\frac{n}{3}, \frac{n}{3}), (\frac{2n}{3}, \frac{2n}{3})$ in Eq. (C.10) are not allowed for $n = 3\mathbb{Z}$, the triplet with $p = p'$ is absent and replaced by three singlets instead.
D Clebsch-Gordan Coefficients

Under the action of \( a, c, \) and \( d \), the \( \Delta(3n^2) \) triplet fields \( \vec{\varphi} \) and \( \vec{\varphi}' \) transform as

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
\varphi_2 \\
\varphi_3 \\
\varphi_1
\end{pmatrix},
\begin{pmatrix}
\eta^l \varphi_1 \\
\eta^k \varphi_2 \\
\eta^{-k-l} \varphi_3
\end{pmatrix}_c
\longrightarrow
\begin{pmatrix}
\eta^l \varphi_1 \\
\eta^k \varphi_2 \\
\eta^{-k-l} \varphi_3
\end{pmatrix}_d,
\tag{D.1}
\]

and

\[
\begin{pmatrix}
\varphi'_1 \\
\varphi'_2 \\
\varphi'_3
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
\varphi'_2 \\
\varphi'_3 \\
\varphi'_1
\end{pmatrix},
\begin{pmatrix}
\eta''^l \varphi'_1 \\
\eta''^k \varphi'_2 \\
\eta'^{-k-l} \varphi'_3
\end{pmatrix}_c
\longrightarrow
\begin{pmatrix}
\eta''^l \varphi'_1 \\
\eta''^k \varphi'_2 \\
\eta'^{-k-l} \varphi'_3
\end{pmatrix}_d,
\tag{D.2}
\]

respectively.

(i) \( n \neq 3 \mathbb{Z} \).

In terms of the component fields, Eq. (5.4) is written as

\[
\varphi'_1 \varphi_1 + \omega^{-r} \varphi'_2 \varphi_2 + \omega^r \varphi'_3 \varphi_3.
\tag{D.3}
\]

For \((k', l') = (-k, -l)\) this expression is invariant under \( c \) and \( d \). Under the action of \( a \), Eq. (D.3) is transformed to

\[
\varphi'_2 \varphi_2 + \omega^{-r} \varphi'_3 \varphi_3 + \omega^r \varphi'_1 \varphi_1 = \omega^r (\varphi'_1 \varphi_1 + \omega^{-r} \varphi'_2 \varphi_2 + \omega^r \varphi'_3 \varphi_3).
\tag{D.4}
\]

The Clebsch-Gordan coefficients are thus given as

\[
\langle \mathbf{3}_{(k', l')}, \mathbf{3}_{(k, l)} | \mathbf{1}_r \rangle = \omega^{r(1-i)} \delta_{r,i} \delta_{(k')^{-1}(k')^{-1}} \tag{D.5}
\]

with \( i = 1, 2, 3 \) denoting the component of the triplet \( \vec{\varphi} \).

Turning to the three-dimensional representations build out of the product of two triplets, let us consider the three vectors

\[
\begin{pmatrix}
\varphi'_1 \varphi_1 \\
\varphi'_2 \varphi_2 \\
\varphi'_3 \varphi_3
\end{pmatrix},
\begin{pmatrix}
\varphi'_1 \varphi_1 \\
\varphi'_2 \varphi_2 \\
\varphi'_3 \varphi_3
\end{pmatrix},
\begin{pmatrix}
\varphi'_1 \varphi_1 \\
\varphi'_2 \varphi_2 \\
\varphi'_3 \varphi_3
\end{pmatrix},
\tag{D.6}
\]

which under \( c \) transform as

\[
\begin{pmatrix}
\eta'^{l+1} \varphi'_1 \varphi_1 \\
\eta^{k+1} \varphi'_2 \varphi_2 \\
\eta^{-k-l-t} \varphi'_3 \varphi_3
\end{pmatrix},
\begin{pmatrix}
\eta'^{l+k} \varphi'_1 \varphi_2 \\
\eta'^{l-k-l} \varphi'_2 \varphi_3 \\
\eta^{-k-l+t} \varphi'_3 \varphi_1
\end{pmatrix},
\begin{pmatrix}
\eta'^{-l} \varphi'_1 \varphi_3 \\
\eta'^{k} \varphi'_2 \varphi_1 \\
\eta'^{-k-t} \varphi'_3 \varphi_2
\end{pmatrix}.
\tag{D.7}
\]

Under the action of \( a \), the components of the three-dimensional vectors in Eq. (D.6) are permuted cyclically. Hence, up to cyclic permutations, these vectors constitute the triplet representations \( \mathbf{3}_{(k'', l'')} \) of \( \Delta(3n^2) \) with \((k'', l'')\) defined by Eq. (D.7).
The ambiguity of labeling the three-dimensional representations is taken care of by introducing the parameter \( p \); as in Eq. (D.10) we can write

\[
\begin{pmatrix} k'' \\ l'' \end{pmatrix} = M^p \left[ \begin{pmatrix} k' \\ l' \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix} \right],
\]

for the first of the three vectors in Eq. (D.6). The corresponding three-dimensional representation reads either

\[
\begin{pmatrix} \varphi'_1 \varphi_1 \\ \varphi'_2 \varphi_2 \\ \varphi'_3 \varphi_3 \end{pmatrix}_{p=0} , \quad \text{or} \quad \begin{pmatrix} \varphi'_3 \varphi_3 \\ \varphi'_2 \varphi_2 \\ \varphi'_1 \varphi_1 \end{pmatrix}_{p=1}.
\]

Therefore, we get the Clebsch-Gordan coefficient as

\[
\langle 3'_{(k',l')} , 3'_3 | 3''_{(k'',l'')} \rangle = \delta_{i',i''}^{(3)} \delta_{i',i''}^{(3)} \delta_{i',i''}^{(3)} M^p \left[ (k' + k') \right].
\]

with \( p \) defined in Eq. (D.8). The superscript \( ^{(3)} \) on the Kronecker delta indicates that we calculate modulo 3. Generalizing this result to the other two vectors of Eq. (D.6) we find

\[
\langle 3'_{(k',l')} , 3'_3 | 3''_{(k'',l'')} \rangle = \delta_{i',i''}^{(3)} \delta_{i',i''}^{(3)} \delta_{i',i''}^{(3)} M^p \left[ (k' + k') \right].
\]

If \( (k', l') \neq (-k, -l) \), one can build three \( (q = 0, 1, 2) \) different three-dimensional representations out of the product of two triplets. In the case where \( (k', l') = (-k, -l) \), only \( q = 1, 2 \) yield triplet representations; the remaining degrees of freedom are the three singlet states of Eq. (D.3).

(ii) \( n = 3 \mathbb{Z} \).

The one-dimensional representations obtained from the product of two triplets are given in Eqs. (5.12) and (5.13). In terms of the component fields, they take the form

\[
\varphi'_1 \varphi_1 + \omega^{-r} \varphi'_2 \varphi_2 + \omega^r \varphi'_3 \varphi_3 , \quad \text{(D.12)}
\]

\[
\varphi'_2 \varphi_2 + \omega^{-r} \varphi'_3 \varphi_3 + \omega^r \varphi'_1 \varphi_1 , \quad \text{(D.13)}
\]

\[
\varphi'_3 \varphi_3 + \omega^{-r} \varphi'_1 \varphi_1 + \omega^r \varphi'_2 \varphi_2 . \quad \text{(D.14)}
\]

Analogous to Eq. (D.4), the action of \( a \) changes these expression only by an overall factor of \( \omega^r \). With regard to the action of the generator \( c \), we have to specify the values of \( k', l', k, l \). We treat the two cases of Section 5 separately.

(a) In this case [cf. Eq. (5.12)], we have \((-k', -l') = (k, l) = (0, \pm n/3)\). Recalling that \( n^{n/3} = \omega \) and \( \omega^3 = 1 \), Eqs. (D.12)-(D.14), under the action of \( c \), transform
\[ \phi_1' \phi_1 + \omega^{-r} \phi_2' \phi_2 + \omega^r \phi_3' \phi_3, \quad (D.15) \]

\[ \omega^{\pm 1} (\phi_2' \phi_1 + \omega^{-r} \phi_3' \phi_2 + \omega^r \phi_1' \phi_3), \quad (D.16) \]

\[ \omega^{\mp 1} (\phi_3' \phi_1 + \omega^{-r} \phi_1' \phi_2 + \omega^r \phi_2' \phi_3), \quad (D.17) \]

where the two signs in the exponents correspond to the two possible values for \( l = \pm n/3 \). Hence, the expression in Eq. (D.12) gives the \( \mathbf{1}_{r,0} \) representation. For \( l = +n/3 \), the \( \mathbf{1}_{r,1} \) representation is given by Eq. (D.13) and the \( \mathbf{1}_{r,2} \) representation by Eq. (D.14). This is opposite for the case where \( l = -n/3 \).

The Clebsch-Gordan coefficients are thus given as

\[ \langle 3_{(k',l')}^i, 3_{(k,l)}^i | \mathbf{1}_{r,s} \rangle = \omega^r (1-i) \delta_{\nu,\pm i} \delta_{(k',l')_s,(-k,l')_s}, \quad (D.18) \]

with \((k, l) = (0, \pm n/3)\).

(b) Here, \((k', l')\) are given in Eq. (5.17); they depend on \( s \) and the mapping \( \sim \) (i.e. the value of \( p \)). For \( p = 0 \), Eq. (D.12) transforms as

\[ \omega^s (\phi_1' \phi_1 + \omega^{-r} \phi_2' \phi_2 + \omega^r \phi_3' \phi_3), \quad (D.19) \]

under the action of \( c \). It therefore constitutes the one-dimensional representation \( \mathbf{1}_{r,s} \). Similarly, for \( p = 1/2 \), the \( \mathbf{1}_{r,s} \) representation is given by Eq. (D.14) \( \text{or} \) (D.13), respectively. Thus the Clebsch-Gordan coefficients are

\[ \langle 3_{(k',l')}^i, 3_{(k,l)}^i | \mathbf{1}_{r,s} \rangle = \omega^r (1-i) \delta_{\nu,\pm i} \delta_{(k',l')_s,(-k,l')_s}, \quad (D.20) \]

The three-dimensional representations are constructed as in case (i). Eq. (D.11) shows the corresponding Clebsch-Gordan coefficients. In subcase (a), the parameter \( p \) can be directly related to \( q \). This can be seen as follows: In order to obtain triplet representations, we must have \((k', l') = (k, l)\) as discussed below Eq. (C.22). Taking \((0, \pm n/3)\) as the standard labeling for the three-dimensional representations, we can determine \( p \) for a given value of \( q \) explicitly

\[ \begin{pmatrix} k'' \\ l'' \end{pmatrix} = M^p \left[ \begin{pmatrix} k' \\ l' \end{pmatrix} + M^q \begin{pmatrix} k \\ l \end{pmatrix} \right] = M^p \left[ \begin{pmatrix} 0 \\ l \end{pmatrix} + M^q \begin{pmatrix} 0 \\ l \end{pmatrix} \right]. \quad (D.21) \]

\( p \) is obtained by the condition that \( k'' = 0 \). This yields the relation \( p = q \), for subcase (a).

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