On integral graphs obtained by dual Seidel switching

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Abstract

In this paper we apply dual Seidel switching to the Star graphs and to the Odd graphs, which gives two infinite families of integral graphs. In particular, we obtain three new 4-regular integral connected graphs.

Keywords: integral graph; 4-regular graph; dual Seidel switching; Star graph; Odd graph

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1. Introduction

A graph is integral if all eigenvalues of its adjacency matrix are integers \cite{2, 14}. The spectrum of a graph is the multiset of eigenvalues with their multiplicities, and we use $\sigma(\Gamma)$ to denote the spectrum of a graph $\Gamma$. In this paper we deal with regular integral graphs. Some constructions of non-regular integral graphs can be found in \cite{13, 14}.

It was proved by D. Cvetković \cite{6} in 1975 that the set of connected regular integral graphs of any fixed degree is finite. Classification of 3-regular integral connected graphs was given in 1976 by F. C. Bussemaker, D. Cvetković \cite{4} and A. J. Schwenk \cite{14}. There are only 13 cubic integral connected graphs.

There is no complete classification of 4-regular integral connected graphs. In what follows, we briefly observe known results on their classification.

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In 1998, D. Cvetković, S. Simić, and D. Stevanović [5] found 1888 possible spectra of 4-regular bipartite integral graphs, more than 500 of which do not exist as it was shown in [13]. They also published a list of 65 known 4-
regular connected integral graphs. All 24 connected 4-regular integral graphs that do not contain ±3 in the spectrum were determined by D. Stevanović [11] in 2003. In the same paper the possible values for the number $n$ of vertices in 4-regular integral graphs were given as $8 \leq n \leq 1260$, except for 5 identified spectra. In 2007, the upper bound for $n$ was improved [17] so that now we have $8 \leq n \leq 560$, and the number of possible spectra of connected 4-regular integral graphs was decreased down to 828. Moreover, all 828 feasible spectra of connected 4-regular integral graphs, and all 47 connected 4-regular integral graphs with up to 24 vertices were listed. The largest of these 828 spectra has 560 vertices, and actually there are only 12 spectra with more than 360 vertices.

Let us note that the number of possible spectra of connected 4-regular non-
bipartite integral graphs is at most 424. This follows from the fact that if $G$ is a connected non-bipartite 4-regular integral graph, then a direct product of a non-
bipartite graph $G$ and $K_2$ is a connected bipartite 4-regular integral graph [1]. The spectra of 14 non-bipartite connected 4-regular integral graphs found by decomposing the known 47 bipartite connected 4-regular integral graphs are given in [17].

In 2015, M. Minchenko and I. M. Wanless [12] investigated 4-regular integral
Cayley graphs. It was shown that up to isomorphism, there are 32 connected 4-regular integral Cayley graphs (17 of them are bipartite) and 27 connected 4-regular integral arc-transitive graphs (16 of them are bipartite, and 16 of them are Cayley graphs). Complete catalogues of these graphs are available by users.monash.edu.au/~iwanless/data/graphs/IntegralGraphs.html.

In this paper we apply dual Seidel switching [9] to the Star graphs and to the Odd graphs, which gives two infinite families of integral graphs. In particular, we obtain three new 4-regular integral connected graphs.

2. Preliminaries

2.1. Dual Seidel switching

For any simple graph $\Gamma$ with adjacency matrix $A(\Gamma)$ and an order 2 automorphism $\varphi$ of $\Gamma$ interchanging only non-adjacent vertices, we have

$$PA(\Gamma)P^T = A(\Gamma),$$

where $P$ is the permutation matrix corresponding to the automorphism $\varphi$. It is easy to verify that $PA(\Gamma)$ is a symmetric (0,1)-matrix with zero diagonal and thus can be viewed as an adjacency matrix of some simple graph. The resulting graph is said to be obtained from $\Gamma$ by dual Seidel switching induced by $\varphi$ ([9] and [8, Theorem 3.1]). Note further that $(PA(\Gamma))^2 = (A(\Gamma))^2$. In particular, if $\Gamma$ is integral, then a graph obtained from $\Gamma$ by the dual Seidel switching is integral as well.
2.2. Cayley graphs

Let $G$ be a group and $S$ be an inverse-closed identity-free subset in $G$. We define the left Cayley graph $\text{Cay}_L(G, S)$ (resp. right Cayley graph $\text{Cay}_R(G, S)$) whose vertices are the elements of the group $G$ and with two vertices $x, y$ being adjacent whenever $y^{-1}x \in S$ (resp. $xy^{-1} \in S$) holds. For an element $\pi \in G$, let $\varphi^\ell_\pi$ and $\varphi^r_\pi$ denote the left and the right shifts by the element $\pi$. Let $L_G = \{ \varphi^\ell_\pi \mid \pi \in G \}$ and $R_G = \{ \varphi^r_\pi \mid \pi \in G \}$ be the groups of left and right shifts $G$, respectively.

Lemma 1. The following statements hold.
(1) $L_G$ is a group of automorphisms of $\text{Cay}_L(G, S)$;
(2) $R_G$ is a group of automorphisms of $\text{Cay}_R(G, S)$;
(3) For an element $\pi \in G$, the mapping $\varphi^r_\pi$ is an automorphism of $\text{Cay}_L(G, S)$ if and only if $\pi S\pi^{-1} = S$ holds;
(4) For an element $\pi \in G$, the mapping $\varphi^\ell_\pi$ is an automorphism of $\text{Cay}_R(G, S)$ if and only if $\pi S\pi^{-1} = S$ holds.

Proof. (1) For any vertices $x, y \in G$ and element $\pi \in G$, we have
\[
\varphi^\ell_\pi(x) \sim_L \varphi^\ell_\pi(y) \Leftrightarrow \exists s \in S \ \varphi^\ell_\pi(y)^{-1}\varphi^\ell_\pi(x) = s \Leftrightarrow \\
\exists s \in S \ (\pi y)^{-1}\pi x = s \Leftrightarrow \exists s \in S \ y^{-1}x = s \Leftrightarrow x \sim_L y
\]
(2) Similar to item (1).
(3) For any vertices $x, y \in G$ and element $\pi \in G$, we have
\[
\varphi^r_\pi(x) \sim_L \varphi^r_\pi(y) \Leftrightarrow \exists s \in S \ \varphi^r_\pi(y)^{-1}\varphi^r_\pi(x) = s \Leftrightarrow \\
\exists s \in S \ (y\pi)^{-1}\pi x = s \Leftrightarrow \exists s \in S \ y^{-1}x = s \Leftrightarrow x \sim_L y
\]
(4) Similar to item (3). □

Lemma 2. For elements $\pi_{\ell}, \pi_r \in G$, the following statements hold.
(1) $\varphi_{\pi_{\ell}, \pi_r}$ is the composition of $\varphi^\ell_\pi$ and $\varphi^r_\pi$.
(2) $\varphi_{\pi_{\ell}, \pi_r}$ is an automorphism of $\text{Cay}_L(G, S)$ if and only if $\pi_r S\pi_{\ell}^{-1} = S$ holds.
(3) $\varphi_{\pi_{\ell}, \pi_r}$ is an automorphism of $\text{Cay}_R(G, S)$ if and only if $\pi_{\ell} S\pi_r^{-1} = S$ holds.

Proof. (1) It follows from the definitions.
(2) It follows from Lemma 1(3).
(3) It follows from Lemma 1(4). □
2.3. Odd graphs

For a positive integer \( m \), the Odd graph, denoted by \( O_{m+1} \), on a \((2m+1)\)-set \( X \) is the graph whose vertex set is the set of \( m \)-subsets of \( X \), where two \( m \)-sets are adjacent if and only if they are disjoint. It is easy to see that every permutation of \( X \) induces an automorphism of \( O_{m+1} \). The following lemma gives the complete information about the spectrum and the automorphism group of the graph \( O_{m+1} \).

**Lemma 3** ([3], Proposition 9.1.7). The following statements hold.

1. The eigenvalues of \( O_{m+1} \) are \((-1)^i(m+1-i)\) with multiplicity \( \binom{2m+1}{i} - \binom{2m+1}{i-1} \), where \( i \) runs over \( \{0, 1, \ldots, m\} \);
2. The automorphism group of \( O_{m+1} \) consists of the automorphisms induced by the permutations of \( X \).

3. Dual Seidel switching and Star graphs

Let \( n \) be a positive integer, \( n \geq 3 \). Consider the symmetric group \( G = \text{Sym}_n \) and put \( S = \{(1i) \mid i \in \{2, \ldots, n\}\} \). The left Star graph (resp. right Star graph) is the Cayley graph \( \text{Cay}_L(\text{Sym}_n, S) \) (resp. \( \text{Cay}_R(\text{Sym}_n, S) \)).

**Lemma 4.** For an element \( \pi \in G \), the equality \( \pi S \pi^{-1} = S \) holds if and only if \( \pi \) is an element from \( \text{Stab}_G(1) \), where \( \text{Stab}_G(1) \) is the stabilizer of 1 in \( G \).

**Proof.** Let \( \pi \) be a permutation from \( \text{Stab}_G(1) \), which means that \( \pi(1) = 1 \). Note that, for any \( i \in \{2, \ldots, n\} \), we have \( \pi(1i)\pi^{-1} = (1j) \), where \( j = \pi(i) \). Thus, \( \pi S \pi^{-1} = S \) for all \( \pi \in \text{Stab}_G(1) \).

Let \( \pi \) be a permutation that does not belong to \( \text{Stab}_G(1) \) and let \( i \) be an element from \( \{2, \ldots, n\} \) such that \( \pi^{-1}(1) \neq i \). Then the permutation \( \pi(1i)\pi^{-1} \) does stabilize 1 and thus does not belong to \( S \), which means that \( \pi S \pi^{-1} \neq S \). \( \square \)

**Lemma 5.** For \( \pi_\ell, \pi_r \in \text{Sym}_n \), if \( \pi_\ell, \pi_r \) satisfies

1. \( \pi_\ell, \pi_r \) are of order 2;
2. \( \pi_\ell, \pi_r \) have different parity;
3. \( \pi_r S \pi_r^{-1} = S \);
4. \( \pi_\ell \) is not conjugate to any element in \( \pi_r S \),

then \( \varphi_{\pi_\ell, \pi_r} \) is an order 2 automorphism of the left Star graph \( \text{Cay}_L(\text{Sym}_n, S) \) interchanging only non-adjacent vertices from different parts in bipartition of \( \text{Cay}_L(\text{Sym}_n, S) \).

**Proof.** For all vertices \( x \in \text{Sym}_n \), by condition (2), the vertices \( x \) and \( \pi_\ell x \pi_r \) belong to different parts in bipartition of \( \text{Cay}_L(\text{Sym}_n, S) \). In view of Lemma 2(2), it suffices to show that \( x \) and \( \pi_\ell x \pi_r \) are non-adjacent in \( \text{Cay}_L(\text{Sym}_n, S) \). Suppose there exists a vertex \( x \) such that \( x \) and \( \pi_\ell x \pi_r \) are adjacent, which means that \( \pi_r x^{-1} \pi_\ell x = (1i) \) for some \( i \in \{2, \ldots, n\} \). Thus we have \( x^{-1} \pi_\ell x = \pi_r(1i) \), a contradiction with condition (4). \( \square \)
Corollary 1. For a positive integer $n \geq 5$, $\varphi(2\,4), (2\,3)(4\,5)$ is an order 2 automorphism of the left Star graph $\text{Cay}_L(\text{Sym}_n, S)$ interchanging only non-adjacent vertices.

Proof. It suffices to show that, for all $x \in \text{Sym}_n$ and $i \in \{2, \ldots, n\}$, that $x^{-1}(2\,4)x \neq (2\,3)(4\,5)(1\,i)$. Note that, $(2\,3)(4\,5)(1\,i)$ is either a product of a transposition and a cycle of length 3 (the case $2 \leq i \leq 5$) or a product of three disjoint transpositions (the case $i \geq 6$). Since the conjugation preserves the cyclic structure of a permutation, we are completed. □

Lemma 6 ([11],Theorem 6.2.24). Let $G$ be a graph of order $n$ with adjacency matrix $A$. The following are equivalent:

1. $A$ is irreducible;
2. $(I + A)^{n-1}$ is positive;
3. $G$ is connected.

Theorem 1. Let $\varphi$ be an automorphism of left Star graph $\text{Cay}_L(\text{Sym}_n, S)$ satisfying the four conditions in Lemma 5. Then the graph obtained from $\text{Cay}_L(\text{Sym}_n, S)$ by dual Seidel switching induced by $\varphi$ consists of two isomorphic connected components.

Proof. We denote by $A$ the adjacency matrix of $\text{Cay}_L(\text{Sym}_n, S)$. Since $\text{Cay}_L(\text{Sym}_n, S)$ is bipartite, its adjacency matrix $A$ has the following form

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

where $B$ is the reduced adjacency matrix of $\text{Cay}_L(\text{Sym}_n, S)$ with rows indexed by odd permutations of $\text{Sym}_n$ and columns indexed by even permutations of $\text{Sym}_n$. Let $P$ denote the permutation matrix corresponding to automorphism $\varphi$ of $\text{Cay}_L(\text{Sym}_n, S)$. Note that $\varphi$ only interchanges nonadjacent vertices from different parts of bipartition of $\text{Cay}_L(\text{Sym}_n, S)$. Then we may suppose that $\varphi = (1i_1)(2i_2)\cdots(ai_a)$, where $a = \frac{n}{2}$ and \{i_1, i_2, \ldots, i_a\} = \{a+1, a+2, \ldots, 2a\}.

Let $Q$ denote the permutation matrix of order $a$ corresponding to the following $a$-permutation

$$\begin{pmatrix} 1 & 2 & \cdots & a \\ i_1-a & i_2-a & \cdots & i_a-a \end{pmatrix}.$$ 

Then

$$P = \begin{pmatrix} 0 & Q^T \\ Q & 0 \end{pmatrix}.$$

Since $PAP = A$, we have $B^T = QBQ$ and then

$$PA = \begin{pmatrix} Q^TB^T & 0 \\ 0 & QB \end{pmatrix} = \begin{pmatrix} BQ & 0 \\ 0 & QB \end{pmatrix}.$$

Note that $Q^T(QB)Q = BQ$, namely $QB$ and $BQ$ are permutation similar.
Let $H$ denote the graph obtained from $\text{Cay}_L(Sym_n, S)$, with $Alt_n$ as vertex set and two vertices being adjacent whenever they have common neighbor in $\text{Cay}_L(Sym_n, S)$. Clearly $H$ is connected and let $A_1$ denote the adjacency matrix of $H$, $A_1$ is irreducible and then $(I + A_1)^{n!/2-1} > 0$. Note that $B^T B \geq I + A_1$. Thus $(QB)^{n!/2-1} = (B^T B)^{n!/2-1} \geq (I + A_1)^{n!/2-1} > 0$. So $QB$ must be irreducible. Therefore the graph (with $PA$ as its adjacency matrix) obtained by applying dual Seidel switching on $\text{Cay}_L(Sym_n, S)$ has two isomorphic connected components. □

4. Dual Seidel switching and Odd graphs

In this section, we apply dual Seidel switching to the Odd graphs and construct an infinite family of integral graphs. In particular, we construct two new 4-regular integral graphs.

For a positive integer $t$, put $\tau_t := (1 2) \ldots (2t - 1 2t)$.

**Lemma 7.** Given a positive integer $m$, $m \geq 2$, the following statements hold.
(1) For any $t \in \{1, \ldots, m - 1\}$, the permutation $\tau_t$ induces an involution $\varphi_t$ of $O_{m+1}$ that interchanges only non-adjacent vertices;
(2) The permutation $\tau_m$ induces an involution $\varphi_m$ of $O_{m+1}$ that interchanges adjacent vertices as well as non-adjacent vertices.

**Proof.** (1) The image of any vertex $Y_1 = \{s_1, s_2, \ldots, s_m\}$ under the involution $\varphi_t$ is the vertex $Y_2 = \{\tau_t(s_1), \tau_t(s_2), \ldots, \tau_t(s_m)\}$. If they are adjacent, then the two $m$-subsets are disjoint. We must have $\tau_t(s_i) \not\in Y_1$ for every $1 \leq i \leq m$. So no element of $Y_1$ is fixed by $\tau_t$ and $s_i, s_j$ cannot be in the same 2-cycle of $\tau_t$ for $1 \leq i \neq j \leq m$. This forces $t \geq m$. Contradiction.

(2) The involution $\varphi_m$ interchanges the vertices $\{1, 3, \ldots, 2m - 1\}$ and $\{2, 4, \ldots, 2m\}$, which are adjacent. □

It follows from Lemma 4(1) that the smallest eigenvalue of $O_{m+1}$ is $-m$ with multiplicity $2m$, and $O_{m+1}$ has no eigenvalue $m$.

For any positive integers $i, j$, $1 \leq i, j \leq 2m + 1, i \neq j$, let us introduce a partition the vertex set of $O_{m+1}$ into $V_{i,j}, V_{i,j}', V_{i,j}''$, where

- $V_{i,j} := \{m$-subsets of $X$ that contain both $i$ and $j\}$,
- $V_{i,j}'' := \{m$-subsets of $X$ that contain $i$ and do not contain $j\}$,
- $V_{i,j} := \{m$-subsets of $X$ that do not contain $i$ and contain $j\}$,
- $V_{i,j}'' := \{m$-subsets of $X$ that do not contain $i$ or $j\}$,

and define a function $f_{i,j} := V(O_{m+1}) \rightarrow \mathbb{R}$ by the following rule: for any $Y \in V(O_{m+1})$,

$$f_{i,j}(Y) := \begin{cases} 1, & Y \in V_{i,j}'; \\ -1, & Y \in V_{i,j}''; \\ 0, & Y \in V_{i,j} \cup V_{i,j}''. \end{cases}$$
For any vertex subset \( W \subseteq V \), we denote by \( 1_W \) the characteristic function of \( W \), namely
\[
1_W(v) = \begin{cases} 
1, & \text{if } v \in W; \\
0, & \text{otherwise}.
\end{cases}
\]

One can see that we have \( f_{i,j} = 1_{V_i} - 1_{V_{i,j}} \).

**Lemma 8.** For any integers \( m, i, j \), where \( m \geq 2 \) and \( 1 \leq i, j \leq m, i \neq j \), the following statements hold.

1. The partition of the vertex set of \( O_{m+1} \) into \( V_{i,j}, V_{i,j}^c, V_{i,j}^+ \) is equitable with quotient matrix
\[
\begin{pmatrix}
0 & 0 & 0 & m+1 \\
0 & m & 1 & 0 \\
m-1 & 1 & 1 & 0
\end{pmatrix};
\]
2. The function \( f_{i,j} \) is an \(-m\)-eigenfunction of \( O_{m+1} \).

**Proof.** (1) Straightforward; (2) It follows from item (1). \( \square \)

**Lemma 9.** Eigenfunctions \( f_{1,2m+1}, f_{2,2m+1}, \ldots, f_{2m,2m+1} \) form a basis of the \(-m\)-eigenspace of \( O_{m+1} \).

**Proof.** We regard the eigenfunctions as vectors in the space \( \mathbb{R}^{V(O_{m+1})} \). Let us consider the Gram matrix of these vectors. Since
\[
\langle f_{i,2m+1}, f_{i,2m+1} \rangle = \langle 1_{V_{i,2m+1}} - 1_{V_{i,2m+1}}, 1_{V_{i,2m+1}} - 1_{V_{i,2m+1}} \rangle = |V_{i,2m+1}| + |V_{i,2m+1}| = 2^{2m-1}
\]
for \( i = 1, 2, \ldots, 2m \) and
\[
\langle f_{i,2m+1}, f_{j,2m+1} \rangle = \langle 1_{V_{i,2m+1}} - 1_{V_{i,2m+1}}, 1_{V_{j,2m+1}} - 1_{V_{j,2m+1}} \rangle = |V_{i,2m+1} \cap V_{j,2m+1}| + |V_{i,2m+1} \cap V_{j,2m+1}|
\]
\[
= \binom{2m-2}{m-2} + \binom{2m-2}{m-1}
= \binom{2m-1}{m-1}
\]
for \( 1 \leq i \neq j \leq 2m \). Therefore the Gram matrix \( G = \binom{2m-1}{m-1}(J + I) \), where \( J \) is the all-one matrix. So \( G \) is non-singular. Hence the \( 2m \) eigenfunctions \( f_{1,2m+1}, f_{2,2m+1}, \ldots, f_{2m,2m+1} \) are linearly independent. By Lemma 3(1), they form a basis of the \(-m\)-eigenspace of \( O_{m+1} \). \( \square \)

Given positive integers \( m \geq 2 \) and \( 1 \leq t \leq m-1 \), we denote by \( O'_{m+1} \) the graph obtained from \( O_{m+1} \) by dual Seidel switching w.r.t. the involution \( \varphi_t \) of \( O_{m+1} \) induced by the permutation \( \tau_t \).
**Theorem 2.** Let $m$ be a positive integer, $m \geq 2$. Then the following statements hold.

1. For any integer $t$, $1 \leq t \leq m - 1$, the graph $O_{m+1}^t$ has eigenvalue $m$ with multiplicity $t$.
2. The $m - 1$ graphs $O_{m+1}^t$ ($1 \leq t \leq m - 1$) are integral and pairwise non-isomorphic.

**Proof.** (1) For real square matrix $A$, the spectrum of $A^2$ is determined by the spectrum of $A$ by squaring the eigenvalues and summing up the multiplicity of opposite numbers. Since $\varphi_t$ is an involution, the adjacency matrices of $O_{m+1}$ and $O_{m+1}^t$ share the same square. We will show (1) by determining the $-m$-eigenspace and $m$-eigenspace of $O_{m+1}^t$. Note that $m$ is not an eigenvalue of $O_{m+1}$ and the multiplicity of $-m$ of $O_{m+1}$ is $2m$. We study the action of $\varphi_t$ on the eigenfunctions in Lemma 9. One can see that $\varphi_t V_{i,j} = V_{\tau_i(i), \tau_i(j)}$ (and similarly for $V_{\tau_i(i), \tau_i(j)}$ and $V_{\tau_i(j), \tau_i(i)}$). So we have $\varphi_t f_{i,2m+1} = f_{\tau_i(i),2m+1}$ for $1 \leq t \leq m - 1$. Let $A$ be the adjacency matrix of $O_{m+1}$ and $B$ the adjacency matrix of $O_{m+1}^t$. They are related by $B = PA$ where $P = PT$ is the permutation matrix of the involution $\varphi_t$. For $1 \leq i \leq t$, we have $B(f_{2i-1,2m+1} - f_{2i,2m+1}) = PA(f_{2i-1,2m+1} - f_{2i,2m+1}) = -mP(f_{2i-1,2m+1} - f_{2i,2m+1}) = m(f_{2i-1,2m+1} - f_{2i,2m+1})$. For $t < i$, we have $B f_{i,2m+1} = -m f_{i,2m+1}$. We've constructed $t$ eigenfunctions of the eigenvalue $m$ and $(2m - t)$ eigenfunctions of the eigenvalue $-m$ for the graph $O_{m+1}^t$. It is clear that these eigenfunctions are all linearly independent. Since $t + (2m - t) = 2m$, the $m$-eigenspace and $-m$-eigenspace of $O_{m+1}^t$ are determined.

(2) It follows directly from (1). □

The following theorem determines the spectrum of the integral graphs found in Theorem 2.

**Theorem 3.** The spectrum of $O_{m+1}^t$ is determined as follows. The eigenvalue $(-1)^{i+1}(m + 1 - i)$ is of multiplicity $nf_i$ and the eigenvalue $(-1)^i(m + 1 - i)$ is of multiplicity $(2m + 1) - (2m + 1) - nf_i$ where

$$nf_i = \frac{1}{2}(\# \{ i \text{ - subsets not fixed by } \tau_i \} - \# \{ (i - 1) \text{ - subsets not fixed by } \tau_i \}).$$

**Proof.** Let $V_A$ be the collection of $m$-subsets that contain $A$. Let $W_i$ be the subspace spanned by characteristic function $1_{V_A}$ where $A$ runs over all $i$-subsets (in fact they are a basis of $W_i$). We have $W_{i-1} \subset W_i$. The $(-1)^i(m + 1 - i)$-eigenspace of $O_{m+1}$ is in fact $U_i := W_i \cap W_{i-1}^\perp$. Note that $v = \begin{pmatrix} m + 1 \nabla i \\ m - 1 \nabla i \end{pmatrix}$ gives a decomposition $\mathbb{R}^\binom{m}{i} = F \oplus N$, where $\varphi_f f = f$ for every $f \in F$ and $\varphi_n n = -n$ for every $n \in N$. Let $F_i = U_i \cap F$ and $N_i = U_i \cap N$. Then $N_i$ is the $(-1)^{i+1}(m + 1 - i)$-eigenspace of $O_{m+1}$ and $F_i$ is the $(-1)^{i}(m + 1 - i)$-eigenspace of $O_{m+1}^t$. To determine the dimension of $N_i$, we consider $WN_i = W_i \cap N$. We have $N_i = WN_i \cap U_i$ and $WN_i = N_0 \oplus N_1 \oplus \cdots \oplus N_i$. So $\dim N_i = \dim WN_i - \dim WN_{i-1}$. Note that $\dim WN_i$ can be directly computed by considering the action of $\varphi_t$ on the basis of $W_i$. □
5. Concluding remarks

In this paper, we applied dual Seidel switching to the Star graphs and to the Odd graphs, which gives two infinite families of integral graphs. In particular, Theorem 1 gives a new 4-regular graph with spectrum \{-3^7, -2^{13}, -1^3, 0^{15}, 1^1, 2^{15}, 3^5, 4^1\} and Theorem 2 gives two new 4-regular graphs with spectra \{-3^5, -2^{4}, -1^9, 1^5, 2^{10}, 3^1, 4^1\} and \{-3^4, -2^6, -1^8, 1^6, 2^8, 3^2, 4^1\}.

Note that Theorem 2 exhaust all involutions of the Odd graphs that interchange only non-adjacent vertices, while Theorem 1 gives an example of such an involution of the Star graph. We are wondering if the dual Seidel switching can be fruitfully applied to other known 4-regular graphs.

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