Dissipative self-gravitating Bose-Einstein condensates with arbitrary nonlinearity as a model of dark matter halos

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We develop a general formalism applying to Newtonian self-gravitating Bose-Einstein condensates (BECs) at zero temperature [1]. At the scale of galaxies, Newtonian gravity can be used so the evolution of the wave function is governed by the Gross-Pitaevskii-Poisson (GPP) equations (see, e.g., [28]). Using the Madelung transformation [80], the GPP equations can be written under the form of a damped quantum Euler equation associated with a quantum Smoluchowski equation. These equations satisfy an $H$-theorem for a free energy functional constructed with a generalized entropy. We specifically consider the entropy associated with the logotropic equation of state. We derive the virial theorem corresponding to the generalized Gross-Pitaevskii equation, damped quantum Euler equation, and quantum Smoluchowski equation. Using a Gaussian ansatz, we obtain a simple equation governing the dynamical evolution of the size of the condensate. We develop a mechanical analogy associated with this gross dynamics. We highlight a specific model of dark matter halos corresponding to a generalized Gross-Pitaevskii equation with a logarithmic nonlinearity and a cubic nonlinearity. It corresponds to a damped quantum Euler equation associated with a mixed entropy combining the Boltzmann and Tsallis entropies. This formalism may find application in the context of dark matter halos. We introduce a specific applications of our formalism to dark matter halos will be developed in future papers.

I. INTRODUCTION

Bose-Einstein condensates (BECs) play an important role in condensed matter physics [1]. Recently, it has been suggested that they could also play a major role in astrophysics and cosmology (see [22, 41, 54] for recent reviews). Indeed, dark matter halos could be giant self-gravitating BECs at zero temperature [22, 54]. At the scale of galaxies, Newtonian gravity can be used so the evolution of the wave function is governed by the Gross-Pitaevskii-Poisson (GPP) equations (see, e.g., [23, 57]). Using the Madelung transformation [80], the GPP equations can be written under the form of hydrodynamic equations, the so-called quantum Euler-Poisson (EP) equations. These equations are similar to the equations of cold dark matter (CDM) except that they include an anisotropic quantum pressure (or quantum potential) accounting for the Heisenberg uncertainty principle and an isotropic pressure due to the self-interaction (scattering). For usual BECs, described by the Gross-Pitaevskii (GP) equation with a cubic nonlinearity, the equation of state is quadratic, $P = \frac{\hbar^2}{2m} |\psi|^2 \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{2m} a_s^2 |\psi|^2$, where $m$ is the mass of the bosons and $a_s$ is their scattering length [81]. At large (cosmological) scales, quantum effects are negligible and one recovers the classical hydrodynamic equations of CDM which are remarkably successful in explaining the large-scale structure of the universe [2]. At small (galactic) scales, the BEC model differs from the CDM model because of the pressure due to quantum mechanics. Gravitational collapse is prevented by the pressure arising from the Heisenberg uncertainty principle or by the pressure arising from the repulsive scattering of the bosons. Dark matter halos can then reach an equilibrium state with a smooth core density. On the other hand, the BEC model has a finite Jeans length that provides a sharp small-scale cut-off in the matter power spectrum. Therefore, quantum mechanics may be a way to solve the problems of the CDM model such as the cusp problem and the missing satellite problem. For that reason, the BEC model is a good candidate to this problem.

Specific applications of our formalism to dark matter halos will be developed in future papers.

1 The CDM model encounters many problems at the scale of galactic or sub-galactic structures. Indeed, CDM simulations lead to $r^{-1}$ cuspy density profiles at galactic centers (in the scale of the order of 1 kpc and smaller) while most rotation curves indicate a smooth core density [84]. On the other hand, the predicted number of satellite galaxies around each galactic halo is far beyond what we see around the Milky Way [85]. These problems might be solved, without altering the virtues of the CDM model, if the dark matter is composed of BECs [23]. The wave properties of the dark matter may stabilize the system against gravitational collapse preventing halo cores instead of cuspy density profiles in agreement with observations. The resulting coherent configuration may be understood as a ground state of some gigantic bosonic atom where the boson particles are condensed in a single macroscopic quantum state $\psi(r)$. In the BEC model, the formation of dark matter structures at small scales is suppressed by quantum mechanics.
describe dark matter.

However, the self-gravitating BEC model faces apparent difficulties. First of all, the GPP equations are conservative (dissipationless) equations, so it is not clear at first sight how they can relax towards a steady state representing a dark matter halo. If we ignore this difficulty for a moment and consider stable steady state solutions of the GPP equations, we can construct models of dark matter halos and determine their mass-radius relationship. This has been done in Refs. [35, 36] for an arbitrary value of the scattering length $a_s$ connecting the non-interacting limit ($a_s = 0$) to the Thomas-Fermi (TF) limit valid when $GM^2ma_s/\hbar^2 \gg 1$. However, these results are not consistent with the observations of large dark matter halos. When the bosons are noninteracting, one finds that the radius of the halo should decrease with their mass as $R \propto M^{-1}$ and when the bosons are self-interacting, one finds that the halos should have the same radius, independently of their mass. These results are in contradiction with the observations that reveal that the radius of dark matter halos increases with their mass as $R \propto M^{1/2}$ corresponding to a constant surface density (see [86, 89, 90]). This apparent paradox can be solved (see Appendix F of [90]) by considering that the stable steady state solution of the GPP equations (soliton) describes only the core of the halos and that this core is surrounded by an envelope in which the density profile decays approximately as $r^{-3}$ at large distances like the Navarro-Frank-White (NFW) [83] and Burkert [84] density profiles. This core-halo structure has been evidenced in the numerical simulations of [53, 54, 73]. In conclusion, in order to solve the apparent difficulties of the self-gravitating BEC model, we need to find a source of dissipation leading to a relaxation mechanism, and understand the formation of an envelope surrounding the solitonic core.

A solution is provided by the concept of gravitational cooling that was introduced by Seidel and Suen [96] in the context of boson stars. This is a dissipationless mechanism similar in some respect to the concept of violent relaxation introduced by Lynden-Bell [97] for collisionless self-gravitating systems but ending on a unique final state independent of the initial condition. A self-gravitating BEC at $T = 0$, described by the GPP equations, that is not initially in a steady state undergoes gravitational collapse (Jeans instability), displays damped oscillations, and finally settles into a QSS (virialization) by radiating part of the scalar field [94, 95]. For example, if the BEC is initially in an excited state (that is unstable), it spontaneously evolves towards the ground state (that is stable) by ejecting scalar radiation. This cooling mechanism allows the halo to get rid of its excess kinetic energy necessary to form a compact bosonic core. This process may also be at work during hierarchical clustering. As a result of gravitational cooling, dark matter halos take a core-halo structure with a condensed core (soliton/BEC) with an equation of state involving an effective temperature $T_{\text{eff}}$ (soliton) and a halo of scalar radiation (waves) that is approximately isothermal like in the process of violent collisionless relaxation [97]. The halo of scalar radiation is similar to an isothermal envelope with an equation of state $P = \rho k_B T_{\text{eff}}/m$ involving an effective temperature $T_{\text{eff}}$. In that case, the density decreases as $r^{-2}$ at large distances leading to flat rotation curves. Gravitational cooling explains how self-gravitating bosons can rapidly thermalize and acquire a large effective temperature $T_{\text{eff}}$ even if $T = 0$ fundamentally. Therefore, although the true thermodynamic temperature is $T = 0$, everything happens as if the system had a core-halo structure with a core at $T = 0$ (BEC/soliton) and a halo with an effective temperature $T_{\text{eff}} \neq 0$. We emphasize that $T_{\text{eff}}$ is an effective temperature, not a thermodynamic temperature.

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2 The good properties of the BEC model are nevertheless not sufficient to vindicate that model. Other dark matter models based, e.g., on a fermionic [84, 85] or on a logotropic [91, 92] equation of state also give relevant results.

3 They correspond to the ground state of the GPP equations for which the wave function has no node. Excited states for which the wave function has nodes are unstable.

4 Collisionless self-gravitating systems such as elliptical galaxies [57] and dark matter halos made of massive neutrinos [83, 90] are described by the Vlasov equation which is a conservative equation. A spatially homogeneous collisionless self-gravitating system undergoes gravitational collapse (Jeans instability) and forms regions of overdensity. When the density has sufficiently grown, these regions collapse under their own gravity at first in free fall. Then, as nonlinear gravitational effects become important at higher densities, they undergo damped oscillations due to an exchange of kinetic and potential energy. They heat up and finally settle into a quasi stationary state (QSS) with a core-halo structure on a coarse-grained scale (virialization). The system is able to form a dense core by sending some of the particles (stars or neutrinos depending on the context) at large distances. This process is related to phase mixing and nonlinear Landau damping. The resulting Lynden-Bell distribution is similar to the Fermi-Dirac distribution. Elliptical galaxies are nondegenerate [57]. Fermionic dark matter halos may be degenerate [84, 90]. In that case, the QSS has a core-halo structure with a completely degenerate core at $T = 0$ (fermion ball) and an isothermal atmosphere with an effective temperature $T_{\text{eff}}$. The density is finite in the core and decreases as $r^{-2}$ in the halo leading to flat rotation curves. Actually, the halo cannot be exactly isothermal otherwise it would have an infinite mass. The density rather decreases as $r^{-3}$ at large distances like in the NFW [83] and Burkert [84] profiles. This steeper decay may be due to incomplete relaxation [57], tidal effects from nearby galaxies such as those accounted for in the fermionic King model [84, 85], or external stochastic perturbations. Violent relaxation [57] explains how collisionless self-gravitating systems can rapidly thermalize and reach a statistical equilibrium state with a large effective temperature $T_{\text{eff}}$ even if the initial temperature is low.

Hierarchical clustering is the mechanism by which small dark matter halos merge and form larger halos in a bottom-up structure formation scenario. It is believed that dark matter halos acquire a NFW profile in the envelope as a result of successive mergings.
temperature. Bosonic dark matter halos are fundamentally described by the GPP equations at $T = 0$ (it is shown in Appendix F of Ref. [90] that the temperature $T$ of the halos is always much smaller than the condensation temperature $T_c$ whatever their size). However, we propose that, because of gravitational cooling, bosonic dark matter halos acquire an envelope of scalar radiation that is similar to an isothermal atmosphere with an effective temperature $T_{\text{eff}}$. In the analogy with the process of violent relaxation of collisionless self-gravitating systems, the bosonic core (BEC/soliton) corresponds to the fermion ball and the halo made of scalar radiation corresponds to the isothermal halo predicted by Lynden-Bell’s theory. Actually, the halo cannot be exactly isothermal for the reason given in footnote 5. In reality, the density in the halo decreases as $r^{-3}$, similarly to the NFW [53] and Burkert [54] profiles, instead of $r^{-2}$ (isothermal sphere [101]). This extra-confinement may be due to incomplete relaxation, tidal effects, stochastic perturbations... Several types of halos are possible depending on their size. Dwarf dark matter halos are compact objects that are completely condensed without an atmosphere. Therefore, their size is equal to the size of the BEC/soliton. By contrast, large dark matter halos are extended objects with a core-halo structure. They have a condensed core (BEC/soliton) surrounded by an extended atmosphere made of scalar radiation with a density profile decaying as $r^{-3}$ at large distances like the NFW and Burkert profiles. It is the atmosphere that fixes their proper size. The atmosphere can be much larger than the size of the soliton. The presence of the halo of scalar radiation explains why the radius of the dark matter halos increases with their mass. In this way, there is no paradox with the BEC model at $T = 0$.

Because of gravitational cooling, a self-gravitating BEC at $T = 0$ reaches a steady state with a condensed core (soliton) and an approximately isothermal atmosphere made of scalar radiation. In this paper, we propose to heuristically model the process of gravitational cooling by a generalized GP equation including a source of dissipation (damping) and an arbitrary nonlinearity. This equation may be viewed as an effective description of the system’s dynamics on a coarse-grained scale. In other words, it provides a coarse-grained parametrization of the (fined-grained) GP equation at $T = 0$. By using Madelung’s transformation, we show that the generalized GP equation is equivalent to a damped quantum Euler equation involving a friction force proportional and opposite to the velocity and a pressure force associated with an equation of state related to the nonlinearity present in the generalized GP equation. In the strong friction limit, we obtain a quantum Smoluchowski equation. These equations satisfy an $H$-theorem for a free energy functional associated with a generalized entropy. A logarithmic nonlinearity $\ln |\psi|^2$ in the GP equation leads to an isothermal equation of state associated with the Boltzmann entropy. A power law nonlinearity $|\psi|^2 \gamma / |\psi|^2$ leads to a polytropic equation of state associated with the Tsallis entropy. We also consider an hyperbolic nonlinearity $\psi^2 / |\psi|^2$ leading to the logotropic equation of state associated with a logarithmic entropy. We highlight a specific model of dark matter halos corresponding to the generalized GPP equations with a logarithmic nonlinearity and a cubic nonlinearity. It corresponds to the damped quantum isothermal-polytropic EP equations associated with a mixed entropy combining the Boltzmann and Tsallis entropies. We propose that this model provides an effective coarse-grained model of dark matter halos experiencing gravitational cooling. It leads to dark matter halos with an equation of state $P = \rho k_B T_{\text{eff}} / m + 2 \pi a_k \hbar^2 / m^3$ presenting a condensed core (soliton) and an isothermal halo. This model is not perfect since the density in the halo should decrease as $r^{-3}$ (NFW/Burkert) instead of $r^{-2}$ (isothermal), but it can be interesting in a first approach. The $r^{-3}$ (NFW/Burkert) profile may result from a more complicated physics such as incomplete relaxation, tidal effects, stochastic forcing... Tidal effects can be taken into account heuristically by introducing a confining potential in the GP equation. Alternatively, the $r^{-3}$ (NFW/Burkert) profile could be accounted for by using a more complicated nonlinearity in the generalized GP equation (or, equivalently, a more complicated equation of state in the corresponding Euler equation).

The paper is organized as follows. In section III we introduce the generalized GPP equations, derive their hydrodynamic representation, and establish the general condition of hydrostatic equilibrium. In section IV we develop a generalized thermodynamic formalism and derive an $H$-theorem. In section V we derive the virial theorem. In section VI we consider particular forms of generalized GPP and quantum EP equations, and determine their associated equations of state, generalized entropies, and equilibrium distributions. In section VII we highlight a specific model of dark matter halos corresponding to the generalized GPP equations with a logarithmic nonlinearity and a cubic nonlinearity, equivalent to the damped quantum isothermal-polytropic EP equations. In section VIII we compare our results with previous works. In section IX we make a Gaussian ansatz and obtain a simplified equation governing the dynamical evolution of the size of the condensate. We develop a mechanical analogy associated with this gross dynamics and obtain a general analytical expression for the mass-radius relation of dark matter halos. We study their stability and determine their pulsation period. This paper introduces a general formalism appropriate to Newtonian physics.

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7 In this sense, the generalized GP equation represents the counterpart of the relaxation equation for the coarse-grained distribution function introduced in Refs. [28–104] in the context of Lynden-Bell’s theory of violent relaxation [22]. This relaxation equation provides a coarse-grained parametrization of the (fined-grained) Vlasov equation.
self-gravitating BECs. This formalism covers a great variety of situations. Specific applications to dark matter halos will be developed in future works (in preparation). However, the generalized GPP equations that we study in this paper are interesting in their own right and may find applications for other systems beyond the context of dark matter halos.

II. SELF-GRAVITATING BOSE-EINSTEIN CONDENSATES

A. The Gross-Pitaevskii-Poisson equations

We consider a system of $N$ bosons with mass $m$ interacting via a binary potential $u(|\mathbf{r} - \mathbf{r}'|)$ [102]. At $T = 0$, all the bosons condense into the same quantum ground state and the system is described by one order parameter $\psi(\mathbf{r}, t)$ called the condensate wave function. In the mean-field approximation, this gas of interacting BECs is governed by the time-dependent self-consistent field equations (or mean-field Schrödinger equation) [102,106]:

$$i\hbar \frac{\partial \psi}{\partial t}(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \Delta \psi(\mathbf{r}, t) + m\Phi_{\text{tot}}(\mathbf{r}, t)\psi(\mathbf{r}, t),$$

(1)

$$\Phi_{\text{tot}}(\mathbf{r}, t) = \int \rho(\mathbf{r}', t)u(|\mathbf{r} - \mathbf{r}'|)\,d\mathbf{r'},$$

(2)

$$\rho(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2,$$

(3)

$$\int |\psi(\mathbf{r}, t)|^2\,d\mathbf{r} = M = Nm.$$  

(4)

Equation (4) is the normalization condition, Eq. (3) gives the density of the BEC, Eq. (2) determines the associated potential, and Eq. (1) determines the evolution of the wave function. We assume that the potential of interaction can be written as $u = u_{\text{LR}} + u_{\text{SR}}$, where $u_{\text{LR}}$ refers to long-range interactions and $u_{\text{SR}}$ to short-range interactions. For self-gravitating BECs, the potential of long-range interactions is the gravitational potential $u_{\text{LR}} = -G/|\mathbf{r} - \mathbf{r}'|$, where $G$ is the constant of gravity. On the other hand, following Gross [103,105] and Pitaevskii [106], we assume that the short-range interactions correspond to binary collisions that can be modeled by the effective potential $u_{\text{SR}} = g\delta(\mathbf{r} - \mathbf{r}')$ [107,108], where the coupling constant (or pseudo-potential) $g$ is related to the s-wave scattering length $a_s$ of the bosons through $g = 4\pi a_s\hbar^2/m^3$ [81]. For the sake of generality, we allow $a_s$ to be positive or negative ($a_s > 0$ corresponds to a short-range repulsion and $a_s < 0$ corresponds to a short-range attraction). Under these conditions, the total potential can be written as $\Phi_{\text{tot}} = \Phi + h(\rho)$ where $\Phi(\mathbf{r}, t) = -G \int \rho(\mathbf{r}', t)/|\mathbf{r} - \mathbf{r}'|\,d\mathbf{r'}$ is the gravitational potential that is the solution of the Poisson equation $\Delta \Phi = 4\pi G\rho$ and $h(\rho) = gp = g|\psi|^2$ is an effective potential modelling short-range interactions. Regrouping these results, we obtain the Gross-Pitaevskii-Poisson (GPP) equations

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi_{\text{tot}}\psi + \frac{4\pi a_s\hbar^2}{m^2}|\psi|^2\psi,$$

(5)

$$\Delta \Phi = 4\pi G|\psi|^2.$$  

(6)

We note that the GP equation (5) involves a cubic nonlinearity. As mentioned in the Introduction, dark matter halos could be self-gravitating BECs described by the GPP equations (5) and (6).

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8 The condensation of the bosons takes place when their thermal (de Broglie) wavelengths $\lambda_T = (2\pi\hbar^2/mk_B T)^{1/2}$ overlap, that is, when the thermal wavelength is greater than the mean inter-particle distance $l = n^{-1/3}$ (n is the number density of the bosons). This leads to the inequality $\lambda_T < l$, $n\lambda_T^3 > 1$ or $T < T_c$ where $T_c \sim 2\pi\hbar^2 n^{2/3}/mk_B$ is the critical condensation temperature (up to a numerical proportionality factor of order unity).
B. The generalized Gross-Pitaevskii-Poisson equations

In this paper, we consider the generalized GPP equations

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m [\Phi + h(|\psi|^2) + \Phi_{\text{ext}}] \psi - \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi, \tag{7}
\]

\[
\Delta \Phi = S_d G |\psi|^2. \tag{8}
\]

A physical interpretation of these equations will be given in Sec. VII. These equations generalize the GPP equations [5] and [6] in several ways:

(i) We have written these equations in a space of dimension \(d\) (the dimension \(d = 3\) corresponds to spherical halos, the dimension \(d = 2\) corresponds to filaments, and the dimension \(d = 1\) corresponds to sheets). In that case, the Poisson equation is written as \(\Delta \Phi = S_d G \rho\), where \(S_d = 2\pi^{d/2}/\Gamma(d/2)\) is the surface of a hypersphere of unit radius in a \(d\)-dimensional space (the gravitational constant \(G\) depends on the dimension of space \(d\) but, for convenience, we shall not write this dependence explicitly). We recall that \(S_3 = 4\pi \text{ in } d = 3, S_2 = 2\pi \text{ in } d = 2, \text{ and } S_1 = 2 \text{ in } d = 1\).

The gravitational potential can be explicitly written as

\[
\Phi(r, t) = -\frac{G}{d - 2} \int \frac{\rho(r', t)}{|r - r'|^{d - 2}} dr' \quad (d \neq 2),
\]

\[
\Phi(r, t) = G \int \rho(r', t) \ln |r - r'| dr' \quad (d = 2).
\]

(ii) The last term in Eq. 7 represents a source of dissipation measured by the friction coefficient \(\xi\) (this interpretation will become clear in Sec. II C where we introduce a hydrodynamic representation of the generalized GPP equations). The brackets denote spatial averaging: \(\langle X \rangle = \frac{1}{M} \int \rho X \, dr\).

(iii) We have introduced an arbitrary nonlinearity determined by the effective potential \(h(|\psi|^2)\) instead of the usual quadratic potential \(h(|\psi|^2) = g|\psi|^2\) describing pair contact interactions [103–106]. In this way, we can describe a larger class of systems. We recall that the GP equation can be derived from the Klein-Gordon (KG) equation

\[
\square \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + 2 \frac{dV}{d|\varphi|^2} \varphi - i \frac{\xi m}{\hbar} \left[ \ln \left( \frac{\varphi}{\varphi^*} \right) - \left\langle \ln \left( \frac{\varphi}{\varphi^*} \right) \right\rangle \right] \varphi = 0 \tag{11}
\]

in the nonrelativistic limit \(c \to +\infty\) (we have generalized the KG equation by introducing dissipative effects). In that case, the effective potential \(h(|\psi|^2)\) in the GP equation is related to the self-interaction potential \(V(|\varphi|^2)\) in the KG equation by (see [51] and Appendix C of [91]):

\[
h(|\psi|^2) = \frac{dV}{d|\psi|^2}, \quad \text{i.e.} \quad h(\rho) = V'(\rho). \tag{12}
\]

As a result, we can rewrite the generalized GP equation [7] as

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \left( \Phi + \frac{dV}{d|\psi|^2} + \Phi_{\text{ext}} \right) \psi - \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \tag{13}
\]

(iv) We have assumed that the particles are subjected to an external potential \(\Phi_{\text{ext}}(r)\). For illustration, we shall consider the harmonic potential

\[
\Phi_{\text{ext}} = \frac{1}{2} \omega_0^2 r^2. \tag{14}
\]

When \(\omega_0^2 = -\Omega^2 < 0\), this potential mimics the effect of a solid-body rotation of the system (this analogy is exact in \(d = 2\)). When \(\omega_0^2 > 0\), this potential mimics the effect of a confining trap. In astrophysics, where the GPP equations describe dark matter halos, this trapping potential could account for tidal interactions arising from neighboring galaxies.
C. The Madelung transformation

We use the Madelung transformation to rewrite the generalized GP equation (7) under the form of hydrodynamic equations. We write the wave function as

$$\psi(r, t) = \sqrt{\rho(r, t)} e^{iS(r, t)/\hbar},$$

(15)

where $\rho(r, t)$ is the density and $S(r, t)$ is the real action. We have

$$\rho = |\psi|^2 \quad \text{and} \quad S = -\frac{i\hbar}{2} \ln \left( \frac{\psi}{\psi^*} \right).$$

(16)

We note that the dissipative term in the GP equation (7) can be written as $\xi (S - \langle S \rangle) \psi$. Following Madelung, we introduce the velocity field

$$u = \frac{\nabla S}{m}.$$  

(17)

Since the velocity is potential, the flow is irrotational: $\nabla \times u = 0$. Substituting Eq. (15) into Eq. (7) and separating real and imaginary parts, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$

(18)

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + m [\Phi + h(\rho) + \Phi_{\text{ext}}] + Q + \xi (S - \langle S \rangle) = 0,$$

(19)

where

$$Q = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right].$$

(20)

is the quantum potential which takes into account the Heisenberg uncertainty principle. The first equation is similar to the equation of continuity in hydrodynamics. It accounts for the local conservation of mass $M = \int \rho \, d\mathbf{r}$. The second equation has a form similar to the classical Hamilton-Jacobi equation with an additional quantum term and a source of dissipation. It can also be interpreted as a generalized Bernoulli equation for a potential flow. Taking the gradient of Eq. (19), and using the well-known identity of vector analysis $(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left( \frac{\mathbf{u}^2}{2} \right) - \mathbf{u} \times (\nabla \times \mathbf{u})$ which reduces to $(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla (\mathbf{u}^2/2)$ for an irrotational flow, we obtain an equation similar to the Euler equation with a linear friction and a quantum force

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla h - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q - \xi \mathbf{u}. $$

(21)

This equation shows that the effective potential $h$ appearing in the GP equation can be interpreted as an enthalpy in the hydrodynamic equations. We can also write Eq. (21) under the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \varphi - \nabla \varphi_{\text{ext}} - \frac{1}{m} \nabla Q - \xi \mathbf{u},$$

(22)

where $P(r, t)$ is a pressure. Since $h(r, t) = h[\rho(r, t)]$, the pressure $P(r, t) = P[\rho(r, t)]$ is a function of the density, i.e., the flow is barotropic. The equation of state $P(\rho)$ is determined by the potential $h(\rho)$ through the relation

$$h'(\rho) = \frac{P'(\rho)}{\rho},$$

(23)

which can be viewed as a Gibbs-Duhem relation $dP = \rho dh$ (we shall see later that $h$ is one component of the chemical potential). Equation (23) can be integrated into

$$P(\rho) = \rho h(\rho) - V(\rho) = \rho V'(\rho) - V(\rho),$$

(24)
where \( V \) is a primitive of \( h \) (this notation is consistent with Eq. (12) where \( V \) represents the potential of self-interaction in the KG equation that reduces to the GP equation in the nonrelativistic limit \( c \to +\infty \)). The speed of sound is
\[
c_s^2 = P'(\rho) = \rho V''(\rho).
\]
In conclusion, the generalized GPP equations are equivalent to the hydrodynamic equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q - \xi \mathbf{u},
\]
\[
\Delta \Phi = S_d G \rho.
\]
For the harmonic potential defined by Eq. (14), we have \( \nabla \Phi_{\text{ext}} = \omega_0^2 \mathbf{r} \). Using the continuity equation (25), the Euler equation (26) can be rewritten as
\[
\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla ((\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla P - \rho \nabla \Phi - \rho \nabla \Phi_{\text{ext}} - \frac{\rho}{m} \nabla Q - \xi \rho \mathbf{u}.
\]
From that equation, one can introduce a momentum tensor (see Appendix A). We shall refer to these equations as the damped quantum barotropic EP equations.

Finally, if we neglect the advection term \( \nabla ((\rho \mathbf{u} \otimes \mathbf{u}) \) in Eq. (25), but retain the term \( \partial (\rho \mathbf{u}) / \partial t \), and combine the resulting equation with the continuity equation (25), we obtain the quantum barotropic telegraphic equation
\[
\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q \right).
\]
It can be seen as a generalization of the quantum barotropic Smoluchowski equation (30) taking inertial (or memory) effects into account.

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9. Because of the complex nature of the wave function, it is not possible to take the strong friction limit directly in the generalized GP equation (4). We must necessarily split this equation into its real and imaginary parts (which can be done by means of the Madelung transformation) and take the limit \( \xi \to +\infty \) in the damped Euler equation (26) which corresponds to the gradient of the real part of the generalized GP equation (4).

10. The classical barotropic SP equations also describe a gas of self-gravitating Brownian particles in the overdamped limit (109, 110). These particles experience a random force in addition to the gravitational interaction. In that context, the barotropic Smoluchowski equation can be interpreted as a nonlinear Fokker-Planck (NFP) equation associated with stochastic processes (111, 112). It is interesting to note that overdamped BECs and overdamped Brownian particles are described by similar equations. However, we emphasize that (besides the presence of the quantum potential) their physical interpretation is different. For example, in the case of Brownian particles, the pressure term in the Smoluchowski equation leading to (nonlinear) diffusion comes from a (multiplicative) random force. By contrast, in the case of BECs, it comes from the effective potential \( h(|\psi|^2) \) accounting for short-range interactions between the particles. As we show in Secs. A and C a logarithmic effective potential leads to normal (linear) diffusion while a pair contact potential leads to anomalous (quadratic) diffusion. We also note that the damped quantum Euler equation (26) can be rigorously justified for dissipative BECs while the damped Euler equation is not rigorously justified for Brownian particles except in the strong friction limit \( \xi \to +\infty \) where it reduces to the barotropic Smoluchowski equation (see the discussion in (113, 114)). Indeed, the hydrodynamic equations describing dissipative BECs do not involve viscous terms (because of their superfluid nature) while the hydrodynamic equations describing Brownian particles generally do.
D. The energy

If we define the energy by

\[ E = -\left( \frac{\partial S}{\partial t} \right)_{\xi=0} \]  (33)

and use the Hamilton-Jacobi equation (19), we obtain

\[ E(r,t) = \frac{1}{2} m u^2 + m [\Phi + h(\rho) + \Phi_{\text{ext}}] + Q. \]  (34)

This is the sum of the kinetic energy, the gravitational potential, the enthalpy, the external potential, and the quantum potential. We note that the damped quantum barotropic Euler equation (26) can be written as

\[ \frac{\partial u}{\partial t} = -\nabla E - \xi u. \]  (35)

E. The quantum force

The quantum potential (20) first appeared in the work of Madelung [80] and was rediscovered by Bohm [115]. For that reason, it is sometimes called “the Bohm potential”.

The “quantum force” by unit of mass writes

\[ \mathbf{F}_Q = -\frac{1}{m} \nabla Q. \]  (36)

We note the identity

\[ (\mathbf{F}_Q)_i = -\frac{1}{m} \partial_i Q = -\frac{1}{\rho} \partial_i P_{ij}, \]  (37)

where \( P_{ij} \) is the quantum stress (or pressure) tensor defined by

\[ P_{ij}^{(1)} = -\frac{\hbar^2}{4m^2 \rho} \partial_i \partial_j \ln \rho \quad \text{or} \quad P_{ij}^{(2)} = \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \delta_{ij} \Delta \rho \right). \]  (38)

This tensor is manifestly symmetric: \( P_{ij} = P_{ji} \). The identity (37) shows that the quantum force \(-\nabla Q\) is equivalent to the force produced by an anisotropic pressure tensor \( P_{ij} \). In comparison, the effective potential \( h(\rho) \) is equivalent to an isotropic pressure \( P(\rho) \). The tensors defined by Eq. (38) are related to each other by

\[ P_{ij}^{(1)} = P_{ij}^{(2)} + \frac{\hbar^2}{4m^2} (\delta_{ij} \Delta \rho - \partial_i \partial_j \rho). \]  (39)

They differ by a tensor \( \chi_{ij} = \delta_{ij} \Delta \rho - \partial_i \partial_j \rho \) satisfying \( \partial_j \chi_{ij} = 0 \). Contracting the indices, we obtain

\[ P_{ii}^{(1)} = -\frac{\hbar^2}{4m^2} \rho \Delta \ln \rho, \quad P_{ii}^{(2)} = \frac{\hbar^2}{4m^2} \left[ \frac{(\nabla \rho)^2}{\rho} - d \Delta \rho \right], \]  (40)

and the relation

\[ P_{ii}^{(1)} = P_{ii}^{(2)} + (d - 1) \frac{\hbar^2}{4m^2} \Delta \rho. \]  (41)

According to Eq. (37) we note that

\[ \langle \mathbf{F}_Q \rangle = \int \rho \mathbf{F}_Q \, d\mathbf{r} = 0 \]  (42)

so there is no resultant of the quantum force.

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11 A relativistic version of the quantum potential appears in the works of de Broglie [116–118] and London [119] who developed a hydrodynamic representation of the KG equation independently from Madelung [80].
F. Vortices

In the Madelung transformation, the velocity field defined by \( \mathbf{u} = \nabla S/m \) is potential. This implies that the flow is irrotational:

\[
\nabla \times \mathbf{u} = 0 \quad \forall \mathbf{r} \quad \text{where} \quad \rho(\mathbf{r}) \neq 0.
\]

(43)

This relation is valid only at the points where \( \mathbf{u} = \nabla S/m \) is well defined, i.e., at the points where the wave function (or the density) does not vanish. When the wave function vanishes, its phase does not have any meaning and neither \( S \) nor \( \nabla S \) is well defined (the velocity is singular). At such points, known as nodal points, \( \nabla \times \mathbf{u} \) does not vanish in general, leading to the appearance of singular vortices. If we consider the circulation of the velocity around a nodal point, we have

\[
\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = \frac{1}{m} \oint \nabla S \cdot d\mathbf{l} = \frac{1}{m} \oint dS = 2\pi n \frac{\hbar}{m} \quad n = \pm 1, \pm 2, \ldots
\]

(44)

since the phase \( S/\hbar \), when it exists, is defined up to a multiple of \( 2\pi \). This relation shows that the circulation around a nodal point is quantized in units of \( \hbar/m \). The integer \( n \) is the circulation number of the vortex. Using the Stokes theorem, we have

\[
\Gamma = \int \nabla \times \mathbf{u} \, dS = 2\pi n \frac{\hbar}{m}.
\]

(45)

Therefore, the vorticity \( \nabla \times \mathbf{u} \) vanishes everywhere except on certain singular lines where it has singularities of the \( \delta \)-type. This allows for the existence of point vortices with quantized circulation \( nh/m \). This result was stated by Onsager in a footnote [120], and by Feynman [121, 122], in the context of superfluidity. Actually, the quantization of the circulation of singular vortices was discovered by Dirac [123] in a more general context in which an electromagnetic field may be present.

G. Time-independent GP equation

If we consider a wave function of the form

\[
\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt/\hbar},
\]

(46)

where \( \phi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})} \) is real, and substitute Eq. (46) into Eqs. (7) and (8), we obtain the time-independent GPP equations

\[
- \frac{\hbar^2}{2m} \Delta \phi + m(\Phi + h(\rho) + \Phi_{\text{ext}}) \phi = E\phi,
\]

(47)

\[
\Delta \Phi = S_d \nabla^2 \phi^2.
\]

(48)

Equations (47) and (48) define an eigenvalue problem for the wave function \( \phi(\mathbf{r}) \) where the eigenvalue \( E \) is the energy (eigenenergy). The fundamental eigenmode corresponds to the smallest value of \( E \). For this mode, the wave function \( \phi(\mathbf{r}) \) is spherically symmetric and has no node so that the density profile decreases monotonically with the distance. Dividing Eq. (47) by \( \phi(\mathbf{r}) \) and using \( \rho = \phi^2 \), we get

\[
m\Phi + m h(\rho) + m \Phi_{\text{ext}} + Q = E.
\]

(49)

This relation can also be derived from the damped quantum Hamilton-Jacobi equation (19) by setting \( S = -Et \). We note that dissipative effects do not alter the time-independent solutions of the GPP equations because \( S = -Et \) is uniform so that \( \xi(S - \langle S \rangle) = 0 \). We also note that \( E(\mathbf{r}, t) = -\partial S/\partial t = E \) when \( S = -Et \), so that Eq. (49) corresponds to the static value (\( \mathbf{u} = 0 \)) of the energy defined by Eq. (34).
H. Hydrostatic equilibrium

The time-independent GP equation (49) can also be obtained from the damped quantum barotropic Euler equation (22) since it is equivalent to the generalized GP equation. The equilibrium state of the damped quantum barotropic Euler equation (22), obtained by taking $\partial_t = 0$ and $u = 0$, satisfies

$$\nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q = 0. \quad (50)$$

This equation generalizes the usual condition of hydrostatic equilibrium by incorporating the contribution of the quantum potential. From the hydrodynamic representation, we clearly understand why frictional effects do not influence the equilibrium state since they vanish when $u = 0$. Equation (50) describes the balance between the gravitational attraction, the external potential, the quantum potential arising from the Heisenberg uncertainty principle, and the pressure due to short-range interactions (scattering). This equation is equivalent to Eq. (49). Indeed, integrating Eq. (50) using Eq. (23), we obtain Eq. (49) where the eigenenergy $E$ appears as a constant of integration. On the other hand, combining Eq. (50) with the Poisson equation (27), we obtain the fundamental differential equation of hydrostatic equilibrium including the quantum potential

$$- \nabla \cdot \left( \frac{\nabla P}{\rho} \right) + \frac{\hbar^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = S_d G \rho + \Delta \Phi_{\text{ext}}. \quad (51)$$

For the harmonic potential (14), we get $\Delta \Phi_{\text{ext}} = d \omega^2_0$. Some interesting limits can be mentioned:

(i) In the absence of short-range interactions ($P = 0$), the equations of hydrostatic equilibrium (50) and (51) reduce to

$$\rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q = 0, \quad (52)$$

$$\frac{\hbar^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = S_d G \rho + \Delta \Phi_{\text{ext}}. \quad (53)$$

(ii) In the TF limit where we can neglect the quantum potential ($Q \approx 0$), Eqs. (50) and (51) reduce to the classical equations of hydrostatic equilibrium

$$\nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} = 0, \quad (54)$$

$$- \nabla \cdot \left( \frac{\nabla P}{\rho} \right) = 4\pi G \rho + \Delta \Phi_{\text{ext}}. \quad (55)$$

Remark: Using Eq. (24), we can rewrite Eq. (51) under the form

$$\Delta \left( \frac{1}{\rho} + \frac{Q}{m} + \Phi_{\text{ext}} \right) = -S_d G \rho. \quad (56)$$

This equation can also be directly obtained by substituting Eq. (49) into Eq. (48).

III. THERMODYNAMICS OF SELF-GRAVITATING BECS

In this section (and later in Sec. V), we develop a thermodynamical formalism associated with the generalized GPP equations (7) and (8). We stress from the start that this thermodynamical formalism is effective since we are basically considering a boson gas at $T = 0$. However, a strong analogy with thermodynamics arises from the nonlinear term in the generalized GP equation (7) giving rise to a pressure force in the Euler equation (24). We must keep in mind, however, that this pressure has not a thermal origin.
A. The free energy

The free energy associated with the generalized GPP equations (7) and (8), or equivalently with the damped quantum barotropic EP equations (25)-(27), can be written as

\[ F = \Theta_c + \Theta_Q + U + W + W_{\text{ext}}. \] (57)

The first two terms in Eq. (57) correspond to the total kinetic energy

\[ \Theta = \frac{1}{m} \left\langle \psi \left| -\frac{\hbar^2}{2m} \Delta \right| \psi \right\rangle = -\frac{\hbar^2}{2m^2} \int \psi^* \Delta \psi \, d\mathbf{r} = \frac{\hbar^2}{2m^2} \int \left| \nabla \psi \right|^2 \, d\mathbf{r}. \] (58)

Using the Madelung transformation, the kinetic energy can be decomposed into the classical kinetic energy

\[ \Theta_c = \int \rho \frac{u^2}{2} \, d\mathbf{r} \] (59)

and the quantum kinetic energy

\[ \Theta_Q = \frac{\hbar^2}{8m^2} \int \frac{\left( \nabla \rho \right)^2}{\rho} \, d\mathbf{r}. \] (60)

Using Eq. (20), integrating by parts, and assuming that the boundary term can be neglected, we get

\[ \frac{1}{m} \int \rho Q \, d\mathbf{r} = -\frac{\hbar^2}{2m^2} \int \sqrt{\rho} \Delta \sqrt{\rho} \, d\mathbf{r} = \frac{\hbar^2}{2m^2} \int \left( \nabla \sqrt{\rho} \right)^2 \, d\mathbf{r} = \frac{\hbar^2}{8m^2} \int \frac{\left( \nabla \rho \right)^2}{\rho} \, d\mathbf{r}. \] (61)

Therefore, the quantum kinetic energy can be rewritten as

\[ \Theta_Q = \frac{1}{m} \int \rho Q \, d\mathbf{r}. \] (62)

It can be interpreted as a potential energy associated with the quantum potential \( Q \).\(^\text{13}\) The third term in Eq. (57) is the internal energy

\[ U = \int \rho \int \rho' \frac{P(\rho')}{\rho^2} \, d\rho' \, d\mathbf{r}. \] (63)

The density of internal energy \( \rho u \) satisfies the first law of thermodynamic \( du = -Pd(1/\rho) \). We note that the internal energy is defined up to a term of the form \( AM + B \) where \( M \) is the total mass and \( A \) and \( B \) are constants. In the following, we shall use the expression of the internal energy given by

\[ U = \int \left[ \rho h(\rho) - P(\rho) \right] \, d\mathbf{r} = \int V(\rho) \, d\mathbf{r}, \] (64)

which can be obtained from Eq. (63) by a part integration, using Eqs. (23) and (24). For a given self-interaction potential \( V(\rho) \), the enthalpy \( h(\rho) \), the pressure \( P(\rho) \), and the internal energy \( U \) are completely determined by Eqs. (12), (24) and (64). The fourth term in Eq. (57) is the gravitational potential energy

\[ W = \frac{1}{2} \int \rho \Phi \, d\mathbf{r}. \] (65)

The fifth term in Eq. (57) is the external potential energy

\[ W_{\text{ext}} = \int \rho \Psi_{\text{ext}} \, d\mathbf{r}. \] (66)

\(^{12}\) This functional was introduced by von Weizsäcker \[124\] and is related to the Fisher \[122\] entropy \( S_F = (1/m) \int (\nabla \rho)^2 / \rho \, d\mathbf{r} \) by the relation \( \Theta_Q = (\hbar^2/8m) S_F \). Actually, the functional \[60\] was already introduced by Madelung \[80, 126\] under the equivalent form \( \Theta_Q = -(\hbar^2/2m^2) \int \sqrt{\rho} \Delta \sqrt{\rho} \, d\mathbf{r} \) [see Eq. (61)].

\(^{13}\) This is not obvious since \( Q \) is a function of the density (it is not an external potential).
For the harmonic potential, we have
\[ W_{\text{ext}} = \frac{1}{2} \omega_0^2 I, \tag{67} \]
where
\[ I = \int \rho r^2 \, dr \tag{68} \]
is the moment of inertia. We note that \( I = M \langle r^2 \rangle \) where \( \langle r^2 \rangle \) measures the dispersion of the particles (or the size of the condensate). Regrouping all these results, the free energy can be explicitly written as
\[ F = \int \rho \frac{u^2}{2} \, dr + \frac{1}{m} \int \rho Q \, dr + \int V(\rho) \, dr + \frac{1}{2} \int \rho \Phi \, dr + \int \rho \Phi_{\text{ext}} \, dr. \tag{69} \]
On the other hand, the free energy associated with the quantum barotropic SP equations and is given by
\[ F = \Theta_Q + U + W + W_{\text{ext}} = \frac{1}{m} \int \rho Q \, dr + \int V(\rho) \, dr + \frac{1}{2} \int \rho \Phi \, dr + \int \rho \Phi_{\text{ext}} \, dr \tag{70} \]
since the classical kinetic energy \( \Theta_c \), which is of order \( O(\xi^{-2}) \), can be neglected in the overdamped limit \( \xi \to +\infty \).

**B. Difference between the average energy and the free energy**

The average value \( \langle E \rangle \) of the energy \( E(r,t) \) defined by Eq. (34) is given by
\[ N \langle E \rangle = \int \rho \frac{u^2}{2} \, dr + \int \rho \Phi \, dr + \int \rho h(\rho) \, dr + \int \rho \Phi_{\text{ext}} \, dr + \frac{1}{m} \int \rho Q \, dr, \tag{71} \]
i.e.
\[ N \langle E \rangle = \Theta_c + \Theta_Q + \int \rho h(\rho) \, dr + 2W + W_{\text{ext}}. \tag{72} \]
It coincides with the average value of the energy operator (see Appendix B). In a static state where \( E(r,t) = E \), we have \( \langle E \rangle = E \), where \( E \) is the eigenenergy. On the other hand, comparing Eqs. (69) and (71), we find that
\[ F = N \langle E \rangle - \frac{1}{2} \int \rho \Phi \, dr + \int [V(\rho) - \rho h(\rho)] \, dr. \tag{73} \]
Using Eq. (24), we get
\[ F = N \langle E \rangle - \frac{1}{2} \int \rho \Phi \, dr - \int P \, dr = N \langle E \rangle - W - \int P \, dr. \tag{74} \]
In general the free energy is different from the average energy:
\[ F \neq N \langle E \rangle. \tag{75} \]
It is only in the case of the linear Schrödinger equation \( (\Phi = P = 0) \) that \( F = N \langle E \rangle \). For nonlinear Schrödinger equations \( (P \neq 0) \), or for systems with long-range interactions \( (\Phi \neq 0) \), \( F \) differs from \( N \langle E \rangle \) by a nontrivial functional \( -\frac{1}{2} \int \rho \Phi \, dr - \int P \, dr \). As a result, \( F \) and \( \langle E \rangle \) have different properties in general.\[ ^{14} \]

\[ ^{14} \] One exception is when \( P = \rho k_B T/m \) and \( \Phi = 0 \) because, in that case, \( F \) and \( N \langle E \rangle \) just differ by a constant \( -Nk_B T \). This corresponds to the logarithmic GP equation discussed in Sec. V A.
C. The \( H \)-theorem

It is shown in Appendices \([\text{C}\) and \([\text{D}\) that the time derivative of the free energy \((57)\) satisfies the identity

\[
\dot{F} = -\xi \int \rho u^2 \, dr = -2\xi \Theta_c. \tag{76}
\]

The local free energy equation is given in Appendix \([\text{E}\). We have to consider two situations:

(i) For dissipationless systems \((\xi = 0)\), Eq. \((70)\) shows that the GPP equations, or the quantum barotropic EP equations, conserve the free energy \((\dot{F} = 0)\).\(^{15}\) In that case, it can be shown from general arguments \([127]\) that a minimum of free energy at fixed mass determines a steady state of the GPP equations, or quantum barotropic EP equations, that is formally nonlinearly dynamically stable.

(ii) For dissipative systems \((\xi > 0)\), Eq. \((76)\) shows that the generalized GPP equations, or the damped quantum barotropic EP equations, decrease the free energy \((\dot{F} \leq 0)\). When \(\dot{F} = 0\), Eq. \((70)\) implies that \(u = 0\). From the Euler equation \((26)\), we obtain the condition of hydrostatic equilibrium \((50)\). Therefore, Eq. \((70)\) forms an \( H \)-theorem for the generalized GPP equations or for the damped quantum barotropic EP equations: \(\dot{F} \leq 0\) and \(\dot{F} = 0\) if, and only if, the system is at equilibrium. In that case, \(F\) is called a Lyapunov functional. From Lyapunov’s direct method, one can show that the system will relax, for \(t \to +\infty\), towards an equilibrium state that is a (local) minimum of free energy at fixed mass. Maxima or saddle points of free energy are unstable. If several local minima of free energy exist, the selection depends on the initial condition and on a notion of basin of attraction.

The free energy associated with the quantum barotropic SP equations \((30)\) and \((31)\) is given by Eq. \((70)\). Its time derivative satisfies the identity

\[
\dot{F} = -\frac{1}{\xi} \int \frac{1}{\rho} \left( \nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q \right)^2 \, dr. \tag{77}
\]

This identity can be obtained from the quantum barotropic SP equations (see Appendix \([\text{D}\)). It can also be directly obtained from Eq. \((76)\) by using Eq. \((29)\) which is valid in the strong friction limit. When \(\dot{F} = 0\), Eq. \((77)\) implies that the term in parenthesis vanishes, leading to the condition of hydrostatic equilibrium \((50)\). Therefore, Eq. \((77)\) forms an \( H \)-theorem for the quantum barotropic SP equations.

Remark: Since the dissipative \((\xi \neq 0)\) GPP equations, the damped quantum barotropic EP equations, and the quantum SP equations are relaxation equations, they can be used as numerical algorithms to compute stable equilibrium states of the conservative \((\xi = 0)\) GPP equations, or quantum barotropic EP equations. This can be very useful on a practical point of view because it is generally not easy to solve the time-independent equations directly and be sure that the solution is stable.\(^{16}\)

D. The equilibrium state

According to the previous discussion, the equilibrium state of the generalized GPP equations, or quantum barotropic EP equations, is the solution of the minimization problem

\[
F(M) = \min_{\rho, u} \{ F[\rho, u] \mid M \text{ fixed} \}. \tag{78}
\]

A critical point of free energy at fixed mass is determined by the variational principle

\[
\delta F - \frac{\mu}{m} \delta M = 0, \tag{79}
\]

where \(\mu\) is a Lagrange multiplier taking into account the mass constraint. Using the results of Appendix \([\text{C}\) this variational problem gives \(u = 0\) (the equilibrium state is static) and the condition

\[
m\Phi + m\Phi_{\text{ext}} + mh(\rho) + Q = \mu. \tag{80}
\]

\(^{15}\) For dissipationless systems, \(F\) is called the total energy \(E_{\text{tot}}\) of the system, not the free energy. However, for convenience, we shall always refer to \(F\) as the free energy.

\(^{16}\) See Appendix E of \([128]\) and \([129]\) for numerical algorithms under the form of relaxation equations that can be used to construct stable steady states of the Vlasov-Poisson and 2D Euler-Poisson equations.
Taking the gradient of Eq. (80), and using Eq. (23), we recover the condition of hydrostatic equilibrium (50). Equation (80) is also equivalent to the time-independent GP equation (49) provided that we make the identification

$$\mu = E. \quad (81)$$

This shows that the Lagrange multiplier (chemical potential) in the variational problem associated with Eq. (78) can be identified with the eigenenergy $E$. Inversely, the eigenenergy $E$ may be interpreted as a chemical potential.

According to Eq. (80), the equilibrium state is given by

$$\rho = h^{-1} \left( \frac{\mu}{m} - \frac{Q}{m} - \Phi - \Phi_{\text{ext}} \right). \quad (82)$$

When $Q = \Phi = 0$, this equation determines the equilibrium distribution $\rho(\mathbf{r})$. More generally, Eq. (82) is a differential, or an integrodifferential, equation. Considering the second order variations of free energy, we find that the equilibrium is stable if, and only if,

$$\delta^2 F = \frac{1}{2} \int \delta \rho \delta \Phi \, d\mathbf{r} + \frac{1}{2} \int \frac{\hbar^2}{8m^2} \int \left[ \frac{1}{\rho} \left( \frac{\Delta \rho}{\rho} - \frac{(\nabla \rho)^2}{\rho^2} \right) \right] \, d\mathbf{r} > 0, \quad (83)$$

for all perturbations that conserve mass: $\int \delta \rho \, d\mathbf{r} = 0$. This inequality can also be written as

$$\delta^2 F = \frac{1}{2} \int \delta^2 \rho \, d\mathbf{r} + \frac{1}{2} \int \frac{\hbar^2}{8m^2} \int \left[ \nabla \left( \frac{\delta \rho}{\sqrt{\rho}} \right) \right] \, d\mathbf{r} + \frac{\hbar^2}{8m^2} \int \frac{\Delta \sqrt{\rho}}{\rho^{3/2}} (\delta \rho)^2 \, d\mathbf{r} > 0. \quad (84)$$

### E. The Poincaré theorem

In the minimization problem of Sec. 1119, the chemical potential (or eigenenergy) $\mu/m = E/m = \partial F/\partial M$ is the quantity conjugate to the mass $M$ (constraint) with respect to the free energy $F$ (thermodynamical potential). Therefore, if we plot $\mu = E$ as a function of $M$, we can determine the stability of the system by a direct application of the Poincaré theory of linear series of equilibria [130]. According to the Poincaré theorem, a change of stability can only occur at a turning point of mass, or at a bifurcation point, in the series of equilibria. Therefore, if we know a limit in which the configuration is stable, we can use the Poincaré theorem to deduce the stability of the whole series of equilibria. In general, the series of equilibria becomes unstable at the first turning point of mass, corresponding to the maximum mass $M_{\text{max}}$. Furthermore, since $\delta F = 0$ at a turning point of mass where $\delta M = 0$ [see Eq. (79)], the curve $F(M)$ presents cusps.

### F. Functional derivatives

In this section, we show that the hydrodynamic equations associated with the generalized GPP equations can be expressed in terms of functional derivatives of the free energy. This reveals their Hamiltonian structure (in the conservative case) and facilitates the derivation of the $H$-theorem (in the dissipative case). The same is true for the generalized GPP equations as shown in Appendix E.

Using Eq. (17), the free energy (83) can be written as

$$F = \int \rho \left( \frac{\nabla S}{2m^2} \right)^2 \, d\mathbf{r} + \frac{1}{m} \int \rho Q \, d\mathbf{r} + \frac{1}{2} \int V(\rho) \, d\mathbf{r} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + \int \rho \Phi_{\text{ext}} \, d\mathbf{r}. \quad (85)$$

Taking the functional derivatives of the free energy (85) with respect to $\rho$ and $S$, and using the relations of Appendix C we obtain

$$\frac{\delta F}{\delta \rho} = \frac{\mathbf{u}^2}{2} + \Phi + \Phi_{\text{ext}} + \frac{Q}{m} = \frac{E(\mathbf{r}, t)}{m} \quad \text{and} \quad \frac{\delta F}{\delta S} = -\frac{1}{m} \nabla \cdot (\rho \mathbf{u}). \quad (86)$$

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17 See Refs. [131, 132] for the application of the Poincaré theorem in connection to the thermodynamical stability of self-gravitating systems and [53, 55, 71] for the application of the Poincaré theorem in connection to the dynamical stability of self-gravitating BECs.
Therefore, the hydrodynamic equations (18) and (19) can be rewritten as

\[
\frac{\partial \rho}{\partial t} = m \frac{\delta F}{\delta S}, \quad \frac{\partial S}{\partial t} = -m \frac{\delta F}{\delta \rho} - \xi (S - \langle S \rangle).
\]  

(87)

For conservative systems (\(\xi = 0\)), these equations can be interpreted as Hamilton equations for the density \(\rho\) and its canonical action \(S\). This shows that the free energy \(F\) represents the true Hamiltonian of the system (see also Appendices B and F). This formulation directly implies the conservation of the free energy \(F\) since

\[
\dot{F} = \int \frac{\delta F}{\delta \rho} \frac{\partial \rho}{\partial t} \, d\mathbf{r} + \int \frac{\delta F}{\delta S} \frac{\partial S}{\partial t} \, d\mathbf{r} = 0.
\]

(88)

For dissipative systems, one directly recovers the \(H\)-theorem (76) from Eq. (87) since

\[
\dot{F} = -\xi \int \frac{\delta F}{\delta S} (S - \langle S \rangle) \, d\mathbf{r} = \frac{\xi}{m} \int \nabla \cdot (\mathbf{u} S) \, d\mathbf{r} = -\frac{\xi}{m} \int \mathbf{u} \cdot \nabla S \, d\mathbf{r} = -\xi \int \rho \mathbf{u}^2 \, d\mathbf{r}.
\]

(89)

On the other hand, taking the functional derivative of the free energy (69) with respect to \(\mathbf{u}\), we obtain

\[
\frac{\delta F}{\delta \mathbf{u}} = \rho \mathbf{u}.
\]

(90)

Therefore, the hydrodynamic equations (25) and (26) can be rewritten as

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \left( \frac{\delta F}{\delta \mathbf{u}} \right), \quad \frac{\partial \mathbf{u}}{\partial t} = -\nabla \left( \frac{\delta F}{\delta \rho} \right) - \xi \mathbf{u}.
\]

(91)

One directly recovers the \(H\)-theorem (76) from Eq. (91) since

\[
\dot{F} = \int \frac{\delta F}{\delta \rho} \frac{\partial \rho}{\partial t} \, d\mathbf{r} + \int \frac{\delta F}{\delta \mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{r} = -\int \frac{\delta F}{\delta \rho} \nabla \cdot \left( \frac{\delta F}{\delta \mathbf{u}} \right) \, d\mathbf{r} - \int \frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \left( \frac{\delta F}{\delta \rho} \right) \, d\mathbf{r} - \xi \int \frac{\delta F}{\delta \mathbf{u}} \cdot \mathbf{u} \, d\mathbf{r} = -\xi \int \rho \mathbf{u}^2 \, d\mathbf{r}.
\]

(92)

We note that the two first terms of the second equality cancel out after an integration by parts. In the strong friction limit \(\xi \to +\infty\), Eqs. (90) and (91) can be combined into a single equation

\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \rho \nabla \frac{\delta F}{\delta \rho} \right),
\]

(93)

which is equivalent to the quantum barotropic Smoluchowski equation (50). Equation (93) is called a flow gradient equation in the mathematical literature. In this equation, the free energy \(F\) is given by Eq. (70). Again, one directly recovers the \(H\)-theorem (77) from Eq. (93) since

\[
\dot{F} = \int \frac{\delta F}{\delta \rho} \frac{\partial \rho}{\partial t} \, d\mathbf{r} + \frac{1}{\xi} \int \frac{\delta F}{\delta \rho} \nabla \cdot \left( \rho \nabla \frac{\delta F}{\delta \rho} \right) \, d\mathbf{r} = -\frac{1}{\xi} \int \rho \left( \nabla \frac{\delta F}{\delta \rho} \right)^2 \, d\mathbf{r}.
\]

(94)

Remark: We note the identity

\[
\langle E \rangle = \frac{\delta F}{\delta N} \, d\mathbf{r},
\]

(95)

obtained from Eq. (86), which shows the intrinsic difference between \(\langle E \rangle\) and \(F\). It is the Hamiltonian structure of the hydrodynamic equations (87), (91) and (93) [see also Eqs. (B9) and (B12)] that justifies to consider the free energy \(F\) as a Hamiltonian, or as a Lyapunov functional, instead of the average energy \(\langle E \rangle\). We have seen in Sec. III D that the eigenenergy \(E\) can be regarded as a chemical potential \(\mu = \delta F/\delta N\). Therefore, the average energy \(\langle E \rangle\), which is related to the free energy by Eq. (95), can be regarded as a time-dependent chemical potential \(\mu(t)\) that becomes constant at equilibrium. Similarly, the local energy \(E(r, t)\), which is related to the free energy by Eq. (50), can be regarded as an out-of-equilibrium chemical potential \(\mu(r, t)\) that becomes uniform and constant at equilibrium.
IV. THE VIRIAL THEOREM

A. General case

From the damped quantum barotropic EP equations \((25)-(27)\), we can derive the time-dependent scalar virial theorem (see Appendix G):

\[
\frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} = 2(\Theta_c + \Theta_Q) + d \int P \, dr + W_{ii} + W_{ii}^{\text{ext}}. \tag{96}
\]

In the strong friction limit \(\xi \to +\infty\), corresponding to the quantum barotropic SP equations \((30)\) and \((31)\), we get

\[
\frac{1}{2} \xi \dot{I} = 2\Theta_Q + d \int P \, dr + W_{ii}. \tag{97}
\]

At equilibrium \((\ddot{I} = \dot{I} = \Theta_c = 0)\), the virial theorem becomes

\[
2\Theta_Q + d \int P \, dr + W_{ii} + W_{ii}^{\text{ext}} = 0. \tag{98}
\]

On the other hand, the free energy \((57)\) reduces to

\[
F = \Theta_Q + U + W + W_{\text{ext}}. \tag{99}
\]

Multiplying Eq. \((49)\) by \(\rho\) and integrating over the whole domain, we obtain

\[
NE = \Theta_Q + \int \rho h \, dr + 2W + W_{\text{ext}}. \tag{100}
\]

Comparing Eqs. \((99)\) and \((100)\), and using Eq. \((24)\), we get

\[
F = NE - W - \int P \, dr. \tag{101}
\]

Equations \((100)\) and \((101)\) are the equilibrium forms of Eqs. \((72)\) and \((74)\). Eliminating \(\int P \, dr\) between Eqs. \((98)\) and \((101)\), we find that

\[
F = NE - W + \frac{2}{d} \Theta_Q + \frac{1}{d} W_{ii} + \frac{1}{d} W_{ii}^{\text{ext}}. \tag{102}
\]

B. Harmonic potential

For the harmonic potential \((14)\), the time-dependent virial theorem can be written as (see Appendix G):

\[
\frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_c + \Theta_Q) + d \int P \, dr + W_{ii}. \tag{103}
\]

In the strong friction limit \(\xi \to +\infty\), it reduces to

\[
\frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_Q + d \int P \, dr + W_{ii}. \tag{104}
\]

At equilibrium \((\ddot{I} = \dot{I} = \Theta_c = 0)\), the virial theorem, the free energy, and the energy take the form

\[
2\Theta_Q + d \int P \, dr + W_{ii} - \omega_0^2 I = 0, \tag{105}
\]

\[
F = \Theta_Q + U + W + \frac{1}{2} \omega_0^2 I, \tag{106}
\]
\[ NE = \Theta_Q + \int \rho h \, dr + 2W + \frac{1}{2} \omega_0^2 I. \] (107)

**Remark:** If we consider the nongravitational limit \((G = 0)\) and the case \(\xi = 0\) where the free energy \(F\) is conserved, we can combine Eqs. (57) and (103) to obtain the exact equation

\[ \frac{1}{2} \dddot{I} + 2\omega_0^2 I = 2F - 2U + d \int P \, dr = 2F - (d + 2) \int V(\rho) \, dr + \int \rho V'(\rho) \, dr. \] (108)

An application of this equation will be given in Sec. V B 3.

**V. PARTICULAR EQUATIONS OF STATE AND GENERALIZED ENTROPIES**

The free energy (57) associated with the generalized GPP equations (14) and (5), or with the damped quantum barotropic EP equations (25)-(27), can be written as

\[ F = E_s + U, \] (109)

where \(U\) is the internal energy (64) and

\[ E_s = \Theta_c + \Theta_Q + W + W_{\text{ext}} \] (110)

is the energy that includes the classical kinetic energy \(\Theta_c\), the quantum kinetic energy \(\Theta_Q\), the gravitational potential energy \(W\), and the external potential energy \(W_{\text{ext}}\). It can be written explicitly as

\[ E_s = \int \rho \frac{u^2}{2} \, dr + \frac{1}{m} \int \rho Q \, dr + \frac{1}{2} \int \rho \Phi \, dr + \int \rho \Phi_{\text{ext}} \, dr. \] (111)

The effective potential \(h(\rho)\) which accounts for short-range interactions (collisions) between the bosons in the GP equation (7) determines a barotropic equation of state \(P(\rho)\) in the quantum Euler equation (26) through the relations (23) and (24). Inversely, for a given equation of state \(P = P(\rho)\), we can obtain the corresponding effective potential \(h(\rho)\). The effective potential and the equation of state determine the internal energy \(U\) through Eq. (64). As we shall see, the internal energy can be interpreted as the opposite of a generalized (effective) entropy \(S_{\text{eff}}\) multiplied by a generalized (effective) temperature \(T_{\text{eff}}\). Consequently, the free energy can be put in the standard form

\[ F = E_s - T_{\text{eff}} S_{\text{eff}}. \] (112)

In this section, we construct generalized forms of GP equations associated with specific equations of state, and determine the corresponding entropies. The case of composite models is discussed in Appendix H.

**A. Isothermal equation of state: Boltzmann entropy**

1. **Gross-Pitaevskii equation**

The isothermal equation of state (133):

\[ P = \frac{k_B T}{m}, \quad c_s^2 = \frac{k_B T}{m}, \] (113)

is associated with an effective potential of the form

\[ h(\rho) = \frac{k_B T}{m} \ln \rho, \quad V(\rho) = \frac{k_B T}{m} \rho (\ln \rho - 1). \] (114)

The corresponding generalized GP equation is

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{1}{2} m \omega_0^2 r^2 \psi + 2k_B T \ln |\psi| \psi - \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi^*}{\psi} \right) - \ln \left( \frac{\psi}{\psi^*} \right) \right] \psi. \] (115)
It has a logarithmic nonlinearity. The internal energy is given by
\[ U_B = \frac{k_B T}{m} \int \rho (\ln \rho - 1) \, dr. \] (116)

The free energy and the average energy are
\[ F_B = \Theta_c + \Theta_Q + W + W_{\text{ext}} + U_B, \] (117)
\[ N\langle E \rangle = \Theta_c + \Theta_Q + 2W + W_{\text{ext}} + U_B + Nk_B T. \] (118)

They satisfy the relation \( F_B = N\langle E \rangle - W - Nk_B T \). The free energy can be written as
\[ F_B = E^* - TS_B, \] (119)
where \( T \) is the temperature and
\[ S_B = -k_B \int \frac{\rho}{m} (\ln \rho - 1) \, dr \] (120)
is the Boltzmann entropy.

Remark: In order to have a dimensionless quantity in the logarithm, we can replace Eqs. (114), (116) and (120) by
\[ h(\rho) = \frac{k_B T}{m} \ln \frac{\rho}{\rho_0}, \quad U_B = \frac{k_B T}{m} \int \rho \left[ \ln \left( \frac{\rho}{\rho_0} \right) - 1 \right] \, dr, \quad S_B = -k_B \int \frac{\rho}{m} \left[ \ln \left( \frac{\rho}{\rho_0} \right) - 1 \right] \, dr, \] (121)
where \( \rho_0 \) is a reference density.

2. Hydrodynamic representation

The generalized GP equation (115) is equivalent to the damped quantum isothermal Euler equations
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \] (122)
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{k_B T}{m} \nabla \ln \rho - \nabla \Phi - \omega_0^2 \mathbf{r} - \frac{1}{m} \nabla Q - \xi \mathbf{u}. \] (123)

In the strong friction limit \( \xi \to +\infty \), we obtain the quantum Smoluchowski equation
\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi + \rho \omega_0^2 \mathbf{r} + \frac{\rho}{m} \nabla Q \right). \] (124)

We note that it corresponds to a normal classical diffusion.

3. Hydrostatic equilibrium

The condition of hydrostatic equilibrium writes
\[ \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi + \rho \omega_0^2 \mathbf{r} + \frac{\rho}{m} \nabla Q = 0. \] (125)

Combined with the Poisson equation (27), we obtain
\[ -\frac{k_B T}{m} \Delta \ln \rho + \frac{k^2}{2m^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = S_{dG} \rho + d\omega_0^2. \] (126)

In the TF approximation \( (Q = 0) \), the foregoing equation reduces to
\[ -\frac{k_B T}{m} \Delta \ln \rho = S_{dG} \rho + d\omega_0^2. \] (127)
4. The equilibrium state

The minimization of the Boltzmann free energy at fixed mass (see Sec. [III.D]) leads to the equation

\[ Q + m\Phi + \frac{1}{2}m\omega_0^2 r^2 + k_BT \ln \rho = \mu. \] (128)

This equation is equivalent to the condition of hydrostatic equilibrium. It can be rewritten as

\[ \rho = e^{-\beta(m\Phi+Q+\frac{1}{2}m\omega_0^2 r^2 - \mu)}, \] (129)

which can be interpreted as a generalized Boltzmann distribution including the contribution of the quantum potential. In the TF approximation \((Q = 0)\), we recover the Boltzmann distribution.

Remark: If we use the definitions of Eq. (121), we obtain

\[ \rho = \rho_0 e^{-\beta(m\Phi+Q+\frac{1}{2}m\omega_0^2 r^2 - \mu)}. \] (130)

5. The virial theorem

The scalar virial theorem writes

\[ \frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_Q + \Theta_Q) + dNk_BT + W_{ii}. \] (131)

In the strong friction limit, we get

\[ \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_Q + dNk_BT + W_{ii}. \] (132)

At equilibrium, the virial theorem, the free energy and the eigenenergy reduce to

\[ 2\Theta_Q + dNk_BT + W_{ii} - \omega_0^2 I = 0, \] (133)

\[ F_B = \Theta_Q + W + \frac{1}{2} \omega_0^2 I + U_B, \] (134)

\[ NE = \Theta_Q + U_B + Nk_BT + 2W + \frac{1}{2} \omega_0^2 I. \] (135)

Remark: In the strong friction limit \( \xi \to +\infty \), in the TF approximation where we can neglect the quantum potential \((Q = 0)\), and in \( d = 2 \) dimensions where the virial of the gravitational force \( W_{ii} \) is given by Eq. (G25), the virial theorem \((132)\) takes the form

\[ \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2Nk_BT - \frac{GM^2}{2}. \] (136)

If we introduce the critical temperature\(^{18}\)

\[ k_BT_c = \frac{GMm}{4}, \] (137)

it can be rewritten as

\[ \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2Nk_B(T - T_c). \] (138)

---

\(^{18}\) The dimension \( d = 2 \) is critical for isothermal self-gravitating systems.
At equilibrium,
\[ \omega_0^2 I = 2Nk_B(T - T_c). \]  
(139)

Remarkably, Eq. (138) is a closed equation. Therefore, although we cannot solve the SP equations analytically, it turns out that we can obtain the evolution of the moment of inertia \( I(t) \), or equivalently the evolution of the mean square displacement \( \langle r^2 \rangle(t) = I(t)/M \), analytically. Indeed, we obtain
\[ \langle r^2 \rangle(t) = \langle r^2 \rangle_0 e^{-2\omega_0^2 t/\xi} + \frac{2k_B}{m\omega_0^2}(T - T_c) \left( 1 - e^{-2\omega_0^2 t/\xi} \right). \]  
(140)

If \( \omega_0 = 0 \), we get
\[ \langle r^2 \rangle(t) = \frac{4k_B}{\xi m}(T - T_c)t + \langle r^2 \rangle_0. \]  
(141)

The mean square displacement behaves like in a pure diffusion process, \( \langle r^2 \rangle = 4D_{\text{eff}}t + \langle r^2 \rangle_0 \), with an effective diffusion coefficient
\[ D_{\text{eff}} = \frac{k_BT}{\xi m} \left( 1 - \frac{T_c}{T} \right). \]  
(142)

This exact result generalizes the Einstein relation to the case of 2D Brownian particles in gravitational interaction. The original Einstein relation \( D = \frac{k_B T}{\xi m} \) is recovered for \( T_c = 0 \), i.e., in the absence of gravitational interaction (\( G = 0 \)). A detailed discussion of this model can be found in [135, 136].

B. Polytropic equation of state: Tsallis entropy

1. Gross-Pitaevskii equation

The polytropic equation of state [133]:
\[ P = K\rho^\gamma, \quad \gamma = 1 + \frac{1}{n}, \quad c_s^2 = K\gamma^{-1}, \]  
(143)
is associated with an effective potential of the form
\[ h(\rho) = \frac{K\gamma}{\gamma - 1}\rho^{\gamma - 1}, \quad V(\rho) = \frac{K}{\gamma - 1}\rho^\gamma. \]  
(144)

We note that \( P(\rho) = (\gamma - 1)V(\rho) \). The corresponding generalized GP equation is
\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \psi + m\Phi \psi + \frac{1}{2}m\omega_0^2 r^2 \psi + \frac{K\gamma m}{\gamma - 1}|\psi|^{2(\gamma - 1)}\psi - \frac{\hbar}{2}\xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \]  
(145)

It has a power-law nonlinearity. The internal energy is given by
\[ U_\gamma = \frac{K}{\gamma - 1} \int \rho^\gamma \, dr. \]  
(146)
The free energy and the average energy are
\[ F_\gamma = \Theta_c + \Theta_Q + W + W_{\text{ext}} + U_\gamma, \]  
(147)
\[ N(E) = \Theta_c + \Theta_Q + 2W + W_{\text{ext}} + \gamma U_\gamma. \]  
(148)

They satisfy the relation \( F_\gamma = N(E) - W - (\gamma - 1)U_\gamma \). The free energy can be written as
\[ F_\gamma = E_s - KS_\gamma, \]  
(149)
where $K$ is the polytropic temperature and

$$S_\gamma = -\frac{1}{\gamma - 1} \int \rho^\gamma \, d\mathbf{r}$$

is the Tsallis, or power-law, entropy of index $\gamma$.

**Remark:** In order to recover the results of Sec. V A 1 in the limit $\gamma \to 1$, we can define

$$h(\rho) = \frac{K}{\gamma - 1} \left( \frac{\rho}{\rho_0} \right)^{\gamma - 1} - 1, \quad V(\rho) = \frac{K}{\gamma - 1} \left( \frac{\rho}{\rho_0} - \gamma \rho \right),$$

or

$$U_\gamma = \frac{K}{\gamma - 1} \int (\rho^\gamma - \gamma \rho) \, d\mathbf{r}, \quad S_\gamma = -\frac{1}{\gamma - 1} \int (\rho^\gamma - \gamma \rho) \, d\mathbf{r},$$

(151)

$$h(\rho) = \frac{K}{\gamma - 1} \left( \frac{\rho}{\rho_0} \right)^{\gamma - 1} - 1, \quad V(\rho) = \frac{K \rho_0}{\gamma - 1} \left( \frac{\rho}{\rho_0} \right)^{\gamma - \gamma \rho},$$

or

$$U_\gamma = \frac{K \rho_0}{\gamma - 1} \int \left[ \left( \frac{\rho}{\rho_0} \right)^\gamma - \gamma \frac{\rho}{\rho_0} \right] \, d\mathbf{r}, \quad S_\gamma = -\frac{\rho_0}{\gamma - 1} \int \left[ \left( \frac{\rho}{\rho_0} \right)^\gamma - \gamma \frac{\rho}{\rho_0} \right] \, d\mathbf{r}.$$

(153)

(154)

We note that the entropy defined by Eqs. (152) and (154) slightly differs from the usual Tsallis entropy because of the factor $\gamma$ in front of $\rho$. On the other hand, in order to recover the correct dimensions of temperature and entropy when $\gamma \to 1$, we must define the polytropic temperature $T_\gamma$ such that $K = k_B T_\gamma / m$ and the Tsallis entropy $S_\gamma$ such that the Tsallis free energy writes $F_\gamma = E_\ast - T_\gamma S_\gamma$. This amounts to multiplying the entropies (152) and (154) by $k_B / m$. When $\gamma \to 1$, we recover Eqs. (120) and (121).

2. Hydrodynamic representation

The generalized GP equation (145) is equivalent to the damped quantum polytropic Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

(155)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{K}{\gamma - 1} \nabla \rho^{\gamma - 1} - \nabla \Phi - \omega_0^2 \rho \mathbf{r} - \frac{1}{m} \nabla Q - \xi \mathbf{u}.$$

(156)

In the strong friction limit $\xi \to +\infty$, we obtain the quantum polytropic Smoluchowski equation

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( K \nabla \rho^\gamma + \rho \nabla \Phi + \rho \omega_0^2 \rho \mathbf{r} + \frac{\rho}{m} \nabla Q \right).$$

(157)

3. Hydrostatic equilibrium

The condition of hydrostatic equilibrium writes

$$K \nabla \rho^\gamma + \rho \nabla \Phi + \rho \omega_0^2 \mathbf{r} + \frac{\rho}{m} \nabla Q = 0.$$

(158)

Combined with the Poisson equation (27), we obtain

$$-\frac{K}{\gamma - 1} \Delta \rho^{\gamma - 1} + \frac{h^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = S_d \rho + d \omega_0^2.$$

(159)

In the TF approximation ($Q = 0$), the foregoing equation reduces to

$$-\frac{K}{\gamma - 1} \Delta \rho^{\gamma - 1} = S_d \rho + d \omega_0^2.$$

(160)
4. The equilibrium state

The minimization of the Tsallis free energy at fixed mass (see Sec. III D) leads to the equation

$$Q + m\Phi + \frac{1}{2}m\omega_0^2r^2 + \frac{K\gamma m}{\gamma - 1}r^{-\gamma} = \mu.$$  \hspace{1cm} (161)

This equation is equivalent to the condition of hydrostatic equilibrium. It can be rewritten as

$$\rho = -\frac{\gamma - 1}{K\gamma m\rho_0^{\gamma - 1}} \left( m\Phi + Q + \frac{1}{2}m\omega_0^2r^2 - \mu \right)^{1/(\gamma - 1)},$$  \hspace{1cm} (162)

which can be interpreted as a generalized Tsallis distribution including the contribution of the quantum potential. In the TF approximation ($Q = 0$), we recover the Tsallis distribution.

Remark: If we use the definition of Eq. (153), we get

$$\rho = \rho_0 \left[ 1 - \frac{\gamma - 1}{K\gamma m\rho_0^{\gamma - 1}} \left( m\Phi + Q + \frac{1}{2}m\omega_0^2r^2 - \mu \right)^{1/(\gamma - 1)} \right]$$  \hspace{1cm} (163)

which returns the Boltzmann distribution \ref{153} when $\gamma \to 1$.

5. The virial theorem

The scalar virial theorem writes

$$\frac{1}{2}I + \frac{1}{2}\xi I + \omega_0^2I = 2(\Theta_c + \Theta_Q) + d(\gamma - 1)U_\gamma + W_{ii},$$  \hspace{1cm} (164)

where we have used the identity $\int P \, d\mathbf{r} = (\gamma - 1)U_\gamma$. In the strong friction limit, we get

$$\frac{1}{2}\xi I + \omega_0^2I = 2\Theta_Q + d(\gamma - 1)U_\gamma + W_{ii}.$$  \hspace{1cm} (165)

At equilibrium, the virial theorem, the free energy and the eigenenergy reduce to

$$2\Theta_Q + d(\gamma - 1)U_\gamma + W_{ii} - \omega_0^2I = 0,$$

$$F_\gamma = \Theta_Q + U_\gamma + W + \frac{1}{2}\omega_0^2I,$$  \hspace{1cm} (166)

$$NE = \Theta_Q + \gamma U_\gamma + 2W + \frac{1}{2}\omega_0^2I,$$  \hspace{1cm} (167)

where we have used the identity $\int \rho \, d\mathbf{r} = \gamma U_\gamma$ valid for a polytropic equation of state.

Remark: If we consider the nongravitational limit ($G = 0$) and the case of conservative systems ($\xi = 0$) for which the free energy $F_\gamma$ is conserved, we can combine Eqs. (144) and (164) to obtain

$$\frac{1}{2}I + 2\omega_0^2I = 2F_\gamma + [d(\gamma - 1) - 2]U_\gamma.$$  \hspace{1cm} (168)

This is a particular case of Eq. (108). For the critical index $\gamma_c = 1 + 2/d$ (corresponding to $n_c = d/2$) \ref{137}, we obtain a closed equation for the moment of inertia

$$\frac{1}{2}I + 4\omega_0^2I = 4F_{\gamma_c}.$$  \hspace{1cm} (170)

It has the solution $I(t) = A \cos(2\omega_0t + \phi) + F_{\gamma_c}/\omega_0^2$. This result is valid for repulsive ($K > 0$) and attractive ($K < 0$) interactions. For $d = 2$, we find $\gamma_c = 2$ which corresponds to the standard BEC.

Remark: If we consider classical polytropes ($h = 0$) without external potential ($\omega_0 = 0$) we can combine Eqs. (166) and (167) to obtain, at equilibrium, and for $d \neq 2$

$$F_\gamma = \frac{\gamma - \gamma_4/3}{\gamma - 1}W = \left( 1 - \frac{n}{n_3} \right)W,$$  \hspace{1cm} (171)

where $\gamma_4/3 = 2 - 2/d$ (corresponding to $n_3 = d/(d - 2)$). In $d = 3$, we get $F_\gamma = (\gamma - 4/3)W/(\gamma - 1) = (1 - n/3)W$. According to the Poincaré argument \ref{132}, a necessary (but not sufficient) condition of nonlinear dynamical stability is that $F < 0$ (we assume $n > 0$). Since $W < 0$ \cite{129}, we conclude that the system is unstable for $\gamma < 4/3$ (i.e. $n > 3$).
C. Standard BEC: quadratic entropy

1. Gross-Pitaevskii equation

For a standard BEC in $d = 3$, the generalized GP equation writes
\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{1}{2} m \omega_0^2 r^2 \psi + \frac{4\pi a_s h^2}{m} |\psi|^2 \psi - \frac{i\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \] (172)

It has a cubic nonlinearity. The effective potential is given by
\[ h(\rho) = \frac{4\pi a_s h^2}{m^3} \rho, \quad V(\rho) = \frac{2\pi a_s h^2}{m^3} \rho^2. \] (173)

where $a_s$ is the s-scattering length of the bosons. This potential arises from close contact interactions (see Sec. II A).

The corresponding equation of state is
\[ P = \frac{2\pi a_s h^2}{m^3} \rho, \quad c_s^2 = \frac{4\pi a_s h^2}{m^3} \rho. \] (174)

This is a polytropic equation of state of the form of Eq. 143 with $n = 1, \gamma = 2$ and $K = g/2 = 2\pi a_s h^2/m^3$. It is quadratic. The internal energy is given by
\[ U_2 = \frac{2\pi a_s h^2}{m^3} \int \rho^2 \, d\tau. \] (175)

The free energy and the average energy are
\[ F_2 = \Theta_c + \Theta_Q + W + W_{\text{ext}} + U_2, \] (176)
\[ N(E) = \Theta_c + \Theta_Q + 2W + W_{\text{ext}} + 2U_2. \] (177)

They satisfy the relation $F_2 = N\langle E \rangle - W - U_2$. The free energy can be written as
\[ F_2 = E_s - KS_2, \] (178)

where $K$ is a generalized temperature and
\[ S_2 = -\int \rho^2 \, d\tau \] (179)

is the Tsallis entropy of index $\gamma = 2$. We shall call it the quadratic entropy. This quadratic functional is similar to the enstrophy in two-dimensional turbulence [138].

2. Hydrodynamic representation

The generalized GP equation (172) is equivalent to the damped quantum Euler equations
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \] (180)
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{4\pi a_s h^2}{m^3} \nabla \rho - \nabla \Phi - \omega_0^2 \mathbf{r} - \frac{1}{m} \nabla Q - \xi \mathbf{u}. \] (181)

In the strong friction limit $\xi \to +\infty$, we obtain the quantum Smoluchowski equation
\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{2\pi a_s h^2}{m^3} \nabla \rho^2 + \rho \nabla \Phi + \rho \omega_0^2 \mathbf{r} + \frac{\rho}{m} \nabla Q \right). \] (182)

We note that it corresponds to an anomalous classical diffusion.

---

19 This equation of state was first derived by Bogoliubov [102] from a hard spheres model.
3. Hydrostatic equilibrium

The condition of hydrostatic equilibrium writes

\[
\frac{2\pi a_s h^2}{m^3} \nabla^2 \rho^2 + \rho \nabla \Phi + \rho \omega_0^2 r + \frac{\rho}{m} \nabla Q = 0. \tag{183}
\]

Combined with the Poisson equation, we obtain

\[
-4\pi a_s h^2 \Delta \rho + \frac{h^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4\pi G \rho + 3\omega_0^2. \tag{184}
\]

In the TF approximation \((Q = 0)\), the foregoing equation reduces to

\[
-4\pi a_s h^2 \Delta \rho = 4\pi G \rho + 3\omega_0^2. \tag{185}
\]

4. The equilibrium state

The minimization of the quadratic free energy at fixed mass (see Sec. III D) leads to the equation

\[
Q + m\Phi + \frac{1}{2}m\omega_0^2 r^2 + \frac{4\pi a_s h^2}{m^2} \rho = \mu. \tag{186}
\]

This equation is equivalent to the condition of hydrostatic equilibrium. It can be rewritten as

\[
\rho = \frac{m^2}{4\pi a_s h^2} \left( \mu - m\Phi - Q - \frac{1}{2} m\omega_0^2 r^2 \right), \tag{187}
\]

which can be interpreted as a generalized Tsallis distribution of index \(\gamma = 2\) including the contribution of the quantum potential. In the TF approximation \((Q = 0)\), we recover the Tsallis distribution of index \(\gamma = 2\) which is just an affine relationship.

5. The virial theorem

The scalar virial theorem writes

\[
\frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_Q + \Theta) + 3U_2 + W. \tag{188}
\]

In the strong friction limit, we get

\[
\frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_Q + 3U_2 + W. \tag{189}
\]

At equilibrium, the virial theorem, the free energy and the eigenenergy reduce to

\[
2\Theta_Q + 3U_2 + W - \omega_0^2 I = 0, \tag{190}
\]

\[
F_2 = \Theta_Q + W + \frac{1}{2} \omega_0^2 I + U_2, \tag{191}
\]

\[
NE = \Theta_Q + 2U_2 + 2W + \frac{1}{2} \omega_0^2 I. \tag{192}
\]
D. Logotropic equation of state: logarithmic entropy

1. Gross-Pitaevskii equation

The logotropic equation of state\(^\text{20}\)
\[ P = A \ln \rho, \quad c_s^2 = \frac{A}{\rho}, \]  \hspace{1cm} (193)
is associated with an effective potential of the form
\[ h(\rho) = -\frac{A}{\rho}, \quad V(\rho) = -A \ln \rho - A. \]  \hspace{1cm} (194)
The corresponding generalized GP equation is
\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{1}{2} m\omega_0^2 r^2 \psi - A m \frac{1}{|\psi|^2} \psi - i \hbar \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \langle \ln \left( \frac{\psi}{\psi^*} \rangle \right) \right] \psi. \]  \hspace{1cm} (195)
It has a hyperbolic nonlinearity. The internal energy is given by
\[ U_L = -A \int \ln \rho \, d\mathbf{r}. \]  \hspace{1cm} (196)
We have omitted a constant term proportional to the volume. The free energy and the average energy are
\[ F_L = \Theta_c + \Theta_Q + W + W_{\text{ext}} + U_L, \]  \hspace{1cm} (197)
\[ N \langle E \rangle = \Theta_c + \Theta_Q + 2W + W_{\text{ext}}. \]  \hspace{1cm} (198)
They satisfy the relation \( F_L = N \langle E \rangle - W + U_L. \) The free energy can be written as
\[ F_L = E_s - AS_L, \]  \hspace{1cm} (199)
where \( A \) is the logotropic temperature and
\[ S_L = \int \ln \rho \, d\mathbf{r} \]  \hspace{1cm} (200)
is the logarithmic entropy.\(^\text{91, 92, 140}\).

Remark: In order to have a dimensionless quantity in the logarithm, we can replace Eqs. (193), (196) and (200) by
\[ P = A \ln \left( \frac{\rho}{\rho_0} \right), \quad U_L = -A \int \ln \left( \frac{\rho}{\rho_0} \right) \, d\mathbf{r}, \quad S_L = \int \ln \left( \frac{\rho}{\rho_0} \right) \, d\mathbf{r}, \]  \hspace{1cm} (201)
where \( \rho_0 \) is a reference density.

2. Hydrodynamic representation

The generalized GP equation is equivalent to the damped quantum logotropic Euler equations
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \]  \hspace{1cm} (202)
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = A \nabla \left( \frac{1}{\rho} \right) - \nabla \Phi - \omega_0^2 \mathbf{r} - \frac{1}{m} \nabla Q - \xi \mathbf{u}. \]  \hspace{1cm} (203)
In the strong friction limit \( \xi \to +\infty \), we obtain the quantum logotropic Smoluchowski equation
\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( A \nabla \ln \rho + \rho \nabla \Phi + \rho \omega_0^2 \mathbf{r} + \frac{\rho}{m} \nabla Q \right). \]  \hspace{1cm} (204)

\(^{20}\) The logotropic equation of state was introduced in \text{139} in astrophysics. It was further discussed in \text{140} in the context of generalized thermodynamics and NFP equations. More recently, it was used in \text{91, 92} in cosmology. It can be viewed as a polytropic equation of state of the form \( P = K(\rho) - 1 \) with \( \gamma \to 0 \) and \( K \to \infty \) in such a way that \( A = \gamma K \) is finite \text{140}.\)
3. Hydrostatic equilibrium

The condition of hydrostatic equilibrium writes

\[ A \nabla \ln \rho + \rho \nabla \Phi + \rho \omega_0^2 r + \frac{\rho}{m} \nabla Q = 0. \] (205)

Combined with the Poisson equation [27], we obtain

\[ A \Delta \left( \frac{1}{\rho} \right) + \frac{\hbar^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = S_d \rho + d \omega_0^2. \] (206)

In the TF approximation \((Q = 0)\), the foregoing equation reduces to

\[ A \Delta \left( \frac{1}{\rho} \right) = S_d \rho + d \omega_0^2. \] (207)

4. The equilibrium state

The minimization of the logotropic free energy at fixed mass (see Sec. III D) leads to the equation

\[ Q + m \Phi + \frac{1}{2} m \omega_0^2 r^2 - \frac{A m}{\rho} = \mu. \] (208)

This equation is equivalent to the condition of hydrostatic equilibrium. It can be rewritten as

\[ \rho = \frac{A m}{m \Phi + Q + \frac{1}{2} m \omega_0^2 r^2 - \mu}, \] (209)

which can be interpreted as a generalized logotropic distribution including the contribution of the quantum potential. In the TF approximation \((Q = 0)\), we recover the logotropic distribution [91, 92, 139, 140]. For \( \Phi = 0 \), it reduces to the Lorentzian distribution.

5. The virial theorem

The scalar virial theorem writes

\[ \frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_c + \Theta_Q) - dU_L + W_{ii}, \] (210)

were we have used the identity \( \int P \, d\mathbf{r} = -U_L \) valid for a logotropic equation of state. In the strong friction limit, we get

\[ \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2 \Theta Q - dU_L + W_{ii}. \] (211)

At equilibrium, the virial theorem, the free energy and the eigenenergy reduce to

\[ 2 \Theta_Q - dU_L + W_{ii} - \omega_0^2 I = 0, \] (212)

\[ F_L = \Theta_Q + U_L + W + \frac{1}{2} \omega_0^2 I, \] (213)

\[ NE = \Theta_Q + 2W + \frac{1}{2} \omega_0^2 I. \] (214)
E. An improved form of Tsallis entropy

For a given equation of state \( P(\rho) \), the effective potential \( h(\rho) \) is defined up to an additive constant \( C_1 \) and the potential \( V(\rho) \) is defined up to a function of the form \( C_1 \rho + C_2 \). In the previous sections, we have determined the constants \( C_1 \) and \( C_2 \) in order to obtain the simplest expressions of \( h \) and \( V \). However, the expressions of \( h \) and \( V \) that we have given in Sec. V B (polytropes) do not exactly return the results of Sec. V A (isothermal distributions) in the limit \( \gamma \to 1 \) nor the results of Sec. V D (logotropes) in the limit \( \gamma \to 0 \), \( K \to +\infty \) and \( A = K\gamma \) finite because the constants \( C_1 \) and \( C_2 \) have not been calibrated for that purpose. In order to obtain unified results, we can define\(^{21}\)

\[
h(\rho) = \frac{K\gamma}{\gamma - 1} \left[ (\frac{\rho}{\rho_0})^{\gamma - 1} - 1 \right], \quad V(\rho) = \frac{K\rho_0}{\gamma - 1} \left[ \left( \frac{\rho}{\rho_0} \right)^{\gamma} - \frac{\rho}{\rho_0} + \gamma - 1 \right],
\]

\[
U_\gamma = \frac{K\rho_0}{\gamma - 1} \int \left[ \left( \frac{\rho}{\rho_0} \right)^{\gamma} - \frac{\rho}{\rho_0} + \gamma - 1 \right] \, dr,
\]

\[
S_\gamma = - \frac{\rho_0}{\gamma - 1} \int \left[ \left( \frac{\rho}{\rho_0} \right)^{\gamma} - \frac{\rho}{\rho_0} + \gamma - 1 \right] \, dr.
\]

For \( \gamma \to 1 \) and \( K = k_BT/m \), they reduce to

\[
h(\rho) = \frac{k_BT}{m} \ln \left( \frac{\rho}{\rho_0} \right), \quad V(\rho) = \frac{k_BT}{m} \rho \left[ \ln \left( \frac{\rho}{\rho_0} \right) - 1 \right] + \frac{k_BT}{m} \rho_0,
\]

\[
U = \frac{k_BT}{m} \int \left\{ \rho \left[ \ln \left( \frac{\rho}{\rho_0} \right) - 1 \right] + \rho_0 \right\} \, dr, \quad S = - \int \left\{ \rho \left[ \ln \left( \frac{\rho}{\rho_0} \right) - 1 \right] + \rho_0 \right\} \, dr.
\]

For \( \gamma \to 0 \) and \( K \to +\infty \) with \( A = K\gamma \) finite, they reduce to

\[
h(\rho) = -A \left( \frac{\rho_0}{\rho} - 1 \right), \quad V(\rho) = -A\rho_0 \left[ \ln \left( \frac{\rho}{\rho_0} \right) - \frac{\rho}{\rho_0} + 1 \right],
\]

\[
U = -A\rho_0 \int \left[ \ln \left( \frac{\rho}{\rho_0} \right) - \frac{\rho}{\rho_0} + 1 \right] \, dr, \quad S \sim \gamma\rho_0 \int \left[ \ln \left( \frac{\rho}{\rho_0} \right) - \frac{\rho}{\rho_0} + 1 \right] \, dr.
\]

We can interpret the entropy \( S_\gamma \) given by Eq. (216) as an improved form of Tsallis entropy that unifies the entropies associated with isothermal, polytropic and logotropic equations of state.

VI. A GENERALIZED BEC MODEL OF DARK MATTER HALOS

A. A generalized Gross-Pitaevskii equation

We propose to give a special emphasis to the following GPP equations

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{1}{2}m\omega_0^2 \psi^2 + 4\pi a_s \hbar^2 \left[ \left( \frac{\psi}{\psi^*} \right)^2 \psi + 2k_BT \ln |\psi| - \frac{\hbar^2}{2m} \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \int \left( \frac{\psi}{\psi^*} \right) \right] \right],
\]

\[
\Delta \Phi = 4\pi G|\psi|^2,
\]

that could provide a relevant BEC model of dark matter halos. This model takes into account the Heisenberg uncertainty principle (quantum potential), the self-gravity of the system, an external (harmonic) potential, the self-interaction of the bosons (scattering), an effective temperature, and a source of dissipation. The effective potential entering into the GP equation (221) is given by

\[
h(\rho) = \frac{k_BT}{m} \ln \rho + \frac{4\pi a_s \hbar^2}{m^3} \rho.
\]

\(^{21}\) We have introduced a reference density \( \rho_0 \) in the equations in order to have dimensionless quantities in the powers and in the logarithms, but we can take \( \rho_0 = 1 \) to make the connection with the results obtained in the previous sections.
It leads to an equation of state

\[ P = \frac{k_B T}{m} + \frac{2\pi a_s \hbar^2}{m^3} \rho^2 \]  

(224)

which has a linear part and a quadratic part. The linear part corresponds to the effective temperature and the quadratic part corresponds to the self-interaction of the bosons. This is a composite model with a core-halo structure (see Appendix H). The quadratic equation of state dominates in the core where the density is high and the isothermal equation of state dominates in the halo where the density is low. This leads to dark matter halos with a solitonic/BEC core surrounded by an isothermal envelope (see Sec. VII B). The internal energy is given by

\[ U = U_B + U_2, \]  

(225)

where \( U_B \) is given by Eq. (116) and \( U_2 \) is given by Eq. (175). The free energy and the average energy are

\[ F = \Theta_c + \Theta_Q + W + W_{\text{ext}} + U_B + U_2, \]  

(226)

\[ N\langle E \rangle = \Theta_c + \Theta_Q + 2W + W_{\text{ext}} + U_B + Nk_B T + 2U_2. \]  

(227)

They satisfy the relation \( F = N\langle E \rangle - W - Nk_B T - U_2 \).

**B. Mixed entropy: Boltzmann and Tsallis**

The free energy can be written as

\[ F = E_* - TS_B - KS_2, \]  

(228)

where \( S_B \) is the Boltzmann entropy (120) and \( S_2 \) in the Tsallis entropy (179) of index \( \gamma = 2 \) (quadratic entropy). We can also write

\[ F = E_* - KS_{\text{mix}} \]  

(229)

where

\[ S_{\text{mix}} = -\int \rho^2 \, d\mathbf{r} - \lambda \int \rho (\ln \rho - 1) \, d\mathbf{r} \]  

(230)

is a mixed entropy combining the Tsallis and Boltzmann entropies, and

\[ \lambda = \frac{k_B T}{K m} \]  

(231)

is the ratio between the ordinary temperature \( T \) and the polytropic temperature \( K = 2\pi a_s \hbar^2 / m^3 \). Alternatively, we can choose to include the internal energy \( U_2 \) in an “augmented” energy

\[ E_0 \equiv E_* + U_2 = \Theta_c + \Theta_Q + W + W_{\text{ext}} + U_2 \]  

(232)

and write the free energy as

\[ F = E_0 - TS_B, \]  

(233)

so that it only involves the Boltzmann entropy and the ordinary temperature.

*Remark:* Instead of considering a polytrope of index \( \gamma = 2 \), we could consider a polytrope of arbitrary index \( \gamma \). In that case, the mixed entropy takes the form

\[ S_{\text{mix}} = \frac{1}{\gamma - 1} \int \rho^\gamma \, d\mathbf{r} - \lambda \int \rho (\ln \rho - 1) \, d\mathbf{r}. \]  

(234)
C. Hydrodynamic representation

The generalized GPP equations (221) and (222) are equivalent to the damped quantum isothermal-polytropic EP equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \]  
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{k_B T}{m} \nabla \rho - \frac{4 \pi a_s \hbar^2}{m^3} \nabla \rho - \nabla \Phi - \frac{\omega_0^2}{m} \nabla Q - \xi u, \]  
\[ \Delta \Phi = 4 \pi G \rho. \]

In the strong friction limit \( \xi \to +\infty \), we obtain the quantum isothermal-polytropic SP equations

\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \frac{2 \pi a_s \hbar^2}{m^3} \nabla \rho^2 + \rho \nabla \Phi + \rho \omega_0^2 \nabla Q - \frac{\rho}{m} \nabla Q \right), \]
\[ \Delta \Phi = 4 \pi G \rho. \]

D. Hydrostatic equilibrium

The condition of hydrostatic equilibrium writes

\[ \frac{k_B T}{m} \nabla \rho + \frac{2 \pi a_s \hbar^2}{m^3} \nabla \rho^2 + \rho \nabla \Phi + \rho \omega_0^2 \nabla Q = 0. \]

Combined with the Poisson equation (27), we obtain

\[ -\frac{k_B T}{m} \Delta \ln \rho - \frac{4 \pi a_s \hbar^2}{m^3} \Delta \rho + \frac{1}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4 \pi G \rho + 3 \omega_0^2. \]

In the TF approximation \( Q = 0 \), we get

\[ -\frac{k_B T}{m} \Delta \ln \rho - \frac{4 \pi a_s \hbar^2}{m^3} \Delta \rho = 4 \pi G \rho + 3 \omega_0^2. \]

E. The equilibrium state

The minimization of the mixed free energy at fixed mass (see Sec. [111]) leads to the equation

\[ Q + m \Phi + \frac{1}{2} m \omega_0^2 r^2 + k_B T \ln \rho + \frac{4 \pi a_s \hbar^2}{m^2} \rho = \mu. \]

This equation is equivalent to the condition of hydrostatic equilibrium. It can be rewritten as

\[ \rho = \frac{m^2 k_B T}{4 \pi |a_s| \hbar^2 W} \left[ \frac{4 \pi |a_s| \hbar^2}{m^2 k_B T} e^{-\beta (m \Phi + Q + \frac{1}{2} m \omega_0^2 r^2 - \mu)} \right]. \]

In the case of repulsive interactions \( (a_s > 0) \), \( W(z) \) is the Lambert function defined implicitly by the equation \( W e^W = z \). Therefore, Eq. (244) can be interpreted as a generalized Lambert distribution including the contribution of the quantum potential. In the TF approximation \( Q = 0 \) we recover the Lambert distribution. In the case of attractive interactions \( (a_s < 0) \), \( W(z) \) is a new function defined implicitly by the equation \( W e^{-W} = z \). In that case, the density (244) is defined only for

\[ \frac{4 \pi |a_s| \hbar^2}{m^2 k_B T} e^{-\beta (m \Phi + Q + \frac{1}{2} m \omega_0^2 r^2 - \mu)} \leq \frac{1}{e}. \]
and Eq. (243) exhibits two different solutions. This interesting feature will be studied in a specific paper.\footnote{The same discussion applies to Sec. 7.2. of [112].}

Remark: It is actually possible to generalize these results to the case of a polytrope of arbitrary index $\gamma$ instead of $\gamma = 2$. In that case, we obtain

$$\rho = \left\{ \frac{k_B T}{|K| \gamma m} W \left[ \frac{|K| \gamma m}{k_B T} e^{-\beta (\gamma - 1) (m \Phi + Q + \frac{1}{2} m \omega_0^2 r^2 - \mu)} \right] \right\}^{\frac{1}{\gamma - 1}}.$$  \hspace{1cm} (246)

When $K > 0$, there is only one solution. When $K < 0$, there are two solutions when the term in bracket is less than $1/e$ and no solution otherwise.

F. The virial theorem

The scalar virial theorem writes

$$\frac{1}{2} \ddot{I} + \frac{1}{2} \dot{\xi} \dot{I} + \omega_0^2 I = 2(\Theta_c + \Theta_Q) + 3 N k_B T + 3 U_2 + W.$$  \hspace{1cm} (247)

In the strong friction limit, we get

$$\frac{1}{2} \dot{\xi} \dot{I} + \omega_0^2 I = 2 \Theta_Q + 3 N k_B T + 3 U_2 + W.$$  \hspace{1cm} (248)

At equilibrium, the virial theorem, the free energy and the eigenenergy reduce to

$$2 \Theta_Q + 3 N k_B T + 3 U_2 + W - \omega_0^2 I = 0,$$  \hspace{1cm} (249)

$$F = \Theta_Q + W + \frac{1}{2} \omega_0^2 I + U_B + U_2,$$  \hspace{1cm} (250)

$$NE = \Theta_Q + U_B + N k_B T + 2 U_2 + 2 W + \frac{1}{2} \omega_0^2 I.$$  \hspace{1cm} (251)

VII. DISCUSSION

A. Comparison with other works

The generalized BEC model (221) and (222) includes two new terms with respect to the standard BEC model (5) and (6): a temperature term $T$ and a friction term $\xi$. The standard BEC model is recovered when $T = \xi = 0$. In this section, we connect the generalized GP equation (221) to nonlinear Schrödinger equations that have been introduced in the past from different arguments.

A wave equation with a logarithmic nonlinearity $-b \ln |\psi|$ similar to the one present in Eq. (221), namely

$$i \hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - 2b \ln |\psi| \psi,$$  \hspace{1cm} (252)

has been introduced by Bialynicki-Birula & Mycielski\footnote{Bialynicki-Birula & Mycielski\cite{Bialynicki-Birula1975} have introduced a wave equation with a logarithmic nonlinearity $-b \ln |\psi|$ similar to the one present in Eq. (221), namely}

$$i \hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - 2b \ln |\psi| \psi.$$  \hspace{1cm} (252)

In that case, the logarithmic nonlinearity has a fundamental origin and the coefficient $b$ is interpreted as a fundamental constant of physics. In the interpretation of Bialynicki-Birula & Mycielski\cite{Bialynicki-Birula1975}, the Schrödinger equation is an approximation of this nonlinear wave equation. The coefficient $b$ can be positive or negative. When it is positive, it plays the role of an attractive interaction that can balance the repulsion due to the quantum potential and lead to a stationary solution of the wave equation with a Gaussian profile and a finite width called a gausson. The radius of the gausson is $R = \hbar/(2mb)^{1/2}$. The logarithmic nonlinearity may be a way to prevent the spreading of the wave packet in the Schrödinger equation. Of course, the constant $b$ must be
sufficiently small in order to satisfy the constraints set by laboratory experiments. Bialynicki-Birula and Mycielski \[141\] obtained \( b < 4 \times 10^{-10} \text{eV} \) which implies a bound to the electron soliton spatial width of \( 10 \mu \text{m} \). Shull et al. \[142\] obtained \( b < 3.4 \times 10^{-13} \text{eV} \). Finally, an upper limit \( b < 3.3 \times 10^{-15} \text{eV} \) was obtained by Gähler et al. \[143\] from precise measurements of Fresnel diffraction with slow neutrons. This implies a bound to the electron soliton spatial width of 3 mm. For \( m \rightarrow +\infty \), the density probability becomes a delta-function and the particle is localized. This means that a particle with a sufficiently big mass has a classical motion. In our approach, the logarithmic potential \( k_B T \ln |\psi| \) in Eq. (221) is obtained by looking for the generalized Schrödinger equation that leads, through the Madelung transformation, to a quantum Euler equation including a pressure term with an isothermal equation of state. Our approach gives another interpretation to the logarithmic Schrödinger equation (252) from a hydrodynamic representation of the wave equation (see Sec. II C) associated with an effective thermodynamic formalism (see Sec. III). Comparing Eqs. (221) and (252) we find that \( b = -k_B T \) so that a positive coefficient \( b \) corresponds to a negative effective temperature.

A Schrödinger equation with a damping term similar to Eq. (221), namely

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi - \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi, \tag{253}
\]

has been introduced by Kostin \[144\]. He derived it from the Heisenberg-Langevin equation describing a quantum Brownian particle interacting with a thermal bath environment. In our approach, the damping term is obtained by looking for the Schrödinger equation that leads, through the Madelung transformation, to a quantum Euler equation with a linear friction force proportional and opposite to the velocity. This gives another interpretation to the generalized Schrödinger equation (221) from a hydrodynamic representation of the wave equation (see Sec. II C) associated with an effective thermodynamic formalism (see Sec. III).

In Ref. \[146\], we have derived a generalized Schrödinger equation similar to Eq. (221) that unifies the logarithmic Schrödinger equation (252) and the damped Schrödinger equation (253). We have shown that the temperature and friction terms in the generalized Schrödinger equation (221) have a common origin and that they can be obtained from a unified description based on Nottale’s theory of scale relativity \[147\]. They satisfy a sort of fluctuation-dissipation theorem. In this approach, one can show that \( \xi \sim 1 \) while \( b = -k_B T \sim h \). Therefore \( b = -k_B T \sim h \) has its origin in quantum mechanics (it vanishes in the classical limit \( \hbar \rightarrow 0 \)) while \( \xi \sim 1 \) survives in the classical limit. The fact that \( b = -k_B T \) is proportional to the Planck constant may explain its small value and why it is not detectable in earth experiments. We also note, parenthetically, that the scaling \( T \sim h \) is similar to the one arising in the expression of the Hawking temperature \( k_B T = hc/4\pi R = hc^3/8\pi GM \) of the black holes. This may suggest a relation to quantum gravity.

### B. Physical interpretation: coarse-grained parametrization of gravitational cooling

The fluid equations (235)-(237) associated with the dissipative GPP equations (221) and (222) generalize the hydrodynamic equations of the CDM model by accounting for a quantum force due to the self-interaction of the bosons, an external harmonic potential, a temperature, a pressure force due to the self-interaction of the bosons, an external harmonic potential, a temperature, and a friction. The hydrodynamic equations of the CDM model by accounting for a quantum force due to the Heisenberg uncertainty principle, a pressure force due to the self-interaction of the bosons, an external harmonic potential, a temperature, and a friction.

### References

23 The motivation of Bialynicki-Birula & Mycielski \[141\] to introduce a logarithmic nonlinearity in the Schrödinger equation is that this term still satisfies the additivity property for noninteracting subsystems while solving the spreading of the wave packet problem. As we have seen in our effective thermodynamic formalism, this logarithmic nonlinearity is associated with the Boltzmann entropy. We note that a power-law nonlinearity \( -|\psi|^2(\gamma-1) \) corresponds to a polytropic equation of state, also solves the spreading of the wave packet problem but does not satisfy the additivity property for noninteracting subsystems. It is associated with the Tsallis entropy that has been introduced precisely in order to deal with non-extensive and non-additive systems \[144\]. Therefore, the Tsallis entropy may find applications in relation to nonlinear Schrödinger equations with a power-law nonlinearity.
We should write $\psi$ instead of $\psi$ to distinguish the coarse-grained wave function that is the solution of Eqs. (221) and (222) from the fine-grained wave function that is the solution of Eqs. (9) and (10) but, for commodity, we shall denote them with the same symbol.

In some cases, there is no equilibrium state (see, e.g., [35, 36, 76]). A complete study of these different behaviors will be given in a forthcoming paper (in preparation). In this interpretation, the flat rotation curves of the galaxies would have a fundamental origin related to the nonzero value of $b$.

In Appendix I, we propose an improved model in which the temperature $T$ fixes the size of the halos. The presence of the atmosphere solves the apparent paradox (see Appendix F of [90]) that the density of a self-gravitating isothermal halo decreases for $r < r_0$ as shown in [89, 90] in the context of the King model. In the present model, this confinement can be taken into account by the harmonic potential. Another possibility would be to change the nonlinearity in the generalized GPP equations (221) and (222), or equivalently the equation of state of the damped quantum EP equations (235)-(237), so as to yield a density profile decreasing as $r^{-3}$ at large distances similarly to the NFW and Burkert profiles. However, an isothermal halo can account heuristically for an atmosphere of scalar radiation. Therefore, Eqs. (221) and (222) can provide a relevant parametrization of dark matter halos experiencing gravitational cooling. As mentioned in the Introduction, it is the atmosphere of scalar radiation that fixes the size of the halos. The presence of the atmosphere solves the apparent paradox (see Appendix F of [90]) that BEC halos at $T = 0$ should all have the same radius (in the self-interacting case) or that their radius should decrease with their mass as $R \propto M^{-1}$ (in the non-interacting case), in contradiction with the observations that reveal that their radius increases with their mass as $R \propto M^{1/2}$ corresponding to a constant surface density (see [53], [84], [90]). In Appendix I we propose an improved model in which the temperature $T(t)$ changes with time in order to exactly conserve the energy $E_0$ of the (fine-grained) GPP equations (9) and (10).

C. Speculations: fundamental wave equations

We would like to close this discussion by making speculations. We have previously justified the generalized BEC model (221) and (222) as an effective coarse-grained description of a dissipationless BEC at $T = 0$ described by Eqs. (9) as undergoing a process of gravitational cooling. In that case, $\xi$ is an effective friction which accounts for the relaxation process. On the other hand, $T$ is an effective temperature which represents the temperature of the halo made of scalar radiation. Alternatively, following Bialynicki-Birula & Mycielski [141], we may argue that the generalized GP equation (221) is a fundamental equation of physics from which the standard GP equation (3) is an approximation. The new terms in Eq. (221) may arise from the interaction of the system with an external medium, a sort of aether. In that case, $\xi$ represents the friction with the aether and $T$ represents the temperature of the aether. The constant $b = -k_B T$ in Eq. (221) may naturally account for the nonzero temperature of the halos and for the flat rotation curves of the galaxies. In this interpretation, the flat rotation curves of the galaxies would have a fundamental origin related to the nonzero value of $b = -k_B T$ in the generalized GP equation (221). This interpretation can be correct only if $b$ is very small so that it is not detectable in earth laboratory experiments and manifests itself only on...
astrophysical or cosmological scales. This actually sets a constraint on the mass m of the bosons that may compose dark matter halos as we now show.

An isothermal halo leads to a flat rotation curve with an asymptotic circular velocity \( v_c^2(r) = GM(r)/r \) given by

\[
v_c^2 \to \frac{2k_B T}{m}.
\]  

(254)

For a typical galaxy such as the Medium Spiral, the circular velocity is of the order of \( v_c = 150 \text{ km/s} \). In Ref. [90], we have estimated the mass of the bosons that may compose dark matter halos by assuming that the smallest observed halos such as Willman 1 are completely condensed (ground state). For noninteracting bosons, we found \( m = 2.57 \times 10^{-20} \text{ eV}/c^2 \). From Eq. [254], we obtain \( T = 3.73 \times 10^{-23} \text{ K} \) and \( k_B T = -b = 3.22 \times 10^{-27} \text{ eV} \). This value of \( b \) is below the experimental bounds found in [141, 143] so that an ultra small boson mass is acceptable.\(^{27}\) For self-interacting bosons, we found a maximum boson mass \( m = 1.69 \times 10^{-2} \text{ eV}/c^2 \). From Eq. [254], we obtain \( T = 2.45 \times 10^{-5} \text{ K} \) and \( k_B T = -b = 2.115 \times 10^{-9} \text{ eV} \). This value of \( b \) is above the experimental bounds found in [141, 143] so that a too large boson mass must be rejected. Using the constraint \( k_B T = -b < 3.3 \times 10^{-15} \text{ eV} \) \(^{143}\), we get \( T < 3.83 \times 10^{-11} \text{ K} \) leading to a boson mass \( m < 2.64 \times 10^{-8} \text{ eV}/c^2 \).

It would be fascinating to find that the flat rotation curves of dark matter halos arise from a fundamental nonlinear term in the Schrödinger equation that cannot be detected in earth experiments but that would have an effect at large (cosmological) scales. This is not completely impossible if the mass of the bosons is ultra small (as shown above) so that \( T \) and \( b \) are also very small, beyond experimental reach on earth. There is nevertheless is difficulty with this scenario. Indeed, if the temperature of the halos is related to a fundamental constant \( b \) in the Schrödinger equation, i.e., if \( T \) is a fundamental constant representing the temperature of the aether, the halos should all have the same temperature, or the same value of \( v_c \), which is not the case. Indeed, it appears that the temperature of the halos increases linearly with their size, namely \( T \propto R \) (see [80, 83, 90].)\(^{28}\) We do not know at that stage how to escape this difficulty. This suggests that our interpretation of the generalized BEC model [221] and [222] as an effective equation describing gravitational cooling is more relevant than its interpretation in terms of a fundamental wave equation. In the first interpretation, \( T \) and \( \xi \) are effective coefficients determined by the efficiency of gravitational cooling allowing them to change from halo to halo. However, even in this case, we have to explain why the (effective) temperature increases linearly with the size of the halos.

Remark: In Ref. [91, 92], we have shown that many properties of dark matter halos (in particular the universal value of their surface density \( \Sigma_0 \)) can be remarkably well explained by assuming that their equation of state is logotropic (see Sec. 7.1). As a result, we are led to postulating that the fundamental nonlinear wave equation (from which the Schrödinger equation would be an approximation) is the hyperbolic GP equation

\[
\frac{i\hbar}{2m} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi - A m \frac{1}{|\psi|^2} \psi
\]  

(255)

instead of the logarithmic GP equation [222]. In this equation, \( A \) should be regarded as a new fundamental constant of physics (superseeding the cosmological constant). In [91, 92], we have shown that its value is \( A = B \rho_A c^2 = 1.3 \times 10^{-9} \text{ g m}^{-1} \text{s}^{-2} \), where \( \rho_A = 6.72 \times 10^{-24} \text{ g m}^{-3} \) is the cosmological density and \( B = 3.53 \times 10^{-3} \) is a dimensionless constant approximately equal to \( B \approx 1/[123 \ln(10)] \) where 123 is the famous number appearing in the ratio between the Planck and the cosmological density: \( \rho_P / \rho_A \sim 10^{123} \). We note that the energy scale associated with the last term in Eq. (255) is of the order of \( A m |\psi|^2 \sim (\rho_A / \rho) mc^2 \). In view of the smallness of \( \rho_A \), this term is completely negligible for the typical values of the density \( \rho \gg \rho_A \) considered in laboratory experiments. However, this term becomes important on cosmological scales where \( \rho \sim \rho_A \) and could explain the structure of the core of dark matter halos. When applied to dark matter halos, the logotropic model [91, 92] yields a universal rotation curve that coincides, up to the halo radius \( r_h \), with the empirical Burkert \(^{54}\) profile that fits a lot of observational rotation curves. However, the density of a self-gravitating logotropic halo decreases for \( r \to +\infty \) as \( \rho(r) \sim (A/\sqrt{2G})^{1/2} r^{-1} \) (corresponding to an accumulated mass \( M(r) \sim (A\pi/2G)^{1/2} r^2 \)). Therefore, this profile has an infinite mass \( 1.40 \times 10^{14} \text{ M}_\odot \).

As explained previously, the confinement of dark matter halos may be due to complicated physical processes such as incomplete relaxation, evaporation, stochastic forcing from the external environment etc. As a result, the density profiles of the halos decrease at large distances as \( r^{-3} \) like the NFW \(^{83}\) and Burkert \(^{54}\) profiles instead of \( r^{-1} \) as predicted by the logotropic model [91, 92]. This extra confinement could be taken into account in our model by

\(^{27}\) We note that the results of [141, 143] assume \( b > 0 \) while, in the present case, \( b < 0 \). Therefore, our comparisons are only indicative.

\(^{28}\) It is an observational evidence that the dark matter halos have the same surface density \( \Sigma_0 = \rho_0 r_h = 141 \text{ M}_\odot/\text{pc}^2 \) \(^{143}\). This implies that their mass and temperature scale with the radius as \( M_h \sim \rho_0 r_h^3 \propto r_h^2 \) and \( k_B T \sim GM_h m/r_h \propto r_h \).
introducing a harmonic potential in Eq. (255) or by using a more complicated nonlinearity in the generalized GP equation, or a more complicated equation of state in the corresponding EP equations, that yields a logotropic profile in the core and a profile in the halo decreasing as $r^{-3}$ at large distances, like for the NFW [83] and Burkert profiles [84].

VIII. THE GAUSSIAN ANSATZ

A. The generalized Gross-Pitaevskii-Poisson equations

In order to obtain exact results, the generalized GPP equations (7) and (8) must be solved numerically. However, we can obtain approximate analytical results by making a Gaussian ansatz for the wave function. In order to be sufficiently general, we consider the GPP equations

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{1}{2} m \omega_0^2 r^2 \psi + \frac{K}{\gamma - 1} |\psi|^{2(\gamma - 1)} \psi + 2k_B T \ln |\psi| \psi - i\hbar \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \langle \ln \left( \frac{\psi}{\psi^*} \right) \rangle \right] \psi,$$  

(256)

$$\Delta \Phi = S_d G |\psi|^2,$$  

(257)

in a space of dimension $d$. They take into account the Heisenberg uncertainty principle (quantum potential), the self-gravity of the system, an external harmonic potential, a power-law self-interaction, an effective temperature, and a source of dissipation. All the relevant equations associated with this model can be obtained from the results of Secs. [VA] and [VB] by adding the contribution of the isothermal and polytropic equations of state as we did in Sec. [VI] for the particular case $\gamma = 2$ (see the remark at the end of Sec. [VI B] and Appendix [H]). For example, the equation of state writes

$$P = K \rho^\gamma + \frac{k_B T}{m}.$$  

(258)

The free energy and the average energy are given by

$$F = \Theta_e + \Theta_Q + W + W_{\text{ext}} + U_B + U,$$  

(259)

$$N \langle E \rangle = \Theta_e + \Theta_Q + 2W + W_{\text{ext}} + U_B + Nk_B T + \gamma U.$$  

(260)

They satisfy $F = N \langle E \rangle - W - Nk_B T - (\gamma - 1)U$. The scalar virial theorem takes the form

$$\frac{1}{2} \dot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_e + \Theta_Q) + dNk_B T + d(\gamma - 1)U + W_{ii}.$$  

(261)

In the strong friction limit, we get

$$\frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_Q + dNk_B T + d(\gamma - 1)U + W_{ii}.$$  

(262)

At equilibrium, the virial theorem, the free energy and the eigenenergy reduce to

$$2\Theta_Q + dNk_B T + d(\gamma - 1)U + W_{ii} - \omega_0^2 I = 0,$$  

(263)

$$F = \Theta_Q + W + W_{\text{ext}} + U_B + U,$$  

(264)

$$NE = \Theta_Q + 2W + W_{\text{ext}} + U_B + Nk_B T + \gamma U.$$  

(265)
B. The free energy

We shall calculate the free energy functional \( \mathcal{F} \) by making a Gaussian ansatz for the wave function

\[
\psi(r,t) = \left[ \frac{2M}{S_d \Gamma(d/2) R(t)^d} \right]^{1/2} e^{-\frac{r^2}{2R(t)^2}} e^{imH(t)r^2/2\hbar} e^{iS_0(t)/\hbar},
\]

where \( R(t) \) measures the size of the system (wave packet). It corresponds to the typical radius of the BEC. The wave function is normalized such that \( \int |\psi|^2 \, dr = M \). \( \Gamma(x) \) is the Gamma function: \( \Gamma(3/2) = \sqrt{\pi}/2 \) in \( d = 3 \), \( \Gamma(1) = 1 \) in \( d = 2 \), and \( \Gamma(1/2) = \sqrt{\pi} \) in \( d = 1 \). Comparing Eq. (266) with Eq. (15), we find that the density and the action are given by

\[
\rho(r,t) = \frac{2M}{S_d \Gamma(d/2) R(t)^d} e^{-\frac{r^2}{2R(t)^2}}
\]

and

\[
S(r,t) = \frac{1}{2} mH(t)r^2 + S_0(t).
\]

The velocity defined by Eq. (17) is then given by

\[
u(r,t) = H(t)r.
\]

It is proportional to the radial distance \( r \) with a proportionality constant \( H(t) \) depending only on time. It is shown in Appendix J that Eqs. (267) and (268) constitute an exact solution of the continuity equation (25) provided that \( H = \dot{R}/R \).

Equation (269) is similar to the Hubble parameter in cosmology (see Sec. VIII H). Using the Gaussian ansatz, one can show that (see Appendix K):

\[
I = \alpha M R^2, \quad \Omega_c = \frac{1}{2} \alpha M \left( \frac{dR}{dt} \right)^2, \quad \Omega_Q = \frac{\hbar^2 M}{m^2 R^2}, \quad U = \frac{K \zeta}{\gamma - 1} \left( \frac{M}{R^d} \right)^\gamma R^d,
\]

\[
UB = -a \frac{k_BT}{m} M \ln R + C, \quad W_{\text{ext}} = \frac{1}{2} \omega_0^2 \alpha M R^2,
\]

\[
W = -\frac{\nu}{d - 2} \frac{GM^2}{R^{d-2}} \quad (d \neq 2), \quad W = \frac{1}{2} GM^2 \ln R + W_0 \quad (d = 2),
\]

\[
W_{ii} = -\nu \frac{GM^2}{R^{d-2}},
\]

with the coefficients

\[
\alpha = \frac{d}{2}, \quad \sigma = \frac{d}{4}, \quad \zeta = \left[ \frac{2}{S_d \Gamma(d/2)} \right]^{\gamma - 1} \frac{1}{\gamma^{d/2}}, \quad \nu = \frac{1}{\Gamma(d/2)2^{d/2}},
\]

\[
C = \frac{k_BT}{m} M \left( \ln \left( \frac{2M}{S_d \Gamma(d/2)} \right) - 1 - \alpha \right), \quad W_0 = \frac{1}{4} (\ln 2 - \gamma_E)GM^2 = 0.0289828...GM^2.
\]

We have assumed that \( \gamma > 0 \) otherwise the internal energy \( U \) is infinite within the Gaussian ansatz. For \( d = 2 \), we note that \( \nu = 1/2 \).

Using these results, the free energy functional (259) can be written as a function of \( R \) and \( \dot{R} \) (for a fixed mass \( M \)) as

\[
F = \frac{1}{2} \alpha M \left( \frac{dR}{dt} \right)^2 + V(R),
\]
with
\[ V(R) = \frac{\hbar^2 M}{m^2 R^2} - \frac{\nu}{d-2} \frac{GM^2}{R^{d-2}} + \frac{1}{2} \omega_0^2 \alpha M R^2 + \frac{\zeta}{\gamma - 1} \frac{K M^\gamma}{R^{d(\gamma - 1)}} - d \frac{k_B T}{m} \ln R + C \quad (d \neq 2), \] (277)

\[ V(R) = \frac{\hbar^2 M}{m^2 R^2} + \frac{1}{2} GM^2 \ln R + \frac{1}{2} \omega_0^2 \alpha M R^2 + \frac{\zeta}{\gamma - 1} \frac{K M^\gamma}{R^{d(\gamma - 1)}} - 2 \frac{k_B T}{m} \ln R + W_0 \quad (d = 2). \] (278)

Equation (276) can be interpreted as the total energy of a fictive particle with effective mass \( \alpha M \) and position \( R \) moving in a potential \( V(R) \). The first term is the classical kinetic energy \( \Theta \), and the second term is the potential energy \( V \) including the quantum kinetic energy \( \Theta_q \), the gravitational potential energy \( W \), the external potential energy \( W_{\text{ext}} \), and the internal energies \( U \) and \( U_B \). We shall come back to this mechanical analogy in Sec. VIIID.

Remark: The average energy \( \langle E \rangle \) can be obtained similarly from Eq. (276). At equilibrium, it reduces to the eigenenergy (265) which, within the Gaussian ansatz, takes the form

\[ NE = \frac{\hbar^2 M}{m^2 R^2} - \frac{2\nu}{d-2} \frac{GM^2}{R^{d-2}} + \frac{1}{2} \omega_0^2 \alpha M R^2 + \frac{\zeta}{\gamma - 1} \frac{K M^\gamma}{R^{d(\gamma - 1)}} - d \frac{k_B T}{m} \ln R + Nk_B T + C \quad (d \neq 2), \] (279)

\[ NE = \frac{\hbar^2 M}{m^2 R^2} + GM^2 \ln R + \frac{1}{2} \omega_0^2 \alpha M R^2 + \frac{\zeta}{\gamma - 1} \frac{K M^\gamma}{R^{d(\gamma - 1)}} - 2 \frac{k_B T}{m} \ln R + Nk_B T + W_0 \quad (d = 2). \] (280)

C. The mass-radius relation

We have seen in Sec. IIIID that a stable equilibrium state of the generalized GPP equations is a minimum of free energy \( F[\alpha, u] \), given by Eq. (269), at fixed mass \( M \). Within the Gaussian ansatz, we are led to determining the minimum of the function \( F(R, \dot{R}) \), given by Eq. (276), at fixed mass \( M \). Clearly, we must have \( \dot{R} = 0 \), implying that a minimum of energy at fixed mass is a static state. Then, we must determine the minimum of the potential energy \( V(R) \). Computing the first derivative of \( V(R) \) giving

\[ V'(R) = -2\frac{\hbar^2 M}{m^2 R^3} + \frac{\nu GM^2}{R^{d-1}} + \omega_0^2 \alpha M R - d \frac{K R^{d(\gamma - 1) + 1}}{K M^\gamma} - d \frac{k_B T}{m R}, \] (281)

and writing \( V'(R) = 0 \), we obtain the mass-radius relation

\[ -2\frac{\hbar^2 M}{m^2 R^3} + \frac{\nu GM^2}{R^{d-1}} + \omega_0^2 \alpha M R - d \frac{K R^{d(\gamma - 1) + 1}}{K M^\gamma} - d \frac{k_B T}{m R} = 0 \] (282)

or, equivalently,

\[ \omega_0^2 R^{d(\gamma - 1) + 2} + \nu GM R^{d(\gamma - 1) + 2 - d} - 2\frac{\hbar^2 M}{m^2 R^{d(\gamma - 1) - 2}} - dK M^\gamma - d \frac{k_B T}{m R^{d(\gamma - 1)}} = 0. \] (283)

These relations may also be obtained from the equilibrium virial theorem (263) by making the Gaussian ansatz (see Sec. VIIID). A critical point of \( V(R) \), satisfying \( V'(R) = 0 \), is a free energy minimum if, and only if, \( V''(R) > 0 \). Computing the second derivative of \( V(R) \) and using the mass-radius relation (282), we get

\[ V''(R) = 6\frac{\hbar^2 M}{m^2 R^4} - (d - 1)\nu \frac{GM^2}{R^{d-1}} + \omega_0^2 M + [d(\gamma - 1) + 1]d \frac{K R^{d(\gamma - 1) + 2}}{K M^\gamma} + d \frac{k_B T}{m R^{2}}. \] (284)

D. The virial theorem

Using the Gaussian ansatz, the time-dependent virial theorem (201) can be written as

\[ \frac{1}{2} \alpha M \frac{d^2 R^2}{dt^2} + \frac{1}{2} \xi \alpha M \frac{dR^2}{dt} + \omega_0^2 \alpha M R^2 = \alpha M \left( \frac{dR^2}{dt} \right)^2 + 2\frac{\hbar^2 M}{m^2 R^2} + d \frac{k_B T}{m} + d \frac{K M^\gamma}{R^{d(\gamma - 1)}} - \nu \frac{GM^2}{R^{d-2}}. \] (285)
Since
\[
\frac{d^2 R^2}{dt^2} = 2R \frac{d^2 R}{dt^2} + 2 \left( \frac{dR}{dt} \right)^2 ,
\] (286)
we note the nice cancelation of terms in Eq. (285) leading to the final equation
\[
\alpha M \frac{d^2 R}{dt^2} + \xi \alpha M \frac{dR}{dt} + \alpha \omega_0^2 M R = 2a \frac{\hbar}{m^2 R^3} + \frac{dMk_B T}{mR} + d\xi \frac{K M^\gamma}{R^{d(\gamma-1)+1}} - \nu \frac{GM^2}{R^{d-1}}. \tag{287}
\]
The equilibrium virial theorem ($\ddot{R} = \dot{R} = 0$) returns the mass-radius relation (282) obtained from the condition $dV/dR = 0$. In fact, the time-dependent virial theorem (287) can be written as
\[
\alpha M \frac{d^2 R}{dt^2} + \xi \alpha M \frac{dR}{dt} = -\frac{dV}{dR}. \tag{288}
\]
This equation describes the damped motion of a fictive particle with effective mass $\alpha M$ and position $R$ in a potential $V(R)$. From Eq. (288), we find that the rate of change of the free energy $F = \Theta c + V$ defined by Eq. (276) is given by
\[
\frac{dF}{dt} = -\xi \alpha M \left( \frac{dR}{dt} \right)^2 \leq 0. \tag{289}
\]
This equation can be obtained directly from the $H$-theorem (76) by making the Gaussian ansatz. For $\xi > 0$ the free energy is dissipated and for $\xi = 0$ it is conserved. In this mechanical analogy, a stable equilibrium state corresponds to a minimum of $V(R)$ as we have previously indicated. In the dissipative case ($\xi > 0$), the system relaxes towards an equilibrium state for $t \to +\infty$ by exhibiting damped oscillations due to the friction. As explained in Sec. VII, this may account for the process of gravitational cooling. In the dissipationless case ($\xi = 0$), the system oscillates permanently, without reaching equilibrium, except if it is prepared in a very particular initial state.\textsuperscript{29}

In the strong friction limit $\xi \to +\infty$, the equation of motion (288) reduces to
\[
\xi \alpha M \frac{dR}{dt} = -\frac{dV}{dR}. \tag{290}
\]
Furthermore, the free energy is $F = V$. Therefore, we obtain
\[
\frac{dF}{dt} = \frac{dV}{dt} = V'(R) \frac{dR}{dt} = -\frac{1}{\xi \alpha M} \left( \frac{dV}{dR} \right)^2 \leq 0. \tag{291}
\]
The free energy decreases and the system relaxes towards an equilibrium state which corresponds to the minimum of $V(R)$. In that case, the relaxation is overdamped, showing no oscillation.

Remark: We can get the equation of dynamics (288) in a different manner, without using the virial theorem. In the dissipationless case, the free energy $F$ is conserved. Canceling the time derivative of Eq. (276), we obtain Eq. (288) with $\xi = 0$. In the dissipative case, taking the time derivative of Eq. (276) and using the $H$-theorem of Eq. (289) which can be obtained from Eq. (70), we obtain Eq. (288) with $\xi \neq 0$.

E. The general solution of the problem

For dissipationless systems ($\xi = 0$), the equation of motion (288) reduces to
\[
\alpha M \frac{d^2 R}{dt^2} = -\frac{dV}{dR}. \tag{292}
\]
\textsuperscript{29} We note that the Gaussian ansatz is not relevant to describe the dynamics of the conservative GPP equations (5) and (6) because it would lead to undamped everlasting oscillations and would not account for the process of gravitational cooling. By contrast, the Gaussian ansatz is able to describe the dynamics of the dissipative GPP equations (256) and (257) that already incorporate the process of gravitational cooling.
The free energy $F$ is conserved ($\dot{F} = 0$), leading to the first integral of motion \[ F(t) = F(0), \] where $F$ is a constant. The general solution of this equation is

$$\int_{R_0}^{R(t)} \frac{dR}{\sqrt{F - V(R)}} = \pm \left( \frac{2}{\alpha M} \right)^{1/2} t, \quad (293)$$

with $+$ for solutions describing an expansion (evaporation) and $-$ for solutions describing a contraction (collapse).

For overdamped systems ($\xi \to +\infty$), the equation of motion (288) reduces to Eq. (291) whose general solution is

$$\int_{R_0}^{R(t)} \frac{dR}{-V'(R)} = \frac{t}{\xi \alpha M}. \quad \text{(294)}$$

F. The pulsation equation

To study the linear dynamical stability of an equilibrium state of Eq. (288), we make a small perturbation about that state and write $R(t) = R_0 + \epsilon(t)$ where $R$ is the equilibrium radius and $\epsilon(t) \ll R$ is the perturbation. Using $V'(R) = 0$ and keeping only terms that are linear in $\epsilon$, we obtain the equation

$$\frac{d^2 \epsilon}{dt^2} + \xi \frac{d\epsilon}{dt} + \omega^2 \epsilon = 0, \quad \text{(295)}$$

where

$$\omega^2 = \frac{1}{\alpha M} V''(R). \quad \text{(296)}$$

Looking for solutions of Eq. (295) under the form $\epsilon \sim e^{\lambda t}$, we obtain $\lambda^2 + \xi \lambda + \omega^2 = 0$ yielding $\lambda_{\pm} = \left( -\xi \pm \sqrt{\xi^2 - 4\omega^2} \right) / 2$. If $\omega^2 > \xi^2 / 2$, the two modes ($\pm$) are damped at a rate $-\xi/2 < 0$ and oscillate with a pulsation $\pm \sqrt{4\omega^2 - \xi^2 / 2}$. If $0 < \omega^2 < \xi^2 / 4$, the two modes are damped at a rate $(-\xi \pm \sqrt{\xi^2 - 4\omega^2}) / 2 < 0$. If $\omega^2 < 0$, one mode is damped at a rate $(-\xi - \sqrt{\xi^2 - 4\omega^2}) / 2 < 0$ and the other one grows at a rate $(-\xi + \sqrt{\xi^2 - 4\omega^2}) / 2 > 0$. If $\omega^2 = \xi^2 / 4$, the perturbation decays $\epsilon = (A_0 + B_0)e^{-\xi t / 2}$. For dissipationless systems ($\xi = 0$), the perturbation oscillates with a pulsation $\pm \omega$ if $\omega^2 > 0$ and grows at a rate $\sqrt{-\omega^2} > 0$ if $\omega^2 < 0$ (the other mode is damped at a rate $-\sqrt{-\omega^2} < 0$). For overdamped systems ($\xi \to +\infty$), the perturbation is damped at a rate $-\omega^2 / \xi < 0$ if $\omega^2 < 0$ and grows at a rate $-\omega^2 / \xi > 0$ if $\omega^2 > 0$. In conclusion, the equilibrium state of Eq. (288) is linearly stable if, and only if, $\omega^2 > 0$ that is to say if, and only if, it is a (local) minimum of the potential energy $V(R)$.

For a self-gravitating BEC satisfying $\omega^2 > \xi^2 / 4$, the radius of the condensate evolves as

$$R(t) = R_0 + (R_0 - R)e^{-\xi t / 2} \cos \left( \frac{\sqrt{4\omega^2 - \xi^2}}{2} t \right), \quad \text{(297)}$$

where $R_0$ is the initial radius and $R$ the equilibrium radius. The BEC undergoes damped oscillations and relaxes towards an equilibrium state for $t \to +\infty$. These damped oscillations may account for the process of gravitational cooling. They can explain how a self-gravitating BEC achieves an equilibrium state by dissipating free energy.

Using Eqs. (284) and (295), we find that the complex pulsation is given by

$$\omega^2 = \omega_0^2 + \frac{6\alpha}{\alpha M^2 R^2} + \left[ d(\gamma - 1) + 1 \right] \frac{K M \gamma^{-1}}{R^d (\gamma - 1) + 2} - \frac{(d - 1) \nu GM}{\alpha R^d} + \frac{d M \lambda B T}{\alpha m R^2}. \quad \text{(298)}$$

Using Eqs. (270) and (273) it can be expressed under the form

$$\omega^2 = \frac{6 \Theta Q + [d(\gamma - 1) + 1] d(\gamma - 1) U + (d - 1) W_{ij} + \omega_0^2 I + d N k_B T}{I}. \quad \text{(299)}$$

Alternative expressions of the pulsation can be obtained by using the equilibrium virial theorem (263) or the equilibrium free energy (264). This may be useful in the dissipationless case ($\xi = 0$) where $F$ is conserved. Let us consider particular cases.
For classical polytropes ($\Theta_Q = T = 0$), the virial theorem reduces to $d(\gamma - 1)U + W_{ii} - \omega_0^2 I = 0$ and the complex pulsation can be written as

$$\omega^2 = (2d - 2 - d\gamma) \frac{W_{ii}}{I} + (d\gamma - d + 2)\omega_0^2.$$  \hfill (300)

For $d = 3$ and $\omega_0 = 0$, using $W_{ii} = W$, we recover the usual Ledoux formula $\omega^2 = (4 - 3\gamma)W/I$ [149].

For classical isothermal spheres ($\Theta_Q = U = 0$), the virial theorem reduces to $W_{ii} - \omega_0^2 I + dNk_BT = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{(d-2)W_{ii} + 2\omega_0^2}{I} \quad \text{or} \quad \omega^2 = \frac{(d-2)dNk_BT}{I} + d\omega_0^2.$$  \hfill (301)

For $d = 2$, we obtain $\omega^2 = 2\omega_0^2$ and the virial theorem leads to the identity [139].

In the noninteracting case ($U = 0$), the virial theorem reduces to $2\Theta_Q + W_{ii} - \omega_0^2 I + dNk_BT = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{(d - 4)W_{ii} + 4\omega_0^2 I - 2dNk_BT}{I}.$$  \hfill (302)

For nongravitational ($G = 0$) polytropes ($T = 0$), the virial theorem reduces to $2\Theta_Q + d(\gamma - 1)U - \omega_0^2 I = 0$ and the complex pulsation can be written as

$$\omega^2 = \frac{d(\gamma - 1)[d(\gamma - 1) - 2]U + 4\omega_0^2 I}{I}.$$  \hfill (303)

For the critical index $\gamma_c = 1 + 2/d$ [137], we obtain $\omega^2 = 4\omega_0^2$ in agreement with Eq. (170). In the TF approximation ($\Theta_Q = 0$), the virial theorem reduces to $d(\gamma - 1)U - \omega_0^2 I = 0$ and the complex pulsation becomes $\omega^2 = [d(\gamma - 1) + 2]\omega_0^2$.

\section{G. The Poincaré theorem}

If we differentiate the mass-radius relation [282] with respect to $R$ and substitute the result into Eq. (283), we obtain

$$V''(R) = \left(\frac{2\sigma h^2}{m^2 R^3} - \frac{2\nu GM}{R^{d-1}} - \omega_0^2 \alpha R + \frac{dK\zeta \gamma M^{-1}}{R^{d(\gamma - 1) + 1}} + \frac{dNk_BT}{mR}\right) \frac{dM}{dR} = 0.$$  \hfill (304)

Using Eqs. (282) and (286), the foregoing equation can be rewritten as \cite{30}

$$\omega^2(R) = -\frac{1}{\alpha M} \left(\frac{2\sigma h^2}{m^2 R^3} - \omega_0^2 \alpha R + \frac{dK\zeta (2 - \gamma)M^{-1}}{R^{d(\gamma - 1) + 1}} + \frac{dNk_BT}{mR}\right) \frac{dM}{dR}.$$  \hfill (305)

This equation relates the square of the complex pulsation $\omega^2$ determining the stability of the system to the slope of the mass-radius relation $M(R)$. We see that the change of stability along the series of equilibria ($\omega^2 = V''(R) = 0$) coincides with the turning point of mass ($M'(R) = 0$) in agreement with the Poincaré theorem.

\textit{Remark:} More generally, the mass-radius relation $M(R)$ is given in implicit form by

$$\frac{dV}{dR}(R, M(R)) = 0.$$  \hfill (306)

Differentiating this relation with respect to $R$, we get

$$\frac{\partial^2 V}{\partial R^2}(R, M) + \left.\frac{\partial^2 V}{\partial R \partial M}\right|_{(R, M)} \frac{dM}{dR} = 0,$$  \hfill (307)

which is the generalization of Eq. [301]. We first note that the turning point of mass ($M'(R) = 0$) corresponds to $\partial^2 V/\partial R^2 = 0$. On the other hand, the turning point of radius ($R'(M) = 0$) corresponds to $\partial^2 V/\partial R \partial M = 0$. A change of stability takes place when $\partial^2 V/\partial R^2$ changes sign, i.e. when $M'(R)$ changes sign while $\partial^2 V/\partial R \partial M$ does not change sign. This happens at a turning point of mass, not at a turning point of radius. As a result, the stability of the system is not directly related to the sign of the slope of the mass-radius relation since there is no change of stability after a turning point of radius although the slope changes.

\footnote{We note that the third term in the parenthesis vanishes for the index $\gamma = 2$ corresponding to the standard BEC model.}
H. Analogy with cosmology

In $d = 3$, the free energy (276) is given by

$$F = \frac{1}{2} \alpha MR^2 + \frac{\hbar^2 M}{m^2 R^2} - \nu \frac{GM^2}{R} + \frac{1}{2} \omega_0^2 \alpha MR^2 + \frac{\zeta}{\gamma - 1} \frac{KM^\gamma}{R^{\gamma(\gamma - 1)}} - 3 \frac{\alpha MR^2 m}{\hbar} \ln R + C. \quad (308)$$

For dissipationless systems ($\xi = 0$), the free energy is conserved so that Eq. (308) can be seen as the first integral of the equation of motion (292). It can be rewritten under the form

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2(F - C)}{\alpha MR^2} - \frac{2\sigma h^2}{\alpha m^2 R^2} + \frac{2\nu GM}{\alpha R^3} - \omega_0^2 - \frac{2K\zeta M^\gamma}{(\gamma - 1)\alpha R^{\gamma(\gamma - 1)}} + \frac{6k_B T \ln R}{\alpha m R^2}. \quad (309)$$

For $h = a_s = T = 0$, it reduces to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2F}{\alpha MR^2} + \frac{2\nu GM}{\alpha R^3} - \omega_0^2. \quad (310)$$

Equation (310) is similar to the Friedmann equation in cosmology

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = -\frac{k c^2}{R^2} + \frac{8\pi G \rho}{3c^2} + \frac{\Lambda}{3}. \quad (311)$$

for a pressureless ($P = 0$) universe whose energy density decreases as $\epsilon \propto R^{-3}$. In this analogy, $R$ plays the role of the scale factor, $H = \dot{R}/R$ plays the role of the Hubble parameter, $-2F/\alpha M$ plays the role of the curvature constant $k c^2$, $2\nu M/\alpha R^3$ plays the role of the mass density $8\pi \epsilon/3c^2$, and $-3\omega_0^2$ plays the role of the cosmological constant $\Lambda$. When $\omega_0^2 < 0$ we obtain the $\Lambda$CDM model ($\Lambda > 0$) and when $\omega_0^2 > 0$ we obtain the anti-$\Lambda$CDM model ($\Lambda < 0$). We can therefore draw certain analogies between the evolution of a self-gravitating BEC and the evolution of a Friedmann-Lemaître-Robertson-Walker (FLRW) universe. This suggests the possibility of reproducing cosmological behaviors in BEC laboratory experiments.

Remark: In a universe filled with a fluid with an equation of state $P = \alpha \epsilon$, the energy density is related to the scale factor by $\epsilon \propto R^{-3(1 + \alpha)}$. If we take into account all the terms in Eq. (309), we find that the quantum potential term is analogous to an energy density $\epsilon \propto 1/R^4$ in cosmology. This corresponds to $\alpha = 1/3$ like for the standard radiation. However, the energy density is negative. On the other hand, the temperature term in Eq. (309) is analogous to an energy density $\epsilon \propto \ln R/R^2$ in cosmology. This corresponds to $\alpha = -1/3$ (up to a logarithmic correction) like for a gas of cosmic strings. Finally, the self-interaction term is analogous to an energy density $\epsilon \propto 1/R^5$ in cosmology corresponding to $\alpha = 2/3$ (to our knowledge, this coefficient has no particular interpretation in cosmology). The energy density is negative when $a_s > 0$ and a positive when $a_s < 0$. A gas of domain walls in cosmology ($\alpha = -2/3$) corresponds to a BEC with $\gamma = 2/3$ ($n = -3$).

IX. CONCLUSION

In this paper, we have developed a general formalism applying to Newtonian self-gravitating BECs. We have introduced and studied the generalized GPP equations (7) and (8). We have given their main properties, derived a hydrodynamic representation of these equations, established the virial theorem, and showed that these equations are consistent with a generalized thermodynamic formalism. In particular, they satisfy an $H$-theorem for a free energy functional associated with a generalized entropy. We have shown how the generalized free energy and the equation of state are related to the nonlinearity in the generalized GP equation and we have given several illustrative examples. Finally, by using a Gaussian ansatz for the wave function, we have shown how the generalized GPP equations (7) and (8) could be reduced to a simple dynamical equation giving the evolution of the size $R(t)$ of the BEC. We have

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31 A short account of the early development of modern cosmology is given in the Introduction of [150].
established very general equations that can describe many situations of physical and astrophysical interest. In our following papers (in preparation), these equations will be studied in detail.

The generalized GPP equations (7) and (8) may provide a model of dark matter halos. The generalized GP equation (7) includes a source of dissipation and an arbitrary nonlinearity \( h(|\psi|^2) \). Concerning the nonlinearity, we have given a special emphasis to the logarithmic nonlinearity \( 2k_B T \ln |\psi| \) where \( T \) plays the role of an effective temperature. This leads to a particularly interesting BEC dark matter model described by the generalized GPP equations (221) and (222). In this model, the system has a core-halo structure with a soliton/BEC core and an isothermal halo. Furthermore, the dissipation term ensures that the system relaxes towards an equilibrium state.

We have given three possible justifications of the generalized GPP equations:

(i) The generalized GPP equations (7) and (8) may be justified by physical processes. The dissipation could be due to non ideal effects or to the interaction of the system with an external medium, and the nonlinear potential \( h(|\psi|^2) \) could account for the self-interaction of the bosons. For bosons interacting via short-range interactions, the effective potential is quadratic, given by \( h(|\psi|^2) = (4\pi a_s \hbar^2 / m^3)|\psi|^2 \), but in more general situations other forms of potentials can emerge such as the logarithmic potential \( h(|\psi|^2) = (2k_B T / m) \ln |\psi| \). In that interpretation, \( T \) would be a formal measure of the collisionless interactions inside the BEC at zero (thermal) temperature.

(ii) The generalized GPP equations (7) and (8) may provide an effective model of gravitational cooling.\(^{32}\) In that interpretation, they can be viewed as a coarse-grained description of the (fine-grained) GPP equations (5) and (6). The friction term accounts for the relaxation process (dissipation of free energy) and the nonlinear term accounts for the formation of a halo of scalar radiation. This interpretation may be particularly relevant in the case of a logarithmic nonlinearity leading to an isothermal halo, with a density profile decreasing as \( r^{-2} \), that is relatively close to the NFW/Burkert profile at large distances. This model could be improved to match exactly the \( r^{-3} \) (NFW/Burkert) profile at large distances by introducing a more complicated nonlinearity in the generalized GPP equations (7) and (8), or by introducing an external potential to confine the system.

(iii) In Ref. \(^{146}\), we have derived the generalized GP equation (7) with a logarithmic nonlinearity from Nottale’s theory of scale relativity relying on a fractal spacetime \(^{147}\). In that interpretation, the friction and the temperature are obtained from a unified formalism and they correspond to the real and imaginary parts of a complex friction coefficient arising in a scale-covariant equation of dynamics. These terms can be the properties of a fractal spacetime or an aether. The dissipation is due to the friction with the aether and the temperature represents the temperature of the aether. In this interpretation, the generalized GP equation (221) could be a fundamental equation of physics from which the standard GP and Schrödinger equations would be approximations. In that case, \( \xi \) and \( T \) would be fundamental constants. We have, however, pointed out a difficulty with this interpretation in relation to the non constancy of the temperature of dark matter halos.

**Appendix A: The momentum tensor**

The equation of continuity (25) can be written as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{A1}
\]

where \( \mathbf{j} = \rho \mathbf{u} \) is the density current. Using Eqs. (15)–(17), the density current can be expressed in terms of the wave function as

\[
\mathbf{j} = \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*). \tag{A2}
\]

As a result, Eq. (A1) takes the form

\[
\frac{\partial |\psi|^2}{\partial t} + \frac{\hbar}{2im} \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) = 0. \tag{A3}
\]

On the other hand, the quantum Euler equation (28) can be written as

\[
\frac{\partial \mathbf{j}}{\partial t} = -\nabla(\rho \mathbf{u} \otimes \mathbf{u}) - \nabla P - \rho \nabla \Phi - \rho \nabla \Phi_{\text{ext}} - \frac{\rho}{m} \nabla Q - \xi \mathbf{j}. \tag{A4}
\]

\(^{32}\) In particular, they are able to account for the damped oscillations of a system experiencing gravitational cooling \(^{34,36}\). This damping is apparent on the simplified equation of motion (257) derived within the Gaussian ansatz.
Introducing the quantum pressure tensor (38), we find that the equation for the density current is given by
\[ \frac{\partial j_i}{\partial t} = -\partial_j T_{ij} - \rho \partial_i \Phi - \rho \partial_i \Phi_{\text{ext}} - \xi_j, \] (A5)
where
\[ T_{ij} = \rho u_i u_j + P \delta_{ij} + P_{ij} \] (A6)
is the momentum tensor. Using Eq. (38), we get
\[ T_{ij} = \rho u_i u_j + P \delta_{ij} - \hbar^2 \frac{1}{4m^2} \partial_i \partial_j \ln \rho \] (A7)
or, alternatively,
\[ T_{ij} = \rho u_i u_j + \left( P - \hbar^2 \frac{1}{4m^2} \Delta |\psi|^2 \right) \delta_{ij} + \hbar^2 \frac{1}{4m^2} \partial_i \rho \partial_j \rho. \] (A8)
Using Eqs. (15) and (17), we find after straightforward algebra that
\[ \frac{\hbar^2}{4m^2} \partial_i \rho \partial_j \rho = \frac{\hbar^2}{4m^2} |\psi|^2 (\psi^* \partial_i \psi + \psi \partial_i \psi^*) (\psi^* \partial_j \psi + \psi \partial_j \psi^*) \] (A9)
and
\[ \rho u_i u_j = -\frac{\hbar^2}{4m^2} |\psi|^2 (\psi^* \partial_i \psi - \psi \partial_i \psi^*) (\psi^* \partial_j \psi - \psi \partial_j \psi^*). \] (A10)
Therefore
\[ \rho u_i u_j + \frac{\hbar^2}{4m^2} \frac{1}{\rho} \partial_i \rho \partial_j \rho = \frac{\hbar^2}{m^2} \text{Re} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_j} \right). \] (A11)
Regrouping these results, the momentum tensor can be expressed in terms of the wave function as
\[ T_{ij} = \frac{\hbar^2}{m^2} \text{Re} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_j} \right) + \left( P - \frac{\hbar^2}{4m^2} \Delta |\psi|^2 \right) \delta_{ij}. \] (A12)

Appendix B: The energy operator and the Hamiltonian

The generalized GP equation (7) can be written as
\[ i\hbar \frac{\partial \psi}{\partial t} = \hat{E} \psi - \frac{i}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi \] (B1)
with the energy operator given by
\[ \hat{E} = -\hbar^2 \Delta + m[\Phi + \hbar(|\psi|^2) + \Phi_{\text{ext}}]. \] (B2)
Its average value is
\[ \langle E \rangle = \frac{1}{m} \langle \psi | \hat{E} | \psi \rangle = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 \, d\mathbf{r} + \int |\psi|^2 \Phi_{\text{ext}} \, d\mathbf{r} + \int |\psi|^2 \Phi \, d\mathbf{r} + \int |\psi|^2 \hbar(|\psi|^2) \, d\mathbf{r}. \] (B3)
Using the results of Sec. IIIA, we see that the average value of the energy operator coincides with the average value of the energy given by Eq. (11). It differs from the free energy (39) which can be written as
\[ F = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 \, d\mathbf{r} + \int |\psi|^2 \Phi_{\text{ext}} \, d\mathbf{r} + \frac{1}{2} \int |\psi|^2 \Phi \, d\mathbf{r} + \int V(|\psi|^2) \, d\mathbf{r}. \] (B4)
We have

\[ F = N\langle E \rangle - \frac{1}{2} \int |\psi|^2 \Phi \, dr + \int [V(|\psi|^2) - |\psi|^2 h(|\psi|^2)] \, dr, \]  

(B5)

where the term in brackets is the opposite of the pressure [see Eq. (32)]. Equation (B5) is equivalent to Eqs. (48) and (44).

Taking the first variation of the free energy (B4), we get

\[ \delta F = \frac{1}{m} \int \left[ -\frac{\hbar^2}{2m} \Delta \psi^* + m\Phi \psi^* + m\Phi^* \psi + m h(|\psi|^2) \psi \right] \delta \psi \, dr + \text{c.c.} \]  

(B6)

The term in brackets coincides with the energy operator (B2) applied on \( \psi^* \). Indeed,

\[ m \frac{\delta F}{\delta \psi^*} = \hat{E}\psi. \]  

(B7)

This relation can be compared with Eq. (86). We also note that

\[ N\langle E \rangle = \int \frac{\delta F}{\delta \psi^*} \psi^* \, dr \]  

which can be compared with Eq. (95).

Let us first consider conservative systems (\( \xi = 0 \)). The generalized GP equation (B1) can be rewritten as

\[ i\hbar \frac{\partial \psi}{\partial t} = m \frac{\delta F}{\delta \psi}. \]  

(B9)

This expression shows that \( F \) represents the true Hamiltonian of the system. Indeed, in terms of the wavefunction \( \psi(r,t) \) and its canonical momentum \( \pi(r,t) = i\hbar \psi^*(r,t) \), the generalized GP equation is exactly reproduced by the Hamilton equations

\[ \frac{\partial \psi}{\partial t} = m \frac{\delta F}{\delta \pi}, \quad \frac{\partial \pi}{\partial t} = -m \frac{\delta F}{\delta \psi}. \]  

(B10)

This formulation directly implies the conservation of the free energy \( F \) since

\[ \dot{F} = \int \frac{\delta F}{\delta \psi} \frac{\partial \psi}{\partial t} \, dr + \int \frac{\delta F}{\delta \pi} \frac{\partial \pi}{\partial t} \, dr = 0. \]  

(B11)

For dissipative systems, the generalized GP equation (B1) can be rewritten as

\[ i\hbar \frac{\partial \psi}{\partial t} = m \frac{\delta F}{\delta \psi^*} - i\frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \]  

(B12)

From Eqs. (B7) and (B12), we easily obtain the identity

\[ \dot{F} = -\xi \int \frac{\hbar^2}{4m^2} |\psi|^2 \left| \nabla \ln \left( \frac{\psi}{\psi^*} \right) \right|^2 \, dr, \]  

(B13)

which coincides with the \( H \)-theorem (76).

Finally, considering the minimizing of the free energy \( F[\psi] \) given by Eq. (B4) at fixed mass \( M \) given by Eq. (4), and writing the variational principle as

\[ \delta F - \frac{\mu}{m} \delta \int |\psi|^2 \, dr = 0, \]  

(B14)

where \( \mu \) is a Lagrange multiplier (chemical potential), we obtain the time-independent GP equation (47) with \( \mu = E \) (see Sec. III D for a detailed discussion). This variational principle was introduced by Schrödinger [151] in his first paper on wave mechanics. Actually, this is how he originally obtained the fundamental eigenvalue equation (47). A short account of the early development of quantum mechanics is given in the Introduction of [77].

Remark: For nonlinear Schrödinger equations, the true Hamiltonian of the system is \( F \), not \( N\langle E \rangle \). As a result, it is the free energy \( F \) that is conserved (\( \dot{F} = 0 \)) for dissipationless systems (\( \xi = 0 \)) and that satisfies an \( H \)-theorem (\( \dot{F} \leq 0 \)) for dissipative systems (\( \xi \neq 0 \), not the average energy \( N\langle E \rangle \)). For the standard (linear) Schrödinger equation with \( h = \Phi = 0 \), we have \( F = N\langle E \rangle \). For the logarithmic Schrödinger equation (115) with \( \Phi = 0 \), since \( F \) and \( N\langle E \rangle \) only differ by a constant \( Nk_B T \) [see Sec. V A 1], we can also interpret \( N\langle E \rangle \) as the Hamiltonian of the system. However, this identification is not true anymore for other nonlinear Schrödinger equations.
Appendix C: Variation of the energies

In this Appendix, we detail the first and second order variations of the different functionals that compose the free energy (57).

The first and second order variations of the classical kinetic energy (59) are

\[ \delta \Theta_c = \int \frac{u^2}{2} \delta \rho \, dr + \int \rho u \cdot \delta u \, dr, \quad \delta^2 \Theta_c = \frac{1}{2} \int \rho (\delta u)^2 \, dr + \int \delta \rho u \cdot \delta u \, dr. \]  \hfill (C1)

The first and second order variations of the quantum kinetic energy (60) are

\[ \delta \Theta_Q = \frac{1}{m} \int Q \delta \rho \, dr, \quad \delta^2 \Theta_Q = \frac{\hbar^2}{8m^2} \int \frac{1}{\rho} \left[ \left( \frac{\Delta \rho}{\rho} - \frac{(\nabla \rho)^2}{\rho^2} \right) (\delta \rho)^2 + (\nabla \delta \rho)^2 \right] \, dr. \]  \hfill (C2)

The first and second order variations of the internal energy (64) are

\[ \delta U = \int h(\rho) \delta \rho \, dr, \quad \delta^2 U = \frac{1}{2} \int h'(\rho)(\delta \rho)^2 \, dr. \]  \hfill (C3)

The first and second order variations of the gravitational potential energy (65) are

\[ \delta W = \int \Phi \delta \rho \, dr, \quad \delta^2 W = \frac{1}{2} \int \delta \rho \delta \Phi \, dr. \]  \hfill (C4)

The first and second order variations of the external potential energy (66) are

\[ \delta W_{\text{ext}} = \int \Phi_{\text{ext}} \delta \rho \, dr, \quad \delta^2 W_{\text{ext}} = 0. \]  \hfill (C5)

The calculations leading to these relations are straightforward except, maybe, the ones leading to the first relation of Eq. (C2). We give below two different derivations of this relation:

(i) From Eqs. (20), (61) and (62) we directly obtain

\[ \delta \Theta_Q = \frac{\hbar^2}{m^2} \int \nabla \sqrt{\rho} \cdot \nabla \sqrt{\rho} \delta \rho \, dr = \frac{\hbar^2}{m^2} \int \delta \sqrt{\rho} \Delta \sqrt{\rho} \, dr = -\frac{\hbar^2}{2m^2} \int \delta \rho \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \, dr = \frac{1}{m} \int Q \delta \rho \, dr. \]  \hfill (C6)

(ii) From the first equality of Eq. (20), we find that

\[ \delta Q = -\frac{\hbar^2}{2m\rho} \nabla \cdot (\sqrt{\rho} \nabla \sqrt{\rho} - \delta \sqrt{\rho} \nabla \sqrt{\rho}) = -\frac{\hbar^2}{4m\rho} \nabla \cdot (\rho \delta \nabla \ln \rho). \]  \hfill (C7)

This implies the identity

\[ \int \rho \delta Q \, dr = 0, \]  \hfill (C8)

from which we get \( \delta \Theta_Q = \frac{1}{m} \int \rho \delta Q \, dr + \frac{1}{m} \int Q \delta \rho \, dr = \frac{1}{m} \int Q \delta \rho \, dr \) leading to the first relation of Eq. (C2). We note that Eq. (C7) is the equivalent of the tensorial equation (37) with Eq. (38) except that it applies to a perturbation \( \delta \) instead of a space derivative \( \partial_i \).

Appendix D: The H-theorem

In this Appendix, we establish the \( H \)-theorem associated with the damped quantum barotropic EP equations (25)-(27) that are equivalent to the generalized GPP equations (7) and (8).

Taking the time derivative of the free energy (57), and using the results of Appendix C we get

\[ \dot{F} = \int \left( \frac{u^2}{2} + \frac{Q}{m} + h(\rho) + \Phi + \Phi_{\text{ext}} \right) \frac{\partial \rho}{\partial t} \, dr + \int \rho u \cdot \frac{\partial u}{\partial t} \, dr. \]  \hfill (D1)
Substituting the continuity equation \((25)\) into Eq. \((11)\), integrating by parts, and using the Euler equation \((26)\) together with Eq. \((23)\) to simplify some terms, we obtain

\[
\dot{F} = \int \rho \mathbf{u} \cdot \left[ \nabla \left( \frac{u^2}{2} \right) - \xi \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} \right] \, dr. \tag{D2}
\]

Using the identity \((\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left( \frac{u^2}{2} \right) - \mathbf{u} \times (\nabla \times \mathbf{u})\), the foregoing equation can be rewritten as

\[
\dot{F} = \int \rho \mathbf{u} \cdot \left[ \mathbf{u} \times (\nabla \times \mathbf{u}) - \xi \mathbf{u} \right] \, dr. \tag{D3}
\]

Since \(\mathbf{u}\) is a potential flow, it is irrotational \((\nabla \times \mathbf{u} = 0)\), so that Eq. \((D3)\) reduces to Eq. \((70)\). Actually, we note that this result remains valid even if \(\mathbf{u}\) is not a potential flow since \(\mathbf{u} \cdot [\mathbf{u} \times (\nabla \times \mathbf{u})] = 0\) in any case.\(^3\)

In the strong friction limit, the free energy is given by Eq. \((70)\). Taking its time derivative and using the results of Appendix \(C\) we get

\[
\dot{F} = \int \left( \frac{Q}{m} + h(\rho) + \Phi + \Phi_{\text{ext}} \right) \frac{\partial \rho}{\partial t} \, dr. \tag{D4}
\]

Substituting the quantum barotropic Smoluchowski equation \((30)\) into Eq. \((D4)\), integrating by parts, and using Eq. \((23)\), we obtain Eq. \((77)\).

Appendix E: Local free energy equation

The free energy \((69)\) can be written as

\[
F = \int \rho e \, dr, \tag{E1}
\]

where \(e(r, t)\) is the free energy density given by

\[
e = \frac{u^2}{2} + \frac{Q}{m} + \frac{V(\rho)}{\rho} + \frac{\Phi}{2} + \Phi_{\text{ext}}. \tag{E2}
\]

Using the equation of continuity \((25)\) and the damped quantum Euler equation \((26)\), we obtain the local free energy equation

\[
\frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho e \mathbf{u}) = -\nabla \cdot (P \mathbf{u}) - \frac{1}{2} \rho \mathbf{u} \cdot \nabla \Phi + \frac{\rho}{m} \frac{\partial Q}{\partial t} + \frac{1}{2} \rho \frac{\partial \Phi}{\partial t} - \xi \rho u^2. \tag{E3}
\]

According to Eq. \((C7)\), we have

\[
\frac{\rho}{m} \frac{\partial Q}{\partial t} = -\nabla \cdot \mathbf{J}_Q, \tag{E4}
\]

where

\[
\mathbf{J}_Q = \frac{\hbar^2}{2m^2} \left( \sqrt{\rho} \frac{\partial \nabla \sqrt{\rho}}{\partial t} - \frac{\partial \sqrt{\rho}}{\partial t} \nabla \sqrt{\rho} \right) = \frac{\hbar^2}{4m^2} \rho \frac{\partial \nabla \ln \rho}{\partial t} \tag{E5}
\]

is the quantum current. Therefore, Eq. \((E3)\) can be rewritten as

\[
\frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho e \mathbf{u}) + \nabla \cdot \mathbf{J}_Q + \nabla \cdot (P \mathbf{u}) = -\frac{1}{2} \rho \mathbf{u} \cdot \nabla \Phi + \frac{1}{2} \rho \frac{\partial \Phi}{\partial t} - \xi \rho u^2. \tag{E6}
\]

\(^3\) We could arrive directly at Eq. \((76)\) from Eq. \((11)\) by using Eq. \((35)\) but we wanted to be more general so that our derivation also applies to nonpotential flows (this may be useful in other circumstances).
The $H$-theorem directly results from this equation. Indeed, taking the time derivative of the free energy \( F \), and using Eq. (E7), we get
\[
\dot{F} = \frac{1}{2} \int \rho \mathbf{u} \cdot \nabla \Phi \, d\mathbf{r} + \frac{1}{2} \int \rho \frac{\partial \Phi}{\partial t} \, d\mathbf{r} - \xi \int \rho \mathbf{u}^2 \, d\mathbf{r}.
\] (E7)

Using the Poisson equation (27) and the equation of continuity (25), and integrating by parts, we find that
\[
\frac{1}{2} \int \rho \frac{\partial \Phi}{\partial t} \, d\mathbf{r} = \frac{1}{2} \int \Delta \Phi \frac{\partial \Phi}{\partial t} \, d\mathbf{r} = \frac{1}{2} \int \Phi \frac{\partial \rho}{\partial t} \, d\mathbf{r} = \frac{1}{2} \int \rho \mathbf{u} \cdot \nabla \Phi \, d\mathbf{r}.
\] (E8)

Substituting this identity into Eq. (E7), we obtain the $H$-theorem (76).

We can also consider the energy \( E (\mathbf{r}, t) \) defined by Eq. (34). We note that
\[
E_m - e = \frac{1}{2} \Phi + h (\rho) - V (\rho).
\] (E9)

Proceeding as before, we obtain the local energy equation
\[
\frac{\partial}{\partial t} (\rho E_m) + \nabla \cdot (\rho E_m \mathbf{u}) + \nabla \cdot \mathbf{J}_Q = \frac{\partial P}{\partial t} + \rho \frac{\partial \Phi}{\partial t} - \xi \rho \mathbf{u}^2,
\] (E10)

where we used Eq. (23) in the course of the calculations. Taking the time derivative of Eq. (71) giving the average value of \( E \), and using Eq. (E10), we obtain
\[
Nd \langle E \rangle \frac{d}{dt} = - \xi \int \rho \mathbf{u}^2 \, d\mathbf{r} + \int \rho \frac{\partial \Phi}{\partial t} \, d\mathbf{r} + \frac{d}{dt} \int P \, d\mathbf{r}.
\] (E11)

This relation can also be obtained by taking the time derivative of Eq. (74) and using Eq. (76). It confirms that when \( P \neq \rho k_B T / m \) and \( \Phi \neq 0 \) the average energy \( \langle E \rangle \) is not conserved, even when \( \xi = 0 \), contrary to the free energy \( F \).

The previous results remain valid in the strong friction limit \( \xi \to +\infty \) with \( u \) given by Eq. (29).

Remark: In Ref. [146], we have established the identity
\[
\frac{\partial}{\partial t} \left( \rho \frac{\partial Q}{\partial t} \right) + \nabla \cdot \left( \rho \frac{\partial \mathbf{u}_Q}{\partial t} \right) = 0,
\] (E12)

where \( \mathbf{u}_Q = (\hbar / 2m) \nabla \ln \rho \) is the quantum (or osmotic) velocity. Therefore, the quantum current can be written as
\[
\mathbf{J}_Q = \frac{\hbar}{2m} \frac{\partial \mathbf{u}_Q}{\partial t}.
\] (E13)

From Eqs. (E12) and (E13), we obtain the identity [see also Eq. (C8)]:
\[
\int \rho \frac{\partial Q}{\partial t} \, d\mathbf{r} = 0.
\] (E14)

Appendix F: Lagrangian of a self-gravitating BEC

In this Appendix, we discuss the Lagrangian structure of the generalized GPP equations (7) and (8) and of the corresponding hydrodynamic EP equations (25)-(27). We take \( \xi = 0 \) for simplicity. The Lagrangian of the generalized GPP equations is
\[
L = \int \left\{ \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{\hbar^2}{2m^2} |\nabla \psi|^2 - \frac{1}{2} \Phi |\psi|^2 - \Phi_{\text{ext}} |\psi|^2 - V (|\psi|^2) \right\} \, d\mathbf{r}.
\] (F1)

We can view the Lagrangian (F1) as a functional of \( \psi, \dot{\psi} \) and \( \nabla \psi \). The action is \( S = \int L \, dt \). The least action principle \( \delta S = 0 \), which is equivalent to the Lagrange equation
\[
\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{\psi}} \right) + \nabla \cdot \left( \frac{\delta L}{\delta \nabla \psi} \right) - \frac{\delta L}{\delta \psi} = 0,
\] (F2)
returns the GP equation (7). The free energy is obtained from the transformation

$$F = \int \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \, dr - L$$

leading to

$$F = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 \, dr + \frac{1}{2} \int \Phi |\psi|^2 \, dr + \int \Phi_{\text{ext}} |\psi|^2 \, dr + \int V(|\psi|^2) \, dr.$$ 

The first term is the kinetic energy, the second term is the gravitational energy, the third term is the external potential energy and the fourth term is the self-interaction energy. Using the Lagrange equations, one can show that the free energy is conserved (see also Appendix B).

Using the Madelung transformation, we can rewrite the Lagrangian in terms of hydrodynamic variables. According to Eqs. (15) and (16) we have

$$\frac{\partial S}{\partial t} = -\frac{i\hbar}{2|\psi|^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right)$$

and

$$|\nabla \psi|^2 = \frac{1}{4\rho} (\nabla \rho)^2 + \frac{\rho}{\hbar^2} (\nabla S)^2.$$ 

Substituting these identities into Eq. (F1), we get

$$L = -\int \left\{ \frac{\rho}{m} \frac{\partial S}{\partial t} + \frac{\rho}{2m^2} (\nabla S)^2 + \frac{\hbar^2}{8m^2} \frac{(\nabla \rho)^2}{\rho} + \frac{1}{2} \rho \Phi + \rho \Phi_{\text{ext}} + V(\rho) \right\} \, dr.$$ 

We can view the Lagrangian (F7) as a functional of $S$, $\dot{S}$, $\nabla S$, $\rho$, $\dot{\rho}$, and $\nabla \rho$. The Lagrange equation for the action

$$\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{S}} \right) + \nabla \cdot \left( \frac{\delta L}{\delta \nabla S} \right) - \frac{\delta L}{\delta S} = 0$$

returns the equation of continuity (18). The Lagrange equation for the density

$$\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{\rho}} \right) + \nabla \cdot \left( \frac{\delta L}{\delta \nabla \rho} \right) - \frac{\delta L}{\delta \rho} = 0$$

returns the quantum Hamilton-Jacobi (or Bernoulli) equation (19) leading to the quantum Euler equation (22). The free energy is obtained from the transformation

$$F = -\int \frac{\rho}{m} \frac{\partial S}{\partial t} \, dr - L$$ 

leading to

$$F = \int \frac{1}{2} \rho u^2 \, dr + \int \frac{\hbar^2}{8m^2} \frac{(\nabla \rho)^2}{\rho} \, dr + \frac{1}{2} \int \rho \Phi \, dr + \int \rho \Phi_{\text{ext}} \, dr + \int V(\rho) \, dr.$$ 

The first term is the classical kinetic energy, the second term is the quantum kinetic energy, the third term is the gravitational energy, the fourth term is the external potential energy and the fifth term is the self-interaction energy. Using the Lagrange equations, one can show that the free energy is conserved (see also Sec. III C).

We now consider the model of Sec. VIII. With the Gaussian ansatz of Eq. (266), we obtain

$$\int \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \, dr = -\int \frac{\rho}{m} \frac{\partial S}{\partial t} \, dr = -\frac{1}{2} \alpha MR^2 \dot{H}$$

and

$$F = \frac{1}{2} \alpha MR^2 H^2 + V(R),$$
where $V(R)$ is given by Eqs. (277) and (278). Substituting these expressions into Eq. (F3) or into Eq. (F10), we obtain the effective Lagrangian

$$L(H, \dot{H}, R) = -\frac{1}{2} \alpha M R^2 (\dot{H} + H^2) - V(R).$$  \hspace{1cm} (F14)$$

We can view the Lagrangian (F14) as a function of $H$, $\dot{H}$ and $R$. The Lagrange equation for $H$

$$\frac{\partial}{\partial H} \left( \frac{\delta L}{\delta \dot{H}} \right) - \frac{\delta L}{\delta H} = 0$$  \hspace{1cm} (F15)$$

returns Eq. (269). The Lagrange equation for $R$

$$\frac{\delta L}{\delta R} = 0,$$ \hspace{1cm} (F16)$$

together with Eq. (269), returns the equation of motion (292). We also note that, using Eq. (F13) the free energy (F13) can be written in the form of Eqs. (276).

Appendix G: Derivation of the virial theorem

1. General case

In this Appendix, we establish the virial theorem associated with the damped quantum barotropic EP equations (25)-(27) that are equivalent to the generalized GPP equations (7) and (8). This generalizes the results of [4, 35].

Taking the time derivative of the moment of inertia tensor

$$I_{ij} = \int \rho x_i x_j \, dr$$  \hspace{1cm} (G1)$$

and using the continuity equation (25), we obtain after an integration by parts

$$\dot{I}_{ij} = \int \rho (x_i u_j + x_j u_i) \, dr.$$ \hspace{1cm} (G2)$$

Taking the time derivative of Eq. (G2), we get

$$\ddot{I}_{ij} = \int x_i \frac{\partial}{\partial t} (\rho u_j) \, dr + (i \leftrightarrow j),$$ \hspace{1cm} (G3)$$

where $\frac{\partial}{\partial t} (\rho u_j)$ is given by Eq. (28). This equation can be rewritten as

$$\frac{\partial}{\partial t} (\rho u_j) = -\frac{\partial}{\partial x_k} (\rho u_k u_j) - \frac{\partial P}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_j} - \rho \frac{\partial \Phi_{ext}}{m} \frac{\partial}{\partial x_j} - \xi \rho u_j.$$ \hspace{1cm} (G4)$$

To obtain $\dddot{I}_{ij}$, we need to evaluate six terms. The first term is the kinetic energy tensor

$$-\int x_i \frac{\partial}{\partial x_k} (\rho u_k u_j) \, dr = \int \rho u_i u_j \, dr.$$ \hspace{1cm} (G5)$$

The second term is the pressure tensor

$$-\int x_i \frac{\partial P}{\partial x_j} \, dr = \delta_{ij} \int P \, dr.$$ \hspace{1cm} (G6)$$

The third term is the gravitational potential energy tensor

$$W_{ij} = -\int \rho x_i \frac{\partial \Phi}{\partial x_j} \, dr.$$ \hspace{1cm} (G7)$$
It can be written in an alternative form as follows. Substituting the expression of the gravitational force

\[ F = -\nabla \Phi = -G \int \rho(r', t) \frac{r - r'}{|r - r'|^d} dr' \]  \hspace{1cm} (G8)

into Eq. (G7), we obtain

\[ W_{ij} = -G \int \rho(r, t) \rho(r', t) x_i x_j \frac{x_j - x'_j}{|r - r'|^d} dr dr'. \]  \hspace{1cm} (G9)

Interchanging the prime and unprimed variables, we get

\[ W_{ij} = G \int \rho(r, t) \rho(r', t) x'_i x'_j \frac{x_j - x'_j}{|r - r'|^d} dr dr'. \]  \hspace{1cm} (G10)

Taking the half-sum of the foregoing expressions, we find that

\[ W_{ij} = -\frac{G}{2} \int \rho(r, t) \rho(r', t) \frac{(x_i - x'_i)(x_j - x'_j)}{|r - r'|^d} dr dr'. \]  \hspace{1cm} (G11)

Under this form, the gravitational potential energy tensor is manifestly symmetric: \( W_{ij} = W_{ji} \). The fourth term is the external potential energy tensor

\[ W_{ij}^{\text{ext}} = -\int \rho x_i \frac{\partial \Phi^{\text{ext}}}{\partial x_j} dr. \]  \hspace{1cm} (G12)

The fifth term is the quantum potential energy tensor

\[ W_{ij}^{Q} = -\int x_i \frac{\rho m}{\partial x_j} Q dr. \]  \hspace{1cm} (G13)

Substituting Eq. (37) into Eq. (G13), we get

\[ W_{ij}^{Q} = -\int x_i \partial_k P_{jk} dr = \int P_{ij} dr, \]  \hspace{1cm} (G14)

where \( P_{ij} \) is the quantum pressure tensor defined by Eq. (38). Since \( P_{ij} \) is symmetric, the quantum potential energy tensor is also symmetric: \( W_{ij}^{Q} = W_{ji}^{Q} \). The sixth term is the frictional tensor

\[ -\int x_i \xi u_j dr = -\xi \int \rho x_i u_j dr. \]  \hspace{1cm} (G15)

Substituting these results into Eq. (G3), we obtain the tensorial virial theorem

\[ \frac{1}{2} \ddot{\mathbf{I}}_{ij} + \frac{1}{2} \dot{\varepsilon} \dot{I}_{ij} = \int \rho u_i u_j dr + \delta_{ij} \int P dr + W_{ij}^{Q} + W_{ij} + \frac{1}{2} (W_{ij}^{\text{ext}} + W_{ji}^{\text{ext}}). \]  \hspace{1cm} (G16)

According to Eq. (G14), it can also be written as

\[ \frac{1}{2} \ddot{\mathbf{I}}_{ij} + \frac{1}{2} \dot{\varepsilon} \dot{I}_{ij} = \int \rho u_i u_j dr + \delta_{ij} \int P_{ij} dr + \delta_{ij} \int P dr + W_{ij} + \frac{1}{2} (W_{ij}^{\text{ext}} + W_{ji}^{\text{ext}}). \]  \hspace{1cm} (G17)

Contracting the indices, we obtain the scalar virial theorem

\[ \frac{1}{2} \ddot{I} + \frac{1}{2} \dot{\varepsilon} \dot{I} = 2(\Theta_c + \Theta_Q) + d \int P dr + W_{ii} + W_{ii}^{\text{ext}}. \]  \hspace{1cm} (G18)

This equation involves the virial of the quantum force

\[ W_{ii}^{Q} = -\int \frac{\rho}{m} r \cdot \nabla Q dr, \]  \hspace{1cm} (G19)
the virial of the gravitational force

\[ W_{ii} = -\int \rho \mathbf{r} \cdot \nabla \Phi \, d\mathbf{r}, \quad (G20) \]

and the virial of the external force

\[ W_{ii}^{\text{ext}} = -\int \rho \mathbf{r} \cdot \nabla \Phi_{\text{ext}} \, d\mathbf{r}. \quad (G21) \]

According to Eqs. (G14), (40) and (60), the virial of the quantum force is equal to twice the quantum kinetic energy

\[ W_{ii}^{Q} = \int P_{ii} \, d\mathbf{r} = \frac{\hbar^2}{4m^2} \int \frac{(\nabla \rho)^2}{\rho} \, d\mathbf{r} = 2\Theta_{Q}, \quad (G22) \]

leading to the second term in the right hand side of Eq. (G18). From Eq. (G11), we find that the virial of the gravitational force is given by

\[ W_{ii} = -\frac{G M^2}{2}, \quad (d = 2) \quad (G25) \]

where \( W \) is the gravitational potential energy

\[ W = -\frac{G}{2(d-2)} \int \frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r} - \mathbf{r}'|^{d-2}} \, d\mathbf{r} d\mathbf{r}'. \quad (G26) \]

In the strong friction limit \( \xi \to +\infty \) in which \( u = O(1/\xi) \), the tensorial virial theorem (G16) becomes

\[ \frac{1}{2} \xi \dot{I}_{ij} = \delta_{ij} \int P \, d\mathbf{r} + W_{ii}^{Q} + W_{ij} + \frac{1}{2}(W_{ij}^{\text{ext}} + W_{ji}^{\text{ext}}). \quad (G27) \]

According to Eq. (G14), it can also be written as

\[ \frac{1}{2} \xi \dot{I}_{ij} = \int P_{ij} \, d\mathbf{r} + \delta_{ij} \int P \, d\mathbf{r} + W_{ij} + \frac{1}{2}(W_{ij}^{\text{ext}} + W_{ji}^{\text{ext}}). \quad (G28) \]

Contracting the indices, we obtain the scalar virial theorem

\[ \frac{1}{2} \xi \dot{I} = 2\Theta_{Q} + d \int P \, d\mathbf{r} + W_{ii} + W_{ii}^{\text{ext}}. \quad (G29) \]

At equilibrium (\( \dot{I}_{ij} = I_{ij} = 0 \) and \( u = 0 \)), the tensorial virial theorem reduces to

\[ \delta_{ij} \int P \, d\mathbf{r} + W_{ij}^{Q} + W_{ij} + \frac{1}{2}(W_{ij}^{\text{ext}} + W_{ji}^{\text{ext}}) = 0. \quad (G30) \]

According to Eq. (G14), it can also be written as

\[ \int P_{ij} \, d\mathbf{r} + \delta_{ij} \int P \, d\mathbf{r} + W_{ij} + \frac{1}{2}(W_{ij}^{\text{ext}} + W_{ji}^{\text{ext}}) = 0. \quad (G31) \]
Contracting the indices, we obtain the equilibrium scalar virial theorem

\[ 2\Theta_Q + d \int P \, d\mathbf{r} + W_{ii} + W_{ii}^{\text{ext}} = 0. \] (G32)

For a spherically symmetric system, according to the Gauss theorem, we have

\[ \nabla \Phi = \frac{GM(r,t)}{r^{d-1}} \mathbf{e}_r, \] (G33)

where

\[ M(r,t) = \int_0^r \rho(r',t) S_d r'^{d-1} \, dr', \] (G34)

is the mass contained within the sphere of radius \( r \). This is Newton’s law in \( d \) dimensions. Using Eq. (G34), the virial of the gravitational force (G20) can be written as

\[ W_{ii} = -S_d G \int \rho(r,t) M(r,t) r \, dr = - \int \frac{GM(r,t)}{r^{d-2}} \, dM(r,t). \] (G35)

In \( d = 2 \), we immediately recover Eq. (G25). In \( d \neq 2 \), using Eq. (G24), we obtain the formula

\[ W = - \frac{1}{d-2} \int \rho(r,t) \frac{GM(r,t)}{r^{d-2}} S_d r^{d-1} \, dr, \] (G36)

which is useful to calculate the gravitational potential energy of a spherically symmetric distribution of matter (see Appendix K).

### 2. Harmonic potential

For the harmonic potential (14), we have

\[ W_{ij}^{\text{ext}} = -\omega_0^2 I_{ij}, \quad W_{ii}^{\text{ext}} = -\omega_0^2 I = -2W_{\text{ext}}. \] (G37)

The harmonic potential energy tensor is manifestly symmetric: \( W_{ij}^{\text{ext}} = W_{ji}^{\text{ext}} \). The tensorial virial theorem can be written as

\[ \frac{1}{2} \ddot{I}_{ij} + \frac{1}{2} \xi \dot{I}_{ij} + \omega_0^2 I_{ij} = \int \rho u_i u_j \, d\mathbf{r} + \delta_{ij} \int P \, d\mathbf{r} + W_{ij}^Q + W_{ij}, \] (G38)

or, equivalently, as

\[ \frac{1}{2} \ddot{I}_{ij} + \frac{1}{2} \xi \dot{I}_{ij} + \omega_0^2 I_{ij} = \int \rho u_i u_j \, d\mathbf{r} + \int P_{ij} \, d\mathbf{r} + \delta_{ij} \int P \, d\mathbf{r} + W_{ij}. \] (G39)

The scalar virial theorem can be written as

\[ \frac{1}{2} \dot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2(\Theta_c + \Theta_Q) + d \int P \, d\mathbf{r} + W_{ii}. \] (G40)

In the strong friction limit \( \xi \to +\infty \), the tensorial virial theorem takes the form

\[ \frac{1}{2} \xi \dot{I}_{ij} + \omega_0^2 I_{ij} = \delta_{ij} \int P \, d\mathbf{r} + W_{ij}^Q + W_{ij}, \] (G41)

or, equivalently,

\[ \frac{1}{2} \xi \dot{I}_{ij} + \omega_0^2 I_{ij} = \int P_{ij} \, d\mathbf{r} + \delta_{ij} \int P \, d\mathbf{r} + W_{ij}, \] (G42)

and the scalar virial theorem takes the form

\[ \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_Q + d \int P \, d\mathbf{r} + W_{ii}. \] (G43)
The equilibrium tensorial virial theorem can be written as
\[ \delta_{ij} \int P \, d\mathbf{r} + W_{ij}^{Q} + W_{ij} - \omega_{ij}^{2} I_{ij} = 0 \] (G44)
or, equivalently, as
\[ \int P_{ij} \, d\mathbf{r} + \delta_{ij} \int P \, d\mathbf{r} + W_{ij} - \omega_{ij}^{2} I_{ij} = 0. \] (G45)

The equilibrium scalar virial theorem can be written as
\[ 2\Theta_{Q} + d \int P \, d\mathbf{r} + W_{ii} - \omega_{ii}^{2} I_{ij} = 0. \] (G46)

Appendix H: Composite models

In Sec. V, we have considered particular classes of generalized GPP equations associated with particular equations of state and particular entropies. In this Appendix, we consider composite models. Specifically, we consider generalized GPP equations where the effective potential \( h = \sum h_{i} \) is a sum of potentials \( h_{i} \) such as those studied in Sec. V. In that case, \( P = \sum p_{i}, U = \sum U_{i}, \) and \( F = E_{s} - \sum T_{i}S_{i}. \) For example, if \( P = K_{1}I_{1} + K_{2}I_{2} + A\ln \rho + \rho k_{B}T/m, \) we get \( F = E_{s} - K_{1}S_{1} - K_{2}S_{2} - AS_{L} - TS_{B}. \) We emphasize, however, that the density \( \rho \) associated with the equation of state \( P = \sum p_{i} \) (or the free energy \( F = E_{s} - \sum T_{i}S_{i} \)) is not the sum of the densities \( \rho_{i} \) associated with the equations of state \( p_{i} \) (or the free energies \( F_{i} = E_{s} - T_{i}S_{i} \)) taken individually: \( \rho \neq \sum \rho_{i}. \)

Let us give some examples of physical interest in the context of dark matter halos. The equations of state
\[ P = K_{1}\rho_{1} + K_{2}\rho_{2}, \quad \frac{1}{P} = \frac{1}{K_{1}\rho_{1}} + \frac{1}{K_{2}\rho_{2}} \] (H1)
describe a composite halo with a polytropic core and a polytropic halo. The equation of state
\[ P = K\rho^{\gamma} + \rho \frac{k_{B}T}{m} \] (H2)
with \( \gamma > 1 \) (resp. \( \gamma < 1 \)) describes a composite halo with a polytropic (resp. isothermal) core and an isothermal (resp. polytropic) halo. Symmetrically, the equation of state
\[ \frac{1}{P} = \frac{1}{K\rho^{\gamma}} + \frac{m}{\rho k_{B}T} \] (H3)
with \( \gamma > 1 \) (resp. \( \gamma < 1 \)) describes a composite halo with an isothermal (resp. polytropic) core and a polytropic (resp. isothermal) halo.

More specifically, the equation of state
\[ P = \frac{2\pi a_{s}h^{2}}{m^{3}}\rho^{2} + K\rho^{\gamma} \] (H4)
with \( \gamma < 2 \) describes a composite halo with a BEC core and a polytropic halo while the equation of state
\[ P = \frac{2\pi a_{s}h^{2}}{m^{3}}\rho^{2} + \rho \frac{k_{B}T}{m} \] (H5)
describes a composite halo with a BEC core and an isothermal halo.

Finally, the equation of state
\[ \frac{1}{P} = \frac{1}{A\ln \rho} + \frac{m}{\rho k_{B}T} \] (H6)
describes a composite halo with a logotropic core and an isothermal halo while the equation of state
\[ P = A\ln \rho + \rho \frac{k_{B}T}{m} \] (H7)
describes a composite halo with an isothermal core and a logotropic halo. Similarly, the equations of state
\[ \frac{1}{P} = \frac{1}{A\ln \rho} + \frac{1}{K\rho^{\gamma}} \quad \text{and} \quad P = A\ln \rho + K\rho^{\gamma} \] (H8)
describe a composite halo with a logotropic core and a polytropic halo, or the converse.
Appendix I: A damped logarithmic Gross-Pitaevskii equation that conserves the energy of the standard Gross-Pitaevskii equation

The free energy associated with the generalized GPP equations (221) and (222) with a constant temperature $T$ can be written as $F = E_0 - T S_B$ (see Eq. (233)). When $\xi = 0$ and $T = 0$, the energy $E_0$ corresponds to the free energy of the standard GPP equations (5) and (6) and it is conserved. When $\xi \neq 0$ and $T \neq 0$, the energy $E_0$ is not conserved anymore. However, we can consider a model in which the temperature $T(t)$ varies with time in order to conserve $E_0$ exactly. This model can be relevant if, as advocated in the Introduction and in Sec. VII, the generalized GPP equations (221) and (222) provide a coarse-grained representation of the GPP equations (5) and (6), and an effective model of gravitational cooling. In that case, we can argue that the energy $E_0$ associated with the GPP equations (fine-grained) should be conserved by the generalized GPP equations (coarse-grained). This idea is similar to the one developed in the context of the theory of violent relaxation in Ref. [98]. Physically, the energy of the core decreases until it reaches the ground state and, in parallel, the energy lost by the core heats up the halo so that its temperature $T(t)$ increases so as to conserve the total (core + halo) energy $E_0 = E_{\text{core}}(t) + E_{\text{halo}}(t)$.

Taking the time derivative of Eq. (232), and proceeding as in Appendix D, we find that

$$\dot{E}_0 = -\frac{k_B T(t)}{m} \int u \cdot \nabla \rho \, dr - \xi \int \rho u^2 \, dr. \quad (I1)$$

Taking the time derivative of the Boltzmann entropy (120) and using the continuity equation (25), we get

$$\dot{S}_B = -\frac{k_B}{m} \int u \cdot \nabla \rho \, dr. \quad (I2)$$

Recalling Eq. (59), we find that Eq. (I1) can be rewritten as

$$\dot{E}_0 - T(t) \dot{S}_B = -2\xi \Theta_c. \quad (I3)$$

If we now impose that $\dot{E}_0 = 0$ according to the arguments given above, we find that the temperature must evolve as

$$T(t) = \frac{2\xi \Theta_c(t)}{\dot{S}_B(t)}. \quad (I4)$$

In this manner, the (coarse-grained) GPP equations (221) and (222) with a time-dependent temperature $T(t)$ given by Eq. (I1) relax towards a steady state with a core-halo structure while conserving the energy $E_0$ of the (fine-grained) GPP equations (5) and (6). This model may provide an improved coarse-grained parametrization of the process of gravitational cooling.

Remark: In the strong friction limit $\xi \to +\infty$, the previous equations remain valid with $u$ given by Eq. (29), where the pressure $P$ contains only the contribution (174) arising from the self-interaction, not from the temperature.

Appendix J: Particular solution of the continuity equation

In this Appendix, we derive a particular exact solution of the continuity equation (25). We stress, however, that this solution is generally not an exact solution of the Euler equation (26).

We consider a density profile of the form

$$\rho(r, t) = \frac{M}{R(t)} \, dx \, f \left[ \frac{r}{R(t)} \right] \quad (J1)$$

with $\int f(x) \, dx = 1$ so as to ensure the normalization condition (conservation of mass). We also consider a velocity profile of the form

$$u(r, t) = H(t) \, r. \quad (J2)$$

The continuity equation (25) can be rewritten as

$$\frac{\partial \ln \rho}{\partial t} + \nabla \cdot u + u \cdot \nabla \ln \rho = 0. \quad (J3)$$
From Eqs. (J1) and (J2), we obtain
\[ \frac{\partial \ln \rho}{\partial t} = -\frac{\dot{R}}{R} \mathbf{x} \cdot \nabla \ln f - d \frac{\dot{R}}{R}, \quad \nabla \ln \rho = \frac{1}{R} \nabla \ln f, \quad \nabla \cdot \mathbf{u} = dH. \] (J4)
Substituting the foregoing relations into Eq. (J3), we get
\[ (H - \frac{\dot{R}}{R}) (d + \mathbf{x} \cdot \nabla \ln f) = 0. \] (J5)
This equation is satisfied for all \( r \) if, and only if,
\[ H(t) = \frac{\dot{R}}{R}. \] (J6)
Using Eqs. (17) and (J2), we find that the action is given by
\[ S(r, t) = \frac{1}{2} mH(t) r^2 + S_0(t). \] (J7)
Therefore, the wave function is given by Eq. (15) where \( \rho(r, t) \) is given by Eq. (J1) and \( S(r, t) \) is given by Eq. (J7).

Appendix K: Details of the calculation of the free energy with the Gaussian ansatz

We consider a Gaussian density profile of the form
\[ \rho = Ae^{-r^2/R^2}. \] (K1)
The associated mass is
\[ M = A \int_0^{+\infty} e^{-r^2/R^2} S_{d\mathbf{r}} d^{-1} dr = \frac{1}{2} AS_d R^d \int_0^{+\infty} e^{-t(d-2)/2} dt = \frac{1}{2} AS_d R^d \Gamma \left( \frac{d}{2} \right), \] (K2)
where we have made the change of variables \( t = r^2/R^2 \). Equation (K2) determines the normalization constant through the relation
\[ A = \frac{2M}{S_d R^d \Gamma \left( \frac{d}{2} \right)}. \] (K3)
The moment of inertia (68) is given by
\[ I = A \int_0^{+\infty} e^{-r^2/R^2} r^2 S_{d\mathbf{r}} d^{-1} dr = \frac{1}{2} AS_d R^{d+2} \int_0^{+\infty} e^{-t(d-2)/2} dt = \frac{1}{2} AS_d R^{d+2} \Gamma \left( \frac{d}{2} + 1 \right) = \frac{d}{2} M R^2, \] (K4)
where we have made the change of variables \( t = r^2/R^2 \) to get the second equality, and used the identity \( \Gamma(x+1) = x\Gamma(x) \) and Eq. (K3) to get the fourth equality.
Substituting Eq. (268) into Eq. (59), we find that the classical kinetic energy can be written as
\[ \Theta_c = \frac{I}{2R^2} \left( \frac{dR}{dt} \right)^2 = \frac{d}{4} M \left( \frac{dR}{dt} \right)^2, \] (K5)
where we have used Eq. (K4) to get the second equality.
Substituting Eq. (K1) into Eq. (60), we find that the quantum kinetic energy can be written as
\[ \Theta_Q = \frac{\hbar^2 I}{2m^2 R^2} = \frac{d\hbar^2 M}{4m^2 R^2}, \] (K6)
where we have used Eq. (K4) to get the second equality.
Integrating by parts, we get (271).

Using the expression of \( \gamma \) given by Eq. (K3), we obtain the result of Eq. (272).

Using simple integrations by parts, we get (65), where we have made the change of variables \( \tilde{r} = \sqrt{\gamma} r \) to get the second equality. Replacing \( A \) by its expression given by Eq. (K3), we obtain the result of Eq. (270).

Substituting Eq. (K1) into Eq. (116), we find that the internal energy of a fluid with an isothermal equation of state is given by

\[
U_B = \frac{k_B T}{m} \int \rho \left( \ln A - \frac{r^2}{R^2} - 1 \right) \, dr = \frac{k_B T}{m} M (\ln A - 1) - \frac{k_B T}{m} \frac{I}{R^2}.
\]  

Using the identity

\[
U \gamma = \frac{\sqrt{\gamma}}{R} e^{-\gamma r^2 / R^2} \, d\gamma \quad \text{and} \quad \gamma > 0
\]

we have made the change of variables \( \tilde{r} = \sqrt{\gamma} r \) to get the second equality. Replacing \( A \) by its expression given by Eq. (K3), we obtain the result of Eq. (270).

Using the expression of \( \gamma \) given by Eq. (K3) and the expression of \( I \) given by Eq. (K4), we obtain the result of Eq. (271).

The potential energy of a harmonic potential is given by

\[
W_{\text{ext}} = \frac{1}{2} a_0^2 I = \frac{d}{4} a_0^2 M R^2,
\]

where we have used Eq. (K4) to get the second equality.

To compute the gravitational potential energy in \( d \neq 2 \), we use Eq. (K36). For the Gaussian profile (K1), we obtain

\[
W = -\frac{1}{d-2} S_d G A \int_0^{+\infty} e^{-r^2 / R^2} M(r, t) r \, dr.
\]

Integrating by parts, we get

\[
W = -\frac{1}{d-2} S_d G A \int_0^{+\infty} e^{-r^2 / R^2} M'(r) \, dr - \frac{1}{d-2} S_d G A^2 \frac{R^2}{2} \int_0^{+\infty} e^{-2r^2 / R^2} r d-1 \, dr = -\frac{1}{d-2} S_d G A M R^2 2^{1+d/2},
\]

where we have made the change of variables \( \tilde{r} = \sqrt{2} r \) to get the third expression and used Eq. (K2). Replacing \( A \) by its expression given by Eq. (K3), we obtain the result of Eq. (272). In \( d = 2 \), if we use the formula [see Eqs. (10) and (65)]

\[
W = \frac{1}{2} G \int \rho(r)\rho(r') \ln |r - r'| \, drdr',
\]

and make the change of variables \( x = r / R \), we obtain the result of Eq. (272) with

\[
W_0 = \frac{GM^2}{2\pi^2} \int e^{-(x^2 + x'^2)} \ln |x - x'| \, dx dx'.
\]

Using the identity

\[
\ln |x - x'| = \ln x_\succ - \sum_{m=1}^{+\infty} \frac{1}{m} \left( \frac{x_\prec}{x_\succ} \right)^m \cos \left[ m(\theta - \theta') \right]
\]

where \( x_\succ \) (resp. \( x_\prec \)) denotes the largest (smallest) value of \( x \) and \( x' \), we obtain

\[
\int e^{-(x^2 + x'^2)} \ln |x - x'| \, dx dx' = 4\pi^2 \int_0^{+\infty} dx x \ln x e^{-x^2} \int_0^x dx' x' e^{-x'^2} + 4\pi^2 \int_0^{+\infty} dx x e^{-x^2} \int_x^{+\infty} dx' x' \ln x' e^{-x'^2}.
\]

Using simple integrations by parts, we get

\[
\int e^{-(x^2 + x'^2)} \ln |x - x'| \, dx dx' = 4\pi^2 \int_0^{+\infty} e^{-x^2} \left( 1 - e^{-x^2} \right) x \ln x \, dx = \frac{\pi^2}{2} (\ln 2 - \gamma_E) = 0.572099...
\]

where \( \gamma_E = 0.57721566... \) is the Euler constant. This leads to the result of Eq. (275).
Appendix L: The generalized Jeans problem

In this Appendix, we study the linear dynamical stability of an infinite homogeneous self-gravitating BEC described by the generalized GPP equations (2) and (3). This is a generalization of the classical Jeans problem [152] to a quantum dissipative system. We use the hydrodynamic representation (25)-(27) of the generalized GPP equations (7) and (8). We consider an infinite homogeneous equilibrium state with \( \rho_{eq}(r, t) = \rho \) and \( \mathbf{u}_{eq}(r, t) = \mathbf{0} \) and \( S_{eq}(r, t) = -\mathbf{E}t \). In the presence of an external harmonic potential (14), the quantum Euler equation (26) reduces to the condition of hydrostatic equilibrium \(-\nabla \Phi - \omega_{0}^{2} \mathbf{r} = \mathbf{0}\). Considering the Poisson equation (27), this condition can be satisfied only when \( \omega_{0}^{2} < 0 \), \( \Phi = |\omega_{0}|^{2} r^{2}/2 \), and \( \rho = d |\omega_{0}|^{2} / S_{d} G \). Then, we have \( \mathbf{E} = m \hbar \rho \). In the absence of external potential, we make the Jeans swindle [101] (see [153] for a mathematical justification).

Considering a small perturbation about the equilibrium state and linearizing the hydrodynamic equations (25)-(27), we obtain

\[
\frac{\partial \delta}{\partial t} + \nabla \cdot \mathbf{u} = 0, \tag{L1}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} = -c_{s}^{2} \nabla \delta - \nabla \Phi + \frac{\hbar^{2}}{4m^{2}} \nabla (\Delta \delta) - \xi \mathbf{u}, \tag{L2}
\]

\[
\Delta \delta \Phi = S_{d} G \rho \delta, \tag{L3}
\]

where \( c_{s}^{2} = P'(\rho) \) is the speed of sound and \( \delta(r, t) = \delta \rho(r, t)/\rho \) the density contrast. We note that the external potential does not appear in the linearized equations. Taking the time derivative of Eq. (L1) and the divergence of Eq. (L2), we obtain a single equation for the density contrast

\[
\frac{\partial^{2} \delta}{\partial t^{2}} + \xi \frac{\partial \delta}{\partial t} = -\frac{\hbar^{2}}{4m^{2}} \Delta^{2} \delta + c_{s}^{2} \Delta \delta + S_{d} G \rho \delta. \tag{L4}
\]

Expanding the solutions into plane waves of the form \( \delta(r, t) \propto \exp[i(k \cdot r - \omega t)] \), we obtain the dispersion relation

\[
\omega^{2} + i\xi \omega = \frac{\hbar^{2} k^{4}}{4m^{2}} + c_{s}^{2} k^{2} - S_{d} G \rho \quad \Rightarrow \quad \omega = -i\xi \frac{k}{2} \pm \sqrt{\frac{\xi^{2}}{4} + \frac{\hbar^{2} k^{4}}{4m^{2}} + c_{s}^{2} k^{2} - S_{d} G \rho}. \tag{L5}
\]

For \( \xi = h = 0 \), we recover the celebrated Jeans dispersion relation [152]. For \( \xi = G = 0 \), we recover the Bogoliubov dispersion relation [102]. For \( \xi = 0 \), we recover the dispersion relation studied in Sec. V of [33]. The general dispersion relation (L5) will be studied in detail in a forthcoming paper (in preparation).

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