GENERICALLY FREE REPRESENTATIONS III:
EXCEPTIONALLY BAD CHARACTERISTIC

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Abstract. In parts I and II, we determined which irreducible representations $V$ of a simple linear algebraic group $G$ are generically free for $\text{Lie}(G)$, i.e., which $V$ have an open subset consisting of vectors whose stabilizer in $\text{Lie}(G)$ is zero, with some assumptions on the characteristic of the field. This paper settles the remaining cases, which are of a different nature because $\text{Lie}(G)$ has a more complicated structure and there need not exist general dimension bounds of the sort that exist in good characteristic. In a later work, we combine these results with those of Guralnick–Lawther–Liebeck to show that for any irreducible module for a simple algebraic group, there is a generic stabilizer (as a group scheme) and gives a classification of the generic stabilizers in all cases.

Let $G$ be a simple algebraic group over an algebraically closed field $k$. In case $k = \mathbb{C}$, it has been known for 40+ years which irreducible representations $V$ of $G$ are generically free, i.e., have the property that the stabilizer in $G$ of a generic $v \in V$ is the trivial group scheme. Recent applications of this to the theory of essential dimension have motivated the desire to extend these results to arbitrary $k$. We did this in previous papers — [GG17c], [GLL18], [GG17a], and [GG17b] — except for a handful of cases that we address here, completing the solution to the problem. In particular we prove the following, which was announced at the end of [GG17a].

Theorem A. Let $\rho : G \to \text{GL}(V)$ be a faithful irreducible representation of a simple algebraic group over an algebraically closed field $k$.

(1) $G_v$ is finite étale for generic $v \in V$ if and only if $\dim V > \dim G$ and $(G, V)$ does not appear in Table 1.

(2) $G$ acts generically freely on $V$ if and only if $\dim V > \dim G$ and $(G, V)$ appears in neither Table 1 nor Table 2.

The remaining cases of the theorem that need to be covered in this paper are where char $k$ is special$^1$ for $G$, meaning that $G$ has type $G_2$ and char $k = 3$ or $G$ has type $B_n$ ($n \geq 2$), $C_n$ ($n \geq 2$), or $F_4$ and char $k = 2$. These are the cases where the Dynkin diagram of $G$ has a multiple bond of valence char $k$. Equivalently, these are the cases where $G$ possesses so called “special” isogenies, which are neither central nor the Frobenius, cf. [BT73, §3].

$^1$This choice of vocabulary imitates [Ste63]; we have written instead the more illuminating “exceptionally bad characteristic” in the title. The hypothesis “char $k$ special” is properly more restrictive than “char $k$ very bad”, in that 2 is very bad but not special (i.e., not exceptionally bad) for type $G_2$. 

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We say that $V$ is faithful if $\ker \rho$ is the trivial group scheme, so Theorem A does not consider representations that factor through a special isogeny of $G$. Nonetheless, we do consider such representations in detail in this paper.

In a sequel work, [GGL18], we combine Theorem A with the results of [GLL18] to prove the existence of a stabilizer in general position for every action of a simple algebraic group on an irreducible representation.

Regarding the proof of Theorem A, the (abstract) subgroup $G_v(k)$ has been calculated in [GLL18]. To determine whether $G_v$ is trivial as a group scheme, it remains to determine whether the stabilizer $g_v := \{ x \in g \mid d\rho(x)v = 0 \}$ of $v$ in the Lie algebra $g$ of $G$ is zero. To allow for the possibility that $\rho$ is not faithful, we typically ask whether $g_v$ equals $\ker d\rho$, i.e., is as small as possible; when that is the case, we say that $g$ acts virtually freely on $V$, or $V$ is virtually free.

Large, possibly reducible representations. In [GG17a], we proved a general bound when $G$ is simple and $\text{char } k$ is not special: if $V^g = 0$ and $\dim V$ is big enough, then $g$ acts virtually freely on $V$. However, Example 7.2 shows that such a result does not hold when $\text{char } k$ is special. Because the true results are of a varied nature, we do not include a summary statement here; see Proposition 3.1, Lemma 5.2, Proposition 5.4, and Proposition 6.4 for precise statements. Note that these results have no requirements that $G$ acts irreducibly or faithfully.

Irreducible representations. Recall that every irreducible representation $V$ of $G$ has a highest weight $\lambda$, and we consider separately three cases. Specifically, we may write $\lambda = \lambda_0 + p\lambda_1$ where $p := \text{char } k \neq 0$ and $\lambda_0$ is restricted, i.e., when $\lambda_0$ is expressed as a sum $\sum \omega c_\omega \omega$ where the sum runs over the fundamental dominant weights $\omega$, we have $0 \leq c_\omega < p$ for all $\omega$. (In case $\text{char } k = 0$, all dominant weights are, by definition, restricted.) We first treat the case where $\lambda$ is itself restricted, i.e., when $\lambda_1 = 0$.

**Theorem B.** Let $G$ be a simple linear algebraic group over a field $k$. Let $\rho: G \to \text{GL}(V)$ be a restricted irreducible representation for $G$. Then $g$ acts virtually freely on $V$ if and only if $\dim V > \dim G$, except for those cases where $(G, V, k)$ appears in Table 1 or 3.

We prove here the case of the theorem where $\text{char } k$ is special for $G$, in which case Table 1 plays no role in Theorem B. The case where $\lambda_1$ is not special for $G$ is settled in [GG17b, Th. A].

We remark that, in the setting of Theorem B and on the level of abstract groups, $G_v(k)$ is always finite when $\dim V > \dim G$ by [GLL18].

When $\lambda_0 = 0$, $V$ is a Frobenius twist of the irreducible module of $G$ with highest weight $\lambda_1$, so $g$ acts trivially on $V$. The next theorem treats the remaining case, where $\lambda_0$ and $\lambda_1$ are both nonzero, and is slightly more general besides.

**Theorem C.** Let $G$ be a simple linear algebraic group over a field $k$. If $\rho: G \to \text{GL}(V)$ is an irreducible representation that is tensor decomposable, then for generic $v \in V$, $\text{Lie}(G)_v = \ker d\rho$.

We prove here the case of the theorem where $\text{char } k$ is special for $G$. The other cases are [GG17b, Th. B].
Theorems B and C are proved by applying the bounds on large representations proved earlier to reduce to considering representations of small dimension. There are only a finite number of those to consider, all of which are enumerated in [Lüb01]. These can then be inspected one by one. Only a few cases need to be inspected by hand, many of which can be settled by a quick computer calculation identical to those done in [GG17b].

| $G$ | char $k$ | weight | dim $V$ | dim $\mathfrak{g}_v$ | $G$ | char $k$ | weight | dim $V$ | dim $\mathfrak{g}_v$ |
|-----|---------|--------|--------|----------------|-----|---------|--------|--------|----------------|
| $A_7$ | 2 | $\wedge^4$ | 70 | 4 | $C_4$ | 3 | 0100 | 40 | 2 |
| $A_8$ | 3 | $\wedge^3$ or $\wedge^6$ | 84 | 3 | $B_2$ | 5 | 11 | 12 | 1 |
| $D_8$ | 2 | half-spin | 128 | 4 |

Table 1. Irreducible and restricted representations $V$ of simple $G$ with $\dim V > \dim G$ that are not virtually free for $\mathfrak{g}$. For each, the stabilizer $\mathfrak{g}_v$ of a generic $v \in V$ is a toral subalgebra, and $\dim \mathfrak{g}_v$ is given for the case where $G$ is simply connected.

| $G$ | char $k$ | $V$ | dim $V$ | $G$ | char $k$ | $V$ | dim $V$ |
|-----|---------|-----|--------|-----|---------|-----|--------|
| $A_1$ | $\neq$ 2, 3 | $S^3$ | 4 | $A_2$ | $\neq$ 2, 3 | $S^3$ | 10 |
| $A_1$ | $\neq$ 2, 3 | $S^4$ | 5 | $A_3$ | $\neq$ 2 | $L(2\omega_2)$ | 19 or 20 |
| $A_8$ | $\neq$ 3 | $\wedge^3$ | 84 | $A_7$ | $\neq$ 2 | $\wedge^4$ | 70 |
| $A_3$ | 3 | $L(\omega_1 + \omega_2)$ | 16 | $A_\ell$ | $p \neq 0$ | $L(\omega_1 + p\omega_2)$, $(\ell + 1)^2$ |
| $B_\ell$ ($\ell \geq 2$) | $\neq$ 2 | $L(2\omega_\ell)$ | | $C_4$ | $\neq$ 2 | “spin” | 41 or 42 |
| $D_\ell$ ($\ell \geq 4$) | $\neq$ 2 | $L(2\omega_\ell)$ | | $D_8$ | $\neq$ 2 | half-spin | 128 |

Table 2. Irreducible faithful representations $V$ of simple $G$ with $\dim V > \dim G$ such that $G_v$ is finite étale and $\neq 1$ for generic $v \in V$, up to graph automorphism.

| $G$ | char $k$ | $V$ | dim $V$ | dim ker $\rho$ |
|-----|---------|-----|--------|---------------|
| any | any | $k$ | 1 | $\dim G$ |
| Sp$_6$ | 2 | spin | 8 | 14 |
| Sp$_8$ (but not PSp$_8$) | 2 | spin | 16 | 27 |
| Sp$_{10}$ | 2 | spin | 32 | 44 |
| Sp$_{12}$ or PSp$_{12}$ | 2 | spin | 64 | 65 |

Table 3. The restricted irreducible representations of simple $G$ with $\dim V \leq \dim G$ that are virtually free for $\mathfrak{g}$.

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1. Structure of $\mathfrak{g}$ and $V$

Structure of $\mathfrak{g}$. We refer to [His84] and [Hog82] for properties of $\mathfrak{g} := \text{Lie}(G)$ when $G$ is simple. For example, when $G$ is simply connected, we have: (1) $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is a reducible $G$-module if and only if $\text{char } k$ is special, and (2) $\mathfrak{g}$ has a unique proper maximal $G$-submodule, which we denote by $\mathfrak{m}$. Statement (2) can be seen by direct computation or because $\mathfrak{g}$ is a Weyl module of $G$ in the sense of [Jan03], the one whose highest weight is the highest root.

Supposing now that $\text{char } k$ is special for $G$ and $\pi$ is the very special isogeny, we put $\mathfrak{n} := \ker d\pi$, a $G$-invariant ideal, see [CGP15, §7.1] or [Ste63, §10] for a concrete description of $\pi$. (We note that $\mathfrak{n}$ is the subalgebra in $\mathfrak{g}$ generated by the short root subalgebras, and it need not contain the center, e.g., in case $G = \text{Sp}_{2\ell}$ with odd $\ell \geq 3$.) It follows that

$$V^n := \{v \in V \mid d\rho(x)v = 0 \text{ for all } x \in \mathfrak{n}\}$$

is a $G$-invariant submodule of $V$.

Examining the tables in [His84] and [Hog82], we find the following:

**Lemma 1.1.** Let $G$ be a simple and simply connected split algebraic group over a field $k$ whose characteristic is special for $G$. If $\mathfrak{h}$ is a $G$-invariant submodule of $\mathfrak{g}$, then either $\mathfrak{h} \supseteq \mathfrak{n}$ or $\mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{g})$. \hfill \square

Irreducible representations of $G$ when $\text{char } k \neq 0$. Fix a pinning for $G$, which includes the data of a maximal torus $T$ and a choice of simple roots $\Delta$. Then irreducible representations $\rho: G \to \text{GL}(V)$ (up to equivalence) are in bijection with the set of dominant weights $\lambda \in T^\ast$, i.e., those $\lambda$ such that $\langle \lambda, \delta \rangle \geq 0$ for all $\delta \in \Delta$.

Supposing now that $p := \text{char } k \neq 0$ and writing $\lambda = \lambda_0 + p\lambda_1$ for dominant weights $\lambda_0, \lambda_1$ with $\lambda_0$ restricted, the irreducible representation $L(\lambda)$ with highest weight $\lambda$ is equivalent to $L(\lambda_0) \otimes L(\lambda_1)^{[p]}$, the tensor product of $L(\lambda_0)$ with a Frobenius twist of $L(\lambda_1)$. As a representation of $\mathfrak{g}$ (forgetting about the action of $G(\mathfrak{k})$), this is the direct sum of $\dim L(\lambda_1)$ copies of $L(\lambda_0)$.

Irreducible representations of $G$ when $\text{char } k$ is special. Now suppose that $\text{char } k$ is special for $G$, so in particular $\Delta$ has two root lengths. For any weight $\lambda = \sum c_\delta \omega_\delta$, we may write $\lambda = \lambda_s + \lambda_\ell$ where $\lambda_s = \sum_{\text{short}} c_\delta \omega_\delta$ and $\lambda_\ell = \sum_{\text{long}} c_\delta \omega_\delta$, equivalently, $\lambda_s$ vanishes on the long simple roots and $\lambda_\ell$ vanishes on the short simple roots. Steinberg [Ste63] shows that $L(\lambda) \cong L(\lambda_\ell) \otimes L(\lambda_s)$ and that furthermore $L(\lambda_\ell)$ factors through the very special isogeny.

Suppose now that $\lambda$ is restricted. Then $L(\lambda_\ell)$ is irreducible as a representation of $\mathfrak{n}$ [Ste63, p. 52], so Lemma 1.1 shows that the kernel of this representation is contained in $\mathfrak{z}(\mathfrak{g})$. Similarly, as an $\mathfrak{n}$-module, $L(\lambda)$ is a direct sum of $\dim L(\lambda_\ell)$ copies of $L(\lambda_s)$, and again the kernel of the representation is contained in $\mathfrak{z}(\mathfrak{g})$.

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**Table 4.** Dynkin diagrams of the non-simply-laced simple root systems, with simple roots numbered as in [Lüb01]

- $B_\ell$: 
  - $1$ \rightarrow $2$ \rightarrow $3$ \rightarrow $\cdots$ \rightarrow $\ell$

- $C_\ell$: 
  - $1$ \rightarrow $2$ \rightarrow $3$ \rightarrow $\cdots$ \rightarrow $\ell$

- $F_4$: 
  - $1$ \rightarrow $2$ \rightarrow $3$ \rightarrow $4$

- $G_2$: 
  - $1$ \rightarrow $2$
In summary, for \( \lambda \) restricted, we have either (1) \( \lambda_e = 0 \) and \( \ker d\rho \supseteq \mathfrak{n} \), or (2) \( \lambda_e \neq 0 \) and \( \ker d\rho \subseteq \mathfrak{s}(\mathfrak{g}) \).

2. KEY BACKGROUND

Put \( \mathfrak{g} := \text{Lie}(G) \) and choose a representation \( \rho: G \to \text{GL}(V) \). For \( x \in \mathfrak{g} \), put
\[
V^x := \{ v \in V \mid d\rho(x)v = 0 \}
\]
and \( x^G \) for the \( G \)-conjugacy class \( \text{Ad}(G)x \) of \( x \). Recall the following from [GG17a, §1].

Lemma 2.1. For \( x \in \mathfrak{g} \),
\[
(2.2) \quad x^G \cap \mathfrak{g}_v = \emptyset \quad \text{for generic } v \in V
\]
is implied by:
\[
(2.3) \quad \dim x^G + \dim V^x < \dim V,
\]
which is implied by:
\[
(2.4) \quad \text{There exists } e > 0 \text{ and } x_1, \ldots, x_e \in x^G \text{ such that the subalgebra } \mathfrak{s} \text{ of } \mathfrak{g}
\]
genenerated by \( x_1, \ldots, x_e \) has \( V^s = 0 \) and \( e \cdot \dim x^G < \dim V \).

We use this as follows. Choose a subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). To verify that \( \mathfrak{g}_v \subseteq \mathfrak{h} \) for generic \( v \in V \), it suffices to verify, for all \( x \in \mathfrak{g} \setminus \mathfrak{h} \), that \( x \notin \mathfrak{g}_v \), for which we may check (2.3) or (2.4).

Moreover, assuming \( \mathfrak{h} \) is restricted, we can write \( x \) as a sum of elements that are toral (i.e., \( x^{[p]} = x \)) or nilpotent (i.e., \( x^{[p]} = 0 \) for some \( n \geq 1 \)), and it follow that it suffices to check that \( x \notin \mathfrak{g}_v \) for such elements. If furthermore \( \mathfrak{h} \) consists of semisimple elements, we need only check that \( x \notin \mathfrak{g}_v \) for \( x \in \mathfrak{g} \setminus \mathfrak{h} \) with \( x^{[p]} \in \{0, x\} \).

3. TYPE \( F_4 \) OR \( G_2 \)

Suppose \( G \) has type \( F_4 \) or \( G_2 \) and \( \text{char } k = 2 \) or \( 3 \) respectively. The maximal ideal \( \mathfrak{m} \) equals the kernel \( \mathfrak{n} \) of the very special isogeny; it is the unique nonzero and proper ideal of \( \mathfrak{g} \). Both \( \mathfrak{n} \) and \( \mathfrak{g}/\mathfrak{n} \), as Lie algebras, are the simple quotient \( \mathfrak{g}/\mathfrak{s}(\mathfrak{g}) \), where \( \mathfrak{g} = \mathfrak{spin}_8 \) or \( \mathfrak{g}_2 \) respectively.

It is not true that large faithful modules are virtually free for these groups, see Example 7.2, but we do have the following:

Proposition 3.1. Let \( G \) be a simple algebraic group of type \( F_4 \) or \( G_2 \) over a field \( k \) such that \( \text{char } k = 2 \) or \( 3 \) respectively. Let \( \rho: G \to \text{GL}(V) \) be a representation of \( G \).

(1) If \( \ker d\rho = \mathfrak{n} \) and \( \dim V > 192 \) or \( 36 \) respectively, then for generic \( v \in V \), \( \text{Lie}(G)_v = \mathfrak{n} \).

(2) If \( V \) has a \( G \)-subquotient \( X \) with \( X^n = 0 \) and \( \dim X > 240 \) or \( 48 \) respectively, then for generic \( v \in V \), \( \text{Lie}(G)_v = 0 \).

Proof. Suppose first that \( \ker d\rho = \mathfrak{n} \) and let \( x \in \mathfrak{g} \setminus \mathfrak{n} \) be nilpotent or toral. For \( x \) nilpotent, put \( \pi(x) \) for the image of \( x \) in \( \mathfrak{g}/\mathfrak{n} \). The closure of \( \pi(x)^G \) contains a root element \( y \in \mathfrak{g}/\mathfrak{n} \), and at most \( 4 \) or \( 3 \) \( G \)-conjugates of \( y \) (hence also of \( x \)) suffice to generate \( \mathfrak{g}/\mathfrak{n} \), see [GG17a, Prop. 9.4, 5.4]; call this number \( e \). If \( \dim V > e \cdot (\dim G - \text{rk} G) = 192 \) or \( 36 \), then (2.4) holds and the class of \( x \) does not meet \( \mathfrak{g}_v \) for generic \( v \). If \( x^{[p]} = x \), then we may reduce to a nilpotent element...
by passing to closures as in [GG17a, §3] and obtain the same bound. Since every nilpotent and toral element of \( g \) belongs to \( n \), we have \( g_v \subseteq n \), proving (1).

Next suppose that there exists a subquotient \( X \) as in (2). It suffices to prove the claim in case \( V = X \), cf. [GG17a, Prop. 2.4]. Arguing as in the preceding paragraph with the roles of \( n \) and \( g \) replaced by 0 and \( n \), we find that for \( \dim V > 192 \) or 36, \( \dim n_v = 0 \). Relative to a chosen maximal torus in \( G \), the long root subalgebras generate a subalgebra \( h \) isomorphic to \( \text{spin}_8 \) or \( \mathfrak{sl}_3 \) that surjects onto \( g/n \) under the natural quotient map, and as in the previous paragraph, \( e \) generic conjugates generate a subalgebra containing \( h \). As a representation of \( h \), \( g \) is a sum of \( h \) and three inequivalent 8-dimensional representations (for type \( F_4 \) through \( \lambda \)) restricted highest weight nilpotent \( x \in g \) generate a subalgebra \( h \) isomorphic to \( \text{spin}_8 \) or \( \mathfrak{sl}_3 \) that surjects onto \( g/n \) under the natural quotient map, and as in the previous paragraph, \( e \) generic conjugates generate a subalgebra containing \( h \). As a representation of \( h \), \( g \) is a sum of \( h \) and three inequivalent 8-dimensional representations (for type \( F_4 \) or two inequivalent 3-dimensional representations (for type \( G_2 \)), so \( e + 1 \) generic conjugates of a nonzero nilpotent \( x \in h \) will generate \( g \). Therefore, it suffices to take \( \dim V > 240 \) or 48. □

**Restricted irreducible representations.** Let \( G = G_2 \) and \( \text{char } k = 3 \) or \( G = F_4 \) and \( \text{char } k = 2 \), and suppose \( \rho : G \to \text{GL}(V) \) is an irreducible representation with restricted highest weight \( \lambda \). We aim to prove Theorem B in this case.

If \( \dim V \leq \dim G \), then by A.49 and A.50 in [Lüb01], \( V \) is either the natural module (of dimension 7 or 26, respectively) or the irreducible quotient \( g/n \) of the adjoint representation. For \( \rho \) the natural module, \( \ker d\rho = 0 \) and a generic vector has stabilizer of type \( A_2 \) or \( D_4 \) respectively (of dimension 8 or 28 respectively). Note that this stabilizer has dimension larger than \( \dim g/n \), so it meets \( n \), the image of \( g \) under the very special isogeny. It follows that composing the natural representation with the very special isogeny gives a representation with \( \ker d\rho = n \) that is not virtually free; this is \( g/n \).

If \( \dim V > 48 \) or 240 respectively, then \( V \) is virtually free by Proposition 3.1. Table A.50 in [Lüb01] shows that we have considered all restricted irreducible representations of \( F_4 \), so the proof of Theorem B is complete in that case.

For \( G_2 \), there are two remaining possibilities for \( \rho \), according to Table A.49. The first has dimension 27 and \( \ker d\rho = 0 \). We observe that the representation factors through \( \mathfrak{so}_7 \), for which the action is virtually free by [GG17a, Lemma 13.1], or one can verify that this \( \rho \) is virtually free using a computer.

The last possibility for \( \rho \) is obtained by composing the representation in the preceding paragraph with the very special isogeny. This representation is virtually free by the considerations in the previous paragraph, completing the proof of Theorem B for \( G \) of type \( G_2 \).

4. A Heisenberg Lie algebra

Let \( G = \text{Spin}_{2n+1} \) over a field \( k \) of characteristic 2. The short root subalgebras of \( g \) generate a “Heisenberg” Lie algebra \( h \) of dimension \( 2n+1 \) such that \( [h, h] \) is the 1-dimensional center \( z(h) \). The algebra \( h \) is the image of \( \mathfrak{sl}_2^{\times n} \) under a central isogeny \( \text{SL}_2^{\times n} \to \text{SL}_2^{\times n} / Z \) where \( Z \) is isomorphic to \( \mu_2^{\times (n-1)} \), and the quotient \( h/z(h) \) is the image of \( \mathfrak{sl}_2^{\times n} \to \mathfrak{psl}_2^{\times n} \).

For \( G = \text{Sp}_{2n} \) over the same k, the very special isogeny \( \pi : \text{Sp}_{2n} \to \text{Spin}_{2n+1} \) has image \( h \), and so we may identify \( h \) with \( g/\ker d\pi \).

**Lemma 4.1.** Suppose \( \rho : G \to \text{GL}(V) \) is a representation of \( G = \text{Spin}_{2n+1} \) or \( \text{Sp}_{2n} \). In the latter case, assume additionally that \( d\rho \) vanishes on \( \ker d\pi \).

1. If \( 4n + \dim V \pi(h) < \dim V \), then \( \dim xG + \dim Vx < \dim V \) for all nonzero \( x \in h \).
Lemma 5.2. Let \( V^h = 0 \) and \( 4n^2 < \dim V \), then \( \dim x^G + \dim V^x < \dim V \) for all noncentral \( x \in \mathfrak{h} \).

Proof. For \( x \) nonzero central, \( \dim x^G + \dim V^x = \dim V^3(b) \), verifying (1), so suppose \( x \) is noncentral.

In case (1), there is a \( g \in G(k_{\text{alg}}) \) so that \([x, x^g]\) is nonzero central in \( \mathfrak{h} \), so \( \dim V^x \leq \frac{1}{2}(\dim V + \dim V^3(b)) \). As \( \dim x^G < 2n + 1 \), the claimed inequality follows.

In case (2), \( 2n \) conjugates of \( x \) generate \( \mathfrak{h} \), and therefore to prove the claim it suffices to note that \( 2n \cdot \dim x^G < \dim V \) \cite[Lemma 1.1]{GG17a}.

5. Type B

For \( G = \text{Spin}_{2n+1} \) for some \( n \geq 2 \) over a field \( k \) of characteristic 2, the Lie algebra \( \mathfrak{g} \) is uniserial where the short root subalgebra \( n \) is the Heisenberg Lie algebra \( \mathfrak{h} \) from \&sect;4.

Lemma 5.1. Consider representations \( V \) and \( W \) of a Lie algebra \( L \). For nilpotent \( x \in L \), \( \dim(V \otimes W)^x \leq (\dim V^x)(\dim W) \).

Proof. Put \( \psi : L \rightarrow \mathfrak{gl}(V) \) and \( \zeta : L \rightarrow \mathfrak{gl}(W) \) for the two actions. For each \( t \in k, t \zeta \) is a representation of the Lie algebra \( k \mathfrak{t} \); since \( x \) is nilpotent the ones with \( t \neq 0 \) are all equivalent. Therefore, writing \( V_t \) for the representation \( \psi \otimes (t \zeta) \), the dimension of \( (V_t)^x \) is constant for \( t \neq 0 \). Now \( V_0 \) is a direct sum of \( \dim W \) copies of \((V, \psi)\), so \( \dim(V_0)^x = (\dim V^x)(\dim W) \). On the other hand, by upper semicontinuity of dimension, \( \dim(V_0)^x \geq \dim(V_t)^x \) for \( t \neq 0 \).

Lemma 5.2. Let \( G = \text{Spin}_{2n+1} \) for some \( n \geq 2 \) over a field \( k \) of characteristic 2. If \( V \) is a representation of \( G \) such that \( V^3(\mathfrak{g}) = 0 \) and (1) \( \dim V > 4n^2 + 4n \) or (2) \( n \geq 8 \), then (a) \( \dim x^G + \dim V^x < \dim V \) for all noncentral \( x \in \mathfrak{g} \) such that \( x[2] \in \{0, x\} \) and (b) \( V \) is virtually free for \( \mathfrak{g} \).

Proof. We first claim that \( \dim V^x \leq \frac{1}{4} \dim V \). By passing to closures, it suffices to do so in case \( x \) is nilpotent. If \( V \) is a restricted irreducible representation, then it is \( L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_2) \) where \( L(\lambda_t) \) is the spin representation. If \( \lambda_t = 0 \) then the claim holds if \( n = 2 \) (because \( \dim V = 4 \)) and if \( n \geq 3 \) by \cite[Prop. 2.1(i)]{GG17c}. If \( \lambda_t \neq 0 \), the claim is proved by Lemma 5.1. If \( V \) is irreducible but not restricted, then its highest weight is \( \lambda_0 + p\lambda_1 \) where \( \lambda_0 \) is nonzero and restricted, so \( V \cong \oplus^{\dim L(\lambda_1)} L(\lambda_0) \) as a \( \mathfrak{g} \)-module and again the claim holds. Finally, for a composition series \( 0 = V_0 \subset V_1 \subset \cdots \subset V_r = V \), we have \((V_i/V_{i-1})^3(\mathfrak{g}) = 0 \) because \( \mathfrak{g}(\mathfrak{g}) \) acts semisimply on \( V \) and \( \dim(V')^x \leq \frac{n}{2} \dim V \) for \( V' := \oplus V_i/V_{i-1} \), so also for \( V \), proving the claim.

Now fix \( x \in \mathfrak{g} \) as in the statement. If \( x[2] = 0 \), then the image \( \bar{x} \in \mathfrak{so}_{2n+1} \) of \( x \) — as a \((2n+1)\)-by-\((2n+1)\) matrix — has even rank \( 2r \) and \( \dim \bar{x}^G = 2r(2n-2r+1) \) by the formulas from \cite{Hes79} or \cite{LS12}, compare \cite[Example 9.5]{GG17a}. Thus \( \dim x^G = \dim 2r(2n-2r+1) \), because the map \( \mathfrak{spin}_{2n+1} \rightarrow \mathfrak{so}_{2n+1} \) is injective on nilpotents. (Indeed, if \( x \) and \( x+z \) are square-zero and \( z \) is central in \( \mathfrak{spin}_{2n+1} \), then \( 0 = (x+z)^{[2]} = z^{[2]} \), so \( z = 0 \).) If \( x[2] = x \), then the centralizer of \( x \) has type \( D_r \times B_{n-r} \) for some \( r \) and we find the same formula for \( \dim x^G \). The dimension is maximized for \( r = (2n+1)/4 \), so \( \dim x^G \leq n^2 + n \). Thus, (1) implies (a).

Suppose (2), \( n \geq 8 \). If \( V \) is the spin representation, then \( \dim x^G + \dim V^x < \dim V \) as in \cite{GG17c}. If \( V \) is restricted irreducible but not the spin representation,
i.e., \( V \cong (\text{spin}) \otimes L(\lambda) \), then \( \dim V \geq 2n2^n > 4n^2 + 4n \) and again (a) holds. Claim (a) follows for general \( V \) as in the first paragraph of this proof.

(a) implies (b) as recalled in §2. \( \square \)

The very special isogeny \( G = \text{Spin}_{2n+1} \to \text{Sp}_{2n} \) is another way of viewing the alternating statement that the alternating bilinear form on the natural module of \( G \) is \( G \)-invariant. It factors through \( \text{SO}_{2n+1} \), and the image \( g/n \) of \( g \) in \( \mathfrak{sp}_{2n} \) is isomorphic to the derived subalgebra of \( \mathfrak{so}_{2n} \), which is a simple \( G \)-module (i.e., \( n = m \)) if \( n \) is odd and has a 1-dimensional center if \( n \) is even (i.e., \( n \) has codimension 1 in \( m \)).

**Example 5.3.** For \( n \) even, the composition series for \( g := \mathfrak{spin}_{2n+1} \) is \( 0 \subset \mathfrak{g}(g) \subset n \subset m \subset g \), where the successive quotients have dimension 1, \( 2n, 1, 2n^2 - n - 2 \) [His84]. The long root subalgebras of \( g \) (determined by a choice of maximal torus in \( G \)) generate a subalgebra \( \mathfrak{spin}_{2n} \) of type \( D_n \). In view of the preceding paragraph, \( \mathfrak{spin}_{2n} \) maps on to the image \( g/n \) of the very special isogeny. In particular, \( m = n + \mathfrak{spin}_{2n} \) and \( g = \mathfrak{spin}_{2n} + n \) as vector spaces.

**Proposition 5.4.** Let \( V \) be a representation of \( G := \text{Spin}_{2n+1} \) for some \( n \geq 2 \) and assume \( \text{char } k = 2 \). If \( \dim V/V^g > \max\{5n^2 + 10n, 6n^2 + 8n - 8\} \), then a generic \( v \in V \) has \( g_v \subseteq m \).

**Proof.** We have \( (V/V^g)^g = 0 \) (see, for example, [GG17a, Lemma 10.3]), so we may assume that \( V^g = 0 \).

Let \( x \in g \setminus m \) and suppose \( x[2] - x \in n \) or \( x[2] \in n \). Let \( y \in \mathfrak{so}_{2n} \) denote the image of \( x \) — which is noncentral because \( x \not\in m \) — and put \( r \) for the rank of the \( 2n \)-by-\( 2n \) matrix \( y \). As in the proof of [GG17a, Cor. 9.6], we have (1) \( \max\{4, \lceil n/r \rceil \} \) conjugates of \( y \) suffice to generate \( [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}] \), and therefore \( e := \max\{5, \lceil n/r \rceil + 1\} \) conjugates generate \( g \), and (2) the class of \( y \) in \( \mathfrak{so}_{2n} \) has dimension at most \( 4r(n - r) \), so \( \dim x^G \leq 4r(n - r) + 2n \).

We now give an upper bound for the product \( e(4r(n - r) + 2n) \). If \( n \leq 4r \), then \( e = 5 \) and the product is maximized at \( r = n/2 \), where it takes the value \( 5n^2 + 10n \). If \( n \geq 34 \), then \( e < n/r + 2 \). The derivative with respect to \( r \) of the product is \( (4rn^2 - 2n^2 - 16rn^3)/r^3 \). The numerator is non-positive (with maximum 0 at \( r = 8 \) and \( n = 64 \)), so the product is maximized at \( r = 1 \), where it is \( 6n^2 + 8n - 8 \).

As \( V^g = 0 \), the hypothesis on \( \dim V \) guarantees that \( \dim x^G + \dim V^x < \dim V \) for \( x \), so \( x \not\in g_v \), for generic \( v \in V \).

Now suppose \( x \in g \setminus m \) is nilpotent. If \( x \in g_v \), then \( x[2] \in g_v \), and iterating we find a power of \( x \) outside \( m \) whose square is in \( m \), a contradiction. Therefore \( g_v \setminus m \) contains no toral nor nilpotent elements, and so must be empty. \( \square \)

**Lemma 5.5.** Let \( \rho : \text{Spin}_{2n+1} \to \text{GL}(V) \) be a representation over a field of characteristic 2. If \( \rho \) factors through the very special isogeny, \( V^{\text{spin}_{2n+1}} = 0 \), and \( \dim V > 4n^2 \), then \( \text{spin}_{2n+1} \) and \( \mathfrak{so}_{2n+1} \) act virtually freely on \( V \).

**Proof.** By hypothesis, \( \rho \) factors through \( \text{Sp}_{2n} \). The image of \( \mathfrak{so}_{2n+1} \) in \( \mathfrak{sp}_{2n} \) in \( \mathfrak{so}_{2n} \), the unique maximal \( \text{Sp}_{2n} \)-invariant ideal in \( \mathfrak{sp}_{2n} \) and the Lie algebra of a subgroup \( \text{SO}_{2n} \subset \text{Sp}_{2n} \). Now the image of \( \mathfrak{spin}_{2n+1} \) is \( [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}] \), so \( V^{\mathfrak{so}_{2n}} = V^{[\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]} = V^{\text{spin}_{2n+1}} = 0 \), where the first equality is by [GG17a, Lemma 10.3(1)]. Theorem A in [GG17a] gives that \( \mathfrak{so}_{2n} \) acts virtually freely on \( V \), so \( \mathfrak{so}_{2n+1} \) and \( \text{spin}_{2n+1} \) do as well. \( \square \)
Example 5.6. Let \( \rho : G \to \text{GL}(V) \) be the irreducible quotient \( L(\omega_n) \) of the natural module of \( G := \text{Spin}_{2n+1} \) over a field \( k \) of characteristic 2. It has \( \ker d\rho = n \), \( V^g = 0 \), and \( \dim V = 2n \). For \( W := \mathfrak{o}^g V \) with \( c > 2n \), Lemma 5.5 says that \( W \) is virtually free. (Compare [GG17b, Example 8.2] for the case char \( k \neq 2 \).)

Restricted irreducible representations, part I. Let \( V \) be a restricted irreducible representation of a group \( G \) of type \( B_n \) for some \( n \geq 2 \) over a field \( k \) of characteristic 2; we aim to prove Theorem B for this \( G \). The highest weight \( \lambda = \sum c_i \omega_i \) of \( V \) has \( c_i \in \{0,1\} \) for all \( i \). If \( \lambda = 0 \), equivalently \( \ker d\rho = \mathfrak{g} \), then there is nothing to do.

In this part I, we assume that \( c_1 \neq 0 \), in which case \( \ker d\rho = 0 \) and \( G \cong \text{Spin}_{2n+1} \). Further, \( V^{\text{Lie}(\mathfrak{g})} = 0 \) as in Lemma 5.2 and in particular we may assume \( n \leq 7 \).

Example 5.7 (spin representation). The spin representation \( V := L(\omega_1) \) of \( G := \text{Spin}_{2n+1} \) is virtually free if and only if \( n \geq 7 \) [GG17c], if and only if \( \dim V > \dim G \).

We remark that one can check with a computer that for \( n = 2, 3, 4, 5, 6 \), a sum of \( 4, 4, 3, 2, 2 \) copies of \( V \) is generically free for \( \text{spin}_{2n+1} \).

For each of \( 2 \leq n \leq 7 \), there is a table of restricted irreducible representations of \( G \) in [Lüb01] and we find that the only ones that are not the spin representation nor satisfy the bounds of Lemma 5.2(2) are, for \( n = 3 \) or 2, the representation with highest weight \( \omega_1 + \omega_n \) of dimension 48 or 16 respectively. It is easy to generate random vectors in \( V \) with stabilizer 0, so \( V \) is virtually free and \( \dim V > \dim G \).

Restricted irreducible representations, part II. We continue now the proof of Theorem B for type \( B \), except now we assume that \( c_1 = 0 \). Thus, the highest weight \( \lambda \) belongs to the root lattice and the representation \( \rho \) factors through not just \( \text{SO}_{2n+1} \) but \( \text{Sp}_{2n} \). We abuse notation by writing \( G = \text{Spin}_{2n+1} \).

To prove Theorem B, we must verify that \( \text{spin}_{2n+1} \) (resp., \( \mathfrak{so}_{2n+1} \)) operates virtually freely on \( V \) if and only if \( \dim V > \dim G \). Note that it suffices to show that (1) \( \text{spin}_{2n+1} \) does not act virtually freely when \( \dim V \leq \dim G \) and (2) \( \mathfrak{so}_{2n+1} \) does act virtually freely when \( \dim V > \dim G \).

Example 5.8 (natural representation). Here we treat \( V := L(\omega_n) \). Note that \( \dim V = 2n < \dim G \) and \( \mathfrak{n} V = 0 \). Note that \( V \) is the natural module for \( \mathfrak{d} \) and the generic stabilizer in \( \mathfrak{d} \) is thus isomorphic to \( \mathfrak{so}_{2n-1} \) and so in \( \mathfrak{g} \), the generic stabilizer is \( \mathfrak{n} + \mathfrak{so}_{2n-1} \).

Example 5.9 (adjoint representation). Here we treat \( V := L(\omega_{n-1}) \), the irreducible quotient of the Weyl module \( \text{spin}_{2n+1} \). The long root subalgebras of \( \text{spin}_{2n+1} \) relative to a maximal torus generate a subalgebra \( \mathfrak{d} = \text{Lie}(\text{Spin}_{2n}) \) of type \( D_n \) and \( \text{spin}_{2n+1} = \mathfrak{d} + \mathfrak{n} \). This \( V \) is the irreducible quotient of the \( \mathfrak{d} \)-module \( \mathfrak{d} \). By [GG17a, Example 13.2], the generic stabilizer in \( \mathfrak{d} \) is \( \text{Lie}(T) \) for \( T \) a maximal torus and so in \( \mathfrak{g} \), the generic stabilizer is \( \mathfrak{n} + \text{Lie}(T) \).

Applying Lemma 5.5, we may assume that \( \dim V \leq 4n^2 \). If \( n > 11 \), then \( \dim V < n^3 \) and by [Lüb01], \( V \) is either the \( 2n \)-dimensional representation \( L(\omega_n) \) or it is \( L(\omega_{n-1}) \) as in the preceding examples.

So assume \( 2 \leq n \leq 11 \). Examining the tables in [Lüb01], we find that the only possibilities for \( V \) that have not yet been considered are the cases where \( n = 4 \) or 5, \( \lambda = \omega_{n-2} \), and \( \dim V = 48 \) or 100 respectively. In both cases, \( \mathfrak{n} \) is in the kernel of the representation and computer calculations with Magma as in [GG17b] produces a vector in \( V \) with stabilizer \( \mathfrak{n} \). Note that \( \dim V > \dim G \) in both cases.
Example 5.10 (Spinₐ). For later reference, we examine more carefully the spin representation V of G = Spinₐ. The stabilizer Gᵥ of a generic vector v ∈ V is isomorphic to Spinₐ [GG17c] and is contained in a long root subgroup Spinₗ in G in such a way that the composition Spinₐ ⊂ Spinₗ ⊂ Spin₉ → SO₉ is injective. (See for example [Var01] for a discussion of the first inclusion in the case k = ℝ.) In particular, 3(σ(Tᵥ)) ̸= 3(σ(nₐ)).

As in Example 5.3, 3(σ(nₐ)) ⊂ 3(σ(nₐ)) ⊂ m. (The first term has dimensions 1 and 2.) We claim that furthermore 3(σ(Tᵥ)) ⊂ 3(σ(nₐ)) so 3(σ(Tᵥ)) + 3(σ(nₐ)) = 3(σ(Tᵥ)). To see this, restrict V to Spinₗ to find a direct sum V₁ ⊕ V₂ of inequivalent 8-dimensional irreducible representations. One V₁ restricts to be the spin representation of Spinₗ and the other is uniserial with composition factors of dimension 1, 6, 1, corresponding to the inclusion SO₇ ⊂ SO₈. From this we can read off the action of the central µ₂ of Spinₗ on V and we find that it is central in Spinₗ, proving the claim.

We have spinₗ ∩ n = 3(σ(nₐ)) and spinₗ ∩ m = 3(σ(nₐ)). Using that gᵥ = spinₐ, we conclude that nᵥ = 0 and mᵥ = 3(σ(Tᵥ)).

6. Type C

For G = Sp₂₂ with n ≥ 2 over a field k of characteristic 2, the Lie algebra g has derived subalgebra m = so₂ of codimension 2n; it is the unique maximal G-invariant ideal in g. The short root subalgebra n = ker dσ has codimension 1 in m and is [m, m]. The quotient g⟨n⟩ is the Heisenberg Lie algebra from the section 4. The unique maximal ideal m also contains 3(g) (dimension 1). The center 3(g) is contained in n if and only if n is even.

Example 6.1. Suppose b is a nondegenerate alternating bilinear form on a finite-dimensional vector space V over a field F (of any characteristic). This gives an "adjoint" involution σ : Endₓ(V) → Endₓ(V) such that b(Tᵥ, v') = b(v, σ(T)v') for all T ∈ Endₓ(V) and v, v' ∈ V. If x ∈ Endₓ(V) is such that σ(x) = ±x, then we find an equation b(xv, v′) = ±b(v, xv′). By taking v ∈ ker x or (im x)⊥ and allowing v′ to vary over V we find each of the containments between ker x and (im x)⊥, i.e., ker x = (im x)⊥. If additionally x² = x (i.e., x is a projection), then V is an orthogonal direct sum (ker x) ⊕ (im x).

If x ∈ Endₓ(V) is such that σ(x)x = 0, then b(xv, xv′) = 0 for all v, v' ∈ V and we find that im x ⊆ (im x)⊥, i.e., im x is totally singular.

We first make some remarks about nilpotent elements of square 0.

Lemma 6.2. Let x ∈ sp₂₂ for n ≥ 3 have x[2] = 0 and rank r. Then dim xsp₂₂ ≤ r(2n + 1) − r².

Proof #1. Let C be the set of nilpotent elements of square 0 and rank r in sp₂₂, so x ∈ C(k). Note that im x is totally singular and ker x = (im x)⊥ by the preceding example.

Consider the map C → X, where X is the variety of totally singular spaces of dimension r, sending x → im x. This is a surjection by the remark above. The fiber corresponds to the set of x with a given image and kernel. For the parabolic subgroup P stabilizing x, the radical of Lie(P) has a natural filtration of length 2 where the bottom term (of dimension r(r+1)/2) has an open subvariety isomorphic to the fiber of C → X.
The dimension of $P$ is the sum of the dimension of a Levi subgroup $\text{Sp}_{2(n-r)} \text{SL}_r \mathbb{G}_m$ and the two terms in the filtration, of dimension $r(r+1)/2$ and $2r(n-r)$. In summary, 
\[
\dim C = \dim X + r(r+1)/2 = \dim \text{Sp}_{2n} - \dim P + r(r+1)/2 = 2r(n-r) + r(r+1),
\] as claimed.

Note that $C$ consists of a single class if $r$ is odd and two classes if $r$ is even (corresponding in the nil radical to either skew or symmetric but not skew). In particular, the formula is valid for the largest class of square 0 and rank $r$. \hfill \Box

**Proof #2.** There are two possibilities for the conjugacy class of $x$ in case $r$ is even, see [Hes79, 4.4] or [LS12, p. 70]. We focus on the larger class; regardless of the parity of $r$ we may assume that the restriction of the natural module to $x$ includes a 2-dimensional summand denoted by $V(2)$ in [LS12]. For this $x$, the function denoted by $\chi$ in the references amounts to $1 \mapsto 0$ and $2 \mapsto 1$. The formulas in these references now give that the centralizer of $x$ in $\text{Sp}_{2n}$ has dimension
\[
(2n-r) + \sum_{i=1}^{2n-r} 2(i-1) + \sum_{i=r+1}^{2n-r} (i-1) = 2n-r + \left(\frac{2n-r}{2}\right) + \left(\frac{r}{2}\right). \hfill \Box
\]

**Lemma 6.3.** Let $G = \text{Sp}_{2n}$ for some $n \geq 3$ over a field $k$ of characteristic 2, and let $x \in \mathfrak{sp}_{2n}$.

1. If $x$ is toral of rank $r \leq n$ (so 0 is the largest eigenvalue), then we can generate a subalgebra containing $n$ with at most $\max\{4, [2n/r]\}$ conjugates.
2. If $x^{[2]} = 0$, and $x$ has even rank $r$, then we can generate a subalgebra containing $n$ with at most $\max\{4, [2n/r]\}$ conjugates. Conjugates suffice.
3. If $x^{[2]} = 0$ and $x$ has rank 1, then $\max\{8, 2n\}$ conjugates of $x$ generate a subalgebra containing $n$.
4. If $x^{[2]} = 0$ and $x$ has odd rank $2s + 1 \geq 3$, then $\max\{4, [n/s]\}$ conjugates of $x$ generate a subalgebra containing $n$.

**Proof.** Any toral element is contained in $n$ and so [GG17a, Prop. 9.4] implies (1). If $x^{[2]} = 0$ and $x$ has even rank $r$, then the closure of $x^{\text{Sp}_{2n}}$ contains an element $y \in \mathfrak{m}$ by the description of the classes above. In this case, we apply [GG17a, 9.1] and (2) follows.

In case (3), we choose $y$ conjugate to $x$ such that $(x+y)^{[2]} = 0$ and $x+y$ has rank 2. By (2), $\max\{4, n\}$ conjugates of $x+y$ which generate a subalgebra containing $n$, whence (3) holds.

In case (4), the description of the classes given in the proof of Lemma 6.2 show that the closure of $x^{\text{Sp}_{2n}}$ contains a $y$ of rank 2 such that $y^{[2]} = 0$. By (2), $\max\{4, [n/s]\}$ conjugates of $y$ suffice to generate a subalgebra containing $n$, so the same holds for $x$. \hfill \Box

**Concrete description of some Lie algebras.** We may view $\text{Sp}_{2n}$ as the subgroup of $\text{GL}_{2n}$ preserving the bilinear form $b(v, v') := v^\top J v'$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Lie algebra $\mathfrak{sp}_{2n}$ of $\text{Sp}_{2n}$ consists of matrices \( \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \) for $A, B, C \in \mathfrak{gl}_n$ such that $B^\top = B$ and $C^\top = C$. In the notation of Example 6.1, $\sigma(g) = -J g^\top J$ and $\mathfrak{sp}_{2n}$ consists of those $x \in \mathfrak{gl}_{2n}$ such that $\sigma(x) + x = 0$.

The group $\text{GSp}_{2n}$ of similarities of $b$ is the sub-group-scheme of $\text{GL}_{2n}$ generated by $\text{Sp}_{2n}$ and the scalar transformations. Its Lie algebra consists of those $x \in \mathfrak{gl}_{2n}$...
such that \( \sigma(x) + x \in kI_{2n} \), i.e., \( x \) of the form \( \left( \begin{array}{cc} A & B \\ C & -A^\top \end{array} \right) \) for \( A, B, C \in \mathfrak{gl}_n \) and \( \mu \in k \), where \( B^\top = B \) and \( C^\top = C \). Then \( \mathfrak{sp}_{2n} \subseteq \mathfrak{gsp}_{2n} \), the quotient \( \mathfrak{gsp}_{2n}/\mathbb{G}_m \cong \mathfrak{sp}_{2n}/\mu_2 \) is the adjoint group \( \mathbb{PSp}_{2n} \), and the natural map \( \mathfrak{gsp}_{2n} \to \mathfrak{psp}_{2n} \) is surjective.

The preceding two paragraphs apply to any field \( k \); we now explicitly assume \( \text{char } k = 2 \) and describe the toral elements in \( \mathfrak{gsp}_{2n} \), i.e., those \( x \) such that \( x^2 = x \).

As such an \( x \) is a projection, it gives a decomposition of \( k_{2n} \) as a direct sum of vector spaces \( (\ker x) \oplus (\text{im } x) \). If \( x \) belongs to \( \mathfrak{sp}_{2n} \), then this is an orthogonal decomposition as in Example 6.1 where the restrictions of \( b \) to ker \( x \) and im \( x \) are nondegenerate. Otherwise, \( \sigma(x) + x = I_n \), so

\[
b(v,xv') = b((I_n - \sigma(x))v,xv') = b(v,v') - b(v,x^2v') = 0 \quad (v,v' \in V).
\]

That is, im \( x \) is totally isotropic. Analogously, \( 1 - x \in \mathfrak{gsp}_{2n} \) is toral and so ker \( x = \text{im}(1-x) \) is also totally isotropic. In sum, we find that ker \( x \) and im \( x \) are maximal totally isotropic subspaces.

**Proposition 6.4.** Let \( (V,\rho) \) be a representation of \( G = \mathbb{Sp}_{2n} \) or \( \mathbb{PSp}_{2n} \) with \( n \geq 3 \) over a field \( k \) of characteristic 2. If \( V^n = 0 \) and

\[
\dim V > \begin{cases} 48 & \text{if } n = 3; \\
72 & \text{if } n = 4 \text{ and } G = \mathbb{Sp}_{2n}; \\
80 & \text{if } n = 4 \text{ and } G = \mathbb{PSp}_{2n}; \\
6n^2 - 6n & \text{if } n > 4,
\end{cases}
\]

then \( \mathfrak{g} \) acts virtually freely on \( V \).

In the statement, in case \( G = \mathbb{PSp}_{2n} \), \( \rho \) induces a representation of \( \mathbb{Sp}_{2n} \) and so it still makes sense to speak of the action of \( \mathfrak{n} \subseteq \mathfrak{sp}_{2n} \) on \( V \).

**Proof.** Suppose first that \( G = \mathbb{Sp}_{2n} \). Since \( V^n = 0 \), ker \( \rho \subseteq \mathfrak{n} \), i.e., ker \( \rho \subseteq \mathfrak{z}(\mathfrak{g}) \). Suppose \( x \in \mathfrak{g} \) is noncentral and \( x^{[2]} \in \{0,x\} \): we want to show that there are \( x_1,\ldots,x_e \in x^{[2]} \) that generate a subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{n} \) and such that \( e \cdot \dim x^{[2]} \) is at most the right side of the displayed expression, which is \( 6(n - 1) \max\{4,n\} \).

If \( x \) belongs to \( \mathfrak{so}_{2n} \), then \( e \cdot \dim x^{[2]} \leq 4n^2 \) by [GG17a, Cor. 9.6]. So we may assume that \( x^{[2]} = 0 \) and \( x \) has odd rank \( r = 2s + 1 \), where \( \dim x^{[2]} \) is bounded as in Lemma 6.2.

If \( r = 1 \), then applying Lemma 6.3(3) gives that \( e \cdot \dim x^{[2]} \leq 4n \cdot \max\{4,n\} \).

If \( r = 3 \), then \( \dim x^{[2]} \leq 6(n - 1) \), providing exactly the desired bound.

If \( r \geq 5 \) (so \( n \geq 5 \)), we bound \( \lceil n/s \rceil < (n + s)/s \). The expression \( 2(n + s)(2s + 1)(n - s)/s \) is a decreasing function of \( s \) so it is maximized at \( s = 2 \) where we obtain \( 5n^2 - 20 \), which is less than \( 6n^2 - 6n \).

Now suppose that \( G = \mathbb{PSp}_{2n} \). For \( \mathbb{GSp}_{2n} := (\mathbb{Sp}_{2n} \times \mathbb{G}_m)/\mu_2 \), the group of similarities of the bilinear form, we have a natural surjection \( \mathbb{GSp}_{2n} \to \mathbb{PSp}_{2n} \) whose differential \( \mathfrak{gsp}_{2n} \to \mathfrak{psp}_{2n} \) is surjective and has central kernel \( k \), the scalar matrices. Let \( x \in \mathfrak{gsp}_{2n} \) be noncentral such that \( x^{[2]} \in \{0,x\} \). If \( x \) belongs to \( \mathfrak{sp}_{2n} \), we have already verified that \( x \) cannot lie in \( (\mathfrak{sp}_{2n})_e \) for generic \( v \in V \). So assume that \( x \) is toral and does not belong to \( \mathfrak{sp}_{2n} \), i.e., is the projection on a maximal totally isotropic subspace as in the paragraph preceding the statement. Up to conjugacy we may assume that \( x \) is \( y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). The nilpotent linear transformation \( y' := \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \) belongs to \( \mathfrak{sp}_{2n} \) and does not commute with \( y \). Lemma 6.3 gives
that 4 conjugates of $y'$ suffice to generate a subalgebra of $\mathfrak{sp}_{2n}$ containing $\mathfrak{n}$, and therefore the same is true for $y$. On the other hand, $\dim y^{G_{\text{Sp}}_{2n}} = n^2 + n$, giving $e \cdot \dim y^{G_{\text{Sp}}_{2n}} \leq 4n^2 + 4n$. \hfill \square

**Restricted irreducible representations.** Let $V$ be a restricted irreducible representation of an algebraic group $G$ of type $C_n$ with $n \geq 3$ over $k$ of characteristic 2; we aim to prove Theorem B for this $G$. The highest weight $\lambda = \sum_{i=1}^{n} c_i \omega_i$ has $c_i \in \{0, 1\}$ for all $i$. If $\lambda = 0$, equivalently $\ker d\rho = \mathfrak{g}$, then there is nothing to do.

**Example 6.5** ("spin" representation). Put $G' := \text{Spin}_{2n+1}$. Composing the very special isogeny $\text{Sp}_{2n} \to G'$ with the (injective) spin representation of $G'$ of dimension $2^n$ yields the irreducible representation $(V, \rho) = L(\omega_1)$ of $\text{Sp}_{2n}$. For $n \geq 7$, $\dim V > \dim G$ and the stabilizer $(\mathfrak{g'})_v$ of a generic $v \in V$ is zero [GG17c, Th. 1.1], so $(\mathfrak{sp}_{2n})_v = \ker d\rho$.

For $3 \leq n < 7$, $\dim V < \dim \text{Sp}_{2n}$ and one can check using a computer that there exist vectors in $V$ whose stabilizer is $\ker d\rho$, therefore, $\mathfrak{sp}_{2n}$ acts virtually freely on $V$. Alternatively, for $n = 3, 5, 6$ and generic $v \in V$, $(\text{Spin}_{2n+1})_v$ is $G_2$, $\text{SL}_5 \rtimes \mathbb{Z}/2$, $(\text{SL}_3 \times \text{SL}_3) \rtimes \mathbb{Z}/2$ respectively [GG17c]; each of these has a Lie algebra that is a direct sum of simples, and so it can not meet the solvable ideal that is the image of $\mathfrak{sp}_{2n}$ in $\mathfrak{spin}_{2n+1}$.

For $n$ even, the representation factors through $\text{PSp}_{2n}$. For $n \geq 8$ and $n = 6$, $\mathfrak{psp}_{2n}$ acts virtually freely as in the preceding two paragraphs. For $n = 4$, the image of $\mathfrak{psp}_8$ in $\mathfrak{spin}_9$ is the maximal proper Spin$_9$-submodule, for which a generic vector $v$ in the spin representation has a 1-dimensional stabilizer (Example 5.10), i.e., $\dim(\mathfrak{psp}_8)_v / \ker d\rho = 1$, i.e., $\mathfrak{psp}_8$ does not act generically freely.

We may therefore assume that $V$ is neither $k$ nor one of the representations considered in the preceding example (i.e., we assume $\lambda \neq 0, \omega_1$). As explained in §1, $V^n = 0$. Our task is to prove that $\mathfrak{g}$ acts virtually freely on $V$ if and only if $\dim V > \dim G$. As for type $B$, it suffices to prove (1) $\mathfrak{sp}_{2n}$ does not act virtually freely if $\dim V \leq \dim G$ and (2) $\mathfrak{psp}_{2n}$ acts virtually freely if $\dim V > \dim G$.

**Example 6.6** (natural representation). Here we treat $V := L(\omega_n)$. In this case $\text{Sp}_{2n}$ is transitive on all nonzero vectors and so we see the generic stabilizer is the derived subalgebra of the maximal parabolic subalgebra that is the stabilizer of a 1-dimensional space.

**Example 6.7** ("$\wedge^2$" representation). Let $V := L(\omega_{n-1})$, which has $\dim V < \dim G$. We will show that the generic stabilizer in $\text{Sp}_{2n}$ is $A_1 \times \cdots \times A_1$ (more precisely the stabilizer of $n$ pairwise orthogonal two dimensional non degenerate subspaces), so $V$ is not virtually free.

Let $M$ be the natural module for $G$, of dimension $2n$. Let $W = \wedge^2 M$ which we can identify with the set of skew adjoint operators on $M$, i.e., the linear maps $T: M \to M$ with $(Tv, v) = 0$ for all $v \in M$. If $n$ is odd, then $W \cong k \oplus V$. If $n$ is even, then $W$ is uniserial with 1-dimensional socle and head with $V$ as the nontrivial composition factor. The unique submodule $W_0$ of codimension 1 in each case consists of those elements $T$ with reduced trace equal to 0.

It follows as in [GG07] or [GG15, Example 8.5] that a generic element $T$ of $W$ (or $W_0$) is semisimple and has $n$ distinct eigenvalues each of multiplicity 2. It follows that, as a group scheme, the stabilizer of such an element is as given and so the result holds for $n$ odd.
Assume that $n \geq 4$ is even. Note that a generic element $T$ of $W$ also has the property that the $n(n-1)/2$ differences of the eigenvalues of $T$ are distinct. It follows easily that the stabilizer of such an element in $V = W_0/W^G$ is the same as in $W_0$ and the result follows.

As these highest weights $\lambda$ have been handled, we ignore them below. If $\dim V$ is large, then $\mathfrak{g}$ acts virtually freely on $V$ by Proposition 6.4. When $n > 11$, $V$ is $L(\omega_n)$ or $L(\omega_{n-1})$ by [Lüb01], so we may assume $3 \leq n \leq 11$. Therefore we are reduced to considering Tables A.32–A.40 in [Lüb01], one table for each value of $n$. In those tables, the representations $V$ not already handled in Examples 6.6–6.5 and not handled by Prop. 6.4 are: $L(\omega_{n-2})$ ("$\wedge^3$") for $C_n$ with $n = 4, 5$ and the representation $L(\omega_1 + \omega_3)$ of $C_3$ of dimension 48. A computer check verifies that $\mathfrak{g}$ acts virtually freely on these representations.

7. A large representation that is not virtually free

In this section, $G$ is a simple and simply connected algebraic group over $k$ and $\text{char } k$ is special for $G$. The main theorem of [GG17a] includes a statement of the following type: If $\text{char } k$ is not special and $V$ is a $G$-module such that $\dim V/\mathfrak{g}V$ is big enough, then $\mathfrak{g}$ acts virtually freely on $V$. Example 7.2 below shows that such a result does not hold when $\text{char } k$ is special.

As in §1, we put $\mathfrak{n}$ for the kernel of the (differential of the) very special isogeny in $\mathfrak{g} = \text{Lie}(G)$.

Lemma 7.1. $\mathfrak{n}$ is not virtually free as an $\mathfrak{n}$-module.

Proof. If $G$ has type $G_2$ or $F_4$, then $\mathfrak{n}$ is the simple quotient of $\mathfrak{sl}_3$ or $\mathfrak{spin}_8$ by its center. Thus $\mathfrak{n}$ acts on $\mathfrak{n}$ with trivial kernel and the stabilizer of a generic element of $\mathfrak{n}$ is the image of a maximal torus in $\mathfrak{sl}_3$ or $\mathfrak{spin}_8$, of dimension 1 or 2.

If $G$ has type $C_n$ with $n \geq 3$, then $\mathfrak{n} = [\mathfrak{so}_{2n}, \mathfrak{so}_{2n}]$ and a similar argument applies.

If $G$ has type $B_n$ with $n \geq 2$, then $\mathfrak{n} = \mathfrak{h}$, the Heisenberg algebra from §4. For a generic $h \in \mathfrak{h}$, the map $x \mapsto [x, h] \in \mathfrak{z}(\mathfrak{h})$ is a nonzero linear map, so has kernel of codimension 1. □

Example 7.2. Let $W$ be a representation of $G$ such that $W^\theta = 0$, $\mathfrak{n}W = 0$, and such that $\mathfrak{g}_w = \mathfrak{n}$ for generic $w \in W$. For $G$ of type $B_n$, such a $W$ is provided in Example 5.6. For $G$ of type $C_n$, $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{h} \subset \mathfrak{spin}_{2n+1}$, and we take $W$ to be a direct sum of copies of the spin representation as in Example 5.6. For $G$ of type $G_2$ or $F_4$, we take $W$ to be a sum of two copies of $\mathfrak{g}/\mathfrak{n}$.

Let $V = \mathfrak{n} \oplus \bigoplus W$ for $W$ as in the preceding example. Then $V^\theta = \mathfrak{n}^\theta$ (and even $V^\theta = 0$ for $G = G_2$ or $F_4$ and $\dim V/\mathfrak{g}V > m \dim V$ can be made arbitrarily large by increasing $m$. On the other hand, for a generic vector $v = (n, w_1, \ldots, w_m) \in V$, the stabilizer $\mathfrak{g}_v$ equals $\mathfrak{n}_v$, so by Lemma 7.1 $\mathfrak{g}$ does not act virtually freely on $V$.

8. Proof of Theorem C

Theorem B has been proved in the preceding sections, and we now proceed to the proof of Theorem C.

Consider the irreducible representation $L(\lambda)$ of a simple $G$ with highest weight $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0$, $\lambda_1$ are nonzero and dominant and $\lambda_0$ is restricted. If $\lambda_0 = 0$ then $\mathfrak{g}$ acts trivially and we are done.
If $\lambda_1 = 0$, then $V$ is a restricted irreducible representation. Write $\lambda_0 = \lambda_s + \lambda_\ell$ as in §1. As $V$ is assumed tensor decomposable, it follows that char $k$ is special and $\lambda_s, \lambda_\ell \neq 0$ [Sei87, 1.6]. Therefore $\dim V > \dim G$, and $\mathfrak{g}$ acts virtually freely on $V$ by Theorem B.

So assume $\lambda_0, \lambda_1 \neq 0$. The representation $L(\lambda)$ of $\mathfrak{g}$ is a sum of $\dim L(\lambda_1)$ copies of $L(\lambda_0)$. Therefore, replacing $\lambda_1$ with a dominant weight such that $\dim L(\lambda_1) = m(G)$, we may assume that $V$ is $L(\lambda_0)^{\otimes m(G)}$ as a $\mathfrak{g}$-module. By the preceding paragraph, we may assume that $(\lambda_0)_s = 0$ or $(\lambda_0)_\ell = 0$.

If $\dim L(\lambda_0) = m(G)$, then $L(\lambda)$ is equivalent to $L(\lambda_0)^{\otimes \dim V}$ for some $i \geq 1$ as $\mathfrak{g}$-modules and we are done by [GG17a, Lemma 8.1].

**Types $G_2$ and $F_4$.** As $m(G_2) = 7$ (for char $k = 3$) and $m(F_4) = 26$ (for char $k = 2$), $\dim V \geq 49$, 676 respectively. Then Proposition 3.1 gives that $V$ is virtually free for $\mathfrak{g}$ for $G$ of type $G_2$ or $F_4$.

**Type $C_n$ for some $n \geq 3$.** For $G$ of type $C_n$ with $n \geq 3$ and char $k = 2$, $m(G) = 2n$ so we may assume that $\dim V > 4n^2$.

Suppose first that $(\lambda_0)_s = 0$, i.e., $L(\lambda_0)$ is the spin representation. If $G \neq \text{PSp}_8$, then $L(\lambda_0)$ is virtually free for $\mathfrak{g}$. For $G = \text{PSp}_8$, the spin representation factors through $\text{Spin}_n$, for which a sum of three copies is virtually free by Example 5.7 (and in fact a sum of two copies is virtually free for $\text{sp}_{ps}$).

Now suppose that $(\lambda_0)_\ell = 0$, i.e., $V^n = 0$. Put $b$ for the bound appearing Proposition 6.4; by that result we are reduced to considering those $\lambda_0$ such that $(\lambda_0)_\ell = 0$ and $2n < \dim L(\lambda_0) \leq b/(2n)$. No such representations exist, see Tables 2 and A.33–A.40 in [Liub01].

**Type $B_n$ for $n \geq 2$.** $G$ of type $B_n$, with $n \geq 2$ and char $k = 2$, $m(G) = 2n$, so we may assume that $\dim V > 4n^2$, i.e., $\dim L(\lambda_0) > 2n$.

If $(\lambda_0)_s = 0$, then ker $d$ and we are done by Lemma 5.5.

So assume $(\lambda_0)_s \neq 0$, hence $(\lambda_0)_s = \omega_1$. Then Lemma 5.2 applies and we may assume that $n \leq 7$ and $\dim L(\lambda_0) \leq 2n + 2$. Examining Tables A.22–A.27 in [Liub01] we see that the only possibility for $\lambda_0$ is 100 for $B_3$, the spin representation of dimension 8, where $V \cong L(\lambda_0)^{\otimes 6}$ is a generically free representation of $\text{spin}$, as in Example 5.7.

This completes the proof of Theorem C.

\[ \square \]

**9. Proof of Theorem A**

Theorem A now follows quickly from what has gone before. We repeat the argument given at the end of part I for the convenience of the reader.

As $\rho$ is assumed faithful, it cannot be one of the representations in Table 3. In case $\rho$ is restricted, Theorem A(1) is then part of Theorem B. If $\rho$ is not restricted, then (as it cannot be a Frobenius twist of a representation) it is a tensor product of an irreducible restricted representation and a Frobenius twist, and $\rho$ is virtually free by Theorem C.

For Theorem A(2), we must enumerate in Table 2 those representations $V$ such that $\dim V > \dim G$, $V$ does not appear in Table 1, and the group of points $G_\rho(k)$ is not trivial. Those $V$ with the latter property are enumerated in [GLL18], completing the proof of Theorem A(2).

\[ \square \]
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