IMPROVED $L^2$ ESTIMATE FOR GRADIENT SCHEMES AND SUPER-CONVERGENCE OF THE TPFA FINITE VOLUME SCHEME

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Abstract. The gradient discretisation method is a generic framework that is applicable to a number of schemes for diffusion equations, and provides in particular generic error estimates in $L^2$ and $H^1$-like norms. In this paper, we establish an improved $L^2$ error estimate for gradient schemes. This estimate is applied to a family of gradient schemes, namely, the Hybrid Mimetic Mixed (HMM) schemes, and yields an $O(h^2)$ super-convergence rate in $L^2$ norm, provided local compensations occur between the cell points used to define the scheme and the centers of mass of the cells. To establish this result, a modified HMM method is designed by just changing the quadrature of the source term; this modified HMM enjoys a super-convergence result even on meshes without local compensations. Finally, the link between HMM and Two-Point Flux Approximation (TPFA) finite volume schemes is exploited to partially answer a long-standing conjecture on the super-convergence of TPFA schemes.

Keywords: super-convergence, two-point flux approximation finite volumes, hybrid mimetic mixed methods, gradient schemes.

AMS subject classifications: 65N08, 65N12, 65N15.

1. Introduction

When applying a numerical scheme to an elliptic partial differential equation, the expected rate of convergence is directly dependent on the interpolation properties of the approximation space. For example, when using a piecewise constant approximation, as in many finite volume methods, the expected rate of convergence in $L^2$ norm is $O(h)$, where $h$ is the mesh size. Super-convergence is the phenomenon that occurs when a numerical method displays a better convergence rate than the expected one.

Let us consider the linear elliptic second-order problem

\[
\begin{aligned}
-\text{div}(A\nabla \pi) &= f \text{ in } \Omega, \\
\pi &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

where

\begin{align*}
\Omega &\subset \mathbb{R}^d \ (d \geq 1) \text{ is a bounded domain, } f \in L^2(\Omega), \\
A : \Omega &\rightarrow \mathcal{M}_d(\mathbb{R}) \text{ is measurable, bounded, uniformly elliptic,} \\
&\text{ and } A(x) \text{ is symmetric for a.e. } x \in \Omega.
\end{align*}

Problem (1.1) is understood in the usual weak sense, that is:

Find $\pi \in H^1_0(\Omega)$ such that, for all $v \in H^1_0(\Omega)$, $a(\pi, v) = (f, v), \quad (1.3)$
where \((\cdot, \cdot)\) is the scalar product in \(L^2(\Omega)\) and

\[
a(v, w) = \int_\Omega A \nabla v \cdot \nabla w \, dx \quad \forall v, w \in H^1_0(\Omega).
\]

Under the assumption (1.2), Problem (1.3) has a unique solution.

For many (low-order) finite volume methods, \(O(h)\) error estimates in the \(L^2\) norm and a discrete \(H^1\) norm are known, and several super-convergence results have been numerically observed for the \(L^2\) norm without being proved yet (see [16] and references therein).

The Two-Point Flux Approximation (TPFA) finite volume scheme is a very popular scheme used for decades in reservoir simulation [38]. It has been fully analysed in [27], and extensively tested in a number of situations. Due to its construction, classical TPFA benchmarks are conducted on 2D meshes made of acute triangles [5, 14]. This scheme uses piecewise constant approximations and hence the expected rate of convergence in \(L^2\) norm is \(O(h)\). However, the aforementioned numerical tests have shown that this piecewise constant approximation provides an \(O(h^2)\) estimate of the value of the solution at the circumcenters of the triangles. Such a super-convergence result was never proved theoretically.

The main contribution of this paper is to give a rigorous proof of this super-convergence. Precisely, we establish an \(O(h^2)\) estimate in \(L^2\) norm of the difference between the solution to the TPFA scheme and the piecewise constant projection of the exact solution constructed from its values at the circumcenters of the triangles. Our result covers all 2D meshes encountered in TPFA benchmarking.

Previous works have established some relations between TPFA and \(RT_0-P_0\), provided particular choices of numerical integrations are used [3, 33]. These relations are therefore not exact algebraic equivalence, and do not allow one to deduce the super-convergence for TPFA from the super-convergence for \(RT_0-P_0\) [15]. A relation, not based on numerical integration, between TPFA on triangles and \(RT_0-P_0\) mixed finite elements has been established in [12], but has a limited scope, since the source term \(f\) must vanish [1] (see also [11, 40]). If the source term is not zero, then \(RT_0-P_0\) can be reformulated as a finite volume method, which is different from TPFA since the source term is involved in the definition of the fluxes. We refer to [39] for a thorough study of mixed finite element methods interpreted as finite volume methods, and related fluxes and properties.

In any case, these various relations between TPFA and \(RT_0-P_0\) do not seem to directly lead to a proof of the observed super-convergence of TPFA. This is due to variations in the choice of approximation points. The TPFA interpretation of \(RT_0-P_0\) requires to introduce new cell unknowns located at the circumcenters of the triangles, which do not correspond to the standard \(RT_0-P_0\) cell unknowns, located at the centers of mass of the triangles, for which the super-convergence is proved.

In [2], a relationship is established on Voronoi meshes between the TPFA scheme and a generalised mixed-hybrid mimetic finite difference method (with cell and face centers moved away from the centers of mass). A super-convergence of this method is established, under the assumption that certain lifting operators exist. This existence is only checked in the case of rectangular cells (for which TPFA amounts to a finite difference scheme).
It should also be mentioned that some post-processing techniques can provide, under certain circumstances, an $O(h^2)$ convergence in $L^2$ norm for functions reconstructed from the solutions to finite volume approximations. One of these post-processing technique, using two TPFA schemes on two dual meshes, is described in [37]. These quadratic convergences of post-processed solutions however do not say anything specific on the super-convergence of the original finite volume scheme. The super-convergence result for TPFA established in the present paper holds without post-processing, for the natural unknown at the circumcenter of the triangles, and on all the kinds of triangular meshes used in benchmarking. This result therefore appears to solve a long-standing conjecture on this popular finite volume method, on 2D triangular meshes as encountered in practical test-cases.

The technique used to prove the super-convergence of TPFA is an indirect one. We use the fact that, on 2D triangular grids, the TPFA scheme is an HMM method. HMM schemes, defined in [22], is a family of methods that includes mixed-hybrid mimetic finite difference (hMFD) schemes [4, 9, 34], mixed finite volume schemes [17] and hybrid finite volume (“SUSHI”) schemes [28]. The construction of an HMM scheme requires to choose one point inside each mesh cell. When this point is at the center of mass of the cell, HMM schemes boil down to hMFD schemes and super-convergence is then known [4, 6, 7, 13]. But when this cell point is moved away from the center of mass, super-convergence is less clear and can possibly fail, as we show in a numerical test. On triangular grids, the TPFA scheme is an HMM method precisely when these cell points are not located at the centers of mass, but at the circumcenters of the cells. Establishing the super-convergence of TPFA through its identification as an HMM scheme therefore requires first to obtain a super-convergence result for HMM methods with cell points located away from centers of mass of the cells.

This super-convergence for HMM schemes is obtained through a new, improved $L^2$ estimate for gradient schemes. A gradient scheme for, say, (1.1) is obtained by selecting a family of discrete space and operators, called a gradient discretisation (GD), and by substituting, in the weak formulation of (1.1), the continuous space and operators with these discrete ones. This method is called the gradient discretisation method (GDM). The vast possible choice of GD makes the GDM a generic framework for the convergence analysis of many numerical methods, which include finite elements, mixed finite elements, finite volume, mimetic finite difference methods, HMM, etc. for diffusion, Navier–Stokes, elasticity equations and some other models. We refer to [24] for an analysis of the methods covered by this framework, and to [19, 20, 23, 25, 26, 29, 30] for a few models on which the convergence analysis can be carried out within this framework; see also the monograph [21] for a complete presentation of the GDM for various boundary conditions. Each specific scheme corresponds to a certain choice of GD, and the convergence analysis conducted in the GDM applies to all choices of GD and thus, to all the schemes covered by the framework. A generic error estimate has been established for the GDM applied to (1.1). This estimate gives the standard $O(h)$ rate of convergence in $H^1$ norm for the low-order methods covered by the GDM, such as the HMM schemes [23].

To summarise, the contributions of this paper are

(i) an improved $L^2$ estimate for gradient schemes, in any dimension $d$,
(ii) a modified HMM scheme with unconditional super-convergence, in dimension \( d \leq 3 \),
(iii) a super-convergence result for HMM, in dimension \( d \leq 3 \), and
(iv) a super-convergence result for TPFA, in dimension \( d = 2 \) on triangular meshes as encountered in benchmarks.

The improved \( L^2 \) error estimate for gradient schemes involves, as in the Aubin–Nitsche trick, the solution to a dual problem. Applied to HMM schemes, this new estimate provides an \( O(h^2) \) super-convergence result when some form of local compensation occurs; that is, the cell points may be away from the centers of mass, but not too far away on average over a few neighbouring cells. The proof of the super-convergence of TPFA then consists in checking that, for triangular meshes used in TPFA benchmarkings, this local compensation always occur. A by-product of the proof of super-convergence for HMM schemes is the design of a modified HMM scheme, in which only the right-hand side is modified. This modified HMM has the same matrix, and same computational cost as the original HMM since only the quadrature of the source term is modified; but yields super-convergence for any choice of cell points, even when the standard HMM scheme fails to super-converge.

The paper is organised as follows. The description of the TPFA scheme and of the meshes used in benchmarking are provided in Subsection 1.1 at the end of this introduction. This section also states our main result, that is the super-convergence of TPFA. Section 2 recalls the principle of the GDM and Section 3 establishes the improved \( L^2 \) estimate. In Section 4, the construction of HMM method is recalled and a modified HMM method is designed. In Section 5, we state and prove a new \( L^2 \) error estimate for HMM, that involves patches of cells. When these patches can be chosen so that a compensation occurs, within each patch, between the cell points and the centers of mass, this new \( L^2 \) estimate provides the super-convergence of HMM. The proof of the super-convergence of TPFA is given at the end of Section 5. Numerical results provided in Section 6 show that in the absence of patches as above, super-convergence may fail for HMM schemes but holds for the modified HMM scheme. A conclusion, recalling the main results, is given in Section 7. Section 8, an appendix, gathers various results: a proper analysis of approximate diffusion tensors \( A \) in the construction of gradient schemes; some technical results used in the rest of the paper; and a discussion on the implementation of the HMM method, the modified HMM method, and their corresponding fluxes.

Two remarks are in order to conclude this introduction. First, we consider here homogeneous Dirichlet boundary conditions in (1.1) only for the sake of simplicity. The gradient scheme framework has been developed for all classical boundary conditions [21] and our technique applies to other boundary conditions with minor modifications. Secondly, although we only apply it to HMM and TPFA schemes, the improved \( L^2 \) error estimate that we establish in the context of the GDM could certainly lead to super-convergence results for other schemes contained in this framework, such as discrete duality finite volumes, some multi-point flux approximation finite volumes, etc.

1.1. Super-convergence for TPFA. Consider a TPFA-admissible mesh \( \mathcal{T} \) as in [27]. \( \mathcal{T} \) is therefore a partition of \( \Omega \) into polygonal cells \( \mathcal{M} \) together with a choice of points \( (x_K)_{K \in \mathcal{M}} \) in the cells such that, denoting by \( \mathcal{E}_K \) the edges of \( K \in \mathcal{M} \),

- for any neighboring cells \( K \) and \( L \) in \( \mathcal{M} \), if \( \sigma \in \mathcal{E}_K \cap \mathcal{E}_L \) then \( (x_K - x_L) \perp \sigma \).
• for any cell \( K \in \mathcal{M} \), if \( \sigma \in \mathcal{E}_K \) and \( \sigma \subset \partial \Omega \) then \((x_K + \mathbb{R}^+ n_{K,\sigma}) \cap \sigma \neq \emptyset\), where \( n_{K,\sigma} \) is the normal to \( \sigma \) pointing outward \( K \).

Let \( \mathcal{E}_{\text{int}} \) be the set of edges interior to \( \Omega \) and \( \mathcal{E}_{\text{ext}} \) be the set of edges lying on \( \partial \Omega \). If \( A \) is an isotropic tensor, that is, \( A(x) = a(x) \text{Id} \), for some \( a(x) \in (0, \infty) \), the TPFA method for (1.1) on \( \mathcal{T} \) reads \([16, 27]\):

\[
\forall K \in \mathcal{M}, \quad \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}} \tau_{\sigma} (u_K - u_L) + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}} \tau_{\sigma} u_K = \int_K f(x) \, dx, \quad (1.4)
\]

where \( L \) is the cell on the other side of \( \sigma \) if \( \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}} \), and, with \( a_K \) being the average value of \( a \) on \( K \),

\[
\forall \sigma \in \mathcal{E}_{\text{int}}, \quad \text{if} \ K \neq L \quad \text{are the cells on both sides of} \ \sigma, \quad \tau_{\sigma} = |\sigma| \frac{a_K a_L}{a_K d_{L,\sigma} + a_L d_{K,\sigma}},
\]

\[
\forall \sigma \in \mathcal{E}_{\text{ext}}, \quad \text{if} \ K \quad \text{is the cell such that} \ \sigma \in \mathcal{E}_K, \quad \tau_{\sigma} = |\sigma| \frac{a_K}{d_{K,\sigma}}.
\]

In 2D, a classical way to construct meshes satisfying the orthogonality property \((x_K, x_L) \bot \sigma\) is to partition \( \Omega \) into a conforming triangulation with acute triangles, and to take each \( x_K \) as the circumcenter of \( K \). Three triangulation constructions are widely used in TPFA benchmarkings, see e.g. \([5, 14]\): subdivision, reproduction by symmetry, or reproduction by translation. Actually, we are not aware of any reported benchmark on TPFA that uses different mesh constructions.

**Definition 1.1 (Classical TPFA triangulation).** Let \( \Omega \) be a polygonal bounded open set of \( \mathbb{R}^2 \). A classical TPFA triangulation of \( \Omega \) is a conforming acute triangulation \( \mathcal{T} \) of \( \Omega \) such that, for all \( K \in \mathcal{M} \), \( x_K \) is the circumcenter of \( K \), and that is constructed in one of the following ways (illustrated for a square domain \( \Omega \) in Figure 1):

- **Subdivision:** an initial triangulation \( \mathcal{T}_0 \) of \( \Omega \) is chosen, and then subdivided by creating on each edge an identical number of equally spaced points, by joining the corresponding points on different edges, and by adding the interior points resulting from intersections of the lines thus created.
- **Reproduction by symmetry:** an initial triangulation \( \mathcal{T}_0 \) of the unit square is chosen, this unit square is shrunk by a factor \( N \) and reproduced in the entire domain by symmetry.
- **Reproduction by translation:** an initial triangulation \( \mathcal{T}_0 \) of the unit square is chosen, this unit square is shrunk by a factor \( N \) and reproduced in the entire domain by translation.

**Theorem 1.2 (Super-convergence for TPFA on triangles).**

Let the assumptions (1.2), \( A = a \text{Id} \) for some \( a : \Omega \to (0, \infty) \), and that \( d = 2 \) hold. Also assume that (1.1) has the optimal \( H^2 \) regularity property (see (4.5)), and that \( f \in H^1(\Omega) \) and \( \pi \) is the solution to (1.3). Let \( \mathcal{T} \) be a classical TPFA triangulation of \( \Omega \) in the sense of Definition 1.1. If \( u = (u_K)_{K \in \mathcal{M}} \) is the solution of the TPFA scheme on \( \mathcal{T} \) then there exists \( C \), depending only on \( \Omega, a, \) and \( \mathcal{T}_0 \), such that

\[
\| u - \pi_p \|_{L^2(\Omega)} \leq C \| f \|_{H^1(\Omega)} h_M^2. \quad (1.5)
\]

Here, \( u \) is identified with a piecewise constant function on \( \mathcal{M} \), and \( \pi_p \) is defined by

\[
\forall K \in \mathcal{M}, \quad \pi_p = \pi(x_K) \text{ on } K. \quad (1.6)
\]
2. THE GRADIENT DISCRETISATION METHOD FOR ELLIPTIC PDEs

In a nutshell, the gradient discretisation method (GDM) consists in writing a scheme—called a gradient scheme—by replacing the continuous space and operators by discrete counterparts in the weak formulation of the PDE. These discrete space and operators are provided by a gradient discretisation (GD).

**Definition 2.1 (Gradient discretisation).** A gradient discretisation (for homogeneous Dirichlet conditions) is a triplet $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ of:

- A finite-dimensional space $X_{\mathcal{D},0}$ of degrees of freedom, that accounts for the zero boundary condition,
- A linear mapping $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$ which reconstructs a function from the degrees of freedom,
- A linear mapping $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ which reconstructs a gradient from the degrees of freedom. It is chosen such that $\|\nabla_{\mathcal{D}} \cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Given a gradient discretisation $\mathcal{D}$, the corresponding gradient scheme for (1.3) is:

Find $u_{\mathcal{D}} \in X_{\mathcal{D},0}$ such that, for all $v_{\mathcal{D}} \in X_{\mathcal{D},0}$, $a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}) = (f, \Pi_{\mathcal{D}} v_{\mathcal{D}})$, \hspace{0.5cm} (2.1)

where

$$a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}) := \int_\Omega A \nabla_{\mathcal{D}} u_{\mathcal{D}} \cdot \nabla_{\mathcal{D}} v_{\mathcal{D}} \, dx.$$  

The accuracy of a gradient scheme is measured by three quantities which are defined now. The first one is a discrete Poincaré constant $C_\mathcal{D}$, which ensures the coercivity of the method.

$$C_\mathcal{D} := \sup_{w \in X_{\mathcal{D},0}} \frac{\|\Pi_{\mathcal{D}} w\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} w\|_{L^2(\Omega)^d}}. \hspace{0.5cm} (2.2)$$

The second quantity is the interpolation error $S_\mathcal{D}$, which measures what is called the GD-consistency of the gradient discretisation (which is known as the interpolation error in the context of finite element methods).

$$\forall \phi \in H^1_0(\Omega), \quad S_\mathcal{D}(\phi) = \min_{w \in X_{\mathcal{D},0}} \left( ||\Pi_{\mathcal{D}} w| - \phi|_{L^2(\Omega)} + ||\nabla_{\mathcal{D}} w - \nabla \phi||_{L^2(\Omega)^d} \right). \hspace{0.5cm} (2.3)$$
Finally, we measure the limit-conformity (or defect of conformity) of a gradient discretisation through $W_D$ defined by

$$\forall \psi \in H_{\text{div}}(\Omega), \ W_D(\psi) = \max_{w \in X_D, o} \frac{\| W_D(\psi, w) \|}{\| \nabla_D w \|_{L^2(\Omega)}}, \quad (2.4)$$

where $H_{\text{div}}(\Omega) = \{ \psi \in L^2(\Omega)^d : \text{div}(\psi) \in L^2(\Omega) \}$ and

$$W_D(\psi, w) = \int_{\Omega} (\Pi_D w \text{div}(\psi) + \nabla_D w \cdot \psi) \, dx. \quad (2.5)$$

Using these quantities, the following basic stability and error estimates can be established [21, 29].

**Theorem 2.2.** Let $D$ be a gradient discretisation, $\pi$ be the solution to (1.3) and $u_D$ be the solution to (2.1). Then there exists $C > 0$ depending only on $A$ and an upper bound of $C_D$ such that

$$\| \Pi_D u_D \|_{L^2(\Omega)} + \| \nabla_D u_D \|_{L^2(\Omega)^d} \leq C \| f \|_{L^2(\Omega)} \quad (2.6)$$

and

$$\| \Pi_D u_D - \pi \|_{L^2(\Omega)} + \| \nabla_D u_D - \nabla \pi \|_{L^2(\Omega)^d} \leq C \text{WS}_D(\pi), \quad (2.7)$$

where

$$\text{WS}_D(\pi) = W_D(A \nabla \pi) + S_D(\pi). \quad (2.8)$$

**Remark 2.3** (Rates of convergence). For all classical low-order methods based on meshes, $O(h)$ estimates can be obtained, under classical regularity assumptions on $A$ and $u$, on $W_D(A \nabla \pi)$ and $S_D(\pi)$ (see [21]). The estimate (2.7) then gives a linear rate of convergence for these methods.

### 3. Improved $L^2$ error estimate for gradient schemes

As mentioned in the introduction, we follow the Aubin–Nitsche idea to establish an improved $L^2$ error estimate for gradient schemes. We therefore need to define the adjoint problem of (1.3), and consider its approximation by the GDM.

The weak formulation for the dual problem with source term $g \in L^2(\Omega)$ is:

Find $\varphi_g \in H^1_0(\Omega)$ s.t., for all $w \in H^1_0(\Omega)$, $a(w, \varphi_g) = (g, w). \quad (3.1)$

The gradient scheme corresponding to (3.1) is stated as:

Find $\varphi_{g,D} \in X_{D,0}$ s.t., for all $w_D \in X_{D,0}$, $a_D(w_D, \varphi_{g,D}) = (g, \Pi_D w_D). \quad (3.2)$

In order to state the improved $L^2$ error estimate, we need to define a measure of the interpolation error that, contrary to $S_D$, separates the orders of approximation for the function and its gradient. If $\alpha > 0$, $\phi_D \in X_{D,0}$ and $\phi \in H^1_0(\Omega)$, let

$$I_{D,\alpha}(\phi, \phi_D) = \| \Pi_D \phi_D - \phi \|_{L^2(\Omega)} + \alpha \| \nabla_D \phi_D - \nabla \phi \|_{L^2(\Omega)^d}. \quad (3.3)$$

**Theorem 3.1** (Improved $L^2$ error estimate for gradient schemes).

Assume (1.2), and let $\pi$ be the solution to (1.3). Let $D$ be a gradient discretisation in the sense of Definition 2.1, and let $u_D$ be the solution to the gradient scheme (2.1). Define

$$g = \frac{\Pi_D u_D - \pi}{\| \Pi_D u_D - \pi \|_{L^2(\Omega)}} \in L^2(\Omega)$$

...
and let \( \varphi_g \) be the solution to \((3.1)\). Let \( \mathcal{P}_D : H^2(\Omega) \cap H^1_0(\Omega) \to X_{D,0} \) be a mapping that selects, for each \( \phi \in H^2(\Omega) \cap H^1_0(\Omega) \), an element \( \mathcal{P}_D \phi \in X_{D,0} \). Then, there exists \( C > 0 \) depending only on \( \Omega \), \( A \) and an upper bound of \( C_D \) such that

\[
\left\| \mathcal{P}_D u_D - \pi \right\|_{L^2(\Omega)} \\
\leq C \left( \alpha^{-1} \mathcal{I}_{D,\alpha}(\pi, \mathcal{P}_D \pi) + W_S(\pi) \right) \left[ \alpha^{-1} \mathcal{I}_{D,\alpha}(\varphi_g, \mathcal{P}_D \varphi_g) + W_S(\varphi_g) \right] \\
+ \mathcal{I}_{D,\alpha}(\pi, \mathcal{P}_D \pi) + \| f \|_{L^2(\Omega)} \mathcal{I}_{D,\alpha}(\varphi_g, \mathcal{P}_D \varphi_g) \\
+ \left| \tilde{W}_D(A \nabla \pi, \mathcal{P}_D \varphi_g) \right| + \left| \tilde{W}_D(A \nabla \varphi_g, \mathcal{P}_D \pi) \right|,
\]

(3.4)

where \( \mathcal{I}_{D,\alpha} \) is defined by \((3.3)\), \( W_S \) is defined by \((2.8)\), and \( \tilde{W}_D \) is defined by \((2.5)\).

**Remark 3.2** (Dominating terms). Following Remark 2.3, for low-order methods (with mesh size \( h \)) it is expected that \( W_S(\psi) = O(h) \) if \( \psi \in H^2(\Omega) \) and \( A \) is Lipschitz-continuous. Hence, for a given gradient scheme, Theorem 3.1 provides a super-convergence result, under the \( H^2 \) maximal regularity, if we can find a mapping \( \mathcal{P}_D \) (usually, an interpolant) such that \( \mathcal{I}_{D,h}(\psi, \mathcal{P}_D \psi) = O(h^2) \) and \( \tilde{W}_D(\psi, \mathcal{P}_D \psi) = O(h^2) \) for all \( \psi \in H^2(\Omega) \cap H^1_0(\Omega) \) and all \( \psi \in H^1(\Omega)^d \). This is the strategy followed in Section 5 to establish super-convergence results for HMM schemes.

**Remark 3.3** (\( P_1 \) finite elements). Conforming and non-conforming \( P_1 \) finite elements are gradient schemes \([21, 24]\), for which the basic estimate \((2.7)\) provides an \( O(h) \) rate of convergence in \( L^2 \) norm. The improved estimate \((3.4)\) allows to recover, for these methods, the expected \( O(h^2) \) rate of convergence.

### 3.1. Preliminary lemmas

To prove Theorem 3.1, we need two technical lemmas. The first one measures the error committed when replacing the continuous bilinear form \( a(\cdot, \cdot) \) with the discrete bilinear form \( a_D(\cdot, \cdot) \).

**Lemma 3.4.** Let the assumption \((1.2)\) hold and let \( \psi, \phi \in H^1_0(\Omega) \) be such that \( \text{div}(A \nabla \psi) \in L^2(\Omega) \) and \( \text{div}(A \nabla \phi) \in L^2(\Omega) \). Then, for any \( \psi_D, \phi_D \in X_D \), it holds true that:

\[
|a(\psi, \phi) - a_D(\psi_D, \phi_D)| \leq E_D(\psi, \phi, \psi_D, \phi_D),
\]

where

\[
E_D(\psi, \phi, \psi_D, \phi_D) = \left| \tilde{W}_D(A \nabla \psi, \phi_D) \right| + \left| \tilde{W}_D(A \nabla \phi, \psi_D) \right| \\
+ \| \text{div}(A \nabla \psi) \|_{L^2(\Omega)} I_{D,\alpha}(\psi, \psi_D) \\
+ \| \text{div}(A \nabla \phi) \|_{L^2(\Omega)} I_{D,\alpha}(\phi, \phi_D) \\
+ \| A \|_{\infty} \alpha^{-2} I_{D,\alpha}(\psi, \psi_D) I_{D,\alpha}(\phi, \phi_D).
\]

(3.6)

**Proof.** Let

\[
T = a(\psi, \phi) - a_D(\psi_D, \phi_D) \\
= \int_\Omega A \nabla \psi \cdot \nabla \phi \, dx - \int_\Omega A \nabla \psi_D \cdot \nabla \phi_D \, dx.
\]

Introduce \( \nabla_D \psi_D \) in the first term in the above integral to obtain

\[
T = \int_\Omega A(\nabla \psi - \nabla_D \psi_D) \cdot \nabla \phi \, dx + \int_\Omega A \nabla_D \psi_D \cdot (\nabla \phi - \nabla_D \phi_D) \, dx.
\]
Now introduce $\nabla \psi$ in the second term to obtain
\[
T = \int_{\Omega} A(\nabla \psi - \nabla_D \psi_D) \cdot \nabla \phi \, dx
+ \int_{\Omega} A(\nabla_D \psi_D - \nabla \psi) \cdot (\nabla \phi - \nabla_D \phi_D) \, dx
+ \int_{\Omega} A \nabla \psi \cdot (\nabla \phi - \nabla_D \phi_D) \, dx =: T_1 + T_2 + T_3. \tag{3.7}
\]
The term $T_1$ is re-written as
\[
T_1 = \int_{\Omega} A \nabla \psi \cdot \nabla \phi \, dx - \int_{\Omega} A \nabla_D \psi_D \cdot \nabla \phi \, dx
= -\int_{\Omega} \psi \text{div}(A \nabla \phi) \, dx - \overline{W}_D(A \nabla \phi, \psi_D) + \int_{\Omega} \text{div}(A \nabla \phi) \Pi_D \psi_D \, dx, \tag{3.8}
\]
and this leads to
\[
|T_1| \leq |\overline{W}_D(A \nabla \phi, \psi_D)| + \|\text{div}(A \nabla \phi)\|_{L^2(\Omega)} \|\psi - \Pi_D \psi_D\|_{L^2(\Omega)}
\leq |\overline{W}_D(A \nabla \phi, \psi_D)| + \|\text{div}(A \nabla \phi)\|_{L^2(\Omega)} I_{D,\alpha}(\psi, \psi_D). \tag{3.9}
\]
The term $T_2$ is estimated as
\[
|T_2| \leq \|A\|_{\infty} \|\nabla \psi - \nabla_D \psi_D\|_{L^2(\Omega)} \|\nabla \phi - \nabla_D \phi_D\|_{L^2(\Omega)}
\leq \|A\|_{\infty} \alpha^{-2} I_{D,\alpha}(\psi, \psi_D) I_{D,\alpha}(\phi, \phi_D). \tag{3.10}
\]
The term $T_3$ is estimated exactly as $T_1$ interchanging the roles of $(\psi, \psi_D)$ and $(\phi, \phi_D)$, which leads to
\[
|T_3| \leq |\overline{W}_D(A \nabla \psi, \phi_D)| + \|\text{div}(A \nabla \psi)\|_{L^2(\Omega)} I_{D,\alpha}(\phi, \phi_D). \tag{3.11}
\]
A substitution of (3.9)-(3.11) in (3.7) yields the required estimate in (3.5).

The following trivial lemma will enable us to evaluate the distance between $P_D \overline{\pi}$ and $w_D$ (and similar for $\varphi_y$) in the proof of Theorem 3.1.

**Lemma 3.5.** Under the assumption (1.2), let $\overline{\pi}$ be the solution to (1.3). Let $D$ be a gradient discretisation in the sense of Definition 2.1, and denote the solution to the corresponding gradient scheme (2.1) by $v_D$. Then, there exists $C > 0$ depending only on $A$ and an upper bound of $C_D$ such that, for all $v_D \in X_{D,0},$
\[
\|\Pi_D v_D - \Pi_D w_D\|_{L^2(\Omega)} \leq I_{D,\alpha}(\overline{\pi}, v_D) + C \text{WS}_D(\overline{\pi}), \tag{3.12}
\]
and
\[
\|\nabla_D v_D - \nabla_D w_D\|_{L^2(\Omega)} \leq \alpha^{-1} I_{D,\alpha}(\overline{\pi}, v_D) + C \text{WS}_D(\overline{\pi}). \tag{3.13}
\]

**Remark 3.6.** Estimate (3.12) will not be used in the sequel, but is stated for the sake of completeness since it is required when dealing with PDEs with lower order terms such as $-\text{div}(A \nabla \overline{\pi}) + \overline{\pi} = f$.

**Proof.** A use of triangle inequality after introducing $\overline{\pi}$ as an intermediate term, along with the definition of $I_{D,\alpha}(\overline{\pi}, v_D)$ and the estimate (2.7) in Theorem 2.2, yields (3.12). Similarly, (3.13) can be established by introducing $\nabla \overline{\pi}$ as an intermediate term. \qed
3.2. Proof of the improved $L^2$ estimate. We now turn to the proof of Theorem 3.1.

From (3.1) with $w = \pi$ and (3.2) with $w_D = u_D$,
\[
(g, \pi - \Pi_D u_D) = a(\pi, \varphi_g) - a_D(u_D, \varphi_g, \nu).
\] (3.14)

Since $\text{div}(A\nabla \pi) = -f \in L^2(\Omega)$ and $\text{div}(A\nabla \varphi_g) = -g \in L^2(\Omega)$, a use of (3.5) in (3.14) leads to
\[
\|\pi - \Pi_D u_D\|_{L^2(\Omega)} = (g, \pi - \Pi_D u_D)
\leq a_D(P_D \pi, P_D \varphi_g) - a_D(u_D, \varphi_g, \nu) + E_D(\pi, \varphi_g, P_D \pi, P_D \varphi_g)
\leq a_D(P_D \pi - u_D, P_D \varphi_g - \varphi_g, \nu) + a_D(u_D, P_D \varphi_g - \varphi_g, \nu)
+ a_D(P_D \pi - u_D, \varphi_g, \nu) + E_D(\pi, \varphi_g, P_D \pi, P_D \varphi_g)
=: T_1 + T_2 + T_3 + E_D(\pi, \varphi_g, P_D \pi, P_D \varphi_g).
\] (3.15)

In the rest of this proof, we denote $A \lesssim B$ for $A \leq CB$ with $C$ depending only on $\Omega$, $A$ and an upper bound of $C_D$. Using the Cauchy–Schwarz inequality and (3.13), the term $T_1$ can be estimated as
\[
|T_1| \lesssim \|\nabla_D P_D \pi - \nabla_D u_D\|_{L^2(\Omega)} \|\nabla_D P_D \varphi_g - \nabla_D \varphi_g, \nu\|_{L^2(\Omega)}^4
\leq \left[\alpha^{-1}I_{D,\alpha}(\pi, P_D \pi) + W_{D_\lambda}(\pi)\right]\left[\alpha^{-1}I_{D,\alpha}(\varphi_g, P_D \varphi_g) + W_{D_\lambda}(\varphi_g)\right].
\] (3.16)

Consider the term $T_2$ now. Simple manipulations and a use of (3.2) lead to
\[
T_2 = a_D(u_D, P_D \varphi_g) - a_D(u_D, \varphi_g, \nu)
= - a_D(P_D \pi - u_D, P_D \varphi_g) + a_D(P_D \pi - u_D, \varphi_g, \nu) + a_D(P_D \pi, P_D \varphi_g - \varphi_g, \nu)
= - a_D(P_D \pi - u_D, P_D \varphi_g - (u_D, \Pi_D(P_D \pi - u_D])) + a_D(P_D \pi, P_D \varphi_g - \varphi_g, \nu)
= - T_{2,1} + T_{2,2}.
\] (3.17)

Since $-\text{div}(A\nabla \varphi_g) = g$, we write
\[
T_{2,1} = \int_\Omega \left[\nabla_D(P_D \pi - u_D) \cdot A \nabla \varphi_g - g \Pi_D(P_D \pi - u_D)\right] dx
+ \int_\Omega A \nabla_D(P_D \pi - u_D) \cdot (\nabla_D P_D \varphi_g - \nabla \varphi_g) dx
= \tilde{W}_D(A \nabla \varphi_g, P_D \pi - u_D) + \int_\Omega A \nabla_D(P_D \pi - u_D) \cdot (\nabla_D P_D \varphi_g - \nabla \varphi_g) dx.
\]

We then use (2.4) and (3.13) and the definition of $I_{D,\alpha}(\varphi_g, P_D \varphi_g)$ to obtain
\[
|T_{2,1}| \lesssim \tilde{W}_D(A \nabla \varphi_g) \|\nabla_D P_D \pi - \nabla_D u_D\|_{L^2(\Omega)}^4
+ \|\nabla_D P_D \pi - \nabla_D u_D\|_{L^2(\Omega)}^2 \|\nabla_D P_D \varphi_g - \nabla \varphi_g\|_{L^2(\Omega)}^4
\leq \left[\alpha^{-1}I_{D,\alpha}(\varphi_g, P_D \varphi_g) + W_{D_\lambda}(\varphi_g)\right]\left[\alpha^{-1}I_{D,\alpha}(\pi, P_D \pi) + W_{D_\lambda}(\pi)\right].
\] (3.18)

From (3.5), (3.1) and (3.2), the term $T_{2,2}$ can be estimated as follows:
\[
|T_{2,2}| \leq |a(\pi, \varphi_g) - a_D(P_D \pi, \varphi_g, \nu)| + E_D(\pi, \varphi_g, P_D \pi, P_D \varphi_g)
\leq |(\pi - \Pi_D P_D \pi, \varphi_g)| + E_D(\pi, \varphi_g, P_D \pi, P_D \varphi_g)
\leq \|g\|_{L^2(\Omega)}I_{D,\alpha}(\pi, P_D \pi) + E_D(\pi, \varphi_g, P_D \pi, P_D \varphi_g).
\] (3.19)
A substitution of (3.18) and (3.19) into (3.17) leads to an estimate for $T_2$:

$$|T_2| \lesssim \left[ |\alpha|^{-1} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + W_{SD}(\varphi_g) \right] \left[ |\alpha|^{-1} I_{D,\alpha}(\pi, P_D\pi) + W_{SD}(\pi) \right]$$

$$+ \|g\|_{L^2(\Omega)} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + E_D(\pi, \varphi_g, P_D\pi, P_D\varphi_g).$$

(3.20)

The term $T_3$ is similar to $T_2$, upon swapping the primal and dual problems (both continuous and discrete), that is $(f, \pi, u_D, g, \varphi_g, \varphi_g, D) \leftrightarrow (g, \varphi_g, \varphi_g, D, f, \pi, u_D)$. Hence, with these substitutions in (3.20), we see that

$$|T_3| \lesssim \left[ |\alpha|^{-1} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + W_{SD}(\varphi_g) \right] \left[ |\alpha|^{-1} I_{D,\alpha}(\pi, P_D\pi) + W_{SD}(\pi) \right]$$

$$+ \|f\|_{L^2(\Omega)} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + E_D(\pi, \varphi_g, P_D\pi, P_D\varphi_g).$$

(3.21)

Since $\|g\|_{L^2(\Omega)} = 1$, a substitution in (3.15) of the estimates (3.16), (3.20) and (3.21) for $T_1$, $T_2$ and $T_3$ leads to

$$\|\pi - \Pi_D\pi\|_{L^2(\Omega)} \lesssim \left[ |\alpha|^{-1} I_{D,\alpha}(\pi, P_D\pi) + W_{SD}(\pi) \right] \left[ |\alpha|^{-1} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + W_{SD}(\varphi_g) \right]$$

$$+ I_{D,\alpha}(\pi, P_D\pi) + \|f\|_{L^2(\Omega)} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + E_D(\pi, \varphi_g, P_D\pi, P_D\varphi_g).$$

The proof of Theorem 3.1 is complete by recalling the definition (3.6) of $E_D$, by using

$$|\alpha|^{-2} I_{D,\alpha}(\pi, P_D\pi) I_{D,\alpha}(\varphi_g, P_D\varphi_g)$$

$$\leq \left[ |\alpha|^{-1} I_{D,\alpha}(\pi, P_D\pi) + W_{SD}(\pi) \right] \left[ |\alpha|^{-1} I_{D,\alpha}(\varphi_g, P_D\varphi_g) + W_{SD}(\varphi_g) \right],$$

and by noticing that $-\text{div}(A\nabla \varphi_g) = g$ (that has $L^2(\Omega)$ norm equal to 1) and that $-\text{div}(A\nabla \pi) = f$.

**Remark 3.7.** The symmetry of $A$, assumed in (1.2), is not a restrictive hypothesis in applications. However, Theorem 3.1 and all subsequent results in this paper are valid if $A$ is not symmetric. In some terms, $A$ simply needs to be replaced with $A^T$: in $\bar{W}(A\nabla \varphi_g, P_D\pi)$ in (3.4); in $\bar{W}(A\nabla \phi, \psi_D)$ and $\text{div}(A\nabla \phi)$ in (3.6) and (3.8); etc.

4. Super-convergence for a modified HMM scheme

Here, we recall some notations and the definition of the HMM scheme, which is a gradient scheme for a specific gradient discretisation. Then, we design a modified HMM scheme with a better quadrature rule for the source term and we prove that this modified scheme super-converges, without any assumption on the mesh or regularity assumption on $f$. In the next section, we use this super-convergence of the modified HMM scheme to obtain, under the assumption that $f \in H^1(\Omega)$, a super-convergence for HMM schemes in the case where, on average on patches of cells, the “cell points” $P$ of Definition 4.1 are not far from the centers of mass of the cells.

4.1. Polytopal meshes and definition of the HMM gradient discretisation.

Let us recall the definition of the gradient discretisations that correspond to HMM schemes, starting with the definition of a polytopal mesh (we follow [24], without including the vertices which are not useful to our purpose).

**Definition 4.1** (Polytopal mesh). Let $\Omega$ be a bounded polytopal open subset of $\mathbb{R}^d$ $(d \geq 1)$. A polytopal mesh of $\Omega$ is $T = (M, E, P)$, where:
(1) \( \mathcal{M} \) is a finite family of non empty connected polytopal open disjoint subsets of \( \Omega \) (the cells) such that \( \overline{\Omega} = \bigcup_{K \in \mathcal{M}} K \). For any \( K \in \mathcal{M} \), \( |K| > 0 \) is the measure of \( K \) and \( h_K \) denotes the diameter of \( K \).

(2) \( \mathcal{E} \) is a finite family of disjoint subsets of \( \overline{\Omega} \) (the edges of the mesh in 2D, the faces in 3D), such that any \( \sigma \in \mathcal{E} \) is a non empty open subset of a hyperplane of \( \mathbb{R}^d \) and \( \sigma \subset \overline{\Omega} \). Assume that for all \( K \in \mathcal{M} \) there exists a subset \( \mathcal{E}_K \) of \( \mathcal{E} \) such that the boundary of \( K \) is \( \bigcup_{\sigma \in \mathcal{E}_K} \sigma \). We then denote by \( \mathcal{M}_\sigma = \{ K \in \mathcal{M} : \sigma \in \mathcal{E}_K \} \), and assume that, for all \( \sigma \in \mathcal{E} \), \( \mathcal{M}_\sigma \) has exactly one element and \( \sigma \subset \partial \Omega \), or \( \mathcal{M}_\sigma \) has two elements and \( \sigma \subset \Omega \). Let \( \mathcal{E}_{\text{int}} \) be the set of all interior faces, i.e. \( \sigma \in \mathcal{E} \) such that \( \sigma \subset \Omega \), and \( \mathcal{E}_{\text{ext}} \) the set of boundary faces, i.e. \( \sigma \in \mathcal{E} \) such that \( \sigma \subset \partial \Omega \). The \((d-1)\)-dimensional measure of \( \sigma \in \mathcal{E} \) is \( |\sigma| \), and its center of mass is \( \overline{\sigma} \).

(3) \( \mathcal{P} = (x_K)_{K \in \mathcal{M}} \) is a family of points of \( \Omega \) indexed by \( \mathcal{M} \) and such that, for all \( K \in \mathcal{M} \), \( x_K \in K \). Assume that any cell \( K \in \mathcal{M} \) is strictly \( x_K \)-star-shaped, meaning that if \( x \in K \) then the line segment \([x_K, x]\) is included in \( K \).

For all \( K \in \mathcal{M} \), denote the center of mass of \( K \) by \( \overline{x}_K \) and, if \( \sigma \in \mathcal{E}_K \), denote the (constant) unit vector normal to \( \sigma \) outward to \( K \) by \( n_{K,\sigma} \). Also, let \( d_{K,\sigma} \) be the signed orthogonal distance between \( x_K \) and \( \sigma \) (see Fig. 2), that is:

\[
d_{K,\sigma} = (x - x_K) \cdot n_{K,\sigma} \quad \forall x \in \sigma
\]  

(4.1)

(note that \( (x - x_K) \cdot n_{K,\sigma} \) is constant for \( x \in \sigma \)). The fact that \( K \) is strictly star-shaped with respect to \( x_K \) is equivalent to \( d_{K,\sigma} > 0 \) for all \( \sigma \in \mathcal{E}_K \). For all \( K \in \mathcal{M} \) and \( \sigma \in \mathcal{E}_K \), we denote by \( D_{K,\sigma} \) the cone with apex \( x_K \) and base \( \sigma \), that is \( D_{K,\sigma} = \{ tx_K + (1-t)y : t \in (0,1), y \in \sigma \} \).

The size of the discretisation is \( h_M = \sup\{ h_K : K \in \mathcal{M} \} \) and the regularity factor is

\[
\theta_T = \max_{\sigma \in \mathcal{E}_{\text{int}}, \mathcal{M}_\sigma = \{ K, K' \}} \frac{d_{K,\sigma}}{d_{K',\sigma}} + \max_{K \in \mathcal{M}} \left( \max_{\sigma \in \mathcal{E}_K} \frac{h_K}{d_{K,\sigma}} + \text{Card}(\mathcal{E}_K) \right).
\]  

(4.2)

An upper bound on \( \theta_T \) imposes three geometrical conditions: the orthogonal distance \( d_{K,\sigma} \) between \( x_K \) and \( \sigma \in \mathcal{E}_K \) must be comparable to the diameter \( h_K \) of \( K \); the orthogonal distances \( d_{K,\sigma} \) and \( d_{K',\sigma} \) between \( \sigma \) and its two neighbouring cell points \( x_K \) and \( x_{K'} \) must have similar magnitudes (see Figure 2); and there is a global upper bound on the number of faces of each cell.

![Figure 2. A cell K of a polytopal mesh.](image)
Definition 4.2 (HMM gradient discretisation). Let $T$ be a polytopal mesh of $\Omega$ as per Definition 4.1. An HMM gradient discretisation $D = (X_D, \Pi_D, \nabla_D)$ is defined by:

1. $X_{D,0} = \mathbb{R}^M \times \mathbb{R}^{E_{int}} \times \{0\}^{E_{ext}}$ is the space of degrees of freedom in the cell and on the interior faces of the mesh:
   
   $X_{D,0} = \{v = (v_K)_{K \in M}, (v_\sigma)_{\sigma \in E} : v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}, v_\sigma = 0 \text{ if } \sigma \in E_{ext}\}$.

2. $\Pi_D : X_{D,0} \to L^2(\Omega)$ is the following piecewise constant reconstruction on the mesh:
   
   $\forall v \in X_{D,0}, \forall K \in M, \Pi_D v = v_K$ on $K$.

3. $\nabla_D : X_{D,0} \to L^2(\Omega)^d$ reconstructs piecewise constant gradients on the cones $(D_{K,\sigma})_{K \in M, \sigma \in E_K}$:
   
   $\forall v \in X_{D,0}, \forall K \in M, \forall \sigma \in E_K$,  
   
   $\nabla_D v = \nabla_K v + \frac{\sqrt{d}}{d_{K,\sigma}} [\mathcal{L}_K R_K(v)]_\sigma n_{K,\sigma}$ on $D_{K,\sigma}$,  

   (4.3)

where:

- $\nabla_K v = \frac{1}{|K|} \sum_{\sigma \in E_K} |\sigma| v_\sigma n_{K,\sigma}$,

- $R_K : X_{D,0} \to \mathbb{R}^{E_K}$ is given by $R_K(v) = (R_{K,\sigma}(v))_{\sigma \in E_K}$ with $R_{K,\sigma}(v) = v_\sigma - v_K - \nabla_K v \cdot (x_\sigma - x_K)$,

- $\mathcal{L}_K$ is an isomorphism of the space $\text{Im}(R_K)$.

If $D$ is an HMM gradient discretisation, define $\zeta_D$ as the smallest positive number such that, for all $K \in M$ and all $v \in X_{D,0}$,

$$\zeta_D^{-1} \sum_{\sigma \in E_K} |D_{K,\sigma}| \left| \frac{R_{K,\sigma}(v)}{d_{K,\sigma}} \right|^2 \leq \sum_{\sigma \in E_K} |D_{K,\sigma}| \left| \frac{[\mathcal{L}_K R_K(v)]_\sigma}{d_{K,\sigma}} \right|^2 \leq \zeta_D \sum_{\sigma \in E_K} |D_{K,\sigma}| \left| \frac{R_{K,\sigma}(v)}{d_{K,\sigma}} \right|^2.$$  

(4.4)

It is proved in [23] that, if $A$ is piecewise constant on the mesh, for any HMM scheme $\mathcal{S}$ (as defined in [22]) on a polytopal mesh $T$, there exists a choice of isomorphisms $(\mathcal{L}_K)_{K \in M}$ such that $\mathcal{S}$ is the gradient scheme corresponding to the gradient discretisation $D$ given by Definition 4.2.

Remark 4.3. If $A$ is not piecewise constant, the HMM method corresponds to a modified gradient scheme which consists in writing (2.1) with $A$ replaced by its $L^2$ projection on piecewise constant matrix-valued functions on $M$. As shown in Section 8.1 in the appendix, this modification of the gradient scheme preserves the basic rates of convergence in Theorem 2.2, as well as the super-convergence results established below (see Theorems 4.6 and 5.3). In the following, we will therefore only consider the standard gradient scheme (2.1), having ensured that the results established for this one also apply to the original HMM methods even if $A$ is not piecewise constant (which would be an unreasonable constraint given the assumption (4.5) to come).
As for conforming and non-conforming finite element methods, the super-convergence estimate for HMM schemes requires a higher regularity of the solution. In particular, a convergence rate of order 2 is obtained under the following $H^2$ regularity assumption (which holds if $A$ is Lipschitz continuous and $\Omega$ is convex).

For all $f \in L^2(\Omega)$, the solution $\pi$ to (1.1) belongs to $H^2(\Omega)$ and
\[
\|\pi\|_{H^2(\Omega)} + \|A\nabla\pi\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},
\] with $C$ depending only on $\Omega$ and $A$.

Under this assumption, the solution $\varphi_g$ to (3.1) also satisfies the $H^2$ regularity (with $f$ replaced with $g$ in the estimate).

**Remark 4.4** ($H^{1+\delta}$ super-convergence). If we only assume an $H^{1+\delta}$ regularity property instead of (4.5), then standard interpolation techniques can be applied in the proofs below, and we still obtain some super-convergence results (albeit of reduced order, typically $h^{2\delta}_{\mathcal{M}}$ instead of $h^2_{\mathcal{M}}$, and possibly by strengthening the regularity assumptions on $f$ if $\delta$ is small).

In the rest of this section, we use the following notation:
\[
A \lesssim B \text{ means that } A \leq C B \text{ with } C \text{ depending only on } \Omega, A \text{ and an upper bound of } \theta_T + \zeta_D. \quad (4.6)
\]

### 4.2. A modified HMM scheme with better source term approximation.

We define here the modified HMM scheme and prove its super-convergence.

**Definition 4.5** (Modified HMM gradient discretisation). Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ be an HMM gradient discretisation in the sense of Definition 4.2. The modified HMM gradient discretisation is $\mathcal{D}^* = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}^*}, \nabla_{\mathcal{D}})$, where the reconstruction $\Pi_{\mathcal{D}^*}$ is defined by
\[
\forall v \in X_{\mathcal{D},0}, \forall K \in \mathcal{M}, \forall x \in K, \Pi_{\mathcal{D}^*}v(x) = \Pi_{\mathcal{D}}v(x) + \nabla_{K}v \cdot (x - x_K). \quad (4.7)
\]
We notice that using $\mathcal{D}^*$ in the gradient scheme (2.1) only modifies the discretisation of the source term, and not the scheme’s matrix. This modified HMM scheme is therefore only marginally more expensive than a standard HMM scheme. Moreover, it enjoys a better super-convergence result than the HMM method, since this super-convergence (i) does not require $\mathcal{P}$ in Definition 4.1 to be close on average to the centers of mass of the cells, (ii) does not require the $H^1$ regularity of $f$, and (iii) gives an $O(h^{2\delta}_{\mathcal{M}})$ approximation of the solution $\pi$ to (1.3) rather than its piecewise constant projection. The influence on super-convergence of the choice of quadrature rule for the source term was already noticed in [36] for the TPFA scheme in dimension 1.

**Theorem 4.6** (Super-convergence for the modified HMM method). Assume (1.2), (4.5), and that $d \leq 3$. Let $\bar{u}$ be the solution to (1.3). Take $\mathcal{T}$ as a polytopal mesh in the sense of Definition 4.1. Let $u_{\mathcal{D}^*}$ be the solution of the gradient scheme (2.1) for the modified HMM gradient discretisation $\mathcal{D} = \mathcal{D}^*$ defined above. Then, recalling the notation (4.6),
\[
\|\Pi_{\mathcal{D}^*}u_{\mathcal{D}^*} - \bar{u}\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} h^2_{\mathcal{M}} \quad (4.8)
\]
and
\[
\|\nabla u_{\mathcal{D}^*} - \nabla \bar{u}\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} h_{\mathcal{M}}. \quad (4.9)
\]
Let us recall a few results that will be used in the proof of this theorem. From [21, Lemma 12.4]:

\[ \forall v \in X_{D,0}, \forall K \in M, \int_K \nabla_D v(x) \, dx = |K| \nabla_K v. \quad (4.10) \]

As a consequence,

\[ |\nabla_K v| \leq |K|^{-\frac{1}{2}} \|\nabla_D v\|_{L^2(K)}. \quad (4.11) \]

By [21, Propositions 12.14 and 12.15],

\[ C_D \lesssim 1, \quad \forall \phi \in H^2(\Omega) \cap H^1_0(\Omega), \ P_D(\phi) \lesssim \|\phi\|_{H^2(\Omega)} h_M, \quad \forall \psi \in H^1(\Omega)^d, \ W_D(\psi) \lesssim \|\psi\|_{H^1(\Omega)} h_M. \quad (4.12, 4.13, 4.14) \]

Using (4.11), we readily check that

\[ \forall v \in X_{D,0}, \|P_D v - \Pi_D v\|_{L^2(\Omega)} \leq h_M \|\nabla_D v\|_{L^2(\Omega)^d}. \quad (4.15) \]

Therefore, as a consequence of (4.12)–(4.14) and of [21, Remark 7.49],

\[ C_{D^*} \lesssim 1, \quad \forall \phi \in H^2(\Omega) \cap H^1_0(\Omega), \ S_{D^*}(\phi) \lesssim \|\phi\|_{H^2(\Omega)} h_M, \quad \forall \psi \in H^1(\Omega)^d, \ W_{D^*}(\psi) \lesssim \|\psi\|_{H^1(\Omega)} h_M. \quad (4.16, 4.17, 4.18) \]

We can now turn to the proof of the super-convergence result for the modified HMM gradient scheme based on \( D^* \).

**Proof of Theorem 4.6.** Properties (4.17)–(4.18) show that, for all \( \phi \in H^2(\Omega) \cap H^1_0(\Omega) \) such that \( A\nabla \phi \in H^1(\Omega)^d \),

\[ W_{D^*}(\phi) \lesssim (\|A\nabla \phi\|_{H^1(\Omega)^d} + \|\phi\|_{H^2(\Omega)}) h_M. \quad (4.19) \]

Hence, estimate (4.9) is a consequence of (2.7) in Theorem 2.2, of (4.16) and of (4.5). To prove (4.8), we use the improved \( L^2 \) estimate for gradient schemes (Theorem 3.1). Let us assume that we find a mapping \( \mathcal{P}_{D^*} : H^2(\Omega) \cap H^1_0(\Omega) \to X_{D,0} \) such that, for any \( \phi \in H^2(\Omega) \cap H^1_0(\Omega) \),

\[ I_{D^*,h_M}(\phi, \mathcal{P}_{D^*} \phi) \lesssim \|\phi\|_{H^2(\Omega)} h_M^2 \quad (4.20) \]

and

\[ \forall \psi \in H^1(\Omega)^d, \quad |W_{D^*}(\psi, \mathcal{P}_{D^*} \phi)| \lesssim \|\phi\|_{H^2(\Omega)} \|\psi\|_{H^1(\Omega)} h_M. \quad (4.21) \]

Then, the proof of (4.8) is concluded by applying Theorem 3.1 with this interpolant \( \mathcal{P}_{D^*} \) (and \( \alpha = h_M \)), by using (4.5) on \( \mathbf{p} \) and \( \varphi_g \) (recall that \( \|g\|_{L^2(\Omega)} = 1 \)), and by invoking (4.19). We now turn to the construction of \( \mathcal{P}_{D^*} \) and to the proof of its properties.

If \( \phi \in H^2(\Omega) \), then \( \phi \) is continuous (since \( d \leq 3 \)) and we can therefore set \( \mathcal{P}_{D^*} \phi = (((\phi_K)_{K \in M, (\phi_\sigma)_{\sigma \in E_K}}) \in X_{D,0} \) with

\[ \forall K \in M, \phi_K = \phi(x_K); \quad \forall \sigma \in E, \phi_\sigma = \frac{1}{|\sigma|} \int_\sigma \phi(x) \, dx. \]

**Step 1:** *Proof of (4.20).*

Let \( K \in M \). By [21, Lemma B.1], \( K \) is star-shaped with respect to all points in the ball \( B(x_K, \min_{\sigma \in E_K} d_{K,\sigma}) \supset B(x_K, \theta^{-1} h_K) \). Hence we can apply Lemma 8.5 (see Appendix) with \( V = K \). Let \( L_\phi \) be the affine map given by this lemma.
Let $G_K = (G_K^s, (G_K^s)_{s \in E_K})$ be a $\mathbb{P}_1$-exact gradient reconstruction on $K$ upon $S = (x_K, (\mathbf{x}_s)_{s \in E_K})$ (see Definition 8.4 with $U = K$). We let

$$\ell = (L_\phi(x_K), (L_\phi(\mathbf{x}_s))_{s \in E_K}).$$

(4.22)

With abuse of notations, write $G_K P_{\mathbb{P}} \phi$ for $G_K(\phi_K, (\phi_s)_{s \in E_K})$. Since $G_K$ is a $\mathbb{P}_1$-exact gradient reconstruction, we have $G_K \ell = \nabla L_\phi$ and thus $G_K P_{\mathbb{P}} \phi - \nabla \phi = G_K P_{\mathbb{P}} \phi - G_K \ell + \nabla L_\phi - \nabla \phi$. By property of the norm $\|G_K\|$ in Definition 8.4,

$$\|G_K P_{\mathbb{P}} \phi - \nabla \phi\|_{L^2(K)\ell^*} \leq h_K^{-1} |K|^{\frac{n}{2}} \|G_K\| \max \left( |L_\phi(x_K) - \phi_K|, \max_{s \in E_K} |L_\phi(\mathbf{x}_s) - \phi_s| \right) + \|\nabla L_\phi - \nabla \phi\|_{L^2(K)\ell^*}. \quad (4.23)$$

Since $L_\phi$ is affine and $\mathbf{x}_s$ is the center of mass of $\sigma$, by definition of $\phi_s$ and using estimate (8.13), we have

$$|L_\phi(\mathbf{x}_s) - \phi_s| = \frac{1}{|\sigma|} \int_{\sigma} (L_\phi(x) - \phi(x)) \, ds(x) \lesssim h_K^2 |K|^{-\frac{1}{2}} \|\phi\|_{H^2(K)}.$$  

We also have $|L_\phi(x_K) - \phi_K| \lesssim h_K^2 |K|^{-\frac{1}{2}} \|\phi\|_{H^2(K)}$. Plugging into (4.23) and using (8.14), this gives

$$\|G_K P_{\mathbb{P}} \phi - \nabla \phi\|_{L^2(K)\ell^*} \lesssim (1 + \|G_K\|) h_K \|\phi\|_{H^2(K)}. \quad (4.24)$$

The restriction $(\nabla_{\mathbb{P}})_{K}$ of $\nabla_{\mathbb{P}}$ to the degrees of freedom $(v_{K}, (v_s)_{s \in E_K})$ in the cell $K$ is a $\mathbb{P}_1$-exact gradient reconstruction on $K$ upon $S$ [24, Section 3.6] and satisfies $\|\nabla_{\mathbb{P}}\|_{K} \lesssim 1$ (see [21, Lemma 12.8]). Hence, applying (4.24) to $G_K = (\nabla_{\mathbb{P}})_{K}$, squaring the resulting inequality, and summing over $K \in \mathcal{M}$, we find

$$\|\nabla_{\mathbb{P}} P_{\mathbb{P}} \phi - \nabla \phi\|_{L^2(\Omega)\ell^*} \lesssim h_{\mathcal{M}} \|\phi\|_{H^2(\Omega)}. \quad (4.25)$$

Let us now estimate $\Pi_{\mathbb{P}} P_{\mathbb{P}} \phi - \phi$. We still take $\ell$ as defined by (4.22). Since $\nabla_{\mathbb{P}}$ is a $\mathbb{P}_1$-exact gradient reconstruction upon $(x_K, (\mathbf{x}_s)_{s \in E_K})$ (see [21, Lemma B.6]), we have $\nabla_{\mathbb{P}} \ell = \nabla L_\phi$ and thus, for any $x \in K$, by the definition (4.7) of $\Pi_{\mathbb{P}}$,

$$\Pi_{\mathbb{P}} \ell(x) = L_\phi(x_K) + \nabla L_\phi \cdot (x - x_K) = L_\phi(x).$$

Thus, using the estimate (8.13), we have, for all $x \in K$,

$$\begin{align*}
\|\Pi_{\mathbb{P}} P_{\mathbb{P}} \phi(x) - \phi(x)\| & \leq \|\Pi_{\mathbb{P}} P_{\mathbb{P}} \phi(x) - \Pi_{\mathbb{P}} \ell(x)\| + |L_\phi(x) - \phi(x)| \\
& \leq |\phi(x_K) - L_\phi(x_K)| + h_K |\nabla_{\mathbb{P}} P_{\mathbb{P}} \phi - \nabla \phi| + |\nabla \phi| \\
& \lesssim h_K^2 |K|^{-\frac{1}{2}} \|\phi\|_{H^2(K)} + h_K \|\nabla_{\mathbb{P}} P_{\mathbb{P}} \phi - \nabla \phi| + h_K \|\nabla \phi - \nabla L_\phi|.
\end{align*}$$

Taking the $L^2(K)$ norm leads to

$$\begin{align*}
\|\Pi_{\mathbb{P}} P_{\mathbb{P}} \phi - \phi\|_{L^2(K)} & \lesssim h_K^2 \|\phi\|_{H^2(K)} + h_K \|\nabla_{\mathbb{P}} P_{\mathbb{P}} \phi - \nabla \phi\|_{L^2(K)\ell^*} \\
& \quad + h_K \|\nabla \phi - \nabla L_\phi\|_{L^2(K)\ell^*}. \quad (4.26)
\end{align*}$$

It is easy to see that the $\mathbb{P}_1$-exact gradient reconstruction $G_K = \nabla K$ satisfies $\|G_K\| \lesssim 1$ (see for example [21, Lemma B.6]), or use (4.11) and $\|(\nabla_{\mathbb{P}})_{K}\| \lesssim 1$ as mentioned previously). Hence, applying (4.24) to $G_K = \nabla K$ and using (8.14), we obtain

$$\|\Pi_{\mathbb{P}} P_{\mathbb{P}} \phi - \phi\|_{L^2(K)} \lesssim h_K^2 \|\phi\|_{H^2(K)}.$$  

Squaring and summing over $K \in \mathcal{M}$ yields

$$\|\Pi_{\mathbb{P}} P_{\mathbb{P}} \phi - \phi\|_{L^2(\Omega)} \lesssim h_{\mathcal{M}} \|\phi\|_{H^2(\Omega)},$$

which, combined with (4.25), concludes the proof of (4.20).
Step 2: Proof of (4.21).

Since \( \phi \in H^{1}_{0}(\Omega) \), by Stokes formula we can write

\[
\tilde{W}_{D}^{*}(\psi, P_{D}^{*} \phi) = \int_{\Omega} (\Pi_{D}^{*} P_{D}^{*} \phi \nabla \psi + \nabla D P_{D}^{*} \phi \cdot \psi) \, dx
\]

\[
= \int_{\Omega} ((\Pi_{D}^{*} P_{D}^{*} \phi - \phi) \nabla \psi + (\nabla D P_{D}^{*} \phi - \nabla \phi) \cdot \psi) \, dx. \tag{4.27}
\]

Using (4.20), this gives

\[
\left| \tilde{W}_{D}^{*}(\psi, P_{D}^{*} \phi) \right| \lesssim h_{M}^{2} \|\phi\|_{H^{1}(\Omega)} \|\nabla \psi\|_{L^{2}(\Omega)}
\]

\[
+ \sum_{K \in M} \left| \int_{K} (\nabla D P_{D}^{*} \phi - \nabla \phi) \cdot \psi \, dx \right|. \tag{4.28}
\]

We now work on the last term in this estimate. By (4.10) and choice of \( \phi_{\sigma} \) we have

\[
\int_{K} \nabla D P_{D}^{*} \phi \, dx = |K| \nabla K P_{D}^{*} \phi = \sum_{\sigma \in E_{K}} |\sigma| \phi_{\sigma} n_{K,\sigma}
\]

\[
= \sum_{\sigma \in E_{K}} \int_{\sigma} \phi_{\sigma} n_{K,\sigma} \, dx = \int_{K} \nabla \phi \, dx. \tag{4.29}
\]

Hence, if \( \psi_{K} = \frac{1}{|K|} \int_{K} \psi(x) \, dx \),

\[
\left| \int_{K} (\nabla D P_{D}^{*} \phi - \nabla \phi) \cdot \psi \, dx \right| = \left| \int_{K} (\nabla D P_{D}^{*} \phi - \nabla \phi) \cdot (\psi - \psi_{K}) \, dx \right|
\]

\[
\leq \|\nabla D P_{D}^{*} \phi - \nabla \phi\|_{L^{2}(K)} \|\psi - \psi_{K}\|_{L^{2}(K)}. \tag{4.30}
\]

It is quite classical (see, e.g., [21, Lemma B.7]) that, for all \( \psi \in H^{1}(K) \), if \( \psi_{K} = \frac{1}{|K|} \int_{K} \psi(x) \, dx \) then

\[
\|\psi - \psi_{K}\|_{L^{2}(K)} \lesssim h_{K} \|\psi\|_{H^{1}(K)}. \tag{4.30}
\]

Applying this to each component of \( \psi \) and using (4.24) with \( G_{K} = (\nabla D)_{K} \) yields

\[
\left| \int_{K} (\nabla D P_{D}^{*} \phi - \nabla \phi) \cdot \psi \, dx \right| \lesssim h_{K}^{2} \|\phi\|_{H^{2}(K)} \|\psi\|_{H^{1}(K)}. \tag{4.31}
\]

Summing over \( K \), using the Cauchy–Schwarz inequality, and plugging the result in (4.28), we obtain (4.21).

\[
\square
\]

5. Super-convergence result for HMM and TPFA schemes

If \( (x_{K})_{K \in M} \) are the centers of mass of the cells, HMM methods are hMFD methods and the super-convergence is therefore known. In some instances, however, it is interesting to choose \( x_{K} \) not at the center of mass of \( K \). On triangles, for example, choosing \( x_{K} \) as the circumcenter of \( K \) allows to recover the two-point flux approximation finite volume scheme. This ensures that the scheme satisfies a discrete maximum principle, has a very small matrix stencil, etc. Our aim is to show that, if \( (x_{K})_{K \in M} \) are not the centers of mass, but if we can create patches of cells over which, on average, these points are (close to) the centers of mass, then a super-convergence result still occurs for HMM methods. We first define this notion of patches of cells. Recall that a set \( X \) is star-shaped with respect to a subset \( Y \) if, for all \( x \in X \) and all \( y \in Y \), the segment \([x, y]\) is contained in \( X \).
Definition 5.1 (Patching of cells). Let $\mathcal{T}$ be a polytopal mesh of $\Omega$ in the sense of Definition 4.1. A patching of the cells of $\mathcal{T}$ is a family $\mathcal{P}$ of disjoint sets of cells (the patches), such that, for each patch $Pa \in \mathcal{P}$, letting $U_{Pa} := \cup_{K \in Pa} K$, there exists a ball $B_{Pa} \subset U_{Pa}$ such that $U_{Pa}$ is star-shaped with respect to $B_{Pa}$.

We then define:

- $\Omega_{\mathcal{P}} = \cup_{Pa \in \mathcal{P}} \cup_{K \in Pa} K$ the region of $\Omega$ covered by the patches, and $\Omega_{\mathcal{P}}^c = \Omega \setminus \Omega_{\mathcal{P}}$ its complement,
- the regularity factor by

$$\mu_{\mathcal{P}} = \max_{Pa \in \mathcal{P}} \mathrm{Card}(Pa) + \max_{Pa \in \mathcal{P}} \max_{K \in Pa} \frac{h_K}{\text{diam}(B_{Pa})}$$ (5.1)

where $\text{Card}(Pa)$ is the number of cells in $Pa$,
- the maximum norm of the patch-averaged vector between the centers of masses and the cell points by

$$e_{\mathcal{P}} = \max_{Pa \in \mathcal{P}} \left| \frac{1}{|U_{Pa}|} \sum_{K \in Pa} |K| (\mathbf{u}_K - \mathbf{x}_K) \right|.$$ (5.2)

Remark 5.2. For any $Pa \in \mathcal{P}$, since $U_{Pa}$ is connected, the diameter of $U_{Pa}$ (and thus of $B_{Pa}$) is bounded above by $\mu_{\mathcal{P}} h_K$ for all $K \in Pa$. Hence, bounding above $\mu_{\mathcal{P}}$ requires in particular that, for any $Pa \in \mathcal{P}$, the diameter of $B_{Pa}$ is comparable to the diameter of any $K \in Pa$.

Using this notion, we state our super-convergence result for HMM schemes. The result varies from some "classical" ones as it does not directly involve the $L^2$ projection of the exact function on the piecewise constant functions, but pointwise values of the exact function. This is in a sense expected, as the $L^2$ projection is an appropriate operator only when $(x_K)_{K \in \mathcal{M}}$ are the centers of mass (only case when the $L^2$ projection of $\mathbf{u}$ is $h^2_K$ close to the pointwise values $\mathbf{u}(x_K)$).

In the following theorem and its proof, we use the notation:

$$A \lesssim B$$ means that $A \leq CB$ with $C$ depending only on $\Omega$, $A$ and an upper bound of $\theta_T + \zeta_D + \mu_{\mathcal{P}}$ (5.2)

and we call a "strip of width $\rho > 0$" a set of the form $S_H(\rho) := \{x \in \Omega : \text{dist}(x,H) \leq \rho\}$, where $H$ is an hyperplane of $\mathbb{R}^d$.

Theorem 5.3 (Super-convergence for HMM schemes).

Under the assumptions (1.2), (4.5), and that $d \leq 3$, let $f \in H^1(\Omega)$ and $\mathbf{u}$ be the solution to (1.3). Choose $\mathcal{T}$ to be a polytopal mesh in the sense of Definition 4.1. Let $\mathcal{P}$ be a patching of the cells of $\mathcal{T}$ such that $\Omega_{\mathcal{P}}$ is contained in the union of $r_{\mathcal{P}}$ strips of width $\rho_{\mathcal{P}}$.

Take $\mathcal{D}$ an HMM gradient discretisation on $\mathcal{T}$ (see Definition 4.2) and let $u_\mathcal{D}$ be the solution of the corresponding gradient scheme (2.1). Let $\mathbf{u}_\mathcal{P}$ be the piecewise constant function on $\mathcal{M}$ equal to $\mathbf{u}(x_K)$ on $K \in \mathcal{M}$ (see (1.6)). Then,

$$\|\Pi_\mathcal{D} u_\mathcal{D} - \mathbf{u}_\mathcal{P}\|_{L^2(\Omega)} \lesssim \|f\|_{H^1(\Omega)} \left(h_{\mathcal{M}}^2 + e_{\mathcal{P}} + r_{\mathcal{P}} \rho_{\mathcal{P}} h_{\mathcal{M}}\right).$$ (5.3)

In particular, if $\mathcal{P}$ is such that $e_{\mathcal{P}} \lesssim h_{\mathcal{M}}^3$, $r_{\mathcal{P}} \lesssim 1$ and $\rho_{\mathcal{P}} \lesssim h_{\mathcal{M}}$, then the following super-convergence estimate occurs:

$$\|\Pi_\mathcal{D} u_\mathcal{D} - \mathbf{u}_\mathcal{P}\|_{L^2(\Omega)} \lesssim \|f\|_{H^1(\Omega)} h_{\mathcal{M}}^2.$$
Remark 5.4 (Super-convergence for $\mathbb{RT}_0$). The $\mathbb{RT}_0 - \mathbb{P}_0$ mixed finite element method is a particular instance of an HMM scheme with $(x_K)_{K \in \mathcal{M}}$ being the centers of mass of the cells [10]. Hence, Theorem 5.3 with the trivial patching $\Psi = \mathcal{M}$ yields a super-convergence result for the $\mathbb{RT}_0 - \mathbb{P}_0$ scheme. This result was previously established in [15].

Proof. The proof hinges on two tricks. In Step 1, letting $u_D$ be the solution to the modified HMM scheme, we show that $\|\Pi_D u_D - \Pi_D u_D^\ast\|_{L^2(\Omega)}$ is of order $O(h_M^2 + e_M + r_M \rho_p h_M)$. In Step 2, we introduce a weighted projection $\pi^w_M$ such that $\Pi_D u_D^\ast = \pi^w_M(\Pi_D^w u_D^\ast)$ and $\|\pi^w_M u - \pi^w_M\|_{L^2(\Omega)} = O(h_M^2)$. Combined with the result from Step 1 and the super-convergence property (4.8) of the modified HMM method, this concludes the proof.

Step 1: Comparison of the solutions to the schemes for $\mathcal{D}$ and $\mathcal{D}^\ast$.

Let $u_D^\ast$ be the solution to (2.1) with $\mathcal{D}^\ast$ instead of $\mathcal{D}$. Subtracting the two gradient schemes corresponding to $\mathcal{D}^\ast$ and $\mathcal{D}$ we see that, for all $v_D \in X_{\mathcal{D}, 0}$,

\[
\int_{\Omega} A(x) (\nabla_D u_D^\ast - \nabla_D u_D)(x) \cdot \nabla_D v_D(x) \, dx = \int_{\Omega} f(x) (\Pi_D^w v_D - \Pi_D v_D)(x) \, dx. \tag{5.4}
\]

Let $e_K = x_K - x_K$. Using the definition (4.7) of $\Pi_D^w$, we can write

\[
\int_{\Omega} f(x) (\Pi_D^w v_D - \Pi_D v_D)(x) \, dx = \sum_{K \in \mathcal{M}} \left( \int_{K} f(x) (x - x_K) \, dx \right) \cdot \nabla_K v_D
\]

\[
= \sum_{K \in \mathcal{M}} \left( \int_{K} f(x) (x - x_K) \, dx \right) \cdot \nabla_K v_D + \sum_{K \in \mathcal{M}} \left( \int_{K} f(x) e_K \, dx \right) \cdot \nabla_K v_D. \tag{5.5}
\]

Let $g \in L^2(\Omega)$ and $v_D$ be the solution to the gradient scheme (2.1) with right-hand side $g$ instead of $f$. Combining (5.4), (5.5) and the definition of $v_D$, we find

\[
\int_{\Omega} g(x) (\Pi_D u_D^\ast - \Pi_D u_D)(x) \, dx = \int_{\Omega} f(x) (\Pi_D^w v_D - \Pi_D v_D)(x) \, dx
\]

\[
= \sum_{K \in \mathcal{M}} \left( \int_{K} f(x) (x - x_K) \, dx \right) \cdot \nabla_K v_D + \sum_{K \in \mathcal{M}} \left( \int_{K} f(x) e_K \, dx \right) \cdot \nabla_K v_D
\]

\[
= T_1 + T_2. \tag{5.6}
\]

Since $\overline{x}_K$ is the center of mass of $K$, we have $\int_K (x - \overline{x}_K) \, dx = 0$. Hence, letting $f_K$ to be the average value of $f$ on $K$, using (4.30) and (4.11), $T_1$ is estimated:

\[
T_1 = \sum_{K \in \mathcal{M}} \left( \int_{K} (f(x) - f_K)(x - \overline{x}_K) \, dx \right) \cdot \nabla_K v_D
\]

\[
\leq \sum_{K \in \mathcal{M}} h_K \|f - f_K\|_{L^2(K)} |K|^{1/2} \|\nabla_K v_D\|
\]

\[
\lesssim h_M^2 \sum_{K \in \mathcal{M}} \|f\|_{H^1(K)} \|\nabla_D v_D\|_{L^2(K)}
\]

\[
\lesssim h_M^2 \|f\|_{H^1(\Omega)} \|\nabla_D v_D\|_{L^2(\Omega)} \lesssim h_M^2 \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}. \tag{5.7}
\]
In the last line, we used the discrete Cauchy–Schwarz inequality and the stability property (2.6) in Theorem 2.2.

We now estimate $T_2$. Let $\tau$ be the solution to (1.3) with right-hand side $g$ instead of $f$, and recall that $v_\mathcal{D}$ is the solution to the gradient scheme for this continuous problem. Hence, $\nabla_K \tau$ should be close to $\nabla \tau$ on the cell $K$. To use this proximity, write

$$T_2 = \sum_{K \in \mathcal{M}} \int_K f(x) e_K \cdot (\nabla_K v_\mathcal{D} - \nabla \tau(x)) \, dx + \sum_{K \in \mathcal{M}_\mathcal{P}} \int_K (f \nabla \tau)(x) \cdot e_K \, dx$$

$$+ \sum_{K \notin \mathcal{M}_\mathcal{P}} \int_K (f \nabla \tau)(x) \cdot e_K \, dx$$

$$= T_{2,1} + T_{2,2} + T_{2,3},$$

where $\mathcal{M}_\mathcal{P} = \cup_{P_a \in \mathcal{P}} \mathcal{P}_a$ is the set of cells covered by the patches. Letting $\nabla \mathcal{M}_\mathcal{V}$ be the piecewise constant function on $\mathcal{M}$ equal to $\nabla_K \tau$ on each $K \in \mathcal{M}$, the bound $\sup_{K \in \mathcal{M}} |e_K| \leq h_M$ gives

$$|T_{2,1}| \leq h_M \|f\|_{L^2(\Omega)} \|\nabla \mathcal{M}_\mathcal{V} \nabla - \nabla \tau\|_{L^2(\Omega)^d}.$$

By (4.10), $\nabla \mathcal{M}_\mathcal{V} = \pi_\mathcal{M}(\nabla \mathcal{V} \nabla)$, where $\pi_\mathcal{M}$ is the $L^2$ projector on piecewise constant functions (here, it is used component-wise). Since $\pi_\mathcal{M}$ has norm 1, the estimates (2.7), (4.12)–(4.14), (4.30) and the $H^2$ regularity property (4.5) for $\tau$ yield

$$|T_{2,1}| \leq h_M \|f\|_{L^2(\Omega)} \left( \|\pi_\mathcal{M}(\nabla \mathcal{V} \nabla - \nabla \tau)\|_{L^2(\Omega)^d} + \|\pi_\mathcal{M}(\nabla \mathcal{V} - \nabla \tau)\|_{L^2(\Omega)^d} \right)$$

$$\leq h_M \|f\|_{L^2(\Omega)} \left( \|\nabla \mathcal{V} \nabla - \nabla \tau\|_{L^2(\Omega)^d} + h_M \|\nabla \tau\|_{H^2(\Omega)} \right)$$

$$\leq \frac{h_M^2}{2} \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

The sum $T_{2,2}$ is estimated by using the patches. Let $P_a \in \mathcal{P}$, $K \in \mathcal{P}_a$, and apply Lemma 8.6 (see Appendix) twice to $(U, V, O) = (K, B_{Pa}, U_{Pa})$ and $(U, V, O) = (U_{Pa}, B_{Pa}, U_{Pa})$. Owing to Remark 5.2 and using the upper bound on $\theta_\tau$,

$$\text{diam}(U_{Pa}) \leq h_K, \quad \text{diam}(U_{Pa})^d \leq \text{diam}(B_{Pa})^d \leq |B_{Pa}| \leq |U_{Pa}|$$

and $|K| \leq h_K \leq \text{diam}(B_{Pa})^d \leq |B_{Pa}|$.

Hence,

$$\left| \frac{1}{|K|} \int_K (f \nabla \tau)(x) \, dx - \frac{1}{|U_{Pa}|} \int_{U_{Pa}} (f \nabla \tau)(x) \, dx \right|$$

$$\leq \left| \frac{1}{|K|} \int_K (f \nabla \tau)(x) \, dx - \frac{1}{|B_{Pa}|} \int_{B_{Pa}} (f \nabla \tau)(x) \, dx \right|$$

$$+ \left| \frac{1}{|B_{Pa}|} \int_{B_{Pa}} (f \nabla \tau)(x) \, dx - \frac{1}{|U_{Pa}|} \int_{U_{Pa}} (f \nabla \tau)(x) \, dx \right|$$

$$\leq \left( \frac{\text{diam}(U_{Pa})^{d+1}}{|K|} + \frac{\text{diam}(U_{Pa})^{d+1}}{|B_{Pa}|} \right) \|f \nabla \tau\|_{W^{1,1}(U_{Pa})^d}$$

$$\leq \frac{h_M}{|K|} \|f \nabla \tau\|_{W^{1,1}(U_{Pa})^d}.$$
Since the patches are pairwise disjoint and $|\sum_{K \in P_\mathcal{M}} |K| e_K| \leq |U_{\mathcal{M}}| e_{\mathcal{M}}$, by the $H^2$ regularity property (4.5),

$$T_{2,2} = \sum_{\mathcal{P} \in \Psi} \sum_{K \in P_\mathcal{M}} |K| |e_K| \cdot \left( \frac{1}{|K|} \int_K (f \nabla \tau)(x) \, dx \right)$$

$$\lesssim \sum_{\mathcal{P} \in \Psi} \sum_{K \in P_\mathcal{M}} h_M^2 \|f \nabla \tau\|_{W^{1,1}((U_{\mathcal{P}}))}^d$$

$$+ \sum_{\mathcal{P} \in \Psi} \left( \frac{1}{|U_{\mathcal{P}}|} \int_{U_{\mathcal{P}}} (f \nabla \tau)(x) \, dx \right) \cdot \sum_{K \in P_\mathcal{M}} |K| |e_K|$$

$$\lesssim h_M^2 \|f \nabla \tau\|_{W^{1,1}((U_{\mathcal{P}}))}^d + e_{\mathcal{P}} \|f \nabla \tau\|_{L^1((U_{\mathcal{P}}))}^d$$

$$\lesssim (h_M^2 + e_{\mathcal{P}}) \|f\|_{H^1(\Omega)} \|\nabla\|_{H^1(\Omega)} \lesssim (h_M^2 + e_{\mathcal{P}}) \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}. \quad (5.10)$$

To estimate $T_{2,3}$, let $(S_{\mathcal{H}}(\rho_{\mathcal{P}}))_{1 \leq i \leq r_{\mathcal{P}}}$ be the strips covering $\Omega^*_\mathcal{P}$. Since $|e_K| \leq h_M$, Lemma 8.8 yields

$$|T_{2,3}| \leq h_M \int_{\Omega_{\mathcal{P}}} \|f \nabla \tau\| \, dx \leq h_M \sum_{i=1}^{r_{\mathcal{P}}} \|f \nabla \tau\|_{L^1(S_{\mathcal{H}}(\rho_{\mathcal{P}}))}^d$$

$$\lesssim r_{\mathcal{P}} \rho_{\mathcal{P}} h_M \|f \nabla \tau\|_{W^{1,1}(\Omega)}^d \leq r_{\mathcal{P}} \rho_{\mathcal{P}} h_M \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}.$$

Plugging this estimate, (5.9) and (5.10) in (5.8) gives

$$T_2 \lesssim (h_M^2 + e_{\mathcal{P}} + r_{\mathcal{P}} \rho_{\mathcal{P}} h_M) \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}$$

which, combined with (5.7) and (5.6), leads to

$$\int_{\Omega \setminus \Omega_{\mathcal{P}}} g(x)(\Pi_{\mathcal{D}} u_{\mathcal{D}^*} - \Pi_{\mathcal{D}} u_{\mathcal{D}^*})(x) \, dx \lesssim (h_M^2 + e_{\mathcal{P}} + r_{\mathcal{P}} \rho_{\mathcal{P}} h_M) \|f\|_{H^1(\Omega)} \|g\|_{L^2(\Omega)}.$$

Take the supremum over $g \in L^2(\Omega)$ with norm 1 to find

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}^*} - \Pi_{\mathcal{D}} u_{\mathcal{D}^*}\|_{L^2(\Omega)} \lesssim (h_M^2 + e_{\mathcal{P}} + r_{\mathcal{P}} \rho_{\mathcal{P}} h_M) \|f\|_{H^1(\Omega)}. \quad (5.11)$$

**Step 2:** a weighted projection, and conclusion.

Let $(w_K)_{K \in \mathcal{M}}$ be the functions given by Lemma 8.7 (see Appendix). We define $\pi^w_{\mathcal{M}} : L^2(\Omega) \to L^2(\Omega)$ by

$$\forall \phi \in L^2(\Omega), \forall K \in \mathcal{M}, \ (\pi^w_{\mathcal{M}} \phi)|_K = \frac{1}{|K|} \int_K \phi(x)w_K(x) \, dx.$$

We have $\|\pi^w_{\mathcal{M}}\|_{L^2(\Omega) \to L^2(\Omega)} \leq \max_{K \in \mathcal{M}} \|w_K\|_{L^\infty(\Omega)} \lesssim 1$. Moreover, by definition of $w_K$ and of $\Pi_{\mathcal{D}^*}$, we have, on any $K \in \mathcal{M}$,

$$\pi^w_{\mathcal{M}}(\Pi_{\mathcal{D}^*} u_{\mathcal{D}^*}) = \frac{1}{|K|} \int_K (\Pi_{\mathcal{D}^*} u_{\mathcal{D}^*} + \nabla K u_{\mathcal{D}^*} \cdot (x - x_K))w_K(x) \, dx = \Pi_{\mathcal{D}} u_{\mathcal{D}^*}.$$

Hence, by (4.8),

$$\|\Pi_{\mathcal{D}^*} u_{\mathcal{D}^*} - \pi^w_{\mathcal{M}}\Pi_{\mathcal{D}^*}\|_{L^2(\Omega)} = \|\pi^w_{\mathcal{M}}(\Pi_{\mathcal{D}^*} u_{\mathcal{D}^*} - \Pi_{\mathcal{D}^*})\|_{L^2(\Omega)}$$

$$\lesssim \|\Pi_{\mathcal{D}^*} u_{\mathcal{D}^*} - \Pi_{\mathcal{D}^*}\|_{L^2(\Omega)} \lesssim h_M^2 \|f\|_{L^2(\Omega)}. \quad (5.12)$$

Using (8.17) in Lemma 8.7, we have $\|\pi_{\mathcal{P}} - \pi^w_{\mathcal{M}}\Pi_{\mathcal{D}^*}\|_{L^2(\Omega)} \lesssim h_M^2 \|\Pi_{\mathcal{D}^*}\|_{H^2(\Omega)}$. Combined with (5.12) and using the $H^2$ regularity property (4.5), this gives

$$\|\Pi_{\mathcal{D}^*} u_{\mathcal{D}^*} - \pi_{\mathcal{P}}\|_{L^2(\Omega)} \lesssim h_M^2 \|f\|_{L^2(\Omega)}. \quad (5.13)$$

The conclusion follows from this estimate and from (5.11).
We can now prove the super-convergence of TPFA schemes on triangular meshes.

**Proof of Theorem 1.2.** On a classical TPFA triangulation as in Definition 1.1, the TPFA scheme is an HMM scheme $\mathcal{D}$, with $\zeta_\mathcal{D}$ depending only on an upper bound of $\theta_\mathcal{T}$ [22, Section 5.3]. Theorem 5.3 therefore applies. Notice that $\theta_\mathcal{T}$ is bounded by a constant only depending on $\mathcal{T}_0$. To conclude the proof, we describe, for each type of TPFA triangulation, a patching $\Psi$ such that (i) $e_\Psi = 0$, (ii) $\mu_\Psi \leq C$ (see (5.1)), where $C$ depends only on an upper bound of $\theta_\mathcal{T}$, and (iii) $\Omega_\Psi$ is contained in $r$ strips of size $Mh_M$, where $r$ and $M$ only depend on $\mathcal{T}_0$. Theorem 1.2 then follows immediately from the second conclusion in Theorem 5.3. In the rest of this proof, we use the same notation $e_K = \pi_K - \mathbf{x}_K$ as in the proof of Theorem 5.3.

**Subdivision.** Most triangles of $\mathcal{T}$ can be patched in pairs forming rhombuses. Two of such pairs are illustrated in grey in Figure 1, left. In such a rhombus, each triangle $K_i$, $i = 1, 2$, is the symmetric of the other with respect to the center of the rhombus. As a consequence, $e_{K_i} = -e_{K_2}$ and the corresponding patching satisfies $e_\Psi = 0$. The region $\Omega_\Psi^c$ consists of one layer of triangles around each edge of the initial triangulation $\mathcal{T}_0$ – examples of such triangles are the dotted triangles in Figure 1 (left) – and is therefore contained in a fixed number of strips of width $h_M$.

**Reproduction by symmetry.** The cells are patched in four contiguous reproductions of $\mathcal{T}_0$, as shown in grey in Figure 1 (center). The symmetries ensure that, with such a patching, $e_\Psi = 0$. For an odd number of symmetries, $\Omega_\Psi^c = \emptyset$. For an even number of symmetries, $\Omega_\Psi^c$ is made of the reproductions of $\mathcal{T}_0$ along two edges of $\Omega$, and is thus contained in two strips of size $Mh_M$ with $M$ depending only on $\mathcal{T}_0$.

**Reproduction by translation.** Obtaining a conforming triangulation of $\Omega$ with a reproduction by translation of $\mathcal{T}_0$ imposes some symmetry properties on this initial triangulation. The vertices on the left side of $\mathcal{T}_0$ must match the vertices on the right side of $\mathcal{T}_0$, and similarly for the vertices on the top and bottom sides. Applying Lemma 8.9 in the appendix to the unit square $Q$ shows that, for such a triangulation, $\sum_{K \in \mathcal{T}_0} |K|e_K = 0$. Indeed, each left boundary edge of $\mathcal{T}_0$ is matched by a right boundary edge (same for top/bottom), and for these edges the quantities $|v_0 - \pi_Q|$ are identical whilst $n_{Q,\sigma}$ are opposite. Hence, the patching $\Psi$ made of the reproductions of the initial triangulation, as shown in grey in Figure 1 (right), satisfies $e_\Psi = 0$ and $\Omega_\Psi^c = \emptyset$.

**Remark 5.5.** The property $\sum_{K \in \mathcal{T}_0} |K|e_K = 0$, used in the proof above for meshes obtained by reproduction by translation, occurs with initial triangulations of other polygons than the unit square. For example, if a conforming tessellation of a region is created by translating an elementary triangulation $\mathcal{T}_0$ of an hexagon $Q$, the edges of $\mathcal{T}_0$ on the opposite boundaries of this hexagon must match and $\sum_{K \in \mathcal{T}_0} |K|e_K = 0$.

### 6. Numerical tests

In all the following tests, we consider (1.1) with $\Omega = (0,1)^2$, $A = \text{Id}$, $\pi(x,y) = 16x(1-x)y(1-y)$ and $f = -\Delta \pi$. We measure the following relative $L^2$ errors of the HMM or TPFA schemes on $\pi$ and its gradient:

\[
\text{err}_D(\pi) = \frac{\|\Pi_D u_D - \pi_P\|_{L^2(\Omega)}}{\|\pi_P\|_{L^2(\Omega)}} \quad \text{and} \quad \text{err}_D(\nabla \pi) = \frac{\|\nabla D u_D - (\nabla \pi)_P\|_{L^2(\Omega)}}{\|\nabla \pi\|_{L^2(\Omega)}}.
\]
where \((\nabla \pi)_P\) is the piecewise constant function equal, for all \(K \in \mathcal{M}\), to \(\nabla \pi(x_K)\) on \(K\). These errors are plotted against the mesh size \(h_M\). To test the super-convergence of the modified HMM method of Section 4.2, we use the following measure which, according to (5.13) (a direct consequence of the super-convergence result (4.8) of \(u_{D^*}\)), should decrease as \(h_M^2\), even in the absence of local compensation:

\[
\text{err}_{D^*}(\pi) = \frac{\|\Pi_D u_{D^*} - \Pi_P\|_{L^2(\Omega)}}{\|\Pi_P\|_{L^2(\Omega)}}.
\]

6.1. HMM method. The super-convergence for HMM schemes with \(P\) given by the centers of mass of the cells has already been numerically illustrated in a number of test cases, see e.g. [32]. We rather focus here on two cases where the points in \(P\) are shifted away from the centers of mass.

Test 1: Local compensation
We consider a cartesian grid in which, every other cell, \(x_K\) is shifted to the top-right or bottom-left of the centers of mass; see Figure 3, left. Grouping the cells by neighbouring pairs, as represented by the greyed area in this figure, gives a patching \(P\) such that \(e_P = 0\) and \(\Omega_P = \emptyset\). Theorem 5.3 therefore predicts the \(O(h_M^2)\) estimate on \(\text{err}_{D}(\nabla \pi)\) that is observed in Figure 3, right. The modified HMM method is also super-convergent and, quite naturally, beats the error of the HMM method (by a factor 2).

![Figure 3](image-url)

Figure 3. Test 1: position of the points \(x_K\) (left), and rates of convergence (right) for the HMM and modified HMM methods.

Test 2: Loss of super-convergence for HMM schemes
Still using a cartesian grid, the positions of \(x_K\) are inspired by the counter-example of [36] to super-convergence for TPFA in dimension 1. These positions are presented in Figure 4, left. The rates observed on the right of the figure show that the super-convergence of HMM is lost, which seems to indicate that Theorem 5.3 is relatively optimal, i.e. that even for very simple grids, HMM is not super-convergent if some local compensations do not occur. As expected, the modified HMM method remains super-convergent for this case.
6.2. TPFA finite volumes on triangles. We illustrate here the result of Theorem 1.2, considering three families of triangulations corresponding to the classical TPFA triangulations as in Definition 1.1. Many previous numerical tests (see, e.g., [14, 31]) have numerically demonstrated the super-convergence of TPFA on such meshes but, to our knowledge, no complete rigorous proof of this phenomenon has been provided so far. All numerical results show a clear order 2 rate of convergence, confirming Theorem 1.2.

7. Conclusion

The contributions of the paper can be summarised as follows. We first establish an improved $L^2$ estimate for gradient schemes, in any dimension $d$, which is more precise than the known ones of [21, 29]. This estimate yields better rates for a number of gradient schemes. Secondly, a modified HMM scheme with unconditional super-convergence (in dimension $d \leq 3$) is introduced. This modified scheme uses a piecewise linear, instead
of piecewise constant, approximation of the test functions. This approximation was introduced in [10], but only as a post-processing tool. By using this approximation in the design of the modified HMM method, we create a method that is superconvergent for any choice of the cell points as a consequence of the improved $L^2$ estimate for gradient schemes.

The next contribution is a new $L^2$ error estimate for HMM, that involves patches of cells. When these patches can be chosen so that a compensation occurs, within each patch, between the cell points and the centers of mass, this new $L^2$ estimate provides the super-convergence of HMM. The numerical results show that in the absence of patches, super-convergence may fail for HMM schemes, but holds true for the modified HMM scheme. Moreover the super-convergence is recovered if local compensation occurs.

Finally, perhaps the main contribution of this work, we prove the super-convergence of the TPFA finite volume scheme on the kinds of meshes used in 2D benchmarking of this method. This result is a consequence of all the previous ones. Numerical tests confirming this super-convergence are presented.

8. Appendix

8.1. Gradient schemes with approximate diffusion. Let $D$ be a gradient discretisation in the sense of Definition 2.1. As per (2.1), the corresponding gradient scheme for (1.1) is

$$\text{Find } u_D \in X_{D,0} \text{ such that, for all } v_D \in X_{D,0},$$

$$\int_{\Omega} A \nabla_D u_D \cdot \nabla_D v_D \, dx = \int_{\Omega} f \Pi_D v_D \, dx.$$  \hspace{1cm} (8.1)

For low-order methods, it is however customary to replace $A$ with a piecewise approximation on the mesh. More precisely, if $T$ is a polytopal mesh of $\Omega$, we denote by $A_M$ the $L^2$ projection of $A$ on the piecewise constant (matrix-valued)
functions on \( \mathcal{M} \), that is
\[
\forall K \in \mathcal{M}, \ A_M = \frac{1}{|K|} \int_K A(x) \, dx \text{ on } K, \tag{8.2}
\]
and we consider the modified gradient scheme
\[
\begin{align*}
\int_{\Omega} A_M \nabla_D \tilde{u}_D \cdot \nabla_D v_D \, dx &= \int_{\Omega} f \Pi_D v_D \, dx + \int_{\Omega} (A - A_M) \nabla_D \tilde{u}_D \cdot \nabla_D v_D \, dx. \tag{8.3}
\end{align*}
\]
The following two propositions show that, for low order methods (for which it is expected that WS\(_D(\mathcal{P}) = \mathcal{O}(h_M)\)), both the basic rate of convergence and the rate of super-convergence are not degraded by considering (8.3) instead of (8.1). In the following, we use the notation \( A \lesssim B \) as a shorthand for “\( A \leq CB \) for some \( C \) depending only on \( A \) and \( \Omega \)”.

**Proposition 8.1.** Let \( D \) be a gradient discretisation of \( \Omega \) and \( \mathcal{T} \) a polytopal mesh of \( \Omega \). Assume that (1.2) holds, \( A \) is Lipschitz-continuous on each \( K \in \mathcal{M} \), and \( A_M \) is defined by (8.2). If \( v_D \) and \( u_D \) are, respectively, the solutions to (8.1) and (8.3), then
\[
\| \Pi_D \tilde{u}_D - \Pi_D u_D \|_{L^2(\Omega)} + \| \nabla_D \tilde{u}_D - \nabla_D u_D \|_{L^2(\Omega)^d} \lesssim h_M \| f \|_{L^2(\Omega)}. \tag{8.4}
\]
As a consequence,
\[
\| \Pi_D \tilde{u}_D - \pi \|_{L^2(\Omega)} + \| \nabla_D \tilde{u}_D - \nabla \pi \|_{L^2(\Omega)^d} \lesssim WS_D(\pi) + h_M \| f \|_{L^2(\Omega)}. \tag{8.5}
\]
**Proof.** We first notice that, once it is known that \( A \) is Lipschitz-continuous on each \( K \in \mathcal{M} \), the maximum of the Lipschitz constants of \( (A|_K)_{K \in \mathcal{M}} \) is actually independent of \( \mathcal{M} \). This entails
\[
\| A - A_M \|_{L^\infty(\Omega)} \lesssim h_M. \tag{8.6}
\]
We have, for \( v_D \in X_{D,0} \),
\[
\int_{\Omega} A \nabla_D \tilde{u}_D \cdot \nabla_D v_D \, dx = \int_{\Omega} f \Pi_D v_D \, dx + \int_{\Omega} (A - A_M) \nabla_D \tilde{u}_D \cdot \nabla_D v_D \, dx.
\]
Subtracting the gradient scheme (8.1) and using (8.6), we infer
\[
\int_{\Omega} A (\nabla_D \tilde{u}_D - u_D) \cdot \nabla_D v_D \, dx = \int_{\Omega} (A - A_M) \nabla_D \tilde{u}_D \cdot \nabla_D v_D \, dx \lesssim h_M \| \nabla_D \tilde{u}_D \|_{L^2(\Omega)^d} \| \nabla_D v_D \|_{L^2(\Omega)^d}. \tag{8.7}
\]
It is clear that \( \tilde{u}_D \) still satisfies the stability property (2.6) (to verify this, take \( v_D = \tilde{u}_D \) in (8.3), use the definition (2.2) of \( C_D \), and the fact that \( A_M \) is uniformly coercive with the same coercivity constant as \( A \)). Hence, choosing \( v_D = \tilde{u}_D - u_D \) in (8.8),
\[
\| \nabla_D \tilde{u}_D - \nabla_D u_D \|_{L^2(\Omega)^d} \lesssim h_M \| f \|_{L^2(\Omega)}. \tag{8.8}
\]
The proof of (8.4) is complete by recalling the definition (2.2) of \( C_D \). The estimate (8.5) follows from a triangle inequality (introducing \( \Pi_D v_D \) and \( \nabla_D v_D \)), (8.4), and the estimate (2.7) in Theorem 2.2. \( \square \)

**Proposition 8.2.** Under the assumptions of Proposition 8.1, let us moreover suppose that the \( H^2 \) regularity property (4.5) holds. We also assume that, for all \( \phi \in H^1_0(\Omega) \cap H^2(\Omega) \),
\[
WS_D(\phi) \lesssim (\| A \phi \|_{L^2(\Omega)^d} + \| \phi \|_{H^2(\Omega)}) h_M. \tag{8.9}
\]
Let \( u_D \) and \( \tilde{u}_D \) be, respectively, the solutions to (8.1) and (8.3). Then,

\[
\|\Pi_D u_D - \Pi_D \tilde{u}_D\|_{L^2(\Omega)} \lesssim h_M^2 \|f\|_{L^2(\Omega)}.
\]  \tag{8.10}

**Proof.** Let \( g \in L^2(\Omega) \), \( \varphi_g \) be the solution to (1.1) with \( f \) instead of \( f \), and \( \varphi_{g, D} \) be the solution to (8.1) with \( g \) instead of \( f \). By (2.7), (8.9) and (4.5),

\[
\|\nabla_D \varphi_{g, D} - \nabla \varphi_g\|_{L^2(\Omega)^d} \lesssim h_M \|g\|_{L^2(\Omega)}.
\]

Using \( v_D = \varphi_{g, D} \) in (8.7) and recalling (8.6) therefore leads to

\[
\int g \Pi_D(\tilde{u}_D - u_D) \, dx
= \int (A - A_M) \nabla_D \tilde{u}_D \cdot \nabla_D \varphi_{g, D} \, dx
= \int (A - A_M) \nabla_D \tilde{u}_D \cdot (\nabla_D \varphi_{g, D} - \nabla \varphi_g) \, dx
+ \int (A - A_M) \nabla_D \tilde{u}_D \cdot \nabla \varphi_g \, dx
\lesssim h_M^2 \|\nabla_D \tilde{u}_D\|_{L^2(\Omega)^d} \|g\|_{L^2(\Omega)} + \int (A - A_M) (\nabla_D \tilde{u}_D - \nabla \tilde{u}_D) \cdot \nabla \varphi_g \, dx
+ \int (A - A_M) \nabla \tilde{u}_D \cdot \nabla \varphi_g \, dx
\lesssim h_M^2 \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} + \int (A - A_M) \nabla \tilde{u}_D \cdot \nabla \varphi_g \, dx. \tag{8.11}
\]

In the last line, we have used the standard stability property \( \|\nabla \varphi_g\|_{L^2(\Omega)^d} \lesssim \|g\|_{L^2(\Omega)} \), and Proposition 8.1 along with (8.9) and (4.5). We now estimate the last term in (8.11). By definition of \( A_M \),

\[
\int (A - A_M) \nabla \tilde{u}_D \cdot \nabla \varphi_g \, dx = \sum_{i,j=1}^d \int (A - A_M)_{i,j} \partial_i \tilde{u}_D \partial_j \varphi_g \, dx
= \sum_{i,j=1}^d \int (A - A_M)_{i,j} [\partial_j \tilde{u}_D \partial_i \varphi_g - \pi_M(\partial_j \tilde{u}_D \partial_i \varphi_g)] \, dx,
\]

where \( \pi_M \) denotes projection on piecewise constant functions on \( M \). By classical estimates (see e.g. [21, Lemma B.7]),

\[
\|\partial_j \tilde{u}_D \partial_i \varphi_g - \pi_M(\partial_j \tilde{u}_D \partial_i \varphi_g)\|_{L^1(\Omega)} \lesssim \|\partial_j \tilde{u}_D \partial_i \varphi_g\|_{W^{1,1}(\Omega)} h_M \lesssim \|\tilde{u}_D\|_{H^2(\Omega)} \|\varphi_g\|_{H^2(\Omega)} h_M.
\]

Hence, using (8.6) and the \( H^2 \) regularity property (4.5),

\[
\int (A - A_M) \nabla \tilde{u}_D \cdot \nabla \varphi_g \, dx \lesssim h_M^2 \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.
\]

Plugging this estimate into (8.11) and taking the supremum of the resulting inequality over \( g \in L^2(\Omega) \) of norm 1 concludes the proof of (8.10).

**Remark 8.3.** Similar results could be obtained for higher-order methods, by considering as \( A_M \) the \( L^2 \) projection on piecewise polynomial functions on \( M \).
8.2. Technical results. The following definition appears in [21, 24], in a slightly more general context (here, it is restricted to the Hilbertian case).

**Definition 8.4 (P₁-exact gradient reconstruction).** Let $U$ be a bounded subset of $\mathbb{R}^d$ with non-zero measure, and let $S = \{x_i\}_{i \in I} \subset \mathbb{R}^d$ be a finite family of points. A $P_1$-exact gradient reconstruction on $U$ upon $S$ is a family $\mathcal{G} = (G^i)_{i \in I}$ of functions in $L^2(U)^d$ such that, for any affine mapping $\ell : \mathbb{R}^d \to \mathbb{R}$ and a.e. $x \in U$,

$$
\sum_{i \in I} \ell(x_i) G^i(x) = \nabla \ell.
$$

The norm of $\mathcal{G}$ is defined by

$$
\|\mathcal{G}\| = \text{diam}(U) |U|^{-\frac{1}{2}} \left\| \sum_{i \in I} |G^i| \right\|_{L^2(U)}. \quad (8.12)
$$

For any family $\xi = (\xi_i)_{i \in I}$ of real numbers, define $\mathcal{G}\xi = \sum_{i \in I} \xi_i G^i \in L^2(U)^d$, and notice that

$$
\|\mathcal{G}\xi\|_{L^2(U)^d} \leq \text{diam}(U)^{-1} |U|^{\frac{1}{2}} \|\mathcal{G}\| \max_{i \in I} |\xi_i|.
$$

The following lemma is a specific case of [21, Lemma A.3]. The polynomial $L_\phi$ in this lemma is similar to an averaged Taylor polynomial as in [8].

**Lemma 8.5 (Approximation of $H^2$ functions by affine functions).** Let $d \leq 3$ and assume that $V \subset \mathbb{R}^d$ is bounded and star-shaped with respect to all points in a ball $B$. Choose $\theta \geq \text{diam}(V)/\text{diam}(B)$ and $\phi \in H^2(V) \cap C(V)$.

Then, there exists $C_1 > 0$, depending only on $d$ and $\theta$, and an affine function $L_\phi : V \to \mathbb{R}$ such that

$$
\sup_{x \in V} |\phi(x) - L_\phi(x)| \leq C_1 \text{diam}(V)^2 |V|^{-\frac{1}{2}} \|\phi\|_{H^2(V)} \quad (8.13)
$$

and

$$
\|\nabla L_\phi - \nabla \phi\|_{L^2(V)^d} \leq C_1 \text{diam}(V) \|\phi\|_{H^2(V)}. \quad (8.14)
$$

The next lemma estimates the difference between the averages of a function on two neighbouring sets.

**Lemma 8.6.** Let $U$, $V$ and $O$ be open sets of $\mathbb{R}^d$ such that, for all $(x, y) \in U \times V$, $[x, y] \subset O$. There exists $C_2$ only depending on $d$ such that, for all $\phi \in W^{1,1}(O)$,

$$
\left| \frac{1}{|U|} \int_U \phi(x) \, dx - \frac{1}{|V|} \int_V \phi(x) \, dx \right| \leq \frac{C_2 \text{diam}(O)^{d+1}}{|U| |V|} \int_O \|\nabla \phi(x)\| \, dx.
$$

**Proof.** Since $C^\infty(O) \cap W^{1,1}(O)$ is dense in $W^{1,1}(O)$, we can assume that $\phi \in C^\infty(O) \cap W^{1,1}(O)$. We then write, by Taylor’s expansion, $\phi(x) - \phi(y) = \int_0^1 \nabla \phi(tx + (1-t)y) \cdot (x-y) \, dt$ for $(x, y) \in U \times V$, and thus

$$
\left| \frac{1}{|U|} \int_U \phi(x) \, dx - \frac{1}{|V|} \int_V \phi(x) \, dx \right| \leq \frac{\text{diam}(O)}{|U| |V|} \int_U \int_V \int_0^1 |\nabla \phi(tx + (1-t)y)| \, dt \, dy \, dx. \quad (8.15)
$$
Let us fix $y \in V$ and apply the change of variable $x \in U \rightarrow z = tx + (1-t)y \in O$. This gives
\[\int_U \int_0^1 \left| \nabla \phi(tx + (1-t)y) \right| \, dt \, dx \, dy \leq \int_0^1 \left| \nabla \phi(z) \right| \int_I(z,y) t^{-d} \, dt \, dy \, dz \quad (8.16)\]
where $I(z,y) = \{ t \in [0,1] \mid \exists x \in U, \, tx + (1-t)y = z \}$. As in Step 1 of the proof of [18, Lemma 6.6], we see that
\[\int_V \int_I(z,y) t^{-d} \, dt \, dy \leq \frac{C_3}{d-1} \text{diam}(O)^d\]
where $C_3$ is the surface of the unit sphere in $\mathbb{R}^d$. The proof is complete by substituting this inequality into (8.16) and plugging the result in (8.15). \hfill \square

The existence of the functions $w_K$ mentioned in the following lemma has been first established in [22, Lemma A.1]. We provide here some additional estimates on these functions.

**Lemma 8.7.** Let $d \leq 3$ and $\mathcal{T}$ be a polytopal mesh of $\Omega$ in the sense of Definition 4.1. There exist affine functions $(w_K)_{K \in M}$, and $C_4$ depending only on $d$ and an upper bound of $\theta_T$, such that, for all $K \in \mathcal{M}$,
\[\int_K w_K(x) \, dx = |K|, \quad \int_K x w_K(x) \, dx = |K|x_K, \quad \|w_K\|_{L^\infty(K)} \leq C_4,\]
and, for all $\phi \in H^2(K)$,
\[\left| \phi(x_K) - \frac{1}{|K|} \int_K \phi(x) w_K(x) \, dx \right| \leq C_4 h_K^2 |K|^{-\frac{1}{2}} \|\phi\|_{H^2(K)}. \quad (8.17)\]

**Proof.** Consider the function given by $w_K(x) = 1 + \xi \cdot (x - \bar{x}_K)$, where $\xi$ is the vector such that $J_K \xi = |K|(x_K - \bar{x}_K)$, with $J_K$ the $d \times d$ matrix given by
\[J_K = \int_K (x - \bar{x}_K)(x - \bar{x}_K)^T \, dx.\]

Let us now establish the estimate on $w_K$. We refer to Figure 7 for an illustration of the reasoning. Up to a change of coordinate system, we can assume that $\bar{x}_K$ lies on the hyperplane $H_0 = \{ x : x_d = 0 \}$, and that $\xi$ is orthogonal to $H_0$ and points towards the direction $x_d > 0$. By definition of $\theta_T$, $K$ contains a cube $Q_K$ centered at $x_K$ and of length $C_5 h_K$, where $C_5$ only depends on $d$ and an upper bound of $\theta_T$.

Let $R_+$ be the upper and lower thirds of $Q_K$, that is $R_+ = \{ x \in Q_K : (x-x_K)_d > \frac{C_5 h_K}{6} \}$ and $R_- = \{ x \in Q_K : (x-x_K)_d < -\frac{C_5 h_K}{6} \}$.

Since $Q_K \setminus (R_+ \cup R_-)$ has width $\frac{C_5 h_K}{3}$, one of the regions $R_+$ or $R_-$ (let us assume $R_+$), must lie entirely outside the band of width $\frac{C_5 h_K}{4}$ around $x_d = 0$. If $x \in R_+$, we then have $| (x - \bar{x}_K) \cdot \xi | = \text{dist}(x, H_0) |\xi| \geq \frac{C_5 h_K}{3} |\xi|$. Hence,
\[|K|h_K|\xi| \geq |K|(x_K - \bar{x}_K) \cdot \xi = J_K \xi \cdot \xi = \int_K ((x - \bar{x}_K) \cdot \xi)^2 \, dx \]
\[\geq \int_{R_+} ((x - \bar{x}_K) \cdot \xi)^2 \, dx \geq |R_+| \frac{C_5^2 h_K^2}{9} |\xi|^2.\]
We have $|R_+| = \frac{c_5 h_K^d}{3} \geq C_6^d |K|$, where $C_6$ only depends on $d$ and an upper bound of $\theta_T$. Hence, $|\xi| \leq 9C_5^{-2d}C_6^{-1}h_K^{-1}$ and, for all $x \in K$,

$$|w_K(x)| \leq 1 + h_K |\xi| \leq 1 + 9C_5^{-2d}C_6^{-1}. \quad (8.18)$$

This concludes the proof of the estimate on $\|w_K\|_{L^\infty(K)}$.

To prove (8.17), we use Lemma 8.5 with $V = K$ (since $K$ is star-shaped and $d \leq 3$, we have $\phi \in C(K)$). A triangle inequality gives

$$\left| \phi(x) - \frac{1}{|K|} \int_K \phi(x) w_K(x) \, dx \right| \leq |\phi(x_K) - L\phi(x_K)| + |L\phi(x_K) - \frac{1}{|K|} \int_K L\phi(x) w_K(x) \, dx|$$

$$+ \frac{1}{|K|} \int_K |L\phi(x) - \phi(x)| w_K(x) \, dx. \quad (8.19)$$

We have $L\phi(x) = L\phi(x_K) + \nabla L\phi \cdot (x - x_K)$ and thus

$$\frac{1}{|K|} \int_K L\phi(x) w_K(x) \, dx = L\phi(x_K) + \frac{1}{|K|} \nabla L\phi \cdot \int_K (x - x_K) w_K(x) \, dx = L\phi(x_K).$$

Hence, (8.19) and the properties of $L\phi$ give

$$\left| \phi(x_K) - \frac{1}{|K|} \int_K \phi(x) w_K(x) \, dx \right| \leq C_7 h_K^2 |K|^{-\frac{3}{2}} \|\phi\|_{H^2(K)} (1 + \|w_K\|_{L^\infty(K)})$$

where $C_7$ only depends on $d$ and an upper bound of $\theta_T$. The estimate (8.17) is complete by using (8.18).

The following result was used in the proof of the super-convergence of HMM schemes (Theorem 5.3), to estimate a residual on the part of the domain not covered by the patches. It is a $W^{1,1}$ hyperplanar version of Ilin’s inequality in $H^1$ [35].

**Lemma 8.8.** Let $\Omega$ be an open set with a Lipschitz boundary, $H$ be an hyperplane, $\rho \in (0, \text{diam}(\Omega))$ and $S_H(\rho) = \{x \in \Omega : \text{dist}(x, H) \leq \rho\}$. Then there exists $C$ depending only on $\Omega$ such that, for all $\phi \in W^{1,1}(\Omega)$,

$$\|\phi\|_{L^1(S_H(\rho))} \leq C \rho \|\phi\|_{W^{1,1}(\Omega)}. \quad (8.20)$$
Lemma 8.9. Let \( Q \) be a polygonal subset of \( \mathbb{R}^2 \), with center of mass \( \mathbf{x}_Q \), and let \( T^Q = (\mathcal{M}^Q, \mathcal{E}^Q, \mathcal{P}^Q) \) be a conforming triangulation of \( Q \) into triangles. For each \( T \in \mathcal{M}^Q \) we denote by \( c_T \) the circumcenter of \( T \) and by \( \mathbf{x}_T \) the center of mass of \( T \). If \( \sigma \in \mathcal{E}^Q \) is an edge of the triangulation, we denote by \( v^1_\sigma \) and \( v^2_\sigma \) the two endpoints of \( \sigma \). Then,

\[
\sum_{T \in \mathcal{M}^Q} |T| (c_T - \mathbf{x}_T) = \sum_{\sigma \in \mathcal{E}^Q_{\text{ext}}} |\sigma| \frac{|v^1_\sigma - \mathbf{x}_Q|^2 + |v^2_\sigma - \mathbf{x}_Q|^2}{4} n_{Q,\sigma},
\]

where \( n_{Q,\sigma} \) is the outer normal to \( Q \) on \( \sigma \).

Proof. Let us first establish the following formula, for any triangle \( T \):

\[
|T| c_T = \sum_{\sigma \in \mathcal{E}_T} |\sigma| \frac{|v^1_\sigma|^2 + |v^2_\sigma|^2}{4} n_{T,\sigma}.
\]

We use the notations in Figure 8. Any vector \( \xi \in \mathbb{R}^2 \) can be written

\[
\xi = -\frac{1}{2|T|} \left( |\sigma_3| (\xi \cdot (a_3 - a_1)) n_{T,\sigma_3} + |\sigma_2| (\xi \cdot (a_2 - a_1)) n_{T,\sigma_2} \right)
\]

(take the dot product of each side this inequality with the two linearly independent vectors \( a_3 - a_1 \) and \( a_2 - a_1 \)). We apply this relation to \( \xi = c_T \) and use the characterisations

\[
\left( c_T - \frac{a_1 + a_3}{2} \right) \cdot (a_3 - a_1) = 0 \quad \text{and} \quad \left( c_T - \frac{a_1 + a_2}{2} \right) \cdot (a_2 - a_1) = 0
\]
of $c_T$ to obtain

$$-2|T|c_T = |\sigma_3|[c_T \cdot (a_3 - a_1)]n_{T,\sigma_3} + |\sigma_2|[c_T \cdot (a_2 - a_1)]n_{T,\sigma_2}$$

$$= |\sigma_3| \left( \left( \frac{a_1 + a_3}{2} \right) \cdot (a_3 - a_1) \right) n_{T,\sigma_3}$$

$$+ |\sigma_2| \left( \left( \frac{a_1 + a_2}{2} \right) \cdot (a_2 - a_1) \right) n_{T,\sigma_2}$$

$$= \frac{1}{2} \left( |\sigma_3| [a_3^2 - a_1^2] n_{T,\sigma_3} + |\sigma_2| [a_2^2 - a_1^2] n_{T,\sigma_2} \right).$$

Since $|\sigma_1|n_{T,\sigma_1} + |\sigma_2|n_{T,\sigma_2} + |\sigma_3|n_{T,\sigma_3} = 0$, we infer

$$4|T|c_T = |a_1|^2 |\sigma_2|n_{T,\sigma_2} + |\sigma_3|n_{T,\sigma_3} + |a_2|^2 |\sigma_1|n_{T,\sigma_1} + |\sigma_3|n_{T,\sigma_3}$$

$$+ |a_3|^2 |\sigma_1|n_{T,\sigma_1} + |\sigma_2|n_{T,\sigma_2}.$$ 

Gathering this sum by edges contributions $|\sigma_i|n_{T,\sigma_i}$ concludes the proof of (8.22).

The proof of (8.21) is now trivial. We have $|Q|z_Q = \sum_{T \in \mathcal{M}^Q} |T|z_T$ and, without loss of generality, we can assume that this quantity is equal to 0 (we translate $Q$ so that its center of mass is 0). By summing (8.22) over $T \in \mathcal{M}^Q$ and gathering the right-hand side by edges, we find

$$\sum_{T \in \mathcal{M}^Q} |T|c_T = \sum_{\sigma \in \mathcal{E}^{in}} |\sigma| \left( \frac{|v_1^T|^2 + |v_2^T|^2}{4} (n_{T,\sigma} + n_{T',\sigma}) \right) + \sum_{\sigma \in \mathcal{E}^{ext}} |\sigma| \left( \frac{|v_1^Q|^2 + |v_2^Q|^2}{4} n_{Q,\sigma} \right).$$

In the first sum, $T$ and $T'$ are the triangles on each side of $\sigma$, and thus $n_{T,\sigma} + n_{T',\sigma} = 0$. The proof of (8.21) is complete. 

8.3. Implementation and fluxes of the HMM and modified HMM methods. We give here some elements for implementing the HMM method (the gradient scheme (2.1) based on the gradient discretisation in Definition 4.2), and the modified HMM method of Section 4.2, which also leads us to discuss their interpretation as finite volume methods for appropriate choices of fluxes.
8.3.1. HMM method. The following is drawn from [22, 28], and solely recalled for ease of reference. The fluxes \( (F_{K,\sigma}(u))_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} \) of the HMM method are defined, for \( u \in X_{D,0} \), by

\[
\forall K \in \mathcal{M}, \forall v = (v_K, (v_\sigma)_{\sigma \in \mathcal{E}_K}), \quad \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u)(v_K - v_\sigma) = \int_K A(x) \nabla_D u(x) \nabla_D v(x) \, dx \tag{8.23}
\]

(note that, on \( K \), \( \nabla_D v \) only depends on \( (v_K, (v_\sigma)_{\sigma \in \mathcal{E}_K}) \)). Then, \( u \in X_{D,0} \) is a solution to the HMM scheme if and only if the following balance and conservativity of fluxes are satisfied:

\[
\forall K \in \mathcal{M}, \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u) = \int_K f(x) \, dx, \tag{8.24}
\]

\[
\forall \sigma \in \mathcal{E}_{\text{int}}, \text{ if } \mathcal{M}_\sigma = \{K, L\} \text{ then } F_{K,\sigma}(u) + F_{L,\sigma}(u) = 0. \tag{8.25}
\]

These equations are respectively obtained by taking, in (2.1), a test function \( v \) that is equal to 1 on \( K \) and zero at all other degrees of freedom, and a test function \( v \) that is equal to 1 on \( \sigma \) and zero at all other degrees of freedom. The HMM method is best implemented through (8.24)–(8.25), once a practical local formula for the fluxes is obtained.

**Remark 8.10** (Formula for the fluxes). **Thanks to** (4.10), assuming that \( A \) is constant equal to \( A_K \) on \( K \), we have

\[
\int_K A(x) \nabla_D u(x) \nabla_D v(x) \, dx = |K| A_K \nabla_K u \cdot \nabla_K v + R_K(v)^T \mathbb{B}_K R_K(u), \tag{8.26}
\]

where \( \mathbb{B}_K \) is a \( \text{Card}(\mathcal{E}_K) \times \text{Card}(\mathcal{E}_K) \) symmetric positive definite matrix related to \( \mathcal{L}_K \). If \( \mathcal{L}_K = \alpha_K \text{Id} \) (usual choice), then \( \mathbb{B}_K = \alpha_K^2 \text{Id} \text{diag}(\frac{\partial f}{\partial x_\sigma} A_K \mathbf{n}_K, \cdot \mathbf{n}_K, \cdot) \). To implement the HMM method in practice, one chooses \( \mathbb{B}_K \). \( \mathcal{L}_K \) is only a tool for the analysis of the method.

Owing to (8.23) and (8.26), the fluxes can be written \( (F_{K,\sigma}(u))_{\sigma \in \mathcal{E}_K} = \mathbb{W}_K(u_K - u_\sigma)_{\sigma \in \mathcal{E}_K} \). Here, \( \mathbb{W}_K \) is the square matrix of size \( \text{Card}(\mathcal{E}_K) \) defined by

\[
\mathbb{W}_K = |K| \mathcal{G}_K^T A_K \mathcal{G}_K + \mathcal{R}_K^T \mathbb{B}_K \mathcal{R}_K,
\]

where \( \mathcal{G}_K \) is the \( d \times \text{Card}(\mathcal{E}_K) \) matrix with columns \( \frac{\partial f}{\partial x_\sigma} \mathbf{n}_K, \cdot \), and \( \mathcal{R}_K = \mathbb{I}_K - \mathbb{X}_K \mathcal{G}_K \) with \( \mathbb{I}_K \) the \( \text{Card}(\mathcal{E}_K) \times \text{Card}(\mathcal{E}_K) \) identity matrix and \( \mathbb{X}_K \) the matrix with rows \( (\mathbb{X}_\sigma - x_K)^T \)_{\sigma \in \mathcal{E}_K}.

8.3.2. Modified HMM method. Given that the modification is only on \( \Pi_D \) (cf. (4.7)), as previously mentioned the matrix of the modified HMM method is identical to the matrix of the HMM method. The fluxes of the modified HMM method are therefore still defined by (8.23). If \( v \) is equal to 1 at the degree of freedom corresponding to \( K \) and to zero at all other degrees of freedom, then \( \Pi_D v = 1 = \Pi_D v \). Hence, the rows of the source-term corresponding to cell degrees of freedom are also unchanged with respect to the HMM method. This means that the balance of fluxes (8.24) remains.

The only changes, from the HMM to the modified HMM scheme, in the source-term are in the rows corresponding to interior edge unknowns. Taking \( v \) equal to 1 on \( \sigma \) (such that \( \mathcal{M}_\sigma = \{K, L\} \)) and zero at all other degrees of freedom, we have
\[ \nabla_{K} v = \frac{|\sigma|}{|K|} n_{K,\sigma} \] (and similarly for \( \nabla_{L} v \)) and therefore the conservativity equation (8.25) is modified into
\[ F_{K,\sigma}(u) + F_{L,\sigma}(u) = \frac{|\sigma|}{|K|} \int_{K} f(x) n_{K,\sigma} \cdot (x - x_{K}) \, dx + \frac{|\sigma|}{|L|} \int_{L} f(x) n_{L,\sigma} \cdot (x - x_{L}) \, dx. \] (8.27)

The fluxes of the modified HMM method are therefore no longer conservative, and the modified HMM method is not a finite volume scheme.

**Remark 8.11** (Preserving the conservativity). Two options exist to preserve the conservativity of the modified HMM method. The first one is to re-define the fluxes by setting, for all \( K \in \mathcal{M} \) and all \( \sigma \in \mathcal{E}_{K} \),
\[ F_{K,\sigma}^{*}(u) = F_{K,\sigma}(u) - \frac{|\sigma|}{|K|} \int_{K} f(x) n_{K,\sigma} \cdot (x - x_{K}) \, dx. \]
Then, from (8.27) we deduce that \( F_{K,\sigma}^{*}(u) + F_{L,\sigma}^{*}(u) = 0 \) for all interior edge \( \sigma \). Moreover, since \( \sum_{\sigma \in \mathcal{E}_{K}} |\sigma| n_{K,\sigma} = 0 \), we have \( \sum_{\sigma \in \mathcal{E}_{K}} F_{K,\sigma}^{*}(u) = \sum_{\sigma \in \mathcal{E}_{K}} F_{K,\sigma}(u) = \int_{K} f(x) \, dx \), i.e. the new fluxes still satisfy the balance equation.

Another option is to modify the reconstruction \( \Pi_{D} \) by taking, instead of \( \nabla_{K} \), a linearly exact gradient reconstruction based on the cell degrees of freedom, and not using any edge degrees of freedom. The corresponding new modified HMM scheme is then naturally conservative, but the source term in the balance equation is modified (it involves \( f \) in neighbouring cells).

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