REnormalization Group Flow
In a General Gauge Theory

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ABSTRACT

The renormalization group flow in a general renormalizable gauge theory with a simple
gauge group in 3+1 dimensions is analyzed. The flow of the ratios of the Yukawa couplings
and the gauge coupling is described in terms of a bounded potential, which makes it
possible to draw a number of non-trivial conclusions concerning the asymptotic structure
of the theory. A classification of possible flow patterns is given.

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1 Beta functions of a general gauge theory

When mass effects can be neglected, the behaviour of a quantum field theory under scale
transformations can be described conveniently in terms of “running couplings” defined
via beta functions. Whereas it is quite straightforward to calculate the beta functions, it
is usually not so simple to deduce from them the structure of the renormalization group
flow, especially when there is a large number of coupling constants. In the present work we
will show that even for a very general theory it is possible to make a number of nontrivial
statements concerning its asymptotic behaviour.

We consider a gauge theory with a simple gauge group $G$ and with Weyl fermions $\psi_i$ and
real scalars $\phi_\alpha$ transforming under some (generically reducible) representations $R_F$
and $R_B$ of $G$. The Lagrangian of this model is

$$\begin{align*}
-\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + i \bar{\psi}_i \tilde{\sigma}^\mu D_\mu \psi_i + \frac{1}{2} D_\mu \phi_\alpha D^\mu \phi_\alpha - \frac{1}{2} \bar{\psi}_i \psi_j Y^{\alpha}_{ij} \phi_\alpha - \frac{1}{2} \bar{\psi}_i \psi_j Y^{\alpha*}_{ij} \phi_\alpha - \frac{1}{24} V^{\alpha\beta\gamma\delta} \phi_\alpha \phi_\beta \phi_\gamma \phi_\delta,
\end{align*}$$

(1)

where we have omitted gauge fixing, ghost, $\phi^3$ and mass terms. Gauge invariance forces
the Yukawa couplings $Y^{\alpha}_{ij}$ and the quartic scalar couplings $V^{\alpha\beta\gamma\delta}$ to fulfill the relations

$$\begin{align*}
Y^{\alpha}_{kj}(T_F)^a_{ki} + Y^{\alpha}_{ik}(T_F)^a_{kj} + Y^{\alpha}_{ij}(T_B)^a_{\alpha} = 0
\end{align*}$$

(2)

and

$$\begin{align*}
V^{\varepsilon\gamma\delta}(T_B)^a_{\varepsilon\alpha} + V^{\alpha\varepsilon\delta}(T_B)^a_{\alpha\gamma} + V^{\alpha\varepsilon\gamma}(T_B)^a_{\alpha\delta} = 0,
\end{align*}$$

(3)

respectively. $T_F$ and $T_B$ are the generators of $R_F$ and $R_B$. These equations could be
solved by decomposing the group representations according to

$$\begin{align*}
\psi_i \rightarrow \psi^I_i, \quad \phi_\alpha \rightarrow \phi^A_\alpha,
\end{align*}$$

where $I, A$ run through the sets of irreducible representations spanned by $i = i(I)$ and $\tilde{\alpha} = \tilde{\alpha}(A)$. Then the couplings can be written as

$$\begin{align*}
Y^{\alpha}_{ij} \rightarrow Y^{\alpha I}_{A ij} = \sum_k (Z^{(k)})^A_{IJ} (\Lambda^{(k)})_{\tilde{\alpha}}^{\tilde{\beta}}
\end{align*}$$

(4)

where the $\Lambda^{(k)}$ are tensors satisfying an analogue of Eq. (2) whereas the $(Z^{(k)})^A_{IJ}$ are the usual unrestricted coupling constants. In the same way also the $V^{\alpha\beta\gamma\delta}$ can be decomposed. It will turn out, however, that for a general analysis it is more convenient not to make use of this decomposition.

The one– and two–loop contributions to the beta functions of this model have been
calculated in the $R_\xi$ gauge in dimensional regularization with the $\overline{MS}$ scheme [4]. The
gauge beta function is given by

$$\begin{align*}
\beta(g) = \mu \frac{dg}{d\mu} = -\frac{g^3}{6(4\pi)^2} (22c_g - 4S_F - S_B) - \frac{g^3}{(4\pi)^4 d_g^2} \text{Tr}(C_F Y^{\beta\dagger} Y^{\beta})
+ \frac{g^5}{3(4\pi)^4} [6Q_F + 6Q_B + c_g (10S_F + S_B - 34c_g)] + O(g^7).
\end{align*}$$

(5)
\(C = T^a T^a\) is the quadratic Casimir Operator, \(d_g\) the dimension of \(G\), \(S = \frac{1}{d_g} \text{Tr} C\) the Dynkin index, and \(Q = \frac{1}{d_g} \text{Tr} C^2\). The counting of orders in \(g\) is such that \(Y = O(g)\), \(V = O(g^2)\). The other beta functions are

\[
\beta(Y^\alpha) = \frac{dY^\alpha}{d\mu} = \frac{1}{2(4\pi)^2} [4Y^\beta Y^{\alpha\dagger} Y^\beta + Y^{\alpha\dagger} Y^\beta +Y^\beta Y^{\alpha\dagger} Y^\alpha +Y^\beta \text{Tr}(Y^{\alpha\dagger} Y^\beta +Y^{\alpha\dagger} Y^\alpha) -6g^2(Y^\alpha C_F +C_F Y^\alpha)] +O(g^5) \tag{6}
\]

(\text{using matrix notation for the fermionic indices}) and

\[
(4\pi)^2 \beta(V^{\alpha\beta\gamma\delta}) = V^{\alpha\beta\lambda\epsilon} V^{\gamma\delta\lambda\epsilon} +V^{\alpha\gamma\lambda\epsilon} V^{\beta\delta\lambda\epsilon} +V^{\alpha\delta\lambda\epsilon} V^{\beta\gamma\lambda\epsilon}
\]

\[
+\frac{1}{2} \text{Tr}(Y^{\alpha\dagger} Y^\lambda +Y^{\lambda\dagger} Y^\alpha -3g^2 C_B^{\alpha\lambda})V^{\lambda\beta\gamma\delta}
\]

\[
+\frac{1}{2} \text{Tr}(Y^{\beta\dagger} Y^\lambda +Y^{\lambda\dagger} Y^\beta -3g^2 C_B^{\beta\lambda})V^{\alpha\lambda\gamma\delta}
\]

\[
+\frac{1}{2} \text{Tr}(Y^{\gamma\dagger} Y^\lambda +Y^{\lambda\dagger} Y^\gamma -3g^2 C_B^{\gamma\lambda})V^{\alpha\beta\delta\lambda}
\]

\[
+\frac{1}{2} \text{Tr}(Y^{\delta\dagger} Y^\lambda +Y^{\lambda\dagger} Y^\delta -3g^2 C_B^{\delta\lambda})V^{\alpha\beta\gamma\lambda}
\]

\[
+3g^4 \{(T_B^a, T_B^b)^{\alpha\beta\gamma\delta} +\{T_B^a, T_B^b\}^{\alpha\gamma} \{T_B^a, T_B^b\}^{\beta\delta} +\{T_B^a, T_B^b\}^{\alpha\delta} \{T_B^a, T_B^b\}^{\beta\gamma}\}
\]

\[
-2 \text{Tr}[(Y^{\alpha\dagger} Y^\beta +Y^{\beta\dagger} Y^\alpha)(Y^{\gamma\dagger} Y^\delta +Y^{\delta\dagger} Y^\gamma) + (Y^{\alpha\dagger} Y^\gamma +Y^{\gamma\dagger} Y^\alpha)(Y^{\beta\dagger} Y^\delta +Y^{\delta\dagger} Y^\beta) + (Y^{\alpha\dagger} Y^\delta +Y^{\delta\dagger} Y^\alpha)(Y^{\beta\dagger} Y^\gamma +Y^{\gamma\dagger} Y^\beta)] +O(g^6). \tag{7}
\]

In order to solve the system of ordinary differential equations (\text{3}) – (\text{6}) it is convenient to change variables to \(t = \ln(\mu/\mu_0), G = (g/4\pi)^2, y = Y/g\) and \(v = V/g^2\). Then the evolution of \(G\) is governed by

\[
\frac{dG}{dt} = -\lambda G^2 -2C^2 \left(34c_g^2 -10c_g S_F -c_g S_B -6Q_F -6Q_B + \frac{3}{d_g} \text{Tr}(C_F y^{\beta\dagger} y^\beta)\right) + O(G^4). \tag{8}
\]

where

\[
\lambda = (22c_g -4S_F -S_B)/3. \tag{9}
\]

To lowest order the solution is given by

\[
G^{-1}(t) = G_0^{-1} +\lambda t +O(G), \tag{10}
\]

where \(G_0 = G(t = 0)\) (similar notation will be used again). \(y\) and \(v\) evolve according to

\[
\frac{dy^a}{dt} = \frac{G}{2}(4y^\beta y^{\alpha\dagger} y^\beta +y^\alpha y^{\beta\dagger} y^\beta +y^\beta y^{\alpha\dagger} y^\alpha +y^\alpha y^{\beta\dagger} y^\alpha) \text{Tr}(y^{\alpha\dagger} y^\beta +y^{\beta\dagger} y^\alpha) -y^a D\text{-}Dy^a) +O(G^2), \tag{11}
\]

where

\[
D_{ij} = 6(C_F)_{ij} -\lambda \delta_{ij}/2, \tag{12}
\]
and

\[
G^{-1} \frac{dv^{\alpha\beta\gamma\delta}}{dt} = v^{\alpha\beta\lambda\epsilon} v^{\gamma\delta\lambda\epsilon} + v^{\alpha\gamma\lambda\epsilon} v^{\beta\delta\lambda\epsilon} + v^{\alpha\delta\lambda\epsilon} v^{\beta\gamma\lambda\epsilon}
\]

\[
+ \frac{1}{2} \text{Tr}(y^{\alpha\dagger} y^{\lambda\dagger} + y^{\lambda\dagger} y^{\alpha}) - 3 C^{\alpha\lambda}_B + \frac{\lambda}{4} \delta^{\alpha\lambda} v^{\beta\gamma\delta} \\
+ \frac{1}{2} \text{Tr}(y^{\beta\dagger} y^{\lambda} + y^{\lambda\dagger} y^{\beta}) - 3 C^{\beta\lambda}_B + \frac{\lambda}{4} \delta^{\beta\lambda} v^{\alpha\gamma\delta} \\
+ \frac{1}{2} \text{Tr}(y^{\gamma\dagger} y^{\lambda} + y^{\lambda\dagger} y^{\gamma}) - 3 C^{\gamma\lambda}_B + \frac{\lambda}{4} \delta^{\gamma\lambda} v^{\alpha\beta\delta} \\
+ \frac{1}{2} \text{Tr}(y^{\delta\dagger} y^{\lambda} + y^{\lambda\dagger} y^{\delta}) - 3 C^{\delta\lambda}_B + \frac{\lambda}{4} \delta^{\delta\lambda} v^{\alpha\beta\gamma} \\
+ 3 \{T^a_B, T^b_B\}^{\alpha\beta} \{T^a_B, T^b_B\}^{\gamma\delta} + \{T^a_B, T^b_B\}^{\alpha\gamma} \{T^a_B, T^b_B\}^{\beta\delta} + \{T^a_B, T^b_B\}^{\alpha\delta} \{T^a_B, T^b_B\}^{\beta\gamma} \\
- 2 \text{Tr}[(y^{\alpha\dagger} y^{\beta\dagger} + y^{\beta\dagger} y^{\alpha})(y^{\gamma\dagger} y^{\delta\dagger} + y^{\delta\dagger} y^{\gamma}) \\
+ (y^{\alpha\dagger} y^{\gamma\dagger} + y^{\gamma\dagger} y^{\alpha})(y^{\beta\dagger} y^{\delta\dagger} + y^{\delta\dagger} y^{\beta}) \\
+ (y^{\alpha\dagger} y^{\delta\dagger} + y^{\delta\dagger} y^{\alpha})(y^{\beta\dagger} y^{\gamma\dagger} + y^{\gamma\dagger} y^{\beta})] + O(G),
\]

(13)

respectively. The forms of these equations suggest to define a new evolution parameter \( \tau \) by

\[
d\tau = \frac{G}{2} dt \quad \text{and} \quad \tau_0 = 0. \tag{14}
\]

If \( \lambda \neq 0 \), then

\[
\tau = -\frac{1}{2\lambda} \ln(G/G_0) + O(G \ln G), \tag{15}
\]

whereas for \( \lambda = 0 \)

\[
\tau = \frac{G t}{2} + O(G^3). \tag{16}
\]

# 2 Potentials

In the center of our further discussion of the flow of \( y \) there will be a “potential” \( U \) defined by

\[
U = \frac{1}{3} \sum_{ijkl} |S_{ijkl}|^2 + \sum_{\alpha\beta} (I^{\alpha\beta})^2 + \sum_{ij} |M_{ij}|^2 \tag{17}
\]

with

\[
S_{ijkl} = y_{ij}^{\alpha\dagger} y_{kl}^\alpha + y_{ik}^{\alpha\dagger} y_{lj}^\alpha + y_{il}^{\alpha\dagger} y_{jk}^\alpha, \tag{18}
\]

\[
I^{\alpha\beta} = \text{Im} \text{Tr} y^{\alpha\dagger\beta}, \tag{19}
\]

and \( M = y^{\alpha\dagger} y^\alpha - D \).

(20)

It can easily be checked that, at lowest order in \( G \),

\[
\frac{2}{G} \frac{dy_{ij}^\alpha}{dt} = \frac{dy_{ij}^\alpha}{d\tau} = \frac{\partial U}{\partial y_{ij}^{\alpha\dagger}}, \tag{21}
\]

(21)
implying
\[
\frac{dU}{d\tau} = \frac{\partial U}{\partial y^{ij}_a} \frac{dy^{ij}_a}{d\tau} + \frac{\partial U}{\partial y^{ij}_b} \frac{dy^{ij}_b}{d\tau} = 2|\frac{\partial U}{\partial y^{ij}_a}|^2 \geq 0 \quad (22)
\]

Thus \(U\) increases or decreases with increasing or decreasing \(\tau\), along paths of steepest ascent or descent, respectively. \(U\) plays a role similar to that of \(c\) in Zamolodchikov’s \(c\)-theorem [2]. In fact, potentials for the renormalization group flow have been considered already 20 years ago [3]. In general one has to define a metric with respect to which a gradient flow is defined. In our particular case it turns out to be just the Euclidean metric on \(y\)-space. The sign of \(U\) is chosen in such a way that \(\mu, t, \tau, U\) are monotonically increasing functions of one another.

From the explicit form of Eq. (17) it is obvious that \(U\) is non-negative, and it is not hard to show that the entries of \(M\) become large when one of the \(y^{ij}_a\) becomes large. Therefore \(U\) has a global minimum, i.e. an infrared fixed point, at some finite value of \(y\). If \(R_F\) or \(R_B\) contains isomorphic irreducible representations more than once, Eqs. (11) and (13) are invariant under some global symmetries relating these irreducible representations and the set of fixed points generically will not be a unique point, but rather the orbit of such a symmetry. In fact it is also possible that the space of minima of \(U\) has a degeneracy that is not related to a global symmetry. An example is given by the model of Ref. [4], where a continuous set of solutions of \(U = 0\) was found. Since the eigenvalues of \(\text{Tr}\ y^{\alpha_i} y^{\beta_j}\) would be invariant under global symmetries, solutions with different eigenvalues that were found cannot be related by symmetries. It would be interesting to find out whether local minima at different values of \(U\) are possible and whether the space of global minima is connected.

It will also be important how fast the minimum is approached in the limit \(\tau \to \infty\). If the Hessian \(H\) of \(U\) at \(y_{FP}\) has maximal rank (i.e. if \(y_{FP}\) is a non-degenerate critical point), all components of \(\Delta y = y - y_{FP}\) will tend to zero exponentially in \(\tau\) because of
\[
\frac{d\Delta y}{d\tau} = H\Delta y + O((\Delta y)^2). \quad (23)
\]
As we have seen, the minimum often has flat directions, implying that the Hessian cannot have maximal rank. If the minimum locus is locally a submanifold of \(y\)-space (this is not fulfilled, for example, for the origin for \(U = y_1^2 y_2^2\)), and if the rank of the Hessian is the maximal rank \(N\) (the dimension of \(y\)-space) minus the number \(N_{\text{flat}}\) of flat directions, one can locally define coordinates by the following procedure: Define \(N_{\text{flat}}\) coordinates parametrizing the minimum locus and label each point in some suitably chosen open set by the \(N_{\text{flat}}\) coordinates of the fixed point to which it flows and by \(N - N_{\text{flat}}\) linear combinations of the original coordinates on which the Hessian is positive definite. Under the flow, the first \(N_{\text{flat}}\) coordinates do not change, whereas the other \(N - N_{\text{flat}}\) coordinates approach the fixed point exponentially in \(\tau\). Thus \(||\Delta y||\) is again exponentially bounded. If \(N_{\text{flat}} + \text{rank}(H) < N\), we cannot apply these arguments. For example, if \(d\Delta y/d\tau \sim \text{const.} \times (\Delta y)^3\), then \(\Delta y \sim \text{const.} \times (\text{const.} - \tau)^{-1/2}\).

We have written \(U\) in terms of three highly symmetric structures which were discovered
in Ref. 3 (see also 4) in the context of a search for finite theories. A case of special interest occurs when each of these structures vanishes separately. In particular this is the case for one loop finite supersymmetric theories 3, 7. For any supersymmetric theory $S_{ijkl}$ will vanish: The Yukawa couplings come in two types, namely interactions involving gauginos, which are proportional to generators of the gauge group, and interactions within the chiral sector, which are determined by totally symmetric constants $d_{ijk}$. Within one type of couplings contributions corresponding to the real and imaginary parts of the scalars cancel, whereas the mixed terms vanish because of the invariance condition on $d_{ijk}$ (their contribution to $S_{ijkl}$ is just the analogue of the l.h.s. of Eq. (2)). The conditions $I = 0$ and $M = 0$ are not fulfilled automatically and lead to the well-known one-loop finiteness conditions for supersymmetric theories 3. An example for a non-supersymmetric theory with simultaneous vanishing of $S_{ijkl}$, $I_{\alpha\beta}$ and $M_{ij}$ is given in Ref. 4. There are many models, however, where it is not possible to solve $U = 0$ 9.

It is easy to see that $U$ has at most one local maximum: Consider the one parameter family of couplings $y_{ij}^\alpha(z) = z(y_0)^{ij}_\alpha$ with a fixed set of values $(y_0)^{ij}_\alpha$. Then $U(y_{ij}^\alpha(z))$ is of the form $a|z|^4 + b|z|^2 + c$ with $a > 0$, which has at most one local maximum, namely at $z = 0$. Therefore any maximum of $U$ can only be located at $y_{ij}^\alpha = 0$ for all $\alpha, i, j$. Clearly $y = 0$ is always a critical point. The Hessian there is diagonal (in the $d_B \times d_B^2$-dimensional $y$-space) with diagonal elements corresponding to $y_{ij}^\alpha$ given by $-(D_{ii} + D_{jj})$. This point really represents a maximum if and only if the sub-matrix of $D$ corresponding to Yukawa–interacting fermions is positive definite. If this is the case, $y = 0$ is approached according to

$$y_{ij}^\alpha \sim \text{const.} \times e^{-(D_{ii}+D_{jj})\tau}$$ (24)

(no summation over repeated indices).

In addition to this maximum and the set of minima, $U$ might (and usually will) also have saddle points. Particular examples are points where the Yukawa couplings for some of the particles vanish while the Yukawa couplings for the other particles correspond to some minimum of the restricted $U$. Although saddle points are unstable fixed points of the flow both in the infrared and in the ultraviolet limit, getting close to such a point (and thereby getting small beta functions) can considerably decelerate the flow, such that a realistic theory will often be close to a saddle point along large portions of the flow. A theory flowing exactly into a saddle point would require exactly fine-tuned initial conditions. In the absence of reasons for such a fine-tuning, this will occur with zero probability. A possible reason might be a global symmetry which could impose constraints on the couplings. In such a case one should consider the potential $U$ only over the subspace of $y$-space defined by these constraints. Then the couplings run to extrema of the constrained potential. A typical example is supersymmetry which prevents $y$ from running to 0 in the ultraviolet limit.

Let us now discuss the behavior of $v$: We consider Eq. (13) for fixed $y = y_{FP}$, neglecting higher orders in $G$. It is possible to integrate to get a potential again:

$$U_v = v^{\alpha\beta\gamma\delta}v^{\gamma\delta\epsilon\lambda}v^{\epsilon\lambda\alpha\beta} + 2\frac{1}{2} \text{Tr}(y^{\alpha y^\lambda} + y^{\lambda y^\alpha}) - 3C_{\alpha\lambda}^\alpha + \frac{\lambda}{4} \delta^{\alpha\lambda} v^{\alpha\beta\gamma\delta} v^{\lambda\beta\gamma\delta}$$
Thus perturbation theory breaks down. If the starting point is in the region of attraction of the local maximum of \( U \), all \( y_{ij}^{\alpha} \) will go to zero. Near the fixed point,

\[
y_{ij}^{\alpha} \sim const. \times e^{-\tau(D_{ij} + D_{ij})} \sim const. \times G^{(D_{ij} + D_{ij})/(2\lambda)}
\]  

(28)

In contrast to the potential for the Yukawa couplings, \( U_v \) is unbounded, because it is cubic. We can show, however, that for any given set of \( y \)'s it has at most a single local minimum: Assume there are two minima \( v_1, v_2 \) and consider the line \( v(s) = v_1 + s\Delta v \) with \( \Delta v = v_2 - v_1 \). Then \( s = 0 \) and \( s = 1 \) are minima of the function \( U_v(v(s)) \) which is (at most) cubic in \( s \). Therefore \( U_v \) must be constant along \( v(s) \). Now consider \( v(s) \) near \( s = 0 \). If every neighborhood of \( s = 0 \) contains points that are not minima of \( U_v \), then every neighborhood of \( v_1 \) (in the full \( v \)-space) will contain points \( v' \) with \( U(v') < U(v_1) \), in contradiction to the assumption that \( v_1 \) is a local minimum of \( U_v \). If, on the other hand, there is a neighborhood of \( s = 0 \) containing only minima of \( U_v \), then \( d(v_1 + s\Delta v)/d\tau \) must vanish identically. The coefficient of \( s^2 \) in \( d(v_1 + s\Delta v)/d\tau \) is proportional to \( \Delta v^2 \beta\gamma \). Summation over \( \alpha = \gamma \) and \( \beta = \delta \) gives

\[
2 \sum_{\alpha<\beta<\gamma<\delta} (\Delta v^{\alpha\beta\gamma\delta})^2 + \sum_{\gamma<\delta} (\sum_{\alpha} \Delta v^{\alpha\gamma\delta})^2 = 0,
\]

(26)

implying \( \Delta v = 0 \), i.e. \( v_1 = v_2 \). The same arguments can be used to show that there is at most a single local maximum.

We are now able to start a detailed case by case discussion of the asymptotic behavior (in the regime of validity of perturbation theory) of a theory described by the Lagrangian \( \mathcal{L} \) both in the infrared and in the ultraviolet limit.

### 3 Classification of flow patterns

**A) \( \lambda > 0 \)**

When \( \tau \to -\infty \), \( G \) becomes large and perturbation theory is no longer valid. So, if there is any fixed point within the range of validity of perturbation theory, it must occur for \( \tau \to +\infty \), where \( G \sim e^{-2\lambda\tau} \). If the \( y \)'s escape attraction by the ultraviolet fixed point or if there is no ultraviolet fixed point, then the \( y \)'s go to infinity. One might wonder whether \( Y = gy \) could still remain finite. That this is not so follows from the fact that the \( y \)'s reach infinity within finite \( \tau \):

\[
\frac{dy}{d\tau} \sim const. \times y^3 \quad \text{implies} \quad -y^{-2} \sim const. \times (\tau - const.)
\]

(27)

Thus perturbation theory breaks down.
Then, at lowest order in $G$, the evolution of $v$ will be determined by Eq. (13) at $y = 0$. A simple example for a non-trivial fixed point of $v$ is the case of just one real scalar which is a singlet under the gauge group: At $y = 0$

\[ G^{-1} \frac{dv}{dt} = 3v^2 + \lambda v. \] (29)

There is an ultraviolet fixed point at $v_{FP} = -\lambda/3 = -(22c_g - 4S_F)/9$, which is approached according to

\[ v - v_{FP} \sim \text{const.} \times e^{-2\lambda \tau} \sim \text{const.} \times G. \] (30)

In general $v - v_{FP}$ will behave like some positive power of $G$. Describing the evolution of couplings in terms of some other coupling is just the idea of the “reduction of coupling constants” program [10]. The exponents of $G$ encountered in $y$ and $v - v_{FP}$, which are not necessarily integral, correspond precisely (modulo the fact that different models were considered) to the non-integral exponents found in [10].

On the other hand there are many theories without a fixed point for $v$, due to the following argument (adapted from Ref. [11]): A little calculation shows that at $y = 0$

\[ G^{-1} \frac{dv^{\alpha\gamma\gamma}}{dt} = (v^{\alpha\lambda\epsilon} - 6C^{\lambda\epsilon}B^{\alpha})^2 + 2v^{\alpha\gamma\lambda\epsilon}v^{\alpha\gamma\lambda\epsilon} - 12d_gQB + 6d_gSB(\lambda + 2SB - c_g) - \frac{d_B\lambda^2}{4}, \] (31)

which is positive definite for many theories. In such a case, or whenever we start outside the domain of attraction of an ultraviolet fixed point, for large $\tau$

\[ \frac{dv}{d\tau} \sim \text{const.} \times v^2, \quad v^{-1} \sim \text{const.} \times (\text{const.} - \tau), \] (32)

i.e. we leave the region of validity of perturbation theory within finite $\tau$.

**B) $\lambda < 0$**

Here $G$ becomes large for $\tau \to +\infty$, so that a perturbative fixed point is possible only in the infrared limit $\tau \to -\infty$. In this case $y$ will always approach some infrared fixed point. According to the discussion after Eq. (23), if $N_{\text{flat}} + \text{rank}(H)$ is equal to the dimension of $y$-space, $y - y_{FP}$ behaves like some exponential of $\tau$ and therefore like a positive power of $G$. If this is not the case, it is possible that

\[ y - y_{FP} \sim \text{const.} \times (\text{const.} - \tau)^{-1/2} \sim \text{const.} \times (\ln(G^{-1}))^{-1/2}, \] (33)

which is certainly a very slow approach to the fixed point. If a fixed point for $v$ (at $y = y_{FP}$) exists and the starting values are in its domain of attraction, $v$ will also go to the fixed point.

As an example consider the case of $N$ chiral fermions in some complex representation $R$ of the gauge group and as many fermions in the conjugate representation $\bar{R}$, together with a single scalar singlet. By a biunitary transformation one can diagonalize the couplings,
so that \( y_{i\bar{j}} = Z_{I(i)} \delta_{i\bar{j}} \) (cf. Eq. (1)), where \( I \) runs from 1 to \( N \). Then \( U \) turns out to be of the form
\[
U = a(\sum_{I} |Z_{I}|^{2})^{2} + b \sum_{I} |Z_{I}|^{4} - c \sum_{I} |Z_{I}|^{2} + d
\]  
(34)

with positive constants \( a, b, c, d \). \( U \) is easily minimized with the result that at the fixed point all \( |Z_{I}|^{2} \) are equal to some constant depending on \( N \) and the dimension and Casimir eigenvalue of \( R \). At \( y = y_{FP} \), \( dv/d\tau \) is positive for \( v \to \pm \infty \) and negative for \( v = 0 \), i.e. it must have two zeroes. The larger of the two values of \( v \) for which \( dv/d\tau \) is zero corresponds to a minimum of \( U \).

\( C) \lambda = 0 \)

The behaviour of \( G \) is dictated by
\[
\frac{1}{8G^{2}} \frac{dG}{d\tau} = -2c_{g}^{2} + c_{g}S_{F} + Q_{F} + Q_{B} - \frac{1}{2d_{g}} \text{Tr}(C_{F}y^{\beta^{\dagger}y}^{\beta}) + O(G),
\]  
(35)

where we have used \( \lambda = 0 \), i.e. \( S_{B} = 22c_{g} - 4S_{F} \), in order to eliminate \( S_{B} \). \( y \) evolves according to Eq. (11) with \( D = 6C_{F} \).

Let us first consider the ultraviolet limit \( \tau \to +\infty \): If no Yukawa interacting fermionic singlets are present, \( y = 0 \) is a local maximum of \( U \). If there are fermionic singlets, or if the starting configuration is outside the domain of attraction, \( y \) will go to infinity within finite \( \tau \). Let us now assume that perturbation theory remains valid for \( \tau \to +\infty \). Then \( \lim_{\tau \to +\infty} y = 0 \) and stability of \( G \), determined by Eq. (35), at \( y = 0 \), implies \( Q_{B} \leq 2c_{g}^{2} \) and \( S_{F} \leq 2c_{g} \). Reinserting the latter inequality into \( \lambda = 0 \), we get \( S_{B} \geq 14c_{g} \). Putting this into Eq. (31), we see that \( v^{\alpha\alpha\gamma\gamma} \) will go to infinity within finite \( \tau \). We conclude that for \( \lambda = 0 \) there is no perturbative ultraviolet fixed point. Of course, all this is again only true if there is no exact fine tuning of the initial values which would allow the couplings to stay in the minimum or in a saddle point of the potential. Such a case is considered in Ref. [12].

For \( \tau \to -\infty \), \( y \) will go to some infrared fixed point. The behavior of \( G \) is determined by Eq. (35) with \( \text{Tr}(C_{F}y^{\beta^{\dagger}y}^{\beta}) \) evaluated at the one–loop fixed point for \( y \). If \( U \) has flat directions, \( y_{FP} \) depends continually on the initial conditions. If all flat directions correspond to symmetries of \( U \), or if the fixed point is at \( U = 0 \), \( \text{Tr}(C_{F}y^{\beta^{\dagger}y}^{\beta}) \) is nevertheless stable under small changes in the initial conditions. If the r.h.s. of Eq. (35) is negative, perturbation theory will break down; if it is positive, \( G = 0 \) will be an infrared fixed point, approached as \( G \sim (\text{const.} - t)^{-1/2} \). If \( U \) takes its minimum at vanishing \( S_{ijkl}, I_{\alpha\beta} \) and \( M_{ij} \), then \( \text{Tr}(C_{F}y^{\beta^{\dagger}y}^{\beta}) = 6 \text{Tr} C_{F}^{2} = 6d_{g}Q_{F} \) and
\[
\frac{1}{8G^{2}} \frac{dG}{d\tau} = -2c_{g}^{2} + c_{g}S_{F} - 2Q_{F} + Q_{B} + h.o.
\]  
(36)

The behavior of \( v \) depends on whether there is a fixed point for \( v \) at \( y = y_{FP} \) and on the initial conditions. Of course the theory is stable in the infrared limit only if \( v \) approaches such a fixed point.
The most interesting case occurs when the r.h.s. of Eq. (35), evaluated at $y = y_{FP}$, vanishes. Then
\[ \frac{d_g}{4G^2} \frac{dG}{d\tau} = \text{Tr}(C_F(y^\beta \beta^\dagger y^\beta - y_{FP}^\beta y_{FP}^\dagger)) + O(G) \]
\[ = \text{Tr}(C_F(\Delta y^\beta \beta^\dagger \Delta y^\beta + y_{FP}^\beta \beta^\dagger \Delta y^\beta + \Delta y^\beta \beta^\dagger y_{FP}^\dagger)) + O(G), \]  
(37)
where $\Delta y = y - y_{FP}$, so it is crucial for the further discussion how the fixed point is approached. If there are components of $y$ behaving like $(\text{const.} - \tau)^{-1/2}$, $G$ will run to 0 or $\infty$. If, however, $N_{\text{flat}} + \text{rank}(H)$ is equal to the dimension of $y$-space, all components of $\Delta y$ will tend to zero exponentially in $\tau$. Again the same type of behaviour has been encountered in the context of “reduction of coupling constants” [13]. The r.h.s. of Eq. (37) receives an exponential inhomogeneity; for suitable (but not exactly fine-tuned) initial values $G$ remains finite for $\tau \to -\infty$ at the present order in perturbation theory. In a next step one has to consider higher order corrections to the evolution of $y$. Using the vanishing of the first two orders of $\beta(g)$, one gets
\[ \frac{dy}{d\tau} = \frac{\partial U}{\partial y^\dagger} + \frac{2}{gG} \beta^{(2)}(Y) + O(G^2) \]  
(38)
with $\beta^{(2)}(Y) = gG^2 f(y, v)$. The ansatz $y = y_{FP} + Gy^{(1)} + \ldots$ yields
\[ G \frac{dy^{(1)}}{d\tau} = G(Hy^{(1)} + 2f(y_{FP}, v_{FP})) + h.O. \]  
(39)
Thus $y^{(1)}$ goes to $-2H^{-1}f(y_{FP}, v_{FP})$ exponentially with exactly the same rate as $\Delta y$ goes to zero. In a similar way $v_{FP}$ gets shifted by a term of the order of $G$. The value of $y^{(1)}$ is relevant for the $G^4$-term (which is then dominant) in $\beta(G)$. The same analysis can be repeated until eventually $G$ starts to run like some power of $\tau$. One would expect that this should always happen at some order in perturbation theory. This is however not the case for supersymmetric theories, due to the theorem of Ref. [14] that in such theories the vanishing of all beta functions at $N$ loop level implies vanishing of the gauge beta function at $N + 1$ loop level.

Summing up our results, we can distinguish the following four cases:

- Perturbation theory remains valid neither in the infrared nor in the ultraviolet limit.
- The theory is unstable in the infrared limit but shows asymptotic freedom: For $t \to \infty$, $g$ behaves according to $g \sim t^{-1/2}$. The Yukawa couplings go to zero as powers of $g$ with exponents greater than 1; the behavior of the quartic scalar couplings is determined by a nontrivial fixed point of $V/g^2$.
- The theory is unstable in the ultraviolet limit. In the infrared limit $g$ goes to zero, $Y/g$ and $V/g^2$ approach nontrivial fixed points.
- There is no ultraviolet stable fixed point; in the infrared limit all couplings approach a fixed point.
There is no theory with stable fixed points both in the infrared and in the ultraviolet limit.

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