Algebraisable versions of topological predicate logic, Part 1

*Topological logic via cylindric algebras*

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Abstract

Motivated by questions like: which spatial structures may be characterized by means of modal logic, what is the logic of space, how to encode in modal logic different geometric relations, topological logic provides a framework for studying the confluence of the topological semantics for S4 modalities, based on topological spaces rather than Kripke frames, with the S4 modality induced by the interior operator.

Following research initiated by Sgro, and further pursued algebraically by Georgescu, we prove an interpolation theorem and an omitting types theorem for various extensions of predicate topological logic and Chang’s modal logic. Our proof is algebraic addressing expansions of cylindric algebras using interior operators and boxes, respectively. Then we proceed like is done in abstract algebraic logic by studying algebraisable extensions of both logics; obtaining a plethora of results on the amalgamation property for various subclasses of their algebraic counterparts, which are varieties. As a sample, we show that the free algebras of infinite dimensions enjoy several weak forms of interpolation, a property equivalent to the fact that the class of simple algebras have the amalgamation property, but they fail the usual Craig interpolation property, because the whole variety fails to have the amalgamation property. Such interpolation properties fail for finite dimensions $> 1$.

Notions like atom-canonicity and complete representations are approached for finite dimensional topological cylindric algebras. The logical consequences of our algebraic results are carefully worked out for infinitary extensions of Chang’s predicate modal logic and finite versions thereof, by restricting to $n$ variables, $n$ finite, viewed as a propositional multi-dimensional modal logic, and $n$ products of bimodal whose frames are of the form $(U, U \times U, R)$ where $R$ is a pre-order, endowed with diagonal constants. We show that for any finite $n > 2$ such modal
logics, though canonical hence Kripke complete, are necessarily non-finitely axiomatizable, furthermore, any axiomatization must contain infinitely many propositional variables, infinitely many diagonal constants, and infinitely many non-canonical sentences; hence they are only barely canonical. In particular, they are not Sahlqvist axiomatizable; and even more they cannot be axiomatized by modal formulas with first order corepondances on their Kripke frames. For \( n \leq 2 \), such logics are are finitely axiomatizable by Sahlqvist modal formulas, they are decidable for \( n = 1 \) and undecidable for \( n = 2 \). We shall also deal with guarded versions of such topological \( n \) modal logics (by relativizing states to guards) proving that they have the finite model property, are decidable, and finitely axiomatizable, for any finite \( n \).

The paper has four parts, this is the first.\(^1\)

1 Introduction

1.1 Universal logic

Universal logic is the field of logic that is concerned with giving an account of what features are common to all logical structures. If slogans are to be taken seriously, then universal logic is to logic what universal algebra is to algebra. The term “universal logic” was introduced in the 1990s by Swiss Logician Jean Yves Beziau but the field has arguably existed for many decades. Some of the works of Alfred Tarski in the early twentieth century, on metamathematics and in algebraic logic, for example, can be regarded undoubtedly, in retrospect, as fundamental contributions to universal logic. Indeed, there is a whole well established branch of algebraic logic, that attempts to deal with the universal notion of a logic. Pioneers in this branch include Andréka and Németi\(^2\) and Blok and Pigozzi\(^3\). The approach of Andréka and Németi though is more general, since, unlike the approach in \(^9\) which is purely syntactical, it allows semantical notions stimulated via so-called ‘meaning functions’\(^7\), to be defined below. Another universal approach to many cylindric-like algebras was implemented in \(^55\) in the context of the very general notion of what is known in the literature as systems of varieties definable by a Monk’s schema \(^45\) \(22\).

One aim of universal logic is to determine the domain of validity of such and such metatheorem (e.g. the completeness theorem, the Craig interpolation theorem, or the Orey-Henkin omitting types theorem of first order logic) and to give general formulations of metatheorems in broader, or even entirely other

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\(^1\)Topological logic, Chang modal logic, cylindric algebras, representation theory, amalgamation, congruence extension, interpolation Mathematics subject classification: 03B50, 03B52, 03G15.
contexts. This is also done in algebraic logic, by dealing with modifications and variants of first order logic resulting in a natural way during the process of \textit{algebraisation}, witness for example the omitting types theorem proved in \cite{59}.

This kind of investigation is extremely potent for applications and helps to make the distinction between what is really essential to a particular logic and what is not.

During the 20th century, numerous logics have been created, to mention only a few: intuitionistic logic, modal logic, topological logic, spatial logic, dynamic logic, many-valued logic, fuzzy logic, relevant logic, para-consistent logic, non monotonic logic, etc. Universal logic is not a new addition (not a new logic), it is rather a way of unifying this multiplicity of logics by developing general means and concepts that can encompass all hitherto existing logics allowing a uniform treatment of their meta theories, so in this respect it resembles category theory whose main concern is to highlight adjoint situations in various branches of mathematics.

Universal logic also helps to clarify basic concepts explaining what is an extension and what is a deviation of a given logic, what does it mean for a logic to be equivalent, stronger, or interpretable into another one. It allows to give precise definitions of notions often discussed by philosophers like truth-functionality, extensionality, logical form, etc. But such issues are at the heart of research in algebraic logic as well.

\subsection{1.2 Algebraic logic}

Traditionally, algebraic logic has focused on the algebraic investigation of particular classes of algebras, the most famous are Tarski’s cylindric algebras and Halmos’ polyadic algebras, whether or not they could be connected to some known assertional system by means of the Lindenbaum-Tarski method of forming algebras of formulas. Viewing the set of formulas as an algebra with operations induced by the logical connectives, logical equivalence is a congruence relation on the formula algebra.

However, when such a connection could be established, there was interest in investigating the relationship between various metalogical properties of the logistic system and the algebraic properties of the associated class of algebras (obtaining what are sometimes called “bridge theorems”); so in a way algebraic logic can be viewed as the natural interface between logic (in a broad sense) and universal algebra.

For example, it was discovered at quite an early stage of the development of the subject that there is a natural relation between the interpolation theorems of intuitionistic, intermediate propositional calculi, and the amalgamation properties of varieties of Heyting algebras, due to several authors, including Tarski, Jonsson, Rasiowa, Sikorski and others. Similar connections
were investigated between interpolation theorems in the classical predicate calculi and congruence extension properties and amalgamation results in varieties of cylindric and polyadic algebras; pioneers in this connection include Comer, Johnson, Diagneault and Pigozzi [48, 41, 36, 35, 34].

Interpolation theorems require the presence of at least a partial order, but the congruence extension property for an algebra, and for that matter the amalgamation property for a class of algebras are more universal notions, and lend themselves to wider contexts.

Quoting Pigozzi from [41], “It is always exciting for a mathematician when close connections are discovered between seemingly distant notions and results from two different branches of mathematics, and this is especially true when the notions and results involved are important ones and the focus of considerable research in their respective areas. Thus for instance, some recent developments have brought to light close and unexpected connections between two groups of results - metalogical interpolation theorems of which the first and best known is Craig’s interpolation theorem for first order logic and the algebraic theorems to the effect that certain classes of algebras have the amalgamation property.”

The framework of the work of Pigozzi in [41] was cylindric algebras an equational formalism of first order logic. On the other hand, Georgescu has shown that the strongly related representation theory of Halmos’ polyadic algebras can be applied to prove a completeness theorem for many predicate logics, like tense logic, $S_4$ modal logic, intuitionistic logic, Chang $S_5$ modal logic and topological logic [16, 18, 15, 14, 17].

In this paper, among many other things, we carry out an analogous investigation but for algebraizable extensions and/or versions and modifications of predicate topological logic and Chang modal logic, using the well developed machinery of the theory of cylindric algebras, and ‘bridge theorems’ in abstract algebraic logic.

1.3 Topological logic and Chang’s modal logic

Topological logic was introduced by Makowsky and Ziegler [37, 39], and Sgro [68]. Such logics have a classical semantics with a topological flavour, addressing spatial logics and their study was approached using algebraic logic by Georgescu [16], the task that we further pursue in this paper. Topological logics are apt for dealing with logic and space; the overall point is to take a common mathematical model of space (like a topological space) and then to fashion logical tools to work with it.

One of the things which blatantly strikes one when studying elementary topology is that notions like open, closed, dense are intuitively very transparent, and their basic properties are absolutely straightforward to prove. However, topology uses second order notions as it reasons with sets and subsets
of ‘points’. This might suggest that like second order logic, topology ought to be computationally very complex. This apparent dichotomy between the two paradigms vanishes when one realizes that a large portion of topology can be formulated as a simple modal logic, namely, $S4$. This is for sure an asset for modal logics tend to be much easier to handle than first order logic let alone second order.

The project of relating topology to modal logic begins with work of Alfred Tarski and J.C.C McKinsey [69]. Strictly speaking Tarski and McKinsey did not work with modal logic, but rather with its algebraic counterpart, namely, Boolean algebras with operators which is the approach we adopt here; the operators they studied where the closure operator induced on what they called the algebra of topology, certainly a very ambitious title, giving the impression that the paper aspired to completely algebraise topology.

In retrospect McKinsey and Tarski showed, that the Stone representation theorem for Boolean algebras extend to algebras with operators to give topological semantics for classical propositional modal logic, in which the ‘necessity’ operation is modelled by taking the interior (dual operation) of an arbitrary subset of topological space. Although the topological completeness of $S4$ has been well known for quite a long time, it was until recently considered as some exotic curiosity, but certainly having mathematical value. It was in the 1990-ies that the work of McKinsey and Tarski, came to the front scene of modal logic (particularly spatial modal logic), drawing serious attention of many researchers and inspiring a lot of work stimulated basically by questions concerning the ‘modal logic of space’; how to encode in modal logic different geometric relations? A point of contact here between topological spaces, geometry, and cylindric algebra theory is the notion of dimension.

From the modern point of view one introduces a basic modal language with a set $\mathbf{At}$ of atomic propositions, the logical Boolean connectives $\land$, $\neg$ and a modality $I$ to be interpreted as the interior operation. Let $X$ be a topological space. The modal language $\mathcal{L}_0$ is interpreted on such a space $X$ together with an interpretation map $i : \mathbf{At} \to \wp(X)$. For atomic $p \in \mathbf{At}$, $i(p)$ says which points satisfy $p$. We do not require that $i(p)$ is open. $(X, i)$ is said to be a topological model. Then $i$ extends to all $\mathcal{L}_0$ formulas by interpreting negation as complement relative to $X$, conjunction as intersection and $I$ as the interior operation. In symbols we have:

\[
\begin{align*}
    i(\neg \phi) &= X \sim i(\phi), \\
    i(\phi \land \psi) &= i(\phi) \cap i(\psi), \\
    i(I(\phi)) &= \text{int}i(\phi).
\end{align*}
\]

The main idea here is that the basic properties of the Boolean operations on sets as well as the salient topological operations like interior and its dual the
closure, correspond to to schemes of sentences. For example, the fact that the interior operator is idempotent is expressed by

\[ i((II\phi) \leftrightarrow (I\phi)) = X. \]

The natural question to ask about this language and its semantics is: Can we characterize in an enlightening way the sentences \( \phi \) with the property that for all topological models \((X, i)\), \(i(\phi) = X\); these are the topologically valid sentences. They are true at all points in all spaces under whatever interpretation. More succintly, do we have a nice completeness theorem?

Tarski and McKinsey proved that the topologically valid sentences are exactly those provable in the modal logic \( \textbf{S4} \). \( \textbf{S4} \) has a seemingly different semantics using standard Kripke frames. Now \( X \) is viewed as the set of possible worlds. In \( \textbf{S4} \), \( I \) is read as all points which the current point relates to. To get a sound interpretation of \( \textbf{S4} \) we should require that the current point is related to itself. Therefore we are led to the notion of a pre-ordered model. A pre-ordered model is defined to be a triple \((X, \leq, i)\) where \((X, \leq)\) is a pre-order and \( i : \text{At} \to \wp(X) \) where

\[ i(I(\phi)) = \{ x : \{ y : x \leq y \} \subseteq i(\phi) \}. \]

Temporally world \( x' \in X \) is a successor of world \( x \in X \) if \( x \leq x' \), \( x \) and \( x' \) are equivalent worlds if further \( x \leq x' \) and \( x' \leq x \). We have a completely analogous result here; \( \phi \) is valid in pre-ordered models if \( \phi \) is provable in \( \textbf{S4} \).

One can prove the equivalence of the two systems using only topologies on finite sets. Let \((X, \leq)\) be a pre-order. Consider the Alexandrov topology on \( X \), the open sets are the sets closed upwards in the order. This gives a topology, call it \( O_{\leq} \). A correspondence between topological models and pre-ordered models can thereby be obtained, and as it happens we have for any pre-ordered model \((X, \leq, i)\), all \( x \in X \), and all \( \phi \in \mathcal{L}_0 \)

\[ x \models \phi \text{ in } (X, \leq, i) \iff x \models \phi \text{ in } (X, O_{\leq}, i). \]

Using this result together with the fact that sentences satisfiable in \( \textbf{S4} \) have finite topological models, thus they are Alexandov topologies, one can show that the semantics of both systems each is interpretable in the other; they are equivalent. We can summarize the above discussion in the following neat theorem, that we can and will attribute to McKinsey, Tarski and Kripke; this historically is not very accurate. For a topological space \( X \) and \( \phi \) an \( \textbf{S4} \) formula we write \( X \models \phi \), if \( \phi \) is valid topologically in \( X \) (in either of the senses above). For example, \( w \models \Box \phi \) iff for all \( w' \) if \( w \leq w' \), then \( w' \models \phi \), where \( \leq \) is the relation \( x \leq y \) iff \( y \in \text{cl} \{x\} \).

**Theorem 1.1.** (McKinsey-Tarski-Kripke) Suppose that \( X \) is a dense in itself metric space (every point is a limit point) and \( \phi \) is a modal \( \textbf{S4} \) formula. Then the following are equivalent:

1. \( X \models \phi \).
(1) $\phi \in S_4$.
(2) $\models \phi$.
(3) $X \models \phi$.
(4) $\mathbb{R} \models \phi$.
(5) $Y \models \phi$ for every finite topological space $Y$.
(6) $Y \models \phi$ for every Alexandrov space $Y$.

One can say that finite topological space or their natural extension to Alexandrov topological spaces reflect faithfully the $S_4$ semantics, and that arbitrary topological spaces generalize $S_4$ frames. On the other hand, every topological space gives rise to a normal modal logic. Indeed $S_4$ is the modal logic of $\mathbb{R}$, or any metric that is separable and dense in itself space, or all topological spaces, as indicated above. Also a recent result is that it is also the modal logic of the Cantor set, which is known to be Baire isomorphic to $\mathbb{R}$.

But, on the other hand, modal logic is too weak to detect interesting properties of $\mathbb{R}$, for example it cannot distinguish between $[0, 1]$ and $\mathbb{R}$ despite their topological disimilarities, the most striking one being compactness; $[0, 1]$ is compact, but $\mathbb{R}$ is not.

To make $S_4$ stronger and more expressive, one can enrich the modal language. Hybrid languages are such; they have proposition letters called nominals and global modality. Nominals denote singleton sets and global modality allows to say that a formula holds somewhere. In Hybrid modal logic one can say that the closure of any singleton is itself, by $\diamond i \rightarrow i$ ($i$ a nominal) which is valid in $\mathbb{R}$ but not in spaces that are not $T_1$. Hence $T_1$ is definable by nominals and the Hybrid logic of $\mathbb{R}$ is not that of any topological space and so it is stronger than $S_4$.

One can also enrich the language of $S_4$ with a modal operator $[a]$ giving it a temporal dimension; $[a]$ interpreted as 'next'. If $X$ is a topological space and $f : X \to X$ is a continous function, then the pair $(X, f)$ is called a dynamic space over $X$. If $f$ is the identity function, then this is a static space; it is nothing more than $S_4$, because the 'next' world is only the same world. The field of dynamic topological logic dealing with dynamic spaces over topological spaces, modalizing dynamical systems, is quite an active field of research; providing a unifying framework for studying the confluence of three rich research areas: the topological semantics for $S_4$, topological dynamics, and temporal logic.

Definition 1.2. A dynamic topological model on $X$ consists of a dynamic space $(X, f)$ over $X$ and a valuation $i$ of propositional variables to subsets of
$X$ such that
\[
i(\neg \phi) = X \sim i(\phi),
\]
\[
i(\phi \land \psi) = i(\phi) \cap i(\psi),
\]
\[
i(I(\phi)) = \text{int} i(\phi)
\]
\[
i(\lfloor a \rfloor \phi) = f^{-1}(i(a)).
\]

The resulting modal logic is called $S4C$. We have a completeness theorem here as well:

**Theorem 1.3.** For any formula $\phi$ the following are equivalent:

1. $S4C \vdash \phi$
2. $\phi$ is topologically valid
3. $\phi$ is true in any finite topological space

But here there are derivable formulas that are not valid in $\mathbb{R}$. However, such dynamic topological logics, have a very interesting completeness theorem, namely, that for any formula that is *not* derivable, there exists a countermodel in $\mathbb{R}^n$ for $n$ sufficiently large, where the upper bound of the dimension, namely $n$, is characterized by the modal depth of such a formula. The techniques used suggest that such a modality, or perhaps a similar one, may be used characterize the geometric notion of dimension, but further research is needed.

Topological interpretations of propositional topological logic were recently extended in a natural way to arbitrary theories of full first order logic by Awodey and Kishida using so-called *topological sheaves* to interpret domains of quantification [43].

They prove that $S4V$ (predicate $S4$ logic) is complete with respect to such extended topological semantics, using techniques related to recent work in topos theory. Indeed, historically Sheaf semantics was first introduced by topoi theorists for higher order intuitionistic logic, and has been applied to first order modal logic, by both modal and categorical logicians.

Sheaves or pre-sheaves taken over a possible world structure- most notably Kripke sheaves over a Kripke frame can be regarded as extending the structure to the first order level with variable domains of individuals; the modality arises naturally from a geometric morphism between the topos of such sheaves of the associated world structures. The completeness proof in essence is a translation of a Henkin construction; implementing a so-called ‘de modalization process’

Given a first order modal language, the construction gives a first order non-modal language and a surjective interpretation from the former to the latter, along this interpretation we can have a non-modal version of a given modal theory. So the modal predicate language is reduced to an ordinary predicate
one by eliminating the $S4$ operation, but its models of consistent theories built by a Henkin usual construction are interpretable, or rather give rise to, models of the original predicate modal language.

In this paper we also study algebraically a predicate version of the modal topological logic described above. One way of doing this is that we deal with the same syntax in [43], but we alter the semantics, dealing with usual Kripke semantics, proving a stronger result, namely, an interpolation theorem; we also touch on dynamic topological predicate logics.

But next we proceed differently; the modus operandi, and the overall goals of our work are different, too.

We assume that the models carry a topology, but now models are more complex; they are structures for first order logic. Consider such a structure $M$ for a given first order language in a certain signature having a sequence of variables of order type $\omega$ and assume that its underlying set $M$ is endowed with a topology. Then the set of all assignments satisfying a formula $\phi$ interpreted the usual Tarskian way can be seen as an $\omega$-ary relation on $M$, call it $\phi^M$.

Unlike the approach adopted in [43], where there is only one $S4$ modality, here for each $k < \omega$, we add to the syntax an operation $I_k$ interpreted at a formula $\phi$ as those sequences $s$ satisfying $\phi$ except that at the $k$th co-ordinate we require that $s(k)$ is in the interior of the $k$th component of $\phi^M$, so we get a smaller set than $\phi^M$. So here we have $\omega$ many modalities, not just one, each acting on one component of the set of sequences satisfying a given formula; when we deal with only finitely many variables $m$, we will have $m$ modalities, but we shall also look at cylindrifiers as diamonds dealing with $2m$ multi dimensional propositional modal logic.

A completeness and an omitting types theorem are proved algebraically by Georgescu in [16] for usual first order logic (with infinitely many variables) with such semantics involving the interior operators induced by a topology on the base of models.

But as it happens, there is also a modal approach to topological predicate logic [11, 16, 39]. Each interior operator can again be viewed as a modality $\square_i$, called Chang’s modal operator and its semantics is specified by a Chang system for a model $M$, which is a function $V : M \rightarrow \phi(\phi(M))$. The semantics is now defined as follows:

$$s \in \square_i \phi^M \iff \{u \in M : s^i_u \in \phi^M\} \in V(s_i).$$

Here $s^i_u$ is the function that is like $s$ everywhere except that its value at $i$ is $u$. If $M$ carries a topology $\mathcal{O}$ say, then this gives a natural Chang system defined by $V(x) = \mathcal{O}$, for all $x \in M$. A completeness theorem was also proved by Georgescu [18] for Chang’s modal logics using polyadic algebras. So one can view topological modals as nice semantics for Chang’s $S4$ modal logic.
1.4 The process of algebraisation

We go further in the analysis carried out in [16, 18] using also an algebraic approach, but on a wider scale, proving stronger and much more results. In the above cited references Georgescu uses the representation theory of \textit{locally finite polyadic algebras with equality}, here we use the representation theory of \textit{dimension complemented cylindric algebras}, which is not only of a strictly wider scope, but is actually much simpler. Both cases reflect a \textit{Henkin construction}, but in the case of polyadic algebras the procedure is much more complex. One starts with an algebra \textit{dilates it}, meaning embedding it into a reduct of an algebra having infinitely extra dimensions, fixes some of the extra dimensions obtaining a \textit{free or rich extension} of $\mathfrak{A}$, and then ‘constants’ are stimulated as \textit{algebraic endomorphism} on the dilated algebra, and these endomorphisms are used to eliminate cylindrifiers, witness [16, p.449].

In cylindric algebras one also dilates the algebra, but then cylindrifiers are eliminated by the \textit{spare dimensions} via certain Boolean ultrafilters (which we call Henkin ultrafilters; that correspond exactly to Henkin’s notion of \textit{rich theories}). In this case a constant is not a complicated algebraic entity like an endomorphism, but it can be viewed as simply \textit{an index in the dilated dimension} which conforms more to Henkin’s notion of expanding the language by adding constants or witnesses for existential formulas. This makes life much easier and also the construction lends itself to more general contexts.

Indeed our results address possibly infinitary extensions of topological first order logic, and Chang’s modal logic. We not only prove completeness and an omitting types theorem for such logics, but we also prove an interpolation theorem, analogous to the Craig interpolation theorem for first order logic, but in a more general setting.

The results in [16, 18], are special cases of two of our three results proved for topological logic and Chang’s modal logic. The new interpolation theorem proved here which is not approached at all in the two cited references, is next elaborated upon in a \textit{universal algebraic way} as done in \textit{abstract algebraic logic}.

From the algebraic point of view, we depart from the so-called \textit{locally finite} and \textit{dimension complemented} algebras. An algebra $\mathfrak{A}$ is locally finite if the \textit{dimension set} of every element in $\mathfrak{A}$. The dimension set of an element in $\mathfrak{A}$ reflects the number of variables in the formula of the corresponding Tarski-Lindenbaum algebra of formulas.

An algebra is dimension complemented if the complement of the dimension set of every element is infinite; this reflects, in turn, that infinitely many variables lie outside the formula corresponding to the element, but the possibility remains that this formula contains infinitely many variables, so such logics have an infinitary flavour. In fact, they can be seen as an instance of the so-called \textit{finitary logics with infinitary predicates} [7, 9, 11, 22, 59], finitary here, in turn, points out to the fact that quantification is only allowed on finitely
many variables, as is the case with first order logic.

This is a natural generalization of first order logic, for in many classical theorems of first order logic, like Gödel’s completeness theorem, Craig interpolation theorem and the Orey-Henkin omitting types theorem, the proof does not depend on the fact that every formula contains many (free) variables but rather on the weaker fact that \textit{infinitely many variables} lie outside each formula, because in such a case witnesses for existential formulas in Henkin constructions can, like the case with first order logic, always be found.

But locally finite algebras, the algebraic counterpart of topological predicate logic, and for that matter the larger class of dimension complemented algebras, have some serious defects when treated as the sole subject of research in an autonomous algebraic theory.

In universal algebra one prefers to deal with \textit{equational classes} of algebras i.e. classes of algebras characterized by systems of postulates, in which every postulate has the form of an equation (an identity). Such classes are also referred to as \textit{varieties}.

Classes of algebras which are not varieties are often introduced in discussions as specialized subclasses of varieties. One often treats fields as a special case of rings. This is due to the tradition that in algebra, mainly the equational language and thus equational logic is used. Thus, finding an equational form for an algebraic entity is always a value on its own right.

Another reason for this preference, is the fact that every variety is closed under certain general closure operations frequently used to construct new algebras from given ones. We mean here the operations of forming subalgebras, homomorphic images and direct products. By a well known theorem of Garrett Birkhoff, varieties are precisely those classes of algebras that have all three of these closure properties. Local finiteness does not have the form of an identity, nor can it be equivalently replaced by any identity or system of identities, nor indeed any set of first order axioms. This follows from the simple observation that the ultraproduct of infinitely many locally finite algebras is not, in general, locally finite, and a first order axiomatizable class is necessarily closed under ultraproducts. The same applies to the class of dimension complemented algebras.

The definition of local finiteness contains an assumption which considerably restricts the scope of the definition and thus it is very tempting to just drop it, and see what happens. As is the case with Tarski’s cylindric algebras, a lot does. We hope, and in fact we think, that the reader will be convinced of this bold declaration after reading the paper.

Indeed, the restrictive character of this notion becomes obvious when we turn our attention to \textit{cylindric set algebras}; these are concrete having having top element a \textit{cartesian square}, namely, a set of the form $^\alpha U$, $\alpha$ an ordinal is the dimension; the Boolean operations are the usual operations of intersection and
complementation with respect to $\alpha U$ and cylindrifiers and diagonal elements are defined reflecting the semantics of existential quantifiers and equality. If for $s, t \in \alpha U$ and $i < \alpha$, $s \equiv_i t$ means that $t(j) = s(j)$ for all $j \neq i$, then the $i$th cylindrifier is defined via

$$c_i X = \{ s \in \alpha U : \exists t \in X (s \equiv_i t) \}, X \subseteq \alpha U,$$

and the $i, j$ diagonal via

$$d_{ij} = \{ s \in \alpha U : s_i = s_j \}.$$

We find that there are such set algebras of all dimensions, and set algebras that are not locally finite are easily constructed.

We thereby simply remove the condition of local finiteness and also we will have occasion to deal with topological cylindric algebras of finite dimension extending many deep results proved for cylindric algebras, and proving new ones.

For the infinite dimensional case we study the corresponding minimal algebraizable extension of both predicate topological logic and Chang’s modal logic, that necessarily allow infinitary predicates. The condition of local finiteness in the infinite dimensional case is not warranted from the algebraic point of view because it is a property that cannot be expressed by first order formulas, let alone equations or quasi-equations.

Roughly, minimal extension here means this (algebraizable) logic corresponding to the quasi-variety generated by the class of algebras arising from ordinary topological predicate logic, namely, the class of locally finite algebras. This correspondence is taken in the sense of Blok and Pigozzi associating quasi-varieties to algebraizable logics [9].

In algebraizable extensions of first order logic studied by Henkin, Monk and Tarski and Blok and Pigozzi in [22, 9], and even earlier by Andréka and Németi [3], the notion of a formula schema plays a key role. If we have a set of formulas $F$ say, then a formula schema is an element of $F$. An instance of a formula schema is obtained by substituting formulas for the formula variables, i.e for atomic formulas, in this formula schema. A formula schema is called type-free valid if all its instances are valid. This is a new notion of validity defined in [22, Remark 4.3.65].

A drawback at least from the algebraic point of view for ordinary first order logic, and for that matter predicate topological logic is the following: There are type-free valid formula schemas $\psi$, say of first order logic that are not uniformly provable. Though each instance of $\psi$ is provable, these proofs vary from one instance to the other. We cannot give a uniform proof of all these instances in spite of there being a uniform cause $\psi$ of their validity.

The reason for this phenomena is that the standard formalism of first order logic is not structural in the sense of [9]. In fact, this formalism is not even
substitutional in the sense of [7, definition 4.7 (ii) p.72]. This means that a formula resulting from substituting formulas for atomic formulas in any valid formula, may not be valid. To remedy this “defect” one can give a structural formalism of first order logic.

Following [3, 7] a logic is a quadruple \((F, \mathcal{K}, \text{mng}, \models)\) where \(F\) is a set (of formulas) in a certain signature, \(\mathcal{K}\) is a class of structure \(\text{mng}\) is a function with domain \(F \times \mathcal{K}\) and \(\models \subseteq F \times F\). Intuitively, \(\mathcal{K}\) is the class of structures for our language \(\text{mng}(\phi, M)\) is the interpretation of \(\phi\) in \(M\), possibly relativized, and \(\models\) is the pure semantical relation determined by \(\mathcal{K}\). This of course is too broad a definition. An algebraizable logic is defined next.

**Definition 1.4.** A logic \((F, \mathcal{K}, \text{mng}, \models)\) with formula algebra \(\mathfrak{F}\) of signature \(t\) is algebraizable if

1. A set \(Cn\mathfrak{L}\) the logical connectives fixed and each \(c \in Cn\mathfrak{L}\) finite rank determining the signature \(t\),
2. There is set \(P\) called atoms such that \(\mathfrak{F}\) is the term algebra or absolutely free algebra over \(P\) with signature \(t\),
3. \(\text{mng}_M = \langle \text{mng}(\phi, M) : \phi \in F \rangle \in \text{Hom}(\mathfrak{F})\),
4. There is a derived binary connective \(\leftrightarrow\) and a nullary connective \(\top\) that is compatible with the meaning functions, so that for all \(\psi, \phi \in F\), we have \(\text{mng}(\phi) = \text{mng}(\psi)\) iff \(M \models \phi \leftrightarrow \psi\) and \(M \models \phi\) if \(M \models \phi \leftrightarrow \top\),
5. For each \(h \in \text{Hom}(\mathfrak{F}, \mathfrak{F})\), \(M \in \mathcal{K}\), there is an \(N \in \mathcal{K}\) such \(\text{mng}_N = \text{mng}_M \circ h\), so that validity is preserved by homomorphisms.

Item (5) is what guarantees that instances of valid formulas remain valid for a homomorphism applied to a formula \(\phi\) amounts to replacing the atomic formulas in \(\phi\) by formula schemes. This is a crucial property for a logic to allow algebraization.

To form the algebraic counterpart of such a logic, which is a quasi-variety, there are essentially two conceptually different means. One can define it syntactically using quasi-equations via a Hilbert style axiomatization involving type free valid schemas that translate to quasi-equations in the signature \(t\) [7, 9]. Or alternatively one can proceed semantically, defining the algebraic counterpart as the quasi-variety generated by the 'meaning algebras \(\{\text{mng}_M(\mathfrak{F}) : M \in \mathcal{K}\}\). These two notions in general are distinct, but in favourable circumstances they can coincide; indeed this is the case when we have a completeness theorem [7]. Structural formalism of first order logic and non finite Hilbert-style complete axiomatizations go hand in hand. Such issues will be approached in some depth below; where we show that this phenomena persists in the new topological context.
1.5 Sample of results

For the algebraisable version of topological logic we show that the corresponding algebraic counterpart, call it $V$, is not only a quasi-variety, but is in fact a variety, that is an expansion of the variety of representable cylindric algebras of infinite dimensions by interior operators.

A plethora of results on representability and amalgamation for $V$ are proved. For example we show that the variety of representable algebras coincides with the class of algebras having the neat embedding property, lifting a famous result of Henkin proved for cylindric algebras when we count in interior operators.

In universal algebra and indeed in the newly born field of universal logic a crucial and extremely fruitful role is played by the fact that certain global properties of varieties, like the variety $V$ above, typically amalgamation properties are mirrored in corresponding local properties of their free algebras, typically congruence extension properties and even equational consequence relations in the variety itself, which in turn corresponds to various forms of interpolation when we happen to have an order, like the Boolean order, a condition that holds in our subsequent investigations. The synthesis of these characterizations provides an illuminating and potentially very useful bridge between the paradigms of algebra and logic, with results enriching both.

In this paper all results in the late [36], on interpolation, congruence extension properties on free algebras and various forms of amalgamation on classes of algebras are obtained for $V$. As a sample we show that the class of semi-simple algebras have the amalgamation property but $V$ itself does not, and the former result is equivalent to the fact the free algebras satisfy a natural weak form of interpolation, call it $WIP$. From the second result we can infer that the free algebras do not satisfy the usual Craig interpolation property; in fact, it turns out that they do not satisfy an interpolation property strictly weaker than the Craig interpolation property, but of course strictly stronger than the $WIP$. Sharp results on non-finite axiomatizability are obtained for several subvarieties of $V$ whose members have a neat embedding property, to be clarified below. Entirely analogous results are obtained for the variety corresponding to the algebraisable extension of Chang’s predicate $S4$ and $S5$ modal logic.

We shall also show that several approximations of the variety of representable algebras cannot be axiomatized by a finite schema of equations. Such varieties are defined via the notion of neat reducts an old venerable notion in the theory of cylindric algebras. Given $\alpha < \beta$, the $\alpha$ neat reduct of a $\beta$ dimensional algebra is a subalgebra of the reduct of $\mathcal{B}$ obtained by discarding all operations indexed by $\beta \sim \alpha$ and keeping only $\alpha$ dimensional elements. Denoting the class of topological cylindric algebra of dimension $\mu$ by $TCA_\mu$, the $\alpha$ neat reduct of $\mathcal{B} \in TCA_\beta$ is denoted by $\xi_{\alpha} TCA_\beta$; the latter is a $TCA_\alpha$. A classical
result of Monk (which we prove an analogue thereof for topological cylindric algebras) says that for cylindric algebras $CA_n$ If $\alpha > 2$, $S\forall_\alpha CA_{\alpha+n} \neq RCA_{\alpha+n}$ for all $n \in \omega$, where $RCA_{\alpha}$ denotes the class of representable $CA_{\alpha}$. On the other hand, a classical result of Henkin, which is a strong algebraic extension of Gödel’s completeness theorem, proved using a Henkin construction too, says that $S\forall_\alpha CA_{\alpha+\omega} = RCA_{\alpha}$, which we prove for $TCA_{\alpha}$.

We also prove the following result extending a recent result of the present author and Robin Hirsch for several cylindric-like algebras, namely:

**Theorem 1.5.** Let $\alpha > 2$ be an ordinal. Then for any $r \in \omega$, for any finite $k \geq 1$, for any $l \geq k + 1$ (possibly infinite), there exist $B^r \in S\forall_\alpha TCA_{\alpha+k} \sim S\forall_\alpha TCA_{\alpha+k+1}$ such $\Pi_{r \in \omega} B^r \in S\forall_\alpha TCA_{\alpha+l}$. In particular, for any such $k$ and $l$, and for $\alpha$ finite, $S\forall_\alpha TCA_{\alpha+k+l}$ is not finitely axiomatizable over $S\forall_\alpha TCA_{\alpha+k}$, and for infinite $\alpha$, $S\forall_\alpha TCA_{\alpha+l}$ is not axiomatizable by a finite schema over $S\forall_\alpha TCA_{\alpha+k}$.

In contrast we introduce another variety of topological polyadic algebras of infinite dimensions; the term polyadic refers to the fact that the signature of this new class contains all substitutions, so is closer to the polyadic paradigm, and prove that such a variety can be axiomatized by a finite schema and it further enjoys the super amalgamation property.

1.6 Product of modal logics

We shall also deal rather extensively with topological logic with only finitely many variables, corresponding to finite dimensional topological cylindric algebras of dimension $m$ say, with $m \in \omega$. Such a logic can be viewed as a predicate logic with $m$ variables enriched by $m$ modalities, or as a propositional multi-dimensional modal logic with $2^m$ modalities; call it $L_m$.

We show that for $m > 2$ ($m$ finite), $L_m$ is not finitely axiomatizable, it is undecidable, it is undecidable to tell whether a finite frame is a frame for $L_m$, $L_m$ fails Craig interpolation and Beth definability, and $L_m$ fails the omitting types theorem in a very strong sense, even if we allow clique guarded semantics. We shall address deeply decidability issues for such logics, by viewing them as product modal logics.

One of the main reasons for the praise of modal logics in computer science is their robust decidability, which is preserved under forming combinations of modal logics like products, as long as there are no interaction axioms or constraints (fusions). This situation however changes drastically as soon as some kind of interaction between the modalities is imposed. In fact, straightforward constructions of combined modal logics from the simple 1-dimensional ones will almost certainly result in computationally complex logics. The fact that all three dimensional modal logics are undecidable can be intuitively explained
by the undecidability of the product $S5^3$ and its relation to the undecidable fragment of first order logic with 3 variables; represented algebraically by $CA_3$. But unlike $CA_2$, even some two dimensional modal logics are undecidable, like products of transitive frames.

Such a view will enable us to show that unlike first order logic with two variables, the topological logic $L_2$ with two variables is undecidable and does not have the finite model property. It will then readily follows the equational theory of $TRCA_2$ is undecidable, a significant point of deviation from cylindric algebras.

The ‘two dimensional undecidability result’ will be done by encoding tilings; that is encoding the $\mathbb{N} \times \mathbb{N}$ grid using the the two interior operators, which is the standard technique of proving undecidability for many modal logics. But we will also show that $L_2$ is finitely axiomatizable, but does not have the finite model property. There are refutable formulas that cannot be refuted in finite Kripke models. The latter result holds too for $L_n$ when $n \geq 3$, but this is utterly unsurprising.

Products of modal logics, like temporal, spatial, epitemistic logics or multi-dimensional modal languages interpreted in various product-like frames are very natural and clear formalisms arising in both pure logic and numeros applications, like multi-agent systems. For example, dynamic topological logic dealt with earlier can be interpreted semantically in products of the form $(T, <) \times (W, R)$ where $(T, <)$ models the flow of time and $(W, R)$ is a frame for $S4$ representing the topological space, with the $S4$ box being also interpreted as the interior operator. By interpreting $W$ as a domain of objects that can change over time, one can view such product frames as models for finite variable fragments of first order temporal and modal logics.

We shall also deal with guarded versions of $L_m$ by relativizing the set of worlds or states, obtaining a finite variable fragment of predicate topological logic having nice modal behaviour. We show that such logics (with any number of finitely many variables) is finitely axiomatizable, have the finite model property, is decidable (in a strong way; in fact the universal theory of its modal algebras is decidable), and have the interpolation property.

1.7 Concluding

The algebraic facet of this paper can be seen as a refresher to proofs of many deep results proved for cylindric algebras, and also new ones for cylindric algebras by passing to reducts of topological cylindric algebras by discarding the interior operators. Such results include the deep results of Andréka [1] on the complexity of universal axiomatizations of the variety of representable cylindric algebras, which lift mutatis mutandis to the ‘topological addition’, the answer to problem 2.12 in [21] given in [26], together with its infinite
analogue, the main result in [46], which is the solution to problem 4.4 in [22], and all the results in [10, 34, 35, 36, 48, 49] answering all open problems in Pigozzi’s landmark paper [41] and more, several results in [2] confirming three conjectures of Tarski’s on cylindric algebras, formulated in the language of category theory, a theme initiated in [58].

We use advanced sophisticated machinery of cylindric algebra theory, like so called rainbow constructions [25, 26, 29, 27], obtaining new results, strengthening results in [8, 25, 29] both algebraic and metalogical which we formulate for topological logics with finitely many variables, and finally we use tilings twice to prove undecidability of topological logics with more than one variable.

Relativizing states, we also deal with finite variable fragments of such topological logics as multi-modal logics, and guarded fragments of finite variable predicate topological logic. Using games we show that such logics are finitely axiomatizable, and using a model-theoretic result of Herwig and the well-developed duality theory between Kripke frames and complex algebras, we show that such logics having $n$ variables, are also decidable; the universal theory of their modal algebras is decidable, and have the definability properties of Beth and Craig, for each finite $n$.

Due to the length of the paper it is divided into four parts.

(1) Part one: **Topological logic via cylindric algebras.**

(2) Part two: **Amalgamation, interpolation and congruence extension properties in topological algebras.**

(3) Part three: **Logical consequences for extensions of predicate topological logic.**

(4) Part four: **Logical consequences for finite variable fragments of first order logic.**

Each part can be read separately modulo cross references to other parts.

**On the notation of all parts, some required basics in Topology**

We follow more or less standard notation. But for the reader’s convenience, we include the following list of notation that will be used throughout the paper.

An **ordinal** $\alpha$ is transitive set (i.e., any member of $\alpha$ is also a subset of $\alpha$) that is well-ordered by $\in$. Every well-ordered set is order isomorphic to a unique ordinal. For ordinals $\alpha, \beta$, $\alpha < \beta$ we means $\alpha \in \beta$. An ordinal is therefore the set of all smaller ordinals, so for a finite ordinal $n$ we have $n = \{0, 1, \ldots, n-1\}$ and the least infinite ordinal is $\omega = \{0, 1, 2, \ldots\}$. 
A cardinal is an ordinal not in bijection with any smaller ordinal, briefly an initial ordinal and the cardinality \( |X| \) of a set \( X \) is the unique cardinal in bijection with \( X \). Cardinals are ordinals and are therefore ordered by \( < \) (i.e., \( \in \)). The first few cardinals are \( 0 = \emptyset, 1, 2, \ldots, \omega \) (the first infinite ordinal), \( \omega_1 \) (the first uncountable cardinal). A set will be said to be countable if it has cardinality \( \leq \omega \), uncountable otherwise, and countable infinite if it has cardinality \( \omega \). \( 2^\omega \) denotes the power of the continuum.

For a set \( X \), \( \wp(X) \) denotes the set of all subsets of \( X \), i.e. the powerset of \( X \). \( A^B \) denotes the set of functions from \( A \) to \( B \). If \( f \in A^B \) and \( X \subseteq A \) then \( f \upharpoonright X \) denotes the restriction of \( f \) to \( X \). We denote by \( \text{dom} f \) and \( \text{rng} f \) the domain and range of a given function \( f \), respectively. \( A \sim B \) is the set \( \{ x \in A : x \notin B \} \).

We frequently identify a function \( f \) with the sequence \( \langle f_x : x \in \text{dom} f \rangle \). We write \( fx \) or \( f_x \) or \( f(x) \) to denote the value of \( f \) at \( x \). We define composition so that the right-hand function acts first, thus for given functions \( f, g \), \( f \circ g(x) = f(g(x)) \), whenever the left-hand side is defined, i.e. when \( g(x) \in \text{rng} f \).

For a non-empty set \( X \), \( f(X) \) denotes the image of \( X \) under \( f \), i.e. \( f(X) = \{ f(x) : x \in X \} \). If \( X \) and \( Y \) are sets then \( X \subseteq \omega Y \) denotes that \( X \) is a finite subset of \( Y \).

Algebras will be denoted by gothic letters, and when we write \( \mathfrak{A} \) then we will tacitly assuming that \( A \) will denote the universe of \( \mathfrak{A} \). However, in some occasions we will identify (notationally) an algebra and its universe.

If \( U \) is an ultrafilter over \( \wp(I) \) and if \( \mathfrak{A}_i \) is some structure (for \( i \in I \)) we write either \( \Pi_{i \in I} \mathfrak{A}_i / U \) or \( \Pi_{i \in I} \mathfrak{A}_i \) for the ultraproduct of the \( \mathfrak{A}_i \) over \( U \). Fix some ordinal \( n \geq 2 \). For \( i,j < n \) the replacement \( [i \mid j] \) is the map that is like the identity on \( n \), except that \( i \) is mapped to \( j \) and the transposition \( [i,j] \) is the like the identity on \( n \), except that \( i \) is swapped with \( j \). We will refer to maps from \( \tau : n \rightarrow n \) as transformations. A transformation is finite if the set \( \{ i < n : \tau(i) \neq i \} \) is finite, so if \( n \) is finite then all transformations \( n \rightarrow n \) are finite. It is known, and indeed not hard to show, that any finite permutation is a product of transpositions and any finite non-injective map is a product of replacements. A transformation is infinitary if it is not finite.

A Topological space \( X \) is a pair \( (X, \tau) \) where \( X \) is a set and \( \tau \) a collection of subsets of \( X \) such that \( \emptyset, X \in \tau \) and \( \tau \) is closed under arbitrary unions and finite intersections. Such a collection is called a topology on \( X \) and its members are called open sets. The complements of open sets are called closed sets. Clearly, both \( \emptyset, X \) are closed and arbitrary intersections and finite unions of closed sets are closed. For \( A \subseteq X \), we denote by \( \text{int} A \) the interior of \( A \), which is the largest open set contained in \( A \).

\( X = (X, \tau) \) is discrete if \( \tau = \wp(X) \). Note that \( X \) is discrete if and only if \( \text{int} A = A \) for every \( A \subseteq X \). The space \( X \) is almost discrete if for all \( A \in \tau \), \( \text{cl}(A) = \text{int cl}(A) \), where \( \text{cl}(A) \), the smallest closed set containing \( A \), is the
closure of $A$. Notice that the operations $\text{int}$ and $\text{cl}$ are dual; for a topological space with underlying set $X$, and $A \subseteq X$, we have $\text{cl}(A) = X \sim [\text{int} \sim A]$.

A set of the form $\bigcap_{n \in \mathbb{N}} U_n$, where $U_n$ are open sets, is called a $G_\delta$ set, and a set of the form $\bigcup_{n \in \mathbb{N}} F_n$, where $F_n$ are closed sets, is called an $F_\sigma$ set.

Let $X$ be the underlying set of a topological space. A set $A \subseteq X$ is called nowhere dense if its closure $\text{cl}(A)$ has empty interior. (This means equivalently that $X \sim \text{cl}(A)$ is dense). So $A$ is nowhere dense iff $\text{cl}(A)$ is nowhere dense. A set $A \subseteq X$ is meager or of first category if $A$ is the countable union of nowhere dense sets. The complement of a meager set is called comeager. So a set is comeager iff it contains the intersection of a countable family of dense open sets.

**Theorem 1.6.** Let $X$ be a topological space. The following statements are equivalent:

1. Every nonempty open set in $X$ is nonmeager.
2. Every comeager set in $X$ is dense.
3. The intersection of countably many dense open sets in $X$ is dense.

**Definition 1.7.** A topological space is called a Baire space if it satisfies any of the equivalent conditions of the above proposition.

**Theorem 1.8** (The Baire Category theorem). Every completely metrizable space is Baire. Every locally compact Hausdorff space is Baire.

For operators on classes of algebras: $S$ stands for the operation of forming subalgebras, $H$ for the operation of forming homomorphic images, $P$ for the operation of forming products, and $\text{Up}$ for the operation of forming ultraproducts. In particular, a class $K$ is a variety iff $\text{HSP}K = K$ and $K$ is a quasi-variety if $\text{SPUp}K = K$.

## 2 Basics

Let $\alpha$ be an arbitrary ordinal $> 0$. Cylindric set algebras are algebras whose elements are relations of a certain pre-assigned arity, the dimension, endowed with set-theoretic operations that utilize the form of elements of the algebra as sets of sequences. $\mathcal{B}(X)$ denotes the Boolean set algebra $\langle \wp(X), \cup, \cap, \sim, \emptyset, X \rangle$.

Let $U$ be a set and $\alpha$ an ordinal; $\alpha$ will be the dimension of the algebra. For $s, t \in ^\alpha U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq ^\alpha U$ and $i, j < \alpha$, let

$$c_i X = \{s \in ^\alpha U : \exists t \in X(t \equiv_i s)\}$$
and
\[ d_{ij} = \{ s \in {}^\alpha U : s_i = s_j \} \].

\( (\mathfrak{A}(\alpha^U), c_i, d_{ij})_{i,j<\alpha} \) is called the full cylindric set algebra of dimension \( \alpha \) with unit (or greatest or top element) \( {}^\alpha U \). \( {}^\alpha U \) is called a cartesian space.

Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if \( M \) is a first order structure in a first order language \( L \) with \( \alpha \) many variables, then one manufactures a cylindric set algebra based on \( M \) as follows. Let
\[ \phi^M = \{ s \in {}^\alpha M : M \models \phi[s] \}, \]
(here \( M \models \phi[s] \) means that \( s \) satisfies \( \phi \) in \( M \)), then the set \( \{ \phi^M : \phi \in \text{Fm}^L \} \) is a cylindric set algebra of dimension \( \alpha \). Indeed
\[
\phi^M \land \psi^M = (\phi \land \psi)^M,
\]
\[
{}^\alpha M \sim \phi^M = (\neg \phi)^M,
\]
\[
c_i(\phi^M) = \exists v_i \phi^M,
\]
\[
d_{ij} =: (x_i = x_j)^M.
\]

Instead of taking ordinary set algebras, as in the case of cylindric algebras, with units of the form \( {}^\alpha U \), one may require that the base \( U \) is endowed with some topology. This enriches the algebraic structure. For given such an algebra, for each \( k < \alpha \), one defines an interior operator on \( \varphi(\alpha^U) \) by
\[ I_k(X) = \{ s \in {}^\alpha U ; s_k \in \text{int}\{ a \in U : s^k_a \in X \} \} , X \subseteq {}^\alpha U. \]
Here \( s^k_a \) is the sequence that agrees with \( s \) except possibly at \( k \) where its value is \( a \). This gives a topological cylindric set algebra of dimension \( \alpha \). The dual operation of \( I_k \) is \( \text{Cl}_k \) defined by
\[ \text{Cl}_k(X) = \{ s \in {}^\alpha U ; s_k \in \text{Cl}(\{ a \in U : s^k_a \in X \} ) , X \subseteq {}^\alpha U. \]
Notice that when \( U \) has the indiscrete topology, then \( \text{Cl}_k(X) = c_k X \).

A more general semantics is provided by the Chang systems:

**Definition 2.1.** A Chang system is a pair \((U, V)\), where \( U \) is a non-empty set and
\[ V : U \rightarrow \varphi(\varphi(U)). \]

Given such a system, one can introduce unary operations, called box operators on \( \varphi(\alpha^U) \) as follows:
\[ s \in \square_i X \iff \{ u \in U : s^i_u \in X \} \in V(s_i) , X \subseteq {}^\alpha U. \]
The interior operators, as well as the box operators can also be defined on \textit{weak spaces}, that is, sets of sequences agreeing co-finitely with a given fixed sequence. This makes a difference only when $\alpha$ is infinite. We mention the case of interior operators, the box operators are defined entirely analogously using Chang systems.

A \textit{weak space of dimension} $\alpha$ is a set of the form \begin{equation*}
\{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}
\end{equation*}
for a given fixed in advance $p \in {}^\alpha U$. Now for $k < \alpha$, define

$$I_k(X) = \{s \in {}^\alpha U(p) : \{s_k \in \text{int}\{u \in U : s_k^u \in X\}\}.$$

But we can even go further. Such operations also extend to the class of representable algebras $\mathcal{CAS}$, briefly $\text{RCA}_\alpha$. $\text{RCA}_\alpha$ is defined to be the class $\text{SPCs}_\alpha$. This class is also equal to $\text{SPW}_\alpha$, and it is known that $\text{RCA}_\alpha$, is a variety, hence closed under $H$, though infinitely many schema of equations are required to axiomatize it \cite{1}, witness also theorem 4.2 below.

An algebra in $\text{RCA}_\alpha$ is isomorphic to a set algebra with universe $\mathcal{P}(V)$; the top element $V$ is a generalized space which is a set of the form $\bigcup_{i \in I} {}^\alpha U_i$, $I$ a set $U_i \neq \emptyset$ ($i \in I$), and $U_i \cap U_j = \emptyset$ for $i \neq j$. The class of all such concrete algebras is denoted by $\mathcal{Gs}_\alpha$. We refer to $A \in \mathcal{Gs}_\alpha$ as a generalized set algebra of dimension $\alpha$.

So let $A \in \text{RCA}_\alpha$, and assume that $A \cong B$ where $B \in \mathcal{Gs}_\alpha$ has top element the generalised space $V$. The base of $V$ is the set $U = \bigcup_{s \in V} \text{rng}s$. Then one defines the interior operator $I_k$ on $B$ by:

$$I_k(X) = \{s \in V : s_k \in \text{int}\{a \in U : s_k^a \in X\}\}, X \subseteq V.$$

and, for that matter the box operator relative to a Chang system $V : U \rightarrow \varphi(\varphi(U))$ as follows

$$s \in \square_k(X) \iff \{a \in U : s_k^a \in X\} \in V(s_k), X \subseteq V.$$

The following lemma is very easy to prove, so we omit the proof. Formulated only for set algebras, it also holds for weak set algebras.

**Lemma 2.2.** For any ordinal $\mu > 1$, $\mathfrak{A} \in \mathcal{Cs}_\mu$ and $k < \mu$, let $I_k$ and $\square_k$ be as defined above. Then if $\mathfrak{A} \in \mathcal{Cs}_\alpha$ has top element $^\alpha U$ and $\beta > \alpha$, then the following hold for any $Y \subseteq {}^\alpha U$ and $k < \alpha$:

1. $I_k(Y) \subseteq Y$, $\square_k(Y) \subseteq Y$,

2. If $f : \varphi(\alpha U) \rightarrow \varphi(\beta U)$ is defined via

$$X \mapsto \{s \in \beta U : s \upharpoonright \alpha \in X\},$$

then $f(I_k X) = I_k(f(X))$ and $f(\square_k X) = \square_k(f(X))$, for any $X \subseteq {}^\alpha U$.  

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For an algebra \( A \) in \( G_{S_{\alpha}} \) with top element \( V \) and base \( U \), let \( A^t \) be the TPCA \( \alpha \) obtained when \( U \) is endowed with the some topology \( \tau \), equivalently the Chang algebra (this will turn out to be an \( S_{\alpha} \) Chang algebra, to be defined shortly) corresponding to the Chang system \( F : U \to \wp(\wp(U)) \), defined via \( F(x) = \tau(x \in U) \).

**Lemma 2.3.** Let \( A, B \) be in \( G_{S_{\alpha}} \) such that \( A \subseteq B \) have the same top element giving the same base \( U \). If \( U \) is endowed with any topology; then \( A^t \subseteq B^t \). If \( A \cong B \), then \( A^t \cong B^t \).

In cylindric algebra theory a subdirect product of set algebras is isomorphic to a generalized set algebra. We show that this phenomena persists when the bases carry topologies; we need to describe the topology on the base of the resulting generalized set algebra in terms of the topologies on the bases of the set algebras involved in the subdirect product.

**Definition 2.4.** Let \( \{ X_i : i \in I \} \) be a family of topological spaces indexed by \( I \). Let \( X = \bigcup X_i \) be the disjoint union of the underlying sets. For each \( i \in I \) let \( \phi_i : X_i \to X \) be the canonical injection. The coproduct on \( X \) is defined as the finest topology on \( X \) for which the canonical injections are continuous.

That is a subset \( U \) of \( X \) is open in the coproduct topology on \( X \) iff its preimage \( \phi_i^{-1}(U) \) is open in \( X_i \) for each \( i \in I \) iff its intersection with \( X_i \) is open relative to \( X_i \) for each \( i \in I \).

**Theorem 2.5.** Let \( B \) be the \( G_{S_{\alpha}} \) with unit \( V = \bigcup_{i \in I} \alpha U_i \) where \( U_i \cap U_j = \emptyset \) and base \( \bigcup_{i \in I} U_i \) carrying a topology. Assume that \( B \) has universe \( \wp(V) \). Let \( A_i \) be the \( C_{S_{\alpha}} \) with base \( U_i \), \( U_i \) having the subspace topology and universe \( \wp(\alpha U_i) \). Then \( f : B \to \prod_{i \in I} A_i \) defined by \( X \mapsto (X \cap \alpha U_i : i \in I) \) is an isomorphism of cylindric algebras; furthermore it respects the interior operators stimulated by the topologies on the bases.

\( T_{C_{S_{\alpha}}}(T_{G_{S_{\alpha}}}) \) denotes the class of topological (generalized) set algebras.

**Theorem 2.6.** \( SPT_{C_{S_{\alpha}}} \subseteq T_{G_{S_{\alpha}}} \).

*Proof.* Suppose that \( C \subseteq \prod_{i \in I} \mathcal{D}_i \) each \( \mathcal{D}_i \in T_{C_{S_{\alpha}}} \) with base \( U_i \neq 0 \), and \( U_i \cap U_j = \emptyset \). Each \( U_i \) has a topology. Let \( f \) be as in the previous theorem. Then \( f^{-1} : C \to T_{C_{S_{\alpha}}} \) is an isomorphism into a \( T_{G_{S_{\alpha}}} \) whose base \( \bigcup_{i \in I} U_i \) carry the coproduct topology. \( \square \)

Now such algebras lend itself to an abstract formulation aiming to capture the concrete set algebras; or rather the variety generated by them.

This consists of expanding the signature of cylindric algebras by unary operators, or modalities, one for each \( k < \alpha \), satisfying certain identities.
The axiomatizations we give are actually simpler than those stipulated by Georgescu in [16, 18], although locally finite polyadic algebras and locally finite cylindric algebras are equivalent. We use only substitutions corresponding to replacements; in the case of dimension complemented algebras all substitutions corresponding to finite transformations are term definable from these [21]. This makes axiom (A8) on p.1 of [16] superfluous.

In [16, 18] representation theorems are proved for locally finite polyadic algebras; here we extend this theorem in three ways. We prove a strong representation theorem for dimension complemented algebras, this is a strictly larger class. The logic corresponding to such algebras allow infinitary predicates. We prove an interpolation and an omitting types for such logics, too. The constructions used are standard Henkin constructions; for luckily the ‘expanded’ semantics allows such proofs.

We start with the standard definition of cylindric algebras [21, Definition 1.1.1]:

**Definition 2.7.** Let $\alpha$ be an ordinal. A cylindric algebra of dimension $\alpha$, a $\text{CA}_\alpha$ for short, is defined to be an algebra

$$\mathfrak{C} = \langle C, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j \in \alpha}$$

obeying the following axioms for every $x, y \in C$, $i, j, k < \alpha$

1. The equations defining Boolean algebras,
2. $c_i0 = 0$,
3. $x \leq c_ix$,
4. $c_i(x \cdot c_iy) = c_ix \cdot c_iy$,
5. $c_ic_jx = c_jc_ix$,
6. $d_{ii} = 1$,
7. if $k \neq i, j$ then $d_{ij} = c_k(d_{ik} \cdot d_{jk})$,
8. If $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

For a cylindric algebra $\mathfrak{A}$, we set $q_ix = -c_i - x$ and $s^i_j(x) = c_i(d_{ij} \cdot x)$. Now we want to abstract equationally the prominent features of the concrete interior operators defined on cylindric set and weak set algebras. We expand the signature of $\text{CA}_\alpha$ by a unary operation $I_i$ for each $i \in \alpha$. In what follows $\oplus$ denotes the operation of symmetric difference, that is, $a \oplus b = (\neg a + b) \cdot (\neg b + a)$. For $\mathfrak{A} \in \text{CA}_\alpha$ and $p \in \mathfrak{A}$, $\Delta p$, the dimension set of $p$, is defined to be the set \{ $i \in \alpha : c_ip \neq p$ \}. In polyadic terminology $\Delta p$ is called the support of $p$, and if $i \in \Delta p$, then $i$ is said to support $p$ [16, 18].
Definition 2.8. A topological cylindric algebra of dimension $\alpha$, $\alpha$ an ordinal, is an algebra of the form $(\mathfrak{A}, I_i)_{i<\alpha}$ where $\mathfrak{A} \in \mathcal{CA}_\alpha$ and for each $i < \alpha$, $I_i$ is a unary operation on $A$ called an interior operator satisfying the following equations for all $p, q \in A$ and $i, j \in \alpha$:

1. $q_i(p \oplus q) \leq q_i(I_ip \oplus I_iq)$,
2. $I_ip \leq p$,
3. $I_ip \cdot I_ip = I_i(p \cdot q)$,
4. $p \leq I_i I_ip$,
5. $I_11 = 1$,
6. $c_k I_ip = I_ip, k \neq i, k \notin \Delta p$,
7. $s_j I_ip = I_js_j I_ip, j \notin \Delta p$.

The class of all such topological cylindric algebras are denoted by $\mathcal{TCA}_\alpha$.

We do the same task axiomatizing the properties of Chang’s modal operators, or boxes, equationally.

Definition 2.9. A Chang cylindric algebra of dimension $\alpha$, $\alpha$ an ordinal, is an algebra of the form $(\mathfrak{A}, \Box_i)_{i\in\alpha}$ where $\mathfrak{A} \in \mathcal{CA}_\alpha$ and for each $i < \alpha$, $\Box_i$ is a unary operator on $\mathfrak{A}$, called a modality, satisfying the following equations for all $p, q \in A$ and $i, j \in \alpha$:

1. $q_i(p \oplus q) \leq q_i(\Box_ip \oplus \Box_iq)$,
2. $s_j \Box_ip = \Box_is_j p, j \notin \Delta p$.

Consider the following equations expressible in the signature of Chang algebras of dimension $\alpha$; where $i \in \alpha$:

1. $\Box_i1 = 1$,
2. $\Box_ip \leq p$,
3. $\Box_ip \cdot \Box_ip = \Box_i(p \cdot q)$,
4. $c_k \Box_ip = \Box_ip, k \neq i, k \notin \Delta p$,
5. $\Box_ip \leq \Box_i \Box_ip$,
6. $\neg \Box_i \neg p \leq \Box_i \neg \Box_i \neg p$. 

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The $S_4$ Chang algebras of dimension $\alpha$ are defined as the Chang algebras of dimension $\alpha$ with properties equivalent to items (1) – (5) and the $S_5$ Chang algebras of dimension $\alpha$ are the Chang cylindric algebras of dimension $\alpha$ satisfying items (1) – (6). Notice that the $S_4$ Chang algebras are equivalent to the topological cylindric algebras of the same dimension. For $\mathcal{B} = (\mathcal{A}, I_i)_{i<\alpha} \in TCA_\alpha$ we write $\mathcal{N}_{\mathcal{A}} \mathcal{B}$ for $\mathcal{A}$. Notice too that every $CA_\alpha$ can be extended to a $TCA_\alpha$, by defining for all $i < \alpha$, $I_i$ to be the identity function.

Topological algebras in the form we defined are not Boolean algebras with operators because the interior operators do not distribute over the Boolean join.

But we could have just as well worked with the dual operators, in which case we land in the realm of Boolean algebras with operators. From the point of view of multi modal logic such operators are the diamonds and the interior operators are the boxes.

But in all cases algebras dealt with are not completely additive; cylindrifiers are completely additive but the interior operators are not as shown next.

**Example 2.10.** Let $\mathfrak{A} = \wp(\omega \mathbb{N})$ with the co-finite topology on $\mathbb{N}$. Let $X_n = \{n\} \times \omega \mathbb{N}$. Then

$$I_0 X_n = \emptyset,$$

and so

$$\bigcup I_0 X_n = \emptyset$$

But

$$\bigcup_{n \in \omega} X_n = \mathbb{N}$$

hence

$$I_0(\bigcup X_n) = \mathbb{N} \neq \bigcup I_0 X_n.$$ 

Second observation is that the interior operators are not term definable, for if $U$ is an infinite set and $\mathfrak{A}$ is the full set algebra with base $U$ of dimension $\alpha > 1$, then if $U$ has the discrete topology and $i < \alpha$, then $I_i X = X$ for any $X \in \mathfrak{A}$, which is not the case when $U$ has the indiscrete topology. In other words the cylindric structure does not uniquely define the interior operators.

We do not know whether one can construct a set algebra with base $U$ and two non-homeomorphic topologies on $U$ such that the induced interior operators gives rise to isomorphic topological cylindric set algebras.

If $\mathfrak{A}$ is a set algebra with base $U$, this is concretely reflected by giving $U$ the discrete topology. Viewed otherwise, if $\mathfrak{A}$ is in $RCA_\alpha$, then this expansion is also representable by giving the base $U$ of the $Gs_\alpha$ representing $\mathfrak{A}$ the discrete topology. This simple observation will turn out immensely useful to obtain results about $TCA_\alpha$ by bouncing it back to the cylindric part. This works in the case of transferring negative results for cylindric algebras to the topological
paradigm, but does not help much in case we are encountered with a positive result. For example as we shall see, though the equational theory of $\text{RCA}_2$ is known to be decidable, it will turn out that the equational theory of the class of representable topological cylindric algebras of dimension 2 is not.

Such an observation also holds for $S5$ Chang algebras, too because a discrete space is obviously almost discrete.

We stipulate that each and every result, with no single exception, proved for $TCA_\alpha$ can be obtained using the same methods for Chang algebras, $S4$ Chang algebras and $S5$ Chang algebras.

An algebra $B$ is locally finite (dimension complemented) if $\text{Rad}_\alpha B$ is such. We denote by $\text{TLf}_\alpha$ and $\text{TDC}_\alpha$, the classes of locally finite and dimension complemented cylindric topological algebras of dimension $\alpha$, respectively. That is, $B \in \text{TDC}_\alpha$, if $\Delta x \neq \alpha$ for every $x \in B$; this turns out, in the infinite dimensional case, equivalent to $\alpha \sim \Delta x$ is infinite for every $x \in B$. On the other hand, $B \in \text{TLf}_\alpha$ if $\Delta x$ is finite for all $x \in B$ (recall that $\Delta x = \{ i \in \alpha : c_i x \neq x \}$). For finite dimension obviously every algebra is locally finite.

We also need the notion of compressing dimensions and, dually, dilating them; expressed by the notion of neat reducts.

**Definition 2.11.** (1) Let $\alpha < \beta$ be ordinals and $B \in TCA_\beta$. Then $\text{Nr}_\alpha B$ is the algebra with universe $\text{Nr}_\alpha A = \{ a \in A : \Delta a \subseteq \alpha \}$ and operations obtained by discarding the operations of $B$ indexed by ordinals in $\beta \sim \alpha$. $\text{Nr}_\alpha B$ is called the neat $\alpha$ reduct of $B$. If $A \subseteq \text{Nr}_\alpha B$, with $B \in TCA_\beta$, then we say that $B$ is a $\beta$ dilation of $A$, or simply a dilation of $A$.

(2) An injective homomorphism $f : A \rightarrow \text{Nr}_\alpha B$ is called a neat embedding; if such an $f$ exists, then we say that $A$ neatly embeds into its dilation $B$. In particular, if $A \subseteq \text{Nr}_\alpha B$, then $A$ neatly embeds into $B$ via the inclusion map.

Note that the algebra $\text{Nr}_\alpha B$ is well defined; it is closed under the cylindric operations; this is well known and indeed easy to show, and it also closed under all the interior operators $I_i$ for $i < \alpha$, for if $x \in \text{Nr}_\alpha B$, and $k \in \beta \sim \alpha$, then by axiom (6) of definition 2.8

$k \notin \alpha \supseteq \Delta x \cup \{ i \} \supseteq \Delta(I_i(x))$, hence $c_k(I_i(x)) = I_i(x)$.

A piece of notation used throughout. If $A$ is an algebra and $X \subseteq A$, then $\text{Sz}^3 X$ denotes the subalgebra of $A$ generated by $X$.

**Theorem 2.12.** Let $\alpha \geq \omega$. If $A \in \text{TDC}_\alpha$ and $\beta > \alpha$, then there exists $B \in TCA_\beta$ such that $A \subseteq \text{Nr}_\alpha B$ and for all $X \subseteq A$, $\text{Sz}^3 X = \text{Nr}_\alpha \text{Sz}^3 X$.

**Proof.** Exactly like the proof in [21] Theorem 2.6.49] defining the interior operators the obvious way. \hfill $\square$
Recall that for a class $K$, $S$ stands for the operation of forming subalgebras of $K$, and $PK$ that of forming direct products.

**Definition 2.13.** Let $\delta$ be a cardinal. Let $\alpha$ be an ordinal. Let $\mathfrak{A} \alpha \mathfrak{r}_\delta$ be the absolutely free algebra on $\delta$ generators and of type $\text{TCA}_\alpha$. For an algebra $\mathfrak{A}$, we write $R \in \text{Co}\mathfrak{A}$ if $R$ is a congruence relation on $\mathfrak{A}$. Let $\rho \in ^\delta \phi(\alpha)$. Let $L$ be a class having the same signature as $\text{TCA}_\alpha$. Let

$$Cr_\delta^{(\rho)} L = \bigcap \{ R : R \in \text{Co}_\alpha \mathfrak{r}_\delta, \alpha \mathfrak{r}_\delta / R \in \text{SP}L, c_k^{\mathfrak{r}_\delta} \eta / R = \eta / R \text{ for each } \eta < \delta \text{ and each } k \in \alpha \setminus \rho(\eta) \}$$

and

$$\mathfrak{r}_\delta^{(\rho)} L = \alpha \mathfrak{r}_\beta / Cr_\delta^{(\rho)} L.$$

The ordinal $\alpha$ does not figure out in $Cr_\delta^{(\rho)} L$ and $\mathfrak{r}_\delta^{(\rho)} L$ though it is involved in their definition. However, $\alpha$ will be clear from context so that no confusion is likely to ensue.

**Definition 2.14.** Assume that $\delta$ is a cardinal, $L \subseteq \text{TCA}_\alpha$, $\mathfrak{A} \in L$, $x = \langle x_\eta : \eta < \beta \rangle \in ^\delta A$ and $\rho \in ^\delta \phi(\alpha)$. We say that the sequence $x L$-freely generates $\mathfrak{A}$ under the dimension restricting function $\rho$, or simply $x$ freely generates $\mathfrak{A}$ under $\rho$, if the following two conditions hold:

(i) $\mathfrak{A} = \mathfrak{g}^\delta \text{rng} x$ and $\Delta^\delta x_\eta \subseteq \rho(\eta)$ for all $\eta < \delta$.

(ii) Whenever $\mathfrak{B} \in L$, $y = \langle y_\eta : \eta < \delta \rangle \in ^\delta B$ and $\Delta^\delta y_\eta \subseteq \rho(\eta)$ for every $\eta < \delta$, then there is a unique homomorphism $h$ from $\mathfrak{A}$ to $\mathfrak{B}$ such that $h \circ x = y$.

It can be proved without much difficulty that in the above characterization the existence of a unique homomorphism $h$ from $\mathfrak{A}$ to $\mathfrak{B}$ such that $h \circ x = y$ can be replaced by the existence of a unique injective homomorphism $h$ from $\mathfrak{A}$ to $\mathfrak{B}$ such that $h \circ x = y$.

**Lemma 2.15.** Let $\alpha \geq \omega$ and let $\rho : \mu \to \phi(\alpha)$ such that $\mathfrak{r}_\mu^\rho \text{TCA}_\alpha \in \text{TDC}_\alpha$. Then for any ordinal $\beta > \alpha$, the sequence $x = \langle \eta / Cr_\mu^\rho \text{MA}_\beta : \eta < \mu \rangle \text{TCA}_\alpha$-freely generates $\mathfrak{r}_\alpha \mathfrak{r}_\mu^\rho (\text{TCA}_\beta)$ under $\rho$.

**Proof.** Let $\mathfrak{C} \in \text{TCA}_\alpha$ and let $y : \mu \to \mathfrak{C}$ be a homomorphism such that $\Delta y_\eta \subseteq \rho y_\eta$ for all $\eta < \mu$. Then we can assume that $\text{rng} y$ generates $\mathfrak{C}$, so that $\mathfrak{C} \in \text{Dc}_\alpha$, hence $\mathfrak{C} \in \mathfrak{N}_\alpha \text{TCA}_\beta$. Accordingly, let $\mathfrak{C}' \in \text{TCA}_\beta$ be such that $\mathfrak{C} = \mathfrak{N}_\alpha \mathfrak{C}'$. Then clearly $y \in \mu \mathfrak{C}'$ and $\Delta y_\eta \subseteq \alpha$ for all $\eta < \mu$. Let $D = \mathfrak{r}_\mu^\rho (\text{TCA}_\beta)$. Then by freeness there exists a homomorphism $h$ from $D$ to $\mathfrak{C}'$ such that $h \circ x = y$. Clearly $h$ is a homomorphism from $\mathfrak{N}_\alpha D$ to $\mathfrak{N}_\alpha \mathfrak{C}'$, hence it is a homomorphism from $\mathfrak{g}^{\text{rng} x \mathfrak{D}} \text{rng} x$ to $\mathfrak{g}^{\text{rng} \mathfrak{C}' \mathfrak{E}} h(\text{rng} x)$. Since $\text{rng} x \subseteq \mathfrak{N}_\alpha \mathfrak{D}$, we have $h$ is a homomorphism from $\mathfrak{g}^{\text{rng} \mathfrak{D}} \text{rng} x = \mathfrak{N}_\alpha \mathfrak{g}^{\text{rng} x}$ to $\mathfrak{C}$, such that $h(\eta / Cr_\mu^\rho \text{TCA}_\beta) = a_\eta$ and we are done. In particular, we have $\mathfrak{N}_\alpha \mathfrak{r}_\mu^\rho \text{TCA}_\beta \cong \mathfrak{r}_\mu^\rho (\text{TCA}_\alpha)$.
3 Completeness, Interpolation and Omitting types

In this section $\alpha$ will be an infinite ordinal. To prove our first (completeness) theorem, we formulate and prove several lemmas. Properties of substitutions reported in [21] are freely used. For example, for every $\mathfrak{A} \in \mathcal{T}Dc_\alpha$ and every finite transformation $\tau$ we have a unary operation $s_\tau$ that happens to be a Boolean endomorphism on $\mathfrak{A}$ [21, Theorem 1.11.11].

Lemma 3.1. (1) Let $\mathfrak{C} \in \mathcal{CA}_\alpha$ and let $F$ be a Boolean filter on $\mathfrak{C}$. Define the relation $E$ on $\alpha$ by $(i,j) \in E$ if and only if $d_{ij} \in F$. Then $E$ is an equivalence relation on $\alpha$.

(2) Let $\mathfrak{C} \in \mathcal{CA}_\alpha$ and $F$ be a Boolean filter of $\mathfrak{C}$. Let $V = \{\tau \in \alpha : |\{i \in \alpha : \tau(i) \neq i\}|<\omega\}$. For $\sigma, \tau \in V$, write

$\sigma \equiv_E \tau$ iff $(\forall i \in \alpha)(\sigma(i), \tau(i)) \in E$.

and let $\bar{E} = \{(\sigma, \tau) \in 2^V : \sigma \equiv_E \tau\}$. Then $\bar{E}$ is an equivalence relation on $V$. Let $W = V/\bar{E}$. For $h \in W$, write $h = \tau/\bar{E}$ for $\tau \in V$ such that $\tau(j)/\bar{E} = h(j)$ for all $j \in \alpha$. Let $f(x) = \{\bar{\tau} \in W : s_\tau x \in F\}$. Then $f$ is well defined.

Furthermore, $W$ can be identified with the weak space $\alpha[U/\bar{E}]^{(p)}$ where $p = (p(i)/E : i < \alpha)$ via $\tau/\bar{E} \mapsto [\tau]$, where $[\tau](i) = \tau(i)/\bar{E}$. Accordingly, we write $W = \alpha[U/E]^{(p)}$.

Definition 3.2. Let $\mathfrak{A}$ be an algebra having a cylindric reduct of dimension $\alpha$. A Boolean ultrafilter $F$ of $\mathfrak{A}$ is said to be Henkin if for all $k < \alpha$, for all $x \in A$, whenever $c_k x \in F$, then there exists $l \not\in \Delta x$ such that $s^k_l x \in F$.

Lemma 3.3. Let everything be as in the previous lemma, and assume that $F$ is a Henkin ultrafilter. Then $f$ as defined in the previous lemma is a $\mathcal{CA}$ homomorphism.

Proof. [16].

Definition 3.4. Let everything be as in the hypothesis of lemma 3.5. For $s \in W$ and $k < \alpha$ we write $s^k_u$ for $s^k_{u/E}$. For $k \in \alpha$, then $I_k$ is the (interior) operator on $\varphi(W)$ defined by $I_k(X) = \{s \in W : s_k \in \text{int}\{u \in U : s^k_u \in X\}\}$. Similarly, if $V : U/E \to \varphi(\varphi(U/E))$ is a Chang system then $\Box_k$ is defined on $\varphi(W)$ by $s \in \Box_k(X) \iff \{u/E \in U/E : s^k_u \in X\} \in V[s(i)/E]$. 

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Lemma 3.5. \( (1) \) Assume that \( C \in \text{TD}_{\alpha}, F \) is a Henkin ultrafilter of \( C \) and \( a \in F \). Then there exist a non-empty set \( U, p \in \alpha U, \) a topology on \( U/E \) and a homomorphism \( f : C \to (\wp(W), I_i)_{i<\alpha} \) with \( f(a) \neq 0 \), where \( W = \alpha[U/E]^\wp \), with \( E \) as defined in lemma 3.1 and \( I_i \) \( (i < \alpha \) is the concrete interior operator defined in 3.4.

\( (2) \) Assume that \( C \) is an \( S_5 \) dimension complemented Chang algebra, \( F \) is a Henkin ultrafilter of \( C \) and \( a \in F \). Then there exist a Chang system \( V : U/E \to \wp(\wp(U/E)), p \in \alpha U, \) and a homomorphism \( f : C \to (\wp(W), \Box_i)_{i<\alpha} \) with \( f(a) \neq 0 \), where \( W = \alpha[U/E]^\wp \) and the concrete box operators are defined from \( V \) as in 3.4.

Proof. We prove the first item. The proof of the second item is the same. Let \( W = \alpha[\alpha/E]^\wp \). Define, as we did before, \( f : A \to \wp(W) \) via
\[
p \mapsto \{ \bar{r} \in W : s_r p \in F \}.
\]
For \( i \in \alpha \) and \( p \in A \), let
\[
O_{p,i} = \{ k/E \in \alpha/E : s^k_i I(i) p \in F \}.
\]
Let
\[
\mathcal{B} = \{ O_{p,i} : i \in \alpha, p \in A \}.
\]
Then it is easy to check that \( \mathcal{B} \) is the base for a topology on \( \alpha/E \).

To define the interior operations, we set for each \( i < \alpha \)
\[
J_i : \wp(W) \to \wp(W)
\]
by
\[
[x] \in J_i X \iff \exists U \in \mathcal{B}(x_i/E \in U \subseteq \{ u/E \in \alpha/E : [x]^i u \in X \})
\]
where \( X \subseteq V \). Note that \( [x]^i u = [x]^i u \). We now check that \( f \) preserves the interior operators \( J_i \) \( (i < \alpha \), too. We need to show
\[
\psi(I_i p) = J_i(\psi(p)).
\]
The reasoning is like [16]; the difference is that in [16], the constants denoted by \( x_i \) are endomorphisms on \( A \); the value \( x_i \) at \( j \) corresponds in our adopted approach to \( s^j_i \) where \( u = x_i (j) \). Let \( [x] \) be in \( \psi(I_i p) \). Let
\[
\sup(x) = \{ k \in \alpha : x_k \neq k \}.
\]
Then, by definition, \( s_x I_p \in F \). Hence
\[
s^i_{x_i} I_{s^1_{x_1}} \ldots s^n_{x_n} p \in F,
\]
It follows that \( x \) used for cylindric algebras to prove the interpolation theorems \([49, 48]\) works is that in these references the interpolation property was proved for difference between the coming proof and the proofs in the two cited references when the algebras are endowed with interior operators. The only significant

\[
\sup(x) \sim \{i\} = \{j_1, \ldots, j_n\}.
\]

Let

\[
y = [j_1|x_1] \circ \ldots [j_n|x_n].
\]

Then \( x_i/E \in \{u/E : s^i_u I(i)syp \in F\} \in q \). But \( I_syp \leq syp \), hence

\[
U = \{u/E : s^i_u I(i)syp \in F\} \subseteq \{u/E : s^i_u syp \in F\}.
\]

It follows that \( x_i/E \in U \subseteq \{u/E : x^i_u \in \Psi(p)\} \). Thus \([x] \in J_i\psi(p)\).

Now we prove the other direction. Let \([x] \in J_i\Psi(p)\). Let \( U \in \mathfrak{B} \) be such that

\[
x_i/E \in U \subseteq \{u/E : x^i_u \in \alpha/E : s^i_u syp \in F\}.
\]

Assume that \( U = O_{r,j} \), where \( r \in \mathfrak{A} \) and \( j \in \alpha \). Let \( u \in \alpha \sim [\Delta p \cup \Delta r \cup \{i, j\}] \). By dimension complementedness such a \( u \) exists. Then we have:

\[
s^i_u I^r_j r \in F \iff s^i_u s^i_u s_p \in F,
\]

\[
s^i_u I^r_j r \cdot s^i_u s_p \in F \iff s^i_u I^r_j r \in F.
\]

But \( s^i_u I^r_j r = s^i_u I^r_j s^i_r \), so we have

\[
s^i_u I^r_j s^i_r \cdot s^i_u s^i_r \cdot s_p \in F,
\]

\[
s^i_u [I^r_j s^i_r \cdot s^i_u s_p] \in F,
\]

\[
q[i][I^r_j s^i_r \cdot s^i_u s_p] \in F,
\]

\[
q[i][I^r_j s^i_r \cdot I^r_j s^i_u s_p] \in F,
\]

\[
s^i_u [I^r_j s^i_r \cdot I^r_j s^i_u s_p] \in F,
\]

\[
s^i_u I^r_j r \cdot s^i_u I^r_j s^i_u s_p \in F.
\]

But \( s^i_u I^r_j r \in F \), hence \( s^i_u I^r_j s^i_u s_p \in F \), and so \( x \in \Psi(I)p \) as required.

For the second part, define \( V : U/E \rightarrow \phi(\phi(U/E)) \) by

\[
V(m/E) = \{\{j/E \in U/E : s^i_j p \in F\} : p \in A, i \in I, s^i_m \square(i)p \in F\}.
\]

Using the above reasoning together with the reasoning in [18] p. 46-47, it can be checked that \( V \) is as required.

Having lemma [3.3] at hand, we can now show that Henkin constructions used for cylindric algebras to prove the interpolation theorems [49] [48] works when the algebras are endowed with interior operators. The only significant difference between the coming proof and the proofs in the two cited references is that in these references the interpolation property was proved for countable.
Definition 3.6. An algebra $\mathfrak{A} \in \text{TCA}_\alpha$ has the interpolation property if for all $X_1, X_2 \subseteq \mathfrak{A}$, if whenever $a \in \mathcal{G}g^3 X_1$ and $c \in \mathcal{G}g^3 X_2$ are such that $a \leq c$, then there exists $b \in \mathcal{G}g^3 (X_1 \cap X_2)$ such that $a \leq b \leq c$, in which case we say that $b$ is an interpolant of $a$ and $c$ or even simply an interpolant.

Theorem 3.7. Let $\alpha$ be an infinite ordinal. Let $\beta$ be a cardinal. Let $\rho : \beta \to \wp(\alpha)$ such that $\alpha \sim \rho(i)$ is infinite for all $i \in \beta$. Then $\mathfrak{A}_{\beta} \cap \text{TCA}_\alpha$ has the interpolation property.

Proof. Let $\mathfrak{A} = \mathfrak{A}_{\beta} \cap \text{TCA}_\alpha$. Let $a \in \mathcal{G}gX_1$ and $c \in \mathcal{G}gX_2$ be such that $a \leq c$. We want to find an interpolant in $\mathcal{G}g^3 (X_1 \cap X_2)$. By lemma 2.15 let $\mathfrak{B} \in \text{TCA}_\kappa$, $\kappa$ a regular cardinal, such that $\mathfrak{A} = \text{Fr}_\kappa \mathfrak{B}$. Assume that no such interpolant exists in $\mathfrak{A}$, then no interpolant exists in $\mathfrak{B}$, because if $b$ is an interpolant in $\mathcal{G}g^3 (X_1 \cap X_2)$, then there exists a finite set $\Gamma \subseteq \kappa$, such that $\mathfrak{A} = \text{Fr}_\kappa \mathfrak{B}$. Arrange $\kappa \times \mathcal{G}g^3 X_1$ and $\kappa \times \mathcal{G}g^3 X_2$ into $\kappa$-termed sequences:

$(\langle k_i, x_i \rangle : i \in \kappa)$ and $(\langle l_i, y_i \rangle : i \in \kappa)$ respectively.

Since $\kappa$ is regular, we can define by recursion $\kappa$-termed sequences of witnesses:

$\langle u_i : i \in \kappa \rangle$ and $\langle v_i : i \in \kappa \rangle$

such that for all $i \in \kappa$ we have:

$u_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}$

and

$v_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}$.

For a Boolean algebra $\mathfrak{C}$ and $Y \subseteq \mathfrak{C}$, we write $f^{\mathfrak{C}} Y$ to denote the Boolean filter generated by $Y$ in $\mathfrak{C}$. Now let

$Y_1 = \{a\} \cup \{-c, x_i + s^k_{x_i} x_i : i \in \kappa\}$,

$Y_2 = \{-c\} \cup \{-c, y_i + s^k_{y_i} y_i : i \in \kappa\}$,

$H_1 = f^{\mathfrak{B}g^3 (X_1)} Y_1$, $H_2 = f^{\mathfrak{B}g^3 (X_2)} Y_2$,

$H = f^{\mathfrak{B}g^3 (X_1 \cap X_2)} [(H_1 \cap \mathcal{G}g^3 (X_1 \cap X_2)) \cup (H_2 \cap \mathcal{G}g^3 (X_1 \cap X_2))].$
Then $H$ is a proper filter of $\mathcal{S}g^\mathfrak{g}(X_1 \cap X_2)$ \cite{19}. Proving that $H$ is a proper filter of $\mathcal{S}g^\mathfrak{g}(X_1 \cap X_2)$, let $H^*$ be a (proper Boolean) ultrafilter of $\mathcal{S}g^\mathfrak{g}(X_1 \cap X_2)$ containing $H$. We obtain ultrafilters $F_1$ and $F_2$ of $\mathcal{S}g^\mathfrak{g}X_1$ and $\mathcal{S}g^\mathfrak{g}X_2$, respectively, such that

$$H^* \subseteq F_1, \quad H^* \subseteq F_2$$

and (**)

$$F_1 \cap \mathcal{S}g^\mathfrak{g}(X_1 \cap X_2) = H^* = F_2 \cap \mathcal{S}g^\mathfrak{g}(X_1 \cap X_2).$$

Now for all $x \in \mathcal{S}g^\mathfrak{g}(X_1 \cap X_2)$ we have

$$x \in F_1 \text{ if and only if } x \in F_2.$$  

Also from how we defined our ultrafilters, $F_i$ for $i \in \{1, 2\}$ are Henkin, that is, they satisfy the following condition:

(*) For all $k < \mu$, for all $x \in \mathcal{S}g^\mathfrak{g}X_1$ if $e_kx \in F_i$ then $e_lx$ is in $F_i$ for some $l \notin \Delta x$. We obtain ultrafilters $F_1$ and $F_2$ of $\mathcal{S}g^\mathfrak{g}X_1$ and $\mathcal{S}g^\mathfrak{g}X_2$, respectively, such that

$$H^* \subseteq F_1, \quad H^* \subseteq F_2$$

and (**)

$$F_1 \cap \mathcal{S}g^\mathfrak{g}(X_1 \cap X_2) = H^* = F_2 \cap \mathcal{S}g^\mathfrak{g}(X_1 \cap X_2).$$

Now for all $x \in \mathcal{S}g^\mathfrak{g}(X_1 \cap X_2)$ we have

$$x \in F_1 \text{ if and only if } x \in F_2.$$  

Fix $m \in \{1, 2\}$. the definition of the representations here slightly differs from the definition in \S \ref{3.3} for the equivalence relation $E$ is now defined on $\beta$ the dilated dimension, but this does not alter the proof that the maps to be defined using the hitherto constructed Henkin ultrafilters are homomorphisms. In more detail, let $V = \alpha \beta(id)$. $E$ denotes the equivalence relation on $\beta$ defined via $(i, j) \in E$ iff $d_{ij} \in F_m$. Now define for $\sigma, \tau \in V$, $\sigma \bar{E} \tau$ iff $d_{\sigma(i), \tau(i)} \in F_m$ for all $i \in \alpha$. Let $W = V/\bar{E}$. For $h \in W$, write $h = \bar{\tau}$ for $\tau \in V$ such that $\tau(j)/E = h(j)$ for all $j \in \alpha$. Define for $i < \alpha$, and $X \subseteq W = \alpha(\beta/\bar{E})id$ the $i$th interior operator

$$I_i(X) = \{ s \in W : s_i \in \{u/E \in \beta/E : s_u \in X\}\}.$$ 

Now define, as in lemma \S \ref{3.5} $f_m : \mathcal{S}g^\mathfrak{g}X_m \rightarrow (\wp(W), I_i)_{i<\alpha}$ by

$$f_m(a) = \{ \bar{\tau} \in W : s^\mathfrak{g}_{\tau \cup id} \in F_m \}.$$ 

It can be checked exactly as before that $f_m$ is a homomorphism.

Without loss of generality, we can assume that $X_1 \cup X_2 = X$. We have $f_1$ and $f_2$ agree on $X_1 \cap X_2$. So that $f_1 \cup f_2$ defines a function on $X_1 \cup X_2$. By
dimension restricted freeness, it follows that there is a homomorphism $f$ from $\mathfrak{A}$ to $(\wp(W), I_i)_{i<\alpha}$ such that $f_1 \cup f_2 \subseteq f$. Then $\tilde{I}d \in f(a) \cap f(-c) = f(a \cdot -c)$. This is so because $s_{\tilde{I}d}a = a \in F_1$ $s_{\tilde{I}d}(-c) = -c \in F_2$. But this contradicts the premise that $a \leq c$.

The representability of $TDc_\alpha$s can be discerned below the surface of the previous proof, so that the representability result in [16] is a special case. In more detail, we have:

**Corollary 3.8.** Every algebra $\mathfrak{A} \in TDc_\alpha$ is representable.

*Proof.* Let $\mathfrak{A}$ be given and $a \neq 0$ be in $\mathfrak{A}$. Let $\kappa$ be a regular cardinal $\geq \max\{|\alpha|, |A|\}$. Let $\mathfrak{B} \in TCA_\kappa$ be such that $\mathfrak{A} = \mathfrak{N}_r\mathfrak{B}$. Let $\langle (k_i, x_i) : i \in \kappa \rangle$ be an enumeration of $\kappa \times B$. Since $\kappa$ is regular, we can define by recursion a $\kappa$-termed sequence $\langle u_i : i \in \kappa \rangle$ such that for all $i \in \kappa$ we have: $u_i \in \kappa \sim (\Delta a \cup \bigcup_{j \leq i} \Delta x_j \cup \{u_j : j < i\})$. Let $Y = \{a\} \cup \{-c_{k_i}x_i + s_{k_i}^jx_i : i \in \kappa\}$. Let $H$ be the filter generated by $Y$; then $H$ is proper, take the maximal filter containing $H$ and $a$, and define $\psi(b) = \{r \in W : s_rb \in F\}$ where $b \in B$ and $W$ is as defined in the previous proof. Then $\psi(a) \neq 0$, and $\psi$ establishes the representability of $\mathfrak{B}$, hence of $\mathfrak{A}$. □

### 3.1 Omitting types

Now we prove an omitting types theorem for $TDc_\alpha$ and $TLf_\alpha$ when $\alpha$ is a countable infinite ordinal; also generalizing the result in [16] which addresses only topological locally finite algebras. An omitting types theorem for Chang modal logic is not proved in [18].

The proof adopted herein, we find is much simpler than the proof in [16]; and it resorts to the Baire category theorem for compact Hausdorff spaces as is often the case with ‘omitting types constructions’ though they are rarely presented this way.

The proof is similar to the proof of [59, Theorem 3.2.4] having at our disposal lemma 3.5. We omit the parts of the proof that overlap with those in [59]. But we still need some preparing to do.

Given $\mathfrak{A} \in TCA_\alpha$, $X \subseteq \mathfrak{A}$ is called a *finitary type*, if $X \subseteq \mathfrak{N}_r\mathfrak{A}$ for some $n \in \omega$. It is non-principal if $\prod X = 0$.

A representation of $\mathfrak{A} \in TDc_\alpha$ is a non-zero homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$ where $\mathfrak{B}$ is a weak set algebra. If $\mathfrak{A}$ is simple then $f$ is necessarily an isomorphism. $X \subseteq \mathfrak{A}$ is omitted by $f$ if $\bigcap_{x \in X} f(x) = \emptyset$, otherwise it is realized by $f$.

Let $covK$ be the least cardinal $\kappa$ such that the real line can be covered by $\kappa$ no-where dense sets. $covK$ is a cardinal closely related to the number of
omitting types and to independent set theoretic axioms like Martin’s axiom re-
stricted to countable Boolean algebras. It also has topological re-incarnations, 
closely related to the Baire category theorem, witness [59] for a discussion of 
properties of this cardinal.

Let $A$ be any Boolean algebra. The set of ultrafilters of $A$ is denoted by $\mathcal{U}(A)$. The Stone topology makes $\mathcal{U}(A)$ a compact Hausdorff space. We denote this space by $A^*$. Recall that the Stone topology has as its basic open sets the 
sets $\{N_x : x \in A\}$ where $N_x = \{F \in \mathcal{U}(A) : x \in F\}$.

Let $x \in A$, $Y \subseteq A$ and suppose that $x = \sum Y$. We say that an ultrafilter $F \in \mathcal{U}(A)$ preserves $Y$ iff $x \in F$ implies that $y \in F$ for some $y \in Y$.

Now let $A \in \mathcal{T}l_f \omega$. For each $i \in \omega$ and $x \in A$ let 
$\mathcal{U}_{i,x} = \{F \in \mathcal{U}(A) : F \text{ preserves } \{s^i_j x : j \in \omega\}\}$. 

Then 
$\mathcal{U}_{i,x} = \{F \in \mathcal{U}(A) : F \text{ preserves } \{s^i_j x : j \in \omega\}\} = \bigcap_{j<\omega} N_{s^i_j x}$. 

Let 
$\mathcal{H}(A) = \bigcap_{i \in \omega, x \in A} \mathcal{U}_{i,x}(A) \cap \bigcap_{i \neq j} N_{-d_{ij}}$. 

It is clear that $\mathcal{H}(A)$ is a $G_\delta$ set in $A^*$.

For $F \in \mathcal{U}(A)$, let 
rep$_F(x) = \{\tau \in {}^{<\omega}A : s^\alpha_x x \in F\}$, for all $x \in A$. Here for $\tau \in {}^{<\omega}A$, $s^\alpha_x x$ by definition is $s^\alpha_{\tau|\Delta x} x$. The latter is well 
defined because $|\Delta x| < \omega$.

When $a \in F$, then rep$_F$ is a representation of $A$ such that rep$_F(a) \neq 0$. Notice that here we do not have a notion of quotient involved here defined via 
the diagonal elements. Preservation of diagonal elements is guaranteed by the 
fact that $-d_{ij} \in F$. As before, it is easy to check that the cylindrifiers are 
preserved as well because the ultrafilter is Henkin.

The following theorem is due to Sági [42], establishing a one to one corre-
pondance between representations of locally finite cylindric algebras and 
Henkin ultrafilters. C$_{reg}^\omega$ denotes the class of regular set algebras; a a set 
algebra with top element $^\alpha U$ is such, if whenever $f, g \in {}^\alpha U$, $f \upharpoonright \Delta x = g \upharpoonright \Delta x$, and $f \in X$ then $g \in X$. This reflects the metalogical property that if two 
assignments agree on the free variables occuring in a formula then both satisfy 
the formula or none does.
Theorem 3.9. If \( F \in \mathcal{H}(\mathfrak{A}) \), then \( \text{rep}_F \) is a homomorphism from \( \mathfrak{A} \) onto an element of \( \mathcal{L}_\omega \cap \mathcal{C}_\omega^{\text{reg}} \) with base \( \omega \). Conversely, if \( h \) is a homomorphism from \( \mathfrak{A} \) onto an element of \( \mathcal{L}_\omega \cap \mathcal{C}_\omega^{\text{reg}} \) with base \( \omega \), then there is a unique \( F \in \mathcal{H}(\mathfrak{A}) \) such that \( h = \text{rep}_F \).

The next theorem is due to Shelah, and will be used to show that in certain cases uncountably many non-principal types can be omitted.

Theorem 3.10. Suppose that \( T \) is a theory, \( |T| = \lambda \), \( \lambda \) regular, then there exist models \( \mathfrak{M}_i : i < \lambda^2 \), each of cardinality \( \lambda \), such that if \( i(1) \neq i(2) < \chi \), \( \bar{a}_{i(l)} \in M_{i(l)} \), \( l = 1, 2 \), \( \text{tp}(\bar{a}_{i(1)}) = \text{tp}(\bar{a}_{i(2)}) \), then there are \( p_i \subseteq \text{tp}(\bar{a}_{i(i)}) \), \( |p_i| < \lambda \) and \( p_i \vdash \text{tp}(\bar{a}_{i(i)}) \) (\( \text{tp}(\bar{a}) \) denotes the complete type realized by the tuple \( \bar{a} \)).

Proof. \[65\] Theorem 5.16, Chapter IV. \( \square \)

We shall use the algebraic counterpart of the following corollary obtained by restricting Shelah’s theorem to the countable case:

Corollary 3.11. For any countable theory, there is a family of \( < \omega^2 \) countable models that overlap only on principal types.

Theorem 3.12. (1) Let \( \mathfrak{A} \in \mathcal{T}\text{Dc}_\omega \) be countable. Assume that \( \kappa < \text{cov}K \). Let \( (\Gamma_i : i \in \kappa) \) be a set of non-principal types in \( \mathfrak{A} \). Then there is a topological weak set algebra \( (\mathcal{B}, I_i)_{i<\omega} \), that is, \( \mathcal{B} \) has top element a weak space, and a homomorphism \( f : \mathfrak{A} \rightarrow (\mathcal{B}, I_i)_{i<\omega} \), such that for all \( i \in \kappa \), \( \bigcap_{x \in X_i} f(x) = \emptyset \), and \( f(a) \neq 0 \).

(2) If \( \mathfrak{A} \in \mathcal{T}\text{Lf}_\omega \), and \( (\Gamma_i : i \in \kappa) \) is a family of finitary non-principal types then there is a topological set algebra \( (\mathcal{B}, I_i)_{i<\omega} \), that is, \( \mathcal{B} \) has top element a cartesian square, and \( \mathcal{B} \in \mathcal{C}_\omega^{\text{reg}} \cap \mathcal{L}_\omega \) together with a homomorphism \( f : \mathfrak{A} \rightarrow (\mathcal{B}, I_i)_{i<\omega} \) such that \( \bigcap_{x \in X_i} f(x) = \emptyset \), and \( f(a) \neq 0 \).

If the types are maximal then \( \text{cov}K \) can be replaced by \( 2^\omega \), so that \( < 2^\omega \) types can be omitted.

Proof. (1) For the first part, we have by \[21\] 1.11.6 that

\[(\forall j < \alpha)(\forall x \in A) c_j x = \sum_{i \in \alpha \setminus \Delta x} s^j_i x.\]  \( (1) \)

Now let \( V \) be the weak space \( \omega^{(\text{Id})} = \{ s \in \omega^\omega : |\{ i \in \omega : s_i \neq i \} | < \omega \} \).

For each \( \tau \in V \) for each \( i \in \kappa \), let

\[ X_{i,\tau} = \{ s_x x : x \in X_i \}. \]
Here $s_\tau$ is the unary operation as defined in [21, 1.11.9]. For each $\tau \in V$, $s_\tau$ is a complete boolean endomorphism on $A$ by [21, 1.11.12(iii)]. It thus follows that

$$\forall \tau \in V \forall i \in \kappa \prod_{x \in A} X_{i,\tau} = 0 \quad (2)$$

Let $S$ be the Stone space of the Boolean part of $A$, and for $x \in A$, let $N_x$ denote the clopen set consisting of all Boolean ultrafilters that contain $x$. Then from $\prod_2$ it follows that for $x \in A$, $j < \beta$, $i < \kappa$ and $\tau \in V$, the sets

$$G_j, x = N_{c_j x} \setminus \bigcup_{i \notin \Delta x} N_{s_i^j x} \text{ and } H_{i, \tau} = \bigcap_{x \in X_i} N_{s_{\tau x}}$$

are closed nowhere dense sets in $S$. Also each $H_{i, \tau}$ is closed and nowhere dense. Let

$$G = \bigcup_{j \in \beta} \bigcup_{x \in B} G_{j, x} \text{ and } H = \bigcup_{i \in \kappa} \bigcup_{\tau \in V} H_{i, \tau}.$$

By properties of $cov K$, it can be shown $H$ is a countable collection of nowhere dense sets. By the Baire Category theorem for compact Hausdorff spaces, we get that $H(A) = S \sim H \cup G$ is dense in $S$. Accordingly, let $F$ be an ultrafilter in $N_n \cap X$. By the very choice of $F$, it follows that

$$\forall j < \beta \forall x \in B (c_{j, x} \in F \implies (\exists j \notin \Delta x) s_i^j x \in F.) \quad (3)$$

and

$$\forall i < \kappa \forall \tau \in V (\exists x \in X_i) s_{\tau x} \notin F. \quad (4)$$

Let $V = \omega^{\omega Id}$ and let $W$ be the quotient of $V$ as defined above. That is $W = V/E$ where $\tau E \sigma$ if $d_{\tau(i), \sigma(i)} \in F$ for all $i \in \omega$.

Define $f$ as before by

$$f(x) = \{ \bar{r} \in W : s_{\tau x} \in F \}, \text{ for } x \in A.$$ and the interior operators for each $i < \alpha$ by

$$J_i : \wp(W) \rightarrow \wp(W)$$

by

$$[x] \in J_i X \iff \exists U \in B(x_i/E \in U \subseteq \{ u/E \in \alpha/E : [x]_{u/E} \in X \}).$$

where $X \subseteq W$; here $W$ and $E$ are as defined in lemmas, 3.1. ?? and $B$ is the base for the topology on $U/E$ defined as in the proof of theorem 3.5. Then by lemma 3.5 $f$ is a homomorphism such that $f(a) \neq 0$ and it can be easily checked that $\bigcap_i f(X_i) = \emptyset$ for all $i \in \kappa$, hence the desired conclusion.
Definition 3.13. (1) A two theorems $A$ the existence of an atomic representation $A \rightarrow \text{a}$ as follows. Denote if $X$ its direct algebraic counterpart, there are principal. This implies that will be countable and simple, and we refer to an isomorphism $B$ family of countable locally finite set algebras, each with countable base, $X$ is necessarily principal. That is there exist a family of countable locally finite set algebras, each with countable base, call it $(\mathcal{B}_i : i < 2^\omega)$, and isomorphisms $f_i : \mathcal{A} \rightarrow \mathcal{B}_i$ such that if $X$ is an ultralimit in $\mathfrak{N} \mathcal{A}$, for which there exists distinct $k, l \in 2^\omega$ with $\bigcap f_i(X) \neq \emptyset$ and $\bigcap f_j(X) \neq \emptyset$, then $X$ is principal, so that from corollary 3.11 such representations overlap only on maximal principal types. By theorem 3.9 there exists a family $(F_i : i < 2^\omega)$ of Henkin ultrafilters such that $f_i = h_{F_i}$, and by theorem 3.10 we can assume that $h_{F_i}$ is a $\text{TCA}_\alpha$ isomorphism as follows. Denote $F_i$ by $G$. For $p \in \mathcal{A}$ and $i < \alpha$, let $O_{p,i} = \{ k \in \alpha : s^k_I(i) p \in G \}$ and let $B = \{ O_{p,i} : i < \alpha, p \in \mathcal{A} \}$. Then $B$ is the base for a topology on $\alpha$ and the concrete interior operations are defined for each $i < \alpha$ via $J_i : \varphi(\alpha \alpha) \rightarrow \varphi(\alpha \alpha)$

$$x \in J_i X \iff \exists U \in B(x_i \in U \subseteq \{ u \in \alpha : x_u^i \in X \})$$

where $X \subseteq W$.

Assume, for contradiction, that there is no representation (model) that omits $F$. Then for all $i < 2^\omega$, there exists $F$ such that $F$ is realized in $\mathcal{B}_{j_i}$. Let $\psi : 2^\omega \rightarrow \varphi(F)$, be defined by $\psi(i) = \{ F : F$ is realized in $\mathcal{B}_{j_i} \}$. Then for all $i < 2^\omega$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq k$, $\psi(i) \cap \psi(k) = \emptyset$, for if $F \in \psi(i) \cap \psi(k)$ then it will be realized in $\mathcal{B}_{j_i}$ and $\mathcal{B}_{j_k}$, and so it will be principal. This implies that $|F| = 2^\omega$ which is impossible.

From the omitting types theorem proved in theorem 3.10(2), one can infer the existence of an atomic representation of a neatly atomic $\text{TLF}_\omega$. In the coming two theorems $\mathcal{A}$ will be countable and simple, and we refer to an isomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$, $\mathcal{B}$ a set algebra as a representation of $\mathcal{A}$.

Definition 3.13. (1) $\mathcal{A} \in \text{TLF}_\omega$ is neatly atomic if $\mathfrak{N} \mathcal{A}$ is atomic for every $n \in \omega$. 37
(2) A representation \( f : \mathcal{A} \rightarrow \mathcal{B} \) with base \( U \) is an atomic representation of \( \mathcal{A} \) if \( \bigcup \{ f(x) : x \in \text{At} \mathfrak{U}_n \mathcal{A} \} = \omega U \) for every \( n \in \omega \).

**Theorem 3.14.** If \( \mathcal{A} \) is a simple neatly atomic countable \( \text{TLf}_\omega \), then \( \mathcal{A} \) has an atomic representation \( f : \mathcal{A} \rightarrow \mathcal{B} \). Furthermore, if \( (Y_i : i \in I) \) is a family of non-principal finitary types then \( \bigcap_{x \in Y_i} f(x) = \emptyset \).

**Proof.** The first part follows from the omitting types theorem by taking \( X_n \) to be the set of co-atoms of \( \mathfrak{U}_n \mathcal{A} \) and then finding a representation that omits those. For the second part. Let \( i \in I \). Assume that \( X_i \subseteq \mathfrak{U}_n \mathcal{A} \). Let \( Z_i = \{ -y : y \in Y_i \} \). Then \( \sum Z_i = 1 \). So for any atom \( x \in \mathfrak{U}_n \mathcal{A} \), we have \( x \cdot \sum Z_i = x \neq 0 \). Hence there exists \( z \in Z_i \), such that \( x \cdot z \neq 0 \). But \( x \) is an atom, hence \( x \cdot z = x \) and so \( x \leq z \). We have shown that for every atom \( x \in \mathfrak{U}_n \mathcal{A} \), there exists \( z \in Z_i \) such that \( x \leq z \). It follows immediately, that \( \omega U = \bigcup f(x) : x \in \text{At} \mathfrak{U}_n \mathcal{A} \} \leq \bigcup_{z \in Z_i} f(z) \), and so, \( \bigcap_{y \in Y_i} f(y) = \emptyset \), and we are done. \( \square \)

### 4 Notions of Representability

\( \text{TC}_{\alpha} \) denotes the class of set algebras and \( \text{TWs}_{\alpha} \) denotes the class of weak set algebra. Recall that \( \mathcal{A} \in \text{TC}_{\alpha} \) if it has top element \( \alpha U \), \( U \) carries a topology and the interior operators are defined as in definition 3.4, while \( \mathcal{A} \) is in \( \text{TWs}_{\alpha} \) if \( \mathcal{A} \) has unit a weak space \( \alpha U(p) \) and the interior operator also defined as in definition 3.4. For \( \alpha < \omega \), \( \text{TC}_{\alpha} = \text{TWs}_{\alpha} \).

We choose to define the class of representable algebras as follows (we will see that there are other possible equivalent definitions when \( \alpha \) is infinite, namely, to take set algebras with square units. For the finite dimensional case this is obviously equivalent).

**Definition 4.1.** \( \mathcal{A} \in \text{TCA}_{\alpha} \) is representable if it is isomorphic to a subdirect product of weak set algebras of dimension \( \alpha \).

In the next theorem to allow uniform treatment of the finite and infinite dimensional case, we always consider weak set algebras, which is the same as set algebras for the finite dimension case. In this case we have for any \( p \in \alpha U \), \( \alpha U = \alpha U(p) \). We also write for any ordinal \( \alpha, \alpha + \omega \) which is just \( \omega \) when \( \alpha \) is finite.

**Theorem 4.2.**

(1) For \( \alpha \geq \omega \) if \( \mathcal{A} \in \text{TDC}_{\alpha} \) and \( \mathfrak{M}_{\alpha} \mathcal{A} \) is representable, then \( \mathcal{A} \) is representable, too.

(2) For any ordinal \( \alpha \), \( \text{RTCA}_{\alpha} = \mathfrak{M}_{\alpha} \text{TCA}_{\alpha + \omega} \).

(3) For any pair of infinite ordinals \( \alpha < \beta \), \( \mathfrak{M}_{\alpha} \text{TCA}_{\beta} \) is a variety. In particular, \( \text{RTCA}_{\alpha} \) is a variety. Furthermore, \( \text{RTCA}_{\alpha} = \bigcap_{k \in \omega} \mathfrak{M}_{\alpha} \text{TCA}_{\alpha + k} \).

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(4) $\text{RTCA}_\alpha = \text{SPTWs}_\alpha$. In particular, $\text{SPTWs}_\alpha$ is closed under $H$.

(5) $\text{HSPTCs}_\alpha \subseteq \text{RTCA}_\alpha$.

(6) For finite $\alpha$, $\text{TRCA}_\alpha$ is a discriminator variety, that is not completely additive, hence is not conjugated.

(7) Assume that $\alpha$ is an infinite ordinal. Then for any class $K$, such that $L_f \alpha \subseteq K \subseteq \text{RTCA}_\alpha$, we have $\text{SUP}_K = \text{RTCA}_\alpha$. In particular, $\text{HSP}_K = \text{RTCA}_\alpha$.

(8) $\text{RTCA}_\alpha = \text{HSPTCs}_\alpha$.

(9) For $\alpha > 2$, $\text{RTCA}_\alpha$ cannot be axiomatized by a set of universal formulas containing only finitely many variables.

(10) For any pair of ordinals $1 < \alpha < \beta$ the class of neat reducts $\mathfrak{Nr}_\alpha\text{TCA}_\beta$ is not elementary.

Proof. For the first item. Assume for simplicity that $\mathfrak{A}$ is simple (has no proper congruences) and that $h : \mathfrak{N}_{\alpha}\mathfrak{A} \to \mathfrak{B}$ is an isomorphism into a weak set algebra $\mathfrak{B}$. The general case follows easily from this special case. Then theorem 3.9 provides a Henkin ultrafilter $F$ such that $h = \text{rep}_F$. The interior operators are then represented as in the proof of theorem 3.5. The same $h$ establishes the required isomorphism. For the other items the proofs for the $\text{CA}$ case lift without much difficulty [21, Theorems 2.6.32, 2.6.35, 2.6.52] and [19, 1] taking into account lemma 2.2. We give a sample. That for any pair of ordinals $\alpha < \beta$, $S\mathfrak{Nr}_\alpha\text{CA}_\beta$ is a variety is exactly like the $\text{CA}$ case. To show that $\text{RTCA}_\alpha \subseteq S\mathfrak{Nr}_\alpha\text{TCA}_\alpha + \omega$, it suffices to consider algebras in $\text{TWS}_\alpha$, since $S\mathfrak{Nr}_\alpha\text{CA}_\alpha + \omega$ is closed under $\text{SP}$. Let $\mathfrak{A} \in \text{TWS}_\alpha$ and assume that $\mathfrak{A}$ has top element $\alpha U(p)$. Let $\beta = \alpha + \omega$ and let $p^* \in \beta U$ be a fixed sequence such that $p^* \upharpoonright \alpha = p$. Let $\mathfrak{C}$ be the $\text{TCA}_\beta$ with top element $\beta U(p^*)$; cylindrifiers and diagonal elements are defined the usual way and the interior operators induced by the topology on $U$. Define $\psi : \mathfrak{A} \to \mathfrak{C}$ via

$$X \mapsto \{ s \in \beta U(p^*) : s \upharpoonright \alpha \in X \}. $$

Then by lemma 2.2, $\psi$ is a homomorphism, further it is injective, and as easily checked, $\psi$ is a neat embedding that is $\psi(\mathfrak{A}) \subseteq \mathfrak{Nr}_\alpha \mathfrak{C}$. Maybe the hardest part is to show that if $\mathfrak{A} \in S\mathfrak{Nr}_\alpha\text{CA}_\alpha + \omega$ then it is representable. But this follows from the fact that we can assume that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, where $\mathfrak{B} \in \text{Dc}_\alpha + \omega$, and then using theorem 3.8 baring in mind that a neat reduct of a representable algebra is representable.

For item (7), let $k$ be finite $> 1$. Take $\mathfrak{A}$ obtained by splitting an atom in a set algebra into $k + 1$ atoms as done in [1], expanded by interior operators
defined as identity operator. $\mathfrak{A}$ will have a non representable cylindric reduct but its $k$ generated subalgebras will be representable. Expand such representations by the identity functions and stimulate their representation using the discrete topology on the base.

Finally for non elementarity, in [49], for any ordinal $\alpha > 1$ two weak cylindric set set algebras having top element $V$ are constructed such that $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B} \notin \mathfrak{Nr}_\alpha \mathfrak{CA}_{\alpha + 1}$, $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{A} \in \mathfrak{Nr}_\alpha \mathfrak{CA}_{\alpha + \omega}$. Give the base of $\mathfrak{B}$ the discrete topology, then clearly the resulting expanded structure, call it $\mathfrak{B}^*$, by the induced (identity) interior operators is still not in $\mathfrak{Nr}_\alpha \mathfrak{TCA}_{\alpha + 1}$. Now we want to expand the algebra $\mathfrak{A}$ to an algebra in $\mathfrak{TCA}_\alpha$ in such a way to preserve elementary equivalence, so we do not have a choice but to expand it with the identity interior operations, call the resulting algebra $\mathfrak{A}^*$. But we also want an $\alpha + \omega$ dimensional topological cylindric algebra such that $\mathfrak{A}^*$ neatly embeds into $\mathfrak{D}$ and exhausts its $\alpha$ dimensional element, that is the neat embedding is onto $\mathfrak{Nr}_\alpha \mathfrak{D}$. As above we can assume that $\mathfrak{A} = \mathfrak{Nr}_\alpha \mathfrak{C}$, $\mathfrak{C} \in \mathfrak{TDC}_{\alpha + \omega}$. Give the base of any representation $\mathfrak{D}$ say, of $\mathfrak{C}$ the discrete topology forming $\mathfrak{D}^*$. Then $\mathfrak{A}^* = \mathfrak{Nr}_\alpha \mathfrak{D}^*$, $\mathfrak{B}^* \notin \mathfrak{Nr}_\alpha \mathfrak{TCA}_{\alpha + 1}$ and $\mathfrak{A}^* \equiv \mathfrak{B}^*$.

4.1 Rainbows, atom-canonicity

In this section we deal only with finite dimensional algebras. Throughout, unless otherwise explicity indicated, $n$ will be a finite ordinal $> 2$.

Notions like Dedekind-MacNeille completions, and atom-canonicity [26] are problematic for $\mathfrak{TRCA}_\alpha$ because the interior operators are not completely additive. However, in some cases the interior operators can turn out completey additive (e.g when they are equal to the identity map). If $\mathfrak{A}$ is such an atomic algebra, that is an algebra whose interior operators are completely additive, then the Dedekind-MacNeille completion exists and it is the complex algebra of its atom structure, in symbols $\mathcal{C}m\mathfrak{At} \mathfrak{A}$. This prompts the following definition:

**Definition 4.3.** A variety $V$ of Boolean algebras with operators is atom-canonical, if for every completely additive atomic $\mathfrak{A} \in V$, its Dedekind-MacNeille completion, namely, $\mathcal{C}m\mathfrak{At} \mathfrak{A}$ is also in $V$.

So using constructions for cylindric algebras one can construct such a representable algebra whose Dedekind-MacNeille completion is not representable, a task done by Hodkinson for cylindric algebras [29], but now we considerably sharpen Hodkinson’s result by passing to the $\mathfrak{Sc}$ reducts of certain topological cylindric algebras to be constructed. For $\mathfrak{A} = (A, +, \cdot, -, c_i, d_{ij}, I_i)_{i, j \in \alpha} \in \mathfrak{TCA}_\alpha$ its $\mathfrak{Sc}$ reduct denoted by $\mathfrak{Rd}_\mathfrak{sc} \mathfrak{A}$ is the algebra $\mathfrak{F} = (A, +, \cdot, -, c_i, s_{ij})_{i, j \in \alpha}$ where $s_{ij}x = c_j(x \cdot d_{ij})$ for $i \neq j$ and $s_{ii}x = x$. Such algebras are called Pinter’s substitution algebras; they are also diagonal-free reducts of $\mathfrak{CAs}$. Here reducts are taken in the generalized sense, so that the opeartions of the reduct are term definable in the expansion.
We emphasize that the next result cannot be obtained by lifting the relation algebra case [26, lemmas 17.32, 17.34, 17.35, 17.36] to cylindric algebras using Hodkinson’s construction in [30]. Hodkinson constructs from every atomic relation algebra an atomic cylindric algebra of dimension \( n \), for any \( n \geq 3 \), but the relation algebras does not embed into the \( \text{Ra} \) reduct of the constructed cylindric algebra when \( n \geq 6 \). If it did, then the \( \text{RA} \) result would lift as indeed is the case with \( n = 3 \). We instead start from scratch. We use a rainbow cylindric algebra.

In [27] the rainbow cylindric algebra of dimension \( n \) on a graph \( \Gamma \) is denoted by \( R(\Gamma) \). We consider \( R(\Gamma) \) to be in \( \text{TCA}_n \) be expanding its signature with \( n \) operators each interpreted as the identity map. In what follows we consider \( \Gamma \) to be the indices of the reds, and for a complete irreflexive graph \( \mathcal{G} \), by \( \text{TCA}_{\mathcal{G},\Gamma} \) we mean the rainbow topological cylindric algebra \( R(\Gamma) \) of dimension \( n \), where \( \mathcal{G} = \{ g_i : 1 \leq i < n - 1 \} \cup \{ g_0 : i \in \mathcal{G} \} \).

More generally, we consider a rainbow topological cylindric algebra based on relational structures \( A, B \), to be the rainbow algebra with signature the binary colours (binary relation symbols) \( \{ r_{ij} : i, j \in B \} \cup \{ w_i : i < n - 1 \} \cup \{ g_i : 1 \leq i < n - 1 \} \cup \{ g_0^i : i \in A \} \) and \( n - 1 \) shades of yellow (\( n - 1 \) ary relation symbols) \( \{ y_S : S \subseteq \omega A, \text{ or } S = A \} \).

We look at models of the rainbow theorem as coloured graphs [25]. This class is denoted by \( \text{CRG} \).

A coloured graph is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some \( n - 1 \) hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

**Definition 4.4.** Let \( i \in A \), and let \( M \) be a coloured graph consisting of \( n \) nodes \( x_0, \ldots, x_{n-2}, z \). We call \( M \) an \( i \)-cone if \( M(x_0, z) = g_0^i \) and for every \( 1 \leq j \leq n - 2 \), \( M(x_j, z) = g_j \), and no other edge of \( M \) is coloured green. \((x_0, \ldots, x_{n-2})\) is called the center of the cone, \( z \) the apex of the cone and \( i \) the tint of the cone.

The class of coloured graphs \( \text{CRG} \) are

- \( M \) is a complete graph.
- \( M \) contains no triangles (called forbidden triples) of the following types:

\[
(g, g', g^*), \quad (g_i, g_i, w_i), \quad \text{any } 1 \leq i < n - 1 \quad (5)
\]
\[
(gp_i, g_0^k, w_0) \quad \text{any } j, k \in A \quad (6)
\]
\[
(r_{ij}, r^*_{ij}, r^*_{ik}) \quad i, j, j', k', i^*, k^* \in B, \quad (7)
\]

unless \( i = i^*, j = j' \) and \( k' = k^* \) (8)

and no other triple of atoms is forbidden.
• If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge $(a_i, a_j) \ i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow.

• If $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an $i$ cone with apex $\delta$, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade $y_S$ then $i \in S$.

One then can define a polyadic equality atom structure of dimension $n$ from the class CRG. It is a rainbow atom structure. Rainbow atom structures are what Hirsch and Hodkinson call atom structures built from a class of models \cite{27}. Our models are, according to the original more traditional view \cite{25} coloured graphs. So let CRG be the class of coloured graphs as defined above. Let

$$\text{At} = \{a : n \to M, M \in \text{CRG} : a \text{ is surjective}\}.$$ 

We write $M_a$ for the element of At for which $a : n \to M$ is a surjection. Let $a, b \in \text{At}$ define the following equivalence relation: $a \sim b$ if and only if

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = M_b(b(i), b(j))$ whenever defined,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = M_b(b(k_0), \ldots, b(k_{n-2}))$ whenever defined.

Let $\text{At}$ be the set of equivalences classes. Then define

$$[a] \in E_{ij} \text{ iff } a(i) = a(j).$$

$$[a] T_i [b] \text{ iff } a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}.$$ 

This, as easily checked, defines a $\text{CA}_n$ atom structure. The complex algebra of this atom structure is denote by $\text{CA}_{A,B}$ where $A$ is the greens and $B$ is the reds. For interior operators define

$$[a] I_i [b] \text{ iff } a \sim b;$$

this defines an atom structure of a $\text{TCA}_n$, we denote the resulting complex algebra $\mathfrak{C}m\text{At}$ by $\text{TCA}_{A,B}$.

Consider the following two games on coloured graphs, each with $\omega$ rounds, and limited number of pebbles $m > n$. They are translations of $\omega$ atomic games played on atomic networks of a rainbow algebra using a limited number of nodes $m$. Both games offer $\forall$ only one move, namely, a cylindrifier move.

From the graph game perspective both games \cite{25} p.27-29 build a nested sequence $M_0 \subseteq M_1 \subseteq \ldots$ of coloured graphs.

First game $G^m$: $\forall$ picks a graph $M_0 \in \text{CRG}$ with $M_0 \subseteq m$ and $\exists$ makes no response to this move. In a subsequent round, let the last graph built be $M_i$. $\forall$ picks
• a graph $\Phi \in \text{CRG}$ with $|\Phi| = n$,
• a single node $k \in \Phi$,
• a coloured graph embedding $\theta : \Phi \setminus \{k\} \to M_i$. Let $F = \phi \setminus \{k\}$. Then $F$ is called a face. $\exists$ must respond by amalgamating $M_i$ and $\Phi$ with the embedding $\theta$. In other words she has to define a graph $M_{i+1} \in C$ and embeddings $\lambda : M_i \to M_{i+1}$ $\mu : \phi \to M_{i+1}$, such that $\lambda \circ \theta = \mu \restriction F$.

$F^m$ is like $G^m$, but $\forall$ is allowed to reuse nodes.

$F^m$ has an equivalent formulation on atomic networks of atomic algebras.

Let $\delta$ be a map. Then $\delta[i \to d]$ is defined as follows. $\delta[i \to d](x) = \delta(x)$ if $x \neq i$ and $\delta[i \to d](i) = d$. We write $\delta_i^j$ for $\delta[i \to \delta_j]$.

**Definition 4.5.** Let $2 < n < \omega$. Let $\mathcal{C}$ be an atomic $\text{CA}_n$. An atomic network over $\mathcal{C}$ is a map $N : {}^n \Delta \to \text{At}\mathcal{C}$, where $\Delta$ is a non-empty set called a set of nodes, such that the following hold for each $i,j < n$, $\delta \in {}^n \Delta$ and $d \in \Delta$:

- $N(\delta^j_i) \leq d_{ij}$
- $N(\delta[i \to d]) \leq c_i N(\delta)$

**Definition 4.6.** Let $2 \leq n < \omega$. For any $\text{Sc}_n$ atom structure $\alpha$ and $n < m \leq \omega$, we define a two-player game $F^m(\alpha)$, each with $\omega$ rounds.

Let $m \leq \omega$. In a play of $F^m(\alpha)$ the two players construct a sequence of networks $N_0, N_1, \ldots$ where $\text{nodes}(N_i)$ is a finite subset of $m = \{j : j < m\}$, for each $i$.

In the initial round of this game $\forall$ picks any atom $a \in \alpha$ and $\exists$ must play a finite network $N_0$ with $\text{nodes}(N_0) \subseteq m$, such that $N_0(\bar{d}) = a$ for some $\bar{d} \in {}^n \text{nodes}(N_0)$.

In a subsequent round of a play of $F^m(\alpha)$, $\forall$ can pick a previously played network $N$ an index $l < n$, a face $F = \langle f_0, \ldots, f_{n-2} \rangle \in {}^{n-2} \text{nodes}(N)$, $k \in m \sim \{f_0, \ldots, f_{n-2}\}$, and an atom $b \in \alpha$ such that

$$b \leq c_i N(f_0, \ldots, f_i, x, \ldots, f_{n-2}).$$

The choice of $x$ here is arbitrary, as the second part of the definition of an atomic network together with the fact that $c_i(c_i x) = c_i x$ ensures that the right hand side does not depend on $x$.

This move is called a cylindrifier move and is denoted

$$(N, \langle f_0, \ldots, f_{n-2} \rangle, k, b, l)$$
or simply by \((N, F, k, b, l)\). In order to make a legal response, \(\exists\) must play a network \(M \supseteq N\) such that \(M(f_0, \ldots, f_{i-1}, k, f_{i+1}, \ldots, f_{n-2}) = b\) and \(\text{nodes}(M) = \text{nodes}(N) \cup \{k\}\).

\(\exists\) wins \(F^m(\alpha)\) if she responds with a legal move in each of the \(\omega\) rounds. If she fails to make a legal response in any round then \(\forall\) wins.

In what follows by \(S_cK\), where \(K\) is a class having a Boolean reduct, we understand the class of complete subalgebras of \(K\), that is \(\mathfrak{A} \in S_cK\) if there exists \(\mathfrak{B} \in K\) such that \(\mathfrak{A} \subseteq \mathfrak{B}\) and for all \(X \subseteq \mathfrak{A}\) whenever \(\sum^\mathfrak{A} X = 1\), then \(\sum^\mathfrak{B} X = 1\).

**Theorem 4.7.** Let \(K\) be any class between \(S_c\) and \(CA\). Let \(n < m\), and let \(\mathfrak{A}\) be an atomic \(K_n\). If \(\mathfrak{A} \in S_cM_nK_m\), then \(\exists\) has a winning strategy in \(F^m(\mathfrak{A})\).

**Proof.** [24, Theorem 33]. Strictly speaking this theorem is proved for relation algebras, but the proof easily lifts to the \(CA\) case.

**Theorem 4.8.** For any finite \(n > 2\), any class \(K\) between \(S\mathfrak{M}_nTCA_{n+3}\) and \(TRCA_n\) is not atom-canonical.

**Proof.** We blow up and blur a finite rainbow cylindric algebra namely \(R(\Gamma)\) where \(\Gamma\) is the complete irreflexive graph \(n + 1\), and the greens are \(G = \{g_i : 1 \leq i < n - 1\} \cup \{g_i^0 : 1 \leq i \leq n + 1\}\), we denote this finite algebra endowed by the \(n\) identity interior operators by \(TCA_{n+1,n}\).

Let \(\mathfrak{A}\) be the rainbow atom structure similar to that in [29] except that we have \(n+1\) greens and only \(n\) indices for reds, so that the rainbow signature now consists of \(g_i : 1 \leq i < n - 1\), \(g_i^0 : 1 \leq i \leq n + 1\), \(w_i : i < n - 1\), \(r^t_{kl} : k < l \in n\), \(t \in \omega\), binary relations and \(y_S, S \subseteq n + 1\), \(n - 1\) ary relations.

We also have a shade of red \(\rho\); the latter is a binary relation but is outside the rainbow signature, though it is used to label coloured graphs during a certain game devised to prove representability of the term algebra [29], and in fact \(\exists\) can win the \(\omega\) rounded game and build the \(n\) homogeneous model \(M\) by using \(\rho\) whenever she is forced a red, as will be shown in a while.

So \(\mathfrak{A}\) is obtained from the rainbow atom structure of the algebra \(\mathfrak{A}\) defined in [29] section 4.2 starting p. 25] truncating the greens to be finite (exactly \(n+1\) greens). In [29] it shown that the complex algebra \(Cm\mathfrak{A}\) is not representable; the result obtained now, because the greens are finite but still outfit the red, is sharper; it will imply that \(Cm\mathfrak{A} \not\in S\mathfrak{M}_nTCA_{n+3}\).

The logics \(L^n, L^n_{\omega\omega}\) are taken in the rainbow signature (without \(\rho\)).

Now \(\mathfrak{M}\mathfrak{A} \in TRCA_n\); this can be proved like in [29]. Strictly speaking the cylindric reduct of \(\mathfrak{M}\mathfrak{A}\) can be proved representable like in [29]; giving, as usual, the base of the representation the discrete topology we get representability of the interior operators as well. The colours used for coloured graphs involved in building the finite atom structure of the algebra \(TCA_{n+1,n}\) are:
• greens: $g_i$ $(1 \leq i \leq n - 2)$, $g_0$, $1 \leq i \leq n + 1$,
• whites: $w_i : i \leq n - 2$,
• reds: $r_{ij} : i < j \in n$,
• shades of yellow: $y_S : S \subseteq n + 2$.

with forbidden triples

$$\begin{align*}
(g_i, g'_i, g^*) \quad & \text{any } 1 \leq i \leq n - 2 \\
(g_i^j, g_i^k, w_0) \quad & \text{any } 1 \leq j, k \leq n + 1 \\
(r_{ij}, r_{jk'}, r_{i'k'}) \quad & i,j,i',k',i^*,j^* \in n,
\end{align*}$$

unless $i = i^*$, $j = j'$ and $k' = k^*$.

and no other triple is forbidden.

A coloured graph is red if at least one of its edges is labelled red. For brevity write $r$ for $r_{ij}(j < k < n)$. If $\Gamma$ is a coloured graph using the colours in $\text{AtTCA}_{n+1,n}$, and $a : n \rightarrow \Gamma$ is in $\text{AtTCA}_{n+1,n}$, then $a' : n \rightarrow \Gamma'$ with $\Gamma' \in \text{CGR}$ is a copy of $a : n \rightarrow \Gamma$ if $|\Gamma| = |\Gamma'|$, all non red edges and $n - 1$ tuples have the same colour (whenever defined) and for all $i < j < n$, for every red $r$, if $(a(i), a(j)) \in r$, then there exits $l \in \omega$ such that $(a'(i), a'(j)) \in r'$. Here we implicitly require that for distinct $i, j, k < n$, if $(a(i), a(j)) \in r, (a(j), a(k)) \in r'$, $(a(i), a(k)) \in r''$, and $(a'(i), a'(j)) \in r'_1$, $(a'(j), a'(k)) \in [r']^{[i]}$ and $(a'(i), a'(k)) \in [r''^{[i]}], then l_1 = l_2 = l_3 = l$, say, so that $(r', [r']^{[i]}, [r''^{[i]}])$ is a consistent triangle in $\Gamma'$. If $a' : n \rightarrow \Gamma'$ and $\Gamma'$ is a red graph using the colours of the rainbow signature of $\text{At}$, whose reds are $\{r_{kj} : k < j < n, l \in \omega\}$, then there is a unique $a : n \rightarrow \Gamma$, $\Gamma$ a red graph using the red colours in the rainbow signature of $\text{TCA}_{n+1,n}$, namely, $\{r_{kj} : k < j < n\}$ such that $a'$ is a copy of $a$. We denote $a$ by $o(a')$, $o$ short for original; $a$ is the original of its copy $a'$.

For $i < n$, let $T_i$ be the accessibility relation corresponding to the $i$th cylindrifier in $\text{At}$. Let $T_i^s$, be that corresponding to the $i$th cylindrifier in $\text{TCA}_{n+1,n}$. Then if $c : n \rightarrow \Gamma$ and $d : n \rightarrow \Gamma'$ are surjective maps $\Gamma, \Gamma'$ are coloured graphs for $\text{TCA}_{n+1,n}$, that are not red, then for any $i < n$, we have

$$(\Gamma, \Delta) \in T_i \iff ([\Gamma], [\Delta]) \in T_i^s.$$
Now we deal with the last case, when the two graphs involved are red. Now assume that \( a' : n \to \Gamma \) is as above, that is \( \Gamma \in \text{CGR} \) is red, \( b : n \to \Gamma' \) and \( \Gamma' \) is red too, using the colours in the rainbow signature \( \mathcal{At} \).

Say that two maps \( a : n \to \Gamma, \ b : n \to \Gamma', \) with \( \Gamma \) and \( \Gamma' \in \text{CGR} \) having the same size are \( r \) related if all non red edges and \( n-1 \) tuples have the same colours (whenever defined), and for all every red \( r \), whenever \( i < j < n, \ l \in \omega, \) and \( (a(i), a(j)) \in i^r, \) then there exists \( k \in \omega \) such that \( (b(i), b(j)) \in r^k. \) Let \( i < n. \) Assume that \(( [o(a')], [o(b)] ) \in T^*_i \). Then there exists \( c : n \to \Gamma \) that is \( r \) related to \( a' \) such that \( [c]T_i[b]. \) Conversely, if \( [c]T_i[b], \) then \( [o(c)]T_i[o(b)] \).

Hence, by complete additivity of cylindrifiers, the map \( \Theta : \mathcal{At}(\text{TCA}_{n+1,n}) \to \mathcal{E}m\mathcal{At} \) defined via

\[
\Theta([a]) = \begin{cases} 
[a'] : a' \text{ copy of } a & \text{if } a \text{ is red}, \\
[a] & \text{otherwise}.
\end{cases}
\]

induces an embedding from \( \text{TCA}_{n+1,n} \) to \( \mathcal{E}m\mathcal{At} \), which we denote also by \( \Theta \).

We first check preservation of diagonal elements. If \( a' \) is a copy of \( a, \ i, j < n, \) and \( a(i) = a(j), \) then \( a'(i) = a'(j) \).

We next check cylindrifiers. We show that for all \( i < n \) and \( [a] \in \mathcal{At}(\text{TCA}_{n+1,n}) \) we have:

\[
\Theta(c_i[a]) = \bigcup \{ \Theta([b]) : [b] \in \mathcal{At}\text{TCA}_{n+1,n}, [b] \leq c_i[a] \} = c_i\Theta([a]).
\]

Let \( i < n. \) If \( [b] \in \mathcal{At}\text{TCA}_{n+1,n}, [b] \leq c_i[a], \) and \( b' : n \to \Gamma, \ \Gamma \in \text{CGR}, \) is a copy of \( b, \) then there exists \( a' : n \to \Gamma', \ \Gamma' \in \text{CGR}, \) a copy of \( a \) such that \( b' \upharpoonright n \setminus \{i\} = a' \upharpoonright n \setminus \{i\}. \) Thus \( \Theta([b]) \leq c_i\Theta([a]). \)

Conversely, if \( d : n \to \Gamma, \ \Gamma \in \text{CGR} \) and \( [d] \in c_i\Theta([a]), \) then there exist \( a' \) a copy of \( a \) such that \( d \upharpoonright n \setminus \{i\} = a' \upharpoonright n \setminus \{i\}. \) Hence \( o(d) \upharpoonright n \setminus \{i\} = a \upharpoonright n \setminus \{i\}, \) and so \( [d] \in \Theta(c_i[a]), \) and we are done.

But now we can show that \( \forall \) can win the game \( F^{n+3} \) on \( \mathcal{At}(\text{TCA}_{n+1,n}) \) in only \( n + 2 \) rounds as follows. Viewed as an Ehrenfeucht–Fraïssé forth game pebble game, with finitely many rounds and pairs of pebbles, played on the two complete irreflexive graphs \( n + 1 \) and \( n, \) in each round \( 0, 1 \ldots n, \) \( \forall \) places a new pebble on an element of \( n + 1. \) The edge relation in \( n \) is irreflexive so to avoid losing \( \exists \) must respond by placing the other pebble of the pair on an unused element of \( n. \) After \( n \) rounds there will be no such element, and she loses in the next round. Hence \( \forall \) can win the graph game on \( \mathcal{At}(\text{TCA}_{n+1,n}) \) in \( n + 2 \) rounds using \( n + 3 \) nodes.

In the game \( F^{n+3} \) \( \forall \) forces a win on a red clique using his excess of greens by bombarding \( \exists \) with \( \alpha \) cones having the same base \( (1 \leq \alpha \leq n + 2). \)

In his zeroth move, \( \forall \) plays a graph \( \Gamma \) with nodes \( 0, 1, \ldots, n-1 \) and such that \( \Gamma(i, j) = w_0(i < j < n - 1), \) \( \Gamma(i, n - 1) = g(i = 1, \ldots, n - 2), \) \( \Gamma(0, n - 1) = g_0, \) and \( \Gamma(0, 1, \ldots, n - 2) = y_{n+2}. \) This is a 0-cone with base \( \{0, \ldots, n - 2\}. \) In the
following moves, $\forall$ repeatedly chooses the face $(0, 1, \ldots, n-2)$ and demands a node $\alpha$ with $\Phi(i, \alpha) = g_i$, $(i = 1, \ldots, n-2)$ and $\Phi(0, \alpha) = g_0^\alpha$, in the graph notation – i.e., an $\alpha$-cone, without loss $n-1 < \alpha \leq n+1$, on the same base. $\exists$ among other things, has to colour all the edges connecting new nodes $\alpha, \beta$ created by $\forall$ as apexes of cones based on the face $(0, 1, \ldots, n-2)$, that is $\alpha, \beta \geq n-2$. By the rules of the game the only permissible colours would be red. Using this, $\forall$ can force a win in $n+2$ rounds, using $n+3$ nodes without needing to re-use them, thus forcing $\exists$ to deliver an inconsistent triple of reds.

Let $B = TCA_{n+1,n}$. Then $\mathcal{Rd}_{scB}$ is outside $SM\mathcal{N}_n\mathcal{S}c_{n+3}$ for if it was in $SM\mathcal{N}_n\mathcal{S}c_{n+3}$, then being finite it would be in $S_c\mathcal{N}_n\mathcal{S}c_{n+3}$ because $\mathcal{Rd}_{scB}$ is the same as its canonical extension $D$, say, and $D \in S_c\mathcal{N}_n\mathcal{S}c_{n+3}$. But then by theorem 4.7, $\exists$ would have won.

Hence $\mathcal{Rd}_{scCmAt} \notin SM\mathcal{N}_n\mathcal{S}c_{n+3}$, because $\mathcal{Rd}_{scB}$ is embeddable in it and $SM\mathcal{N}_n\mathcal{S}c_{n+3}$ is a variety; in particular, it is closed under forming subalgebras. It now readily follows that $\mathcal{Rd}_{scCmAt} \notin SM\mathcal{N}_n\mathcal{S}c_{n+3}$.

Finally $\mathcal{Rd}_{dfA}$ is not completely representable, because if it were then $A$, generated by elements whose dimension sets $< n$, as a $TCA_n$ would be completely representable and this induces a representation of its Dedekind-MacNeille completion $CmAtA$.

\[ \square \]

4.2 Non-finite axiomatizability

Now we deal with fine non-finite axiomatizability results. We use the construction in [26] together with the lifting argument used in [28] to prove a very strong non-finite axiomatizability results expressed by excluding finite schema. Of course $RTCA_\alpha$ cannot be finitely axiomatizable for the simple reason that its signature has infinitely many operations. This is the case with $TCA_\alpha$, too, but one cannot help but ‘sense’ that $TCA_\alpha$ is axiomatized by some finite schema; and indeed it is.

The axiomatization is finitary in a two sorted sense, one for ordinals $< \alpha$ and the other for the first order situation. This can be formulated in such a way that there is a strict finite set of equations in the signature of $TCA_\omega$ such that the axiomatization of $TCA_\alpha$, for any ordinal $\alpha \geq \omega$, consists of all $\alpha$ instances of such equations. Such a situation is best formulated in the context of systems of varieties definable by a Monk’s schema [22, Definitions, 5.6.11-5.6.12]. This is not the case for $RTCA_\alpha$, and each of its approximations $SM\mathcal{N}_\alpha TCA_{\alpha+k}$, $k \geq 2$, as we proceed to show.

For this purpose we show that for any ordinal $\alpha > 2$, for any $r \in \omega$, and for any $k \geq 1$, there exists $B^r \in SM\mathcal{N}_\alpha TCA_{\alpha+k} \sim SM\mathcal{N}_\alpha TCA_{\alpha+k+1}$ such that $\Pi_{r/\forall} B^r \in TRCA_\alpha$, for any non-principal ultrafilter on $\omega$. We will use quite sophisticated constructions of Hirsch and Hodkinson for relation and cylindric
algebras reported in [26].

Assume that $3 \leq m \leq n < \omega$. For $r \in \omega$, let $C_m = \mathcal{C}(H_{m+1}(\mathcal{A}(n, r), \omega))$ as defined in [26, definition 15.3]. We denote $C_m$ by $\mathcal{C}(m, n, r)$. Then the following hold:

**Lemma 4.9.**  
(1) For any $r \in \omega$ and $3 \leq m \leq n < \omega$, we have $\mathcal{C}(m, n, r) \in \mathcal{C}_r \mathcal{C}_m \mathcal{C}_n$. $\mathcal{C}(m, n, r) \notin \mathcal{S}_r \mathcal{C}_m \mathcal{C}_n$ and $\mathcal{C}(m, n, r) \in \mathcal{R}_m \mathcal{C}_n$. Furthermore, for any $k \in \omega$, $\mathcal{C}(m, m + k, r) \cong \mathcal{R}_m \mathcal{C}(m + k, m + k, r)$.

(2) If $3 \leq m < n$, $k \geq 1$ is finite, and $r \in \omega$, there exists $x_n \in \mathcal{C}(n, n + k, r)$ such that $\mathcal{C}(m, m + k, r) \cong \mathcal{R}_m \mathcal{C}(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for all $i, j < m$.

**Proof.**  
(1) Assume that $3 \leq m \leq n < \omega$, and let $\mathcal{C}(m, n, r) = \mathcal{C}(H_{m+1}(\mathcal{A}(n, r), \omega))$, be as defined in [26, Definition 15.4]. Here $\mathcal{A}(n, r)$ is a finite Monk-like relation algebra [26, Definition 15.2] which has an $n + 1$-wide $m$-dimensional hyperbasis $H_{m+1}(\mathcal{A}(n, r), \omega)$ consisting of all $n + 1$-wide $m$-dimensional wide $\omega$ hypernetworks [26, Definition 12.21]. For any $r$ and $3 \leq m \leq n < \omega$, we have $\mathcal{C}(m, n, r) \in \mathcal{C}_r \mathcal{C}_m \mathcal{C}_n$. Indeed, let $H = H_{m+1}(\mathcal{A}(n, r), \omega)$. Then $H$ is an $n + 1$-wide $n$ dimensional hyperbasis, so $\mathcal{A} H \in \mathcal{C}_n$. But, using the notation in [26, Definition 12.21 (5)], we have $H_{m+1}(\mathcal{A}(n, r), \omega) = H_{m+1}$. Thus $\mathcal{C}(m, n, r) = \mathcal{C}(H_{m+1}(\mathcal{A}(n, r), \omega)) = \mathcal{C}(H_{m+1}) = \mathcal{C}(m, n, r) \cong \mathcal{R}_m \mathcal{C}(n, n + k, r)$.

The second part is proved in [26, Corollary 15.10], and the third in [26, exercise 2, p. 484]. We consider $\mathcal{C}(m, n, r)$ expanded by $m$ unary operations, namely, each equal to the identity.

(2) Let $3 \leq m < n$. Take $x_n = \{ f : \leq n + k \to \mathcal{A}(n + k, r) \mid \omega : m \leq j < n \to \exists i < m, f(i, j) = 1d \}$. Then $x_n \in C(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for distinct $i, j < m$. Furthermore $I_n : \mathcal{C}(m, m + k, r) \cong \mathcal{R}_m \mathcal{C}(n, n + k, r)$, via the map, defined for $S \subseteq H_{m+1}(\mathcal{A}(m + k, r), \omega)$, by $I_n(S) = \{ f : \leq n + k + 1 \to \mathcal{A}(n + k, r) \cup \omega : f \mid \leq m + k + 1 \in S \}$, $\forall j(m \leq j < n \to \exists i < m, f(i, j) = 1d \}$. \qed
An analogous result to the coming theorem is proved for cylindric algebras in [61], with a precursor in [28]. The argument used in all three proofs is basically a lifting argument initiated by Monk [22, Theorem 3.2.67].

**Theorem 4.10.** Let \( \alpha > 2 \) be an ordinal. Then for any \( r \in \omega \), for any finite \( k \geq 1 \), for any \( l \geq k + 1 \) (possibly infinite), there exist \( \mathcal{B}^r \in \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k+1} \sim \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k} \) such \( \prod_{r \in \omega} \mathcal{B}^r \in \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+l} \). In particular, for any such \( k \) and \( l \), and for \( \alpha \) finite, \( \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+l} \) is not finitely axiomatizable over \( \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k} \), and for infinite \( \alpha \), \( \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+l} \) is not axiomatizable by a finite schema over \( \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k} \).

**Proof.** By lemma 4.9 we can assume that \( \alpha \) is infinite. We first show that \( \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k+1} \neq \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k} \); furthermore we construct infinitely many algebras that witness the strictness of the inclusion, one for each \( r \in \omega \). Their ultraproduct relative to any non-principal ultrafilter on \( \omega \) will be representable.

We use the algebras \( \mathfrak{E}(m, n, r) \) in theorem 4.9 in the signature of \( \text{TCA}_m \), by interpreting the interior operator \( I_i \) for each \( i < m \) as the identity function, so we still have \( \mathfrak{E}(m, m + k, r) \cong \mathfrak{sm}_{\alpha}\mathfrak{E}(m + k, m + k, r) \) for any \( k \in \omega \).

Fix \( r \in \omega \). Let \( I = \{ \Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega \} \). For each \( \Gamma \in I \), let \( M_\Gamma = \{ \Delta \in I : \Gamma \subseteq \Delta \} \), and let \( F \) be an ultrafilter on \( I \) such that \( \forall \Gamma \in I, M_\Gamma \in F \). For each \( \Gamma \in I \), let \( \rho_\Gamma \) be a one to one function from \( |\Gamma| \) onto \( \Gamma \).

Let \( \mathfrak{E}^r_\Gamma \) be an algebra similar to \( \text{TCA}_\alpha \) such that

\[
\mathfrak{E}^\rho_\Gamma \mathfrak{E}^r_\Gamma = \mathfrak{E}(|\Gamma|, |\Gamma| + k, r).
\]

Let

\[
\mathcal{B}^r = \prod_{r \in \omega} \mathfrak{E}^r_\Gamma.
\]

Then it can be proved like in [28] that

1. \( \mathcal{B}^r \in \mathfrak{Sm}_{\alpha} \text{TCA}_{\alpha+k+1} \),
2. \( \mathfrak{E}^\rho_\Gamma \mathcal{B}^r \not\in \mathfrak{Sm}_{\alpha} \text{CA}_{\alpha+k+1} \),
3. \( \Pi_{r \in \omega} \mathcal{B}^r \in \text{RTCA}_\alpha \).

For the first part, for each \( \Gamma \in I \) we know that \( \mathfrak{E}(|\Gamma| + k, |\Gamma| + k, r) \in \text{TCA}_{|\Gamma|+k} \) and \( \mathfrak{sm}_{|\Gamma|} \mathfrak{E}(|\Gamma| + k, |\Gamma| + k, r) \cong \mathfrak{E}(|\Gamma|, |\Gamma| + k, r) \). Let \( \sigma_\Gamma \) be a one to one function \( (|\Gamma| + k) \to (\alpha + k) \) such that \( \rho_\Gamma \subseteq \sigma_\Gamma \) and \( \sigma_\Gamma(|\Gamma| + i) = \alpha + i \) for every \( i < k \). Let \( \mathfrak{A}_\Gamma \) be an algebra similar to a \( \text{CA}_{\alpha+k} \) such that \( \mathfrak{E}^\rho_\Gamma \mathfrak{A}_\Gamma = \mathfrak{E}(|\Gamma| + k, |\Gamma| + k, r) \). Then, clearly \( \Pi_{r \in \omega} \mathfrak{A}_\Gamma \in \text{TCA}_{\alpha+k} \).

We prove that \( \mathcal{B}^r \subseteq \mathfrak{Sm}_{\alpha} \Pi_{r \in \omega} \mathfrak{A}_\Gamma \). Recall that \( \mathcal{B}^r = \Pi_{r \in \omega} \mathfrak{E}^r_\Gamma \) and note that \( \mathfrak{E}^r_\Gamma \subseteq A_\Gamma \) (the universe of \( \mathfrak{E}^r_\Gamma \) is \( C(|\Gamma|, |\Gamma| + k, r) \), the universe of \( \mathfrak{A}_\Gamma \) is
Thus (using a standard Los argument) we have: \( \Pi_{\Gamma/F} \mathcal{E}_r \cong \Pi_{\Gamma/F} \mathcal{N}_\Gamma \mathcal{A}_\Gamma = \mathcal{N}_\alpha \Pi_{\Gamma/F} \mathcal{A}_\Gamma \), proving \( \Pi \).

Now we prove \( \Pi \). For this assume, seeking a contradiction, that \( \mathcal{B}^r \in S\mathcal{N}_\alpha \text{TCA}_{\alpha+k+1} \), then \( \mathcal{N}_\alpha \mathcal{B}^r \subseteq \mathcal{N}_\Gamma \mathcal{E} \), where \( \mathcal{E} \in \text{CA}_{\alpha+k+1} \). Let \( 3 \leq m < \omega \) and \( \lambda : m + k + 1 \rightarrow \alpha + k + 1 \) be the function defined by \( \lambda(i) = i \) for \( i \leq m \) and \( \lambda(m + i) = \alpha + i \) for \( i < k + 1 \). Then \( \mathcal{N}_\alpha \mathcal{E} \in \text{CA}_{m+k+1} \) and \( \mathcal{N}_m \mathcal{N}_\alpha \mathcal{B}^r \subseteq \mathcal{N}_m \mathcal{N}_\alpha \mathcal{B}^r \).

For each \( \Gamma \in I \), let \( I_{\Gamma} \) be an isomorphism

\[
\mathcal{E}(m, m + k, r) \cong \mathcal{N}_{\lambda(\rho)} \mathcal{N}_m \mathcal{E}((\Gamma |, | \Gamma| + k , r)),
\]

Exists by item (2) of lemma \( \text{[4.9]} \). Let \( x = (x_{\Gamma} : \Gamma) / F \) and let \( \iota(b) = (I_{\Gamma} b : \Gamma) / F \) for \( b \in \mathcal{E}(m, m + k, r) \). Then \( \iota \) is an isomorphism from \( \mathcal{E}(m, m + k, r) \) into \( \mathcal{N}_\alpha \mathcal{N}_m \mathcal{B}^r \). Then by \( \text{[21]} \) theorem 2.6.38 we have \( \mathcal{N}_\alpha \mathcal{N}_m \mathcal{N}_\alpha \mathcal{B}^r \in S\mathcal{N}_m \text{CA}_{m+k+1} \). It follows that \( \mathcal{E}(m, m + k, r) \in S\mathcal{N}_m \text{CA}_{m+k+1} \) which is a contradiction and we are done.

Now we prove \( \Pi \) putting the superscript \( r \) to use. Recall that \( \mathcal{B}^r = \Pi_{\Gamma/F} \mathcal{E}^r \), where \( \mathcal{E}^r \) has the type of \( \text{TCA}_\alpha \) and \( \mathcal{N}_\alpha \mathcal{E}^r = \mathcal{E}(\Gamma, | \Gamma| + k , r) \). We know from item (1) of lemma \( \text{[4.9]} \) that \( \Pi_{\Gamma/F} \mathcal{N}_\alpha \mathcal{E}^r = \Pi_{\Gamma|F} \mathcal{E}(\Gamma, | \Gamma| + k , r) \subseteq \mathcal{N}_\alpha \mathcal{A} \mathcal{C}_\Gamma \), for some \( \mathcal{A} \mathcal{C}_\Gamma \in \text{TCA}_{\alpha+1} \).

Let \( \lambda_\Gamma : | \Gamma| + k + 1 \rightarrow \alpha + k + 1 \) extend \( \rho_\Gamma : | \Gamma| \rightarrow \Gamma \) (\( \leq \alpha \)) and satisfy

\[
\lambda_\Gamma(| \Gamma| + i) = \alpha + i
\]

for \( i < k + 1 \). Let \( k + 1 \leq \ell \leq \omega \). Let \( \mathcal{F}_\Gamma \) be a \( \text{TCA}_{\alpha+1} \) type algebra such that \( \mathcal{N}_\alpha \mathcal{F}_\Gamma = \mathcal{N}_\ell \mathcal{A}_\Gamma \). As before, \( \Pi_{\Gamma/F} \mathcal{F}_\Gamma \in \text{TCA}_{\alpha+1} \). And

\[
\Pi_{\Gamma/F} \mathcal{B}^r = \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathcal{E}^r
\]

\[
\cong \Pi_{\Gamma/F} \Pi_{\Gamma/F} \mathcal{E}^r
\]

\[
\subseteq \Pi_{\Gamma/F} \mathcal{N}_\Gamma \mathcal{A}_\Gamma
\]

\[
\cong \Pi_{\Gamma/F} \mathcal{N}_\Gamma \mathcal{A}_\Gamma
\]

\[
\subseteq \mathcal{N}_\alpha \Pi_{\Gamma/F} \mathcal{F}_\Gamma,
\]

Hence, we get that \( \Pi_{\Gamma/F} \mathcal{B}^r \in S\mathcal{N}_\alpha \text{TCA}_{\alpha+1} \) and we are done.
Now we show that for \( k \geq 1 \) and \( l \geq k + 1 \), there is no finite set of equations in the language of \( \text{TCA}_\omega \), such that its \( \alpha \) instances axiomatize \( \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + l \) over \( \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + k \).

By an \( \alpha \) instance of an equation in the signature of \( \text{TCA}_\omega \) is meant the following. If \( \rho : \omega \to \alpha \) is an injection, then \( \rho \) extends recursively to a function \( \rho^+ \) from \( \text{TCA}_\alpha \) terms to \( \text{TCA}_\alpha \) terms. On variables \( \rho^+(v_k) = v_k \), and for compound terms like \( c_k \tau \), where \( \tau \) is a \( \text{TCA}_\omega \) term, and \( k < \omega \), \( \rho^+(c_k \tau) = c_{\rho(k)} \rho^+(\tau) \). For an equation \( e \) of the form \( \sigma = \tau \) in the language of \( \text{TCA}_\omega \), \( \rho^+(e) \) is the equation \( \rho^+(\sigma) = \rho^+(\tau) \) in the language of \( \text{TCA}_\alpha \). This last equation, namely, \( \rho^+(e) \) is called an \( \alpha \) instance of \( e \) obtained by applying the injection \( \rho \).

Let \( k \geq 1 \) and \( l \geq k + 1 \). Assume for contradiction that \( \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + l \) is axiomatizable by a finite schema over \( \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + k \). We can assume without loss that there is only one equation in the signature of \( \text{TCA}_\omega \), such that all its \( \alpha \) instances, axiomatize \( \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + l \) over \( \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + k \). So let \( \sigma \) be such an equation and let \( E \) be its \( \alpha \) instances; so that for any \( A \in \mathcal{S} \mathcal{N} \mathcal{R}_\alpha \text{TCA} + k \) we have \( A \models E \) iff \( A |_{\sigma} = \sigma \).

For \( r \in \omega \), let \( v_r \in \alpha \), be an injection such that \( \mu_r(i) = v_r(i) \) for each index \( i \) appearing in \( \sigma \), and let \( A_r = \mathcal{R}^{v_r} \mathcal{B}^r \). Now \( \Pi_{r/\omega} A_r \models \sigma \). But then

\[
\{ r \in \omega : A_r \models \sigma \} = \{ r \in \omega : \mathcal{B}^r \models \sigma_r \} \in U,
\]

contradicting that \( \mathcal{B}^r \) does not model \( \sigma_r \) for all \( r \in \omega \).

\[\square\]

4.3 Decidability issues

**Theorem 4.11.** It is undecidable to tell whether a finite \( \text{TCA}_n \, n > 2 \) is representable or not.

**Proof.** Let \( n > 2 \). Assume that there is a desicion procedure to tell. Let \( \mathfrak{A} \) be a given simple finite \( \text{CA}_n \). Consider \( \mathfrak{A} \) as a \( \text{TCA}_n \) expanded with interior operators defined as the identity map, call it \( \mathfrak{A}^+ \). Then we can decide whether \( \mathfrak{A}^+ \) is representable or not as a \( \text{TCA}_n \). But \( \mathfrak{A}^+ \) is representable iff \( \mathfrak{A} = \mathfrak{R}^{d_a} \mathfrak{A}^+ \) is representable, hence we get a decision procedure that tells whether \( \mathfrak{A} \) as a \( \text{CA}_n \) is representable or not. This contradicts \( [30] \).

\[\square\]

The following corollary is a consequence of the above lemma and of the undecidability result that we have just proved, witness \([26, \text{corollary 18.16, theorem 18.27}]\) for similar results for relation algebras.

For a class \( K \) of algebras, the class \( K \cap \text{Fin} \) denotes the class of finite members of \( K \).
Corollary 4.12. Let $2 < n < \omega$. Then the following hold

(1) The set of isomorphism types of finite algebras in $\text{TCA}_n$ with only infinite representations is not recursively enumerable.

(2) The equational theory of $\text{RTCA}_n$ is undecidable.

(3) The equational theory of $\text{RTCA}_n \cap \text{Fin}$ is undecidable.

(4) The variety $\text{RTCA}_n$ is not finitely axiomatizable even in $m$th order logic.

References

[1] H. Andréka, Complexity of equations valid in algebras of relations. Ann Pure and App Logic 89(1997) p. 149-209.

[2] H. Andréka, M. Ferenczi and I. Németi (Editors), Cylindric-like Algebras and Algebraic Logic, Bolyai Society Mathematical Studies and Springer-Verlag, 22 (2012).

[3] H. Andréka, T. Gregely H. and I. Németi, On universal algebraic constructions of logics. Studia Logica, 36 (1977) p.9-47.

[4] H. Andréka, J.D. Monk., I. Németi.I. (editors) Algebraic Logic, North-Holland, Amsterdam, 1991.

[5] Andréka,H., Németi,I., A simple purely algebraic proof of the completeness of some first order logics. Algebra Universalis, 5 (1975) p.8-15.

[6] Andréka,H.,Nemeti,I., On Systems of varieties definable by schemes of equations. Algebra Universalis, 11 (1980) p. 105-116.

[7] Andréka, H., Németi, I., Sain, I Algebraic Logic (2000). In Handbook of Philosophical Logic, Editors Gabbay et all.

[8] H. Andréka, I. Németi and T. Sayed Ahmed, Omitting types for finite variable fragments and complete representations. Journal of Symbolic Logic 73 (2008) p. 65-89

[9] Blok,W.J., and Pigozzi,D. Algebraizable logics. Memoirs of American Mathematical Society, 77(396), 1989.

[10] Comer S.D. Classes without the amalgamation property Pacific journal of Mathematics, 28(2)(1969), p.309-318.
[11] Chang *Modal model theory*. Proceedings of the Cambridge Summer School in mathematical logic, Lecture Notes 337 (1974) p. 599-617.

[12] A. Daigneault, *Freedom in polyadic algebras and two theorems of Beth and Craig*. Michigan Math. J. 11(1963), p. 129-135.

[13] A. Daigneault and J.D. Monk, *Representation Theory for Polyadic algebras*. Fund. Math. 52(1963), p.151-176.

[14] G. Georgescu, *A representation theorem for tense polyadic algebras*. Mathematica, Tome 21 44 (2) (1979) p.131-138.

[15] G. Georgescu *Modal polyadic algebras*. Bull. Math Soc. Sci Math R, S Romaina (1979) 23 p.49-64

[16] G. Georgescu *Algebraic analysis of topological logic*. Mathematical Logic Quarterly (28) p.447-454 (1982) 52(5)(2006) p.44-49.

[17] G. Georgescu, *A representation theorem for polyadic Heyting algebras*. Algebra Universalis, 14 (1982) , p.197-209.

[18] G. Georgescu *Chang’s modal operators in Algebraic Logic*. Studia Logica 42(1), (1983) p.43-48

[19] P. Halmos, *Algebraic Logic*. Chelsea Publishing Co., New York, (1962.)

[20] L. Henkin, *An extension of the Craig-Lyndon interpolation theorem*. Journal of Symbolic Logic 28(3) (1963), p.201-216

[21] L. Henkin, J.D. Monk and A. Tarski, *Cylindric Algebras Part I*. North Holland, 1971.

[22] L. Henkin, J.D. Monk and A. Tarski, *Cylindric Algebras Part II*. North Holland, 1985.

[23] H. Herrlich, G. Strecker, *Category theory*. Allyn and Bacon, Inc, Boston (1973)

[24] R. Hirsch, *Relation algebra reducts of cylindric algebras and complete representations*, Journal of Symbolic Logic, 72(2) (2007), p.673-703.

[25] Hirsch R., Hodkinson, I. *Complete representations in algebraic logic* Journal of Symbolic Logic, 62, 3 (1997) 816-847

[26] R. Hirsch and I. Hodkinson, *Relation algebras by games*. Studies in Logic and the Foundations of Mathematics, volume 147 (2002)
[27] R. Hirsch and I. Hodkinson, *Completions and complete representations in algebraic logic*. In [2]

[28] R. Hirsch and T. Sayed Ahmed, *The neat embedding problem for algebras other than cylindric algebras and for infinite dimensions*. Journal of Symbolic Logic (in press).

[29] I. Hodkinson, *Atom structures of relation and cylindric algebras*. Annals of pure and applied logic, 89(1997), p.117-148.

[30] I. Hodkinson, *A construction of cylindric and polyadic algebras from atomic relation algebras*. Algebra Universalis, 68 (2012), p. 257-285.

[31] A.S. Kechris, *Classical Descriptive Set Theory*, Springer Verlag, New York (1995).

[32] M. Khaled and T. Sayed Ahmed, *Classes of Algebras not closed under completions*. Bulletin Section of Logic 38 (2009) p. 29-43.

[33] J. Madarász *Interpolation and Amalgamation; Pushing the Limits. Part I* Studia Logica, 61, (1998) p. 316-345.

[34] J. Madárasz and T. Sayed Ahmed, *Amalgamation, interpolation and epimorphisms*. Algebra Universalis 56 (2) (2007), p. 179 - 210,

[35] J. Madárasz and T. Sayed Ahmed, *Neat reducts and amalgamation in retrospect, a survey of results and some methods. Part 2: Results on amalgamation*. Logic Journal of IGPL 17, (2009), p.755-802.

[36] J. Madárasz and T. Sayed Ahmed *Amalgamation, interpolation and epimorphisms in algebraic logic*. In [2], p.91-104

[37] Makowski and Ziegler *Topological model theory with an interior operator*. Preprint

[38] L. Maksimova *Amalgamation and interpolation in normal modal logics*. Studia Logica 50(1991) p.457-471.

[39] J. Marowski and A. Marcia *Completeness theorem for modal model theory with Montague Chang semantics*. This Zeisachr 23 (1977) 97-104.

[40] G. Metcalfe, F. Montagna , and C. Tsinakis *Amalgamation and interpolation in ordered algebras*. (2012) pre-print.

[41] D. Pigozzi, *Amalgamation, congruence extension, and interpolation properties in algebras*. Algebra Universalis. 1(1971), p.269-349.

54
[42] G. Sági and D. Sziráki, Some Variants of Vaught’s Conjecture from the Perspective of Algebraic Logic, Logic Journal of the IGPL, published online January 5, 2012.

[43] S. Awodey and K. Kishida Topology and modality, the topology of first order models. Review of Symbolic Logic 1 (2008) p. 146-166.

[44] I. Sain Searching for a finitizable algebraization of first order logic. 8, Logic Journal of IGPL. Oxford University, Press (2000) no 4, p.495–589.

[45] I. Sain and R. Thompson, Strictly finite schema axiomatization of quasi-polyadic algebras, in Algebraic Logic H. Andréka, J. D. Monk and I. Németi (Editors), North Holland (1990) p.539-571.

[46] T. Sayed Ahmed, The class of neat reducts is not elementary. Logic Journal of IGPL, 9(2001)p. 593-628.

[47] T. Sayed Ahmed, A model-theoretic solution to a problem of Tarski. Math Logic Quarterly, Vol. 48 (2002), pp. 343-355.

[48] T. Sayed Ahmed, On Amalgamation of Reducts of Polyadic Algebras. Algebra Universalis 51 (2004), p.301-359.

[49] T. Sayed Ahmed, An interpolation theorem for first order logic with infinitary predicates. Logic journal of IGPL, 15(1) (2007), p.21-32.

[50] T. Sayed Ahmed, Weakly representable atom structures that are not strongly representable, with an application to first order logic, Mathematical Logic Quarterly, 54(3)(2008) p. 294-306.

[51] Sayed Ahmed, T., On finite axiomatizability of expansions of cylindric algebras. Journal of Algebra, Number Theory, Advances and Applications, 1(2010), p.19-40.

[52] T. Sayed Ahmed, The class of polyadic algebras has the super amalgamation property Mathematical Logic Quarterly 56(1)(2010), p.103-112

[53] T. Sayed Ahmed, Classes of algebras without the amalgamation property. Logic Journal of IGPL, 19 (2011), p. 87-104.

[54] T. Sayed Ahmed Representability and amalgamation in Heyting polyadic algebras, Studia Mathematica Hungarica, 48(4)(2011), p. 509-539.

[55] T. Sayed Ahmed, Amalgamation in universal algebraic logic. Stududia Mathematica Hungarica, 49 (1) (2012), p. 26-43.
[56] T. Sayed Ahmed, *Three interpolation theorems for typeless logics.* Logic Journal of *IGPL* 20(6) (2012), p.1001-1037.

[57] T. Sayed Ahmed, *Epimorphisms are not surjective even in simple algebras.* Logic Journal of *IGPL*, 20(1) (2012), p. 22-26.

[58] T. Sayed Ahmed, *Neat embeddings as adjoint situations.* Published on Line, Synthese. DOI 10.1007/s11229-013-0344-7.

[59] T. Sayed Ahmed *Completions, complete representations and omitting types.* In [2].

[60] T. Sayed Ahmed, *Neat reducts and neat embeddings in cylindric algebras.* In [2].

[61] T. Sayed Ahmed *Results on neat embeddings with applications to algebraizable extensions of first order logic.* Submitted.

[62] T. Sayed Ahmed *An algebraic approach to topological logic and Chang’s modal logic using cylindric algebras, Part 2* Amalgamation, interpolation and congruence extension properties in topological algebras.

[63] T. Sayed Ahmed *An algebraic approach to topological logic and Chang’s modal logic using cylindric algebras, Part 3* Some more algebra; finite dimensional topological cylindric algebras

[64] T. Sayed Ahmed *An algebraic approach to topological logic and Chang’s modal logic using cylindric algebras, Part 4* Logical consequences.

[65] S. Shelah, *Classification theory: and the number of non-isomorphic models* Studies in Logic and the Foundations of Mathematics. (1990).

[66] Simon *Non-representable algebras of relations.* Ph.D dissertation, Mathematical institute, of the Hungarian Academy of Sciences (1997).

[67] Sgro *The interior operator logic and product topologies.* Trans. Amer. Math. Soc. 258(1980) p. 99-112.

[68] Sgro *Completeness theorems for topological models.* Annals of Mathematical Logic (1977) p.173-193.

[69] Tarski and Mckinsey *The Algebra of topology* Annals of mathematics 45(1944) p. 141-191