The centripetal force law and the equation of motion for a particle on a curved hypersurface

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It is pointed out that the current form of extrinsic equation of motion for a particle constrained to remain on a hypersurface is in fact a half-finished version for it is established without regard to the fact that the particle can never depart from the geodesics on the surface. Once the fact be taken into consideration, the equation takes that same form as that for centripetal force law, provided that the symbols are re-interpreted so that the law is applicable for higher dimensions. The controversial issue of constructing operator forms of these equations is addressed, and our studies show the quantization of constrained system based on the extrinsic equation of motion is favorable.

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I. INTRODUCTION

The study of the centripetal force law (CFL) can be traced back at least to Newton in 1684 in his manuscript entitled "On the motion of bodies in an orbit", [1] which was crucial step toward his law of universal gravitation. In classical mechanics, it is the gravity that provides the centripetal force responsible for astronomical orbits. However, in modern physics, there is no gravity but the curved space-time, and the astronomical orbits are nothing but geodesics in it. Though there is no gravity but geodesics, is there any force law that bears resemblance to the CFL for an orbit on a curved space? If yes, is it any beneficial to resolve the problem associated with the quantization of the constrained system?

For the motion of a particle on a curved hypersurface is an exactly solvable model to examine various problems such as higher-dimensional gravity, [2] dark energy/matter problem, [3] and quantization of constrained motions, [4–6] etc., [7] we are familiar with both the geodesic equation from the intrinsically curved surface and the equation of motion from the extrinsically Euclidean space, [5, 6] but no relationship in between has been seriously explored. From the point of geometry, the geodesic equation can be independent from the extrinsic world, but the extrinsic equation of motion for a particle can never be independent from the geodesics because the particle can not move unless follows one of them.

For a non-relativistic, mass \( \mu \), particle that is constrained to remain on a hypersurface described by a constraint \( f(\mathbf{r}) = 0 \) in \( N \) dimensional Euclidean space \( E^N \) spanned by \( N \) mutually orthogonal unit vectors \( \mathbf{e}_i \) (\( i = 1, 2, ..., N \)), where \( f(\mathbf{r}) \) is some smooth function of position \( \mathbf{r} = x^i \mathbf{e}_i \) and the Einstein summation convention of sum over repeated indices is hereafter assumed, there are two equations of motion for the particle. One is well-known, given by the differential equation for a geodesic line \( C \) determined by,

\[
\frac{d^2 u^\mu(s)}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} = 0,
\]

where \( \{u^\mu\} (\mu = 1, 2, ..., N - 1) \) are the intrinsic \( N - 1 \) local coordinates, and \( s \) stands for the arc-length along \( C \), and \( \Gamma^\mu_{\alpha\beta} \) are Christoffel symbols of the second kind. Another is quite known, [5, 6] given by the differential of velocity \( \mathbf{v} \equiv d\mathbf{r}/dt \) with respect to time \( t \),

\[
\frac{d}{dt} \mathbf{v} = -\mathbf{n} (\mathbf{v} \cdot \nabla \mathbf{n} \cdot \mathbf{v}),
\]

where \( \mathbf{n} \equiv \nabla f(\mathbf{r})/|\nabla f(\mathbf{r})| \) is the local unit normal vector on the surface at point \( \mathbf{r} = (x^1\{u^\mu\}, x^2\{u^\mu\}, x^3\{u^\mu\}, ..., x^N\{u^\mu\}) \), and \( \nabla \equiv \mathbf{e}^i \partial/\partial x^i \) is the usual gradient operator. Though both equations [1] and [2] share a salient feature that neither contains the mass \( \mu \), the possible difference between them is...
more interesting and challenging. Some authors claim both to be identical but without justification. In order to obtain a proper form of Eq. (2) in quantum mechanics, it is usually assumed that the equation (2) has direct correspondence in quantum mechanics once the velocity is rewritten in terms of momentum $v = p/\mu$, with possible ordering distributions of the momentum $p$ and position-dependent functions $\nabla n$ and $n$, in Heisenberg picture,

$$\frac{d}{dt}p = -n \left( \frac{p \cdot \nabla n \cdot p}{\mu} \right).$$

However, on one hand, Eq. (1) is purely from intrinsic geometry, from which we know that the Dirac quantization of constrained systems cannot be fulfilled throughout. On the other hand, Eq. (3) contains mutually dependent components of the momentum $p$ because the motion lies on the tangential plane to the surface so that $n \cdot p = 0$, thus in quantum mechanics, we have inequivalent forms of (3). In other words, Eq. (1) under-describes the motion of the particle, which must be enlarged, while the Eq. (3) over-describes it, which must be used with some constraints. Therefore, a proper form of equation of motion for the particle in classical mechanics is worthy to be investigated. Whereas in quantum mechanics the meaning of Eq. (3) is under dispute, Ikegami from a larger sense concluded that from the constraint equation $f(r) = 0$ we could not build up a satisfactory theory and we must start from another constraint equation $df(r)/dt = 0$, but Weinberg thought that Eq. (3) should be as true as it is in classical mechanics. Though no explicit use of the Dirac formalism for a constrained system in present paper, our explorations are in fact within it because we further develop the results given by it.

In the following section II, the generalized form of the CFL which incorporates both Eqs. (2) and (3) is given. In section III, the quantization problem of the constrained motion on the surface is addressed. In final section IV, a brief conclusion and discussion is presented.

II. THE GENERALIZED CFL UNIFIES BOTH INTRINSIC AND EXTRINSIC EQUATION OF MOTION

For our purpose, let us first recall the celebrated CFL $a = v^2/r$ for the particle moves on a planar curve, especially on the 2D circle of radius $r$, and it can readily be rewritten in terms of the curvature $\kappa(= 1/r)$ and the Hamiltonian $H \equiv p^2/2\mu = \mu v^2/2$ for the free motion without any external force imposed,

$$\frac{d}{dt}p = -2H\kappa n.$$  

In fact, the Eq. (4) holds true in general provided that $\kappa$ symbolizes the first curvature of the geodesic $C$ on the hypersurface and Hamiltonian $H$ applies to the free particle on the surface.

From the differential geometry for the hypersurface, at the point $\{u^\alpha\}$ on the surface $f(r) = 0$, we can define the vectors of tangential space $r_\alpha(= dr/du^\alpha)$ and the unit normal vector $n$, and these vectors $\{r_\alpha, n\}$ form a complete set of the coordinates in the vicinity of the surface in the $E^N$, other than the fixed Cartesian one $\{e_j\}$. The first and second fundamental quantities are $g_{\alpha\beta} \equiv r_\alpha \cdot r_\beta$ and $b_{\alpha\beta} \equiv r_\alpha \cdot n = -r_\alpha \cdot n_\beta$, respectively. The equations of motion for $r_\alpha$ and $n$ are in the $E^N$, i.e.

$$\frac{\partial r_\alpha}{\partial u^\beta} = \Gamma^\nu_{\alpha\beta} r_\nu + b_{\alpha\beta} n,$$

$$\frac{\partial n}{\partial u^\alpha} = -b^\alpha_{\beta} r_\beta.$$  

Furthermore, from the differential geometry for curves $r(s)$ lying on the surface, we can define, respectively, the unit tangential $\alpha$ and its derivative with respect to $s$ in the following,

$$\alpha \equiv \frac{dr(s)}{ds} = \frac{\partial r}{\partial u^\alpha} \frac{du^\alpha}{ds} \equiv r_\alpha \frac{du^\alpha}{ds},$$

$$\frac{d^2 r}{ds^2} = \frac{d\alpha}{ds} = \frac{d}{ds} \left( r_\alpha \frac{du^\alpha}{ds} \right).$$

First, we limit the curve $r(s)$ to be the geodesic line $C$. On one hand, the first curvature $\kappa$ is defined by,

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First, we limit the curve $r(s)$ to be the geodesic line $C$. On one hand, the first curvature $\kappa$ is defined by,

$$\frac{d^2 r}{ds^2} = \kappa n.$$
where vector \( \mathbf{m} \) is a unit normal vector of the curve, which by the convention of the geometry is identical to \(-\mathbf{n}\) (Recall that any normal section of a surface is a geodesic [11]). On the other hand, the right-handed side of the Eq. (6b) becomes, from (5a) and (1),

\[
\frac{d}{ds} \left( r_\alpha \frac{du^\alpha}{ds} \right) = \frac{\partial r_\alpha}{\partial u^\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} + r_\alpha \frac{d^2 u^\alpha(s)}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} \frac{dr_\mu}{ds} \]

(8a)

\[
= (\Gamma^\mu_{\alpha\beta} r_\mu + b_{\alpha\beta} n) \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} - \Gamma^\mu_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} r_\mu \]

(8b)

\[
= b_{\alpha\beta} \frac{du^\alpha(s)}{ds} \frac{du^\beta(s)}{ds} n. \]

(8c)

Substituting (7) and (8c) into both sides of (6b), we get, with noting \( \mathbf{m} = -\mathbf{n} \), the relation between curvature \( \kappa \) and the second fundamental quantities \( b_{\alpha\beta} \),

\[
\kappa = -b_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}. \]

(9)

We see that the curvature \( \kappa \) of the curve is related to the second fundamental quantities, the extrinsic geometric ones, of the surface. In geometry, \(-\mathbf{m}n\) is a geometric invariant under parameter transformation \( \{u^\alpha\} \rightarrow \{u'^\alpha\} \).

Secondly, taking derivative of orthogonal relation \( \mathbf{n} \cdot \mathbf{r}_\gamma = 0 \) with respect to any local coordinate \( u^\alpha \) with noting \( \mathbf{n} = n(r(u)) \), we find,

\[
0 = \frac{\partial}{\partial u^\alpha} (\mathbf{n} \cdot \mathbf{r}_\gamma) = r_\alpha \cdot \nabla n \cdot \mathbf{r}_\gamma + n \frac{\partial}{\partial u^\alpha} r_\gamma = r_\alpha \cdot \nabla n \cdot \mathbf{r}_\gamma + b_{\alpha\gamma}, \text{i.e., } b_{\alpha\gamma} = -r_\alpha \cdot \nabla n \cdot \mathbf{r}_\gamma, \]

(10)

where Eq. (5a) is used in simplifying \( n \partial \mathbf{r}_\gamma/\partial u^\alpha \). Substituting \( b_{\alpha\beta} = -r_\alpha \cdot \nabla n \cdot \mathbf{r}_\beta \) into Eq. (4), we have another form of the curvature \( \kappa \),

\[
\kappa = -b_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = r_\beta \cdot \nabla n \cdot r_\alpha \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = \frac{dr}{ds} \cdot \nabla n \cdot \frac{dr}{ds}. \]

(11)

Thirdly, since \( \alpha = \frac{dr}{ds} \) is the unit tangential vector along the curve \( C \) and so is the ratio \( p/\mu = v/v \), the expression \( p \cdot \nabla n \cdot p/\mu \) can be written within the frameworks of the Hamiltonian mechanics and differential geometry, which is given by,

\[
\frac{p \cdot \nabla n \cdot p}{\mu} = \left( \frac{p \cdot \nabla n \cdot p}{p} \right) \frac{p^2}{\mu} = \left( \frac{dr}{ds} \cdot \nabla n \cdot \frac{dr}{ds} \right) \frac{p^2}{\mu} = 2\kappa H. \]

(12)

Substituting it into (3), we in final reach the Eq. (4). Thus, the CFL (4) really holds true universally.

For the particle is constrained on an arbitrary \( N - 1 \) dimensional space curve \( f(r(u(s))) = 0 \), the similar fashion gives the CFL (4) as well. Under the coordinate transformation: \( \{u^\alpha\} \rightarrow \{u'^\alpha\} \), two equations (1) and (4) are completely different for the former transforms accordingly whereas the latter keeps invariant. In other words, Eq. (1) is geometric invariant whereas the Eq. (4) is not, though both are covariant.

III. ON THE QUANTIZATION PROBLEM OF THE CONSTRAINED MOTION

Why did we use the frameworks of Hamiltonian mechanics and differential geometry to represent our result (4)? It not only formulates the seemingly different physics laws into a compact and unified form, but also sheds new light on the curvature-induced additional energy \( V_q \) in quantum mechanics. In quantum mechanics, the equation of motion (4) takes following commutator version,

\[
[p, H] = i\hbar (\kappa n H + H \kappa n). \]

(13)

Since \([x, H] = i\hbar p/\mu \) gives the geometric momentum \( p \), this relation (13) requires that quantum Hamiltonian include an additional term \( V_q \) such as \( H = p^2/2\mu + V_q \), otherwise the equation (13) would go violated. For instance, for an \( N - 1 \) dimensional sphere in \( E^N \), the additional energy is \( V_q = (N - 1)(N - 3)\hbar^2/8mr^2 \), which is exactly the geometric energy potential predicted by the confining potential technique (5), for the system under consideration. The additional energy has been confirmed by experiments (14, 15) and may play some roles in understanding of our present universe.
However, whether the Eq. (13) holds true in general is dubious. This is because Eq. (13) contains the first curvature $\kappa$ of the geodesic curve which depends on a single coordinate, which represents a classical orbit, and its operator version is hard to be true except the curvature is a constant. To reveal an even deeper difficulty, let us consider a particular situation that the parametrization is chosen so that the motion has unit speed (the so-called unit speed parametrization), the coordinate parameterizing the geodesic can be taken as the time in classical mechanics. But in quantum mechanics, the time remains a parameter without operator counterpart while the coordinate could be quantized. So, Eq. (13) can not take effect unless in classical limit.

Now let us turn to the Eq. (3). As we stressed in the first section, this equation over-describes the motion of the particle on the surface: The dependence of Eq. (3) on components of position and momentum is ambiguous, because we are free to choose not only independent coordinates but also momenta for we have two constrained conditions $f(r) = 0$ and $n \cdot p = 0$. It may not be a shortcoming, though. Instead, the over-description has a remarkable advantage that includes the results predicted by the confining potential technique. Thus, we see that, even from the point of operator algebra, a complete formulation of the quantization of the constrained motion is still an open problem, though our approach supports Weinberg whose point is the Eq. (3) holds true in quantum mechanics, and disfavors Ikegami et. al. whose point is that (3) identical to Eq. (1) and the satisfactory theory from constraint equation $f(r) = 0$ is not reachable with Dirac formalism for quantizing a constrained system.

IV. CONCLUSIONS AND DISCUSSIONS

The extrinsic equation of motion (3) for the particle on a hypersurface is only meaningful for the particle moves along the geodesic $C$, so in classical mechanics the current form (3) of the equations in literature is only a half-finished version. The final version of the equation takes a compact form that turns out to be that for CFL once it is generalized. Therefore, even there is no gravity but geodesics on a curved space, there is a force law that bears striking resemblance to the CFL. However, in quantum mechanics, the operator equation corresponding the CFL has a limited meaning, the equation corresponding (3) opens a wider door to establish a satisfactory quantum theory within the Dirac formalism for quantizing a constrained system.

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