Improved LayerWise Theory method application to stress analysis for composite tube in pure bending

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Abstract. An improved approach using LayerWise theory (LWT) is developed for the stress analysis in pure bending composite tubes. The variables in the circumferential direction are separated from the local displacement, so the eigenvalue techniques are avoided. Then the composite tube is discretized only in the radial direction, and the related displacement in the local displacement is assumed as the simple interpolation polynomial function. Based on the principle of minimum potential energy, the local nodal variables are solved to calculate the strains and stresses for the composite tube. Several composite tubes are analyzed to illustrate the proposed approach feasibility. Besides, the exact solution by unified connected parameter method is employed for comparison. They are in good agreement with each other.

1. Introduction
The composite tubes have proved to be very useful for many structures. As the main load-bearing components, their design requires high-precision stress calculations. However, the classic laminated theory cannot be satisfied due to the large arc curved surface while the numerical simulation with redefined finite element method is quite time-consuming. The methods in elastic theory have been developed to calculate the stressed state of composite tubes in pure bending. Lekhnitskii [1] has provided the solution for the composite tube with only one cylindrically orthotropic layer under bending moment. Jolicoeur and Cardou [2] and Zhang and Hoa [3] generalized this solution to the composite tube with several layers of any winding angles. However, these methods in elastic theory are very complicated. Besides, the particular technique has to be employed to deal with the singular parameters for the layers of some specific winding angles. Hoa et al [4] used the LayerWise theory method (LWT) [5] to calculate the cantilever tube. The global displacement along the axis of the composite tube is theoretical. The composite tube is discretized in the radial direction, and the related variables are assumed as the simple functions. However, the eigenvalue techniques have to be used to obtain the local displacement in the circumferential direction. Thai C H etc. [6] utilized LWT and increase human-made constraints to calculate stress and strain in composite laminates. Xing Y F etc. [7] extended the application of the LWT theory to the calculation of stability problems and dynamic problems of composite laminates. In this paper, a new LWT-based approach is developed to calculate the composite tube of pure bending. The displacement in the circumferential direction is separated from the local displacement. So, the composite tube is discretized only in the radial direction. Based on the principle of minimum potential energy, the local nodal displacements are solved to calculate the strains and stresses for the composite tube.
2. New approach with Layer Wise theory

2.1. Variable separation from the local displacements
It is convenient to consider the composite tube with the cylindrical coordinate system, and the related constitutive equation between strain and stress can be expressed as

\[ \varepsilon = S \sigma \]

in which

\[
\begin{bmatrix}
\dot{\varepsilon}_r \\
\dot{\varepsilon}_\theta \\
\dot{\varepsilon}_z \\
\gamma_{r\theta} \\
\gamma_{r\varphi} \\
\gamma_{\theta\varphi}
\end{bmatrix} = \begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\sigma_z \\
\tau_{r\theta} \\
\tau_{r\varphi} \\
\tau_{\theta\varphi}
\end{bmatrix} = S = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\
S_{12} & S_{22} & S_{23} & S_{24} & 0 & 0 \\
S_{13} & S_{23} & S_{33} & S_{34} & 0 & 0 \\
S_{14} & S_{24} & S_{34} & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & S_{56} \\
0 & 0 & 0 & 0 & S_{56} & S_{66}
\end{bmatrix}
\]

The reduced elastic parameters are also employed as [1]

\[ \beta_{ij} = S_{ij} - \frac{S_{ij} S_{ii}}{S_{jj}} \quad (3) \]

It is a generalized plane strain problem for the composite tubes of pure bending. Lekhnitskii defined it in [1] as follows:

\[ D = S_{13} \sigma_r + S_{23} \sigma_\theta + S_{33} \sigma_z + S_{34} \sigma_{\theta\varphi} \]

where \( D \) is a function of \( r \) and \( \theta \), and used this expression with (1) and (3), to rewrite the six equations of the strain tensor, which were then integrated. In case of moments \( M_x \neq 0 \) and \( M_y = 0 \), the following results are obtained for \( D \) and respective displacements [2]:

\[ D = \kappa_z r \sin \theta \]

and

\[ u_r = -\frac{z^2}{2} \kappa_z \sin \theta + \bar{U}(r, \theta) + u'_r \]
\[ u_\theta = -\frac{z^2}{2} \kappa_z \cos \theta + \bar{V}(r, \theta) + u'_\theta \]
\[ w = \kappa_z z \sin \theta + \bar{W}(r, \theta) + w' \]

In equation (6), \( \bar{U}, \bar{V}, \bar{W} \) are local displacements related to the warping of the cross-section, satisfying [2]

\[ \frac{\partial \bar{U}}{\partial r} = \beta_{12} \sigma_r + \beta_{13} \sigma_\theta + \beta_{14} \tau_{r\theta} + \frac{C_{13}}{C_{33}} D \]
\[ \frac{\partial \bar{V} - U}{r \partial \theta} = \frac{\bar{V}}{r} = \beta_{12} \sigma_r + \beta_{22} \sigma_\theta + \beta_{24} \tau_{r\theta} + \frac{C_{23}}{C_{33}} D \]
\[ \frac{\partial \bar{V} + \partial \bar{U}}{r \partial \theta} = \frac{\bar{V}}{r} = \beta_{56} \tau_{r\theta} + \beta_{66} \tau_{r\theta} \]
\[ \frac{\partial \bar{W}}{\partial r} = \beta_{56} \tau_{r\theta} + \beta_{66} \tau_{r\theta} \]
\[ \frac{\partial \bar{W}}{r \partial \theta} = \beta_{14} \sigma_r + \beta_{24} \sigma_\theta + \beta_{44} \tau_{r\theta} + \frac{C_{34}}{C_{33}} D \]
Since we only consider the pure bending of the composite tube, the tensile and torsion stresses are not included in equation (6). Two stress functions are used to express the strain-compatibility equations. According to their non-homogeneous terms, separating variables can be obtained for the two stress functions, and the further stresses can be expressed as the following form:

\[
\sigma_r = s_r(r) \sin \theta; \sigma_\theta = s_\theta(r) \sin \theta; \sigma_z = s_z(r) \sin \theta; \\
\tau_{r\theta} = s_{r\theta}(r) \sin \theta; \tau_{rz} = s_{rz}(r) \cos \theta; \tau_{r\theta} = s_{r\theta}(r) \cos \theta
\]  

(8)

Using the unified connected parameter method, the stresses in equation (8) can be derived in theory, and the displacements in equation (6) can be obtained finally. However, not only very complicated are the procedures, but also the particular technique should be employed to deal with some singular parameters. The LWT-based approach is a better candidate to calculate the composite tube. In the LWT-based approach, since some variables of the displacement are determined, it is only necessary to consider the other variables in the further discretization. We attempt to separate the variables in the circumferential direction from the local displacement in equation (6) to improve the LWT-based approach. Substituting equation (7) into equation (6) yields [2]:

\[
\frac{\partial U}{\partial r} = (\beta_{31} S_r + \beta_{32} S_\theta + \beta_{34} S_{r\theta} + \frac{C_{31}}{C_{33}} \kappa_r) r \sin \theta \\
\frac{\partial \tilde{V}}{r \partial \theta} + \frac{\tilde{U}}{r} = (\beta_{32} S_r + \beta_{33} S_\theta + \beta_{35} S_{r\theta} + \frac{C_{32}}{C_{33}} \kappa_r) r \sin \theta \\
\frac{\partial \tilde{V}}{\partial r} + \frac{\partial \tilde{U}}{r \partial \theta} - \frac{\tilde{V}}{r} = (\beta_{35} S_r + \beta_{36} S_{r\theta}) \cos \theta \\
\frac{\partial \tilde{W}}{\partial r} = (\beta_{34} S_r + \beta_{36} S_{r\theta}) \cos \theta \\
\frac{\partial \tilde{W}}{r \partial \theta} = (\beta_{31} S_r + \beta_{34} S_\theta + \beta_{35} S_{r\theta} + \frac{C_{34}}{C_{33}} \kappa_r) r \sin \theta
\]

(9)

According to the non-homogeneous terms in equation (9), one can obtain the following:

\[
\tilde{U} = U(r) \sin \theta \\
\tilde{V} = V(r) \cos \theta \\
\tilde{W} = W(r) \cos \theta
\]

(10)

So far, the variables of the displacement in the axial direction are determined via equation (6), and those in the circumferential direction are defined via equation (10). Therefore it is necessary to discretize the composite tube only in the radial direction, i.e., versus \( r \) value.

2.2. The numerical solution in the radial direction

Based on the separating variables for the displacement of pure bending composite tube, the LWT is employed for further analysis. The tube is discretized only in the radial direction into \((N-1)\) elements including \( N \) nodes. And the related variables are assumed as the simple interpolation polynomial function as follows:
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\[ U(r) = \sum_{i=1}^{2} U_{i}^e \phi_{i}^e \]

\[ V(r) = \sum_{i=1}^{2} V_{i}^e \phi_{i}^e \]

\[ W(r) = \sum_{i=1}^{2} W_{i}^e \phi_{i}^e \]

where \[ \phi_{i}^e = \frac{r_{i}^{e} - r}{r_{i}^{e} - r_{i}^{e}} \] (12)

Upon the substitution of equation (11) into equations (10) and (6), the strains can be calculated as follows:

\[ \varepsilon_r = e_r(r) \sin \theta; \varepsilon_\theta = e_\theta(r) \sin \theta; \varepsilon_z = e_z(r) \sin \theta \]

\[ \gamma_{r \theta} = e_{r \theta}(r) \sin \theta; \gamma_{r z} = e_{r z}(r) \cos \theta; \gamma_{\theta z} = e_{\theta z}(r) \cos \theta \]

where

\[ e_r = \sum_{i=1}^{2} U_{i}^e (\phi_i^e)'; e_\theta = \sum_{i=1}^{2} \left( \frac{\phi_i^e}{r} U_{i}^e - \frac{\phi_i^e}{r} V_{i}^e \right); e_z = \kappa_r r \]

\[ e_{r \theta} = -\sum_{i=1}^{2} \frac{\phi_i^e}{r} W_{i}^e; e_{r z} = \sum_{i=1}^{2} (\phi_i^e) W_{i}^e \]

\[ e_{\theta z} = \sum_{i=1}^{2} \left( \frac{\phi_i^e}{r} U_{i}^e - \frac{\phi_i^e}{r} V_{i}^e \right) + \sum_{i=1}^{2} (\phi_i^e) V_{i}^e \]

The relationship between the nodal displacements and strains can be derived from equation (14),

\[ e = B_e q_e \]

\[ s = C e = CB_e q_e \]

in which

\[ s = [s_r \ s_\theta \ s_z \ s_{r \theta} \ s_{r z} \ s_{\theta z}]^T \]

\[ C = S^{-1} \]

As is well known, the principle of minimum potential energy can be expressed as:

\[ \iiint_{\Omega_0} \delta E d\Omega + \delta P = 0 \]

whereas the strain energy can be calculated as follows.
\[ \int \int \int_{\Omega_0} \delta E d\Omega = \pi l \sum \int_{e} s^T \delta e rdr \]  
(21)

and the virtual work can be calculated as follows

\[ \delta P = M \frac{\partial}{\partial \zeta} (\delta u_r \sin \theta + \delta u_\theta \cos \theta) |_{\zeta} = -Ml \delta \kappa_x \]  
(22)

Substitution of equations (15) and (17) into equation (20) yields

\[ Kq = f \]  
(23)

where

\[ f = \begin{pmatrix} 0 & \ldots & 0 & Ml \end{pmatrix}_{1 \times (3N+1)}^T \]  
(24)

\[ q = \begin{pmatrix} U_1 & V_1 & W_1 & \ldots & U_N & V_N & W_N & \kappa_{x}^N \end{pmatrix}_{1 \times (3N+1)}^T \]  
(25)

while \( K \) is assessed via the element stiffness matrix as follows.

\[ K_e = \pi l \int_{e} B^T CB_e rdr \]  
(26)

Now, the composite tube is assembled after its discretization. Next, the nodal displacements in equation (23) can be derived to calculate the stresses and strains in the composite tube.

3. Examples of numerical calculation

The stress analysis of three composite tubes subjected to pure bending with bending moments \( M_x = 0.3 \text{kN}\cdot\text{m} \) and \( M_y = 0 \). The material parameters are listed in table 1. The proposed approach is adopted to solve this problem. The exact solution by unified connected parameter method for the same problem is also employed to verify it. The stress distributions are depicted in figures 1-3. The numerical results obtained by the proposed approach and the exact solution by the unified connected parameter method are in good agreement. So the current approach can be used to obtain a satisfactory solution for the pure bending of composite tubes.

| E1/GPa | E2/GPa | E3/GPa | G12/GPa | G13/GPa | G23/GPa | v12 | v13 | v23 |
|--------|--------|--------|---------|---------|---------|-----|-----|-----|
| 155    | 12.1   | 12.1   | 4.4     | 4.4     | 3.2     | 0.248| 0.248| 0.458|

Table 1. Composite material (Carbon/Epoxy) properties

Figure 1. Stresses in composite tube: [90°]
4. Conclusions
An improved LayerWise Theory-based approach is developed to calculate the stresses for composite tubes of pure bending. Firstly, the variables in the circumferential direction are separated from the local displacement. Secondly, the composite tube is only discretized in the radial direction, and the related variables in the local displacement are assumed as the simple interpolation polynomial function. Thirdly, based on the principle of minimum potential energy, the local nodal displacements are solved to calculate the strains and stresses for the composite tube. Several composite tubes are analyzed to illustrate the proposed approach. Besides, the exact solutions by unified connected parameter method are employed for comparison. They are in good agreement with each other.

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