ON THE FIRST EIGENVALUE OF INVARIANT KÄHLER METRICS

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Abstract. Given a simply connected compact generalized flag manifold \( M \) together with its invariant Kähler Einstein metric \( \bar{g} \), we investigate the functional given by the first eigenvalue of the Hodge Laplacian on \( C^\infty(M) \) restricted to the space of invariant Kähler metrics. We give sufficient and necessary conditions so that the metric \( \bar{g} \) is a critical point for this functional. Moreover we prove that when \( M \) is a full flag manifold, the metric \( \bar{g} \) is critical if and only if \( M = SU(3)/T^2 \) and in this case \( \bar{g} \) is a maximum.

1. Introduction

The first eigenvalue \( \lambda_1(g) \) of the Laplacian \( \Delta_g \) acting on smooth functions on a Riemannian manifold \( (M, g) \) is a very important, widely investigated geometrical object (see e.g.\[6\]). Given a compact manifold \( M \) and a class of Riemannian metrics \( \mathcal{R} \) on \( M \), it is a natural and interesting problem to investigate the extrema and boundedness of the functional \( \mathcal{R} \ni g \mapsto \lambda_1(g) \). It is known that the functional \( \lambda_1 \) is unbounded when \( \mathcal{R} \) is the space of metrics of fixed volume and \( \dim M \geq 3 \) (see \[9\]). In the Kähler setting, if a compact Kähler manifold \( (M, g) \) admits a holomorphic isometric embedding into some complex projective space, then the functional \( \lambda_1 \) is bounded on the space of Kähler metrics whose Kähler form belongs to the Kähler class \([\omega_g]\) (\[8\]). Similarly, when \( (M, g) \) is Hodge the functional \( \lambda_1 \) is bounded on the space of all Kähler metrics \( g' \) which are Kähler w.r.t. a suitable complex structure \( J' \) on \( M \) and whose Kähler form is \( \omega_g \) (see \[15\]).

More recently Biliotti and Ghigi (\[3\]) showed that on a compact Hermitian symmetric space \( N \) of type ABCD endowed with the standard invariant metric \( g_0 \) with \( \text{Ric}(g_0) = g_0 \) we have that \( \lambda_1(g) \leq 2 = \lambda_1(g_0) \), where \( g \) is any Kähler metric with Kähler class \([\omega_g] \in 2\pi c_1(M) \). When \( N \) is irreducible, this is equivalent to saying that the functional \( g \mapsto \lambda_1(g) \) restricted to the space of Kähler metrics with fixed volume attains its maximum at the Kähler Einstein metric. This last reformulation is consistent with the fact that on a compact homogeneous Riemannian manifold \( (M, g) \) with irreducible isotropy representation the invariant metric \( g \) is extremal for the functional \( \lambda_1 \) on the space of all Riemannian metrics \( g' \) on \( M \) with \( \text{vol}(M, g') = \text{vol}(M, g) \) (see \[10\]). An appropriate notion of extremality for the functional \( \lambda_1 \) has been defined in \[10\] and in this paper we will investigate it in the case of a compact simply connected homogeneous Kähler manifold \( (M, g, J) \) when \( \mathcal{R} \) is given by the set of all Kähler invariant metrics of fixed volume.

When \( (M, g, J) \) is a compact connected Kähler manifold which is homogeneous under the action of a compact semisimple Lie group \( G \), it is well known that \( M \) admits a unique invariant Kähler Einstein metric \( \bar{g} \) with \( \text{Ric}(\bar{g}) = \bar{g} \). It is also known that \( \lambda_1(\bar{g}) = 2 \) and that the corresponding eigenspace in \( C^\infty(M) \) can be described in terms of the Lie algebra \( \mathfrak{g} \) of \( G \). If we now consider the set \( \mathcal{K}_o \) of all \( G \)-invariant Kähler metrics \( g' \) on \( M \) with \( \text{vol}(M, g') = \text{vol}(M, \bar{g}) \), we can ask the question when \( \bar{g} \) is an extremal metric for the functional \( \lambda_1 \) on \( \mathcal{K}_o \).

In our first main result, Theorem \[3.3\] we will give a sufficient and necessary condition for the metric \( \bar{g} \) to be extremal in terms of invariant data depending only on the complex structure \( J \) of \( M \) and the Lie algebra \( \mathfrak{g} \), which is supposed to be simple and to coincide with the algebra of the
full isometry group of \( \bar{g} \) (this last condition can be assumed without loss of generality). We will also see that there exist flag manifolds whose Kähler Einstein metric is extremal.

Our second result focuses on the special cases of full flag manifolds, namely spaces of the form \( G/T \), where \( G \) is a compact simple Lie group and \( T \) is maximal torus of \( G \). These spaces admit precisely one invariant complex structure up to diffeomorphism and can also be described as the quotient space \( G^C/B \), where \( B \) is a Borel subgroup of the complexification \( G^C \) of \( G \). In this case we prove in Theorems 3.8 that the Kähler Einstein metric is extremal for \( \lambda_1 \) on \( \mathcal{K}_o \) if and only if \( G = SU(3) \); moreover in this case we also show that the \( \lambda_1|\mathcal{K}_o \) attains its maximum precisely at \( \bar{g} \).

In Section 2 we recall some standard facts about flag manifolds together with the definition and main properties of extremality for \( \lambda_1 \) and in Section 3 we prove our main results.

**Notation.** For a compact Lie group, we denote its Lie algebra by the corresponding lowercase gothic letter. If a group \( G \) acts on a manifold \( M \), for every \( X \in \mathfrak{g} \) we denote by \( X^* \) the corresponding vector field on \( M \) induced by the \( G \)-action.

## 2. Preliminaries

Given a compact manifold \( M \) together with the set \( \mathcal{R}_o \) of all Riemannian metrics on \( M \) with fixed volume, the functional \( \lambda_1: \mathcal{R}_o \rightarrow \mathbb{R} \) which assigns to each metric \( g \in \mathcal{R}_o \) the first eigenvalue of the Laplacian \( \Delta_g \) acting on \( C^\infty(M) \) is continuous (see [3]). If \( g_t \) is an analytic curve of metrics the function \( \lambda_1(g_t) \) is not differentiable but still has left and right derivatives in \( t \). Indeed given a metric \( \bar{g} \) whose first eigenvalue \( \lambda_1(\bar{g}) \) has multiplicity \( m \) and given any analytic curve \( g_t \) with \( t \in (-\varepsilon, \varepsilon) \) and \( g_0 = \bar{g} \), it is known (see [3]) that there exist \( L^2(g_t) \)-orthonormal functions \( u^1_1, \ldots, u^m_1 \in C^\infty(M) \) and \( \Lambda^1_1, \ldots, \Lambda^m_1 \in \mathbb{R} \) depending analytically on \( t \) such that for every \( j = 1, \ldots, m \) we have \( \Lambda^j_1 = \lambda_1(\bar{g}) \) and

\[
\Delta_g u^j_t = \Lambda^j_1 \cdot u^j_t \quad \forall \ t \in (-\varepsilon, \varepsilon).
\]

Then for \( t \) small enough, we have \( \lambda_1(g_t) = \min_{1 \leq i \leq m} \{ \Lambda^i_1 \} \) and

\[
\frac{d}{dt}|_{t=0} \lambda_1(g_t) = \min_{1 \leq i \leq m} \{ \frac{d}{dt}|_{t=0} \Lambda^i_1 \}, \quad \frac{d}{dt}|_{t=0} - \lambda_1(g_t) = \max_{1 \leq i \leq m} \{ \frac{d}{dt}|_{t=0} \Lambda^i_1 \}.
\]

On the other hand Berger ([3]) computed the derivative

\[
\frac{d}{dt}|_{t=0} \Lambda^i_1 = - \int_M \langle q(u_i), h \rangle \, d\mu_g,
\]

where \( h \) is the symmetric tensor given by \( \frac{d}{dt}|_{t=0} g_t, u_i := u^i_0 \) and for every \( u \in C^\infty(M) \)

\[
q(u) = du \otimes du + \frac{1}{4} \Delta_g u^2 \cdot \bar{g}.
\]

In [10] the following definition of extremality has been given

**Definition 2.1.** A metric \( \bar{g} \) is said to be \( \lambda_1 \)-extremal if for every analytic deformation \( g_t \) of \( \bar{g} \) in \( \mathcal{R}_o \) we have

\[
\frac{d}{dt}|_{t=0} \lambda_1(g_t) \leq 0 \leq \frac{d}{dt}|_{t=0} - \lambda_1(g_t).
\]

Therefore we have the following expressions

\[
\frac{d}{dt}|_{t=0} \lambda_1(g_t) = - \max_{1 \leq i \leq m} \{ \int_M \langle q(u_i), h \rangle \, d\mu_g \}, \quad \frac{d}{dt}|_{t=0} - \lambda_1(g_t) = - \min_{1 \leq i \leq m} \{ \int_M \langle q(u_i), h \rangle \, d\mu_g \}.
\]

We now focus on the case where the manifold \( M \) is a compact homogeneous Kähler manifold. We consider a compact connected semisimple Lie group \( G \) and a compact subgroup \( H \) which
coincides with the centralizer in $G$ of a torus. The homogeneous space $M = G/H$ is a generalized flag manifold and it can be equipped with invariant Kähler structures. We will now state some of the main properties of generalized flag manifolds, referring to [11] for a more detailed exposition.

We fix a maximal abelian subalgebra $t \subseteq \mathfrak{h}$ and the $B$-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $B$ denotes the Cartan-Killing form of $\mathfrak{g}$. The subspace $\mathfrak{m}$ can be naturally identified with the tangent space $T_oM$ where $o := [H] \in G/H$. If $R$ denotes the root system of $\mathfrak{g}^C$ relative to the Cartan subalgebra $t^C$, for every root $\alpha \in R$ the corresponding root space is given by $\mathfrak{g}_\alpha = \mathbb{C} \cdot E_\alpha$ and

$$\mathfrak{h}^C = t^C \bigoplus_{\alpha \in \mathfrak{r}_\mathfrak{h}} \mathfrak{g}_\alpha, \quad m^C = \bigoplus_{\alpha \in \mathfrak{r}_m} \mathfrak{g}_\alpha,$$

where $\mathfrak{r}_\mathfrak{h} \subset R$ is a closed subsystem of roots and $\mathfrak{r}_m := R \setminus \mathfrak{r}_\mathfrak{h}$. The roots in $\mathfrak{r}_\mathfrak{h}$ are characterized by the fact that they vanish on the center $c \subseteq t$. Observe that $(\mathfrak{r}_\mathfrak{h} + \mathfrak{r}_m) \cap R \subseteq \mathfrak{r}_m$.

Any $G$-invariant complex structure $J$ on $M$ induces an endomorphism $J \in \text{End} (\mathfrak{m})$ with $J^2 = -Id$. If we extend $J$ to $\mathfrak{m}^C$ and we decompose $\mathfrak{m}^C = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}$ into the sum of the $\pm i$ - eigenspaces of $J$, then the integrability of $J$ is equivalent to the fact that $q := \mathfrak{h}^C \oplus \mathfrak{m}^{1,0}$ is a subalgebra, actually a parabolic subalgebra of $\mathfrak{g}^C$. Moreover it can be shown that $G$-invariant structures are in bijective correspondence with the invariant orderings of $\mathfrak{r}_m$, namely subsets $\mathfrak{r}_m^+ \subset \mathfrak{r}_m$ such that $\mathfrak{r}_m$ is the disjoint union $\mathfrak{r}_m = \mathfrak{r}_m^+ \cup (-\mathfrak{r}_m^+)$ and:

$$(\mathfrak{r}_\mathfrak{h} + \mathfrak{r}_m^+) \cap R \subset \mathfrak{r}_m^+, \quad (\mathfrak{r}_m^+ + \mathfrak{r}_m^+) \cap R \subset \mathfrak{r}_m^+,$$

the correspondence being given by $\mathfrak{m}^{1,0} = \bigoplus_{\alpha \in \mathfrak{r}_m^0} \mathfrak{g}_\alpha$. Invariant orderings are then in one-to-one correspondence with Weyl chambers in the center $c$ of $\mathfrak{h}$, namely connected components of the set $c \setminus \bigcup_{\alpha \in \mathfrak{r}_m} \ker(\alpha|_c)$, and an invariant ordering in $\mathfrak{r}_m$ can be combined with an ordering in $\mathfrak{r}_\mathfrak{h}$ to provide a standard ordering in $\mathfrak{r}$.

If we fix an invariant complex structure $J$ on $M$ (hence a Weyl chamber $C$ in $c$), we can endow $M$ with many $G$-invariant Kähler metrics which are Hermitian w.r.t. $J$. Actually, it can be proved that $G$-invariant symplectic structures, namely $G$-invariant non-degenerate closed two-forms, are in one-to-one correspondence with elements in the Weyl chambers in $c$. Indeed, if $\omega \in \Lambda^2(\mathfrak{m})$ is a symplectic form, then there exists $\xi \in \text{some Weyl chamber in } c$ such that

$$\omega(X, Y) = B(\text{ad}_\xi X, Y), \quad X, Y \in \mathfrak{m}. \tag{2.3}$$

Moreover $\omega$ is the Kähler form of a Kähler metric $\rho$ w.r.t. the complex structure $J$ (i.e. $\rho := \omega(\cdot, J \cdot)$ defines a Kähler metric) if and only if $\xi \in C$.

The functional

$$\delta_m := \sum_{\alpha \in \mathfrak{r}_m^0} \alpha \tag{2.4}$$

plays a very important role. Indeed, its dual $\hat{\delta}_m = \sum_{\alpha \in \mathfrak{r}_m^0} H_\alpha \in t^C$, where $H_\alpha$ denotes the $B$-dual of the root $\alpha$, lies in $ic$ while the element $\eta_m := -i \delta_m$ belongs to $C$ and it represents (via the correspondence (2.3)) the Ricci form of every invariant metric which is Kähler w.r.t. the invariant complex structure $J$ (see e.g. [7], p. 627). It then follows that the element $\eta_m$ itself defines via (2.3) an invariant Kähler metric which is the unique invariant Kähler Einstein metric $\bar{g}$ with $Ric(\bar{g}) = \bar{g}$. It is a well known fact (see e.g. [12], p.96) that $\lambda_1(\bar{g}) = 2$ and that the relative eigenspace $E$ is isomorphic to the Lie algebra of all Killing fields on $(M, \bar{g})$ via the isomorphism

$$E_1 \ni f \mapsto J\text{grad}(f). \tag{2.5}$$
3. The main results

Keeping the same notations as in the previous section, we consider a compact homogeneous space \( M = G/H \), where \( G \) is a semisimple compact Lie group and \( H \) is the centralizer in \( G \) of a torus, endowed with an invariant complex structure \( J \). The set of all Kähler metrics (w.r.t. the complex structure \( J \)) is then parametrized by the points in the open Weyl chamber \( C \) in the center \( \mathfrak{c} \) of \( \mathfrak{h} \) corresponding to the complex structure \( J \). If we now consider the hypersurface \( K_o \subset C \) given by those points in \( C \) which corresponds to invariant Kähler metrics with the same volume as the Kähler Einstein metric \( \bar{g} \), we are interested in the functional \( \lambda_1^{IK} : K_o \to \mathbb{R} \). In particular we would like to study the question when the metric \( \bar{g} \) is \( \lambda_1^{IK} \)-extremal, namely

**Definition 3.1.** The Kähler Einstein metric \( \bar{g} \) is said to be \( \lambda_1^{IK} \)-extremal if for every analytic curve \( g_t \) in \( K_o \), \( t \in (-\epsilon, \epsilon) \), with \( g_0 = \bar{g} \), we have

\[
\frac{d}{dt}\bigg|_{t=0} \lambda_1(g_t) \leq 0 \leq \frac{d}{dt}\bigg|_{t=0} - \lambda_1(g_t).
\]

Given any analytic variation \( g_t \) as in the definition, we put \( h := \frac{d}{dt}|_{t=0} g_t \). The symmetric tensor \( h \) is also \( G \)-invariant and therefore the scalar product \( \langle h, \bar{g} \rangle \) is a constant on \( M \). Moreover, since \( \text{vol}(g_t) = \text{vol}(\bar{g}) \) we see that \( \int_M \langle h, \bar{g} \rangle \, d\mu_{\bar{g}} = 0 \), hence \( \langle h, \bar{g} \rangle = 0 \). It is clear that the set of all symmetric tensors which are tangent vectors to variations \( g_t \) can be identified with \( \mathfrak{c} \) and those which correspond to variations of constant volume with the hyperplane \( Y := \ker \text{Tr} \subset \mathfrak{c} \), where \( \text{Tr}(h) = \langle h, \bar{g} \rangle \).

If we now consider the quadratic form \( Q_h \) on the space \( E \) (see (2.1)) given by

\[
Q_h(u) = \int_M \langle du \otimes du + \frac{1}{4} \Delta_{\bar{g}}(u^2) \cdot \bar{g}, h \rangle \, d\mu_{\bar{g}},
\]

we see that it can be simplified to

\[
(3.6) \quad Q_h(u) = \int_M \langle du \otimes du, h \rangle \, d\mu_{\bar{g}}.
\]

We now recall that the space \( E \) is isomorphic to the Lie algebra of Killing vector fields on \((M, \bar{g})\).

Now if \( G = \text{loc} G_1 \times \ldots \times G_k \) is the decomposition of \( G \) into a product of simple factors, then \( H \) splits accordingly as \( H = \text{loc} H_1 \times \ldots \times H_k \) for \( H_i \subset G_i \) and \( M \) is biholomorphically isometric to the product of irreducible homogeneous spaces \( M = M_1 \times \ldots \times M_k \), \( M_i := G_i/H_i \), endowed with invariant complex structures and Kähler Einstein metrics \( \bar{g}_i \) for \( i = 1, \ldots, k \). The Lie algebra of holomorphic automorphisms \( \text{aut}(M, J) \) coincides with the complexification of the algebra \( \mathfrak{i}(M, \bar{g}) \) of isometries of \((M, \bar{g})\) and it splits as \( \bigoplus_{i=1}^k \text{aut}(M_i, J_i) \), where again \( \text{aut}(M_i, J_i) = \mathfrak{i}(M_i, \bar{g}_i)^C \). The inclusion \( \mathfrak{g}_i \subset \mathfrak{i}(M_i, \bar{g}_i) \) is proper in a few cases which are classified by Onishchik (see [14]) and also in these cases the algebra of infinitesimal isometries is simple. Therefore without loss of generality we can suppose that each factor \( \mathfrak{g}_i \) coincides with the algebra \( \mathfrak{i}(M_i, \bar{g}_i) \).

We first deal with the case where \( G \) is simple. We note that on \( E \) the \( G \)-invariant \( L^2(\bar{g}) \) inner product is given by \( \kappa \cdot B \) for a suitable non-zero constant \( \kappa \), so that the \( L^2 \)-orthonormal basis \( u_1, \ldots, u_m \) of \( E \) is also \( B \)-orthogonal. For \( h \in \mathfrak{c} \) the quadratic form \( Q_h \) on \( E \) given by

\[
Q_h(u) := \int_M \langle du \otimes du, h \rangle \, d\mu_{\bar{g}}
\]

is also \( G \)-invariant and therefore there exists \( \phi \in \mathfrak{c}^* \) such that

\[
(3.7) \quad Q_h = \phi(h) \cdot B.
\]
It then follows that
\[
\max_{1 \leq i \leq m} Q_h(u_i, u_i) = \min_{1 \leq i \leq m} Q_h(u_i, u_i) = \frac{1}{\kappa} \phi(h).
\]
By the definition (3.1) and (2.2) we get the following

**Lemma 3.2.** Let \( G \) be simple and suppose that \( G \) coincides (locally) with the group of isometries of \((M, \bar{g})\). Then the Kähler Einstein metric is \( \Lambda_1^{K1} \)-extremal if and only if the functional \( \phi \) vanishes identically on the hyperplane \( Y \), i.e. if and only if there exists a constant \( c \in \mathbb{R} \) so that
\[
(3.8) \quad \phi = c \cdot \text{Tr}.
\]

We now elaborate (3.8) in terms of algebraic data in \( \mathfrak{g} \). We consider the standard Weyl basis \( \{E_\alpha\}_{\alpha \in \mathbb{R}} \) of the root spaces \( \mathfrak{g}_\alpha \) (see e.g. [11], p. 421) and we construct a \( \bar{g} \)-orthonormal basis of \( \mathfrak{m} \) by considering for \( \alpha \in R_+^\mathfrak{m} \)
\[
v_\alpha := \frac{1}{\sqrt{2 B(\alpha, \delta_{\mathfrak{m}})}} (E_\alpha - E_{-\alpha}), \quad w_\alpha := Jv_\alpha = \frac{i}{\sqrt{2 B(\alpha, \delta_{\mathfrak{m}})}} (E_\alpha + E_{-\alpha}).
\]
Now if \( h \) is a \( G \)-invariant symmetric Hermitian \((0,2)\)-tensor given by \( h(v, v) = B([\xi_h, v], Jv) \) for every \( v \in \mathfrak{m} \) and for some \( \xi_h \in \mathfrak{c} \), we have that
\[
(3.9) \quad \text{Tr}(h) = 2 \sum_{\alpha \in R_+^\mathfrak{m}} h(v_\alpha, v_\alpha) = 2i \sum_{\alpha \in R_+^\mathfrak{m}} \frac{\alpha(\xi_h)}{B(\alpha, \delta_{\mathfrak{m}})}.
\]
In order to compute the functional \( \phi \) we consider a \((-B)\)-orthonormal basis \( \{v_j\}_{1 \leq j \leq N} \) of \( E \), where we recall that the map \( T : E \rightarrow \mathfrak{g} \) given by \( T(v) = J \text{grad} v \) is an isomorphism. We then consider
\[
\sum_{j=1}^N Q_h(v_j) = -N \cdot \phi(h),
\]
where \( N = \dim \mathfrak{g} \). We now observe that the symmetric bilinear form \( b := \sum_j dv_j \otimes dv_j \) is \( G \)-invariant and therefore the function \( \langle b, h \rangle \) is constant. Hence we get
\[
-N \cdot \phi(h) = \text{vol}(M, \bar{g}) \cdot \langle b, h \rangle.
\]
Now in order to compute the scalar product \( \langle b, h \rangle \), we fix a \( \bar{g} \)-orthonormal basis \( e_1, \ldots, e_n \) \((n = \dim_{\mathbb{R}} M)\) at \( T[\mathfrak{h}]M \) and compute
\[
\langle b, h \rangle = \sum_{i,j} (dv_j(e_i))^2 h(e_i, e_i) = \sum_{ij} h(\bar{g}(e_i, \text{grad} v_j)e_i, \bar{g}(e_i, \text{grad} v_j)e_i) = \sum_{j} h(J \text{grad} v_j, J \text{grad} v_j) = \sum_{j} h(T(v_j), T(v_j)),
\]
so that \( \langle b, h \rangle \) is simply given by the trace \( \text{Tr}_{-B}(h) \) of \( h \in S^2(\mathfrak{m}^*) \) w.r.t. the inner product \(-B\) on \( \mathfrak{m} \). Hence
\[
\langle b, h \rangle = \sum_{\alpha \in R_+^\mathfrak{m}} h(E_\alpha - E_{-\alpha}, E_\alpha - E_{-\alpha}) = \sum_{\alpha \in R_+^\mathfrak{m}} i B([\xi_h, E_\alpha - E_{-\alpha}], E_\alpha + E_{-\alpha}) = 2i \sum_{\alpha \in R_+^\mathfrak{m}} \alpha(\xi_h).
\]
Now condition (3.8) can be written as
\[
\sum_{\alpha \in R_+^\mathfrak{m}} \alpha = \mu \cdot \sum_{\alpha \in R_+^\mathfrak{m}} \frac{\alpha}{B(H_\alpha, \delta_{\mathfrak{m}})}
\]
for some real constant $\mu$. By contracting both members with $\delta_m$, we obtain
\[ ||\delta_m||^2 = \mu \cdot \dim \mathbb{C} M.\]
Therefore we have proved the following.

**Theorem 3.3.** Let $G$ be a compact and simple Lie group and let $M = G/H$ be a flag manifold endowed with a $G$-invariant complex structure $J$. Let $g$ be the Kähler Einstein with $\text{Ric}_g = g$ and suppose that $G$ coincides (locally) with the full isometry group of $\bar{g}$.

The metric $\bar{g}$ is $\lambda^K I$-extremal if and only if
\[ \alpha \in R^+ \quad \delta_m |_{c} = \frac{||\delta_m||^2}{\dim \mathbb{C} M} \sum_{\alpha \in R^+} \frac{\alpha|_{c}}{B(\alpha, \delta_m)} \]
where $c$ is the center of the Lie algebra $\mathfrak{h}$ of $H$.

**Remark 3.4.** The condition of $\lambda^K I$-extremality makes sense when $\dim \mathfrak{c} \geq 2$, hence when $b_2(M) \geq 2$.

**Remark 3.5.** Equation (3.10) provides a simple and computable condition to be tested in a flag manifold $M$. Actually it is an equation involving only the $T$ roots as we now explain. If $p : (it)^{\ast} \to (ic)^{\ast}$ denotes the restriction map, a $T$-root is by definition the image $p(\alpha)$ for some root $\alpha \in R$. The set $R_T := p(R)$ of $T$-roots is not a root system and it is in one-to-one correspondence with the set of irreducible $H$-submodules $m_j$ of $m^C$, $j = 1, \ldots, t$, where each $\rho \in R_T$ corresponds to the submodule $\sum_{\alpha \in R_m, p(\alpha) = \rho} g_{\alpha}$ (see e.g. [2, 16]). We denote by $R_T^+ = p(R^+) = p(R_m^+)$ the set of $T$-roots which are the image of positive roots, say $R_T^+ = \{\rho_1, \ldots, \rho_{\ell}\}$, where each $\rho_j$ corresponds to an irreducible submodule $m_j$ of complex dimension $m_j$, $j = 1, \ldots, \ell$. Then $t = 2\ell$ and $m^C = \bigoplus_{i=1}^{\ell} m_j \oplus \bar{m}_j$. Moreover, for each $\rho \in R_T^+$, the quantity $B(\alpha, \delta_m)$ with $p(\alpha) = \rho$ does not depend on $\alpha$ with $p(\alpha) = \rho$ and it is denoted by $B(\rho, \delta_m)$. Then condition (3.10) can be rewritten as
\[ \sum_{j=1}^{\ell} \left( \frac{\mu}{\beta_j} - 1 \right) m_j \rho_j = 0, \]
where $\beta_j := B(\rho_j, \delta_m)$ and $\mu = ||\delta_m||^2 / \dim \mathbb{C} M$.

**Example 3.6.** We consider the flag manifold $M = SU(3n)/S(U(n) \times U(n) \times U(n))$, $n \geq 1$, endowed with the complex structure $J$ corresponding to the standard positive root system $R^+ = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq 3n\}$ of $\mathfrak{su}(3n)$. We have that $R_m^+ = \{\epsilon_i - \epsilon_j | 1 \leq i \leq n, n + 1 \leq j \leq 3n\} \cup \{\epsilon_i - \epsilon_j | n + 1 \leq i \leq 2n, 2n + 1 \leq j \leq 3n\}$ and $\delta_m = 2n(\sum_{i=1}^{n} \epsilon_i - \sum_{i=2n+1}^{3n} \epsilon_i)$. There are precisely three positive $T$-roots and six irreducible $H$-irreducible submodules of $m^C$ of complex dimension $n^2$. We have $\rho_1 = \epsilon_1 - \epsilon_{n+1}$, $\rho_2 = \epsilon_1 - \epsilon_{2n+1}$ and $\rho_3 = \epsilon_{n+1} - \epsilon_{2n+1}$. If we normalize the Cartan Killing form of $\mathfrak{su}(3n)$ in such a way that $||\alpha||^2 = 2$ for every root $\alpha$, then $\beta_1 = \beta_3 = 2n$ and $\beta_2 = 4n$ and $\mu = \frac{8}{3}n$. Then condition (3.11) reads
\[ \frac{1}{3}(\rho_1 - \rho_2 + \rho_3) = 0, \]
so that Theorem 3.3 applies and the Kähler Einstein metric on $(M, J)$ is $\lambda^K I$-extremal.

On the other hand we will see in Theorem 3.8 that there are many flag manifolds whose Kähler Einstein metric is not $\lambda^K I$-extremal.
We now turn to the reducible case, namely when \( M = M_1 \times \ldots \times M_k \), where \( M_j = G_j/H_j \) and \( G_j \) are compact simple Lie groups for \( j = 1, \ldots, k \). The invariant complex structure \( J \) is the product of \( G_j \)-invariant complex structures \( J_j \) on \( M_j \) as well as the Kähler Einstein metric \( \bar{g} \) which is isometric to the product metric \( \bar{g}_1 \times \ldots \times \bar{g}_k \). We prove the following

**Theorem 3.7.** The Kähler Einstein metric \( \bar{g} \) on \( M = M_1 \times \ldots \times M_k \) is \( \lambda_1^{KI} \)-extremal if and only each \( \bar{g}_i \) is \( \lambda_1^{KI} \)-extremal for \( i = 1, \ldots, k \).

**Proof.** If we consider an analytic curve of \( G \)-invariant Kähler metric \( g_t \) with \( g_0 = \bar{g} \), then it splits as a product of invariant metrics \( g_t^{(i)} \) on \( M_i \) and \( h := \frac{d}{dt}|_{t=0} g_t \) also splits as \( h = h_1 \times \ldots \times h_k \) with \( h_i \) being \( J_i \)-Hermitian \( G_i \)-invariant symmetric tensors on \( M_i \) for \( i = 1, \ldots, k \). If \( \text{vol}(M, g_t) \) is constant, then \( \sum_i \text{Tr}_{\bar{g}_i}(h_i) = 0 \). The space \( E \cong \mathfrak{g} \) splits accordingly as \( E = \bigoplus_{i=1}^k E_i \) with \( E_i \cong \mathfrak{g}_i \) and \( \lambda_1(g_i) = \min_{1 \leq j \leq k} \{ \lambda_1(g_t^{(j)}) \} \).

It is clear that choosing a variation \( g_t \) in only one factor \( M_i \), the \( \lambda_1^{KI} \)-extremality of \( \bar{g} \) implies the one of \( \bar{g}_i \) for each \( i = 1, \ldots, k \).

Vice versa given an analytic deformation of \( \bar{g} \) as above, we consider the quadratic forms \( Q_{h_i}^{(i)} \) on \( E_i \) given by

\[
Q_{h_i}^{(i)}(u) = \int_{M_i} \langle du \otimes du, h_i \rangle \ d\mu_{\bar{g}_i} = \phi_i(h_i) \cdot B_i
\]

for some \( \phi_i \in \mathfrak{c}_i^* \), where \( B_i \) denotes the Cartan-Killing form on \( E_i \cong \mathfrak{g}_i \). We also consider the \( L^2(\bar{g}_i) \)-orthonormal sets \( \{ u_{1(i)}^{(i)}, \ldots, u_{N_i(i)}^{(i)} \} \subset E_i \) for \( i = 1, \ldots, k \) and \( N_i = \dim \mathfrak{g}_i \) corresponding to the variations \( g_t^{(i)} \) as explained in §2. Then by Theorem 3.3 we have \( \phi_i(h_i) = c_i \cdot \text{Tr}(h_i) \) and therefore

\[
\frac{d}{dt}|_{t=0^+} \lambda_1(g_t) = -\max_{1 \leq i \leq k} \{ Q_{h_i}^{(i)}(u_{1(i)}^{(i)}), \ldots, Q_{h_i}^{(i)}(u_{N_i(i)}^{(i)}) \} = -\max_{1 \leq i \leq k} \frac{c_i}{\kappa_i} \text{Tr}(h_i),
\]

where again we denote by \( \kappa_i \) the negative constant such that \( \kappa_i \cdot B_i \) is equal to the \( L^2(\bar{g}_i) \)-inner product on \( E_i \). Now, we note that \( \kappa_i < 0 \) and \( c_i > 0 \) for all \( i \). Since \( \sum_i \text{Tr}(h_i) = 0 \), there exists \( i \) so that \( \text{Tr}(h_i) \leq 0 \) and therefore \( \frac{d}{dt}|_{t=0^+} \lambda_1(g_t) \leq 0 \). Similarly we get \( \frac{d}{dt}|_{t=0^-} \lambda_1(g_t) \geq 0 \), hence \( \bar{g} \) is \( \lambda_1^{KI} \)-extremal. \( \square \)

### 3.1. The case of full flag manifolds.

In this subsection we focus on the case of a full flag manifold, namely a homogeneous space \( M = G/T \) where \( G \) is compact simple, \( T \) is a maximal torus in \( G \) and \( \dim T \geq 2 \). It is known (see e.g. [3]) that two invariant complex structures on \( M \) are biholomorphic and therefore we fix one invariant complex structure \( J \) with corresponding Weyl chamber \( C \). The main result is the following

**Theorem 3.8.** If \( G \) is a compact simple classical group, the \( G \)-invariant Kähler Einstein metric on \( (G/T, J) \) is \( \lambda_1^{KI} \)-extremal if and only \( G = SU(3) \).

Moreover when \( M = SU(3)/T^2 \), the eigenvalue functional \( \lambda_1 : \mathcal{K}_0 \to \mathbb{R} \) attains its maximum at \( \bar{g} \).

**Proof.** In order to prove the first assertion, we will go through the classical simple Lie algebras checking condition \( \langle \delta, \alpha \rangle = |\alpha| \|^2 \) whenever \( \alpha \) is a simple root (see e.g. [11]).

When \( \mathfrak{g} = \mathfrak{su}(n) \), we consider the standard set of roots \( R = \{ \epsilon_{i,j} := \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n \} \) where the positive roots are given by \( R^+ = \{ \epsilon_{i,j} \mid i < j \} \). We may also normalize the Cartan Killing form
by setting $||\alpha||^2 = 2$ for every root $\alpha$. It is immediate to compute $\delta = \sum_{j=1}^{n}(n + 1 - 2j)e_j$, so that

$$||\delta||^2 = \frac{1}{3}n(n - 1)(n + 1).$$

If condition (3.10) holds and $\bar{\alpha}$ is a simple root, then

(3.12)

$$2 = \frac{||\delta||^2}{\text{dim}_C M} \sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle} = \frac{2}{3}(n + 1) \sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle}.$$ 

If we select $\bar{\alpha} = \epsilon_{1,2}$ then the set $A := \{ \alpha \in R^+; \langle \alpha, \bar{\alpha} \rangle \neq 0 \}$ is given by $A = \{ \epsilon_{1,2}, \ldots, \epsilon_{1,n}, \epsilon_{2,3}, \ldots, \epsilon_{2,n} \}$ so that

$$\sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle} = 1 + \frac{1}{2} \sum_{j=2}^{n-1} j - \frac{1}{2} \sum_{j=1}^{n-2} j = \frac{n}{2(n-1)},$$

so that condition (3.12) reads $6(n - 1) = n(n + 1)$, i.e. $n = 2, 3$. On the other hand if $n = 3$ then $\delta = 2\epsilon_{1,3}$ and $\sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle} = \frac{3}{4} \epsilon_{1,3}$, so that condition (3.10) holds true.

When $g = so(2n + 1)$ ($n \geq 2$) we consider the set of simple roots $\{\omega_i - \omega_{i+1}, \omega_n, i = 1, \ldots, n - 1 \}$ and the set $R^+ = \{\omega_i \pm \omega_j, 1 \leq i < j \leq n \}$ for an orthonormal basis $\{\omega_i\}$ of $t$. We have $\delta = \sum_{i=1}^{n}(2n - 2i + 1)\omega_i$ and therefore $||\delta||^2 = \frac{1}{3}n(4n^2 - 1)$. If we consider the simple root $\bar{\alpha} = \omega_n$, the set of positive roots $A = \{ \alpha \in R^+; \langle \alpha, \bar{\alpha} \rangle \neq 0 \}$ is given by $A = \{ \omega_i \pm \omega_n; i < n \}$ and therefore condition (3.10) implies

$$\frac{\text{dim}_C M}{||\delta||^2} = \sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle} = \sum_{i=1}^{n-1} \frac{1}{2(n - i + 1)} - \sum_{i=1}^{n-1} \frac{1}{2(n - i)} + 1 = \frac{1}{2} \left( 1 + \frac{1}{n} \right).$$

Since $\text{dim}_C M = n^2$, we have $6n^2 = (4n^2 - 1)(1 + n)$, which has no integer solution $n \geq 2$.

When $g = sp(n)$ ($n \geq 3$), we fix a standard system of simple roots $\{\omega_1 - \omega_2, \ldots, \omega_{n-1} - \omega_n, 2\omega_n\}$ with system of positive roots given by $R^+ = \{\omega_i \pm \omega_j, 1 \leq i < j \leq n \}$. Then $\delta = 2\sum_{i=1}^{n}(n + 1 - i)\omega_i$ and $||\delta||^2 = \frac{2}{3}n(n + 1)(2n + 1)$. If we choose the simple root $\alpha = 2\omega_n$ the set of positive roots $A = \{ \alpha \in R^+; \langle \alpha, \bar{\alpha} \rangle \neq 0 \}$ is given by $A = \{ \omega_i \pm \omega_n; i < n \}$ and therefore condition (3.10) implies

$$4 \frac{\text{dim}_C M}{||\delta||^2} = \sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle} = \sum_{i=1}^{n-1} \frac{1}{n + 2 - i} - \sum_{i=1}^{n-1} \frac{1}{n - i} + 1 = \frac{1}{n + 1} + \frac{1}{n} - \frac{1}{2}.$$ 

Noting that $\text{dim}_C M = n^2$ and that the right hand side of the above equation is positive only for $n = 3$, we immediately see that we have no integer solution.

When $g = so(2n)$ ($n \geq 3$), we fix a standard system of simple roots $\{\omega_1 - \omega_2, \ldots, \omega_{n-1} - \omega_n, \omega_{n-1} + \omega_n\}$ with system of positive roots given by $R^+ = \{\omega_i \pm \omega_j, 1 \leq i < j \leq n \}$. Then $\delta = 2\sum_{i=1}^{n}(n - i)\omega_i$ and $||\delta||^2 = \frac{2}{3}n(n - 1)(2n - 1)$. If we choose the simple root $\alpha = \omega_1 - \omega_2$ the set of positive roots $A = \{ \alpha \in R^+; \langle \alpha, \bar{\alpha} \rangle \neq 0 \}$ is given by $A = \{ \omega_1 - \omega_2, \omega_1 \pm \omega_i, \omega_2 \pm \omega_i; 3 \leq i \leq n \}$ and therefore condition (3.10) implies

$$2 \frac{\text{dim}_C M}{||\delta||^2} = \sum_{\alpha \in R^+} \frac{\langle \alpha, \bar{\alpha} \rangle}{\langle \alpha, \delta \rangle} = 1 + \frac{1}{2} \left( \sum_{i=3}^{n} \frac{1}{2n - 1 - i} \right) - \frac{1}{2} \left( \sum_{i=3}^{n} \frac{1}{2n - 2 - i} \right) = \frac{1}{n - 1} - \frac{1}{4n - 10} + \frac{1}{2}.$$ 

Since $\text{dim}_C M = n^2 - n$, the previous equation is equivalent to $3(n - 1)(2n - 5) = (2n - 1)(n^2 - n - 3)$, which has no solution for $n \geq 3$. 

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In order to prove the last assertion regarding $M = SU(3)/T^2$, we fix some notations. We consider the abelian subalgebra $t = \{\text{diag}(ia, ib, -i(a + b)); \ a, b \in \mathbb{R}\} \cong \mathbb{R}^2$ and a standard basis of the complement $m = t^\perp$ w.r.t. the Cartan Killing form $B$, which is given by $B(X, Y) = 6 \text{Tr}(XY)$:

$$X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad Y_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$  (3.13)

Given the set of simple roots $\{\epsilon_{1,2}, \epsilon_{2,3}\}$, we fix the Weyl chamber $C = \{(a, b) \in \mathbb{R}^2 \cong t; -\frac{5}{2} < b < a\}$. It corresponds to the invariant complex structure $J$ such that $JX_i = Y_i, i = 1, 2, 3$. The element $\delta = 2 \epsilon_{1,3}$ has a $B$-dual given by $\xi = \frac{1}{2} \text{diag}(i, 0, -i)$, i.e. the point $(\frac{i}{2}, 0) \in \mathbb{R}^2$.

Any $\xi \in C$ determines an invariant Kähler metric $g_\xi$ according to (2.3). The $\text{Ad}(T^2)$-submodules $m_i := \text{Span}\{X_i, Y_i\}$ are mutually inequivalent and therefore they are $g_\xi$-orthogonal. Moreover we easily get $g_\xi(X_i, X_i) = g_\xi(Y_i, Y_i) (i = 1, 2, 3)$ and $g_\xi(X_1, X_1) = 12(a - b); g_\xi(X_2, X_2) = 12(2a + b); g_\xi(X_3, X_3) = 12(a + 2b)$.

If we now put $s := a - b$ and $t := a + 2b$, then the set $K_o$ of invariant Kähler metrics whose volume is equal to $vol(M, g)$ is given by

$$K_o \cong \{(s, t) | s, t > 0, \ st(s + t) = \frac{2}{27}\}.$$  

In order to estimate the first eigenvalue of $g_\xi$, we recall some well-known facts about the Laplacian of an invariant metric on a homogeneous space. Indeed, if $\{v_1, \ldots, v_6\}$ is an orthonormal basis of $m$ w.r.t. an invariant metric $g$ on $SU(3)/T^2$ and if $\rho : SU(3) \to SU(V)$ is an irreducible representation on a Hermitian vector space $V$ with $V^{T^2} = \{v \in V; \ hv = v \ \forall h \in T^2\} \neq \{0\}$, then the operator

$$D_\rho := \sum_{i=1}^{6} \rho(v_i)^2$$

leaves $V^{T^2}$ invariant and its spectrum $\text{Spec}(D_\rho, V^{T^2})$ is contained in the spectrum of $\Delta_\rho$ acting on $C^\infty(M)$ (see e.g. [13]).

We will consider the irreducible representation $\text{Ad}$ of $SU(3)$ given by the adjoint representation $V = su(3)^C$. In order to compute $D_\text{Ad}$ relative to the metric $g_{(s,t)} \in K_o$, we fix the $g_{(s,t)}$-orthonormal basis given by

$$v_1 = \frac{1}{\sqrt{12s}}X_1, \quad v_2 = \frac{1}{\sqrt{12(s + t)}}X_2, \quad v_3 = \frac{1}{\sqrt{12t}}X_3, \quad v_i = Jv_{i-3}, \quad i = 4, 5, 6.$$  

The space $V^{T^2}$ coincides with the Lie subalgebra $t^C$ which can be identified with $\mathbb{C}^2$ via $(z_1, z_2) \mapsto \text{diag}(z_1, z_2, -z_1 - z_2)$. A lengthy but straightforward computation shows that the endomorphism $D_\text{Ad}$ of $t^C$ can be represented by the matrix

$$D_\text{Ad} = \frac{1}{3} \begin{pmatrix} \frac{3s + t}{s(s + t)} & -\frac{t}{s(s + t)} \\ -\frac{t}{s(s + t)} & \frac{2s + t}{s(t)} \end{pmatrix}$$

and since we are considering metrics in $K_o$, i.e. with $st(s + t) = \frac{2}{27}$, it can be rewritten as

$$D_\text{Ad} = \frac{9}{2} \begin{pmatrix} t(3s + t) & -t^2 \\ s^2 - t^2 & (2s + t)(s + t) \end{pmatrix}.$$  

Its eigenvalues are given by $\frac{9}{2}(t^2 + s^2 + 3st \pm \sqrt{t^4 + s^4 - s^2t^2})$. Note that we get eigenvalues $\{2, 3\}$ for $s = t = \frac{1}{3}$, which corresponds to the Kähler Einstein metric $\bar{g}$.  

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Using a software, e.g. Maple, it can be shown that the function \( f(s, t) = \frac{9}{2}(t^2 + s^2 + 3st - \sqrt{t^4 + s^4 - s^2t^2}) \) restricted to the curve \( \{ st(s + t) = \frac{2}{27} \} \) attains its maximum value 2 at \( s = t = \frac{1}{3} \). This shows in particular that \( \lambda_1(g(s,t)) < 2 \) for \( (s, t) \in K_{\alpha_1} \), \( (s, t) \neq (\frac{1}{3}, \frac{1}{3}) \) and our last claim is proved.

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