Integrable inhomogeneous spin chains in generalized Lunin-Maldacena backgrounds

Matheus Jatkoske Lazo

UFSM/Unipampa
What is the actual relevance of spring theory for real world physics?
String Theory?

What is the real world? Particles? Fields? Singularities?
String Theory?

Born-Infeld

A first attempt to a world without particles and singularities.

\[ L = m \frac{v^2}{2} \implies L = mc^2 \left( \sqrt{1 - \frac{v^2}{c^2}} - 1 \right) \]

\[ L_e = \frac{E^2}{2} \implies L_e = b^2 \left( \sqrt{1 - \frac{E^2}{b^2}} - 1 \right) \]

Born Proc. R. Soc. A 143 110 (1934)

Born and Infeld Proc. R. Soc. A 144 425 (1934)
String Theory?

The born of String Theory

Particles as oscillations of space-time.

\[ S = \frac{T}{2} \int d\sigma d\tau \eta_{\mu\nu} \left[ \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau} - \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X^\nu}{\partial \sigma} \right] \]

Green, Schwarz and Witten, *Superstring Theory* (1987)

Polchinski, *String Theory* (1998)
String Theory?

What is the actual relevance of spring theory for real world physics?

- A candidate theory of everything.
- Productive and vital influence upon mathematics.

The AdS/CFT

- A new formalism for QFT.
- Integrable Models: mathematics and physics.
String Theory?

What is the actual relevance of string theory for real world physics?

- A candidate theory of everything.
- Productive and vital influence upon mathematics.

The AdS/CFT

- A new formalism for QFT.
- Integrable Models: mathematics and physics.
The AdS/CFT correspondence

The Maldacenas conjecture relates operators, states, correlation functions and dynamics of a ten dimensional $AdS_5 \times S^5$ string theory to a four dimensional super Yang-Mills (SYM).

\begin{align*}
|0; J\rangle & \leftrightarrow \text{Tr}Z^J, \\
a_0^{i\dagger}|0; J\rangle & \leftrightarrow \text{Tr}\phi_i Z^J, \quad (Z = \phi_1 + i\phi_2) \\
a_n^{i\dagger}a_{-n}^{j\dagger}|0; J\rangle & \leftrightarrow \sum_l \text{Tr}\phi_i Z^l \phi_j Z^{J-l}
\end{align*}

The AdS/CFT correspondence relates the weak coupling constant regime, in the gauge theory, with the strong coupling constant ones, in the string theory. We have:

\[ g_{YM}^2 N = \lambda = \frac{R^4}{\alpha'^2} = 4\pi g_s N, \]

Maldacena, *Adv. Theor. Math. Phys.* 2 231 (1998).

Gubser, Klebanov and Polyakov, *Phys. Lett.* B428 105 (1998).

Witten, *Adv. Theor. Math. Phys.* 2 253 (1998).
The Dilatation Operator

This conjecture has been notoriously hard to test.

spectrum of string states $\leftrightarrow$ spectrum of the Dilatation Operator

Renormalization of scalar operators.

$O(\Psi) = \Psi^{\alpha_1 \cdots \alpha_L} \text{Tr}(\Phi_{\alpha_1} \cdots \Phi_{\alpha_L})$,

$O_{\text{ren}}^A = Z_B^A \hat{O}^B$

where

$\Gamma = \frac{dZ}{d \ln \Lambda} \frac{1}{Z}$
The Dilatation Operator

The one loop approximation and the integrability

$$\Gamma(g) = L + g^2 H(g) + O\left(\frac{1}{N}\right),$$

where $H(g)$ can be identified to an integrable $SO(6)$ Heisemberg spin chain.

Minahan and Zarembo, *JHEP* **0303** 013 (2003)
The Dilatation Operator

What is the integrability importance of the Dilatation operator?

- AdS/CFT: the best way to test the Maldacena’s correspondence.
- QCD: is possible to construct a dual theory to QCD?
- Condensed Matter: generator of non-trivial integrable spin chains.
The Dilatation Operator

Generalization of the one loop Dilatation Operator

\[ H = \sum_{j=1}^{L} \left( \sum_{\alpha \neq \beta = 1} \gamma^{\alpha \beta}_{\alpha \beta}(j, j+1) E^{\beta \alpha j}_{j+1} E^{\alpha \beta j}_{j+1} + \sum_{\alpha, \beta = 1} \gamma^{\alpha \beta}_{\alpha \beta}(j, j+1) E^{\alpha \alpha j}_{j+1} E^{\beta \beta j}_{j+1} \right) \]

\[ \phi_{\alpha} \phi_{\beta} \rightarrow \phi_{\beta} \phi_{\alpha} \]

\[ \phi_{\alpha} \phi_{\beta} \rightarrow \phi_{\beta} \phi_{\alpha} \]
The Matrix Product Ansatz

The eigenvalue equation

\[ HO(\Psi) = \varepsilon O(\Psi) \]

The Matrix Product Ansatz

\[ O(\Psi) = \Psi^{\alpha_1 \ldots \alpha_L} \text{Tr}(\phi_{\alpha_1} \cdots \phi_{\alpha_L}) , \]

\[ \Psi^{\alpha_1 \ldots \alpha_L} = \text{Tr}(A_{1}^{(\alpha_1)} A_{2}^{(\alpha_2)} \cdots A_{L}^{(\alpha_L)}) \quad (\alpha_j = 1, \ldots, N). \]

Lazo, submitted to PLB
Lazo, Physica A \textbf{374} 655 (2007)

Alcaraz and Lazo, J. Phys. A: Math. Gen. \textbf{37} L1 (2004)
The Matrix Product Ansatz

The $U(1)^2$ sector

$\phi_1$ and $\phi_2$

Referential state

$\text{Tr}(\phi_1 \cdots \phi_1) \implies A_j^{(1)} \equiv A$

From the eigenvalue equation

$\epsilon_n \text{Tr}(A_{x_1} A_{x_2} \cdots A_{x_{j-1}} A_{x_j}^{-x_j-1} A_{x_{j+1}} A_{x_{j+1}}^{-x_{j+1}-1} A_{x_{j+2}} \cdots A_{x_n}^{-x_n})$

$= \sum_{j=1}^n \left[ \Gamma_{1/2}^{1/2} (x_j - 1, x_j) \text{Tr}(\cdots A_{x_j} A_{x_j}^{-x_j-1} A_{x_{j+1}} A_{x_{j+1}}^{-x_{j+1}-1} A_{x_{j+2}} \cdots) \\
+ \Gamma_{1/2}^{1/2} (x_j, x_j + 1) \text{Tr}(\cdots A_{x_j} A_{x_j}^{-x_j-1} A_{x_{j+1}} A_{x_{j+1}}^{-x_{j+1}-1} A_{x_{j+2}} \cdots) \\
+ \left( \Gamma_{2/1}^{1/2} (x_j, x_j + 1) + \Gamma_{1/2}^{1/2} (x_j - 1, x_j) \right) \text{Tr}(\cdots) \right]$
The Matrix Product Ansatz

Defining

\[ A^{(2)}_x = \sum_{j=1}^{n} A^{(2)}_{x,k_j} A, \quad A^{(2)}_{x,k_j} A = g(x, x + 1) e^{ik_j} A A^{(2)}_{x+1,k_j}, \]

we obtain

\[ \varepsilon_n = \sum_{j=1}^{n} \left( \Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}} e^{ik_j} + \Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}} e^{-ik_j} \right) + n \left( \Gamma^{1 \frac{2}{2}}_{1 \frac{1}{2}} + \Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}} \right) + (L - 2n) \Gamma^{1 \frac{1}{1}}_{1 \frac{1}{1}}, \]

where we need to impose

\[ g(x, x + 1) = \frac{\Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}}}{\Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}}(x, x + 1)} = \frac{\Gamma^{1 \frac{2}{2}}_{1 \frac{1}{2}}(x, x + 1)}{\Gamma^{1 \frac{2}{2}}_{1 \frac{1}{2}}}, \]

and

\[ \Gamma^{1 \frac{1}{1}}_{1 \frac{1}{1}}(x, x + 1) = \Gamma^{1 \frac{1}{1}}_{1 \frac{1}{1}}, \quad \Gamma^{1 \frac{2}{2}}_{1 \frac{1}{2}}(x, x + 1) = \Gamma^{1 \frac{2}{2}}_{1 \frac{1}{2}}, \quad \Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}}(x, x + 1) = \Gamma^{2 \frac{1}{2}}_{1 \frac{1}{2}}. \]
The Matrix Product Ansatz

From the eigenvalue equation

$$\sum_{j,l=2}^{n} \left[ \Gamma_{1 \ 2}^{1 \ 2} + \Gamma_{1 \ 2}^{2 \ 1} e^{ik_j + k_l} + (\Gamma_{2 \ 2}^{1 \ 2} + \Gamma_{1 \ 1}^{1 \ 1} - \Gamma_{2 \ 2}^{2 \ 1}(x, x + 1))e^{ik_j} \right] A^{(2)}_{x,k_j} A^{(2)}_{x+1,k_l} = 0,$$

where we need to impose

$$\Gamma_{2 \ 2}^{2 \ 2}(x, x + 1) = \Gamma_{2 \ 2}^{2 \ 2}.$$

We obtain

$$A^{(2)}_{x,k_j} A^{(2)}_{x+1,k_l} = S(k_j, k_l) A^{(2)}_{x,k_l} A^{(2)}_{x+1,k_j}, \quad A^{(2)}_{x,k_j} A^{(2)}_{x+1,k_j} = 0,$$

$$S(k_j, k_l) = -\frac{\Gamma_{1 \ 2}^{1 \ 2} + \Gamma_{1 \ 2}^{2 \ 1} e^{ik_j + k_l} + (\Gamma_{2 \ 2}^{1 \ 2} + \Gamma_{1 \ 1}^{1 \ 1} - \Gamma_{2 \ 2}^{2 \ 1}(x, x + 1))e^{ik_l}}{\Gamma_{2 \ 2}^{1 \ 2} + \Gamma_{1 \ 2}^{2 \ 1} e^{ik_j + k_l} + (\Gamma_{2 \ 2}^{1 \ 2} + \Gamma_{1 \ 1}^{1 \ 1} - \Gamma_{2 \ 2}^{2 \ 1}(x, x + 1))e^{ik_j}}.$$
The Matrix Product Ansatz

The model

\[
\Gamma_{\frac{1}{2}1}^{21}(x, x + 1) = \frac{\Gamma_{\frac{1}{2}1}^{21}\Gamma_{\frac{1}{2}1}^{12}}{\Gamma_{\frac{1}{2}1}^{12}(x, x + 1)} \quad \Gamma_{\alpha \beta}^{\alpha \beta}(x, x + 1) = \Gamma_{\alpha \beta}^{\alpha \beta}
\]

Algebraic relations

\[
A_{x,k_j}^{(2)} A = g(x, x + 1)e^{ik_j} A A_{x+1,k_j}^{(2)},
\]

\[
A_{x,k_j}^{(2)} A_{x+1,k_l}^{(2)} = S(k_j, k_l) A_{x,k_l}^{(2)} A_{x+1,k_j}^{(2)}, \quad A_{x,k_j}^{(2)} A_{x+1,k_j}^{(2)} = 0.
\]

Eigenstates and eigenvalues

\[
O(\Psi) = Tr(A_1^{(\alpha_1)} A_2^{(\alpha_2)} \cdots A_L^{(\alpha_L)}) Tr(\phi_{\alpha_1} \cdots \phi_{\alpha_L}), \quad A_x^{(2)} = \sum_{j=1}^{n} A_{x,k_j}^{(2)} A.
\]

\[
\varepsilon_n = \sum_{j=1}^{n} \left( \Gamma_{\frac{1}{2}1}^{21} e^{ik_j} + \Gamma_{\frac{1}{2}1}^{12} e^{-ik_j} \right) + n \left( \Gamma_{\frac{1}{2}1}^{12} + \Gamma_{\frac{1}{2}1}^{21} \right) + (L - 2n)\Gamma_{\frac{1}{2}1}^{11},
\]
The Matrix Product Ansatz

To fix the spectral parameters

\[
\text{Tr} \left[ A_{1,k_1}^{(2)} \cdots A_{j-1,k_{j-1}}^{(2)} A_{j,k_j}^{(2)} A_{j+1,k_{j+1}}^{(2)} \cdots A_{k_n}^{(2)} A^L \right] =
\]

\[-e^{ik_j L} \prod_{x=1}^{L} g(x,x+1) \prod_{l=1}^{n} S(k_j,k_l) \text{Tr} \left[ A_{1,k_1}^{(2)} \cdots A_{k_n}^{(2)} A^L \right].\]

The Bethe equation

\[e^{-ik_j L} = \prod_{x=1}^{L} g(x,x+1) \prod_{l=1}^{n} S(k_l,k_j).\]
Conclusions and generalizations

- How much we can learn about real systems from integrability?
- Insight for the construction of a dual theory to QCD.
- Applications in condensed matter physics.
- The solution of the Bethe equation.
- Generalization for higher loops Dilatation Operators.
- A solution for open strings.
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\[
A^{(\alpha)}_{k_j} A^{(\beta)}_{k_l} = \sum_{\alpha', \beta'=2}^N S_{\beta'}^{\alpha'} (k_j, k_l) A^{(\alpha')}_{k_l} A^{(\beta')}_{k_j}, \quad A^{(\alpha)}_{k_j} A^{(\beta)}_{k_j} = 0 \quad (l \neq j = 1, \ldots, n),
\]

\[
S^{\alpha \alpha}_{\alpha \alpha} (k_j, k_l) = \frac{\Gamma_1^{1/2} + \Gamma_2^{1/2} e^{i(k_j+k_l)} + (\Gamma_1^{1/2} + \Gamma_2^{1/2} - \Gamma_1^{1/2} - \Gamma_2^{1/2}) e^{ik_l}}{\Gamma_1^{1/2} + \Gamma_2^{1/2} e^{i(k_j+k_l)} + (\Gamma_1^{1/2} + \Gamma_2^{1/2} - \Gamma_1^{1/2} - \Gamma_2^{1/2}) e^{ik_j}},
\]

\[
S^{\alpha \beta}_{\beta \alpha} (k_j, k_l) = -\frac{C_{\alpha, \beta} (k_l, k_j) C_{\beta, \alpha} (k_j, k_l) - \Gamma_1^{1/2} + \Gamma_2^{1/2} e^{i(k_j+k_l)}}{C_{\alpha, \beta} (k_j, k_l) C_{\beta, \alpha} (k_j, k_l) - \Gamma_1^{1/2} + \Gamma_2^{1/2} e^{2ik_j}},
\]

\[
S^{\alpha \beta}_{\alpha \beta} (k_j, k_l) = -\frac{g(\alpha) \Gamma_1^{1/2} \Gamma_2^{1/2} e^{i(k_j+k_l)}}{C_{\alpha, \beta} (k_j, k_l) C_{\beta, \alpha} (k_j, k_l) - \Gamma_1^{1/2} + \Gamma_2^{1/2} e^{2ik_j}},
\]

\[
C_{\alpha, \beta} (k_j, k_l) = \Gamma_1^{1/2} + \Gamma_2^{1/2} e^{i(k_j+k_l)} + (\Gamma_1^{1/2} + \Gamma_2^{1/2} - \Gamma_1^{1/2} - \Gamma_2^{1/2}) e^{ik_j}.
\]

\[
\sum_{\gamma, \gamma', \gamma''=2}^N S^{\gamma}_{\gamma'} (k_1, k_2) S^{\gamma''}_{\gamma} (k_1, k_3) S^{\gamma'}_{\gamma''} (k_2, k_3)
\]

\[
= \sum_{\gamma, \gamma', \gamma''=2}^N S^{\gamma'}_{\gamma''} (k_2, k_3) S^{\gamma}_{\gamma''} (k_1, k_3) S^{\gamma'}_{\gamma} (k_1, k_2).
\]
\( U(1)^3 \)

\[(A.1)\] \( \Gamma^1_{\alpha 1} \Gamma^1_{\beta 1} = \Gamma^1_{\beta 1} \Gamma^1_{\alpha 1} = \Gamma^\beta_{\alpha \beta} \Gamma^\alpha_{\beta \alpha} = t_{\alpha \beta 1} t_{\beta \alpha 1}, \quad t_{\alpha \alpha 1} = t_{\beta \beta 1} = 0, \]

\[(A.2)\] \( \Gamma^1_{\alpha 1} \Gamma^1_{\beta 1} = \Gamma^1_{\beta 1} \Gamma^1_{\alpha 1} = \Gamma^\beta_{\alpha \beta} \Gamma^\alpha_{\beta \alpha} = t_{\alpha \beta 1} t_{\beta \alpha 1}, \quad t_{\alpha \alpha 1} = t_{\alpha \beta \beta} = 0, \]

\[(A.3)\] \( \Gamma^1_{\alpha 1} \Gamma^1_{\beta 1} = \Gamma^1_{\beta 1} \Gamma^1_{\alpha 1} = \Gamma^\beta_{\alpha \beta} \Gamma^\alpha_{\beta \alpha} = t_{\alpha \beta 1} t_{\beta \alpha 1}, \quad t_{1 \beta \alpha} = t_{\alpha 1 \beta} = t_{\beta \alpha 1}, \)

\[(A.4)\] \( \Gamma^1_{\alpha 1} \Gamma^1_{\beta 1} = \Gamma^1_{\beta 1} \Gamma^1_{\alpha 1} = \Gamma^\beta_{\alpha \beta} \Gamma^\alpha_{\beta \alpha} = 0, \quad t_{\alpha \beta 1} = t_{\beta \alpha 1} = t_{\beta \beta 1} = t_{\alpha \alpha 1}, \)

\[(A.5)\] \( \Gamma^1_{\alpha 1} = g \Gamma^1_{\beta 1}, \quad \Gamma^1_{\alpha 1} = \Gamma^1_{\beta 1} = 0, \quad \Gamma^\beta_{\alpha \beta} = 0, \quad t_{\alpha \beta 1} = t_{\alpha \alpha 1}, \)

\[(A.6)\] \( \Gamma^1_{\alpha 1} = g \Gamma^1_{\beta 1}, \quad \Gamma^1_{\alpha 1} = \Gamma^1_{\beta 1} = 0, \quad \Gamma^\beta_{\alpha \beta} = 0, \quad t_{\beta \alpha 1} = t_{\alpha \alpha 1}, \)

\[(A.7)\] \( \Gamma^1_{\alpha 1} = g \Gamma^1_{\beta 1}, \quad \Gamma^1_{\alpha 1} = \Gamma^1_{\beta 1} = 0, \quad \Gamma^\beta_{\alpha \beta} = 0, \quad t_{\alpha \beta 1} = t_{\beta \beta 1}, \)

\[(A.8)\] \( \Gamma^1_{\alpha 1} = g \Gamma^1_{\beta 1}, \quad \Gamma^1_{\alpha 1} = \Gamma^1_{\beta 1} = 0, \quad \Gamma^\beta_{\alpha \beta} = 0, \quad t_{\beta \alpha 1} = t_{\beta \beta 1}, \)

\[(A.9)\] \( \Gamma^1_{\alpha 1} = \Gamma^1_{\alpha 1} = \Gamma^1_{\beta 1} = \Gamma^1_{\beta 1} = 0. \)

\[t_{\alpha \beta \gamma} = \Gamma^\alpha_{\alpha \beta} \Gamma^\gamma_{\beta \gamma} + \Gamma^\gamma_{\alpha \gamma} - \Gamma^\alpha_{\alpha \gamma} - \Gamma^\gamma_{\beta \gamma} \]