Phase Transition in the Kolkata Paise Restaurant problem

Antika Sinha
Department of Computer Science, Asutosh College, Kolkata-700026, India

Bikas K. Chakrabarti
Saha Institute of Nuclear Physics, Kolkata-700064, India
S.N. Bose National Centre for Basic Sciences, Kolkata-700106, India and
Economic Research Unit, Indian Statistical Institute, Kolkata-700108, India

Abstract: A novel phase transition behavior is observed in the Kolkata Paise Restaurant (KPR) problem where large number ($N$) of agents or customers or players collectively (and iteratively) learn to choose among the $N$ restaurants where she would expect to be alone that evening and would get the only dish available there that evening (or may get randomly picked up if more than one agent arrive there that evening). The players evolve their strategy such that the publicly available information about past crowd in different restaurants can be utilized and each of them is able to make the best minority choice. Following two crowd-avoiding strategies (I) for equally ranked restaurants and (II) where each restaurant has an unique rank, each of the $n_i(t-1)$ number of agents arriving at the $i$-th (ranked) restaurant in the last $(t-1)$-th evening chooses this $t$-th evening with weight $[n_i(t-1)]^{-\alpha}$ to go back to the same $i$-th restaurant for Strategy I and to the $(i-1)$-th ranked restaurant for Strategy II and with weight $1/(N-1)$ for the other restaurants (ranking with periodic boundary condition). We study the steady state ($t$-independent) utilization fraction $f$: $(1-f)$ giving the steady state (wastage) fraction of restaurants going without any customer in any particular evening. We find, near $\alpha_c = 0_+$, $(1-f) \propto (\alpha - \alpha_c)^\beta$ with $\beta \simeq 0.8$ and the convergence time $\tau$ (for $f(t)$ becoming independent of $t$) varies as $\tau \propto (\alpha - \alpha_c)^{-\gamma}$, with $\gamma \simeq 1.3$ for both the strategies (I) and (II). More importantly we find that the agents following Strategy (II), after convergence time $\tau$, form constantly moving spontaneous queues of most-probable size $N$ ($N \to \infty$ in KPR) as $\alpha \to \alpha_c$; $\alpha_c = 0_+$, leading to the most efficient social utilization.
I. INTRODUCTION

We study the steady state statistics and the phase transition behavior of the Kolkata Paise Restaurant (KPR) problem \[1,2\]. KPR is a many-agent (player) and many-choice repeated game, where the agents collectively learn from past mistakes, how to share best the limited resources. In this kind of games, each agent tries to anticipate and choose her own strategy, every time (learning from the publicly available past informations) in parallel mode (unguided; in absence of any dictator or non-playing instructor). The restaurants are assumed to prepare every evening fixed meal plates which are equally priced (hence no budget constraint for customers). Only the crowd avoidance abilities determines the individual success in securing meal on any or successive evenings.

Specifically, we consider here the case of \(N\) restaurants and \(N\) agents or customers or players who decide every evening (on the basis of informations about the past evenings, available to everyone), which restaurant to choose such that she will be alone there and will get the meal. Each restaurant is assumed, for simplicity, to prepare one dish every evening (generalization does not help getting any further insight at this stage). For more than one person arriving any restaurant any evening, a randomly chosen one will get the meal and rest (arriving there) will not get any that evening. We will also consider the case of ranking of these restaurants here.

Although every evening each of the \(N\) restaurants prepares one dish and in-principle everyone is entitled to a dish every evening, overcrowding due to stochasticity of choices make the probability of success for each customer less than unity in such (democratic choice) games. We measure the social efficiency by the utilization fraction \(f(t)\) on any day (evening) \(t\) as

\[
f(t) = [1 - \sum_{i=1}^{N} \delta(n_i(t))/N],
\]

with \(\delta(n) = 1,0\) for \(n = 0,1\) respectively; \(n_i(t)\) denotes the number of agents arriving at the \(i\)-th (rank) restaurant on \(t\)-th evening. The fraction \((1-f(t))\) gives the fraction of social wastage or the fraction of restaurants going without any customer on the \(t\)-th day. The objective of social learning strategies here is to achieve \(f(t) = 1\) preferably in finite convergence time \((\tau)\), i.e., for \(t \geq \tau\), or at least as \(t \to \infty\) (see e.g., \[2,3\]).

Indeed, a dictated solution is extremely simple and very efficient: the dictator asks everyone to form a queue and visit the restaurants according to their respective positions in the queue and then asks them to shift their positions by one step (rank) in the next evening (assuming periodic boundary condition). Everyone gets the food: No wastage, i.e., the steady state \((t\)-independent\))
utilization fraction $f = 1$, and that too from the first evening (convergence time $\tau$ is zero). This is true even when the restaurants have ranks (agreed by all the agents or customers). However, in reality (in democracy), this dictated solution is not acceptable and each agent would like to (learn from past experience and) decide on her own every evening which restaurant to choose such that she is alone there and gets the dish. The more successful such collective learning, the more is the utilization fraction of the services. Question is, what is the maximum utilization fraction value ($f$) and convergence time ($\tau$) of such ‘learned’ democratic choices (due to individually learned and chosen strategies) for a large society ($N \rightarrow \infty$). Note that the dictated solution gives full utilization ($f = 1$) and that too in zero convergence time ($\tau = 0$) for any $N$.

Assuming that no past history of restaurant occupancy is available, i.e., no learning, let us consider the process of randomly choosing any of the $N$ restaurants by $\lambda N$ agents (we consider $\lambda = 1$ in KPR game later). Then the probability of choosing any restaurant by $m$ ($> 1$) agents on any evening is

$$\rho(m) = \binom{\lambda N}{m} p^m (1-p)^{\lambda N-m}, \quad (2a)$$

where $p = \frac{1}{N}$; giving $\rho(m) = [\lambda^m/(m!)] exp(-\lambda)$ as $N \rightarrow \infty$. Hence the average fraction of restaurants not chosen on any evening is $\rho(m = 0) = exp(-\lambda)$, and average fraction of restaurants filled or utilized on any evening, is given by [1]

$$f = 1 - exp(-\lambda) \simeq 0.63, \quad f or \lambda = 1. \quad (2b)$$

As we mentioned earlier, the agents would try to learn from the past mistakes in making their respective choices and can improve her chance to be in the minority with efficient learning strategy. We study here the dynamics of the game with two such stochastic learning strategies which allow for considerable increase in the eventual (steady state) value of the utilization fraction ($f$). Specifically, we study here two learning strategies (Strategy I and Strategy II), giving the probabilities to choose going back or not to the same restaurant visited last evening or to a better ranked restaurant, depending on the last evening’s crowd-size in the chosen restaurant. We find interesting phase transition behavior with these strategies. This transition behavior is qualitatively different from the transitions observed [5, 6] earlier with agents’ sticking probability to any chosen restaurant and with limited resources ($\lambda < 1$) in KPR and similar models.
II. LEARNING STRATEGY

In the following, we study numerically the dynamics of KPR game played by $N$ agents (interchangeably called players or agents) such that each evening (interchangeably called day or time) each of the agents employ some stochastic (learning) strategy, based on the past crowd information in different restaurants (available to each players), helping her to choose among $N$ restaurants, maximizing her chance of arriving at a vacant restaurant that evening and to get the dish. We study here the following learning strategies: I and II.

**Strategy I:** Let all the restaurants be equally ranked. The strategy here is that any agent tries to go back to the same restaurant as chosen in the earlier evening (day) with a probability decreasing with an inverse power of the crowd size arriving there last evening and goes to any other restaurant randomly with the rest of the probability. In other words, on day $t$, an agent goes back to her last day’s visited restaurant $k$ with probability

$$p_k^{(I)}(t) = [n_k(t - 1)]^{-\alpha}, \alpha > 0$$

(3a)

if she was one of the $n_k(t - 1)$ agents or players arriving there ($k$-th restaurant) last day. She chooses one among the rest of the $(N - 1)$ equi-probable restaurants $k' (\neq k)$ with probability

$$p_{k'}^{(I)}(t) = \frac{(1 - p_k^{(I)}(t))}{(N - 1)}.$$  

(3b)

**Strategy II:** Next we consider the case where every agent agrees to the (nondegenerate) ranking of each restaurant and learn collectively (from past crowd level information, available to every one) so that they can reach as close to the efficient solution as the dictated solution mentioned in the Introduction: every one forms a queue and moves by an unit rank in each successive evening. Here in absence of any dictator, the agents collectively learn and modify Strategy I such that the probability $p_k^{(II)}$, to go to the $k$-th ranked restaurant on the $t$-th evening, is as follows:

$$p_k^{(II)}(t) = [n_{k+1}(t - 1)]^{-\alpha},$$

(4a)

for those who arrived in the restaurant with rank $k + 1$ in the last evening and

$$p_{k'}^{(II)}(t) = \frac{(1 - p_k^{(II)}(t))}{(N - 1)}.$$  

(4b)

for choosing any of the other restaurants ($k' \neq k$).
Note that for $\alpha = 0$, every evening she will return to the first day’s chosen and visited restaurant (in Strategy I) or to the earlier ranked restaurant (in Strategy II). The dynamics on successive days become trivial for both the strategies I and II. However $\alpha = 0_+$ case can be extremely non-trivial and, as we will see, has interesting transition and other behaviors.

III. NUMERICAL RESULTS

In this simulation study using Strategy I, with $N = 500$ and $\lambda = 1$, we found the steady state (for $t \geq \tau(\alpha)$, the convergence time) value of the aggregated utilization fraction $f(I)$ (estimated using (1)) becomes unity for $\alpha \to \alpha_c$, where $\alpha_c = 0_+$. We find the power laws $(1-f(I)) \sim (\alpha-\alpha_c)^{\beta(I)}$, $\beta(I) = 0.80 \pm 0.05$ (see Fig. 1) for the utilization fraction and $\tau_R(I) \sim (\alpha-\alpha_c)^{-\gamma_R(I)}$ with $\gamma_R(I) = 1.18 \pm 0.07$ for and $\tau_H(I) \sim (\alpha-\alpha_c)^{-\gamma_H(I)}$, $\gamma_H(I) = 1.12 \pm 0.06$ for the ‘half-life’ defined later (Fig. 2a and 2b). All simulations are done with $N = 500$ and the steady averages up to a maximum number of steps (days/evenings) of order $10^5$. For estimating $\tau(I)$ values, we looked for the convergence time ($\tau_R(I)$) for $f(I)(t)$ to attain the steady state value $f(I)$ (within a small fore-assigned error margin). We also look for the half lifetime ($\tau_H(I)$) for $f(I)(t)$ to start from it’s initial random value ($f(I)(0) \simeq 0.63$, see Eq. 2b) to its half value (still greater than the steady state value) at $\tau_H(I)$.

![Figure 1](image-url)

**FIG. 1.** Plot of $(1-f(I))$ against $\alpha$ ($f(I)$ is steady state social utilization fraction following learning Strategy I), showing that $(1-f(I))$ varies as power of $\alpha$, if on day $t$ each agent returns to $(t-1)$-th day’s visited restaurant with probability $[n_i(t-1)]^{-\alpha}$, else goes to any of $(N-1)$ restaurant randomly. We found that for $\alpha$ values near $\alpha_c(= 0_+)$, a power law holds for $(1-f(I)) \sim (\alpha-\alpha_c)^{\beta(I)}$ where $\beta(I) = 0.80 \pm 0.05$ (the straight line fit is from maximum likelihood estimation). The inset plot shows direct relationship between $(1-f(I))$ and $\alpha$. 
FIG. 2. Plots of $\tau_{RI}^{(I)}$ (a) and $\tau_{HL}^{(I)}$ (b) against $\alpha$, following learning Strategy I ($\tau_{RI}^{(I)}$ denotes the convergence time to attain steady state within a preassigned error margin, whereas $\tau_{HL}^{(I)}$ refers to the time to reach half of the initial value of the order parameter $f^{(I)}$). Starting from an initial state (say, the random choice case) the dynamics proceed as follows: if at day $t$ each of the $n_i(t-1)$ agents return to $(t-1)$-th day’s visited restaurant ($i$) with probability $[n_i(t-1)]^{-\alpha}$, else goes to any of ($N-1$) restaurant randomly. For $\alpha$ values near $\alpha_c = 0$+, the power law $\tau^{(I)} \sim (\alpha - \alpha_c)^{-\gamma^{(I)}}$ holds, with (a) $\gamma^{(I)}_R = 1.18 \pm 0.07$ for $\tau_{RI}^{(I)}$ and (b) $\gamma^{(I)}_{HL} = 1.12 \pm 0.06$ for $\tau_{HL}^{(I)}$ (the straight line fits are from maximum likelihood estimation). The inset plots show direct functional relationship between $\tau_{RI}^{(I)}$ (a) or $\tau_{HL}^{(I)}$ (b) and $\alpha$.

FIG. 3. Plot of $(1 - f^{(II)})$ against $\alpha$ ($f^{(II)}$ is steady state social utilization fraction following learning Strategy II), showing that $(1 - f^{(II)})$ varies as power of $\alpha$, if on day $t$ each agent visits one rank higher than the $(t-1)$-th day’s visited restaurant with probability $[n_i(t-1)]^{-\alpha}$, else goes to any of ($N-1$) restaurant randomly. We found that for $\alpha$ values near $\alpha_c (= 0_+)$, a power law holds for $(1 - f^{(II)}) \sim (\alpha - \alpha_c)^{\beta^{(II)}}$ where $\beta^{(II)} = 0.80 \pm 0.05$ (the straight line fit is from maximum likelihood estimation). The inset plot shows direct relationship between $(1 - f^{(II)})$ and $\alpha$. 
FIG. 4. Plots of $\tau^{(II)}_R$ (a) and $\tau^{(II)}_{HL}$ (b) against $\alpha$, following learning Strategy II ($\tau^{(II)}_R$ denotes the average convergence time to attain steady state, whereas $\tau^{(II)}_{HL}$ refers to the time to reach half of the initial value of the order parameter $f^{(II)}$). Starting from an initial state (say, the random choice case) the dynamics proceed as follows: if at day $t$ each of the $n_i(t-1)$ agents return to $(t-1)$-th day’s visited restaurant $(i)$ with probability $[n_i(t-1)]^{-\alpha}$, else goes to any of $(N-1)$ restaurant randomly. For $\alpha$ values near $\alpha_c = 0_+$, the power law $\tau^{(II)} \sim (\alpha - \alpha_c)^{-\gamma^{(II)}}$ holds, with (a) $\gamma^{(II)}_R \simeq 1.18 \pm 0.07$ for $\tau^{(II)}_R$ and (b) $\gamma^{(II)}_{HL} \simeq 1.12 \pm 0.06$ for $\tau^{(II)}_{HL}$ (the straight line fits are from maximum likelihood estimation). The inset plots show direct relationship between $\tau^{(II)}_R$ (a) or $\tau^{(II)}_{HL}$ (b) and $\alpha$.

FIG. 5. Normalized frequency $P(S)$ of occurrence of moving queues of size $S$ (number of agents normalized by $N$) with Strategy II (for different restaurants) for different values of $\alpha$. $\alpha = 0.1$ for (a) and $\alpha = 0.01$ for (b). One clearly observes that the average queue size becomes of order $N$ ($<S> \simeq 1.0$) for $\alpha \rightarrow 0_+$ (closest to the case of dictatorial solution).

For Strategy II, when the restaurants each have a nondegenerate rank, we also measured the
normalized (by \(N\)) queue length \(S\) of the moving queue, given by the number of agents forming a continuous queue (without any intermediate restaurant vacant) and the queue head moves one step (rank) each day (assuming the agent in rank 1 restaurant today moves to the \(N\)-th rank restaurant next day). We again find the same power laws \((1-f^{(II)}) \sim (\alpha - \alpha_c)^{\beta^{(II)}}, \beta^{(II)} \simeq 0.80\) (see Fig. 3) for the utilization fraction and \(\tau_{R}^{(II)} \sim (\alpha - \alpha_c)^{-\gamma_{R}^{(II)}}\) with \(\gamma_{R}^{(II)} = 1.18 \pm 0.07\) for and \(\tau_{HL}^{(II)} \sim (\alpha - \alpha_c)^{-\gamma_{HL}^{(II)}}, \gamma_{HL}^{(II)} = 1.12 \pm 0.06\) for the ‘half-life’ see later; (Fig. 4a and 4b). We also look for the half lifetime \(\tau_{HL}^{(II)}\) for \(f^{(II)}(t)\) to start from its initial random value \(f^{(II)}(0) \simeq 0.63\), see Eq. 2b) to its half value (still greater than the steady state value) at \(\tau_{HL}^{(II)}\).

![FIG. 6. Normalized frequencies \(P(S)\) of the highest queue length \(S\) (normalized the number of restaurants or agents \(N\)) for some typical values of \(\alpha\) in (4a,4b) for Strategy II.](image)

The numerical results shown in Figs. 4 and 5 are for \(N = 500\) (queue size \(S\) is normalized by \(N\)) and the frequencies \(P(S)\) of the \(S\) size queue of agents forming a continuous line (without any discontinuity of restaurants in between) any evening averaged over 50,000 days (evenings) after convergence time \(\tau^{(II)}\) for that value of \(\alpha\) \((P(S)\) is normalized of getting \(S\) size queues over this 50,000 days). We clearly see (see Figs. 5 and 6) that the most probable queue size grows to order \(N\) \((S \leq 1.0)\) as \(\alpha \rightarrow \alpha_c\) \((\alpha_c = 0.1)\) and the maximum queue length approaches to \(S = N\) as \(\alpha \rightarrow \alpha_c\) (see Fig. 5).
IV. SUMMARY & DISCUSSION

The KPR problem is an iterative, many choice, many player game, where the players try to learn from their past mistakes and from the publicly available information regarding the crowd-sizes etc. of all the restaurants in the past, to choose one where she is expected to be alone today. We consider the cases here where number (N) of agents (players) equal the number of resources (restaurants). We showed [1] that no learning leads to a societal resource utilization fraction \( f = 1 - \exp(-1) \simeq 0.63 \). We study the collective learning induced phase transition in the KPR problem where \( N \) customers or players choose every evening one of the \( N \) restaurants with the hope that she gets the only dish available in that restaurant that evening, if she reaches there alone (or one is randomly picked up by the restaurant if more than one customer arrive there, and the rest do not get any dish that evening).

The social utilization fraction \( f(t) = 1 - \sum_{i=1}^{N} \frac{\delta(n_i(t))}{N} \), where \( n_i(t) \) represents the number of customers who choose the \( i \)-th restaurant for \( t \)-th evening (\( \delta(n) = 1 \) for \( n = 0 \) and \( \delta(n) = 0 \) for \( n > 0 \)), have been studied numerically here. We considered first the case of equally ranked restaurants and next the case where each of the restaurants have an unique rank and those ranks are agreed by all the agents. The learning strategies employed by the agents are strategies I and II respectively. The steady state corresponds to the case where the average utilization \( f(t) = f \) becomes \( t \)-independent. We studied the game dynamics using Monte Carlo simulation, when each agent chooses the \( i \)-th restaurant with probability \( p_i(t) = [n_i(t-1)]^{-\alpha} \) for Strategy I and \( p_i(t) = [n_{i-1}(t-1)]^{-\alpha} \) for Strategy II, where she was one of the \( n_i(t-1) \) agents who landed up last evening \( (t-1) \), and chooses all the other restaurants \((j \neq i \) for Strategy I and \( j \neq i-1 \) for Strategy II) with probability \( p_j(t) = [1-p_i(t)]/(N-1) \). We show numerically for the steady state \((1-f) \sim (\alpha - \alpha_c)^\beta \), \( \beta \simeq 0.8 \) and \( \alpha_c = 0_+ \) (see Figs. 1 and 3), while the convergence time \( \tau \) (time taken to reach a \( \tau \) independent value of \( f \), on average) grows as \( (\alpha - \alpha_c)^{-\gamma} \), with \( \gamma \simeq 1.2 \) (see Figs. 2 and 4) for both strategies I and II.

With learning Strategy II, we show that constantly moving queues of the agents with most probable size \( \sim O(N) \) are formed spontaneously in time \( \tau \). This feature of the uniform access of the restaurant ranks across the agents for Strategy II is comparable to that in the dictated solution discussed earlier (where of course \( \tau = 0 \)).

Earlier works [1, 3, 4] had already proposed several strategies to maximize the average social utilization fraction \( f \) (see refs. [7,10] for further applications of the model in practical situations). We hope, further studies on this phase transition behavior for the practical cases considered in
refs. \cite{7-10} will contribute significantly in social science. However such studies also face the formidable challenges to accommodate effectively the ranking of the restaurants, heterogeneity in their capacities and learning in finite convergence time to form the queues spontaneously in the case of ranked restaurants.

ACKNOWLEDGEMENT:

We are thankful to Arnab Chatterjee for helpful suggestions.

[1] A S Chakrabarti, B K Chakrabarti, A Chatterjee, and M Mitra, The kolkata paise restaurant problem and resource utilization, *Physica A: Statistical Mechanics and its Applications*, 388(12): 2420–2426 (2009).
[2] B K Chakrabarti, A Chatterjee, A Ghosh, S Mukherjee, and B Tamir, *Econophysics of the Kolkata Restaurant Problem and Related Games: Classical and Quantum Strategies for Multi-agent, Multi-choice Repetitive Games*, Springer (2017).
[3] A Ghosh, A Chatterjee, M Mitra, and B K Chakrabarti, Statistics of the kolkata paise restaurant problem, *New Journal of Physics*, 12(7): 075033 (2010).
[4] A Ghosh, A S Chakrabarti, and B K Chakrabarti, Kolkata paise restaurant problem in some uniform learning strategy limits, In *Econophysics and Economics of Games, Social Choices and Quantitative Techniques*, pages 3–9, Springer (2010).
[5] A Ghosh, D De Martino, A Chatterjee, M Marsili, and B K Chakrabarti, Phase transitions in crowd dynamics of resource allocation, *Physical Review E*, 85(2):021116, (2012).
[6] A Ghosh, A Chatterjee, A S Chakrabarti, and B K Chakrabarti, Zipf’s law in city size from a resource utilization model, *Physical Review E*, 90(4):042815, (2014).
[7] T Park and W Saad, Kolkata Paise Restaurant game for resource allocation in the Internet of Things, In *2017 51st Asilomar Conference on Signals, Systems, and Computers; IEEE Xplore*, pages 1774–1778, IEEE (2017). DOI: 10.1109/ACSSC.2017.8335666.
[8] P Yang, K Iyer, and P Frazier, Mean Field Equilibria for Resource Competition in Spatial Settings, *Stochastic Systems*, 8(4):307–334 (2018).
[9] L Martin and P Karaenke, The vehicle for hire problem: A Generalized Kolkata Paise Restaurant Problem, [https://mediatum.ub.tum.de/doc/1437330/1437330.pdf](https://mediatum.ub.tum.de/doc/1437330/1437330.pdf) (2017).
[10] L Martin, Extending kolkata paise restaurant problem to dynamic matching in mobility markets, *Junior Management Sc.*, 4(1):1–34 (2019). DOI: [https://jums.ub.uni-muenchen.de/JMS/article/view/5032](https://jums.ub.uni-muenchen.de/JMS/article/view/5032)