On the Convergence of Differentially Private Federated Learning on Non-Lipschitz Objectives, and with Normalized Client Updates

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Abstract
There is a dearth of convergence results for differentially private federated learning (FL) with non-Lipschitz objective functions (i.e., when gradient norms are not bounded). The primary reason for this is that the clipping operation (i.e., projection onto an $\ell_2$ ball of a fixed radius called the clipping threshold) for bounding the sensitivity of the average update to each client’s update introduces bias depending on the clipping threshold and the number of local steps in FL, and analyzing this is not easy. For Lipschitz functions, the Lipschitz constant serves as a trivial clipping threshold with zero bias. However, Lipschitzness does not hold in many practical settings; moreover, verifying it and computing the Lipschitz constant is hard. Thus, the choice of the clipping threshold is non-trivial and requires a lot of tuning in practice. In this paper, we provide the first convergence result for private FL on smooth convex objectives for a general clipping threshold – without assuming Lipschitzness. We also look at a simpler alternative to clipping (for bounding sensitivity) which is normalization – where we use only a scaled version of the unit vector along the client updates, completely discarding the magnitude information. The resulting normalization-based private FL algorithm is theoretically shown to have better convergence than its clipping-based counterpart on smooth convex functions. We corroborate our theory with synthetic experiments as well as experiments on benchmarking datasets.

1 Introduction
Collaborative machine learning (ML) schemes such as federated learning (FL) [MMR+17] are growing at an unprecedented rate. In contrast to the conventional centralized paradigm of training, wherein all the data is stored in a central database, FL (and in general, a collaborative ML scheme) enables training ML models from decentralized and heterogeneous data through collaboration of many participants, e.g., mobile devices, each with different data and capabilities. In a standard FL setting, there are $n$ clients (e.g., mobile phones or sensors), each with their own decentralized data, and a central server that is trying to train a model, parameterized by $\mathbf{w} \in \mathbb{R}^d$, using the clients’ data. Suppose the $i^{th}$ client has $m$ training examples/samples $\{x^{(i)}_1, \ldots, x^{(i)}_m\} := \mathcal{D}_i$, drawn from some distribution $\mathcal{P}_i$. Then the $i^{th}$ client has an objective function $f_i(\mathbf{w})$ which is the average loss, w.r.t. some loss function $\ell$, over its $m$ samples, and the

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1In general, each client may have different number of training examples. We consider the case of equal number of examples per client for ease of exposition.
central server tries to optimize the average\footnote{In general, this average is a weighted one with the weight of a client being proportional to the number of samples in that client.} loss \( f(w) \), over the \( n \) clients, i.e.,

\[
f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w), \quad \text{where} \quad f_i(w) := \frac{1}{m} \sum_{j=1}^{m} \ell(x^{(i)}_j, w).
\] (1)

The setting where the data distributions of all the clients are identical, i.e. \( P_1 = \ldots = P_n \), is known as the “homogeneous” setting. Other settings are known as “heterogeneous” settings. We quantify heterogeneity in more detail in Section 2 (see Definition 1).

The key algorithmic idea of FL is Federated Averaging commonly abbreviated as \texttt{FedAvg} [MMR+17]. In \texttt{FedAvg}, at every round, the server randomly chooses a subset of the clients and sends them the latest global model. These clients then undertake multiple steps of local updates (on the global model received from the server) with their respective data based on (stochastic) gradient descent, and then communicate back their respective updated local models to the server. The server then averages the clients’ local models to update the global model (hence the name \texttt{FedAvg}). \texttt{FedAvg} forms the basis of more advanced federated optimization algorithms. For the sake of completeness, we state \texttt{FedAvg} in Algorithm 3 (Appendix A). The convergence of \texttt{FedAvg} as well as other FL algorithms depends heavily on the number of local updates as well as the degree of data heterogeneity – specifically, for the same number of local updates, the convergence worsens as the amount of heterogeneity increases.

Despite the locality of data storage in FL, information-sharing opens the door to the possibility of sabotaging the security of personal data through communication. Hence, it is crucial to devise effective, privacy-preserving communication strategies that ensure the integrity and confidentiality of user data. Differential privacy (DP) [DMNS06] is a popular privacy-quantifying framework that is being incorporated in the training of ML models. In particular, DP focuses on a learning algorithm’s sensitivity to an individual’s data; a less sensitive algorithm is less likely to leak individuals’ private details through its output. This idea has laid the foundation for designing a simple strategy to ensure privacy by adding random Gaussian or Laplacian noise to the output, where the noise is scaled according to the algorithm’s sensitivity to an individual’s data. We talk about DP in more detail in Section 2.

There has been a lot of work on differentially private optimization in order to enable private training of ML models. In this regard, DP-SGD [ACG+16] is the most widely used private optimization algorithm in the centralized setting. It is essentially the same as regular SGD, except that Gaussian noise is added to the average of the “clipped” per-sample gradients (or updates) for privacy. There is a natural extension of DP-SGD to the federated setting based on \texttt{FedAvg}, wherein the server receives a noise-perturbed average of the “clipped” client updates [GKN17,TAM19]; this is called \texttt{DP-FedAvg} (with clipping) and it is stated in Algorithm 1. Specifically, if the original update is \( u \), then its clipped version is \( u \min(1, \frac{C}{\|u\|_2}) \), for some threshold \( C \); notice that this is the projection of \( u \) onto an \( \ell_2 \) ball of radius \( C \) centered at the origin. Clipping is performed to bound the sensitivity of the average update to each individual update, which is required to set the variance of the added Gaussian noise; specifically, the noise variance is proportional to \( C^2 \).

While the privacy aspect of DP-SGD and its variants, both in the centralized and federated setting, is typically the main consideration, the optimization aspect – particularly due to clipping – is not given that much attention. Specifically, the average of the clipped updates is biased and the amount of bias depends on the clipping threshold \( C \) – the higher the value of \( C \), the lower the bias, and vice-versa. But as mentioned before, the noise variance is proportional to \( C^2 \). Thus, the choice of the clipping threshold \( C \) is associated with an intrinsic tension between the bias and variance of the noise-perturbed average of the clipped updates, which impacts the rate of convergence.

To provide convergence guarantees for DP-SGD, most prior works assume that the per-sample losses are Lipschitz (i.e., they have bounded gradients); under this assumption, setting \( C \) equal
to the Lipschitz constant results in zero bias, making the convergence analysis trivial. But in practice, we cannot ascertain the Lipschitzness property, let alone figuring out the Lipschitz constant, due to which the choice of the clipping threshold is not trivial and requires a lot of tuning. So ideally, we would like to have convergence results for non-Lipschitz functions. However, there aren’t too many in the literature, primarily because analyzing the clipping bias is not easy, and more so in the federated setting due to multiple local updates. A few works in the centralized setting do provide some results for the non-Lipschitz case by making more relaxed assumptions [CWH20, WXDX20, KLZ21, BWLS21]; we discuss these in Section 3. However, in the more challenging federated setting with multiple local update steps, there is no result even for the convex non-Lipschitz case. In this work, we provide the first convergence result for differentially private federated convex optimization with a general clipping threshold, while not assuming Lipschitzness or making any other relaxed assumption; see Theorems 2 and 3. Moreover, prior works do not consider whether performing multiple local update steps is indeed beneficial (or not) for private optimization; we make the first attempt to analyze this theoretically. Informally, under an extra assumption, our result indicates that multiple local update steps are beneficial if the degree of heterogeneity of the data is dominated by poor choice of initialization (of the model parameters) for training. See (c) in the discussion after Theorem 2.

Further, we also propose a simpler alternative (compared to clipping) for bounding the sensitivity which is to always normalize the individual client updates; specifically, if the original update is $u$, then its normalized version is $u(C/\|u\|_2)$, for some appropriate scaling factor $C$. Surprisingly, this simpler option has not been considered by prior works on private optimization. The resultant private FL algorithm, where we replace clipping by normalization, is summarized in Algorithm 2 and we call it DP-NormFedAvg. We explain why/how/when the simpler alternative of normalization will offer better convergence than clipping in private optimization both theoretically (see Section 5.1) as well as intuitively (see Section 5.2); we also elaborate on this while summarizing our contributions next.

Our main contributions are summarized next:

(a) In Theorem 2, we provide a convergence result for DP-FedAvg with clipping (Algorithm 1) which is the first convergence result for differentially private federated convex optimization with a general clipping threshold and without assuming Lipschitzness, followed by a simplified (but less tight) convergence result in Theorem 3. Based on our derived result, we also attempt to quantify the benefit/harm of performing multiple local update steps in private optimization. Informally, under an extra assumption (Assumption 1), we show that multiple local updates are beneficial if the effect of poor initialization (of the model parameters) outweighs the effect of data heterogeneity by a factor depending on the privacy level; see (c) in the discussion after Theorem 2.

(b) In Section 5, we present DP-NormFedAvg (Algorithm 2) where we replace update clipping by the simpler alternative of update normalization (i.e., sending a scaled version of the unit vector along the update) for bounding the sensitivity. We provide a convergence result for DP-NormFedAvg in Theorem 4 and compare it against the result of DP-FedAvg with clipping (i.e., Theorem 2), showing that when the effect of poor initialization of the model parameters is more severe than the degree of data heterogeneity and/or if we can train for a large number of rounds, we expect the simpler alternative of normalization to offer better convergence than clipping in private optimization; see Remark 1 and Section 5.1 for details. Intuitively, this happens because normalization has a higher signal (i.e., update norm) to noise ratio than clipping; this aspect is discussed in detail in Section 5.2.

(c) We demonstrate the superiority of normalization over clipping via experiments on a synthetic quadratic problem in Section 5.3 as well as on three benchmarking datasets, viz., Fashion MNIST [XRV17], CIFAR-10 and CIFAR-100 in Section 6. For our synthetic experiment, we
show that normalization has a higher signal to noise ratio than clipping (as mentioned above) in Figure 2 and that the trajectory of normalization (projected in 2D space) reaches closer to the optimum of the function than the trajectory of clipping in Figure 3. In the experiments on benchmarking datasets, for \( \varepsilon = 5 \), the improvement offered by normalization over clipping w.r.t. the test accuracy is more than 2.8\% for CIFAR-100, 2.1\% for Fashion MNIST and 1.5\% for CIFAR-10; see Table 1.

2 Preliminaries

In this work, we are able to naturally quantify the effect of heterogeneity on convergence as follows.

**Definition 1 (Heterogeneity).** Let \( w^* \in \arg\min_{w' \in \mathbb{R}^d} f(w') \) and \( \Delta_i^* := f_i(w^*) - \min_{w' \in \mathbb{R}^d} f_i(w') \geq 0 \). Then the heterogeneity of the system is quantified by some increasing function of the \( \Delta_i^* \)'s.

The above way of quantifying heterogeneity shows up naturally in our convergence results for private FL assuming that the \( f_i 's \) are convex and smooth. The exact function of \( \Delta_i^* \)'s (quantifying heterogeneity) depends on the algorithm as well as data, and this will become clear when we present the convergence results. Also note that if the per-client data distributions (i.e., \( P_i 's \) ) are similar, then we expect the \( \Delta_i^* \)'s to be small indicating smaller heterogeneity.

**Differential Privacy (DP):** Suppose we have a collection of datasets \( D_c \) and a query function \( h : D_c \rightarrow \mathcal{X} \). Two datasets \( D, D' \in D_c \), and \( D' \neq D \), are said to be neighboring if they differ in exactly one sample, and we denote this by \( |D - D'| = 1 \). A randomized mechanism \( \mathcal{M} : \mathcal{X} \rightarrow \mathcal{Y} \) is said to be \((\varepsilon, \delta)-DP\), if for any two neighboring datasets \( D, D' \in D_c \), and for any measurable subset of outputs \( R \in \mathcal{Y},
\[
\mathbb{P}(\mathcal{M}(h(D)) \in R) \leq e^\varepsilon \mathbb{P}(\mathcal{M}(h(D')) \in R) + \delta.
\]

When \( \delta = 0 \), it is commonly known as pure DP. Otherwise, it is known as approximate DP.

Adding random Gaussian noise to the output of \( h(.) \) above is the customary approach to provide DP; this is known as the Gaussian mechanism and we formally define it below.

**Definition 2 (Gaussian mechanism [DR+14]).** Suppose \( \mathcal{X} \) (i.e., the range of the query function \( h \) above) is \( \mathbb{R}^p \). Let \( \Delta_2 := \sup_{D, D' \in D_c : |D - D'| = 1} \| h(D) - h(D') \|_2 \). If we set
\[
\mathcal{M}(h(D)) = h(D) + Z,
\]
where \( Z \sim \mathcal{N}(0_p, \frac{2 \log(1.25/\delta)\Delta_2^2}{\varepsilon^2} I_p) \), then the mechanism \( \mathcal{M} \) is \((\varepsilon, \delta)-DP\).

The Gaussian mechanism is also employed in private optimization [ACG+16].

**Definition 3 (Lipschitz).** A function \( g : \Theta \rightarrow \mathbb{R} \) is said to be \( G \)-Lipschitz if \( \sup_{\theta \in \Theta} \| \nabla g(\theta) \|_2 \leq G \).

**Definition 4 (Smoothness).** A function \( g : \Theta \rightarrow \mathbb{R} \) is said to be \( L \)-smooth if for all \( \theta, \theta' \in \Theta \),
\[
\| \nabla g(\theta) - \nabla g(\theta') \|_2 \leq L \| \theta - \theta' \|_2.
\]
If \( g \) is twice differentiable, then for all \( \theta, \theta' \in \Theta :\)
\[
g(\theta') \leq g(\theta) + \langle \nabla g(\theta), \theta' - \theta \rangle + \frac{L}{2} \| \theta' - \theta \|_2^2.
\]

**Definition 5 (A Key Quantity).** All the theoretical results in this paper are expressed in terms of the following key quantity:
\[
\rho := \frac{\sqrt{qd \log(1/\delta)}}{n\varepsilon},
\]

(3)
where \((\varepsilon, \delta)\)-DP is the desired privacy level, \(n\) is the number of samples, \(d\) is the parameter dimension and \(q\) is the absolute constant in Theorem 1. Further, all our results are for the non-vacuous privacy regime, i.e., when \(\varepsilon\) is finite and \(\delta < 1\), where \(\rho > 0\). Finally, we also assume that \(n\) is sufficiently large so that \(\rho < 1\).

Note that \(\rho\) increases as the level of privacy increases (i.e., \(\varepsilon\) and \(\delta\) decrease), and vice versa.

**Notation:** Throughout the rest of this paper, we denote the \(\ell_2\) norm simply by \(\|\cdot\|\) (omitting the subscript 2). Vectors and matrices are written in boldface. We denote the uniform distribution over the integers \(\{0, \ldots, a\}\) (where \(a \in \mathbb{N}\)) by \(\text{unif}[0, a]\). The function \(\text{clip}: \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d\) is defined as:

\[
\text{clip}(z, c) := z \min\left(1, \frac{c}{\|z\|}\right).
\]

\(K\) is the number of communication rounds or the number of global updates, \(E\) is the number of local updates per round, and \(r\) is the number of clients that the server accesses in each round.

The proofs of all theoretical results are in the Appendix.

### 3 Related Work

**Differentially private optimization:** Most differentially private optimization algorithms for training ML models (both in the centralized and federated settings) are based off of DP-SGD, wherein the optimizer receives a Gaussian noise-perturbed average of the clipped per-sample gradients (to guarantee DP), and the moments accountant method \([\text{ACG}+16]\). Similar to and/or related to DP-SGD, there are several papers on private optimization algorithms in the centralized setting \([\text{CM08}, \text{CMS11}, \text{KST12}, \text{SCS13}, \text{DJW13}, \text{BST14}, \text{TGTZ15}, \text{WLK}+17, \text{ZMW}+17, \text{WYX}+18]\) as well as in the federated and distributed (without multiple local updates) setting \([\text{GKN}17, \text{ASY}+18, \text{TAM}+19, \text{LLSS}+19, \text{PKM}+21, \text{GDD}+21]\). \(\text{DP-FedAvg with clipping}\) \([\text{GKN}17, \text{TAM}+19]\) (stated in Algorithm 1) is the most standard private algorithm in the federated setting. Among these previously mentioned works in the centralized setting, the ones that do provide convergence guarantees assume Lipschitzness and they set the clipping threshold equal to the Lipschitz constant, obtaining a suboptimality gap (i.e., \(\mathbb{E}[f(w_{\text{priv}})] - \min_w f(w)\), where \(w_{\text{priv}}\) is the output) in the convex case of \(O(\rho)\), where \(\rho = O\left(\frac{\sqrt{d \log(1/\delta)}}{n \varepsilon}\right)\) is the key quantity defined in Definition 5. In fact, \([\text{BST14}]\) show that in the convex Lipschitz case, the \(O(\rho)\) suboptimality gap is tight. However, as mentioned in Section 1, Lipschitzness is not a very practical assumption, due to which it is important to obtain convergence guarantees under weaker assumptions where there is no trivial clipping threshold. To that end, there a few results in the centralized setting that do not make the simplistic Lipschitzness assumption, but instead make more relaxed assumptions such as gradients having bounded moments \([\text{WXDX}20, \text{KLZ}+21]\) or the stochastic gradient noise having a symmetric probability distribution function \([\text{CWH}20]\). Also, \([\text{BWLS21}]\) analyze full-batch DP-GD from the NTK perspective for deep learning models. In comparison, there are hardly any convergence results for **private federated optimization** (which is harder to analyze due to multiple local steps) of non-Lipschitz objectives; \([\text{ZCH}+21]\) provide a complicated result for the nonconvex case, but surprisingly there is no result for the convex case. In addition, the role of multiple local steps in **private FL** has not been theoretically studied.

**Normalized gradient descent (GD) and related methods:** In the centralized setting, \([\text{HLSS15}]\) propose (Stochastic) Normalized GD. This is based on a similar idea as \(\text{DP-NormFedAvg}\) – instead of using the (stochastic) gradient, use the unit vector along the (stochastic) gradient for the update. Extensions of this method incorporating momentum \([\text{YGG}+17, \text{YLR}+19, \text{CM}+20]\) have been shown to significantly improve the training time of very large models such as BERT in the
centralized setting. In the FL setting, \cite{CGH+21} propose Normalized \texttt{FedAvg}, where the server uses a normalized version of the average of client updates (and not the average of normalized client updates, which is what we do) to improve training. However, it must be noted here that these works perform (some kind of) normalization to accelerate non-private training, whereas we are proposing normalization as an alternative sensitivity bounding mechanism to improve private training compared to the usual mechanism of clipping.

\section{Convergence of Vanilla DP-FedAvg with Client-Update Clipping}

First, we focus on the most standard version of \texttt{DP-FedAvg} involving client-update clipping, which is summarized in Algorithm \ref{alg:DP-FedAvg}. The primary difference from \texttt{FedAvg} is in lines 9, 10 and 12 of Algorithm \ref{alg:FedAvg}. Each client in the selected subset of clients sends its clipped update plus zero-mean Gaussian noise (for differential privacy) to the server; since Gaussian noise is additive, we can add it at the clients itself. The server then computes the mean of the noisy clipped client updates that it received (i.e., \(a_k\)) and then uses it to update the global model similar to \texttt{FedAvg}, except with a potentially different global learning rate \(\bar{\beta}_k\) than the local learning rate \(\eta_k\). Since each \(\zeta_k^{(i)}\) (i.e., noise added at client \(i\)) is \(N(0,\sigma^2 I_\ell)\), the average noise at the server is \(N(0_d, \sigma^2 1_d)\).

Using the moments accountant method of \cite{ACG+16}, we now specify the value of \(\sigma^2\) required to make Algorithm \ref{alg:DP-FedAvg} \((\varepsilon, \delta)\)-DP.

\begin{theorem}[\cite{ACG+16}]
For any \(0 < \varepsilon < \mathcal{O}\left(\frac{r^2 K}{n^2}\right)\), Algorithm \ref{alg:DP-FedAvg} will be \((\varepsilon, \delta)\)-DP as long as

\[
\sigma^2 = qKC^2 \frac{\log(1/\delta)}{n^2 \varepsilon^2},
\]

where \(q > 0\) is an absolute constant.
\end{theorem}

Note that the original DP-SGD algorithm of \cite{ACG+16} returns the last iterate (i.e., \(w_{K}\)) as the output, and Theorem 1 in their paper guarantees that the last iterate is \((\varepsilon, \delta)\)-DP by setting \(\sigma^2\) as per eq. \([5]\). But if the last iterate is \((\varepsilon, \delta)\)-DP, then so is any other iterate (due to additivity of the privacy cost), from which Theorem \ref{alg:DP-FedAvg} follows.

We now present the abridged convergence result for Algorithm \ref{alg:DP-FedAvg} the full version and proof can be found in Appendix \ref{app:convergence}.

\begin{theorem}[Convergence of \texttt{DP-FedAvg} with Clipping: Convex Case] Suppose each \(f_i\) is convex and \(L\)-smooth over \(\mathbb{R}^d\). Let \(\hat{C} := \frac{C}{\gamma}\), where \(C\) is the clipping threshold used in Algorithm \ref{alg:DP-FedAvg}. For any \(w^*\in\mathbb{R}^d\) where \(C\) is the clipping threshold used in Algorithm \ref{alg:DP-FedAvg}, the following convergence guarantees:

\[
E\left[\frac{1}{n} \sum_{i=1}^{n} \left(1(\|u_k^{(i)}\| \leq \hat{C}E) \left(2-\frac{\rho E}{\alpha} - \frac{\rho^2 E^2}{\alpha^2}\right)(f_i(w_k) - f_i(w^*)) + 1(\|u_k^{(i)}\| > \hat{C}E)(\frac{3\hat{C}}{8LE}\|u_k^{(i)}\|)\right)\right] \\
\leq \hat{C}\left(\frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L}\right) + \frac{3E}{\alpha} \left[\frac{1}{n} \sum_{i=1}^{n} 1(\|u_k^{(i)}\| \leq \hat{C}E)\Delta_i^*\right] \rho ,
\]

where \(\hat{C} := A(\text{effect of initialization})\) and \(\hat{C} := B(\text{effect of heterogeneity})\).

In the above result, we remind the reader that \(u_k^{(i)}\) is the \(i^{\text{th}}\) client's update at a random round number \(\tilde{k}\) (see line 9 in Algorithm \ref{alg:DP-FedAvg}). Also, this result is for the non-vacuous privacy regime, where \(\rho := \sqrt{\frac{qd \log(1/\delta)}{nx}} > 0\).

\end{theorem}
Algorithm 1 DP-FedAvg (with clipping)

1: **Input:** Initial point $w_0$, number of rounds of communication $K$, number of local updates per round $E$, local learning rates $\{\eta_k\}_{k=0}^{K-1}$, global learning rates $\{\beta_k\}_{k=0}^{K-1}$, clipping threshold $C$, number of participating clients in each round $r$ and noise variance $\sigma^2$.
2: for $k = 0, \ldots, K - 1$ do
3: Server sends $w_k$ to a random set $S_k$ of clients, formed by sampling each client $\in [n]$ with probability $r/n$.
4: for client $i \in S_k$ do
5: Set $w_{k,0}^{(i)} = w_k$.
6: for $\tau = 0, \ldots, E - 1$ do
7: Update $w_{k,\tau+1}^{(i)} \leftarrow w_{k,\tau}^{(i)} - \eta_{k}\nabla f_i(w_{k,\tau}^{(i)})$.
8: end for
9: Let $u_k^{(i)} = \frac{w_k - w_k^{(i)},E}{\eta}$ and $g_k^{(i)} = \text{clip}(u_k^{(i)},C) = u_k^{(i)} \min\left(1, \frac{C}{\|w_k^{(i)}\|}\right)$. // ($u_k^{(i)}$ is client $i$'s update.)
10: Send $(g_k^{(i)} + \zeta_k^{(i)})$ to the server, where $\zeta_k^{(i)} \sim \mathcal{N}(0, r^2 \sigma^2 1_d)$.
11: end for
12: Update $w_{k+1} = w_k - \beta_k a_k$, where $a_k = \frac{1}{r} \sum_{i \in S_k} (g_k^{(i)} + \zeta_k^{(i)})$.
13: Return $w_{\text{priv}} = w_k$, where $k \sim \text{unif}[0, K - 1]$.
14: end for

We also present a simplified, but less tight, convergence result based on Theorem 2; its proof is in Appendix C.

**Theorem 3 (Simplified version of Theorem 2).** With $\gamma = \mathcal{O}(L\|w_0 - w^*\|)$ and $\alpha = \mathcal{O}(1)$, the convergence result of Theorem 2 can be simplified to:

$$
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \min\left(f_i(w_k) - f_i(w^*), \mathcal{O}\left(\frac{C}{L} \|\nabla f_i(w_k)\|\right)\right)\right] \leq \mathcal{O}\left(\hat{C}\|w_0 - w^*\| + E\left(\frac{1}{n} \sum_{i=1}^{n} \Delta_i^*\right)\right) \rho. \quad (7)
$$

We now delineate the key implications of Theorem 2.

(a) **Convergence without assuming Lipschitzness:** Note that Theorem 2 does not assume any $f_i$ to be Lipschitz. To our knowledge, this is the first convergence result for private federated convex optimization with a general clipping threshold, and without assuming Lipschitz.

Let us also see what happens in the Lipschitz case. For that, suppose each $f_i$ is $G$-Lipschitz over $\mathbb{R}^d$. So if we set $\hat{C} = G$, then $\|u_k^{(i)}\| \leq \hat{C}E$ for all $k$. Now, using the fact that $\left(2 - \frac{\rho E}{\alpha} - \frac{\rho^2 E^2}{\alpha^2}\right) \geq \frac{5}{4}$ for $E \leq \frac{\alpha}{2\rho}$, the convergence result in Theorem 2 (i.e., eq. (4)) reduces to:

$$
\mathbb{E}[f(w_k)] - f(w^*) \leq \frac{4}{5} \left(\mathcal{O}\left(\frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{\hat{C}}\right) + \frac{3E}{2\alpha} \left(\frac{1}{n} \sum_{i=1}^{n} \Delta_i^*\right)\right) \rho. \quad (8)
$$

Thus with $E = \mathcal{O}(1)$, our bound matches the lower bound of [BST14] for the centralized convex and Lipschitz case with respect to the dependence on $\rho$.

(b) **Effect of initialization and heterogeneity:** Observe that our convergence result depends on two things: (i) term A in eq. (6), i.e. the distance of the initialization $w_0$ from the optimum $w^*$, and (ii) term B in eq. (8), i.e. the degree of heterogeneity which itself depends on the $\Delta_i^*$'s.
Additionally, each (ii)

(c) Effect of multiple local steps: Characterizing whether having multiple local steps, i.e. $E > 1$, is beneficial or detrimental is not obvious from Theorem 2. The RHS of eq. (6) seems to suggest that the convergence result gets worse as we increase $E$ – but this is whilst keeping $\hat{C}$ fixed. The intricacy here is that for the “same amount of clipping”, the required value of $\hat{C}$ is a non-increasing function of $E$. To make this more precise, let us consider two values of $E$, say $E_1$ and $E_2$ where $E_1 < E_2$. Suppose the corresponding clipping thresholds that we use are $C_1 = \hat{C}E_1$ and $C_2 = \hat{C}E_2$, respectively. Now if we wish to have $1(||u_{k(i)}|| \leq \hat{C}E_1) = 1(||u_{k(i)}|| \leq \hat{C}E_2)$, i.e. the “same amount of clipping” with $E_1$ and $E_2$, then $\hat{C} \leq \hat{C}_1$. This is because we are doing gradient descent on convex functions locally, due to which $\frac{\|u_{k(i)}\|}{E}$ is a non-increasing function of $E$. However, quantifying the extent of “non-increasingness” of $\frac{\|u_{k(i)}\|}{E}$ – which allows us to provide choices of $\hat{C}$ as a function of $E$ – is not easy (and perhaps not possible) without making more assumptions other than convexity and smoothness. So, we now make a couple of extra assumptions (one of which is the standard Lipschitzness assumption) in Assumption 1, which then allows us to illustrate and quantify the “non-increasingness” of $\hat{C}$ as a function of $E$ in Proposition 1.

Assumption 1. (i) For any $w \in \mathbb{R}^d$ and each $i \in [n]$, we have that:

$$\|\nabla f_i(w - \eta \nabla f_i(w)) - \nabla f_i(w)\| \geq \eta \lambda \|\nabla f_i(w)\|,$$

for some $0 < \lambda \leq L$ and $\eta \leq \frac{\eta}{2\mu}$.

(ii) Additionally, each $f_i$ is $G$-Lipschitz over $\mathbb{R}^d$.

Assumption 1 (i) can be also interpreted as a lower bound on the norm of the product of the Hessian matrix and the gradient vector. This is because for small enough $\eta$, we have:

$$\|\nabla f_i(w - \eta \nabla f_i(w)) - \nabla f_i(w)\| = \Theta(\eta\|\nabla^2 f_i(w)\|\nabla f_i(w)\|).$$

So basically for Assumption 1 (i) to hold, we are assuming $\|\nabla^2 f_i(w)\| \geq \Omega(\|\nabla f_i(w)\|)$; note that a similar assumption has been made in [DKL18]. Also, Assumption 1 (ii) is a weaker assumption than strong convexity. This is because strong convexity would imply that $\|\nabla^2 f_i(w)v\| \geq \mu\|v\|$ for any $v \in \mathbb{R}^d$ and some $\mu > 0$, while we assume this to hold only for $v = \nabla f_i(w)$ (with $\mu = \Omega(\lambda)$).

**Proposition 1 (\hat{C} under Assumption 1).** Set $\alpha = 1$ in Theorem 2; this imposes the constraint $E \leq \frac{1}{2\mu}$. Then, under Assumption 1 (i) choosing $\hat{C} = G \left( 1 - \frac{11(E-1)\eta}{64} \right)$ in Theorem 2 ensures that $1(||u_{k(i)}|| > \hat{C}E) = 0 \forall k$, i.e. no clipping happens for all $E$.

The proof of Proposition 1 is in Appendix D. Notice that $\hat{C}$, which is set so that the same (= zero) amount of clipping happens for all $E \leq \frac{1}{2\mu}$, is a non-increasing (more specifically, a decreasing) function of $E$ as mentioned before. It is worth mentioning here that the value of $\hat{C}$ in Proposition 1 is not the tightest possible value, but even for the tightest value, the non-increasingness will hold.

To summarize, there is a tradeoff involved as far as the number of local steps $E$ is concerned. Increasing $E$ allows us to reduce $\hat{C}$ which mitigates the effect of initialization, i.e. term A in eq. (6), but at the cost of increasing the effect of heterogeneity, i.e. term B in eq. (6). To illustrate
this tradeoff, let us plug in our choice of $\hat{C}$ derived in Proposition 1 in the convergence result of Theorem 2 with $\gamma = L\|\mathbf{w}_0 - \mathbf{w}^*\|$ (this choice is just for simplicity). After a bit of simplification, this yields:

$$
\mathbb{E}[f(\mathbf{w}_k)] - f(\mathbf{w}^*) \leq \left(2G\|\mathbf{w}_0 - \mathbf{w}^*\| + \frac{6E}{5}\left\{ \frac{1}{n} \sum_{i=1}^{n} \Delta_i^* - \frac{11G\|\mathbf{w}_0 - \mathbf{w}^*\|}{48}\left(\frac{\lambda^2}{E^2}\right) \right\} \right)\rho. \quad (11)
$$

Equation (11) tells us that if $\frac{1}{n} \sum_{i=1}^{n} \Delta_i^* < O\left(G\left(\frac{\lambda^2}{E^2}\right)\right)\|\mathbf{w}_0 - \mathbf{w}^*\|\rho$, which in plain English basically means that if the degree of heterogeneity is less than the product of the distance of the initialization from the optimum and $\rho = O\left(\frac{\sqrt{\log(1/\delta)}}{n\epsilon}\right)$ (and some other data-dependent constants), then having a large value of $E$ is beneficial; in particular, setting the maximum permissible value of $E$, which is $\frac{1}{\rho\delta}$, is the best (in terms of smallest suboptimality gap). Otherwise, having a small value of $E$ is better; specifically, setting $E = 1$ is the best. From Theorem 2, recall that for $\alpha = 1$, $K = \left(\frac{2\gamma}{CE}\right)\frac{1}{\rho\delta}$; so the higher we set $E$, the fewer the number of communication rounds needed.

From the above discussion, one should not form the opinion that a poor initialization, i.e., a $\mathbf{w}_0$ such that $\|\mathbf{w}_0 - \mathbf{w}^*\|$ is large, is advantageous in private FL. This is because choosing such a $\mathbf{w}_0$ will increase the first term within the big round brackets in eq. (11), i.e. $2G\|\mathbf{w}_0 - \mathbf{w}^*\|$, which happens to be the dominant term – leading to a high suboptimality gap by default.

5 DP-NormFedAvg: DP-FedAvg with Client-Update Normalization (instead of Clipping)

We define the normalization function norm : $\mathbb{R}^d - \{\mathbf{0}_d\} \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ as:

$$
norm(\mathbf{z}, c) := \frac{c\mathbf{z}}{\|\mathbf{z}\|}. \quad (12)
$$

where $c$ is the scaling factor. The parameter $c$ is analogous to the clipping threshold in the clip(.) function. Also, note that $\|\norm(\mathbf{z}, c)\| \leq c$ holds.

Here we propose to normalize client-updates instead of clipping them, i.e., we propose to change line 9 of Algorithm 1 as follows:

$$
g_k^{(i)} = \norm(u_k^{(i)}, c). \quad (13)
$$

We call the resultant algorithm DP-NormFedAvg because it involves normalizing updates in DP-FedAvg. For completeness, we state it in Algorithm 2; note the normalization step in line 9.

The abridged convergence result of DP-NormFedAvg is presented next. The full version and proof can be found in Appendix E.

**Theorem 4 (Convergence of DP-NormFedAvg: Convex Case).** In the same setting and with the same choices as Theorem 2 DP-NormFedAvg (i.e., Algorithm 2) has the following convergence guarantee:

$$
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_k^{(i)}\| \leq \tilde{C}E) \left( 2 - \frac{\rho^2 E^2}{\alpha^2} \right) \left( \frac{\tilde{C}E}{\|u_k^{(i)}\|} \right) (f_i(\mathbf{w}_k) - f_i(\mathbf{w}^*)) + \mathbb{I}(\|u_k^{(i)}\| > \tilde{C}E) \left( 3\tilde{C} \frac{\|u_k^{(i)}\|}{8LE} \right) \right] \\
\leq \tilde{C} \left( \frac{L\|\mathbf{w}_0 - \mathbf{w}^*\|^2}{\gamma} + \frac{\gamma}{L} \right) \rho + \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_k^{(i)}\| \leq \tilde{C}E) \left( \frac{\tilde{C}^2}{2\alpha L} + \left( \frac{\tilde{C}E}{\|u_k^{(i)}\|} \right) \frac{\Delta_i^* \rho E}{\alpha^2} \right) \right] \rho, \quad (14)
$$

with $\tilde{k} \sim \text{unif}[0, K - 1]$.

We now provide some insights on the convergence rate of Theorem 4 by comparing it with that of DP-FedAvg with clipping (i.e., Theorem 2).
**Algorithm 2 DP-NormFedAvg**

1: **Input:** Initial point $w_0$, number of rounds of communication $K$, number of local updates per round $E$, local learning rates $\{\eta_k\}_{k=0}^{K-1}$, global learning rates $\{\beta_k\}_{k=0}^{K-1}$, scaling factor $C$, number of participating clients in each round $\tau$, and noise variance $\sigma^2$.

2: **for** $k = 0, \ldots, K - 1$ **do**

3: Server sends $w_k$ to a random set $S_k$ of clients, formed by sampling each client $\in [n]$ with probability $r/n$.

4: **for** client $i \in S_k$ **do**

5: Set $w_{k,0}^{(i)} = w_k$.

6: **for** $\tau = 0, \ldots, E - 1$ **do**

7: Update $w_{k,\tau+1}^{(i)} \leftarrow w_{k,\tau}^{(i)} - \eta_k \nabla f_i(w_{k,\tau}^{(i)})$.

8: **end** for

9: Let $u_k^{(i)} = \frac{w_k - w_{k,\tau}^{(i)}}{\eta_k E}$ and $g_k^{(i)} = \text{norm}(u_k^{(i)}, C) = \frac{C u_k^{(i)}}{\|u_k^{(i)}\|}$ // (Normalization instead of Clipping.)

10: Send $(g_k^{(i)} + \zeta_k^{(i)})$ to the server, where $\zeta_k^{(i)} \sim \mathcal{N}(0, r \sigma^2 I_d)$.

11: **end** for

12: Update $w_{k+1} \leftarrow w_k - \beta_k a_k$, where $a_k = \frac{1}{r} \sum_{i \in S_k} (g_k^{(i)} + \zeta_k^{(i)})$.

13: Return $w_{\text{priv}} = w_k$, where $k \sim \text{unif}[0, K - 1]$.

14: **end** for

### 5.1 Theoretical Comparison of DP-FedAvg with Clipping and DP-NormFedAvg

Per Theorem 2, recall that the convergence rate of DP-FedAvg with clipping (i.e., Algorithm 1) is:

$$
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C} E) \left( 2 - \frac{\rho E}{\alpha} - \frac{\rho^2 E^2}{\alpha^2} \right) \left( f_i(w_k) - f_i(w^*) + \mathbb{1}(\|u_k^{(i)}\| > \hat{C} E) \left( \frac{3 \hat{C}}{8 L E} \|u_k^{(i)}\| \right) \right) \right] 
\leq \hat{C} \left( \frac{L \|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \right) \rho + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C} E) \left( \frac{3 \hat{C}}{8 L E} \|u_k^{(i)}\| \right) \right] \rho, \quad (15)
$$

with $k \sim \text{unif}[0, K - 1]$. In comparison, the convergence rate of DP-NormFedAvg (i.e., Algorithm 2), under the same setting, is:

$$
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C} E) \left( 2 - \frac{\rho E}{\alpha} - \frac{\rho^2 E^2}{\alpha^2} \right) \left( f_i(w_k) - f_i(w^*) + \mathbb{1}(\|u_k^{(i)}\| > \hat{C} E) \left( \frac{3 \hat{C}}{8 L E} \|u_k^{(i)}\| \right) \right) \right] 
\leq \hat{C} \left( \frac{L \|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \right) \rho + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C} E) \left( \frac{\hat{C}^2}{2 L} + \frac{\hat{C} E}{\|u_k^{(i)}\|} \right) \right] \rho. \quad (16)
$$

The terms that are different in eq. (15) and eq. (16) have been colored. Let us consider the same choice of $\hat{C}$ and optimum $w^*$ for both algorithms. As discussed earlier, the convergence rate depends on: (i) distance of the initialization $w_0$ from the optimum $w^*$ (specifically, term A in both equations), and (ii) the degree of heterogeneity which is itself a function of the $\Delta_i$'s (specifically, term B in eq. (15) and eq. (16), respectively).
Note that the LHS of eq. (16) is larger than the LHS of eq. (15). Thus, the effect of term A, i.e. the effect of initialization, on convergence is smaller in the case of normalization than clipping. Next, recalling that we must set \( \hat{C} \geq 4 \sqrt{L \max_{j \in [n]} \Delta_j^*} \) in both Theorem 2 and 4, let us choose \( \hat{C} = c \sqrt{L \max_{j \in [n]} \Delta_j^*} \) with \( c \geq 4 \) in both cases. Then:

\[
B_2 = E \left[ \frac{1}{n} \sum_{i=1}^{n} 1(\|u_k(i)\| \leq \hat{C}E) \left\{ \left( \frac{c^2 E}{2 \alpha} \right) \max_{j \in [n]} \Delta_j^* + \left( \frac{\hat{C}E}{\|u_k(i)\|} \right) \left( \frac{\Delta_i^* \rho E^2}{\alpha^2} \right) \right\} \right] \rho. \tag{17}
\]

Now observe that for \( c \geq 4 \), \( B_2 \) is less than \( B_2 \). However, the LHS of eq. (16) is more than that of eq. (15). So in general, it is difficult to predict whether the effect of heterogeneity on convergence is smaller in the case of clipping or normalization.

But the effect of heterogeneity can be mitigated arbitrarily for both clipping and normalization by increasing \( \alpha \), i.e. increasing the number of rounds \( K \) arbitrarily (recall that we set \( K = \left( \frac{2 \alpha}{\hat{C}E} \right)^{\frac{1}{\rho}} \)). So asymptotically, i.e. for \( \alpha \to \infty \) or \( K \to \infty \), the effect of heterogeneity gets killed and only the effect of initialization matters, where we expect normalization to outperform clipping. It is worth mentioning here that the previous discussion is not specific to the federated setting and also applies to the centralized setting (i.e., \( E = 1 \)).

We summarize all the above discussion in the following remark.

**Remark 1 (Normalization versus Clipping).** Compared to clipping, normalization is associated with a smaller effect of initialization on convergence. However, in general, it is difficult to characterize whether the effect of heterogeneity is smaller for normalization or clipping. The good thing is that for both clipping and normalization, the effect of heterogeneity gets killed asymptotically, i.e. when the number of communication rounds (\( K \)) tends to \( \infty \).

Hence, for problems that do not have a high degree of heterogeneity and the effect of initialization is more severe (for e.g., by poor random initialization) and/or if we can train for a very large number of rounds, we expect normalization to offer better convergence than clipping in private optimization.

It is also worth pointing out that clipping can be equivalent to normalization in certain scenarios. Specifically, suppose the client update norms are lower bounded by \( C_{low} \); then, clipping with threshold \( C \leq C_{low} \) is equivalent to normalization with the same scaling factor.

In Section 5.2 we provide a more intuitive argument as to why update normalization can offer better convergence than update clipping in terms of their signal (viz., update norm) to noise ratios, and also relate it to the previous theoretical comparison in Section 5.1.

### 5.2 Intuitive Explanation of why Normalization can Outperform Clipping

Intuitively, clipping has the following issue with respect to optimization - as the client update norms decrease and fall below the clipping threshold, the norm of the added noise (which has constant expectation proportional to the clipping threshold, regardless of the client update norms) can become arbitrarily larger than the client update norms, which should inhibit convergence. This issue is not as grave in DP-NormFedAvg because its update-normalization step ensures that the noise norm cannot become arbitrarily larger than the normalized update’s norm (even if the original update’s norm is small). In other words, the signal (which is the update norm) to noise ratio of clipping eventually falls below that of normalization.

The mathematical manifestation of the aforementioned argument can be also seen in the convergence bounds of clipping (i.e., eq. (15)) and normalization (i.e., eq. (16)) in Section 5.1 Specifically, note that the coefficient of \( 1(\|u_k(i)\| \leq \hat{C}E) \) (i.e., when the update norm \( \|u_k(i)\| \) is less
We set the clipping threshold \(\hat{C}E\) in the LHS of eq. (16) as at least \(\frac{CE}{\|v_k^{(i)}\|}\) \((\geq 1)\) times more than the corresponding term in the LHS of eq. (15); this amplification is a consequence of the improvement in signal to noise ratio (SNR) of normalization over clipping. On the other hand, the coefficient of \(I(\|v_k^{(i)}\| > \hat{C}E)\) (i.e., when the update norm is more than the clipping threshold) in the LHS of eq. (16) is exactly the same as the corresponding term in the LHS of eq. (15); this is because normalization and clipping are equivalent when \(\|v_k^{(i)}\| > \hat{C}E\). Now, as discussed in Section 5.1 the RHS of both eq. (15) and eq. (16) become the same asymptotically with a large number of rounds as the effect of heterogeneity dies down. Thus, the asymptotic convergence of normalization (i.e., eq. (16)) is better than that of clipping (i.e., eq. (15)).

Let us now see some experimental results on a synthetic problem illustrating the superiority of normalization over clipping.

### 5.3 Empirical Comparison of \textit{DP-FedAvg with Clipping and DP-NormFedAvg on a Synthetic Problem}

We consider \(f_i(w) = \frac{1}{2}(w - w^*)^TQ_i(w - w^*)\), where \(i \in [100]\) (so, \(n = 100\)) and \(w \in \mathbb{R}^{200}\) (so, \(d = 200\)). Further, \(w^*\) is drawn i.i.d. from \(\mathcal{N}(0, I_{200})\) and \(Q_i = A_iA_i^T\), where \(A_i\) is a \(200 \times 200\) matrix whose entries are drawn i.i.d from \(\mathcal{N}(0, 1/200);\) hence, \(Q_i\) is a PSD matrix with bounded maximum eigenvalue, due to which \(f_i\) is convex and smooth.

We set \((\varepsilon, \delta) = (5, 10^{-5}), K = 500\) and \(E = 20\) for this set of experiments. We consider two different initializations with different distances from the global optimum \(w^*\):

1. \(\textbf{I1}: \ w_0 = w^* + z, \) and
2. \(\textbf{I2}: \ w_0 = w^* + \frac{\varepsilon}{5}, \)

where each coordinate of \(z\) is drawn i.i.d. from the continuous uniform distribution with support \((0, 1)\). We set \(\eta_k = \beta_k = \eta\) for all rounds \(k\), and also have full-device participation. In Figure 1, we plot the function suboptimality (i.e., \(f(w_k) - f(w^*)\) at round number \(k\)) of \textit{DP-FedAvg} with Clipping and \textit{DP-NormFedAvg} for different values of \(\eta\) and clipping threshold/scaling factor \(C\), for I1 and I2; specifically, “Clip(\(\eta\))” and “Norm(\(\eta\))” in the legend denote \textit{DP-FedAvg} with Clipping and \textit{DP-NormFedAvg} with \(\eta_k = \beta_k = \eta\) for all rounds \(k\), respectively. In Figure 2, for each round \(k\), we plot the corresponding SNR := \[
\frac{\frac{1}{n} \sum_{i \in S_k} g_k^{(i)}}{\frac{1}{n} \sum_{i \in S_k} \xi_k^{(i)}}
\]
where \(g_k^{(i)}\) and \(\xi_k^{(i)}\) are the clipped/normalized per-client update and per-client noise, respectively, as defined in Algorithm 1 and 2. We only show the SNR plots for one value of \(\eta\) as the trend for other values of \(\eta\) is similar (and to avoid congestion). All plots are averaged over three independent runs. For a fair comparison, in each run, the exact same noise vectors (sampled randomly at each round) are used in both algorithms.

The thing to note in Figure 1 is that for \(C = \{50, 100\}\) and all values of \(\eta\), normalization attains an appreciably lower function suboptimality than clipping. For \(C = 40\), normalization is just slightly better. The SNR values in Figure 2 also follow a similar trend – the improvement in SNR for normalization compared to clipping is much higher for \(C = \{50, 100\}\) than \(C = 40\). We only show results up to \(C = 40\) as for smaller values of \(C\), clipping and normalization are equivalent. As discussed at the end of Section 5.1 after Remark 1, recall that if the client update norms are lower bounded by \(C_{low}\), then clipping with threshold \(C \leq C_{low}\) is equivalent to normalization with the same scaling factor.

For further illustration, in Figure 3, we plot the smoothed 2D projection of the trajectories of \textit{DP-FedAvg} with clipping and \textit{DP-NormFedAvg} for two of the cases of Figure 1. From here, we can see that \textit{DP-NormFedAvg} reaches closer to the optimum than \textit{DP-FedAvg} with clipping.

These plots corroborate our previous theoretical predictions and intuition. We also show the superiority of normalization over clipping via experiments on actual datasets in Section 6.
We consider the task of private multi-class classification to compare DP-FedAvg with clipping.
against DP-NormFedAvg; for brevity, we will often call them just clipping and normalization, respectively. Our experiments are performed on three benchmarking datasets, Fashion-MNIST 
[XRV17] (abbreviated as FMNIST henceforth), CIFAR-10 and CIFAR-100, where the first two datasets have 10 classes each and the last one has 100 classes.

Specifically, we consider logistic regression on FMNIST, CIFAR-10 and CIFAR-100 with ℓ₂-regularization; the weight decay value in PyTorch for ℓ₂-regularization is set to 1e-4. For FMNIST, we flatten each image into a 784-dimensional vector and use that as the feature vector. For CIFAR-10 and CIFAR-100, we use 512-dimensional features extracted from the last layer of a ResNet-18 [HZRS16] model pretrained on ImageNet. Similar to [MMR+17], we simulate a heterogeneous setting by distributing the data among the clients such that each client can have data from at most five classes. The exact procedure is described in Appendix G. For the CIFAR-10 and CIFAR-100 (respectively, FMNIST) experiment, the number of clients n is set to 5000 (respectively, 3000), with each client having the same number of samples. The number of participating clients in each round is set to r = 0.2n for all datasets, with 20 local client updates per-round.

We consider two privacy levels: ε = {5, 1.5} with δ = 10⁻⁵; note that ε = 5 (respectively, 1.5) corresponds to the low (respectively, high) privacy regime. For clipping and normalization, the values of C that we tune over are {500, 250, 125, 62.5, 31.25, 15.625}. The details about the learning rate schedule can be found in Appendix G. In Table 1, we show the comparison between clipping and normalization (in terms of test accuracy) for the two aforementioned privacy levels as well as vanilla FedAvg (without any privacy) as the baseline. The results reported here are the best ones for each algorithm by tuning over C and the learning rates, and have been averaged over three different runs.

In all cases, normalization is clearly superior to clipping. It is worth noting that the improvement obtained with normalization is more for the low privacy regime (i.e., ε = 5).

7 Conclusion

In this work, we provide the first convergence result for DP-FedAvg with clipping (which is the most standard algorithm for differentially private FL) in the convex case, and without assuming Lipschitzness. We also propose DP-NormFedAvg which normalizes client updates rather than clipping them (which is the customary approach for bounding sensitivity). Theoretically, we argue that DP-NormFedAvg should have better convergence than DP-FedAvg with clipping for
Table 1: Average test accuracy over the last 5 rounds for (a) FMNIST, (b) CIFAR-10 and (c) CIFAR-100. Recall that “Clipping” and “Normalization” denote $\text{DP-FedAvg}$ with Clipping and $\text{DP-NormFedAvg}$, respectively. The accuracy of $\text{FedAvg}$, which is our baseline without privacy, is at the bottom.

| Algo. | (5, 10^{-8})-DP | (1.5, 10^{-8})-DP |
|-------|-----------------|-------------------|
| Clipping | 75.59% | 56.90% |
| Normalization | **77.72%** | **57.80%** |
| FedAvg (w/o privacy) | 83.43% |

(a) FMNIST

| Algo. | (5, 10^{-5})-DP | (1.5, 10^{-5})-DP |
|-------|-----------------|-------------------|
| Clipping | 82.63% | 81.53% |
| Normalization | **84.21%** | **82.42%** |
| FedAvg (w/o privacy) | 85.64% |

(b) CIFAR-10

| Algo. | (5, 10^{-5})-DP | (1.5, 10^{-5})-DP |
|-------|-----------------|-------------------|
| Clipping | 56.53% | 41.33% |
| Normalization | **59.36%** | **42.76%** |
| FedAvg (w/o privacy) | 64.61% |

(c) CIFAR-100

problems that do not have a high degree of heterogeneity and the effect of poor initialization is more severe, and/or if we can train for a large number of rounds. Intuitively, this happens because normalization has a higher signal (i.e., update norm) to noise ratio than clipping. We also show the superiority of normalization over clipping via several experiments.

Several avenues of future work are possible. One of them is to provide principled recommendations on how to set the clipping threshold. Another one is to explore the feasibility of using adaptive and/or round-dependent clipping thresholds. It would be also nice to come up with meaningful additional assumptions that hold in practice, in order to simplify and/or improve our convergence results.

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Appendix

A  The FedAvg Algorithm

For the sake of completeness, here we state the famous FedAvg algorithm of [MMR+17] (with local updates using full gradients).

Algorithm 3 FedAvg [MMR+17]

1: Input: Initial point $w_0$, number of rounds of communication $K$, number of local updates per round $E$, local learning rates $\{\eta_k\}_{k=0}^{K-1}$ and number of participating clients in each round $r$.
2: for $k = 0, \ldots, K-1$ do
3: Server sends $w_k$ to a random set $S_k$ of $r$ clients chosen uniformly at random.
4: for client $i \in S_k$ do
5: Set $w_{k,0}^{(i)} = w_k$.
6: for $\tau = 0, \ldots, E - 1$ do
7: Update $w_{k,\tau+1}^{(i)} \leftarrow w_{k,\tau}^{(i)} - \eta_k \nabla f_i(w_{k,\tau}^{(i)})$.
8: end for
9: Send $w_k - w_{k,E}^{(i)}$ to the server.
10: end for
11: Update $w_{k+1} \leftarrow w_k - \frac{1}{r} \sum_{i \in S_k} (w_k - w_{k,E}^{(i)})$.
// (The above is equivalent to $w_{k+1} \leftarrow \frac{1}{r} \sum_{i \in S_k} w_{k,E}^{(i)}$ so the clients might as well just send the $w_{k,E}^{(i)}$'s.)
12: end for

B  Full Version of Theorem 2 and its Proof

Theorem 5 (Full version of Theorem 2). Suppose each $f_i$ is convex and $L$-smooth over $\mathbb{R}^d$. Let $\hat{C} := \frac{C}{E}$, where $C$ is the clipping threshold used in Algorithm 4. For any $w^* \in \arg\min_{w \in \mathbb{R}^d} f(w)$ and $\Delta_i^* := f_i(w^*) - \min_{w \in \mathbb{R}^d} f_i(w) \geq 0$, Algorithm 4 with $C \geq \sqrt{\frac{L \max_{j \in [n]} \Delta_j^*}{\gamma}}$, $\beta_k = \eta_k = \eta = \left(\frac{\gamma}{C\sqrt{K}}\right)\frac{1}{\rho}$ and $K > \left(\frac{2\gamma}{CE}\right)\frac{1}{\rho}$, where $\gamma > 0$ is a constant of our choice, has the following convergence guarantee:

$$
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_k^{(i)}\| \leq \hat{C}E) \left(2 - \frac{2\gamma}{EK\rho} - \frac{4\gamma^2}{E^2K^2\rho^2} \right)(f_i(w_k) - f_i(w^*)) + \mathbb{I}(\|u_k^{(i)}\| > \hat{C}E) \left(\frac{3\hat{C}\|u_k^{(i)}\|}{8LE}\right)\right] \\
\leq \hat{C} \left(\frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \rho + \left(\frac{2\gamma}{EK\rho}\right) \left(1 + \frac{2\gamma}{EK\rho}\right) \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_k^{(i)}\| \leq \hat{C}E) \Delta_i^*\right),
$$

with $\hat{k} \sim \text{unif}[0, K - 1]$.

Specifically, with $K = \left(\frac{2\alpha\gamma}{CE}\right)\frac{1}{\rho}$ and $E \leq \frac{\alpha}{2\rho}$, where $\alpha \geq 1$ is another constant of our choice,
Algorithm 1 has the following convergence guarantee:

$$
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \left(1 \left\| w_k^{(i)} \right\| \leq \hat{C} \mathbb{E}\left(2 - \frac{\rho E}{\alpha} - \frac{\rho^2 E^2}{\alpha^2}\right) (f_i(w_k) - f_i(w^*)) + 1 \left\| w_k^{(i)} \right\| > \hat{C} \mathbb{E}\left(\frac{3\tilde{C}\|u_k^{(i)}\|}{8LE}\right)\right)\right] 
\leq \hat{C} \left(\frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L}\right) + \left(\frac{3E}{2\alpha}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} 1(\|w_k^{(i)}\| > \hat{C} \mathbb{E}) \Delta_i^*\right].
$$

B.1 Proof of Theorem 5

Proof. Let us set $\eta_k = \beta_k = \eta$ for all $k \geq 0$.

The update rule of the global iterate is:

$$w_{k+1} = w_k - \eta \left(\frac{1}{r} \sum_{i \in S_k} \text{clip}(u_k^{(i)}, C) + \zeta_k\right),$$

where $\zeta_k = \frac{1}{r} \sum_{i \in S_k} \xi_k^{(i)} \sim \mathcal{N}(\tilde{\theta}_k, \frac{qK \log(1/\delta)C^2}{\eta_r^2} \mathbf{I}_d)$ and

$$u_k^{(i)} = \frac{w_k - w_k^{(i), E}}{\eta} = \sum_{i=0}^{E-1} \nabla f_i(w_k^{(i)}).$$

Taking expectation with respect to the randomness in the current round, we get for any $w^* \in \arg\min_{w' \in \mathbb{R}^d} f(w')$:

$$\mathbb{E}[\|w_{k+1} - w^*\|^2] = \mathbb{E} \left[\left\| w_k - \eta \left(\frac{1}{r} \sum_{i \in S_k} \text{clip}(u_k^{(i)}, C) + \zeta_k\right) - w^*\right\|^2\right]$$

$$= \|w_k - w^*\|^2 - 2\eta \mathbb{E}_{S_k} \left[\frac{1}{r} \sum_{i \in S_k} \langle \text{clip}(u_k^{(i)}, C), w_k - w^* \rangle\right] + \eta^2 \mathbb{E}_{S_k} \left[\left\| \frac{1}{r} \sum_{i \in S_k} \text{clip}(u_k^{(i)}, C) + \zeta_k\right\|^2\right]$$

$$= \|w_k - w^*\|^2 + \frac{1}{n} \sum_{i=1}^{n} -2\eta \langle \text{clip}(u_k^{(i)}, C), w_k - w^* \rangle + \eta^2 \mathbb{E}_{S_k} \left[\left\| \frac{1}{r} \sum_{i \in S_k} \text{clip}(u_k^{(i)}, C)\right\|^2\right]$$

$$+ \eta^2 \left(\frac{qK \log(1/\delta)C^2}{\eta_r^2}\right)$$

$$\leq \|w_k - w^*\|^2 + \frac{1}{n} \sum_{i=1}^{n} -2\eta \langle \text{clip}(u_k^{(i)}, C), w_k - w^* \rangle + \eta^2 \mathbb{E}_{S_k} \left[\left\| \text{clip}(u_k^{(i)}, C)\right\|^2\right]$$

$$+ \eta^2 \left(\frac{qK \log(1/\delta)C^2}{\eta_r^2}\right)$$

$$= \|w_k - w^*\|^2 + \frac{1}{n} \sum_{i=1}^{n} \left\{ -2\eta \langle \text{clip}(u_k^{(i)}, C), w_k - w^* \rangle + \eta^2 \left\| \text{clip}(u_k^{(i)}, C)\right\|^2\right\}$$

$$+ \eta^2 \left(\frac{qK \log(1/\delta)C^2}{\eta_r^2}\right).$$

Note that eq. (23) is obtained by using Fact 2. Let us examine $A_i$ for each $i$. 

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Case 1: $\|u_k^{(i)}\| > C$. So we have $\text{clip}(u_k^{(i)}, C) = \frac{C}{\|u_k^{(i)}\|} u_k^{(i)}$. Thus,

$$A_i = -2\eta C\langle u_k^{(i)}, w_k - w^* \rangle + \eta^2 C^2$$

$$= -\frac{C}{\|u_k^{(i)}\|} \left( \|w_k - w^*\|^2 + \eta^2 \|u_k^{(i)}\|^2 - \|w_k - \eta u_k^{(i)} - w^*\|^2 \right) + \eta^2 C^2,$$

where the last step follows by using the fact for any two vectors $a$ and $b$, $\langle a, b \rangle = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|a - b\|^2)$. Next, notice that $w_k - \eta u_k^{(i)} = w_k^{(i)}$. Since $f_i$ is convex, we use Lemma 1 to get:

$$\|w_k - w^*\|^2 - \|w_k^{(i)} - w^*\|^2 \geq \frac{\eta}{2L} \sum_{\tau=0}^{E-1} \|\nabla f_i(w_k^{(i)}_{\tau})\|^2 - 2\eta E \Delta_i^*,$$

for $\eta \leq \frac{1}{2L}$ with $\Delta_i^* := f_i(w^*) - \min_{w \in \mathbb{R}^d} f_i(w') \geq 0$. But:

$$\|u_k^{(i)}\|^2 = \left\| \sum_{\tau=0}^{E-1} \nabla f_i(w_k^{(i)}_{\tau}) \right\|^2 \leq E \sum_{\tau=0}^{E-1} \|\nabla f_i(w_k^{(i)}_{\tau})\|^2.$$

The inequality above follows from Fact 2. Using this in eq. (27), we get:

$$\|w_k - w^*\|^2 - \|w_k^{(i)} - w^*\|^2 \geq \frac{\eta}{2LE} \|u_k^{(i)}\|^2 - 2\eta E \Delta_i^*.$$ 

Plugging this back in eq. (26), we get:

$$A_i \leq -C \left( \eta^2 + \frac{\eta}{2LE} \right) \|u_k^{(i)}\|^2 + 2\eta \left( \frac{C}{\|u_k^{(i)}\|} \right) \frac{\eta}{16L} + \eta^2 C^2,$$

for $\eta \leq \frac{1}{2L}$.

Let us choose $C^2 \geq 16LE^2 \max_{j \in [n]} \Delta_j^*$. Then, we have $E \Delta_i^* \leq \frac{C^2}{16LE} \leq \frac{C\|u_k^{(i)}\|}{16LE}$. Using this in eq. (30), we get:

$$A_i \leq -C \left( \eta^2 + \frac{\eta}{2LE} \right) \|u_k^{(i)}\|^2 + \frac{\eta C}{8LE} \|u_k^{(i)}\|^2 + \eta^2 C^2$$

$$= -\frac{3\eta C}{8LE} \|u_k^{(i)}\|^2 + \eta^2 C (C - \|u_k^{(i)}\|)$$

$$\leq -\frac{3\eta C}{8LE} \|u_k^{(i)}\|^2,$$

for $C \geq 4E \sqrt{L \max_{j \in [n]} \Delta_j^*}$ and $\eta \leq \frac{1}{2L}$.

Case 2: $\|u_k^{(i)}\| \leq C$. So we have $\text{clip}(u_k^{(i)}, C) = u_k^{(i)}$. Thus,

$$A_i = -2\eta \langle u_k^{(i)}, w_k - w^* \rangle + \eta^2 \|u_k^{(i)}\|^2 \leq -2\eta \langle u_k^{(i)}, w_k - w^* \rangle + 2\eta^2 LE^2 (f_i(w_k) - f_i^*),$$

for $\eta \leq \frac{1}{2L}$.
for $\eta L \leq 1$; the inequality $\|u_k^{(i)}\|^2 \leq 2LE^2(f_i(w_k) - f_i^*)$ (for $\eta L \leq 1$) is obtained from Lemma 2.

Now:

$$B_i = \langle u_k^{(i)}, w_k - w^* \rangle$$

$$= \sum_{\tau=0}^{E-1} \langle \nabla f_i(w_{k,\tau}^{(i)}), w_k - w^* \rangle$$

$$= \sum_{\tau=0}^{E-1} \{ \langle \nabla f_i(w_{k,\tau}^{(i)}), w_{k,\tau}^{(i)} - w^* \rangle + \langle \nabla f_i(w_{k,\tau}^{(i)}), w_k - w_{k,\tau}^{(i)} \rangle \}$$

$$\geq \sum_{\tau=0}^{E-1} \{ f_i(w_{k,\tau}^{(i)}) - f_i(w^*) + \langle \nabla f_i(w_k), w_k - w_{k,\tau}^{(i)} \rangle + \langle \nabla f_i(w_{k,\tau}^{(i)}), w_k - w_{k,\tau}^{(i)} \rangle \}$$

$$\geq \sum_{\tau=0}^{E-1} \{ f_i(w_{k,\tau}^{(i)}) - f_i(w^*) + f_i(w_k) - f_i(w_{k,\tau}^{(i)}) - L\|w_k - w_{k,\tau}^{(i)}\|^2 \}$$

$$= E(f_i(w_k) - f_i(w^*)) - L \sum_{\tau=0}^{E-1} \|w_k - w_{k,\tau}^{(i)}\|^2.$$  (38)

Note that eq. (38) follows from the convexity of $f_i$, while eq. (39) follows by once again using the convexity of $f_i$, the smoothness of $f_i$, as well as the Cauchy-Schwarz inequality.

Again, from Lemma 2, we have

$$\|w_k - w_{k,\tau}^{(i)}\|^2 \leq 2\eta^2 L^2\tau^2(f_i(w_k) - f_i^*),$$  (41)

for $\eta L \leq 1$. Using eq. (41) in eq. (40), we get

$$B_i \geq E(f_i(w_k) - f_i(w^*)) - 2\eta^2 L^2 \sum_{\tau=0}^{E-1} \tau^2(f_i(w_k) - f_i^*) \geq E(f_i(w_k) - f_i(w^*)) - 2\eta^2 L^2 E^3(f_i(w_k) - f_i^*).$$  (42)

Now using eq. (42) in eq. (34), we get

$$A_i \leq -2\eta E(f_i(w_k) - f_i(w^*)) + 4\eta^3 L^2 E^3(f_i(w_k) - f_i^*) + 2\eta^2 L^2 E^2(f_i(w_k) - f_i^*)$$

$$= -\eta E\left(2 - 2\eta LE - 4\eta^2 L^2 E^2\right)(f_i(w_k) - f_i(w^*)) + (2\eta^2 L^2 E^2 + 4\eta^3 L^2 E^3)\Delta_i^*.$$  (43)

for $\eta \leq \frac{1}{L}$.

Combining the results of Case 1 and 2, i.e. eq. (33) and eq. (43), we get

$$A_i \leq \mathbb{1}(\|u_k^{(i)}\| > C) \left(\frac{3\eta C}{8LE}\|u_k^{(i)}\|\right)$$

$$+ \mathbb{1}(\|u_k^{(i)}\| \leq C) \left(\frac{\eta E\left(2 - 2\eta LE - 4\eta^2 L^2 E^2\right)(f_i(w_k) - f_i(w^*)) + (2\eta^2 L^2 E^2 + 4\eta^3 L^2 E^3)\Delta_i^*}{C^2}\right),$$  (44)

for $C \geq 4\sqrt{L \max_{j \in [n]} \Delta_j^*}$ and $\eta \leq \frac{1}{2L}$. Let us define $\hat{C} := \frac{C}{E}$. Then eq. (44) can be re-written as:

$$A_i \leq -\eta E\left\{ \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E)\left(2 - 2\eta LE - 4\eta^2 L^2 E^2\right)(f_i(w_k) - f_i(w^*))\right\}$$

$$+ \mathbb{1}(\|u_k^{(i)}\| > \hat{C}E)\left(\frac{3\hat{C}E}{8LE}\|u_k^{(i)}\|\right) - \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E)(2\eta LE + 4\eta^2 L^2 E^2)\Delta_i^*.$$  (45)
where $\hat{C} \geq 4\sqrt{L \max_{i \in [n]} \Delta_i^*}$ and $\eta \leq \frac{1}{2\varepsilon}$. Now using eq. (45) in eq. (24), we get:

$$
\mathbb{E}[\|w_{k+1} - w^*\|^2] \leq \|w_k - w^*\|^2 - \frac{\eta E}{n} \sum_{i=1}^{n} \{ \mathbb{1}(\|u_k^{(i)}\| > \hat{C}E) \left( \frac{3\hat{C}}{8LE} \|u_k^{(i)}\| \right) \\
+ \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \left( (2 - 2\eta LE - 4\eta^2 L^2 E^2)(f_i(w_k) - f_i(w^*)) \right) \}
+ \eta E(2\eta LE + 4\eta^2 L^2 E^2) \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \Delta_i^* \right) + \eta^2 E^2 \hat{C}^2 \left( \frac{K_d \log(1/\delta)}{n^2 \varepsilon^2} \right).
$$

(46)

Solving the above recursion after taking expectation throughout and some rearranging, we get:

$$
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \left( (2 - 2\eta LE - 4\eta^2 L^2 E^2)(f_i(w_k) - f_i(w^*)) \right) \right\} + \mathbb{1}(\|u_k^{(i)}\| > \hat{C}E) \left( \frac{3\hat{C}}{8LE} \|u_k^{(i)}\| \right) \right] \\
\leq \frac{\|w_0 - w^*\|^2}{\eta EK} + \eta E \hat{C}^2 \left( \frac{qd \log(1/\delta)}{n^2 \varepsilon^2} \right) + \frac{2\eta LE(1 + 2\eta LE)}{K} \mathbb{E} \left[ \sum_{k=0}^{K-1} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \Delta_i^* \right) \right].
$$

(47)

Let us choose $\eta = \frac{\gamma}{C E \sqrt{qd \log(1/\delta)}}$ for some constant $\gamma > 0$. Note that we must have $K > \frac{2\gamma}{C \hat{C} \sqrt{qd \log(1/\delta)}}$ for our condition of $\eta L \leq \frac{1}{2}$ to be satisfied. With that, we get:

$$
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \left( (2 - 2\gamma \sqrt{qd \log(1/\delta)} - \frac{4\gamma^2}{C^2 K^2 qd \log(1/\delta)}) (f_i(w_k) - f_i(w^*)) \right) \right\} + \mathbb{1}(\|u_k^{(i)}\| > \hat{C}E) \left( \frac{3\hat{C}}{8LE} \|u_k^{(i)}\| \right) \right] \leq \left( \frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \right) \hat{C} \sqrt{qd \log(1/\delta)}
+ \left( \frac{2\gamma}{C \sqrt{qd \log(1/\delta)}} \right) \left( 1 + \frac{2\gamma}{C K \sqrt{qd \log(1/\delta)}} \right) \mathbb{E} \left[ \left( \frac{1}{K} \sum_{k=0}^{K-1} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \Delta_i^* \right) \right) \right].
$$

(48)

with $\hat{C} \geq 4\sqrt{L \max_{i \in [n]} \Delta_i^*}$.

The above equation is equivalent to:

$$
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \left( (2 - 2\gamma \sqrt{qd \log(1/\delta)} - \frac{4\gamma^2}{C^2 K^2 qd \log(1/\delta)}) (f_i(w_k) - f_i(w^*)) \right) \right\} + \mathbb{1}(\|u_k^{(i)}\| > \hat{C}E) \left( \frac{3\hat{C}}{8LE} \|u_k^{(i)}\| \right) \right] \leq \left( \frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \right) \hat{C} \sqrt{qd \log(1/\delta)}
+ \left( \frac{2\gamma}{C \sqrt{qd \log(1/\delta)}} \right) \left( 1 + \frac{2\gamma}{C K \sqrt{qd \log(1/\delta)}} \right) \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\|u_k^{(i)}\| \leq \hat{C}E) \Delta_i^* \right) \right],
$$

(49)

where $\tilde{k} \sim \text{unif}[0, K - 1]$. Let us set $K = \frac{2\alpha}{C \hat{C} \left( \frac{n^2 \varepsilon^2}{\sqrt{qd \log(1/\delta)}} \right)^2}$ in eq. (49), where $\alpha \geq 1$ is a constant.
of our choice and \( E \leq \frac{\sigma}{2\sqrt{qd\log(1/\delta)}} \). That gives us:

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \mathbb{I}(\|u_{k}^{(i)}\| \leq \hat{C}E) \left( 2 - \left( \frac{E\sqrt{qd\log(1/\delta)}}{\alpha n} \right)^{2} \right) \right\} \right] \\
+ \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_{k}^{(i)}\| \leq \hat{C}E) \Delta_{i}^{*} \right] \\
\leq \hat{C}(E\|w_{0} - w^{*}\|^{2} + \gamma \frac{\sqrt{qd\log(1/\delta)}}{n\varepsilon} + \frac{\gamma}{L}) \\
+ \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_{k}^{(i)}\| \leq \hat{C}E) \Delta_{i}^{*} \right] \frac{E\sqrt{qd\log(1/\delta)}}{\alpha n} \left( 1 + \frac{E\sqrt{qd\log(1/\delta)}}{\alpha n} \right) \leq \frac{1}{2} \text{ from our constraint on } E.
\]

The final result follows by substituting \( \rho = \frac{\sqrt{qd\log(1/\delta)}}{n\varepsilon} \). \( \blacksquare \)

### C Proof of Theorem 3:

**Proof.** First, using

\[
\min \left( \frac{2 - \rho E}{\alpha} - \frac{\rho^{2}E^{2}}{\alpha^{2}} \right) (f_{i}(w_{k}) - f_{i}(w^{*})) , \frac{3\hat{C}}{8LE}\|u_{k}^{(i)}\| \leq \hat{C}(2 - \frac{\rho E}{\alpha} - \frac{\rho^{2}E^{2}}{\alpha^{2}}) (f_{i}(w_{k}) - f_{i}(w^{*})) + \mathbb{I}(\|u_{k}^{(i)}\| > \hat{C}E) \frac{3\hat{C}}{8LE}\|u_{k}^{(i)}\|, \quad (53)
\]

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\|u_{k}^{(i)}\| \leq \hat{C}E) \Delta_{i}^{*} \right] \leq \frac{1}{n} \sum_{i=1}^{n} \Delta_{i}^{*} , \quad \left( 2 - \frac{\rho E}{\alpha} - \frac{\rho^{2}E^{2}}{\alpha^{2}} \right) = \mathcal{O}(1) \text{ as } E \leq \frac{\alpha}{2\rho}, \quad \text{and plugging in } \gamma = \mathcal{O}(L\|w_{0} - w^{*}\|) \text{ and } \alpha = \mathcal{O}(1) \text{ in Theorem 2, we get:}
\]

\[
\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \min \left( f_{i}(w_{k}) - f_{i}(w^{*}) , \mathcal{O}\left( \frac{\hat{C}}{LE}\|u_{k}^{(i)}\| \right) \right) \right] \leq \mathcal{O}\left( \hat{C}\|w_{0} - w^{*}\| + \mathbb{E}\left( \frac{1}{n} \sum_{i=1}^{n} \Delta_{i}^{*} \right) \right). \quad (51)
\]

Now we need to lower bound \( \|u_{k}^{(i)}\| \) in terms of \( \|\nabla f_{i}(w_{k})\| \). To that end, note that:

\[
\|u_{k}^{(i)}\|^{2} = \left\| \frac{1}{E-1} \sum_{\tau=0}^{E-1} \nabla f_{i}(w_{k,\tau}) \right\|^{2} \|
\]

\[
= \sum_{\tau,\tau'} \frac{1}{2} \left( \|\nabla f_{i}(w_{k,\tau})\|^{2} + \|\nabla f_{i}(w_{k,\tau'})\|^{2} - \|\nabla f_{i}(w_{k,\tau}) - \nabla f_{i}(w_{k,\tau'})\|^{2} \right) \|
\]

\[
= \mathbb{E} \sum_{\tau=0}^{E-1} \|\nabla f_{i}(w_{k,\tau})\|^{2} - \sum_{\tau < \tau'} \|\nabla f_{i}(w_{k,\tau}) - \nabla f_{i}(w_{k,\tau'})\|^{2}. \|
\]

Equation (53) follows from the fact that for any two vectors \( a \) and \( b \), \( \langle a, b \rangle = \frac{1}{2}(\|a\|^{2} + \|b\|^{2} - \|a - b\|^{2}) \). Next, by using the \( L \)-smoothness of \( f_{i} \), we have for \( \tau < \tau' \):

\[
\|\nabla f_{i}(w_{k,\tau}) - \nabla f_{i}(w_{k,\tau'})\| \leq L\|w_{k,\tau}^{(i)} - w_{k,\tau'}^{(i)}\| \|
\]

\[
= \eta L \left\| \sum_{t=\tau}^{\tau'-1} \nabla f_{i}(w_{k,t}) \right\| \|
\]

\[
\leq \eta L \sum_{t=\tau}^{\tau'-1} \|\nabla f_{i}(w_{k,t})\|. \|
\]
But from Lemma 3, we have that \( \|\nabla f_i(w_{k,t})\| \leq \|\nabla f_i(w_{k,t-1})\| \leq \cdots \leq \|\nabla f_i(w_{k,0})\| = \|\nabla f_i(w_k)\| \). Using this in eq. (57), we get:

\[
\|\nabla f_i(w_{k,t}) - \nabla f_i(w_{k,t-1})\| \leq \|\nabla f_i(w_k)\| \leq \eta L(\tau' - \tau)\|\nabla f_i(w_k)\| \leq \eta L E\|\nabla f_i(w_k)\|. \tag{58}
\]

Plugging this into eq. (54), we get:

\[
\|u_k^{(i)}\|^2 \geq E \sum_{\tau=0}^{E-1} \|\nabla f_i(w_{k,\tau})\|^2 - \sum_{\tau < \tau'} \eta^2 L^2 E^2 \|\nabla f_i(w_k)\|^2 \tag{59}
\]

\[
\geq E \sum_{\tau=0}^{E-1} \|\nabla f_i(w_{k,\tau})\|^2 - \frac{\eta^2 L^2 E^4}{2} \|\nabla f_i(w_k)\|^2. \tag{60}
\]

Further, for any \( \tau \geq 1 \):

\[
\|\nabla f_i(w_k)\| \leq \|\nabla f_i(w_{k,\tau})\| + \|\nabla f_i(w_{k,t}) - \nabla f_i(w_{k,\tau})\| \tag{61}
\]

\[
\leq \|\nabla f_i(w_{k,\tau})\| + L\|w_k - w_{k,\tau}\|. \tag{62}
\]

Recall that \( \eta = \frac{\alpha}{2L} \) and \( E \leq \frac{\alpha}{2\rho} \) in Theorem 2, due to which \( \eta L \leq \frac{1}{4} \). Thus, we can apply Lemma 4 in eq. (62) to obtain:

\[
\|\nabla f_i(w_k)\| \leq \|\nabla f_i(w_{k,\tau})\| + 2\eta L \tau \|\nabla f_i(w_k)\|. \tag{63}
\]

Now using the fact that \( \eta L \tau \leq \eta L E \leq \frac{1}{4} \) above, we get:

\[
\|\nabla f_i(w_{k,\tau})\| \geq \frac{\|\nabla f_i(w_k)\|}{2} \forall \tau \geq 1. \tag{64}
\]

Plugging this back in eq. (60) and using the fact that \( \eta L E \leq \frac{1}{4} \), we get:

\[
\|u_k^{(i)}\|^2 \geq \frac{E^2}{4} \left(1 - 2\eta^2 L^2 E^2\right) \|\nabla f_i(w_k)\|^2 \geq \frac{7E^2}{32} \|\nabla f_i(w_k)\|^2. \tag{65}
\]

So, we have:

\[
\|u_k^{(i)}\| \geq O(E\|\nabla f_i(w_k)\|). \tag{66}
\]

Using this in eq. (51) gives us the final result.

\[
\boxed{}
\]

## D Proof of Proposition 1

**Proof.** First, note that with \( \eta_k = \eta \), we have:

\[
\|u_k^{(i)}\| = \left\| \frac{w_k - w_{k,E}}{\eta} \right\| \tag{67}
\]

\[
= \left\| \sum_{\tau=0}^{E-1} \nabla f_i(w_{k,\tau}) \right\| \tag{68}
\]

\[
\leq \sum_{\tau=0}^{E-1} \|\nabla f_i(w_{k,\tau})\|. \tag{69}
\]
Now using the result of Lemma 3 and applying our assumption that $\|\nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)})\| \geq \eta \lambda \|\nabla f_i(w_{k,\tau}^{(i)})\|$ in it, we get:

$$\|\nabla f_i(w_{k,\tau+1}^{(i)})\|^2 \leq \left(1 - \frac{2\eta \lambda^2}{L} \left(1 - \frac{\eta L}{2}\right)\right) \|\nabla f_i(w_{k,\tau}^{(i)})\|^2. $$ (70)

Plugging in $\eta = \frac{\rho}{2L}$ above, we get:

$$\|\nabla f_i(w_{k,\tau+1}^{(i)})\| \leq \sqrt{1 - \frac{\lambda^2}{L^2} \rho \left(1 - \frac{\rho}{4}\right)} \|\nabla f_i(w_{k,\tau}^{(i)})\|$$ (71)

$$\leq \left(1 - \frac{\lambda^2}{2L^2} \rho \left(1 - \frac{\rho}{4}\right)\right) \|\nabla f_i(w_{k,\tau}^{(i)})\|$$ (72)

$$\leq \left(1 - \frac{3\lambda^2}{8L^2} \rho\right) \|\nabla f_i(w_{k,\tau}^{(i)})\|. $$ (73)

For notational convenience, let $\hat{\rho} := \frac{3\lambda^2}{8L^2} \rho$. Then from eq. (73), we get:

$$\|\nabla f_i(w_{k,\tau}^{(i)})\| \leq (1 - \hat{\rho})\|\nabla f_i(w_{k,0}^{(i)})\|. $$ (74)

Using this in eq. (69), we get:

$$\|u_k^{(i)}\| \leq \sum_{\tau=0}^{E-1} (1 - \hat{\rho})^{\tau}\|\nabla f_i(w_{k,0}^{(i)})\| = \left(1 - \frac{(1 - \hat{\rho})^E}{\hat{\rho}}\right) \|\nabla f_i(w_{k,0}^{(i)})\| \leq \frac{1 - (1 - \hat{\rho})^E}{\hat{\rho}^B(E)} G. $$ (75)

Recall that $E \leq \frac{1}{2\rho} \Delta$ due to which we have $E \hat{\rho} \leq \frac{1}{4}$. So using Fact 3 in eq. (75), we get:

$$B(E) \leq GE\left(1 - \frac{11(E - 1)\hat{\rho}}{24}\right) = GE\left(1 - \frac{11(E - 1)\hat{\rho}}{64}\left(\frac{\lambda^2}{L^2}\right)\right). $$ (76)

So if we set $\tilde{C} = \frac{1 - 11(E - 1)\hat{\rho}}{64}\left(\frac{\lambda^2}{L^2}\right) = G\left(1 - \frac{11(E - 1)\hat{\rho}}{64}\left(\frac{\lambda^2}{L^2}\right)\right)$, then we will have no clipping as $\|u_k^{(i)}\| \leq \tilde{C}E$ always. 

\section*{E Full Version of Theorem 4 and its Proof}

**Theorem 6 (Full version of Theorem 4)**. Suppose each $f_i$ is convex and $L$-smooth over $\mathbb{R}^d$. Let $\hat{C} := \frac{C}{E}$, where $C$ is the scaling factor used in Algorithm 2. For any $w^* \in \arg\min_{w^i} f(w)$ and $\Delta_i^* := f_i(w^*) - \min_{w^i} f_i(w)$, $0 \geq 0$, Algorithm 2 with $\hat{C} \geq \frac{4\sqrt{L} \max_{j \in [n]} \Delta_j^*}{\gamma}$, $\beta_k = \eta_k = \eta = \left(\frac{\gamma \lambda_{\text{CEK}}}{4\hat{C}E}\right)^{\frac{1}{2}} \gamma$ and $K > \left(\frac{C \hat{C}}{\hat{C}E}\right)^{\frac{1}{2}} \gamma$, where $\gamma > 0$ is a constant of our choice, has the following convergence guarantee:

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\|u_k^{(i)}\| \leq \hat{C}E\right) \left(\frac{4\gamma^2}{\hat{C}^2 K^2 \rho^2} \left(\frac{\hat{C}E}{\|u_k^{(i)}\|} \right) \left(f_i(w_k^{(i)}) - f_i(w^*)\right) + \|u_k^{(i)}\| > \hat{C}E\right) \left(\frac{3\hat{C}E}{8LE} \|u_k^{(i)}\|\right)\right] \leq \tilde{C} \frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \partial + \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\|u_k^{(i)}\| \leq \hat{C}E\right) \left(\frac{\gamma \hat{C}}{LK \rho \|u_k^{(i)}\|} + \left(\frac{\hat{C}E}{\|u_k^{(i)}\|} \right) \frac{4\gamma^2 \Delta_i^*}{\hat{C}^2 K^2 \rho^2}\right)\right],$$

with $\tilde{k} \sim \text{unif}[0, K - 1]$. Further, this result holds for any $w^* = \arg\min_{w^i} f(w)$.
Specifically, with $K = \frac{2\alpha \gamma}{(CE)} \frac{1}{\rho^2}$ and $E \leq \alpha \frac{\rho}{2\rho}$, where $\alpha \geq 1$ is another constant of our choice, Algorithm 3 has the following convergence guarantee:

$$
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \|u_k^{(i)}\| \leq \hat{CE} \left( 2 - \frac{\rho^2 E^2}{\alpha^2} \right) \left( \|u_k^{(i)}\| \right) (f_i(w_k) - f_i(w^*)) + \|u_k^{(i)}\| > \hat{CE} \left( \frac{3\hat{C} \|u_k^{(i)}\|}{8LE} \right) \right\} \right]
\leq \hat{C} \left( \frac{L\|w_0 - w^*\|}{\gamma} + \frac{\gamma}{L} \right) \rho + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \|u_k^{(i)}\| \leq \hat{CE} \left( \frac{\hat{C}^2}{2\alpha L} + \left( \frac{\hat{C} \|u_k^{(i)}\|}{\alpha^2} \right) \right) \right\} \right] \rho.
$$

E.1 Proof of Theorem 6

Proof. Let us again set $\eta_k = \beta_k = \eta$, for all $k \geq 0$.

Everything remains the same till eq. (24) in the proof of Theorem 5 with clip(.) replaced by norm(.).

$$
\mathbb{E}[\|w_{k+1} - w^*\|^2] \leq \|w_k - w^*\|^2 + \frac{1}{n} \sum_{i=1}^{n} \left\{ -2\eta (\text{norm}(u_k^{(i)}, C), w_k - w^*) + \eta^2 \text{norm}(u_k^{(i)}, C)^2 \right\}
\leq \frac{\hat{C}^2}{2\alpha L} + \left( \frac{\hat{C} \|u_k^{(i)}\|}{\alpha^2} \right).
$$

Again, let us examine $A_i$ for each $i$. Also, as used in the proof of Theorem 5, let $\hat{C} = \frac{C}{\rho}$.

Case 1: $\|u_k^{(i)}\| > \hat{CE}$. Everything remains the same as Case 1 in the proof of Theorem 5. Thus,

$$
A_i \leq -\frac{3\eta \hat{C}}{8L} \|u_k^{(i)}\|,
$$

for $\eta L \leq \frac{1}{2}$ and $\hat{C} \geq 4\sqrt{L \max_{j \in [n]} \Delta_j}$.

Case 2: $\|u_k^{(i)}\| \leq \hat{CE}$. Here:

$$
A_i \leq \left( \frac{\hat{CE}}{\|u_k^{(i)}\|} \right) \left( -2\eta (u_k^{(i)}, w_k - w^*) \right) + \eta^2 \hat{C}^2 E^2.
$$

For ease of notation henceforth, let us define:

$$
z_k^{(i)} := \left( \frac{\hat{CE}}{\|u_k^{(i)}\|} \right).
$$

The bound for $B_i$ remains the same as the one in the proof of Theorem 5 (in eq. 42), i.e.,

$$
B_i \geq E(f_i(w_k) - f_i(w^*)) - 2\eta^2 L^2 E^3 (f_i(w_k) - f_i^*)
$$

for $\eta L \leq 1$. Using this in eq. (79), we get:

$$
A_i \leq -2\eta E z_k^{(i)} \left\{ (f_i(w_k) - f_i(w^*)) - 2\eta^2 L^2 E^2 (f_i(w_k) - f_i^*) \right\} + \eta^2 \hat{C}^2 E^2
= -2\eta E z_k^{(i)} \left\{ (f_i(w_k) - f_i(w^*)) - 2\eta^2 L^2 E^2 (f_i(w_k) - f_i^*) \right\} + \eta^2 \hat{C}^2 E^2.
$$

Combining the results of Case 1 and 2, i.e. eq. (78) and eq. (82), we get:

$$
A_i \leq \eta E \left\{ \mathbb{I}(\|u_k^{(i)}\| \leq \hat{CE}) \left( 4\eta^2 L^2 E^2 \Delta_i z_k^{(i)} + \eta \hat{C}^2 E \right) \right\}
- \mathbb{I}(\|u_k^{(i)}\| > \hat{CE}) (2 - 4\eta^2 L^2 E^2) z_k^{(i)} (f_i(w_k) - f_i(w^*)) - \mathbb{I}(\|u_k^{(i)}\| > \hat{CE}) \left( \frac{3\hat{C} \|u_k^{(i)}\|}{8LE} \right).
$$
for $\eta L \leq \frac{1}{2}$ and $\tilde{C} \geq 4\sqrt{L \max_{j \in [n]} \Delta_j^*}$.

Now using the above bound in eq. (77), plugging in $z_k^{(i)} = \frac{\tilde{C}E}{\|w_k^{(i)}\|}$, and following the same process and choice of $\eta = \frac{\gamma}{\tilde{C}E \sqrt{qd \log(1/\delta)}}$ that we used in Theorem 3 we get:

$$\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \mathbb{I}(\|w_k^{(i)}\| \leq \tilde{C}E) \left( 2 - \frac{4\gamma^2}{C^2K^2 qd \log(1/\delta)} \frac{n\epsilon^2}{\tilde{C}E} \right) (f_i(w_k) - f_i(w^*)) \right. \right.$$

$$+ \left. \mathbb{I}(\|w_k^{(i)}\| > \tilde{C}E) \left( 3\tilde{C}E \frac{\|w_k^{(i)}\|}{8LE} \right) \right\} \leq \left( \frac{L\|w_0 - w^*\|^2}{\gamma} + \frac{\gamma}{L} \tilde{C} \sqrt{qd \log(1/\delta)} \right) + \left( \frac{\gamma C}{L} \sqrt{qd \log(1/\delta)} + \frac{\gamma^2 C_1^*}{C^2K^2 qd \log(1/\delta)} \frac{n\epsilon^2}{\tilde{C}E} \right) \left( \frac{\|w_k^{(i)}\|}{\|w_k^{(i)}\|} \right),$$

(84)

with $\tilde{k} \sim \text{unif} \{0, K - 1\}$ and $K > \frac{2\gamma \tilde{C}E \sqrt{qd \log(1/\delta)}}{n\epsilon}$ (so that $\eta LE \leq \frac{1}{2}$). Now setting $K = \frac{2\gamma \tilde{C}E \sqrt{qd \log(1/\delta)}}{n\epsilon}$ above gives us the final result. □

F Lemmas and some Facts used in the Proofs

**Lemma 1.** Suppose $f_i$ is convex and $L$-smooth over $\mathbb{R}^d$. Let us set $\eta_k \leq \frac{1}{2L}$ for round $k$ of Algorithm 2 and 3 Then:

$$\|w_k^{(i)} - w^*\|^2 \leq \|w_k - w^*\|^2 - \frac{\eta_k}{2L} \sum_{\tau=0}^{E-1} \|\nabla f_i(w_k^{(i)}\|)^2 + 2\eta_k E \Delta^*_i,$$

where $\Delta^*_i := f_i(w^*) - \min_{w' \in \mathbb{R}^d} f_i(w').$

**Proof.** Let us define $f_i^* \equiv \min_{w' \in \mathbb{R}^d} f_i(w')$. Then, $\Delta^*_i = f_i(w^*) - f_i^*.$

For any $\tau \geq 0$, we have:

$$\|w_{k,\tau+1}^{(i)} - w^*\|^2 = \|w_{k,\tau}^{(i)} - w^*\|^2 - 2\eta_k (\nabla f_i(w_{k,\tau}^{(i)}), w_{k,\tau}^{(i)} - w^*) + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)}\|)^2$$

$$\leq \|w_{k,\tau}^{(i)} - w^*\|^2 - 2\eta_k f_i(w_{k,\tau}^{(i)}) - f_i(w^*) + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)}\|)^2$$

$$\leq \|w_{k,\tau}^{(i)} - w^*\|^2 - 2\eta_k (f_i(w_{k,\tau}^{(i)}) - f_i^*) + \eta_k (f_i(w^*) - f_i^*) + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)}\|)^2$$

$$\leq \|w_{k,\tau}^{(i)} - w^*\|^2 - \frac{\eta_k}{L} \|\nabla f_i(w_{k,\tau}^{(i)}\|)^2 + 2\eta_k \Delta^*_i + \eta_k^2 \|\nabla f_i(w_{k,\tau}^{(i)}\|)^2.$$  

(85)

Equation (85) follows by using the fact that each $f_i$ is convex. Equation (87) follows using Fact 1.

Now if we set $\eta_k \leq \frac{1}{2L}$, then we get:

$$\|w_{k,\tau+1}^{(i)} - w^*\|^2 \leq \|w_{k,\tau}^{(i)} - w^*\|^2 - \frac{\eta_k}{2L} \|\nabla f_i(w_{k,\tau}^{(i)}\|)^2 + 2\eta_k \Delta^*_i.$$  

(88)

Doing this recursively for $\tau = 0$ through to $\tau = E - 1$ and adding everything up gives us the desired result. □

**Lemma 2.** Suppose each $f_i$ is $L$-smooth over $\mathbb{R}^d$ and $f_i^* := \min_{w' \in \mathbb{R}^d} f_i(w')$. Let us set $\eta_k \leq \frac{1}{L}$ for round $k$ of Algorithm 2 and 3 Then:

$$\|w_k - w_k^{(i)}\|^2 \leq 2\eta_k L^2 (f_i(w_k) - f_i^*) \forall \tau \geq 1.$$

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Thus,
\[\|u_k^{(i)}\|^2 \leq 2LE^2(f_i(w_k) - f_i^*).\]

Proof.

\[
\|w_k - w_{k,\tau}^{(i)}\|^2 = \left\| \eta_k \sum_{t=0}^{\tau-1} \nabla f_i(w_{k,t}^{(i)}) \right\|^2 \leq \eta_k^2 \tau \sum_{t=0}^{\tau-1} \|\nabla f_i(w_{k,t}^{(i)})\|^2, \tag{89}
\]

where the last step follows from Fact 2. Next, since \( f_i \) is \( L \)-smooth, we have using Fact 1
\[
\|\nabla f_i(w_{k,t}^{(i)})\|^2 \leq 2L(f_i(w_{k,t}^{(i)}) - f_i^*). 
\]

Applying this in eq. (89), we get:
\[
\|w_k - w_{k,\tau}^{(i)}\|^2 \leq 2\eta_k^2 L \tau \sum_{t=0}^{\tau-1} (f_i(w_{k,t}^{(i)}) - f_i^*). \tag{90}
\]

But using the \( L \)-smoothness of \( f_i \), we have for any \( t \geq 1 \):
\[
f_i(w_{k,t}^{(i)}) - f_i^* = f_i(w_{k,t-1}^{(i)} - \eta_k \nabla f_i(w_{k,t-1}^{(i)})) - f_i^*
\leq (f_i(w_{k,t-1}^{(i)}) - f_i^*) - \eta_k \|\nabla f_i(w_{k,t-1}^{(i)})\| \|\nabla f_i(w_{k,t-1}^{(i)})\|^2 + \frac{\eta_k^2 L}{2} \|\nabla f_i(w_{k,t-1}^{(i)})\|^2
\leq (f_i(w_{k,t-1}^{(i)}) - f_i^*) - \frac{\eta_k}{2} \|\nabla f_i(w_{k,t-1}^{(i)})\|^2, \tag{91}
\]

for \( \eta_k L \leq 1 \). Doing this recursively (and recalling that \( w_{k,0}^{(i)} = w_k \)), we get:
\[
f_i(w_{k,t}^{(i)}) - f_i^* \leq (f_i(w_k) - f_i^*) - \frac{\eta_k}{2} \sum_{t'=0}^{t-1} \|\nabla f_i(w_{k,t'}^{(i)})\|^2 \leq f_i(w_k) - f_i^*. \tag{94}
\]

Plugging this in eq. (90), we get:
\[
\|w_k - w_{k,\tau}^{(i)}\|^2 \leq 2\eta_k^2 L \tau^2 (f_i(w_k) - f_i^*). \tag{95}
\]

The upper bound on \( \|u_k^{(i)}\|^2 \) follows by recalling that \( u_k^{(i)} = (w_k - w_{k,E}^{(i)})/\eta_k \).

Lemma 3. Suppose each \( f_i \) is \( L \)-smooth over \( \mathbb{R}^d \). Then for both Algorithm 1 and 2 we have:
\[
\|\nabla f_i(w_{k,\tau+1}^{(i)})\|^2 \leq \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 - \left( \frac{2}{\eta_k L} - 1 \right) \|\nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)})\|^2, \tag{96}
\]

for any \( i \in [n], k \in \{0, \ldots, K-1\} \) and \( \tau \in \{0, \ldots, E-1\} \).

Proof. Since each \( f_i \) is \( L \)-smooth, we have by using the co-coercivity of the gradient:
\[
\langle \nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)}), w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)} \rangle \geq \frac{1}{L} \|\nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)})\|^2. \tag{96}
\]

Now using the fact that \( w_{k,\tau+1}^{(i)} - w_{k,\tau}^{(i)} = -\eta_k \nabla f_i(w_{k,\tau}^{(i)}) \) above, we get:
\[
L(\nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)}), -\eta_k \nabla f_i(w_{k,\tau}^{(i)})) \geq \|\nabla f_i(w_{k,\tau+1}^{(i)})\|^2 + \|\nabla f_i(w_{k,\tau}^{(i)})\|^2
- 2\langle \nabla f_i(w_{k,\tau+1}^{(i)}), \nabla f_i(w_{k,\tau}^{(i)})) \rangle. \tag{97}
\]
Rearranging the above a bit, we get:
\[
(2 - \eta_k L)\langle \nabla f_i(w_{k,\tau+1}^{(i)}), \nabla f_i(w_{k,\tau}^{(i)}) \rangle \geq \|\nabla f_i(w_{k,\tau+1}^{(i)})\|^2 + (1 - \eta_k L)\|\nabla f_i(w_{k,\tau}^{(i)})\|^2.
\]
(98)
But, we also have:
\[
\langle \nabla f_i(w_{k,\tau+1}^{(i)}), \nabla f_i(w_{k,\tau}^{(i)}) \rangle = \frac{1}{2} \left( \|\nabla f_i(w_{k,\tau+1}^{(i)})\|^2 + \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 - \|\nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)})\|^2 \right).
\]
(99)
Using this in eq. (98) and simplifying a bit, we get:
\[
\|\nabla f_i(w_{k,\tau+1}^{(i)})\|^2 \leq \|\nabla f_i(w_{k,\tau}^{(i)})\|^2 - \left( \frac{2}{\eta_k L} - 1 \right)\|\nabla f_i(w_{k,\tau+1}^{(i)}) - \nabla f_i(w_{k,\tau}^{(i)})\|^2.
\]
(100)
This completes the proof.

**Lemma 4.** Suppose each $f_i$ is $L$-smooth over $\mathbb{R}^d$. Let us set $\eta_k \leq \frac{1}{2L}$ for round $k$ of Algorithm 1 and 2. Then:
\[
\|w_k - w_{k,\tau}^{(i)}\| \leq 2\eta_k \tau\|\nabla f_i(w_k)\| \forall \tau \geq 1.
\]

The reader might be wondering that Lemma 2 also bounds $\|w_k - w_{k,\tau}^{(i)}\|$, so why do we need this lemma? The difference is that this lemma provides a stronger bound at the cost of a stronger requirement on $\eta_k$, whereas Lemma 2 provides a weaker bound but it imposes a weaker requirement on $\eta_k$. This lemma is used only in the proof of Theorem 3, while Lemma 2 is used in the proofs of Theorems 2 and 4.

**Proof.**
\[
\|w_k - w_{k,\tau}^{(i)}\| = \left\| \eta_k \sum_{t=0}^{\tau-1} \nabla f_i(w_{k,t}^{(i)}) \right\| \leq \eta_k \sum_{t=0}^{\tau-1} \|\nabla f_i(w_{k,t}^{(i)})\|.
\]
(101)
But:
\[
\|\nabla f_i(w_{k,t}^{(i)})\| = \|\nabla f_i(w_{k,t}^{(i)}) - \nabla f_i(w_k) + \nabla f_i(w_k)\|
\leq \|\nabla f_i(w_k)\| + \|\nabla f_i(w_{k,t}^{(i)}) - \nabla f_i(w_k)\|
\leq \|\nabla f_i(w_k)\| + L\|w_{k,t}^{(i)} - w_k\|.
\]
(102)
Putting eq. (102) back in eq. (101), we get:
\[
\|w_k - w_{k,\tau}^{(i)}\| \leq \eta_k \tau\|\nabla f_i(w_k)\| + \eta_k L \sum_{t=0}^{\tau-1} \|w_{k,t}^{(i)} - w_k\|.
\]
(103)
We claim that $\|w_k - w_{k,\tau}^{(i)}\| \leq 2\eta_k \tau\|\nabla f_i(w_k)\|$ for $\eta_k LE \leq 1/2$. We shall prove this by induction. Let us first check the base case of $\tau = 1$. Observe that:
\[
\|w_k - w_{k,1}^{(i)}\| = \eta_k \|\nabla f_i(w_k)\| \leq 2\eta_k \|\nabla f_i(w_k)\|.
\]
Hence, the base case is true. Assume the hypothesis holds for $t \in \{0, \ldots, \tau - 1\}$. Let us now put our induction hypothesis into eq. (103) to see if the hypothesis is true for $\tau$ as well.
\[
\|w_k - w_{k,\tau}^{(i)}\| \leq \eta_k \tau\|\nabla f_i(w_k)\| + \eta_k L \sum_{t=0}^{\tau-1} 2\eta_k t\|\nabla f_i(w_k)\|
\leq \eta_k \tau\|\nabla f_i(w_k)\| + (\eta_k L)\eta_k \tau^2\|\nabla f_i(w_k)\|
\leq \eta_k \tau\|\nabla f_i(w_k)\| + \eta_k \tau (\eta_k L\tau)\|\nabla f_i(w_k)\|
\leq \eta_k \tau\|\nabla f_i(w_k)\| + 0.5\eta_k \tau\|\nabla f_i(w_k)\| < 2\eta_k \tau\|\nabla f_i(w_k)\|.
\]
The second last inequality is true because $\eta_k L \tau \leq \eta_k L E \leq \frac{1}{2}$, per our choice of $\eta_k$.

Thus, the hypothesis holds for $\tau$ as well. So by induction, our claim is true. ■

**Fact 1** ([N+18]). For an $L$-smooth function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ with $h^* = \min_{x \in \mathbb{R}^d} h(x)$ and $L > 0$, $\|\nabla h(x)\|^2 \leq 2L(h(x) - h^*)$.

**Fact 2.** For any $p > 1$ vectors $\{y_1, \ldots, y_p\}$, $\|\sum_{i=1}^{p} y_i\|^2 \leq p \sum_{i=1}^{p} \|y_i\|^2$.

Fact 2 follows from Jensen’s inequality.

**Fact 3.** Suppose $x \in (0, 1)$. Then for any positive integer $m$ such that $mx \leq \frac{1}{4}$, we have:

$$\frac{1 - (1 - x)^m}{x} \leq m \left(1 - \frac{11(m-1)}{24} x\right). \quad (104)$$

**Proof.** Using the Binomial expansion, we have:

$$(1 - x)^m \geq 1 - mx + \frac{m(m-1)}{2} x^2 - \frac{m(m-1)(m-2)}{6} x^3. \quad (105)$$

Thus,

$$\frac{1 - (1 - x)^m}{x} \leq m \left\{ 1 - \frac{(m-1)}{2} x + \frac{(m-1)(m-2)}{6} x^2 \right\} \quad (106)$$

$$\leq m \left\{ 1 - \frac{(m-1)}{2} x + \frac{(m-1)}{6} x \left( \frac{m-2}{4m} \right) \right\} \quad (107)$$

$$\leq m \left(1 - \frac{11(m-1)}{24} x\right). \quad (108)$$

Here, eq. (107) follows from the fact that $mx \leq \frac{1}{4}$. ■

**G Experimental Details**

First, we explain the procedure we have used to generate heterogeneous data for our FL experiments in Section 6. For each dataset (individually), the training data was first sorted based on labels and then divided into $5n$ equal data-shards, where $n$ is the number of clients. Splitting the data in this way ensures that each shard contains data from only one class for all datasets (and because $n$ was chosen appropriately). Now, each client is assigned 5 shards chosen uniformly at random without replacement which ensures that each client can have data belonging to at most 5 distinct classes.

Next, we specify the learning rate schedule for our experiments in Section 6. We use $\beta_k = \eta_k$ for all $k$. We employ the learning rate scheme suggested in [Bot12] where we decrease the local learning rate by a factor of 0.99 after every round, i.e. $\eta_k = (0.99)^k \eta_0$. We search the best initial local learning rates $\eta_0$ over $\{10^{-3}, 2 \times 10^{-3}, 4 \times 10^{-3}, 8 \times 10^{-3}, 1.6 \times 10^{-2}, 3.2 \times 10^{-2}, 6.4 \times 10^{-2}\}$ in each case. Server momentum = 0.8 is also applied (at the server).