SPECIAL PRIME FANO FOURFOLDS OF DEGREE 10 AND INDEX 2

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Abstract. We analyze (complex) prime Fano fourfolds of degree 10 and index 2. Mukai gave a complete geometrical description; in particular, most of them are contained in a Grassmannian $G(2, 5)$. They are all unirational and, as in the case of cubic fourfolds, some are rational, as already remarked by Roth in 1949.

We show that their middle cohomology is of K3 type and that their period map is dominant, with smooth 4-dimensional fibers, onto a 20-dimensional bounded symmetric period domain of type IV. Following Hassett, we say that such a fourfold is special if it contains a surface whose cohomology class does not come from the Grassmannian $G(2, 5)$. Special fourfolds correspond to a countable union of hypersurfaces (the Noether-Lefschetz locus) in the period domain, labelled by a positive integer $d$. We describe special fourfolds for some low values of $d$. We also characterize those integers $d$ for which special fourfolds do exist.

1. Introduction

One of the most vexing classical questions in complex algebraic geometry is whether there exist irrational smooth cubic hypersurfaces in $\mathbb{P}^5$. They are all unirational, and rational examples are easy to construct (such as Pfaffian cubic fourfolds) but no smooth cubic fourfold has yet been proven to be irrational. The general feeling seems to be that the question should have an affirmative answer but, despite numerous attempts, it is still open.

In a couple of very interesting articles on cubic fourfolds ([H1], [H2]), Hassett adopted a Hodge-theoretic approach and, using the period map (proven to be injective by Voisin in [V]) and the geometry of the period domain, a 20-dimensional bounded symmetric domain of type IV, he related geometrical properties of a cubic fourfold to arithmetical properties of its period point.

We do not solve the rationality question in this article, but we investigate instead similar questions for another family of Fano fourfolds (see §2 for their definition). Again, they are all unirational (see §3), and rational examples were found by Roth (see [R], and also [P] and [7]), but no irrational examples are known.

We prove in §4 that the moduli stack $\mathcal{Z}_{10}$ associated with these fourfolds is smooth of dimension 24 (Proposition 4.1) and that the period map is smooth and dominant onto, again, a 20-dimensional bounded symmetric domain of type IV (Theorem 4.4). We identify

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the underlying lattice in \[\S5\]. Then, following \[H1\], we define in \[\S6.1\] hypersurfaces in the period domain which parametrize “special” fourfolds \(X\), whose period point satisfy a non-trivial arithmetical property depending on a positive integer \(d\), the discriminant. As in \[H1\], we characterize in Proposition \[6.5\] those integers \(d\) for which the non-special cohomology of a special \(X\) is essentially the primitive cohomology of a K3 surface; we say that this K3 surface is associated with \(X\). Similarly, we characterize in Proposition \[6.6\] those \(d\) for which the non-special cohomology of a special \(X\) is the non-special cohomology of a cubic fourfold in the sense of \[H1\].

In \[\S7\] we give geometrical constructions of special fourfolds for \(d \in \{8, 10, 12\}\); in particular, we discuss some rational examples (already present in \[R\] and \[P\]). When \(d = 10\), the associated K3 surface (in the sense of Proposition \[6.5\]) does appear in the construction; when \(d = 12\), so does the associated cubic fourfold (in the sense of Proposition \[6.6\]) and they are even birationally isomorphic.

In \[\S8\] we characterize the positive integers \(d\) for which there exist (smooth) special fourfolds of discriminant \(d\). As in \[H1\], our construction relies on the surjectivity of the period map for K3 surfaces. Finally, we summarize in \[\S9\] some of our results on the period map and ask a couple of questions about its geometry.

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2. Prime Fano fourfolds of degree 10 and index 2

Let \(X\) be a (smooth) prime Fano fourfold of degree 10 (i.e., of “genus” 6) and index 2; this means that \(\text{Pic}(X)\) is generated by the class of an ample divisor \(H\) such that \(H^4 = 10\) and \(-K_X \equiv 2H\). Then \(H\) is very ample and embeds \(X\) in \(\mathbb{P}^8\) as follows (\[M2\]; \[IP\], Theorem 5.2.3).

Let \(V_5\) be a 5-dimensional vector space (our running notation is \(V_k\) for any \(k\)-dimensional vector space). Let \(G(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)\) be the Grassmannian in its Plücker embedding and let \(CG \subset \mathbb{P}(\mathbb{C} \oplus \wedge^2 V_5) \simeq \mathbb{P}^{10}\) be the cone, with vertex \(v = \mathbb{P}(\mathbb{C})\), over \(G(2, V_5)\). Then

\[
X = CG \cap \mathbb{P}^8 \cap Q,
\]

where \(Q\) is a quadric. There are two cases:

- either \(v \notin \mathbb{P}^8\), in which case \(X\) is isomorphic to the intersection of \(G(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)\) with a hyperplane (the projection of \(\mathbb{P}^8\) to \(\mathbb{P}(\wedge^2 V_5)\)) and a quadric;
- or \(v \in \mathbb{P}^8\), in which case \(\mathbb{P}^8\) is a cone over a \(\mathbb{P}^7 \subset \mathbb{P}(\wedge^2 V_5)\) and \(X\) is a double cover of \(G(2, V_5) \cap \mathbb{P}^7\) branched along its intersection with a quadric.
The varieties obtained by the second construction will be called “of Gushel type” (after Gushel, who studied the 3-dimensional case in [G]). They are specializations of varieties obtained by the first construction.

Let $\mathcal{X}_{10}$ be the irreducible moduli stack for (smooth) prime Fano fourfolds of degree 10 and index 2, let $\mathcal{X}^G_{10}$ be the (irreducible closed) substack of those which are of Gushel type, and let $\mathcal{X}^{\sigma}_{10} := \mathcal{X}_{10} \setminus \mathcal{X}^G_{10}$.

Let $G := G(2, V_5)$ and let $X := G \cap P^8 \cap Q$ be a fourfold of type $\mathcal{X}^{\sigma}_{10}$. We have

$$\dim(\mathcal{X}_{10}) = 24 \quad , \quad \dim(\mathcal{X}^G_{10}) = 22.$$

3. Unirationality

Let $G := G(2, V_5)$ and let $X := G \cap P^8 \cap Q$ be a fourfold of type $\mathcal{X}^{\sigma}_{10}$. We give a new proof of the classical fact that $X$ is unirational.

The hyperplane $P^8$ is defined by a non-zero skew-symmetric form $\omega$ on $V_5$, and the singular locus of $W := G \cap P^8$ is isomorphic to $G(2, \text{Ker}(\omega))$. Since $X$ is smooth, this singular locus must be finite, hence $\omega$ must be of maximal rank and $W$ is also smooth. The variety $W$ is the unique del Pezzo fivefold of degree 5 ([IP], Theorem 3.3.1). It is also the odd isotropic Grassmannian $G_{\omega}(2, V_5)$, which parametrizes isotropic 2-planes for the form $\omega$.

If $V_1 \subset V_5$ is the kernel of $\omega$, the 3-plane $P(V_1 \wedge V_5)$ of lines passing through $[V_1] \in P(V_5)$ is contained in $W = G \cap P^8$, hence $X$ contains $\Sigma_0 := P(V_1 \wedge V_5) \cap Q$, a “$\sigma$-quadric” surface.

It is the only irreducible $\sigma$-quadric contained in $X$.

**Proposition 3.1.** Any fourfold $X$ of type $\mathcal{X}^{\sigma}_{10}$ is unirational. More precisely, there is a rational double cover $P^4 \dasharrow X$.

**Proof.** If $p \in \Sigma_0$, the associated $V_{2,p} \subset V_5$ contains $V_1$, hence its $\omega$-orthogonal is a hyperplane $V_{2,p}^\perp \subset V_5$. The $(P^1 \times P^1)$-bundle $Y := \bigcup_{p \in \Sigma_0} P(V_{2,p}) \times P(V_{2,p}^\perp / V_{2,p})$ over $\Sigma_0$ is then a rational fourfold.

A general point of $Y$ defines a flag $V_1 \subset V_{2,p} \subset V_3 \subset V_{2,p}^\perp \subset V_5$ hence a line in $G(2, V_5)$ passing through $p$ and contained in $P^8$. This line meets $X \setminus \Sigma_0$ at a unique point, and this defines a rational map $Y \dasharrow X$.

This map has degree 2: if $x$ is general in $X$, lines in $G(2, V_5)$ through $x$ meet $P(V_1 \wedge V_5)$ in points $p$ such that $V_{2,p} = V_1 + V_x'$, with $V_x' \subset V_{2,x}$, hence the intersection is $P(V_1 \wedge V_{2,x})$. This is a line, therefore it meets $\Sigma_0$ in two points. \hfill $\square$

4. Cohomology and the local period map

As in [E] set $G := G(2, V_5)$ and let $W := G \cap P^8$ be a smooth hyperplane section of $G$.

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1This means that the lines in $P(V_5)$ parametrized by $\Sigma_0$ all pass through a fixed point. Since $X$ is smooth, it contains no 3-planes by the Lefschetz theorem, hence $\Sigma_0$ is indeed a surface.

2If $\Sigma \subset X$ is an irreducible $\sigma$-quadric, its span $P(V_1' \wedge V_5)$ is contained in the isotropic Grassmannian $G_{\omega}(2, V_5)$, hence $V_1'$ is the kernel of $\omega$ and $\Sigma = \Sigma_0$. 

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4.1. The Hodge diamond of $X$. The inclusion $W \subset G$ induces isomorphisms
\[(1) \quad H^k(G, \mathbb{Z}) \cong H^k(W, \mathbb{Z}) \quad \text{for all } k \in \{0, \ldots, 5\}.\]
The Hodge diamond for a fourfold $X := W \cap Q$ of type $\mathcal{X}^0_{10}$ was computed in [IM], Lemma 4.1; its upper half is as follows:
\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 1 & 0 & \\
0 & 0 & 0 & 0 \\
0 & 1 & 22 & 1 & 0
\end{array}
\]
When $X$ is of Gushel type, the Hodge diamond remains the same. In all cases, the rank-2 lattice $H^4(G, \mathbb{Z})$ embeds into $H^4(X, \mathbb{Z})$ and we define the vanishing cohomology $H^4(X, \mathbb{Z})_{\text{van}}$ as the orthogonal (for the intersection form) of the image of $H^4(G, \mathbb{Z})$ in $H^4(X, \mathbb{Z})$. It is a lattice of rank 22.

4.2. The local deformation space. We compute the cohomology groups of the tangent sheaf $T_x$.

**Proposition 4.1.** For any fourfold $X$ of type $\mathcal{X}^0_{10}$, we have
\[H^0(X, T_x) = 0 \quad \text{for} \quad p \neq 1\]
and $h^1(X, T_x) = 24$. In particular, the group of automorphisms of $X$ is finite and the local deformation space Def($X$) is smooth of dimension 24.

**Proof.** For $p \geq 2$, the conclusion follows from the Kodaira-Akizuki-Nakano theorem, since $T_x \cong \Omega^3_X(2)$.

Assume first that $X$ is not of Gushel type, so that $X = W \cap Q$.

Let us prove $H^0(X, T_x) = 0$. We have inclusions $X \subset W \subset G$. The conormal exact sequence $0 \to \mathcal{O}_X(-2) \to \Omega^1_W|_X \to \Omega^1_X \to 0$ induces an exact sequence
\[0 \to \Omega^2_X \to \Omega^1_W(2)|_X \to T_x \to 0.\]
Since $H^1(X, \Omega^2_X)$ vanishes, it is enough to show $H^0(X, \Omega^1_W(2)|_X) = 0$. Since $H^1(W, \Omega^3_W) = 0$, it is enough to show that $H^0(W, \Omega^3_W(2))$, or equivalently its Serre-dual $H^5(W, \Omega^3_W(-2))$, vanishes.

The conormal exact sequence of $W$ in $G$ induces an exact sequence
\[0 \to \Omega^1_W(-3) \to \Omega^2_G(-2)|_W \to \Omega^2_W(-2) \to 0.\]
The desired vanishing follows since $H^5(G, \Omega^2_G(-2)) = H^6(G, \Omega^2_G(-3)) = 0$ by Bott’s theorem.

Assume now that $X$ is of Gushel type, so we have a double cover $\pi_X : X \to W' := G \cap \mathbb{P}^7$ branched along the intersection of $W'$ with a quadric. We have an exact sequence
\[(3) \quad 0 \to T_x \to \pi^*_XT_{W'} \to \mathcal{O}_R(R) \to 0,
\]
where $R \subset X$ is the ramification of $\pi_X$. Since $\Omega^1_G(-2)|_{W'}$ is acyclic ([DIM], proof of Proposition 3.3), we have ([DIM], (4.7))
\[H^0(W', T_{W'}) \cong H^4(W', \Omega^1_{W'}(-2))^\vee = 0.\]
Since $H^0(X, \pi_X^*T_{W'}) \simeq H^0(W', T_{W'}) \oplus H^0(W', T_{W'}(-1))$, we obtain the desired vanishing in the Gushel case.

Since $X$ is (anti)canonically polarized, this vanishing implies that its group of automorphisms is a discrete subgroup of $\text{PGL}(9, \mathbb{C})$, hence is finite. Finally, we leave the computation of $\chi^1(X, T_X) = -\chi(X, T_X) = -\chi(X, \Omega^1_X(-2))$ to the reader. □

**Remark 4.2.** When $X$ is of Gushel type, we have with the notation above $H^1(R, \mathcal{O}_R(2)) = 0$ by Kodaira vanishing, hence an exact sequence

$$0 \to H^0(R, \mathcal{O}_R(2)) \to H^1(X, T_X) \to H^1(W', T_{W'}) \oplus H^1(W', T_{W'}(-1)) \to 0.$$  

Moreover,

$$H^1(W', T_{W'}) \simeq H^3(W', \Omega^1_{W'}(-3))^\vee = 0.$$  

Similarly, $H^1(W', T_{W'}(-1)) \simeq L^\vee$, where $L \subset \wedge^2 V_5^\vee$ is the 2-dimensional vector space that defines the $\mathbb{P}^8 = L^\perp \subset \wedge^2 V_5$ that defines $W'$. The kernel of the map $H^1(X, T_X) \to H^1(W', T_{W'}(-1))$ describes the tangent space to the Gushel locus.

### 4.3. The local period map.

Let $X$ be a fourfold of type $\mathcal{X}_{10}$ and let $\Lambda$ be a fixed lattice isomorphic to $H^4(X, \mathbb{Z})_{\text{van}}$. By Proposition 1.1, $X$ has a smooth (simply connected) local deformation space $\text{Def}(X)$ of dimension 24. By (2), the Hodge structure of $H^4(X)_{\text{van}}$ is of K3 type hence we can define a morphism

$$\text{Def}(X) \to \mathbb{P}(\Lambda \otimes \mathbb{C})$$

with values in the smooth 20-dimensional quadric

$$\mathcal{Q} := \{ \omega \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (\omega \cdot \omega) = 0 \}.$$  

We show below (Theorem 4.3) that the restriction $p : \text{Def}(X) \to \mathcal{Q}$, the local period map, is a submersion.

Recall from §3 that the hyperplane $\mathbb{P}^8$ is defined by a skew-symmetric form on $V_5$ whose kernel is a one-dimensional subspace $V_1$ of $V_5$.

**Lemma 4.3.** There is an isomorphism $H^1(W, \Omega^3_W(-2)) \simeq V_5/V_1$.

**Proof.** From the normal exact sequence of the embedding $W \subset G$, we deduce the exact sequences

(4) \hspace{1cm} 0 \to \Omega^1_W \to \Omega^2_G(1)|_W \to \Omega^2_W(1) \to 0

(5) \hspace{1cm} 0 \to \Omega^2_W(1) \to \Omega^3_G(2)|_W \to \Omega^3_W(2) \to 0.

By Bott’s theorem, $\Omega^2_G(1)$ is acyclic, so that we have

$$H^q(W, \Omega^2_G(1)|_W) \simeq H^{q+1}(G, \Omega^2_G) \simeq \delta_{q,1} V_1.$$  

On the other hand, by (1), we have $H^q(W, \Omega^1_W) \simeq \delta_{q,1} V_1$. Therefore, we also get, by (1), $H^q(W, \Omega^2_W(1)) \simeq \delta_{q,1} V_1$.

By Bott’s theorem again, $\Omega^3_G(1)$ is acyclic hence, using (3), we obtain

$$H^q(W, \Omega^3_G(2)|_W) \simeq H^q(G, \Omega^3_G(2)) \simeq \delta_{q,1} V_5.$$  

This finishes the proof of the lemma. □
**Theorem 4.4.** For any fourfold $X$ of type $\mathcal{X}_{10}$, the local period map $p : \text{Def}(X) \rightarrow \mathcal{O}$ is a submersion.

**Proof.** The tangent map to $p$ at the point $[X]$ defined by $X$ is the morphism

$$H^1(X, T_X) \rightarrow \text{Hom}(H^{3,1}(X), H^{3,1}(X)/H^{3,1}(X))$$

$$\simeq \text{Hom}(H^1(X, \Omega^3_X), H^2(X, \Omega^2_X))$$

defined by the natural pairing $H^1(X, T_X) \otimes H^1(X, \Omega^3_X) \rightarrow H^2(X, \Omega^2_X)$ (by (2), $H^1(X, \Omega^3_X)$ is one-dimensional).

Assume first that $X$ is not of Gushel type, so that, keeping the notation above, $X$ is a smooth quadratic section of $W$. Recall the isomorphism $T_X \simeq \Omega^3_X(2)$. The normal exact sequence of the embedding $X \subset W$ yields the exact sequence $0 \rightarrow \Omega^3_X \rightarrow \Omega^3_W(2)|_X \rightarrow T_X \rightarrow 0$.

Moreover, the induced coboundary map

$$H^1(X, T_X) \rightarrow H^2(X, \Omega^2_X)$$

coincides with the cup-product by a generator of $H^1(X, \Omega^3_X) \simeq \mathbb{C}$, hence is the tangent map $T_{p,[X]}$. Since $H^{2,1}(X) = 0$ (see (2)), its kernel $K$ is isomorphic to $H^1(X, \Omega^3_W(2)|_X)$.

In order to compute this cohomology group, we consider the exact sequence $0 \rightarrow \Omega^3_W \rightarrow \Omega^3_W(2) \rightarrow \Omega^3_W(2)|_X \rightarrow 0$. Since, by (1), we have $H^1(W, \Omega^3_W) = H^2(W, \Omega^3_W) = 0$, we get

$$K \simeq H^1(X, \Omega^3_W(2)|_X) \simeq H^1(W, \omega^3_W(2)) \simeq V_5/V_1$$

by Lemma 4.3. This concludes the proof of the theorem in this case.

Assume now that $X$ is of Gushel type, so we have a double cover $\pi_X : X \rightarrow W' := G \cap \mathbb{P}^7$ branched along the intersection of $W'$ with a quadric. We consider $X$ as a subvariety of the blow-up $\mathbb{P}W'$ of the vertex of the (projective) cone over $W$. The $\mathbb{P}^1$-bundle $\pi : \mathbb{P}W' \rightarrow W'$ is associated with $\mathcal{E} = \mathcal{O}_W \oplus \mathcal{O}_W(-1)$ (in Grothendieck’s notation). We have $\omega_{\mathbb{P}W'}/W' \simeq \pi^* \det(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}W'}(-2)$, hence $\pi_* \omega_{\mathbb{P}W'}/W' = 0$, while $R^1\pi_* \omega_{\mathbb{P}W'}/W' = \mathcal{O}_W$ by Grothendieck’s duality. The exact sequence

$$0 \rightarrow \pi^* \Omega^3_W \rightarrow \Omega^3_{\mathbb{P}W'} \rightarrow \pi^* \Omega^2_W \otimes \omega_{\mathbb{P}W'}/W' \rightarrow 0$$

(6) gives a long exact sequence

$$\cdots \rightarrow H^i(W', \Omega^3_W) \rightarrow H^i(\mathbb{P}W', \Omega^3_{\mathbb{P}W'}) \rightarrow H^{i-1}(W', \Omega^2_W) \rightarrow \cdots$$

In particular, we obtain, using the fact that $h^{p,q}(W') = 0$ for all $p \neq q$,

$$H^1(\mathbb{P}W', \Omega^3_{\mathbb{P}W'}) = H^2(\mathbb{P}W', \Omega^3_{\mathbb{P}W'}) = 0.$$  

As in the first case above, the kernel $K$ is therefore isomorphic to $H^1(\mathbb{P}W', \Omega^3_{\mathbb{P}W'} \otimes \mathcal{O}_{\mathbb{P}W'}(2))$. Using (5) again, we obtain an exact sequence

$$H^1(W', \Omega^3_W \otimes \text{Sym}^2 \mathcal{E}) \rightarrow K \rightarrow H^1(W', \Omega^2_W \otimes \det(\mathcal{E})).$$

By (DIMl), Proposition 5.3, we have

$$H^1(W', \Omega^3_W \otimes \text{Sym}^2 \mathcal{E}) = H^1(W', \Omega^3_{\mathbb{P}W'}(2)) \simeq H^3(W', \Omega^2_W(-2))^\vee,$$

$$H^1(W', \Omega^2_W \otimes \det(\mathcal{E})) = H^1(W', \Omega^2_W(1)) \simeq H^3(W', \Omega^2_W(-1))^\vee.$$
and these spaces are both 2-dimensional. In particular, $K$ has dimension at most 4. By semi-continuity, there is equality and the theorem is proved. \hfill \square

If $X$ is of Gushel type, we may also consider, inside $\text{Def}(X)$, the locus $\text{Def}^G(X)$ where the deformation of $X$ remains of Gushel type and the restriction

$$p^G : \text{Def}^G(X) \to \mathcal{Q}$$

of the local period map.

The tangent space to $\text{Def}^G(X)$ at $[X]$ corresponds to the subspace of $H^1(X, T_X)$ where the Gushel involution acts trivially. One can describe it as follows: the inclusion $T_X \hookrightarrow \pi_X^* T_W$ induces a map

$$H^1(X, T_X) \to H^1(X, \pi_X^* T_W) \cong H^1(W, T_W) \oplus H^1(W, T_W(-1)) \cong H^1(W, T_W(-1)) \cong H^1(W, \Omega_W^2(2))$$

which can be checked, using (3), to be surjective. Its kernel is the tangent space to the Gushel locus, and it follows from the proof above that the intersection of $K = \text{Ker}(T_{p,[X]})$ with that space has dimension 2.

**Proposition 4.5.** For any smooth $X$ of type $\mathcal{X}^G_{10}$, the kernel of $T_{p^G,[X]}$ is 2-dimensional. In particular, $p^G$ is a submersion at $[X]$.

The fact that the period map is dominant implies a Noether-Lefschetz-type result.

**Corollary 4.6.** If $X$ is a very general fourfold of type $\mathcal{X}_{10}$, or is very general of type $\mathcal{X}^G_{10}$, we have $H^{2,2}(X) \cap H^4(X, \mathbb{Q}) = H^4(G, \mathbb{Q})$ and the Hodge structure $H^4(X, \mathbb{Q})_{\text{van}}$ is simple.

**Proof.** For $H^{2,2}(X) \cap H^4(X, \mathbb{Q})_{\text{van}}$ to be non-zero, the corresponding period must be in one of the (countably many) hypersurfaces $\alpha^\perp \cap \mathcal{Q}$, for some $\alpha \in \text{P}(\Lambda \otimes \mathbb{Q})$. Since the local period map is dominant, this does not happen for $X$ very general (or very general of Gushel type).

For any $X$, a standard argument (see, e.g., [Z], Theorem 1.4.1) shows that the transcendental lattice $(H^4(X, \mathbb{Z})_{\text{van}} \cap H^{2,2}(X))^\perp$ inherits a simple rational Hodge structure. For $X$ very general (or very general of Gushel type), the transcendental lattice is $H^4(X, \mathbb{Z})_{\text{van}}$. \hfill \square

5. The period domain and the period map

5.1. The vanishing cohomology lattice. Let $(L, \cdot)$ be a lattice; we denote by $L^\vee$ its dual Hom$_\mathbb{Z}(L, \mathbb{Z})$. The symmetric bilinear form on $L$ defines an embedding $L \subset L^\vee$. The discriminant group is the finite abelian group $D(L) := L^\vee/L$; it is endowed with the symmetric bilinear form $b_L : D(L) \times D(L) \to \mathbb{Q}/\mathbb{Z}$ defined by $b_L([w], [w']) := w \cdot \mathbb{Q} w'$ (mod $\mathbb{Z}$) ([N], §1, 3°). We define the divisibility $\text{div}(w)$ of a non-zero element $w$ of $L$ as the positive generator of the ideal $w \cdot L \subset \mathbb{Z}$, so that $w/\text{div}(w)$ is primitive in $L^\vee$. We set $w_* := [w/\text{div}(w)] \in D(L)$. If $w$ is primitive, $\text{div}(w)$ is the order of $w_*$ in $D(L)$.
Proposition 5.1. Let $X$ be a fourfold of type $\mathcal{Z}_{10}$. The vanishing cohomology lattice $H^4(X, \mathbb{Z})_{\text{van}}$ is even and has signature $(20, 2)$ and discriminant group $(\mathbb{Z}/2\mathbb{Z})^2$. It is isometric to

$$\Lambda := 2E_8 \oplus 2U \oplus 2A_1.$$  

Proof. By (2), the Hodge structure on $H^4(X)$ has weight 2 and the unimodular lattice $\Lambda_X := H^4(X, \mathbb{Z})$, endowed with the intersection form, has signature $(22, 2)$. Since 22 - 2 is not divisible by 8, this lattice must be odd, hence of type $22\langle 1 \rangle \oplus 2\langle -1 \rangle$, often denoted by $I_{22,2}$ ([S], Chap. V, §2, cor. 1 of th. 2 and th. 4).

The intersection form on $\Lambda_G := H^4(G(2, V_2), \mathbb{Z})|_X$ has matrix $\left( \begin{array}{cc} 2 & 0 \\ 0 & 4 \end{array} \right)$ in the basis $(\sigma_1, |_X, \sigma_2|_X)$. It is of type $2\langle 1 \rangle$ and embeds as a primitive sublattice in $H^4(X, \mathbb{Z})$. The vanishing cohomology lattice $\Lambda_X := H^4(X, \mathbb{Z})_{\text{van}} := \Lambda^0_G$ therefore has signature $(20, 2)$ and $D(\Lambda^0_X) \simeq D(\Lambda_G) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ ([N], Proposition 1.6.1).

An element $x$ of $I_{22,2}$ is characteristic if

$$\forall y \in I_{22,2} \quad x \cdot y \equiv y^2 \pmod{2}.$$  

The lattice $x^\perp$ is then even. One has from [BH], §16.2,

$$c_1(T_X) = 2\sigma_1|_X,$$
$$c_2(T_X) = 4\sigma_2^2|_X - \sigma_2|_X.$$  

Wu’s formula ([W]) then gives

$$\forall y \in \Lambda_X \quad y^2 \equiv y \cdot (c^2_1 + c_2) \equiv y \cdot \sigma_2|_X \pmod{2}.$$  

In other words, $\sigma_2|_X$ is characteristic, hence $\Lambda^0_X$ is an even lattice. As one can see from Table (15.4) in [CS], there is only one genus of even lattices with signature $(20, 2)$ and discriminant group $(\mathbb{Z}/2\mathbb{Z})^2$ (it is denoted by $I_{20,2}^2(2^2)$ in that table); moreover, there is only one isometry class in that genus ([CS], Theorem 21). In other words, any lattice with these characteristics, such as the one defined in (7), is isometric to $\Lambda^0_X$.

One can also check that $\Lambda$ is the orthogonal in $I_{22,2}$ of the lattice generated by the vectors

$$u := e_1 + e_2 \quad \text{and} \quad v' := e_1 + \cdots + e_{22} - 3f_1 - 3f_2$$

in the canonical basis $(e_1, \ldots, e_{22}, f_1, f_2)$ for $I_{22,2}$. Putting everything together, we see that there is an isometry $\gamma : \Lambda_X \cong I_{22,2}$ such that

$$\gamma(\sigma_{1,1}|_X) = u, \quad \gamma(\sigma_2|_X) = v', \quad \gamma(\Lambda^0_X) \simeq \Lambda.$$  

We let $\Lambda_2 \subset I_{22,2}$ be the rank-2 sublattice $\langle u, v' \rangle = \langle u, v \rangle$, where $v := v' - u$. Then $u$ and $v$ both have divisibility 2, $D(\Lambda_2) = \langle u_*, v_* \rangle$, and the matrix of $b_{\Lambda_2}$ associated with these generators is $\left( \begin{array}{cc} 1/2 & 0 \\ 0 & 1/2 \end{array} \right)$. 
5.2. Lattice automorphisms. One can construct \( I_{20,2} \) as an overlattice of \( \Lambda \) as follows. Let \( e \) and \( f \) be respective generators for the last two factors \( A_1 \) of \( \Lambda \) (see (7)). They both have divisibility 2 and \( D(\Lambda) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \), with generators \( e_2 \) and \( f_2 \); the form \( b_\Lambda \) has matrix \(
abla = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \). In particular, \( e_2 + f_2 \) is the only isotropic non-zero element in \( D(\Lambda) \). By [N], Proposition 1.4.1, this implies that there is a unique unimodular overlattice of \( \Lambda \). Since there is just one isometry class of unimodular lattices of signature \((20,2)\), this is \( I_{20,2} \).

Note that \( \Lambda \) is an even sublattice of index 2 of \( I_{20,2} \), so it is the maximal even sublattice \( \{ x \in I_{20,2} \mid x^2 \text{ even} \} \) (it is contained in that sublattice, and it has the same index in \( I_{20,2} \)).

Every automorphism of \( I_{20,2} \) will preserve the maximal even sublattice, so \( O(I_{20,2}) \) is a subgroup of \( O(\Lambda) \). On the other hand, the group \( O(D(\Lambda)) \) has order 2 and fixes \( e_2 + f_2 \). It follows that every automorphism of \( \Lambda \) fixes \( I_{20,2} \), and we obtain \( O(I_{20,2}) \simeq O(\Lambda) \).

Now let us try to extend to \( I_{22,2} \) an automorphism \( \text{Id} \oplus h \) of \( \Lambda_2 \oplus \Lambda \). Again, this automorphism permutes the overlattices of \( \Lambda_2 \oplus \Lambda \), such as \( I_{22,2} \), according to its action on \( D(\Lambda_2) \oplus D(\Lambda) \). By [N], overlattices correspond to isotropic subgroups of \( D(\Lambda_2) \oplus D(\Lambda) \) that map injectively to both factors. Among them is \( I_{22,2} \); after perhaps permuting \( e \) and \( f \), it corresponds to the (maximal isotropic) subgroup

\[
\{0, u_2 + e_2, v_2 + f_2, u_2 + v_2 + e_2 + f_2\}.
\]

Any automorphism of \( \Lambda \) leaves \( e_2 + f_2 \) fixed. So either \( h \) acts trivially on \( D(\Lambda) \), in which case \( \text{Id} \oplus h \) leaves \( I_{22,2} \) fixed hence extends to an automorphism of \( I_{22,2} \); or \( h \) switches the other two non-zero elements, in which case \( \text{Id} \oplus h \) does not extend to \( I_{22,2} \).

In other words, the image of the restriction map

\[
\{g \in O(I_{22,2}) \mid g|_{\Lambda_2} = \text{Id} \} \hookrightarrow O(\Lambda)
\]

is the stable orthogonal group

\[
\bar{O}(\Lambda) := \text{Ker}(O(\Lambda) \to O(D(\Lambda))).
\]

It has index 2 in \( O(\Lambda) \) and a generator for the quotient is the involution \( r \in O(\Lambda) \) that exchanges \( e \) and \( f \) and is the identity on \( \langle e, f \rangle \). Let \( r_2 \) be the involution of \( \Lambda_2 \) that exchanges \( u \) and \( v \). It follows from the discussion above that the involution \( r_2 \oplus r \) of \( \Lambda_2 \oplus \Lambda \) extends to an involution \( r_I \) of \( I_{22,2} \).

5.3. The period domain and the period map. Fix a lattice \( \Lambda \) as in (7); it has signature \((20,2)\). The manifold

\[
\Omega := \{ \omega \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (\omega \cdot \omega) = 0, (\omega \cdot \bar{\omega}) < 0 \}
\]

is a homogeneous space for the real Lie group \( \text{SO}(\Lambda \otimes \mathbb{R}) \simeq \text{SO}(20,2) \). This group has two components, and one of them reverses the orientation on the negative definite part of \( \Lambda \otimes \mathbb{R} \). It follows that \( \Omega \) has two components, \( \Omega^+ \) and \( \Omega^- \), both isomorphic to the 20-dimensional open complex manifold \( \text{SO}_0(20,2)/\text{SO}(20) \times \text{SO}(2) \), a bounded symmetric domain of type IV.

Let \( \mathcal{U} \) be a smooth (irreducible) quasi-projective variety parametrizing all fourfolds of type \( \mathcal{X}_{10} \). Let \( u \) be a general point of \( \mathcal{U} \) and let \( X \) be the corresponding fourfold. The
group $\pi_1(\mathcal{U}, u)$ acts on the lattice $\Lambda_X := H^4(X, \mathbb{Z})$ by isometries and the image $\Gamma_X$ of the morphism $\pi_1(\mathcal{U}, u) \to O(\Lambda_X)$ is called the monodromy group. The group $\Gamma_X$ is contained in the subgroup (see (11))

$$\widetilde{O}(\Lambda_X) := \{ g \in O(\Lambda_X) \mid g|_{\Lambda_G} = Id \}.$$  

Choose an isometry $\gamma : \Lambda_X \cong \mathbb{I}_{22,2}$ satisfying (10). It induces an isomorphism $\tilde{\mathcal{O}}(\Lambda_X) \cong \tilde{\mathcal{O}}(\Lambda)$. The group $\tilde{\mathcal{O}}(\Lambda)$ acts on the manifold $\Omega$ defined above and, by a theorem of Baily and Borel, the quotient $\mathcal{D} := \tilde{\mathcal{O}}(\Lambda)\backslash \Omega$ has the structure of an irreducible quasi-projective variety. One defines as usual a period map $\mathbb{U} \to \mathcal{D}$ by sending a variety to its period; it is an algebraic morphism. It descends to “the” period map

$$\varphi : \mathcal{X}_{10} \to \mathcal{D}.$$  

By Theorem 4.4, $\varphi$ is dominant with 4-dimensional smooth fibers as a map of stacks.

**Remark 5.2.** As in the three-dimensional case ([DIM1]), we do not know whether our fourfolds have a coarse moduli space, even in the category of algebraic spaces. If such a space $X_{10}$ exists, note however that it is singular along the Gushel locus: any fourfold $X$ of Gushel type has a canonical involution; if $X$ has no other non-trivial automorphisms, $X_{10}$ is then locally around $[X]$ the product of a 22-dimensional germ and the germ of a surface node. The fiber of the period map $X_{10} \to \mathcal{D}$ then has multiplicity 2 along the surface corresponding to Gushel fourfolds.

### 6. Special fourfolds

Following [H1], §3, we say that a fourfold $X$ of type $\mathcal{X}_{10}$ is **special** if it contains a surface whose cohomology class “does not come” from $G(2, V_5)$. Since the Hodge conjecture is true (over $\mathbb{Q}$) for Fano fourfolds (more generally, by [CM], for all uniruled fourfolds), this is equivalent to saying that the rank of the (positive definite) lattice $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ is at least 3. The set of special fourfolds is sometimes called the Noether-Lefschetz locus (by Corollary 4.6, a very general $X$ is not special).

#### 6.1. Special loci

For each primitive, positive definite, rank-3 sublattice $K \subset \mathbb{I}_{22,2}$ containing the lattice $\Lambda_2$ defined at the end of §5.1, we define an irreducible hypersurface of $\Omega^+$ by setting

$$\Omega_K := \{ \omega \in \Omega^+ \mid K \subset \omega^\perp \}.$$  

A fourfold $X$ is **special** if and only if its period is in one of these (countably many) hypersurfaces. We now investigate these lattices $K$.

**Lemma 6.1.** The discriminant $d$ of $K$ is positive and $d \equiv 0, 2, 4 \pmod{8}$.

**Proof.** Since $K$ is positive definite, $d$ must be positive. Completing the basis $(u, v)$ of $\Lambda_2$ from §5.1 to a basis of $K$, we see that the matrix of the intersection form in that basis is

$$\begin{pmatrix} 2 & 0 & a \\ 0 & 2 & b \\ a & b & c \end{pmatrix},$$

whose determinant is $d = 4c - 2(a^2 + b^2)$. By Wu’s formula (9) (or equivalently,
since \( v \) is characteristic), we have \( c \equiv a + b \mod 2 \), hence \( d \equiv 2(a^2 + b^2) \mod 8 \). This proves the lemma. \( \square \)

We keep the notation of §5.

**Proposition 6.2.** Let \( d \) be a positive integer such that \( d \equiv 0, 2, \) or \( 4 \mod 8 \) and let \( \mathcal{O}_d \) be the set of orbits for the action of the group

\[
\tilde{O}(\Lambda) = \{ g \in O(I_{22,2}) \mid g|_{\Lambda_2} = \text{Id} \} \subset O(\Lambda)
\]

on the set of primitive, positive definite, rank-3, discriminant-\( d \) sublattices \( K \subset I_{22,2} \) containing \( \Lambda_2 \). Then,

a) if \( d \equiv 0 \mod 8 \), \( \mathcal{O}_d \) has one element, and \( K \simeq \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & d/4 \end{pmatrix} \); 

b) if \( d \equiv 2 \mod 8 \), \( \mathcal{O}_d \) has two elements, which are interchanged by the involution \( r_1 \)

of \( I_{22,2} \), and \( K \simeq \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & (d + 2)/4 \end{pmatrix} \); 

c) if \( d \equiv 4 \mod 8 \), \( \mathcal{O}_d \) has one element, and \( K \simeq \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & (d + 4)/4 \end{pmatrix} \).

In case b), one orbit is characterized by the properties \( K \cdot u = \mathbb{Z} \) and \( K \cdot v = 2\mathbb{Z} \), and the other by \( K \cdot u = 2\mathbb{Z} \) and \( K \cdot v = \mathbb{Z} \).

**Proof.** By a theorem of Eichler (see, e.g., [GHS], Lemma 3.5), the \( \tilde{O}(\Lambda) \)-orbit of a primitive vector \( w \) in the even lattice \( \Lambda \) is determined by its length \( w^2 \) and its class \( w_* \in D(\Lambda) \).

If \( \text{div}(w) = 1 \), we have \( w_* = 0 \) and the orbit is determined by \( w^2 \). The lattice \( \Lambda_2 \oplus \mathbb{Z}w \) is primitive: if \( \alpha u + \beta v + \gamma w = mw' \), and if \( w \cdot w'' = 1 \), we obtain \( \gamma = mw' \cdot w'' \), hence \( \alpha u + \beta v = m((w' \cdot w'')w - w') \) and \( m \) divides \( \alpha, \beta, \) and \( \gamma \). Its discriminant is \( 4w^2 \equiv 0 \mod 8 \).

If \( \text{div}(w) = 2 \), we have \( w_* \in \{ e_*, f_* \} \). Recall from [5.2] that \( \frac{1}{2}(u + e), \frac{1}{2}(v + f) \), and \( \frac{1}{2}(u + v + e + f) \) are all in \( I_{22,2} \). It follows that exactly one of \( \frac{1}{2}(u + w), \frac{1}{2}(v + w) \), and \( \frac{1}{2}(u + v + w) \) is in \( I_{22,2} \), and \( \Lambda_2 \oplus \mathbb{Z}w \) has index 2 in its saturation \( K \) in \( I_{22,2} \). In particular, \( K \) has discriminant \( w^2 \). If \( w_* \in \{ e_*, f_* \} \), this is \( \equiv 2 \mod 8 \); if \( w_* = e_* + f_* \), this is \( \equiv 4 \mod 8 \).

Now if \( K \) is a lattice as in the statement of the proposition, we let \( K^\perp \) be its orthogonal in \( I_{22,2} \), so that the rank-1 lattice \( K^0 := K \cap \Lambda \) is the orthogonal of \( K^\perp \) in \( \Lambda \). From \( K^0 \subset \Lambda \), we can therefore recover \( K^\perp \), then \( K \supset \Lambda_2 \). The preceding discussion applied to a generator \( w \) of \( K^\perp \) gives the statement, except that we still have to prove that there are indeed elements \( w \) of the various types for all \( d \), i.e., we need construct elements in each orbit to show they are not empty.
Let \( u_1, u_2 \) be standard generators for a hyperbolic factor \( U \) of \( \Lambda \). For any integer \( m \), set \( w_m := u_1 + mu_2 \). We have \( w_m^2 = 2m \) and \( \text{div}(w_m) = 1 \). The lattice \( \Lambda_2 \oplus \mathbb{Z}w_m \) is saturated with discriminant \( 8m \).

We have \( (e + 2w_m)^2 = 8m + 2 \) and \( \text{div}(e + 2w_m) = 2 \). The saturation of the lattice \( \Lambda_2 \oplus \mathbb{Z}(e + 2w_m) \) has discriminant \( d = 8m + 2 \), and similarly upon replacing \( e \) with \( f \) (same \( d \)) or \( e + f \) (\( d = 8m + 4 \)).

Let \( K \) be a lattice as above. The image in \( D = \tilde{O}(\Lambda) \setminus \Omega \) of the hypersurface \( \Omega_K \subset \Omega^+ \) depends only on the \( \tilde{O}(\Lambda) \)-orbit of \( K \). Also, the involution \( r \in O(\Lambda) \) induces a non-trivial involution \( r_D \) of \( D \).

**Corollary 6.3.** The periods of the special fourfolds of discriminant \( d \) are contained in

a) if \( d \equiv 0 \pmod{4} \), an irreducible hypersurface \( D_d \subset D \);

b) if \( d \equiv 2 \pmod{8} \), the union of two irreducible hypersurfaces \( D_d \) and \( D_d'' \), which are interchanged by the involution \( r_D \).

Assume \( d \equiv 2 \pmod{8} \) (case b)). Then, \( D_d' \) (resp. \( D_d'' \)) corresponds to lattices \( K \) with \( K \cdot u = \mathbb{Z} \) (resp. \( K \cdot v = \mathbb{Z} \)). In other words, given a fourfold \( X \) of type \( \mathcal{X}_{10} \) whose period point is in \( D_d = D_d' \cup D_d'' \), it is in \( D_d' \) if \( K \cdot \sigma_1^2 \subset 2\mathbb{Z} \), and it is in \( D_d'' \) if \( K \cdot \sigma_{1,1} \subset 2\mathbb{Z} \).

**Remark 6.4.** Zarhin’s argument, already used in the proof of Corollary 4.6, proves that if \( X \) is a fourfold whose period is very general in any given \( D_d \), the lattice \( K = H^4(X, \mathbb{Z}) \cap H^{2,2}(X) \) has rank exactly 3 and the rational Hodge structure \( K^\perp \otimes \mathbb{Q} \) is simple.

### 6.2. Associated K3 surface

As we will see in the next section, K3 surfaces often occur in the geometric description of special fourfolds \( X \) of type \( \mathcal{X}_{10} \). This is related to the fact that, for some values of \( d \), the non-special cohomology of \( X \) looks like the primitive cohomology of a K3 surface.

Following [H1], we determine, in each case of Proposition 6.2, the discriminant group of the non-special lattice \( K^\perp \) and the symmetric form \( b_{K^\perp} = -b_K \). We then find all cases when the non-special lattice of \( X \) is isomorphic (with a change of sign) to the primitive cohomology lattice of a (pseudo-polarized, degree-\( d \)) K3 surface. Although this property is only lattice-theoretic, the surjectivity of the period map for K3 surfaces then produces an actual K3 surface, which is said to be “associated with \( X \).” For \( d = 10 \), we will see in §7.1 and §7.3 geometrical constructions of the associated K3 surface.

Finally, there are other cases where geometry provides an “associated” K3 surface \( S \) (see §7.6), but not in the sense considered here: the Hodge structure of \( S \) is only isogeneous to that of the fourfold. So there might be integers \( d \) not in the list provided by the proposition below, for which special fourfolds of discriminant \( d \) are still related to K3 surfaces (of degree different from \( d \)).

**Proposition 6.5.** Let \( d \) be a positive integer such that \( d \equiv 0, 2, \) or \( 4 \pmod{8} \) and let \( (X, K) \) be a special fourfold of type \( \mathcal{X}_{10} \) with discriminant \( d \). Then,

a) if \( d \equiv 0 \pmod{8} \), we have \( D(K^\perp) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/(d/4)\mathbb{Z}) \);
b) if \( d \equiv 2 \pmod{8} \), we have \( D(K^\perp) \simeq \mathbb{Z}/d\mathbb{Z} \) and we may choose this isomorphism so that \( b_{K^\perp}(1, 1) = -\frac{d+8}{2d} \pmod{\mathbb{Z}} \);

c) if \( d \equiv 4 \pmod{8} \), we have \( D(K^\perp) \simeq \mathbb{Z}/d\mathbb{Z} \) and we may choose this isomorphism so that \( b_{K^\perp}(1, 1) = -\frac{d+2}{2d} \pmod{\mathbb{Z}} \).

The lattice \( K^\perp \) is isomorphic to the opposite of the primitive cohomology lattice of a pseudo-polarized K3 surface (necessarily of degree \( d \)) if and only if we are in case b) or c) and the only odd primes that divide \( d \) are \( \equiv 1 \pmod{4} \).

In these cases, there exists a pseudo-polarized, degree-\( d \), K3 surface \( S \) such that the Hodge structure \( H^2(S, \mathbb{Z})^0(-1) \) is isomorphic to \( K^\perp \). Moreover, if the period point of \( X \) is not in \( \mathcal{D}_8 \), the pseudo-polarization is a polarization.

The first values of \( d \) that satisfy the conditions for the existence of an associated K3 surface are: 2, 4, 10, 20, 26, 34, 50, 52, 58, 68, 74, 82, 100...

**Proof.** Since \( I_{22,2} \) is unimodular, we have \( (D(K^\perp), b_{K^\perp}) \simeq (D(K), -b_K) \) ([N], Proposition 1.6.1). Case a) follows from Proposition 6.2.2.

Let \( e, f, g \) be the generators of \( K \) corresponding to the matrix given in Proposition 6.2. The matrix of \( b_{K^\perp} \) in the dual basis \((e^\vee, f^\vee, g^\vee)\) of \( K^\perp \) is the inverse of that matrix.

In case b), one checks that \( e^\vee + g^\vee \) generates \( D(K) \), which is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \). Its square is \( \frac{1}{2} + \frac{4}{d} = \frac{d+8}{2d} \).

In case c), one checks that \( e^\vee \) generates \( D(K) \), which is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \). Its square is \( \frac{d+2}{2d} \).

The opposite of the primitive cohomology lattice of a pseudo-polarized K3 surface of degree \( d \) has discriminant group \( \mathbb{Z}/d\mathbb{Z} \) and the square of a generator is \( \frac{1}{d} \). So case a) is impossible.

In case b), the forms are conjugate if and only if \( -\frac{d+8}{2d} \equiv \frac{n^2}{d} \pmod{\mathbb{Z}} \) for some integer \( n \) prime to \( d \), or \( -\frac{d+8}{2d} \equiv n^2 \pmod{d} \). Set \( d = 2d' \) (so that \( d' \equiv 1 \pmod{4} \)); then this is equivalent to saying that \( d' - 4 \) is a square in the ring \( \mathbb{Z}/d\mathbb{Z} \). Since \( d' \) is odd, this ring is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/d'\mathbb{Z} \), hence this is equivalent to asking that \(-4\), or equivalently \(-1\), is a square in \( \mathbb{Z}/d'\mathbb{Z} \). This happens if and only if the only odd primes that divide \( d' \) (or \( d \)) are \( \equiv 1 \pmod{4} \).

In case c), the reasoning is similar: we need \( -\frac{d+2}{2d} \equiv \frac{n^2}{d} \pmod{\mathbb{Z}} \) for some integer \( n \) prime to \( d \). Set \( d = 4d' \), with \( d' \) odd. This is equivalent to \( -2 \equiv 2n^2 \pmod{d'} \), and we conclude as above.

As already explained, the existence of the polarized K3 surface \((S, f)\) follows from the surjectivity of the period map for K3 surfaces. Finally, if \( \varphi([X]) \) is not in \( \mathcal{D}_8 \), there are no classes of type \((2,2)\) with square 2 in \( H^4(X, \mathbb{Z})_{\text{van}} \), hence no \((-2)\)-curves on \( S \) orthogonal to \( f \), so \( f \) is a polarization. \( \square \)

6.3. **Associated cubic fourfold.** Cubic fourfolds also sometimes occur in the geometric description of special fourfolds \( X \) of type \( \mathcal{X}_{10} \) (see §7.2). We determine for which values of
the non-special cohomology of $X$ is isomorphic the non-special cohomology of a special cubic fourfold. Again, this is only a lattice-theoretic association, but the surjectivity of the period map for cubic fourfolds then produces a (possibly singular) actual cubic. We will see in §7.2 that some special fourfolds $X$ of discriminant 12 are actually birationally isomorphic to their associated special cubic fourfold.

**Proposition 6.6.** Let $d$ be a positive integer such that $d \equiv 0, 2, or 4 \ (mod \ 8)$ and let $(X, K)$ be a special fourfold of type $\mathcal{X}_{10}$ with discriminant $d$. The lattice $K^\perp$ is isomorphic to the non-special cohomology lattice of a (possibly singular) special cubic fourfold (necessarily of discriminant $d$) if and only if

a) either $d \equiv 2 \ or \ 20 \ (mod \ 24)$, and the only odd primes that divide $d$ are $\equiv \pm 1 \ (mod \ 12)$;

b) or $d \equiv 12 \ or \ 66 \ (mod \ 72)$, and the only primes $\equiv 5 \ (mod \ 12)$.

In these cases, if moreover the period point of $X$ is general in $\mathcal{D}_d$ and $d \neq 2$, there exists a smooth special cubic fourfold whose non-special Hodge structure is isomorphic to $K^\perp$.

The first values of $d$ that satisfy the conditions for the existence of an associated cubic fourfold are: 2, 12, 26, 44, 66, 74, 92, 122, 138, 146, 156, 194...

**Proof.** Recall from [HI], §4.3, that (possibly singular) special cubic fourfolds of positive discriminant $d$ exist for $d \equiv 0 \ or \ 2 \ (mod \ 6)$ (for $d = 2$, the associated cubic fourfold is the (singular) determinantal cubic; for $d = 6$, it is nodal). Combining that condition with that of Lemma 6.1, we obtain the necessary condition $d \equiv 0, 2, 8, 12, 18, 20 \ (mod \ 24)$. Write $d = 24d' + e$, with $e \in \{0, 2, 8, 12, 18, 20\}$.

Then, one needs to check whether the discriminant forms are isomorphic. Recall from [HI], Proposition 3.2.5, that the discriminant group of the non-special lattice of a special cubic fourfold of discriminant $d$ is isomorphic to $\left(\mathbb{Z}/3\mathbb{Z}\right) \times \left(\mathbb{Z}/(d/3)\mathbb{Z}\right)$ if $d \equiv 0 \ (mod \ 6)$, and to $\mathbb{Z}/d\mathbb{Z}$ if $d \equiv 2 \ (mod \ 6)$. This excludes $e = 0$ or 8; for $e = 12$, we need $d' \equiv 1 \ (mod \ 3)$, and for $e = 18$, we need $d' \equiv 0 \ (mod \ 3)$. In all these cases, the discriminant group is cyclic.

When $e = 2$, the discriminant forms are conjugate if and only if $-\frac{d+8}{2d} \equiv n^2 \frac{2d-1}{3d} \ (mod \ \mathbb{Z})$ for some integer $n$ prime to $d$ (Proposition 6.5 and [HI], Proposition 3.2.5), or equivalently, since 3 is invertible modulo $d$, if and only if $\frac{d}{2} + 12 \equiv 3\frac{d+8}{2d} \equiv n^2 \ (mod \ d)$. This is equivalent to saying that $12d' + 13$ is a square in $\mathbb{Z}/d\mathbb{Z} \simeq (\mathbb{Z}/(12d' + 1)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, or that 3 is a square in $\mathbb{Z}/(12d' + 1)\mathbb{Z}$. Using quadratic reciprocity, we see that this is equivalent to saying that the only odd primes that divide $d$ are $\equiv \pm 1 \ (mod \ 12)$.

When $e = 20$, we need $-\frac{d+8}{2d} \equiv n^2 \frac{2d-1}{3d} \ (mod \ \mathbb{Z})$ for some integer $n$ prime to $d$, or equivalently, $\frac{d}{2} + 3 \equiv n^2 \ (mod \ d)$. Again, we get the same condition.

When $e = 12$, we need $9/d$ and $-\frac{d+8}{2d} \equiv n^2 \left(\frac{2}{3} - \frac{3}{d}\right) \ (mod \ \mathbb{Z})$ for some integer $n$ prime to $d$, or equivalently $-12d' - 7 \equiv n^2(16d' + 5) \ (mod \ d)$. Modulo 3, we get that $1 - d'$ must be a non-zero square, hence $3 \mid d'$. Modulo 4, there are no conditions. Then we need $1 \equiv 3n^2 \ (mod \ 2d' + 1)$ and we conclude as above.
Finally, when \( e = 18 \), we need \( 9 \mid d \) and \(-\frac{4+8}{2d} \equiv n^2 \left( \frac{2}{3} - \frac{3}{d} \right) \pmod{2} \) for some \( n \) prime to \( d \), or equivalently \(-12d - 13 \equiv n^2(16d' + 9) \pmod{d} \). Modulo 3, we get \( d' \equiv 2 \pmod{3} \), and then \( 4 \equiv 3n^2 \pmod{4d' + 3} \) and we conclude as above.

At this point, we have a Hodge structure on \( K^+ \) which is, as a lattice, isomorphic to the non-special cohomology of a special cubic fourfold. It corresponds to a point in the period domain \( \mathcal{C} \) of cubic fourfolds. To make sure that it corresponds to a (then unique) smooth cubic fourfold, we need to check that it is not in the special loci \( \mathcal{C}_2 \cup \mathcal{C}_6 \) ([La], Theorem 1.1). If the period point of \( X \) is general in \( \mathcal{D}_d \), the period point in \( \mathcal{C} \) is general in \( \mathcal{C}_d \), hence is not in \( \mathcal{C}_2 \cup \mathcal{C}_6 \) if \( d \notin \{2, 6\} \). \( \square \)

**Remark 6.7.** One can be more precise and figure out explicit conditions on \( \varphi([X]) \) for the associated cubic fourfold to be smooth, but calculations are complicated: if \( d = 6e \) and we are in \( \mathcal{C}_6 \), there is a class \( v \) with \( v^2 = 2 \) and \( v \cdot h = 0 \), so we get a rank-3 lattice of \((2, 2)\)-classes with intersection matrix

\[
\begin{pmatrix}
3 & 0 & 0 \\
0 & 2e & a \\
0 & a & 2
\end{pmatrix}
\]

(with \( a^2 < 4e \)) and the \((2, 2)\)-class \( au - 2ev \) is non-special, hence corresponds to a non-special class in \( X \) with square \( 2e(4e - a^2) \); if we are in \( \mathcal{C}_2 \), there is a class \( v \) with \( v^2 = 6 \) and \( v \cdot h = 0 \), and we proceed similarly. For example, when \( d = 12 \), we find that it is enough to assume \( \varphi([X]) \notin \mathcal{D}_2 \cup \mathcal{D}_4 \cup \mathcal{D}_8 \cup \mathcal{D}_{16} \cup \mathcal{D}_{28} \cup \mathcal{D}_{60} \cup \mathcal{D}_{112} \cup \mathcal{D}_{240} \).

### 7. Examples of special fourfolds

Assume that a fourfold \( X \) of type \( \mathcal{X}_{12} \) contains a smooth surface \( S \). Then, by ([3], (12)),

\[
c(T_X)|_S = 1 + 2\sigma_1|_S + (4\sigma_1^2|_S - \sigma_2|_S) = c(T_S)c(N_{S/X}).
\]

This implies \( c_1(T_S) + c_1(N_{S/X}) = 2\sigma_1|_S \) and

\[
4\sigma_1^2|_S - \sigma_2|_S = c_1(T_S)c_1(N_{S/X}) + c_2(T_S) + c_2(N_{S/X}).
\]

We obtain

\[
(S)^2_X = c_2(N_{S/X}) = 4\sigma_1^2|_S - \sigma_2|_S - c_1(T_S)(2\sigma_1|_S - c_1(T_S)) - c_2(T_S).
\]

Write \([S] = a\sigma_{3,1} + b\sigma_{2,2}\) in \( G(2, V_5) \). Using Noether’s formula, we obtain

\[
(S)^2_X = 3a + 4b + 2K_S \cdot \sigma_1|_S + 2K_S^2 - 12\chi(O_S).
\]

The determinant of the intersection matrix in the basis \((\sigma_{1,1}|_X, \sigma_2|_X, \sigma_{1,1}|_X, [S])\) is then

\[
d = 4(S)^2_X - 2(b^2 + (a - b)^2)
\]

We remark that \( \sigma_2|_X - \sigma_{1,1}|_X \) is the class of the unique \( \sigma \)-quadric surface \( \Sigma_0 \) contained in \( X \) (see ([3], 3)).

#### 7.1. Fourfolds containing a \( \sigma \)-plane (divisor \( \mathcal{D}_{10}' \)).

A \( \sigma \)-plane is a 2-plane in \( G(2, V_5) \) of the form \( \mathbf{P}(V_1 \wedge V_4) \); its class in \( G(2, V_5) \) is \( \sigma_{3,1} \). Fourfolds of type \( \mathcal{X}_{10} \) containing such a 2-plane were already studied by Roth ([R], §4) and Prokhorov ([P], §3).
Proposition 7.1. Inside $\mathcal{X}_{10}$, the family $\mathcal{X}_{\sigma}$ of fourfolds containing a $\sigma$-plane is irreducible of codimension 2. The period map induces a dominant map $\mathcal{X}_{\sigma} \rightarrow \mathcal{P}_{10}^\prime$ whose general fiber has dimension 3 and is rationally dominated by a $\mathbb{P}^1$-bundle over a degree-10 K3 surface.

A general member of $\mathcal{X}_{\sigma}$ is rational.

Proof. A parameter count ([IM], Lemma 3.6) shows that $\mathcal{X}_{\sigma}$ is irreducible of codimension 2 in $\mathcal{X}_{10}$. Let $P \subset X$ be a $\sigma$-plane. From (12), we obtain $(P)^2 = 3$, and from (13), $d = 10$. Since $\sigma^2 : P$ is odd, we are in $\mathcal{P}_{10}^\prime$.

For $X$ general in $\mathcal{X}_{\sigma}$ (see [P], §3, for the precise condition), the image of the projection $\pi_P : X \rightarrow \mathbb{P}^5$ from $P$ is a smooth quadric $Y \subset \mathbb{P}^5$ and, if $\tilde{X} \rightarrow X$ is the blow-up of $P$, the projection $\pi_P$ induces a birational morphism $\tilde{X} \rightarrow Y$ which is the blow-up of a smooth degree-9 surface $\tilde{S}$, itself the blow-up of a smooth degree-10 K3 surface $S$ at one point ([P], Proposition 2).

Conversely, starting from a (general) degree-10 K3 surface $S \subset \mathbb{P}^6$, project it from a point on $S$ to obtain an embedding $\tilde{S} \subset \mathbb{P}^5$. The surface $\tilde{S}$ is then contained in a pencil of quadrics. For each such smooth quadric, one can reverse the construction above and produce a fourfold $X$ containing a $\sigma$-plane (we will go back in more details to this construction during the proof of Theorem 8.1).

There are isomorphisms of polarized integral Hodge structures

$$H^4(\tilde{X}, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \oplus H^2(P, \mathbb{Z})(-1) \simeq H^4(Y, \mathbb{Z}) \oplus H^2(\tilde{S}, \mathbb{Z})(-1) \simeq H^4(Y, \mathbb{Z}) \oplus H^2(S, \mathbb{Z})(-1) \oplus \mathbb{Z}(-2).$$

For $S$ very general, the Hodge structure $H^2(S, \mathbb{Q})_0$ is simple, hence it is isomorphic to the non-special cohomology $K^\perp \otimes \mathbb{Q}$ (where $K$ is the lattice spanned by $H^4(G(2, V_6), \mathbb{Z})$ and $[P]$ in $H^4(X, \mathbb{Z})$). Moreover, the lattice $H^2(S, \mathbb{Z})_0(-1)$ embeds isometrically into $K^\perp$. Since they both have rank 21 and discriminant 10, they are isomorphic. The surface $S$ is thus the (polarized) K3 surface associated with $X$ as in Proposition 6.5. Since the period map for polarized degree-10 K3 surfaces is dominant onto their period domain, the period map for $\mathcal{X}_{\sigma}$ is dominant onto $\mathcal{P}_{10}^\prime$ as well.

With the notation above, the inverse image of the quadric $Y \subset \mathbb{P}^5$ by the projection $\mathbb{P}^8 \rightarrow \mathbb{P}^5$ from $P$ is a rank-6 non-Plücker quadric in $\mathbb{P}^8$ containing $X$, with vertex $P$. We will show in §7.5 that $\mathcal{X}_{\sigma}$ is contained in the irreducible hypersurface of $\mathcal{X}_{10}$ parametrizing the fourfolds $X$ contained in such a quadric.
7.2. **Fourfolds containing a ρ-plane (divisor $\mathcal{D}_{12}$).** A ρ-plane is a 2-plane in $G(2, V_5)$ of the form $\mathbb{P}(\Lambda^2 V_3)$; its class in $G(2, V_5)$ is $\sigma_{2,2}$. Fourfolds of type $\mathcal{X}_{10}$ containing such a 2-plane were already studied by Roth ([R], §4).

**Proposition 7.2.** Inside $\mathcal{X}_{10}$, the family $\mathcal{X}_{\rho}$-plane of fourfolds containing a ρ-plane is irreducible of codimension 3. The period map induces a dominant map $\mathcal{X}_{\rho}$-plane $\to \mathcal{D}_{12}$ whose general fiber is the union of two rational surfaces.

A general member of $\mathcal{X}_{\rho}$-plane is birationally isomorphic to a cubic fourfold containing a smooth cubic surface scroll.

The proof presents a geometrical construction of a general member of $\mathcal{X}_{\rho}$-plane, starting from any smooth cubic fourfold $Y \subset \mathbb{P}^5$ containing a smooth cubic surface scroll $T$. The birational isomorphism $Y \dasharrow X$ is given by the linear system of quadrics containing $T$.

**Proof.** A parameter count ([IM], Lemma 3.6) shows that $\mathcal{X}_{\rho}$-plane is irreducible of codimension 3 in $\mathcal{X}_{10}$. Let $P = \mathbb{P}(\Lambda^2 V_3) \subset X$ be a ρ-plane. From ([12]), we obtain $(P)_X^2 = 4$. From ([13]), we obtain $d = 12$ and we are in $\mathcal{D}_{12}$.

As shown in [R], §4, the image of the projection $\pi_P : X \to \mathbb{P}^5$ from $P$ is a cubic hypersurface $Y$ and the image of the intersection of $X$ with the Schubert hypersurface

$$\Sigma_P = \{ V_2 \subset V_5 \mid V_2 \cap V_3 \neq 0 \} \subset G(2, V_5)$$

is a cubic surface scroll $T$ (contained in $Y$). If $\tilde{X} \to X$ is the blow-up of $P$, with exceptional divisor $E_P$, the projection $\pi_P$ induces a birational morphism $\tilde{\pi}_P : \tilde{X} \to Y$. One checks (with the same arguments as in [P], §3) that all fibers have dimension $\leq 1$ hence that $\tilde{\pi}_P$ is the blow-up of the smooth surface $T$. The image $\tilde{\pi}_P(E_P)$ is the (singular) hyperplane section $Y_0 := Y \cap \langle T \rangle$.

Conversely, a general cubic fourfold $Y$ containing a smooth cubic scroll contains two families (each parametrized by $\mathbb{P}^2$) of such surfaces (see [HT1] and [HT2], Example 7.12). For each such smooth cubic scroll, one can reverse the construction above and produce a smooth fourfold $X$ containing a ρ-plane.

As in ([7,1]) there are isomorphisms of polarized integral Hodge structures

$$H^4(\tilde{X}, Z) \simeq H^4(X, Z) \oplus H^2(P, Z)(-1) \simeq H^4(Y, Z) \oplus H^2(T, Z)(-1).$$

Let $K$ be the lattice spanned by $H^4(G(2, V_5), Z)$ and $[P]$ in $H^4(X, Z)$. For $X$ very general in $\mathcal{X}_{\rho}$-plane, the Hodge structure $K^+ \otimes \mathbb{Q}$ is simple (Remark 6.4), hence it is isomorphic to the Hodge structure $\langle h^2, [T] \rangle^+ \subset H^4(Y, \mathbb{Q})$. Moreover, the lattices $K^+$ and $\langle h^2, [T] \rangle^+ \subset H^4(Y, \mathbb{Z})$, which both have rank 21 and discriminant 12 (see [H1], §4.1.1), are isomorphic. This case fits into the setting of Proposition 6.6: the special cubic fourfold $Y$ is associated with $X$. Finally, since the period map for cubic fourfolds containing a cubic scroll surface is dominant onto the corresponding hypersurface in their period domain, the period map for $\mathcal{X}_{\rho}$-plane is dominant onto $\mathcal{D}_{12}$ as well. \[\Box\]

With the notation above, let $V_4 \subset V_5$ be a general hyperplane containing $V_3$. Then $G(2, V_4) \cap X$ is the union of $P$ and a cubic scroll surface.
Remark 7.3. No fourfold $X$ of type $\mathcal{Z}_{10}$ contains infinitely many 2-planes. Indeed, if there is a one-dimensional family of 2-planes contained in $X$, two general such 2-planes $P$ and $P'$ will be of the same type (i.e., both $\sigma$-planes, or both $\rho$-planes), and $P \cdot P' = P^2 = 3$ (for $\sigma$-planes, as seen in the proof of Proposition 7.1) or 4 (for $\rho$-planes, as seen in the proof of Proposition 7.2). But one checks that any two distinct 2-planes of the same type meet in either 0 or 1 points. So this is a contradiction.

7.3. Fourfolds containing a $\tau$-quadric surface (divisor $\mathcal{D}_1^\prime_0$). A $\tau$-quadric surface in $G(2, V_5)$ is a linear section of $G(2, V_4)$ (its general hyperplane sections are $\tau$-conics); its class in $G(2, V_5)$ is $\sigma_1^2 \cdot \sigma_{1,1} = \sigma_{3,1} + \sigma_{2,2}$.

Proposition 7.4. The closure $\mathcal{Z}_{\tau, \text{quadric}} \subset \mathcal{Z}_{10}$ of the family of fourfolds containing a $\tau$-quadric surface is an irreducible component of $\varphi^{-1}(\mathcal{D}_1^\prime)$. The period map induces a dominant map $\mathcal{Z}_{\tau, \text{quadric}} \to \mathcal{P}_{10}^1$ whose general fiber is rationally dominated by the symmetric square of a degree-10 K3 surface.

A general member of $\mathcal{Z}_{\tau, \text{quadric}}$ is rational.

During the proof, we present a geometrical construction of a general member of $\mathcal{Z}_{\tau, \text{quadric}}$, starting from a general degree-10 K3 surface $S \subset \mathbb{P}^6$ and two general points on $S$: if $S_0 \subset \mathbb{P}^4$ is the (singular) projection of $S$ from these two points, the birational isomorphism $\mathbb{P}^4 \dashrightarrow X$ is given by the linear system of quartics containing $S_0$.

Proof. A parameter count shows that $\mathcal{Z}_{\tau, \text{quadric}}$ is irreducible of codimension 1 in $\mathcal{Z}_{10}$ (one can also use the parameter count at the end of the proof). Let $\Sigma \subset X$ be a smooth $\tau$-quadric surface. From (12), we obtain $\langle \Sigma \rangle = 3$, and from (13), $d = 10$. Since $\sigma_1^2 \cdot \Sigma$ is even, we are in $\mathcal{Z}_{10}$. The family $\mathcal{Z}_{\tau, \text{quadric}}$ is therefore a component of the divisor $\varphi^{-1}(\mathcal{D}_1^\prime)$.

The projection from the 3-plane $\langle \Sigma \rangle$ induces a birational map $X \dashrightarrow \mathbb{P}^4$ (in particular, $X$ is rational!). If $\varepsilon : \tilde{X} \to X$ is the blow-up of $\Sigma$, one checks that it induces a birational morphism $\pi : \tilde{X} \to \mathbb{P}^4$ which is more complicated than just the blow-up of a smooth surface (compare with (17)).

Indeed, since $\Sigma$ is contained in a $G(2, V_4)$, the quartic surface $X \cap G(2, V_4)$ is the union of $\Sigma$ and another $\tau$-quadric surface $\Sigma^\ast$. The two 3-planes $\langle \Sigma \rangle$ and $\langle \Sigma^\ast \rangle$ meet along a 2-plane, hence (the strict transform of) $\Sigma^\ast$ is contracted by $\pi$ to a point. Generically, the only quadric surfaces contained in $X$ are the $\sigma$-quadric surface $\Sigma_0$ (defined in (13)) and the $\tau$-quadric surfaces $\Sigma$ and $\Sigma^\ast$. Using the fact that $X$ is an intersection of quadrics, one checks that $\Sigma^\ast$ is the only surface contracted (to a point) by $\pi$.

Let $\ell' \subset \tilde{X}$ be a line contracted by $\varepsilon$. If $\ell \subset \tilde{X}$ is (the strict transform of) a line contained in $\Sigma^\ast$, it meets $\Sigma$ and is contracted by $\pi$. Since $\tilde{X}$ has Picard number 2, the rays $R^+[\ell]$ and $R^+[\ell']$ are extremal hence span the cone of curves of $\tilde{X}$. These two classes have $(-K_{\tilde{X}})$-degree 1, hence $\tilde{X}$ is a Fano fourfold. Extremal contractions on smooth fourfolds have been classified ([AM], Theorem 4.1.3); in our case, we have:
• \( \pi \) is a divisorial contraction, its (irreducible) exceptional divisor \( D \) contains \( \Sigma^* \), and 
\[ D \equiv 3H - 4E_3 \]
• \( S_0 := \pi(D) \) is a surface with a single singular point \( s := \pi(\Sigma^*) \), where it is locally
the union of two smooth 2-dimensional germs meeting transversely;
• outside of \( s \), the map \( \pi \) is the blow-up of \( S_0 \) in \( \mathbb{P}^4 \).

Let \( \tilde{X} \to \tilde{X} \) be the blow-up of \( \Sigma^* \), with exceptional divisor \( \tilde{E} \), and let \( \tilde{\mathbb{P}}^4 \to \mathbb{P}^4 \) be the
blow-up of \( s \), with exceptional divisor \( \mathbb{P}^3_s \). The strict transform \( \tilde{S}_0 \subset \tilde{\mathbb{P}}^4 \) of \( S_0 \) is the blow-up
of its (smooth) normalization \( S'_0 \) at the two points lying over \( s \) and meets \( \mathbb{P}^3_s \) along the
disjoint union of the two exceptional curves \( L_1 \) and \( L_2 \). There is an induced morphism
\( \tilde{X} \to \tilde{\mathbb{P}}^4 \) which is an extremal contraction (\[AM\], Theorem 4.1.3) hence is the blow-up of
the smooth surface \( \tilde{S}_0 \), with exceptional divisor the strict transform \( \tilde{D} \subset \tilde{X} \) of \( D \); it induces
by restriction a morphism \( \tilde{E} \to \mathbb{P}^3_s \) which is the blow-up of \( L_1 \sqcup L_2 \).

It follows that we have isomorphisms of polarized Hodge structures
\[
\begin{align*}
H^4(\tilde{X}, \mathbb{Z}) &\cong H^4(X, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z})(-1) \oplus H^2(\Sigma^*, \mathbb{Z})(-1) \\
&\cong H^4(\mathbb{P}^4, \mathbb{Z}) \oplus H^2(\mathbb{P}^3, \mathbb{Z})(-1) \oplus \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2] \oplus H^2(S'_0, \mathbb{Z})(-1).
\end{align*}
\]
In particular, we have \( b_2(S'_0) = 24 + 2 + 2 - 1 - 1 - 1 = 24 \) and \( h^{2,0}(S'_0) = h^{3,1}(\tilde{X}) = 1 \);
moreover, the Picard number of \( S'_0 \) is 3 for \( X \) general. The situation is as follows:

To compute the degree \( d \) of \( S_0 \), we consider the (smooth) inverse image \( P \subset \tilde{X} \) of a 2-plane
in \( \mathbb{P}^4 \). It is isomorphic to the blow-up of \( \mathbb{P}^2 \) at \( d \) points, hence \( K_{\tilde{P}}^2 = 9 - d \). On the other
hand, we have by adjunction
\[
K_P \equiv (K_{\tilde{X}} + 2(H - E))|_P \equiv (-2H + E + 2(H - E))|_P = -E|_P,
\]

hence \( K_{\tilde{P}}^2 = E^2(H - E)^2 = 1 \) and \( d = 8 \).

Consider now a general hyperplane \( h \subset \mathbb{P}^4 \). Its intersection with \( S_0 \) is a smooth
connected curve \( C \) of degree 8, and its inverse image in \( \tilde{X} \) is the blow-up of \( h \) along \( C \), with
exceptional divisor its intersection with \( D \). From \[IP\], Lemma 2.2.14, we obtain
\[
D^3 \cdot (H - E) = -2g(C) + 2 + Kh \cdot C = -2g(C) + 2 - 4\deg(C) = -2g(C) - 30,
\]
\[\text{Use } H^4 = 10, H^3E = 0, H^2E^2 = -2, HE^3 = 0, E^4 = 3.\]
from which we get $g(C) = 6$. In particular, $c_1(S'_0) \cdot h = 2$. On the other hand, using a variant of the formula for smooth surfaces in $\mathbb{P}^4$, we obtain
\[ d^2 - 2 = 10d + c_2(S'_0) - c_2(S'_0) + 5c_1(S'_0) \cdot h, \]
hence $c_1^2(S'_0) - c_2(S'_0) = -28$. We can also use the formula from [P], Lemma 2:
\[
\begin{align*}
\hat{D}^4 &= (c_2(\hat{\mathcal{P}}^4) - c_1^2(\hat{\mathcal{P}}^4)) \cdot \hat{S}_0 + c_1(\hat{\mathcal{P}}^4)|_{\hat{S}_0} \cdot c_1(\hat{S}_0) - c_2(\hat{S}_0) \\
&= (-15h^2 - 7[\mathbb{P}^3]^2) \cdot \hat{S}_0 + (-5h^2 + 3[\mathbb{P}^3])|_{\hat{S}_0} \cdot c_1(\hat{S}_0) - c_2(\hat{S}_0) \\
&= (-15h^2 - 7[\mathbb{P}^3]^2) \cdot \hat{S}_0 + (-5h^2 + 3[\mathbb{P}^3])|_{\hat{S}_0} \cdot c_1(\hat{S}_0) - c_2(\hat{S}_0) \\
&= -120 + 14 - 10 - 6 - c_2(\hat{S}_0).
\end{align*}
\]
Since $\hat{D}^4 = D^4 = (3H - 4E)^4 = -150$, we obtain $c_2(S'_0) = 28$, hence $c_2(S'_0) = 26$ and $c_1^2(S'_0) = -2$. Noether’s formula implies $\chi(S'_0, \mathcal{O}_{S'_0}) = 2$, hence $h^1(S'_0, \mathcal{O}_{S'_0}) = 0$. The classification of surfaces implies that $S'_0$ is the blow-up at two points of a K3 surface $S$ of degree 10. By the simplicity argument used before, the integral polarized Hodge structures $H^2(S, \mathbb{Z})_0(-1)$ and $K^\perp$ are isomorphic: $S$ is the (polarized) K3 surface associated with $X$ via Proposition 6).

Conversely, let $S = G(2, V_3) \cap Q' \cap \mathbb{P}^6$ be a general K3 surface of degree 10 and let $p$ (corresponding to $V_2 \subset V_3$) and $p'$ (corresponding to $V'_2 \subset V_3$) be two general points on $S$. If $V_q := V_2 \oplus V'_2$, the intersection $S \cap G(2, V_q)$ is a set of four points $p, p', q, q'$ in the 2-plane $\mathbb{P}(\wedge^2 V_q) \cap \mathbb{P}^6$. Projecting $S$ from the line $pp'$ gives a non-normal degree-8 surface $S_0 := S_{pp'} \subset \mathbb{P}^4$, where $q$ and $q'$ have been identified. Its normalization $\tilde{S}_0$ is the blow-up of $S$ at $p$ and $p'$. Now let $\hat{\mathbb{P}}^4 \to \mathbb{P}^4$ be the blow-up of the singular point of $S_0$, and let $\tilde{X} \to \hat{\mathbb{P}}^4$ be the blow-up of the strict transform of $S_0$ in $\hat{\mathbb{P}}^4$. The strict transform in $\tilde{X}$ of the exceptional divisor $\mathbb{P}^3_s \subset \hat{\mathbb{P}}^4$ can be blown down by $\tilde{X} \to \hat{X}$.

The resulting smooth fourfold $\hat{X}$ is a Fano variety with Picard number 2. One extremal contraction is $\pi : \tilde{X} \to \mathbb{P}^4$. The other extremal contraction gives the desired $X$. This construction depends on 23 parameters (19 for the surface $S$ and 4 for $p, p' \in S$).

\textbf{Remark 7.5.} What happens if we start from the $\tau$-quadric $\Sigma^*$ instead of $\Sigma$? Blowing-up $\Sigma$ and then the strict transform of $\Sigma^*$ is not the same as doing it in the reverse order, but the end products have a common open subset $\tilde{X}^0$ (whose complements have codimension 2). The morphisms $\tilde{X}^0 \to \hat{\mathbb{P}}^4 \to \mathbb{P}^3$ (where the second morphism is induced by projection from $s$) are then the same, because they are induced by the projection of $X$ from the 4-plane $\langle \Sigma, \Sigma^* \rangle$, and the locus where they are not smooth is the common projection $S_1$ in $\mathbb{P}^3$ of the surfaces $S_0 \subset \mathbb{P}^4$ and $S'_0 \subset \mathbb{P}^4$ from their singular points.

This surface $S_1$ is also the projection of the K3 surface $S \subset \mathbb{P}^6$ from the 2-plane spanned by $p, p', q, q'$. The end result is therefore the same K3 surface $S$, but the pair of points is now $q, q'$. If $\iota_S$ is the birational involution on $S^{[2]}$ defined by $p + p' \mapsto q + q'$ one sees that for $\hat{X}$ general in $\mathcal{X}_{\tau\text{-quadric}}$, the fiber of $\mathcal{X}_{\tau\text{-quadric}} \to \mathcal{D}_{10}$ through $[X]$ is birationally isomorphic to the quotient $S^{[2]}/\iota_S$.

\footnote{In [O], Proposition 5.20, O’Grady proves that for $S$ general, the involution $\iota_S$ is birational on the complement of a 2-plane.}
Remark 7.6. No fourfold \( X \) of type \( \mathcal{X}_{10} \) contains infinitely many quadric surfaces. The proof is more involved than in Remark 7.3 but still elementary. Since \( X \) contains a unique \( \sigma \)-quadric, we may assume (using Remark 7.3) that \( X \) contains irreducible \( \tau \)-quadrics \( \Sigma \) and \( \Sigma^* \), contained in \( G(2, V_4) \), with \( G(2, V_4) \cap X = \Sigma \cup \Sigma^* \), that move in positive-dimensional families. Assume \( G(2, V_4') \cap X = \Sigma' \cup \Sigma'^* \), with \( V_4 \neq V_4' \). Since \( \Sigma \) is a linear section of \( G(2, V_4) \), the intersection \( \Sigma \cap \Sigma' \) is a linear section of \( G(2, V_4) \cap G(2, V_4') = G(2, V_4 \cap V_4') \), a 2-plane. Since \( \Sigma \) is irreducible, it is therefore either a line, a point, or the empty set. If \( \Sigma' \) is a deformation of \( \Sigma \) in \( X \), we have \( \Sigma : \Sigma' = (\Sigma^2)_X = 3 \), hence the intersection \( \Sigma \cap \Sigma' \) must be a line \( L \subset G(2, V_4 \cap V_4') \). Similarly, since \( \Sigma^* \cdot \Sigma' = (\sigma_{1,1}|_X - \Sigma) \cdot \Sigma' = -2 \), the intersection \( \Sigma^* \cap \Sigma' \) is a line \( L^* \subset G(2, V_4 \cap V_4') \). We also have

\[
G(2, V_4 \cap V_4') \cap X = (\Sigma \cup \Sigma^*) \cap (\Sigma' \cup \Sigma'^*)
= L \cup L^* \cup (\Sigma \cap \Sigma'^*) \cup (\Sigma^* \cap \Sigma'^*).
\]

If we assume \( L \neq L^* \), since the set in (14) is the intersection of the 2-plane \( G(2, V_4 \cap V_4') \) with a hyperplane and a quadric, it must be \( L \cup L^* \). In that case, since \( L \not\subset \Sigma \), we have \( \Sigma \cap \Sigma'^* = L \) and similarly, \( \Sigma^* \cap \Sigma'^* = L^* \). Then, \( \Sigma' \cap \Sigma = \Sigma'^* \cap \Sigma \) so, by switching the roles of \( \Sigma \) and \( \Sigma' \), we obtain a contradiction. Hence,

\[
L = L^* \subset \Sigma \cap \Sigma^* \cap G(2, V_4 \cap V_4').
\]

The projective lines in \( L \) then sweep out the 2-plane \( P(V_4 \cap V_4') \subset P(V_5) \), which is therefore independent of \( V_4' \); hence of \( \Sigma' \); call it \( P(V_3') \). If \( X \) contains infinitely many quadric surfaces, they must occur as components of \( G(2, V_4') \cap X \) for all \( V_3 < V_4 < V_5 \). The union

\[
\bigcup_{V_3 < V_4 < V_5} G(2, V_4') \cap X
\]

is then the intersection of \( X \) with the Schubert cycle of lines that meet the fixed 2-plane \( P(V_3) \). This is a hyperplane section of \( X \), which is irreducible by Lefschetz theorem. This implies that \( \Sigma \) and \( \Sigma^* \) are interchanged by monodromy; but this is impossible since \( (\Sigma^2)_X = 3 \), whereas \( (\Sigma \cdot \Sigma^*)_X = -2 \).

7.4. Fourfolds containing a cubic scroll (divisor \( \mathcal{D}_{12} \)). We consider rational cubic scroll surfaces obtained as smooth hyperplane sections of the image of a morphism \( P(V_2) \times P(V_2) \rightarrow G(2, V_5) \), where \( V_5 = V_2 \oplus V_3 \); their class in \( G(2, V_5) \) is \( \sigma_1^2 \cdot \sigma_2 = 2\sigma_{3,1} + \sigma_{2,2} \).

Proposition 7.7. The closure \( \overline{\mathcal{X}}_{\text{cubic scroll}} \subset \mathcal{X}_{10} \) of the family of fourfolds containing a cubic scroll surface is the irreducible component of \( \varphi^{-1}(\mathcal{D}_{12}) \) that contains the family \( \mathcal{X}_{\text{cubic scroll}} \).

Proof. Let us count parameters. We have \( 6 + 6 = 12 \) parameters for the choice of \( V_2 \) and \( V_3 \), hence \( a \text{ priori} \) 12 parameters for cubic scroll surfaces in the isotropic Grassmannian \( G_\omega(2, V_5) \). However, one checks that there is a 1-dimensional family of \( V_3 \) which all give the same cubic scroll, so there are actually only 11 parameters. Then, for \( X \) to contain a given cubic scroll \( F \) represents \( h^0(F, \mathcal{O}_F(2, 2)) = 12 \) conditions. It follows that \( \mathcal{X}_{\text{cubic scroll}} \) is irreducible of codimension \( 12 - 11 = 1 \) in \( \mathcal{X}_{10} \).

Let \( F \subset X \) be a cubic scroll. Since \( K_F \) has type \((-1, -2)\), we obtain \( (F)^2_X = 4 \) from (12). From (13), we obtain \( d = 12 \) and we are in \( \mathcal{D}_{12} \). The family \( \mathcal{X}_{\text{cubic scroll}} \) is therefore a component of the hypersurface \( \varphi^{-1}(\mathcal{D}_{12}) \).
In the degenerate situation where \( V_4 = V_2 + V_3 \) is a hyperplane, the associated rational cubic scroll is contained in \( G(2, V_4) \) and is a cubic scroll surface as in the comment right before Remark 7.3. It follows that \( \mathcal{X}_{\rho}\)-plane is contained in the closure of \( \mathcal{X}_{\text{cubic scroll}} \). \( \square \)

7.5. **Fourfolds containing a quintic del Pezzo surface** (divisor \( \mathcal{D}_{10}'' \)). We consider quintic del Pezzo surfaces obtained as the intersection of \( G(2, V_5) \) with a \( \mathbb{P}^5 \); their class is \( \sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2} \) in \( G(2, V_5) \). Fourfolds of type \( \mathcal{X}_{10} \) containing such a surface were already studied by Roth ([R], §4).

**Proposition 7.8.** The closure \( \mathcal{X}_{\text{quintic}} \subset \mathcal{X}_{10} \) of the family of fourfolds containing a quintic del Pezzo surface is the irreducible component of \( \varphi^{-1}(\mathcal{D}_{10}'') \) that contains \( \mathcal{X}_{\rho}\)-plane.

A general member of \( \mathcal{X}_{\text{quintic}} \) is rational.

**Proof.** Let us count parameters. We have \( \dim G(5, \mathbb{P}^8) = 18 \) parameters for the choice of the \( \mathbb{P}^5 \) that defines a del Pezzo surface \( T \). Then, for \( X \) to contain a given quintic del Pezzo surface \( T \) represents \( h^0(\mathbb{P}^5, \mathcal{O}(2)) - h^0(\mathbb{P}^5, \mathcal{I}(T))(2) = 21 - 5 = 16 \) conditions.

Since \( h^0(\mathbb{P}^5, \mathcal{I}(T)(2)) = 6 = h^0(\mathbb{P}^5, \mathcal{I}(T))(2) + 1 \), there exists then a unique (non-Plücker) quadric \( Q \subset \mathbb{P}^5 \) containing \( X \) and \( \mathbb{P}^5 \). This quadric has rank \( \leq 6 \), hence it is a cone with vertex a 2-plane over a (in general) smooth quadric in \( \mathbb{P}^5 \). Such a quadric contains two 3-dimensional families of 5-planes. The intersection of such a 5-plane with \( X \) is, in general, a quintic del Pezzo surface, hence \( X \) contains (two) 3-dimensional families of quintic del Pezzo surfaces. It follows that \( \mathcal{X}_{\text{quintic}} \) has codimension \( 16 - 18 + 3 = 1 \) in \( \mathcal{X}_{10} \).

Let \( T \subset X \) be a quintic del Pezzo surface. From [12], we obtain \( (T)^3_X = 5 \), and from [13], \( d = 10 \). Since \( \sigma_{1,1}, T \) is odd, we are in \( \mathcal{D}_{10}'' \). The family \( \mathcal{X}_{\text{quintic}} \) is therefore a component of the divisor \( \varphi^{-1}(\mathcal{D}_{10}'') \).

The lattice spanned by \( H^4(G(2, V_5), \mathbb{Z}) \) and \([T]\) in \( H^4(X, \mathbb{Z}) \) is the same as for fourfolds containing a \( \sigma \)-plane \( P \), and \([T] = \sigma_2|_X - [P] \). We will now explain this fact geometrically.

If \( X \) contains a quintic del Pezzo surface, we saw that \( X \) is contained in a (non-Plücker) quadric \( Q \subset \mathbb{P}^8 \) of rank \( \leq 6 \). Conversely, if \( X \) is contained in such a quadric, this quadric contains 5-planes and the intersection of such a 5-plane with \( X \) is, in general, a quintic del Pezzo surface.

If follows that \( \mathcal{X}_{\text{quintic}} \) has same closure in \( \mathcal{X}_{10} \) as the set of \( X \) contained in a non-Plücker rank-6 quadric \( Q \). When the vertex of \( Q \) is contained in \( X \), it is a \( \sigma \)-plane, hence \( \mathcal{X}_{\text{quintic}} \) contains \( \mathcal{X}_{\rho}\)-plane.

Finally, note after [R], §5.5), that the general fibers of the projection \( X \to \mathbb{P}^2 \) from \( \langle T \rangle \) are again degree-5 del Pezzo surfaces (they are residual surfaces to \( T \) in the intersection of \( X \) with a 6-plane \( \langle T, x \rangle \), and this intersection is contained in \( \langle T, x \rangle \cap Q \), which is the union of two hyperplanes). It follows from a theorem of Enriques that \( X \) is rational ([E], [SB]). \( \square \)

7.6. **Nodal fourfolds** (divisor \( \mathcal{D}_8 \)). Let \( X \) be a general prime nodal Fano fourfold of index 2 and degree 10. As in the 3-dimensional case ([DIM2], Lemma 4.1), \( X \) is the intersection of a smooth \( W := G(2, V_5) \cap \mathbb{P}^8 \) with a nodal quadric \( Q \), singular at a general point \( O \in W \).
One checks that, as in the case of cubic fourfolds (see [V], §4; [H1], Proposition 4.2.1), the limiting Hodge structure is pure, and the period map extends to the moduli stack $\overline{\mathcal{X}}_{10}$ of our fourfolds with at most one node as

$$\overline{\varphi} : \overline{\mathcal{X}}_{10} \to \mathcal{D}.$$  

**Proposition 7.9.** The closure $\overline{\mathcal{X}}_{\text{nodal}} \subset \overline{\mathcal{X}}_{10}$ of the family of nodal fourfolds is an irreducible component of $\overline{\varphi}^{-1}(\mathcal{D}_8)$.

**Proof.** If $\tilde{X} \to X$ is the blow-up of $O$, the (pure) limiting Hodge structure is the direct sum of $(\delta)$, where $\delta$ is the vanishing cycle, with self-intersection 2, and $H^4(\tilde{X}, \mathbb{Z})$. In the basis $(\sigma_{1,1}|_X, \sigma_2|_X - \sigma_{1,1}|_X, \delta)$, the corresponding lattice $K$ has intersection matrix

$$\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.$$  

The discriminant is 8 and we are in $\mathcal{D}_8$.

The point $O$ defines a pencil of Plücker quadrics, singular at $O$, and the image $W_O$ of $W$ by the projection $p_O : \mathbb{P}^8 \dashrightarrow \mathbb{P}^7_O$ is the base-locus of a pencil of rank-6 quadrics (see [DIM2], §3). One checks that $W_O$ contains the 4-plane $P_O := p_O(T_{W,O})$ and that $W_O$ is singular along a cubic surface contained in $P^4_O$. If $\tilde{P}_O^7 \to P^7_O$ is the blow-up of $P^4_O$, the strict transform $\tilde{W}_O \subset \tilde{P}_O^7 \subset P^7_O \times \mathbb{P}^2$ of $W_O$ is smooth and the projection $\tilde{W}_O \to \mathbb{P}^2$ is a $\mathbb{P}^3$-bundle (this can be checked by explicit computations as in [DIM2], §9.2).

The image $X_O := p_O(X)$ is thus the base locus in $P^7_O$ of a net of quadrics $P$, containing a special line of rank-6 Plücker quadrics. The strict transform $\tilde{X}_O \subset \tilde{W}_O$ of $X_O$ is smooth. The induced projection $\tilde{X}_O \to \mathbb{P}^2$ is a quadric bundle, with discriminant a smooth sextic curve $\Gamma^*_6 \subset \mathbb{P}^2$ (compare with [DIM2], Proposition 4.2) and associated double cover $S \to \mathbb{P}^2$ ramified along $\Gamma^*_6$. It follows that $S$ is a K3 surface with a degree-2 polarization. By [L], Theorem II.3.1, there is an exact sequence

$$0 \to H^4(\tilde{X}_O, \mathbb{Z})_0 \xrightarrow{\Phi} H^2(S, \mathbb{Z})_0(-1) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$  

Both desingularizations $\tilde{X} \to X_O$ and $\tilde{X}_O \to X_O$ are small and their fibers all have dimension $\leq 1$; by [FW], Proposition 3.1, the graph of the rational map $\tilde{X} \dashrightarrow \tilde{X}_O$ induces an isomorphism $H^4(X_O, \mathbb{Z}) \to H^4(\tilde{X}, \mathbb{Z})$ of polarized Hodge structures. This is also the non-special cohomology $K^\perp$, which therefore has index 2 in $H^2(S, \mathbb{Z})_0(-1)$.

When $X$ is general, so is $S$ among degree-2 K3 surfaces, hence the image $\overline{\varphi}(\mathcal{X}_{\text{nodal}})$ has dimension 19. It follows that $\mathcal{X}_{\text{nodal}}$ is an irreducible component of $\overline{\varphi}^{-1}(\mathcal{D}_8)$.

## 8. Construction of special fourfolds

Again following Hassett (particularly [H1], §4.3), we construct special fourfolds with given discriminant. Hassett’s idea was to construct, using the surjectivity of the (extended) period map for K3 surfaces, nodal cubic fourfolds whose Picard group also contains a rank-2 lattice with discriminant $d$ and to smooth them using the fact that the period map remains a submersion on the nodal locus ([V], p. 597). This method should work in our case, but would require first to make the construction of $\mathcal{X}_{10}$ of a nodal fourfold $X$ of type $\mathcal{X}_{10}$ from a
given degree-2 K3 surface more explicit, and second to prove that the extended period map remains submersive at any point of the nodal locus.

We prefer here to use the simpler construction of [17, 1] to prove the following.

**Theorem 8.1.** The image of the period map $\varphi : \mathcal{X}_2^0 \to \mathcal{D}$ meets all divisors $\mathcal{D}_d$, for $d \equiv 0 \pmod{4}$ and $d \geq 12$, and all divisors $\mathcal{D}_d'$ and $\mathcal{D}_d''$, for $d \equiv 2 \pmod{8}$ and $d \geq 10$, except possibly $\mathcal{D}_{18}''$.

Actually, the divisor $\mathcal{D}_{18}''$ also meets the image of the period map: in a forthcoming article, we construct birational transformations that take elements of $\varphi^{-1}(\mathcal{D}_d')$ to elements of $\varphi^{-1}(\mathcal{D}_d'')$.

**Proof.** Our starting point is Lemma 4.3.3 of [11]: let $\Gamma$ be a rank-2 indefinite even lattice containing a primitive element $h$ with $h^2 = 10$, and assume there is no $c \in \Gamma$ with

- either $c^2 = -2$ and $c \cdot h = 0$;
- or $c^2 = 0$ and $c \cdot h = 1$;
- or $c^2 = 0$ and $c \cdot h = 2$.

Then there exists a K3 surface $S$ with Pic($S$) = $\Gamma$ and $h$ is very ample on $S$, hence embeds it in $\mathbf{P}^6$. Assuming moreover that $S$ is not trigonal, e.g., that there is no class $c \in \Gamma$ with $c^2 = 0$ and $c \cdot h = 3$, it has Clifford index 2 and is therefore obtained as the intersection of a Fano threefold $Z := G(2, V_5) \cap \mathbf{P}^6$ with a quadric ([JK, table p. 144]).

In particular, $S$ is an intersection of quadrics, hence its projection from a general point $p$ of $S$ is a (degree-9) smooth surface $\tilde{S}_p \subset \mathbf{P}^5$.

We want to show that the projection $\tilde{Z}_p \subset \mathbf{P}^5$ of $Z$ from $p$ (hence also $\tilde{S}_p$) is contained in a smooth quadric. This can be seen by direct computation: first of all, if $\Pi \subset \mathbf{P}(\wedge^2 V_5^\vee)$ is the 2-plane of hyperplanes that cut out $\mathbf{P}^6$ in $\mathbf{P}(\wedge^2 V_5)$, one has ([PV, Corollary 1.6]

$$\text{Sing}(Z) = \bigcup_{\omega \in \Pi} G(2, \text{Ker}(\omega)).$$

In particular, either $Z$ is smooth (if all forms in $\Pi$ have maximal rank 4), or its singular locus has dimension $\geq 2$, in which case $S$ would be singular. It follows that $Z$ is smooth. In suitable coordinates $(x_i)_{0 \leq i \leq 4}$ on $V_5$, inducing coordinates $(x_{ij})_{0 \leq i < j \leq 4}$ on $\wedge^2 V_5$, it can be defined in $G(2, V_5)$ by the equations

$$x_{12} - x_{03} = x_{02} - x_{14} = x_{01} - x_{24} + x_{23} = 0$$

([1], Technical Lemma (2.5.1)). Next, Aut($Z$) acts on $Z$ with two orbits: the 3-plane of lines passing through the point $[V_4] \in \mathbf{P}(V_5)$, and its complement in $Z$ ([PV, Proposition 5.3]). Since $S$ spans $\mathbf{P}^6$, it must meet the open orbit. Take $p \in S$ in that orbit. As in [1, §3.1], we may assume $p = e_{44}$ and we take $(x_{03}, x_{13}, x_{23}, x_{04}, x_{14}, x_{24})$ as homogeneous coordinates on $\mathbf{P}^6$. Then the degree-4 fourfold $\tilde{Z}_p \subset \mathbf{P}^5$ is the base-locus of the pencil generated by the rank-5 (restrictions of the) Plücker quadrics

$$\Omega_{e_3} : ((x_{24} - x_{23})x_{24} - x_{14}^2 + x_{03}x_{04} = 0) \quad \text{and} \quad \Omega_{e_4} : ((x_{24} - x_{23})x_{23} - x_{13}x_{14} + x_{03}^2 = 0)$$
and a smooth fourfold \( X \) with matrix \( \begin{pmatrix} 10 & 0 \\ 0 & -2e \end{pmatrix} \) in a basis \((h, w)\). When \( e > 1 \), the conditions we need on \( \Gamma \) are satisfied and we obtain a K3 surface \( S \) and a smooth fourfold \( X \) such that \( H^4(X, \mathbb{Z}) \cap H^{2,2}(X) \) contains a lattice \( K_{10} = \langle u, v, w'_{10} \rangle \), with matrix \( \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \) and discriminant 10. Moreover, \( H^2(S, \mathbb{Z})_0(-1) \simeq K_{10}^+ \) as polarized integral Hodge structures. The element \( w \in \Gamma \cap H^2(S, \mathbb{Z})_0 \) corresponds to \( w_X \in K_{10}^+ \cap H^{2,2}(X) \), and \( w_X^2 = -w^2 = 2e \). Therefore, \( H^4(X, \mathbb{Z}) \cap H^{2,2}(X) \) is the lattice \( \langle u, v, w''_{10}, w_X \rangle \), with matrix

\[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 2e \\
\end{pmatrix}.
\]

It contains the lattice \( \langle u, v, w_X \rangle \), with matrix \( \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2e \end{pmatrix} \). Therefore, the period of \( X \) belongs to \( \mathcal{D}_{8e} \), and this proves the theorem when \( d \equiv 0 \pmod{8} \).

It also contains the lattice \( \langle u, v, w''_{10} + w_X \rangle \), with matrix \( \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2e + 3 \end{pmatrix} \) and discriminant \( 8e + 10 \), hence we are also in \( \mathcal{D}_{8e+10} \).

Now let \( e \geq 0 \) and apply Hassett’s lemma with the lattice \( \Gamma \) with matrix \( \begin{pmatrix} 10 & 5 \\ 5 & -2e \end{pmatrix} \) in a basis \((h, g)\). The orthogonal of \( h \) is spanned by \( w := h - 2g \). One checks that primitive classes \( c \in \Gamma \) such that \( c^2 = 0 \) satisfy \( c \cdot h \equiv 0 \pmod{5} \). All the conditions we need are thus satisfied and we obtain a K3 surface \( S \) and a smooth fourfold \( X \) such that \( H^4(X, \mathbb{Z}) \cap H^{2,2}(X) \) contains a lattice \( K_{10} \) of discriminant 10 and \( H^2(S, \mathbb{Z})_0(-1) \simeq K_{10}^+ \) as polarized Hodge structures. Again, \( w \) corresponds to \( w_X \in K_{10}^+ \cap H^{2,2}(X) \) with \( w_X^2 = -w^2 = 8e + 10 \). Set

\[
K := (\Lambda_2 \oplus \mathbb{Z}w_X)^{\text{sat}}.
\]

To compute the discriminant of \( K \), we need to know the ideal \( w_X \cdot \Lambda \). As in the proof of Proposition 6.2, let \( w_{10} \) be a generator of \( K_{10} \cap \Lambda \); it satisfies \( w_{10}^2 = 10 \). Then \( K_{10}^+ \oplus \mathbb{Z}w_{10} \) is a sublattice of \( \Lambda \) and, taking discriminants, we find that the index is 5. Let \( u \) be an element of \( \Lambda \) whose class generates the quotient. We have

\[
w_X \cdot \Lambda = \mathbb{Z}w_X \cdot u + w_X \cdot (K_{10}^+ \oplus \mathbb{Z}w_{10}) = \mathbb{Z}w_X \cdot u + w_X \cdot K_{10}^+ = \mathbb{Z}w_X \cdot u + w \cdot H^2(S, \mathbb{Z})_0.
\]
One checks directly on the K3 lattice that \( w \cdot H^2(S, \mathbb{Z})_0 = 2\mathbb{Z} \). Since \( 5u \in K_{10}^+ \oplus \mathbb{Z}w_{10} \), we have \( 5w_X \cdot u \in 2\mathbb{Z} \), hence \( w_X \cdot u \in 2\mathbb{Z} \). All in all, we have proved \( w_X \cdot \Lambda = 2\mathbb{Z} \) hence the proof of Proposition 6.2 implies that the discriminant of \( K \) is \( w^2_X = 8e + 10 \). Therefore, the period of \( X \) belongs to \( \mathcal{D}_{8e+10} \).

Since the period of \( X \) is in \( \mathcal{D}_{10}'' \), we saw in the proof of Proposition 6.2 that \( w''_{10} := \frac{1}{2}(v + w_{10}) \) is in \( H^4(X, \mathbb{Z}) \). Similarly, either \( w'_{10} := \frac{1}{2}(u + w_X) \) or \( w''_{10} := \frac{1}{2}(v + w_X) \) is in \( K \). Taking intersections with \( w''_{10} \) (and recalling \( w_X \cdot w_{10} = 0 \) and \( v \cdot w_{10} = 1 \)), we see that we are in the first case, hence the period of \( X \) is actually in \( \mathcal{D}_{8e+20} \).

More precisely, \( H^4(X, \mathbb{Z}) \cap H^{2,2}(X) \) is the lattice \( \langle u, v, w''_{10}, w'_X \rangle \), with matrix \[
\begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 2 & 1 & 0 \\
0 & 1 & 3 & 0 \\
1 & 0 & 0 & 2e + 3
\end{pmatrix}.
\]

This lattice also contains the lattice \( \langle u, v, w''_{10} + w'_X \rangle \), with matrix \[
\begin{pmatrix}
2 & 0 & 1 \\
0 & 2 & 1 \\
1 & 1 & 2e + 6
\end{pmatrix}
\] and discriminant \( 8e + 20 \), hence we are also in \( \mathcal{D}_{8e+20} \).

Since we know from §7.2 that the periods of some smooth fourfolds \( X \) of type \( X_{10}^0 \) lie in \( \mathcal{D}_{12} \), this proves the theorem when \( d \equiv 4 \pmod{8} \).

9. Summary and open questions

We summarize the results of §7 in the following diagrams (where the tags next to the arrows describe the general fibers):

\[
\begin{array}{c}
\mathcal{D}_8 \xrightarrow{\mathcal{X}_{\text{nodal}}} \mathcal{D}_10 \xrightarrow{\mathcal{X}_{\tau\text{-quadric}}} K3[3]/\text{inv.} \\
\mathcal{D}_8 \xrightarrow{\mathcal{X}_{\rho\text{-plane}}} \mathcal{D}_12 \xrightarrow{\mathcal{X}_{\text{cubic scroll}}} \mathcal{D}_8 \xrightarrow{\mathcal{X}_{\sigma\text{-plane}}}
\end{array}
\]

where the general members of each of these families are all rational, and

\[
\begin{array}{c}
\mathcal{X}_{\tau\text{-quadric}} \subset H^4(X, \mathbb{Z}) \cap H^{2,2}(X) \\
\mathcal{X}_{\sigma\text{-plane}} \subset \mathcal{X}_{\text{quintic}} \\
\mathcal{X}_{\tau\text{-quadric}} \subset \mathcal{X}_{\text{quintic}} \\
\mathcal{X}_{\rho\text{-plane}} \subset \mathcal{X}_{\text{quintic}} \\
\mathcal{X}_{\rho\text{-plane}} \subset \mathcal{X}_{\text{cubic scroll}} \\
\mathcal{X}_{\rho\text{-plane}} \subset \mathcal{X}_{\text{cubic scroll}}
\end{array}
\]

In the first two of these diagrams, all fourfolds in a given (general) fiber of the period map are of course birationally isomorphic. We think this should be a general fact. More precisely, all fourfolds with the same period should be related by an analog of the conic transformations discussed in [DIMI] in the case of threefolds.

It would be very interesting, as Laza did in [La] for cubic fourfolds, to determine the exact image \( \varphi(\mathcal{X}_{10}) \) in the period domain \( \mathcal{D} \) of the period map. To start with, inspired by the results of [H1], we could ask the following question (see Theorem 8.1).
Question 9.1. Is the image $\varphi(\mathcal{X}_{10}) \subset \mathcal{D}$ of the period map disjoint from the hypersurfaces $\mathcal{D}_2$, $\mathcal{D}_4$, and $\mathcal{D}_8$?

One would also like to identify precisely the set $\varphi(\mathcal{X}_{10})$ in the Baily-Borel compactification $\overline{\mathcal{D}}$ of $\mathcal{D}$. We can then ask the following stronger question.

Question 9.2. Is the image $\varphi(\mathcal{X}_{10}) \subset \overline{\mathcal{D}}$ of the period map in the Baily-Borel compactification $\overline{\mathcal{D}}$ of the period domain $\mathcal{D}$, the complement of the hypersurfaces $\mathcal{D}_2$, $\mathcal{D}_4$, and $\mathcal{D}_8$?

But this seems very far from our present possibilities. In a forthcoming article, we will concentrate instead on the geometrical description of the (4-dimensional) fibers of the period map.

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