A QUASILINEAR PARABOLIC PROBLEM WITH A SOURCE TERM AND A NONLOCAL ABSORPTION

HUI-LING LI, HENG-LING WANG AND XIAO-LIU WANG*

School of Mathematics, Southeast University
Nanjing 210096, China

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Abstract. We investigate a quasi-linear parabolic problem with nonlocal absorption, for which the comparison principle is not always available. The sufficient conditions are established via energy method to guarantee solution to blow up or not, and the long time behavior is also characterized for global solutions.

1. Introduction. In this paper, a quasilinear parabolic problem with a nonlocal term are considered:

\[
\begin{cases}
    u_t = u^p(u_{xx} + u - \bar{u}(t)), & x \in (0, a), \ t > 0, \\
    u_x(0, t) = u_x(a, t) = 0, & t > 0, \\
    u(x, 0) = u_0(x), & x \in [0, a],
\end{cases}
\]

where \( p > 1, a > 0, \bar{u}(t) = \frac{1}{a} \int_0^a u(x) \, dx \). The initial datum \( u_0(x) \in C^{2+\beta}([0, a]) \) \((0 < \beta < 1)\) satisfies \( u_0(0) = u_0(a) = 0 \). From now on, we always assume that \( u_0(x) > 0 \) on \([0, a]\). We say that a nonnegative solution \( u \) of problem (1) blows up at a time \( T \leq \infty \) if it satisfies

\[
\limsup_{t \to T^-} \max_{[0, a]} u(x, t) = \infty.
\]

In recent decades, more and more physical phenomena have been formulated into nonlocal mathematical models ([18, 20]). Property of solution to the following problem of the form

\[
\begin{cases}
    u_t = u^p(\Delta u + k \int_{\Omega} u^q \, dx + \ell u^r), & x \in \Omega, \ t > 0, \\
    u = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \Omega
\end{cases}
\]

has been studied intensively (see, for example, [1, 5, 9, 22]), where \( p, q, r \) and \( k \) are positive constants, the constant \( \ell \in \mathbb{R} \), and \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \). An interesting fact is that property of solution depends critically on the domain \( \Omega \). This is also observed in problems without nonlocal term. For problem (2) with \( k = 0, \ell = r = 1 \) and \( u_0(x) > 0 \), it has been proven that the
size of \( \Omega \) plays an important role in global solvability of solution \( u \) (refer to papers [24, 25, 26, 27] and the references therein). Roughly speaking, related results can be written as follows: if we let \( \lambda_1 \) be the principal eigenvalue of \(-\Delta\) in \( \Omega \) with zero Dirichlet boundary condition, then the following conclusions hold:  

- If \( \Omega \) is large such that \( \lambda_1 < 1 \), then every solution \( u \) blows up in a finite time. In addition, when \( 0 < p < 2 \), blow-up set of \( u \) has positive measure and its blowup rate is exactly \((T - t)^{-1/p}\); when \( p \geq 2 \), blowup occurs essentially faster than the rate \((T - t)^{-1/p}\), and the subset of \( \Omega \) of points of such fast blowup has positive measure. 
- If \( \lambda_1 = 1 \), then all solutions are global in time. Moreover, when \( 0 < p < 3 \), all solutions are bounded and converge to steady states as \( t \to \infty \); when \( p \geq 3 \), there are both bounded solution and unbounded solution, which hinges on \( u_0(x) \). 
- If \( \Omega \) is small such that \( \lambda_1 > 1 \), then each solution exists globally and tends to zero as \( t \to \infty \).

Assume \( u \) is a positive solution of problem (1) and \( T_{\text{max}} \) is its maximal existence time. It is not difficult to find that the integral identity below holds: 

\[
\int_0^t u^{1-p}dx = \int_0^t u_0^{1-p}(x)dx, \quad \forall \ 0 \leq t < T_{\text{max}}.
\]  

Moreover, due to the special structure of nonlocal term, the comparison principle does not always hold for problem (1). These properties are also found in the nonlocal parabolic problem 

\[
\begin{cases}
  u_t - \Delta u = f(u) - \int_{\Omega} f(u) \, dx, & x \in \Omega, \ t > 0, \\
  \partial u/\partial \eta = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]  

where \( \eta \) is the unit outward normal vector on \( \partial \Omega \), and \( f(u) = |u|^p \) or \( f(u) = u|u|^{p-1} \) or \( f(u) \) is of other general form. Solution \( u \) to problem (4) blows up in a finite time provided that \( u_0(x) \) satisfies certain energy condition, please refer to, for example, papers [2, 3, 4, 6, 7, 8], [10]–[13] and [19, 23]).

Another motivation of our study arises from the popular model of curvature flow. Let \( v = u^{p-1} \). Then by the first equation of problem (1) it yields 

\[
v_t = (p - 1)u^2 \left[ (v^{\frac{1}{p-1}})_x + v^{\frac{1}{p-1}} \right] - \frac{1}{a} \int_a^x v^{\frac{1}{p-1}} \, dx,
\]  

which is related to the embedded convex length-preserving curvature flow ([21]), where \( v \) is the smooth curvature function of evolving curves at time \( t \) satisfying \( 2\pi \)-periodic boundary condition. In such a setting, solution \( v \) exists for all time and converges smoothly to a constant eventually.

By applying the Leray-Schauder fixed point theory, problem (1) has at least a positive classical solution \( u \in C^{2,\beta}([0, a] \times [0, T]) \) for some \( T > 0 \), which is actually smooth for \( t \in (0, T) \). At the same time, uniqueness of solution \( u \) is achieved in virtue of the standard parabolic equation theory. Furthermore, such \( u \) can be extended to \([0, T_{\text{max}}]\), where \( T_{\text{max}} \leq \infty \) is so called the maximal existence time of \( u \). In addition, if \( T_{\text{max}} < \infty \), there must hold: \( \lim_{t \to T_{\text{max}}} \max_{[0,a]} u(x,t) = \infty \) (see Remark 2.1) and its derivatives \( u_x \) has regularity estimates ([17]).
Suppose $u$ is a positive solution to problem (1), and its associated energy functional is written as

$$E(t) = \int_0^a \left[ u_x^2 - (u - \bar{u})^2 \right] dx. \quad (5)$$

Now we state our main theorems as follows.

**Theorem 1.1.** Let $E(0) < 0$. Then solution of problem (1) blows up at the maximal existence time $T_{\text{max}} \leq \infty$, and in particular, $T_{\text{max}} < \infty$ when $p > 2$.

**Remark 1.** By Lemma 3.2, one finds that the case $E(0) < 0$ comes to happen only when $a > \pi$.

**Theorem 1.2.** Assume $a \leq \pi$ and $1 < p < 2$, or $a < \pi$ and $p \geq 2$. Then, every solution $u$ of problem (1) is global and uniformly bounded. Moreover, $u$ converges smoothly to a constant for $a \leq \pi$ and $p > 1$, and while, for $a = \pi$ and $1 < p < 2$, for any sequence $\{t_j\}_{j=1}^{\infty} \rightarrow \infty$, there is a subsequence $\{t_{j_k}\}_{k=1}^{\infty}$, such that $u(x, t_{j_k})$ converges to the function $A\cos x + B$ in $C([0, a])$ as $k \rightarrow \infty$, where $A$ and $B$ are constants satisfying $|A| \leq B$.

2. **Proof of Theorem 1.1.** Before proving Theorem 1.1, we establish two lemmas. The first one claims positivity of solution for problem (1).

**Lemma 2.1.** Let $u_0(x) > 0$ on $[0, a]$, and let $u$ be a solution of problem (1) with $T_{\text{max}} \leq \infty$. Then $u > 0$ for all $(x, t) \in [0, a] \times [0, T_{\text{max}})$.

**Proof.** Assume on the contrary that the assertion was not true, and then, there would be a point $(x^*, t^*) \in [0, a] \times (0, T_{\text{max}})$, such that $u(x^*, t^*) = 0$ and $u > 0$ for every $(x, t) \in [0, a] \times (0, t^*)$. Define

$$\phi = \frac{1}{u^{p-1}} - (p-1) \int_0^t \bar{u}(\tau) d\tau, \quad (x, t) \in [0, a] \times [0, t^*). \quad (6)$$

Then by $p > 1$ we have that, for any $0 < t' < t^*$,

$$\begin{aligned}
\phi_t - u^p \phi_{xx} & = \frac{1-p}{u^p} [u_t - u^p (u_{xx} - \bar{u}(t))] - \frac{p(p-1)u_x^2}{u} \\
& \leq (1-p)u \leq 0, \quad (x, t) \in (0, a) \times (0, t'], \\
\phi_x(0, t) & = \phi_x(a, t) = 0, \quad 0 < t \leq t'.
\end{aligned}$$

As a result, the classical comparison principle illustrates that $\phi \leq \max_{[0,a]} \phi(x, 0) = \max_{[0,a]} u_0^{1-p}(x)$ in $[0, a] \times [0, t']$, that is,

$$\begin{aligned}
\frac{1}{u^{p-1}} & \leq (p-1) \int_0^t \bar{u}(\tau) d\tau + \max_{[0,a]} u_0^{1-p}(x) \\
& \leq (p-1) \int_0^{t^*} \bar{u}(\tau) d\tau + \max_{[0,a]} u_0^{1-p}(x) \\
& = M + \max_{[0,a]} u_0^{1-p}(x), \quad \forall \ (x, t) \in [0, a] \times [0, t'].
\end{aligned}$$

It is clear that $M > 0$. It follows that

$$u^{p-1} \geq \frac{1}{M + \max_{[0,a]} u_0^{1-p}(x)} > 0, \quad \forall \ (x, t) \in [0, a] \times [0, t'].$$
Hence, the arbitrary of $t' \in (0, t^*)$ guarantees that
\[
 u^{p-1}(x, t^*) \geq \frac{1}{M + \max_{[0, a]} u_0^{1-p}(x)} > 0, \quad \forall \ x \in [0, a].
\]
The fact that $p > 1$ shows that it contradicts to the assumption that $u(x^*, t^*) = 0$, and we finish the proof of this lemma.

**Remark 2.** Under the hypothesis of Lemma 2.1, one might obtain that if $\limsup_{t \to T_{\max}} u = \infty$, then $\limsup_{t \to T_{\max}} u = \infty$. Indeed, from the standard parabolic theory, the following alternative holds:

\[
 \limsup_{t \to T_{\max}} \max_{[0, a]} u = \infty \quad \text{or} \quad \liminf_{t \to T_{\max}} \inf_{[0, a]} u = 0.
\]
We suppose conversely that $\limsup_{t \to T_{\max}} \max_{[0, a]} u < \infty$, then
\[
 \sup_{[0, T_{\max})} \max_{[0, a]} u =: M^* < \infty,
\]
and thereafter,
\[
 (p - 1) \int_0^{T_{\max}} \bar{u}(\tau)d\tau \leq (p - 1)M^*T_{\max} < \infty.
\]
Define $\phi$ as in (6) over $[0, a] \times [0, T_{\max})$, and adopt a similar argument as above. It yields that for any $t' \in (0, T_{\max})$,
\[
 u^{p-1} \geq \frac{1}{(p - 1)M^*T_{\max} + \max_{[0, a]} u_0^{1-p}(x)} > 0, \quad \forall \ (x, t) \in [0, a] \times [0, t'].
\]
Thus, the equation in problem (1) is uniformly parabolic on $[0, T_{\max})$. And then, we can apply similar arguments as in the proof of Theorem 1.2 (case (i)), and obtain the gradient estimate of $u$. Hence, the standard parabolic theory asserts the existence of a classical solution $u$ to problem (1). In addition, such $u$ can be extended to some $T > T_{\max}$, which contradicts to the definition of $T_{\max}$. Therefore, $\limsup_{t \to T_{\max}} u = \infty$.

**Lemma 2.2.** The energy functional $E(t)$ defined by (5) satisfies
\[
 E'(t) \leq 0, \quad \forall \ 0 < t < T_{\max}.
\]

**Proof.** Multiplying the first equation of problem (1) by $u_t$ and then integrating by part, we see that
\[
 0 \leq \int_0^a \frac{u^2}{u^p} dx = \int_0^a u_t(u_{xx} + u - \bar{u}(t))dx
 = -\int_0^a u_x u_{xt} dx + \frac{1}{2} \frac{d}{dt} \left( \int_0^a u^2 dx - au^2(t) \right)
 = -\frac{1}{2} \frac{d}{dt} \int_0^a [u_x^2 - (u^2 - \bar{u}^2(t))] dx
 = -\frac{1}{2} \frac{d}{dt} \int_0^a [u_x^2 - (u - \bar{u}(t))^2] dx = -\frac{1}{2} E'(t), \quad 0 < t < T_{\max},
\]
and it leads to the desired conclusion. \(\square\)

Now we prove Theorem 1.1.
Proof of Theorem 1.1. Similar as above, from the first equation of problem (1) it follows that
\[
-\int_0^a \frac{u_t}{u^{p-1}} dx = \int_0^a \left[ u^2 - (u - \bar{u}(t))^2 \right] dx = E(t), \quad \forall \ t \in (0, T_{\text{max}}).
\]
Subsequently, for any \(0 < t < T_{\text{max}}\),
\[
E(t) = \begin{cases} 
\frac{1}{p-2} \frac{d}{dt} \int_0^a u^{2-p} dx, & p > 1 \text{ and } p \neq 2, \\
-\frac{d}{dt} \int_0^a \ln u \ dx, & p = 2.
\end{cases}
\]
(7)

When \(1 < p < 2\), Lemma 2.2 and (7) show that
\[
\frac{d}{dt} \int_0^a u^{2-p} dx = (p-2)E(t) \geq (p-2)E(0) > 0, \quad 0 < t < T_{\text{max}}.
\]
Therefore, if \(T_{\text{max}} = \infty\), then it is obvious that \(\int_0^a u^{2-p} dx \to \infty \) as \(t \to \infty\), which implies that \(\max_{x \in [0,a]} u(x,t) \to \infty \) as \(t \to \infty\). If \(T_{\text{max}} < \infty\), just as stated in Remark 2, the conclusion of Theorem 1.1 holds.

When \(p = 2\), adopting similar analysis as above, we arrive at the conclusion of Theorem 1.1. When \(p > 2\), analogously as above, we find that
\[
\frac{d}{dt} \int_0^a u^{2-p} dx = (p-2)E(t) \leq (p-2)E(0) < 0, \quad 0 < t < T_{\text{max}}.
\]
Consequently,
\[
\int_0^a u^{2-p} dx \leq (p-2)E(0)t + \int_0^a u_0^{2-p}(x) dx, \quad 0 < t < T_{\text{max}}.
\]
(8)
Suppose on the contrary that \(T_{\text{max}} = \infty\). Take
\[
t^* = -\frac{1}{(p-2)E(0)} \int_0^a u_0^{2-p}(x) dx.
\]
Then by (8) and Lemma 2.1 we achieve
\[
0 < \int_0^a u^{2-p}(x,t^*) dx = 0.
\]
It is a contradiction, and \(T_{\text{max}} < \infty\) follows immediately. Therefore, the assertion of Theorem 1.1 is valid.

3. Proof of Theorem 1.2. In this section, we establish three lemmas to prove Theorem 1.2. The first one is to establish gradient estimate of solution \(u\). In the following, we use \(C, C_1, C_2, \ldots\) to denote general constants independent of time \(t\).

Lemma 3.1. Suppose \(u\) is a solution to problem (1), which exists on the maximal interval \([0, T_{\text{max}}]\). Then there is a constant \(\tilde{C}\) depending only on \(u_0(x)\), such that
\[
\int_0^a u_x^2 dx \leq \tilde{C} + \int_0^a u^2 dx, \quad \forall \ 0 \leq t < T_{\text{max}}.
\]
Proof. With the help of the definition of \(E(t)\) and Lemma 2.2, it is not difficult to see that for every \(0 \leq t < T_{\text{max}},\)
\[
\tilde{C} := E(0) \geq E(t) = \int_0^a (u_x^2 - u^2) dx + a[\ddot{u}(t)]^2 \geq \int_0^a (u_x^2 - u^2) dx.
\]
Lemma 3.2. Let \( a \leq \pi \), then the energy \( E(t) \) defined by (5) satisfies
\[
E(t) \geq 0, \quad \forall \ t \in [0, T_{\text{max}}).
\]

Proof. For each \( t \in [0, T_{\text{max}}) \), define a \( 2a \)-periodic function \( w \) by \( w(x + 2a, t) = w(x, t) \) and
\[
w(x, t) = \begin{cases} 
    u(x, t), & x \in [0, a], \\
    u(-x, t), & x \in [-a, 0).
\end{cases}
\]

A direct calculation shows that for all \( 0 \leq t < T_{\text{max}} \),
\[
\bar{w}(t) = \frac{1}{2a} \int_{-a}^{a} w(x, t) \, dx = \frac{1}{a} \int_{0}^{a} u(x, t) \, dx = \bar{u}(t),
\]
\[
\int_{-a}^{a} [w_x^2 - (w - \bar{w}(t))^2] \, dx = 2 \int_{0}^{a} [u_x^2 - (u - \bar{u}(t))^2] \, dx = 2E(t).
\]

On the other hand, Poincaré inequality (or Wirtinger inequality) illustrates that
\[
\int_{-a}^{a} (w - \bar{w}(t))^2 \, dx \leq \frac{a^2}{\pi^2} \int_{-a}^{a} w_x^2 \, dx.
\]

Joining with all the above estimates, we obtain
\[
E(t) \geq \frac{\pi^2 - a^2}{2\pi^2} \int_{-a}^{a} u_x^2 \, dx = \frac{\pi^2 - a^2}{\pi^2} \int_{0}^{a} u_x^2 \, dx \geq 0, \quad \forall \ t \in [0, T_{\text{max}}). \tag{9}
\]

Therefore, we complete the proof of Lemma 3.2.

Lemma 3.3. Let \( u \) be a solution to problem (1). Assume \( a \leq \pi \) and \( 1 < p < 2 \). Then \( T_{\text{max}} = \infty \), and there exists a constant \( C > 0 \) independent of \( t \), such that \( u \leq C \) for all \( (x, t) \in [0, a] \times [0, \infty) \). In other words, such solution \( u \) is global and uniformly bounded.

Proof. Set
\[
\varphi(t) = \int_{0}^{a} u^{2-p} \, dx, \quad 0 \leq t < T_{\text{max}}.
\]

Then by (7) and Lemma 3.2 we have
\[
\varphi'(t) = (p - 2)E(t) \leq 0, \quad \forall \ 0 \leq t < T_{\text{max}}.
\]

Consequently,
\[
\int_{0}^{a} u^{2-p} \, dx = \varphi(t) \leq \varphi(0) =: C_1, \quad \forall \ 0 \leq t < T_{\text{max}}. \tag{10}
\]

We claim that all assertions of Lemma 3.3 are valid. In fact, if not, then there would exist a sequence \( \{t_j\}_{j=1}^{\infty} \to T_{\text{max}} \) and its corresponding \( x_j \in [0, a] \), such that
\[
u(x_j, t_j) = \max_{[0,a]} u(x, t_j) \to \infty \quad \text{as} \ j \to \infty. \tag{11}
\]
Lemma 3.1 and (11) guarantee that for all suitable large $j$,
\[
    u(x, t_j) = u(x, t_j) + \int_x^{x_j} u_x(x, t_j) dx \\
    \leq u(x, t_j) + \sqrt{|x - x_j|} \left( \int_x^{x_j} u_x^2(x, t_j) dx \right)^{1/2} \\
    \leq u(x, t_j) + \sqrt{|x - x_j|} \left( \int_0^a u^2(x, t_j) dx + C \right)^{1/2} \\
    \leq u(x, t_j) + \sqrt{|x - x_j|} \left( au^2(x_j, t_j) + C \right)^{1/2} \\
    \leq u(x, t_j) + \sqrt{2a|x - x_j|} u(x_j, t_j), \quad \forall x \in [0, a].
\]

From this we derive that for such large $j$,
\[
    u(x, t_j) > \frac{1}{2} u(x, t_j), \quad \forall x \in \left\{ x \in [0, a] \mid \sqrt{2a|x - x_j|} < 1/2 \right\} =: \Sigma_j.
\]

Note that the measure of $\Sigma_j$ satisfies $|\Sigma_j| \geq \min\{a, 1/(8a)\}$. Thus, the fact that $1 < p < 2$ and the positivity of $u$ demonstrate
\[
    \int_0^a u^{2-p}(x, t_j) dx \geq \int_{\Sigma_j} u^{2-p}(x, t_j) dx = |\Sigma_j| \left( \frac{1}{2} u(x_j, t_j) \right)^{2-p} \to \infty \quad \text{as} \quad j \to \infty.
\]

It contradicts with (10). Consequently,
\[
    u \leq C, \quad \forall (x, t) \in [0, a] \times [0, T_{\text{max}}].
\]

Combining this with Remark 2, proceeding with converse argument we arrive at the conclusion that $T_{\text{max}} = \infty$. Therefore, Lemma 3.3 is concluded. \hfill \Box

The above argument is not suitable for $p \geq 2$. In the following we are going to deal with it by applying another analysis ($a < \pi$ is forced).

**Lemma 3.4.** Assume $u$ is a solution to problem (1). If $a < \pi$ and $p \geq 2$, then there exists a constant $C > 0$ independent of $t$, such that $u \leq C$ for all $(x, t) \in [0, a] \times [0, \infty)$.

**Proof.** By carefully analyzing the proof of Lemma 3.3, we find that to verify this lemma, it only needs to demonstrate the existence of a constant $C$, which does not depend on $t$, such that
\[
    u \leq C, \quad \forall (x, t) \in [0, a] \times [0, T_{\text{max}}]. \quad (12)
\]

From Lemma 2.2 and (9) it follows
\[
    E(0) \geq E(t) \geq \frac{\pi^2 - a^2}{\pi^2} \int_0^a u_x^2 dx, \quad \forall t \in [0, T_{\text{max}}),
\]

which means
\[
    \int_0^a u_x^2 dx \leq C_1, \quad \forall t \in [0, T_{\text{max}}). \quad (13)
\]
By (3) and the mean value theorem, we know that for any $0 \leq t < T_{\text{max}}$, there is an $x_2(t) \in [0, a]$ satisfying
\[ I_0 := \int_0^a u_0^{-p}(x)dx = \int_0^a u_1^{-p}dx = au_2^{-p}(x_2(t), t). \]  
(14)
For such $t$ and $x_2(t)$, we find that, in view of (13) and (14),
\[ u(x_1, t) = u(x_2(t), t) + \int_{x_2(t)}^{x_1} u_x dx \]
\[ \leq u(x_2(t), t) + \sqrt{|x_1 - x_2(t)|} \left( \int_{x_2(t)}^{x_1} u_x^2 dx \right)^{1/2} \]
\[ \leq \left( \frac{a}{I_0} \right)^{1/(p-1)} + \sqrt{a} \left( \int_0^a u_x^2 dx \right)^{1/2} \leq C, \quad \forall \, x_1 \in [0, a]. \]

Therefore, the estimate (12) holds, and the proof of Lemma 3.4 is accomplished. \qed

We are now able to demonstrate Theorem 1.2.

**Proof of Theorem 1.2.** From Lemma 3.3 and Lemma 3.4 it follows that all solutions of problem (1) are global and uniformly bounded by $C$, where the constant $C$ depends only on $u_0(x)$ and is independent of $t$. Lemma 3.1 shows the existence of a constant $C$, independent of $t$, such that
\[ \|u(\cdot, t)\|_{W^{1,2}([0, a])} \leq C, \quad \forall \, 0 \leq t < \infty, \]
and subsequently,
\[ \|u(\cdot, t)\|_{C^{1/2}([0, a])} \leq C, \quad \forall \, 0 \leq t < \infty. \]
As a result, Arzela-Ascoli Theorem asserts that for any sequence $\{t_j\}_{j=1}^\infty$, there exist a subsequence $\{t_{j_k}\}_{k=1}^\infty$ and a function $u^* \in C([0, a])$, such that
\[ \|u(\cdot, t_{j_k}) - u^*(\cdot)\|_{C([0, a])} \to 0 \quad \text{as} \quad k \to \infty. \]  
(15)
To obtain property of $u^*$, we divide the remaining proof into two cases.

**Case (i):** $p \geq 2$ and $a < \pi$. We will demonstrate that $u^*$ is a constant and the convergence is smooth.

**Step 1:** We claim that there exists a constant $C$ depending only on $u_0(x)$, such that
\[ |u_x| \leq C, \quad \forall \, (x, t) \in [0, a] \times [0, \infty). \]  
(16)
Define
\[ \Psi = u_x^2 + u^2, \quad (x, t) \in [0, a] \times [0, \infty), \]
\[ M(t) = \max \left\{ \max_{[0, a] \times [0, t]} u^2, \max_{[0, a]} \Psi(x, 0) \right\}, \quad t \in [0, \infty). \]
For any given $t > 0$, we assume
\[ \Psi(x_0, t_0) = \max_{[0, a] \times [0, t]} \Psi. \]

First, we consider the case $t_0 > 0$. When $x_0 = 0$ or $x_0 = a$, the boundary condition of problem (1) directly yields that $u_x(x_0, t_0) = 0$. When $0 < x_0 < a$, we suppose conversely that $u_x(x_0, t_0) \neq 0$. Then by the definition of $\Psi(x_0, t_0)$ we achieve that at the point $(x_0, t_0)$, it holds: $\Psi_x = 0$, $\Psi_{xx} \leq 0$ and $\Psi_t \geq 0$. Hence,
u_{xx}(x_0,t_0) + u(x_0,t_0) = 0. A direct calculation and Lemma 2.1 tell us that at the point \( (x_0,t_0) \),
\[
0 \leq \Psi_t = u^p \Psi_{xx} - 2u^{p-1} \bar{u}(t)(u^2 + pu_x^2) < u^p \Psi_{xx} \leq 0.
\]
It is impossible, and \( u_x(x_0,t_0) = 0 \) provided that \( t_0 > 0 \). Therefore, by making use of definition of \( \Psi(x_0,t_0) \) and the fact \( u_x(x_0,t_0) = 0 \), we conclude that
\[
\max_{[0,a] \times [0,t]} \Psi = u^2(x_0,t_0) \leq \max_{[0,a] \times [0,t]} u^2.
\]
The second case is \( t_0 = 0 \), that is, \( \max_{[0,a] \times [0,t]} \Psi = \max \Psi(x,0) \).

In a word we have proved
\[
\max_{[0,a] \times [0,t]} \Psi \leq M(t), \quad \forall t \in [0,\infty).
\]
Combining this with Lemma 3.4 we arrive at (16).

**Step 2:** We will prove that there exists a constant \( C_1 > 0 \) depending only on \( u_0(x) \), such that
\[
u \geq C_1, \quad \forall (x,t) \in [0,a] \times [0,\infty).
\]
When \( p = 2 \) and \( a < \pi \), for every \( t > 0 \), we apply the integral identity (3) and then the mean value theorem to obtain the existence of \( x_1(t) \in [0,a] \), such that
\[
I_0 := \int_0^a \frac{dx}{u_0(x)} = \int_0^a \frac{dx}{u} = \frac{a}{u(x_1(t),t)}.
\]
By (16) we have that for every \( 0 \leq x \leq a \),
\[
|\ln u(x_1(t),t) - \ln u(x,t)| = \left| \int_{x}^{x_1(t)} \frac{d\xi}{u} \right| \leq C \int_0^a \frac{dx}{u} = CI_0.
\]
It can be inferred that
\[
u(x,t) \geq e^{-CI_0} u(x_1(t),t) = aI_0^{-1} e^{-CI_0}, \quad \forall (x,t) \in [0,a] \times [0,\infty),
\]
and (17) follows immediately with \( C_1 = aI_0^{-1} e^{-CI_0} \).

When \( p > 2 \) and \( a < \pi \), similar as above, one can find that for every \( t > 0 \), there is an \( x_1(t) \in [0,a] \), such that
\[
I_0 := \int_0^a \frac{dx}{u_0^{p-1}(x)} = \int_0^a \frac{dx}{u^{p-1}} = \frac{a}{u^{p-1}(x_1(t),t)},
\]
\[
u^{2-p}(x,t) \leq (aI_0^{-1})^{\frac{2-p}{p}} + C(p-2)I_0, \quad \forall (x,t) \in [0,a] \times [0,\infty),
\]
with
\[
C_1 := \left[(aI_0^{-1})^{\frac{2-p}{p}} + C(p-2)I_0 \right]^{1/(2-p)}.
\]
Therefore, the estimate (17) holds.

**Step 3:** We are going to prove that for any positive integers \( \ell, m \) and any constant \( \varepsilon > 0 \), there exists a constant \( C_2 \) which depends only on \( u_0(x) \), \( \ell, m \) and \( \varepsilon \), such that
\[
\left|\partial_\ell^\ell \partial_\varepsilon^m u\right| \leq C_2, \quad \forall (x,t) \in [0,a] \times [\varepsilon,\infty).
\]
Note that (17) illustrates that the first equation of problem (1) is uniformly parabolic. Joining (16) with Lemma 3.4, one can employ the classical theory of parabolic type equation to derive the regularity estimate (18), please refer to [14, 15].
for more details. The regularity estimate (18) can also be obtained by using the Maximum principle, see [16] for example.

Step 4: The last step is to prove the limit function $u^*$ in (15) is a constant. Let

$$
\varphi(t) = \begin{cases} 
\frac{1}{p-2} \int_0^a u^{2-p} dx, & p > 2, \\
- \int_0^a \ln u dx, & p = 2.
\end{cases}
$$

Then the upper bound of $\varphi(t)$ follows directly from (17). Thanks to Lemma 3.2 and (7),

$$
\varphi'(t) = E(t) \geq 0, \quad \forall \ 0 \leq t < \infty.
$$

Hence, $\lim_{t \to \infty} \varphi(t) = K < \infty$, and

$$
\int_0^\infty E(t) dt = \lim_{t \to \infty} \int_0^t E(\tau) d\tau = \lim_{t \to \infty} \left( \varphi(t) - \varphi(0) \right) = K - \varphi(0).
$$

(19)

Lemma 2.2 and Lemma 3.2 yield the existence of a constant $L$, such that $\lim_{t \to \infty} E(t) = L \geq 0$. The integral (19) shows that $L = 0$, and thus,

$$
\int_0^a u^2 dx - \int_0^a \left( u - \bar{u}(t) \right)^2 dx \to 0 \quad \text{as} \quad t \to \infty.
$$

(20)

On the other hand, the estimates (18) and (15) guarantee that

$$
\| u(\cdot, t_{jk}) - u^*(\cdot) \|_{C^\ell([0,a])} \to 0 \quad \text{as} \quad k \to \infty,
$$

(21)

where $\ell$ is any given positive integer. Put $t = t_{jk}$ in (20) and take $k \to \infty$. Then by (21), we have

$$
\int_0^a \left( u^*_x \right)^2 dx - \int_0^a \left( u^* - \bar{u}^* \right)^2 dx = 0.
$$

(22)

Similar as in the proof of Lemma 3.2, we first extend $u^*$ onto $[-a, a]$, and then apply the Poincaré inequality (or Wirtinger inequality), and in the final we can acquire that

$$
\int_0^a \left( u^* - \bar{u}^* \right)^2 dx \leq \frac{a^2}{\pi^2} \int_0^a (u^*_x)^2 dx.
$$

(23)

Notice that $a < \pi$. From (22) and (23) it can be deduced that

$$
\int_0^a \left( u^*_x \right)^2 dx \equiv 0,
$$

and thus, $u^*$ is a constant. Therefore, by the integral identity (3) we arrive at

$$
u^* \equiv \left[ a^{-1} \int_0^a u^{1-p(x)} dx \right]^{\frac{1}{1-p}}.
$$

The uniqueness of $u^*$ also implies that the convergence is smooth as $t \to \infty$.

Case (ii): $1 < p < 2$ and $a \leq \pi$. Note that in this case, positive lower bound of $u$ cannot be easily obtained via the method in case (i), and it makes us find a different way to prove the convergence. As $\| u \|_{W^{1,2}([0,a])}$ is uniform bounded, there exists a subsequence of $\{ t_{jk} \}_{k=1}^\infty \to \infty$ (along which $\| u(\cdot, t_{jk}) - u^*(\cdot) \|_{C([0,a])} \to 0$),
also denoted by \( \{ t_{jk} \}_{k=1}^{\infty} \), such that \( u(x, t_{jk}) \) weakly converges to a function \( w \in W^{1,2}([0, a]) \), and the uniqueness of the limit implies \( w = u^* \). Hence,

\[
\| u^* \|_{W^{1,2}([0, a])} \leq \liminf_{k \to \infty} \| u(\cdot, t_{jk}) \|_{W^{1,2}([0, a])}. \tag{24}
\]

In view of \( \| u(\cdot, t_{jk}) - u^*(\cdot) \|_{C([0, a])} \to 0 \), we have

\[
\int_0^a u^2 \, dx = \lim_{k \to \infty} \int_0^a u^2(x, t_{jk}) \, dx.
\]

Thus, by (24) it follows that

\[
\int_0^a (u_x^*)^2 \, dx \leq \liminf_{k \to \infty} \int_0^a u_x^2(x, t_{jk}) \, dx.
\]

We still denote \( \varphi(t) = \frac{1}{p - 2} \int_0^a u^{2-p} \, dx \). By adopting exactly the same argument as in the above step 4, we are able to obtain the estimate (20). From (20) it yields

\[
\int_0^a (u_x^*)^2 \, dx - \int_0^a (u^* - \bar{u})^2 \, dx \leq 0. \tag{25}
\]

For \( a < \pi \), combining the Poincaré inequality with (25), we find

\[
\int_0^a (u_x^*)^2 \, dx \leq 0,
\]

and it means that \( u^* \) is a positive constant. Meanwhile, it illustrates that the solution \( u \) has two-sided positive bounds and the first equation of problem (1) is uniformly parabolic. Therefore, we can repeat the remain proof in case (i), and then show that the convergence is smooth.

For \( a = \pi \), (25) illustrates that the equality in the Poincaré inequality holds, and hence \( u^* = A \cos x + \tilde{A} \sin x + B \) for some constants \( A, \tilde{A} \) and \( B \) with \( \sqrt{A^2 + \tilde{A}^2} \leq B \). From the symmetric extension of problem (1), it follows \( u^* = A \cos x + B \) with \( |A| \leq B \). \hfill \square

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E-mail address: huilingli@seu.edu.cn
E-mail address: 220151335@seu.edu.cn
E-mail address: xlwang@seu.edu.cn