ON CONNECTIVITY, DOMINATION NUMBER AND SPECTRAL RADIUS OF THE PROPER ENHANCED POWER GRAPHS OF FINITE NILPOTENT GROUPS

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Abstract. For a group $G$, the enhanced power graph of $G$ is a graph with vertex set $G$ in which two distinct elements $x, y$ are adjacent if and only if there exists an element $w$ in $G$ such that both $x$ and $y$ are powers of $w$. The proper enhanced power graph is the induced subgraph of the enhanced power graph on the set $G \setminus S$, where $S$ is the set of dominating vertices of the enhanced power graph. In this paper we first characterize the dominating vertices of enhanced power graph of any finite nilpotent group. Thereafter, we classify all nilpotent groups $G$ such that the proper enhanced power graphs are connected and find out their diameter. We also explicitly find out the domination number of proper enhanced power graphs of finite nilpotent groups. Finally, we determine the multiplicity of the Laplacian spectral radius of the enhanced power graphs of nilpotent groups.

1. Introduction

The study of graphs associated with various algebraic structures is a topic of increasing interest during the last two decades. The benefit of studying these graphs is multifold. They help us to (1) characterize the resulting graphs, (2) characterize the algebraic structures with isomorphic graphs, and (3) also to realize the interdependence between the algebraic structures and the corresponding graphs. Besides, these graphs have important applications (see, for example, [2, 20]) and they are related to automata theory [21]. Many different types of graphs, specifically power graph of semigroup [11, 23], group [22], intersection power graph of group [4], enhanced power graph of a group [11, 6], comaximal subgroup graph [15], etc. have been introduced to explore the properties of algebraic structures using graph theory. The concept of a power graph was introduced in the context of semigroup theory by Kelarev and Quin [22].

**Definition 1.1** ([2, 11, 22]). Given a group $G$, the power graph $P(G)$ of $G$ is a simple graph with vertex set $G$ and two vertices $u$ and $v$ are connected by an edge if and only if one of them is the power of another.

Another well-studied graph, called commuting graph associated with a group $G$ has been studied in [9] as a part of the classification of finite simple groups. For more information about the commuting graph, we refer to [3, 9, 17].
Definition 1.2 ([1]). Let $G$ be a group. The *commuting graph* of $G$, denoted by $\mathcal{C}(G)$, is the simple graph whose vertex set is a set of non-central elements of $G$ and two distinct vertices $u$ and $v$ are joined if and only if $u$ and $v$ commutes, that is, $uv = vu$.

In this paper, our topic is the enhanced power graph of a group which is introduced by Alipour et al. in [1] as follows:

Definition 1.3 ([1]). Let $G$ be a group. The *enhanced power graph* of $G$, denoted by $\mathcal{G}_G(G)$, is the graph with vertex set $G$, in which two vertices $u$ and $v$ are joined if and only if there exists an element $w \in G$ such that both $u \in \langle w \rangle$ and $v \in \langle w \rangle$.

Lots of works have been done recently studying various properties of the enhanced power graph of the finite group. The authors in [1] characterized the finite groups such that any arbitrary pair of these three graphs (power, commuting, enhanced) are equal. Besides, in [27], the researchers proved that finite groups with isomorphic enhanced power graphs have isomorphic directed power graphs. Bera et al. in [6] studied the completeness, dominatability and many other interesting properties of the enhanced power graph. Ma and She in [24] derived the metric dimension whereas Hamzeh et al. in [19] derived the automorphism groups of enhanced power graphs of finite groups. In this paper, we study the connectivity, dominatability, diameter and Laplacian spectral radius of enhanced power graphs of groups.

1.1. Basic definitions and notations. For the convenience of the reader and also for later use, we recall some basic definitions and notations about graphs. Let $\Gamma$ be a graph with vertex set $V$. Two elements $u$ and $v$ are said to be adjacent if there is an edge between them. For a vertex $u$, we denote by $N(u)$ the set of vertices which are adjacent to $u$. For a set $V_1 \subseteq V$, define $N(V_1) = \cup_{u \in V_1} N(u)$. A *path* of length $k$ between two vertices $v_0$ and $v_k$ is an alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, v_2, \cdots, v_{k-1}, e_{k-1}, v_k$, where the $v_i$'s are distinct (except possibly the first and last vertices) and $e_i$'s are the edges $(v_i, v_{i+1})$. A graph $\Gamma$ is said to be *connected* if for any pair of vertices $u$ and $v$, there exists a path between $u$ and $v$. The distance between two vertices $u$ and $v$ in a connected graph $\Gamma$ is the length of the shortest path between them and it is denoted by $d(u, v)$. Clearly, if $u$ and $v$ are adjacent, then $d(u, v) = 1$. For a graph $\Gamma$, its *diameter* is defined as $\text{diam}(\Gamma) = \max_{u,v \in V} d(u, v)$. That is, the diameter of graph is the largest possible distance between pair of vertices of a graph.

$\Gamma$ is said to be *complete* if any two distinct vertices are adjacent. For a disconnected graph $\Gamma$, we denote the set of connected components of it by $\mathcal{C}(\Gamma)$.

A vertex of a graph $\Gamma$ is called a *dominating vertex* if it is adjacent to every other vertex. For a graph $\Gamma$, let $\text{Dom}(\Gamma)$ denote the set of all dominating vertices in $\Gamma$. The *vertex connectivity* of a graph $\Gamma$, denoted by $\kappa(\Gamma)$ is the minimum number of vertices that need to be removed from the vertex set $\Gamma$ so that the induced subgraph of $\Gamma$ on the remaining vertices is disconnected. The complete graph with $n$ vertices has vertex connectivity $n - 1$. A set $S \subseteq V(\Gamma)$ is said to be a *dominating set* if every vertex of $V \setminus S$ is adjacent to some vertex of $S$. The minimum possible number of a dominating set is called the *domination number* and it is denoted by $\gamma(\Gamma)$. From the definition of the enhanced power graph, it is clear that the identity of the group is always a dominating vertex. The enhanced power graph is called *dominatable* if it has a dominating vertex other than identity. For more on graph theory we refer [8], [18], [26].
Throughout this paper we consider $G$ as a finite group. $|G|$ denotes the cardinality of the set $G$. For a prime $p$, a group $G$ is said to be a $p$-group if $|G| = p^r, r \in \mathbb{N}$. If $|G| = p^r \ell$ for some prime $p\ell$, then we say that $G$ is a $p\ell$-group. For a subgroup $H$ of $G$, we call $H$ a $p$-order subgroup if $|H| = p$. For two $p$-order subgroups $H_1$ and $H_2$, we say $H_1$ and $H_2$ to be distinct if $|H_1 \cap H_2| = 1$. For a $p$-group $G_p$, let $\exp(G_p)$ be the highest natural number $t$ such that there exists an element of order $p^t$ in $G_p$. For any element $g \in G$, $o(g)$ denotes the order of the element $g$. Let $m$ and $n$ be any two positive integers, then the greatest common divisor of $m$ and $n$ is denoted by $\gcd(m, n)$. The Euler’s phi function $\phi(n)$ is the number of integers $k$ in the range $1 \leq k \leq n$ for which the $\gcd(n, k)$ is equal to 1. The set $\{1, 2, \cdots, n\}$ is denoted by $\mathbb{Z}_n$.

The plan of the paper is as follows. In Section 2 we state our main results, and in Section 3 we mention some earlier known results. In Section 4 we completely characterize the dominating vertices of the enhanced power graphs of finite nilpotent groups. In Section 5 we study about the connectivity and diameter of proper enhanced power graphs of nilpotent groups. The domination number of proper enhanced power graphs is studied in Section 6. Finally, multiplicity of the Laplacian spectral radius of enhanced power graphs is given in Section 7.

2. Main results

In this section, we state and motivate our main results of this paper. If a non-complete graph $\Gamma$ has a dominating vertex, then clearly the graph is connected, has domination number 1 and diameter 2. Therefore, for any graph with a dominating vertex, the properties connectivity, domination number, diameter are not interesting. In this respect, the authors in [1, Question 40] asked about the connectivity of power graphs when all the dominating vertices are removed. Recently, Cameron and Jafari in [10] answered this question for power graphs. Bera et al. in [7] answered the same question for the enhanced power graphs of finite abelian groups. In this paper, we investigate the connectivity, domination number, and diameter for the enhanced power graphs of finite nilpotent groups after the dominating vertices are removed. To seek the answer to this question, define the following graph:

**Definition 2.1.** For a group $G$, the proper enhanced power graph of $G$, denoted by $G_{E}^{**}(G)$, is the graph obtained by deleting all the dominating vertices from the enhanced power graph $G_E(G)$.

Therefore, for studying the proper enhanced power graph, we first need to characterize all the dominating vertices of the graph $G_E(G)$ for a finite group $G$. Cameron and Jafari in [10] characterized the dominating vertices of the power graph for any finite group $G$. Bera et al. in [6] characterized the dominatability of the enhanced power graph for any finite abelian group $G$. In this paper, we at first extend this result to finite nilpotent groups and for that purpose, we first recall the following structure of a finite nilpotent group.

2.1. Nilpotent group. A finite group $G$ is nilpotent if and only if $G \cong P_1 \times \cdots \times P_r$, where for each $i \in [r]$, $P_i$ is a sylow subgroup of order $p_i^{t_i}$ of $G$. So, for a finite nilpotent group $G$, we have the following cases:

(1) No sy low subgroups of $G$ are either cyclic or generalized quaternion.
(2) $G$ has cyclic Sylow subgroups. In this case, $G \cong G_1 \times \mathbb{Z}_n$, where $G_1$ is a nilpotent group having no Sylow subgroups which are either cyclic or generalized quaternion and $\gcd(|G_1|, n) = 1$.

(3) $G$ has a Sylow subgroup isomorphic to generalized quaternion. Here $G \cong G_1 \times Q_{2^k}$, where $G_1$ is described as (2) and $\gcd(|G_1|, 2) = 1$.

(4) $G$ has Sylow subgroups which are cyclic and generalized quaternion. In this case, $G \cong G_1 \times \mathbb{Z}_n \times Q_{2^k}$, where $G_1$ is described as (2) and $\gcd(|G_1|, n) = \gcd(|G_1|, 2) = \gcd(n, 2) = 1$.

We now characterize the dominatability of the enhanced power graph for any finite nilpotent group $G$.

**Theorem 2.1.** Let $G$ be a finite nilpotent group. Then $\text{Dom}(G_E(G)) = \{e\}$ if and only if no Sylow subgroups of $G$ are either cyclic or generalized quaternion.

Subsequently, we characterize all the dominating vertices of the enhanced power graph $G_E(G)$ for any finite nilpotent group $G$. **Theorem 2.1** tells about that. We move on to the connectivity of the proper enhanced power graph $G_E^*(G)$ and our approach depends on whether $G$ has a Sylow subgroup that is generalized quaternion. In this context, we have the following two results which completely characterize the connectivity of the graph $G_E^*(G)$.

**Theorem 2.2.** Let $G$ be a finite nilpotent group that does not have any Sylow subgroups which are generalized quaternion. That is,

1. either $G = G_1$, where $G_1$ is a finite nilpotent group which does not have any Sylow subgroups which are either cyclic or generalized quaternion,
2. or $G = G_1 \times \mathbb{Z}_n$, where $G_1$ is described as (1) and $\gcd(|G_1|, n) = 1$.

Then $G_E^*(G)$ is disconnected if and only if $G_1$ is $p$-group.

**Theorem 2.3.** Let $G$ be a finite nilpotent group having a Sylow subgroup which is generalized quaternion. Then $G_E^*(G)$ is connected.

We also find the diameter of the proper enhanced power graph $G_E^*(G)$ for any finite nilpotent group $G$ in **Theorem 5.9**. Moreover, we improve the earlier known upper bound on the vertex connectivity of the enhanced power graph of a finite abelian group. In this context, our result is **Theorem 5.3**.

It is clear that for any finite group $G$, the proper enhanced power graph $G_E^*(G)$ has no dominating vertex and therefore clearly, $\gamma(G_E^*(G)) > 1$. One of the main goals of this paper is to find the domination number for the proper enhanced power graph $\gamma(G_E^*(G))$. The way we approach towards finding the domination number depends on the connectivity of the enhanced power graph. We first consider the case when $G_E^*(G)$ is disconnected and on this theme, we have the following result.

**Theorem 2.4.** Let $G_1$ be a finite $p$-group which is neither cyclic nor generalized quaternion. Then $\gamma(G_E^*(G_1)) =$ number of distinct $p$-order subgroups of $G_1$. Let $G$ be a nilpotent group such that $G = G_1 \times \mathbb{Z}_n$, where $r \geq 2$ and $\gcd(p, n) = 1$. Then also, $\gamma(G_E^*(G)) =$ number of distinct $p$-order subgroups of $G_1$.

We now shift our attention to the case when $G_E^*(G)$ is connected and here we have the following result.
**Theorem 2.5.** Let $G_1$ be a product of non-cyclic $p$-groups, that is, of the following form:

$$G_1 = P_1 \times P_2 \times \cdots \times P_m$$

where $m \geq 2$ and for each $i \in [m]$, $P_i$ is a $p_i$-group which is neither cyclic nor generalized quaternion. Let $s_i$ be the number of distinct $p_i$-order subgroups of $G_1$. Then,

(1) \[ \gamma(G_{E}^{**}(G_1)) = \min_{1 \leq i \leq m} s_i. \]

Let $G = G_1 \times \mathbb{Z}_n$ with $\gcd(n, |G_1|) = 1$, then also

(2) \[ \gamma(G_{E}^{**}(G)) = \min_{1 \leq i \leq m} s_i. \]

Last but not the least, as an application of finding the dominating vertices of the enhanced power graph we mention a spectral theoretic connection and prove Theorem 7.5.

### 3. Preliminaries

In this section, we first recall some earlier known results on enhanced power graphs which we need throughout the paper. In [6] and [7], Bera et al. studied many interesting properties of enhanced power graphs of finite groups. In fact, they proved the following:

**Lemma 3.1** (Theorem 2.4, [6]). The enhanced power graph $G_{E}(G)$ of the group $G$ is complete if and only if $G$ is cyclic.

**Lemma 3.2** (Theorem 1.1, [7]). Let $G$ be a finite $p$-group such that $G$ is neither cyclic nor generalized quaternion group. Then $\kappa(G_{E}(G)) = 1$.

**Lemma 3.3** (Lemma 2.5, [7]). Let $G$ be a finite group and $x, y \in G \setminus \{e\}$ be such that $\gcd(o(x), o(y)) = 1$ and $xy = yx$. Then, $x \sim y$ in $G_{E}^{*}(G)$.

**Lemma 3.4** (Lemma 2.6, [7]). Let $G$ be a $p$-group. Let $a, b$ be two elements of $G$ of order $p, p^i (i \geq 1)$ respectively. If there is a path between $a$ and $b$ in $G_{E}^{*}(G)$, then $\langle a \rangle \subseteq \langle b \rangle$. In particular, if both $a$ and $b$ have order $p$, then, $\langle a \rangle = \langle b \rangle$.

**Lemma 3.5** (Theorem 3.1, [7]). Let $G$ be a finite abelian $p$-group. Suppose that

$$G = \mathbb{Z}_{p^{t_1}} \times \mathbb{Z}_{p^{t_2}} \times \cdots \times \mathbb{Z}_{p^{t_r}}.$$

where $r \geq 2$ and $1 \leq t_1 \leq t_2 \leq \cdots \leq t_r$. Then, the number of components of $G_{E}^{**}(G)$ is $\frac{p^r - 1}{p - 1}$.

**Lemma 3.6** (Theorem 3.3, [6]). Let $G$ be a non-abelian 2-group. Then the enhanced power graph $G_{e}(G)$ is domatic if and only if $G$ is generalized quaternion group. In this case identity and the unique element of order 2 are dominating vertices.

We next prove some important results which are used to prove our main theorems.

**Theorem 3.7.** Let $G$ be a finite group and $n \in \mathbb{N}$. If $\gcd(|G|, n) = 1$, then $\{(e, a) : a \in \mathbb{Z}_n\} \subseteq \text{Dom}(G_{E}(G \times \mathbb{Z}_n))$. 
Proof. We show that \((e, a)\) is a dominating vertex, where \(e\) is the identity element of the group \(G\) and \(a\) is an arbitrary element of the group \(\mathbb{Z}_n\). Consider an arbitrary vertex \((g, b)\) of the graph \(\mathcal{G}_E(G \times \mathbb{Z}_n)\).

Case 1: Let \(g = e\). Suppose \(a'\) is a generator of the cyclic group \(\mathbb{Z}_n\). Now, \(a, b \in \mathbb{Z}_n\) and \(a'\) is a generator of \(\mathbb{Z}_n\) implies that \((e, b), (e, a) \in \langle (e, a') \rangle\). As a result, \((e, b) \sim (e, a)\) in \(\mathcal{G}_E(G \times \mathbb{Z}_n)\).

Case 2: Let \(g \neq e\) and \(b = e'\), where \(e'\) is the identity of the group \(\mathbb{Z}_n\). We show that \((g, e'), (e, a) \in \langle (g, a) \rangle\). Let \(o(g) = m\). Then \((e, a^m) = (g, a^m) \in \langle (g, a) \rangle\). Now \(\gcd(|G|, n) = 1\) implies that \(\gcd(m, n) = 1\). Therefore, \((e, a) \in \langle (g, a) \rangle\). Again \((e, a^m) \in \langle (g, a) \rangle\) implies that \((e, a) \in \langle (g, a) \rangle\). Here we show that \((g, e') \in \langle (g, a) \rangle\). Now \(\gcd(m, n) = 1\) implies that \(n^{\phi(m)} = m^k + 1, k \in \mathbb{N}\) (by the Euler’s theorem). Therefore, \((g, a)^{n^{\phi(m)}} = (g^{m^k+1}, e') = (g, e')\). Hence, \((g, e') \in \langle (g, a) \rangle\).

Case 3: Let \(g \neq e\) and \(b \neq e'\). We show that \((g, b), (e, a) \in \langle (g, a') \rangle\). Already we have proved that \((g, e') \in \langle (g, a) \rangle\). Now \(a'\) is a generator of \(\mathbb{Z}_n\) implies \((e, b) \in \langle (e, a') \rangle\) (in fact, \(\gcd(m, n) = 1 \Rightarrow k_1m + k_2n = 1\). Therefore, \((g, a')^{k_1m} = (e, a')\)). Hence \((g, b) = (g, e')(e, b) \in \langle (g, a') \rangle\). This completes the proof.

**Proposition 3.8.** Let \(G_1, \ldots, G_r\) be finite groups. Then \(\text{Dom}(\mathcal{G}_E(G_1) \times \cdots \times \mathcal{G}_E(G_r)) \subseteq \text{Dom}(\mathcal{G}_E(G_1)) \times \cdots \times \text{Dom}(\mathcal{G}_E(G_r))\).

**Proof.** Let \((a_1, \ldots, a_r)\) be a dominating vertex of the graph \(\mathcal{G}_E(G_1 \times \cdots \times G_r)\). Let \((b_1, \ldots, b_r)\) be an arbitrary vertex of \(\mathcal{G}_E(G_1 \times \cdots \times G_r)\). Now \((a_1, \ldots, a_r)\) is a dominating vertex, so there exists \((c_1, \ldots, c_r) \in G_1 \times \cdots \times G_r\) such that \((a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \langle (c_1, \ldots, c_r) \rangle\). Therefore, for each \(i \in [r]\), we have \(a_i, b_i \in \langle c_i \rangle\). As a result, \(a_i \sim b_i\). As \(b_i\) is arbitrary, so \(a_i\) is a dominating vertex of \(\mathcal{G}_E(G_i)\). Hence, \((a_1, \ldots, a_r) \in \text{Dom}(\mathcal{G}_E(G_1)) \times \cdots \times \text{Dom}(\mathcal{G}_E(G_r))\).

**Remark 1.** The equality of Proposition 3.8 is not strict, in general. One can take \(G_1 = G_2 = \mathbb{Z}_p\) and in that case, \(\text{Dom}(\mathcal{G}_E(G_1 \times G_2)) = \{(e, e)\}\), whereas \(\text{Dom}(\mathcal{G}_E(G_1)) \times \text{Dom}(\mathcal{G}_E(G_2)) = \mathbb{Z}_p \times \mathbb{Z}_p\).

Now we recall the following result which is already used to prove Theorem 2.3 of [5].

**Lemma 3.9.** Let \(G\) be a \(p\)-group. Then the number of distinct \(p\)-ordered cyclic subgroup is either 1 or greater than equal to 3.

**Proof.** Let \(H_1 = \langle v_1^{(p)} \rangle, \ldots, H_\ell = \langle v_\ell^{(p)} \rangle\) be the collection of all distinct cyclic subgroups of order \(p\) of \(G\). We prove that either \(\ell = 1\) or \(\ell \geq 3\). Let \(\ell = 2\). Since \(G\) is \(p\)-group, the center of \(G\) is \(Z(G)\) is non trivial. Therefore, \(Z(G)\) has a subgroup \(H_{i_1} = \langle v_{i_1}^{(p)} \rangle\) (say) of order \(p\). Let \(H_{i_2} = \langle v_{i_2}^{(p)} \rangle\) be another cyclic subgroup of order \(p\), (as \(\ell = 2\), we can choose more than one subgroup of order \(p\)). Then \(v_{i_1}^{(p)} v_{i_2}^{(p)} = v_{i_2}^{(p)} v_{i_1}^{(p)}\) and \(o(v_{i_1}^{(p)} v_{i_2}^{(p)}) = p\). Take \(H_{i_3} = \langle v_{i_1}^{(p)} v_{i_2}^{(p)} \rangle\) and it is easy to see that \(H_1 \neq H_3\) and \(H_2 \neq H_3\). This completes the proof.

**Lemma 3.10.** Let \(G\) be any group. Let \(v_1, v_2 \in V(\mathcal{G}_E(G))\) such that \(o(v_1) = m_1, o(v_2) = m_2\) and \(m_1 \text{ divides } m_2\). Then \(v_1 \sim v_2\) in \(\mathcal{G}_E(G)\) if and only if \(v_1 \in \langle v_2 \rangle\).
Proof. Let \( v_1 \in \langle v_2 \rangle \). Then clearly \( v_1 \sim v_2 \) in \( G_E(G) \). Conversely suppose that \( v_1 \sim v_2 \) in \( G_E(G) \). So, there exists \( v \in G \) such that \( v_1, v_2 \in \langle v \rangle \). Clearly, for each \( i \in \{1, 2\} \), \( \langle v_i \rangle \) is the unique cyclic subgroup of order \( m_i \) of \( \langle v \rangle \). Again, \( m_1 \) divides \( m_2 \), so \( \langle v_2 \rangle \) must have a subgroup say \( H \) of order \( m_1 \). Now \( \langle v_2 \rangle \) is a subgroup of \( \langle v \rangle \). Therefore, \( H \) is a subgroup of \( \langle v \rangle \) of order \( m_1 \). Again \( \langle v_1 \rangle \) is the unique subgroup of \( \langle v \rangle \) of order \( m_1 \). So, \( H = \langle v_1 \rangle \). Hence \( v_1 \in \langle v_2 \rangle \). □

4. Dominating vertices of \( G_E(G) \) when \( G \) is nilpotent

In Section 5, we will focus on the connectivity of the proper enhanced power graph of finite nilpotent group. To do so first we have to characterize all the dominating vertices of the enhanced power graph. In this portion, we completely classify the dominating vertices of the graph \( G_E(G) \) when \( G \) is nilpotent.

We denote by \( Q_{2^k} \), a generalized quaternion group of order \( 2^k \). Let \( G_1 \) be a nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. Suppose \( e, e'' \) are the identities of \( G_1 \) and \( Q_{2^k} \) respectively. Let \( D_1 = \{(e, x, e'') : x \in \mathbb{Z}_n\} \) and \( D_2 = \{(e, x, y) : x \in \mathbb{Z}_n, y \in Q_{2^k} \text{ and } o(y) = 2\} \). Then the following theorem completely characterizes the dominating vertices of the enhanced power graph of any nilpotent group.

**Theorem 4.1.** Let \( G \) be a finite nilpotent group. Then

\[
\text{Dom}(G_E(G)) = \begin{cases} 
\{e\}, & \text{if } G = G_1 \\
\{(e, x) : x \in \mathbb{Z}_n\}, & \text{if } G = G_1 \times \mathbb{Z}_n \text{ and } \gcd(|G_1|, n) = 1 \\
\{(e, e''), (e, y)\}, & \text{if } G = G_1 \times Q_{2^k} \text{ and } \gcd(|G_1|, 2) = 1 \\
D_1 \cup D_2, & \text{if } G = G_1 \times \mathbb{Z}_n \times Q_{2^k} \text{ and } \\
& \gcd(|G_1|, n) = \gcd(|G_1|, 2) = \gcd(n, 2) = 1.
\end{cases}
\]

Any abelian group is nilpotent. So, by Theorem 4.1, we immediately get the following corollary.

**Corollary 1** (Theorem 1.4, [7]). Let \( G \) be a non-cyclic abelian group such that \( G \) has no cyclic sylow subgroup. If \( n \in \mathbb{N} \), and \( \gcd(|G|, n) = 1 \), then \( \text{Dom}(G_E(G \times \mathbb{Z}_n)) = \{(e, x), \text{ where } e \text{ is the identity of } G \text{ and } x \in \mathbb{Z}_n\} \).

According to the structure (described as in Subsection 2.1) of nilpotent group, we split the proof of Theorem 4.1 into following 4 propositions.

**Proposition 4.2.** Let \( G_1 \) be a nilpotent group such that \( G_1 \) has no sylow subgroups which are either cyclic or generalized quaternion. Then \( \text{Dom}(G_E(G_1)) = \{e\} \), the identity of \( G_1 \).

**Proof.** \( G_1 \) is nilpotent so, \( G_1 \cong P_1 \times \cdots \times P_r \), where each \( P_i \) is a sylow subgroup of order \( p_i^{k_i} \) of \( G_1 \). Let \( g(\neq e) \) be a dominating vertex of the graph \( G_E(G_1) \). Suppose \( p_i \) is a prime divisor of \( o(g) \). Now it is given that each \( P_i \) is neither cyclic nor generalized quaternion. So, the number of distinct cyclic subgroup of order \( p_i \) in \( P_i \) is greater than 1. Let \( H_1 = \langle h \rangle \) and \( H_2 = \langle h' \rangle \) be two distinct cyclic subgroups of order \( p_i \) of \( P_i \). Since \( g \) is a dominating vertex, then \( g \sim h \) and \( g \sim h' \) in \( G_E(G_1) \). Therefore, there exists two cyclic subgroups \( K_1, K_2 \) in \( G_1 \) such that \( g, h \in K_1 \) and \( g, h' \in K_2 \). Now \( g, h \in K_1 \), and \( K_1 \) is cyclic, so \( h \in \langle g \rangle \). Similarly,
Let $G_1$ be a nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. If $n \in \mathbb{N}$, and $\gcd(|G_1|, n) = 1$, then $\text{Dom}(G_E(G_1 \times \mathbb{Z}_n)) = \{(e, e^n), (e, y)\}$, where $e, e^n$ are the identities of $G_1, Q_{2^k}$ respectively and $o(y) = 2$.

Proof. We know that $Q_{2^k}$ has a unique subgroup $\langle y \rangle$ (say) of order 2. Now $\gcd(2, |G_1|) = 1$, so $G_1 \times Q_{2^k}$ has a unique subgroup of order 2, namely $\langle (e, y) \rangle$. We show that $(e, y)$ is a dominating vertex of the graph $G_E(G_1 \times Q_{2^k})$. Let $(a, b)$ be an arbitrary vertex of $G_E(G_1 \times Q_{2^k})$.

Case 1: Let $b \neq e^n$, the identity of the group $Q_{2^k}$. Then $o(a, b)$ has a factor $2^\ell, \ell \geq 1$. So, $\langle (a, b) \rangle$ has a cyclic subgroup of order 2. Now $\langle (e, y) \rangle$ is the unique subgroup of order 2 implies that $(e, y) \in \langle (a, b) \rangle$. Hence $(e, y) \sim (a, b)$.

Case 2: Let $b = e^n$. Here we show that $(a, e^n) \in \langle (a, b') \rangle$, for some $b' \in Q_{2^k}$ such that $o(b') = 2^r, r \geq 1$. It is given that $\gcd(2^k, |G_1|) = 1$, so there exists $k_1, k_2$ such that $2^k k_1 + o(a) k_2 = 1$. Now it is cleared that $(a, b')^{2^k k_1} = (a, e^n)$. So, $(a, e^n) \in \langle (a, b') \rangle$. As before we can show that $(e, y) \in \langle (a, b') \rangle$. As a result, $(e, y) \sim (a, e^n)$. Hence, $(e, y)$ is a dominating vertex. Now by Lemma 3.6 and Proposition 4.2 we can say that $\text{Dom}(G_E(G_1 \times Q_{2^k})) = \{(e, e^n), (e, y)\}$, where $e, e^n$ are the identities of $G_1, Q_{2^k}$ respectively and $o(y) = 2$.

Proposition 4.5. Let $G_1$ be a nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. If $n \in \mathbb{N}$, and $\gcd(|G_1|, n) = \gcd(|G_1|, 2) = \gcd(n, 2) = 1$, then $\text{Dom}(G_E(G_1 \times \mathbb{Z}_n \times Q_{2^k})) = \{(e, x, e^n) : x \in \mathbb{Z}_n \} \cup \{(e, x, y) : x \in \mathbb{Z}_n \text{ and } y \in Q_{2^k} \text{ with } o(y) = 2\}$.

Proof. We show that each element of the set $D = D_1 \cup D_2$ is a dominating vertex, where $D_1 = \{(e, x, e^n) : x \in \mathbb{Z}_n \} \text{ and } D_2 = \{(e, x, y) : x \in \mathbb{Z}_n \text{ and } y \in Q_{2^k} \text{ with } o(y) = 2\}$. Let $(a, b, c)$ be an arbitrary vertex of $G_E(G_1 \times \mathbb{Z}_n \times Q_{2^k})$. First we show that $(e, x, e^n)$ and $(a, b, c) \in \{(a, x', c)\}$. Here we consider two cases.

Case 1: Let $c \neq e^n$. It is cleared that $(e, x, y) = (e, x, e^n)(e, e', y)$, where $e'$ is the identity of $\mathbb{Z}_n$. Let $|G_1| = m$ and $o(c) = 2^\ell, \ell \geq 1$. Now $\gcd(2^\ell, n) = 1$ and $\gcd(m, n) = 1$. So, there exist $k_1, k_2$ and $r_1, r_2$ such that $k_1 2^\ell + k_2 n = 1$ and $r_1 m + r_2 n = 1$. Now

$$
(a, x', c)^{(2^\ell k_1)(r_1 m)} = (a^{2^\ell k_1}, x^{2^\ell k_1}, c^{2^\ell k_1})^{r_1 m} = (a, x', e^n)^{r_1 m} = (e, x', e^n)
$$
So, \((e, x', e'') \in \langle(a, x', c)\rangle\). Again \(x'\) is a generator of \(\mathbb{Z}_n\), therefore, \((e, x, e'') \in \langle(e, x', e'')\rangle \subseteq \langle(a, x', c)\rangle\). Also, it is given that \(\gcd(2^k, m) = 1\). So there exist \(s_1, s_2\) such that \(2^s_1 s_2 + m s_2 = 1\). Therefore, 
\[
(a, x', c)^{(k_2 n)(2^s_1)} = (a, e', e'') \text{ and } (a, x', c)^{(k_2 n)(s_2 m)} = (e, e').
\]
Also, \((e, b, e'') \in \langle(e, x', e'')\rangle\) (as \(b \in \mathbb{Z}_n\) and \(x'\) is a generator of \(\mathbb{Z}_n\)). Moreover, it is proved that \((e, x', e'')\) is a generator of \(\langle(a, x', c)\rangle\). Therefore, \((e, b, e'') \in \langle(a, x', c)\rangle\). Now, \((a, b, c) = (a, e', e'')(e, b, e'')(e, e', c)\). As a result, if \(c \neq e''\), then \((a, b, c) \in \langle(a, x', c)\rangle\). Hence in this case, \((e, x, e'') \sim (a, b, c)\).

Now we show that \((e, x, y) \in \langle(a, x', c)\rangle\). Clearly, \((e, x, e'')(e, e', y) = (e, x, y)\). Already we have shown that \((e, x, e'') \in \langle(a, x', c)\rangle\). So, it is enough to show that \((e, e', y) \in \langle(a, x', c)\rangle\). Note that \(y\) is the unique element of order 2 in \(Q_k\) and \(o(c) = 2^s\). Therefore, \(e' = y\), for some \(r\). Now it is easy to see that \((a, x', c)^{(r s_2 n)(k_2 n)} = (e, e', y)\). Thus \((e, x, y) = (e, x, e'')(e, e', y) \in \langle(a, x', c)\rangle\). Consequently \((e, x, y) \sim (a, b, c)\).

Case 2: Let \(c = e''\). Then \((a, b, e'') = (a, e', e'')(b, e'')\). In this case, continuing the same way as Case 1, we can show that \((e, x, e'')(e, e', y)(a, b, e'') \in \langle(a, x', y)\rangle\). Thus \((e, x, e'') \sim (a, b, e'')\) and \((e, x, y) \sim (a, b, e'')\). Hence \(D \subseteq \text{Dom}(G_E(G_1 \times \mathbb{Z}_n \times Q_{2k}))\). Now, by Lemma \[3.6\] and Proposition \[3.8\] \(\text{Dom}(G_E(G_1 \times \mathbb{Z}_n \times Q_{2k})) = D\).

\[\text{Proof of Theorem 4.1 and Theorem 2.1.}\] Theorem 4.1 follows from Propositions \[4.2\] \[4.3\] \[4.4\] and \[4.5\] Theorem 2.1 directly follows from Theorem 4.1.}

5. Connectivity and diameter of proper enhanced power graph

Finding the vertex connectivity for the power graph and enhanced power graph of a group has been a very interesting problem for the last decade. Many researchers have attempted and found out good bounds for the power graphs. For the power graph of a cyclic group, Chattopadhyay et. al in [12] [13] found the exact vertex connectivity for most of the cyclic groups and has given an upper bound for the rest. Chattopadhyay et al. in [14] have given exact values for the vertex connectivity of particular kinds of some nilpotent groups. Bera et al. in [7] Theorem 1.6] proved the following upper bound on the vertex connectivity for the enhanced power graph of an abelian group.

**Theorem 5.1.** Let \(G\) be a non-cyclic abelian group such that
\[
G \cong \mathbb{Z}_{p_1^{t_1}} \times \cdots \times \mathbb{Z}_{p_1^{t_{k_1}}} \times \mathbb{Z}_{p_2^{t_2}} \times \cdots \times \mathbb{Z}_{p_2^{t_{k_2}}} \times \cdots \times \mathbb{Z}_{p_r^{t_r}} \times \cdots \times \mathbb{Z}_{p_r^{t_{k_r}}},
\]
where \(k_i \geq 1\) and \(1 \leq t_{i1} \leq t_{i2} \leq \cdots \leq t_{ik_i}\), for all \(i \in [r]\). Then
\[
\kappa(G_E(G)) \leq p_1^{t_{i1}} p_2^{t_{i2}} \cdots p_r^{t_{i1}} - \phi(p_1^{t_{i1}} p_2^{t_{i2}} \cdots p_r^{t_{i1}}).
\]

In this section, we prove an improved bound on the vertex connectivity for the enhanced power graph of a finite abelian group. Also, we derive the exact value of the vertex connectivity of the enhanced power graphs of some particular kind of nilpotent groups. Besides, we classify all nilpotent groups \(G\) for which \(G_E(G)\) is connected and find out their diameters.

**Theorem 5.2.** Let \(G\) be an abelian group such that \(G \cong G_1 \times \mathbb{Z}_n\) and \(\gcd(|G_1|, n) = 1\). Let, \(S \subset V(G_E(G))\) such that \(G_E(G_1 \setminus S)\) has exactly \(r\) components \(C_1, C_2, \cdots, C_r\). Then,
$G_E(G \setminus (S \times \mathbb{Z}_n))$ has at least $r$ components. In particular, if $G_E(G_1 \setminus S)$ is disconnected, $G_E(G \setminus (S \times \mathbb{Z}_n))$ is also disconnected.

**Proof.** Define $f : C(G_E(G_1 \setminus S)) \mapsto C(G_E(G \setminus (S \times \mathbb{Z}_n)))$ by

$$f(C_i) = C_i \times \mathbb{Z}_n.$$  

It is enough to show that there is no path in between $C_i \times \mathbb{Z}_n$ and $C_j \times \mathbb{Z}_n$ for $1 \leq i < j \leq r$. Let, there exists an path between $(a_1, b_1)$ and $(a_2, b_2)$ where $a_1 \in C_i$, $a_2 \in C_j$ and $b_1, b_2 \in \mathbb{Z}_n$. If possible, let $(a_1, b_1) \sim (c_1, d_1) \sim (c_2, d_2) \sim \cdots \sim (c_{m-1}, d_{m-1}) \sim (a_2, b_2)$ in $G_E(G \setminus (S \times \mathbb{Z}_n))$ where $c_1, c_2, \ldots, c_{m-1} \in G_1$ and $d_1, d_2, \ldots, d_{m-1} \in \mathbb{Z}_n$. Then $c_1, c_2, \ldots, c_{m-1}$ must be non-zero elements of $G_1$. This proves that $a_1$ and $a_2$ are connected by a path in $G_E(G_1 \setminus S)$ which contradicts the fact that $C_i$ and $C_j$ are distinct connected components of $G_E(G_1 \setminus S)$. The proof is complete.  

We are now in a position to prove the main result of this section.

**Theorem 5.3.** Without loss of generality, we can assume that $G$ is non-cyclic abelian group such that

$$G \cong \mathbb{Z}_{p_1^{t_1}} \times \cdots \times \mathbb{Z}_{t_1 k_1} \times \mathbb{Z}_{p_2^{t_2}} \times \cdots \times \mathbb{Z}_{t_2 k_2} \times \cdots \times \mathbb{Z}_{p_r^{t_r}} \times \cdots \times \mathbb{Z}_{t_r k_r} \times \mathbb{Z}_{t_{r+1}} \times \mathbb{Z}_{t_{r+2}} \times \cdots \times \mathbb{Z}_{t_{i_k}}$$

where $k_i \geq 2$ and $1 \leq t_{i_1} \leq t_{i_2} \leq \cdots \leq t_{i_k}$, for all $i \in [r]$. We then have,

$$\kappa(G_E(G)) \leq p_{s_1} \cdot p_{s_2} \cdots p_{s_n} \cdot (p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n} - \phi(p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n})).$$

**Proof.** Let, $n = p_{s_1}^{t_1 + 1} p_{s_2}^{t_2 + 1} \cdots p_{s}^{t_n + 1} p_{s}^{t_{s_1}}$ and

$$G_1 = \mathbb{Z}_{p_1^{t_1}} \times \cdots \times \mathbb{Z}_{p_1^{t_1 k_1}} \times \mathbb{Z}_{p_2^{t_2}} \times \cdots \times \mathbb{Z}_{p_2^{t_2 k_2}} \times \cdots \times \mathbb{Z}_{p_r^{t_r}} \times \cdots \times \mathbb{Z}_{p_r^{t_r k_r}}.$$ 

Clearly we have $([G_1], n) = 1$. By Theorem 5.1, there exists a set $S$ of cardinality $p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n} - \phi(p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n})$ such that $G_1 \setminus S$ is disconnected. Hence, by Theorem 5.2 $G_E(G \setminus (S \times \mathbb{Z}_n))$ is disconnected. Moreover, the cardinality of $[S \times \mathbb{Z}_n]$ is

$$|S \times \mathbb{Z}_n| = p_{s_1}^{t_1 + 1} p_{s_2}^{t_2 + 1} \cdots p_{s_n}^{t_n} |S|$$

The proof is complete.  

Let $G$ be a non-cyclic abelian group such that

$$G \cong \mathbb{Z}_{p_1^{t_1}} \times \cdots \times \mathbb{Z}_{p_1^{t_1 k_1}} \times \mathbb{Z}_{p_2^{t_2}} \times \cdots \times \mathbb{Z}_{p_2^{t_2 k_2}} \times \cdots \times \mathbb{Z}_{p_r^{t_r}} \times \cdots \times \mathbb{Z}_{p_r^{t_r k_r}} \times \mathbb{Z}_{t_{r+1}} \times \mathbb{Z}_{t_{r+2}} \times \cdots \times \mathbb{Z}_{t_{i_k}}$$

and $\alpha(G)$ and $\beta(G)$ be the bounds for $\kappa(G_E(G))$ in Theorem 5.1 and Theorem 5.3 respectively. That is,

$$\alpha(G) = p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n} - \phi(p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n})$$

and

$$\beta(G) = p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n} + \phi(p_{s_1}^{t_1} p_{s_2}^{t_2} \cdots p_{s_n}^{t_n}).$$

We at first prove that for any abelian group $G$, the bound $\beta(G)$ is better.
Lemma 5.4. For all non-cyclic abelian groups $G$, we have $\beta(G) \leq \alpha(G)$. Moreover, equality happens if and only if $G$ do not have any cyclic sylow $p$-subgroup.

Proof. For any two positive integers $a$ and $b$, we have $a\phi(b) \geq \phi(ab)$ and equality happens if and only if $a = 1$. Now set $a = p_{r+1}^{t_{r+1}} p_{r+2}^{t_{r+2}} \cdots p_{r+s}^{t_{r+s}}$ and $b = p_{1}^{t_1} p_{2}^{t_2} \cdots p_{r}^{t_r}$ and the proof is complete. \qed

Looking at Lemma 5.4, one may think that the improved bound $\beta(G)$ is not a much better bound than the previously known bound $\alpha(G)$ but in the next remark we show that the differences can be much larger.

Remark 2. Theorem 5.3 clearly gives a much a better bound than Theorem 5.1 when $G$ has atleast one sylow $p$-subgroup. As an instance, one can take

$$G = \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_7 \times \mathbb{Z}_{49} \times \mathbb{Z}_{13}.$$ 

In this case, we have

$$\alpha(G) = 3.5.7.13 - \phi(3.5.7.13) = 789 \quad \text{and} \quad \beta(G) = 3.5.7(13 - \phi(13)) = 105.$$ 

Indeed, if we take

$$G = \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_7 \times \mathbb{Z}_{49} \times \mathbb{Z}_p,$$

where $p$ is a large prime, then $\alpha(G) = 3.5.7.p - \phi(3.5.7.p)$ which is very very large, where as $\beta(G)$ remains the same.

In this place, we want to give the exact value of vertex connectivity of enhanced power graphs of some particular nilpotent groups.

Theorem 5.5. Let $G_1$ be a nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. Then $\kappa(G_E(G_1)) = 1$ if and only if $G_1$ is $p$-group.

Proof. Let $G_1$ be $p$-group such that $G_1$ is neither cyclic nor generalized quaternion. Then, by Theorem 3.2, $\kappa(G_E(G_1)) = 1$.

For the converse part, let $G_1$ be a finite group which is not a $p$-group. Let, $p_1, p_2, \cdots, p_r (r \geq 2)$ be the distinct prime factors of $|G_1|$. Let, $a, b \in G_1$ and $o(a) = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $o(b) = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$. We consider the following cases:

Case 1: There exists distinct $i$ and $j$ with $k_i \neq 0$ and $s_j \neq 0$. So, we have $a', b' \in G_1$ such that $o(a') = p_i$, $o(b') = p_j$ and $a' \in \langle a \rangle$ and $b' \in \langle b \rangle$. As $G_1$ nilpotent, we have $a'b' = b'a'$ (as $p_i \neq p_j$). Therefore, by Lemma 3.3, $a' \sim b'$. As a result, we get a path $a \sim a' \sim b' \sim b$. That is, there exists a path of length 3 between $a$ and $b$. We observe that this case takes care of everything except when both $o(a)$ and $o(b)$ are power of the same prime $p_\ell$ for some $\ell \in [r]$. We next consider this case:

Case 2: Let $o(a) = p_\ell^{k_\ell}$ and $o(b) = p_\ell^{s_\ell}$. Let, $c$ be an element of order $p_i$ in $G_1$ with $i \neq \ell$ (as $r \geq 2$). Then by Lemma 3.3, we have $a \sim c \sim b$. Thus, $G^*_E(G_1)$ is connected. This completes the proof. \qed

Corollary 2. Let $G_1$ be a nilpotent non $p$-group having no sylow subgroups which are either cyclic or generalized quaternion. Then, $\text{diam}(G^*_E(G_1)) \leq 3$. 


Theorem 5.6. Let $G_1$ be a finite nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. Let for $n \in \mathbb{N}$, gcd$(|G_1|, n) = 1$. Then $\kappa(G_E(G_1 \times \mathbb{Z}_n)) = n$ if and only if $G_1$ is p-group.

Proof. Let $G_1$ be a finite p-group such that $G_1$ is neither cyclic nor generalized quaternion group and gcd$(|G_1|, n) = 1$. We show that $\kappa(G_E(G_1 \times \mathbb{Z}_n)) = n$. Let $D = \{(e, x) : e \in G_1, x \in \mathbb{Z}_n\}$. Then by Proposition 4.3, Dom$(G_E(G_1 \times \mathbb{Z}_n)) = D$. Now clearly to disconnect the graph we have to delete all vertices in $D$. So, $\kappa(G_E(G_1 \times \mathbb{Z}_n)) \geq n$. Let $(g', e'), (g'', e') \in V(G_E^*(G_1 \times \mathbb{Z}_n))$ such that $\alpha((g', e')) = \alpha((g'', e')) = p$ and $\langle (g', e') \rangle \neq \langle (g'', e') \rangle$. Now we show that there is no path between the vertices $(g', e')$ and $(g'', e')$ in $G_E^*(G_1 \times \mathbb{Z}_n)$. Let $\mathcal{P}$ be a path $v_0 \sim v_1 \sim \cdots \sim v_k$ in $G_E^*(G_1 \times \mathbb{Z}_n)$, where $v_0 = (g', e')$ and $v_k = (g'', e')$. Note that $v_0 \sim v_1$. So, there exists $v_1 \in G_1 \times \mathbb{Z}_n$ such that $v_0, v_1 \in \langle v_1 \rangle$. Now, $v_0 \in \langle v_1 \rangle$ and $\langle v_1 \rangle$ is cyclic. So, $\langle v_0 \rangle$ is the unique cyclic subgroup of order $p$ of $\langle v_1 \rangle$. Again, $v_1 \in \langle v_1 \rangle$ and $p$ divides $\alpha(v_1)$ (Note that $p$ divides the order of each vertex in the graph $G_E^*(G_1 \times \mathbb{Z}_n)$). Therefore, $p$ divides the order of each vertex in the path $\mathcal{P}$. As a result, $v_0 \in \langle v_1 \rangle$. Again, $v_1 \sim v_2$ and $v_0 \in \langle v_1 \rangle$ (already we have shown). Now, proceeding exactly same way as above, it can be shown that $v_0 \in \langle v_2 \rangle$. Continuing this way, we show that $v_0 \in \langle v_k \rangle$. Again, $v_k \sim v_k$. In a similar manner, it can be shown that $v_k \in \langle v_{k-1} \rangle$. As a result $\langle v_0 \rangle = \langle v_k \rangle$ (as $\langle v_{k-1} \rangle$ is cyclic and it has a unique subgroup of order $p$). Thus we choose $v_0, v_k$ such that $\langle v_0 \rangle \neq \langle v_k \rangle$, which is a contradiction. Hence the graph $G_E^*(G_1 \times \mathbb{Z}_n)$ is disconnected and $\kappa(G_E^*(G_1 \times \mathbb{Z}_n)) = n$.

Conversely, let $|G_1| = p_1^{a_1} \times \cdots \times p_t^{a_t}$, where $a_i \geq 1, t \geq 2$ and $p_1, \ldots, p_t$ are primes such that $p_i \neq p_j$, for $i \neq j$. Here we show that the graph $G_E^*(G_1 \times \mathbb{Z}_n)$ is connected. Let $v \in V(G_E^*(G_1 \times \mathbb{Z}_n))$ and we consider the set

$$\text{Div}(v) = \{p_i : p_i \text{ is a prime divisor of both } \alpha(v) \text{ and } |G_1|\}.$$

Clearly for each $p_i \in \text{Div}(v), p_i$ does not divide $n$ (as gcd$(|G_1|, n) = 1$). Let $v', v''$ be two arbitrary vertices of $G_E^*(G_1 \times \mathbb{Z}_n)$. We split the proof of this part into two cases.

Case 1: First suppose that at least one of $\text{Div}(v'), \text{Div}(v'')$ has cardinality greater than or equal to 2. With out loss of generality we assume that $|\text{Div}(v')| \geq 2$. So we can choose $p_i \in \text{Div}(v')$ and $p_j \in \text{Div}(v'')$ such that $p_i \neq p_j$. Let $v_1, v_2 \in V(G_E^*(G_1 \times \mathbb{Z}_n))$ such that $\alpha(v_1) = p_i, \alpha(v_2) = p_j$ and $v_1 \in \langle v' \rangle, v_2 \in \langle v'' \rangle$. As a result $v' \sim v_1$ and $v'' \sim v_2$. Again, $G_1 \times \mathbb{Z}_n$ is nilpotent, so $v_1v_2 = v_2v_1$ and gcd$p_i, p_j = 1$. Therefore by Lemma 3.3, $v_1 \sim v_2$ in $G_E^*(G_1 \times \mathbb{Z}_n)$. Consequently, we get a path $v' \sim v_1 \sim v_2 \sim v''$ in $G_E^*(G_1 \times \mathbb{Z}_n)$.

Case 2: Let $|\text{Div}(v')| = 1$ and $|\text{Div}(v'')| = 1$. Suppose that $p_i \in \text{Div}(v') \cap \text{Div}(v'')$. This implies that $\alpha(v') = p_i^k k_1$ and $\alpha(v'') = p_i^k k_2$ for some positive integers $k_1, k_2$ dividing $n$. Since $G_1$ is non p-group, then there exists $v_5 \in V(G_E^*(G_1 \times \mathbb{Z}_n))$ such that $\alpha(v_5) = p_i$ with $p_i \neq p_j$ and gcd$(p_j, n) = 1$. Thus, gcd$(p_j, \alpha(v')) = 1 = \text{gcd}(p_j, \alpha(v'')).$ As $G_1$ is nilpotent, any two elements of coprime order commute. Hence, the elements $v'$ and $v_5$ commute in $G_E^*(G_1 \times \mathbb{Z}_n)$. Similarly, $v''$ and $v_5$ also commute. So by Lemma 3.3, $v' \sim v_5 \sim v''$ in $G_E^*(G_1 \times \mathbb{Z}_n)$. Consequently, $G_E^*(G_1 \times \mathbb{Z}_n)$ is connected and hence $\kappa(G_E(G_1 \times \mathbb{Z}_n)) > n$. □

Corollary 3. Let $G_1$ be a finite nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. Suppose for $n \in \mathbb{N}$, gcd$(|G_1|, n) = 1$. Then $G_E^*(G_1 \times \mathbb{Z}_n)$ disconnected if and only if $G_1$ is p-group.
Proof. By Proposition 3.3, \( \text{Dom}(G_E(G_1 \times \mathbb{Z}_n)) = \{(e, x) : e \text{ is the identity of } G_1 \text{ and } x \in \mathbb{Z}_n\} \). Clearly, \(|\text{Dom}(G_E(G_1 \times \mathbb{Z}_n))| = n\). Again, by Theorem 5.6, \( k(G_E(G_1 \times \mathbb{Z}_n)) = n \) if and only if \( G_1 \) is a \( p \)-group. i.e., if \( G_1 \) is non \( p \)-group then the proper enhanced power graph \( G_E^*(G_1 \times \mathbb{Z}_n) \) would be connected. Thus \( G_E^*(G_1 \times \mathbb{Z}_n) \) is disconnected if and only if \( G_1 \) is a \( p \)-group.

\[ \square \]

Proof of Theorem 2.2. Proof of this theorem follows from Theorem 5.5 and Corollary 3. \( \square \)

Corollary 4. Let \( G_1 \) be a nilpotent non \( p \)-group having no sylow subgroups which are either cyclic or generalized quaternion. Then \( \text{diam}(G_E^*(G_1 \times \mathbb{Z}_n)) \leq 3 \).

Theorem 5.7. Let \( G_1 \) be a finite nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. Let \( \text{gcd}([G_1], 2) = 1 \). Then \( G_E^*(G_1 \times Q_{2^k}) \) is connected.

Proof. By Proposition 4.4, \( D = \text{Dom}(G_E(G_1 \times Q_{2^k})) = \{(e, e'), (e, y)\} \), where \( o(y) = 2 \). Let \( v_1, v_2 \) be two arbitrary vertices in the graph \( G_E^*(G_1 \times Q_{2^k}) \) such that \( o(v_1) = m_1 \) and \( o(v_2) = m_2 \). Now we have the following cases:

Case 1: Let \( \text{gcd}(m_1, 2) = 1 = \text{gcd}(m_2, 2) \). We choose \( v_3 \in V(G_E^*(G_1 \times Q_{2^k})) \) such that \( o(v_3) = 2^t, t \geq 2 \). Then \( v_1v_3 = v_3v_1 \) and \( v_2v_3 = v_3v_2 \) (as \( G_1 \times Q_{2^k} \) is nilpotent). So, by Lemma 3.3 we get a path \( v_1 \sim v_3 \sim v_2 \) in \( G_E^*(G_1 \times Q_{2^k}) \).

Case 2: Let \( \text{gcd}(m_1, [G_1]) = 1 = \text{gcd}(m_2, [G_1]) \). Similarly as Case 1, we get a path \( v_1 \sim v_4 \sim v_2 \) in \( G_E^*(G_1 \times Q_{2^k}) \), where \( o(v_4) \) divides \([G_1] \) and \( \text{gcd}(o(v_4), 2) = 1 \).

Case 3: Let \( \text{gcd}(m_1, [G_1]) \neq 1 \) and \( \text{gcd}(m_1, 2) \neq 1 \). So, there exists a prime \( p \) (with \( \text{gcd}(p, 2) = 1 \)) such that \( p \) divides both \([G_1] \) and \( m_1 \). As a result, we get a vertex say \( v_5 \in V(G_E^*(G_1 \times Q_{2^k})) \) such that \( o(v_5) = p \) and \( v_5 \in \langle v_1 \rangle \). Now we have the following choices;

1. either \( \text{gcd}(m_2, [G_1]) \neq 1 \) and \( \text{gcd}(m_2, 2) \neq 1 \)
2. or \( \text{gcd}(m_2, [G_1]) \neq 1 \) and \( \text{gcd}(m_2, 2) = 1 \)
3. or \( \text{gcd}(m_2, [G_1]) = 1 \).

For the first choice, there exists a prime \( p' \) with \( \text{gcd}(p', 2) = 1 \) such that \( p' \) divides both \([G_1] \) and \( m_2 \), (it is possible as \( \text{gcd}([G_1], 2) = 1 \)). As a result, we get a vertex \( v_6 \) (say) in \( V(G_E^*(G_1 \times Q_{2^k})) \) such that \( o(v_6) = p' \) and \( v_6 \in \langle v_2 \rangle \). Now we choose a vertex say \( v_7 \in V(G_E^*(G_1 \times Q_{2^k})) \) such that \( o(v_7) = 2^t, t \geq 2 \). Then using Lemma 3.3, we get a path \( v_1 \sim v_5 \sim v_7 \sim v_6 \sim v_2 \) (as \( v_7v_5 = v_5v_7 \) and \( v_7v_6 = v_6v_7 \)). Similarly, for the other choices we can find a path in \( G_E^*(G_1 \times Q_{2^k}) \) between \( v_1 \) and \( v_2 \). This completes the proof. \( \square \)

Corollary 5. Let \( G_1 \) be a nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. Then, \( \text{diam}(G_E^*(G_1 \times Q_{2^k})) \leq 4 \).

Theorem 5.8. Let \( G_1 \) be a nilpotent group having no sylow subgroups which are either cyclic or generalized quaternion. If \( n \in \mathbb{N} \) and \( \text{gcd}([G_1], n) = \text{gcd}([G_1], 2) = \text{gcd}(n, 2) = 1 \), then \( G_E^*(G_1 \times \mathbb{Z}_n \times Q_{2^k}) \) is connected. Here also, \( \text{diam}(G_E^*(G_1 \times \mathbb{Z}_n \times Q_{2^k})) \leq 4 \).

Proof. Proceeding exactly same way as in the proof of Theorem 5.7, it can be shown that the graph \( G_E^*(G_1 \times \mathbb{Z}_n \times Q_{2^k}) \) is connected and \( \text{diam}(G_E^*(G_1 \times \mathbb{Z}_n \times Q_{2^k})) \leq 4 \). \( \square \)

Proof of Theorem 2.3. Follows from Theorem 5.7 and Theorem 5.8. \( \square \)
In Corollaries 2, 4 and 5 and Theorem 5.8 we have seen that the diameter of the proper enhanced power graphs of finite nilpotent groups are always \( \leq 4 \). In the following result we show that when \( G \) does not have any sylow subgroup that is generalized quaternion, the diameter is always exactly 3.

**Theorem 5.9.** Let \( G \) be a noncyclic nilpotent group without any sylow subgroup which is generalized quarternion and \( G^*_E(G) \) is connected. Then \( \text{diam}(G^*_E(G)) = 3 \).

**Proof.** From Theorem 2.2, we can see that the proper enhanced power graph \( G^*_E(G) \) of a finite group \( G \) is connected if and only if

1. either \( G = G_1 \), where \( G_1 = P_1 \times P_2 \times \cdots \times P_m, m \geq 2 \) and each \( P_i \) is neither cyclic nor generalized quaternion,
2. or \( G = G_1 \times Z_n \), where \( G_1 \) is as (1) and \( \gcd(|G_1|, n) = 1 \).

We at first show that for any such group \( G \), the diameter of \( G^*_E(G) \) is \( \leq 3 \). This directly follows from Corollaries 2 and 4. We now show that for any such group \( G \), the diameter of \( G^*_E(G) \) is not 2. At first we assume that \( G = P_1 \times P_2 \times \cdots \times P_m, m \geq 2 \). We produce two elements \( x, y \in G \) such that \( d(x, y) > 2 \). Let \( x = (x_1, x_2, \ldots, x_m) \) and \( y = (y_1, y_2, \ldots, y_m) \), where for each \( i \in [m] \), we choose \( x_i(\neq e_{P_i}) \) and \( y_i(\neq e_{P_i}) \) such that \( x_i \sim y_i \). Clearly \( d(x, y) \neq 1 \). This is possible, since for each \( i \in [m] \), \( P_i \) is a non-cyclic \( p \)-group. If possible, let the distance of \( x \) and \( y \) in the proper enhanced power graph \( G^*_E(G) \) is 2. Then, there exists an element, say \( z = (z_1, z_2, \ldots, z_m) \) such that \( x \sim z \sim y \) and \( z \in G^*_E(G) \). Now \( x \sim z \) implies that there exists some \( u = (u_1, u_2, \ldots, u_m) \) such that both \( x \) and \( z \) are multiple of \( u \). Therefore \( x_1 \sim z_1 \) in the proper enhanced power graph \( G^*_E(P_1) \). In a similar way, \( z_1 \sim y_1 \) in the proper enhanced power graph \( G^*_E(P_1) \) and this forces \( z_1 = e_{P_1} \). Similarly, we can show that \( z_i = e_{P_i} \) for each \( i \in [m] \). Hence \( z = e \), the identity of \( G \) and therefore \( z \notin G^*_E(G) \), contradiction.

We now move on to the case when \( G = G_1 \times Z_n \), where \( G_1 = P_1 \times P_2 \times \cdots \times P_m \) and \( \gcd(p_1 p_2 \ldots p_m, n) = 1 \). Here also, our intention is same, that is, to produce two elements \( x, y \in G \) such that \( d(x, y) > 2 \). Let \( x = (x_1, x_2, \ldots, x_m, e') \) and \( y = (y_1, y_2, \ldots, y_m, e') \) where for each \( i \in [m] \), we choose \( x_i(\neq e_{P_i}) \) and \( y_i(\neq e_{P_i}) \) such that \( x_i \sim y_i \). Let \( z = (z_1, z_2, \ldots, z_m, a) \) be an element such that \( x \sim z \sim y \) and \( z \in G^*_E(G) \). By proceeding similarly as in above, we can show that \( z = (e_{G_{p_1}}, e_{G_{p_2}}, \ldots, e_{G_{p_m}}, a) = (e_{G_1}, a) \) for some \( a \in Z_n \), and therefore \( z \notin G^*_E(G) \), contradiction.

Thus, for any finite nilpotent group \( G \) such that \( G^*_E(G) \) is connected and it does not have any sylow subgroup which is generalized quaternion, we have found two elements whose distance is \( \geq 3 \) and therefore diameter of \( G^*_E(G) \) is \( \leq 3 \), completing the proof. \( \square \)

### 6. Domination number of proper enhanced power graph

In this section, we determine the domination number and diameter of the graph \( G^*_E(G) \) for any finite nilpotent group \( G \). For this purpose, we start with counting the number of components of the proper enhanced power graph \( G^*_E(G) \).

**Theorem 6.1.** Let \( G \) be a finite \( p \)-group which is neither cyclic nor generalized quaternion. Then, the number of components of \( G^*_E(G) \) is same as the number of distinct \( p \)-order subgroups of \( G \).
Proof. Let \( H_1, H_2, \ldots, H_s \) be the distinct \( p \)-order subgroups of \( G \). We claim that \( H_1 \cup N(H_1), H_2 \cup N(H_2), \ldots, H_s \cup N(H_s) \) are disjoint components of \( G^{**}(E) \). Consider any non-identity element \( a \) of \( G \). Clearly, there exists \( r \) such that \( a^r \) is of order \( p \). As \( a^r \in \langle a \rangle \), \( a \) and \( a^r \) are adjacent in \( G^{**}(E) \). So, \( a \) must be in one of the components \( H_1 \cup N(H_1), H_2 \cup N(H_2), \ldots, H_s \cup N(H_s) \). Thus, the number of components is \( \leq s \). By Lemma 6.1, if any two elements of order \( p \) are connected by a path, then one of them must be the multiple of another. Henceforth, there are at least \( s \) many components. This completes the proof. \( \square \)

**Theorem 6.2.** Let \( G \) be a nilpotent group such that \( G \cong G_1 \times \mathbb{Z}_n \), where \( G_1 \) is a finite \( p \)-group which is neither cyclic nor generalized quaternion. Then, the number of components of \( G^{**}(E) \) is same as the number of distinct \( p \)-order subgroups of \( G_1 \).

Proof. By Theorem 6.1, we see that the number of connected components of \( G^{**}(E)(G_1) \) is \( s \) where \( s \) is the number of distinct \( p \)-order subgroups of \( G \). Let, \( C_i = H_i \cup N(H_i) \). Then by Theorem 6.1, \( C_1, C_2, \ldots, C_s \) are the components of \( C(G^{**}(E)(G_1)) \). Define \( f : C(G^{**}(E)(G_1)) \mapsto C(G^{**}(E)(G_1 \times \mathbb{Z}_n)) \) by

\[
f(C_i) = C_i \times \mathbb{Z}_n.
\]

Clearly, the number of components is at most \( s \). Thus it is enough to show that there is no path in between \( C_i \times \mathbb{Z}_n \) and \( C_j \times \mathbb{Z}_n \) for \( 1 \leq i < j \leq s \). This follows in an identical manner to the proof of Theorem 5.2. Therefore, the number of components of \( G^{**}(E) \) is at least \( s \). The proof is complete. \( \square \)

**Proof of Theorem 6.4.** Let \( H_1, H_2, \ldots, H_s \) be the distinct \( p \)-order subgroups of \( G_1 \). From the proof of Theorem 6.1, we see that \( H_1 \cup N(H_1), H_2 \cup N(H_2), \ldots, H_s \cup N(H_s) \) are disjoint components of \( G^{**}(E) \). Thus, the domination number of \( G_1 \) is clearly \( \geq s \). For \( 1 \leq i \leq s \), let \( a_i \) be some element of order \( p \) which is chosen from \( H_i \). Consider the following set

\[
D_1 = \{a_1, a_2, \ldots, a_s\}.
\]

By Lemma 3.4, the component \( C_i \) is dominated by the element \( a_i \) and therefore \( D_1 \) is a dominating set for \( G^{**}(E)(G_1) \). This completes the proof for \( G_1 \).

We now consider the case when \( G = G_1 \times \mathbb{Z}_n \). By Theorem 6.2, the number of components of \( G^{**}(E) \) is at least \( s \). Consider the following set

\[
D_2 = \{(a_1, e'), (a_2, e'), \ldots, (a_s, e')\}
\]

where \( e' \) denotes the identity element of \( \mathbb{Z}_n \) and \( a_i \in H_i \). We claim that the element \( (a_i, e') \) dominates the component \( C_i \times \mathbb{Z}_n \). Let \( (x, y) \in C_i \times \mathbb{Z}_n \). Then, \( (x, y)^n = (x^n, e') \) and as \( \gcd(n, p) = 1 \), we have \( a_i = x^{nr} \) for some \( r \in \mathbb{N} \). Therefore, we have \( (a_i, e') = (x, y)^{rn} \) and this proves our claim. Hence, the set \( D_2 \) dominates \( G^{**}(E) \) and the proof is complete. \( \square \)

For an abelian group \( G \) of order \( p^r \), the number of distinct \( p \)-order subgroups is \( \frac{p^r - 1}{p - 1} \). So, we immediately get the following.

**Theorem 6.3.** Let \( G_1 \) be a finite abelian noncyclic \( p \)-group. Suppose that

\[
G_1 = \mathbb{Z}_{p^t_1} \times \mathbb{Z}_{p^t_2} \times \cdots \times \mathbb{Z}_{p^t_r}.
\]
where \( r \geq 2 \) and \( 1 \leq t_1 \leq t_2 \leq \cdots \leq t_r \). In this case, \( \gamma(G^*_E(G_1)) = \frac{p^r-1}{p-1} \). Let \( G \) be an abelian group such that \( G = G_1 \times \mathbb{Z}_n \), where \( r \geq 2 \) and \( \gcd(p, n) = 1 \). Then also we have \( \gamma(G^*_E(G)) = \frac{p^r-1}{p-1} \).

**Proof of Theorem 2.3** At first, we consider the case when \( G_1 \) is of the following form:

\[
G_1 = P_1 \times P_2 \times \cdots \times P_m
\]

where \( m \geq 2 \) and for each \( i \in [m] \), \( P_i \) is a \( p_i \)-group which is neither cyclic nor generalized quaternion. As in the proof of Theorem 6.3, consider \( G^*_E(P_i) \) and for \( 1 \leq i \leq s_1 \), let \( a_i \) be some element of order \( p_i \) which is chosen from the component \( C_i \). Consider the following set \( D_1 = \{a_1, a_2, \ldots, a_{s_1}\} \) which is a dominating set of \( G^*_E(P_i) \). Let \( D_3 = \{(d, e_{P_2}, \ldots, e_{P_m}) : d \in D_1\} \) where \( e_{P_i} \) denotes the identity of \( P_i \). Let \( (x_1, x_2, x_3, \ldots, x_m) \in G^*_E(G_1) \). Then

\[
(x_1, x_2, \ldots, x_m)^{e_{P_2}^{r_2} \cdot e_{P_3}^{r_3} \cdots e_{P_m}^{r_m}} = (x'_1, e_{P_2}, \ldots, e_{P_m})
\]

for some \( x'_1 \in P_1 \). As \( D_1 \) is a dominating set of \( G^*_E(P_i) \), by Lemma 3.4 there exists \( d' \) such that \( d' = (x'_1)^l \) and hence we have

\[
(d', e_{P_2}, \ldots, e_{P_m}) \sim (x_1, x_2, \ldots, x_m).
\]

Thus \( D_1 \) is a dominating set of \( G^*_E(G_1) \) and therefore

\[
\gamma(G^*_E(G_1)) \leq s_1.
\]

We can similarly show that \( \gamma(G^*_E(G_1)) \leq s_i \) for any \( 1 \leq i \leq m \) and that proves

\[
\gamma(G^*_E(G_1)) \leq \min_{1 \leq i \leq m} s_i.
\]

Let \( D \) be a dominating set of \( \gamma(G^*_E(G_1)) \) of cardinality \( \ell < \min_{1 \leq i \leq m} s_i \). Let \( D = \{(w_{11}, w_{12}, \ldots, w_{1m}), \ldots, (w_{t_1}, w_{t_2}, \ldots, w_{t_m})\} \). Therefore, \( |D| < |D_1| \) for each \( i \) with \( 1 \leq i \leq m \), where \( D_i \) is a dominating set of minimum cardinality for \( G^*_E(P_i) \). Hence, for each \( i \in [m] \), there exists some \( y_i \in P_i \) such that \( y_i \) is not dominated by any of the vertices among \( \{w_{1i}, w_{2i}, \ldots, w_{ti}\} \) in the graph \( G^*_E(P_i) \). Consider the vertex \( u = (y_1, y_2, \ldots, y_m) \). If \( u \) is dominated by some vertex of \( D \) say \( v = (v_1, v_2, \ldots, v_m) \), then we must have

\[
u_1 = e_{P_1}, v_2 = e_{P_2}, \ldots, v_m = e_{P_m}\]

and hence \( v = e_{G_1} \) which contradicts the fact that \( v \in G^*_E(G_1) \). Hence, any dominating set of \( G^*_E(G_1) \) must have cardinality \( \geq \min_{1 \leq i \leq m} s_i \). This proves [1].

We now move on to the case when

\[
G = P_1 \times P_2 \times \cdots \times P_m \times \mathbb{Z}_n
\]

and we show that \( S = \{(d, e_{P_2}, \ldots, e_{P_m}, e_{Z_n}) : d \in D_1\} \) is a dominating set of \( G^*_E(G) \). Let \( (x_1, x_2, \ldots, x_m, x_{m+1}) \in G^*_E(G) \). Then

\[
(x_1, x_2, \ldots, x_m, x_{m+1})^{e_{P_2}^{r_2} \cdot e_{P_3}^{r_3} \cdots e_{P_m}^{r_m} e_{Z_n}} = (x'_1, e_{P_2}, \ldots, e_{P_m}, e_{Z_n})
\]

for some \( x'_1 \in G_{p_1} \). Hence, there exists \( d' \) such that \( d' = x'_1 \) and hence we have

\[
(d', e_{P_2}, \ldots, e_{P_m}, e_{Z_n}) \sim (x_1, x_2, \ldots, x_m, x_{m+1})
\]

Thus \( D_1 \) is a dominating set of \( G^*_E(G) \) and therefore \( \gamma(G^*_E(G_1)) \leq s_1 \). We can similarly show that \( \gamma(G^*_E(G_1)) \leq s_i \) for any \( i \) and that proves that \( \gamma(G^*_E(G)) \leq \min_{1 \leq i \leq m} s_i \).
Finally we are left to show that any dominating set of $G_{E}^{**}(G)$ has cardinality $\geq \min_{1 \leq i \leq m} s_i$. This follows in an identical manner to the proof of the above fact that any dominating set of $G_{E}^{**}(G_1)$ has cardinality $\geq \min s_i$ and therefore we omit this. Hence, (2) is proved. □

**Corollary 6.** Let $G_1$ be a product of non-cyclic abelian $p$-groups, that is, of the following form:

$$G_1 = P_1 \times P_2 \times \cdots \times P_m$$

where $m \geq 2$ and for all $1 \leq i \leq m$, we have

$$P_i = \mathbb{Z}_{p_i^{t_{i1}}} \times \mathbb{Z}_{p_i^{t_{i2}}} \times \cdots \mathbb{Z}_{p_i^{t_{ik_i}}}$$

with $k_i \geq 2$ and $1 \leq t_{i1} \leq t_{i2} \leq \cdots \leq t_{ik_i}$. Then

$$\gamma(G_{E}^{**}(G_1)) = \min_{1 \leq i \leq m} \frac{p_i^{k_i} - 1}{p_i - 1}.$$  

Let $G = G_1 \times \mathbb{Z}_n$ with $\gcd(n, |G_1|) = 1$, then also

$$\gamma(G_{E}^{**}(G)) = \min_{1 \leq i \leq m} \frac{p_i^{k_i} - 1}{p_i - 1}.$$  

7. **Multiplicity of Laplacian spectral radius**

In this section, we find the multiplicity of the Laplacian spectral radius of the enhanced power graph of any finite nilpotent group. For that purpose, we recall the definition of the Laplacian matrix $L(\Gamma) = (L_{i,j})_{n \times n}$ of a graph $\Gamma$ with the vertex set $\{v_1, v_2, \ldots, v_n\}$, where

$$L_{i,j} = \begin{cases} 
    d_i, & \text{if } i = j \\
    -1, & \text{if } i \neq j \text{ and } v_i \sim v_j \\
    0, & \text{if } i \neq j \text{ and } v_i \not\sim v_j
  \end{cases}$$

and $d_i$ is the degree of the vertex $v_i$. For any graph $\Gamma$, the characteristic polynomial $\det(xI - L(\Gamma))$ of $L(\Gamma)$ is called the Laplacian characteristic polynomial of $\Gamma$ and is denoted by $\Theta(\Gamma, x)$. Let $\lambda_1(\Gamma)$ be the largest eigenvalue of $L(\Gamma)$. Let

$$\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$$

be the eigenvalues of the Laplacian matrix $L(\Gamma)$. The highest eigenvalue $\lambda_1(\Gamma)$ is called the *Laplacian spectral radius* of $\Gamma$. We denote by $\eta(\lambda_1(\Gamma))$, the multiplicity of the Laplacian spectral radius or the highest eigenvalue. For a graph $\Gamma$, let $\Gamma^{\text{com}}$ denote the complement of the graph $\Gamma$ and $(\Gamma^{\text{com}})^*$, denote the induced subgraph on $\Gamma^{\text{com}}$ after removing the isolated vertices, if any. Dey in [16] proved the following result which connects the multiplicity of the Laplacian spectral radius of a graph $\Gamma$.

**Theorem 7.1.** Let $\Gamma$ be a simple graph on $n(\geq 3)$ vertices. Then $\eta(\lambda_1(\Gamma)) = |\text{Dom}(\Gamma)|$ if and only if $\Gamma$ is non-complete, $(\Gamma^{\text{com}})^*$ is connected and $\Gamma$ has at least one dominating vertex.

For the sake of completeness, as the result is not yet published, we at first give a short proof of this result. For that we need the following result of Mohar [25].
Theorem 7.2 (Mohar). Let $\Gamma$ be a graph with $n$ vertices. Then $\lambda_1(\Gamma) \leq n$. Equality holds if and only if $\Gamma^{\text{com}}$ is not connected.

The union of graphs $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set is the union of all the edges of $\Gamma_1$ and all the edges of $\Gamma_2$. If $\Gamma_1$ and $\Gamma_2$ are disjoint, that is, they do not have any common vertices, we refer to their union as a disjoint union, and denote it by $\Gamma_1 + \Gamma_2$. If $\Gamma_1$ and $\Gamma_2$ are disjoint, their join $\Gamma_1 \lor \Gamma_2$ is the graph obtained by taking $\Gamma_1 + \Gamma_2$ and adding all edges $\{u, v\}$ with $u \in V(\Gamma_1)$ and $v \in V(\Gamma_2)$.

Mohar in the same paper [25] proved the following result which provides the Laplacian spectrum of the join of two graphs.

Theorem 7.3 (Mohar). Let $\Gamma_1$ and $\Gamma_2$ are disjoint graphs on $n_1$ and $n_2$ vertices. Then,

$$\Theta(\Gamma_1 \lor \Gamma_2, x) = \frac{x(x-n_1-n_2)}{(x-n_1)(x-n_2)} \Theta(\Gamma_1, x-n_2) \Theta(\Gamma_2, x-n_1).$$

For a graph $\Gamma$ with $r$ dominating vertices, let $\Gamma_1$ be the complete graph $K_r$ by taking all the dominating vertices and let $\Gamma'$ be the induced graph on the remaining $n - r$ vertices. Clearly $\Gamma'$ is a graph in $n - r$ vertices with no dominating vertex. By Theorem 7.3, we then have,

$$\Theta(K_r \lor \Gamma', x) = \frac{x(x-n)}{(x-r)(x-(n-r))} \Theta(K_r, x-(n-r)) \Theta(\Gamma', x-r)$$

$$= \frac{x(x-n)}{(x-r)(x-n+r)} (x-n)^{r-1} (x-n+r) \Theta(\Gamma', x-r)$$

$$= \frac{x(x-n)^r}{(x-r)} \Theta(\Gamma', x-r)$$

(3)

The second line follows from the fact that the Laplacian matrix of the complete graph has eigenvalues $n$ with multiplicity $n-1$ and 0 with multiplicity 1. It is easy to note that $\Theta(\Gamma', x-r)$ equals with the characteristic polynomial of the submatrix of $L(\Gamma)$ obtained after deleting rows and columns corresponding to the dominating vertices of $\Gamma$. We then have

(4) \quad \lambda_i(\Gamma) = n \quad \text{for} \quad 1 \leq i \leq r \quad \text{and} \quad \lambda_{r+1}(\Gamma) = \lambda_1(\Gamma') + r

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. We prove the forward implication at first. Let the multiplicity of the highest eigenvalue of $L(\Gamma)$ be the number of dominating vertices of $\Gamma$ and we need to show that $\Gamma$ is non-complete, has a dominating vertex and $(\Gamma^{\text{com}})^s$ is connected. Firstly, the graph $\Gamma$ cannot be complete otherwise the number of dominating vertices would be $n$ and the multiplicity of the largest eigenvalue would be $n-1$. Secondly, $\Gamma$ must have a dominating vertex. Finally, if $(\Gamma^{\text{com}})^s$ is disconnected then by Theorem 7.2, the highest eigenvalue of $L(\Gamma')$ is $n - r$ as it is easy to check that $(\Gamma')^{\text{com}} = (\Gamma^{\text{com}})^s$. Thus, by (4), we have $\lambda_{r+1}(\Gamma) = \lambda_1(\Gamma') + r = n - r + r = n$. Therefore the multiplicity of the highest eigenvalue is greater than the number of dominating vertices, which is a contradiction.
For the other direction, we assume that $\Gamma$ is a simple, non-complete graph with a dominating vertex such that $(\Gamma^\text{com})_*$ is connected. Assume that $\Gamma$ has $r$ dominating vertices and we form $K_r$ and $\Gamma'$ as above. As $(\Gamma')^\text{com} = (\Gamma^\text{com})_*$ and $(\Gamma^\text{com})_*$ is connected, by Theorem 7.2, we have $\lambda_1(\Gamma') < n - r$. Thus $\lambda_{r+1}(\Gamma) = \lambda_1(\Gamma') + r < n - r + r = n$ and now by using (4) the proof is complete. □

Therefore, in order to find out the multiplicity of the Laplacian spectral radius of the enhanced power graph $G^*(E(G))$ for any nilpotent group, we at first prove that $G^**(E(G))^\text{com}$ is connected.

**Theorem 7.4.** Let $G$ be a noncyclic nilpotent group which does not have any cyclic sylow subgroup which is generalized quaternion. Then $G^**(E(G))^\text{com}$ is connected.

**Proof.** Let $a, b$ be vertices of $G^**(E(G))^\text{com}$. We claim that there exists a vertex $c$ such that $c \sim a$ in $G^**(E(G))^\text{com}$ and $c \sim b$ in $G^**(E(G))^\text{com}$. Otherwise, every vertex is dominated by at least one of $a$ and $b$ in $G^*(E(G))$ and hence $\gamma(G^**(E(G))) \leq 2$, which is a contradiction. Thus, our claim is true and hence there exists a path of length 2 between $a$ and $b$ in $G^**(E(G))^\text{com}$, completing the proof. □

Let $G_1$ be a finite nilpotent group having no sylow subgroups which are generalized quaternion or cyclic.

**Theorem 7.5.** Let $G$ be a finite nilpotent group having no sylow subgroup which is generalized quaternion. That is, either $G = G_1$ or $G = G_1 \times \mathbb{Z}_n$, where $\gcd(|G_1|, n) = 1$. Then

$$
\eta(\lambda_1(G^*(E(G)))) = \begin{cases} 
1, & \text{if } G = G_1 \\
n, & \text{if } G = G_1 \times \mathbb{Z}_n \text{ and } \gcd(|G_1|, n) = 1.
\end{cases}
$$

**Proof.** By using Theorems 7.1 and 7.4, we can see that $\eta(\lambda_1(G^*(E(G)))) = |\text{Dom}(\lambda_1(G^*(E(G))))|$ when $G_1$ is a finite nilpotent group which has no sylow subgroup which is generalized quaternion. The proof of this theorem now follows from Theorem 4.1. □

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