ARTICLE

On Approximation Properties of Generalized Lupas Type Operators Based on Polya Distribution with Pochhammer k-Symbol

Övgü Gürel Yılmaz\(^a\), Rabia Aktaş \(^b\), Fatma Taşdelen\(^b\), and Ali Olgun \(^c\)

\(^a\)Recep Tayyip Erdogan University, Department of Mathematics, 53100 Rize, Turkey;
\(^b\)Ankara University, Faculty of Science, Department of Mathematics, 06100, Tandoğan, Ankara, Turkey; \(^c\)Kırıkkale University, Department of Mathematics, Yahşihan 71450, Kırıkkale, Turkey

ARTICLE HISTORY
Compiled December 23, 2020

ABSTRACT
In our present investigation, we are concerned with the Kantorovich variant of Lupas-Stancu operators based on Polya distribution with Pochhammer \(k\)-symbol. We briefly give some basic properties of the generalized operators and by making use of these results, we investigate convergence properties of the studied operators. Furthermore, the rate of convergence of these operators is obtained and Voronovskaja type theorem for the pointwise approximation is established. Then we construct bivariate generalization of the operators and we discuss some convergence properties. Finally, taking into account some illustrative graphics, we conclude our study with the comparison of the rate of convergence between our operators and other operators which are mentioned in the paper.

KEYWORDS
Bersntein operators; Stancu operators; Lupas operators; Kantorovich operators; Polya distribution; modulus of continuity; Lipschitz class; Voronovskaja type theorem; Pochhammer \(k\)-symbol

1. Introduction

In the field of approximation theory, Bernstein operators have a considerable importance since it is the major key for the proof of the Weierstrass approximation theorem. In 1912, Bernstein \([9]\) presented the well-known Bernstein operators of order \(n \in \mathbb{N}\) in the following form

\[
B_n (f; x) = \sum_{m=0}^{n} \binom{n}{m} x^m (1 - x)^{n-m} f \left( \frac{m}{n} \right), \tag{1}
\]

where \(f \in C [0, 1]\) (real valued continuous function on \([0, 1]\)). The fact that they are functional in studying many problems, convenient in computer-aided studies and also they have a simple representation, important generalizations and applications has motivated a great number of authors to study intensively up to now. We refer the readers \([1, 3, 11, 22, 27, 28]\) for some studies about these operators.

Corresponding author. Email: ovgu.gurelyilmaz@erdogan.edu.tr
In the year 1968, Stancu [33] introduced the operators \( P_n^{(\alpha)} : C [0, 1] \to C [0, 1] \) with a nonnegative parameter \( \alpha \),

\[
P_n^{(\alpha)} (f; x) = \sum_{m=0}^{n} p_{n,m}^{(\alpha)} (x) f \left( \frac{m}{n} \right)
\]

(2)

where \( p_{n,m}^{(\alpha)} \) is defined by

\[
p_{n,m}^{(\alpha)} (x) = \binom{n}{m} \frac{\prod_{\nu=0}^{m-1} (x + \nu \alpha) \prod_{\mu=0}^{n-m-1} (1 - x + \mu \alpha)}{(1 + \alpha) (1 + 2 \alpha) \ldots (1 + (n-1) \alpha)}
\]

(3)

for \( n \in \mathbb{N} \). It should be noted that when \( \alpha = 0 \), (2) obviously reduces to the classical Bernstein operators given by (1).

After Stancu’s paper, by taking into account the special choice \( \alpha = \frac{1}{n} \) in (2) in 1987 Lupaş and Lupaş [20] constructed the operators \( P_n^{\left(\frac{1}{n}\right)} : C [0, 1] \to C [0, 1] \) as follows

\[
P_n^{\left(\frac{1}{n}\right)} (f; x) = \frac{2n!}{(2n)!} \sum_{m=0}^{n} \binom{n}{m} (nx)_m (n-nx)_{n-m} f \left( \frac{m}{n} \right),
\]

(4)

where \( (s)_m \) is a rising factorial also known as the Pochhammer symbol namely,

\[
(s)_m = \begin{cases} 
  s (s+1) (s+2) \ldots (s+m-1) & \text{for } m \in \mathbb{N} \\
  1 & \text{for } m = 0, \ s \neq 0
\end{cases}
\]

(5)

where \( s \) is a real or complex number. In 2012, Miclaus [21] reconsidered the operators (4) and in this work, some of the properties of the operators such as moments, the remainder term and the monotocity properties, were recalculated with a different technique and also asymptotic behaviour of the (4) was discussed. Up to now, many operators which are based on Polya distribution have been extensively studied. We refer the reader to the articles [4,5,7,10,12,15,16,25,31] and the references therein.

In 1989, Razi [32] defined the following Kantorovich modification of the Bernstein-Stancu operators \( P_n^{(\alpha)} (f; x) \) given by (2)

\[
K_n^{(\alpha)} (f; x) = (n+1) \sum_{m=0}^{n} p_{n,m}^{(\alpha)} (x) \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} f (t) \, dt.
\]

and studied some approximation properties. For \( \alpha = 0 \), it takes the classical Bernstein Kantorovich operators

\[
K_n (f; x) = (n+1) \sum_{m=0}^{n} \binom{n}{m} x^m (1-x)^{n-m} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} f (t) \, dt.
\]

(6)
In 2016, for $\alpha = \frac{1}{n}$, the Kantorovich modification of Lupaş operators based on Pólya distribution $P_n^{\langle h \rangle} (f; x)$

$$D_n^* \left( \frac{1}{n} \right) (f; x) = (n + 1) \cdot \frac{2n!}{(2n)!} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) (nx)_m (n - nx)_{n-m} \int_{m/n+1}^{m+1/n} f(t) \, dt , \; f \in C[0, 1]$$

(7)

was studied by Agrawal et al. [6] and, local and global approximation properties were obtained. Furthermore, the authors introduced bivariate form of the Kantorovich modification of Lupaş operators based on Pólya distribution defined by (7) as follows

$$D_{n_1, n_2}^* \left( \frac{1}{n_1} \frac{1}{n_2} \right) (f; x, y) = (n_1 + 1)(n_2 + 1) \sum_{m_1 = 0}^{n_1} \sum_{m_2 = 0}^{n_2} p^{(1/n_1, 1/n_2)} (m_1, m_2) \cdot \int_{m_1/n_1+1}^{m_1+1/n_1} \int_{m_2/n_2+1}^{m_2+1/n_2} f(t, s) \, dt \, ds$$

(8)

for $f : C(J^2) \rightarrow C(J^2)$, $J = [0, 1]$ where

$$p^{(1/n_1, 1/n_2)} (x, y) = \frac{2n_1!}{(2n_1)!} \frac{2n_2!}{(2n_2)!} \left( \begin{array}{c} n_1 \\ m_1 \end{array} \right) \left( \begin{array}{c} n_2 \\ m_2 \end{array} \right) \cdot \left((n_1x)_{m_1} (n_1 - n_1x)_{n_1-m_1} (n_2y)_{m_2} (n_2 - n_2y)_{n_2-m_2}\right)$$

and they gave some rates of convergence for these operators.

In 2010, Gadjiev and Ghorbanalizadeh [14] defined a new construction of Bernstein–Stancu type polynomials as

$$B_{n, \alpha, \beta} (h; y) = \left( \frac{n + \beta_2}{n} \right) \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \left(y - \frac{\alpha_2}{n + \beta_2} \right)^m \left( \frac{n + \alpha_2}{n + \beta_2} - y \right)^{n-m} \cdot h \left( \frac{m + \alpha_1}{n + \beta_1} \right)$$

(9)

where $\frac{\alpha_2}{n + \beta_2} \leq y \leq \frac{n + \alpha_2}{n + \beta_2}$, $\alpha_1, \beta_1, \alpha_2, \beta_2$ are positive real number and $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$.

In 2020, inspired by the operator [9] Rahman et al. [30] constructed a Kantorovich type Lupaş–Stancu operators based on Pólya distribution as follows:

$$S_{n, \frac{1}{n}}^{(\alpha, \beta)} (h; y) = (n + \beta_1 + 1) \sum_{m=0}^{n} S_{n, m}^{(\alpha_2, \beta_2)} (y) \int_{m+1/n+\beta_1+1}^{m+1/n+\beta_1} h(u) \, du$$

(10)

where

$$S_{n, m}^{(\alpha_2, \beta_2)} (y) = \left( \begin{array}{c} n \\ m \end{array} \right) \left(y - \frac{\alpha_2}{n + \beta_2} \right)_m \left( \frac{n + \alpha_2}{n + \beta_2} - y \right)_{n-m} \cdot \left( \frac{m + 1}{n} \right)$$

and

$$(y)_m = y \cdot \left( y + \frac{1}{n} \right) \cdot \left( y + \frac{2}{n} \right) \cdots \left( y + \frac{m-1}{n} \right)$$
and \( y \in \left[ \frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2} \right] \) and \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) are positive real number, and \( 0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \). It is clear that for \( \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0 \) it reduces to operator (7). For \( \alpha_2 = \beta_2 = 0 \), it gives the Kantorovich-Stancu generalization of the operators \( D_0^k(\frac{z}{w}) \). Additionally, Rahman et al. [30] defined a bivariate generalization of the Kantorovich-Stancu type Lupaš operators given by (10) as follows

\[
S_{n_1,n_2,1}^{(\alpha,\beta)}(h; y, z) = (n_1 + \beta_1 + 1) (n_2 + \beta_1 + 1) \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} s_{m_1,n_2,m_1,m_2}^{(\alpha_2,\beta_2)}(y, z) \times \int_{\frac{m_1}{n_1+\beta_1}+1}^{\frac{m_1}{n_1+\beta_1}+1} \int_{\frac{m_2}{n_2+\beta_1}+1}^{\frac{m_2}{n_2+\beta_1}+1} h(u, s) \, du \, ds.
\]  

(11)

In 2007, the notion of Pochhammer \( k \)-symbol was first proposed by Diaz and Pariguan in the work [13]. For \( \lambda \in \mathbb{C} \), it is defined by

\[
(\lambda)_{m,k} = \begin{cases} 
\lambda (\lambda + k) (\lambda + 2k) \ldots (\lambda + (m-1)k) ; & m \geq 1 \\
1 ; & m = 0, \, \lambda \neq 0
\end{cases}
\]

(12)

where it is assumed that \( m \in \mathbb{N} \), \( k \) nonnegative real number. It is easy to say that by taking \( k = 1 \), the definition of Pochhammer \( k \)-symbol coincides with the usual Pochhammer symbol which is given by (5). This investigation has revealed many new generalizations with it, such as \( k \)-Gamma function, \( k \)-Beta function, \( k \)-Zeta function, \( k \) generalization of hypergeometric function and so on. For more detail see [13,17,19,23,24].

Our present study is motivated essentially by the theory of Polya distribution of the operators, considered by Lupaš and Lupaš [20], presented in [21] and the Pochhammer \( k \)-symbol given by [13]. Here, we define some slight modifications of Polya distribution for the special case \( \alpha = \frac{k}{n} \), \( k \) nonnegative real number, applying the notion of Pochhammer \( k \)-symbol in the definition of Polya distribution. Our main objective of this article is to investigate such a generalization how affects the rate of convergence of the operators. The structure of the paper reads as follows. In section 2, we first consider Lupaš type generalization \( P_{n,k}^{(\frac{k}{n})}(f;x) \) for the case \( \alpha = \frac{k}{n} \), \( k \) nonnegative real number, of the Bernstein-Stancu operators \( P_n^{(\alpha)}(f;x) \). We compare the operators \( P_{n,k}^{(\frac{k}{n})}(f;x) \) and the operators \( P_n^{(\frac{k}{n})}(f;x) \) with some graphics. We show that for \( 0 \leq k < 1 \) the operators \( P_{n,k}^{(\frac{k}{n})}(f;x) \) give a better approximation than \( P_{n,k}^{(\frac{k}{n})}(f;x) \). Then we introduce Kantorovich-Stancu modification of the Lupaš type operator \( P_{n,k}^{(\frac{k}{n})} \) with Pochammer \( k \)-symbol and present some fundamental results such as moments, central moments. Convergence properties of the new operators are examined. More precisely, we give the theorem about the uniform convergence, estimate the rate of the convergence by means of the classical modulus of continuity and discuss pointwise approximation via Voronovskaja type theorem. Furthermore, in section 3 we also consider a bivariate generalization of Kantorovich-Stancu modification of the Lupaš type operator \( P_{n,k}^{(\frac{k}{n})} \). We give some preliminary results for the bivariate Lupaš-Kantorovich-Stancu type operators and discuss some approximation properties. Finally, taking into
account some illustrative graphics, we conclude our study with the comparison of the rate of convergence between our operators and other operators which are mentioned in the paper for the values of the parameter \( k \).

2. Lupaš type operators by means of Pochhammer \( k \)-symbol

Before we start this section, we briefly describe the Lupaš type operators that play an important role in this paper. Let \( C [0, 1] \) be the space of all real valued continuous functions on \([0, 1]\) endowed with the norm

\[
\| f \|_{C[0,1]} = \sup_{x \in [0,1]} | f(x) | .
\]

Let \( n \in \mathbb{N} \) and \( k \) nonnegative real number. By taking into account the special choice \( \alpha = \frac{k}{n} \) in \([2]\) in view of the notion of Pochhammer \( k \)-symbol, the operators \( P_{n,k}^{\langle \frac{k}{n} \rangle} : C [0, 1] \to C [0, 1] \) is defined by

\[
P_{n,k}^{\langle \frac{k}{n} \rangle} (f; x) = \frac{1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} f \left( \frac{m}{n} \right), \tag{13}
\]

where \((\lambda)_{m,k}\) is a Pochhammer \( k \)-symbol given by \([12]\). The case \( k = 0 \) turns to the classical Bernstein operators. For \( k = 1 \), it gives Lupaš operators given by \([4]\).

Now, we begin our study by giving moments, central moments and rate of convergence for the operators \( P_{n,k}^{\langle \frac{k}{n} \rangle} \). We first give Lemma 2.1, Lemma 2.3 and Corollary 2.2 without proof, which follows from the results given in the paper \([21]\) for \( \alpha = \frac{k}{n} \), \( k \geq 0 \).

Throughout this paper, \( \mathbb{N} \) denotes the set of positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and let us denote the monomials \( e_j (t) = t^j \) for \( j \in \mathbb{N}_0 \).

**Lemma 2.1.** Let \( n \in \mathbb{N} \) and \( k \) be nonnegative real number. Then for the operators \( P_{n,k}^{\langle \frac{k}{n} \rangle} \) defined by \([13]\), we have

\[
P_{n,k}^{\langle \frac{k}{n} \rangle} (e_0; x) = 1, \tag{14}
\]

\[
P_{n,k}^{\langle \frac{k}{n} \rangle} (e_1; x) = x, \tag{15}
\]

\[
P_{n,k}^{\langle \frac{k}{n} \rangle} (e_2; x) = x^2 + \frac{(k + 1) x (1 - x)}{n + k}, \tag{16}
\]

\[
P_{n,k}^{\langle \frac{k}{n} \rangle} (e_3; x) = x^3 + \frac{(3n + 2k - 2) (k + 1) x^2 (1 - x)}{(n + k) (n + 2k)} + \frac{(2k + 1) (k + 1) x (1 - x)}{(n + k) (n + 2k)}, \tag{17}
\]
Corollary 2.2. Let \( n \in \mathbb{N} \) and \( k \) be nonnegative real number. Then the central moments of the operators \( P_{n,k}^{\langle z \rangle} \) are given by

\[
P_{n,k}^{\langle z \rangle} \left( (e_1 - x)^2 ; x \right) = \frac{(k + 1) x (1 - x)}{n + k},
\]

\[
P_{n,k}^{\langle z \rangle} \left( (e_1 - x)^3 ; x \right) = \frac{(k + 1) (2k + 1) x (1 - x) (1 - 2x)}{(n + k) (n + 2k)},
\]

\[
P_{n,k}^{\langle z \rangle} \left( (e_1 - x)^4 ; x \right) = \frac{(k + 1) 3n (-2n + 8n + k (-6 - 2k + 6n + k)) x (1 - x)^2}{n (n + k) (n + 2k) (n + 3k)} + \frac{(k + 1) (n + k) (-1 + 6 (k + 1) n) x (1 - x)}{n (n + k) (n + 2k) (n + 3k)}.
\]  

(18)

Lemma 2.3. Let \( n \in \mathbb{N} \) and \( k \) be nonnegative real number. Then for every \( f \in C \left[ 0, 1 \right] \), we have

\[
\lim_{n \to \infty} P_{n,k}^{\langle z \rangle} (f; x) = f (x)
\]

uniformly in \([0, 1]\).

For \( \alpha = \frac{k}{n}, k \geq 0 \), we can give the next theorem from the results given in [31].

Theorem 2.4. Let \( n \in \mathbb{N} \) and \( k \) be nonnegative real number. Then for every \( f \in C \left[ 0, 1 \right] \), the following inequality holds

\[
\left\| P_{n,k}^{\langle z \rangle} (f; x) - f (x) \right\|_{C \left[ 0, 1 \right]} \leq \frac{3}{2} \omega \left( f, \sqrt{\frac{k + 1}{n + k}} \right)
\]

(21)

for \( k = 1 \), which concludes that

\[
\left\| P_{n,1}^{\langle z \rangle} (f; x) - f (x) \right\|_{C \left[ 0, 1 \right]} \leq \frac{3}{2} \omega \left( f, \sqrt{\frac{2}{n + 1}} \right)
\]

where \( \omega (f, \cdot) \) denotes modulus of continuity of \( f \) defined by

\[
\omega (f, \delta) = \sup_{x, t \in [a, b]} \frac{|f (t) - f (x)|}{|t - x| \leq \delta}
\]

for \( \delta > 0 \).
We compare the rate of convergence between the operators $P_{n,k}^{(\frac{1}{x})}$ in [13] and Lupas\ operators $P_{n}^{(\frac{1}{x})}(f;x)$ with illustrative graphics. It is observed that the operators $P_{n,k}^{(\frac{1}{x})}$ give a better approximation than $P_{n}^{(\frac{1}{x})}(f;x)$ for $0 \leq k < 1$.

**Remark 1.** Let $n \in \mathbb{N}$, $k \geq 0$. Then for every $f \in C[0,1]$, the inequality $P_{n,k}^{(\frac{1}{x})} ((e_1 - x)^2;x) \leq P_{n}^{(\frac{1}{x})} ((e_1 - x)^2;x)$ holds such that $k \leq 1$. It follows that the operators $P_{n,k}^{(\frac{1}{x})}$ provide a better approximation than the classical operators $P_{n}^{(\frac{1}{x})}$.

Now, we demonstrate the behaviour of the approximation for the operators $P_{n,k}^{(\frac{1}{x})}$ by graphical examples.

**Example 2.5.** Let $f(x) = 20x^6 + 3x^3 - 5x^2 + 2x$, $n = 10$ and $k = 0.1$. In Figure 1, we analyse the convergence of the new operators $P_{n,k}^{(\frac{1}{x})}$, the classical operators $P_{n}^{(\frac{1}{x})}$ and the classical Bernstein operators $B_n$ to the function $f$. It is seen that for $k = 0.1$, $P_{n,k}^{(\frac{1}{x})}$ provides a better approximation than the operators $P_{n}^{(\frac{1}{x})}$ to the function $f$.

**Example 2.6.** Let us consider $f(x) = \sin(6\pi x) + 5 \sin\left(\frac{1}{5}\pi x\right)$. Figure 2 illustrates the approximation process of the operators $P_{n,k}^{(\frac{1}{x})}$ for $k = 0.5$ fixed and the special choices of $n = 10, 50$ and $100$. It can be observed that as the value of $n$ increases, the approximation of the operators $P_{n,k}^{(\frac{1}{x})}$ is getting better.

**Example 2.7.** Let $f(x) = \sin(2\pi x) + 2 \sin\left(\frac{1}{7}\pi x\right)$. Figure 3 presents the convergence of $P_{n,k}^{(\frac{1}{x})}$ to the function $f$ for $n = 10$ fixed and $k = 0.1, 0.3, 0.6, 1$ and 3. From this figure, it follows that when $k$ gets smaller towards to zero, approximation is better than others.

![Figure 1](image1.png)

*Figure 1.* Convergence of $P_{n,k}^{(\frac{1}{x})}$, $P_{n}^{(\frac{1}{x})}$ and $B_n$ to the function $f$ for $n = 10$ and $k = 0.1.
Figure 2. Approximation of the operators $P_{n,k}^{(x)}$ for $k = 0.5$ fixed and $n = 10, 50, 100$

Figure 3. Approximation of the operators $P_{n,k}^{(x)}$ for $n = 10$ fixed and $k = 0.1, 0.3, 0.6, 1, 3$
Now we consider a Kantorovich-Stancu modification of Lupaş type operators as follows

\[
    K_n^{(\alpha, \beta, k)}(f; x) = \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \int_{m/k+1}^{m+n+1/k+1} f(t) \, dt, \tag{22}
\]

where \( \alpha, \beta, k \) are nonnegative real number, and \( 0 \leq \alpha \leq \beta \). In the case of \( k = 0 \), it reduces to the Kantorovich-Stancu modification of Bernstein operators. For \( k = 1 \), it gives the special case \( \alpha_2 = \beta_2 = 0 \) of Kantorovich type Lupas–Stancu operators given by (10). For \( k = 1, \alpha = \beta = 0 \), this operator reduces to the Kantorovich modification of the operators \( P_n^{(\frac{1}{2})}(f; x) \) given by (7).

Taking into account the results given for the operators \( P_n^{(\frac{1}{2})}(f; x) \) in Lemma 2.1, we can give the next lemma.

**Lemma 2.8.** Let \( n \in \mathbb{N}, k \geq 0 \). Then for the operators \( K_n^{(\alpha, \beta, k)} \) defined by (22), we have

\[
K_n^{(\alpha, \beta, k)}(e_0; x) = 1, \tag{23}
\]

\[
K_n^{(\alpha, \beta, k)}(e_1; x) = \frac{nx}{n + \beta + 1} + \frac{2\alpha + 1}{2(n + \beta + 1)}, \tag{24}
\]

\[
K_n^{(\alpha, \beta, k)}(e_2; x) = \frac{n^2(n - 1)x^2}{(n + \beta + 1)^2(n + k)} + \frac{(2\alpha + 2 + k)n + (2\alpha + 1)k}{(n + \beta + 1)^2(n + k)} x
\]
\[
+ \frac{3\alpha^2 + 3\alpha + 1}{3(n + \beta + 1)^2}, \tag{25}
\]

\[
K_n^{(\alpha, \beta, k)}(e_3; x) = \frac{(n - 1)(n - 2)n^3x^3}{(n + \beta + 1)^3(n + k)(n + 2k)} + \frac{3n^2(n - 1)((3 + 2\alpha + 2k)n + 2(1 + 2\alpha)k)}{2(n + \beta + 1)^3(n + k)(n + 2k)} x^2
\]
\[
+ \left( \frac{4(1 + 3\alpha(1 + \alpha))k^2n + 6k(2 + k + \alpha(5 + 3\alpha + 2k))n^2}{2(n + \beta + 1)^3(n + k)(n + 2k)} x + \frac{4\alpha^3 + 6\alpha^2 + 4\alpha + 1}{4(n + \beta + 1)^3} \right) x
\]
\[
+ \frac{12(1 + 3\alpha(1 + \alpha))k^2n^2 + k(-27 + 5k + 6\alpha(-11 + \alpha(-5 + 6k)))n^3}{(n + \beta + 1)^3(n + k)(n + 2k)(n + 3k)} \tag{26}
\]

\[
K_n^{(\alpha, \beta, k)}(e_4; x) = \frac{(n - 1)(n - 2)(n - 3)n^4x^4}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)}
\]
\[
+ \frac{2n^3(n - 1)(n - 2)((4 + 2\alpha + 3k)n + 3(1 + 2\alpha)k)}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)} x^3
\]
\[
+ \left( \frac{-12(1 + 3\alpha(1 + \alpha))k^2n^2 + k(-27 + 5k + 6\alpha(-11 + \alpha(-5 + 6k)))n^3}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)} \right) x
\]
\[
+ \frac{-24(1 + 3\alpha(1 + \alpha))k^2n^3 + k(-27 + 5k + 6\alpha(-11 + \alpha(-5 + 6k)))n^4}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)} \tag{27}
\]
\[
3 \left( -5 - 2\alpha(3 + \alpha) + k + 2\alpha(9 + 5\alpha)k + 2(1 + 6\alpha)k^2 \right) n^4 \\
\frac{15 + 6\alpha^2 + 6\alpha(3 + 2k)k(24 + 11k)}{(n + \beta + 1)^3(n + k)(n + 2k)(n + 3k)} n^5 \right) x^2 \\
+ \frac{6(1 + 2\alpha(2 + \alpha(3 + 2\alpha)))^3 n + k^2(23 + 12k + 2\alpha(40 + 51\alpha + 22\alpha^2 + 18(1 + \alpha)k)n^2}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)} \\
+ \frac{3k(7 + 22\alpha + 22\alpha^2 + 8\alpha^3 + 9k + 22ak + 10\alpha^2k + 4k^2 + 8ak^2)n^3}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)} \\
+ \frac{(6 + 2\alpha(7 + 2\alpha(3 + \alpha)) + 15k + 6\alpha(3 + \alpha)k + 8(2 + \alpha)k^2 + 6k^3)n^4}{(n + \beta + 1)^4(n + k)(n + 2k)(n + 3k)} x \\
+ \frac{5\alpha^4 + 10\alpha^3 + 10\alpha^2 + 5\alpha + 1}{5(n + \beta + 1)^4} \tag{28}
\]

**Proof.** Following the same procedure as in Lemma 2.1 and by direct calculations, we can complete proof of the theorem easily. Now, we recall the moments of \( P_{n,k}^{(\alpha,\beta)} \) which are indicated in Lemma 2.1 For \( f(t) = e_0(t) \), we have

\[
K_n^{(\alpha,\beta,k)}(e_0;x) = \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \int_{m+\alpha \over n+\beta+1}^{m+\alpha+1 \over n+\beta+1} dt \\
= \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \left( \frac{1}{n+\beta+1} \right) \\
= \frac{1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \\
= P_n^{(\alpha,\beta)}(e_0;x) \\
= 1
\]

For the case \( f(t) = e_1(t) \), it follows

\[
K_n^{(\alpha,\beta,k)}(e_1;x) = \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \int_{m+\alpha \over n+\beta+1}^{m+\alpha+1 \over n+\beta+1} t dt \\
= \frac{n + \beta + 1}{2(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \left( \frac{(m + \alpha + 1)^2 - (m + \alpha)^2}{(n + \beta + 1)^2} \right) \\
= \frac{2\alpha + 1}{2(n + \beta + 1)} \frac{1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \\
+ \frac{n}{n + \beta + 1} \frac{1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \left( \frac{m}{n} \right)
\]
\[
K_n^{(\alpha,\beta,k)} (e_2; x) = \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \int_{\frac{m}{n+\beta+1}}^{\frac{m+1}{n+\beta+1}} t^2 dt
\]

\[
= \frac{1}{n+\beta+1} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \left( (m + \alpha + 1)^3 - (m + \alpha)^3 \right)
\]

\[
= \frac{n^2}{(n+\beta+1)^2 (n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \left( \frac{m}{n^2} \right)
\]

\[
= \frac{3(n+\beta+1)^2}{n(n+\beta+1)^2} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \left( \frac{m}{n} \right) + (\alpha + 1)^3 - \alpha^3 \cdot \frac{1}{3(n+\beta+1)^2} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k}
\]

\[
= \frac{n^2}{(n+\beta+1)^2} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \left\{ x^2 + \frac{k+1}{n+k} x (1-x) \right\} + \frac{(2\alpha + 1) n}{(n+\beta+1)^2} P_{n,k}^{(x)} (e_1; x)
\]

\[
+ \frac{(\alpha + 1)^3 - \alpha^3}{3(n+\beta+1)^2} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k}
\]

The proof for \( f(t) = e_i (t), i = 3, 4 \) is quite similar as others, hence results are given as follows

\[
K_n^{(\alpha,\beta,k)} (e_3; x) = \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n - nx)_{n-m,k} \int_{\frac{m}{n+\beta+1}}^{\frac{m+1}{n+\beta+1}} t^3 dt
\]

\[
= \frac{1}{4(n+\beta+1)^3} \left\{ 4n^3 P_{n,k}^{(x)} (e_3; x) + 6(2\alpha + 1) n^2 P_{n,k}^{(x)} (e_2; x) + 4(3\alpha^2 + 3\alpha + 1) n P_{n,k}^{(x)} (e_1; x) + ((\alpha + 1)^4 - \alpha^4) P_{n,k}^{(x)} (e_0; x) \right\}
\]
and

\[
K_n^{(\alpha,\beta,k)}(e_4; x) = \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{\frac{m+\alpha+1}{n+\beta+1}} \binom{n}{m} \left( (n)_{m,k} (n - nx)_{n-m,k} \int_t^{t^4} dt \right) \\
= \frac{1}{5(n + \beta + 1)^2} \left\{ 5n^4 P_{n,k}^{(\beta)}(e_4; x) + 10(2\alpha + 1)n^3 P_{n,k}^{(\beta)}(e_3; x) \right.
+ 10(3\alpha^2 + 3\alpha + 1)n^2 P_{n,k}^{(\beta)}(e_2; x) \\
+ 5(4\alpha^3 + 6\alpha^2 + 4\alpha + 1)n P_{n,k}^{(\beta)}(e_1; x) + \left( (\alpha + 1)^5 - \alpha^5 \right) P_{n,k}^{(\beta)}(e_0; x) \left\},
\]

by taking into account \( P_{n,k}^{(\beta)}(e_i; x) \) \( i = 0, 1, 2, 3, 4 \), we obtain the desired results.

\begin{proof}
Exploiting the previous results and doing some simple computations allow the proof of the corollary.
\end{proof}

\begin{corollary}
Let \( n \in \mathbb{N} \), \( k \geq 0 \). Then the central moments of the operators \( K_n^{(\alpha,\beta,k)} \) are given by

\[
K_n^{(\alpha,\beta,k)}(e_1 - x; x) = \frac{2\alpha + 1}{2(n + \beta + 1)} - \frac{\beta + 1}{n + \beta + 1} x, \tag{29}
\]

\[
K_n^{(\alpha,\beta,k)}((e_1 - x)^2; x) = \frac{((\beta + 1)^2 (n + k) - (k + 1) n^2) x^2}{(n + \beta + 1)^2 (n + k)} \\
+ \frac{((k + 1) n^2 - (1 + 2\alpha)(\beta + 1)(n + k)) x}{(n + \beta + 1)^2 (n + k)} \\
+ \frac{3\alpha^2 + 3\alpha + 1}{3(n + \beta + 1)^2}. \tag{30}
\]

Moreover \( K_n^{(\alpha,\beta,k)}((e_1 - x)^4; x) = O\left( \frac{1}{n^4} \right) \) as \( n \to 0 \).
\end{corollary}

\begin{proof}
Exploiting the previous results and doing some simple computations allow the proof of the corollary.
\end{proof}

\begin{lemma}
For \( n \in \mathbb{N} \) and \( k \geq 0 \), we have

\[
K_n^{(\alpha,\beta,k)}((e_1 - x)^2; x) \leq \xi_{n,k}^{\alpha,\beta}
\]

where \( \xi_{n,k}^{\alpha,\beta} = \frac{1}{n+\beta+1} \left\{ \frac{k+1}{4} + \beta + 2\alpha + \frac{\alpha + 1 - 3}{3} \right\} \).
\end{lemma}
Proof. From Corollary 2.9 it follows

\[
K_{n}^{(\alpha, \beta, k)} \left( (e_1 - x)^2 ; x \right) = \frac{1}{(n + \beta + 1)^2} \left\{ x (1 - x) \left( \frac{n^2 (k + 1)}{n + k} - (\beta + 1)^2 \right) + (\beta + 1) (\beta - 2 \alpha) x + \frac{(\alpha + 1)^2 - \alpha^3}{3} \right\} \\
\leq \frac{1}{n + \beta + 1} \left\{ (k + 1) x (1 - x) + \frac{\beta + 1}{n + \beta + 1} (\beta + 2 \alpha) x + \frac{(\alpha + 1)^3 - \alpha^3}{3(n + \beta + 1)} \right\} \\
\leq \frac{1}{n + \beta + 1} \left\{ (k + 1) x (1 - x) + (\beta + 2 \alpha) + \frac{(\alpha + 1)^3 - \alpha^3}{3} \right\} \\
\leq \frac{1}{n + \beta + 1} \left\{ k + 1 + \beta + 2 \alpha + \frac{(\alpha + 1)^3 - \alpha^3}{3} \right\}.
\]

Corollary 2.11. Taking into account Corollary 2.9, we get the following limits as follows,

\[
\lim_{n \to \infty} nK_{n}^{(\alpha, \beta, k)} (e_1 - x; x) = \alpha + \beta + 1 - (\beta + 1) x, \quad (31)
\]
\[
\lim_{n \to \infty} nK_{n}^{(\alpha, \beta, k)} \left( (e_1 - x)^2 ; x \right) = (k + 1) x (1 - x), \quad (32)
\]
\[
\lim_{n \to \infty} n^2 K_{n}^{(\alpha, \beta, k)} \left( (e_1 - x)^4 ; x \right) = 3 (k + 1)^2 x^2 (1 - x)^2. \quad (33)
\]

2.1. Convergence Properties of \( K_n^{(\alpha, \beta, k)} \)

Theorem 2.12. Let \( n \in \mathbb{N}, \ k \geq 0 \). Then for every \( f \in C [0, 1] \), we have

\[
\lim_{n \to \infty} K_{n}^{(\alpha, \beta, k)} (f; x) = f(x)
\]

uniformly in \([0, 1] \).

Proof. Making use of the results in Lemma 2.8, we deduce that

\[
\lim_{n \to \infty} K_{n}^{(\alpha, \beta, k)} (e_i(t); x) = x^i, \quad i = 0, 1, 2
\]

uniformly in \([0, 1] \). According to Korovkin’s theorem, one can easily get the desired result.

Now, we are in a position to give the theorems of the rate of convergence of the operators by virtue of classical modulus of continuity.

For \( \delta > 0 \), the modulus of continuity of \( f \) denoted by \( w (f; \delta) \) is defined to be

\[
\omega (f; \delta) = \sup_{x, t \in [a, b]} \frac{|f(t) - f(x)|}{|t - x| \leq \delta}.
\]

13
Then, for any $\delta > 0$ and each $x \in [a, b]$, it is well known that

$$|f(t) - f(x)| \leq \left(\frac{|t - x|}{\delta} + 1\right) \omega(f; \delta). \quad (36)$$

**Theorem 2.13.** Let $n \in \mathbb{N}$, $k \geq 0$. Then for every $f \in C[0,1]$, we have the following result

$$|K_n^{(\alpha, \beta, k)}(f; x) - f(x)| \leq 2\omega\left(f; \sqrt{K_n^{(\alpha, \beta, k)}((e_1 - x)^2; x)}\right), \quad (37)$$

where $\omega(f; \cdot)$ is modulus of continuity defined by (36).

**Proof.** In virtue of the definition of the operators $K_n^{(\alpha, \beta, k)}$ given by (22) and by the property of modulus of continuity (36), we obtain

$$|K_n^{(\alpha, \beta, k)}(f; x) - f(x)|$$

$$\leq \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \int_{m+1}^{m+1} \frac{|f(t) - f(x)|}{\delta} \, dt$$

$$\leq \frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \int_{m+1}^{m+1} \left(\frac{|t - x|}{\delta} + 1\right) \omega(f; \delta) \, dt$$

$$= \left[ \frac{1}{\delta} \left(\frac{n + \beta + 1}{(n)_{n,k}} \sum_{m=0}^{n} \binom{n}{m} (nx)_{m,k} (n-nx)_{n-m,k} \int_{m+1}^{m+1} |t - x| \, dt \right) + 1 \right] \omega(f; \delta). \quad (38)$$

Using Cauchy Schwarz inequality, the integral can be written as follows

$$\int_{m+1}^{m+1} |t - x| \, dt \leq \frac{1}{\sqrt{n + \beta + 1}} \left(\int_{m+1}^{m+1} (t - x)^2 \, dt \right)^{\frac{1}{2}}. \quad (39)$$

Taking into consideration again Cauchy Schwarz inequality for summation and com-
bining it with (39), we reach

\[
\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) (nx)_{m,k} (n - nx)_{n-m,k} \int_{m+\alpha+1}^{n+\beta+1} (t-x) \, dt
\]

\[
\leq \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) (nx)_{m,k} (n - nx)_{n-m,k} \left(\frac{1}{n + \beta + 1} \int_{m+\alpha+1}^{n+\beta+1} (t-x)^2 \, dt\right)^{1/2}
\]

\[
\leq \frac{(n)_{n,k}}{n + \beta + 1} \left( K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) \right)^{1/2}.
\]  

(40)

Finally, in view of the (40), (38) can be expressed as

\[
\left| K_n^{(\alpha,\beta,k)} (f; x) - f (x) \right| \leq \left[ \frac{1}{\delta} \left( K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) \right)^{1/2} + 1 \right] \omega (f; \delta).
\]

Choosing \( \delta = \left( K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) \right)^{1/2} \), we get desired result as

\[
\left| K_n^{(\alpha,\beta,k)} (f; x) - f (x) \right| \leq 2 \omega \left( f; \sqrt{K_n^{(\alpha,\beta,k)}} \left( (e_1 - x)^2 ; x \right) \right).
\]

Theorem 2.14. If \( f \in C^1 [0,1] \), then

\[
\left| K_n^{(\alpha,\beta,k)} (f; x) - f (x) \right| \leq \nu_1 |f'(x)| + 2 \sqrt{\nu_2} \omega (f'; \sqrt{\nu_2})
\]

where \( \nu_1 = K_n^{(\alpha,\beta,k)} (e_1 - x; x) \) and \( \nu_2 = K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) \).

Proof. Let \( f \in C^1 [0,1] \). For any \( x, t \in [0,1] \)

\[
f (t) - f (x) = f'(x) (t-x) + \int_{x}^{t} (f'(y) - f'(x)) \, dy.
\]

Applying \( K_n^{(\alpha,\beta,k)} \) on the both side

\[
K_n^{(\alpha,\beta,k)} (f (t) - f (x); x) = f'(x) K_n^{(\alpha,\beta,k)} (e_1 - x; x) + K_n^{(\alpha,\beta,k)} \left( \int_{x}^{t} (f'(y) - f'(x)) \, dy \right) ; x).
\]
By the property of modulus continuity (36), we get

\[
\left| \int_x^t \left| f'(y) - f'(x) \right| \, dy \right| \leq \omega(f'; \delta) \left( \frac{(t-x)^2}{\delta} + |t-x| \right),
\]

from which we conclude that

\[
\left| K_n^{(\alpha,\beta,k)} (f; x) - f(x) \right| \leq |f'(x)| \left| K_n^{(\alpha,\beta,k)} (e_1 - x; x) \right| + \left[ \frac{1}{\delta} K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) \right] \omega(f'; \delta).
\]

By the Cauchy Schwarz inequality, we can write

\[
\left| K_n^{(\alpha,\beta,k)} (f; x) - f(x) \right| \leq |f'(x)| \left| K_n^{(\alpha,\beta,k)} (e_1 - x; x) \right| + \left[ \frac{1}{\delta} K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) \right]^{1/2} \omega(f'; \delta).
\]

By taking \( K_n^{(\alpha,\beta,k)} (e_1 - x; x) = \nu_1 \) and \( K_n^{(\alpha,\beta,k)} \left( (e_1 - x)^2 ; x \right) = \nu_2 \), and by choosing \( \delta = \sqrt{\nu_2} \), we arrive at the desired result.

**Theorem 2.15.** Let \( f \) be bounded on \([0, 1]\). Then for any \( x \in (0, 1) \) at which \( f', f'' \) exist, we have

\[
\lim_{n \to \infty} n \left[ K_n^{(\alpha,\beta,k)} (f; x) - f(x) \right] = \frac{1}{2} \left[ (2\alpha + 1 - 2(\beta + 1)x) f'(x) + (k + 1)x (1-x) f''(x) \right].
\]

**Proof.** Let \( x \in [0, 1] \). According to Taylor’s series expansion of the function \( f \) at the point \( x \), we can write

\[
f(s) = f(x) + f'(x)(s-x) + \frac{1}{2} f''(x)(s-x)^2 + \eta(s,x)(s-x)^2
\]

where \( \eta(s,x) \in C[0,1] \) which satisfies

\[
\lim_{s \to x} \eta(s,x) = 0.
\]

Employing the operators \( K_n^{(\alpha,\beta,k)} \) to the both sides of (42) and taking limit for \( n \to \infty \), it follows that

\[
\lim_{n \to \infty} n \left( K_n^{(\alpha,\beta,k)} (f; x) - f(x) \right) = f'(x) \lim_{n \to \infty} n K_n^{(\alpha,\beta,k)} (s-x; x)
\]

\[
+ \frac{1}{2} f''(x) \lim_{n \to \infty} n K_n^{(\alpha,\beta,k)} \left( (s-x)^2 ; x \right)
\]

\[
+ \lim_{n \to \infty} n K_n^{(\alpha,\beta,k)} (\eta(s,x)(s-x)^2; x).
\]
Applying the Cauchy-Schwarz inequality for the last term $K_n^{(\alpha,\beta,k)}(\eta(s,x)(s-x)^2;x)$, we have

$$K_n^{(\alpha,\beta,k)}(\eta(s,x)(s-x)^2;x) \leq \sqrt{K_n^{(\alpha,\beta,k)}(\eta^2(s,x);x)} \sqrt{K_n^{(\alpha,\beta,k)}((s-x)^4;x)}.$$ 

From the property of (43) and from the limit (33), we obtain

$$\lim_{n \to \infty} nK_n^{(\alpha,\beta,k)}(\eta(s,x)(s-x)^2;x) = 0.$$ 

Finally taking into account (31) and (32), we arrive at

$$\lim_{n \to \infty} n \left( K_n^{(\alpha,\beta,k)}(f;x) - f(x) \right) = \frac{1}{2} \left[ (2\alpha + 1 - 2(\beta + 1)x)f'(x) + (k + 1)x(1-x)f''(x) \right],$$

which is required result.

**Remark 2.** If we get $\alpha = \beta = 0$, $k = 1$ in Theorem 2.15 we obtain the Voronovskaja type theorem given for the operators $D_n^{*\left(\frac{1}{n}\right)}$ by Agrawal et al. [6].

Now, we continue by analyzing the approximation properties of the operators $K_n^{(\alpha,\beta,k)}$ taking into account the graphical examples.

**Example 2.16.** Let $f(x) = x^3 \sin(4\pi x)$, $n = 50$, $\alpha = \beta = 0$ and $k = 0.2$. Behaviours of the approximation for the modified operators $K_n^{(\alpha,\beta,k)}$, the operators $D_n^{*\left(\frac{1}{n}\right)}$ which was given in [6] and the classical Kantorovich operators $K_n$ are illustrated in Figure 4. We choose the same function in [6] for the better comparison and in Figure 4, one can see that for $k = 0.2$, $K_n^{(\alpha,\beta,k)}$ provides a better approximation than the operators $D_n^{*\left(\frac{1}{n}\right)}$ to the function $f$ but the classical Kantorovich operators $K_n$ have better approximation than others.

![Figure 4](image-url)
Example 2.17. Consider \( f(x) = 2x^2 \sin(2\pi x) \) and \( k = 0.3 \). Figure 5 presents the approximation process of the operators \( K_n^{(\alpha,\beta,k)} \) for the special choices of \( n = 30, 90, 150 \) and \( \alpha = \beta = 0 \). It is clearly seen that as the value of \( n \) increases, the approximation of the operators \( K_n^{(\alpha,\beta,k)} \) is getting better.

![Figure 5](image1)

**Figure 5.** Approximation of the operators \( K_n^{(\alpha,\beta,k)} \) for \( \alpha = \beta = 0 \) and \( k = 0.3 \) fixed and \( n = 30, 90, 150 \)

Example 2.18. Let \( f(x) = x^5 \left(x - \frac{1}{4}\right) \sin(\pi x) \). Figure 6 shows the convergence of \( K_n^{(\alpha,\beta,k)} \) to the function \( f \) for \( n = 20 \) fixed and \( k = 0.3, 0.6, 0.9, 1.2 \) and 1.5 and \( \alpha = \beta = 1 \). As the value of \( k \) decrease towards to zero, the approximation of the operators \( K_n^{(\alpha,\beta,k)} \) is getting better.

![Figure 6](image2)

**Figure 6.** Approximation of the operators \( K_n^{(\alpha,\beta,k)} \) for \( n = 20, \alpha = \beta = 1 \) fixed and \( k = 0.3, 0.6, 0.9, 1.2, 1.5 \)
3. Bivariate Generalization of the Lupaş-Kantorovich-Stancu type operators by means of Pochhammer $k$-symbol

In the year 2016, Agrawal et al. \cite{6} introduced bivariate Kantorovich variant of the operators in view of the operators given by Lupaş and Lupaş \cite{20} and they investigated the rate of convergence by means of the modulus of continuity and proved the Voronovskaja type asymptotic theorem for the bivariate Lupaş-Kantorovich operators. Also in the same year, Agrawal et al. \cite{5} constructed the bivariate Lupaş-Durrmeyer operators and they examined some approximation properties of these operators using Peetre’s K-functional and discussed the asymptotic behaviour of the operators. Moreover, they considered the Generalized Boolean Sum operators of Lupaş–Durrmeyer type operators and the rate of convergence theorems for the operators were given in that study. In 2018, inspired by the concept of Özarslan and Duman \cite{29}, Kajla and Miclaus \cite{16} generalized the Lupaş-Kantorovich type operators based on Polya distribution. They gave the degree of approximation and discussed the Voronovskaja type theorem for the bivariate operators. Subsequently, Agrawal and Gupta \cite{4} introduced the Kantorovich variant of $q$-analogue of the Stancu operators defined by Nowak \cite{26} and they presented bivariate operators in their work. Recently, Rahman et al. \cite{30} has presented the bivariate extension of Kantorovich variant of Lupaş operators based on Polya distribution with shifted knots $\alpha_i, \beta_i, i = 1, 2$. They proposed some approximation properties of these operators by means of the modulus of continuity, Peetre’s $K$-functional and provided some illustrative graphics and numerical examples.

In this section, taking into account Polya distribution, we present bivariate Lupaş Kantorovich-Stancu type operators via Pochhammer $k$-symbol mainly motivated by the work \cite{6}. Also we give rate of convergence via modulus of continuity and the Lipschitz class functions for the bivariate case and we prove the Voronovskaja type theorem for the modified operators. Finally we try to illustrate the approximation process with some graphics. Let $J^2 = J \times J$ where $J = [0, 1]$. $C(J^2)$ be the space of all real-valued continuous functions on $J^2$ endowed with the norm

$$\|g\|_{C(J^2)} = \sup_{x,y \in J} |g(x, y)|.$$  

Let $C^2(J^2)$ be the space of all continuous functions $g \in C(J^2)$ such that $g_x, g_y, g_{xx}, g_{yy}$ belong to $C(J^2)$. The norm of $g \in C^2(J^2)$ is defined by

$$\|g\|_{C^2(J^2)} = \|g\|_{C(J^2)} + \sum_{i=1}^{2} \left( \|\frac{\partial^i g}{\partial x^i}\|_{C(J^2)} + \|\frac{\partial^i g}{\partial y^i}\|_{C(J^2)} \right).$$

We construct Lupaş-Kantorovich-Stancu type operators in the bivariate case as
follows

\[
K_{n_1,n_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2,k_1,k_2)}(f; x, y) = \frac{(n_1 + \beta_1 + 1)(n_2 + \beta_2 + 1)}{(n_1)_{n_1,k_1}(n_2)_{n_2,k_2}} \\
\times \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \binom{n_1}{m_1} \binom{n_2}{m_2} (n_1x)_{m_1,k_1} (n_1 - n_1x)_{n_1 - m_1,k_1} \\
\times (n_2y)_{m_2,k_2} (n_2 - n_2y)_{n_2 - m_2,k_2} \int \int f(t, s) dt ds,
\]

where \(n_1, n_2 \in \mathbb{N}\) and \(k_1, k_2, \alpha_1, \alpha_2, \beta_1, \beta_2\) are nonzero real numbers provided 0 \(\leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2\).

Now, we establish the following Lemmas. It is obvious that the extension \(K_{n_1,n_2}^{(\alpha,\beta,k)}\) given above is coincide with the operators in [6] when \(k_1 = 1\) and \(\alpha_i = 0, \beta_i = 0\) for \(i = 1, 2\). Also in [30] taking \(\alpha_2 = \beta_2 = 0\), the operators in [30] reduce to the \(K_{n_1,n_2}^{(\alpha,\beta,1)}\).

In the following, for the simplicity we use the notation \(K_{n_1,n_2}^{(\alpha,\beta,k)}\) instead of \(K_{n_1,n_2}^{(\alpha_1,\alpha_2,\beta_1,\beta_2,k_1,k_2)}\) which is given as

\[
K_{n_1,n_2}^{(\alpha,\beta,k)}(f; x, y) := K_{n_1,n_2}^{(\alpha,\beta,k)}(f; x, y),
\]

where \(\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), k = (k_1, k_2)\).

First, we state some basic lemmas which are required to prove approximation properties of \(K_{n_1,n_2}^{(\alpha,\beta,k)}\). Let us denote the monomials \(e_{i,j}(x, y) = x^i y^j\) for \((i, j) \in \mathbb{N}_0 \times \mathbb{N}_0\).

**Lemma 3.1.** Let \(n_1, n_2 \in \mathbb{N}\) and \(k_1, k_2\) be nonnegative real numbers. From Lemma 2.8 for the operators \(K_{n_1,n_2}^{(\alpha,\beta,k)}\) defined by (44), we have

\[
K_{n_1,n_2}^{(\alpha,\beta,k)}(e_{00}; x, y) = 1, 
\]

\[
K_{n_1,n_2}^{(\alpha,\beta,k)}(e_{10}; x, y) = \frac{2\alpha_1 + 1}{2(n_1 + \beta_1 + 1)} + \frac{n_1x}{n_1 + \beta_1 + 1}, \tag{46}
\]

\[
K_{n_1,n_2}^{(\alpha,\beta,k)}(e_{01}; x, y) = \frac{2\alpha_2 + 1}{2(n_2 + \beta_2 + 1)} + \frac{n_2y}{n_2 + \beta_2 + 1}, \tag{47}
\]

\[
K_{n_1,n_2}^{(\alpha,\beta,k)}(e_{20}; x, y) = \frac{n_1^2}{(n_1 + \beta_1 + 1)^2} \left[ x^2 + x(1 - x) \left( \frac{k_1 + 1}{n_1 + k_1} \right) \right] \\
+ \frac{(2\alpha_1 + 1)n_1x}{(n_1 + \beta_1 + 1)^2} + \frac{(\alpha_1 + 1)^3 - \alpha_1^3}{3(n_1 + \beta_1 + 1)^2}, \tag{48}
\]

\[
K_{n_1,n_2}^{(\alpha,\beta,k)}(e_{02}; x, y) = \frac{n_2^2}{(n_2 + \beta_2 + 1)^2} \left[ y^2 + y(1 - y) \left( \frac{k_2 + 1}{n_2 + k_2} \right) \right] \\
+ \frac{(2\alpha_2 + 1)n_2y}{(n_2 + \beta_2 + 1)^2} + \frac{(\alpha_2 + 1)^3 - \alpha_2^3}{3(n_2 + \beta_2 + 1)^2}. \tag{49}
\]

**Corollary 3.2.** Let \(n_1, n_2 \in \mathbb{N}, k_1, k_2 \geq 0\). From Lemma 3.1, the central moments of
the operators \( K^{(\alpha,\beta,k)}_{n_1,n_2} \) are given by

\[
K^{(\alpha,\beta,k)}_{n_1,n_2} (e_{10} - x; x, y) = \frac{2\alpha_1 + 1}{2(n_1 + \beta_1 + 1)} \cdot \frac{(\beta_1 + 1)x}{n_1 + \beta_1 + 1},
\]
(50)

\[
K^{(\alpha,\beta,k)}_{n_1,n_2} (e_{01} - y; x, y) = \frac{2\alpha_2 + 1}{2(n_2 + \beta_2 + 1)} \cdot \frac{(\beta_2 + 1)y}{n_2 + \beta_2 + 1},
\]
(51)

\[
K^{(\alpha,\beta,k)}_{n_1,n_2} \left( (e_{10} - x)^2; x, y \right) = \frac{1}{(n_1 + \beta_1 + 1)^2} \left\{ x(1-x) \left( \frac{a_1^2 (k_1 + 1)}{n_1 + k_1} - (\beta_1 + 1)^2 \right) + (\beta_1 + 1)(\beta_1 - 2\alpha_1)x + \frac{(\alpha_1 + 1)^2 - \alpha_1^3}{3} \right\},
\]
(52)

\[
K^{(\alpha,\beta,k)}_{n_1,n_2} \left( (e_{01} - y)^2; x, y \right) = \frac{1}{(n_2 + \beta_2 + 1)^2} \left\{ y(1-y) \left( \frac{a_2^2 (k_2 + 1)}{n_2 + k_2} - (\beta_2 + 1)^2 \right) + (\beta_2 + 1)(\beta_2 - 2\alpha_2)y + \frac{(\alpha_2 + 1)^3 - \alpha_2^3}{3} \right\}.
\]
(53)

**Lemma 3.3.** Let \( n_1, n_2 \in \mathbb{N} \), \( k_1, k_2 \geq 0 \). Then for every \( f \in C(J^2) \),

\[
\lim_{n_1,n_2 \to \infty} (e_{ij}; x, y) = e_{ij},
\]

for \( (i, j) \in \{(0,0), (0,1), (1,0)\} \) and

\[
\lim_{n_1,n_2 \to \infty} K^{(\alpha,\beta,k)}_{n_1,n_2} (e_{20} + e_{02}; x, y) = e_{20} + e_{02},
\]

uniformly on \( J^2 \), \( J = [0,1] \).

**Theorem 3.4.** Let \( n_1, n_2 \in \mathbb{N} \), \( k_1, k_2 \geq 0 \). Then for every \( f \in C(J^2) \), we have

\[
\lim_{n_1,n_2 \to \infty} \left\| K^{(\alpha,\beta,k)}_{n_1,n_2} (f) - f \right\| = 0.
\]
(54)

**Proof.** According to the Korovkin theorem for the bivariate case given in [34], by applying the results given in Lemma 3.3 we get the desired result. \( \square \)

For any function \( f \in C(J^2) \), the complete modulus of continuity for bivariate case is defined as follows:

\[
\bar{\omega} (f; \delta) = \sup \left\{ \left| f(t,s) - f(x,y) \right| : (t,s) \in J^2 \text{ and } \sqrt{(t-x)^2 + (s-y)^2} \leq \delta \right\}
\]
(55)

Moreover, the partial moduli of continuity of \( f \) with respect to \( x \) and \( y \) is given by

\[
\omega_1 (f; \delta) = \sup \left\{ \left| f(x_1,y) - f(x_2,y) \right| : y \in J \text{ and } \left| x_1 - x_2 \right| \leq \delta \right\},
\]
(56)

\[
\omega_2 (f; \delta) = \sup \left\{ \left| f(x,y_1) - f(x,y_2) \right| : x \in J \text{ and } \left| y_1 - y_2 \right| \leq \delta \right\}.
\]
(57)
Theorem 3.5. Let $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \geq 0$. Then for every $f \in C(J^2)$ for all $(x, y) \in J^2$, we have the following result

$$
\left| K_{n_1,n_2}^{(\alpha,\beta,k)}(f; x, y) - f(x,y) \right| \leq 2\tilde{\omega} \left( f; \sqrt[\alpha,\beta]{\xi_{n_1,k_1}^{(\alpha,\beta,1)} + \xi_{n_2,k_2}^{(\alpha,\beta,2)}} \right),
$$

where

$$
\xi_{n_i,k_i}^{(\alpha,\beta,i)} = \frac{1}{n_i + \beta_i + 1} \left( \frac{k_i + 1}{4} + \beta_i + 2\alpha_i + \frac{(\alpha_i + 1)^3 - \alpha_i^3}{3} \right), \quad i = 1, 2 \quad (58)
$$

and $\tilde{\omega}(f; \cdot)$ is the complete modulus of continuity defined by (55).

Proof. Taking in view the definition of the operators (44) and using the complete modulus of continuity (55), from the Cauchy–Schwarz inequality it follows that

\begin{align*}
\left| K_{n_1,n_2}^{(\alpha,\beta,k)}(f; x, y) - f(x,y) \right|
\leq & K_{n_1,n_2}^{(\alpha,\beta,k)} ( |f(t, s) - f(x, y)| ; x, y ) \\
\leq & \tilde{\omega} (f; \delta_{n_1,n_2}) \left( 1 + \frac{1}{\delta_{n_1,n_2}} K_{n_1,n_2}^{(\alpha,\beta,k)} \left( (t - x)^2 + (s - y)^2 ; x, y \right) \right) \\
\leq & \tilde{\omega} (f; \delta_{n_1,n_2}) \left( 1 + \frac{1}{\delta_{n_1,n_2}} \left( K_{n_1,n_2}^{(\alpha,\beta,k)} \left( (t - x)^2 + (s - y)^2 ; x, y \right) \right)^{1/2} \right) \\
\leq & \tilde{\omega} (f; \delta_{n_1,n_2}) \left( 1 + \frac{1}{\delta_{n_1,n_2}} \left( K_{n_1}^{(\alpha_1,\beta,1)} \left( (t - x)^2 ; x \right) + K_{n_2}^{(\alpha_2,\beta,2)} \left( (s - y)^2 ; y \right) \right)^{1/2} \right).
\end{align*}

From Lemma 2.10 we find that

\begin{align*}
\left| K_{n_1,n_2}^{(\alpha,\beta,k)}(f; x, y) - f(x,y) \right|
\leq & \tilde{\omega} (f; \delta_{n_1,n_2}) \left( 1 + \frac{1}{\delta_{n_1,n_2}} \left( \xi_{n_1,k_1}^{(\alpha,\beta,1)} + \xi_{n_2,k_2}^{(\alpha,\beta,2)} \right)^{1/2} \right)
\end{align*}

where

$$
\xi_{n_1,k_1}^{(\alpha,\beta,1)} = \frac{1}{n_1 + \beta_1 + 1} \left( \frac{k_1 + 1}{4} + \beta_1 + 2\alpha_1 + \frac{(\alpha_1 + 1)^3 - \alpha_1^3}{3} \right),
$$

$\xi_{n_2,k_2}^{(\alpha,\beta,2)} = \frac{1}{n_2 + \beta_2 + 1} \left( \frac{k_2 + 1}{4} + \beta_2 + 2\alpha_2 + \frac{(\alpha_2 + 1)^3 - \alpha_2^3}{3} \right).$

Taking

$$
\delta_{n_1,n_2} = \sqrt{\xi_{n_1,k_1}^{(\alpha,\beta,1)} + \xi_{n_2,k_2}^{(\alpha,\beta,2)}},
$$

we reach the required result. \qed

Theorem 3.6. Let $n_1, n_2 \in \mathbb{N}$, $k_1, k_2 \geq 0$. Then for every $f \in C(J^2)$ for all $(x, y) \in J^2$,
J^2, we have the following result

\[ \left| K^{(\alpha,\beta,k)}_{n_1,n_2} (f; x, y) - f (x, y) \right| \leq 2 \left( \omega_1 \left( f; \sqrt{\xi_{n_1,k_1}} \right) + \omega_2 \left( f; \sqrt{\xi_{n_2,k_2}} \right) \right), \]

where \( \omega_1 (f;.) \) and \( \omega_2 (f;.) \) are the partial moduli of continuity of \( f \) defined by (56) and (57), respectively and \( \xi_{n_1,k_1}^{(\alpha,\beta)} \) is as in Theorem 3.5.

**Proof.** Directly from (44) and the using the Cauchy Schwarz inequality, we easily obtain

\[ \left| K^{(\alpha,\beta,k)}_{n_1,n_2} (f; x, y) - f (x, y) \right| \leq K^{(\alpha,\beta,k)}_{n_1,n_2} (\| f (t, s) - f (x, y) \|; x, y) \]

\[ \leq K^{(\alpha,\beta,k)}_{n_1,n_2} (\| f (t, s) - f (t, y) \|; x, y) + K^{(\alpha,\beta,k)}_{n_1,n_2} (\| f (t, y) - f (x, y) \|; x, y) \]

\[ \leq K^{(\alpha,\beta,k)}_{n_1,n_2} (\omega_1 (f; |t - x|); x, y) + K^{(\alpha,\beta,k)}_{n_1,n_2} (\omega_2 (f; |s - y|); x, y) \]

\[ \leq \omega_1 (f; \delta_{n_1}) \left( 1 + \frac{1}{\delta_{n_1}} K^{(\alpha,\beta,k)}_{n_1,n_2} (|t - x|; x) \right) \]

\[ + \omega_2 (f; \delta_{n_2}) \left( 1 + \frac{1}{\delta_{n_2}} K^{(\alpha,\beta,k)}_{n_2,n_2} (|s - y|; y) \right) \]

\[ \leq \omega_1 (f; \delta_{n_1}) \left( 1 + \frac{1}{\delta_{n_1}} \left( K^{(\alpha,\beta,k)}_{n_1,n_2} (|t - x|^2; x) \right)^{1/2} \right) \]

\[ + \omega_2 (f; \delta_{n_2}) \left( 1 + \frac{1}{\delta_{n_2}} \left( K^{(\alpha,\beta,k)}_{n_2,n_2} (|s - y|^2; y) \right)^{1/2} \right) \]

From Lemma 2.10

\[ \left| K^{(\alpha,\beta,k)}_{n_1,n_2} (f; x, y) - f (x, y) \right| \leq \omega_1 (f; \delta_{n_1}) \left( 1 + \frac{1}{\delta_{n_1}} \left( \xi_{n_1,k_1}^{(\alpha,\beta)} \right)^{1/2} \right) \]

\[ + \omega_2 (f; \delta_{n_2}) \left( 1 + \frac{1}{\delta_{n_2}} \left( \xi_{n_2,k_2}^{(\alpha,\beta)} \right)^{1/2} \right) \]

Taking \( \delta_{n_1} = \sqrt{\xi_{n_1,k_1}^{(\alpha,\beta)}} \) and \( \delta_{n_2} = \sqrt{\xi_{n_2,k_2}^{(\alpha,\beta)}} \), we complete the proof. \( \Box \)

We continue by recalling the definition of the Lipschitz class for bivariate function of \( f \). It is known that a function \( f \) belongs to \( Lip_M (\gamma_1, \gamma_2) \) if it satisfies

\[ |f (t, s) - f (x, y)| \leq M |t - x|^\gamma_1 |s - y|^\gamma_2, \]

where \( M > 0, 0 < \gamma_1 \leq 1, \ 0 < \gamma_2 \leq 1 \). We are in a position to prove the rate of convergence for the bivariate operators by virtue of the Lipschitz class.

**Theorem 3.7.** Let \( M > 0, 0 < \gamma_1 \leq 1, \ 0 < \gamma_2 \leq 1 \) and \( f \in Lip_M (\gamma_1, \gamma_2) \). Then

\[ \left| K^{(\alpha,\beta,k)}_{n_1,n_2} (f) - f \right| \leq M \left( \lambda_{n_1,k_2}^{(\alpha_1,\beta_1)} \right)^{\gamma_1} \left( \lambda_{n_2,k_2}^{(\alpha_2,\beta_2)} \right)^{\gamma_2}, \]
Applying the Hölder’s inequality, we easily get
\[
\lambda_{n_1, k_1}^{(\alpha, \beta)} = \|K_{n_1}^{(\alpha, \beta, k_1)} ((t - \cdot)^2 ; \cdot)\| \quad \text{and} \quad \lambda_{n_2, k_2}^{(\alpha, \beta)} = \|K_{n_2}^{(\alpha, \beta, k_2)} ((s - \cdot)^2 ; \cdot)\|.
\]

**Proof.** Taking into account \( f \in \text{Lip}_M (\gamma_1, \gamma_2) \), this allows us to write
\[
\begin{align*}
|K_{n_1, n_2}^{(\alpha, \beta, k_1)} (f; x, y) - f(x, y)| &
\leq K_{n_1, n_2}^{(\alpha, \beta, k_1)} (|f(t, s) - f(x, y)| ; x, y) \\
&
\leq MK_{n_1, n_2}^{(\alpha, \beta, k_1)} (|t - x|^{\gamma_1} |s - y|^{\gamma_2} ; x, y) \\
&
= MK_{n_1}^{(\alpha, \beta, k_1)} (|t - x|^{\gamma_1} ; x) K_{n_2}^{(\alpha, \beta, k_2)} (|s - y|^{\gamma_2} ; y).
\end{align*}
\]
Applying the Hölder’s inequality, we easily get
\[
|K_{n_1, n_2}^{(\alpha, \beta, k_1)} (f; x, y) - f(x, y)| \leq M \left(\frac{1}{2} \right)^{\frac{2 - \gamma_1}{2}} \left(\frac{1}{2} \right)^{\frac{\gamma_2}{2}} \left(\lambda_{n_1, k_1}^{(\alpha, \beta)} \lambda_{n_2, k_2}^{(\alpha, \beta)} \right).
\]

**Theorem 3.8.** Let \( n_1, n_2 \in \mathbb{N} \), \( k_1, k_2 \geq 0 \). If \( f \) has partial derivatives \( f_x \) and \( f_y \) on \( J^2 \), then for all \((x, y) \in J^2 \), we have
\[
\|K_{n_1, n_2}^{(\alpha, \beta, k_1)} (f) - f\| \leq \|f_x\|_{C(J^2)} \left(\lambda_{n_1, k_1}^{(\alpha, \beta)} \right)^{\frac{1}{2}} + \|f_y\|_{C(J^2)} \left(\lambda_{n_2, k_2}^{(\alpha, \beta)} \right)^{\frac{1}{2}},
\]
where \( \lambda^{(\alpha, \beta)}_{n_1, k_1} \) and \( \lambda^{(\alpha, \beta)}_{n_2, k_2} \) are given as in Theorem 3.7.

**Proof.** Since \( f \in C^1 (J^2) \), we can write as follows
\[
f(t, s) - f(x, y) = \int_x^t f_u (u, s) \, du + \int_y^s f_v (x, v) \, dv. \tag{59}
\]

Taking \( K_{n_1, n_2}^{(\alpha, \beta, k_1)} \) on both sides of (59), we obtain
\[
|K_{n_1, n_2}^{(\alpha, \beta, k_1)} (f; x, y) - f(x, y)| \leq K_{n_1, n_2}^{(\alpha, \beta, k_1)} \left(\int_x^t f_u (u, s) \, du ; x, y\right) + K_{n_1, n_2}^{(\alpha, \beta, k_1)} \left(\int_y^s f_v (x, v) \, dv ; x, y\right).
\]

With the help of the inequalities which are given as follows
\[
\left|\int_x^t f_u (u, s) \, du\right| \leq \|f_x\|_{C(J^2)} |t - x| \quad \text{and} \quad \left|\int_y^s f_v (x, v) \, dv\right| \leq \|f_y\|_{C(J^2)} |s - y|,
\]
we easily reach
\[
|K_{n_1, n_2}^{(\alpha, \beta, k_1)} (f; x, y) - f(x, y)| \leq \|f_x\|_{C(J^2)} K_{n_1}^{(\alpha, \beta, k_1)} (|t - x| ; x) + \|f_y\|_{C(J^2)} K_{n_2}^{(\alpha, \beta, k_2)} (|s - y| ; y).
\]
By considering Cauchy Schwarz inequality, from Corollary 3.2, we can deduce the desired result as

$$\left| K^{(\alpha, \beta, k)}_{n_1,n_2} (f; x, y) - f(x, y) \right| \leq \| f_x \|_{C(J^2)} \left( K^{(\alpha_1, \beta_1, k_1)}_{n_1} \left( (t - x)^2 ; x \right) \right)^{\frac{1}{2}} \left( K^{(\alpha_1, \beta_1, k_1)}_{n_1} \left( \epsilon_0 ; x \right) \right)^{\frac{1}{2}}$$
$$+ \| f_y \|_{C(J^2)} \left( K^{(\alpha_2, \beta_2, k_2)}_{n_2} \left( (s - y)^2 ; y \right) \right)^{\frac{1}{2}} \left( K^{(\alpha_2, \beta_2, k_2)}_{n_2} \left( \epsilon_0 ; y \right) \right)^{\frac{1}{2}}$$
$$\leq \| f_x \|_{C(J^2)} \left( \lambda^{(\alpha_1, \beta_1)}_{n_1,k_1} \right)^{\frac{1}{2}} + \| f_y \|_{C(J^2)} \left( \lambda^{(\alpha_2, \beta_2)}_{n_2,k_2} \right)^{\frac{1}{2}}.$$ 

\[ \tag{60} \]

**Theorem 3.9.** For \( f \in C^2(J^2) \), then

$$\lim_{n \to \infty} n \left( K^{(\alpha, \beta, k)}_{n,n} (f; x, y) - f(x, y) \right) = \left( \alpha_1 + \frac{1}{2} - (\beta_1 + 1) x \right) f_x (x, y) + \left( \alpha_2 + \frac{1}{2} - (\beta_2 + 1) y \right) f_y (x, y)$$
$$+ \frac{1}{2} (k_1 + 1) x (1 - x) f_{xx} (x, y) + \frac{1}{2} (k_2 + 1) y (1 - y) f_{yy} (x, y)$$

uniformly on \( J^2 \).

**Proof.** Let \((x, y) \in J^2\) be arbitrary. In view of the Taylor’s series expansion of the function \( f \) at the point \((x, y)\), we obtain

$$f(t, s) = f(x, y) + f_x(x, y)(t-x) + f_y(x, y)(s-y)$$
$$+ \frac{1}{2} f_{xx}(x, y)(t-x)^2 + 2f_{xy}(x, y)(t-x)(s-y)$$
$$+ f_{yy}(x, y)(s-y)^2 + \Omega(t, s; x, y) \sqrt{(t-x)^4 + (s-y)^4} \tag{60}$$

for \((t, s) \in J^2\) where \(\Omega(t, s; x, y) \in C(J^2)\) and \(\lim_{(t, s) \to (x, y)} \Omega(t, s; x, y) = 0\).

Applying the operators \( K^{(\alpha, \beta, k)}_{n,n} \) to the both sides of (60), it follows

\[
\lim_{n \to \infty} n \left( K^{(\alpha, \beta, k)}_{n,n} (f; x, y) - f(x, y) \right)
= f_x(x, y) \lim_{n \to \infty} n K^{(\alpha_1, \beta_1, k_1)}_n ((t - x); x) + f_y(x, y) \lim_{n \to \infty} n K^{(\alpha_2, \beta_2, k_2)}_n ((s - y); y)
+ \frac{1}{2} f_{xx}(x, y) \lim_{n \to \infty} n K^{(\alpha_1, \beta_1, k_1)}_n ((t - x)^2; x) + f_{xy}(x, y) \lim_{n \to \infty} n K^{(\alpha_2, \beta_2, k_2)}_n ((t - x)(s - y); x, y)
+ \frac{1}{2} f_{yy}(x, y) \lim_{n \to \infty} n K^{(\alpha_2, \beta_2, k_2)}_n ((s - y)^2; y)
+ \lim_{n \to \infty} n K^{(\alpha, \beta, k)}_n \left( \Omega(t, s; x, y) \sqrt{(t-x)^4 + (s-y)^4}; x, y \right). \tag{61} \]

Using the Hölder’s inequality to the last term of the right side of the equation (61),
we reach

\[
\left| K_{n,n}^{(\alpha,\beta,k)} \left( \Omega (t, s; x, y) \sqrt{(t-x)^4 + (s-y)^4}; x, y \right) \right|
\]

\[
\leq \left( K_{n,n}^{(\alpha,\beta,k)} \left( \Omega^2 (t, s; x, y); x, y \right) \right)^{\frac{1}{2}} \left( K_{n,n}^{(\alpha,\beta,k)} \left( (t-x)^4 + (s-y)^4; x, y \right) \right)^{\frac{1}{2}}
\]

\[
\leq \left( K_{n,n}^{(\alpha,\beta,k)} \left( \Omega^2 (t, s; x, y); x, y \right) \right)^{\frac{1}{2}} \left( K_{n,n}^{(\alpha,\beta,k)} \left( (t-x)^4; x \right) + K_{n,n}^{(\alpha,\beta,k)} \left( (s-y)^4; y \right) \right)^{\frac{1}{2}}.
\]

Since $K_{n,n}^{(\alpha,\beta,k)} \left( \Omega^2 (t, s; x, y); x, y \right) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on $J^2$ from Theorem 3.4, by using limit which is given by (33), we have

\[
\lim_{n \rightarrow \infty} n K_{n,n}^{(\alpha,\beta,k)} \left( \Omega (t, s; x, y) \sqrt{(t-x)^4 + (s-y)^4}; x, y \right) = 0.
\]

In view of Corollary 2.11, since

\[
\lim_{n \rightarrow \infty} n K_{n,n}^{(\alpha,\beta,k)} ((t-x) (s-y); x, y) = \lim_{n \rightarrow \infty} n K_{n,n}^{(\alpha_1,\beta_1,k_1)} (t-x; x) K_{n,n}^{(\alpha_2,\beta_2,k_2)} (s-y; y) = 0,
\]

we thus find that

\[
\lim_{n \rightarrow \infty} n \left( K_{n,n}^{(\alpha,\beta,k)} (f; x, y) - f (x, y) \right)
\]

\[
= \left( \alpha_1 + \frac{1}{2} - (\beta_1 + 1) \right) f_x (x, y) + \left( \alpha_2 - \frac{1}{2} + (\beta_2 + 1) \right) f_y (x, y)
\]

\[
+ \frac{1}{2} (k_1 + 1) x (1-x) f_{xx} (x, y) + \frac{1}{2} (k_2 + 1) y (1-y) f_{yy} (x, y),
\]

which completes the proof. 

\[\square\]

**Remark 3.** The case $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, $k_1 = k_2 = 1$ in Theorem 3.9 presents the Voronovskaja type theorem given for bivariate operators $D_n^{* \left( \frac{1}{n_1}, \frac{1}{n_2} \right)}$ by Agrawal et al. [6].

**Example 3.10.** Let $f (x, y) = 2x^2 y \cos \left( \frac{5\pi x}{2} \right)$, $n_1 = n_2 = 10$ and $k_1 = k_2 = 0.2$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. Convergence for the bivariate generalized operators $K_{n_1,n_2}^{(\alpha,\beta,k)}$ (green) and the bivariate operators $D_{n_1,n_2}^{* \left( \frac{1}{n_1}, \frac{1}{n_2} \right)}$ (yellow) which was given in [6] to the function $f$ (red) is demonstrated in Figure 7. It can be noted that for $k_1, k_2 = 0.2$, the approximation by the operators $K_{n_1,n_2}^{(\alpha,\beta,k)}$ is better than the operators $D_{n_1,n_2}^{* \left( \frac{1}{n_1}, \frac{1}{n_2} \right)}$ to the function $f$.

**Example 3.11.** Consider $f (x, y) = 2x \cos (3\pi (x + y))$, $k_1 = k_2 = 0.4$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. Figure 8 presents the approximation process of the bivariate operators $K_{n_1,n_2}^{(\alpha,\beta,k)}$ to the function $f$ (red) for $n_1 = n_2 = 10$, 20, 40 (yellow, green, blue, respectively). It is clearly seen that as the values of $n_1$, $n_2$ increase, the approximation of the operators $K_{n_1,n_2}^{(\alpha,\beta,k)}$ is getting better.

**Example 3.12.** Let $f (x, y) = 7x^5 \left( x - \frac{1}{4} \right) \sin (2\pi y)$ (red). Figure 9 shows the convergence of $K_{n_1,n_2}^{(\alpha,\beta,k)}$ to the function $f$ for $n_1 = n_2 = 10$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ fixed.
and $k_1, k_2 = 0.3, 0.9, 1.2$ (yellow, green, blue, respectively). As the value of $k_1, k_2$ decreases towards to zero, the approximation of the bivariate operators $K_{n_1, n_2}^{(\alpha, \beta, k)}$ is getting better.

**Figure 7.** Convergence of $K_{n_1, n_2}^{(\alpha, \beta, k)}$ and $D_{n_1, n_2}^{*(\frac{n_1}{n_2}, \frac{n_2}{n_2})}$ to the function $f$

**Figure 8.** Approximation of the operators $K_{n_1, n_2}^{(\alpha, \beta, k)}$, for $n_1 = n_2 = 10, 20, 40$
Acknowledgements

This work of the first, second and fourth authors was supported by Scientific Research Projects Coordination Unit of Kirikkale University. Project number 2020/045.

References

[1] Acar, T., Aral, A., Raşa, I., Iterated Boolean Sums of Bernstein Type Operators, Numerical Functional Analysis and Optimization, 1-13, (2020).
[2] Acu, A. M., Gonska, H., Perturbed Bernstein-type operators, Analysis and Mathematical Physics, 10(4),1-26, (2020).
[3] Acu, A.M., Manav, N., Sofonea, D.F., Approximation properties of λ-Kantorovich operators, Journal of Inequalities and Applications, 2018: 202, (2018).
[4] Agrawal, P. N., Gupta, P., q-Lupaş Kantorovich operators based on Pólya distribution, Ann. Univ. Ferrara 64, 1–23, (2018).
[5] Agrawal, P.N., Ispir, N., Kajla, A., GBS Operators of Lupaş–Durrmeyer type based on Pólya Distribution, Results Math. 69(3–4), 397–418, (2016).
[6] Agrawal, P.N., Ispir, N., Kajla, A., Approximation properties of Lupas–Kantorovich operators based on Polya distribution, Rend. Circ. Mat. Palermo, 65, 185–208, (2016).
[7] Agrawal, P.N., Ispir, N., Kajla, A., Approximation properites of Bézier-summation integral type operators based on Pólya-Bernstein functions, Appl. Math. Comput. 259, 533–539, (2015).
[8] Barbosu, D., Kantorovich Stancu type operators, J. Inequal.Pure Appl. Math., 5(3), Article ID 53, (2004).
[9] Bernstein, S.N., Demonstration du theoreme de Weierstrass Fondee sur le calcul des probabilites, Comp. Comm. Soc. Mat. Charkow Ser. 13, no. 2, 1–2, (1912).
[10] Cárdenas-Morales, D., Gupta, V., Two families of Bernstein–Durrmeyer type operators, Applied Mathematics and Computation, 248, 342-353, (2014).
[11] Çetin, N., Başçanbaz-Tunca, G., Approximation by a new complex generalized Bernstein operators, An. Univ. Oradea Fasc. Mat, 26(2), 127–139, (2019).
[12] Deo, N., Dhamija, M., Miclăuş, D., Stancu–Kantorovich operators based on inverse Pólya–Eggenberger distribution, Applied Mathematics and Computation, 273, 281-289, (2016).
[13] Díaz, R., Pariguan, E., On hypergeometric functions and Pochhammer $k$-symbol, Divulgaciones Matemáticas, Vol. 15(2), 179-192, (2007).
[14] Gadjiév, A.D., Ghorbanalizadeh, A.M., Approximation properties of a new type Bernstein–Stancu polynomials of one and two variables, Appl. Math. Comput. 216(3), 890–901, (2010).
[15] Gupta, V., Rassias, T., Lupas–Durrmeyer operators based on Pólya distribution, Banach J. Math. Anal. 8(2), 146–155, (2014).
[16] Kajla, A., Miclăuş, D., Some smoothness properties of the Lupas-Kantorovich type operators based on Pólya distribution, Filomat, 32(11), 3867-3880, (2018).
[17] Li, S.F., Dong, Y., $k$-Hypergeometric series solutions to one type of non-homogeneous $k$-hypergeometric equations., Symmetry, 11, 262, (2019).
[18] Kokologiannaki, C. G., Properties and inequalities of generalized $k$-gamma, beta and zeta functions, Int. J. Contemp. Math. Sciences, Vol. 5(14), 653-660, (2010).
[19] Krasniqi V., A limit for the $k$-Gamma and $k$-Beta Function, Int. Math.Forum, 5. N33., (2010).
[20] Lupas, L., Lupas, A., Polynomials of binomial type and approximation operators. Stud. Univ. Babes-Bolyai Math. 32(4), 61–69 (1987)
[21] Miclaus, D., The revision of some results for Bernstein Stancu type operators, Carpathian J. Math, 28(2), 289–300, (2012).
[22] Mohiuddine, S.A., Özger, F., Approximation of functions by Stancu variant of Bernstein Kantorovich operators based on shape parameter $\alpha$, RACSAM, 114:70, (2020).
[23] Mubeen, S., Rehman, A., A note on $k$-gamma function and pochhammer $k$-symbol, Journal of Informatics and Mathematical Sciences 6(2), 93-107, (2014).
[24] Mubeen, S., $k$-Analogue of Kummer’s first formula, J. Inequal. Spec. Funct., 3(3), 41–44, (2012).
[25] Neer, T., Agrawal, P. N., A genuine family of Bernstein-Durrmeyer type operators based on Pólya basis functions, Filomat, 31(9), 2611-2623, (2017).
[26] Nowak, G., Approximation properties for generalized q-Bernstein polynomials. J. Math. Anal. Appl. 350, 50–55, (2009).
[27] Opris, A.A., Approximation by modified Kantorovich Stancu operators, Journal of Inequalities and Applications, 2018:346, (2018).
[28] Ostrovska, S., Turan, M., The distance between two limit $q$-Bernstein operators, Rocky Mountain Journal of Mathematics, 50(3), 1085-1096, (2020).
[29] Özarslan, M.A., Duman, O., Smoothness properties of modified Bernstein-Kantorovich operators, Numer. Funct. Anal. Optim. 37, 92–105, (2016).
[30] Rahman, S., Mursaleen, M., Khan, A., A Kantorovich variant of Lupas–Stancu operators based on Pólya distribution with error estimation, RACSAM, 114:75, (2020).
[31] Razi, Q., Approximation of functions by Bernstein type operators, Master Thesis, Aligarh Muslim University, Aligarh, India, 1983.
[32] Razi, Q., Approximation of a function by Kantorovich type operators, Mat. Vesnik, 3, 183-192, (1989).
[33] Stancu, D. D., Approximation of functions by a new class of linear polynomial operators, Rev. Roumaine Math. Pures Appl, 13(8), 1173-1194, (1968).
[34] Volkov, V.I., On the convergence of sequences of linear positive operators in the space of two variables, Dokl. Akad. Nauk. SSSR (N.S.), 115, 17–19, (1957).

29