Length of $\mathcal{D}_X f^{-\alpha}$ in the Isolated Singularity Case

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Abstract. Let $f$ be a convergent power series of $n$ variables having an isolated singularity at $0$. Set $(X, 0) = (\mathbb{C}^n, 0)$. For $\alpha \in \mathbb{Q}$, we show that the length of the $\mathcal{D}_X$-module $\mathcal{D}_X f^{-\alpha}$ coincides with $\nu_\alpha + \rho \delta_\alpha + 1$. Here $\nu_\alpha$ is the dimension of the graded piece $\text{Gr}_V^\alpha$ of the $V$-filtration on the saturation of the Brieskorn lattice modulo the image of $N := \partial_t - \alpha$ on $\text{Gr}_V^\alpha$ of the Gauss-Manin system, $\rho$ is the number of local irreducible components of $f^{-1}(0)$ (where $\rho = 1$ if $n > 2$), and $\delta_\alpha := 1$ if $\alpha \in \mathbb{Z}_{>0}$, and $0$ otherwise. This generalizes an assertion by T. Bitoun and T. Schedler in the weighted homogeneous case, where the saturation coincides with the Brieskorn lattice and $N = 0$. In the semi-weighted-homogeneous case, the above formula implies certain sufficient conditions for their conjecture about the length of $\mathcal{D}_X f^{-1}$ to hold or to fail.

Introduction

Let $f \in \mathbb{C}\{x\}$ be a convergent power series of $n$ variables having an isolated singularity at $0$. To calculate the monodromy on the vanishing cohomology in an algebraic way, E. Brieskorn [Br 70] introduced the Brieskorn lattice $\mathcal{H}''_f$, which is a free $\mathbb{C}\{t\}$-module of rank $\mu_f$, and is endowed with a regular singular Gauss-Manin connection, where $\mu_f$ is the Milnor number of $f$. B. Malgrange [Ma 75] proved that the reduced Bernstein-Sato polynomial $b_f(s)/(s+1)$ coincides with the minimal polynomial of the action of $-\partial_t$ on $\mathcal{H}''_f/t\mathcal{H}''_f$ with $\mathcal{H}''_f$ the saturation of $\mathcal{H}''_f$ defined by

$$\mathcal{H}''_f := \sum_{j \in \mathbb{N}} (\partial_t)^j \mathcal{H}''_f \subset \mathcal{D}_f.$$ 

Here $\mathcal{D}_f := \mathcal{H}''_f[\partial_t]$ is the Gauss-Manin system, which is the localization of $\mathcal{H}''_f$ by the action of $\partial_t^{-1}$. The latter operator is well defined on $\mathcal{H}''_f$, see [1.12] below.

Recently T. Bitoun and T. Schedler studied the length of the quotient $\mathcal{D}_X$-modules $\mathcal{D}_X f^{-\alpha}/\mathcal{D}_X f^{-\alpha+1}$ for $\alpha \in \mathbb{Q}$ in the weighted homogeneous case, where $(X, 0) = (\mathbb{C}^n, 0)$. They found that it coincides with the sum of $\delta_{\alpha,1}$ (Kronecker delta) and the multiplicity $n_{f,\alpha}$ of the spectral number $\alpha$, assuming $n \geq 3$, see [BiSc 18] Cor. 1.20 and Thm. 2.1] (and Remark 2.2a below). This follows also from [Sa 21 Cor. 1]. In general this multiplicity can be defined by

$$(1) \quad n_{f,\alpha} = \dim_\mathbb{C} \text{Gr}^\alpha_V(\mathcal{H}''_f/t\mathcal{H}''_f),$$

following [Va 81], where $V$ is the $V$-filtration of Kashiwara and Malgrange indexed by $\mathbb{Q}$ so that the action of $N := \partial_t - \alpha$ is nilpotent on $\text{Gr}_V^\alpha$, see [1.1] and Remark [1.2a] below. In the weighted homogeneous case, this $V$-filtration can be induced by the weighted degree of differential forms and we have $\mathcal{H}''_f = \mathcal{H}''_f$ (see Remarks [1.1] and [1.2a] below). Hence the spectral numbers of $f$ coincide with the roots of the reduced Bernstein-Sato polynomial up to sign forgetting the multiplicities.

In general, set $\tilde{\mathcal{H}}''_f^{(\alpha)} := \text{Gr}^\alpha_V \tilde{\mathcal{H}}''_f$, $\mathcal{G}_f^{(\alpha)} := \text{Gr}^\alpha_V \mathcal{G}_f$, and

$$(2) \quad \tilde{\nu}_{f,\alpha} := \dim_\mathbb{C} \text{Im}(\tilde{\mathcal{H}}''_f^{(\alpha)} \to \mathcal{G}_f^{(\alpha)}/\mathcal{G}_f^{(\alpha)}),$$

In the weighted homogeneous case, we have $\tilde{\nu}_{f,\alpha} = \sum_{j \in \mathbb{N}} n_{f,\alpha-j}$, since $\tilde{\mathcal{H}}''_f = \mathcal{H}''_f$ and $N = 0$. Let $r_f$ be the number of local irreducible components of $Z := f^{-1}(0) \subset X$. Note that $r_f = 1$.
if \( n \geq 3 \). Set \( \tilde{\alpha} \) := 1 if \( \alpha \in \mathbb{Z}_{>0} \), and 0 otherwise. We denote by \( \ell_{\mathcal{D}_{X,0}}(\mathcal{M}) \) the length of a holonomic \( \mathcal{D}_{X,0} \)-module \( \mathcal{M} \) in general. We show in this paper the following.

**Theorem 1.** For \( \alpha \in \mathbb{Q} \), we have the equality

\[
\ell_{\mathcal{D}_{X,0}}(\mathcal{D}_{X,0}f^{-\alpha}) = \tilde{\nu}_{f,\alpha} + r_f \tilde{\delta}_{\alpha} + 1,
\]

where \( \tilde{\nu}_{f,\alpha}, r_f \tilde{\delta}_{\alpha} \), and 1 respectively correspond to regular holonomic \( \mathcal{D}_{X,0} \)-modules supported on 0, local irreducible components of \((Z,0)\), and \((X,0)\).

This follows easily from [Sa 21, Thm. 1 and Prop. 1] (that is, Theorem 1.4, Proposition 1.4 in this paper), see [2.1] below.

Let \( g \) be a weighted homogeneous polynomial with an isolated singularity at 0. Let \( h \) be a monomial with weighted degree \( d' > 1 \), where the weighted degree of \( g \) is 1. Set \( f := g + h \), and \( \alpha_f = \alpha_g := \sum_{i=1}^{n} w_i \) with \( w_i \) the weights of variables \( x_i \). Let \( x^{(k)} (k \in [1, \mu_f]) \) be monomials whose images in the Jacobian ring \( \mathbb{C}[x]/(\partial g) \) form a \( \mathbb{C} \)-basis. Here \( (\partial g) \) denotes the Jacobian ideal of \( g \), and \( \mu_f (= \mu_g) \) is the Milnor number of \( f, g \). Let \( \alpha_{f,k} \) be the sum of \( \alpha_f \) and the weighted degree of \( x^{(k)} \), that is, \( \alpha_{f,k} = \sum_{i=1}^{n} (\nu_i + 1) w_i \) if \( x^{(k)} = \prod_{i} x_i^{\nu_i} \). Calculating the Gauss-Manin connection in the semi-weighted-homogeneous case, we get the following as a corollary of Theorem 1.

**Theorem 2.** (i) If \( \alpha < \alpha_f + d' \), we have \( \tilde{\nu}_{f,\alpha} = \tilde{\nu}_{g,\alpha} \), hence the equality (1) holds for \( f \) with \( \tilde{\nu}_{f,\alpha} \) replaced by \( \tilde{\nu}_{g,\alpha} \) and \( r_f \) by \( r_g \) (even if \( h = x^{(k)} \) for some \( k \in [1, \mu_f] \)).

(ii) If there are \( x^{(k')}, x^{(k'')} \) \((k', k'' \in [1, \mu_f])\) with \( x^{(k')} h = x^{(k'')} \), put \( \alpha = \alpha_{f,k''} - 1 = \alpha_{f,k'} + d' - 1 \). Then we have the inequality

\[
\ell_{\mathcal{D}_{X,0}}(\mathcal{D}_{X,0}f^{-\alpha}) \geq \tilde{\nu}_{g,\alpha} + r_f \tilde{\delta}_{\alpha} + 2,
\]

where the equality holds if \( \alpha_{f,k'} = \alpha_f \), that is, if \( x^{(k')} = 1 \).

This gives certain sufficient conditions for a conjecture in [BiSc 18, Conj. 1.7] to hold or to fail, see also [MuOl 22]. Here \( n_{f,\alpha} = n_{g,\alpha} = \# \{ k \in [1, \mu_g] \mid \alpha_{f,k} = \alpha \} \) (see Remarks 1.2b–c below) with \( r_f = r_g \), and \( \tilde{\nu}_{g,\alpha} = \sum_{j \in \mathbb{N}} n_{g,\alpha-j} \), since \( g \) is weighted homogeneous. In the case \( g = x^4 + y^4 + z^4 \) for instance, setting \( h = x^2 y^2 z^2 \) and \( x^2 y^2 z \), the hypotheses of (i) and (ii) are satisfied with \( \alpha = 1 \), \( \alpha_f = \frac{3}{2} \), and \( d' = \frac{3}{2} \) and \( \frac{5}{2} \) respectively (compare with [MuOl 22]). More generally, Theorem 2(ii) implies that, for any homogeneous polynomial \( g \) in \( n \) variables of degree \( d > n \geq 3 \) having an isolated singularity at 0, there is always a semi-homogeneous deformation \( f = g + h \) such that the above conjecture fails, where \( \text{deg } h = 2d - n < n(d-2) \) (since \( (n-2)(d-1) > 2 \)).

It does not seem easy to determine \( \ell_{\mathcal{D}_{X,0}}(\mathcal{D}_{X,0}f^{-\alpha}) \) in case the assumption of (ii) is satisfied except the case \( \alpha = \alpha_f + d' - 1 \) even though the nilpotent operator \( N \) on the right-hand side of (3) vanishes in the semi-weighted-homogeneous case. This applies even to the case where \( n_{f,\alpha} = 1 \) or 0 \((\forall \alpha)\), for instance, if \( g = \sum_{i=1}^{n} x_i^{a_i} \) with \( a_i \) mutually prime (here the problem is equivalent to the determination of \( b_{f,s} \)). Note also that the assumption of (ii) is not a necessary condition for the inequality (5) to hold, see Example 2.2 below.

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In Section 1 we review some basics of Brieskorn lattices, spectral numbers, and Bernstein-Sato polynomials for isolated singularities as well as a formula for \( \mathcal{D}_xf^{-\alpha} / \mathcal{D}_xf^{-\alpha+1} \) in [Sa 21, Thm. 1]. In Section 2, we prove the main theorems as well as the assertions in Section 1.
1. Preliminaries

In this section we review some basics of Brieskorn lattices, spectral numbers, and Bernstein-Sato polynomials for isolated singularities as well as a formula for \( \mathcal{D}_X f^{-\alpha}/\mathcal{D}_X f^{-\alpha+1} \) in [Sa21, Thm. 1].

1.1. Brieskorn lattices and spectral numbers. Let \( f \in \mathbb{C}\{x\} \) be a convergent power series of \( n \) variables with \( n \geq 2 \). Set \((X, 0) := (\mathbb{C}^n, 0)\). The Brieskorn lattice [Br70] is defined by

\[
\mathcal{H}''_f := \Omega^n_{X,0}/df \wedge d\Omega^{n-2}_{X,0}.
\]

This is a free module of rank \( \mu_f \) (with \( \mu_f \) the Milnor number) over \( \mathbb{C}\{t\} \) and \( \mathbb{C}\{\partial_t^{-1}\} \), see [Sa89, 1.8]. The actions of \( t \) and \( \partial_t^{-1} \) are defined respectively by multiplication by \( f \)

\[
\partial_t^{-1}[\omega] = [df \wedge \eta] \quad \text{if} \quad d\eta = \omega \quad (\omega \in \Omega^n_{X,0}, \eta \in \Omega^{n-1}_{X,0}).
\]

Here the well-definedness follows from the Poincaré lemma. Set

\[
\mathcal{G}_f := \mathcal{H}''_f[\partial_t],
\]

which is the localization of \( \mathcal{H}''_f \) by the action of \( \partial_t^{-1} \). This is a regular holonomic \( \mathcal{D}_X \)-module with quasi-unipotent monodromy, and is called the Gauss-Manin system. It has the \( V \)-filtration of Kashiwara and Malgrange indexed by \( \mathbb{Q} \) so that \( \partial_t t - \alpha \) is nilpotent on \( \text{Gr}^n_k \mathcal{G}_f \), see for instance [Sa89, Sa94]. This filtration was originally indexed by \( \mathbb{Z} \), see [Sa83, Sa84, §3.4] about the reason for which it must be indexed by \( \mathbb{Q} \).

Remark 1.1. In the weighted homogeneous case, we have the Euler field

\[
\xi := \sum_{i=1}^n w_i x_i \partial_{x_i} \quad \text{with} \quad \xi(f) = f.
\]

Using the interior product \( \iota_\xi \) and the Lie derivation \( L_\xi \) (see for instance [BaSa22, §1.4]), it is easy to see that the Brieskorn lattice is stable by the action of \( \partial_t \), and the \( V \)-filtration on it is induced by the weighted degree of differential forms (where \( \deg_w x_i = \deg_w dx_i = w_i \), see [Sa88a]) which is given by the Lie derivation. Recall that there are well-known relations

\[
d\iota_\xi \omega = L_\xi \omega, \quad df \wedge \iota_\xi \omega = f \omega \quad (\omega \in \Omega^n_X).
\]

1.2. Spectral numbers. The spectrum \( \text{Sp}_f(t) = \sum_{\alpha \in \mathbb{Q}} n_{f, \alpha} t^\alpha \) was first introduced in [St77] for isolated singularities. (There was some confusion about the monodromy, see [DiSa14].) The multiplicities \( n_{f, \alpha} \) can be given by

\[
n_{f, \alpha} = \dim_{\mathbb{C}} \text{Gr}^\alpha_f (\mathcal{H}''_f/\partial_t^{-1} \mathcal{H}''_f)
\]

\[
= \dim_{\mathbb{C}} \text{Gr}^\alpha_f (\mathbb{C}\{x\}/(\partial f)).
\]

The first equality follows for instance from [ScSt85, Va81] (see also [Sa89, JKSY22, §1]), and the second one from the isomorphism

\[
\mathcal{H}''_f/\partial_t^{-1} \mathcal{H}''_f \cong \mathbb{C}\{x\}/(\partial f),
\]

trivializing \( \Omega^n_{X,0} \), where \( (\partial f) \) denotes the Jacobian ideal of \( f \). We call \( \alpha \) a spectral number of \( f \) if \( n_{f, \alpha} \neq 0 \). It is well known (see for instance [St77, Sa89]) that there is the symmetry

\[
n_{f, \alpha} = n_{f, -\alpha} \quad (\alpha \in \mathbb{Q}).
\]

Remark 1.2a. The equality (2) in the introduction follows from (1.2.1), since we can replace \( \partial_t^{-1} \mathcal{H}''_f \) in the second term by \( t \mathcal{H}''_f \) passing to the graded pieces of the monodromy filtration.
W associated with the nilpotent operator \( N := \partial t - \alpha \) on \( \text{Gr}_V \mathcal{G}_f \) as in [Va81]. Note however that \( \partial_t^{-1} \mathcal{N}^n_f \neq t \mathcal{N}^n_f \) (although \( \partial_t^{-1} \mathcal{N}^n_f = t \mathcal{N}^n_f \)) unless \( f \) is weighted homogeneous.

**Remark 1.2b.** It is well known (see for instance [Va82]) that the spectrum \( \text{Sp}_f(t) \) is invariant under a \( \mu \)-constant deformation of \( f \) (e.g., under a semi-weighted-homogeneous deformation of a weighted homogeneous polynomial with an isolated singularity).

**Remark 1.2c.** In the weighted homogeneous case, the \( V \)-filtration on the Jacobian ring \( \mathbb{C}\{x\}/(\partial f) \) can be given by using the weighted degree as in Remark 1.1. Here we have a shift coming from the trivialization of the highest differential forms \( \Omega^n_{X,0} \), which give free resolutions of the Jacobian rings \( \mathbb{C}\{x\}/(\partial g) \) respectively.

**Remark 1.2d.** Let \( f = g + h \) be a semi-weighted-homogeneous deformation of a weighted homogeneous polynomial \( g \) with an isolated singularity, where the weighted degrees of \( g \) and \( h \) are 1 and strictly larger than 1 respectively. Let \( V \) be the decreasing filtration on \( \mathbb{C}\{x\} \) defined by the condition that the weighted degree is at least \( \alpha \). This induces the \( V \)-filtration on the Gauss-Manin system \( \mathcal{G}_f \) (see Remark 1.1 for \( \mathcal{G}_g \)). This can be shown by using the \( V \)-adic completion together with the Mittag-Leffler condition (see [Gro61, Prop. 13.2.3]) as in [Sa88a]. Here we have the canonical isomorphism

\[
\text{Gr}_V^\bullet(\mathbb{C}\{x\}/(\partial f)) = \text{Gr}_V^\bullet(\mathbb{C}\{x\}/(\partial g)),
\]

where the weighted degree is shifted by \( \sum_{i=1}^n w_i \) as in Remark 1.2c. This isomorphism can be verified by using the filtered shifted Koszul complexes

\[
((\Omega^n_{X,0}, V), df \wedge [n]), \quad ((\Omega^n_{X,0}, V), dg \wedge [n]),
\]

which give filtered free resolutions of the Jacobian rings \( \mathbb{C}\{x\}/(\partial f), \mathbb{C}\{x\}/(\partial g) \) respectively.

**1.3. Bernstein-Sato polynomials.** Let \( f \) and \( \mathcal{M}_f \), \( \mathcal{G}_f \) be as in 1.1. We have the saturation \( \mathcal{M}_f \subset \mathcal{G}_f \) defined as in 1.1 in the introduction. Let \( b_f(s) \) be the Bernstein-Sato polynomial of \( f \). Set

\[
\tilde{b}_f(s) := b_f(s)/(s+1).
\]

This is called the reduced (or microlocal) Bernstein-Sato polynomial, see [Sa94]. Note that the roots of \( \tilde{b}_f(s) \) are strictly negative rational numbers, see [Ka76]. By B. Malgrange [Ma75], we have the following equality as is explained in the introduction:

\[
\tilde{b}_f(s) = \text{Mini. Poly}(-\partial_t t \in \text{End}_\mathbb{C}(\mathcal{M}_f / t \mathcal{M}_f)).
\]

Comparing this with (2) in the introduction, we see that the roots of \( \tilde{b}_f(s) \) coincides with the spectral numbers up to sign in the weighted homogeneous case (where \( \mathcal{M}_f = \mathcal{M}_f^\alpha \)).

**1.4. Description of the quotient \( \mathcal{D}_X f^{-\alpha}/\mathcal{D}_X f^{-\alpha+1} \).** For \( X, f \) as in 1.1 we denote by \( i_f : X \hookrightarrow Y := X \times \mathbb{C} \) the graph embedding for \( f \). Set

\[
\mathcal{B}_f := ((i_f)^\sharp \mathcal{O}_X)_0 = \mathcal{O}_X[\partial_t][\delta(t-f)],
\]

where \((i_f)^\sharp \) is the direct image as \( \mathcal{D} \)-module, see for instance [Sa21] §1. It is a regular holonomic \( \mathcal{D}_{Y,0} \)-module, and has the \( V \)-filtration of Kashiwara and Malgrange indexed by \( \mathbb{Q} \) so that \( N := \partial_t t - \alpha \) is nilpotent on \( \text{Gr}_V^\alpha \), where \( t \) is the coordinate of \( \mathbb{C} \). Put

\[
\mathcal{M}_f^{(\alpha)} := \text{Gr}_V^\alpha \mathcal{B}_f \quad (\alpha \in \mathbb{Q}).
\]
These are regular holonomic $\mathcal{D}_{X,0}$-modules (corresponding to the $\lambda$-eigenspace of the nearby cycle complex $\psi_{t,\lambda}C_{X,[n-1]}$ with $\lambda := e^{-2\pi i \alpha}$). Using the above property of $\partial_t$ on $\text{Gr}_{V}^j$, we can deduce the isomorphisms

\begin{equation}
(1.4.1) \quad t : \mathcal{M}_j^{(\alpha)} \xrightarrow{\sim} \mathcal{M}_j^{(\alpha+1)}, \quad \partial_t : \mathcal{M}_j^{(\alpha+1)} \xrightarrow{\sim} \mathcal{M}_j^{(\alpha)} \quad (\alpha \neq 0).
\end{equation}

As is well known (see for instance [Ma 75]), there is a natural inclusion

$$\mathcal{N}_f := \mathcal{D}_{X,0}[s]f^s \hookrightarrow \mathcal{B}_f,$$

where $s$ and $f^s$ are identified with $-\partial_t$ and $\delta(t \cdot f)$ respectively, and the Bernstein-Sato polynomial $b_f(s)$ coincides with the minimal polynomial of the action of $s = -\partial_t$ on the quotient regular holonomic $\mathcal{D}_{X,0}$-module

\begin{equation}
(1.4.2) \quad \mathcal{N}_f/\mathcal{T}_f,
\end{equation}

where the action of $t$ on $\mathcal{N}_f$ is defined by $s \mapsto s+1$, see [Ka 76]. Note that the quotient filtration $V$ on $\mathcal{N}_f/\mathcal{T}_f$ is a finite filtration (since $\mathcal{N}_f/\mathcal{T}_f$ is holonomic), and $-\alpha$ is a root of $b_f(s)$ if and only if $\text{Gr}_V^i(\mathcal{N}_f/\mathcal{T}_f) \neq 0$. Set

$$\mathcal{N}_f^{(\alpha)} := \text{Gr}_V^i\mathcal{N}_f \subset \mathcal{M}_j^{(\alpha)} \quad (\alpha \in \mathbb{Q}).$$

We define the filtration $G$ on $\mathcal{M}_j^{(\beta)}$ for $\beta \in (0, 1]$ so that

\begin{equation}
(1.4.3) \quad t^j : G_j.\mathcal{M}_j^{(\beta)} \xrightarrow{\sim} \mathcal{N}_f^{(\beta+j)} \subset \mathcal{M}_j^{(\beta+j)} \quad (j \in \mathbb{N}),
\end{equation}

where $G_j.\mathcal{M}_j^{(\beta)} = 0$ for $j < 0$. This definition coincides with the one in [Sa 21], since there is the inclusion

\begin{equation}
(1.4.4) \quad \mathcal{N}_f \subset V^>0 \mathcal{B}_f,
\end{equation}

by negativity of the roots of $b_f(s)$, see [Ka 76]. We have also the isomorphisms

\begin{equation}
(1.4.5) \quad \partial_t^j : \mathcal{N}_f^{(\beta+j)} \xrightarrow{\sim} G_j.\mathcal{M}_j^{(\beta)} \quad (j \in \mathbb{N}),
\end{equation}

since the holonomic $\mathcal{D}_{X,0}$-submodule $\mathcal{N}_f^{(\beta+j)}$ is stable by the automorphism $t^j \partial_t^j$ of $\mathcal{M}_j^{(\beta+j)}$ (and an injective endomorphism of a holonomic $\mathcal{D}$-module is an automorphism). Indeed, we have

\begin{equation}
(1.4.6) \quad t^j \partial_t^j = \prod_{k=1}^j (\partial_t - k) = \prod_{k=1}^j (N+\beta+j-k) \quad \text{on} \quad \mathcal{M}_j^{(\beta+j)},
\end{equation}

and $\mathcal{N}_f^{(\beta+j)} \subset \mathcal{M}_j^{(\beta+j)}$ is stable by the action of $N = -s-\beta-j$ on $\mathcal{M}_j^{(\beta+j)}$.

We have the following

**Theorem 1.4 ([Sa 21] Thm 1]).** There are isomorphisms of $\mathcal{D}_{X,0}$-modules

\begin{equation}
(1.4.7) \quad \mathcal{D}_{X,0}f^{-\beta-j}/\mathcal{D}_{X,0}f^{-\beta-j+1} = \text{Gr}_j^G(\mathcal{M}_j^{(\beta)}/N.\mathcal{M}_j^{(\beta)}) \quad (\beta \in (0, 1], j \in \mathbb{N}).
\end{equation}

Here $G$ denotes also the quotient filtration on $\mathcal{M}_j^{(\beta)}/N.\mathcal{M}_j^{(\beta)}$ so that we have the strict surjection

\begin{equation}
(\mathcal{M}_j^{(\beta)}, G) \twoheadrightarrow (\mathcal{M}_j^{(\beta)}/N.\mathcal{M}_j^{(\beta)}, G).
\end{equation}

Note that $N$ is not necessarily strictly compatible with $G$ in general, see [Sa 21].

We need also the following for the proof of Theorem 1.

**Proposition 1.4 ([Sa 21] Prop 1]).** For $\beta \in (0, 1]$, we have

\begin{equation}
(1.4.8) \quad \text{Supp} \text{Gr}_j^G\mathcal{M}_j^{(\beta)} \subset \{0\} \quad \text{if} \quad (\beta, j) \neq (1, 0).
\end{equation}
More precisely, setting \( \tilde{\mathcal{M}}_j^{(1)} := \mathcal{M}_j^{(1)}/\ker N \), we have the isomorphisms
\[
\mathrm{Gr}_j^{G} \mathcal{M}_j^{(1)} \sim \mathrm{Gr}_j^{G} \tilde{\mathcal{M}}_j^{(1)},
\]
(1.4.9)
\[
\mathrm{Gr}_j^{G} (\mathcal{M}_j^{(1)}/N.\mathcal{M}_j^{(1)}) \sim \mathrm{Gr}_j^{G} (\tilde{\mathcal{M}}_j^{(1)}/N.\tilde{\mathcal{M}}_j^{(1)}) \quad (j \geq 1),
\]
(1.4.10)
and the short exact sequences of regular holonomic \( D_{X,0} \)-modules
\[
0 \to \mathcal{M}_j \to \mathrm{Gr}_0^{G} \mathcal{M}_j^{(1)} \to \mathrm{Gr}_0^{G} \tilde{\mathcal{M}}_j^{(1)} \to 0,
\]
(1.4.11)
\[
0 \to \mathcal{M}_j^{IC} \to \mathrm{Gr}_0^{G} (\mathcal{M}_j^{(1)}/N.\mathcal{M}_j^{(1)}) \to \mathrm{Gr}_0^{G} (\tilde{\mathcal{M}}_j^{(1)}/N.\tilde{\mathcal{M}}_j^{(1)}) \to 0.
\]
(1.4.12)
Here \( \mathcal{M}_j \), \( \mathcal{M}_j^{IC} \) are regular holonomic \( D_{X,0} \)-modules such that \( \mathrm{DR}_X(\mathcal{M}_j) = \mathbb{C}_Z[n-1] \) and \( \mathrm{DR}_X(\mathcal{M}_j^{IC}) = \mathrm{IC}_Z \mathbb{C} \) with \( \mathrm{IC}_Z \mathbb{C} \) the intersection complex of \( Z := f^{-1}(0) \subset X \), and these \( D_{X,0} \)-modules are extended to \( D_X \)-modules by shrinking \( X \) sufficiently.

2. Proof of the main theorems

In this section we prove the main theorems using the assertions in Section 1.

2.1. Proof of Theorem 11 It is well known that we have a canonical isomorphism of regular holonomic \( D_{\mathbb{C},0} \)-modules
\[
\mathcal{H}^0 \mathrm{DR}_X(\mathcal{B}_f) = \mathcal{B}_f,
\]
(2.1.1)
since \( \mathrm{DR}_X(\mathcal{B}_f) \) is \( \mathbb{C} \)-rigid, see for instance \( \text{Sa 84, ScSt 85, Sa 88a} \). Moreover, the \( V \)-filtration on the complex \( \mathrm{DR}_X(\mathcal{B}_f) \) is \( \mathbb{V} \)-strict, and induces the \( V \)-filtration on \( \mathcal{B}_f \). This can be verified by using for instance an argument similar to the proof of \( \text{Sa 88b, Prop. 3.4.8} \), see also \( \text{DiSa 12, §4.11, JKS 22, §1.4} \).

We denote \( \mathcal{H}^j \mathrm{DR}_X \) by \( \mathcal{H}^j_{\mathrm{DR}} \) to simplify the notation. In the notation of \( \text{L.4} \) set
\[
\mathcal{C}_f := \mathcal{B}_f/\mathcal{N}_f.
\]
From (1.4.11) we can deduce that
\[
\text{Supp } \mathcal{C}_f \subset \{0\},
\]
(2.1.2)
since \( \text{Supp } \mathcal{M}_j^{(\alpha)} \subset \{0\} \) \( (\alpha \leq 0) \). Hence \( \mathcal{C}_f \) is isomorphic to an infinite direct sum of (copies of) \( \mathcal{B}_{(0)} := \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}] \) by a special case of Kashiwara’s equivalence as in \( \text{Ma 75} \), see also \( \text{Sa 22} \). We then get for \( j \in \mathbb{Z} \) the commutative diagram of short exact sequences
\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{H}^j_{\mathrm{DR}} V^\alpha \mathcal{N}_f & \to & \mathcal{H}^j_{\mathrm{DR}} V^\alpha \mathcal{B}_f & \to & \mathcal{H}^j_{\mathrm{DR}} V^\alpha \mathcal{C}_f & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}^j_{\mathrm{DR}} \mathcal{N}_f & \to & \mathcal{H}^j_{\mathrm{DR}} \mathcal{B}_f & \to & \mathcal{H}^j_{\mathrm{DR}} \mathcal{C}_f & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{H}^j_{\mathrm{DR}} \mathcal{N}_f/V^\alpha \mathcal{N}_f & \to & \mathcal{H}^j_{\mathrm{DR}} \mathcal{B}_f/V^\alpha \mathcal{B}_f & \to & \mathcal{H}^j_{\mathrm{DR}} \mathcal{C}_f/V^\alpha \mathcal{C}_f & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]
(2.1.3)
since \( \mathcal{H}^j_{\mathrm{DR}} \mathcal{C}_f = 0 \ (j \neq 0) \) by (2.1.2) (similarly with \( \mathcal{C}_f \) replaced by \( V^\alpha \mathcal{C}_f \), \( \mathcal{C}_f/V^\alpha \mathcal{C}_f \)) and \( \mathrm{DR}_X(\mathcal{N}_f)^j = 0 \ (j > 0) \). Indeed, these imply the exactness of the rows and the right column,
and that of the middle column follows from the strictness of the filtration \( V \) on the complex \( DR_X(\mathcal{N}_f) \) explained above. We then get the exactness of the left column by the nine lemma.

The last exactness means the strictness of the filtration \( V \) on the complex \( DR_X(\mathcal{N}_f) \), and the diagram implies the strict injectivity of the canonical morphism
\[
(\mathcal{H}^0_{\text{DR}}\mathcal{N}_f, V) \hookrightarrow (\mathcal{H}^0_{\text{DR}}\mathcal{B}_f, V) = (\mathcal{G}_f, V),
\]
where the last isomorphism follows from (2.1.1) and a remark after it. Then \( \mathcal{H}^0_{\text{DR}}\mathcal{N}_f \) is identified with the image of \( \Omega^n_{X,0} \) in \( \mathcal{G}_f \), since the de Rham complex is the Koszul complex for the actions of the \( \partial_v \) fixing the coordinates \( x_1, \ldots, x_n \). Moreover the image of \( \Omega^n_{X,0} \) is \( \mathcal{H}^n_f \), since that of \( \Omega^n_{X,0} \) is \( \mathcal{H}^n_f \). We thus get the canonical isomorphisms
\[
(\mathcal{H}^n_f, V) = (\mathcal{H}^0_{\text{DR}}\mathcal{N}_f, V).
\]

We have also the following isomorphism for \( \alpha \in \mathbb{Q} \) using a similar diagram whose columns are the short exact sequence \( 0 \to V^{>\alpha} \to V^\alpha \to \text{Gr}^\alpha_V \to 0 \):
\[
\mathcal{H}^n_{f}^{\alpha}) = \mathcal{H}^0_{\text{DR}}\mathcal{N}_f^{\alpha})
\]
\[
\mathcal{G}_f^{\alpha}) = \mathcal{H}^0_{\text{DR}}\mathcal{N}_f^{\alpha}).
\]

Here the last isomorphism follows from a remark after (2.1.1).

In the case \( \alpha \notin \mathbb{Z} \), the equality (1) then follows from Theorem 1.4 together with (1.4.8) in Proposition 1.4. Indeed, the functor \( \mathcal{H}^0_{\text{DR}} \) induces an equivalence between the abelian category of regular holonomic \( \mathcal{D}_X \)-modules supported on \( \{0\} \) and that of finite dimensional \( \mathbb{C} \)-vector spaces (in particular, it commutes with the image of morphisms). Note that the Verdier-type extension theorem is used in the proof of Theorem 1.4 where the image of \( N \) corresponds to the minimal extension, which has length 1. This is the reason for which we have to consider module the image of \( N \) on the right-hand side of (1.4.7).

In the case \( \alpha \in \mathbb{Z} \), we can replace \( \mathcal{H}^{(1)}_f \) with \( \mathcal{H}^{(1)}_f \) (which corresponds to the vanishing cycle complex \( \phi_{f,v}^\infty \mathbb{C}_X[n-1] \)) using (1.4.9) in Proposition 1.4. Indeed, the image by the functor \( \mathcal{H}^0_{\text{DR}} \) does not change by this (using (1.4.9), (1.4.11)). On the other hand, the length of the \( \mathcal{D}_{X,0} \)-module on the right-hand side of (1.4.7) decreases by \( r_f \delta_{\alpha,1} \) (using (1.4.10), (1.4.12)). The assertion then follows by an argument similar to the case \( \alpha \notin \mathbb{Z} \). This completes the proof of Theorem 1.

2.2. Proof of Theorem 2. By Theorem 1, it is enough to calculate the saturation \( \mathcal{H}^n_f \). We denote by \( [x^\nu dx] \) the image of \( x^\nu dx \) in \( \mathcal{H}^n_f \). Here \( x^\nu := \prod_{i=1}^n x_{v_i}^{w_i} \) for \( \nu = (\nu_i) \in \mathbb{N}^n \), and \( dx := dx_1 \wedge \cdots \wedge dx_n \). Setting \( v(x^\nu) = v(\nu) := \sum_{i=1}^n w_i(\nu_i+1) \), we see that
\[
(\partial_t-v(\nu))[x^\nu dx] = c_{\nu} \partial_t [hx^\nu dx],
\]
(2.2.1)
that is,
\[
(t-v(\nu)\partial_t^{-1})[x^\nu dx] = c_{\nu} [hx^\nu dx],
\]
with \( c_{\nu} \in \mathbb{C}^* \) by the same argument as in [Sa89], [Sa21] (using (1.1.2), (1.1.3) with the Euler field \( \xi \) satisfying \( \xi(g) = g \)). Combining this with Theorem 1, Remark 1.2(1) and (1.2.1), we then get the assertions (i) and (ii). This finishes the proof of Theorem 2.

Example 2.2. Let \( a_i \in [1, 6] \) be mutually prime odd positive integers. Set
\[
f = g + h \quad \text{with} \quad g = \sum_{i=1}^6 x_i^{2a_i}, \quad h = \prod_{i=1}^6 x_i^{(a_i-1)/2}.
\]
Then the non-shifted weighted degree of \( h \) is strictly larger than 1, that is
\[
\sum_{i=1}^6 \frac{a_i-1}{2a_i} = \frac{3}{2} - \frac{1}{4} \sum_{i=1}^6 \frac{1}{a_i} > 1,
\]
(2.2.2)
since
\[ \sum_{i=1}^{6} \frac{1}{a_i} \leq \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} < 1. \]
Moreover we have
\[ (2.2.3) \quad v(h^2) = 3, \]
and 3 is the only integral spectral number of g, that is, there is no \( \nu = (\nu_i) \in \mathbb{N}^6 \) with \( \nu_i \in [0, 2a_i - 2] \) \( (i \in [1, 6]) \) and \( v(\nu) \in \mathbb{Z} \) except the case \( \nu_i = a_i - 1 \) \( (i \in [1, 6]) \), since the \( a_i \) are mutually prime. So \( f \) gives an example where the assumption of Theorem 2 \( (ii) \) is not a necessary condition for the inequality \( (5) \) to hold. (Here we apply \( (2.2.1) \) to \([dx]\) and \([hdx]\).)

**Remark 2.2a.** Let \( \pi: \tilde{Z} \to Z \) be a desingularization such that \( E := \pi^{-1}(0) \) is a divisor with normal crossing. Then we have the equality
\[ (2.2.4) \quad \dim_{\mathbb{C}} H^{n-2}(E, \mathcal{O}_E) = n_{f,1}. \]

Indeed, the mapping cone of the inclusion
\[ (2.2.5) \quad \Omega^*_Z(\log E)(-E) \hookrightarrow \Omega^*_Z(\log E) \]
gives the Hodge complex of the link \( L_{Z,0} \) of \( Z \) at 0 forgetting the weight filtration, where the Hodge filtration is defined by the truncations \( \sigma \geq p \). By the strictness of the Hodge filtration \( F \), this implies that
\[ (2.2.6) \quad \dim_{\mathbb{C}} H^{n-2}(E, \mathcal{O}_E) = \dim_{\mathbb{C}} \text{Gr}^0_H H^{n-2}(L_{Z,0}, \mathbb{C}). \]

On the other hand, it is known (see for instance [FPS21 (2.3.1)]) that
\[ (2.2.7) \quad \dim_{\mathbb{C}} \text{Gr}^0_H H^{n-2}(L_{Z,0}, \mathbb{C}) = \dim_{\mathbb{C}} \text{Gr}^1_H H^{n-1}(F_f, \mathbb{C})_1, \]
where \( H^{n-1}(F_f, \mathbb{C})_1 \) denotes the unipotent monodromy part of the vanishing cohomology with \( F_f \) the Milnor fiber. (Note that \( \text{Gr}^1_H H^{n-1}(F_f, \mathbb{C})_1 \) is contained in the kernel of \( \text{Gr}_F N \), since \( \text{Gr}^0_H H^{n-1}(F_f, \mathbb{C})_1 = 0 \).) By the definition of spectral numbers (see for instance [JKSY22 §1.1]) we thus get the equality \( (2.2.3) \) using the symmetry of spectral numbers \( (1.2.3) \).

**Remark 2.2b.** It is known that the multiplicity of the minimal spectral number is 1, see [DiSa12 §4.11]. This implies that \( n_{f,1} = 1 \) if \( Z \) has a non-rational Du Bois singularity at 0. In this case we have a positive answer to the conjecture in [BiSc18 Conj. 1.7], since the Brieskorn lattice is a (B)-lattice in the sense of [Sa89].

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