Statistical entropy of three-dimensional q-deformed Kerr-de Sitter space

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Abstract

A quantum deformation of three-dimensional de Sitter space was proposed in hep-th/0407188. We use this to calculate the entropy of Kerr-de Sitter space, using a canonical ensemble, and find agreement with the semiclassical result.

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I. INTRODUCTION

Black holes are known to carry entropy proportional to their horizon area \([1, 2]\). One of the main successes of string theory has been to provide a microscopic interpretation of this entropy \([3]\), at least in certain cases. Since the semiclassical arguments for this horizon area are not specific to horizons surrounding black holes, they should also apply to cosmological horizons \([4, 5, 6]\). It would be of great interest to have microscopic state-counting arguments for such situations.

The natural place to investigate cosmological horizons is de Sitter space, the maximally symmetric spacetime with constant positive cosmological constant \(\Lambda \) \([7, 8, 9, 10, 11]\). Current observations suggest that our universe is now \(\Lambda\)-dominated, and thus asymptotically de Sitter in the future \([12, 13, 14]\).

Following the great success of the AdS/CFT correspondence \([15]\), there have been suggestions of a dual conformal field theory, living on the conformal boundary of de Sitter \([8, 16]\). Unlike the anti-de Sitter case, this boundary is spacelike, and time-translation in the bulk corresponding to scale transformation on the boundary \([17, 18, 19]\). It also has two disconnected components, and it is not clear whether the boundary theory should live on one or both \([7]\).

Since the area of an observer’s cosmological horizon is finite, de Sitter has finite Bekenstein-Hawking entropy. This immediately causes problems with finding a state counting interpretation, since the isometry group is non-compact \([20]\) and hence only has infinite dimensional unitary representations. This apparent contradiction is even stronger if the dimension of the Hilbert space is also finite \([21, 22, 23]\). It has been suggested that the correct inner product is not the naive local one, which changes the notion of unitarity \([24, 25]\). (Another approach is given in \([26, 27]\).)

Some of these difficulties might be tamed by noncommutative geometry \([28, 29]\). The approach used in \([30]\) is to deform the group of isometries into a quantum group. This was further studied in \([31, 32, 33]\) and is the approach followed here. Quantum deformations of the Lorentz group in various dimensions are studied in \([34, 35]\) and in \([36, 37]\).

The plan of the paper is as follows. Section \([\text{III}]\) of this paper is mostly a recap of \([30, 31]\); section \([\text{III}]\) contains a novel calculation of the entropy. The interpretation of the result is discussed and concluding remarks made in section \([\text{IV}]\)
II. Q-DEFORMED DE SITTER SPACE

Three-dimensional de Sitter space can be defined as the hyperboloid \(-(x^0)^2 + \sum_{i=1}^{3}(x^i)^2 = \ell^2\) in Minkowski space. This is a spacetime of constant curvature, representing a vacuum with positive cosmological constant \(\Lambda = 1/\ell^2\).

The isometries of this hyperboloid are just the rotations and boosts of the embedding space, which generate the Lorentz group \(SO(3,1)\). We will focus on the Lie algebra, rather than the global properties of the group. The complex combinations of generators \(X_i = J_i + iK_i\) (left) and \(\overline{X}_i = J_i - iK_i\) (right) each obey the \(su(2)\) commutation relation \([X_i, X_j] = i\epsilon_{ijk}X_k\), and commute with each other.

In the complex algebra, the fact that \(J\) and \(K\) are Hermitian is encoded in the star operation \(J^* = J, K^* = K\), and so \(X_i^* = \overline{X}_i\). The use of this star-structure specifies that we are dealing with the non-compact real form \(so(3,1)\). We will also use the basis given by:

\[
\begin{align*}
L_0 &= X_1 \\
L_1 &= X_2 - iX_3 \\
L_{-1} &= -X_2 - iX_3
\end{align*}
\]

These generators form the \(n = 0 \pm 1\) part of the Virasoro algebra \([L_m, L_n] = (m-n)L_{m+n}\), but with real form

\[L^*_n = -\overline{L}_n.\]  

A field in de Sitter space will transform under isometries in some representation of this algebra. Since the group is non-compact, there are no finite-dimensional unitary representations, thus any field has infinitely many modes. For a field of mass \(m > \ell\) the representation is in the principal series \([38, 39, 40]\).

It was proposed in \([30]\) that the Lie algebra of isometries should be deformed to a quantum group (Hopf algebra) \([41, 42]\). Taking the deformation parameter to be a root of unity

\[q = e^{2\pi i/N}\]

limits the dimension of an irreducible representation to at most \(N\). In particular, the deformed versions of non-compact algebras can have finite dimensional unitary representations, which become infinite in the classical limit \(q \rightarrow 1\) \([43, 44]\). This was done explicitly for dS_2’s.
so(2, 1) principal series in [30]. The relation between $N$ and gravity quantities will be fixed momentarily.

In dS$_3$ however there is a complication which does not arise in dS$_2$: even the deformed algebra cannot have non-trivial unitary representations [31]. Suppose $|\psi\rangle$ is an eigenstate of $L_0$ and $\overline{L}_0$ in a unitary representation. Then the state $L_{\pm 1}|\psi\rangle$ has zero norm, since $L_1^\ast$ does not lower the eigenvalue $L_1$ raised. So the representation must be trivial. (In the infinite-dimensional principal series representation, such a $|\psi\rangle$ lies outside the Hilbert space.)

Similar problems with unitarity are found in [37] in attempting to deform this and higher Lorentz groups, and multi-parameter families of deformations were studied in [36].

These algebraic problems are related to the problem of defining an inner product for fields on de Sitter space, which in turn induces a particular adjoint. The standard local Klein-Gordon one, which induces (1). Witten proposed to use the path integral from asymptotic past to future, with an extra insertion of CPT [24]. Choosing the parity operation to be $P x_3 = -x_3$, [31] showed that this induces

$$L_n^\dagger = -L_n, \quad \overline{L}_n^\dagger = -\overline{L}_n$$

(2)
or $X_{1,2}^\dagger = -X_{1,2}$, $X_3^\dagger = X_3$ and the same on the right. This amounts to using the (non-compact) split real form $su(1, 1) \oplus su(1, 1)$, instead of $so(3, 1)$. [48]

With this real form, the natural deformation of the algebra to use is

$$U_q(su(1, 1)) \oplus U_q(su(1, 1)).$$

The quantum group $U_q(su(1, 1))$ has unitary representations of dimension $N$. These are representations without highest weight, having $(X_\pm)N \neq 0$, and are called cyclic representations (B in [41]). It was shown in [30] that the parameters of a cyclic representation can be chosen so as to give the same Casimirs as the classical $su(1, 1) = so(2, 1)$ principal series, and in [31] that a left-right product of two cyclic representations has the correct Casimirs to match the $so(3, 1)$ principal series.

The geodesics lying in the embedding space’s 0-1 plane are the north and south poles of de Sitter space. The south pole is $r = 0$ in the static coordinate patch, whose metric is

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1 - r^2/\ell^2} + r^2 d\phi^2.$$  (3)

The generator of time translations here is

$$-i\partial_t = K_1 = -i(L_0 + \overline{L}_0).$$
At the antipodal point $-x^\mu$ this generates instead reverse time translation. (This is the standard situation for a thermofield double, the canonical example of which is Rindler space.)

In these coordinates the horizon is at $r = \ell$. It has Hawking temperature $T = 1/2\pi$ which can be derived most transparently for our purposes by tracing over modes living behind the horizon (which have negative frequency) to produce southern density matrix $\rho_{\text{south}} \propto e^{-\beta K_1}$.

In the classical (principal series) case this operator has a continuous spectrum, while a single irreducible cyclic representation of the quantum group it has eigenvalues spaced approximately $1/\ell$ apart. So it was proposed in [31] that the appropriate quantum representations are not the cyclic representations $\mathcal{B}$, of dimension $N$, but rather reducible representations $\bigoplus_{i=1}^N \mathcal{B}_i$ of dimension $N^2$. There is one phase parameter of the cyclic representation not fixed by matching the principal series’s Casimirs, and the sum is over different choices of this phase. In the resulting twisted representation, $-i(L_o + \mathcal{T}_0)$ has eigenvalues spaced $\sim 1/N\ell$, thus tending to a continuum in the classical limit.

The natural choice for $N$ is the de Sitter radius in Planck units: we set

$$N = \frac{\ell}{G}.$$ 

The maximum eigenvalue of $J_1$, the generator of rotations about the south pole, is of order $N$, so this can be viewed as allowing only those rotations which move the most distant points by at least one Planck length. (The same $N$ would be obtained by the argument used in [33]. There, the semiclassical entropy is used to obtain an estimate of the mass gap, which is then matched to the spacing of the Hamiltonian’s eigenvalues.)

Note that the dimension of the twisted representation, $N^2 = \ell^2/G^2$, is equal to the ratio of the Planck density $1/G^3$ to the vacuum energy density $\Lambda/G$. Thus the dimension of the Hilbert space associated with a single twisted representation is essentially one bit per unit Planck volume. In the next section, the horizon entropy (one bit per unit Planck area) is identified as the entropy of a thermal ensemble inside the full Hilbert space built out of tensor products of these representations.
Kerr-de Sitter is obtained by placing a spinning point mass at the origin of the static patch, changing the metric to

\[ ds^2 = -N dt^2 + \frac{dr^2}{N} + r^2 \left( d\phi - \frac{4GJ}{r^2} dt \right)^2 \]  

(4)

where

\[ N = M - \frac{r^2}{\ell^2} + \frac{16G^2 J^2}{r^2}. \]

The point has mass \( E = (1 - M)/8G \) and angular momentum \( J \). There is still only one horizon, at radius

\[ r = r_+ = \frac{1}{2} \left( \sqrt{\tau} + \sqrt{\bar{\tau}} \right) \ell \]

where \( \tau = M + i8GJ/\ell \). It carries entropy

\[ S = \frac{A}{4G} = \frac{\pi \ell}{4G} \left( \sqrt{\tau} + \sqrt{\bar{\tau}} \right). \]  

(5)

This space is a quotient of pure dS\(_3\) by a discrete group, so is locally the same. In particular, it has the same \( \Lambda \) and the same Lie algebra of isometries. We therefore use the same quantum deformation of this algebra, including the same \( \alpha \).

As for rotating black holes in flat space [46], the rotation creates an angular potential \( \Omega \) conjugate to \( J \), in addition to the temperature \( T \). The Boltzmann factor becomes [25]

\[ e^{-\frac{M+\Omega J}{T}} = e^{\beta i L_0 + \bar{\beta} \bar{L}_0}, \]

where the complex inverse temperature is given by

\[ \beta = \frac{1 + i \Omega}{T} = \frac{2\pi \ell}{\sqrt{\tau}}. \]

Now consider a field living in the above twisted representation. We propose that the microscopic CFT is formulated in terms of elementary degrees of freedom in the twisted representation discussed above. We will also assume that we are in a regime where free-field calculations in the CFT suffice to give a good approximation to the entropy. In this case, the multiparticle thermodynamic averages are as follows [49]:

\[ \langle -i L_0 \rangle = \sum_{-iL_0 > 0} e^{\beta i L_0} (-i L_0) \approx N \ell \int_0^{\infty} dE \frac{N \ell}{\beta^2}. \]
The states \(-iL_0 < 0\) are the ones traced over to produce this thermal behaviour. (\(\sim\) here means equal up to numerical factors of order 1.) Similarly \(\langle -i\bar{L}_0 \rangle \sim N\ell/\beta\). Notice that unitarity (1) is restored at the level of thermal expectation values: \(\langle i\bar{L}_0 \rangle\) is the complex conjugate of \(\langle iL_0 \rangle\), matching \((iL_0)^* = +i\bar{L}_0\).

Using the relationships \(\langle -iL_0 \rangle = -\frac{\partial}{\partial \beta} \log Z\) and \(\langle -i\bar{L}_0 \rangle = -\frac{\partial}{\partial \beta} \log Z\), where \(Z\) is the partition function, we can then calculate the entropy as follows:

\[
S = \left(1 - \beta \frac{\partial}{\partial \beta} - \overline{\beta} \frac{\partial}{\partial \overline{\beta}} \right) \log Z \sim N\ell \left(\frac{1}{\beta} + \frac{1}{\overline{\beta}}\right)
\]

matching the semiclassical result (5) up to an overall numerical factor [50].

IV. CONCLUSIONS

In this paper, a quantum deformed CFT has been proposed as the holographic dual to a theory of gravity in a de Sitter background. This accounts for the Bekenstein-Hawking horizon entropy of de Sitter spacetime. The functional dependence on three independent parameters: cosmological constant, mass, and angular momentum, was reproduced precisely. This should be regarded as a very interesting success of this approach, as other approaches to quantizing gravity in a de Sitter background lead to divergent horizon entropy. One of the interesting features of this construction is that the entropy is really to be thought of as a thermal entropy, or more precisely the entropy in a canonical ensemble with fixed temperature and angular potential. On the other hand, the microcanonical entropy, with fixed total mass and angular momentum, will not agree with the canonical ensemble. Instead, the fundamental degrees of freedom are non-unitary with respect to the standard inner product of quantum fields in de Sitter space, which leads to imaginary angular momenta. Only when they are combined as an ensemble with fixed angular potential does the average total angular momentum agree with the macroscopic value of the Kerr de Sitter space.

These facts point to the instability of de Sitter spacetime. The analog of heat baths are needed for the fixed temperature and angular potential ensemble to make sense. The fact that we cannot ignore the presence of these heat baths at large \(N\) and obtain agreement with the microcanonical ensemble suggests that the CFT is not a complete self-contained description of quantum gravity in a de Sitter background. Including other degrees of freedom becomes a necessity, which then opens the door to the complete theory describing more than...
just asymptotically de Sitter spacetime.

It would be interesting to understand whether these facts relate to the metastability of de Sitter backgrounds in string theory \[47\]. The hope is that the type of formulation of dS/CFT described in the present work will provide an effective description of physics around such backgrounds for timescales smaller than the lifetime of the de Sitter phase.

Acknowledgments

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[48] Throughout this paper we use ⋆ for the so(3,1) involution [11], and † for this one.

[49] Here we assume Boltzmann statistics for simplicity, Fermi-Dirac or Bose-Einstein changes overall numerical factors only. Note also that for convenience the trivial tensor product is used rather than the usual coproduct of the quantum group [42]. We have checked via
numerical calculation that this does not change the expression for the entropy for large $N$.

[50] In AdS/CFT there is a similar discrepancy in overall numerical coefficient in comparing the weakly coupled CFT calculation at finite temperature with the Bekenstein-Hawking entropy of the black hole.