ON THE $K$-THEORY OF ELLIPTIC CURVES

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Abstract. Let $A$ be the coordinate ring of an affine elliptic curve (over an infinite field $k$) of the form $X - \{p\}$, where $X$ is projective and $p$ is a closed point on $X$. Denote by $F$ the function field of $X$. We show that the image of $H^\bullet(\text{GL}_2(A), \mathbb{Z})$ in $H^\bullet(\text{GL}_2(F), \mathbb{Z})$ coincides with the image of $H^\bullet(\text{GL}_2(k), \mathbb{Z})$. As a consequence, we obtain numerous results about the $K$-theory of $A$ and $X$. For example, if $k$ is a number field, we show that $r_m(\text{K}_2(A) \otimes \mathbb{Q}) = 0$, where $r_m$ denotes the $m$th level of the rank filtration.

1. Introduction

Computing the $K$-theory of a scheme $X$ is a very difficult task. Even the simplest case $X = \text{Spec } k$, where $k$ is a field, is not completely solved, although a great deal is known. The next case to consider is when $X$ is a curve over $k$, and it is here that the complexity grows rapidly. Some curves of genus zero present no real difficulty thanks to the fundamental theorem: $K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R)$ for $R$ regular. The $K$-theory of elliptic curves, on the other hand, has proved to be much more elusive.

A great deal of recent work has focused on the construction of specific elements in the $K$-theory of elliptic curves, particularly in the second group $K_2$. This program goes back to the work of S. Bloch [2], who constructed a regulator map on $K_2$ and used it to find nontrivial elements. A. Beilinson [1] generalized this construction and made a number of conjectures relating the dimension of $K_2 \otimes \mathbb{Q}$ with the values of $L$-functions on the curve. More recently, Goncharov–Levin [6], Rolshausen–Schappacher [10], and Wildeshaus [15] have made further progress in this area.

In this paper we consider the following situation. Let $E$ be an affine elliptic curve defined by the Weierstrass equation $F(x, y) = 0$, where

$$F(x, y) = y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6.$$ 

Here, the $a_i$ lie in an infinite field $k$. Denote by $\overline{E}$ the projective curve $E \cup \{\infty\}$ and by $F$ the function field of $\overline{E}$. Denote by $A$ the affine coordinate ring of $E$; it is a Dedekind domain with field of fractions $F$. We have $A^\times = k^\times$. 

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Consider the obvious embedding \( i : GL_2(A) \to GL_2(F) \). The main result of this paper is the following.

**Theorem 1.1.** The image of the map 
\[
i_\ast : H_\bullet(GL_2(A),\mathbb{Z}) \to H_\bullet(GL_2(F),\mathbb{Z})
\]
coincides with the image of 
\[
(i|_{GL_2(k)})_\ast : H_\bullet(GL_2(k),\mathbb{Z}) \to H_\bullet(GL_2(F),\mathbb{Z}).
\]

This is a consequence of an explicit computation of the homology of \( PGL_2(A) \) due to the author [3] (recalled in Section 4 below). The proof of Theorem 1.1 is given in Section 3.

**Remark.** Theorem 1.1 and its corollaries in Sections 2 and 3 are valid also for singular cubic curves \( F(x,y) = 0 \). We shall point out the necessary modifications needed to prove this below.

From this result we deduce a number of facts about the \( K \)-theory of \( E \) and \( \overline{E} \). Recall the rank filtration of the rational \( K \)-theory \( K_\bullet(R)_\mathbb{Q} := K_\bullet(R) \otimes \mathbb{Q} \) of a ring \( R \):
\[
r_mK_n(R)_\mathbb{Q} = \text{im}\{H_n(GL_m(R),\mathbb{Q}) \to H_n(GL(R),\mathbb{Q})\} \cap K_n(R)_\mathbb{Q}.
\]

**Corollary 1.2.** The image of the map \( r_2K_n(A)_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q} \) coincides with the image of \( r_2K_n(k)_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q} \).

In particular, when \( n = 2 \) we see that the image of \( r_2K_2(A)_\mathbb{Q} \to r_2K_2(F)_\mathbb{Q} \) coincides with the image of \( K_2(k)_\mathbb{Q} \).

**Remark.** This corollary is valid for any field \( k \). Indeed, if \( k \) is finite, then the rational homology \( H_\bullet(GL_2(A),\mathbb{Q}) \) vanishes in positive degrees (as does \( H_\bullet(GL_2(k),\mathbb{Q}) \)) from which it follows that \( r_2K_2(A)_\mathbb{Q} = 0 \).

Define a filtration \( r_\ast K_\bullet(\overline{E})_\mathbb{Q} \) by pulling back the rank filtration of \( K_\bullet(A)_\mathbb{Q} \):
\[
r_mK_n(\overline{E})_\mathbb{Q} := (f_\ast)^{-1}(r_mK_n(A)_\mathbb{Q}),
\]
where \( f : E \to \overline{E} \) is the inclusion and \( f_\ast : K_\bullet(\overline{E}) \to K_\bullet(E) = K_\bullet(A) \) is the induced map in \( K \)-theory. Then we obviously have the following result.

**Corollary 1.3.** The image of \( r_2K_n(\overline{E})_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q} \) coincides with the image of \( r_2K_n(k)_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q} \).

We study the filtration \( r_\ast \) in greater detail in Section 3. In Section 3 we specialize to the case where \( k \) is a number field. In this case, we show that \( r_2K_2(A)_\mathbb{Q} = 0 \).

In the case \( n = 2 \), results of Nesterenko–Suslin [3] imply that \( r_3K_2(A)_\mathbb{Q} = K_2(A)_\mathbb{Q} \). A description of the homology of \( PGL_3(A) \) (or \( GL_3(A) \)) would provide a great deal of insight into the structure of \( K_2(\overline{E})_\mathbb{Q} \), especially over a number field. Such a computation remains elusive, however.

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2. The Rank Filtration

The rational \(K\)-groups of affine schemes admit the rank filtration mentioned in the introduction. Since \(BGL(R)^+\) is an \(H\)-space, the Milnor–Moore Theorem \([8]\) implies that the Hurewicz map

\[ h : \pi_\bullet(BGL(R)^+) \otimes \mathbb{Q} \rightarrow H_\bullet(GL(R), \mathbb{Q}) \]

is injective with image equal to the primitive elements of the homology. The rank filtration is the increasing filtration defined by

\[ r_m K_n(R) = \mathrm{im}\{H_n(GL_m(R), \mathbb{Q}) \rightarrow H_n(GL(R), \mathbb{Q})\} \cap K_n(R). \]

By Theorem 2.7 of \([9]\), the map \(H_2(GL_3(A), \mathbb{Z}) \rightarrow H_2(GL(A), \mathbb{Z})\) is surjective so that \(r_3 K_2(A) = K_2(A)\). The rank filtration of \(K_2(A)\) then has the form

\[ 0 = r_1 K_2(A) \subseteq r_2 K_2(A) \subseteq r_3 K_2(A) = K_2(A) \]

(the vanishing of \(r_1\) is a consequence of the vanishing of \(r_1 K_2(k)\) for infinite fields \([9]\), and the fact that \(A^\times = k^\times\)).

Define an increasing filtration \(r_\bullet\) of \(K_n(\overline{E})\) as follows. Let \(f : E \rightarrow \overline{E}\) be the canonical inclusion and denote by \(f^\ast\) the induced map on \(K\)-theory. We define \(r_m K_n(\overline{E})\) by

\[ r_m K_n(\overline{E}) = (f^\ast)^{-1}(r_m K_n(A)). \]

There is a commutative diagram

\[ \begin{array}{ccc}
  r_m K_n(\overline{E}) & \rightarrow & r_m K_n(A) \\
  \downarrow & & \downarrow \\
  r_m K_n(F) & \rightarrow & r_m K_n(F) \\
\end{array} \] \tag{1}

**Proposition 2.1.** The image of \(r_2 K_n(A) \rightarrow r_2 K_n(F)\) coincides with the image of \(r_2 K_n(k) \rightarrow r_2 K_n(F)\).

**Proof.** By Theorem \([11]\), the image of

\[ i_* : H_\bullet(GL_2(A), \mathbb{Z}) \rightarrow H_\bullet(GL_2(F), \mathbb{Z}) \]

coincides with the image of \((i|_{GL_2(k)})_*\). Consider the commutative diagram

\[ \begin{array}{ccc}
  H_n(GL_2(A), \mathbb{Q}) & \rightarrow & H_n(GL(A), \mathbb{Q}) \\
  \downarrow & & \downarrow \\
  H_n(GL_2(F), \mathbb{Q}) & \rightarrow & H_n(GL(F), \mathbb{Q}) \\
\end{array} \]

It follows that the image of \(H_n(GL_2(A), \mathbb{Q})\) in \(H_n(GL(F), \mathbb{Q})\) coincides with the image of \(H_n(GL_2(k), \mathbb{Q})\); i.e., the image of \(r_2 K_n(A) \rightarrow r_2 K_n(F)\) coincides with the image of \(r_2 K_n(k)\). \(\square\)

**Corollary 2.2.** The image of \(r_2 K_n(\overline{E}) \rightarrow r_2 K_n(F)\) coincides with the image of \(r_2 K_n(k)\).

**Proof.** This follows by considering the diagram \([11]\). \(\square\)
3. THE NUMBER FIELD CASE

Suppose that the ground field $k$ is a number field. By localizing the projective curve at its generic point we obtain the following exact sequence for $K_2$

$$0 \rightarrow K_2(E)_\mathbb{Q} \rightarrow K_2(F)_\mathbb{Q} \xrightarrow{T} \bigoplus_P K_1(k(P))_\mathbb{Q}$$

where $P$ varies over the closed points of $E$ and $k(P)$ is the residue field at $P$. The map $T$ is the tame symbol (see, e.g., [10]).

Remark. It is not known for a single curve if $K_2(E)_\mathbb{Q}$ is finite dimensional. Beilinson has conjectured that the dimension of this space is related to special values of $L$-functions on $E$. This conjecture was modified by Bloch and Grayson [3] to predict that the dimension is the number of infinite places of $k$ plus the number of primes $p \subset \mathcal{O}_k$ where $E$ has split multiplicative reduction modulo $p$. For a discussion of this see, for example, [10].

We also have the localization sequence for $A$:

$$\cdots \rightarrow K_{i+1}(F) \rightarrow \bigoplus_{p \text{ maximal}} K_i(A/p) \rightarrow K_i(A) \rightarrow K_i(F) \rightarrow \cdots .$$

Since $A/p$ is a finite extension of $k$ for all $p$, the groups $K_{2m}(A/p)$ are torsion. It follows that we have an exact sequence

$$0 \rightarrow K_{2m}(A)_\mathbb{Q} \rightarrow K_{2m}(F)_\mathbb{Q} \rightarrow \bigoplus_P K_{2m-1}(A/p)_\mathbb{Q} .$$

Proposition 3.1. If the ground field $k$ is a number field, then the map $K_2(E)_\mathbb{Q} \rightarrow K_2(A)_\mathbb{Q}$ is injective.

Proof. This follows by considering the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & K_2(A)_\mathbb{Q} \\
\uparrow & & \| \\
0 & \rightarrow & K_2(E)_\mathbb{Q} & \rightarrow & K_2(F)_\mathbb{Q} .
\end{array}
$$

Proposition 3.2. If $k$ is a number field, then $r_2K_2(A)_\mathbb{Q} = 0 = r_2K_2(E)_\mathbb{Q}$.

Proof. The map $K_2(A)_\mathbb{Q} \rightarrow K_2(F)_\mathbb{Q}$ is injective. But by Proposition 2.1, the image of $r_2K_2(A)_\mathbb{Q}$ coincides with the image of $r_2K_2(k)_\mathbb{Q} = K_2(k)_\mathbb{Q} = 0$.

As a consequence we see that any nontrivial elements of $K_2(A)_\mathbb{Q}$ (and hence of $K_2(E)_\mathbb{Q}$) must come from $H_2(GL_3(A), \mathbb{Q})$. Thus, to prove that $K_2(E)_\mathbb{Q}$ is a finite dimensional vector space, it suffices to show that the image of $H_2(GL_3(A), \mathbb{Q})$ in $H_2(GL_3(F), \mathbb{Q}) = H_2(GL(F), \mathbb{Q})$ is finite dimensional.
4. The Homology of $PGL_2(A)$

The remainder of the paper is devoted to the proof of Theorem 1.1. We begin by recalling the calculation of $H_4(PGL_2(A),\mathbb{Z})$ given in [7]. The proof uses the action of $PGL_2(A)$ on a certain Bruhat–Tits tree $\mathcal{X}$.

We use the description of $\mathcal{X}$ given by Takahashi [14]. Recall that $A$ is the coordinate ring of the affine curve $E$ with function field $F$. The field $F$ has transcendence degree 1 over $k$ and is equipped with the discrete valuation at $\infty$, $v_\infty$. Denote by $O_\infty$ the valuation ring and by $t = x/y$ the uniformizer at $\infty$. Denote by $L$ the field of Laurent series in $t$ and let $v$ be the valuation on $L$ defined by $v(\sum_{n \geq n_0} a_n t^n) = n_0$. The ring $A$ can be embedded in $L$ in such a way that $v(x) = -2$ and $v(y) = -3$; we identify $A$ with its image in $L$. Note that this embedding induces an embedding $F \to L$ and that the completion of $F$ with respect to $v_\infty$ is $L$. We therefore have a commutative diagram

$$GL_2(A) \longrightarrow GL_2(F) \downarrow \downarrow GL_2(L).$$

Let $G = GL_2(L)$ and $K = GL_2(k[[t]])$. Denote by $Z$ the center of $G$. The Bruhat–Tits tree $\mathcal{X}$ is defined as follows. The vertex set of $\mathcal{X}$ is the set of cosets $G/KZ$. Two cosets $g_1KZ$ and $g_2KZ$ are adjacent if

$$g_1^{-1}g_2 = \begin{pmatrix} t & b \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mod KZ$$

for some $b \in k$. The graph $\mathcal{X}$ is a tree [12], p. 70. Note that $GL_2(A)$ acts on $\mathcal{X}$ without inversion and that the center of $GL_2(A)$ (which is equal to $k^\times$) acts trivially on $\mathcal{X}$. It follows that the quotient $PGL_2(A) \backslash \mathcal{X}$ is defined. We describe a fundamental domain $\mathcal{D} \subset \mathcal{X}$ for the action (i.e., $\mathcal{D} \cong PGL_2(A) \backslash \mathcal{X}$).

If $f_1, f_2 \in L$, denote by $\phi(f_1, f_2)$ the vertex $\begin{pmatrix} f_1 & f_2 \\ 0 & 1 \end{pmatrix}KZ$. Denote by $F_x(l, m)$ and $F_y(l, m)$ the partial derivatives at $(l, m)$ of the Weierstrass equation $F(x, y)$. Define two sets $E_1$ and $E_2$ as follows:

$$E_1 = \{(l, m) : F(l, m) = 0 \text{ and } F_y(l, m) = 0\} \cup \{\infty\}$$

and

$$E_2 = \{(l, m) : F(l, m) = 0 \text{ and } F_y(l, m) \neq 0\}.$$
Figure 1. $F(l, y) = 0$ has no rational solutions

Figure 2. $F(l, y) = 0$ has a unique rational solution

Observe that $E = E_1 \cup E_2$. Define vertices of $X$ by

$$
o = \phi(t, t^{-1});$$

$$v(l) = \begin{cases} 
\phi(t^2, t^{-1} + lt) & \text{if } l \in k \\
\phi(1, t^{-1}) & \text{if } l = \infty;
\end{cases}$$

$$c(p, n) = \begin{cases} 
\phi(t^n + \frac{y-m}{x-l}) & \text{if } p = (l, m) \in E \\
\phi(t^{-n}, 0) & \text{if } p = \infty;
\end{cases}$$

$$e(p) = \begin{cases} 
\phi(t^4, \frac{y-m}{x-l} + \frac{F(l,m)}{y-m}) & \text{if } p = (l, m) \in E_1 \\
\phi(1,0) & \text{if } p = \infty.
\end{cases}$$

We are now ready to describe the subgraph $D$. For each $l \in k \cup \{\infty\}$, the vertex $v(l)$ is adjacent to $o$. Denote by $D(l)$ the connected component of $D - \{o\}$ which contains $v(l)$. The $D(l)$ fall into three types.

1. Suppose $F(x, y) = 0$ has no rational solution with $x = l$. Then $D(l)$ consists only of $v(l)$ (see Figure 1).

2. Suppose $l = \infty$ or $F(x, y) = 0$ has a unique rational solution with $x = l$. Let $p$ be the point at infinity of $E$ or the rational point corresponding to the solution. Note that $p$ is a point of order 2. Then $D(l)$ consists of an infinite path $c(p, 1), c(p, 2), \ldots$ and an extra vertex $e(p)$ (see Figure 2).

3. Suppose $F(x, y) = 0$ has two different solutions such that $x = l$. Let $p, q$ be the corresponding points on $E$. Then $D(l)$ consists of two infinite paths $c(p, 1), c(p, 2), \ldots$ and $c(q, 1), c(q, 2), \ldots$ (see Figure 3).

The infinite path $c(p, 1), c(p, 2), \ldots$ is called a cusp. Note that there is a one-to-one correspondence between cusps and the rational points of $E$.

**Theorem 4.1 (Takahashi).** The graph $D$ is a fundamental domain for the action of $GL_2(A)$ on $X$ (and hence is also a fundamental domain for the action of $PGL_2(A)$).

**Remark.** The theorem is true also for singular curves $C$ given by $F(x, y) = 0$ with the following modification. If the curve is singular at $p = (l, m)$, then the vertex $e(p)$ is the same as $c(p, 2)$. In this case, then, the tree $D(l)$ consists only of the cusp $c(p, 1), c(p, 2), \ldots$. The proofs of the following results for
C then go through unchanged except that the summands in the homology decomposition of $H_\bullet(PGL_2(k[C]), \mathbb{Z})$ corresponding to singular points are $H_\bullet(k^\times, \mathbb{Z})$ instead of $H_\bullet(PGL_2(k), \mathbb{Z})$.

Since $X$ is contractible, we have a spectral sequence with $E^1$–term

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset D} H_q(\Gamma_\sigma, \mathbb{Z}) \Rightarrow H_{p+q}(PGL_2(A), \mathbb{Z})$$

where $\Gamma_\sigma$ is the stabilizer of the $p$–simplex $\sigma$ in $PGL_2(A)$. We shall discuss the stabilizers in detail in the next section. For the purpose of computing homology, the next result is sufficient (see [14], Theorem 5). If $F(l, y) = 0$ has no rational solution, denote by $k(\omega)$ the quadratic extension of $k$ in which $F(l, \omega) = 0$.

**Proposition 4.2.** Up to isomorphism, the stabilizers $\Gamma_\sigma$ are as follows:

- $\Gamma_o = \{1\}$
- $\Gamma_v(l) \cong \begin{cases} k(\omega)^\times/k^\times & \text{in case (1)} \\ k & \text{in case (2)} \\ k^\times & \text{in case (3)} \end{cases}$
- $\Gamma_{c(p,q)} \cong \left\{ \begin{pmatrix} p & v \\ 0 & q \end{pmatrix} : p, q \in k^\times, v \in k^n \right\}/k^\times$
- $\Gamma_{c(p)} \cong PGL_2(k)$.

The stabilizer of an edge is the intersection of its vertex stabilizers (one of which is contained in the other).
By Theorem 1.11 of [9], the inclusion of the diagonal subgroup into $\Gamma_{c(p,n)}$ induces an isomorphism in homology. This leads to the proof of the following, which is the main result of [7].

**Theorem 4.3.** For all $i \geq 1$,

$$H_i(PGL_2(A), \mathbb{Z}) \cong \bigoplus_{l \in k \cup \{\infty\}} H_i(PGL_2(k), \mathbb{Z})$$

- $F(l,y) = 0$ has unique sol.
- $\bigoplus_{l \in k} H_i(k^\times, \mathbb{Z})$ for $F(l,y) = 0$ has two sol.
- $\bigoplus_{l \in k} H_i(k(\omega)^\times / k^\times, \mathbb{Z})$ has no sol.

**Remark.** This theorem holds also in degrees $\leq 2$ if $k$ is a finite field with at least 4 elements. For in this case, the inclusion of the diagonal subgroup into $\Gamma_{c(p,n)}$ induces a homology isomorphism in degrees $\leq 2$; see [11], p. 204.

In the next section, we shall compute the image of the map $H_\bullet(PGL_2(A), \mathbb{Z}) \rightarrow H_\bullet(PGL_2(F), \mathbb{Z})$.

5. **The Map** $H_\bullet(PGL_2(A), \mathbb{Z}) \rightarrow H_\bullet(PGL_2(F), \mathbb{Z})$

To compute the image of $H_\bullet(PGL_2(A), \mathbb{Z})$ in $H_\bullet(PGL_2(F), \mathbb{Z})$, we must examine the various $\Gamma_v(l)$ and $\Gamma_e(p)$. In other stabilizers for $l \neq \infty$ are not subgroups of $PGL_2(k)$, although they are isomorphic to such. We have the following result.

**Theorem 5.1.** For each $l \in k$, the stabilizers $\Gamma_v(l)$ and $\Gamma_e(p)$ ($p = (l, m)$) are conjugate in $PGL_2(F)$ to subgroups of $PGL_2(k)$.

**Corollary 5.2.** The image of $j_* : H_\bullet(PGL_2(A), \mathbb{Z}) \rightarrow H_\bullet(PGL_2(F), \mathbb{Z})$ coincides with the image of $H_\bullet(PGL_2(k), \mathbb{Z})$.

**Proof.** It is well-known (see [3], p. 48) that conjugation induces the identity on homology. It follows that if $H_1, H_2$ are conjugate subgroups of a group $G$, then the images of $H_\bullet(H_1, \mathbb{Z}) \rightarrow H_\bullet(G, \mathbb{Z})$ coincide. Since each stabilizer which appears in the homology decomposition of $PGL_2(A)$ is conjugate in $PGL_2(F)$ to a subgroup of $PGL_2(k)$, the result follows.

**Proof of Theorem 5.1.** To keep the notation as simple as possible, we only prove the case $\Gamma_v(0)$ and in the case $F(0, 0) = 0 = F_y(0, 0), \Gamma_{y(0,0)}$. All other
cases are similar (but notationally more complex). For \( r_1, \ldots, r_4 \in k \) define
\[
M_2(r_1, r_2) = \begin{pmatrix} r_2 y + r_1 & -r_2 \left( \frac{y^2 + a_3 y - a_6}{x} \right) \\ r_2 x & -r_2 y - a_3 r_2 + r_1 \end{pmatrix}
\]
and
\[
M_4(r_1, r_2, r_3, r_4) = \begin{pmatrix} r_4 xy + r_3 (x^2 + a_2 x + a_4) + r_2 y + r_1 & -r_4 y^2 - r_3 y (x + a_2) + a_4 r_4 (x + a_2) + r_2 (x^2 + a_2 x + a_4 - a_1 y) \\ r_4 x^2 + r_3 (y + a_1 x) + r_2 x + a_4 r_4 & -r_4 xy - r_3 (x^2 + a_2 x + a_4) + r_2 y + a_1 a_4 r_4 + a_4 r_3 + r_1 \end{pmatrix}.
\]
According to Proposition 9 of [14], the stabilizer of \( v(0) \) in \( GL_2(\mathcal{A}) \) is
\[
\tilde{\Gamma}_{v(0)} = \{ M_2(r_1, r_2) : r_1 (-a_3 r_2 + r_1) - a_6 r_2^2 \neq 0 \},
\]
and of \( e(0, 0) \) is
\[
\tilde{\Gamma}_{e(0, 0)} = \{ M_4(r_1) : r_1 (a_4 r_3 + r_1) + (-a_2 a_4 r_4 + a_1 a_4 r_3 + a_4 r_2 a_1 r_1) a_4 r_4 \neq 0 \}.
\]
Consider the following identity:
\[
\begin{pmatrix} x & -y \\ 0 & 1 \end{pmatrix} M_2(r_1, r_2) \begin{pmatrix} 1/x \\ 0 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_1 - a_3 r_2 \end{pmatrix} = N_2(r_1, r_2).
\]
It follows that
\[
\begin{pmatrix} x & -y \\ 0 & 1 \end{pmatrix} \tilde{\Gamma}_{v(0)} \begin{pmatrix} 1/x \\ 0 \end{pmatrix} = \{ N_2(r_1, r_2) : \det N_2(r_1, r_2) \neq 0 \} = \tilde{\Gamma}.
\]
Note that the subgroup \( \tilde{\Gamma} \) lies in \( GL_2(k) \) and that \( g = \begin{pmatrix} x & -y \\ 0 & 1 \end{pmatrix} \) is an element of \( GL_2(F) \). It follows that \( g\tilde{\Gamma}_{v(0)} g^{-1} = \tilde{\Gamma}/k^\times \subset PGL_2(k) \) inside \( PGL_2(F) \). Moreover, we can demonstrate the isomorphism of Proposition [12] as follows. If \( F(0, y) = 0 \) has no rational solutions, then define a map \( \Gamma \to k(\omega)^\times \) by \( N_2(r_1, r_2) \mapsto r_1 + r_2 \omega \). One checks easily that this is an isomorphism. If \( F(0, y) = 0 \) has two solutions, say \( u, v \in k \), then it is easy to see that
\[
\begin{pmatrix} u-1 & uv \\ u(v-1) & u(v-1) \end{pmatrix} \tilde{\Gamma} \begin{pmatrix} u-1 \\ u(v-1) \end{pmatrix}^{-1} = D(k)
\]
where \( D(k) \subset GL_2(k) \) is the subgroup of diagonal matrices. Finally, if \( F(0, y) = 0 \) has a unique solution, say \( u \in k^\times \), then
\[
\begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix} \tilde{\Gamma} \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}^{-1} = B(k)
\]
where \( B(k) \) is the upper triangular subgroup of \( GL_2(k) \).
For the group $\Gamma_{e(0,0)}$ we have
\[
\begin{pmatrix}
x & -y \\
a_4 & 1 - \frac{y^2}{a_4 x}
\end{pmatrix} \Gamma_{e(0,0)} \begin{pmatrix}
x & -y \\
a_4 & 1 - \frac{y^2}{a_4 x}
\end{pmatrix}^{-1} = GL_2(k)
\]
from which it follows that $\Gamma_{e(0,0)}$ is conjugate to $PGL_2(k)$ inside $PGL_2(F)$. (Note that since $E$ is smooth, $a_4 \neq 0$.)

\section*{6. Proof of Theorem 1.1}

We now prove Theorem 1.1. Corollary 5.2 shows that $H_\bullet(PGL_2(A),\mathbb{Z})$ has image equal to the image of $H_\bullet(PGL_2(k),\mathbb{Z})$ in $H_\bullet(PGL_2(F),\mathbb{Z})$. Consider the following commutative diagram
\[
\begin{array}{ccl}
1 & \rightarrow & k^\times \\
\downarrow & & \downarrow \\
1 & \rightarrow & F^\times \\
\downarrow & & \downarrow \\
1 & \rightarrow & GL_2(A) & \rightarrow & PGL_2(A) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & & & \\
& & & & & & \\
E^2_{p,q}(A) & = & H_p(PGL_2(A),H_q(k^\times)) & \Rightarrow & H_{p+q}(GL_2(A),\mathbb{Z}) \\
E^2_{p,q}(F) & = & H_p(PGL_2(F),H_q(F^\times)) & \Rightarrow & H_{p+q}(GL_2(F),\mathbb{Z}).
\end{array}
\]

Since the extensions are central, the groups $H_q(k^\times)$ (resp. $H_q(F^\times)$) are trivial $PGL_2(A)$ (resp. $PGL_2(F)$) modules. Hence we have the following commutative diagram of universal coefficient sequences
\[
\begin{array}{ccl}
& & H_p(PGL_2(A),H_q(k^\times)) & \Rightarrow & Tor^2(H_{p-1}(PGL_2(A)),H_q(k^\times)) \\
H_p(PGL_2(F),H_q(F^\times)) & \Rightarrow & Tor^2(H_{p-1}(PGL_2(F)),H_q(F^\times)).
\end{array}
\]

By Corollary 5.2, we see that the image of $E^2_{p,q}(A) \rightarrow E^2_{p,q}(F)$ coincides with the image of $E^\infty_{p,q}(k) \rightarrow E^\infty_{p,q}(F)$. It follows that the same is true of the $E^\infty$ terms:
\[
\text{im}\{E^\infty_{p,q}(A) \rightarrow E^\infty_{p,q}(F)\} = \text{im}\{E^\infty_{p,q}(k) \rightarrow E^\infty_{p,q}(F)\}.
\]

Thus, the image of $H_\bullet(GL_2(A),\mathbb{Z})$ in $H_\bullet(GL_2(F),\mathbb{Z})$ coincides with the image of $H_\bullet(GL_2(k),\mathbb{Z})$.

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