Bergman representative coordinate and constant holomorphic curvature

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Abstract

Linking two uniformization results of the Lu and Suita types, we study the Bergman representative coordinate on bounded pseudoconvex domains in \( \mathbb{C}^n \) with the Bergman metric of constant holomorphic sectional curvature, and characterize such domains that are biholomorphic to a ball possibly less a relatively closed pluripolar set. Sufficient conditions for the continuous extension of the biholomorphisms to the closures are given in terms of the Bergman kernel and applied particularly to the Riemann maps of bounded, simply-connected planar domains.

1 Introduction

Bergman discovered that the Riemann map associated to a bounded simply-connected domain \( D \) in the complex plane \( \mathbb{C} \) can be expressed very simply in terms of his kernel function \( K(z, w) \). For some fixed \( p \in D \), after a dilation and translation,

\[
K^{-1}(z, p) \left. \frac{\partial}{\partial t} \right|_{t=p} K(z, t)
\]

is a biholomorphic map from \( D \) onto the unit disc around the origin (see [4] and [10, Chap. VI]). Bergman in [9] introduced his representative coordinates as a tool of generalizing the Riemann mapping theorem to \( \mathbb{C}^n, n \geq 1 \). Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain whose Bergman metric is denoted by \( g \). Relative to a point \( p \in \Omega \), the Bergman representative coordinate \( T(z) = (w_1, ..., w_n) \) is defined as

\[
w_{\alpha}(z) := \sum_{j=1}^{n} g^{\bar{\alpha}j}(p) \left( K(z, p)^{-1} \left. \frac{\partial}{\partial t} \right|_{t=p} K(z, t) - \left. \frac{\partial}{\partial t} \right|_{t=p} \log K(t, t) \right), \quad \alpha = 1, \cdots, n, \quad (1.1)
\]

where \( K(\cdot, p) \) is the Bergman kernel and \( g^{\bar{\alpha}j} \) are the entries of the inverse of the matrix \([g_{\alpha j}]\) associated with the Bergman metric. Since the possible obstruction in (1.1) is that \( K(\cdot, p) \) may have zeros, \( T(z) \) is only generally defined and holomorphic outside the zero set of \( K(\cdot, p) \). Studying zeros of the Bergman kernel attracted much interest, and domains for which the Bergman kernel is zero-free are known as Lu Qi-Keng domains (or those satisfying the Lu Qi-Keng conjecture, cf. [16]), after Lu’s well-known paper [44] on his uniformization theorem. In that paper, Lu proved that for a bounded domain in \( \mathbb{C}^n \) with a complete Bergman metric of constant holomorphic...
sectional curvature, the Bergman representative coordinate is a biholomorphism that maps $\Omega$ to a Euclidean ball. Alternative proofs of Lu’s theorem are available by following Bergman’s key idea that biholomorphic mappings become linear when represented in his representative coordinates, cf. [33, 58]. See also [20] for a simplification of Lu’s proof by Cheung and the second author. Lu’s theorem also played a decisive role in the resolution of the Cheng conjecture, which asserts that for a smoothly bounded strictly pseudoconvex domain, the Bergman metric is Kähler-Einstein if and only if the domain is biholomorphic to a ball, and was recently confirmed by Huang and Xiao [39], after the previous works of Fu and the second author [31] and Nemirovski and Shafikov [45]. In all the above proofs, the completeness of the Bergman metric was crucial. Since a bounded domain which is complete with respect to the Bergman metric is pseudoconvex (see [17], and the Bergman completeness holds for any bounded pseudoconvex domain with Hölder boundary in $\mathbb{C}^n$ (see [1, 19]), in this paper we mostly focus on domains with worse boundaries.

On the other hand, the past decade has witnessed the remarkable progresses around the sharp $L^2$ extension theorems of the Ohsawa-Takegoshi type (see [11, 13, 37, 50], the survey papers [14, 48, 49, 59], and the references therein), which particularly solved a long-standing conjecture of Suita [53]. The Green’s function on a hyperbolic Riemann surface $X$ induces the logarithmic capacity $c_\beta$, which is defined as

$$c_\beta(z_0) = \lim_{z \to z_0} \exp(G(z, z_0) - \log |w(z)|),$$

where $w$ is a fixed local coordinate in a neighborhood of $z_0 \in X$ with $w(z_0) = 0$. Denote by $K$ the Bergman kernel on the diagonal for holomorphic 1-forms on $X$. Suita’s conjecture (now a theorem of Błocki, and Guan and Zhou) asserts that for any surface $X$ as above, it holds that

$$\pi K \geq c_\beta^2, \quad (1.2)$$

and equality in (1.2) holds true at some $z \in X$ if and only if $X$ is biholomorphic to a disc possibly less a relatively closed polar set. Here, a polar set is the local singularity set of a subharmonic function. Since the possible polar part is negligible for $L^2$ holomorphic 1-forms, the biholomorphism can be expressed in terms of the Bergman representative coordinate (see [47, Chap. 4]). Due to an identity $(\log c_\beta)_{zz} = \pi K$ proved by Suita in the same paper [53], his conjecture has geometric interpretation: the inequality part in (1.2) is equivalent to that the Gaussian curvature of the invariant metric $c_\beta^2(z)dz \otimes d\overline{z}$ always has an upper bound $-4$; the equality/rigidity part, which says that the curvature attains $-4$ at some $z \in X$, guarantees that the surface is necessarily as asserted with the curvature identically $-4$. Similar curvature property is satisfied for the Carathéodory metric (see [54, 57]); additional results on metrics of constant Gaussian curvature were obtained by the first author in [24]. Notice that neither boundary regularity nor completeness of the metric is assumed in the above Suita type problems. Moreover, Błocki and Zwonek [15] obtained a multidimensional version of the Suita conjecture (see also [2, 36] for related comparison results).

Linking the above two uniformization results of the Lu and Suita types, we aim to provide a good curvatural characterization of pseudoconvex domains that are biholomorphic to a ball possibly less a relatively closed pluripolar set. This will not only extend Lu’s theorem towards the Bergman-incomplete situation but also generate higher dimensional results analogous to the equality part of Suita’s conjecture. In [28], the authors initiated such an attempt by using ideas from Lu’s original paper as well as Calabi’s concept of diastasis, and substituted the Bergman
completeness with other conditions before dropping it completely. There, the authors showed that a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose Bergman metric has negative constant holomorphic sectional curvature is a Lu Qi-Keng domain, and such $\Omega$ must be biholomorphic to a ball possibly less a relatively closed pluripolar set if there exists some point $p \in \Omega$ such that

1) $|K(z,p)|$ is bounded from above by a finite constant $C > 0$ for any $z \in \Omega$, and

2) the Bergman representative coordinate $T$ defined at $p$ is continuous up to $\overline{\Omega}$.

In this paper, we are able to drop the above second condition for domains in $\mathbb{C}$.

**Theorem 1.1.** For a domain $\Omega \subset \mathbb{C}$ whose Bergman metric has Gaussian curvature identically equal to $-2$, it holds that

1) If there exists some point $p \in \Omega$ such that $|K(z,p)|$ is bounded from above by a finite constant $C_1 > 0$ for any $z \in \Omega$, then the Bergman representative coordinate $T(z)$ relative to $p$ is biholomorphic from $\Omega$ to a disc possibly less a relatively closed polar set, and $T$ extends continuously up to $\overline{\Omega}$.

2) Under the same assumption as in 1), if $|K(z,p)|$ is bounded from below by a finite constant $C_2 > 0$ for any $z \in \Omega$, then $T$ extends to a homeomorphism of the closures.

Theorem 1.1 improves Theorem 1.3 in [28] for the case of planar domains. The conditions in Theorem 1.1 relate to an important result of Kerzman [41], who used Kohn’s theory of the $\partial$-Neumann problem to show that on a bounded strictly pseudoconvex domain $\Omega$ with $C^\infty$-smooth boundary, for each fixed $p \in \Omega$, the Bergman kernel $K(\cdot,p)$ is $C^\infty$ up to the boundary. In [41, p.151-152], he gave an example of a simply-connected planar domain whose Bergman kernel $K(\cdot,p)$ blows up to infinity at some boundary point; see also [30] for an example of Fornæss.

Let $D$ be a bounded simply-connected domain in $\mathbb{C}$, and let $S : \mathbb{D} \to D$ be the Riemann map with $S(0) = 0$ and $S'(0) > 0$, where $\mathbb{D}$ denotes the unit disc around the origin. It is well known that $S$ extends continuously up to $\overline{\mathbb{D}}$ if and only if $\partial D$ is a continuous curve, and a celebrated theorem of Carathéodory [18] states that $S$ extends to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{D}$ if and only if $\partial D$ is a Jordan curve. In the latter case, if the Jordan curve is $C^\infty$-smooth, then $S$ and all its derivatives extend continuously to $\overline{\mathbb{D}}$ (see [6]). The corollary below gives sufficient conditions for the extension of the Riemann map to the closures.

**Corollary 1.2.** Let $D$ be a bounded, simply-connected domain in $\mathbb{C}$. Then, for any $p \in D$, the Bergman representative coordinate $T(z)$ relative to $p$ is biholomorphic from $D$ to a disc $\mathbb{D}_r := \{w \in \mathbb{C} : |w|^2 < 2g^{-1}(p)\}$, where $g$ is the Bergman metric of $D$. Moreover,

1) If there exists some point $p \in D$ such that $|K(z,p)|$ is bounded from above by a finite constant $C_1 > 0$ for any $z \in D$, then $T$ extends continuously up to $\overline{D}$.

2) If there exists some point $p \in \Omega$ such that $|K(z,p)|$ is bounded from below by a finite constant $C_2 > 0$ for any $z \in D$, then the inverse map of $T$ extends continuously up to $\overline{\mathbb{D}_r}$; in particular, $\partial D$ is a continuous curve.

3) Under the same assumption as in 1), if $|K(z,p)|$ is bounded from below by a finite constant $C_2 > 0$ for any $z \in D$, then $T$ extends to a homeomorphism of the closures, $T : \overline{D} \to \overline{\mathbb{D}_r}$; in particular, $\partial D$ is a Jordan curve.
Condition 3) in Corollary 1.2 is only sufficient but not necessary for the continuous extension to the closures, in view of Kerzman’s example in [41]. Corollary 1.2 also yields a criteria (see Proposition 3.2) for a simply-connected planar domain $D$ to satisfy
\[
\inf_{z \in D} |K(z, p)| = 0,
\]
which may be compared with two examples of Webster in [56]. The conditions in Theorem 1.1 and Corollary 1.2 are entirely about the Bergman kernel, without assuming any topological condition.

Historically, the Bergman representative coordinate provides a right tool to analyze the extension of biholomorphic maps to the boundary. Fefferman [29] proved that biholomorphic maps between two bounded domains in $\mathbb{C}^n$ with smooth, strictly pseudoconvex boundaries admit smooth extensions to the closures of the domains. Previously, it was known that such maps extend to homeomorphisms of the closures (see the papers of Henkin [38] and Vormoor [55]). Webster [56], and Nirenberg, Webster and Yang [46] gave new proof of Fefferman’s mapping theorem, whose original proof used the asymptotic expansion of the Bergman kernel and an analysis of the boundary behavior of the geodesics of the Bergman metric. Later, the Bergman-style coordinates were applied by Bell and Ligocka [7, 43] to prove that subelliptic estimates for the $\bar{\partial}$-Neumann problem imply boundary regularity of biholomorphic maps. See also [3, 22, 30] for the extensions of biholomorphic maps involving more general domains. For more applications of the Bergman representative coordinates, see the papers [4, 8, 12, 23, 34, 42, 58] and the references therein.

For higher dimensional domains, we improve our results in [28] by considering the following technical condition, which is similar to the so-called Condition $R$, cf. [5, 7].

**Definition 1.3.** A domain $\Omega$ in $\mathbb{C}^n$, $n \geq 1$, is said to satisfy Condition ($B$) if there exists some point $p \in \Omega$ such that for each $j \in \{1, \ldots, n\}$,
\[
\left| \frac{\partial}{\partial z_j} K(z, p) \right| \leq C|K(z, p)|, \quad \text{for any } z \in \Omega,
\]
where $C > 0$ is a finite constant.

For a bounded domain $\Omega$ satisfying Condition ($B$), it follows from our Lemma 2.2 that $|K(z, p)|$ is also bounded from above by a finite constant for all $z \in \Omega$, if $K(\cdot, p)$ has no zero set. This further implies that Condition ($B$) is not a biholomorphic invariant (see Remark 3.4). Our next theorem is stated as follows.

**Theorem 1.4.** Let $\Omega \subset \mathbb{C}^n$, $n \geq 1$, be a bounded pseudoconvex domain whose Bergman metric $g$ has its holomorphic sectional curvature identically equal to a negative constant $-c^2$.

1) If $\Omega$ satisfies Condition ($B$) at some point $p \in \Omega$, then the Bergman representative coordinate $T(z)$ relative to $p$ is biholomorphic from $\Omega$ to a ball
\[
B := \{ w \in \mathbb{C}^n : \sum_{a, \beta=1}^n w_\alpha g_{a\bar{\beta}}(p) \overline{w_\beta} < 2c^{-2} \} \quad (1.3)
\]
possibly less a relatively closed pluripolar set, where $n = 2c^{-2} - 1$, and $T$ extends continuously up to $\overline{\Omega}$. 4
2) Under the same assumption as in 1), if \(|K(z, p)|\) is bounded from below by a finite constant \(C_2 > 0\) for any \(z \in \Omega\), then \(T\) extends to a homeomorphism of the closures.

The pseudoconvexity in Theorem 1.4 is a necessary assumption. To see this, remove from \(B^n, n \geq 2\), a compact subset \(G\) of Lebesgue \(\mathbb{R}^{2n}\)-measure zero such that \(B^n \setminus G\) is connected. Then, the Bergman metric on \(B^n \setminus G\) extends to \(B^n\) by Hartogs’ extension theorem. So, the assertion of Theorem 1.4 fails if \(G\) is not pluripolar.

We say a set \(E\) is pluripolar if there exists a plurisubharmonic function \(\varphi\) in \(\mathbb{C}^n\) such that \(\varphi = -\infty\) on \(E\). In view of a result [52] of Siciak, a pluripolar set is negligible for \(L^2\) holomorphic functions. It is known that a bounded \(L^2\)-domain of holomorphy is pseudoconvex and its boundary contains no pluripolar part, cf. [40, 51]; see Example 2.4 for a pseudoconvex domain which is not an \(L^2\)-domain of holomorphy. Theorem 1.4 directly yields the following corollary.

**Corollary 1.5.** Let \(\Omega \subset \mathbb{C}^n\) be a bounded \(L^2\)-domain of holomorphy such that the holomorphic sectional curvature of the Bergman metric on \(\Omega\) is identically equal to a negative constant \(-c^2\).

1) If \(\Omega\) satisfies Condition (B) at some point \(p \in \Omega\), then the Bergman representative coordinate \(T(z)\) relative to \(p\) is biholomorphic from \(\Omega\) to a ball \(B\), and \(T\) extends continuously up to \(\overline{\Omega}\).

2) Under the same assumption as in 1), if \(|K(z, p)|\) is bounded from below by a finite constant \(C_2 > 0\) for any \(z \in \Omega\), then \(T\) extends to a homeomorphism of the closures, \(\tilde{T} : \overline{\Omega} \to \overline{B}\).

The assumptions on the Bergman kernel in Part 2) of both Theorem 1.4 and Corollary 1.5 can be further weakened as stated in our Theorem 2.5 and Corollary 2.6. Our last result gives an estimate of the Bergman kernel on bounded domains in \(\mathbb{C}^n\) with the Bergman metric of constant holomorphic sectional curvature.

**Theorem 1.6.** Let \(\Omega\) be a bounded domain whose Bergman metric has constant holomorphic sectional curvature in \(\mathbb{C}^n, n \geq 1\). Then, for any \(p \in \Omega\), there exists its small neighborhood \(U\) such that

\[
\sup_{\zeta \in \overline{U}} |K(z, \zeta)| \leq 2 |K(z, p)|, \quad \text{for any } z \in \Omega. \tag{1.4}
\]

For any domain in \(\mathbb{C}^n\) admitting the Bergman metric, we show in Proposition 4.1 that if there exists a point \(p\) and its neighborhood \(U\) such that (1.4) holds true, then the Bergman representative coordinate relative to \(p\) becomes a bounded holomorphic map.

The organization of the paper is as follows. In Section 2, after deriving some inequalities for domains whose Bergman metric has constant holomorphic sectional curvature, we prove our multidimensional results. In Section 3, we obtain one dimensional results, including those on the extension of the Riemann map to the boundaries of simply-connected planar domains. In Section 4, we study the boundedness of the Bergman representative coordinate.

### 2 Proofs of multidimensional results

Recall that a bounded domain \(\Omega \subset \mathbb{C}^n\) is called a Lu Qi-Keng domain if for any \(p \in \Omega\), its Bergman kernel \(K(\cdot, p)\) has no zero set, cf. [16, 44]. The authors in [28] showed that if its (not necessarily complete) Bergman metric \(g\) has constant holomorphic sectional curvature \(-c^2\), then the domain
\( \Omega \) is Lu Qi-Keng and the Bergman representative coordinate \( T \) relative to \( p \) maps \( \Omega \) to a ball \( \mathcal{B} \) defined by (1.3). Previously, Lu’s theorem in [44] yields the same conclusions under the additional assumption that \( \Omega \) is Bergman complete. One key step in [28] was the use of Calabi’s concept of diastasis on a bounded domain \( \Omega \subset \mathbb{C}^n \). Fix a point \( z_0 \in \Omega \) and let \( A_{z_0} := \{ z \in \Omega \mid K(z, z_0) = 0 \} \) be the zero set of the Bergman kernel \( K(\cdot, z_0) \). Since \( A_{z_0} \) is an analytic variety, as domains \( \Omega \setminus A_{z_0} \) and \( \Omega \) have the same Bergman kernel \( K \) and Bergman metric \( g \). Consider on \( \Omega \setminus A_{z_0} \) the Kähler potential

\[
\Phi_{z_0}(z) := \log \frac{K(z, z)K(z_0, z_0)}{|K(z, z_0)|^2}
\tag{2.1}
\]

for the Bergman metric \( g = \partial \bar{\partial} \Phi_{z_0} \), and we call the function \( \Phi_{z_0}(z) \) the Bergman-Calabi diastasis relative to \( z_0 \). The idea in [28, 44] was to investigate locally the Taylor expansion of the Kähler potentials for the Bergman metric. More precisely, at any \( p \in \Omega \) with

\[
g_{\alpha \beta}(p) = \delta_{\alpha \beta},
\tag{2.2}
\]

there exists a neighborhood \( U_p \) such that the Bergman kernel on the diagonal can be decomposed as

\[
K(z, z) = \left( 1 - \frac{c^2}{2} |T(z)|^2 \right)^{-\frac{n}{2}} e^{f(T(z))+\bar{f}(\bar{T}(z))}, \quad z \in U_p,
\tag{2.3}
\]

where \( f \) is holomorphic on \( U_p \). Let \( \Omega' := \{ z \in \Omega \setminus A_p : T(z) \in \mathbb{B}^n \} \) denote the set of points in \( \Omega \setminus A_p \) that are mapped into a ball \( \mathbb{B}^n(0, \sqrt{2}c^{-1}) := \{ (w_1, ..., w_n) : |w|^2 < 2c^2 \} \), where \( A_p \) is the zero set of \( K(\cdot, p) \). Then, by (2.3) and the theory of power series, one may duplicate the variable with its conjugate so that the full Bergman kernel can be complex analytically continued as

\[
K(z, z_0) = \left( 1 - \frac{c^2}{2} \sum_{\alpha=1}^n w_\alpha(z)w_\alpha(z_0) \right)^{-\frac{n}{2}} e^{f(T(z))+\bar{f}(\bar{T}(z_0))}, \quad z, z_0 \in U_p.
\tag{2.4}
\]

Moreover, for any \( z, z_0 \in U_p \), it holds that

\[
\Phi_{z_0}(z) = \frac{-2}{c^2} \log \left[ \left( 1 - \frac{c^2}{2} |T(z)|^2 \right) \left( 1 - \frac{c^2}{2} |T(z_0)|^2 \right) \left( 1 - \frac{c^2}{2} \sum_{\alpha=1}^n w_\alpha(z)w_\alpha(z_0) \right)^{-2} \right], \quad z \in U_p.
\]

Then, the symmetry of the Bergman-Calabi diastasis defined in (2.1) further yields that

\[
\Phi_p(z_0) = \Phi_{z_0}(p) = \frac{-2}{c^2} \log \left( 1 - \frac{c^2}{2} |T(z_0)|^2 \right).
\]

Since on \( \Omega' \) both \( \Phi_p(z) \) and \( \frac{-2}{c^2} \log \left( 1 - \frac{c^2}{2} |T(z)|^2 \right) \) are well-defined, these two real-analytic functions coincide on \( U_p \), and thus are identical to each other on \( \Omega' \). That is,

\[
\Phi_p(z) = \frac{-2}{c^2} \log \left( 1 - \frac{c^2}{2} |T(z)|^2 \right), \quad z \in \Omega'.
\tag{2.5}
\]

One can show by contradiction that no point in \( \Omega \setminus A_p \) is mapped outside the ball \( \mathbb{B}^n(0, \sqrt{2}c^{-1}) \) by \( T \), so \( \Omega' = \Omega \setminus A_p \). Therefore, (2.5) in fact holds on \( \Omega \setminus A_p \). By the Riemann removable singularity
theorem, $T$ extends across the analytic variety $A_p$ to the whole domain $\Omega$ with $|T(z)|^2 \leq 2c^{-2}$, and the maximum modulus principle yields that $|T(z)|^2 < 2c^{-2}$ on $\Omega$. The above explicit formula of $\Phi_p(z)$ also guarantees that the zero set $A_{x_0} = \emptyset$, as shown in [28], so $\Omega$ is a Lu Qi-Keng domain.

Based on these facts, in this section, we first obtain the following estimates.

**Proposition 2.1.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain whose Bergman metric $g$ has its holomorphic sectional curvature identically equal to a negative constant $-c^2$. Then, for any $p \in \Omega$ satisfying (2.2), there exists a finite constant $C_p > 0$ such that for each $\alpha, j = 1, \ldots, n$, it holds that

$$\left| K(z, p) \frac{\partial w_\alpha(z)}{\partial z_j} \right| \leq \left| \frac{\partial^2}{\partial z_j \partial \bar{t}_\alpha} \right|_{t=p} K(z, t) + C_p \left| \frac{\partial K(z, p)}{\partial z_j} \right|, \quad \forall z \in \Omega. \quad (2.6)$$

**Proof.** By the above discussion, the Bergman representative coordinate $T$ relative to $p$ maps $\Omega$ to a ball $\mathbb{B}^n(0, \sqrt{2}c^{-1})$. It then follows that

$$\sum_{j=1}^n K(z, p)^{-1} \left| \frac{\partial}{\partial t_j} \right|_{t=p} K(z, t) - \left| \frac{\partial}{\partial t_j} \right|_{t=p} \log K(t, t) |^2 < 2c^{-2}, \quad \forall z \in \Omega,$$

which further implies that

$$\sum_{j=1}^n K(z, p)^{-1} \left| \frac{\partial}{\partial t_j} \right|_{t=p} K(z, t) \leq \sum_{j=1}^n \left| \frac{\partial}{\partial t_j} \right|_{t=p} \log K(t, t) + n\sqrt{2}c^{-1} =: C_p, \quad \forall z \in \Omega.$$

Here $C_p$ is a finite positive constant depending on $p$ since the Bergman kernel is locally uniformly bounded. Thus, for each $j = 1, \ldots, n$,

$$\left| \frac{\partial}{\partial t_j} \right|_{t=p} K(z, t) \leq C_p |K(z, p)|, \quad \forall z \in \Omega. \quad (2.7)$$

By the definition (1.1), we know that for each $\alpha, j = 1, \ldots, n$,

$$\frac{\partial w_\alpha(z)}{\partial z_j} = \frac{\partial}{\partial z_j} \left\{ K(z, p)^{-1} \left| \frac{\partial}{\partial t_\alpha} \right|_{t=p} K(z, t) \right\}$$

$$= \frac{K(z, p)}{K(z, p)^2} \left| \frac{\partial^2}{\partial z_j \partial t_\alpha} \right|_{t=p} K(z, t) - \left| \frac{\partial K(z, p)}{\partial z_j} \frac{\partial}{\partial t_\alpha} \right|_{t=p} K(z, t)$$

This combined with (2.7) will yield that for any $z \in \Omega$,

$$\left| K(z, p) \frac{\partial w_\alpha(z)}{\partial z_j} \right| \leq \left| \frac{\partial^2}{\partial z_j \partial t_\alpha} \right|_{t=p} K(z, t) + \left| \frac{\partial K(z, p)}{\partial z_j} \frac{\partial}{\partial t_\alpha} \right|_{t=p} K(z, t)$$

$$\leq \left| \frac{\partial^2}{\partial z_j \partial t_\alpha} \right|_{t=p} K(z, t) + C_p \left| \frac{\partial K(z, p)}{\partial z_j} \right|.$$
Next, we will show the following boundedness lemma for domains satisfying Condition (B).

**Lemma 2.2.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n, n \geq 1$. Suppose $\Omega$ satisfies Condition (B) at some point $p \in \Omega$ and $K(z, p) \neq 0$ for any $z \in \Omega$. Then, there exists a finite constant $C_1 > 0$ such that for any $z \in \Omega$,

\[
|K(z, p)| \leq C_1, \\
\sup_{w \in U} \left| \frac{\partial}{\partial z_j} K(z, w) \right| \leq C_1, \quad \forall j \in \{1, \ldots, n\},
\]

where $U$ is a small neighborhood of $p$.

**Proof.** For the first inequality, since $K(\cdot, p)$ is zero free, let $h(z) := 2 \log |K(z, p)|$ be a pluriharmonic function on $\Omega$. Then, Condition (B) implies that for each $j \in \{1, \ldots, n\}$,

\[
\left| \frac{\partial h}{\partial z_j} (z) \right| = \left| \frac{\partial \log(K(z, p)K(z, p))}{\partial z_j} \right| = \left| \frac{\partial}{\partial z_j} K(z, p) \right| \leq C, \quad \forall z \in \Omega,
\]

By the mean value theorem for several variables and the Cauchy-Schwarz inequality, it holds that $h$ is Lipschitz continuous and there exists a finite constant $C_0 > 0$ such that $|h| < C_0$ on $\Omega$. Therefore, $|K(z, p)| = e^{\frac{h(z)}{2}} \leq e^{\frac{h(z)}{2}} \leq e^{\frac{C_0}{2}}, \quad \forall z \in \Omega.$

For the second inequality, note that for each fixed $z \in \Omega$ and $j \in \{1, \ldots, n\}$,

\[
\frac{\partial K(w, z)}{\partial z_j} K(w, z)
\]

is holomorphic in $w$ and

\[
\left| \frac{\partial K(z, w)}{\partial z_j} K(z, w) \right| = \left| \frac{\partial K(w, z)}{\partial z_j} K(w, z) \right|.
\]

By the continuity, for any $\epsilon \in (0, 1)$, there exists a neighborhood $U$ of $p$ such that

\[
\left| \frac{\partial K(w, z)}{\partial z_j} K(w, z) - \frac{\partial K(p, z)}{\partial z_j} K(p, z) \right| \leq \epsilon, \quad \text{whenever } w \in U.
\]

Thus, by Condition (B), it holds that

\[
\sup_{w \in U} \left| \frac{\partial K(w, z)}{\partial z_j} K(w, z) \right| \leq \left| \frac{\partial K(p, z)}{\partial z_j} K(p, z) \right| + \epsilon \leq C + 1.
\]

Since $z$ is arbitrary, we have completed the proof by taking $C_1 := \max\{e^{\frac{C_0}{2}}, C + 1\}$. 

\[\square\]

**Lemma 2.2** implies that Condition (B) is not satisfied for the examples of Fornæss [30] and Kerzman [41] (see also Remark 3.4). One can also check that in Kerzman’s example the determinant of the Jacobian of a Bergman representative coordinate is unbounded (see [28]). We will need the following theorem to prove our Theorem 1.4.
Theorem 2.3. [28] Let $\Omega \subset \mathbb{C}^n$ be a bounded domain whose Bergman metric has its holomorphic sectional curvature identically equal to a negative constant $-c^2$. Then, for some $z_0 \in \Omega$, $\Phi_{z_0}(z)$ blows up to infinity at $\partial \Omega$ if and only if $\Omega$ is biholomorphic to a ball and $n = 2/c^2 - 1$.

Under the constant holomorphic sectional curvature assumption, two proofs of the above equivalence using and not using Lu’s theorem were given in [28]. For a general bounded domain $\Omega$, as demonstrated in [28, Proposition 3.1], the fact that for some $z_0 \in \Omega$, $\Phi_{z_0}$ blows up to infinity at $\partial \Omega$ does not imply the completeness of the Bergman metric of $\Omega$.

Proof of Theorem 1.4. Part 1) We first assume that $\Omega$ satisfies Condition $(B)$ at some $p$ with (2.2). Choose a small neighborhood $U$ of $p$ as in Lemma 2.2 and take a small polydisc $\mathbb{D}^n(p; r_p) \subset U$ for some $0 < r_p \ll 1$. By Cauchy’s integral formula for derivatives and (2.9), for each $\alpha, j = 1, \ldots, n$, it holds that

$$
\left| \frac{\partial}{\partial t_{\alpha}} \right|_{t=p} \left. \frac{\partial}{\partial z_j} K(t, z) \right| \leq \frac{1}{2\pi i} \int_0^{2\pi} \frac{\partial}{\partial z_j} K(p + r_p e^{i\theta}, z) \frac{r_p e^{i\theta}}{r_p^2 K(p + r_p e^{i\theta}, z)} d\theta
$$

$$
\leq \frac{1}{r_p} \sup_U \left| \frac{\partial K(z, \cdot)}{\partial z_j} \right| K(p, z)
$$

$$
\leq \frac{C_1}{r_p}, \quad \forall z \in \Omega.
$$

Thus, by (2.7),

$$
\left| \frac{\partial^2}{\partial z_j \partial t_{\alpha}} \right|_{t=p} K(t, z) = \left| \frac{\partial}{\partial t_{\alpha}} \right|_{t=p} \left. \frac{\partial}{\partial z_j} K(t, z) \right| + \frac{\partial}{\partial t_{\alpha}} \left. \frac{\partial}{\partial z_j} K(p, z) \right|_{t=p} \frac{\partial}{\partial t_{\alpha}} \left. \frac{\partial}{\partial z_j} K(p, z) \right|_{t=p}
$$

$$
\leq \frac{C_1}{r_p} + C \cdot C_p, \quad \forall z \in \Omega.
$$

Therefore, by (2.6), it holds that

$$
\left| \frac{\partial w_{\alpha}(z)}{\partial z_j} \right| \leq \left| \frac{\partial^2}{\partial z_j \partial t_{\alpha}} \right|_{t=p} K(z, t) \left| + C_p \left| \frac{\partial K(z, p)}{\partial z_j} \right|_{t=p} \left. \frac{\partial}{\partial t_{\alpha}} \frac{\partial}{\partial z_j} K(z, p) \right|_{t=p}
$$

$$
\leq \frac{C_1}{r_p} + 2C \cdot C_p, \quad \forall z \in \Omega.
$$

Since $w_{\alpha}(z)$ is a bounded holomorphic function whose partial derivatives are bounded from above by some finite positive constant on $\Omega$, the Bergman representative coordinate $T(z) = (w_1, \ldots, w_n)$ defined by (1.1) is Lipschitz. Thus, $T(z)$ is continuous up to $\overline{\Omega}$ by an elementary calculus argument involving the uniform continuity.

Let $w \in \partial \Omega$ be an arbitrary boundary point such that

$$
\limsup_{z \to w} K(z, z) = \infty.
$$
By Lemma 2.2, there exists a finite constant $C_1 > 0$ such that
\[
\limsup_{\Omega \ni z \to w} \Phi_p(z) \geq \limsup_{\Omega \ni z \to w} \log \frac{K(z, z)K(p, p)}{C_1^2} = \infty. \tag{2.10}
\]
Since $T(z)$ is continuous up to $\overline{\Omega}$, for any two sequences of points $(z_j)_{j \in \mathbb{N}}, (w_j)_{j \in \mathbb{N}} \subset \Omega$, both approaching $w$, it holds that
\[
\lim_{j \to \infty} T(z_j) = \lim_{j \to \infty} T(w_j). \tag{2.11}
\]
By the explicit formula of the Bergman-Calabi diastasis
\[
\Phi_p(z) = -\frac{2}{c^2} \log \left(1 - \frac{c^2}{2}|T(z)|^2\right), \quad z \in \Omega, \tag{2.12}
\]
we know that
\[
\lim_{\Omega \ni z \to w} \Phi_p(z) = \infty.
\]
For any boundary point $q \in \partial \Omega$ such that
\[
\limsup_{\Omega \ni z \to q} K(z, z) < \infty, \tag{2.13}
\]
a result of Pflug and Zwonek [51] says that $q \in \operatorname{int}(\overline{\Omega})$ and there exists a neighborhood $U$ of $q$ such that $P := U \setminus \Omega$ is a pluripolar set. Taking all boundary points $q_j$ that satisfy (2.13), we get the corresponding neighborhoods $U_j$ and pluripolar sets $P_j$. Then the (bounded) domain
\[
\tilde{\Omega} := \bigcup_j U_j \cup \Omega
\]
has the same Lebesgue measure as $\Omega$ due to the pluripolarity. In view of [51], $\partial \tilde{\Omega}$ coincides with the non-pluripolar part of $\partial \Omega$. Moreover, as domains $\tilde{\Omega}$ and $\Omega$ have the same Bergman metric (with constant holomorphic sectional curvature). To see this, notice that $P_j = U_j \cap \partial \tilde{\Omega}$ is relatively closed in $U_j$. So, restricting any function $f \in L^2 \cap \mathcal{O}(\Omega)$ to $U_j \setminus P_j$, we get a function $F \in L^2 \cap \mathcal{O}(U_j)$ such that $F = f$ on $U_j \setminus P_j$. Hence one has an $L^2$ holomorphic extension to $\Omega \cup U_j$, and consequently to $\tilde{\Omega}$. By the above discussions, we know that the Bergman-Calabi diastasis $\Phi_p$ blows up to infinity at $\partial \tilde{\Omega}$, since $w$ is arbitrary. Then, Theorem 2.3 guarantees that $\tilde{\Omega}$ is biholomorphic to a ball and $n = 2/c^2 - 1$. Define the set
\[
E := \bigcup_j P_j = \bigcup_j U_j \cap \partial \tilde{\Omega} = \tilde{\Omega} \cap \partial \Omega,
\]
which is relatively closed in $\tilde{\Omega}$. Since the biholomorphic (pre)images and countable union of pluripolar sets are still pluripolar, we conclude that $T(z)$ is biholomorphic from $\Omega$ to a ball $\mathbb{B}^n(0, \sqrt{2c^{-1}})$ possibly less a relatively closed pluripolar set $E$.

If $\Omega$ satisfies Condition $(B)$ at some general point $p \in \Omega$, let $g_{\alpha\beta}(p)$ be the positive-definite Hermitian matrix associated with the Bergman metric at $p$. One performs a linear transformation $F$ from $\Omega$ to $\Omega^1$ such that the Bergman metric $g^1$ on $\Omega^1$ satisfies $g^1_{\alpha\beta}(F(p)) = \delta_{\alpha\beta}$. Since $F$ is a biholomorphism, the Bergman metric on $\Omega^1$ also has its holomorphic sectional curvature identically equal to a negative constant $-c^2$. Moreover, the Bergman kernels on $\Omega$ and $\Omega^1$ differ by a multiple
constant, which is the determinant square of \( g_{\alpha\beta}(p) \), due to the transformation rule. Therefore, \( \Omega^1 \) satisfies Condition (B) at the point \( F(p) \). By the previous argument, the Bergman representative coordinate \( T^1(z) \) relative to \( F(p) \) is biholomorphic from \( \Omega^1 \) to a ball \( \mathbb{B}^n \) possibly less a relatively closed pluripolar set, and extends continuously up to \( \overline{\Omega}^1 \). Finally, the composition map \( F^{-1} \circ T^1 \circ F \) is the Bergman representative coordinate \( T(z) \) relative to \( p \) and it is biholomorphic from \( \Omega \) to a ball \( B \) defined in (1.3) possibly less a relatively closed pluripolar set \( E \). The continuous extension up to \( \overline{\Omega} \) follows due to the linearity of \( F \).

**Part 2)** Since the pluripolar set \( E \) is negligible for \( L^2 \) holomorphic functions, the transformation formula of the Bergman kernel yields that

\[
K(z,p) = D_T(z) \cdot D_T(p) \cdot K_{B\setminus E}(T(z), \overline{0}) = D_T(z) \cdot K_B(0, \overline{0}) = \frac{D_T(z)}{v(B)}, \quad z \in \Omega,
\]

where \( D_T \) is the determinant of the complex Jacobian of \( T \) and \( v(\cdot) \) denotes the Euclidean volume. Let \( S \) be the inverse map of \( T \). By the inverse function theorem, the complex Jacobian of \( S \) is the inverse of the complex Jacobian of \( T \), and

\[
JS = (JT)^{-1} = \frac{\text{Adj}(JT)}{D_T},
\]

where \( \text{Adj}(JT) \) is the adjugate matrix of \( JT \), the complex Jacobian of \( T \). If \( |K(z,p)| \) is also bounded from below by a positive constant for any \( z \in \Omega \), then so is \( D_T(z) \). Therefore, each component of \( S \) is a bounded holomorphic function whose partial derivatives are bounded from above by some finite constant on \( B \setminus E \). We conclude that \( S \) is Lipschitz and thus continuous up to \( B \setminus E \), which implies that \( T \) extends to a homeomorphism of the closures.

Corollary 1.5 follows directly from Theorem 1.4 as the boundary of a bounded \( L^2 \)-domain of holomorphy, which is the domain of existence of some \( L^2 \) holomorphic function, contains no pluripolar part, cf. [40, 51]. The following example is provided by Peter Pflug, and we present it here with his kind permission.

**Example 2.4.** \( \mathcal{D} := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1, z \neq 0\} \) is an example of a *pseudoconvex* domain that is not an \( L^2 \)-domain of holomorphy. To see this, let \( \mathbb{B}^2 \) be the unit ball in \( \mathbb{C}^2 \) and let \( E := \{(0,w) \in \mathbb{C}^2 : |w| < 1\} \) be a relatively closed pluripolar set of \( \mathbb{B}^2 \). Then, \( \mathcal{D} = \mathbb{B}^2 \setminus E \) and \( \mathcal{D} \) is not an \( L^2 \)-domain of holomorphy since all \( L^2 \) holomorphic functions on \( \mathcal{D} \) extend across \( E \). Indeed, \( \mathcal{D} \) is pseudoconvex. For instance, the function \( f(z,w) = 1/|z| \) is holomorphic on \( \mathcal{D} \), but \( f \) cannot extend across \( E \) on which it blows up.

To conclude the biholomorphisms in Theorem 1.4 and Corollary 1.5, \( \Omega \) satisfying Condition (B) can be weakened to \( \Omega \) being biholomorphic to a domain \( \Omega_1 \) satisfying Condition (B). Moreover, to conclude the homeomorphisms of the closures, the assumptions on the Bergman kernel can be further weakened as follows.

**Theorem 2.5.** Let \( \Omega \subset \mathbb{C}^n, n \geq 1 \), be a bounded pseudoconvex domain whose Bergman metric has its holomorphic sectional curvature identically equal to a negative constant \( -c^2 \). If there exists some point \( p \in \Omega \) such that for any \( z \in \Omega \),

\[
\left| \frac{\partial}{\partial z_j} K(z,p) \right| \leq C_1, \quad |K(z,p)| \geq C_2, \quad \forall j \in \{1, \ldots, n\},
\]

then...
where $C_1, C_2 > 0$ are finite constants, then the Bergman representative coordinate $T(z)$ relative to $p$ is biholomorphic from $\Omega$ to a ball $B$ defined in (1.3) possibly less a relatively closed pluripolar set, where $n = 2c^{-2} - 1$, and $T$ extends to a homeomorphism of the closures.

**Proof.** We first assume that $\Omega$ satisfies (2.15) at some $p$ with (2.2). Similar to the proof of (2.9), one can show by (2.15) that there exists a small neighborhood $U$ of $p$ such that for each $j \in \{1, \ldots, n\}$,

$$
\sup_{w \in U} \left| \frac{\partial}{\partial z_j} K(z, w) \right| \leq C_1 + 1, \quad \forall z \in \Omega.
$$

Then, by (2.6) and (2.15), it follows that for each $\alpha, j = 1, \ldots, n$,

$$
C_1 \left| \frac{\partial w_\alpha(z)}{\partial z_j} \right| \leq \left| \frac{\partial^2}{\partial z_j \partial t_\alpha} \right|_{t=p} K(z, t) + C_p C_1, \quad \forall z \in \Omega.
$$

(2.16)

Similar to the proof of Theorem 1.4, Part 1), one uses Cauchy’s integral formula to show that for each $\alpha, j = 1, \ldots, n$ and for any $z \in \Omega$,

$$
\left| \frac{\partial^2}{\partial z_j \partial t_\alpha} \right|_{t=p} K(z, t) = \left| \frac{\partial}{\partial t_\alpha} \right|_{t=p} \frac{\partial K(t, z)}{\partial z_j} \leq \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{r_p^2} \frac{\partial K(p + r_p e^{i\theta}, z)}{\partial z_j} r_p e^{i\theta} i d\theta \right| \leq \frac{1}{r_p} \sup_{U} \left| \frac{\partial K(\cdot, z)}{\partial z_j} \right| \leq \frac{1}{r_p} (C_1 + 1),
$$

where $0 < r_p \ll 1$ and the small polydisc $D^n(p; r_p) \subset U$. This combined with (2.16) will imply that for each $\alpha, j = 1, \ldots, n$ and for any $z \in \Omega$,

$$
\left| \frac{\partial w_\alpha(z)}{\partial z_j} \right| \leq C_2^{-1} \left( \frac{1}{r_p} (C_1 + 1) + C_p C_1 \right) < \infty.
$$

Since $w_\alpha(z)$ is a bounded holomorphic function whose partial derivatives are bounded from above by some finite constant on $\Omega$, the Bergman representative coordinate $T(z) = (w_1, ..., w_n)$ defined by (1.1) is Lipschitz and thus continuous up to $\Omega$. Since (2.15) implies that $|K(z, p)|$ is bounded from above by a finite constant $C_3 > 0$ for any $z \in \Omega$, the remaining part of the proof is the same as that of Theorem 1.4. In particular, if $\Omega$ satisfies (2.15) at some general point $p \in \Omega$, we may use a similar argument as in the proof of Theorem 1.4, Part 1) to conclude that the Bergman representative coordinate $T(z)$ relative to $p$ is biholomorphic from $\Omega$ to a ball $B$ defined in (1.3) possibly less a relatively closed pluripolar set $E$, and it extends continuously up to $\Omega$; to further conclude that $T$ extends to a homeomorphism of the closures, one uses the same argument as in the proof of Theorem 1.4, Part 2).

\[\square\]

Theorem 2.5 directly yields the following corollary.
Corollary 2.6. Let $\Omega \subset \mathbb{C}^n$ be a bounded $L^2$-domain of holomorphy such that the holomorphic sectional curvature of the Bergman metric on $\Omega$ is identically equal to a negative constant $-c^2$. If there exists some point $p \in \Omega$ such that (2.15) holds, then the Bergman representative coordinate $T(z)$ relative to $p$ is biholomorphic from $\Omega$ to a ball $B$ defined in (1.3), where $n = 2c^{-2} - 1$, and $T$ extends to a homeomorphism of the closures, $\bar{T} : \bar{\Omega} \to \bar{B}$.

Condition (2.15) is weaker than the assumptions on the Bergman kernel in Part 2) of both Theorem 1.4 and Corollary 1.5, in view of (2.8).

3 Proofs of one dimensional results

In this section, we prove our one dimensional results for planar domains in $\mathbb{C}$. Our arguments continue from the authors’ previous works [24, 28]. We begin with the following lemma.

Lemma 3.1. Let $\Omega \subset \mathbb{C}$ be a domain whose Bergman metric $g(z)dz \otimes d\bar{z} := (\log K(z, z))_z dz \otimes d\bar{z}$ has Gaussian curvature identically equal to $-2$. Then, for any $p \in \Omega$, the Bergman representative coordinate $T(z)$ relative to $p$ satisfies

$$|T'(z)| \leq \frac{2\pi}{g(p)} |K(z, p)|, \quad z \in \Omega.$$ 

Proof. By the discussion in Section 2 (cf. [24, 28]), the zero set $A_z = \emptyset$ and the explicit formula (2.12) of the Bergman-Calabi diastasis relative to $p$ is given by

$$\Phi_p(z) = -2 \log \left(1 - \frac{1}{2} |T(z)|^2 g(p)\right), \quad z \in \Omega,$$

which yields

$$g(z) = |T'(z)|^2 g(p) \left(1 - \frac{1}{2} |T(z)|^2 g(p)\right)^{-2}, \quad z \in \Omega.$$ 

Therefore, by the above two formulas and (2.1), it holds that

$$\frac{|K(z, p)|^2}{K(z, z)K(p, p)} = e^{-\Phi_p(z)} = \left(1 - \frac{1}{2} |T(z)|^2 g(p)\right)^2 = \frac{|T'(z)|^2 g(p)}{g(z)}, \quad z \in \Omega. \quad (3.1)$$

From [24], we know that for any $w \in \Omega$,

$$\pi K(w, w) \geq c_B^2 w \geq \frac{g(w)}{2}, \quad (3.2)$$

where $c_B$ is the analytic capacity defined as

$$c_B(w) := \sup \{|h'(w)| : h \text{ is holomorphic on } \Omega \text{ with } h(w) = 0 \text{ and } |h| \leq 1\}.$$ 

Notice that the first inequality of (3.2) was proved by Suita [53]. Therefore, by (3.1) and (3.2), we get

$$|K(z, p)| = \sqrt{\frac{g(p)}{g(z)} K(z, z) K(p, p) |T'(z)|} \geq \frac{g(p)}{2\pi} |T'(z)|, \quad z \in \Omega.$$
Some rigidity theorems related to capacities were given in [26, 27].

**Proof of Theorem 1.1.** For Part 1), by assumption, we know that $|T'(z)|$ is bounded from above by a finite constant $C_1 > 0$ for any $z \in \Omega$. Therefore, we conclude that $T$ is Lipschitz and thus continuous up to $\overline{\Omega}$. Similar to the proof of Theorem 1.4, Part 1), we know that $T$ is biholomorphic from $\Omega$ to a disc $\mathbb{D}_r := \{ \eta \in \mathbb{C} : |\eta|^2 < 2g^{-1}(p) \}$ possibly less a relatively closed polar set $P$.

For Part 2), the transformation formula of the Bergman kernel under biholomorphism yields

$$K(z, p) = T'(z) \cdot \overline{T'(p)} \cdot K_{\mathbb{D}_r \setminus \eta}(T(z), \overline{0}) = \frac{T'(z)}{2\pi} T'(z), \quad z \in \Omega, \quad (3.3)$$

where the second equality holds due to that the polar set is negligible for $L^2$ holomorphic functions. If $|K(z, p)|$ is additionally bounded from below by a finite constant $C_2 > 0$ for any $z \in \Omega$, then the inverse map of $T$ will be also Lipschitz and thus continuous up to $\overline{\mathbb{D}_r \setminus P}$. Therefore, we conclude that $T$ extends to a homeomorphism of the closures, $\tilde{T} : \overline{\Omega} \to \overline{\mathbb{D}_r \setminus P}$.

Any bounded, simply-connected planar domain $D$ is biholomorphic to a disc by the Riemann mapping theorem, cf. [32]. We will use the Bergman kernel to give sufficient conditions for the extension of the Riemann map to the closures.

**Proof of Corollary 1.2.** By the uniqueness of the Riemann map, the Bergman representative coordinate $T(z)$ defined by (1.1) is biholomorphic from $D$ to $\mathbb{D}$, such that $T(p) = 0$ and $T'(p) = 1$. See, for instance, [10, Chap. VI] or [44]. Also, (3.3) holds for any $z \in D$.

For 1), by (3.3), $|T'(z)|$ is bounded from above by the finite constant $2\pi g^{-1}(p)C_1 > 0$ for any $z \in D$. Therefore, we conclude that $T$ is Lipschitz and thus continuous up to $\overline{D}$.

For 2), denote by $\tau : \mathbb{D}_r \to D$ the inverse biholomorphic map of $T$. By (3.3) and the inverse function theorem, $|\tau'(z)|$ is bounded from above by the finite constant $g(p)(2\pi C_2)^{-1} > 0$ for any $z \in \mathbb{D}_r$. Therefore, we conclude that $\tau$ is also Lipschitz and thus continuous up to $\overline{\mathbb{D}_r}$. Moreover, $\tau(\partial \mathbb{D}_r) \subset \partial D$. Thus $\tau$ maps $\overline{\mathbb{D}_r}$ onto $\overline{D}$ and so $\tau(\partial \mathbb{D}_r) = \partial D$, which shows that $\partial D$ is a continuous curve.

For 3), if $|K(z, p)|$ is bounded from both above and below by finite positive constants for any $z \in D$, then both $T$ and its inverse $\tau$ extend continuously up to the closures. Therefore, $T$ extends to a homeomorphism of $\overline{\mathbb{D}_r}$ onto $\overline{D}$, which implies that $\partial D$ is a Jordan curve.

In Kerzman’s example in [41], the boundary of the domain $D$ is a Jordan curve, but $|K(z, p)|$ is not bounded from above by a finite constant for any $z \in D$. Consequently, Condition 3) in Corollary 1.2 is only sufficient but not necessary for the continuous extension to the closures. On the other hand, although $D$ is a Lu Qi-Keng domain, namely for any $p \in D$, the Bergman kernel $K(., p)$ has no zero set, we have the following observation.

**Proposition 3.2.** Let $D \subset \mathbb{C}$ be a bounded, simply-connected domain whose boundary is not a continuous curve. Then, for any $p \in D$, neither $|K(z, p)|$ nor $|T'(z)|$ is bounded from below by a finite constant $C > 0$ for any $z \in D$, i.e.,

$$\inf_{z \in D} |K(z, p)| = 0 = \inf_{z \in D} |T'(z)|.$$
Proof. Assume the contrary that for some \( p \in D \),
\[
\inf_{z \in D} |K(z, p)| > 0.
\]
Then by Part 2) of Corollary 1.2, \( \partial D \) is necessary continuous, which is a contradiction. The same thing happens to \( T'(z) \) in view of (3.3), and we have completed the proof.

Example 3.3. An example of a bounded, simply-connected planar domain \( D \) whose boundary is not a continuous curve is given below in Figure 1. In this example, the boundary \( \partial D \) consists of the comb space. Here, the boundary \( \partial D \) being a continuous curve means that there exists a continuous function \( \gamma : [0, 1] \to \mathbb{C} \) such that \( \gamma(0) = \gamma(1) \) and \( \gamma([0, 1]) = \partial D \). But since the comb space is not locally connected, it follows that \( \partial D \) is not continuous, cf. [21, Chapter 14].

![Figure 1: A simply-connected domain with discontinuous boundary](image)

Remark 3.4. Condition (B) is not a biholomorphic invariant. First, it is satisfied for the unit disc \( \mathbb{D} \), whose Bergman kernel is written as

\[
K(z, p) = \frac{1}{\pi} \frac{1}{(1 - \overline{z}p)^2} \quad \text{and} \quad \frac{\partial}{\partial z} K(z, p) = \frac{2\overline{p}}{\pi(1 - z\overline{p})^3}.
\]

For any \( p \in \mathbb{D} \), choose
\[
c > \frac{2}{1 - |p|}.
\]
Then, for any \( z \in \mathbb{D} \), it follows that
\[
\left| \frac{\partial}{\partial z} K(z, p) \right| \leq \frac{2}{\pi |1 - \overline{z}p|} = |K(z, p)| \frac{2}{|1 - z\overline{p}|} < |K(z, p)| \frac{2}{1 - |p|} < c|K(z, p)|.
\]
However, Condition (B) is not satisfied for the examples of Fornæss [30] and Kerzman [41], where \( |K(z, p)| \) are unbounded as \( z \) approaches certain boundary point; we can see from Lemma 2.2 that if a bounded, simply-connected domain \( D \) in \( \mathbb{C} \) satisfies Condition (B), then \( |K(z, p)| \) is bounded from above by a finite constant for all \( z \in D \).
4 Bounded Bergman representative coordinates

In this section, we study the boundedness of the Bergman representative coordinate.

**Proposition 4.1.** Let $\Omega$ be a domain admitting the Bergman metric in $\mathbb{C}^n$, $n \geq 1$, and let $K(z, w)$ denote the Bergman kernel of $\Omega$. Assume that there exists a point $p \in \Omega$ such that

$$\sup_{\zeta \in U} |K(\zeta, z)| \leq C |K(p, z)|, \quad \forall z \in \Omega, \quad (4.1)$$

where $U$ is a neighborhood of $p$ and $C > 0$ is a finite constant. Then, the Bergman representative coordinate $T(z)$ relative to $p$ maps $\Omega$ holomorphically to a bounded domain in $\mathbb{C}^n$.

**Proof.** We first assume that $\Omega$ satisfies (4.1) at some $p$ with (2.2). Let $A_p$ be the zero set of the Bergman kernel $K(\cdot, p)$. Then, for each fixed $j = 1, \ldots, n$,

$$w_j(z) := K(z, p)^{-1} \left| \frac{\partial}{\partial t_j} \left|_{t=p} \right. \right. K(z, t) - \left. \left. \frac{\partial}{\partial t_j} \right|_{t=p} \right. \log K(t, t), \quad z \in \Omega \setminus A_p.$$ 

Take a small polydisc $\mathbb{D}^n(p; r_p) \subset U$ for some $0 < r_p < 1$. For each $j = 1, \ldots, n$, by Cauchy’s integral formula for derivatives and (4.1), it holds that

$$\left| \frac{\partial}{\partial t_j} \right|_{t=p} K(z, t) = \left| \frac{1}{2\pi \sqrt{-1}} \int_{\{|t_j - p_j| = r_p\}} \frac{K((p_1, \ldots, t_j, \ldots, p_n), z)}{(t_j - p_j)^2} dt_j \right|$$

$$= \frac{1}{2\pi r_p} \int_0^{2\pi} K((p_1, \ldots, p_j + r_p e^{i\theta}, \ldots, p_n), z) d\theta$$

$$\leq \frac{1}{r_p} \sup_U |K(\cdot, z)|$$

$$\leq \frac{C}{r_p} |K(p, z)|, \quad \forall z \in \Omega \setminus A_p.$$

Therefore, for any $z \in \Omega \setminus A_p$,

$$|w_j(z)| \leq \left| K(z, p)^{-1} \left| \frac{\partial}{\partial t_j} \right|_{t=p} \right. K(z, t) - \left. \left. \frac{\partial}{\partial t_j} \right|_{t=p} \right. \log K(t, t) \leq \frac{C}{r_p} + C_p.$$

By the Riemann removable singularity theorem, the bounded holomorphic function $w_j$ extends across the analytic variety $A_p$ to the whole domain $\Omega$, which completes the proof.

If $\Omega$ satisfies (4.1) at some general point $p \in \Omega$, let $g_{\alpha\beta}(p)$ be the positive-definite Hermitian matrix associated with the Bergman metric at $p$. One performs a linear transformation $F$ from $\Omega$ to $\Omega^1$ such that the Bergman metric $g^1$ on $\Omega_1$ satisfies $g^1_{\alpha\beta}(F(p)) = \delta_{\alpha\beta}$. Since $F$ is a biholomorphism, the Bergman kernels on $\Omega$ and $\Omega^1$ differ by a multiple constant, which is the determinant square of $g_{\alpha\beta}(p)$, due to the transformation rule. Therefore, $\Omega^1$ satisfies (4.1) with respect to the point $F(p)$. By the previous argument, the Bergman representative coordinate $T^1(z)$ relative to $F(p)$ maps $\Omega^1$ holomorphically to a bounded domain in $\mathbb{C}^n$. Finally, the composition map $F^{-1} \circ T^1 \circ F$ is the Bergman representative coordinate $T(z)$ relative to $p$ and it is holomorphic from $\Omega$ to a bounded domain in $\mathbb{C}^n$. 

\[\square\]
Using Proposition 4.1, we will prove Theorem 1.6.

**Proof of Theorem 1.6.** Assume that the holomorphic sectional curvature of the Bergman metric $g$ is identically $-c^2$. Note that if $U$ is small enough, then it is convex and the set $T(U)$ is contained in a ball $B$ defined in (1.3). For any $\zeta \in U$, by the Cauchy-Schwarz inequality and mean value theorem, there exists a point $\eta = sz + (1-s)p \in U$, where $s \in (0,1)$, such that

$$|K(\zeta, t) - K(p, t)| \leq |\nabla_w|_{w=\eta} K(w, t)| \cdot |\zeta - p|,$$

where $\nabla_w := (\frac{\partial}{\partial w_1}, ..., \frac{\partial}{\partial w_n})$ denotes the complex gradient operator. Therefore, by (2.7), it holds for any $\zeta \in U$ and $z \in \Omega$ that

$$|K(\zeta, z)| \leq |K(p, z)| + \sqrt{n}C_\eta|K(\eta, z)| \cdot |\zeta - p|,$$

where

$$C_\eta := \sum_{j=1}^n \left| \frac{\partial}{\partial z_j} \log K(z, z) \right| + n\sqrt{2}c^{-1}$$

is a finite positive constant depending on $\eta$. Thus, $C_U := \sup_{\zeta \in U} C_\zeta > 0$ is also a finite constant as the Bergman kernel is locally uniformly bounded. Choosing a smaller neighborhood $U_1$ of $p$ such that $\sqrt{n}C_U|\zeta - p| < \frac{1}{2}$ whenever $\zeta \in U_1$, we will get

$$|K(\zeta, z)| \leq |K(p, z)| + \frac{1}{2}|K(\eta, z)| \leq |K(p, z)| + \frac{1}{2} \sup_{U_1} |K(\cdot, z)|, \quad \forall \zeta \in U_1, \forall z \in \Omega.$$

Since the above right hand side is independent of $\zeta$, it follows that

$$\sup_{U_1} |K(\cdot, z)| \leq 2|K(p, z)|, \quad \forall z \in \Omega.$$

For simplicity, still denote $U_1$ by $U$, and we have completed the proof. \hfill \Box

We remark that (1.4) indeed holds on symmetric bounded domains, cf. [25, Proposition 2.2]. For general simply-connected complete Kähler manifolds with sectional curvatures bounded between negative constants, one expects a good hold of the Bergman kernel as pointed out by Greene and Wu in [35, Chap. 8].

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