Stationary broken parity states in nonvariational models

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Abstract

We demonstrate that selected, commonly used continuum models of active matter exhibit nongeneric behavior: each model supports asymmetric but stationary localized states even when the gradient structure of the model is broken by activity. As a result such states only begin to drift following a drift-transcritical bifurcation as the activity increases. We explain the origin of this nongeneric behavior, elucidate the model structure responsible for it, and identify the types of additional terms that render the model generic, i.e. with asymmetric states that appear via drift-pitchfork bifurcations and are at rest at most at isolated values of the activity parameter. We give detailed illustrations of our results using numerical continuation of spatially localized structures in both generic and nongeneric cases.

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I. INTRODUCTION

The onset of spontaneous motion, i.e. the transition from a resting state to directed motion, is observed in many hydrodynamic [32, 80] systems. Examples include moving interfaces in directional solidification [80], traveling spatially extended waves in a partially filled horizontal annulus with a rotating outer wall [57], traveling localized structures in binary fluid convection [60] and spatiotemporal chaos in directional viscous fingering [71]. Analogous behavior is also reported in e.g., optical [1, 33, 41, 76] and reaction-diffusion [35, 45, 66, 68] systems. Furthermore, motion is particularly essential for “living” systems like bacteria, animals, or (artificial) microswimmers which transform different forms of energy into self-propelled directed motion, giving rise to a wide variety of models for active media [7, 48]. In these models the transition between resting and traveling states is of particular interest [50, 63, 78, 82, 98].

The common property of all these systems is that they are permanently out-of-equilibrium, and so “active” as opposed to “passive”. Passive systems, i.e., systems described by kinetic equations that have a variational structure [20, 61, 62], can be written in the form of gradient dynamics of an underlying energy (or Lyapunov) functional \( F \). Such models describe evolution towards stable or metastable equilibria corresponding to global and local minima of the energy functional, respectively. Hence, no persistent time-periodic behavior such as traveling or standing oscillations is possible and all asymptotic states are time-independent, i.e., stationary. Important examples include the Allen-Cahn, Cahn-Hilliard, Swift-Hohenberg and phase field crystal models [10, 12, 22, 24, 37].

In contrast, active systems are open systems as energy and/or matter flows through them: for active particles chemical energy is provided, drives their motion and is continuously dissipated via friction [48]. Such dissipative structures are described by kinetic equations that are nonvariational, i.e., only part of the model can be brought into a gradient dynamics form. In consequence, time-periodic behavior is possible.

In various models the nonvariational character is due to nonreciprocal coupling of the different order parameter fields [2, 29, 30, 50, 74, 78, 97], i.e. the coupling breaks Newton’s third law [40]. Examples include chemical or biophysical systems where particles of different species react chemically while particles of the same species only interact passively, e.g. via an attractive/repulsive interaction [2]. In various recent studies the role of such nonreciprocal interactions in generating time-periodic behavior and suppressing coarsening has been elucidated [29, 74, 97].
In particular, the emergence of time-dependent, e.g. chiral, states in these systems can be related to spectral singularities called exceptional points and described in terms of nonreciprocal phase transitions [30].

Many active models assume a linear coupling [29, 50, 74, 78, 97]. This assumption is related to the physical intuition that active systems are often perturbations of passive ones, resulting at leading order in a linear coupling. It is generally assumed that such linear nonvariational coupling suffices to capture the relevant qualitative behavior and that this behavior can be related to the “dead” limit corresponding to the passive system whose thermodynamic behavior is often well understood [7]. Here we show that certain commonly used couplings in fact result in nongeneric behavior in the onset of motion.

In general activity by itself does not automatically lead to motion. The onset of spontaneous motion is generally associated with broken parity symmetry, with the resulting asymmetry determining the direction and speed of propagation [15, 57, 82]. In translation-invariant systems a steady-state, pattern-forming bifurcation creates a periodic state $u(x)$ with definite parity under the spatial reflection $x \to -x$ [16]: $u$ is even [odd] under this reflection, i.e. $u(x) = u(-x)$ [$u(x) = -u(-x)$]. In multicomponent systems, a mixed parity symmetry is also possible when certain components exhibit even parity while others have odd parity [63]. Asymmetric states, i.e. states which are neither symmetric nor antisymmetric, are typically generated via a secondary spontaneous parity-breaking bifurcation of a symmetric state and in translation-invariant active systems such bifurcations generally lead to the onset of drift [26, 31–33] and are therefore referred to as drift-pitchfork bifurcations. In other systems forced symmetry-breaking plays a similar role. Examples include Taylor vortices in the presence of through-flow [73], convection in the presence of through-flow [56] or rotation [21], the dynamics due to third order dispersion [9, 33], the sliding of a drop on an incline [25, 69], and chemotactic motion in an imposed chemical gradient [75].

Symmetry considerations often suffice to explain the qualitative behavior accompanying the transition to drift. Coulet et al. [15] proposed coupled amplitude and phase equations that explain why asymmetric domains propagate. Since the drift-pitchfork bifurcation can be either supercritical or subcritical the transition to drift in experiments may occur in two different ways: In the supercritical case, the asymmetric state is stable at birth, and the drift velocity increases continuously as the square root of the distance of the control parameter from its critical value. In contrast, in the subcritical case, the symmetric state changes dramatically when the parity-breaking bifurcation is crossed, resulting in a state of finite asymmetry and speed. Furthermore, subcriticality
creates bistability between asymmetric drifting and symmetric resting states, thereby permitting nucleation of inclusions of the broken-parity state within the symmetric one. These spatiotemporal grain boundaries (defects) act as sources and sinks of traveling waves which locally propagate in opposite directions \([31, 55, 64]\), leading to wavelength relaxation \([15, 32]\). This bifurcation scenario represents the simplest case of drift onset, involving the minimum number of bifurcations, although more complex scenarios where the onset of drift occurs via a tertiary or higher parity-breaking bifurcation have also been reported. This is the case, for example, for drifting localized states \([42, 65]\).

States in inhomogeneous systems may also drift but with a drift speed that is no longer constant in time. This drift sets in via a depinning transition \([70, 84]\) that is triggered when the degree of asymmetry of a stationary pinned state exceeds a threshold value. Examples include stick-slip motion, e.g. for droplets on an incline pinned by a chemical or topographic defect \([6, 86]\), or the orientation of liquid-crystal molecules when forced by a spatially periodic optical feedback \([36]\). These cases are excluded from the present study.

It is usually assumed that in nonvariational translation-invariant systems all broken parity states necessarily drift \([15, 42]\). In other words, all steady states in active systems have to be parity-symmetric (even parity) or parity-antisymmetric (odd parity). Note that this does not imply that such states cannot be time-periodic. Indeed, fixed-parity standing waves are also frequently found \([43]\). Although this assumption holds for scalar systems \([16]\), recent work has identified several active systems described by the coupled dynamics of two or more order parameter fields that allow for parity-asymmetric steady states. Moreover, the onset of drift may now occur via a drift-transcritical bifurcation. Examples are steady localized asymmetric states in active phase field crystal (PFC) models \([38, 63, 65]\) and in Cahn-Hilliard models with nonreciprocal (i.e., non-variational) coupling \([28]\). In these examples the steady asymmetric states are all unfortunately unstable, and the question remains whether in other systems such states can become linearly stable and thus dynamically accessible. To address this question we must first understand the reasons behind the existence of such states, whether stable or unstable, in particular models of active matter and reconcile this fact with the general expectation that all asymmetric states in nonvariational systems should move.

In the present work we are mostly interested in steady, resting states or in states that are at rest in a moving frame. We identify “hidden” features of many active models that render such models nongeneric and allow nonvariational systems to have steady but asymmetric states. In
passing we also present examples that prove the existence of asymmetric, linearly stable resting states. We start by providing several examples of active models with steady asymmetric states. In particular, Sections II A and II B review and extend existing results for an active PFC model and a nonreciprocal two-field Cahn-Hilliard model, respectively, while Section II C discusses a model with nonvariationally coupled conserved and nonconserved dynamics. Section II D then provides results on a PFC model with 3 order parameter fields describing a mixture of passive and active particles while Section II E considers a simple reaction-diffusion system. After presenting these examples, we provide in Section III a general theory that identifies a class of active multicomponent models that allow for steady asymmetric states. We show that these models are characterized by certain interaction terms between the different order parameter fields and revisit each model in Section III D to explain how generic behavior can be restored. Although one might characterize such models as “nongeneric”, we emphasize that they arise naturally in a number of systems. In Section IV we employ a two-field coupled Swift-Hohenberg model to summarize the essential interplay between parity symmetry and generic behavior. Although many of the resting asymmetric states present in the models are unstable, our examples include cases where they become linearly stable and so are dynamically accessible (see Sections II D and IV). Finally, Section V discusses the implications of our work for the development of generic models for active systems. Note that data sets and plot functions for all figures as well as examples of Matlab codes for the employed numerical path continuation are provided on the open source platform zenodo.

II. ASYMMETRIC STEADY STATES IN ACTIVE MODELS

In this section we present several examples of nonvariational models where resting parity-asymmetric states are found. Our models are selected as representative examples of different classes of models, focusing on models with conserved, nonconserved and mixed dynamics. Our first two examples, an active PFC model and an active two-field Cahn-Hilliard model with nonreciprocal interactions, exhibit conserved dynamics for the density fields (and mixed dynamics for the polarization field in the PFC case). As our next example we consider a system that models the active interaction between a large-scale (i.e., long-wave) and small-scale (i.e., finite wave-length) instability represented by nonreciprocally coupled Cahn-Hilliard and Swift-Hohenberg equations. The behavior of these two-variable models is then compared to a three-variable PFC system, consisting of two coupled conserved density fields, one of which couples to a polarization field that
renders the model active. Such models are common in the description of pattern formation in physico-chemical and biophysical nonequilibrium systems. All of the selected models feature a linear nonvariational coupling between different order parameter fields and exhibit a thermodynamically well-understood dead limit if this coupling is turned off. We conclude this section by comparing this behavior to that of a standard reaction-diffusion model with nonconserved dynamics, the FitzHugh–Nagumo model.

A. Active phase field crystal model

The first model we consider is the active PFC model studied in [14, 50, 51, 63–65]. In the passive limit, the PFC model (aka conserved Swift-Hohenberg model) provides a simple mean-field description of crystallization processes [22, 23, 85] in terms of gradient dynamics of a single conserved order parameter field, the density \( \phi(x, t) \). The underlying free energy \( \mathcal{F}^{\text{SH}} \) is of Swift-Hohenberg type [10, 18]. This passive model is made active by coupling the density dynamics to the dynamics of a (vector) polarization field \( P(x, t) \) [50]. The linear coupling employed represents the leading order active term and can be derived from a microscopic dynamical density functional theory via a combined gradient and Taylor expansion [93].

In one spatial dimension the active PFC model reads

\[
\begin{align*}
\partial_t \phi &= \partial_{xx} \left( \frac{\delta \mathcal{F}^{\text{SH}}}{\delta \phi} \right) - v_0 \partial_x P, \\
\partial_t P &= \partial_{xx} \left( \frac{\delta \mathcal{F}^P}{\delta P} \right) - D_r \frac{\delta \mathcal{F}^P}{\delta P} - v_0 \partial_x \phi
\end{align*}
\]

with the functionals

\[
\begin{align*}
\mathcal{F}^{\text{SH}} &= \int dx \left[ \phi \left( \epsilon + (1 + \partial_{xx})^2 \right) \phi + \frac{1}{4} \left( \bar{\phi} + \phi \right)^4 \right], \\
\mathcal{F}^P &= \int dx \left[ \frac{1}{2} c_1 P^2 + \frac{1}{4} c_2 P^4 \right].
\end{align*}
\]

Here the coupling parameter \( v_0 \) represents activity and is referred to as the self-propulsion velocity. The dynamics of the polarization \( P(x, t) \) contains translational and rotational diffusion captured by the mixed conserved and nonconserved gradient dynamics, with \( D_r \) being the rotational diffusivity. The free energy of the density field \( \phi \) defines a typical length scale which is set to 1, and depends on the effective temperature parameter \( \epsilon \) and the background density parameter \( \bar{\phi} \). Note that by convention \( \int dx \phi = 0 \). The free energy density of the polarization field represents a...
parabolic (for \( c_1 > 0 \)) or a double-well potential (for \( c_1 < 0 \)); \( c_2 \geq 0 \) in both cases. In the former case diffusion reduces polar order, in the latter spontaneous polarization arises. Computing the functional derivatives in Eqs. (1) yields

\[
\begin{align*}
\partial_t \phi &= \partial_{xx} \left\{ \epsilon + (1 + \partial_{xx})^2 \right\} \phi + (\bar{\phi} + \phi)^3 - v_0 \partial_x P, \\
\partial_t P &= \partial_{xx} (c_1 P + c_2 P^3) - D_r (c_1 P + c_2 P^3) - v_0 \partial_x \phi.
\end{align*}
\]

(3)

Based on the scalar and vector character of \( \phi \) and \( P \), respectively, the model (3) is invariant under the parity symmetry transformation \( \mathcal{R}_1 : (x, \phi, P) \mapsto (-x, \phi, -P) \). According to this symmetry, steady states that are invariant under \( \mathcal{R}_1 \), i.e. with \( (\phi(x), P(x)) = (\phi(-x), -P(-x)) \), are referred to as parity-symmetric. Note that if \( \bar{\phi} = 0 \) the model also obeys inversion symmetry \( \mathcal{R}_2 : (\phi, P) \mapsto (-\phi, -P) \).

For \( v_0 = 0 \), the two equations (3) decouple and one recovers the (passive) PFC model. In contrast, for \( v_0 \neq 0 \) the system is active and can no longer be written in the form of gradient dynamics. Thus for \( v_0 \neq 0 \) we expect to find time-dependent dynamics and, in particular, traveling structures, resulting in a rich variety of complex behavior [50] that can be represented in intricate bifurcation diagrams [63, 64]. Here, we focus on a subset of the extensive results presented in [63], namely, on the bifurcation behavior of the resting localized states \( \text{RLS}_{\text{odd}} \) and \( \text{RLS}_{\text{even}} \) as summarized in Fig. 1. All bifurcation diagrams we show for this and subsequent models are obtained using numerical continuation [24] employing the Matlab package \textit{pde2path} [90]. As a solution measure we always use the \( L_2 \) norm

\[
||\delta u|| = \sqrt{\sum_i \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left( u_i - \bar{u}_i \right)^2},
\]

(4)

where the \( u_i \) correspond to the order parameter fields of the model, \( \bar{u}_i \) is their mean value and \( L \) is the domain size.

Figure 1(a) shows branches of resting and traveling localized states that form a slanted snaking bifurcation structure where solid and dashed lines indicate linearly stable and unstable states, respectively. The snaking structure consists of two intertwined branches of steady parity-symmetric states \( \text{RLS}_{\text{odd}} \) and \( \text{RLS}_{\text{even}} \) with an odd (dark blue) or even (light blue, see e.g. profile in Fig. 1(b)) number of peaks. The interconnecting rung-like branches (red) consist of steady asymmetric states (see e.g., profile in Fig. 1(c)). For the passive PFC model, the presence of steady rung states is a direct consequence of the gradient structure of the model. However, it is unexpected that these remain at rest even in the active case \( v_0 \neq 0 \). Beside the steady states one also finds branches
of traveling states (brown branches, see e.g. profile in Fig. 1(d)) that bifurcate from the branches of steady symmetric states via drift-pitchfork bifurcations (empty triangle symbol), and from the steady asymmetric states via drift-transcritical bifurcations (filled triangle symbol), see the inset of Fig. 1(a) for a magnification. In the latter case the resting state is no longer parity-symmetric (even or odd) and the direction of drift of the emerging states is determined by the preexisting asymmetry of the resting state.

We emphasize that this behavior is surprising: Contrary to the expectation that asymmetric states drift in active systems, branches of resting and traveling asymmetric states coexist in this active PFC model and, moreover, are related via drift-transcritical bifurcations that should not exist in generic systems. Since this behavior is contrary to theoretical predictions one may suspect that the model cannot be generic. This hypothesis is strengthened when we take a resting asymmetric state in Fig. 1(a) and turn on the spontaneous polarization parameter \(c_2\). Panel (e) shows the velocity \(v\) of the asymmetric state in panel (c) as a function of \(c_2\) and shows that the state begins to move as soon as \(c_2 \neq 0\). For \(c_2 > 0\) the velocity \(v\) first decreases, then increases again, but remains small and negative while approaching a plateau (see the inset). For \(c_2 < 0\) the velocity is positive and much larger; \(v\) increases monotonically as \(c_2\) decreases until a fold where two branches of traveling states merge. Following the upper branch for increasing \(c_2\) we find that the branch approaches close to \(c_2 = 0\) when \(v \approx 0.15\) before doubling back. The resulting complex behavior is omitted. Figure 1 (a) also reveals the presence of several Hopf bifurcations, some on the branches of even and odd localized states, and one other on the branch of drifting asymmetric states (open diamond symbols). The latter generates a quasiperiodic state. The Hopf bifurcations on \(\text{RLS}_{\text{odd}}\) and \(\text{RLS}_{\text{even}}\) are created via Bogdanov-Takens bifurcations and so first arise with zero frequency at the folds of the slanted snaking structure. With increasing activity \(v_0\) these bifurcations move inwards, decreasing the parameter range where RLS are linearly stable, and may ultimately annihilate, thereby rendering all steady localized states unstable. We have not explored the resulting dynamical states, but refer to Section II B for a detailed study of a similar problem.

In the following sections we pursue the question whether this nongeneric behavior is exclusive to the active PFC model or is a common feature of other active models.
FIG. 1: Localized states in the active phase field model (3). (a) Bifurcation diagram showing the L₂ norm (4) with $u₁ = \phi$ and $u₂ = P$, as a function of the parameter $\bar{\phi}$. The primary bifurcation from the trivial state (black line) generates periodic states (dark purple line), followed by a secondary bifurcation to a pair of stationary parity-symmetric localized states with odd and even numbers of peaks (resp., $RLS_{\text{odd}}$ and $RLS_{\text{even}}$) are shown as dark and light blue lines. Tertiary branches of asymmetric resting rung states (red lines) and asymmetric traveling states (brown line) are also shown. Linear stability is indicated via solid lines, while pitchfork, drift-pitchfork, drift-transcritical and Hopf bifurcations are indicated via circle, empty triangle, filled triangle and diamond symbols, respectively. Unstable steady and time-periodic states are indicated as dashed and dashed-dotted lines, respectively. Profiles in (b)-(d) show sample profiles at locations indicated by crosses in panel (a). Parameters: $\varepsilon = -1.5$, $c₁ = 0.1$, $D_r = 0.5$, $c₂ = 0$, $v₀ = -0.16475$ (note that the sign of $v₀$ is given incorrectly in [63]). The domain size is $L = 100$. Panel (e) shows that the steady rung state of panel (c) starts to move when $c₂ \neq 0$. 

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B. Nonreciprocal Cahn-Hilliard model

The nonreciprocal Cahn-Hilliard model has been analyzed in several recent studies \cite{7, 28, 29, 74, 97} and provides a description of mixtures of active colloids. The model is based on the Cahn-Hilliard equation, a passive model originally proposed to describe phase separation of isotropic solid or binary fluid phases \cite{12, 13}. Owing to mass conservation the model takes the form of a pair of coupled continuity equations driven by gradients in the corresponding chemical potentials. Since it captures many qualitative features of phase separation it and its variants and extensions are widely applied in, e.g., biophysics and soft matter contexts. Variants can be distinguished by the features of the original model that are modified or broken. For example, in the convective Cahn-Hilliard model the parity symmetry is broken, a property that is used to describe material flux across the system boundaries \cite{88, 94}. Active, i.e. nonvariational, extensions of the model are used to describe motility-induced phase separation of active Brownian particles \cite{72, 81}.

The nonreciprocal Cahn-Hilliard model consists of two order parameter fields, both described by a Cahn-Hilliard equation, and augmented by a linear coupling to the other field:

\[
\frac{\partial \phi_1}{\partial t} = \frac{\partial^2}{\partial x^2} \left( -\frac{\partial^2 \phi_1}{\partial x^2} + f_1'(\phi_1) - (\rho + \alpha) \phi_2 \right),
\]

\[
\frac{\partial \phi_2}{\partial t} = \frac{\partial^2}{\partial x^2} \left( -\kappa \frac{\partial^2 \phi_2}{\partial x^2} + f_2'(\phi_2) - (\rho - \alpha) \phi_1 \right),
\]

where

\[
f_1 = \frac{1}{2} a \phi_1^2 + \frac{1}{4} \phi_1^4,
\]

\[
f_2 = \frac{1}{2} (a + a \Delta) \phi_2^2 + \frac{1}{4} \phi_2^4
\]

represent double-well potentials. Mass conservation implies that at all times

\[
\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi_1 = \bar{\phi},
\]

\[
\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi_2 = \bar{\phi} + \Delta \bar{\phi},
\]

where \(L\) is the fixed domain size and \(\bar{\phi}\) and \(\bar{\phi} + \Delta \bar{\phi}\) represent constant mean densities.

The model (5) is invariant with respect to \(\mathcal{R}_1 : x \mapsto -x\) and, if both mean densities vanish, i.e. if \(\bar{\phi} = \Delta \bar{\phi} = 0\), it is also inversion-symmetric, i.e. invariant under \(\mathcal{R}_2 : (\phi_1, \phi_2) \mapsto (-\phi_1, -\phi_2)\). Compared to the active PFC model of the previous section the parity symmetry \(\mathcal{R}_1\) takes a different representation due to the scalar character of both density fields in the model. Hence, parity-
symmetric and -antisymmetric states are those which are invariant under $R_1$ and $R_1 \circ R_2$, respectively.

If $|\alpha| < |\rho|$ the model (5) can be derived from the gradient of an effective bounded free energy, and so corresponds to a passive case. A thermodynamic analysis then reveals two and three-state coexistence regions that also survive for low but nonzero activity [29, 74]. The model is referred to as “nonreciprocal” since the parameter $\alpha$, the antisymmetric part of the coupling, represents a “run-and-chase” interaction between the two fields/species $\phi_1$ and $\phi_2$, i.e. it breaks Newton’s third law [40]. When $|\alpha| > |\rho|$ the effective free energy is no longer bounded and the system becomes active. In this case the phase separation mechanism of the classical Cahn-Hilliard model may be arrested either by newly stable shorter wavelength steady states (stationary suppression of coarsening) or by the onset of motion [29, 97]. Furthermore, the large-scale linear instability may be replaced by a (small-scale) Turing-type instability that allows for localized states.

Here, we revisit recently published results [28] on localized structures in this model, focusing on the presence of asymmetric steady states in the active case $|\alpha| > |\rho|$. Figure 2(a) presents a bifurcation diagram for $\Delta \phi = 0$ using the mean density $\bar{\phi}$ as a control parameter. Solid and dashed lines indicate linearly stable and unstable states, respectively; dashed-dotted lines indicate unstable time-periodic states. The reciprocal and nonreciprocal interaction strengths are $\rho = 1.35$ and $\alpha = 1.65$, respectively. In this case a linear stability analysis reveals that the homogeneous state exhibits a small-scale instability [28] leading to a supercritical branch of stationary periodic states (dark purple). Owing to its spatial period we refer to this branch as the $n = 11$ branch. However, this branch loses stability almost immediately at a secondary bifurcation generating two branches of stationary symmetric localized states $R_{LS_{odd}}$ (dark blue) and $R_{LS_{even}}$ (light blue). These emerge subcritically and are organized in a slanted snaking structure with alternating stable segments, much as found for the active PFC model (Fig. 1); every second fold one pattern wavelength is added symmetrically on either side of the structure, resulting in the addition of two new peaks. This process continues as one follows $R_{LS_{odd}}$ and $R_{LS_{even}}$ to larger norm until the whole domain is filled with peaks (see e.g. the profile of the four peak state in panel (b)) and the RLS branches terminate on a branch of periodic solutions, here the $n = 9$ branch (light purple). Within the slanted snaking structure one finds interconnecting branches of asymmetric rung states that emerge and terminate in pitchfork bifurcations near the $R_{LS_{odd}}$ and $R_{LS_{even}}$ folds (see e.g. profile of a four peak state in panel (c)). Just as in the active PFC model, here, too, the asymmetric rung states are time-independent and remain so as $\alpha$ increases. Nevertheless, increasing nonreciprocity
FIG. 2: (a) The $L_2$ norm (4) as a function of the mass $\bar{\phi}$ for $\alpha = 1.65$, $\Delta \bar{\phi} = 0$. Steady symmetric localized states with odd and even numbers of peaks (resp., RLS$_{\text{odd}}$ and RLS$_{\text{even}}$) are shown as dark and light blue lines. Also included are the four lowest branches of steady asymmetric localized states (red dashed lines), the periodic steady states with $n = 11$ peaks (dark purple line) and $n = 9$ peaks (light purple line), and the uniform state (black line). Panels (b) to (c) show sample profiles at parameters marked by bold symbols in panel (a). The remaining parameters are $\kappa = 0.14$, $a = 1.25$, $a_{\Delta} = -1.9$, $\rho = 1.35$. The domain size is $L = 20\pi$. Panel (d) shows the existence of Hopf bifurcations (diamond symbols) near the ninth fold of the RLS$_{\text{odd}}$ branch for $\alpha = 1.78$ and $a = 0.9$ and unchanged values of the other parameters. The resulting branches of standing oscillations are represented by dashed-dotted brown lines, with sample solutions shown in panels (e)-(g). The number of unstable eigenvalues along the steady state branches is indicated by $+$ symbols. All line styles and symbols are as in Fig. 1.
of the coupling does have a qualitative effect. Hopf bifurcations (diamond symbols) arise on all
the branches shown in Fig. 2 as $\alpha$ increases and the segments of stable steady states shrink with in-
creasing activity [28]. Within the snaking structure, Hopf bifurcations arise via Bogdanov-Takens
bifurcations at the folds and at the pitchfork bifurcations to the asymmetric states, respectively.
Figure 2(d) shows an example of this behavior for $\alpha = 1.78$ and focuses on a small parameter
region within the snaking structure where two successive Hopf bifurcations appear on the branch
of symmetric states while one Hopf bifurcation appears on the branch of asymmetric states. These
bifurcations change the linear stability behavior of the steady states in this parameter region. In
panel (d) the $+$ symbols indicate the number of unstable eigenvalues calculated via numerical
stability analysis on the full domain, while the thin black lines emerging from these bifurcations
represent the resulting branches of standing oscillations (cf. panels (e)-(g)). Note that panel (f)
represents an in-phase oscillation of the two fronts while (g) shows a similar oscillation that is
exactly out of phase. Such oscillations are expected when an even parity state loses stability in a
Hopf bifurcation; moreover, we expect that for wider even parity states the two Hopf bifurcations
will approach one another, becoming almost degenerate. Panel (e) shows an asymmetric though
standing oscillation originating from the branch of asymmetric steady states.

Thus in this case increased activity does not lead to the onset of drift but is instead responsible
for the presence of a pair of new branches of (unstable) time-periodic solutions near the left folds
(as well as for asymmetric oscillations), thereby reducing the stability interval of the existing
steady states. Since the upper Hopf bifurcation in (d) is subcritically passed, this bifurcation does
not generate stable oscillations but leads instead to the elimination of a pair of peaks, one on either
side of the structure, and a transition to a stable 7 peak state. In contrast, near the right folds
Hopf bifurcations are only present when the RLS are short. Hopf bifurcations within the snaking
structure have also been found in other systems (see e.g. [8]), but not on branches of asymmetric
solutions. This may be because in our case the asymmetric states remain steady, while in generic
active systems such states necessarily drift and a Hopf bifurcation from such states would lead to
two-frequency localized states [67].

Next, we examine how drift-transcritical bifurcations arise in this model. As for the active PFC
model, we need to find a coexistence between asymmetric steady and drifting states so we can
track these states. For this purpose we begin with the inversion-symmetric case $\bar{\phi} = \Delta \bar{\phi} = 0$,
i.e. the model (5) is invariant with respect to $R_1$ and $R_2$, choose a rather large $\alpha$, so that time-
periodic behavior is preferred, and trace this back to drift-pitchfork bifurcations from symmetric
or antisymmetric steady states. We then gently break the inversion symmetry of the system so that
the antisymmetric states lose their antisymmetry and become asymmetric, and ask whether these
states remain at rest and the corresponding drift-pitchfork bifurcation becomes a transcritical one.

![Bifurcation diagrams](image)

FIG. 3: (a) The $L_2$ norm (4) as a function of the effective temperature $a$ for the two coupled Cahn-Hilliard
equations (5) in the inversion-symmetric case $\bar{\phi} = \Delta \bar{\phi} = 0$. Panel (b) highlights a parameter region of (a)
where even (panel (d)) and odd (panel (e)) parity states connect the primary branches $1^-$ and $2^+$. Panel
(c) shows the same parameter range as (b), but for the case when the inversion symmetry is weakly broken
($\bar{\phi} = 0.002$, $\Delta \bar{\phi} = 0$). Insets in (b) and (c) highlight the branch of drifting states [panel (g)] that connect
branches of steady parity symmetric and asymmetric states [panel (f)] via drift-pitchfork [empty triangles]
and drift-transcritical (filled triangles) bifurcations, respectively. Other parameters are $\alpha = 1.65$, $\rho = 1.35$,
$\kappa = 1$, $a_\Delta = -1.9$. The domain size is $L = 4\pi$. Linestyles and symbols are as in Fig. 1.

Figure 3 presents the bifurcation diagram in the inversion-symmetric case. Here, the effective
temperature $a$ is used as the main control parameter. For the given parameters, the model exhibits a
large-scale instability, i.e., at the first primary bifurcation the fully phase-separated state ($n_1 = 1^+$) appears. We label each branch with a number indicating the spatial period of the corresponding
solution and additionally an index $\pm$ indicating to which eigenvalue of the dispersion relation does
the associated primary bifurcation belong. Interestingly, all $+$ branches arise subcritically, even
though no quadratic (in general destabilizing) nonlinearities appear in the individual Eqs. (5). Nevertheless, as explained in more detail in Ref. [29], subcritical behavior can occur in a two-component system with nonreciprocal coupling even in this case. In the context of this work, we focus on the secondary and tertiary bifurcations and the corresponding branches that emerge. First there are two pairs of similar branches that connect the $1^-$ branch to the $2^+$ branch and the $2^-$ branch to the $3^+$ branch via pitchfork bifurcations. The former are also highlighted in a zoom (panel (b)) where we see that one drift-pitchfork (triangle symbol) and one Hopf bifurcation (diamond symbol) occur on each branch [see inset for magnification]. The corresponding states exhibit even (light purple line in (b), profiles in (d)) and odd (red line in (b), profiles in (e)) parity, respectively, and are therefore at rest (cf. Section B). Two drift-pitchforks, one on each of the secondary branches, break the even or odd symmetry of these states, resulting in traveling asymmetric states (dashed-dotted brown line in (b), see inset for magnification), as one would expect for asymmetric states in nonvariational systems. Moreover, one finds that the resulting branch of asymmetric traveling states connects the two secondary branches of even and odd parity symmetric steady states, cf. [17, 19].

Next, we set $\bar{\phi} \neq 0$, thereby breaking the inversion symmetry. The corresponding bifurcation diagram is shown in Fig. 3(c) and reveals the effect of breaking of the inversion symmetry on the diagram in panel (b). The figure shows that the broken inversion symmetry has different consequences for the secondary bifurcations on the $1^-$ and $2^+$ branches. In particular, the upper pitchfork bifurcation on the $1^-$ branch that gives rise to the light purple branch in panel (b) becomes imperfect in (c), resulting in a saddle-node bifurcation on the light purple branch $1_b^-$ in (c) and a single-valued branch $1_a^-$ (dark blue line in (c)). Both $1_a^-$ and $1_b^-$ still connect to the $2^+$ branch (light blue line) but the termination points now differ. This is a consequence of the splitting of the degenerate pitchfork bifurcation on the $2^+$ branch in (b) into three separate pitchfork bifurcations, two of which remain on the $2^+$ branch while the third one takes place on the $1_a^-$ branch just prior to its termination on the $2^+$ branch.

The splitting of the branch of even parity states (light purple line in (b)) into two different branches that remain parity-symmetric in (c) also doubles the number of the corresponding drift-pitchfork and Hopf bifurcations (triangle and diamond symbols on the light purple and dark blue branches in (c), respectively). Importantly, the branch of antisymmetric states in (b) is still present in (c), but now represents asymmetric, but still steady, states (red line in (c), profile in (f)) that arise in the two pitchfork bifurcations on the $1_a^-$ branch. The Hopf bifurcation on the red branch of (b) is
also present in (c), and likewise for the drift bifurcation (filled triangle symbol) representing onset of motion where two branches of traveling asymmetric states emerge (see dashed dotted brown lines in insets in (c), profile in (g)). However, this bifurcation is now a transcritical drift bifurcation, and one of the emerging branches exhibits a saddle-node bifurcation (see bottom inset in (c)) before each branch of the resulting traveling states connects to one of the two now distinct branches of symmetric steady states via a drift-pitchfork bifurcation (triangle symbols). That is, owing to the broken inversion symmetry, steady antisymmetric states become steady asymmetric states while the drift-pitchfork bifurcation splits into a drift-transcritical and a saddle-node bifurcation.

With increasing inversion symmetry breaking, i.e. for increasing $\bar{\phi}$, the branch of steady asymmetric states (red branch in (c)) shrinks until the corresponding pitchfork bifurcations annihilate and the drift-transcritical bifurcation vanishes.

C. Coupled Cahn-Hilliard and Swift-Hohenberg equations

Next, we examine a two-variable model that couples the Cahn-Hilliard equation to the non-conserved Swift-Hohenberg equation, the former of which has already been introduced in the previous example. The latter one has been investigated in Section II A in its conserved version and coupled to a polarisation field. The classical (nonconserved) Swift-Hohenberg equation was originally introduced to describe pattern selection in Rayleigh-Bénard convection [83]. The equation was subsequently identified as a simple but useful model capable of providing a qualitative understanding of various pattern-forming systems. Applications run from reaction-diffusion systems [34] to laser physics [46, 87] and fluid dynamics [52, 77], and even to growth processes in ecosystems [53]. The bistable SH equation captures essential properties such as wavelength selection and the properties of spatially localized states such as fronts and pulses and their pinning and depinning. It also points to the important role played by the Maxwell point present here because of the gradient structure of the equation. In particular, there have been numerous studies of the properties of localized structures and of the so-called homoclinic snaking these structures exhibit in both one and two dimensions [11, 47, 95].

The Cahn-Hilliard and Swift-Hohenberg models are minimal models for two different stationary instabilities of the trivial homogeneous state. The Swift-Hohenberg model describes a non-conserved quantity exhibiting a small-scale instability with a finite critical wavenumber which determines the emerging pattern. In contrast, the Cahn-Hilliard equation models a large-scale in-
stability of a conserved quantity where demixing results in a phase-separated state. A model that couples the Swift-Hohenberg equation to the Cahn-Hilliard equation thus describes interaction between pattern formation and phase separation. Near the codimension-2 point, where both instabilities occur with comparable growth rates the pattern-forming process competes with demixing as observed e.g. in Marangoni convection in thin liquid films [91], resulting in states that may consist of different spatial scales and exhibit additional secondary instabilities [54].

Here we employ a linear interaction between the two fields which can again be separated into reciprocal and nonreciprocal parts controlled by the parameters \(\rho\) and \(\alpha\), respectively. The coupled model reads

\[
\begin{align*}
\frac{\partial \phi_1}{\partial t} &= \frac{\partial^2}{\partial x^2} \left( -\kappa \frac{\partial^2 \phi_1}{\partial x^2} + f_1'(\phi_1) + (\rho + \alpha) \phi_2 \right), \\
\frac{\partial \phi_2}{\partial t} &= - \left[ \left( q^4 - r + 2q^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4} \right) \phi_2 + f_2'(\phi_2) + (\rho - \alpha) \phi_1 \right]
\end{align*}
\] (8)

with

\[
\begin{align*}
f_1 &= \frac{a}{2} \phi_1^2 + \frac{1}{4} \phi_1^4, \\
f_2 &= - \frac{\delta}{3} \phi_2^3 + \frac{1}{4} \phi_2^4.
\end{align*}
\] (9)

Since the individual equations both obey gradient dynamics, the coupled model becomes active, i.e. nonvariational, only when \(|\alpha| > |\rho|\). The dynamics of \(\phi_1\) remains mass conserving, and at all times

\[
\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi_1 = \bar{\phi}_1.
\] (10)

Figure 4 presents a bifurcation diagram and solution profiles of various steady and drifting states, including steady asymmetric states (red lines). The parameter \(a\) is used as the control parameter, the characteristic wavenumber of the Swift-Hohenberg equation is set to \(q = 1.3\); with \(\rho = 0.2 < \alpha = 0.5\) the model (8) is nonvariational, i.e. we expect time-periodic behavior.

For a given mean density \(\bar{\phi}_1\) the homogeneous state is \((\bar{\phi}_1, \bar{\phi}_2)\) where \(\bar{\phi}_2\) solves \((q^4 - r)\bar{\phi}_2 + f_2'(\bar{\phi}_2) + (\rho - \alpha) \bar{\phi}_1 = 0\). With \(\bar{\phi}_1 = 0.8\) this equation gives \(\bar{\phi}_2 \approx -0.067\) and the resulting homogeneous state (black line) is stable for \(a \approx -1.99\), where it becomes unstable with respect to a large-scale instability inherited from the Cahn-Hilliard equation. This instability leads to a subcritical branch of period-one solutions (dark blue line) that localize and gain stability in a fold at
FIG. 4: (a) The L2 norm $|\delta u|$ as a function of $a$ for the nonvariational coupled CH and SH equations (8) with reciprocal and nonreciprocal interaction parameters $\rho = 0.2$ and $\alpha = 0.5$, respectively. The homogeneous state (black line) corresponds to $(\bar{\phi}_1, \bar{\phi}_2) \approx (-0.8, -0.067)$ where $\bar{\phi}_1$ is also the mean value of $\phi_1$ for all the other states. Branches of steady asymmetric states [panel (d)] are indicated via red lines. Linestyles and symbols are as in Fig. 1. The remaining parameters are $\kappa = 0.04$, $r = -0.5$, $\delta = 3.2$ and $q = 1.3$. The domain size is $L = 5\pi$.

$a \approx -0.93$. The resulting stable solutions are characterized by a pronounced peak in the $\phi_1$-field at a location where $\phi_2$ is a minimum (cf. panel (b)). Note that due to an effective run-and-chase interaction ($\rho + \alpha > 0$, $\rho - \alpha < 0$), $\phi_2$ [\phi_1] favors alignment [anti-alignment] with $\phi_1$ [\phi_2]; since $|\bar{\phi}_1| > |\bar{\phi}_2|$ $\phi_1$ is dominant and it is therefore reasonable that anti-alignment is preferred. For decreasing $a$ these one-peak solutions lose stability in a Hopf bifurcation (empty diamond) at $a \approx -1.28$, followed by drift-pitchfork (empty triangle) and pitchfork (empty circle) bifurcations at $a \approx -1.40$ and $a \approx -1.48$, respectively; these states are subsequently restabilized via a
second drift-pitchfork bifurcation, followed by a pair of fold bifurcations and a further pitchfork bifurcation at $a \approx -1.64$. With further decrease of $a$ the one-peak states remain stable but acquire a pair of side peaks, i.e. they become symmetric three-peak states (not shown). Having described the branch of one-peak solution we now return to the aforementioned two drift-pitchfork bifurcations and the pitchfork bifurcation between them (see upper right inset in panel (a)). The diagram shows that the drift-pitchfork bifurcations are connected by a branch of drifting asymmetric states (dashed dotted brown line, panel (c)). In contrast, the pitchfork bifurcation leads to a branch of steady asymmetric states (red line, panel (d)) which collides with the branch of drifting states in a drift-transcritical bifurcation (filled triangle, see smaller inset for magnification). Both branches of asymmetric states exhibit Hopf bifurcations which lead to modulated waves and standing asymmetric oscillations that emerge from the drifting and steady asymmetric states, respectively. The red branch finally connects via a pitchfork bifurcation to a symmetric two-peak state (light purple line, panel (e)), thereby stabilizing the two-peak states for lower values of $a$ before they lose stability again via two successive Hopf bifurcations. Tracking the light purple branch back, i.e. for increasing $a$, we see that it exhibits a fold but does not connect to the homogeneous state. Instead it turns around at small amplitude in a sharp fold. The bifurcation behavior that follows resembles that of the one-peak states, occurs in a similar parameter range, but is much more complex. We omit these details. Other two-peak states emerge from the second primary bifurcation of the trivial state, but are also omitted here.

Next, we consider the bifurcation behavior connected to the branch emerging in the third primary bifurcation from the homogeneous state. The resulting branch consists of five equispaced peaks (light blue line, panel (f)) instead of the three-peak states one might expect for a large-scale instability. This is due to the chosen parameters that render the system close to a codimension-2 point where both large- and small-scale instability occur simultaneously. The light blue branch also emerges subcritically this time with five unstable eigenvalues. The branch gains linear stability through a fold and two degenerate, i.e. double, pitchfork bifurcations, both of which ultimately lead to asymmetric states. The behavior near the first one, at $a \approx -2.30$, is enlarged in the corresponding inset. The pitchfork bifurcation of the five-peak periodic state generates a subcritical branch bifurcating towards decreasing $a$ values [dark purple line] consisting of a reflection-symmetric five-peak state with unequal spacing. The dark purple branch turns back at the left fold and back again at the next fold on the right, followed by a pitchfork and a Hopf bifurcation. Further complex bifurcation behavior occurs beyond these points and is omitted. At the aforementioned
pitchfork bifurcation the reflection symmetry of the solution is broken and a new branch of steady asymmetric states emerges (red line, panel (g)). This branch also oscillates back and forth through four folds before extending to larger $a$ values. Near the fifth fold where the branch turns back again two drift-transcritical bifurcations (filled triangles) occur where the steady states connect to drifting asymmetric states. Beyond the sixth fold the asymmetric steady states branch terminates in a pitchfork bifurcation to symmetric states, the corresponding branch is omitted (which is not the dark blue branch).

In addition, we have found various steady asymmetric states that show asymmetric peak-to-peak separations, e.g. at the second pitchfork bifurcation of the dark blue branch where the three-peak state regains stability such a steady asymmetric state occurs (red line) where one of the outer peak becomes broader, the other narrower. Since this asymmetric state has a similar norm as its symmetric pendant, both branches, symmetric and asymmetric one, proceed close together, almost parallel. Here, and for the other examples we do not show, the symmetric state is more stable than the asymmetric one. In this case, the asymmetric state also exhibits a Hopf bifurcation, which renders it even more unstable. Note that in contrast to the aforementioned steady asymmetric states these states exist in a large parameter range.

Much of the structure just described owes its existence to the presence of a subcritical Turing bifurcation (where the light blue branch emerges) as described further in [44] and a strong quadratic nonlinearity ($\delta = 3.2$) that creates a coupling of neutral and pattern mode [49]. Evidently the fact that the Turing bifurcation is preceded by additional primary bifurcations to large-scale states is a consequence of the coupling to the Cahn-Hilliard equation.

D. Coupled passive and active PFC models

In this section we describe the properties of a three-variable model, comprising a passive PFC model coupled to an active one:

$$
\begin{align*}
\partial_t \phi_1 &= \partial_{xx} \left\{ \left[ \epsilon + (1 + \partial_x^2) \right]^2 \phi_1 + \left( \bar{\phi}_1 + \phi_1 \right)^3 + c \phi_2 \right\} - v_0 \partial_x P, \\
\partial_t P &= \partial_{xx} \left( c_1 P + c_2 P^3 \right) - D_t (c_1 P + c_2 P^3) - v_0 \partial_x \phi_1, \\
\partial_t \phi_2 &= \partial_{xx} \left\{ \left[ \epsilon + (1 + \partial_x^2) \right]^2 \phi_2 + \left( \bar{\phi}_2 + \phi_2 \right)^3 + c \phi_1 \right\}. 
\end{align*}
$$

The two densities $\phi_1$, $\phi_2$ are coupled via the parameter $c$. The latter obeys gradient dynamics, while the former is nongradient when the activity parameter $v_0 \neq 0$. As for the active PFC
model in Section (II A), Eqs. (11) are parity-symmetric in the sense of being invariant w.r.t. $\mathcal{R}_1: (x, \phi_1, P, \phi_2) \mapsto (-x, \phi_1, -P, \phi_2)$.

Figure 5(a) summarizes the steady states of Eqs. (11) at low activity, $v_0 = 0.1$, as a function of the conserved mean density $\bar{\phi} \equiv \bar{\phi}_1 = \bar{\phi}_2$. The homogeneous state is stable for low $\bar{\phi}$ but loses stability in a pitchfork bifurcation at $\bar{\phi} \approx -0.75$. From this bifurcation a branch of periodic states with $n = 8$ emerges supercritically. Shortly thereafter the periodic states are destabilized in another pitchfork bifurcation, from which two branches of localized states, with odd and even numbers of peaks, emerge. These exhibit homoclinic snaking with new peaks added near the left folds. Near these the rung branches of unstable, resting, asymmetric states emerge in pitchfork bifurcations, as in the previous examples. However, at this choice of parameters, there is at least one segment of stable, resting, asymmetric states on each rung branch generated via a pair of folds. Similar restabilization has been observed for the two-dimensional Swift-Hohenberg equation [4]. Panels (b) and (c) of Fig. 5 show two distinct types of stable, resting, asymmetric states. In panel (b) [(c)] there is an additional $\phi_1$-[$\phi_2$-]peak at the left interface of a three-peak localized state. Both states are located on the same rung branch, highlighted in the magnified inset in the main panel (a). Panel (d) shows a third kind of resting, asymmetric state: Here, in contrast to 'normal' localized states, the peaks of the two fields no longer coincide but are a wavelength apart. The peaks in each field are locked to the oscillatory tail of the other field. This state forms an isola with stable states between two saddle-node bifurcations. The most narrow stable localized state is found on the RLS_{odd} branch [panel (e)] and consists of a single large amplitude peak of $\phi_2$ and a single low amplitude peak of $\phi_1$, both at the same location.

The saddle-node bifurcations that frame each stable segment on the rung branches can be tracked using two-parameter continuation (Fig. 6). Figure 7 shows that stable, resting, asymmetric localized states are present when $v_0 \neq 0$ and $\bar{\phi}_2 \neq \bar{\phi}_1$; the latter condition results in the splitting of the rung states.

E. Reaction-diffusion System

We mention finally the properties of the standard reaction-diffusion model of FitzHugh–Nagumo (FHN) type. The model consists of two nonconserved variables and their dynamics is determined by diffusion and local nonlinear kinetics. Without diffusion the coupled ordinary differential equations are known as the FitzHugh–Nagumo model originally used to describe nerve
FIG. 5: Bifurcation diagram (left) and solution profiles (right) for the active PFC system (11). Panel (a) shows the $L_2$ norm (4) as a function of the conserved mean density $\bar{\phi} \equiv \bar{\phi}_1 = \bar{\phi}_2$. The homogeneous state (black) is destabilized in a primary bifurcation at $\bar{\phi} \approx -0.75$. From this bifurcation, the branch of periodic states (purple) with $n = 8$ peaks emerges supercritically. The branch of periodic states is destabilized by a secondary bifurcation to RLS$\text{odd}$ (dark blue) and RLS$\text{even}$ (light blue). The branches of localized states are connected by branches (red) of resting, asymmetric localized states. Panels (b)-(e) show stable resting states at the locations indicated by the corresponding labels in the main panel (a). The solution profiles show the fields $\phi_1$ (solid red) and $\phi_2$ (dashed dark blue). Linestyles are as in Fig. 1. The remaining parameters are $r = -1.5$, $q_1 = q_2 = 1$, $c = -0.2$, $v_0 = 0.1$, $c_1 = 0.1$, $c_2 = 0$, $D_r = 0.5$. The domain size is $L = 16\pi$. 

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FIG. 6: Two-parameter continuation of the saddle-node bifurcations that frame the stable, resting, asymmetric localized states in Fig. 5 in the $(\bar{\phi}_1, \bar{\phi}_2)$ plane. Paths of a given color are from the same ladder branch. The lower [upper] thin gray diagonal line shows the continuation path in Fig. 5 [Fig. 7]. Stable, resting, asymmetric localized states are present between each pair of same-color lines cut by the diagonal lines. The remaining parameters are the same as in Fig. 5.

Impulses [27, 58]. The corresponding reaction-diffusion model describes an activator-inhibitor system which can be used to study e.g. collision of nonlinear waves in excitable media [3], self-sustained oscillations in semiconductor amplifiers [5], and diffusion-driven (Turing) instabilities resulting in the emergence of self-organized pattern formation [59, 89].

The FHN model is neither variational, nor mass-conserving. Despite this, we argue that some versions of the model are also nongeneric, in the same sense as the models considered hitherto. Moreover, we can use similar arguments (see the following section) to understand the results
FIG. 7: (a) One-parameter continuation along $\bar{\phi}_1$ when $\bar{\phi}_2 = \bar{\phi}_1 + 0.15$ following the upper gray line in Fig. 6. The right panels show solution profiles at the locations labeled in the main panel (a). Panels (b) and (c) show stable, symmetric RLS$_{\text{odd}}$ states. Panels (d) and (e) show two stable, resting, asymmetric states. Linestyles, norm and the remaining parameters are the same as in Fig. 5.

described below.
We consider here the two-variable model

\[
\begin{align*}
\partial_t u_1 &= D \partial_{xx} u_1 + f(u_1) - u_2 + \kappa + \varepsilon u_2^3, \\
\delta \partial_t u_2 &= \partial_{xx} u_2 - u_2 + u_1,
\end{align*}
\]  

(12)

where we choose \( f(u_1) = \lambda u_1 - u_1^2 \) as e.g. in Ref. [79]. In Eqs. (12) \( u_1 \) and \( u_2 \) denote the activator and inhibitor, respectively, and \( D \) the ratio of their diffusion constants; \( \kappa \) is a source term for the activator and it is used as the control parameter. The time scale ratio \( \delta \) drops out when considering steady states. When \( \varepsilon = 0 \) the local kinetics are of FitzHugh–Nagumo type; the additional term \( \varepsilon u_2^3 \) represents a generalization that corresponds to an amplification [weakening] of the inhibition for large \( u_2 \) values and \( \varepsilon > 0 \) [\( \varepsilon < 0 \)]. For our purposes the additional term is key because it destroys the resting asymmetric localized states present when \( \varepsilon = 0 \).

The equations are invariant w.r.t. \( R_1 : x \mapsto -x \) only. As a result the parity-symmetric spatially localized states present in this model are again of two types, RLS\(_{\text{odd}}\) and RLS\(_{\text{even}}\), with odd (dark blue) and even (light blue) number of peaks, respectively, but this time organized within a standard snaking structure with no tilt. The resulting bifurcation diagram is shown in Fig. 8 for \( \varepsilon = 0 \). In this case the presence of LS requires bistability between the trivial and the periodic states. Moreover, for this parameter value, the system also exhibits resting, linearly unstable asymmetric rung states (red), despite its nongradient structure. Furthermore, these same states begin to drift as soon as \( \varepsilon \neq 0 \) as depicted in panel (e).

In the following section we analyze the nongeneric behavior exhibited by the five explicit models described in this section and explain its origin.

III. WHEN DO STEADY ASYMMETRIC STATES EXIST?

A. A multi-component model

We now propose a general form of a family of multi-component models that allows for resting asymmetric states even when it is nonvariational, i.e. when the system is active. These multi-component models typically have a “dead limit”, i.e. a limiting case exists in which the system is variational. We show that within the proposed model family any resting asymmetric state that exists at a specific parameter value is part of a whole branch of such states. In particular, any steady asymmetric state that exists in the variational limit continues to do so as a resting state in the nonvariational case, at least over a finite parameter range.
FIG. 8: The reaction-diffusion system (12) exhibits standard homoclinic snaking, shown in panel (a) in terms of the $L_2$ norm (4) as a function of the parameter $\kappa$. The homogeneous state (black) is destabilized in a primary bifurcation at $\kappa \approx -0.045$. From this bifurcation, a branch of periodic states (purple) with $n = 8$ peaks, shown in panel (b), emerges subcritically. The branch of periodic states is further destabilized in a secondary bifurcation where branches of localized states with an odd (dark blue) and an even (light blue) number of peaks emerge. Panel (c) shows an example of the former, with $u_1$ shown in solid red and $u_2$ in dashed dark blue. The branches of localized states are connected by branches (red) of unstable, resting, asymmetric localized states, shown in panel (d). The resting asymmetric states such as (d) start moving as soon as $\varepsilon \neq 0$, as shown in panel (e). The locations of the solution profiles are indicated by corresponding labels b-d in the main panel (a). Linestyles are as in Fig. 1. The remaining parameters are $D = 0.14$, $\lambda = 1.01$, $\varepsilon = 0$, $\delta = 0.3$. The domain size is $L = 40$.

We assume a general $M$-component order parameter field $\mathbf{u} = \{u_i\}$ that consists of $M_1$ non-
conserved components \( n_i \) and \( M_2 \) conserved components \( c_i \). We assume that the dynamics of the \( i \)-th component can be written as

\[
\frac{\partial}{\partial t} u_i = \begin{cases} 
\frac{\partial}{\partial t} n_i = - \left( \frac{\delta G}{\delta n_i} + \alpha_i \frac{\delta G_C}{\delta n_i} \right) & \text{for } i = 1, \ldots, M_1, \\
\frac{\partial}{\partial t} c_i = \frac{\partial}{\partial x \cdot x} \left( \frac{\delta G}{\delta c_i} + \alpha_i \frac{\delta G_C}{\delta c_i} \right) & \text{for } i = M_1 + 1, \ldots, M_1 + M_2, 
\end{cases}
\]

(13)

where \( G \) and \( G_C \) are two functionals containing self-interaction and coupling terms, respectively. Furthermore, we introduce the operator \( D_{xx}^{(i)} \) to avoid an explicit distinction between conserved and nonconserved fields. For this purpose we simply define \( D_{xx}^{(i)} = -1 \) if \( i \leq M_1 \) and \( D_{xx}^{(i)} = \frac{\partial}{\partial x \cdot x} \) if \( i > M_1 \). Equation (13) then takes the compact form

\[
\frac{\partial}{\partial t} u_i = D_{xx}^{(i)} \left( \frac{\delta G}{\delta u_i} + \alpha_i \frac{\delta G_C}{\delta u_i} \right).
\]

(14)

Here we only consider steady states. Because in this formulation mobilities do not alter the steady states of the system, we do not include them. Since \( G \) only consists of self-interaction terms it can be written as a sum of independent contributions, i.e.

\[
G[u] = \sum_i G_i[u_i].
\]

If \( \alpha_i = 1 \ \forall i \) we obtain the variational case with the Lyapunov functional \( \hat{G} = G + G_C \). More precisely, even for arbitrary but purely positive entries, i.e. \( \alpha_i > 0 \), the system corresponds to a variational model since we can rewrite Eq. (14) as

\[
\frac{\partial}{\partial t} u_i = \alpha_i D_{xx}^{(i)} \left( \alpha_i^{-1} \frac{\delta G}{\delta u_i} + \frac{\delta G_C}{\delta u_i} \right),
\]

(15)

describing gradient dynamics with the Lyapunov functional \( \hat{G} = \sum_i \alpha_i^{-1} G_i[u_i] + G_C \) and prefactors \( \alpha_i \) that act as an effective (positive definite) mobility matrix. Note that for purely negative values one can incorporate the negative sign into the coupling terms \( \frac{\delta G_C}{\delta u_i} \), i.e. the system is still variational. In contrast, Eq. (14) becomes nonvariational if the set of \( \alpha_i \) contains both positive and negative values. This is because in this case the functional \( \hat{G} \) is no longer bounded from below and the effective mobility matrix is indefinite. As a result, the system needs not evolve to a steady state, a reflection of its active character.

In the following we take Eq. (14) as given and consider steady states \( u^0 \) for which we replace the left hand side by zero. We integrate the steady state equation for the conserved fields once
and assume that the integration constant, the flux, is zero.\footnote{The active PFC model is an example where the flux is not zero. We show in Section III D 1 why the results of the present section still apply.} Evaluating the result together with the spatial derivative of the steady state equations of the nonconserved fields we obtain

$$0 = \sum_j L_{ij} \partial_x u_j^0,$$

(16)

where $L$ is the operator with components

$$L_{ij} = \frac{\delta^2 G}{\delta u_i \delta u_j} + \alpha_i \frac{\delta^2 G_C}{\delta u_i \delta u_j} \delta_{ij} + \alpha_i \frac{\delta^2 G_C}{\delta u_i \delta u_j} = \delta_{ii}^G \delta_{ij} + \alpha_i L_{ij}^C.$$

(17)

Note that $L$ is evaluated at the steady state $\mathbf{u} = \mathbf{u}^0$.

**B. Existence of asymmetric steady states**

We proceed as follows. We assume that there exists an asymmetric state for a given choice of $\alpha_i = \alpha_i^0$. A natural example would be the asymmetric states that exist in the variational limit. We then consider infinitesimal but arbitrary changes $\delta \alpha_i$ in the parameters and show that the resulting slightly modified, but still asymmetric, state remains steady.

**Step 1: Derive the solvability condition**

We assume that for some initial parameters, $\alpha_i = \alpha_i^0$, an asymmetric steady state $\mathbf{u}^0$ exists. Next, we consider the changed parameters $\alpha_i = \alpha_i^0 + \delta \alpha_i$ resulting in a slightly modified steady state $\mathbf{u} = \mathbf{u}^0 + \delta \mathbf{u}$. We insert both expressions into Eq. (14) with $\partial_t u_i = 0$ and linearize the result:

$$\sum_j D^{(i)}_{xx} (L_{ij} \delta u_j) = - \delta \alpha_i D^{(i)}_{xx} \frac{\delta G_C}{\delta u_i}.$$

(18)

Now, if we can always find a steady state correction $\delta \mathbf{u}$ that solves Eq. (18) then there exists a branch of asymmetric steady states. The solvability condition (which is sufficient) for this to be the case is

$$0 = \sum_i \delta \alpha_i \langle \delta u_i^\dagger, D^{(i)}_{xx} \frac{\delta G_C}{\delta u_i} \rangle$$

(19)

with the adjoint zero eigenvector $\delta \mathbf{u}^\dagger$ solving the adjoint linear homogeneous equation

$$\sum_j (L^\dagger)_{ij} D^{(j)}_{xx} \delta u_j^\dagger = 0 \quad \forall i.$$  

(20)
The expression \( \langle \ldots, \ldots \rangle \) denotes spatial integration over the finite domain \([−L_2, L_2]\) with periodic boundary conditions.\(^2\)

**Step 2: Find the adjoint zero eigenvector**

We can write the adjoint linear operator \( \mathcal{L}^\dagger \) as

\[
(\mathcal{L}^\dagger)_{ij} = \mathcal{L}_{ji} = \mathcal{L}_i^g \delta_{ij} + \alpha_j \mathcal{L}_i^{GC} = \frac{\alpha_j}{\alpha_i} (\mathcal{L}_i^{G} \delta_{ij} + \alpha_i \mathcal{L}_i^{GC}) = \frac{\alpha_j}{\alpha_i} \mathcal{L}_{ij}, \tag{21}
\]

where we have used the self-adjointness of \( \mathcal{L}_i^g \) and \( \mathcal{L}_i^{GC} \). Inserting Eq. (21) into Eq. (20) we obtain

\[
\sum_j \frac{\alpha_j}{\alpha_i} \mathcal{L}_{ij} D^{(j)}_{xx} \delta u_j^\dagger = 0 \iff \sum_j \mathcal{L}_{ij} \alpha_j D^{(j)}_{xx} \delta u_j^\dagger = 0. \tag{22}
\]

Comparing the adjoint linear equation (22) with Eq. (16) we see that a permitted adjoint zero eigenvector can be found as

\[
\alpha_j D^{(j)}_{xx} \delta u_j^\dagger = N \partial_x u_0^j \tag{23}
\]

with one arbitrary (normalization) constant \( N \). In the variational case \( \alpha_j = \alpha \) the adjoint zero eigenvector is the translation mode for the nonconserved fields and the second spatial integral of the translation mode for the conserved fields, respectively. In the nonvariational case each component of the adjoint zero eigenvector is still proportional to the (independent) translation mode of the corresponding field. However, the ratio of these modes is given by the inverse ratio of the (active) coupling parameters, i.e.

\[
\frac{D^{(j)}_{xx} \delta u_j^\dagger}{D^{(i)}_{xx} \delta u_i^\dagger} = \frac{\alpha_j \partial_x u_0^i}{\alpha_i \partial_x u_0^j}. \tag{24}
\]

Moreover, the adjoint zero eigenvector has to satisfy the boundary conditions, i.e. \( \delta u_i^\dagger(x = \frac{L_2}{2}) = \delta u_i^\dagger(x = -\frac{L_2}{2}) \). This is trivially true for the nonconserved part since \( \delta n_i^\dagger = -\frac{N}{\alpha_i} \partial_x n_i^0 \). For the conserved part we integrate Eq. (23) twice and find

\[
\delta c_i^\dagger(x) = \frac{N}{\alpha_i} \left( \int_{-\frac{L_2}{2}}^x dx' c_i^0(x') + E_i x + F_i \right). \tag{25}
\]

\(^2\) Our analysis also applies to an infinite domain. Domains with other boundary conditions do not allow stationary drifting states and are not considered here.
Then $\delta c_i^\dagger(x = \frac{L}{2}) \equiv \delta c_i^\dagger(x = -\frac{L}{2})$ yields

$$
\frac{N}{\alpha_i} \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' c_i^0(x') + E_i \frac{L}{2} + F_i \right) = \frac{N}{\alpha_i} \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' c_i^0(x') - E_i \frac{L}{2} + F_i \right)
$$

$$
\Rightarrow E_i = -\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' c_i^0(x').
$$

Thus the periodic boundary conditions determine the integration constants $E_i$ as the (negative) mean value of the steady conserved field $c_i^0$. The remaining integration constants $F_i$ remain undetermined.

Note that if some coupling parameters are zero, the factor $\frac{1}{\alpha_i}$ in Eq. (21) is problematic. This partially coupled case needs to be considered separately as done in Appendix A.

**Step 3: Show that the adjoint zero eigenvector solves the solvability condition**

We are left to show that the adjoint zero eigenvector solves the solvability condition (19). For this purpose we note that for any functional $G[u] = \int dx \, f(u, \partial_x u, \partial_{xx} u, \ldots)$

$$
0 = \int dx \, \partial_x f = \langle \partial_x u, \frac{\partial G}{\partial u} \rangle \equiv \sum_i \langle \partial_x u_i, \frac{\partial G}{\partial u_i} \rangle. \tag{26}
$$

For the self-interaction terms contained in $G (= \sum_i G_i[u_i])$, we therefore have

$$
0 = \langle \partial_x u_i, \frac{\partial G_i}{\partial u_i} \rangle \quad \forall \, i, \tag{27}
$$

here evaluated on the steady state $u = u^0$.

Next, we take the steady state Eq. (14) and multiply its $i$-th component by the $i$-th component of the adjoint zero eigenvector $\delta u_i^\dagger$, yielding

$$
0 = \langle \delta u_i^\dagger, D_{xx}^{(i)} \left( \frac{\partial G}{\partial u_i} + \alpha_i \frac{\partial G_C}{\partial u_i} \right) \rangle = \langle D_{xx}^{(i)} \delta u_i^\dagger, \frac{\partial G}{\partial u_i} \rangle + \alpha_i \langle \delta u_i^\dagger, D_{xx}^{(i)} \frac{\partial G_C}{\partial u_i} \rangle. \tag{28}
$$

To obtain the first term we twice integrated by parts assuming that the boundary terms vanish.\(^3\) In the fully coupled case ($\alpha_i \neq 0 \, \forall \, i$) we insert Eq. (23) for the first term in Eq. (28) and use Eq. (27) to obtain

$$
0 = \langle \delta u_i^\dagger, D_{xx}^{(i)} \frac{\partial G_C}{\partial u_i} \rangle \quad \forall \, i. \tag{29}
$$

In view of Eqs. (27)–(29) each summand in Eq. (19) vanishes, i.e. the adjoint zero eigenvector solves the solvability condition. This result extends to the partially coupled case as well (Appendix A).

\(^3\) For the active PFC model this assumption is not valid. We consider this special case separately in Section III D 1.
In summary, any asymmetric steady state of Eq. (14) that exists for a particular set of parameters remains at rest when these parameters are continuously changed (at least for some finite range of these parameters). In particular, the asymmetric states of the passive system in general do not begin to drift when the system becomes active unless the assumed form (14) is broken. Evidently this is a necessary but not sufficient requirement for the appearance of motion. Note that the drifting asymmetric states in the models discussed in Section II are found on branches that have no counterpart in the passive limit.

C. Onset of motion

Besides the knowledge that resting states exist, it is also of great interest to determine the onset of motion. For this we follow Ref. [63] and consider an expansion in the drift velocity $v$ of the form

$$u(x, t) = u^0(x + vt) + vu^1(x + vt) + O(v^2),$$

as appropriate for steadily drifting states in the vicinity of the drift bifurcation. In the comoving frame $\xi = x + vt$, $\partial_x = \partial_\xi$ and $\partial_t = v\partial_\xi$, and Eq. (14) yields

$$\sum_j D_{xx}^{(i)} L_{ij} u_j^1 = \partial_x u_0^0$$

at leading order in $v$. Here $\partial_x u^0$ is the Goldstone mode. This inhomogeneous equation can be solved for $u^1$ provided the solvability condition

$$0 = \langle \partial_x u^0, \delta u^\dagger \rangle = -\langle u^0, \partial_x \delta u^\dagger \rangle$$

holds, where $\delta u^\dagger$ is the adjoint zero eigenvector determined in the previous section [see Eq. (23)].

For the fully coupled case we have determined in the previous section that

$$D_{xx}^{(i)} \delta u_i^\dagger = \frac{N}{\alpha_i} \partial_x u_0^0,$$

i.e. the nonconserved components are

$$\delta n_i^\dagger = -\frac{N}{\alpha_i} \partial_x n_0^0$$

and the conserved components are

$$\partial_x \delta c_i^\dagger = \frac{N}{\alpha_i} \left( c_i^0 + E_i \right) = \frac{N}{\alpha_i} \left( c_i^0 - \frac{1}{L} \int_{-L/2}^{L/2} dx' c_i^0 \right).$$
Then the solvability condition (32) becomes

\[ 0 = \sum_{i=1}^{M_1} -\frac{N}{\alpha_i} \langle \partial_x n_0^i, \partial_x n_0^i \rangle - \sum_{i=M_1+1}^{M_1+M_2} \frac{N}{\alpha_i} \langle c_i^0, c_i^0 + E_i \rangle \]  

(36)

\[ \Leftrightarrow 0 = \sum_{i=1}^{M_1} \frac{1}{\alpha_i} \int_{-L/2}^{L/2} dx |\partial_x n_0^i|^2 + \sum_{i=M_1+1}^{M_1+M_2} \frac{1}{\alpha_i} \left( \frac{1}{L} \int_{-L/2}^{L/2} dx |c_i^0|^2 - \left( \frac{1}{L} \int_{-L/2}^{L/2} dx c_i^0 \right)^2 \right) \]  

(37)

\[ \Leftrightarrow 0 = \sum_i \frac{1}{\alpha_i} \left( \left\langle (\partial_x n_0^i)^2 \right\rangle - \left\langle \partial_x n_0^i \right\rangle^2 \right) + \sum_i \frac{1}{\alpha_i} \left( \left\langle (c_i^0)^2 \right\rangle - \left\langle c_i^0 \right\rangle^2 \right). \]  

(38)

Equation (38) tells us that motion sets in at the particular point where the sum of the variances weighted by the corresponding inverse coupling parameter vanishes. These are the variances of the first spatial derivative of the nonconserved fields and those of the conserved fields themselves.

The above prediction relies on the assumed model structure, Eq. (14), and will be employed in the following section. Note that in the variational case ($\alpha_i = \alpha$) Eq. (38) can never be satisfied since all variances are positive, i.e. there is no onset of motion, an observation in line with expected behavior. The onset of motion in the partially coupled case is discussed in Appendix A.

The expression for the onset of motion calculated above holds for any steady state, i.e. it applies equally to (anti)symmetric and asymmetric steady states; it does not apply to time-dependent states, including steadily drifting states, i.e. for these states condition (38) may be fulfilled though the velocity is nonzero. In the former case the bifurcation is a drift-pitchfork bifurcation provided $u^1$ breaks the parity symmetry of $u^0$. In the latter case $u^0$ is already asymmetric, and the onset of motion then corresponds to a drift-transcritical bifurcation. Both types of bifurcation can be found in the models considered in Section II. Based on the new insights of the present section we now revisit these models.

D. How do the above models satisfy the proposed structure, Eq. (14)?

1. Active phase field crystal model

The active phase field crystal model (3) does not seem to fit the proposed form Eq. (14). Since the density-like order parameter $\phi$ is conserved (with mean density $\int_{-L/2}^{L/2} dx \phi = 0$) its dynamics
can be written according to Eq. (14) as
\[ \partial_t \phi = \partial_{xx} \left( \left[ \epsilon + (1 + \partial_{xx})^2 \right] \phi + (\bar{\phi} + \phi)^3 - v_0 \int^x dx' P \right) . \] (39)

The coupling term must have a similar structure in the equation for the polarization field, i.e. we write somewhat unusually
\[ \partial_t P = \partial_{xx} \left( c_1 P + c_2 P^3 - D_r \int \int dx' dx'' \left( c_1 P + c_2 P^3 \right) - v_0 \int^x dx' \phi \right) . \] (40)

Now, for the existence of steady asymmetric states it is crucial that \( c_2 = 0 \). Then we can identify
\[ G = \mathcal{F}_{SH} + \mathcal{F}_P + D_r \frac{c_1}{2} \int dx \left( \int^x dx' P \right)^2 , \]
\[ \mathcal{G}_C = \int dx \left( P \int^x dx' \phi \right) , \]
\[ \alpha_1 = v_0 \quad \text{and} \quad \alpha_2 = -v_0 \]

and in this ways the active PFC model does fit into the model family of Eq. (14). Furthermore, for steady states \((\phi^0, P^0)\) we find via integration of Eq. (40) over the whole domain
\[ \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' P^0 = 0 , \] (41)

while Eqs. (39) and (40) yield (still for \( c_2 = 0 \)),
\[ 0 = \partial_x \left( \left[ \epsilon + (1 + \partial_{xx})^2 \right] \phi^0 + (\bar{\phi} + \phi)^3 - v_0 \int^x dx' P^0 \right) = \mathcal{L}_{11} \partial_x \phi^0 + v_0 \mathcal{L}_{12} \partial_x P^0 , \]
\[ J_P = \partial_x \left( c_1 P^0 - D_r \int \int dx' dx'' c_1 P^0 - v_0 \int^x dx' \phi^0 \right) \]
\[ = \mathcal{L}_{22} \partial_x P^0 - v_0 \mathcal{L}_{12} \partial_x \phi^0 , \] (43)

by analogy with Eq. (16). Here \( J_P \) is a constant of integration that is nonzero due to the nonlocal term in Eq. (43), in contrast to Eq. (16). We determine this constant (the flux) by integrating over the domain:
\[ J_P = -\frac{D_c c_1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' P^0(x') . \] (44)

Despite this difference our general result remains true. First, regarding the adjoint zero eigenvector from Eq. (25) we have
\[ \delta \phi^\dagger = \frac{1}{v_0} \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' \phi^0 + E_1 x + F_1 \right) , \] (45)
\[ \delta P^\dagger = -\frac{1}{v_0} \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' P^0 + E_2 x + F_2 \right) \] (46)
with

\[
E_1 = -\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' \phi^0 = 0 ,
\]

\[
E_2 = -\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' P^0 = 0 .
\]

As in the general derivation the constant \( F_1 \) remains undetermined since the adjoint linear equation only contains first and second derivatives of \( \delta \phi^\dagger \). However, owing to the double integral in Eq. (40) \( \delta P^\dagger \) occurs by itself and we must then choose

\[
F_2 = \frac{J_P}{D_i c_1}
\]

so that the adjoint zero eigenvector solves the adjoint linear equation with the correct, periodic boundary conditions as described by Eqs. (42) and (43). Second, for step 3 of the derivation in the previous section we need the result

\[
0 = \langle \delta u_i^\dagger, D_{xx}^{(i)} \frac{\delta G}{\delta u_i} \rangle = \langle D_{xx}^{(i)} \delta u_i^\dagger, \frac{\delta G}{\delta u_i} \rangle ,
\]

i.e. we assume that the boundary terms vanish. To verify this assumption we note that

\[
\partial_{xx} \frac{\delta G}{\delta P} \sim \partial_{xx} \int \int dx' dx'' P^0 = P^0
\]

so that

\[
\langle \delta P^\dagger, P^0 \rangle = \langle \int_{-\frac{L}{2}}^{\frac{L}{2}} dx P^0 + F_2, P^0 \rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' P^0 + F_2 \right) P^0
\]

\[
= \frac{1}{2} \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx P^0 \right)^2 + F_2 \int_{-\frac{L}{2}}^{\frac{L}{2}} dx P^0 = 0
\]

and hence Eq. (48) still holds. Thus after these additional considerations the active PFC model with \( c_2 = 0 \) also falls into the class of models for which we may expect steady asymmetric states, as already found in Section II A.

In contrast, when \( c_2 \neq 0 \) we cannot write the kinetic equations in the form of Eq. (14) since a term proportional to \( \int \int dx' dx'' P^3 \) cannot be written as a functional derivative, and this is so for similar terms as well. We conclude that for \( c_2 \neq 0 \) all asymmetric states necessarily drift.

To emphasize that resting behavior is not caused by the linearity of Eq. (40) when \( c_2 = 0 \) we also considered the problem

\[
\partial_t P = \partial_{xx} \left( c_1 P + c_2 P^3 - D_i c_1 \int \int dx' dx'' P - v_0 \int x' \phi \right)
\]
and still observed asymmetric steady states.

According to Eq. (38), the steady states in either case lose stability to drift when
\[
\langle (φ^0)^2 \rangle - \langle (P^0)^2 \rangle = \int_{-L}^{L} dx \left( |φ^0|^2 - |P^0|^2 \right) = 0. \tag{52}
\]
This equation predicts correctly the location of the drift-pitchfork and the drift-transcritical bifurcations, indicated by open and filled triangles in Fig. 1(a).

2. Nonreciprocal Cahn-Hilliard model

The nonreciprocal Cahn-Hilliard model (5) can be written in the proposed form (14) with
\[
G = F^{CH}_1 + F^{CH}_2, \tag{53}
\]
where
\[
F^{CH}_i = \int dx \kappa_i \left( \frac{∂^2}{∂x^2} φ_i \right)^2 + f'_i(φ_i), \tag{54}
\]
\[
G_C = -\int dx φ_1 φ_2, \tag{55}
\]
\[
α_1 = ρ + α, \quad α_2 = ρ - α \tag{56}
\]
and \(κ_1 = 1\), \(κ_2 = κ\). According to Eq. (38) the onset of motion occurs when a steady state satisfies\(^4\)
\[
\sum_i \frac{1}{α_i} \left( \langle (φ^0_i)^2 \rangle - \langle φ^0_i \rangle^2 \right) = 0
\]
\[
⇔ \int_{-L}^{L} dx |φ^0_i|^2 - \frac{1}{L} \left( \int_{-L}^{L} dx φ^0_i \right)^2 + \frac{ρ + α}{ρ - α} \left( \int_{-L}^{L} dx |φ^0_i|^2 - \frac{1}{L} \left( \int_{-L}^{L} dx φ^0_i \right)^2 \right) = 0. \tag{57}
\]
This condition predicts correctly the location of the drift-pitchfork bifurcations shown as open triangles in Fig. 3(b,c) and of the drift-transcritical bifurcation shown as a filled triangle in Fig. 3(c). It also confirms that no drift bifurcations occur on the branches shown in Fig. 2 (a,d).

A different nonreciprocal Cahn-Hilliard model is studied in Ref. [78]. After rescaling, this model takes the form
\[
\frac{∂φ_1}{∂t} = \frac{∂^2}{∂x^2} \left( -\frac{∂^2 φ_1}{∂x^2} + f'_1(φ_1) \right) + α_1 φ_2, \tag{58}
\]
\[
\frac{∂φ_2}{∂t} = \frac{∂^2}{∂x^2} \left( -κ \frac{∂^2 φ_2}{∂x^2} + f'_2(φ_2) \right) + α_2 φ_1.
\]
\(^4\) The corresponding expression in [28] is missing the term \(⟨φ^0_i⟩^2\).
Here the nonvariational coupling breaks both conservation laws. Despite this the model can still be written in the proposed form (14) using the functional $G$ given above and adapting only $G_C$: 

$$G_C = \int dx \left( \phi_1 \int^{x'} dx' \int^{x''} dx'' \phi_2 \right). \quad (59)$$

As for the active PFC model integration of the steady state version of Eqs. (58) shows on the one hand that the masses of both fields vanish,

$$0 = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi^0_1 = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi^0_2,$$

and on the other hand yields the two nonzero constant fluxes $J_i$ with

$$J_1 = \frac{\alpha_1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int^{x'} dx' \phi^0_2(x'), \quad (60)$$

$$J_2 = \frac{\alpha_2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int^{x} dx' \phi^0_1(x'). \quad (61)$$

Again, the nonzero fluxes determine the constants $F_i$ for the adjoint zero eigenvectors [see Eq. (25)]:

$$\delta \phi^i_1 = \int dx' \phi^0_1 + F_i = \int dx' \phi^0_i + \frac{J_i}{\alpha_i}. \quad (62)$$

The second issue that occurs in the active PFC model related to Eq. (48) does not occur in this case, since the integral term is the coupling term in contrast to the active PFC model. We conclude that our general result also holds for this nonreciprocal nonconserved Cahn-Hilliard model. According to Eq. (38) the onset of motion occurs when

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left( |\phi^0_1|^2 + \frac{\rho + \alpha}{\rho - \alpha} |\phi^0_2|^2 \right) = 0,$$

a prediction that can in principle be checked following the computations in [78].

3. Coupled Cahn-Hilliard and Swift-Hohenberg equations

The model (8) fits the model family (14) with

$$G = \mathcal{F}^{\text{CH}} + \mathcal{F}^{\text{SH}},$$

where

$$\mathcal{F}^{\text{SH}} = \int dx \left( -\frac{\phi_2}{2} \left[ r - (1 + \partial_{xx})^2 \right] \phi_2 + f_2(\phi_2) \right),$$

$$G_C = \int dx \phi_1 \phi_2,$$

$$\alpha_1 = \rho + \alpha, \quad \alpha_2 = \rho - \alpha.$$

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with $F^{\text{CH}}$ as in Eq. (54). In contrast to the previous models in which all variables exhibit conserved
dynamics, here $\phi_1$ is conserved but $\phi_2$ is not. Thus $\phi_1$ takes the role of $c_1$ and $\phi_2$ takes the role of
$n_1$ in Eq. (13). Since Eq. (38) still applies we see that steady states become unstable with respect
to drift when

$$\frac{\rho + \alpha}{\rho - \alpha} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx |\partial_x \phi_2^0|^2 + \int_{-\frac{L}{2}}^{\frac{L}{2}} dx |\phi_1^0|^2 - \frac{1}{L} \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi_1^0 \right)^2 = 0. \quad (63)$$

This condition correctly predicts all drift-pitchfork and drift-transcritical bifurcations shown in
Fig. 4 (a).

4. Coupled passive and active PFC model

If $c_2 = 0$ we can rewrite Eqs. (11) in a similar way to what we have done for the active PFC
model in Sec. III D 1:

$$\partial_t \phi_1 = \partial_{xx} \left[ (\epsilon + (1 + \partial_{xx})^2) \phi_1 + (\phi_1 + \phi_2)^3 + c \phi_2 - v_0 \int^x dx' P \right], \quad (64)$$

$$\partial_t P = \partial_{xx} \left[ c_1 P - D_1 c_1 \int dx' dx'' P - v_0 \int^x dx' \phi_1 \right], \quad (65)$$

$$\partial_t \phi_2 = \partial_{xx} \left[ (\epsilon + (1 + \partial_{xx})^2) \phi_2 + (\phi_2 + \phi_2)^3 + c \phi_1 \right]. \quad (66)$$

We see that these equations fit the form (14) with

$$G = \sum_{i=1,2} F_i^{\text{SH}} + F_P, \quad (67)$$

$$F_P = \int dx \left[ \frac{c_1}{2} P^2 - \frac{c_1}{2} D_1 \left( \int^x dx' P \right)^2 \right], \quad (68)$$

$$G_C = \int dx \left[ c \phi_1 \phi_2 + v_0 \left( P \int^x dx' \phi_1 \right) \right], \quad (69)$$

$$\alpha_1 = \alpha_3 = 1 \quad \text{and} \quad \alpha_2 = -1. \quad (70)$$

Similar considerations to those used in the active PFC model regarding the double integral term in
the polarization equation apply here as well, and help us understand the existence of asymmetric
steady states in this model. Owing to the vanishing of the mean densities and the polarization in
steady state, i.e.

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \phi_i^0 = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx P^0 = 0, \quad (71)$$

the onset of motion occurs when the steady state satisfies

\[
\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left( |\phi_1^0|^2 - |P^0|^2 + |\phi_2^0|^2 \right) = 0 ,
\]  

(72)

cf. Eq. (38). This condition confirms that none of the steady states shown in Figs. 5 and 7 exhibit a drift instability. For larger activities such instabilities do occur and are also reliably predicted by Eq. (72) (not shown).

5. Reaction-diffusion system

Remarkably, even the FitzHugh–Nagumo model, Eqs. (12) with \( \varepsilon = 0 \), can be written in the form (14), this time with

\[
G = G_1 + G_2 = \int dx \left( \frac{D}{2} (\partial_x u_1)^2 - \frac{\lambda}{2} u_1^2 + \frac{1}{4} u_1^4 - \kappa u_1 + \frac{1}{2} (\partial_x u_2)^2 + \frac{1}{2} u_2^2 \right) ,
\]  

(73)

\[ G_C = \int dx \, u_1 u_2 , \]  

(74)

\[ \alpha_1 = 1 \text{ and } \alpha_2 = -1 . \]  

(75)

This observation explains the presence of resting asymmetric states in this model, such as state (d) in Fig. 8. According to Eq. (38) stationary drift sets in when a steady state satisfies

\[
\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left( |\partial_x u_1^0|^2 - |\partial_x u_2^0|^2 \right) = 0 .
\]  

(76)

Since none of the steady states shown in Fig. 8 meets this condition, no onset of motion is observed. However, when \( \varepsilon \neq 0 \) the equation structure (14) is broken and consequently all asymmetric states now drift as verified in Fig. 8(e).

We mention that the class of models exhibiting steady asymmetric states can be extended further, as described in Appendix C. However, this generalization is not required for any of the models studied in the present work.

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IV. HOW IS GENERIC BEHAVIOR RESTORED?

Now that we have been able to explain the presence of asymmetric steady states in linearly coupled models, let us show how generic behavior may be recovered. For this purpose, we consider a model consisting of two nonlinearly coupled Swift-Hohenberg (SH) equations. First, we verify that steady asymmetric states are still observed for nonlinear couplings that respect the specific structure considered in the previous section. Subsequently, we show that all steady asymmetric states of this type immediately begin to move when this structure is weakly broken. We then use the insight from the study of this model to suggest different ways in which the models of Section II can be adapted and/or extended to restore generic behavior in these cases, too.

We begin with introducing two (nonlinearly) coupled Swift-Hohenberg equations as a simple example that captures the properties of resting states in active systems arising from symmetry and those arising from the special form of the coupling:

\begin{align*}
\partial_t \phi_1 &= \left[ r_1 - (q_1^2 + \partial_{xx}) \right] \phi_1 + \delta_1 \phi_1^3 + b_1 \phi_2^3 - \phi_1^5 + \alpha_1 \left( \phi_2 + \beta \phi_1 \phi_2^2 + \gamma \partial_{xxx} \phi_2 \right), \\
\partial_t \phi_2 &= \left[ r_2 - (q_2^2 + \partial_{xx}) \right] \phi_2 + \delta_2 \phi_2^3 + b_2 \phi_2^3 - \phi_2^5 + \alpha_2 \left( \phi_1 + \left( \beta + \tilde{\beta} \right) \phi_1^3 \phi_2 - \left( \gamma + \tilde{\gamma} \right) \partial_{xxx} \phi_1 \right).
\end{align*}

(77)

Based on our general considerations we expect to find steady asymmetric states whenever \( \tilde{\beta} = \tilde{\gamma} = 0 \), since Eqs. (77) then take the form of two nonconserved fields that can be written in the form (14) with

\begin{align*}
G &= G_1^{\text{SH}} + G_2^{\text{SH}}, \\
G_i^{\text{SH}} &= \int dx \left( -\frac{\phi_i^3}{3} + \left( q_i^2 + \partial_{xx} \right) \phi_i + \frac{b_i}{4} \phi_i^4 + \frac{1}{6} \phi_i^6 \right), \\
G_C &= -\int dx \left( \phi_1 \phi_2 + \frac{\beta}{2} \phi_1^2 \phi_2^2 + \gamma \phi_1 \partial_{xxx} \phi_2 \right).
\end{align*}

If we turn off the nonlinear and dispersive coupling terms, we obtain two linearly coupled Swift-Hohenberg equations similar to the SH equations studied in Ref. [78] where the authors focused on the onset of motion but did not report the existence of any steady asymmetric states. Beside the additional coupling terms we also consider the possibility of breaking the inversion symmetry, i.e. taking \( \delta_i \neq 0 \), but employ a cubic-quintic nonlinearity to allow for a subcritical primary bifurcation even in the inversion-symmetric case. Note that we permit the wavenumbers \( q_i \) to differ. The choice of these wavenumbers does not affect the symmetry properties of the model.
but can be leveraged to accentuate the consequences of changing the parameters that do, as done in Fig. 9 below.

We begin with the reflection and inversion-symmetric case, i.e. we set \( \delta_1 = \delta_2 = \gamma = \tilde{\gamma} = 0 \). Furthermore we set \( \tilde{\beta} = 0 \). Equations (77) are then symmetric with respect to the reflection \( R_1 : x \mapsto -x \) and the inversion \( R_2 : (\phi_1, \phi_2) \mapsto (-\phi_1, -\phi_2) \). Note that the gradient structure is broken when \( \alpha_1 \) and \( \alpha_2 \) have opposite signs although the form of Eq. (14) is maintained as \( \tilde{\beta} = \tilde{\gamma} = 0 \). Furthermore, we define \( r_1 = r \) and \( r_2 = r + r_\Delta \) and use \( r \) as the main control parameter which in many applications of the SH equation plays the role of an effective temperature.

The bifurcation diagram in Fig. 9(a) confirms that steady asymmetric states exist and form the rung states of the snakes-and-ladders structure typical of SH-like equations. At the primary bifurcation of the trivial state (black line) a periodic state with 5 peaks emerges subcritically (dark purple line). This state is further destabilized in the first secondary bifurcation generating two distinct branches of localized states that ultimately form the homoclinic snaking structure and eventually terminate back on the 5-peak branch. Note that each of the two snaking branches corresponds to two overlapping branches of symmetry-related states [39]: the light blue [dark blue] branch consists of states that are invariant under \( R_1 \) and hence of even parity [point-symmetric states invariant under \( R_1 \circ R_2 \) and hence of odd parity]. Both snaking branches change their stability at successive folds and nearby pitchfork bifurcations. The latter give rise to branches of asymmetric states (red dashed lines), i.e. states that are neither reflection nor point-symmetric. Note that these states are at rest even though the gradient structure is broken. They cannot, however, be realized dynamically since they are always unstable.

Next, we investigate whether the different localized states start to move when \( \tilde{\beta} \) or \( \tilde{\gamma} \) are nonzero. We consider the states at the locations denoted by the three cross symbols in panel (a) and vary \( \tilde{\gamma} \) and \( \tilde{\beta} \) in panels (e) and (f), respectively. For \( \tilde{\gamma} \neq 0 \) the additional dispersive coupling term \( \tilde{\gamma} \partial_{xxx} \phi_1 \) breaks the reflection symmetry and the proposed form in Eq. (14). In this case all states lose their even or odd parity symmetry and start to move [cf. the three colored lines in panel (d)] with the velocity increasing monotonically with increasing \( \tilde{\gamma} \). This is similar to the behavior in the one-field case when reflection symmetry is broken [9]. Note that the speed of the originally symmetric states only depend on \( |\tilde{\gamma}| \) while the drift of the originally asymmetric states is (weakly) sensitive to the sign of \( \tilde{\gamma} \). However, this effect is not visible to the eye.

In panel (f) we see that for nonzero \( \tilde{\beta} \) the asymmetric state begins to move (red dashed line), in contrast to the even and odd symmetric localized states (dark blue line) which remain at rest.
FIG. 9: Two coupled SH equations (77) exhibit snaking behavior with resting symmetric and asymmetric states when $\tilde{\beta} = \tilde{\gamma} = 0$. Shown is the $L_2$ norm (4) in the cases of (a) reflection and inversion symmetry ($\gamma = \delta_i = 0$), (b) reflection symmetry but broken inversion symmetry ($\delta_1 = 0.02, \delta_2 = 0.04, \gamma = 0$), (c) inversion symmetry but broken reflection symmetry ($\delta_i = 0, \gamma = 0.8$) and (d) broken reflection and broken inversion symmetry ($\delta_1 = 0.02, \delta_2 = 0.04, \gamma = 0.8$) as a function of the parameter $r$. Panels (e)-(h) show the velocity $v$ of specific localized states marked by cross, plus and circle symbols in panels (a), (b) and (c), respectively, as a function of $\tilde{\gamma}$ (e) and $\tilde{\beta}$ [(f), (g) and (h)]. Red and dark red lines indicate steady asymmetric states. The remaining parameters are $\eta_{\text{H}} = -0.3, q_1 = 1, q_2 = 1.3, \alpha_1 = -0.1, \alpha_2 = 0.125, \beta = 0.2$. The domain size is $L = 10\pi$. Linestyles are as in Fig. 1.
for any $\tilde{\beta}$. Since the additional terms $\sim \phi_i \phi_j^2$ preserve both inversion and reflection symmetry, the states with the corresponding symmetries remain at rest even though the form (14) of the model is broken. The absence of movement due to symmetries is explained in Appendix B. This behavior is also reported in Ref. [8] where nonvariational terms are added to the original SH equation. There, the model retains reflection symmetry and, hence, all symmetric states are steady for arbitrary parameter values.

Next, we investigate the role of the inversion symmetry $R_2$. Panel (b) shows a bifurcation diagram with the same parameters as in (a) except that we turn on the quadratic terms by setting $\delta_1 = 0.02$ and $\delta_2 = 0.04$ thereby breaking the inversion symmetry. In particular, the small values of $\delta_1$ and $\delta_2$ allow a direct comparison with the symmetric case in panel (a). It can be observed that the original snaking structure splits at the pitchfork bifurcations of the odd parity states (see inset) resulting in $Z$- and $S$-shaped branches (red lines). These pitchfork bifurcations correspond in (a) to spontaneous breaking of the point symmetry at the transition from antisymmetric to asymmetric states. In (b) this symmetry is no longer present and so cannot be broken. Instead, all antisymmetric states in (a) become asymmetric in (b). Note that sections of stable asymmetric solutions can be found on the $Z$-shaped structures, which in (a) represent the stable regions of the odd parity states.

The spatial reflection symmetry $R_1$ is still preserved in (b), so that even parity states and their pitchfork bifurcations to the asymmetric states persist. However, compared to (a), the degeneracy due to inversion symmetry is now removed and the symmetric states break up into two branches (light purple). These are RLS$_{even}$ and RLS$_{odd}$ states discussed earlier (recall that the terms even and odd now refer to the number of peaks and not to the symmetry of the state under $R_1$). If we increase the coefficients of the quadratic terms, the folds of the $S$- and $Z$-shaped structures annihilate and one recovers snaking behavior similar to that in (a) although the corresponding states are qualitatively different, just as in the passage from the cubic-quintic to the quadratic-cubic SH equation [39]. In this context, however, it is striking that we have identified stable resting but asymmetric solutions, a finding we attribute to the special form of the problem, Eq. (14). If this form is broken while keeping the reflection symmetry, we see in panel (g) that the asymmetric states drift, while the symmetric ones remain at rest. Here the speed of the asymmetric states depends strongly on the sign of $\tilde{\beta}$.

In panel (c) we show a bifurcation diagram for $\gamma \neq 0$, but $\delta_i = 0$, i.e. this case represents the case of broken reflection symmetry while keeping the inversion symmetry. Here the even and odd
parity states from (a) both lose their symmetry and become asymmetric but remain at rest because
the form (14) is preserved. As a result of the broken reflection symmetry all pitchfork bifurcations
become imperfect and the intertwined branches reconnect (see inset) resulting in a branch of purely
unstable asymmetric states (dashed dark red line) and a second branch of alternately stable and
unstable asymmetric states (red line). Both branches are degenerate due to the inversion symmetry
of the model and so represent states with both signs of asymmetry. If the model structure (14) is
broken via \( \tilde{\beta} \neq 0 \) all asymmetric states start to drift [see panel (h)]; the sign of \( \nu \) is selected by the
asymmetry of the state considered.

Finally, in panel (d) we break both inversion and reflection symmetries by setting \( \gamma = 0.8, \)
\( \delta_1 = 0.02 \) and \( \delta_2 = 0.04 \), thereby destroying the degeneracy due to inversion symmetry present in
panel (c) and that due to the reflection symmetry in panel (b). Thus all pitchfork bifurcations of
the even parity states in (b) also become imperfect in (d) (see inset). As a result the connections
between the different states are changed compared to the other cases. On the one hand there is the
red branch that consistently snakes upwards, resembling the red branch in panel (c). In contrast
to panel (c) where the dark red branch behaves analogously, here the dark red branch first heads
upwards but then makes a loop downwards before resuming its upward growth. This is difficult
to see in the figure since the splitting of the original odd parity branch is quite small. As a result
all localized states (red and dark red lines) are asymmetric but, despite all the broken symmetries,
they are still at rest. This is because \( \tilde{\beta} \) and \( \tilde{\gamma} \) remain zero.

The above examples illustrate clearly the considerations that are required to determine whether
or not a particular model will exhibit generic behavior. When applied to the model system studied
in Section II A these considerations explain why the addition of the term \( c_2 P^3 \) to Eqs. (3) results
in drifting asymmetric states [Fig. 1(e)]. For the nonreciprocal Cahn-Hilliard model (Section II B)
related higher order interaction terms play a similar role, except in special cases, as illustrated
by the role of of the parameters \( \beta \) and \( \tilde{\beta} \) in Eqs. (77), and likewise for the (linearly) coupled
Cahn-Hilliard and Swift-Hohenberg equations in Section II C. There one could also imagine a
different coupling, a linear coupling that breaks the conservation of the Cahn-Hilliard field. Such
a coupling also restores generic behavior. The coupled passive and active PFC model (11) is
a simplified version of a more general model whose structure follows from dynamical density
functional theory [92]. Such a model not only includes nonlinear terms in the polarization field
(corresponding to \( c_2 \neq 0 \)) but also feature nonlinear variational coupling terms between all three
fields and variational gradient coupling terms involving the densities. Based on our theoretical
study it is clear that any variational coupling term between the polarization field and any of the density fields would restore generic behavior, while any variational coupling between the densities, including either nonlinearities or gradients, would fail to do so. Furthermore, as discussed for the active PFC model without passive particles, the presence of resting asymmetric states (red branches in Figs. 5 and 7) also requires the absence of a nonlinearity in the mixed conserved and nonconserved dynamics of the polarization field \( P \) (i.e. \( c_2 = 0 \)).

Finally, we now also understand why the nonlinear coupling \( \varepsilon u_2^3 \) in the reaction-diffusion model in Eq. (12) precludes resting asymmetric states. Based on our earlier examples one might suppose that linear coupling terms yield nongeneric behavior provided they respect some basic requirements as the conservation law of the uncoupled field. This is indeed so for two-component models. Consider, however, a simple three-component reaction-diffusion model with linear cross-coupling of the form

\[
\begin{align*}
\partial_t u & \sim \alpha v + \beta w \\
\partial_t v & \sim \alpha' u + \gamma w = \frac{\alpha'}{\alpha} \left( \alpha u + \frac{\alpha}{\alpha'} \gamma w \right) \\
\partial_t w & \sim \beta' u + \gamma' v = \frac{\beta'}{\beta} \left( \beta u + \frac{\beta}{\beta'} \gamma' v \right)
\end{align*}
\]

Thus for \( \beta = \gamma' = 0 \) one obtains a three-component FitzHugh-Nagumo model with a second activator field \( w \) [96]. However, despite the linearity of the coupling terms the nongeneric model structure of (14) is only fulfilled if \( \frac{\alpha}{\alpha'} \frac{\beta'}{\beta} \frac{\gamma}{\gamma'} = 1 \), a relation that need not hold in general.
V. CONCLUSION

We have seen that a number of models of active systems in common use are nongeneric in the sense that they admit stationary asymmetric states, and identified the origin of this unexpected behavior. In particular, we have shown that the statement “asymmetry in the presence of activity necessarily results in a drift” is only true for one-component systems. For active, multicomponent systems we have identified the conditions under which asymmetric states can be steady and conversely the conditions under which such states necessarily drift. These conditions extend the gradient structure responsible for stationary states in passive systems to a related structure that encompasses nongradient and hence active systems.

We have seen that in generic active systems standing oscillations emerge from symmetric steady states in Hopf bifurcations, while traveling waves arise in drift pitchfork bifurcations. When asymmetric steady states are present (i.e. in nongeneric systems) the onset of drift is instead associated with a drift-transcritical bifurcation while Hopf bifurcations yield oscillating asymmetric states. We have presented explicit theoretical predictions for these transitions and confirmed their veracity through detailed numerical continuation and linear stability analysis. Owing to these bifurcations time-periodic states tend to dominate the system behavior at large activity. Drift-transcritical bifurcations cannot occur in single-component systems and have not been observed in multicomponent systems that do not match the structure of Eq. (14) or the more general Eq. (C2) proposed and analyzed in Appendix C. The possibility of further generalizations cannot be excluded.

It is evident that in order to avoid models with nongeneric behavior one must pay heed to the considerations detailed in the present work. Several models of active systems in common use suffer from nongeneric behavior and their properties are therefore very sensitive to the inclusion of small additional terms that restore genericity. We believe that this is a serious shortcoming of such models and that models with generic behavior are always preferable, particularly when relating the predictions of such models to experimental observations. It is striking that nongeneric behavior occurs even in the FHN model that is not, in contrast to the other examples, a model that is seen as an active extension of a passive one. Instead the nongeneric character of this model derives from the nonreciprocal, activator-inhibitor interaction satisfying Eq. (14). In fact linear coupling between two species with gradient dynamics generally satisfies Eq. (14), although this is not necessarily the case for systems with additional components.
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Appendix A: Existence of steady asymmetric states for partially coupled systems

In Section III B we showed that stationary asymmetric states of Eq. (14) that exist for a particular parameter set stay at rest for infinitesimally shifted parameters. However, the derivation implicitly assumes that the coupling parameters $\alpha_i$ are nonzero, since otherwise some expressions diverge, see e.g. Eqs. (21) and (25). However, this case is essential since, as already explained, Eq. (14) represents a passive model as long as all coupling parameters have the same sign. In other words, setting a coupling parameter to zero defines the transition from a passive to an active system. Moreover, since steady asymmetric states are a natural feature of passive systems, an understanding of the transition to an active system is crucial.

We assume, therefore, that one specific coupling parameter is zero, i.e. $\alpha_k = 0$ for some $k$ and $\alpha_i \neq 0$ for all $i \neq k$. Note that the dynamics of the corresponding field $u_k$ are uncoupled, i.e. they are not influenced by the other state variables $u_i$ although the latter do depend on $u_k$. We call the resulting case a partially coupled system.

The main difference that occurs in the partially coupled case relates to the linearized equations and the construction of the adjoint zero eigenvector. From Eq. (16) we find that

$$0 = L_{kk}^G \partial_x u_k^0.$$  \hspace{1cm} (A1)

The adjoint linear equation (20) is thus solved by

$$D^{(j)}_{xx} \delta u_j = N \partial_x u_j^0 \delta_{jk},$$  \hspace{1cm} (A2)

where $N$ is a normalization constant. In other words, the adjoint zero eigenvector has only one nonzero entry and represents the single translation mode for $u_k^0$ (or its second integral if the field is conserved). In fact, this result represents the appropriate limit of the adjoint zero eigenvector found in the fully coupled case [Eq. (23)] since for $\alpha_k \to 0$ the ratio $D^{(j)}_{xx} \delta u_j / D^{(i)}_{xx} \delta u_i$ diverges [see Eq. (24)].

What is not affected by setting $\alpha_k = 0$ is that we need to show that the solvability condition (19) is satisfied for any parameter shift $\delta \alpha_i$, $\forall i$. From the steady state equations for the coupled fields it follows that

$$0 = \langle \partial_x u_i^0, \delta G_i \rangle + \alpha_i \langle \partial_x u_i^0, \delta G_C \rangle \Rightarrow \langle \partial_x u_i^0, \delta G_C \rangle = 0 \quad \forall i \neq k.$$  \hspace{1cm} (A3)

This equation, together with the identity (26), leads to

$$0 = \sum_{i \neq k} \langle \partial_x u_i^0, \delta G_C \rangle + \langle \partial_x u_k^0, \delta G_C \rangle = \langle \partial_x u_k^0, \delta G_C \rangle,$$  \hspace{1cm} (A4)
i.e., the solvability condition (19) for the adjoint zero eigenvector (A2). As a result Eq. (19) holds even when $\delta \alpha_k \neq 0$, showing that steady states still remain at rest when the system transitions into an active one.

**Onset of motion:**

The onset of motion in the partially coupled case (for one zero coupling parameter) follows from Eq. (32). Employing the adjoint zero eigenvector given by Eq. (A2) this condition reduces to

$$0 \overset{!}{=} \begin{cases} \partial_x n_k^0 & \text{if } k \leq M_1 \\ \langle (c_k^0)^2 \rangle - \langle (c_k^0)^2 \rangle & \text{if } k > M_1. \end{cases} \quad (A5)$$

Recall again that the partially coupled case represents the transition from the passive to the active system, i.e. if Eq. (A5) is fulfilled when $\alpha_k = 0$ then it indicates a higher-dimensional bifurcation point at which a whole branch of drifting asymmetric states emerges when $\alpha_k$ passes zero. In [29] [see Scenario SubMinus] this situation leads to the emergence of two drift-pitchfork, two Hopf bifurcations, and one fold.

**Appendix B: Parity-symmetric states do not drift**

We now consider a general dynamic system with

$$\partial_t u = F[u, \partial_x], \quad (B1)$$

where the vector field $F$ involves both linear and nonlinear (integro)-differential operators, represented by positive and negative powers of $\partial_x$. Since we want to consider resting and steadily drifting states, we assume that Eq. (B1) is translation-invariant, i.e. that $F$ is independent of the spatial variable $x$ itself. When Eq. (B1) is a passive system with gradient structure, the corresponding free energy decreases over time and the system evolves towards a state at rest. On the other hand, when the gradient structure is absent, states that drift at constant speed are allowed, although parity-symmetric states remain at rest. To demonstrate this fact, consider the reflection $R_1: x \to -x$. Under this transformation $u(x) \to R_1(u)(-x)$, where $R_1(u) = r_1u = (r_1 u_1, r_1 u_2, \ldots, r_1 u_M)$ and $r_1 \in \{-1, +1\}$ represent different (one-dimensional) representations of the parity symmetry. We say that Eq. (B1) is equivariant under $R_1$ whenever $R_1 F[R_1(u), -\partial_x] = F[u, \partial_x]$ and parity-symmetric states $u^0(x)$ satisfy $R_1(u^0)(-x) = u^0(x)$. In particular, if $r_1 = +1 \ [r_1 = -1] \ \forall i$ we say that $u^0(x)$ has even [odd] parity symmetry.
To show that drifting parity-symmetric states cannot exist, we argue by contradiction, i.e. we consider a drifting parity-symmetric state \( u^0(x, t) \). Such a state is steady in the comoving frame and by hypothesis satisfies
\[
0 = F[u^0(\xi), \partial_\xi] + v_\xi u^0(\xi),
\]
where \( \xi \equiv x - vt \). Applying the parity transformation \( R_1 \) to this equation and noting that \( R_1^{-1} = R_1 \) gives
\[
0 = F[u^0(-\xi), -\partial_\xi] - v_\xi u^0(-\xi) = R_1 \left( F[u^0(\xi), \partial_\xi] + v_\xi u^0(\xi) \right).
\]
Thus
\[
0 = F[u^0(\xi), \partial_\xi] - v_\xi u^0(\xi).
\]
Comparison of Eqs. (B2) and (B3) shows that \( v = 0 \), i.e. that steady states with parity symmetry do not drift. This relation applies to most of the steady states in the main text, as all primary pattern-forming states are necessarily parity-symmetric and many of the secondary bifurcations also respect this symmetry.

In contrast, for asymmetric states, i.e. states without any parity symmetry, the above argument suggests that drift is no longer prohibited and indeed in translation-invariant systems one often finds that asymmetry is associated with a nonzero, constant drift speed. However, as shown in the main text, parity symmetry is not necessary for vanishing drift velocity.

**Appendix C: Further generalization of Eq. (14)**

The existence of asymmetric steady states in Section III was established based on Eq. (14) which includes multi-component models with conserved and nonconserved order parameters. A further generalization is possible and leads to models with the structure
\[
\partial_t u = \left( \nabla Q(u) \nabla - M(u) \right) \cdot \delta G \frac{\delta G}{\delta u},
\]
where \( Q(u) \) and \( M(u) \) are symmetric, positive definite matrices that may depend on the field variables. Equation (13) is special case with \( Q \) and \( M \) diagonal and entries \( 1, \ldots, M_1 \) zero in \( Q \) and entries \( M_1 + 1, \ldots, N \) zero in \( M \). As in the main text, the functional \( G \) consists of noncoupling and coupling terms with prefactors \( \alpha_i \) that represent activity in the system. Separating these
contributions, Eq. (C1) can be written as
\[
\partial_t u = (\nabla Q \nabla - M) \cdot \left( \frac{\delta G_{nc}}{\delta u} + \alpha \cdot \frac{\delta G_c}{\delta u} \right)
\]
(C2)

with \( G_{nc} = \sum_i G_i[u_i] \)
(C3)

and a diagonal matrix \( \alpha \) with constant entries that can be positive, negative or zero. In the following we establish the existence of steady asymmetric states in this system, following the procedure in Section III.

Steady states of Eq. (C1) satisfy
\[
\frac{\delta G_{nc}}{\delta u} + \alpha \cdot \frac{\delta G_c}{\delta u} = 0,
\]
(C4)

with possible chemical potentials \( \mu_i \) for mass conservation included in \( G_i[u_i] \) as additional terms \(-\mu_i u_i\).

As before, nonzero fluxes which can occur in special cases (see the active PFC model) are excluded. It follows that
\[
\mathbf{L} \cdot \partial_x u^0 \equiv \left( \frac{\delta^2 G_{nc}}{\delta u^2} \bigg|_{u^0} + \alpha \cdot \frac{\delta^2 G_c}{\delta u^2} \bigg|_{u^0} \right) \cdot \partial_x u^0 = 0.
\]
(C5)

a. Step 1: Derive the solvability condition, Eq. (C7)

We assume that for some initial parameters, \( \alpha_i = \alpha_i^0 \), an asymmetric steady state \( u^0 \) exists. Next, we consider a changed parameter value, \( \alpha_i = \alpha_i^0 + \delta \alpha_i \) resulting in a slightly modified steady state \( u = u^0 + \delta u \). We insert both expressions into Eq. (C2) with \( \partial_t u = 0 \) and linearize
\[
(\nabla Q \nabla - M) \cdot \mathbf{L} \cdot \delta u = - (\nabla Q \nabla - M) \cdot \delta \alpha \cdot \frac{\delta G_c}{\delta u}.
\]
(C6)

Note that a second term \((\nabla Q' \delta u \nabla - M' \delta u) \cdot \frac{\delta G}{\delta u} \) vanishes since \( \frac{\delta G}{\delta u} = 0 \) for any steady state, cf. Eq. (C4).

The solvability condition (which is sufficient) for Eq. (C6) is
\[
0 = \{\delta u^1, (\nabla Q \nabla - M) \cdot \delta \alpha \cdot \frac{\delta G_c}{\delta u}\}
\]
(C7)

with the adjoint zero eigenvector \( \delta u^1 \) solving the adjoint linear homogeneous equation
\[
\mathbf{L}^\dagger \cdot (\nabla Q \nabla - M) \cdot \delta u^1 = 0.
\]
(C8)

Due to positivity of \( Q(u) \) and \( M(u) \) the terms \( \nabla Q(u) \nabla \cdot \frac{\delta G}{\delta u} \) and \( -M(u) \cdot \frac{\delta G}{\delta u} \) have the same sign, i.e. they cannot cancel each other for steady states.
b. Step 2: Find the adjoint zero eigenvector Eq. (C11)

Writing the linear operator as $\mathcal{L} = D + \alpha \cdot \delta^2 G_C \delta u \delta u$, where $D = \delta^2 G_C \delta u \delta u$ is a diagonal matrix, we can write the adjoint linear operator $\mathcal{L}^\dagger$ as

$$
\mathcal{L}^\dagger = D + \delta^2 G_C \delta u \delta u \cdot \alpha = \alpha^{-1} \cdot \mathcal{L} \cdot \alpha
$$

(C9)

where we have used the self-adjointness of the matrix of second variations. Inserting Eq. (C9) into Eq. (C8) we obtain

$$
\alpha^{-1} \cdot \mathcal{L} \cdot \alpha \cdot (\nabla Q \nabla - M) \cdot \delta u^\dagger = 0
$$

$\Leftrightarrow \mathcal{L} \cdot \alpha \cdot (\nabla Q \nabla - M) \cdot \delta u^\dagger = 0$

(C10)

Comparing the adjoint linear equation (C10) with Eq. (C5) we see that one adjoint zero eigenvector can be found as

$$
\alpha \cdot (\nabla Q \nabla - M) \cdot \delta u^\dagger = N \partial_x u^0
$$

(C11)

with an arbitrary (normalization) constant $N$. Since $Q$ and $M$ are known functions of $u^0$ this equation can be solved for $\delta u^\dagger$ using Green’s function methods.

c. Step 3: Show that the adjoint zero eigenvector solves the solvability condition

The final step cannot be done in matrix representation since we need to consider each steady state equation separately. Following the same derivation as in the main text one can show that each summand of the solvability condition vanishes (C7). This is necessary to argue that a steady state can be found for any parameter change $\delta \alpha$.

1. Onset of motion

Following the same expansion for small velocities as in the main text the onset of motion occurs for

$$
(\nabla Q \nabla - M) \cdot \mathcal{L} \cdot u^1 = \partial_x u^0
$$

(C12)

with the translational Goldstone mode $\partial_x u^0$. The solvability condition for Eq. (C12) is

$$
0 \equiv \langle \partial_x u^0, \delta u^\dagger \rangle = \langle \partial_x u^0, (\nabla Q \nabla - M)^{-1} \cdot \alpha^{-1} \cdot \partial_x u^0 \rangle
$$

(C13)
since the adjoint zero eigenvector solves Eq. (C11). Unfortunately further simplification of this condition appears not to be possible, and the case detailed in the main text may be the only one where this is possible.