The Cabello Nonlocal Argument is Stronger Control than the Hardy Nonlocal Argument for Detecting Post-Quantum Correlations

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In this paper, we study the Hardy nonlocal argument (HNA) and the Cabello nonlocal argument (CNA) under the Information Causality (IC), Macroscopic Locality (ML) and Local Orthogonality (LO) principles in the context of Local Randomness. We see that, in the context of all the possibilities of local randomness, the gap between the quantum mechanics and the above principles, in the CNA is larger than the HNA. Therefore the CNA is stronger control than the HNA for detecting post-quantum nosignalling correlations.

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I. INTRODUCTION

It is known that the value of Bell-CHSH [1,2] quantity, exceeding the local hidden variable theorems (LHVT) bound, quantifies a measure of nonlocality. Therefore, measurement of nonlocality within the quantum mechanics (QM) is limited by the Cirel’son bound 2√2 [3] and the bound of nonlocality imposed by the no-signaling (NS) principle (PR box) is 4 [4]. On the other hand, Hardy [5,6] introduced a non-locality theorem to prove Bell’s theorem without inequality. The Hardy’s argument of “nonlocality without inequality” has been considered to be the “best version of Bell’s theorem” [7]. After that the authors [8–10] shown that the Hardy nonlocality argument (HNA) is a special case of the Cabello nonlocality argument (CNA). One important difference between HNA and CNA is that a mixed two-qubit entangled state can never exhibit HNA, but they can exhibit CNA [11]. We know that the maximum success probability for the HNA and CNA in LHVTs are zero, but in the QM, the corresponding values are 0.09 [5,6,12] and 0.11 [13] respectively. On the other hand, it has been shown that the upper bound of success probability of HNA and CNA in the general NS theorem is 0.5 [10]. The natural question arise is that: why is QM not more nonlocal than it is? Why quantum not violate Bell’s inequality (BI) and HNA and CNA more than the number 2√2, 0.09 and 0.11 respectively?

In the recent years, several attempts have been taken to get an answer to this fundamental question by using some physical or information theoretical principles. Some of these principles are non trivial communication complexity [14,15], Information Causality (IC) [16], Macroscopic Locality (ML) [17], Exclusivity principle [18], Relativistic Causality in the Classical limit [19,20], the uncertainty principle along with steering [21] and complementarity principle [22]. It is interesting, some of these principles like IC and ML can explain the Cirel’son bound of Bell-CHSH, but they can not explain the limit HNA and CNA in the quantum theory [24,23]. It has been shown that under applying the IC, the upper bound on the success probability of both HNA and CNA is almost 0.2071 [23], and under applying the ML condition they are 0.2062 [24]. Very soon, it has been shown that for a more than two subsystem correlations, no bipartite physical principle (like IC and ML) can completely recognize the full set of QM correlations [25,26]. In order the solve this problem, some new principles like Local Orthogonality (LO) is introduced by Fritz and et.al [27]. Next, the authors in ref. [24] found that the maximum success probability of HNA under the LO principle is 0.177 which is closer to the QM value, but in the case CNA, it is same as that obtained from IC principle (0.207).

In the other hand, Gazi and et.al [28] shown that in terms of local randomness (LR) of the observable for the Hardy’s correlations, there is also a gap between QM and IC condition and this gap in the context of CNA is much larger than the HNA case [29]. In this paper, we extend this approach to ML and LO principles and get some interesting results.

This article is organized as follows. In Sec.II, we overview the HNA and the CNA under the no-signaling condition. In Sec.III and IV, we briefly review the IC, ML and LO principles and obtain the maximum success probability HNA and CNA under these principles in the context of LR, and we bring our conclusions in sec V. The all inequalities are sufficient for obtaining the upper bound of the HNA and CNA under LO principle are presented in the appendices A and B. Finally the details of the MATLAB programs are shown in the appendix C.

II. HNA AND CNA UNDER THE NS CONDITION

Consider the convex set of bipartite no-signaling correlations with binary inputs and binary outputs for each
party in a $2^4$ dimensional vector space. Let $P_{ab\mid xy}$ denote the joint probabilities, where $a, b \in \{0, 1\}$ and $x, y \in \{0, 1\}$ denote outputs and inputs of two parties (we call them Alice and Bob) respectively. The joint probabilities satisfy the positivity constraints:

$$0 \leq P_{ab\mid xy} \leq 1 \quad \forall a, b, x, y \in \{0, 1\} \quad (1)$$

and also satisfy the normalization constraints:

$$\sum_{a, b=0}^1 P_{ab\mid xy} = 1 \quad \forall x, y \in \{0, 1\}. \quad (2)$$

On the other hand, since the parties (Alice and Bob) are spatially separate, causality and relativity imply that the joint probabilities satisfy the no-signaling (NS) correlations. The NS conditions imposing that the choice of measurement by Alice (or Bob) can not affect the outcome distributions of Bob (or Alice). In the other words, the marginal conditional probabilities $p_{ax}$ and $p_{by}$ must be independent of $y$ and $x$, respectively. The NS constraints can be express:

$$\sum_b P_{ab\mid x0} = \sum_b P_{ab\mid x1} = p_{a\mid x} \quad \forall a, x \in \{0, 1\} \quad (3)$$

$$\sum_a P_{ab\mid y0} = \sum_a P_{ab\mid y1} = p_{b\mid y} \quad \forall b, y \in \{0, 1\}. \quad (4)$$

The full set of normalization and the NS correlations form an eight dimensional polytope structure [23] that we call it, NS polytope. We can represent these types of correlations by a $4 \times 4$ correlation matrix with 8 independent parameters:

$$
\begin{pmatrix}
e_1 & f_1 & e_1 & f_1 - e_1 \\
e_2 & f_2 & e_2 & f_2 - e_2 \\
e_3 & f_2 & e_3 & f_2 - e_3 \\
e_4 & f_4 & e_4 & f_4 - e_4
\end{pmatrix}
$$

where the parameters $0 \leq e_k \leq 1$, $0 \leq f_i \leq 1$ and $0 \leq g_i \leq 1$ ($k = 1, 2, 3, 4$ and $i = 1, 2$) make the elements of matrix and the positivity constraint is guaranteed by the condition:

$$\max\{0, g + f - 1\} \leq e \leq \min\{f, g\}.$$ 

The sets of LHVT and QM correlations are strictly contained within the NS polytope, but notice that the set of LHVT correlations like NS correlation forms a convex polytope, whereas the set of quantum correlations is convex but is not a polytope [23]. The eight dimensional NS polytope has 24 vertices contains 16 local vertices:

$$P_{ab\mid xy}^{\alpha \beta \gamma} = \delta_{(a=\alpha x \oplus \delta)} \delta_{(b=\gamma y \oplus \delta)} \quad (6)$$

and the eight nonlocal vertices:

$$P_{ab\mid xy}^{\alpha \beta \gamma} = \frac{1}{2} \delta_{a=\alpha x \oplus \beta y \oplus \gamma} \quad (7)$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and $\oplus$ is addition modulo 2.

Now, we consider joint probabilities satisfying the Hardy-Cabello argument:

$$
\begin{align*}
P_{ab\mid xy} &= q_1, \\
P_{a'\mid b\mid xy} &= 0, \\
P_{ab\mid x'\mid y} &= 0, \\
P_{a'\mid b\mid xy} &= q_4.
\end{align*}
$$

where $\bar{a}$ denotes complement of $a$ ($\bar{a} = 1 \oplus a$). These equations form the basis of the Hardy-Cabello nonlocality argument. It can easily be shown that these equations contradict local realism if $q_1 < q_4$. Whenever $q_1 = 0$, the Cabello’s argument reduces to the Hardy’s argument.

In the remaining part of this paper, without loss of generality we consider the following form of the Hardy-Cabello correlation:

$$P_{ab\mid xy} = c_1 P_{a\mid b\mid xy}^{110} + c_2 P_{a\mid b\mid xy}^{101} + c_3 P_{a\mid b\mid xy}^{100} + c_4 P_{a\mid b\mid xy}^{111} \quad (9)$$

where $\sum_{k=0}^3 c_k = 1$. By placing the local matrices $P_{a\mid b\mid xy}^{110}, P_{a\mid b\mid xy}^{101}, P_{a\mid b\mid xy}^{100}, P_{a\mid b\mid xy}^{111}$, using Eq (5) and the nonlocal matrix $P_{a\mid b\mid xy}^{100}$ from Eq (7) in the above equation, We get a $4 \times 4$ Hardy correlation nonlocal matrix:

$$
\begin{pmatrix}
c_3 & c_4 & 0 & c_{1,2.5} \\
c_{3,4} & 0 & c_{1,5} & c_{1,2.4} \\
c_5 & c_6 & 0 & 0 \\
c_5 & c_6 & 0 & 0
\end{pmatrix}
+ \frac{1}{2}
\begin{pmatrix}
c_6 & 0 & 0 & c_6 \\
c_6 & 0 & 0 & c_6 \\
c_6 & 0 & 0 & c_6 \\
c_6 & 0 & 0 & c_6
\end{pmatrix}
$$

where $c_{i,j,k} \equiv c_i + c_j + c_k$. The Cabello’s nonlocality argument ($q_1 \neq 0$), can be written as a convex combination of the above 6 vertices which satisfies Hardy’s conditions along with another four local vertices $P_{a\mid b\mid xy}^{100}, P_{a\mid b\mid xy}^{010}, P_{a\mid b\mid xy}^{011}$ and one nonlocal vertex $P_{a\mid b\mid xy}^{110}$:

$$
P_{a\mid b\mid xy} = P_{a\mid b\mid xy}^H + c_7 P_{a\mid b\mid xy}^{100} + c_8 P_{a\mid b\mid xy}^{101} + c_9 P_{a\mid b\mid xy}^{110} + c_{11} P_{a\mid b\mid xy}^{111} \quad (11)$$

where the expression $P_{a\mid b\mid xy}^H$ is given in Eq (9) and coefficients $c_i$’s satisfy $\sum_{i=1}^{11} c_i = 1$. Also by using Eqs (5) and (7) the correlation matrix for the Cabello’s non-signaling boxes can be written as [23, 24]:

$$
\begin{pmatrix}
c_{3,8,10} & c_{4,7,9} & 0 & c_{1,2.5} \\
c_{3,4} & c_{7,8,9,10} & c_2 & c_{1,5} \\
c_8 & c_{5,7} & c_{3,10} & c_{1,2.4,9} \\
c_5 & c_{5,7,8} & c_{2,3,4} & c_{1,9,10}
\end{pmatrix}
+ \frac{1}{2}
\begin{pmatrix}
c_6,11 & 0 & 0 & c_{6,11} \\
c_6 & c_{11} & c_{11} & c_6 \\
c_6 & c_{11} & c_{11} & c_6 \\
c_6 & c_{11} & c_{6,11} & 0
\end{pmatrix}
\quad (12)$$
It has worth, notice here that while the upper bound of the success probability of the HNA and the CNA in QM is 0.09 [12] and 0.11 [13], respectively, but under imposing the NS condition [10, 34], by substituting $c_i = \delta_{i,6}$, they are same and equal to 0.5 (please see the TABLE I). Now, in the next section, we briefly review the IC, ML and LO principles.

III. HARDY’S AND CABELLO’S NON-LOCALITY UNDER IC, ML AND LO PRINCIPLES

A. Information Causality Principle

Pawłowski and et.al [16] introduced Information Causality (IC) principle for understanding the quantum mechanical bound on nonlocal correlations. Alice and Bob, who are separated in space, have access to non-signaling resources such as shared randomness, entanglement or PR boxes. Alice receives a randomly generated N-bit string $\vec{x} = (x_0, x_1, ..., x_{N-1})$ while Bob is asked to guess Alice’s $i$-th bit where $i$ is randomly chosen from the set $\vec{y} = \{0, 1, 2, ..., N - 1\}$. Alice then sends $M$ classical bits to Bob ($M < N$) and let Bob’s answer be denoted by $\beta_i$. Then, the amount of the information about the variable $x_i$ of Alice’s input potentially gained by Bob is measured by:

$$I = \sum I(x_i : \beta | y = i) \geq N - \sum_{k=1}^{N} h(p_k)$$

(13)

where $I(x_i : \beta | y = i)$ is Shanon mutual information between $x_i$ and $\beta$, and $p_k$ is the probability that $x_i = \beta$ and inequality can be proved by Fano inequality. The statement of the IC is that the total potential information [16] about Alice’s bit string $\vec{x}$ accessible to Bob cannot exceed the volume of message received from Alice, i.e., $I = \sum_{i=1}^{N} I(X_i : \beta_i) \leq M$. Both classical and quantum correlations have been proved to satisfy the IC condition [16]. It was further shown that, if Alice and Bob share arbitrary two inputs and two outputs no-signaling correlations corresponding to conditional probabilities $P_{ab|x,y}$, then by applying a protocol by Van Dam [32] and Wolf and Wullschleger [53], one can derive a necessary condition for respecting the IC principle. The mathematical form of necessary conditions from Alice to Bob ($A \rightarrow B$) can be expressed as:

$$\sum_{y=0,1} \left[ \sum_{a,b} P(a \oplus b = xy \oplus ax \oplus \beta y \oplus \gamma | xy) - 1 \right]^2 \leq 1$$

(14)

and the necessary conditions to satisfy IC from Bob to Alice ($B \rightarrow A$) can be expressed as:

$$\sum_{x=0,1} \left[ \sum_{y=0,1} P(a \oplus b = xy \oplus ax \oplus \beta y \oplus \gamma | xy) - 1 \right]^2 \leq 1$$

(15)

where $\alpha, \beta, \gamma \in \{0, 1\}$. It is important to note that the above conditions are only a sufficient condition [14] for non-violating the IC principle. It means that a violation of [14] or [15] implies a violation of IC but the converse may not be true [34]. Notice that, the upper bound of success probability of the case HNA under imposing (NS+IC) is equal to 0.2071 which is same as that for the case of CNA [22] (please see the TABLE I).

B. Macroscopic Locality Principle

Miguel Navascues and et.al [37, 38] introduced the macroscopic locality (ML) principle. The principle of ML states that two parties (Alice and Bob) performing coarse-grained extensive measurements over independent correlated pairs of physical systems can always interpret their observations with a classical theory. They proposed a way to approximate quantum correlations $Q$. They show that, there exist a series of correlation sets which, asymptotically converge on the set of quantum correlations. i.e. $Q$. The first step in this series of correlation sets, i.e. $Q_1$ exactly coincides the set correlations respecting ML principle and this set strictly contains the quantum set $Q$, thus showing the insufficiency of ML principle in distinguishing all post-quantum correlations. Also, Navascues and et.al [37, 38] show that for a bipartite system with two binary inputs and two binary outputs exists a the necessary and sufficient criteria for respecting ML given by the condition:

$$| \sum_{x,y=0,1} (-1)^{x+y} \sin^{-1}(\frac{C_{xy} - C_xC_y}{\sqrt{(1 - C_x^2)(1 - C_y^2)})} | \leq \pi$$

(16)

where $C_{xy} = \sum_{a,b} (-1)^{a+b}p_{ab|xy}$. $C_x = \sum_{a,b} (-1)^a p_{ab|xy}$ and $C_y = \sum_{a,b} (-1)^b p_{ab|xy}$.

S. Das and et.al [24] numerically showed that, the maximum of success probability HNA turn out to be $\approx 0.2062$ which is same as that for CNA under imposing (NS + ML) condition (please see the TABLE I).

C. Local Orthogonality Principle

T. Fritz and et.al [24] considered a pair of events $e = (a_1, ..., a_n|x_1, ..., x_n)$ and $e' = (a_1', ..., a_n'|x_1', ..., x_n')$ in a Bell scenario involving n, particle, with m different inputs $x_i \in \{0, 1, ..., m - 1\}$ and d possible outputs $a_i \in \{0, 1, ..., d - 1\}$ (where $i \in \{0, 1, ..., n - 1\}$). Next, they called that these two different events $e$ and $e'$ are locally orthogonal, if involve different outcomes of the
TABLE I: The Maximum value of HNA and CNA under imposing the NS, IC, ML and LO conditions. Our numerically calculation by MATLAB software exactly confirm all the previous results.

| Case | LHV T | QM | NS | NS + IC | NS + ML | NS + LO |
|------|-------|----|----|---------|---------|---------|
| HNA  | 0.09  | 0.5 | 0.2071 | 0.206 | 0.177 |
| CNA  | 0.11  | 0.5 | 0.2071 | 0.206 | 0.2071 |

same measurement by at least one party. The collection of events \{e_i\} are called orthogonal set, if they are pairwise orthogonal and according to the LO principle, the sum of probabilities of pairwise orthogonal event cannot exceed 1.

\[ \sum_i p(e_i) \leq 1 \] (17)

Also, we can represent a set of local orthogonal events \{e_i\} as a graph, where an event corresponds to a vertex and a pair of orthogonal vertex defines an edge and a set of orthogonal vertex forms a clique. So, any clique in the orthogonality graph of events is equivalent to an LO inequality (Eq 17). On the other hand, it is worth that mention here, unlike IC and ML principles, there exist correlations such that single copy of those satisfy LO inequality, but two or more copies of those correlations violate LO principle. Also, the one copy of bipartite scenario, LO constraint is equivalent to NS conditions [27].

More recently, S. Das and et.al [24] considered two orthogonal graph, that first graph contains 169 vertices correspond to two copies of binary inputs-outputs Hardy correlation and the second graph contains 196 vertices corresponded to two copies of binary inputs-outputs Cabello correlation. Next, they showed that for getting the maximum success of HNA (CNA) under the full set of resulting LO inequalities, it is sufficient to maximize it under a small subset of LO inequalities contains 10 (13) inequalities. (please see the Appendices A and B). Next, they [24] obtained that by applying the NS principle and 10 LO inequalities, the maximum success probability in the case HNA turns out to be 0.177 but it is not same as the bound of CNA (please see the TABLE I).

IV. THE CONTEXT OF LOCAL RANDOMNESS

The local randomness condition imposes that the marginal probabilities of all possible outcomes on Alice’s (Bob’s) side for the x(y) input, are equal [28]. So, in the case of two inputs and two outputs bipartite correlations, an input x on Alice’s side is locally random if for any choice of Bob’s input y, we have:

\[ P_{a=0|x} = \sum_b P_{b|x,y} = \frac{1}{2} \text{ (we show with } 0_A \text{) (18)} \]
\[ P_{a=1|x} = \sum_b P_{b|x,y} = \frac{1}{2} \text{ (we show with } 1_A \text{). (19)} \]

Similarly, an input y on Bob’s side is locally random if for any choice of Alice’s input x, we have:

\[ P_{b=0|y} = \sum_a P_{a|y,x} = \frac{1}{2} \text{ (we show with } 0_B \text{) (20)} \]
\[ P_{b=1|y} = \sum_a P_{a|y,x} = \frac{1}{2} \text{ (we show with } 1_B \text{). (21)} \]

In this stage, we want to find the maximum success probability of HNA and CNA under IC principle inequalities, ML principle inequality and under the full set of resulting LO inequalities, in the context of all the possibilities of local randomness.

Now, let us introduce the following maximization:

**Problem**

Maximize \[ q_4 = \frac{x}{y} \text{ ( the case of HNA ) or } \]

Maximize \[ q_4 - q_1 = \frac{x+y}{y} - c_{7,9,10} \text{ (the case of CNA )} \]

Subject to the constraints:

i) The positivity: Eq(1)

ii) The normalization: Eq(2)

iii) The NS conditions: Eqs(3,4)

iv) The IC inequalities: Eqs(11,12,15)

or

The ML inequality: Eq(16)

or

The LO inequalities: Eqs(22,23) in the Appendix A for the case of HNA or

The LO inequalities: Eqs(22,23) in the Appendix B for the case of CNA

v) Without considering LR

vi) Under consideration LR: Eqs(18,24)

The optimal value of this problem gives us, an upper bound of the HNA and CNA for all possible choice of inputs that can be locally random. We solve this optimization problem by using a program in MATLAB software (please see the appendix C). We present the our results for every choice of collection \{0_a, 1_A, 0_B, 1_B\} in the TABLES II, III.

Numerical calculation shows that the maximum value of CNA under the restrictions of IC, ML and LO is strictly larger than HNA under consideration LR. The distance between two random points \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) is defined by: \[ d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \], then we see that the distance of LO correlation from the QM correlations in both cases HNA and CNA is much larger than HNA (please see the TABLE IV).
TABLE II: The numerical Maximum value of the HNA for the corresponding choice of inputs to be locally random and satisfy IC, ML and LO principles.

| Case | Locally random inputs | Max(HNA)$_{QM}$ | Max(HNA)$_{NS}$ | Max(HNA)$_{NS+IC}$ | Max(HNA)$_{NS+ML}$ | Max(HNA)$_{NS+LO}$ |
|------|-----------------------|-----------------|-----------------|---------------------|---------------------|---------------------|
| 1    | 0, 1, 0, 1           | 0               | 0               | 0                   | 0                   | 0                   |
| 2    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1002              | 0                   |
| 3    | 0, 1, 0, 1           | 0               | 0               | 0.002               | 0                   | 0                   |
| 4    | 0, 0, 1, 1           | 0               | 0               | 0.2071              | 0.0998              | 0                   |
| 5    | 1, 0, 1, 1           | 0               | 0               | 0.0016              | 0                   | 0                   |
| 6    | 0, 1, 1             | 0               | 0               | 0.2071              | 0.0967              | 0                   |
| 7    | 0, 1, 1             | 0               | 0               | 0.2071              | 0.0967              | 0                   |
| 8    | 1, 0, 1             | 0               | 0               | 0                   | 0                   | 0                   |
| 9    | 0, 0, 1             | 0               | 0               | 0.2071              | 0.2                 | 0                   |
| 10   | 0, 0, 1             | 0               | 0               | 0.2071              | 0.12                | 0                   |
| 11   | 1, 0, 0             | 0               | 0               | 0.2071              | 0.1045              | 0.1250              |
| 12   | 0, 1, 1, 0, 1       | 0.0875          | 0.5             | 0.2071              | 0.1965              | 0.1760              |
| 13   | 0, 1, 1, 0, 1       | 0.0556          | 0.5             | 0.2071              | 0.1776              | 0.1344              |
| 14   | 0, 0, 1, 1, 1       | 0.0556          | 0.5             | 0.2071              | 0.2038              | 0.1692              |
| 15   | 0, 0, 1, 1, 1       | 0.0875          | 0.5             | 0.2071              | 0.1879              | 0.1508              |
| 16   | NO LR               | 0.09            | 0.5             | 0.2071              | 0.2063              | 0.1770              |

TABLE III: The numerical Maximum value of the CNA for the corresponding choice of inputs to be locally random and satisfy IC, ML and LO principles.

| Case | Locally random inputs | Max(CNA)$_{QM}$ | Max(CNA)$_{NS}$ | Max(CNA)$_{NS+IC}$ | Max(CNA)$_{NS+ML}$ | Max(CNA)$_{NS+LO}$ |
|------|-----------------------|-----------------|-----------------|---------------------|---------------------|---------------------|
| 1    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.2071              | 0.2071              |
| 2    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1976              | 0.2071              |
| 3    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1845              | 0.2070              |
| 4    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1963              | 0.2                 |
| 5    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1976              | 0.2071              |
| 6    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1850              | 0.2071              |
| 7    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.2034              | 0.2071              |
| 8    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1976              | 0.1990              |
| 9    | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1940              | 0.1954              |
| 10   | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1976              | 0.2071              |
| 11   | 0, 1, 0, 1           | 0               | 0               | 0.2071              | 0.1845              | 0.2070              |
| 12   | 0, 1, 0, 1           | 0.0992          | 0.5             | 0.2071              | 0.2063              | 0.2071              |
| 13   | 0, 1, 0, 1           | 0.0716          | 0.5             | 0.2071              | 0.1964              | 0.2071              |
| 14   | 0, 1, 0, 1           | 0.0992          | 0.5             | 0.2071              | 0.2038              | 0.2071              |
| 15   | 0, 1, 0, 1           | 0.0992          | 0.5             | 0.2071              | 0.1991              | 0.2071              |
| 16   | NO LR               | 0.1078          | 0.5             | 0.2071              | 0.2063              | 0.1978              |

V. CONCLUSION

In this article, we study all the possibilities of local randomness in the HNA and the CNA respected by the principle of non-violation of IC, ML and LO inequalities. We obtain that without considering local randomness, the maximum success probability of the case HNA under imposing NS, IC, ML is equal to the case of CNA. Therefore, in this stage, there is no benefit between them. Next, we get interesting results, after considering all possibilities of the local randomness. We see that the gap between QM and the above principles, in the context of
TABLE IV: The Euclidean distance IC, ML and LO correlations from QM correlations under consideration LR

| Case  | $d(IC + NS, QM)$ | $d(ML + NS, QM)$ | $d(LO + NS, QM)$ |
|-------|------------------|------------------|------------------|
| HNA   | 0.6239           | 0.4233           | 0.2377           |
| CNA   | 0.7362           | 0.6701           | 0.7183           |

the Cabello nonlocal argument is larger than the Hardy’s case. This difference gap is interesting because may be relevant to assessing the viability of “information causality”, “macroscopic locality” and “local orthogonality” as partial candidate explanations for why QM correlations are weaker than generalized non-signalling correlations. So, in the case of CNA, the number of non-quantum correlation definitely obey the IC, ML, LO condition is more than the HNA case. Therefore, we can conclude our work that the optimal success probability of CNA in QM is stronger control than the HNA for detecting post-quantum no signaling correlations. However, it remains to see, in the future, whether some stronger necessary condition can explain the upper bound of nonlocality in QM.

VI. APPENDIX A

The following inequalities are sufficient for obtaining the upper bound of the HNA under LO principle\[24\]. The inequalities can be shown, in terms of variables $e_k$, $m_i$ and $n_i$ ($k \in \{1, 2, 3, 4\}$ and $i \in \{1, 2\}$):

\[
\begin{align*}
&c_1 = c_3 + \frac{c_6}{2} \\
&c_2 = c_{3,4} + \frac{c_6}{2} \\
&c_3 = \frac{c_6}{2} \\
&e_3 = 0 \\
&f_1 = e_2 \\
&f_2 = e_2 + e_3 \\
g_1 = e_3 \\
g_2 = e_2 + e_2
\end{align*}
\]

and the Hardy correlation matrix\[10\]. So, we have:

\[
\begin{align*}
&\sum_{i=1}^{2} (e_1 + e_i - f_1 - g_2) + (1 + e_2 + e_3 - f_1 - f_2 - g_2) e_2 \\
&- e_1 + (e_1 + f_2 + g_2 - 1) g_2 \leq 0
\end{align*}
\]

VII. APPENDIX B

The following inequalities are sufficient for obtaining the upper bound of CNA under LO principle\[24\]. The inequalities can be shown, in terms of variables $e_k$, $m_i$ and $n_i$ ($k \in \{1, 2, 3, 4\}$ and $i \in \{1, 2\}$):

\[
\begin{align*}
&c_1 = c_3 + \frac{c_6}{2} \\
&c_2 = c_{3,4} + \frac{c_6}{2} \\
&c_3 = \frac{c_6}{2} \\
&e_3(1 + e_1 - f_1 - g_2) + (1 + e_2 + e_3 - f_1 - f_2 - g_2) e_2 \\
&- e_1 + (e_1 + f_2 + g_2 - 1) g_2 \leq 0
\end{align*}
\]

The relation between the 8 independent parameter and the coefficients $c_i$ is obtained by comparing the matrix

The relation between the 8 independent parameters and the coefficients $c_i$ is obtained by comparing the matrix
So, we have:

\[ e_1 = c_{3,8,10} + \frac{c_6 + c_{11}}{2} \]

\[ e_2 = c_{3,4} + \frac{c_6}{2} \]

\[ e_3 = c_8 + \frac{c_6}{2} \]

\[ e_4 = 0 \]

\[ f_1 = e_1 + c_{4,7,9} \]

\[ f_2 = e_3 + c_{5,7} + \frac{c_{11}}{2} \]

\[ g_1 = e_1 \]

\[ g_2 = e_2 + c_2 + \frac{c_{11}}{2} \]

**VIII. APPENDIX C**

A. The MATLAB program to find the maximum violation HNA under IC, ML and LO under imposing LR

```matlab
fun=@(x)(-1)*(x(6)/2);

aeq0a1=[1 0 0 0 0 0.5];

aeq0a2=[0 0 0 0 1 0.5];

aeq0a=[aeq0a1;aeq0a2];

aeq1a1=[0 0 0 0 1 0.5];

aeq1a2=[0 0 0 0 1 0.5];

aeq1a=[aeq1a1;aeq1a2];

aeq0b1=[0 0 0 0 1 0.5];

aeq0b2=[0 0 0 0 1 0.5];

aeq0b=[aeq0b1;aeq0b2];

aeq1b1=[0 0 0 0 1 0.5];

aeq1b2=[0 0 0 0 1 0.5];

aeq1b=[aeq1b1;aeq1b2];

aeqcon=ones(1,6);

beqlr1=0.5*ones(8,1);

beqlr2=0.5*ones(6,1);

beqlr3=0.5*ones(4,1);

beqlr4=0.5*ones(2,1);

Aeq0=aeqcon;

beq0=1;

Aeq1=[aeqcon;aeq0a1;aeq0b;aeq1b];

beq1=[1;beqlr1];

Aeq2=[aeqcon;aeq0a1;aeq0b];

beq2=[1;beqlr2];

Aeq3=[aeqcon;aeq0a1;aeq1b];

beq3=[1;beqlr2];

Aeq4=[aeqcon;aeq0a1;aeq0b;aeq1b];

beq4=[1;beqlr2];

Aeq5=[aeqcon;aeq0a1;aeq1b];

beq5=[1;beqlr2];

Aeq6=[aeqcon;aeq0b;aeq1b];

beq6=[1;beqlr3];

Aeq7=[aeqcon;aeq0a1;aeq1b];

beq7=[1;beqlr3];

Aeq8=[aeqcon;aeq0a1;aeq1b];

beq8=[1;beqlr3];

Aeq9=[aeqcon;aeq0a1;aeq0b];

beq9=[1;beqlr3];

Aeq10=[aeqcon;aeq0a1;aeq1b];

beq10=[1;beqlr3];

Aeq11=[aeqcon;aeq1a;aeq0b];

beq11=[1;beqlr3];

Aeq12=[aeqcon;aeq0a];

beq12=[1;beqlr4];

Aeq13=[aeqcon;aeq1a];

beq13=[1;beqlr4];

Aeq14=[aeqcon;aeq0b];

beq14=[1;beqlr4];

Aeq15=[aeqcon;aeq1b];

beq15=[1;beqlr4];

ub=ones(6,1);

lb=zeros(6,1);

x0=0.2*[1 1 1 1 1 0];

ml = @mlhardy;

ic = @ichardy;

lo = @lohardy;

fminlo0=fmincon(fun,x0,[],[],Aeq0,beq0,lb,ub,lo);

fminic0=fmincon(fun,x0,[],[],Aeq0,beq0,lb,ub,ic);

fminml0=fmincon(fun,x0,[],[],Aeq0,beq0,lb,ub,ml);

fminlo1=fmincon(fun,x0,[],[],Aeq1,beq1,lb,ub,lo);

fminic1=fmincon(fun,x0,[],[],Aeq1,beq1,lb,ub,ic);

fminml1=fmincon(fun,x0,[],[],Aeq1,beq1,lb,ub,ml);

fminlo2=fmincon(fun,x0,[],[],Aeq2,beq2,lb,ub,lo);

fminic2=fmincon(fun,x0,[],[],Aeq2,beq2,lb,ub,ic);

fminml2=fmincon(fun,x0,[],[],Aeq2,beq2,lb,ub,ml);

fminlo3=fmincon(fun,x0,[],[],Aeq3,beq3,lb,ub,lo);

fminic3=fmincon(fun,x0,[],[],Aeq3,beq3,lb,ub,ic);

fminml3=fmincon(fun,x0,[],[],Aeq3,beq3,lb,ub,ml);

fminlo4=fmincon(fun,x0,[],[],Aeq4,beq4,lb,ub,lo);

fminic4=fmincon(fun,x0,[],[],Aeq4,beq4,lb,ub,ic);

fminml4=fmincon(fun,x0,[],[],Aeq4,beq4,lb,ub,ml);

fminlo5=fmincon(fun,x0,[],[],Aeq5,beq5,lb,ub,lo);

fminic5=fmincon(fun,x0,[],[],Aeq5,beq5,lb,ub,ic);

fminml5=fmincon(fun,x0,[],[],Aeq5,beq5,lb,ub,ml);

fminlo6=fmincon(fun,x0,[],[],Aeq6,beq6,lb,ub,lo);

fminic6=fmincon(fun,x0,[],[],Aeq6,beq6,lb,ub,ic);

fminml6=fmincon(fun,x0,[],[],Aeq6,beq6,lb,ub,ml);

fminlo7=fmincon(fun,x0,[],[],Aeq7,beq7,lb,ub,lo);

fminic7=fmincon(fun,x0,[],[],Aeq7,beq7,lb,ub,ic);

fminml7=fmincon(fun,x0,[],[],Aeq7,beq7,lb,ub,ml);

fminlo8=fmincon(fun,x0,[],[],Aeq8,beq8,lb,ub,lo);

fminic8=fmincon(fun,x0,[],[],Aeq8,beq8,lb,ub,ic);

fminml8=fmincon(fun,x0,[],[],Aeq8,beq8,lb,ub,ml);

fminlo9=fmincon(fun,x0,[],[],Aeq9,beq9,lb,ub,lo);

fminic9=fmincon(fun,x0,[],[],Aeq9,beq9,lb,ub,ic);

fminml9=fmincon(fun,x0,[],[],Aeq9,beq9,lb,ub,ml);

fminlo10=fmincon(fun,x0,[],[],Aeq10,beq10,lb,ub,lo);

fminic10=fmincon(fun,x0,[],[],Aeq10,beq10,lb,ub,ic);

fminml10=fmincon(fun,x0,[],[],Aeq10,beq10,lb,ub,ml);

fminlo11=fmincon(fun,x0,[],[],Aeq11,beq11,lb,ub,lo);

fminic11=fmincon(fun,x0,[],[],Aeq11,beq11,lb,ub,ic);

fminml11=fmincon(fun,x0,[],[],Aeq11,beq11,lb,ub,ml);

fminlo12=fmincon(fun,x0,[],[],Aeq12,beq12,lb,ub,lo);

fminic12=fmincon(fun,x0,[],[],Aeq12,beq12,lb,ub,ic);

fminml12=fmincon(fun,x0,[],[],Aeq12,beq12,lb,ub,ml);
```

x0=0.2*[1 1 1 1 1 0]; ml = @mlhardy;

ic = @ichardy;

lo = @lohardy;
function \[c, ceq]=ichardy(x)
\begin{align*}
p11 &= x(3) + x(6)/2; 
p12 &= x(4); 
p13 &= 0; 
p14 &= x(1) + x(2) + x(5) + x(6)/2; 
p21 &= x(3) + x(4) + x(6)/2; 
p22 &= 0; 
p23 &= x(2); 
p24 &= x(1) + x(5) + x(6)/2; 
p31 &= x(6)/2; 
p32 &= x(5); 
p33 &= x(3); 
p34 &= x(1) + x(2) + x(4) + x(6)/2; 
p41 &= 0; 
p42 &= x(5) + x(6)/2; 
p43 &= x(3) + x(2) + x(4) + x(6)/2; 
p44 &= x(1); 
P1 &= (p11 + p14 + p31 + p34)/2; 
P1I &= (p21 + p24 + p42 + p43)/2; 
E1 &= (2 * P1I - 1); 
E2 &= (2 * P1I - 1); 
c1 &= E1^2 + E2^2 - 1; 
\text{\(Q1\)} &= (p11 + p14 + p21 + p24)/2; 
\text{\(Q1I\)} &= (p31 + p34 + p42 + p43)/2; 
F1 &= (2 * \text{\(Q1I\)} - 1); 
F2 &= (2 * \text{\(Q1I\)} - 1); 
c2 &= F1^2 + F2^2 - 1; 
\text{ceq} &= [c1; c2];
\end{align*}

\text{ceq} = []; 
\]

3. @lohardy

function \[c, ceq]=lohardy(x)
\begin{align*}
p11 &= x(3)+x(6)/2; 
p12 &= x(4); 
p13 &= 0; 
p21 &= x(3)+x(4)+x(6)/2; 
p23 &= x(2); 
p31 &= x(6)/2; 
p32 &= x(5); 
p41 &= 0; 
c1 &= p11; 
c2 &= p21; 
c3 &= p31; 
n1 &= p23+c2; 
m1 &= p32+c3; 
d1 &= c3^2 + 2 * c1 * n1 - c1^2 - n1^2; 
d2 &= 2 * c1 * c3 - c1^2 - (c2 - n1) * (m1 + n1 - 1); 
d3 &= c2 * (c3 + m1) + (c3 - m1) * n1 - c2^2; 
d4 &= c3 * (1 + m1) + c2 * (m1 + n1) - c2 - m1 - c3 * n1; 
d5 &= c1 * (m1 + c3) + c3 * (c3 + n1) - m1 + n1; 
d6 &= c3 + c1 * c3 + c2 * (1 - c1 - c3 + m1 + n1) - c1 * (m1 + n1); 
d7 &= c3 + c3^2 + c2 * (m1 + n1) - c2 - c3 * (m1 + n1); 
d8 &= c2^2 + 2 * c2 * n1 - c2^2 - n1^2; 
d9 &= c3 * (1 + c3) + c1 * (c2 + m1) - c1 - c3 * (c2 + m1); 
d10 &= c3^2 + m1 * (-1 + m1 + n1) - (c1 - c2)^2 - c3 * (-1 +
The MATLAB program to find the maximum violation CNA under IC, ML and LO under imposing LR

```matlab
fun=@(x)(-1)*(0.5*(x(6)-x(11))-x(6)-x(7)-x(9)-x(10));
x0=[1/8 1/8 1/8 1/8 1/8 1/8 1/20 1/20 1/20 1/20];
A=(-1)*[0 0 0 0 0 0 1 1 1 1 0.5];
b=-eps;
aeql=ones(11,1);
Aeq0a=aeqcon;
Aeq0b=aeq0b;
Aeq1a=aeq1a;
Aeq1b=aeq1b;
beq1=1;
Aeq0a=[1 1 0 0 0.5 1 1 1 1 0.5];
Aeq0b=[0 0 0 0 0 0 0 0 0 0 0];
beq1=[1 1 1 1 0.5 0 0 0 0 0 0.5];
beq2=beq1;
beq12=[1 1 1 1 0.5 1 1 1 1 0.5];
beq13=[1 1 1 1 0.5 1 1 1 1 0.5];
beq14=beq13;
beq15=beq14;
Aeq12=[aeqcon;aeq1a;aeq0b;aeq1b];
Aeq13=[aeqcon;aeq1a];
Aeq14=[aeqcon;aeq0b];
Aeq15=[aeqcon;aeq1b];
Aeq16=[aeqcon;aeq0b;aeq1b];
Aeq17=[aeqcon;aeq1a;aeq0b];
beq13=[1;beqlr4];
beq14=[1;beqlr4];
beq15=[1;beqlr4];
ub=ones(11,1);
lo=zeros(11,1);
ic=@iccabelo;
ml=@mlcabelo;
```
\[ f_{\text{minlo1}} = f_{\text{mincon}}(\text{fun}, x_0, [], [], \text{Aeq15}, \text{beq15}, \text{lb}, \text{ub}, \text{ml}) \]
\[ p_{32} = x(7) + x(8) + x(9) + x(10) + x(11)/2; \]
\[ p_{33} = x(1) + x(11)/2; \]
\[ p_{34} = x(1) + x(5) + x(6)/2; \]
\[ p_{35} = x(8) + x(6)/2; \]
\[ p_{36} = x(5) + x(7) + x(11)/2; \]
\[ p_{37} = x(3) + x(10) + x(11)/2; \]
\[ p_{38} = x(1) + x(2) + x(4) + x(9) + x(6)/2; \]
\[ p_{10} = 0; \]
\[ p_{11} = x(5) + x(7) + x(8) + x(6)/2; \]
\[ p_{12} = x(3) + x(4) + x(7) + x(9)/2; \]
\[ p_{13} = 0; \]
\[ p_{14} = x(3) + x(4) + x(6)/2; \]
\[ p_{15} = x(3) + x(4) + x(6)/2; \]
\[ p_{16} = x(3) + x(4) + x(6)/2; \]
\[ c_{1} = E_{II}^2 + E_{II}^2 - 1; \]
\[ c_{2} = E_{III}^2 + E_{IV}^2 - 1; \]
\[ c_{3} = F_{II}^2 + F_{II}^2 - 1; \]
\[ c_{4} = F_{IV}^2 + F_{IV}^2 - 1; \]
\[ c_{1} = [c_{1}c_{2}c_{3}c_{4}]; \]
\[ \text{ceq} = []; \]

1. \text{iiccabelo}

\[ ; \text{function} [c, \text{ceq}] = \text{iiccabelo}(x) \]
\[ p_{11} = x(3) + x(8) + x(10) + x(11)/2; \]
\[ p_{12} = x(3) + x(4) + x(7) + x(9)/2; \]
\[ p_{13} = 0; \]
\[ p_{14} = x(3) + x(4) + x(6)/2; \]
\[ c_{1} = E_{II} + E_{II} - 1; \]
\[ E_{III} = (p_{11} + p_{14} + p_{21} + p_{24} - 1); \]
\[ E_{IV} = (p_{31} + p_{34} + p_{42} + p_{43} - 1); \]
\[ c_{1} = E_{II}^2 + E_{II}^2 - 1; \]
\[ c_{2} = E_{III}^2 + E_{IV}^2 - 1; \]
\[ c_{3} = F_{II}^2 + F_{II}^2 - 1; \]
\[ c_{4} = F_{IV}^2 + F_{IV}^2 - 1; \]
\[ c_{1} = [c_{1}c_{2}c_{3}c_{4}]; \]
\[ \text{ceq} = []; \]

2. \text{mlcabelo}

\[ ; \text{function} [c, \text{ceq}] = \text{mlcabelo}(x) \]
\[ p_{11} = x(3) + x(8) + x(10) + x(11)/2; \]
\[ p_{12} = x(3) + x(4) + x(7) + x(9)/2; \]
\[ p_{13} = 0; \]
\[ p_{14} = x(3) + x(4) + x(5) + x(6)/2; \]
\[ c_{1} = E_{II} + E_{II} - 1; \]
\[ E_{III} = (p_{11} + p_{14} + p_{21} + p_{24} - 1); \]
\[ E_{IV} = (p_{31} + p_{34} + p_{42} + p_{43} - 1); \]
\[ c_{1} = E_{II}^2 + E_{II}^2 - 1; \]
\[ c_{2} = E_{III}^2 + E_{IV}^2 - 1; \]
\[ c_{3} = F_{II}^2 + F_{II}^2 - 1; \]
\[ c_{4} = F_{IV}^2 + F_{IV}^2 - 1; \]
\[ c_{1} = [c_{1}c_{2}c_{3}c_{4}]; \]
\[ \text{ceq} = []; \]
[1] J. S. Bell, Physics 1, 195 (1964).
[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[3] B. S. Tsirelson, Lett. Math. Phys. 4, 93 (1980).
[4] S. Popescu and D. Rohrlich, Found. Phys. 24, 379 (1994).
[5] L. Hardy, Phys. Rev. Lett. 68, 2981 (1992).
[6] L. Hardy, Phys. Rev. Lett. 71, 1665 (1993).
[7] N. D. Mermin, Am. J. Phys. 62, 880 (1994).
[8] A. Cabello, Phys. Rev. A 65, 032108 (2002).
[9] Lin-mei Liang and Cheng-zu Li, Phys. Lett. A 335, 371-373 (2005).
[10] S. K. Choudhary, S. Ghosh, G. Kar, S. Kunkri, R. Rahaman and A. Roy, Quantum Information and Computation, Vol. 10, No. 9 & 10 0859-0871 (2010).
[11] G. Kar, Phys. Lett. A 228, 119 (1997).
[12] R. Rabelo, Y. Z. Law and V. Scarani, Phys. Rev. Lett. 109, 180401 (2012).
[13] S. Kunkri, S. K. Choudhary, A. Ahanj and P. Joag, Phys. Rev. A 73, 022346 (2006).
[14] W. van Dam, Ph.D. thesis, University of Oxford (2000).
[15] G. Brassard, H. Buhrman, N. Linden, A. A. Methot, A. Tapp, and F. Unger, Phys. Rev. Lett. 96, 250401 (2006).
[16] M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter and M. Zukowski, Nature 461, 1101 (2009).
[17] M. Navascues and H. Wunderlich, Proc. Roy. Soc. Lond. A 466, 881-890 (2009).
[18] A. Cabello, Phys. Rev. A 90, 062125 (2014).
[19] D. Rohrlich, arXiv: 1407.8530.
[20] D. Rohrlich, arXiv: 1408.3125.
[21] J. Oppenheim and S. Wehner, Science 330, 1072 (2010).
[22] M. Banik, Md. Rajjak Gazi, S. Ghosh and G. Kar, Phys. Rev. A 87, 052125 (2013).
[23] A. Ahanj, S. Kunkri, A. Rai, R. Rahaman and P. S. Joag, Phys. Rev. A 81, 032103 (2010).
[24] S. Das, M. Banik, Md. Rajjak Gazi, A. Rai and S. Kunkri, Phys. Rev. A 88, 062101 (2013).
[25] R. Gallego, L. E. Wurdinger, A. Acin and M. Navascues, Phys. Rev. Lett. 107, 210403 (2011).
[26] S. Das, M. Banik, A. Rai, M. Rajjak Gazi, and S. Kunkri, Phys. Rev. A 87, 012112 (2013).
[27] T. Fritz, A. B. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier and A. Acin, Nature Communications 4, 2263 (2013).
[28] M. D. R. Gazi, A. Rai, S. Kunkri and R. Rahaman, J. Phys. A: Math. Theor. 43, 452001 (2010).
[29] G. Zoka and A. Ahanj, Quantum Stud.: Math. Found. Volume 3, Issue 2, pp 135145 (2016).
[30] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu and D. Roberts, Phys. Rev. A 71, 022101 (2005).
[31] J. Allcock, N. Brunner, M. Pawlowski and V. Scarani, Phys. Rev. A 80, 040103(R) (2009).
[32] W. van Dam, e-print arXiv:quant-ph/0501159.
[33] S. Wolf and J. F. Wullschleger, e-print arXiv:quant-ph/0502030v1 (2005).
[34] J. Allcock, N. Brunner, M. Pawlowski and V. Scarani, Phys. Rev. A 80, 040103(R) (2009).
[35] L. Landau, Found. Phys. 18, 449 (1988).
[36] L. Masanes, quant-ph/0309137.
[37] M. Navascues, S. Pironio and A. Acin, Phys. Rev. Lett. 98, 010401 (2007).
[38] M. Navascues, S. Pironio and A. Acin, New J. Phys. 10, 073013 (2008).
[39] Jose L. Cereceda, Found. Phys. Lett. 13, 427 (2000).