THE TERWILLIGER ALGEBRA OF THE HALVED CUBE

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Abstract. Let $D \geq 3$ denote an integer. For any $x \in \mathbb{F}_2^D$ let $w(x)$ denote the Hamming weight of $x$. Let $X$ denote the subspace of $\mathbb{F}_2^D$ consisting of all $x \in \mathbb{F}_2^D$ with even $w(x)$. The $D$-dimensional halved cube $\frac{1}{2}H(D, 2)$ is a finite simple connected graph with vertex set $X$ and $x, y \in X$ are adjacent if and only if $w(x - y) = 2$. Fix a vertex $x \in X$. The Terwilliger algebra $T = T(x)$ of $\frac{1}{2}H(D, 2)$ with respect to $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by the adjacency matrix $A$ and the dual adjacency matrix $A^* = A^*(x)$ where $A^*$ is a diagonal matrix with $A^*_{yy} = D - 2w(x - y)$ for all $y \in X$. In this paper we decompose the standard $T$-module into a direct sum of irreducible $T$-modules.

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1. Preliminaries

Throughout this paper, we adopt the following conventions: An algebra is meant to be a unital associative algebra. A subalgebra has the same unit as the parent algebra. Let $\mathbb{C}$ denote the complex number field. Given a finite set $Y \neq \emptyset$, let $\text{Mat}_Y(\mathbb{C})$ denote the algebra consisting of the square matrices over $\mathbb{C}$ indexed by $Y$ and let $\mathbb{C}^Y$ denote the vector space consisting of all column vectors over $\mathbb{C}$ indexed by $Y$.

Let $\Gamma$ denote a finite simple connected graph with vertex set $X \neq \emptyset$. Let $\partial$ denote the path-length distance function of $\Gamma$. The diameter $D$ of $\Gamma$ is defined by

$$D = \max_{x, y \in X} \partial(x, y).$$

Given any $x \in X$ let

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\} \quad \text{for } i = 0, 1, \ldots, D.$$ 

For short we abbreviate $\Gamma(x) = \Gamma_1(x)$. We call $\Gamma$ distance-regular whenever for all $h, i, j \in \{0, 1, \ldots, D\}$ and all $x, y \in X$ with $\partial(x, y) = h$ the number $|\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of $x$ and $y$. If $\Gamma$ is distance-regular, then the numbers $a_i, b_i, c_i$ for all $i = 0, 1, \ldots, D$ defined by

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

for any $x, y \in X$ with $\partial(x, y) = i$ are called the intersection numbers of $\Gamma$. Here $\Gamma_{-1}(x)$ and $\Gamma_{D+1}(x)$ are empty sets.

Now assume that $\Gamma$ is distance-regular. For all $i = 0, 1, \ldots, D$ the $i^{th}$ distance matrix $A_i$ of $\Gamma$ is a 0-1 matrix in $\text{Mat}_X(\mathbb{C})$ defined by

$$(A_i)_{xy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{if } \partial(x, y) \neq i \\
1 & 
\end{cases} \quad \text{for all } x, y \in X.$$
We abbreviate $A = A_1$ and $A$ is called the *adjacency matrix* of $\Gamma$. Let $a_i, b_i, c_i \ (0 \leq i \leq D)$ denote the intersection numbers of $\Gamma$. Observe that

$$A_i A = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad \text{for all} \ i = 0, 1, \ldots, D,$$

where $b_{-1} A_{-1}$ and $c_{D+1} A_{D+1}$ are interpreted as the zero matrix in $\text{Mat}_X(\mathbb{C})$. The *Bose–Mesner algebra* $\mathcal{M}$ of $\Gamma$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$. Note that the matrices $\{A_i\}_{i=0}^D$ form a basis for $\mathcal{M}$. Let $\circ$ denote the entrywise product. Since

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D),$$

it follows that $\mathcal{M}$ is closed under $\circ$.

Since $A$ is a real symmetric matrix and $\mathcal{M}$ has dimension $D + 1$, it follows that $A$ has $D + 1$ distinct real eigenvalues $\theta_0, \theta_1, \ldots, \theta_D$. There are unique $E_0, E_1, \ldots, E_D \in \text{Mat}_X(\mathbb{C})$ satisfying

$$\sum_{i=0}^D E_i = I, \quad A E_i = \theta_i E_i \quad \text{for all} \ i = 0, 1, \ldots, D,$$

where $I$ denotes the identity matrix in $\text{Mat}_X(\mathbb{C})$. Note that $\{E_i\}_{i=0}^D$ form a basis for $\mathcal{M}$. The element $E_i$ is called the *primitive idempotent* of $\Gamma$ associated with $\theta_i$ for $i = 0, 1, \ldots, D$.

The distance-regular graph $\Gamma$ is said to be *Q-polynomial* with respect to the ordering $\{E_i\}_{i=0}^D$ if there are unique $a_i^*, b_i^*, c_i^* \in \mathbb{C}$ for all $i = 0, 1, \ldots, D$ with $b_D^* = c_0^* = 0$, $b_i^* c_i^* \neq 0$ for all $i = 1, 2, \ldots, D$ such that

$$E_i \circ E_1 = \frac{1}{|X|} (b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1}) \quad \text{for all} \ i = 0, 1, \ldots, D,$$

where $b_{-1}^* E_{-1}$ and $c_{D+1}^* E_{D+1}$ are interpreted as the zero matrix in $\text{Mat}_X(\mathbb{C})$. If this is the case, then $a_i^*, b_i^*, c_i^*$ for all $i = 0, 1, \ldots, D$ are called the *dual intersection numbers* of $\Gamma$.

Now assume that $\Gamma$ is a $Q$-polynomial distance-regular graph. Fix a vertex $x \in X$. For $i = 0, 1, \ldots, D$ the $i^{th}$ *dual distance matrix* $A_i^* = A_i^*(x)$ of $\Gamma$ respect to $x$ is the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ defined by

$$(A_i^*)_{y,y} = |X|(E_i)_{xy} \quad \text{for all} \ y \in X.$$ 

We abbreviate $A^* = A_1^*$ and $A^*$ is called the *dual adjacency matrix* of $\Gamma$ with respect to $x$. Let $a_i^*, b_i^*, c_i^* \ (0 \leq i \leq D)$ denote the dual intersection numbers of $\Gamma$. Observe that

$$A_i^* A^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \quad \text{for all} \ i = 0, 1, \ldots, D,$$

where $b_{-1}^* A_{-1}^*$ and $c_{D+1}^* A_{D+1}^*$ are interpreted as the zero matrix in $\text{Mat}_X(\mathbb{C})$. The *dual Bose–Mesner algebra* $\mathcal{M}^* = \mathcal{M}^*(x)$ of $\Gamma$ with respect to $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A^*$. Note that $\{A_i^*\}_{i=0}^D$ form a basis for $\mathcal{M}^*$.

The *Terwilliger algebra* $\mathcal{T} = \mathcal{T}(x)$ of $\Gamma$ with respect to $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^* \ [7, 9]$. The vector space $\mathbb{C}^X$ has a natural $\mathcal{T}$-module structure and it is called the *standard $\mathcal{T}$-module*. Since $\mathcal{T}$ is finite-dimensional the irreducible $\mathcal{T}$-modules are finite-dimensional. Since $\mathcal{T}$ is closed under the conjugate-transpose map the algebra $\mathcal{T}$ is semisimple. Hence the algebra $\mathcal{T}$ is isomorphic to

$$\bigoplus \text{End}(V),$$

irreducible $\mathcal{T}$-modules $V$. 
where the direct sum is over all non-isomorphic irreducible $\mathcal{T}$-modules $V$. Since the standard $\mathcal{T}$-module $C^X$ is faithful it follows that $C^X$ contains all irreducible $\mathcal{T}$-modules up to isomorphism.

The paper is organized as follows: In §2 we state the main results of this paper on the Terwilliger algebra of the halved cube (Proposition 2.2 and Theorems 2.3–2.5). In §3 we recall some results concerning the Terwilliger algebra of the hypercube. In §4 we relate the Terwilliger algebra of the halved cube to a subalgebra of the Terwilliger algebra of the hypercube. In §5 we give the proofs of our main results.

2. Statement of results

Let $D \geq 3$ denote an integer. Let $\mathbb{F}_2$ denote the finite field of order two. Let $\mathbb{F}_2^D$ denote a $D$-ary Cartesian product of $\mathbb{F}_2$. For any $x \in \mathbb{F}_2^D$ the Hamming weight $w(x)$ of $x$ is the number of ones in $x$. Let $X$ denote the subspace of $\mathbb{F}_2^D$ consisting of all $x \in \mathbb{F}_2^D$ with even $w(x)$.

**Definition 2.1.** The $D$-dimensional halved cube $\frac{1}{2}H(D, 2)$ is a finite simple connected graph with vertex set $X$ and $x, y \in X$ are adjacent if and only if $w(x - y) = 2$.

Note that $\frac{1}{2}H(D, 2)$ is a distance-regular graph of diameter $\lfloor \frac{D}{2} \rfloor$ whose intersection numbers $[1, 3, 9]$ are

$$a_i = 2i(D - 2i), \quad b_i = \left(\frac{D - 2i}{2}\right), \quad c_i = \left(\frac{2i}{2}\right) \quad \text{for } i = 0, 1, \ldots, \lfloor \frac{D}{2} \rfloor.$$ Let $A$ denote the adjacency matrix of $\frac{1}{2}H(D, 2)$. The eigenvalues of $A$ are

$$\theta_i = \frac{1}{2}((D - 2i)^2 - D) \quad \text{for } i = 0, 1, \ldots, \lfloor \frac{D}{2} \rfloor.$$ Let $E_i$ denote the primitive idempotent of $\frac{1}{2}H(D, 2)$ associated with $\theta_i$ for $i = 0, 1, \ldots, \lfloor \frac{D}{2} \rfloor$.

Note that $\frac{1}{2}H(D, 2)$ is $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^{\lfloor \frac{D}{2} \rfloor}$ whose dual intersection numbers $[1, 3, 9]$ are

$$a_i^* = 0, \quad b_i^* = D - i, \quad c_i^* = i \quad \text{for } i = 0, 1, \ldots, \lfloor \frac{D}{2} \rfloor - 1,$$

$$a_{\lfloor \frac{D}{2} \rfloor}^* = \begin{cases} 0 & \text{if } D \text{ is even}, \\ \frac{D+1}{2} & \text{if } D \text{ is odd}, \end{cases} \quad b_{\lfloor \frac{D}{2} \rfloor}^* = 0, \quad c_{\lfloor \frac{D}{2} \rfloor}^* = \begin{cases} D & \text{if } D \text{ is even}, \\ \frac{D-1}{2} & \text{if } D \text{ is odd}. \end{cases}$$

Fix a vertex $x \in X$. Let $A^* = A^*(x)$ denote the dual adjacency matrix of $\frac{1}{2}H(D, 2)$ with respect to $x$. The diagonal matrix $A^*$ is given by

$$A^*_{yy} = D - 2w(x - y) \quad \text{for all } y \in X.$$ Let $\mathcal{T} = \mathcal{T}(x)$ denote the Terwilliger algebra of $\frac{1}{2}H(D, 2)$ with respect to $x$. The main results of this paper are as follows:

**Proposition 2.2.** (i) For any even integer $k$ with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$, there exists a $(\lfloor \frac{D}{2} \rfloor - k + 1)$-dimensional irreducible $\mathcal{T}$-module $M_k$ that has a basis with respect to which the matrix
Theorem 2.3. The standard \( \mathcal{T} \)-module \( \mathbb{C}^X \) is isomorphic to

\[
\bigoplus_{k=0}^{[\frac{D}{2}]} \left( \frac{D-2k+1}{D-k+1} \binom{D}{k} \cdot M_k \right) \oplus \bigoplus_{k=2}^{[\frac{D}{2}]} \left( \frac{D-2k+3}{D-k+2} \binom{D}{k-1} \cdot N_k \right).
\]

Theorem 2.4. The \( \mathcal{T} \)-modules

\[
M_k \quad \text{for all even integers } k \text{ with } 0 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor,
\]

\[
N_k \quad \text{for all even integers } k \text{ with } 2 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor.
\]
are all irreducible $\mathcal{T}$-modules up to isomorphism. Moreover these $\mathcal{T}$-modules are mutually non-isomorphic.

**Theorem 2.5.** The algebra $\mathcal{T}$ is isomorphic to

$$\bigoplus_{k \text{ is even}} \operatorname{Mat} \left( \left[ \frac{D}{2} \right]_{k+1} \right) \oplus \bigoplus_{k \text{ is even}} \operatorname{Mat} \left( \left[ \frac{D}{2} \right]_{-k+1} \right).$$

In particular $\dim \mathcal{T} = \left( \left[ \frac{D}{3} \right] + 3 \right) + \left( \left[ \frac{D}{3} \right] + 1 \right)$.

Note that the notations of this section will be used in the rest of this paper.

3. **The Terwilliger algebra of the hypercube**

**Definition 3.1.** The $D$-dimensional hypercube $H(D, 2)$ is a finite simple connected graph with vertex set $\mathbb{F}_2^D$ and $x, y \in \mathbb{F}_2^D$ are adjacent if and only if $w(x - y) = 1$.

Note that $H(D, 2)$ is a distance-regular graph of diameter $D$ whose intersection numbers $[1, 3, 9]$ are $a_i = 0$, $b_i = D - i$, $c_i = i$ for $i = 0, 1, \ldots, D$.

For $i = 0, 1, \ldots, D$ the $i$th distance matrix $A_i$ of $H(D, 2)$ is a 0-1 matrix in $\operatorname{Mat} \left( \mathbb{F}_2^D \right)$ given by

$$\text{(3)} \quad (A_i)_{xy} = 1 \quad \text{if and only if} \quad w(x - y) = i$$

for all $x, y \in \mathbb{F}_2^D$. The eigenvalues of $A = A_1$ are

$$\theta_i = D - 2i \quad \text{for} \quad i = 0, 1, \ldots, D.$$

Let $E_i$ denote the primitive idempotent of $H(D, 2)$ associated with $\theta_i$ for $i = 0, 1, \ldots, D$.

The distance-regular graph $H(D, 2)$ is $Q$-polynomial with respect to the ordering $\{ E_i \}_{i=0}^D$ whose dual intersection numbers $[1, 9]$ are $a_i^* = 0$, $b_i^* = D - i$, $c_i^* = i$ for $i = 0, 1, \ldots, D$.

Let $A_i^* = A_i^*(x)$ denote the $i$th dual distance matrix of $H(D, 2)$ with respect to $x$ for $i = 0, 1, \ldots, D$. The diagonal matrix $A^* = A_1^*$ is given by

$$\text{(4)} \quad A_{yy}^* = D - 2w(x - y) \quad \text{for all} \quad y \in \mathbb{F}_2^D.$$

Let $t$ denote an indeterminate over $\mathbb{C}$. In view of the intersection numbers and the dual intersection numbers of $H(D, 2)$, we consider the polynomials $\{ v_i(t) \}_{i=0}^D$ given by the following recurrence relation:

$$tv_i(t) = (D - i + 1)v_{i-1}(t) + (i + 1)v_{i+1}(t) \quad \text{for all} \quad i = 1, 2, \ldots, D - 1$$

with $v_0(t) = 1$ and $v_1(t) = t$. Recall that for any nonzero $q \in \mathbb{C}$ and any integer $n \geq 1$, the Krawtchouk polynomials are

$$K_i(t; q, n) = \sum_{j=0}^{i} (-q)^j \binom{i}{j} \binom{i}{j} \quad \text{for all} \quad i = 0, 1, \ldots, n.$$
Applying the three-term recurrence of the Krawtchouk polynomials [§9.11] it is routine to verify that

\[(5) \quad v_i(t) = \binom{D}{i} \cdot K_i \left( \frac{D - t}{2}; 2, D \right) \quad \text{for all } i = 0, 1, \ldots, D.\]

The following lemma is immediate from the construction of \(\{v_i(t)\}_{i=0}^{D} \).

**Lemma 3.2.** (i) \(A_i = v_i(A)\) for all \(i = 0, 1, \ldots, D\).

(ii) \(A_i^* = v_i(A^*)\) for all \(i = 0, 1, \ldots, D\).

Let \(\mathcal{T} = \mathcal{T}(x)\) denote the Terwilliger algebra of \(H(D, 2)\) with respect to \(x\). We now recall some results on \(\mathcal{T}\).

**Proposition 3.3** (Corollary 6.8, [4]). For any integer \(k\) with \(0 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor\), there exists a \((D - 2k + 1)\)-dimensional irreducible \(\mathcal{T}\)-module \(L_k\) that has a basis with respect to which the matrices representing \(A\) and \(A^*\) are

\[
\begin{pmatrix}
0 & \gamma_1 & \ldots & 0 \\
\beta_0 & 0 & \gamma_2 & \ldots \\
\beta_1 & 0 & \ldots & \gamma_{D-2k-1} \\
0 & \beta_{D-2k} & \ldots & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
\theta_0^* & \theta_1^* & \ldots & 0 \\
\theta_0 & \theta_1 & \ldots & \theta_{D-2k}^*
\end{pmatrix},
\]

respectively, where

\[
\beta_i = i + 1 \quad (0 \leq i \leq D - 2k - 1),
\]

\[
\gamma_i = D - i - 2k + 1 \quad (1 \leq i \leq D - 2k),
\]

\[
\theta_i^* = D - 2(i + k) \quad (0 \leq i \leq D - 2k).
\]

**Theorem 3.4** (Theorem 10.2, [4]). The standard \(\mathcal{T}\)-module \(\mathbb{C}^{i+2}\) is isomorphic to

\[
\bigoplus_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \frac{D - 2k + 1}{D - k + 1} \binom{D}{k} \cdot L_k.
\]

**Theorem 3.5** (Corollary 6.9, [4]). The \(\mathcal{T}\)-modules \(L_k\) for all integers \(k\) with \(0 \leq k \leq \left\lfloor \frac{D}{2} \right\rfloor\) are all irreducible \(\mathcal{T}\)-modules up to isomorphism. Moreover these \(\mathcal{T}\)-modules are mutually non-isomorphic.

**Theorem 3.6** (Theorem 14.4, [4]). The algebra \(\mathcal{T}\) is isomorphic to

\[
\bigoplus_{k=0}^{\left\lfloor \frac{D}{2} \right\rfloor} \text{Mat}_{D-2k+1}(\mathbb{C}).
\]

In particular \(\text{dim } \mathcal{T} = \binom{D+3}{3}\).

Note that \(H(D, 2)\) is an example of the \(D\)-dimensional Hamming graphs. In the case of the \(D\)-dimensional Hamming graphs, the decomposition formula for the standard modules was recently given in [5, Theorem 1.7] and [2, p. 17].
4. The Terwilliger Algebra of the Halved Cube

Given a nonempty subset \( Y \) of \( \mathbb{F}_2^D \), we identify \( \mathbb{C}^Y \) as a subspace of \( \mathbb{C}^{\mathbb{F}_2^D} \) via the natural injection \( \mathbb{C}^Y \to \mathbb{C}^{\mathbb{F}_2^D} \) and we view any \( M \in \text{Mat}_{\mathbb{F}_2}(\mathbb{C}) \) as the linear map \( \mathbb{C}^Y \to \mathbb{C}^Y \) that sends \( v \) to \( Mv \) for all \( v \in \mathbb{C}^Y \). Recall from \(|\text{§}|\) that \( A \) denotes the adjacency matrix of \( \frac{1}{2}H(D, 2) \).

**Lemma 4.1.** \( A = A_2|_{\mathbb{C}^X} \).

*Proof.* Immediate from Definition \(|\text{2.|}| \) and \(|\text{3.}|\).

By Lemma \(|\text{3.|}| \) we have \( A_2 = v_2(A) \). Combined with Lemma \(|\text{4.|}| \) it follows that the eigenvalues of \( A \) are \( v_2(\theta_i) = \theta_i \) for \( i = 0, 1, \ldots, \lfloor \frac{D}{2} \rfloor \). Recall from \(|\text{§}|\) that \( E_i \) denotes the primitive idempotent of \( \frac{1}{2}H(D, 2) \) associated with \( \theta_i \) for \( i = 0, 1, \ldots, \lfloor \frac{D}{2} \rfloor \).

**Lemma 4.2.**

(i) Suppose that \( D \) is odd. Then
\[
E_i = (E_i + E_{D-i})|_{\mathbb{C}^X} \quad \text{for } i = 0, 1, \ldots, \frac{D-1}{2}.
\]

(ii) Suppose that \( D \) is even. Then
\[
E_i = \begin{cases} 
(E_i + E_{D-i})|_{\mathbb{C}^X} & \text{for } i = 0, 1, \ldots, \frac{D}{2} - 1, \\
E_{\frac{D}{2}}|_{\mathbb{C}^X} & \text{for } i = \frac{D}{2}.
\end{cases}
\]

*Proof.* (i): Let \( E'_i = E_i + E_{D-i} \) for \( i = 0, 1, \ldots, \frac{D-1}{2} \). Observe that \( \sum_{i=0}^{\frac{D-1}{2}} E'_i = \sum_{i=0}^{\frac{D}{2}} E_i \) is equal to the identity matrix and \( A_2 E'_i = \theta_i E'_i \) for \( i = 0, 1, \ldots, \frac{D-1}{2} \). It follows that \( E'_i \) is a polynomial in \( A_2 \) for \( i = 0, 1, \ldots, \frac{D-1}{2} \). Therefore \( \mathbb{C}^X \) is \( E'_i \)-invariant. Combined with Lemma \(|\text{4.|}| \) this yields that \( E_i = E'_i|_{\mathbb{C}^X} \) for \( i = 0, 1, \ldots, \frac{D-1}{2} \).

(ii): Similar to the proof of Lemma \(|\text{4.|}| \) (i).

**Lemma 4.3.** Let \( n \) denote an integer with \( 0 \leq n \leq D \). Then
\[
v_{D-1}(D - 2n) = (-1)^n(D - 2n).
\]

*Proof.* Using \(|\text{5.|}| \) yields that \( v_{D-1}(D - 2n) \) is equal to
\[
\sum_{j=0}^{D-1} (-2)^j(D - j) \binom{n}{j} = D \sum_{j=0}^{D-1} (-2)^j \binom{n}{j} + 2n \sum_{j=0}^{D-2} (-2)^j \binom{n-1}{j}.
\]

It follows from the binomial formula that \( \sum_{j=0}^{k} (-2)^j \binom{k}{j} = (-1)^k \) for any integer \( k \geq 0 \). The lemma follows by evaluating \(|\text{6.|}| \) by using the above equation.

Recall from \(|\text{§}|\) that \( A^* = A^*(x) \) denotes the dual adjacency matrix of \( \frac{1}{2}H(D, 2) \) with respect to \( x \).

**Lemma 4.4.** \( A^* = A^*|_{\mathbb{C}^X} \).

*Proof.* Let \( y \in X \) be given. By Lemma \(|\text{4.|}| \) and since \( D \geq 3 \) we have
\[
A_{yy}^* = 2^{D-1}(E_1 + E_{D-1})_{yy} = \frac{1}{2}(A^* + A_{D-1}^*)_{yy}.
\]

Recall from \(|\text{1.|}| \) that \( A_{yy}^* = D - 2w(x - y) \). By Lemma \(|\text{3.|}| \) (ii) we have
\[
(A_{D-1}^*)_{yy} = v_{D-1}(D - 2w(x - y)).
\]
Since $x, y \in X$ the integer $w(x - y)$ is even. Combined with Lemma 4.3 it follows that $(A^*_D)_{yy}^D = A^*_y$. Therefore $A^*_y = A^*_y$ for all $y \in X$. The lemma follows. □

Recall from [2] that $\mathcal{T} = \mathcal{T}(x)$ stands for the Terwilliger algebra of $\frac{1}{2}H(D, 2)$ with respect to $x$. Define $\mathcal{S}$ to be the subalgebra of $\mathcal{T}$ generated by $A_2$ and $A^*$. In light of Lemmas 4.1 and 4.4 we may identify $\mathcal{T}$ as the algebra consisting of all elements $M|_{C^X}$ for all $M \in \mathcal{S}$.

5. Proofs of Proposition 2.2 and Theorems 2.3–2.5

Lemma 5.1. Let $L$ denote a $\mathcal{T}$-submodule of $\mathbb{C}^D$. Then the following hold:

(i) $L \cap C^X$ is a $\mathcal{T}$-submodule of $C^X$.

(ii) $L = (L \cap C^X) \oplus (L \cap \mathbb{C}^D \setminus X)$.

Proof. (i): Observe that $C^X$ is an $\mathcal{S}$-module. Since $\mathcal{S}$ is a subalgebra of $\mathcal{T}$ it follows that $L$ is an $\mathcal{S}$-module. Hence $L \cap C^X$ is an $\mathcal{S}$-module. The statement (i) follows.

(ii): Since $A^*$ is diagonalizable in $\mathbb{C}^D$ it follows that $A^*|_L$ is diagonalizable. By [4] the eigenvectors of $A^*$ in $\mathbb{C}^D$ lie in $C^X$ or $\mathbb{C}^D \setminus X$. The statement (ii) follows. □

Given an even integer $k$ with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$, let $[A] = [A]_k$ and $[A^*] = [A^*]_k$ denote the two matrices given in (1), respectively. Given an even integer $k$ with $2 \leq k \leq \lfloor \frac{D}{2} \rfloor$, let $\langle A \rangle = \langle A \rangle_k$ and $\langle A^* \rangle = \langle A^* \rangle_k$ denote the two matrices given in (2), respectively.

Let $k$ be any integer with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$. Recall the $\mathcal{T}$-module $L_k$ from Proposition 3.3. We regard $L_k$ as a $\mathcal{T}$-submodule of $\mathbb{C}^D$. It follows from Lemma 5.1(i) that $L_k \cap C^X$ is a $\mathcal{T}$-module.

Proof of Proposition 2.2. (i): Let $k$ be an even integer with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$. Since the subdiagonal and superdiagonal entries of $[A]$ are nonzero and the diagonal entries of $[A^*]$ are mutually distinct, the $\mathcal{T}$-module $M_k$ is irreducible.

Let $\{v_i\}_{i=0}^{D-2k}$ denote the basis for the $\mathcal{T}$-module $L_k$ from Proposition 3.3. Using (4) yields that

$\{v_{2i}\}_{i=0}^{\lfloor \frac{D}{2} \rfloor - k}$

form a basis for $L_k \cap C^X$. Using Proposition 3.3 along with Lemmas 4.1 and 4.4 a direct calculation yields that the matrices representing $A$ and $A^*$ with respect to the basis (7) for $L_k \cap C^X$ are exactly $[A]$ and $[A^*]$, respectively. The existence of $M_k$ follows.

(ii): Let $k$ be an even integer with $2 \leq k \leq \lfloor \frac{D}{2} \rfloor$. Since the subdiagonal and superdiagonal entries of $\langle A \rangle$ are nonzero and the diagonal entries of $\langle A^* \rangle$ are mutually distinct, the $\mathcal{T}$-module $N_k$ is irreducible.

Let $\{v_i\}_{i=0}^{D-2k+2}$ denote the basis for the $\mathcal{T}$-module $L_{k-1}$ from Proposition 3.3. Using (4) yields that

$\{v_{2i+1}\}_{i=0}^{\lfloor \frac{D}{2} \rfloor - k}$

form a basis for $L_{k-1} \cap C^X$. Using Proposition 3.3 along with Lemmas 4.1 and 4.4 a direct calculation yields that the matrices representing $A$ and $A^*$ with respect to the basis (8) for $L_{k-1} \cap C^X$ are exactly $\langle A \rangle$ and $\langle A^* \rangle$, respectively. The existence of $N_k$ follows. □
Proof of Theorem 2.3. Applying Lemma 5.1(ii) to Theorem 3.4 the $\mathcal{T}$-module $C^X$ is isomorphic to
\[ \bigoplus_{k=0}^{\lfloor \frac{D}{2} \rfloor} \frac{D - 2k + 1}{D - k + 1} \binom{D}{k} \cdot (L_k \cap C^X). \]
From the proof of Proposition 2.2 we see that the $\mathcal{T}$-module $L_k \cap C^X$ is isomorphic to
\[ \begin{cases} M_k & \text{for all even integers } k \text{ with } 0 \leq k \leq \lfloor \frac{D}{2} \rfloor, \\ N_{k+1} & \text{for all odd integers } k \text{ with } 1 \leq k < \lceil \frac{D}{2} \rceil. \end{cases} \]
Note that the $\mathcal{T}$-module $L_k \cap C^X = \{0\}$ if $k = \frac{D}{2}$ is odd. By the above comments the result follows.

Proof of Theorem 2.4. Since the standard $\mathcal{T}$-module $C^X$ contains all irreducible $\mathcal{T}$-modules up to isomorphism, the first assertion is immediate from Theorem 2.3.

The $\mathcal{T}$-modules $M_k$ for all even integers $k$ with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$ are non-isomorphic since their dimensions are all distinct. Similarly the $\mathcal{T}$-modules $N_k$ for all even integers $k$ with $2 \leq k \leq \lceil \frac{D}{2} \rceil$ are non-isomorphic. Now suppose that there are two even integers $k$ and $k'$ with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$ and $2 \leq k' \leq \lceil \frac{D}{2} \rceil$ such that the $\mathcal{T}$-module $M_k$ is isomorphic to $N_{k'}$.

Since $\dim M_k = \frac{D}{2} - k + 1$ and $\dim N_{k'} = \frac{D}{2} - k' + 1$ by Proposition 2.2 it follows that $D$ is even and $k = k'$. Hence $[A^*] = (A^*)$. Since the diagonal entries of $[A^*]$ are mutually distinct, the diagonal entries of $[A]$ are identical to those of $\langle A \rangle$. This leads to $k = \frac{D}{2} + 1$, a contradiction. The second assertion follows.

Proof of Theorem 2.5. Since $\mathcal{T}$ is a finite-dimensional semisimple algebra, it follows from Theorem 2.4 that $\mathcal{T}$ is isomorphic to
\[ \bigoplus_{k=0}^{\lfloor \frac{D}{2} \rfloor} \text{End}(M_k) \oplus \bigoplus_{k=2}^{\lceil \frac{D}{2} \rceil} \text{End}(N_k). \]
By Proposition 2.2 the algebra $\text{End}(M_k)$ is isomorphic to $\text{Mat}_{\lfloor \frac{D}{2} \rfloor - k + 1}(\mathbb{C})$ for all even integers $k$ with $0 \leq k \leq \lfloor \frac{D}{2} \rfloor$ and the algebra $\text{End}(N_k)$ is isomorphic to $\text{Mat}_{\lceil \frac{D}{2} \rceil - k + 1}(\mathbb{C})$ for all even integers $k$ with $2 \leq k \leq \lceil \frac{D}{2} \rceil$. Hence $\dim \mathcal{T}$ is equal to
\[ \sum_{k=0}^{\lfloor \frac{D}{2} \rfloor} \left( \left\lfloor \frac{D}{2} \right\rfloor - k + 1 \right)^2 + \sum_{k=2}^{\lceil \frac{D}{2} \rceil} \left( \left\lceil \frac{D}{2} \right\rceil - k + 1 \right)^2 = \left( \left\lfloor \frac{D}{2} \right\rfloor + 3 \right) + \left( \left\lceil \frac{D}{2} \right\rceil + 1 \right). \]
The result follows.

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