The final state of black strings and p-branes, and the Gregory–Laflamme instability

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Abstract
It is shown that the usual entropy argument for the Gregory–Laflamme (GL) instability for some appropriate black strings and p-branes gives surprising agreement up to a few per cent. This may provide strong support to GL’s horizon fragmentation, which would produce the array of higher-dimensional Schwarzschild-type black holes finally. On the other hand, another estimator for the size of the black-hole end-state relative to the compact dimension indicates a second-order (i.e., smooth) phase transition for some other appropriate compactifications and total dimension of spacetime wherein the entropy argument is not appropriate. In this case, Horowitz–Maeda-type non-uniform black strings or p-branes can be the final state of the GL instability.

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1. Introduction
The four-dimensional Schwarzschild black hole in Einstein gravity is well known to be stable classically under linearized perturbations [1]. Recently, it has been shown that this extends to hold for higher dimensional cases [2]. However, Gregory and Laflamme discovered that the black strings and p-branes of 10D low energy string theory, which have hypercylindrical horizons $\text{Sch}_4 \times \mathcal{V}_p$ instead of compact hyperspherical ones $\text{Sch}_{n+p}$, are found to be unstable as the compactification scale, say $L$, of extended directions becomes larger than the order of the horizon radius $r_+$—the so-called Gregory–Laflamme (GL) instability [3]. In GL’s original work, they explained the instability by arguing that a black string $\text{Sch}_4 \times L$ has a lower entropy than a 5D Schwarzschild black hole $\text{Sch}_5$ with the same total mass when $L > r_+$, in the context of microcanonical ensemble; they also argued that this lends support to the horizon fragmentation, which would eventually produce an array of black holes. However, it is widely

Another explanation, based on $D$–$\bar{D}$ pair annihilations, is also known though it gives only the order of magnitudes [4].
believed that this entropy argument for the classical stability should not be taken seriously since it estimates a wrong onset point of the instability—this means the black string can be classically stable even if its entropy is smaller than that of a 5D Schwarzschild black hole for some regime of $L$—though it provides some plausibility argument [5–8]. Moreover, the GL fragmentation scenario was disproved under very weak assumptions, including the classical black-hole area theorem, by Horowitz and Maeda (HM) and a non-uniform black string as the final state of the GL instability [9] is considered accordingly.

In this paper, I will show that this widespread belief is not quite correct. If one properly applies the entropy argument to the black string solution $Sch_n \times L$ of 10D low energy string theory, one can estimate the onset point of the GL instability up to 2.4% discrepancy. For $p$-brane solutions, the thing depends on the geometry of the compactification of $p$-branes. I consider two typical methods of compactification: thin-torus compactification and $p$-dimensional isotropic-torus compactification. For the former case, the discrepancy grows as $p$ grows ($n$ decreases) up to 35% discrepancy for $p = 6$ ($n = 4$). But, for the latter case, the discrepancy is quite reduced up to 0.5–2.4%. This may provide strong support for the GL horizon fragmentation, which would finally produce the array of higher-dimensional Schwarzschild-type black holes. On the other hand, another estimator for the size of the black-hole end-state relative to the compact dimension indicates a second-order (i.e., smooth) phase transition for some other appropriate compactifications and total dimension of spacetime wherein the entropy argument is not appropriate: for the black strings $Sch_n \times L$, this occurs for $n$ as large as $n > 12$ and for the (isotropic) black $p$-branes $Sch_n \times L^p$, this occurs for $n$ as low as $n < 6$ with a fixed total dimension of spacetime $d = 10$. In this case, HM-type non-uniform black strings or $p$-branes can be the final state of the GL instability instead.

2. The black string instability

The black string and $p$-branes I am specifically interested in are those introduced by Horowitz and Strominger [10] in 10D low energy string theory with a metric given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\Omega_{n-2}^2 + dx^i dx_i,$$

where

$$N^2 = 1 - \frac{16\pi G M_{(n)}}{(n - 2)\Omega_{n-2} r^{n-3}},$$

$n = 4, \ldots, 10$, and index $i$ runs from 1 to $p = 10 - n$. $M_{(n)}$ is the mass of the $n$-dimensional black holes, and $\Omega_{n-2}$ is the area of the unit sphere $S^{n-2}$ [11]. This is always a solution of the Einstein equation in $d = n + p = 10$ dimensions for compact as well as non-compact string or brane directions if the string or brane directions are completely factorized. But, this particular solution does not exist for 0-brane (i.e., 10D black hole) and we must consider deformations of the ordinary Schwarzschild solution due to non-compact dimensions in general [8, 12, 13]. But let me approximate the 10D black hole by the ordinary $Sch_{10}$ metric, with 10D radial coordinate $R$.

In order to compute the transition point between the black strings or branes and the 10D black hole of the same mass due to the entropy difference, in the context of a microcanonical ensemble, we need to know details of compactified dimensions. In this section let me first consider the simplest one, a black string, and consider simply a $S^1$-compactification. To this end, let me note that the masses and entropies of the black string and the 10D black hole, with
the horizon radii $r_+ \text{ and } R_+$, are, respectively

\begin{align*}
M_{b.s.} &= \frac{7\pi^3 r_+ L}{48G}, \quad S_{b.s.} = \frac{\pi^4 r_+^2 L}{12G}, \\
M_{b.h.(10)} &= \frac{16\pi^3 R_+^7}{105G}, \quad S_{b.h.(10)} = \frac{8\pi^4 R_+^8}{105G}.
\end{align*}

Now for the same mass of the black string and the 10D black hole, the condition of an unstable black string due to having smaller entropy than a 10D black hole is

\[ L \geq \left( \frac{8}{7} \right) \left( \frac{\Omega_S}{\Omega_{10}} \right)^{\frac{1}{2}} r_+ \approx 2.661 r_+. \]  

(3)

Note also that

\[ L \geq \left( \frac{8}{7} \right) \left( \frac{\Omega_S}{\Omega_{10}} \right)^{\frac{1}{2}} R_+ \approx 2.328 R_+, \]  

(4)

such that the 10D black hole can easily fit in the compact dimension $S^1$. In terms of the wave number $k$ for the unstable perturbation [14], (3) can be re-expressed as

\[ k \leq k_S, \quad k_S = \frac{2\pi}{L_S} \approx 2.361 r_+^{-1}, \]  

(5)

where $L_S$ is the entropy estimator—‘equal entropy for equal mass’ estimator—of the minimum length of compact dimension for the GL instability. This agrees with GL’s numerical analysis for the classical instability under linearized perturbations $k \leq k_{GL}$, $k_{GL} \approx 2.306 r_+^{-1}$ up to 2.4% discrepancy\(^2\). This good agreement is rather surprising since thermodynamic instability based on global entropy arguments, which have quantum origins, does not generally imply a classical instability.

3. The black $p$-brane instability I: thin-torus compactification

The generalization of the string instability of the previous section to arbitrary $p$-branes ($2 \leq p \leq 6$) in 10D low energy string theory requires knowledge of the compactification. In this section, I first consider a thin-torus compactification with horizons $S_{ch} \times L \times V_{p-1}$, which has one compact dimension $S^1$ with length $L$ and a very tiny volume $V_{p-1} \ll L^{p-1}$ for other compact dimensions. Since the effect of small compact dimensions would be tiny, I would approximate this system by the black strings $S_{ch} \times L$ in $(n+1)$ dimensions effectively, such that the transition problems between $p$-branes and 10D black holes are reduced to those of black strings $S_{ch} \times L$ and $(n+1)$D black holes. To this end, similarly to the previous section, let me approximate the $(n+1)$D black holes by the ordinary $S_{ch,n+1}$ metric, with $(n+1)$D radial coordinate $\hat{R}$. Then, the masses and entropies of the black string $S_{ch} \times L$ and the $(n+1)$D black hole are, respectively,

\begin{align*}
M_{b.s.(n)} &= \frac{(n-2)\Omega_{n-2} r_+^{n-3} L}{16\pi G}, \quad S_{b.s.(n)} = \frac{\Omega_{n-2} r_+^{n-2} L}{4G}, \\
M_{b.h.(n+1)} &= \frac{(n-1)\Omega_{n-1} R_+^{n-2} L}{16\pi G}, \quad S_{b.h.(n+1)} = \frac{\Omega_{n-1} R_+^{n-1} L}{4G}.
\end{align*}

\(^2\) 2$\mu$ in GL’s analysis is what I have called $k$ [14]. I use values of $\mu$ recently obtained by Hirayama et al [15], which is more accurate than the original GL’s analysis; I thank G Kang for informing me about this updated data. Similar data has also been obtained by Sorkin [16, 17] in a different context of Gubser [14] and Wiseman [18]; I thank E Sorkin for kindly sending his data.
Now, for the same mass of the black string and the \((n+1)\)D black hole, the condition of an unstable black string due to smaller entropy than the \((n+1)\)D black hole is

\[
L \geq \left( \frac{n-1}{n-2} \right)^{\frac{n-1}{2}} \frac{\Omega_{n-1}}{\Omega_{n-2}} r_+ \equiv 2\pi k_S^{-1},
\]

where \(k_S\) is the entropy estimator of the maximum wave number for an unstable perturbation. Note also that

\[
L \geq \left( \frac{n-1}{n-2} \right)^{-\frac{n-2}{2}} \frac{\Omega_{n-1}}{\Omega_{n-2}} R_+ \equiv 2f R_+,
\]

where \(f = (\frac{n-1}{n-2})^{\frac{n-2}{2}} \Omega_{n-1}/2\Omega_{n-2}\) denotes how a \((n+1)\)D black hole can fit in the compact direction \(S^1\), such that \(f \geq 1\) is required for a safe fitting. The values computed for \(k_S\) and \(f\) are listed and compared with GL’s data \(k_{GL}\) in table 1, figure 1 and table 2, figure 2, respectively. Table 1 and figure 1 show that the discrepancy grows as \(p\) grows (\(n\) decreases) up to 35\% discrepancy for \(p = 6\) (\(n = 4\)). But, in the light of the entropy argument the discrepancy would only imply that the crude approximation that I have taken for the Sch\((n+1)\) metric as the \((n+1)\)D black-hole solution even with one compactified dimension \(S^1\), and/or the thin-torus limit of the compactification, which treats one specific direction differently from others, becomes bad as the dimension of the compactification \(p\) increases. So, this indicates that a better approximation which treats all the compact directions equally is needed. This will be done in the next section. But, before this, let me note the following.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(n\) & GL’s data & Isotropic torus & Thin torus \\
\hline
4 & 0.880 & 0.857 (−3\%) & 1.185 (+35\%) \\
5 & 1.27 & 1.206 (−5\%) & 1.491 (+17\%) \\
6 & 1.58 & 1.524 (−4\%) & 1.748 (+11\%) \\
7 & 1.85 & 1.820 (−2\%) & 1.973 (+7\%) \\
8 & 2.088 & 2.098 (+0.5\%) & 2.176 (+4.2\%) \\
9 & 2.306 & 2.361 (+2.4\%) & 2.361 (+2.4\%) \\
\hline
\end{tabular}
\caption{Table of the entropy estimator \(k_S\) for isotropic-torus and thin-torus compactifications in comparison with GL’s data \(k_{GL}\). The values in the brackets denote their discrepancies with GL’s data \((r_+ \equiv 1)\); the + or − sign represents whether it is bigger (+) or smaller (−) than GL’s.}
\end{table}
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Figure 2. Plot of $f$ as a function of $n$ for black string ($n = 9$) and $p$-branes ($n = 10 - p$). The thin and the thick lines represent the values calculated for thin-torus and isotropic-torus compactifications, respectively. The crossing occurs at about $n = 9$.

Table 2. Comparison of $f$ for isotropic torus and thin torus. The values marked * are the two nearest dimensions to the critical dimension $n_c$.

| $n$ | Isotropic torus | $n$ | Thin torus |
|-----|----------------|-----|------------|
| 4   | 0.916          | 8   | 1.238      |
| 5   | 0.977*         | 9   | 1.164      |
| 6   | 1.031*         | 10  | 1.102      |
| 7   | 1.079          | 11  | 1.049      |
| 8   | 1.123          | 12  | 1.003*     |
| 9   | 1.164          | 13  | 0.962*     |

First, the widespread belief [5–8] that the entropy argument for the classical instability should not be taken seriously originated from the big discrepancy of 35% with GL’s numerical analysis for the $n = 4$ case, which is found to be the worst case in the thin-torus, i.e., string, approximation of the $p$-branes of the 10D low energy string theory. But my analysis shows that this is not quite correct since the $n = 4$ case is not truly a black string but a 6-brane exactly, which would deform the black string picture quite a lot; one should have considered the $n = 9$ case to discuss the black string and compare with GL’s data.

Second, note that results (6) and (7) can be applied for any dimension $n$ to analyse the transition from a black string $S_{ch} \times L$ to a $(n + 1)$D black hole $S_{ch_{n+1}}$, though I have introduced this set-up to approximate the $p$-brane solutions in 10D string theory. Then, it is interesting to observe that there is a critical dimension $n_c = 12$ above which $f < 1$ such that the $(n + 1)$D black hole cannot fit in the compact direction $S^1$; in this case, approximating the ordinary $S_{ch_{n+1}}$ as the final state solution needs some important correction due to the compact dimension [13], such as the black string can evolve into a different final state, presumably a non-uniform black string, between the (uniform) black string and the black hole. This indicates that the order of the phase transition between the uniform and the non-uniform black strings changes from first (i.e., sudden transition) to second order (i.e., smooth transition) at the critical dimension $n_c$. Recently, another estimator for the critical dimension has been considered by Sorkin [16], but one finds a very good agreement between these two estimators.

3 This is sharply in contrast to the equations for the linear perturbations [3], which depend only on the sum of the Kaluza–Klein mass squared, i.e., $\mu^2 = \sum_{i=1}^{p} \mu_i^2$, and are blind to the dimensionality $p$ of the brane’s world volumes as long as $\mu^2 \neq 0$. 
4. The black $p$-brane instability II: isotropic $p$-dimensional torus compactification

As a correction to the thin-torus compactification of the previous section, I will consider an isotropic $p$-dimensional torus compactification $S_{\text{ch}} \times V_p$ where all compactified directions are treated equally. To this end, let me approximate a 10D black hole by the ordinary $S_{\text{ch}10}$ metric, with 10D black radial coordinate $R$, similarly to section 2. Then, the masses and entropies of a black $p$-brane and a 10D black hole are, respectively,

$$M_{\text{b.b.}}(n) = \left(\frac{n-2}{\Omega_{n-2}}\right)^{\frac{1}{n-2}} r_+^{n-3} V_p \Omega_{n-2}, \quad S_{\text{b.b.}}(n) = \frac{\Omega_{n-2} r_+^{n-2} V_p}{4G},$$

$$M_{\text{b.h.}}(10) = \frac{16\pi^3 R_+^7}{105G}, \quad S_{\text{b.h.}}(10) = \frac{8\pi^4 R_+^8}{105G}.$$

Now, for the same mass of the black $p$-branes and the 10D black hole, the condition of an unstable black $p$-brane due to smaller entropy than the 10D black hole is

$$V_p \geq \left(\frac{8}{n-2}\right)^{\frac{8}{n-2}} \left(\frac{\Omega_8}{\Omega_{n-2}}\right) r_+^{10-n} \Omega_{n-2}, \quad (8)$$

while

$$V_p \geq \left(\frac{8}{n-2}\right)^{\frac{n-2}{n-2}} \left(\frac{\Omega_8}{\Omega_{n-2}}\right) R_+^{10-n} \Omega_{n-2}. \quad (9)$$

Moreover, since I am considering a $p$-dimensional torus with equal length $L = (V_p)^{1/n}$, (8) and (9) can be re-expressed in terms of $L$ and the associated entropy estimator of the maximum wave number $k_S$ for the unstable perturbation, as

$$L \geq \left(\frac{8}{n-2}\right)^{\frac{4}{n-2}} \left(\frac{\Omega_8}{\Omega_{n-2}}\right) r_+ \equiv 2\pi k_S^{-1}, \quad (10)$$

and

$$L \geq \left(\frac{8}{n-2}\right)^{\frac{n-2}{n-2}} \left(\frac{\Omega_8}{\Omega_{n-2}}\right) R_+ \equiv 2f R_+. \quad (11)$$

The values computed for $k_S$ and $f$ are listed and plotted also in table 1, figure 1 and table 2, figure 2, respectively, in comparison with GL’s data $k_{\text{GL}}$ and the results for the thin-torus compactification. Table 1 and figure 1 show that the discrepancy of the thin-torus approximation has been quite reduced in the isotropic torus and the worst case is about 4–5% for $p = 4, 5$ ($n = 6, 5$); all the other cases have been less than about 2–3% and the best case is 0.5% for $p = 2$ ($n = 8$); moreover, compared to the growing discrepancy for the thin torus as $p$ grows ($n$ decreases), that of the isotropic torus is almost stable so that this improved approximation is fairly good.

On the other hand, according to the result for $f$ in table 2 and figure 2, there is a critical dimension $n_c = 6$ below which $f < 1$ such that the 10D black hole cannot fit in the compact dimension $S^1$. This implies that approximating the ordinary $S_{\text{ch}10}$ as the final state of the black $p$-branes needs some important corrections due to the compactified dimension such that the uniform black $p$-brane can evolve into a different final state, presumably a non-uniform black $p$-brane. So, the relatively big discrepancies for $n = 5, 6$ would not be so surprising in light of the entropy argument; but it is a remarkable fact that $n = 4$ case has a relatively good agreement with GL’s data with 3% discrepancy even though it does not have to. Hence, by taking into account this additional fact to the result of $k_S$ in table 1 and figure 1, the true discrepancy in this approximation would be quite small and the reliable results would have a discrepancy of only about 0.5%–2.4% by excluding the $n = 4, 5, 6$ cases.
Furthermore, this also indicates a smooth decay of an unstable (uniform) black p-brane $S_n \times L^p$ to the non-uniform state for $n$ as low as 4 or 5. This is in contrast to the decay of a black string, where $n$ as large as $n > 12$ is required for a smooth decay. More recently another estimator for the critical dimension, following Sorkin [16], has also been considered by Kol and Sorkin [19] and they found a very good agreement with mine again; moreover, they found interestingly that $d = 10$ is the smallest total dimension of the spacetime to allow a smooth decay of an unstable black brane to a non-uniform state.

5. Discussion

I have shown that the usual entropy argument for the GL instability for some appropriate black strings and p-branes gives surprising agreement up to a few per cent. This may provide strong support to GL’s horizon fragmentation, which would produce finally an array of—single in my analysis—higher-dimensional Schwarzschild-type black holes; this result is remarkable in that the end point of the unstable evolution, which is by its nature ‘nonlinear’, crucially affects the onset of the instability calculation, which is by its nature ‘linear’.

On the other hand, another estimator for the size of the black-hole end-state relative to the compact dimension indicates a second-order (i.e., smooth) phase transition for some other appropriate compactifications and total dimension of spacetime wherein the entropy argument is not appropriate. For the black strings $S_n \times L$, this occurs for $n$ as large as $n > 12$ and for the (isotropic) black p-branes $S_n \times L^p$, this occurs for $n$ as low as $n < 6$ with a fixed total dimension of spacetime $d = 10$. In this case, HM-type non-uniform black strings or p-branes can be a natural final state of the GL instability. This result agrees quite well with the analysis of Kol and Sorkin wherein a different estimator has been considered [16, 19].

Note added. After the first appearance of this paper, I was informed by E Sorkin that my analysis of the instability for thin-torus compactification is very similar to that of [16] which uses a single dimensionless parameter $\tilde{\mu} \equiv GM/L^{n-2}$ instead of $r_+R_+L$. Afterward, I checked that his result (9) on the critical values of $\tilde{\mu}$ for the onset of an instability agrees exactly with my result (6). And his analysis [16, 17] on the black string perturbation shows quite good agreement with the thin-torus set-up up to 0.2–1% discrepancy for $10 \leq n \leq 12$, in contrast to $n \leq 9$, where the isotropic-torus set-up is more favourable (figure 3); this indicates another critical dimension at $n = 9$ which agrees with Kol’s ‘merger
point’, where the string and black-hole branches merge [20]. The increasing discrepancy above $n = 12$ is not so surprising since this is the regime where the naive thin-torus set-up does not have to be correct, due to $f < 1$, such that the black string can evolve into a different final state, presumably a non-uniform black string, as in $n < 6$ cases of isotropic-torus set-up in section 4.

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