PREPROJECTIVE ALGEBRAS OF \( d \)-REPRESENTATION FINITE SPECIES WITH RELATIONS

CHRISTOFFER SÖDERBERG

Abstract. In this article we study the properties of preprojective algebras of representation finite species. To understand the structure of a preprojective algebra, one often studies its Nakayama automorphism. A complete description of the Nakayama automorphism is given by Brenner, Butler and King when the algebra is given by a path algebra. We generalize this result to the species case.

We show that the preprojective algebra of a representation finite species is an almost Koszul algebra. With this we know that almost Koszul complexes exist. It turns out that the almost Koszul complex for a representation finite species is given by a mapping cone of a chain map, which is homogeneous of degree 1 with respect to a certain grading. We also study a higher dimensional analogue of representation finite hereditary algebras called \( d \)-representation finite algebras. One source of \( d \)-representation finite algebras comes from taking tensor products. By introducing a functor called the Segre product, we manage to give a complete description of the almost Koszul complex of the preprojective algebra of a tensor product of two species with relations with certain properties, in terms of the knowledge of the given species with relations. This allows us to compute the almost Koszul complex explicitly for certain species with relations more easily.

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1. Introduction

In the paper [Gab73] Gabriel gathered results regarding the problem to determine which algebras were of finite representation type. One of these results, known as Gabriel’s Theorem [Gab72], says that a path algebra over an algebraically closed field is representation finite if and only if its underlying diagram is a disjoint union of Dynkin diagrams of type ADE. Gabriel also sketched the proof in [Gab73] of how one could extend one direction of Gabriel’s Theorem to \( \mathcal{K} \)-species when \( \mathcal{K} \) is a perfect field, i.e. a \( \mathcal{K} \)-species is representation finite if its diagram is a disjoint union of Dynkin diagrams of type ADE. The general case was later solved completely by Dlab and Ringel in [DR75]. They proved that a species is representation finite if and only if its diagram is a finite disjoint union of Dynkin diagrams of type ABCDEFG. Thus by Dlab and Ringel we have a complete set of representation finite species. Given a hereditary finite dimensional \( \mathcal{K} \)-algebra \( \Lambda \) we can consider its
preprojective algebra $\Pi(\Lambda)$. It is a graded algebra with degree 0 equal to $\Lambda$ and as a $\Lambda$-module it gives an additive generator of the category of preprojective $\Lambda$-modules. Moreover, $\Lambda$ is representation finite if and only if $\Pi(\Lambda)$ is a finite dimensional self-injective algebra. In this paper we study the properties of preprojective algebras for representation finite species, such as the Nakayama permutation, Nakayama automorphism and Koszul properties of the preprojective algebra.

The study of the preprojective algebra of a quiver shows up in all kind of topics in mathematics such as cluster algebras [GLS13], quiver varieties [Nak94], quantum groups [Lus91] [KS97], and many more topics. This is a motivation to study the Koszul properties of the preprojective algebra $\Pi(\Lambda)$ of some representation finite species $S$. In the case when $S$ is not representation finite, the Koszul properties of $\Pi(S)$ are studied in [AHI+22]. When $S$ is representation finite, $\Pi(S)$ is a Frobenius algebra, and so we have that $D\Pi(S) \cong \Pi(S)$ as left $\Pi(S)$-modules. This isomorphism yields an isomorphism of $\Pi(S)$-$\Pi(S)$-bimodules by introducing a twist on $\Pi(S)$ denoted by $\gamma$ and it is called the Nakayama automorphism. The paper [BBK02] gave a complete description of the Nakayama automorphism $\gamma$ of the preprojective algebra of a path algebra of a Dynkin quiver. Later Grant [Gra19] gave a partial description of the Nakayama automorphism of the preprojective algebra of any hereditary $K$-algebra using the Nakayama permutation $\sigma$ and the Auslander-Reiten translation. In this paper we compute $\sigma$ for all Dynkin diagrams in Theorem 3.1 so we can use Grant’s description to give an explicit description of $\gamma$. This is formulated in the following theorem.

**Theorem.** (Theorem 5.3) Let $S$ be a species, with Dynkin diagram $\Delta$, over division algebras $F \subset G$. The Nakayama automorphism $\gamma$ of $\Pi(S)$ is given by

$$
\gamma(y^i_\alpha) = \begin{cases} 
g^i_{\sigma(\alpha)} , & \text{if } \alpha \in Q_1 \\
\text{sgn}(\sigma(\alpha))g^i_{\sigma(\alpha)} , & \text{if } \alpha \notin Q_1. 
\end{cases}
$$

The proof of the classification of representation finite species by Dlab and Ringel shows that every representation finite species is isomorphic to a species, whose underlying diagram is a Dynkin diagram, over two division algebras $F \subset G$, therefore this theorem applies to all representation finite species. This sums up the first part of this paper.

In the second part of the paper we change focus from studying tensor algebras of species to algebras given by species with relations. More specifically, we study so called $d$-representation finite algebras. We say that a finite dimensional $K$-algebra $\Lambda$ with $\operatorname{gldim}\Lambda \leq d$ is a $d$-representation finite algebra if there exists a $d$-cluster tilting $\Lambda$-module. A hereditary representation finite algebra is 1-representation finite. For a $d$-representation finite algebra $\Lambda$, there exists a positive integer $l_P$ for each indecomposable projective $\Lambda$-module $P$ such that $P_\Lambda \cong (P_\Lambda)_{l_P}$ is an indecomposable injective $\Lambda$-module. Here $\tau_\Lambda$ is the higher dimensional analogue of the Auslander-Reiten translation. When $l_P = l$ for all indecomposable projective $\Lambda$-modules $P$ we say that $\Lambda$ is $l$-homogeneous. If $\Lambda_1$ is a $d_1$-representation finite $l$-homogeneous algebra for each $i \in \{1, 2\}$ and $K$ is a perfect field, then $\Lambda_1 \otimes_K \Lambda_2$ is $l$-homogeneous and $(d_1 + d_2)$-representation finite [HII11 Corollary 1.5]. For the case when $d_1 = d_2$, McMahon and Williams showed that the $(2d + 1)$-preprojective algebra of $\Lambda_1 \otimes_K \Lambda_2$ has a $d$-cluster tilting module [MW21]. In general, $\Lambda_1 \otimes_K \Lambda_2$ is not $(d_1 + d_2)$-representation finite if we drop the assumption that $\Lambda_1$ and $\Lambda_2$ are $l$-homogeneous. Relaxing the assumptions on $\Lambda_1$ and $\Lambda_2$ Pasquale proved that if $\Lambda_1$ is an acyclic $d_1$-complete algebra for each $i \in \{1, 2\}$ then $\Lambda_1 \otimes_K \Lambda_2$ is $(d_1 + d_2)$-complete [Pas19]. If $\Lambda_1$ is a $d_1$-representation infinite algebra such that $\Lambda_1 / J_i$ is semi-simple for each $i \in \{1, 2\}$, then $\Lambda_1 \otimes_K \Lambda_2$ is a $(d_1 + d_2)$-representation infinite algebra [HIO14 Theorem 2.10]. The preprojective algebra of such algebras were studied in [TB20]. In this paper we will further study the former case, that $\Lambda_1$ is a $d_1$-representation finite $l$-homogeneous algebra, but we also assume that $\Lambda_1$ is a Koszul algebra for each $i \in \{1, 2\}$ and focus on the Koszul properties of the preprojective algebra of $\Lambda_1 \otimes_K \Lambda_2$.

Let $\Pi(\Lambda)$ be the $(d + 1)$-preprojective algebra introduced by Iyama and Oppermann in [IO13]. Iyama and Oppermann proved that $\Pi(\Lambda)$ is self-injective if $\Lambda$ is $d$-representation finite. The $(d + 1)$-preprojective algebra $\Pi(\Lambda)$ was further studied by Grant and Iyama in [GI20] and they showed that when $\Lambda$ is a $d$-representation finite Koszul algebra, the $(d + 1)$-preprojective algebra is an almost Koszul algebra in the sense of [BBK02].
In this setting the homologies of the almost Koszul complex for \( \Pi(\Lambda) \) can be described using the Nakayama automorphism. We investigate various properties for \( \Pi(\Lambda_1) \) and \( \Pi(\Lambda_2) \) that are related to the same properties for \( \Pi(\Lambda_1 \otimes \Lambda_2) \). For example, we investigate how one can describe the almost Koszul complexes of \( \Pi(\Lambda_1 \otimes \Lambda_2) \) using the knowledge of the almost Koszul complexes of \( \Pi(\Lambda_1) \) and \( \Pi(\Lambda_2) \). This is formulated in the following theorem.

**Theorem.** (Theorem [10.9]) Let \( \kappa \) be a perfect field. Let \( \Lambda_i \) be an acyclic \( d_i \)-representation finite \( l \)-homogeneous Koszul algebra for each \( i \in \{1, 2\} \). Let \( S^{\Lambda_i} \) be a simple \( \Pi(\Lambda_i) \)-module and let \( \varphi^{\Lambda_i} : Q^{\Lambda_i}_0 \to R^{\Lambda_i}_0 \) be an almost quasi-isomorphism as in Theorem [10.7] such that \( C(\varphi^{\Lambda_1}) \) is the almost Koszul complex for \( S^{\Lambda_1} \). The complex

\[
C(\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2}) : \text{Tot}(Q^{\Lambda_1}_0 \otimes_S Q^{\Lambda_2}_0) \to \text{Tot}(R^{\Lambda_1}_0 \otimes_S R^{\Lambda_2}_0))
\]

is the almost Koszul complex for \( S^{\Lambda_1} \otimes_S S^{\Lambda_2} \in \Pi(\Lambda_1 \otimes \kappa \Lambda_2) - \text{mod} \).

Here, the \( C(\varphi) \) denotes the mapping cone of \( \varphi \) and \( \otimes_S \) is the Segre product defined in Section [9]. We define almost quasi-isomorphisms in Section [10] and we define acyclic algebras in Section [2]. This theorem allows us to have total control of the almost Koszul complexes in \( \Pi(\Lambda_1 \otimes \kappa \Lambda_2) \), and thus one of the corollaries is the following.

**Corollary.** (Corollary [10.10]) Let \( \kappa \) be a perfect field. Let \( \Lambda_i \) be an acyclic \( d_i \)-representation finite \( l \)-homogeneous Koszul algebra for each \( i \in \{1, 2\} \). If \( \Pi(\Lambda_i) \) is an \((p_i, q_i)\)-almost Koszul algebra, then \( \Pi(\Lambda_1 \otimes \kappa \Lambda_2) \) is an \((p_1 + p_2 - 1 + 1, q_1 + q_2 - 1)\)-almost Koszul algebra.

Note that given the assumptions in the above corollary \( \Lambda_1 \otimes \kappa \Lambda_2 \) is a \((d_1 + d_2)\)-representation finite \( l \)-homogeneous Koszul algebra and therefore we can apply the corollary iteratively. Moreover, for a species \( S \) over a Dynkin diagram \( \Delta \) with Coxeter number \( h \), the preprojective algebra \( \Pi(S) \) is a \((h - 2, 2)\)-almost Koszul algebra (Corollary [6.13]).

The article has the following structure. In Section [2] we introduce preliminary concepts and notations. In Section [3] we show that the Nakayama permutation only depends on the underlying diagram and we compute the Nakayama permutation for all Dynkin diagrams of type ABCDEFG. In Section [4] we compare different descriptions of the preprojective algebra. In Section [5] we give a description of the Nakayama automorphism for species of Dynkin type BCFG, and we extend the results from [BBK02] to species of Dynkin type ADE. In Section [6] we extend results from [BBK12]. In [BBK12] it is shown that the preprojective algebra of a path algebra is almost Koszul if it is of Dynkin type ADE and otherwise it is Koszul. We partially extend the former result to the species case, i.e. the preprojective algebra of a species is almost Koszul if the diagram is a Dynkin diagram of type ABCDEFG. In Section [7] we introduce \( d \)-representation finite algebras together with the \((d + 1)\)-preprojective algebra, also known as the higher preprojective algebra. In Section [5] we investigate various properties of the tensor product of algebras relevant to the paper. In Section [9] we define the Segre product of graded algebras and modules and state some basic properties and the Künneth formula for the Segre product. In Section [10] we investigate the structure of the almost Koszul complexes. In Section [11] we finish the paper by computing some examples to illustrate how every main theorem comes together to yield concrete results.

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## 2. Preliminaries

Let \( \kappa \) be a field. We first define species as it is given in [Ber11, Rin76, Gab73].

**Definition 2.1.** (Species) Let \( Q \) be a finite quiver. A species \( S = (D_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1} \) is a collection of division rings \( D_i \) and \( M_\alpha \in D_i - D_j \)-mod, where \( \alpha : i \to j \), such that \( \text{Hom}_{D_j}(M_\alpha, D_j) \cong \text{Hom}_{D_j}(M_\alpha, D_j) \) as \( D_j - D_i \)-modules. We say that a species \( S \) is a \( \kappa \)-species if all \( D_i \) are finite dimensional over a common central subfield \( \kappa \) and all \( M_\alpha \) are finite dimensional over \( \kappa \) satisfying \( \lambda m = m \lambda \) for all \( m \in M_\alpha, \lambda \in \kappa \).
Definition 2.2. For a species $S$ let $D = \bigoplus_{s \in Q_0} D_s$ and $M = \bigoplus_{\alpha \in Q_1} M_{\alpha} \in D$-$D$-mod. We define the tensor algebra $T(S)$ to be the tensor ring $T(D, M)$. More explicitly,
\[ T(S) = T(D, M) = D \oplus \bigoplus_{k \geq 1} M^\otimes D^k. \]

Remark 2.3. In some contexts it is useful to allow for a more general definition of a species by requiring that $D$ has to be Morita equivalent to a sum of division rings, and not necessarily a division ring. In Section 11 we will briefly discuss the fact that tensor products of tensor algebras of species is not necessarily a tensor algebra over a species by Definition 2.1, but it will be a species according to this more general definition. Another generalization of species called Pro-species is studied in [Kui17, LY15].

In this paper we assume that all our species are $\mathbb{K}$-species. Since $T(S)$ is a tensor algebra, $T(S)$ has a natural $\mathbb{Z}$-grading which we will refer to as the path length grading. Also note that if $S$ is a $\mathbb{K}$-species then $T(S)$ is a $\mathbb{K}$-algebra. One could relax the assumption on the species from being a $\mathbb{K}$-species to be polynomial identity species, i.e. when the division algebras $D_s$ are polynomial identity algebras. In this setting, it does not have to exist a common subfield $\mathbb{K}$ such that all $D_s$ are finitely generated over $\mathbb{K}$. Hence we cannot use [DR75] Theorem B. Therefore, for convenience, we assume that all species are $\mathbb{K}$-species.

We denote $e_i$ the identity element in $D_i$. The set $\{e_1, \ldots, e_{|Q_0|}\}$ is a complete set of pairwise orthogonal primitive idempotents for $T(S)$.

If $Q$ is a quiver with multiple arrows $\alpha, \alpha' : i \to j$ we consider a modified quiver $Q'$ which is obtained by replacing $\alpha$ and $\alpha'$ with $\beta : i \to j$. Then if $S$ is a species over $Q$, we can modify $S$ to a species $S'$ over $Q'$ by setting $M_\beta = M_\alpha \oplus M_{\alpha'}$. We see that
\[ T(S) = T(D, M) = T(D, M') = T(S'). \]
Therefore we assume that $Q$ has no multiple arrows.

A simple example of a species would be $S = (D_i, M_\alpha)_{\alpha \in Q_0, \alpha \in Q_1}$ where $Q : 1 \to 2 \to 3$ and
\[
D_i = \begin{cases} 
\mathbb{C}, & \text{if } i = 1 \\
\mathbb{R}, & \text{if } i = 2, 3
\end{cases} \\
M_\alpha = \begin{cases} 
\mathbb{C}, & \text{if } \alpha : 1 \to 2 \\
\mathbb{R}, & \text{if } \alpha : 2 \to 3
\end{cases}.
\]

We often write $S$ as a decorated quivers given by decorated subquivers of the form
\[ D_{s(\alpha)} \xrightarrow{M_{s(\alpha)}} D_{t(\alpha)} \]
for all $\alpha \in Q_1$. So in our example above we would write
\[ \mathbb{C} \xrightarrow{\mathbb{C}} \mathbb{R} \xrightarrow{\mathbb{R}} \mathbb{R}. \]

When $Q$ is a finite and acyclic quiver and $S$ a species over $Q$, the tensor algebra $(S)$ is a basic finite dimensional $\mathbb{K}$-algebra. Moreover, since rad($T(S)$) $\cong \bigoplus_{s(\alpha) = 1} \dim_{D_{s(\alpha)}} (M_{s(\alpha)})^\gamma$, and thus projective, we can apply [Lam08] Theorem 2.35] to see that $T(S)$ is indeed hereditary.

Definition 2.4. Let $S$ be a species over an acyclic quiver $Q$. Then the diagram $\Delta$ of $S$ is defined to have its vertices as $Q_0$ and for every $\alpha \in Q_1$ we have an edge with valuation
\[ (\dim_{D_{s(\alpha)}} (M_{s(\alpha)}), \dim_{D_{t(\alpha)}} (M_{t(\alpha)})). \]
If there is an edge with valuation $(1, k)$ then we write $s(\alpha) \xrightarrow{(k)} t(\alpha)$, if we have the valuation $(k, 1)$ then we write $t(\alpha) \xrightarrow{(k)} s(\alpha)$ and if we have the valuation $(1, 1)$ we simply write the edge without valuation, i.e.
\[ s(\alpha) \xrightarrow{} t(\alpha). \]
Definition 2.5. A representation $V = (V_i, \phi_i)$ over $S$ is a collection of $D_i$-modules $V_i$ and $D_{t(\alpha)}$-module morphisms  
$$\phi_i : M_\alpha \otimes_{V_i(\alpha)} V_{s(\alpha)} \to V_{t(\alpha)}.$$ 
A morphism $f : V \to V'$ between two representations $V$ and $V'$ over a species $S$ is given by a family of morphisms $f_i : V_i \to V'_i$ such that 
$$M_\alpha \otimes_{V_i(\alpha)} V_{s(\alpha)} \xrightarrow{1 \otimes f_i(\alpha)} M_\alpha \otimes_{V_i(\alpha)} V'_{s(\alpha)} \xrightarrow{\phi'_i} V'_{t(\alpha)}$$ 
commutes for all $\alpha \in Q_1$. The set of representations over $S$ together with the morphism forms an abelian category and is denoted by $\mathcal{R}ep S$. The full subcategory of $\mathcal{R}ep S$ consisting of representations $V$ where $\dim K V_i < \infty$ for all $i \in Q_0$ is denoted by $\mathcal{R}ep S$. 

In this paper we will focus on the module category $T(S)\text{-mod}$, which is the category consisting of finitely generated $T(S)$-modules, and usually when we relate representations over $S$ and modules over $T(S)$ we use the following result.

Proposition 2.6. [DR75 Proposition 10.1] Let $S$ be a species. The category $\mathcal{R}ep S$ is equivalent to the category $T(S)\text{-Mod}$.

Here $T(S)\text{-Mod}$ is the category consisting of all $T(S)$-modules. When we restrict ourselves to a finite quiver we have the following.

Corollary 2.7. [Ber11 Corollary 2.2] Let $S$ be a species over $Q$, where $Q$ is a finite quiver. The category $\mathfrak{r}ep S$ is equivalent to the category $T(S)\text{-mod}$.

We say that a species $S$ is representation finite if there is only a finite number of finitely generated indecomposable $T(S)$-modules up to isomorphism. Gabriel characterized all representation finite species $S$ of the form $D_i = M_\alpha = \mathbb{F}$ for some field $\mathbb{F}$ in [Gab80]. Gabriel’s result was extended by Dlab and Ringel in the following theorem.

Theorem 2.8. [DR75 Theorem B] A species $S$ is representation finite if and only if $\Delta$ is a finite disjoint union of Dynkin diagrams.

The Dynkin diagrams are given in Figure 1.

Given a species $S$ where $\Delta$ is a Dynkin diagram. Then reading from the proof of [DR75 Theorem B] we can assume that $S$ is of the form where $D_i, M_\alpha \in \{F, G\}$, where $F$ and $G$ are division algebras satisfying $F \subseteq G$. If $\Delta$ is of type ADE, we call $\Delta$ simply laced, then $F = G$. When $\Delta$ is non-simply laced, the valuation $k = \dim K G / \dim K F$. More explicitly, each case is described in Figure 2. In this description we did not specify the orientation of $Q$. This is due to the fact that the orientation of $Q$ only determines the module structure on all $M_\alpha$'s.

For a species $S$ the Auslander-Reiten quiver $\Gamma_S$ is defined by setting the vertices to be isomorphism classes of indecomposable objects in $T(S)$ mod. Given two indecomposable objects in $X, Y \in T(S)$ mod, then there is an arrow $[X] \xrightarrow{d_{XY}} [Y]$ in $\Gamma_S$ if 
$$\text{rad}(X, Y) / \text{rad}^2(X, Y) \neq 0$$
and the valuation is given by 
$$d_{XY} = \dim K \text{rad}(X, Y) / \text{rad}^2(X, Y).$$
This construction is introduced in [ARS95] with a slight modification. In [ARS95] the valuation $(a, b)$ indicates that there is a minimal right almost split morphism $X^a \oplus M \to Y$ where $X$ is not a summand of $M$, and a minimal left almost split morphism $X \to Y^b \oplus N$ where $Y$ is not a summand of $N$. To see the connection between these two valuations, we introduce the following.
Definition 2.9. Let $M \in \Lambda \text{--mod}$ be indecomposable for some $\mathbb{K}$-algebra $\Lambda$. Then we define

$$\delta(M) = \dim_{\mathbb{K}}(\text{End}_{\Lambda}(M)/\text{rad}_{\Lambda}(M)).$$

Remark 2.10. Let $S$ be a species where $\Delta$ is a Dynkin diagram. Then

$$\delta(P_i) = \dim_{\mathbb{K}}(D_i) = \begin{cases} \dim_{\mathbb{K}}(F), & \text{if } D_i = F \\ \dim_{\mathbb{K}}(G), & \text{if } D_i = G \end{cases}.$$

Now $(a, b) = (d_{XY}/\delta(X), d_{XY}/\delta(Y))$.

We define a functor $\tau = D \circ \text{Tr}$, where $D = \text{Hom}_{\mathbb{K}}(-, \mathbb{K})$ denotes the $\mathbb{K}$-dual and $\text{Tr}$ is the Auslander-Bridger transpose [ARS95, Chapter 4.1]. By [ARS95, Proposition 1.9] there is an equivalence

$$T(S) \text{--mod} \xrightarrow{\tau} \tau^{-1} T(S) \text{--mod}.$$

If $S$ is a species over an acyclic quiver $Q$, then $T(S)$ is hereditary and $\tau = D\text{Ext}^1_{T(S)}(-, T(S))$ is even defined on the category $T(S) \text{--mod}$, and the inverse is given by $\tau^{-1} = \text{Ext}^1_{T(S)}(DT(S), T(S)) \otimes_{T(S)} -$. We call $\tau$ the Auslander-Reiten translation of $\Gamma_S$.

Recall that for every species we have a complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ induced by the identity elements in $D_1, D_2, \ldots, D_{|Q_0|}$. With these idempotents we can write down the indecomposable projective modules $P_i$ and the indecomposable injective modules $I_i$ as

$$P_i = T(S)e_i, \quad I_i = D(e_i T(S)).$$

Figure 1. Dynkin Diagrams
For a species $S$ we define the preprojective component of $\Gamma_S$ to be the full subquiver of $\Gamma_S$ consisting of all $X \in (\Gamma_S)_0$ such that $\tau^a X$ is projective for some $a \in \mathbb{Z}_{\geq 0}$. Similarly, we define the preinjective component of $\Gamma_S$ to be the full subquiver of $\Gamma_S$ consisting of all $Y \in (\Gamma_S)_0$ such that $\tau^{-b} Y$ is injective for some $b \in \mathbb{Z}_{\geq 0}$.

**Proposition 2.11.** Let $S$ be a species with a Dynkin diagram $\Delta$. Then there exists a permutation $\sigma : Q_0 \to Q_0$ and integers $l_i$ such that

$$P_{\sigma(i)} = \tau^{l_i - 1} I_i.$$

**Proof.** Since $S$ is representation finite $\Gamma_S$ has finitely many vertices. Therefore, by [ARS95, Theorem 2.1], the preprojective component of $\Gamma_S$ coincides with the preinjective component of $\Gamma_S$. 

**Definition 2.12.** Let $S$ be a species with Dynkin diagram $\Delta$. We call $\sigma$ in Proposition 2.11 the Nakayama permutation of $\Delta$. Moreover, if $l_i = l$ for some integer $l \in \mathbb{Z}$, then we call $S$ $l$-homogeneous.

With these integers $l_i$ we can explicitly describe the Auslander-Reiten quiver for representation finite species. Let $S$ be a representation finite species over a quiver $Q$. Then by [ARS95, Proposition 1.15], the projective modules in $\Gamma_S$ form a subquiver which is isomorphic to the opposite of $Q$. In fact, the quiver $Q$, the valuations in $\Delta$ and the length of each $\tau$-orbit (i.e. the numbers $l_i$) is enough information to write down $\Gamma_S$. This follows from [ARS95, Theorem 2.1] together with the fact that $\Gamma_S$ is the preprojective component of $\Gamma_S$.

We also introduce the Coxeter transformation. Since the Coxeter transformation is defined on the Grothendieck group we first define the Grothendieck group.

**Definition 2.13.** Let $S$ be a species. We define the Grothendieck group $K_0(T(S)-\text{mod}) = [T(S)-\text{mod}]/R$, where $[T(S)-\text{mod}]$ is the free abelian group on the isomorphism classes $[M]$ of finitely generated $T(S)$-modules.
$M$ and $R$ is the subgroup generated by expressions $[A] + [C] - [B]$ whenever there is an exact sequence $0 \to A \to B \to C \to 0$ of $T(S)$-modules.

First we note that $\{(D_i)_{i=0}^{Q_i}\}$ is a basis for $K_0(T(S)\text{-mod})$ by [ARS95] Chapter 1, Theorem 1.7. We also have two other basis given by $\{(P_i)_{i=0}^{Q_i}\}$ and $\{(I_i)_{i=0}^{Q_i}\}$ by [ARS95] Chapter 8, Lemma 2.1.

Definition 2.14. Let $S$ be a species. We define the Coxeter transformation $c : K_0(T(S)\text{-mod}) \to K_0(T(S)\text{-mod})$ by $[P_i] \mapsto -[I_i]$.

We call a non-zero element in $K_0(T(S)\text{-mod})$ positive, respectively negative, if the coordinates are greater than or equal to 0, respectively less than or equal to 0 in the basis $\{(D_i)_{i=0}^{Q_i}\}$. For hereditary $\mathbb{K}$-algebras the Coxeter transformation is closely related to the Auslander-Reiten translation, and since $T(S)$ is a hereditary $\mathbb{K}$-algebra when $Q$ is finite and acyclic we have the following proposition.

Proposition 2.15. [ARS95] Chapter 8, Proposition 2.2] Let $S$ be a species over a finite acyclic quiver $Q$. We have the following.

1. If $M \in T(S)\text{-mod}$ is a non-projective indecomposable module, then $c[M] = [\tau M]$.
2. Let $M$ be a non-projective indecomposable $T(S)$-module. Then $M$ is projective if and only if $c[M]$ is negative.
3. If $M$ is an indecomposable $T(S)$-module, then $c[M]$ is either positive or negative.
4. If $M \in T(S)\text{-mod}$ is a non-injective indecomposable module, then $c^{-1}[M] = [\tau^{-1} M]$.
5. Let $M$ be an indecomposable $T(S)$-module. Then $M$ is injective if and only if $c^{-1}[M]$ is negative.
6. If $M$ is an indecomposable $T(S)$-module, then $c^{-1}[M]$ is either positive or negative.

The second and fifth condition in Proposition 2.15 yields a connection to the Auslander-Reiten quiver by considering all of the elements $[M]$, where $M$ is indecomposable $T(S)$-module, in $K_0(T(S)\text{-mod})$.

For a $\mathbb{K}$-algebra $\Lambda$ we denote the bounded derived category of $\Lambda$-mod by $D^b(\Lambda\text{-mod})$.

Definition 2.16. Let $\Lambda$ be a finite dimensional $\mathbb{K}$-algebra with $\text{gldim}(\Lambda) < \infty$.

1. We define the Nakayama functor $\nu : D^b(\Lambda\text{-mod}) \to D^b(\Lambda\text{-mod})$ as the composition $\nu = D \circ R\text{Hom}_{\Lambda\text{-mod}}(\Lambda (-), \Lambda)$.
2. The Auslander-Reiten translation in the derived category is defined as $\nu_1 = \nu \circ [-1]$, where $[-1]$ is the shift functor.

Let $\phi$ be an endomorphism of a $\mathbb{K}$-algebra $\Lambda$. Using the derived tensor product we obtain an endomorphism functor $\Lambda_\phi \otimes^L_{\Lambda} -$ of $D^b(\Lambda\text{-mod})$, where $\Lambda_\phi$ is the $\Lambda$-$\Lambda$-bimodule $\Lambda$ where the right action is twisted by $\phi$, i.e. the right action is given by $\cdot b = a \phi(b)$.

Definition 2.17. [HI11] Definition 0.3] Let $\Lambda$ be a finite dimensional $\mathbb{K}$-algebra with $\text{gldim}(\Lambda) < \infty$. We say that $\Lambda$ is twisted $\frac{1}{l}$-Calabi-Yau if there is an isomorphism $\nu^l \simeq [m] \circ (\Lambda_\phi \otimes^L_{\Lambda} -)$ of functors for some integer $l \neq 0$ and an algebra endomorphism $\phi$ of $\Lambda$. In this case $\phi$ is always an automorphism.

We also need to define acyclic algebras since our main results in Section 10 are only proved for acyclic algebras.

Definition 2.18. [Pas19] Definition 2.5] We say that $\Lambda$ is cyclic if there exist indecomposable projective $\Lambda$-modules $P_1, \ldots, P_m$ with non-zero non-isomorphisms $P_1 \to P_2, \ldots, P_{m-1} \to P_m, P_m \to P_1$ for some $m \geq 1$. We call $\Lambda$ acyclic if it is not cyclic.

Remark 2.19. Let $S$ be a species over the quiver $Q$. The tensor algebra $T(S)$ is acyclic if and only if $Q$ is acyclic.
3. Properties for Representation Finite Species

In this section we compute the Nakayama permutation for all Dynkin diagrams. As an application we determine which species are \( l \)-homogeneous for each integer \( l \). The Nakayama permutation is described in the following main theorem of this section.

**Theorem 3.1.** Let \( S \) be a species with Dynkin diagram \( \Delta \). Then the Nakayama permutation is given by:

\[
\sigma_A(i) = n + 1 - i, \quad \text{if } \Delta = A_n \\
\sigma_D(i) = i, \quad \text{if } \Delta = D_{2n} \\
\sigma_D(i) = \begin{cases} 
2, & \text{if } i = 1 \\
1, & \text{if } i = 2 \\
i, & \text{otherwise}
\end{cases}, \quad \text{if } \Delta = D_{2n+1} \\
\sigma_{E_6}(i) = \begin{cases} 
6 - i, & \text{if } i \neq 6 \\
6, & \text{if } i = 6
\end{cases}, \quad \text{if } \Delta = E_6 \\
\sigma_{E_7}(i) = i, \quad \text{if } \Delta = E_7 \\
\sigma_{E_8}(i) = i, \quad \text{if } \Delta = E_8 \\
\sigma_{\Delta}(i) = i, \quad \text{if } \Delta = B_n, C_n, F_4, G_2
\]

We will later see the importance of this theorem when we describe the Nakayama automorphism in Section 5.

Before we prove this theorem we will state a couple of lemmas.

**Lemma 3.2.** Let \( S \) be a species. If there is an arrow 

\[ [P_i] \xrightarrow{d_{ij}} [P_j] \]

in \( \Gamma_S \), then there is an arrow 

\[ [I_i] \xrightarrow{d_{ij}} [I_j] \]

in \( \Gamma_S \).

**Proof.** Let \( S \) be a species. Note that 

\[ D : \text{T}(S)\text{-mod} \xrightarrow{\sim} \text{T}(S)^{op}\text{-mod}. \]

Therefore, we have that 

\[ \text{Hom}_{T(S)}(T(S)e_i, T(S)e_j) \cong e_i T(S)e_j \cong \text{Hom}_{T(S)^{op}}(e_j T(S), e_i T(S)) \cong \text{Hom}_{T(S)}(D(e_i T(S)), D(e_j T(S))). \]

The claim now follows from the structure of \( \Gamma_S \) described in Section 2. \( \square \)

**Lemma 3.3.** Let \( \Delta \) be a Dynkin diagram. The Nakayama permutation \( \sigma \) of \( \Delta \) induces a diagram automorphism of \( \Delta \).

**Proof.** Let \( S \) be a species with diagram \( \Delta \). Assume that there exist an arrow \([P_i] \rightarrow [P_j]\) in \( \Gamma_S \). We need to show that every \([P_{\sigma(i)}] \rightarrow [P_{\sigma(j)}]\) or \([P_{\sigma(j)}] \rightarrow [P_{\sigma(i)}]\) in \( \Gamma_S \). Note that there is an arrow \([I_i] \rightarrow [I_j]\) by Lemma 3.2. By Proposition 2.11 there exists an integer \( l_i \) such that \( \tau_i^{-1} I_i = P_{\sigma(i)} \). We only need to show that \( l_i \leq l_j \leq l_i + 1 \). Indeed, if \( l_j = l_i \) then there is an arrow \([P_{\sigma(i)}] \rightarrow [P_{\sigma(j)}]\) in \( \Gamma_S \), and if \( l_j = l_i + 1 \) then there is an arrow \([P_{\sigma(j)}] \rightarrow [P_{\sigma(i)}]\) in \( \Gamma_S \), which is immediate since \( \tau_i^{-1} I_j \) is not projective, then there exists an almost split sequence ending in \( \tau_i^{-1} I_j \) by [ARS95] Chapter 5, Corollary 2.4.

Assume towards contradiction that \( l_j > l_i + 1 \). Then \( \tau_i I_j \) is not a projective module. Let \( f : \tau_i I_j \rightarrow \tau_i^{-1} I_i \) be an irreducible map. Using the fact that \( T(S) \) is hereditary, we get that \( \text{Im} f \) is projective. Thus \( \tilde{f} : \tau_i I_j \rightarrow \text{Im} f \) is a split epimorphism, which contradicts the fact that \( \tau_i I_j \) is indecomposable and non-projective. A similar argument can be used when \( l_j < l_i \). Thus \( l_i \leq l_j \leq l_i + 1 \). \( \square \)
The following proposition is a special case of [Sch85, Theorem 10.1 and 10.5].

**Proposition 3.4.** Let $S$ be a species. There is a short exact sequence of $T(S)$-$T(S)$-bimodules

$$0 \rightarrow \bigoplus_{\alpha \in Q_1, s(\alpha) = i} T(S)e_{t(\alpha)} \otimes_{D_{t(\alpha)}} M_{\alpha} \otimes_{D_{s(\alpha)}} e_{s(\alpha)}T(S) \xrightarrow{d} \bigoplus_{i \in Q_0} T(S)e_i \otimes_{D_i} e_iT(S) \xrightarrow{\text{mult}} T(S) \rightarrow 0,$$  

(3.4.1)

where $d(a \otimes b \otimes c) = ab \otimes c - a \otimes bc$ and $\text{mult}$ denotes the natural multiplication.

**Lemma 3.5.** Let $\Delta$ be a Dynkin diagram of type ADE. The Nakayama permutation $\sigma$ only depends on $\Delta$.

**Proof.** It is enough to show that the Coxeter transformation only depends on the Dynkin diagram $\Delta$, since we can determine the structure of $\Gamma_S$ using the Coxeter transformation by Proposition 2.15. The description of the Auslander-Reiten quivers by [Gab80] for the ADE cases gives us the information that $\sigma$ only depends on $\Delta$ when $T(S)$ is a path algebra over a quiver $Q$ with diagram $\Delta$. This reduces our problem to showing that for a given $S$ with diagram $\Delta$, $\sigma$ only depends on $Q$ and not the species $S$ over $Q$.

Let $S$ be a species with diagram $\Delta$. Applying $\otimes_{T(S)} D_i$ on the sequence (3.4.1) yields and exact sequence

$$0 \rightarrow \bigoplus_{\alpha \in Q_1, s(\alpha) = i} T(S)e_{t(\alpha)} \otimes_{D_{t(\alpha)}} M_{\alpha} \rightarrow \bigoplus_{i \in Q_0} T(S)e_i \rightarrow D_i \rightarrow 0.$$  

Moreover, this is the minimal projective resolution of $D_i$, which implies that $[P_i] + \sum_{\alpha \in Q_1, s(\alpha) = i} [P_{t(\alpha)}] = [D_i]$. The construction of this sequence only depends on $Q$, therefore the coordinates of $[P_i]$ only depends on $Q$ in $K_0(T(S)-\text{mod})$. By the dual argument we show that the coordinates of $[I_i]$ only depends on $Q$ in $K_0(T(S)-\text{mod})$. Hence $\Gamma_S$ only depends on $Q$. \hfill $\Box$

**Lemma 3.6.** Let $\Lambda$ be a finite dimensional $k$-algebra. If $M$ is an indecomposable and non-projective $\Lambda$-module then $\delta(M) = \delta(\tau M)$.

**Proof.** Recall that $\tau : \Lambda-\text{mod} \xrightarrow{\sim} \Lambda-\text{mod}$. In particular,

$$\text{End}_\Lambda(M) \cong \text{End}_\Lambda(\tau M).$$

Let $\alpha \in \text{End}_\Lambda(M)$ be such that the residue class of $\alpha$ is zero in $\text{End}_\Lambda(M)$. In other words, we are in the situation

$$M \xrightarrow{\alpha} M \xrightarrow{\beta} P \xrightarrow{\gamma} M,$$

where $\alpha = \gamma \beta$ and $P$ projective (not necessarily indecomposable). Since $M$ is not projective we have that $\beta$ and $\gamma$ are not split. Therefore $\beta \in \text{rad}(M, P)$ and $\gamma \in \text{rad}(P, M)$. This implies that $\alpha \in \text{rad}(M, M)$. In other words, every endomorphism of $M$ that factors through a projective object is in the radical. Therefore we have the following

$$\text{End}_\Lambda(M)/\text{rad(End}_\Lambda(M)) \cong \text{End}_\Lambda(M)/\text{rad(End}_\Lambda(M)).$$

A similar argument can be used to show that

$$\text{End}_\Lambda(\tau M)/\text{rad(End}_\Lambda(\tau M)) \cong \text{End}_\Lambda(\tau M)/\text{rad(End}_\Lambda(\tau M)).$$

Hence, composing the three isomorphism gives us the result. \hfill $\Box$

We finally have everything we need to prove Theorem 3.1.
Proof. (Theorem 3.1) Let $S$ be a species where $\Delta$ is a Dynkin diagram. With Lemma 3.5 we can use the description of the Auslander-Reiten quivers by \cite{Gab80} for the ADE cases.

Now let $\Delta$ be non-simply laced. We define $N(i)$ to be the number of neighbours of $i$ in $\Delta$. Note that $\Delta$ satisfies $N(i) \leq 2$ for all $i \in \Delta_0$. Since there is only one arrow of valuation higher than one in $\Delta$, using Lemma 3.2 and Lemma 3.6 we have that the Nakayama permutation fixes the vertices that are connected to that arrow. Now we use an inductive argument to show that the rest of the vertices are also fixed. Let $i \xrightarrow{(k)} j$ be the arrow in $\Delta$ such that $k \neq 1$ and $\delta(P_i) = k\delta(P_j)$. Then $\sigma(i) = i$ and $\sigma(j) = j$ as discussed. By Lemma 3.3 neighbours must be mapped to neighbours. In other words, $j'$, a neighbour of $i$ different from $j$, must be mapped to $j'$ or $j$, and since $j$ is fixed $j'$ must be fixed too. Inductively we can use this argument to show that every vertex is fixed. □

The following corollary is a generalization of \cite[Proposition 3.2 a)]{HI11}.

**Corollary 3.7.** Let $S$ be a species where $\Delta$ is a Dynkin diagram. Then $S$ is $l$-homogeneous if $Q$ is stable under $\sigma$. Moreover, for the different cases, the integer $l$ is

| $\Delta$ | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $l$      | $\frac{n+1}{2}$ | $n$    | $n$    | $n-1$  | 6      | 9      | 15     | 6      | 3      |

Proof. By Lemma 3.5 the ADE cases follow from Gabriel’s description of the AR-quivers \cite{Gab80}.

In the non-simply laced cases $\sigma$ is the identity by Theorem 3.1 and so $S$ is always $l$-homogeneous by \cite[Proposition 2.1]{HI11}.

The number of indecomposable modules in $T(S)\text{-mod}$ is the same as the number of positive roots for $\Delta$. Thus $m = \frac{nh}{2}$ is the number of indecomposable modules, where $h$ is the Coxeter number for $\Delta$. Since $S$ is $l$-homogeneous, $l = \frac{m}{n}$ because $l$ is the number of indecomposable module in each $\tau$-orbit. Hence $l = \frac{h}{2}$, which gives the values in the above table. □

4. Preprojective Algebras

In this section we introduce the preprojective algebra of a hereditary representation finite $K$-algebra.

**Definition 4.1.** [BGL87] Let $\Lambda$ be a hereditary representation finite $K$-algebra. The preprojective algebra $\Pi(\Lambda)$ is defined by

$$\Pi(\Lambda) = \bigoplus_{i=0}^{\infty} \Pi_i$$

(4.1.1)

where $\Pi_i = \text{Hom}_\Lambda(\Lambda, \tau^{-i}\Lambda)$, with multiplication

$$\Pi_i \times \Pi_j \to \Pi_{i+j},$$

$$(u, v) \mapsto uv = (\tau^{-i}(v) \circ u : \Lambda \to \tau^{-(i+j)}\Lambda).$$

**Remark 4.2.** The preprojective algebra has a natural $\mathbb{Z}$-grading due to (4.1.1). We will call this grading the $\ast$-grading to avoid confusion. As it turns out in Section 10 the $\ast$-grading will play an important role when describing the almost Koszul complexes.

In the paper \cite{DR80} there is a more explicit description of the preprojective algebra of a species. Let $S$ be a species and let $\alpha \in Q_1$. There exist $x_1, \ldots, x_n \in M_\alpha$ and $f_1, \ldots, f_n \in \text{Hom}_{D^b(\alpha)}(M_\alpha, D_n(\alpha))$ such that for every $x \in M_\alpha$ we have

$$x = \sum_{i=1}^{n} f_i(x)x_i.$$
Definition 4.3. Let $x_i$ and $f_i$ be as above. Then, the element
\[ c_\alpha = \sum_{i=1}^n x_i \otimes_{D_s(\alpha)} f_i \in M_\alpha \otimes_{D_s(\alpha)} \Hom_{D_s(\alpha)}^{D_{s_0}}(M_\alpha, D_{s_0}) \]
is called the Casimir element of $M_\alpha \otimes_{D_s(\alpha)} \Hom_{D_s(\alpha)}^{D_{s_0}}(M_\alpha, D_{s_0})$.

The Casimir element does not depend on the choice of $x_i$ and $f_i$ by [DR80, Lemma 1.1].

Definition 4.4. [DR80] Let $S$ be a species where $\Delta$ is a Dynkin diagram.

1. The double quiver $\overline{Q}$ is defined to be $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \cup Q_1^*$ where
   \[ Q_1^* = \{ \alpha^*: j \to i \mid \alpha: i \to j \in Q_1 \}. \]

2. Let $\overline{S}$ be the species over $\overline{Q}$ where $\overline{D}_i = D_i$ for all $i \in \overline{Q}_0$ and $\overline{M}_\alpha = M_\alpha$ when $\alpha \in Q_1$ and $\overline{M}_\alpha^* = \Hom_{\Pi(\alpha)}^{D_{s_0}}(M_\alpha, D_{s_0})$.

3. For each $\alpha \in Q_1$, let $c_\alpha$ be the Casimir element of $\overline{M}_\alpha \otimes_{D_s(\alpha)} \overline{M}_\alpha^*$. Define
   \[ c = \sum_{\alpha \in Q_1} \sgn(\alpha)c_\alpha, \]
   where
   \[ \sgn(\alpha) = \begin{cases} 1, & \alpha \in Q_1 \\ -1, & \text{else.} \end{cases} \]

We define the preprojective algebra of $S$ as $\Pi(S) = T(\overline{S})/\langle c \rangle$.

Remark 4.5. Let $S = (D, M)$ be a species. Given a decomposition
\[ M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_i \]
we may consider $T(S)$ as a $\mathbb{Z}$-graded algebra by setting elements in $D$ to have degree zero and setting elements in $M_i$ to have degree $i$. This is equivalent to choosing a grading on the quiver $Q$ of $S$ by setting the arrow in $Q$ associated to $M_i$ to have degree $i$. Note that for such a graded species $S$ the quiver $Q$ may have multiple arrows between two vertices, but there will not be multiple arrows of the same degree.

We will consider several different gradings for our species. For instance, setting $M = M_1$ we get a grading on $T(S)$ which corresponds to the path length grading on $T(S)$. We sometimes consider the preprojective algebra $\Pi(S) = T(\overline{S})/\langle c \rangle$ as a $\mathbb{Z}^2$-graded algebra. The first grading is induced by the decomposition $\overline{M} = \overline{M}_1$, and the second grading is induced by the decomposition
\[ \overline{M} = \overline{M}_0 \oplus \overline{M}_1, \quad \overline{M}_0 = \bigoplus_{\alpha \in Q} \overline{M}_\alpha, \quad \overline{M}_1 = \bigoplus_{\alpha^* \in Q^*} \overline{M}_\alpha^*. \]
In other words, the first grading corresponds to the path length grading and the second grading corresponds to the $*$-grading defined in Remark 4.3.

For a representation finite species $S$ over division algebras $F \subseteq G$, let $d_{s(\alpha)} = \dim_{D_{s(\alpha)}} \overline{M}_\alpha$ and $d_{t(\alpha)} = \dim_{D_{s(\alpha)}} \overline{M}_\alpha$. We choose elements $y^1_\alpha, \ldots, y_{\max(d_{s(\alpha)}, d_{t(\alpha)})}^\alpha \in \overline{M}_\alpha$ such that $\{ y_1^\alpha, \ldots, y_{d_{s(\alpha)}}^\alpha \}$ is a basis for $\overline{M}_\alpha$ when $\overline{M}_\alpha$ is viewed as a $D_{s(\alpha)}$-module and $\{ y_1^\alpha, \ldots, y_{d_{t(\alpha)}}^\alpha \}$ is a basis for $\overline{M}_\alpha$ when $\overline{M}_\alpha$ is viewed as a $D_{t(\alpha)}$-module. Moreover, we do this such that $c = \sum_{i=1}^{\dim_{D_{s(\alpha)}} \overline{M}_\alpha} y_\alpha \otimes_{D_{s(\alpha)}} y_\alpha^*$ is a Casimir element.

Example 4.6. Let $S$ be the species
\[ C \subseteq \mathbb{R}_1 \subseteq \mathbb{R}_2. \]
Then $\mathcal{S}$ is given as

$$
\begin{array}{c}
\mathbb{C} \\ \xrightarrow{C}
\end{array}
\xrightarrow{C^*} 
\begin{array}{c}
\mathbb{R}_1 \\ \xrightarrow{R}
\end{array}
\xrightarrow{R^*} 
\begin{array}{c}
\mathbb{R}_2.
\end{array}
$$

Pick the basis $1_C, i_C$ in $\mathbb{C}$ and $1_R$ in $\mathbb{R}$, and let us denote the dual basis with $*$ as a superscript. The preprojective algebra of $S$ is

$$
\Pi(S) = T(\mathcal{S})/(c),
$$

where

$$
c = -1_C^* \otimes_{\mathbb{R}} 1_C - i_C^* \otimes_{\mathbb{R}} i_C + 1_C \otimes_C 1_C^* - 1_R^* \otimes_{\mathbb{R}} 1_R + 1_R \otimes_{\mathbb{R}} 1_R^* .
$$

Let us also describe $\Pi(T(S))$. The Auslander-Reiten quiver $\Gamma_S$ is given by

$$
\begin{array}{c}
P_1 \quad \tau^{-1}P_1 \quad \tau^{-1}P_2 \quad \tau^{-1}P_3 \quad I_1 \quad I_2 \quad I_3
\end{array}
$$

where the red arrows denotes the morphisms of $*$-degree 1. Elements in $\Pi(T(S))$ can be seen as linear combinations of paths in $\Gamma_S$. Thus we see similarities between $\Pi(S)$ and $\Pi(T(S))$ by comparing $Q_1$ with the arrows between projective modules, $Q_1^*$ with the red arrows from projective modules and the relations $\langle c \rangle$ with the relations in $\Gamma_S$ coming from the Auslander-Reiten sequences.

With Example 4.6 as motivation we now derive a relation between $\Pi(T(S))$ and $\Pi(S)$. In Section 2 we gave an explicit description of the Auslander-Reiten quiver $\Gamma_S$ for a representation finite species $S$. Thus we can represent all indecomposable modules in $\Gamma_S$ as translates of indecomposable projective modules.
be the subquiver of $Q$ consisting of all outgoing and ingoing arrows to $i$. By [ARS95, Proposition 1.15] we have a corresponding picture in $\Gamma_S$.

Let us now consider the almost split sequence starting at $P_i$.

$$\bigoplus_{\alpha \in Q_1, \text{ } t(\alpha) = i} P_i^{d_{P_i, P_{s(\alpha)}} / \delta(P_{s(\alpha)})}$$

$$f_i \quad g_i \quad -f'_i \quad \tau^{-1}P_i$$

(4.6.1)

We write $f_i = [f_{\alpha}]_{\alpha \in Q_1}$, where $f_{\alpha} = [f_{\alpha}^1, f_{\alpha}^2, \ldots, f_{\alpha}^{d_{P_{s(\alpha)}, P_{t(\alpha)}} / \delta(P_{t(\alpha)})}]$. We express $g_i, f'_i$ and $g'_i$ in a similar fashion. From the fact that (4.6.1) is an almost split sequence, we know that $g'_i \circ f_i - f'_i \circ g_i = 0$. We want to compare this equality to $e_i ce_i$, where $c$ is the Casimir element for $\Pi(S)$. Let us write out both expressions and compare them.

$$g'_i \circ f_i - f'_i \circ g_i = \sum_{\alpha \in Q_1, \text{ } t(\alpha) = i} d_{P_i, P_{s(\alpha)}} / \delta(P_{s(\alpha)}) \sum_{k=1}^{d_{P_{s(\alpha)}, P_{t(\alpha)}} / \delta(P_{t(\alpha)})} d_{P_i, P_{s(\alpha)}} / \delta(P_{s(\alpha)}) \sum_{\alpha \in Q_1, \text{ } s(\alpha) = i} d_{P_{s(\alpha)}, P_{t(\alpha)}} / \delta(P_{s(\alpha)}) \sum_{k=1}^{d_{P_{s(\alpha)}, P_{t(\alpha)}} / \delta(P_{t(\alpha)})} f'_{\alpha}^k \circ g'_{\alpha}^k =$$

$$= \sum_{\alpha \in Q_1, \text{ } t(\alpha) = i} \sum_{k=1}^{d_{P_i, P_{s(\alpha)}} / \delta(P_{s(\alpha)})} f_{\alpha}^k g_{\alpha}^k - \sum_{\alpha \in Q_1, \text{ } s(\alpha) = i} \sum_{k=1}^{d_{P_{s(\alpha)}, P_{t(\alpha)}} / \delta(P_{s(\alpha)})} f'_{\alpha}^k \circ g'_{\alpha}^k$$

(4.6.2)

Note that the second equality holds since the multiplication in the $\Pi(T(S))$ is in the reversed order compared to composition. On the other hand

$$e_i ce_i = \sum_{\alpha \in Q_1, \text{ } t(\alpha) = i} \dim D_{s(\alpha)} M_\alpha \sum_{k=1}^{d_{P_i, P_{s(\alpha)}} / \delta(P_{s(\alpha)})} y_{\alpha}^k \oplus D_{s(\alpha)} y_{\alpha}^k - \sum_{\alpha \in Q_1, \text{ } s(\alpha) = i} \dim D_{t(\alpha)} M_\alpha \sum_{k=1}^{d_{P_{s(\alpha)}, P_{t(\alpha)}} / \delta(P_{s(\alpha)})} y_{\alpha}^k \oplus D_{t(\alpha)} y_{\alpha}^k.$$ 

(4.6.3)

First note that if $\alpha : i \to j$ then

$$\text{rad}_{T(S)}(P_j, P_i) = \text{Hom}_{T(S)}(P_j, P_i) \cong e_j T(S) e_i = M_\alpha,$$
and also that $\text{rad}^2_{T(S)}(P_i, P_j) = 0$ since $i$ and $j$ are neighbours in $Q$ and $\Delta$ is a tree. Therefore

$$\dim_{D_{\text{tr}(\alpha)}} \overline{M}_\alpha = \dim_k \text{rad}_{T(S)}(P_j, P_i)/\dim_k D_{t(\alpha)} = d_{p_{\text{tr}(\alpha)}, P_i}/\delta(P_{t(\alpha)}).$$

Similarly, if $\alpha : j \rightarrow i$ then

$$\dim_{D_{\text{tr}(\alpha)}} \overline{M}_\alpha = d_{i, P_{q(\alpha)}}/\delta(P_{q(\alpha)}).$$

Hence we know the ranges of the sums in (4.6.2) and (4.6.3) are the same. Let us now fix $f_i$ for all $i \in Q_0$, for example one can choose $f_i^k = -y_i^k$. Using the equivalence $\tau : T(S)\text{-mod} \rightarrow T(S)\text{-mod}$, and the fact that $\tau^{-1} : T(S)\text{-mod} \rightarrow T(S)\text{-mod}$ is given by $\tau^{-1} = \text{Ext}_{T(S)}^1(DT(S), T(S)) \otimes_{T(S)} -$, we know that $f_i^k \in \tau^{-1}(\text{rad}_{T(S)}(T(S), T(S)))$ for all $i$, and therefore we can choose $f_i^k = \tau^{-1}(f_i^k)$. Now choose $g_i$ and $g_i'$ such that (4.6.1) becomes an almost split sequence for each $i \in Q_0$. Comparing (4.6.2) and (4.6.3) we can construct a well-defined isomorphism stated in the following proposition.

**Proposition 4.7.** Let $S$ be a species, where $\Delta$ is a Dynkin diagram, over division algebras $F \subset G$. There is an isomorphism $\Pi(S) \xrightarrow{\sim} \Pi(T(S))$, where $\Pi(T(S))$ is the preprojective algebra of $T(S)$ defined in [HGL87], given by

$$\Pi(S) \rightarrow \Pi(T(S))$$

$$e_i \mapsto 1_{D_i}$$

$$y_i^k \mapsto \begin{cases} f_i^k & \text{ if } \alpha \in Q_1 \\ g_i^k & \text{ if } \alpha \in Q_1^* \end{cases}.$$

In the path algebra case, i.e. when $\Delta$ is of Dynkin type ADE, the above result is proven by Ringel in [Rin98, Theorem A]. It is also proven for a generalization of species called phylum in [GKKP20, Theorem 7.6]. A species $S$ give rise to a phylum by considering the adjoint functors

$$X = \left( \bigoplus_{\alpha \in Q_1} \overline{M}_\alpha \right) \otimes_{D_{\text{tr}(\alpha)}} - : D\text{-mod} \rightarrow D\text{-mod},$$

$$Y = \left( \bigoplus_{\alpha \in Q_1} \overline{M}_\alpha^* \right) \otimes_{D_{\text{tr}(\alpha)}} - : D\text{-mod} \rightarrow D\text{-mod}.$$

**Remark 4.8.** Recall that $\Pi(S)$ is $\mathbb{Z}^2$-graded. We can define a $\mathbb{Z}^2$-grading on $\Pi(T(S))$, motivated by Proposition 4.7 in the following way. We define a $\mathbb{Z}^2$-grading on $\Pi(T(S))$ by setting the first grading to be

$$\Pi(T(S))_0 = T(S)_0 = D,$$

$$\Pi(T(S))_1 = T(S)_1 \oplus \bigoplus_{\alpha \in Q_1} \text{Hom}_{T(S)}(T(S)e_{s(\alpha)}, \tau^{-1}(T(S)e_{t(\alpha)})},$$

and letting the second grading be the $*$-grading. Then the morphism in Proposition 4.7 is compatible with the $\mathbb{Z}^2$-gradings since it induces isomorphisms in the degrees $(0, 0)$, $(1, 0)$ and $(1, 1)$ in which both $\Pi(S)$ and $\Pi(T(S))$ are generated.

5. Nakayama Automorphism

In this section we study the Nakayama automorphism of the preprojective algebra $\Pi(\Lambda)$, where $\Lambda$ is a basic hereditary representation finite $k$-algebra. The existence of a Nakayama automorphism is a consequence of $\Pi(\Lambda)$ being a Frobenius algebra. In the paper [BBK02] they give an explicit formula for the Nakayama automorphism for preprojective algebras $\Pi(KQ)$, where $Q$ is a quiver of Dynkin type ADE. Thus showing that $\Pi(KQ)$ is self-injective. Iyama and Oppermann later showed, in a more general setting, that the preprojective algebra
of a representation finite hereditary $K$-algebra is self-injective in the paper [IO13]. An alternative proof of this statement was given in [Gra19].

**Theorem 5.1.** [BBK02 Theorem 4.8] [Gra19 Theorem 3.1] Let $\Lambda$ be basic, hereditary and representation finite, then $\Pi(\Lambda)$ is a Frobenius algebra.

For a Frobenius algebra $\Pi(\Lambda)$ we know that $\Pi(\Lambda)$ isomorphisms $\Pi(\Lambda)$ as left $\Pi(\Lambda)$-modules, but this isomorphism can be made into an isomorphism of $\Pi(\Lambda)$-$\Pi(\Lambda)$-bimodules by introducing a twist on $D\Pi(\Lambda)$, i.e. $\Pi(\Lambda) \cong D\Pi(\Lambda)^\gamma$ as $\Pi(\Lambda)$-$\Pi(\Lambda)$-bimodules. This $\gamma$ is unique up to an inner automorphism, and it is called the Nakayama automorphism of $\Pi(\Lambda)$.

From [Gra19] we also have a description of the Nakayama automorphism $\gamma$. By [Gra19, Proposition 3.4] we have an isomorphism

$$\text{Hom}_\Lambda(P_i, \tau^{-r} P_j) \xrightarrow{\sim} D\text{Hom}_\Lambda(\tau^{-r} P_j, I_i) \xrightarrow{\sim} D\text{Hom}_\Lambda(P_j, \tau^r I_i) \xrightarrow{\sim} \text{Hom}_\Lambda(\tau^r I_i, I_j).$$

(5.1.1)

In the [BGL87] perspective, let $f : P_i \to \tau^{-r} P_j$ be a map. Choosing an isomorphism $P_{\sigma(i)} \cong \tau^{h_i} I_i$ and combining with the isomorphism in (5.1.1) we define

$$\gamma(f) : P_{\sigma(i)} \to \tau^{h_i \lambda - r} P_{\sigma(j)}.$$

Note that if $f = \text{Id}_{P_i}$, then $r = 0$ and $i = j$ and therefore $\gamma(\text{Id}_{P_i}) = \text{Id}_{P_{\sigma(i)}}$.

**Theorem 5.2.** [Gra19] Theorem 3.8] The morphism $\gamma$ defined above is a Nakayama automorphism of $\Pi(\Lambda)$.

The aim of this section is to prove the following result.

**Theorem 5.3.** Let $S$ be a species, where $\Delta$ is a Dynkin diagram, over division algebras $F \subset G$. There is a Nakayama automorphism $\gamma$ of $\Pi(S)$ given by

$$\gamma(y^i_{\alpha}) = \begin{cases} \frac{y^i_{\sigma(\alpha)}}{\sgn(\sigma(\alpha))}y^i_{\sigma(\alpha)}, & \text{if } \alpha \in Q_1, \\ \frac{y^i_{\sigma(\alpha)}}{\sgn(\sigma(\alpha))}f y^i_{\sigma(\alpha)}, & \text{if } \alpha \not\in Q_1. \end{cases}$$

and $\gamma(e_i) = e_{\sigma(i)}$ for all $i \in Q_0$.

Note that Theorem 5.3 is a generalization of [BBK02 Corollary 4.7].

**Proof.** We prove the following statements separately.

(1) There is a Nakayama automorphism $\gamma$ of $\Pi(S)$ given by

$$\gamma(y^i_{\alpha}) = \begin{cases} \frac{y^i_{\sigma(\alpha)}}{\sgn(\sigma(\alpha))}f y^i_{\sigma(\alpha)}, & \text{if } \alpha \in Q_1, \\ \frac{y^i_{\sigma(\alpha)}}{\sgn(\sigma(\alpha))}f y^i_{\sigma(\alpha)}, & \text{if } \alpha \not\in Q_1. \end{cases}$$

for some constant $f \in F$, and $\gamma(e_i) = e_{\sigma(i)}$ for all $i \in Q_0$.

(2) $f = 1$.

**Proof of (1):** We will divide the proof in two parts. The first part will consist of the proof for when $\Delta$ is simply laced, and the other part will cover when $\Delta$ is non-simply laced.

Assume that $\Delta$ is simply laced. Since $\Delta$ is a Dynkin diagram of type ADE we have that $D_i = F$ for all $i \in Q_0$ and also $M_\alpha = F$ for all $\alpha \in Q_1$. Using Theorem 5.2 together with the isomorphism in Proposition 4.7 we see that the Nakayama automorphism is defined on the generators as

$$\gamma(y_\alpha) = f_\alpha y_{\sigma(\alpha)} f_\alpha^2,$$
where \( f_1, f_2 \in F \), and note that \( \gamma(e_i) = e_{\sigma(i)} \). Since \( M_\alpha = F \) for all \( \alpha \in Q_1 \), we have that \( \dim_{D_\sigma(\alpha)} \mathcal{M}_\alpha = \dim_F F = 1 \) and thus we can choose \( y_\alpha = 1_F \) as a basis for \( \mathcal{M}_\alpha \). Everything commutes with \( 1_F \) in \( F \) and therefore

\[
\gamma(y_\alpha) = f_\alpha y_{\sigma(\alpha)},
\]

where \( f_\alpha = f_1 f_2 \). Since the Nakayama automorphism is unique up to an inner automorphism, we can use inner automorphisms to simplify \( \gamma \). We define a map

\[
g(-, -) : Q_0 \times D^\times \to \text{Inn}(\Pi(S)),
\]

\[
(j, d = (d_1, d_2, \ldots, d_{|Q_0|})) \mapsto x \mapsto \left( \sum_{i \in Q_0, i \neq j} e_i + d_j^{-1} e_j \right) x \left( \sum_{i \in Q_0, i \neq j} e_i + d_j e_j \right).
\]

We can compute explicitly that \( g(t(\beta), f_\beta) \circ \gamma \) is a Nakayama automorphism given by

\[
(g(t(\beta), f_\beta) \circ \gamma)(y_\alpha) = \begin{cases} y_\alpha, & \text{if } \alpha = \beta \\ f_\beta f_\alpha y_\alpha, & \text{if } s(\alpha) = t(\beta) \\ f_\beta^{-1} f_\alpha y_\alpha, & \text{if } t(\alpha) = t(\beta) \\ f_\alpha y_\alpha, & \text{else} \end{cases}
\]

(5.3.1)

Note that by replacing \( \gamma \) with \( g(t(\beta), f_\beta) \circ \gamma \) we change \( f_\beta \) to 1 and leave \( f_\alpha \) unchanged for all \( \alpha \) not adjacent to \( t(\beta) \). We can also remove \( f_\beta \) by considering the Nakayama automorphism \( g(s(\beta), f_\beta^{-1}) \circ \gamma \).

Since \( Q \) is a tree we can enumerate \( Q_0 \) by picking a root 0 and then proceed to label the other vertices by 1, 2, \ldots such that for each \( i \geq 1 \) there exists an arrow \( \alpha : i \rightarrow j \in Q_1 \) or \( \alpha : j \rightarrow i \in Q_1 \), where \( j < i \), and let us denote said arrow by \( \alpha_i \). We claim that this induces an enumeration on the arrows in \( Q \). It is enough to show that \( \alpha_i \) is unique since \( Q \) has \(|Q_0| - 1 \) arrows. Let \( \Delta_i \) be the full subdiagram of \( \Delta \) with vertices \( \{0, 1, \ldots, i - 1\} \), where \( i \leq |Q_0| \). Now assume that there is an edge \( a \in \Delta \) between \( i \) and \( j \) and an edge \( a' \in \Delta \) between \( i \) and \( j' \), where \( j, j' < i \). Since \( \Delta_i \) is connected there is a path \( \omega \) from \( j \) to \( j' \). Extending \( \omega \) with \( a \) and \( a' \) yields a cycle in \( \Delta \) which is absurd since \( Q \) is a tree. This proves uniqueness of \( \alpha_i \).

We can now use induction on the arrows \( y_{\alpha_i} \) to simplify \( \gamma \). Assume that \( \gamma(y_{\alpha_i}) = y_{\sigma(\alpha_i)} \) for all \( i \leq n \). Without loss of generality we can assume that \( t(y_{\alpha_{n+1}}) = n + 1 \), since the proof for \( s(y_{\alpha_{n+1}}) = n + 1 \) uses the same argument. Let us now consider \( g(n + 1, f_{\alpha_{n+1}}) \circ \gamma \). By (5.3.1) we see that \( g(n + 1, f_{\alpha_{n+1}}) \circ \gamma)(y_{\alpha_{n+1}}) = y_{\sigma(\alpha_{n+1})} \).

Since \( y_{\alpha_{n+1}} \) is the unique arrow connecting \( \{1, \ldots, n\} \) to \( n + 1 \) we still have that \( g(n + 1, f_{\alpha_{n+1}}) \circ \gamma)(y_{\alpha_i}) = y_{\sigma(\alpha_i)} \) for all \( i \leq n \). Therefore, by inductively modifying \( \gamma \) we get

\[
\gamma(y_\alpha) = \begin{cases} y_{\sigma(\alpha)}, & \text{if } \alpha \in Q_1 \\ f_\alpha y_{\sigma(\alpha)}, & \text{if } \alpha \notin Q_1 \end{cases}
\]

It is left to show that \( f_\alpha \) does not depend on \( \alpha \) up to a sign. This is shown by comparing \( c \) and \( \gamma(c) \). Recall that

\[
c = \sum_{\alpha \in Q_1} \text{sgn}(\alpha) y_\alpha \otimes_{D_\sigma(\alpha)} y_\alpha^*.
\]

Now applying \( \gamma \) gives

\[
\gamma(c) = \sum_{\alpha \in Q_1} y_{\sigma(\alpha)} \otimes_{D_\sigma(\alpha)} f_\alpha^* y_{\sigma(\alpha)^*} - \sum_{\alpha \in Q_1} f_\alpha y_{\sigma(\alpha)^*} \otimes_{D_\sigma(\alpha^*)^*} y_{\sigma(\alpha)}
\]

Note that \( \gamma \) can be viewed as a graded morphism where the grading is given by \( \deg(y_\alpha) = 1 \) for all \( \alpha \in \overline{Q_1} \) and \( \deg(e_i) = 0 \) for all \( i \in Q_0 \). Let \( \gamma' : T(\mathcal{S}) \to T(\mathcal{S}) \) be defined by having the same constants as \( \gamma \). Since \( \gamma' \) induces \( \gamma \) we get that \( \gamma' \) sends \( (c) \) to \( (c) \). We get that

\[
\gamma'(e_i e_i^*) = c_{\sigma(i)} \gamma'(c) e_{\sigma(i)} e_{\sigma(i)^*} T(\mathcal{S})_2 e_{\sigma(i)} \cap (c) = c_{\sigma(i)} T(\mathcal{S})_0 e_{\sigma(i)}
\]
since \( c \) has degree 2. Now \( e_{\sigma(i)} T(S)_0 = T(S)_0 e_{\sigma(i)} = D_i = F \) yields
\[
\gamma'(e_{i} c_{i} e_{i}) \in F\langle e_{\sigma(i)} c_{\sigma(i)} e_{\sigma(i)} \rangle.
\]
Let \( f_i \in F \) be a constant such that
\[
\gamma'(e_{i} c_{i} e_{i}) = \sum_{\substack{c \in \mathbb{Q}_1, \ s(\alpha) = i \text{ or } t(\alpha) = i}} - \text{sgn}(\alpha) f_i y_{\sigma(\alpha)} \otimes D_{\sigma(\alpha)} y_{\sigma(\alpha)^*}.
\]
Thus
\[
f_{\alpha^*} = - \text{sgn}(\alpha) \text{sgn}(\sigma(\alpha)) f_i = - \text{sgn}(\alpha^*) \text{sgn}(\sigma(\alpha^*)) f_i = \text{sgn}(\sigma(\alpha^*)) f_i
\]
for all \( \alpha \in Q_1^* \) such that \( s(\alpha) = i \) or \( t(\alpha) = i \). Since \( Q \) is connected, \( f_i = f_j \) for all \( i, j \in Q_0 \), and therefore
\[
\gamma(y_{\alpha}) = \begin{cases} y_{\sigma(\alpha)}, & \text{if } \alpha \in Q_1 \\ \text{sgn}(\alpha) y_{\sigma(\alpha)}, & \text{if } \alpha \notin Q_1, \end{cases}
\]
for all \( f \in F \).

Now we assume that \( \Delta \) is non-simply laced. We want to reduce the problem so that we can reuse the logic we used for when \( \Delta \) is simply laced. Similarly to the simply laced case, we can assume that \( y_{\alpha}^1 = 1_F \). Since \( \Delta \) is a Dynkin diagram of \( BCFG \), there exists a unique arrow \( a_0 \in Q_1 \) such that \( M_{a_0} = f G \sigma G \) or \( M_{a_0} = g G f \). We will assume the latter since both cases use similar arguments. Then we know \( \dim D_{\sigma(\alpha)} M_{\alpha} = 1 \) for all \( \alpha \in Q_1 \). Therefore we can write
\[
\gamma(y_{\alpha}^0) = d_{\alpha} y_{\sigma(\alpha)},
\]
for all \( \alpha \in Q_1 \) where \( d_{\alpha} \in D_{\sigma(\alpha)} \).

Let us now enumerate the vertices and apply the same argument as in the simply laced case to simplify \( \gamma \). Pick \( 0 = s(a_0) \) as our root and let the other vertices be defined in the same way as before. Then the same induction argument works for this case, since \( \dim D_{\sigma(\alpha)} M_{\alpha} = 1 \) and \( \dim D_{s(\beta)} M_{\beta} = \dim D_{s(\beta)} M_{\beta} = 1 \) for all \( \alpha \neq \beta \in Q_1 \), and we can write
\[
\gamma(y_{\alpha}^i) = \begin{cases} y_{\sigma(\alpha)}^i, & \text{if } \alpha \in Q_1 \\ d_{\alpha} y_{\sigma(\alpha)}^i, & \text{if } \alpha \notin Q_1. \end{cases}
\]

Now we want to show that \( d_{\alpha} \in F \) for all \( \alpha \in Q_1 \) and that \( d_{\alpha} \) does not depend on \( \alpha \). Let \( \gamma' \) be defined in the same way as in the simply laced case. We again compare \( c \) and \( \gamma'(c) \). Recall that
\[
c = \sum_{\alpha \in Q_1} \sum_{i=1}^{\dim D_{s(\alpha)}} (\overline{\gamma}_{\alpha}^i) \sum_{i=1}^{\dim D_{s(\alpha)}} \text{sgn}(\alpha) y_{\alpha}^i \otimes D_{\sigma(\alpha)} y_{\sigma(\alpha)}^i.
\]
Now applying \( \gamma' \) gives
\[
\gamma'(c) = \sum_{\alpha \in Q_1} \sum_{i=1}^{\dim D_{s(\alpha)}} (\overline{\gamma}_{\alpha}^i) \otimes D_{\sigma(\alpha)} d_{\alpha^*} y_{\sigma(\alpha)}^i \otimes D_{\sigma(\alpha)} d_{\alpha^*} y_{\sigma(\alpha)}^i - \sum_{\alpha^* \in Q_1^*} \sum_{i=1}^{\dim D_{s(\alpha^*)}} (\overline{\gamma}_{\alpha^*}^i) \sum_{i=1}^{\dim D_{s(\alpha^*)}} d_{\alpha^*} y_{\sigma(\alpha)}^i \otimes D_{\sigma(\alpha)} y_{\sigma(\alpha)}.
\]
The same argument as in the simply laced case is used to show that \( \gamma'(c) \in \langle c \rangle \). More explicitly,
\[
\gamma'(e_{i} c_{i} e_{i}) = -d_{e_{\sigma(i)} c_{\sigma(i)}},
\]
for some \( d_i \in D_i \). If we show that \( \gamma(c_{\alpha}) = d_{\alpha^*} c_{\sigma(\alpha)} \) and \( \gamma(c_{\alpha^*}) = d_{\alpha} c_{\sigma(\alpha)} \) for all \( \alpha \in Q_1 \), then we can use the same argument as in the simply laced case to say that \( d_{\alpha^*} = d \text{sgn}(\sigma(\alpha^*)) \) for some \( d \in G \), and also use that \( d \in D_i \) for all \( i \in Q_0 \) to say that \( d \in F \).
Now we prove that \( \gamma'(c_{\alpha}) = d_{\alpha} \cdot c_{\sigma(\alpha)} \) and \( \gamma'(c_{\alpha *}) = d_{\alpha} \cdot c_{\sigma(\alpha) *} \) for all \( \alpha \in Q_1 \). The latter is immediate from

\[
\gamma'(c_{\alpha *}) = \gamma \left( \sum_{i=1}^{\dim D_{\sigma(\alpha) *} (M_\alpha)} y_{\alpha i}^* \otimes D_{\sigma(\alpha)} y_{\alpha i}^* \right) = \sum_{i=1}^{\dim D_{\sigma(\alpha) *} (M_\alpha)} d_{\alpha} \cdot y_{\sigma(\alpha)}^* \otimes D_{\sigma(\alpha)} y_{\alpha i}^* = d_{\alpha} \cdot c_{\sigma(\alpha) *}.
\]

To prove that \( \gamma'(c_{\alpha}) = d_{\alpha} \cdot c_{\sigma(\alpha)} \) we first do a similar computation

\[
\gamma'(c_{\alpha}) = \gamma \left( \sum_{i=1}^{\dim D_{\sigma(\alpha)} (M_\alpha)} y_{\alpha i} \otimes D_{\sigma(\alpha)} y_{\alpha i} \right) = \sum_{i=1}^{\dim D_{\sigma(\alpha)} (M_\alpha)} y_{\sigma(\alpha)}^i \otimes D_{\sigma(\alpha)} d_{\alpha} \cdot y_{\alpha i}^* = \sum_{i=1}^{\dim D_{\sigma(\alpha)} (M_\alpha)} d_{\alpha}^i \cdot y_{\sigma(\alpha)}^i \otimes D_{\sigma(\alpha)} y_{\alpha i}^*,
\]

where \( d_{\alpha i}^* \) is defined via \( d_{\alpha i}^* \cdot y_{\sigma(\alpha)} = y_{\sigma(\alpha)} \cdot d_{\alpha i}^* \). By \( 5.3.2 \) we get \( d_{\alpha i}^* = d_{\alpha j}^* \) for all \( i, j = 1, \ldots, \dim D_{\sigma(\alpha)} (M_\alpha) \). Moreover, since \( y_{\alpha}^1 = 1_F \) we have \( d_{\alpha i}^* = d_{\alpha *}. \) Hence \( \gamma'(c_{\alpha}) = d_{\alpha} \cdot c_{\sigma(\alpha)}. \)

**Proof of (2):** For this part we will use an alternative description of Frobenius algebras that uses a bilinear form. In our case, there is a non-degenerate bilinear form \( \beta : \Pi(S) \times \Pi(S) \rightarrow \mathbb{K} \) such that \( \beta(xy, z) = \beta(x, yz) \) and \( \beta(x, y) = \beta(y, \gamma(x)) \) for all \( x, y, z \in \Pi(S) \). Let \( \alpha : i \rightarrow j \in Q_1 \) and fix a non-zero element \( u_i \in \text{Soc}(\Pi(S) e_i) \). Since \( \Pi(S) y_{\alpha}^1 \subset \Pi(S) e_i \) and that \( \text{Soc}(\Pi(S) e_i) \) is simple, we have that \( \text{Soc}(\Pi(S) e_i) \subset \Pi(S) y_{\alpha}^1 \). In other words, \( u_i = v_1 y_{\alpha}^1 \) for some element \( v_1 \in \Pi(S) \). Dually, we have that \( \text{Soc}(\Pi(S) e_i) \subset y_{\sigma(\alpha) *} \Pi(S) \) and thus \( u_i = y_{\sigma(\alpha) *} v_2 \) for some element \( v_2 \in \Pi(S) \). Consider the element \( u_j = y_{\sigma(\alpha)}^1 v_1 \). We now claim that we are done if we can show that

\[
u_j = -\text{sgn}(\alpha) \cdot \text{sgn}(\sigma(\alpha)) v_2 y_{\alpha}^1.
\]

To summarise

\[
u_1 y_{\alpha}^1 = u_i = y_{\sigma(\alpha) *} v_2, \\
u_{\sigma(\alpha)}^1 v_1 = u_j = -\text{sgn}(\alpha) \cdot \text{sgn}(\sigma(\alpha)) v_2 y_{\alpha}^1.
\]

Indeed, if we show that \( \beta(z, u_i) = \beta(z, u_i f) \), then since \( \beta \) is non-degenerate we have that \( u_i = u_i f \) and therefore \( f = 1 \). Note that if \( \beta(z, u_i) = 0 \) for all \( z \in \Pi(S) \) if and only if \( \beta(z, u_i) = 0 \) for all \( z \in D_{\sigma(i)} \) since \( u_i \in \text{Soc}(\Pi(S) e_i) \). Hence without loss of generality we can assume that \( z \in D_{\sigma(i)} \). Also recall that \( y_{\sigma(\alpha)}^1 = 1_F \),
and therefore \( z y^1_{\sigma(\alpha')} = y^1_{\sigma(\alpha')} z' \) and \( z' y^1_{\sigma(\alpha)} = y^1_{\sigma(\alpha)} z \) for some \( z' \in D \). The computation

\[
\beta(z,u_i) = \beta(z,y^1_{\sigma(\alpha)},v) = \beta(y^1_{\sigma(\alpha)},v) = \beta(y^1_{\sigma(\alpha)},z'v) =
\]

\[
\begin{cases}
\beta(z,v_y^1), & \text{if } \sigma(\alpha^*) \in Q_1 \\
\beta(-z,v_y^1,sgn(\alpha^*) y^1_{\alpha}), & \text{if } \sigma(\alpha^*) \not\in Q_1
\end{cases}
\]

\[
\begin{cases}
\beta(z',-sgn(\alpha^*) u), & \text{if } \sigma(\alpha^*) \in Q_1 \\
\beta(z', u), & \text{if } \sigma(\alpha^*) \not\in Q_1
\end{cases}
\]  

\[
\begin{cases}
\beta(z',-sgn(\alpha^*) v_y^1), & \text{if } \sigma(\alpha^*) \in Q_1 \\
\beta(z', y^1_{\alpha},v_y^1), & \text{if } \sigma(\alpha^*) \not\in Q_1
\end{cases}
\]  

(5.3.3)

shows that \( f = 1 \). It is left to show that the claim holds in each case.

In the simply laced case we can use the calculations done in \cite{BBK02}. Let \( S \) be a species of type ADE such that \( \Pi(S) = \mathbb{K}Q \) and let \( \alpha : i \to j \in \mathcal{Q} \). In the proof of \cite{BBK02} Proposition 4.5 (b)] where they computed the constant \( C = -1 \), they chose \( u_i \) and \( u_j \) such that \( v_1 y^1_{\alpha} = y^1_{\alpha^*} v_2 \) and \( y^1_{\sigma(\alpha)} v_1 = u_j = C \ rho(\alpha^*) \rho(\alpha^*) v_2 y^1_{\alpha^*} = -sgn(\alpha^*) sgn(\alpha^*) v_2 y^1_{\alpha^*} \) for some elements \( v_1,v_2 \in \Pi(S) \). The sign \( -sgn(\alpha^*) sgn(\alpha^*) \) only depends on the different signs appearing in front of terms in the Casimir element \( c \). Since the orientation of \( Q \) decides the signs appearing in \( c \) we can use the same computations to prove \( u_j = -sgn(\alpha^*) sgn(\alpha^*) v_2 y^1_{\alpha^*} \), for arbitrary species of type ADE. Thus the claim holds.

Before we prove the claim for the non-simply laced cases we will introduce some notation. Recall that \( y^1_{\alpha} = 1_F \) for all \( \alpha \in \mathcal{Q} \). Fix \( \alpha \in \mathcal{Q} \) such that \( D_{\sigma(\alpha)} = G \) and assume that \( 2 \dim_{\mathbb{K}} F = \dim_{\mathbb{K}} G \). Let \( a,a' \in \mathbb{G} \) be such that \( y^1_{\alpha} = y^1_{\alpha} a \) and \( y^1_{\alpha} = a' y^1_{\alpha} \). By [DRS0] Lemma 1.1] the Casimir elements \( c_\alpha \) and \( c_{\alpha^*} \) does not depend on the chase of basis. In particular,

\[
c_{\alpha} = y^1_{\alpha} \otimes_G y^1_{\alpha^*} = y^2_{\alpha} \otimes_G y^2_{\alpha^*} = y^1 a \otimes_G a' \otimes_G y^1_{\alpha^*} = y^1 a d \otimes_G y^1_{\alpha^*}
\]

and thus \( a' = a^{-1} \). Note that

\[
ac_{\alpha} a^{-1} = a(y^1_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^1_{\alpha} + y^2_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^2_{\alpha}) a^{-1} =
\]

\[
a(y^1_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^1_{\alpha} + a^{-1} y^1_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^1_{\alpha}) a^{-1} =
\]

\[
a y^1_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^1_{\alpha} + y^1_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^1_{\alpha} = (5.3.5)
\]

We define \( y^0_{\alpha} = y^1_{\alpha} a^{-1} \) and \( y^0_{\alpha} = a y^1_{\alpha} \), so we can rewrite (5.3.5) as

\[
ac_{\alpha} a^{-1} = y^1_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^1_{\alpha} + y^0_{\alpha} \otimes_{D_{\sigma(\alpha)}} y^0_{\alpha} \]

(5.3.6)

Also note that

\[
y^1_{\alpha} \otimes_{G} y^1_{\alpha^*} = y^1_{\alpha} \otimes_{G} y^0_{\alpha^*}, \quad y^1_{\alpha} \otimes_{G} y^2_{\alpha^*} = y^0_{\alpha} \otimes_{G} y^0_{\alpha^*} \]

(5.3.7)

We want to represent the elements in \( \Pi(S) \) in the \cite{BGS} perspective and thus we introduce the following notation:

(1) The element \( y^0_{\alpha} \) will be represented by an undecorated edge from \( s(\alpha) \) to \( t(\alpha) \).
(2) The element $y^2\alpha$ will be represented by an edge from $s(\alpha)$ to $t(\alpha)$ decorated with a black circle.
(3) The element $y^0\alpha$ will be represented by an edge from $s(\alpha)$ to $t(\alpha)$ decorated with a red circle.
(4) The element $y^3\alpha$ will be represented by an edge from $s(\alpha)$ to $t(\alpha)$ decorated with disk.

In this notation we can rewrite (5.3.4) and $c_\alpha^*$ as $c_\alpha = \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix}$ and $c_\alpha^* = \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix}$.

(5.3.6) as

\[ac_\alpha a^{-1} = \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix} + \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix}\]

and (5.3.7) as

\[\begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix} = \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix}, \quad \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix} = \begin{pmatrix} t(\alpha) \\ s(\alpha) \end{pmatrix}.

\]

Remark 5.4. Note that we will only use (1) and (2) for all of the non-simply laced cases, (3) will be used for all of the non-simply laced cases except for type $G$ and (4) will only be used in type $G$.

For example, if $S$ is a species over $Q : 1 \alpha \rightarrow 2 \beta \rightarrow 3$, then the element $y^1\beta y^1\beta y^1\alpha$ will be represented by

\[
\begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix},
\]

where the numbers on the left hand side indicates the vertices in $Q_0$. If the element $y^0\beta y^1\beta y^2\alpha$ would exist it would be represented by

\[
\begin{pmatrix} \beta \\ \alpha \end{pmatrix}.
\]

In all of the non-simply laced cases we label the vertices in the same way as in Figure 1.

Let $S$ be a species of type $B$. Computing $v_k e_k e_k$, for all $k \in \{1, 2, \ldots, n\}$, under the isomorphism in Proposition 4.7 we have, in our notation, that the relations in $\Pi(S)$ are given by

\[
\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} k + 2 \\ k + 1 \end{pmatrix} = \epsilon_k \begin{pmatrix} k + 2 \\ k \end{pmatrix}, \quad \begin{pmatrix} n \\ n - 1 \end{pmatrix} = 0
\]
where $\epsilon_k$ is a sign that depends on the orientation of $Q$. Let $\alpha : n \to n-1 \in \mathcal{Q}_1$. Consider the non-zero elements

$$u_n = \begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad u_{n-1} = \begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

in $\text{Soc}(\Pi(S))$. Reading from Figure 3 where $\epsilon = \prod_{k=1}^{n-2} \epsilon_k$, we get that

$$y_1^\alpha v_1 = u_n = v_2 y_1^\alpha, \quad y_1^\alpha v_1 = u_{n-1} = -v_2 y_1^\alpha,$$

for some elements $v_1, v_2 \in \Pi(S)$. Hence $f = 1$.

Let $S$ be a species of type C. Similarly as in type B the elements $\epsilon_k e_k$ for $k \in \{2, 3, \ldots, n\}$ give rise to the relations in $\Pi(S)$ given by

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \epsilon_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} k+2 \\ k+1 \end{pmatrix} = \epsilon_k \begin{pmatrix} k+2 \\ k+1 \end{pmatrix}, \quad \begin{pmatrix} n \\ n-1 \end{pmatrix} = 0$$

where $\epsilon_k$ is a sign that depends on the orientation of $Q$. Similarly to the type B case we assume that $\alpha : n \to n-1 \in \mathcal{Q}_1$ and consider the non-zero elements

$$u_n = \begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad u_{n-1} = \begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

in $\text{Soc}(\Pi(S))$. Let $\epsilon = \prod_{k=1}^{n-2} \epsilon_k$. The computation in Figure 4 show that

$$y_1^\alpha v_1 = u_n = v_2 y_1^\alpha, \quad y_1^\alpha v_1 = u_{n-1} = -v_2 y_1^\alpha,$$

for some elements $v_1, v_2 \in \Pi(S)$. Which in turn implies that $f = 1$. 


\[
\begin{align*}
\mathbf{u}_{n-1} &= \begin{pmatrix}
n \\
n - 1 \\
n - 2 \\
3 \\
2 \\
1
\end{pmatrix} = \epsilon \\
\mathbf{n} &= \begin{pmatrix}
n \\
n - 1 \\
n - 2 \\
3 \\
2 \\
1
\end{pmatrix} = -\epsilon \\
\mathbf{u}_{n-1} &= \begin{pmatrix}
n \\
n - 1 \\
n - 2 \\
3 \\
2 \\
1
\end{pmatrix} = -\epsilon \\
\mathbf{n} &= \begin{pmatrix}
n \\
n - 1 \\
n - 2 \\
3 \\
2 \\
1
\end{pmatrix} = \epsilon
\end{align*}
\]

Figure 3. Computation for \( f = 1 \) in type B
Let $S$ be a species of type F. Similarly as in Type B, the preprojective algebra $\Pi(S)$ is described with the relations

\begin{align*}
\begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix} &= 0, \\
\begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix} &= \epsilon \\
\begin{pmatrix} n \\ n-1 \\ n-2 \\ 3 \\ 2 \\ 1 \end{pmatrix} &= 0.
\end{align*}

(5.4.1)
where $\epsilon$ and $\epsilon'$ are signs that depend on the orientation of $Q$. Let $\alpha : 4 \to 3 \in Q_1$ and choose the following non-zero elements

$$\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}. $$

In this case it is not clear why $u_3$ and $u_4$ are non-zero and therefore we will begin by giving the outline of the proof why $u_3$ and $u_4$ are non-zero. The proof for $u_3$ is similar to the proof for $u_4$ and thus we only show that $u_4$ is non-zero. Using the relations (5.4.1) we have four possible candidates for $u_4$ which is

$$\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}. $$

First we note that

$$\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

and similarly $u_4^4 = 0$. Reading from Figure 5 we have that $u_3 = 0$. The computation in Figure 6 shows that

$$\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$ 

Thus we can conclude that

$$y_{\alpha} v_1 = u_4 = v_2 y_{\alpha}, \quad y_{\alpha} v_1 = u_3 = -v_2 y_{\alpha}. $$
for some elements \( v_1, v_2 \in \Pi(S) \). Hence \( f = 1 \).

Let \( S \) be a species of type G. Similarly as in type B, its preprojective algebra \( \Pi(S) \) is described using the relations

\[
\left( \begin{array}{c}
1 \\
2
\end{array} \right) + \left( \begin{array}{c}
1 \\
2
\end{array} \right) + \left( \begin{array}{c}
1 \\
2
\end{array} \right) = 0, \quad \left( \begin{array}{c}
1 \\
2
\end{array} \right) = 0.
\]

Let \( \alpha : 1 \to 2 \) and choose the non-zero elements

\[
u_1 = \left( \begin{array}{c}
2 \\
1
\end{array} \right), \quad v_2 = \left( \begin{array}{c}
2 \\
1
\end{array} \right).
\]

Similarly as in the previous cases the computation

\[
u_1 = \left( \begin{array}{c}
2 \\
1
\end{array} \right) = - \left( \begin{array}{c}
2 \\
1
\end{array} \right) = \left( \begin{array}{c}
2 \\
1
\end{array} \right) = - \left( \begin{array}{c}
2 \\
1
\end{array} \right)
\]

shows that

\[
y_{\alpha}v_1 = u_1 = v_2y_{\alpha}^*, \quad y_{\alpha}v_1 = u_2 = -v_2y_{\alpha}^*.
\]

for some elements \( v_1, v_2 \in \Pi(S) \) and thus \( f = 1 \). \qed
Figure 6. Computation for $f = 1$ in type F
6. Koszul algebras

The goal of this section is to show that the preprojective algebra of a representation finite species of Dynkin type $\Delta$ with Coxeter number $h$ is $(h-2,2)$-almost Koszul. We will introduce the definition of the preprojective algebra using the derived category, which in turn will be used to compute the almost Koszul complex for $D_i \in \Pi(S)\text{-mod}.$

Definition 6.1. [BGS96 Def. 1.1.2 and Def. 1.2.1] Let $\Lambda = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \Lambda_i$ be a graded $k$-algebra, where $\Lambda_0$ is semi-simple. We say that $\Lambda$ is a Koszul algebra, if there exists a graded exact complex

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda_0 \rightarrow 0,$$

of $\Lambda$-modules where $P_i$ is projective and is generated by its component of degree $i$ for all $i \geq 0$.

Lemma 6.2. Let $S$ be a species. Then $T(S)$ is a Koszul algebra.

Proof. Applying $- \otimes_{T(S)} T(S)_0$ to (3.4.1) yields an exact sequence of $T(S)$-modules

$$P_\bullet : 0 \rightarrow \bigoplus_{\alpha \in Q_1} T(S) e_{i(\alpha)} \otimes_{D_{i(\alpha)}} M_\alpha \xrightarrow{\text{mult}} \bigoplus_{i \in Q_0} T(S)e_i \rightarrow T(S)_0 \rightarrow 0. \tag{6.2.1}$$

This is the Koszul complex of $T(S)$, in other words, $T(S)$ is a Koszul algebra. \hfill \square

We will now shift our focus to the bounded derived category of $T(S)$. As we will see in this section, we can describe the preprojective algebra in the bounded derived category and use the theory of Auslander-Reiten triangles due to [Hap87] to compute the almost Koszul complex for all simple $\Pi(S)$-modules. Using $\nu_1$ we have an alternate description of the preprojective algebra $\Pi(S)$ as follows

$$\Pi(S) = \bigoplus_{i \geq 0} \text{Hom}_{D^b(T(S)\text{-mod})}(T(S), \nu_1^{-i}T(S)).$$

This definition is indeed equivalent to the definition given by [BGL87]. This can be seen by noting that $T(S)\text{-mod}$ can be viewed as a subcategory of $D^b(T(S)\text{-mod})$ and the fact that

$$\text{Hom}_{D^b(T(S)\text{-mod})}(T(S), \nu_1^{-i}T(S)) \cong \text{Hom}_{T(S)}(T(S), \tau^{-i}T(S))$$

for all $i \in \mathbb{Z}_{\geq 0}$. We denote the Auslander-Reiten quiver for $D^b(T(S)\text{-mod})$ by $\Gamma_{D(S)}$. The relation between $T(S)\text{-mod}$ and $D^b(T(S)\text{-mod})$ can be seen via $\Gamma_{D(S)}$. More explicitly, we can describe $\Gamma_{D(S)}$ using $\Gamma_S$ as follows. The vertex set is

$$(\Gamma_{D(S)})_0 = \bigcup_{i \in \mathbb{Z}} (\Gamma_S[i])_0,$$

and the arrow set is

$$(\Gamma_{D(S)})_1 = V \cup \bigcup_{i \in \mathbb{Z}} (\Gamma_S[i])_1,$$

where $V$ is a set of arrows connecting $\Gamma_S[i]$ with $\Gamma_S[i+1]$ for all $i \in \mathbb{Z}$. A description of $V$ will be given with the following example. Let $S$ be the species $\mathbb{C} \xrightarrow{2} \mathbb{R} \xrightarrow{2} \mathbb{R}$ over the quiver $1 \rightarrow 2 \rightarrow 3$. The quiver $\Gamma_{D(S)}$ is

\[
\begin{array}{cccccccc}
\cdots & I_1[-1] & \alpha_1 & P_1 & \nu_1^{-1}P_1 & I_1 & P_1[1] & \\
\cdots & I_2[-1] & \alpha_2 & P_2 & \nu_1^{-1}P_2 & I_2 & P_2[1] & \\
I_3[-1] & P_3 & \nu_1^{-1}P_3 & I_3 & P_3[1] & \cdots
\end{array}
\]
Here the red arrows are the arrows in $V$. Note that if there is an arrow $P_s \to P_t$ with valuation $d_{P_s, P_t}$, then there is an arrow $I_{[i]} \to P_s[i+1]$ with valuation $d_{I_{[i]}, P_s[i+1]} = d_{P_s, P_t}$, for all $i \in \mathbb{Z}$ and these give all arrows in $V$.

The almost split sequences in $T(S) - \text{mod}$ relate to Auslander-Reiten triangles in $D^b(T(S) - \text{mod})$ in the sense that for any almost split sequence

$$0 \to M \to X \to N \to 0$$

in $T(S) - \text{mod}$, there is an Auslander-Reiten triangle of the form

$$M \to X \to N \to M[1]$$

in $D^b(T(S) - \text{mod})$. We will use this relation between almost split sequences and Auslander-Reiten triangles to prove that representation finite species are twisted fractionally Calabi-Yau.

**Lemma 6.3.** Let $S$ be a representation finite species of Dynkin type $\Delta$ with Coxeter number $h$. Then $l_i + l_{\sigma(i)} = h$ for all $i \in Q_0$. In particular, $\nu_1^{-h} P_i = P_i[2]$.

**Proof.** Let $a_i = l_i + l_{\sigma(i)}$. By Theorem 3.1 we know that $\sigma^2 = \text{Id}$. Thus $\nu_1^{-h} P_i = P_i[2]$ for all $i \in Q_0$. Now if there is an arrow from $i \to j$ in $Q$ then $\text{Hom}_{T(S)}(P_j, P_i) \neq 0$, which means that

$$0 \neq \text{Hom}_{T(S)}(\nu_1^{-a_i} P_j, \nu_1^{-a_i} P_i) = \text{Hom}_{T(S)}(P_j, P_i[2(a_j - a_i)])$$

and so $a_j = a_i$. Hence $a_i = a$ for all $i \in Q_0$ and for some constant $a \in \mathbb{Z}$. On the other hand, setting $n = |Q_0|$, we get

$$na = \sum_{i \in Q_0} a_i = 2 \sum_{i \in Q_0} l_i = 2n \frac{h}{2} = nh$$

(6.3.1)

and therefore $a = h$. The third equality in (6.3.1) holds since there are $n \frac{h}{2}$ indecomposable $T(S)$-modules and that $l_i$ is the number of indecomposable $T(S)$-modules in the $\tau$-orbit of $P_i$. \hfill $\square$

**Proposition 6.4.** Let $S$ be a representation finite species of Dynkin type $\Delta$ with Coxeter number $h$. Then $T(S)$ is twisted $\frac{h-2}{h}$-Calabi-Yau.

**Proof.** By Proposition 4.3 it is enough to show that $\nu^h T(S) \cong T(S)[h-2]$. This can be seen via the following computation

$$\nu^h T(S) \cong (\nu_1 \circ [1]^h) T(S) = \nu_1^h T(S)[h] \cong T(S)[h-2].$$

The third isomorphism holds by Lemma 6.3. \hfill $\square$

Next we show for every representation finite species $S$ that all paths in $\Gamma_D(S)$ from $X$ to $Y$ have the same length $W(X, Y)$. To do this we first introduce integers $W(i, j)$ for all $i, j \in Q_0$. We will define $W$ in several steps.

First we define $W(\omega)$ where $\omega$ is a path in $\mathcal{Q}$. Given a path $\omega$ in $\mathcal{Q}$ we can split it into a product of arrows in $\mathcal{Q}_1$ as follows

$$\omega = \alpha_m \alpha_{m-1} \ldots \alpha_1, \quad \alpha_i \in \mathcal{Q}_1.$$

Let us define

$$W(\omega) = \sum_{i=1}^m W(\alpha_i), \quad W(\alpha_i) = \begin{cases} -1, & \text{if } \alpha_i \in Q_1 \\ 1, & \text{if } \alpha_i \not\in Q_1 \end{cases} \quad (6.4.1)$$

**Proposition 6.5.** Let $Q$ be an acyclic quiver and let $\omega$ and $\omega'$ be two paths from $i$ to $j$ in $\mathcal{Q}$. Then $W(\omega) = W(\omega')$. Thus we write $W(i, j) = W(\omega)$. 


Proof. First we show that if \( \omega_{\text{cyc}} \) is a cycle in \( \mathcal{Q} \) then \( W(\omega_{\text{cyc}}) = 0 \). First we write

\[
\omega_{\text{cyc}} = \alpha_m \alpha_{m-1} \cdots \alpha_1,
\]

where \( \alpha_i \in \mathcal{Q}_i \) for all \( i \in \{1, \ldots, m\} \). Since \( Q \) is acyclic and without multiple arrows there exists \( k \in \{1, \ldots, m\} \) such that \( \alpha_k = \alpha_{k-1}^* \). Thus

\[
W(\omega_{\text{cyc}}) = W(\omega'_{\text{cyc}}),
\]

where

\[
\omega'_{\text{cyc}} = \alpha_m \alpha_{m-1} \cdots \alpha_{k+1} \alpha_k \cdots \alpha_1.
\]

Since \( \omega'_{\text{cyc}} \) is a cycle in \( \mathcal{Q} \) we have that \( W(\omega'_{\text{cyc}}) = 0 \) by induction on the length of \( \omega_{\text{cyc}} \).

Now let \( \omega \) and \( \omega' \) be path from \( i \) to \( j \) in \( \mathcal{Q} \). Then \( \omega \omega' \) is a cycle, and therefore \( W(\omega \omega') = 0 \). Rewriting the left hand side we get

\[
0 = W(\omega \omega') = W(\omega') + W(\omega') = W(\omega') - W(\omega).
\]

Since \( T(S) \) is hereditary we know that the projective \( T(S) \)-modules in \( \Gamma_{D(S)} \) form a subquiver which is isomorphic to \( Q^* \). So if there is a path in \( \Gamma_{D(S)} \) from \( P_i \) to \( P_j \), it must have length \( W(i, j) \). We can define \( W \) on all indecomposable objects in \( \mathcal{D}^b(T(S)\mod) \) preserving this property by observing that any path in \( \Gamma_{D(S)} \) from \( i \) to \( j \) has length 2, for all indecomposable objects \( X \in \mathcal{D}^b(T(S)\mod) \). Let \( X, Y \in \mathcal{D}^b(T(S)\mod) \). Then since \( S \) is representation finite there exist \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \) such that \( X = \nu_{i}^{-k_1} P_i \) and \( Y = \nu_{j}^{-k_2} P_j \) for some \( i, j \in Q_0 \). We define

\[
W(X, Y) = 2(k_2 - k_1) + W(i, j).
\]

Let \( Z = \nu_{i}^{-k_3} P_{m} \) for some \( m \in Q_0 \). We claim that \( W(X, Z) = W(X, Y) + W(Y, Z) \). Indeed, we have that

\[
W(X, Z) = 2(k_3 - k_1) + W(i, m) = 2(k_2 - k_1) + W(i, j) + 2(k_3 - k_2) + W(j, m) + W(i, m) = 2(k_2 - k_1) + W(i, j) + W(j, m) - W(i, j) = W(X, Y) + W(Y, Z) + W(i, m) - W(j, m) - W(i, j).
\]

Now let \( \omega \) be a path in \( \mathcal{Q} \) from \( i \) to \( j \) and \( \omega' \) be a path in \( \mathcal{Q} \) from \( j \) to \( m \). Then \( \omega \omega' \) is a path in \( \mathcal{Q} \) from \( i \) to \( m \), and by Proposition \[6.5\], \( W(\omega') = W(\omega') + W(\omega) = W(j, m) + W(i, j) \) which proves our claim.

Proposition 6.6. Let \( X, Y \in \mathcal{D}^b(T(S)\mod) \) be indecomposable modules. Any path from \( X \) to \( Y \) in \( \mathcal{D}^b(T(S)\mod) \) has length \( W(X, Y) \).

Proof. As above, \( X = \nu_{i}^{-k_1} P_i \) and \( Y = \nu_{j}^{-k_2} P_j \) for some \( i, j \in Q_0 \) and \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \). By construction of \( W \) the claim holds when \( k_1 = k_2 = 0 \). Let \( Z \in \mathcal{D}^b(T(S)\mod) \) and let \( k_3 \) be such that \( Z = \nu_{i}^{-k_3} P_l \) for some \( l \in Q_0 \). Assume that \( \alpha : l \to j \in \mathcal{Q}_i \) and that there is a path of length 1 from \( Z \) to \( Y \). We need to show that \( W(X, Y) = W(X, Z) + 1 \). Since there is a path of length 1 from \( Z \) to \( Y \) we get that \( |k_2 - k_3| \leq 1 \) and thus we have two cases. Either \( k_3 = k_2 \) or \( k_3 = k_2 - 1 \). If \( k_3 = k_2 - 1 \) then \( \alpha \notin Q_1 \) and therefore

\[
W(X, Y) = W(X, Z) + W(Z, Y) = W(X, Z) + W(l, j) = W(X, Z) + 1.
\]

If \( k_3 = k_2 - 1 \) then \( \alpha \in Q_1 \) and thus

\[
W(X, Y) = W(X, Z) + W(Z, Y) = W(X, Z) + 2(k_2 - k_3) + W(l, j) = W(X, Y) + 2 - 1 = W(X, Y) + 1.
\]

Now the claim follows by induction on the length of the path from \( X \) to \( Y \).

Proposition 6.7. Let \( X \in \mathcal{D}^b(T(S)\mod) \). Then \( W(X, X[1]) = h \).

Proof. Without loss of generality we can assume that \( X = P_i \) since

\[
W(X, X[1]) = W(\nu_i^h X, \nu_i^h X[1])
\]

for all \( k \in \mathbb{Z} \). By Lemma \[6.3\] we have that \( \nu_{i}^{-h} P_i = P_i[2] \) for all \( i \in Q_0 \). Now \( W(P_i, P_i[2]) = W(P_i, \nu_{i}^{-h} P_i) = 2h \) and \( W(P_i, P_i[2]) = 2W(P_i, P_i[1]) \) give the claim.
Proof. To construct the sequence (6.9.1) we start by considering Auslander-Reiten triangles in
\[ D \]
where
\[ \nu \]

Proposition 6.8. Let \( S \) be a representation finite species. Then
\[ \deg \text{Soc}(\Pi(S)) = h - 2. \]

Proof. If we shift (6.7.1) to the right we get
\[ P_i \to I_i \to N \to P_i[1]. \]

Up to multiplication by a scalar the first morphism in (6.8.1) is given by
\[ P_i \to \text{Top}(P_i) = \text{Soc}(I_i) \to I_i, \]
and it is an element of \( \Pi(S)e_i \). Since (6.8.1) is obtained by shifting an Auslander-Reiten triangle starting at \( I_i \), any radical map from \( I_i \) can be factored through \( I_i \to N \). The exactness of the triangle yields that any radical morphism composed with (6.8.2) is the zero map. Hence (6.8.2) lies in the socle of \( \Pi(S)e_i \). It is left to show that the \( \text{Soc}(\Pi(S)e_i) \) is \( D \)-generated by (6.8.2). Since \( \Pi(S) \) is self-injective
\[ \dim_{\mathcal{D}^b(\Pi(S)e_i)} = 1, \]
for all \( i \in Q_0 \). This proves the proposition. \( \square \)

Proposition 6.9. Let \( S \) be a species of Dynkin type \( \Delta \). For every \( i \in Q_0 \), there is an exact sequence of left \( \Pi(S) \)-modules
\[ 0 \to D_{\sigma(i)} \to \Pi(S)e_i \to \bigoplus_{\alpha \in Q_1, s(\alpha)=i} (\Pi(S)e_{t(\alpha)})^{\dim_{\mathcal{D}^b(\alpha)}} \to \Pi(S)e_i \to D_i \to 0, \]
where \( D_i \) is the simple module at \( i \in Q_0 \).

Proof. To construct the sequence (6.9.1), we start by considering Auslander-Reiten triangles in \( \mathcal{D}^b = \mathcal{D}^b(\mod-T(S)) \) (i.e. we for the moment switch to considering right \( T(S) \)-modules).

By [DR80] Theorem, the preprojective algebra does not depend on the orientation of \( Q \). Therefore, we choose an orientation of \( Q \) such that \( P_i \) is simple. Consider the Auslander-Reiten triangle
\[ P_i \to \bigoplus_{\alpha \in Q_1, s(\alpha)=i} P^{\text{d}P_i}_{t(\alpha), \nu_1^{2j}P_i} \to \nu_1^{-1}P_i \to P_i[1]. \]

The fact that the middle term of (6.9.2) is a direct sum of projective modules is a consequence of \( P_i \) being simple. Note that this triangle consists of two length 1 morphisms and one length \( h - 2 \) morphism \( \nu_1^{2j}P_i \to P_i[1] \), which follows from the discussion for (6.7.1). Let \( 0 \leq j \leq k \). We apply \( \text{Hom}_\mathcal{D}(-, \nu_1^{-j}P_k) \) on (6.9.2) to get the long exact sequence
\[ \cdots \to \bigoplus_{\alpha \in Q_1, s(\alpha)=i} \text{Hom}_\mathcal{D}\left(P^{\text{d}P_i}_{t(\alpha), \nu_1^{2j}P_i}[1], \nu_1^{-j}P_k\right) \to \text{Hom}_\mathcal{D}\left(P_i[1], \nu_1^{-j}P_k\right) \to \]
\[ \to \text{Hom}_\mathcal{D}\left(\nu_1^{-1}P_i, \nu_1^{-j}P_k\right) \to \bigoplus_{\alpha \in Q_1, s(\alpha)=i} \text{Hom}_\mathcal{D}\left(P^{\text{d}P_i}_{t(\alpha), \nu_1^{2j}P_i}, \nu_1^{-j}P_k\right) \]
\[ \to \text{Hom}_\mathcal{D}\left(P_i, \nu_1^{-j}P_k\right) \to \text{Hom}_\mathcal{D}\left(\nu_1^{-1}P_i[-1], \nu_1^{-j}P_k\right) \to \bigoplus_{\alpha \in Q_1, s(\alpha)=i} \text{Hom}_\mathcal{D}\left(P^{\text{d}P_i}_{t(\alpha), \nu_1^{2j}P_i}[-1], \nu_1^{-j}P_k\right) \to \cdots \]
\[ (6.9.3) \]
which follows from the fact that we chose our orientation of $0$, the direct sum

$$\text{coker} \rightarrow 0$$

Let us now take the direct sum of (6.9.6) for values $k$.

$$\text{coker} \left( \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow \text{Hom}_{\mathcal{D}^b}(P_t[1], \nu_{1}^{-1} P_k) \right) \rightarrow$$

$$\text{Hom}_{\mathcal{D}^b}(P_t[1], \nu_{1}^{-1} P_k) \rightarrow \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow$$

$$\text{Hom}_{\mathcal{D}^b}(P_t[1], \nu_{1}^{-1} P_k) \rightarrow \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow \cdots$$

We can therefore consider the non-brutal truncation of (6.9.3), i.e.

$$0 \rightarrow \text{coker} \left( \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow \text{Hom}_{\mathcal{D}^b}(P_t[1], \nu_{1}^{-1} P_k) \right) \rightarrow$$

$$\text{Hom}_{\mathcal{D}^b}(P_t[1], \nu_{1}^{-1} P_k) \rightarrow \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow$$

Also note that

$$\text{Hom}_{\mathcal{D}^b}(\nu_{1}^{-1} P_t[-1], P_k) = \begin{cases} D_i, & \text{if } j = 0 \text{ and } k = i, \\ 0, & \text{otherwise} \end{cases}$$

which follows from the fact that we chose our orientation of $Q$ such that $P_t$ is simple. Since $\text{Hom}_{\mathcal{D}^b}(T(S)[-1], T(S)) = 0$, the direct sum

$$\bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) = 0.$$  

(6.9.8)

Let us now take the direct sum of (6.9.6) for values $k \in Q_0$ and $0 \leq j \leq l_k$ to get the following sequence

$$0 \rightarrow \bigoplus_{k \in Q_0} l_k \rightarrow \text{coker} \left( \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow \text{Hom}_{\mathcal{D}^b}(P_t[1], \nu_{1}^{-1} P_k) \right) \rightarrow$$

$$\bigoplus_{k \in Q_0} l_k \text{Hom}_{\mathcal{D}^b}(\nu_{1}^{-1} P_t, \nu_{1}^{-1} P_k) \rightarrow \bigoplus_{\alpha \in Q_i} \text{Hom}_{\mathcal{D}^b}(P_t^d_{\alpha}, \nu_{1}^{-1} P_k) \rightarrow$$

$$\bigoplus_{k \in Q_0} l_k \text{Hom}_{\mathcal{D}^b}(P_t, \nu_{1}^{-1} P_k) \rightarrow \bigoplus_{k \in Q_0} l_k \text{Hom}_{\mathcal{D}^b}(\nu_{1}^{-1} P_t[-1], P_k) \rightarrow 0 \rightarrow \cdots.$$  

(6.9.9)
From the definition of the preprojective algebra we know
\[ \bigoplus_{k \in \mathbb{Q}} \bigoplus_{j=0}^{l_k-1} \text{Hom}_{\text{D} \text{b}}(P_i, \nu_1^{-j} P_k) = \Pi(S)e_i, \]
as a vector space. Naturally, we can view it as an \( \Pi(S) \)-module, and since all of the maps in (6.9.9) are given by composition by certain morphisms, we can view the sequence as being a sequence of \( \Pi(S) \)-modules. Using the fact that \( \text{Hom}_{\text{D} \text{b}}(T(S), T(S)[1]) = 0 \) it follows that
\[ \text{Hom}_{\text{D} \text{b}}(P_i, \nu_1^{-l} P_k) = 0. \]
Using (6.9.5), (6.9.7), (6.9.8) and the fact that \( d_t^t(P_t(\alpha)) \), \( P_i \)/\( \delta(P_t(\alpha)) = \dim_{D_t(\alpha)} M_\alpha \)
we can rewrite (6.9.9) to
\[ 0 \rightarrow D_{\sigma(i)} \rightarrow \Pi(S)e_i \rightarrow \bigoplus_{\alpha \in \mathbb{T}} (\Pi(S)e_{t(\alpha)})^{\dim_{D_t(\alpha)} M_\alpha} \rightarrow \Pi(S)e_i \rightarrow D_i \rightarrow 0. \]

Remark 6.10. Note that taking the \( * \)-degree 0 part of the complex in Proposition 6.9 yields the Koszul complex (6.2.1).

Remark 6.11. In the case where the species \( S \) is not of finite representation type, then \( \Pi(S) \) is Koszul and the Koszul sequence is similar to 6.9.1 but with the last term removed. This is formulated in [AHI +22, Proposition 8.8].

Definition 6.12. [BBK02, Definition 3.1] Let \( \Lambda = \bigoplus_{i=0}^p \Lambda_i \) be a graded \( \mathbb{K} \)-algebra, where \( \Lambda_0 \) is semi-simple. We say that \( \Lambda \) is an almost Koszul, or \((p,q)\)-almost Koszul algebra, if there exists a graded exact complex
\[ 0 \rightarrow W \rightarrow P_q \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda_0 \rightarrow 0, \]
of \( \Lambda \)-modules where \( P_i \) is projective and is generated by its component of degree \( i \) for all \( i = 0, \ldots, q \). Here \( W = \Lambda_p \otimes_{\Lambda_0} (P_q)_q \).

Corollary 6.13. If \( S \) is a species of Dynkin type \( \Delta \), then \( \Pi(S) \) is \((h-2,2)\)-almost Koszul.

Proof. From the definition of the preprojective algebra it is easy to see that it is indeed a quadratic algebra. The sequence in Proposition 6.9 is the almost Koszul complex for \( S \). All the morphism are of degree 1 except for the fourth morphism which is of degree \( h-2 \). This tells us that the third syzygy is \( S_{\sigma(i)} \), and therefore \( \Pi(S) \) is \((h-2,2)\)-almost Koszul.

7. Higher Preprojective Algebras

In this section we recall the higher analogue of the preprojective algebra and the \( l \)-homogeneous property for \( d \)-representation finite algebras.

In the setup of higher Auslander-Reiten theory [Iya07, Iya08], The notion of being representation finite and hereditary is generalized by the following definition.

Definition 7.1. Let \( \Lambda \) be a finite dimensional \( \mathbb{K} \)-algebra of global dimension \( d \). We say that \( \Lambda \) is \( d \)-representation finite if there exists a \( d \)-cluster tilting module \( M \in \Lambda \text{-mod} \), i.e.
\[ \text{add}(M) = \{ X \in \Lambda \text{-mod} \mid \text{Ext}^i_\Lambda(M, X) = 0 \text{ for any } 0 < i < n \} = \{ X \in \Lambda \text{-mod} \mid \text{Ext}^i_\Lambda(X, M) = 0 \text{ for any } 0 < i < n \}. \]

Before we introduce the higher analogue of being \( l \)-homogeneous, we need to introduce the higher Auslander-Reiten translation together with some of its properties, which are stated in the following proposition.
Proposition 7.2. [Iya11 Proposition 1.3b)] Let $\Lambda$ be a $d$-representation finite algebra. Let
\[
\tau_d = \text{Tor}_d^\Lambda(\Lambda, -) \cong D\text{Ext}_\Lambda^d(-, \Lambda) : \Lambda\text{-mod} \to \Lambda\text{-mod},
\]
\[
\tau_d^{-1} = D\text{Tor}_d^\Lambda(D-, \Lambda) \cong \text{Ext}_\Lambda^d(\Lambda, -) : \Lambda\text{-mod} \to \Lambda\text{-mod}.
\]
Let $P_1, \ldots, P_n$ be the isomorphism classes of indecomposable projective $\Lambda$-modules, and let $I_i$ be the indecomposable injective corresponding to $P_i$.

(1) There exists a permutation $\sigma$ and positive integers $l_1, \ldots, l_a$ such that $\tau_d^{l_i-1}I_i = P_{\sigma(i)}$ for all $1 \leq i \leq a$.
(2) There exists a unique basic $d$-cluster tilting $\Lambda$-module $M$, which is given as the direct sum of the following mutually non-isomorphic indecomposable $\Lambda$-modules.
\[
\begin{align*}
I_1, \quad \tau_d I_1, \quad \ldots & \quad \tau_d^{l_1-2} I_1, \quad \tau_d^{l_1-1} I_1 = P_{\sigma(1)} \\
I_2, \quad \tau_d I_2, \quad \ldots & \quad \tau_d^{l_2-2} I_2, \quad \tau_d^{l_2-1} I_2 = P_{\sigma(2)} \\
\cdots & \quad \cdots & \quad \cdots & \quad \cdots & \quad \cdots \\
I_a, \quad \tau_d I_a, \quad \ldots & \quad \tau_d^{l_a-2} I_a, \quad \tau_d^{l_a-1} I_a = P_{\sigma(a)}
\end{align*}
\]
(3) We have mutually quasi-inverse equivalences $\tau_d : \text{add}(M/\Lambda) \cong \text{add}(M/D\Lambda)$ and $\tau_d^{-1} : \text{add}(M/D\Lambda) \cong \text{add}(M/\Lambda)$.

Definition 7.3. [HI11 Definition 1.2] Let $\Lambda$ be a $d$-representation finite algebra. We say that $\Lambda$ is $l$-homogeneous if $l_i = l$ for all $i$.

In the paper [IO13] Iyama and Oppermann introduced the $(d+1)$-preprojective algebra of an algebra of global dimension $d$, which is the higher analogue of Definition 4.1.

Definition 7.4. [IO13 Definition 2.11] Let $\Lambda$ be a $d$-representation finite algebra. We denote the $(d+1)$-preprojective algebra, or the higher preprojective algebra, of $\Lambda$ by $\Pi(\Lambda)$. It is defined as
\[
\Pi(\Lambda) = \bigoplus_{i \geq 0} \Pi_i = \bigoplus_{i \geq 0} \text{Hom}_\Lambda(\Lambda, \tau_d^{-i}\Lambda),
\]
with multiplication
\[
\Pi_i \times \Pi_j \to \Pi_{i+j},
\]
\[
(u, v) \mapsto uv = (\tau_d^{-i}(v) \circ u : \Lambda \to \tau_d^{-(i+j)}\Lambda).
\]

As for the preprojective algebra in Definition 4.1, the $(d+1)$-preprojective algebra is naturally $\mathbb{Z}$-graded by setting elements in $\Pi_i$ to have degree $i$. This grading will be referred to as the $\ast$-grading for $\Pi(\Lambda)$.

Definition 7.5. Let $\Lambda$ be a $K$-algebra. We define $\nu_d = \nu \circ [-d] : \mathcal{D}^b(\Lambda\text{-mod}) \to \mathcal{D}^b(\Lambda\text{-mod})$, where $\nu$ the is the Nakayama functor.

Remark 7.6. For a $d$-representation finite algebra $\Lambda$, we can also define the $(d+1)$-preprojective algebra using $\nu_d^{-1}$ instead of $\tau_d^{-1}$, i.e.
\[
\Pi(\Lambda) = \bigoplus_{i \geq 0} \text{Hom}_{\mathcal{D}^b(\Lambda\text{-mod})}(\Lambda, \nu_d^{-i}\Lambda). \quad (7.6.1)
\]

We have the following important result by Grant and Iyama.

Theorem 7.7. [GI20 Theorem 4.21(b)] Let $\Lambda$ be a Koszul $d$-hereditary algebra, and $\Pi$ its $(d+1)$-preprojective algebra with the $(d+1)$-total grading given in [GI20] Definition 4.15. If $\Lambda$ is $d$-representation finite, then $\Pi$ is almost Koszul.
8. Tensor Product of algebras

In this section we study tensor products of $d$-representation finite algebras. The motivation comes from the paper [HI11] that showed that if $\Lambda_i$ is $l$-homogeneous and $d_i$-representation finite for $i \in \{1, 2\}$, then $\Lambda_1 \otimes_K \Lambda_2$ is again $l$-homogeneous and $(d_1 + d_2)$-representation finite when $K$ is a perfect field. For this reason we assume that $K$ is perfect from this point. The goal of this section is to prove that the higher preprojective algebra $\Pi(\Lambda_1 \otimes_K \Lambda_2)$ is an almost Koszul algebra given certain conditions. This is formulated in Theorem 8.4. This shows existence of almost Koszul complexes and in Section 10 we give a more complete description of the almost Koszul complexes given certain assumptions on the algebras $\Lambda_1$ and $\Lambda_2$.

For completeness we will first prove the folklore statement that says that the tensor product of two Koszul algebras is a Koszul algebra. This is proven for connected $K$-algebras in [PP05, Chapter 3.1, Corollary 1.2].

**Lemma 8.1.** Let $\Lambda_1$ and $\Lambda_2$ be two Koszul algebras. Then $\Lambda_1 \otimes_K \Lambda_2$ is Koszul.

**Proof.** Let $P^\Lambda_1$ and $P^\Lambda_2$ be the Koszul complex for $(\Lambda_1)_0$ and $(\Lambda_2)_0$ respectively. Consider the following double complex

$$
\cdots \longrightarrow P^\Lambda_1 \otimes_K P^\Lambda_2 \longrightarrow 0 \\
\rightarrow \downarrow \quad \downarrow \\
\cdots \rightarrow P^\Lambda_1 \otimes_K P^\Lambda_2 \longrightarrow 0 \\
\downarrow \quad \downarrow \\
0 \quad 0 \\
\downarrow \quad \downarrow \\
\cdots \rightarrow P^\Lambda_1 \otimes_K P^\Lambda_2 \longrightarrow 0 \\
\end{array}
$$

Denote this complex by $P^{\Lambda_1} \otimes_K P^{\Lambda_2}$. Note that all the squares in this complex commute, and therefore we can take the total complex. At degree $i$ we have

$$\text{Tot}(P^{\Lambda_1} \otimes_K P^{\Lambda_2})_i = \bigoplus_{m+n=i} P^\Lambda_1 \otimes_K P^\Lambda_2$$

and the morphism, denoted by $d_i$, from $\text{Tot}(P^{\Lambda_1} \otimes_K P^{\Lambda_2})_i$ is

$$d_i = 
\begin{bmatrix}
    p^\Lambda_1 \otimes 1 & -1 \otimes p^\Lambda_2 \\
    p^\Lambda_1 \otimes 1 & 1 \otimes p^\Lambda_2 \\
    \vdots & \vdots \\
    p^\Lambda_1 \otimes 1 & (-1)^i \otimes p^\Lambda_2
\end{bmatrix}
$$

Since all squares in $P^{\Lambda_1} \otimes_K P^{\Lambda_2}$ commute, $\text{Tot}(P^{\Lambda_1} \otimes_K P^{\Lambda_2})$ is exact except at degree 0. Moreover, $H_0(\text{Tot}(P^{\Lambda_1} \otimes_K P^{\Lambda_2})) = (\Lambda_1)_0 \otimes_K (\Lambda_2)_0 = (\Lambda_1 \otimes_K \Lambda_2)_0$. The morphisms in $\text{Tot}(P^{\Lambda_1} \otimes_K P^{\Lambda_2})$ have degree 1, and thus $\text{Tot}(P^{\Lambda_1} \otimes_K P^{\Lambda_2})$ is the Koszul complex of $(\Lambda_1 \otimes_K \Lambda_2)_0$. \hfill \Box

Herschend and Iyama gave a description of the Calabi-Yau properties of tensor products of fractionally Calabi-Yau algebras which is applicable to our case.

**Proposition 8.2.** [HI11 Proposition 1.4] Let $K$ be a perfect field. If $\Lambda_i$ is $\frac{m_i}{l_i}$-CY (respectively, twisted $\frac{m_i}{l_i}$-CY) for each $i \in \{1, 2\}$, then $\Lambda_1 \otimes_K \Lambda_2$ is $\frac{m}{l}$-CY (respectively, twisted $\frac{m}{l}$-CY) for the least common multiple $l$ of $l_1, l_2$ and

$$m = l \left( \frac{m_1}{l_1} + \frac{m_2}{l_2} \right).$$
Corollary 8.3. \cite{HII1} Corollary 1.5] Let $\mathbb{K}$ be a perfect field and $l$ a positive integer. If $\Lambda_i$ is $l$-homogeneous $d_i$-representation finite for each $i \in \{1, 2\}$, then $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$ is an $l$-homogeneous $(d_1 + d_2)$-representation-finite algebra with a $(d_1 + d_2)$-cluster tilting module $\bigoplus_{i=0}^{l-1} (\tau_{d_1}^{-i} \Lambda_1 \otimes_{\mathbb{K}} \tau_{d_2}^{-i} \Lambda_2)$.

Combining Lemma \ref{lem:preproj} and Theorem \ref{thm:preproj} yields the following result.

Theorem 8.4. Let $\Lambda_i$ be an $l$-homogeneous and $d_i$-representation finite Koszul algebra. Then the $(d_1 + d_2 + 1)$-preprojective algebra $\Pi(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2)$ is an almost Koszul algebra.

9. Properties of the Segre Product

In section \ref{sec:almostkoszul} we describe the almost Koszul complexes in $\Pi(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2)$ using the almost Koszul complexes in $\Pi(\Lambda_1)$ and $\Pi(\Lambda_2)$, where $\Lambda_1$ and $\Lambda_2$ are $d_{\Lambda_1}$- and $d_{\Lambda_2}$-representation finite Koszul algebras. This is done by using the Segre product, so we devote this section to define the Segre product for graded algebras.

Definition 9.1. \cite{PP05} Chapter 3.2, Definition 1] Given two graded $\mathbb{K}$-algebras $\Lambda_1$ and $\Lambda_2$.

(1) As a vector space, we can decompose $\Lambda_i = \bigoplus_{k \in \mathbb{Z}} \Lambda_{i,k}$, where $\Lambda_{i,k}$ is the space of elements of degree $k$ in $\Lambda_i$, for $i \in \{1, 2\}$. We define the Segre product of $\Lambda_1$ and $\Lambda_2$ as

$$\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2 = \bigoplus_{k \in \mathbb{Z}} \Lambda_{1,k} \otimes_{\mathbb{K}} \Lambda_{2,k}.$$ 

(2) Let $X \in \Lambda_1 - \text{mod}^Z$ and $Y \in \Lambda_2 - \text{mod}^Z$. We define the Segre product of $X$ and $Y$ as

$$X \otimes_{\mathbb{K}} Y = \bigoplus_{k \in \mathbb{Z}} X_k \otimes_{\mathbb{K}} Y_k \in \Lambda_1 \otimes_{\mathbb{K}} \Lambda_2 - \text{mod}^Z,$$

where the subscript $k$ denotes the graded part at $k$.

(3) Let $f_i : X_i \rightarrow Y_i$ where $X_i, Y_i \in \Lambda_i - \text{mod}^Z$ for each $i \in \{1, 2\}$. We define the Segre product of $f_1$ and $f_2$ as

$$f_1 \otimes_{\mathbb{K}} f_2 = \bigoplus_{k \in \mathbb{Z}} f_{1,k} \otimes_{\mathbb{K}} f_{2,k} : \bigoplus_{k \in \mathbb{Z}} X_{1,k} \otimes_{\mathbb{K}} X_{2,k} \rightarrow \bigoplus_{k \in \mathbb{Z}} Y_{1,k} \otimes_{\mathbb{K}} Y_{2,k},$$

where the subscript $k$ denotes the graded part at $k$.

The first application of the Segre product will be used to describe the preprojective algebra of $\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2$.

Lemma 9.2. Let $\Lambda_i$ be an $l$-homogeneous $d_{\Lambda_i}$-representation finite algebra for each $i \in \{1, 2\}$. Then

$$\Pi(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2) \cong \Pi(\Lambda_1) \otimes_{\mathbb{K}} \Pi(\Lambda_2).$$

Moreover, this isomorphism is compatible with the $*$-grading.

Proof. We will use the definition for the preprojective algebra given in Remark \ref{rem:preproj} From \cite{HIO13} Lemma 2.11] we have

$$\nu_{d_{\Lambda_1} + d_{\Lambda_2}}(X \otimes_{\mathbb{K}} Y) = \nu_{d_{\Lambda_1}}(X) \otimes \nu_{d_{\Lambda_2}}(Y),$$

where $X \in D^b(\Lambda_1 - \text{mod})$ and $Y \in D^b(\Lambda_2 - \text{mod})$. Therefore

$$\Pi(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2) = \bigoplus_{i \geq 0} \text{Hom}_{D^b(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2 - \text{mod})}(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2, \nu_{d_{\Lambda_1} + d_{\Lambda_2}}^i(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2)) \cong$$

$$\cong \bigoplus_{i \geq 0} \text{Hom}_{D^b(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2 - \text{mod})}(\Lambda_1 \otimes_{\mathbb{K}} \Lambda_2, \nu_{d_{\Lambda_1}}^{-i}(\Lambda_1) \otimes_{\mathbb{K}} \nu_{d_{\Lambda_2}}^{-i}(\Lambda_2)) \cong$$

$$\cong \bigoplus_{i \geq 0} \text{Hom}_{D^b(\Lambda_1 - \text{mod})}(\Lambda_1, \nu_{d_{\Lambda_1}}^{-i}(\Lambda_1)) \otimes_{\mathbb{K}} \text{Hom}_{D^b(\Lambda_2 - \text{mod})}(\Lambda_2, \nu_{d_{\Lambda_2}}^{-i}(\Lambda_2)) = \Pi(\Lambda_1) \otimes_{\mathbb{K}} \Pi(\Lambda_2).$$

We also prove the K"unneth formula for the Segre product with the use of the following lemma.
**Lemma 9.3.** The Segre product $-\otimes_{K}-$ is bi-exact.

**Proof.** Let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \quad (9.3.1)$$

be an exact sequence of graded $\Lambda_2$-modules. Let $M$ be a graded $\Lambda_1$-module. We have to show that

$$0 \rightarrow M \otimes_{K} X \rightarrow M \otimes_{K} Y \rightarrow M \otimes_{K} Z \rightarrow 0$$

is exact. It is enough to show that the sequence is exact at all degrees, i.e.

$$0 \rightarrow (M \otimes_{K} X)_i \rightarrow (M \otimes_{K} Y)_i \rightarrow (M \otimes_{K} Z)_i \rightarrow 0$$

but then note from the definition of the Segre product that this is equivalent to taking the tensor product over $K$, since $(M \otimes_{K} -)_i = M_i \otimes_{K} (-)_i$, which is exact, thus $M \otimes_{K} -$ is exact. A similar argument can show that $-\otimes_{K} -$ is exact in the first argument. \hfill \Box

Now we want to extend this functor to complexes. As with the usual tensor product, we extend the definition to complexes by taking the total complex of a certain double complex.

**Definition 9.4.** Let $\Lambda_1$ and $\Lambda_2$ be two graded $K$-algebras. Let $X \in C(\Lambda_1)$ and $Y \in C(\Lambda_2)$ be two complexes, then consider the double complex

$$\cdots \rightarrow X_i \otimes_{K} Y_i \xrightarrow{d_X \otimes 1_Y} X_{i-1} \otimes_{K} Y_i \xrightarrow{1_X \otimes d_Y} X_{i-1} \otimes_{K} Y_{i-1} \rightarrow \cdots$$

We define

$$\text{Tot}(-\otimes_{K}-) : C(\Lambda_1) \times C(\Lambda_2) \rightarrow C(\Lambda_1) \otimes_{K} C(\Lambda_2)$$

such that $\text{Tot}(X \otimes_{K} Y)$ is the total complex of (9.4.1).

**Theorem 9.5.** ([CE56] Theorem VI.3.1) Let $\Lambda_1$ and $\Lambda_2$ be two graded $K$-algebras. Let $X \in C(\Lambda_1)$ and $Y \in C(\Lambda_2)$ be two complexes. Then

$$H(\text{Tot}(X \otimes_{K} Y)) \cong \text{Tot}(H(X) \otimes_{K} H(Y)).$$

Explicitly

$$H_n(\text{Tot}(X \otimes_{K} Y)) \cong \bigoplus_{i+j=n} H_i(X) \otimes_{K} H_j(Y).$$

**Proof.** Setting $T = -\otimes_{K}-$ in [CE56] Theorem IV.8.1] together with Lemma 9.3 yields the result. \hfill \Box
10. Almost Koszul Complex for Higher Species

If we have an \( l \)-homogeneous \( d_i \)-representation finite Koszul algebra \( \Lambda_i \) for each \( i \in \{1, 2\} \), then we have seen earlier that \( \Lambda_1 \otimes \mathbb{K} \Lambda_2 \) is a \( (d_1 + d_2) \)-representation finite \( l \)-homogeneous Koszul algebra and by Theorem 8.4 the higher preprojective algebra \( \Pi(\Lambda_1 \otimes \mathbb{K} \Lambda_2) \) is almost Koszul. In this section we describe the almost Koszul complexes in \( \Pi(\Lambda_1 \otimes \mathbb{K} \Lambda_2) \) using the almost Koszul complexes in \( \Pi(\Lambda_1) \) and \( \Pi(\Lambda_2) \). This is formulated in Theorem 10.9. Before doing so we need to introduce some preliminary notions.

**Definition 10.1.** Let \( \mathcal{A} \) be an abelian category. Let \( f : Q \to R \) be a morphism between two complexes in \( C(A) \). Then we define the mapping cone of \( f \) as

\[
C(f)_i = Q_{i-1} \oplus R_i
\]

with the differential

\[
d_i^{C(f)} = \begin{bmatrix} -d_i^Q & 0 \\ f_{i-1} & d_i^R \end{bmatrix}.
\]

Let \( C^b(A, m) \) be the full subcategory of \( C^b(A) \) consisting of complexes of length \( m \), i.e. if \( X \in C^b(A, m) \) then

\[
X_i = X_m \to X_{m-1} \to \cdots \to X_0 \to 0.
\]

**Definition 10.2.** Let \( \mathcal{A} \) be an abelian category and let \( f : Q \to R \) be a morphism in \( C^b(A, m) \). We say that \( f \) is an almost quasi-isomorphism if \( H_i(f) \) is an isomorphism when \( 0 < i < m \), and \( H_0(f) \) and \( H_m(f) \) are a monomorphism and an epimorphism respectively.

**Lemma 10.3.** Let \( \mathcal{A} \) be an abelian category, and let \( f : Q \to R \) be a morphism in \( C^b(A, m) \). Then \( H_i(C(f)) = 0 \) for \( 0 < i < m \) if and only if \( f \) is an almost quasi-isomorphism.

**Proof.** We can assume \( \mathcal{A} \) is a module category of some ring due to the full imbedding theorem [Mit64, Theorem 4.4]. First we assume that \( f \) is an almost quasi-isomorphism. It is enough to show that for all \( i, H_i(C(f)) = 0 \) if \( H_{i-1}(f) \) is a monomorphism and \( H_i(f) \) is an epimorphism. Let \( (x, y) \in \ker(d_i^{C(f)}) \subset Q_{i-1} \oplus R_i \). This means that

\[
d_i^Q(x) = 0
\]

\[
f_{i-1}(x) + d_i^R(y) = 0
\]

The second equation says that \( f_{i-1}(x) \in \text{Im}(d_i^R) \). Using the fact that \( H_{i-1}(f) \) is a monomorphism yields that \( x \in \text{Im}(d_i^Q) \). Let \( s \in Q_i \) be such that \( -d_i^Q(s) = x \). Then \( f_i(s) - y \in \ker(d_i^R) \) as is easily shown by the following computation

\[
d_i^R(f_i(s) - y) = f_{i-1}(d_i^Q(s)) - d_i^R(y) = -f_{i-1}(x) - d_i^R(y) = 0.
\]

Since \( H_i(f) \) is an epimorphism there exists a \( z \in \ker(d_i^R) \) such that

\[
f_i(z) = f_i(s) - y + d_i^{R_{i+1}}(t)
\]

for some \( t \in R_{i+1} \). By construction we have that \( d_i^{C(f)}(s - z, t) = (x, y) \), and thus \( H_i(C(f)) = 0 \).

Now for the other direction, assume that \( H_i(C(f)) = 0 \) if \( 0 \leq i < m \). It is enough to show that \( H_{i-1}(f) \) and \( H_i(f) \) are a monomorphism and an epimorphism, respectively. Assume for contradiction that \( H_{i-1}(f) \) is not a monomorphism. This means that we can choose an \( x \in \ker(d_{i-1}^Q) \) such that \( x \notin \text{Im}(d_i^Q) \) and \( H_{i-1}(f)(x) = 0 \). Then there exists \( y \in R_i \) such that \( (x, y) \in \ker(d_i^{C(f)}) \subset Q_{i-1} \oplus R_i \). By construction, \( (x, y) \notin \text{Im}(d_i^{C(f)}) \), contradicting the assumption \( H_i(C(f)) = 0 \). We have now shown that \( H_{i-1}(f) \) is a monomorphism for all \( 0 < i \leq m \). We assume for contradiction that \( H_i(f) \) is not an epimorphism. Then there is a \( y \in \ker(d_i^R) \setminus \text{Im}(d_i^{R_{i+1}}) \) such that \( y \notin f_i(\ker(d_{i+1}^Q)) \). By the construction of \( y \) we see that \( y \notin \text{Im}(d_i^{R_{i+1}}) + \text{Im}(f_i) \), therefore \( (0, y) \notin \text{Im}(d_i^{C(f)}) \) and \( (0, y) \in \ker(d_i^{C(f)}) \), contradicting the fact that \( H_i(C(f)) = 0 \). Thus \( f \) is an almost quasi-isomorphism. \( \square \)
In \cite{GI20} $\Pi(\Lambda)$ is described as $\Pi(\Lambda) = T_\Lambda(E)$, where $E = \text{Ext}^d_{\Lambda}(DA, \Lambda)$ which coincides with our definition as $\text{Ext}^d_{\Lambda}(DA, \Lambda) \cong \tau_d^{-1}A \cong \text{Hom}_{D\Lambda}(\Lambda-\text{mod})(A, \nu_d^{-1}(\Lambda))$. In the case when $\Lambda$ is Koszul we can compute $E$ by a graded projective resolution and so $E$ is graded and generated in degree $-d$ \cite[Proposition 4.16]{GI20}. This gives a natural $\mathbb{Z}^2$-grading on $\Pi(\Lambda) = T_\Lambda(E)$ where

$$\Pi(\Lambda)_{jk} = (E^{\otimes k})_{lj}.$$ 

In order to get a grading where $\text{rad}(\Pi(\Lambda))$ is the positive degree part and $\Pi(\Lambda)/\text{rad}(\Pi(\Lambda))$ is the degree 0 part. Grant and Iyama introduce the $(d+1)$-total grading by

$$\Pi(\Lambda) = \bigoplus_{(d+1)k+j=l} \Pi(\Lambda)_{jk}$$ 

so that $\Pi(\Lambda)_0 = \Lambda_0$ and $\Pi(\Lambda)_1 = \Lambda_1 \oplus E_{-d}$ and so on. From now on, when $\Lambda$ is Koszul we always consider $\Pi(\Lambda)$ as $\mathbb{Z}^2$-graded

$$\Pi(\Lambda) = \bigoplus_{l,k \geq 0} \Pi(\Lambda)_{lk},$$ 

where $l$ refers to the $(d+1)$-total grading and $k$ refers to the $*$-grading.

Note that if $\Lambda = T(S)$ where $S$ is some representation finite species, then, using Definition 4.4, the grading on $\Pi(\Lambda)$ is such that the elements in $\overline{M_\alpha}$ and $\overline{M_\alpha^*}$ are of degree $(1, 0)$ and $(1, 1)$ respectively for all $\alpha \in Q_1$.

Now let $\Lambda = \Lambda_1 \otimes_K A_2$ and $d = d_1 + d_2$. The Segre product $\Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2)$ with respect to the $*$-grading has in addition a grading coming from the $(d_1+1)$- respectively $(d_2+1)$-total grading on $\Pi(\Lambda_1)$ and $\Pi(\Lambda_2)$ respectively. However, it does not correspond to the $(d+1)$-total grading on $\Pi(\Lambda_1 \otimes_K A_2)$. To fix this we regard $\Pi(\Lambda_1 \otimes_K A_2)$ as a $\mathbb{Z}^2$-graded by

$$(\Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2))_{lk} = \bigoplus_{l_1+l_2-k=l} \Pi(\Lambda_1)_{l_1,k} \otimes_K \Pi(\Lambda_2)_{l_2,k},$$

where $l_i$ refers to the $(d_i+1)$-total grading on $\Pi(\Lambda_i)$ and $k$ refers to the $*$-grading.

**Proposition 10.4.** Let $\Lambda_i$ be a $d_i$-representation finite $l_i$-homogeneous Koszul algebra for each $i \in \{1, 2\}$. Then

$$\Pi(\Lambda_1 \otimes_K A_2) \cong \Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2)$$

as $\mathbb{Z}^2$-graded algebras.

**Proof.** The proof is similar to Lemma 9.2. In fact letting $E_i = \text{Ext}^{d_i}_{\Lambda_i}(DA_i, \Lambda_i) \cong \text{Hom}_{D\Lambda_i}(\Lambda_i-\text{mod})(A_i, \nu_d^{-1}(\Lambda_i))$ and $E = \text{Ext}^d_{\Lambda}(DA, \Lambda) \cong \text{Hom}_{D\Lambda}(\Lambda-\text{mod})(A, \nu_d^{-1}A)$ the isomorphism

$$\text{Hom}_{D\Lambda}(\Lambda-\text{mod})(A_1, \nu_d^{-1}(\Lambda_1)) \otimes_K \text{Hom}_{D\Lambda_2}(\Lambda_2-\text{mod})(A_2, \nu_d^{-1}(\Lambda_2)) \cong \text{Hom}_{D\Lambda}(\Lambda-\text{mod})(A, \nu_d^{-1}A)$$

yields an isomorphism $E_1 \otimes E_2 \cong E$ of $\Lambda$-$\Lambda$-bimodules. Thus we have an isomorphism

$$T_\Lambda(E) \cong T_\Lambda(E_1 \otimes_K E_2) = T_{\Lambda_1}(E_1) \otimes_K T_{\Lambda_2}(E_2)$$

which is compatible with the natural $\mathbb{Z}^2$-gradings on $T_\Lambda(E)$ and $T_{\Lambda_1}(E_1) \otimes_K T_{\Lambda_2}(E_2)$. The above isomorphism induces

$$\Pi(\Lambda)_{lk} = \bigoplus_{(d+1)k+j=l} T_\Lambda(E)_{jk} \cong \bigoplus_{(d+1)k+j=l} \bigoplus_{j_1+j_2=j} T_{\Lambda_1}(E_1)_{j_1,k} \otimes_K T_{\Lambda_2}(E_2)_{j_2,k} = \bigoplus_{(d+1)k+j_1+j_2=l} T_{\Lambda_1}(E_1)_{j_1,k} \otimes_K T_{\Lambda_2}(E_2)_{j_2,k}.$$ 

But $(d+1)k + j_1 + j_2 = l$ if and only if $(d_1+1)k + j_1 + (d_2+1)k + j_2 - k = l$, so for $l_i = (d_i+1)k + j_i$ we have

$$\Pi(\Lambda)_{jk} = \bigoplus_{l_i+l_2-k=l} \Pi(\Lambda)_{l_i,k} \otimes_K \Pi(\Lambda)_{l_2,k}.$$
as desired. \hfill \Box

When \( \Lambda \) is an \( d \)-representation finite algebra Pasquali studied \( d \)-almost split sequences and proved the following result.

**Theorem 10.5.** [Pas17 Theorem 2.4] Let \( \Lambda \) be a \( d \)-representation finite \( \mathbb{k} \)-algebra, and let \( C_{d+1} \in \text{add}(\tau_{-i}^{-1}A) \) be an indecomposable non-injective module for some integer \( i \in \mathbb{Z} \), and let

\[
C_{\bullet} = 0 \rightarrow C_{d+1} \xrightarrow{f_{d+1}} C_d \rightarrow \cdots \rightarrow C_1 \xrightarrow{f_1} C_0 \rightarrow 0
\]

be the corresponding \( d \)-almost split sequence. Then there are complexes \( Q_{\bullet} \in \mathcal{C}(\text{add}(\tau_{-i}^{-1}A)) \), \( R_{\bullet} \in \mathcal{C}(\text{add}(\tau_{-i}^{-1}(1+i)A)) \) and a morphism \( \varphi : Q_{\bullet} \rightarrow R_{\bullet} \) such that \( C_{\bullet} = C(\varphi) \).

**Remark 10.6.** Note that the morphism \( \varphi \) in Theorem 10.5 is of \( \ast \)-degree 1 which directly follows from the definition of \( \ast \)-degree. In fact, given a morphism \( \varphi \) Pasquali wrote down criterion when \( C(\varphi) \) is a \( d \)-almost split sequence which is formulated in [Pas17 Lemma 2.7]. For the purpose in this article we only need existence of \( \varphi \) for a given \( d \)-almost split sequence.

As in [BBK02], we consider almost split sequences in order to construct almost Koszul complexes. Grant and Iyama generalized this idea in [GI20] to \( d \)-representation finite algebras. They introduced the functor

\[
H^Z = \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{D^b(\Lambda - \text{mod}^Z)}(-, \nu_d^i \Lambda(j)) : \mathcal{U}^Z \rightarrow \text{mod}^Z - \Pi,
\]

where

\[
\mathcal{U}^Z = \text{add}\{\nu_d^i \Lambda(j) \mid i,j \in \mathbb{Z}\} \subset D^b(\Lambda - \text{mod}^Z).
\]

Note that \( H^Z(\Lambda) = \Pi(\Lambda) \in \text{mod}^Z - \Pi(\Lambda) \) so the essential image of \( H^Z \) is exactly the projective objects in \( \text{mod}^Z - \Pi(\Lambda) \). However, the \( \mathbb{Z}^2 \)-grading on \( H^Z(\Lambda) \) does not correspond to the \( \mathbb{Z}^2 \)-grading in Remark 10.4 since the action of \( (i,j) \) on \( H^Z \) is given by \( \nu_d^i \mu_d^j \). In order to relate these two gradings, we introduce the functor

\[
F' : \text{mod}^Z - \Pi(\Lambda) \rightarrow \text{mod}^Z - \Pi(\Lambda)
\]

\[
\bigoplus_{i,j \in \mathbb{Z}} X_{i,j} \mapsto \bigoplus_{s,t \in \mathbb{Z}} X'_{s,t}
\]

where \( X'_{s,t} = X_{-s,(d+1)t-s} \). Then the first part of the grading will correspond to the \((d+1)\)-total grading and the second part will correspond to the \( \ast \)-grading. In the following theorem we are only interested in the \( \ast \)-grading, and thus we introduce another functor

\[
F : \text{mod}^Z - \Pi(\Lambda) \rightarrow \text{mod}^Z - \Pi(\Lambda)
\]

\[
\bigoplus_{i,j \in \mathbb{Z}} X_{i,j} \mapsto \bigoplus_{t \in \mathbb{Z}} \left( \bigoplus_{s \in \mathbb{Z}} X'_{s,t} \right)
\]

In other words, \( F \) forgets the \((d+1)\)-total grading.

**Theorem 10.7.** Let \( \Lambda \) be an acyclic \( d \)-representation finite Koszul algebra and let \( S \) be a simple \( \Pi(\Lambda) - \text{mod}^Z \), then there exist complexes \( R_{\bullet}, Q_{\bullet} \in \mathcal{C}(\Pi(\Lambda) - \text{mod}^Z, d) \) and an almost quasi-isomorphism \( \varphi : Q_{\bullet} \rightarrow R_{\bullet} \) such that \( \deg^\ast(\varphi) = 1 \) and \( C(\varphi) \) is the almost Koszul complex for \( S \).

**Proof.** We follow the proof of [GI20 Theorem 4.21]. They showed that for each simple \( S \in \Pi - \text{mod}^Z \) there is a \( d \)-almost split sequence

\[
0 \rightarrow C_{d+1} \xrightarrow{f_{d+1}} C_d \rightarrow \cdots \rightarrow C_1 \xrightarrow{f_1} C_0 \rightarrow 0,
\]

in \( \mathcal{U}^Z \) such that

\[
F \circ H(C_{d+1}) \xrightarrow{F \circ H(f_{d+1})} F \circ H(C_d) \rightarrow \cdots \rightarrow F \circ H(C_1) \xrightarrow{F \circ H(f_1)} F \circ H(C_0) \rightarrow 0.
\]
is the almost Koszul complex for $S$. In particular, it only has non-zero homology in position 0 and $d + 1$.

Since the functors $F$ and $H$ are additive, it is enough to show that (10.7.1) can be written as a mapping cone of some morphism $\varphi : \mathcal{C}_i \to R_i$. Moreover, $Q_i \in \text{add}(\nu_d^{-1})$ and $R_i \in \text{add}(\nu_d^{-1+1})$ and $\deg(\varphi) = 1$. We know that [1] is an equivalence and therefore every $d$-almost split sequence with terms in $(\Lambda - \text{mod}^\circ)[k]$ can also be written as a mapping cone of some morphism which is homogeneous of $\varphi$-degree 1 for every $k \in \mathbb{Z}$. Now, since $\nu_d$ is an equivalence, it is left to show that for every $d$-almost sequence $C_i$ in $\mathcal{U}^\circ$ ending in a projective module $P \in \Lambda - \text{mod}^\circ$, i.e. $C_{d+1} = P$, there exists an integer $i \in \mathbb{Z}$ such that $\nu_i(C_i)$ is a $d$-almost split sequence with terms in $(\Lambda - \text{mod}^\circ)[k]$ for some integer $k \in \mathbb{Z}$. When $P$ is not an injective module it is clear that $C_i$ is a $d$-almost split sequence in $\Lambda - \text{mod}^\circ$. Now suppose that $P = P_i = \Lambda e_i$ is a graded projective injective $\Lambda$-module. Then $\nu_i P_i = P_{\sigma(i)}[-d]$. If we can show that there exists $j \in \mathbb{Z}$ such that $P_{\sigma(i)}$ is not an injective module we are done. Assume that such an integer $j \in \mathbb{Z}$ does not exist. In other words, $P_{\sigma(i)}$ is a graded projective-injective $\Lambda$-module for all $j \in \mathbb{Z}$. Since $\Lambda$ is a finite dimensional $\mathbb{K}$-algebra, we know that there exists an integer $n$ such that $\sigma^n(i) = i$, and because we have non-zero morphisms

$$P_{\sigma(i)} \to \text{Top}(P_{\sigma(i)}) = \text{Soc}(I_{\sigma(i)}) \to I_{\sigma(i)} = P_i$$

which is a contradiction since $\Lambda$ is acyclic. Hence (10.7.1) can be written as a mapping cone of some morphism $\varphi$ with $\deg(\varphi) = 1$.

Finally observe $F \circ H(C(\varphi)) = C(F \circ H(\varphi))$, which holds because $F$ and $H$ are additive. Now from the definition of $F$ and $H$ we see that $\deg(F \circ H(\varphi)) = 1$. The fact that $F \circ H(\varphi)$ is an almost quasi-isomorphism comes from the fact that (10.7.2) only has non-zero homology at position 0 and $d + 1$ and Lemma 10.3.

It is still unclear if we need to assume $\Lambda$ is acyclic since there are no known examples of a $d$-representation finite algebra $\Lambda$ which is not an acyclic algebra. Therefore we make the following conjecture.

**Conjecture 10.8.** Theorem 10.7 still holds if we drop the acyclic assumption on $\Lambda$.

**Theorem 10.9.** Let $\Lambda_i$ be an acyclic $d_i$-representation finite $l$-homogeneous Koszul algebra for each $i \in \{1, 2\}$. Let $S_{\Lambda_i} : \Pi(\Lambda_i)$-mod be a simple $\Pi(\Lambda_i)$-module and let $\varphi^{\Lambda_i} : Q^{\Lambda_i} \to R^{\Lambda_i}$ be an almost quasi-isomorphism as in Theorem 10.7 such that $C(\varphi^{\Lambda_i})$ is the almost Koszul complex for $S_{\Lambda_i}$. The complex

$$C(\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2}) : \text{Tot}(Q_{\Lambda_1} \otimes \mathbb{K} Q_{\Lambda_2}) \to \text{Tot}(R_{\Lambda_1} \otimes \mathbb{K} R_{\Lambda_2}))$$

is the almost Koszul complex for $S_{\Lambda_1} \otimes \mathbb{K} S_{\Lambda_2} \in \Pi(\Lambda_1 \otimes \mathbb{K} \Lambda_2) - \text{mod}^\circ$.

**Proof.** Existence of $\varphi^{\Lambda_i} : Q^{\Lambda_i} \to R^{\Lambda_i}$ is due to Theorem 10.7. Lemma 9.2 implies that $C(\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2})) \in \text{C}(\Pi(\Lambda_1 \otimes \mathbb{K} \Lambda_2) - \text{proj})$, since $C(\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2}))_i \in \text{add}(\Pi(\Lambda_1 \otimes \mathbb{K} \Pi(\Lambda_2)))$ for all $i \in \mathbb{Z}$. We show that $\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2})$ is an almost quasi-isomorphism. First note that $\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2})_0 = \varphi_0^{\Lambda_1} \otimes \varphi_0^{\Lambda_2}$ and $\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2})_{d_{A_1} + d_{A_2} + 1} = \varphi_{d_{A_1} + 1}^{\Lambda_1} \otimes \varphi_{d_{A_2}}^{\Lambda_2}$ and so on the level of homology they induce a monomorphism and an epimorphism respectively. Now let $0 < i < d_{A_1} + d_{A_2} + 1$. We need to show that

$$\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2})_i = \begin{bmatrix}
\varphi_0^{\Lambda_1} \otimes \varphi_0^{\Lambda_2} \\
\varphi_1^{\Lambda_1} \otimes \varphi_1^{\Lambda_2} \\
\vdots \\
\varphi_{i-1}^{\Lambda_1} \otimes \varphi_{i-1}^{\Lambda_2} \\
\varphi_i^{\Lambda_1} \otimes \varphi_i^{\Lambda_2}
\end{bmatrix}$$

induces an isomorphism on the level of homology. Theorem 9.5 tells us that

$$H_i(\text{Tot}(\varphi^{\Lambda_1} \otimes \varphi^{\Lambda_2})) : \bigoplus_{j+k=i} H_j(Q_{\Lambda_1}^\circ) \otimes \mathbb{K} H_k(Q_{\Lambda_2}^\circ) \to \bigoplus_{j+k=i} H_j(R_{\Lambda_1}^\circ) \otimes \mathbb{K} H_k(R_{\Lambda_2}^\circ),$$
and since $\varphi^A$ and $\varphi^A_2$ are both almost quasi-isomorphisms, it is enough to show that all $\varphi^A_0 \otimes \varphi^A_2$, $\varphi^A_1 \otimes \varphi^A_2$, $\varphi^A_{d_1} \otimes \varphi^A_2$ and $\varphi^A_1 \otimes \varphi^A_{d_2}$ induces an isomorphism on the level of homology. This is due to the fact that $\varphi^A_0$, $\varphi^A_{d_1}$, and $\varphi^A_{d_2}$ are the only morphisms that do not induce isomorphisms on the level of homology. Let us start with $\varphi^A_1 \otimes \varphi^A_2$. First note that elements in $H_1(R^A_2)$, for $i \neq 0$, have $\ast$-degree at least 1 since

$$H_1(\varphi^A_2) : H_1(Q_2^\ast) \to H_1(R_2^\ast)$$

and the fact that $\deg^\ast(\varphi^A_2) = 1$. Therefore the elements in $H_0(R^A_1) \otimes H_1(R^A_2)$ have $\ast$-degree at least 1. Also note that $H_0(\varphi^A_1)$ surjects onto the $\ast$-degree at least 1 part of $H_0(R^A_1)$ because

$$H_0(C(\varphi^A_1)) = \ker(H_0(\varphi^A_1))$$

is a simple concentrated at $\ast$-degree 0. Thus using that $H_0(\varphi^A_1)$ is a monomorphism we get that $\varphi^A_1 \otimes \varphi^A_2$ induces an isomorphism on the level of homology. A similar argument shows that $\varphi^A_1 \otimes \varphi^A_2$ induces an isomorphism on the level of homology.

For the other two cases, $\varphi^A_{d_1} \otimes \varphi^A_2$ and $\varphi^A_1 \otimes \varphi^A_{d_2}$ we need that $L_i$ is $l$-homogeneous. The $l$-homogeneous property of $L_i$ ensures that the socle of each projective module is in $\ast$-degree $l - 1$, and therefore we can use the dual version of the argument as follows. We only show that $\varphi^A_1 \otimes \varphi^A_2$ induces an isomorphism on the level of homology since the argument for $\varphi^A_1 \otimes \varphi^A_2$ is similar. The elements in $H_i(Q_2^\ast)$, for $i \neq d_{\Lambda_1}$, have $\ast$-degree at most $l - 1$ since

$$H_i(\varphi^A_2) : H_i(Q_2^\ast) \to H_i(R_2^\ast)$$

and the fact that $\deg^\ast(\varphi^A_2) = 1$. Thus the elements in $H_{d_{\Lambda_1}}(R^A_1) \otimes H_{d_{\Lambda_2}}(R^A_2)$ have $\ast$-degree at most $l - 1$. Now using that

$$H_{d_{\Lambda_1}}(\varphi^A_1) = \ker(H_{d_{\Lambda_1}}(\varphi^A_1))$$

is concentrated in $\ast$-degree $l$, the morphism $\varphi^A_{d_1} \otimes \varphi^A_2$ will induce an isomorphism on the level of homology. Thus $\text{Tot}(\varphi^A_1 \otimes \varphi^A_2)$ is an almost quasi-isomorphism.

To conclude that this in fact is the almost Koszul complex for $S^{\Lambda_1} \otimes_{K} S^{\Lambda_2}$ we need to show that

$$H_0(C(\text{Tot}(\varphi^A_1 \otimes \varphi^A_2))) = S^{\Lambda_1} \otimes_{K} S^{\Lambda_2},$$

and compute the homology at $d_{\Lambda_1} + d_{\Lambda_2} + 1$. This is shown by again using the fact that

$$H_0(\varphi^A_1) : H_0(Q^\ast_1) \to H_0(R^\ast_1)$$

surjects onto the $\ast$-degree $\geq 1$ part of $H_0(R^\ast_1)$ and the fact that

$$H_0(C(\text{Tot}(\varphi^A_1 \otimes \varphi^A_2))) = \ker(H_0(\varphi^A_1) \otimes H_0(\varphi^A_2)).$$

Note that we did not have to use the fact that both $\Lambda_1$ and $\Lambda_2$ are $l$-homogeneous to compute the homology at 0, but to compute the homology at $d_{\Lambda_1} + d_{\Lambda_2} + 1$, we need to use $l$-homogeneous property to ensure that elements in $H_{d_{\Lambda_1}}(C(\varphi^A_1))$ and $H_{d_{\Lambda_2}}(C(\varphi^A_2))$ have $\ast$-degree $l$. We want to show that

$$\ker(H_{d_{\Lambda_1} + d_{\Lambda_2} + 1}(C(\text{Tot}(\varphi^A_1 \otimes \varphi^A_2)))) = H_{d_{\Lambda_1}}(C(\varphi^A_1)) \otimes_{K} H_{d_{\Lambda_2}}(C(\varphi^A_2)).$$

(10.9.1)

First note that

$$\overline{H}_{d_{\Lambda_1}}(\varphi^A_1) : H_{d_{\Lambda_1}}(Q^\ast_1) / H_{d_{\Lambda_1} + 1}(C(\varphi^A_1)) \cong H_{d_{\Lambda_1}}(R^\ast_1)$$

is an isomorphism, and since both $\Lambda_1$ and $\Lambda_2$ are $l$-homogeneous the morphism

$$\overline{H}_{d_{\Lambda_1} + d_{\Lambda_2}}(\text{Tot}(\varphi^A_1 \otimes \varphi^A_2)) : (H_{d_{\Lambda_1}}(Q^\ast_1) \otimes_{K} H_{d_{\Lambda_2}}(Q^\ast_2)) / (H_{d_{\Lambda_1} + 1}(C(\varphi^A_1)) \otimes_{K} H_{d_{\Lambda_2} + 1}(C(\varphi^A_2))) \cong H_{d_{\Lambda_1} + d_{\Lambda_2}}(R^\ast_1 \otimes R^\ast_2)$$

is also an isomorphism. Hence (10.9.1) holds. □
Corollary 10.10. Let $\Lambda_i$ be an acyclic $d_i$-representation finite $l$-homogeneous Koszul algebra for each $i \in \{1, 2\}$. If $\Pi(\Lambda_i)$ is an $(p_i, q_i)$-almost Koszul algebra, then $\Pi(\Lambda_1 \otimes_K \Lambda_2)$ is an $(p_1 + p_2 - l + 2, q_1 + q_2 - 1)$-almost Koszul algebra.

Proof. Let $\varphi^{A_1} : Q^{A_1}_1 \to R^{A_1}_1$ be a morphism such that $C(\varphi)$ is the almost Koszul complex for $\Pi(\Lambda_i)_0$. By Theorem 10.9 $C(Tot((\varphi^{A_1} \otimes \varphi^{A_2}))$ is the almost Koszul complex for $\Pi(\Lambda_1 \otimes_K \Lambda_2)_0$. Since $q_i$ describes the length of the almost Koszul complex, we can use the definition of $Tot(- \otimes_K -)$ to find the length of the almost Koszul complex $C(Tot((\varphi^{A_1} \otimes \varphi^{A_2}))$ to be $q_1 + q_2 - 1$.

By Theorem 10.9 the last projective module of the almost Koszul complex

$$0 \to Q^{A_1}_1 \otimes_K Q^{A_2}_1 \to \cdots \to R^{A_1}_1 \otimes_K R^{A_2}_1 \to C(\varphi^{A_1} \otimes \varphi^{A_2}) \to 0$$

is the almost Koszul complex for $\Pi(\Lambda_1 \otimes_K \Lambda_2)_0$. Here

$$f = \begin{bmatrix} \varphi^{A_1}_1 \otimes \varphi^{A_2}_1 \\ \varphi^{A_1}_q \otimes \varphi^{A_2}_2 \end{bmatrix}.$$ 

Thus we can compute the kernel as

$$\ker f = \ker(\varphi^{A_1}_1 \otimes \varphi^{A_2}_q) \cap \ker(1_{R^{A_1}_1 \otimes_K R^{A_2}_q} \circ C(\varphi^{A_2})) \cap \ker(\varphi^{A_1}_q \otimes 1_{Q^{A_2}_q})$$

and hence $\ker f = H_{q_1}(C(\varphi^{A_1})) \otimes_K H_{q_2}(C(\varphi^{A_2}))$. The algebras $\Lambda_1$ and $\Lambda_2$ being $l$-homogeneous ensures that $H_{q_1}(C(\varphi^{A_1}))$ and $H_{q_2}(C(\varphi^{A_2}))$ are concentrated in $*$-degree $l$, which implies that

$$\ker f = H_{q_1}(C(\varphi^{A_1})) \otimes_K H_{q_2}(C(\varphi^{A_2})) = H_{q_1}(C(\varphi^{A_1})) \otimes_K H_{q_2}(C(\varphi^{A_2}))$$

It is left to show that $\deg \ker f = p_1 + p_2 + q_1 + q_2 - l + 1$, but this follows from Proposition 10.4. \qed

Proposition 10.11. Let $\Lambda_i$ be a $d_i$-representation finite $l$-homogeneous Koszul algebra. Let $\gamma_i$ be the Nakayama automorphism for $\Pi(\Lambda_i)$. Then $\gamma_1 \otimes \gamma_2$ is the Nakayama automorphism for $\Pi(\Lambda_1 \otimes_K \Lambda_2)$.

Proof. By Lemma 9.2 we have to show that

$$\Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2) \cong D(\Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2))_{\gamma_1 \otimes \gamma_2}. \quad (10.11.1)$$

First recall that the Nakayama automorphism is an isomorphism of graded modules, where the grading is defined via the path length. This is seen as an application of [1111 Proposition 2.4] together with the fact that inner automorphisms does not change the grading. Here we defined the grading on $DII(\Lambda_i)$ to be the induced grading from $\Pi(\Lambda_i)$. We want to ensure that the Nakayama automorphism is a morphism of $*$-graded modules, in particular, we want that $\deg^* (\gamma_i) = 0$. Since $\Lambda_1$ and $\Lambda_2$ are $l$-homogeneous we can apply [1111 Theorem 2.3] to show $\deg^*(\gamma_i) = 0$. Thus $\deg^*(\gamma_1 \otimes \gamma_2) = 0$, which implies that $\gamma_1 \otimes \gamma_2$ will be an automorphism on $\Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2)$. At degree $i$ we have

$$(\Pi(\Lambda_1) \otimes_K \Pi(\Lambda_2))_i \cong (\Pi(\Lambda_1)_i \otimes_K \Pi(\Lambda_2)_i) \cong (DII(\Lambda_1)_{\gamma_1})_i \otimes_K (DII(\Lambda_2)_{\gamma_2})_i \cong (DII(\Lambda_1)_{\gamma_1} \otimes_K DII(\Lambda_2)_{\gamma_2})_i \cong (DII(\Lambda_1)_{\gamma_1} \otimes_K DII(\Lambda_2)_{\gamma_2})_{\gamma_1 \otimes \gamma_2}.$$ 

Since it holds for every $i \in \mathbb{Z}$, we can conclude that (10.11.1) holds. \qed

11. Computations for Tensor Products of Species with Relations

We devote this section to examples we get when we apply the theory we developed in this paper. We can generate a lot of examples if we start with representation finite species. It is important that these species are $l$-homogeneous due to Corollary 8.3 and thanks to Corollary 3.7 we have complete set of $l$-homogeneous representation finite species. For all representation finite species $S$, we have an explicit description of the almost Koszul complexes for simple $\Pi(S)$-modules and the Nakayama automorphism. By Corollary 8.3 tensor products
between representation finite \( l \)-homogeneous species \( S^1 \) and \( S^2 \) are 2-representation finite \( l \)-homogeneous species with relations which we will describe completely in Corollary 11.7. Applying Theorem 8.4 we know that \( \Pi(T(S^1) \otimes_K T(S^2)) \) is an almost Koszul algebra. Since we established earlier that we already know the almost Koszul complexes for all representation finite species, we can apply Theorem 10.9 to fully describe the almost Koszul complexes for the simple \( \Pi(T(S^1) \otimes_K T(S^2)) \)-modules. We also have a description of the Nakayama automorphism of \( \Pi(T(S^1) \otimes_K T(S^2)) \) due to Proposition 10.11. Let us now compute some explicit examples to illustrate this.

**Example 11.1.** Let \( S \) be the species in Example 4.6 over \( Q : 1 \rightarrow 2 \rightarrow 3 \). The Auslander-Reiten quiver \( \Gamma_S \) is

\[
\begin{array}{c}
P_1 \xrightarrow{2} P_2 \xrightarrow{\tau^{-1}P_1} I_1 \\
P_2 \xrightarrow{2} \tau^{-1}P_2 \xrightarrow{2} I_2 \xrightarrow{\star} P_3 \xrightarrow{\tau^{-1}P_3} I_3 \xrightarrow{1} P_4 \xrightarrow{2} \tau^{-1}P_4 \xrightarrow{1} I_4
\end{array}
\]

By Proposition 6.9 the almost Koszul complex for the simple \( \Pi(S) \)-module \( D_1 \) is

\[
R_* : 0 \rightarrow P_1 \rightarrow P_2 \oplus P_2 \rightarrow P_1 \rightarrow 0,
\]

where \( H_0(R_*) = D_1 \) and \( H_2(R_*) = D_3 \). The map \( P_1 \rightarrow P_2 \) in (11.1.1) corresponds to the morphism \( P_1 \rightarrow \tau^{-1}P_2 \) in the Auslander-Reiten quiver \( \Gamma_S \), and so has \( \star \)-degree 1. Therefore we can view (11.1.1) as the mapping cone of the morphism

\[
\begin{array}{c}
0 \rightarrow P_1 \rightarrow 0 \\
0 \rightarrow P_2 \oplus P_2 \rightarrow P_1 \rightarrow 0.
\end{array}
\]

**Example 11.2.** Let \( Q \) be the quiver

\[
\begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \\
1 \rightarrow 2 \rightarrow 4
\end{array}
\]

and let \( S \) be a species over \( Q \) such that \( T(S) = \mathbb{R}Q \). The Auslander-Reiten quiver \( \Gamma_S \) is

\[
\begin{array}{c}
P_3 \xrightarrow{\tau^{-1}P_3} P_2 \xrightarrow{\tau^{-1}P_2} P_1 \rightarrow \tau^{-1}P_2 + \tau^{-1}P_1 \rightarrow I_2 \rightarrow I_1 \\
P_4 \xrightarrow{\tau^{-1}P_4} P_2 \xrightarrow{\tau^{-1}P_2} P_1 \rightarrow \tau^{-1}P_2 + \tau^{-1}P_1 \rightarrow I_2 \rightarrow I_1
\end{array}
\]

By Proposition 6.9 the almost Koszul complex for the simple \( \Pi(S) \)-module \( D_2 \) is

\[
R_* : 0 \rightarrow P_2 \rightarrow P_2 \oplus P_3 \oplus P_4 \rightarrow P_2 \rightarrow 0,
\]

where \( H_0(R_*) = D_2 \) and \( H_2(R_*) = D_2 \). As in Example 11.1 we can deduce that the morphism \( P_1 \rightarrow P_2 \) and \( P_2 \rightarrow P_3 \oplus P_4 \) are the only morphisms of \( \star \)-degree 1 by looking at \( \Gamma_S \). Hence we can view (11.2.1) as the
mapping cone of

\[
\begin{array}{c}
0 \\ \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \rightarrow P_2 \rightarrow P_1 \rightarrow 0 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \rightarrow P_3 \oplus P_4 \rightarrow P_2 \rightarrow 0.
\end{array}
\]

**Example 11.3.** Let \( S^1 \) and \( S^2 \) be the species from Example 11.1 and Example 11.2 respectively. Reading from the table in Corollary 3.7, or by studying their Auslander-Reiten quivers, we see that both \( S^1 \) and \( S^2 \) are 3-homogeneous. Therefore \( T(S^1) \otimes_K T(S^2) \) is a 3-homogeneous 2-representation finite algebra. We can use the almost Koszul complexes in Example 11.1 and Example 11.2 together with Theorem 10.9 to compute the almost Koszul complex for the simple module \( D_1^1 \otimes_K D_2^2 \) in \( \Pi(T(S^1) \otimes_K T(S^2)) - \text{mod} \). Let us define \( P_{ij} = P_{ij}^1 \otimes_K P_{ij}^2 \), then the almost Koszul complex is given as the mapping cone of

\[
\begin{array}{c}
0 \\ \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \rightarrow P_{12} \rightarrow P_{11} \rightarrow 0 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \rightarrow P_{23} \oplus P_{23} \oplus P_{24} \oplus P_{24} \rightarrow P_{22} \oplus P_{22} \oplus P_{13} \oplus P_{14} \rightarrow P_{12} \rightarrow 0.
\end{array}
\]

**Example 11.4.** Let \( S^1 \) and \( S^2 \) be the species from Example 11.1 and Example 11.2 respectively. Let \( S^3 \) be the species over the quiver

\[
Q^3 : 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5
\]

such that \( T(S^3) = \mathbb{R}Q \). Similarly as in Example 11.1 and Example 11.2, using Proposition 6.9 we have the almost Koszul complex for \( D_1^3 \) in \( \Pi(S^3) - \text{mod} \) described as the mapping cone of

\[
\begin{array}{c}
0 \\ \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \rightarrow P_4 \rightarrow P_5 \rightarrow 0 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \rightarrow P_3 \rightarrow P_3 \rightarrow 0.
\end{array}
\]

By Corollary 3.7, we have that the species \( S^3 \) is 3-homogeneous. Thus, by Corollary 8.3, the tensor product \( \Lambda \cong T(S^1) \otimes_K T(S^2) \otimes_K T(S^3) \) is a 3-homogeneous 3-representation finite algebra.

Let us now compute the almost Koszul complex for \( D_1^1 \otimes_K D_2^2 \otimes_K D_1^3 \) in \( \Pi(\Lambda) - \text{mod} \). Applying Theorem 10.9 we get the following almost Koszul complex for \( D_1^1 \otimes_K D_2^2 \otimes_K D_1^3 \) in \( \Pi(\Lambda) - \text{mod} \) as the mapping cone of

\[
\begin{array}{c}
0 \\ \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \rightarrow P_{124} \rightarrow P_{114} \oplus P_{225} \rightarrow P_{115} \rightarrow 0 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \rightarrow P_{233} \oplus P_{233} \oplus P_{243} \oplus P_{243} \rightarrow P_{223} \oplus P_{223} \oplus P_{133} \oplus P_{143} \oplus P_{244} \rightarrow P_{224} \oplus P_{224} \oplus P_{224} \oplus P_{134} \oplus P_{144} \rightarrow P_{124} \rightarrow 0
\end{array}
\]

where \( P_{ijk} = P_{ijk}^1 \otimes_K P_{ijk}^2 \otimes_K P_{ijk}^3 \).

We have now illustrated with some example how to compute the almost Koszul complex using Proposition 6.9 and Theorem 10.9. Let us now go back to Example 11.3 and compute the preprojective algebra more explicitly and describe its Nakayama automorphism. To describe the Segre products and tensor products between species with relations we need the following proposition.

**Proposition 11.5.** For \( k \in \{1, 2\} \), let \( S^k \) be a graded species as in Remark 4.5, with \( M^k = M_0^k \oplus M_1^k \). Moreover, let \( R^k \subset T(S^k) \) be a homogeneous ideal generated in degree 0 and 1. Then

\[
T(S^1)/R^1 \otimes_K T(S^2)/R^2 \cong T(S)/R,
\]

where \( T(S) = T(D, M) \), \( D = D^1 \otimes_K D^2 \) and \( M = (M_0^1 \otimes_K D^2) \oplus (D^1 \otimes_K M_0^2) \oplus (M_1^1 \otimes_K M_1^2) \), and

\[
R = (R_{00}^1 \otimes 1_{D^2}, 1_{D^1} \otimes R_{00}^1, R_{10}^1 \otimes K M_0^1, M_1^1 \otimes K R_1^2, | \alpha \otimes 1_{D^2}, 1_{D^1} \otimes \alpha' | : \alpha \in M_0^1, \alpha' \in M_0^2).
\]
where
\[ (\alpha \otimes 1_{D^2}, 1_{D^1} \otimes \alpha') = (\alpha \otimes 1_{D^2}) \otimes_{D^1 \otimes K} D^2 \cdot (1_{D^1} \otimes \alpha') - (\alpha \otimes 1_{D^2}) \otimes_{D^1 \otimes K} D^2 \cdot (1_{D^1} \otimes \alpha'). \]

**Remark 11.6.** In Definition 2.1 we require that \( D^1 \otimes_K D^2 \) is a direct sum of division rings, but in general it will only be Morita equivalent to a direct sum of division rings. Therefore we consider the more general definition of a species in Remark 2.3. For example, if \( D^1 = D^2 = \mathbb{H} \), then \( D^1 \otimes_K D^2 \) is isomorphic to the space of 4 \times 4-matrices over \( \mathbb{R} \), which in turn is Morita equivalent to \( \mathbb{R} \).

**Proof.** Since \( K \) is perfect we have that \( D^1 \otimes_K D^2 \) is semi-simple. First we check that \( S \) is a species. It is enough to show that \( S \) is dualisable. Let \( i \) be in \( Q_1 \) and \( m \) be in \( Q_1^* \). Since \( S^1 \) and \( S^2 \) are dualisable, we have graded isomorphisms
\[
\phi_i^1 : \text{Hom}_{D^1} (M^1_1, D^1_i) \rightarrow \text{Hom}_{D^1} (M^1_1, D^1_i)
\]
\[
\psi_i^2 : \text{Hom}_{D^1} (M^2_1, D^2_i) \rightarrow \text{Hom}_{D^1} (M^2_1, D^2_i)
\]

Now let \( \psi_{D^1} : \text{Hom}_{D^1} (D^1_1, D^1_2) \rightarrow \text{Hom}_{D^1} (D^2_1, D^2_2) \) be the isomorphism that sends \( L_d \) to \( R_d \), where \( L_d \) and \( R_d \) denote left and right multiplication by \( d \in D^k \) respectively, for all \( D^k \). Consider the following maps
\[
\phi_i^1 \otimes \psi_i^2 : \text{Hom}_{D^1 \otimes K} (M^1_1 \otimes_K D^1_1, D^2_i) \rightarrow \text{Hom}_{D^1 \otimes K} (M^1_1 \otimes_K D^2_1, D^2_i),
\]
\[
\psi_i^1 \otimes \phi_i^2 : \text{Hom}_{D^1 \otimes K} (M^2_1 \otimes_K D^2_1, D^2_i) \rightarrow \text{Hom}_{D^1 \otimes K} (M^2_1 \otimes_K D^1_1, D^2_i),
\]
\[
\phi_i^1 \otimes \phi_i^2 : \text{Hom}_{D^1 \otimes K} (M^1_1 \otimes_K D^1_1, D^2_i) \rightarrow \text{Hom}_{D^1 \otimes K} (M^2_1 \otimes_K D^2_1, D^2_i).
\]

The morphisms in (11.6.1) are graded isomorphisms since all of them are tensor product of two graded isomorphisms. This shows that \( S \) is dualisable.

Let \( X_1 \) be the exact sequence
\[
0 \rightarrow R^1_1 \rightarrow T(S^1) \rightarrow X_1 \rightarrow 0
\]
where \( X_1 = T(S^1) \). Then \( T(S^1)/R^1 = H_0(X_1) \). We shift our focus to the total complex \( \text{Tot}(X_1 \otimes_K X_2) \) which is explicitly
\[
0 \rightarrow R^1 \otimes_{D^1} T(S^2) \rightarrow T(S^1) \otimes_{D^1} R^2 \otimes_{D^2} T(S^2) \rightarrow \text{Tot}(X_1 \otimes_{D^1} X_2)
\]
\[
\xrightarrow{f} T(S^1 \otimes_{D^1} T(S^2) \otimes_{D^2} T(S^2)) \rightarrow 0.
\]
Here we see that \( \text{coker}(f) = T(S^1)/R^1 \otimes_{D^1} T(S^1)/R^2 \) by using the Künneth formula in Theorem 9.5.

By the structure of the tensor product
\[
(\alpha_1 \otimes 1)(1 \otimes \alpha_2) = \alpha_1 \otimes \alpha_2 = (1 \otimes \alpha_2)(\alpha_1 \otimes 1),
\]

where \( \alpha_1 \in M_1^i \). Therefore we have a natural epimorphism
\[
\xi : T(S) \rightarrow T(S^1) \otimes_{D^1} T(S^2),
\]
with kernel \( R' = ((\alpha_1 \otimes 1)(1 \otimes \alpha_2) - (1 \otimes \alpha_2)(\alpha_1 \otimes 1)) \in M_0^{i1} \). To see this we first note that every element in \( T(S)/R' \) can be written as a linear combination of elements of the form
\[
(\alpha_0 \otimes 1)(1 \otimes \beta_0)(\gamma_1 \otimes \delta_1)(\alpha_1 \otimes 1)(1 \otimes \beta_1) \cdots (\gamma_N \otimes \delta_N)(\alpha_N \otimes 1)(1 \otimes \beta_N),
\]
where \( \alpha_0 \in M_1^1, \beta_k \in M_2^k, \gamma_1 \otimes \delta_1 \in M_1^1 \otimes_{D^1} M_2^1, \) and \( k \in \mathbb{Z}_{\geq 0} \) and some \( a_k, b_k, n \in \mathbb{Z}_{\geq 0} \). Let \( \alpha \) and \( \beta \) be bases of \( M^1 \) and \( M^2 \) respectively, chosen such that \( \alpha \) and \( \beta \) consists of homogeneous elements. Then we can create a generating set \( G \subset T(S)/R \) consisting of non-zero elements of the form (11.6.3) where \( \alpha_k \in \alpha_{\otimes a_k} \) and \( \beta_k \in \beta_{\otimes b_k} \) are of degree 0 and \( \gamma_k \in \alpha_{\otimes \delta_k} \beta \) is of degree 1. The image of \( G \) under
\[
\tilde{\xi} : T(S)/R' \rightarrow T(S^1) \otimes_{D^1} T(S^2),
\]
is a linearly independent set. Since \( |G| = |\tilde{\xi}(G)| \) we have that \( G \) is linearly independent and hence a basis of \( T(S)/R' \). Therefore (11.6.4) is injective. Thus (11.6.4) is bijective and hence an isomorphism.
Using (11.6.4) we have

\[
\begin{array}{c}
T(S^1) \otimes_K R^2 \oplus R^1 \otimes_K T(S^2) \\
\xrightarrow{\tilde{f}} T(S^1) \otimes_K T(S^2)
\end{array}
\]

where \( \tilde{f} \) is defined so the diagram commutes. Then \( T(S)/R \cong \text{coker} \tilde{f} \) which proves the proposition. \( \square \)

**Corollary 11.7.** Let \( S^i \) be a species with relations \( R^i \). Then

\[
T(S^1)/R^1 \otimes_K T(S^2)/R^2 \cong T(S)/R,
\]

where \( T(S) = T(D = D^1 \otimes_K D^2, M = (M^1 \otimes_K D^2) \oplus (D^1 \otimes_K M^2)) \) and

\[
R = \langle [R^i \otimes 1_{D^2}, 1_{D^1} \otimes R^2], [\alpha \otimes 1_{D^2}, 1_{D^1} \otimes \alpha'] \mid \alpha \in M^1, \alpha' \in M^2 \rangle,
\]

where

\[
[\alpha \otimes 1_{D^2}, 1_{D^1} \otimes \alpha'] = (\alpha \otimes 1_{D^2}) \otimes 1_{D^1} \otimes D^2 \cdot (1_{D^1} \otimes \alpha') - (\alpha \otimes 1_{D^2}) \otimes 1_{D^1} \otimes (1_{D^1} \otimes \alpha').
\]

**Proof.** Define a grading on \( T(S^i)/R^i \) such that everything lies in degree 0 and then apply Proposition 11.5. \( \square \)

**Example 11.8.** Let \( S^1 \) and \( S^2 \) be the species from Example 11.1 and Example 11.2 respectively. Due to [DR80] we have a complete description of \( \Pi(S^1) \) and \( \Pi(S^2) \). The preprojective algebra of \( S^1 \) is given by

\[
\Pi(S^1) = T(S^1)/\langle c_1 \rangle,
\]

where

\[
S^1 : C_1^1 \xrightarrow{C} R_2^1 \xrightarrow{R^*} R_3^1
\]

and

\[
c_1 = \sum_{\alpha \in \bar{Q}_1^1} \text{sgn}(\alpha)c_\alpha.
\]

The preprojective algebra of \( S^2 \) is given by

\[
\Pi(S^2) = T(S^2)/\langle c_2 \rangle,
\]

where

\[
S^2 : R_1^2 \xrightarrow{R} R_2^2 \xrightarrow{R^*} R_3^2 \xrightarrow{R^*} R_4^2,
\]

and

\[
c_2 = \sum_{\alpha \in \bar{Q}_1^2} \text{sgn}(\alpha)c_\alpha.
\]

The subscripts in (11.8.1) and (11.8.2) denotes the positions in \( S^1 \) and \( S^2 \) respectively.
By Lemma 9.2 we know that $\Pi(T(S^1) \otimes_k T(S^2)) = \Pi(S^1) \triangleleft_k \Pi(S^2)$ and using Proposition 11.5 we can describe $\Pi(T(S^1) \otimes_k T(S^2))$ as the species $T(S)/R$, where $S$ given by the diagram

![Diagram](image)

(11.8.3)

Here the dotted lines represents the $\star$-degree 1 part of $M$ in $S = (D,M)$. The bimodule associated to an arrow $\alpha$ in (11.8.3) is $C$ if either the source or the target of $\alpha$ is $C$, otherwise it is $R$. For each arrow in (11.8.3) we have a relation. More explicitly, the relations are given by

$$R = \langle c_1 \otimes_k M_2^1, M_2^1 \otimes_k c_2, [\alpha \otimes 1_{D^2}, 1_{D^1} \otimes \alpha'] | \alpha \in M_1^1, \alpha' \in M_2^2 \rangle.$$

We can describe the Nakayama automorphism of $\Pi(T(S^1) \otimes_k T(S^2))$ by using Theorem 5.3 together with Proposition 10.11. Let $\gamma_i$ be the Nakayama automorphism of $\Pi(S^i)$ for each $i \in \{1, 2\}$. Then by Proposition 10.11 $\gamma = \gamma_1 \triangleleft \gamma_2$ is the Nakayama automorphism for $\Pi(S^1 \triangleleft_k S^2)$. Recall that

$$\gamma_i(y^k_\alpha) = \begin{cases} y^k_{\sigma(\alpha)}, & \text{if } \alpha \in Q_1 \\ \text{sgn}(\sigma(\alpha)) y^k_{\sigma(\alpha)}, & \text{if } \alpha \notin Q_1 \end{cases},$$

for each $i \in \{1, 2\}$. Since the Nakayama permutation is trivial for both $C_3$ and $D_4$ by Theorem 3.1 we get that $\gamma = \text{Id}_{\Pi(T(S^1) \otimes_k T(S^2))}$. 
References

[AHI+22] Toshitaka Aoki, Akihiro Higashitani, Osamu Iyama, Ryoichi Kase, and Yuya Mizuno. Fans and Polytopes in Tilting Theory. 2022. Preprint. arXiv: 2203.15213.

[ARS95] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. Representation Theory of Artin Algebras. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[ASS06] Ibrahim Assem, Andrzej Skowronski, and Daniel Simson. Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory, volume 1 of London Mathematical Society Student Texts. Cambridge University Press, 2006.

[BBK02] Sheila Brenner, Michael C. R. Butler, and Alastair D. King. Periodic Algebras Which are Almost Koszul. Algebras and Representation Theory, 5:331–368, 2002.

[Ber11] Carl F. Berg. Structure Theorems for Basic Algebras. 2011. Preprint. arXiv: 1102.1100.

[BGL87] Dagmar Baer, Werner Geigle, and Helmut Lenzing. The Preprojective Algebra of a Tame Hereditary Artin Algebra. Communications in Algebra, 15(1-2):425–457, 1987.

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. Koszul Duality Patterns in Representation Theory. J. Amer. Math. Soc., 9(2):473–527, 1996.

[CE56] Henry Cartan and Samuel Eilenberg. Homological Algebra, volume 41. Princeton University Press, 1956.

[DK94] Vlastimil Dlab and Claus Michael Ringel. On Algebras of Finite Representation Type. Journal of Algebra, 33(2):306–394, 1975.

[DR75] Vlastimil Dlab and Claude Michael Ringel. The Preprojective Algebra of a Modulated Graph. In Representation theory II, pages 216–231. Springer Berlin Heidelberg, 1980.

[Far05] Rolf Farnsteiner. Self-Injective Algebras: The Nakayama Permutation. Lecture Notes, available at http://www.mathematik.uni-bielefeld.de/~sek/selected.html, 2005.

[Gab72] Peter Gabriel. Unzerlegbare Darstellungen I. Manuscripta mathematica, 6:71–104, 1972.

[Gab73] Peter Gabriel. Indecomposable Representations II. Symposia Mathematica, Vol XI, pages 81–104, 1973.

[Gab80] Peter Gabriel. Auslander-Reiten Sequences and Representation-Finite Algebras. In Representation theory I, pages 1–71. Springer, 1980.

[GK20] Joseph Grant and Osamu Iyama. Higher Preprojective Algebras, Koszul Algebras, and Superpotentials. Compositio Mathematica, 156(12):2588–2627, Dec 2020.

[GKKP20] Nan Gao, Julian Külshammer, Sondre Kvamme, and Chrysostomos Psaroudakis. A Functorial Approach to Monomorphism Categories for Species I. Preprint. arXiv: 1907.04657, 2020.

[GLS13] Christof Geiss, Bernard Leclerc, and Jan Schröer. Cluster Algebras in Algebraic Lie Theory. Transformation Groups, 18(1):149–178, Feb 2013.

[Gra19] Joseph Grant. The Nakayama Automorphism of a Self-Injective Preprojective Algebra. Bulletin of the London Mathematical Society, 52(1):137–152, Dec 2019.

[Hap87] Dieter Happel. On the Derived Category of a Finite-Dimensional Algebra. Commentarii Mathematici Helvetica, 62(2):339–389, 1987.

[HI11] Martin Herschend and Osamu Iyama. n-Representation-Finite Algebras and Twisted Fractionally Calabi-Yau Algebras. Bulletin of the London Mathematical Society, 43(3):449–466, Jun 2011.

[HIO14] Martin Herschend, Osamu Iyama, and Steffen Oppermann. n-Representation Infinite Algebras. Advances in Mathematics, 252:292–342, 2014.

[IO13] Osamu Iyama and Steffen Oppermann. Stable Categories of Higher Preprojective Algebras. Advances in Mathematics, 244:23–68, 2013.

[Iya07] Osamu Iyama. Higher-Dimensional Auslander–Reiten Theory on Maximal Orthogonal Subcategories. Advances in Mathematics, 210(1):22–50, 2007.

[Iya08] Osamu Iyama. Auslander-Reiten Theory Revisited. Trends in Representation Theory of Algebras and Related Topics, 03 2008.

[Iya11] Osamu Iyama. Cluster Tilting for Higher Auslander Algebras. Advances in Mathematics, 226(1):1–61, 2011.

[Kü17] Julian Külshammer. Pro-Species of Algebras I: Basic Properties. Algebras and Representation Theory, 20, 10 2017.

[KS97] Masaki Kashiwara and Yoshishita Saito. Geometric Construction of Crystal Bases. Duke Mathematical Journal, 89(1):9–36, 1997.

[Lam98] Tsit-Yuen Lam. Lectures on Modules and Rings, volume 189. Springer-Verlag, 1998.

[Lus91] George Lusztig. Quivers, Perverse Sheaves, and Quantized Enveloping Algebras. Journal of the American Mathematical Society, 4:365–421, 1991.

[LY15] Fang Li and Chang Ye. Representations of Frobenius-Type Triangular Matrix Algebras. 2015. Preprint. arXiv: 1511.08040.

[Mit64] Barry Mitchell. The Full Imbedding Theorem. American Journal of Mathematics, 86(3):619–637, 1964.
[MW21] Jordan McMahon and Nicholas J. Williams. The Combinatorics of Tensor Products of Higher Auslander Algebras of Type $a$. *Glasgow Mathematical Journal*, 63(3):526–546, 2021.

[Nak94] Hiraku Nakajima. Instantons on ALE Spaces, Quiver Varieties, and Kac-Moody Algebras. *Duke Mathematical Journal*, 76, 11 1994.

[Pas17] Andrea Pasquali. Tensor Products of Higher Almost Split Sequences. *Journal of Pure and Applied Algebra*, 221(3):645–665, 2017.

[Pas19] Andrea Pasquali. Tensor Products of $n$-Complete Algebras. *Journal of Pure and Applied Algebra*, 223(8):3537–3553, 2019.

[PP05] Alexander Polishchuk and Leonid Positselski. *Quadratic algebras*, volume 37. American Mathematical Soc., 2005.

[Rin76] Claus Michael Ringel. Representations of K-Species and Bimodules. *Journal of algebra*, 41(2), 1976.

[Rin98] Claus Michael Ringel. The Preprojective Algebra of a Quiver. *Algebras and Modules, II (Geiranger, 1996)*, 24:467–480, 1998.

[Sch85] Aidan Harry Schofield. *Representations of Rings Over Skew Fields*, volume 92. Cambridge University Press, 1985.

[Thi20] Louis-Philippe Thibault. Preprojective Algebra Structure on Skew-Group Algebras. *Adv. Math.*, 365:107033, 43, 2020.