UNITARY REPRESENTATIONS OF SU_q(2)
ON THE PLANE FOR q ∈ R^+ OR GENERIC q ∈ S^1*

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Abstract
Some time ago, Rideau and Winternitz introduced a realization of the quantum algebra su_q(2) on a real two-dimensional sphere, or a real plane, and constructed a basis for its representations in terms of q-special functions, which can be expressed in terms of q-Vilenkin functions. In their study, the values of q were implicitly restricted to q ∈ R^+. In the present paper, we extend their work to the case of generic values of q ∈ S^1 (i.e., q values different from a root of unity). In addition, we unitarize the representations for both types of q values, q ∈ R^+ and generic q ∈ S^1, by determining some appropriate scalar products.

1 Introduction
Rideau and Winternitz introduced a realization of the quantum algebra su_q(2) on the plane and constructed a basis for its irreducible representations (irreps) in terms of q-Vilenkin functions, so called because, for q = 1, they reduce to functions introduced by Vilenkin and related to Jacobi polynomials. This realization was used to set up su_q(2)-invariant Schrödinger equations in the usual framework of quantum mechanics. Although not explicitly stated

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in Ref. [1], the values of the deformation parameter \( q \), considered there, are restricted to \( q \in \mathbb{R}^+ \). Though important both from the \( q \)-special function viewpoint, and from that of their applications in quantum mechanics, the question of the \( su_q(2) \) irrep unitarity was also left unsolved.

The purpose of a previous paper [4] and of the present lecture is twofold: firstly, to find basis functions of the representation for generic \( q \in \mathbb{S}^1 \) (i.e., for \( q \) different from a root of unity), and secondly, to unitarize the representations for both \( q \in \mathbb{R}^+ \), and generic \( q \in \mathbb{S}^1 \). As a consequence, the orthonormality relations of the \( q \)-Vilenkin and related functions are established [4]. In Sec. 2, the representations of \( su_q(2) \) obtained in Ref. [1] are briefly reviewed. Their basis functions are determined in Sec. 3. The unitarization of the representations is dealt with in Sec. 4. Sec. 5 contains the conclusion.

2 Representations of \( su_q(2) \) on the plane

The \( su_q(2) \) generators \( H_3, H_+, H_- \) satisfy the commutation relations [1]

\[
[H_3, H_\pm] = \pm H_\pm, \quad [H_+, H_-] = [2H_3]_q \equiv \frac{q^{2H_3} - q^{-2H_3}}{q - q^{-1}},
\]

and the Hermiticity properties \( H_3^\dagger = H_3 \), and \( H_\pm^\dagger = H_{\mp} \). One can construct a Casimir operator \( C = H_+ H_- + [H_3]_q [H_3 - 1]_q = H_- H_+ + [H_3]_q [H_3 + 1]_q \). The generators \( H_3, H_+, H_- \) can be realized by the following operators, acting on the plane, more precisely, on functions \( f(z, \overline{z}) \) of a complex variable \( z \) and its complex conjugate \( \overline{z} \):

\[
\begin{align*}
H_3 &= -T + \overline{T} - N, \\
H_+ &= -z^{-1} [T]_q q^{T-(N/2)} - q^{T+(N/2)} \overline{T} - N]_q, \\
H_- &= z [T + N]_q q^{T-(N/2)} + q^{T+(N/2)} \overline{T}^{-1} [T]_q,
\end{align*}
\]

where \( T = z \partial_z \), and \( \overline{T} = \overline{z} \partial_{\overline{z}} \).

Basis functions \( \Psi_{M,N}(z, \overline{z}) \) for the \((2J+1)\)-dimensional irrep of \( su_q(2) \) satisfy the relations [3]

\[
\begin{align*}
H_3 \Psi_{M,N} &= M \Psi_{M,N}, \\
H_\pm \Psi_{M,N} &= ([J \mp M]_q [J \pm M + 1]_q)^{1/2} \Psi_{M\pm 1,N}, \\
C \Psi_{M,N} &= [J][J + 1]_q \Psi_{M,N}, \quad M = \{-J, -J + 1, \cdots, J\}, \quad |N| \leq J,
\end{align*}
\]

where \( J, M \) and \( N \) are simultaneously integers or half-integers. Let us remark that, when \( q \in \mathbb{S}^1 \), the existence of such a representation implies that \([n]_q\) does not vanish unless \( n = 0 \), hence that \( q \) is not a root of unity.

Following Ref. [3], let us write \( \Psi_{M,N}(z, \overline{z}) \) as

\[
\Psi_{M,N}(z, \overline{z}) = N_{M,N} J_q(\eta) q^{-N/2} R_{M,N}(\eta) \overline{z}^M, \quad \eta = z \overline{z}.
\]

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Here, $N^J_{MNq}$ is a constant, which can be expressed as

$$N^J_{MNq} = \frac{1}{\sqrt{2\pi}} \left(\frac{[J + N]_q! [2J + 1]_q!}{[J - N]_q!}\right)^{1/2} \left(\frac{[J + M]_q!}{[J - M]_q! [2J]_q!}\right)^{1/2} \gamma(J, N, q),$$

(5)
in terms of some yet undetermined normalization constant $\gamma(J, N, q)$. The $q$-factorials are defined by $[x]_q! \equiv [x]_q[x - 1]_q \ldots [1]_q$ if $x \in \mathbb{N}^+$, $[0]_q! \equiv 1$, and $([x]_q!)^{-1} \equiv 0$ if $x \in \mathbb{N}^-$. The function $R^J_{MNq}(\eta)$ involved in Eq. (5) is a polynomial

$$R^J_{MNq}(\eta) = \sum_k \frac{[J - N]_q! [J - M]_q! (-\eta)^k}{[k]_q! [J - M - k]_q! [J - N - k]_q! [M + N + k]_q!},$$

(6)
the summation over $k$ being restricted by the condition that all the arguments of the factorials in the denominator be positive. The function $Q_{Jq}(\eta)$ involved in Eq. (4) is a polynomial

$$Q_{Jq}(\eta) = \frac{1}{\sqrt{2\pi}} \left(\frac{[J + N]_q! [2J + 1]_q!}{[J - N]_q!}\right)^{1/2} \left(\frac{[J + M]_q!}{[J - M]_q! [2J]_q!}\right)^{1/2} \gamma(J, N, q),$$

(7)
whose solution, only determined up to an arbitrary multiplicative factor $f_{Jq}(\eta)$ such that $f_{Jq}(\eta^2) = f_{Jq}(\eta)$, will be discussed in the next section.

The functions (5) are related to the $q$-Vilenkin functions, defined by

$$P^J_{MNq}(\xi) = i^{2J-M-N} \left(\frac{[J + M]_q! [J + N]_q!}{[J - M]_q! [J - N]_q!}\right)^{1/2} \eta^{(M+N)/2} Q_{Jq}(\eta) R^J_{MNq}(\eta),$$

(8)
where $\eta = (1 + \xi)/(1 - \xi)$. For integer $J$ values, the functions $\Psi^J_{M0q}$ are proportional to $q$-spherical harmonics, while $P_{Jq}(\xi) \equiv P^J_{M0q}(\xi)$ are $q$-analogues of Legendre polynomials.

In the $q \to 1$ limit, the su$_q(2)$ realization (2) goes into a standard su(2) realization, and choosing $Q_{J1}(\eta) = (1 + \eta)^{-J}$, the functions $\Psi^J_{MN1}$ form an orthonormal set with respect to the scalar product

$$\langle \psi_1 | \psi_2 \rangle = 2 \int \frac{dzd\overline{z}}{(1 + z^2)^2} \psi_1(z, \overline{z}) \psi_2(z, \overline{z}).$$

(9)

### 3 Determination of $Q_{Jq}(\eta)$

Following Ref. [1], as a solution of Eq. (4), we may consider the function

$$Q_{Jq}(\eta) = \Phi_0 \left( q^{2J}; -; q^2, -q^{-2J} \eta \right) = \Phi_0 \left( q^{-2J}; -; q^{-2}, -q^{-2} \eta \right),$$

(10)
where $\Phi_0$ is a basic hypergeometric series in the notations of Ref. [3].
For \( q \in \mathbb{R}^+ \), use of the \( q \)-binomial theorem \([6]\) leads to the expressions

\[
Q_{Jq}(\eta) = \prod_{k=0}^{\infty} \frac{(1+q^{2k}\eta)}{(1+q^{2k+2}\eta)}, \quad \text{if } 0 < q < 1,
\]
\[
Q_{Jq}(\eta) = \prod_{k=0}^{\infty} \frac{(1+q^{-2J-2k-2}\eta)}{(1+q^{-2k-2}\eta)}, \quad \text{if } q > 1.
\]

For integer \( J \) values, both expressions reduce to the inverse of a polynomial,

\[
Q_{Jq}(\eta) = \prod_{k=0}^{J-1} \frac{1}{1+\eta q^{-2J+2k}}.
\]

For half-integer \( J \) values, we are left with convergent infinite products.

- For generic \( q \in S^1 \), and integer \( J \) values, Eq. (12) still remains a valid solution of Eq. (7). For half-integer \( J \) values, however, the infinite products contained in Eq. (11) are divergent. We have therefore to look for another solution to Eq. (7). For such a purpose, by setting \( K_{Jq}(\eta) = \ln Q_{Jq}(\eta) \), we linearize Eq. (7) into

\[
K_{Jq}(q^2\eta) - K_{Jq}(\eta) = \ln \frac{1+q^{-2J}\eta}{1+\eta},
\]

whose solution can be written in the form \( K_{Jq}(\eta) = L_q(q^{-2J-1}\eta) - L_q(q^{-1}\eta) \), where \( L_q \) is solution of

\[
L_q(q\eta) - L_q(q^{-1}\eta) = \ln(1+\eta).
\]

In Ref. [4], we demonstrated

**Lemma 3.1** For \( 0 < \eta < \infty \), and \( q = e^{i\tau} \) different from a root of unity, the function

\[
L_q(\eta) = \frac{1}{2\pi i} \int_0^\infty \frac{dt}{t(1+t)} \ln \left(1+\eta t^{\tau/\pi}\right), \quad \text{if } 0 < \tau < \pi,
\]
\[
L_q(\eta) = -\frac{1}{2\pi i} \int_0^\infty \frac{dt}{t(1+t)} \ln \left(1+\eta t^{-\tau/\pi}\right), \quad \text{if } -\pi < \tau < 0
\]

is a solution of Eq. (14).

The results of the present section can be collected into

**Proposition 3.2** The function \( Q_{Jq}(\eta) \), appearing on the right-hand side of Eq. (4), is given by Eq. (17) for integer \( J \) values, and either \( q \in \mathbb{R}^+ \) or generic \( q \in S^1 \), and by Eq. (11) for half-integer \( J \) values and \( q \in \mathbb{R}^+ \). For half-integer \( J \) values, and generic \( q \in S^1 \), it can be expressed as

\[
Q_{Jq}(\eta) = \exp \left( L_q(q^{-2J-1}\eta) - L_q(q^{-1}\eta) \right),
\]

where \( L_q(\eta) \) admits the integral representation given in Lemma 3.1.
4 Unitarization of the representations

In the present section, we will determine a new scalar product \( \langle \psi_1 | \psi_2 \rangle_q \) that unitarizes the realization (4) of \( su_q(2) \), and goes over into the old one \( \langle \psi_1 | \psi_2 \rangle \), defined in Eq. (9), whenever \( q \to 1 \).

4.1 The case where \( q \in \mathbb{R}^+ \)

Let us make the following ansatz for \( \langle \psi_1 | \psi_2 \rangle_q \),

\[
\langle \psi_1 | \psi_2 \rangle_q = \int_0^\infty d\rho \int_0^{2\pi} d\phi \left( A_q \psi_1(\rho, \phi, q) f_1(\rho, q) \right. \\
\left. \quad + \psi_1(\rho, \phi, q) f_2(\rho, q) \right) q^{\alpha_1 \rho \partial_\rho} \psi_2(\rho, \phi, q), \quad z = \rho e^{i\phi}.
\]

Here \( \alpha_1, \alpha_2 \) and \( f_1(\rho, q), f_2(\rho, q) \) are some yet undetermined constants and functions, and \( A_q \equiv q^{-2\alpha_2 \rho^2} \) is the operator that changes \( q \) into \( q^{-1} \), when acting on any function of \( q \).

It is easy to check that \( H_3 \) satisfies the equation \( \langle \psi_1 | H_3 \psi_2 \rangle_q = \langle H_3 \psi_1 | \psi_2 \rangle_q \). Let us impose the condition \( \langle \psi_1 | H_+ \psi_2 \rangle_q = \langle H_- \psi_1 | \psi_2 \rangle_q \). We obtain

\[
a_1 = -1, \quad a_2 = 1, \quad f_1(z, q) = B_1(z, q), \quad f_2(z, q) = B_2(z, q),
\]

in terms of two undetermined constants \( B_1(q) \), and \( B_2(q) \).

Let us now further restrict the sesquilinear form (18) by imposing that it is Hermitian: \( \langle \psi_1 | \psi_2 \rangle_q = \langle \psi_2 | \psi_1 \rangle_q \). We get the relation

\[
B_2(q) = B_1(q).
\]

All the Hermiticity conditions on the \( su_q(2) \) generators are satisfied by the form defined in Eqs. (18), (19), (20), and the functions \( \Psi_{J,N,q}^{J,N,q}(z, \bar{z}) \), defined in Eq. (4), are orthogonal with respect to such a form.

To make \( \langle \psi_1 | \psi_2 \rangle_q \) into a scalar product, it only remains to impose that it is a positive definite form, which amounts to the condition \( \langle \Psi_{J,N,q}^{J,N,q} \rangle_q^2 = 1 \). A straightforward calculation of this squared norm leads to

\[
\frac{\ln q}{q - q^{-1}} \left( B_1(q) \gamma(J, N, q^{-1}) \gamma(J, N, q) + B_1(q) \gamma(J, N, q) \gamma(J, N, q^{-1}) \right) = 1.
\]

We can choose \( \gamma(J, N, q) = 1 \), and \( B_1(q) = (q - q^{-1})(2 \ln q)^{-1} \). When \( q \to 1 \), we find that \( \langle \psi_1 | \psi_2 \rangle_q \to \langle \psi_1 | \psi_2 \rangle \), where the latter is given by Eq. (9), as it should be.

The results obtained can be summarized as follows:

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Proposition 4.1 For $q \in \mathbb{R}^+$, the scalar product

\[
\langle \psi_1 | \psi_2 \rangle_q = \frac{q - q^{-1}}{2 \ln q} \int dz \, d\bar{z} \left( \psi_1(z, \bar{z}, q) \frac{q^{-1}}{1 + z \bar{z}} \psi_2(1 - q^{-1} z, q^{-1} \bar{z}, q) \right.
\]

\[
+ \left. \psi_1(z, \bar{z}, q) \frac{q}{1 + z \bar{z}} \psi_2(qz, q \bar{z}, q^{-1}) \right),
\]

unitarizes the $su_q(2)$ realization (3), where $N$ may take any integer or half-integer value. The functions $\Psi_{JMq}(z, \bar{z})$, defined in Eq. (4), where $J = |N|$, $|N| + 1, \ldots, M = -J, -J + 1, \ldots, J$, and $\gamma(J, N, q) = 1$, form an orthonormal set with respect to such a scalar product.

4.2 The case where $q \in S^1$

Whenever $q \in S^1$, the scalar product obtained when $q \in \mathbb{R}^+$ does not work. In Ref. [4], by a treatment analogous to that of the previous subsection, we established

Proposition 4.2 For generic $q \in S^1$, the scalar product

\[
\langle \psi_1 | \psi_2 \rangle_q = \frac{q - q^{-1}}{2 \ln q} \int dz \, d\bar{z} \left( \psi_1(z, \bar{z}, q) \frac{q^{-1}}{1 + z \bar{z}} \psi_2(1 - q^{-1} z, q^{-1} \bar{z}, q) \right.
\]

\[
+ \left. \psi_1(z, \bar{z}, q^{-1}) \frac{q}{1 + z \bar{z}} \psi_2(qz, q \bar{z}, q^{-1}) \right)
\]

unitarizes the $su_q(2)$ realization (3), where $N$ may take any integer or half-integer value. The functions $\Psi_{JMq}(z, \bar{z})$, defined in Eq. (4), where $J = |N|$, $|N| + 1, \ldots, M = -J, -J + 1, \ldots, J$, and $\gamma(J, N, q) = 1$, form an orthonormal set with respect to such a scalar product.

5 Conclusion

In the present communication, we did extend the study of the $su_q(2)$ representations on the plane, carried out by Rideau and Winternitz [1], in two ways. Firstly, we did prove that such representations exist not only for $q \in \mathbb{R}^+$, but also for generic $q \in S^1$. For such a purpose, we did provide an integral representation for the functions $Q_{Jq}(\eta)$, entering the definition of the $q$-Vilenkin functions, whenever $J$ takes any half-integer value.

Secondly, we did unitarize the representations by determining appropriate scalar products for both ranges of $q$ values. Such scalar products are expressed in terms of ordinary integrals, instead of $q$-integrals, as is usually the case [7].

The resulting orthonormality relations for the $q$-Vilenkin and related functions [4] should play an important role in applications to quantum mechanics, such as those considered in Ref. [8].
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