LAYERS OF THE CORADICAL FILTRATION

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Abstract. Under suitably nice conditions, given a coalgebra object in a tensor category we compute the layers of its coradical (socle) filtration.

1. Statement of Main Result

Let $k$ be a field and $C$ a semisimple pointed tensor category over $k$ (precise definitions are given in Section 2). Recall that pointed means that every simple object of $C$ is invertible. For instance $C$ could be the category of finite-dimensional (super) vector spaces. Let $C$ be a coalgebra object in the cocompletion of $C$. Then $C$ is a bicomodule over itself via its comultiplication morphism. We prove a result on one aspect of this structure.

As a $C$-bicomodule, $C$ has an ascending Loewy series, i.e. its socle filtration:

$$0 = \sigma_0(C) \subseteq \sigma_1(C) \subseteq \sigma_2(C) \subseteq \cdots$$

This is often called the coradical filtration of $C$, and it is in fact a filtration of $C$ by coalgebras. We seek to describe the layers of this filtration, as a bicomodule over $C$. But we need a few assumptions on $C$.

Before we state the assumptions, we recall a few constructions. First, every simple right comodule $L$ of $C$ has an injective envelope $I(L)$, which is a right comodule that is an object of the cocompletion of $C$. It too has a socle filtration $\sigma_\bullet(I(L))$ as a right comodule. Next, given a right $C$-comodule $V$ and an object $S$ of $C$, the tensor product $S \otimes V$ has the natural structure of a right comodule. Finally, if $V$ is a right $C$-comodule, then its left dual $V^*$ is a left $C$-comodule and if $W$ is a right comodule then the tensor product $V^* \otimes W$ has the natural structure of a bicomodule, which we denote by writing $V^* \boxtimes W$.

Here are the assumptions we place on $C$: for the third assumption we fix $n \in \mathbb{N}$ with $n \geq 1$, meaning that this assumption depends on $n$. It is possible that a given coalgebra $C$ only satisfies (C3–$n$) for certain $n$.

(C1) If $L$ is a simple right $C$-comodule and $S$ is a simple object of $C$, then $L$ and $S \otimes L$ are not isomorphic as comodules unless $S \cong 1$.

(C2) If $L, L'$ are simple right $C$-comodules then $L^* \boxtimes L'$ is a simple bicomodule and further every simple bicomodule is of this form up to isomorphism.

(C3–$n$) If $L, L'$ are simple right comodules, then, $[\sigma_n(I(L)) : L'] < \infty$.

Note that if $V$ and $L$ are right $C$-comodules of finite length, and $L$ is simple, we write $[V : L]$ number of times $L$ appears in a composition series of $V$.

Finally, we observe that (C3–$n$) holds for all $n$ if for all simple right comodules $L, L'$ the following hold:

(a) $\text{Ext}^1(L, L')$ is finite-dimensional; and
Indeed, in this case one can prove by induction that \( \sigma_n(I(L)) \) is of finite length for any simple module \( L \): for \( n = 1 \) it is clear, and we have the inequality:

\[
[\sigma_{n+1}(I(L))/\sigma_n(I(L)) : L''] \leq \dim \text{Ext}^1(L'', \sigma_n(I(L))) \\
\leq \sum_{[\sigma_n(I(L)):L'] \neq 0} \dim \text{Ext}^1(L'', (L')^{\oplus[\sigma_n(I(L)):L']}).
\]

The first inequality is clear, and the second equality follows from the fact that \( \dim \text{Ext}^1(L'', Z) \leq \dim \text{Ext}^1(L'', X) + \dim \text{Ext}^1(L'', Y) \) whenever we have a short exact sequence \( 0 \to X \to Z \to Y \to 0 \). By our assumptions (a) and (b), the RHS is finite. Thus

\[
[\sigma_{n+1}(I(L)) : L''] \leq [\sigma_{n+1}(I(L))/\sigma_n(I(L)) : L''] + [\sigma_n(I(L)) : L''] < \infty.
\]

Our inequalities further show that \( [\sigma_{n+1}(I(L)) : L''] \neq 0 \) only if there exists simple right comodules \( L_1, \ldots, L_{n-1} \) such that \( \text{Ext}^1(L_1, L) \neq 0, \text{Ext}^1(L_2, L_1) \neq 0, \ldots, \text{Ext}^1(L_n, L_n) \neq 0 \). By assumption (b), only finitely many simple right comodules satisfy this property. Thus \( \sigma_{n+1}(I(L)) \) will be of finite length.

1.1. Main result. The main theorem we prove is:

**Theorem 1.1.** Assuming (C1)-(C3-n), if \( i \leq n \) then for simple right comodules \( L, L' \) we have

\[
[\sigma_i(C)/\sigma_{i-1}(C) : L^* \boxtimes L'] = [\sigma_i(I(L))/\sigma_{i-1}(I(L)) : L'].
\]

In particular if (C3-n) holds for all \( n \), then the above equality holds for all \( i \).

The following statement is clearly equivalent.

**Theorem 1.2.** Assuming (C1)-(C3-n), if \( i \leq n \) then for simple right comodules \( L, L' \) we have

\[
[\sigma_i(C) : L^* \boxtimes L'] = [\sigma_i(I(L)) : L'].
\]

In particular if (C3-n) holds for all \( n \), then the above equality holds for all \( i \).

In the case of \( i = 2 \) we obtain a generalization of a corollary of the Taft-Wilson theorem for pointed coalgebras over a field.

**Corollary 1.3.** Assuming (C1)-(C2), if \( L, L' \) are simple right comodules such that \( \text{Ext}^1(L, L') \) is finite-dimensional then we have

\[
[\sigma_2(C)/\sigma_1(C) : L^* \boxtimes L'] = \dim \text{Ext}^1(L', L).
\]

The finiteness assumption in (C3-n) is clearly necessary in order to state the theorems. The assumptions (C1) and (C2) are necessary for obtaining a clear description of the simple bicomodules of \( C \). If \( A \) is a simple finite-dimensional \( G \)-graded algebra over an algebraically closed field of characteristic zero for a group \( G \), and the center of \( A \) contains a non-scalar element, then the assumptions (C1) and (C2) will fail for \( C = A^* \) the dual coalgebra of \( A \), as an object of \( G \)-graded vector spaces.
1.2. Applications. The conditions (C1)-(C3) hold for many examples. The case of original motivation and interest to the author appears in Section 6 of [She]. Namely, let $G$ be a quasireductive supergroup, that is one for which $G_0$ is reductive, and suppose that it has an even Cartan subgroup. Then $G \times G$ acts on $G$ by left and right multiplication, and this induces an action of $G \times G$ on $\mathbb{C}[G]$. One seeks a nice description of the structure of this $G \times G$-module; this is the natural generalization of the Peter-Weyl Theorem to the super setting. Note that the structure of $\mathbb{C}[G]$ as a $G$-module, where $G$ acts by left translation, was given in [Ser].

The above situation is exactly given by the setup of this paper, where $\mathcal{C}$ is the category of finite-dimensional super vector spaces over $\mathbb{C}$ and $\mathcal{C}$ is the coalgebra $\mathbb{C}[G]$ ($\mathbb{C}[G]$ is in fact a Hopf algebra). In this case, from Theorem 1.2, one obtains a beautiful description of the Loewy layers of $\mathbb{C}[G]$ viewed as a $G \times G$-module.

More generally, if $\mathcal{C}$ is the category of finite-dimensional vector spaces, then (C1) automatically holds, and if $\mathcal{C}$ is a coalgebra over an algebraically closed field $k$, then (C1)-(C2) hold$^1$. More generally, if $G$ is a group, $k$ an algebraically closed field of characteristic zero or characteristic $p$ where $p$ is coprime to the order of each finite subgroup of $G$, and $\mathcal{C}$ is the category of $G$-graded vector spaces over $k$, then (C1) and (C2) become equivalent. This follows as a corollary of the main results of [BZS], that a finite-dimensional $G$-graded simple algebra $B$ is a matrix algebra over $k$ if and only if the center of $B$ is $k$.

1.3. Outline of paper. In Section 2 we state formal constructions related to coalgebras and comodules in tensor categories, with [EGNO] being our main reference. In Section 3 we state basic results about the matrix coefficient morphism. Section 4 goes into the existence and structure of injective comodules, and Section 5 explains the structure of the coalgebra as a right comodule. The statements and proofs of these results are known and go back to [Ser]. Finally Section 6 examines the structure of $\mathcal{C}$ as a bicomodule, concluding with Theorem 6.6.

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2. Setup and Preliminaries

2.1. We follow the definitions and terminology from [EGNO]. Let $k$ be a field and $\mathcal{C}$ a semisimple pointed tensor category over $k$. In other words we assume:

1. $\mathcal{C}$ is a locally finite semisimple $k$-linear abelian category;
2. $\mathcal{C}$ is rigid monoidal such that $(-) \otimes (-)$ is a biexact bilinear bifunctor, and $\text{End}(1) \cong k$;
3. every simple object of $\mathcal{C}$ is invertible.

By Thm. 2.11.5 of [EGNO], such categories are always isomorphic (as monoidal categories) to a category $\text{vec}(G, \omega)$, the category of finite-dimensional $G$-graded vector spaces (where $G$ is a group) with associativity isomorphism determined by the 3-cocycle $\omega \in Z^3(G, k^\times)$. Note that we do not assume $(\mathcal{C}, \otimes)$ is braided.

$^1$Thank you to Nicolás Andruskiewitsch for explaining why this is true.
2.2. For an object $V$ of $\mathcal{C}$, we write $V^*$ for its left dual, $ev_V : V^* \otimes V \to 1$ for the evaluation morphism and $\text{coev}_V : 1 \to V \otimes V^*$ for the coevaluation morphism. If $W$ is a subobject of $V$, we write $W^\perp$ for the subobject of $V^*$ given by the kernel of the epimorphism $V^* \to W^*$. If $f : W \to V$ is an arbitrary morphism then we have a commutative diagram which will be used later on:

\[
\begin{array}{ccc}
W^* \otimes W & \xrightarrow{ev_W} & 1 \\
\downarrow{f^* \otimes 1} & & \downarrow{ev_V} \\
V^* \otimes W & \xrightarrow{1 \otimes f} & V^* \otimes V
\end{array}
\] (2.1)

2.3. We consider the cocomplete abelian category $\hat{\mathcal{C}}$ constructed from $\mathcal{C}$, as described in [Sta]. Note that here if $\mathcal{C} \cong \text{vec}(G, \omega)$, then $\hat{\mathcal{C}} \cong \text{Vec}(G, \omega)$ which is the category of $G$-graded vector spaces of arbitrary dimension. We have a fully faithful embedding $\mathcal{C} \to \hat{\mathcal{C}}$ admitting the usual universal property. Further, in this case $\hat{\mathcal{C}} \times \hat{\mathcal{C}}$ is a cocomplete abelian category with a natural fully faithful functor $\mathcal{C} \times \mathcal{C} \to \hat{\mathcal{C}} \times \hat{\mathcal{C}}$ that satisfies the desired universal property. Thus in particular $\otimes$ extends to a biexact bilinear functor $\hat{\mathcal{C}} \times \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ which we continue to write as $\otimes$ by abuse of notation.

2.4. Let $C$ be a coalgebra object in $\hat{\mathcal{C}}$. This means $C$ comes equipped with morphisms $\Delta : C \to C \otimes C$ and $\epsilon : C \to 1$ such that

$$(\Delta \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \epsilon) \circ \Delta = \text{id}_C.$$ By thinking of $\hat{\mathcal{C}}$ as $\text{Vec}(G, \omega)$, a standard argument shows that $\hat{\mathcal{C}}$ is a direct limit of subcoalgebras objects of $\mathcal{C}$.

2.5. An object $V \in \hat{\mathcal{C}}$ is said to be a right $C$-comodule (resp. left $C$-comodule) if it is equipped with a morphism $a_V = a : V \to V \otimes C$ (resp. $a_V = a : V \to C \otimes V$) such that

$$(a \otimes \text{id}_C) \circ a = (\text{id}_V \otimes \Delta) \circ a, \quad (\text{id}_C \otimes a) \circ a = (\Delta \otimes \text{id}_V) \circ a$$ and

$$(\text{id}_V \otimes \epsilon) \circ a = \text{id}_V, \quad (\epsilon \otimes \text{id}_V) \circ a = \text{id}_V.$$ An object $V \in \hat{\mathcal{C}}$ is a $C$-bicomodule if it is both a left and right comodule with comodule structure morphisms $a_{V,l}$ and $a_{V,r}$ such that $(\text{id}_C \otimes a_{V,r}) \circ a_{V,l} = (a_{V,l} \otimes \text{id}_C) \circ a_{V,r}$. Observe that $C$ is naturally a left and right comodule via $a_{C,r} = a_{C,l} = \Delta$, such that it obtains the structure of a $C$-bicomodule.

Again by a standard argument, any $C$-(bi)comodule $V$ will be a sum of sub-(bi)comodule objects in $\mathcal{C}$. In particular, simple (bi)comodules are always objects of $\mathcal{C}$.

2.6. Consider the category $\text{Mod}_C$ (resp. $\text{cMod}$) of right $C$-comodules (resp. left $C$-comodules) with morphisms between two objects $V, W$ being morphisms in $\hat{\mathcal{C}}$ respecting comodule structure morphisms. Let $\text{cmod}$ (resp. $\text{mod}_C$) denote the full subcategory of right $C$-comodules (resp. left $C$-comodules) in $\mathcal{C}$. We also have the categories $\text{cMod}_C$ and $\text{cmod}_C$ of $C$-bicomodules in $\hat{\mathcal{C}}$ and $\mathcal{C}$ respectively. By our assumption that $\otimes$ is biexact, these categories are all abelian. Further, $\text{cmod}$, $\text{mod}_C$ and $\text{mod}_C$ are locally finite, and thus the Jordan-Holder and Krull-Schmidt theorems are valid. The categories $\text{cMod}$, $\text{Mod}_C$ and $\text{cMod}_C$ are cocomplete, and we have natural inclusion functors
mod}_C \to \text{Mod}_C, \text{Cmod} \to \text{CMod}, \text{and}_C \text{mod}_C \to \text{CMod}_C\) that have the usual universal properties as cocompletions.

2.7. Given a right (resp. left) \(C\)-comodule \(V\) and an object \(S \in \hat{C}\), we may construct a new right (resp. left) \(C\)-comodule \(S \otimes V\) (resp. \(V \otimes S\)) with comodule morphism \(a_{S \otimes V} = \text{id}_S \otimes a_V\) (resp. \(a_{V \otimes S} = a_V \otimes \text{id}_S\)). This defines an endofunctor of the categories \(\text{Mod}_C\) and \(\text{CMod}_C\), and it preserves \(\text{mod}_C\) and \(\text{Cmod}\) if \(S\) is in \(C\). We observe that if \(S\) is simple (and thus invertible) then this functor defines automorphisms of these abelian categories, and thus it takes simple comodules to simple comodules.

2.8. Given a right \(C\)-comodule \(V\) and left \(C\)-comodule \(W\) we may construct a \(C\)-bicomodule \(V \flat W\), which is \(V \otimes W\) as an object of \(\hat{C}\) and has left and right comodule structures as described in 2.7. This satisfies the necessary commutativity condition to be a bicomodule.

**Lemma 2.1.** Suppose that \(V\) is a right \(C\)-comodule, \(W\) a left \(C\)-comodule, and \(S\) is an object of \(\hat{C}\). Then we have a canonical isomorphism of bicomodules

\[
(W \otimes S) \boxtimes V \cong W \boxtimes (S \otimes V).
\]

**Proof.** Indeed, the associativity isomorphism coming from the monoidal structure of \(\hat{C}\) provides us with such an isomorphism. \(\square\)

**Corollary 2.2.** With the same hypotheses as Lemma 2.1 and assuming that \(S\) is a simple object of \(C\), we have a canonical isomorphism

\[
(W \otimes S^*) \boxtimes (S \otimes V) \cong W \boxtimes V.
\]

**Proof.** Apply Lemma 2.1 and the invertibility isomorphism \(S^* \otimes S \cong 1\). \(\square\)

2.9. Let \(V\) be an object in \(\text{mod}_C\) and \(V^*\) its left dual in \(C\). Then \(V^*\) has the natural structure of a left \(C\)-comodule by

\[a_{V^*} = (\text{ev}_V \otimes \text{id}_C \otimes \text{id}_{V^*}) \circ (\text{id}_{V^*} \otimes a_V \otimes \text{id}_{V^*}) \circ (\text{id}_{V^*} \otimes \text{coev}_V).\]

This construction is functorial, so that we have a contravariant functor \((-)^* : \text{mod}_C \to \text{Cmod}\). This functor is an antiequivalence with inverse taking the right dual of a comodule, \(V \mapsto V^*\). We observe that if \(W\) is a right subcomodule of \(V\) then \(W^\perp\) is naturally a left subcomodule of \(V^*\).

3. **Matrix Coefficients**

3.1. For this section, all objects are assumed to be in \(C\), i.e. they are of finite length. Given an object \(V\) of \(\text{mod}_C\), by 2.9 and 2.8 we obtain a \(C\)-bicomodule given by \(V^* \boxtimes V\). Define the matrix coefficients morphism \(c_V : V^* \boxtimes V \to C\) by

\[c_V = (\text{ev}_V \otimes \text{id}_C) \circ (\text{id}_{V^*} \otimes a_V) = (\text{id}_C \otimes \text{ev}_V) \circ (a_{V^*} \otimes \text{id}_V).\]
Lemma 3.1. Suppose that $f : W \to V$ is a morphism of right $C$-comodules. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
V^* \boxtimes V & \overset{c_V}{\longrightarrow} & C \\
\downarrow f^* \otimes 1 & & \downarrow c_W \\
V^* \boxtimes W & \overset{1 \otimes f}{\longrightarrow} & W^* \boxtimes W
\end{array}
$$

Proof. Indeed, this follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
W^* \otimes W & \overset{\text{id}_{W^*} \otimes a_W}{\longrightarrow} & W^* \otimes W \otimes C \\
\downarrow \text{id}_{W^*} \otimes a_W & & \downarrow \text{id}_{W^*} \otimes a_W \\
V^* \otimes W \otimes C & \overset{\text{ev}_V \otimes \text{id}_C}{\longrightarrow} & V^* \otimes V \otimes C \\
\downarrow \text{id}_{V^*} \otimes a_W & & \downarrow \text{id}_{V^*} \otimes a_V \\
V^* \otimes W & \longrightarrow & V^* \otimes V
\end{array}
$$

The top left square is obviously commutative. The bottom right square is commutative because $f : W \to V$ is a morphism of comodules. The top right square is simply (2.1) tensored with $C$, and thus is commutative.

Corollary 3.2. Suppose that $V$ is a right $C$-comodule, with $W$ a sub-comodule of $V$. Then $W^\perp \boxtimes W \subseteq \ker c_V$.

Proof. Clear from previous lemma.

Lemma 3.3. Suppose that $V$ is a right $C$-comodule, with $W$ a sub-comodule of $V$ and $U$ a quotient of $V$. Then $\text{Im} c_W$ and $\text{Im} c_U$ are sub-bicomodules of $\text{Im} c_V$.

Proof. Apply the commutative squares obtained from Lemma 3.1 for the two morphisms $W \to V$ and $V \to U$.

Corollary 3.4. If $V$ is a right $C$-comodule and $W$ is a subquotient of $V$ as a comodule, then $\text{Im} c_W$ is a sub-bicomodule of $\text{Im} c_V$.

Lemma 3.5. Let $V$ be a right $C$-comodule, and suppose that $W_1, W_2$ are subcomodules such that $W_1 + W_2 = V$. Then $\text{Im} c_V = \text{Im} c_{W_1} + \text{Im} c_{W_2}$. Similarly, if $U_1, U_2$ are quotients comodules of $V$ such that the map $V \to U_1 \oplus U_2$ is injective, then $\text{Im} c_V = \text{Im} c_{U_1} + \text{Im} c_{U_2}$.

Proof. We apply Lemma 3.3 to the epimorphism $W_1 \oplus W_2 \to V$ and monomorphism $V \to U_1 \oplus U_2$, and use Corollary 3.2 to find that $c_{W_1 \oplus W_2}$ (resp. $c_{U_1 \oplus U_2}$) factors through $c_{W_1} \oplus c_{W_2}$ (resp. $c_{U_1} \oplus c_{U_2}$).
3.2. Given a finite-length right $C$-subcomodule $V$ of $C$, let $\epsilon_V : V \to 1$ be the restriction of $\epsilon$ to $V$ and $\epsilon_V^* : 1 \to V^*$ its dual. Then the following is a commutative diagram of right $C$-comodules:

$$
\begin{array}{ccc}
V & \xrightarrow{\epsilon_V \otimes 1} & V^* \otimes V \\
& \searrow_{\epsilon_V} & \downarrow \\
& & C
\end{array}
$$

Thus $V$ is a right subcomodule of the image of $\epsilon_V$ in $C$. Since $C$ is the sum of its finite length right sub-comodules, it follows that $C = \sum \text{Im} \epsilon_V$, where the sum runs over all right $C$-comodules in $C$.

4. Socle Filtration and Injectives

4.1. The objects of $\text{Mod}_C$, $\text{cMod}$, and $\text{cMod}_C$ admit socle filtrations. Using the same notation as Green in [Gre], we write $\sigma_i(V)$ for the $i$th term in the socle filtration of an object $V$. In this case we have that $V$ is the direct limit of its socle filtration. If the socle filtration of an object $V$ is finite (which happens in particular if $V$ is of finite-length, i.e. is in $C$), then we write $\ell\ell(V)$ for the length of the socle filtration, the Loewy length of $V$. In this case, $\ell\ell(V)$ is the length of every minimal semisimple filtration of $V$. Further, then $V$ also has a radical filtration which is a descending filtration whose $i$th term we write as $\rho^i(V)$, and whose length is also $\ell\ell(V)$. Recall that $\rho^1(V) := \rho(V)$ is defined to be the minimal subcomodule of $V$ such that $V/\rho(V)$ is semisimple, and we define the filtration inductively by $\rho^1(V) = \rho(\rho^{i-1}(V))$.

**Lemma 4.1.** If $V$ is of finite length, then $\sigma_i(V)^\perp = \rho^i(V^*)$ and $\rho^i(V)\perp = \sigma_i(V^*)$.

**Proof.** Follows from the fact that dualizing is an antiequivalence of comodule categories. □

4.2. The socle filtration on $C$ as a $C$-bicomodule is often called the coradical filtration of $C$, and is sometimes written $C_i := \sigma_i(C)$. The goal of this paper is to give a description of the layers of the coradical filtration of $C$.

4.3. Define the functor $F_C : \hat{C} \to \text{Mod}_C$ by $F_C(S) = S \otimes C$ (see 2.7).

**Lemma 4.2.** The functor $F_C$ is right adjoint to the forgetful functor $\text{Mod}_C \to \hat{C}$.

**Proof.** The proof follows the same ideas as in (1.5a) of [Gre]. □

4.4.

**Lemma 4.3.** The categories $\text{cMod}$ and $\text{Mod}_C$ have enough injectives.

**Proof.** Given a right $C$-comodule $V$, $F_C(V)$ is injective by Lemma 4.2 and the morphism $a_V : V \to F_C(V)$ is a monomorphism of right $C$-comodules. □

**Lemma 4.4.** The direct sum of injective comodules is injective.

**Proof.** The proof in (1.5b) of [Gre] carries through to our case. □
Given a right \( C \)-comodule \( V \), an injective envelope of \( V \) is the data of an injective right comodule \( I \) with a monomorphism \( V \to I \) which induces an isomorphism \( \sigma(V) \cong \sigma(I) \). An injective envelope is unique up to isomorphism if it exists. Using Brauer’s idempotent lifting process as described in [Gre], we can prove that injective envelopes always exist. Choose for each simple right comodule \( L \) an injective envelope \( I(L) \). We now have:

**Corollary 4.5.** The indecomposable injective right comodules are exactly those of the form \( I(L) \) for a simple right comodule \( L \). Thus the injective right comodules are exactly the direct sums of injective indecomposables \( I(L) \).

5. **Structure of \( C \) as a right comodule**

We now make some assumptions on \( C \) and its right comodule category.

(C1) We suppose that if \( L \) is a simple right \( C \)-comodule and \( S \) is a simple object of \( C \), then \( L \) and \( S \otimes L \) are not isomorphic as comodules unless \( S \cong 1 \).

(C2) We assume that if \( L, L' \) are right \( C \)-comodules then \( L^* \boxtimes L' \) is a simple bicomodule, and further every simple bicomodule is of this form.

**Remark 5.1.** Assumption (C2) implies that every semisimple bicomodule is semisimple as a right comodule. In particular, if \( V \) is a bicomodule of finite length then its Loewy length as a right comodule is less than or equal to its Loewy length as a bicomodule.

Assumption (C1) is saying that the action of the Picard group of \( C \) (the group of invertible objects of \( C \) modulo isomorphism, under tensor product) on the set of simple comodules is free. Thus we may, and do, choose representatives of each orbit, \( \{ L_\alpha \}_\alpha \). In other words the simple right comodules \( L_\alpha \) have the property that if \( L_\alpha \cong S \otimes L_\beta \) for a simple object \( S \) of \( C \), then \( \alpha = \beta \) and \( S \cong 1 \). Further if \( L \) is a simple right comodule then there exists an \( \alpha \) and a simple object \( S \) of \( C \) such that \( L \cong S \otimes L_\alpha \).

**Lemma 5.2.** Every simple bicomodule may be written as \( L_\alpha^* \boxtimes L \) for a unique \( \alpha \) and a unique simple right comodule \( L \), up to isomorphism.

**Proof.** By (C2) the simple bicomodules are all of the form \( (L')^* \boxtimes L'' \) for some simple right comodules \( L', L'' \). Choose \( \alpha \) such that \( L' \cong S \otimes L_\alpha \). Then by Lemma 2.1 \( (L')^* \boxtimes L'' \cong (L_\alpha^* \otimes S^*) \boxtimes L'' \cong L_\alpha^* \boxtimes (S^* \otimes L'') \). Setting \( L = S^* \otimes L'' \) we have proven the first half of our claim.

The proof of uniqueness of \( L \) is a little trickier. We prove the following statement which implies it: if \( L \) is a simple right comodule, \( L' \) a simple left comodule, and \( S \) is a simple object of \( C \), then if \( (L' \otimes S) \boxtimes L \cong L' \boxtimes L \) then \( S \cong 1 \). Write \( G = \text{Pic}(C) \) for the Picard group of \( C \), that is the group of simple (hence invertible) objects of \( C \) up to isomorphism under tensor product. For each \( g \in G \), choose a representative simple object \( S_g \), and let \( h \in G \) be the class of \( S \) so that \( S \cong S_h \). Finally, write \( \phi : (L' \otimes S) \boxtimes L \to L' \boxtimes L \) for a given isomorphism of bicomodules.

Now write as objects of \( C \) isotypic decompositions \( L' = \bigoplus_g T_g \), where \( T_g \cong S_g^{\oplus n_g} \) and \( L = \bigoplus_g U_g \) where \( U_g \cong S_g^{\oplus m_g} \). The isomorphism \( \phi \) of bicomodules gives rise to an isomorphism of right comodules

\[
\bigoplus_g (T_g \otimes S) \otimes L \cong \bigoplus_g T_g \otimes L
\]
and an isomorphism of left comodules

\[ \bigoplus_{g} L' \otimes (S \otimes U_g) \cong \bigoplus_{g} L' \otimes U_g. \]

By (C1), this must induce isomorphisms of right comodules

\[ (T_g \otimes S) \otimes L \cong T_{gh} \otimes L, \quad (5.1) \]

i.e. \( \phi \) must take \( (T_g \otimes S) \otimes L \) into \( T_{gh} \otimes L \) for all \( g \in G \), and similarly of left comodules

\[ L' \otimes (S \otimes U_g) \cong L' \otimes U_{hg}, \quad (5.2) \]

i.e. \( \phi \) must take \( L' \otimes (S \otimes U_g) \) into \( L' \otimes U_{hg} \) for all \( g \in G \). However for \( g, h, k \in G \), (5.1) implies that \( \phi \) induces an isomorphism

\[ T_g \otimes S \otimes U_k \cong T_{gh} \otimes U_k \]

while (5.2) implies that \( \phi \) induces an isomorphism

\[ T_{gh} \otimes S \otimes U_{h^{-1}k} \cong T_{gh} \otimes U_k. \]

Thus we learn that \( \phi \) takes both \( T_g \otimes S \otimes U_k \) and \( T_{gh} \otimes S \otimes U_{h^{-1}k} \) isomorphically to \( T_{gh} \otimes U_k \); but since these are objects of \( C \), and hence of finite length, this forces an equality of subobjects of \( L' \boxtimes L \), namely that \( T_g \otimes S \otimes U_k = T_{gh} \otimes S \otimes U_{h^{-1}k} \). But these subobjects are distinct unless \( gh = g \) and \( h^{-1}k = k \) i.e. \( h \) must be the identity, and so \( S \cong 1 \) as desired.

**Lemma 5.3.** If \( S \) is a simple object of \( C \) and \( V \) is in \( \text{mod}_C \), then \( \text{Im} c_V = \text{Im} c_{S \otimes V} \). In particular if \( L \) is a simple right comodule and \( L \cong S \otimes L_\alpha \), then \( \text{Im} c_L = \text{Im} c_{L_\alpha} \).

**Proof.** By Corollary 2.2 we have \( V^* \boxtimes V \cong (S \otimes V)^* \boxtimes (S \otimes V) \), and this isomorphism of bicomodules respects the matrix coefficient morphisms. \( \square \)

**Proposition 5.4.** We have \( \sigma(C) := \sigma_1(C) = \bigoplus_{\alpha} L_\alpha^* \boxtimes L_\alpha \) as bicomodules.

**Proof.** For each \( \alpha \) we have a nonzero, and thus injective, morphism \( c_{L_\alpha} : L_\alpha^* \boxtimes L_\alpha \to \sigma(C) \). Since the simple bicomodules \( L_\alpha^* \boxtimes L_\alpha, L_\beta^* \boxtimes L_\beta \) are non-isomorphic for distinct \( \alpha, \beta \) by Lemma 5.2, we obtain an inclusion \( \bigoplus_{\alpha} L_\alpha^* \boxtimes L_\alpha \subseteq \sigma(C) \). Conversely, a simple sub-bicomodule \( W \) of \( C \) must be semisimple as a right \( C \)-comodule by Remark 5.1, and thus if \( W = \bigoplus_i L_i \) for simple right comodules \( L_i \) then \( W \subseteq \sum_i \text{Im} c_{L_i} \). Now Lemma 5.3 completes the proof. \( \square \)

**Corollary 5.5.** We have an isomorphism of right comodules:

\[ C \cong \bigoplus_{\alpha} L_\alpha^* \otimes I(L_\alpha) \]

**Proof.** By Proposition 5.4 these right comodules have isomorphic socles. Since injectives are determined by their socles, we are done. \( \square \)
6. LAYERS OF THE CORADICAL FILTRATION

6.1. We would like to prove that if $V$ is a finite-length right $C$-comodule then $\ell(\text{Im } c_V) = \ell(V)$, i.e. the Loewy length of $\text{Im } c_V$ as bicomodule is equal to the Loewy length of $V$ as a right comodule. First we prove a lemma.

**Lemma 6.1.** Suppose that $W$ is a right comodule of finite length with simple socle $L$. Choose a splitting $\tilde{L} \subseteq W^*$ (in $C$) of the epimorphism $W^* \to L^*$ so that we obtain a right subcomodule $\tilde{L} \otimes W$ of $W^* \boxtimes W$. Then the restriction of $c_W$ to $\tilde{L} \otimes W$ is injective.

*Proof.* Since this restriction defines a morphism of right comodules $\tilde{L} \otimes W \to C$, it suffices to show that it is injective on the socle $\sigma(\tilde{L} \otimes W) = \tilde{L} \otimes L$. However $\tilde{L} \otimes L$ is a splitting of the head of the bicomodule $W^* \boxtimes L$, and the restriction of $c_W$ to $W^* \boxtimes L$ has $L^\bot \boxtimes L = \rho(W^* \boxtimes L)$ in its kernel by Corollary 3.2, and thus factors through $(W^* \boxtimes L)/(L^\bot \boxtimes L) \cong L^* \boxtimes L \xrightarrow{\sigma} C$. In summary we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{L} \otimes L & \longrightarrow & W^* \otimes L \\
\sigma_W|_{\tilde{L} \otimes L} \uparrow & & \downarrow \sigma_L \\
C & \longrightarrow & L^* \boxtimes L
\end{array}
\]

Since the composition $\tilde{L} \otimes L \to L^* \boxtimes L$ is an isomorphism and $c_L$ is injective we are done. \hfill \Box

**Lemma 6.2.** Let $V$ be a finite-length right comodule with $\ell(\ell(V)) = n$. Then

\[F_k = \sum_{i+j=k} \rho^{n-i}(V^*) \boxtimes \sigma_j(V) = \sum_{i+j=k} \sigma_{n-i}(V) \boxtimes \sigma_j(V)\]

is a semisimple filtration of $V^* \boxtimes V$ such that $F_1 = 0$ and $F_{2n} = V^* \boxtimes V$.

*Proof.* The tensor product of semisimple filtrations is again a semisimple filtration. \hfill \Box

We now observe that $F_n = \sum \sigma_i(V) \boxtimes \sigma_i(V) \subseteq \ker c_V$ and thus $F_\bullet$ induces a semisimple filtration of $\text{Im } c_V$ of length at most $n = \ell(\ell(V))$. It follows that $\ell(\text{Im } c_V) \leq \ell(\ell(V))$.

On the other hand $V$ contains a subquotient $W$ with $\ell(W) = \ell(V)$ such that $W$ has a simple socle. Since $\text{Im } c_W \subseteq \text{Im } c_V$, if we can show that $\ell(\text{Im } c_W) \geq \ell(W) = \ell(V)$ then we will have that $\ell(\text{Im } c_V) = \ell(V)$.

By Lemma 6.1 we know that $\text{Im } c_W$ contains a right subcomodule of the form $\tilde{L} \otimes W$ for an object $\tilde{L}$ of $C$, and thus its Loewy length as a right comodule is at least $\ell(\ell(W))$, which by Remark 5.1 implies its Loewy length as a bicomodule is at least $\ell(\ell(W))$. We have now finished showing:

**Proposition 6.3.** For a finite length right comodule $V$, $\ell(V) = \ell(\text{Im } c_V)$.

**Proposition 6.4.** We have

\[\sigma_i(C) = \sum_{\ell(V) \leq i} \text{Im } c_V.\]

*Proof.* By Proposition 6.3, $\text{Im } c_V$ has Loewy length equal to that of $V$, so if $\ell(V) \leq i$ then $\text{Im } c_V = \sigma_i(\text{Im } c_V) \subseteq \sigma_i(C)$. Conversely if $V \subseteq \sigma_i(C)$ is a right sub-comodule then by Remark 5.1 $\ell(V) \leq i$ and so $V \subseteq \text{Im } c_V \subseteq \sigma_i(C)$. Since $\sigma_i(C)$ is the sum of its right subcomodules, we are done. \hfill \Box
6.2. From now on, fix \( n \in \mathbb{N} \) with \( n \geq 1 \). We make a finiteness assumption on the comodule category \( \text{mod}_C \).

(C3-n) Assume that if \( L, L' \) are simple right comodules, then \([\sigma_n(I(L)) : L'] < \infty\).

Note that (C3-n) implies (C3-m) whenever \( m \leq n \).

6.3. For each pair of simple right comodules \( L, L' \) and for each \( i \leq n \) we define \( H_{L',L}^i \) to be the right subcomodule of \( \sigma_i(I(L')) \) that is generated by a splitting of the isotypic component of \( L \) in \( \sigma_i(I(L'))/\sigma_{i-1}(I(L')) \). In particular it is zero if and only if \( [\sigma_i(I(L'))/\sigma_{i-1}(I(L')) : L] = 0 \). By (C3-n), \( H_{L',L}^i \) is a finite length right comodule.

Further we have for \( i \leq n \)

\[
\sigma_i(I(L')) = \sum_{L \text{ simple}} \sum_{j \leq i} H_{L',L}^j.
\] (6.1)

Write \( H_{\alpha,L}^i := H_{I_{\alpha},L}^i \).

Lemma 6.5. For \( i \leq n \),

\[
\sigma_i(C) = \sum_{\alpha} \sum_{L \text{ simple}} \sum_{j \leq i} \text{Im} \ c_{H_{\alpha,L}^j}.
\]

Proof. Since \( \ell\ell(H_{\alpha,L}^j) \leq j \leq i \), by Proposition 6.4 it suffices to show that \( \text{Im} \ c_V \) is contained in the RHS whenever \( V \) is a right comodule of Loewy length less than or equal to \( i \). However in this case \( \text{Im} \ c_V = \sum \text{Im} \ c_W \) where the sum runs over quotients of \( V \) with simple socles. Note that \( \ell\ell(W) \leq \ell\ell(V) \leq i \) for all such \( W \). On the other hand, if \( W \) has a simple socle \( L' \) then after potentially twisting \( W \) by a simple object \( S \) (which won’t change \( \text{Im} \ c_W \)) we may assume \( L' \cong L_\alpha \) for some \( \alpha \), and then \( I(L_\alpha) \) is the injective envelope of \( W \). If \( \ell\ell(W) \leq i \) then \( W \subseteq \sigma_i(I(L_\alpha)) \) under an embedding of \( W \) in \( I(L_\alpha) \). Therefore by 6.1,

\[
W \subseteq \sum_{L \text{ simple}} \sum_{j \leq i} H_{\alpha,L}^j.
\]

and so there exists finitely many simple right comodules \( L_1, \ldots, L_n \) such that

\[
W \subseteq \sum_{k, j \leq i} H_{\alpha_{k},L_k}^j
\]

and hence

\[
\text{Im} \ c_W \subseteq \sum_{k, j \leq i} \text{Im} \ c_{H_{\alpha_{k},L_k}^j}.
\]

\[ \square \]

6.4. We may now state the main theorem.

Theorem 6.6. For \( i \leq n \),

\[
[\sigma_i(C)/\sigma_{i-1}(C) : L^* \boxtimes L'] = [\sigma_i(I(L))/\sigma_{i-1}(I(L)) : L'].
\]
Proof. The case of $i = 1$ is Proposition 5.4. If $n = 1$ then the theorem is proven.

Otherwise if $n > 1$ we consider the case $i > 1$. We use Lemma 6.5 and study the contribution of $\text{Im } c_{H_{\alpha,L}}^i$ for a fixed simple comodule $L$. Write $V_i = H_{\alpha,L}^i$ and $V_{i-1} = \sigma_{i-1}(H_{\alpha,L}^i)$ so that $V_i/V_{i-1}$ is a sum of copies of $L$. Consider the sub-bicomodule $W = V_i^* \otimes V_{i-1} = (V_i/L_\alpha)^* \otimes V_i$ of $V_i^* \otimes V_i$. We see from the arguments of Lemma 3.1 that

$$c_{V_i}(W) \subseteq \text{Im } c_{V_{i-1}} + \text{Im } c_{V_i/L_\alpha},$$

and since $\ell \ell(V_{i-1}), \ell \ell(V_i/L_\alpha) \leq i - 1$, we find that $c_{V_i}(W) \subseteq \sigma_{i-1}(C)$. We have

$$(V_i^* \otimes V_i)/W \cong L_\alpha^* \otimes V_i/V_{i-1}$$

and so we have epimorphisms

$$L_\alpha^* \otimes V_i/V_{i-1} \cong V_i^* \otimes V_i/W \rightarrow \text{Im } c_{V_i}/c_{V_i}(W) \rightarrow \text{Im } c_{V_i}/(\sigma_{i-1}(C) \cap \text{Im } c_{V_i}). \quad (*)$$

We aim to show this composition (*) is in fact an isomorphism. To this end, choose a splitting $\tilde{L}_\alpha$ of $V_i^* \rightarrow L_\alpha$ so that we get a right subcomodule $\tilde{L}_\alpha \otimes V_i$ of $V_i^* \otimes V_i$. By Lemma 6.1, the restriction of $c_{V_i}$ to $\tilde{L}_\alpha \otimes V_i$ will be injective. Further, as a right comodule we have

$$\sigma_{i-1}(\tilde{L}_\alpha \otimes V_i) = \tilde{L}_\alpha \otimes V_{i-1}.$$ 

Thus by Remark 5.1

$$\sigma_{i-1}(C) \cap c_{V_i}(\tilde{L}_\alpha \otimes V_i) \subseteq c_{V_i}(\tilde{L}_\alpha \otimes V_{i-1}).$$

Conversely $\tilde{L}_\alpha \otimes V_{i-1} \subseteq W$ and therefore

$$c_{V_i}(\tilde{L}_\alpha \otimes V_{i-1}) \subseteq \sigma_{i-1}(C) \cap c_{V_i}(\tilde{L}_\alpha \otimes V_i)$$

which implies these are equal. It follows that we obtain an injection of right comodules

$$L_\alpha^* \otimes V_i/V_{i-1} \rightarrow \text{Im } c_{V_i}/(\sigma_{i-1}(C) \cap \text{Im } c_{V_i})$$

and so (*) is an isomorphism. What this shows is that the contribution of $\text{Im } c_{H_{\alpha,L}}^i$ to $\sigma_i(C)/\sigma_{i-1}(C)$ is exactly $L_\alpha^* \otimes V_i/V_{i-1}$. By Lemma 5.2 it follows that

$$\sigma_i(C)/\sigma_{i-1}(C) = \bigoplus_{\alpha} L_\alpha^* \otimes \sigma_i(I(L_{\alpha}))/\sigma_{i-1}(I(L_{\alpha})).$$

Thus we have proven the theorem whenever $L \cong L_\alpha$ for some $\alpha$. For the general case we write $L \cong S \otimes L_\alpha$ for some $\alpha$ and some simple object $S$ of $C$ and derive the result from Lemma 2.1. 

We now obtain a generalization of the Taft-Wilson theorem for pointed coalgebras over a field.

**Corollary 6.7.** Assume (C1)-(C2) and that $L, L'$ are simple right comodules such that $\dim \text{Ext}^1(L, L') < \infty$. Then

$$[\sigma_2(C)/\sigma_1(C) : L^* \otimes L'] = \dim \text{Ext}^1(L, L').$$
REFERENCES

[BZS] Y.A. Bahturin, M.V. Zaicev, and S.K. Sehgal. *Finite-dimensional simple graded algebras*, Sbornik: Mathematics, Vol. 199, no. 7 (2008): 965–983.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, American Mathematical Soc., Vol. 205 (2016).

[Gre] J.A. Green. *Locally finite representations*, Journal of Algebra Vol. 41, no. 1 (1976): 137–171.

[Ser] V. Serganova. *Quasireductive supergroups*, New developments in Lie theory and its applications, Vol. 544 (2011): 141–159.

[She] A. Sherman. *Spherical supervarieties*, Annales de l’Institut Fourier, Vol. 71, no. 4 (2021): 1449–1492.

[Sta] H.B. Stauffer. *The completion of an abelian category*, Transactions of the American Mathematical Society, Vol. 170 (1972): 403–414.

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