One-way quantum deficit and quantum coherence in the anisotropic XY chain

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In this study, we investigate pairwise non-classical correlations measured using a one-way quantum deficit as well as quantum coherence in the XY spin-1/2 chain in a transverse magnetic field for both zero and finite temperatures. The analytical and numerical results of our investigations are presented. In the case when the temperature is zero, it is shown that the one-way quantum deficit can characterize quantum phase transitions as well as quantum coherence. We find that these measures have a clear critical point at $\lambda = 1$. When $\lambda \leq 1$, the one-way quantum deficit has an analytical expression that coincides with the relative entropy of coherence. We also study an XX model and an Ising chain at the finite temperatures.

Keywords: One-way quantum deficit, quantum coherence, quantum phase transitions, XY chain

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I. INTRODUCTION

Quantum entanglement is considered to be a resource in many quantum information processing tasks [1, 2], and entangled states have been experimentally created using 14 trapped ions [3], five superconducting qubits [4], and optical systems [5–7]. Quantum entangled states can be used as resources in quantum cryptography [8], quantum dense coding [9], quantum communications [10], and quantum key distribution [11]. Nonetheless, in the past few decades, it is realized that quantum correlations beyond entanglement also play essential roles in quantum information processing [12]. Many measures have been proposed to quantify quantum correlations in physical systems, e.g., quantum discord [13, 14], one-way quantum deficit [15], measurement induced disturbance [16], geometric discord [17], quantum dissonance [18], and measurement induced non-locality [19].

In recent years, quantum coherence has also attracted considerable attention [20]; consequently, reasonable measures of quantum coherence have been discussed extensively [21]. As an analogy of quantum entanglement, quantum coherence may be also considered as a resource to characterize the classical-quantum boundary [22].

The study of various quantum correlation measures in the ground states of spin models has been an active area of research. Entanglement in the finite size XY chain has been investigated [23]. Multi-particle entanglement in an anisotropic XY model in a transverse field has been explored by using different criteria for detecting the entanglement [24, 25], which shows that it obeys a scaling behavior near the critical point of the quantum phase transition in the model. Interest in the subject has increased since the introduction of quantum discord [13], which is one of the most important quantum correlations that characterizes the quantum phase transition [26]. Following quantum discord, many other measures have been introduced to explore the correlations in the context of the XY model. The quantum phase transition is studied using local quantum uncertainty and Wigner-Yanase skew information [27]. It has been shown that single-spin coherence reliably identifies the quantum phase transition in the thermal ground state of the anisotropic spin-1/2 XY chain in the transverse magnetic field. Geometric discord is used to characterize the quantum phase transition for the XY model [28]. There are many followed results dedicated to the quantum phase transition in other spin chain models [29], such as XXZ, XY, Lipkin-Meshkov-Glick (LMG), XY with the Dzyaloshinskii-Moriya (DM) interaction [30], both in the thermodynamic limit and in few body cases.

In this article, we consider more general quantum correlations in the XY model. The one-way quantum deficit is one of the popular measures that can characterize and quantify the quantum correlations [15]. Nevertheless, the one-way quantum deficit has not been studied with regards to characterizing the XY spin-1/2 chain. Quantum coherence based on $l_1$ norm and relative entropy measures is also a basic and important method for characterizing quantum systems [21]. We capture the quantum phase transition using these measures to study quantum phase transitions for the XY chain in a transverse field.

The article is organized as follows. In Section II, we recall the basic notation and concepts of the one-way quantum deficit and quantum coherence. The spin-1/2 anisotropic XY chain is introduced in Section III. The numerical results regarding the quantum phase transition are presented in Section IV. Finally, we present our conclusions in Section V.
II. ONE-WAY QUANTUM DEFICIT AND QUANTUM COHERENCE

Let us first review the basic definitions of the one-way quantum deficit and quantum coherence.

One-way quantum deficit The one-way quantum deficit is defined as the difference in the von Neumann entropy of a bipartite state, $\rho_{AB}$, before and after a measurement is performed, without a loss of generality, on particle $A$ [31],

$$\Delta = \min_{\Pi_A} S\left(\sum_i \Pi_A(i\rho_{AB})\right) - S(\rho_{AB}),$$

(1)

where $\Pi_A$ is the measurement on subsystem $A$ and $S(\rho_{AB}) = -\text{Tr}\rho_{AB} \log \rho_{AB}$ is the von Neumann entropy. Throughout the article, log is in base 2. The minimum is taken over all local measurements $\Pi_A$.

Quantum coherence We consider the $l_1$ norm and relative entropy of coherence measures in this article. For a fixed basis set $\{\ket{i}\}$, the set of incoherent states $I$ is the set of quantum states with diagonal density matrices, with respect to this basis. For an arbitrary quantum state

$$\rho = \sum_{i,j} \rho_{i,j} \ket{i}\bra{j},$$

(2)

the $l_1$ norm coherence, $C_{l_1}(\rho)$, of the state $\rho$ is defined by

$$C_{l_1}(\rho) = \sum_{i\neq j} |\rho_{i,j}|,$$

(3)

which is the sum of the absolute values of all the non-diagonal entries for $\rho$.

The relative entropy of the coherence is defined as

$$C_{rel}(\rho) = S(\rho_{\text{diag}}) - S(\rho),$$

(4)

where $S(\cdot)$ is the von Neumann entropy and $\rho_{\text{diag}}$ denotes the state obtained from $\rho$ by deleting all the off-diagonal elements.

III. ANISOTROPIC XY CHAIN

The Hamiltonian, $H$, of the one-dimensional anisotropic spin-$\frac{1}{2}$ XY chain in a transverse magnetic field is given by

$$H = -\sum_{j=0}^{N-1} \left\{ \frac{\Lambda}{2} [(1 + \gamma)\sigma_j^x\sigma_{j+1}^x + (1 - \gamma)\sigma_j^y\sigma_{j+1}^y] + \sigma_j^z \right\},$$

(5)

where $\sigma_j^k(k=x,y,z)$ is the $k$th component of the spin-$1/2$ Pauli operator acting on the $j$th spin, $\gamma$ is the degree of anisotropy (for simplicity we take this to be $0 \leq \gamma \leq 1$), and $\Lambda$ is the strength of the inverse of the external transverse magnetic field. In this study, we focus on the infinite chain case, $N \to \infty$. When $\gamma = 0$, the Hamiltonian reduces to the $XX$ chain, and the Ising model in transverse field when $\gamma = 1$.

The diagonalization procedure for the XY model includes the well-established techniques of Jordan-Wigner and the Bogoliubov transformation [32, 33]. By considering the thermal ground state, the reduced density operator for sites $0$ and $n$ can be described by

$$\rho_{0n} = \frac{1}{4} \left\{ I_{0n} + \langle \sigma^z \rangle (\sigma_0^z + \sigma_n^z) + \sum_{k=x,y,z} (\sigma_0^k \sigma_n^k) \right\},$$

(6)

where $I_{0n}$ is the identity matrix acting on the state space of sites $0$ and $n$. Here, $n$ indicates for the distance between two spins. The two-spin reduced density matrix is only dependent on the distance between the spins $n = |j-i|$, with $j, i$ denoting two different spins. The Hamiltonian exhibits global $\mathbb{Z}_2$ symmetry. The density matrix is of the alphabet $X$ form,

$$\rho_{0n} = \frac{1}{4} \begin{pmatrix}
1 + 2\langle \sigma^z \rangle + \langle \sigma_0^x \sigma_n^x \rangle & 0 & 0 & 0 \\
0 & 1 - \langle \sigma_0^x \sigma_n^x \rangle & \langle \sigma_0^y \sigma_n^y \rangle & \langle \sigma_0^z \sigma_n^z \rangle - \langle \sigma_0^z \sigma_n^z \rangle \\
0 & \langle \sigma_0^y \sigma_n^y \rangle & 1 - \langle \sigma_0^y \sigma_n^y \rangle & 0 \\
\langle \sigma_0^z \sigma_n^z \rangle - \langle \sigma_0^z \sigma_n^z \rangle & \langle \sigma_0^z \sigma_n^z \rangle & 0 & 1 - 2\langle \sigma^z \rangle + \langle \sigma_0^z \sigma_n^z \rangle \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

(7)

Owing to the fact that the system is invariant under translations, the entries of the two-site reduced-density depend only on the distance, $n$. The transverse magnetization is given by

$$\langle \sigma^z \rangle = -\int_0^{\pi} \frac{(1 + \lambda \cos \phi) \tanh(\beta \omega_{\phi})}{2\pi \omega_{\phi}} d\phi,$$

(8)

where $\omega_{\phi} = \sqrt{(\gamma \lambda \sin \phi)^2 + (1 + \lambda \cos \phi)^2}/2$ and $\beta = 1/(kT)$ with $k$ being the Boltzmann’s constant and $T$ the absolute temperature. The two-point correlation func-
tion is given by

\[
\langle \sigma_0^x \sigma_n^x \rangle = \begin{vmatrix} F_{-1} & F_{-2} & \cdots & F_{-n} \\ F_0 & F_{-1} & \cdots & F_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-2} & F_{n-3} & \cdots & F_{-1} \end{vmatrix},
\]

(9)

\[
\langle \sigma_0^y \sigma_n^y \rangle = \begin{vmatrix} F_1 & F_0 & \cdots & F_{n+2} \\ F_2 & F_1 & \cdots & F_{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ F_n & F_{n-1} & \cdots & F_1 \end{vmatrix},
\]

(10)

and

\[
\langle \sigma_0^z \sigma_n^z \rangle = \langle \sigma^z \rangle^2 - F_n F_{-n},
\]

(11)

with

\[
F_n = \int_0^\pi \frac{\tanh(\beta \omega_0)}{2\pi \omega_0} \{ \cos(n\phi)(1 + \lambda \cos \phi) - \gamma \lambda \sin(n\phi) \sin \phi \} d\phi.
\]

(12)

Tracing out one of the two spins, we have the reduced density matrix of a single-spin,

\[
\varrho_0 = \varrho_i = \frac{1}{2} \begin{pmatrix} 1 + \langle \sigma^z \rangle & 0 \\ 0 & 1 - \langle \sigma^z \rangle \end{pmatrix},
\]

(13)

where \(\langle \sigma^z \rangle\) is the transverse magnetization in Eq.(8). All of the single-spin reduced density matrices are of the same form (13).

### IV. BEHAVIORS OF CORRELATIONS

From the one-way quantum deficit defined in Eq.(1), we have

\[
S(\varrho_{0n}) = -\sum_{i=0}^1 (\eta_i \log \eta_i + \xi_i \log \xi_i),
\]

(14)

where

\[
\eta_i = [1 + \langle \sigma_0^z \sigma_n^z \rangle + (-1)^i \sqrt{4\langle \sigma^z \rangle^2 + (\langle \sigma_0^z \sigma_n^z \rangle - \langle \sigma_0^z \sigma_n^z \rangle^2)}]/4,
\]

and

\[
\xi_i = [1 - \langle \sigma_0^z \sigma_n^z \rangle + (-1)^i \langle \sigma_0^z \sigma_n^z \rangle + \langle \sigma_0^z \sigma_n^z \rangle]/4.
\]

(15)

(16)

We perform the complete set of orthonormal projective measurements, \(\Pi_n\), on the \(n\)th nearest spins, with \(\Pi_n^\dagger \Pi_n = V_i^\dagger V_i\), where \(i \in \{0, 1\}\) and

\[
V = \begin{pmatrix} \cos(\theta/2) & e^{i\varphi} \sin(\theta/2) \\ -e^{-i\varphi} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.
\]

(17)

Thus, we obtain the first term of Eq.(1) as follows

\[
S(\sum_{i} \Pi_n^\dagger (\varrho_{0n})) = -\sum_{i=0}^1 \sum_{i,j=0} (\xi_{ij} \log \xi_{ij}),
\]

(18)

where

\[
\xi_{ij} = [1 + (-1)^i \langle \sigma^z \rangle \cos \theta + (-1)^j \sqrt{(\langle \sigma_0^z \sigma_n^z \rangle^2 \cos^2 \varphi + \langle \sigma_0^y \sigma_n^y \rangle^2 \sin^2 \varphi}] \sin^2 \theta + \langle \sigma^z \rangle + (-1)^i \langle \sigma_0^z \sigma_n^z \rangle \cos \theta]}/4.
\]

(19)
FIG. 2. (Color online) The first (second) row shows the quantum correlations (QCs) of the first neighbors spins for $XX$ (Ising) model. From left to right are the figures for relative entropy (Re-QC), $l_1$ norm ($l_1$-QC) and the one-way quantum deficit (QD), with respect to temperature ($kT$) and the magnetic field $\lambda$.

Therefore, the one-way quantum deficit is given by $\{\theta, \varphi\}$

$$\Delta = \min_{\theta, \varphi} \left[ -\sum_{i,j=0}^{1} \xi_{ij} \log \xi_{ij} + \sum_{i=0}^{1} (\eta_i \log \eta_i + \xi_i \log \xi_i) \right].$$  

(20)

The $l_1$ norm of the coherence can be directly shown to be

$$C_{l_1}(\rho_{0n}) = \langle \sigma_0^x \sigma_n^x \rangle.$$  

(21)

The relative entropy of the coherence is given by

$$C_{rel}(\rho_{0n}) = \sum_{i=0}^{1} (\zeta_i \log \zeta_i + \eta_i \log \eta_i + \xi_i \log \xi_i) - 2\varepsilon \log \varepsilon,$$

(22)

with

$$\zeta_i = [1 + \langle \sigma_0^z \sigma_n^z \rangle + (-1)^i 2 \langle \sigma_i^z \rangle] / 4$$

(23)

and $\varepsilon = [1 - \langle \sigma_0^z \sigma_n^z \rangle] / 4$.

Remarkably, the numerical analysis implies that the extremization is achieved when $\lambda \leq 1$. The one-way quantum deficit can be represented by an analytical expression by choosing $\theta = \varphi = 0$, which is in coincidence with the relative entropy of the coherence $\Delta = C_{rel}(\rho_{0n})$. In the region where $\lambda > 1$, it is only possible to obtain the numerical solution for the one-way quantum deficit.

A. Quantum phase transition and correlations at zero temperature

The one-way quantum deficit, relative entropy of coherence and $l_1$ norm for the first, second and fifth nearest neighbors in the thermal ground state (6) near zero temperature are depicted in Fig.1. The figures in the first row show the quantum coherence by the $l_1$ norm, the relative entropy of coherence and the one-way quantum deficit for $n = 1, 2$ and 5, respectively, from left to right, with $\gamma = 0.5$. The figures in the second row are the first order derivations of the corresponding quantum correlations with respect to the parameter $\lambda$.

As expected, all these three types of quantum correlations decrease as the distance, $n$, increases. Nonetheless, we see a clear difference between the regions where $\lambda < 1$ and $\lambda > 1$. It is evident that the first order derivations of the corresponding quantum correlations with respect to the parameter, $\lambda$, are singular at the quantum phase transition point $\lambda = 1$. Namely, these quantum correlations can be used to characterize the quantum phase transition in this quantum system.

Some properties are obvious: the quantum coherence via the $l_1$ norm is greater than both the relative entropy of coherence and the one-way quantum deficit. The one-way quantum deficit coincides with the relative entropy of coherence in the region where $\lambda \leq 1$. While in the region where $\lambda > 1$, the $l_1$ norm is greater than the relative entropy of coherence, which is greater than one-way quantum deficit. A critical point appears at $\lambda = 1$. The first order derivations of these quantum correlations show the quantum phase transition clearly: they change sharply at $\lambda = 1$ for $n = 1, 2$ and 5, with $\gamma = 0.5$. 


B. Quantum phase transition and correlations at finite temperature

We now consider the thermal state of the XX ($\gamma = 0$) chain at finite temperatures. In this case, we plot the three kinds of quantum correlations for the first nearest neighbors, as shown in Fig.2. Similar results can be obtained for the other nearest neighbors, $n$. It can be seen that for a given $\lambda$ ($\lambda > 1$), the quantum correlations decrease as the temperature increases. However, in a given region of $\lambda$ ($\lambda \leq 1$), the quantum correlations indeed increase as the temperature increases when near the critical value of $\lambda = 1$. The quantum phase transition phenomenon disappears as the temperature increases.

For the case of the transverse Ising model ($\gamma = 1$) at finite temperatures, as can be seen in the figures in the second row of Fig.2, the one-way quantum deficit increases when $\lambda$ increases until it is nearly 1. It then decreases as $\lambda$ increases, even in the high temperature region. Meanwhile the coherence always increases when $\lambda$ increases for a given temperature.

V. CONCLUSIONS

In this study, we investigated the pairwise quantum correlations in the thermodynamic limit of the anisotropic XY spin-1/2 chain in the presence of an external transverse magnetic field. The cases of both zero and finite temperatures have been considered. We have shown that the one-way quantum deficit, the $l_1$ norm, and the relative entropy of coherence can be used to characterize the quantum phase transition for the anisotropic XY chain. It has been shown that in the region where $\lambda \leq 1$, the one-way quantum deficit has an analytical expression that coincides with the relative entropy of coherence. The critical point is at $\lambda = 1$. The XX chain and the Ising models have also been studied at finite temperature. These results can highlight the investigation of the relations between quantum correlations and quantum phase transition.

VI. ACKNOWLEDGMENTS

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