Lower Bounds on Unambiguous Automata
Complementation and Separation via
Communication Complexity

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Abstract

We use results from communication complexity, both new and old ones, to prove lower bounds for problems on unambiguous finite automata (UFAs). We show:

1. Complementing UFAs with \( n \) states requires in general at least \( n\tilde{\Omega}(\log n) \) states, improving on a bound by Raskin.

2. There are languages \( L_n \) such that both \( L_n \) and its complement are recognized by NFAs with \( n \) states but any UFA that recognizes \( L_n \) requires \( n\tilde{\Omega}(\log n) \) states, refuting a conjecture by Colcombet on separation.

1 Preliminaries

Finite Automata

An NFA is a quintuple \( \mathcal{A} = (Q, \Sigma, \delta, I, F) \), where \( Q \) is the finite set of states, \( \Sigma \) is the finite alphabet, \( \delta \subseteq Q \times \Sigma \times Q \) is the transition relation, \( I \subseteq Q \) is the set of initial states, and \( F \subseteq Q \) is the set of accepting states. We write \( q \xrightarrow{a} r \) to denote that \((q, a, r) \in \delta \). A finite sequence \( q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n \) is called a run; it can be summarized as \( q_0 \xrightarrow{a_1 \cdots a_n} q_n \). The NFA \( \mathcal{A} \) recognizes the language \( L(\mathcal{A}) := \{ w \in \Sigma^* \mid \exists q_0 \in I . \exists f \in F . q_0 \xrightarrow{f} f \} \). The NFA \( \mathcal{A} \) is a DFA if \( |I| = 1 \) and for every \( q \in Q \) and \( a \in \Sigma \) there is exactly one \( q' \) with \( q \xrightarrow{a} q' \). The NFA \( \mathcal{A} \) is a UFA if for every word \( w = a_1 \cdots a_n \in \Sigma^* \) there is at most one accepting run for \( w \), i.e., a run \( q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n \) with \( q_0 \in I \) and \( q_n \in F \). Clearly, any DFA is a UFA.

Notation \( \tilde{O} \) and \( \tilde{\Omega} \)

We use the notation \( \tilde{O}(f(n)) \) and \( \tilde{\Omega}(f(n)) \) to hide polylogarithmic factors; i.e., \( \tilde{O}(f(n)) = f(n) \log^{\tilde{O}(1)} f(n) \) and \( \tilde{\Omega}(f(n)) = f(n) / \log^{\tilde{O}(1)} f(n) \).
2 UFA Complementation

Given two finite automata $A_1, A_2$, recognizing languages $L_1, L_2 \subseteq \Sigma^*$, respectively, the state complexity of union (or intersection, or complement, etc.) is how many states may be needed for an automaton that recognizes $L_1 \cup L_2$ (or $L_1 \cap L_2$, or $\Sigma^* \setminus L_1$, etc.). It depends on the type of automaton considered, such as NFAs, DFAs, or UFAs.

The state complexity has been well studied for various types of automata and language operations, see, e.g., [9] and the references therein for some known results. For example, it was shown in [7] that complementing an NFA with $n$ states may require $\Theta(2^n)$ states. However, the state complexity for UFAs is not yet fully understood. It was shown only in 2018 by Raskin [11] that the state complexity for UFAs and complement is not polynomial:

**Proposition 2.1** ([11]). For any $n \in \mathbb{N}$ there exists a UFA $A$ with $n$ states and unary alphabet $\Sigma$ (i.e., $|\Sigma| = 1$) such that any NFA that recognizes $\Sigma^* \setminus L(A)$ has at least $n^{(\log \log \log n)^{\Omega(1)}}$ states.

This super-polynomial blowup (even for unary alphabet and even if the output automaton is allowed to be ambiguous) refuted a conjecture that it may be possible to complement UFAs with a polynomial blowup [3]. A non-trivial upper bound (for general alphabets and outputting a UFA) was shown by Jirásek et al. [9]:

**Proposition 2.2** ([9]). Let $A$ be a UFA with $n \geq 7$ states that recognizes a language $L \subseteq \Sigma^*$. Then there exists a UFA with at most $n \cdot 2^{0.786n}$ states that recognizes the language $\Sigma^* \setminus L$.

An almost tight analysis [8] of Jirásek et al.’s construction yields a slight improvement:

**Proposition 2.3** ([8]). Let $A$ be a UFA with $n \geq 0$ states that recognizes a language $L \subseteq \Sigma^*$. Then there exists a UFA with at most $\sqrt{n+1} \cdot 2^{n/2}$ states that recognizes the language $\Sigma^* \setminus L$.

In this section we improve the lower bound from Proposition 2.1:

**Theorem 2.4.** For infinitely many $N \in \mathbb{N}$ there is a UFA $A$ with $N$ states and alphabet $\Sigma = \{0, 1\}$ and finite $L(A) \subseteq \Sigma^*$ such that any NFA that recognizes $\Sigma^* \setminus L(A)$ has at least $N^{\tilde{O}(\log N)}$ states.

Like Proposition 2.1, the lower bound holds even for NFAs (not just UFAs) that recognize the complement language. Unlike Proposition 2.1, the lower bound in Theorem 2.4 uses a binary alphabet, i.e., $|\Sigma| = 2$.

In the rest of the section we prove Theorem 2.4. The proof uses concepts and results from communication complexity, in particular a recent result from [1].
2.1 Communication Complexity

Let $D = C_1 \lor \cdots \lor C_m$ be an $n$-variate boolean formula in disjunctive normal form (DNF). DNF $D$ has width $k$ if every $C_i$ is a conjunction of at most $k$ literals. We call such $D$ a $k$-DNF. For conjunctive normal form (CNF) formulas the width and $k$-CNFs are defined analogously. DNF $D$ is said to be unambiguous if for every input $x \in \{0, 1\}^n$ at most one of the conjunctions $C_i$ evaluates to true, $C_i(x) = 1$. For any boolean function $f : \{0, 1\}^n \to \{0, 1\}$ define

- $C_1(f)$ as the least $k$ such that $f$ can be written as a $k$-DNF;
- $C_0(f)$ as the least $k$ such that $f$ can be written as a $k$-CNF;
- $UC_1(f)$ as the least $k$ such that $f$ can be written as an unambiguous $k$-DNF.

Note that $C_0(f) = C_1(\neg f)$. The following is a recent result [11]:

**Theorem 2.5** ([1] Theorem 1). For infinitely many $n$ there exists a boolean function $f : \{0, 1\}^n \to \{0, 1\}$ with $UC_1(f) = n^{\Omega(1)}$ and $C_0(f) = \Omega(UC_1(f)^2)$.

In words, for infinitely many $k$ there is an unambiguous $k$-DNF such that any equivalent CNF requires width $\tilde{\Omega}(k^2)$. The bound is almost tight, as every unambiguous $k$-DNF has an equivalent $k^2$-CNF; see [6] Section 3.

We need results on two-party communication complexity; see [10] for the standard textbook. Consider a “two-party” function $F : X \times Y \to \{0, 1\}$. A set $A \times B \subseteq X \times Y$ (with $A \subseteq X$ and $B \subseteq Y$) is called a rectangle. Rectangles $R_1, \ldots, R_k$ cover a set $S \subseteq X \times Y$ if $\bigcup R_i = S$. For $b \in \{0, 1\}$, the cover number $Cov_b(F)$ is the least number of rectangles that cover $F^{-1}(b)$.

The nondeterministic (resp., co-nondeterministic) communication complexity of $F$ is defined as $N_1(F) := \log_2 Cov_1(F)$ (resp., $N_0(F) := \log_2 Cov_0(F)$). Note that $N_0(F) = N_1(\neg F)$. The nondeterministic communication complexity can be interpreted as the number of bits that two parties, holding inputs $x \in X$ and $y \in Y$, respectively, need to communicate in a nondeterministic (i.e., based on guessing and checking) protocol in order to establish that $F(x, y) = 1$; see [10] Chapter 2 for details.

The following is a “lifting” theorem, which allows us to transfer lower bounds on the DNF width of a boolean function to the nondeterministic communication complexity of a two-party function.

**Theorem 2.6** ([6] Theorem 4). For any $n \in \mathbb{N}$ there is a function $g : \{0, 1\}^b \times \{0, 1\}^b \to \{0, 1\}$ with $b = \Theta(\log n)$ such that for any function $f : \{0, 1\}^n \to \{0, 1\}$ the function $F : \{0, 1\}^{bn} \times \{0, 1\}^{bn} \to \{0, 1\}$ defined by

$$F((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = f(g(x_1, y_1), \ldots, g(x_n, y_n))$$

for $x_i, y_j \in \{0, 1\}^b$

satisfies $N_0(F) = \Omega(C_0(f) \cdot b)$ (and thus also $N_1(F) = \Omega(C_1(f) \cdot b)$).
Finally, we need the following simple lemma:

**Lemma 2.7.** If a two-party function $F : \{0, 1\}^m \times \{0, 1\}^m \rightarrow \{0, 1\}$ admits an NFA with $s$ states, i.e., there is an NFA $A$ with $s$ states and $L(A) = \{xy \in \{0, 1\}^{2m} \mid F(x, y) = 1\}$, then $Cov_1(F) \leq s$.

**Proof.** Let $A = (Q, \Sigma, \delta, I, F)$ be an NFA with $L(A) = \{xy \in \{0, 1\}^{2m} \mid F(x, y) = 1\}$. We show that $F^{-1}(1)$ is covered by at most $|Q|$ rectangles. Indeed, $F^{-1}(1)$ equals

$$
\bigcup_{q \in Q} \left( \{x \in \{0, 1\}^m \mid \exists q_0 \in I . q_0 \xrightarrow{x} q\} \times \{y \in \{0, 1\}^m \mid \exists f \in F . q \xrightarrow{y} f\} \right).
$$

(Alternatively, in terms of a nondeterministic protocol, the first party, holding $x \in \{0, 1\}^m$, produces a run for $x$ from an initial state to a state $q$ and then sends the name of $q$, which takes $\log_2 |Q|$ bits, to the other party. The other party then produces a run for $y$ from $q$ to an accepting state.) \hfill \Box

### 2.2 Proof of Theorem 2.4

For $n \in \mathbb{N}$, let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be the function from Theorem 2.5, i.e., $f$ has an unambiguous $k$-DNF with $k = n^{\Omega(1)}$ (hence, $\log n = O(\log k)$) and $C_0(f) = \widetilde{\Omega}(k^2)$. Let $g : \{0, 1\}^b \times \{0, 1\}^b \rightarrow \{0, 1\}$ with $b = \Theta(\log n)$ and $F : \{0, 1\}^{bn} \times \{0, 1\}^{bn} \rightarrow \{0, 1\}$ be the two-party functions from Theorem 2.6 with $F((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = f(g(x_1, y_1), \ldots, g(x_n, y_n))$. The UFA $A$ from the statement of Theorem 2.4 will recognize $F^{-1}(1)$.

First we argue that $F$ has an unambiguous DNF of small width. Indeed, $g$ and $\neg g$ have unambiguous 2$bk$-DNFs, which can be extracted from the deterministic decision tree of $g$. By plugging these unambiguous 2$bk$-DNFs for $g$ and $\neg g$ into the unambiguous $k$-DNF for $f$ (and “multiplying out”), one obtains an unambiguous 2$bk$-DNF, say $D$, for $F$.

Over the $2bn$ variables of $F$, there exist at most $(2(2bn) + 1)^{2bk}$ different conjunctions of at most $2bk$ literals. So $D$ consists of at most $n^{O(2bk)}$ conjunctions. From $D$ we obtain a UFA $A$ that recognizes $F^{-1}(1) \subseteq \{0, 1\}^{2bn}$, as follows. Each initial state of $A$ corresponds to a conjunction in $D$. When reading the input $x \in \{0, 1\}^{2bn}$, the UFA checks that the corresponding assignment to the variables satisfies the conjunction represented by the initial state. This check requires at most $O(bn)$ states for each initial state. Thus, $A$ has at most $n^{O(2bk)} = 2^\widetilde{O}(k) = N$ states in total.

On the other hand, by Theorem 2.6 we have $N_0(F) = \Omega(C_0(f) \cdot b) = \widetilde{\Omega}(k^2)$. So by Lemma 2.7 any NFA that recognizes $F^{-1}(0)$ has at least $2^{\widetilde{O}(k^2)}$ states. Any NFA that recognizes $\{0, 1\}^* \setminus L(A)$ can be transformed into an NFA that recognizes $F^{-1}(0) = \{0, 1\}^{2bn} \setminus L(A)$ by taking a product with a DFA that has $2bn + 2$ states. It follows that any NFA that recognizes $\{0, 1\}^* \setminus L(A)$ has at least $2^{\widetilde{O}(k^2)/(2bn + 2)} = 2^{\widetilde{O}(k^2)} = N^{\widetilde{O}(\log N)}$ states. \hfill \Box
3 Separation of Regular Languages by UFAs

In [3 Conjecture 2], Colcombet conjectured that for any NFAs $\mathcal{A}_1, \mathcal{A}_2$ with $L(\mathcal{A}_1) \cap L(\mathcal{A}_2) = \emptyset$ there is a polynomial-sized UFA $\mathcal{A}$ with $L(\mathcal{A}_1) \subseteq L(\mathcal{A})$ and $L(\mathcal{A}) \cap L(\mathcal{A}_2) = \emptyset$. Related separability questions are classical in formal language theory and have attracted renewed attention; see, e.g., [5] and the references therein. Separating automata have also been used recently to elegantly describe quasi-polynomial time algorithms for solving parity games in an automata theoretic framework; see [2 Chapter 3] and [3].

In this section we refute the above-mentioned conjecture by Colcombet, even when $L(\mathcal{A}_2) = \Sigma^* \setminus L(\mathcal{A}_1)$:

**Theorem 3.1.** For any $N \in \mathbb{N}$ there are NFAs $\mathcal{A}_1, \mathcal{A}_2$ with $N$ states and alphabet $\Sigma = \{0, 1\}$ and finite $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2) = \Sigma^* \setminus L(\mathcal{A}_1)$ such that any UFA that recognizes $L(\mathcal{A}_1)$ has at least $N^{\Omega(\log N)}$ states.

Loosely speaking, in our construction, NFAs $\mathcal{A}_1, \mathcal{A}_2$ recognize (sparse) set disjointness and its complement. For $n \in \mathbb{N}$ write $[n] := \{1, \ldots, n\}$ and define for $k \leq n$

$$\text{Disj}_k^n := \{ (S, T) | S \subseteq [n], T \subseteq [n], |S| = |T| = k, S \cap T = \emptyset \}.$$ 

Define also $\langle \text{Disj}_k^n \rangle := \{ \langle S \rangle \langle T \rangle | (S, T) \in \text{Disj}_k^n \}$ where $\langle S \rangle \in \Sigma^n = \{0, 1\}^n$ is such that the $i$th letter of $\langle S \rangle$ is 1 if and only if $i \in S$, and similarly for $\langle T \rangle$. Note that $\langle S \rangle, \langle T \rangle$ each contain $k$ times the letter 1. To prove Theorem 3.1 it suffices to prove the following lemma.

**Lemma 3.2.** For any $n \in \mathbb{N}$ let $k := \lceil \log_2 n \rceil$. There are NFAs $\mathcal{A}_1, \mathcal{A}_2$ with $n^{O(1)}$ states and alphabet $\Sigma = \{0, 1\}$ and $L(\mathcal{A}_1) = \langle \text{Disj}_k^n \rangle$ and $L(\mathcal{A}_2) = \Sigma^* \setminus \langle \text{Disj}_k^n \rangle$. Any UFA that recognizes $\langle \text{Disj}_k^n \rangle$ has at least $n^{\Omega(\log n)}$ states.

In the rest of the section we prove Lemma 3.2. We use known results from communication complexity to show that any UFA for $\langle \text{Disj}_k^n \rangle$ needs super-polynomially many states. We will give a self-contained proof of the existence of polynomial-sized NFAs for $\langle \text{Disj}_k^n \rangle$ and its complement, but the main argument also comes from communication complexity, as we remark below at the end of the section.

3.1 Communication Complexity

Recall from Section 2.1 the notions of rectangles and rectangles covering a set. For a two-party function $F : X \times Y \to \{0, 1\}$, the partition number $\text{Par}_1(F)$ is the least number of pairwise disjoint rectangles that cover $F^{-1}(1)$. Note that $\text{Cov}_1(F) \leq \text{Par}_1(F)$. The unambiguous communication complexity of $F$ is defined as $U_1(F) := \log_2 \text{Par}_1(F)$. Note that $N_1(F) \leq U_1(F)$. Denote by $M(F) \in \{0, 1\}^{X \times Y}$ the communication matrix,
with entries \( M(F)_{x,y} = F(x,y) \). Denote by \( \text{rank}(M) \) the rank over the reals of a matrix \( M \). The following lemma, the “rank bound”, is often used for lower bounds on the deterministic communication complexity (a concept we do not need here), but it holds even for unambiguous communication complexity:

**Lemma 3.3.** Let \( F : X \times Y \to \{0,1\} \). Then \( \text{rank}(M(F)) \leq \text{Par}_1(F) \).

**Proof.** For \( k = \text{Par}_1(F) \), let \( A_1 \times B_1, \ldots, A_k \times B_k \) be pairwise disjoint rectangles that cover \( F \). Each \( A_i \times B_i \) defines a rank-1 matrix \( M(i) \in \{0,1\}^{X \times Y} \) with \( M(i)_{x,y} = 1 \) if and only if \( x \in A_i \) and \( y \in B_i \). It follows from the pairwise disjointness that \( M(F) = \sum_{i=1}^k M(i) \). Hence \( \text{rank}(M(F)) \leq \sum_{i=1}^k \text{rank}(M(i)) = k = \text{Par}_1(F) \).

The following lemma and its proof are analogous to Lemma 2.7.

**Lemma 3.4.** If a two-party function \( F : \{0,1\}^m \times \{0,1\}^m \to \{0,1\} \) admits a UFA with \( s \) states, i.e., there is a UFA \( A \) with \( s \) states and \( L(A) = \{xy \in \{0,1\}^{2m} \mid F(x,y) = 1\} \), then \( \text{Par}_1(F) \leq s \).

**Proof.** Let \( A = (Q, \Sigma, \delta, I, F) \) be a UFA with \( L(A) = \{xy \in \{0,1\}^{2m} \mid F(x,y) = 1\} \). We show that \( F^{-1}(1) \) is covered by at most \( |Q| \) pairwise disjoint rectangles. Indeed, \( F^{-1}(1) \) equals

\[
\bigcup_{q \in Q} \left( \{x \in \{0,1\}^m \mid \exists q_0 \in I . q_0 \xrightarrow{q} q\} \times \{y \in \{0,1\}^m \mid \exists f \in F . q \xrightarrow{f} y\} \right)
\]

and the rectangles do not overlap, as \( A \) is unambiguous. \( \Box \)

### 3.2 Proof of Lemma 3.2

First we prove the statement on UFAs. Write \( \binom{[n]}{k} := \{S \subseteq [n] \mid |S| = k\} \). Let \( F : \binom{[n]}{k} \times \binom{[n]}{k} \to \{0,1\} \) be the two-party function with \( F(S,T) = 1 \) if and only if \( (S,T) \in \text{Disj}_k^m \). It is shown, e.g., in [10, Example 2.12] that the communication matrix \( M(F) \) has full rank \( \binom{m}{k} \). Let \( F' : \{0,1\}^n \times \{0,1\}^n \to \{0,1\} \) be such that \( F'(x,y) = 1 \) if and only if \( xy \in \text{Disj}_k^m \). Then \( M(F) \) is a principal submatrix of \( M(F') \), so \( \binom{m}{k} \leq \text{rank}(M(F')) \). Using Lemmas 3.3 and 3.4 it follows that any UFA, say \( A \), that recognizes \( \text{Disj}_k^m \) has at least \( \binom{m}{k} \) states. With \( k := \lceil \log_2 n \rceil \), it follows that \( A \) has \( n^{\Omega(\log n)} \) states.

It is easy to see that there is an NFA, \( A_2 \), with \( n^{O(1)} \) states and \( L(A_2) = \Sigma^* \setminus \text{Disj}_k^m \). Indeed, we can assume that the input is of the form \( \langle S \rangle \langle T \rangle \); otherwise \( A_2 \) accepts. NFA \( A_2 \) guesses \( i \in [n] \) such that \( i \in S \cap T \) and then checks it.

Finally, we show that there is an NFA, \( A_1 \), with \( n^{O(1)} \) states and \( L(A_1) = \text{Disj}_k^m \). We can assume that the input is of the form \( \langle S \rangle \langle T \rangle \); otherwise \( A_1 \) rejects. NFA \( A_1 \) “hard-codes” polynomially many sets \( Z_1, \ldots, Z_t \subseteq [n] \). It
guesses \( i \in \ell \) such that \( S \subseteq Z_i \) and \( Z_i \cap T = \emptyset \) and then checks it. It remains to show that there exist \( \ell = n^{O(1)} \) sets \( Z_1, \ldots, Z_\ell \subseteq [n] \) such that for any \((S,T) \in \text{Disj}_k^n\) there is \( i \in [\ell] \) with \( S \subseteq Z_i \) and \( Z_i \cap T = \emptyset \). The argument uses the probabilistic method and is due to [12]; see also [10, Example 2.12]. We reproduce it here due to its elegance and brevity.

Fix \((S,T) \in \text{Disj}_k^n\). Say that a set \( Z \subseteq [n] \) separates \((S,T)\) if \( S \subseteq Z \) and \( Z \cap T = \emptyset \). A random set \( Z \subseteq [n] \) (each \( i \) is in \( Z \) with probability \( \frac{1}{2} \)) separates \((S,T)\) with probability \( 2^{-2k} \). Thus, choosing \( \ell := \left\lceil 2^{2k \ln \left( \frac{n}{k} \right)} \right\rceil = n^{O(1)} \) random sets \( Z \subseteq [n] \) independently, the probability that none of them separates \((S,T)\) is

\[
(1 - 2^{-2k})^\ell < e^{-2^{-2k \ell}} \leq \left( \frac{n}{k} \right)^{-2}.
\]

By the union bound, since \( |\text{Disj}_k^n| < \left( \frac{n}{k} \right)^2 \), the probability that there exists \((S,T) \in \text{Disj}_k^n\) such that none of \( \ell \) random sets separates \((S,T)\) is less than 1. Equivalently, the probability that for all \((S,T) \in \text{Disj}_k^n\) at least one of \( \ell \) random sets separates \((S,T)\) is positive. It follows that there are \( Z_1, \ldots, Z_\ell \subseteq [n] \) such that each \((S,T) \in \text{Disj}_k^n\) is separated by some \( Z_i \).

The proof above is based on known arguments from communication complexity. Indeed, they show, for \( k = \lceil \log_2 n \rceil \) and the function \( F \) from above, that \( U_1(F) \in \Omega(\log^2 n) \) and \( N_0(F) \in O(\log n) \) and \( N_1(F) \in O(\log n) \). This gap is in a sense the largest possible, as \( U_1(F) = O(N_0(F) \cdot N_1(F)) \) holds for all two-party functions \( F \). We even have \( D(F) = O(N_0(F) \cdot N_1(F)) \), where \( D(F) \geq U_1(F) \) is the deterministic communication complexity [10, Theorem 2.11].

\section{4 Conclusions}

In the main results, Theorems 2.4 and 3.1, we have obtained super-polynomial but quasi-polynomial lower bounds on UFA complementation and separation. These bounds are not known to be tight; indeed, in both cases the best known upper bound is exponential. At the same time, we have transferred techniques from communication complexity relatively directly. More concretely, both main theorems hinge on a finite language \( \{xy \mid F(x,y) = 1\} \) where \( F \) is a two-party function whose communication complexity is in a sense extreme. This suggests two kinds of opportunities for future work:

- Can other techniques from communication complexity improve the lower bounds further? Perhaps by somehow iterating a two-party function or via multi-party communication complexity?

- Can techniques for proving upper bounds on communication complexity be adapted to prove upper bounds on the size of automata?
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