DIFFERENTIAL INCLUSIONS, NON-ABSOLUTELY CONVERGENT INTEGRALS AND THE FIRST THEOREM OF COMPLEX ANALYSIS

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Abstract. In the theory of complex valued functions of a complex variable arguably the first striking theorem is that pointwise differentiability implies \( C^\infty \) regularity. As mentioned in Ahlfors [Ah 78] there have been a number of studies [Po 61], [Pl 59] proving this theorem without use of complex integration but at the cost of considerably more complexity. In this note we will use the theory of non-absolutely convergent integrals to firstly give a very short proof of this result without complex integration and secondly (in combination with some elements of the theory of elliptic regularity) provide a far reaching generalization.

One of the first and most striking theorems about the analysis of complex valued functions of a complex variable is that merely from considering the class of pointwise complex differentiable functions on an open set we instantly find ourself in the category of \( C^\infty \) functions.

Theorem 1. Given open set \( \Omega \subset \mathbb{C} \). Suppose \( f : \Omega \to \mathbb{C} \) is a complex differentiable at every point. Then \( f \) is \( C^\infty \) on \( \Omega \).

Typically Theorem 1 is proved via the method of complex integration. The first step is to prove that the integral of a differentiable function over the boundary of a rectangle inside a ball is zero, this was first proved by Goursat [Go 01]. The existence of an anti-derivative is then concluded, Cauchy’s integral formula follows and it is shown that you can differentiate through the integral of Cauchy’s integral formula infinitely many times and hence the function is \( C^\infty \).

On the first paragraph of page 101 of Ahlfors’s standard text [Ah 78], he writes that many important properties of analytic functions are difficult to prove without use of complex integration. Ahlfors states that only recently it has been possible to prove continuity of the gradient (or the existence of higher gradients) without the use of complex integration. He refers to articles of Plunkett [Pl 59], and Porcelli and Connell [Po 61] both of which rely on a topological theorem of Whyburn [Wh 58]. Ahlfors notes that both these proofs are much more complicated than the original proof.

It turns out that generalizations of Goursat’s theorem have a long history and one line of generalization provides an alternative proof of the theorem that in essence does not require complex integration. This line of research was started by Montel [Mo 07] and further developed by Looman [Lo 23] and Menchoff [Me 36] (their theorem receives a very clear exposition in Saks [Sa 37]) and later by Tolstov [To 42]. Although not explicitly stated in these works the method of proof was essentially to construct a Denjoy type integral to integrate the divergence of a differentiable vector field [Pr 12]. The explicit application of the theory of non-absolutely convergent integrals appears to have first been made in [Ju-No 90] by Jurkat and Nonnenmacher, they used a kind of Perron integral in the plane to prove a generalization of Goursat’s theorem due to Besicovitch [Be 31].

The purpose of this note is firstly to use the theory of non-absolutely convergent integrals to provide the shortest proof of Theorem 1 the proof we provide is also independent of the theory of complex line integrals. Secondly by rephrasing Theorem 1 in terms of differential inclusions

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\end{itemize}
we will provide a far reaching generalization of this result by again applying the theory of non absolutely convergent integrals and some simple elliptic regularity estimates that are afforded to us by the Fourier transform.

First some background. Note that the statement that $f(x + iy) = u(x, y) + iv(x, y)$ is pointwise complex differentiable on $\Omega \subset \mathbb{C}$ is equivalent to the statement that the vector valued function $\tilde{f}(x, y) = (u(x, y), v(x, y))$ is pointwise differentiable and satisfies the differential inclusion

$$D\tilde{f}(x, y) \in \{(a - b) : a, b \in \mathbb{R}\} =: \mathcal{L} \text{ for any } x + iy \in \Omega. \quad (1)$$

The set $\mathcal{L}$ has no rank-1 connections, by this we mean that if $A, B \in \mathcal{L}$ and $\text{rank}(A - B) = 1$ then $A = B$. It turns out this is a crucial property that implies regularity of differential inclusions. We will establish the following generalization of Theorem\[1\]

**Theorem 2.** Suppose $u : \Omega \rightarrow \mathbb{R}^m$ is differentiable on open set $\Omega \subset \mathbb{R}^n$ and $L \subset M^{n \times n}$ is a subspace without rank-1 connections. If $Du(x) \in L$ for every $x \in \Omega$ then $u$ is $C^\infty$.

As noted one of the main ideas we will need to establish Theorem\[2\] is the use of the theory of non-absolutely continuous integrals. The main point about these integrals is that they allow us to integrate all derivatives of functions and thus they provide a stronger form of the fundamental theorem of calculus. On the real line this was accomplished by Denjoy [De 12a, De 12b] and later by Perron [Pe 14] by somewhat different ideas. Later in fifties in Kursweil [Ku 57] and Henstock [He 68] developed a different approach (sometimes known as the Gauge integral) that allowed for an easier generalization to higher dimensions. Recently there has been a strong line of generalization by Maly [Ma 11] who developed a theory of integration with respect to a distribution that he refers to as the CU integral. For our purposes we require any integral that in addition to basic linearity and finiteness properties satisfies

$$\int \phi_i(z) d\lambda_n = 0 \text{ for any } \phi \in C_c(\mathbb{R}^n : \mathbb{R}), i \in \{1, 2, \ldots n\}. \quad (2)$$

These properties are satisfied on the real line by any of the integrals mentioned. They are also satisfied in the plane by a version of the Perron integral [Ju-No 90]. They are satisfied in all dimension by the CU integral of [Ma 11]. The easiest integral to reference that satisfies these properties in all dimensions is the Gauge integral. In particular the divergence theorems of Mawhin [Ma 81a] and Pfeffer [Pf 86] show that such an integral satisfies (2). For this reason we carefully detail the results on the Gauge integral that we need. In order to highlight the role of non-absolutely convergent integrals are playing in the arguments we will denote these integrals as $\int \ldots d\lambda_n$. Where as when the standard Lebesgue integral suffices we will denote these integrals by $\int \ldots dx$.

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### 0.1. Preliminaries

As in [Pf 92] page 134 we define a cell in $\mathbb{R}^n$ to be a set of the form $A = \Pi_{i=1}^m A_i$ where $A_i := [\xi_i, \eta_i]$ where we insist that the interior of this set (denoted $A^0$) is non empty.

As outlined on page 210 [Pf 93], a figure is a finite (possibly empty) union of cells. On page 215 [Pf 93] a set $T \subset \mathbb{R}^m$ is said to be thin if and only if it is the union of countably many sets whose $(m - 1)$ dimensional Hausdorff measure is finite. We let $\mathcal{R}_a(A, \lambda_m)$ denote the set of Gauge integrable functions and $\mathcal{R}_a(A, \lambda_m)$ be the set of extended real valued functions that agree with some function in $\mathcal{R}_a(A, \lambda_m)$ a.e.. The only result on the Gauge integral we need is the following powerful theorem (Theorem 11.4.10 [Pf 93])
Theorem 3. Let $T$ be a thin set and let $v$ be a continuous vector field on a figure $A$ that is differentiable in $A^0 \setminus T$. Then $\text{div} v \in R_\times(A, \lambda_m)$

$$\int_A \text{div} v \, d\lambda_m = \int_{\partial A} v \cdot n \, dH^{m-1}. \quad (3)$$

Since $A$ is continuous on the figure $A$ which is a closed set, the right-hand side of $(3)$ can be considered as any kind of surface integral, in particular as an integral with respect to Hausdorff measure. The left-hand side is a Gauge integral however and under the hypothesis has to be considered in that way. It is immediate from Theorem 3 that the Gauge integral satisfies $(2)$.

1. Elliptic estimates for differential inclusions

As far as we are aware the following estimates are folk law, we learned of them from [St 99], however the proofs are only sketched in [St 99] and the precise estimates we need are not stated so we prove the results in detail. As noted in [St 99] many well known results in Elliptic regularity follow from the rigidity implied by the differential inclusion $Du \in L$ where $L$ is a subspace containing no rank-1 connections and $u$ is a Sobolev function. The reason our Theorem 2 is distinct from Theorem 4 (which will be stated later in this section) is that the space of differentiable functions is distinct from the space of Sobolev functions, however in our opinion the heart of the matter is the powerful estimates of this section.

Lemma 1. Suppose $\Omega$ is a Lipschitz domain and $L$ is a subspace without rank-1 connections. If $v \in W^{1,2}(\Omega : \mathbb{R}^m)$ is such that $Dv \in L$ a.e. then for any sub-domain $U \subset \subset \Omega$ there exists constant $c = c(U)$ s.t.

$$\int_U |Dv|^2 \, dz \leq c \int_\Omega |v|^2 \, dz. \quad (4)$$

Proof of Lemma 1. Let $A : M^{m \times n} \to M^{m \times n}$ be the projection onto $L^\perp$. So if $A(a \otimes b) = 0$ then $a \otimes b \in L$ which is a contradiction because $0 \in L$ and so we would have a rank-1 connection in $L$. Now by homogeneity there exists constant $\lambda$ such that

$$|A(a \otimes b)| \geq \lambda |a \otimes b|. \quad (5)$$

We will represent $A$ by a matrix in the following way. Let $e_{i,j} \in M^{m \times n}$ be a matrix whose entry in the $(i,j)$-th position is 1 and zero otherwise. So the set $\{e_{i,j} : i = 1,2,\ldots,m, j = 1,2,\ldots,n\}$ is a basis for $M^{m \times n}$. For $B,C \in M^{m \times n}$ define the matrix inner product $B : C = \sum_{i=1}^m \sum_{j=1}^n B_{ij} C_{ij}$.

Now define $A_{(i,j),(l,k)} = e_{i,j} : \langle Ae_{l,k} \rangle$. Now for vector valued function $w$ we let $w_1,w_2,\ldots,w_m$ denote the co-ordinate function of $w$. We let $w_{l,k}$ denote the $k$-th partial derivatives of the $l$-th coordinate. So

$$A(e_{l,k}) = \sum_{i=1}^m \sum_{j=1}^n A_{(i,j),(l,k)} e_{i,j}. \quad (6)$$

Maly [Ma 11] defines a class of weakly differentiable functions that include both differentiable functions and Sobolev functions, this yields the intriguing possibility of a broader analysis of functions spaces.
Let $F$ denote the Fourier transform. Let $\phi \in C_0^\infty(\Omega : \mathbb{R})$ be such that $\phi(z) = 1$ for $z \in U$ and $\|\phi\|_{L^\infty(\Omega)} \leq 1$. Let $w(x) = v(x)\phi(x)$. So

$$[F(A(Dw))] (\xi) = \left[ F \left( \sum_{i=1}^m \sum_{k=1}^n A(\omega_{i,k}e_{i,k}) \right) \right] (\xi) = \left[ F \left( \sum_{i=1}^m \sum_{k=1}^n \omega_{i,k} \left( \sum_{l=1}^m \sum_{j=1}^n A(i,j),(l,k) e_{i,j} \right) \right) \right] (\xi) = \sum_{i=1}^m \sum_{k=1}^n \left( \sum_{l=1}^m \sum_{j=1}^n \omega_{i,k} A(i,j),(l,k) \right) e_{i,j} = \sum_{i=1}^m \sum_{k=1}^n \left( \sum_{l=1}^m \sum_{j=1}^n 2\pi i \xi_k \hat{\omega}_l(\xi) A(i,j),(l,k) \right) e_{i,j} = 2\pi i \sum_{l=1}^m \sum_{k=1}^n \xi_k \hat{\omega}_l(\xi) \left( \sum_{i=1}^m \sum_{j=1}^n A(i,j),(l,k) e_{i,j} \right) = 2\pi i A \left( \sum_{i=1}^m \sum_{k=1}^n \xi_k \hat{\omega}_l(\xi) e_{i,k} \right)$$

$$= 2\pi i A(\xi \otimes \hat{\omega}(\xi)). \quad (7)$$

Since $Dv(x) \in L$ a.e.

$$A(Dw) = A(Dv \phi + v \otimes \nabla \phi) = A(v \otimes \nabla \phi). \quad (8)$$

Now by Parseval’s theorem and the fact $A$ is a projection

$$\int |v(x)|^2 |\nabla \phi(x)|^2 dx \geq \int |A(Dw)|^2 dx \geq \int |[F(A(Dw))] (\xi)|^2 d\xi \geq \int |2\pi i A(\xi \otimes \hat{\omega}(\xi))|^2 d\xi \geq 4\pi^2 \lambda^2 \int |\xi \otimes \hat{\omega}(\xi)|^2 d\xi. \quad (9)$$

Now $[F(Dw)] (\xi) = 2\pi i \xi \otimes \hat{\omega}(\xi)$. So again by Parseval’s theorem have

$$\int |v(x)|^2 |\nabla \phi(x)|^2 dx \geq \lambda^2 \int |[F(Dw)] (\xi)|^2 d\xi \geq \lambda^2 \int |Dw(x)|^2 dx. \quad (10)$$

As $Dw(x) = Dv(x)\phi(x) + v(x) \otimes \nabla \phi(x)$. Hence

$$\|\nabla \phi\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \geq \lambda \|Dv\|_{L^2(U)} - \lambda \|\nabla \phi\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \quad (11)$$
which establishes \(\blacksquare\). \(\blacksquare\)

**Theorem 4.** Suppose \(u \in W^{1,2}(\Omega : \mathbb{R}^m)\) with \(Du(x) \in L\) a.e. \(x \in \Omega\) then \(u \in C^\infty\).

**Proof of Theorem 4.** This follows from Lemma 1 relatively easily. Let \(i \in \{1,2,\ldots,n\}\). Let \(\delta > 0\) and \(\Pi_0 := \Omega \setminus N_\delta(\Omega)\), \(\Pi_1 := \Pi_0 \setminus N_\delta(\Pi_0)\). For any \(h \in (0,\delta)\) we have \(w_h(x) := \frac{u(x+he_\iota)-u(x)}{h}\) we have \(Dw_h(x) \in L\) a.e. \(x \in \Pi_0\). So by Lemma 1 we have

\[
\int_{\Pi_1} |Dw_h|^2 \, dz \leq c \int_{\Pi_0} |w_h|^2 \, dz.
\]

Since \(u \in W^{1,2}(\Omega : \mathbb{R}^m)\), \(w_h \stackrel{L^2(\Pi_0)}{\rightarrow} u_j\) and \(u_j \in L^2(\Omega)\) so \(\int_{\Pi_0} |w_h| \, dz \leq c\) for all small enough \(h > 0\). Since \(i\) is arbitrary this implies \(Du \in W^{1,2}(\Pi_1)\) and

\[
\int_{\Pi_1} |D^2 u|^2 \, dz \leq c \int_{\Pi_0} |Du|^2 \, dz.
\]

Now as \(Dw_h(z) \in L\) a.e. \(z\) and \(Dw_h \stackrel{L^2(\Pi_0)}{\rightarrow} Du_j\) so we know \(Du_j \in L\) a.e. \(x \in \Pi_0\). So we can repeat the argument for the function \(u_j\) and gain control of the third order derivatives. By repeating in this way the result follows from Sobolev embedding theorem. \(\blacksquare\)

2. **Proof of Theorem 2**

**Step 1.** We will show that denoting \(u^\varepsilon := u * \rho_\varepsilon\) we have

\[
Du^\varepsilon(x) \in L\text{ for any } x \in \Omega \setminus N_\varepsilon(\partial \Omega).
\]

**Proof of Step 1.** Given \(\psi \in C^\infty(\Omega : M^{m \times n})\). Define the \(m\) vector

\[
[A_\varepsilon(\psi)](z) := \begin{pmatrix}
\psi_{1,1}(z) \\
\psi_{1,2}(z) \\
\vdots \\
\psi_{m,1}(z)
\end{pmatrix}.
\]

Let \(w \in C^1(\Omega' : \mathbb{R}^m)\) where \(\Omega' \subset \Omega\) and \(\text{spt} \psi \subset \Omega'\). Note that for any \(i \in \{1,2,\ldots,m\}\)

\[
\text{div} (w_1(z)\psi_{1,1}(z), w_1(z)\psi_{1,2}(z), \ldots w_1(z)\psi_{i,n}(z)) = \sum_{j=1}^n \left( w_{i,j}(z) \psi_{i,j}(z) + w_1(z) \frac{\partial \psi_{i,j}}{\partial z_j}(z) \right).
\]

Note

\[
\int \text{div} (w_1(z)\psi_{1,1}(z), w_1(z)\psi_{1,2}(z), \ldots w_1(z)\psi_{i,n}(z)) \, dz = 0.
\]

So putting this together with (13) we have

\[
\int \sum_{i=1}^m \sum_{j=1}^n w_{i,j}(z)\psi_{i,j}(z) \, dz = - \int \sum_{i=1}^m \sum_{j=1}^n w_1(z) \frac{\partial \psi_{i,j}}{\partial z_j}(z) \, dz = - \int w(z) \cdot (A_\varepsilon(\psi))(z) \, dz.
\]
Let $\phi \in C^\infty_c(\Omega : M^{m \times n})$ with $\phi(z) \in L^1$ for all $z \in \Omega$. Take $\epsilon > 0$ sufficiently small to that $spt \phi \subset \Omega \setminus N_\epsilon(\partial \Omega)$

$$\int u^\epsilon(z) \cdot [\Lambda_\epsilon(\phi)](z)dz = \int \left( \int u(x) \rho_\epsilon(x-z)dx \right) \cdot ([\Lambda_\epsilon(\phi)](z))dz$$

$$= \int \int u(x) \rho_\epsilon(x-z) \cdot ([\Lambda_\epsilon(\phi)](z))dzdx$$

$$= \int \int \sum_{i=1}^m \sum_{j=1}^n u_i(x) \rho_\epsilon(x-z) \frac{\partial \phi_{ij}(z)}{\partial z_j}dzdx$$

But notice

$$\int \sum_{i=1}^m \sum_{j=1}^n (u_i(x) \rho_\epsilon(x-z)) \frac{\partial \phi_{ij}(z)}{\partial z_j}dz = \sum_{i=1}^m \sum_{j=1}^n \int u_i(x) \frac{\partial \rho_\epsilon}{\partial z_j}(x-z) \phi_{ij}(z)dz$$

$$= \sum_{i=1}^m \sum_{j=1}^n - \int u_i(x) \frac{\partial \rho_\epsilon}{\partial x_j}(x-z) \phi_{ij}(z)dz.$$ \hspace{1cm} (17)

Now note by (2) $\int \frac{\partial}{\partial x_j} (u_i(x) \rho_\epsilon(x-z)) d\lambda_n(x) = 0$. So

$$\int \frac{\partial u_i}{\partial x_j}(x) \rho_\epsilon(x-z)d\lambda_n(x) = - \int u_i(x) \frac{\partial \rho_\epsilon}{\partial x_j}(x-z)dx.$$ \hspace{1cm} (18)

Putting these things together we have

$$\int u^\epsilon(z) \cdot [\Lambda_\epsilon(\phi)](z)dz \overset{\text{(17),(16)}}{=} - \int \int \sum_{i=1}^m \sum_{j=1}^n u_i(x) \frac{\partial \rho_\epsilon}{\partial x_j}(x-z) \phi_{ij}(z)dxdz$$

$$\overset{\text{(18)}}{=} \int \int \sum_{i=1}^m \sum_{j=1}^n \frac{\partial u_i}{\partial x_j}(x) \rho_\epsilon(x-z) \phi_{ij}(z)d\lambda_n(x)dz$$

$$= \int Du(x) : \phi \rho_\epsilon(x) d\lambda_n(x)$$

$$= 0.$$ \hspace{1cm} (19)

since $\phi \ast \rho_\epsilon(x) \in L$ for any $x$.

Thus by (15) equation (19) this implies that $\int Du^\epsilon(z) : \phi(z)dz = 0$ for any $\phi \in C^\infty(\Omega : L^1)$ which implies (12).

**Proof of Theorem 2 completed.** Let $U \subset \subset V \subset \subset \Omega$. Let $\epsilon_n \to 0$. Now by Lemma 1 and Step 1 we know

$$\int_U |Du^{\epsilon_n}|^2 dz \leq c \int_V |u^{\epsilon_n}|^2 dz.$$ \hspace{1cm} (20)

Since $u^{\epsilon_n} \overset{L^2(V)}{\rightharpoonup} u$ so there exists constant $c_0$ such that $\int_U |u^{\epsilon_n}|^2 dx \leq c_0$ for all $n$. So $u^{\epsilon_n}$ is a bounded sequence in $W^{1,2}(U)$ and we can extract a weakly converging subsequence $u^{\epsilon_{n_k}}$ that converge to $u$. So $u \in W^{1,2}(U)$. So by Theorem 4 this implies $u \in C^\infty(\Omega)$. As U is an arbitrary subset of $\Omega$ this implies $u \in C^\infty(\Omega)$. □

3. **A simple proof of Theorem 1**

Theorem 1 is a very special case of Theorem 2 that we have just proved. However in this section we will show the theory of non-absolutely convergent integrals and Weyl’s lemma allows us a very short direct proof of Theorem 1.
Theorem 5. Given open set \( \Omega \subset \mathbb{C} \). Suppose \( f : \Omega \to \mathbb{C} \) is a complex differentiable at every point. Let \( u, v : \Omega \to \mathbb{R} \) be defined by \( f(x + iy) = u(x, y) + iv(x, y) \). We will show \( u, v \) weakly satisfy Laplace’s equation.

Proof of Theorem 5. Firstly since \( f(x + iy) = u(x, y) + iv(x, y) \) is complex differentiable, for any vector \( (r, s) \) we have

\[
f'(x + iy) = \lim_{h \to 0} \frac{u((x, y) + h(r, s)) - u(x, y)}{h} + i \frac{v((x, y) + h(r, s)) - v(x, y)}{h}
\]

and so \( u, v \) are pointwise differentiable in \( \Omega \). Now definition of complex differentiability we have

\[
f'(x + iy) = u_x(x, y) + iv_x(x, y) = -iu_y(x, y) + v_y(x, y).
\]

So \( u, v \) satisfy the Cauchy Riemann equations. Take \( \phi \in C_c^\infty(\Omega) \). Note \( \text{div}(\phi_y \overline{v}, \phi_x v) = -\phi_y \overline{v_x} + \phi_x v_y \) and \( (\phi_y \overline{v}, \phi_x v) \) is a differentiable vector field.

Take \( R > 0 \) such that \( \Omega \subset Q_R(0) \), by (20) (or by Theorem 5) we have that

\[
0 = \int_{Q_R(0)} \text{div}(\phi_y \overline{v}, \phi_x v) \, d\lambda_2 = \int_{\Omega} -\phi_y \overline{v_x} + \phi_x v_y \, d\lambda_2 = \int_{\Omega} \phi_y u_y + \phi_x u_x \, d\lambda_2. \tag{21}
\]

Now note that vector field \( (\phi_x u, \phi_y u) \) is differentiable and

\[
0 = \int_{Q_R(0)} \text{div}(\phi_x u, \phi_y u) \, d\lambda_2 = \int_{\Omega} \triangle u + \phi_x u_x + \phi_y u_y \, d\lambda_2 \tag{21} = \int_{\Omega} \triangle u \, d\lambda_2. \tag{22}
\]

Note that the integral on the right-hand side of (22) is the integral of a continuous function as an integral it is equal to the Lebesgue integral. So specifically we have shown

\[
\int_{\Omega} \triangle u \, dz = 0 \text{ for any } \phi \in C_c^\infty(\Omega) \tag{23}
\]

So \( u \) weakly satisfy Laplace’s equation.

Arguing in the same way. Take \( \phi \in C_c^\infty(\Omega) \). Note the vector field \( (\phi_y v, -\phi_x v) \) is differentiable so applying (20) (or Theorem 5) we have

\[
0 = \int_{Q_R(0)} \text{div}(\phi_y v, -\phi_x v) \, d\lambda_2 = \int_{\Omega} \phi_y \overline{v_x} + \phi_x \overline{v_y} \, d\lambda_2. \tag{24}
\]

And again by differentiability of the vector field \( (\phi_x v, \phi_y v) \) we have

\[
0 = \int_{Q_R(0)} \text{div}(\phi_x v, \phi_y v) \, d\lambda_2 = \int_{\Omega} \triangle v + \phi_x v_x + \phi_y v_y \, d\lambda_2 = \int_{\Omega} \triangle v \, dz \tag{24}
\]

So again \( v \) weakly satisfies Laplace’s equation. □

3.1. Proof of Theorem 1 continued. We write \( f(x + iy) = u(x, y) + iv(x, y) \), by Theorem 5 \( u \) and \( v \) weakly satisfy Laplace’s equation. So by Weyl’s lemma \( u, v \) are \( C^\infty \). So as \( f'(x + iy) = u_x(x, y) + iv_x(x, y) \) and \( u_x, v_x \) satisfy the Cauchy Riemann equations so \( f' \) is complex differentiable. In the same way all orders of derivate of \( f \) exists. □

4. Appendix

4.1. Weyl’s lemma. Weyl’s lemma is well known but how easy and elementary its proof is perhaps is less well dissipated. For completeness we briefly outline the three main points.

Firstly if we have a weakly harmonic function \( u \) defined on \( \Omega \) then the convolution \( v = u * \rho_x \) is harmonic on \( \Omega \setminus N_c(\partial \Omega) \), this follow from the definition of weak harmonicity and differentiating through the integral of the convolution.

The second point is that it follows from the first point that weakly harmonic functions are uniformly approximated by harmonic functions and hence must satisfy the mean value theorem.
The third point is that if function \( u \) satisfies the mean value theorem then letting \( \rho \) be a radial symmetric convolution kernel we have
\[
  u * \rho_c(z) = \int u(x)\rho_c(z-x)dx = \int_0^\infty \int_{\partial B(z)} u(x)\rho_c(z-x)dH^{n-1}dr \\
  = \int_0^\infty \rho_c(z-re_1)u(z)H^{n-1}(\partial B_r(z))dr = u(z)\int \rho_c(z-x)dx = u(z).
\]
So \( u \) is \( C^\infty \) and this completes the sketch.

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