COMPARISON OF QUIVER VARIETIES, LOOP GRASSMANNIANS
AND NILPOTENT CONES IN TYPE A

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Abstract. In type A we find equivalences of geometries arising in three settings: Nakajima’s (“framed”) quiver varieties, conjugacy classes of matrices and loop Grassmannians. These are all given by explicit formulas. In particular, we embedd the framed quiver varieties into Beilinson-Drinfeld Grassmannians. This provides a compactification of Nakajima varieties and a decomposition of affine Grassmannians into Nakajima varieties. As an application we provide a geometric version of symmetric and skew ($GL(m), GL(n)$) dualities.

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1. Introduction

In type A (only) we relate three geometric objects related to representations: Nakajima’s quiver varieties, conjugacy classes of matrices and Beilinson-Drinfeld Grassmannians. The main result is an identification of certain normal slices to standard stratifications in the
three settings, in particular the singularities are the same in all three cases. We also identify natural resolutions of slices.\footnote{These observations clearly do not literally extend beyond type A. For instance, the closures of orbits in the loop Grassmannian are normal and this is not true for nilpotent orbits.} An affine version of this paper has been constructed in \cite{BF1}.

We embed Nakajima’s quiver varieties into the Beilinson-Drinfeld Grassmannians and this provides a compactification of Nakajima’s quiver varieties and a decomposition of loop Grassmannians into a disjoint union of quiver varieties. For nilpotent orbits we construct new transverse slices naturally related to loop Grassmannians. As an application we construct a geometric version of both the skew and the symmetric version of the \((GL_m, GL_n)\) duality.

1.0.1. \textit{Loop Grassmannians and matrices.} To go from a loop Grassmannian \(\mathcal{G}\) of \(G = GL_m\) to matrices in \(\mathfrak{gl}(N)\) we consider a subvariety \(\mathcal{T}\) of \(\mathcal{G}\) such that all lattices \(L \in \mathcal{T}\), contain the standard lattice \(L_0 \overset{\text{def}}{=} \mathbb{C}[[[z]]]^{\otimes m}\). Then a choice of vector space trivializations \(\iota_L : \mathbb{C}^N \xrightarrow{\cong} L/L_0\) for \(L \in \mathcal{T}\), translates the operator \(z\) on lattices \(L \in \mathcal{T}\), into a family of matrices \(x_L\) of size \(N\). It turns out that when \(\mathcal{T}\) is a standard normal slice to a \(G[[z]]\)-orbit in \(\mathcal{G}\) then there is a natural trivialization \(\iota\) and the corresponding family of matrices is a (nonstandard) normal slice to a nilpotent orbit in \(\mathfrak{gl}(N)\). So, one exchanges a setting with a fixed operators \(z\) on \(\mathbb{C}((z))^{\otimes m}/L_0\) and a variable invariant subspace \(L/L_0\) to a setting of a fixed space \(\mathbb{C}^N\) and a variable operator \(x_L\). The origin of our constructions is Lusztig’s embedding of the nilpotent cone for \(GL_n\) into the loop Grassmannian of \(GL_n\) \cite{Lusztig}.

1.0.2. \textit{Quiver varieties and nilpotent orbits.} The relationship between these was conjectured by Nakajima \cite{Nakajima} and proved by Maffei \cite{Maffei}. Our treatment of this relation is close to Maffei’s work. However, while he uses Slodowy’s normal slices to nilpotent orbits we use different slices suggested by the relation to the loop Grassmannians, and this makes the construction canonical and explicit while Maffei’s approach is based on an existence result.

1.0.3. \textit{Isomorphism of quiver varieties and loop Grassmannians.} These two objects are in principle related by combining \cite{1.0.1} and \cite{1.0.2}. Having an explicit formula is a deeper problem which is resolved in the appendix \cite{10} by Vasily Krylov through a completely new approach of using moduli of bundles over \(\mathbb{P}^2\) (following ideas of Nakajima).

1.0.4. Some of the results of this paper have appeared in an earlier version \cite{MV}. The new results are the explicit isomorphism of quiver varieties and loop Grassmannians (see \cite{10}) and a much better understanding of the relation of loop Grassmannians and nilpotent cones \cite{IV}. Beyond this the paper has been extensively rewritten.
1.1. The Isomorphism Theorem. Here we formulate the main result more precisely. We work over the field of complex numbers $\mathbb{C}$.

We consider a quiver $(I, H)$ of type $A_{n-1}$. Any dimension vector $d = (d_1, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}$ for this quiver determines integers $m \overset{\text{def}}{=} d_1 + \cdots + d_{n-1}$ and $N \overset{\text{def}}{=} \sum_{j=1}^{n-1} j d_j$. These give rise to the loop Grassmannian $\mathcal{G}$ for $GL_m$ and the nilpotent cone $N$ for $\mathfrak{gl}_N$.

In the quiver setting, using a choice of $d, v$ in $\mathbb{N}^{n-1}$ and a central element $c = (c_1, \ldots, c_{n-1})$ of the Lie algebra $\mathfrak{gl}(v) \overset{\text{def}}{=} \bigoplus_{i=1}^{n-1} \mathfrak{gl}(v_i, \mathbb{C})$, Nakajima [N1, N2] constructs a map of quiver varieties $M_c(v, d) \rightarrow \tilde{M}_c(v, d)$. These quiver data $(d, v)$ are also reformulated as a pair of dominant coweights $\lambda$ and $\mu$ for $G = GL_m$ which we can think of as a pair of partitions $\lambda, \mu$ of $N$ (see [5.1.1] for the combinatorics of data).

For the loop Grassmannian setting the loop group $LG = G(\mathbb{C}[[z]]) = GL(m, \mathbb{C}[[z]])$ contains the “disc group” $L^{>0}G = G(\mathbb{C}[[z]])$ and the negative congruence subgroup $L^{<0}G$ (the kernel of the evaluation of $G(\mathbb{C}[[z]]) \rightarrow \text{at } z^{-1} = 0$).

Any coweight $\eta$ of $GL_m$ defines a point in the loop Grassmannian $\mathcal{G}$, denoted $L^\eta$ or just $\eta$. It generates the “disc group” orbit $\mathcal{G}_\lambda = L^{>0}G \eta \subseteq \mathcal{G}$. Then $L^{<0}G \cdot \eta$ is a normal slice to $L^{>0}G \eta$.

For the coweights $\lambda, \mu$ above one has $L^{>0}G \lambda \subseteq L^{>0}G \mu$. In the setting of matrices we view $\lambda, \mu$ as partitions of $N$, so they provide nilpotent orbits $O_\lambda \subseteq O_\mu$ in $\mathfrak{gl}_N$. We will denote by $T_\lambda$ certain “regular” normal slice to $O_\lambda$.

The following theorem (announced in [MVY]) is a common generalization of (some of) the results of Kraft-Procesi [KP], Lusztig [L1], and Nakajima [N1]. The lower row contains an affine quiver variety and intersections of normal slices (in nilpotent matrices and the loop Grassmannian) with larger orbits. The three vertical maps are resolutions. Here, the map $\overline{\mathcal{G}_\mu} \overset{m}{\rightarrow} \overline{O_\mu}$ is a certain Springer resolution of the closure of the nilpotent orbit $O_\mu$ and $\pi : \mathcal{G}_\mu \rightarrow \overline{L^{>0}G \cdot \mu}$ is a certain “convolution map”.

1.1.1. Theorem. There exist natural algebraic isomorphisms $\phi, \tilde{\phi}, \psi, \tilde{\psi}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Quiver varieties} & \text{Nilpotent orbits} & \text{Loop Grassmannians} \\
\text{for } A_{n-1} & \text{for } \mathfrak{gl}_N & \text{for } GL_m \\
\mathcal{M}_0(v, d) & (T_\lambda \cap \overline{O_\mu}) \times \overline{O_\mu} & \mathcal{G}_\mu \times \mathcal{G} \\
\overset{\phi}{\sim} & \overset{\psi}{\sim} & \overset{\tilde{\phi}}{\sim} \\
\downarrow p & \downarrow m & \downarrow \pi \\
\mathcal{M}_0(v, d) & T_\lambda \cap \overline{O_\mu} & \overline{L^{>0}G \cdot \mu} \cap L^{<0}G \cdot \lambda \\
\end{array}
\]
1.1.2. Deformation along $Z[gl(v)]$. The above theorem corresponds to the case when the central element $c$ of $gl(v)$ is zero. For arbitrary $c$ the nilpotent orbits deform to general conjugacy classes, and the loop Grassmannian deforms to the fiber of the Beilinson-Drinfeld Grassmannian $G_{\mathbb{A}^{n}}$ at the point $(0, c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{n-1}) \in \mathbb{A}^{n}$. In this generality the theorem is formulated as Theorem 5.2.

1.2. The ‘regular’ normal slice $T$ to a nilpotent orbit. The existence of isomorphisms such as $\phi$, $\tilde{\phi}$ – using Slodowy slices – was conjectured by Nakajima [N1] and established by Maffei [Maf]. In our modifications the normal slice $T$ is not the Slodowy’s slice – it originates from the relation to loop Grassmannian given by $\psi$.\footnote{As we will see in 3.2.5 $T$ is more generic or “regular” than the Slodowy slice.}

Our transverse slice $T$ allows several new results. First, the isomorphism $\phi$ is given by simple explicit formulas, cf. 8.1.2 as opposed to an inductive existence result used in [Maf]. This leads to an explicit formula for the isomorphism $\psi \circ \phi$ between quiver varieties and pieces of the loop Grassmannians in the appendix. Second, we are able to decompose a loop Grassmannian into a disjoint union of quiver varieties, cf. 5.3.3. Finally, our construction provides a natural environment for the geometric $(GL_m, GL_n)$ duality, cf. section 9.

Example. In order to illustrate the difference, let us give an example for $N = 5$ and a nilpotent element $x$ with Jordan blocks of sizes 3 and 2. If we fix the basis in which the matrix of $x$ has the Jordan canonical form, then the two slices are affine spaces

\[
\text{Slodowy’s slice} = \begin{pmatrix}
a_1 & 1 & 0 & 0 & 0 \\
a_2 & a_1 & 1 & b_1 & 0 \\
a_3 & a_2 & a_1 & b_2 & b_1 \\
c_1 & 0 & 0 & d_1 & 1 \\
c_2 & c_1 & 0 & d_2 & d_1
\end{pmatrix}, \quad \text{our slice} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & a_3 & a_2 \\
0 & 0 & 0 & b_2 & b_1 \\
c_2 & c_1 & 0 & d_2 & d_1
\end{pmatrix}.
\]

In terms of a corresponding $sl(2)$-triple $\{x, h, y\}$ both slices are of the form $x + C$ for an $h, y$-invariant subspace $C$ complementary to $[gl_N, x]$ in $gl_N$ on which $h$ acts with non-positive integral eigenvalues. For Slodowy’s slice $C = Z_{gl_N}(y)$, so here $y$ acts by zero. For our slice the action of $y$ on $C$ is “as close to regular nilpotent as possible”, cf. 3.2.5.

1.2.1. Organization of the paper. Section 2 recalls quiver varieties of type A. In section 3 we define the “regular” normal slices $T$ to nilpotent orbits. In section 4 our slice $T$ is related to the Beilinson-Drinfeld family of loop Grassmannians. In section 5 we formulate the Isomorphism Theorem and its consequences in full generality (with a deformation parameter $c$). Section 6 proves relation between quiver varieties and matrices in a special case. The general case is related to this special case in section 7. This allows us to finish the proof of the Isomorphism Theorem in section 8. In section 9 we give a geometric interpretation of $(GL_m, GL_n)$ dualities. Finally, the Appendix (section 10) provides an explicit formula for the isomorphism $\psi \circ \phi$ between quiver varieties and loop Grassmannians.
1.2.2. Glossary of notation. As the paper compares three settings we provide here some basic notations. The relation between the three kind of data is explained in 5.1.

(I) Quiver data $n,v,d,c$. Here $n$ “means” the quiver $A_{n-1}$. Vectors $v,d \in \mathbb{N}^{n-1}$ are data for two Nakajima quiver varieties related by a map $p : \mathcal{M}^c(v,d) \to \mathcal{M}_0^c(v,d)$ (with image $\mathcal{M}_1^c(v,d)$). Here we view $c \in Z[\oplus \mathfrak{gl}(v_i)]$ as a deformation parameter.

(II) Nilpotent orbits data $N,\lambda,\mu$. The partitions $\lambda,\mu$ of $N$ give nilpotent orbits $\mathcal{O}_\lambda \subseteq \overline{\mathcal{O}_\mu}$ in $\mathfrak{gl}(D)$ for a vector space $D$ of dimension $N$. By $T_\lambda = T_x$ we denote the “regular” normal slice to $\mathcal{O}_\lambda$ at $x \in \mathcal{O}_\lambda$.

(III) Loop Grassmannian data $m,\lambda,\mu$. Here $m$ “means” the group $G = GL_m$ with the loop Grassmannian $\mathcal{G} = \mathcal{G}(GL_m)$. We will realize $GL_m$ as $GL(U)$ for a vector space $U$ with a basis $e_1,\ldots,e_m$. Then the loop Grassmannian of $G$ consists of lattices in $U_K = U \otimes K$ where $K \overset{\text{def}}{=} \mathbb{C}((z)) \supseteq O \overset{\text{def}}{=} \mathbb{C}[[z]]$ (see 4.1.1). Any coweight $\eta \in Z^m$ of $GL_m$ defines a lattice $L_\eta = \oplus z^{-m} e_i \in \mathcal{G}$.

For a coweight $\sigma \in Z^m$ the orbit $\mathcal{O}_\lambda = L^\sigma \cdot G \cdot L_\lambda$ of the negative congruence subgroup is the standard normal slice to $\mathcal{O}_\lambda$ in $\mathcal{G}$ at $L_\lambda$.

Res(Resolution datum $a$. In the quiver setting $\mathcal{M}^c(v,d)$ is a resolution of the image $\mathcal{M}_1^c(v,d)$ of $p^c : \mathcal{M}^c(v,d) \to \mathcal{M}_0^c(v,d)$.

A decomposition $a \in \mathbb{N}^n$ of $N = \sum a_i$ defines resolutions in matrices and loop Grassmannians. It gives a partial flag variety $\mathcal{F}^a$ of $GL(D)$ consisting of systems of subspaces $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = D$ with $\dim(F_i/F_{i-1}) = a_i$. Its cotangent bundle $\tilde{\mathcal{N}}^a \overset{\text{def}}{=} T^* \mathcal{F}^a$ is (see 3.1.2) a resolution of the closure $\overline{\mathcal{O}_\mu}$ of a nilpotent orbit $\mathcal{O}_\mu$ where $\mu$ is the dual of the partition $\hat{\mu}$ obtained by ordering the terms of $a = (a_1,\ldots,a_n)$.

In the loop Grassmannian context, $a$ defines a convolution space $\mathcal{G}_\mu^a \overset{\text{def}}{=} \mathcal{G}_{\omega_1} \star \cdots \star \mathcal{G}_{\omega_n}$ for the fundamental coweights $\omega_i$. This is a partial resolution of $\mathcal{G}_\mu$.

Deformation data $c,b$. A quiver datum $c \in Z[\oplus \mathfrak{gl}(v_i)]$ deforms quiver varieties from $\mathcal{M}^c(v,d)$ to $\mathcal{M}^c(v,d)$.

The corresponding “spectral datum” $b \in A^{(n)}$ deforms resolutions in matrices and in loop Grassmannians. In matrices, $b$ deforms resolution $\tilde{\mathcal{N}}^a \to \overline{\mathcal{O}_\mu}$ of the closure of a nilpotent orbit to a resolution of the closure of a general conjugacy class $\mathfrak{g}^{a,b} \to \overline{\mathcal{O}_{E,\hat{\mu}}}$ (see 3.1.5). Here, $E$ is the set of eigenvalues of any element $x \in \mathcal{O}_{E,\hat{\mu}}$ and $\hat{\mu}$ is a family of partitions $\mu^e$, $e \in E$, describing the nilpotent operators $x - e$ on generalized $e$-eigenspace $D_e$ of $x$ in $D$. In the loop Grassmannian $b$ deforms the partial resolution $\mathcal{G}_\mu^a \to \mathcal{G}_\mu$.

Various notation. By $G_m = \mathbb{C}^*$ we denote the multiplicative group. For an algebraic variety $X$ whose connected components $X_c$ are of pure dimension we denote by $\mathcal{H}(X)$ its top-dimensional Borel-Moore homology $\oplus_c H_{2\dim(X_c)}^{BM}(X_c)$. 


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2. Quiver varieties of type A

2.1. Representations of quivers. We start with the Dynkin quiver $(I, \Omega)$ of type $A_{n-1}$ with vertices $I = \{1, \ldots, n-1\}$ and the arrows $\Omega$ given by $1 \to \cdots \to n-1$.

Passing to the Nakajima (framed) version of the quiver involves first doubling the arrows to $H = \Omega \sqcup \bar{\Omega}$ where $\Omega \mapsto \bar{\Omega}$, $\omega \mapsto \bar{\omega}$, is the reversal of orientation. For an arrow $h \in H$ we denote by $h' \in I$ its initial vertex and by $h'' \in I$ its terminal vertex.

The data for framed quiver varieties are two $I$-graded vector spaces $V = \bigoplus_{i \in I} V_i$ and $D = \bigoplus_{i \in I} D_i$. Their dimension vectors $v, d \in \mathbb{N}^I$ define a vector space $M(v, d) = \bigoplus_{h \in H} \text{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i \in I} \text{Hom}(D_i, V_i) \oplus \bigoplus_{i \in I} \text{Hom}(V_i, D_i)$.

Following Lusztig and Maffei we will consider an element in $M(v, d)$ as a quadruple $(x, \overline{x}, p, q)$ with

\begin{equation}
\begin{aligned}
x &= (x_h)_{h \in \Omega} \in \bigoplus_{h \in \Omega} \text{Hom}(V_{h'}, V_{h''}), \quad \overline{x} = (x_h)_{h \in \overline{\Omega}} \in \bigoplus_{h \in \overline{\Omega}} \text{Hom}(V_{h'}, V_{h''}), \\
p &= (p_i)_{i \in I} \in \bigoplus_{i \in I} \text{Hom}(D_i, V_i), \quad q = (q_i)_{i \in I} \in \bigoplus_{i \in I} \text{Hom}(V_i, D_i).
\end{aligned}
\end{equation}

The total of the $x$-data is denoted $\tilde{x} = (x, \overline{x}) = (x_h)_{h \in H}$. For any $j > i$ we define the following polynomial operators in $(\tilde{x}, p, q)$:

\begin{equation}
\begin{aligned}
p_{j \to i} &\overset{\text{def}}{=} \overline{x}_i \cdots \overline{x}_{j-1} p_j : D_j \to V_i \quad \text{and} \quad q_{i \to j} \overset{\text{def}}{=} q_j x_{j-1} \cdots x_i : V_i \to D_j.
\end{aligned}
\end{equation}

2.1.1. The group $G(V) = \prod_{i \in I} GL(V_i)$ acts on $M(v, d)$ so that for $g = (g_i)_{i \in I}$

\begin{equation}
g(\tilde{x}, \overline{x}, p, q) \overset{\text{def}}{=} (g_{i+1} x_i g_i^{-1}, g_i x_i g_{i+1}^{-1}, g_i p_i, q_i g_i^{-1})_{i \in I}.
\end{equation}

We are interested in fibers of the corresponding moment map

\[ \mu : M(v, d) \to \mathfrak{g}(V) \overset{\text{def}}{=} \text{Lie}[G(V)] \cong \bigoplus \mathfrak{l}(V_i), \]
at elements \( c = (c_1, \ldots, c_{n-1}) \in \mathbb{Z}[gl(V)] \). The fiber \( M^c(v, d) \defeq \mu^{-1}(c) \) consists of all \((x, \pi, p, q)\) such that
\[
\begin{align*}
    c_i + \pi_i x_i &= x_{i-1} \pi_{i-1} + p_i q_i & \text{for } 2 \leq i \leq n - 2; \quad \text{and} \\
    c_1 + \pi_1 x_1 &= p_1 q_1, & c_{n-1} = x_{n-2} \pi_{n-2} + p_{n-1} q_{n-1}.
\end{align*}
\]

2.2. Generators of \( G(V) \)-invariant polynomials on \( M^c(v, d) \). The following is version of a result of Lusztig using some ideas of Maffei.

2.2.1. Theorem. The algebra of invariant functions \( \mathcal{O}(M^c(v, d))^{G(V)} \) is generated by polynomials of the form \( \chi(q_{l \to j} p_{i \to l}) \), where \( 1 \leq l \leq i, j < n \), and \( \chi \) is a linear form on \( \text{Hom}(D_i, D_j) \).

Proof. Lusztig [L3 1.2] considered two kinds of invariant polynomials in \( \mathcal{O}(M(v, d))^{G(V)} \).
(a) Any cycle \( \gamma = (h_1, h_2, \ldots, h_r) \) in \( (I, H) \) defines a polynomial \( \text{Tr}(V_{h_l}, \tilde{x}_h, \tilde{x}_{h_{r-1}} \cdots \tilde{x}_{h_1}) \).
(b) A path \( (h_1, h_2, \ldots, h_r) \) in \( (I, H) \) and a linear form \( \chi \) on \( \text{Hom}(D_{h_l'}, D_{h_r'}) \) define a polynomial \( \chi(q_{h_l} \tilde{x}_{h_l}, \tilde{x}_{h_{r-1}} \cdots \tilde{x}_{h_1} p_{h_r'}) \).

He proved [L3 Theorem 1.3, 5.8] that the algebra \( \mathcal{O}(M^c(v, d))^{G(V)} \) is generated by restrictions of invariant polynomials of types (a) and (b) above.

It just remains to switch from Lusztig’s generators to Maffei’s generators [Maf] using the following lemma. □

Lemma. Let \( \mathcal{P} \) be the space of operators on \( V \) which are polynomials in products \( q_{l \to j} p_{i \to l} \) for \( 1 \leq l \leq i, j \leq n - 1 \),
(a) For any cycle \( h_1, \ldots, h_r \) in \( (I, H) \) the restriction of the polynomial \( \text{Tr}(\tilde{x}_{h_l}, \tilde{x}_{h_{r-1}} \cdots \tilde{x}_{h_1}) \) to \( M^c(v, d) \subseteq M(v, d) \) is of the form \( \text{Tr}(P) \) for some \( P \in \mathcal{P} \).
(b) For any path \( h_1, \ldots, h_r \) in \( (I, H) \) and a linear form \( \chi \) on \( \text{Hom}(D_{h_l'}, D_{h_r'}) \) the restriction of the polynomial \( \chi(q_{h_l} \tilde{x}_{h_l} \cdots \tilde{x}_{h_1} p_{h_r'}) \) to \( M^c(v, d) \) is of the form \( \chi(P) \) for some \( P \in \mathcal{P} \).

Proof. Easily follows from equations (3) for \( M^c(v, d) \subseteq M(v, d) \). □

2.3. Nakajima’s quiver varieties [N2 3.12]. The affine invariant theory quotient of \( M^c(v, d) \) by \( G(V) \) is denoted
\[
\mathfrak{M}^c_0(v, d) \defeq M^c(v, d)/G(V) = \text{Spec}[\mathcal{O}(M^c(v, d))^{G(V)}].
\]

2.3.1. The stable part \( M^c_s(v, d) \subseteq M^c(v, d) \). Following Nakajima [N2] and Lusztig [L3 2.11] we say that a quadruple \( (x, \pi, p, q) \) is stable if for any \( I \)-graded subspace \( V' \) of \( V \) preserved by \( x \) and \( \pi \) and containing \( \text{Im}(p) \), we have \( V' = V \). The subset of all stable quadruples in \( M^c(v, d) \) is denoted by \( M^c_s(v, d) \).
Lemma. [Mař Lemma 14] An element \((x, p, q)\) of \(M^c(v, d)\) is stable if and only if for all \(1 \leq i \leq n - 1\)

\[
\text{Im}(x_{i-1}) + \sum_{j=i}^{n-1} \text{Im}(p_{j-i}) = V_i.
\]

2.3.2. The quiver variety \(M^c(v, d)\). This is the geometric quotient of the stable part \(M^c_s(v, d)\) by \(G(V)\). In particular the set of \(C\)-points is the quotient set \(M^c_s(v, d)/G(V)\).

Below we only consider such \((v, d)\) that \(M^c(v, d)\) is nonempty (for explicit conditions on \((v, d)\) see [N2, 10], [Mař Lemma 7]).

2.3.3. \(M^c_1(v, d)\) and the Lagrangian \(L(v, d)\). There is a canonical map \(p : M^c(v, d) \to M^c_0(v, d)\).

Following Maffei we denote its image

\[M^c_1(v, d) \overset{\text{def}}{=} \text{Im}(p) \subset M^c_0(v, d).\]

Finally, we consider the central fiber of \(p\) which is a Lagrangian \(L^c(v, d) = p^{-1}(0)\) in \(M^c(v, d)\), and its top-dimensional Borel-Moore homology \(\mathcal{H}(L(v, d))\) (see 1.2.2).

2.4. Nakajima’s construction of \(SL(n)\)-modules. In this subsection \(c = 0\).

Theorem. [N2 10.ii] For any \(d \in \mathbb{N}^{n-1}\) the sum \(\bigoplus_v \mathcal{H}(\mathcal{L}(v, d))\) has the structure of a simple \(SL(n)\)-module \(V^c_d\) with the highest weight \(\hat{\lambda}\) (meaning really \(\sum_{i=1}^{n-1} d_i \omega_i\) for the fundamental weights \(\omega_i\)). The summand \(\mathcal{H}(\mathcal{L}(v, d))\) is the weight space for the weight \(d - Cv\), where \(C\) is the Cartan matrix of type \(A_{n-1}\).

In particular, the weight space \(V^c_d(\mathcal{L}(v, d))\) has a basis \(\text{Irr} \ \mathcal{L}(v, d)\) of irreducible components of \(p^{-1}(0)\). Lusztig [L4] calls this basis semicanonical.

2.4.1. From \(SL(n)\) to \(GL_n\). We may consider \(\bigoplus_v \mathcal{H}(\mathcal{L}(v, d))\) as a representation \(V^c_{\hat{\lambda}}\) of \(GL_n\) with highest weight \(\hat{\lambda}\), where \(\hat{\lambda} = \hat{\lambda}(d) = (\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n)\) is a partition of \(N = \sum_{j=1}^{n-1} j d_j\) defined as follows: \(\hat{\lambda}_i = \sum_{j=i}^{n} d_j\) (here \(d_n = 0\)). Then \(\mathcal{H}(\mathcal{L}(v, d))\) is the weight space \(V^c_{\hat{\lambda}}(a)\), where \(a_i = v_{n-1} + \sum_{j=i}^{n} (d - Cv)_j\) (here \((d - Cv)_n = 0\)), cf. [N1 8.3].

3. Conjugacy classes of matrices

In this section we fix a vector space \(N\) of dimension \(N\), let \(G = GL(N)\) and \(g = gl(N) = \text{End}(N)\). In 3.1 we recall degenerations of conjugacy classes in \(gl(N)\) and the corresponding resolutions. In 3.2 we consider normal slices to nilpotent orbits.
3.1. Resolutions and degenerations of closures of conjugacy classes.

For an operator \( x \in \text{End}(N) \) we list invariants \( \mathcal{E}_x, \mu_x, \mu_x, M_x \) of the conjugacy class \( \mathcal{O}_x \). We use \( (\mathcal{E}_x, \mu_x) \) to describe \( \mathcal{O}_x \). An ordering \( (a, b) \) of \( M_x \) gives a resolution \( \widetilde{\mathcal{F}}^{a,b}_x \) of \( \mathcal{O}_x \) and a degeneration of \( \mathcal{O}_x \) to \( \mathcal{O}_x \) for the nilpotent orbit attached to the partition \( \mu_x \). We also recall two related constructions of representations of \( \mathfrak{gl}_n \).

3.1.1. The variety \( \mathcal{F}_n \) of \( n \)-step flags. Let \( N \) be the nilpotent cone in \( g = \text{End}(N) \). The variety \( \mathcal{F}_n = \mathcal{F}_n(N) \) of \( n \)-step flags in \( N \) consists of systems of subspaces \( F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = N) \) (it was studied by Ginzburg [CG]). The \( n \)-term decompositions \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) of \( N = \sum_{i=1}^n a_i \), parameterize the connected components \( \mathcal{F}^a = \{ F \in \mathcal{F}_n; \dim(F_i/F_{i-1}) = a_i \} \) of \( \mathcal{F}_n \).

3.1.2. Fibers of moment maps. The cotangent bundle \( \widetilde{\mathcal{N}}_n \) consists of all \((x, F) \in g \times \mathcal{F}_n \) such that \( x F_i \subseteq F_{i-1} \). It lies inside the “extended cotangent bundle” \( \mathcal{G}_n = \mathcal{F}_n(N) \) given by all \((x, F) \) with \( x F_i \subseteq F_i \). The moment map for the \( G \)-action on \( \mathcal{G}_n \) is the projection \( \mathcal{N}_n \to g \) denoted \( m \). Its image \( \mathcal{N}_n \) is given by the equation \( x^n = 0 \). We also denote \( \mathcal{N}_n^\alpha = T^\ast \mathcal{F}_n^\alpha \) and in the extended version \( \mathcal{G}_n^\alpha = T^\ast \mathcal{F}_n^\alpha \). The restrictions of \( m, \tilde{m} \) to \( \mathcal{N}_n^\alpha, \mathcal{G}_n^\alpha \) are denoted \( m^\alpha, \tilde{m}^\alpha \).

Lemma. \( \mathcal{N}_n^\alpha \) is a resolution of the closure of the nilpotent orbit \( \mathcal{O}_\mu \) such that the dual partition \( \tilde{\mu} \) is given by reordering \( a = (a_1, \ldots, a_n) \).

Proof. From any \( y \in \mathcal{O}_\mu, \mu_p \) is the number of Jordan blocks of size \( \geq p \), so this is the jump \( \mu_p = \dim(K_p/K_{p-1}) \) in the increasing filtration \( K_i = \text{Ker}(y^i) \). We now look at this from the point of view of the decreasing filtration \( \mathcal{K}(p) \) of \( \text{Ker}(y) = K_1 \) defined by \( \mathcal{K}(p) \) is the image of \( y^p : Gr_p(N) \to Gr_q(N) \) are embeddings since \( yk \in K_i \) iff \( k \in K_{i+1} \). So, \( \mathcal{K}(p) \) is the image of \( y^{p-1} : Gr_p(\mathcal{N}) \to Gr_1(\mathcal{N}) \), Now we have \( K_1 = \mathcal{K}(1) = \mathcal{K}(2) = \cdots \) and \( \dim(\mathcal{K}(p)) = \dim(Gr_p(N)) = \mu_p \). Since \( \mathcal{K}(p) \) only depends on \( \mu_p \) we define \( \mathcal{K}_i = \mathcal{K}(p) \) when \( \mu_p = i \) in order to erase repetitions in our filtration.

Now, consider any reordering \( a = (a_1, \ldots, a_n) \) of the partition \( \mu_p \). We will inductively define a filtration \( F \) starting with \( F_1 = \mathcal{K}_{a_1} \). The reason is that \( \mathcal{K}(a_1) \) is the only \( y \)-invariant subspace \( S \) of \( N \) such that the operator induced by \( y \) on \( N/S \) corresponds to a partition whose dual is obtained from \( \mu \) by removing \( a_1 \).

Now we pass from \( N \) to \( N/F_{a_1} \) and repeating this procedure. In this way we construct a unique filtration \( F \) such that \( (y, F) \in \mathcal{N}^\alpha \). (For example, if \( a = \tilde{\mu} \) then \( F_i = \text{Ker}(x^i) \).)
Now it remains to notice equality of dimensions
\[
\dim(\widetilde{N}^a) = 2 \dim(T^*F^a) = \sum_{i \neq j} a_i a_j = N^2 - \sum_{i=1}^n a_i^2 = N^2 - \sum_{i=1}^n |\widetilde{\mu}_i|^2 = \dim(\mathcal{O}_\mu).
\]

3.1.3. Data \((\mathcal{E}, \widetilde{\mu})\) for conjugacy classes in \(\text{End}(\mathbb{N})\). To an operator \(x\) on \(\mathbb{N}\) we assign the set of its eigenvalues \(\mathcal{E} = \mathcal{E}_x \subseteq \mathbb{A}^1\) and a collection \(\widetilde{\mu}_x = \widetilde{\mu} = (\mu^e)_{e \in \mathcal{E}}\) of partitions \(\mu^e\) corresponding to the nilpotent operator \(x - e\) on the generalized \(e\)-eigenspace \(N_e\) of \(x\). We will call such \(\widetilde{\mu}\) an \(\mathcal{E}\)-bipartition of \(N\) since \(\sum_{e \in \mathcal{E}} |\mu^e| = N\).

The data \((\mathcal{E}, \widetilde{\mu})\) determines the conjugacy class \(\mathcal{O}_x\) through \(x\) so we also denote it \(\mathcal{O}_{\mathcal{E}, \widetilde{\mu}}\). This parameterizes all conjugacy classes as \(\mathcal{E}\) goes through subsets of \(\mathbb{A}^1\) of \(\leq n\) elements and \(\widetilde{\mu}\) through \(\mathcal{E}\)-bipartitions of \(N\).

3.1.4. Invariants \(n_x, \mu_x, M_x\) of an operator \(x\). A bipartition \(\widetilde{\mu} = (\mu^e)_{e \in \mathcal{E}}\) defines the dual bipartition \(\widetilde{\mu}^\vee \overset{\text{def}}{=} ((\mu^e)^\vee)_{e \in \mathcal{E}}\). It also defines a single partition \(\mu\) of \(N\) such that its dual \(\mu^\vee\) is the partition obtained by ordering the multiset of terms in the bipartition \(\widetilde{\mu}^\vee\).

For any eigenvalue \(e\) of \(x\) denote by \(n_e\) the size of the largest Jordan blocks of \(x\) associated with eigenvalues \(e\), i.e., the number of steps in the filtration of \(N_e\) by \(\ker[(x - e)^i]\). This is also the number of terms in the dual partition \((\mu^e)^\vee = (\mu_1^e \geq \cdots \geq \mu_n^e > 0)\). The sum \(n_x = \sum_{e \in \mathcal{E}} n_e\) is the total number of terms in the bipartition \((\widetilde{\mu}_x)^\vee\) and it is also the degree of the minimal polynomial of \(x\).

Let us say that a filtration \(F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{N})\) is a Jordan filtration for \(x\) if there is an ordering \(b = (b_1, ..., b_n)\) on the multiset \(\mathcal{E}_x\) of roots of the minimal polynomial of \(x\), such that \(F_i = \ker[\prod_{1 \leq j \leq i} (x - b_j)]\). (The length of any Jordan filtration is \(n_x\).) The graded piece \(F_i/F_{i-1}\) is described by its dimension \(a_F(i) = \dim(F_i/F_{i-1})\) and by the scalar \(b_F(i) = b_i\) (the action of \(x\)).

We denote by \(M_x\) the multiset of pairs \((a_F(i), b_F(i)), 1 \leq i \leq n_x\), for any Jordan filtration \(F\) (it is independent of \(F\)).

The multiset \(A_x\) is given by all first components of elements of \(M_x\), i.e., by dimensions of graded pieces of any Jordan filtration. Notice that if \(x\) lies in the orbit \(\mathcal{O}_{\mathcal{E}, \widetilde{\mu}}\) then \(M_x\) can be described as the multiset of terms in the dual bipartition \(\widetilde{\mu}^\vee \overset{\text{def}}{=} ((\mu^e)^\vee)_{e \in \mathcal{E}}\) of \(N\).

3.1.5. Resolutions and degenerations. For an operator \(x\) on \(\mathbb{N}\) define \(\mathcal{E}, \widetilde{\mu}\) and \(M_x\) as above so that \(\mathcal{O}_x = \mathcal{O}_{\mathcal{E}, \widetilde{\mu}}\). An ordering \((a_1, b_1; ..., a_n, b_n)\) of the multiset \(M_x\) will be viewed as a pair of vectors \((a, b) = (a_1, ..., a_n; b_1, ..., b_n)\) (hence \(b_i \in \mathcal{E}\) and \(a_i \in \mathbb{N}\)). The component \(a\) is an ordering of the multiset \(A_x\) of dimensions of graded pieces in a Jordan filtration

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3 A multiset structure on a set \(A\) is a multiplicity function \(A \rightarrow \mathbb{N}\), i.e., an isomorphism type of surjections \(A_1 \rightarrow A\) with finite fibers. An ordering on the multiset \(A_1 \rightarrow A\) is an ordering on \(A_1\).
while \( b = (b_1, \ldots, b_n) \) remembers the eigenvalues of \( x \) on the graded pieces. Notice that any Jordan filtration \( F \) gives an ordering \((a_F, b_F)\) and all orderings arise in this way.

The first component \( a \) defines \( \tilde{N}^a = T^*F^a \subseteq \tilde{g}^a = \tilde{T}^*F^a \) and then \( b \) defines

\[
\tilde{g}^{a,b} \overset{\text{def}}{=} \{(x, F) \in \tilde{g}^a; \text{\( x \) acts on } F_i/F_{i-1} \text{ as } b_i\}.
\]

**Lemma.** (a) The variety \( \tilde{g}^{a,b} \) is smooth and connected. Its moment map image is \( \overline{O}_{E,\tilde{\mu}} \) and the moment map \( \tilde{g}^{a,b} \to \overline{O}_{E,\tilde{\mu}} \) is projective and finite over \( O_{E,\tilde{\mu}} \).

(b) \( \tilde{g}^{a,b} \to \overline{O}_{E,\tilde{\mu}} \) is a deformation of \( \tilde{N}^a \to \overline{O}_\mu \) which is the case \( b = 0 \).

(c) For the Lusztig stratification \( \mathcal{L}_x \) that contains \( O_x = O_{E,\tilde{\mu}}, \mathcal{L}_x \cap N = \overline{O}_{\mu_s} \).

**Proof.** (a) Variety \( \tilde{g}^{a,b} \) is a torsor for the vector bundle \( T^*F^a \) over \( F^a \), so it is smooth and connected.

Let us see that the fiber at \( x \in O_{E,\tilde{\mu}} \), is finite and nonempty. Since \( x \)-invariant filtrations \( F_i \) decomposes into an \( E \)-sum of filtrations on generalized eigenspaces, the claim reduces to the case when \( x \) has one eigenvalue and then it may as well be nilpotent. But now we are in the situation of lemma 3.1.2.

The remaining claims follow from equality of dimensions of \( \tilde{g}^{a,b} \) and \( O_{E,\tilde{\mu}} \). Since we know that \( \dim(\tilde{g}^{a,b}) = \dim(\tilde{N}^a) = \dim(O_{\mu}) \), we just need

\[
\dim(O_{E,\tilde{\mu}}) = \dim(\mathfrak{gl}(N)) - \dim[Z_{\mathfrak{gl}(N)}(x)] = N^2 - \sum_{\substack{e \in E \cap r \in E \cap \mathfrak{z}}} \dim[Z_{\mathfrak{gl}(N_e)}(x - e)]
\]

\[
= N^2 - \sum_{\substack{e \in E \cap r \in E \cap \mathfrak{z}}} \sum_{\substack{e \in E \cap r \in E \cap \mathfrak{z}}^t} |\tilde{\mu}_e|^2 = \dim(O_{\mu}).
\]

(b) Torsor \( \tilde{g}^{a,b} \) deforms as \( b \to 0 \) to \( \tilde{N}^a \) itself. At the same time the moment map image \( \overline{O}_{E,\tilde{\mu}} = \tilde{m}(\tilde{g}^{a,b}) \) moves to \( \mathfrak{m}(\tilde{N}^a) = \overline{O}_\mu \) and \( \dim(O_{\mu}) = \dim(\tilde{N}^a) = \dim(\tilde{g}^{a,b}) = \dim(O_{E,\tilde{\mu}}) \).

(c) Notice that \( \mathcal{L}_x \) is the union of all \( O_{(E),\mu} \) where \( \mu : E \to \mathbb{A}^1 \) is injective. Denote

\[
\tilde{g}^{a,\mathfrak{A}^n} \overset{\text{def}}{=} \{(x, F) \in \tilde{g}^a; \text{\( x \) acts on } F_i/F_{i-1} \text{ as a scalar} \} = \bigcup_{b \in A(n)} \tilde{g}^{a,b'}.
\]

This is a closed subspace of \( \tilde{g}^a \) and its moment map image \( \tilde{m}(\tilde{g}^{a,\mathfrak{A}^n}) \) is by (a) the closure of \( \mathcal{L}_x \). Now, \( \tilde{m}(\tilde{g}^{a,\mathfrak{A}^n}) \cap N = \tilde{m}(\tilde{g}^{a,0}) = \tilde{m}(\tilde{N}^a) = \overline{O}_{\mu_s}. \)

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4 Here \( O_{E,\tilde{\mu}} \) degenerates to \( O_{\mu} \) so that in each Jordan block eigenvalue degenerates to zero but extensions are created between blocks corresponding to different eigenvalues. An example is a regular semisimple orbit degenerating to the regular nilpotent orbit.

5 The Lusztig stratification of \( g \) is the stratification induced by the \( G \)-action. The stratum through \( x \) with a Jordan decomposition \( x = s + n \) is \( \mathcal{L}_x \overset{\text{def}}{=} G(Z_r(I) + n) \) where \( I \overset{\text{def}}{=} Z_g(s) \) and \( Z_r(I) = \{ y \in I; Z_g(y) = I \} \). (cf. [Min 5.5] and references therein.)
3.1.6. **Two geometric bases of representations of \( \mathfrak{gl}_n \).** At \( x \in \mathcal{N}_n \) we denote the fibers of moment map by \( \mathcal{F}^a_{n,x} \overset{\text{def}}{=} m^{-1}_a x \subseteq \mathcal{F}^a_n \) and \( \mathcal{F}_{n,x}^a \overset{\text{def}}{=} \tilde{m}^{-1}_a x \subseteq \mathcal{F}^a_n \). Let also \( \mathcal{H}(Z) \) denote the top Borel-Moore homology of \( Z \).

**Proof.** Let \( \lambda \) be a partition of \( N \) with at most \( n \) terms and let \( x, y \) be nilpotent operators on \( N \) of types \( \lambda \) and \( \lambda' \). Then the irreducible \( \mathfrak{gl}(n) \)-module \( V_{\lambda} \) with the highest weight \( \lambda \) has realizations as \( \mathcal{H}(\mathcal{F}_{n,y}^a) \) (see [CG 4.2]) and as \( \mathcal{H}(\mathcal{F}_{n,x}^a) \) (see [BG]).

3.2. **Normal slices to nilpotent orbits.** We will say that a normal (transverse) slice in \( \mathfrak{g} \) to a nilpotent orbit \( \alpha \) at a point \( x \in \alpha \), is a submanifold \( S \) of \( \mathfrak{g} \) such that

- (IN) (Infinitesimal normality) The tangent space \( T_x(\mathfrak{g}) \) equals \( T_x(\alpha) \oplus T_x(S) \).
- (C) (Contraction.) There is an action of \( G_m \) on \( S \) which contracts it to \( x \) and preserves intersections with the Lusztig strata in \( \mathfrak{g} \).

3.2.1. **Lemma.** For a normal slice \( S \) at an element \( x \) of a nilpotent orbit \( \alpha \):

1. \( S \cap \alpha = \{x\} \); and \( G S \) is open in \( \mathfrak{g} \);
2. \( S \) meets Lusztig stratum \( \beta \) iff \( \alpha \subseteq \beta \);
3. \( S \) meets conjugacy classes transversally;
4. for any partial flag variety \( \mathcal{P} = G/P \) the base changes \( \tilde{T}^*\mathcal{P} \times \mathfrak{g} S \) and \( T^*\mathcal{P} \times \mathfrak{g} S \) of the slice \( S \) are smooth and connected.

**Proof.** (1) Since nilpotent orbits are Lusztig strata, \( G_m \) contracts \( \alpha \cap S \) to \( x \). So if \( \alpha \cap S \) were more then a point then it would have positive dimension contradicting property (IN). Next, \( G S \) is open at the point \( x \) by (IN) and therefore at each \( y \in S \) by (C).

(2) If \( S \) meets \( \beta \) then the contraction of \( \beta \cap S \) to \( x \) shows that \( \beta \) contains \( x \) and \( \alpha \). On the other hand, if \( \alpha \subseteq \beta \) then the neighborhood \( G(S \subseteq \mathfrak{g}) \) of \( \alpha \) meets \( \beta \).

(3) To see that for \( y \in S \), the sum \( \Sigma_y = T_y(G \mathfrak{g}) + T_{u y}(S) \) is all of \( \mathfrak{g} \), notice that for \( u \in G_m \ni u \to 0 \) we have \( z = u \to x \). Now \( \dim(\Sigma_y) = \dim(\Sigma_{x}) \geq \dim(\Sigma_x) = \dim(\mathfrak{g}) \).

(4) Near \( y \in S \), \( \mathfrak{g} \) is a product of \( S \) and the orbit \( G \cdot y \), so smoothness is inherited from that of \( T^*\mathcal{P} \) and \( \tilde{T}^*\mathcal{P} \). The \( G_m \)-action lifts to \( \tilde{T}^*\mathcal{P} \times \mathfrak{g} S \supseteq T^*\mathcal{P} \times \mathfrak{g} S \) and contracts the spaces to the fiber at \( x \). Connectedness then follows from connectedness of (generalized) Springer fibers cf. [Sp].

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\[ \overset{\text{Lusztig stratification of } \mathfrak{g} \text{ is the stratification induced by the } G \text{-action. The stratum through } x \text{ with a Jordan decomposition } x = s + n \text{ is } L_x \overset{\text{def}}{=} G(Z_r(l) + n) \text{ where } l \overset{\text{def}}{=} Z_\mathfrak{g}(s) \text{ and } Z_r(l) = \{ y \in l; Z_\mathfrak{g}(y) = l \}. \text{ (cf. } \text{[Mir 5.5]} \text{ and references therein.)} \]
3.2.2. Lemma. Consider a vector subspace \( C \subseteq \mathfrak{g} \) complementary to \( T_x(\alpha) = [\mathfrak{g}, x] \). For the submanifold \( S = x + C \) to be a normal slice to the orbit \( G x \) at \( x \) it suffices that there exists \( h \in \mathfrak{g} \) which is (i) semisimple, (ii) integral (i.e., eigenvalues of \( \text{ad} \ h \) are integral), (iii) \([h, x] = 2x\), (iv) \( h \) preserves \( C \) and (v) the eigenvalues of \( h \) in \( C \) are \( \leq 1 \).

Proof. Such \( h \) lifts to a homomorphism \( \iota : G_m \rightarrow G \). Now the action of \( G_m \) on \( \mathfrak{g} \) by \( s * y = s^2 \cdot \iota(s)^{-1} y \), \( s \in G_m \), \( y \in \mathfrak{g} \); preserves \( x + C \) and contracts it to \( x \).

3.2.3. Normal slice \( T_x \). For a nilpotent \( x \) we will use an \( sl(2) \)-triple \( \{x, h, y\} \) to construct a subspace \( C \) with \( (h, C) \) as in the lemma 3.2.2. The subspaces \( C \) will be of a special form: (i) \( C \) is \( y \)-invariant, (ii) the \( h \)-eigenvalues in \( C \) are \( \leq 0 \). For instance the Slodowy slice corresponds to the case \( C = Z_q(y) \).

We will use the \( sl(2) \)-decomposition of the vector space \( \mathcal{N} \) as a sum \( \mathcal{N} = \bigoplus_i M_i \otimes L_i \) where \( L_i = L_i^{sl(2)} \) is a simple \( sl(2) \)-module of highest weight \( i \), and \( M_i = \text{Hom}_{sl(2)}(L_i, \mathcal{N}) \). Then \( \mathfrak{g} = \text{End}(\mathcal{N}) = \bigoplus_{i,j} \text{Hom}(M_j, M_i) \otimes L_j^* \otimes L_i \), and we choose

\[
C = \bigoplus_{i,j} \text{Hom}(M_j, M_i) \otimes \text{Ker}(y, L_i) \otimes \text{Ker}(y^{i+1}, L_j^*) \subseteq \text{End}(\mathcal{N}).
\]

Notice that \( C \) generates \( \text{End}(\mathcal{N}) \) as a \( \mathbb{C}[y] \)-module and it is a minimal subspace with this property (it is easy to see that this holds for the subspace \( \text{Ker}(y^{i+1}, L_j) \otimes \text{Ker}(y, L_i) \) of \( L_j \otimes L_i \)). Therefore, \( \dim(C) \) is the number of Jordan blocks for the operator \( \text{ad} \ y \) on \( \mathfrak{g} \), i.e., \( \dim([Z_q(y)]) \).

It is elementary to see that \( h, C \) satisfy the conditions of Lemma 3.2.2 and thus \( T_x = x + C \) is a normal slice.

3.2.4. Normal slice to a pronilpotent. Let \( z \) be regular nilpotent operator on the vector space \( \mathcal{O} \) with a cyclic line \( \mathcal{L} \). On the multiple \( \mathcal{N} = U \otimes \mathcal{O} \) we consider the operator \( x = 1 \otimes z \). Then the formula for the subspace \( C \) is

\[
\text{End}(U) \otimes \mathcal{L} \otimes \mathcal{O}^* \cong \text{Hom}(U \otimes \mathcal{O}, U \otimes \mathcal{L}).
\]

Notice that this gives a normal slice \( T_x = x + C \) even if \( e \) is pronilpotent, say the \( z \)-multiplication on \( \mathcal{O} = \mathbb{C}[[z]] \).

3.2.5. “As close to regular nilpotent as possible”. If \( x \) is a nilpotent of type \( \lambda \) then \( \mathcal{N} \cong \bigoplus_{i=1}^m L_{\lambda_i-1} \) hence \( \text{End}(\mathcal{N}) = \bigoplus_{i,j=1}^m L_{\lambda_i-1} \otimes L_{\lambda_j-1}^* \). Then \( C = \bigoplus_{i,j=1}^m \text{Ker}(y^{\lambda_i}, L_{\lambda_j-1}^*) \otimes \text{Ker}(y, L_{\lambda_i-1}) \).

As a \( y \)-module \( \text{Ker}(y^{\lambda_i}, L_{\lambda_j-1}^*) \otimes \text{Ker}(y, L_{\lambda_i-1}) \cong \text{Ker}(y^{\lambda_i}, L_{\lambda_j-1}^*) \cong L_{\min(\lambda_i, \lambda_j)-1} \) is a single Jordan block of size \( \min(\lambda_i, \lambda_j) \). So, if \( \lambda = 1^m \cdot \ldots \cdot s^m \) then the \( y \)-action on \( C \) contains the Jordan block of size \( i \) with multiplicity \( 2(m_i + \cdots + m_s) - 1 \). Then the partition \( \lambda_C \) corresponding to \( y \) on \( C \) is the largest possible partition for the action of \( y \) on any \( y \)-invariant subspace of \( \mathfrak{g}_{\leq 0}^b \) complementary to \( T_x(\alpha) = [\mathfrak{g}, x] \). (For instance, if \( x \)
is regular, then \( y \) restricted to our \( C \) will be regular.) This is opposite from Slodowy’s situation where \( y \) acts on \( C = \text{Ker}(y, g) \) as 0. For this reason we will call \( T_x \) the “regular” normal slice.

### 3.2.6. Slice \( T_x \) in a Jordan basis and \( C \subseteq \text{Hom}(N, U) \). Again, here \( x \) is a nilpotent operator of type \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_m) \) on \( N \). Let us choose a Jordan basis \( e_{k,i} \) for \( 1 \leq k \leq \lambda_i \) of \( N \), i.e., such that \( x e_{k,i} = e_{k-1,i} \). Let \( U \subseteq N \) be the span of the generators \( e_{\lambda_i,i}, 1 \leq i \leq m \) of Jordan blocks. Then \( C \) is a subspace of \( \text{Hom}(N, U) \) consisting of all \( \gamma \in \text{Hom}(N, U) \) such that for the matrix elements \( \gamma_{l,j} \) corresponding to the component \( C e_{l,j} \rightarrow C e_{\lambda_i,i} \), one has \( \gamma_{l,j}^{i} = 0 \) for \( l > \lambda_i \). For example, if \( \lambda = (\lambda_1 \geq \lambda_2) = (3, 2) \) the operators in \( T_x \) will have the form (in the basis \( e_{k,i} \))

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\gamma_{1,1} & \gamma_{1,1} & \gamma_{1,1} & \gamma_{1,2} & \gamma_{2,2} \\
0 & 0 & 0 & 0 & 1 \\
\gamma_{2,1} & \gamma_{2,1} & 0 & \gamma_{1,2} & \gamma_{2,2}
\end{pmatrix}
\]

(8)

### 3.2.7. Intersection of the normal slice \( T_x \) with the conjugacy class \( O_x \). The slice \( T_x \) will sometimes be denoted by \( T_\lambda \). Consider \( \overline{O_\mu} \supseteq O_\lambda \) and let \( \alpha \) be any permutation of terms in \( \mu \). The restriction of the moment map \( m^a : \tilde{N}^a \rightarrow N \) to the slice \( T_x \) is the space \( \tilde{T}_x^a \defeq T_x \times_N \tilde{N}^a \).

**Lemma.** Consider an element \( x \) of a nilpotent orbit \( O_\lambda \).

(a) Let \( O_\lambda \subseteq \overline{O_\mu} \) and let \( \alpha = (a_1, \ldots, a_n) \) be related to \( \mu \) as above. Then the variety \( \tilde{T}_x^a \) is smooth and connected with the moment map image \( T_x \cap \overline{O_\mu} \). The map \( m_\alpha : \tilde{T}_x^a \rightarrow T_x \cap \overline{O_\mu} \) is projective and generically finite.

(b) \( T_x \) meets a conjugacy class \( O_\lambda \) iff it meets the Lusztig stratum \( \mathcal{L}_x \) through \( x \), i.e., iff \( x \in \overline{\mathcal{L}_x} \). If \( \mathcal{O}_x = O_{E,E} \) as in [3.1.5], this is equivalent to the relation \( \mathcal{O}_x \subseteq \overline{\mathcal{E}_{\mu}} \) of two nilpotent orbits. Then the variety \( \tilde{T}_x^{a,b} \defeq T_x \times_{\overline{O_{i,j}}} \tilde{N}^{a,b} \) is smooth and connected, its moment map image is \( T_x \cap \overline{O_{i,j}} \) and it is finite over \( T_x \cap \mathcal{O}_{i,j} \).

**Proof.** (a) Smoothness and connectedness follow from lemma 3.2. The claims about the map \( m_\alpha : \tilde{T}_x^a \rightarrow T_x \cap \overline{O_\mu} \) follow from the corresponding claims for the map \( \tilde{N}^a \rightarrow \overline{O_\mu} \).

(b) If \( T_x \) meets \( \mathcal{O}_x \) then it certainly meets \( \mathcal{L}_x \). If \( T_x \) contains a point \( y \in \mathcal{L}_x \) we present its conjugacy class as \( O_{(i,j),i,j} \) for some injection \( i : E \rightarrow \mathbb{A} \). Since the \( u \)-action is given by conjugation (which preserves \( O_{(i,j),i,j} \)) and by multiplication with the scalar \( u^2 \), we see that \( u \ast y \) is in \( O_{u^2(i,j),i,j} \). Therefore, as \( u \rightarrow 0 \) this orbit degenerates to \( \overline{O_{\mu}} \), hence \( x = \lim_{u \rightarrow 0} u \ast y \) lies in \( \overline{O_{\mu}} \).
In the opposite direction, if \( x \in \overline{O}_{\mu} \) then \( T_x \) meets \( O_{\mu} \) by lemma 3.2.2. Since \( G T_x \subseteq \mathfrak{g} \) is open, the deformation of \( O_{\mu} \) through all \( O_{c \in (c), \tilde{\mu}} \) for \( c \in G_m \) implies that \( T_x \) meets \( O_{c \in (c), \tilde{\mu}} \) for small \( c \), and then also for \( c = 1 \) by using the \( G_m \)-action on \( T_x \). So, \( T_x \) meets \( O_{\tilde{\mu}} = O_x \). The remaining claims now follow as in similar proofs above. \( \square \)

4. **Beilinson-Drinfeld Grassmannians**

In this section \( U \) is a vector space of dimension \( m > 0 \) and \( G \) denotes the group \( GL(U) \). We will fix a basis \( e_1, \ldots, e_m \) of \( U \), hence \( G = GL_m \).

In [4.1] we recall the *Beilinson-Drinfeld* loop Grassmannians \( \mathcal{G}_{C(n)} \) of groups \( GL_m \). We construct a map from loop Grassmannians to linear operators in 4.2 and a map in the opposite direction in 4.3. This leads to identification of normal slices in these two settings [4.4]. This comparison is extended to resolutions of slices in 4.5. Finally, the representation theoretic aspect ("geometric Satake correspondence") is in 4.6.

4.1. **Grassmannians** \( \mathcal{G}_{C(n)} \). Let \( C \) be a smooth curve or the formal neighborhood of a finite subset of a smooth curve. The loop Grassmannian construction for \( G = GL(U) \) yields over each symmetric power \( C^{(n)} \) a reduced ind-scheme \( \mathcal{G}_{C^{(n)}} \to C^{(n)} \). The \( \mathbb{C} \)-points in the fiber at \( b \in C^{(n)} \) are vector bundles \( \mathcal{L} \) over \( C \) which coincide with \( U \otimes O_C \) off \( b \).

Here, "off \( b \)" means over \( C - b \), for the support subset \( b \subseteq C \) of \( b \) (we will denote \( C - b \) just by \( C - b \)). The condition implies that the vector bundle \( \mathcal{L} \) is a subbundle of the quasi-coherent sheaf \( j_* j^*(U \otimes O_C) \) for \( j : C - b \to C \). We will call pairs \( (b, \mathcal{L}) \) above "\( C \)-lattices" (for \( U \)) (we sometimes omit \( b \)). The fiber of \( \mathcal{G}_{C^{(n)}} \) at a fixed \( b \in C^{(n)} \) is denoted \( \mathcal{G}_b \). Over any curve \( C \) we denote the lattice \((\emptyset, U \otimes O_C) \) by \( \mathcal{L}_0 \).

4.1.1. **Local loop Grassmannians.** For a point \( c \in C \), the fiber \( \mathcal{G}_c \) equals \( \mathcal{G}_{\tilde{c}} \) for the formal neighborhood \( \tilde{c} \) of \( c \). For a choice of a formal parameter \( z \) at \( c \) let \( O \overset{\text{def}}{=} \mathbb{C}[z] \subseteq K \overset{\text{def}}{=} \mathbb{C}((z)) \) be the formal power series and Laurent series in \( z \). Then \( \tilde{\mathcal{L}} \)-lattices \( \mathcal{L} \) are identified with the \( O \)-submodules \( L = \Gamma(\tilde{c}, \mathcal{L}) \) of \( U_K = U \otimes_{\mathbb{C}} K \), such that the map \( L \otimes_{\mathcal{O}} K \to U_K \) is an isomorphism. This identifies \( \mathcal{G}_c = \mathcal{G}_{\tilde{c}} \) with its group theoretic realization \( \mathfrak{g} \overset{\text{def}}{=} G(\mathfrak{K})/G(\mathfrak{O}) \).

To a coweight \( \lambda \in \mathbb{Z}^m \) of \( G \), one attaches the lattice \( L_{\lambda} = \bigoplus_{i=1}^m \mathbb{C}[z] \cdot z^{-\lambda_i} e_i \in \mathfrak{g} \) and its orbit \( \mathcal{G}_\lambda \overset{\text{def}}{=} G(O) \cdot L_{\lambda} \). This parameterizes the \( G(O) \)-orbits in \( \mathfrak{g} \) by coweights \( \lambda \) which are dominant (i.e., \( \lambda_1 \geq \cdots \geq \lambda_m \)).

For \( N \in \mathbb{Z} \), the connected component \( N_{\mathfrak{g}} \) of \( \mathfrak{g} \) consists of all lattices \( L \) such that \( \dim(L) \overset{\text{def}}{=} \dim(L/L \cap L_0) - \dim(L_0/L \cap L_0) = N \). We will also use \( \mathfrak{g}^+ \subseteq \mathfrak{g} \) consisting of all lattices \( L \) that contain \( L_0 = U \otimes \mathcal{O} \). The \( G(O) \)-orbits \( \mathcal{G}_{\mu} \) in the intersection \( N_{\mathfrak{g}} \mathfrak{g}^+ \overset{\text{def}}{=} N_{\mathfrak{g}} \cap \mathfrak{g}^+ \) correspond to partitions \( \mu \) of \( N \) into at most \( m \) parts. For such \( \mu \) the orbit \( \mathcal{G}_{\mu} \) consists of lattices \( L \) such that \( z \) acts on \( L/L_0 \) as a nilpotent of type \( \mu \).
Remark. The factorization property of loop Grassmannians says that at any \( b \in C(n) \) the fiber factors into a power of the local loop Grassmannian

\[
\mathcal{G}_b \cong \prod_{c \in \mathbb{Z}} \mathcal{G}_c.
\]

4.1.2. Convolution spaces \( \widetilde{\mathcal{G}}^a_{\mu} \). Any \( G(\mathbb{O}) \)-orbit \( \mathcal{G}_\mu \subseteq \mathcal{N} \mathcal{G} \) given by a partition \( \mu \) of \( N \) (into \( \leq m \) parts). A choice of a reordering \( a = (a_1, \ldots, a_n) \) of the dual partition \( \hat{\mu} \) defines a variety of \( n \)-step flags of lattices in \( U_K \):

\[
\widetilde{\mathcal{G}}^a_{\mu} \overset{\text{def}}{=} \{ L_0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \in \mathcal{G}_\mu; \dim L_i/L_{i-1} = a_i \text{ and } \tau(L_i) \subseteq L_{i-1} \}
\]

Since each \( a_i \) is \( \leq m \) it corresponds to a fundamental coweight \( \omega_{a_i} \) of \( GL_m \) and \( \mathcal{G}_{\omega_{a_i}} \cong Gr_p(m) \). Then \( \widetilde{\mathcal{G}}^a_{\mu} \) can be described as the convolution scheme \( \mathcal{G}_{\omega_{a_1}} \ast \cdots \ast \mathcal{G}_{\omega_{a_n}} \).

Remarks. (0) The convolution map \( \pi^a_{\mu} : \widetilde{\mathcal{G}}^a_{\mu} \rightarrow \overline{\mathcal{G}}^\mu_{\mu}, \pi^a_{\mu}(L_0, \ldots, L_n) = L_n \), is a partial resolution of singularities \([\text{MV}i1]\).

(1) The fiber \( (\pi^a_{\mu})^{-1}(L) \) at a point \( L \) in \( \overline{\mathcal{G}}^\mu_{\mu} \) is by definition isomorphic to the Springer-Ginzburg fiber \( \mathcal{F}_x^a \) defined in \([3,1,2]\) for the operator \( x \) on \( D = L/L_0 \), given by the \( z \)-action on \( L/L_0 \).

4.2. From loop Grassmannians to linear operators. This transition will use the first cohomology of vector bundles over \( \mathbb{P} = \mathbb{P}^1 \). We will denote \( \mathbb{A} = \mathbb{A}^1 \) and \( \mathbb{A}_- = \mathbb{P} - 0 \).

4.2.1. Normal slices. Denote \( \mathcal{O}^- = \mathbb{C}[z^{-1}] \), the negative congruence subgroup \( L^{<0}G \) is the kernel of the evaluation \( G(\mathcal{O}^-) \rightarrow G \) at \( z^{-1} = 0 \). The standard normal slice to \( \mathcal{G}_\Lambda \) at \( L_\Lambda \) is the orbit \( \mathcal{T}_\Lambda \overset{\text{def}}{=} L^{<0}G \cdot L_\Lambda \).

4.2.2. Complementary lattices and the map \( \Phi \). At infinity of \( \mathbb{P} = \mathbb{P}^1 \) we will fix a lattice \( (\infty, \mathcal{J}) \), it defines an open submoduli \( \mathcal{U}^\mathcal{J}_{(\infty)} \) of lattices \( (b, \mathcal{L}) \in \mathcal{S}_{(\infty)} \) which are complementary to \( \mathcal{J} \) in the sense that over the open subset \( \mathbb{A} - b \) where \( \mathcal{L} = \mathcal{L}_0 = \mathcal{J} \), the subspaces \( \Gamma(\mathbb{A}, \mathcal{L}) \) and \( \Gamma(\mathbb{P}^1 - b, \mathcal{J}) \) of \( \Gamma(\mathbb{A} - b, \mathcal{L}_0) \) are complementary.

For \( b \in \mathbb{A}^{(n)} \) consider the formal neighborhood \( \widehat{b} \) of \( b \) in \( \mathbb{A} \) and the punctured formal neighborhood \( \tilde{b} = \widehat{b} - b \). Then for any lattice \( L \in \mathcal{U}_b^\mathcal{J} \overset{\text{def}}{=} \mathcal{U}^\mathcal{J}_{(\infty)} \cap \mathcal{S}_b \) we have \( \Gamma(\tilde{b}, \mathcal{L}) \oplus \Gamma(\mathbb{P}^1 - b, \mathcal{J}) = \Gamma(\widehat{b}, \mathcal{J}) = U \otimes \mathcal{O}(\widehat{b}) \). Then the corresponding trivialization \( \Gamma(\tilde{b}, \mathcal{L}) \cong Y \overset{\text{def}}{=} U \otimes \mathcal{O}(\widehat{b})/\Gamma(\mathbb{P}^1 - b, \mathcal{J}) \) will associate to the lattice \( L \) the linear operator \( \Phi(L) \) on \( Y \) corresponding to the action of \( z \) on \( \Gamma(\tilde{b}, \mathcal{L}) \).

\footnote{In terms of \( \varpi : G(\mathcal{X}) \rightarrow G(\mathbb{K})/G(\mathbb{O}) = \mathcal{S} \) for \( A \subseteq \mathcal{S} \) let us denote \( A \overset{\text{def}}{=} \varpi^{-1}A \). Then the convolution of \( G(\mathbb{O}) \)-invariant \( A_1 \subseteq \mathcal{S} \) is \( A_1 \ast \cdots \ast A_q \overset{\text{def}}{=} A_1 \times G(\mathbb{O})\cdots \times G(\mathbb{O})A_q \times G(\mathbb{O})^{pt} \).}
Example. Any \( \mathbb{C} \)-form \( U \) of \( U_K \), gives \( L = U \otimes \mathbb{O} \) and \( L^\perp = U \otimes \mathbb{O}^\perp \) which are lattices at \( b = 0 \in A \) and at \( \infty \in A_\infty \). Let us choose \( J = L^\perp(-\infty) = z^{-1}L_\infty \) so that the open submoduli \( U^j_{A(n)} \) is a neighborhood of \( L \) (since \( L \) is complementary to \( z^{-1}L_\infty \) in \( U[z^\pm] \)).

In particular, when \( U = U \) we have \( L = L_0 \) and \( J = U \otimes \mathbb{O}_F(-\infty) \). Then the moduli \( U^j_0 \) is a normal slice \( \mathcal{T}_0 \) from \([4.2.1]\). So, we will think of the moduli \( U^j_{A(n)} \) as generalizations of normal slices.

4.2.3. Cohomological trivializations. At \( b \in A^{(n)} \) a choice of a lattice \( K \in \mathcal{G}_b \) defines \( \mathcal{G}^K_b \) consisting of lattices \( L \in \mathcal{G}_b \) that contain \( K \). Then \( U^j_{b,K} = U^j_{A(n)} \cap \mathcal{G}^K_b \) is finite dimensional. Now \( K \) modifies \( J \) to a vector bundle \( J_K \) on \( \mathbb{P} \) by gluing \( J \) on \( \mathbb{P} - b \) and \( K \) over \( A \). We will assume that \( K \subseteq L_0 \).

Since \( K = L_0 \) is a lattice at any \( b \in A^{(n)} \), we have an open submoduli \( U^j_{L_0} \) of \( \mathcal{G}^0_{L_0} \).

Lemma. (a) For any lattice \( L \in \mathcal{G}^K_b \) there is a canonical map \( \Gamma(A, L/K) \to H^1(\mathbb{P}^1, J_K) \).

(b) This map is an isomorphism iff \( L \in U^j_{L_0} \).

Proof. Over \( A - b \) lattices \( L, J, K, J_K \) equal \( L_0 \), and on \( A \) we have \( J = L_0 \supseteq K \), so

\[
\frac{\Gamma(A, L)}{\Gamma(A, K)} = \frac{\Gamma(A - b, J_K)}{\Gamma(A - b, J_K)} = \frac{\Gamma(A - b, J_K) + \Gamma(\mathbb{P} - b, J_K)}{\Gamma(A, J_K)} = H^1(\mathbb{P}, J_K),
\]

where we use the open cover \( A, \mathbb{P} - b \) of \( \mathbb{P} \). This composition is an isomorphism iff \( \Gamma(A, L) \oplus \Gamma(\mathbb{P} - b, J_K) = \Gamma(A - b, J_K) \). This is just the requirement \( \Gamma(A, L) \oplus \Gamma(\mathbb{P} - b, J) = \Gamma(A - b, J) \) because over \( \mathbb{P} - b \) one has \( J_K = J \).

Corollary. (a) There is a canonical map \( \Phi^K_b : U^j_{L_0} \to gl[H^1(\mathbb{P}, J_K)] \) that sends a lattice \( L \) to the action of \( z \) on \( \Gamma(A, L/K) \xrightarrow{\cong} H^1(\mathbb{P}, J_K) \).

(b) The restriction of the map \( \Phi \) from \([4.2.2]\) to \( U_b \) is the limit of maps \( \Phi^K_b \) as lattices \( \mathcal{K} \in \mathcal{G}_b \) become small.

4.2.4. Example: neighborhood \( U^\sigma \) of \( L_\sigma \). In particular, to a coweight \( \sigma \in \mathbb{Z}^m \) we associate a \( \mathbb{C} \)-form \( U_\sigma = \bigoplus_i z^{-1}\mathbb{C} e_i \) and therefore also the lattices \( L_\sigma = \bigoplus_i z^{-1}\mathbb{O} e_i \) at 0 and \( L^{-}_\sigma \) at \( \infty \). As above we use \( J = z^{-1}L_\infty \) to define a neighborhood \( U_{A(n)}^\sigma = U^j_{A(n)} \) of \( L_\sigma \).

Now, \( U_{A(n)}^{\sigma, L_0} = U_{A(n)}^j \cap \mathcal{G}^0_{L_0} \) contains \( L_\sigma \) iff \( \sigma_i \geq 0 \). If so, we can reorder \( \sigma \) into a partition \( \lambda \) and then the operator \( \Phi(L_\sigma) \) is nilpotent of type \( \lambda \) (the same as \( z \) on \( L_\sigma/L_0 \)). In this setting the corollary gives the map

\[
\Phi : U_{A(n)}^{\sigma, L_0} \to gl(L_\sigma/L_0).
\]
4.3. From nilpotent operators to lattices. For this transition we first provide a \( \mathbb{C} \)-linear formula in (4.3.1) and then a \( K \)-linear formula in (4.3.2).

Let \( U, L, L^- \) etc, be as in the example (4.2.2). Let \( \text{Hom}(L, L) = \mathfrak{gl}(L) \) denote the continuous linear operators. For instance, its subspace \( \text{Hom}(L, U) \) consists of operators \( L \to U \) that vanish on some sublattice \( K \subseteq L \).

4.3.1. Isomorphism of a pronilpotent slice to \( z \) and a neighborhood of \( L \). Notice that \( T_z \overset{\text{def}}{=} z + \text{Hom}(L, U) \) is the “regular” normal slice in \( \mathfrak{gl}(L) \) to the pronilpotent operator \( z \) (3.2.4). Let \( N_z \subseteq \mathfrak{gl}(L) \) consist of all operators \( A \) for which there is a sublattice \( K \subseteq L \) such that \( A = z \) on \( K \) and \( A \) is nilpotent on \( L/K \).

**Proposition.** (a) The following map is an isomorphism of the pronilpotent part of the normal slice to the coadjoint orbit at \( z \in \mathfrak{gl}(L) \) to a neighborhood of \( L \) in \( \mathfrak{g} \): \( ^8 \)

\[
\Psi_0 : T_z \cap N_z \to U^{z^{-1}L^-}, \quad \text{by} \quad \Psi_0(z + \varphi) \overset{\text{def}}{=} [1 + \sum_{k=1}^{\infty} z^{-k} \varphi(z + \varphi)^{k-1}]L.
\]

(b) All lattices \( L \in U^{z^{-1}L^-} \) are trivialized as topological vector spaces via \( L \cong U_K/L^- \cong L \). For \( g \in [z + \text{Hom}(L, U)] \cap N_z \), under this isomorphism \( \Psi_0(g) \cong L \), the action of \( z \) on \( \Psi_0(g) \) corresponds to the operator \( g \) on \( L \).

(c) \( \Psi_0(z + \varphi) \cap L \) is the largest lattice inside \( \text{Ker}(\varphi) \).

**Proof.** (a) Since we know one lattice \( L \) in \( U^{z^{-1}L^-} \), all lattices \( L \in U^{z^{-1}L^-} \) are graphs \( (1 + f)L \) of continuous \( \mathbb{C} \)-linear maps \( f : L \to z^{-1}L^- \). Here, \( z^{-1}L^- = \oplus_{k>0} z^{-k}U \) decomposes \( f \) into \( \sum_{k>0} z^{-k}f_k \) with \( f_k \in \text{Hom}(L, U) \).

For which \( f \)'s is \( (1 + f)L \) a lattice? Invariance of \( (1 + f)L \) under \( z \) means that for each \( v \in L = \prod_{i \geq 0} z^iU \) one has \( z(1 + f)v = (1 + f)w \) for some \( w \in L \), i.e., \( zv + \sum_{i \geq 0} z^{-i}f_{i+1}v = w + \sum_{k>0} z^{-k}f_kw \). The summand in \( L \) is \( zv + f_1v = w \) and for \( p > 0 \) the summand in \( z^{-p}U \) is \( z^{-p}f_{p+1}v = z^{-p}f_pv \). Then \( g = z + f_1 \) lies in \( \mathfrak{gl}(L) \) and \( z \)-invariance means that:

(i) the only choice for \( w \) is \( gv \) and
(ii) \( f_1 \) freely determines \( f \) by \( f_p = f_1w^{p-1} \), for \( p > 1 \). Now, \( (1 + f)L = (1 + \sum_{k>0} z^{-k}f_k)L \) is a lattice iff \( f_k = f_1(z + f_1)^{k-1} \) is zero on \( L \) for some \( k \).

We have chosen \( g \) so that \( z(1 + f) = (1 + f)g \), i.e., the action of \( g \) on \( L \) corresponds under \( 1 + f \) to the action of \( z \) on \( (1 + f)L \).

Since \( f_1 \) is continuous it vanishes on sufficiently small sublattices \( K \subseteq L \). Then on \( K \) we have \( g = z \) and \( (1 + f)K = K \). So, the condition on \( f_k \) reduces to vanishing on \( W \) for some complementary subspace \( K \oplus W = L \). If \( 1 + f \) gives a lattice then \( g \) is nilpotent on \( L/K \) since it behaves the same as \( z \) on \( (1 + f)L/K \). Conversely if \( g \) is nilpotent on \( L/K \) then \( f_1g^p \) vanishes on \( W \cong L/K \) for large \( p \).

\( ^8 \) Here, \( \varphi \in \text{Hom}(L, U) \subseteq \mathfrak{gl}(U) \) and we use the composition \( L \overset{(z+\varphi)^{k-1}}{\to} L \overset{z}{\to} U \overset{z^{-k}}{\to} z^{-k}U \).
(b) just says that for $v \in L$, $(1 + f)v - v$ lies in $z^{-1}L^-$ which is clear.

c) By definition, $\Psi_0(z + \phi)$ is $(1 + f)L$ with $f_1 = \phi$ and $f = \sum_{k=1}^{\infty} z^{-k}\phi(z + \phi)^{k-1}$. Since $fL \subseteq z^{-1}L^-$ we have $(1 + f)L \cap L = \text{Ker}(f)$. Then $\text{Ker}(f)$ is an intersection of kernels of maps $z^{-k}\phi(z + \phi)^{k-1} : L \to z^{-k}U$. Case $k = 1$ gives $\text{Ker}(f) \subseteq \text{Ker}(\phi)$. Also, on any sublattice $K$ of $\text{Ker}(\phi)$ we have $(z + \phi)K = zK \subseteq K$ hence $\phi(z + \phi)^{k-1}K = 0$. □

**Remark.** The isomorphism in the proposition translates a moving lattice $L \in U^L$ with the “constant” operator $z$, into a moving operator $g$ on a fixed vector space $L$.

4.3.2. A rational formula for isomorphism $\Psi_0$. For a continuous operator $\phi : L \to U$ and $a \in \mathbb{N}$ we denote by $\phi_a$ the linear operator on $U$ such that the restriction $\phi : z^aU \to U$ is $\phi_a \circ z^{-a}$. Now we can attach to $\phi$ an $\text{End}(U)$-valued Laurent polynomial $\tilde{\phi} \overset{\text{def}}{=} \sum_{i \geq 0} z^{-i}\phi_i$. It acts on $U_K = U_K$ by the $K$-linear extension of all $\phi_a$. In terms of the projectors $p_k$ to the summands $z^kU$ of $U_K$, one has $\phi = p_0\tilde{\phi}$.

**Lemma.** For $f = \sum_{l \geq 0} z^{-l-1}\phi(z + \phi)^l$ we have

$$(1 + f)L = (1 - z^{-1}\tilde{\phi})^{-1}L.$$  

**Proof.** It suffices to check that $1 + f$ equals $(1 + z^{-1}\tilde{\phi})^{-1}$ on $U \subseteq L$. The reason is that as $\mathbb{C}[z]$-modules, $(1 - z^{-1}\tilde{\phi})^{-1}L$ and $(1 + f)L$ are generated by $(1 - z^{-1}\tilde{\phi})^{-1}U$ and $(1 + f)U$. The first claim is obvious. By proposition 4.3.1.b the second claim is really that $U$ generates $L$ as a $\mathbb{C}[[g]]$-module. This in turn holds because on $L$ we have $g = z$ modulo $U$.

Now we will calculate $1 + f = 1 + \sum_{l \geq 0} z^{-l-1}\phi(z + \phi)^l$ on $U$. Notice that each $\phi(z + \phi)^l$ is a sum of products $\phi z^{a_1}\cdots \phi z^{a_s}$ with $a_1 + \cdots + a_s = (l + 1) - s$. Each factor $\phi z^a$ preserves $U$ and acts on it as the linear operator $\phi_a$.

So, on $U$ operator $z^{-l-1}\phi(z + \phi)^l$ is the sum over all $a_s$ as above of $z^{-s-\sum_i^s a_i} \prod_1^s \phi a_j$. Since $\phi_a$ is $K$-linear this can be rewritten as $\prod_{s \geq 0} \sum_{a_s \in \mathbb{N}} \prod_{s \geq 0} \sum_{a_s \in \mathbb{N}} z^{-1}(z^{-a_i}\phi a_j) = \sum_{0}^{\infty} (z^{-1}\tilde{\phi})^s = (1 - z^{-1}\tilde{\phi})^{-1}$. □

4.4. Isomorphism of slices in loop Grassmannians and in matrices. Here we restrict the isomorphism from 4.3 to finite dimensional slices. The local case single loop Grassmannian is in In 4.4.1 and the global case (Beilinson-Drinfeld Grassmannian) in 4.4.2.
4.4.1. Loop Grassmannian slice. As in example 4.2.4 any coweight $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{Z}^m$ defines the $\mathbb{C}$-form $U_{\sigma} = \oplus z^{\sigma_i} \mathbb{C} e_i$ of $U_K$, which in turn generates the $\mathbb{O}$-module $L = L_{\sigma} \in \mathcal{G}$ and an $\mathcal{O}^-$-module $L^- = L_{\sigma}^- \oplus z^{-\sigma_i} \mathcal{O}^- e_i$. Then $U_{\sigma} \overset{\text{def}}{=} U_{z^{\sigma_i}} \cap \mathcal{G}_0$ is a neighborhood of $L_{\sigma}$ in $\mathcal{G}_0 = \mathcal{G}$. Recall the standard normal slice $\mathcal{T}_{\sigma} = L^{<0}G \cdot L_{\sigma}$ to $\mathcal{G}_0 = G(\mathcal{O}) \cdot L_{\sigma}$ at $L_{\sigma}$ (4.2.1).

For $N \in \mathbb{Z}$ denote by $\mathcal{G}^N = \mathcal{G}^{z^N L_0} \subseteq \mathcal{G}$ the lattices that contain $z^N L_0$. We will consider $N$ such that $L_{\sigma}$ lies in $\mathcal{G}^N$, i.e., in $U_{\sigma}^N \overset{\text{def}}{=} U_{\sigma} \cap U^N$. Denote by $x = z^N$ the action of $z$ on the vector space $N \overset{\text{def}}{=} L_{\sigma}/z^N L_0$. When we order $\sigma - N \overset{\text{def}}{=} (\sigma_1 - N, \ldots, \sigma_n - N)$ into a partition $\lambda = \lambda_N$ then $x$ lies in the nilpotent orbit $\mathcal{O}_\lambda$ in the nilpotent cone $N \subseteq \mathfrak{gl}(N)$. Let $T_x$ be the "regular" normal slice $T_x \subseteq \mathfrak{gl}(N)$ to $\mathcal{O}_\lambda$ 3.2.3.

**Theorem.** (a) The isomorphism $\Psi_0$ from 4.3.1 restricts to an isomorphism of nilpotent operators in the "regular" normal slice $T_x$ (to the nilpotent orbit through $x$) and the $N$-part $\mathcal{T}_{\sigma} \cap \mathcal{G}^N$ of the slice in $\mathcal{G}$:

$$\Psi_0 : T_x \cap \mathcal{N} \xrightarrow{\sim} \mathcal{T}_{\sigma} \cap \mathcal{G}^N.$$  

(b) The inverse of this isomorphism is a restriction of the isomorphism $\Phi$ from 1.2.

(c) If $\mathcal{O}_\lambda \subseteq \overline{\mathcal{O}_\mu}$ for a nilpotent orbit $\mathcal{O}_\mu$ in $\mathfrak{gl}(N)$ then $\Psi_0$ restricts to an isomorphism of intersections $\Psi_{0\mu} : T_x \cap \overline{\mathcal{O}_\mu} \xrightarrow{\sim} \mathcal{T}_{\sigma} \cap \mathcal{G}_{\mu+N}$.

**Remark.** The normal slice $\mathcal{T}_{\sigma}$ is the union of all $\mathcal{T}_{\sigma} \cap \mathcal{G}^N$ as $N \to \infty$. This identifies it with the "regular" normal slice to the orbit of the pronilpotent operator $z$ on $L_{\sigma}$ 3.2.4.

**Proof.** (c) follows from (a) since $\Psi_0(T_x \cap \mathcal{O}_\mu) \subseteq \mathcal{T}_{\sigma} \cap \mathcal{G}_{\mu+N}$ (according to the proposition 4.3.1) the nilpotent operators $g$ on $L_{\sigma}/z^N L_0$ and $z$ on $\Psi_0(g)/z^N L_0$ have the same type).

(a) According to the proposition 4.3.1c, for $g = z + \varphi$, $\Psi_0(g)$ contains $z^N L_0$ iff $\varphi = 0$ on $z^N L_0$, i.e., $\varphi : \mathcal{N} \to \mathcal{U}_s$ for $N = L_{\sigma}/z^N L_0$. Then we can consider $g$ as an operator on $\mathcal{N}$.

Observe that $\mathcal{T}_{\sigma} = L^{<0}G \cdot L_{\sigma}$ is a torsor for $L^{<0}G \cap \mathcal{O}(L^{<0}G)$, so it lies in the neighborhood $\mathcal{O}(L^{<0}G) \cdot L_{\sigma}$ of $L_{\sigma}$ which is exactly $U^\sigma$. It remains to show that for $\varphi \in \text{Hom}(N, U_\sigma)$ such that $g = x + \varphi$ is nilpotent, the corresponding lattice $L = \Psi_0(g)$ in $U_{\sigma} \cap \mathcal{G}_N$ lies in $\mathcal{T}_{\sigma}$ iff $\varphi$ lies in the subspace $C \overset{\text{def}}{=} T_x - x$ of $\text{Hom}(N, U_\sigma)$.

Group $\mathcal{R}$ of loop rotations is the group $G_m$ acting on $U((z)) = U_K$ by $s \bullet z^k e_i \overset{\text{def}}{=} (sz)^k e_i$. It acts on the space $\mathcal{G}$ of lattices and on $U_K$. One knows ([MVII]) that a lattice $L \in \mathcal{G}$ is in $L^{<0}G \cdot L_{\sigma} \overset{\text{def}}{=} \mathcal{T}_{\sigma}$ if and only if $\lim_{s \to \infty} s \bullet L = L_{\sigma}$.

Let us write $\Psi_0(g)$ as $(1+f)L_{\sigma}$ for $f = \sum_{i=1}^{\infty} z^{-k} \varphi(z + \varphi)$ as in the proof of proposition 4.3.1. Since $L_{\sigma}$ is fixed by $\mathcal{R}$, we have $s \bullet (1+f)L_{\sigma} = (1+s \bullet f)L_{\sigma}$. Therefore, $\lim_{s \to \infty} s \bullet \Psi_0(g) = L_{\sigma}$.
is equivalent to \( \lim_{s \to \infty} s \cdot f = 0 \). Since 
\[
 s \cdot f = \sum_{k=1}^{\infty} (sz)^{-k} (s \cdot \varphi)(sz + (s \cdot \varphi))^{k-1} = \sum_{k=1}^{\infty} z^{-k} s^{-1} (s \cdot \varphi) (z + s^{-1}(s \cdot \varphi))^{k-1}.
\]
So, \( \lim_{s \to \infty} s \cdot f = 0 \) is equivalent to \( \lim_{s \to \infty} s^{-1}(s \cdot \varphi) = 0 \).

Here, \( \varphi : L_{\sigma} \to U_{\sigma} \) has the \((i, j)\)-component maps in \( \text{Hom}(z^{\sigma_i}O, z^{-\sigma_j}C) \cong \oplus_{p \geq 0} z^{\sigma_i - \sigma_j - p}C \).

So, the condition is that the \((i, j, p)\) component of \( \varphi \) vanishes unless \( p \geq \sigma_i - \sigma_j \). However, this condition on \( \varphi \) is just the description of \( C \subseteq \text{Hom}(N, U_{\sigma}) \) from \(3.2.6\).

Finally, (b) is the claim about the action of \( z \) on \( \Psi_0(g)/z^N L_0 \). One asks that via the natural isomorphism \( \Psi_0(g)/z^N L_0 \cong L_{\sigma}/z^N L_0 \) this operator is \( g = z + \varphi \). But this is exactly the proposition \(4.3.1\). \( \square \)

4.4.2. Beilinson-Drinfeld Grassmannian. Let again \( \sigma \in \mathbb{Z}^m \) and \( N \in \mathbb{Z} \) be such such that \( L_{\sigma} \supseteq z^N L_0 \). Here we remove the nilpotency restriction in the isomorphism \( \Psi_0 : T_x \cap N \xrightarrow{\cong} T_\sigma \cap \mathbb{G}_a \) in theorem \(4.4.1\) by extending the target from the loop Grassmannian \( \mathbb{G}_a \) to the global loop Grassmannian \( \mathbb{G}_{A(n)} \). We use the rational formula for \( \Psi_0 \) \(4.3.2\) that involves transformation \( \varphi \mapsto \tilde{\varphi} \).

**Lemma.** The isomorphism of slices \( \Psi_0 \) from theorem \(4.4.1\) extends to an embedding

\[
\Psi : T_x \hookrightarrow U_{\mathbb{G}_{A(n)}},
\]

that sends an operator \( g = x + \varphi \in T_x \) to the lattice \( \Psi(g) = (b, L) \) with \( L = (1 - z^{-1}\tilde{\varphi})^{-1} L_{\sigma} \).

**Remark.** Here parameters \( n \) and \( b \) are not unique. For instance when \( N = 0 \) then a natural choice is \( n = \sum \sigma_i \) and \( b = \text{Spec}(g) \).

**Proof.** Let \( g = z + \varphi \in T_x \subseteq \text{End}(N) \) for \( N = L_{\sigma}/z^N L_0 \), so that \( \varphi : N \to U_{\sigma} \). \( L = (1 - z^{-1}\tilde{\varphi})^{-1} L_{\sigma} \) is clearly a lattice. Most of the argument is the same as above but now in it is in the setting where expansion \( (1 - z^{-1}\tilde{\varphi})^{-1} = 1 + f \) for \( f = \sum_{k>0} z^{-k}\varphi(z + \varphi)^{k-1} \) (here \( \varphi = p_0\tilde{\varphi} \), is a formal power series. This expansion shows that \( L \) is complementary to \( z^{-1}L_{\sigma} \), also and \( L \subseteq z^N L_0 \) because \( \varphi = 0 \) on \( z^N L_0 \).

The identity \( z(1 + f) = (1 + f)g \) shows that the characteristic polynomial \( \chi \) of \( g \) kills the restriction of the operator \( z \) to \( L/z^N L_0 \). So, \( \chi(g) \Psi(g) \subseteq z^N L_0 \) hence \( z^N L_0 \subseteq L \subseteq \chi(g)^{-1} z^N L_0 \) shows that \( L = L_0 \) off \( \{0\} \cup \text{Spec}(g) \subseteq \mathbb{A} \) (so \( b = \text{Spec}(g z^N) \) works).

Finally, \( \Psi \) is an embedding because for \( \Phi \) from \(4.2.2\) the composition \( \Phi \circ \Psi \) is identity on the slice \( T_x \subseteq \mathfrak{gl}(N) \). \( \square \)
Example. The map $\phi$ is a generalization of the modification of line bundles by points. A special case is the embedding of $gl(n)$ into $S_{\Lambda(n)}$ by $x \mapsto \frac{1}{z-x}O^{\otimes n}$ that lies above $gl_n/GL_n \cong \Lambda(n)$. Its restriction to the nilpotent cone in $\text{End}(U)$ recovers Lusztig’s embedding into the local loop Grassmannian $G$ at $0 \in \Lambda[2]$ which is the origin of this paper.

4.5. Comparison of resolutions of slices. Let $x \in O_\lambda \subseteq \overline{O_\mu}$ and let $\alpha$ be any reordering of terms in the dual partition $\hat{\mu}$. Recall the partial resolution $\pi_{\mu}^\alpha : \tilde{G}_\mu^{\alpha} \to \overline{G}_\mu$ from 4.1.2 and $\tilde{T}_x^\alpha \overset{\text{def}}{=} T_x \times_N \tilde{N}^\alpha$ from 3.2.7. We can lift the map $\psi_\mu : T_x \cap \overline{O_\mu} \overset{\psi_\mu}{\longrightarrow} \mathcal{I}_\sigma \cap \overline{G}_\mu$ to the map

$$
\tilde{\psi}_\mu : \tilde{T}_x^\alpha = T_x \times_N \tilde{N}^\alpha \longrightarrow (\mathcal{I}_\sigma \cap \overline{G}_\mu) \times_{\overline{G}_\mu} \tilde{G}_\mu
$$

since any $(x + \varphi)$-invariant $n$-step flag in $N$ will give rise to an $n$-step flag of lattices in $L = \psi(x + \varphi)$.

Corollary. Maps $(\tilde{\psi}_\mu, \psi_\mu)$ form an isomorphism of resolutions

$$
\begin{array}{ccc}
\tilde{T}_x^\alpha & \overset{\tilde{\psi}_\mu}{\longrightarrow} & (\mathcal{I}_\sigma \cap \overline{G}_\mu) \times_{\overline{G}_\mu} \tilde{G}_\mu \\
\downarrow m_\alpha & & \downarrow \pi_\alpha^\mu \\
T_x \cap \overline{O_\mu} & \overset{\psi_\mu}{\longrightarrow} & \mathcal{I}_\sigma \cap \overline{G}_\mu.
\end{array}
$$

(10)

4.6. Perverse sheaves on loop Grassmannians. In this subsection $G$ denotes $GL_m$ or $PGL_m$.

Let $\mathcal{P}_{G(O)}(S_G)$ be the category of $G(O)$-equivariant perverse sheaves on $S_G$. We will denote by $IC_\mu = IC(\overline{G}_\mu)$ the intersection cohomology complex on the closure of the orbit $\overline{G}_\mu$. There is a tensor product (convolution) construction $\boxtimes$ [MV1, MV2] which makes the category $\mathcal{P}_{G(O)}(S_G)$ into a tensor category. Let $\text{Rep}(G)$ be the category of rational representations of the Langlands dual group $\hat{G}$ over $\mathbb{C}$.

4.6.1. Geometric Satake correspondence [MV2, 7.1]. The global cohomology functor is an equivalence of tensor categories $H^* : (S, \sim) : (\mathcal{P}_{G(O)}(S), \boxtimes) \overset{\sim}{\longrightarrow} (\text{Rep}(\hat{G}), \otimes)$. Under this equivalence the intersection cohomology sheaf $IC_\mu = IC(\overline{G}_\mu)$ corresponds to the irreducible representation $V^G_\mu$ of $G$ with the highest weight $\mu$.

4.6.2. Multiplicity spaces $\text{Hom}_{GL_m}(V^G_{a_1} \otimes \cdots \otimes V^G_{a_n}, V^G_\lambda)$. By Satake equivalence this space is realized as $\text{Hom}_{p_{G(O)}(S)}(IC_{a_1} \ast \cdots \ast IC_{a_n}, IC_\lambda)$. The convolution $IC_{a_1} \ast \cdots \ast IC_{a_n}$ is the direct image of the IC-sheaf $IC(S_{a_1} \ast \cdots \ast S_{a_n})$ of the smooth simply connected space $S_{a_1} \ast \cdots \ast S_{a_n}$ under the map $\pi_{a_1}^\mu : S_{a_1} \ast \cdots \ast S_{a_n} \to \overline{G}_\mu$ from 4.1.2 (see [MV2]). The map
\[\pi_\mu^a\] is semismall and therefore the multiplicity space is the highest Borel-Moore homology \(\mathcal{H}[(\pi_\mu^a)^{-1}(L_\lambda)]\) of the fiber at \(L_\lambda\) (see [MV12]).

5. Formulation of the Isomorphism Theorem

5.1. Combinatorial data.

5.1.1. From quiver data \((v,d)\) to \(GL_n\) weights \(\lambda, \mu\) and \(a\). Let \(d = (d_1, \ldots, d_{n-1})\) and \(v = (v_1, \ldots, v_{n-1})\) be two \((n-1)\)-tuples of non-negative integers. We will transform these “quiver data” into \(GL_n\) weights.

1. Let \(N = \sum_{j=1}^{n-1} j d_j\) and let \(m = \sum_{j=1}^{n-1} d_j\).
2. Let \(\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)\) be a partition of \(N\) for \(\tilde{\lambda}_i = \sum_{j=i}^{n} j d_j\).
3. Let \(\mu\) be the dual partition.
4. Let \(a = (a_1, \ldots, a_n)\) be defined as follows, cf. [N1, 8.3],
   \[
   a_i = v_{n-1} + \sum_{j=i}^{n-1} (d - Cv) j
   \]
   where \(C\) is the Cartan matrix of type \(A_{n-1}\).
5. Let \(\tilde{\mu}\) be the partition obtained from \(a\) by permutation and let \(\mu\) be the dual partition.

Here, \(\tilde{\lambda}\) can be viewed as a dominant weight of \(GL_n\) and \(a\) is a weight of \(V_{\lambda}^{GL(N)}\), cf. 2.4.1.

5.1.2. From the quiver datum \(c\) to the spectral datum \(b\) \(\in A^n\). We will encode an element \(c = (c_1, \ldots, c_{n-1})\) of the center of \(\mathfrak{g}(V)\) (where \(\mathfrak{g}(V) = \prod_{i=1}^{n-1} \mathfrak{gl}(V_i)\) and \(\text{dim } V_i = v_i\)), as an element \(b = (b_1, \ldots, b_n)\) of \(A^n\) where

\[
b_1 = 0 \quad \text{and} \quad b_i = c_1 + \cdots + c_{i-1} \quad \text{for} \quad 2 \leq i \leq n.
\]

We can encode the image \(b\) of \(b\) in \(\Lambda^{(n)}\) by the polynomial \(P(t) \overset{\text{def}}{=} \prod_{i=1}^{n} (t - b_i)\).

5.1.3. From the \(GL_n\) weight \(a\) and the spectral datum \(b \in A^n\) to the conjugacy class \(O_{E, \tilde{\mu}}\) in \(\mathfrak{gl}(N)\). The eigenvalues of an element \(x \in \mathfrak{g}_{E, \tilde{\mu}}\) will form the set \(E = \{b_i; \ 1 \leq i \leq n\}\), the support of the spectral datum \(b\). The partition \(\mu^e\) that describes the Jordan blocks with eigenvalue \(e\) will be given in terms of the weight \(a \in \mathbb{Z}^n\) that was defined from the quiver data \((v,d)\) in [11]. First let \(I_e \overset{\text{def}}{=} \{i \in [1,n]; \ b_i = e\}\) and then let \(a_e \overset{\text{def}}{=} (a_i)_{i \in I_e}\). Now the dual partition \((\mu^e)^v\) is obtained by ordering \(a^e\).

Notice that this is the conjugacy class denoted \(\mathfrak{g}_{E, \tilde{\mu}}\) in [3.1.3] where \(\tilde{\mu}\) is the \(E\)-bipartition \(\{\mu^e\}_{e \in E}\).
5.1.4. Now we can formulate our main theorem. For notation on quiver varieties see 2.3, on Springer-Ginzburg resolutions see 3.1, on transverse slices see 3.2.6, and finally on Beilinson-Drinfeld Grassmannians see 4.1.

5.2. **Theorem.** Let \( N, m, v, d, a, c, b, \varepsilon, \lambda, \tilde{\mu} \) be as above. There exist algebraic isomorphisms \( \phi, \tilde{\phi} \) and algebraic immersions \( \psi, \tilde{\psi} \) such that the following diagram commutes:

\[
\begin{array}{c}
\mathcal{M}^c(v, d) \xrightarrow{\bar{\phi}} T_{\lambda} \times \mathcal{G}^{a,b} \xrightarrow{\bar{\psi}} \mathcal{G}^a_b(P) \\
\mathcal{M}^c_1(v, d) \xrightarrow{\phi} T_{\lambda} \cap \mathcal{G}^{\varepsilon,\tilde{\mu}} \xrightarrow{\psi} \mathcal{G}_{N,b}(P).
\end{array}
\]

5.3. **Consequences and Remarks.** We comment on the case \( c = 0 \), consider two applications of our Isomorphism Theorem and check an equality of dimensions.

5.3.1. **Case \( c = 0 \).** Here, we can describe the images of the maps \( \psi \) and \( \tilde{\psi} \) and obtain a more precise result stated in the introduction and [MVy]. In particular, \((\psi \circ \phi)(0) = L_{\lambda} \in \mathcal{G}\) and \(\tilde{\psi} \circ \tilde{\phi}\) restricts to an isomorphism

\[
\tilde{\psi} \circ \tilde{\phi} : \mathcal{L}(v, d) \simeq \pi^{-1}(L_{\lambda}).
\]

We believe that one should be able to generalize these statements for arbitrary \( c \).

5.3.2. **A compactification of quiver varieties.** The closure in \( \mathcal{G}_{N,b}(P) \) of the image of \( \mathcal{M}^c_1(v, d) \) under the map \( \psi \circ \phi \) gives us a compactification of \( \mathcal{M}^c_1(v, d) \). Analogously, the closure in \( \mathcal{G}^a_b(P) \) of the image of \( \mathcal{M}^c(v, d) \) under the map \( \tilde{\psi} \circ \tilde{\phi} \) gives us a compactification of the quiver variety \( \mathcal{M}^c(v, d) \).

5.3.3. **A decomposition of the loop Grassmannian.** The following is a corollary of the main theorem. Here \( c = 0 \).

**Corollary.** We can decompose \( \mathcal{G}_{\mu} \) into the following disjoint union:

\[
\mathcal{G}_{\mu} = \bigsqcup_{y \in G \cdot \lambda} \mathcal{M}^c_0(v, d)_y,
\]

where \( \lambda \) varies over the set of dominant coweights of \( G \), \( G \cdot \lambda \) is the \( G \)-orbit of \( L_{\lambda} \) in \( \mathcal{G} \), and \( \mathcal{M}^c_0(v, d)_y \) is a copy of quiver variety \( \mathcal{M}^c_0(v, d) \) for every point \( y \in G \cdot \lambda \), with \( v, d \) obtained from \( \lambda, \mu \) by reversing the procedures of 5.1.1.
Proof. The dominant cocharacters $\lambda \in X^+_*(T)$ parameterize the $G(\mathbb{C}[z^{-1}])$-orbits $G(\mathbb{C}[z^{-1}]) \cdot L_\lambda$ in $G$ and the maps $L^{<0}G \times G \cdot L_\lambda \to G(\mathbb{C}[z^{-1}]) \cdot L_\lambda$ are isomorphisms. Then:

$$\mathfrak{G}_\mu = \bigsqcup_{\lambda \in X^+_*(T)} (L^{<0}G \cdot y) \cap \mathfrak{G}_\mu \cong \bigsqcup_{\lambda \in X^+_*(T)} \mathfrak{M}_0^c(v, d)_y$$

since every $(L^{<0}G \cdot y) \cap \mathfrak{G}_\mu$, for $y \in G \cdot \lambda$ is isomorphic to a copy of $\mathfrak{M}_0^c(v, d)$. □

5.3.4. Remarks. (1) An “affine analogue” of our construction has appeared in the paper [BF1].

(2) We would also like to mention another example of a decomposition of an infinite Grassmannian into a disjoint union of quiver varieties. Generalizing a result of G. Wilson [W], V. Baranovsky, V. Ginzburg, and A. Kuznetsov [BGK] constructed a decomposition of (a part of) adelic Grassmannian into a disjoint union of deformed versions of quiver varieties $\mathfrak{M}^c(v, d)$ associated to affine quivers of type A.

5.3.5. Dimensions for $c = 0$. First of all we’ll check that the varieties $\mathfrak{M}^c(v, d)$ and $\tilde{T}_x^\mu$ have the same dimension. According to Nakajima [N2, Corollary 3.12] $\mathfrak{M}^c(v, d)$, if nonempty, is a smooth variety of dimension $t^v(2d - Cv)$ where $C$ is the Cartan matrix of type $A_{n-1}$. If $\tilde{\lambda}$ and $\tilde{\mu}$ are defined by $v, d$ as in 5.1.1 then we have

$$\dim \mathfrak{M}^c(v, d) = t^v(2d - Cv) = 2 \sum_{i=1}^{n-1} v_i d_i - 2 \sum_{i=1}^{n-1} v_i^2 + 2 \sum_{i=1}^{n-1} v_i v_{i+1}$$

$$= \sum_{i=1}^{n-1} [\tilde{\lambda}_i^2 - (\tilde{\mu}_i)^2] = \dim \tilde{T}_x^\mu.$$ (17)

6. Quiver varieties and conjugacy classes of matrices

Here we consider a particular case of the Isomorphism Theorem. We choose $d = (N, 0, \ldots, 0)$ and $v = (v_1, \ldots, v_{n-1})$ so that $N \geq v_1 \geq v_2 \geq \cdots \geq v_{n-1} \geq 0$. It is explained in 8.1.1 how the general case of the theorem reduces to this special setting by the “Main lemma” (section 7).

Define algebraic morphisms $\tilde{\varphi} : \mathfrak{M}^c(v, d) \to \tilde{g}^{a,b}$ and $\phi : \mathfrak{M}^c_1(v, d) \to \tilde{O}_{\varepsilon, \tilde{\mu}}$ by:

$$\tilde{\varphi} : (x, \overline{x}, p, q) \mapsto (q_1 p_1, \{0\} \subseteq \text{Ker}(p_1) \subseteq \text{Ker}(x_1 p_1) \subseteq \text{Ker}(x_{n-1} \ldots x_1 p_1)), $$

$$\phi : (x, \overline{x}, p, q) \mapsto q_1 p_1.$$ (18)

The following theorem is a common generalization of results of [KP] and [N1], cf. [CB].
6.1. **Theorem.** The maps \( \phi, \tilde{\phi} \) defined above are isomorphisms of algebraic varieties and the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{M}^c(v, d) & \xrightarrow{\phi} & \mathcal{M}^c_1(v, d) \\
p \downarrow & & \downarrow \tilde{m} \\
\tilde{g}^a, b & \xrightarrow{\tilde{\phi}} & \tilde{g}^a, b \\
\end{array}
\]

(19)

**Proof.** Following the logic of [N2, Maf], it is not hard to check that \( \tilde{\phi} \) is a bijective morphism between two smooth varieties of the same dimension and thus an isomorphism. The map \( \phi \) is a closed immersion and it is surjective since both \( p \) and \( \tilde{m} \) are surjective. \( \square \)

6.1.1. In the case when all the numbers \( 0, c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{n-1} \) are pairwise distinct, then the quiver variety \( \mathcal{M}^c(v, d) \) is isomorphic to the conjugacy class of semisimple elements with eigenvalues \( b_i = \sum_{1 \leq k < i} c_k \), \( 1 \leq i \leq n \), where \( b_i \) appears with multiplicity \( a_i = v_{i-1} - v_i \) (we use \( v_0 = N \) and \( v_n = 0 \)).

Quiver variety \( \mathcal{M}^c_0(v, d) \) is also isomorphic to a conjugacy class which is generally different from the conjugacy class considered above. The two classes coincide when the \( SL(n) \) weight \( d - Cv \) is dominant, i.e. when \( a_1 \geq a_2 \geq \cdots \geq a_n \).

7. **Main Lemma**

This lemma provides an isomorphism \( \Phi : M^c(v, d) \xrightarrow{\sim} S \) with a subspace \( S \) of “transversal” elements of \( M^c(v, d) \) for certain parameters \( (\tilde{v}, \tilde{d}) \). This map restricts to an isomorphism \( \Phi^* : M^c(v, d) \rightarrow S^* \) of subspaces of stable elements. The point is that the new parameters \( \tilde{v}, \tilde{d} \) are of the special kind considered in section 6.

7.1. **D’après Maffei.** We borrow Maffei’s [Maf] notations and conventions. Let \( v = (v_1, \ldots, v_{n-1}) \) and \( d = (d_1, \ldots, d_{n-1}) \) be two \((n - 1)\)-tuples of integers and let us define \((n - 1)\)-tuples \( \tilde{v} \) and \( \tilde{d} \) as follows:

\[
\begin{align*}
\tilde{d}_1 & \equiv \sum_{j=1}^{n-1} jd_j \quad \text{and} \quad \tilde{d}_i \equiv 0, \; \text{for} \; i > 1, \\
\tilde{v}_i & \equiv v_i + \sum_{j=i+1}^{n-1} (j - i)d_j.
\end{align*}
\]

(20)

Our goal is to construct a map from \( M^c(v, d) \) to \( M^c(\tilde{v}, \tilde{d}) \), that is we have to send a quadruple \( (x, \pi, p, q) \in M^c(v, d) \) to a quadruple \( (\tilde{A}, \tilde{B}, \tilde{\gamma}, \tilde{\delta}) \in M^c(\tilde{v}, \tilde{d}) \). First of all, the
Let $D_i$ be a copy of $D_j$.

$$
\tilde{D}_1 = \bigoplus_{1 \leq k \leq n-1} D_j^{(k)} \quad \text{and} \quad \tilde{D}_i = 0, \text{ for } i > 1,
$$

\begin{equation}
\widetilde{V}_i = V_i \bigoplus \bigoplus_{1 \leq k \leq j-i \leq n-i-1} D_j^{(k)}.
\end{equation}

We need the following subspaces of $\widetilde{V}_i$.

\begin{equation}
D_i' = \bigoplus_{i+1 \leq j \leq n-1} D_j^{(k)} \quad D_i^+ = \bigoplus_{i+2 \leq j \leq n-1} D_j^{(k)} \quad D_i^- = \bigoplus_{1 \leq k \leq j-i-1} D_j^{(k)}.
\end{equation}

In order to make the notation more homogeneous we set $\widetilde{V}_0 \overset{\text{def}}{=} \tilde{D}_1$, $\tilde{A}_0 = \tilde{\gamma}_1$, $\tilde{B}_0 = \tilde{\delta}_1$.

We will name the blocks of the maps $\tilde{A}_i$ and $\tilde{B}_i$ as follows

\begin{equation}
\begin{align*}
\pi_{D_j^{(h)}} \tilde{A}_i |_{D_j^{(h')}} &= i_{j,h}^{j',h'} & \pi_{D_j^{(h)}} \tilde{B}_i |_{D_j^{(h')}} &= s_{j,h}^{j',h'} \\
\pi_{D_j^{(h)}} \tilde{A}_i |_{V_i} &= i_{j,h}^V & \pi_{D_j^{(h)}} \tilde{B}_i |_{V_{i+1}} &= s_{j,h}^V \\
\pi_{V_{i+1}} \tilde{A}_i |_{D_j^{(h')}} &= i_{V, h}^{j',h'} & \pi_{V_i} \tilde{B}_i |_{D_j^{(h')}} &= s_{V, h}^{j',h'}
\end{align*}
\end{equation}

We define also the following operator $z_i$ on $D_i'$

\begin{equation}
z_i |_{D_j^{(i)}} = 0 \quad \text{and} \quad z_i |_{D_j^{(h)}} = Id_{D_j} : D_j^{(h)} \to D_j^{(h-1)}
\end{equation}

7.1.1. Following Maffei let us introduce the following degrees:

\begin{equation}
\begin{align*}
\text{deg}(i_{j,h}^{j',h'}) &= \min(h - h' + 1, h - h' + 1 + j' - j), \\
\text{deg}(s_{j,h}^{j',h'}) &= \min(h - h', h - h' + j' - j).
\end{align*}
\end{equation}

7.1.2. A quadruple $(\tilde{A}, \tilde{B}, \tilde{\gamma}, \tilde{\delta}) \in M^c(\tilde{v}, \tilde{d})$ is called transversal if it satisfies the following two groups of relations for $0 \leq i \leq n - 2$
(1) first group (Maffei)

\[
i_{j',h'}^{i,j,h} = 0 \quad \text{if } \deg(t_{j,h}^{j',h'}) < 0
\]

\[
i_{j',h'}^{i,j,h} = 0 \quad \text{if } \deg(t_{j,h}^{j',h'}) = 0 \text{ and } (j', h') \neq (j, h + 1)
\]

\[
i_{j',h'}^{i,j,h} = Id_{D_j} \quad \text{if } \deg(t_{j,h}^{j',h'}) = 0 \text{ and } (j', h') = (j, h + 1)
\]

\[
i_{j',h'}^{i,j,h} = 0 \quad \text{if } h' \neq 1
\]

\[
i_{j',h'}^{i,j,h} = 0 \quad \text{if } \deg(s_{j,h}^{j',h'}) < 0
\]

\[
i_{j',h'}^{i,j,h} = 0 \quad \text{if } \deg(s_{j,h}^{j',h'}) = 0 \text{ and } (j', h') \neq (j, h)
\]

\[
i_{j',h'}^{i,j,h} = Id_{D_j} \quad \text{if } \deg(s_{j,h}^{j',h'}) = 0 \text{ and } (j', h') = (j, h)
\]

\[
i_{j',h'}^{i,j,h} = 0 \quad \text{if } h \neq j - i
\]

\[
i_{j',h'}^{i,j,h} = 0
\]

(2) second group

(27) \[\pi_{D(j)} \tilde{B}_i \tilde{A}_i |_{D(j')} - x_i = 0 \quad \text{unless } h = j - i\]

Let us denote the set of all transversal elements in \(M^c(\tilde{v}, \tilde{d})\) by \(S\). The set of all stable transversal elements is denoted by \(S^* = S \cap M^c(\tilde{v}, \tilde{d})\).

7.1.3. We will need more notation. First of all denote

(28) \[b_i^j = c_{i+2} + \cdots + c_j \quad \text{for } -1 \leq i \leq n - 3, \text{ and } i + 2 \leq j \leq n - 1.\]

Now we introduce some invariant polynomials of \(q_{i \to j}p_{j \to i}\) as follows. First,

(29) \[P(i, 1, j) = q_{i+2 \to j}p_{j \to i+2}\]

and for \(2 \leq h' \leq j - i - 1\)

(30) \[P(i, h', j) = q_{i+h'+1 \to j}p_{j \to i+h'+1} + (-1)^{j-i-h'-1}\sigma_{j-i-h'}(b_{i+2}^i, \ldots, b_{j-1}^j)\]

\[+ \sum_{k=1}^{j-i-h'-1} (-1)^k \sigma_k(b_{i+2}^i, \ldots, b_{i+h'-1+k}^i)q_{i+h'+1+k \to j}p_{j \to i+h'+1+k}\]

where \(\sigma_k\) is the \(k\)-th elementary symmetric function.

7.2. Main Lemma. We can now formulate our main lemma
Lemma.

(i) There exists a unique $G(V)$-equivariant map $\Phi : M^c(v, d) \to S$ such that

$$\pi_{V_{i+1}} \tilde{A}_i|_{V_i} = x_i \quad \pi_{V_i} \tilde{B}_i|_{V_{i+1}} = \pi_i$$

$$q^{i+1,1}_V = p_{i+1} \quad s^{V}_{i+1,1} = q_{i+1}$$

(ii) The blocks of $\tilde{A}_i, \tilde{B}_i$ not defined in the equations (26) and (31) are described as follows:

$$\tilde{q}^{i',1}_{j',h'} = p_{j' \to i+1} \quad \tilde{s}^{V}_{j,j-i} = q_{i+1 \to j}$$

When $j' \neq j$ we have

$$\tilde{q}^{i',h'}_{j,h} = 0 \quad \text{if} \ (j', h') \neq (j, h + 1)$$

$$\tilde{s}^{i',h'}_{j,h} = 0 \quad \text{if} \ (j', h') \neq (j, h) \text{ and } h \neq j - i$$

and

$$\tilde{s}^{i',h'}_{j,j-i} = q_{i+h'+1 \to j}p_{j' \to i+h'+1}$$

When $j = j'$ we have

$$\tilde{q}^{i',h'}_{j,h} = \begin{cases} 0, & \text{if } h' = 1 \\ (-1)^{h-h'+1} \left( h - 1 \right) c_{i+1}^{h-h'+1}, & \text{if } 2 \leq h' \leq h + 1 \end{cases}$$

And finally,

$$\tilde{s}^{i',h'}_{j,h} = \begin{cases} \left( \frac{h-1}{h' - 1} \right) c_{i+1}^{h-h'}, & \text{if } h \neq j - i \\ \left( \frac{h}{h' - 1} \right) c_{i+1}^{h-h'}, & \text{if } 1 \leq h' \leq h, \text{ and } h = j - i \end{cases}$$

(iii) For $x \in M^c(v, d)$ we have $\Phi(x) \in S^s$ if and only if $x \in M^c_s(v, d)$. Thus the restriction of $\Phi$ to the stable points provides the $G(V)$-equivariant map $\Phi^s : M^c(v, d) \to S^s$

(iv) The maps $\Phi$ and $\Phi^s$ are isomorphisms of algebraic varieties.

Proof. Following Maffei, we prove the lemma by decreasing induction on $i$. If $i = n - 2$ the maps $\tilde{A}_{n-2}$ and $\tilde{B}_{n-2}$ are completely defined by the relations (31) and (26) and it is easy to see that $\tilde{A}_{n-2} = \tilde{B}_{n-2} = c_{n-1}$.

Assume that $\tilde{A}_k, \tilde{B}_k$ are defined for $k > i$ by the formulas in the lemma.

We have the following equations for $\tilde{A}_i$ and $\tilde{B}_i$:

$$\tilde{A}_i \tilde{B}_i = \tilde{B}_{i+1} \tilde{A}_{i+1} + c_{i+1}$$

$$\pi_{D_j^{(h)}} \tilde{B}_i \tilde{A}_i|_{D_j^{(h')}} - z_i = 0 \quad \text{unless } h = j - i.$$
Observe that
\[ \pi_{V_{i+1}} \tilde{A}_i \tilde{B}_i |_{V_{i+1}} = A_i B_i + p_{i+1} q_{i+1} = B_{i+1} A_{i+1} + c_{i+1} = \pi_{V_{i+1}} \tilde{A}_{i+1} \tilde{B}_{i+1} |_{V_{i+1}} + c_{i+1}. \]

Then, in agreement with formulas (32)
\[ \pi_{V_{i+1}} \tilde{A}_i \tilde{B}_i |_{D_j^{(h)}} = \pi_{V_{i+1}} \tilde{A}_{i+1} \tilde{B}_{i+1} |_{D_j^{(h)}} = \delta_{h,1} B_{i+1} p_{j-i+2} = \delta_{h,1} p_{j-i+1}, \]
\[ \pi_{D_j^{(h)}} \tilde{A}_i \tilde{B}_i |_{V_{i+1}} = \pi_{D_j^{(h)}} \tilde{A}_{i+1} \tilde{B}_{i+1} |_{V_{i+1}} = \delta_{h,j-i-1} q_{i+2-j} A_{i+1} = \delta_{h,j-i-1} q_{i+1-j}. \]

where \( \delta_{p,q} \) is the Kronecker symbol.

Now, in order to simplify the notation a bit we set \( \tau_{j,h} \) where \( \tau \) for \( j \)
\( \delta \)
(39)
\[ \pi_{V_{i+1}} \tilde{A}_i \tilde{B}_i |_{V_{i+1}} = \pi_{V_{i+1}} \tilde{A}_{i+1} \tilde{B}_{i+1} |_{V_{i+1}} = \delta_{h,j-i-1} q_{i+2-j} A_{i+1} = \delta_{h,j-i-1} q_{i+1-j}. \]

where \( \delta_{p,q} \) is the Kronecker symbol.

Now, in order to simplify the notation a bit we set \( \tau_{j,h} \) and \( \delta_{j,h} \)

Case I: \( j \neq j' \). In this case the equation (37) and translates into the following equations for \( \tau_{j,h} \) and \( \delta_{j,h} \):
\[ \tau_{j,h}^{\prime \prime} + \sum_{h' < h'' < h+1, h' - j < h'' < h+1-j} \tau_{j,h}^{\prime \prime} s_{j,h}^{\prime \prime} + s_{j,h}^{\prime} = 0, \]
\[ \text{if } h \neq j - i - 1 \]
\[ \tau_{j,h}^{\prime \prime} + \sum_{h' - j < h'' < h+1-j} \tau_{j,h}^{\prime \prime} s_{j,h}^{\prime \prime} + s_{j,h}^{\prime} = 0, \]
\[ \text{if } h = j - i - 1. \]

while the equation (38) translates into the following equations for \( \tau_{j,h}^{\prime \prime} \) and \( \delta_{j,h}^{\prime \prime} \):
\[ \tau_{j,h}^{\prime \prime} + \sum_{h' - j' < h'' < h+1-j} \tau_{j,h}^{\prime \prime} s_{j,h}^{\prime \prime} + s_{j,h}^{\prime} = 0. \]

We claim that the system of equations (39) and (40) has a unique solution indicated in the statement of the lemma. We will prove this claim by induction on \( h \) and \( h' \).

First of all, observe that from the equation (40) we have \( \tau_{j,1}^{\prime} = 0. \)

We make two induction assumptions \((k \geq 1)\):
\[ (1) \ \tau_{j,k+1}^{\prime \prime} = 0 \text{ for all } (h', h) \text{ such that } h' \leq h \leq k \text{ for all } j \neq j' \text{ at the same time.} \]
\[ (2) \ s_{j,k+1}^{\prime \prime} = 0 \text{ for all } (h', h) \text{ such that } h' < h \leq k + 1 \leq j - i \text{ for all } j \neq j' \text{ at the same time.} \]

Induction Step 1. Consider the equation (40) for \( h = k + 1 \). By assumption (2) we have \( s_{j,k+1}^{\prime \prime} = 0 \) and \( s_{j,k+1}^{\prime} = 0 \) for \( j'' \neq j \). If \( j'' = j \), then \( j'' \neq j' \) and by assumption (1) \( \tau_{j,k+1}^{\prime \prime} = 0 \) for \( h'' \leq k \). Now from equation (40) we see that \( \tau_{j,k+1}^{\prime \prime} = 0 \) for \( h' \leq k + 1 \).

Induction Step 2. Consider the equation (39) for \( h = k + 1 \). By induction step (1) \( \tau_{j,k+1}^{\prime \prime} = 0 \) and \( \tau_{j,k+1}^{\prime} = 0 \) for \( j'' \neq j \). If \( j'' = j \), then \( j'' \neq j' \) and by assumption (2) \( s_{j,k+2}^{\prime \prime} = 0 \). Now from equation (39) we see that \( s_{j,k+2}^{\prime \prime} = 0 \) for \( h' < k + 2 \).

Finally, if \( h + 1 = j - i \), then the equations (39) and the induction steps 1 and 2 yield:
\[ s_{j,j-i}^{\prime \prime} = q_{i+h'+1-j'} P_{j'-i+j'+1}. \]
Case II: $j = j'$. In this case we fix $j$ and simplify the notation further a bit, by setting $t_{h'} \overset{\text{def}}{=} t_{j,h'}^{j}$ and $s_{h'} \overset{\text{def}}{=} s_{j,h'}^{j}$. Now, taking into account Case I, the equation (37) and translates into the following equations for $t_{h'}$ and $s_{h'}$:

\begin{equation}
(42) \quad s_{h+1}^{h'} + \sum_{h' < h'' < h+1} t_{j,h}^{h''} s_{h'}^{h''} + t_{h'}^{h'} = \begin{cases}
0, & \text{if } h \neq j - i - 1 \text{ and } h \neq h' \\
c_{i+1}, & \text{if } h \neq j - i - 1 \text{ and } h = h' \\
P(i, h', j), & \text{if } h = j - i - 1 \text{ and } h \neq h' \\
P(i, h', j) + c_{i+1}, & \text{if } h = j - i - 1 \text{ and } h = h'
\end{cases}
\end{equation}

(In order to compute the right hand side, we need to use the following combinatorial formula

\[ \sigma_a(c, c + b_1, \ldots, c + b_p) = \sum_{l=0}^{a} c^l \binom{p - a + l + 1}{l} \sigma_{a-l}(b_1, \ldots, b_p) \]

for $a, p \in \mathbb{Z}$, $1 \leq a \leq p$. We assume here that $\sigma_0(b_1, \ldots, b_p) = 1$.)

The equation (38) translates into the following equations for $t_{j,h}^{j'}$ and $s_{j,h}^{j'}$, $h < j - i$:

\begin{equation}
(43) \quad t_{h'}^{h'} + \sum_{h' - 1 < h'' < h} s_{h'}^{h''} t_{h''}^{h'} + s_{h'}^{h' - 1} = 0
\end{equation}

Again, we claim that the system of equations (42) and (43) has a unique solution indicated in the statement of the lemma. Again, we will prove this claim by induction on $h$ and $h'$.

First of all, observe that from the equation (43) we have $t_{1}^{1} = 0$.

We make two induction assumptions ($k \geq 1$):

1. $t_{h}^{h'}$ is given by equations (33) for all $(h', h)$ such that $h' \leq h \leq k$.
2. $s_{h}^{h'}$ is given by equations (34) for all $(h', h)$ such that $h' < h \leq k + 1 \leq j - i$.

Proceeding by induction as in Case I and using the formula (for $b, l \in \mathbb{Z}$, $0 \leq b \leq l - 2$)

\[ \sum_{a=b}^{l} (-1)^{l-a} \binom{l}{a} \binom{a+1}{b+1} = 0 \]

it is easy to see that all $t_{h}^{h'}$ and $s_{h+1}^{h'}$ are given by formulas (33) and (34) respectively.

We have proved the assertions (i) and (ii) of the lemma. The assertion (iii) follows from the construction and Lemma 2.3.1 exactly as in [Maf, Lemma 19]. The assertion (iv) follows from the construction, cf. [Maf, Lemma 19]. □

7.2.1. It is important for us to record the formula for $\widetilde{B}_0 \Delta_0 = \widetilde{\delta}_1 \gamma_1$. To simplify notation, we set

\[ b_l \overset{\text{def}}{=} b_{l-1} = c_1 + \cdots + c_l, \]

\[ b_l \overset{\text{def}}{=} b_{l-1} = c_1 + \cdots + c_l, \]
and
\[ P'(h', j) \overset{\text{def}}{=} \sum_{k=1}^{j-h'-1} (-1)^k \sigma_k(b_1, \ldots, b_{h'-2+k}) q_{h'+k \to j} P_{j \to h'+k} + (-1)^{j-h'} \sigma_{j-h'}(b_1, \ldots, b_{j-1}). \]

Now we have
\[ \text{Now recall the definition (cf. 3.2.6) of the transverse slice} \]
\[ \sigma \]

Finally, let us record the specialization of the above formula for the case \( c = 0 \). Clearly, in this case \( P'(h', j) = 0 \) and we have
\[ (\tilde{\delta}_1 \tilde{\gamma}_1)_{j', h'} = \begin{cases} \text{Id}_{D_j}, & \text{if } h' = h + 1, \quad j' = j, \\ q_{h' \to j} P_{j' \to h'}, & \text{if } h = j, \\ 0, & \text{otherwise}. \end{cases} \]

where \( \sigma_k \) is the \( k \)-th elementary symmetric function, and we assume that the value of \( \sigma_k \) at the empty collection of variables is zero.

8. Proof of the Isomorphism Theorem

In this section we complete the proof of the Isomorphism Theorem (Theorem 5.2.) The immersions \( \psi \) and \( \tilde{\psi} \) were constructed in section 4.4.

8.1. The isomorphisms \( \phi \) and \( \tilde{\phi} \). The argument in this subsection is for the case \( c = 0 \). The argument for a general \( c \) is completely analogous. In the proof we mostly follow the logic of [Maf].

**Lemma.** Let \( (\tilde{A}, \tilde{B}, \tilde{\gamma}, \tilde{\delta}) \in S \) and let \( \tilde{g} \in G(\tilde{V}) \) be such that \( \tilde{g}(\tilde{A}, \tilde{B}, \tilde{\gamma}, \tilde{\delta}) \in S \). Then \( \tilde{g}_i(V_i) \subseteq V_i \) and if we denote \( g_i = \tilde{g}_i|_{V_i} \) we have
\[ \tilde{g}(\tilde{A}, \tilde{B}, \tilde{\gamma}, \tilde{\delta}) = g(\tilde{A}, \tilde{B}, \tilde{\gamma}, \tilde{\delta}). \]

**Proof.** The proof is lifted verbatim from [Maf], Lemma 22. \( \square \)

8.1.1. Reduction to the situation from section 6. Let \( D = \tilde{D}_1 \) as in 7.1. Then \( \text{dim } D = N = \tilde{d}_1 \overset{\text{def}}{=} \sum_{j=1}^{n-1} j d_j \). Observe that \((\tilde{v}, \tilde{d})\) as constructed in 7.1 must satisfy the conditions of section 5 in order for \( \mathfrak{M}^c(\tilde{v}, \tilde{d}) \) and \( \mathfrak{M}^c(\tilde{v}, \tilde{d}) \) to be nonempty, cf. [Maf] 1.4 and therefore, if nonempty, \( \mathfrak{M}^c(\tilde{v}, \tilde{d}) \approx \mathcal{O}_\mu \) and \( \mathfrak{M}^c(\tilde{v}, \tilde{d}) \approx T^*F^a \) (for \( c = 0 \)), where \( \mu, a \) are defined as in 5.1.1. (For a general \( c \) the nilpotent orbit \( \mathcal{O}_\mu \) deforms into a general conjugacy class, cf. 3.1.5, 5.1.3.) Now recall the definition (cf. 3.2.6) of the transverse slice \( T_x \) to the orbit.
We denote $\mathcal{O}_\lambda$ where $\lambda$ is obtained from $(v, d)$ as in 5.1.1. Let $T_{x, \mu} = T_x \cap \mathcal{O}_\mu$ be as in 3.2.6 and let $\tilde{T}_x^{a}$ be as in 3.2.7.

8.1.2. Now we will construct the maps $\phi_0$ and $\tilde{\phi}$ completing the following commutative diagrams.

$$
\begin{array}{ccc}
M^c(v, d) & \xrightarrow{\phi} & S \\
\downarrow & & \downarrow \\
\mathcal{M}_0(v, d) & \xrightarrow{\phi_0} & \mathcal{M}_0^c(\tilde{v}, \tilde{d}) \\
\downarrow & & \downarrow \\
M^c(v, d) & \xrightarrow{\phi^*} & S^c \\
\downarrow & & \downarrow \\
\mathcal{M}^c(v, d) & \xrightarrow{\tilde{\phi}} & \mathcal{M}^c(\tilde{v}, \tilde{d})
\end{array}
$$

(47)

We denote $\phi \overset{\text{def}}{=} \phi_0|_{\mathcal{M}_1^c(v, d)} : \mathcal{M}_1^c(v, d) \to \mathcal{M}_0^c(\tilde{v}, \tilde{d})$. Since $\mathcal{M}_1^c(\tilde{v}, \tilde{d}) \simeq \mathcal{O}_\mu$ an element of $\mathcal{M}_1^c(v, d)$ will be sent by $\phi$ to an operator $y + f \in \text{End}(D)$, where $y$ is nilpotent of type $\lambda$ and $f$ is given by the explicit formulas (15) (and (14) for arbitrary $c$). A simple inspection shows that Im $\phi \subseteq T_{x, \mu}$, and Im $\tilde{\phi} \subseteq \tilde{T}_x^{a}$.

8.1.3. Lemma. The map $\phi$ is a closed immersion.

Proof. It is enough to prove that $\phi_0$ is closed immersion. Recall that

$$
\begin{align*}
\mathcal{M}_0^c(v, d) &= M^c(v, d)/G(V) = \text{Spec} \mathcal{O}(M^c(v, d))^{G(V)}, \\
\mathcal{M}_0^c(\tilde{v}, \tilde{d}) &= M^c(\tilde{v}, \tilde{d})/G(\tilde{V}) = \text{Spec} \mathcal{O}(M^c(\tilde{v}, \tilde{d}))^{G(\tilde{V})}.
\end{align*}
$$

(48)

We will prove that the restriction map $\phi^* : \mathcal{O}(M^c(\tilde{v}, \tilde{d}))^{G(\tilde{V})} \to \mathcal{O}(M^c(v, d))^{G(V)}$ is surjective.

By Theorem 2.2.1 the algebra $\mathcal{O}(M^c(\tilde{v}, \tilde{d}))^{G(\tilde{V})}$ is generated by $\bar{\chi}(\delta_1 \gamma_1)$ where $\bar{\chi}$ is a linear form on $\text{Hom}(\tilde{D}_1, \tilde{D}_1)$. If $\tilde{\delta}_1 \gamma_1$ is of the form (15) and

$$
\bar{\chi} = \chi \in \text{Hom}(D_{\delta'}^{(h')}, D_{\delta}^{(j)})^* \subseteq \text{Hom}(\tilde{D}_1, \tilde{D}_1)^*,
$$

then for $1 \leq h' \leq \min(j, j')$ we have

$$
\bar{\chi}(\tilde{\delta}_1 \gamma_1) = \chi(\pi_{D_{\delta'}^{(h')}}(\tilde{\delta}_1 \gamma_1)|_{D_{\delta}^{(j)}}) = \chi(q_{h' \rightarrow j} p_{j' \rightarrow h'}),
$$

which are all the generators of the algebra $\mathcal{O}(M^c(v, d))^{G(V)}$ according to the Theorem 2.2.1 \qed

8.1.4. Lemma. The map $\tilde{\phi} : \mathcal{M}^c(v, d) \to \tilde{T}_x^{a}$ is proper and injective.
Proof. We have the following diagrams

\[
\begin{array}{cccc}
\mathcal{M}^c(v, d) & \xrightarrow{\tilde{\phi}} & \tilde{T}_x^a & \mathcal{M}^c(v, d) \\
p & \downarrow & m_a & \downarrow p \\
\mathcal{M}_0^c(v, d) & \xrightarrow{\phi_0} & \mathcal{M}_0^c(\tilde{v}, \tilde{d}) & \mathcal{M}_1^c(v, d) \\
 & \downarrow m_a & & \downarrow m_a \\
\mathcal{M}^c(v, d) & \xrightarrow{\phi} & T_{x, \mu} & M^c(v, d, \tilde{\phi})_{\tilde{\phi}} \xrightarrow{\phi} \tilde{T}_x^a
\end{array}
\]

Since \( \phi \) is a closed immersion and the morphisms \( p \) and \( m_a \) are projective, we see that \( \tilde{\phi} \) is proper. Since all orbits in \( M^c(v, d) \) and \( M^c(\tilde{v}, \tilde{d}) \) are closed, \( \tilde{\phi} \) is injective. \( \square \)

8.1.5. Lemma. The map \( \tilde{\phi} : \mathcal{M}^c(v, d) \to \tilde{T}_x^a \) is an isomorphism of algebraic varieties.

Proof. Since \( \tilde{\phi} \) is a proper injective morphism between connected smooth varieties of the same dimension, \( \tilde{\phi} \) is an analytic isomorphism and therefore an algebraic isomorphism. \( \square \)

Lemma. The map \( \phi : \mathcal{M}_1^c(v, d) \to T_{x, \mu} \) is an isomorphism of algebraic varieties.

Proof. Since \( m_a \) is surjective, from the diagram \( [49] \) we see that \( \phi \) is surjective. Since \( \phi \) is a surjective closed immersion, and both \( \mathcal{M}_1^c(v, d) \) and \( T_{x, \mu} \) are reduced varieties over \( \mathbb{C} \), \( \phi \) is an algebraic isomorphism. \( \square \)

9. Application to representation theory: \((\mathfrak{gl}_m, \mathfrak{gl}_n)\)-duality

Here we observe that the relationship between quiver varieties and loop Grassmannians provides a natural framework for \((GL_m, GL_n)\) duality. In this section we use notation \( V = \mathbb{C}^m \), \( W = \mathbb{C}^n \) and denote by \( V_\lambda = V_{\lambda}^{\mathfrak{gl}_m} \) and \( W_\lambda = W_\lambda^{\mathfrak{gl}_n} \) the irreducible representation of \( \mathfrak{gl}_m \) and \( \mathfrak{gl}_n \) with highest weights \( \lambda \) and \( \tilde{\lambda} \) (with weight spaces \( V_\lambda(a) \) etc).

9.1. Skew \((\mathfrak{gl}_m, \mathfrak{gl}_n)\) duality. For the \( \mathfrak{gl}_m \times \mathfrak{gl}_n \) bimodule \( V \otimes W \) we have the following decomposition \([\text{Ho}, 4.1.1]\):

\[
\wedge^N (V \otimes W) = \bigoplus_\lambda V_\lambda \otimes W_{\tilde{\lambda}},
\]

where \( \lambda \) goes through all partitions of \( N \) which fit into the \( n \times m \) box.

9.1.1. Considering \( V \otimes W \) as a \( \mathfrak{gl}_m \) module \( V \otimes \mathbb{C}^n \), we have the following decomposition:

\[
\wedge^N (V \otimes W) = \bigoplus_{a_1 + \cdots + a_n = N} \wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V.
\]

Considered as a representation of the torus \( (\mathbb{C}^*)^n \subseteq \mathfrak{gl}_n \) the vector space \( \wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V \) has weight \( a = (a_1, \ldots, a_n) \). Thus decompositions \([50]\) and \([51]\) imply that

\[
\text{Hom}_{\mathfrak{gl}_m} (\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda) \simeq W_\lambda(a).
\]
9.1.2. Geometric skew duality. We construct a based version of the isomorphism (52), i.e., a geometric skew \((GL_n, GL_m)\) duality. More precisely, with \(N, v, d, a, \lambda\) as in 5.1.1 we identify the right hand side with \(H(\pi^{-1}(L_\lambda))\), where \(L_\lambda\) is a lattice in the loop Grassmannian \(\mathcal{G}\), and the left hand side with \(H(L_\lambda)\) by Theorem 2.4. The identification of irreducible components \(\text{Irr} \pi^{-1}(L_\lambda) = \text{Irr} L_\lambda\), which follows from the isomorphism (14) matches the natural basis of the space of intertwiners \(\text{Hom}_{GL_m}(\wedge^a V \otimes \cdots \otimes \wedge^a V, V_\lambda)\) arising from the loop Grassmannian construction (i.e., \(\text{Irr} L_\lambda\)), and the natural basis of the weight space \(W_\lambda(a)\) in the Nakajima construction (i.e., \(\text{Irr} L_\lambda\)). Altogether:

\[
\text{Hom}_{GL_m}(\wedge^a V \otimes \cdots \otimes \wedge^a V, V_\lambda) \simeq H(\pi^{-1}(L_\lambda)) \simeq H(L_\lambda) \simeq W_\lambda(a).
\]

Dually, we have

\[
(53) \quad \text{Hom}_{gl_m}(\wedge^c W \otimes \cdots \otimes \wedge^c W, W_\lambda) = V_\lambda(c).
\]

9.2. Symmetric \((GL_m, GL_m)\) duality. Analogously, if we consider the \(N\)-th symmetric power \(\text{Sym}^N(V \otimes V)\) of the \(gl_m \times gl_m\) bimodule \(V \otimes V\), we have the following decomposition (a particular case of [Ho, 2.1.2]):

\[
(54) \quad \text{Sym}^N(V \otimes V) = \bigoplus_\lambda V_\lambda \otimes V_\lambda,
\]

where the sum is over all partitions \(\lambda\) of \(N\) with at most \(m\) parts.

Considering \(V \otimes V\) as a \(gl_m\) module \(V \otimes \mathbb{C}^m\), we have the following decomposition:

\[
(55) \quad \text{Sym}^N(V \otimes V) = \bigoplus_{c_1 + \cdots + c_m = N} \text{Sym}^{c_1} V \otimes \cdots \otimes \text{Sym}^{c_m} V.
\]

Thus decompositions (54) and (55) imply the following formula

\[
(56) \quad \text{Hom}_{gl_m}(\text{Sym}^{c_1} V \otimes \cdots \otimes \text{Sym}^{c_m} V, V_\lambda) = V_\lambda(c),
\]

where \(V_\lambda(c)\) is the weight space corresponding to weight \(c\) of the \(gl_m\) highest weight module \(V_\lambda\).

Combining the equations (53) and (56) we get

\[
(57) \quad \text{Hom}_{gl_m}(\wedge^c W \otimes \cdots \otimes \wedge^c W, W_\lambda) = \text{Hom}_{gl_m}(\text{Sym}^{c_1} V \otimes \cdots \otimes \text{Sym}^{c_m} V, V_\lambda).
\]

9.2.1. Geometric symmetric duality. Geometry allows us to find a based isomorphism of the left and right hand side of (57). We let \(N, v, d, a, \lambda\) be as in 5.1.1 except that we use notation \(c\) for \(a\). First of all, the quiver tensor product theorems of Malkin [Mal] and Nakajima [N3] say that a natural basis of the multiplicity space in the left hand side of (57) is given by the relevant irreducible components \(\text{Irr}_{rel}[\mathcal{G}^{c,x}]\) of the “Spaltenstein fiber” \(\mathcal{G}^{c,x}\) at \(x\), defined by

\[
\mathcal{G}^{c,x} \overset{\text{def}}{=} \{ F \in \mathcal{F}; \ x(F_i) \subseteq F_i \text{ and } x \text{ acts on } F_i/F_{i-1} \text{ as a regular nilpotent } \}.
\]
Now consider another convolution space
\[
\tilde{G}^c = G_{c_1 \omega_1} \ast \cdots \ast G_{c_m \omega_1} = \{ L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m; \, \dim L_i / L_{i-1} = c_i, \, z |_{L_i / L_{i-1}} \text{ is a regular nilpotent} \}
\]
where \( \omega_1 \) is the first fundamental weight of \( GL_m \). We have a map \( \pi : \tilde{G}^c \to G \) defined by 
\[
\pi : (L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n) \mapsto L = L_n.
\]
Consider \( \pi^{-1}(L_\lambda) \) for \( L_\lambda \in G \). It follows from the Geometric Satake Correspondence that the set of relevant irreducible components
\[
\text{Irr} \pi^{-1}(L_\lambda) \text{ indexes a basis in the right hand side of (57).}
\]
It is clear that the varieties \( \pi^{-1}(L_\lambda) \simeq \tilde{g}^c ; x \). This gives us a bijection \( \text{Irr} \tilde{g}^c \simeq \text{Irr} \pi^{-1}(L_\lambda) \). Summarizing:
\[
\text{Hom}_{GL_n}(\wedge^{c_1} W \otimes \cdots \otimes \wedge^{c_m} W, W_\lambda) \simeq \mathcal{H}(\tilde{g}^c) \simeq \mathcal{H}(\pi^{-1}(L_\lambda)) \simeq \text{Hom}_{GL_m}(\text{Sym}^{c_1} V \otimes \cdots \otimes \text{Sym}^{c_m} V, V_\lambda).
\]

9.2.2. Remark. The second author has greatly benefited from a class taught by W. Wang at Yale [Wa1]. The “geometric symmetric duality” above has a lot in common with the construction described in [Wa2] and we believe that the “geometric skew duality” construction answers a question posed by Weiqiang Wang.

10. APPENDIX BY VASILY KRYLOV. EXPlicit isomorphism of quivers and loop Grassmannians

In this section we describe another approach to constructing in type A an isomorphism to be denoted \( \zeta \) between (regular parts of) quiver varieties and slices in the affine Grassmannian. We prove that the isomorphism \( \zeta \) coincides with the above morphism \( \psi \circ \phi \). This approach enables us to compute the morphism \( \psi \circ \phi \) in explicit terms (see Theorem 10.1.1). Let us briefly explain how the isomorphism \( \psi \circ \phi \) can be constructed geometrically.

Our formula will actually define \( \zeta \) only on the “regular” part (certain open dense subvariety) of a quiver variety. We check that here it coincides with the isomorphism \( \psi \circ \phi \) and therefore \( \zeta \) extends uniquely to the quiver variety. The construction of \( \zeta \) is in some sense 2-dimensional since over the regular part we actually identify the above two moduli with a certain moduli of vector bundles on \( \mathbb{P}^2 \).
Let $\mathcal{M}(v, d)$ be a quiver variety with dimension vectors $v, d$. Let $\lambda, \mu$ be as in Subsection 5.1. Let variety $\mathcal{M}_0^{\text{reg}}(v, d)$ be the regular part of $\mathcal{M}(v, d)$ (see Subsection 10.2.1 for the definition). Let us denote $V := \bigoplus_i V_i$, $D := \bigoplus_i D_i$. Let us consider the isomorphism $\phi$ of $\mathcal{M}$ from the results of [BF1] it follows that we have constructed $\psi$ and $\phi$ are as in the previous section. Let us consider the natural inclusion from $\mathcal{M}_0^{\text{reg}}(v, d)$ to $\mathcal{M}_0^{\text{reg}}(V, D)$. It follows from Nakajima’s results that there exists a $C^*$-action on $\mathcal{M}(V, D)$ such that $\mathcal{M}_0^{\text{reg}}(v, d)$ is a connected component of the fixed point subvariety $(\mathcal{M}_0^{\text{reg}}(V, D))^C^*$.

From the $ADHM$ description (see Subsection 10.2.2) it follows that the variety $\mathcal{M}_0^{\text{reg}}(v, d)$ is isomorphic to the moduli space of principal $GL(D)$-bundles on $\mathbb{P}^2$ with fixed trivialization at the line at infinity and fixed second Chern class.

Thus $\mathcal{M}_0^{\text{reg}}(v, d)$ can be thought as a moduli space (to be denoted by $\text{Bun}_{GL_m, w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m)$) of certain $C^*$-equivariant vector bundles on $\mathbb{P}^2$ satisfying some conditions (see Subsection 10.2.3 for the precise statement). From the results of [BF1] it follows that we have an isomorphism

$$\text{Bun}_{GL_m, w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m) \simeq (L^{\leq 0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu).$$

Thus we get a desired isomorphism between $\mathcal{M}_0^{\text{reg}}(v, d)$ and $(L^{\leq 0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu)$.

The appendix is organized as follows:

In Subsection 10.2 we give a construction of an isomorphism between regular parts of quiver varieties and regular parts in slices and compute it explicitly.

In Subsection 10.3.1 we prove that the isomorphism we have constructed is given by the formula (10.1.1) below.

In Subsection 10.3.2 we prove Theorem (10.1.1) below by showing that the isomorphism we have constructed coincides with the morphism $\psi \circ \phi$ (restricted to the regular part).

10.1. Recall that $\mathcal{G}_{GL_m} = GL_m(K)/GL_m(O)$. For a cocharacter $\lambda$ of $GL_m$ we denote by $z^\lambda$ the corresponding element in $T(K)/T(O) = \mathcal{G}_T \subset \mathcal{G}_{GL_m}$ where $T$ is the diagonal torus in $GL_m$.

Recall the isomorphisms $\phi: \mathcal{M}_0^{\text{reg}}(v, d) \xrightarrow{\sim} T_\lambda \cap \overline{\mathcal{U}}_\mu$ (see Section 8, Subsection 3.3) and $\psi: T_\lambda \cap \overline{\mathcal{U}}_\mu \xrightarrow{\sim} L^{\leq 0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu$ (see Subsection 4.3) where $\lambda, \mu$ are as in Subsection 5.1. It follows from [45] that the isomorphism $\phi$ can be described as follows. Recall a vector space $\tilde{D}_j := \bigoplus_{1 \leq k \leq j \leq n-1} D_j^{(k)}$ of Section 7.1 here $D_j^{(k)}$ is a copy of $D_j$. For any $f \in \text{End}(D)$ we consider its blocks $f_{j,h}^{j',h'}: D_j^{(h')} \rightarrow D_j^{(h)}$. Then we have

$$f_{j,h}^{j',h'} = \begin{cases} \text{Id}_{D_j} & \text{if } h' = h + 1, \ j' = j \\ q_{j}x_{j-1} \ldots x_{h'+1}x_{h}x_{h'}x_{h'+1} \ldots x_{j-1}p_{j} & \text{if } h = j, \\ 0, & \text{otherwise.} \end{cases}$$

(60)
10.1.1. **Theorem.** The isomorphism $\psi \circ \phi$ is given by the formula:

\[(61) \quad (x_i, \bar{x}_i, p, q_i) \mapsto z^{-w_0 \lambda}(1 + z^{-1} \sum_{n, l=0}^{\infty} z^{-n} q^n x^l p),\]

where $(x, \bar{x}, p, q) := (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i)$. The proof will be given in Subsection 10.3.

10.2. **Geometric interpretation of the formula (61).** In this Section we give a geometric interpretation of the morphism from $\mathcal{M}_0^q(v, d)$ to $L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu$ given by the formula (61). We prove that the variety $\mathcal{M}_0^q(v, d)$ can be identified with a certain moduli space $\text{Bun}_{GL_{m-1}, w_0(\lambda)}^{-w_0(\mu)}$ of $\mathbb{C}^*$-equivariant bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with a fixed trivialization on $\mathbb{P}^1 \times \mathbb{P}^1$ (see Section 10.2.8). Under this identification the morphism given by the formula (61) sends a vector bundle $E \in \text{Bun}_{GL_{m-1}, w_0(\lambda)}^{-w_0(\mu)}$ to a point $z^{-w_0(\lambda)} \cdot E|_{(1,1) \times \mathbb{P}^1}$ of the Affine Grassmannian $\mathcal{G}$ (here one should note that the $\mathbb{C}^*$-equivariance of $E$ allows us to uniquely extend the trivialization of $E$ on $\mathbb{P}^1 \times \mathbb{P}^1$ to the trivialization of $E|_{(1,1) \times \mathbb{P}^1}$ away from $(1 : 1) \times (0 : 1)$, hence, $E|_{(1,1) \times \mathbb{P}^1}$ defines a point of the Affine Grassmannian $\mathcal{G}$).

10.2.1. A **variety** $\mathcal{M}_0^{reg}(v, d)$ and cocharacters $\lambda, \mu$. A quadruple $(x, \bar{x}, p, q) \in M^c(v, d)$ is called costable if for any $I$-graded subspace $V'$ of $V$ contained in $\text{Ker}(q)$ and preserved by $x$ and $\bar{x}$ we have $V' = 0$. Denote by $M_0^{reg}(v, d)$ the set of stable and costable quadruples in $M^c(v, d)$. Let $\mathcal{M}_0^{reg}(v, d)$ [N5, Section 3] be the quotient of $M_0^{reg}(v, d)$ by the free action of $GL(V)$. The morphism $p : M_0^c(v, d) \to M_0^{reg}(v, d)$ embeds $M_0^{reg}(v, d)$ as an open subvariety in $M_0^c(v, d)$.

For the dimension vectors $(v, d)$ we define the following dominant cocharacters $\lambda, \mu$ of $GL(D)$: the cocharacter $\lambda$ acts with eigenvalue $t^i$ on $D_i$ and the cocharacter $\mu$ acts with eigenvalue $t^i$ on the subspace of dimension $v_{i+1} + v_{i-1} + d_i - 2v_i$. Note that this definition is in accordance with Subsection 5.4.

10.2.2. **ADHM description.** Let $\text{Bun}_{GL_m}(\mathbb{A}^2)$ denote the moduli space of principal $GL_m$-bundles on $\mathbb{P}^2$ of second Chern class $a$ with a trivialization at the line at infinity $l_{\infty}$.

We set $\mathcal{M}_0^{reg}(V, D) = \{(x, \bar{x}, p, q) \in \mu^{-1}(0) \mid \text{stable and costable} \}/GL_m$, where $(x, \bar{x}, p, q)$ are Jordan quiver quadruples:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{p} & \bullet \\
\downarrow & & \downarrow \\
W & & V
\end{array}
\]
dim \( V = a \), \( \dim D = m \)

The ADHM description \([\mathcal{N}4]\) Theorem 2.1] identifies \( \text{Bun}_{GL_m}(\mathbb{A}^2) \) with \( \mathcal{M}_{0}^{\text{reg}}(V, D) \).

The vector bundle \( E_{(x, \bar{x}, p, q)} \) corresponding to a quadruple \((x, \bar{x}, p, q)\) can be obtained as the middle cohomology of the following monad:

\[
\begin{align*}
V \otimes \mathcal{O}_{\mathbb{P}^2} & \quad \oplus \\
V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \quad D \otimes \mathcal{O}_{\mathbb{P}^2} \quad b = \left[ \begin{array}{ccc}
-z_0\bar{x} - z_1 & z_0x - z_1 & z_0p
\end{array} \right]
\end{align*}
\]

10.2.3. Torus fixed points. Recall the notations of \([\mathcal{B}1]\) Subsection 4.4]. Consider the action of \( \mathbb{C}^* \) on \( \mathbb{A}^2 \) which sends \((x, y)\) to \((t^{-1}x, ty)\). Note that \( GL_m \times \mathbb{C}^* \) acts on \( \text{Bun}_{GL_m}(\mathbb{A}^2) \): the first factor acts by changing a trivialization at \( l_\infty \) and the second factor via its action on \( \mathbb{A}^2 \). Now for every cocharacter \( \rho_\lambda : \mathbb{C}^* \to GL_m \) obtain the diagonal action of \( \mathbb{C}^* \) on \( \text{Bun}_{GL_m}(\mathbb{A}^2) \). Let \( \text{Bun}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m) \) denote the fixed point set of this action. The point \((0, 0) \in \mathbb{A}^2\) is fixed under the \( \mathbb{C}^* \)-action. So for every \( E \in \text{Bun}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m) \) the group \( \mathbb{C}^* \) acts on the fiber \( E_{(0, 0)} \) of \( E \) at the point \((0, 0) \in \mathbb{A}^2 \). Let us denote by \( \text{Bun}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m) \) the subvariety of \( \text{Bun}_{GL_m}(\mathbb{A}^2/\mathbb{G}_m) \) formed by all \( E \in \text{Bun}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m) \) such that \( \mathbb{C}^* \) acts on \( E_{(0, 0)} \) by the cocharacter \( \rho_\mu \). (Here \( \mu \) and \( \lambda \) are the dominant cocharacters of \( GL_m \) and \( \rho_\mu \) and \( \rho_\lambda \) denote the corresponding conjugancy classes of morphisms \( \mathbb{C}^* \to GL_m \).) Also let us denote by \( \text{Bun}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m) \) the variety \( \text{Bun}_{GL_m,\lambda}^{\mu,\alpha}(\mathbb{A}^2/\mathbb{G}_m) \) (according to \([\mathcal{B}1]\) Theorem 5.2(1)) for \( a \neq \frac{\mu}{2} - \frac{\lambda}{2} \) the variety \( \text{Bun}_{GL_m,\lambda}^{\mu,\alpha}(\mathbb{A}^2/\mathbb{G}_m) \) is empty).

10.2.4. Induced torus action on \( \mathcal{M}_{0}^{\text{reg}}(V, D) \). The \( \mathbb{C}^* \)-action on \( \text{Bun}_{GL_m}(\mathbb{A}^2) \) corresponding to a cocharacter \( \lambda \) defines an action on \( \mathcal{M}_{0}^{\text{reg}}(V, D) \) via the ADHM isomorphism \([10.2.2]\).

10.2.5. Lemma. This action can be described as follows:

\((x, \bar{x}, p, q) \mapsto (t^{-1}x, t\bar{x}, pp\rho_\lambda(t)^{-1}, \rho_\lambda(t)q)\).

Proof. Take \( t \in \mathbb{C}^* \). Consider a vector bundle \( tE_{(x, \bar{x}, p, q)} \) that is obtained from \( E_{(x, \bar{x}, p, q)} \) by the action of \( t \). It can be described as the middle cohomology of the following monad:

\[
\begin{align*}
V \otimes \mathcal{O}_{\mathbb{P}^2} & \quad \oplus \\
V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \quad D \otimes \mathcal{O}_{\mathbb{P}^2} \quad b = \left[ \begin{array}{ccc}
nz_0\bar{x} - t^{-1}z_2 & z_0x - tz_1 & z_0p
\end{array} \right]
\end{align*}
\]
We have to emphasize that the trivialization of \( tE(x, \bar{x}, p, q) \) at infinity is \( \rho_\lambda(t) \).

We have the following commutative diagram giving the isomorphism between the monad for \( tE(x, \bar{x}, p, q) \) and the one for the quadruple \( (t^{-1}x, t\bar{x}, pp_\lambda(t)^{-1}, \rho_\lambda(t)q) \):

\[
\begin{array}{cccccc}
V \otimes \mathcal{O}_{P^2}(-1) & \xrightarrow{Id} & V \otimes \mathcal{O}_{P^2} & \oplus & V \otimes \mathcal{O}_{P^2} & \oplus & D \otimes \mathcal{O}_{P^2} & \xrightarrow{\rho_\lambda(t)} & V \otimes \mathcal{O}_{P^2}(1) \\
V \otimes \mathcal{O}_{P^2}(-1) & \xrightarrow{Id} & V \otimes \mathcal{O}_{P^2} & \oplus & V \otimes \mathcal{O}_{P^2} & \oplus & D \otimes \mathcal{O}_{P^2} & \xrightarrow{\rho_\lambda(t)} & V \otimes \mathcal{O}_{P^2}(1)
\end{array}
\]

The morphism on cohomology induces the isomorphism between corresponding vector bundles with trivializations. 

10.2.6. An isomorphism between \( \bigsqcup_{v, \sum v_i = a} M_0^{reg}(v, d) \) and \( \text{Bun}_{GL_m, -w_0(\lambda)}^a \).

Let \( \bigsqcup_{v, \sum v_i = a} M_0^{reg}(v, d) \) be the disjoint union of quiver varieties of type A with vertices numbered by integers with a framing of dimension \( d \) and \( \sum v_i = a \). We define a morphism \[ \Theta_{v, d} : M_0^{reg}(v, d) \to M_0^{reg}(V, D) \simeq \text{Bun}_{GL_m}(A^2), \]

\[ \Theta_{v, d}(x_i, \bar{x}_i, p_i, q_i) = (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i). \]

The maps \( \Theta_{v, d} \) for different \( v \) induce the map \( \tilde{\Theta}_d : \bigsqcup_{v, \sum v_i = a} M_0^{reg}(v, d) \to \text{Bun}_{GL_m}^a(A^2) \).

10.2.7. Lemma. \( \tilde{\Theta}_d \) induces an isomorphism between \( \bigsqcup_{v, \sum v_i = a} M_0^{reg}(v, d) \) and \( \text{Bun}_{GL_m, -w_0(\lambda)}^a \) where \( \lambda \) is as in Subsection 10.2.1.

Proof. We describe the inverse map. Let \((x, \bar{x}, p, q)\) be a fixed point under the \( \mathbb{C}^* \)-action on \( M_0^{reg}(V, D) \) corresponding to \(-w_0(\lambda)\). Then using Lemma 10.2.5 we have that for every \( t \in \mathbb{C}^* \) there exists \( \rho_V(t) \in GL(V) \) such that

\[
(t^{-1}x, t\bar{x}, pp_{-w_0(\lambda)}(t)^{-1}, \rho_{-w_0(\lambda)}(t)q) = (\rho_V(t)x\rho_V(t)^{-1}, \rho_V(t)\bar{x}\rho_V(t)^{-1}, \rho_V(t)p, q\rho_V(t)^{-1}).
\]

Note that \( \rho_V(t) \) is uniquely determined by \( t \) because of the freeness of \( GL(V) \)-action on stable and costable quadruples. In particular \( \rho_V \) defines a cocharacter of \( GL(V) \). We decompose \( V \) into a direct sum \( \oplus V_i \) (where \( V_i \) is the \( t^{-i} \)-eigenspace of \( \rho_V \) and similarly
decompose $D$ into a direct sum $\oplus D_i$ with respect to $\rho_{-w_0(\lambda)}$. It is easy to see that the condition (62) implies that $\forall i \in \mathbb{Z}$, $x(V_i) \subset V_{i+1}$, $\bar{x}(V_i) \subset V_{i-1}$, $p(D_i) \subset V_i$, $q(V_i) \subset D_i$. So $(x, \bar{x}, p, q)$ defines a point in a quiver variety of type A with vertices numbered by integers such that $\sum_{i=-\infty}^{+\infty} v_i = a$, and the framing is $d$. The inverse map is constructed.

\[\square\]

10.2.8. An isomorphism between $\mathcal{M}_0^{\text{reg}}(v, d)$ and $\text{Bun}_{GL_m, -w_0(\lambda)}^{\text{reg}}$.  

10.2.9. Lemma. $\tilde{\Theta}_d$ induces the isomorphism [N5, Section 4]

$$\Theta : \mathcal{M}_0^{\text{reg}}(v, d) \cong \text{Bun}_{GL_m, -w_0(\lambda)}^{\text{reg}}$$

where $\lambda, \mu$ are as in Subsection 10.2.1.

Proof. It is enough to prove that $\mathbb{C}^*$ acts on the fiber of $E \in \Theta(\mathcal{M}_0^{\text{reg}}(v, d))$ at the origin by $\rho_{\mu}$. Let us denote the cocharacter corresponding to the framing by $\rho_d$ ($\rho_d = \rho_{-w_0(\lambda)}$) and let $\rho_v$ be the cocharacter of $GL(\oplus V_i)$ that acts with eigenvalue $t^{-i}$ on the space $V_i$.

Let $(x, \bar{x}, p, q) := \Theta(x, \bar{x}, p, q)$ and $E_{(x, \bar{x}, p, q)}$ be the corresponding vector bundle. The bundle $tE_{(x, \bar{x}, p, q)}$ is the middle cohomology of the following monad:

\[
\begin{array}{c}
V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \\
\oplus V \otimes \mathcal{O}_{\mathbb{P}^2} \\
\oplus D \otimes \mathcal{O}_{\mathbb{P}^2}
\end{array}
\begin{array}{c}
z_0x - tz_1 \\
z_0\bar{x} - t^{-1}z_2 \\
z_0q
\end{array}
\begin{array}{c}
-(z_0\bar{x} - t^{-1}z_2) \\
z_0x - tz_1 \\
z_0p
\end{array}
\begin{array}{c}
V \otimes \mathcal{O}_{\mathbb{P}^2}(1)
\end{array}
\]

The map $(\rho_v(t), t\rho_v(t) \oplus t^{-1}\rho_v(t) \oplus \rho_d(t), \rho_v(t))$ provides an isomorphism between the monads corresponding to $E_{(x, \bar{x}, p, q)}$ and $tE_{(x, \bar{x}, p, q)}$. In particular this map induces a $\mathbb{C}^*$-module structure on the fiber at the origin of the monad corresponding to $E_{(x, \bar{x}, p, q)}$:

Now we can calculate the action of $\mathbb{C}^*$ on cohomology of this complex. The cocharacter corresponding to this action is the difference of the cocharacters on $V \oplus V \oplus D$ and the double cocharacter on $V$. It means that the desired cocharacter acts with an eigenvalue $t^{-i}$ on a subspace of dimension $v_{i-1} + v_{i+1} + d_i - 2v_i$. So this cocharacter is $-w_0(\mu)$. \[\square\]
10.2.10. An isomorphism \( \text{Bun}_{GL_m,-w_0(\lambda)}^\mu(\mathbb{A}^2/G_m) \simeq (L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu) \).

Recall the construction of the isomorphism

\[
\eta : \text{Bun}_{GL_m,-w_0(\lambda)}^\mu(\mathbb{A}^2/G_m) \simeq (L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu)
\]

[BF1, Theorem 5.2].

Let us think about vector bundles on \( \mathbb{P}^2 \) with trivialization on the line at infinity (and fixed rank and second Chern class) as of bundles on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with trivialization on \( \mathbb{P}^1 \times \infty \cup \infty \times \mathbb{P}^1 \).

The morphism \( \eta \) is constructed as follows: a bundle \( E \in \text{Bun}_{GL_m,-w_0(\lambda)}^\mu(\mathbb{A}^2/G_m) \) has to be trivial on \( \mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0) \). (It is trivial on the line \( \mathbb{P}^1 \times \infty \) hence on a neighborhood of that line. It means that the number of horizontal jumping lines has to be finite. Using invariance of \( E \) under \( \mathbb{C}^* \)-action we see that the only jumping line must be \( \mathbb{P}^1 \times 0 \). Alternatively we can look at the monad corresponding to \( E \) and see that we can write down an explicit trivialization of \( E \) restricted on \( \mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0) \) (we will do it)). But \( E \) is also trivializes on the line \( \infty \times \mathbb{P}^1 \). We can uniquely extend this trivialization to the whole variety \( \mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0) \). Now we restrict \( E \) with the trivialization to the line \( (1 : 1) \times \mathbb{P}^1 \). We get a point in the affine Grassmannian \( G_{GL_m} \). Finally, we apply \( z^{-w_0(\lambda)} \) to this point to obtain the desired point in the slice.

10.2.11. An isomorphism between \( \mathcal{M}_0^{reg}(v,d) \) and \( (L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu) \). Composing the isomorphisms from subsections 10.2.8 and 10.2.10 we obtain an isomorphism

\[
\eta \circ \Theta : \mathcal{M}_0^{reg}(v,d) \simeq (L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu).
\]

10.2.12. Lemma. The isomorphism \( \eta \circ \Theta \) can be described as follows:

\[
(x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-w_0\lambda}(1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q x^n \bar{x}^l p),
\]

where \( (x, \bar{x}, p, q) := (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i) \).

Proof. (Similar to [Hen, Proposition 4.8]) Take \((x_i, \bar{x}_i, p_i, q_i) \in \mathcal{M}_0^{reg}(v,d)\). The vector bundle \( E_{(x,\bar{x},p,q)} \) can be described as the middle cohomology of the following monad [BF2, Subsection 2.4]:

\[
\begin{align*}
V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1) & \xrightarrow{a} V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \\
& \xrightarrow{b} V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0) \\
& \xrightarrow{c} V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}
\end{align*}
\]
where \(((y : t), (z : h))\) are the coordinates on \(\mathbb{P}^1 \times \mathbb{P}^1\). Let \((\infty, \infty) := ((1 : 0), (1 : 0))\). We want to describe the trivialization of \(E_{(x, \bar{x}, p, q)}\) restricted to \(\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)\). For this it suffices to construct a map \(D \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1) \rightarrow \text{Ker}(b)|_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}\) transversal to \(\text{Im}(a)|_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}\). It is easy to see that the map:

\[
V \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}(0, -1) \\
\oplus \\
D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}(1, 0) \\
\tau_1 = \begin{bmatrix}
(hx - z)^{-1} & 0 \\
0 & \text{Id}
\end{bmatrix}
\]

satisfies the requirement.

Note that \(\tau_1\) is well defined because \(hx\) is nilpotent (\(\bar{x} = \oplus \bar{x}_i\), and \(\bar{x}_i\) sends \(V_i\) to \(V_{i-1}\), so that \(\oplus \bar{x}_i\) acts nilpotently on \(\oplus V_i\), hence \(hx - z\) is invertible when restricted to \(\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)\) (since \(z \neq 0\) on \(\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)\) and \(hx\) is nilpotent).

For the same reasons the map:

\[
V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1) \\
\oplus \\
D \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0) \\
\tau_2 = \begin{bmatrix}
(y - tx)^{-1} & 0 \\
0 & \text{Id}
\end{bmatrix}
\]

induces the trivialization of \(E_{(x, \bar{x}, p, q)}\) restricted to \((\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1\). Note that these two trivializations agree at the point \((\infty, \infty)\) and extend the trivialization of \(E_{(x, \bar{x}, p, q)}\) restricted to two infinite lines. Now we can construct \(\eta(E_{(x, \bar{x}, p, q)})\). To this end we have to calculate the transition function \((\tau_1^{-1} \circ \tau_2)|_{((1:1) \times (\mathbb{P}^1 \setminus 0, \infty))}\) it is the point in \(G_{\text{GL}_m}\) corresponding to \(E_{((1:1) \times (\mathbb{P}^1 \setminus 0, \infty))}\) and the trivialization induced by

\[
\tau_1 : D \otimes \mathcal{O}_{((1:1) \times (\mathbb{P}^1 \setminus 0, \infty))} \rightarrow D \otimes \mathcal{O}_{((1:1) \times (\mathbb{P}^1 \setminus 0, \infty))}.
\]

Let us compute \(\tau_1^{-1} \circ \tau_2\) on the fiber at a point \((y_0 : t_0), (z_0 : h_0) := g\). On the fiber of \(\tau_1, \tau_2\) at \(g\) we have the following morphisms:
\[
\begin{align*}
D & \xrightarrow{(\tau_1)|_g} V \oplus V \oplus D & \xleftarrow{(\tau_2)|_g} D
\end{align*}
\]

They induce isomorphisms:
\[
\begin{align*}
D & \xrightarrow{(\tau_1)|_g} \text{Ker}(b) / \text{Im}(a) & \xleftarrow{(\tau_2)|_g} D
\end{align*}
\]

For a vector \( w \in D \) we want to find \( \tau_1^{-1} \circ \tau_2(w) \) i.e. a vector \( \tilde{w} \in D \) such that \( \tau_2(w) - \tau_1(\tilde{w}) \in \text{Im}(a) \). It means that there exists a vector \( u \in V \) such that \( \tau_2(w) - \tau_1(\tilde{w}) = a(u) \). It gives us the system of equations:

\[
\begin{align*}
(z - h\bar{x})^{-1}p(\tilde{w}) &= tx(u) - yu \\
(y - tx)^{-1}p(w) &= h\bar{x}(u) - zu \\
w - \tilde{w} &= thq(u)
\end{align*}
\]

(63)

\[
\begin{align*}
u &= (h\bar{x} - z)^{-1}(y - tx)^{-1}p(w) \\
\tilde{w} &= w - thq(u)
\end{align*}
\]

(64)

Hence \( \tilde{w} = w - thq(h\bar{x} - z)^{-1}(y - tx)^{-1}p(w) \).

So

\[ (\tau_1^{-1} \circ \tau_2)|_{(1:1) \times (\mathbb{P}^1 \setminus \{0_2, \infty_2\})} = (1 + q(\bar{x} - z)^{-1}(x - 1)^{-1}p). \]

Thus we obtained the following point in \( S_{GL_m} \): \( 1 + \sum_{n,l=0}^{\infty} z^n q\bar{x}^n x^l p \). It remains to multiply it by \( z^{-w_0(\lambda)} \). We have calculated \( \eta(E_{(x,\bar{x},p,q)}) \). Thus

\[ \eta \circ \Theta(x_i, \bar{x}_i, p_i, q_i) = z^{-w_0(\lambda)}(1 + \sum_{n,l=0}^{\infty} z^n q\bar{x}^n x^l p). \]

Note that from (63) it also follows \( w = \tilde{w} + thq(y - tx)^{-1}(h\bar{x} - z)^{-1}p(w) \). So

\[
(1 + \sum_{n,l=0}^{\infty} z^n q\bar{x}^n x^l p)^{-1} = (1 - \sum_{n,l=0}^{\infty} z^n q\bar{x}^n x^l p)^{-1}. 
\]

(65)
10.3. Proof of Theorem 10.1.1

10.3.1. Lemma. The isomorphism $\psi \circ \phi$ restricted to $\mathcal{M}^{reg}(v,d)$ induces an isomorphism between $\mathcal{M}^{reg}_0(v,d)$ and $(L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu)$ and is given by the formula:

\[(x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-u_0\lambda} (1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p)\]

where $(x, \bar{x}, p, q) := (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i)$.

Proof. In Subsection 10.2 we proved that the map $\eta \circ \Theta : (x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-u_0\lambda} (1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p)$ is an isomorphism between $\mathcal{M}^{reg}_0(v,d)$ and $(L^{<0}GL_m \cdot \lambda \cap L^{\geq 0}GL_m \cdot \mu)$. Let us think of $G_{GL_m}$ as of the moduli space of lattices $L \subset D(K)$. Then the above isomorphism sends $(x_i, \bar{x}_i, p_i, q_i)$ to the lattice $L := z^{-u_0\lambda} (1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p)(L_0) = (1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-l} q \bar{x}^n x^l p)(L_\sigma)$, where $\sigma$ is a permutation of $\lambda$. According to Subsection 4.3 this lattice $L$ is uniquely determined by a $\mathbb{C}$-linear map $f : L_\sigma / L_0 \to L_\sigma^- / L_0^-$. Let us compute $f_1$.

Recall the definition of $f$. Denote the projection of $L_\sigma \oplus L_\sigma^-$ to $L_\sigma$ (resp. $L_\sigma^-$) along $L_\sigma^-$ (resp. $L_\sigma$) by $\pi_\sigma$ (resp. $\pi^-_\sigma$). Note that $\pi_\sigma$ induces the isomorphism $\pi : L \overset{\sim}{\to} L_\sigma$.

\[f := \pi^-_\sigma \circ \pi^{-1}\]

We have the following commutative diagram:

\[(67)\]

Let $f = \sum_{k=1}^{\infty} z^{-k-1} f_k$ as in Subsection 4.3. For a vector $z^{-h'}e_{j'}$, $e_{j'} \in D_{j'}$,
\[ \pi^{-1}(z^{-h}e_{j'}) = z^{-h}e_{j'} + \sum_{k=1}^{\infty} z^{-k-1} f_k(z^{-h}e_{j'}) = z^{-h}e_{j'} + \sum_{j} z^{-j-1} w_j + \sum_{k=2}^{\infty} z^{-k-1} f_k(z^{-h}e_{j'}) \]

for some \( w_j \in D_j \).

Conjugating (65) by \( z^{-w_0(\lambda)} \) we note that the map
\[ 1 - z^{-1} \sum_{n,l=0}^{\infty} z^{-l} q x^n \bar{x}^p : L_b \oplus L_b^- \rightarrow L_b \oplus L_b^- \]
is inverse to the map
\[ 1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-l} q \bar{x}^n x^p : L_b \oplus L_b^- \rightarrow L_b \oplus L_b^- . \]

Now we see that the diagram (67) gives us the condition
\[ (1 - z^{-1} \sum_{n,l=0}^{\infty} z^{-l} q x^n \bar{x}^p)(z^{-h}e_{j'}) + \sum_{j} z^{-j-1} w_j + \sum_{k=2}^{\infty} z^{-k-1} f_k(z^{-h'}e_{j'}) \in L_{\sigma} . \]

Straightforward computation shows that this condition implies \( w_j = q x^{j-h'} \bar{x}^{j-h'} p(e_{j'}) \). So
\[ w_j = q x_{j-1} \ldots x_{h'} \bar{x}_{h'} \ldots \bar{x}_{j'-1} p_{j'}(e_{j'}) . \]

From that and (60) it follows that the maps \( f_1 \) for \( L \) and for \( \psi \circ \phi(x_i, \bar{x}_i, p_i, q_i) \) are the same. So
\[ \psi \circ \phi(x_i, \bar{x}_i, p_i, q_i) = L = \eta \circ \Theta(x_i, \bar{x}_i, p_i, q_i) . \]

10.3.2. **Proof of Theorem 10.1.1**

**Proof.** According to Lemma 10.3.1 the morphism \( \psi \circ \phi \) restricted to the dense open subvariety \( \mathcal{M}_0^{v,d}(v, d) \subset \mathcal{M}_0(v, d) \) is given by the formula
\[ (x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-w_0 \lambda}(1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^p) . \]

Now continuity of the map \( \psi \circ \phi \) implies Theorem 10.1.1. \( \square \)

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