A NOTE ON SET THEORETICAL SOLUTIONS OF THE YANG-BAXTER EQUATION WITH TRIVIAL RETRACTION

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Abstract. We show that every finite non-degenerate set theoretical solution to the YBE whose retraction is a flip linearizes to a twist of the flip solution by roots of unity. This generalizes a result of Gateva-Ivanova and Majid. To prove the result we use a new invariant associated to a solution, its Lie algebra. We show also that a solution retracts to a flip solution if and only if its Lie algebra is abelian.

Keywords: Set theoretical YBE, Lie algebras, non commutative algebras.

1. Introduction

A set theoretical solution to the Yang-Baxter equation is a pair $(X,r)$ where $X$ is a set and $r : X \times X \to X \times X$ is a map satisfying

$$r \times \text{Id}_X \circ \text{Id}_X \times r \circ r \times \text{Id}_X = \text{Id}_X \times r \circ r \times \text{Id}_X \circ \text{Id}_X \times r.$$

Their study was suggested by Drinfeld in [Dri92], and begun in earnest with the seminal works of Gateva-Ivanova and van den Bergh [GIvdB98] and Etingof, Schedler and Soloviev [ESS99]. To study these solutions one usually constructs associated algebraic objects, such as monoids, groups, quadratic algebras, Hopf algebras (all introduced in the cited papers) and more recently homology groups [LV17], braces [Rumo7], and skew-braces [GV17], among many others. For an overview of the literature and open problems in the area, and its connections to other areas of mathematics, we refer the reader to [Ven19].

The space of all set theoretical solutions of the YBE is vast and one usually focuses on special classes to avoid drowning in it. A typical result will prove that the nice combinatorial properties of the solutions are reflected in nice properties of the associated algebraic objects. Common families to study are involutive solutions, or solutions of small size, etc. This paper is concerned with a family of solutions which is close to trivial in the sense that its retraction is a flip solution (see section 2 for definitions). These solutions have attracted much interest recently, mostly in the involutive case (i.e. $r^2 = \text{Id}_{X \times X}$), see for example [Rumo22],[PZD20],[PZD21],[P23].

We present two results on solutions whose retraction is a flip solution, but are not assumed to be involutive. First, we show that the multipermutation level is reflected in
the structure of the linearized solution, which must be a flip solution twisted by roots of unity. This result was already observed by Gateva-Ivanova and Majid for involutive square-free solutions in [GIM11] Theorem 4.9]. Second, we introduce a new algebraic invariant, the Lie algebra associated to a solution, and show that a solution retracts to a flip solution if and only if its associated Lie algebra is abelian.

**Convention.** All vector spaces and tensor products are over \( \mathbb{C} \). Throughout this paper \( X \) will denote a finite set.

2. Generalities on solutions

2.1. Quadratic sets. Following [Glo04][GIM08], a quadratic set is a pair \((X, r)\) where \( X \) is a set and \( r : X \times X \rightarrow X \times X \) is a bijective map; its cardinality is the cardinality of \( X \). For a fixed quadratic set \((X, r)\) and \( x, y \in X \) we denote by \( L_x, R_y \) the functions defined implicitly by

\[
 r(x, y) = (L_x(y), R_y(x)).
\]

It is sometimes useful to see \( L_x \) and \( R_y \) as defining actions of \( X \) on itself, so we also use the alternative notation

\[
 r(x, y) = (x \triangleright y, x \triangleleft y)
\]

The quadratic set is non-degenerate if the maps \( L_x \) and \( R_y \) are bijective for each \( x \in X \), it is involutive if \( r^2 = \text{Id}_X \), it is square-free if \( r(x, x) = (x, x) \) for all \( x \in X \), and finally it is finite if the set \( X \) is finite. As per our convention, in this paper we will only consider finite quadratic sets.

Set \( r_1 = r \times \text{Id}_X \) and \( r_2 = \text{Id}_X \times r \), which are functions of \( X \times X \times X \) to itself. The quadratic set \((X, r)\) is a solution to the set theoretical Yang-Baxter equation, or to be brief just a solution, if \( r_1 \circ r_2 \circ r_1 = r_2 \circ r_1 \circ r_2 \).

2.2. Examples. Suppose \( X \) is any finite set. If we choose arbitrary functions \( f, g : X \rightarrow X \) then \( r : X \times X \rightarrow X \times X \) given by \( r(x, y) = (f(y), g(x)) \) is a quadratic set. It is a solution if and only if \( f \) and \( g \) commute, it is non degenerate if and only if both \( f \) and \( g \) are bijective, and it is involutive if and only if \( g = f^{-1} \). These solutions are known as permutation solutions and were suggested by Lyubashenko [Dri92]. The flip solution on \( X \) is the permutation solution with \( f = g = \text{Id}_X \), that is the function \( r(x, y) = (y, x) \).

We will use the following two examples in the sequel. For \( n \in \mathbb{N} \) we denote by \( \mathbb{n} \) the set \( \{1, 2, \ldots, n\} \). Take \( s : \mathbb{2} \times \mathbb{2} \rightarrow \mathbb{2} \times \mathbb{2} \) given by

\[
 s \begin{array}{c|cc}
 1 & 1 & 2 \\
 2 & (2, 2) & (1, 2) \\
\end{array}
\]

This solution is involutive and non-degenerate but not square-free. It is a permutation solution with \( f = g = (12) \).
Now take \( t : 8 \times 8 \rightarrow 8 \times 8 \) to be the solution with
\[
\begin{array}{c|ccc|ccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\mathcal{L}_x & e & \ (34)(78) & 34 & \ (34)(78) & e & \\
\mathcal{R}_x & e & \ (34)(56) & e & \ (34)(56) & \\
\end{array}
\]
A long but direct computation shows that this is a solution, which is clearly not a multipermutation solution. Since \( t^2(3, 5) = t(5, 3) = (4, 6) \), \( t \) is not an involutive solution, and since \( t(3, 3) = (4, 4) \) it is not square-free.

**Remark.** In the literature the expression trivial solution is reserved for flip solutions with \( |X| \geq 2 \), while the only solution with \( |X| = 1 \) is a one-element solution. To avoid clashing with the established notation we will use the expression “flip solutions” to encompass both trivial and one-element solutions.

### 2.3. Retractions

We define an equivalence relation in \( X \) denoted by \( \sim \), where \( x \sim y \) if and only if \( \mathcal{L}_x = \mathcal{L}_y \) and \( \mathcal{R}_x = \mathcal{R}_y \). We denote by \( \overline{X} \) the equivalence class of \( x \) and by a slight abuse of notation we write \( \mathcal{L}_{\overline{X}} = \mathcal{L}_x \) and \( \mathcal{R}_{\overline{X}} = \mathcal{R}_x \). The following lemma is standard. It was proved for non-degenerate involutive solutions by Etingof, Schedler and Soloviev [ESS09], Subsection 3.2]. We denote by \( \overline{X} \) the quotient \( X / \sim \).

**Lemma 2.1.** Suppose \((X, r)\) is a quadratic set. Then \((X, r)\) is a solution if and only if the following equalities hold in \( \text{End}(V) \)
\[
\mathcal{L}_{\overline{X}} \circ \mathcal{L}_{\overline{y}} = \mathcal{L}_{\overline{x \circ y}} \circ \mathcal{L}_{\overline{x \circ y}};
\mathcal{R}_{\overline{X}} \circ \mathcal{R}_{\overline{y}} = \mathcal{R}_{\overline{x \circ y}} \circ \mathcal{R}_{\overline{x \circ y}};
\mathcal{R}_{\overline{x \circ y}}(x) \circ \mathcal{L}_{\overline{x \circ y}}(y) = \mathcal{L}_{\overline{x \circ y} \circ y}(x \circ y).
\]
If \( r \) is non-degenerate, it induces a map \( \overline{r} : \overline{X} \times \overline{X} \rightarrow \overline{X} \times \overline{X} \) given by \( \overline{r}(\overline{x}, \overline{y}) = (\overline{x \circ y}, \overline{x \circ y}) \) and the quadratic set \((\overline{X}, \overline{r})\) is also a solution.

**Proof.** For each \( x, y, z \in X \) we have
\[
r_1 r_2 r_1(x, y, z) = (\mathcal{L}_{\overline{x \circ y}}(\mathcal{L}_{\overline{x \circ y}}(z)), \mathcal{R}_{\overline{x \circ y}}(\mathcal{L}_{\overline{x \circ y}}(y)), \mathcal{R}_{\overline{x \circ y}}(\mathcal{R}_{\overline{x \circ y}}(x)));
\]
\[
r_2 r_1 r_2(x, y, z) = (\mathcal{L}_{\overline{x \circ y}}(\mathcal{L}_{\overline{x \circ y}}(z)), \mathcal{R}_{\overline{x \circ y}}(\mathcal{L}_{\overline{x \circ y}}(y)), \mathcal{R}_{\overline{x \circ y}}(\mathcal{R}_{\overline{x \circ y}}(x))).
\]
Since \( y = \mathcal{R}_{\overline{x \circ y}}(y) \) the equalities in the statement are equivalent to the fact that \( r \) is a solution to the YBE.

Suppose now that \( r \) is non-degenerate if \( \overline{X} = X \) then \( \overline{x \circ y} = \overline{x \circ y} \). Thus
\[
\mathcal{L}_{\overline{x \circ y}} = \mathcal{L}_{\overline{x}} \circ \mathcal{L}_{\overline{y}} \circ (\mathcal{L}_{\overline{x \circ y}})^{-1} = \mathcal{L}_{\overline{x}} \circ \mathcal{L}_{\overline{y}} \circ (\mathcal{L}_{\overline{x \circ y}})^{-1} = \mathcal{L}_{\overline{x \circ y}}
\]
and similarly \( \mathcal{R}_{\overline{x \circ y}} = \mathcal{R}_{\overline{x \circ y}} \). Thus \( \overline{x \circ y} \) depends only on \( \overline{y} \) and \( \overline{x} \). Similarly \( \overline{y \circ x} = \overline{y \circ x} \) and this result only depends on \( \overline{y} \) and \( \overline{x} \). This implies that the map \( \overline{r} : \overline{X} \times \overline{X} \rightarrow \overline{X} \times \overline{X} \) given by \( \overline{r}(\overline{x}, \overline{y}) = (\overline{x \circ y}, \overline{x \circ y}) \) is well-defined. Since \( r \) is a solution, so is \( \overline{r} \). \( \square \)

The solution \( \text{Ret}(X, r) = (\overline{X}, \overline{r}) \) is called the *retraction* of \((X, r)\). We say that a finite solution \((X, r)\) is retractable if \( |\overline{X}| < |X| \). It may happen that the retraction of a retractable solution is itself retractable, so we set inductively \( \text{Ret}^{n+1}(X, r) = \text{Ret}^{n}(\overline{X}, \overline{r}) \).
The multipermutation level of \((X, r)\), denoted \(\text{mpl}(X, r)\), is the infimum of all \(n \in \mathbb{N}\) such that \(\text{Ret}^n(X, r)\) is a solution of cardinality 1. Notice that a solution is a permutation solution if and only if it is of multipermutation level 1.

**Example(s).** Any permutation solution retracts to a one-element solution (which under our convention is a flip solution). Example t from \([2,3]\) has \(\overline{1} = \overline{2}, \overline{3} = \overline{4}, \overline{r} = \overline{6}\) and \(\overline{7} = \overline{8}\). The solution \(\text{Ret}(X, r)\) is the flip solution over the set of classes.

**Remark.** There is ambiguity in the notation \(L_\tau\), as it could be a function from \(X\) to itself or from \(\overline{X}\) to itself. This issue will not arise in the sequel as we will always work at the level of \(X\) and its linearization \(V\).

### 2.4. Associated vector space.

We denote by \(V\) the \(C\)-span of \(X\). The space \(V\) is a Hermitian space with the unique inner product making \(X\) a quadratic set the maps \(L_\tau\) and \(R_\tau\) induce linear endomorphisms of \(V\), which we denote by the same symbols. Finally, we define the map \(V\) so the desired equality holds over a basis of \(V\) and its linearization \(V\).

**Formulas.** From now on we assume that \((X, r)\) is a non-degenerate solution. To avoid overly complicated notation we denote by \([\overline{x}]\) the class of \(x\) in \(\text{Ret}^2(X, r)\), and by \([\overline{x}]\) the class of \([\overline{x}]\) in \(\text{Ret}^3(X, r)\). Since \(x \triangleright y\) depends only on the class of \(x\) it makes sense to write \(\overline{x} \triangleright y\), and by a similar reasoning \(\overline{x} \triangleright \overline{y} = [\overline{x}] \triangleright [\overline{y}]\), etc. We can thus rewrite the conditions on the previous lemma as

\[
L_{[\overline{x}]} \circ L_{[\overline{y}]} = L_{[\overline{x} \triangleright \overline{y}]} \circ L_{[\overline{y}]};
\]

\[
R_{[\overline{x}]} \circ R_{[\overline{y}]} = R_{[\overline{x} \triangleright \overline{y}]} \circ R_{[\overline{y}]};
\]

\[
R_{[\overline{x} \triangleright \overline{y}]} \circ L_{[\overline{x}]} \circ p_{[\overline{y}]} = L_{([\overline{x}] \triangleright [\overline{y}])] \circ R_{[\overline{y}]} \circ p_{[\overline{y}]}.
\]

We also point out that

\[
L_{[\overline{x}]} \circ p_{[\overline{y}]} = p_{[\overline{x}] \triangleright [\overline{y}]} \circ L_{[\overline{x}]};
\]

\[
R_{[\overline{x}]} \circ p_{[\overline{y}]} = p_{[\overline{x}] \triangleright [\overline{y}]} \circ R_{[\overline{y}]};
\]

\[
p_{[\overline{x}]} \circ p_{[\overline{y}]} = p_{[\overline{x} \triangleright [\overline{y}]]} = \delta_{\overline{x} \triangleright [\overline{y}]}.
\]

Finally, we can write \(R\) as

\[
R \circ \tau = \sum_{\overline{x} \in X} \sum_{\overline{y} \in X} L_{[\overline{x}]} \circ p_{[\overline{y}]} \circ R_{[\overline{y}]} \circ p_{[\overline{y}]},
\]

where \(\tau\) is the flip map given by \(\tau(v \otimes w) = w \otimes v\). To see this, notice that given \(v, w \in X\) we have

\[
\left( \sum_{\overline{x} \in X} \sum_{\overline{y} \in X} L_{[\overline{x}]} \circ p_{[\overline{y}]} \circ R_{[\overline{y}]} \circ p_{[\overline{y}]} \right) \tau(v \otimes w)
\]

\[
= \sum_{\overline{x} \in X} \sum_{\overline{y} \in X} L_{[\overline{x}]} \circ p_{[\overline{y}]}(w) \otimes R_{[\overline{y}]} \circ p_{[\overline{y}]}(v)
\]

\[
= L_{[\overline{y}]}(w) \otimes R_{[\overline{y}]}(v) = R(v \otimes w),
\]

so the desired equality holds over a basis of \(V \otimes V\).
3. Multipermutation level 2

Recall that here and below \((X, r)\) denotes a finite non-degenerate solution.

3.1. Formulas for multipermutation level 2. If \(\text{mpl}(X, r) \leq 2\) then \([\mathcal{Y}] = X\). Also there exist \(f, g: \overline{X} \rightarrow \overline{X}\) such that \(\tau(\overline{X}, \mathcal{Y}) = (f(\mathcal{Y}), g(\mathcal{Y}))\) so the formulas above simplify to

\[
\begin{align*}
\mathcal{L}_\tau \circ \mathcal{L}_\tau &= \mathcal{L}_f(\mathcal{Y}) \circ \mathcal{L}_g(\mathcal{Y}); & \mathcal{R}_\tau \circ \mathcal{R}_\tau &= \mathcal{R}_g(\mathcal{Y}) \circ \mathcal{R}_f(\mathcal{Y}); & \mathcal{R}_f(\mathcal{Z}) \circ \mathcal{L}_\tau &= \mathcal{L}_g(\mathcal{Z}) \circ \mathcal{R}_\tau; \\
\mathcal{L}_\tau \circ p_\tau &= p_f(\mathcal{Y}) \circ \mathcal{L}_\tau & \mathcal{R}_\tau \circ p_\tau &= p_g(\mathcal{Y}) \circ \mathcal{R}_\tau
\end{align*}
\]

In particular, if \(\text{Ret}(X, r)\) is a flip solution, or equivalently if \(f = g = \text{Id}_X\), then the maps \(\mathcal{L}_\tau, \mathcal{R}_\tau, p_\tau\) commute for all \(x, y, z \in X\).

3.2. The following result is a generalization of [GIM11] Theorem 4.9 to non-involutive solutions. Also, we replace the hypothesis that \(r\) is square-free and of multipermutation level 2 with the weaker hypothesis that the first retraction is a flip solution.

**Proposition 3.1.** Let \((X, r)\) be a finite non-degenerate solution, and suppose \(\text{Ret}(X, r)\) is a flip solution. Then there exist a basis \(\{v_1, \ldots, v_n\}\) of \(V\) and a set of roots of unity \(q = (q_{i,j})_{1 \leq i, j \leq n} \subset \mathbb{C}\) such that \(R(v_i \otimes v_j) = q_{i,j} v_i \otimes v_j\).

**Proof.** Since \(p_\tau\) is a projector it is diagonalizable, and its eigenvalues are either 1 or 0. Since \(\mathcal{L}_\tau\) and \(\mathcal{R}_\tau\) are induced by bijections of the set \(X\), they are also diagonalizable and their eigenvalues are roots of unity. As mentioned above the fact that \(\text{Ret}(X, r)\) is a flip solution means that the maps \(\mathcal{L}_\tau, \mathcal{R}_\tau\) commute, and they commute with \(p_\tau\), for all \(\tau, \mathcal{Y}, \mathcal{Z} \in \overline{X}\). Also since \(X\) is the disjoint union of the classes \(\overline{Z}\) the projectors \(p_\tau\) are orthogonal and add up to the identity, so they also commute with each other.

By a standard result of linear algebra the fact that these linear operators are diagonalizable and commute implies that they are simultaneously diagonalizable, i.e. there exists a basis of \(V\) consisting of eigenvectors of all these maps, say \(\{v_1, \ldots, v_n\}\). Write \(\lambda_{i,\tau}, \mu_{i,\tau}\) for the eigenvalue of \(v_i\) with respect to \(\mathcal{L}_\tau\), resp. \(\mathcal{R}_\tau\). Using the fact that \(R = \left(\sum_{\tau, \mathcal{Y}, \mathcal{Z} \in \overline{X}} \mathcal{L}_\tau \circ p_\tau \otimes \mathcal{R}_\tau \circ p_\tau\right) \circ \tau\) and that for each \(i\) there is exactly one \(\overline{Z}(i)\) such that \(p_{\overline{Z}(i)}(v_i) \neq 0\) we get

\[
R(v_i \otimes v_j) = \lambda_{i,\overline{Z}(i)} \mu_{j,\overline{Z}(i)} v_i \otimes v_j
\]

Since \(\lambda_{i,\overline{Z}(i)}\) and \(\mu_{j,\overline{Z}(i)}\) are roots of unity, so is their product and we are done. \(\square\)

Notice that this proof is the only place where we use that the base field is \(\mathbb{C}\), to guarantee that the minimal polynomial of the linear transformation induced by a permutation splits.

3.3. The algebra \(Q(X, r)\) associated to a solution is the quotient of the tensor algebra \(T(V)\) by the ideal generated by the image of the map \(\text{Id}_{V \otimes V} - R\). Following [Man88] this algebra is called the Yang-Baxter algebra associated to the solution. As shown in [GlydBog1] Corollary 1.5], this algebra is integral whenever \((X, r)\) is involutive. In
fact by [KVA19, Theorem 4.5] if \((X, r)\) is finite, bijective and left non-degenerate then \(Q(X, r)\) is integral if and only if \((X, r)\) is involutive.

In the special case where \((X, r)\) is a finite non-degenerate solution whose retraction is a flip solution then by Proposition \ref{prop:flip_solution}, its Yang-Baxter algebra \(Q(X, r)\) is isomorphic to the \(q\)-polynomial algebra \(C_q[v_1, \ldots, v_n]\), where \(q = (q_{ij})_{1 \leq i, j \leq n}\) are the roots of unity from the statement, and the relations \(v_i v_j = q_{ij} v_j v_i\) hold for all \(i, j\). This allows us to check that this is a domain if and only if \((X, r)\) is involutive very easily.

**Corollary 3.2.** Let \((X, r)\) be a finite non-degenerate solution, and suppose \(\text{Ret}(X, r)\) is a flip solution. Then \(Q(X, r)\) is finite over its center. Furthermore, \(Q(X, r)\) is a domain if and only if \((X, r)\) is involutive.

**Proof.** As we have already observed, \(Q(X, r)\) is the quotient of \(T(V)\) by the relations \(v_i v_j = q_{ij} v_j v_i\). Now if \(d_i\) is the minimal natural number such that \(q_{i,j}^{d_i} = 1\) for all \(j\), then \(y_i = v_i^{d_i}\) is a central element in \(Q(X, r)\). Thus \(\Gamma = \mathbb{C}[y_1, \ldots, y_n] \subset Q(X, r)\) is a central subalgebra and \(Q(X, r)\) is a finite module over \(\Gamma\).

The algebra \(C_q[v_1, \ldots, v_n]\) is integral if and only if \(q_{i,j} = q_{j,i}^{-1}\) for all \(1 \leq i, j \leq n\) (see for example [MR01, Chapter 1 §6]). Now

\[
R^2(v_i \otimes v_j) = q_{i,j} q_{j,i} v_i \otimes v_j,
\]

so \((X, r)\) is involutive if and only if \(q_{i,j} = q_{j,i}^{-1}\). Thus if \(Q(X, r)\) is integral then \((X, r)\) is involutive, and if \((X, r)\) is involutive we only need to show that \(q_{i,i} = 1\).

Let \(v_i\) be any of the vectors in the eigenbasis from the proposition. Then \(v_i\) lies in the span of \(\pi\) for one and only one \(\pi \in \mathcal{X}\). The hypothesis that \(\text{Ret}(X, r)\) is a flip solution implies that \((\pi, r|_{\pi \times \pi})\) is a solution, and indeed a permutation solution since \(r|_{\pi \times \pi} = \mathcal{L}_\pi|_{\pi} \times \mathcal{R}_\pi|_{\pi}\). Furthermore, if \((X, r)\) is involutive then so is \((\pi, r|_{\pi \times \pi})\), and this implies \(\mathcal{R}_\pi|_{\pi} = (\mathcal{L}_\pi|_{\pi})^{-1}\). This equality also holds for the linear extensions of these maps, and so in the notation of the proposition

\[
R(v_i \otimes v_i) = \mathcal{L}_\pi(v_i) \otimes \mathcal{R}_\pi(v_i) = \mathcal{L}_\pi(v_i) \otimes (\mathcal{L}_\pi)^{-1}(v_i) = \lambda_i \pi v_i \otimes \lambda_i^{-1} v_i = v_i \otimes v_i,
\]

so \(q_{i,i} = 1\). \(\square\)

The Yang-Baxter algebra has been studied intensively, notably by Gateva-Ivanova, see for example the articles [Glvd98, Gl04, G104b, G104b, G118, G123] and the references therein. In particular [Gl18] gives conditions on the algebraic structures of an involutive solution \((X, r)\) (not necessarily finite) which determine the multipermutation level of \((X, r)\), and the last sections contains several results on multipermutation solutions of level 2.

**Remark.** Similar results hold for the Hopf algebra associated to the solution as defined in [ESS99, section 2.9]). These results appear in the MsC thesis of Agustín Muñoz [Muñ20].
3.4. Examples. Recall the involutive, non square-free solution $(2, s)$ from [2.2]. Taking $v_1 = x_1 + x_2$ and $v_2 = v_1 - v_2$ the corresponding linear solution is given by

| $R$  | $v_1$  | $v_2$  |
|------|--------|--------|
| $v_1$| $v_1 \otimes v_1$| $-v_2 \otimes v_1$|
| $v_2$| $-v_1 \otimes v_2$| $v_2 \otimes v_2$|

The corresponding Yang-Baxter algebra is then the polynomial algebra in anticommuting variables.

Now consider solution $(8, t)$, which has multipermutation level 2 and is neither involutive nor square free. By Corollary [3.3] $Q(X, r)$ is a $q$-polynomial ring but not a domain. Using the notation from the proposition we have eigenvectors

- $v_1 = x_1$;
- $v_2 = x_2$;
- $v_3 = x_3 + x_4$;
- $v_4 = x_3 - x_4$;
- $v_5 = x_5 + x_6$;
- $v_6 = x_5 - x_6$;
- $v_7 = x_7 + x_8$;
- $v_8 = x_7 - x_8$.

All $\lambda_{3, \pi}$ and $\mu_{i, \tau}$ are 1 except

- $\lambda_{4, \pi} = \lambda_{4, \pi} = \lambda_{8, \pi} = -1$
- $\mu_{4, \pi} = \mu_{4, \pi} = \mu_{6, \tau} = -1$

Thus for example $q_{3, 8} = \lambda_{3, \pi}h_{8, \pi} = 1$ but $q_{8, 3} = \lambda_{8, \pi}h_{3, \pi} = -1$. This implies $v_8v_3 = v_3v_8 = -v_8v_3 = -v_3v_8$ from which follows that $v_8v_3 = v_3v_8 = 0$.

4. A Lie algebra associated to a solution

Throughout this section $(X, r)$ denotes a finite non-degenerate solution. We associate to $(X, r)$ a Lie algebra contained in $\mathfrak{gl}(V)$ and show that it characterizes solutions whose retracion is a flip solution.

**Definition 4.1.** The Lie algebra associated to the solution $(X, r)$, denoted by $\mathfrak{g}(X, r)$, is the Lie subalgebra of $\mathfrak{gl}(V)$ generated by the operators $\{L_\pi \circ p_\pi, R_\tau \circ p_\tau | \pi, \tau \in \Pi\}$.

4.1. From the formulas in [2.5] we deduce the commutation formulas

- $[L_\pi \circ p_\pi, L_\pi \circ p_\pi] = [\pi]^{\langle \pi \rangle} \circ p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ L_\pi \circ L_\pi - p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ L_\pi \circ L_\pi$,
- $[R_\tau \circ p_\tau, R_\tau \circ p_\tau] = [\pi]^{\langle \pi \rangle} \circ p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ R_\tau \circ R_\tau - p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ R_\tau \circ R_\tau$,
- $[L_\pi \circ p_\pi, R_\tau \circ p_\pi] = p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ L_\pi \circ R_\tau - p_\pi \circ [\pi]^{\langle \pi \rangle} \circ p_\pi \circ R_\tau \circ L_\pi$.

It also follows that the linear solution $R$ is a solution to the classical YBE in $\mathfrak{g}(X, r)$. The crucial fact in the proof of Proposition [3.1] is that the operators $\{L_\pi \circ p_\pi, R_\tau \circ p_\pi\}$ commute when Ret$(X, r)$ is a flip solution, so in this case $R$ is a solution to the YBE in an abelian Lie algebra.

**Theorem 4.2.** Let $(X, r)$ be a finite non-degenerate solution and let $\mathfrak{g} = \mathfrak{g}(X, r)$. Then the following are equivalent.

(a) The solution Ret$(X, r)$ is a flip solution.
(b) The span of the set $\{L_\pi, R_\tau, p_\pi\}$ is an abelian Lie subalgebra of $\mathfrak{gl}(V)$.
(c) The algebra \( g(X, r) \) is abelian.

Proof. The implication \((\text{II}) \Rightarrow (\text{I})\) was shown in the proof of Proposition 3.1, and is also easy to deduce from the commutation formulas displayed above. The implication \((\text{I}) \Rightarrow (\text{II})\) is immediate.

Now suppose \( g(X, r) \) is abelian. Choose arbitrary elements \( x, y, z \in X \) and choose \( w \in X \) such that \( y = w \circ z \). Since \([L_x \circ p_{[p]}, R_x \circ p_{[p]}] = 0\) we deduce from the formulas in [2,5] that

\[
p_{[x \circ y]} \circ L_x \circ R_{[x]} = p_{[y]} \circ p_{([x \circ y], [x], [x \circ y] \circ [x], [y] \circ [x])} \circ R_{[y]} \circ L_x.
\]

Since \( L_x \) and \( R_{[x]} \) are automorphisms of \( V \), the morphism on the left hand side of the equality is nonzero and its image is the subspace of \( V \) spanned by \([x] \circ [y]\). Thus the map on the right hand side is also nonzero, which implies that \([y] = ([x] \circ [y]) \circ ([x] \circ [y])\), and its image is the subspace spanned by \([y]\), which implies that \([y] = ([x] \circ [y])\). These imply that \([y]\) is a fixed point for the action of \([x]\) on the left and for the action of \([x]\) on the right.

Since \( x, y, z \) were chosen arbitrarily, this means that \([L_{[x]} = R_{[x]} = \text{Id}_{[y]}\) and hence \( \text{Ret}(X, r) \) is a flip solution. \(\square\)

We would like to have a more intrinsic definition of \( g(X, r) \) but so far this is the best description we have. Computer experiments show that for all \( |X| \leq 8 \) and \( r \) involutive the derived algebra of \( g(X, r) \) is semisimple of type \( A \). Thus the multipermutation level is not directly related to the solubility or nilpotency of \( g(X, r) \), as one might suppose from the statement. We intend to return to this matter in future work.

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