INCORRIGIBLE REPRESENTATIONS

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Abstract. As a consequence of his numerical local Langlands correspondence for $GL(n)$, Henniart deduced the following theorem: If $F$ is a nonarchimedean local field and if $\pi$ is an irreducible admissible representation of $GL(n, F)$, then, after a finite sequence of cyclic base changes, the image of $\pi$ contains a vector fixed under an Iwahori subgroup. This result was indispensable in all proofs of the local Langlands correspondence. Scholze later gave a different proof, based on the analysis of nearby cycles in the cohomology of the Lubin-Tate tower.

Let $G$ be a reductive group over $F$. Assuming a theory of stable cyclic base change exists for $G$, we define an incorrigible supercuspidal representation $\pi$ of $G(F)$ to be one with the property that, after any sequence of cyclic base changes, the image of $\pi$ contains a supercuspidal member. If $F$ is of positive characteristic then we define $\pi$ to be pure if the Langlands parameter attached to $\pi$ by Genestier and Lafforgue is pure in an appropriate sense. We conjecture that no pure supercuspidal representation is incorrigible. We sketch a proof of this conjecture for $GL(n)$ and for classical groups, using properties of standard $L$-functions; and we show how this gives rise to a proof of Henniart’s theorem and the local Langlands correspondence for $GL(n)$ based on V. Lafforgue’s Langlands parametrization, and thus independent of point-counting on Shimura or Drinfel’d modular varieties.

This paper is intended as part of a sequel to the author’s paper arXiv:1609.03491 with G. Böckle, S. Khare, and J. Thorne, and will be incorporated into a future joint paper with the three authors.

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1. Introduction

A little-known but indispensable step in every proof of the local Langlands correspondence for $G = GL(n)$ over a $p$-adic field is the following splitting theorem of Henniart:

**Theorem 1.1.** [He90] [Henniart] Let $F_0$ be a non-archimedean local field, $n$ a positive integer, and $\pi_0$ a supercuspidal representation of $G(F_0)$. There is a finite sequence of cyclic extensions $F_0 \subset F_1 \subset \cdots \subset F_r$ such that, if we define $\pi_i$ by induction as the representation of $G(F_i)$ obtained as the cyclic base change from $F_{i-1}$ of $\pi_{i-1}$, then $\pi_r$ is not supercuspidal.

Since the local Langlands correspondence is a bijection

$$\mathcal{L}_F : \mathcal{G}(n,F) \to \mathcal{A}(n,F)$$

between the set $\mathcal{A}(n,F)$ of (equivalence classes of) $n$-dimensional (Frobenius semisimple, smooth) representations of the Weil-Deligne group of $F = F_i$ and the set $\mathcal{G}(n,F)$ of (equivalence classes of) irreducible admissible representations of $G(F)$, in both cases with coefficients in a field of characteristic 0, this theorem is in fact an immediate consequence of the following properties:

(a) The existence of the local Langlands correspondence;
(b) The property that $\pi$ is supercuspidal if and only if $\mathcal{L}_F(\pi)$ is irreducible, and
(c) The property that cyclic base change from $\mathcal{G}(n,F_{i-1})$ to $\mathcal{G}(n,F_i)$ is taken under $\mathcal{L}_F$ to restriction of Weil-Deligne representations from $F_{i-1}$ to $F_i$.

Indeed, since $\mathcal{L}_{F_0}(\pi_0)$ is necessarily trivial on an open subgroup of the absolute inertia group of $F_0$, after a finite sequence of cyclic extensions as in Theorem 1.1 $\mathcal{L}_{F_r}(\pi_r)$ is no longer irreducible, and thus $\pi_r$ cannot be supercuspidal. One knows more: we can find $r$ such that, writing $\mathcal{L}_{F_r}(\pi_r) = (\sigma_r, N_r)$, where $\sigma_r$ is an $n$-dimensional representation of the Weil group $W_{F_r}$ of $F_r$ and $N$ is a nilpotent operator normalized by $\sigma_r(W_{F_r})$ and satisfying the Weil-Deligne relation with regard to $N$, we may assume that $\sigma_r$ is unramified. Another property of the local Langlands correspondence then implies the following strengthening of Theorem 1.1.

**Theorem 1.2.** We can find a sequence $F_{i-1} \subset F_i$ as in the notation of Theorem 1.1 such that $\pi_r$ contains a vector invariant by an Iwahori subgroup of $G(F_r)$.

In fact, Henniart proves a stronger version of this theorem in [He90]; the irreducibility of $\mathcal{L}_{F_0}(\pi_0)$ easily implies that $N_r = 0$, and moreover that $\sigma_r$ is diagonal. We keep the weaker version here because it is the most that can be expected when $GL(n)$ is replaced by a more general group $G$, and even then it appears one has to find a way to exclude cuspidal unipotent representations (on which more below).
Henniart derived these two theorems in [He90] as a consequence of his numerical correspondence [He88]. This was the construction of bijections $\mathcal{L}_F' : \mathcal{G}(n, F) \rightarrow \mathcal{A}(n, F)$ that were not known to satisfy all the desirable properties of the local Langlands correspondence – in fact, there were many such correspondences, only one of which could be the right one – but that did satisfy certain hereditary properties with respect to cyclic base change. The proof of the numerical correspondence involved many steps, notably passage between $p$-adic fields and local fields of characteristic $p$, and the application of Laumon’s local Fourier transform to the latter. Most important, perhaps, is that the supercuspidal representations in $\mathcal{G}(n, F)$ of fixed (minimal) conductor could be counted, because the Jacquet-Langlands correspondence places these representations in bijection with (a subset of the) irreducible representations of the multiplicative group of a central division algebra of dimension $n^2$ over $F$, and thus reduces the counting problem to the enumeration of representations of finite groups. Then, by a long induction, Henniart is able to show that this number coincides with a similar number on the $\mathcal{A}(n, F)$ side of the correspondence. We wish to draw particular attention to this step because it is unique to reductive groups $G$ of type $A$; it has no analogue for other classes of groups.

The term “base change” in the above theorems is ambiguous. When $F_0$ is of characteristic 0 the base change of $\pi_{i-1}$ to $GL(n, F_i)$ asserts an explicit relation between the distribution character of $\pi_{i-1}$ and the twisted character of $\pi_i$ with respect to the cyclic group $Gal(F_i/F_{i-1})$; this was established by Arthur and Clozel by a combination of local and global means, using the full strength of the (twisted) trace formula for number fields. These methods have not been completely developed over global fields of positive characteristic, and there are no complete references even for the case of $GL(n)$, although it is known to experts. Thus when $F_0$ is of positive characteristic cyclic base change for $\mathcal{G}(n, \bullet)$ is simply defined by property (c) above, whereas (c) is proved directly for $p$-adic fields using formal properties of base change.

The proofs of the full local Langlands correspondence for $GL(n)$ in [LRS93] (for $F$ of positive characteristic) or in [HT01, He00] (for $p$-adic fields) all rely on the numerical correspondence of [He88]. The proofs for $p$-adic fields build more precisely on an inductive argument based on Henniart’s Theorem 1.1 which is used in an argument of Henniart (Theorem 3.3 of [BHK98]) to show that maps from $\mathcal{G}(n, F)$ to $\mathcal{A}(n, F)$ that satisfy the formal properties of those constructed in [H97, HT01] (using $p$-adic uniformization) or in [He00] (using formal properties of representations of global Weil groups) are necessarily bijections.

Although, like the proofs in [HT01, He00], Peter Scholze’s proof of the local Langlands correspondence for $GL(n)$ makes use of the construction of certain automorphic representations attached to representations of global Weil groups, his approach is based on rather different principles. In particular, Scholze makes no use at all of Henniart’s numerical correspondence,
and does not need to pass between characteristic 0 and characteristic $p$. Nevertheless, his proof does use Theorem 1.2, but he proves it by a geometric argument, based on a novel analysis of nearby cycles on the tower of coverings of the Lubin-Tate formal moduli space of deformations of 1-dimensional formal groups. Like Henniart’s numerical correspondence, this method has no obvious generalization to groups other than $GL(n)$. While the representations of many $G(F)$ can be realized on the cohomology of the $p$-adic period domains of Rapoport-Zink, and while there is work in progress of Fargues and Scholze that aims to construct a partial correspondence on the cohomology of spaces with actions by any $G(F)$ that can be constructed in the category of diamonds, the geometric properties that allow Scholze to analyze the nearby cycles seem specific to the Lubin-Tate tower, and thus to $GL(n)$.

The purpose of this note is to explain a third method to prove Theorems 1.1 and 1.2 that do not depend on the uniquely favorable properties of $GL(n)$. The method is based on a hypothetical combination of a local parametrization of representations of groups over local fields, analogous to that provided in positive characteristic by the work of Alain Genestier and Vincent Lafforgue, in the setting of the latter’s global parametrization of automorphic representations, with analytic techniques based on the trace formula and on integral representations of $L$-functions.

To illustrate the method, we begin with a proof of Theorem 1.1 in characteristic $p$, assuming the extension to function fields of the results of $[AC]$ on base change. This is followed by a sketch of the argument deducing the local Langlands correspondence from Theorem 1.1. This qualifies as a new proof, in that, instead of using the Lefschetz formula for the action of correspondences on moduli spaces of shtukas, and point counting, as in $[LRS93, Laf02]$ (or the proofs for $p$-adic fields, all of which refer back to Shimura varieties), we use Vincent Lafforgue’s geometric study of these moduli spaces. The trace formula is required nevertheless in order to prove base change. Since the trace formula for function fields is still a work in progress, the proof must be considered conditional; however, the specialists assure us that there is no essential difficulty. The first two sections are largely devoted to showing that the proof indicated here, which is based on the constructions of $[GLa, Laf12]$, on the Godement-Jacquet $L$-function, and on a (not yet available) theory of global and local base change, is independent of the earlier constructions of $[LRS93]$ as well as $[Laf02]$.

The next section treats classical groups, using the doubling method of Piatetski-Shapiro and Rallis to control the poles of local $L$-functions. We sketch a proof of the analogue of Henniart’s theorem 1.1 for the subclass of pure supercuspidal representations, which are introduced in order to exclude the problematic cuspidal unipotent representations. Base change is more difficult to define for groups other than $GL(n)$, and most of the section is devoted to formulating a definition of base change that is independent of the known case of $GL(n)$, is sufficiently plausible to be a likely consequence of
imminent developments in the trace formula, but is also sufficiently robust for the application to the splitting theorem. For this Lafforgue’s methods, and the local-global compatibility proved in [GLa], are crucial.

In a final section we show how an analogue of Henniart’s theorem can be derived from the Hiraga-Ichino-Ikeda conjecture on formal degrees [III08]. Although this is a preliminary version of a paper that should be incorporated in future joint work with Böckle, Khare, and Thorne, it has already benefited from helpful comments and suggestions by G. Henniart and L. Lomelí. I am especially grateful to J.-L. Waldspurger, who pointed out numerous imprecisions in an earlier version of the manuscript, and who was nevertheless generous enough to read and suggest improvements to the first arXiv version as well.

2. The splitting theorem for $GL(n)$

Let $X$ be a smooth projective curve over the finite field $k$ of characteristic $p$. Let $K = k(X)$ denote its function field. Let $G_n$ be the group $GL(n)$, viewed as a reductive algebraic group over $K$. We let $C$ be an algebraically closed coefficient field of characteristic zero; it could be $C$ or it could be $\mathbb{Q}_\ell$ with $\ell \neq p$. All automorphic representations of the groups we will consider, as well as all irreducible representations of groups over local fields, will have coefficients in $C$

**Hypothesis 2.1.** The results of [AC] are valid over $K$. In particular, let $K'/K$ be a finite extension, and let $K''/K'$ be a cyclic extension of prime degree $q$. Let $\Pi'$ be a cuspidal automorphic representation of $G_{n,K'}$. Then there is an automorphic representation $\Pi'' = BC_{K''/K'}(\Pi')$ of $G_{n,K''}$ with the following properties:

(a) Let $v \in |X_{K'}|$ be a place of $K'$ and suppose $\Pi'_v$ is unramified. Let $w$ be a place of $K''$ dividing $v$. Then $\Pi''_w$ is the unramified principal series representation of $GL(n,K''_w)$ obtained by unramified base change from $\Pi'_v$. In other words, if $K'^{un}_v$, resp. $K''^{un}_w$, is the maximal unramified extension of $K'_v$, resp. of $K''_w$, and if $\sigma_v : Gal(K'^{un}_v/K'_v) \to GL(n,C)$ is the Satake parameter of $\Pi'_v$, then $\Pi''_w$ is the unramified principal series representation with Satake parameter $\sigma_v |_{Gal(K''^{un}_w/K''_w)}$.

(b) Let $\alpha : Gal(K''/K') \to C^\times$ be a non-trivial character. Then $\Pi''$ is cuspidal if and only if $\Pi' \otimes \alpha \neq \Pi'$. If $\Pi' \otimes \alpha = \Pi'$ then $q$ divides $n$; letting $d = n/q$, there is a cuspidal automorphic representation $\Pi_0$ of $GL(d)_{K''}$ such that

$$\Pi'' \sim \Pi_0 \boxplus \Pi_0^q \boxplus \cdots \boxplus \Pi_0^{q-1},$$

where $\tau \in Gal(K''/K')$ is any non-trivial element, and $\boxplus$ is the Langlands sum.

Other results of [AC] will be used in the course of the discussion, and they will be listed at the end of the following section. We want to avoid using
the full strength of [AC]. in fact. The strong multiplicity one theorem for
\( GL(n) \), and the more general classification theorem of Jacquet and Shalika,
guarantees that \( \Pi'' \) is uniquely determined by (a) of Hypothesis 2.1. This
implies in particular that, for any place \( v \in |X_{K'}| \), there is a map associating
\( \Pi''_w \) to \( \Pi'_v \), whether or not \( \Pi'_v \) is ramified. It is by no means obvious that
\( \Pi''_w \) is independent of the global representation \( \Pi' \); Arthur and Clozel prove
that this is the case, but we prefer not to include this in the hypotheses,
because the situation for more general groups is more complicated. Thus
we introduce the following ad hoc definition:

**Definition 2.1.** Let \( F \) be a local field of characteristic \( p > 0 \), and let \( \pi \)
be an irreducible representation of \( GL(n, F) \). Let \( K' = k(X') \) be a global
function field and let \( v \in |X'| \) be a place of \( K' \) that admits an isomorphism
\( K'_v \xrightarrow{\sim} F \). Let \( \Pi' \) be a cuspidal automorphic representation of \( GL(n, K') \) such
that \( \Pi'_v \xrightarrow{\sim} \pi \). Let \( F'/F \) be a cyclic extension of prime degree \( q \), and let
\( K''/K' \) be an extension of degree \( q \) in which \( v \) is inert, and such that, letting \( w \)
denote the prime of \( K'' \) dividing \( v \), we have an isomorphism \( K''_w \xrightarrow{\sim} F' \).
Define

\[
BC^{\Pi',K''}_{F'/F}(\pi) = BC_{K''/K'}(\Pi')_w
\]
to be the local component at \( w \) of \( BC_{K''/K'}(\Pi') \).

Note that this only defines local base change for representations that
occur as local components of cuspidal automorphic representations. This
includes supercuspidal representations with central characters of finite order
(see Theorem 2.2 below), and this suffices for our purpose.

2.2. **Parametrization.** In order to define the version of local base change
that we need, we start with supercuspidal representations. For the moment
we work in the generality of [Laf12]. Thus let \( K \subset K' \subset K'' \) as above,
and let \( G \) be any connected reductive algebraic group over \( K \). We fix a point
\( v \in |X_K| \) and let \( F' = K'_v \) denote the corresponding local field, \( G = G(F') \).
Let \( \pi_v \) be a supercuspidal representation of \( G(F') \) with central character of
finite order.

**Theorem 2.2** (Gan-Lomelí). [GLo] There a cuspidal automorphic repre-
sentation \( \pi \) of \( G_K' \) with central character of finite order, such that the local
componet \( \pi_v \) is the given representation.

Gan and Lomelí prove a much more refined result, with strong restrictions
on the ramification of \( \pi \) away from \( v \), but this simple version will suffice
for our purposes. Before we return to the case of \( GL(n) \), we quote the main
results of [Laf12] and [GLa]. For this, let \( F \) be any of the fields
\( K, K', K'', K'_v, K''_w \), and let \( {}^LG = G \rtimes WF \) be the \( L \)-group over \( F \), where \( \hat G \)
is the Langlands dual group of \( G \), with coefficients in the field \( C \), and \( WF \)
is the (global or local) Weil group.

If \( F \) is a local field, we let \( G = G(F) \). If \( F'/F \) is any extension, we let
\( \mathcal G(G, F') \) denote the set of irreducible admissible representations of \( G(F') \).
We let $A(G, F')$ denote the set of admissible Langlands parameters

$$\phi : WD_{F'} \to L^G$$

with the usual properties; in particular, the restriction of $\phi$ to $W_{F'}$, composed with the tautological map $L^G \to W_{F'}$, is the identity map from $W_{F'}$ to the subgroup $W_{F'} \subset W_F$. Let $A(G, F')^{ss}$ denote the set of equivalence classes of (Frobenius semisimple, smooth) homomorphisms of the Weil group of $F'$ to $L^G$. There is a natural map $A(G, F') \to A(G, F')^{ss}$ given by forgetting the image of the nilpotent operator $N$.

If $F$ is one of the global fields $K, K', K''$, we let $G_0(G_F)$ denote the set of cuspidal automorphic representations of $G_F$ with central character of finite order. We let $A^{ss}(G_F)$ denote the set of equivalence classes of compatible families of completely reducible $\ell$-adic representations, for $\ell \neq p$:

$$\rho_\ell : \text{Gal}(\bar{F}/F) \to L^G(\overline{\mathbb{Q}_\ell}).$$

The term completely reducible is understood to mean that if $\rho_\ell(\text{Gal}(\bar{F}/F)) \cap \hat{G}(\overline{\mathbb{Q}_\ell})$ is contained in a parabolic subgroup $P \subset \hat{G}(\overline{\mathbb{Q}_\ell})$, then it is contained in a Levi subgroup of $P$.

If $\nu : G \to \mathbb{G}_m$ is an algebraic character, with $\mathbb{G}_m$ here designating the split 1-dimensional torus over $F$, the theory of $L$-groups provides a dual character $\hat{\nu} : \mathbb{G}_m \to \hat{G} \subset L^G$. If $Z$ is the center of $G$, and if $c : \mathbb{G}_m \to Z \subset G$ is a homomorphism, then the theory of $L$-groups provides an algebraic character $L^G c : L^G \to \mathbb{G}_m$.

**Theorem 2.3.** (i) [Laf12] [Théorème 0.1] There is a map

$$L^{ss}_K = L^{ss}_{G, K} : G_0(G_K) \to A^{ss}(G_K)$$

with the following property: if $v$ is a place of $K$ and $\Pi \in G_0(G_K)$ is a cuspidal automorphic representation such that $\Pi_v$ is unramified, then $L^{ss}_K(\Pi)$ is unramified at $v$, and $L^{ss}_K(\Pi) |_{W_{K_v}}$ is the Satake parameter of $\Pi_v$.

(ii) Suppose $\nu : G \to \mathbb{G}_m$ is an algebraic character. Suppose

$$\chi : A^1_K / K^\times \to C^\times = GL(1, C)$$

is any continuous character of finite order. For any $\Pi \in G_0(G_K)$, let $\Pi \otimes \chi \circ \nu$ denote the twist of $\Pi$ by the character $\chi \circ \nu$ of $G(A)$. Then

$$L^{ss}(\Pi \otimes \chi \circ \nu) = L^{ss}(\Pi) \cdot \hat{\nu}\hat{\chi}$$

where $\hat{\nu}$ is as above and where $\hat{\chi} : \text{Gal}(\bar{K}/K)^{ab} \to GL(1, C)$ is the character corresponding to $\chi$ by local class field theory.

(iii) [GLa] [Théorème 0.1] Let $v$ be a place of of $K$ and let $F = K_v$. Then the semisimplification of the restriction of $L^{ss}_K(\Pi)$ to $W_{K_v}$ depends only on

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1Here and below we will mainly refer to the restriction of a global Galois parameter to the local Weil group, rather than to the local Galois group, because the unramified Langlands correspondence relates spherical representations to unramified homomorphisms of the local Weil group to the $L$-group. But the difference is inessential.
$F$ and $\Pi_v$ and not on the rest of the automorphic representation $\Pi$, nor on the global field $K$.

Proof. Point (i) is the main result of [Laf12], and point (iii) is the main result of [GLa]. Point (ii) follows from point (i): it is true locally at almost all places, by the compatibility with the unramified local correspondence, and thus it is true everywhere by Chebotarev density. □

The following corollary is implicit in [GLa].

Corollary 2.3. Let $F$ be a local field of characteristic $p$. There is a map

$$L_{F}^{ss} : G(G,F) \to A(G,F)^{ss}$$

with the following properties.

(i) If $\pi \in G(G,F)$ is an unramified principal series representation, then $L_{F}^{ss}(\pi)$ is its Satake parameter.

(ii) More generally, $L_{F}^{ss}$ is compatible with parabolic induction, in the following sense. Suppose $M \subset G$ is the Levi subgroup of an $F$-rational parabolic subgroup $P$ of $G$, and let $i_{M} : L M \to L G$ be the corresponding morphism. Let $\sigma \in G(M,F)$ and let $\pi$ be an irreducible constituent of $\text{Ind}_{P(F)}^{G(F)}(\sigma)$ (normalized induction). Then

$$L_{G,F}^{ss}(\pi) = i_{M}(L_{M,F}^{ss}(\sigma)).$$

(iii) The map $L_{F}^{ss}$ is compatible with twisting by characters, in the following sense. Let $\nu : G \to \mathbb{G}_m$ be an algebraic character. Suppose $\chi : F^{\times} \to C^{\times}$ is a continuous character, and $\hat{\chi} : W_{F} \to GL(1,C)$ is the representation corresponding to $\chi$ by local class field theory. Then for any $\pi \in G(G,F')$, we have

$$L_{F'}^{ss}(\pi \otimes \chi \circ \nu) = L_{F'}^{ss}(\pi) \cdot \hat{\chi} : W_{F'} \to L G.$$

(iv) The map $L_{F}^{ss}$ is compatible with central characters, in the following sense. Let $c : \mathbb{G}_m \to Z \subset G$ and $L c$ be as in the discussion above. Suppose $\pi \in G(G,F)$ has central character $\xi_\pi$, and let $\xi_{c,\pi} = \xi_{c} \circ \xi : Z(F) \to C^{\times}$. Then $L_{c} \circ L_{F}^{ss}(\pi)$ is the character attached to $\xi_{c,\pi}$ by local class field theory.

(v) Suppose $K = k(X)$ is a global function field and $v \in |X|$ is a place such that $F \sim \to K_v$. Then $L_{F}^{ss}$ is compatible with the map $L_{K}^{ss}$: if $\Pi$ is any cuspidal automorphic representation of $G_K$ with finite central character, then $L_{F}^{ss}(\Pi_v)$ is equivalent to the semisimplification of $L_{K}^{ss}(\Pi) \mid_{W_{F_v}}$.

Proof. Suppose $\pi \in G(G,F)$ is a supercuspidal representation with central character of finite order. Let $K = k(X)$ be a global function field with a place $v$ such that $F \sim \to K_v$. By Theorem 2.2 there is a cuspidal automorphic representation $\Pi$ with central character of finite order, such that $\Pi_v \equiv \pi$. Then we define $L_{F}^{ss}(\pi)$ to be the semisimplification of $L_{K}^{ss}(\Pi) \mid_{W_{F_v}}$, as required in point (v). It follows from point (iii) of Theorem 2.3 that this is well defined.
More generally, if \( \pi \in \mathcal{G}(G,F) \) is any supercuspidal representation, we may find a \( C \)-valued character \( \chi \) of \( G(F) \) such that \( \pi \otimes \chi \) has central character of finite order. We apply the previous step to \( \pi \otimes \chi \) and define

\[
L_{FS}^\pi(\pi \otimes \chi) = L_{FS}^\pi(\pi \otimes \chi) \otimes \hat{\chi}^{-1}.
\]

That this is well-defined follows from point (ii) of Theorem \[2.3\].

Now suppose \( \pi \) is an irreducible constituent of \( \text{Ind}_{G(F)}^G \mathcal{P}(F) \sigma \) and define \( L_{FS}^G(\pi) \) by (ii). We need to show that this definition is compatible with (v); but this follows from the final assertion of Theorem 0.1 of \[GLa\].

Finally, points (iii) and (iv) are true at all unramified places, by construction; hence they are true everywhere by Chebotarev density.

The following Theorem contains the main local-global compatibility property of base change.

**Theorem 2.4.** \([Laf12, GLa]\) Let \( F \) be a local field of characteristic \( p \), and let \( F'/F \) be a cyclic extension of prime degree \( q \). Let \( \pi \) be an irreducible representation of \( GL(n,F) \) and let \( K', v, \Pi', K'' \) be as in Definition \[2.1\].

Then

\[
L_{FS}^{\Pi'(K'/F')}(\pi) = L_{FS}^{\Pi(K/F)}(\pi)|_{W_{F'}}.
\]

**Proof.** If \( \pi \) is spherical then this follows from (i) of Theorem \[2.3\]. The general case then follows from Chebotarev’s density theorem. \(\square\)

**2.4. Proof of Theorem 1.1.** Now we return to the case of \( GL(n) \). Because Definition \[2.1\] of base change is not purely local, we need to reformulate Theorem \[1.1\] to take this into account.

**Theorem 2.5.** Let \( F_0 \) be a non-archimedean local field of positive characteristic \( p \), \( n \) a positive integer, and \( \pi_0 \) a supercuspidal representation of \( GL(n,F_0) \). Choose a global function field \( K_0 \) with a place \( v_0 \) such that \( K_0,v_0 \sim F_0 \), and a cuspidal automorphic representation \( \Pi_0 \) of \( GL(n,A_{K_0}) \), as in Theorem \[2.2\], such that \( \Pi_{0,v_0} \sim \pi_0 \). There is a finite sequence of cyclic extensions of prime degree, \( F_0 \subset F_1 \subset \cdots \subset F_r \), with the following property. Let \( K_0 \subset K_1 \subset \cdots \subset K_r \) be any sequence of cyclic extensions, with \( v_0 \) inert in \( K_r \), such that, for each \( i \), \( K_i,v_i \equiv F_i \), where \( v_i \) is the prime of \( K_i \) above \( v \). Define \( \Pi_i \) inductively as \( BC_{K_i/K_{i-1}}(\Pi_{i-1}) \), and let \( \pi_i = \Pi_{i,v_i} \). Then \( \pi_r \) is not supercuspidal.

**Proof.** The proof is based on properties of the Godement-Jacquet \( L \)-function for \( GL(n) \). We fix a prime \( \ell \neq p \) and view \( A^{ss}(G_{K_i}) \) as sets of representations of the respective Galois groups on \( n \)-dimensional vector spaces over \( \overline{\mathbb{Q}}_\ell \). Let \( F_r/F_0 \) be a Galois extension such that \( L_{FS}^{\pi_0}(\pi_0) \) restricts to a (semisimple)
unramified extension of $Gal(F_0/F_r)$. Since Galois groups of local fields are
solvable, we can find a finite sequence of cyclic extensions of prime degree
$F_i/F_{i−1}$, $i = 1, . . . , r$. Now let $K_0$, $Π_0$, and $K_0 ⊂ K_1 ⊂ ⋯ ⊂ K_r$, be as in
the statement of the theorem.

It follows from (v) of Corollary 2.3 that the semisimplification of the re-
striction of $L^{ss}_K(Π_r)$ to $Gal(K_{r,v_r}/K_{r,v_r})$ is unramified. Note that $L^{ss}_K(Π_r)$
is a semisimple representation of the global Galois group, but its restric-
tion to the local Galois group need not be semisimple. Nevertheless, by
Grothendieck’s monodromy theorem, it follows that

• The image of $L^{ss}_K(Π_r)|_{Gal(K_{r,v_r}/K_{r,v_r})}$, viewed as a subgroup of
$GL(n, ℓ)$, fixes a line in $ℓ^m$, and acts on this line by an unramified
character, say $α_r$. This implies that

$$L(2.6) \quad \text{The local Euler factor } L_{v_r}(s, L^{ss}_K(Π_r) ⊗ α_r^{-1}) \text{ has a pole at } s = 0.$$ 

Now fix a global Hecke character $β$ of $GL(1)_r$ and consider the Godement-
Jacquet $L$-function

$$L(2.7) \quad L(s, Π_r, β) = \prod_w L_w(s, Π_{r,w}, β_w).$$

There is a finite set $S$ of places $w$ of $K_r$ such that, for $w ∉ S$ $Π_{r,w}$ is
unramified, and it follows from (i) of Theorem 2.3 that

$$L(w, s, Π_{r,w}, β_w) = L_w(s, L^{ss}_K(Π_r) ⊗ β_w).$$

We assume $S$ contains $v_r$. Moreover, for any Hecke character $β$, viewed
alternately as an automorphic representation of $GL(1)$ or as an $ℓ$-adic Galois
character, $L(s, Π_r, β)$ and the Artin $L$-function $L(s, L^{ss}_K(Π_r) ⊗ β)$ satisfy
functional equations

$$L(s, Π_r, β) = ε(s, Π_r, β) · L(1 − s, Π_{r}^∨, β^{-1});$$

$$L(s, L^{ss}_K(Π_r) ⊗ β) = ε(s, L^{ss}_K(Π_r) ⊗ β) · L(1 − s, L^{ss}_K(Π_r)^{∨} ⊗ β^{-1}).$$

The local factors of the automorphic and Galois functional equations are
equal outside $S$ by (2.7). Moreover, by stability of $γ$-factors [JS85, De73], if
$β$ is sufficiently ramified at all places in $S$ other than $v_r$, the local factors at
such places are also equal. It follows as in the usual argument that

$$L_{v_r}(s, Π_{r,v_r}, β_{v_r}) = L_{v_r}(s, L^{ss}_K(Π_r) ⊗ β_{v_r}).$$

Now we can choose a global character $β$ such that $β_{v_r} = α_r^{-1}$ and such
that $β$ is sufficiently ramified at all places in $S$ other than $v_r$. By (2.6)
and (2.8), it follows that $L_{v_r}(s, Π_{r,v_r}, β_{v_r})$ has a pole at $s = 0$. But if $τ$
is a supercuspidal representation of $GL(n, F_r)$, then the Godement-Jacquet
local Euler factor $L(s, τ, χ) = 1$ for any character $χ$. Thus $Π_{r,v_r}$ is not
supercuspidal. \qed
3. Sketch of a proof of the local Langlands correspondence

Admitting Hypothesis 2.1, Theorem 2.5 can be used as the starting point of a new proof, or at least a “new proof,” of the local Langlands correspondence for $GL(n)$ in the equal characteristic case. Indeed, the inductive arguments of section 12 of [Sch13] are based entirely on this base change argument and local class field theory – Scholze obtains the bijection after reproving Henniart’s Theorem 1.1 while Henniart obtained the result in the other direction, starting from his numerical correspondence. That the correspondence preserves $L$ and $\varepsilon$ factors of pairs is automatic by global arguments over function fields, specifically the fact that the $L$-functions of Galois representations are known to satisfy the expected functional equations; this has already been used to prove Theorem 2.5.

The sense in which this proof is actually new needs to be spelled out, of course. The result has been known since [LRS93] and can also be derived from [Laf02], and many of the intermediate arguments used to deduce the proof are the same in all cases.

In order to convince the reader that it is possible to prove the local Langlands correspondence for $GL(n)$ without using trace formulas to count points on moduli spaces, we indicate the steps of the proof, starting from Theorem 2.5. This will be a line-by-line review of Scholze’s proof in [Sch13]. We write $\mathcal{L}_{n,F}^{ss}$ for the local parametrization denoted $L_{n,F}^{ss}$ in the previous section.

**Step 3.1.** As $n$ and $F$ vary, the parametrizations $\pi \mapsto \mathcal{L}_{n,F}^{ss}(\pi)$ define a functorial extension of class field theory, in the sense of [Sch13], Theorem 12.1.

**Proof.** There are five conditions to check.

(i) When $\mathcal{L}_{1,F} = \mathcal{L}_{1,F}^{ss}$ coincides with local class field theory.

This is a special case of (iii) of Theorem 2.3.

(ii) The map $\pi \mapsto \mathcal{L}_{n,F}^{ss}(\pi)$ commutes with parabolic induction in the sense that, if $\pi$ is a subquotient of a representation $I(\pi_1 \otimes \cdots \otimes \pi_r)$ induced from a parabolic subgroup with Levi factor $\prod_{i=1}^r GL(n_i)$, then
\[
\mathcal{L}_{n,F}^{ss}(\pi) = \bigoplus_{i=1}^r \mathcal{L}_{n_i,F}^{ss}(\pi_i).
\]
This is the final point of [GL3], Theorem 0.1.

(iii) If $\chi \in \mathcal{A}(1,F)$ then $\mathcal{L}_{n,F}^{ss}(\pi \otimes \chi \circ \text{det}) = \mathcal{L}_{n,F}^{ss}(\pi) \otimes \mathcal{L}_{1,F}(\chi)$. Since this is true for unramified representations, it follows in general by the Chebotarev density argument already used.

(iv) Compatibility with base change: if $\pi \in \mathcal{A}(n,F)$ is a supercuspidal representation and $F'/F$ is a cyclic extension of prime degree, then (for any $K,K''$, $\Pi'$ as in Definition 2.1)
\[
\mathcal{L}_{F'}^{ss}(BC_{F'/F}^{\Pi',K''}(\pi)) = \mathcal{L}_{F'}^{ss}(\pi) |_{W_{F'}}.
\]
This is Theorem 2.4
(v) If $L_{n, F}(\pi)$ is unramified then $\pi$ has an Iwahori fixed vector.

In particular, the proof of Theorem 1.2 for $GL(n)$ is a step in the proof of the local Langlands correspondence. Condition (v) follows from Theorem 2.5, but the proof is not immediate and requires a separate step.

\[ \square \]

**Step 3.2. Proof of condition (v) of Step 3.1**

**Proof.** Suppose $n = \sum_{i=1}^r n_i$, $P \subset GL(n)$ a parabolic subgroup with Levi factor $\prod_i GL(n_i)$, and $\pi$ is a constituent of $Ind_{P(F)}^{GL(n, F)} \otimes_{i=1}^r \sigma_i$, where $\sigma_i$ is a supercuspidal representation of $GL(n_i, F)$ such that $L_{n_i, F}(\sigma_i)$ is unramified. It follows from condition (ii) of Step 3.1 that $L_{n_i, F}(\sigma_i)$ is unramified. On the other hand, if each $\sigma_i$ has an Iwahori-fixed vector then every irreducible constituent of $Ind_{P(F)}^{GL(n, F)} \otimes_{i=1}^r \sigma_i$ has an Iwahori fixed vector. It follows that it suffices to treat the case where $\pi$ is supercuspidal and $L_{n, F}(\pi)$ is unramified, which we assume henceforward; the conclusion will be that $n = 1$.

By Theorem 2.5 we know that there is a sequence of cyclic extensions $F_0 \subset F_1 \subset \cdots \subset F_r$ such that the base change $\pi_r$ of $\pi$ to $F_r$ is an unramified principal series representation. By induction on $r$ and on $n$ we are thus reduced to verifying the following statement: Suppose $\pi \in A(n, F)$ is supercuspidal and suppose $F'/F$ is a cyclic extension of prime degree such that $\Pi = BC_{F'/F}(\pi)$ is an unramified principal series representation. Suppose moreover that $L_{n, F}(\pi)$ is unramified. Then $\pi$ is an unramified principal series representation (and in particular $n = 1$).

We follow the argument in Theorem 12.3 of [Sch13] (which goes back to Henniart). By Step 3.3 it follows that

$$\Pi = \bigoplus_{j=0}^{d-1} \tau^j(\chi)$$

where $\chi$ is a character of $GL(1, F')$ such that $\tau(\chi) \neq \chi$. However, by compatibility with local class field theory and parabolic induction (conditions (i) and (ii) of Step 3.1) we know that $\chi$ is unramified. Since $Gal(F'/F)$ acts trivially on unramified characters, $\pi$ could not have been supercuspidal, and indeed it follows that $\pi$ is necessarily a principal series representation. Since $L_{n, F}(\pi)$ is unramified, it follows again from conditions (i) and (ii) of Step 3.1 that $\pi$ is an unramified principal series representation. \[ \square \]

**Step 3.3. Let $\pi \in A(n, F)$ be supercuspidal and suppose $F'/F$ is a cyclic extension of prime degree such that $\Pi = BC_{F'/F}(\pi)$ is not supercuspidal. Then there is a divisor $m$ of $n$, with $n = md$, and a supercuspidal representation $\Pi_1$ of $GL(m, F')$, such that

$$\Pi = \bigoplus_{j=0}^{d-1} \tau^j(\Pi_1)$$
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where $\tau$ is a generator of $\text{Gal}(F'/F)$, and such that $\tau(\Pi_1) \neq \Pi_1$.

Conversely, if $\Pi_i$ is a supercuspidal representation of $GL(m,F')$ such that $\tau(\Pi_1) \neq \Pi_1$, then $\Pi$, defined as above, descends to a discrete series representation $\pi$ of $GL(n,F)$.

Proof. The first statement is Lemma 6.10 of Chapter I of Arthur-Clozel, [AC]. The proof of this result, which appears in the local part of [AC] depends on the existence of cyclic base change, which we have admitted for function fields (Hypothesis 2.1). However, a close look at the proof of Theorem 6.2 of Chapter I of [AC], on which Lemma 6.10 depends, indicates that it suffices to use the Deligne-Kazhdan simple trace formula, and this is already available for function fields.

The second statement follows from Theorem 6.2 (b) of Chapter I of [AC]. There it is proved that there is at least one $\pi \in A(n,F)$ whose base change to $GL(n,F)$ is $\Pi$. Moreover, the hypothesis $\tau(\Pi_1) \neq \Pi_1$ implies that $\Pi_1$ is $\sigma$-discrete, and an examination of the proof of (b) on p. 53 of [AC] shows that $\pi$ must belong to the discrete series. □

Remark 3.4. The proof of Lemma 6.10 also depends on the Bernstein-Zelevinsky classification of representations of $GL(n,F)$, and thus on the complete local theory for $GL(n)$. The normalization of the twisted trace formula in Chapter I, §2 of [AC] makes use of the Whittaker model. The arguments here thus do not extend to groups other than $GL(n)$; however, at no point have we used the Lefschetz formula to study the cohomology of moduli spaces.

Step 3.5. [Local Langlands bijection] The map $\pi \mapsto L_{n,F}^{\text{ss}}(\pi)$ restricts to a bijection between supercuspidal representations of $GL(n,F)$ and irreducible $n$-dimensional representations of the Weil group $W_F$.

Proof. At this point we are ready to argue as in the proof of Theorem 12.3 of [Sch13]. Scholze’s argument is a formal consequence of Steps 3.1 and 3.3. In particular, Scholze’s induction argument (a version of which goes back to [He88]) implies that the representation $\Pi_{m-1}$ that appears in the middle of his proof of Theorem 12.3, and that is guaranteed to belong to the discrete series by Step 3.3, is necessarily supercuspidal. □

Step 3.6. [Compatibility of local factors] The bijection of Step 3.5 preserves $L$ and $\varepsilon$ factors of pairs.

Proof. This is simpler in positive characteristic than for $p$-adic fields, because the $L$-function of a representation of the Galois group of a function field is always known to have a meromorphic continuation with functional equation. Thus the stability of local $\gamma$-factors can be used to verify the compatibility of local factors, as in the proof of Theorem 2.5. □

3.6.1. Comments on base change in positive characteristic. The book [AC] makes use of the full strength of the twisted trace formula for $GL(n)$. At
present there is no reference for the twisted trace formula in positive characteristic. As mentioned above, it seems likely that Hypothesis 2.1 can be verified using only a twisted version of the Deligne-Kazhdan simple trace formula. In particular, the vague version of base change asserted in Hypothesis 2.1, which suffices to prove the splitting theorem, can probably be proved on the basis of results available in the literature.

On the other hand, the version of local base change used in Step 3.3 appears to require the precise characterization of base change by local character identities. In order to deduce the local Langlands correspondence, one should therefore add this to characterization to Hypothesis 2.1.

4. Hypothetical structures

Let $G$ be a connected reductive group over the global field $K$, which need not be a function field. Labesse [Lab99] has established a version of the following hypothesis over number fields; in what follows, we assume it is also available for function fields.

**Hypothesis 4.1.** (i) Let $K'/K$ be a finite cyclic extension. Let $G_0(G, K)$, resp. $G_0(G, K')$, denote the set of cuspidal automorphic representations of $G_K$, resp. $G_{K'}$. There is a (non-empty) subset $G_{\text{simple}}(G, K) \subset G_0(G, K)$ with the following property. For any $\Pi \in G_{\text{simple}}(G, K)$, there is a non-empty set $BC(\Pi) \subset G_0(G, K')$ with the following property: for every place $v$ of $K$ at which both $\Pi$ and $K'$ are unramified, and for any place $v'$ of $K'$ dividing $v$, we have $\Pi_v \sim BC(\Pi_v)$ for any $\Pi' \in BC(\Pi)$, where $BC(\Pi_v)$ is the base change of $\Pi_v$ under the local Langlands correspondence for unramified representations.

(ii) Suppose $P \subset G$ is a rational parabolic subgroup with Levi factor $M$. Suppose the global base change maps are defined for appropriate subsets $G_{\text{simple}}(M, K) \subset G_0(M, K)$ and $G_{\text{simple}}(G, K) \subset G_0(G, K)$. Then the maps $\Pi \mapsto BC(\Pi)$ are compatible locally everywhere with parabolic induction from $M$ to $G$.

The subscript _simple_ refers to the use of Arthur’s simple trace formula (STF) to establish global base change. In [Lab99], $K$ is a number field and the subset $G_{\text{simple}}(G, K)$ consists of representations that are Steinberg at some number of places (at least two). This provides a sufficient condition to permit the application of the STF – in other words to eliminate the difficult parabolic terms from both sides of the invariant trace formula. The Steinberg condition also eliminates the endoscopic terms, thus substantially simplifying the comparison of trace formulas that implies Labesse’s result. We assume

**Hypothesis 4.2.** Labesse’s method works over function fields as well and we assume $G_{\text{simple}}(G, K)$ always includes at least the representations in Labesse’s class.
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It seems likely that this Hypothesis, combined with the general theory of automorphic representations of $GL(n)$, can provide a substitute for Hypothesis 2.1 in the discussion above for $GL(n)$, but we have not checked the details.

Since we are not making any assumptions about packets for groups other than $GL(n)$, we introduce the following definition.

**Definition 4.1.** Let $F'/F$ be a finite cyclic extension of non-archimedean local fields. Let $\pi$ and $\pi'$ be irreducible admissible representations of $G(F)$ and $G(F')$, respectively. We say that $\pi'$ is a base change of $\pi$ if there exists a finite cyclic extension $K'/K$ of global fields, a place $v$ of $K$ such that $K_v \sim F$ and $K'_v \sim F'$, and a cuspidal automorphic representation $\Pi \in \mathcal{G}_{\text{simple}}(G,K)$ such that $\Pi_v \sim \pi$ and, for some $\Pi' \in BC(\Pi)$, we have $\Pi'_v \sim \pi'$.

More generally, if $F = F_0 \subset F_1 \subset \cdots \subset F_r = F'$ is a sequence of finite cyclic extensions, with $\pi$ and $\pi'$ as above., we say that $\pi'$ is a base change of $\pi$ if there is a sequence $\pi_i$, with $\pi_0 = \pi$ and $\pi_r = \pi'$, such that $\pi_i$ is a base change of $\pi_{i-1}$ for each $i \geq 1$ in the above sense.

The set of base changes of $\pi$ defined in this way thus may depend on the choice of global extension $K'/K$, as well as on the intermediate extensions in the setting of the second paragraph. In particular, the set of base changes is potentially infinite.

**Definition 4.3.** Let $\pi \in \mathcal{G}(G,F)$ be a supercuspidal representation of $G(F)$. We say $\pi$ is incorrigible if, for any sequence $F = F_0 \subset F_1 \subset \cdots \subset F_r = F'$ of cyclic extensions, there is a supercuspidal representation $\pi' \in \mathcal{G}(G,F')$ that is a base change of $\pi$.

Say an admissible irreducible representation $\sigma$ of $G$ is pure if its Genestier-Lafforgue parameter $\phi_{\sigma} : \Gamma_F \to L(G(C)$ has the property that, for any Frobenius element $Frob \in \Gamma_F$, the eigenvalues of $\phi_{\sigma}(Frob)$ are all Weil numbers of the same weight. Corresponding to Theorems 1.1 and 1.2 we have the following Conjectures for general groups.

**Conjecture 4.4.** There are no pure incorrigible supercuspidal representations.

The purity hypothesis is meant to exclude cuspidal unipotent representations from consideration.

The following refinement of Conjecture 4.4 should be a consequence of a version of the local Langlands conjecture for $G$ that includes compatibility with parabolic induction.

**Conjecture 4.5.** Let $\pi \in \mathcal{G}(G,F)$ be a pure supercuspidal representation. There is a finite sequence of cyclic extensions $F = F_0 \subset F_1 \subset \cdots \subset F_r = F'$ such that every base change of $\pi$ to $F'$ contains an Iwahori-fixed vector.
5. Classical groups

With local base change defined loosely as in the previous section, we show that the doubling method of Garrett and Piatetski-Shapiro-Rallis, as refined by Lapid and Rallis in [LR05], allows us to prove Conjecture 4.4 over local fields of positive characteristic. We assume $G$ is a split classical group over the global function field $K$, of the form $Sp(2n)$ or $SO(V)$ for a vector space $V$ over $K$ with a non-degenerate symmetric bilinear form $b_V$. We let $F = K_v$ for some place $v$.

**Theorem 5.1.** Assume Hypotheses 4.1 and 4.2 hold for $G$ over $K$. Then $G$ has no incorrigible pure supercuspidal representations.

**Proof.** We use the local theory of the doubling integral, as worked out in complete detail by Lapid and Rallis in [LR05] (see Remark 5.1). If $\pi$ is an irreducible admissible representation of $G(E)$ for some local field $E$, $\psi : E \to \mathbb{C}^\times$ a continuous character, and $\omega : E^\times \to \mathbb{C}^\times$ a continuous (quasi)-character, we let $L(s, \pi, \omega)$ and $\varepsilon(s, \pi, \omega, \psi)$ denote the local $L$ and $\varepsilon$ factors associated to the data by the doubling method of [PSR87]. Similarly, if $L$ is a global field, $\Pi$ is a cuspidal automorphic representation of $G(L)$, and $\chi$ is a Hecke character of $L^\times \backslash A_L^\times$, then we let $L(s, \Pi, \chi) = \prod_w L(s, \Pi_w, \chi_w)$, where $w$ runs over places of $L$, and define the global $\varepsilon$ factor similarly. We fix a global additive character $\Psi = \otimes_w \psi_w : K_{ad}/K \to \mathbb{C}^\times$.

For any finite separable extension $K'/K$ we let $\Psi_{K'} = \Psi \circ Tr_{K'/K}$ (but in general we drop the subscript $K'$).

Now Theorem 4 of [LR05], together with the results listed below, implies that the local and global $L$-functions provided by the doubling method satisfy the following conditions.

(i) Let $\rho : L^\times G \to GL(V_\rho)$ denote the standard representation of the Langlands $L$-group of $G$, and let $d(\rho) = \dim V_\rho$. Suppose $\sigma$ is an unramified representation of $G(K)$ for some $w$ and $\alpha$ is an unramified character of $K_w^\times$. Then $L(s, \sigma, \alpha) = L(s, \sigma, \alpha, \rho)$ is the unramified Langlands Euler factor attached to $\pi$, $\chi$, and $\rho$.

(ii) The global $L$-function $L(s, \pi, \chi) = \prod_w L(s, \pi_w, \chi_w)$ has a meromorphic continuation with at most finitely many poles, and satisfies a global functional equation

$$L(1-s, \pi^\vee, \chi^{-1}) = \varepsilon(s, \pi, \chi)L(s, \pi, \chi).$$

(iii) Fix $w$ and define the local gamma-factor

$$\gamma(s, \pi_w, \chi_w, \psi_w) = \varepsilon(s, \pi_w, \chi_w, \psi_w) \frac{L(1-s, \pi_w^\vee, \chi_w^{-1})}{L(s, \pi_w, \chi_w)}.$$ 

Given two irreducible admissible representations $\pi_{1,w}$ and $\pi_{2,w}$, there is an integer $N$ such that, for all $\chi_w$ of conductor at least $N$, we
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have
\[ \gamma(s, \pi_1, \chi, \psi) = \gamma(s, \pi_2, \chi, \psi). \]

(iv) When \( \pi \) is supercuspidal and \( \omega \) is a unitary character, then the local factor \( L(s, \pi, \omega) \) is holomorphic for \( \Re(s) \geq \frac{1}{2} \). In particular, there is no cancellation between the poles of the numerator and denominator of the local gamma-factor \( \gamma(s, \pi, \omega) \).

(v) There is an integer \( d < d(\rho) \) such that, if \( \pi \) is supercuspidal then the there are at most \( d \) characters \( \omega \) for which the local factor \( L(s, \pi, \omega) \) has a pole, and each such pole is simple. In particular, the local factor \( L(s, \pi, \omega) \) has at most \( d \) poles (counted with multiplicity).

Point (iii) is the stability property proved by Rallis and Soudry ([RS05]; see also [Br08] for the analogous case of unitary groups). Point (iv) is a consequence of Proposition 5 of [LR05]. Point (v) is addressed at the end of the proof of Theorem 5.2 of [Y]. It is shown there that all the poles of the local Euler factor of a supercuspidal representation are poles of the local Euler factor there denoted \( b(s, \chi) \). An examination of the list on p. 667 of [Y] confirms that \( b(s, \chi) \) has fewer than \( d(\rho) \) (fewer than \( \frac{d(\rho)+1}{2} \), in fact) and that they are all simple. (See Remark 5.2 below.)

Now suppose \( \pi \) is pure supercuspidal. By a simple reduction, using property (iv) of Corollary 2.3, we may assume \( \pi \) has central character of finite order. Now suppose \( \pi \) is also incorrigible. Following 2.2, we can globalize \( \pi \) to a cuspidal automorphic representation \( \Pi \) of \( G_K \) with central character of finite order. Let \( \phi_\Pi : \Gamma_K \to L^G(C) \) denote its semisimple Langlands-Lafforgue parameter, \( \phi_\pi : \Gamma_F \to L^G(C) \) the local parameter, \( \rho \circ \phi_\pi^{ss} \) the Frobenius-semisimplification of its composition with \( \rho \). Since \( \pi \) is pure, Grothendieck’s monodromy theorem for representations of Weil groups of local fields implies that the image of \( \rho \circ \phi_\pi^{ss} \) is of finite order, and necessarily solvable. We can thus find a sequence \( K = K_0 \subset K_1 \subset \ldots K_i \subset K_{i+1} \subset K_f = K' \) of cyclic Galois extensions such that for every prime \( v' \) of \( K' \) dividing \( v \), the image of the decomposition group \( \Gamma_{v'} \) under the restriction of \( \rho \circ \phi_\pi^{ss} \) to \( \Gamma_K \) is trivial. Since \( \pi \) is incorrigible, and by Hypothesis 4.1 (i), it follows from our hypotheses that there is an automorphic representation \( \pi' \) of \( G_{K'} \) which is supercuspidal at some prime \( v' \) of \( K' \) dividing \( v \), and whose Langlands-Lafforgue parameter satisfies

\[ \phi_{\pi'} = \phi_\pi |_{\Gamma_{K'}}. \]

Now \( \rho \circ \phi_\Pi \Gamma_{K'} \) is a continuous representation of the Galois group of the global function field \( K' \). Thus for any Hecke character \( \chi \), \( L(s, \rho \circ \phi_\Pi, \chi) \) satisfies a functional equation of Galois type. By the stability property (iii), the local \( \gamma \) factors \( \gamma(s, \chi, \psi) \) at all places \( w \) of \( K' \) coincide with their Galois analogues. In particular, letting \( \omega_1 \) denote the trivial character of \( K' \), the local Euler factor

\[ L(s, \pi', \omega_1) = L(s, \phi_{\pi'} |_{\Gamma_{K'}}, \omega_1) \]

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has a pole of multiplicity \(d(\rho)\) at \(s = 1\). But this contradicts point (v). Thus \(\pi'\) cannot be supercuspidal. \(\square\)

Remark 5.1. There are no references for the doubling method over function fields. The proof of Theorem 5.1 assumes that it works in exactly the same way for function fields as in [PSR87, LR05]. The Langlands-Shahidi method has been developed by Lomelí over function fields. Section 7 of [Lo18] contains details about the Langlands-Shahidi local factors for classical groups; note that special care has to be taken in characteristic 2. Since we have no information a priori about generic representations (the results of [GV] on the tempered packet conjecture rely on Arthur’s results in [A], which we have deliberately chosen not to use), the Langlands-Shahidi method is not available to us.

Remark 5.2. The possible poles of local Euler factors of supercuspidal representations of unitary groups are determined in Theorem 6.2 of [HKS]. They correspond as expected to the classification of cuspidal unipotent representations. In particular, it follows from the characterization in [HKS] that the local Euler factor of a pure supercuspidal representation has at most a single simple pole. (It follows from the actual classification that a supercuspidal representation whose standard local \(L\)-factor has even one pole cannot be pure, but it would be circular to admit the actual classification at this stage of the argument.) The case of orthogonal and symplectic groups is not made explicit in [Y] but the argument of [HKS], applied to his computations, suffices to verify the claim.

Of course, if we are assuming Hypothesis 4.1 then we could also assume the full stable twisted trace formula, as in Arthur’s book [A], and then Proposition 5.1 follows from the case of \(GL(n)\) by Arthur’s trace formula arguments. However, base change, in the form of Hypothesis 4.2, is considerably simpler to manage.

6. Formal degree and incorrigible representations

The article [HI108] of Hiraga, Ichino, and Ikeda proposed an explicit conjectural formula for the formal degrees of discrete series representations of reductive groups over local fields and a related formula for the Plancherel measure. The formula has been proved in a great many cases but remains open, with few results known for exceptional groups – which is hardly surprising, since the conjecture is formulated in terms of the local Langlands parametrization. In this final section we show that Conjecture 4.4 for groups over fields of positive characteristic is a simple consequence of the Hiraga-Ichino-Ikeda conjecture. For the local Langlands parametrization we take the one defined in [GLa].

The careful reader may object that [HI108] only states a conjecture for groups over local fields of characteristic zero, but Ichino has assured us
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that the conjecture can be stated just as well for groups over fields of positive characteristic. The assumption of characteristic zero was only made in [HII08] because the proofs given there in specific examples were based on methods that at the time were only available for $p$-adic fields. (Presumably the article [Lo18] allows for an extension of the proofs in [HII08] to positive characteristic.)

We recall a simplified version of the conjecture of Hiraga-Ichino-Ikeda.

**Conjecture 6.1** ([HII08], Conjecture 1.4). Let $\phi : \text{Gal}(\bar{F}/F) \to L^G$ be an elliptic tempered Langlands parameter. Then for any $\pi$ in the $L$-packet of $\phi$, the formal degree $d(\pi)$ is a non-zero constant multiple of the $\gamma$-factor

$$|\gamma(0, Ad \circ \phi, \psi)| = |\varepsilon(0, Ad \circ \phi, \psi)| \frac{L(1 - s, Ad \circ \hat{\phi})}{L(s, Ad \circ \phi)}.$$

In the statement, $Ad$ is the adjoint representation of $L^G$ on its Lie algebra and $\hat{\phi}$ is the parameter of the contragredient $\hat{\pi}$. The formula in [HII08] is completely explicit; the non-zero constant reflects the place of $\pi$ in its $L$-packet.

**Proposition 6.1.** Conjecture 6.1 for supercuspidal representations implies Conjecture 4.4.

**Proof.** Suppose $\phi$ is pure unramified and tempered. Then the centralizer $Z_{\hat{G}}(\phi)$ of $\phi$ in $\hat{G}$ is of positive dimension, so $Ad \circ \phi$ acts trivially on a positive-dimensional subspace of the Lie algebra of $\hat{G}$. On the other hand, since $Ad \circ \phi$ is pure of weight 0, the factor $L(1 - s, Ad \circ \phi)$ has no pole at $s = 0$. It then follows that $L(s, Ad \circ \phi)$ has a pole at $s = 0$, so $|\gamma(0, Ad \circ \phi, \psi)| = 0$. Thus no multiple of $|\gamma(0, Ad \circ \phi, \psi)|$ can be the formal degree of a discrete series representation.

Now suppose $\pi$ is a pure supercuspidal representation of $G(F)$, with Genestier-Lafforgue parameter $\phi$. Let $F'/F$ be a (solvable) Galois extension such that $\phi' = \phi|_{F'}$ is unramified. Any base change $\pi'$ of $\pi$ to $G(F')$ then has Genestier-Lafforgue parameter $\phi'$, by the Chebotarev density argument we have already used. Thus by the argument of the previous paragraph, the Hiraga-Ichino-Ikeda Conjecture 6.1 implies that $\pi'$ cannot be a discrete series representation.

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