CLASSICAL LIMIT OF QUANTUM BORCHERDS-BOZEC ALGEBRAS

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Abstract. Let $g$ be a Borcherds-Bozec algebra, $U(g)$ be its universal enveloping algebra and $U_q(g)$ be the corresponding quantum Borcherds-Bozec algebra. We show that the classical limit of $U_q(g)$ is isomorphic to $U(g)$ as Hopf algebras. Thus $U_q(g)$ can be regarded as a quantum deformation of $U(g)$. We also give explicit formulas for the commutation relations among the generators of $U_q(g)$.

Introduction

The quantum Borcherds-Bozec algebras were introduced by T. Bozec in his research of perverse sheaves theory for quivers with loops [1, 2, 3]. They can be treated as a further generalization of quantum generalized Kac-Moody algebras. Even though they use the same Borcherds-Cartan data, the construction of the quantum groups are quite different.

More precisely, the quantum Borcherds-Bozec algebras have more generators and defining relations than quantum generalized Kac-Moody algebras. For each simple root $\alpha_i$ with imaginary index, there are infinitely many generators $e_{il}, f_{il}(l \in \mathbb{Z}_{>0})$ whose degrees are $l$ multiples of $\alpha_i$ and $-\alpha_i$. Bozec deals with these generators by treating them as similar positions as divided powers $\theta_i^{(l)}$ in Lusztig algebras.

Bozec gave the general definition of Lusztig sheaves for arbitrary quivers (possibly with multiple loops) and constructed the canonical basis for the positive half of a quantum Borcherds-Bozec algebra in terms of simple perverse sheaves (cf. [15]). In [2], he studied the crystal basis theory for quantum Borcherds-Bozec algebras. He defined the notion of Kashiwara operators and abstract crystals, which provides an important framework for Kashiwara’s grand-loop argument (cf. [11]). He also gave a geometric construction of the crystal for the negative half of a quantum Borcherds-Bozec algebra based on the theory of Lusztig perverse sheaves associated to quivers with loops (cf. [12, 9]), and gave a geometric

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realization of generalized crystals for the integrable highest weight representations via Nakajima’s quiver varieties (cf. [16, 10]).

For a Kac-Moody algebra \( g \), G. Lusztig showed that the integrable highest weight module \( L \) over \( U(\mathfrak{g}) \) can be deformed to those integrable highest weight module \( L \) over \( U_q(\mathfrak{g}) \) in such a way that the dimensions of weight spaces are invariant under the deformation (cf. [14, 6]). Let \( \mathcal{A} = \mathbb{Q}[q, q^{-1}] \) be the Laurent polynomial rings, Lusztig constructed a \( \mathcal{A} \)-subalgebra \( U_\mathcal{A} \) of \( U_q(\mathfrak{g}) \) generated by divided powers and \( k_i^\pm \), and defined a \( U_\mathcal{A} \)-submodule \( L_\mathcal{A} \) of \( L \). He proved that \( F_0 \otimes_{\mathcal{A}} L_\mathcal{A} \) is isomorphic to \( L \) as \( U(\mathfrak{g}) \)-modules, where \( F_0 = \mathcal{A} / I \) and \( I \) is the ideal of \( \mathcal{A} \) generated by \( (q - 1) \).

In [5, Chapter 3], J. Hong and S.-J. Kang modified Lusztig’s approach to show that the \( U_q(\mathfrak{g}) \) is a deformation of \( U(\mathfrak{g}) \) as a Hopf algebra and show that a highest weight \( U(\mathfrak{g}) \)-module admits a deformation to a highest weight \( U_q(\mathfrak{g}) \)-module. They used the \( \mathbb{A}_1 \)-form of \( U_q(\mathfrak{g}) \) and highest weight \( U_q(\mathfrak{g}) \)-module, where \( \mathbb{A}_1 \) is the localization of \( \mathbb{Q}[q] \) at the ideal \((q - 1)\). We can see that \( \mathcal{A} = \mathbb{Q}[q, q^{-1}] \subseteq \mathbb{A}_1 \).

In this paper, we study the classical limit theory of quantum Borcherds-Bozec algebras. We first review some basic notions of Borcherds-Bozec algebras and quantum Borcherds-Bozec algebras. For their representation theory, the readers may refer to [7, 8]. As we show in Appendix, the commutation relations between \( e_{il} \) and \( f_{jk} \) are rather complicated. For the aim of classical limit, we need another set of generators. Thanks to Bozec, there exists an alternative set of primitive generators in \( U_q(\mathfrak{g}) \), which we denote by \( s_{il} \) and \( t_{il} \). They satisfy a simpler set of commutation relations

\[ s_{il} t_{jk} - t_{jk} s_{il} = \delta_{ij} \delta_{lk} \tau_{il} (K_i^l - K_i^{-l}) \]

for some constants \( \tau_{il} \in \mathbb{Q}(q) \). Using Lusztig’s approach, we prove that these generators also satisfy the Serre-type relations (cf. [13, Chapter 1]).

In Section 3, we define the \( \mathbb{A}_1 \)-form of quantum Borcherds-Bozec algebras and their highest weight representations. We show that the triangular decomposition of \( U_q(\mathfrak{g}) \) carries over to \( \mathbb{A}_1 \)-form. In Section 4, we study the process of taking the limit \( q \to 1 \). Let \( U_1 = \mathbb{Q} \otimes_{\mathbb{A}_1} U_{\mathbb{A}_1} \) be a \( \mathbb{Q} \)-algebra, where \( U_{\mathbb{A}_1} \) is the \( \mathbb{A}_1 \)-form of \( U_q(\mathfrak{g}) \). We prove that the classical limit \( U_1 \) of \( U_q(\mathfrak{g}) \) is isomorphic to the universal enveloping algebra \( U(\mathfrak{g}) \) as Hopf algebras, and when we take the classical limit, the Verma module and highest weight modules of \( U_q(\mathfrak{g}) \) tend to those Verma module and highest weight modules of \( U(\mathfrak{g}) \), respectively. Finally, we give the concrete commutation relations between the generators \( e_{il} \) and \( f_{jk} \) of \( U_q(\mathfrak{g}) \) in Appendix, they have an interesting combinatorial structure.

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1. Borcherds-Bozec algebras

Let $I$ be an index set possibly countably infinite. An integer-valued matrix $A = (a_{ij})_{i,j \in I}$ is called an even symmetrizable Borcherds-Cartan matrix if it satisfies the following conditions:

(i) $a_{ii} = 2, 0, -2, -4, \ldots$,
(ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
(iii) there is a diagonal matrix $D = \text{diag}(r_i \in \mathbb{Z}_{>0} | i \in I)$ such that $DA$ is symmetric.

Set $I^{re} := \{i \in I | a_{ii} = 2\}$, the set of real indices and $I^{im} := \{i \in I | a_{ii} \leq 0\}$, the set of imaginary indices. We denote by $I^{iso} := \{i \in I | a_{ii} = 0\}$ the set of isotropic indices.

A Borcherds-Cartan datum consists of

(a) an even symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$,
(b) a free abelian group $P^\vee = (\bigoplus_{i \in I} \mathbb{Z} h_i) \oplus (\bigoplus_{i \in I} \mathbb{Z} d_i)$, the dual weight lattice,
(c) $\mathfrak{h} = \mathbb{Q} \otimes \mathbb{Z} P^\vee$, the Cartan subalgebra,
(d) $P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subseteq \mathbb{Z}\}$, the weight lattice,
(e) $P^\vee = \{h_i \in P^\vee | i \in I\}$, the set of simple coroots,
(f) $\Pi = \{\alpha_i \in P | i \in I\}$, the set of simple roots, which is linearly independent over $\mathbb{Q}$ and satisfies

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_i) = \delta_{ij} \quad \text{for all } i, j \in I.$$ 

(g) for each $i \in I$, there is an element $\Lambda_i \in P$ such that

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d_j) = 0 \quad \text{for all } i, j \in I.$$ 

The $\Lambda_i(i \in I)$ are called the fundamental weights.

We denote by

$$P^+ := \{\lambda \in P | \lambda(h_i) \geq 0 \text{ for all } i \in I\}$$

the set of dominant integral weights. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ is called the root lattice. Set $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $Q_- = -Q_+$. For $\beta = \sum k_i \alpha_i \in Q_+$, we define its height to be $\text{ht}(\beta) := \sum k_i$. 
There is a non-degenerate symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \lambda) = r_i \lambda(h_i) \text{ for all } \lambda \in \mathfrak{h}^*,$$

and therefore we have

$$(\alpha_i, \alpha_j) = r_ia_{ij} = r_ia_{ji} \text{ for all } i, j \in I.$$  

For $i \in I^r$, we define the simple reflection $\omega_i \in GL(\mathfrak{h}^*)$ by

$$\omega_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \text{ for } \lambda \in \mathfrak{h}^*.$$  

The subgroup $W$ of $GL(\mathfrak{h}^*)$ generated by $\omega_i$ ($i \in I^r$) is called the Weyl group of $g$. One can easily verify that the symmetric bilinear form $(\ , \ )$ is $W$-invariant.

Let $I^\infty := (I^r \times \{1\}) \cup (I^m \times \mathbb{Z}_{>0})$. For simplicity, we will often write $i$ for $(i, 1)$ if $i \in I^r$.

**Definition 1.1.** The Borcherds-Bozec algebra $g$ associated with a Borcherds-Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the Lie algebra over $\mathbb{Q}$ generated by the elements $e_{il}, f_{il} ((i, l) \in I^\infty)$ and $\mathfrak{h}$ with defining relations

$$[h, h'] = 0 \text{ for } h, h' \in \mathfrak{h},$$

$$[e_{ik}, f_{jl}] = k \delta_{ij} \delta_{kl} h_i \text{ for } i, j \in I, k, l \in \mathbb{Z}_{>0},$$

$$[h, e_{jl}] = l\alpha_j(h)e_{jl}, \quad [h, f_{jl}] = -l\alpha_j(h)f_{jl},$$

$$(ad e_{ij})^{1-lai}(e_{jl}) = 0 \text{ for } i \in I^r, i \neq (j, l),$$

$$(ad f_{ij})^{1-lai}(f_{jl}) = 0 \text{ for } i \in I^r, i \neq (j, l),$$

$$[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \text{ for } a_{ij} = 0.$$  

Let $U(g)$ be the universal enveloping algebra of $g$. Since we have the following equations in $U(g)$

$$(adx)^m(y) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^{m-k}yx^k \text{ for } x, y \in U(g), m \in \mathbb{Z}_{>0},$$

we obtain the presentation of $U(g)$ with generators and relations given below.

**Proposition 1.2.** The universal enveloping algebra $U(g)$ of $g$ is an associative algebra over $\mathbb{Q}$ with unity generated by $e_{il}, f_{il} ((i, l) \in I^\infty)$ and $\mathfrak{h}$ subject to the following defining
relations
\[ hh' = h'h \quad \text{for} \quad h, h' \in \mathfrak{h}, \]
\[ e_{ik}f_{jl} - f_{jl}e_{ik} = k \delta_{ij} \delta_{kl} h_i \quad \text{for} \quad i, j \in I, k, l \in \mathbb{Z}_{>0}, \]
\[ he_{jl} - e_{jl}h = l\alpha_{jl}(h) e_{jl}, \quad hf_{jl} - f_{jl}h = -l\alpha_{jl}(h) f_{jl}, \]
\[ (1.2) \]
\[ \sum_{k=0}^{1-l_{aij}} (-1)^k \binom{1-l_{aij}}{k} e_i^{1-l_{aij}-k} e_{jl} e_i^k = 0 \quad \text{for} \quad i \in I^e, i \neq (j, l), \]
\[ \sum_{k=0}^{1-l_{aij}} (-1)^k \binom{1-l_{aij}}{k} f_i^{1-l_{aij}-k} f_{jl} f_i^k = 0 \quad \text{for} \quad i \in I^e, i \neq (j, l), \]
\[ e_{ik}e_{jl} - e_{jl}e_{ik} = f_{ik}f_{jl} - f_{jl}f_{ik} = 0 \quad \text{for} \quad a_{ij} = 0. \]

The universal enveloping algebra \( U(\mathfrak{g}) \) has a Hopf algebra structure given by
\[ \Delta(x) = x \otimes 1 + 1 \otimes x, \]
\[ \varepsilon(x) = 0, \]
\[ S(x) = -x \quad \text{for} \quad x \in \mathfrak{g}, \]
where \( \Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) is the comultiplication, \( \varepsilon : U(\mathfrak{g}) \to \mathbb{Q} \) is the counit, and \( S : U(\mathfrak{g}) \to U(\mathfrak{g}) \) is the antipode.

Furthermore, by the Poincaré-Brikhoff-Witt Theorem, the universal enveloping algebra also has the triangular decomposition
\[ (1.3) \]
\[ U(\mathfrak{g}) \cong U^-(\mathfrak{g}) \otimes U^0(\mathfrak{g}) \otimes U^+(\mathfrak{g}), \]
where \( U^+(\mathfrak{g}) \) (resp. \( U^0(\mathfrak{g}) \) and \( U^-(\mathfrak{g}) \)) be the subalgebra of \( U(\mathfrak{g}) \) generated by the elements \( e_{il} \) (resp. \( \mathfrak{h} \) and \( f_{il} \)) for \( (i, l) \in I^\infty \).

In [7], Kang studied the representation theory of the Borcherds-Bozec algebras. We list some results that we will use later.

**Proposition 1.3.** [7] \( \]
(a) Let \( \lambda \in P^+ \) and \( V(\lambda) = U(\mathfrak{g})v_\lambda \) be the irreducible highest weight \( \mathfrak{g} \)-module. Then we have
\[ f_i^{\lambda(h_i)+1}v_\lambda = 0 \quad \text{for} \quad i \in I^e, \]
\[ f_i v_\lambda = 0 \quad \text{for} \quad (i, l) \in I^\infty \quad \text{with} \quad \lambda(h_i) = 0. \]
(b) Every highest weight \( \mathfrak{g} \)-module with highest weight \( \lambda \in P^+ \) satisfying (1.5) is isomorphic to \( V(\lambda) \).
Let $q$ be an indeterminate and set
\[ q_i = q^{r_i}, \quad q^{(i)} = q^{\frac{\alpha_i \cdot \alpha_i}{2}}. \]
Note that $q_i = q^{(i)}$ if $i \in I^r$. For each $i \in I^r$ and $n \in \mathbb{Z}_{\geq 0}$, we define
\[ [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \left[ \frac{n}{k} \right]_i = \frac{[n]_i!}{[k]_i! [n-k]_i!} \]
and set $F_A = \{ x \in F \mid |x| \in A \}$.

Let $\mathcal{F} = \mathbb{Q}(q) \langle f_{il} \mid (i, l) \in I^\infty \rangle$ be the free associative algebra over $\mathbb{Q}(q)$ generated by the symbols $f_{il}$ for $(i, l) \in I^\infty$. By setting $\deg f_{il} = -l\alpha_i$, $\mathcal{F}$ becomes a $\mathbb{Q}_-$-graded algebra. For a homogeneous element $u$ in $\mathcal{F}$, we denote by $|u|$ the degree of $u$, and for any $A \subseteq \mathbb{Q}_-$, set $\mathcal{F}_A = \{ x \in \mathcal{F} \mid |x| \in A \}$.

We define a twisted multiplication on $\mathcal{F} \otimes \mathcal{F}$ by
\[ (x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-(|x_2|_l|y_1|_l)} x_1 y_1 \otimes x_2 y_2, \]
and equip $\mathcal{F}$ with a co-multiplication $\delta$ defined by
\[ \delta(f_{il}) = \sum_{m+n=l} q_{il}^{-mn} f_{im} \otimes f_{in} \text{ for } (i, l) \in I^\infty. \]

Here, we understand $f_{i0} = 1$ and $f_{il} = 0$ for $l < 0$.

**Proposition 2.1.** [1, 2] For any family $\nu = (\nu_{il})_{(i, l) \in I^\infty}$ of non-zero elements in $\mathbb{Q}(q)$, there exists a symmetric bilinear form $(\ , \ )_L : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{Q}(q)$ such that
(a) $(x, y)_L = 0$ if $|x| \neq |y|$,  
(b) $(1, 1)_L = 1$,  
(c) $(f_{il}, f_{il})_L = \nu_{il}$ for all $(i, l) \in I^\infty$,  
(d) $(x, yz)_L = (\delta(x), y \otimes z)_L$ for all $x, y, z \in \mathcal{F}$.

Here, $(x_1 \otimes x_2, y_1 \otimes y_2)_L = (x_1, y_1)_L(x_2, y_2)_L$ for any $x_1, x_2, y_1, y_2 \in \mathcal{F}$.

From now on, we assume that
\[ \nu_{il} \in 1 + q\mathbb{Z}_{\geq 0}[[q]] \text{ for all } (i, l) \in I^\infty. \]

Then, the bilinear form $(\ , \ )_L$ is non-degenerate on $\mathcal{F}(i) = \bigoplus_{l \geq 1} \mathcal{F}_{-l\alpha_i}$ for $i \in I^{im} \setminus I^{iso}$. 
Let \( \hat{U} \) be the associative algebra over \( \mathbb{Q}(q) \) with \( \mathbf{1} \) generated by the elements \( q^h \) \( (h \in P^\vee) \) and \( e_{il}, f_{il} \) \( ((i,l) \in I^\infty) \) with defining relations

\[
q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee,
\]

\[
q^h e_{jl} q^{-h} = q^{\alpha_j(h)} e_{jl}, \quad q^h f_{jl} q^{-h} = q^{-\alpha_j(h)} f_{jl} \quad \text{for } h \in P^\vee, (j,l) \in I^\infty.
\]

(2.2)

\[
\sum_{k=0}^{1-l a_{ij}} (-1)^k \left[ 1 - \frac{la_{ij}}{k} \right] e_i^{1-l a_{ij}-k} e_{jl}^{k} = 0 \quad \text{for } i \in I^e, i \neq (j,l),
\]

\[
\sum_{k=0}^{1-l a_{ij}} (-1)^k \left[ 1 - \frac{la_{ij}}{k} \right] f_i^{1-l a_{ij}-k} f_{jl}^{k} = 0 \quad \text{for } i \in I^e, i \neq (j,l),
\]

\[
e_{ik} e_{jl} - e_{jl} e_{ik} = f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \quad \text{for } a_{ij} = 0.
\]

We extend the grading by setting \( |q^h| = 0 \) and \( |e_{il}| = l \alpha_i \).

The algebra \( \hat{U} \) is endowed with the co-multiplication \( \Delta: \hat{U} \to \hat{U} \otimes \hat{U} \) given by

\[
\Delta(q^h) = q^h \otimes q^h,
\]

\[
\Delta(e_{il}) = \sum_{m+n=l} q^{mn}_{(i)} e_{im} \otimes K_i^{-m} e_{in},
\]

\[
\Delta(f_{il}) = \sum_{m+n=l} q^{-mn}_{(i)} f_{im} K_i^n \otimes f_{in},
\]

(2.3)

where \( K_i = q^{h_i}(i \in I) \).

Let \( \hat{U}^{\leq 0} \) be the subalgebra of \( \hat{U} \) generated by \( f_{il} \) and \( q^h \), for all \( (i,l) \in I^\infty \) and \( h \in P^\vee \), and \( \hat{U}^+ \) be the subalgebra generated by \( e_{il} \) for all \( (i,l) \in I^\infty \). In [1], Bozec showed that one can extended \( (\ , \ )_L \) to a symmetric bilinear form \( (\ , \ )_L \) on \( \hat{U} \) satisfying

\[
(q^h, 1)_L = 1, \quad (q^h, f_{il})_L = 0,
\]

(2.4)

\[
(q^h, K_j)_L = q^{-\alpha_j(h)},
\]

\[
(x, y)_L = (\omega(x), \omega(y))_L \quad \text{for all } x, y \in \hat{U}^+,
\]

where \( \omega: \hat{U} \to \hat{U} \) is the involution defined by

\[
\omega(q^h) = q^{-h}, \quad \omega(e_{il}) = f_{il}, \quad \omega(f_{il}) = e_{il} \quad \text{for } h \in P^\vee, (i,l) \in I^\infty.
\]

For any \( x \in \hat{U} \), we shall use the Sweedler’s notation, and write

\[
\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.
\]
Following the Drinfeld double process, we define \( \tilde{U} \) as the quotient of \( \hat{U} \) by the relations
\[
\sum (a(1), b(2)) L \omega(b(1)) a(2) = \sum (a(2), b(1)) L a(1) \omega(b(2)) \quad \text{for all } a, b \in \hat{U}^{\leq 0}
\]

**Definition 2.2.** Given a Borcherds-Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\), the quantum Borcherds-Bozec algebra \(U_q(g)\) is defined to be the quotient algebra of \(\tilde{U}\) by the radical of \( (\ , \ )_L \) restricted to \(\tilde{U}^- \times \tilde{U}^+\).

Let \(U^+(\text{resp. } U^-)\) be the subalgebra of \(U_q(g)\) generated by \(e_{il}\) (resp. \(f_{il}\)) for all \((i, l) \in I^\infty\). We will denote by \(U^0\) the subalgebra of \(U_q(g)\) generated by \(q^h\) for all \(h \in P^\vee\). It is easy to see that \(q^h(h \in P^\vee)\) is a \(\mathbb{Q}(q)\)-basis of \(U^0\).

In [8], Kang and Kim showed that the co-multiplication \(\Delta : \hat{U} \rightarrow \hat{U} \otimes \hat{U}\) passes down to \(U_q(g)\) and with this, \(U_q(g)\) becomes a Hopf algebra. They also proved the quantum Borcherds-Bozec algebra has a triangular decomposition.

**Theorem 2.3.** [8] The the quantum Borcherds-Bozec algebra \(U_q(g)\) has the following triangular decomposition:
\[
U_q(g) \cong U^- \otimes U^0 \otimes U^+.
\]

By the defining relation (2.5), we obtain complicated commutation relations between \(e_{il}\) and \(f_{jk}\) for \((i, l), (j, k) \in I^\infty\). We will derive explicit formulas for these complicated commutation relations in Appendix A. But, as we already see in (1.2), the commutation relations in the universal enveloping algebra \(U(g)\) of Borcherds-Bozec algebra \(g\) are rather simple
\[
e_{ik} f_{jl} - f_{jl} e_{ik} = k \delta_{ij} \delta_{kl} h_i \quad \text{for } i, j \in I, k, l \in \mathbb{Z}_{>0}.
\]

Thanks to Bozec, there exists another set of generators in \(U_q(g)\) called primitive generators. They satisfy a simpler set of commutation relations, and we shall prove that these generators also satisfy all the defining relations of \(U_q(g)\) described in (2.2).

We denote by \(C_l\) (resp. \(P_l\)) the set of compositions (resp. partitions) of \(l\), and denote by \(\eta : U_q(g) \rightarrow U_q(g)\) the \(\mathbb{Q}\)-algebra homomorphism defined by
\[
\eta(e_{il}) = e_{il}, \eta(f_{il}) = f_{il}, \eta(q^h) = q^{-h}, \eta(q) = q^{-1} \quad \text{for } h \in P^\vee, (i, l) \in I^\infty.
\]

As usual, let \(S : U_q(g) \rightarrow U_q(g)\) and \(\epsilon : U_q(g) \rightarrow \mathbb{Q}(q)\) be the antipode and the counit of \(U_q(g)\), respectively. Then, we have the following proposition.

**Proposition 2.4.** [1, 2] For any \(i \in I^m\) and \(l \geq 1\), there exist unique elements \(t_{il} \in U^-_{-l\alpha_i}\) and \(s_{il} = \omega(t_{il})\) such that
(1) $Q(q) \langle f_{il} \mid l \geq 1 \rangle = Q(q) \langle t_{il} \mid l \geq 1 \rangle$ and $Q(q) \langle e_{il} \mid l \geq 1 \rangle = Q(q) \langle s_{il} \mid l \geq 1 \rangle$,
(2) $(t_{il}, z)_L = 0$ for all $z \in Q(q) \langle f_{i1}, \cdots, f_{il-1} \rangle$,
$(s_{il}, z)_L = 0$ for all $z \in Q(q) \langle e_{i1}, \cdots, e_{il-1} \rangle$.
(3) $t_{il} - f_{il} \in Q(q) \langle f_{ik} \mid k < l \rangle$ and $s_{il} - e_{il} \in Q(q) \langle e_{ik} \mid k < l \rangle$,
(4) $\eta(t_{il}) = t_{il}$, $\eta(s_{il}) = s_{il}$,
(5) $\delta(t_{il}) = t_{il} \otimes 1 + 1 \otimes t_{il}$, $\delta(s_{il}) = s_{il} \otimes 1 + 1 \otimes s_{il}$,
(6) $\Delta(t_{il}) = t_{il} \otimes 1 + K_i^l \otimes t_{il}$, $\Delta(s_{il}) = s_{il} \otimes K_i^{-l} + 1 \otimes s_{il}$,
(7) $S(t_{il}) = -K_i^{-l}t_{il}$, $S(s_{il}) = -s_{il}K_i^l$.

If we set $\tau_{il} = (t_{il}, t_{il})_L = (s_{il}, s_{il})_L$, we have the following commutation relations in $U_q(\mathfrak{g})$

$$s_{il}t_{jk} - t_{jk}s_{il} = \delta_{ij}\delta_{lk}\tau_{il}(K_i^l - K_i^{-l}).$$

Assume that $i \in I^{im}$ and let $c = (c_1, \cdots, c_m)$ be an element in $C_I$ or in $P_I$. We set

$$t_{i,c} = \prod_{j=1}^m t_{icj} \text{ and } s_{i,c} = \prod_{j=1}^m s_{icj}.$$ 

Notice that $\{t_{i,c} \mid c \in C_I\}$ is a basis of $\mathcal{F}_{-\lambda_i}$.

For $i \in I^{iso}$ and $c, c' \in P_I$, if $c \neq c'$, then by induction, we have

$$(t_{i,c}, t_{i,c'})_L = (s_{i,c}, s_{i,c'})_L = 0.$$ 

For $i \in I^{im}\setminus I^{iso}$ and $c, c' \in C_I$, if the partitions obtained by rearranging $c$ and $c'$ are not equal, then we have

$$(t_{i,c}, t_{i,c'})_L = (s_{i,c}, s_{i,c'})_L = 0.$$ 

For each $i \in I^{re}$, we also use the notation $t_{i1}$ and $s_{i1}$. Here we set

$$t_{i1} = f_{i1}, \quad s_{i1} = e_{i1}.$$ 

Sometimes, we simply write $t_i$ (resp. $s_i$) instead of $t_{i1}$ (resp. $s_{i1}$) in this case. By mimicking Definition 1.2.13 in [13], we have the following definition.

**Definition 2.5.** For every $(i, l) \in I^\infty$, we define the linear maps $e'_{i,l}, e''_{i,l} : \mathcal{F} \rightarrow \mathcal{F}$ by

$$e'_{i,l}(1) = 0, \quad e'_{i,l}(t_{jk}) = \delta_{ij}\delta_{lk} \text{ and } e'_{i,l}(xy) = e'_{i,l}(x)y + q^{(|x|_\alpha)}xe'_{i,l}(y)$$
$$e''_{i,l}(1) = 0, \quad e''_{i,l}(t_{jk}) = \delta_{ij}\delta_{lk} \text{ and } e''_{i,l}(xy) = q^{(|y|_\alpha)}e''_{i,l}(x)y + xe''_{i,l}(y)$$

for any homogeneous elements $x, y$ in $\mathcal{F}$. 

Proposition 2.6.

(a) For any \( x, y \in \mathcal{F} \), we have
\[
(t_{il}y, x)_L = \tau_{il}(y, e'_{i,l}(x))_L, \quad (yt_{il}, x)_L = \tau_{il}(y, e''_{i,l}(x))_L
\]
(b) The maps \( e'_{i,l} \) and \( e''_{i,l} \) preserve the radical of \( (\ _, \_)_L \).
(c) Let \( x \in U^- \), we have

(i) If \( e'_{i,l}(x) = 0 \) for all \( (i, l) \in I^\infty \), then \( x = 0 \).
(ii) If \( e''_{i,l}(x) = 0 \) for all \( (i, l) \in I^\infty \), then \( x = 0 \).

Proof. (a) For any homogeneous element \( x \in \mathcal{F} \). We first show that
\[
\delta(x) = t_{il} \otimes e'_{i,l}(x) + \sum_{w \neq (i,l)} t_w \otimes y_w, \tag{2.12}
\]
where if \( w = (j_1, t_1) \cdots (j_r, t_r) \) is a word in \( I^\infty \), \( t_w = t_{(j_1,t_1)} \cdots t_{(j_r,t_r)} \) and \( y_w \) is an element in \( \mathcal{F} \) depending on \( w \).

Since \( e'_{i,l} \) is a linear map, it is enough to check (2.12) by assuming that \( x \) is a monomial in \( t_{jk} \). Fix \( (i, l) \in I^\infty \). We use induction on the number of \( t_{il} \) that appears in \( x \). If \( x \) contains no \( t_{il} \), then \( e'_{i,l}(x) = 0 \) and there is no term of the form \( t_{il} \otimes - \). Now assume that \( x \) contains \( t_{il} \), then we can write \( x = x_1 t_{il} x_2 \) for some monomials \( x_1, x_2 \) such that \( x_1 \) doesn’t contains \( t_{il} \). So we have
\[
e'_{i,l}(x) = e'_{i,l}(x_1 t_{il} x_2) = q^{(|x_1|,\alpha_i)} x_1 e'_{i,l}(t_{il} x_2) = q^{(|x_1|,\alpha_i)} x_1 x_2 + q^{(-|\alpha_i|,\alpha_i)} t_{il} e'_{i,l}(x_2). \tag{2.13}
\]

On the other hand
\[
\delta(x) = \delta(x_1)(t_{il} \otimes 1 + 1 \otimes t_{il}) \delta(x_2). \tag{2.14}
\]
By induction hypothesis, the term \( t_{il} \otimes - \) only appear in
\[
(1 \otimes x_1)(t_{il} \otimes 1)(1 \otimes x_2) + (1 \otimes x_1)(1 \otimes t_{il})(t_{il} \otimes e'_{i,l}(x_2)), \tag{2.15}
\]
which is equal to
\[
t_{il} \otimes q^{(|x_1|,\alpha_i)} x_1 x_2 + t_{il} \otimes q^{-(|x_1|,\alpha_i)} x_1 t_{il} e'_{i,l}(x_2) = t_{il} \otimes q^{(|x_1|,\alpha_i)} x_1 x_2 + q^{-|\alpha_i|,\alpha_i} t_{il} e'_{i,l}(x_2). \tag{2.16}
\]
This shows (2.12).

Similarly, we can show that
\[
\delta(x) = e''_{i,l}(x) \otimes t_{il} + \sum_{w \neq (i,l)} z_w \otimes t_w. \tag{2.17}
\]
Since \( e'_{i,l} \) and \( e''_{i,l} \) are linear maps, the equations (2.12) and (2.17) hold for any \( x, y \in \mathcal{F} \).
For any $c \in \mathcal{C}_d$, we have $(t_{il}, t_{ic})_L = \delta_{(l),c} \tau_{il}$. Thus

\[(t_{iy}, x)_L = \tau_{il}(y, e'_{il}(x))_L, \quad (yt_{il}, x)_L = \tau_{il}(y, e''_{il}(x))_L\]

for any $x, y \in \mathcal{T}$.

(b) Since $\tau_{il} = (t_{il}, t_{il})_L \neq 0$, our assertion follows.

(c) Note that each monomial ends with some $t_{jk}$'s. By (a), if $e''_{il}(x) = 0$ for all $(i, l) \in I^\infty$, then $x$ belongs to the radial of $(, )_L$, which is equal to 0 in $U^-$. $\square$

For any $i \in I^\text{re}$ and $n \in \mathbb{N}$, we set

\[ i_i^{(n)} = \frac{t_i^n}{[n]_!}. \]

By a similar argument as [13, 1.4.2], we have the following Lemma.

**Lemma 2.7.** We have

\[(2.19) \quad \delta(t_i^{(n)}) = \sum_{p+p'=n} q_i^{-pp'} t_i^{(p)} \otimes t_i^{(p')} \]

for any $i \in I^\text{re}$ and $n \in \mathbb{N}$.

**Proposition 2.8.** For any $i \in I^\text{re}$, $(j, l) \in I^\infty$, and $i \neq (j, l)$, we have

\[ \sum_{p+p'=1-\alpha_{ij}} (-1)^{p} t_i^{(p)} t_j l_i^{(p')} = 0 \]

in $U_q(\mathfrak{g})$.

**Proof.** If $i \in I^\text{re}$, we have $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Set

\[ R_{i,(j,l)} = \sum_{p+p'=1-\alpha_{ij}} (-1)^{p} t_i^{(p)} t_j l_i^{(p')}. \]

By (2.6), we only need to show that $e''_{\mu}(R_{i,(j,l)}) = 0$ for all $\mu \in I^\infty$. It is clear that

\[ e''_{\mu}(R_{i,(j,l)}) = 0 \quad \text{if} \quad \mu \neq i, (j, l). \]

By the definition of $e''_{i}$, we have

\[ e''_{i}(t_i^{(p)} t_j l_i^{(p')}) = q^{(\alpha_i, -p' \alpha_j)} e''_{i}(t_i^{(p)}) t_j l_i^{(p')} + t_i^{(p)} t_j e''_{i}(t_i^{(p')}) \]

\[ = q^{-p'(\alpha_i, \alpha_j)} q^{(1-p') \alpha_j} t_i^{(p'-1)} t_j l_i^{(p')} + q^{(1-p') \alpha_j} t_i^{(p')} t_j l_i^{(p'-1)}. \]
Thus

\[ e''_i(R_{i,j,l}) = \sum_{p+p'=1-l_{a_{ij}}} (-1)^p q^{-p(\alpha_i,\alpha_i)} q^{-(\alpha_i,\alpha_{ij})} q_i^{(1-p)} t_i^{(p-1)} t_{jl} t_i^{(p')} \]

\[ + \sum_{p+p'=1-l_{a_{ij}}} (-1)^p q_i^{(1-p')} t_i^{(p)} t_{jl} t_i^{(p'-1)} \]

\[ = \sum_{0 \leq p \leq 1-l_{a_{ij}}} (-1)^p q^{-(1-l_{a_{ij}}-p)(\alpha_i,\alpha_i)} q^{-(\alpha_i,\alpha_{ij})} q_i^{(1-p)} t_i^{(p-1)} t_{jl} t_i^{(1-l_{a_{ij}}-p)} \]

\[ + \sum_{0 \leq p \leq 1-l_{a_{ij}}} (-1)^p q_i^{(l_{a_{ij}}+p)} t_i^{(p)} t_{jl} t_i^{(-l_{a_{ij}}-p)}. \]

The coefficient of \( t_i^{(p)} t_{jl} t_i^{(-l_{a_{ij}}-p)} \) in the first sum of (2.21) is

\[ (-1)^{p+1} q^{-(l_{a_{ij}}-p)(\alpha_i,\alpha_i)} q^{-(\alpha_i,\alpha_{ij})} q_i^{-(1-p)} \]

\[ = (-1)^{p+1} q^{(\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)})+(p)(\alpha_i,\alpha_i)-(l(\alpha_i,\alpha_j)+(-p)(\alpha_i,\alpha_i))} \]

\[ = (-1)^{p+1} q^{(\alpha_i,\alpha_j)+\frac{p(\alpha_i,\alpha_i)}{2}} \]

\[ = (-1)^{p+1} q_i^{(l_{a_{ij}}+p)}. \]

Hence, we have \( e''_i(R_{i,j,l}) = 0. \)

By the definition of \( e''_j \), we have

\[ e''_{jl}(t_i^{(p)} t_{jl} t_i^{(p')}) = q^{-(\alpha_j,\alpha_{ij})} e''_j(t_i^{(p)} t_{jl}) t_i^{(p')} = q^{-(\alpha_j,\alpha_{ij})} t_i^{(p)} t_i^{(p')} \]

So

\[ e''_{jl}(R_{i,j,l}) = \sum_{0 \leq p \leq 1-l_{a_{ij}}} (-1)^{1-l_{a_{ij}}-p} q^{-(\alpha_j,\alpha_{ij})} t_i^{(1-l_{a_{ij}}-p)} t_i^{(p')}. \]

By [13, 1.3.4], we obtain

\[ \sum_{0 \leq p' \leq 1-l_{a_{ij}}} (-1)^{1-l_{a_{ij}}-p'} q^{-(\alpha_j,\alpha_{ij})} t_i^{(-l_{a_{ij}}-p')} \left[ 1 - \frac{l_{2(\alpha_i,\alpha_j)}}{p'} \right] = 0. \]

Hence, we get \( e''_{jl}(R_{i,j,l}) = 0. \) This finishes the proof.
By the above arguments, we have primitive generators $t_{il}((i,l) \in \mathcal{I}^\infty)$ in $U^-$ of degree $-l\alpha_i$ and $s_{il}((i,l) \in \mathcal{I}^\infty)$ in $U^+$ of degree $l\alpha_i$ satisfying

\begin{equation}
(2.25) \quad s_{il}t_{jk} - t_{jk}s_{il} = \delta_{ij}\delta_{kl}\tau_{il}(K^l_i - K^{-l}_i),
\end{equation}

\begin{equation}
(2.26) \quad \sum_{k=0}^{1-l\alpha_i} (-1)^k \left[ 1 - l\alpha_{ij} \right] t_i^{1-l\alpha_{ij}-k} t_{jl}^k = 0 \quad \text{for } i \in \mathcal{I}_re, i \neq (j,l).
\end{equation}

By using the involution $\omega$, we get

\begin{equation}
(2.27) \quad q^h t_{ij} q^{-h} = q^{-l\alpha_j(h)} t_{ij}, \quad q^h s_{ij} q^{-h} = q^{l\alpha_j(h)} s_{ij} \quad \text{for } h \in P^\vee, (j,l) \in \mathcal{I}^\infty,
\end{equation}

and

\begin{equation}
(2.28) \quad [t_{ik}, t_{jl}] = [s_{ik}, s_{jl}] = 0 \quad \text{for } a_{ij} = 0.
\end{equation}

3. $A_1$-form of the quantum Borcherds-Bozec algebras

We consider the localization of $\mathbb{Q}[q]$ at the ideal $(q - 1)$:

\begin{equation}
(3.1) \quad A_1 = \{ f(q) \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 1 \} = \{ g/h \mid g, h \in \mathbb{Q}[q], h(1) \neq 0 \}
\end{equation}

Let $\mathbb{J}_1$ be the unique maximal ideal of the local ring $A_1$, which is generated by $(q - 1)$. Then we have an isomorphism of fields

\[ A_1/\mathbb{J}_1 \cong \mathbb{Q}, \quad f(q) + \mathbb{J}_1 \mapsto f(1). \]

Note that, for $i \in \mathcal{I}_re$, $[n]_i$ and $\left[ \begin{array}{c} n \\ k \end{array} \right]_i$ are elements of $\mathbb{Z}[q, q^{-1}] \subseteq A_1$. For any $h \in P^\vee$, $n \in \mathbb{Z}$, we formally define

\[ (q^h; n)_q = \frac{q^h q^n - 1}{q - 1} \in U^0. \]
**Definition 3.1.** We define the $A_1$-form, denote by $U_{A_1}$ of the quantum Borcherds-Bozec algebra $U_q(g)$ to be the $A_1$-subalgebra generated by the elements $s_{it}$, $T_{il}$, $q^h$ and $(q^h; 0)_q$, for all $(i, l) \in I^\infty$ and $h \in P^\vee$, where

\begin{equation}
T_{il} = \frac{1}{\tau_{il} q_{il}^2 - 1} t_{il} \text{ for } (i, l) \in I^\infty.
\end{equation}

Let $U_{A_1}^+$ (resp. $U_{A_1}^-$) be the $A_1$-subalgebra of $U_{A_1}$ generated by the elements $s_{it}$ (resp. $T_{il}$) for $(i, l) \in I^\infty$, and $U_{A_1}^0$ be the subalgebra of $U_{A_1}$ generated by $q^h$ and $(q^h; 0)_q$ for $(h \in P^\vee)$.

**Lemma 3.2.**

(a) $(q^h; n)_q \in U_{A_1}^0$ for all $n \in \mathbb{Z}$ and $h \in P^\vee$.
(b) $K_i^l - K_i^{-l} q_{i}^{2l - 1} \in U_{A_1}^0$.

**Proof.** It is straightforward to check that

\begin{equation}
(q^h; n)_q = q^n (q^h; 0)_q + \frac{q^n - 1}{q - 1},
\end{equation}

\begin{equation}
\frac{K_i^l - K_i^{-l}}{q_{i}^{2l - 1}} = \frac{q - 1}{q_{i}^{2l - 1}} (1 + K_i^{-l}) \frac{K_i^l - 1}{q - 1}.
\end{equation}

The lemma follows. \hfill \Box

The next proposition shows that the triangular decomposition (2.6) of $U_q(g)$ carries over to its $A_1$-form.

**Proposition 3.3.** We have a natural isomorphism of $A_1$-modules

\begin{equation}
U_{A_1} \cong U_{A_1}^- \otimes U_{A_1}^0 \otimes U_{A_1}^+
\end{equation}

induced from the triangular decomposition of $U_q(g)$.

**Proof.** Consider the canonical isomorphism $\varphi : U_q(g) \cong U^- \otimes U^0 \otimes U^+$ given by multiplication. By (2.25) and (2.27), we have the following commutation relations

\begin{align}
s_{it}(q^h; 0)_q &= (q^h; -l\alpha_i(h)) q_{s_{it}}, \\
(q^h; 0)_q T_{il} &= T_{il}(q^h; -l\alpha_i(h)), \\
s_{it} T_{jk} - T_{jk} s_{it} &= \delta_{ij} \delta_{lk} \frac{K_i^l - K_i^{-l}}{q_{i}^{2l - 1}}.
\end{align}

Combining with (3.2), we can see that the image of $\varphi$ lies inside $U_{A_1}^- \otimes U_{A_1}^0 \otimes U_{A_1}^+$. \hfill \Box
The representation theory of quantum Borcherds-Bozec algebras has been studied by Kang and Kim in [8]. In the following sections, we will use some notions defined in [8], which are similar to those in classical representation theory of quantum groups.

Fix \( \lambda \in P \), let \( V^q \) be a highest weight \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). Then we have the \( \Lambda_1 \)-form for the highest weight modules.

**Definition 3.4.** The \( \Lambda_1 \)-form of \( V^q \) is defined to be the \( U_{\Lambda_1} \)-module \( V_{\Lambda_1} = U_{\Lambda_1} v_\lambda \).

By the definition of highest weight module and \( V_{\Lambda_1} \), it is easy to see that \( V_{\Lambda_1} = U_{\Lambda_1}^- v_\lambda \). The highest weight \( U_q(\mathfrak{g}) \)-module \( V^q \) has the weight space decomposition

\[
V^q = \bigoplus_{\mu \leq \lambda} V^q_{\mu},
\]

where \( V^q_{\mu} = \{ v \in V^q \mid q^{\alpha_1} v = q^{\mu(h)} v \ \text{for all} \ h \in P^+ \} \). For each \( \mu \in P \), we define the weight space \( (V_{\Lambda_1})_{\mu} = V_{\Lambda_1} \cap V^q_{\mu} \). The following proposition shows that \( V_{\Lambda_1} \) also has the weight space decomposition.

**Proposition 3.5.** \( V_{\Lambda_1} = \bigoplus_{\mu \leq \lambda} (V_{\Lambda_1})_{\mu} \)

**Proof.** The proof is the same as [5, Proposition 3.3.6]. \( \square \)

**Proposition 3.6.** For each \( \mu \in P \), the weight space \( (V_{\Lambda_1})_{\mu} \) is a free \( \Lambda_1 \)-module with \( \text{rank}_{\Lambda_1}(V_{\Lambda_1})_{\mu} = \dim_{\mathbb{Q}(q)} V^q_{\mu} \).

**Proof.** We first show that \( (V_{\Lambda_1})_{\mu} \) is finite generated as an \( \Lambda_1 \)-module. Since we have \( V_{\Lambda_1} = U_{\Lambda_1}^- v_\lambda \), every element in \( V_{\Lambda_1} \) is a polynomial of \( T_{i_1} \cdots T_{i_p} v_\lambda \) with coefficients in \( \Lambda_1 \). Assume that \( \lambda = \mu + \alpha \) for some \( \alpha \in \mathbb{Q}_+ \). Then for each \( v \in \Lambda_1 \) with weight \( \mu \), \( v \) must be a \( \Lambda_1 \)-linear combination of \( \{ T_{i_1} \cdots T_{i_p} v_\lambda \mid l_1 \alpha_1 + \cdots + l_p \alpha_p = \alpha \} \), which is a finite set.

Let \( \{ T_{l} v_\lambda \} \) be a \( \mathbb{Q}(q) \)-basis of \( V^q_{\mu} \), where \( T_l \) are monomials in \( T_{i_l} \). The set \( \{ T_{l} v_\lambda \} \) certainly belongs to \( (V_{\Lambda_1})_{\mu} \) and is also \( \Lambda_1 \)-linearly independent. So we have \( \text{rank}_{\Lambda_1}(V_{\Lambda_1})_{\mu} \geq \dim_{\mathbb{Q}(q)} V^q_{\mu} \). Let \( \{ u_1, \cdots, u_p \} \) be an \( \Lambda_1 \)-linearly independent subset of \( (V_{\Lambda_1})_{\mu} \). Consider a \( \mathbb{Q}(q) \)-linear dependence relation

\[
c_1(q) u_1 + \cdots + c_p(q) u_p = 0, \ c_k(q) \in \mathbb{Q}(q) \ \text{for} \ k = 1, \cdots, p.
\]

Multiplying some powers of \( q - 1 \) if needed, we may assume that all \( c_k(q) \in \Lambda_1 \), which implies that \( c_k(q) = 0 \) for all \( k = 1, \cdots, p \). Hence \( u_1, \cdots, u_p \) are linearly independent over \( \mathbb{Q}(q) \) and \( \text{rank}_{\Lambda_1}(V_{\Lambda_1})_{\mu} \leq \dim_{\mathbb{Q}(q)} V^q_{\mu} \), which completes the proof. \( \square \)

**Corollary 3.7.** The \( \mathbb{Q}(q) \)-linear map \( \varphi : \mathbb{Q}(q) \otimes_{\Lambda_1} V_{\Lambda_1} \rightarrow V^q \) given by \( c \otimes v \mapsto cv \) is an isomorphism.
4. Classical limit of quantum Borcherds-Bozec algebras

Define the \( \mathbb{Q} \)-linear vector spaces
\[
U_1 = (A_1/j_1) \otimes_{A_1} U_{A_1} \cong U_{A_1}/j_1 U_{A_1},
\]
\[
V^1 = (A_1/j_1) \otimes_{A_1} V_{A_1} \cong V_{A_1}/j_1 V_{A_1}.
\]
Then \( V^1 \) is naturally a \( U_1 \)-module. Consider the natural maps
\[
U_{A_1} \to U_1 = U_{A_1}/j_1 U_{A_1},
\]
\[
V_{A_1} \to V^1 = V_{A_1}/j_1 V_{A_1}.
\]
The passage under these maps is referred to as taking the classical limit. We will denote by \( \overline{\imath} \) the image of \( \imath \) under the classical limit. Notice that \( q \) is mapped to 1 under these maps.

For each \( \mu \in P \), set
\[
V^1_\mu = (A_1/j_1) \otimes_{A_1} (V_{A_1})_\mu.
\]
Then we have

**Proposition 4.1.**

(a) \( V^1 = \bigoplus_{\mu \leq \lambda} V^1_\mu \).

(b) For each \( \mu \in P \), \( \dim \mathbb{Q} V^1_\mu = \text{rank}_{A_1} (V_{A_1})_\mu = \dim \mathbb{Q} (q) V^q_\mu \).

Let \( \overline{\imath} \in U_1 \) denote the classical limit of the element \( (q^h; 0)_q \in U_{A_1} \). As in [5], we have the following lemma.

**Lemma 4.2.**

(i) For all \( h \in P^\vee \), we have \( \overline{q^h} = 1 \).

(ii) For any \( h, h' \in P^\vee \), \( \overline{h + h'} = \overline{h} + \overline{h'} \). Hence, we have \( \overline{nh} = n\overline{h} \) for \( n \in \mathbb{Z} \).

Define the subalgebras \( U^0_1 = \mathbb{Q} \otimes U^0_{A_1} \) and \( U^\pm_1 = \mathbb{Q} \otimes U^{\pm}_{A_1} \). The next theorem shows that we can define a surjective homomorphism from the universal enveloping algebra \( U(g) \) to \( U_1 \), and as a \( U(g) \)-module, \( V^1 \) is a highest weight module with highest weight \( \lambda \in P \) and highest weight vector \( \overline{\nu} \).

**Theorem 4.3.**

(a) The elements \( \overline{s_{il}}, \overline{T_{il}}((i, l) \in I^\infty) \) and \( \overline{\imath} = (i \in P^\vee) \) satisfy the defining relations of \( U(g) \). Hence there exists a surjective \( \mathbb{Q} \)-algebra homomorphism \( \psi : U(g) \to U_1 \) sending \( e_{il} \) to \( \overline{s_{il}} \), \( f_{il} \) to \( \overline{T_{il}} \), \( h \) to \( \overline{\imath} \). In particular, the \( U_1 \)-module \( V^1 \) has a \( U(g) \)-module structure.

(b) For each \( \mu \in P \), \( h \in P^\vee \), the element \( \overline{\imath} \) acts on \( V^1_\mu \) as scalar multiplication by \( \mu(h) \). So \( V^1_\mu \) is the \( \mu \)-weight space of the \( U(g) \)-module \( V^1 \).
(c) As a $U(\mathfrak{g})$-module, $V^1$ is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v}_\lambda$.

**Proof.** (a) Since\[ \frac{K_i^l - K_i^{-l}}{q_i^2 - 1} = \frac{q - 1}{q_i^2 - 1} (1 + K_i^{-l}) \frac{K_i^l - 1}{q - 1}, \]
when we take classical limit, we get\[ \frac{K_i^l - K_i^{-l}}{q_i^2 - 1} = \frac{1}{2r_i} \cdot 2 \cdot l_i \overline{h}_i = l_i \overline{h}_i. \]
By (2.25), we have the following equation in $U_1$
\[ \overline{s}_{il} \overline{T}_{jk} - \overline{T}_{jk} \overline{s}_{il} = \delta_{ij} \delta_{lk} \overline{h}_i, \]
and it is the same as the commutation relations in $U(\mathfrak{g})$.

Since we have\[ q^h s_{jl} = q^{l \alpha_j(h)} s_{jl} q^h, \quad q^h T_{jl} = q^{-l \alpha_j(h)} T_{jl} q^h \quad \text{for } h \in P^\vee, (j,l) \in I^\infty, \]
we get\[ \frac{q^h - 1}{q - 1} s_{il} = s_{il} \frac{q^{l \alpha_i(h)} q^h - 1}{q - 1} \]
and
\[ \frac{q^h - 1}{q - 1} s_{il} - s_{il} \frac{q^h - 1}{q - 1} = s_{il} \frac{q^{l \alpha_i(h)} - 1}{q - 1} q^h. \]
Thus $\overline{h} \overline{s}_{il} - \overline{s}_{il} \overline{h} = l \alpha_i(h) \overline{s}_{il}$. Similarly, we have
\[ \overline{h} \overline{T}_{il} - \overline{T}_{il} \overline{h} = -l \alpha_i(h) \overline{T}_{il}. \]

It is easy to check the commutation relations
\[ [\overline{T}_{ik}, \overline{T}_{jl}] = [\overline{s}_{ik}, \overline{s}_{jl}] = 0 \quad \text{for } a_{ij} = 0. \]

For $i \in I^\text{re}$, we have
\[ [n_{i}]_i = n \quad \text{and} \quad \binom{n}{k}_i = \binom{n}{k} \]
Hence the remaining Serre relations follow.

(b) For $v \in (V_{\overline{h}_1})_\mu$ and $h \in P^\vee$, we have $(q^h; 0) v = \frac{q^{\mu(h) - 1}}{q - 1} v$. Hence when we take the classical limit, we obtain $\overline{h} v = \mu(h) v$.

(c) As a $U(\mathfrak{g})$-module, by (2), we have $h \overline{v}_\lambda = \overline{h} \overline{v}_\lambda = \lambda(h) \overline{v}_\lambda$ in $V^1$ for all $h \in P^\vee$. For each $(i,l) \in I^\infty$, $s_{il} \overline{v}_\lambda$ is zero. Therefore, $V^1 = U_1 \overline{v}_\lambda = U^- (\mathfrak{g}) \overline{v}_\lambda$ and hence $V^1$ is a highest weight module with highest weight $\lambda \in P$ and highest weight vector $\overline{v}_\lambda$. □
Combining Proposition 4.1 (b) and Theorem 4.3 (b), we have \( \text{ch} V^1 = \text{ch} V^q \). For a dominant integral weight \( \lambda \in P^+ \), the irreducible highest weight \( U_q(\mathfrak{g}) \)-module \( V^q(\lambda) \) has the following property.

**Proposition 4.4.** [8] Let \( \lambda \in P^+ \) and \( V^q(\lambda) \) be the irreducible highest weight module with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). Then the following statements hold.

(a) If \( i \in I^r \), then \( f_i^{\lambda(h_i)+1} v_\lambda = 0 \).

(b) If \( i \in I^m \) and \( \lambda(h_i) = 0 \), then \( f_{ik} v_\lambda = 0 \) for all \( k > 0 \).

We now conclude that the classical limit of the irreducible highest weight \( U_q(\mathfrak{g}) \)-module \( V^q(\lambda) \) is isomorphic to the irreducible highest \( U(\mathfrak{g}) \)-module \( V(\lambda) \).

**Theorem 4.5.** If \( \lambda \in P^+ \) and \( V^q \) is the irreducible highest weight \( U_q(\mathfrak{g}) \)-module \( V^q(\lambda) \) with highest weight \( \lambda \), then \( V^1 \) is isomorphic to the irreducible highest weight module \( V(\lambda) \) over \( U(\mathfrak{g}) \) with highest weight \( \lambda \).

**Proof.** By Proposition 4.4, if \( i \in I^r \), then \( T_i^{\lambda(h_i)+1} v_\lambda = 0 \); if \( i \in I^m \) and \( \lambda(h_i) = 0 \), then \( T_{ik} v_\lambda = 0 \) for all \( k > 0 \). Therefore, \( V^1 \) is a highest weight \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \) and highest weight vector \( \overline{v}_\lambda \) satisfying:

(a) If \( i \in I^r \), then \( f_i^{\lambda(h_i)+1} \overline{v}_\lambda = T_i^{\lambda(h_i)+1} \overline{v}_\lambda = 0 \).

(b) If \( i \in I^m \) and \( \lambda(h_i) = 0 \), then \( f_{ik} \overline{v}_\lambda = T_{ik} \overline{v}_\lambda = 0 \) for all \( k > 0 \).

Hence \( V^1 \cong V(\lambda) \) by Proposition 1.3. \( \square \)

By Proposition 4.1 (b), the character of \( V^q(\lambda) \) is the same as the character of \( V(\lambda) \), which is given by (see, [7, 3])

\[
\text{ch} V(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda) + \rho - \rho(S_\lambda)}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim g_\alpha}}
\]

\[
(4.5) = \frac{\sum_{w \in W} \sum_{s \in F_\lambda} \epsilon(w) \epsilon(s) e^{w(\lambda + \rho - s) - \rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim g_\alpha}}.
\]

**Theorem 4.6.** The classical limit \( U_1 \) of \( U_q(\mathfrak{g}) \) is isomorphic to the universal enveloping algebra \( U(\mathfrak{g}) \) as \( \mathbb{Q} \)-algebras.

**Proof.** By Theorem 4.3 (a), we already have an epimorphism \( \psi : U(\mathfrak{g}) \to U_1 \) sending \( e_{il} \) to \( \overline{e}_{il} \), \( f_{il} \) to \( \overline{f}_{il} \), \( h \) to \( \overline{h} \), respectively. So it is sufficient to show that \( \psi \) is injective.

We first show that the restriction \( \psi_0 \) of \( \psi \) to \( U^0(\mathfrak{g}) \) is an isomorphism of \( U^0(\mathfrak{g}) \) onto \( U^0_1 \). Note that \( \psi_0 \) is certainly surjective. Since \( \chi = \{ h_i \mid i \in I \} \cup \{ d_i \mid i \in I \} \) is a \( \mathbb{Z} \)-basis of the free \( \mathbb{Z} \)-lattice \( P^\vee \), it is also a \( \mathbb{Q} \)-basis of the Cartan subalgebra \( \mathfrak{h} \). Thus any element of
$U^0(g)$ may be written as a polynomial in $\chi$. Suppose $g \in \text{Ker}\psi_0$. Then, for each $\lambda \in P$, we have

$$0 = \psi_0(g) \cdot \varpi_\lambda = \lambda(g)\varpi_\lambda,$$

where $v_\lambda$ is a highest weight vector of a highest weight $U_q(g)$-module of highest weight $\lambda$ and $\lambda(g)$ denotes the polynomial in $\{\lambda(x) \mid x \in \chi\}$ corresponding to $g$. Hence, we have $\lambda(g) = 0$ for every $\lambda \in P$. Since we may take any integer value for $\lambda(x)(x \in \chi)$, $g$ must be zero, which implies that $\psi_0$ is injective.

Next, we show that the restriction of $\psi$ to $U^-(g)$, denote by $\psi_-$, is an isomorphism of $U^-(g)$ onto $U^+_1$. Suppose $\text{Ker}\psi_- \neq 0$, and take a non-zero element $u = \sum a_\zeta f_\zeta \in \text{Ker}\psi_-$, where $a_\zeta \in \mathbb{Q}$ and $f_\zeta$ are monomials in $f_{il}$'s $(i, l) \in I^\infty$. Let $N$ be the maximal length of the monomials $f_\zeta$ in the expression of $u$ and choose a dominant integral weight $\lambda \in P^+$ such that $\lambda(h_i) > N$ for all $i \in I$. The kernel of the $U^-(g)$-module homomorphism $\varphi : U^-(g) \to V^1$ given by $x \mapsto \psi(x) \cdot \varpi_\lambda$ is the left ideal of $U^-(g)$ generated by $f_{i}^{\lambda(h_i)+1}(i \in I^e)$ and $f_{il}$ for $i \in I^m$ with $\lambda(h_i) = 0$. Because of the choice of $\lambda$, it is generated by $f_{i}^{\lambda(h_i)+1}$ for all $i \in I^e$.

Therefore, $u = \sum a_\zeta f_\zeta \notin \text{Ker}\varphi$. That is, $\psi_-(u) \cdot \varpi_\lambda = \psi(u) \cdot \varpi_\lambda \neq 0$, which is a contradiction. Therefore, $\text{Ker}\psi_- = 0$ and $U^-(g)$ is isomorphic to $U^+_1$.

Similarly, we have $U^+(g) \cong U^+_1$. Hence, by the triangular decomposition, we have the linear isomorphisms

$$U(g) \cong U^-(g) \otimes U^0(g) \otimes U^+(g) \cong U^+_1 \otimes U^-_1 \otimes U^+_1 \cong U_1,$$

where the last isomorphism follows from Proposition 3.3. It is easy to see that this isomorphism is actually an algebra isomorphism. \hfill \Box

We now show that $U_1$ inherits a Hopf algebra structure from that of $U_q(g)$. It suffices to show that $U_{\hbar_1}$ inherits the Hopf algebra structure from that of $U_q(g)$. Since we have

\begin{equation}
\begin{aligned}
\Delta(T_{il}) &= T_{il} \otimes 1 + K_{il} \otimes T_{il}, \quad \Delta(s_{il}) = s_{il} \otimes K_{il}^{-1} + 1 \otimes s_{il}, \\
\Delta(q^h) &= q^h \otimes q^h, \\
S(T_{il}) &= -K_{il}^{-1}T_{il}, \quad S(s_{il}) = -s_{il}K_{il}, \quad S(q^h) = q^{-h}, \\
\epsilon(T_{il}) &= \epsilon(s_{il}) = 0, \quad \epsilon(q^h) = 1,
\end{aligned}
\end{equation}
we get
\begin{equation}
\Delta((q^h;0)_q) = \frac{q^h \otimes q^h - 1 \otimes 1}{q - 1} = (q^h;0)_q \otimes 1 + q^h \otimes (q^h;0)_q,
\end{equation}
\begin{equation}
S((q^h;0)_q) = (q^{-h};0)_q,
\end{equation}
\begin{equation}
\epsilon((q^h;0)_q) = 0.
\end{equation}

Hence the maps \( \Delta: U_{\mathbb{A}_1} \rightarrow U_{\mathbb{A}_1} \otimes U_{\mathbb{A}_1}, \epsilon: U_{\mathbb{A}_1} \rightarrow \mathbb{A}_1, \) and \( S: U_{\mathbb{A}_1} \rightarrow U_{\mathbb{A}_1} \) are all well-defined and \( U_{\mathbb{A}_1} \) inherits a Hopf algebra structure from that of \( U_q(\mathfrak{g}) \).

Let us show that the Hopf algebra structure of \( U_1 \) coincides with that of \( U(\mathfrak{g}) \) under the isomorphism we have been considering. Taking the classical limit of the equations in (4.6) and in (4.7), we have
\begin{equation}
\Delta(T_{\mathfrak{gl}}) = T_{\mathfrak{gl}} \otimes 1 + 1 \otimes T_{\mathfrak{gl}}, \quad \Delta(\bar{\mathfrak{gl}}) = \bar{\mathfrak{gl}} \otimes 1 + 1 \otimes \bar{\mathfrak{gl}}, \quad \Delta(\bar{h}) = \bar{h} \otimes 1 + 1 \otimes \bar{h},
\end{equation}
\begin{equation}
S(T_{\mathfrak{gl}}) = -T_{\mathfrak{gl}}, \quad S(\bar{\mathfrak{gl}}) = -\bar{\mathfrak{gl}}, \quad S(\bar{h}) = -\bar{h},
\end{equation}
\begin{equation}
\epsilon(T_{\mathfrak{gl}}) = \epsilon(\bar{\mathfrak{gl}}) = \epsilon(\bar{h}) = 0.
\end{equation}

This coincides with (1.3). Therefore, we have the following corollary.

**Corollary 4.7.** The classical limit \( U_1 \) of \( U_q(\mathfrak{g}) \) inherits a Hopf algebra structure from that of \( U_q(\mathfrak{g}) \) so that \( U_1 \) and \( U(\mathfrak{g}) \) are isomorphic as Hopf algebras over \( \mathbb{Q} \).

Since \( U^-(\mathfrak{g}) \cong U_1^- \), by the same argument in [5, Theorem 3.4.10], we have the following theorem when we take the classical limit on the Verma module over \( U_q(\mathfrak{g}) \).

**Theorem 4.8.** [5] If \( \lambda \in P \) and \( V^q \) is the Verma module \( M^q(\lambda) \) over \( U_q(\mathfrak{g}) \) with highest weight \( \lambda \), then its classical limit \( V^1 \) is isomorphic to the Verma module \( M(\lambda) \) over \( U(\mathfrak{g}) \) with highest weight \( \lambda \).
APPENDIX A.

We shall provide an explicit commutation relations for $e_{ik}$ and $f_{jl}$, for $(i, k), (j, l) \in I^\infty$ in $U_q(g)$. Recall that, we have the co-multiplication formulas

$$\Delta(f_{jl}) = \sum_{m+n=l} q_{(i)}^{-mn} f_{im} K_i^n \otimes f_{in}.$$ 

Then, the defining relation (2.5) yields the following lemma.

**Lemma A.1.** [8] For any $i, j \in I$ and $k, l \in \mathbb{Z}_{>0}$, we have

(a) If $i \neq j$, then $e_{ik}$ and $f_{jl}$ are commutative.

(b) If $i = j$, we have the following relations in $U_q(g)$ for all $k, l > 0$

$$\sum_{m+n=k} q_{(i)}^{n(m-s)} \nu_{in} e_{is} f_{im} K_i^{n-s} = \sum_{m+n=k} q_{(i)}^{n(m-s)} \nu_{in} f_{im} e_{is} K_i^n. \tag{A.1}$$

Since we have

$$K_i^n e_{im} K_i^{-n} = q_{(i)}^{2mn} e_{im},$$

$$K_i^n f_{im} K_i^{-n} = q_{(i)}^{-2mn} f_{im}.$$ 

We can modify the equations (A.1) as the following form

$$\sum_{m+n=k} q_{(i)}^{n(s-m)} \nu_{in} K_i^{n-s} e_{is} f_{im} = \sum_{m+n=k} q_{(i)}^{n(s-m)} \nu_{in} K_i^n f_{im} e_{is}. \tag{A.2}$$

If $i \in I^\text{re}$, then $k = l = 1$ and $m = s$, so there are only one commutation relation in this case

$$e_i f_i + \nu_1 K_i^{-1} = f_i e_i + \nu_1 K_i. \tag{A.3}$$

If $i \in I^\text{im}$ (we omit the notation “$i$” in this case for simplicity), we first assume that $k = l$. By (A.2), we have

$$k = l = 1, \quad e_1 f_1 + \nu_1 K^{-1} = f_1 e_1 + \nu_1 K,$$

$$k = l = 2, \quad e_2 f_2 + \nu_1 K^{-1} e_1 f_1 + \nu_2 K^{-2} = f_2 e_2 + \nu_1 K f_1 e_1 + \nu_2 K^2,$$

$$\ldots$$

$$k = l = n, \quad e_n f_n + \nu_1 K^{-1} e_{n-1} f_{n-1} + \ldots + \nu_{n-1} K^{1-n} e_1 f_1 + \nu_n K^{-n} = f_n e_n + \nu_1 K f_{n-1} e_{n-1} + \ldots + \nu_{n-1} K^{n-1} f_1 e_1 + \nu_n K^n. \tag{A.4}$$

By direct calculation, we can write \( e_n f_n - f_n e_n \) in the following way
\[
e_n f_n - f_n e_n = \alpha_1 f_{n-1} e_{n-1} + \alpha_2 f_{n-2} e_{n-2} + \cdots + \alpha_{n-1} f_1 e_1 + \alpha_n,
\]
where
\[
\begin{align*}
\alpha_1 &= \nu_1(K - K^{-1}), \\
\alpha_2 &= \nu_2(K^2 - K^{-2}) - \nu_1 K^{-1} = \nu_2(K^2 - K^{-2}) - \nu_1^2 K^{-1}(K - K^{-1}), \\
\alpha_3 &= \nu_3(K^3 - K^{-3}) - \nu_1 K^{-1} = \nu_3(K^3 - K^{-3}) - \nu_1 K^{-1} + (\nu_1^3 - \nu_1^2)K^{-2}(K - K^{-1}), \\
\alpha_n &= \nu_n(K^n - K^{-n}) - \nu_1 K^{-1} = \nu_n(K^n - K^{-n}) - \nu_1 K^{-1} - \nu_2 K^{-2}(K - K^{-1}) - \cdots - \nu_{n-1} K^{-(n-1)} K^{-1}.
\end{align*}
\]

If \( m \in \mathbb{N} \) and \( c = (c_1, \ldots, c_d) \) is a composition of \( m \) (i.e. \( c \in \mathcal{C}_m \)), then we set \( \nu_c = \prod_{k=1}^d \nu_k \) and \( \|c\| = d \).

By induction, we have
\[
e_n f_n = \sum_{p=1}^{n} \left\{ \sum_{r=1}^{p} \left[ \nu_r \vartheta_{p-r} K^{r-p}(K^{r} - K^{-r}) \right] \right\} f_{n-p} e_{n-p} + f_n e_n,
\]
where \( \vartheta_m = \sum_{c \in \mathcal{C}_m} (-1)^{\|c\|} \nu_c \). For example, \( \vartheta_4 = \nu_4^4 - 3\nu_1^2 \nu_2 + 2\nu_1 \nu_3 + \nu_2^2 - \nu_4 \).

Next, we assume that \( k - l = t \), then \( m - s = t \). By (A.2), we get
\[
\sum_{n=0}^{l} q_{(i)}^{-n} \nu_n K^{-n} e_{l-n} f_{k-n} = \sum_{n=0}^{l} q_{(i)}^{-n} \nu_n K^n f_{k-n} e_{l-n}.
\]
Hence, we have
\[
e_l f_k + q_{(i)}^{-l} \nu_1 K^{-1} e_{l-1} f_{k-1} + \cdots + q_{(i)}^{-(l-1)} \nu_{l-1} K^{-(l-1)} e_{1} f_{t+1} + q_{(i)}^{-l} \nu_l K^{-l} f_t = f_k e_l + q_{(i)}^{l} \nu_1 K f_{k-1} e_{l-1} + \cdots + q_{(i)}^{l} \nu_{l-1} K^{l-1} f_{t+1} e_1 + q_{(i)}^{l} \nu_l K^l f_t.
\]
We substitute \( K \) by \( q_{(i)}^l K \) in formula (A.6) and obtain
\[
e_l f_k = \sum_{p=1}^{l} \left\{ \sum_{r=1}^{p} \left[ \nu_r \vartheta_{p-r} (q_{(i)}^l K)^{r-p} ((q_{(i)}^l K)^r - (q_{(i)}^l K)^{-r}) \right] \right\} f_{k-p} e_{l-p} + f_k e_l.
\]
Finally, we assume that \( l - k = t \), then \( s - m = t \). By (A.2), we get
\[
\sum_{n=0}^{k} q_{(i)}^{n} \nu_{n} K^{-n} e_{l-n} f_{k-n} = \sum_{n=0}^{k} q_{(i)}^{-n} \nu_{n} K^{n} f_{k-n} e_{l-n}.
\]
Hence, we have
\[
e_{l} f_{k} + q_{(i)}^{t} \nu_{1} K^{-1} e_{l-1} f_{k-1} + \cdots + q_{(i)}^{(l-1)t} \nu_{l-1} K^{-(l-1)} e_{l+1} f_{1} + q_{(i)}^{lt} \nu_{l} K^{-l} e_{l} = f_{k} e_{l} + q_{(i)}^{-t} \nu_{1} K f_{k-1} e_{l-1} + \cdots + q_{(i)}^{-(l-1)t} \nu_{l-1} K^{l(l-1)} f_{1} e_{l+1} + q_{(i)}^{-lt} \nu_{l} K^{l} e_{l}.
\]
We substitute \( K \) by \( q_{(i)}^{-t} K \) in formula (A.6) and obtain
\[
(A.8) \quad e_{l} f_{k} = \sum_{p=1}^{k} \left\{ \sum_{r=1}^{p} \left[ \nu_{r} \vartheta_{p-r} (q_{(i)}^{-t} K)^{r-p} ((q_{(i)}^{-t} K)^{r} - (q_{(i)}^{-t} K)^{-r}) \right] \right\} f_{k-p} e_{l-p} + f_{k} e_{l}.
\]
Combine the formulas (A.6), (A.7), and (A.8), we have the following statement.

**Proposition A.2.** For \( i \in \mathbb{N}^{m} \), we have the following commutation relations for all \( k, l > 0 \)
\[
e_{l} f_{k} - f_{k} e_{l} = \sum_{p=1}^{\min\{k, l\}} \left\{ \sum_{r=1}^{p} \left[ \nu_{r} \vartheta_{l-p-r} (q_{(i)}^{-k-l} K_{1})^{r-p} ((q_{(i)}^{-k-l} K_{1})^{r} - (q_{(i)}^{-k-l} K_{1})^{-r}) \right] \right\} f_{k-p} e_{l-r} + f_{k} e_{l}.
\]
Where \( \vartheta_{l-p-r} = \sum_{c \in c_{p-r}} (-1)^{\|c\|} \nu_{ic} \).

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