**STRENGTHENED FRACTIONAL SOBOLEV TYPE INEQUALITIES IN BESOV SPACES**

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**ABSTRACT.** The purpose of this article is twofold. The first is to strengthen fractional Sobolev type inequalities in Besov spaces via the classical Lorentz space. In doing so, we show that the Sobolev inequality in Besov spaces is equivalent to the fractional Hardy inequality and the iso-capacitary type inequality. Secondly, we will strengthen fractional Sobolev type inequalities in Besov spaces via capacitary Lorentz spaces associated with Besov capacities. For this purpose, we first study the embedding of the associated capacitary Lorentz space to the classical Lorentz space. Then, the embedding of the Besov space to the capacitary Lorentz space is established. Meanwhile, we show that these embeddings are closely related to the iso-capacitary type inequalities in terms of a new-introduced fractional $(\beta, p, q)$-perimeter. Moreover, characterizations of more general Sobolev type inequalities in Besov spaces have also been established.

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1. Introduction

The Sobolev inequality plays a significant role in harmonic analysis, mathematical physics, and PDEs. Interested readers are referred to see [8, 14, 19, 23, 31] and the references therein for more about Sobolev type inequalities. Fractional calculus and fractional PDEs have become extremely popular in mathematics, physics, engineering science and other areas, see [11, 21] for instance. Fractional Sobolev inequalities have been studied in many references, see [15, 16, 17, 24, 29, 34, 41, 39, 37, 38, 43] for instance.

Denote by $C^\infty_0(\mathbb{R}^n)$ the set of all compactly supported infinitely differentiable functions. Let $1 \leq p < n/\beta$. The following fractional Sobolev type inequality in Besov spaces $\Lambda_\beta^{p,p}(\mathbb{R}^n)$ holds:

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-p\beta}} \, dx \right)^{\frac{n-p\beta}{np}} \leq ||f||_{\Lambda_\beta^{p,p}(\mathbb{R}^n)} \quad \forall \ f \in C^\infty_0(\mathbb{R}^n),
\]

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(1.1)
see \cite[Theorem 7.34]{4} and \cite[Theorem 6.5]{33}. Motivated by \cite{40,41}, in this paper, we will strengthen the inequality \eqref{1.1} as follows:

\begin{equation}
\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}} \leq \left(\int_0^\infty V(O_t(f))^{\frac{np}{n-p}} dt\right)^{1/p} \lesssim \|f\|_{L_{p,q}^\alpha(\mathbb{R}^n)} \quad \forall \ f \in C_0^\infty(\mathbb{R}^n).
\end{equation}

Here $O_t(f) = \{x \in \mathbb{R}^n : |f(x)| > t\}$, and $V(O)$ denotes the volume of $O$ which is defined as the Lebesgue integral of the characteristic function $1_O$ on $\mathbb{R}^n$, i.e.,

$$V(O) := \int_{\mathbb{R}^n} 1_O(x)dx.$$ 

On the other hand, we will show that fractional Sobolev type inequalities in Besov spaces can be strengthened by a Choquet integral with respect to Besov capacities. That is, the second term in (1.2)

\begin{equation}
\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}} \leq \left(\int_0^\infty \left(C^{p,p}_\beta(O_t(f))\right)^{\frac{np}{n-p}} dt\right)^{1/p} \lesssim \|f\|_{\Lambda_{p,q}^\alpha(\mathbb{R}^n)} \quad \forall \ f \in C_0^\infty(\mathbb{R}^n).
\end{equation}

We will also show that (1.2) is equivalent to the iso-capacitary type inequality in terms of Besov capacities. While (1.3) implies the iso-capacitary type inequalities in terms of Besov capacities and a new introduced fractional $(\beta, p, q)$–perimeter $P_{\beta,q}^{p,q}(\cdot)$.

Moreover, we will show that the general Sobolev type inequality

\begin{equation}
\|f\|_{L_{p,q}^\alpha(\mathbb{R}^n)} \leq \|f\|_{\Lambda_{p,q}^\alpha(\mathbb{R}^n)},
\end{equation}

established in \cite[Theorem 7.34]{4}, is equivalent to a new fractional Hardy inequality, an iso-capacitary inequality, and can be strengthened by capacitary Lorentz norms when $q > p$.

Here, for $\beta \in (0, n)$, $p \in (0, n/\beta)$ and $q > 0$, $\Lambda_{p,q}^\beta(\mathbb{R}^n)$ are defined as the closure of all $C_0^\infty$ functions $f$ with $\|f\|_{\Lambda_{p,q}^\beta(\mathbb{R}^n)} < \infty$. The norm $\|f\|_{\Lambda_{p,q}^\beta(\mathbb{R}^n)}$ is defined as follows.

$$\|f\|_{\Lambda_{p,q}^\beta(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \|\Delta_h^k f\|_{L_p}^{p} |h|^{-\beta|\Delta^k|} dh\right)^{1/q}.$$

Here $k = 1 + |\beta|, \beta = |\beta| + |\beta|$ with $|\beta| \in \mathbb{Z}_+$, $|\beta| \in (0, 1)$ and

$$\Delta_h^k f(x) = \begin{cases} \Delta^k_h f(x), & k > 1; \\ f(x+h) - f(x), & k = 1. \end{cases}$$

For a compact set $K \subset \mathbb{R}^n$, the Besov capacity $C_\beta^{p,q}(K)$ is defined as

$$C_\beta^{p,q}(K) := \inf \left\{ \|f\|^p_{\Lambda_{p,q}^\beta(\mathbb{R}^n)} : f \in C_0^\infty(\mathbb{R}^n) \text{ and } f \geq 1_K \right\}$$

and for any set $E \subset \mathbb{R}^n$, one defines

$$C_\beta^{p,q}(E) := \inf_{\text{open } O \supseteq E} \sup_{\text{compact } K \subset O} \left\{ C_\beta^{p,q}(K) \right\},$$

where $1_E$ denotes the characteristic function of $E$. The Besov capacity $C_\beta^{p,q}(\cdot)$ has been studied in \cite{2,5} for $1 < q < \infty$, in \cite{43} for $p = q \in (0, 1)$.

This paper is organized as follows. In Section 2, we will show that Besov spaces can be embedded to capacitary Lorentz spaces. In Section 3 we first prove that fractional Sobolev type inequalities are equivalent to fractional Hardy inequalities and iso-capacitary inequalities. Then, we get strengthened Sobolev inequalities by Lorentz norms. In Section 4, firstly, we will study the embeddings of capacitary
Lorentz spaces to the classical Lorentz space, and that of Besov space to capacitary Lorentz spaces. Finally, we strengthen fractional Sobolev type inequalities by capacitary Lorentz norms.

Some notations:

- \( U \approx V \) means that there is a constant \( C > 0 \) such that \( C^{-1}V \leq U \leq CV \). If \( U \leq V \), then we write \( U \leq V \). Similarly, we write \( V \geq U \) if \( V \geq CU \).
- Let \( k \in \mathbb{N} \cup \{\infty\} \). The symbol \( \mathcal{C}^k(\mathbb{R}^n) \) denotes the class of all functions \( f : \mathbb{R}^n \to \mathbb{R} \) with \( k \) continuous partial derivatives. For any subset \( E \subseteq \mathbb{R}^n \), denote by \( 1_E \) the characteristic function of \( E \).

2. Capacitary Lorentz Spaces

In this paper, we need to use Lorentz/Lebesgue spaces associated with a nonnegative Radon measure \( \mu \), or the Besov capacity \( C_{\beta}^{p,q}(\cdot) \), or the Netrusov capacity. The Netrusov capacity \( H_{d,\vartheta}(\cdot) \) with \( (\varepsilon, d, \theta) \in (0, \infty) \times (0, \infty) \times (0, \infty) \) is defined as, see [32],

\[
H_{d,\vartheta}(\cdot) = \inf \left( \sum_{i=0}^{\infty} (m_i 2^{-id})^{\frac{1}{\theta}} \right),
\]

where the infimum is taken over all countable coverings of \( K \subset \mathbb{R}^n \) by balls whose radii \( r_j \) do not exceed \( \varepsilon \), while \( m_i \) is the number of balls from this covering whose radii \( r_j \) belong to the intervals \( (2^{-i-1}, 2^{-i}] \), \( i = 0, 1, 2, \ldots \). When \( \theta = 1 \), the Netrusov capacity \( H_{d,1}(\cdot) \) is the classical Hausdorff capacity \( H_d(\cdot) \).

Denote by \( \nu \) either a nonnegative Radon measure \( \mu \) on \( \mathbb{R}^n \), or the Besov capacity \( C_{\beta}^{p,q}(\cdot) \) with \( 0 < p, q < \infty \), or the Netrusov capacity \( H_{d,\vartheta}(\cdot) \). For \( 0 < p_0, q_0 < \infty, L^{p_0,q_0}(\mathbb{R}^n, \nu) \) and \( L^{p_0}(\mathbb{R}^n, \nu) \) denote the Lorentz space and the Lebesgue space of all functions \( g \) on \( \mathbb{R}^n \), respectively, for which

\[
\|g\|_{L^{p_0,q_0}(\mathbb{R}^n, \nu)} = \left( \int_0^{\infty} (\nu(\{g > t\}))^{p_0} \, dt \right)^{\frac{1}{q_0}} < \infty
\]

and

\[
\|g\|_{L^{p_0}(\mathbb{R}^n, \nu)} = \left( \int_{\mathbb{R}^n} |g(x)|^{p_0} \, d\nu \right)^{\frac{1}{p_0}} < \infty,
\]

respectively. Moreover, we denote by \( L^{p_0,\infty}(\mathbb{R}^n, \nu) \) the set of all \( \nu \)-measurable functions \( g \) on \( \mathbb{R}^n \) with

\[
\|g\|_{L^{p_0,\infty}(\mathbb{R}^n, \nu)} = \sup_{s>0} (\nu(\{g > s\}))^{\frac{1}{p_0}} < \infty.
\]

The following result is standard. For the readers’ convenience, we provide the proof here.

**Lemma 2.1.** For \( 0 < q_0 \leq r < \infty \) & \( p_0 > 0 \), there holds

\[
(2.1) \quad L^{p_0,q_0}(\mathbb{R}^n, \nu) \hookrightarrow L^{p_0,r}(\mathbb{R}^n, \nu) \hookrightarrow L^{p_0,\infty}(\mathbb{R}^n, \nu).
\]

**Proof.** Since \( \nu(\{f > s\}) \) is monotone decreasing in \( t \), we have

\[
\frac{d}{dt} \left( \int_0^r (\nu(\{f > s\}))^{\frac{r}{p_0}} \, ds \right)^{\frac{p_0}{r}} \leq p_0 \nu(\{f > s\})^{\frac{p_0}{r}}
\]

and

\[
(s^{p_0} \nu(\{f > s\}))^{\frac{r}{p_0}} \leq \left( \int_0^s (\nu(\{f > t\}))^{\frac{r}{p_0}} \, dt \right)^{\frac{p_0}{r}} \leq \int_0^s (\nu(\{f > t\}))^{\frac{r}{p_0}} \, dt \quad \forall \ s > 0,
\]

which implies \( \|f\|_{L^{p_0,\infty}(\mathbb{R}^n, \nu)} \leq \|f\|_{L^{p_0,r}(\mathbb{R}^n, \nu)} \). Thus, we have \( L^{p_0,r}(\mathbb{R}^n, \nu) \hookrightarrow L^{p_0,\infty}(\mathbb{R}^n, \nu) \).

On the other hand, since

\[
\|f\|_{L^{p_0,r}(\mathbb{R}^n, \nu)} = \left( \int_0^s (\nu(\{f > t\}))^{\frac{s}{p_0}} \, ds \right)^{\frac{p_0}{s}},
\]

we have

\[
\|f\|_{L^{p_0,\infty}(\mathbb{R}^n, \nu)} \leq \|f\|_{L^{p_0,r}(\mathbb{R}^n, \nu)} \leq \|f\|_{L^{p_0,\infty}(\mathbb{R}^n, \nu)}.
\]
we have

\[ \|f\|_{L^{p,r}(\mathbb{R}^n; \nu)} = \left( \int_0^\infty \lambda^{r-q_0+q_0} (\nu(O_\lambda(f)))^{r/p_0-q_0/p_0+q_0/p_0} \frac{d\lambda}{\lambda} \right)^{1/r} \leq \left( \sup_{\lambda > 0} (\lambda \nu(O_\lambda(f)))^{1/p_0} \right)^{1/q_0/r} \left( \int_0^\infty \lambda^{q_0} (\nu(O_\lambda(f)))^{q_0/p_0} \frac{d\lambda}{\lambda} \right)^{1/r} \leq \|f\|_{L^{p_0,q_0}(\mathbb{R}^n; \nu)}^{1/q_0/r} \left( \int_0^\infty \lambda^{q_0} (\nu(O_\lambda(f)))^{q_0/p_0} \frac{d\lambda}{\lambda} \right)^{1/r} \leq \|f\|_{L^{p_0,q_0}(\mathbb{R}^n; \nu)}^{1/q_0/r}, \]

which gives \( \|f\|_{L^{p_0,q_0}(\mathbb{R}^n; \nu)} \leq \|f\|_{L^{p_0,q_0}(\mathbb{R}^n; \nu)} \) and thus \( L^{p_0,q_0}(\mathbb{R}^n; \nu) \hookrightarrow L^{p_0,q_0}(\mathbb{R}^n; \nu) \). \( \square \)

When \( \nu = C^{p,q}_\beta(\cdot) \), we call \( L^{p_0,q_0}(\mathbb{R}^n; C^{p,q}_\beta(\cdot)), L^{p_0,q_0}(\mathbb{R}^n; C^{p,q}_\beta(\cdot)), \) and \( L^{p_0}(\mathbb{R}^n; C^{p,q}_\beta(\cdot)) \) the associated capacitary Lorentz/Lebesgue space. We can show that Besov spaces are embedded to the associated capacitory Lorentz spaces. Let \( p \lor q = \max\{p, q\} \) and \( p \land q = \min\{p, q\} \).

**Proposition 2.2.** Let \( \beta \in (0, 1), p = q \in (n/(n + \beta), 1) \), or \( (\beta, p, q) \in (0, n) \times [1, n/\beta) \times (1, \infty) \), \( p \lor q \leq r \leq \infty \). There holds

\[ (2.2) \quad \|f\|_{L^{p,p}(\mathbb{R}^n; C^{p,q}_\beta)} \leq \|f\|_{L^{p_0,q_0}(\mathbb{R}^n)}, \quad \forall f \in C^{p,q}_0(\mathbb{R}^n). \]

**Proof.** The case \( r = p \lor q \) is due to Maz’ya [30] when \( p = q > 1 \). When \( 1 \leq p \leq q < \infty, 0 < \beta < 1 \), Wu [43] proved (2.2). Adams-Xiao [3] established (2.2) when \( 0 < \beta < \infty, (p, q) \in (1/n/\beta) \times (1, \infty) \). Xiao-Zhai [43] showed that (2.2) holds when \( 0 < \beta < 1, n/(n + \beta) < p = q < 1 \). The case \( r > p \lor q \) follows from (ii) and the inclusion (2.1) directly. \( \square \)

Proposition 2.2 is very important in studying Sobolev type inequalities and Carleson embeddings problems, see [5,2,7,27,43] for instance. We will use Proposition 2.2 to establish the main results in this paper. Moreover, it helps us to get the following result which generalizes Admas’ inequalities to general fractional Besov space. In [1], Adams proved that

\[ \int_0^\infty H^{(\infty)}_{n-k}(O_\lambda(f)) \, dt \leq \|f\|_{L^1(\mathbb{R}^n)}, \quad \forall f \in C^0(\mathbb{R}^n), \]

which was generalized by Xiao in [38] to endpoint Besov spaces:

\[ \int_0^\infty H^{(\infty)}_{n-\beta}(O_\lambda(f)) \, dt \leq \|f\|_{H^{1,1}(\mathbb{R}^n)}, \quad \forall f \in C^0(\mathbb{R}^n), \]

when \( \beta \in (0, n) \). Here, we will get a similar inequality for \( H^{(\infty)}_{n-\beta}(\mathbb{R}^n) \) and the Netrusov capacity.

**Proposition 2.3.** Let \( \beta \in (0, 1), p = q \in (n/(n + \beta), 1) \), or \( (p, q) \in (1/n/\beta) \times (1, \infty) \). If \( r \in [p \lor q, \infty] \), then there holds

\[ \|f\|_{L^{p,r}(\mathbb{R}^n; H^{(\infty)}_{n-\beta,q/p})} \leq \|f\|_{H^{(\infty)}_{n-\beta,q/p}(\mathbb{R}^n)}, \quad \forall f \in C^0(\mathbb{R}^n). \]

**Proof.** It follows from [3] Theorem 2] and [62] Theorem 2] that \( C^{p,q}_\beta(\cdot) \approx H^{(\infty)}_{n-\beta,q/p}(\cdot) \). Thus, (ii) of Proposition 2.2 finishes the proof. \( \square \)

3. **Strengthened Fractional Sobolev Inequalities by Lorentz Spaces**

We will discuss the case \( p = q \) and \( p \neq q \) separately.
3.1. **The case** $p = q$. We will show that the fractional Sobolev type inequality in $\dot{L}^{p,q}_\beta(\mathbb{R}^n)$ can be strengthened. Our first result holds for general $p$ and $q$. Recall $p \vee q = \max\{p, q\}$ and $p \wedge q = \min\{p, q\}$.

**Theorem 3.1.** Let $\beta \in (0, 1)$ and $p = q \in (n/(n + \beta), 1)$, or $(\beta, p, q) \in (0, n) \times [1, n/\beta) \times (0, \infty)$. Let $p_0 \geq q_0 \geq p \vee q$, and $\mu$ be a non-negative Radon measure. Then the following statements are equivalent.

(i) For any $f \in C^0_0(\mathbb{R}^n)$,

$$\|f\|_{L^{p_0,q_0}(\mathbb{R}^n, \mu)} \leq \|f\|_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)}.$$ 

(ii) For any $f \in C^0_0(\mathbb{R}^n)$,

$$\|f\|_{L^{p_0}(\mathbb{R}^n)} \leq \|f\|_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)}.$$ 

(iii) For any $f \in C^\infty_0(\mathbb{R}^n)$,

$$\|f\|_{L^{p_0,\infty}(\mathbb{R}^n, \mu)} \leq \|f\|_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)}.$$ 

(iv) For any bounded domain $O \subset \mathbb{R}^n$ with $C^\infty$ boundary $\partial O$,

$$(\mu(O))^{\beta/p_0} \leq C^\beta_{p,q}(O).$$

**Proof.** Since the Lorentz space $L^{q}(\mathbb{R}^n; \mu)$ is increasing with respect to $q$ and $p_0 \geq q_0$, we have the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(iv). Assume that (iii) is true. Given a bounded domain $O \subset \mathbb{R}^n$, for any non-negative $f \in C^\infty_0(\mathbb{R}^n)$ with $f \geq 1$ on $O$, i.e., $O \subset \overline{O}(f)$, we have

$$\mu(O) \leq \mu(\overline{O}(f)) \leq \|f\|^{p_0}_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)},$$

which implies (iv) by taking infimum on $f$ on the right hand side of (3.1).

(iv)$\Rightarrow$(i). Assume that (iv) holds. Then, one has $(\mu(O))^{\beta/q_0} \leq (C^\beta_{p,q}(O))^{q_0/p}$ and hence,

$$\|f\|_{L^{p_0,\infty}(\mathbb{R}^n, \mu)} = \left(\int_0^{\infty} (\mu(\overline{O}(f)))^{q_0/p} dt^{p_0} \right)^{1/q_0} \leq \left(\int_0^{\infty} (C^\beta_{p,q}(O(\overline{f})))^{q_0/p} dt^{p_0} \right)^{1/q_0} \leq \|f\|_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)},$$

where in the last inequality of (3.2), we have used Proposition 2.2 since $q_0 \geq p \vee q$. This indicates (i). \qed

Theorem 3.1 itself is very important because it characterizes a Radon measure such that the Sobolev type inequality holds. The case of $p = q = q_0$ of Theorem 3.1 has been studied in [38, 45, 43]. In [22], Frank and Seiringer proved that the fractional Sobolev inequality can be deduced from the fractional Hardy inequality, see [22, Theorem 4.1]. Using the case $p = q = q_0$ and $p_0 = np/(n - \beta)$ of Theorem 3.1, we prove the following theorem which shows the equivalent of the fractional Sobolev inequality, the fractional Hardy inequality, and the iso-capacitary inequality.

**Theorem 3.2.** Let $\beta \in (0, 1)$, $1 \leq p < n/\beta$. Then the following statements are equivalent.

(i) The analytic inequality:

$$\left(\int_0^{\infty} (V(f))(d\mu)^{n/p} dt^p \right)^{1/p} \leq \|f\|_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)} \quad \forall \ f \in C^\infty_0(\mathbb{R}^n).$$

(ii) The fractional Sobolev inequality:

$$\left(\int_{\mathbb{R}^n} |f(x)|^{n/p} d\mu \right)^{1/n} \leq \|f\|_{\dot{L}^{p,q}_\beta(\mathbb{R}^n)} \quad \forall \ f \in C^\infty_0(\mathbb{R}^n).$$
(iii) The fractional Hardy inequality:

\[ \left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{\alpha p}} \, dx \right)^{1/p} \leq \|f\|_{L^{n/p,p}(\mathbb{R}^n)} \quad \forall \ f \in C_0^\infty(\mathbb{R}^n). \]

(iv) The iso-capacitary inequality: for any bounded domain \( O \subset \mathbb{R}^n \) with \( C^\infty \) boundary \( \partial O \),

\[ (V(O))^{\frac{n-p}{n}} \leq C_{\beta}^{p,p}(\bar{O}). \]

Moreover, (3.3), (3.4), (3.5) and (3.6) are all true.

**Proof.** In Theorem 3.1, let \( q_0 = p = q \), \( p_0 = np/n - p\beta \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R}^n \). We can see that the equivalence of (i) \( \iff \) (ii) \( \iff \) (iv) holds. We will only provide a proof of (i) \( \iff \) (iii).

For (i) \( \implies \) (iii), we assume that (3.3) holds. If \( A \subset \mathbb{R}^n \) is a Borel set of finite Lebesgue measure, we define \( A^\# \), the symmetric rearrangement of the set \( A \), to be the open ball centered at the origin whose volume is that of \( A \). The symmetric-decreasing rearrangement, \( f^\# \), of a function \( f \) is defined as follows. The symmetric-decreasing rearrangement of a characteristic function of a set is obvious, namely, \( 1_A^\# = 1_{A^\#} \). Now, if \( f : \mathbb{R}^n \to \mathbb{C} \) is a Borel measurable function vanishing at infinity, we define

\[ f^\#(x) := \int_0^\infty 1_{|f| > t}(x) \, dt \]

as the symmetric-decreasing rearrangement of \( f \).

Firstly, we show that there exists a constant \( C \) such that

\[ \left( \int_{\mathbb{R}^n} \frac{|f^\#(x)|^p}{|x|^{\alpha p}} \, dx \right)^{1/p} = C\|f\|_{L^{n/p,p}(\mathbb{R}^n)}, \]

which is equivalent to [22] Lemma 4.3 since the Lorentz norm is invariant under the symmetric decreasing rearrangement. For the convenience, we provide the proof here. In fact, for \( t > 0 \), define

\[ O_t(f) = \{ x \in \mathbb{R}^n : |f(x)| > t \}. \]

Then \( V(O_t(f^\#)) = V(O_t((f^\#)^p)) \). It follows from the equimeasurability of the functions \( f \) and \( f^\# \) that \( V(O_t(f^\#)) = V(O_t((f^\#)^p)) \), which implies

\[ \left( \int_0^\infty (V(O_t(f)))^{\frac{n-p\beta}{n}} \, dt \right)^{1/p} = \left( \int_0^\infty (V(O_t((f^\#)^p)))^{\frac{n-p\beta}{n}} \, dt \right)^{1/p} = \left( \int_0^\infty (V(O_t((f^\#)^p)))^{\frac{n-p\beta}{n}} \, dt \right)^{1/p}. \]

Since \( (f^\#)^p \) is a non-negative symmetric decreasing function, \( |O_t((f^\#)^p)| \) is the same as the volume of a ball \( B_{r(t)} \) with radius

\[ r(t) = \left( \frac{V(|O_t((f^\#)^p))|}{\omega_n} \right)^{1/n} \]

with \( \omega_n \) the surface area of the unit sphere \( S^{n-1} \). Thus, there exists a constant \( C \) such that

\[ \left( \int_{\mathbb{R}^n} \frac{|f^\#(x)|^p}{|x|^{\alpha p}} \, dx \right)^{1/p} = C\|f\|_{L^{n/p,p}(\mathbb{R}^n)} \]

since \( \| \cdot \|_{L^{n/p,p}(\mathbb{R}^n)} \) is invariant under the symmetric decreasing rearrangement.
Note that [28, Theorem 3.4] reads as
\[
\int_{\mathbb{R}^n} f(x)g(x)\,dx \leq \int_{\mathbb{R}^n} f^\#(x)g^\#(x)\,dx.
\]
Notice that \((\Phi \circ |f|^p)^\# = \Phi \circ (f^\#)^p\), where \(\Phi(t) = t^{\nu}\) is non-decreasing for \(t > 0\). Since \(|x|^{-\rho\beta}\) is symmetric-decreasing, it follows from (3.7) and (3.3) that
\[
(\int_{\mathbb{R}^n} |f(x)|^p \frac{1}{|x|^{\rho\beta}}\,dx)^{1/p} \leq (\int_{\mathbb{R}^n} |f^\#(x)|^p \frac{1}{|x|^{\rho\beta}}\,dx)^{1/p} \leq \|f\|_{L^{p/(p-\rho\beta)}} \leq \|f\|_{\Lambda_{p}^{\rho\beta}(\mathbb{R}^n)},
\]
which gives us (3.5). Thus (iii) is true.

For (iii) \(\Rightarrow\) (i), we assume (3.5) holds. Since the Lorentz norm is invariant under the symmetric decreasing rearrangement, using (3.3) one has
\[
\left( \int_0^\infty V(O_{t}(f)) \frac{n-\eta t}{n} \,dt \right)^{1/p} = \left( \int_0^\infty (V(O_{t}(f^\#)) \frac{n-\eta t}{n} \,dt \right)^{1/p} = \left( \int_{\mathbb{R}^n} |f^\#(x)|^p \frac{1}{|x|^{\rho\beta}}\,dx \right)^{1/p} \leq \|f^\#\|_{\Lambda_{p}^{\rho\beta}(\mathbb{R}^n)} \leq \|f\|_{\Lambda_{p}^{\rho\beta}(\mathbb{R}^n)},
\]
where in the last step, we have used [6, Theorem 9.2] which states that the symmetric decreasing rearrangement is continuous under the fractional Besov norm \(\Lambda_{p}^{\rho\beta}(\mathbb{R}^n)\).

\[\square\]

**Remark 3.3.**
(i) The sharp constant of inequalities (3.4) is only known when \(p = 1\), see [22, Theorem 4.1] or [9, Theorem 4.10].
(ii) The requirement \(\beta \in (0,1)\) is only used when proving the equivalence of (i) and (iii). For the equivalence of (i) \(\iff\) (ii) \(\iff\) (iv), \(\beta\) can take values in \((0,n)\).

Since \(p < \frac{np}{n-\rho\beta}\), we have
\[
L^{\frac{np}{n-\rho\beta}}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-\rho\beta}}(\mathbb{R}^n).
\]
Therefore, (3.8) and (3.4) imply the following strengthened fractional Sobolev type inequality in Besov spaces.

**Corollary 3.4.** Let \(\beta \in (0,n)\) and \(1 \leq p \leq n/\beta\). For any \(f \in C_0^\infty(\mathbb{R}^n)\), there holds
\[
(3.9) \quad \left( \int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-\rho\beta}}\,dx \right)^{1/p} \leq \left( \int_0^\infty V(O_{t}(f)) \frac{n-\eta t}{n} \,dt \right)^{1/p} \leq \|f\|_{\Lambda_{p}^{\rho\beta}(\mathbb{R}^n)}.
\]

3.2. **The case** \(p \neq q\). The inequality (3.9) strengthens the fractional Sobolev type inequality in Besov space \(\Lambda_{p}^{\rho\beta}(\mathbb{R}^n)\) when \(p = q\). In the following, we consider the case \(q \neq p\). Firstly, we need to establish the following result similar to Theorem 3.1 without the requirement of \(p_0 \geq p \lor q\).

**Proposition 3.5.** Let \((\beta,p,q) \in (0,n) \times [1, n/\beta] \times (1,\infty)\). Let \(p_0 > 0\) and \(q_0 \geq p \lor q\), \(p_0 > 0\) and \(\mu\) be a non-negative Radon measure. Then the following statements are equivalent.
(i) For any \(f \in C_0^\infty(\mathbb{R}^n)\),
\[
\|f\|_{L^{p_0,q_0}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\Lambda_{p_0}^{\rho,q_0}(\mathbb{R}^n)},
\]
(ii) For any \(f \in C_0^\infty(\mathbb{R}^n)\),
\[
\|f\|_{L^{p_0,q_0}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{\Lambda_{p_0}^{\rho,q_0}(\mathbb{R}^n)}.
\]
(iii) For any bounded domain \(O \subset \mathbb{R}^n\) with \(C^\infty\) boundary \(\partial O\),
\[
(\mu(O))^{p_0/p_0} \leq C_{\beta}^{|p-q|}(O).
\]

**Proof.** Since the Lorentz spaces \(L^{p_0,q_0}(\mathbb{R}^n; \mu)\) is increasing with respect to \(q_0\), we have the implications (i) \(\implies\) (ii).

(ii) \(\implies\) (iii). Assume that (ii) is true. Given a bounded domain \(O \subset \mathbb{R}^n\), for any non-negative \(f \in C_0^\infty(\mathbb{R}^n)\) with \(f \geq 1\) on \(O\), i.e., \(O \subset \partial \Omega(f)\), we have
\[
\mu(O) \leq \mu(\partial \Omega(f)) \lesssim \| f \|_{\Lambda_{\beta}^{p,q}(\mathbb{R}^n)},
\]
which implies (iii) by taking infimum on \(f\).

(iii) \(\implies\) (i). Assume that (iv) holds. Then, one has \((\mu(O))^{q_0/p_0} \leq (C_{\beta}^{|p-q|}(O))^{q_0/p_0}\) and hence,
\[
\| f \|_{L^{p_0,q_0}(\mathbb{R}^n; \mu)} = \left( \int_0^\infty (\mu(O))^{q_0/p_0} \, dt \right)^{1/q_0} \leq \left( \int_0^\infty (C_{\beta}^{p,q}(O))^{q_0/p_0} \, dt \right)^{1/q_0} \leq \| f \|_{\Lambda_{\beta}^{p,q}(\mathbb{R}^n)},
\]
where in the last inequality of (3.10), we have used Proposition 2.2 since \(q_0 \geq p \vee q\). This indicates (i).

**Theorem 3.6.** Let \((\beta, p, q) \in (0, n) \times [1, n/\beta) \times [1, \infty)\). Then the following statements are equivalent.

(i) The analytic inequality:
\[
\left( \int_0^\infty (V(O))^{a(p)(q)/(np)} \, dt \right)^{1/np} \lesssim \| f \|_{\Lambda_{\beta}^{p,q}(\mathbb{R}^n)} \quad \forall f \in C_0^\infty(\mathbb{R}^n).
\]

(ii) The iso-capacitary inequality: for any bounded domain \(O \subset \mathbb{R}^n\) with \(C^\infty\) boundary \(\partial O\),
\[
(V(O))^{\frac{np}{n-\beta}} \lesssim C_{\beta}^{p,q}(\mathbb{R}^n).
\]

(iii) The fractional Hardy inequality: for \(\gamma = n(1 - (p \vee q)/p) + \beta(p \vee q),\)
\[
\left( \int_{\mathbb{R}^n} \frac{|f(x)|^{p \vee q}}{|x|^\gamma} \, dx \right)^{1/(p \vee q)} \lesssim \| f \|_{\Lambda_{\beta}^{p,q}(\mathbb{R}^n)} \quad \forall f \in C_0^\infty(\mathbb{R}^n).
\]

Moreover, when \(q > p\), (3.11), (3.12) and (3.13) are all true.

**Proof.** The equivalence of (3.11) and (3.12) is a special of Proposition 3.5 when \(\mu\) is the Lebesgue measure, \(q_0 = p \vee q\), and \(p_0 = np/n - \beta\). It follows from [3.14 Theorem 7.34] that \(\Lambda_{\beta}^{p,q}(\mathbb{R}^n) \hookrightarrow L^{a(p)(q)/(np)}(\mathbb{R}^n)\) when \(q \in [1, \infty)\). Thus, (3.11), (3.12) and (3.13) are true when \(q > p\).

Below we prove (i) is equivalent to (iii). We first assume that (i) holds. For any Borel set \(A \subset \mathbb{R}^n\) with finite Lebesgue measure, denote by \(A^*\) the symmetric rearrangement of \(A\), which is the open ball centered at the origin whose volume is that of \(A\). Let \(f^*\) denote the symmetric-decreasing rearrangement of a function \(f\). For \(t > 0\), define \(O_t(f) := \{ x \in \mathbb{R}^n : |f(x)| > t \}\). It is easy to see that \(V(O_t(f^*)) =\)
\( V(O_{1/p}(f)), \ p > 1. \) Then we can get
\[
\left( \int_0^\infty (V(O_t(f)))^{\frac{(n-p)(p;q)}{np}} \, dt \right)^{\frac{1}{p^*q}} \\
= \left( \int_0^\infty (V(O_t(f^#)))^{\frac{(n-p)(p;q)}{np}} \, dt \right)^{\frac{1}{p^*q}} \\
\approx \left( \int_0^\infty (V(O_t((f^#)^{p;q})))^{\frac{(n-p)(p;q)}{np}} \, dt \right)^{\frac{1}{p^*q}}.
\]

Denote by \( B_{r(t)} \) the ball centered at the origin with the radius
\[
r(t) := \left( V(O_t((f^#)^{p;q})) \right)^{\frac{1}{n}}.
\]

Then
\[
\left( \int_{\mathbb{R}^n} \frac{|f^#(x)|^{p;q}}{|x|^p} \, dx \right)^{\frac{1}{p^*q}} = \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{|x|^p} \, dx \, dt \right)^{\frac{1}{p^*q}} \\
= \left( \int_0^\infty (V(O_t((f^#)^{p;q})))^{\frac{n-p}{n}} \, dt \right)^{\frac{1}{p^*q}}.
\]

Using [28, Theorem 3.4], we obtain
\[
\left( \int_{\mathbb{R}^n} \frac{|f(x)|^{p;q}}{|x|^p} \, dx \right)^{\frac{1}{p^*q}} \leq \left( \int_{\mathbb{R}^n} \frac{|f^#(x)|^{p;q}}{|x|^p} \, dx \right)^{\frac{1}{p^*q}} \\
\leq \left( \int_0^\infty (V(O_t((f^#)^{p;q})))^{\frac{n-p}{n}} \, dt \right)^{\frac{1}{p^*q}} \\
\approx \left( \int_0^\infty (V(O_t(f^{p;q})))^{\frac{n-p}{n}} \, dt \right)^{\frac{1}{p^*q}} \\
\approx \left( \int_0^\infty (V(O_t(f)))^{\frac{n-p}{n}} \, dt^{p;q} \right)^{\frac{1}{p^*q}} \\
\leq \|f\|_{\Lambda^p_{p,q}(\mathbb{R}^n)},
\]
which gives (3.13).

Conversely, if (iii) holds, then via a similar procedure, we can deduce that
\[
\left( \int_0^\infty (V(O_t(f)))^{\frac{(n-p)(p;q)}{np}} \, dt \right)^{\frac{1}{p^*q}} \\
\approx \left( \int_0^\infty (V(O_t(f^#)))^{\frac{(n-p)(p;q)}{np}} \, dt \right)^{\frac{1}{p^*q}} \\
\approx \left( \int_{\mathbb{R}^n} \frac{|f^#(x)|^{p;q}}{|x|^p} \, dx \right)^{\frac{1}{p^*q}} \\
\leq \|f\|_{\Lambda^p_{p,q}(\mathbb{R}^n)},
\]
which proves (i). \( \square \)
Remark 3.7. When $q > p$, Theorem 3.6 implies that the Sobolev inequality
\begin{equation}
\| f \|_{L^{\frac{n p}{n-p q}}(\mathbb{R}^n)} \leq \| f \|_{\Lambda^{p q}_p(\mathbb{R}^n)},
\end{equation}
established in [4] Theorem 7.34, is equivalent to the iso-capacitary inequality: $(V(O))^{n-p q} \leq C^{p q}_p(\overline{O})$, and the fractional Hardy inequality:
\begin{equation}
\left( \int_{\mathbb{R}^n} \frac{|f(x)|^q}{|x|^{n(q/p-q) + \beta}} \, dx \right)^{1/q} \leq \| f \|_{\Lambda^{p q}_p(\mathbb{R}^n)}
\end{equation}
which is more general than the fractional Hardy inequality in $\dot{\Lambda}^{p q}_p(\mathbb{R}^n)$. In the next section, we will strengthen (3.14) by capacitary Lorentz norms.

4. Strengthened Fractional Sobolev Inequalities by Capacitary Lorentz Spaces

4.1. Embeddings of Capacitary Lorentz Spaces to Lorentz Spaces. In this section, we prove that the second term in (3.9) will be replaced by capacitary Lorentz norms.

Theorem 4.1. Let $\beta \in (0, n)$, $p \geq 1$, $q > 0$, $1 < r < \infty$. Let $p_0 \geq 1$, $q_0 > 0$, and $\mu$ be a non-negative Radon measure. Then the following statements are equivalent.

(i) The embedding:
\begin{equation}
\| f \|_{L^{r, q_0}(\mathbb{R}^n)} \leq \| f \|_{L^{p_0, q_0}(\mathbb{R}^n)} \quad \forall f \in C_0^\infty(\mathbb{R}^n).
\end{equation}

(ii) The iso-capacitary inequality:
\begin{equation}
(\mu(O))^{p_0/r} \leq C^{p_0 q_0}_p(\overline{O}),
\end{equation}
holds for any bounded domain $O \subset \mathbb{R}^n$ with $C^\infty$ boundary $\partial O$.

Proof. Suppose that (4.2) is true. For any $f \in C_0^\infty(\mathbb{R}^n)$, (4.2) implies
\begin{equation}
(\mu(O(f)))^{p_0/r} \leq C^{p_0 q_0}_p(O(f))
\end{equation}
and so
\begin{align}
\| f \|_{L^{r, q_0}(\mathbb{R}^n)} &= \left( \int_0^\infty \mu(O(f))^{p_0/r} \, dt^{q_0} \right)^{1/q_0} \\
&\leq \left( \int_0^\infty (C^{p_0 q_0}_p(O(f)))^{p_0/r} \, dt^{q_0} \right)^{1/q_0}.
\end{align}

Thus, (4.1) holds.

Now, assume that (4.1) is true. For any bounded domain $O \subset \mathbb{R}^n$ with $C^\infty$ boundary $\partial O$, denote by $\text{dist}(x, E)$ the Euclidean distance of a point $x$ to a set $E$. For any $\varepsilon \in (0, 1)$, define
\begin{equation}
f_\varepsilon(x) = \begin{cases} 
1 - \varepsilon^{-1} \text{dist}(x, \overline{O}), & \text{dist}(x, \overline{O}) < \varepsilon; \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

Thus, $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and
\begin{equation}
O(f_\varepsilon) \subset U_1 := \{ x \in \mathbb{R}^n : \text{dist}(x, \overline{O}) < 1 \}
\end{equation}
and so $\mu(O(f_\varepsilon)) \leq \mu(U_1) < \infty$. Then, the dominated convergence theorem implies
\begin{align}
\lim_{\varepsilon \to 0^+} \| f_\varepsilon \|_{L^{r, q_0}(\mathbb{R}^n)} &= \lim_{\varepsilon \to 0^+} \left( \int_0^1 (\mu(O(f_\varepsilon)))^{p_0/r} \, dt^{q_0} \right)^{1/q_0} \\
&= \left( \int_0^1 \lim_{\varepsilon \to 0^+} (\mu(O(f_\varepsilon)))^{p_0/r} \, dt^{q_0} \right)^{1/q_0} \\
&= (\mu(O))^{1/r}.
\end{align}
On the other hand, (4.1) implies
\[ \|f_e\|_{L^{q_0}}(\mathbb{R}^n, \mu) = \left( \int_0^\infty (\mu(O(f_e)))^{q_0/p} \, dt^{q_0} \right)^{1/q_0} \leq \left( \int_0^1 \left( C_{p,q}^{p,q}(O_\varepsilon(f_e)) \right)^{q_0/p} \, dt^{q_0} \right)^{1/q_0} \leq \left( C_{p,q}^{p,q}(O_\varepsilon) \right)^{1/p} , \]
where the last inequality is due to
\[ \{ x \in \mathbb{R}^n : |f_e(x)| \geq t \} \subset O_\varepsilon := \{ x \in \mathbb{R}^n : \text{dist}(x, \overline{O} < \varepsilon) \} . \]
Letting \( \varepsilon \to 0^+ \) gives us (4.2) due to (4.3). \( \square \)

**Theorem 4.2.** Let \( \beta \in (0, n), q > 0, 1 \leq p < n/\beta, \) and \( q_0 > 0. \) Then the following two statements are equivalent.
(i) The embedding:
\[ (4.4) \quad \|f\|_{L^{n/p}}(\mathbb{R}^n) \leq \|f\|_{L^{p,q_0}(\mathbb{R}^n, \mathcal{C}_{\beta}^{p,q})} \quad \forall f \in \mathcal{C}_{0}^{p,q}(\mathbb{R}^n). \]
(ii) The iso-capacitary inequality
\[ (4.5) \quad (V(O))^{n/p} \leq C_{p,q}^{p,q}(O) \quad \forall \text{bounded domain } O \subset \mathbb{R}^n \text{ with } C^\infty \text{boundary } \partial O . \]
Moreover, when \( q \geq p, \) both (4.4) and (4.5) are true.

**Proof.** The truth of the iso-capacitary inequality (4.5) was established in Theorem 3.2 for \( p = q, \) and Theorem 3.6 for \( q > p. \) So, (4.5) implies the truth of (4.4) if we establish the equivalence of (4.4)-4.5 which is the special case of Theorem 4.1 when \( r = np/(n-p\beta), p_0 = p \) and \( \mu \) is the Lebesgue measure on \( \mathbb{R}^n. \) \( \square \)

### 4.2. Embeddings of Besov Spaces to Capacitary Lorentz Spaces.
Below we will show that the embedding \( \Lambda^{p,q}_{\beta}(\mathbb{R}^n) \hookrightarrow L^{p,q_0}(\mathbb{R}^n, \mathcal{C}_{\beta}^{p,p}) \) implies the iso-capacitary inequality in term of a new introduced fractional \((\beta, p, q)\)-perimeter \( P_{\beta}^{p,q}(E) : \)
\[ \left( C_{p,q}^{p,q}(O) \right)^{1/p} \leq 2P_{\beta}^{p,q}(O) \]
for \( p \in [1, n/\beta), \beta \in (0, 1) \) and all bounded domain \( O \subset \mathbb{R}^n \) with \( C^\infty \text{boundary } \partial O. \) Here \( P_{\beta}^{p,q}(O) \) is defined as follows.

**Definition 4.3.** Let \( p, q > 0. \) For any \( E \subset \mathbb{R}^n, \) let \( E^c = \mathbb{R}^n \setminus E. \) The fractional \((\beta, p, q)\)-perimeter \( P_{\beta}^{p,q}(E) \) is defined as
\[ (4.6) \quad P_{\beta}^{p,q}(E) = \left( \int_E \left( \int_{E^c} \frac{dx}{|x-y|^{(n+pq\beta)/q}} \right)^{q/p} \, dy \right)^{1/q} . \]

When \( p = q = 1, \) \( P_{\beta}^{1,1}(E) = P_{\beta}(E) \) which is the fractional \( \beta \)-perimeter \( P_{\beta}(E) \) defined as
\[ P_{\beta}(E) := \frac{1}{2} \|1_E\|_{\Lambda^{1,1}_{\beta}(\mathbb{R}^n)} = \int_E \int_{E^c} \frac{dxdy}{|x-y|^{|\beta|+1}} . \]
The regularity of set with minimal fractional perimeter \( P_{\beta}(E), \) the approximation of \( P_{\beta}(E) \) to the classical perimeter and other geometric properties of \( P_{\beta}(E) \) have been studied in [9, 12, 7, 10, 13, 15, 55, 36, 22].
The fractional perimeter $P_p(E)$ has been applied to study other embeddings in \cite{27}. When $p = q > 1$ and $p \beta \in (0, 1)$, $P_p^{\beta}(E) = (P_p(E))^{1/p}$.

**Theorem 4.4.** Let $\beta \in (0, n)$, $1 \leq p < n/\beta$, $q > 0$, and $p \vee q \leq q_0 < \infty$.

(i) The following embedding holds:

$$
\|f\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{A^p_q(\mathbb{R}^n)} \quad \forall f \in C^\infty_0(\mathbb{R}^n).
$$

(ii) If $\beta \in (0, 1)$, then the iso-capacitary inequality holds: for any bounded domain $O \subset \mathbb{R}^n$ with $C^\infty$-boundary $\partial O$, there holds the following geometric inequality:

$$
(C^{p,q}_\beta(\partial O))^{1/p} \lesssim P^{p,q}_\beta(O).
$$

**Proof.** For (i), Proposition 2.2 implies the truth of \cite{47}. Now we prove (ii). For $\varepsilon > 0$ and a bounded domain $O \subset \mathbb{R}^n$ with $C^\infty$ boundary $\partial O$, denote

$$
O_\varepsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \overline{O}) < \varepsilon\}
$$

and

$$
f_\varepsilon(x) := \begin{cases} 
1 - \varepsilon^{-1}\text{dist}(x, \overline{O}), & \text{dist}(x, \overline{O}) < \varepsilon; \\
0, & \text{otherwise}.
\end{cases}
$$

Then, $f_\varepsilon(x) = 1$ for all $x \in \overline{O}$ and so $\overline{O} \subset O_\varepsilon$ for all $\varepsilon \in (0, 1)$ and $t \in (0, 1)$. Thus,

$$
(C^{p,q}_\beta(\partial O))^{1/p} \lesssim \left(\int_0^\infty \left(\int_{O_\varepsilon} \left(\frac{1}{|y|^{(p+q)/p}} \text{dist}(x, \overline{O})^p \right)^{q/p} dy\right)^{1/q} dt\right) \lesssim \|f_\varepsilon\|_{A^p_q(\mathbb{R}^n)}.
$$

Since $f_\varepsilon \to 1_\partial$ as $\varepsilon \to 0^+$, the dominated convergent theorem implies

$$
(C^{p,q}_\beta(\partial O))^{1/p} \lesssim \lim_{\varepsilon \to 0^+} \|f_\varepsilon\|_{A^p_q(\mathbb{R}^n)} = \|1_\partial\|_{A^p_q(\mathbb{R}^n)} = 2 P^{p,q}_\beta(O).
$$

Thus, \cite{48} holds.

**Remark 4.5.** When $p = q = 1$ and $q_0 = n/(n-\beta)$, Xiao \cite{41} showed that \cite{47} and \cite{48} are true, sharp and equivalent using the general co-area formula

$$
\|f\|_{A^1_q(\mathbb{R}^n)} = 2 \int_0^\infty P_\beta(O_t(f)) dt.
$$

### 4.3. Strengthened Fractional Sobolev Inequalities by Capacitary Lorentz Spaces.

Based on Theorems 4.2 and 4.4 we can strengthen the fractional Sobolev inequality

$$
\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-p\beta}} dx\right)^{\frac{n-p\beta}{np}} \lesssim \|f\|_{A^p_q(\mathbb{R}^n)}
$$

and the isoperimetric type inequality

$$
(V(O))^{1-\beta/n} \lesssim (2P^{p,q}_\beta(O))^p.
$$

**Corollary 4.6.** Let $\beta \in (0, n)$ and $1 \leq p < n/\beta$, $p \vee q \leq q_0 < \infty$. 

(i) There holds the analytic inequality:
\[(4.9) \quad \|f\|_{L^{\frac{np}{n-p\beta}}(\mathbb{R}^n)} \leq \|f\|_{L^{p,q}(\mathbb{R}^n)} \leq \|f\|_{\Lambda^{p,q}_{\beta}(\mathbb{R}^n)} \quad \forall f \in C_0^{\infty}(\mathbb{R}^n).\]

When \(\beta \in (0, 1)\), (i) implies the following geometric inequality.

(ii) The geometric inequality:
\[\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-p\beta}} \, dx\right)^{\frac{n-p\beta}{np}} \leq \left(\int_0^\infty \left(\mathcal{C}^{p,q}_\beta(O(f))\right)^{\frac{np}{n-p\beta}} \, dt\right)^{\frac{n-p\beta}{np}} \leq \|f\|_{\Lambda^{p,q}_{\beta}(\mathbb{R}^n)}\]

which strengthens the fractional Sobolev inequality \((1.1)\). When \(p = q = 1\), \((4.10)\) was established by Xiao in \([40, 41]\).

(ii) When \(q_0 = q \geq p \geq 1\), \((4.9)\) implies
\[(4.10) \quad \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-p\beta}} \, dx\right)^{\frac{n-p\beta}{np}} \leq \left(\int_0^\infty \left(\mathcal{C}^{p,q}_\beta(O(f))\right)^{\frac{np}{n-p\beta}} \, dt\right)^{\frac{n-p\beta}{np}} \leq \|f\|_{\Lambda^{p,q}_{\beta}(\mathbb{R}^n)}\]

which strengthens the Sobolev type inequality
\[\|f\|_{L^{\frac{np}{n-p\beta}}(\mathbb{R}^n)} \leq \|f\|_{\Lambda^{p,q}_{\beta}(\mathbb{R}^n)}\]

established in \([4]\) Theorem 7.34).

(iii) When \(p = 1\), in \([25]\), \((V(O)\frac{np}{n}) \leq 2p_{\beta,1}(O)\) was proved to be equivalent to the fractional Sobolev inequality
\[
\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{np}{n-p\beta}} \, dx\right)^{\frac{n-p\beta}{np}} \leq \|f\|_{\Lambda^{p,q}_{\beta}(\mathbb{R}^n)}.
\]

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