ReLU nets adapt to intrinsic dimensionality beyond the target domain

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Abstract

We study the approximation of two-layer compositions \( f(x) = g(\phi(x)) \) via deep ReLU networks, where \( \phi \) is a nonlinear, geometrically intuitive, and dimensionality reducing feature map. We focus on two complementary choices for \( \phi \) that are intuitive and frequently appearing in the statistical literature. The resulting approximation rates are near optimal and show adaptivity to intrinsic notions of complexity, which significantly extend a series of recent works on approximating targets over low-dimensional manifolds. Specifically, we show that ReLU nets can express functions, which are invariant to the input up to an orthogonal projection onto a low-dimensional manifold, with the same efficiency as if the target domain would be the manifold itself. This implies approximation via ReLU nets is faithful to an intrinsic dimensionality governed by the target \( f \) itself, rather than the dimensionality of the approximation domain. As an application of our approximation bounds, we study empirical risk minimization over a space of sparsely constrained ReLU nets under the assumption that the conditional expectation satisfies one of the proposed models. We show near-optimal estimation guarantees in regression and classifications problems, for which, to the best of our knowledge, no efficient estimator has been developed so far.

Keywords: deep ReLU nets, approximation theory, curse-of-dimensionality, empirical risk minimization, low-dimensional adaptivity, function-adaptive complexity, noisy manifold models

1 Introduction

In the past decade, neural networks have emerged as powerful tools to construct state-of-the-art solutions for various different data analysis tasks. Much of this progress is of empirical nature and not well founded in solid mathematical theory. This has led to a reemerging interest into developing a theoretical understanding of complicated deep network models, with a focus on approximation theory, generalization performance, and the role of the training algorithm.

Approximation theory of neural networks with rectified linear units (ReLUs) has seen much progress in recent years. A breakthrough has been achieved in [71] who shows that deep ReLU nets can efficiently approximate the square function \( x \mapsto x^2 \). Despite the seemingly simplistic nature of the statement, it immediately implies efficient approximation of general multiplication [71] and thus paves the way to approximate polynomials, affine representation systems such as wavelets, rational functions, or generally functions with different smoothness properties [11,19,52,61,67,71,72]. For instance, it is now well-known that functions \( f : [0,1]^D \to \mathbb{R} \), whose derivatives of order up to \( \alpha - 1 \) exist and are Lipschitz continuous, can be approximated up to accuracy \( \epsilon \) by ReLU networks with \( \mathcal{O}(\epsilon^{-\frac{D}{2}}) \) nonzero parameters [71].

Although these results imply optimal approximation efficiency under the assumption of a continuous map from \( f \) to the network parameters [15], they suggest a rather pessimistic scaling of the number of
parameters with the dimension $D$ of the target domain. Indeed, such results are not informative in the high-dimensional regime as the number of required parameters for representing $f : [0,1]^D \to \mathbb{R}$ quickly explodes. As clarified by optimality results and lower bounds [15], the exponential dependency is a bottleneck shared among all approximation techniques and therefore it can not be avoided by traditional smoothness assumptions on the target $f$ [48]. Instead, complementary structural assumptions have become increasingly important as a remedy in the high-dimensional regime.

The most popular approach is to study approximation of $f$ on intrinsically low dimensional domains [13,46,57,60]. For instance, the approximation domain might be represented as a $d$-dimensional smooth embedded submanifold $\mathcal{M} \subset \mathbb{R}^D$, in which case a ReLU net with $\cal{O}(\epsilon^{-\tilde{\beta}})$ nonzero parameters suffices to approximate $f$ up to accuracy $\epsilon$ [13,57,60]. In light of the prevalent assumption that real-world data is inherently structured and intrinsically low-dimensional, such results are more informative than general approximation guarantees over $[0,1]^D$. Furthermore, they can be used in tandem with techniques from statistical learning to establish nontrivial generalization bounds about empirical risk minimizers over sparsely-constrained ReLU nets [57].

An alternative setting assumes that the target $f$ can be written as a composite function $f(x) = g_L \circ \ldots \circ g_1(x)$ with building blocks $g_1, \ldots, g_L$ that are easier to approximate than the composed target $f$ [39,40,43,54–56]. While such an assumption seems natural in view of the hierarchical nature of ReLU nets, they often tend to be abstract and thus challenging to validate or compare with real-world settings. Still, many composite models lead to significantly improved approximation guarantees, where the dimension $D$ of the approximation rate $\alpha/D$ is replaced by notions of layer-to-layer connectivity.

1.1 Main goals

Our main goal in this work is to extend ReLU approximation and estimation results from low-dimensional target domains to cases where low-dimensionality is encoded in the joint input-output relation $x \mapsto f(x)$. Borrowing tools from composite functions, we formulate the target as a two-layer composition $f(x) = g(\phi(x))$, where $\phi(x)$ is a geometrically intuitive, dimensionality reducing, nonlinear feature map that encodes low-dimensional sufficient information for allowing perfect approximation of $f$. We abstain from considering increasingly deep compositions of functions to preserve a geometrical intuition about our models.

To make the above more rigorous, our focus is on targets $f$ where the feature map $\phi$ is an orthogonal projection onto a $d$-dimensional connected compact Riemannian submanifold $\mathcal{M}$ of $\mathbb{R}^D$. In this case we can write the target $f : \mathcal{A} \subset [0,1]^D \to \mathbb{R}$ as

$$f(x) = g(\pi_\mathcal{M}(x)) \quad \text{where} \quad \pi_\mathcal{M}(x) \in \arg\min_{z \in \mathcal{M}} \|x - z\|_2,$$

(1)

where we additionally assume that $\mathcal{M}$ can be chosen in a way so that the approximation domain $\mathcal{A}$ is contained in a turbular region around $\mathcal{M}$, which guarantees uniqueness and a degree of regularity of the projection $\pi_\mathcal{M}$ (the details follow in Section 2). Our model significantly extends the standard setting [13,46,57,60], where the approximation domain $\mathcal{A}$ equals $\mathcal{M}$ and $\pi_\mathcal{M}$ acts as the identity. Furthermore, Model (1) is completely adaptive to the target function because it suffices that low-dimensionality is encoded in the mapping $x \mapsto f(x)$. Hence, differently to [13,46,57,60], we may observe vastly different approximation rates when approximating functions of varying complexity on the same approximation domain.

Model (1) borrows ideas typically associated with sufficient dimension reduction literature in statistics [35]. Assuming $(X, Y)$ is a joint distribution of features and responses, sufficient dimension reduction studies estimation of conditional expectations following a model $\bb{E}[Y|X = x] = g(\phi(x))$ for unknown functions $g, \phi$. In the past two decades the case $\phi(x) = Ax$, where $A$ is some unknown $d \times D$ matrix (sometimes also referred to as multi-index models), has been well understood, culminating in the development of efficient estimators with optimal guarantees under certain assumptions [36]. The developed techniques do not trivially transfer to the nonlinear case (1) however such that computationally and statistically efficient estimation of (1) is, to the best of our knowledge, an open problem.

In addition to Model (1), we consider a complementary model where $f$ depends on distances to a collection of finite or low-dimensional sets $\mathcal{C}_1, \ldots, \mathcal{C}_M$. Mathematically, we assume $f : [0,1]^D \to \mathbb{R}$ can be written as

$$f(x) = \sum_{\ell=1}^{M} g_\ell(\text{ndist}(x; C_\ell))^2 \quad \text{with} \quad \text{ndist}(x; C_\ell)^2 := D^{-1/2} \min_{z \in C_\ell} \|x - z\|_2^2,$$

(2)
where \( D^{-1/2} = \text{diam}([0,1]^D) \) is a normalization factor for the Euclidian distance over \([0,1]^D\). The low-dimensionality is encoded by assuming that sets \( C_\varepsilon \) can be covered with \( \varepsilon \)-nets with respect to \( \|\cdot\|_2 \) consisting of \( O(\varepsilon^{-d}) \) points. Our analysis concentrates on the case, where \( M \) is assumed negligible compared to \( D \), which is reasonable for instance if \( M \) represents the number of classes in a classification problem. Our results and proofs however can be easily extended to \( M \) growing with \( D \), leading to different complexity dependence on \( D \) in the corresponding approximation result (Theorem 8). Under additional sparsity assumptions, which assert that for each \( x \) only a smaller subset of \( g_i \)'s is active, it should also be possible to recover Theorem 8 without changing the dependence on \( D \).

### 1.2 Motivating examples

To further motivate Models (1) and (2) we now provide some additional intuition about their expressive power in the context of common statistical estimation problems. We denote a feature vector by \( X \) and the dependent output by \( Y \) in the following.

**Noisy manifold regression** Manifold regression relies on the assumption that the distribution of the feature vector \( X \) is supported on a lower-dimensional manifold \( \mathcal{M} \) and it allows for establishing estimation rates governed by the intrinsic manifold dimensionality. In the past two decades the setting was frequently used to explain empirical success of common statistical estimator such as kNN [33], local polynomials [8], or kernel regression [73] in the high-dimensional regime. However, the manifold assumption has also been criticized as too stringent and rarely exactly observable in practice [25, 26]. This suggests a gap between our perception of real-world data and the idealized setting that is used to establish meaningful guarantees in high dimensions.

Model (1) offers an alternative to introduce a relaxed manifold hypothesis that naturally closes this gap. Namely, we may assume \( X \) to be concentrated in a tube around the manifold \( \mathcal{M} \) and that the target satisfies \( \mathbb{E}[Y|X] = g(\pi_M(X)) \). This implies that \( \mathbb{E}[Y|X] \) is statistically independent of the 'off-manifold' behavior encoded in \( X - \pi_M(X) \), see Figure 1a, and \( X - \pi_M(X) \) can be viewed as noise for the task of predicting \( Y \). Our results capture such assumptions and show near-optimal estimation rates for empirical risk minimizers (ERMs) over spaces of ReLU nets. In particular, the rates match those attained in manifold regression under the classic manifold assumption.

**Adaptivity to function complexity** Consider \( \mathcal{M} \) as a swiss role as in Figures 1b-1c, where colors indicate values of two different Lipschitz functions. Based on manifold regression results [8, 33, 73], kNN, local polynomial regression, or kernel regression estimate both functions at a \( N^{-2/(2+\dim(\mathcal{M}))} \) rate, where \( N \) is the size of the data sample. When comparing the complexity of the functions in 1b and 1c however, they are not the similar since we can express \( f \) in 1b as \( f(x) = g(\pi_\gamma(x)) \), where \( \gamma \) is the red curve (a one-dimensional manifold) as shown in the illustration. In other words, there exists a submanifold \( \gamma \subset \mathcal{M} \) with \( 1 = \dim(\gamma) < \dim(\mathcal{M}) = 2 \) induced by the function, which suffices to perfectly represent the target \( f \). Our results show that ERMs over spaces of ReLU nets are faithful to this function-adaptive notion of complexity since we prove an improved estimation rate \( N^{-2/(2+\dim(\mathcal{M}))} \approx N^{-2/3} \) for the case in Figure 1b compared to \( N^{-2/(2+\dim(\mathcal{M}))} \approx N^{-1/2} \) for the case 1c.

**Classification problems with class attractors** Model (2) is useful for modelling clustered data or classification problems such as depicted in Figure 1d. Namely, we may model data by assuming the corresponding label depends on the proximity to a finite number of class attractors, and let \( C_1 \) and \( C_2 \) contain these attractors, see the bold dots in Figure 1d. The sets \( C_i \) are trivially covered by a finite number of points (the attractors themselves), which indicates intrinsic dimension \( d = 0 \). Our results imply near-optimal univariate classification guarantees under the Hinge loss, where the estimation rate depends on smoothness properties and margin conditions [68] of the conditional class probability, see (4) for \( d = 0 \). While our theory concentrates on binary classification for simplicity, it can be extended to multi-class problems with a sample complexity scaling linearly with the number of classes in a situation as described above. Furthermore, our proofs easily extend to other norms than \( \|\cdot\|_2 \), provided they can be efficiently approximated by ReLU nets. Interesting examples include \( \|\cdot\|_\infty \) or \( \|\cdot\|_1 \), or corresponding \( \{1,2,\infty\} \) seminorms that take only into account a subset of the coordinates of \( x \).
1.3 Contributions

We first study in Section 3 approximation of targets $f(x)$ under Models (1) and (2) by ReLU networks. Assuming functions $g,g_1,...,g_M$ are $\alpha$-Hölder continuous, either with respect to the geodesic metric on $\mathcal{M}$ in Model (1), or with respect to $|\cdot|$ on $\mathbb{R}$ in Model (2), we show that ReLU nets approximate

- Model (1) to accuracy $\varepsilon$ with $O\left(\log(\varepsilon^{-1})\varepsilon^{-d/\alpha}\right)$ nonzero parameters;

- Model (2) to accuracy $\varepsilon$ with $O\left(\log(\varepsilon^{-1})\varepsilon^{-\max\left\{1/\alpha,d\right\}}\right)$ nonzero parameters.

Following [15, 72], the first result is, up to logarithmic factors, optimal because the problem class includes approximation of $\alpha$-Hölder functions on $\mathbb{R}^d$ and the depth of the network grows only logarithmically with $\varepsilon^{-1}$(see Theorem 6). Similarly, the second is optimal up to log factors for $1 \geq \alpha d$, because the problem class includes approximation of univariate $\alpha$-Hölder functions and we have a similar network depth behavior (see Theorem 8).

Since our inspiration for Models (1) and (2) is drawn from statistical literature, we derive in Section 4 generalization guarantees of empirical risk minimizers (ERMs) over a space of sparsely-constructed ReLU nets under the assumption that the conditional expectation follows Model (1) or (2). We first consider in Section 4.1 regression problems and empirical risk minimization with $\ell_2$-loss. Having access to $N$ data samples and a correctly tuned ReLU function space with corresponding ERM $\hat{\Phi}_N$, the resulting estimation rate is, up to log-factors,

$$\mathbb{E}\left(\left(\hat{\Phi}_N(X) - g(\phi(X))\right)^2\right) \lesssim \begin{cases} N^{-\frac{d\beta}{\alpha\beta+1}}, & \text{ for Model (1),} \\ N^{-\frac{2\alpha}{\alpha+\max\{1,\alpha d\}}}, & \text{ for Model (2).} \end{cases}$$

These rates are optimal whenever our approximation guarantees are optimal [64].

In Section 4.2 we switch to binary classification problems using the Hinge-loss function and assuming Tsybakov’s margin condition [2] with exponent $\beta > 0$. Denoting $f^*(x) = \text{sign}(2g(\phi(x)) - 1)$ as the Bayes classifier, i.e. the optimally performing classifier [16], the excess risk is, up to log-factors, bounded by

$$\mathbb{E}\left(\mathbb{P}\left(\text{sign}(\hat{\Phi}_N(X))Y \neq 1\right) - \mathbb{P}\left(f^*(X)Y \neq 1\right)\right) \lesssim \begin{cases} N^{-\frac{\alpha(\beta+1)}{\alpha(d+1)\beta+1}}, & \text{ for Model (1),} \\ N^{-\frac{\alpha(\beta+1)}{\alpha(d+1)(\alpha\beta+1)+\max\{1,\alpha d\}}}, & \text{ for Model (2).} \end{cases}$$

Figure 1: Motivating examples for Models (1) and (2): 1a depicts a regression problem under a noisy manifold assumption. Our results show that ReLU empirical risk minimizers (ERMs) achieve similar rates for such data sets compared to estimation under a traditional manifold assumption. 1b - 1c show data with marginal distribution $X$ supported on the swiss role and colors indicating function values. In Figure 1b, the function additionally depends on a single intrinsic coordinate of the swiss role and our results suggest ReLU ERMs adapt to such function-adaptive complexity measures. 1d shows a classification problem which can be modeled conveniently using (2), where $C_1$ and $C_2$ contain class attractors, i.e. the bold dots in the illustration. Our results show that ReLU ERMs achieve univariate classification guarantees.
Again, these rates are optimal whenever our approximation guarantees are optimal [2].

We stress that our estimation guarantees follow as rather straight-forward corollaries from approximation guarantees and recent advancements of empirical risk minimization over ReLU nets [29,56,66]. This is why we view Section 3 as our main contribution. Nevertheless, it is worthwhile to know that approximation guarantees lead to near-optimal generalization guarantees for Models (1) and (2), especially because there is to date no computationally and statistically efficient estimator for Model (1).

Summing up, the manuscript extends recent results [13,46,57,60], who study approximation and estimation on low-dimensional target domains, to the case where low-dimensionality is encoded in the map \( x \mapsto f(x) \) itself. We cover settings in [13,57,60] as a special case and achieve similar guarantees, which indicates that low-dimensional approximation domains may represent an overly simplified setting leading to unnecessarily narrow problem classes for ReLU nets.

### 1.4 Related work

Approximation theory of neural networks started over three decades ago with shallow neural networks and well-known universal approximation theorems [14,27,34], which state that spaces of infinitely wide shallow nets generated by non-polynomial activations are dense in continuous functions on any compact domain. These results have been successively refined over the years [4,5,41,44,53,59], where one of the most prominent results characterized the number of required weights in terms of first moments of the Fourier transform [4]. Complementing the rich literature on shallow networks, more recent work [39,43,54,55] concentrates on the benefit of network depth for expressivity. By now it is well understood that deeper networks are more expressive for approximating hierarchical or composite functions because of their ability to adapt to certain notions of intrinsic dimension of the function. For instance, [55] shows that a composite function \( f(x) = g_L \circ \ldots \circ g_1(x) \) on \( \mathbb{R}^D \), where each coordinate function of \( g_{r+1} \) depends on at most \( d \) outputs of \( g_r \), requires \( O(\varepsilon^{-d/\alpha}) \) nonzero parameters when approximated by a deep network, while a shallow networks requires \( O(\varepsilon^{-D/\alpha}) \) nonzero parameters in general. Hence, deep nets are able to adapt to layer-to-layer connectivity.

In recent years, approximation theory of ReLU nets has become particularly popular due to their increased usage in practice. A breakthrough was achieved in [71] who proved efficient approximation of general \( C^\alpha \)-smooth functions in \( L_\infty \)-norm with \( O(\log(\varepsilon^{-1})\varepsilon^{-D/\alpha}) \) nonzero parameters. This is optimal if either the weight assignment is constrained to be continuous, or the network depth is restricted to grow only moderately compared to the overall number of parameters [72]. Furthermore, the results have been continuously refined by adding width constraints on the approximating network [23,24,51] and by studying different function classes or approximation metrices [11,19,52,61,67]. Specifically, [19,52] gradually build up the framework of ReLU calculus and thus serve as useful entry points to the literature.

[52] also addressed the curse of dimensionality concern by showing that it can be circumvented for two-layer composite functions \( f = g \circ \tau \) with \( \tau : [1/2,1/2]^D \to [1/2,1/2]^d \) if \( \tau \) is more regular compared to \( g \). Moreover, [60] early on studied approximation properties of ReLU nets over manifolds \( \mathcal{M} \) and showed the number of nonzero parameters scales with the manifold dimension. Such results have been refined [13], and extended to low-dimensional Minkowski sets [46], or band-limited functions [45]. We also mention [42] that introduces an abstract general framework for developing approximation guarantees that overcome the curse of dimensionality.

With the exception of [46], the referenced works concentrate on approximation guarantees for neural networks. In the past three years, it has become popular to combine approximation guarantees with statistical learning theory to establish nontrivial guarantees for empirical risk minimizers (ERMs) over sparsely constrained ReLU net function spaces. Many of these contributions, as well as our own, are inspired by [7,56], who show that the ERM optimally regresses composite functions \( f(x) = g_L \circ \ldots \circ g_1(x) \), i.e. achieves an estimation rates that depends on the maximal layer-to-layer connectivity similar to [55]. We also refer to earlier works [30,31], who studied related problems earlier, and to [29,49] who show optimal classification guarantees for plug-in classifiers defined via a ReLU nets under Hinge-loss and different margin conditions.

To remedy the curse of dimensionality, [57] extended [56] to feature distributions supported on a lower-dimensional manifold and shows minimax optimal rates. Similarly, [46] considers lower-dimensional Minkowski sets, and [65,66] consider functions in mixed or anisotropic Besov spaces with highly inhomogeneous smoothness behavior. The latter approach also eases the curse of dimensionality if functions are much smoother (potentially even constant) in many directions in the ambient space. Furthermore, viewing the problem dimensionality as a property of the target function through anisotropic smoothness is a view intrinsic to the function and is thus well in agreement with our results about Models (1) and (2).
Lastly, we note that a common critique shared about studying ERMs over sparsely constrained ReLU nets is that sparsity is typically not enforced in practice, see for instance the discussion articles and rejoinder accompanying [56]. This has been recently addressed to some extent by [32] who study estimation of hierarchical functions over fully connected ReLU spaces and by [50] who use fully connected ReLU nets and a penalized ERM formulation to control the function space complexity.

1.5 Organization of the paper

Section 2 introduces some necessary preliminary concepts about differential geometry and ReLU networks. Section 3 presents our main findings on approximation properties of ReLU nets. We include brief and geometrically intuitive proof sketches for our main results. Statistical guarantees for empirical risk minimizers over ReLU networks are presented in Section 4. Section 5 and 6 present full proof details for approximation results and statistical guarantees respectively. Appendix 7 contains proofs of basic differential geometric statements given in Section 2 and further results on basic ReLU calculus.

1.6 General notation

For $N \in \mathbb{N}$ we let $[N] := \{1, \ldots, N\}$. $\text{cl}(B)$ denotes the closure of a set $B$ and $\text{Im}(M)$ denotes the image of an operator $M$. $|A|$ denotes the absolute if $A \in \mathbb{R}$, the length if $A$ is an interval, and the cardinality if $A$ is a finite set. We denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. The ReLU activation function is denoted $(t)_+ = \max\{0, t\}$.

$\|\cdot\|_p$ denotes the standard Euclidean $p$-norm for vectors and $\|\cdot\|_2$ denotes the spectral norm for matrices. We denote $\text{dist}(z; A) := \inf_{p \in A} \|z - p\|_2$ for $z \in \mathbb{R}^D$ and $A \subset \mathbb{R}^D$. $B_r(z)$ denotes the standard $\|\cdot\|_2$-ball of radius $r$ around $z$, while $B_{M, r}(v)$ denotes the geodesic ball of radius $r$ around $v \in M$. $\|A\|_0$ counts the number of nonzero entries of a matrix $A$. $L_p(A)$ contains function with finite $p$-th order Lebesgue norm $\|\cdot\|_{L_p(A)}$.

We use $A \lesssim_p B$, respectively, $A \gtrsim_p B$, if there exists a constant $C_p$ depending on a quantity $p$ such that $A \leq C_p B$, respectively $A \geq C_p B$. Furthermore, we write $A \asymp_p B$ if $A \lesssim_p B$ and $A \gtrsim_p B$.

2 Preliminaries

In this section we introduce some concepts about differential geometry, packing numbers, and ReLU calculus to rigorously define Models (1)- (2) in the next section. Most of the concepts should be well-known and are thus included for self-containedness. A summary of the introduced terminology is given in Table 1.

2.1 Differential geometry and packing numbers for sets

We begin by defining some geometric quantities that are used later to properly define Model 1. Let $\mathcal{M}$ be a nonempty, connected, compact, $d$-dimensional Riemannian submanifold of $\mathbb{R}^D$. A manifold $\mathcal{M}$ has an associated medial axis that contains points $x \in \mathbb{R}^D$ with set-valued orthogonal projection $\arg\min_{z \in \mathcal{M}} \|x - z\|_2$ and is thus defined as

$$\text{Med}(\mathcal{M}) := \{z \in \mathbb{R}^D : \exists q \neq q \in \mathcal{M}, \|p - z\|_2 = \|q - z\|_2 = \text{dist}(z; \mathcal{M})\}. \quad (5)$$

The local reach or local feature size [9] describes the minimum distance needed to travel from a point $v \in \mathcal{M}$ to the closure of the medial axis. We define it accordingly by

$$\tau_{\mathcal{M}}(v) := \text{dist}(v; \text{Med}(\mathcal{M})) \quad (6)$$

and also introduce the global reach $\tau_{\mathcal{M}} := \inf_{v \in \mathcal{M}} \tau_{\mathcal{M}}(v)$ as the minimal local reach.

The local reach is important in the context of Model (1) because it is used to formulate a well-defined approximation problem. Namely, assuming the approximation target $f : A \subseteq [0, 1]^D \rightarrow \mathbb{R}$ admits a manifold $\mathcal{M} \subset \mathbb{R}^D$ so that $f(x) = g(\pi_{\mathcal{M}}(x))$, the manifold $\mathcal{M}$ must be such that $\pi_{\mathcal{M}}$ is unique and has some degree of regularity over the approximation domain $A$. Certainly, this requires $A \cap \text{Med}(\mathcal{M}) = \emptyset$ (otherwise $f$ can be set-valued), but, as we shall see below, we need the stronger requirement $\text{dist}(A; \text{Med}(\mathcal{M})) > c > 0$ for some $c$ to guarantee Lipschitz $\pi_{\mathcal{M}}$ on $A$. 

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We can extend this quantity to the blow-up well defined according to Lemma 1. The geodesic metric can be used to bound perturbations between nearby tangent orthoprojectors and local reaches. For the former, (10) shows the guarantee given by

\[ d_M(q) := d_M(\pi_M(x), \pi_M(x')) \]

geodesic metric on \( M \) extended to \( M(q) \) by \( d_M(q)(x, x') := d_M(\pi_M(x), \pi_M(x')) \)

geodesic ball of radius \( r \) around \( v \in M \)

Table 1: Summary about the notation introduced in Section 2

To make matters more precise, it is convenient to work with a blow-up of \( M \), which we define for \( q \in [0, 1] \) by

\[ M(q) := \{ x \in \mathbb{R}^D : x = v + u, \; v \in M, \; u \in \ker(A(v)^\top), \; \|u\|_2 < qt_M(v) \}, \]

where \( A(v) \in \mathbb{R}^{D \times d} \) is a columnwise orthonormal basis of the tangent space of \( M \) at \( v \in M \). Then we can analyze the properties of \( \pi_M \) constrained to \( M(q) \) and show that \( \pi_M \) is \((1 - q)^{-1}\)-Lipschitz on \( M(q) \) for \( q \in [0, 1) \). Consequently, if \( f : A \subseteq [0, 1]^D \rightarrow \mathbb{R} \) admits a manifold \( M \) so that \( f(x) = g(\pi_M(x)) \) and \( A \subset M(q) \) for some \( q \in [0, 1) \), the \( L_\infty(A) \)-approximation problem is well-posed and regularity of \( f \) is determined by the regularity of \( g \).

Lemma 1. Let \( q \in [0, 1) \) and \( \tau_M > 0 \). Let \( x \in M(q) \) have decomposition \( x = v + u \) for \( v \in M \) and \( u \in \ker(A(v)^\top) \) with \( \|u\|_2 < qt_M(v) \). Then \( \pi_M(x) \) is uniquely determined by \( \pi_M(x) = v \).

Proof. The proof is deferred to Section 7.1 in the Appendix.

Lemma 2. Let \( M \) be a connected compact \( d \)-dimensional Riemannian submanifold \( M \) of \( \mathbb{R}^D \) and let \( q \in [0, 1) \). The projection \( \pi_M \) is \((1 - q)^{-1}\)-Lipschitz on \( M(q) \), i.e. we have

\[ \|\pi_M(x) - \pi_M(x')\|_2 \leq \frac{1}{1 - q}\|x - x'\|_2 \quad \text{for all} \quad x, x' \in M(q). \]

Proof. The proof is deferred to Section 7.1 in the Appendix.

Compact Riemannian manifolds \( M \) are geodesically complete and the Hopf-Rinow theorem thus implies the existence of a length-minimizing geodesic \( \gamma : [t, t'] \rightarrow M \) between \( \gamma(t) = v \) and \( \gamma(t') = v' \), where the length is measured by \( |\gamma| = \int_t^{t'} \|\dot{\gamma}(s)\|_2 \, ds \). This allows to define the geodesic metric on \( M \) given by

\[ d_M(v, v') := \inf \{|\gamma| : \gamma \in C^1([t, t']), \; \gamma : [t, t'] \rightarrow M, \; \gamma(t) = v, \; \gamma(t') = v' \}. \]

We can extend this quantity to the blow-up \( M(q) \) using \( d_M(q)(x, x') := d_M(\pi_M(x), \pi_M(x')) \), which is well defined according to Lemma 1. The geodesic metric can be used to bound perturbations between nearby tangent orthoprojectors and local reaches. For the former, (10) shows the guarantee

\[ \|A(v)A(v)^\top - A(v')A(v')^\top\|_2 \leq \frac{1}{\tau_M} d_M(v, v'), \]

where \( \tau_M \) is the packing number of a set with non-unique projections $\pi_M$ and \( \Phi \).
whereas the local reach satisfies a 1-Lipschitz condition \[9\]

\[|\tau_M(v) - \tau_M(v')| \leq \|v - v'\|_2 \leq d_M(v, v').\] (11)

Lastly, we have to introduce maximally \(\delta\)-separated sets and packing numbers for metric sets.

**Definition 3** ([69, Section 4.2]). Let \(C\) be a set endowed with a metric \(\Delta\) and let \(\delta > 0\). We say \(Z \subset C\) is \(\delta\)-separated if for any \(z \neq z' \in Z\) we have \(\Delta(z, z') > \delta\). \(Z\) is maximally separated if adding any other point in \(Z\) destroys the separability property. The size of the largest maximally separated set is called the packing number and denoted by \(P(\delta, Z, \Delta)\).

The packing numbers of \(M\) and \(C\)'s are important for Models (1) and (2) because they define the intrinsic dimension \(d\) of the problem and replace the ambient dimension \(D\) in our approximation results. For the distance-based model (2) this is asserted by an assumption. For the projection-based model (1) it is a consequence from low-dimensionality of the manifold \(M\).

**Lemma 4.** Let \(M\) be a \(d\)-dimensional compact connected Riemannian submanifold \(M\) of \(\mathbb{R}^D\) and let \(\delta \in (0, \frac{1}{2}\tau_M)\). Let \(\text{Vol}(M)\) denote the standard volume of the manifold \(M\). Then we have

\[
P(\delta, M, d_M) \leq \frac{3^d \text{Vol}(M) d^\frac{d}{2}}{\delta^d}, \tag{12}
\]

**Proof.** The bound on the packing number \(P(\delta, M, d_M)\) can be found for instance in [3, 47]. \(\square\)

### 2.2 ReLU calculus

ReLU calculus is a systematic framework for developing approximation results for networks with ReLU activation function \((t)_+ := 0 \lor t = \max\{0, t\}\). The framework has been developed in recent years [11, 19, 52, 71] and we informally recall some basic results and history in the following. At the end of the section we additionally provide a table with ReLU approximation results of some special functions, which will be needed for our ReLU constructions to approximate Models (1) and (2) in the next section.

We adopt the following definition of ReLU networks.

**Definition 5** ([19, Definition 2.1]). Let \(L \geq 2\) and \(N_0, \ldots, N_L \in \mathbb{N}_{>0}\). A map \(\Phi : \mathbb{R}^{N_0} \to \mathbb{R}^{N_L}\) is called a ReLU network if there exist matrices \(A_\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}\) and vectors \(b_\ell \in \mathbb{R}^{N_\ell}\) for \(\ell \in [L]\) so that \(\Phi(x) = W_L y_{L-1} + b_L\), where \(y_\ell\) is recursively defined by \(y_0 := x\) and

\[y_\ell := (A_\ell y_{\ell-1} + b_\ell)_+ \text{ for } \ell \in [L-1].\]

Furthermore, we define \(L(\Phi) := L\) as the number of layers, \(W(\Phi) := \max_{\ell=0,\ldots,L} N_\ell\) as the maximum width, \(P(\Phi) := \sum_{\ell=1}^{L} \|A_\ell\|_0 + \|b_\ell\|_0\) as the number of free parameters, and

\[B(\Phi) := \max\{|(b_\ell)_i|, |(A_\ell)_{ij}| : i \in N_\ell, j \in N_{\ell-1}, \ell \in [L]\}\]

as a bound for the absolute value over all parameters.

ReLU calculus gradually builds up approximation results for maps of increasing complexity. The first step is to endow the space of ReLU nets with basic operations of concatenation and linear combinations. Specifically, for ReLU nets \(\Phi_1, \ldots, \Phi_M\) and scalars \(a_1, \ldots, a_M \in \mathbb{R}\), [19, Lemma 2.5 and Lemma 2.7] (recalled in Lemma 23 and 24 in the Appendix) show that there exist ReLU networks that exactly realize the maps

\[x \mapsto \Phi_1 \circ \Phi_2(x), \quad x \mapsto (\alpha \Phi_1(x), \ldots, \alpha \Phi_M(x)), \quad x \mapsto \sum_{i=1}^{M} \alpha_i \Phi_i(x),\]

provided the input and output dimensions are matching. Furthermore, the dimensions of the realizing networks are controlled by dimensions of individual networks \(\Phi_1, \ldots, \Phi_M\).

The next step is the approximation of the square functions \(x \mapsto x^2\), which has been established in [71] and later be refined in [19]. The latter shows that there exists a ReLU net with width 4 and depth \(O(\log(\varepsilon^{-1}))\) that approximates \(x \mapsto x^2\) up to accuracy \(\varepsilon\). The approximation of \(x \mapsto x^2\) is an important
stepping stone of ReLU calculus because it immediately translates to general multiplication by leveraging the identity
\[ xy = \frac{1}{2} \left( x^2 + y^2 - (x - y)^2 \right). \]

Having access to the multiplication allows for approximating approximation systems such as polynomials [19,71]. Then, local Taylor expansions may be used to extend approximation results to functions with certain smoothness properties such as \( C^\alpha([0,1]^D) \). We refer to [11,19,52,71] for more details on these results. Furthermore, Table 2 contains a list of basic approximation results that are relevant in our approximation guarantees in the next section.

### 3 Approximation theory

In this section we make the informal descriptions of Models (1) and (2) precise and we present our approximation guarantees. We give brief and geometrically intuitive proof sketches with full proof details deferred to Section 5.

#### 3.1 Model (1): projection onto an embedded manifold

We consider the following setting.

Model 1 The target \( f : \mathcal{A} \subseteq [0,1]^D \to \mathbb{R} \) can be written as \( f(x) = g(\pi_\mathcal{M}(x)) \) for a connected, compact, nonempty, d-dimensional manifold \( \mathcal{M} \) with \( \tau_\mathcal{M} > 0, \mathcal{A} \subseteq \mathcal{M}(q) \subseteq [0,1]^D \) for some \( q \in [0,1] \), and where \( \pi_\mathcal{M}(x) := \arg \min_{z \in \mathcal{M}} \| x - z \|_2 \). Function \( g : \mathcal{M} \to [0,1] \) satisfies for an \( \alpha \in (0,1] \) and \( L \geq 0 \) the regularity condition

\[ |g(v) - g(v')| \leq L d_\mathcal{M}^\alpha(v, v') \quad \text{for all} \quad v, v' \in \mathcal{M}. \]  

Note that Model 1 postulates the relation \( f(x) = g(\pi_\mathcal{M}(x)) \), but additionally requires the approximation domain \( \mathcal{A} \) to be contained in the turbular region \( \mathcal{M}(q) \) around the manifold, see the definition in Equation (7). As explained in the previous section, the latter requirement is to ensure that \( \pi_\mathcal{M} \) is \((1-q)^{-1}\)-Lipschitz on \( \mathcal{A} \), which then implies \( \alpha \)-Hölder continuity of the composition \( f = g \circ \pi_\mathcal{M} \). Consequently, \( L_\infty(\mathcal{A}) \)-approximation of \( f \) is well-posed.

**Theorem 6.** In the setting of Model 1, there exist \( C, \varepsilon_0 > 0 \) independent of \( D \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) there exists a ReLU net \( \Phi \) with \( L(\Phi) \lesssim C D \log(D) \log^2(\varepsilon^{-1}) \varepsilon^{-d}, P(\Phi) \lesssim C D \log(D) \log^2(\varepsilon^{-1}) \varepsilon^{-d}, W(\Phi) \lesssim C D e^{-d}, \) and \( B(\Phi) \lesssim C \varepsilon^{-2} \) and \( D > 0 \) and

\[ \sup_{x \in \mathcal{A}} |f(x) - \Phi(x)| \lesssim C \varepsilon^\alpha. \]  

**Theorem 6** proves efficient approximation of \( f \) to accuracy \( \varepsilon \) using \( O(\log^2(\varepsilon^{-1})\varepsilon^{-d/\alpha}) \) nonzero parameters. According to [15,72] the rate is optimal apart from logarithmic factors because the depth of \( \Phi \) grows only moderately like \( \log(\varepsilon^{-1}) \) and the problem class contains \( \alpha \)-Hölder functions on \( \mathbb{R}^d \).

A source of concern about the approximating ReLU net in Theorem 6 might be the coefficient growth behavior \( B(\Phi) \in O(\varepsilon^{-1}) \). We note however that [19, Proposition A.1] implies the existence of an equivalent network \( \hat{\Phi} \) with \( B(\hat{\Phi}) \leq 2 \) and \( P(\hat{\Phi}) \lesssim \log(\varepsilon^{-1}) P(\Phi) \) that exactly realizes \( \Phi \). Thus, we still achieve near-optimal rates when bounding the weights in absolute by 2.

| map | metric | \( L(\phi) \) | \( W(\phi) \) | \( P(\phi) \) | \( B(\phi) \) | Reference |
|-----|--------|-------------|-------------|-------------|-------------|---------|
| \( t \to t^2 \) | \( L_\infty([0,1]) \) | \( O(\log(\varepsilon^{-1})) \) | 4 | \( O(L(\Phi)) \) | 4 | [19] |
| \( x \to \|x\|_2^2 \) | \( L_\infty(B_R(0)) \) | \( O(\log(R^2\varepsilon^{-1})) \) | 4D | \( O(D(L(\Phi)) \) | \( 4R^2 \lor R^{-1} \) | Lem. 26 |
| \( (x,t) \to tx \) | \( L_\infty([-B,B]^{D+1}) \) | \( O(\log(B^2\varepsilon^{-1})) \) | 12D | \( O(D(L(\Phi)) \) | \( 4 \lor 2[B^2] \) | Lem. 27 |
| \( t \to t^{-1} \) | \( L_\infty([-1,1]) \) | \( O(B^2 \log^2(B\varepsilon^{-1})) \) | 16 | \( O(L(\Phi)) \) | \( 8 \lor B^{-1} \) | Lem. 28 |
| \( x \to \min_{t \leq 1} x_t \) | \( L_\infty((x : B^{-1} \leq \|x\|_1 \leq B)) \) | \( O(B^2 \log^2(B\varepsilon^{-1})) \) | \( O(D) \) | \( O(D(L(\Phi)) \) | \( O(B^2) \) | Lem. 29 |
| \( t \to \sign(t) \) | \( L_2(R) \) | \( 2[\log_2(D)] \) | 3[D/2] | 11D[\log_2(D)] | \( 1 \) | Lem. 30 |
| \( \varepsilon \) | | 2 | 2 | \( 7 \) | \( \varepsilon^{-1} \) | Lem. 31 |
The constants $\varepsilon_0$ and $C$ depend on characteristics of the manifold such as its reach $\tau_M$, dimension $d$, and volume $\text{Vol}(M)$, and on parameter $q$ that is linked to the approximation domain $A$. We omit to explicitly track these effects because finding optimal and thus informative dependencies is technical and not the focus of this work. We stress however that $\varepsilon_0 \to 0$ and $C \to \infty$ as $q \to 1$, ie. if we aim for approximating $f$ over $A$ contained in $\mathcal{M}(q)$ with $q \to 1$. This is expected because the Lipschitz constant of $\pi_M$ restricted to $\mathcal{M}(q)$ diverges as $q \to 1$ (Lemma 2) and thus the approximation problem becomes increasingly ill-posed.

**Proof sketch** Let $\{z_1, \ldots, z_K\}$ be an arbitrary maximal separated $\varepsilon$-net of $\mathcal{M}$ with cardinality bounded by $K \lesssim C \varepsilon^{-d}$ according to Lemma 4. The proof follows three steps to explicitly construct the approximating network, which is illustrated in Figure 2. First we construct a partition of unity $\{\eta_i : \mathcal{M}(q) \to [0, 1] : i \in [K]\}$ of $\mathcal{M}(q)$ that satisfies the localization property

$$\sup_{x \in \mathcal{M}(q) : \eta_i(x) \neq 0} d_{\mathcal{M}(q)}(x, z_i) \lesssim_C \varepsilon,$$

ie. $\eta_i$ is supported on the preimage $\pi^{-1}_M(B_{M,C\varepsilon}(z_i))$ with $B_{M,C\varepsilon}(z_i)$ being a small geodesic ball of radius $C\varepsilon$ around $z_i$. Secondly, we show that functions $\eta_i$ can be efficiently approximated by small-size ReLU networks $\Theta_i$ and finally we approximate the target $f$ by the linear combination

$$\Phi(x) \approx \sum_{i=1}^K g(z_i)\Theta_i(x).$$

The main challenges of the proof center around the first two steps because the functions $\eta_i$ need to be defined in a way so that they are efficiently approximable by ReLU nets. More precisely, each individual $\eta_i$ should be approximated by a ReLU net up to some accuracy $\varepsilon$ with at most $O(\text{polylog}(\varepsilon^{-1}))$ parameters because we need $K \in O(\varepsilon^{-d})$ such sub-networks in total.

The way we do this is by locally approximating the extended geodesic metric $d_{\mathcal{M}(q)}(x, x') := d_{\mathcal{M}}(\pi_M(x), \pi_M(x'))$ by basic features of translations $x-z_i$ and tangent space projections $A(z_i)^\top(x-z_i)$ of the input vector $x$. Namely, Proposition 13 shows the metric equivalence

$$\|A(z_i)^\top(x-z_i)\|_2 \lesssim_C d_{\mathcal{M}(q)}(x, z_i) \lesssim \frac{1}{1-p} \|A(z_i)^\top(x-z_i)\|_2,$$

conditional on taking $x \in \mathcal{M}(q) \cap B_{\tau_M(z_i)}(z_i)$ with $\|A(z_i)^\top(x-z_i)\|_2 \lesssim (1-p)\tau_M$ for some $p \in [q, 1)$. Therefore, we can approximate $d_{\mathcal{M}(q)}(x, z_i)$ by first checking if $x$ is contained in $B_{\tau_M(z_i)}(z_i) \cap \{x \in \mathcal{M}(q) : \|A(z_i)^\top(x-z_i)\|_2 \lesssim (1-p)\tau_M\}$, and, after confirmation, use the elementary function $\|A(z_i)^\top(x-z_i)\|_2$ to evaluate $d_{\mathcal{M}(q)}(x, z_i)$ according to (17). Importantly, both tasks can be performed by computing elementary features $\|A(z_i)^\top(x-z_i)\|_2$ and $\|A(z_i)^\top(x-z_i)\|_2$ of the input $x$.

![Figure 2: Schematic ReLU construction used to approximate Model 1. At each node we illustrate the feature of $x$ that is being approximated by the network. Green nodes can be exactly realized (assuming the previous layer is exact) with finite width layers, whereas blue nodes are approximated to accuracy $O(\varepsilon)$ using $O(\text{polylog}(\varepsilon^{-1}))$ layers.](image-url)
Leveraging the thresholding nature of ReLU nets, this motivates the construction
\[
\tilde{\eta}(x) = \left(1 - \frac{\|x - z_i\|_2}{p\tau_M(z_i)}\right)^2 - \left(\frac{\|A(z_i)^T (x - z_i)\|_2}{h\varepsilon}\right)^2 + \eta(x) = \frac{\tilde{\eta}(x)}{\|\tilde{\eta}(x)\|_1},
\]
where \(h\) is a bandwidth parameter suitably chosen as a function of \(q\) and \(\tau_M\). We immediately see that \(\tilde{\eta}(x) \neq 0\) implies
\[
\|x - z_i\|_2 < p\tau_M(z_i) \quad \text{and} \quad \|A(z_i)^T (x - z_i)\|_2 < h\varepsilon.
\]
Thus, as soon as \(\varepsilon < h^{-1}(1 - p)\tau_M\), (17) applies and gives \(d_M(q)(x, z_i) \lesssim C \varepsilon\) on the support of \(\tilde{\eta}\).

Auxiliary functions \(\tilde{\eta}\) can be approximated to accuracy \(O(\varepsilon)\) using \(O((\text{polylog}(\varepsilon^{-1}))\) parameters due to the fact that the square and thus also the squared \(\ell_2\)-norm are efficiently approximable (Lemma 26). Approximating the \(\ell_1\)-normalization to form \(\eta\) is slightly more involved (Lemma 29) and requires additional uniform upper and lower bounds on \(\|\tilde{\eta}(x)\|_1\). This can however be achieved for appropriately chosen bandwidth parameter \(h\).

In the final step of the proof, the error \(|\Phi(x) - f(x)|\) is bounded by the \(\alpha\)-Hölder continuity of \(g\) and the approximation result about \(\sum_{i=1}^K |\Theta_i(x) - \eta(x)|\) in Lemma 16.

Following the high-level proof strategy for Theorem 6, we see that the projection \(\pi_M\) can be approximated by replacing \(g(z_i)\) in (16) with \(z_i\). We include a corresponding result below as it may be of independent interest.

**Theorem 7.** Let \(q \in [0, 1)\) and let \(M\) be a nonempty, connected, compact \(d\)-dimensional manifold with \(\tau_M > 0\) and \(M(q) \subseteq [0, 1]^D\). There exist \(\varepsilon_0\) and \(C > 0\) independent of \(D\) such that for all \(\varepsilon \in (0, \varepsilon_0)\) there exists a ReLU net \(\Phi\) with \(\|\Phi\| \lesssim C D \log(D) \log^2(\varepsilon^{-1})\), \(P(\Phi) \lesssim C D \log(D) \log^2(\varepsilon^{-1})\varepsilon^{-d}\), \(W(\Phi) \lesssim C D \varepsilon^{-d}\), and \(\|\Phi\|_\infty \lesssim C \varepsilon^{-1}\), \(B(\Phi) \lesssim C \varepsilon^{-1} \lor D\), and
\[
\sup_{x \in M(q)} \|\pi_M(x) - \Phi(x)\|_\infty \lesssim C \varepsilon. \tag{18}
\]

### 3.2 Model (2): distance-based target functions

We now consider the second model.

**Model 2** Let \(C_1, \ldots, C_M \subseteq [0, 1]^D\) be nonempty closed sets and assume there exists \(\delta_0 > 0\) such that
\[
P(\sqrt{D}\delta, C, \|\cdot\|_2) \lesssim \delta^{-d} \quad \text{for all } \delta < \delta_0 \text{ and } \ell \in [M].
\]
Furthermore assume
\[
f(x) = \sum_{\ell=1}^M g_\ell(n\text{dist}(x; C_\ell)^2), \quad \text{where} \quad \text{dist}(x; C)^2 = D^{-1} \text{dist}(x; C)^2,
\]
and \(g_\ell : [0, 1] \to [0, 1]\) satisfies for \(\alpha \in (0, 1]\)
\[
|g_\ell(t) - g_\ell(t')| \leq L |t - t'|^\alpha \quad \text{for all } \quad t, t' \in [0, 1]. \tag{19}
\]
Note that dimension dependent factors in the packing number, respectively, the definition of \(n\text{dist}(\cdot; C)\) normalize the \(\|\cdot\|_2\)-distances by \(\text{diam}([0, 1]^D) = \sqrt{D}\).

**Theorem 8.** In the setting of Model 2 there exists \(C\) depending on \(M, L, \alpha\), and \(d\) such that for any \(\varepsilon \in (0, (6L\delta_0)^{1/\alpha})\) there exists a ReLU network \(\Phi\) with \(\|\Phi\| \lesssim C D \log(D) \varepsilon^{-1}\), \(W(\Phi) \lesssim C D \varepsilon^{-(1+\alpha)d}\), \(P(\Phi) \lesssim C D \log(D) \varepsilon^{-(1+\alpha)d}\), and \(B(\Phi) \leq 1\) with
\[
\sup_{x \in [0, 1]^D} |f(x) - \Phi(x)| \leq \varepsilon^\alpha. \tag{20}
\]

Theorem 8 proves efficient approximation of \(f\) up to accuracy \(\varepsilon\) using \(O(\log(\varepsilon^{-1}))\varepsilon^{-(1+\alpha)d}\) parameters. For \(d \leq \alpha^{-1}\) this is optimal up to the log-factor because the network depth grows moderately like \(O(\log(\varepsilon^{-1}))\) and the problem class includes approximation of univariate \(\alpha\)-Hölder functions with optimal parameter bound \(O(\varepsilon^{-1/\alpha})\) [15, 72]. If \(d > \alpha^{-1}\) on the other hand, we are currently not aware of the optimal approximation rate.
We therefore extensively discuss our results in Section 4.3 to put them into an appropriate statistical

The final layer of the distance approximation subnetwork realizes

where one approximates function

Theorem 8

Proof sketch for Theorem 8 Let us first consider the case $M = 1$. We construct two sub-networks, where one approximates function $g_1$ and the other approximates $x \mapsto \text{ndist}(x; C_1)^2$. Approximation of an $\alpha$-Hölder function $g_1$ is well understood and we construct the corresponding sub-network using [56, Theorem 5] (see Theorem 25). To approximate the normalized distance $\text{ndist}(x; C_1)^2$, we first take a maximally $\sqrt{D\varepsilon}$-separated set $\{z_1, \ldots, z_K\} \subset C_1$, whose cardinality is bounded by the packing number bound $K \lesssim \varepsilon^{-d}$ according to the assumptions. Then, for each $i \in [K]$ we construct sub-networks, which first compute normalized differences $D^{-1/2}(x - z_i)$, and then approximates their norms $D^{-1} \|x - z_i\|_2^2$. The final layer of the distance approximation subnetwork realizes $\min_{\delta \in [\varepsilon]} D^{-1} \|x - z_i\|_2^2$ to approximate $\text{ndist}(x; C_1)^2$. The dimensions of a network approximating the minimum operator is provided in Lemma 30. The entire network to approximate $g_1(\text{ndist}(x; C_1))$ is summarized in Figure 3.

The case $M > 1$ follows immediately from $M = 1$ because we can create subnetworks $\Phi_i$ for each $g_i(\text{ndist}(x; C_i))$ and use $\Phi(x) = \sum_{i=1}^M \Phi_i(x)$ according to Lemma 24.

Inspecting the high-level proof strategy of Theorem 8, we notice that Model 2 can be extended by considering distance functions and packings with respect to metrics other than $\|\cdot\|_2$. Namely, for any metric $\Delta$, which can be approximated by ReLU nets to accuracy $\varepsilon$ with $O(\text{polylog}(\varepsilon^{-1}))$ parameters, and for which $C$ satisfies $P(\delta, C, \Delta) \lesssim \delta^{-d}$, similar results hold. Interesting examples include $\|\cdot\|_1$ or $\|\cdot\|_{\infty}$ because they can be realized exactly by small ReLU nets.

We note that carefully tracking the influence of $M$ on the network dimensions in the proofs in Section 5.2 shows the number of nonzero parameters $P(\Phi)$ and the width grow like $M^{(1/\alpha)\delta d}$, while the number of layers grows like $\log(M)$. Thus, having $M$ depending on $D$ may lead to worse polynomial scaling with $D$ in the network dimension bounds in Theorem 8. We believe however that the case with $M$ and $D$ of similar order should be treated under additional sparsity assumptions, which restrict how many $g_i$s are active at most for a fixed $x$. Such consideration would lead to more informative bounds that are reminiscent of Theorem 8, i.e. that avoid high-order polynomial dependencies on $D$.

4 Implications on empirical risk minimization guarantees

We now present statistical guarantees for the empirical risk minimizer (ERM) over a class of sparsely-constrained ReLU networks under the assumption that the conditional expectation $f(x) = \mathbb{E}[Y | X = x]$ follows Model 1 or Model 2. Section 4.1 deals with regression problems, where the ERM is learned by minimizing the average squared loss over the training data. In Section 4.2 we consider binary classification under Tsybakov’s margin condition [2] with an ERM learned using the Hinge loss. We establish that the ERM adapts to the intrinsic dimensionality, and additionally to Tsybakov’s margin condition in binary classification.

Albeit technical, our results follow as fairly straight-forward corollaries from approximation bounds in Section 3 and recent results in [29, 56, 66] about statistical learning theory for ERMs over ReLU network spaces. Therefore, the proof details are deferred to Section 6.

The guarantees presented in Sections 4.1 and 4.2 suggest an astonishing adaptivity of ReLU nets to an intrinsic problem complexity encoded in the joint distribution $(X, Y)$. Similar to the approximation case, this significantly extends earlier results on ERMs over sparsely-constrained ReLU nets where low-dimensionality is encoded directly in the feature vector $X$ [46, 57]. As has been pointed out in the discussion article accompanying [56] however, studying ERMs over sparsely-constrained ReLU spaces blends out the training procedure and thus does not capture all difficulties of neural networks in practice. We therefore extensively discuss our results in Section 4.3 to put them into an appropriate statistical

Figure 3: Schematic ReLU construction used to approximate Model 2 for $M = 1$. At each node we denote the feature of $x$ that is being approximated. Green nodes can be exactly realized (assuming previous layer is exact) using finitely many layers, whereas blue nodes are approximated to accuracy $O(\varepsilon)$. The number of layers required to approximate $g_i$, i.e. the last node, is $O(\varepsilon^{-\alpha})$. 

# features: $\mathbb{R}^{KD}$ # features: $\mathbb{R}^K$

$\sqrt{D}$

# features: $\mathbb{R}$

$g_i(\text{ndist}(x; C_i)^2)$

$\text{ndist}(x; C_i)^2$
4.1 Regression

Let \((X, Y) \in [0, 1]^D \times \mathbb{R}\) be a random vector and let \(f(x) = \mathbb{E}[Y|X = x]\) denote the regression function. We assume the response \(Y\) is generated by the model

\[
Y = f(X) + \zeta, \quad \zeta \sim \mathcal{N}(0, 1), \tag{21}
\]

where \(f\) follows either Model 1 or Model 2.

Given access to a data set \(\{(X_i, Y_i) : i \in [N]\}\) of \(N\) independent copies of \((X, Y)\), empirical risk minimization defines an estimator as the minimizer of an empirical risk over a suitable function space. Namely, we consider the space of ReLU networks

\[
\mathcal{F}(L_N, W_N, P_N, B_N) := \{\Phi \text{ is a ReLU-net with } L(\Phi) \leq L_N, W(\Phi) \leq W_N, P(\Phi) \leq P_N, B(\Phi) \leq B_N, \|\Phi\|_{[0, 1]^D} \leq 1\}, \tag{22}
\]

and the corresponding ERM

\[
\Phi_N \in \arg\min_{\Phi \in \mathcal{F}(L_N, W_N, P_N, B_N)} \sum_{i=1}^{N} (\Phi(X_i) - Y_i)^2. \tag{23}
\]

For simplicity we assume that the minimum is attained but similar results can be achieved if this is not the case (see discussion in Section 4.3).

The parameters \(L_N, W_N, P_N, B_N\) bound the dimension of admissible ReLU networks and control the complexity of the function space \(\mathcal{F}(L_N, W_N, P_N, B_N)\) (see Lemma 19 in Section 6.1). To achieve optimal statistical estimation rates, we need to balance the bias and variance of the ERM, which amounts to choosing \(L_N, W_N, P_N\) and \(B_N\) as properly scaled functions of the sample size \(N\). The universal bound \(\|\Phi\|_{L_\infty([0, 1]^D)} \leq 1\) is used for technical simplicity to avoid dealing with potentially unbounded predictions. Note that a given ReLU net \(\Phi\) can always be concatenated with a two-layer ReLU \(\mathcal{T}\) that realizes a thresholding to \([-1, 1]\) so that \(\|\mathcal{T} \circ \Phi\|_{L_\infty([0, 1]^D)} \leq 1\).

**Theorem 9.** Let \((X, Y)\) be a random pair so that \(X \in A \subseteq [0, 1]^D\) almost surely, \(Y\) is generated as in (21) and \(f(x) = \mathbb{E}[Y|X = x]\) satisfies Model 1 with approximation domain \(A\). There exist \(N_0\) and \(C\) independent of \(D\) and \(N\) such that for \(N > N_0\) and the choices \(L_N \asymp C \log(D) \log^2(\epsilon^{-1})\), \(P_N \asymp D \log(D) \log^2(\epsilon^{-1}) \epsilon^{-d}\), \(W_N \asymp D \epsilon^{-d}, B_N \asymp C \epsilon^{-2}\), where

\[
\epsilon_N := \log \frac{2}{\sqrt{\pi}} (N^{-\frac{1}{2+d}}), \tag{24}
\]

the ERM (23) satisfies

\[
\mathbb{E}(\Phi_N(X) - f(X))^2 \lesssim_C D \log^{3}(D) \log \frac{\log N}{\epsilon_N} (N)^{-\frac{1}{2+d}}. \tag{25}
\]

**Theorem 10.** Assume Model 2 for \(f(x) = \mathbb{E}[Y|X = x]\). Furthermore assume \(X \in [0, 1]^D\) almost surely and \(Y\) is generated by (21). There exists \(N_0\) and \(C\) independent of \(D\) and \(N\) such that for \(N > N_0\) and the choices \(L_N \asymp C \log(D \epsilon^{-1})\), \(P_N \asymp D \log(2 \epsilon^{-1}) D \epsilon^{-d}\), \(W_N \asymp D \epsilon^{-d}, B_N \leq 1\), where

\[
\epsilon_N := \log (N) \frac{1}{\sqrt{4 \pi (1+d)}} N^{-\frac{1}{2+d}}, \tag{26}
\]

the empirical risk minimizer (23) satisfies

\[
\mathbb{E} \left( \Phi_N(X) - f(X) \right)^2 \lesssim_C D \log^{3}(D) \log \frac{\log N}{\epsilon_N} (N)^{-\frac{1}{2+d}}. \tag{27}
\]

Rate (25) is, up to the log-factor, minimax optimal according to [64] because the considered problem class includes nonparametric regression of \(\alpha\)-Hölder functions on \([0, 1]^d\). Similarly, (27) is nearly rate-optimal whenever \(1 \geq ad\) because the problem class includes univariate nonparametric regression. For \(ad > 1\), we are currently not aware of the minimax rate of Model 2.
4.2 Classification

We consider binary classification with \((X,Y) \in [0,1]^D \times \{-1,1\}\) and class labels generated by

\[
P(Y = 1|X = x) = f(x), \quad \text{respectively,} \quad \mathbb{E}[Y|X = x] = 2f(x) - 1,
\]

where \(f\) follows Model 1 or Model 2. The goal in classification is to compute an estimator that achieves small misclassification error measured by \(R(h) := \mathbb{P}(\text{sign}(h(X)) \neq Y)\). The optimal classifier is known as the Bayes classifier and can be expressed as \(f^*(x) = \text{sign}(2f(x) - 1)\) [16]. Since we can not perform better than the Bayes classifier, we are interested in guarantees about the excess risk \(R(h) - R(f^*)\).

The 0/1-loss used in the definition of the risk \(R\) gives rise to NP-hard minimization problems in empirical risk minimization [6]. Therefore, in practice the 0/1-loss is replaced by a convex surrogate loss such as the logistic loss, Hinge loss, quadratic loss or others [6]. For many typical surrogate losses the estimator still achieves optimal classification rate [6]. In the following we concentrate on the Hinge loss function, which is defined by \(\ell(\Phi(X),Y) = (1 - \Phi(X)Y)_+\). The corresponding ERM is

\[
\hat{\Phi}_N \in \arg\min_{\Phi \in \mathcal{F}(L_N,W_N,P_N,B_N)} \sum_{i=1}^N (1 - \Phi(X)_+),
\]

where \(\mathcal{F}(L_N,W_N,P_N,B_N)\) is as in (22). As in the previous section, network dimensions \(L_N, W_N, P_N\) and \(B_N\) are chosen as functions of the sample size \(N\) to optimally balance the bias and variance of the estimator.

Following (28), the conditional expectation associated with \((X,Y)\) is \(\mathbb{E}[Y|X = x] = 2f(x) - 1\) and thus an easy but potentially naive way of achieving classification guarantees is to treat the problem in the regression framework of Section 4.1. For instance, computing the empirical risk minimizer \(\hat{\Phi}_N(x)\) corresponding to \(\ell_2\)-loss as in (23), we immediately obtain [16, Theorem 2.2]

\[
\left( \mathcal{R}(\hat{\Phi}_N) - \mathcal{R}(f^*) \right)^2 \lesssim \text{MSE}(\hat{\Phi}_N(x), g(\phi(x))).
\]

Then, by using a variant of the oracle inequality Lemma 20 for responses bounded as \(|Y| \leq 1\), see e.g. [22, Theorem 11.4, 11.5], the same learning rates as in Theorems 9 and 10 follow for the squared excess risk (30) [2,70].

It is now well-known however that classification behaves differently than regression because the classification task is less challenging for points \(x\) far away from the decision boundary \(\{x : f(x) = 1\}\). Namely, we can still predict the correct label despite making errors in estimating the conditional expectation \(2f(x) - 1\) if we are sufficiently far away from the decision boundary. As a consequence faster rates can be achieved under suitable margin conditions [37,38,68].

In the following we concentrate on Tsybakov’s margin condition [37,68]

\[
P(|2f(x) - 1| \leq t) \leq C_\beta t^\beta \quad \text{for some } \beta \in (0,\infty) \text{ and } C_\beta > 0,
\]

which bounds the probability mass of \(x\) close to the decision boundary \(\{x : 2f(x) = 1\}\). We show that the classifier \(\text{sign}(\hat{\Phi}_N(x))\), where \(\hat{\Phi}_N(x)\) is the ERM (29), leverages assumption (31) in addition to the structural information about \(f\). Optimal classification rates under margin conditions but without imposing Model 1 or Model 2 have been recently proven in [29].

**Theorem 11.** Let \((X,Y)\) be a random pair so that \(X \in A \subseteq [0,1]^D\) almost surely and \(f(x) = \frac{1}{2}(\mathbb{E}[Y|X = x] - 1)\) satisfies Model 1 with approximation domain \(A\). Furthermore assume margin condition (31) for \(\beta \in (0,\infty]\). There exist \(N_0\) and \(C\) independent of \(D\) and \(N\) such that for \(N > N_0\) and the choices \(L_N \asymp C D \log(D) \log^2(\varepsilon_N^{-1})\), \(P_N \asymp C D \log(D) \log^2(\varepsilon_N^{-1})\varepsilon_N^d\), \(W_N \asymp C D \varepsilon_N^{-d}\), \(B_N \asymp C \varepsilon_N^{-1}\), \(\varepsilon_N \coloneqq \log \frac{\alpha}{\alpha(\alpha + 1)} (N)N^{-\frac{1}{\alpha(\alpha + 1)}}\),

\[
\mathcal{R}(\hat{\Phi}_N) - \mathcal{R}(f^*) \lesssim C D \log^3(D) \log^2(\varepsilon_N^{-1})(N)N^{-\frac{1}{\alpha(\alpha + 1)}}.
\]

The empirical risk minimizer (29) satisfies

\[
\mathcal{R}(\hat{\Phi}_N) \lesssim C D \log^3(D) \log^2(\varepsilon_N^{-1})(N)N^{-\frac{1}{\alpha(\alpha + 1)}}.
\]

**Theorem 12.** Assume Model 2 for \(f(x) = \frac{1}{2}(\mathbb{E}[Y|X = x] - 1)\), \(X \in [0,1]^D\) almost surely, and margin condition (31) for \(\beta \in (0,\infty]\). There exists \(N_0\) and \(C\) independent of \(D\) and \(N\) such that for all \(N > N_0\) and the choices \(L_N \asymp C D \log(D) \varepsilon_N^{-1}\), \(P_N \asymp C D \log(D) \varepsilon_N^{-1}\varepsilon_N^{-d}\), \(W_N \asymp C D \varepsilon_N^{-d}\), \(B_N \asymp C \varepsilon_N^{-1}\), \(\varepsilon_N \coloneqq \log \frac{\alpha}{\alpha(\alpha + 1)} (N)N^{-\frac{1}{\alpha(\alpha + 1)}}\),

\[
\mathcal{R}(\hat{\Phi}_N) \lesssim C D \log^3(D) \log^2(\varepsilon_N^{-1})(N)N^{-\frac{1}{\alpha(\alpha + 1)}}.
\]

\[
\mathcal{R}(\hat{\Phi}_N) \lesssim C D \log^3(D) \log^2(\varepsilon_N^{-1})(N)N^{-\frac{1}{\alpha(\alpha + 1)}}.
\]
the empirical risk minimizer (29) satisfies
\[
\mathcal{R}(\hat{\Phi}_N) - \mathcal{R}(\Phi^*) \lesssim_C D \log^3(D) \log \frac{3n(\beta+1)}{\alpha(\beta+1)} (N) N^{-\alpha(\beta+1)}.
\]
Rate (33) is, up to the log-factors, minimax optimal according to [2] because the problem class contains classification functions, where \( f \) is \( \alpha \)-Hölder and supported on \( \mathbb{R}^d \). The same holds for (33) in the case \( \alpha d \leq 1 \).

4.3 Statistical context, pitfalls, and remedies

Statistical context Adaptivity of statistical estimators to low-dimensionality is a well-known phenomenon in the case where the low-dimensionality is encoded in the distribution \( X \). For instance, in the classical manifold regression setting, where \( X \) is supported on a low-dimensional manifold \( \mathcal{M} \), many common estimators such as kNN [33], piecewise polynomials [8], or kernel estimators [73], are known to adapt automatically to the intrinsic dimension of the embedded manifold \( \mathcal{M} \) and to achieve better statistical estimation rates. It thus not surprising that similar results hold for the ERM over a suitably tuned ReLU function space as shown recently in [46, 57].

The setting covered in this work is however more challenging because the intrinsic dimensionality is only encoded in the joint distribution of \((X, Y)\). Our results suggest that the settings [46, 57] are overly simplified for the case of ReLU estimators because similar rates can be established for a much broader class of models.

Concentrating on Model 1, we are currently not aware of any other practical estimator that achieves estimation rates as suggested by Theorem 9, respectively, Theorem 11. In the simplified setting, where \( \mathcal{M} \) corresponds to a linear subspace, the problem has been extensively studied under the name sufficient dimension reduction and multi-index models in the statistics, and practical estimators with optimal or near-optimal guarantees (under some assumptions that often hold in practice) have been developed [35, 36]. As a step towards nonlinear models, [28] recently proved efficient estimation of \( f \) in the case where \( \mathcal{M} \) is a one-dimensional curve, \( g \) is monotone along the curve, and exact values \( Y = f(X) \) are observed. There is however no trivial way to extend the techniques to higher-dimensional manifolds or to extend sufficient dimension reduction techniques to nonlinear manifolds. In fact, statistically and computationally efficient estimators for Model 1 are, to the best of our knowledge, unknown.

Pitfalls In view of the preceding discussion it is tempting to think that our results solve the open problem because statistically efficient estimation has been proven. This is however not entirely correct because we do not take into account computational costs and the practicability of ERMs over sparsely-constrained ReLU nets. Following discussion articles accompanying the work [56], which pioneered some of the statistical learning techniques that we use in this work, the general approach studied in this section has the following main pitfalls:

Pitfall 1 Computing the global ERM (29) may be NP-hard.

Pitfall 2 In practice we use fully connected, overparametrized ReLU nets instead of sparsely-constrained ReLU nets. A ReLU space with fully connected nets that contains networks constructed in Section 3 is however too rich and does not imply optimal statistical rates.

Pitfall 3 The choice of the optimization method plays an important part in deep learning, but is blended out when studying ERMs.

Remedies Pitfall 1 can be remedied by considering slightly more technical analysis. Namely, it is easy to modify the regression guarantees in Theorems 9 and 10 to an arbitrary estimator \( \hat{\Phi}_N \in \mathcal{F}(L_N, W_N, P_N, B_N) \) by adding to (25), respectively, (27), the term
\[
\Delta_N := \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} (Y_i - \hat{\Phi}_N(X_i))^2 - \inf_{\Phi \in \mathcal{F}(L_N, W_N, P_N, B_N)} \frac{1}{N} \sum_{i=1}^{N} (Y_i - \Phi(X_i))^2 \right],
\]
which measures the discrepancy between optimal empirical loss and achieved empirical loss. This has already been pointed out in [56] and our proof in Section 6.1 actually proves the more general statement. We believe that similar results hold for the classification case but did not attempt to prove it rigorously.

Pitfall 2 can potentially be avoided by replacing \( \mathcal{F}(L_N, W_N, P_N, B_N) \) with a fully connected space of ReLU nets of suitable dimensions and then adding penalty terms to optimal programs (23) or (29). Such
an approach has been recently used in [50], where the authors define clipped \( \ell_1 \)-penalty terms applied to the weights of the network. Another idea is to study the effect of pruning networks, i.e. to first compute an estimator contained in a space of fully-connected ReLU nets, and then project the estimator onto \( \mathcal{F}(L_N, W_N, P_N, B_N) \) by thresholding weights or by using other more sophisticated techniques [58]. If the empirical loss remains small for the pruned network, then \( \Delta_N \) is small and we still obtain meaningful statistical guarantees. 

Finally, we believe that Pitfall 3 reveals an important issue in our understanding of generalization of deep neural networks, namely of how the optimizer influences the generalization performance of the network. It is conjectured that the choice of the optimizer leads to implicit regularization, which chooses a network of minimal complexity in a suitable sense. While this is well understood for other problems such as linear regression [20], logistic regression [62], or matrix factorization [21], little is known in the case of deep networks. Nonetheless, we believe that the importance of optimization for deep learning does not stand in contrast with statistical guarantees presented in this work. Rather, we believe that optimization might give the answer why explicitly enforcing low-complexity network structure through sparse or penalized networks is typically not necessary in practice.

5 Proofs for Section 3

5.1 Proofs of Theorems 6 and 7

We begin with the proof of Theorem 6 and keep in mind that there exists \( q \in [0, 1) \) so that \( \mathcal{A} \subset \mathcal{M}(q) \). The first step of the proof is to construct partition of unity functions \( \eta_i : \mathcal{A}(q) \to [0, 1] \) satisfying the localization property \((15)\). To this end, we first have to rigorously prove the equivalence between the geodesic metric and \( \|\hat{A}(z_i)\top(x-z_i)\|_2 \) as was mentioned in \((17)\).

Proposition 13. Let \( \mathcal{M} \) be a connected compact \( d \)-dimensional Riemannian submanifold \( \mathcal{M} \) of \( \mathbb{R}^p \) and let \( q \in [0, 1) \). For \( x \in \mathcal{M}(q) \) with \( v = \pi_{\mathcal{M}}(x) \) and arbitrary \( z \in \mathcal{M} \) we have

\[
\|A(z)\top(x-z)\|_2 \leq \left(1 + \frac{\text{dist}(x; \mathcal{M})}{\tau_{\mathcal{M}} \vee (\tau_{\mathcal{M}}(v) - d_{\mathcal{M}}(v, z))}\right) d_{\mathcal{M}}(z, v). \quad (37)
\]

Let now \( p \in [q, 1) \) arbitrary. Then for \( x \in B_{p\tau_{\mathcal{M}}(z)}(z) \) with \( \|A(z)\top(x-z)\|_2 < \frac{1-p}{4} \tau_{\mathcal{M}} \), we have

\[
d_{\mathcal{M}}(z, v) \leq \frac{3}{1-p} \|A(z)\top(x-z)\|_2. \quad (38)
\]

Proof. Throughout the proof we denote \( P(z) = A(z)A(z)\top \) as the orthoprojector onto the tangent space of \( \mathcal{M} \) at \( z \in \mathcal{M} \). For \((37)\) we use \( P(v)(x-v) = 0 \) from Lemma 1, \( \|z-v\|_2 \leq d_{\mathcal{M}}(z, v) \), and the tangent perturbation bound \((10)\) applied to the geodesic path \( \gamma_{x \to v} \) from \( x \) to \( v \) with reach bound \( \tau_{\gamma_{x \to v}} = \inf_{y \in \text{Im}(\gamma_{x \to v})} \tau_{\mathcal{M}}(y) \) to compute

\[
\|P(z)(x-z)\|_2 \leq \|P(z)(v-z)\|_2 + \|P(z)(x-v)\|_2 \leq d_{\mathcal{M}}(v, z) + \|P(z) - P(v)\|_2 \|x-v\|_2 \\
\leq d_{\mathcal{M}}(v, z) + \frac{\text{dist}(x; \mathcal{M})}{\tau_{\gamma_{x \to v}}} d_{\mathcal{M}}(v, z).
\]

Furthermore, by the 1-Lipschitz property of the local reach, see \((11)\), we have

\[
\tau_{\gamma_{x \to v}} = \inf_{y \in \text{Im}(\gamma_{x \to v})} \tau_{\mathcal{M}}(y) \geq \tau_{\mathcal{M}}(v) - \sup_{y \in \text{Im}(\gamma_{x \to v})} |\tau_{\mathcal{M}}(y) - \tau_{\mathcal{M}}(v)| \geq \tau_{\mathcal{M}}(v) - d_{\mathcal{M}}(v, z).
\]

Since the global bound \( \tau_{\gamma_{x \to v}} \geq \tau_{\mathcal{M}} \) holds due to \( \text{Im}(\gamma_{x \to v}) \subset \mathcal{M} \), we obtain

\[
\|P(z)(x-z)\|_2 \leq \left(1 + \frac{\text{dist}(x; \mathcal{M})}{\tau_{\mathcal{M}} \vee (\tau_{\mathcal{M}}(v) - d_{\mathcal{M}}(v, z))}\right) d_{\mathcal{M}}(v, z).
\]

For the opposite direction \((38)\) we let \( \omega := \|P(z)(x-z)\|_2 \) and \( \tilde{x} := x + Q(z)(x-z) \), where \( Q(z) := \text{Id} - P(z) \). By construction we have \( P(z)(\tilde{x}) = 0 \) and

\[
\|x - \tilde{x}\|_2 = \|x - z - Q(z)(x-z)\|_2 = \|P(z)(x-z)\|_2 = \omega.
\]
Furthermore, since $x \in B_{p\tau_M(z)}(z)$, $\omega < \frac{1-p}{3} \tau_M$, and $\tau_M \leq \tau_M(z)$, we can bound

$$||\hat{x} - z||_2 \leq ||x - z||_2 + ||x - \hat{x}||_2 \leq p\tau_M(z) + \omega < p\tau_M(z) + \frac{1-p}{3} \tau_M < \bar{p}\tau_M(z),$$

for $\bar{p} = \frac{1+2p}{3} < 1$. We thus have the decomposition $\hat{x} = z + (\hat{x} - z)$ for $z \in M$, $\hat{x} - z \perp \text{Im}(P(z))$, and $||\hat{x} - z||_2 < \bar{p}\tau_M(z)$. Lemma 1 implies $z = \pi_M(\hat{x})$ and $\hat{x} \in M(\bar{p})$. Using now the Lipschitz property of $\pi_M$ in Lemma 2 and $x \in M(q) \subset M(\bar{p})$, $\hat{x} \in M(\bar{p})$, it follows that

$$\|v - z\|_2 = \|\pi_M(x) - \pi_M(\hat{x})\|_2 \leq \frac{1}{1 - \bar{p}} \|x - \hat{x}\|_2 = \frac{3}{2(1 - p)} \omega.$$

To translate this to a bound for $d_M(v, z)$, we further note

$$\|v - z\|_2 \leq \frac{3}{2(1 - p)} \omega < \frac{1}{2} \tau_M,$$

so that we can apply [18, Lemma 3] to bound

$$d_M(v, z) \leq \tau_M - \tau_M \sqrt{1 - \frac{2\|v - z\|_2}{\tau_M}} \leq \|v - z\|_2 + \frac{2\|v - z\|_2^2}{\tau_M} \leq 2\|v - z\|_2.$$

Based on the metric equivalence in Proposition 13 we now construct the partition of unity function (not yet approximated by ReLU nets). We require the following technical Lemma, which states that the cardinality of an intersection of a maximally $\delta$-separated set. For any $v \in M$ and $p$ with $p\delta \in (0, \frac{1}{4} \tau_M)$ we have

$$|Z \cap B_{M,p\delta}(v)| \lesssim_d p^d.$$  

**Proof.** We first note that $Z \cap B_{M,p\delta}(v)$ is still a $\delta$-separated set of the geodesic ball $B_{M,p\delta}(v)$, which implies $|Z \cap B_{M,p\delta}(v)| \leq P(\delta, B_{M,p\delta}(v), d_M)$ by Lemma 4. Since the reach of the geodesic ball $B_{M,p\delta}(v)$ is also bounded by $\tau_M$, we can apply [3, 47] as in Lemma 4 to get

$$P(\delta, B_{M,p\delta}(v), d_M) \leq \frac{3^d \text{Vol}(B_{M,p\delta}(v)) d^{\frac{d}{2}}}{\delta d}.$$

The result follows by applying the bound [12, Proposition 1.1] giving

$$\text{Vol}(B_{M,p\delta}(v)) \leq C_d \left(\frac{\tau_M}{\tau_M - 2p\delta}\right)^d (p\delta)^d \leq 2^d C_d p^d \delta^d,$$

where $C_d$ is the volume of the Euclidean unit ball in $\mathbb{R}^d$. \qed

**Proposition 15.** Let $M$ be a connected compact $d$-dimensional Riemannian submanifold $M$ of $\mathbb{R}^D$ and let $q \in [0, 1]$. Let $Z = \{z_1, \ldots, z_{|Z|}\} \subset M$ be a maximal $\delta$-separated set of $M$ with respect to $d_M$. Define bandwidth parameters $p := \frac{1}{2}(1 + q)$ and $h := \frac{6}{1 - 9p}$ and functions $\tilde{\eta}, \eta : M(q) \rightarrow \mathbb{R}^{|Z|}$ entrywise by

$$\tilde{\eta}(x) = \left(1 - \frac{||x - z||_2^2}{p\tau_M(z_i)}\right)^2 \left(\frac{||A(z_i)^T(x - z)||_2}{h\delta}\right)^2 + \text{ and } \eta(x) = \frac{\tilde{\eta}(x)}{||\tilde{\eta}(x)||_1}.$$

There exists a universal constant $C$ such that if $\delta \in (0, C(1 - q)^2 \tau_M)$ we have

$$\sup_{x \in M(q), \tilde{\eta}(x) \neq 0} d_M(q)(x, z_i) \leq \frac{72}{(1 - q)^3} \delta,$$

(1 - q) \lesssim ||\eta(x)||_1 \lesssim_d (1 - q)^{-2d}. (41)
Proof. Denote \( v = \pi_{\mathcal{M}}(x) \). We will a few times require in the following the bandwidth ratio
\[
\frac{3h}{1-p} = \frac{36(q+1)}{(1-q)^2} \in \left[ \frac{36}{(1-q)^2}, \frac{72}{(1-q)^2} \right].
\]
By construction \( \eta_i(x) \neq 0 \) implies \( x \in B_{p\tau_M(z_i)}(z_i) \) and \( \|A(z_i)^T(x-z_i)\|_2 < h\delta \). Thus, as soon as \( \delta < \frac{1}{36} \tau_M \), which is implied by \( \delta < \frac{1}{36}(1-q)^2\tau_M \), we have \( \|A(z_i)^T(x-z_i)\|_2 \leq \frac{1-p}{3} \tau_M \). Applying Proposition 13 gives (40) by
\[
d_{\mathcal{M}(q)}(x,z_i) = d_{\mathcal{M}}(v,z_i) \leq \frac{3h}{1-p} \delta \leq \frac{72}{(1-q)^2} \delta.
\]
We now concentrate on the lower bound in (41). Denote \( j \in \arg\min_{i \in |Z|} d_{\mathcal{M}(q)}(x,z_i) \). Since \( Z \) is a maximal \( \delta \)-separated set of \( \mathcal{M} \), we have \( d_{\mathcal{M}(q)}(x,z_j) \leq \delta \). Eqn. (37) in Proposition 13 implies
\[
\|A(z_j)^T(x-z_j)\|_2 \leq \left( 1 + \frac{\text{dist}(x;\mathcal{M})}{\tau_M(v) - \delta} \right) \delta \leq \left( 1 + \frac{q\tau_M(v)}{\tau_M(v) - \delta} \right) \delta = \left( 1 + q \frac{1}{1 - \frac{\delta}{\tau_M(v)}} \right) \delta \leq (1 + 2q)\delta \leq 3\delta,
\]
provided \( \delta < \frac{1}{2} \tau_M \).

Using the triangle inequality to get \( \|x - z_j\|_2 \leq \delta + \|x - v\|_2 \) and the 1-Lipschitz continuity of \( \tau_M(\cdot) \) in (11) to further bound \( \|x - v\|_2 \leq q\tau_M(v) \leq q(\delta + \tau_M(z_j)) \), it follows that
\[
\|r_M(z_j)\| \leq \|x - z_j\|_2 \leq \frac{\|A(z_j)^T(x-z_j)\|_2}{h\delta} \leq \frac{\frac{1}{1-q} \tau_M}{\delta} \leq \frac{q}{p} \leq \frac{q}{\tau_M} \leq \frac{q}{p} \leq \frac{q}{\tau_M} \leq \frac{4}{\tau_M} \delta.
\]
Inserting the definition of the bandwidth parameter \( h \), we thus obtain
\[
1 - \frac{\|x - z_j\|_2}{\|A(z_j)^T(x-z_j)\|_2} \geq 1 - \frac{q}{p} - \frac{4}{\tau_M} \delta \geq \frac{1}{2} \left( 1 - \frac{q}{p} \right) - \frac{4}{\tau_M} \delta.
\]
This is bounded from below by \( \frac{1}{4}(1-qp^{-1}) \) as soon as
\[
\delta < \frac{\tau_M}{16} \left( 1 - \frac{q}{p} \right) = \frac{\tau_M}{16} \frac{1-q}{1+q}, \quad \text{which is implied by} \quad \delta < \frac{(1-q)\tau_M}{16}.
\]
Since squaring one of the subtracted terms in (42) reduces their size, we get the lower bound \( \|\tilde{\eta}(x)\|_1 \geq \tilde{\eta}_i(x) \geq \frac{1}{4}(1-qp^{-1}) \geq 1/8(1-q) \).

For the upper bound on \( \|\tilde{\eta}(x)\|_1 \) we notice that \( \tilde{\eta}_i(x) \neq 0 \) implies by Proposition 13
\[
\|A(z_j)^T(x-z_j)\|_2 \geq \frac{1-p}{3} d_{\mathcal{M}(q)}(z_i,x) \quad \text{provided} \quad \delta < \frac{(1-q)^2}{36} \tau_M.
\]
Thus, \( \tilde{\eta}_i(x) \neq 0 \) implies \( d_{\mathcal{M}(q)}(x,z_i) \leq 3h(1-p)^{-1}\delta \), ie. all \( z_i \)'s contributing to \( \|\tilde{\eta}\|_1 \) are contained within a geodesic ball of radius \( 3h(1-p)^{-1} \delta \) around \( v \). As soon as \( \frac{3h}{1-p} \delta < \frac{1}{2} \tau_M \), which is implied by \( \delta < 288(1-q)^2\tau_M \), we can then use Lemma 14 to bound
\[
\|Z \cap B_{\mathcal{M}(q,v)}(v)\|_d \leq \left( \frac{3h}{1-p} \right)^d \frac{72^d}{(1-q)^{2d}}.
\]
Since each \( \tilde{\eta}_i(x) \) is individually bounded by 1, the upper bound on \( \|\tilde{\eta}(x)\|_1 \) in (41) follows. \( \square \)

To conclude the proof of Theorems 6 and 7 we need to show that \( \eta \) can be efficiently approximated up to accuracy \( \varepsilon \) with ReLU net of small complexity. For simplicity, we neglect further tracking of the influence of parameter \( q \), while keeping in mind that constants diverge when \( q \to 1 \) as indicated in Propositions 13 and 15.

Lemma 16. Assume the setting of Proposition 15 and let \( \mathcal{M}(q) \subseteq [0,1]^D \). There exist \( C, \varepsilon_0 \) depending on \( q, d, \tau_M, \text{Vol}(\mathcal{M}) \) so that for all \( \varepsilon \in (0, \varepsilon_0) \) we can construct a ReLU-net \( \Phi \) with \( L(\Phi) \leq C \log^2(D\delta^{-1}\varepsilon^{-1}) \),
\[
P(\Phi) \leq C D \log(D) \log^2(\delta^{-1}\varepsilon^{-1}) \delta^{-d}, \quad W(\Phi) \leq C D \delta^{-d}, \quad \text{and} \quad B(\Theta) \leq C \delta^{-2} \vee D \text{ such that}
\]
\[
\sup_{x \in \mathcal{M}(q)} \|\eta(x) - \Phi(x)\|_1 \leq \varepsilon.
\]
1. Approximating \(\tilde{\eta}:\) Let \(\Theta\) be a ReLU net that approximates \(\|\|_2^2\) over \(B_{\sqrt{D}}(0)\) to accuracy \(\tilde{\varepsilon} := C_{q,d}\) \(|Z|^{-1}\delta^2\varepsilon\) (existence proven in Lemma 26) for suitably chosen constant \(C_{q,d}\) depending on \(q, d\) and \(\tau, M\). Furthermore, let \(\Psi_i\) realize \(x \mapsto x - z_i\), and \(\Gamma_i\) realize \(x \mapsto A(z_i)^	op(x - z_i)\). For bandwidth parameters \(p, h\) as in Proposition 15, we then define a ReLU network \(\tilde{\Phi}_i(x) = \left(1 - \frac{\Theta(\Psi_i(x))}{(p\tau_M(z_i))^2} - \frac{\Theta(\Gamma_i(x))}{(h\delta)^2}\right)_+\).

Comparing \(\tilde{\Phi}_i\) with \(\tilde{\eta}\) we obtain by 1-Lipschitzness of the ReLU and the triangle inequality

\[
\sup_{x \in M(q)} \left| \tilde{\Phi}_i(x) - \tilde{\eta}(x) \right| \leq \frac{\Theta(\Psi_i(x))}{p\tau_M(z_i)^2} + \frac{\Theta(\Gamma_i(x))}{h\delta^2} = \frac{1}{h^2} \left( \frac{1}{2} \frac{\varepsilon}{\tau_M(z_i)^2} \right) + \frac{1}{h^2} \left( \frac{\varepsilon}{\tau_M(z_i)^2} \right) \leq \frac{\varepsilon}{4|Z|},
\]

where we used \(x - z_i \in B_{\sqrt{D}}(0)\) since \(x, z_i \in [0, 1]^D\), and that \(p, \tau_M, h, c_q\) just depend on \(q, d, \tau, M\) such that we can choose \(C_{q,d}\) in the definition of \(\tilde{\varepsilon}\) suitable to achieve the bound. To compute the complexity of \(\tilde{\Phi}_i\) we apply the rules of ReLU concatenation and linear combination in Lemma 23 and 24, and the complexity bounds in Lemma 26. We have \(L(\Theta \circ \Psi_i) \leq L(\Theta) + L(\Psi_i) \leq L(D\varepsilon^{-1})\), \(W(\Theta \circ \Psi_i) \leq D\), \(P(\Theta \circ \Psi_i) \leq D\), and the same bounds hold for \(\Theta \circ \Gamma_i\). Thus, by the rules of ReLU linear combination in Lemma 24 (the additional ReLU activation in the last layer does not matter asymptotically) we have

\[
L(\tilde{\Phi}_i) \leq 2\log(D\varepsilon^{-1}), \quad W(\tilde{\Phi}_i) \leq D, \quad P(\tilde{\Phi}_i) \leq D, \quad B(\tilde{\Phi}_i) \leq \frac{1}{h^2} \leq D \
\]
We note that the proof of Lemma 16 reveals the importance of uniform lower and upper bounds on $\|\tilde{\eta}(x)\|_1$, since they are required for approximating the normalization factor $\|\tilde{\eta}(x)\|_1$ efficiently with a small-size ReLU net. To conclude the proof of Theorem 6, we combine Lemma 16 and the $\alpha$-Hölder property of $g$.

**Proof of Theorem 6** Let $Z := \{z_1, \ldots, z_K\}$ be a maximal separated $\varepsilon$-net of $\mathcal{M}$ with $K := |Z| \lesssim_C \varepsilon^{-d}$ by Lemma 4 and let $g(Z) = (g(z_1), \ldots, g(z_K)) \in \mathbb{R}^K$. According to Lemma 16, we can construct a network $\Theta : \mathbb{R}^D \to \mathbb{R}^K$, which approximates the partition of unity function $\eta(x)$ in (40) over $\mathcal{A} \subseteq \mathcal{M}(q)$ up to accuracy $\varepsilon^α$. To approximate the target $f$ we define the net

$$\Psi(x) := \sum_{i=1}^K (g(z_i)\Theta_i(x))_+ - (g(z_i)\Theta_i(x))_- = \langle g(Z), \Theta(x) \rangle,$$

where we used the representation $u = (u)_+ - (u)_-$ to comply with the requirement that only the last layer of the network $\Psi$ is linear (recall Definition 5 of ReLU nets). Taking arbitrary $x \in \mathcal{A} \subseteq \mathcal{M}(q)$, we can first bound the error with the triangle inequality by

$$|f(x) - \Psi(x)| = |g(\pi_M(x)) - g(Z)\Theta(x)| \leq |g(\pi_M(x)) - \langle g(Z), \eta(x) \rangle| + |\langle g(Z), \eta(x) - \Theta(x) \rangle| \leq \|g(\pi_M(x))\|_2 \|\eta(x) - \Theta(x)\|_2 + \varepsilon^α,$$

where $\|\cdot\|_2$ is the $\alpha$-Hölder norm and we used $\langle \eta, \eta(x) \rangle = 1$, $\|g(Z)\|_\infty \leq 1$. The first term can be bounded by Hölder’s inequality and Proposition 15 according to

$$\|g(\pi_M(x))\|_2 \|\eta(x) - \Theta(x)\|_2 \lesssim_C \varepsilon^α,$$

which implies the approximation error bound. To bound the complexity of $\Phi$ we note that the network is a concatenation of $\Theta$ with a two-layer network that has first layer weights $\pm g(Z) \in \mathbb{R}^K$ and second layer weights $\pm 1$. By the rules of ReLU concatenation in Lemma 23 and since $\Theta$ dominates the complexity compared to the two-layer network, it follows that $L(\Phi) \leq L(\Theta)$, $W(\Phi) \leq W(\Theta)$ and $P(\Phi) \leq P(\Theta)$. The weights of the two layer network are bounded by 1, which implies $B(\Phi) \leq B(\Theta)$. The result thus follows from dimensions bounds in Lemma 16.

**Proof of Theorem 7** The proof follows along the lines of the proof of Theorem 6 with the difference that the constructed network (46) is now given by

$$\Phi(x) := (Z\Theta(x))_+ - (-Z\Theta(x))_+ = Z\Theta(x),$$

for $Z = [z_1| \ldots |z_K] \in \mathbb{R}^{D \times K}$, and $\Theta : \mathbb{R}^D \to \mathbb{R}^K$ approximates $\eta(x)$ over $\mathcal{M}(q)$ up to accuracy $\varepsilon$ (as in the previous proof, we use $u = (u)_+ - (u)_-$ to comply with the requirement that only the last layer of $\Phi$ can be linear according to Definition 5). To bound the approximation error we first use the triangle inequality to get

$$\|\Psi(x) - \pi_M(x)\|_\infty = \|Z\Theta(x) - \pi_M(x)\|_\infty \leq \|Z\Theta(x) - Z\eta(x)\|_\infty + \|Z\eta(x) - \pi_M(x)\|_\infty.$$

Then, by Hölder’s inequality with $p = 1$ and $q = \infty$, the first term is bounded by

$$\|Z\Theta(x) - Z\eta(x)\|_\infty \leq \max_{i,j} |Z_{ij}| \|\Theta(x) - \eta(x)\|_1 \leq \varepsilon,$$

where we used $Z_{ij} \in [0,1]^D$ and thus $\max_{i,j} |Z_{ij}| \leq 1$. For the second term let $\tilde{Z} = [z_1 - \pi_M(x)| \ldots |z_K - \pi_M(x)] \in \mathbb{R}^{D \times K}$. Since $\sum_{i=1}^K \eta_i(x) = 1$, we have

$$\|Z\eta(x) - \pi_M(x)\|_\infty = \|\tilde{Z}\eta(x)\|_\infty \leq \max_{i,j} |z_i - \pi_M(x)|_2 \leq \max_{i,j} d_M(z_i, \pi_M(x)) \lesssim_C \varepsilon,$$

where we used Proposition 15 in the last inequality. Since $\Phi$ is a concatenation of networks $\Theta$ and a two layer ReLU-network with first layer weights $Z - Z \in \mathbb{R}^{D \times K}$ and second layers weights $\pm 1$, and since $K \lesssim_C \varepsilon^{-d}$, we have $L(\Phi) \approx L(\Theta)$, $W(\Phi) \approx W(\Theta)$, and $P(\Phi) \approx P(\Theta)$ by the rules of ReLU concatenation in Lemma 23. Furthermore, all additional weights of $\Phi$ are bounded by 1, hence $B(\Phi) \leq B(\Theta)$. 

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5.2 Proof of Theorem 8

As sketched in Section 3.2 the proof of Theorem 8 requires approximation of the normalized distance function \( x \mapsto \text{ndist}(x; C)^2 \) for a closed subset \( C \subset \mathbb{R}^D \) and approximation of \( g \). The result then follows from ReLU concatenation and linear combination (Lemma 23 and 24). We first prove the approximation of the normalized distance as a separate result.

Lemma 17. Let \( C \subset [0,1]^D \) be a nonempty closed subset of \( \mathbb{R}^D \) and assume there exists \( \delta_0 > 0 \) so that \( P(\sqrt{D}\delta, C, \| \cdot \|_2) \lesssim \delta^{-d} \) for all \( \delta \in (0, \delta_0) \). For any \( \varepsilon \in (0,3\delta_0) \) there exists a ReLU network \( \Phi \) with \( L(\Phi) \lesssim d \log(D\varepsilon^{-1}) \), \( W(\Phi) \lesssim \varepsilon^{-d} \), \( P(\Phi) \lesssim D \log(D\varepsilon^{-1}) \varepsilon^{-d} \), and \( B(\Phi) \leq 1 \) satisfying

\[
\sup_{x \in [0,1]^D} |\text{ndist}(x; C)^2 - \Phi(x)| \leq \varepsilon. \tag{47}
\]

Proof. Let \( Z \subset C \) be a maximal separated \( \sqrt{D\varepsilon} \)-net of \( C \), which has cardinality bounded according to \(|Z| \lesssim \varepsilon^{-d}\) as soon as \( \varepsilon < 3\delta_0 \). Let \( \Phi \) be a ReLU network that approximates \( \| \cdot \|_2^2 \) up to accuracy \( \frac{\varepsilon}{3} \) on \( B_1(0) \) (Lemma 26), let \( \Theta_1 \) be a ReLU net that realizes \( \Theta_i(x) = \frac{x_i}{\sqrt{D}} \) for \( i \in |Z| \), and let \( \Gamma : \mathbb{R}^{|Z|} \to \mathbb{R} \) be a network that realizes \( \Gamma(u) = \min_{i \in |Z|} u_i \) (Lemma 30). Then we set \( \Phi(x) = \Gamma(\Theta_1(x), \ldots, \Theta_1(x)|_Z(x)) \).

Using the triangle inequality, we first split the error as

\[
|\text{ndist}(x; C)^2 - \Phi(x)| \leq \min_{z \in Z} \left\| \frac{x - z}{\sqrt{D}} \right\|_2^2 - \Phi(x) + \frac{1}{\sqrt{D}} \min_{z \in Z} \| x - z \|_2^2 - \text{dist}(x; C)^2 . \tag{48}
\]

For the first term we have \( \frac{x - z}{\sqrt{D}} \in B_1(0) \) for any \( z \in Z \) and thus

\[
\min_{z \in Z} \left\| \frac{x - z}{\sqrt{D}} \right\|_2^2 - \Phi(x) = \min_{z \in Z} \left\| \frac{x - z}{\sqrt{D}} \right\|_2^2 - \min_{z \in Z} \Theta_1 \left( \frac{x - z}{\sqrt{D}} \right) \leq \max_{z \in Z} \left\| \frac{x - z}{\sqrt{D}} \right\|_2^2 - \Phi \left( \frac{x - z}{\sqrt{D}} \right) \leq \frac{\varepsilon}{3}.
\]

For the second term in \((48)\) we note that there exists \( v(x) \in C \) satisfying \( \| x - v(x) \|_2^2 = \text{dist}(x; C)^2 \) because \( C \) is closed and nonempty. Then we compute

\[
\min_{z \in Z} \| x - z \|_2^2 - \text{dist}(x; C)^2 = \min_{z \in Z} \left( \| x - z \|_2^2 - \| x - v(x) \|_2^2 \right) \leq 2\sqrt{D} \min_{z \in Z} \| z - v(x) \|_2 \leq \frac{2}{3} D\varepsilon,
\]

where we used the inverse triangle inequality and the fact that \( Z \) is a \( \sqrt{D\varepsilon} \)-net of \( C \). Since the error is normalized by \( D^{-1} \), see \((48)\), the result follows. It remains to bound complexity of the network in terms of \( \varepsilon \) and \( Z \). Using the rules of concatenation and linear combinations of networks in Lemma 23 and 24 we have

\[
L(\Phi) = L(\Gamma) + L(\Phi(1), \ldots, \Phi(|Z|))) = L(\Gamma) + L(\Phi(1)) = L(\Gamma) + L(\Phi) + L(\Theta_1)
\lesssim \log(|Z|) + \log(D\varepsilon^{-1}) + 2 \lesssim d \log(D\varepsilon^{-1}) + \log(D\varepsilon^{-1}) \lesssim d \log(D\varepsilon^{-1})
\]

\[
W(\Phi) = \max\{W(\Gamma), W(\Phi(1), \ldots, \Phi(|Z|)))\} \leq |Z| W(\Phi(1)) \lesssim D\varepsilon^{-d},
\]

\[
P(\Phi) \lesssim P(\Gamma) + P(\Phi(1), \ldots, \Phi(|Z|))) \lesssim P(\Gamma) + |Z| P(\Phi(1)) \lesssim P(\Gamma) + |Z| P(\Phi) + |Z| P(\Theta_1)
\lesssim |Z| \log(|Z|) + |Z| D \log(D\varepsilon^{-1}) \lesssim d \log(D\varepsilon^{-1}) + |Z| \lesssim \varepsilon^{-d} D \log(D\varepsilon^{-1})
\]

\[
B(\Phi) \leq B(\Gamma) \lor B((\Theta_1), \ldots, \Phi(|Z|))) \leq 1.
\]

To prove Theorem 8 we now combine Lemma 17 with Theorem 25 in the Appendix, which provides ReLU approximation results for functions in the smoothness class

\[
C^\alpha_k(L) := \left\{ f : [0,1]^k \to \mathbb{R} : \sum_{|\alpha| < \alpha} \| \varphi^\alpha f \|_\infty + \sum_{|\alpha| = |\beta|} \| \varphi^\alpha f(x) - \varphi^\alpha f(y) \|_{\infty - |\alpha|} \leq L \right\}.
\]

By the assumptions in Model 2, we have \( g_\ell \in C^\alpha_k(1 + L) \) for all \( \ell \in [M] \).
Proof of Theorem 8. Consider the case $M=1$ first and let $g_1=g$, $C_1=C$. Let $\Psi: \mathbb{R}^D \to \mathbb{R}$ be the ReLU net approximating $x \mapsto \text{ndist}(x; C)^2$ up to accuracy $\frac{\varepsilon}{2}$ according to Lemma 17, with $\varepsilon^{\alpha} < 6\delta_0 L$, and let $\Theta: \mathbb{R} \to \mathbb{R}$ be a ReLU net that realizes $\Theta(f) = 1 + t \geq (1-t)$, $\forall t \in (0, 1)$. Furthermore, by Theorem 25 there exists a ReLU network $\Omega$ which approximates $g$ to accuracy $\frac{\varepsilon}{2}$ over $[0, 1]$. We define the overall approximation by $\Phi(x) := \Omega(\Psi(x))$ and compute

$$
\left| g(\text{ndist}(x; C)^2) - \Phi(x) \right| = \left| g(\text{ndist}(x; C)^2) - \Omega(\Psi(x)) \right| 
\leq \left| g(\text{ndist}(x; C)^2) - g(\Theta(\Psi(x))) \right| + \left| g(\Theta(\Psi(x))) - \Omega(\Theta(\Psi(x))) \right| 
\leq \left| g(\text{ndist}(x; C)^2) - g(\Theta(\Psi(x))) \right| + \frac{\varepsilon^{\alpha}}{2},
$$

where we used $\Theta(\Psi(x)) \in [0, 1]$ by construction and the approximation guarantees about $\Omega$ in the last step. For the first term in (51) on the other hand, we use the $\alpha$-Hölder property of $g$ to get

$$
\left| g(\text{ndist}(x; C)^2) - g(\Theta(\Psi(x))) \right| \leq L \left| \text{dist}(x; C)^2 - \Theta(\Psi(x)) \right| = L \left| \text{dist}(x; C)^2 - 1 \right| \leq \frac{\varepsilon^{\alpha}}{2},
$$

where the second to last inequality is an equality if $\Psi(x) < 1$, and follows from $\text{ndist}(x; C)^2 \leq 1$ (since $x \in [0, 1]^D$ and $C \subset [0, 1]^D$ if $\Psi(x) \geq 1$. To bound the complexity of $\Phi$ we will use the rules of concatenation according to Lemma 23. We have

$$
L(\Phi) \leq L(\Omega) + L(\Theta) + L(\Psi) \lesssim C \log(\varepsilon^{-1}) + 2 + \log(D\varepsilon^{-1}) \lesssim C \log(D\varepsilon^{-1}),
$$

$$
W(\Phi) \leq \max\{W(\Omega), W(\Theta), W(\Psi)\} \lesssim C (\varepsilon^{\alpha})^{-1/\alpha} + D\varepsilon^{-\alpha d} \lesssim D\varepsilon^{-1(1+\alpha d)},
$$

$$
P(\Phi) \lesssim P(\Omega) + P(\Theta) + P(\Psi) \lesssim C \log_2(e^1)(\varepsilon^{\alpha})^{-1} + \varepsilon^{-\alpha d} D\log(\varepsilon^{-1}) \lesssim C D \log(D\varepsilon^{-1})^{-1(1+\alpha d)},
$$

$$
B(\Phi) \leq \max\{B(\Omega), B(\Theta), B(\Psi)\} \lesssim 1.
$$

For the case $M > 1$ we construct networks $\Phi_i$ approximating $g_i(\text{ndist}(x; C_i))$ to accuracy $\varepsilon^{\alpha}/M$ and then use $x \mapsto \sum_{i=1}^M \Phi_i(x)$, which can be realized by a ReLU net according to Lemma 24. Dimension bounds for the approximating network also follows from Lemma 24. 

\[\square\]

6 Proofs for Section 4

6.1 Proofs of Theorems 9 and 10

The proof of regression results Theorem 9 and 10 follows classic steps of analyzing performances of ERMs. Namely, we first use Lemma 20 below to separate the generalization error $E(\Phi_N(x) - f(x))^2$ into an estimation error, which decreases with growing sample size and increases with increasing complexity of the ReLU space $F_N := F(L_N, W_N, P_N, B_N)$, and an approximation error, which decreases when increasing the complexity of $F_N$. Bounding the approximation error follows directly from applying approximation guarantees in Theorems 6 and 8. For the estimation error on the other hand, we bound the complexity of $F_N$ by Lemma 19, where we measure the complexity by covering numbers as introduced in Definition 18. The proof concludes by verifying that choices $L_N$, $W_N$, $P_N$, and $B_N$ in Theorems 9 and 10 balance estimation error and approximation error optimally and that the resulting error matches the claimed rate.

We briefly collect the necessary tools from the literature and then proof the results.

Definition 18. Let $\delta > 0$. For a function space $\mathcal{F} = \{f : [0,1]^D \to \mathbb{R}\}$ we define the covering number $N(\delta, \mathcal{F}, \|\cdot\|_\infty) \in \mathbb{N}$ as the minimum number of elements $\{f_i : i \in [N(\delta, \mathcal{F}, \|\cdot\|_\infty)]\}$, which do not necessarily belong to $\mathcal{F}$, such that

$$
\sup_{f \in \mathcal{F}} \min_{i \in [N(\delta, \mathcal{F}, \|\cdot\|_\infty)]} \|f - f_i\|_\infty \leq \delta.
$$

Lemma 19 ([66, Lemma 5]). The covering number of $\mathcal{F}(L, W, P, B)$ as defined in (22) satisfies

$$
\log(N(\delta, \mathcal{F}(L, W, P, B), \|\cdot\|_\infty)) \leq 2PL \log((B + 1)(W + 1) + S \log(\delta^{-1} L))
$$
Lemma 20 ([56, Lemma 4]). Let \( D_N = \{(X_1, Y_1), \ldots, (X_N, Y_N)\} \) denote \( N \) independent copies of a random pair \((X, Y) \in \mathbb{R}^{D \times 1}\), where \( Y_i = f(X_i) + \zeta_i \) for some \( f : [0, 1]^D \to [0, 1] \) and \( \zeta_i \sim \mathcal{N}(0, 1) \). Let \( \mathcal{F} \subset \{ f : [0, 1]^D \to [0, 1] \} \) and let \( \hat{f} \) be any estimator taking values in \( \mathcal{F} \). Define the expected discrepancy between the global ERM and \( \hat{f} \) as
\[
\Delta_N := \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \left( Y_i - \hat{f}(X_i) \right)^2 - \inf_{f \in \mathcal{F}} \frac{1}{N} \sum_{i=1}^{N} (Y_i - f(X_i))^2 \right]
\]
If \( \mathcal{N}(\omega, \mathcal{F}, \| \cdot \|_\infty) \geq 3 \), then we have for any \( \omega \in (0, 1) \)
\[
\mathbb{E} \left( \hat{f}(X) - f(X) \right)^2 \lesssim \inf_{h \in \mathcal{F}} \mathbb{E} (h(X) - f(X))^2 + \frac{\log(\mathcal{N}(\omega, \mathcal{F}, \| \cdot \|_\infty))}{N} + \omega + \Delta_N.
\]

Proof of Theorem 9 We first apply Lemma 20 with \( \omega = \varepsilon_N^2 \) to split the generalization error into approximation error and estimation error. To control the approximation error, Theorem 6 implies the existence of a ReLU net \( \Phi_N \in \mathcal{F}_N \) with
\[
\mathbb{E}(\Phi_N(X) - f(X))^2 \lesssim \| \Phi_N - f \|_{L_\infty(A)}^2 \leq C \varepsilon_N^2 = \log \frac{\log N}{\varepsilon_N^2} (N) N^{-\frac{2}{\varepsilon_N}}.
\]
Note here that the additional constraint \( \| \Phi_N \|_{L_\infty([0,1]^D)} \subset [0,1] \) can be achieved by thresholding the network in Theorem 6 without losing approximation accuracy since \( \text{Im}(f) \subset [0,1] \). For the estimation error we have to bound the covering number of \( \mathcal{F}_N \). Using Lemma 19 we obtain
\[
\log(\mathcal{N}(\omega, \mathcal{F}_N, \| \cdot \|_\infty)) \lesssim P_N L_N \log(B_N W_N) + P_N \log(\omega^{-1} L_N).
\]
Inserting the complexity bounds and the definition of \( \varepsilon_N \) the first term in (52) is bounded by
\[
P_N L_N \log(B_N W_N) \lesssim C D \log^3(D) \log^5(\varepsilon_N^{-1}) \varepsilon_N^{-d} \lesssim C D \log^3(D) \log^5(1 - \varepsilon_N^{-d}) (N) N \frac{d}{\varepsilon_N} \lesssim C D \log^3(D) \log^5 \frac{\log N}{\varepsilon_N^2} (N)(N) N \frac{d}{\varepsilon_N}.
\]
For the second term in (52) with \( \omega = \varepsilon_N^2 \), we obtain
\[
P_N \log(\omega^{-1} L_N) \leq P_N L_N \log(\omega^{-1}) \lesssim C D \log(D) \log^4(\varepsilon_N^{-1}) \varepsilon_N^{-d} \log(\varepsilon_N^2) \lesssim C D \log(D) \log^5(\varepsilon_N^{-1}) \varepsilon_N^{-d},
\]
which is by a \( \log(D) \)-factor smaller than (53). The result now follows from plugging these bounds into Lemma 20, i.e.
\[
\mathbb{E}(\Phi_N(X) - f(X))^2 \lesssim \inf_{h \in \mathcal{F}_N} \mathbb{E}(h(X) - f(X))^2 + \frac{\log(\mathcal{N}(\omega, \mathcal{F}_N, \| \cdot \|_\infty))}{N} + \omega + \Delta_N
\]
\[
\lesssim C \varepsilon_N^2 + D \log^3(D) \log^5 \frac{\log N}{\varepsilon_N^2} (N)(N) N \frac{d}{\varepsilon_N^2} + \Delta_N
\]
\[
\lesssim C \varepsilon_N^2 + D \log^3(D) \log^5 \frac{\log N}{\varepsilon_N^2} (N)(N) N \frac{d}{\varepsilon_N^2} + \Delta_N.
\]
Furthermore, for the empirical risk minimizer we have \( \Delta_N = 0 \).

Proof of Theorem 10 We provide only the crucial steps since the proof is similar to the proof of Theorem 9. Theorem 8 guarantees, for sufficiently large \( N > N_0 \) so that \( \varepsilon_N < \varepsilon_0 \) (\( \varepsilon_0 \) from Theorem 8), existence of a network \( \Phi_N \in \mathcal{F}_N \) with
\[
\mathbb{E}(\Phi_N(X) - f(X))^2 \lesssim \| \Phi_N - f \|_{L_\infty([0,1]^D)}^2 \leq \varepsilon_N^2 = \log \frac{\log N}{\varepsilon_N^2} (N) N^{-\frac{2}{\varepsilon_N}}.
\]
The comments about thresholding apply as in the proof of Theorem 9. For the estimation error we reuse (52). Inserting the complexity bounds on \( L_N \) and so forth with \( \varepsilon_N \) as in (26), the first term is bounded by
\[
P_N L_N \log(B_N W_N) \lesssim C D \log^3(D) \log^3(\varepsilon_N^{-1}) \varepsilon_N^{-(1+d)} \lesssim C D \log^3(D) \log^5 \frac{\log N}{\varepsilon_N^2} (N)(N) N \frac{d}{\varepsilon_N^2}.
\]
Furthermore, with \( \omega = \varepsilon 2N \), \( P_N \log(\omega^{-1}L_N) \) is of lower order in \( D \) and similar order in \( N \). Plugging bounds on approximation error and the covering number into Lemma 20 yields

\[
\mathbb{E} \left( \Phi_N(X) - f(X) \right)^2 \lesssim \inf_{h \in \mathcal{F}_N} \mathbb{E}(h(X) - f(X))^2 + \frac{\log(N(\omega, \mathcal{F}_N, \| \cdot \|_\infty))}{N} + \omega + \Delta_N
\]

\[
\lesssim C e^\frac{2N}{6} + D \log^3(D) \log \left( \frac{N}{\omega} \right) 2^{-\frac{1}{2}} N + \Delta_N
\]

\[
\leq \log \left( \frac{N}{\omega} \right) 2^{-\frac{1}{2}} N + \Delta_N.
\]

The result follows from \( \Delta_N = 0 \) for the empirical risk minimizer. \( \square \)

### 6.2 Proofs of Theorems 11 and 12

Similar results to Theorems 11 and 12 without enforcing Model 1 or Model 2 have been recently shown in [29]. As our model assumptions influence only the analysis of the approximation error, we can rely on the following oracle inequality, which has been derived for the model free case [29].

**Theorem 21** ([29, Theorem 6]). Let \((X, Y)\) follow model (28), \( f^* \) denote the Bayes classifier, and assume the margin condition (31) for \( \beta \in (0, \infty) \). Let \( f_N \) be the empirical risk minimizer over a function class \( \mathcal{F}_N \) with \( t(t, s) = (1 - ts)_+ \). Assume the function spaces \( \mathcal{F}_N \subset \{ f : [0, 1]^D \to [-1, 1] \} \) satisfy

1. there exist functions \( h_N \in \mathcal{F}_N \) and a sequence \( a_N \) tending to 0 as \( N \to \infty \) such that

\[
\mathbb{E}(\ell(h_N(X), Y) - \ell(f^*(X), Y)) \leq a_N,
\]

2. there exists a sequence \( \omega_N \) such that \( \log(N(\omega_N, \mathcal{F}_N, \| \cdot \|_\infty)) \leq N \omega_N^{\frac{2 + \beta}{2}} \).

Then, as soon as \( N \geq \beta (a_N \vee \omega_N) \), we have the bound

\[
\mathbb{E} \left( \ell(f_N(X), Y) - \ell(f^*(X), Y) \right) \lesssim a_N \vee \omega_N.
\]

**Proof.** This is a reformulated version of [29, Theorem 6], which uses the relation

\[
\text{log}(N(\delta_N, \mathcal{F}_N, \| \cdot \|_\infty)) \geq H_B(\delta_N, \mathcal{F}_N, \| \cdot \|_2),
\]

where \( H_B \) is the bracketing number of \( \mathcal{F}_N \) with respect to \( \| \cdot \|_2 \). \( \square \)

To apply Theorem 21 we need to check Conditions 1 and 2 for the ReLU space \( \mathcal{F}_N := \mathcal{F}(L_N, W_N, P_N, B_N) \). A bound for the Hinge-loss approximation error (55) follows from \( L_\infty \)-approximation errors of the class probability \( f(x) = \mathbb{P}(Y = 1|X = x) \) and the margin condition.

**Lemma 22.** Let \( A \subseteq \mathbb{R}^D \) and let \((X, Y)\) follow (28) with class probability \( f(x) = \mathbb{P}(Y = 1|X = x) \) and \( X \in A \) almost surely. Assume margin condition (31) and that there exists a ReLU net \( \Psi \) that approximates \( f \) in \( L_\infty(A) \) to accuracy \( \varepsilon > 0 \). Then there exists a ReLU net \( \Phi \) with \( L(\Phi) \leq L(\Psi) + 2, W(\Phi) \leq W(\Psi) + 2, P(\Phi) \leq 2P(\Psi) + 14 \) and \( B(\Phi) \leq B(\Psi) + \frac{14}{\varepsilon} \) such that

\[
\mathbb{E}(\ell(\Phi(X), Y) - \ell(f^*(X), Y)) \lesssim C_{\beta} \varepsilon^{\beta + 1}.
\]

**Proof.** Recall that \( f^*(x) = \text{sign}(2f(x) - 1) \) is the Bayes classifier and let \( \Theta(x) \) be a ReLU net that satisfies \( |\Theta(t) - \text{sign}(t)| \leq 1 - 2\varepsilon, 2\varepsilon \) as in Lemma 31. Set \( \Phi(x) := \Theta(2\Psi(x) - 1) \). For any \( x \) such that \( |2f(x) - 1| > 4\varepsilon \) we have \( |2\Psi(x) - 1| > 2\varepsilon \) and it follows that \( \Phi(x) = f^*(x) \) for \( x \) with \( |2f(x) - 1| > 4\varepsilon \). Using \( |\Phi(Y) \leq 1, \{ f^*(X) \} \} \leq 1, \) and \( \mathbb{E}[Y|X = x] = 2f(x) - 1 \) we have

\[
\mathbb{E}(\ell(\Phi(X), Y) - \ell(f^*(X), Y)) = \mathbb{E}[(1 - \Phi(Y) + (1 - f^*(X)_Y) + \mathbb{E}[f^*(X)Y - \Phi(Y)]
\]

\[
= \mathbb{E}[(f(X) - \Phi(Y))|X = x]
\]

\[
= \mathbb{E}[(f(X) - \Phi(Y))(2f(x) - 1)].
\]

Combining this with \( \Phi(x) = f^*(x) \) whenever \( |2f(x) - 1| > 4\varepsilon \) and margin condition (31), it follows that

\[
\mathbb{E}(\ell(\Phi(X), Y) - \ell(f^*(X), Y)) \leq 2\mathbb{E}[|2f(x) - 1| |2f(x) - 1| \leq 4\varepsilon] \mathbb{P}(|2f(x) - 1| \leq 4\varepsilon)
\]

\[
\leq 8\varepsilon \mathbb{P}(|2f(x) - 1| \leq 4\varepsilon) \leq C_{\beta} \varepsilon^{\beta + 1}.
\]

The dimensions of the network \( \Phi \) follow from the rules of concatenation in Lemma 23. \( \square \)
Proof of Theorem 11  The proof is an application of Theorem 21 with approximation error $a_N$ bounded by Theorem 6 and Lemma 22, and complexity of $F_N$ bounded by Lemma 19. For the approximation error, using Theorem 6 and Lemma 22, we can guarantee the existence of a network $\Theta_N$ in $F_N$ satisfying
\[
E(\ell(\Theta_N(X), Y) - \ell(f^*(X), Y)) \leq C_N \varepsilon_N^{\alpha N(\beta+1)} = \log \frac{\alpha N(\beta+1)}{1+N^{1+\beta}} (N) N^{-\frac{\alpha N(\beta+1)}{1+N^{1+\beta}}} =: a_N,
\] (57)
which is smaller than the excess risk bound (33) by factors of $D$. To verify the second condition in Theorem 21, we let $\omega_N \gtrsim_C D \log^3(D) \varepsilon_N^{\alpha N(\beta+1)}$ and compute the complexity of the function space $F_N$ by Lemma 19. For the first term in Lemma 19 we have
\[
P_N L_N \log(B_N W_N) \lesssim_C D \log^3(D) \log^5(\varepsilon_N)^{-1} \varepsilon_N^{-d}.
\] (58)
Using the choice of $\omega_N$, the second term in Lemma 19 is bounded by
\[
P_N \log(\omega_N^{-1} L_N) \leq P_N L_N \log(\omega_N^{-1}) \lesssim_C D \log(D)^2 \log^4(\varepsilon_N^{-1}) \varepsilon_N^{-d} \log(\varepsilon_N^{-N(\beta+1)})
\lesssim_C D \log(D)^2 \log^5(\varepsilon_N^{-1}) \varepsilon_N^{-d},
\]
which is always smaller than (58) due to a smaller log($D$) power. Therefore
\[
\log(N(\omega_N, F_N, \|\cdot\|_\infty)) \lesssim_C D \log^3(D) \log^5(\varepsilon_N)^{-1} \varepsilon_N^{-d} \lesssim_C D \log^3(D) \log^5(\varepsilon_N)^{\alpha N(\beta+2)} (N) N^{-\frac{\alpha N(\beta+2)}{1+\alpha N(\beta+2)}}.
\]
The right hand side of Condition 2 in Theorem 21 reads
\[
N^\frac{\alpha N(\beta+2)}{1+\alpha N(\beta+2)} \gtrsim C ND \log^3(D) \varepsilon_N^{\alpha N(\beta+2)} \gtrsim_C D \log^3(D) \log^5(\varepsilon_N)^{\alpha N(\beta+2)} (N) N^{-\frac{\alpha N(\beta+2)}{1+\alpha N(\beta+2)}}.
\]
which matches the upper bound for $\log(N(\omega_N, F_N, \|\cdot\|_\infty))$, i.e. the second condition in Theorem 21 is satisfied. Lastly, by the definition we have $\omega_N \gtrsim_C D \log^3(D) a_N$, where $a_N$ is the approximation error (57), and furthermore $N \gtrsim a_N^{-1/\beta+2/\beta+1}$ for large enough $N \geq N_0$. Theorem 21 implies
\[
E \left( \ell(\Phi_N(X), Y) - \ell(f^*(X), Y) \right) \lesssim a_N \vee \omega_N = \omega_N \gtrsim_C D \log^3(D) \varepsilon_N^{\alpha N(\beta+1)}.
\]
Finally, the result follows by Zhang’s inequality [63, Theorem 2.31],
\[
\mathcal{R}(\Phi_N) - \mathcal{R}(f^*) \leq E \left( \ell(\Phi_N(X), Y) - \ell(f^*(X), Y) \right),
\]
for the Hinge loss $\ell(t, s) = (1 - ts)_+$.

Proof of Theorem 12  The proof is conceptually similar to the proof of Theorem 11. First, we use Theorem 8 and Lemma 22 bounding the approximation error, i.e. for guaranteeing the existence of $\Theta_N \in F_N$ satisfying
\[
E(\ell(\Theta_N(X), Y) - \ell(f^*(X), Y)) \lesssim \varepsilon_N^{\alpha N(\beta+1)} = \log \frac{\alpha N(\beta+1)}{1+N^{1+\beta}} (N) N^{-\frac{\alpha N(\beta+1)}{1+N^{1+\beta}}} =: a_N,
\] (59)
which is smaller than the claimed excess risk bound (35) by factors of $D$. To verify the second condition in Theorem 21, we let $\omega_N \gtrsim_C D \log^3(D) \varepsilon_N^{\alpha N(\beta+1)}$ and bound the complexity of $F_N$ using Lemma 19. As in the proof of Theorem 11, the choice of $\omega_N$ implies that the second term in Lemma 19 is always bounded by the first term, i.e.
\[
P_N \log(\omega_N^{-1} L_N) \lesssim P_N L_N \log(B_N L_N).
\]
Therefore, we have
\[
\log(N(\omega_N, F(L_N, W_N, P_N, B_N, 1), \|\cdot\|_\infty)) \lesssim D \log^3(D) \log^3(\varepsilon_N)^{-1} \varepsilon_N^{-1/\beta+2} \lesssim D \log^3(D) \log^3(\varepsilon_N)^{\alpha_N^{-1/\beta+2}} (N) N^{-\frac{\alpha N^{-1/\beta+2}}{1+\alpha N^{-1/\beta+2}}}.
\]
This also matches the right hand side in Condition 2 of Theorem 21 since
\[
N^\frac{1}{\beta+2} \gtrsim C ND \log^3(D) \varepsilon_N^{\alpha N^{-1/\beta+2}} \gtrsim_C D \log^3(D) \log^3(\varepsilon_N)^{\alpha N^{-1/\beta+2}} (N) N^{-\frac{\alpha N^{-1/\beta+2}}{1+\alpha N^{-1/\beta+2}}}.
\]
Lastly, we have $a_N \lesssim \omega_N$. Furthermore, $N \gtrsim a_N^{-1/\beta+2/\beta+1}$ as soon as $N$ is large enough, and thus Theorem 21 implies
\[
E \left( \ell(\Phi_N(X), Y) - \ell(f^*(X), Y) \right) \lesssim a_N \vee \omega_N = \omega_N \gtrsim_C D \log^3(D) \varepsilon_N^{\alpha N(\beta+1)}.
\]
The result follows by Zhang’s inequality [63, Theorem 2.31] as in the proof of Theorem 11.
7 Appendix

7.1 Proofs for Section 2.1

In this section we provide the proofs for some statements from differential geometry given in Section 2.1.

Proof of Lemma 1. We first note that \( \text{dist}(x;M) \leq \|x-v\|_2 = \|u\|_2 < q\tau_M(v) \leq \tau_M(v) \), which implies \( x \not\in \text{Med}(M) \), and thus there exists a unique projection \( \pi_M(x) \) according to the construction of \( \text{Med}(M) \). To show \( \pi_M(x) = v \), we consider a proof by contradiction. Let \( \pi_M(x) \neq v \) and denote

\[
l(v) := \sup_{t \geq 0} \left\{ \pi_M \left( v + t \frac{u}{\|u\|_2} \right) = v \right\}.
\]

We have \( l(v) > 0 \) since \( u \perp \text{Im}(A(v)) \) and \( \tau_M > 0 \) (see for instance [47, Section 4]) but also \( l(v) < q\tau_M(v) \) since \( \pi_M(x) \neq v \). By [17, 6. in Theorem 4.8] we thus have \( w := v + l(v) \frac{u}{\|u\|_2} \not\in \text{Int}(\text{Med}(M)^C) \) (recall \( \text{Med}(M)^C \) is the set of points in \( \mathbb{R}^D \) with unique projection), or in other words, for any \( \varepsilon > 0 \) and the corresponding Euclidean ball \( B_{\varepsilon}(w) \), there exists

\[
y \in B_{\varepsilon}(w) \cap (\text{Int}(\text{Med}(M)^C))^C = B_{\varepsilon}(w) \cap \text{cl}(\text{Med}(M)).
\]

Using the existence of such \( y \)'s for every \( \varepsilon > 0 \), we get

\[
\tau_M(v) = \text{dist}(v;\text{Med}(M)) \leq \|y-x\|_2 \leq \|y-w\|_2 + \|w-x\|_2 \leq \|y-w\|_2 < q\tau_M(v) + \varepsilon
\]
or \( (1-q)\tau_M(v) < \varepsilon \). Letting \( \varepsilon \to 0 \) and recalling \( \tau_M(v) \geq \tau_M > 0 \), \( q < 1 \), this is a false statement. Hence \( \pi_M(x) = v \).

\[\Box\]

Proof of Lemma 2. The statement is a slight modification of [1, Lemma B.1, ii)], respectively, [17, Theorem 4.8]. Using [1, Lemma B.1, i)], we first note that for any \( x \in M(q) \) and \( v \in M \) we have

\[
\langle x - \pi_M(x), \pi_M(x) - v \rangle \geq -\frac{\|\pi_M(x) - v\|^2}{2\tau_M(\pi_M(x))}, \tag{60}
\]

(the statement is trivially true if \( x \in M \) contrary to what the assumptions in [1, Lemma B.1, i]) suggest). Taking arbitrary \( x, x' \in M(q) \) we obtain by Cauchy-Schwartz inequality and (60)

\[
\|x - x'\|_2 \|\pi_M(x) - \pi_M(x')\|_2 \geq \langle x - x', \pi_M(x) - \pi_M(x') \rangle
\]
\[
= \langle x - \pi_M(x) + \pi_M(x) - \pi_M(x') + \pi_M(x') - x', \pi_M(x) - \pi_M(x') \rangle
\]
\[
\geq \|\pi_M(x) - \pi_M(x')\|^2 \left( 1 - \frac{1}{2} \frac{\|x - \pi_M(x')\|^2}{\tau_M(\pi_M(x'))} - \frac{1}{2} \frac{\|x' - \pi_M(x)\|^2}{\tau_M(\pi_M(x))} \right)
\]
\[
= \|\pi_M(x) - \pi_M(x')\|^2 (1 - q),
\]

where we used \( x, x' \in M(q) \) in the last inequality. The result follows from division by \( \|\pi_M(x) - \pi_M(x')\|_2 \) and \( (1-q) \).

\[\Box\]

7.2 Additional result from ReLU calculus

This section extends the ReLU calculus introduction of Section 2.2 and additionally presents some specific results that are used in proofs in Sections 5 and 6. We begin by recalling basic operations of concatenation and linear combination.

Lemma 23 (Concatenation [19, Lemma 2.5]). Let \( \Phi_1 : \mathbb{R}^{N_0} \to \mathbb{R}^{N_{L_1}} \) and \( \Phi_2 : \mathbb{R}^{N_{L_1}} \to \mathbb{R}^{N_{L_2}} \) be two ReLU nets. There exists a ReLU net \( \Psi : \mathbb{R}^{N_0} \to \mathbb{R}^{N_{L_2}} \) with \( \Psi(x) = \Phi_2(\Phi_1(x)) \) and \( L(\Psi) = L(\Phi_1) + L(\Phi_2) \), \( W(\Psi) = \max\{W(\Phi_1), W(\Phi_2), 2N_{L_1}\} \), \( P(\Psi) = 2(P(\Phi_1) + P(\Phi_2)) \), and \( B(\Psi) \leq B(\Phi_1) \lor B(\Phi_2) \) so that \( \Psi(x) = \Phi_2(\Phi_1(x)) \).

Lemma 24 (Linear combination [19, Lemma 2.7]). Let \( \{\Phi_i : i \in [N]\} \) be a set of ReLU networks with similar input dimension \( N_0 \). There exist ReLU networks \( \Psi_1 \) and \( \Psi_2 \) with \( L(\Psi_j) = \max_{i \in [N]} L(\Phi_i) \),
There exists \( \varepsilon \) a ReLU network \( \Phi \) realizing the maps \( B \). We use [56, Theorem 5] for the case of an arbitrary degree polynomials [11]. Then we can study the approximation of smooth functions by using local Taylor expansions, which has been presented in several different works in the literature [11,56,71].

We use [56, Theorem 5] for the case of an \( \alpha \)-Hölder smooth function \( g : [0,1] \to \mathbb{R} \) in Section 5.2.

**Theorem 25** ([56, Theorem 5]). Define the class of functions \( C_1^0(D,L) \) by

\[
C_1^0(D,L) := \left\{ f : D \subseteq \mathbb{R}^k \to \mathbb{R} \mid \sum_{|\alpha| < \alpha} \| \partial^\alpha f \|_\infty + \sum_{|\alpha| = \alpha} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{\|x - y\|^{\alpha - |\alpha|}} \leq L \right\}
\]

There exists \( \varepsilon_0 > 0 \) depending on \( \alpha, L, k \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and any \( f \in C_1^0([0,1]^k,L) \) there exists a ReLU network \( \Phi \) with \( L(\Phi) \lesssim_{\alpha,k} \log_\alpha(\varepsilon^{-1}) \), \( W(\Phi) \lesssim_{\alpha,k} \varepsilon^{-\frac{k}{2}} \), \( P(\Phi) \lesssim_{\alpha,k} \log_\alpha(\varepsilon^{-1}) \), \( B(\Phi) \leq 1 \) and

\[
\sup_{x \in [0,1]^k} |f(x) - \Phi(x)| \lesssim_{1V,L,\alpha,k} \varepsilon.
\]

**Proof.** We slightly reformulate the result [56, Theorem 5]. Using the notation therein, we choose \( N = \varepsilon^{-\frac{k}{2}} \) and \( m = \lceil -\frac{k}{\alpha} \log_\alpha(\varepsilon) \rceil \) that satisfy \( m \geq 1 \) and \( N \geq (\alpha + 1)^k \lor (L + 1)^k \) for sufficiently small \( \varepsilon < \varepsilon_0 \). Thus we can apply the Theorem to obtain

\[
\sup_{x \in [0,1]^k} |f(x) - \Phi(x)| \lesssim_{1V,L,\alpha,k} N^{-m} + N^{-\frac{k}{2}} \leq \varepsilon^{-\frac{k}{2}} 2^{\frac{\alpha + k}{\alpha} \log_\alpha(\varepsilon)} + \varepsilon \leq \varepsilon^{-\frac{k}{2}} \varepsilon^{\frac{\alpha + k}{\alpha}} + \varepsilon \lesssim \varepsilon.
\]

The dimensions of the network are also bounded according to [56, Theorem 5]. Namely, we have \( L(\Phi) \lesssim_{\alpha,k} m \), \( W(\Phi) \lesssim_{\alpha,k} N \leq \varepsilon^{-\frac{k}{2}} \), \( P(\Phi) \lesssim_{\alpha,k} Nm \lesssim_{\alpha,k} \varepsilon^{-\frac{k}{2}} \log_\alpha(\varepsilon^{-1}) \) and \( B(\Phi) \leq 1 \).

The remainder of this section contains approximation results about some special functions, which are required for the construction of ReLU nets in Theorems 6 - 8:

- approximation of the \( \ell_2 \)-norm in Lemma 26;
- vector-valued multiplication in Lemma 27;
- division \( t \mapsto t^{-1} \) in Lemma 28;
- \( \ell_1 \)-norm and \( \ell_1 \)-normalization in Lemma 29;
- the minimum operator \( (x_1, \ldots, x_K) \mapsto \min\{x_1, \ldots, x_K\} \) in Lemma 30;
- the sign-function in Lemma 31.

**Lemma 26** (Squared \( \ell_2 \)-norm). Let \( \varepsilon, R > 0 \). There exists a ReLU network \( \Phi : \mathbb{R}^D \to \mathbb{R} \) with \( L(\Phi) \lesssim \log(RDe^{-1}) \), \( W(\Phi) \leq 4D \), \( P(\Phi) \lesssim D \log(RDe^{-1}) \) and \( B(\Phi) \leq 4 (R^2 \lor R^{-1}) \) such that

\[
\sup_{x \in B_R(0)} \left\| x - \Phi(x) \right\|_2^2 \leq \varepsilon^2.
\]

**Proof.** We first note that the univariate function \( t \mapsto t^{-1} |t| \) can be realized by a ReLU network with weight matrices \( A_1 = [1, -1]^\top \), \( A_2 = [R^{-1}, R^{-1}] \) since

\[
A_2 (A_1 t)_+ = A_2 [(t)_+, (-t)_+]^\top = R^{-1} ((t)_+ + (-t)_+) = R^{-1} |t|.
\]

We denote this network by \( \Phi \). Furthermore, following [19, Proposition 3.1], there exists a ReLU network \( \Gamma \) that approximates the univariate square \( t \mapsto t^2 \) up to arbitrary accuracy \( \varepsilon R^{-2} D^{-1} > 0 \) on \([0,1]\). Now
define $\Theta = \Gamma \circ \Psi$ (ReLU concatenation) and set $\Phi(x) := \sum_{i=1}^D R^2 \Theta(x_i)$ (ReLU linear combination). For arbitrary $x \in B_R(0)$ we have

$$\|x\|_2^2 - \Phi(x) \leq \sum_{i=1}^D |x_i^2 - R^2 \Theta(x_i)| = R^2 \sum_{i=1}^D \left(\frac{x_i}{R}\right)^2 - \Gamma \left(\frac{x_i}{R}\right) \leq \varepsilon.$$ 

The dimensions of network $\Phi$ are bounded according to the rules in Lemma 23 and 24. Specifically, using $L(\Gamma) \leq \log(RD\varepsilon^{-1}) \leq \log(RD\varepsilon^{-1})$, $W(\Gamma) = 4$, $P(\Gamma) \leq \log(R^2D\varepsilon^{-1}) \leq \log(RD\varepsilon^{-1})$ and $B(\Gamma) \leq 4$ as in [19, Proposition 3.1], we obtain

$$L(\Phi) = L(\Theta) + L(\Psi) \leq \log(RD\varepsilon^{-1}) + 2 \leq \log(RD\varepsilon^{-1}),$$

$$W(\Phi) = D(2 \lor W(\Theta)) = D(2 \lor \max\{W(\Gamma), W(\Psi)\}) \leq 4D,$$

$$P(\Phi) = D(P(\Theta) + W(\Theta) + 1) \leq D(P(\Gamma) + P(\Psi)) \leq DP(\Gamma) \leq D \log(RD\varepsilon^{-1}),$$

$$B(\Phi) \leq \max\{1, R^2, B(\Theta)\} \leq \max\{1, R^2, B(\Gamma), B(\Psi)\} \leq \max\{R^2, R^{-1}, 4\} \leq 4 (R^2 \lor R^{-1}).$$

Lemma 27 (Multiplication). Let $\varepsilon \in (0, \frac{1}{4})$ and $a > 0$. There exists a ReLU network $\Phi : \mathbb{R}^D \to \mathbb{R}$ with $\|\Phi\| \leq \log(a^2\varepsilon^{-1})$, $W(\Phi) \leq 12D$, $P(\Phi) \leq D \log(a^2\varepsilon^{-1})$ and $B(\Phi) \leq 4 \lor 2[a^2]$ with

$$\sup_{\|x\|_2 \leq a, |y| \leq a} \|\Phi(x, y) - xy\|_\infty \leq \varepsilon.$$ 

Proof. By [19, Proposition 3.2] there exists a ReLU net $\Psi : \mathbb{R}^2 \to \mathbb{R}$ approximating $xy$ up to arbitrary accuracy $\varepsilon$ on $[-a, a]^2$. We set $\Phi(x, y) = (\Psi(x_1, y), \ldots, \Psi(x_D, y))$, which can be realized by a ReLU net according to Lemma 24. Furthermore, using dimension bounds from [19, Proposition 3.2], we get $L(\Phi) = L(\Psi) \leq \log(a^2\varepsilon^{-1})$, $W(\Phi) \leq DW(\Psi) \leq 12D$, $P(\Phi) = D(P(\Psi) + W(\Psi) + 1) \leq D \log(a^2\varepsilon^{-1})$ and $B(\Phi) \leq 1 \lor B(\Psi) \leq 4 \lor [a^2]$.}

Lemma 28 (Division). Let $\varepsilon \in (0, 1)$ and $a \in \mathbb{R}_{\geq 1}$. There exists a network $\Phi : \mathbb{R} \to \mathbb{R}$ with $\|\Phi\| \leq a^2\log^2(\frac{a}{\varepsilon})$, $W(\Phi) \leq 16$, $P(\Phi) \leq a^2\log^2(\frac{a}{\varepsilon})$ and $B(\Phi) \leq 8 \lor \frac{1}{\varepsilon}$, so that

$$\sup_{t \in [\frac{a}{\varepsilon}, a]} \left|\Phi(t) - \frac{1}{t}\right| \leq \varepsilon.$$ 

Proof. We combine the proof of [67, Lemma 3.6] and [19, Proposition 3.3]. Set $c = \frac{1}{4}$ and $r = [a^2 \ln(\frac{2a}{\varepsilon})]$. Following [67], we can write $t^{-1} = c \sum_{i=1}^\infty (1 - ct)^i$ and, after cutting the series at $i = r$, we obtain the approximation error

$$\left|\frac{1}{t} - c \sum_{i=1}^r (1 - ct)^i\right| = \left|c \sum_{i=r+1}^\infty (1 - ct)^i\right| \leq \frac{\varepsilon}{2}.$$ 

Now let $p(t) = c \sum_{i=1}^r z_i$ so that $p(1 - ct) = c \sum_{i=1}^r (1 - ct)^i$. Notice that $0 \leq 1 - ct \leq 1$ since $t \in [a^{-1}, a]$ and $c = a^{-1}$, so using [67, Proposition 3.3], we can approximate $p$ over $[0, 1]$ to accuracy $\frac{\varepsilon}{2}$ with a network $\Psi$ adhering to the dimension bounds $L(\Psi) \leq r(\log([c]) + \log(\frac{r}{c}))$, $W(\Psi) \leq 16$, $P(\Psi) \leq r(\log([c]) + \log(\frac{r}{c}))$, and $B(\Psi) \leq c \lor 8$. Therefore, we get for any $t \in [a^{-1}, a]$

$$\left|\frac{1}{t} - \Psi(1 - ct)\right| \leq \left|\frac{1}{t} - p(1 - ct)\right| + |p(1 - ct) - \Psi(1 - ct)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$ 

Lastly, we can further simplify the bounds on $L(\Psi)$ and $P(\Psi)$ by using the definition of $r$ and recognizing $\log(r) \leq \log(a) + \log(\ln(a) + \ln(\varepsilon^{-1})) \leq \log(a) + \log(\varepsilon^{-1})$ to get

$$\log([c]) + \log(\frac{r}{c}) = \log([c]) + \log(r) + \log(\frac{1}{\varepsilon}) \leq \log(a) + \log(\frac{1}{\varepsilon})$$

$$= r \log(\frac{a}{\varepsilon}) \leq a^2 \log^2(\frac{a}{\varepsilon}).$$

\qed
Lemma 29 ($\ell_1$-normalization). Let $a \geq 1$, $\varepsilon \in (0, \frac{1}{2})$. There exists a ReLU network $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ with $L(\Phi) \lesssim a^2 \log^2 \left( \frac{a}{\varepsilon} \right)$, $W(\Phi) \lesssim D$, $P(\Phi) \lesssim a^2 D \log^2 \left( \frac{a}{\varepsilon} \right)$, and $B(\Phi) \lesssim a^2$ such that

$$\sup_{\frac{1}{2} \leq \|x\|_1 \leq a} \left\| \Phi(x) - \frac{x}{\|x\|_1} \right\|_\infty \leq \varepsilon.$$

Proof. We combine four networks: a network realizing the identity, a network realizing the 1-norm, a network realizing approximate division based on Lemma 28, and lastly, a network realizing approximate multiplication based on Lemma 27. The identity map $\text{Id}_D : \mathbb{R}^D \to \mathbb{R}^D$ can be realized by a two-layer net $\Psi$ with zero biases and weight matrices

$$A_{h,1} = \begin{pmatrix} \text{Id}_D & -\text{Id}_D \end{pmatrix}, \quad A_{h,2} := (\text{Id}_D, -\text{Id}_D),$$

whereas $x \mapsto \|x\|_1$ can be realized by a two-layer ReLU net $\Theta$ with zero biases and weight matrices

$$A_{g,1} = \begin{pmatrix} \text{Id}_D & -\text{Id}_D \end{pmatrix}, \quad A_{g,2} := (1, \ldots, 1) \in \mathbb{R}^{1 \times 2D}.$$

Furthermore, let $\Gamma$ denote a ReLU net approximating univariate division on $[a^{-1}, a]$ up to accuracy $\frac{\varepsilon}{2a}$, whose existence has been shown in Lemma 28, and let $\Omega$ denote a ReLU net approximating $(x, y) \mapsto y/x$ on $[-2a, 2a]^{D+1}$ to accuracy $\frac{\varepsilon}{2}$. Then we set $\Phi(x) = \Omega((\Psi(x), \Gamma(\Theta(x))))$, which satisfies

$$\left\| \Phi(x) - \frac{x}{\|x\|_1} \right\|_\infty \leq \|\Omega(x, \Gamma(\|x\|_1)) - x\Gamma(\|x\|_1)\|_\infty + \left\| x\Gamma(\|x\|_1) - \frac{x}{\|x\|_1} \right\|_\infty \leq \varepsilon,$$

where we used $\|x\|_1 \leq a$ in the last inequality. To compute the dimensions of $\Phi$, first note that the concatenation rules in Lemma 23 imply

$$L(\Gamma \circ \Theta) = L(\Theta) + L(\Gamma) \lesssim 2 + a^2 \log^2 \left( \frac{a}{\varepsilon} \right) \lesssim a^2 \log^2 \left( \frac{a}{\varepsilon} \right),$$

$$W(\Gamma \circ \Theta) = \max\{W(\Theta), W(\Gamma), 2\} \leq 2D \lor 16,$$

$$P(\Gamma \circ \Theta) = 2P(\Gamma) + 2P(\Theta) \lesssim a^2 \log^2 \left( \frac{a}{\varepsilon} \right) + D,$$

$$B(\Gamma \circ \Theta) = B(\Gamma) \lor B(\Theta) \leq 8 \lor a^{-1}.$$

Then, using linear combination and concatenation rules of ReLU nets in Lemma 23, 24 we obtain

$$L(\Phi) = L(\Omega) + L((\Psi(x), \Gamma \circ \Theta)) \lesssim \log \left( \frac{a^2}{\varepsilon} \right) + 2 + a^2 \log^2 \left( \frac{a}{\varepsilon} \right) \lesssim a^2 \log^2 \left( \frac{a}{\varepsilon} \right),$$

$$W(\Phi) = W(\Omega) \lor W((\Psi(x), \Gamma \circ \Theta)) \lesssim 12D \lor (4 + W(\Psi) + W(\Gamma \circ \Theta)) \lesssim D,$$

$$P(\Phi) \lesssim P(\Omega) + P((\Psi(x), \Gamma \circ \Theta)) \lesssim P(\Omega) + P(\Psi) + P(\Gamma \circ \Theta) + L(\Gamma \circ \Theta) + W(\Psi) + W(\Gamma \circ \Theta) \lesssim D \log(a^2 \varepsilon^{-1}) + D + a^2 \log^2 \left( \frac{a}{\varepsilon} \right) \lesssim a^2 D \log^2 \left( \frac{a}{\varepsilon} \right),$$

$$B(\Phi) = B(\Omega) \lor B((\Psi, \Gamma \circ \Theta)) \lesssim \max\{a^2, B(\Psi), B(\Gamma \circ \Theta)\} \lesssim a^2.$$

Lemma 30. Let $K \geq 2$. There exists a ReLU network $\Phi_K : \mathbb{R}^K \to \mathbb{R}$ with $L(\Phi_K) \leq 2[\log_2(K)]$, $W(\Phi_K) \leq 3[K/2]$, $P(\Phi_K) \leq 11K[\log_2(K)]$ and $B(\Phi_K) \leq 1$ such that $\Phi_K(x) = \min_{i \in [K]} x_i$.

Proof. Without loss of generality we assume $K$ is even as we can otherwise just replace $x$ by repeating one of its arguments without changing the bounds on the dimension of the network. We proof the statement by induction. For $K = 2$ we set

$$A_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = (1, -1, -1)$$

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such that $\Phi_2(x) = (x_1)_+ - (-x_1)_+ - (x_1 - x_2)_+ = x_1 - (x_1 - x_2)_+ = x_1 \wedge x_2$. Clearly, $L(\Phi_2) = 2$, $W(\Phi_2) = 3$, $P(\Phi_2) = 7$, and $B(\Phi_2) = 1$ and the induction start is proven. Let us now assume the statement holds up to $K - 1$ and we prove it for $K$. Set $\Phi_K = \Phi_2(\Phi_2(x_1, x_2), \ldots, \Phi_2(x_{K-1}, x_K))$, which exactly realizes $\min_{x \in [K]} x_i$. To compute the dimensions of the network we use the rules of concatenation and parallelization as in Lemma 23 and 24. This gives

$$L(\Phi_K) = L \left( \Phi_2 \right) + L(\Phi_2) = 2 \left\lceil \log_2 \left( \frac{K}{2} \right) \right\rceil + 2 = 2 \left\lceil \log_2 (K) - 1 \right\rceil + 2 = 2 \left\lceil \log_2 (K) \right\rceil,$$

$$W(\Phi_K) = \max \left\{ W(\Phi_2), W(\Phi_2), \ldots, W(\Phi_2) \right\} \leq \frac{K}{2} W(\Phi_2) \leq 3 \frac{K}{2},$$

$$P(\Phi_K) = 2P(\Phi_2) + 2P(\Phi_2, \ldots, \Phi_2) \leq 11K \left\lceil \log_2 \left( \frac{K}{2} \right) \right\rceil + K(P(\Phi_2) + W(\Phi_2) + 1) \leq 11K \left\lceil \log_2 (K) \right\rceil - 11K + 11K,$$

and $B(\Phi_K) \leq 1$. 

**Lemma 31 (Sign-function).** Let $\varepsilon > 0$. There exists a ReLu net $\Phi$ with $L(\Phi) = 2$, $W(\Phi) = 2$, $P(\Phi) = 7$ and $B(\Phi) = \varepsilon^{-1}$ such that $|\Phi(x) - \text{sign}(x)| \leq (1 - \frac{\varepsilon}{2}) \mathbb{1}_{[-\varepsilon, \varepsilon]}(x)$.

**Proof.** Define the ReLU net

$$\Phi(x) := \left( \frac{x}{\varepsilon} + 1 \right)_+ - \left( \frac{x}{\varepsilon} - 1 \right)_+ - 1,$$

constructed by the weight matrices and biases

$$A_1 = \left( \begin{array}{c} \varepsilon^{-1} \\ -1 \end{array} \right), \quad b_1 = \left( \begin{array}{c} 1 \\ -1 \end{array} \right), \quad A_2 = (1, 1), \quad b_2 = (-1).$$

It is straight-forward to verify that $x < -\varepsilon$ implies $\Phi(x) = -1$ while $x > \varepsilon$ implies $\Phi(x) = 1$. Furthermore, for $x \in [-\varepsilon, \varepsilon]$ we have $\Phi(x) = \frac{x}{\varepsilon}$ and thus

$$|\Phi(x) - \text{sign}(x)| = \left| \frac{x}{\varepsilon} - \text{sign}(x) \right| = 1 - \frac{x}{\varepsilon}.$$

\[\square\]

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