Development of analytical solution for thermo-mechanical stresses of multi-layered hollow cylinder for the application of underground hydrogen storage

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Abstract. Hydrogen fuel has been playing an important role nowadays, it is considered as clean fuel and the hydrogen element is highly abundant. Storage of hydrogen in geological formations has been widely applied particularly for large-scale energy storage that required to meet high energy demand. There are different types of underground hydrogen storage methods, one of the most popular and reliable method is to store the hydrogen in depleted oil and gas wells. The structure of the depleted oil and gas well usually constructed by several layers of different materials such as formations, cement and casing. To ensure the well integrity is maintained, a reliable analytical solution to estimate the development of stress under pressure and thermal loading is required. In this paper, a more effective method to estimate the thermo-mechanical stresses of multi-layered hollow cylinder well is developed. The analytical solution is established by considering the well as generalized plane strain problem. The derivation of the equations and the algorithm to obtain results for four layered gas well were demonstrated. In general, the results obtained from the proposed analytical solution are well in agreement with the numerical results. The variations of results obtained from analytical solution and numerical solution are only less than 0.1%

1. Introduction
The growing concerns over climate change has driven the electricity power sector to absorb electricity generated from various renewable sources and other alternative like hydrogen (H2) [1]. Energy generated from renewable energy sources is generally random and inconsistent, while hydrogen can be derived from renewable sources and store in a form of energy to solve the fluctuation problem in energy production [2, 3]. As hydrogen is known as an efficient energy carrier, it possesses a potential of replacing up to 60% of the nature gas used for non-industrial activities [1]. To meet high demand of energy, large scale of hydrogen storage in the geological reservoirs has been widely used [3]. There are different types of underground hydrogen storage structures depending on the geological characteristic. The structure of the depleted oil and gas well usually constructed by several layers of different materials such as formations, cement and casing [4]. As the well is often subjected to extreme conditions and various loadings, the structural integrity plays an important role for its successful operation. Yeo et al. have reported a recursive algorithm that gives an exact solution on stresses and displacements in a thermo-mechanically loaded multi-layered hollow cylinder wall under plane strain assumption [8]. Analysis of a gas well wall section normally involves axial strain due to structural weight above the wall
section [7]. Therefore, this paper proposes an effective recursive algorithm that gives an exact solution on stresses in a thermo-mechanically loaded multi-layered hollow cylindrical well wall under generalized plane strain assumption. The proposed algorithm is verified by comparison with the results generated by numerical tool.

2. Heat conduction equation and thermo-mechanical stresses in multi-layered hollow cylindrical well

2.1. Priori assumptions
The priori assumptions used in this study are: (1) the heat transfer is in a steady state condition; (2) the temperature and pressure subjected to the well are constant; (3) the stress and displacement equations are derived based on small displacement and generalized plane strain condition; (4) all the layers for the multi-layered hollow cylinder wall are perfectly bonded together and without sliding.

2.2. Geometry and material properties
A multi-layered hollow cylindrical well that has n-layers is shown in Figure 1. It is subjected to pressures and temperatures on the outer surface ($r_n$) and inner surface ($r_0$). Geometrically, $r_i$ represents the outer radius of $i$-th layer. The notations for the material properties of $i$-th layer are depicted in Figure 1, i.e. the Poisson’s ratio $\nu_i$, elastic modulus $E_i$, the thermal conductivity $k_i$, and the coefficient of thermal expansion $\alpha_i$.

![Figure 1. Multi-layered hollow cylinder well that is subjected to temperature and pressure on the inner and outer surface.](image)

2.3. Boundary conditions
The boundary conditions on the inner and outer surfaces of the hollow cylinder, are prescribed as,

$$T_i(r_0) = T_0 = T_{int}, \quad \sigma_{rr,1}(r_0) = -q_0 = -T_{int}, \quad T_n(r_n) = T_n = T_{ext}, \quad \sigma_{rr,n}(r_n) = -q_n = -P_{ext}.$$  

While the interface can be expressed as $T_i(r_i) = T_{i+1}(r_i)$, $q_i(r_i) = q_{i+1}(r_i)$, $\sigma_{rr,i}(r_i) = \sigma_{rr,i+1}(r_i)$, $u_{r,i}(r_i) = u_{r,i+1}(r_i)$. 

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2.4. Heat conduction equations

Based on the Fourier’s law of heat conduction [5], the heat flow equation for multi-layered hollow cylinder can be written as,

\[ Q = q_i A_i = \frac{2\pi k_i L (\bar{T}_{i-1} - \bar{T}_i)}{ln(r_i/r_{i-1})} \]  

(1)

when \( \bar{T}_0 \) and \( \bar{T}_n \) are known, \( \bar{T}_i \) can be found by using following equation,

\[ \bar{T}_i = \bar{T}_{i-1} - \frac{Q ln(r_i/r_{i-1})}{2\pi k_i L} \]  

(2)

where \( i = 1, 2, 3 \ldots n \) and \( Q \) can be determined when \( \bar{T}_0 \) and \( \bar{T}_n \) are known.

2.5. Stress and displacement equations for hollow cylinder

According to Lame’s solution [6], the stress and displacement equations can be written as

\[ u_r(r) = \beta_i C_i r + \frac{\lambda_i}{r} (D_i + I_i) \]; \[ u_\theta(r) = 0; \] \[ u_z(r) = C_{i} z \frac{L}{E} = \epsilon_{zz} Z \]  

(3)

\[ \sigma_{rr}(r) = C_i + \frac{D_i + I_i}{r^2} + \varphi_i; \] \[ \sigma_{\theta\theta}(r) = C_i - \frac{D_i + I_i}{r^2} + \varphi_i - \frac{E \alpha_i \theta_i}{(1-\nu_i)} \]  

(4)

\[ \sigma_{zz}(r) = 2\nu_i C_i + \frac{(1-\nu_i) \epsilon_{zz}}{\beta_i} - \frac{E \alpha_i \theta_i}{(1-\nu_i)} \]

where \( I_i = I_i(r) = \frac{E \alpha_i}{1-\nu_i} \int_{r_{i-1}}^{r_i} \theta_i r dr \), \( \beta_i = \frac{(1+\nu_i)(1-2\nu_i)}{E_i}, \) \( \lambda_i = \frac{(1+\nu_i)}{E_i}, \) \( \varphi_i = \nu_i \epsilon_{zz}/\beta_i, \) \( \theta_i = T_i - T_r \) with \( T_r \) being the reference temperature and \( C \) and \( D \) are the integration constant.

For the calculation of displacements and radial stresses across the hollow cylindrical wall, the constants \( C_i \) and \( D_i \) may be determined by boundary conditions for stress and displacement. It yields

\[ \beta_i C_i r_i \lambda_i \left( \frac{D_i + I_i}{r_i} \right) = \beta_{i+1} C_{i+1} r_i \lambda_i \left( \frac{D_i + I_i}{r_i} \right) \]

(6)

where \( I_i^0 = I_i^0(r_i) = -\frac{E \alpha_i}{1-\nu_i} \int_{r_i}^{r_{i-1}} \theta_i r dr = 0 \); \( I_i = I_i(r_i) = -\frac{E \alpha_i}{1-\nu_i} \int_{r_i}^{r_{i-1}} \theta_i r dr \)

Solving (5) and (6) results in the constants \( C_{i+1} \) and \( D_{i+1} \). Also, for the radial stresses of two adjacent layers written as contact pressure,

\[ D_i = \frac{\gamma_i r_i^2 \left( p_{i-1} g_i - p_{i+1} g_i + \varphi_i (s_{i+1} + \beta_i + \gamma_i (\lambda_i + \beta_i)) - \varphi_i (s_{i+1} + \beta_i + \gamma_i (\lambda_i + \beta_i)) \right)}{s_{i-1} g_i} \]

\[ \frac{l_s I_i + l_2 I_i + \gamma_i (\lambda_i + \beta_i)}{s_{i-1} g_i} \]

\[ C_i = \frac{-p_{i-1} s_i + p_{i+1} (\lambda_i + \beta_i) - \varphi_i (s_i + \beta_i + \gamma_i (\lambda_i + \beta_i)) + \varphi_i (s_i + \beta_i + \gamma_i (\lambda_i + \beta_i)) \gamma_i (\lambda_i + \beta_i)}{s_{i-1} g_i} \]

\[ \frac{l_s I_i + l_2 I_i + \gamma_i (\lambda_i + \beta_i)}{r_i^2 (s_{i-1} g_i)} \]

(7)
where \( S_i = y_i y_{i+1}(\lambda_i - \beta_{i+1}) + (\lambda_{i+1} - \lambda_i)y_i \); \( G_i = \lambda_{i+1} - \beta_i + (\beta_i - \beta_{i+1})y_{i+1} \);

\[
y_{i+1} = \frac{r_i^2}{r_{i+1}^2}
\]

The radial stress on outer surface of \( i \)-layer has the contact pressure as following,

\[
\sigma_{rr,i}(r_i) = -p_i \Rightarrow C_i + \frac{d_i+1}{r_i^2} + \varphi_i = -p_i
\]

Next, substituting (7) into (9) gives the relationship between the adjacent contact pressures as

\[
p_{i+1} = \frac{p_{i-1}(S_i-y_{i}G_i)-p_i(S_i-G_i)}{(1-\gamma_i)(\lambda_{i+1}-\beta_{i+1})} - \frac{I_{i+1} y_{i+1}(\lambda_{i+1}-\beta_{i+1})(1-\gamma_i)+I_i(S_i/G_i)}{r_i^2(1-\gamma_i)(S_i-G_i)} - \frac{(1-\gamma_{i+1})(\nu_i-v_{i+1})\varepsilon_{zz}}{p_0(\lambda_{i+1}-\beta_{i+1})}
\]

Also, substituting (10) into (7) results in the constants \( D_i \) and \( C_i \) as

\[
D_i = \frac{y_i r_i^2}{1-\gamma_i} \left( p_i - p_{i-1} + \frac{I_i}{r_i^2} \right)
\]

\[
C_i = \frac{1}{1-\gamma_i} \left( y_i p_{i-1} - p_i - \frac{l_i}{r_i^2} \right) - \varphi_i
\]

By proposing two recurrence relations, \( D_i \) and \( C_i \) as well as \( p_i \) can be written in terms of \( p_0 \) and \( p_n \),

\[
c_{i+1} = \frac{c_{i-1}(S_i-y_{i}G_i)-c_i(S_i-G_i)}{(1-\gamma_i)(\lambda_{i+1}-\beta_{i+1})}
\]

\[
d_{i+1} = \frac{d_{i-1}(S_i-y_{i}G_i)-d_i(S_i-G_i)}{(1-\gamma_i)(\lambda_{i+1}-\beta_{i+1})} - \frac{I_{i+1} y_{i+1}(\lambda_{i+1}-\beta_{i+1})(1-\gamma_i)+I_i(S_i/G_i)}{p_0 r_i^2(1-\gamma_i)(S_i-G_i)} - \frac{(1-\gamma_{i+1})(\nu_i-v_{i+1})\varepsilon_{zz}}{p_0(\lambda_{i+1}-\beta_{i+1})}
\]

where \( i = 1, 2, \ldots, (n-1) \).

Introducing \( p_i \) in terms of the recurrence coefficients results in

\[
p_i = c_i p_1 + d_i p_0
\]

For given initial values of \( c_0 = 0, c_1 = 1, d_0 = 1, d_1 = 0 \).

Solving \( p_i \) in terms of the recurrence relations and the boundary values gives

\[
p_i = \frac{c_i}{c_n} p_n + \left( d_i - \frac{d_n}{c_n} c_i \right) p_0
\]

Considering for the case of a given generalized plane strain condition where \( \varepsilon_{zz} \) is constant, the corresponded axial force can be found by,

\[
\sum_{i=1}^{n} \sigma_{zz,i} \pi (r_i^2 - r_{i-1}^2) = F_{zz}
\]

\[
F_{zz} = \sum_{i=1}^{n} r_i^2 (1-\gamma_i) \left\{ \frac{2v_i}{(1-\gamma_i)} \left( y_i p_{i-1}^0 - p_i^0 - \frac{l_i}{r_i^2} \right) - \frac{E_i a_i \theta_i}{(1-\nu_i)} \right\} + \left[ \frac{2v_i}{(1-\gamma_i)} (y_i \omega_{i-1} - \omega_i) + E_i \right] \varepsilon_{zz}
\]

3. Results and discussion
To validate the proposed analytical solutions, a depleted oil and gas well reported in Hartman et al. [7] is used in present analysis. The well consists of four-layered section which formed by alternating layer of steel casing and cement. The material properties are summarized in Table 1 and the gas well geometry is presented in Table 2. For results comparison purpose, the gas well is modelled in finite element analysis software ANSYS. A two-dimensional axisymmetric cylindrical model has been developed. Element of the model is set to be quadrilateral and PLANE13 type. All layers are glued together to ensure the mesh connectivity and a converged mesh size of 0.0001 is applied. The thermo-mechanical coupled problem is defined based on loading stated in Table 2. Under generalized plane strain condition, axial displacement of 0.005mm is applied to the model.

**Table 1. Material properties [7]**

| Material | Young’s Modulus, E (MPa) | Poisson’s Ratio, ν | Heat Conductivity, k (Wm⁻¹K⁻¹) | Heat Expansion Coefficient, α (K⁻¹) |
|----------|--------------------------|------------------|---------------------------------|-------------------------------------|
| Steel    | 2 x 10⁵                   | 0.2              | 45                              | 1.1 x 10⁻⁵                          |
| Cement   | 1.4 x 10⁴                 | 0.35             | 1.5                             | 1.3 x 10⁻⁵                          |

**Table 2. Geometry and loading subjected to the well [7]**

| Layer | Material | Inner Radius (m) | Outer Radius (m) | Internal Pressure (MPa) | External Pressure (MPa) | Internal Temperature (°C) | External Temperature (°C) |
|-------|----------|------------------|------------------|-------------------------|-------------------------|--------------------------|--------------------------|
| 1     | Steel    | 0.113            | 0.125            | 12                      | 10                      | 47.3                     | 45                       |
| 2     | Cement   | 0.125            | 0.155            |                         |                         |                          |                          |
| 3     | Steel    | 0.155            | 0.175            |                         |                         |                          |                          |
| 4     | Cement   | 0.175            | 0.205            |                         |                         |                          |                          |

**Figure 2. Temperature distribution across the well**

The temperature distribution results produced from FEA and the proposed exact solution are compared as shown in Figure 2. It can be seen that the results determined by using the analytical heat conduction solution are greatly matched with the FEA results. On the other hand, the radial stress, hoop stress and axial stress calculated using proposed analytical solution are shown in Figure 3, and the results are
compared against the FEA. In general, all stresses results generated using proposed analytical solution show in good agreement with the FEA results. Besides that, the relative errors for temperature and stresses computed by the proposed algorithm and FEA are found to be lower than 0.1%. The advantage of using the proposed algorithm will be more discernible as the number of section layer increases. This is because getting a solution analytically does not required meshing, therefore it saves computing cost compared to solving the problem numerically.

![Graph](image)

**Figure 3.** Results of radial, tangential and axial stresses

4. Conclusion
   In this paper, an effective analytical solution for prediction of stresses induced by thermal and mechanical loading has been proposed. The results generated by the proposed analytical solutions are in good agreement with the FEA ANSYS simulation results. The error for comparison between results generated by proposed algorithm and FEA is less than 0.1%. The proposed analytical solutions can serve as a simple and inexpensive tool to predict the thermo-mechanical stresses across the cylindrical underground gas well which constructed by layers of steel casing and cement.

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6. References
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