Screened Perturbation Theory

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Abstract

A new perturbative scheme is proposed for the evaluation of the free energy density of field theories at finite temperature. The screened loop expansion takes into account exactly the phenomenon of screening in thermal propagators. The approach is tested in the $N$-component scalar field theory at 2-loop level and also at 3-loop in the large $N$ limit. The perturbative series generated by the screened loop expansion shows much better numerical convergence than previous expansions generated in powers of the quartic coupling.

1 Introduction

The weak coupling expansion of the QCD free energy density does show very bad convergence properties. The coefficients of the usual perturbative series are of alternating sign and of increasing magnitude $[1, 2, 3, 4, 5]$. Only in the TeV temperature range can one find a satisfactory numerical convergence rate $[6]$. This is in great contrast to numerical calculations of this quantity (or of its derivative, the internal energy) in Monte Carlo simulations. These calculations indicate that deviations from the high temperature ideal gas limit are within 15% already for temperatures about $5T_c$ ($T_c$ being the critical temperature) $[7]$.

In the effective theory approach of Braaten and Nieto the contribution of different scales, $2\pi T$, $gT$ and $g^2T$ to the QCD free energy was separated$[8]$. It was noticed that the apparent bad convergence of perturbation theory for the QCD free energy is due to the poor convergence of the perturbative contribution to the free energy from the momentum interval $gT << k << 2\pi T$ for temperatures few times the critical $[6]$. One of the possible reasons for this breakdown might be the fact that scales $2\pi T$ and $gT$ are not actually separated in the above temperature range. This observation lead us to the conjecture that the convergence of perturbation series might be improved if it is reorganized in a way that does not assume separation of the scales $gT$ and $2\pi T$.

There are several mass scales in QCD, what makes the practical application of this conjecture more complicated than it looks like at first sight. On the other
The bad convergence of the perturbative free energy density series is also seen in the $\phi^4$ scalar field theory and our aim is to check the above conjecture in this case. In the scalar field theory the only relevant mass scale is the Debye screening mass. This scale is taken into account exactly in our approach which we thus call screened perturbation theory.

2 Loop Expansion with Screened Propagators

We consider an $O(N)$ symmetric scalar field theory with the following Lagrangian:

$$L = L_0 + L_{\text{int}},$$
$$L_0 = \frac{1}{2}((\partial \phi_i)^2 + m^2 \phi_i^2),$$
$$L_{\text{int}} = -\frac{1}{2}m^2 \phi_i^2 + \frac{g^2}{2N}(\phi_i^2)^2 + \frac{g^2}{2N}(Z_2 - 1)(\phi_i^2)^2. \quad (1)$$

Following Refs. 9 and 10 we have introduced and subtracted a thermal mass term with $m \sim O(g)$ which modifies the bare propagator and thus reorganizes the perturbative series. The coupling constant renormalisation factor is given by\[11\]

$$Z_2 = 1 + \frac{3g^2 N}{(4\pi)^2} \frac{N + 8}{18\epsilon} + O(g^4), \quad (2)$$

and we have left out the field renormalisation factor $Z_1$, which is unity up to $O(g^4)$.

Our aim is to perform a loop expansion with massive propagators without making any assumption on the magnitude of $m$, i.e. to evaluate the free energy and other thermodynamic quantities "exactly" in $m$. This means that we intend to perform the loop expansion starting from a massive ideal gas.

It should be noticed that in the present approach as well as in Ref.9 and 10 static and non-static modes are both resummed. In Ref.12 it was shown that when the perturbative series is organized in powers of $g$, i.e when one takes into account that $m \sim O(g)$ the resummation of all modes is equivalent to the resummation of the static mode only. In our case, however, these two resummation schemes will lead to different results.

For finite $N$ we have carried out our program up to 2-loop level, for large $N$ it is possible to estimate also the 3-loop contribution\[14\]. The contribution of different diagrams to the free energy can be found in Ref. 14, where the necessary technical details are also given. Here we just mention that the contribution of all diagrams to the free energy are proportional to 1-loop sum-integrals, except the basketball diagram which is, however, suppressed in the large $N$ limit. The first problem which arises in the present approach is that 1-loop integrals entering the free energy contain divergent and scale dependent terms, with coefficients proportional to positive powers of the screening mass and therefore
can not be subtracted or canceled by the renormalisation procedure applied at \( T = 0 \). There is also a scale dependent term \( \sim g^4 T^4 \) in the 3-loop free energy, but this is canceled by renormalisation of the coupling constant.

The next question to be discussed is the choice of the screening mass. In the conventional resummation schemes \( m \) was chosen to be equal to its 1-loop value [5, 9, 10, 13]. In our approach the free energy is evaluated exactly with respect to the screening mass. Therefore, to be consistent, the screening mass itself should be evaluated "exactly" in \( m \), which means that it should be determined from the 1-loop gap equation, which becomes exact in the large \( N \) limit [14, 16]. The screening mass obtained this way agrees well with the 1-loop value for \( g \leq 1 \) while it is only half as large as the 1-loop value for \( g \sim 10 \).

In our numerical investigation we have studied the cases \( N = 1 \), and \( N = 10 \) for the large \( N \) limit. The free energy for both cases is shown in Fig.1. The screening masses have been computed from the 1-loop gap equation. For \( N = 1 \) 1-loop and 2-loop results of screened perturbation theory are shown together with \( \mathcal{O}(g^2) \) and \( \mathcal{O}(g^3) \) result of the conventional perturbation theory. For \( N = 10 \) 1-loop, 2-loop and 3-loop results of the screened loop expansion are shown. For comparison we also show here the conventional \( \mathcal{O}(g^4) \) result. Lower orders \( \mathcal{O}(g^2) \) and \( \mathcal{O}(g^3) \) do show alternating behaviour similar for the \( N = 1 \) case.

**Figure 1:** The free energy density of scalar field theory as a function of the scalar self-coupling \( g \) in units of the free energy of a massless ideal gas \( (N \pi T^4/90) \). For \( N = 1 \) 1-loop (dashed line) and 2-loop (solid line) results of the screened loop expansion are shown versus the \( \mathcal{O}(g^2) \) and \( \mathcal{O}(g^3) \) results of the conventional approach. For \( N = 10 \) the 1-loop (dotted line), 2-loop (dashed line) and 3-loop (solid line) results are shown versus the \( \mathcal{O}(g^4) \) result of the conventional approach.
3 Discussion

As it was discussed in the previous section at each order of the screened loop expansion divergent and scale dependent terms appear, but they are canceled by higher loop contributions \cite{14}. However divergent and scale dependent terms are defined up to a constant, which defines the renormalisation scheme. In Ref. 14 \textit{MS} was used and terms like $\ln \bar{\mu}/T$ were omitted. We can have some insight into the sensitivity of the free energy on the scheme keeping the scale dependent terms and studying the dependence of the final result on the choice of the renormalisation scale. In Fig. 2a the free energy evaluated in 1-to-3 loop approximation is shown as function of the scalar self-coupling for $\bar{\mu} = 1.5$ and $N = 10$, the area painted in grey shows the variation of the free energy as the scale is varied from $\bar{\mu} = 1.5$ to $\bar{\mu} = 3$. In Fig. 2b the free energy is shown for $g = 10$ as a function of $\bar{\mu}$. For these figures the leading result for the screening mass was used. It should be noticed that scale dependent part $\sim T^4$ in the 3-loop contribution is canceled by the renormalisation of the coupling constant. As one can see from the figures the scale dependence weakens as one goes from the 1-loop to the 3-loop level free energy. The reason for this is that at 3-loop level the conjectured cancellation of the scale dependent terms proportional to $m^6$ does occur.

Now let us turn to the question of the choice of the screening mass. One may think that good convergence of the present approach is due to the fact that terms like $g^3$ are replaced by $m^3$ and $m$ being determined from the gap equation is small. However, one can see from Fig. 2b that the convergence of screened perturbation theory is also good when the leading order mass is used in the
calculation of the free energy. The difference between the 3-loop free energy evaluated with the leading mass and that evaluated with the mass determined from the 1-loop gap equation is less than 1%.

In conclusion, we managed to improve substantially the convergence of the perturbative series of the $\phi^4$ theory using the screened perturbation expansion and we hope that some of the lessons learned in the scalar field theory remain valid also in QCD.

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