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A Geometric Realization of Confinement

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We study the geometric realization of the Higgs phenomenon in type II string compactifications on Calabi–Yau manifolds. The string description is most directly phrased in terms of confinement of magnetic flux, with magnetic charged states arising from D-branes wrapped around chains as opposed to cycles. The rest of the closed cycle of the D-brane worldvolume is manifested as a confining flux tube emanating from the magnetic charges, in the uncompactified space. We also study corrections to hypermultiplet moduli for type II compactifications, in particular for type IIA near the conifold point.

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1. Introduction

A tremendous amount of progress has been made recently in understanding nonperturbative aspects of string theory. One central tool in these developments has been the window on nonperturbative physics opened by the study of solitonic states and their subsequent microscopic description in terms of D-branes [1]. In compactified string theory, a by now familiar application of these ideas is to consider the physics associated with wrapping various \( p \)-branes around nontrivial homology cycles in spacetime. For instance, such considerations have led to an understanding of enhanced gauge symmetry [2], conifold singularities [3], topology-changing conifold transitions [4], and many other striking developments.

In this paper we initiate a study which is a natural outgrowth of the above considerations. In particular, attempts to describe the inverse of the transition studied in [4], i.e. a geometric description of un-Higgsing, naturally leads us to consider D-branes which wrap around \emph{chains} as opposed to cycles. That is, we wrap D-branes around submanifolds with nontrivial boundaries. We find that they correspond to magnetically charged particles in the confining phase. Just as a free quark in the confining phase does not make sense in isolation, D-branes wrapped around chains do not make sense in isolation either. However, we can consider pairs of such magnetically charged objects, separated in the uncompactified spacetime, and we find that pairs of such chains can be assembled into cycles whose image in spacetime includes magnetic flux tubes. In this way, although we mainly consider the abelian case here, we find a simple geometric interpretation of gauge confinement which we believe has bearing on a geometric interpretation of confinement even in the non-abelian case.

In [4] conifold transitions in type II string theory were found by moving to a singular point in the vector multiplet moduli space and passing on to a new Higgs branch by giving appropriate vacuum expectation values to new massless hypermultiplets which arise. This corresponds geometrically to degenerating the complex structure of a Calabi–Yau in the type IIB theory and then performing a small resolution, or, by using mirror symmetry, to degenerating the Kähler structure of a Calabi–Yau in type IIA and then performing a desingularizing deformation. These descriptions, though, only cover half of the story. Abstractly, one can wonder about the “reverse” processes in which one first moves in the hypermultiplet moduli space and then finds a new Coulomb branch. For ease of reference, we will call the transitions studied in [4] \emph{vector}-conifold transitions and the reverse process
hyper-conifold transitions, where the prefix indicates the type of parameters initially being varied. Concretely, hyper-conifold transitions arise by initially degenerating the Kähler structure of a type IIB string or, by mirror symmetry, the complex structure of a type IIA string. It is not a trivial change of perspective to consider these reverse processes from [4] for two reasons. First, whereas the vector multiplet moduli space is governed by special geometry and receives no quantum corrections, we know far less about the quantum-corrected quaternionic geometry of the hypermultiplet moduli space. Second, type IIA theory has even-dimensional branes while type IIB has odd-dimensional branes. In [4], odd-cycles were degenerated in the type IIB context (equivalently, even-cycles were degenerated in the type IIA context) and hence a local description in terms of particles was obtained by wrapping the appropriate odd (even) branes around the vanishing cycles. In hyper-conifold processes, even (odd) cycles are degenerating in the type IIB (IIA) context and hence we, at first sight, lose a particle description and are forced to understand some of the physics of tensionless strings. In the present context, we unravel the physics of these tensionless strings and reinterpret pairs of them in terms of magnetically confined flux tubes. We will see that a key part is played by wrapping branes on chains, giving rise to magnetically charged states that are linked together via such nearly-tensionless strings. The fact that magnetically charged objects are linked by confining flux tubes is reflected in the geometric fact that chains must be attached to other chains in order to make a closed three-cycles on which branes can be wrapped.

In section 2 we explain the geometric description of conifold transitions in somewhat greater detail than was done in [4] as this is required for our present study. In section 3 we discuss some simple field theory phenomena whose string theory counterparts comprise section 4. In the latter section we begin by following the fate of massive magnetic solitons through the vector-conifold transitions of [4]. This naturally yields magnetic confinement as the physical interpretation of our mathematical description in section 2. We develop this picture in some detail which leads us to a deeper understanding of the reverse process of hyper-conifold transitions. Moreover in this geometric approach to confinement we find that the confined charge and the confined flux tube are unified into a single geometric object in higher dimension whose projection in spacetime leads to an apparent asymmetry between the charge and the flux tube. In section 5 we elaborate on [4] and on section 2 by delineating, at the level of conformal field theory, the location in hypermultiplet moduli space where these hyper-conifold transitions occur and therefore clarify the attachment of the associated Coulomb branch. In section 6 we describe some aspects of quantum string corrections for hypermultiplet moduli motivated from the description of the above transition. Finally in section 7 we end with some conclusions and speculations.
2. The Geometry of Conifold Transitions

We begin by reviewing the mathematics of conifold transitions, following both the original mathematics literature \[5–8\], and subsequent discussions of these transitions in the physics literature \[9–12\]. The transition can be approached by varying either the complex structure or the Kähler structure, and we shall discuss both perspectives.

The set of possible complex structures on a given Calabi–Yau threefold can be parameterized, in favorable circumstances, by the coefficients in the defining equation or equations of that threefold. A familiar example of this is the case of the quintic hypersurface in \(\mathbb{CP}^4\), in which 101 of the coefficients of the quintic polynomial can be used as parameters. At general values of these parameters, the solution set describes a smooth manifold, but at special values, the solution set will have singularities.

The simplest singularities which one encounters are nodes, that is, singularities which can be locally described at the singular point by an equation whose constant and linear terms at that point vanish, and whose quadratic terms define a quadratic of maximal rank. This phenomenon—of acquiring a single node—happens at (complex) codimension one in the complex structure moduli space, since there is one algebraic condition on the parameters which characterizes when the solution set fails to be nonsingular. The set of points where a node has been acquired is called the conifold locus in the moduli space.

Associated to any node which is acquired in this fashion is a so-called vanishing cycle. This is a three-cycle which exists on the nearby nonsingular Calabi–Yau manifolds, and which can be seen topologically by intersecting the Calabi–Yau manifolds with a small ball in the ambient space which surrounds the singular point. The boundary of that intersection (i.e. the intersection with a small sphere) will have topology \(S^2 \times S^3\), and in fact the singular Calabi–Yau space looks topologically like a real cone over \(S^2 \times S^3\) near the singular point. However, in the nearby nonsingular Calabi–Yau manifolds, the \(S^2 \times S^3\) will be the boundary of some \(B^3 \times S^3\), which is a neighborhood of the “vanishing cycle” \(0 \times S^3\).

If we call the homology class of the vanishing cycle \(\gamma_1\), then it is possible to find a basis \(\gamma_0, \gamma_1, \ldots, \gamma_k\) for a Lagrangian subspace of \(H_3(X, \mathbb{Z})\) and a holomorphic three-form \(\Omega\) so that \(\int_{\gamma_0} \Omega\) is single-valued and non-zero near the conifold locus, and the corresponding periods \(Z^1, \ldots, Z^k\) defined by \(Z^j = \int_{\gamma_j} \Omega\) give multi-valued local coordinates on the complex structure moduli space near the conifold locus. The period \(Z^1 = \int_{\gamma_1} \Omega\) is zero along the entire conifold locus, which is why \(\gamma_1\) is called a vanishing cycle.
The vanishing cycle is expected to have a unique representation as a supersymmetric three-cycle \[13\] (i.e., there should be a unique “special Lagrangian submanifold” \[14,15\] in this homology class near the node). Since for such a representative, the volume is proportional to the absolute value of the period, the supersymmetric vanishing cycle gets smaller and smaller as the conifold locus is approached, and the cycle literally vanishes in the limit.

In the conifold transition, the singularity of the limiting Calabi–Yau space is resolved by a type of blowing up, in which the node is replaced by a holomorphically embedded \(S^2\). (This is known as a small resolution of the node.) Topologically, the new \(S^2\) has a neighborhood of the form \(S^2 \times B^4\) whose boundary is \(S^2 \times S^3\); this is the same \(S^2 \times S^3\) (up to homotopy) as was obtained by intersection with a ball in the ambient space.

The space which one obtains by blowing up the singularity in this way is always a complex manifold, but it will fail to be a Kähler manifold if only a single node is involved, as we will see later. In fact, even if we have a Calabi–Yau space with several nodes, the condition for the existence of a Kähler metric after blowing up these nodes is a somewhat subtle one: the vanishing cycle of each node must be homologous to a nonzero linear combination of vanishing cycles of the other nodes.\[\textit{1}\]

The areas of the holomorphic two-spheres are new classes in cohomology after the blowup (although not all of them will be independent). In fact, the reverse of the blowing up process can be regarded as a variation of the Kähler parameters in which all of the areas of the holomorphic two-spheres are sent to zero, producing again the singular Calabi–Yau space. Thus, when we approach the transition from one side we see vanishing supersymmetric three-spheres, and when we approach it from the other side we see shrinking holomorphic two-spheres.

For example, if we vary the defining equation of a quintic hypersurface in \(\mathbb{CP}^4\) until it contains some fixed \(\mathbb{CP}^2\), the limiting quintic has sixteen nodes along the \(\mathbb{CP}^2\). If we remove from \(\mathbb{CP}^2\) small balls around each of the nodes, we find a four-chain whose boundary is the sum of the sixteen vanishing three-spheres. This provides the homology

\[\text{\textit{1\ These homology relations can be represented by four-chains whose boundaries are linear combinations of vanishing cycles; in the limit, these four-chains become four-cycles which can actually be represented by complex submanifolds (of complex codimension one). Blowing up those submanifolds resolves the singularities, with the blown up space naturally embedded in }X \times \mathbb{CP}^1 \times \ldots \times \mathbb{CP}^1; \text{ it inherits a Kähler metric from that of the ambient space.}\]
relation needed to satisfy the Kählerity condition. The resolved manifold is obtained by explicitly blowing up the $\mathbb{CP}^2$ within $\mathbb{CP}^4$.

More generally, we can consider a conifold transition which begins with a Calabi–Yau which acquires $N$ nodes along some multi-conifold locus in the complex structure moduli space, such that there are $M$ homology relations among the vanishing cycles, with each vanishing cycle involved in at least one of the relations. Due to the homology relations, any three-cycle which meets one of these vanishing cycles must meet at least two of them. We refer to such cycles as “magnetic” three-cycles (complementary to the “electric” vanishing cycles). Since there are $N - M$ independent homology classes of vanishing cycles, there must be an $(N - M)$-dimensional space of such “magnetic” cycles. In our example with $N = 16$ and $M = 1$, if the vanishing cycles are represented by $\gamma_1, \ldots, \gamma_{16}$ such that $\gamma_1 + \cdots + \gamma_{16}$ is homologous to zero, then there are $15$ “magnetic” cycles $\beta_1, \ldots, \beta_{15}$ with $\gamma_i \cdot \beta_j = \delta_{ij}$ for $i \leq 15$, and $\gamma_{16} \cdot \beta_j = -1$.

The “magnetic” three-cycles remain three-cycles in the singular limit, but once we blow the space up, each of these three-cycles becomes a three-chain whose boundary is a combination of the shrinking holomorphic two-spheres (the combination being determined by which of the vanishing cycles the three-cycle met). This process is illustrated in figure 1, in which a three-cycle (meeting two homologous vanishing cycles) opens up into a three-chain after the transition. These three-chains provide $N - M$ homology relations among the shrinking two-spheres, so that the areas of those spheres generate only $M$ independent new Kähler classes. It is now clear why there must be at least two vanishing cycles initially: if there were only one, there would be a dual “magnetic” cycle meeting that vanishing cycle, which would become a three-chain after the transition whose boundary would be the two-sphere, which would thus be homologically trivial. But a holomorphic two-sphere on a Kähler manifold can never be homologically trivial.

Viewed from the opposite direction, we have $N$ shrinking two-cycles with $N - M$ homology relations given by three-chains, and $M$ dual “magnetic” four-cycles. At the transition, the two-cycles have shrunk to nothing, and the three-chains have closed up to three-cycles; after the transition, there are new “vanishing” three-cycles dual to the three-cycles which came from three-chains, and the old four-cycles open up into four-chains which specify the homology relations among the vanishing three-cycles.

If we now consider the type IIB theory compactified on these Calabi–Yau manifolds (as was done in [4]), the vector-conifold transition begins with the variation of complex structure to acquire $N$ nodes. Due to the $M$ homology relations, this multi-conifold locus
occurs at complex codimension $N - M$ in the complex structure moduli space. Along that locus, the Dirichlet three-branes which wrap the vanishing three-cycles become massless; that is, we get $N$ new massless hypermultiplets at the transition. The cohomology classes which they span are associated with $N - M$ $U(1)$’s, and the $D$-terms admit $M$ flat directions; going to the associated Higgs branch we find that we have geometrically carried out the conifold transition. The expectation values for the surviving hypermultiplets correspond to the areas of the holomorphic two-spheres—there are $M$ independent classes of these.

The Dirichlet three-branes wrapping the $N - M$ “magnetic” three-cycles remain massive during this process (with some finite mass), and it is natural to ask what happens to them after the transition, since the three-cycles have turned into three-chains. In fact, this is the key question to ask if one wishes to understand the hyper-conifold transition, and we will return to it in detail once we have studied an analogous question in field theory.
3. Some Field Theory Considerations

In this section we consider some aspects of the low energy field theory description of conifold transitions whose string theory counterparts will be the subject of section 4.

In the previous section we have seen that in the type IIB vector-conifold transition on the quintic described in [4] we go to a locus in the quintic vector moduli space where sixteen new hypermultiplets arise from three-branes wrapped on degenerate three-cycles. These sixteen hypermultiplets are charged under fifteen $U(1)$ gauge factors, and it is this unit numerical difference which is responsible for the single flat direction, in the low energy field theory description, which allows passage to the new Higgs branch. Due to the flat direction, the Higgs branch has an extra hypermultiplet, and it comprises the hypermultiplet moduli space of $(86, 2)$ Calabi–Yau, which can be realized as a complete intersection in $\mathbb{CP}^4 \times \mathbb{CP}^1$.

For ease of discussion of this low energy field theory process, we consider the simpler case of two hypermultiplets $H^{(a)}$, $a = 1, 2$, each charged under a single $U(1)$. We imagine that these hypermultiplets arise from wrapping three-branes around two degenerating three-cycles $A_1$ and $A_2$ which satisfy the homology relation $A_1 + A_2 = 0$. This implies that their charges, $q^a$ under the $U(1)$ are 1 and $-1$ respectively. Recall that each hypermultiplet contains two complex scalar fields $h^{(a)\alpha}$, $\alpha = 1, 2$, giving us a total of eight real scalar fields. The relevant part of the $N = 2$ supersymmetric Lagrangian describing these fields is

$$ |\mathcal{D}H^a|^2 + D^{\alpha\beta}D_{\alpha\beta} + ... \tag{3.1} $$

where the $D^{\alpha\beta}$ are

$$ D^{\alpha\beta} = \sum_a q^a (h^{*\alpha}h^{(a)\beta} + h^{*\beta}h^{(a)\alpha}), \tag{3.2} $$

where $h^{*\alpha} = e^\gamma_{\alpha\gamma}h^{(a)\gamma}$ with an overline denoting complex conjugation, and $\mathcal{D}$ is the gauge covariant derivative with connection $A$.

Setting the scalar potential to zero involves three real constraints $D^{\alpha\beta} = 0$, together with one $U(1)$ gauge invariance. These conditions collectively reduce the original configuration space by what is known as a hyper-Kähler quotient. Briefly, whereas a Kähler manifold is equipped with a Kähler two-form, a hyper-Kähler manifold is endowed with three such forms. Such manifolds always have real dimension $4d$ for an integer $d$. If the latter space also admits a $U(1)$ action under which these three two-forms are invariant, there is a natural way to construct a new hyper-Kähler space with dimension $4(d - 1)$. Following [10], each two-form gives rise to a moment map whose vanishing locus reduces
the dimension of the space by one real unit, and choosing a transverse slice to the gauge orbits yields the fourth real constraint. It is not hard to show that the $4(d-1)$ dimensional space which results inherits a hyper-Kähler structure from that on the original space.

In the present context, our original configuration space is the hyper-Kähler space $\mathbb{C}^4$ consisting of the four complex scalar fields $h^{(a)\alpha}$. A convenient hyper-Kähler structure arises from the three two-forms

$\begin{align*}
   & dh^{(1)1} \bar{dh}^{(1)2} + dh^{(2)1} \bar{dh}^{(2)2} \\
   + & dh^{(1)1} \bar{dh}^{(1)2} + dh^{(2)1} \bar{dh}^{(2)2} + dh^{(2)2} \bar{dh}^{(2)2} \\
   + & dh^{(1)2} \bar{dh}^{(1)1} + dh^{(2)2} \bar{dh}^{(2)1}
\end{align*}
$ (3.3)

and the natural $U(1)$ action arises from gauge transformations. Applying the procedure indicated above, the three moment maps are precisely the three $D$-terms in (3.2) and hence the hyper-Kähler quotient thus yields the moduli space of gauge inequivalent vacuum configurations.

Concretely, this hyper-Kähler quotient space has complex dimension two, which corresponds to the scalar degrees of freedom in a single hypermultiplet. Thus, the moduli space of vacua consists of a single flat direction in the potential. Explicitly, this flat direction arises from the four parameter family of solutions (up to a $U(1)$ gauge transformation)

$\begin{align*}
   & |h^{(1)\alpha}| = |h^{(2)\alpha}| \\
   \theta^{(1)2} - \theta^{(1)1} = \theta^{(2)2} - \theta^{(2)1}
\end{align*}$ (3.6)

where we have written

$\begin{align*}
   h^{(a)\alpha} = r^{(a)\alpha} e^{i\theta^{(a)\alpha}}.
\end{align*}$ (3.8)

A convenient solution to these conditions is the one chosen in [4], namely

$\begin{align*}
   h^{(a)\alpha} = v^{\alpha}
\end{align*}$ (3.9)

for any choice of the complex two-vector $v^{\alpha}$. For nonzero values of $v^{\alpha}$ we therefore spontaneously break the $U(1)$ gauge symmetry, one hypermultiplet disappears via the Higgs mechanism, and we are left with one net hypermultiplet.

We were careful to indicate above that the vacuum configuration arising via such a hyper-Kähler quotient need not be a manifold—it can have singularities. There are two
convenient ways to see this. First, the choice we made in (3.9) only partially fixes the
gauge freedom. Namely, \( v^\alpha \) and \(-v^\alpha \) are choices in (3.9) which differ by a \( U(1) \) gauge
transformation on our fields with gauge parameter \( \pi \). Thus, we need to impose a further
\( \mathbb{Z}_2 \) discrete invariance on (3.9) to have the set of gauge inequivalent vacua. The vacuum
configuration space is thus \( \mathbb{C}^2/\mathbb{Z}_2 \). A second way of seeing this is to work directly with
gauge invariant combinations of our fundamental fields. Namely, let
\( x = h^{(1)*} h^{(2)+}, y = h^{(2)*} h^{(1)+}, z = h^{(1)*} h^{(1)+}, t = h^{(2)*} h^{(2)+} \), each of which is gauge invariant. These four
combinations manifestly satisfy the constraint \( xy = zt \). Furthermore, the constraint (3.2)
implies \( z = t \). Thus, we arrive at the vacuum space \( \mathbb{C}^3/(xy = z^2) \) which again is \( \mathbb{C}^2/\mathbb{Z}_2 \).

Whenever we Higgs a \( U(1) \) gauge factor, we have the possibility of generating the
string-like topological defects of Nielsen and Olesen—that is, cosmic strings. At first sight,
in the presence of two hypermultiplets, one might think that we would have the possibility
of two kinds of cosmic strings associated with nontrivial windings of each of the two fields.
Indeed, this is true and each would correspond to a global cosmic string, neither of which
has finite energy due to infrared divergences. However, pairs of these global strings where
both fields undergo appropriate monodromy at infinity can be reinterpreted as a finite
energy cosmic string of the Higgsed \( U(1) \). More explicitly, we take the hypermultiplets to
have the asymptotic behavior
\[ h^{(a)\alpha} = C^\alpha e^{iq^a \phi} \tag{3.10} \]
where we express our fields as functions of cylindrical coordinates \((r, \phi, z)\) in \( \mathbb{R}^3 \), and the
string points in the \( z \)-direction. Finite energy per unit length requires the gauge field \( A_\mu \)
to have the asymptotic form
\[ A_\phi \sim \frac{1}{r} \tag{3.11} \]
in order to cancel the infrared diverging kinetic contributions to the energy. This means
that
\[ \oint \vec{A} \, d\vec{l} = 1 \tag{3.12} \]
and hence the cosmic string carries a single unit of magnetic flux.

The situation in such a configuration is that the original \( U(1) \) gauge symmetry is
Higgsed by the non-zero vacuum expectation values. The only remnant of the original
gauge symmetry is the magnetic flux confined in the core of the cosmic string, where
the massive vector multiplet becomes massless. If for instance we followed a magnetic
monopole through the symmetry breaking, we would find that its flux becomes trapped into
such tubes—a finite total energy configuration can arise from following a monopole/anti-monopole pair which can reside at the endpoints of these cosmic flux tubes after passing through the phase transition.

There are two relevant scales which set the size of a cosmic string: the Higgs mass and the symmetry breaking induced vector meson mass. One subtle feature in the field theory manifestation of these ideas as they arise in the fundamental string context, as we shall see in the next section, is that the Higgs potential is exactly flat. It would be interesting to study how field theory corrections serve to capture all of the fundamental string phenomena we will encounter in the next section, but as we only use field theory as a qualitative guide, we shall not do so here. Rather, the only extension of the field theory discussion we shall need is the generalization to multiple $U(1)$ factors, which we now describe.

The generalization of this discussion to any number of hypermultiplets charged under a smaller number of $U(1)$ gauge groups is immediate. For instance, in the case of a transition like the vector-conifold transition on the quintic in which there is one homology relation between the $N$ vanishing three-cycles, we will have $N$ hypermultiplets $H^{(a)}$, $a = 1, ..., N$ charged under a $U(1)^{N-1}$ gauge symmetry, $U(1)_I$, $I = 1, ..., N - 1$’s with charges

$$q^{(a)}_I = \delta^{(a)}_I$$  \hfill (3.13)

for $1 \leq a \leq N - 1$ and

$$q_N^I = -1$$  \hfill (3.14)

for all $I$. The vacuum manifold is again found by minimizing the potential—a procedure which as above applies a hyper-Kähler quotient construction ($N - 1$ times) to the original hyper-Kähler $\mathbb{C}^{2N}$ field configuration space. Specifically, this yields the conditions

$$|h^{(1)\alpha}| = |h^{(N)\alpha}|$$  \hfill (3.15)

$$\theta^{(1)2} - \theta^{(1)1} = \theta^{(N)2} - \theta^{(N)1}$$  \hfill (3.16)

We have $4N$ real scalars, $3(N - 1)$ real constraints and $N - 1$ gauge invariances thereby leaving a single hypermultiplet flat direction which can be conveniently parameterized as in (3.9) by

$$h^{(a)\alpha} = v^{\alpha}$$  \hfill (3.17)
where \( a \) now takes values from 1 to \( N \). As in our simple two-field case discussed earlier, the vacuum configuration has an orbifold quotient singularity at the origin. As above, we can see this in two ways. First, the choices \( v^\alpha \) and \( \omega v^\alpha \) where \( \omega \) is a fundamental \( N^{th} \) root of unity are related by a discrete subgroup of our original \( U(1)^{N-1} \) symmetry; namely, the diagonal element with gauge parameter \( 2\pi/N \) for each \( U(1) \) factor. Second, we can again work directly with gauge invariant combinations of our fields: 

\[
x = (\prod_{i=1}^{N-1} h^{(i)1}) h^{(N)2} \quad y = (\prod_{i=1}^{N-1} h^{(i)2}) h^{(N)1}.
\]

These manifestly satisfy the constraint

\[
z_1 z_2 \cdots z_N = xy
\]

and the \( D \)-terms equations imply \( z_i = z_j \) for all \( i \) and \( j \), thereby yielding the vacuum moduli space \( \mathcal{O}^3/(z^N = xy) \). From either point of view, therefore, the moduli space is \( \mathcal{O}^2/\mathbb{Z}_N \).

As in our two-field case, in the Higgs phase we can again form cosmic strings with the same constraints as previously, giving us \( N-1 \) distinct string solutions, each carrying a single unit of magnetic flux of one of the \( U(1) \) gauge groups. In this way, then, each of the dual magnetic fields are confined into vortex lines in the Higgs phase. Monopole/anti-monopole configurations connected by these cosmic strings provide the fate of free Coulomb phase magnetically charged particles when the symmetry breaking phase transition occurs.

In the next section we shall see this simple field theory discussion realized through the geometry of conifold transitions.

4. Branes on Chains and Gauge Confinement

In physical terms, the vector-conifold transition is most conveniently described as a Higgsing of a gauge symmetry. If we wish to understand the reverse transition—the hyper-conifold transition—we thus have to understand un-Higgsing. On the face of it this might seem like an easy task; all we have to do is identify, in the Higgs phase, a vector multiplet which has become massive by ‘eating’ a hypermultiplet. This however is not a simple matter because such a massive state is not a reduced BPS supersymmetry multiplet and thus there is no reason for its stability. Thus we cannot identify it with any stable particle or soliton in our string theory setup. How else can we detect the existence of the Higgs phase? In the Higgs phase we have a broken gauge symmetry and the corresponding short range force due to electrically charged objects. On the other hand, if we have massive magnetically charged objects in the Higgs phase they will be confined and their
magnetic flux will be confined to a tube. These magnetic flux tubes are simply the cosmic strings of the broken local \( U(1) \) gauge symmetries. The existence of massless vectors and hypermultiplets—which are generically massive in the Higgs phase—can be directly seen as propagating massless modes moving up and down the cosmic string.

So the question to ask is whether we have any magnetically charged states in our string theory context? Indeed we do. Let us for simplicity start with the case where we have one \( U(1) \) and two homologous \( S^3 \) vanishing cycles. By a choice of convention we call the three-branes wrapped around the vanishing \( S^3 \)'s electrically charged states with respect to this \( U(1) \). As discussed in section 2, there is a dual three-cycle which intersects each of these \( S^3 \)'s at one point. If we consider a three-brane wrapped around such a dual cycle we get a particle which is magnetically charged with respect to the \( U(1) \). By choosing a minimal volume cycle in this dual homology class, the wrapped configuration is a BPS magnetically charged state, \( m \). Of course, in this un-Higgsed phase what we call electric and magnetic is a matter of convention. In particular, if we put a magnetically charged BPS state \( m \) at \( x \) in the uncompactified spacetime, and its anti-particle state \( \overline{m} \), which is the three-brane wrapped with opposite orientation about the same three-cycle, at \( y \) then there is a Coulomb potential for this configuration proportional to \( 1/|x - y| \). Let us keep these magnetically charged states \( m \) and \( \overline{m} \) fixed at \( x \) and \( y \) respectively and pass to the Higgs phase. At this point the physics suggests that the magnetic charges should get screened and hence there should be a linearly growing energy for moving them apart. In particular the energy of the same configuration should now go like \( T|x - y| \) where \( T \) is the tension of the cosmic string connecting them. Note that the total energy in this configuration should also include the bare mass of the magnetically charged particles. If we call the bare mass \( M_0 \), we expect that the energy for this configuration should go like

\[
E = 2M_0 + T|x - y|.
\]

We would like to see how this is realized geometrically.

As discussed in section 2, in the Higgs phase the two \( S^3 \)'s are replaced by two \( S^2 \)'s which are in the same homology class. This homology relation implies that there is a three-chain \( C_3 \) whose boundary is the difference of these \( S^2 \)'s:

\[
\partial C_3 = S_1^2 - S_2^2
\]

where we have denoted the two \( S^2 \)'s by \( S_i^2 \), with \( i = 1, 2 \). Moreover, in the Higgs phase, the magnetic three-cycle which intersects the original \( S^3 \)'s is naturally identified with this
three-chain. On the other hand we now face the dilemma that we cannot wrap a three-brane around a three-chain. This of course is similar to the statement that we do not expect to have a free magnetically charged state in the phase where the magnetic charge is confined. However we should be able to consider $m(x) - \overline{m}(y)$ configurations in this context. Indeed there is a very natural geometric construction which precisely does this.

Consider the total space $\mathbb{R}^3 \times X$ where $\mathbb{R}^3$ is the uncompactified space and $X$ is the Calabi–Yau. We will now construct a three-cycle in this total space which will correspond to the configuration $m(x) - \overline{m}(y)$. The three-cycle in question is comprised of four three-chains which are joined together along their boundaries (see figure 2, in which the base represents spacetime and the vertical direction represents the internal space). Let $I$ denote the straight line from $x$ to $y$. The four three-chains are

$$m = x \times C_3, \quad f = I \times S_1^2, \quad \overline{m} = y \times C_3^*, \quad \overline{f} = I^* \times S_2^2$$

where we have used $^*$ to denote the same chain with the opposite orientation. Note that the two chains we denoted by $m$ and $\overline{m}$, which project to $x$ and $y$ in $\mathbb{R}^3$ respectively, are precisely the objects we would like to identify with magnetically charged states $m$ and $\overline{m}$.

Figure 2. Four three-chains assembled into a three-cycle, yielding a confined $m\overline{m}$ pair connected by a flux tube.
at these two points. Moreover, the image of $f$ and $\tilde{f}$ in $\mathbb{R}^3$ is simply the straight line $I$ from $x$ to $y$. Thus the image of this three-chain in $\mathbb{R}^3$ is simply two points $x$ and $y$ connected by a line $I$. The identification with the $m(x) - \overline{m}(y)$ configuration suggests that the $f \tilde{f}$ image on $\mathbb{R}^3$ should be viewed as the magnetic flux of the cosmic string. We will now provide evidence for this identification of the magnetic flux tube.

In the Higgs phase, the hypermultiplet moduli which take us away from the transition point can be viewed as coming from the complexified Kähler deformations of the Calabi–Yau as well as the moduli of RR gauge fields. The RR gauge fields consist of an antisymmetric two-form and an antisymmetric four-form field, which after decomposing with a harmonic form on an internal $S^2$, gives rise in the four uncompactified dimensions to a scalar and an antisymmetric two-form $C_{\mu\nu}$ respectively. There is a cosmic string that $C_{\mu\nu}$ couples to. In particular if we dualize $C$ to a scalar

$$dC = *d\phi,$$

the cosmic string in question is characterized by $\phi \to \phi + 1$. In fact it is easy to give a geometric description of such a string. Since $C_{\mu\nu}$ came from the reduction of the four-form gauge potential, and that couples to a Dirichlet three-brane, the corresponding cosmic string is simply a three-brane wrapped around the holomorphic $S^2$. In the example at hand we have two $S^2$’s and so we will get two cosmic strings from wrapping the three-brane around each of the $S^2$’s. These cosmic strings which are BPS saturated do not have finite energy per unit length. This is in fact true of any global cosmic string. In the Higgs phase the BPS saturated strings correspond to global strings and not to cosmic strings of a broken local gauge symmetry—the latter do not carry a BPS charge \([19]\). There is an easy way to see that the global cosmic string we are dealing with is not quite the local cosmic string we are interested in, to which we will now turn.

As discussed in section 3 it is useful to write the hypermultiplet degrees of freedom in terms of composites of the charged fields so that they are $U(1)$ neutral. In particular, if $(h^{(1)1}, \overline{h}^{(1)2})$ denote the hypermultiplet degrees of freedom coming from one vanishing $S^3$ and $(h^{(2)1}, \overline{h}^{(2)2})$ those coming from the other, we recall that $x = h^{(1)1}\overline{h}^{(2)2}$, $y =$

---

2 The situation here is the same as the seven-brane of type IIB, which by itself will have infinite energy. However in the type IIB by utilizing the fact that the dual scalar has $SL(2, \mathbb{Z})$ monodromy we can get finite energy solutions by taking seven-brane configurations which are relatively non-local \([17]\) by using the stringy cosmic string geometry \([18]\).
\( h^{(2)1}h^{(1)2} = h^{(1)1}h^{(2)2} \), and that relation \( xy = zt \) becomes \( xy = z^2 \), after imposing the vanishing of the potential. As will be discussed in more detail in section 6, the moduli space of hypermultiplets near this region is two ‘cosmic strings’ coming together at \( z = 0 \), where the two cycles of the elliptic fiber are identified with RR gauge fields on \( S^2 \) and a dual four-cycle. In particular the RR gauge field corresponding to the vanishing \( S^2 \) corresponds to going around \( x, y \) in phase and keeping \( z \) fixed:

\[
\begin{align*}
x &\to x \exp(i\phi), \\
y &\to y \exp(-i\phi), \\
z &\to z.
\end{align*}
\]

In terms of the charged fields we can interpret this transformation as either

\[
h^{(1)\alpha} \rightarrow e^{i\phi}h^{(1)\alpha}, \quad h^{(2)\alpha} \rightarrow h^{(2)\alpha}
\]

or the same form with the two hypermultiplets exchanged: \( 1 \leftrightarrow 2 \). Thus the BPS string we have obtained from wrapping the three-brane around each of the two-cycles should correspond to one of these two possibilities. This means that each of these BPS strings can be viewed as a global Nielsen-Olesen string \(^{[20]}\). But for a cosmic string of the Higgsed \( U(1) \), both hypermultiplets undergo phase rotations in such a way that all gauge invariant objects are invariant and, in particular, \( x, y, z \) undergo no monodromy. Thus we identify the cosmic string of the local \( U(1) \) with the superposition of the BPS cosmic strings coming from each of the \( S^2 \)'s with opposite orientation. This string, therefore, carries no net BPS charge. In fact the \( f\tilde{f} \) string in the above three-brane configuration is clearly the same as this cosmic string, which we identify with the confined magnetic flux.

It is also interesting to connect the classical mass formula we derived for the confined magnetic states with the energy of the three-brane configuration we constructed. The total energy of the three-brane wrapped around the three-cycle we considered is clearly the volume of the three-cycle, which consists of the volume of four three-chains. The volume of the two three-chains which correspond to the two magnetically charged states are naturally identified with the bare masses of the two particles. The volume of the other two chains is \( 2|x - y|T' \) where \( T' \) is the area of each of the holomorphic \( S^2 \)'s. Noting that two global strings give one cosmic string of the local \( U(1) \) and the the tension of the global string is identified with \( T' \), we see that this is the same as \( |x - y|T \). We have thus seen a simple explanation of the mass formula for the classical magnetically charged states separated by a large distance.
Another interesting aspect is the issue of zero modes on the cosmic string. The counting of such zero modes can be directly accomplished in the D-brane picture. Explicitly, we note that on a Dirichlet-three-brane world volume there is an $N = 4 U(1)$ gauge theory \cite{21,22} whose degrees of freedom correspond to open string states with ends lying on the three-brane. Moreover, after wrapping the three-brane around an $S^2$ in the CY we get a twisting of this $N = 4$ Yang–Mills \cite{23}, which has the same degrees of freedom as the dimensional reduction of $N = 1$ Yang–Mills in four dimensions down to two. Thus from each of the BPS strings we get two massless bosonic and two massless fermionic modes and thus putting the two together we get a total of four bosonic and four fermionic zero modes. It would be interesting to understand this from a field theory viewpoint.

Clearly, the discussion above can be generalized to many other cases. For example if we have a $U(1)^{N−1}$ gauge group which we identify with the Cartan subalgebra of $SU(N)$, and we have $N$ charged hypermultiplets corresponding to the fundamental weight in the weight lattice of $SU(N)$, we can Higgs this system and end up with one massless hypermultiplet. The magnetic charges will belong to the dual lattice, which is simply the root lattice of $SU(N)$. We expect to have $N − 1$ basic types of cosmic strings to be identified with $N − 1$ broken $U(1)$ gauge symmetries. In the geometric description of the Higgs/confining phase we have $N$ homologous vanishing $S^2$’s. There are $N$ basic BPS saturated cosmic strings (they form the fundamental of $SU(N)$). The cosmic strings of the broken $U(1)$’s are to be identified with the combination of BPS/anti-BPS strings. They form the adjoint weights of $SU(N)$, and they are generated by $N − 1$ elements, which we identify with $N − 1$ basic cosmic strings of the higgsed $U(1)^{N−1}$. The magnetically charged objects in the confining phase correspond to the three-chains whose boundaries are the differences of two homologous $S^2$’s and so they correspond to the adjoint weights of $SU(N)$. Pairs of magnetically charged states with opposite charge get connected with magnetic flux tubes which are the corresponding cosmic strings carrying the adjoint weights of $SU(N)$.

5. Moduli Space Attachment

In this section we give a more precise discussion of how the topologically distinct Calabi–Yau moduli spaces connected by conifold transitions actually attach. For concreteness, we shall focus on the quintic hypersurface and the $(86, 2)$ model, although our discussion is more general. In fact, we still lack a first principles argument which ensures that the vector-conifold transition discussed in \cite{4} and reviewed in section 2 does in fact
take us to the Calabi–Yau space given by the complete intersection of a bidegree (4, 1) and (1, 1) in $\mathbb{CP}^4 \times \mathbb{CP}^1$. There is essentially airtight circumstantial evidence for this being the case: the confluence between the mathematics of degenerate deformations followed by resolving small resolutions and the physics of degenerate points in Coulomb sector the moduli space followed by a resolving Higgs mechanism, is very convincing. In this section we add to this evidence through more global considerations which allow us to explicitly see the attachment of the quintic Kähler moduli space within the Kähler moduli space of the (86, 2) model. Furthermore, and somewhat surprisingly, we see a dramatic shift in the phase boundaries from the prediction of the linear sigma model to the reality of the nonlinear model, which plays a key part in yielding a consistent moduli space attachment.

To begin the discussion, let’s recall that the vector-conifold transition we are discussing most naturally arises for fixed and large Kähler class on the quintic, where we trust conclusions drawn from low energy field theory considerations. However, as BPS states are the central participants in the transition, and since they are stable under local variation of parameters, the transition can actually take place for a range of values of the Kähler class on the quintic.\footnote{We note that in changing the Kähler class for which we perform a vector-conifold transition, we may pass through marginal stability curves for BPS states. Nonetheless, we expect to be able to perform the transitions at generic points.} Thus, we actually have a one-parameter family of vector-conifold transitions, taking us into the two-dimensional Kähler moduli space of the (86, 2) model. The one-dimensional Kähler moduli space of the quintic therefore must attach to a one-dimensional locus inside the Kähler moduli space of the (86, 2) model.

To understand this attachment, let’s first review the structure of the Kähler moduli space of the (86, 2) model. The realization of this model originally given in \[11\] is as a bidegree (4, 1) (1, 1) complete intersection in $\mathbb{CP}^4 \times \mathbb{CP}^1$. For our purposes, though, there is an equivalent but more convenient realization of this Calabi–Yau. Namely, the degenerate complex structure we pass to in the quintic moduli space is of the form

$$x_4 G_4(x_1, \ldots, x_5) + x_5 H_4(x_1, \ldots, x_5) = 0 \quad (5.1)$$

where $(x_1, \ldots, x_5)$ are homogeneous $\mathbb{CP}^4$ coordinates with $G_4$ and $H_4$ being homogeneous quadrics. The small resolution we perform to pass to the (86, 2) model can be realized by blowing up the $\mathbb{CP}^2$ given by $x_4 = x_5 = 0$. Explicitly, we do this by introducing additional homogeneous variables $y_1$ and $y_2$ satisfying $x_4 y_1 = x_5 y_2$. In this ambient toric space, which
is now a $\mathbb{C}P^4$ blown up along a $\mathbb{C}P^2$, the resolved model is realized as a hypersurface. We can explicitly realize this procedure torically by augmenting the toric data for (the bundle $\mathcal{O}(-5)$ over) $\mathbb{C}P^4$,

\[
v_0 = (0,0,0,0,1), v_1 = (1,0,0,0,1), v_2 = (0,1,0,0,1), v_3 = (0,0,1,0,1),
    v_4 = (0,0,0,1,1), v_5 = (-1,-1,-1,-1,1),
\]

by the additional point

\[
v_6 = (-1,-1,-1,0,1) = v_4 + v_5.
\]  

In this toric variety, the equation of the Calabi–Yau, obtained by forming the dual Newton polyhedron, is precisely of the form

\[
\sum_{(a,b) \neq (0,0)} x_4^a x_5^b x_6^{(a+b-1)} F_{-a-b}^a b(x_1, ..., x_3) = 0.
\]  

Clearly, when the new variable $x_6$ is nonzero, we can use one of the $\mathbb{C}^*$ actions defining the toric variety to scale it to one, in which case (5.4) becomes (5.1). This establishes an isomorphism between the complement of the singular locus on the quintic (5.1) and the complement of $x_6 = 0$ on (5.4). When $x_6 = 0$, though, we see that there are more points solving (5.4) than (5.1), reflecting the fact that we have blown-up the singular locus.

The relation of this realization to the more standard complete intersection form is immediate. Namely, the ambient toric variety involves the bidegree $(1,1)$ equation $x_4 y_1 = x_5 y_2$, which together with the hypersurface equation (5.1) yields the two homogeneous equations in the complete intersection form. Having established their equivalence, we shall henceforth use the hypersurface realization for convenience.

The toric data (5.2)(5.3) — which is equivalent to a linear sigma model formulation — allows us, using the methods of [24,25], to describe the blown up space in terms of seven fields and a $U(1) \times U(1)$ action with charges given by

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $p$ |
|---|-------|-------|-------|-------|-------|-------|-----|
| $U(1)_1$ | $1$   | $1$   | $1$   | $1$   | $1$   | $0$   | $-5$|
| $U(1)_2$ | $0$   | $0$   | $0$   | $1$   | $1$   | $-1$  | $-1$|

(5.5)
There are four phases in this model, which can be identified as the original (86, 2) model, a flopped Calabi–Yau counterpart, a hybrid model, and the Landau–Ginsburg phase. The phase diagram is shown in figure 3.

To avoid confusion, we emphasize that this Kähler moduli space is only the NS-NS base of the full hypermultiplet moduli space; the full space includes RR toroidal fibers.

Where in this phase diagram do we expect the quintic Kähler moduli space to attach? In the large radius limit of the geometric part of the phase diagram, $r_1 \to \infty$, we expect [26] that the quintic will attach at the center of the flop—the place where the $S^2$ homology class, whose size varies by the $r_2$ modulus, vanishes. In [27] it was shown that in the large radius limit, there are no stringy corrections to linear sigma model coordinates associated with flops, and hence we would naively expect the attachment to pass through the point $p_1$, as indicated. On the other hand, in the “small radius limit” of $r_1 \to -\infty$ (with $r_2$ held at $-\infty$), we would expect the Landau–Ginsburg point of the quintic to attach to the Landau–Ginsburg point of the (86, 2) model, at $p_2$. At first sight, therefore, it is hard to see how the one-dimensional Kähler locus of the quintic attaches to both of these points.

The resolution to this problem requires a more careful study of the flop transition taking us from phase I to phase II. Torically, this flop transition is shown in figure 4. The key fact to notice is that this flop makes us of the point $v_0 = (0, 0, 0, 0, 1)$ which corresponds to the anticanonical line bundle field in the linear sigma model, represented...
by the two-dimensional chiral field $p$ \cite{24}. Unlike the flops studied in \cite{24,27}, therefore, the flop we encounter here is not contained within the compact part of the ambient toric variety, but makes use of the noncompact part as well. We must therefore redo the analysis of \cite{27} to determine the relationship between the true stringy volume of the $S^2$ and the linear sigma model coordinate $r_2$.

To do so, we pass to the two-dimensional complex structure moduli space of the mirror to the $(86, 2)$ model, which can be obtained from the methods of \cite{28}. Using the methods described in \cite{27}, we can write down local coordinates valid in a neighborhood of the two large complex structure points, mirror to points $p_3$ and $p_4$ in figure 3. As the moduli space is naturally expressed in the form

$$(\Pi^7 - F_\Delta)/(\Pi^5)$$

with $F_\Delta$ determined by the corresponding triangulation, we can express the local coordinates in terms of $\Pi^5$ invariant combinations of the initial $\Pi^7$ toric coordinates, $a_0, ..., a_6$. Written in this form, the coordinates are

$$z(I)_1 = \frac{a_1a_2a_3a_6}{a_0^4}, \quad z(I)_2 = \frac{a_4a_5}{a_0a_6}$$

and

$$z(II)_1 = \frac{a_1a_2a_3a_4a_5}{a_0^5}, \quad z(II)_2 = \frac{a_0a_6}{a_4a_5}.$$
Calabi–Yau, along the rational curve at large complex structure (mirror to \( r_1 \to \infty \)). Following the methods of [28] as applied in [27] we find the following equation

\[
z(1 - z) \frac{\partial^2 f}{\partial z^2} + (1 - 2z) \frac{\partial f}{\partial z} = 0 \tag{5.9}
\]

for the periods of the holomorphic three-form on the Calabi–Yau. The two solutions to this equation with trivial and logarithmic behavior near \( z = 0 \) will be denoted \( f_0 \) and \( f_1 \) respectively. The important point to note is that this equation differs from eqn (63) of [27] due to the fact that this flop involves the noncompact direction. Thus, whereas the solution of the flop equation from [27] gave

\[
f_1 = \log(z), \quad f_0 = \text{constant} \tag{5.10}
\]

and hence the mirror map

\[
B + iJ = \frac{1}{2\pi i} \log(z) \tag{5.11}
\]

from which we see that there are no sigma model corrections, the situation now is different. It is immediate to solve (5.9) and find

\[
f_1 = \log \frac{z}{1 - z}, \quad f_0 = \text{constant}. \tag{5.12}
\]

This yields the mirror map

\[
B + iJ = \frac{1}{2\pi i} \log \frac{z}{1 - z}. \tag{5.13}
\]

We therefore see that there are strong sigma model corrections near \( z = 1 \) in this case.

| \( z \) | \( r_2 \) | \( z/(z - 1) \) | \( J \) |
|---|---|---|---|
| \( p_3 \) | 0 | \( \infty \) | 0 | \( \infty \) |
| \( p_1 \) | 1 | 0 | \( \infty \) | \( -\infty \) |
| \( p_4 \) | \( \infty \) | \( -\infty \) | 1 | 0 |

Table 1.

Let us now use this solution to measure the volume of the flopping \( S^2 \) curve in the \( (86, 2) \) model, with \( r_1 \) fixed at \( \infty \). We directly see the results in table 1. Notice that the location of the center of the flop \( \int_{S^2} J = 0 \) has undergone a dramatic shift from the linear sigma model prediction and now resides on the lower toric boundary component, at \( p_4 \). This means, recalling that the diagram of figure 3 is just the real part of a complex phase
diagram, that the phase diagram at $r_1 \to \infty$ has undergone a shift from the linear sigma model prediction. That prediction, illustrated in figure 5, was that the boundary of the region of convergence of the sigma model centered at $p_3$ would pass through $p_1$, so one might expect that $p_1$ would be the center of the flop. However, the shifted prediction, illustrated in figure 6, is that the boundary of the region of convergence actually passes through $p_4$ instead. We see that the phase boundary has substantially shifted so that the two phases join along what appears to be the lower horizontal boundary in figure 3. It is on this locus that the string volume of the $S^2$ has shrunken to zero, not along the locus $r_2 = 0$ of figure 3. This therefore resolves the previous puzzle, as this locus also includes the respective Landau–Ginsburg points.

![Figure 5. Phase boundaries for the linear sigma model.](image)

We can also quite directly see that this adjoining locus is manifestly identified with the Kähler moduli space of the quintic. Again, it is easiest to work in the mirror description where we intend to identify this locus with the one-dimensional complex structure moduli space of the mirror quintic.

To do so, let’s recall that the complex structure moduli space of the $(2,86)$ model, as implicitly indicated above, arises from considering the possible coefficients in the defining equation. If we wish to describe the equation completely, we would need 91 homogeneous variables, which is a very cumbersome form to work with. However, the essence of the defining equation is captured by the equation for the hypersurface within $(\mathbb{C}^*)^4$ (ignoring
the boundary of the toric variety). In convenient coordinates \( t_1, t_2, t_3, t_4 \) that equation can be written

\[
a_0 + a_1 t_1 + a_2 t_2 + a_3 t_3 + a_4 t_4 + a_5 t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} + a_6 t_1^{-1} t_2^{-1} t_3^{-1} = 0.
\] (5.14)

If we set \( a_6 = 0 \), we get precisely the corresponding equation for the \((1,101)\) model:

\[
a_0 + a_1 t_1 + a_2 t_2 + a_3 t_3 + a_4 t_4 + a_5 t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} = 0.
\] (5.15)

To put this in a more familiar form, if we define

\[
t_1 = z_1^4 z_2^{-1} z_3^{-1} z_4^{-1} z_5^{-1}, \ldots, t_4 = z_1^{-1} z_2^{-1} z_3^{-1} z_4^{-1} z_5^{-1},
\] (5.16)

then the equation takes the form

\[
z_1^{-1} \ldots z_5^{-1} (a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^5 + \ldots a_5 z_5^5) = 0
\] (5.17)

(of which a \((\mathbb{Z}_5)^3\) quotient must be taken since fifth powers occur in the Jacobian determinant of the change of variables; note that the factor of \( z_1^{-1} \ldots z_5^{-1} \), which only affects the boundary of the toric variety, should be removed).

Thus, in this limit of \( a_6 = 0 \) we manifestly recover the one dimensional complex structure moduli space of the mirror quintic.

To make contact with our previous discussion, let’s recall that the monomial-divisor mirror map of [29,30] tells us that the attachment locus \( a_6 = 0 \) corresponds to the locus \( r_2 = -\infty \), which is precisely attachment locus found previously.

We have thus accurately found where in the moduli space the analysis of the previous sections actually occurs.
6. Stringy Quantum Corrections

As discussed in detail in [3,31], for type II compactifications on Calabi–Yau threefolds, even though the vector multiplets do not receive quantum corrections, the hypermultiplet moduli space can. We ask the question of whether there are such corrections in our setup in the Higgs/confining phase. Indeed as we will now argue, the transitions we have talked about require that such corrections exist. Moreover, we will be able to say what these corrections are at least in the limit where gravitational effects are ignored ($M_p \to \infty$).

The hypermultiplet states we have been studying are mainly associated to the Kähler moduli of a type IIB string on a Calabi–Yau manifold. Equivalently, they correspond to the complex structure moduli of a Calabi–Yau manifold for a compactification of a type IIA string. Let us focus on the latter interpretation. If the Calabi–Yau has an $n$ dimensional complex structure moduli space, then the hypermultiplet moduli space is a $4n + 4$ real dimensional quaternionic manifold. This space consists of $2n$ real dimensions parameterizing complex structure moduli, $2n + 2$ real dimensions counting the moduli of three-form gauge potential of type IIA which we can associate with a $2n + 2$ dimensional “intermediate Jacobian” torus corresponding to the periods of the holomorphic three form of the Calabi–Yau, and two dimensions associated with the dilaton and axion of type II strings. In more detail, the $2n$-dimensional base $\mathcal{M}$ specifying the complex structure moduli is the NS-NS base discussed in the previous section. A useful bundle can be built on this space whose fibers consist of $H^3(M_x, \mathbb{C})$ with $x$ denoting a point in $\mathcal{M}$. A natural modification of this bundle, both from mathematics and from physics, is to consider modding out the fibers by integral shifts in cohomology $H^3(M_x, \mathbb{Z})$, thereby yielding the so called bundle of intermediate Jacobians, whose fibers are $H^3(M_x, \mathbb{C})/H^3(M_x, \mathbb{Z})$. The latter, in fact, are complex tori as the Hodge decomposition $H^3(M) = H^{3,0}(M) + H^{2,1}(M) + \text{c.c.}$ yields a natural complex structure on the fibers. Without the axion-dilaton system, then, the hypermultiplet moduli space is a bundle of complex $n + 1$ (real $2n + 2$) tori fibered over the $n$ complex dimensional base. The axion-dilaton is included as another complex scalar, which transforms in a complex line bundle over the moduli space, thereby yielding the total $4n + 4$ dimensional quaternionic moduli space.

The perturbative metric on this space has been discussed in detail in [32,33]. We expect non-perturbative corrections to change the geometry of this quaternionic manifold. In principle there could be two types of corrections: one type which is visible even if we take the $M_p \to \infty$ limit and the other type consisting of these gravitational corrections.
This is the case, for example, with the $N = 2$ Coulomb branch of the heterotic strings where the field theory results $^{34}$ get embedded in string theory and can be isolated from the further gravitational corrections $^{35,36}$. In particular, we consider the limit in which $M_p \to \infty$ at the same time as we approach the conifold point. Note that $M_p$ is related to the string coupling constant of type IIA theory which we write as $S + \bar{S}$. In the limit we are considering, we take $S + \bar{S} \gg 1$ at the same time as we approach the conifold point. In this connection it is helpful to recall that in the rigid case the quaternionic manifold is replaced by a hyper-Kähler manifold.

A particular example where we expect there to be strong corrections is for the type IIA theory near the conifold point where Euclidean membranes wrapped around vanishing three-cycles are expected to deform the geometry of the quaternionic moduli space $^{13}$. Let us review the singularity of the hypermultiplet moduli in this context. The most relevant part of the singularity is associated with one complex modulus of the Calabi–Yau, which we will denote by $z$, together with a two dimensional toroidal subspace of the intermediate Jacobian, corresponding to the vanishing cycle and its dual cycle. If we denote the complex structure of this torus by $\tau$, the statement that we have a vanishing three-cycle at $z = 0$ is, via monodromy considerations, the same as the statement that the geometry of the singular space is described by the elliptic fibration

$$\tau(z) = \frac{1}{2\pi i} \log z. \quad (6.1)$$

This description sounds very similar to the stringy cosmic string considered in $^{18}$ where at $z = 0$ there is a cosmic string. Recall that in $^{18}$ a real two-dimensional torus is fibered non-trivially over a one-complex dimensional base, and, in the simplest case, the torus degenerates as in (6.1) near $z = 0$, and hence we presently find ourselves in an analogous situation. Actually more is true: If we write the hyper-Kähler metric on this space using the result $^{33}$, the leading singularity agrees identically with the metric found in $^{18}$ and the Kähler form given by

$$k = \tau_2 d\zeta d\bar{\zeta} + \partial \bar{\partial} \frac{(\zeta - \bar{\zeta})^2}{2(S + \bar{S})\tau_2},$$

where

$$\tau_2 = \text{Im} \tau = \frac{1}{4\pi} \log z \bar{z}$$

and $\zeta$ is a local holomorphic coordinate on the torus with $\zeta \sim \zeta + 1 \sim \zeta + \tau$. As discussed in $^{18}$, this metric is hyper-Kähler everywhere, however its Riemann tensor has singularities.
at $z = 0$, which is the point where the string theory perturbation theory breaks down. Note that the Kähler class of the fiber torus is $1/(S + \bar{S})$ and so in the weak coupling limit it goes to zero.

The case mainly dealt with in this paper, namely type IIB with $N$ homologous simultaneously vanishing two-cycles, corresponds to type IIA with $N$ homologous simultaneously vanishing three-cycles. In this case the relevant singular part of the hypermultiplet moduli is again the same as given above, except that now

$$\tau = \frac{N}{2\pi i} \log z. \quad (6.2)$$

This is a configuration analogous to $N$ stringy cosmic strings which have come together at $z = 0$.

The mathematical version of this degeneration follows immediately from the intermediate Jacobian formalism introduced above. Namely, let’s call the $N$ homologous vanishing three-cycles $\gamma_1, .., \gamma_N$. Consider encircling the locus in $\mathcal{M}$ where this degeneration occurs. By the theorem of [37], if $\gamma'$ is any other three-cycle, it experiences the monodromy transformation

$$\gamma' \to \gamma' + \sum_{i=1}^{N} (\gamma' \int \gamma_i) \gamma_i \quad (6.3)$$

upon traversing such an encircling path. Now, since all of the $\gamma_i$ are homologous, if $\gamma'$ is a dual cycle to the common homology class $[\gamma]$ of the $\gamma_i$, (6.3) implies that

$$[\gamma'] \to [\gamma'] + N[\gamma]. \quad (6.4)$$

Again, therefore, we see that this monodromy relation implies that the local form of $\tau$ must be as in (6.2).

From the field theory analysis, which should give an accurate picture of the hypermultiplet moduli in the limit in which we turn off gravity, we expect the relevant singularity of the space be $\mathbb{C}^2/\mathbb{Z}_N$. The question we now wish to address is how this fits with the perturbative description of this space given above. This is relatively clear in that as $N$ cosmic strings come together, the corresponding family of tori acquires an $A_{N-1}$ singularity in the fiber at $z = 0$. As discussed in [18], if we perturb the metric to give the Kähler class of

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4 We use the “Weierstrass model” for this family, in which the size of the torus is the only allowed Kähler parameter.
the fiber a finite size, there is a unique Ricci-flat hyper-Kähler representative of the perturbed metric; this will reproduce the expected singularity behavior $\mathcal{O}^2/\mathbb{Z}_N$. This differs from the metric we gave at tree-level (with $(S + \bar{S}) \to \infty$), but the corrections needed to reproduce a metric of the form $\mathcal{O}^2/\mathbb{Z}_N$ are uniquely determined by the value of the Kähler class of the fiber which we identify with $1/(S + \bar{S})$. We conjecture that this is exactly the quantum string corrected metric. This in particular implies that the singularity of the metric which appeared for $z = 0$ along the whole torus for $N > 1$ is replaced by a point in the fiber with $\mathcal{O}^2/\mathbb{Z}_N$ singularity in accord with field theory expectations. Note that at very weak coupling the exact metric becomes arbitrarily close to the stringy cosmic string metric. In this limit the hyper-Kähler metric of K3 becomes the stringy cosmic string metric. This actually is very much the sense in which F-theory on K3 is connected with M-theory on K3 upon compactification on a circle [17].

Even though we did not directly study the $N = 1$ case in this paper, i.e. the type IIA near a conifold singularity, given the fact that the Euclidean membrane corrections are of the same type [13], we are led to conjecture that we get the same resolution in this case as well. Note however that in this case the actual moduli space will have no left over singularity.

In fact we can motivate this discussion in another way. We use the results in [39,40] to replace the conifold in type IIA given by

$$xy = z, \quad uv = z - \mu$$

with type IIB with a five-brane at $z = \mu$ which is wrapped around the elliptic fiber given by $x, y$ over the $z$ plane. Now we use the fact that at strong coupling the symmetric five-brane is equivalent to Dirichlet five-brane of the D-string. If we dualize twice on the fiber of the $T^2$ the moduli problem at hand is given by a three-brane in the geometry $xy = z$. The moduli of this theory should be identified with positions of the three-brane, which can move along the space $xy = z$; the smooth metric is the same as that on the $xy = z$ fibration, which has a manifestly non-singular hyper-Kähler metric as we discussed above.

As mentioned above this is the leading correction in the limit of ignoring gravitational effects. When we turn on gravity we do expect that the qualitative features we have found will persist but the actual metric will get corrected in a non-universal way, depending on how the vanishing cycle sits in the Calabi–Yau, and the total space will be a quaternionic manifold.

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5 The quantum corrections we have found in this four-dimensional string theory are similar to the ones encountered in certain three-dimensional field theory systems recently studied [38].
7. Extensions and Speculation

It may appear that the geometric description of confinement we have found only corresponds to abelian confinement. However, we should recall that in the work of [34] the confinement of \( N = 1 \) \( SU(2) \) gauge theory can be continuously connected, upon adding a massive adjoint, to the Higgsing of the magnetic charge.\(^6\) In fact we believe the geometric picture we have found fits well with the geometrization of the Seiberg–Witten system found in [41]. In particular, it was found that the BPS states of the \( N = 2 \) Yang–Mills theory correspond to wrapping of one-branes around cycles of a Riemann surface. This one-brane itself can be viewed as part of the three-brane of type IIB partially wrapped around two-cycles in the internal space. The electric BPS states correspond to wrappings of this one-brane around the A-cycle and magnetic BPS states to wrappings around the B-cycle. In the limit where the B-cycle vanishes one gets massless magnetic monopoles. If we add the mass deformation to the adjoint breaking \( N = 2 \rightarrow N = 1 \) it is natural to expect that the Seiberg–Witten torus opens up around the pinched point. Following the discussions of this paper, the electrically charged BPS states are confined as they now correspond to wrappings of one-chains. In particular, pairs of them should have the same confining description as we have found above—namely, one-branes which wrap homologically trivial one-cycles in the total space, with the part of the one-brane passing through uncompactified space being identified as the flux tube between the electrically charged states (see figure 7, in which once again the base represents spacetime and the vertical direction represents the internal space)).

We can also extend the discussion of magnetically charged states we have found beyond the cases which lead to topological transitions. Namely, if we wrap D-branes around chains which end on supersymmetric cycles (which in principle can have more than two boundaries) we can identify them with charged states in the confining phase. In principle there are a large number of them in Calabi–Yau compactifications. For example if we consider type IIB on quintic threefold there are generically 2875 degree one holomorphic curves. Since they are all in the same homology class we can connect pairs of them with three-chains and identify these with charged states in the confining phase. Or, for example, we can have a three-chain which ends on three two-cycles, two corresponding to degree one maps and one corresponding to a degree two rational curve, with appropriate orientations.

\(^6\) The picture of confinement we present here was initiated during conversations with Nathan Seiberg.
Figure 7. A Seiberg–Witten torus opening up to a sphere, with one of its one-cycles becoming a chain.

Note that even though we have mainly concentrated on electric/magnetic particles in this paper, the same considerations apply to higher $p$-branes. In particular when the $p$-brane is in the Higgs phase the dual $(d - p - 4)$-brane is confined \cite{12} and this can be geometrically realized along the lines we have discussed. In particular, the confined $p$-brane will correspond to a $(p + r)$-brane wrapped on an $r$-chain. An interesting example of this arises in compactifications of M-theory on Calabi–Yau threefolds down to five dimensions. In fact this would be related to what we have studied in the strong coupling limit of type IIA theory on the same Calabi–Yau. If we consider type IIA where some homologous $S^3$’s vanish, by wrapping the M-theory five-brane around four-chains whose boundaries are vanishing $S^3$’s we get strings in five dimensions which are confined by nearly tensionless membranes flux sheets corresponding to wrapping pairs of five-brane around pairs of vanishing $S^3$’s.

It is also possible to study other types of Coulomb/Higgs transitions along these lines. In particular the transitions involving the small $E_8$ instantons in connection with vanishing $E_8$ del Pezzo in CY should be extremely interesting to study in connection with the considerations of this paper.

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