CANONICAL BASIS AND MACDONALD POLYNOMIALS

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To Howard Garland on his 60th birthday

Abstract. In the basic representation of $U_q(\widehat{sl}_2)$ realized via the algebra of symmetric functions we compare the canonical basis with the basis of Macdonald polynomials with $t = q^2$. We show that the Macdonald polynomials are invariant with respect to the bar involution defined abstractly on the representations of quantum groups. We also prove that the Macdonald scalar product coincides with the abstract Kashiwara form. This implies, in particular, that the Macdonald polynomials form an intermediate basis between the canonical basis and the dual canonical basis, and the coefficients of the transition matrix are necessarily bar invariant. We also discuss the positivity and integrality of these coefficients. For level $k$, we expect a similar relation between the canonical basis and Macdonald polynomials with $q^2 = t^k$.

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1. Introduction.

Since G. Lusztig [L] and M. Kashiwara [K] introduced canonical bases of highest weight representations of Kac–Moody algebras, there has been a significant effort to find their explicit algebraic description. The problem becomes especially intriguing for the simplest affine Lie algebra $\widehat{sl}_2$. It is known that the two basic representations of $\widehat{sl}_2$ have a realization in the space of symmetric functions in infinitely many variables $\mathbb{C}[a_{-1}, a_{-2}, \ldots]$, tensored with the extra space (of “sectors”) $\oplus \mathbb{C} e_n, n \in \mathbb{Z}$ or $n \in \mathbb{Z} + \frac{1}{2}$ [FK], [S], and this construction admits a $q$–deformation [FJ]. Therefore the canonical basis yields a class of symmetric polynomials depending on a parameter $q$. Moreover, integrable representations of level $k$ for $\widehat{sl}_2$, having in general more complicated structure, contain natural subspaces generated by $e(z)^k, f(z)^k$, that also have a symmetric space realization [DF]. Again, this construction admits a $q$–deformation and leads to a two parameter family of symmetric functions.

A few years before the discovery of the canonical basis, I. Macdonald [M] found a remarkable two parameter family of symmetric functions that includes as a special case practically all known classical symmetric functions. A representation theoretic interpretation of Macdonald polynomials in terms of certain spherical functions for the quantum groups $U_q(sl_n)$ was given in [EK]. A vertex operator approach to Hall-Littlewood and some Macdonald polynomials can be found in [J1], [J2]. Macdonald polynomials can also be viewed as a basis of the symmetric space $\mathbb{C}[a_{-1}, a_{-2}, \ldots]$, and it is natural to question its relation to the canonical basis for the quantum affine algebra $U_q(\widehat{sl}_2)$.

To compare the two bases we recall a characterization of the canonical basis given by Kashiwara [K]. The elements of the canonical basis are determined up to a sign by the following properties: 1) they are invariant with respect to a bar involution, an operation defined on any highest weight representation of the quantum
affine algebra associated to a Kac–Moody algebra, 2) they are orthogonal modulo $q^{-1}\mathbb{Z}[q^{-1}]$ with respect to a bilinear form, defined by Kashiwara on any highest weight representation and, 3) they belong to the lattice of divided powers of the quantum group over $\mathbb{Z}[q, q^{-1}]$. It turns out that the basis of Macdonald polynomials satisfy similar (and in a certain sense simpler) properties than the ones that characterize the canonical basis. We show that the Macdonald polynomials form an “intermediate basis” between the dual canonical basis and the canonical basis with respect to the Kashiwara form. We also show that the transition matrix between the basis of Macdonald polynomials and the canonical basis (after a minor rescaling) is bar invariant, integral, and independent of the lattice point $e_n$. We conjecture its positivity.

In this paper we only consider the level 1 representations, and this corresponds to the relation $t = q^2$ (or $q = t^2$) between the two parameters of the Macdonald polynomials. An arbitrary level $k$ representation will be considered in a sequel to this paper. In Section 2, we recall basic facts about the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ and its representations. We also give a definition and characterization of the canonical basis. In Section 3, we give the loop–like presentation of the quantum affine algebra and realization of basic representations in the space of symmetric functions. We then introduce certain intertwining operators for $U_q(\hat{\mathfrak{sl}}_2)$ and express them via quantum vertex operators. In Section 4, we recall the definition and properties of Macdonald polynomials and identify their generating functions as “one–half” of quantum vertex operators.

In the next three sections we study the properties of the Macdonald polynomials that correspond to the three characteristic properties of the canonical basis, and establish how the two structures are related. It is the quantum vertex operator, viewed as an intertwining operator for $U_q(\hat{\mathfrak{sl}}_2)$, and as a generating function for Macdonald polynomials, which provides a bridge between the two theories. In Section 5, we prove the invariance of Macdonald polynomials (rescaled by appropriate powers of $q$, depending on sector) under bar conjugation. In Section 6, we establish the coincidence of the Macdonald and Kashiwara forms. This implies that the Macdonald polynomials are orthogonal with respect to the Kashiwara form, and that the transition matrix between the dual canonical basis and the canonical basis for a fixed weight admits a decomposition

$$A(q)^t D(q) A(q),$$

where $A(q) = \{a_{\lambda,\mu}\}$ is a bar invariant matrix, and $D(q)$ is diagonal and bar invariant. The matrix $A(q)$ is precisely the transition matrix between the canonical basis and the basis of Macdonald polynomials. The coincidence of the Macdonald and Kashiwara forms implies that the matrix $A(q)$ depends trivially on the choice of a sector. In Section 7, we show that the $\mathbb{Z}[q, q^{-1}]$–lattice spanned by the dual integral Macdonald polynomials (rescaled Macdonald polynomials introduced in $[A]$) form a lattice in the basic representation which is invariant under the divided powers action. Finally, in Section 8, we show that the integral Macdonald polynomials belong to the lattice of divided powers of the quantum group over $\mathbb{Z}[q, q^{-1}]$. This implies the integrality of the rescaled matrix coefficients, and we also conjecture their positivity.

Our results on the relation between canonical basis and Macdonald polynomials open new avenues of research in both subjects. On one hand, Macdonald polynomials suggest a definition of “intermediate canonical basis” in any integrable highest
2. Preliminary material.

2.1. Quantum algebras. Let \( A = \mathbb{Z}[q, q^{-1}] \). For \( n \in \mathbb{N} \) we define
\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1]\ldots[1], \quad [0]! = 1,
\]
Let the (derived) quantum affine algebra \( U_q(\widehat{\mathfrak{sl}_2}) \) be generated over \( \mathbb{C}(q) \) by \( E_i, F_i, K_i^{\pm 1}, i = 0, 1 \), with relations:
\[
K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},
\]
\[
E_i^{(3)} E_j - [3] E_i^{(2)} E_j E_i + [3] E_i E_j E_i^{(2)} + E_j E_i^{(3)} = 0, \quad i \neq j,
\]
\[
F_i^{(3)} F_j - [3] F_i^{(2)} F_j F_i + [3] F_i F_j F_i^{(2)} + F_j F_i^{(3)} = 0, \quad i \neq j,
\]
where \( a_{ii} = -a_{ij} = 2 \) for \( i \neq j \).

Let \( E_i^{(n)} = \frac{E_i^n}{[n]!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]!} \). The quantum algebras have an \( A \)-form \( U_A \), generated over \( A \) by \( E_i^{(n)}, F_i^{(n)}, \) and \( K_i^{\pm 1} \). We define a Hopf algebra structure on \( U_q(\widehat{\mathfrak{sl}_2}) \) with coproduct \( \Delta \), antipode \( S \), and counit \( \eta \) given by:
\[
\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{\pm 1}, \quad \Delta(K_i) = K_i \otimes K_i,
\]
\[
S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1},
\]
\[
\eta(E_i) = 0, \quad \eta(F_i) = 0, \quad \eta(K_i) = 1.
\]

This coproduct structure is not unique. Given an (anti)–automorphism of the algebra \( \sigma \), the map \( (\sigma \otimes \sigma) \circ \Delta \circ \sigma^{-1} \) is also a coproduct.

2.2. Representations. The free abelian group \( X \) on \( \{ \Lambda_0, \Lambda_1, \delta \} \) is called the weight lattice with fundamental weights \( \Lambda_i, \ i = 0, 1 \). Define the simple roots \( \alpha_0, \alpha_1 \) by
\[
\alpha_0 + \alpha_1 = \delta, \quad \Lambda_1 = \Lambda_0 + (\alpha_1/2).
\]

For \( \lambda \in X^+ \), let \( V = V(\lambda) \) be the irreducible integrable representation of \( U_q(\widehat{\mathfrak{sl}_2}) \) with highest weight \( \lambda \). We denote by \( V(\lambda)_A \) the image of the canonical map \( U_A \to V_\lambda, \ V \) is graded by \( X^+ \) and decomposes as \( V = \bigoplus_\nu V_\nu \). An element \( x \in V_\nu \) is said to be of weight \( \nu \). Denote by \( \hat{V} \) the completion of \( V \) with respect to this homogeneous grading.

In addition to the above highest weight representations, we will also make use of the two dimensional evaluation representation \( \mathbb{1} \) of \( U_q(\mathfrak{sl}_2) \). Let \( U_q(\mathfrak{sl}_2) \) be the Hopf
subalgebra of $U_q(\mathfrak{sl}_2)$ generated over $\mathbb{Q}(q)$ by $E_1, F_1$ and $K_1^{\pm 1}$. Let $V_1 = \mathbb{C}v_+ \oplus \mathbb{C}v_-$. Define the standard two dimensional representation of $U_q(\mathfrak{sl}_2)$ by:

$$E_1 v_+ = v_+, \quad E_1 v_- = 0, \quad F_1 v_+ = 0, \quad F_1 v_- = q^{\pm 1} v_+.$$ \( \tag{2.1} \)

Let $z$ be an indeterminate and consider $V_z = V \otimes \mathbb{C}[z, z^{-1}]$. We define a $U_q(\mathfrak{sl}_2)$ action as follows:

$$E_0(v_z \otimes z^m) = (F_1 v_z) \otimes z^{m+1}, \quad E_1(v_z \otimes z^m) = (E_1 v_z) \otimes z^m,$$

$$F_0(v_z \otimes z^m) = (E_1 v_z) \otimes z^{m-1}, \quad F_1(v_z \otimes z^m) = (F_1 v_z) \otimes z^m,$$

$$K_1(v_z \otimes z^m) = (K_1 v_z) \otimes z^m, \quad K_0(v_z \otimes z^m) = K_1^{-1}(v_z \otimes z^m),$$

where $\varepsilon = \pm$.

**Definition 1.** The algebra $U_q(\mathfrak{sl}_2)$ at level $k$ is defined as

$$U_{q,k}(\mathfrak{sl}_2) = \frac{U_q(\mathfrak{sl}_2)}{(C - q^k)U_q(\mathfrak{sl}_2)}.$$

There is a natural surjection from any representation $V$ of $U_q(\mathfrak{sl}_2)$ to a corresponding representation of the algebra $U_{q,k}(\mathfrak{sl}_2)$.

2.3. Canonical Basis. Introduce the $\mathbb{Q}(q)$–linear anti–involution $\rho$, and the $\mathbb{Q}$–linear involution $\overline{\cdot}$ of $U_q(\mathfrak{sl}_2)$ by:

$$\rho(E_i) = qK_i F_i, \quad \rho(F_i) = qK_i^{-1} E_i, \quad \rho(K_i) = K_i,$$

$$E_i = E_i, \quad F_i = F_i, \quad K_i = K_i^{-1}, \quad \overline{c} = q^{-1}c.$$ \( \tag{2.2} \) \( \tag{2.3} \)

**Proposition 1.** For $\lambda \in X^+$, let $V = V(\lambda)$. There is a unique bilinear form $\langle \ , \, \rangle : V \times V \to \mathbb{Q}(q)$ such that:

$$\langle v_\lambda, v_\lambda \rangle = 1,$$

$$\langle ux, y \rangle = (x, \rho(u)y) \text{ for all } x, y \in V \text{ and } u \in U_q.$$ \( \tag{2.4} \) \( \tag{2.5} \)

This bilinear form is symmetric. If $x \in V_\nu, y \in V_{\nu'}$ with $\nu \neq \nu'$, then $\langle x, y \rangle = 0$.

We cite the following characterization of the canonical basis given by Kashiwara:

**Theorem 1.** Let $b \in V(\lambda)$. Then either $b$ or $-b$ is in the canonical basis if and only if

1. $b = b$, and
2. $(b, b') = \delta_{b, b'} \text{ mod } q^{-1} \mathbb{Z}[q^{-1}]$. 

3. Realization of basic representation and intertwiners

3.1. Loop–like realization. The algebra $U_q(\mathfrak{sl}_2)$ has another set of loop–like generators.

**Proposition 2.** $U_q(\mathfrak{sl}_2)$ is isomorphic to the algebra generated over $\mathbb{Q}(q)$ on the generators $a_n$ ($n \in \mathbb{Z} \setminus \{0\}$), $x_k^\pm$ ($k \in \mathbb{Z}$), $C^\pm 1$, $K^{\pm 1}$ and the following
relations:

\[ [a_n, a_m] = \delta_{m,-n} \frac{n}{2n} \frac{C^n - C^{-n}}{q - q^{-1}} \]
\[ K a_n K^{-1} = a_n, \ K x_k^\pm K^{-1} = q^{\pm 2} x_k^\pm, \]
\[ [a_1, x_k^\pm] = \pm C^{(l)/2} x_k^\pm, \]
\[ x_{k+1}^\pm x_l^\pm - q^{\pm 2} x_l^\pm x_{k+1}^\pm = q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm, \]
\[ [x_k^+, x_l^-] = \frac{C^{(l-1)/2} \psi_k + 1 - C^{(l-k)/2} \phi_{k+1}}{q - q^{-1}}, \]

where \( C \) is a central element and

\[ \sum_{k=0}^\infty \psi_k z^{-k} = K \exp \left( (q - q^{-1}) \sum_{k=1}^\infty \frac{[2k]}{k} a_k z^{-k} \right), \]
\[ \sum_{k=0}^\infty \varphi_k z^k = K^{-1} \exp \left( -(q - q^{-1}) \sum_{k=1}^\infty \frac{[2k]}{k} a_{-k} z^k \right). \]

The isomorphism is determined by mapping the respective generators as follows:

\[ K_1 \mapsto K, \ E_1 \mapsto x_0^+, \ F_1 \mapsto x_0^-, \]
\[ K_0 \mapsto CK^{-1}, \ E_{01}K_1 \mapsto x_1^-, \ K_1^{-1}F_0 \mapsto x_1^+. \]

The isomorphism is explained completely in terms of a braid action on \( U_q(\widehat{\mathfrak{sl}_2}) \) in [3]. Forming generating series from the loop–like generators by

\[ X^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n-1}, \]

the defining relations are written as:

\[ [a_k, X^\pm(z)] = \pm C^{(k)/2} z^k X^\pm(z), \]
\[ (z - q^{\pm 2} w) X^\pm(z) X^\pm(w) + (w - q^{\pm 2} z) X^\pm(w) X^\pm(z) = 0, \]
\[ [X^+(z), X^-(w)] = K \exp \left[ (q - q^{-1}) \sum_{k=1}^\infty \frac{[2k]}{k} a_k C^{k/2} z^{-k} \right] \frac{\delta(z/w)}{(q - q^{-1}) z w} \]
\[ - K^{-1} \exp \left[ -(q - q^{-1}) \sum_{k=1}^\infty \frac{[2k]}{k} a_{-k} C^{k/2} z^k \right] \frac{\delta(C z/w)}{(q - q^{-1}) z w}, \]

where as usual \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \).

3.2. Basic Representation. Now we consider \( U_{q,1}(\widehat{\mathfrak{sl}_2}) \). For \( i = 0, 1 \), let \( V(\Lambda_i) \) be the unique irreducible highest weight representation with highest weight \( \Lambda_i \).

Let

\[ V'(\Lambda_i) = \mathbb{C}[a_{-n}, \ n > 0] \oplus (\oplus_{n \in \mathbb{Z}} \mathbb{C} e^{\Lambda_i + n \alpha}), \ i = 0, 1. \]
For \(a_n(n \neq 0), \ e^\alpha, \ \partial\) define an action on \(V'(\Lambda_i)\) as follows:

\[
(3.9) \quad a_n(f \otimes e^\beta) = a_n f \otimes e^\beta, \text{ if } n < 0, \\
= [a_n, f] \otimes e^\beta, \text{ if } n > 0, \\
e^\alpha(f \otimes e^\beta) = f \otimes e^{\beta+\alpha}, \\
\partial(f \otimes e^\beta) = (\alpha, \beta) f \otimes e^\beta,
\]

where \(f \in \mathbb{C}[a_{-n}, \ n > 0]\) and \(\beta = \Lambda_i + n\alpha\).

**Theorem 2.** The representations \(V'(\Lambda_i)\) and \(V(\Lambda_i)\) are isomorphic. The action of the loop-like generators on \(V'(\Lambda_i)\) are determined by the following relations:

\[
K = q^\beta, \ C = q, \\
X^\pm(z) = \exp(\pm \sum_{n=1}^{\infty} a_{-n} q^{\frac{n}{2}} (q^n + q^{-n}) z^n) \times \\
\exp(\mp \sum_{n=1}^{\infty} a_{n} q^{\frac{n}{2}} (q^n + q^{-n}) z^{-n}) e^{\pm \alpha} z^{\pm \partial}.
\]

3.3. Quantum Vertex Operators. The tensor product of any two representations of \(U_q(\widehat{\mathfrak{sl}_2})\) is defined via a coproduct. Vertex operators of type I are intertwining homomorphisms of the following form:

\[
(3.11) \quad \Phi^{(1-i,i)}(z) : V(\Lambda_i) \rightarrow V(\Lambda_{1-i}) \otimes V_z.
\]

The precise meaning of this map is as follows:

\[
\Phi^{(1-i,i)}(z) = \sum_{\varepsilon = +, -} \Phi^{(1-i,i)}_\varepsilon(z) \otimes v_\varepsilon,
\]

where \(\Phi^{(1-i,i)}_\varepsilon(z) = \sum_{n \in \mathbb{Z}} \Phi^{(1-i,i)}_{\varepsilon,n} z^{-n}\).

Each \(\Phi^{(1-i,i)}_{\varepsilon,n}\) is defined to be a linear map

\[
\Phi^{(1-i,i)}_{\varepsilon,n} : V(\Lambda_i) \rightarrow V(\Lambda_{1-i})
\]

which intertwines \(U_q(\widehat{\mathfrak{sl}_2})\) in the sense that:

\[
(3.12) \quad \sum_\varepsilon \Phi^{(1-i,i)}_{\varepsilon,n} x v \otimes (v_\varepsilon \otimes z^{-n}) = \Delta(x) \{ \sum_\varepsilon \Phi^{(1-i,i)}_{\varepsilon,n} v \otimes (v_\varepsilon \otimes z^{-n}) \}
\]

for all \(x \in U_q(\widehat{\mathfrak{sl}_2})\) and \(v \in V(\Lambda_i)\). Equivalently

\[
\Delta(x) \circ \Phi(z) = \Phi(z) \circ x \quad \text{for } x \in U_q(\widehat{\mathfrak{sl}_2}).
\]

Let \(|\Lambda_0\rangle \in V(\Lambda_0)\) and \(|\Lambda_1\rangle \in V(\Lambda_1)\) be the respective highest weight vectors. These operators are further normalized so that

\[
\Phi_-(z)|\Lambda_0\rangle = |\Lambda_1\rangle \otimes v_- + \text{lower weight terms in the first component}, \\
\Phi_+(z)|\Lambda_1\rangle = |\Lambda_0\rangle \otimes v_+ + \text{lower weight terms in the first component}.
\]

Composition of vertex operators is natural. For example,

\[
(3.14) \quad \Phi^{(0,1)}(z_1) \Phi^{(1,0)}(z_2) : V(\Lambda_0) \rightarrow V(\Lambda_0) \otimes V_{z_1} \otimes V_{z_2}.
\]
Defining $\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1)$ we have

$$\Phi := \Phi^{(1-i,i)}(z) : \mathcal{H} \rightarrow \mathcal{H} \otimes V_z,$$

Define the “dual” vertex operators as the unique intertwiners of the form:

$$\Phi^*(1-i,i)(z) : V(\Lambda_1-i) \otimes V_z \rightarrow V(\Lambda_i), \quad (3.15)$$

with the components

$$\Phi^*_n(1-i,i)(z)v = \Phi^*_n(1-i,i)(v \otimes v),$$

and the normalization given by:

$$\Phi^*_+(z)|\Lambda_0\rangle = |\Lambda_1\rangle \otimes v_+ + \text{lower weight terms in the first component},$$

$$\Phi^*_-(z)|\Lambda_1\rangle = |\Lambda_0\rangle \otimes v_- + \text{lower weight terms in the first component}.$$

**Theorem 3.** [FR] Fix a coproduct $\Delta$ of $U_q(\hat{sl}_2)$. The vertex operator $\Phi^{(1-i,i)}(z)$ exists and is uniquely determined by highest weight normalization. The product of vertex operators

$$\Phi_{\epsilon_1}(z_1) \ldots \Phi_{\epsilon_n}(z_n) \quad (3.16)$$

has analytic matrix elements in the region $|z_1| \gg \cdots \gg |z_n|$, and extends to a meromorphic function in $(\mathbb{C} \setminus \{0\})$. Its highest component to highest component matrix elements satisfy the quantum Knizhnik–Zamolodchikov equation.

**3.4. Construction of type I vertex operators.** In [JM] the type I vertex operator corresponding to the coproduct $\Delta$ is constructed. We recall

**Proposition 3.** [JM] The vertex operators $\Phi_\pm$ have the following explicit expressions in $\mathcal{H}$:

$$\Phi^{(1-i,i)}_-(z) = \exp\left(\sum_{n=1}^{\infty} \frac{q^{7n/2}}{n} a_{-n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{q^{-5n/2}}{n} a_n z^{-n}\right) e^{\alpha/2} (-q^3 z)^{(\Theta+1)/2}, \quad (3.17)$$

$$\Phi^{(1-i,i)}_+(z) = \Phi^{(1-i,i)}_-(z)x_0 - qx_0 \Phi^{(1-i,i)}_-(z), \quad (3.18)$$

where $I$ is the operator on $V(\Lambda_i)$ such that $I(i) = i$.

In order to calculate the bar action on $\mathcal{H}$, we define $\overline{\Delta}$ to be the coproduct conjugated with the bar operator $\overline{(2,3)}$. Explicitly:

$$\overline{\Delta}(K_i) = K_i \otimes K_i, \quad \overline{\Delta}(E_i) = K_i^{-1} \otimes E_i + E_i \otimes 1,$$

$$\overline{\Delta}(F_i) = F_i \otimes K_i + 1 \otimes F_i, \quad i = 0, 1. \quad (3.19)$$

We now construct the vertex operators $\overline{\Phi}_+, \overline{\Phi}_-$ corresponding to the coproduct $\overline{\Delta}$. 
Let \( \hat{x}_k^+ = x_k^+ K \), \( \hat{x}_k^- = K^{-1} x_k^- \). Using (3.12) with \( x \) equal to \( K, E_i, F_i \) (\( i = 0,1 \)) respectively, the following must hold:

\[
\begin{align*}
(3.20) & \quad K \Phi_\pm K^{-1} = q^{\mp 1} \Phi_\pm, \\
(3.21) & \quad \Phi_+(z) \hat{x}_0^+ = q \hat{x}_0^+ \Phi_+(z) + q^{-1} \Phi_-(z), \\
(3.22) & \quad \Phi_-(z) \hat{x}_0^+ = q^{-1} \hat{x}_0^+ \Phi_-(z), \\
(3.23) & \quad \Phi_+(z) \hat{x}_0^- = \hat{x}_0^- \Phi_+(z), \\
(3.24) & \quad \Phi_-(z) \hat{x}_0^- = \hat{x}_0^- \Phi_-(z) + qK^{-1} \Phi_+(z), \\
(3.25) & \quad \Phi_+(z) \hat{x}_1^+ = q^{-1} \hat{x}_1^+ \Phi_+(z) + q^{-2} z^{-1} \Phi_-(z), \\
(3.26) & \quad \Phi_-(z) \hat{x}_1^+ = q \hat{x}_1^+ \Phi_-(z), \\
(3.27) & \quad \Phi_+(z) \hat{x}_1^- = \hat{x}_1^- \Phi_+(z), \\
(3.28) & \quad \Phi_-(z) \hat{x}_1^- = \hat{x}_1^- \Phi_-(z) + z \Phi_+(z) K_1.
\end{align*}
\]

We show these relations together with (3.13) determine \( \Phi_\pm(z) \) uniquely. Assuming (3.20) through (3.28) we show

**Lemma 1.**

\[
\begin{align*}
(3.29) & \quad X^-(w) \Phi_+(z) - q \Phi_+(z) X^-(w) = 0, \\
(3.30) & \quad [a_n, \Phi_+(z)] = \frac{-q^{-n/2}}{q^n + q^{-n}} z^n \Phi_+(z), \\
(3.31) & \quad [a_{-n}, \Phi_+(z)] = \frac{-q^{-(n/2)}}{q^n + q^{-n}} z^{-n} \Phi_+(z).
\end{align*}
\]

**Proof.** For the proof of (3.30) and (3.31), first calculate:

\[
\begin{align*}
[a_1, \Phi_+] &= \left( C^{1/2} K^{-1} \frac{1}{q + q^{-1}} [x_0^+, x_1^-], \Phi_+ \right) = C^{1/2} K^{-1} \left( \frac{1}{q + q^{-1}} \frac{1}{q + q^{-1}} K^{-1} [x_0^+, x_1^-] \Phi_+ - \Phi_+ K^{-1} [x_0^+, x_1^-] \right) \\
&= C^{1/2} K^{-1} \left( \frac{1}{q + q^{-1}} (x_0^+ x_1^- - x_1^- x_0^+ \Phi_+ - K^{-1} [x_0^+, x_1^-]) \right) \\
&= C^{1/2} K^{-1} \left( \frac{1}{q + q^{-1}} (x_0^+ x_1^- - x_1^- x_0^+ \Phi_+ - x_1^- x_0^+ - x_0^+ x_1^-) \right) \\
&= C^{1/2} K^{-1} \left( \frac{1}{q + q^{-1}} (x_0^+ x_1^- - x_1^- x_0^+ \Phi_+ - K^{-1} [x_0^+, x_1^-]) - q^{-1} x_0^+ x_1^- + q^{-1} x_0^+ x_1^- \right) \\
&= C^{1/2} K^{-1} \left( \frac{1}{q + q^{-1}} (-q^{-1} K^{-1} \Phi_+ x_1^- + q^{-2} K^{-1} x_1^- \Phi_+) \right) \\
&= C^{1/2} K^{-1} \left( \frac{1}{q + q^{-1}} (-q^{-1} K^{-1} \Phi_+ x_1^- + q^{-2} x_0^+ \Phi_+) \right) = -C^{1/2} q^{-1} \Phi_+ z = \frac{-q^{-1/2} z}{q + q^{-1}} \Phi_+.
\end{align*}
\]

Similarly we have:

\[
[a_{-1}, \Phi_+] = \frac{-q^{-1/2} z^{-1}}{q + q^{-1}} \Phi_+.
\]

The relations (3.23) and (3.27) together with repeated use of (3.2) now yield

\[
(3.32) \quad X^-(w) \Phi_+(z) - q \Phi_+(z) X^-(w) = 0.
\]
Let
\[ A(w) = \exp((q - q^{-1}) \sum_{n=1}^{\infty} \frac{[2n]}{n} a_n q^{-n/2} w^{-n}), \]
\[ B(w) = \exp(-(q - q^{-1}) \sum_{n=1}^{\infty} \frac{[2n]}{n} a_n q^{-n/2} w^n). \]

We have the following two identities:
\[ (q - q^{-1})[X^-(w), x^+_0] = -w^{-1} K A(w) + w^{-1} K^{-1} B(w), \]
\[ (q - q^{-1})[X^-(w), x^+_{-1}] = -w^{-2}(-q^{-1} K A(w) + q K^{-1} B(w)). \]

Using (3.21) and (3.25) we obtain
\[ \text{(3.34)} \]

We have the following two identities:
\[ (q - q^{-1})[X^-(w), x^+_{-1}] = -w^{-2}(-q^{-1} K A(w) + q K^{-1} B(w)). \]

Using (3.21) and (3.25) we obtain
\[ \text{(3.35)} \]

Bracketing (3.35) with \((q - q^{-1})X^-(w)\) we obtain the identity
\[ qz(\Phi_+ x^+_0 - q^{-1} x^+_{-1} + \Phi_+) - (\Phi_+ x^+_{0} - q x^+_0 + \Phi) = 0. \]

Expanding this identity with respect to \(w\) we obtain (3.30) and (3.31) to complete the proof.

**Proposition 4.** The commutation relations (3.29), (3.31), (3.32), and (3.21) determine the operators \(\Phi_\pm(z)\).

\[ \Phi_+^{1-i,j}(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{q^{-n/2}}{n} a_n z^n \right) \exp\left(\sum_{n=1}^{\infty} \frac{q^{-n/2}}{n} a_n z^{-n} \right) \]
\[ \times e^{-\alpha/2}(-q)^{(-\partial + i)}/(-q)^{i-1}, \]
\[ \Phi_-(z) = K[\Phi_+(z), x^+_{-1}]. \]

**Proof.** The equations (3.32) and (3.33), together with the normalization (3.13) for \(\Phi_-(z)\) determine the operator \(\Phi_+(z)\) uniquely as given in (3.37). To complete the proof, the remaining relations must be checked. For (3.20) and (3.21), we use the following formulas, which are derived from the defining expression for \(\Phi_-\) (3.21).

\[ X^+(w) \Phi_+(z) = \frac{1}{w - q^{-1} z} : X^+(w) \Phi_+(z) : |z| \leq |qw|, \]
\[ \Phi_+(z) X^+(w) = \frac{1}{w - q z} : \Phi_+(z) X^+(w) : |q^{-1} w| \leq |z|, \]
\[ X^+(w_1) X^+(w_2) = w_1^2(1 - \frac{w_2}{w_1})(1 - \frac{q^{-1} w_2}{w_1}) : X^+(w_1) X^+(w_2) :. \]

Here we have introduced the normally ordered product on the \(\{a_n, \partial, \alpha | n \in \mathbb{Z} \setminus \{0\}\} \)

by
\[ a_k a_l = a_k a_l \text{ if } k < 0, \]
\[ a_l a_k = a_l a_k \text{ if } k > 0, \]
\[ : \alpha \partial : = : \partial \alpha : = \alpha \partial. \]
We note the following identities, where the path of integration is the boundary of suitable two dimensional disk about the origin:

\[ x_m^+ \Phi_+(z) x_n^+ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 w_1^n w_2^m X^+(w_1) \Phi_+(z) X^+(w_2) \]

\[ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 w_1^n w_2^m \frac{1}{w_1 - q^{-1}z} w_2^2 (1 - \frac{w_2}{w_1}) \]

\[ \times (1 - \frac{q^{-2}w_2}{w_1}) \frac{1}{w_2 - qz} : X^+(w_1) X^+(w_2) \Phi_+(z) : \]

\[ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 w_1^n w_2^m q^{-2}w_2 (1 - \frac{w_2}{w_1}) \]

\[ \times \left[ \frac{q^2}{w_2 - qz} - \frac{1}{w_1 - q^{-1}z} \right] : X^+(w_1) X^+(w_2) \Phi_+(z) : . \]

And similarly:

\[ x_m^n \Phi_+(z) x_n^+ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 X^+(w_1) X^+(w_2) \Phi_+(z) w_1^m w_2^n \]

\[ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 q^{-2}w_1^{m+1} w_2^n - w_1^m \frac{w_2 q^{-2}w_1 (1 - \frac{w_2}{w_1})}{w_1 - q^{-1}z} \]

\[ \times : X^+(w_1) X^+(w_2) \Phi_+(z) : . \]

And

\[ \Phi_+(z) x_m^+ x_n^+ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 \Phi_+(z) X^+(w_1) X^+(w_2) w_1^m w_2^n \]

\[ = \frac{1}{(2\pi i)^2} \int \int dw_1 dw_2 q^{-2}w_1^{m+1} w_2^n - w_1^m \frac{w_2 q^{-2}w_1 (1 - \frac{w_2}{w_1})}{w_1 - q^{-1}z} \]

\[ \times : X^+(w_1) X^+(w_2) \Phi_+(z) : . \]

A formula for (3.39) is determined explicitly via (3.21). Now (3.26) and (3.22) are seen to hold by using the defining expression for \( \Phi_- \) (3.27). Also, we can now check explicitly (3.24) and (3.28). For example,

\[ [\Phi - (z), x^-] = [q \Phi_+(z) x_0^+ - q^2 x_0^+ \Phi_+(z), x^-] \]

\[ = [2] C^{-1/2} (q \Phi_+(z) K a_1 - q^2 K a_1 \Phi_+(z)) = z \Phi_+(z) K, \]

gives (3.26).

\[ \square \]

3.5. Vertex operator action on the basic representation. We consider the action of \( \Phi_\pm(z), \Phi_\pm(z) \) on \( \mathcal{H}_A = V(\Lambda_0)_A \oplus V(\Lambda_1)_A \). This is given as follows:
Proposition 5. Let $m \geq 0, i = 0, 1$.

(3.44) \((q)^{\partial -1}\Phi_{-}(z)e^{m_\alpha}|\Lambda_i\rangle = \exp\left(\sum_{n=1}^{\infty} \frac{q^{-5n/2}}{n}a_{-n}z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{q^{3n/2}}{n}a_{n}z^{-n}\right)
\times e^{\alpha/2}(-q)^{(\partial+3I)/2}z^{(\partial+I)/2}e^{m_\alpha}|\Lambda_i\rangle,\) where if $m = 0, i = 1$,

(3.45) \((-q)^{1-\partial}\Phi_{-}(z)e^{\pm m_\alpha}|\Lambda_i\rangle = \exp\left(\sum_{n=1}^{\infty} \frac{q^{7n/2}}{n}a_{-n}z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{q^{-5n/2}}{n}a_{n}z^{-n}\right)
\times e^{\alpha/2}(-q)^{(\partial+3I)/2}z^{(\partial+I)/2}e^{\pm m_\alpha}|\Lambda_i\rangle.

The action of $\Phi_{+}$ and $\Phi_{-}$ is as follows:

(3.46) \((-q)^{-\partial}\Phi_{+}(z)e^{\pm m_\alpha}|\Lambda_i\rangle = \exp\left(-\sum_{n=1}^{\infty} \frac{q^{-n/2}}{n}a_{-n}z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{q^{n/2}}{n}a_{n}z^{-n}\right)
\times e^{-\alpha/2}(-q)^{(-\partial+3I)/2}z^{(-\partial+I)/2}e^{\pm m_\alpha}|\Lambda_i\rangle,

(3.47) \((-q)^{\partial}\Phi_{+}(z)e^{-m_\alpha}|\Lambda_i\rangle = \exp\left(-\sum_{n=1}^{\infty} \frac{q^{1n/2}}{n}a_{-n}z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{q^{-9n/2}}{n}a_{n}z^{-n}\right)
\times e^{-\alpha/2}(-q)^{(-\partial+3I)/2}z^{(-\partial+I)/2}e^{-m_\alpha}|\Lambda_i\rangle,\) where if $m = 0, i = 0$.

Proof. The proof of (3.47) follows directly from (3.17). We consider (3.44).

\[\Phi_{-}(z)e^{m_\alpha}|\Lambda_i\rangle = K[\Phi_{+}(z), x_{i+1}^{+}]e^{m_\alpha}|\Lambda_i\rangle = K \frac{1}{2\pi i} \int dw \Phi_{+}(z), X^{+}(w)|e^{m_\alpha}|\Lambda_i\rangle\]

using (3.44) and (3.41)

\[= -K \frac{1}{2\pi i} \int \frac{dw}{(w-q^{-1}z)} :\Phi_{+}(z)X^{+}(w) : e^{m_\alpha}|\Lambda_i\rangle\]

\[= -K :\Phi_{+}(z)X^{+}(w) : e^{m_\alpha}|\Lambda_i\rangle\]

\[= \exp\left(\sum_{n=1}^{\infty} \frac{q^{-5n/2}}{n}a_{-n}z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{q^{3n/2}}{n}a_{n}z^{-n}\right)
\times e^{\alpha/2}(-q)^{(\partial+3I)/2}z^{(\partial+I)/2}e^{m_\alpha}|\Lambda_i\rangle,\]

which gives (3.44), (3.46) and (3.47) are similar. \hfill \Box

4. Macdonald Polynomials

4.1. Partitions. As usual, by a partition we mean a sequence of non-negative integers in decreasing order

(4.1) \[\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots\]

containing finitely many non-zero terms. The number of non-zero $\lambda_i$ is called the length of $\lambda$, denoted by $\ell(\lambda)$, and each $\lambda_i$ is called a part. The sum $|\lambda| = \sum \lambda_i$ is called the weight of $\lambda$. Given a partition $\lambda$, if the part $i > 0$ appears $m_i$ times we write

\[\lambda = (1^{m_1}2^{m_2} \cdots r^{m_r} \cdots).\]
We define the integer
\[ z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! . \]
The dual partition \( \lambda' \) is defined by setting its parts as
\[ \lambda'_i = \text{Card}\{ j : \lambda_j \geq i \} . \]
We define the dominance partial ordering on partitions by setting \( \lambda \leq \mu \) if \( \sum_i \lambda_i = \sum_i \mu_i \) and \( \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \) for every \( k \). That this is a partial order is made clear by considering the two partitions \((1^3, 3)\), \((2^1)\), which are incomparable.

Denote by \( \Lambda^\Sigma \) the ring of symmetric functions in countably many variables \( \{ x_i \mid i \geq 1 \} \). Let \( S^\infty \) be the permutations of \( \mathbb{N} \) which fix a cofinite set. For a partition \( \lambda \) define the monomial symmetric functions
\[ m_\lambda = \sum_{\alpha=s\lambda, s \in S^\infty} \prod_i x_{s(i^\lambda_i)} . \]
(4.2)
For each \( r \geq 1 \) the \( r \)-th power sum is
\[ p_r = \sum_{i \geq 1} x_i^r . \]
The \( p_r \) \( (r \geq 1) \) form a polynomial basis of \( \Lambda_\mathbb{Q}^\Sigma = \Lambda^\Sigma \otimes \mathbb{Q} \). Define
\[ p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \]
for each partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \). The \( p_\lambda \), \( \lambda \) a partition, form a \( \mathbb{Q} \)-basis of \( \lambda_\mathbb{Q}^\Sigma \).

### 4.2. Properties of Macdonald polynomials
We recall some basic facts about Macdonald polynomials. We refer the reader to Chapter VI of \[M\] for further information. Let \( \mathbb{C}(q, t) \) be the field of rational functions in \( q \) and \( t \). Define a scalar product \( \langle \cdot, \cdot \rangle \) on \( \Lambda_\mathbb{Q}^\Sigma = \Lambda^\Sigma \otimes \mathbb{Q} \) by
\[ \langle p_\lambda, p_\mu \rangle = \langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} \prod_i \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} . \]
(4.5)
4.2.1. Let \( \{ x_i \}_{i \geq 0} \) be an infinite set of indeterminates.

**Theorem 4.** \[M\] Let \( \lambda \) be a partition. There exists a unique family of symmetric functions \( P_\lambda(x; q, t) \in \mathbb{C}(q, t)[x_1, x_2, \ldots] \) which satisfy the following properties:
1. \( P_\lambda \) is symmetric with respect to the \( x_i, i \geq 1 \).
2. \( P_\lambda = m_\lambda + \sum_{\lambda \leq \mu} K_{\lambda,\mu} m_\mu \), where \( K_{\lambda,\mu} \in \mathbb{Q}(q, t) \).
3. The \( P_\lambda \) are pairwise orthogonal relative to the scalar product \( \langle \cdot, \cdot \rangle \) and
\[ \langle P_\lambda, P_\mu \rangle = b^{-1}_{\lambda}(q, t) . \]
Here
\[ b_{\lambda}(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)} + 1 t^{l(s)}}{1 - q^{a(s)} + 1} . \]
(4.6)
where for each square \( s = (i, j) \) in the diagram of \( \lambda \) we have
\[ a(s) = a_\lambda(s) = \lambda_i - j, \]
\[ l(s) = l_\lambda(s) = \lambda'_j - i . \]
We also denote the numerator and denominator of \( b_{\lambda} \) by \( c_{\lambda}(q, t) \) and \( c'_{\lambda}(q, t) \) respectively. We refer to the form above as Macdonald’s form. Setting \( Q_\lambda(x; q, t) = \)
Let $b_\lambda(q,t)P\lambda(x; q, t)$ we see $Q\lambda$ is dual to $P\lambda$ with respect to Macdonald’s form. If $z$ is an indeterminate define

$$(z; q)_\infty \equiv \prod_{j \geq 0} (1 - z q^j), \quad \xi(z) = \frac{(q^2 z; q^4)_\infty}{(q^4 z; q^4)_\infty}.$$

A direct calculation shows:

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{[n]}{2n}\right] z^n\right) = \frac{(q^3 z; q^4)_\infty}{(qz; q^4)_\infty},$$

and from here we have immediately

$$\xi(z) = (1 - z)/\xi(z), \quad (4.7)$$

where $\cdot$ is extended to rational functions in $z$ by setting $q z = q^{-1} z$. It is known that the $P\lambda$ can be expressed via generating series as follows:

$$\Pi(x, y; q, t) := \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \exp\left(\sum_{n \geq 0} \frac{1 - t^n}{1 - q^n} \sum_i x_i^n \sum_j y_j^n\right)$$

$$= \sum_{\lambda} P\lambda(x_i; q, t) Q\lambda(y_j; q, t). \quad (4.8)$$

The Macdonald polynomials also satisfy:

$$P\lambda(x; q, t) = P\lambda(x; q^{-1}, t^{-1}), \quad (4.10)$$

$$Q\lambda(x; q, t) = (qt^{-1})^{\lambda}(Q\lambda(x; q^{-1}, t^{-1}), \quad (4.11)$$

The power sums $p_r = \sum x_i^r$ form a $Q$-basis of the ring of symmetric functions in the $x_i$.

**Definition 2.** Let $P\lambda(p_n; q, t)$ to be the $Q(q, t)$ polynomial in $p_i, i > 0$ for which $P\lambda(p_n; q, t) = P\lambda(x^n; q, t)$.

**Proposition 6.** Let $N > 0$. Let $z_i, i \geq 0$ be indeterminates. The Macdonald polynomials satisfy the following generating series:

$$\exp\left(\sum_{n \geq 1} (-1)^{n-1} \frac{b_n}{n} \sum_{j=1}^{N} z_j^n\right) = \sum_{\lambda, \mu \leq N} P\lambda(z; q, t) P\mu(b_n; t, q). \quad (4.12)$$

**Proof.** This follows from [M] (page 310), where we restrict to the ring of symmetric polynomials in $z_1, \ldots, z_N$. \hfill \Box

Finally, introduce the involution of $C(q, t)[x_1, x_2, \ldots]$ by setting

$$\omega_{q, t}(p_n) = \frac{1 - q^n}{1 - t^n} (-1)^{n-1} p_n. \quad (4.13)$$

Then (c.f. [M] VI 5.1),

$$\omega_{q, t} P\lambda(x; q, t) = Q\lambda(x; t, q). \quad (4.14)$$
5. Invariance of Macdonald polynomials under bar action

Following (2.3) we induce a bar action on $\mathcal{H}$ by

**Definition 3.** Let $u \in U_q(\widehat{sl}_2)$, for the highest weight vector $|\Lambda_i\rangle$ of $V(\Lambda_i)$, $i = 0, 1$, define $u|\Lambda_i\rangle = \overline{u}|\Lambda_i\rangle$.

This implicitly defines a bar action on $V(\Lambda_i)$, and it is clear that

**Proposition 7.** For $\Phi(z): \mathcal{H} \rightarrow \mathcal{H} \otimes V(z)$ we have

$$\Phi(z)u|\Lambda_0\rangle + v|\Lambda_1\rangle = \overline{\Phi(z)}(\overline{u}|\Lambda_0\rangle + \overline{v}|\Lambda_1\rangle),$$

where $u, v \in U_q(\widehat{sl}_2)$.

In Proposition 3 the application of a vertex operator involves a multiplication by a power of $q$. In order to simplify the statement of results we introduce the following normalization:

**Definition 4.** Let $m \geq 0, i = 0, 1$.

$$v_{m,i} = (e^{\alpha/2}(-q)^{(\beta+3\ell)/2})2^m|\Lambda_i\rangle = (-q)^{m(m+1+i)}e^{m\alpha}|\Lambda_i\rangle,$$

$$v_{-m,i} = (e^{-\alpha/2}(-q)^{-3\beta+3\ell/2})2^m|\Lambda_i\rangle = (-q)^{m(m-i)}e^{-m\alpha}|\Lambda_i\rangle,$$

where when $m = 0$, we have $i = 1$ in the first case and $i = 0$ in the second case.

As usual, denote by $[n]$ the largest integer less than or equal to $n$.

**Proposition 8.** Let $i = 0, 1$, $N \geq 1$, $n = \lfloor \frac{N+i+1}{2} \rfloor$, $m = \lfloor \frac{N+i}{2} \rfloor$, $k = \frac{1+(-1)^{N+i+1}}{2}$.

$$(-q)^{\delta-1}\Phi_-(z_1)(-q)^{\delta-1}\Phi_-(z_2)\ldots(-q)^{\delta-1}\Phi_-(z_N)|\Lambda_i\rangle =$$

$$\prod_{1<i<j}(-1-z_i/z_j)\prod_{1<i<j}\left(\frac{\xi(z_i/z_j)}{z_i/z_j}\right)^{N}z_j^{\frac{N+i+1}{2}}\exp\left(\sum_{n=1}^{\infty}\frac{q^{n/2}}{n}a_n\sum_{j=1}^{N}z_j^n\right)v_{m,k},$$

where if $m = 0$, $i = 1$,

$$(-q)^{1-\delta}\Phi_-(z_1)(-q)^{1-\delta}\Phi_-(z_2)\ldots(-q)^{1-\delta}\Phi_-(z_N)|\Lambda_i\rangle =$$

$$\prod_{1<i<j}\left(\frac{\xi(z_i/z_j)}{z_i/z_j}\right)^{N}z_j^{\frac{N+i+1}{2}}\exp\left(\sum_{n=1}^{\infty}\frac{q^{n/2}}{n}a_n\sum_{j=1}^{N}z_j^n\right)v_{m,k},$$

and

$$(-q)^{-\delta}\Phi_+(z_1)(-q)^{-\delta}\Phi_+(z_2)\ldots(-q)^{-\delta}\Phi_+(z_N)|\Lambda_i\rangle =$$

$$\prod_{1<j}z_j^{-\frac{i+i+1}{2}}(1-z_j/z_i)\prod_{1<i<j}\left(\frac{\xi(z_i/z_j)}{z_i/z_j}\right)^{N}z_j^{\frac{1-i+1}{2}}\exp\left(\sum_{n=1}^{\infty}\frac{q^{-n/2}}{n}a_n\sum_{j=1}^{N}z_j^n\right)v_{m,k},$$

where if $m = 0$, $i = 0$,

$$(-q)^{\delta}\Phi_+(z_1)(-q)^{\delta}\Phi_+(z_2)\ldots(-q)^{\delta}\Phi_+(z_N)|\Lambda_i\rangle =$$

$$\prod_{1<j}z_j^{-\frac{i+i+1}{2}}(1-z_j/z_i)\prod_{1<i<j}\left(\frac{\xi(z_i/z_j)}{z_i/z_j}\right)^{N}z_j^{\frac{1-i+1}{2}}\exp\left(\sum_{n=1}^{\infty}\frac{q^{1-n/2}}{n}a_n\sum_{j=1}^{N}z_j^n\right)v_{m,k}.$$

**Proof.** These follow by direct calculation from Proposition 3. ∎
Let
\[ c(z) = \prod_{i<j} \xi(\frac{z^i_j}{z^i}) \prod_{j=1}^N z_j^{\frac{\lfloor m + j - 1 \rfloor}{2}}, \quad d(z) = \prod_{i<j} \xi(\frac{z^i_j}{z^i}) \prod_{j=1}^N z_j^{\frac{\lfloor m - j + 1 \rfloor}{2}}. \]

Using the generating function (3.12), the formulas (3.44), (3.45), (3.46) and (3.47) give the following identities:

**Proposition 9.** Let \( N, n, m, k \) be as in the previous proposition.

\[
(-q)^{\ell - 1} \Phi_-(z_1)(-q)^{\ell - 1} \Phi_-(z_2)\ldots(-q)^{\ell - 1} \Phi_-(z_N)|\Lambda_i\)
\[
= c(z) \sum_{l(\lambda) \leq N} P_\lambda(\{q^{-3+(1/2)}z_j\})_{j=1}^N; q^4, q^2) P_{\lambda'}((-1)^{n-1}a_{-\cdot}; q^2, q^4)v_{m,k},
\]
\[
(-q)^{1-\ell} \Phi_-(z_1)(-q)^{1-\ell} \Phi_-(z_2)\ldots(-q)^{1-\ell} \Phi_-(z_N)|\Lambda_i\)
\[
= c(z) \sum_{l(\lambda) \leq N} P_\lambda(\{q^{3+(1/2)}z_j\})_{j=1}^N; q^4, q^2) P_{\lambda'}((-1)^{n-1}a_{-\cdot}; q^2, q^4)v_{m,k},
\]

and

\[
(-q)^{\ell - 1} \Phi_+(z_1)(-q)^{\ell - 1} \Phi_+(z_2)\ldots(-q)^{\ell - 1} \Phi_+(z_N)|\Lambda_i\)
\[
= \overline{d(z)} \sum_{l(\lambda) \leq N} P_\lambda(\{q^{-3+(5/2)}z_j\})_{j=1}^N; q^4, q^2) P_{\lambda'}((-1)^{n}a_{-\cdot}; q^2, q^4)v_{-m,k},
\]
\[
(-q)^{1-\ell} \Phi_+(z_1)(-q)^{1-\ell} \Phi_+(z_2)\ldots(-q)^{1-\ell} \Phi_+(z_N)|\Lambda_i\)
\[
= \overline{d(z)} \sum_{l(\lambda) \leq N} P_\lambda(\{q^{3+(5/2)}z_j\})_{j=1}^N; q^4, q^2) P_{\lambda'}((-1)^{n}a_{-\cdot}; q^2, q^4)v_{-m,k}.
\]

From these and \[4.10\] we immediately obtain:

**Proposition 10.** Let \( m \geq 0, i = 0, 1 \). Let \( \Delta = 3 \min(2m + i - (\ell(\lambda') - 1), 0) \).

\[
(\ell - \Delta)q^{\ell(\lambda')/2}P_\lambda((-1)^{n-1}a_{-\cdot}; q^2, q^4)v_{m,i} = (\ell - \Delta)q^{\ell(\lambda)/2}P_\lambda((-1)^{n-1}a_{-\cdot}; q^2, q^4)v_{m,i},
\]
\[
(q^\Delta)q^{\ell(\lambda)/2}P_\lambda((-1)^{n}a_{-\cdot}; q^2, q^4)v_{-m,i} = (q^\Delta)q^{\ell(\lambda')/2}P_\lambda((-1)^{n}a_{-\cdot}; q^2, q^4)v_{-m,i}.
\]

where when \( m = 0 \) we have \( i = 1 \) in the first case and \( i = 0 \) in the second case.

**Proof.** In both cases the bar–invariance follows directly from Proposition \[2\] when \( l(\lambda') \leq m - 1/2 \). An extra factor of \( q^\Delta \) appears for \( P_\lambda \) in sectors which don’t satisfy this inequality. Fix an arbitrary partition \( \lambda \). By Proposition \[3\] for large enough \( m \), we have \( \Delta = 0 \). Now apply the \( \overline{\Phi}_+(w) \) action of (3.46) to both sides.
Furthermore, there is also a form with the property (2.5) on $V$ and by starting with such a form for the two dimensional representation determined by $(x, y)$ and extending it to $V$ the Macdonald polynomials coincide in our realization of the ring of symmetric functions in the representation $H$. As with the bar invariance in the previous section, the $P_\lambda$ are multiplied by a power of $q$, depending on the sign of $m$ sector determined by $v_{m,i}$.

We note that the bilinear form of Proposition 1 on the representations $V(\Lambda_1)$ and $V(\Lambda_0)$ extends naturally to $H$ by requiring $V(\Lambda_1)$ and $V(\Lambda_0)$ to be orthogonal. Furthermore, there is also a form with the property (2.3) on $V_z$. This is obtained by starting with such a form for the two dimensional representation $V$ of $U_q(\mathfrak{sl}_2)$ and extending it to $V_z = V_1 \otimes C[z, z^{-1}]$ by setting $(v \otimes z^n, w \otimes z^m) = (v, w) \delta_{n,m}$.

**Definition 5.** The form $(\cdot, \cdot) : H \otimes V_z \otimes H \otimes V_z \rightarrow \mathbb{Q}(q)$ is the unique bilinear form determined by $(x \otimes v z^n, y \otimes w z^m) = (x, y)_h(v, w) \delta_{m,n}$.

Explicit expressions for $\Phi_\varepsilon^*(z)$ (see (3.15)) in terms of $\Phi_\varepsilon(z)$ are readily calculated. We recall:

**Proposition 11.** [JM] Let $u, v \in H$. We have

$$\Phi_\varepsilon^*(z) = (-q)^{i+(\varepsilon-1)/2}\Phi_\varepsilon(-q^{-2}z),$$

$$\overline{\Phi}_\varepsilon(z) = (-q)^{-i+(1-\varepsilon)/2}\overline{\Phi}_\varepsilon(q^2 z).$$
We have

**Proposition 12.** Let \( u, v \in \mathcal{H} \). We have

\[
(\Phi(z) u, v) = (\Phi(z)(z^{-1}) u, v), \\
(\Phi(z) u, v) = (\Phi(z)(z^{-1}) u, v).
\]

**Proof.** A direct calculation shows that \( \rho \) commutes with both coproducts: i.e. \( \Delta \circ \rho = \rho \circ \Delta \) and \( \Delta \circ \rho = \rho \circ \Delta \). By definition, the vertex operators \( \Phi, \Phi^* \) (see (3.11), (3.15)) are intertwiners for \( \Delta, \hat{\Delta} \) respectively. Now we prove the proposition by induction on the component degree of \( H \otimes \rho \). We note that when \( u \) and \( v \) the statement easily verifiable. We check that if the proposition holds for \( v \in \mathcal{H} \) with arbitrary \( u \in \mathcal{H} \otimes \rho \), then it holds for \( uv \) where \( a \in U_q(sl_2) \). Let \( u_1, u_2 \in \mathcal{H}, v \in \mathcal{V} \).

\[
(u_1 \otimes v, \Phi(z)u_2) = (u_1 \otimes v, \Delta(a)\Phi(z)u_2) = ((\rho \otimes \rho)\Delta(a)(u_1 \otimes v), \Phi(z)u_2) = (\Delta\rho(a)(u_1 \otimes v), \Phi(z)u_2) = (\Phi^*(z^{-1})(\Delta\rho(a))(u_1 \otimes v), u_2) = (\rho(a)\Phi^*(z^{-1})(u_1 \otimes v), u_2) = (\Phi^*(z^{-1})u_1 \otimes v, au_2).
\]

We note that when \( au_2 \) is \( \Lambda_1 \) or \( \Lambda_0 \) the proposition clearly holds, and this completes the induction. \( \square \)

**Lemma 2.** Let \( m \geq 0, i = 0, 1 \).

(a) \( \Phi_-(z^{-1})(-q)^{2i-2}\Phi_-(w)v_{m,i} = \frac{(q^{-2}z^w q^{-4})\infty}{(q^{-2}z^w q^{-4})\infty} \times \exp(-\sum_{n=1}^{\infty} \frac{q^{2n/2}}{n} a_{-n} z^{-n}) \exp(-\sum_{n=1}^{\infty} \frac{q^{1n/2}}{n} a_{-n} w^n) v_{m,i}, \)

(b) \( \Phi_+(z^{-1})(-q)^{2i}\Phi_+(w)v_{m,i} = \frac{(q^2z^w, q^4)\infty}{(q^2z^w, q^4)\infty} \times \exp(\sum_{n=1}^{\infty} \frac{q^{2n/2}}{n} a_{-n} z^{-n}) \exp(\sum_{n=1}^{\infty} \frac{q^{-2n/2}}{n} a_{-n} w^n) v_{m,i}. \)

**Proof.** This is a direct calculation using Proposition \( \square \)

**Proposition 13.** Let \( m \geq 0, i = 0, 1 \). Let \( \Delta = 3 \min(2m + i - (\ell(\lambda') - 1), 0) \). Then

(a) \( (q^{[\lambda]/2-\Delta} P_{\lambda'}(-1)(-1)^{n-1} a_{-n}; q^{-2}, q^{-4}) v_{m,i} = \delta_{\lambda,\mu} b_{\lambda}(q^2, q^{-4}) = \delta_{\lambda,\mu} b_{\lambda}^{-1}(q^{-2}, q^4), \)

(b) \( (q^{[\lambda]/2+\Delta} P_{\lambda'}((-1)^n a_{-n}; q^2, q^4) v_{m,i} = \delta_{\lambda,\mu} b_{\lambda}(q^2, q^4) = \delta_{\lambda,\mu} b_{\lambda}^{-1}(q^{-2}, q^4), \)

where when \( m = 0 \) we have \( i = 1 \) in the first case and \( i = 0 \) in the second case.
Proof. We check (b). As in the previous section first we will restrict to the case where \( l(\lambda) < \left\lfloor (m - i)/2 \right\rfloor \) and then we will extend to the general case.

\[
((-q)^{\delta} \Phi_+(z_1)(-q)^{\delta} \Phi_+(z_2)\ldots(-q)^{\delta} \Phi_+(z_N)|_{\Lambda_i}),
(-q)^{\delta} \Phi_+(w_1)(-q)^{\delta} \Phi_+(w_2)\ldots(-q)^{\delta} \Phi_+(w_N)|_{\Lambda_i})
\]

\[
(d(z) \prod_{l(\lambda) \leq N} P_{\lambda}(\{q^{3+(5/2)} z_j \}_{j=1}^{N} : q^4, q^3) P_{\lambda'}((-1)^n a_{-n}; q^2, q^4) v_{-m, j},
(d(w) \prod_{l(\mu) \leq N} P_{\mu}(\{q^{3+(5/2)} w_j \}_{j=1}^{N} : q^4, q^3) P_{\mu'}((-1)^n a_{-n}; q^2, q^4) v_{-m, j})
\]

\[
(d(z) d(w) \prod_{1 \leq i, j \leq m} (q^{2+5/2} z_i | q^4) \prod_{1 \leq i, j \leq m} (q^{2+5/2} w_i | q^4)
\]

\[
= d(z) d(w) \sum_{l(\lambda) \leq N} P_{\lambda}(q^{3} z_i : q^4, q^3) Q_{\lambda}(q^{3} w_i : q^4, q^2).
\]

Now we check the general case. Fix a partition \( \lambda \). We know that for \( m \) large enough part (b) holds. Pick the largest \( m \) for which the result doesn't hold for \( P_{\lambda} \) and \( v_{-m+1, j}, \) where \( j = 0, 1 \). Applying Lemma 3 we have

\[
((-q)^{\delta} \Phi_-(z_0)(-q)^{\delta} \Phi_+(z_1)(-q)^{\delta} \Phi_+(z_2)\ldots(-q)^{\delta} \Phi_+(z_N)|_{\Lambda_i}),
(-q)^{\delta} \Phi_-(w_0)(-q)^{\delta} \Phi_+(w_1)(-q)^{\delta} \Phi_+(w_2)\ldots(-q)^{\delta} \Phi_+(w_N)|_{\Lambda_i})
\]

\[
= (d(z) \prod_{l(\lambda) \leq N+1} P_{\lambda}(q^{3+(5/2)} z_0, \{q^{3+(5/2)} z_j \}_{j=1}^{N} : q^2, q^4) P_{\lambda'}((-1)^n a_{-n}; q^4, q^2) v_{-m+1, j},
(d(w) \prod_{l(\mu) \leq N+1} P_{\mu}(q^{3+(5/2)} w_0, \{q^{3+(5/2)} w_j \}_{j=1}^{N} : q^2, q^4) P_{\mu'}((-1)^n a_{-n}; q^4, q^2) v_{-m+1, j})
\]

\[
= d(z) d(w) \sum_{\lambda, \mu} q^{5/2(|\lambda|+|\mu|)} P_{\lambda}(q^{3} z_0, q^{3} z_i : q^4, q^2) P_{\mu}(q^{3} w_0, q^{3} w_i : q^4, q^2)
\]

\[
\times (P_{\lambda'}((-1)^n a_{-n}; q^4, q^2) v_{-m+1, j}, P_{\mu'}((-1)^n a_{-n}; q^4, q^2) v_{-m+1, j}),
\]

where \( \tilde{w}_0 = q^{-2} w_0 \) and \( \tilde{z}_0 = q^{-2} z_0 \) and

\[
d(z) = d(z) \prod_{i=1}^{N} \frac{q^{6} z_i / z_0; q^4}{q^{4} z_i / z_0; q^4})
\]
But, as before, the left hand side also equals:
\[
((q)^3\Phi - (z_0)(-q)^3\Phi_+ (z_1)(-q)^3\Phi_+ (z_2)... (-q)^3\Phi_+ (z_N)|\Lambda_i),
\]
\[
((q)^3\Phi - (w_0)(-q)^3\Phi_+ (w_1)(-q)^3\Phi_+ (w_2)... (-q)^3\Phi_+ (w_N)|\Lambda_i))
\]
\[
= ((-q)^3\Phi_+ (z_1)(-q)^3\Phi_+ (z_2)... (-q)^3\Phi_+ (z_N)|\Lambda_i),
\]
\[
\Phi_+ (z_0)^{-1}((-q)^2\Phi_+ (w_0)(-q)^2\Phi_+ (w_1)(-q)^2\Phi_+ (w_2)... (-q)^2\Phi_+ (w_N)|\Lambda_i))
\]
\[
= \tilde{d}(z)\tilde{d}(w)\sum_{\lambda\in\mathcal{N}} P_{\lambda}(q^{\lambda_3}z_0, q^{\lambda_3}z_1, q^{\lambda_3}z_2, \ldots)Q_{\lambda}(q^{\lambda_4}w_0, q^{\lambda_4}w_1, q^{\lambda_4}w_2, \ldots).
\]

The result now follows as above. □

7. Lattice of dual Macdonald Polynomials

Let \(\hat{a}_{-n} = \omega_n q^2(a_{-n}) = (-q)^n\frac{[2n]}{[n]}a_{-n}\). Then, changing variables, we have:
\[
\mathcal{P}_{\lambda}(a_{-n}; q^2, q^4) = \mathcal{Q}_{\lambda}(\hat{a}_{-n}; q^4, q^2).
\]

Definition 6. Let
\[
\mathcal{Q}_{\lambda}v_{m,i} = \begin{cases} q^{[\lambda]/2}\mathcal{Q}_{\lambda}(\hat{a}_{-n}; q^4, q^2)v_{m,i} & \text{if } m > 0 \text{ or } m = 0 \text{ and } i = 1, \\ q^{-[\lambda]/2}\mathcal{Q}_{\lambda}(\hat{a}_{-n}; q^{-4}, q^{-2})v_{m,i} & \text{if } m < 0 \text{ or } m = 0 \text{ and } i = 0. \end{cases}
\]

As a result of the previous section we have:

Proposition 14. The basis of the \(\mathcal{Q}_{\lambda}v_{m,i}\) is orthogonal with respect to \((\ldots, \ldots)\) and has dual \(b_{\lambda}(q^4, q^2)\mathcal{P}_{\lambda}v_{m,i} = \mathcal{Q}_{\lambda}v_{m,i}\).

Definition 7. Let \(J_{\lambda}(q, t) = (c_{\lambda}(q, t))^{-1}P_{\lambda}(q, t) = (c_{\lambda}(q, t))^{-1}\mathcal{Q}_{\lambda}(q, t)\) be the dual integral Macdonald polynomials. Let \(\mathcal{J}\) (resp. \(\mathcal{J}^*\)) be 
\[
\mathcal{J} = c_{\lambda}(q^4, q^2)\mathcal{P} \text{ (resp. } c_{\lambda}^*(q^4, q^2)^{-1}\mathcal{P}).
\]

Then we have
\[
(J_{\lambda}(q, t), J_{\mu}^*(q, t)) = \delta_{\lambda, \mu}.
\]

where \(J_{\lambda}(q, t)\) is the integral Macdonald function \(\text{[M]}, \text{p. 352}\) defined by 
\[
J_{\lambda}(q, t) = c_{\lambda}(q, t)P_{\lambda}(q, t) = c_{\lambda}(q, t)Q_{\lambda}(q, t).
\]

Let
\[
\mathcal{L} = \bigoplus_{\lambda} \mathcal{J}_{\lambda}(a_{-n}; q^2, q^4)v_{m,i}.
\]

We consider the action of the interwiners on elements of \(\mathcal{L}\). From Proposition 11 it follows that the coefficient of \(\mathcal{P}((-1)^{n-1}a_{-n}; q^2, q^4)v_{m,j}\) is of \(P_{\lambda}\{(q^{3+(1/2)}z_j)^N; q^4, q^2\}\).

We show that the four interwiners considered so far leave the lattice \(\mathcal{L}\) invariant.

Recall \(\text{[M], p.345}\) that
\[
J_{\lambda}(x, z) = \sum_{\mu\subseteq\lambda} J_{\lambda/\mu}(x)J_{\mu}(z)
\]

where \(x, z\) are infinite sets of indeterminates and the \(J_{\lambda/\mu}(x)\) are the skew integral Macdonald polynomials. As before, we will also consider the restriction to the ring of symmetric functions in a finite number of indeterminates.
Fix a partition \( \lambda \). Consider the action of \( \Phi_-(z_0) \) on \( \mathcal{L} \). By Proposition 3, we have
\[
(-q)^{\ell-1}\Phi_-(z_0)(-q)^{\ell-1}\Phi_-(z_1)(-q)^{\ell-1}\Phi_-(z_2)...(-q)^{\ell-1}\Phi_-(z_N)|\Lambda_i) = \sum_{\ell(\mu) \leq N} J_\mu(\{q^{-3+1/2}z_j\}_{j=0}^N; q^4, q^2) J_{\mu'}^\ast((-1)^{n-1}a_n; q^2, q^4)v_{m,j}
\]
(7.5) \( \overline{c}(z) \)

Now using (7.4), (7.5)
\[
= \sum_{\ell(\mu) \leq N+1} \sum_{\lambda \subset \mu} J_\lambda(\{z_j\}_{j=1}^N; q^4, q^2) J_{\mu/\lambda}(z_0; q^4, q^2) J_{\mu'}^\ast((-1)^{n-1}a_n; q^2, q^4)v_{m,i}.
\]

Specializing for a specific \( \lambda \) where \( \ell(\lambda) \leq N \) we have:

**Lemma 3.**
\[
\Phi_-(z_0)J^\ast((-1)^{n-1}a_n; q^2, q^4)v_{m,j}
\]
\[
= \sum_{\mu' ; \lambda \subset \mu, \ell(\mu) \leq N+1} (J_{\mu/\lambda}(z_0; q^4, q^2) J_{\mu'}^\ast((-1)^{n-1}a_n; q^2, q^4)v_{m,i}).
\]

Now from [M], page 340, we have
\[
J_{\mu/\lambda}(z_0; q^4, q^2) = \sum_{\lambda'} f_{\mu',\lambda'}^\lambda J_{\lambda'},
\]
where \( f_{\mu',\lambda'}^\lambda(q^4, q^2) \in \mathbb{Z}[q, q^{-1}] \). Together with similar calculation for the other cases, this proves

**Lemma 4.** \( \mathcal{L} \) is invariant under the action of the vertex operators.

The following lemma is inspired by [M] (see also [3]).

**Lemma 5.**
\[
\prod_{1 \leq i < j \leq N} (z_i - q^{-2}z_j) = \sum_{w \in S_N} (-q)^{\ell(w)} z^w(\delta) + \sum a_{\gamma_1,...,\gamma_n} z_1^{\gamma_1} z_2^{\gamma_2} ... z_n^{\gamma_n},
\]
where \( \delta = (N-1, N-2, ..., 0) \) and for each monomial on the right hand side, some \( \gamma_i = \gamma_j \) for \( i \neq j \) and \( a_{\gamma} \in \mathbb{Z}[q^{-2}] \), \( a_{(1)} = 0 \).

**Proof.** We have
\[
\prod_{i < j} (z_i - q^{-2}z_j) = \sum_{\gamma} (-q)^{d(\gamma)} z^{\gamma_1} z_2^{\gamma_2} ... z_N^{\gamma_n},
\]
where the summation runs through all \( N \times N \) matrices \( (\gamma_{ij}) \) of 0's and 1's such that
\[
\gamma_{ii} = 0, \ \gamma_{ij} + \gamma_{ji} = 1 \text{ if } i \neq j,
\]
\[\text{and } d(\gamma) = \sum_{i < j} \gamma_{ij}, \ \gamma_i = \sum_j \gamma_{ij}.\]

When the \( \gamma_i \) are all distinct we have \( z_1^{\gamma_1} z_2^{\gamma_2} ... z_N^{\gamma_n} = z_1^{\gamma_w(1)} ... z_N^{\gamma_w(N)} \) for some permutation \( w \in S_N \) and \( \gamma_{\mu(i)} = \mu_i + (n-i), \ (1 \leq i \leq N) \),
for some partition $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0$. We claim that all $\mu_i$ are actually 0, from which it will follow $z_1^{\gamma_1} \cdots z_N^{\gamma_N} = z^{w(\delta)}$. In fact, if $s_{ij} = \gamma_{w(i),w(j)}$, for $1 \leq k \leq N$,

$$0 \leq \mu_1 + \cdots + \mu_k = \frac{k}{2} k(k - 1) + k(N - k) - \sum_{i=1}^{k} (N - i),$$

from which it follows that each $\mu_i = 0$. Notice that the last inequality is equality if and only if $s_{ij} = \gamma_{w(i),w(j)} = 1$ for all pairs $i < j$. Then, for each distinct $(\gamma_1, \ldots, \gamma_N)$, we have:

$$d(S) = \sum_{i < j} s_{ij} = \sum_{i < j} \gamma_{w(j),w(i)} = \ell(w),$$

and $d(S)$ is the number of pairs $i < j$ in $\{1, \ldots, N\}$ such that $w(i) > w(j)$. \hfill \Box

**Proposition 15.** The lattice $\mathcal{L}$ is invariant under $x_k^{\pm(N)}$. In particular, $\mathcal{L}$ is invariant under the action of $U_A$.

**Proof.** By a modification of (1.13), the dual integral Macdonald polynomials are also generated by the vertex operators $\Phi_{\pm}(z)$. The generators $x_0^+, x_{-1}^+, x_1^-$ satisfy the following commutation relations:

$$[\Phi_+(z), x_1^-]_{q^{-1}} = 0,$$
$$[\Phi_-(z), x_0^+] = 0,$$
$$[\Phi_- (z), x_0^+] = 0,$$
$$[\Phi_- (z), x_{-1}^+] = 0,$$
$$[\Phi_+(z), x_{-1}^+] = q^{-1}z^{-1}K^{-1}\Phi_-(z),$$
$$[\Phi_-(z), x_1^-]_{q^{-1}} = q^2z\Phi_+(z).$$

By Lemma 4 it follows that $\mathcal{L}$ is invariant under the $\Phi_{\pm}(z)$ action, so it will be sufficient to show that $x_k^{\pm(N)}e^{\alpha_0\ell} |\Lambda_j\rangle \in \mathcal{L}$. We compute that

$$x_k^{\pm N} e^{\alpha_0\ell} |\Lambda_j\rangle = \frac{1}{(2\pi i)^N} \int X^+(z_1)X^+(z_2) \cdots X^+(z_N)z_1^k \cdots z_N^k \; dz \; e^{\alpha_0\ell} |\Lambda_j\rangle$$

$$= \frac{1}{(2\pi i)^N} \int \exp\left(\sum_{n=1}^{\infty} \frac{(q^n + q^{-n})q^{-n/2}}{n} a_n(z_1^n + \cdots + z_N^n)\right)$$

$$\times \prod_{i < j}(z_i - z_j)(z_i - q^{-1}z_j)^{2m+k+i} e^{(m+N)\alpha} |\Lambda_j\rangle \; dz,$$

where we abbreviate $z = z_1 \cdots z_N, dz = dz_1 \cdots dz_N$, and the integration is over the boundary of suitable multidimensional disk about 0. Observe that the integrand divided by $\prod_{i < j}(z_i - q^{-2}z_j)$ is an anti-symmetric function in $z_1, \ldots, z_N$. Invoking
Lemma 3 for $\prod_{i<j}(z_i - q^{-2}z_j)$, we see that considering antisymmetry, the terms $z_1^{\gamma_1}z_2^{\gamma_2} \ldots z_N^{\gamma_N}$ (for which some $\gamma_i = \gamma_j$) make no contribution to the integral. Then

$$x_k^+ e^{\alpha_i} |A_i\rangle = \sum_{w \in S_N} \frac{dz}{(2\pi i)^N} \frac{1}{\left(\sum_{n=1}^{\infty}(q^n + q^{-n})q^{-n/2}\right)} a_n(z_1^n + \cdots + z_N^n)$$

$$\times \prod_{i<j}(z_i - z_j)(-q)^{-\ell(w)z_{w(\delta)+(2m+k+i)}1_{\{m+N\}a}|A_i\rangle$$

$$= \left(\sum_{w \in S_N} q^{-2\ell(w)} \frac{dz}{(2\pi i)^N} \frac{1}{\left(\sum_{n=1}^{\infty}(q^n + q^{-n})q^{-n/2}\right)} a_n(z_1^n + \cdots + z_N^n)\right)$$

$$\times \prod_{i<j}(z_i - z_j)z^{\delta+(2m+k+i)}1_{\{m+N\}a}|A_i\rangle,$$

where $\delta = (N-1,\ldots,0)$ and $1 = (1,1,\ldots,1)$.

By the orthogonality of $J_\lambda$ (4.8) we have

$$\exp\left(\sum_{n=1}^{\infty}(q^n + q^{-n})q^{-n/2}\right) a_n(z_1^n + \cdots + z_N^n) = \sum_{\ell(\lambda) \leq N} J_\lambda^*(a_{-n};q^2,q^4)J_\lambda(z_i;q^2,q^4)q^{-|\lambda|}.$$

The integrality of $J_\lambda$ (see for example [GT]) implies that

$$J_\lambda(z_i;q^2,q^4) = \sum_{\lambda \geq \mu} a_{\lambda \mu} s_\mu(z_i), \ a_{\lambda \mu} \in \mathbb{Z}[q].$$

Then, since $\sum_{w \in S_N} q^{-2\ell(w)} = q^{-\binom{N}{2}}[N]!$, we have

$$x_k^+ (N) e^{\alpha_i} |A_i\rangle = q^{-\binom{N}{2}} \sum_{\ell(\lambda) \leq N} J_\lambda^*(a_{-n};q^2,q^4) \int \frac{dz}{(2\pi i)^N} J_\lambda(z_i;q^2,q^4)q^{-|\lambda|}$$

$$\times \prod_{i<j}(z_i - z_j)z^{\delta+(2m+k+i)}1_{\{m+N\}a}|A_i\rangle$$

$$= q^{-\binom{N}{2}} \sum_{\ell(\lambda) \leq N} J_\lambda^*(a_{-n};q^2,q^4)q^{-|\lambda|}e_{\{m+N\}a}|A_i\rangle \cdot \sum_{\lambda \geq \mu} a_{\lambda \mu}$$

$$\times \int \frac{dz}{(2\pi i)^N} s_\mu(z_i) \prod_{i<j}(z_i - z_j)z^{\delta+(2m+k+i)}1,$$

where the last integral is an integer by the integrality of Schur functions. Therefore it follows $x_k^+ (N) e^{\alpha_i} |A_i\rangle \in \mathcal{L}$. The case of $x_k^- (N)$ can be proved similarly.

8. CANONICAL BASIS AND MACDONALD POLYNOMIALS

Definition 8. Let $m \in \mathbb{Z}$, $i = 0, 1$. Define $A_{(n,m,i)}$ to be the subspace of $\mathcal{H}$ spanned by $\hat{Q}_\lambda v_{m,i}$, where $|\lambda| = n$.

Clearly $\mathcal{H} = \oplus_{n,m,i} \hat{Q}(q)A_{(n,m,i)}$ and the canonical basis respects this grading.

Proposition 16. Let $b \in \mathcal{H}_A$ be an element of the canonical basis of $\mathcal{H}$ in $A_{(n,m,i)}$. Write $b = p(a_{-k}) \otimes v_{m,i}$ where $p$ is a polynomial of degree $n$. Then for any $m'$, up to a sign, $b' = p(a_{-k}) \otimes v_{m',i}$ is also an element of the canonical basis of $\mathcal{H}$.
Proof. This follows from the coincidence of Macdonald’s and Kashiwara’s forms. By the characterization of the canonical basis, \( b = p(a_{-k}) \otimes v_{m, i} \) is in \( U(A) \), bar–invariant, and \( (b, b) = 1 + q^{-1}f(q^{-1}) \). The same holds for \( b' = p(a_{-k}) \otimes v'_{m', i} \).

The integrality result (see, for example, [GT]) for the two variable Kostka matrix \( K(q, t) \) implies:

\[
J_{\lambda}(q, t) = \sum_{\mu \leq \lambda, |\mu| = |\lambda|} v_{\lambda, \mu}(q, t)m_{\mu}, \quad v_{\lambda, \mu} \in \mathbb{Z}[q],
\]

where the \( m_{\mu} \) are the monomial symmetric functions.

Combining this with

\[
s_{\lambda} = \sum_{\mu \leq \lambda} K_{\lambda, \mu} m_{\mu},
\]

where \( (K_{\lambda, \mu}) \) is the usual Kostka matrix, and \( s_{\lambda} \) are the Schur functions, we see that

\[
J_{\lambda}(q, t) = \sum_{\mu \leq \lambda} w_{\lambda, \mu}(q, t)s_{\mu}, \quad \text{where } w_{\lambda, \mu} \in \mathbb{Z}[q, t].
\]

**Definition 9.** Let \( m \in \mathbb{Z}, \ i = 0, 1 \). Let the Schur polynomials \( \tilde{s}_{\lambda}v_{m, i} = \tilde{P}_{\lambda}(\tilde{a}_{-n}; q, q)v_{m, i} \).

By results of [CT] it is known that the \( \tilde{s}_{\lambda}v_{m, i} \) are contained in the lattice of divided powers, and it follows from (8.3) that \( \tilde{J}_{\lambda} \) is contained in the lattice of divided powers.

There is a natural order (see [LT]) on the canonical basis. Since the canonical basis respects the grading \( A((n, m, i)) \) of \( H \), we can consider in each graded component the transition matrix between the dual canonical basis and the integral Macdonald polynomials, given by:

\[
\tilde{J}_{\lambda}v_{m, i} = \sum_{\mu, \sum_{\mu} = n} a_{\mu, \lambda}(q)B_{\mu}^* \quad \text{where } a_{\mu, \lambda} \in \mathbb{Z}[q, q^{-1}], \quad \text{and } B_{\mu}^* \text{ are elements of the dual canonical basis in } A((n, m, i))
\]

Let \( C \) be the diagonal matrix consisting of \( c_{\lambda}(q^4, q^2) \).

**Proposition 17.** Let \( A = C^{-1}(a_{\mu, \lambda}) \).

1. The matrix \( A \) consists of bar invariant elements.
2. \( A \) consists of polynomials in \( q \) and \( q^{-1} \) with integral coefficients.

**Proof.** 1) follows from the previous section. 2) follows from the discussion above.

From Proposition [4], we see that the polynomials \( \tilde{J}_{\lambda}v_{m, i} \) form a quasi–orthogonal basis of \( H \). However, they are not elements of the canonical basis except in the case where \( \lambda \) is the empty partition.

**Conjecture.** The matrix \( A \) is upper unitriangular with coefficients in \( \mathbb{N}[q, q^{-1}] \).

Fix a weight and sector as in Proposition [4]. With respect to this \( A((n, m, i)) \) let \( B \) (resp. \( B^* \)) denote the canonical basis (dual canonical basis) of \( H \). Let \( J \) (resp.
When it is done correctly, the explicit formulas for the coefficients of $A$ Grassmanians, i.e. those corresponding to maximal parabolic subgroups in shows that its coefficients are identified with the Kazhdan–Lusztig polynomials for dimensional construction when $n \to \infty$, which is a very delicate matter. However, when it is done correctly, the explicit formulas for the coefficients of $A(q)$ should be even more elementary than the ones for $A_n(q)$. This provides some assurance that the symmetric functions corresponding to the canonical basis have a simple enough description relative to Macdonald polynomials. At the present moment, however, we do not know to which symmetric functions they correspond.

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