FROM GROUPS TO CLUSTERS

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Abstract. We construct a new class of symmetric algebras of tame representation type that are also the endomorphism algebras of cluster-tilting objects in 2-Calabi-Yau triangulated categories, hence all their non-projective indecomposable modules are $\Omega$-periodic of period dividing 4. Our construction is based on the combinatorial notion of triangulation quivers, which arise naturally from triangulations of oriented surfaces with marked points.

This class of algebras contains the algebras of quaternion type introduced and studied by Erdmann with relation to certain blocks of group algebras. On the other hand, it contains also the Jacobian algebras of the quivers with potentials associated by Fomin-Shapiro-Thurston and Labardini-Fragoso to triangulations of closed surfaces with punctures, hence our construction may serve as a bridge between the modular representation theory of finite groups and the theory of cluster algebras.

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Introduction

The aim of this survey is to report on new connections between the representation theory of finite groups and the theory of cluster algebras.

Blocks of group algebras form an important class of indecomposable, symmetric finite-dimensional algebras. Blocks of finite representation type are Morita equivalent to Brauer tree algebras and are well understood. In order to understand blocks of tame representation type, Erdmann [20] introduced the classes of algebras of dihedral, semi-dihedral and quaternion type, which are defined by properties of their Auslander-Reiten quiver, proved that blocks with dihedral, semi-dihedral

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or generalized quaternion defect group belong to the respective class of algebras and classified the possible quivers with relations these algebras may have.

One of these classes consists of the algebras of quaternion type, which are indecomposable, symmetric algebras of tame representation type having non-singular Cartan matrix, with the property that any indecomposable non-projective module is $\Omega$-periodic of period dividing 4, where $\Omega$ is Heller’s syzygy functor. This class of algebras is closed under derived equivalences [48].

2-Calabi-Yau triangulated categories with cluster-tilting objects arise in the additive categorification of cluster algebras with skew-symmetric exchange matrices [4, 19, 55, 56]. The role of the clusters in the cluster algebra is played by the cluster-tilting objects, whose endomorphism algebras, called 2-CY-tilted algebras, have remarkable representation theoretic and homological properties [20, 57].

We show that symmetric algebras $\Lambda$ that are in addition 2-CY-tilted have interesting structural properties analogous to those of the algebras of quaternion type; firstly, the functor $\Omega^4$ is isomorphic to the identity functor on the stable module category $\text{mod}\, \Lambda$ (Proposition 2.16); secondly, such algebras tend to come in derived equivalence classes (Proposition 2.21). More precisely, if $\Lambda = \text{End}_C(T)$ for a cluster-tilting object $T$ in a 2-Calabi-Yau triangulated category $C$, then the 2-CY-tilted algebra $\Lambda' = \text{End}_C(T')$ is derived equivalent to $\Lambda$ for any other cluster-tilting object $T'$ obtained from $T$ by a finite sequence of Iyama-Yoshino [50] mutations.

Motivated by this analogy one is naturally led to ask whether the algebras of quaternion type can be realized as 2-CY-tilted algebras, and even more generally, what are the symmetric algebras that are also 2-CY-tilted?

In this survey we provide an affirmative answer to the first question and attempt to answer the second question, first by classifying the symmetric, 2-CY-tilted algebras of finite representation type and then by constructing a new class of symmetric, 2-CY-tilted algebras of tame representation type. Note that there are also many wild symmetric, 2-CY-tilted algebras, but we will not discuss them here. Let us describe the main results along with the structure of this survey.

In Section 2 we review some basic notions including blocks, stable categories, symmetric algebras, periodic modules and the definition of algebras of quaternion type. We also introduce the algebras of quasi-quaternion type, which are defined similarly to the algebras of quaternion type, the only difference being the omission of the condition that the Cartan matrix is non-singular.

In Section 3 we investigate symmetric 2-CY-tilted algebras. We start by recalling the definition and basic properties of 2-CY-tilted algebras. Since many of them arise as Jacobian algebras of quivers with potentials [4, 25, 56], we review this notion as well, and introduce the notion of hyperpotential [66] which is useful over ground fields of positive characteristic. Then we present results concerning the periodicity of modules and derived equivalences for these algebras.

A classification of symmetric, 2-CY-tilted, indecomposable algebras of finite representation type which are not simple is presented in Section 6. We show that these algebras are precisely the Brauer tree algebras with at most two simple modules (Theorem 6.3).

Then, we construct a large class of symmetric, 2-CY-tilted algebras of tame representation type (Theorem 7.1). Our construction is based on the combinatorial notion of triangulation quivers, which are quivers with the property that for each vertex the set of incoming arrows and that of outgoing arrows have cardinality 2, together with bijections between these sets that combine to yield a permutation on the set of all arrows which is of order dividing 3. Triangulation quivers can be built from ideal triangulations of surfaces with marked points in a way which is
analogous to, but different than the construction of the adjacency quivers of Fomin, Shapiro and Thurston \cite{36} arising in their work on cluster algebras from surfaces.

The ingredients behind our construction are presented in Sections 3, 4, 5 and 7. Section 3 forms the combinatorial heart of this survey. We introduce ribbon quivers and the dual notion of ribbon graphs, define the subclass of triangulation quivers, and present a block decomposition of the latter into three basic building blocks. Section 4 explains how triangulations of marked surfaces give rise to triangulation quivers. We discuss the differences and similarities to adjacency quivers and provide a dimer model perspective on these constructions.

In Section 5 we introduce two classes of algebras which turn out to be important for our study, one consists of the well known Brauer graph algebras \cite{3, 8, 53}, while the other is the newly defined \textit{triangulation algebras}. Roughly speaking, a Brauer graph algebra arises from any ribbon quiver and auxiliary data given in the form of scalars and positive integer multiplicities, whereas a triangulation algebra arises from any triangulation quiver with similar auxiliary data.

In Section 7 we investigate triangulation algebras in more detail and prove that they are finite-dimensional, tame, symmetric, 2-CY-tilted algebras and hence of quasi-quaternion type. By using Iyama-Yoshino mutations of cluster-tilting objects \cite{50}, we are able to construct even more, derived equivalent, algebras with the same properties. The finite-dimensionality of the triangulation algebras relies on computations inside complete path algebras of quivers, whereas the proof of their representation type uses the observation that apart from a few exceptions, the triangulation algebras are deformations of the corresponding Brauer graph algebras (Proposition 7.13).

Our construction yields new symmetric 2-CY-tilted algebras in addition to the ones constructed by Burban, Iyama, Keller and Reiten \cite{22} arising from the stable categories of maximal Cohen-Macaulay modules over odd-dimensional isolated hypersurface singularities. Moreover, it provides new insights on the important problem of classifying the self-injective algebras with periodic module categories, as the algebras we construct are instances of new tame symmetric algebras with periodic modules which seem not to appear in the classification announced by Erdmann and Skowroński \cite[Theorem 6.2]{33}.

In Section 8 we prove that our newly constructed class of algebras contains two known classes of algebras as subclasses. Firstly, it contains all the members in Erdmann’s lists of algebras of quaternion type (Theorem 8.4). Since these algebras turn out to be 2-CY-tilted, this gives a new proof of the fact that they are indeed of quaternion type, which was first shown in \cite{32} by constructing bimodule resolutions. As a consequence, we are able to characterize all the blocks of group algebras that are 2-CY-tilted algebras (Proposition 8.10).

In order to illustrate the advantage of this new point of view on the algebras of quaternion type, we discover new algebras of quaternion type which seem not to appear in the existing lists (Proposition 8.11).

Secondly, our newly constructed class of algebras contains also all the Jacobian algebras of the quivers with potentials associated by Labardini-Fragoso \cite{64} to triangulations of closed surfaces with punctures. As a consequence, we deduce that the latter algebras are finite-dimensional of quasi-quaternion type and their derived equivalence class depends only on the surface and not on the particular triangulation (Corollary 8.14, see also \cite{70}).

Our newly constructed class contains also all the symmetric algebras of tubular type $(2, 2, 2, 2)$ and their socle deformations classified in \cite[13]{12} (Proposition 8.11).
In Section 9 we introduce a notion of mutation on triangulation quivers and compare it to various other notions of mutations existing in the literature, including flips of triangulations, Kauer’s elementary moves [53] for Brauer graph algebras and mutations of quivers with potentials [25]. We observe that the Brauer graph algebras arising from different triangulations of the same marked surface are derived equivalent (Corollary 9.13), a result which has also been obtained by Marsh and Schroll [74], however the algebras they consider in the case of surfaces with non-empty boundary are different. Analogously, under mild conditions the triangulation algebras of triangulation quivers related by a mutation are derived equivalent (Proposition 9.17).

Finally we outline an application to the theory of quivers with potentials. Non-degenerate potentials are important in various approaches to the categorification of cluster algebras [26, 77]. It was proved by Derksen, Weyman and Zelevinsky [25] that over an uncountable field, any quiver without loops or 2-cycles has at least one non-degenerate potential. For certain classes of quivers, a non-degenerate potential is unique [11, 69]. On the other hand, we construct infinitely many families of quivers, each having infinitely many non-degenerate potentials with pairwise non-isomorphic Jacobian algebras (Corollary 9.20).

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A report containing these results [67] was written during my visit to the IHES at Bures-sur-Yvette in the spring of 2014. In a subsequent visit during Spring 2015 some aspects of the theory were refined. I would like to thank the IHES for the hospitality and the inspiring atmosphere.

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1. Motivation: blocks of tame representation type

1.1. Group algebras. Let $G$ be a finite group and $K$ be a field. The group algebra $KG$ can be written as a direct product of indecomposable rings, which are called blocks. By Maschke’s theorem, if the characteristic of $K$ does not divide the order of $G$, then $KG$, and hence each block, is semi-simple. In particular, when $K$ is also algebraically closed, each block is isomorphic to a matrix ring over $K$.

However, when the characteristic of $K$, denoted here and throughout the paper by $\text{char} K$, divides the order of $G$, a block may not be semi-simple anymore. The defect group of a block $B$ measures how far it is from being semi-simple. It may be defined as a minimal subgroup $D$ of $G$ such that any $B$-module is $D$-projective (i.e. it is isomorphic to a direct summand of $W \otimes_{KD} KG$ for some $KD$-module $W$). A defect group is a $p$-subgroup of $G$ (where $p = \text{char} K$), determined up to
A defect group of the principal block (the block which the trivial $KG$-module $K$ belongs to) is a $p$-Sylow subgroup of $G$, and a block is semi-simple if and only if its defect group is trivial. We refer to the survey article [73] for further details.

Many aspects of the representation theory of a block are controlled by its defect group. One such important aspect is the representation type. Indeed, if $B$ is a block with defect group $D$ over an algebraically closed field of characteristic $p$, then $B$ is of finite representation type if and only if $D$ is cyclic [47], while $B$ is of tame (but not finite) representation type if and only if $p = 2$ and $D$ is either dihedral, semi-dihedral of generalized quaternion group [13]. In all other cases, $B$ is of wild representation type.

Blocks of finite representation type, that it, blocks with cyclic defect group, are Morita equivalent to Brauer tree algebras [24, 51] and hence are well understood. In order to understand blocks of tame representation type (over algebraically closed fields), Erdmann introduced families of symmetric algebras defined by properties of their Auslander-Reiten quivers. These are the algebras of dihedral, semi-dihedral and quaternion type. She showed that a block with dihedral (respectively, semi-dihedral, generalized quaternion) defect group is an algebra of the corresponding type and moreover she classified the quivers with relations these algebras may possibly have [29].

In this section we focus on the algebras of quaternion type and start by reviewing the relevant notions.

1.2. Stable categories and periodicity. Let $A$ be a finite-dimensional algebra over a field $K$. Denote by mod $A$ the category of finitely generated right $A$-modules, and by $D^b(A) = D^b(\text{mod } A)$ its bounded derived category. The latter contains as triangulated subcategory the category per $A$ of perfect complexes whose objects are bounded complexes of finitely generated projective $A$-modules. The Verdier quotient $D^b(A)/\text{per } A$ is known as the singularity category of $A$, see [76]. Its name comes from the fact that it vanishes precisely when $A$ has finite global dimension (i.e. $A$ is “smooth”) [76, Remark 1.9], as in this case any $A$-module has a finite projective resolution, thus any object in $D^b(A)$ is isomorphic to a perfect complex.

Assume that the algebra $A$ is self-injective, i.e. $A$ is injective as left and right module over itself, and consider the stable module category mod $A$ whose objects are the same as those of mod $A$ and the space of morphisms between any two objects $M, N \in \text{mod } A$ is given by

$$\text{Hom}_A(M, N) = \text{Hom}_A(M, N)/\mathcal{P}(M, N)$$

where $\mathcal{P}(M, N)$ consists of all the morphisms $M \to N$ in mod $A$ which factor through some projective module over $A$.

By a result of Happel [44, Theorem I.2.6], the additive category mod $A$ is triangulated. Moreover, by a theorem of Rickard [80, Theorem 2.1], mod $A$ can be identified with the singularity category of $A$.

Let $M \in \text{mod } A$ and consider a projective cover $P_M$ of $M$. Define a module $\Omega M$ by the exact sequence (in mod $A$)

$$0 \to \Omega M \to P_M \to M \to 0.$$ 

The syzygy $\Omega M$ is well defined in the category mod $A$ and gives rise to Heller’s syzygy functor $\Omega : \text{mod } A \to \text{mod } A$, see [45]. Sometimes, when we want to stress the role of the algebra $A$, we will denote the syzygy functor by $\Omega_A$ instead of $\Omega$.

Similarly, by taking an injective envelope $I_M$ of $M$ and the exact sequence

$$0 \to M \to I_M \to \Omega^{-1}M \to 0$$

The syzygy $\Omega^{-1}M$ is well defined in the category mod $A$ and gives rise to Heller’s syzygy functor $\Omega : \text{mod } A \to \text{mod } A$, see [45]. Sometimes, when we want to stress the role of the algebra $A$, we will denote the syzygy functor by $\Omega_A$ instead of $\Omega$. 

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Similarly, by taking an injective envelope $I_M$ of $M$ and the exact sequence

$$0 \to M \to I_M \to \Omega^{-1}M \to 0$$
one can define the cosyzygy functor $\Omega^{-1}: \text{mod} A \to \text{mod} A$, which is an inverse of $\Omega$. The suspension functor of the triangulated category $\text{mod} A$ is given by $\Omega^{-1}$.

An important class of self-injective algebras is formed by the symmetric algebras, which we now define. First, observe that for a finite-dimensional algebra $A$ over $K$, the vector space $DA = \text{Hom}_K(A, K)$ is an $A$-$A$-bimodule.

**Definition 1.1.** A finite-dimensional $K$-algebra $A$ is symmetric if $A \simeq DA$ as $A$-$A$-bimodules. Here and throughout the paper, the symbol $\simeq$ denotes isomorphism.

We recall a few alternative characterizations of symmetric algebras. In order to formulate them, we need the notion of a Calabi-Yau triangulated category which is given below.

**Definition 1.2.** Let $d \in \mathbb{Z}$. A $K$-linear triangulated category $T$ with suspension $\Sigma$ and finite-dimensional morphism spaces is $d$-Calabi-Yau if there exist functorial isomorphisms $\text{Hom}_T(X, Y) \simeq \text{DHom}_T(Y, \Sigma^d X)$ for all $X, Y \in T$.

**Proposition 1.3.** The following conditions are equivalent for a finite-dimensional $K$-algebra $A$.

(a) $A$ is symmetric;

(b) There exists a symmetrizing form on $A$, that is, a $K$-linear map $\lambda: A \to K$ whose kernel does not contain any non-trivial left ideal of $A$ and moreover $\lambda(xy) = \lambda(yx)$ for any $x, y \in A$;

(c) The triangulated category $\text{per} A$ is $0$-Calabi-Yau.

(d) $A$ is isomorphic to the endomorphism algebra of an object in a triangulated $0$-Calabi-Yau category.

**Proof.** The equivalence of (a) and (b) is standard, see e.g. [89, Theorem IV.2.2]. The implication (a) $\Rightarrow$ (c) follows from the fact that for any finite-dimensional algebra $A$ one has $\text{Hom}_{\text{D}^b(A)}(X, Y) \simeq \text{DHom}_{\text{D}^b(A)}(Y, \Sigma X)$ for any $X \in \text{per} A$ and $Y \in \text{D}^b(A)$, see the proof of [44, Theorem I.4.6]. For the implication (b) $\Rightarrow$ (c), observe that $A \simeq \text{End}_{\text{per} A}(A)$. For (c) $\Rightarrow$ (a), note that if $X$ is an object in a $0$-Calabi-Yau triangulated category $T$ and $A = \text{End}_T(X)$, then the functorial isomorphism $\text{Hom}_T(X, X) \simeq \text{DHom}_T(X, X)$ implies that $A \simeq DA$ as $A$-$A$-bimodules. \hfill $\Box$

**Corollary 1.4 ([89, Theorem IV.4.1]).** Let $A$ be a symmetric algebra and $e \in A$ an idempotent. Then the algebra $eAe$ is also symmetric.

**Proof.** One has $D(eAe) \simeq e(DA)e$; alternatively, use Proposition 1.3 for the algebra $eAe \simeq \text{End}_{\text{per} A}(eA)$. \hfill $\square$

**Example 1.5.** Any group algebra $KG$ is symmetric. Indeed, a symmetrizing form on $KG$ is given by

$$\lambda(\sum_{g \in G} a_g g) = a_1.$$

It follows from Corollary 1.4 that any block of a group algebra is also symmetric.

**Example 1.6.** If $A$ is any finite-dimensional $K$-algebra, the bimodule structure on $DA$ allows to define a symmetric algebra whose underlying vector space is $A \oplus DA$ called the trivial extension algebra of $A$ and denoted by $T(A)$. The elements of
$T(A)$ are pairs $(a, \mu)$ where $a \in A$ and $\mu \in DA$. Addition and multiplication are given by the formulae:

$$(a, \mu) + (a', \mu') = (a + a', \mu + \mu')$$

$$(a, \mu) \cdot (a', \mu') = (aa', a\mu + \mu a')$$

for $a, a' \in A$ and $\mu, \mu' \in DA$. The symmetrizing form on $T(A)$ is given by $\lambda(a, \mu) = \mu(1)$.

**Remark 1.7.** The stable category of a symmetric algebra $A$ is $(-1)$-Calabi-Yau, i.e.

$$\text{Hom}_A(M, N) \cong D \text{Hom}_A(N, \Omega M)$$

for $M, N \in \text{mod } A$, see for example [32, Proposition 1.2] and the end of [4, §1].

**Definition 1.8.** A module $M \in \text{mod } A$ is $\Omega$-periodic if $\Omega^r M \cong M$ for some integer $r > 0$.

The category $\text{mod } A$ has Auslander-Reiten sequences, and when $A$ is symmetric there is a close connection between the Auslander-Reiten translation $\tau$ on $\text{mod } A$ and the syzygy $\Omega$, namely $\tau = \Omega^2$. In particular, a module is $\Omega$-periodic if and only if it is $\tau$-periodic.

**Example 1.9.** Let $n \geq 1$ and consider the algebra $A = K[x]/(x^n)$. It is a commutative, local, symmetric algebra over $K$ of finite representation type whose indecomposable modules are given by $M_i = x^i A$ for $0 \leq i < n$. The module $M_0 = A$ is projective, and the exact sequence

$$0 \to x^{n-i} A \to A \xrightarrow{x^{-1}} x^i A \to 0$$

shows that $\Omega(M_i) = M_{n-i}$ for any $0 < i < n$. Hence $\Omega^2 M \cong M$ for any $M \in \text{mod } A$. Note that if $\text{char } K = p$ and $n = p^e$ for some $e \geq 1$, then $A \cong KG$ for $G = \mathbb{Z}/p^e\mathbb{Z}$ and its defect group equals $G$.

1.3. **Algebras of quaternion type.** In this section we assume that the ground field $K$ is algebraically closed. The algebras of quaternion type were introduced by Erdmann, and we refer to the articles [28] and the monograph [29] for a detailed presentation.

**Definition 1.10 ([28]).** A finite-dimensional algebra $A$ is of quaternion type if

(i) $A$ is symmetric, indecomposable as a ring;

(ii) $A$ has tame (but not finite) representation type;

(iii) $\Omega^4 M \cong M$ for any $M \in \text{mod } A$;

(iv) $\det C_A \neq 0$, where $C_A$ denotes the Cartan matrix of $A$.

Recall that the Cartan matrix of a basic algebra $A$ is the $n$-by-$n$ matrix with integer entries given by $(C_A)_{i,j} = \dim_K e_i A e_j$, where $e_1, e_2, \ldots, e_n$ form a complete set of primitive orthogonal idempotents in $A$. The motivation behind condition (iv) lies in the fact that if $B$ is a block over a field of characteristic $p$, then the determinant of its Cartan matrix is a power of $p$.

Let $n \geq 3$. The generalized quaternion group $Q_{2^n}$ is given by generators and relations as follows:

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2, y^4 = 1, y^{-1}xy = x^{-1} \rangle.$$ 

In particular, for $n = 3$ we recover the usual quaternion group with 8 elements.

Erdmann proved the following facts:

(a) Blocks of group algebras with generalized quaternion defect groups are of quaternion type.
(b) An algebra of quaternion type is Morita equivalent to an algebra in 12 families of symmetric algebras given by quivers with relations. In particular, an algebra of quaternion type has at most three isomorphism classes of simple modules.

The lists of the quivers with relations of point (b) can be found in [29, pp. 303–306] or in the survey articles [33, Theorem 5.5] and [87, Theorem 8.4]. Later, Holm [48] presented a derived equivalence classification of the algebras appearing in these lists and proved that these algebras are indeed tame. Finally, in [32] Erdmann and Skowroński showed that the algebras in these lists have the required periodicity property and hence they are indeed of quaternion type.

Example 1.11. One of the families in Erdmann’s list consists of local algebras whose quiver is

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\end{array}
\]

with the relations
\[
\alpha^2 = (\beta\alpha)^{m-1}\beta, \quad \beta^2 = (\alpha\beta)^{m-1}\alpha, \quad \alpha\beta^2 = \alpha^2\beta = \beta\alpha^2 = \beta^2\alpha = 0
\]
depending on an integer parameter \(m \geq 2\). When the ground field \(K\) is algebraically closed, \(\text{char } K = 2\) and \(m = 2^n - 2\) for some \(n \geq 3\), this algebra is the group algebra \(KQ_{2^n}\) of the generalized quaternion group \(Q_{2^n}\).

1.4. Algebras of quasi-quaternion type. It seems natural to lift the restriction on the Cartan determinant in the definition of algebras of quaternion type and consider a wider class of algebras, which we call algebras of quasi-quaternion type.

Definition 1.12. A finite-dimensional algebra \(A\) is of quasi-quaternion type if:

(i) \(A\) is symmetric, indecomposable as a ring;
(ii) \(A\) has tame (but not finite) representation type;
(iii) \(\Omega^4 M \cong M\) for any \(M \in \mod A\);

Remark 1.13. Since \(\tau = \Omega^2\), the stable Auslander-Reiten quiver of an algebra of quasi-quaternion type consists of tubes of ranks 1 and 2.

In analogy with Erdmann’s description of the algebras of quaternion type, the following problem arises naturally.

Problem 1.14. Describe the algebras of quasi-quaternion type.

Algebras of quasi-quaternion type are in particular tame symmetric algebras with periodic modules. A classification of the latter algebras has been announced in [33, Theorem 6.2], see also [87, Theorem 8.7]. However, many of the algebras of quasi-quaternion type to be constructed in Section 7 seem to be missing from the aforementioned classification.

Since the derived equivalence of self-injective algebras implies their stable equivalence [39, Corollary 2.2] and stable equivalence preserves representation type [62], an argument as in Prop. 2.1 and Prop. 2.2 of [48] yields the following observation.

Proposition 1.15. Any algebra which is derived equivalent to an algebra of quasi-quaternion type is also of quasi-quaternion type.

One approach to guarantee the condition (iii) in the definition of algebras of quasi-quaternion type is to show that the algebra \(A\) is periodic as \(A\)-\(A\)-bimodule with period dividing 4, that is, \(\Omega^4_A(A) \cong A\), where \(A^e = A^{op} \otimes_K A\). This is usually done using a projective resolution of \(A\) as a bimodule over itself. In fact, such strategy is used in [32] to prove that the algebras in Erdmann’s list are of quaternion type.
We suggest an alternative approach using 2-Calabi-Yau categories. It turns out that symmetric algebras that are also the endomorphism algebras of cluster-tilting objects in such categories always satisfy the periodicity condition (iii). We explain this in the next section.

2. Symmetric 2-CY-tilted algebras

In this section we study properties of symmetric algebras that are also 2-CY-tilted, i.e. being isomorphic to the endomorphism algebras of cluster-tilting objects in 2-Calabi-Yau triangulated categories.

We start by recalling the definition and basic properties of 2-CY-tilted algebras. Since many of them arise as Jacobian algebras of quivers with potentials, we review this notion as well, and introduce the notion of hyperpotential which is useful over ground fields of positive characteristic. Then we present two new results whose details will appear elsewhere; the first concerns the periodicity of modules over symmetric 2-CY-tilted algebras (Proposition 2.16), and the second concerns derived equivalences of neighboring 2-CY-tilted algebras (Proposition 2.21).

As a consequence of the first result, we deduce that indecomposable, tame, symmetric, 2-CY-tilted algebras are of quasi-quaternion type. For more background on 2-CY-tilted algebras, we refer the reader to the survey article [79].

2.1. 2-CY-tilted algebras. Let \( \mathcal{C} \) be a \( K \)-linear triangulated category with suspension \( \Sigma \). We assume:

- \( \mathcal{C} \) has finite-dimensional morphism spaces.
- \( \mathcal{C} \) is Krull Schmidt (i.e. any object has a decomposition into a finite direct sum of indecomposables which is unique up to isomorphism and change of order).
- \( \mathcal{C} \) is 2-Calabi-Yau.

Such triangulated categories \( \mathcal{C} \) arise in the additive categorification of cluster algebras, see the survey [55]. The role of the clusters in a cluster algebra is played by cluster-tilting objects in the category \( \mathcal{C} \).

**Definition 2.1.** An object \( T \in \mathcal{C} \) is cluster-tilting if:

(i) \( \text{Hom}_\mathcal{C}(T, \Sigma T) = 0 \);

(ii) For any \( X \in \mathcal{C} \) with \( \text{Hom}_\mathcal{C}(T, \Sigma X) = 0 \), we have that \( X \in \text{add} T \), where \( \text{add} T \) denotes the full subcategory of \( \mathcal{C} \) consisting of the objects isomorphic to finite direct sums of summands of \( T \).

**Definition 2.2.** An algebra is called 2-CY-tilted if it is isomorphic to an algebra of the form \( \text{End}_\mathcal{C}(T) \) with \( \mathcal{C} \) as above and \( T \) a cluster-tilting object in \( \mathcal{C} \).

The cluster categories associated to quivers without oriented cycles [19] were the first instances of triangulated 2-Calabi-Yau categories with cluster-tilting object. They are constructed as orbit categories of the bounded derived category of the path algebra of the quiver with respect to a suitable auto-equivalence [54]. The corresponding endomorphism algebras of cluster-tilting objects are called cluster-tilted algebras [20]. Self-injective cluster-tilted algebras were classified by Ringel [84]; there are very few such algebras as all of them are of finite representation type and up to Morita equivalence there are at most two such algebras having a given number of non-isomorphic simple modules. In particular, except for the quiver \( A_1 \) with one vertex whose cluster-tilted algebra equals the ground field, cluster-tilted algebras are never symmetric.

More generally, 2-CY-tilted algebras were investigated by Keller and Reiten [57]. The next proposition records the relevant properties we need.
Proposition 2.3. Let \( \Lambda \) be a 2-CY-tilted algebra. Then:

(a) [57, Prop. 2.1] \( \Lambda \) is Gorenstein of dimension at most 1, i.e. the projective dimension of any injective module and the injective dimension of any projective module are at most 1;

(b) [57, Theorem 3.3] The singularity category of \( \Lambda \) is 3-Calabi-Yau.

Given a 2-CY-tilted algebra \( \Lambda \), there is a procedure to construct new 2-CY-tilted algebras from idempotents of \( \Lambda \). The corresponding statement for cluster-tilted algebras has been shown in [21, Theorem 2.13], see also [23, Theorem 5]. The general case follows from Calabi-Yau reduction [50], see also [17, §II.2]. For the convenience of the reader, we give the short proof.

Proposition 2.4. Let \( \Lambda \) be a 2-CY-tilted algebra and let \( e \in \Lambda \) be an idempotent. Then the algebra \( \Lambda/\Lambda e \Lambda \) is 2-CY-tilted.

Proof. Let \( \Lambda = \text{End}_C(T) \) where \( C \) is a triangulated 2-Calabi-Yau category and \( T \) is a cluster-tilting object in \( C \). Let \( T' \) be the summand of \( T \) corresponding to the idempotent \( e \). The category \( C' = \{ X \in C : \text{Hom}_C(X, \Sigma T') = 0 \}/(\text{add } T') \) is a triangulated 2-Calabi-Yau category by [50, Theorem 4.7] and \( T \) is a cluster-tilting object in \( C' \) by [50, Theorem 4.9]. Finally, \( \text{End}_{C'}(T) \simeq \Lambda/\Lambda e \Lambda \). □

2.2. Quivers with potentials. Thanks to the works of Amiot [4] and Keller [56], a rich source of 2-Calabi-Yau triangulated categories with cluster-tilting object is provided by quivers with potentials whose Jacobian algebras are finite-dimensional. Quivers with potentials and their Jacobian algebras were defined and studied by Derksen, Weyman and Zelevinsky [25].

A quiver is a finite directed graph. Formally, it is a quadruple \( Q = (Q_0, Q_1, s, t) \) where \( Q_0 \) and \( Q_1 \) are finite sets (of vertices and arrows, respectively) and \( s, t : Q_1 \to Q_0 \) are functions specifying for each arrow its starting and terminating vertex, respectively.

The path algebra \( KQ \) has the set of paths of \( Q \) as a basis, with the product of two paths being their concatenation, if defined, and zero otherwise. The complete path algebra \( \hat{KQ} \) is the completion of \( KQ \) with respect to the ideal generated by all the arrows of \( Q \). It is a topological algebra, with a topological basis given by the paths of \( Q \). Thus, an element in \( \hat{KQ} \) is a possibly infinite linear combination of paths. We denote by \( \overline{I} \) the closure of an ideal \( I \) in \( \hat{KQ} \).

Example 2.5. The path algebra of the quiver with one vertex and one loop at that vertex is the ring \( K[x] \) of polynomials in one variable, whereas the complete path algebra is the ring \( K[[x]] \) of power series in one variable.

A cycle in \( Q \) is a path that starts and ends at the same vertex. One can consider the equivalence relation on the set of cycles given by rotations, i.e.

\[ \alpha_1 \alpha_2 \ldots \alpha_n \sim \alpha_i \ldots \alpha_n \alpha_1 \ldots \alpha_{i-1} \]

for a cycle \( \alpha_1 \alpha_2 \ldots \alpha_n \) and \( 1 \leq i \leq n \).

The zeroth continuous Hochschild homology \( \text{HH}_0(\hat{KQ}) \) is \( \hat{KQ}/[\hat{KQ}, \hat{KQ}] \), i.e. the quotient of \( \hat{KQ} \) by the closure of the subspace spanned by all the commutators of elements in \( \hat{KQ} \). It has a topological basis given by the equivalence classes of cycles of \( Q \) modulo rotation.

Definition 2.6 ([25 Definition 3.1]). A potential on \( Q \) is an element in \( \text{HH}_0(\hat{KQ}) \).

In explicit terms, a potential is a (possibly infinite) linear combination of cycles in \( Q \), considered up to rotations.

A pair \((Q, W)\) where \( Q \) is a quiver and \( W \) is a potential on \( Q \) is called a quiver with potential.
For any arrow $\alpha$ of $Q$, there is a cyclic derivative map $\partial_\alpha : \text{HH}_0(\hat{K}Q) \to \hat{K}Q$ which is the unique continuous linear map whose value on each cycle $\alpha_1 \alpha_2 \ldots \alpha_n$ is given by

$$\partial_\alpha (\alpha_1 \alpha_2 \ldots \alpha_n) = \sum_{i : \alpha_i = \alpha} \alpha_{i+1} \ldots \alpha_n \alpha_1 \ldots \alpha_{i-1},$$

where the sum goes over all indices $1 \leq i \leq n$ such that $\alpha_i = \alpha$.

**Definition 2.7 ([25]).** Let $(Q, W)$ be a quiver with potential. Its Jacobian algebra $\mathcal{P}(Q, W)$ is the quotient of the complete path algebra $\hat{K}Q$ by the closure of its ideal generated by the cyclic derivatives $\partial_\alpha W$ with respect to the arrows $\alpha$ of $Q$:

$$\mathcal{P}(Q, W) = \hat{K}Q / (\partial_\alpha W : \alpha \in Q_1).$$

**Remark 2.8.** When the potential $W$ is a finite linear combination of cycles, one can also consider a non-complete version of the Jacobian algebra, namely, the quotient of the path algebra $K_Q$ by its ideal generated by the cyclic derivatives of $W$. While in many cases this variation gives the same result, the next example shows that in general these two notions differ.

**Example 2.9.** Let $Q$ be the quiver

$$\begin{array}{c}
\bullet_1 \\
\gamma \\
\downarrow \\
\alpha \\
\bullet_3 \\
\end{array}$$

$$\begin{array}{c}
\bullet_2 \\
\beta \\
\end{array}$$

with the potential $W = \alpha \beta \gamma - \alpha \beta \gamma \alpha \beta \gamma$. Let $\mathcal{J}$ be the closure of the ideal generated by the cyclic derivatives of $W$, so that $\mathcal{P}(Q, W) = \hat{K}Q / \mathcal{J}$.

Computing the cyclic derivative with respect to the arrow $\gamma$, we get

$$\partial_\gamma W = \alpha \beta - \alpha \beta \gamma \alpha \beta - \alpha \beta \gamma \alpha \beta = \alpha \beta - 2\alpha \beta \gamma \alpha \beta,$$

hence $\alpha \beta - 2\alpha \beta \gamma \alpha \beta \in \mathcal{J}$. Therefore, for any $n \geq 1$,

$$\alpha \beta - (2\alpha \beta \gamma)^n \alpha \beta = \sum_{i=0}^{n-1} (2\alpha \beta \gamma)^i (\alpha \beta - 2\alpha \beta \gamma \alpha \beta) \in \mathcal{J}.$$

Since $\mathcal{J}$ is closed, this implies that $\alpha \beta \notin \mathcal{J}$. Moreover, one can verify that $\mathcal{P}(Q, W) \simeq K_Q / (\alpha \beta, \beta \gamma, \gamma \alpha)$. In particular, we see that in the presentation of the Jacobian algebra as quiver with relations, the relations are not necessarily the cyclic derivatives of the potential.

Consider now the non-complete Jacobian algebra $A$ and assume that $\text{char } K \neq 2$. Since $\alpha \beta = 2\alpha \beta \gamma \alpha \beta$ in $A$, one has

$$2\alpha \beta \gamma = 4\alpha \beta \gamma \alpha \beta = (2\alpha \beta \gamma)^2, \quad (e_1 - 2\alpha \beta \gamma)^2 = e_1^2 - 4\alpha \beta \gamma + 4\alpha \beta \gamma \alpha \beta \gamma = e_1 - 2\alpha \beta \gamma, \quad 2\alpha \beta \gamma (e_1 - 2\alpha \beta \gamma) = (e_1 - 2\alpha \beta \gamma) 2\alpha \beta \gamma = 0,$$

hence the idempotents in $A$ corresponding to the paths of length zero are no longer primitive; for example, $e_1$ can be written as a sum $e_1 = (e_1 - 2\alpha \beta \gamma) + 2\alpha \beta \gamma$ of two orthogonal idempotents. Using these idempotents one can verify that the algebra $A$ decomposes into a direct sum of $\mathcal{P}(Q, W)$ and the matrix ring $M_3(K)$.

For a quiver with potential $(Q, W)$, Ginzburg ([12] §4.2] has defined a dg-algebra $\Gamma(Q, W)$ which is concentrated in non-positive degrees and its zeroth cohomology is isomorphic to the Jacobian algebra, i.e. $H^0(\Gamma(Q, W)) \simeq \mathcal{P}(Q, W)$. In [56, Theorem 6.3], Keller shows that the Ginzburg dg-algebra $\Gamma = \Gamma(Q, W)$ is homologically smooth and bimodule 3-Calabi-Yau, that is, $\text{RHom}_{\Gamma}(\Gamma, \Gamma^e) \simeq \Gamma[-3]$ in $\mathcal{D}(\Gamma^e)$, where $\Gamma^e = \Gamma^{op} \otimes_K \Gamma$. 
Given a dg-algebra $\Gamma$ which is concentrated in non-positive degrees, homologically smooth, bimodule 3-Calabi-Yau and whose zeroth cohomology $H^0(\Gamma)$ is finite-dimensional, Amiot constructs in [4, §2] a triangulated 2-Calabi-Yau category with a cluster-tilting object whose endomorphism algebra is $H^0(\Gamma)$. She then applies this construction to $\Gamma(Q, W)$ for quivers with potentials $(Q, W)$ whose Jacobian algebra is finite-dimensional to obtain the generalized cluster category associated with $(Q, W)$ [4, Theorem 3.5].

**Proposition 2.10** ([4, Corollary 3.6]). Any finite-dimensional Jacobian algebra of a quiver with potential is 2-CY-tilted.

A notion of equivalence of quivers with potentials was introduced by Derksen, Weyman and Zelevinsky [25]. Let $Q$ be a quiver. Any continuous algebra automorphism $\varphi : \hat{K}Q \to \hat{K}Q$ induces a continuous linear automorphism, denoted $\varphi$, of the topological vector space $\text{HH}_0(\hat{K}Q) = \hat{K}Q/[\hat{K}Q, \hat{K}Q]$. For a vertex $i$ of $Q$, denote by $e_i$ the path of length zero at $i$. It is an idempotent of the algebra $\hat{K}Q$.

**Definition 2.11** ([25, Definition 4.2]). Two potentials $W$ and $W'$ on $Q$ are right equivalent if there exists a continuous algebra automorphism $\varphi : \hat{K}Q \to \hat{K}Q$ satisfying $\varphi(e_i) = e_i$ for each $i \in Q_0$ and $W' = \varphi(W)$ in $\text{HH}_0(\hat{K}Q)$.

The Ginzburg dg-algebras of right equivalent potentials are isomorphic and hence also their Jacobian algebras [58, Lemmas 2.8 and 2.9]. If the latter are finite-dimensional, then the associated 2-Calabi-Yau categories are equivalent as triangulated categories, since they depend only on the corresponding Ginzburg dg-algebras.

**Example 2.12.** Consider the quiver with potential $(Q, W)$ of Example 2.9 and let $W' = \alpha \beta \gamma$ be another potential on $Q$. A continuous algebra automorphism of $\hat{K}Q$ fixing each $e_i$ is determined by its value on the arrows. The endomorphism $\varphi$ whose value on the arrows is given by

$\varphi(\alpha) = \alpha - \alpha \beta \gamma \alpha$, $\varphi(\beta) = \beta$, $\varphi(\gamma) = \gamma$

is an automorphism of $\hat{K}Q$; indeed,

$\varphi^{-1}(\alpha) = \alpha + \alpha \beta \gamma \alpha + 2(\alpha \beta \gamma)^2 \alpha + 5(\alpha \beta \gamma)^3 \alpha + 14(\alpha \beta \gamma)^4 \alpha + \ldots$

(where the coefficients are the Catalan numbers). Moreover, $\varphi(W') = W$, hence the potentials $W$ and $W'$ are right equivalent.

**2.3. Hyperpotentials.** The following extension of the notion of a potential, introduced in [66], allows to prove that certain algebras defined over ground fields of positive characteristic are 2-CY-tilted. This will be particularly important when considering blocks of group algebras.

**Definition 2.13** ([66]). A hyperpotential on $Q$ is an element in $\text{HH}_1(\hat{K}Q)$. In explicit terms, it is a collection of elements $(\rho_\alpha)_{\alpha \in Q_1}$ in $\hat{K}Q$ indexed by the arrows of $Q$ satisfying the following conditions:

1. If $\alpha : i \to j$ then $\rho_\alpha \in e_j e_i \hat{K}Q e_i$. In other words, $\rho_\alpha$ is a (possibly infinite) linear combination of paths starting at $j$ and ending at $i$.
2. $\sum_{\alpha \in Q_1} \alpha \rho_\alpha = 0$ in $\hat{K}Q$.

The Jacobian algebra of $(\rho_\alpha)_{\alpha \in Q_1}$ is the quotient of $\hat{K}Q$ by the closure of the ideal generated by the elements $\rho_\alpha$,

$\mathcal{P}(Q, (\rho_\alpha)_{\alpha \in Q_1}) = \hat{K}Q / (\rho_\alpha : \alpha \in Q_1)$. 

Any potential \( W \) gives rise to a hyperpotential by taking its cyclic derivatives 
\((\partial_\alpha W)_{\alpha \in Q}\). This is essentially Connes’ map \( B \) from \( HC_0(\hat{K}Q) \) to \( HH_1(\hat{K}Q) \). Conversely, when char \( K = 0 \), any hyperpotential arises in this way, see the discussion at the end of [56] §6.1.

It is possible to define a Ginzburg dg-algebra for a hyperpotential and follow Keller’s proof to show that it has the same homological properties as in the case of potentials, see [66]. Therefore Amiot’s construction applies and we deduce the following.

**Proposition 2.14.** Any finite-dimensional Jacobian algebra of a quiver with hyperpotential is 2-CY-tilted.

**Example 2.15.** Consider the algebra \( A = K[x]/(x^n) \) of Example 1.9 over a field \( K \) with characteristic \( p \geq 0 \), and consider the quiver \( Q \) consisting of one vertex and one loop, denoted \( x \), at that vertex. If \( p \) does not divide \( n + 1 \), then for any \( c \in K^\times \), the algebra \( A \) is the Jacobian algebra of the potential \( W = cx^{n+1} \) on \( Q \). However, if \( p \) divides \( n + 1 \), then \( A \) is not a Jacobian algebra of a potential on \( Q \). Nevertheless, the sequence consisting of the single element \( x^n \) is always a hyperpotential on \( Q \), hence \( A \) is 2-CY-tilted regardless of the characteristic of \( K \).

2.4. **Periodicity.** A large class of symmetric 2-CY-tilted algebras has been constructed by Burban, Iyama, Keller and Reiten [22]. In their construction, the ambient 2-Calabi-Yau triangulated categories are the stable categories of maximal Cohen-Macaulay modules over odd dimensional isolated hypersurface singularities. These categories are also 0-Calabi-Yau since the square of the suspension functor is isomorphic to the identity. Therefore, the endomorphism algebra of any object is symmetric (cf. Proposition 1.3).

The next proposition provides a partial converse. We start with one cluster-tilting object in a 2-Calabi-Yau category \( \mathcal{C} \) whose endomorphism algebra \( \Lambda \) is symmetric and study the implications this has on the structure of \( \mathcal{C} \) and \( \text{mod} \Lambda \).

**Proposition 2.16.** Let \( \Lambda \) be a finite-dimensional symmetric algebra that is also 2-CY-tilted, i.e. \( \Lambda = \text{End}_{\mathcal{C}}(T) \) for some cluster-tilting object \( T \) within a triangulated 2-Calabi-Yau category \( \mathcal{C} \) with suspension functor \( \Sigma \).

(a) The functor \( \Omega^4 \) on the stable module category \( \text{mod} \Lambda \) is isomorphic to the identity, hence all non-projective \( \Lambda \)-modules are \( \Omega \)-periodic with period dividing 4.

(b) The functor \( \Sigma^2 \) acts as the identity on the objects of \( \mathcal{C} \).

(c) Assume that \( \Lambda \) is a Jacobian algebra of a hyperpotential. Then this hyperpotential is rigid if and only if \( \Lambda \) is semi-simple.

For part (a), note that rigid quivers with potentials have been defined in [25] Definitions 3.4 and 6.10 in terms of vanishing of the deformation space of their Jacobian algebras. This definition carries over without any modification to hyperpotentials. In particular, a hyperpotential with finite-dimensional Jacobian algebra \( \Lambda \) is rigid if and only if \( HH_0(\Lambda) = \Lambda/[[\Lambda, \Lambda]] \) is spanned by the images of the primitive idempotents corresponding to the vertices.

Let us give the short proof of part (a). We note that parts (a) and (b) of the proposition have also been recently observed by Valdivieso-Diaz [90].

**Proof** of part (a). On the one hand, \( \Lambda \) is symmetric, hence \( \text{mod} \Lambda \) is \((-1)\)-Calabi-Yau (Remark 1.7). On the other hand, \( \Lambda \) is 2-CY-tilted, hence \( \text{mod} \Lambda \) is 3-Calabi-Yau (Prop. 2.10). The uniqueness of the Serre functor implies that the fourth power of the suspension on \( \text{mod} \Lambda \) is isomorphic to the identity functor, and since the suspension is \( \Omega^{-1} \), we get the result. \( \square \)
Example 2.17. Let \( n \geq 1 \) and consider the algebra \( A = K[x]/(x^n) \). It is symmetric and 2-CY-tilted (Example 2.15). By Proposition 2.16 \( \Omega_A^2 M \simeq M \) for any \( M \in \text{mod } A \). Indeed, in this case even \( \Omega_A^2 M \simeq M \), see Example 1.9.

As a direct consequence of Proposition 2.16 and Definition 1.12 we obtain the next statement.

Corollary 2.18. An indecomposable, symmetric, 2-CY-tilted algebra of tame representation type is of quasi-quaternion type.

2.5. Derived equivalences. In this section all cluster-tilting objects are assumed to be basic, i.e. they decompose into a direct sum of non-isomorphic indecomposable objects. Iyama and Yoshino [50] have shown that there is a well-defined notion of mutation of (basic) cluster-tilting objects in a triangulated 2-Calabi-Yau category \( C \).

Proposition 2.19 ([50, Theorem 5.3]). Let \( T \) be a cluster-tilting object in \( C \), let \( X \) be an indecomposable summand of \( T \) and write \( T = \bar{T} \oplus X \). Then there exists a unique indecomposable object \( X' \) of \( C \) which is not isomorphic to \( X \) such that \( T' = \bar{T} \oplus X' \) is a cluster-tilting object in \( C \).

The cluster-tilting object \( T' \) in the proposition is called the Iyama-Yoshino mutation of \( T \) at \( X \). The algebras \( \Lambda = \text{End}_C(T) \) and \( \Lambda' = \text{End}_C(T') \) are said to be neighboring 2-CY-tilted algebras.

Let \((Q, W)\) be a quiver with potential and let \( k \) be a vertex in \( Q \) such that no 2-cycle (i.e. a cycle of length 2) passes through \( k \). Derksen, Weyman and Zelevinsky have defined in [25, §5] the mutation of \((Q, W)\) at \( k \), which is a quiver with potential denoted \( \mu_k(Q, W) \). Buan, Iyama, Reiten and Smith have shown in [18] that under some mild conditions the notions of Iyama-Yoshino mutation and mutation of quivers with potentials are compatible. This is expressed in the next proposition.

Proposition 2.20 ([18, Theorem 5.2]). Let \( T \) be a cluster-tilting object in \( C \). Assume that \( \text{End}_C(T) \simeq \mathcal{P}(Q, W) \) for some quiver with potential \((Q, W)\) and that \( \text{End}_C(T) \) satisfies the vanishing condition. Let \( k \) be a vertex of \( Q \) such that no 2-cycle passes through \( k \), let \( X \) be the corresponding indecomposable summand of \( T \) and let \( T' \) be the Iyama-Yoshino mutation of \( T \) at \( X \). Then \( \text{End}_C(T') \simeq \mathcal{P}(\mu_k(Q, W)) \).

For the precise formulation of the vanishing condition we refer the reader to [18], but for our purposes it is sufficient to note that this condition holds when the algebra \( \text{End}_C(T) \) is self-injective, and in particular when it is symmetric.

Neighboring 2-CY-tilted algebras are nearly Morita equivalent in the sense of Ringel [55], that is, there is an equivalence of categories

\[ \text{mod } \Lambda/\text{add } S \simeq \text{mod } \Lambda'/\text{add } S' \]

where \( S \) (respectively, \( S' \)) is the simple module which is the top of the indecomposable projective \( \Lambda \)-module (respectively, \( \Lambda' \)-module) corresponding to the summand \( X \) of \( T \) (respectively, \( X' \) of \( T' \)), provided there are “no loops”, i.e. any non-isomorphism \( X \rightarrow X \) (or \( X' \rightarrow X' \)) factors through \( \text{add } T \), see [57, Proposition 2.2]. However, neighboring 2-CY-tilted algebras are not necessarily derived equivalent, see for example [71, Example 5.2].

The next statement concerns the derived equivalence of neighboring 2-CY-tilted algebras. It is an improvement of [71, Theorem 5.3] which has turned out to be a very useful tool in derived equivalence classifications of various cluster-tilted algebras and Jacobian algebras [6, 7, 68]. The derived equivalences are instances of (refined version of) good mutations introduced in our previous work [71]. Before formulating the result, we recall some relevant notions.
Let $\Lambda$ be a basic algebra and $P$ an indecomposable projective $\Lambda$-module. Consider the silting mutations in the sense of Aihara and Iyama [2] of $\Lambda$ at $P$ within the triangulated category $\text{Per}\Lambda$ of perfect complexes, which are the following two-term complexes

\begin{equation}
U^-_P(\Lambda) = (P \to Q') \oplus Q, \quad U^+_P(\Lambda) = (Q'' \to P) \oplus Q,
\end{equation}

where $Q', Q'' \in \text{add} Q$, the maps are left (resp., right) (add $Q$)-approximations and $Q, Q', Q''$ are in degree 0. These two-term complexes of projective modules are known also as Okuyama-Rickard complexes. In [71] we considered these complexes in relation with our definition of mutations of algebras.

An algebra is weakly symmetric if for any simple module, its projective cover is isomorphic to its injective envelope. Symmetric algebras are weakly symmetric and if $\Lambda$ is weakly symmetric, then the complexes $U^-_P(\Lambda)$ and $U^+_P(\Lambda)$ are tilting complexes.

**Proposition 2.21.** Let $T$ be a cluster-tilting object in a triangulated 2-Calabi-Yau category $C$, let $X$ be an indecomposable summand of $T$ and let $T'$ be the Iyama-Yoshino mutation of $T$ at $X$. Consider the algebras $\Lambda = \text{End}_C(T)$ and $\Lambda' = \text{End}_C(T')$. Let $P$ be the indecomposable projective $\Lambda$-module corresponding to $X$ and let $P'$ be the indecomposable projective $\Lambda'$-module corresponding to $X'$.

(a) If $U^-_P(\Lambda)$ and $U^+_P(\Lambda')$ are tilting complexes (over $\Lambda$ and $\Lambda'$, respectively), then

\[ \text{End}_{\mathcal{D}^b(\Lambda)} U^-_P(\Lambda) \simeq \Lambda' \quad \text{and} \quad \text{End}_{\mathcal{D}^b(\Lambda')} U^+_P(\Lambda') \simeq \Lambda. \]

(b) If $U^+_P(\Lambda)$ and $U^-_P(\Lambda')$ are tilting complexes (over $\Lambda$ and $\Lambda'$, respectively), then

\[ \text{End}_{\mathcal{D}^b(\Lambda)} U^+_P(\Lambda) \simeq \Lambda' \quad \text{and} \quad \text{End}_{\mathcal{D}^b(\Lambda')} U^-_P(\Lambda') \simeq \Lambda. \]

(c) If $\Lambda$ is weakly symmetric, then $\Lambda'$ is also weakly symmetric by [10 §4.2], hence all the complexes $U^-_P(\Lambda)$, $U^+_P(\Lambda)$, $U^-_P(\Lambda')$ and $U^+_P(\Lambda')$ are tilting complexes and

\[ \text{End}_{\mathcal{D}^b(\Lambda)} U^-_P(\Lambda) \simeq \Lambda' \simeq \text{End}_{\mathcal{D}^b(\Lambda')} U^+_P(\Lambda). \]

In particular, $\Lambda$ and $\Lambda'$ are derived equivalent.

(d) If $\Lambda$ is symmetric then $\Lambda'$ is symmetric.

We note that there are related works by Dugas [27] concerning derived equivalences of symmetric algebras and by Mizuno [75] concerning derived equivalences of self-injective quivers with potential.

As the category of perfect complexes over a symmetric algebra is 0-Calabi-Yau, the derived equivalences in part (c) can be considered as 0-CY analogs of the derived equivalences of Iyama-Reiten [49] and Keller-Yang [58 Theorem 6.2] for 3-CY-algebras.

**Definition 2.22.** Let $T$ be a cluster-tilting object in a triangulated 2-Calabi-Yau category $C$. A cluster-tilting object $T'$ in $C$ is reachable from $T$ if it can be obtained from $T$ by finitely many Iyama-Yoshino mutations at indecomposable summands.

**Corollary 2.23.** Let $T$ be a cluster-tilting object in a triangulated 2-Calabi-Yau category $C$ and assume that $\Lambda = \text{End}_C(T)$ is (weakly) symmetric. Then for any cluster-tilting object $T'$ in $C$ that is reachable from $T$, the algebra $\Lambda' = \text{End}_C(T')$ is (weakly) symmetric and derived equivalent to $\Lambda$.

**Remark 2.24.** There are examples of triangulated 2-Calabi-Yau categories $C$ with a cluster-tilting object $T$ such that $\Sigma T$ is not reachable from $T$, see [69 §3] and [78].
Example 4.3]. Interestingly, in all of these examples the algebra \( \text{End}_C(T) \) is symmetric. Note, however, that \( \text{End}_C(\Sigma T) \simeq \text{End}_C(T) \) and in particular these algebras are derived equivalent.

We can rephrase part (c) of Proposition 2.21 as follows.

**Corollary 2.25.** Let \( \Lambda \) be a weakly symmetric 2-CY-tilted algebra and let \( P \) be an indecomposable projective \( \Lambda \)-module. Then the two algebras \( \text{End}_{D^b(\Lambda)}U_P(\Lambda) \) and \( \text{End}_{D^b(\Lambda)}U_P^+(\Lambda) \) are isomorphic, 2-CY-tilted and derived equivalent to \( \Lambda \).

We see that derived equivalences of a particular kind preserve the property of an algebra being symmetric 2-CY-tilted. One may ask whether this is still true for arbitrary derived equivalences.

**Question 2.26.** Let \( \Lambda \) be a symmetric 2-CY-tilted algebra and let \( \Lambda' \) be an algebra derived equivalent to \( \Lambda \). Is \( \Lambda' \) also 2-CY-tilted?

One may also ask if a converse to Proposition 2.16(a) holds.

**Question 2.27.** Let \( \Lambda \) be a symmetric algebra such that \( \Omega^4_\Lambda M \simeq M \) for any \( M \in \text{mod} \Lambda \). Is \( \Lambda \) then 2-CY-tilted?

Observe that by Proposition 1.15 and Proposition 2.16 an affirmative answer to Question 2.24 will yield an affirmative answer to Question 2.26. We note that the answer to Question 2.24 is positive in the following cases: \( \Lambda \) is of finite representation type (Theorem 6.3); \( \Lambda \) is tame with non-singular Cartan matrix (Theorem 8.3); or \( \Lambda \) is tame of polynomial growth (Proposition 8.11).

3. RIBBON QUIVERS AND TRIANGULATION QUIVERS

In this section we develop a theory of ribbon quivers and ribbon graphs, with an emphasis on a particular class of ribbon quivers called triangulation quivers. The connections to ideal triangulations of marked surfaces and dimer models will be explained in Section 4. Ribbon quivers and triangulation quivers are the combinatorial ingredients underlying the definition of Brauer graph algebras and triangulation algebras which will be introduced in Section 5 and studied later in this survey. The combinatorial statements in this section will be stated without proofs, and the details will appear elsewhere.

3.1. Ribbon quivers. Recall from Section 2.2 that a quiver \( Q \) is quadruple \( Q = (Q_0, Q_1, s, t) \) where \( Q_0, Q_1 \) are finite sets and \( s, t : Q_1 \to Q_0 \).

**Definition 3.1.** A **ribbon quiver** is a pair \( (Q, f) \) consisting of a quiver \( Q \) and a permutation \( f : Q_1 \to Q_1 \) on its set of arrows satisfying the following conditions:

(i) At each vertex \( i \in Q_0 \) there are exactly two arrows starting at \( i \) and two arrows ending at \( i \);

(ii) For each arrow \( \alpha \in Q_1 \), the arrow \( f(\alpha) \) starts where \( \alpha \) ends.

Note that loops are allowed in \( Q \). A loop at a vertex is counted both as an incoming and outgoing arrow at that vertex.

**Example 3.2.** Consider a ribbon quiver \( (Q, f) \) with one vertex. Condition (i) implies that \( Q \) must have two loops as in the following picture

\[
\alpha \bigcirc \bigcirc \beta
\]

and condition (ii) is empty in this case, so that \( f \) equals one of the two permutations \( f_1 \) or \( f_2 \) on \( Q_1 \) given in cycle form by \( f_1 = (\alpha)(\beta) \) and \( f_2 = (\alpha \beta) \). In particular we see that the underlying quiver does not determine the ribbon quiver structure.
Let \((Q, f)\) be a ribbon quiver. Since at each vertex of \(Q\) there are exactly two outgoing arrows, there is an involution \(\alpha \mapsto \bar{\alpha}\) on \(Q_1\) mapping each arrow \(\alpha\) to the other arrow starting at the vertex \(s(\alpha)\). Composing it with \(f\) gives rise to the permutation \(g: Q_1 \to Q_1\) given by \(g(\alpha) = \overline{f(\alpha)}\) so that for each arrow \(\alpha\), the set \(\{f(\alpha), g(\alpha)\}\) consists of the two arrows starting at the vertex which \(\alpha\) ends at.

Denote by \(Q_1^f\) and \(Q_1^g\) the subsets of arrows fixed by \(f\) and \(g\), respectively, i.e. \(Q_1^f = \{\alpha \in Q_1 : f(\alpha) = \alpha\}\) and \(Q_1^g = \{\alpha \in Q_1 : g(\alpha) = \alpha\}\). The set of loops in \(Q\) thus decomposes as a disjoint union \(Q_1^f \cup Q_1^g\).

Given a quiver \(Q\) satisfying condition \(\bar{\phi}\) in the definition, the data of the permutation \(f\) is equivalent to the data of the permutation \(g\). Thus from now on when considering a ribbon quiver \((Q, f)\) we will freely refer to the involution \(\alpha \mapsto \bar{\alpha}\) and the permutation \(g\) as defined above.

**Lemma 3.3.** Let \(\alpha \in Q_1\). Then \(f^{-1}(\alpha) = g^{-1}(\bar{\alpha})\) and \(gf^{-2}(\alpha) = fg^{-2}(\bar{\alpha})\).

**Definition 3.4.** Let \((Q, f)\) be a ribbon quiver and define \(g: Q_1 \to Q_1\) by \(g(\alpha) = \overline{f(\alpha)}\). The dual of \((Q, f)\) is the ribbon quiver \((Q, g)\).

**Example 3.5.** In Example 3.2, \(\alpha = \beta\) and \(\beta = \alpha\), so in cycle form \(q_1 = (\alpha \beta) = f_2\) and \(q_2 = (\alpha)(\beta) = f_1\). Hence \((Q_1, f_1)\) and \((Q_1, f_2)\) are dual to each other.

**Definition 3.6.** Let \((Q, f)\) and \((Q', f')\) be ribbon quivers with \(Q = (Q_0, Q_1, s, t)\), \(Q' = (Q_0', Q_1', s', t')\). Recall that a pair of bijections \(\varphi_0: Q_0 \to Q_0'\) and \(\varphi_1: Q_1 \to Q_1'\) is an isomorphism between the quivers \(Q\) and \(Q'\) if \(\varphi_0s = s'\varphi_1\) and \(\varphi_0t = t'\varphi_1\). If, in addition, \(\varphi_1f = f'\varphi_1\) and \(\varphi_1(\bar{\alpha}) = \overline{\varphi_1(\alpha)}\) for any \(\alpha \in Q_1\), we say that \((\varphi_0, \varphi_1)\) is isomorphism between the ribbon quivers \((Q, f)\) and \((Q', f')\).

Ribbon quivers are closely related to ribbon graphs. To avoid confusion, we shall use the term “node” for the graph in order to distinguish it from a vertex in the quiver. Informally speaking, a ribbon graph is a graph consisting of nodes and edges together with a cyclic ordering of the edges around each node. This can be made more formal in the next definition.

**Definition 3.7.** A ribbon graph is a triple \((H, \iota, \sigma)\) where \(H\) is a finite set, \(\iota\) is an involution on \(H\) without fixed points and \(\sigma\) is a permutation on \(H\).

The elements of \(H\) are called half-edges. A ribbon graph gives rise to a graph \((V, E)\) (possibly with loops and multiple edges between nodes) as follows. The set \(V\) of nodes consists of the cycles of \(\sigma\) and the set \(E\) of edges consists of the cycles of \(\iota\). An edge \(e \in E\) can be written as \((h, \iota(h))\) for some \(h \in H\). The \(\iota\)-cycles that \(h\) and \(\iota(h)\) belong to are the nodes that \(e\) is incident to. Moreover, \(\sigma\) induces a cyclic ordering of the edges around each node.

Conversely, given a graph \((V, E)\) with a cyclic ordering of the edges around each node, we think of each edge \(e \in E\) incident to the nodes \(e', e''\) \(\in V\) (which may coincide) as composed of two half-edges \(e'\) and \(e''\), with \(e'\) incident to \(e''\) and \(e''\) incident to \(e'\). This yields a ribbon graph \((H, \iota, \sigma)\) where \(H\) is the set of all half-edges, \(\iota = \prod_{e \in E}(e' e'')\) is the product of all the transpositions \((e' e'')\) for \(e \in E\), and for any half-edge \(h\) incident to a node \(v\), the half-edge \(\sigma(h)\) is the one following \(h\) in the cyclic order around \(v\).

**Example 3.8.** Consider a ribbon graph with one edge. In this case the set \(H\) of half-edges consists of two elements, which we denote by \(\alpha\) and \(\beta\), and the involution \(\iota\) can be written as \(\iota = (\alpha \beta)\) in cycle form. The permutation \(\sigma\) equals one of the two permutations \(\sigma_1\) or \(\sigma_2\) given in cycle form by \(\sigma_1 = (\alpha \beta)\) and \(\sigma_2 = (\alpha)(\beta)\).

The corresponding graphs, with their half-edges labeled, are shown in the picture below. Since \(\sigma_1\) has one cycle, the graph of \((H, \iota, \sigma_1)\), shown to the left, has one
\[(Q, f) \quad (H, i, \sigma) \quad (V, E)\]

| vertex | cycle of \(i\) | edge |
|--------|----------------|------|
| arrow  | element of \(H\) | half-edge |
| \(f\)  | \(i\sigma\)     | cyclic ordering |
| \(g\)  | \(\sigma\)       | node |

Table 1. Dictionary between ribbon quivers and ribbon graphs.

node. Similarly, since \(\sigma_2\) has two cycles, the graph of \((H, i, \sigma_2)\), shown to the right, has two nodes.

\[
\bigcirc_{\beta} \sigma_1 = (\alpha \beta) \quad \bigcirc_{\beta} \sigma_2 = (\alpha)(\beta)
\]

**Definition 3.9.** Let \((H, i, \sigma)\) and \((H', i', \sigma')\) be ribbon graphs. An isomorphism between \(H\) and \(H'\) is a bijection \(\phi: H \to H'\) satisfying \(i'\phi = \phi i\) and \(\sigma'\phi = \phi \sigma\).

Any ribbon quiver \((Q, f)\) gives rise to a ribbon graph \((H, i, \sigma)\) by taking \(H = Q_1\) and defining \(i(\alpha) = \bar{\alpha}\) and \(\sigma(\alpha) = f(\alpha)\) for each \(\alpha \in Q_1\).

Conversely, a ribbon graph \((H, i, \sigma)\) gives rise to a ribbon quiver \((Q, f)\) as follows. Set \(Q_1 = H\) and take \(Q_0\) to be the set of cycles of \(i\). Define the maps \(s, t: Q_1 \to Q_0\) and the permutation \(f: Q_1 \to Q_1\) by letting, for each \(h \in H\), \(s(h)\) to be the \(i\)-cycle that \(h\) belongs to and setting \(t = s\sigma\) and \(f = i\sigma\).

Note that these two constructions are inverses of each other, hence we deduce the following.

**Proposition 3.10.** There is a bijection between the set of isomorphism classes of ribbon quivers and the set of isomorphism classes of ribbon graphs,

\[
\left(\{\text{ribbon quivers}\}/\cong\right) \leftrightarrow \left(\{\text{ribbon graphs}\}/\cong\right).
\]

Under this bijection, the various notions concerning ribbon quivers and ribbon graphs are related as in the dictionary given in Table 1.

**Example 3.11.** We illustrate the bijection between the ribbon quivers with one vertex discussed in Example 3.2 and the ribbon graphs with one edge discussed in Example 3.8. We denote the set of half-edges by \(\{\alpha, \beta\}\) and let \(i = (\alpha \beta)\). The underlying quiver \(Q\) is always

\[
\begin{array}{c}
\alpha \\
\bigcirc \\
\beta
\end{array}
\]

and the corresponding graphs are shown in the right column below.

\[
f_1 = (\alpha)(\beta) \quad \sigma_1 = g_1 = (\alpha \beta) \quad \bigcirc_{\beta}
\]

\[
f_2 = (\alpha \beta) \quad \sigma_2 = g_2 = (\alpha)(\beta) \quad \bigcirc_{\beta}
\]

The data of a graph can be encoded in matrix form in the following way. Let \((V, E)\) be a graph. For a node \(v \in V\), define a vector \(\chi_v \in \mathbb{Z}^E\) by

\[
\chi_v(e) = \begin{cases} 
2 & \text{e is a loop incident to } v, \\
1 & \text{e is incident to } v \text{ but is not a loop,} \\
0 & \text{e is not incident to } v 
\end{cases}
\]

and think of it as a row vector. Obviously, \(\chi_v(e) \geq 0\) and \(\sum_{v \in V} \chi_v(e) = 2\) for any \(e \in E\), so by arranging the vectors \(\chi_v\) as a \(V \times E\) matrix, one gets an integer matrix with non-negative entries whose sum of rows equals the constant vector \((2, 2, \ldots, 2)\). Conversely, any such matrix \(\chi\) gives rise to a graph whose nodes are indexed by the
rows of $\chi$, its edges are indexed by the columns of $\chi$ and the incidence relations are read from the entries $\chi_{\nu}(e)$.

Now let $(Q, f)$ be a ribbon quiver. In the underlying graph $(V, E)$ of the ribbon graph corresponding to $(Q, f)$ under the bijection of Proposition 3.10, the set $V$ corresponds to the set $\Omega_g$ of cycles of the permutation $g$, the set $E$ corresponds to the set $Q_0$ of vertices of $Q$ and the entries of the matrix $\chi$ are given by $\chi_{\omega}(i) = |\{\alpha \in \omega : s(\alpha) = i\}|$ for any $g$-cycle $\omega \in \Omega_g$ and vertex $i \in Q_0$.

3.2. Triangulation quivers.

**Definition 3.12.** A triangulation quiver is a ribbon quiver $(Q, f)$ such that $f^3$ is the identity on the set of arrows.

**Example 3.13.** Considering the ribbon quivers with one vertex of Example 3.2, we see that $(Q, f_1)$ is a triangulation quiver whereas $(Q, f_2)$ is not.

**Remark 3.14.** Given any quiver $Q$ satisfying condition (i) of Definition 3.1, there is always at least one (and in general, many) permutation(s) $f$ on the arrows making $(Q, f)$ a ribbon quiver. Indeed, for each $i \in Q_0$ label by $\alpha, \beta$ the arrows ending at $i$ and by $\gamma, \delta$ the arrows starting at $i$ and set, for instance, $f(\alpha) = \gamma$ and $f(\beta) = \delta$.

However, as the next example demonstrates, there may not exist a permutation $f$ making $(Q, f)$ a triangulation quiver. In other words, the existence of a triangulation quiver $(Q, f)$ imposes some restrictions on the shape of a quiver $Q$.

**Example 3.15.** Up to isomorphism, there are two ribbon quivers whose underlying quiver is the one given below,

```
• 1 → → → → • 2
```

Namely, denoting the arrows from 1 to 2 by $\alpha, \gamma$ and those from 2 to 1 by $\beta, \delta$, the ribbon quivers are given by the permutations $(\alpha\beta)(\gamma\delta)$ and $(\alpha\beta\gamma\delta)$. None of them is a triangulation quiver.

We have seen that not every quiver satisfying condition (i) of Definition 3.1 is an underlying quiver of a triangulation quiver. The next proposition tells us that if such triangulation quiver exists, then it is unique up to isomorphism.

**Proposition 3.16.** Let $(Q, f)$ and $(Q', f')$ be two triangulation quivers. If the quivers $Q$ and $Q'$ are isomorphic, then $(Q, f)$ and $(Q', f')$ are isomorphic as ribbon quivers.

Since the number of triangulation quivers with a given number of vertices is finite, they can be enumerated on a computer. Table 2 lists (up to isomorphism) the connected triangulation quivers with at most three vertices and their corresponding ribbon graphs. Note that the ribbon graph of quiver 2 could have also been drawn as

```
• 1 → → • 2
```

but the drawing in Table 2 emphasizes the relation of this quiver to the punctured monogon, as we shall see in Section 4.

**Remark 3.17.** As the entries in rows 3' and 3'' of Table 2 demonstrate, two different ribbon graphs can have the same underlying graph (in this case a node with three loops).

**Remark 3.18.** In a triangulation quiver $(Q, f)$, the permutations $\alpha \mapsto \bar{\alpha}$ and $\alpha \mapsto f(\alpha)$ are of orders 2 and 3, respectively, hence the group $PSL_2(\mathbb{Z})$, which is the free product of the cyclic groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, acts on the set of arrows $Q_1$. This action is transitive when $Q$ is connected.
The dual of a triangulation quiver \((Q, f)\) need not be a triangulation quiver. However, when it is, then by Proposition 3.16, it must be isomorphic to \((Q, f)\), hence \((Q, f)\) is self dual. The next proposition shows that there are only two connected self dual triangulation quivers.

**Proposition 3.19.** A connected triangulation quiver whose dual is also a triangulation quiver is isomorphic to one of the two triangulation quivers shown in Figure 1.

We call the ribbon graph with two nodes appearing in Figure 1 a punctured monogon, for reasons that will become apparent in Section 4. Similarly, we call the

| Triangulation quiver | Ribbon graph |
|-----------------------|--------------|
| 1                     | ![Triangulation Quiver 1](image1) | ![Ribbon Graph 1](image2) |
| 2                     | ![Triangulation Quiver 2](image3) | ![Ribbon Graph 2](image4) |
| 3a                    | ![Triangulation Quiver 3a](image5) | ![Ribbon Graph 3a](image6) |
| 3b                    | ![Triangulation Quiver 3b](image7) | ![Ribbon Graph 3b](image8) |
| 3'                    | ![Triangulation Quiver 3'](image9) | ![Ribbon Graph 3'](image10) |
| 3''                   | ![Triangulation Quiver 3''](image11) | ![Ribbon Graph 3''](image12) |

**Table 2.** The connected triangulation quivers with at most 3 vertices. We list the triangulation quivers and the corresponding ribbon graphs, where we write the permutation \(f\) in cycle form below each quiver.
Figure 1. The connected self dual triangulation quivers and the corresponding ribbon graphs, a punctured monogon (top) and a tetrahedron (bottom).

Figure 2. Blocks for triangulation quivers. The permutation is given in cycle form below each quiver.

ribbon graph with four nodes appearing in Figure 1 a tetrahedron. In the triangulation quiver corresponding to the tetrahedron there are four f-cycles and four g-cycles, each of length 3, and for any arrow α, each of the arrows α, f(α), ̄α, f(̄α) belongs to a different g-cycle.

3.3. Block decomposition of triangulation quivers. In this section we analyze the structure of triangulation quivers in terms of three types of building blocks. This is similar in spirit to the block decomposition of [36, §13], however the number of blocks in our case is smaller and only full matchings are used.

**Definition 3.20.** A block is one of the three pairs, each consisting of a quiver and a permutation on its set of arrows, shown in Figure 2. A vertex of a block marked with white circle (◦) is called an outlet.

Let $B_1, B_2, \ldots, B_s$ be a collection of blocks. Denote by $V_1, V_2, \ldots, V_s$ their corresponding sets of outlets and let $V = \bigsqcup_{i=1}^s V_i$ be their disjoint union. A matching on $V$ is an involution $θ: V → V$ without fixed points such that $θ(V_i) ∩ V_i$ is empty for each $1 ≤ i ≤ s$ (in other words, an outlet cannot be matched to an outlet in the same block).

Given a collection of blocks and a matching $θ$ on their outlets, construct a quiver $Q$ and a permutation $f$ on its set of arrows as follows; take the disjoint union of
### Table 3. Block decompositions of the triangulation quivers with at most three vertices. The numbers of the quivers refer to Table 2.

| Quiver | Block decomposition |
|--------|---------------------|
| 1      | A, A                |
| 2      | B, A                |
| 3a     | B, B                |
| 3b     | C, C                |
| 3'     | C, A, A, A         |
| 3''    | C, C                |

The blocks and identify each outlet $v \in V$ with the outlet $\theta(v)$ to obtain $Q$. The permutation $f$ on the set of arrows of $Q$ is induced by the permutations on each of the blocks.

**Definition 3.21.** A pair $(Q, f)$ consisting of a quiver $Q$ and a permutation $f$ on its set of arrows is **block-decomposable** if it can be obtained by the above procedure.

**Proposition 3.22.** A block-decomposable pair $(Q, f)$ is a triangulation quiver. Conversely, any triangulation quiver is block-decomposable.

**Example 3.23.** Since each of the blocks of types A and B has only one outlet, there is only one way to match a pair consisting of two such blocks. In contrast, there are two different ways to completely match two blocks of type C, yielding the triangulation quivers $3b$ and $3''$ of Table 2. The block decompositions of the triangulation quivers with at most three vertices are given in Table 3.

**Remark 3.24.** In a block decomposition of a triangulation quiver $(Q, f)$, the blocks of type A are in bijection with the elements of $Q_1^f$, whereas those of type B are in bijection with the elements of $Q_2^f$.

In the theory of cluster algebras, quivers without loops (i.e. cycles of length 1) and 2-cycles (cycles of length 2) play an important role. The block decomposition allows to quickly characterize those triangulation quivers without loops and 2-cycles. Indeed, a loop can only arise from a block of types A or B, whereas a 2-cycle arises either from a block of type B or from gluing two blocks of type C, identifying two pairs of vertices at opposing directions of the arrows. This can be rephrased as follows.

**Proposition 3.25.** Let $(Q, f)$ be a triangulation quiver. Then the length of any non-trivial cycle in $Q$ is at least 3 if and only if the following conditions hold:

(i) There are no arrows fixed by the permutation $f$; and

(ii) the length of any cycle of the permutation $g$ is at least 3.

The block decomposition is also useful in proving the next statement.

**Proposition 3.26.** Let $(Q, f)$ be a triangulation quiver. Then the number of cycles of the permutation $g$ does not exceed the number of vertices of $Q$, and equality holds if and only if $(Q, f)$ is a disjoint union of any of the triangulation quivers 1, 2, 3a or 3b of Table 2.

### 4. Triangulations of marked surfaces and their quivers

In this section we explain how (ideal) triangulations of marked surfaces give rise to triangulation quivers. Marked surfaces were considered by Fomin, Shapiro and Thurston in their work on cluster algebras from surfaces. Let us recall the setup and definitions.
A marked surface is a pair \((S, M)\) consisting of a compact, connected, oriented, Riemann surface \(S\) (possibly with boundary \(\partial S\)) and a finite non-empty set \(M\) of points in \(S\), called marked points, such that each connected component of \(\partial S\) contains at least one point from \(M\). The points in \(M\) which are not on \(\partial S\) are called punctures. We exclude the following surfaces:

- a sphere with one or two punctures;
- an unpunctured digon; (a sphere is a surface of genus 0 with empty boundary, a disc is a surface of genus 0 with one boundary component, an \(m\)-gon is a disc with \(m\) marked points on its boundary, and for \(m = 1, 2, 3\) an \(m\)-gon is called monogon, digon and triangle, respectively).

Up to homeomorphism, \((S, M)\) is determined by the following discrete data:

- the genus \(g\) of \(S\);
- the number \(b \geq 0\) of boundary components;
- the sequence \((n_1, n_2, \ldots, n_b)\) where \(n_i \geq 1\) is the number of marked points on the \(i\)-th boundary component, considered as a multiset;
- the number \(p\) of punctures.

4.1. Triangulation quivers from triangulations. Let \((S, M)\) be a marked surface. An arc \(\gamma\) in \((S, M)\) is a curve in \(S\) satisfying the following:

- the endpoints of \(\gamma\) are in \(M\);
- \(\gamma\) does not intersect itself, except that its endpoints may coincide;
- the relative interior of \(\gamma\) is disjoint from \(M \cup \partial S\);
- \(\gamma\) does not cut out an unpunctured monogon or an unpunctured digon.

Arcs are considered up to isotopy. Two arcs are compatible if there are curves in their respective isotopy classes whose relative interiors do not intersect. A triangulation of \((S, M)\) is a maximal collection of pairwise compatible arcs. The arcs of a triangulation cut the surface \(S\) into ideal triangles. The three sides of an ideal triangle need not be distinct. Sides on the boundary of \(S\) are called boundary segments.

Definition 4.1. Let \(\tau\) be a triangulation of a marked surface \((S, M)\) which is not an unpunctured monogon. Construct a quiver \(Q_\tau\) and \(f_\tau: (Q_\tau)_1 \rightarrow (Q_\tau)_1\) as follows:

- The vertices of \(Q_\tau\) are the arcs of \(\tau\) together with the boundary segments.
- At each vertex corresponding to a boundary segment add a loop \(\delta\) and set \(f(\delta) = \delta\).
- For each ideal triangle in \(\tau\) with sides \(i, j, k\) (which may be arcs or boundary segments) arranged in a clockwise order induced by the orientation of \(S\), add three arrows \(i \xrightarrow{\alpha} j, j \xrightarrow{\beta} k, k \xrightarrow{\gamma} i\) and set \(f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha\) as in Figure 3.

The next statement is immediate from the definitions, observing that in any triangulation \(\tau\), an arc \(\gamma\) of \(\tau\) is either the side of two distinct triangles or there
exists a triangle $\Delta$ such that two of its sides are $\gamma$. In the latter case we say that the triangle $\Delta$ is self-folded and $\gamma$ is its inner side.

**Lemma 4.2.** $(Q_\tau, f_\tau)$ is a triangulation quiver.

**Remark 4.3.** When $(S, M)$ is an unpunctured monogon, a triangulation is empty, there is one boundary segment, and we agree that the associated triangulation quiver is the one with one vertex shown in the top row of Table [2].

**Example 4.4.** Figure 4 shows a triangulation of the square and the corresponding triangulation quiver. There are four boundary segments and for each $1 \leq i \leq 4$ the loop $\delta_i$ corresponds to the boundary segment labeled $i$. The permutation $f$ on the arrows is given in cycle form by $(\alpha_1 \alpha_2 \alpha_3)(\beta_1 \beta_2 \beta_3)(\delta_1)(\delta_2)(\delta_3)(\delta_4)$.

**Remark 4.5.** By using Euler characteristic considerations one sees that if $(S, M)$ is not an unpunctured monogon, then the number of vertices of the triangulation quiver associated to any of its triangulations is

$$6(g-1) + 3(p+b) + 2(n_1 + n_2 + \cdots + n_b),$$

compare [36, Proposition 2.10].

**Remark 4.6.** In terms of the block decomposition of triangulation quivers described in Section 3.3, there is a natural block decomposition of $(Q_\tau, f_\tau)$ induced by the triangulation $\tau$ with bijections

- blocks of type A $\leftrightarrow$ boundary segments,
- blocks of type B $\leftrightarrow$ self-folded triangles in $\tau$,
- blocks of type C $\leftrightarrow$ the other triangles in $\tau$.

In addition, there are also bijections

- cycles of $f_\tau$ of length 1 $\leftrightarrow$ boundary segments,
- cycles of $f_\tau$ of length 3 $\leftrightarrow$ triangles in $\tau$,
- cycles of $g_\tau$ of length 1 $\leftrightarrow$ self-folded triangles in $\tau$,
- cycles of $g_\tau$ $\leftrightarrow$ punctures and boundary components.

We can also obtain the triangulation quiver via a ribbon graph naturally associated to the triangulation. Informally speaking, one thinks of the triangulation as the graph, but some modifications are needed at the boundary components, as in the next definition.

**Definition 4.7.** Let $\tau$ be a triangulation of a marked surface $(S,M)$. Associate to $\tau$ a ribbon graph defined as a graph $(V, E_\tau)$ with cyclic ordering of the edges around each node as follows:
the set \( V \) of nodes consists of the punctures in \( M \) and the connected components of \( \partial S \),

- the set \( E_\tau \) of edges consists of the arcs of \( \tau \) and the boundary segments.

Denote by \( \pi : M \rightarrow V \) the map taking each puncture to itself and each marked point on \( \partial S \) to the boundary component it belongs to. In the graph \((V, E_\tau)\), each edge is incident to the nodes which are the images under \( \pi \) of its endpoints.

The cyclic ordering is determined as follows. If \( v \in V \) is a puncture, then the edges incident to \( v \) are arcs of \( \tau \) and the cyclic ordering of them is the counterclockwise ordering induced by the orientation of \( S \).

If \( v \in V \) is a boundary component, we arrange the set \( \pi^{-1}(v) \) of marked points on \( v \) in a counterclockwise order \( \{q_0, q_1, \ldots, q_{n-1}\} \) such that for each \( 0 \leq i < n \) there is a boundary segment \( \varepsilon_i \) whose endpoints are \( q_i, q_{i+1} \) (where indices are taken modulo \( n \)). The set of edges incident to \( v \) thus consists of the boundary segments \( \varepsilon_i \), which become loops in the graph (see Figure 5), and the arcs incident to any of the marked points \( q_i \). Their cyclic ordering is obtained by taking the arcs incident to \( q_0 \) in the counterclockwise order induced by the orientation of \( S \), then \( \varepsilon_0 \), then the arcs incident to \( q_1 \) in a counterclockwise order, etc.

The next statement is a consequence of the definitions.

**Proposition 4.8.** For any triangulation \( \tau \) of a marked surface \((S, M)\), the ribbon quiver corresponding under the bijection of Proposition 3.10 to the ribbon graph constructed in Definition 4.7 is the triangulation quiver \((Q_\tau, f_\tau)\).

**Example 4.9.** Table 4 lists the marked surfaces whose triangulation quivers have at most three vertices. For each surface, we list the corresponding triangulation quivers (and ribbon graphs) appearing in Table 2.

Note that the unpunctured monogon and unpunctured triangle have only empty triangulations, so for each of these surfaces there is only one quiver. Similarly, a punctured monogon has only one triangulation, consisting of one arc. A sphere with three punctures has two topologically inequivalent triangulations and hence two triangulation quivers.

**4.2. Triangulation vs. adjacency quivers.** The construction of the triangulation quiver of an ideal triangulation resembles that of the adjacency quiver defined in [36, Definition 4.1], however there are several differences:

1. In the triangulation quiver there are vertices corresponding to the boundary segments and not only to the arcs, as in the adjacency quiver.
2. Our treatment of self-folded triangles is different; in the triangulation quiver there is a loop at each vertex corresponding to the inner side of a self-folded triangle.
3. We do not delete 2-cycles that arise in the quiver (e.g. when there are precisely two arcs incident to a puncture).

**Example 4.10.** Consider the triangulation of the square shown in Figure 4. Its triangulation quiver consists of 5 vertices whereas its adjacency quiver is the Dynkin quiver \( A_1 \) (one vertex, no arrows).
| Quiver | Marked surface                  |
|--------|---------------------------------|
| 1      | monogon, unpunctured            |
| 2      | monogon, one puncture           |
| 3a, 3b | sphere, three punctures         |
| 3'     | triangle, unpunctured           |
| 3''    | torus, one puncture             |

Table 4. The marked surfaces whose triangulation quivers have at most three vertices. The numbers of the quivers refer to Table 2.

As Example 4.9 demonstrates, these differences allow to attach triangulation quivers to marked surfaces that do not admit adjacency quivers, such as a monogon, a triangle or a sphere with three punctures. On the other hand, there are situations where the triangulation quiver and the adjacency quiver of a triangulation coincide. By abuse of notation, in the next statements by referring to a triangulation quiver we actually mean the underlying quiver $Q$ of the pair $(Q, f)$. This is not ambiguous in view of Proposition 3.16.

Recall that a surface $S$ is closed if $\partial S$ is empty. If $(S, M)$ is a marked surface and $S$ is closed, then all marked points are punctures. The next statement is a reformulation of our result in [70, §2].

**Lemma 4.11.** Let $(S, M)$ be a closed marked surface which is not a sphere with less than four punctures. Then for any triangulation $\tau$ of $(S, M)$ with at least three arcs incident to each puncture, the triangulation quiver and adjacency quiver associated to $\tau$ coincide.

The condition on $\tau$ in the lemma was called (T3) in [70]. In particular, we get the following corollary (cf. [70, Lemma 5.3]).

**Corollary 4.12.** Let $(S, M)$ be a closed surface with exactly one puncture, i.e. $|M| = 1$. Then for any triangulation $\tau$ of $(S, M)$, the triangulation quiver and the adjacency quiver associated to $\tau$ coincide.

**Example 4.13.** For a torus with empty boundary and one puncture, the adjacency quiver of any triangulation is known as the Markov quiver and is given by the quiver 3'' in the last row of Table 2.

Another difference between triangulation quivers and adjacency quivers concerns the possibility to recover the topology of the underlying marked surface. It is known [36, §12] that a quiver may arise as adjacency quiver of two triangulations of topologically inequivalent marked surfaces. On the other hand, if $(Q_\tau, f_\tau)$ is the triangulation quiver corresponding to a triangulation $\tau$ of a marked surface $(S, M)$, then the topology of $(S, M)$ can be completely recovered from $(Q_\tau, f_\tau)$. Indeed, the cycles of the permutation $g$ on $(Q_\tau)_1$ are in bijection with the punctures and boundary components of $(S, M)$. For each such cycle $\omega$ set $m_\omega = |\{\alpha \in \omega : f(\alpha) = \alpha\}|$. If $m_\omega = 0$, then $\omega$ corresponds to a puncture, otherwise it corresponds to a boundary component with $m_\omega$ marked points on it. In this way we recovered the parameters $p, b$ and the numbers $n_1, \ldots, n_b$. Once these are known, the genus of $S$ can be recovered using Eq. (4.1).

4.3. A dimer model perspective. Dimer models on a torus have been used to construct non-commutative crepant resolutions of toric Gorenstein singularities [14, 16]. Such resolution is a 3-Calabi-Yau algebra which is a (non-complete) Jacobian algebra of a quiver with potential constructed from the dimer model. In this section we explain how triangulations of closed surfaces give rise to a very
A 2-cell in a dimer model (left) and the corresponding vertex with incident arrows (right).

special kind of dimer models, yet the corresponding (complete) Jacobian algebras (which are triangulation algebras to be defined in Section 5.2) have completely different properties, as we shall see in Section 7.

A dimer model on a closed, compact, connected, oriented surface $S$ is a bipartite graph on $S$ whose complement is homeomorphic to a disjoint union of discs. The set of nodes of this graph can thus be written as a disjoint union $V^+ \cup V^-$. We call the elements of $V^+$ white nodes and those of $V^-$ black nodes. Denote by $E$ the set of edges. An edge $e \in E$ defines a pair $(v^+_e, v^-_e) \in V^+ \times V^-$ consisting of the nodes incident to $e$. Each connected component of the complement defines a 2-cell, and an edge is incident to exactly two 2-cells.

Define two permutations $f, g: E \to E$ on the set of edges as follows. For an edge $e \in E$, let $f(e)$ be the edge following $e$ when going clockwise around the node $v^+_e$ and let $g(e)$ be the edge following $e$ when going counterclockwise around the node $v^-_e$, see the left drawing in Figure 6.

A dimer model gives rise to a quiver $Q$ by taking the graph dual to the graph $(V^+ \cup V^-, E)$. The vertices of $Q$ are thus the 2-cells, and the arrows are in bijection with the edges. Let $\alpha$ be an arrow corresponding to an edge $e \in E$. The endpoints of $\alpha$ are the two 2-cells that $e$ is incident to, and $\alpha$ is oriented in such a way that when going forward in the direction of the arrow, the white node $v^+_e$ is seen to the right while the black node $v^-_e$ is to the left, see Figure 6.

The permutations $f, g$ on $E$ induce permutations (denoted by the same letters) on the set of arrows $Q_1$. For any vertex $i \in Q_0$, each of the permutations $f$ and $g$ induces a bijection between the sets of arrows starting at $i$ and those ending at $i$.

Now we restrict attention to dimer models whose 2-cells are quadrilaterals, i.e. consist of exactly four edges. In this case, for each vertex $i$ of the quiver $Q$ there are exactly two arrows starting at $i$ and two arrows ending at $i$, and $(Q, f)$ thus becomes a ribbon quiver. Let us construct the corresponding ribbon graph $(V', E')$ explicitly in terms of the dimer model. We have $V' = V^-$, that is, the nodes of the ribbon graph are the black nodes of the dimer model, and the edges $E'$ of the ribbon graph are in bijection with the quadrilaterals. There are exactly two black nodes incident to each quadrilateral and the corresponding edge $e'$ in the ribbon graph connects these nodes, see Figure 7. The cyclic ordering around each node of $V'$ is induced by the embedding into the oriented surface $S$.

Finally we further restrict to dimer models whose 2-cells are quadrilaterals and moreover their white nodes are trivalent, i.e. each $v^+ \in V^+$ is incident to exactly three edges. In this case the associated ribbon graph $(V', E')$ is a triangulation of the marked surface $(S, V')$ and the ribbon quiver $(Q, f)$ is a triangulation quiver.

Conversely, given a set $M$ of punctures, any triangulation $\tau$ of $(S, M)$ without self-folded triangles gives rise to a dimer model whose white nodes $V^+_\tau$ are the triangles of $\tau$, its black nodes $V^-_\tau$ are the punctures $M$, and there is an edge
Figure 7. A quadrilateral in a dimer model (left) corresponds to an edge in the ribbon graph (right).

Figure 8. Dimer models on a sphere corresponding to the triangulation quiver 3b of Table 2 (left) and that of the tetrahedron (right). Since they arise from triangulations, all 2-cells are quadrilaterals and all white nodes are trivalent.

Table 5. Dictionary between dimer models, ribbon graphs, ribbon quivers and triangulations of a closed surface. A ribbon graph/quer arises when all 2-cells in the dimer model are quadrilaterals, and a triangulation arises when, in addition, all the white nodes $V^+$ are trivalent.

connecting $\Delta \in V^+_{\tau}$ with $v \in V^-_{\tau}$ if and only if $v$ is incident to $\Delta$ in $\tau$. The 2-cells of this dimer model are quadrilaterals (corresponding bijectively to the arcs of $\tau$) and any $\Delta \in V^+_{\tau}$ is trivalent. For example, Figure 8 shows two dimer models; one for the triangulation of a sphere with three punctures corresponding to the triangulation quiver 3b of Table 2 and the other for the tetrahedron of Figure 1 which is a triangulation of a sphere with four punctures.

The various notions concerning dimer models, ribbon graphs, ribbon quivers and triangulations are related as in the dictionary given in Table 5.

5. Brauer graph algebras and triangulation algebras

In this section we introduce two classes of algebras which turn out to be important for our study, one consists of the well known Brauer graph algebras and the other consists of the newly defined triangulation algebras. Roughly speaking, a Brauer graph algebra arises from any ribbon quiver and auxiliary data given in the
form of scalars and positive integer multiplicities, whereas a triangulation algebra arises from any triangulation quiver with similar auxiliary data.

**Definition 5.1.** Let $(Q, f)$ be a ribbon quiver. Recall from Section 5 the permutation $g: Q_1 \to Q_1$ defined by $g(\alpha) = f(\alpha)$ for any $\alpha \in Q_1$. Given a function $\nu$ on the set $Q_1$ of arrows, we write $\nu_\alpha$ instead of $\nu(\alpha)$. We say that $\nu$ is $g$-invariant if $\nu_{g(\alpha)} = \nu_\alpha$ for any $\alpha \in Q_1$. Similarly, we say that $\nu$ is $f$-invariant if $\nu_{f(\alpha)} = \nu_\alpha$ for any $\alpha \in Q_1$.

Since the $g$-cycles are in bijection with the nodes of the corresponding ribbon graph, a $g$-invariant function can thus be regarded as a function on the nodes of that ribbon graph.

Let $(Q, f)$ be a ribbon quiver. For an arrow $\alpha \in Q_1$, set

$$
n_\alpha = \min\{n > 0 : g^n(\alpha) = \alpha\}$$

$$\omega_\alpha = \alpha \cdot g(\alpha) \cdot \ldots \cdot g^{n_\alpha - 1}(\alpha)$$

$$\omega'_\alpha = \alpha \cdot g(\alpha) \cdot \ldots \cdot g^{n_\alpha - 2}(\alpha)$$

The function $\alpha \mapsto n_\alpha$ is obviously $g$-invariant, telling the length of the $g$-cycle $\omega_\alpha$ starting at $\alpha$. The path $\omega'_\alpha$ is “almost” a cycle; when $n_\alpha = 1$ the arrow $\alpha$ is a loop at some vertex $i$ and $\omega'_\alpha$ is understood to be the path of length zero starting at $i$. Similarly, for an arrow $\alpha \in Q_1$, set

$$k_\alpha = \min\{k > 0 : f^k(\alpha) = \alpha\}$$

$$\xi_\alpha = \alpha \cdot f(\alpha) \cdot \ldots \cdot f^{k_\alpha - 1}(\alpha)$$

$$\xi'_\alpha = \alpha \cdot f(\alpha) \cdot \ldots \cdot f^{k_\alpha - 2}(\alpha)$$

The function $\alpha \mapsto k_\alpha$ is obviously $f$-invariant, telling the length of the $f$-cycle $\xi_\alpha$ starting at $\alpha$. The path $\xi'_\alpha$ is “almost” a cycle; when $k_\alpha = 1$ the arrow $\alpha$ is a loop at some vertex $i$ and $\xi'_\alpha$ is understood to be the path of length zero starting at $i$.

**Lemma 5.2.** For any $\alpha \in Q_1$, the paths $\omega'_\alpha$ and $\xi'_\alpha$ are parallel, i.e. they both start at the same vertex and end at the same vertex.

### 5.1. Brauer graph algebras

In this section we fix a field $K$. Brauer graph algebras form a generalization of Brauer tree algebras. They are algebras defined from combinatorial data consisting of a ribbon graph together with multiplicities and scalars associated to its nodes, see [3, 8, 53]. Many authors start with the ribbon graph and construct the quiver with relations of the corresponding Brauer graph algebra, see for example [13, 74]. We prefer to give the definition directly in terms of the associated ribbon quiver.

**Definition 5.3.** Let $(Q, f)$ be a ribbon quiver, and let $m: Q_1 \to \mathbb{Z}_{\geq 0}$ and $c: Q_1 \to K^\times$ be $g$-invariant functions of multiplicities and scalars, respectively. The *Brauer graph algebra* $\Gamma(Q, f, m, c)$ associated to these data is the quotient of the path algebra $KQ$ by the ideal generated by two types of elements; the elements of the first type are the paths $\alpha \cdot f(\alpha)$ for each $\alpha \in Q_1$ (“zero-relations”) and the elements of the second type are the differences $c_\alpha \omega_\alpha - c_\alpha \omega'_\alpha$ (“commutativity-relations”). In other words,

$$\Gamma(Q, f, m, c) = KQ/(\alpha \cdot f(\alpha), c_\alpha \omega_\alpha - c_\alpha \omega'_\alpha)_{\alpha \in Q_1}.$$ 

(It is clearly enough to take one commutativity-relation for each pair of arrows $\alpha$ and $\tilde{\alpha}$, so these relations can be seen as indexed by the vertices of $Q$).
The next proposition is well known. Special biserial algebras have been defined in [SS, §1] and a classification of the indecomposable modules over these algebras, implying that they are of tame representation type, is given in [91, §2].

**Proposition 5.4.** A Brauer graph algebra is finite-dimensional, symmetric, special biserial and hence of tame representation type.

Moreover, it has been recently shown that over an algebraically closed field the classes of symmetric special biserial algebras and that of Brauer graph algebras coincide [85, Theorem 1.1].

We present a few examples of Brauer graph algebras related to group algebras.

**Example 5.5.** Consider the ribbon quiver \((Q, f_1)\) of Example 3.11. In this case there is only one \(g\)-cycle and hence the auxiliary data consists of one multiplicity \(m \geq 1\) and one scalar \(c \in K^\times\). The path algebra \(KQ\) is the free algebra \(K\langle \alpha, \beta \rangle\) on the generators \(\alpha\) and \(\beta\), and the Brauer graph algebra is

\[K\langle \alpha, \beta \rangle/(\alpha^2, \beta^2, c(\alpha\beta)^m - c(\beta\alpha)^m),\]

hence, up to isomorphism, we may set \(c = 1\). When \(m = 1\) and \(\text{char } K = 2\), this algebra is isomorphic the group algebra of Klein’s four-group.

**Example 5.6.** Consider now the ribbon quiver \((Q, f_2)\) of Example 3.11. In this case there are two \(g\)-cycles and hence the auxiliary data consists of two multiplicities \(m, m' \geq 1\) and two scalars \(c, c' \in K^\times\). The Brauer graph algebra is given by

\[K\langle \alpha, \beta \rangle/(\alpha\beta, \beta\alpha, \alpha^m - c'\beta^{m'}),\]

When \(m' = 1\), the arrow \(\beta\) can be eliminated so the relations in the above presentations are no longer minimal and the algebra becomes isomorphic to the algebra \(K[\alpha]/(\alpha^{m+1})\) considered in Example 1.9.

**Example 5.7.** Let’s describe as group algebras some Brauer graph algebras for a few triangulation quivers appearing in Table 2, under the assumption that \(\text{char } K = 2\).

The Brauer graph algebra of the triangulation quiver number 1 with multiplicity 1 was discussed in Example 5.5; it is the group algebra of Klein’s four group.

Assume now that \(K\) contains a primitive third root of unity. Then the Brauer graph algebra of the triangulation quiver number 2 with multiplicities \(m_\alpha = 2\) and \(m_\beta = m_\gamma = m_\eta = 1\) is Morita equivalent to the group algebra of the symmetric group \(S_4\) [29, V.2.5.1], whereas that of the triangulation quiver number 3b with all multiplicities set to 1 is isomorphic to the group algebra of the alternating group \(A_4\) [29, V.2.4.1] (in both cases the scalars take the constant value 1).

We list a few remarks concerning Brauer graph algebras.

**Remark 5.8.** As Example 5.6 shows, if \(\alpha\) is an arrow such that \(n_\alpha = 1\) and \(m_\alpha = 1\), the corresponding commutativity-relation becomes \(c_\alpha^2\alpha - c_\alpha^2\alpha\omega^{n_\alpha}\) and in the presentation of the Brauer graph algebra as quiver with relations the arrow \(\alpha\) can be eliminated at the expense of adding zero-relations of a third kind, namely \(\omega^{n_\alpha}\beta\) for \(\beta \in \{\alpha, g^{-1}(\alpha)\}\). However, in order to keep the presentation unified, we will not eliminate arrows and add the corresponding new relations.

**Remark 5.9.** If \(K\) is algebraically closed, or more generally, if \(K\) contains an \(m_\alpha\)-th root of \(c_\alpha\) for each \(\alpha \in Q_1\), then by considering the automorphism of \(KQ\) defined by choosing from each \(g\)-cycle one arrow \(\alpha\), sending it to \(c_\alpha^{1/m_\alpha}\alpha\) and keeping all other arrows intact, we see that \(\Gamma(Q, f, m, c) \simeq \Gamma(Q, f, m, 1)\) where \(1\) is the constant function \(1_\alpha = 1\).
In the next statements we explicitly compute the Cartan matrix of a Brauer graph algebra and show that it depends only on the multiplicities and the underlying graph of the ribbon graph corresponding to its defining ribbon quiver.

Throughout, we fix a ribbon quiver \((Q, f)\) together with \(g\)-invariant functions \(m: Q_1 \to \mathbb{Z}_{\geq 0}\) and \(c: Q_1 \to K^*\) of multiplicities and scalars, and consider the Brauer graph algebra \(\Gamma = \Gamma(Q, f, m, c)\). For any \(i \in Q_0\), let \(\alpha, \bar{\alpha}\) be the two arrows starting at \(i\). By definition, the images of the paths \(c_\alpha \omega^m_\alpha\) and \(c_{\bar{\alpha}} \omega^m_{\bar{\alpha}}\) in \(\Gamma\) are equal, and we denote their common value by \(z_i \in \Gamma\). The next statement is a consequence of the definition.

**Lemma 5.10.** A basis of \(\Gamma(Q, f, m, c)\) is given by the images of the paths
\[
\{e_i\}_{i \in Q_0} \cup \{\alpha \cdot g(\alpha) \cdot \ldots \cdot g'(\alpha)\}_{\alpha \in Q_1, 0 \leq r < m_\alpha} - 1 \cup \{z_i\}_{i \in Q_0}.
\]

Given the basis of Lemma 5.10 an argument as in [70, §4.4] allows to compute the Cartan matrix and to draw some conclusions. For a \(g\)-cycle \(\omega\) in \(Q_1\), define a row vector \(\chi_\omega \in \mathbb{Z}^{Q_0}\) by \(\chi_\omega(i) = |{\{\alpha \in \omega : s(\alpha) = i\}}|\) for \(i \in Q_0\). Denote by \(\Omega_\omega\) the set of \(g\)-cycles in \(Q_1\). Recall from Section 5.1 that the matrix \((\chi_\omega(i))_{\omega \in \Omega_\omega, i \in Q_0}\) encodes the underlying graph of the ribbon graph corresponding to \((Q, f)\) and hence depends only on that graph. For any \(\omega \in \Omega_\omega\), the square matrix \(\chi_\omega^T\) is symmetric of rank 1 whose \((i, j)\)-entry is \(\chi_\omega(i)\chi_\omega(j)\) for \(i, j \in Q_0\). The \(g\)-invariant function \(m\) on \(Q_1\) induces a function on \(\Omega_\omega\) which will be denoted by the same letter.

**Proposition 5.11.** Let \(C_\Gamma\) be the Cartan matrix of \(\Gamma(Q, f, m, c)\). Then:

\(\begin{align*}
(a) \quad & C_\Gamma = \sum_{\omega \in \Omega_\omega} m_\omega \chi_\omega^T \chi_\omega. \\
(b) \quad & \dim_K \Gamma = \sum_{\omega \in \Omega_\omega} m_\omega |\omega|^2. \\
(c) \quad & The \textit{quadratic form} \(q_{C_\Gamma}: \mathbb{Z}^{Q_0} \to \mathbb{Z}\) defined by \(q_{C_\Gamma}(x) = xC_\Gamma x^T\) takes non-negative even values; in particular it is non-negative definite. \\
(d) \quad & \text{rank } C_\Gamma \leq \min(|Q_0|, |\Omega_\omega|).
\end{align*}\)

The next remark will not be used in the sequel. Nevertheless, we list it here for completeness.

**Definition 5.12.** Given a non-zero power series \(p(x) = \sum_{i=0}^\infty a_i x^i \in K[[x]] \setminus \{0\}\), let \(m = \min\{i \geq 0 : a_i \neq 0\}\). The least order term of \(p(x)\) is \(c_m x^m\).

For any non-trivial cycle \(\omega \in Q\), the evaluation map \(e_{\omega}: K[[x]] \to \hat{KQ}\) sending \(p \in K[[x]]\) to \(p(\omega)\) is a continuous ring homomorphism.

**Remark 5.13.** In analogy with the triangulation algebras to be defined in the next section as quotients of complete path algebras by closed ideals, one could consider an apparently more general, continuous, version of a Brauer graph algebra defined by taking a \(g\)-invariant function \(p: Q_1 \to xK[[x]] \setminus \{0\}\) of non-zero power series without constant term and forming the quotient of \(\hat{KQ}\) by the closure of the ideal generated by zero-relations and commutativity-relations
\[
\hat{KQ}/(\alpha \cdot f(\alpha), p_\alpha(\omega_\alpha) - p_\alpha(\omega_{\bar{\alpha}}))_{\alpha \in Q_1}.
\]

However, it turns out that this algebra is isomorphic to the Brauer graph algebra \(\Gamma(Q, f, m, c)\) where the multiplicities \(m_\alpha\) and scalars \(c_\alpha\) are such that each \(c_\alpha x^{m_\alpha}\) is the least order term of \(p_\alpha(x)\).

5.2. **Triangulation algebras.** In this section we define, for any triangulation quiver together with some auxiliary data, a new algebra called triangulation algebra. Throughout, we fix a field \(K\). We start with a construction of hyperpotentials on ribbon quivers described in the following somewhat technical statement whose proof is omitted.
Proposition 5.14. Let $(Q, f)$ be a ribbon quiver.

(a) Let $p: Q_1 \to K[[x]]$ be $f$-invariant and let $q: Q_1 \to K[[x]]$ be $g$-invariant. Then the collection $(\rho_\alpha)_{\alpha \in Q_1}$ given by

$$\rho_\alpha = p_\alpha(\xi_{f(\alpha)}) \cdot \xi'_{f(\alpha)} = q_\alpha(\omega_{g(\alpha)}) \cdot \omega'_{g(\alpha)}$$

is a hyperpotential on $Q$. 

(b) Let $P: Q_1 \to K[[x]]$ be $f$-invariant and let $R: Q_1 \to K[[x]]$ be $g$-invariant. Consider

$$W = \sum_\alpha P_\alpha(\xi_\alpha) - \sum_\beta R_\beta(\omega_\beta)$$

where the left sum runs over representatives $\alpha$ of the $f$-cycles in $Q$ and the right sum runs over representatives $\beta$ of the $g$-cycles. Then $W$ is a potential on $Q$ and $\partial_\alpha W = \rho_\alpha$ for each $\alpha \in Q_1$, where $\rho_\alpha$ are defined as in part (a) for the functions $p, q: Q_1 \to K[[x]]$ given by $p_\alpha(x) = P'_\alpha(x)$ and $q_\alpha(x) = R'_\alpha(x)$.

Remark 5.15. If $p_\alpha(x)$ and $q_\alpha(x)$ are monomials, that is, there exist $f$-invariant function $f: Q_1 \to \mathbb{Z}_{>0}$ and $g$-invariant function $m: Q_1 \to \mathbb{Z}_{>0}$ such that $p_\alpha(x) = x^{f_\alpha} - 1$ and $q_\alpha(x) = x^{m_\alpha} - 1$ for any $\alpha \in Q_1$, then the (non-complete) Jacobian algebra of the hyperpotential in Proposition 5.14 is the one associated by Bocklandt to a weighted quiver polyhedron [14]. In particular, if $P_\alpha(x) = Q_\alpha(x) = x$ for any $\alpha \in Q_1$, then the potential in Proposition 5.14 is the potential arising from the dimer model corresponding to $(Q, f)$, see Section 4.3.

We are now ready to define what a triangulation algebra is.

Definition 5.16. Let $(Q, f)$ be a triangulation quiver. Let $m: Q_1 \to \mathbb{Z}_{>0}$ and $c: Q_1 \to K^\times$ be $g$-invariant functions of multiplicities and scalars, respectively, and assume that $m_\alpha n_\alpha \geq 2$ for any $\alpha \in Q_1$. Let $\lambda: Q_1 \to K$, i.e. $\lambda$ is an assignment of a scalar $\lambda_\alpha \in K$ for each $\alpha \in Q_1$ such that $f(\alpha) = \alpha$.

The triangulation algebra $\Lambda(Q, f, m, c, \lambda)$ associated to these data is the quotient $\Lambda(Q, f, m, c, \lambda) = KQ/\mathcal{J}$ of the complete path algebra $KQ$ by the closure of the ideal $\mathcal{J}$ generated by the commutativity-relations

$$\mathcal{J} = \left( \{ \alpha \cdot f(\alpha) - c_\alpha \omega_\alpha^{m_\alpha} : \omega'_\alpha \}_{\alpha \in Q_1 : f(\alpha) \neq \alpha} , \right.$$

$$\left. \{ \alpha \cdot f(\alpha) = 1 \}_{\alpha \in Q_1 : f(\alpha) = \alpha} \right)$$

(5.1)

when the set $Q_1^f$ is empty then evidently $\lambda$ does not play any role in the definition).

The data defining a triangulation algebra can be used to define an $f$-invariant function $p: Q_1 \to K[[x]]$ and a $g$-invariant function $q: Q_1 \to K[[x]]$ as follows; set $p_\alpha(x) = x^x - \lambda_\alpha x^3$ if $\alpha \in Q_1^f$ and $p_\alpha(x) = 1$ otherwise. Similarly, set $q_\alpha(x) = c_\alpha x^{m_\alpha} - 1$ for any $\alpha \in Q_1$. We observe that the commutativity-relation in (5.1) corresponding to an arrow $\alpha \in Q_1$ equals the element $\rho_{g^{-1}(\alpha)}$ of the hyperpotential considered in Proposition 5.14 arising from the functions $p$ and $q$. This yields the following basic property of triangulation algebras.

Proposition 5.17. Let $(Q, f)$ be a triangulation quiver. Let $m: Q_1 \to \mathbb{Z}_{>0}$ and $c: Q_1 \to K^\times$ be $g$-invariant functions of multiplicities and scalars, respectively and let $\lambda: Q_1^f \to K$.

(a) The triangulation algebra $\Lambda(Q, f, m, c, \lambda)$ is always a Jacobian algebra of a hyperpotential on $Q$. 

(b) Let $\mu_g = \text{lcm}(\{m_\alpha\}_{\alpha \in Q_1})$ and let

$$
\mu_f = \begin{cases}
6 & \text{if } Q_1^f \text{ is non-empty and } \lambda_\alpha \neq 0 \text{ for some } \alpha \in Q_1^f, \\
3 & \text{if } Q_1^f \text{ is non-empty and } \lambda_\alpha = 0 \text{ for any } \alpha \in Q_1^f, \\
1 & \text{if } Q_1^f \text{ is empty}.
\end{cases}
$$

If $\text{char } K$ does not divide $\mu_f \mu_g$, then $\Lambda(Q, f, m, c, \lambda)$ is a Jacobian algebra of a potential on $Q$.

Additional properties of triangulation algebras will be presented in Section 7.

Since these algebras are given as quotients by closure of ideals generated by commutativity-relations, \textit{a-priori} it is not even clear from the outset if they are finite-dimensional or not. However, it turns out that under some mild conditions on the auxiliary data, this is indeed the case as the closure $\mathcal{J}$ contains sufficiently many paths (that is, zero-relations), see Section 7.1.

It turns out that the concept of triangulation algebra is versatile enough to capture two seemingly unrelated classes of algebras occurring in the literature. Indeed,

- many algebras of quaternion type are in fact triangulation algebras (see Section 8.4); and
- for many triangulations of closed surfaces with punctures, the Jacobian algebras of the quivers with potentials associated by Labardini-Fragoso [64] are triangulation algebras (Section 8.4).

Let us quickly discuss the triangulation algebras on the triangulation quivers with small number of vertices shown in Table 2 and refer to the relevant statements in the sequel. Under some mild conditions on the auxiliary data, the triangulation algebras on the quivers 1, 2, 3a and 3b are algebras of quaternion type (Remark 7.11); triangulation algebras on the quiver 1 are further discussed in Section 5.3 whereas those on the quivers 2 and 3b are considered in Lemma 8.3. A triangulation algebra on the quiver 3′′ with all multiplicities set to 1 coincides with the Jacobian algebra of the quiver with potential associated with a triangulation of a torus with one puncture (a special case of Proposition 8.13; this algebra has been considered in [63], Example 8.2] and [78], Example 4.3).

In order to complete the picture and also to provide some concrete examples, the triangulation algebras on the quivers 3a and 3′ are given in the next two examples.

**Example 5.18.** Let $(Q, f)$ be the triangulation quiver 3a of Table 2 shown in the picture below

$$
\begin{array}{c}
\alpha \xrightarrow{\gamma} \beta \xrightarrow{\eta} \delta \xrightarrow{\xi} \gamma
\end{array}
$$

with $f = (\alpha \beta \gamma)(\xi \eta \delta)$. Then $g = (\alpha)(\beta \delta \eta \gamma)(\xi)$ has three cycles and any $g$-invariant function $\nu$ on $Q_1$ satisfies $\nu_3 = \nu_4 = \nu_3 = \nu_4$, hence it depends on three values which by abuse of notation will be denoted by $\nu_1, \nu_2, \nu_3$ where $\nu_1 = \nu_\alpha, \nu_2 = \nu_\beta$ and $\nu_3 = \nu_\delta$.

The auxiliary data needed to define a triangulation algebra on $(Q, f)$ thus consists of three positive integer multiplicities $m_1, m_2, m_3$ satisfying $m_1, m_3 \geq 2$ and three scalars $c_1, c_2, c_3 \in K^*$ (the function $\lambda$ has empty domain and hence can be ignored). The triangulation algebra $\Lambda(Q, f, m, c, \lambda)$ is the quotient of the complete path algebra of $Q$ by the closure of the ideal generated by the six elements

$$
\begin{align*}
\beta \gamma - c_1 \alpha m_1^{-1}, & \quad \alpha \beta - c_2 (\beta \delta \eta \gamma) m_2^{-1} \beta \delta \eta, & \quad \delta \xi - c_2 (\gamma \beta \eta \gamma) m_3^{-1} \gamma \beta \delta, \\
\eta \delta - c_3 \xi m_3^{-1}, & \quad \xi \eta - c_2 (\eta \gamma \beta \delta) m_3^{-1} \eta \gamma \beta, & \quad \gamma \alpha - c_2 (\delta \eta \gamma \beta) m_3^{-1} \delta \eta \gamma.
\end{align*}
$$
**Example 5.19.** Let \((Q, f)\) be the triangulation quiver \(3'\) of Table 2 shown in the picture below

![Triangulation Quiver](image)

with \(f = (\alpha_1)(\alpha_2)(\alpha_3)(\beta_1\beta_2\beta_3)\). Then \(g = (\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3)\) has one cycle, the set \(Q'\) consists of the arrows \(\alpha_1, \alpha_2, \alpha_3\) and the auxiliary data needed to define a triangulation algebra on \((Q, f)\). The remaining relations. The admissibility condition ensures that the generating relations lie in the square of the ideal generated by all arrows of \(Q\) and three scalars \(\lambda_1, \lambda_2, \lambda_3 \in K\).

The triangulation algebra \(\Lambda(Q, f, m, c, \lambda)\) is the quotient of the complete path algebra of \(Q\) by the closure of the ideal generated by the six elements

\[
\begin{align*}
\beta_i\beta_{i+1} - c(\alpha_i\beta_i\alpha_{i+1}\beta_{i+1}\alpha_{i-1}\beta_{i-1})^{m-1}\alpha_i\beta_i\alpha_{i+1}\beta_{i+1}\alpha_{i-1} - (1 \leq i \leq 3) \\
\alpha_i^2 - \lambda_i\alpha_i^3 - c(\beta_i\alpha_{i+1}\beta_{i+1}\alpha_{i-1}\beta_{i-1}\alpha_i)^{m-1}\beta_i\alpha_{i+1}\beta_{i+1}\alpha_{i-1}\beta_{i-1} - (1 \leq i \leq 3)
\end{align*}
\]

where index arithmetic is taken modulo 3 (i.e. \(3 + 1 = 1\) and \(1 - 1 = 3\)).

**Definition 5.20.** We say that a \(g\)-invariant multiplicity function \(m: Q_1 \to \mathbb{Z}_{>0}\) is **admissible** if \(m_\alpha n_\alpha \geq 3\) for any arrow \(\alpha \in Q_1\).

One needs to check the condition in the definition only for the arrows \(\alpha\) with \(n_\alpha \leq 2\). In particular, these arrows occur as loops or as part of 2-cycles. The admissibility condition thus reads as follows: if \(n_\alpha = 1\) then \(m_\alpha \geq 3\), while if \(n_\alpha = 2\) then \(m_\alpha \geq 2\). Note that when the pair \((n_\alpha, m_\alpha)\) equals \((1, 2)\) or \((2, 1)\) the triangulation algebra is defined but the multiplicity is not admissible, and when it equals \((1, 1)\) the triangulation algebra is not even defined.

**Example 5.21.** For the triangulation quiver \(3\alpha\) of Table 2 considered in Example 5.15, one has \(n_\alpha = n_\xi = 1\) and \(n_\eta = 4\). Hence the multiplicity function \(m\) is admissible if and only if \(m_\alpha \geq 3\) and \(m_\xi \geq 3\).

We conclude this section by a series of remarks concerning the definition of triangulation algebras and possible extensions thereof. The reader might skip these remarks on first reading.

**Remark 5.22.** Since the path \(\omega_\alpha^m n_\alpha^{-1} \cdot \omega_\alpha'\) is of length \(m_\alpha n_\alpha - 1\), the definition of a triangulation algebra makes perfect sense when \(m_\alpha n_\alpha = 2\) for some arrow \(\alpha\), but in this case the right hand side of the corresponding commutativity-relation is just \(c_\alpha \alpha\), so the arrow \(\alpha\) could be eliminated from \(Q\) complicating somewhat the remaining relations. The admissibility condition ensures that the generating relations lie in the square of the ideal generated by all arrows of \(Q\) so that no arrows have to be deleted, compare with Remark 5.8 for Brauer graph algebras.

**Remark 5.23.** When \(\text{char } K \neq 2\), the scalars \(\lambda_\alpha\) occurring in the definition of a triangulation algebra do not play any role, i.e. \(\Lambda(Q, f, m, c, \lambda) \simeq \Lambda(Q, f, m, c, 0)\). This can be shown by considering the automorphism of \(KQ\) sending each arrow \(\alpha\) with \(f(\alpha) = \alpha\) to \(\alpha - (\lambda_\alpha/2)\alpha^2\) and keeping the other arrows unchanged. For the proof ones needs to know the additional zero relations that hold in a triangulation algebra given in Proposition 7.4.

**Remark 5.24.** Even if \(\lambda_\alpha = 0\) for all the arrows \(\alpha \in Q_1\), there may be different \(g\)-invariant functions of scalars \(c, c': Q_1 \to K^\times\) yielding isomorphic triangulation
algebras, that is,
\[ \Lambda(Q, f, m, c, 0) \simeq \Lambda(Q, f, m, c', 0), \]
but in this survey we will not pursue a systematic study of this equivalence relation on invariant functions of scalars.

**Remark 5.25.** It is possible to slightly generalize Definition 5.16 by considering also an \( f \)-invariant function \( a: Q_1 \to K^\times \) of scalars and setting
\[
\mathcal{J} = \left\{ a_\alpha \bar{\alpha} \cdot f(\bar{\alpha}) - c_\alpha \omega_{\alpha}^{m_{\alpha} - 1} \cdot \omega_{\alpha} \right\}_{\alpha: f(\bar{\alpha}) \neq \bar{\alpha}}, \quad
\left\{ a_\alpha \bar{\alpha}^2 - \lambda_\alpha \bar{\alpha}^3 - c_\alpha \omega_{\alpha}^{m_{\alpha} - 1} \cdot \omega_{\alpha} \right\}_{\alpha: f(\bar{\alpha}) = \alpha}
\]
(the current definition uses the constant function \( a = 1 \)).

All the results of Section 7 are valid also in this more general setting, but for simplicity, we chose to present the material without these extra scalars, since in many cases this apparent generalization does not yield any new algebras. Indeed, by using scalar transformation of the arrows and replacing the scalar function \( c \) by another \( g \)-invariant function \( c': Q_1 \to K^\times \), we may always assume that \( a_\alpha = 1 \) for any arrow \( f(\alpha) \neq \alpha \), and if \( \alpha \) is an arrow such that \( f(\alpha) = \alpha \) and the ground field \( K \) contains a third root of \( a_\alpha \), we may assume that \( a_\alpha = 1 \) as well. This holds in particular when \( K \) is algebraically closed.

**Remark 5.26.** One could define an even more general version of a triangulation algebra by utilizing the full power of Proposition 5.14, taking an \( f \)-invariant function \( p: Q_1 \to K[[x]]^\times \) of invertible power series, a \( g \)-invariant function \( q: Q_1 \to K[[x]] \setminus \{0\} \) such that the least order term \( c_\alpha x^{m_{\alpha} - 1} \) of each \( q_\alpha(x) \) satisfies \( m_{\alpha} n_{\alpha} \geq 2 \), and forming the quotient of the complete path algebra \( K\overline{Q} \) by the closure of the ideal \( \mathcal{J} \) given by
\[
\mathcal{J} = \left\{ p_\alpha(\xi_\alpha) \cdot \bar{\alpha} \cdot f(\bar{\alpha}) - q_\alpha(\omega_{\alpha}) \cdot \omega_{\alpha} \right\}_{\alpha \in Q_1}.
\]

However, it turns out by using techniques similar to that in the proof of Theorem 7.14 that if the induced multiplicity function \( m: Q_1 \to \mathbb{Z}_{>0} \) is admissible and \((\overline{Q}, f, m)\) is not exceptional (see Section 5.4 below) then the algebra \( K\overline{Q}/\mathcal{J} \) already occurs as an algebra of the form discussed in Remark 5.25 above, compare with Remark 5.13 for Brauer graph algebras.

Nevertheless, for some triangulation quivers with non-admissible multiplicities this generalized version does yield new algebras, see for example Proposition 8.8 describing some new algebras of quaternion type not appearing in the known lists.

### 5.3. Example – triangulation algebras with one vertex

In this section we work out in some detail the case of triangulation algebras with one vertex. Already in this rather special case, one is able to demonstrate many of the ideas and techniques that apply also in the general case to be treated in Section 7.1.

Recall that the only triangulation quiver \((Q, f)\) with one vertex has two loops \( \alpha \) and \( \beta \) with \( f \) being the identity function (Example 5.10). Hence \( \bar{\alpha} = \beta, \bar{\beta} = \alpha \) and \( \omega_{\alpha} = \alpha \beta, \omega'_{\alpha} = \alpha, \omega_{\beta} = \beta \alpha, \omega'_{\beta} = \beta \). Since there is only one \( g \)-cycle, the multiplicities and scalars are given by an integer \( m \geq 1 \) and some \( c \in K^\times \). In addition, there are parameters \( \lambda_{\alpha}, \lambda_{\beta} \in K \) corresponding to the fixed points of \( f \).

The triangulation algebra is the quotient \( \Lambda = K\overline{(\alpha, \beta)}/\mathcal{J} \) (see the notation in Example 5.10), where the generators of the ideal \( \mathcal{J} \) are given by
\[
\alpha^2 - \lambda_{\alpha} \alpha^3 - c(\alpha \beta)^{m-1} \beta, \quad \beta^2 - \lambda_{\beta} \beta^3 - c(\alpha \beta)^{m-1} \alpha.
\]

If \( m = 1 \), the multiplicity is not admissible and \( \beta \in (\alpha^2, \mathcal{J}), \alpha \in (\beta^2, \mathcal{J}) \), so by induction we get \( \alpha \in (\alpha^n, \mathcal{J}) \) for any \( n \geq 1 \), therefore \( \alpha \in \mathcal{J} \) and similarly for \( \beta \). Hence the image of the arrows \( \alpha, \beta \) in \( \Lambda \) vanishes and \( \Lambda = K \).
If \( m \geq 2 \), the multiplicity is admissible. Define elements \( E_\alpha, E_\beta \in K(\alpha, \beta) \) by
\[
E_\alpha = (1 - \lambda_\alpha \alpha)^{-1} = 1 + \lambda_\alpha \alpha + \lambda_\alpha^2 \alpha^2 + \ldots \\
E_\beta = (1 - \lambda_\beta \beta)^{-1} = 1 + \lambda_\beta \beta + \lambda_\beta^2 \beta^2 + \ldots
\]
Then \( \alpha^2 = E_\alpha(\alpha^2 - \lambda_\alpha \alpha^3) \) and \( \beta^2 = (\beta^2 - \lambda_\beta \beta^3)E_\beta \), hence
\[
\alpha^2 \beta - cE_\alpha(\beta \alpha)^{m-2} \beta \alpha^2 = E_\alpha \left( \alpha^2 - \lambda_\alpha \alpha^3 - c(\beta \alpha)^{m-1} \beta \right) \beta \in \mathcal{J}, \\
\alpha \beta^2 - c\alpha \beta(\alpha \beta)^{m-2} \alpha E_\beta = \alpha \left( (\beta^2 - \lambda_\beta \beta^3) - c(\alpha \beta)^{m-1} \alpha \right) E_\beta \in \mathcal{J},
\]
so \( \alpha^2 \beta - u \alpha^2 \beta \in \mathcal{J} \) and \( \alpha \beta^2 - \alpha^2 \beta v \in \mathcal{J} \) for some (infinite) linear combinations \( u \) and \( v \) of paths of positive lengths. Therefore
\[
\alpha^2 \beta - u \alpha^2 \beta v = (\alpha^2 \beta - u \alpha \beta^2) + u(\alpha^2 \beta - \alpha^2 \beta v) \in \mathcal{J}.
\]
It follows that \( \alpha^2 \beta - u^a \alpha^2 \beta v^n = \sum_{i=0}^{n-1} u^i(\alpha^2 \beta - u \alpha^2 \beta v)v^i \in \mathcal{J} \) for any \( n \geq 1 \), hence \( \alpha^2 \beta \in \mathcal{J} \). Similarly, \( \alpha \beta^2, \beta^2 \alpha, \beta \alpha^2 \in \mathcal{J} \) so their image in the quotient \( \Lambda \) is zero. Therefore, in \( \Lambda \) we have
\[
\alpha^4 = E_\alpha(\alpha^2 - \lambda_\alpha \alpha^3)\alpha^2 = E_\alpha(\beta \alpha)^{m-1} \beta \alpha^2 = 0
\]
and similarly \( \beta^4 = 0 \). We deduce that
\[
(\alpha \beta)^m = (\alpha \beta)^{m-1} \alpha \beta = e^{-1}(\beta^2 - \lambda_\beta \beta^3)\beta = e^{-1} \beta^3 = e^{-1}\beta(\beta^2 - \lambda_\beta \beta^3) \\
= \beta(\alpha \beta)^{m-1} \alpha = (\beta \alpha)^m = (\beta \alpha)^{m-1} \beta \alpha = e^{-1}(\alpha^2 - \lambda_\alpha \alpha^3)\alpha = e^{-1} \alpha^3
\]
(compare with Remark \([7.6]\) and a basis for \( \Lambda \) is given by the \( 4m \) elements
\[
\{1\} \cup \{ (\alpha \beta)^i, (\beta \alpha)^i \}_{0 < i < m} \cup \{ (\alpha \beta)^i \alpha, (\beta \alpha)^i \beta \}_{0 \leq i < m} \cup \{ (\alpha \beta)^m = (\beta \alpha)^m \}
\]
(compare with Proposition \([7.7]\)).
The discussion above shows parts \((\text{a}) \) and \((\text{b}) \) of the next statement. For part \((\text{c}) \), one uses similar considerations whose details will appear elsewhere.

**Proposition 5.27.** Let \( \Lambda \) be a triangulation algebra with one vertex. Then there exist parameters \( m \geq 2, c \in K^\times \) and \( \lambda_\alpha, \lambda_\beta \in K \) such that the following hold.

(a) \( \Lambda = KQ/I \) where \( Q \) is the quiver
\[
\begin{array}{c}
\alpha \\
\text{•} \\
\beta
\end{array}
\]
and \( I \) is the ideal of \( KQ \) generated by the three elements
\[
\alpha^2 - c(\beta \alpha)^{m-1} \beta - c(\beta \alpha)^m \alpha, \beta^2 - c(\alpha \beta)^{m-1} \alpha - c(\alpha \beta)^m \\
\alpha^2 \beta.
\]

(b) The ideal \( I \) is generated by the elements in \((5.2) \) together with the elements
\[
(\alpha \beta)^m - (\beta \alpha)^m, (\alpha \beta)^m \alpha, (\beta \alpha)^m \beta.
\]

(c) The algebra \( \Lambda \) is finite-dimensional and \( \dim_K \Lambda = 4m \).

Comparing the description of \( \Lambda \) with that of the local algebras of quaternion type in \([29\text{ III}]\), we deduce:

**Corollary 5.28.** An algebra of quaternion type with one simple module is a triangulation algebra.
5.4. Exceptional triangulation quivers with multiplicities. In proving the results on triangulation algebras one needs to distinguish two exceptional cases where the triangulation quiver is self dual and the admissible multiplicities are the minimal possible.

**Definition 5.29.** A pair \((Q, f, m)\) consisting of a connected triangulation quiver \((Q, f)\) and a \(g\)-invariant multiplicity function \(m : Q_1 \to \mathbb{Z}_{>0}\) is **exceptional** if \((Q, f)\) is one of the two self-dual triangulation quivers shown in Figure 4 and the function \(m\) is the following:

- \(m_\alpha = 3\) and \(m_\beta = m_\gamma = m_\eta = 1\) in the punctured monogon case (for the labeling of the arrows see row 2 of Table 2);
- \(m_\alpha = 1\) for any arrow \(\alpha\) in the tetrahedron case.

The next statement characterizes the exceptional triangulation quivers with multiplicities among all triangulation quivers with admissible multiplicities.

**Proposition 5.30.** Let \((Q, f)\) be a connected triangulation quiver and \(m : Q_1 \to \mathbb{Z}_{>0}\) an admissible \(g\)-invariant multiplicity function. Then the following conditions are equivalent:

(a) \(((Q, f), m)\) is exceptional;
(b) \(m_\alpha n_\alpha = 3\) for all \(\alpha \in Q_1\);
(c) \((m_\alpha n_\alpha)^{-1} + (m_{f(\alpha)} n_{f(\alpha)})^{-1} + (m_{f^2(\alpha)} n_{f^2(\alpha)})^{-1} = 1\) for some \(\alpha \in Q_1\).

The implications (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (c) are trivial. As with the other combinatorial statements of Section 3, the proof of the implication (c) \(\Rightarrow\) (a) will be given elsewhere.

### 6. Representation-finite symmetric 2-CY-tilted algebras

In this section we classify all the symmetric 2-CY-tilted algebras of finite representation type over an algebraically closed field.

Assume that \(K\) is algebraically closed and consider first Brauer graph algebras.

By invoking the results of Erdmann and Skowroński on the structure of the stable Auslander-Reiten quiver of a self-injective special biserial algebra [31, Theorems 2.1 and 2.2], recalling that \(\tau\)-periodicity and \(\Omega\)-periodicity are equivalent for symmetric algebras, we obtain:

**Proposition 6.1.** The following conditions are equivalent for a Brauer graph algebra \(\Gamma\):

(a) Any indecomposable non-projective \(\Gamma\)-module is \(\Omega_\Gamma\)-periodic;
(b) \(\Gamma\) is of finite representation type.

The class of Brauer graph algebras of finite representation type coincides with that of the Brauer tree algebras. A Brauer tree algebra is a Brauer graph algebra whose underlying ribbon graph is a tree and at most one node has multiplicity greater than 1 (this node is called exceptional).

One of the first applications of Rickard’s Morita theory for derived categories was the derived equivalence classification of the Brauer tree algebras [80, Theorem 4.2]. Rickard proved that any Brauer tree algebra is derived equivalent to a Brauer tree algebra whose graph has a special shape, called a Brauer star, with the same number of edges and same multiplicity of the exceptional node.

Figure 9 shows a Brauer star with \(n\) edges and multiplicity \(m\) of the exceptional node. The corresponding ribbon quiver is shown to the right. The Brauer tree algebra has commutativity-relations of the form \(\beta - \alpha^{nm}\) and zero-relations \(\alpha\beta\) and \(\beta\alpha\), hence the arrows \(\beta\) can be eliminated and one gets a symmetric Nakayama algebra with zero-relations \(\alpha^{nm+1}\). Some properties of this algebra are reviewed in [59, §4]. For the next statement, see paragraphs 4.11 and 4.12 there.
Figure 9. A Brauer star and the corresponding ribbon quiver.

Figure 10. The indecomposable 2-CY-tilted symmetric algebras of finite representation type that are not simple. These algebras are Brauer tree algebras, and for each family we show the ribbon graph with multiplicities (top); the quiver, where we eliminated arrows (middle); and the corresponding hyperpotential (bottom).

Lemma 6.2. Let $\Gamma$ be the Brauer star algebra with $n$ simple modules and multiplicity $m$ of the exceptional node. Let $S$ be a simple $\Gamma$-module. If $m = n = 1$ then $\Omega \Gamma S \simeq S$, otherwise $\Omega \Gamma^n S \simeq S$ but $\Omega \Gamma^i S \not\simeq S$ for any $0 < i < 2n$.

We are now ready to state the classification of symmetric, 2-CY-tilted algebras of finite representation type.

Theorem 6.3. The following conditions are equivalent for an indecomposable, basic, finite-dimensional algebra $\Gamma$ which is not simple.

(a) $\Gamma$ is symmetric, 2-CY-tilted of finite representation type;
(b) $\Gamma$ is symmetric of finite representation type and $\Omega^2 \Gamma M \simeq M$ for any $M \in \text{mod} \Gamma$;
(c) $\Gamma$ is a 2-CY-tilted Brauer graph algebra;
(d) $\Gamma$ is a Brauer graph algebra and $\Omega^1 \Gamma M \simeq M$ for any $M \in \text{mod} \Gamma$;
(e) $\Gamma$ is a Brauer tree algebra with at most two simple modules;
(f) $\Gamma$ belongs to one of the three families of Brauer tree algebras shown in Figure 10.

The implications (a) $\Rightarrow$ (b) and (e) $\Rightarrow$ (d) follow from Proposition 2.16. The implication (b) $\Rightarrow$ (c) is a consequence of Proposition 6.11 and Lemma 6.2. The equivalence (c) $\Leftrightarrow$ (f) is clear. As each of the algebras shown in Figure 10 is a Jacobian algebra of a hyperpotential and hence 2-CY-tilted by Proposition 2.14, this proves the implications (f) $\Rightarrow$ (a) and (f) $\Rightarrow$ (c).

It remains to show that (a) $\Rightarrow$ (e). By Riedtmann [81], the stable Auslander-Reiten quiver of a self-injective algebra of finite representation type has the form $\mathbb{Z}\Delta/\langle \phi \tau^{-1} \rangle$, where $\Delta$ is a Dynkin graph $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$) or $E_n$ ($n = 6, 7, 8$), $\tau$ is the translation of $\mathbb{Z}\Delta$, $\phi$ is an automorphism of $\mathbb{Z}\Delta$ with a fixed vertex and
Following Asashiba [5], these data are encoded in the type \((\Delta, r/(h_\Delta - 1), \ell)\), where \(h_\Delta\) is the Coxeter number of \(\Delta\) and \(\ell\) is the order of \(\phi\). Asashiba [5] classified the self-injective algebras up to derived equivalence and described the possible types that can occur. If \(\Gamma\) is symmetric of finite representation type, then our assumption in (b) implies that \(\tau^2\Gamma\) acts as the identity on the vertices of the stable Auslander-Reiten quiver of \(\Gamma\) and hence either \(r = 2\) and \(t = 1\) or \(r = 1\) and \(t \leq 2\). Comparing this with the list of possible types in [5], one gets that the type of \(\Gamma\) must be \((A_n, r/n, 1)\) for some \(n \geq 1\) and \(r \leq 2\) dividing \(n\). In particular, \(\Gamma\) is derived equivalent to a symmetric Nakayama algebra with \(r \leq 2\) simple modules. By [38, 80], \(\Gamma\) is a Brauer tree algebra.

Remark 6.4. As a consequence of Theorem 6.3, we see that the answer to Question 2.26 and Question 2.27 is affirmative in the representation-finite case.

Remark 6.5. The algebras listed in Figure 10 occur also as the endomorphism algebras of cluster-tilting objects in the 2-Calabi-Yau stable categories of maximal Cohen-Macaulay modules over one dimensional simple hypersurface singularities of types \(A_{2m+1}\) and \(D_{2m+2}\), see Proposition 2.4 and Proposition 2.6 in [22].

7. Triangulation algebras are of quasi-quaternion type

Let \(K\) be a field. Consider a connected triangulation quiver \((Q, f)\) together with the following auxiliary data:

- \(g\)-invariant function \(m: Q_1 \to \mathbb{Z}_{>0}\) of multiplicities;
- \(g\)-invariant function \(c: Q_1 \to K^\times\) of scalars;
- a function \(\lambda: Q_1 \to K\), i.e. a scalar \(\lambda_\alpha \in K\) for each arrow \(\alpha\) with \(f(\alpha) = \alpha\).

Assume that the following conditions hold:

- \(m\) is admissible, i.e. \(m_\alpha n_\alpha \geq 3\) for each arrow \(\alpha\);
- \(((Q, f), m)\) is not exceptional; or
- \(((Q, f), m)\) is exceptional and the scalars \(c: Q_1 \to K^\times\) satisfy
  \[\prod_{\alpha \in Q_1} c_\alpha \neq 1\]
  in the punctured monogon case; or
  \[- c_{\alpha} c_{\bar{\alpha}} c_{f(\alpha)} c_{f(\bar{\alpha})} \neq 1\]
  for some \(\alpha \in Q_1\) in the tetrahedron case.

Denote by \(\Lambda = \Lambda(Q, f, m, c, \lambda)\) the triangulation algebra (Definition 5.16) associated with the above data and by \(\Gamma = \Gamma(Q, f, m, c)\) the Brauer graph algebra (Definition 5.3).

Theorem 7.1. Under the above conditions, we have:

(a) \(\Lambda\) is finite-dimensional.
(b) \(\Lambda\) is symmetric.
(c) \(\Lambda\) is of tame representation type. Moreover, if \(((Q, f), m)\) is not exceptional then the Brauer graph algebra \(\Gamma\) is a degeneration of \(\Lambda\).
(d) \(\Lambda\) is a Jacobian algebra of a hyperpotential and therefore it is 2-CY-tilted, i.e. there is a 2-Calabi-Yau triangulated category \(\mathcal{C}\) and a cluster-tilting object \(T\) in \(\mathcal{C}\) such that \(\Lambda \simeq \text{End}_\mathcal{C}(T)\).
(e) \(\Lambda\) is of quasi-quaternion type.
(f) More generally, for any cluster-tilting object \(T'\) in \(\mathcal{C}\) which is reachable from \(T\), the 2-CY-tilted algebra \(\text{End}_\mathcal{C}(T')\) is derived equivalent to \(\Lambda\) and of quasi-quaternion type.

Remark 7.2. The theorem holds also when the multiplicities are not admissible (but still \(m_\alpha n_\alpha \geq 2\) for any arrow \(\alpha\)), but then more exceptional cases are needed to be taken care of.

Remark 7.3. Remark 4.5 implies that for any positive integer \(n \geq 1\) there exists a triangulation quiver with \(n\) vertices and hence a triangulation algebra with \(n\)
simple modules. The algebras of part (ii) of the theorem thus provide many instances of tame, symmetric, indecomposable algebras with periodic modules which seem to be missing from the classification announced in [33, Theorem 6.2] and [57, Theorem 8.7]. Moreover, they provide counterexamples to [57, Corollary 8.8(3)] which claims to bound the number of simple modules of such algebras of infinite representation type by 10.

Part (ii) of the theorem follows from part (ii), Proposition 5.17 and Proposition 2.14. Part (iii) is a consequence of parts (ii), (iv) and Proposition 2.16. Finally, part (iv) is a consequence of parts (ii), (iv), Corollary 2.23, part (iv) and Proposition 1.14. The ideas behind the proof of parts (iii) and (iv) are explained in the next sections.

7.1. Remarks on finite-dimensionality. We keep the notations as in the preceding section.

Motivated by the dimer model perspective of Section 4.3, a path \( \alpha \cdot f(\alpha) \cdot gf(\alpha) \) may be called zig-zag path. A crucial point in proving part (ii) of Theorem 7.1 is the vanishing of the images of these zig-zag paths in \( \Lambda \), whose proof we sketch below. Let \( \mathcal{J} \) be the ideal defining the triangulation algebra, so that \( \Lambda = KQ/\mathcal{J} \), and let \( \alpha \in Q_1 \) be any arrow. Lemma 3.3 implies that \( gf(\beta) = fg^{-1}(\beta) \) for any arrow \( \beta \), so we can repeatedly use the commutativity-relations defining \( \mathcal{J} \) to deduce that

\[
\alpha \cdot f(\alpha) \cdot gf(\alpha) - u \cdot \alpha \cdot f(\alpha) \cdot gf(\alpha) = v \in \mathcal{J}
\]

for some \( u, v \in KQ \).

If \( ((Q, f), m) \) is not exceptional, then one shows using Proposition 5.30 that \( u \) and \( v \) are linear combinations of paths of positive length, and therefore deduces that \( \alpha \cdot f(\alpha) \cdot gf(\alpha) \in \mathcal{J} \) as done in the case discussed in Section 5.3.

If \( ((Q, f), m) \) is exceptional, then one shows that \( u = C_u + u' \) and \( v = C_v + v' \) where \( u', v' \) are linear combinations of paths of positive length (in the tetrahedron case even \( u' = v' = 0 \)) and \( C_u, C_v \in K^\times \) are scalars satisfying \( C_uC_v \neq 1 \) by our additional assumption on the function \( c: Q_1 \to K^\times \). Hence \( \alpha \cdot f(\alpha) \cdot gf(\alpha) \in \mathcal{J} \) as well.

The next proposition provides a more refined version of part (ii) of Theorem 7.1 by giving a presentation of the triangulation algebra as quiver with relations.

**Proposition 7.4.** Under the hypotheses of Theorem 7.1, the triangulation algebra \( \Lambda(Q, f, m, c, \lambda) \) is the quotient of the path algebra \( KQ \) by the ideal generated by the elements

\[
\begin{align*}
(7.1) \quad & \alpha \cdot f(\alpha) - \epsilon_\alpha \omega_\alpha^{m_\alpha - 1} \cdot \omega_\alpha \alpha \in Q_1 \text{ and } f(\alpha) \neq \bar{\alpha}, \\
(7.2) \quad & \alpha \bar{\alpha}^2 - \epsilon_\alpha \omega_\alpha^{m_\alpha - 1} \cdot \omega_\alpha^f - \epsilon_\alpha \lambda_\alpha \omega_\alpha^{m_\alpha} \alpha \in Q_1 \text{ and } f(\bar{\alpha}) = \bar{\alpha}, \\
(7.3) \quad & \alpha \cdot f(\alpha) \cdot gf(\alpha) \alpha \in Q_1.
\end{align*}
\]

**Remark 7.5.** The ideal of relations in Proposition 7.4 is not changed if the relations of type (7.3) are replaced by

\[
(7.4) \quad \alpha \cdot g(\alpha) \cdot fg(\alpha) \alpha \in Q_1.
\]

Moreover, it turns out that it is enough to specify a zero-relation as in (7.3) or (7.4) for just one arrow \( \alpha \in Q_1 \), as the relations for the other arrows would then follow from the commutativity-relations (7.1) and (7.2).

In the cases where \( ((Q, f), m) \) is exceptional, our assumption on the scalars \( c \) implies that all the zero-relations (7.3) and (7.4) already follow from the relations (7.1) and (7.2), provided that in the punctured monogon case the scalar \( \lambda_\eta \) associated to the loop \( \eta \in Q_1^2 \) vanishes (no additional assumption is needed in the tetrahedron case).
Remark 7.6. Let \( i \in Q_0 \) and let \( \alpha, \bar{\alpha} \) be the two arrows starting at \( i \). Then the images in \( \Lambda \) of the following cycles starting at \( i \) are equal:
\[
\alpha \cdot f(\alpha) \cdot f^2(\alpha) = c_{\alpha} \omega_{\alpha}^{m_{\alpha}} = \bar{\alpha} \cdot f(\bar{\alpha}) \cdot f^2(\bar{\alpha}).
\]
Denote this common value by \( z_i \in \Lambda \).

The next statement is a generalization of our result in [70, §4.1] and its proof is similar.

Proposition 7.7. A basis of the triangulation algebra \( \Lambda(Q, f, m, c, \lambda) \) is given by the images of the paths
\[
\{e_i\}_{i \in Q_0} \cup \{\alpha \cdot g(\alpha) \cdot \ldots \cdot g^r(\alpha)\}_{\alpha \in Q_0, 0 \leq r < m_{\alpha} \cdot n_\alpha - 1} \cup \{z_i\}_{i \in Q_0}.
\]

Example 7.8. In the case of the triangulation quiver with one vertex, the same basis for the triangulation algebra has been constructed in Section 5.3.

Using the basis of Proposition 7.7 one can explicitly compute the Cartan matrix of a triangulation algebra in terms of its defining combinatorial data. As the details of the computation are similar to those given in [70, §4.4], we will state the result without proof.

For a triangulation quiver \((Q, f)\), recall that \( \Omega_g \) denotes the set of \( g \)-cycles in \( Q_1 \). The vectors \( \chi_\omega \in \mathbb{Z}^{Q_0} \) for \( \omega \in \Omega_g \) have been defined at the end of Section 3.1 and singular Cartan matrix must be isomorphic to the trivial extension of a tubular algebra (and hence has at most 10 simple modules).

Remark 7.10. From part (b) of Proposition 7.9 we see that any triangulation algebra with \( n > 10 \) simple modules provides a counterexample to [57, Corollary 8.8(1)] which states that a tame, symmetric algebra of infinite representation type with periodic modules and singular Cartan matrix must be isomorphic to the trivial extension of a tubular algebra (and hence has at most 10 simple modules).

Remark 7.11. Combining part (b) of Proposition 7.9 with part (b) of Theorem 7.1 we deduce that the triangulation algebra \( \Lambda(Q, f, m, c, \lambda) \) is of quaternion type (and not just of quasi-quaternion type) if and only if \((Q, f)\) is any of the triangulation quivers 1, 2, 3a or 3b. Further details will be given in Section 8.1.

Parts (a), (b), (c) follow from the corresponding statements of Proposition 5.11 observing that the basis constructed in Proposition 7.7 for \( \Lambda \) and that constructed in Lemma 5.10 for \( \Gamma \) consist of images (in the respective algebras) of the same set of paths. The inequality \( |\Omega_g| \leq |Q_0| \) in part (b) is a consequence of Proposition 5.2.6 which also implies the “only if” direction of part (c). The “if” direction follows from explicit calculations which are presented in the next example for the purpose of illustration.

Example 7.12. For each of the triangulation quivers 1, 2, 3a and 3b in Table 2 we compute the Cartan matrix of a triangulation algebra on that quiver. Recall that for a \( g \)-cycle \( \omega \), the quantity \( m_\omega \) used in Proposition 7.9 equals any of the values \( m_{\alpha} \) for \( \alpha \in \omega \).
(1) There is one $g$-cycle $(\alpha \beta)$ with $\chi(\alpha \beta) = (2)$. The Cartan matrix is $(4m_{\alpha})$.

(2) There are two $g$-cycles $(\alpha)$ and $(\eta \gamma)$ with $\chi(\alpha) = (1, 0)$ and $\chi(\alpha \gamma) = (1, 2)$, the Cartan matrix is

$$m_\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m_\eta \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and its determinant is $4m_\alpha m_\eta$.

(3a) There are three $g$-cycles $(\alpha)$, $(\beta \delta \gamma)$ and $(\xi)$ with $\chi(\alpha) = (1, 0, 0)$, $\chi(\beta \delta \gamma) = (1, 2, 1)$ and $\chi(\xi) = (0, 0, 1)$, the Cartan matrix is

$$m_\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + m_\beta \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} + m_\xi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and its determinant is $4m_\alpha m_\beta m_\xi$.

(3b) There are three $g$-cycles $(\alpha_1 \beta_1)$, $(\alpha_2 \beta_2)$ and $(\alpha_3 \beta_3)$ with $\chi(\alpha_1 \beta_1) = (1, 0, 0)$, $\chi(\alpha_2 \beta_2) = (0, 1, 1)$ and $\chi(\alpha_3 \beta_3) = (1, 0, 1)$, the Cartan matrix is

$$m_{\alpha_1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + m_{\alpha_2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_{\alpha_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and its determinant is $4m_{\alpha_1} m_{\alpha_2} m_{\alpha_3}$.

7.2. Remarks on tameness. In this section we sketch the proof of part (3) of Theorem 1. We keep the notations as in the preceding sections.

First, we recall the notion of degeneration of algebras appearing in the statement of part (3). For a positive integer $d$, denote by $\text{alg}_d(K)$ the affine variety of associative algebra structures with unit on the vector space $K^d$. The group $\text{GL}_d(K)$ acts on $\text{alg}_d(K)$ by transport of structure and its orbits correspond bijectively to the isomorphism classes of $d$-dimensional $K$-algebras. Given two $d$-dimensional algebras $\Gamma$ and $\Lambda$ viewed as points in $\text{alg}_d(K)$, we say that $\Gamma$ is a degeneration of $\Lambda$ if $\Gamma$ lies in the closure of the $\text{GL}_d(K)$-orbit of $\Lambda$ in the Zariski topology of $\text{alg}_d(K)$.

Assume that $((Q, f), m)$ is not exceptional and let $N = \text{lcm}(m_\alpha n_\alpha)_{\alpha \in Q_1}$, so that $N/(m_\alpha n_\alpha)$ is a positive integer for any $\alpha \in Q_1$. For each $\alpha \in Q_1$, set

$$e_\alpha = 1 - \left((m_\alpha n_\alpha)^{-1} + (m_{f(\alpha)} n_{f(\alpha)})^{-1} + (m_{f(\alpha)} n_{f(\alpha)})^{-1}\right).$$

If $f(\alpha) = \alpha$, set also $e'_\alpha = 1 - 2(m_\alpha n_\alpha)^{-1}$. Note that $e_\alpha$ and $e'_\alpha$ are rational numbers and moreover $N e_\alpha, N e'_\alpha$ are integers by construction.

Since $((Q, f), m)$ is not exceptional, Proposition 4.3 implies that $e_\alpha > 0$ for any arrow $\alpha$ (hence also $e'_\alpha > 0$ for any $\alpha \in Q_1^0$). For $t \in K$, let $I_t$ be the ideal of the path algebra $KQ$ generated by the elements

$$\hat{\alpha} \cdot f(\hat{\alpha}) - c_\alpha t^{N e_\alpha} \omega_\alpha \omega_\alpha^{m_\alpha}, \quad \alpha \in Q_1, \text{and } f(\hat{\alpha}) \neq \hat{\alpha},$$

$$\hat{\alpha}^2 - c_\alpha t^{N e_\alpha} \omega_\alpha \omega_\alpha^{m_\alpha} - c_\alpha \lambda^1 t^{N e'_\alpha} \omega_\alpha \omega_\alpha, \quad \alpha \in Q_1, \text{and } f(\hat{\alpha}) = \hat{\alpha},$$

$$\alpha \cdot f(\alpha) \cdot g(f(\alpha)) - \alpha \cdot g(f(\alpha)) \cdot f(\alpha), \quad \alpha \in Q_1,$$

and let $\Lambda_t = KQ/I_t$.

**Proposition 7.13.** Assume that $((Q, f), m)$ is not exceptional.

Let $\Gamma = \Gamma(Q, f, m, c)$ be the Brauer graph algebra and let $\Lambda = \Lambda(Q, f, m, c, \lambda)$ be the triangulation algebra as in Theorem 7.1. Then:

(a) $\Lambda_0 \simeq \Gamma$.

(b) $\Lambda_1 \simeq \Lambda$.

(c) For any $t \in K^\times$, the automorphism of $KQ$ defined by sending each arrow $\alpha$ to $t^{N/(m_\alpha n_\alpha)} \alpha$ maps $I_1$ onto $I_t$, hence $\Lambda_t \simeq \Lambda$. 
(d) $\Gamma$ is a degeneration of $\Lambda$.

We constructed a one-parameter family of algebras $\{\Lambda_t\}$ for $t \in K$ such that $\Lambda_t \simeq \Lambda$ for $t \neq 0$ and $\Lambda_0 \simeq \Gamma$. Since $\Gamma$ is tame (Proposition 5.1), a degeneration theorem of Geiss [39] implies that $\Lambda$ is also tame.

**Remark 7.14.** In [48, §6] Holm establishes the tameness of the algebras of quaternion type with 2 or 3 simple modules by showing that some of them degenerate to algebras of dihedral type and then applying the result in [39]. Proposition 7.13 can be seen as a generalization of this statement to arbitrary triangulation quivers.

**Example 7.15.** The algebras of quaternion type with one simple module are precisely the algebras listed as items (5) and (5') in the paper [82] by Ringel dealing with the representation type of local algebras, but their representation type was not determined in that paper. Their tameness was later established by Erdmann [29, III.1.2] as a consequence of the result of [15] mentioned in Section 1.1.

Since these algebras are triangulation algebras (see Corollary 5.28), Proposition 7.13 thus yields an alternative proof of their tameness. For the purpose of illustration, let us carry out the explicit calculations.

Recall from Section 5.3 that a triangulation quiver with one vertex has two loops $\alpha$ and $\beta$ with the function $f$ being the identity. Hence there is one $g$-cycle and the auxiliary algebraic data is given by a positive integer multiplicity $m$, which is admissible if $m \geq 2$, and scalars $c \in K^\times$ and $\lambda_\alpha, \lambda_\beta \in K$.

Therefore $n_\alpha = n_\beta = 2$ and $m_\alpha = m_\beta = m$, hence $N = 2m$ and

$$e_\alpha = e_\beta = 1 - \frac{3}{2m}, \quad Ne_\alpha = Ne_\beta = 2m - 3,$$

$$e'_\alpha = e'_\beta = 1 - \frac{2}{2m}, \quad Ne'_\alpha = Ne'_\beta = 2m - 2,$$

so the defining relations of the algebra $\Lambda_t$ (for any $t \in K$) are given by

$$\alpha^2 - ct^{2m-3}(\beta\alpha)^{m-1}\beta - c\lambda_\alpha t^{2m-2}(\beta\alpha)^m, \quad \beta^2 - ct^{2m-3}(\alpha\beta)^{m-1}\alpha - c\lambda_\beta t^{2m-2}(\alpha\beta)^m, \quad \alpha^2, \quad (\alpha\beta)^m.$$

If $t \neq 0$, the linear map defined by sending $\alpha$ to $t\alpha$ and $\beta$ to $t\beta$ induces an isomorphism between the algebras $\Lambda_1$ and $\Lambda_t$, the former being equal to the triangulation algebra associated with the auxiliary data as described in Section 5.3. Therefore the algebra $\Lambda_0$, which is precisely the Brauer graph algebra associated with these data (see Example 5.5), is a degeneration of the corresponding triangulation algebra. It follows that the latter algebra is also tame.

**Remark 7.16.** When $((Q, f), m)$ is exceptional, Proposition 7.13 does not apply but the triangulation algebra is still tame since it is of tubular type [12], see also Section 5.3 below.

8. **Known families of algebras as triangulation algebras**

8.1. **Algebras of quaternion type.** In [29, pp. 303-306], Erdmann gave a list of the possible quivers with relations of the algebras of quaternion type, and asked whether any such algebra is indeed of quaternion type [29, VII.9]. Later, Holm [48, §6] proved that the algebras in this list are of tame representation type. Erdmann and Skowroński [32, Theorem 5.9] proved that these algebras are periodic of period dividing 4 by constructing projective bimodule resolutions for them and deduced that they are indeed of quaternion type. In this section we give an alternative proof of the periodicity of modules for these algebras by showing that all the algebras in Erdmann’s list are 2-CY-tilted.
Remark 8.1. The second family $Q(k,s)(a,c)$ for $k \geq 1$, $s \geq 2$, $k+s \geq 4$ and $a \in K^\times$, $c \in K$.

Assume that the ground field is algebraically closed. We say that an algebra is of possibly quaternion type if it appears in Erdmann’s list. Consider two families of algebras in Erdmann’s list whose quivers with relations are shown in Figure 11, where for the convenience of the reader we tried to keep the notations as close as possible to the original ones. The first family $Q(2B)_1^{k,s}(a,c)$ depends on integer parameters $k \geq 1$, $s \geq 2$ such that $k+s \geq 4$ and scalars $a \in K^\times$ and $c \in K$. If $(k,s) = (1,3)$, one should assume that $a \neq 1$, otherwise one could set $a = 1$. The second family $Q(3K)^{a,b,c}$ depends on three integers $1 \leq a \leq b \leq c$ such that at most one of them equals 1. The scalar $d \in K^\times$ should be set to 1, unless $(a,b,c) = (1,2,2)$ and then $d \neq 1$.

Our presentation slightly deviates from the lists in the existing literature. The next two remarks explain the differences and the motivation behind them.

Remark 8.2. The parameter $d$ for the family $Q(3K)^{a,b,c}$ does not appear in the literature. In fact, in the original tables of [29], the parameters were assumed to satisfy $2 \leq a \leq b \leq c$. Only in [48] one value of 1 was allowed. Note that if $a = 1$ then the two arrows $\kappa$ and $\lambda$ can be eliminated from the quiver and one actually gets the algebras in another family $Q(2A)^{k,c}(c)$ for $k \geq 2$ and $c \in K$. The reason for including these algebras is to have a complete list of the derived equivalence classes of the algebras of (possibly) quaternion type, needed in the proof of Theorem 8.4 below. Thus one has to modify the statements of [32] Proposition 5.8, [33] Theorem 5.7, [48] Theorem 5.1 and [87] Theorem 8.6] accordingly, otherwise the derived equivalence classes of the algebras $Q(2A)^2(c)$ would be missing.

Figure 11. Quivers with relations of some algebras of possibly quaternion type.
Figure 12. Description of some families of algebras of (possibly) quaternion type \([29]\) as triangulation algebras. The labeling of the edges corresponds to that of the vertices in Figure 11. The subscript at each node indicates the corresponding multiplicity whereas the superscript indicates the scalar.

it suffices to specify two zero-relations as in Figure 11. These relations correspond to the two zero-relations occurring in the definition of the family \(Q(3A)^{a,b,c}(d)\).

Lemma 8.3. An algebra in the family \(Q(2B)^{k,s}(a,c)\) or \(Q(3K)^{a,b,c}\) is a triangulation algebra.

Proof (sketch). First, rescaling the arrow \(\alpha\) by a factor of \(a^{1/3}\) we slightly change the presentation of the algebra \(Q(2B)^{k,s}(a,c)\) and get the relations

\[
\begin{align*}
\alpha^2 - a^{k/3}(\beta\gamma\alpha)^{k-1}\beta\gamma - \lambda(\beta\gamma\alpha)^k & \quad \beta\gamma - \eta^{s-1} \\
\beta\eta - a^{k/3}(\alpha\beta\gamma)^{k-1}\alpha\beta & \quad \alpha^2\beta \\
\eta\gamma - a^{k/3}(\gamma\alpha\beta)^{k-1}\gamma\alpha & \quad \gamma^2\alpha
\end{align*}
\]

for some scalar \(\lambda \in K\).

This algebra is isomorphic to the triangulation algebra \(\Lambda(Q, f, m, c, \lambda)\) with the following data:

- The triangulation quiver \((Q, f)\) is isomorphic to quiver 2 in Table 2 with the permutation \(f\) written in cycle form as \((\alpha)(\eta\gamma\beta)\), so that the permutation \(g\) is \((\alpha\beta\gamma)(\eta)\);
- In terms of the corresponding ribbon graph, the \(g\)-invariant functions \(m\) and \(c\) are shown in Figure 12. The multiplicities are admissible when \(s \geq 3\). The triangulation quiver with multiplicities is exceptional precisely when \((k, s) = (1, 3)\), and the assumption that \(a \neq 1\) in this case ensures that Theorem 7.1 holds;
- The scalar \(\lambda_\alpha\) for the loop \(\alpha\) with \(f(\alpha) = \alpha\) is \(\lambda\).

Similarly, the algebra \(Q(3K)^{a,b,c}\) is a triangulation algebra for the following data:

- The triangulation quiver is isomorphic to quiver 3b in Table 2 with the permutation \(f\) written in cycle form as \((\beta\delta\lambda)(\kappa\eta\gamma)\), so that the permutation \(g\) is \((\beta\gamma)(\delta\eta)(\kappa\lambda)\);
- In terms of the corresponding ribbon graph, the \(g\)-invariant multiplicity and scalar functions are shown in Figure 12. The multiplicities are admissible when \(a \geq 2\).

In both cases Theorem 7.1 holds (even when the multiplicities are not admissible) and one deduces the extra zero-relations as in Section 7.1.

\[\square\]

Theorem 8.4. The following assertions are true:

(a) An algebra of possibly quaternion type is 2-CY-tilted.

(b) An algebra of possibly quaternion type is actually of quaternion type.
(c) An algebra of quaternion type is 2-CY-tilted.

Proof (sketch). An algebra of possibly quaternion type has at most three simple modules. The case of one simple module was considered in Corollary 5.28, so let Λ be such an algebra with two or three simple modules. A careful look at the derived equivalences constructed by Holm [48] for algebras of (possibly) quaternion type shows that there exist algebras Λₐ, Λ₁, . . . , Λₙ such that:

• Λ₀ is one of the algebras of possibly quaternion type appearing in Figure 11;
• For each 0 ≤ i < n, there exists an indecomposable projective Λᵢ-module Pᵢ such that Λᵢ₊₁ ≃ End(Uᵢ⁻¹(Λᵢ)) or Λᵢ₊₁ ≃ End(Uᵢ⁺¹(Λᵢ)) (cf. Section 2.5);
• Λₙ ≃ Λ.

The algebra Λ₀ is a triangulation algebra by Lemma 8.3, hence it is 2-CY-tilted by Theorem 7.1(d). By repeatedly applying Corollary 2.25 we see that since Λᵢ is symmetric and 2-CY-tilted, so is Λᵢ₊₁. Therefore Λ ≃ Λₙ is 2-CY-tilted.

Part (b) now follows from Proposition 2.16 and the tameness of the algebras of possibly quaternion type established by Holm [48, §6]. Part (c) is a consequence of Erdmann’s classification, but see the caveat in Proposition 8.8 below. □

The above proof also shows that all the algebras of quaternion type arise as algebras of the form given in Theorem 7.1(f).

Corollary 8.5. Blocks of finite groups with generalized quaternion defect group are 2-CY-tilted.

Remark 8.6. It is actually possible to present all the algebras of (possibly) quaternion type as Jacobian algebras of hyperpotentials and thus deduce an alternative, direct proof of Theorem 8.4.

It is also possible to present more families in the list of algebras of quaternion type as triangulation algebras. However, not all the algebras of quaternion type are triangulation algebras. For example, the algebras in the family Q(3A)ₖ,ₙ have the quiver

which is not a triangulation quiver or obtained from one by deleting arrows.

Remark 8.7. When the ground field is of characteristic zero, Burban, Iyama, Keller and Reiten have shown in [22, §7] that certain algebras of quaternion type occur as endomorphism algebras of cluster-tilting objects in the 2-Calabi-Yau stable categories of maximal Cohen-Macaulay modules over minimally elliptic curve singularities, and hence they are 2-CY-tilted. Moreover, they described these algebras as quotients of the complete path algebra by closed ideals.

These algebras are organized in two families, denoted Aₜ(λ), where q ≥ 2 and λ ∈ K×, and Bₚ,ₚ(λ), where p, q ≥ 1 and λ ∈ K×. The scalar λ could be set to 1 except for the algebras Aₜ(λ) and B₁,₁(λ) corresponding to the simply elliptic singularities, where one should assume λ ≠ 1.

Comparing their definition in [22, §7] with Definition 5.16 we see that the algebras Aₜ(λ) and Bₚ,ₚ(λ) are triangulation algebras with the triangulation quivers numbered 2 and 3a of Table 2, respectively. The corresponding multiplicities and scalars are shown in Figure 13.

As the next proposition shows, by using triangulation quivers and power series as in Remark 5.20 we are able to find algebras of quaternion type which seem not to appear in the known lists. Consider a new family of algebras Q(3A)ₖ,ₙ defined for the integers k > 2 by the quivers with relations given in Figure 14. By computing
their Cartan matrices, one verifies that these algebras do not belong to any of the families in Erdmann’s list.

**Proposition 8.8.** The algebras in the family $\mathcal{Q}(3A)^{k}_3$ are 2-CY-tilted and of quaternion type.

**Proof.** Let $\Lambda$ be the algebra $\mathcal{Q}(3A)^{k}_3$ for some $k > 2$. By slightly modifying the proof of Lemma 5.12 in [48], one shows that $U_P^{−2}P(\Lambda)$ is a tilting complex over $\Lambda$ whose endomorphism algebra is isomorphic to an algebra of the form $\mathcal{Q}(3A)^{k,2}_3$ in Erdmann’s list. The latter algebra has also the form $\mathcal{Q}(3K)^{1,2,k}$. Hence, by [48, Proposition 2.1] and Theorem 8.4, the algebra $\Lambda$ is of quaternion type. □

Corollary 2.25 would then imply that $\Lambda$ is 2-CY-tilted, but let us give a direct proof of this fact. Indeed, the algebra $\Lambda$ is a generalized version of a triangulation algebra, as considered in Remark 5.26, for the triangulation quiver $3b$ of Table 2 with the $f$-invariant invertible power series $p_\alpha(x)$ all set to 1 and the $g$-invariant power series given on the nodes of the corresponding ribbon graph by

where $q(x)$ is any power series such that the least order term of $q(x) − x$ has degree $k − 1$.

All the algebras of quasi-quaternion type constructed so far are 2-CY-tilted. In view of Theorem 6.3 and Theorem 8.4, the following question, which is a reformulation of Question 2.27 in the tame case, arises naturally.

**Question 8.9.** Let $\Lambda$ be an algebra of quasi-quaternion type. Is $\Lambda$ 2-CY-tilted?

### 8.2. 2-CY-tilted blocks.

By using results on the stable Auslander-Reiten quivers of tame blocks [31] and wild blocks [30], Erdmann and Skowroński have characterized the blocks of group algebras whose non-projective modules are periodic [34].

![Figure 13. Description of 2-CY-tilted algebras arising from minimally elliptic curve singularities as triangulation algebras. The subscript at each node indicates the corresponding multiplicity whereas the superscript indicates the scalar.](image1)

![Figure 14. A new family of algebras of quaternion type.](image2)
No. of Algebra Alternative description Marked surface Multiplicities
simplces description
\begin{tabular}{|c|c|c|c|}
\hline
2 & $A_2(\lambda)$ & $\mathbb{Q}(2B)_{\lambda,0}^{1,3}$ & punctured monogon & (1, 3) \\
 & $A_3(\lambda)$ & $\mathbb{Q}(2B)_{\lambda,\lambda}^{1,3}$ & & \\
\hline
3 & $A_1(\lambda)$ & $\mathbb{Q}(3K)_{1,1}^{1,3,2}$ & sphere, 3 punctures & (1, 2, 2) \\
\hline
6 & $T(B_i(\lambda))$ & $[40, \text{Fig. 1}]$, $\mathbb{Q}(3K)_{1,1}^{1,3,2}$ & sphere, 4 punctures & (1, 1, 1, 1) \\
 & $1 \leq i \leq 4$ & $[52, \text{Fig. 1.6}]$ & & \\
\hline
\end{tabular}

Table 6. The symmetric algebras of tubular type $(2, 2, 2, 2)$ and their socle deformations. Each family depends on a parameter $\lambda \in K \setminus \{0, 1\}$.

see also [33, Theorem 5.3]. As a consequence, by invoking Proposition 2.16, Theorem 6.3 and Corollary 8.5 we obtain the following characterization of 2-CY-tilted blocks.

**Proposition 8.10.** Let $B$ be a block of a group algebra over an algebraically closed field of characteristic $p$ with defect group $D$. Then $B$ is a 2-CY-tilted algebra if and only if either:

(a) $D$ is cyclic and $B$ has at most two simple modules; or

(b) $p = 2$ and $D$ is a generalized quaternion group.

**8.3. Symmetric algebras of tubular type $(2, 2, 2, 2)$.** In this section we show that the class of algebras considered in Theorem 7.1 contains all the symmetric algebras of tubular type $(2, 2, 2, 2)$ and their socle deformations. As a consequence, Question 2.27 has a positive answer for the tame symmetric algebras of polynomial growth. For the definitions of the terms in the next proposition we refer the reader to the classification of tame symmetric algebras of polynomial growth by Skowroński [86] and to the surveys [33, 87]. Recall that two self-injective algebras $\Lambda$ and $\Lambda'$ are *socle equivalent* if the factor algebras $\Lambda/\text{soc}\Lambda$ and $\Lambda'/\text{soc}\Lambda'$ are isomorphic.

**Proposition 8.11.** Let $\Lambda$ be a basic, indecomposable, representation-infinite tame symmetric algebra of polynomial growth. Then the following conditions are equivalent:

(a) $\Omega^1_4 M \simeq M$ for any $M \in \text{mod } \Lambda$;

(b) $\Lambda$ is socle equivalent to a symmetric algebra of tubular type $(2, 2, 2, 2)$;

(c) $\Lambda$ is a 2-CY-tilted algebra.

The implication $\text{(a)} \Rightarrow \text{(c)}$ follows from known results in the literature, we refer to [9, Proposition 6.2], [11] or [33, Theorem 6.1]. The implication $\text{(a)} \Rightarrow \text{(b)}$ is a consequence of Proposition 2.16. We prove the implication $\text{(b)} \Rightarrow \text{(c)}$ by using the classification of the tame symmetric algebras of tubular type and their socle deformations in [12, 13], keeping the notation introduced in these papers.

Let $\Lambda$ be socle equivalent to a symmetric algebra of tubular type $(2, 2, 2, 2)$. Then $\Lambda$ may have 2, 3 or 6 simple modules.

In the case of 2 simple modules, the algebra $\Lambda$ is either $A_2(\lambda)$ of [12] or the non-standard $A_3(\lambda)$ of [13], where $\lambda \in K \setminus \{0, 1\}$. We observe that $A_2(\lambda)$ is isomorphic to the algebra $\mathbb{Q}(2B)_{\lambda,0}^{1,3}$ whereas $A_3(\lambda)$ is isomorphic to the algebra $\mathbb{Q}(2B)_{\lambda,\lambda}^{1,3}$, hence both algebras are triangulation algebras by Lemma 8.3.

In the case of 3 simple modules, the algebra $\Lambda$ is $A_1(\lambda)$ of [12], which is isomorphic to $\mathbb{Q}(3K)_{1,1}^{1,3,2}$, so again Lemma 8.3 gives that $\Lambda$ is a triangulation algebra.
In the case of 6 simple modules, by [10] Proposition 5.2 the algebra \( \Lambda \) is the trivial extension algebra of a tubular algebra of type \((2,2,2,2)\), and there are exactly four such algebras, denoted by \( T(B_i(\lambda)) \) for \( 1 \leq i \leq 4 \), see [86] §3.3 or [11] §4.

It is instructive to compare the description of these trivial extension algebras as quivers with relations with the lists of quivers with potentials given in [10] Figure 1 or in [52] Figure 1.6 describing the endomorphism algebras of the cluster-tilting objects within the cluster category associated to a weighted projective line with weights \((2,2,2,2)\), and to see that these are identical.

Moreover, we observe that \( T(B_4(\lambda)) \) is a triangulation algebra for the triangulation quiver whose ribbon graph is the tetrahedron with all multiplicities set to 1. The marked surfaces realizing the symmetric algebras of tubular type \((2,2,2,2)\) and their socle deformations are summarized in Table 6.

8.4. Jacobian algebras from closed surfaces. In this section we explain how Theorem 7.1 implies that the Jacobian algebras of the quivers with potentials associated by Labardini-Fragoso to triangulations of closed surfaces with punctures are of quasi-quaternion type.

In [64], Labardini-Fragoso constructed potentials on the adjacency quivers of triangulations of marked surfaces and proved that flips of triangulations result in mutations of their associated quivers with potentials. Denote by \( Q'_\tau \) the adjacency quiver of a triangulation \( \tau \) of a marked surface \((S, M)\) as defined by Fomin, Shapiro and Thurston [66] Definition 4.1 (we use the notation \( Q'_\tau \) to distinguish it from the underlying quiver \( Q_\tau \) of the triangulation quiver associated to \( \tau \), see Section 4.2) and let \( W_\tau \) be the associated potential on \( Q'_\tau \). The notion of flip occurring in the next proposition is explained later in Section 9.2.

**Proposition 8.12 ([70] Theorem 30).** If a triangulation \( \tau' \) of \((S, M)\) is obtained from \( \tau \) by flipping an arc \( \gamma \), then the quiver with potential \((Q'_\tau, W_\tau')\) is right equivalent to the mutation of \((Q'_\tau, W_\tau)\) at the vertex of \( Q'_\tau \) corresponding to \( \gamma \).

We now assume that the surface \( S \) is closed. In this case the potentials depend on scalars attached to the punctures of \( S \). For “nice” triangulations of \((S, M)\), an equivalent description of the quivers with potentials was given in [70], where we also showed that their Jacobian algebras are finite-dimensional and symmetric. In particular, the scalars can be encoded as a \( g \)-invariant function \( c: Q_1 \to K^\times \) and the Jacobian algebra of the associated potential is a triangulation algebra, where all the multiplicities are set to 1.

**Proposition 8.13 ([70] §2).** Let \( \tau \) be a triangulation of a closed surface which is not a sphere with less than four punctures, and assume that at each puncture there are at least three incident arcs. Then \( Q'_\tau = Q_\tau \), the constant multiplicity function \( 1 \) is admissible, and the Jacobian algebra \( P(Q_\tau, W_\tau) \) is isomorphic to the triangulation algebra \( \Lambda(Q_\tau, f_\tau, 1, c) \).

Let \( (S, M) \) be a closed surface which is not a sphere with less than four punctures.

In [70] §5 we proved the existence of a triangulation \( \tau \) of \((S, M)\) satisfying the condition in Proposition 5.13. Therefore Theorem 7.1 applies for the triangulation algebra \( P(Q_\tau, W_\tau) \). We note that in the case of a sphere with exactly four punctures the ribbon graph of \( \tau \) is a tetrahedron and the corresponding assumption on the scalars attached to the punctures has to be made.

Let \( C \) be the triangulated 2-Calabi-Yau category of Theorem 7.1 such that \( \text{End}_C(T) \cong P(Q_\tau, W_\tau) \) for some cluster-tilting object \( T \) of \( C \). It is well known that any other triangulation \( \tau' \) of \((S, M)\) can be obtained from \( \tau \) by a sequence of flips. Let \( T' \) be the cluster-tilting object of \( C \) obtained from \( T \) by the corresponding sequence of Iyama-Yoshino mutations. Repeated application of Proposition 2.20 and
Proposition 8.12 shows that \( \text{End}_C(T') \simeq \mathcal{P}(Q'_r, W_r) \), hence part 1 of Theorem 7.1 applies and we get the following result.

**Corollary 8.14.** Let \((S, M)\) be a closed surface which is not a sphere with less than four punctures. Then the Jacobian algebras of the quivers with potentials associated to the ideal triangulations of \((S, M)\) are finite-dimensional of quasi-quaternion type and they are all derived equivalent to each other. Moreover, each of these algebras arises as an algebra in part 1 of Theorem 7.1 for a suitable triangulation quiver.

**Remark 8.15.** The tameness of the algebras \( \mathcal{P}(Q'_r, W_r) \) has also been proved in [41] using a different degeneration argument.

**Remark 8.16.** Labardini-Fragoso showed also that the potentials \( W_r \) are non-degenerate [65], but this fact is not needed in order to establish Corollary 8.14.

9. Mutations

Many of the algebras occurring in part 1 of Theorem 7.1 are themselves triangulation algebras. In this section we introduce a notion of mutation for triangulation quivers and study its relations to other notions of mutation in the literature including flips of triangulations, Kauer’s elementary moves for Brauer graph algebras [53], mutations of quivers with potentials [25] and Iyama-Yoshino mutations [50] within the triangulated 2-Calabi-Yau categories appearing in Theorem 7.1.

### 9.1. Mutation of triangulation quivers

A mutation of a triangulation quiver at some vertex is a new triangulation quiver. We first give the definition in the case the vertex we mutate at has no loops.

**Definition 9.1.** Let \((Q, f)\) be a triangulation quiver and let \(k\) be a vertex of \(Q\) without loops. Denote by \(\alpha, \bar{\alpha}\) the two arrows that start at \(k\) and observe that our assumption on \(k\) implies that there are six distinct arrows

\[
\alpha_1 = \alpha, \quad \beta_1 = f(\alpha), \quad \gamma_1 = f^2(\alpha), \quad \alpha_2 = \bar{\alpha}, \quad \beta_2 = f(\bar{\alpha}), \quad \gamma_2 = f^2(\bar{\alpha})
\]

which form two cycles of the permutation \(f\).

The mutation of \((Q, f)\) at \(k\), denoted \(\mu_k(Q, f)\), is the triangulation quiver \((Q', f')\) obtained from \((Q, f)\) by performing the following steps:

1. Remove the two arrows \(\beta_1\) and \(\beta_2\);
2. Replace the four arrows \(\alpha_1, \alpha_2, \gamma_1\) and \(\gamma_2\) with arrows in the opposite direction \(\alpha_1^*\), \(\alpha_2^*, \gamma_1^*\) and \(\gamma_2^*\);
3. Add new arrows \(\delta_1\) and \(\delta_2\) with

\[
s(\delta_{12}) = s(\gamma_1), \quad t(\delta_{12}) = t(\alpha_2), \quad s(\delta_{21}) = s(\gamma_2), \quad t(\delta_{21}) = t(\alpha_1),
\]

see Figure 15(a).
4. Define the permutation \(f'\) on the new set of arrows \(Q'\) by \(f'(\varepsilon) = f(\varepsilon)\) if \(\varepsilon\) is an arrow of \(Q\) which has not been changed, and by

\[
f'(\alpha_1^*) = \gamma_2^*, \quad f'(\gamma_2^*) = \delta_{21}, \quad f'(\delta_{21}) = \alpha_1^*,
\]

\[
f'(\alpha_2^*) = \gamma_1^*, \quad f'(\gamma_1^*) = \delta_{12}, \quad f'(\delta_{12}) = \alpha_2^*.
\]

for the other arrows.

At the level of the underlying quivers, this is similar to Fomin-Zelevinsky mutation [37]. Note, however, that the quivers \(Q\) and \(Q'\) may have 2-cycles.

Next, we define mutation at a vertex with loop.

**Definition 9.2.** Let \((Q, f)\) be a triangulation quiver and let \(k\) be a vertex of \(Q\) with a loop. Denote by \(\alpha, \bar{\alpha}\) the two arrows that start at \(k\) and assume that \(\bar{\alpha}\) is a loop. The mutation of \((Q, f)\) at \(k\), denoted \(\mu_k(Q, f)\), is the triangulation quiver \((Q', f')\) obtained from \((Q, f)\) by performing the following steps:
Figure 15. Mutation of triangulation quivers at the middle vertex \( \circ \); (a) without loops; (b) with a loop fixed by the permutation \( f \).
Some of the other vertices may coincide, and only the arrows that change are shown.

(0) If \( g(\bar{\alpha}) = \bar{\alpha} \), or if \( \alpha \) is also a loop, then set \((Q', f') = (Q, f)\).
Otherwise, there are four distinct arrows
\[ \alpha, \quad \beta = f(\alpha), \quad \gamma = f^2(\alpha), \quad \delta = \bar{\alpha} = f(\bar{\alpha}) \]
which form two cycles of the permutation \( f \).

(1) Replace the four arrows \( \alpha, \beta, \gamma \) and \( \delta \) by arrows in the opposite direction
\[ \alpha^*, \quad \beta^* = f(\alpha^*), \quad \gamma^* = f^2(\alpha^*), \quad \delta^* = \bar{\alpha} = f(\bar{\alpha}) \]
for the other arrows.
Note that the arrow \( \delta^* \) is also a loop at \( k \) so we could have avoided the reversal of \( \delta \). This reversal is done in order to stress the analogy to the general case of Definition 9.1.

Example 9.3. We describe all the mutations of the triangulation quivers appearing in Table 2. For each of the triangulation quivers 1, 2, 3' and 3'', a mutation at any vertex gives an isomorphic triangulation quiver. For the triangulation quiver 3b, a mutation at any vertex is isomorphic to the triangulation quiver 3a. For the triangulation quiver 3a, a mutation at the vertex 2 is isomorphic to the triangulation quiver 3b, whereas a mutation at any of the other vertices gives the triangulation quiver 3a.

Remark 9.4. As can be seen from Figure 15, mutation is an involution. In other words, if \((Q, f)\) is a triangulation quiver and \( k \) is a vertex of \( Q \), then the triangulation quiver \( \mu_k(\mu_k(Q, f)) \) is isomorphic to \((Q, f)\).

The permutation \( f' \) on \( Q'_1 \) defines the permutation \( g' \) by \( g'(\alpha') = \overline{f'(\alpha')} \) for any \( \alpha' \in Q'_1 \). The next statement is a consequence of the definitions.

Lemma 9.5. Let \((Q', f')\) be a mutation of the triangulation quiver \((Q, f)\) at some vertex. Then:
(a) The permutations \( f \) and \( f' \) have the same cycle structure.
(b) The permutations \( g \) and \( g' \) have the same number of cycles.
Example 9.6. Although $g$ and $g'$ have the same number of cycles, the lengths of the cycles may change. For example, for the triangulation quiver 3b of Table 2 the lengths are 2, 2, 2 whereas for the mutated triangulation quiver 3a they are 1, 4, 1.

Lemma [9.5] implies that any $g$-invariant function $\nu$ gives rise to a $g'$-invariant function $\nu'$ on $Q'_1$ with the same image. Explicitly, this is done by setting $\nu'_{\varepsilon} = \nu_{\varepsilon}$ for the arrows in $Q'_1$ that are also in $Q_1$ and

$$\nu'_{\alpha_1} = \nu_{\gamma_1}, \quad \nu'_{\alpha_2} = \nu'_{\gamma_2} = \nu_{\gamma_2}, \quad \nu'_{\delta_1} = \nu_{\gamma_1} = \nu_{\alpha_2}, \quad \nu'_{\delta_2} = \nu_{\gamma_2} = \nu_{\alpha_1}$$

for the other arrows in the case of Definition 9.1 and

$$\nu'_{\alpha_1} = \nu'_{\gamma_1} = \nu_{\delta_1}, \quad \nu'_{\alpha_2} = \nu'_{\gamma_2} = \nu_{\delta_2}, \quad \nu'_{\gamma} = \nu_{\gamma} = \nu_{\delta} = \nu_{\alpha}$$

in the case of Definition 9.2. In particular, any two $g$-invariant functions $m: Q_1 \rightarrow \mathbb{Z}_{>0}$ and $c: Q_1 \rightarrow K^\times$ of multiplicities and scalars on $(Q, f)$ give rise to $g'$-invariant functions of multiplicities $m': Q'_1 \rightarrow \mathbb{Z}_{>0}$ and scalars $c': Q'_1 \rightarrow K^\times$ on $(Q', f')$.

Remark 9.7. Since the lengths of the cycles of $g$ may change under mutation, even if a multiplicity function $m: Q_1 \rightarrow \mathbb{Z}_{>0}$ on $(Q, f)$ was admissible, the multiplicity function $m'$ on $(Q', f')$ may not be admissible anymore.

Example 9.8. Continuing Example 9.6, if $m$ is the multiplicity function for the triangulation quiver 3b taking the constant value 2, then $m'$ takes the constant value 2 on the arrows of the triangulation quiver 3a. Hence $m$ is admissible while $m'$ is not.

Similarly, Lemma [9.5] implies that any function $\theta$ on the set $Q'_1$ of fixed points of $f$ gives rise to a function $\theta'$ on the set $(Q'_1)'$ of fixed points of $f'$. Explicitly, in the case of Definition 9.1 we have $\theta' = \theta$, whereas in the case of Definition 9.2 we have $\theta'_{\gamma_1} = \theta_{\delta_1}$ and $\theta'_{\gamma_2} = \theta_{\delta_2}$ for any arrow $\varepsilon \neq \delta$ with $f(\varepsilon) = \varepsilon$.

9.2. Mutations and flips. Fomin, Shapiro and Thurston have shown in [36, Proposition 4.8] that if two triangulations are related by flipping an arc, then their adjacency quivers are related by a Fomin-Zelevinsky mutation at the vertex corresponding to that arc. In this section we discuss an analogous statement for triangulation quivers.

Let $\tau$ be a triangulation of a marked surface $(S, M)$. If $\gamma$ is an arc of $\tau$ which is not the inner side of a self-folded triangle, then it is possible to replace $\gamma$ by another arc $\gamma'$ to obtain a triangulation $\tau' = \tau \setminus \{\gamma\} \cup \{\gamma'\}$ which is not topologically equivalent to $\tau$, see Figure 16. The triangulation $\tau'$ is called the flip of $\tau$ at $\gamma$.

Lemma 9.9. The triangulation quivers of two triangulations related by a flip at some arc are related by a mutation at the vertex corresponding to that arc.
Proof. First we verify that a vertex corresponding to a flippable arc cannot have loops. Indeed, for a loop \( \alpha \) at some vertex \( k \) we have that either \( f(\alpha) = \alpha \) or \( g(\alpha) = \alpha \). In the former case \( k \) corresponds to a boundary segment, whereas in the latter case it corresponds to an arc which is the inner side of a self-folded triangle.

Now the claim follows by comparing Figure 16 and Figure 15(a) using the construction of triangulation quiver visualized in Figure 3. \( \square \)

Consider now a mutation of a triangulation quiver \((Q, f)\) at a vertex with a loop fixed by the permutation \( f \). The change of the associated ribbon graphs is illustrated in Figure 17. In particular, if \( \tau \) is a triangulation of a marked surface \((S, M)\) and \( k \) is a vertex corresponding to a boundary segment of \((S, M)\), then a mutation of \((Q_\tau, f_\tau)\) at \( k \) is a triangulation quiver \((Q_{\tau'}, f_{\tau'})\) of a new marked surface \((S', M')\) which is obtained from \((S, M)\) as follows: remove the boundary segment corresponding to \( k \) from the boundary component containing it represented by the left node of the ribbon graph in Figure 17 and add it to the component (or puncture) represented by the right node. The arcs of \( \tau' \) are identical to those of \( \tau \).

Adding or removing a boundary segment is equivalent to adding or removing one marked point. Here, it makes sense to consider punctures as boundary components with zero marked points. So, when we remove a boundary segment from a component with just one marked point we get a puncture, and conversely, when we add a boundary segment to a puncture we get a boundary component with one marked point.

This point of view can be made more systematic by using the notion of orbifolds and their triangulations as introduced by Felikson, Shapiro and Tumarkin [35, §4]. The precise details are outside the scope of this survey, but let us just mention that any marked surface \((S, M)\) gives rise to a closed orbifold \( O \) by replacing each boundary component of \((S, M)\) containing \( n \) marked points by a puncture and \( n \) orbifold points, each connected to that puncture by a so-called pending arc. Any triangulation of \((S, M)\) yields a triangulation of the orbifold \( O \).

The transitivity of flips on triangulations of orbifolds implies the next proposition, which provides a partial converse to Lemma 9.5.

Proposition 9.10. Let \( \tau \) be a triangulation of a marked surface \((S, M)\) with \( p \) punctures and \( b \) boundary components, and let \( \tau' \) be a triangulation of a marked surface \((S', M')\) with \( p' \) punctures and \( b' \) boundary components. Then the following conditions are equivalent:

(a) The triangulation quiver \((Q_\tau, f_\tau)\) can be obtained from \((Q_\tau, f_\tau)\) by a finite sequence of mutations;

(b) The topological parameters of the marked surfaces \((S, M)\) and \((S', M')\) satisfy

\[
\text{genus}(S) = \text{genus}(S'), \quad p + b = p' + b', \quad |M| - p = |M'| - p'; \quad (9.1)
\]

(c) The permutations \( f_\tau \) and \( f_{\tau'} \) have the same cycle structure and the permutations \( g_\tau \) and \( g_{\tau'} \) have the same number of cycles.

Remark 9.11. Two closed surfaces \((S, M)\) and \((S', M')\) satisfy (9.1) if and only if they are homeomorphic (i.e. they have the same genus and the same number of punctures).
9.3. Mutations and Kauer moves. Rickard [80, Theorem 4.2] proved that a Brauer tree algebra is derived equivalent to a Brauer star algebra by constructing a tilting complex over the former whose endomorphism algebra is isomorphic to the latter. Later, König and Zimmermann [60] have shown that a Brauer tree can be transformed to a Brauer star by applying a sequence of small changes, replacing one edge at a time. In each such replacement, the Brauer tree algebras of the two trees are related by a tilting complex of length 2 which is of the form given in (2.1), so in particular they are derived equivalent.

In [53], Kauer considered more generally Brauer graph algebras and defined similar moves, which he called elementary moves. For each edge $e$ of a Brauer graph he defined a new graph obtained by replacing $e$ (i.e., taking it out and putting it back in a different place) such that if $\Gamma$ is the Brauer graph algebra corresponding to the original graph and $P$ is the indecomposable projective $\Gamma$-module corresponding to the edge $e$, then $\text{End}_{D^b(\Gamma)} U_P^+ (\Gamma)$ is the Brauer graph algebra corresponding to the new graph.

There are three kinds of elementary moves; the first involves edges that are leaves in the graph (i.e., they are incident to nodes without any additional incident edges); the second involves edges that are loops whose two half-edges are successive in the cyclic ordering around their common node; and the third involves the other edges. In terms of the ribbon quiver, the first case corresponds to vertices with a loop $\alpha$ such that $g(\alpha) = \alpha$; the second to vertices with a loop $\alpha$ such that $f(\alpha) = \alpha$; and the third to vertices without loop.

**Proposition 9.12.** Let $(Q, f)$ be a triangulation quiver, let $k$ be a vertex of $Q$ and let $(Q', f')$ be the mutation of $(Q, f)$ at $k$. Then:

(a) The ribbon graphs of $(Q, f)$ and $(Q', f')$ are related by an elementary move at the edge corresponding to the vertex $k$.

(b) Let $m: Q_1 \to \mathbb{Z}_{>0}$ and $c: Q_1 \to K^\times$ be $g$-invariant functions of multiplicities and scalars, respectively, and let $m': Q'_1 \to \mathbb{Z}_{>0}$ and $c': Q'_1 \to K^\times$ be the $g'$-invariant functions induced from $m$ and $c$. Then the Brauer graph algebras $\Gamma = \Gamma(Q, f, m, c)$ and $\Gamma' = \Gamma(Q', f', m', c')$ satisfy

$$\text{End}_{D^b(\Gamma)} U_P^+ (\Gamma) \simeq \text{End}_{D^b(\Gamma')} U_P^+ (\Gamma')$$

and in particular they are derived equivalent.

If $(Q_\tau, f_\tau)$ is a triangulation quiver arising from a triangulation $\tau$ of a marked surface $(S, M)$, then by Remark 4.6 we can think of the multiplicities and scalars as quantities attached to each puncture and boundary component of $(S, M)$. By combining Proposition 9.10 and Proposition 9.12 we deduce the next corollary which implies in particular that the derived equivalence class of a Brauer graph algebra from a triangulation quiver may depend only on the surface and not on the particular triangulation.

**Corollary 9.13.** Let $(S, M)$ and $(S', M')$ be two marked surfaces whose topological parameters satisfy Eq. (9.1). Let $\tau$ be any triangulation of $(S, M)$ and let $\tau'$ be any triangulation of $(S', M')$. Then:

(a) The triangulation quiver $(Q_\tau', f_\tau')$ can be obtained from $(Q_\tau, f_\tau)$ by a sequence of mutations, hence any $g$-invariant function $\nu$ on $(Q_\tau)_1$ yields a $g$-invariant function $\nu'$ on $(Q_\tau')_1$.

(b) The Brauer graph algebras $\Gamma(Q_\tau, f_\tau, m, c)$ and $\Gamma(Q_\tau', f_\tau', m', c')$ are derived equivalent for any $g$-invariant function of multiplicities $m: (Q_\tau)_1 \to \mathbb{Z}_{>0}$ and scalars $c: (Q_\tau)_1 \to K^\times$.

**Remark 9.14.** It has also been observed by Marsh and Schroll [74] that by viewing triangulations of marked surfaces as ribbon graphs, flips of triangulations become...
elementary moves of Brauer graphs and hence a marked surface gives rise to a collection of derived equivalent Brauer graph algebras. Note that in the case of surfaces with non-empty boundary, the Brauer graph algebras they consider are somewhat different than the algebras considered here.

**Remark 9.15.** Recently, a description of Kauer’s elementary moves in terms of the ribbon quivers has been given in [1].

### 9.4. Mutations and quivers with potentials

Let \((Q, f)\) be a triangulation quiver and let \((Q', f')\) be a mutation of \((Q, f)\) at a fixed vertex \(k\) of \(Q\).

Let \(R: Q_1 \to K[[x]]\) be a \(g\)-invariant function and let \(P: Q'_1 \to K[[x]]\) be a function whose values are power series (i.e. \(P_\alpha(x)\) is a power series for each \(\alpha \in Q_1\) such that \(f(\alpha) = \alpha\)). Consider the potential on \(Q\) defined by

\[
W = \sum_{\alpha : f(\alpha) = \alpha} P_\alpha(\alpha) + \sum_{\alpha : f(\alpha) \neq \alpha} \alpha \cdot f(\alpha) \cdot f^2(\alpha) - \sum_{\beta} R_\beta(\omega_\beta),
\]

where the first sum runs over the fixed points of \(f\), the second runs over representatives \(\alpha\) of the \(f\)-cycles of length 3 the third runs over representatives \(\beta\) of the \(g\)-cycles in \(Q_1\). This is a special case of a potential considered in Proposition 5.14(b), as \(P\) can be extended to an \(f\)-invariant function on all the arrows by setting \(P_\alpha(x) = x\) for any arrow \(\alpha\) with \(f(\alpha) \neq \alpha\).

By the discussion in Section 9.1, the function \(R\) gives rise to a \(g'\)-invariant function \(R'\) and the function \(P\) gives rise to a function \(P'\) on the set \((Q'_1)^{f'}\) of fixed points of \(f'\), hence to the potential on \(Q'\) given by

\[
W' = \sum_{\alpha' : f'(\alpha') = \alpha'} P'_{\alpha'}(\alpha') + \sum_{\alpha' : f'(\alpha') \neq \alpha'} \alpha' \cdot f'(\alpha') \cdot f'^2(\alpha') - \sum_{\beta'} R'_{\beta'}(\omega_{\beta'}),
\]

where the sums run over fixed points \(\alpha'\) of \(f'\), representatives \(\alpha'\) of the \(f'\)-cycles of length 3 and representatives \(\beta'\) of the \(g'\)-cycles in \(Q'_1\).

The next proposition compares \((Q', W')\) with the mutation of the quiver with potential \((Q, W)\) at the vertex \(k\) as defined in [25 §5].

**Proposition 9.16.** Assume that there are no 2-cycles in \(Q\) passing through the vertex \(k\). Then \((Q', W')\) is right equivalent to the mutation of \((Q, W)\) at \(k\).

The assumption in the proposition implies that \(k\) has no loops and therefore the mutation is governed by Definition 9.1. In the notations of that definition, the condition in the proposition is equivalent to the conditions that \(n_{\alpha_1} > 2\), \(n_{\alpha_2} > 2\), \(n_{\beta_1} > 1\) and \(n_{\beta_2} > 1\).

### 9.5. Mutations and triangulation algebras

Let \((Q, f)\) be a triangulation quiver and let \(k\) be a vertex of \(Q\). Let \(m: Q_1 \to \mathbb{Z}_{>0}\) and \(c: Q_1 \to K^\times\) be \(g\)-invariant functions of multiplicities and scalars, respectively and let \(\lambda: Q'_1 \to K\). Assume that:

- \(m\) is admissible;
- if \((Q, f, m, c, \lambda)\) is exceptional, the scalars \(c: Q_1 \to K^\times\) satisfy the conditions stated before Theorem 7.1;
- \(\text{char } K\) does not divide \(\mu_1\mu_2\) (see Proposition 5.17 for the definition);
- there are no 2-cycles in \(Q\) passing through the vertex \(k\).

Consider the triangulation algebra \(\Lambda = \Lambda(Q, f, m, c, \lambda)\). Our first two assumptions imply that Theorem 7.1 holds for \(\Lambda\) and that in particular, \(\Lambda\) is symmetric and there is a triangulated 2-Calabi-Yau category \(\mathcal{C}\) with a cluster-tilting object \(T\) such that \(\Lambda \simeq \text{End}_C(T)\).

Let \((Q', f')\) be the mutation of \((Q, f)\) at \(k\), let \(m': Q'_1 \to \mathbb{Z}_{>0}\) and \(c': Q'_1 \to K^\times\) be the \(g'\)-invariant functions induced from \(m\) and \(c\), and let \(\lambda': (Q'_1)^{f'} \to K\) be the
function on the arrows fixed by $f'$ induced from $\lambda$. Our last assumption implies that the triangulation algebra $\Lambda' = \Lambda(Q', f', m', c', \lambda')$ is well defined (i.e. $m_{\alpha}, n_{\alpha} \geq 2$ for any $\alpha \in Q_1$), but $m'$ is not necessarily admissible.

**Proposition 9.17.** Under the above assumptions, the following assertions hold true:

(a) $\Lambda \simeq \mathcal{P}(Q, W)$, where the potential $W$ takes the form in (9.3) for suitable $g$-invariant function $R : Q_1 \to K[[x]]$ and function $P : Q'_1 \to K[[x]]$.

(b) $\Lambda' \simeq \mathcal{P}(Q', W')$ for the potential $W'$ given in (9.4) with the functions $R'$ and $P'$ corresponding to the functions $R$ and $P$ of part (a).

(c) The quiver with potential $(Q', W')$ is right equivalent to the mutation of the quiver with potential $(Q, W)$ at the vertex $k$.

(d) $\Lambda' \simeq \text{End}_C(T')$, where $T'$ is the Iyama-Yoshino mutation of $T$ with respect to the indecomposable summand corresponding to the vertex $k$.

(e) $\Lambda'$ is derived equivalent to $\Lambda$ and is of quasi-quaternion type. More precisely, we have isomorphisms

\begin{equation}
\text{End}_{\mathcal{D}^b(\Lambda)} U^+_{P_k} (\Lambda) \simeq \Lambda' \simeq \text{End}_{\mathcal{D}^b(\Lambda')} U^+_{P_k} (\Lambda)
\end{equation}

where $P_k$ is the indecomposable projective $\Lambda$-module corresponding to the vertex $k$.

Claim (a) follows by our assumption on $\mu_f \mu_g$. Since $\mu_{f'} = \mu_f$ and $\mu_{g'} = \mu_g$, claim (b) follows in a similar way. Claim (c) is a consequence of the previous claims together with our assumption on the vertex $k$ and Proposition 9.16. Claim (d) follows from (a) and Proposition 2.20. Finally, claim (e) is a consequence of (d) and Proposition 2.21.

**Remark 9.18.** Our assumptions on the characteristic of $K$ and the vertex $k$ are needed in order to use the theory of mutations of quivers with potentials. It seems very likely that the statements in parts (a) and (b) of Proposition 9.17 are still true even if we drop the assumption on the characteristic of $K$ and weaken the assumption on the vertex $k$, requiring only that the triangulation algebra $\Lambda'$ is defined.

### 9.6. Construction of infinitely many non-degenerate potentials

We conclude by presenting an application of the preceding results to the theory of quivers with potentials.

For a mutation $(Q', W')$ of a quiver with potential $(Q, W)$, the underlying quiver $Q'$ may have 2-cycles even if the quiver $Q$ did not have such. Thus, $(Q', W')$ could not be further mutated at the vertices lying on these 2-cycles.

A quiver with potential $(Q, W)$ is non-degenerate if, for any sequence of mutations of quivers with potentials, the underlying quiver does not contain any 2-cycles [25, Definition 7.2]. The existence of non-degenerate potentials is crucial to several approaches to solve various conjectures on cluster algebras, either via the representations of Jacobian algebras and their mutations as in [25], or via the generalized cluster categories [77].

Derksen, Weyman and Zelevinsky proved [25] Corollary 7.4] that if the ground field is uncountable, then over any quiver without loops and 2-cycles there is at least one non-degenerate potential. It is interesting to know when such non-degenerate potential is unique (up to right equivalence). For instance, on quivers without oriented cycles there is only one potential, namely the zero potential. In [41] Theorem 1.4], Geiss, Labardini-Fragoso and Schröer proved that apart from one exception, the adjacency quiver of a triangulation of a marked surface with non-empty boundary has only one non-degenerate potential. More generally, we proved in [69] §4 that a non-degenerate potential is unique on any quiver belonging to the class
Consider a triangulation quiver \((Q, f)\) such that:

(\star) The permutation \(g\) has one cycle and all the cycles of \(f\) are of length 3.

These assumptions imply that the quiver \(Q\) does not have loops or 2-cycles (see Proposition 3.25). Moreover, a potential as in Eq. (9.3) is controlled by one power series \(R(x) \in K[[x]]\), and all the cycles \(\omega_\alpha\) (where \(\alpha\) runs over the arrows of \(Q\)) are rotationally equivalent. Denote by \(\omega\) one of these cycles.

If \((Q', f')\) is a mutation of \((Q, f)\), then by Lemma 9.5 it also satisfies (\(\star\)) and hence Proposition 9.16 can be applied indefinitely to yield the following.

**Proposition 9.19.** Let \((Q, f)\) be a triangulation quiver satisfying condition (\(\star\)). Then for any power series \(R(x) \in xK[[x]]\), the potential \(W_R\) on \(Q\) given by

\[
W_R = -R(\omega) + \sum_\alpha \alpha \cdot f(\alpha) \cdot f^2(\alpha)
\]

(where the sum runs over representatives \(\alpha\) of the \(f\)-cycles) is non-degenerate.

Consider now a triangulation \(\tau\) of a closed surface with exactly one puncture. Then its triangulation quiver \((Q_\tau, f_\tau)\) satisfies condition (\(\star\)) by Remark 4.6. Moreover, the adjacency quiver of \(\tau\) is \(Q_\tau\) by Corollary 4.12.

**Corollary 9.20.** Let \(Q\) be the adjacency quiver of a triangulation of a closed surface with exactly one puncture, and view it as triangulation quiver \((Q, f)\). Then:

(a) For any power series \(R(x) \in xK[[x]]\), the potential \(W_R\) on \(Q\) defined by Eq. (9.6) is non-degenerate.

(b) Let \(R_0(x) = 0\) and \(R_m(x) = x^m\) for \(m \geq 1\). Then

\[
\{W_{R_0}\} \cup \{W_{R_m} : m \geq 1 \text{ is not divisible by char } K\}
\]

is an infinite set of non-degenerate potentials on \(Q\) whose Jacobian algebras are pairwise non-isomorphic.

**Remark 9.21.** For a quiver as in Corollary 9.20 it was known that there are at least two inequivalent non-degenerate potentials, denoted in our notation by \(W_0\) and \(W_x\), see [72, §4.3], [69, §3] and [41, Proposition 9.13]. Note that the Jacobian algebra of \(W_0\) is infinite dimensional whereas that of any \(W_x^m\) with \(m \geq 1\) not divisible by char \(K\) is a triangulation algebra and hence finite-dimensional of quasi-hereditary type. The latter Jacobian algebras are pairwise non-isomorphic because their dimensions are all different; indeed, if the surface has genus \(g \geq 1\) then \(Q\) has \(6g - 3\) vertices (Remark 4.5) and hence the Jacobian algebra of \(W_x^m\) has dimension \(m(12g - 6)^2 = 36m(2g - 1)^2\) (Proposition 7.9).

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