Triangular Ladders $P_{d,2}$ are $e$-positive

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Abstract

In 1995 Stanley conjectured that the chromatic symmetric function of the graphs $P_{d,2}$, which we call triangular ladders, are $e$-positive. In this paper we confirm this conjecture, which is also an unsolved case of the celebrated $(3+1)$-free conjecture. Our method is to follow the generalization of the chromatic symmetric functions by Gebhard and Sagan into to symmetric functions in non-commuting variables, which does satisfy a deletion-contraction property unlike the chromatic symmetric function in commuting variables. We do this by proving a new signed combinatorial formula for all interval graphs in the basis of elementary symmetric functions. Then we prove $e$-positivity for triangular ladders by very carefully defining a sign-reversing involution on our signed combinatorial formula, which leaves us with certain positive terms and further allow us to expand on an already known family of $e$-positive graphs by Gebhard and Sagan.

1 Introduction

The chromatic symmetric function of a simple graph $G$ defined by Richard Stanley [15], $X_G$, is a generalization of the chromatic polynomial defined by Birkoff [2], $\chi_G$, and has received a lot of attention as of late. These symmetric functions carry many properties over from their chromatic polynomials including the number of acyclic orientations [15], but do not satisfy a useful deletion-contraction property that chromatic polynomials do. However, they do have connections to representation theory and algebraic geometry [10], which has been a further motivation in their study and particularly behind the study of their $e$-positivity that is the ability to write the function $X_G$ as a non-negative sum of elementary symmetric functions, and Schur-positivity that is the ability to write $X_G$ as a non-negative sum of Schur functions. In 1995 Stanley [15] conjectured that if a poset is $(3+1)$-free then its incomparability graph is $e$-positive, which is equivalent to the Stanley-Stembridge conjecture in 1993 [17]. This conjecture has been reduced to showing that the incomparability graphs of $(3+1)$ and $(2+2)$-free posets are $e$-positive by Guay-Paquet [8]. These types of graphs are known as interval graphs and have a connection to Jacobi-Trudi matrices [17]. Gasharov [6] has proven that the incomparability graph of $(3+1)$-free poset is Schur-positive, which is weaker than the full conjecture since $e$-positivity implies Schur-positivity.

There have been some partial results on this conjecture. The path and cycle graphs have been shown to be $e$-positive by Stanley in 1995 [15] with a full description of their coefficients in [19]. Coefficients of other graphs have been studied in [11, 12]. Other works have focused on finding graph properties relating to $e$-positivity with an emphasis on induced subgraphs [4, 9, 18]. Shareshian and Wachs [14] defined a generalization of the chromatic symmetric function in the space of quasi-symmetric functions and have generalized the $(3+1)$-free conjecture as well.
as conjectured that these quasi-symmetric functions are $e$-unimodal. This has further been generalized by Ellzey [5] to circular indifference graphs. One new family of interval graphs has been proven to be $e$-positive by Cho and Huh [3] with an old family by Stanley proven to be $e$-positive with a new proof.

In this paper we prove that the graphs $P_{d,2}$ are $e$-positive, which are specifically mentioned in Stanley’s original 1995 paper [p190, 14] where he wrote

“It remains open whether $P_{d,2}$ is $e$-positive”.

In order to do this we follow a different generalization of $X_G$ by Gebhard and Sagan [7] to symmetric functions in non-commuting variables, which does satisfy a deletion-contraction property. Gebhard and Sagan in their paper prove more graphs have an $e$-positive $X_G$ by semi-symmetrizing their chromatic symmetric functions in non-commuting variables. We use ideas in their paper and expand on their proven family of $e$-positive graphs including all $P_{d,2}$, which we call triangular ladders. We do this by proving a new signed combinatorial formula for all interval graphs in the basis of elementary symmetric functions. Then we prove $e$-positivity for triangular ladders by very carefully defining a sign-reversing involution on our signed combinatorial formula, which leaves us with certain positive terms.

In Section 2 we describe the necessary background we need to derive our signed combinatorial formula in the elementary basis including the definition of interval graphs and Gebhard and Sagan’s deletion-contraction property in non-commuting variables. In Section 3 we derive our signed combinatorial formula in the elementary basis for any interval graph. Our method is to repeatedly use the deletion-contraction property on our graphs until we arrive at a single vertex and then reinterpret the coefficients in a combinatorial manner using arc diagrams with arc markings, vertex labels and vertex markings. In Section 4 we apply our signed combinatorial formula to triangular ladders and carefully define a sign-reversing involution in order to prove these graphs are $e$-positive. Lastly, in Section 5 we use the sign-reversing involution along with results by Gebhard and Sagan to show how we can combine complete graphs and triangular ladders to form more $e$-positive graphs.

## 2 Background

In this section we will go over the necessary background needed derive our signed combinatorial formula in the elementary basis. Throughout this paper we will work with simple graphs $G$ with labeled vertices and vertex labels in $[n] = \{1, 2, \ldots, n\}$, and we particularly focus on labeled interval graphs. An interval graph on vertices in $[n]$ is a graph formed by a collection of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_l, b_l]$ where $a_k \leq b_k$ are in $[n]$ and where we define $[a, b] = \{a, a+1, \ldots, b\}$. The interval graph $G$ with the given intervals will have all possible edges from vertex $i$ to $j$ whenever $i, j \in [a_k, b_k]$ for some $k$. There are many equivalent ways to define interval graphs, with some proofs between the equivalent definitions in [5] by Ellzey. In the literature there are special families of interval graphs $P_{n,k}$, which are formed from the intervals $[1, 1+k], [2, 2+k], \ldots, [n-k, n]$ and is the notation Stanley uses in his paper [15]. This notation defines many well-known families of graphs including the complete graphs, $K_n = P_{n,n-1}$, and the paths, $P_n = P_{n,1}$. Since in later sections we will be focusing on one particular family when $k = 2$, we will call the $P_{n,2}$ the triangular ladders, $TL_n$, to help the reader keep in mind our object of interest. Though it is not necessary for the definition, will write tend to write the intervals as $[a_1, 1], [a_2, 2], \ldots, [a_n, n]$, which may be redundant. In Figure 1 we draw the interval graph for intervals $[1, 1], [1, 2], [1, 3], [2, 4], [3, 5], [4, 6], [5, 7]$.

Before we introduce graphs colorings and the chromatic symmetric function in commuting or non-commuting variables let us review the algebra these objects exist in. The algebra of
symmetric functions in non-commuting variables is a sub-algebra of $\mathbb{Q}[[x_1,x_2,\ldots]]$ where variables $x_1,x_2,\ldots$ do not commute and all $f \in \text{NCSym}$ are unchanged by permuting subscripts. There are several classical bases that generate NCSym, including the power-sum basis and the elementary basis, all of which are indexed by set partitions. A set partition, $\pi = B_1/B_2/\cdots/B_l$ of $[n]$, denoted $\pi \vdash [n]$, is a collection of non-empty disjoint subsets $B_i \subseteq [n]$ called blocks that union to form the full set $[n]$. Given $\pi \vdash [n]$ and $\sigma \vdash [m]$ define $\pi|\sigma$ to be the set partition of $[n+m]$ we get from all blocks of $\pi$ together with all blocks of $\sigma$ except all elements in the blocks of $\sigma$ are increased by $n$. For example, $13/2|14/23 = 13/2/47/56$. Rosas and Sagan [13] define all the classical functions and give conversion formulas between them. In this paper we will only work with the power-sum and elementary functions. Given $\pi \vdash [n]$ the power-sum function in non-commuting variables, $p_\pi$, in NCSym is

$$p_\pi = \sum_{(i_1,i_2,\ldots,i_n)} x_{i_1}x_{i_2}\cdots x_{i_n},$$

which is summed over tuples $(i_1,i_2,\ldots,i_n)$ of positive integers where $i_j = i_k$ if $j$ and $k$ are in the same block in $\pi$. Given $\pi \vdash [n]$ the elementary symmetric function in non-commuting variables, $e_\pi$, in NCSym is

$$e_\pi = \sum_{(i_1,i_2,\ldots,i_n)} x_{i_1}x_{i_2}\cdots x_{i_n},$$

which is summed over tuples $(i_1,i_2,\ldots,i_n)$ of positive integers where $i_j \neq i_k$ if $j$ and $k$ are in the same block in $\pi$. These two bases are multiplicative, meaning that if $\pi = \pi_1|\pi_2$ then

$$p_{\pi_1|\pi_2} = p_{\pi_1}p_{\pi_2} \quad \text{and} \quad e_{\pi_1|\pi_2} = e_{\pi_1}e_{\pi_2}.$$  

Bergeron et. al. [1] proved the power-sum equality, but the elementary symmetric function equality comes quickly from the change of basis formula by Rosas and Sagan [13], which is

$$e_\pi = \sum_{\sigma \leq \pi} \mu(\hat{0},\sigma)p_\sigma \quad (1)$$

where $\sigma \leq \pi$ comes from the the poset of set partitions ordered by refinement with $\hat{0} = 1/2/\cdots/n$ being the smallest element in the poset. For more information about this poset and its M"obius function $\mu$ see Stanley’s book [16]. Since the multiplicativity of the elementary basis is vital to many of our proofs we prove it here.

**Lemma 2.1.** For $\pi \vdash [n]$ with $\pi = \pi_1|\pi_2$ where $\pi_1$ and $\pi_2$ are non-empty set partitions we have $e_{\pi_1|\pi_2} = e_{\pi_1}e_{\pi_2}$.

**Proof.** Using the change of basis formula in equation (1) we have

$$e_{\pi_1}e_{\pi_2} = \sum_{\sigma_1 \leq \pi_1} \mu(\hat{0},\sigma_1)p_{\sigma_1} \sum_{\sigma_2 \leq \pi_2} \mu(\hat{0},\sigma_2)p_{\sigma_2} = \sum_{\sigma_1 \leq \pi_1} \sum_{\sigma_2 \leq \pi_2} \mu(\hat{0},\sigma_1)\mu(\hat{0},\sigma_2)p_{\sigma_1|\sigma_2}$$

Figure 1: The triangular ladder graph $TL_7$ on the left and center and $K_4 \cdot TL_4$ on the right.
The chromatic symmetric function in commuting variables is
\[ e_{i} = \sum_{i_{1} < i_{2} < \ldots < i_{j}} x_{i_{1}}x_{i_{2}} \cdots x_{i_{j}} \]
and the *ith power-sum function in commuting variables* is
\[ p_{i} = x_{1}^{i} + x_{2}^{i} + x_{3}^{i} + \cdots \]
where as for an integer partition \( \lambda = \lambda_{1}\lambda_{2} \ldots \lambda_{l} \) we define the *elementary symmetric function*, \( e_{\lambda} \), and the *power-sum function*, \( p_{\lambda} \), to be
\[ e_{\lambda} = e_{\lambda_{1}}e_{\lambda_{2}} \cdots e_{\lambda_{l}} \text{ and } p_{\lambda} = p_{\lambda_{1}}p_{\lambda_{2}} \cdots p_{\lambda_{l}}. \]

These functions do have a close relationship to their relatives in NCSym. To relate them we define for a set partition \( \pi \vdash [n] \) the integer partition \( \lambda(\pi) \vdash n \), which we form by taking the sizes of all the blocks in \( \pi \). For example, \( \lambda(134/25/67) = 322 \). Rosas and Sagan \([13]\) showed that \( \rho(p_{\pi}) = p_{\lambda(\pi)} \) and \( \rho(e_{\pi}) = \pi!e_{\lambda(\pi)} \) where \( \pi! = \lambda(\pi)! = \lambda_{1}!\lambda_{2}! \cdots \lambda_{l}! \) and \( \pi \vdash [n] \) is a set partition. We will call a function \( f \in \Lambda \) *e-positive* if \( f \) can be written as a non-negative sum of elementary symmetric functions.

The symmetric functions in NCSym we study are defined from a graph \( G \) and its proper colorings. A *proper coloring* \( \kappa \) of a graph \( G \) with vertex set \( V \) is a function
\[ \kappa : V \to \{1, 2, \ldots\} \]
such that if \( v_{1}, v_{2} \in V \) form an edge, then \( \kappa(v_{1}) \neq \kappa(v_{2}) \). The chromatic symmetric function in non-commuting variables is defined to be
\[ Y_{G} = \sum_{\kappa} x_{\kappa(v_{1})}x_{\kappa(v_{2})} \cdots x_{\kappa(v_{n})} \]
where the sum is over all proper colorings \( \kappa \) of \( G \) where variables don’t commute. If we let the variables commute then we get the chromatic symmetric function in commuting variables, which we denote \( X_{G} = \rho(Y_{G}) \). We will call a graph \( G \) itself *e-positive* if \( X_{G} \) is e-positive. For example
\[ Y_{G_{n}} = e_{12\ldots n} \text{ and } \rho(Y_{G_{n}}) = X_{K_{n}} = n!e_{n}, \]
so all complete graphs are e-positive.

The main result of the paper is proving a certain family of graphs is e-positive. This family of graphs is formed by combining complete graphs and triangular ladders in a certain way. Given a graph \( G \) with labels in \( [n] \) and a graph \( H \) with labels in \( [m] \) we define their wedge to be the graph \( G \cdot H \) on vertices \( [n + m - 1] \) where the graph on the first \( n \) vertices is isomorphic to \( G \) and the graph on the last \( m \) vertices is isomorphic to \( H \), where by isomorphic we mean...
that the underlying graphs are isomorphic and the vertex labels are in the same relative order. See Figure 1 for an example.

Though the chromatic symmetric function $X_G$ in commuting variables doesn’t satisfy a deletion-contraction property, the $Y_G$, shown by Gebhard and Sagan [7], do satisfy a deletion-contraction property. We define the deletion of an edge $e$ of $G$, $G \setminus e$, to be the graph $G$ with edge $e$ removed. Though contraction can be defined for any edge, for our purposes and for simplicity, we will only define the contraction of an edge $e$ that is between vertices $j$ and $n$. The contracted graph, $G/e$, is the graph $G$ where we identify the vertices $j$ and $n$ and remove any multi-edges or loops created. In order to handle the idea of edge contraction in terms of functions in NCSym we define an induced function. Define the induced monomial to be

$$x_{i_1}x_{i_2} \cdots x_{i_j} \cdots x_{i_{n-1}} \uparrow^n_j = x_{i_1}x_{i_2} \cdots x_{i_j} \cdots x_{i_{n-1}}x_{i_j}$$

where we make an extra copy of the $j$th variable at the end and extend this definition linearly.

Given an integer partition $\pi \vdash [n-1]$ we define for positive $j < n$ that $\pi \oplus_j n \vdash [n]$ is the integer partition $\pi$ but we place $n$ in the same block as $j$. For example, $14/23 \oplus_4 5 = 145/23$. We extend this definition to $j = n$ by letting $\pi \oplus_n n = \pi/n$. Gebhard and Sagan [7] offer that it is not hard to see for $\pi \vdash [n-1]$ and $j < n$ that

$$p_\pi \uparrow^n_j = p_{\pi \oplus_j n}.$$  

For ease of notation later we define

$$p_\pi \uparrow^n_n = p_{\pi \oplus_n n} = p_\pi P_1$$

and extend linearly.

Proposition 2.2 (Deletion-Contraction, Gebhard and Sagan [7] Proposition 3.5). For $G$ with vertices $V = [n]$ and an edge $e$ between vertices $j$ and $n$ we have

$$Y_G = Y_{G \setminus e} - Y_{G/e} \uparrow^n_j.$$  

Though there are nice formulas for inducing in the power-sum basis, the formula for the elementary basis has many terms. However, after symmetrizing, many of these terms cancel out. Gebhard and Sagan defined equivalence classes on set partitions that enable us to partially symmetrize functions. We will say two set partitions $\pi$ and $\sigma$ are $\pi \sim \sigma$ if

1. $\lambda(\pi) = \lambda(\sigma)$ and
2. if $A$ and $B$ are blocks of $\pi$ and $\sigma$ respectively and if $n \in A$ and $n \in B$ then $|A| = |B|$.

Define

$$(\pi) = \{\sigma : \sigma \sim \pi\}.  \tag{3}$$

Say two functions $f, g \in \text{NCSym}$ are $f \equiv_n g$ if the sum of coefficients in the elementary basis in the same equivalence classes are the same. For example, consider the chromatic symmetric function of the path graph on three vertices calculated by Gebhard and Sagan in [7]. We have

$$Y_{P_3} = \frac{1}{2}(e_{12/3} - e_{13/2} + e_{1/23} + e_{123}) \equiv_3 \frac{1}{2}(e_{12/3} + e_{123})$$

because $13/2 \sim 1/23$. Our study in this paper is about whether graphs $G$ themselves are $e$-positive, which is dealing with $X_G$ in full commuting variables. Though it is an abuse of terminology, our goal is to show that $Y_G$ is $e$-positive after partially symmetrizing variables along the lines of these equivalence classes. To formalize this, we call a function $f \in \text{NCSym}$
semi-symmetrized e-positive if \( f \equiv_n g \) for some \( g \in \text{NCSym} \) that can be written as a non-negative sum of elementary symmetric functions in non-commuting variables. We call a graph \( G \) semi-symmetrized e-positive if \( Y_G \) is semi-symmetrized e-positive. It follows that if \( Y_G \) is semi-symmetrized e-positive then certainly \( \rho(Y_G) = X_G \) is e-positive and \( G \) is e-positive. This makes semi-symmetrized e-positivity a stronger condition than e-positivity.

There are two propositions by Gebhard and Sagan that are essential to our proofs. One is a formula for inducing elementary symmetric functions and the other is a relabeling proposition.

**Proposition 2.3** (Gebhard and Sagan [7] Corollary 6.1). For \( \pi \vdash [n-1] \), \( j < n \) and \( b \) the size of the block in \( \pi \) containing \( n-1 \) we have

\[
e_{\pi} \uparrow_{ij} = n^{\frac{1}{b}}(e_{\pi/n} - e_{\pi \uparrow_{ij} n}).
\]

The relabeling proposition considers how permuting vertex labels affects the chromatic symmetric function in NCSym. Given \( \delta \in \mathfrak{S}_n \) and \( f \in \text{NCSym} \) define \( \delta \circ f \) to be the function after we permute the placements of the variables, rather than the subscripts. For example, having \( \delta = 213 \) acting on \( x_1x_2x_1 \) means we switch the first two variables so \( \delta \circ x_1x_2x_1 = x_2x_1x_1 \). Also define for a graph \( G \) on vertices labeled with \([n]\) a new graph \( \delta(G) \), which is \( G \) but we permute the labels of the vertices. We similarly define \( \delta(\pi) \) for \( \pi \vdash [n] \) by permuting the elements in \([n]\). The following is Gebhard and Sagan’s relabeling proposition.

**Lemma 2.4** (Relabeling Proposition, Gebhard and Sagan [7] Proposition 3.3). For a graph \( G \) with distinct vertex labels in \([n]\) and \( \delta \in \mathfrak{S}_n \),

\[Y_{\delta(G)} = \delta \circ Y_G.\]

We will also need a slight generalization of Gebhard and Sagan’s result that easily follows from the relabeling proposition 2.4.

**Lemma 2.5** (Gebhard and Sagan [7] Lemma 6.6). If \( f \equiv_n g \) and \( \delta \in \mathfrak{S}_n \) with \( \delta(n) = n \) then

\[\delta \circ f \equiv_n \delta \circ g.\]

### 3 Formula in the elementary basis

In this section we develop a new formula for intervals graphs in the elementary basis in terms of signed combinatorial objects involving labeled arc diagrams with arc markings and vertex markings. The idea is that we will delete and contract our graph down to a single vertex and then induce the chromatic symmetric function on one vertex back until we get the full function of our interval graph. If we consider this inducing in terms of the power-sum basis we will arrive at an example of Stanley’s broken-circuit theorem ([15] Theorem 2.9). Since our interest is in the elementary basis, we will use Gebhard and Sagan’s [7] formula in Proposition 2.3 for inducing elementary symmetric functions to find a signed combinatorial interpretation of these coefficients. In Section 4 we will show in the case of triangular ladders that we can define a sign-reversing involution on our signed combinatorial objects, which will prove that a new family of graphs, the triangular ladders, is e-positive. We will develop the signed combinatorial formula in stages first determining a recursive formula for the chromatic symmetric function of interval graphs.
Theorem 3.1. Given an interval graph on \( n \) vertices with intervals \([a_1, 1], [a_2, 2], \ldots , [a_n, n]\) let \( G' \) be the same graph on \( n - 1 \) vertices after removing vertex \( n \). Then

\[
Y_G = Y_{G'}Y_{K_1} - \sum_{i=a_n}^{n-1} Y_{G'} \uparrow_i = Y_{G'} \uparrow^n_n - \sum_{i=a_n}^{n-1} Y_{G'} \uparrow^n_i .
\]

Proof. We will prove this by inducting on \( n \), the number of vertices, and \( m \), the number of edges in an interval graph \( G \) defined by the intervals \([a_1, 1], [a_2, 2], \ldots , [a_n, n]\). The base case is \( K_1 \) when \( n = 1 \) and \( m = 0 \). This case is easy to see since everything is equal to \( e_1 \).

Now assume that \( G \) is an interval graph on \( n > 1 \) vertices with \( m \) edges. We will assume that any interval graph \( H \) with \( \tilde{n} \leq n \) vertices and \( \tilde{m} \leq m \) edges with either \( \tilde{n} < n \) or \( \tilde{m} < m \) satisfies the above formula. Define \( G' \) to be the graph \( G \), but we remove vertex \( n \), so \( G' \) satisfies the formula. If \( a_n = n \) then \( G \) is the disjoint union of \( G' \) and \( K_1 \). It is not hard to see that \( G \) satisfies the formula because \( Y_G = Y_{G'}Y_{K_1} \). Say that instead \( a_n < n \). By deletion-contraction in Proposition 2.2 using the edge \( \epsilon \) between vertices \( a_n \) and \( n \) we have

\[
Y_G = Y_{G-\epsilon} - Y_{G/\epsilon} \uparrow^n_{a_n} .
\]

Note that \( G/\epsilon \) is \( G' \). Also, note that \( G - \epsilon \) is also an interval graph with all the same intervals as \( G \), but the interval \([a_n, n]\) changes to \([a_n + 1, n]\). If we remove vertex \( n \) from \( G - \epsilon \) then we get \( G' \). Since \( G - \epsilon \) has less edges than \( G \) by induction we can say

\[
Y_{G-\epsilon} = Y_{G'}Y_{K_1} - \sum_{i=a_n+1}^{n-1} Y_{G'} \uparrow^n_i .
\]

Putting everything all together we get the equation in this proposition. \( \square \)

We will continually use the formula in Theorem 3.1 until we are only inducing from \( K_1 \). This gives us

\[
Y_G = \sum_{i_n=a_n}^{n} \cdots \sum_{i_3=a_3}^{3} \sum_{i_2=a_2}^{2} \sum_{i_1=a_1}^{1} (-1)^{|\{i_j \neq i\}|} Y_{K_1} \uparrow^1_{i_1} \uparrow^3_{i_3} \cdots \uparrow^n_{i_n} . \tag{4}
\]

We will represent each series of inducings \( \uparrow^1_{i_1} \uparrow^3_{i_3} \cdots \uparrow^n_{i_n} \) with an arc diagram. An arc diagram is a drawing on \( n \) vertices in a line numbered from left to right together with a collection of arcs \((i, j)\) with \( i < j \) representing an edge from \( i \) to \( j \). A series of inducings like \( \uparrow^3_{i_3} \uparrow^3_{i_3} \cdots \uparrow^n_{i_n} \) will be represented by the arc diagram on \( n \) vertices with arcs \((i_2, 2), (i_3, 3), \ldots , (i_n, n)\) where if \( i_j = j \) there is no arc, but we may list non-arcs for notational ease. Define an arc \((i, j)\) to be a left arc of \( j \). The collection of arc diagrams just described for an interval graph \( G \) with intervals \([a_1, 1], [a_2, 2], \ldots , [a_n, n]\) are those where

- all vertices have at most one left arc and
- if we have an arc \((i, j)\) then \( i, j \in [a_k, k] \) for some \( k \).

Define this set of arc diagrams to be \( \mathcal{A}(G) \). Note that the sign in equation (4) is determined by \(|\{j \neq i_j\}|\), which is precisely the number of arcs in the arc diagram. For an arc diagram \( D \in \mathcal{A}(G) \) define \( a(D) \) to be the number of arcs in the arc diagram \( D \). See Figure 2 for an example. We will re-represent the series of inducings \( \uparrow^3_{i_3} \uparrow^3_{i_3} \cdots \uparrow^n_{i_n} \), which is associated to some arc diagram \( D \), to be \( \uparrow_D \).

Proposition 3.2. For an interval graph \( G \),

\[
Y_G = \sum_{D \in \mathcal{A}(G)} (-1)^{a(D)} e_1 \uparrow_D .
\]
Figure 2: These are arc diagrams for the interval graph $G$ from intervals $[1, 3], [2, 4], [3, 5], [4, 6], [5, 7]$. From left to right we have elements of $\mathcal{A}(G)$, $\mathcal{A}^r(G)$ and $\mathcal{A}_L(G)$ where in all cases $a(D) = 5$. On the left the associated set partition is 123567/4 and the other two are 1235/4/67.

Proof. This is equation (4), but instead we represent the series of inducings as an arc diagram and use the fact that $Y_{K_1} = e_1$.

Given any arc diagram $D \in \mathcal{A}(G)$ we can use the connected components formed from the dots connected by arcs to form a set partition, $\pi(D)$, if we make all the vertices in each connected component a block. See Figure 2 for an example. One helpful fact about the series of inducings, which will be shown in the next lemma, is that

$$ p_1 \uparrow_D = p_{\pi(D)}. $$

Lemma 3.3. For an interval graph $G$ and arc diagram $D \in \mathcal{A}(G)$ we have $p_1 \uparrow_D = p_{\pi(D)}$. As result, if $D_1$ and $D_2$ are two arc diagrams with $\pi(D_1) = \pi(D_2)$ then

$$ e_1 \uparrow_{D_1} = e_1 \uparrow_{D_2}. $$

Proof. We will first prove that for an arc diagram $D$ associated to an interval graph $G$ with $n$ vertices that $p_1 \uparrow_D = p_{\pi(D)}$ by inducting on $n$. The rest of the statement will follow from this fact.

If $n = 1$ then the only possible arc diagram is a single dot with no arcs and $\pi(D) = 1$ so $p_{\pi(D)} = p_1$. Say that $n > 1$. Consider an arc diagram $D$ with arcs $(i_2, 2), (i_3, 3), \ldots, (i_n, n)$. Let $\bar{D}$ be the arc diagram $D$, but with $n$ removed. By induction $p_1 \uparrow_{\bar{D}} = p_{\pi(\bar{D})}$. Then

$$ p_1 \uparrow_D = p_1 \uparrow_{i_2} \uparrow_{i_3} \cdots \uparrow_{i_n} = p_{\pi(D)} \uparrow_{i_n}. $$

If $n$ is in its own connected component then $D$ has no left arc at $n$ so $i_n = n$. Then $\pi(D) = \pi(\bar{D})/n$ and we know that $p_{\pi(\bar{D})} \uparrow_{i_n} = p_{\pi(\bar{D})} p_1 = p_{\pi(D)/n}$. Consider the case where $n$ is not in its own component in $\pi(D)$. That means $i_n < n$ and $n$ is in the same connected component as $i_n$. Further $\pi(D) = \pi(\bar{D}) \oplus_{i_n} n$. Similarly and using equation (2),

$$ p_1 \uparrow_D = p_1 \uparrow_{i_2} \uparrow_{i_3} \cdots \uparrow_{i_n} = p_{\pi(D)} \uparrow_{i_n} = p_{\pi(D) \oplus_{i_n} n} = p_{\pi(D)}. $$

Because $p_1 \uparrow_D = p_{\pi(D)}$ we know that if $D_1$ and $D_2$ are two arc diagrams with $\pi(D_1) = \pi(D_2)$ then $p_1 \uparrow_{D_1} = p_1 \uparrow_{D_2}$. Because $p_1 = e_1$ we have the rest of our result.

We will note that the formula in Proposition 3.2 is not particularly surprising, because if we instead used $Y_{K_1} = p_1$ and induced in the power-sum basis then we arrive at an example of Stanley’s broken-circuit theorem [15]. Since we are particularly interested in the elementary basis we will be using Gebhard and Sagan’s formula in Proposition 2.3 for inducing elementary symmetric functions, which will require us to semi-symmetrize the chromatic symmetric function.

Because of Lemma 3.3 we can define $e_1 \uparrow_{\pi}$ for any set partition $\pi$ to be equal to $e_1 \uparrow_D$ for any arc diagram $D$ with $\pi(D) = \pi$. Our method will be to continually use Gebhard and
Sagan’s inducing formula in the elementary basis to derive a signed combinatorial formula for a semi-symmetrized $Y_G$, which is distinct from the broken circuit theorem. First we must clarify exactly how we can continually use Gebhard and Sagan’s inducing formula. Gebhard and Sagan prove this in their paper, but we mention the proof again because it is the backbone of our logic.

**Lemma 3.4** (Gebhard and Sagan [7] Lemma 6.2). For $f, g \in \text{NCSym}$ if $f \equiv_{n-1} g$ then $f \uparrow_{n-1}^{n} g \equiv_{n-1}^{n}$.

**Proof.** Note that for $\pi, \sigma \vdash [n-1]$ if $\pi \sim \sigma$ then $\pi/n \sim \sigma/n$ and $\pi \oplus_{n-1} n \sim \sigma \oplus_{n-1} n$. This means if $e_{\pi} \equiv_{n-1} e_{\sigma}$ then $e_{\pi/n} \equiv_{n} e_{\sigma/n}$ and $e_{\pi \oplus_{n} n} \equiv_{n-1} e_{\sigma \oplus_{n-1} n}$, which implies that $e_{\pi} \uparrow_{n-1}^{n} e_{\sigma} \uparrow_{n-1}^{n}$. Extending this linearly gives the result. \[ \square \]

Note that when we induce an elementary symmetric function once, it is equivalent to the subtraction of two elementary symmetric functions after semi-symmetrizing. Since each inducing is associated to an arc in an arc diagram, we will need to track of these two possible terms for each inducing. We will do so by marking arcs with tic marks. Define $\mathcal{A}'(G)$ to be the collection of arc diagrams $D \in \mathcal{A}(G)$, but each arc will be decorated with a tic mark or left alone. See Figure 2 for an example. We will consider each tic mark on arc $(i, j)$ to split the connected component into *pieces*, every dot to the left of $j$, but not including $j$, will be in a different piece then those to the right including $j$. For $D' \in \mathcal{A}'(G)$ we define $\pi(D')$ to be the set partition formed by all these pieces the connected components are broken into. We will also define $t(D')$ to be the number of tic marks on the diagram $D'$. For an arc diagram $D \in \mathcal{A}(G)$ we define $T(D)$ to be all $D' \in \mathcal{A}'(G)$ but the underlying arc diagram is $D$ only.

To get a formula for inducing $e_{1}$ in terms of elementary symmetric functions we will first consider the simple arc diagram $P_{n}$ on $n$ vertices with arcs $(1, 2), (2, 3), \ldots, (n-1, n)$. Each $D' \in \mathcal{A}'(G)$ has an associated set partition $\pi(D')$, but each connected component also has an associated integer composition $\alpha = \alpha_{1} + \alpha_{2} + \cdots + \alpha_{l} \uparrow n$ reading the sizes of the pieces between tic marks in one connected component in $D'$ from left to right. Where $\alpha = \alpha_{1} + \alpha_{2} + \cdots + \alpha_{l} \uparrow n$ is an integer composition of $n$ if $\alpha_{i} \geq 1$ for all $i$ and the sum of the $\alpha_{i}$ is $n$.

**Proposition 3.5.** For the set partition $[n]$ we have

$$ e_{1} \uparrow [n] \equiv_{n} \frac{1}{n!} \sum_{D' \in T(P_{n})} (-1)^{\alpha(D') - t(D')} \alpha\left(\frac{n}{\alpha_{1}, \ldots, \alpha_{l}}\right) e_{\pi(D')} $$

where $\alpha = \alpha_{1} + \alpha_{2} + \cdots + \alpha_{l} \uparrow n$ is the composition associated to the pieces of $D'$.

**Proof.** We will prove this by inducting on $n$. If $n = 1$ then $e_{1} \uparrow [1] = e_{1}$ because $\uparrow [1]$ means no inducing and we are done, so assume that $n > 1$. By induction we know that

$$ e_{1} \uparrow [n-1] \equiv_{n-1} \frac{1}{(n-1)!} \sum_{D' \in T(P_{n-1})} (-1)^{\alpha(D') - t(D')} \alpha\left(\frac{n-1}{\alpha_{1}, \ldots, \alpha_{l}}\right) e_{\pi(D')} $$

where $\alpha = \alpha_{1} + \alpha_{2} + \cdots + \alpha_{l} \uparrow n$ is the composition associated to the pieces of $D'$. By Lemma 3.4 this implies that

$$ e_{1} \uparrow [n] = e_{1} \uparrow [n-1] \uparrow_{n-1}^{n} $$

$$ \equiv_{n-1} \frac{1}{(n-1)!} \sum_{D' \in T(P_{n-1})} (-1)^{\alpha(D') - t(D')} \alpha\left(\frac{n-1}{\alpha_{1}, \ldots, \alpha_{l}}\right) e_{\pi(D')} \uparrow_{n-1}^{n} $$

$$ \equiv_{n-1} \frac{1}{(n-1)!} \sum_{D' \in T(P_{n-1})} (-1)^{\alpha(D') - t(D')} \alpha\left(\frac{n-1}{\alpha_{1}, \ldots, \alpha_{l}}\right) \left(\frac{1}{\alpha_{l}} \left(e_{\pi(D')/n} - e_{\pi(D') \oplus_{n-1} n}\right)\right). $$
We will let the terms with $\pi(D')/n$ and $\pi(D') \oplus n-1$ be associated to $D' \in \mathcal{A}(P_n)$ with a tic mark on arc $(n-1,n)$ or no tic mark on arc $(n-1,n)$ respectively. Manipulating the signs, factorial and multinomial coefficient appropriately gives us the result.

We will use the formula for $e_1 \upharpoonright_{[n]}$ that we just derived to find the formula for $e \upharpoonright D$ for a general arc diagram $D$.

**Proposition 3.6.** For interval graph $G$ and arc diagram $D \in \mathcal{A}(G)$ we have

$$e_1 \upharpoonright_D \equiv_n \frac{1}{n!} \sum_{D' \in \mathcal{T}(D)} (-1)^{a(D')-t(D')} \prod_{i} \alpha_t^{(i)} \left( \frac{n_j}{\alpha_1, \ldots, \alpha_t} \right) e_{\pi(D')}$$

where $\alpha^{(i)} = \alpha^{(i)}_1 + \alpha^{(i)}_2 + \cdots + \alpha^{(i)}_t \equiv n$ are the compositions associated to the pieces of connected components of $D'$ where $\alpha^{(i)}_t$ is the size of the right-most piece.

**Proof.** We will start the proof by considering the particular set partition $\pi = [n_1][n_2] \cdots [n_k]$ and prove this formula for $e_1 \upharpoonright_{\pi}$. Then we will use the multiplicity of the power-sum basis and the formula for inducing the power-sum basis as well as the relabeling proposition 2.4 to show the formula for some particular arc diagram $D$ associated to $\pi$ before finally concluding this formula for a generic arc diagram $D$.

First consider $\pi = [n_1][n_2] \cdots [n_k]$. Using the fact that the elementary basis is multiplicative and Proposition 3.5 we have that

$$e_1 \upharpoonright_{[n]} = e_1 \upharpoonright_{[n_1]} e_1 \upharpoonright_{[n_2]} \cdots e_1 \upharpoonright_{[n_k]}$$

$$\equiv_n \prod_{j=1}^{k} \frac{1}{n_j!} \sum_{D' \in \mathcal{T}(P_{n_j})} (-1)^{a(D')-t(D')} \prod_{j} \alpha_t^{(j)} \left( \frac{n_j}{\alpha_1, \ldots, \alpha_t} \right) e_{\pi(D')}$$

$$\equiv_n \sum_{D' \in \mathcal{T}(P_{n_1}[P_{n_2} \cdots P_{n_k})} \prod_{j=1}^{k} \frac{1}{n_j!} (-1)^{a(D')-t(D')} \alpha_t^{(j)} \left( \frac{n_j}{\alpha_1, \ldots, \alpha_t} \right) e_{\pi(D')}$$

$$\equiv_n \frac{1}{n!} \sum_{D' \in \mathcal{T}(P_{n_1}[P_{n_2} \cdots P_{n_k})} (-1)^{a(D')-t(D')} \prod_{i} \alpha_t^{(i)} \left( \frac{n}{\alpha_1, \ldots, \alpha_t} \right) e_{\pi(D')}$$

where we define for two diagrams $D_1$ on $n$ vertices and $D_2$ on $m$ vertices the diagram $D_1 \upharpoonright D_2$ to be a diagram on $m + n$ vertices, which is isomorphic to $D_1$ on the first $n$ vertices and isomorphic to $D_2$ on the last $m$ vertices with no other arcs. Next we will consider a general set partition $\pi \vdash [n]$. We form an arc diagram $D$ such that we have an arc $(a,b)$ if $a$ and $b$ are listed consecutively in increasing order in a block of $\pi$. Also, there exists a permutation $\delta \in \mathfrak{S}_n$ such that $\delta(n) = n$ and if $(a,b)$ is an arc of $D$ then $\delta(a) = \delta(b) + 1$. This makes $\delta(\pi) = [n_1][n_2] \cdots [n_k]$ for some $n_j$'s and $\delta(D) = P_{n_1}[P_{n_2} \cdots P_{n_k}$ where we permute the placement of the dots. In Lemma 3.3 we showed that $e_1 \upharpoonright_D = p_{\pi}$. We know that

$$\delta(p_{\pi}) = p_{[n_1][n_2] \cdots [n_k]}$$

$$= e_1 \upharpoonright_{[n_1]} e_1 \upharpoonright_{[n_2]} \cdots e_1 \upharpoonright_{[n_k]}$$

$$\equiv_n \frac{1}{n!} \sum_{D' \in \mathcal{T}(P_{n_1}[P_{n_2} \cdots [P_{n_k})} (-1)^{a(D')-t(D')} \prod_{i} \alpha_t^{(i)} \left( \frac{n}{\alpha_1, \ldots, \alpha_t} \right) e_{\pi(D')}$$

Note that $\delta^{-1}(p_{\pi}) = e_1 \upharpoonright_D$. By the relabeling proposition 2.4 and taking $\delta^{-1}$ of the first and last part of the series of equivalences above we get the result for this special diagram $D$ associated to set partition $\pi$. 

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Given any arc diagram \( D \in A(g) \) there is a special arc diagram \( D^* \) associated to \( \pi(D) \) as we defined earlier. However, since we have already shown that \( e_1 \uparrow_D = e_1 \uparrow_{D^*} \) when \( \pi(D) = \pi(D^*) \) in Lemma 3.3 we are done.

We are finally ready to introduce the signed combinatorial formula for a semi-symmetrized \( Y_G \). The idea is to use tic’d arc diagrams and reinterpret the multinomial coefficient in Proposition 3.6 as labels on the vertices with some additional vertex markings. Given an interval graph \( G \) define \( A'_n(G) \) to be the collection of all arc diagrams in \( A'(G) \) with a possible tic mark on each arc as well as a permutation label \( \delta \in S_n \) on the vertices with \( \delta(i) \) on vertex \( i \). We want these labels to be increasing on each piece. Also, on each connected component we will mark one vertex in the most-right piece. We will use a star instead of a dot to show the vertex is marked. Meaning for an arc diagram \( D' \) with possible tic marks that if \( B = \{b_1, b_2, \ldots, b_k\} \) with \( b_1 < b_2 < \cdots < b_k \) is a block of \( \pi(D') \) associated to a piece, then the permutation must have \( \delta(b_1) < \delta(b_2) < \cdots < \delta(b_k) \). If this block was a right-most piece of a connected component then one of the vertices in \( B \) is marked with a star. See Figure 2 for an example. Our signed combinatorial formula is as follows.

**Theorem 3.7.** For an interval graph \( G \),

\[
Y_G \equiv_n \frac{1}{n!} \sum_{D' \in A'_n(G)} (-1)^{t(D')} e_{\pi(D')}.
\]

**Proof.** From Propositions 3.2 and 3.6 we have

\[
Y_G = \sum_{D \in A(G)} (-1)^{a(D)} e_1 \uparrow_D
\]

\[
\equiv_n \frac{1}{n!} \sum_{D \in A(G)} \sum_{D' \in \mathcal{T}(D)} (-1)^{t(D')} \prod_i \alpha_i^{(i)} \left( \alpha_1^{(1)}, \ldots, \alpha_i^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_i^{(2)}, \ldots \right) e_{\pi(D')}.
\]

A multinomial coefficient \( \binom{n}{m_1, m_2, \ldots, m_k} \) can be combinatorially interpreted as a permutation increasing along the first \( m_1 \) indices, increasing along the next \( m_2 \) indices and so on. We will combinatorially interpret the multinomial coefficient in the equation above as permutations \( \delta \in S_n \) such that if \( B = \{b_1, b_2, \ldots, b_k\} \) with \( b_1 < b_2 < \cdots < b_k \) is a block of \( \pi(D') \) associated to a piece then \( \delta(b_1) < \delta(b_2) < \cdots < \delta(b_k) \). The multiplication by \( \prod_i \alpha_i^{(i)} \) will be interpreted by marking a vertex using a star in the right-most piece of each connected component, which each have size \( \alpha_i^{(i)} \).

### 4 Triangular ladders

In this section we will take the ideas from Section 3 and apply them to triangular ladders, \( TL_n \). In [15] Stanley used an iterative technique to solve for the chromatic symmetric function of a graph by solving a system of linear equations using Cramer’s rule, and proved the \( e \)-positivity of the path and cycle graphs. In his paper he mentions that this technique does not work on \( TL_n \) and mentions that proving the \( e \)-positivity remains open. In this section we prove \( TL_n \) is semi-symmetrized \( e \)-positive and so \( e \)-positive for all \( n \). Our method is to use our signed combinatorial formula from Theorem 3.7 and define a sign-reversing involution on the associated signed combinatorial objects. In Section 5 we will use the ideas from this section to prove all wedges of complete graphs and triangular ladders are \( e \)-positive.
Because triangular ladders are interval graphs we can obtain a number of corollaries from Section 3.

**Corollary 4.1.** For \( n \geq 2 \),
\[
Y_{TL_n} = Y_{TL_{n-1}}Y_{K_1} - Y_{TL_{n-1}}^n - Y_{TL_{n-1}}^n - Y_{TL_{n-1}}^n.
\]

**Proof.** Follows quickly from Theorem 3.1. \( \Box \)

Because we are only working with triangular ladders in this section, we will simplify the notation of \( A(TL_n) \) as \( A \) and \( A_L(TL_n) \) as \( A_L' \).

**Corollary 4.2.** For \( n \geq 1 \),
\[
Y_{TL_n} \equiv_n \frac{1}{n!} \sum_{D' \in A_L'} (-1)^{\ell(D')} e_{\pi(D')}.
\]

**Proof.** Follows quickly from Theorem 3.7. \( \Box \)

This section’s focus will be to define a sign-reversing involution that can be applied to the alternating-sign summation in Corollary 4.2. In order to define a sign-reversing involution we will first need to define a *signed set* \( S \), which is a set such that each \( s \in S \) has an associated sign \( \text{sign}(s) \), of +1 or −1. The elements assigned the value +1 are called the positive elements and the elements assigned −1 are the negative elements. Additionally, each element in our signed-set will have a *weight* \( \text{wt}(s) \). A *sign-reversing involution* is a map \( f : S \to S \) that is an involution, \( f \circ f = \text{id} \), and is weight preserving, \( \text{wt}(f(s)) = \text{wt}(s) \). The map \( f \) also has to be sign reversing in that if \( f(s) \neq s \) for \( s \in S \) the sign associated to \( s \) is opposite of the sign associated to \( f(s) \). The consequence of such a map is a pairing between many elements in \( S \) such that each pair shares the same weight and has a positive element and a negative element so that when we add the signs of the pair together we get zero. Not all elements \( s \in S \) will be part of a pairing. This happens when \( f(s) = s \) and we call these elements fixed points. This means regarding summations that
\[
\sum_{s \in S} \text{sign}(s) \text{wt}(s) = \sum_{\text{fixed points } s} \text{sign}(s) \text{wt}(s)
\]
and if all fixed points have positive sign, then we have proven our alternating sum is actually a non-negative sum.

Our sign-reversing involution, \( \varphi : A_L' \to A_L' \), works with the signed set \( A_L' \) where each element \( D' \in A_L' \) has sign \( \text{sign}(D') = (-1)^{\ell(D')} \) and weight \( \text{wt}(D') = e_{\pi(D')} \) where the set partitions are viewed in the light of their equivalence classes. This means that if \( D'_1 \sim D'_2 \) then \( \text{wt}(D'_1) = \text{wt}(D'_2) \). We will define this involution inductively, but we will first need some structure lemmas on the arc diagrams \( D' \in A_L' \), whose underling unmarked and unlabeled arc diagrams are those in \( A \). As a reminder, these arc diagrams in \( A \) are those where

1. \( D \) has at most one left arc at every vertex and
2. every arc is of length 1 or 2

where the *length of an arc* \((i,j)\) is \( j - i \). We define the *length of a diagram* \( D \), \( \ell(D) \), to be the number of vertices minus one.

The structure lemma will revolve around the idea of the *concatenation* of two arc diagrams \( D_1 \) on vertices \([n]\) and \( D_2 \) on vertices \([m]\), which we define to be \( D_1 \cdot D_2 \), the arc diagram on
For an example. In an IC diagram we have one

vertex \( n \) has a left arc of length 2. There will then be a largest integer

\( t \) such that \( D_t \) has a left arc of length two, but vertex \( i \) doesn’t have a left arc.

By induction we are done. Say that vertex \( D_k \) be an arc diagram of

\( \mathcal{A}_L \), where \( n \geq 2 \) vertices will have arcs \((i, i+2)\) for all possible \( i \). We define \( L_m \) to be an IL
diagram of length \( m \). The second we will call an interconnecting diagram. An interconnecting
arc diagram, IC, will be an IL diagram, but we include the arc \((1, 2)\). We define \( C_m \) to be
an IC diagram of length \( m \). See Figure 3 for an example. In an IC diagram we have one
connected component, \( \pi(C_m) = [m + 1] \). In an IL diagram we have two connected components
whose sizes depend on whether the diagram is of odd length or even length.

**Lemma 4.3.** All diagrams \( D \in \mathcal{A} \) with two or more vertices are the concatenation of some IL and IC diagrams of lengths at least one, meaning

\[ D = D_1 \cdot D_2 \cdots \cdot D_k \]

where each \( D_i \) is an IL or IC diagram with \( \ell(D_i) \geq 1 \).

**Proof.** We will prove this by inducting on the length of the arc diagram. Our base case is when
\( D \) is an arc diagram of \( TL_2 \) in which case \( D \) is either an IL or IC diagram of length 1. Let \( D \)
be an arc diagram of \( TL_n \) for \( n > 2 \). Say \( n \) doesn’t have a left arc. Then on vertices \( n - 1 \) and \( n \) we have an IL diagram of length 1 and \( D = \tilde{D} \cdot L_1 \). The arc diagram \( \tilde{D} \) has less than
\( n \) vertices so by induction we are done. Say that vertex \( n \) has a left arc of length one. Then
on vertices \( n - 1 \) and \( n \) we have an IC diagram of length 1 and \( D = \tilde{D} \cdot C_1 \). The arc diagram
\( \tilde{D} \) has less than \( n \) vertices so by induction we are done. Lastly, consider the case where vertex
\( n \) has a left arc of length 2. There will then be a largest integer \( j \) where for all \( i > j \) vertex
\( i \) has a left arc of length two, but vertex \( j \) either has no left arc or a left arc of length one.
In the first case on vertices \( j - 1 \) through \( n \) we have an IL diagram of length \( n - j + 1 \) and
\( D = \tilde{D} \cdot L_{n-j+1} \) where \( \tilde{D} \) has length less than \( n \). By induction we are done. In the second case on vertices \( j - 1 \) through \( n \) we have an IC diagram of length \( n - j + 1 \) and \( D = \tilde{D} \cdot C_{n-j+1} \)
where \( \tilde{D} \) has length less than \( n \). By induction we are done.

We will call the concatenation of \( D \in \mathcal{A} \) into IL and IC diagrams the *decomposition* of \( D \).
When talking about \( D' \in \mathcal{A}_L' \) the decomposition we are referring to is the decomposition of its
underlying arc diagram \( D \in \mathcal{A} \).

We now have all the tools we need to define our sign-reversing involution. Again our signed-
set is \( \mathcal{A}_L' \) where \( D' \in \mathcal{A}_L' \) has sign \( \text{sign}(D') = (-1)^{\ell(D')} \) and weight \( \text{wt}(D') = e_{\pi(D')} \) where the
weight is considered in light of the equivalence classes on set partitions defined by equation (3)
meaning that \( \text{wt}(D'_1) = \text{wt}(D'_2) \) if and only if \( D'_1 \sim D'_2 \). Further, we mean that if \( D'_1 \leftrightarrow D'_2 \) then we want

1. \( \lambda(\pi(D'_1)) = \lambda(\pi(D'_2)) \) and
2. the sizes of the blocks containing \( n \) has the same size.

3. Also, the label on the right-most vertex remains unchanged under the map and

4. \( D'_1 \) has a star marking on its right-most vertex if and only if \( D'_2 \) does as well.

These are the conditions we want our inductive map to satisfy. The motivations for the last two conditions will become apparent in a later discussion. In general, we will match \( D' \) to another diagram that has an arc with a tic mark removed or an arc with a tic mark added. We will be sure to fix the size of the piece attached to vertex \( n \). We will define this involution inductively on \( n \).

When \( n = 1 \) we only have one kind of diagram, a single vertex labeled with 1 and marked with a star. Let diagram

\[
\ast \quad \begin{array}{c}
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\ \ \\
\ \ \\
\ \ \\
\ \ \\
\end{array}
\]

be a fixed point.

Say that \( n > 1 \). By Lemma 4.3 we can write the underlying arc diagram \( D' = \tilde{D} \cdot I \) where \( I \) is an IL or IC diagram of length at least 1. However, before we continue on we must discuss the slight discrepancy between our decomposition of \( D' \), which has vertex star markings and vertex labels, and the diagrams talked about in Lemma 4.3, which do not have any labelings. When we decompose \( D' = A \cdot B \) we will mostly be referring to the decomposition of the underlying arc diagram with possible tic marks ignoring vertex labels and markings. However, there will be times when we will need to refer to properties of \( A \) as if it were also an element of \( \mathcal{A}'_L \). In this case we will keep the underlying vertex labels and think of \( A \) as if its labels have been standardized to a permutation by replacing the \( i \)th smallest label with \( i \geq 1 \). If there is not a vertex star marking in the right-most piece of \( A \) for any reason, we will think of \( A \) as if the right-most vertex is marked with a star. This allows us to inductively map

\[
D' = \tilde{D} \cdot I \mapsto \varphi(\tilde{D}) \cdot I
\]

as long as \( \tilde{D} \) is not a fixed point. Now we will have to discuss here how to reconnect \( \varphi(\tilde{D}) \) to \( I \). First we keep the respective vertex labels that both \( \tilde{D} \) and \( I \) had before the map, but they may be permuted in \( \varphi(\tilde{D}) \). As for the vertex star markings, if \( \varphi(\tilde{D}) \) has its right-most vertex marked with a star, we will remove this if \( I \) contains a star-marked vertex in that piece, or that piece is not the right-most piece in the connected component. Reconnecting \( \varphi(\tilde{D}) \) and \( I \) in terms of labeling makes sense as long as our map always preserves the label of the right-most vertex and makes sense in terms of vertex star markings if a diagram with the right-most vertex marked with a star is mapped to something with the same property. Note that by induction conditions 3 and 4 are satisfied. Also note that because this map fixes the size of the piece connected to the right-most vertex we can say that the piece containing \( n \) has the same size before and after. Since also we maintain the underlying integer partition our output has the same weight as the input. This means conditions 1 and 2 are satisfied as well. Now we only have to consider cases where \( \tilde{D} \) is a fixed point of \( \varphi \).

Before we classify the fixed points and discuss the map in these cases we need a few more definitions. Note that IL diagrams of length 1 naturally break our diagrams into sections. We will define a section to be a diagram without any IL pieces of length 1 in its decomposition. We will say a section satisfies the IC-condition if it contains an IC diagram of length at least two in its decomposition with only IC diagrams of length 1 to its left. See Figure 4.

Recall that we will mark some our vertices with a star. Consider \( D \in \mathcal{A}'_L \) with no tic marks, just like how our fixed-points will be. Say that \( D \) that ends in \( P \), an IL of length at least 1. On vertex \( n - 1 \) we have a right endpoint of a connected component. Because this connected component has only one piece, one vertex in this component will be marked with a
Figure 4: This diagram \( D \in \mathcal{A}_L' \) satisfies the IC-condition, has \( s(D) = 4 \) and \( e(D) = 5 \).

star. We will define \( e(P) = i \) to mean we chose the \( i \)th right-most vertex in the component and marked it with a star. See Figure 4.

Fixed points will end up having ranges of vertices where we cannot have vertices marked with a star. Consider a diagram \( D \in \mathcal{A}_L' \) with no tic marks that ends in an IL diagram. We can count the number of vertices backwards from \( n - 1 \) in the same connected component until we reach a different IL or IC diagram of length at least 2 or we reach the end of the connected component. If we count \( i \) vertices including the most-right vertex of the IL or IC diagram of length at least two we define \( s(D) = i + 1 \). If we counted until the endpoint let \( s(D) \) be the number of vertices connected to \( n - 1 \). For ease when we have a diagram \( D \in \mathcal{A}_L' \) with no tic marks that ends in an IL diagram \( P \), \( D = D \cdot P \), then we will write \( e(P) \) for \( e(D) \) and \( s(P) \) for \( s(D) \) assuming that we are looking at \( P \) in the larger scope of diagram \( D \). See Figure 4.

Now we are ready to describe the fixed-points. Fixed points are diagrams \( D \in \mathcal{A}_L' \) with no tic marks with the following conditions. The diagram in Figure 4 satisfies all five conditions.

FP1. All sections satisfy the IC-condition except for the right-most ending at \( n \) if it is only composed of IC diagrams of length 1.

FP2. All IL diagrams \( P \) of odd length in the decomposition have \( e(P) \notin [a(P)/2 + 1, s(P) - 2] \)

FP3. If the IL diagrams \( P \) of odd length has \( e(P) = s(P) - 1 \) then it is prequeled by \( L_{2j} \cdot C_1^m \), \( j, m \geq 1 \), with \( e(L_{2j}) = s(L_{2j}) \).

FP4. If the IL diagrams \( P \) of odd length in the decomposition ends before \( n \) we have \( e(P) \neq s(P) \).

FP5. If the IL diagram \( P \) in the decomposition is followed by another IL diagram then \( e(P) \neq s(P) \).

To prove our bijection we will first break down all possible \( D' \in \mathcal{A}_L' \) into different cases. After we are sure we have all the cases then we will match up our cases and describe the rest of the involution.

We have already considered the case where \( D' = \tilde{D} \cdot I \) where \( \tilde{D} \) is not a fixed point and \( I \) is an IL or IC diagram of length at least one. Now we only have to consider cases about when \( \tilde{D} \) is a fixed point. Our cases will be determined by what \( I \) is as well as the conditions on \( P \) where \( \tilde{D} = \tilde{D} \cdot P \) and \( P \) is an IC or IL diagram of length at least one or where \( D' = I \) and \( \ell(P) = 0 \).

First we consider the cases where \( P \) satisfies one of the following three conditions. Either \( P = C_1 \) and the right-most section doesn’t satisfy the IC-condition, \( P = L_1 \) with \( e(P) \neq s(P) \) or the length of \( P \) is zero.
\begin{tabular}{|c|l|l|}
\hline
Case & Conditions on $P$ & Conditions on $I$ \\
\hline
1 & $-P = L_1$ with $e(P) \neq s(P)$ or $-P = C_1$ that fails the IC-condition or $-\ell(P) = 0$ & IC with $t(I) \geq 1$ \\
\hline
2 & $-P = L_1$ with $e(P) \neq s(P)$ or $-P = C_1$ that fails the IC-condition or $-\ell(P) = 0$ & IL \\
\hline
3 & $-P = L_1$ with $e(P) \neq s(P)$ or $-P = C_1$ that fails the IC-condition or $-\ell(P) = 0$ & IC with $t(I) = 0$ \\
\hline
\end{tabular}

In all the remaining cases $\ell(P) \geq 1$. Next we consider the cases where $P = L_m$ with $e(P) = s(P)$.

\begin{tabular}{|c|l|l|}
\hline
Case & Conditions on $P$ & Conditions on $I$ \\
\hline
4 & IL, $\ell(P) \geq 1$, $e(P) = s(P)$ & IC, $t(I) \geq 1$ \\
\hline
5 & IL, $\ell(P) \geq 1$ is odd, $e(P) = s(P)$ & IC, $t(I) = 0$ \\
\hline
6 & IL, $\ell(P) \geq 1$ is even, $e(P) = s(P)$ & IC, $t(I) = 0$ \\
\hline
7 & IL, $\ell(P) \geq 1$, $e(P) = s(P)$ & IL \\
\hline
\end{tabular}

Now we are left with the cases where the fixed-point $\tilde{D}$ ends in an IL diagram $P$ of length greater than 1 with $e(P) \neq s(P)$ or $P$ is an IC diagram whose section satisfies the IC-condition. The latter is because if $P$ is an IC diagram with $\ell(P) > 1$ that doesn’t satisfy the IC-condition then $\tilde{D}$ fails FP1 and isn’t a fixed point. In either case $P$ must satisfy the IC-condition. Since so many of our cases has $P$ fall under one of these cases we will say a diagram $\tilde{D} \in A'_L$ satisfies the $*$-condition if

- $\tilde{D}$ is a fixed-point of length at least 1,
- all sections in $\tilde{D}$ satisfy the IC-condition and
- if $\tilde{D}$ ends in an IL diagram then $e(\tilde{D}) \neq s(\tilde{D})$ the IL diagram has length at least two.

In all the remaining cases $\tilde{D}$ satisfies the $*$-condition. We will say $\tilde{D}$ satisfies the $**$-condition if additionally when $\tilde{D}$ ends in $L_{2j} \cdot C^m_1$ with $j, m \geq 1$ then $e(L_{2j}) \neq s(L_{2j})$. We will now start separating cases mostly on based on what happens to $I$. In the next block we consider when $I$ is an IC diagram.

\begin{tabular}{|c|l|l|}
\hline
Case & Conditions on $P$ & Conditions on $I$ \\
\hline
8 & $D = D \cdot P$ satisfies the $*$-condition but fails the $**$-condition & IC, $t(I) = 1$ splits $I$ into $\alpha_1$ vertices before the tic mark and $\alpha_2$ afterwards, $\alpha_1 \geq \alpha_2 \geq 1$ \\
\hline
9 & $D = D \cdot P$ satisfies the $**$-condition & IC, $t(I) = 1$ splits $I$ into $\alpha_1$ vertices before the tic mark and $\alpha_2$ afterwards, $\alpha_1 \geq \alpha_2 \geq 1$ \\
\hline
10 & $D = D \cdot P$ satisfies the $*$-condition & IC, $t(I) = 1$ splits $I$ into $\alpha_1$ vertices before the tic mark and $\alpha_2$ afterwards, $1 \leq \alpha_1 < \alpha_2$ \\
\hline
11 & $D = D \cdot P$ satisfies the $*$-condition & IC, $t(I) \geq 2$ \\
\hline
12 & $D = D \cdot P$ satisfies the $*$-condition & IC, $t(I) = 0$ \\
\hline
\end{tabular}
This completes all cases where \( I \) is an IC diagram. All that is left now is to consider the cases where \( I \) is an IL diagram. Case 13 represents those cases where \( I \) has a tic mark and the remaining cases are those where \( I \) has no tic marks.

| Case | Conditions on \( P \) | Conditions on \( I \) |
|------|----------------------|---------------------|
| 13   | \( D = D \cdot P \) satisfies the *-condition | IL, \( t(I) \geq 1 \) |
| 14   | \( D = D \cdot P \) satisfies the *-condition but fails the **-condition | IL, \( \ell(I) \) is odd, \( t(I) = 0 \), \( e(I) \notin [a(I)/2 + 1, s(I) - 2] \) |
| 15   | \( D = D \cdot P \) satisfies the *-condition but fails the **-condition | IL, \( \ell(I) \) is odd, \( t(I) = 0 \), \( e(I) \notin [a(I)/2 + 1, s(I) - 2] \) |
| 16   | \( D = D \cdot P \) satisfies the **-condition | IL, \( \ell(I) \) is odd, \( t(I) = 0 \), \( e(I) \notin [a(I)/2 + 1, s(I) - 1] \) |
| 17   | \( D = D \cdot P \) satisfies the **-condition | IL, \( \ell(I) \) is odd, \( t(I) = 0 \), \( e(I) \notin [a(I)/2 + 1, s(I) - 1] \) |
| 18   | \( D = D \cdot P \) satisfies the *-condition | IL, \( \ell(I) \) is even, \( t(I) = 0 \) |

We are now ready to define the remaining part of our sign-reversing involution. We will do so by pairing up the cases above and then identifying the remaining cases as fixed points. Recall that we are considering diagrams \( D' \in A'_L \) of the form

\[
D' = \tilde{D} \cdot I = \tilde{D} \cdot P \cdot I,
\]

which has a permutation labeling, \( \tilde{D} = D \cdot P \) is a fixed point so has no tic marks, but \( I \) may or may not have tic marks.

In all these maps we will need to preserve the weight of our diagrams. We will need to have the size of the right-most piece connected to vertex \( n \) to be the same size before and after the map, and also have the underlying integer partition be the same before and after the map. Additionally, we need the label of the right-most vertex to be the same before and after the map and if the right-most vertex is marked with a star we need its image to share that property. We will not outwardly address these four details since they will not be too hard to confirm, and we leave it for the reader.

We instead focus on proving the well definedness of the map, which is far more intricate since it revolves around very precise conditions so needs careful consideration. The following lemma will ease our proof of well definedness later.

**Lemma 4.4.** Consider \( D \) in \( A'_L \) without tic marks.

(i) If \( D = A \cdot B \) is a fixed point then \( A \) is a fixed point.

(ii) If \( D = \tilde{D} \cdot L_m \) with \( m \geq 1 \) is a fixed-point then \( \tilde{D} \) satisfies the *-condition.

(iii) If \( D = \tilde{D} \cdot C_m \) with \( m \geq 1 \) has \( \tilde{D} \) satisfy the *-condition then so does \( D \).

(iv) If \( D = \tilde{D} \cdot L_m \) where \( \tilde{D} \) satisfies the **-condition and \( e(L_m) = s(L_m) \) then \( \tilde{D} \cdot L_m \) is a fixed point.

(v) If \( D = \tilde{D} \cdot C_{m}^1 \) with \( m \geq 1 \) then \( D \) satisfies the **-condition if and only if \( \tilde{D} \) does as well.

(vi) If \( D = \tilde{D} \cdot C_{m}^1 \) with \( m \geq 2 \) then \( \tilde{D} \cdot C_1 \) satisfies the *-condition but fails the **-condition if and only if \( \tilde{D} \cdot C_{m}^1 \) does as well.

**Proof.** Part (i) is true because if \( D = A \cdot B \) is a fixed point, so satisfies all five conditions, then so does \( A \), which also makes \( A \) a fixed point.
Say we have a fixed point $D = \bar{D} \cdot L_m$ with $m \geq 1$. By part (i) we know $\bar{D}$ is also a fixed point. By FP1 all but the right-most section of $D$ satisfy the IC-condition. If $m = 1$ then $\bar{D}$'s right-most section is a single vertex so the second right-most section, which is $\bar{D}$'s right-most section, satisfies the IC-condition. If $m > 1$ then the right-most section of $D$ satisfies the IC-condition, which implies that the right-most section of $\bar{D}$ does as well. Thus, all section in $D$ satisfy the IC-condition. Say that $D$ ends in an IL diagram $L_j$. By FP1 $j > 1$ and by FP5 $\epsilon(L_j) \neq s(L_j)$ so $\bar{D}$ must satisfy the $\ast$-condition. This proves part (ii).

Say $D = \bar{D} \cdot C_m$ with $m \geq 1$ has $\bar{D}$ satisfy the $\ast$-condition. First, we will show that $D$ is a fixed point. We know $\bar{D}$ satisfies the $\ast$-condition so all sections satisfy the IC-condition. Tacking on $C_m$ to the end does not change this, so FP1 is satisfied. Also, tacking on $C_m$ doesn't change the fact that FP2, FP3 and FP5 are satisfied. The only way that FP4 is not satisfied is when $\bar{D}$ ends in $L_k$, $k \geq 1$. Note that because $\bar{D}$ satisfied the $\ast$-condition we must have $\epsilon(L_k) \neq s(L_k)$, so FP4 is satisfied. Thus, $D$ is a fixed point. Because $\bar{D}$ satisfies the $\ast$-condition all sections satisfy the IC-condition. Tacking on $C_m$ does not change this. Because $D$ ends in $C_m$ the last bullet of the $\ast$-condition is satisfied and $D$ satisfies the $\ast$-condition. This completes part (iii).

Say $D = \bar{D} \cdot L_m$ where $\bar{D}$ satisfies the $\ast$-condition and $\epsilon(L_m) = s(L_m)$. We want to show part (iv) by showing that $D$ is a fixed point. Because $\bar{D}$ satisfies the $\ast$-condition all its sections satisfy the IC-condition. If $m = 1$ then all sections of $D$, except the right-most which is a single vertex, satisfy the IC-condition. If $m > 1$ then tacking on an $L_m$ does not change the fact that all sections satisfy the IC-condition, so $D$ satisfies FP1. All IL diagrams of odd length in $\bar{D}$ satisfy FP2 and since we choose $\epsilon(L_m) = s(L_m)$ the $L_m$ at the end also satisfies FP2 if it were odd, so all of $D$ satisfies FP2. Now considering FP3. Because $\bar{D}$ is a fixed point we only have to consider the case when $L_m$ is involved. Because we chose $\epsilon(L_m) = s(L_m)$ we always have $\epsilon(L_m) \neq s(L_m) - 1$, so $D$ satisfies FP3. Because $\bar{D}$ is a fixed point so satisfies FP4 the only way an $L_{2k-1}$ in $D$ may not have $\epsilon(L_{2k-1}) \neq s(L_{2k-1})$ is when it is at the right end of $D$ if one exists there. However, $\epsilon(L_{2k-1}) \neq s(L_{2k-1})$ would still be satisfied since $D$ satisfies the $\ast$-condition so $D$ satisfies FP4. Because $\bar{D}$ is a fixed point the only place FP5 could fail in $D$ is when $L_m$ is involved, meaning that $D$ ends in an $L_k$, $k \geq 1$. Since $D$ satisfies the $\ast$-condition FP5 is certainly satisfied in $D$. Hence, $D$ is a fixed point and we have finished part (iv).

Say that $D = \bar{D} \cdot C_m^j$ with $m \geq 1$ is such that $D$ satisfies the ***-condition. We will show that $\bar{D}$ also satisfies the ***-condition. First, we know that $\bar{D}$ is a fixed point by part (i). Second, since $D$ satisfies the ***-condition it also satisfies the $\ast$-condition, so all sections of $D$ satisfy the IC-condition. Removing $C_m^j$ does not change this so all sections of $D$ satisfy the IC-condition. Consider the case where $\bar{D}$ ends in $L_k$, $k \geq 1$. If $k$ is even then because $D$ satisfies the ***-condition we have $\epsilon(L_k) \neq s(L_k)$. If $k$ was odd then because $D$ is a fixed point by FP4 we also have $\epsilon(L_k) \neq s(L_k)$. Note that if $\bar{D}$ ended in $L_k$ then $k > 1$ because all sections of $D$ satisfy the IC-condition. This proves that $\bar{D}$ satisfies the $\ast$-condition. Say that $\bar{D}$ ends in $L_{2j} \cdot C_1^k$, $j, k \geq 1$. Because $D$ satisfies the ***-condition $s(L_{2j}) \neq \epsilon(L_{2j})$, so $\bar{D}$ satisfies the ***-condition. Conversely, consider the case where $D = \bar{D} \cdot C_m^j$ with $m \geq 1$ where $D$ satisfies the ***-condition and we will show that $D$ does as well. By part (ii) we know that $D$ satisfies the $\ast$-condition. Consider the case where $D$ ends in $L_{2j} \cdot C_1^{m+k}$, $j, m + k \geq 1$. If $k = 0$ then $\bar{D}$ ends in $L_{2j}$ and because $\bar{D}$ satisfied the ***-condition we know $s(L_{2j}) \neq \epsilon(L_{2j})$. If $k \geq 1$ then because $\bar{D}$ ends in $L_{2j} \cdot C_1^k$ and because $\bar{D}$ satisfied the ***-condition we know $s(L_{2j}) \neq \epsilon(L_{2j})$. This proves that $D$ satisfies the ***-condition, which completes part (v).

Say $D = \bar{D} \cdot C_m^j$ with $m \geq 2$ and $\bar{D} \cdot C_1$ satisfies the $\ast$-condition but fails the ***-condition. We will show $D$ does as well. By part (iii) we know $D$ satisfies the $\ast$-condition. Because $\bar{D} \cdot C_1$ fails the ***-condition we can say $\bar{D} \cdot C_1$ ends in $L_{2j} \cdot C_1^k$, $k \geq 1$ with $s(L_{2j}) = \epsilon(L_{2j})$. This implies that $D$ also fails the ***-condition. Conversely, consider $D = \bar{D} \cdot C_m^j$ with $m \geq 2$ where $\bar{D} \cdot C_1$ satisfies the $\ast$-condition but fails the ***-condition. We will show $\bar{D} \cdot C_1$ does as well.
We know $\bar{D} \cdot C_1$ is a fixed point by part (i). Because $D$ satisfies the $*$-condition we know that all sections satisfy the IC-condition and removing $m - 1$ copies of $C_1$ does not change this. Because $\bar{D} \cdot C_1$ doesn’t end in an IL diagram the third bullet of the $*$-condition is irrelevant, which proves that $\bar{D} \cdot C_1$ satisfies the $*$-condition. Because $D$ fails the $**$-condition we have that $D$ ends in $L_{2j} \cdot C_1^{k+l}$, $k + m \geq 2$, and $s(L_{2j}) = e(L_{2j})$. This implies that $\bar{D} \cdot C_1$ ends in $L_{2j} \cdot C_1^{k+1}$, $j,k + 1 \geq 1$, and $s(L_{2j}) = e(L_{2j})$. This means that $\bar{D} \cdot C_1$ fails the $**$-condition and we have completed the proof of part (vi).

Involution part 1: We will map Case 1 with Case 2. In both of these cases we have either $\ell(P) = 0$, $P = L_1$ with $e(\bar{D}) \neq s(\bar{D})$ or $P = C_1$, where $\bar{D}$ fails the IC-condition. In this latter case because we fail the IC-condition we can argue that either $\bar{D} = C_1^m$ or $\bar{D} = F \cdot L_1 \cdot C_1^m$ for $m \geq 1$. Further, we can say in this latter case that because $\bar{D}$ was a fixed point we know $e(F \cdot L_1) \neq s(F \cdot L_1)$, and this matches the other situation where $P = L_1$. In summary, also including the situation where $\ell(P) = 0$, the diagrams in these cases have forms $D' = C_1^m \cdot I$ or $D' = F \cdot L_1 \cdot C_1^m \cdot I$ where $m \geq 0$ and $e(F \cdot L_1) \neq s(F \cdot L_1)$. We will focus on describing the map for $C_1^m \cdot I$, $m \geq 0$, where $I$ is either $C_k$ with $t(I) \geq 1$ as in Case 1 or $L_k$ as in Case 2.

First consider the case where $I = C_k$, $k \geq 1$, which has at least one tic mark. The arc diagram $C_1^m \cdot C_k$ has pieces of sizes $\alpha_1 + \alpha_2 + \cdots + \alpha_l$ where $l \geq 2$ and $\alpha_1 > m$. Let us now determine an integer $j \in [l - 1]$ as follows. This $j$ will indicate how we will split this integer composition into two. If $l = 2$ let $j = 1$. If there exists an $i \in [1, l - 2]$ such that $\alpha_1 + \cdots + \alpha_i \geq \alpha_{i+2} + \cdots + \alpha_l$ then let $j$ be the smallest. If there is no such $i$ then let $j = l - 1$.

Using this $j$ we can define two segments of length 1 arcs. The first segment will have pieces of sizes $\alpha_{j+1} + \cdots + \alpha_l$ and the second segment will have pieces of sizes $\alpha_j + \cdots + \alpha_1$. We label the pieces of each segment with their associated labels in the original diagram. We will interlace these two segments where the first one will end at the most-right vertex and the second one will end at the second most-right vertex. There may be parts of one of these segments that sits outside and to the left of the interlacing, and in this case we leave the arcs as length 1 arcs. The particular choice of $j$ assures us that none of these ‘outside arcs’ of length one have a tic mark. The labels on all the pieces will remain the same, but since we created a new connected component that ends at the second most-right vertex we must choose a vertex in the piece associated to $\alpha_1$ to be marked with a star. Mark the $(\alpha_1 - m)$th most-right vertex in this piece with a star. See Figure 5 for an example. It is not too hard to see that the output is part of Case 2 by the discussion in the last paragraph. Because of the star-marked vertex we can reverse this map.

In more detail and to assure that this map is indeed invertible we will discuss how we map backwards from Case 2 to Case 1. This time say $I = L_k$, $k \geq 1$. The arc diagram $C_1^m \cdot L_k$ has two connected components, one ending at the right-most vertex and one at the second right-most vertex with pieces of sizes $\alpha_j + \cdots + \alpha_l$ and $\alpha_j + \cdots + \alpha_1$ respectively. Because $C_1^m$ doesn’t have a tic mark, this $j$ matches the choice of $j$ in the previous paragraph for the list of numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$. We then map

$$C_1^m \cdot L_k \mapsto C_1^{\alpha_1 - e} \cdot C_{(e-1) + \alpha_2 + \cdots + \alpha_l}$$

where $e = e(C_1^m \cdot L_k) \geq \alpha_1$, $C_1^{\alpha_1 - e}$ has no tic marks, we remove the vertex star-marking on the piece associated to $\alpha_1$ and $C_{(e-1) + \alpha_2 + \cdots + \alpha_l}$ has pieces $e + \alpha_2 + \cdots + \alpha_l$, which has at least two pieces. It is not hard to see this undoes the map given in the paragraph above. See Figure 5 for an example.

Involution part 2: We will map diagrams from Case 4 with diagrams from Case 11. Say $D'$ is a diagram from Case 4. Then $D' = \bar{D} \cdot L_m \cdot C_k$ with $e(\bar{D} \cdot L_m) = s(\bar{D} \cdot L_m)$, $t(C_k) \geq 1$ and $\bar{D} \cdot L_m$ is a fixed point. Say that $C_k$ has pieces of sizes $\alpha_1 + \cdots + \alpha_l$. We map

$$\bar{D} \cdot L_m \cdot C_k \mapsto \bar{D} \cdot C_{m+k}$$
where we remove the star marking on the component connected to the second-most right vertex of \( L_m \) and the we place tic marks on \( C_{m+k} \) in a way that depends on whether \( L_m \) has even or odd length. Say \( L_m \) has even length, so has two components of sizes \( j \) and \( j+1 \). Note that the component of size \( j+1 \) is connected to the left, which we want to preserve. In this case we set \( C_{m+k} \) so it has pieces of sizes \((j+\alpha_1) + j + \alpha_2 + \cdots + \alpha_l\). If instead \( L_m \) has odd length, so has components of sizes \( j \) and \( j \), then we set \( C_{m+k} \) so it has pieces of sizes \( j + (j + \alpha_1 - 1) + \alpha_2 + \cdots + \alpha_l \). In either case we keep the same labels on the same associated pieces. Also, in either case the output satisfies the conditions from Case 11, which are that \( C_{m+k} \) has at least two tic marks and by Lemma 4.4 (ii) we know that \( \bar{D} \) satisfies the \(*\)-condition. This is reversible because the even case corresponds to the first piece of \( C_{m+k} \) being larger than the second. The odd case corresponds to the first piece of \( C_{m+k} \) being weakly smaller than the second. Further because \( \bar{D} \) satisfies the \(*\)-condition and because we set \( e(\bar{D} \cdot L_m) = s(\bar{D} \cdot L_m) \) in the backwards map by (iv) in Lemma 4.4 we have that \( \bar{D} \cdot L_m \) is a fixed point so we are well defined. See Figure 6 for an example.

**Involution part 3:** Here we will map Case 5 to Case 10. Say \( D' \) satisfies Case 5. Then \( D' = \bar{D} \cdot L_{2m-1} \cdot C_k \) where \( m \geq 1 \), \( e(\bar{D} \cdot L_{2m-1}) = s(\bar{D} \cdot L_{2m-1}) \), \( t(C_k) = 0 \) and \( \bar{D} \cdot L_{2m-1} \) is a fixed point. We map

\[
\bar{D} \cdot L_{2m-1} \cdot C_k \mapsto \bar{D} \cdot C_{k+2m-1}
\]

where we remove the star marking on the component connected to the second-most right vertex of \( L_{2m-1} \), \( C_{k+2m-1} \) has two pieces of sizes \( m + (k + m) \) and we keep the same labels on the same associated pieces. This satisfies Case 10 because \( 1 \leq m < k + m \) and \( \bar{D} \) satisfies the \(*\)-condition by Lemma 4.4 (ii). Because of the condition on the two pieces in \( C_{k+2m-1} \) this is reversible. Because \( \bar{D} \) satisfies the \(*\)-condition and because we set \( e(\bar{D} \cdot L_{2m-1}) = s(\bar{D} \cdot L_{2m-1}) \) in the backwards map by (iv) in Lemma 4.4 we have that \( \bar{D} \cdot L_{2m-1} \) is a fixed point so we are well defined. See Figure 7 for an example.

**Involution part 4:** Here we will map Case 7 to Case 13. Say \( D' \) is part of Case 7 so \( D' = \bar{D} \cdot L_m \cdot L_k \) where \( e(\bar{D} \cdot L_m) = s(\bar{D} \cdot L_m) \), \( L_m \) has no tic marks with \( k, m \geq 1 \) and \( \bar{D} \cdot L_m \) is a fixed point. We map

\[
\bar{D} \cdot L_m \cdot L_k \mapsto \bar{D} \cdot L_{m+k},
\]
which is the exact same diagram but we add a length 2 arc between $L_m$ and $L_k$ with a tic mark and remove the star-marking on the piece connected to the second-most right vertex of $L_m$. The output is certainly in Case 13 since $L_{m+k}$ has at least one tic mark and $\bar{D}$ satisfies the $*$-condition by Lemma 4.4 (ii). This map is easily reversible by removing the arc with the tic mark we had just added, which is the first tic mark that appears from left-to-right vertex-wise in $L_{m+k}$. Because $\bar{D}$ satisfies the $*$-condition and because we set $e(D \cdot L_m) = s(D \cdot L_m)$ in the backwards map by (iv) in Lemma 4.4 we have that $\bar{D} \cdot L_m$ is a fixed point so we are well defined. See Figure 8 for an example.

![Figure 8: Example of involution part 4.](image)

**Involution part 5:** Here we will map Case 8 to Case 14. Say $D'$ satisfies Case 8. Then $D' = \bar{D} \cdot C_k$, $k \geq 1$, where $\bar{D}$ satisfies the $*$-condition but fails the $**$-condition and $t(C_k) = 1$ with pieces of sizes $\alpha_1 + \alpha_2$ with $1 \leq \alpha_2 \leq \alpha_1$. This means particularly that $D' = M \cdot L_{2j} \cdot C_1^m \cdot C_k$ where $j, m \geq 1$ and $e(L_{2j}) = s(L_{2j})$. We map

$$M \cdot L_{2j} \cdot C_1^m \cdot C_k \mapsto M \cdot L_{2j} \cdot C_1^m \cdot C_1^{\alpha_1 - \alpha_2} \cdot L_{2\alpha_2 - 1}$$

where there are no tic marks anywhere on $C_1^{\alpha_1 - \alpha_2} \cdot L_{2\alpha_2 - 1}$, we add a star-marking on the component attached to the second right-most vertex of $L_{2\alpha_2 - 1}$ so that $e(L_{2\alpha_2 - 1}) = \alpha_1$ and we keep all the vertex labeling the same in their respective pieces. By definition we know that $s = s(L_{2\alpha_2 - 1}) = \alpha_1 + m + 1$. Since also $\alpha_2 = a(L_{2\alpha_2 - 1})/2 + 1$ we are placing the star-marking so that $\alpha_1 = e(L_{2\alpha_2 - 1}) \in [a/2 + 1, s - 2]$ where $a = a(L_{2\alpha_2 - 1}) = 2\alpha_2$. Because $\bar{D}$ satisfies the $*$-condition but fails the $**$-condition we know by Lemma 4.4 (vi) that $M \cdot L_{2j} \cdot C_1^m \cdot C_1^{\alpha_2 - \alpha_2}$ satisfies the $*$-condition but fails the $**$-condition also. Lastly, because $t(L_{2\alpha_2 - 1}) = 0$ our output lies in Case 14. Note that because of the definition of $s$ we have at least $s - \alpha_2 - 1$ copies of $C_1$ before $L_{2\alpha_2 - 1}$. Because $\alpha_1 = e \in [a/2 + 1, s - 2]$ this means we have at least $\alpha_1 - \alpha_2 + 1$ copies of $C_1$ before $L_{2\alpha_2 - 1}$. This assures us that this is reversible. Since $M \cdot L_{2j} \cdot C_1^m \cdot C_1^{\alpha_2 - \alpha_2}$ satisfies the $*$-condition but fails the $**$-condition, in the backwards map by (vi) in Lemma 4.4 we have that $M \cdot L_{2j} \cdot C_1^m$ also satisfies the $*$-condition but fails the $**$-condition so we are well defined. See Figure 9 for an example.

![Figure 9: Example of involution part 5.](image)

**Involution part 6:** Here we will map Case 9 to Case 16. This is very similar to part 5. Say $D'$ satisfies Case 9. Then $D' = \bar{D} \cdot C_k$, $k \geq 1$, where $\bar{D}$ satisfies the $**$-condition and $C_k$ has $t(C_k) = 1$ with pieces of sizes $\alpha_1 + \alpha_2$ with $1 \leq \alpha_2 \leq \alpha_1$. We map

$$\bar{D} \cdot C_k \mapsto \bar{D} \cdot C_1^{\alpha_1 - \alpha_2} \cdot L_{2\alpha_2 - 1}$$

where there are no tic marks anywhere on $C_1^{\alpha_1 - \alpha_2} \cdot L_{2\alpha_2 - 1}$, we add a star-marking on the component attached to the second right-most vertex of $L_{2\alpha_2 - 1}$ so that $e(L_{2\alpha_2 - 1}) = \alpha_1$ and we keep all the vertex labels the same in their respective pieces. By definition we know that
$s = s(L_{2\alpha_2 - 1}) \geq \alpha_1 + 1$. Because $\alpha_2 = a(L_{2\alpha_2 - 1})/2 + 1$ the star-marking is placed so that $\alpha_1 = e(L_{2\alpha_2 - 1}) \in [a/2 + 1, s - 1]$ where $a = a(L_{2\alpha_2 - 1}) = 2\alpha_2$. Because $\hat{D}$ satisfies the **-condition by Lemma 4.4 (v) we know that $\hat{D} \cdot C_{\alpha_2 - \alpha_2}$ also satisfies the **-condition. Lastly, because $t(L_{2\alpha_2 - 1}) = 0$ we have our output lying in Case 14. Note that because of the placement of the vertex star marking to the right of $s$, but not inside the IL diagram on the right end, (i.e. $e \in [a/2 + 1, s - 1]$) we are assured that at least $s - a/2 - 2 \geq \alpha_1 - \alpha_2 = e - \alpha_2$ copies of $C_1$ occur before $L_{2\alpha_2 - 1}$. This assures that this map is reversible. Because $\hat{D} \cdot C_{\alpha_2 - \alpha_2}$ satisfies the **-condition in the backwards map by (v) in Lemma 4.4 we have $\hat{D}$ also satisfies the **-condition so we are well defined. See Figure 10 for an example.

![Figure 10: Example of involution part 6.](image)

**Involution part 7:** The remaining Cases 3, 6, 12, 15, 17 and 18 are the fixed points of our map. To be sure that these six cases match exactly with the five conditions on fixed points we presented earlier we will discuss how to inductively construct fixed points. By Lemma 4.4 part (i) we know that if $D = \hat{D} \cdot I$ is a fixed point, then $\hat{D}$ is a fixed point. So, we can construct all fixed points by discussing under what conditions can attach an IC or IL diagram $I$ to the right-end of a fixed point $\hat{D}$ so that $\hat{D} \cdot I$ still satisfies all five conditions. Along the way we will discuss why all these cases are exactly the six cases from our map.

First consider if we are attaching $C_k, k \geq 1$, to the right end of $\hat{D}$. The only time $\hat{D} \cdot C_k$ fails one of the five conditions is if $\hat{D}$ ends in $L_{2j - 1}, j \geq 1$, with $e(L_{2j - 1}) = s(L_{2j - 1})$ and we would have failed FP4. In all other circumstances we can attached $C_k$. These other circumstances can be broken into the following four bullets.

- If the right-most section of $\hat{D}$ is $C^m_1$, $m \geq 1$, or $\hat{D}$ has length zero.

This is part of Case 3. Under any other circumstance the right-most section of $\hat{D}$ will satisfy the IC-condition.

- If $\hat{D}$ ends in $C_m, m \geq 1$.

- If $\hat{D}$ ends in $L_{2m}, m \geq 1$.

- If $\hat{D}$ ends in $L_{2m - 1}, m \geq 1$, and $e(L_{2m - 1}) \neq s(L_{2m - 1})$.

The first bullet is part of Case 12. The second bullet is composed of Case 6 and part of Case 12. The third bullet is the remaining part of Case 12 when $m > 1$ and the remaining part of Case 3 when $m = 1$.

Next consider the case where we attach $L_{2k}, k \geq 1$, to the right end of $\hat{D}$. We find that $\hat{D} \cdot L_{2k}$ is only not a fixed point when the right-most component of $\hat{D}$ doesn’t satisfy the IC-condition or when $\hat{D}$ ends in $L_m$ with $e(L_m) = s(L_m)$. We can attached $L_{2k}$ to $\hat{D}$ and keep $\hat{D} \cdot L_{2k}$ as a fixed point under the following circumstances.

- If $\hat{D}$ ends in $C_m, m \geq 1$, and its right-most section satisfies the IC-condition.

- If $\hat{D}$ ends in $L_m, m > 1$, with $e(L_m) \neq s(L_m)$.

Both these bullets together form Case 18. Note that if $\hat{D}$ ends in $L_1$ then attaching $L_{2k}$ will not give us a fixed point because FP1 would not be satisfied.

Lastly, consider attaching $L_{2k - 1}, k \geq 1$, to the right end of $\hat{D}$. First we must have that $e(L_{2k - 1}) \notin [a(L_{2k - 1})/2 + 1, s(L_{2k - 1}) - 2]$, so $\hat{D} \cdot L_{2k - 1}$ satisfies FP2. Also, we must have right-most component of $\hat{D}$ satisfy the IC-condition so $\hat{D} \cdot L_{2k - 1}$ satisfies FP1. Additionally, if we
choose \( e(L_{2k-1}) = s(L_{2k-1}) - 1 \) then \( \tilde{D} \) needs to end in \( L_{2j} \cdot C_1^m, j, m \geq 1 \) with \( e(L_{2j}) = s(L_{2j}) \) so \( \tilde{D} \cdot L_{2k-1} \) satisfies FP3. Finally if \( \tilde{D} \) ends in \( L_j, j \geq 1 \) then \( e(L_j) \neq s(L_j) \) so \( \tilde{D} \cdot L_{2k-1} \) satisfies FP4 and FP5. All together this means we can attach \( L_{2k-1} \) with \( e(L_{2k-1}) = [a(L_{2k-1})/2 + 1, s(L_{2k-1}) - 2] \) to \( \tilde{D} \) all of whose components satisfy the IC-condition under the following three circumstances.

- If we choose \( e(L_{2k-1}) = s(L_{2k-1}) - 1 \) then \( \tilde{D} \) needs to end in \( L_{2j} \cdot C_1^m, j, m \geq 1 \) with \( e(L_{2j}) = s(L_{2j}) \).

This is part of Case 15. In all other circumstances we choose \( e(L_{2k-1}) \neq s(L_{2k-1}) - 1 \).

- If \( \tilde{D} \) ends in \( C_m, m \geq 1 \).
- If \( \tilde{D} \) ends in \( L_m, m \geq 1 \), with \( e(L_m) \neq s(L_m) \).

The first bullet contains \( \tilde{D} \) ending in \( L_{2j} \cdot C_1^m, j, m \geq 1 \) with \( e(L_{2j}) = s(L_{2j}) \), which is the remaining part of Case 15, or \( e(L_{2j}) \neq s(L_{2j}) \), which is part of Case 17. The remaining portion of this bullet is when \( \tilde{D} \) ends in \( C_m \), but doesn’t end in \( L_{2j} \cdot C_1^m, j, m \geq 1 \), which is part of Case 17. The last bullet is the remaining part of Case 17.

**Theorem 4.5.** For \( n \geq 1 \) the triangular ladder \( TL_n \) is semi-symmetrized e-positive and so e-positive.

**Proof.** If we apply the sign reversing involution \( \varphi \) on the signed set \( A'_L \) we can see by the equation in Corollary 4.2 that \( Y_{TL_n} \) is semi-symmetrized e-positive and so e-positive because all fixed points have positive sign because they have no tic marks.

Using the sign-reversing involution we defined in this section we can prove a few more graphs are semi-symmetrized e-positive. We will discuss this in the next section, but we will need the following fact about the sign-reversing involution \( \varphi \) we defined.

**Proposition 4.6.** If we are given a diagram \( D' \in A'_L \) such that \( D' = C_m \cdot \bar{F} \) with \( m \geq 2 \) where \( C_m \) has no tic marks, then \( \varphi(D') = C_m \cdot \bar{F} \) has a similar form. Further, if \( D' = C_m \cdot \bar{F} \) also has labels in \( [m-1] \) on the first \( m-1 \) vertices in increasing order and no star-markings on the first \( m-1 \) vertices then so does its image.

**Proof.** Looking at all seven parts of the involution defined and equation (5), where we also define the involution, we can clearly see that if \( D' \in A'_L \) such that \( D' = C_m \cdot \bar{F} \) with \( m \geq 2 \) where \( C_m \) has no tic marks, then \( \varphi(D') = C_m \cdot \bar{F} \) has a similar form. If we map \( D' \) under equation (5) then our reason is induction. If we map \( D' \) under one of our seven parts we can see that any \( C_m, m \geq 2 \), attached to the first vertex remains unchanged. This also includes the labels on the first \( m-1 \) vertices of \( C_m \).

Now we just have to mention why the last part is true, that if there is no star-markings on the first \( m-1 \) vertices then the image has the same property. While this is true, the reason is hidden in the definition of the s-value, which is actually a bound to where we can place the vertex star-marking in the map. In all parts of the involution the vertex star-marking usually doesn’t change its relative position in its relative piece. When we do add or remove a vertex star-marking we only place it as far left as the vertex associated to the s-value we calculate. By the definition of the s-value, the most-left vertex it could be is the mth vertex since our diagram starts with \( C_m, m \geq 2 \). This implies that if there is no star-markings on the first \( m-1 \) vertices then the image has the same property.
5 More $e$-positive graphs

In this section we will use the involution $\varphi$ and other ideas from Section 4 along with results
from Gebhard and Sagan [7] to expand on the known families of $e$-positive graphs. We will
show that any series of wedges between triangular ladders and complete graphs results in an
$e$-positive graph. This will expand on Gebhard and Sagan’s result.

**Theorem 5.1** (Gebhard and Sagan [7] Theorem 7.6 and 7.8). If a graph $G$ is semi-symmetrized
e-positive then so is $G \cdot K_m$ and $G \cdot TL_4$.

The involution $\varphi$ we defined in Section 4 actually gives us the exact involution we need
to prove an expanded version of Theorem 5.1. First, we need to discuss exactly how we can
do this, so we present the following lemma that is essential to our proof. This lemma, when
$\pi = 1$, is the result we have already proven, that $TL_n$ are semi-summarized $e$-positive.

**Lemma 5.2.** For any $e_\pi$, $\pi \vdash [m]$ with $m, n \geq 1$, the sum

$$\sum_{D \in A(TL_n)} (-1)^{a(D)} e_\pi \uparrow_D$$

is semi-summarized $e$-positive.

**Proof.** In this proof we let $A = A(TL_n)$ and $A'_L = A'_L(TL_n)$. Because the elementary basis is
multiplicative and because of the relabeling proposition 2.4 it suffices to show this formula for
$\pi = [m]$. When $m = 1$ this is exactly showing that the $TL_n$ are semi-summarized $e$-positive,
which we have already done in Theorem 4.5. Let $m \geq 3$. Note that by a reason very similar to
the proof of Theorem 3.7

$$\sum_{D \in A} (-1)^{a(D)} e_{[m]} \uparrow_D \equiv_{m+n} \sum_{D' \in C_{m-1} \cdot A'_L} (-1)^{\ell(D')} e_{[m-1] \mid \pi(D')}$$

where $C_{m-1} \cdot A'_L$ is the set of all arc diagrams $C_{m-1} \cdot D'$ with $D' \in A'_L$ and the labels on $D'$ are
increased by $m-1$, $C_{m-1}$ has no tie marks and no star-vertex markings on the left $m-1$ vertices
and the labels on the first $m-1$ vertices are $[m-1]$ in increasing order. By Proposition 4.6 we
know that $\varphi$ is a sign-reversing involution on the signed set $C_{m-1} \cdot A'_L$, so we are done when
$m \geq 3$. The problem is when $m = 2$. We will define a modified version of the sign-reversing
involution on $\psi : C_1 \cdot A'_L \to C_1 \cdot A'_L$ by using our involution on $\psi : C_2 \cdot A'_L \to C_2 \cdot A'_L$ and
defining an invertible map

$$h : C_1 \cdot A'_L \to \{D' \in C_2 \cdot A'_L : \text{vertex 1 is labeled 1 with no star}\}.$$ 

We define $h$ by taking a $D' \in C_1 \cdot A'_L$ so $D' = C_1 \cdot \bar{D}$ with $\bar{D} \in A'_L$ and defining $A' = C_2 \cdot \bar{D}$
by letting $A'$ be $D'$ on vertices 2 through $n + 1$ but we increase all the labels by 1 and we
also change the arcs on the first two vertices so that $A'$ starts with $C_2$. Also, we label the
first vertex with 1. This map $h(D') = A'$ is easily reversible, so $h$ is a bijection. We define
$\psi : C_1 \cdot A'_L \to C_1 \cdot A'_L$ by mapping $D' \in C_1 \cdot A'_L$ to $h^{-1} \circ \varphi \circ h(D')$. This map is easily
well-defined using Proposition 4.6. \(\square\)

**Theorem 5.3.** If graph $G$ is semi-summarized $e$-positive then so is $G \cdot TL_m$.

**Proof.** First consider a graph $G$ and $G \cdot TL_m$. By a very similar reason as in the proof of
Proposition 3.2 we have

$$Y_{G \cdot TL_m} = \sum_{D \in A(TL_m)} (-1)^{a(D)} Y_G \uparrow_D.$$


Say $G$ is a semi-summarized $e$-positive graph. This means that

$$Y_G \equiv \sum_{(\pi) \subseteq \Pi_n} c(\pi) e_\pi$$

where $c(\pi) \geq 0$. Combining the two equations above gives us

$$Y_{G \cdot TL_m} \equiv n + m - 1 \sum_{(\pi) \subseteq \Pi_n} c(\pi) \sum_{D \in A(TL_m)} (-1)^{a(D)} e(\pi) \uparrow D.$$

Using Lemma 5.2 we have that this is also semi-summarized $e$-positive. □

Putting everything together we have an expanded version of Gebhard and Sagan’s Theorem 5.1.

**Corollary 5.4.** Any graph $G$ such that

$$G = G_1 \cdot G_2 \cdots G_l$$

where $G_i = TL_{n_i}$ or $G_i = K_{n_i}$ is a semi-summarized $e$-positive graph, so is also an $e$-positive graph.

**Proof.** This follow immediately from Theorem 5.1 and Theorem 5.3. □

**References**

[1] N. Bergeron, C. Hohlweg, M. Rosas and M. Zabrocki, Grothendieck bialgebras, partition lattices, and symmetric functions in noncommutative variables, *Electron. J. Combin.* 13 no. 1, 1–19 (2006).

[2] G. Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. of Math.* 14, 43–46 (1912).

[3] S. Cho and J. Huh, On $e$-positivity and $e$-unimodality of chromatic quasisymmetric functions, arXiv:1711.07152.

[4] S. Dahlberg, A. Foley and S. van Willigenburg, Resolving Stanley’s $e$-positivity of claw-contractible-free graphs, arXiv:1703.05770.

[5] B. Ellzey, Chromatic quasisymmetric functions of directed graphs., *Sm. Lothar. Combin.* 78B Art. 74, 1–12 (2017).

[6] V. Gasharov, On Stanley’s chromatic symmetric function and clawfree graphs, *Discrete Math.* 205, 229–234 (1999).

[7] D. Gebhard and B. Sagan, A chromatic symmetric function in noncommuting variables, *J. Algebraic Combin.* 13 no. 3, 227–255 (2001).

[8] M. Guay-Paquet, A modular law for the chromatic symmetric functions of $(3+1)$-free posets, arXiv:1306.2400v1.

[9] A. Hamel, C. Hoang and J. Tuero, Chromatic symmetric functions and H-free graphs, arXiv:1709.03354.

[10] M. Harada and M. Precup, The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture, arXiv:1709.06736.

[11] A. Paunov, Planar graphs and Stanley’s chromatic functions, arXiv:1702.05787.
[12] A. Paunov and A. Szenes, A new approach to e-positivity for Stanley’s chromatic functions, arXiv:1702.05791.

[13] M. Rosas and B. Sagan, Symmetric functions in noncommuting variables, *Trans. Amer. Math. Soc.* 538 no. 1, 215–232 (2006).

[14] J. Shareshian and M. Wachs, Chromatic quasisymmetric functions, *Adv. Math.* 295, 497–551 (2016).

[15] R. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* 111, 166–194 (1995).

[16] R. Stanley, Enumerative Combinatorics Volume 1, *Cambridge University Press* 2011.

[17] R. Stanley and S. Stembridge, On imminants of Jacobi-Trudi matrices and permutations with restricted position, *J. Combin. Theory Ser. A.* 62, 261–279 (1993).

[18] S. Tsujie, The chromatic symmetric functions of trivially perfect graphs and cographs, *Graphs Combin.* 34 no. 5, 1037–1048 (2018).

[19] M. Wolfe, Symmetric chromatic functions, *Pi Mu Epsilon Journal* 10, 643–757 (1998).