Hedonic games formalize coalition formation scenarios where players evaluate an outcome based on the coalition they are contained in. Due to a large number of possible coalitions, compact representations of these games are crucial. We complement known compact representation models by a distance-based approach: Players’ preferences are encoded in a bipolar manner by ordinal preferences over a small set of known neighbouring players, coalitions are represented by adequate preference orders from a player’s perspective, and preferences over coalitions are extended based on a directed form of Hausdorff-Kendall-tau distance between individual preferences and coalitions. We show that this model satisfies desirable axiomatic properties and has reasonable computational complexity in terms of selected individual-based stability notions.

1 Introduction

When basing the decision about whom players share a task with on these players’ preferences, a well-established model are hedonic games (Banerjee, Konishi, and Sönmez, 2001; Bogomolnaia and Jackson, 2002). As a form of a coalition formation game, players aim to partition into a so-called coalition structure. In a hedonic game the individual preferences over such partitions only depend on coalitions, i.e. sets of players, they belong to. The key idea of this paper is to model a distance-based representation of hedonic games and study it from an axiomatic and computational point of view.

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In general, the number of possible coalitions is exponential in the number of players. Hence, from an algorithmic point of view, it is relevant to find reasonable preference representations that are succinct but also expressive (Chalkiadakis, Elkind, and Wooldridge, 2011). In very large settings, especially, it is reasonable to assume that for each player the number of known players is small, e.g., bounded by a constant (Peters, 2016; Fichtenberger, Krivošija, and Rey, 2019). Furthermore, we assume a subdivision of these known co-players into those appreciated (friends) and those disapproved of (enemies) (see, e.g., Dimitrov et al., 2006). A player prefers to share a coalition with their friends rather than their enemies. Additionally, we follow the encoding proposed by Lang et al. (2015) in which each player ranks their friends and enemies, respectively. Their model of how a player compares two coalitions is based on the polarized responsive extension principle which we are going to discuss in Section 5. This encoding allows the expression of a variety of opinions while at the same time players do not have to specify a detailed information, such as a numerical evaluation or knowledge of the whole player set.

The model at hand calculates the distance between a player’s preference and a coalition which expresses the player’s dissatisfaction with this coalition. While distance-based preferences play an important role in decision making, preference extensions in hedonic games have not, to the best of our knowledge, so far, been based on ordinal distances. We attend to the well-studied Kendall-tau distance which counts the minimal necessary swaps to transform one preference into another. In order to allow for indifferences in a player’s preference we consider the Hausdorff-Kendall-tau distance. To this end a comparable encoding of a preference and a coalition is necessary. We show that this model meets desired properties and compare it with other known hedonic game representations in terms of expressibility and computational complexity of stability problems.

1.1 Related Work

Hedonic games have been introduced by Banerjee, Konishi, and Sönmez (2001) and Bogomolnaia and Jackson (2002) and studied from an axiomatic and algorithmic point of view since (see, e.g. Chalkiadakis, Elkind, and Wooldridge, 2011; Aziz and Savani, 2016). Applications can be found in particular instances, such as stable roommate problems (Cechlárová, 2002; Irving, 1985) and group activity selection problems (Darmann et al., 2018). Several convenient representations encode preferences over individual players which are then extended to preferences over coalitions. They include network approaches (Dimitrov et al., 2006), numerical approaches (such as additively separable encodings Bogomolnaia and Jackson, 2002) and ordinal approaches (such as the singleton encoding Cechlárová and Romero-Medina, 2001). More recently, encodings have been considered that do not assume that each player knows every other player in the game, but that they only know a subset of players and consider the others as neutral (Ota et al., 2017; Peters, 2016; Lang et al., 2015), sometimes referred to as FEN-hedonic games. Hedonic games with ordinal preferences and thresholds (Lang et al., 2015; Kerkmann et al., 2020) combine trichotomous preferences with ordinal preferences. In this paper we refer to this encoding. So far, the extensions to coalitions comprise a set of possible and necessary extensions as well as a numerical approach involving Borda scores (Lang
et al., 2015). The additional requirement of a constant number of known players enables computational lower bounds to decrease and we cannot transfer proof techniques to our model immediately. Peters and Elkind (2015) study causes of complexity for hedonic games. Unfortunately, the results are not applicable here. We do, however, fulfill the requirements of a graphical hedonic game (Peters, 2016). Consequently, several problems regarding stability of a game are fixed-parameter tractable with respect to treewidth and degree of the underlying dependency graph. Note that social distance games (Brânzei and Larson, 2011) and distance hedonic games (Flammini et al., 2021) are a different approach concerning distances of players in a graph.

Preference extensions from ordinal preferences over single items to a preference order over subsets of items (see, e.g. Barberà, Bossert, and Pattanaik, 2004) find applications in various topics within the field of Computational Social Choice (Brandt et al., 2016). Distance-based approaches in order to define these extensions are suggested in several contexts, but have not, to the best of our knowledge, yet been modelled for hedonic games. Closely related to our research is the field of committee elections (see, e.g., Falisewski et al., 2017) in which a subset of candidates, the so-called committee, is elected based on voters’ preferences. In this context Brams, Kilgour, and Sanver (2007) proposed to use distances as closeness measures and established a minimax approach, i.e., the election winner minimizes the maximal distance between preferences and committee. As in our work, this method includes the necessity of representing the committee (as in our case the coalition) as well as the preference in the same form. Their approach, however, regards votes which solely express binary opinions (approval or disapproval) of candidates. Thereby, it suffices to make use of the Hamming distance. This approach was extended to other forms of preferences, including linear orders (Baumeister and Dennisen, 2015). Similarly, in judgment aggregation (see, e.g. Endriss, 2016), aiming at a collective judgment set out of individual judgments, distance-based procedures measure the Hamming distance of a complete and consistent judgment set to the individual judgments and return the one minimizing the sum of distances.

We refer to a number of axiomatic properties which are desirable for a hedonic game. These are in parts inspired by different fields of Computational Social Choice as described above and preference-based matching. Fairness properties such as anonymity and nonimposition stem from a voting context. Responsiveness as introduced by Roth (1985) in the context of many-to-one matching markets refers to a guaranteed benefit from replacing a former match by a match with a preferred item. Basic properties for extensions to subsets of any size are defined by Barberà, Bossert, and Pattanaik (2004) and Bossong and Schweigert (2006), whose extension principle is defined for polarized preferences by Lang et al. (2015).

### 1.2 Contribution

We define the model of hedonic games with distance-based preferences in Section 3. In order to extend a player’s opinion encoded by polarized ordinal preferences over a subset of known players (3.1) to a preference order over coalitions which constitute the game (3.4), we represent a coalition also as a preference order from the player’s
perspective \((3.2)\), and compute a distance between these two preference orders \((3.3)\). We support this definition by an easily accessible characterization. In Section \([4]\) we discuss why this particular coalition representation and distance outperforms other approaches. In Section \([5]\) we analyse our model axiomatically: Desired properties are fulfilled, for instance, that a coalition cannot become less favourable when a preferred player enters the coalition. We observe this distance-based approach to be a reasonable and natural completion of existing compact representations which is at the same time flexible enough in terms of expressivity. We study computational aspects of individual-based stability in Section \([6]\) and show that, e.g., Nash stability of a given outcome can be verified in linear time while it is \(\text{NP-complete}\) to decide whether there exists a Nash-stable outcome in a given game.

2 Preliminaries

Our work is based on the game-theoretic study of hedonic games as well as on distance measures which are outlined in the following two subsections.

2.1 Hedonic Games

We consider a class of cooperative games, where players want to partition into subsets of players, so-called coalitions. A hedonic game \(\text{(Bogomolnaia and Jackson, 2002; Banerjee, Konishi, and Sönmez, 2001)}\) is such a coalition formation game \(\langle N, \succeq \rangle\) with a set of \(n = |N|\) players and a preference profile \(\succeq = (\succeq_1, \ldots, \succeq_n)\) such that each player \(i\)'s preference \(\succeq_i\) only depends on the coalitions \(i\) is contained in. The set of such coalitions is denoted by \(\mathcal{N}_i = \{C \subseteq N \mid i \in C\}\). A partition \(\Gamma\) of the players into subsets is called a coalition structure and \(\Gamma(i)\) denotes the coalition in \(\Gamma\) containing player \(i\).

A common goal of these games is to achieve stable coalition structures no player or group of players has an intention to deviate from. A coalition structure \(\Gamma\) is called Nash stable, if there is no player \(i \in N\) who prefers another coalition \(C \cup \{i\}\) with \(C \in \Gamma(i)\) to \(\Gamma(i)\). If a deviation is restricted to coalitions where \(i\) is welcome \((C \cup \{i\} \succeq_j C\) for each \(j \in C\)) and if there is no such deviation, \(\Gamma\) is called individually stable. If additionally \(i\) can only deviate if not bound to a contract in \(i\)'s former coalition \((\Gamma(i) \setminus \{i\} \succeq_j \Gamma(i)\) for each \(j \in \Gamma(i) \setminus \{i\}\)), \(\Gamma\) is called contractually individually stable. Moreover, \(\Gamma\) is called perfect if each \(\Gamma(i)\) is one of player \(i\)'s favourite coalitions \((\Gamma(i) \succeq_j C\) for each \(C \in \mathcal{N}_i\)). For an analysis of stability, usually, two decision problems are distinguished: the verification problem of whether a given coalition structure is stable, and the existence problem of whether a game allows a stable coalition structure.

In order to achieve succinctness, often a player’s opinion is encoded as a preference order \(\succeq\) over the set of players which can be extended to a preference order \(\succeq\) over coalitions. We consider \(\succeq\) to be an ordinal preference order with possible indifferences, i.e., a linear order (reflexive, transitive), that is not necessarily antisymmetric. For two players \(x, y \in N\), \(x\) is weakly preferred to \(y\) if \(x \succeq y\); \(x\) is (strictly) preferred to \(y\) if \(x \succ y\) (i.e., \(x \succeq y\) and not \(y \succeq x\)); and \(x\) and \(y\) are considered indifferent if \(x \sim y\) (i.e., \(x \succeq y\) and \(y \succeq x\)). Similarly, for \(\succeq\), we have \(A \succ B\) if and only if \(A \succeq B\) and not \(B \succeq A\). By \(\succeq(C)\)
we denote the restriction of $\succeq$ to the players within $C \subseteq N$. Let the rank of a player $j$ be the position among equally liked players within a preference order: $j$ is among the top-ranked players (of rank 1) if $j \succeq k$ for each $k \in N$; $j$ is of rank $p$ if there exist $p - 1$ other players $k_1, \ldots, k_{p-1}$ such that $k_1 \succeq \cdots \succeq k_{p-1} \succeq j$, but no $p$ players $\ell_1, \ldots, \ell_p$ such that $\ell_1 \succeq \cdots \succeq \ell_p \succeq j$. Due to indifferences, several players can share the same rank, such that the maximal rank is possibly smaller than $n$.

### 2.2 Distance Measures

We study the distance between two preference orders, i.e., linear orders over the set of players. To be precise, in our setting a player’s opinion is compared to a coalition, the former given in form of a preference order over a subset of players; the latter interpreted as a preference order from the player’s perspective. Before we go into detail in Section 3 we present background on how to compare preference orders in general. A distance measure on a space $A$ is a metric $\text{dist} : A \times A \to \mathbb{R}_{\geq 0}$, i.e. for each $a, b, c \in A$ it fulfills non-negativity ($\text{dist}(a, b) \geq 0$), identity of indiscernibles ($\text{dist}(a, b) = 0 \iff a = b$), symmetry ($\text{dist}(a, b) = \text{dist}(b, a)$), and the triangle inequality ($\text{dist}(a, b) + \text{dist}(b, c) \geq \text{dist}(a, c)$). A directed distance is a quasimetric which does not fulfill symmetry, or a pseudoquasimetric, if it additionally does not fulfill identity of indiscernibles.

For strict preference orders, that is, antisymmetric linear orders, a well-known distance metric is the Kendall-tau distance. Based on Kendall’s measure of rank correlation (Kendall, 1938), it calculates the minimal number of inversions of adjacent players necessary to convert one strict preference order $\succ^\alpha$ into another, $\succ^\beta$:

$$\tau\left(\succ^\alpha, \succ^\beta\right) = |\{(x, y) \in N \times N \mid x \succ^\alpha y \text{ and } y \succ^\beta x\}|.$$

In our context, preference orders allow indifferences (i.e., are not necessarily antisymmetric). Any such order $\succeq$ can be interpreted as an equivalence class containing all strict preferences orders $\succ^\alpha$ which are consistent with $\succeq$ (i.e., $x \succ^\alpha y$ if $x \succeq y$) and each indifference is resolved by some permutation (either $x \succ^\alpha y$ or $y \succ^\alpha x$, if $x \succeq y$ and $y \succeq x$). In order to include indifferences, we consider a Hausdorff distance (Hausdorff, 1927). The Hausdorff–Kendall-tau distance is defined by

$$\tau^\ast(\succ^\pi, \succ^\sigma) = \max \left\{ \min_{\succ^\beta \in \succ^\sigma} \tau\left(\succ^\alpha, \succ^\beta\right), \max_{\succ^\alpha \in \succ^\pi} \min_{\succ^\beta \in \succ^\sigma} \tau\left(\succ^\alpha, \succ^\beta\right) \right\}$$

for two preference orders $\succ^\pi$ and $\succ^\sigma$ interpreted as equivalence classes of strict preferences with maximal ranks $r_\pi$ and $r_\sigma$, respectively. Intuitively, for the first part, a worst-case resolution $\succ^\beta$ of $\succ^\sigma$ is chosen for which the minimal number of swaps between some $\succ^\alpha$ resolving $\pi$ and $\succ^\beta$ is maximized. For our interpretation of a coalition depending on a player’s perspective, we only consider this first part; see Section 3 for details. We denote this directed distance by

$$\overline{\tau}\left(\succ^\pi, \succ^\sigma\right) = \max_{\succ^\beta \in \succ^\sigma} \min_{\succ^\alpha \in \succ^\pi} \tau\left(\succ^\alpha, \succ^\beta\right) = \sum_{p, q \in \{1, \ldots, r_\sigma\}, p \leq q, \{p, q\} \in \{1, \ldots, r_\sigma\}, q \geq q'} n_{pq} \cdot n_{p'q'}, \quad (1)$$

5
where the latter is a characterization (see, e.g., Critchlow, 2012) with the number \( n_{pq} \) of items that are ranked on position \( p \) in \( \geq^p \) and on position \( q \) in \( \geq^q \). While in general \( \tau \) is a pseudoquasimetric, the directed distance as defined in Section 3.3 allows a quasimetric. See Section 5 for details. This matches the Hausdorff-Kendall-tau metric via 
\[
\tau^*(\geq^p, \geq^q) = \max\{ \tau(\geq^p, \geq^q), \tau(\geq^q, \geq^p) \}.
\]

3 The Model

This section presents a compactly encoded model of hedonic games which yet gives credit to each player’s detailed opinion on a subset of candidates. The preference extension is based on calculating the Kendall-tau distances between a player’s preference and coalitions containing this player. Thus, a player compares two coalitions by ranking the coalition with minimal distance, i.e., the minimal dissatisfaction, highest. We create a hedonic game with distance-based preferences as follows: Given a player’s ordinal preferences over the known players, we define

(3.2) a preference-based encoding of a coalition from a player’s perspective and

(3.3) a distance between preference and coalition from a player’s perspective

which determines the extension to preferences over coalitions for each player.

3.1 Preference Encoding

We define the preference encoding similar to hedonic games with ordinal preferences and thresholds (Lang et al., 2015). For each player \( i \) the set of other players is partitioned into accepted players \( N^+_i \) (friends), unaccepted players \( N^-_i \) (enemies), and neutral (e.g. unknown) players \( N^0_i \). This can be represented by a directed graph, the underlying dependency graph with two edge labels \( + \) and \( - \). By \( N_i = N^+_i \cup N^-_i \) we denote \( i \)’s neighbourhood and assume that \( |N_i| = |N^+_i \cup N^-_i| \leq d \) for a constant \( d > 0 \). Additionally, each player \( i \) specifies a partial preference relation \( \succeq_i \) over \( N^+_i \) and a partial preference relation \( \succeq_i \) over \( N^-_i \) such that a ballot contains information of the form \( (\succeq_i(N^+_i) \mid (N^0_i \cup \{i\})_\prec \mid \succeq_i(N^-_i)) \), where \( S_\prec \) for any \( S \subseteq N \) denotes that all players in \( S \) are ranked equally. This induces a preference relation \( \succeq_i \): For players \( j, k \in N \setminus \{i\} \) it holds that \( j \succeq_i k \) if and only if \( j, k \in N^+_i \) and \( j \succeq k \); \( j \in N^+_i \) and \( k \in N^0_i \); \( j \in N^-_i \) and \( k \in N^-_i \); \( j \in N^0_i \) and \( k \in N^-_i \); or \( j, k \in N^-_i \) and \( j \succeq k \). In order to obtain our final preference encoding we conduct two steps. Firstly, since the neutral players in \( N^0_i \) do not have any effect on the evaluation of a coalition we abbreviate the ballot notation by \( (\succeq_i(N^+_i) \mid i \mid \succeq_i(N^-_i)) \). Secondly, in order to facilitate a comparison with a coalition, we subdivide the ballot into a part regarding player \( i \)’s friends and enemies, respectively,

\[
\succeq^+_i = (\succeq_i(N^+_i) \mid i \mid) \quad \text{and} \quad \succeq^-_i = (\mid i \mid \succeq_i(N^-_i)) .
\]

This again induces preference orders, namely \( \succeq^+_i(\succeq_i(N^+_i) \cup \{i\}) \) and \( \succeq^-_i = (\succeq_i(N^-_i) \cup \{i\}) \) in which \( j \succeq^+_i i \) for all \( j \in N^+_i \) and \( j \succeq^-_i j \) for all \( j \in N^-_i \) complements \( \succeq_i(N^+_i) \) (\( \succeq_i(N^-_i) \), respectively).
Example 1. For instance, let i know five other players, a, b, c, d, and e, three of which i likes (a, b, c) and two of which i doesn’t like (d, e), and let i specify preferences with whom to cooperate by (a ↠; b ⊳; c | i | d ↠; e).

3.2 Coalition Encoding

In order to compare a player i’s ballot with a coalition C ∈ N_i, we interpret C from i’s perspective. Intuitively, friends in C and absent enemies are ranked according to i’s opinion, whereas missing good friends and present unfavourable enemies have a reversed impact. By −−→ \Xi_i(S) we denote player i’s reversed preference order over all players in S ⊆ N such that the separate representation is defined matching the form of two ballots

\[ C_i^+ = \left( \Xi_i(\mathcal{C} \cap N_i^+) \mid i \right) \Xi_i(N_i^+ \setminus C) \right) \quad \text{and} \quad C_i^- = \left( \Xi_i(\mathcal{C} \cap N_i^-) \mid i \right) \Xi_i(N_i^- \setminus C) \right) \]

Note that this is again the abbreviated notion, omitting neutral players which are implicitly ranked equally to player i without influencing player i’s preference order over coalitions. By this we allow a partial ballot to be compared to a partial coalition containing the identical subset of players.

Example 2. Continuing Example 1, from i’s perspective, we can now encode coalitions contained in N_i. For a coalition C = {i, a, b, e} (and any combination with further neutral players), we obtain C_i^+ = (a ↠; b | i | c) and C_i^- = (e | i | d). For D = {i, c, d, e}, we have D_i^+ = (c | i | b ↠; a) and D_i^- = (e ↠; d | i |). The coalition containing all neighbouring players, E = {i, a, b, c, d, e} is encoded by E_i^+ = (a ↠; b ⊳; c | i |) and E_i^- = (e ↠; d | i |); F = {i, a, b, c} containing all friends, by F_i^+ = (a ↠; b ⊳; c | i |) and F_i^- = ( | i | d ↠; e).

3.3 Distance Between Preference and Coalition

Given representations of both a player’s preferences and a coalition, we calculate their distance by summing up the separate distances regarding all friends and all enemies. Our model defines this distance between ⪰_i and C ∈ N_i by

\[ \delta(\mathcal{C} \setminus i, C) = \delta^+(\mathcal{C} \setminus i, C) + \delta^-(\mathcal{C} \setminus i, C) \]

with

\[ \delta^+(\mathcal{C} \setminus i, C) = \mathcal{T} \left( \mathcal{C}_i^+, \mathcal{C}_i^- \right) \quad \text{and} \quad \delta^-(\mathcal{C} \setminus i, C) = \mathcal{T} \left( \mathcal{C}_i^-, \mathcal{C}_i^- \right), \]

for which \( \mathcal{C}_i^+ \) and \( \mathcal{C}_i^- \) induce, as defined, equivalence classes containing all antisymmetric linear orders over \( N_i^+ \cup \{i\} \) (\( N_i^- \cup \{i\} \)) dissolving occurring indifferences in the ballots.

Note that we consider the directed distance \( \mathcal{T} \) here, since we have a dependence between the coalition representation and the preferences of a player. Considering the worst case of possible swaps within indifferences in the preference encoding, would contradict this intuition. In fact, it always holds that \( \mathcal{T} \left( \mathcal{C}_i^+, \mathcal{C}_i^- \right) \leq \mathcal{T} \left( \mathcal{C}_i^+, \mathcal{C}_i^- \right) \). Furthermore, note that \( \delta \) could be any other function in \( \delta^+ \) and \( \delta^- \), for instance the Euclidean distance. If not stated otherwise, we consider the 1-norm in this paper.
Example 3. For the instance in Examples 1 and 2, we now obtain the distances from $i$’s preferences to the coalitions by $\delta^+(\succeq_i, C) = 1$, $\delta^-(\succeq_i, C) = 2$, thus, $\delta(\succeq_i, C) = 3$; $\delta(\succeq_i, D) = 4 + 3 = 7$; $\delta(\succeq_i, E) = 0 + 3 = 3$; and $\delta(\succeq_i, F) = 0$.

We can characterize this distance as follows.

**Proposition 4.** It holds that

$$\delta^+(\succeq_i, C) = |N_i^+ \setminus C| + \sum_{f \in N_i^+ \setminus C} |\{ b \in N_i^+ | f \succ_i b \}| \quad (2)$$

and

$$\delta^-(\succeq_i, C) = |C \cap N_i^-| + \sum_{f \in N_i^-} |\{ b \in C \cap N_i^- | f \succ_i b \}| . \quad (3)$$

Figure 1 depicts the characterization of $\delta^+$ (Equation (1), see also Critchlow, 2012) on the left hand side. Note that each row contains at most two non-zero entries, since all equally ranked players are either positioned in the same rank within or not within the coalition. Moreover, each column only has one non-zero entry, since each player in the same rank in the coalition (and outside the coalition) originates from a common rank in the preference. In the bottom row, $i$ marks a fixed position in both orders. We sum up the following parts.

- For each $f \in C \cap N_i^+$ shifting from rank $p$ in the preference to $q$ in the coalition, there is no non-zero entry $n_{p'q'}$ with $p' > p$ and $q' \leq q$.

- Each $f \in N_i^+ \setminus C$ shifting from rank $p$ in the preference to $q$ outside the coalition is multiplied with the entries in the rectangle below and left of $p$ and $q$. These are non-zero for (a) each $b$ shifting from $p' < p$ (i.e., $f$ is preferred to $b$) to any $q'$ (both inside and outside the coalition) and (b) the $i$-marker in the bottom row.

This sums up to Equation (2). Similarly, for $\delta^-$, Equation (3) is obtained, see also Figure 1 right hand side.
3.4 Preference Extension

Now, we make use of this distance notion in order to define our game’s preference extension. A player \( i \in N \) weakly prefers a coalition \( A \in N_i \) to a coalition \( B \in N_i \) if and only if \( A \) is at most as far from \( i \)’s preference order as \( B \). Formally,

\[ A \succeq_i B \iff \delta(\succeq_i, A) \leq \delta(\succeq_i, B). \]

**Example 5.** Finally, for player \( i \) in Examples 1, 2, and 3, we obtain the following preferences over the example coalitions: \( F \) is \( i \)’s favourite coalition with distance 0. Coalition \( C \) is preferred to \( D \) (3 < 7) which is also reasonable since \( C \) contains more and better friends, while \( D \) has an additional enemy. Coalition \( E \) cannot be compared to \( C \) that easily with more friends, but also more enemies. The distance, however, is the same for both \( C \) and \( E \). Therefore, \( C \succeq_i E \) and \( E \succeq_i C \).

Hence, with \( \succeq = (\succeq_1, \ldots, \succeq_n) \) we obtain a hedonic game with distance-based preferences \( \langle N, \succeq \rangle \). In Section 5 we show that this model satisfies desired axiomatic properties.

4 The Model’s Background

In this section we argue the steps toward the above defined model to achieve our aim of finding a Kendall-tau-based preference extension. For a Hausdorff–Kendall-tau approach extending player \( i \)’s polarized ordinal preference encoding \( \langle \succeq_i(N_i^+ \mid \mid s_i \mid \mid N_i^-) \rangle \), we rule out the below described alternative representations. As a starting point we consider a representation of coalition \( C \) where players within and outside of \( C \) remain unranked, such as \((((C \cap N_i) \sim \mid i \mid (N_i \setminus C) \sim))\). This way the number of necessary swaps is minimal for player \( i \)’s best friend and maximal for their least preferred player. Strongly opposing the advantage of a coalition independent of the player’s preference is the following example which shows that adding a friend to a coalition can make the coalition worse.

**Example 6.** Consider player \( i \)’s preference \( \succeq_i = (a \triangleright b \triangleright c \triangleright f\mid i\mid) \). Coalition \( A = \{b, c, i\} \) is preferred to coalition \( A \cup \{f\} \).

Discarding equivalences between all candidates within and outside the coalition leads to the adjusted representation \( C = \langle \succeq_i(C \cap N_i^+) \mid \mid \succeq_i(N_i^- \setminus C) \rangle \). Two different problems arise in this setting. Firstly, still, adding a friend to a coalition can increase player \( i \)’s dissatisfaction as Example 7 shows. Note that reversing the order of players outside the coalition does not change this. Secondly, this ordering contains friends and enemies alike, thus weighting friends higher than enemies, a constraint not going in line with the idea of the model.

**Example 7.** Again consider \( \succeq_i = (a \triangleright b \triangleright c \triangleright f\mid i\mid) \). It holds that \( \{i\} \succ_i \{i, f\} \).

Resulting from this, we consider distances separately for friends and enemies. We subdivide the preference ballot \( \succeq_i \) in the same manner in order to compare ballots which contain the exact same set of players. Keeping the previous coalition representation intact,
a subdivision would lead to the coalition ballots $C_i^+ = (\geq_i (C \cap N_i^+) | i | \geq_i^+ (N_i^+ \setminus C))$ and $C_i^- = (\leq_i^-(C \cap N_i^-) | i | \leq_i^- (N_i^- \setminus C))$. Example 8 shows that adding a friend again increases the distance.

**Example 8.** For $\geq_i = (a \triangleright b \triangleright c \triangleright f | i |)$, $i$ prefers $\{a, i\}$ to $\{a, f, i\}$.

As a last step this postulates that the order of those players who are not in the desired positions is reversed in the coalition, i.e., the order of all friends who are not within the coalition and of all enemies who are within the coalition is reversed. At this point we work with the representations presented in Section 3 for which we show desirable properties in Section 5.

Apart from the representation of preferences and coalitions we make use of a directed distance measure. Next to the above discussed argument that the coalition is represented in relation to the preference, an axiomatic analysis underlines the importance of basing the model on a directed version of the Hausdorff-Kendall-tau distance. Example 9 shows, that again adding a friend to a coalition can make a coalition worse when considering the undirected distance.

**Example 9.** Consider player $i$’s preference $\geq_i = (a \sim_i b \sim_i c \sim_i d \sim_i f | i |)$. Then, player $i$ prefers coalition $\{a, i\}$ to $\{a, f, i\}$ when calculating the distance via the (undirected) Hausdorff-Kendall-tau distance.

This leads us to both the representation of coalitions and preferences defined in Section 3 as well as to the use of a directed distance.

**5 Axiomatic Analysis**

In the following, we study properties of the directed distance $\delta$ between a player’s preference and a coalition as well as the preference extension. Since these properties are from one player $i$’s perspective, we omit the index $i$ in this section.

By definition of $\rightarrow\tau$, $\delta$ naturally satisfies non-negativity and the triangle inequality. In general, the directed distance $\rightarrow\tau$ does not satisfy the identity of indiscernibles. For instance, it holds that $\rightarrow\tau(a \sim b, a \triangleright b) = 0$. Nevertheless, for each $\geq$ over $N_i$ as divided into $\geq^+$ and $\geq^-$ we obtain two unique preference orders $C^+$ and $C^-$ which represent a unique coalition $C \subseteq N_i \cup \{i\}$ (up to neutral players), such that $\delta(\geq, C) = 0$. Indeed, $C$ is $i$’s favourite coalition (up to neutral players) and contains all of $i$’s friends and none of $i$’s enemies.

**Observation 10.** The distance $\delta(\geq, C)$ between a player’s preference order $\geq$ and a coalition $C$ is 0 if and only if $N_i^+ \subseteq C$ and $N^- \cap C = \emptyset$.

Moreover, the comparability of two coalitions is efficient since we assume that for each player the number of known players is bounded by a constant. This conforms to the definition of a graphical hedonic game (Peters, 2016). In Proposition 4 we explicitly state the calculation of the underlying distances.
Observation 11. The distance between a player’s preference and a coalition can be computed in constant time.

Our model satisfies reflexivity \((A \succeq A\) for each \(A \in \mathcal{N}_i\)) and transitivity \((A \succeq B\) and \(B \succeq C\) implies \(A \succeq C\) for each \(A, B, C \in \mathcal{N}_i\)), since the comparison between the coalitions is based on the \(\leq\) relation for natural numbers. As renaming the players has no influence on the preference over coalitions, anonymity is fulfilled by definition. A further property, adapted from the definitions of citizen’s sovereignty and nonimposition in the context of (committee) elections requests the possibility for each coalition to become a player \(i\)’s favourite coalition: Nonimposition holds if for a player \(i\) and each \(C \in \mathcal{N}_i\) there exists a preference order \(\succeq\) such that \(C\) ends up as \(i\)’s most preferred coalition. Considering \(\succeq = (C \sim |i| \mathcal{N}_i \setminus C) \sim\) for some \(C \in \mathcal{N}_i\), \(C^+ = (C \sim |i|)\) and \(C^- = (|i| \mathcal{N}_i \setminus C) \sim\) equal \(\succeq^+_i\) and \(\succeq^-_i\), respectively. Hence, \(\delta(\succeq, C)\) equals 0.

Proposition 12. Hedonic games with distance-based preferences satisfy nonimposition.

5.1 Changes within the Coalition

Among the many options of how to extend a preference order over single players to an order over coalitions, there are some arguably reasonable basic rules for the comparison of two coalitions which only vary in the exchange or addition of one player (see, e.g. Barberà, Bossert, and Pattanaik, 2004). For instance, if \(i\) considers two players to be indifferent, coalitions which differ only in these players are ranked equally by \(i\).

Proposition 13. Let \(i\), \(j\), and \(k\) be players in a hedonic game with distance-based preferences with \(j \sim_i k\). Then, for all coalitions \(C \in \mathcal{N}_i \setminus \{\mathcal{N}_j \cup \mathcal{N}_k\}\), it holds that \(C \cup \{j\} \succeq_i C \cup \{k\}\) and \(C \cup \{k\} \succeq_i C \cup \{j\}\).

In the same way, the idea requires that adding a friend to a coalition never downgrades this coalition while adding an enemy never makes it more preferable.

Proposition 14. From a player \(i\)’s perspective in a hedonic game with distance-based preferences, adding a friend to a coalition always improves this coalition while adding an enemy always makes it less favourable.

Formally, a player’s increasing satisfaction with a coalition \(A \in \mathcal{N}_i\) when adding a friend to this coalition is shown by proving that \(\delta(\succeq, A) > \delta(\succeq, A \cup \{x\})\) for \(x \in \mathcal{N}_i \setminus A\). Via the characterization in Equations 2 and 3 it suffices to observe that \(-1 - |\{b \in \mathcal{N}_i \setminus x \succ b\}| < 0\). Similarly, to show that \(\delta(\succeq, A) < \delta(\succeq, A \cup \{y\})\) for \(y \in \mathcal{N}_i \setminus A\) and \(A \in \mathcal{N}\) it suffices that \(0 < 1 + |\{f \in \mathcal{N}_i \setminus f \succ y\}|\). A weak variant of both findings is also implied by the following general notion. On the basis of these properties and an extension principle for ranked sets of objects by Bosson and Schweigert (2006), Lang et al. (2015) and Kerkmann et al. (2020) define an extension principle for ordinal preference ballots with two thresholds. The general idea is that more and better friends are preferred, while more and worse enemies are less preferred.
Definition 15 (polarized responsive extension principle (Kerkmann et al., 2020)). Let \( \succeq \) be player \( i \)'s preference order over players \( N_i \). Moreover, let \( A \) and \( B \) be two coalitions in \( N_i \). The partial extension principle to preferences \( \succeq^{+0-} \) over coalitions is defined by

\[
A \succeq^{+0-} B \iff \text{there exist two injective functions}
\]

\[
\phi : B \cap N^+ \rightarrow A \cap N^+ \text{ with } \phi(j) \succeq j \text{ for } j \in B \cap N^+ \text{ and}
\]

\[
\psi : A \cap N^- \rightarrow B \cap N^- \text{ with } k \succeq \psi(k) \text{ for } k \in A \cap N^-.
\]

We show that our model is compatible with this extension principle. That is, if two coalitions \( A, B \in N_i \) satisfy \( A \succeq^{+0-} B \), it also holds that \( A \succeq B \). Note that the reverse implication is not required here, since \( \succeq^{+0-} \) allows indecisions between coalitions.

Theorem 16. The preferences \( \succeq \) of a player over coalitions in a hedonic game with distance-based preferences are compatible with \( \succeq^{+0-} \).

Proof. Let \( A \succeq^{+0-} B \) and let \( \phi \) and \( \psi \) be the two injective functions as in Definition 5.1. We want to show that \( A \succeq B \), i.e., \( \delta(\succeq, B) \geq \delta(\succeq, A) \). Firstly, by Equation (3) for the players in \( N^- \) it holds that \( \delta^-(\succeq, B) - \delta^- (\succeq, A) = |B \cap N^-| + \sum_{f \in N^- \setminus f_0} |\{b \in B \cap N^- | f \succ b| - |A \cap N^-| - \sum_{f \in N^-} |\{a \in A \cap N^- | f \succ a|\} |b \in image(\psi) | f \succ b|\}
\]

Furthermore, for each \( f \in N^- \) it holds that if some \( a \in A \cap N^- \) satisfies \( f \succ a \), then \( f \succ \psi(a) \). Hence,

\[
|\{b \in B \cap N^- | f \succ b| - |a \in A \cap N^- | f \succ a|\} |
\]

\[
= |\{b \in B \cap N^- | f \succ b, b \notin image(\psi)\}| + |\{b \in image(\psi) | f \succ b|\}
\]

\[
\geq 0 \iff |\{a \in A \cap N^- | f \succ a|\} | \geq 0.
\]

By (4) and summing up (5) over each \( f \in N^- \), we obtain \( \delta^- (\succeq, B) - \delta^- (\succeq, A) \geq 0 \). Secondly, for the players in \( N^+ \), Equation (2) can be used to argue that \( \delta^+(\succeq, B) - \delta^+(\succeq, A) \geq 0 \). Since \( |A \cap N^+| \geq |B \cap N^+|, |N^+ \setminus B| \geq |N^+ \setminus A| \) is implied. The sum over \( f \in N^+ \setminus B \) can be divided into the sum over all \( f \in N^+ \) minus the sum over \( f \in B \cap N^+ \), such that a similar argument as for \( \delta^- \) can be applied. All in all, it holds that \( \delta(\succeq, B) - \delta(\succeq, A) = \delta^+(\succeq, B) - \delta^+(\succeq, A) + \delta^- (\succeq, B) - \delta^- (\succeq, A) \geq 0 \).

5.2 Changes within the Preference

If a player’s position is changed within a preference order, it is common to assume certain monotonicity properties (see, e.g., Brandt et al., 2016). The following relaxation of monotonicity holds: If a player \( j \)'s position is improved in a preference ranking, a coalition not containing \( j \) can only outperform a previously preferred coalition containing \( j \) by the number of swaps during the improvement.
Theorem 17. Let $\succeq$ be the preference order of player $i$ over the set of players in a hedonic game with distance-based preferences, $j,x$ two players in $N_i$, and $A,B \in N_i$ two coalitions with $j \in A$ and $j \notin B$ and $A \succeq B$. If player $i$ changes their preference $\succeq$ to $\succeq'$ by shifting $j$ to a better position than $x$, it holds that $\delta^+(\succeq,B) - \delta^+(\succeq,A) \geq -\tau^*(\succeq,\succeq')$ and $\delta^-(\succeq,B) - \delta^-(\succeq,A) \geq -\tau^*(\succeq,\succeq')$.

Note that we do not obtain a difference of at most 0. This relates to a classical notion of monotonicity in which a player $j$ is improved within player $i$’s preference. At the same time it is crucial that this improvement is detached from any changes regarding other players. In our setting, however, player $j$’s improvement is immediately associated with a deterioration of at least one other player. Thus, that notion of monotonicity is not applicable.

5.3 Distinction from other Hedonic Games

The notions of $\delta^+$ and $\delta^-$ allow a certain degree of flexibility in the expressivity of hedonic games with distance-based preferences. In its current form, $\delta$ being the 1-norm of $\delta^+$ and $\delta^-$ neither specifies a tendency towards friend appreciation nor enemy aversion, but a combination of both. Either edge case can be expressed by multiplying $\delta^+$ or $\delta^-$ with an appropriate weight.

Hedonic games with distance-based preferences do satisfy additive separability. In fact, equivalent preferences can be encoded by additive utilities $u_i(a) = \delta^+(\succeq^+_i, N_i^+ \setminus \{b\})$ and $u_i(b) = \delta^-(\succeq^-_i, \{b\})$ for players $i, a \in N_i^+$, and $b \in N_i^-$. If $\delta$ is altered to, e.g., the 2-norm, this no longer holds. Then, we can express independent coalition relations such as $\{a,b,c,d,i\} \succ_i \{a,i\}$ as in Example 1, but $\{a,e,i\} \succ_i \{a,b,c,d,e,i\}$.

The model distinguishes from other known extensions of ordinal player preferences. For instance, there exists games for this model (e.g., the relation $\{i,b\} \succ_i \{a,b,c,i\}$ via $a \triangleright i \triangleright b \triangleright c$) which cannot be expressed by $B$- or $W$-preferences (Cechlárková and Romero-Medina, 2001). However, there also exist $B$-preferences ($\{a,i\} \succ_i \{a,b,i\} \succ_i \{b,i\} \succ_i \{i\}$) and $W$-preferences ($\{i\} \succ_i \{c,i\} \succ_i \{d,i\}$ and indifference between $\{d,i\}$ and $\succeq_i \{i,c,d\}$) which cannot be expressed here.

6 Stability

A frequent question for hedonic games is whether a coalition structure is stable in some sense. The best outcome would be a perfect coalition structure, where each player is in their favourite coalition. We obtain Proposition 18 for the verification and the existence problem for perfection. By Observation 10, we know that each player’s favourite coalition $F_i$ satisfies $\delta(\succeq_i, F_i) = 0$. Hence, in order to decide whether a given coalition structure is perfect, it has to be verified whether $\delta(\succeq_i, \Gamma(i)) = 0$ for each $i \in N$ which by Observation 11 can be determined in constant time. For the existence problem, the proof is constructive: Similar to a breadth first search, players are added consecutively to coalitions until they either have a conflict or form a perfect coalition structure. Since
for each player there are at most \(d\) known players, the search runs in \(O(n + d \cdot n) = O(n)\) time.

**Proposition 18.** It can be verified in \(O(n)\) time whether a given coalition structure in a hedonic game with distance-based preferences is perfect. Furthermore, it can be decided in \(O(n)\) time whether a given hedonic game with distance-based preferences allows a perfect coalition structure.

A more likely stable outcome is that of a coalition structure no player has an incentive to deviate from. Example 19 shows that a Nash-stable and an individually stable coalition structure do not always exist. However, it can be shown that a contractually individually stable coalition structure always exists. It can be seen that when a player wants to deviate and is welcome in the new coalition and not bound to the former coalition the sum of all players’ distances decreases which can reach a minimum.

**Example 19.** Consider a game consisting of five players \(a, b, c, d,\) and \(e\) with \(\succeq_a = (b \ | a \ | c \sim d), \ \succeq_b = (c \ | b \ | d \sim e),\) etc. continued rotationally symmetricly. Then for each coalition structure, it can be seen that there is always at least one player who rather wants to play alone or is welcome joining a friend. Thus, no outcome is individually stable and consequently also not Nash-stable.

The verification problem for these stability notions can be decided in polynomial time. For (contractual) individual stability, we need to distinguish between graphs which also have a bounded in-degree (players can only be known by a limited number of players and followers are encoded in the graph) and those which allow an unbounded in-degree (players can have many followers they do not know themselves).

**Theorem 20.** Given a hedonic game with distance-based preferences and a coalition structure \(\Gamma\), it can be verified whether \(\Gamma\) is

- Nash-stable in time in \(O(n)\),
- (contractually) individually stable in time in \(O(n)\) for bounded in-degree,
- (contractually) individually stable in time in \(O(n^2)\) for unbounded in-degree.

**Proof.** Due to the degree bound \(d\), there are at most \(d + 1\) possible coalitions a player \(i\) might want to deviate to, namely those containing a friend of \(i\)’s, \(\Gamma(j)\) for each \(j \in N_i^+\), or the singleton coalition. For each player, we only need to find out whether \(i\) prefers to play in one of those candidate coalitions to \(i\)’s current coalition. This can be determined in time independent from \(n\) by Observation 11. If one player prefers one candidate coalition, \(\Gamma\) is not Nash-stable.

For individual stability, it additionally needs to be verified whether the members of the new coalition welcome \(i\). There is only an impact on those players who know \(i\). For players \(j\) with \(i \in N_j^+\), \(i\) is welcome (see Proposition 14); for those with \(i \in N_j^-\), \(i\) is not welcome. We need to check whether there are any of the latter players in a candidate coalition. If \(i\)’s followers are known and the in-degree is also bounded, this can easily be verified independently from \(n\). If the incoming edges are not encoded or the in-degree
is unbounded, in the worst case we have to ask every player in the candidate coalition whether they don’t like \( i \). This requires an additional factor \( n \) for the running time. The coalition structure \( \Gamma \) is not individually stable, if one player \( i \) prefers one candidate coalition which does not contain any players who consider \( i \) an enemy.

Similarly, for contractual individual stability, it additionally needs to be verified whether \( i \) is not bound to the current coalition. The same distinction for incoming edges needs to be made. If one player \( i \) prefers one candidate coalition which does not contain any players who consider \( i \) an enemy, and \( \Gamma(i) \) does not contain any players who consider \( i \) a friend, \( \Gamma \) is contractually individually stable.

Since hedonic games with distance-based preferences are graphical hedonic games, it follows from a result by Peters (2016, Theorem 5) that if additionally the treewidth of the underlying dependency graph is bounded by a constant, it can be decided in time linear in the number of players whether a Nash-stable coalition structure exists. In general, this problem is NP-complete. The NP-hardness reduction is inspired by another result by Peters (2016, Theorem 8), adapted to our model and a degree bound of 6, but an unbounded treewidth. Note that the same proof holds for \( \delta \) being the 2-norm instead of the 1-norm.

**Theorem 21.** Given a hedonic game with distance-based preferences, it is NP-complete to decide whether a Nash-stable coalition structure exists.

**Proof.** The problem is contained in NP, since, for a chosen coalition structure, it can be verified whether it is Nash-stable in polynomial time in the number of players by Theorem 20.

For the lower bound, we consider a reduction from Exact Cover by Three Sets. We may assume that each element \( x_1, \ldots, x_n \) of an instance occurs in at most three sets \( S_1, \ldots, S_m \). Given such an instance, we create a game with \( 2^n + 4m \) players (\( a_i, b_i \) for each \( x_i \), and \( s_j, t_{j1}, t_{j2}, t_{j3} \) for each \( S_j \)) in linear time with the following preferences for \( a_i, b_i, s_j, \) and \( t_{jk} \), respectively (see also Figure 6).

- For each \( x_i \) there are two players \( a_i \) and \( b_i \),
- for each \( S_j \) there are four players \( s_j \) and \( t_{j1}, t_{j2}, \) and \( t_{j3} \),
- \( a_i \)'s preferences are \( (b_i | a_i | \{s_j | x_i \in S_j \} \sim) \),
- \( b_i \)'s preferences are \( (| b_i | \{a_i, t_{jk} | x_i \text{ is the } k \text{th element in } S_j \} \sim) \),
- \( s_j \)'s preferences are \( (\{b_i, t_{j1}, t_{j2}, t_{j3} | x_i \in S_j \} \sim | s_j |) \), and
- \( t_{jk} \)'s preferences are \( (\{t_{j\ell} | 1 \leq \ell \leq 3, \ell \neq k \} \sim | t_{jk} |) \).

It holds that the original instance allows an exact cover if and only if there exists a Nash-stable coalition structure in the constructed game. If, on the one hand, an exact cover \( C \subseteq \{S_1, \ldots, S_m\} \) exists, the coalition structure \( \{\{a_i \} | 1 \leq i \leq n\} \cup \{\{s_j, b_{j1}, b_{j2}, b_{j3} | S_j = \{x_{j1}, x_{j2}, x_{j3} \}\}, \{t_{j1}, t_{j2}, t_{j3} \} | S_j \in C\} \cup \{\{s_j, t_{j1}, t_{j2}, t_{j3} \} | S_j \notin C\} \) is Nash-stable.
In fact, no \( a_i \) has an incentive to move, since they are indifferent between playing alone and playing with \( b_i \), but also with a corresponding \( s_j \). Each \( b_i \) is as happy as possible, since they are not playing together with \( a_i \) or any \( t_{jk} \). Moreover, each \( s_j \) cannot improve, since \( s_j \) is indifferent between \( \{s_j, b_{j1}, b_{j2}, b_{j3} | S_j = \{x_{j1}, x_{j2}, x_{j3}\}\} \) and \( \{s_j, t_{j1}, t_{j2}, t_{j3}\} \).

The players \( t_{jk} \) don’t want to move, as they are in one of their favourite coalitions.

If, on the other hand, a stable coalition structure exists, we can observe the following: For each \( j, 1 \leq j \leq m \), all three \( t_{jk}, 1 \leq k \leq 3 \) have to play in one coalition, otherwise there exists one \( t_{jk} \) that has an incentive to move to another player of that group. Each \( b_i, 1 \leq i \leq n \), cannot play together with \( a_i \) or \( t_{jk} \), otherwise \( b_i \) would move to the singleton coalition. Therefore, \( s_j \) can only be in a coalition with at most three friends, either the group of \( t_{jk} \) players or corresponding \( b_i \) players. If the latter are less than three \( b_i \) players, \( s_j \) moves to the former group. Thus, \( s_j \)’s coalition contains exactly three friends. If some \( b_i \) does not play together with one corresponding \( s_j \), then \( a_i \) would move to \( b_i \)’s coalition. Hence, there exists an exact cover, namely by those \( S_j \) for which \( s_j \) plays together with those \( b_i \) with \( x_i \in S_j \).

\[ \Box \]

7 Conclusion and Future Work

We have developed a model of hedonic games based on distances between players ordinal preferences and a suitable representation of coalitions. An axiomatic and computational study indicates that this model is a reasonable completion of compact hedonic game representations. For future work, it is interesting to study this model with respect to further stability concepts, such as the core. Furthermore, we are interested in a comparison to a different preference model, e.g., relaxing the constant bound on known players. Additionally, we can extend the model to other distance norms or adapt further
ordinal distance measures. Moreover, the distance-based approach enables a comparison between games, in scenarios which require a distance between games.

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