Abstract

This paper aims at presenting the first steps towards a formulation of the Exact Renormalization Group Equation in the Hopf algebra setting of Connes and Kreimer. It mostly deals with some algebraic preliminaries allowing to formulate perturbative renormalization within the theory of differential equations. The relation between renormalization, formulated as a change of boundary condition for a differential equation, and an algebraic Birkhoff decomposition for rooted trees is explicited.

I Introduction

During the last five decades, renormalization group theory has proven to be a major discovery in theoretical physics whose applications range from high energy physics, its original birthplace, to statistical physics and dynamical systems, thanks to the work of K. Wilson. Originally proposed as a computational device in quantum field theory allowing to compare physical theories defined at different energy scales, it finally turned out to have a deep conceptual significance. Indeed, it is known since more than twenty years that renormalization group arguments allow to define the theory in the sense that one can construct a finite renormalized quantum field theory by asking it to fulfill a differential equation known as the Exact Renormalization Group Equation.

On the other side, renormalization recently triggered a couple of mathematical works that focused on algebraic aspects of the substraction procedure and of the resulting renormalization group invariance. While these works mostly focus on the BPHZ procedure formulated within the minimal substraction procedure in dimensional regularization (see for instance ref. for an application to the renormalization of the wave function), it is obvious that the framework proposed by Connes and Kreimer is versatile enough to encompass the ERGE.

This paper aims at presenting the first steps towards a formulation of the ERGE in the Hopf algebra setting. It mostly deals with some algebraic preliminaries allowing to formulate perturbative renormalization within the theory of differential equations.

In the first part, we present some general results on rooted trees and their interpretation as coefficients of formal power series of non linear operators, in analogy with the theory developed in numerical analysis under the name of B-series. In the second part, we explicit the relation between renormalization, formulated as a change of boundary condition for a differential equation, and an algebraic Birkhoff decomposition for rooted trees.
While this paper mostly presents some elementary facts, a more thorough survey involving the precise relation between trees and Feynman diagrams and their use as computational tools for effective actions will be the subject of a forthcoming publication.

II Trees and power series of non linear operators

This section presents the algebraic tools for the interpretation of the continuous renormalization group in term of Birkhoff decomposition. We introduce the well known Hopf algebra $H$ of rooted trees to generate formal power series of (non linear) operators. Recalling the isomorphism between the group of characters of $H$ and series with non zero constant term, we focus on two particular series: the geometrical serie $f_{\phi_1}[X]$ and the exponential serie $f_{\phi_e}[X]$.

II.1 The Hopf algebra of rooted trees

Most of the material is detailed in ref.[7] and refers to the original work ref.[4]. A rooted tree is a distinguished vertex, the root, together with a set of vertices and non intersecting oriented lines. Any vertex has one and only one incoming line, except the root which has only outgoing lines. The fertility of a vertex is the number of its outgoing lines, its length is the number of lines of the (unique) path that joins it to the root. Two rooted trees are isomorphic if the number of vertices with given length and fertility is the same for all possible choices of lengths and fertilities. Symbols $T$, one for each isomorphism class, together with a unit 1 corresponding to the empty tree generate a complex commutative algebra $H$ with disjoint union as a product.

A simple cut $c$ of a tree $T$ is a (non-empty) subset of its lines (selected for deletions) such that the path from the root to any other vertex includes at most one line of $c$. Deleting the cut lines produces Card($c$) + 1 subtrees: the trunk $R_c(T)$ which contains the original root and the set $P_c(T)$ of the pruned branches. The set of simple cuts of $T$ is written $C(T)$.

$$\begin{align*}
\text{simple cut} & \quad \text{simple cut} & \quad \text{non simple cut}
\end{align*}$$

**Definition II.1.** $H$ is an Hopf algebra with counit $\epsilon = 0$ except $\epsilon(1) = 1$, the antipode

$$S: \quad \bullet \mapsto - \bullet \quad T \mapsto -T - \sum_{c \in C(T)} - S(P_c(T))R_c(T)$$

$\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T), \quad \Delta(1) = 1 \otimes 1.$

**Figure 1:** Example of coproduct.

Among the common quantities associated to a tree $T$, we use the symmetry factor $S_T$ which is the number of isomorphic trees that can be generated by permutation of the branches, the number
of vertices $|T|$ (including the root) and the depth which is the maximum length of the vertices of $T$. The number of vertices is a natural graduation which makes $H$ a graded connected Hopf algebra.

The dual $H^*$ (i.e. the set of linear applications from an Hopf algebra $H$ to $\mathbb{C}$) is an algebra with the convolution product $*$

$$f * g(a) = \langle f \otimes g, \Delta a \rangle $$

for $f, g \in H^*$, $a \in H$ and $\langle , \rangle$ the duality pairing. For those $f$ in $H^*$ for which the transposition of the multiplication $m$ of $H$ takes its values in $H^* \otimes H^*$, one writes $\Delta(f) = \mathcal{t}m(f)$, i.e.

$$\Delta f(a \otimes b) = f(ab) \quad \forall a, b \in H.$$  

A derivation is an element $\delta$ of $H^*$ satisfying

$$\Delta \delta = \delta \otimes \epsilon + \epsilon \otimes \delta.$$  

The algebra generated by the derivations and $\epsilon$ is an Hopf algebra $H^*$ for the product (5) and coproduct (4) (antipode, unit and counit are obtained by transposition of antipode and unit of $H$). A character $\phi$ (algebra morphism from $H$ to $\mathbb{C}$) is a grouplike element of $H^*$

$$\Delta \phi = \phi \otimes \phi.$$  

Characters with product (6) and unity $\epsilon$ form a group $G$. The inverse is given by the antipode

$$\phi^{-1} = \phi \circ S.$$  

When $H$ is graded, with degree $|.|$, derivations are called infinitesimal characters because

$$e^\delta = \sum_{n=0}^{+\infty} \frac{\delta^n}{n!},$$

(well defined since $\delta^n(a)$ vanishes as soon as $n > |a|$) is grouplike (7). The correspondence between characters and infinitesimal characters is one to one:

$$\ln \phi = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (\phi - \epsilon)^n, \quad \phi \in G$$

is a derivation $a$ satisfying $e^{\ln \phi} = \phi$. $b$ Similarly $\ln e^\delta$ coincides with $\delta$. The set of infinitesimal characters is linearly spanned by derivations $Z^T$ that cancel everywhere but on a given generator

$$Z^T(T') \neq 0 \text{ for any } T' \neq T, \quad Z^T(1) = 0.$$  

II.2 Power series of operators

Let $\mathcal{E}$ be a Banach vector space with norm $||.||$, $\mathcal{S}(\mathcal{E})$ the set of smooth applications from $\mathcal{E}$ to $\mathcal{E}$ and $\mathcal{L}^n(\mathcal{E})$ the set of $n$-linear symmetric applications from $\mathcal{E}^n$ to $\mathcal{E}$. Take $X \in \mathcal{S}(\mathcal{E})$. For any $x \in \mathcal{E}$, there exists an infinite sequence of smooth

$$X^n_x \in \mathcal{L}^n(\mathcal{E})$$

$a$Identify

$$\Delta(\ln \phi) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \phi^n \otimes \phi^n$$

$$\ln \phi \circ \epsilon + \epsilon \circ \ln \phi = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (\phi^n \circ \epsilon + \epsilon \circ \phi^n)$$

$b$Developing $e^{\ln \phi} = \phi - \epsilon$, terms in $\phi^p$ cancel by combinatoric for $2 \leq p \leq |a|$. For $p > |a|$, $\phi^p(a)$ vanishes because for graded connected Hopf algebra $\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a' \otimes a''$ with $|a'|,|a''| < |a|$. 


such that for any $y$ in the neighborhood of $x$

$$X(x + y) = X(x) + X'(y) + \ldots + \frac{1}{n!}X^{[n]}(y, \ldots, y) + \mathcal{O}(\|y\|^{n+1}).$$  \hfill (12)

For instance when the norm is associated to real coordinates over a base $e_\mu$, $X$ is a collection of smooth functions $X^\mu$ with derivatives $X'^{\mu_\nu}\ldots$ and (summing on repeated indices)

$$X'_x(y) = X'^{\mu}(x)y^\nu e_\mu, \quad X''_x(y_1, y_2) = X'^{\mu_\nu}(x)y_1^\mu y_2^\nu e_\mu.$$  \hfill (13)

Here we intend to work with a coordinate free notation and we view $X^{[n]}$ as an element of $\mathcal{L}_n(\mathcal{S}(\mathcal{E}))$

$$X^{[n]}(X_1, X_2, \ldots, X_n) : \mathcal{E} \rightarrow \mathcal{E} \quad x \mapsto X^{[n]}(X_1(x), X_2(x), \ldots, X_n(x)).$$  \hfill (14)

One defines a formal power serie in $\hbar$ by Taylor expanding $X$ around the identity

$$X(x + \hbar Y(x)) = X(x) + \hbar X'(Y)(x) + \frac{\hbar^2}{2}X''(Y,Y)(x) + \ldots$$  \hfill (16)

with $Y \in \mathcal{S}(\mathcal{E})$. When $Y = X$, (16) can be written in a nice graphic way by using rooted trees. Explicitly one defines

$$X^\bullet = X, \quad X^\circ = X'(X)$$  \hfill (17)

$$X^{\circ\circ} = \frac{1}{2}X''(X,X), \quad X^{\bullet\bullet} = \frac{1}{3!}X'''(X,X,X)$$  \hfill (18)

so that (16) writes

$$X(\mathbb{I} + \hbar X) = X^\bullet + \hbar X^\circ + \hbar^2 X^{\circ\circ} + \hbar^3 X^{\bullet\bullet} + \ldots$$  \hfill (19)

Formally there is no reason to limit to trees of depth 1 and one associates to any $T$ the operator recursively defined by

$$X^T = \frac{1}{\prod \theta_i!}X^{[\theta_T]}(X^{T_1}, X^{T_1}, \ldots, X^{T_2}, \ldots, X^{T_m}, \ldots),$$  \hfill (20)

$$X^1 = \mathbb{I}$$  \hfill (21)

where the $T_i$’s are the subtrees of $T$ obtained by promoting as roots the vertices of $T$ of length 1 and $n = \sum_{i=1}^{m} \theta_i$ is the fertility of the root of $T$. For instance

$$X^{\circ\circ} = \frac{1}{8}X''(X'(X''(X,X)), X'(X''(X,X))).$$  \hfill (22)

Note that the numerical factor in $X^T$ is $\frac{1}{S_T}$. In the following, we use the shorthand notation

$$X^T = \frac{1}{\prod \theta_i!}X^{[\theta_T]}(X^{\theta_1}).$$  \hfill (23)

When $X$ is a linear operator, $X^{\bullet\bullet} = 0$ as soon as $n > 1$. The following construction, inspired from the theory of Butcher group, is mostly interesting for non linear operators and provide a generalization of Taylor expansion (16) by associating to any smooth $X$ a formal power serie $f[X]$

$$f[X] \doteq \sum_T f_T X^T h^{|T|}$$  \hfill (24)
where \( f_T \) are complex numbers and the sum runs over all generators (including unit 1).

The set \( \mathcal{G} \) of series with \( f_1 = 1 \) is in one to one correspondence with the characters of \( H \). Moreover \( \mathcal{G} \) with the composition of series

\[
f \circ g[X] = \sum_T f_T X^T \left( \sum_T g_T X^T h_{|T|} \right) h_{|T|}^{T} \tag{25}
\]

is a group, isomorphic to \( \mathcal{G}^{op} \), the opposite of the group of characters of \( H \). That \( \mathcal{G} \) is a group is known from the theory of Butcher group. However by using the Hopf algebraic structure of trees we present a proof of the isomorphism

\[
\mathcal{G}^{op} \sim \mathcal{G} \tag{26}
\]

that does not require to compute explicitly the coefficients of the product serie \( \mathcal{G}^{op} \), as this is done in ref. [8] for instance. The main tool for our proof is a slight adaptation to our component free notation of Caley’s relation \( 3 \) between differentials and trees. It appears indeed that the differentials of the \( X^T \)'s are easily computed in a graphic way. In the simplest cases this is immediate from definition \( 20 \), for instance

\[
X'(X^\bullet) = X' \quad \text{and} \quad X''(X^\bullet, X^\bullet) = 2X''.
\]

The differentiation process consists in grafting on a tree as many trees as the order of derivation, with a suitable numerical ponderation reflecting the symmetry of the argument.

**Lemma II.2.** For any trees \( T, T_1, ..., T_l \) and integers \( \beta_1, ..., \beta_l \)

\[
X^T[k](\underbrace{X_{T_1}, X_{T_1}, ..., X_{T_1}}_{\beta_1 \text{ times}}, \underbrace{X_{T_1}, X_{T_1}, ..., X_{T_1}}_{\beta_1 \text{ times}}) = \Pi \beta_i! \sum_{\tilde{T}} n(\tilde{T}; T, \Pi T_i^{\beta_i}) X_{\tilde{T}} \tag{28}
\]

where \( k = \sum_{i=1}^l \beta_i \) and \( n(\tilde{T}; T, \Pi T_i^{\beta_i}) \) is the number of simple cut \( c \) of \( \tilde{T} \) such that

\[
R_c(\tilde{T}) = T \quad \text{and} \quad P_c(\tilde{T}) = \Pi T_i^{\beta_i}. \tag{29}
\]

**Proof.** Let \( X \in \mathcal{S}(E), x \in E \) and \( y = y_1 + ... + y_J \) in a neighborhood of \( x \). For any collection \( \{z_i\}_{i \in \{1, n\}} \) of elements of \( E \),

\[
X_{x+y}[i] = \sum_{\beta=0}^{+\infty} \sum_{|\beta_j|=\beta} \frac{1}{\prod \beta_j!} X^{[\beta]}_x (y_j^\beta, z_i) \tag{32}
\]

\[\text{The formula can be obtained by comparing} \]

\[
X(x + y + z) = X(x + y) + \sum_{n=1}^{+\infty} \sum_{\alpha_n} \frac{1}{\prod \alpha_n} X_{x+y}^{[\alpha_n]} (z_1^{\alpha_1}) \tag{30}
\]

where \( z = z_1 + ... + z_n \) is in a neighborhood of \( x + y \), to

\[
X(x + y + z) = X(x) + \sum_{n=1}^{+\infty} \sum_{|\beta_j|=\beta} \prod \beta_j! \prod \alpha_n X_{x+y}^{[\alpha_n]} (y_j^\beta, z_i^{\alpha_n}). \tag{31}
\]

Choosing all \( \alpha_i = 1 \), the term \( X_{x+y}^{[\alpha_n]} (z_i) \) in \( 30 \) is identified to the sum of terms of \( 31 \) that are linear in each \( z_i \). For instance

\[
X_{x+y_1+y_2} (z_1) = X'_1 (z_1) + X''_1 (z_1, y_1) + X''_1 (z_1, y_2) + X''_1 (z_1, y_1, y_2) + \frac{1}{2} X'''_1 (z_1, y_1, y_1) + \frac{1}{2} X'''_1 (z_1, y_2, y_2) + ...
\]
where the second sum runs on all the configurations of the $\beta_j$’s such that $\sum_{i=1}^{J} \beta_j = \beta$ and we use notation similar as (33). Then

$$X^T(x + y) = \frac{1}{\prod \theta_i!} X_x^{[n]} (X_i^\theta (x + y))$$

(33)

$$= \frac{1}{\prod \theta_i! \sum_{\beta = 0}^{+\infty} \sum_{|\beta_j| = \beta} \frac{1}{\prod \beta_j!} X_x^{[n + \beta]} (y_j^{\beta_j}, X_i^\theta (x + y)).} \tag{34}$$

Inserting

$$X_i(x + y) = X_i(x) + \sum_{r=1}^{+\infty} \sum_{|\beta_j| = \beta} \frac{1}{\prod \beta_j!} X_x^{[r]} (y_j^{\beta_j}) \tag{35}$$

and choosing $\{y_j\}$ of cardinality $J = k$, one identifies $X^T_x^{[k]}(y_j)$ as the terms of (34) linear in each $y_j$, namely those appearing with either $\beta_j$ or $\beta'_j = 1$. Explicitly, for a tree $T$ with depth 1,

$$X^T(X^{\tilde{T}_1}) = \frac{1}{m!} X_x^{[n+1]} (X^{\tilde{T}_1}, X^n) + \frac{1}{(n-1)!} X_x^{[n]} (X' (X^{\tilde{T}_1}), X^{n-1})$$

$$= \sum_{\tilde{T}} n(\tilde{T}; T, \tilde{T}_1) X^{\tilde{T}} \tag{38}$$

for any $\tilde{T}_1$. Similarly for any $k$ and distinct $T_j$’s one obtains

$$X^T_x^{[k]}(X^{\tilde{T}_1}, ..., X^{\tilde{T}_k}) = \sum_{\tilde{T}} n(\tilde{T}; T, \Pi \tilde{T}_j) X^{\tilde{T}}. \tag{39}$$

When some of the $\tilde{T}_j$’s are identical, the extra factor $\frac{1}{\beta_j}$ is absorbed by the definition of $\tilde{T}$ but such terms have to be identified as part of

$$\frac{1}{\beta_j} X^T_x^{[\beta_j]} (X^{\beta_j}). \tag{40}$$

Hence the lemma for any $T$ of depth 1. The final result is obtained recursively on the depth of $T$. ■

**Proposition II.3.** $\mathcal{G}$ is a group, isomorphic to $G^{op}$.

**Proof.** Since $\mathcal{G}$ is in bijection with $G$, we just have to show that

$$\sum_T \phi(T) X^T \left( \sum_{T'} \psi(T') X^{T'} h^{[T]} \right) h^{[T]} = \sum_T (\psi * \phi)(T) X^T h^{[T]} \tag{40}$$

for any $\phi, \psi \in G$. In the l.h.s. $\phi$ is evaluated on generators only. By (40) one checks that the same is true for the r.h.s. Moreover both sides are linear in $\phi$ so this is enough to work with the infinitesimal character

$$\phi = Z^{T_0} \tag{41}$$

$$X^{\phi^{X^\phi}} (X^\phi) = \frac{1}{2} (X^\nu_{\nu\rho} X^\rho X^\nu_{\rho\mu} + \frac{1}{2} X^\nu_{\nu\rho} X^\rho X^\phi e_\mu + \frac{1}{2} X^\nu_{\nu\rho} X^\rho X^\phi X^\delta e_\mu + \frac{1}{2} X^\nu_{\nu\rho} X^\nu_{\delta\rho} X^\phi X^\delta e_\mu) \tag{36}$$

$$= 3X^{\phi^{X^\phi}} + X^{\phi^{X^\phi}} \tag{37}$$

for instance with notation (15)
for a fixed $T_0$. Note that a similar simplification is not available for $\psi$ since $P_c(T)$ may not be a single generator. Then r.h.s. of (40) reduces to
\[ X^{T_0} h^{|T_0|} + \sum_{T \in \tilde{C}(T)} \psi(P_c(T)) X^T h^{|T|} \] (42)
where $\tilde{C}(T)$ is the set of simple cuts of $T$ such that $R_c(T) = T_0$. The l.h.s of (40) developed thanks to (19) writes
\[ X^{T_0} h^{|T_0|} + \sum_{T_1} \psi(T_1) X^{T_1} h^{|T_1|} + \ldots + \sum_{\{T_i^{\beta_i}\}} \prod \psi^{\beta_i} X^{T_0[m]}(T_i^{\beta_i}) h^{\sum |T_i|} + \ldots \] (43)
where sums run over generators distinct from 1 and we use a similar notation as in (23). Thanks to lemma II.2, (43) = (42).

Thanks to this isomorphism one can easily compute some interesting series. Let us write
\[ f_\phi[X] \] (44)
the formal power serie associated to a character $\phi$ by the isomorphism III.3. We define the exponential serie as the one corresponding to the character
\[ \phi_e = e^{Z} \] (45)
defined by (5) and (10). Since (see ref.[4] for the definition of the factorial $T!$ of a tree)
\[ \phi_e(T) = \frac{1}{T!} \] (46)
the exponential serie writes
\[ f_{\phi_e}[X] = \sum_T \frac{1}{T!} h^{|T|} X^T. \] (47)
Another nice example is the geometrical serie
\[ (1 - hX)^{-1} = f_{\phi_1}[X] = \sum_T h^{|T|} X^T \] (48)
obtained by noting that the constant character
\[ \phi_1 : T \mapsto 1 \quad \forall T \] (49)
is the inverse of the character $T \mapsto 0$ except $1 \mapsto 1$, $\bullet \mapsto -1$.

III  Fixed point equation and algebraic Birkhoff decomposition

The geometrical and exponential series III.3, III.4 are interesting to solve fixed point equation. In particular if we view a fixed point equation as the integral form of a differential equation then the change of initial condition (corresponding to the change of the constant in the fixed point equation) is coded into the group $G_2$ of characters of the Hopf algebra of decorated rooted trees. Namely if $\phi \in G_2$ represents the solution for initial condition $x_0$ and $\phi_+ \in G_2$ corresponds to the solution for initial condition $x_1$, then the decomposition
\[ \phi_+ = \phi_- * \phi \] (50)
turns out to define an algebraic Birkhoff decomposition (see definition III.1) similar as the one introduced by Connes and Kreimer in their seminal paper ref.[4].

Using trees to solve fixed point and differential equations is at the heart of the theory of $B$-series (see ref.[1] for a nice overview of such applications). However, to the knowledge of the authors, the interpretation in terms of Birkhoff decomposition is not found in the litterature.
III.1 Change of initial conditions in fixed point equations

Let us start by the fixed point equation

\[ x = x_0 + \hbar X(x) \] (51)

where \( X \in \mathcal{S(E)} \), \( \hbar \) is small and \( x_0 \in \mathcal{E} \). (51) is formally solved by writing

\[ x = (\mathbb{I} - \hbar X)^{-1}(x_0) \] (52)

that is to say, thanks to (48),

\[ x = f_{\phi_1}(X)(x_0) = \sum_T^{|T|} \hbar^{||T||} X^T(x_0). \] (53)

This is the same serie as the one found by recursively developing \( x = x_0 + \hbar X(x_0 + \hbar X(X_0 + ...) \).

Consider now a curve \( x : \mathbb{R} \to \mathcal{E} \) given by

\[ x(t) = x_0 + \hbar \int_{t_0}^t X(x(u))du \] (54)

for some \( X \in \mathcal{S(E)} \). This is the integral form of the differential equation

\[ \frac{dx}{dt} = hX(x), \ x(t_0) = x_0. \] (55)

By recursively developing

\[ x(t) = x_0 + \hbar \int_{t_0}^t X \left( x_0 + \hbar \int_{t_0}^{t_1} X(x_0 + \int_{t_0}^{t_2} ...) dt_2 \right) dt_1 \] (56)

one finds

\[ x(t) = \sum_T^{1} \frac{1}{T!} (t - t_0)^{|T|} \hbar^{||T||} X^T(x_0) \] (57)

that is to say, according to (47),

\[ x(t) = f_{\phi_e[X_0]}(x_0) \] (58)

where

\[ X_0 = (t - t_0)X. \] (59)

One may be tempted to solve formally by writing

\[ x(t) = e^{\hbar X_0}(x_0). \] (60)

The genuine definition of the exponential of an operator \( e \) is not compatible with (58). However defining

\[ e^{\hbar X_0} = f_{\phi_e[X_0]} \] (61)

ensures that (58) equals (60). In other terms rooted trees provide a definition of the exponential of a (non linear) operator which is compatible with the resolution of fixed point equations.

The same differential equation can be directly solved in its integral form (54) by considering the Banach vector space \( \mathcal{E}' \) of curves in \( \mathcal{E} \) and the operator \( \chi \in \mathcal{S(E')} \) defined by

\[ \chi(x) : t \mapsto \int_{t_0}^t X(x(u))du. \] (62)
Considering \( x_0 \) as the curve \( t \mapsto x_0 \), (54) reads as the fixed point equation
\[
x = x_0 + \hbar \chi(x)
\] (63)
whose formal solution is given by (53)
\[
x = (I - \hbar \chi)^{-1}(x_0) = f_{\phi_1}[\chi](x_0).
\] (64)

Trees are especially useful to deal with change of initial condition. Let us fix
\[
x(t_1) = x_1
\] (65)
for a given \( x_1 \in \mathcal{E} \) and \( t_1 \neq t \). Solution (58) becomes
\[
x(t) = f_{\phi_e}[X_1](x_1)
\] (66)
where
\[
X_1 = (t - t_1)X.
\] (67)
In order to keep a trace of the solution with initial condition at \( t_0 \), let us write (66) for \( t = t_0 \),
\[
x_0 = f_{\phi_e}[X_1 - X_0](x_1),
\] (68)
which, inserted in (58)
\[
x(t) = f_{\phi_e}[X_0] \circ f_{\phi_e}[X_1 - X_0](x_1),
\] (69)
and compared to (66) yields
\[
f_{\phi_e}[X_1] = f_{\phi_e}[X_0] \circ f_{\phi_e}[X_1 - X_0].
\] (70)
A similar decomposition can be found for the integral form (54) of the differential equation. The solution for initial condition (67) is
\[
x = f_{\phi_1}[\zeta](x_1)
\] (71)
with
\[
\zeta(x) : t \mapsto \int_{t_1}^{t} X(x(u))du.
\] (72)
Factorizing
\[
(I - \hbar \zeta)^{-1} = (I - \hbar \chi)^{-1} \circ (I - \hbar \xi)^{-1}
\] (73)
where we define
\[
\hbar \xi = I - (I - \hbar \zeta) \circ (I - \hbar \chi)^{-1},
\] (74)
yields by (48)
\[
f_{\phi_1}[\zeta] = f_{\phi_1}[\chi] \circ f_{\phi_1}[\xi].
\] (75)

Both (70) and (75) involves a unique character \( \phi_1 \) or \( \phi_e \) and three distinct operators \( \chi, \zeta, \xi \) or \( X_0, X_1, X_1 - X_0 \). To fix notations we now focus on the decomposition of geometrical series (75) but the following is also true for exponential series. In order to take advantage of the isomorphism (26) one needs series involving the same operator. This can be obtained by considering the group \( G_2 \) of characters of the Hopf algebra \( H_2 \) of decorated rooted trees, i.e. trees whose vertices carry a decoration chosen in a set of cardinality two, in our case of interest either a bullet or a square. To any generator \( T \) of \( H_2 \), one associates \( Y^T \in \mathcal{S}(\mathcal{E}') \) defined in a similar manner as in (20) with the extra rule that bullets are associated to \( \xi \) and squares to \( \zeta \). For instance
\[
Y^\bullet = \zeta, \quad Y^\bullet = \xi
\] (76)
\[
Y^\circ = \zeta''(\xi, \xi).
\] (77)
Let $G_2$ be the group of power series $1 + \sum_T f_T Y^T$. A straightforward adaptation of proposition II.3 shows that

$$G_2 \sim G_2^{op}.$$  \hfill (78)

Let us define two characters of $H_2$

$$\phi_-(T) \doteq \begin{cases} \phi_1(T) & \text{if } T \in H_\bullet, \\ 0 & \text{if } T \notin H_\bullet, \end{cases}, \quad \phi_+(T) \doteq \begin{cases} \phi_1(T) & \text{if } T \in H_\blacksquare, \\ 0 & \text{if } T \notin H_\blacksquare, \end{cases}$$

(79)

where $H_\bullet$ is the set of trees decorated with bullets only and $H_\blacksquare$ those decorated by square only. Then

$$f_{\phi_1}[\xi] = f_{\phi_+}[Y], \quad f_{\phi_1}[\xi] = f_{\phi_-}[Y]$$

are both elements of $G_2^{op}$. By (75) and (78)

$$f_{\phi_1}[\chi] = f_{\phi}[Y]$$

(81)

for the character $\phi \in G_2$ given by \footnote{Remembering (59), $\phi_1^{-1}$ vanishes on all generators of $H_2$ except $\phi_1^{-1}(1) = 1, \phi_1^{-1}(\bullet) = -1$. By (9) one obtains}

$$\phi = \phi_1^{-1} * \phi_+.$$  \hfill (83)

(75) can be written with a unique operator $Y$ and three characters

$$f_{\phi_+}[Y] = f_{\phi} \circ f_{\phi_-}[Y].$$

(84)

(70) can be written as well, by associating bullet to $X_1 - X_0$, square to $X_1$ and defining $\phi_{\pm}$ according to $\phi_e$ instead of $\phi_1$.

(83) is the announced equation (61), establishing that the change of initial condition in a fixed point equation is coded into the group of characters of the algebra of decorated rooted trees. In next section, we show that (83) (or equivalently (84) by identifying characters and series) allows to define a so called algebraic Birkhoff decomposition.

### III.2 Algebraic Birkhoff decomposition

A decomposition of the same kind as (83) appears in renormalization of quantum field theory with minimal substraction scheme and dimensional regularization\footnote{10} the bare theory gives rise to a loop

$$\gamma(z) \in G_F, \quad z \in \mathcal{C}$$

(85)

where $\mathcal{C} \subset \mathbb{C}$ is a small circle around the dimension $D$ of space time and $G_F$ is the group of characters of the Hopf algebra of Feynman diagrams $H_F$. The renormalized theory is the evaluation at $z = D$ of the holomorphic part $\gamma_+$ of the Birkhoff decomposition of $\gamma$. To give a similar interpretation to our decomposition (83), we need to adapt to rooted trees some of the tools of ref.\footnote{10} initially developed for Feynman diagrams.

An important feature is the commutative algebra $\mathcal{A}$ of smooth functions meromorphic inside $\mathcal{C}$ with pole only at $D$. Also important are the subalgebras $\mathcal{A}_+ \subset \mathcal{A}$ of functions holomorphic inside $\mathcal{C}$, and $\mathcal{A}_- \subset \mathcal{A}$ the subalgebra of polynomial in $\frac{1}{z-D}$ without constant term. There exists a projection

$$p_- : \mathcal{A} \rightarrow \mathcal{A}_-$$

(86)

where $n_T = 1$ if there is a simple cut such that $R_e(T) \in H_\blacksquare$ and $P_e(T) = \bullet, n_T = 0$ otherwise.
parallel to \( A_+ \), i.e.
\[
\text{Ker} \; p_- = A_+.
\] (87)

Feynman rules (ponderated by a suitable mass factor) yields an algebra homomorphism
\[
U : H_F \rightarrow A.
\] (88)

The counterterms are given by the algebra homomorphism
\[
C(X) = -p_+ \left( U(X) + \sum C(X')U(X'') \right)
\] (89)
where
\[
\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X'', \quad X \in \tilde{H}_F \doteq \text{Ker} \; \epsilon,
\] (90)
while the renormalized theory is given by the homomorphism
\[
R(X) = (C \ast U)(X).
\] (91)

One checks that
\[
C(\tilde{H}_F) \subset A_-, \quad R(\tilde{H}_F) \subset A_+.
\] (92)

Equation (91) viewed as an equality between algebra homomorphisms,
\[
C \ast U = R,
\] (93)

is called the algebraic Birkhoff decomposition of \( U \). Such a terminology is justified by considering the \( G_F \)-valued loops
\[
\gamma(z) \doteq \chi_z \circ U, \quad \gamma_-(z) = \chi_z \circ C, \quad \gamma_+(z) = \chi_z \circ R
\] (94)
where
\[
\chi_z(f) \doteq f(z) \quad \forall f \in A.
\] (95)

Indeed (91) indicates that
\[
\gamma_+(z) = \gamma_-(z) \gamma(z) \quad \forall z \in \mathcal{C}
\] (96)
for the pointwise product in \( G \). One then shows that (91) is precisely the Birkhoff decomposition of \( \gamma \). Namely, viewing \( \mathcal{C} \) as a subset of the Riemann sphere \( \mathbb{C} \mathbb{P}_1 \), \( \gamma_- \) extends to a \( G_F \)-valued holomorphic maps on \( \mathcal{C}_- \) (the component of the complement of \( \mathcal{C} \) containing \( \infty \)) with \( \gamma_-(\infty) = 0 \) while \( \gamma_+ \) extends to \( G_F \)-valued holomorphic maps on \( \mathcal{C}_+ \) (the other component of the complement of \( \mathcal{C} \)). The reader is invited to consult ref.[4] for a precise definition of the Birkhoff decomposition and its link to the Riemann-Hilbert problem. Let us simply recall that the replacement of \( \gamma \) by \( \gamma_+ \) is a natural principle to extract a finite value from the singular expression \( \gamma(z) \).

The use of the algebra of meromorphic functions on \( \mathcal{C} \) is intimately linked to dimensional regularization scheme. In the framework of the continuous renormalization group, there is no pole given by the dimension of space time. However, given a decomposition of algebra homomorphisms such as (93), satisfying conditions (92), it still makes sense to talk of Birkhoff decomposition, but in the following algebraic sense (taken from ref.[13] or [10]).

**Definition III.1.** Let \( H \) be a commutative Hopf algebra, \( A \) a commutative algebra with a projection \( p_- \) on a subalgebra \( A_- \). An algebra homomorphism \( \gamma : H \rightarrow A \) has an algebraic Birkhoff decomposition if there exists two algebras homomorphisms \( \gamma_+ \), \( \gamma_- \) from \( H \) to \( A \) such that
\[
\gamma_+ \equiv \gamma_- \ast \gamma
\] (97)
where \(*\) is the convolution product \([3]\) and
\[
P_{+} \gamma_{+} = \gamma_{+}, \quad P_{-} \gamma_{-} = \gamma_{-}
\]
(98)
where \(P_{+}\) is the projection on
\[
A_{+} = \text{Ker} \, p_{-}.
\]
(99)

To interpret [33]-[34] as an algebraic Birkhoff decomposition one may be first tempted to consider the algebra of polynomials in \(Y^{T}\) as the equivalent in the continuous renormalization framework of the meromorphic functions. As well characters may be understood as the equivalent of the homomorphisms defined by Feynman rules: in the same way that \(U, C, R\) map a Feynman diagram to a meromorphic function, characters map a decorated rooted tree to a monomial in \(Y^{T}\),
\[
\Phi(T) \doteq \phi(T)Y^{T}, \quad \Phi_{\pm}(T) \doteq \phi_{\pm}(T)Y^{T}.
\]
(100)
Unfortunately \(\Phi, \Phi_{\pm}\) do not define an algebraic Birkhoff decomposition. For instance (see the proof of proposition [11.2] for the computation of \(\phi\))
\[
\Phi_{+}(\circ \bullet) = 0
\]
(101)
\[
(\Phi_{-} \ast \Phi)(\circ \bullet) = \langle \Phi_{-} \otimes \Phi, 1 \otimes \circ \bullet + \bullet \otimes 1 + \bullet \otimes \bullet \rangle
\]
(102)
\[
= -Y^{\circ} + Y^{\circ}Y^{\bullet}.
\]
(103)
A solution would be to define the product of monomials so that \(103\) vanishes,
\[
Y^{\bullet}Y^{\circ} = Y^{\bullet}
\]
(104)
but such a product would not be commutative. However by considering the (commutative) formal product of decorations, it appears that the sum (ponderated by the numerical coefficient) of the products of decorations in \(103\) vanishes (i.e. \(-\circ \bullet + \circ \bullet = 0\)). Consequently we propose a Birkhoff decomposition with value on the the algebra \(A\) of decorations, that is to say the free unital algebra generated by square and bullet
\[
A = \{1, \bullet, \circ\}.
\]
(105)
For \(A_{-}\) we choose the unital subalgebra of \(A\) generated by the bullet
\[
A_{-} = \{1, \bullet\}.
\]
(106)
We note \(p_{-}\) the projection \(A \rightarrow A_{-}\)
\[
p_{-}(1) = 1, \quad p_{-}(\bullet) = \bullet, \quad p_{-}(\circ) = 0
\]
(107)
extended to all \(A\) by algebra homomorphism
\[
p_{-}(\bullet^{n} \circ \bullet^{m} + \bullet^{n'} \circ \bullet^{m'}) = \bullet^{n} + \bullet^{n'}.
\]
(108)
\(A_{+}\) and \(p_{+}\) are defined by [39]. Let \(\Gamma\) be the algebra homomorphism \(H_{2} \rightarrow A\)
\[
\Gamma(1) = 1, \quad \Gamma(T) = |T|_{\circ} |T|_{\bullet}
\]
(109)
where \(|T|_\circ\) is the number of bullets of \(T\) and \(|T|_\square\) is the number of squares. \(\Gamma\) just "counts the decorations", for instance

\[
\Gamma(\begin{array}{c}
\circ \\
\square
\end{array}) = 3. \quad (110)
\]

Finally we define three algebra homomorphisms from \(H_2\) to \(A\),

\[
\gamma(T) = \phi(T)\Gamma(T), \quad \gamma_\pm(T) = \phi_\pm(T)\Gamma(T). \quad (111)
\]

**Proposition III.2.** \(\gamma_+ = \gamma_- * \gamma\) is an algebraic Birkhoff decomposition.

**Proof.** Let \(\overline{H}_\bullet\) be the algebra of trees with bullets only (and similarly \(\overline{H}_\square\)). Then

\[
\begin{align*}
\gamma_-(T) &= \begin{cases} 
\bullet |T|_\circ & \text{for } T \in \overline{H}_\bullet \\
0 & \text{otherwise,}
\end{cases} \\
\gamma_+(T) &= \begin{cases} 
\bullet |T|_\circ & \text{for } T \in \overline{H}_\bullet \\
0 & \text{otherwise,}
\end{cases}
\end{align*} \quad (112)
\]

so that \((\ref{eq:gamma_0})\) is satisfied, as well as \((\ref{eq:gamma_1})\) by construction. By algebra homomorphism, \((\ref{eq:gamma_2})\) has to be checked only on generators. Let us first note that

\[
\gamma(T) = \begin{cases} 
\bullet |T|_\circ & \text{for } T \in \overline{H}_\bullet \\
-\bullet & \text{for } T = \bullet \\
-n_T \bullet |T|_\circ \bullet |T|_\circ & \text{otherwise}
\end{cases} \quad (113)
\]

where \(n_T\) is defined in \((\ref{eq:gamma_2})\). Second, for any \(T \neq 1\)

\[
(\gamma_- * \gamma)(T) = \gamma(T) + \sum_{c \in C(T)} \gamma_-(P_c(T))\gamma(R_c(T)). \quad (114)
\]

This is then not difficult to check \((\ref{eq:gamma_2})\) for \(T \in \overline{H}_\bullet\) or \(T \in \overline{H}_\square\).

For \(T \notin \overline{H}_\bullet \cup \overline{H}_\square\),

\[
(\gamma_- * \gamma)(T) = -n_T \bullet |T|_\circ \bullet |T|_\circ + \sum_{c \in C(T)} \gamma_-(P_c(T))\gamma(R_c(T)). \quad (115)
\]

Assume \(n_T = 0\). Then \((\ref{eq:gamma_2})\) is non zero only if there is at least either a simple cut \(c\) such that

\[
P_c(T) \in \overline{H}_\bullet \text{ and } R_c(T) \in \overline{H}_\square \quad (116)
\]

or a simple cut \(\tilde{c}\) with

\[
P_{\tilde{c}}(T) \in \overline{H}_\bullet \text{ and } n_{R_{\tilde{c}}(T)} \neq 0. \quad (117)
\]

Assume there exists a simple cut \(c\) for which \(P_c(T)\) is a single tree. Then there exist a \(\tilde{c}\) in which \(R_{\tilde{c}}(T)\) is the subtree of \(T\) consisting in \(R_c(T)\) and the root of \(P_{\tilde{c}}(T)\) while \(P_{\tilde{c}}(T)\) is the union of all the subtrees of \(P_c(T)\) obtained by promoting as roots the vertices of length 1 (\(P_{\tilde{c}}(T) \neq \emptyset\) because \(n_T = 0\)). The simple cut \(c\) contributes to \((\ref{eq:gamma_2})\) with a factor

\[
\bullet |P_c(T)|_\bullet |R_c(T)|_\square \quad (118)
\]

whereas \(\tilde{c}\) contributes with a factor

\[
\bullet |P_{\tilde{c}}(T)|_\bullet |R_{\tilde{c}}(T)|_\square. \quad (119)
\]

This is easy to observe that \((\ref{eq:gamma_2}) = - (\ref{eq:gamma_3})\). The same is true if \(P_c(T)\) is a product of \(m\) trees, the only difference being that the contribution of \(c\) is cancelled by the sum of the contributions of the \(m\) \(\tilde{c}\) obtained by grafting alternatively to \(R_c(T)\) each of the roots of \(P_c(T)\). Also, starting from a
\( \hat{c} \), one would similarly notice that its contribution is cancelled by \( c \) so that, finally, (115) vanishes for any \( T \not\in H_\bullet \cup H_\bullet^\circ \) with \( n_T = 0 \). Hence (116).

The same procedure applies when \( n_T \neq 0 \). The term of (115) in \( \gamma(T) \) is cancelled by the sum of the \( n_T \) terms corresponding to the simple cuts with \( P_c(T) = \bullet \).

As announced in (50), we have shown that the change of initial condition in a fixed point equation does correspond to the algebraic Birkhoff decomposition of the algebra morphisms associated by (111) to the characters encoding the solutions. By convention we say that \( \gamma_+ \) is the positive part of the algebraic \( A \)-valued Birkhoff decomposition of \( \gamma \).

### III.3 Continuous renormalization group

Birkhoff decomposition has been introduced in renormalization of quantum field theory by Connes and Kreimer in the framework of minimal substraction scheme and dimensional regularization. Our algebraic Birkhoff decomposition (83) has an interpretation in the framework of the continuous renormalization group of Wilson (20) (see also ref. [16] for a nice introduction). The latest describes the evolution of the parameters of a quantum field theory under a change of the observation scale. When the rescaling is parametrized by a continuous quantity, say the energy, the evolution is governed by flow equation

\[
\Lambda \frac{\partial}{\partial \Lambda} S = \beta(\Lambda, S)
\]

in which \( \Lambda \in \mathbb{R}^{*+} \) is the scale and \( S = S(\Lambda) \) implements the parameters (mainly \( S \) is a functional of the fields and their coupling constants). The initial conditions are encoded by the knowledge of \( S_0 = S(\Lambda_0) \) at a given scale \( \Lambda_0 \). A typical example of continuous renormalization group equations (120) are Polchinski’s equations (15) which gives \( \beta(\Lambda, S) \) for a theory with IR cutoff.

Let us consider the most general context by simply assuming that the theory is described by a smooth operator \( S \)

\[
S : \Lambda \mapsto S(\Lambda) \in \mathcal{E}
\]

where \( \mathcal{E} \cong S(\mathcal{H}) \) and \( \mathcal{H} \) is the (infinite dimensional) vector space spanned by vectors labelled with the parameters of the theory. A scale transformation

\[
\Lambda \to \Lambda^s
\]

with \( s \in \mathbb{R}^{*+} \) induces the transformation (122)

\[
S(\Lambda) \to S(\Lambda^s) = s^\Delta S(\Lambda)
\]

where \( \Delta \in \mathcal{E} \) is the diagonal matrix whose coefficients are the dimensions of the parameters. We define the dimensionless quantities \( x(\Lambda) \in \mathcal{E} \) and \( t \in \mathbb{R} \),

\[
S(\Lambda) = \Lambda^\Delta x(\Lambda), \quad t = \ln(\frac{\Lambda}{\mu})
\]

where \( \mu \) is a parameter of the theory with same dimension as \( \Lambda \). Then (120) yields

\[
\Lambda \frac{\partial x}{\partial \Lambda} = -\Delta x + \Lambda^{-\Delta} \beta(\Lambda, \Lambda^\Delta x).
\]

By dimensional analysis, \( \beta \) is transforming homogeneously under change of scale,

\[
\beta(\Lambda, \Lambda^\Delta x) = \Lambda^\Delta \beta(1, x)
\]

(122)
so that, writing \( hX(x) = \beta(1, x) \) and \( D = -\Delta \),

\[
\frac{\partial x}{\partial t} = Dx + hX(x)
\]

(123)

whose integral form, with initial condition \( x_0 = \Lambda_0^D S_0 \), is

\[
x(t) = e^{(t-t_0)D}x_0 + \hbar \int_{t_0}^t e^{(t-u)D}X(x(u))du.
\]

(124)

Viewing \( x \) as an element of \( \mathcal{E}' \), Banach vector space of applications from \( \mathbb{R}^+ \) to \( \mathcal{E} \), we define

\[
\tilde{x}_0 \in \mathcal{E}' : t \mapsto e^{(t-t_0)D}x_0
\]

(125)

and \( \chi \in \mathcal{S}(\mathcal{E}') \),

\[
\chi(x) : t \mapsto \int_{t_0}^t e^{(t-u)D}X(x(u))du \quad \forall x \in \mathcal{E}'
\]

(126)

so that (124) reads as the fixed point equation

\[
x = \tilde{x}_0 + \hbar \chi(x).
\]

(127)

In the context of Wilson’s continuous renormalization group, \( \Lambda_0 \) is interpreted as an UV cutoff and one is interested in the limits of very high energy scale, i.e. \( t_0 \to +\infty \). However \( \tilde{x}_0 \) is already ill-defined since

\[
e^{(t-t_0)D}x_0 \begin{cases} \text{converges on } \mathcal{H}^+ & \text{as } t_0 \to +\infty \\ \text{is constantly zero on } \mathcal{H}^0 & \\ \text{diverges on } \mathcal{H}^- \end{cases}
\]

(128)

where \( \mathcal{H}^+, \mathcal{H}^0, \mathcal{H}^- \subset \mathcal{H} \) are the proper subspaces of \( D \) corresponding to positive, zero and negative eigenvalues (the corresponding parameters are called relevant, marginal and irrelevant). \( \chi \) as well is ill defined, even in the relevant sector. A solution to ensure the finiteness of \( x(t) \) at high scale consists in fixing the initial conditions for the irrelevant sector at scale \( t_1 \) (equivalently: at scale \( \Lambda_1 \)) distinct from \( t_0 \). Namely one imposes the mixed boundary conditions

\[
x_R = P\tilde{x}_1 + (\mathbb{I} - P)\tilde{x}_0
\]

(129)

where \( P \) is the orthogonal projection from \( \mathcal{H} \) to \( \mathcal{H}^- \) and

\[
\tilde{x}_1 \in \mathcal{E}' : t \mapsto e^{(t-t_1)D}x_1
\]

(130)

Defining

\[
\rho(x) : t \mapsto \int_{t_1}^t e^{(t-u)D}X(x(u))du
\]

(131)

and

\[
\zeta = P\rho + (\mathbb{I} - P)\chi
\]

(132)

allows to write (127) with initial condition \( x_R \)

\[
x(t) = x_R + \hbar \zeta(x).
\]

(133)

This is a well known result that \( x(t) \) computed with mixed boundary condition, namely

\[
x(t) = \int_{t_1}^t [\zeta](x_R)
\]

(134)

remains finite at high energy scale and does not depend on \( x_0 \). Trees have already been used to prove this result (see ref.\cite{9} for instance). To be complete we propose here a simple proof taking advantage of the Hopf algebraic structure.
Proposition III.3. \( \lim_{t_0 \to +\infty} f_{\phi_1}[\zeta](x_R) \) is finite order by order and does not depend on \( x_0 \).

Proof. The idea is to use decorated trees in a similar way as \( (76) \) except that the rule now is

\[
Y^\bullet = \zeta_1 \doteq P \rho, \quad Y^\circ = \zeta_2 \doteq (\mathbb{I} - P) \chi.
\]

Then

\[
\zeta^\circ = (\zeta_1 + \zeta_2)'(\zeta_1 + \zeta_2) = (\zeta_1)'(\zeta_1) + (\zeta_2)'(\zeta_2) + (\zeta_1)'(\zeta_2) + (\zeta_2)'(\zeta_1) = Y^\circ + Y^\bullet + Y^\circ + Y^\bullet
\]

\[
= \sum_{T \in \{\bullet\}_2} Y^T \quad (139)
\]

where \( \{T\}_2 \) is the set of all decorated trees obtained by decoration of the vertices of \( T \in H \), e.g.

\[
\{\bullet\}_2 = \{\bullet, \square\}. \quad (140)
\]

It is not difficult to check that the same is true for any \( T \in H \),

\[
\zeta^T = \sum_{T \in \{T\}_2} Y^T \quad (141)
\]

Thus

\[
f_{\phi_1}[\zeta] = \sum_{T \in H_2} Y^T \quad (142)
\]

Since

\[
\lim_{t_0 \to +\infty} Y^1_1(x_R) = \lim_{t_0 \to +\infty} x_R = \bar{x}_1 \quad (143)
\]

and \( Y^\bullet \) does not depend on \( t_0 \) then

\[
\lim_{t_0 \to +\infty} Y^\bullet(x_R) = Y^\bullet(\bar{x}_1). \quad (144)
\]

Similarly for any \( T \in H_\bullet \), \( \lim_{t_0 \to +\infty} Y^T(x_R) \) is both \( x_0 \) and \( t_0 \) independent. In the same way

\[
\lim_{t_0 \to +\infty} Y^\bullet(x_R) = \lim_{t_0 \to +\infty} (\mathbb{I} - P) \int_{t_0}^t e^{(t-u)P} X(\bar{x}_1(u)) du \quad (145)
\]

is finite because \( X \) is smooth. Similarly for any \( T \in H_\bullet \). For \( T \) decorated with both bullets and squares, combination of \( (144) \) and \( (145) \) ensures that \( Y^T \) is finite and \( x_0 \) independent. \( \blacksquare \)

Now, using decorated rooted trees with the rule \( (76) \), one identifies \( f_{\phi_1}[\zeta](x_R) = f_{\phi_+}[Y](x_R) \) as the renormalized theory, \( f_{\phi_1}[\chi](x_R) = f_{\phi}[Y](x_R) \) as the bare theory and the counterterms are given by \( f_{\phi_-}[Y](x_R) \). By proposition III.2 one finally obtains the main result of this paper:

The bare and renormalized theories define two algebra morphisms \( \gamma \) and \( \gamma_+ \) between the Hopf algebra of decorated rooted trees and the free algebra \( A \) of decorations. \( \gamma_+ \) is the positive part of the algebraic \( A \)-valued Birkhoff decomposition of \( \gamma \).
IV Conclusion

We have presented the first algebraic steps towards an adaption of Connes-Kreimer work to the ERGE. As in the BPHZ procedure with minimal subtraction scheme in dimensional regularization, renormalized and bare theories are linked inside a Birkhoff decomposition. In the BPHZ framework, renormalization corresponds to the projection of meromorphic functions (with pole only at dimension of space time) on their holomorphic part. Continuous renormalization appears as a projection on one decoration. Whether the algebra of decorations is an artefact -hiding deeper connection with the Riemann-Hilbert problem - or is truly meaningful is not perfectly clear at the moment. This will be the object of further investigations, as well as the precise relation between trees and Feynman diagrams and their use as computational tools for effective actions.

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References

1C. Brouder, Runge-Kutta methods and renormalization, Eur. Phys. J. C 12 (2000) 521-534.
2J. C. Butcher, An algebraic theory of integration methods, Math. Comput. 26 (1972) 79-106.
3A. Cayley, On the theory of analytical forms called trees Phil. Mag 13 (1857) 172-6.
4A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm.Math.Phys. 199 (1998) 203 A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem, Commun. Math. Phys. 210 (2000) 249 A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem II: the β-function, diffeomorphisms and the renormalization group, Comm. Math. Phys. 216 (2001) 215
5H. Figueroa, J.M. Gracia Bondia, On the antipode of Kreimer’s Hopf algebra, hep-th/9912170, San Jos, 1999.
6G. Gallavotti, F. Nicolo, Renormalization theory in four dimensional scalar fields 1, Comm. Math. Phys. 100 (1985) 545, Renormalization theory in four dimensional scalar fields 2, Comm. Math. Phys. 101 (1985) 247
7J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa, Elements of noncommutative geometry, Birkhauser (2001).
8E. Hairer, G. Wanner On the Butcher group and general multi-value methods, Computing 13, 1-15 (1974)
9T. Hurd, A renormalization group proof of perturbative renormalizability, Comm. Math. Phys. (1989)
10L. M. Ionescu, M. Marsalli, A Hopf algebra deformation approach to renormalization, hep-th/0307112
11C. Itzykson, J.B. Zuber, Quantum field theory, McGraw-Hill (1985).
12M. Le Bellac, Des phnomnes critiques aux champs de jauge, InterEditions (1990).
13. D. Kastler, *Connes-Moscovici-Kreimer Hopf algebras*, Fields Institute Communications **30** 2001.

14. J. Kogut, K. Wilson, *The renormalization group and the \( \epsilon \)-expansion*, Phys. Rept. 12 (1974) 75

15. F. Girelli, T. Krajewski, P. Martinetti, *Wave-Function renormalization and the Hopf algebra of Connes and Kreimer*, Mod.Phys.Lett. **A16** (2001) 299-303.

16. T. Morris, *Elements of the continuous renormalization group*, hep-th/9802039

17. A. Petermann, E. Stueckelberg, *The renormalization group in quantum theory* Helv.Phys.Acta 24 (1951) 317

18. J. Polchinski, *Renormalization and effective lagrangians*, Nucl. Phys. B 231 (1984) 269

19. V. Rivasseau, *From perturbative to constructive renormalisation*, Princeton University Press (1991)

20. K. G. Wilson, *Renormalization group methods*, Adv. Math. **16** (1975) 170-186.