Existence and nonexistence theorems for global weak solutions to quasilinear wave equations for the elasticity

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Abstract

In this paper, by using the theory of compensated compactness coupled with the kinetic formulation by Lions, Perthame, Souganidis and Tadmor [LPT, LPS], we prove the existence and nonexistence of global generalized (nonnegative) solutions of the nonlinearly degenerate wave equations \( v_{tt} = c(|v|^{s-1}v)_{xx} \) with the nonnegative initial data \( v_0(x) \) and \( s > 1 \). This result is an extension of the results in the second author’s paper [Su], where the existence and the nonexistence of the unique global classical solution were studied with a threshold on \( \int_{-\infty}^{\infty} v_1(x)dx \) and the non-degeneracy condition \( v_0(x) \geq c_0 > 0 \) on the initial data.

Key Words: Global weak solutions; degenerate wave equations; viscosity method; compensated compactness
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1 Introduction

In this paper, we study the global generalized solutions of the nonlinearly degenerate wave equations

\[
v_{tt} = c(|v|^{s-1}v)_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}
\]

with the initial data

\[
(v, v_t)|_{t=0} = (v_0(x), v_1(x)), \quad -\infty < x < \infty. \tag{1.2}
\]

To prove the global existence of solutions to (1.1), we also consider the nonlinear hyperbolic systems of elasticity

\[
v_t - u_x = 0, \quad u_t - c(|v|^{s-1}v)_x = 0 \tag{1.3}
\]

with bounded initial date

\[
(v, u)|_{t=0} = (v_0(x), u_0(x)), \tag{1.4}
\]

where \( v_0(x) \geq 0, s > 1, c = \frac{\theta_2}{s} > 0 \) and \( \theta = \frac{s+1}{2} > 0 \) are constants.

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A function \( v \in L^\infty(\mathbb{R} \times \mathbb{R}^+) \) is called a generalized solution of the Cauchy problem (1.1)-(1.2) if for any test function \( \phi \in C_0^\infty(\mathbb{R} \times [0, \infty)) \),

\[
\int_0^\infty \int_{-\infty}^\infty v(\phi(x,t)u - c|v|^{s-1}v\phi(x,t)_{xx}dxdt + \int_{-\infty}^\infty v_0(x)\phi(x,0)_t - v_1(x)\phi(x,0)dx = 0. \tag{1.5}
\]

A pair of functions \((v(x,t), u(x,t)) \in L^\infty(\mathbb{R} \times \mathbb{R}^+) \times L^\infty(\mathbb{R} \times [0, \infty)) \) is called a generalized solution of the Cauchy problem (1.3)-(1.4) if for any test function \( \phi_i(x,t) \in C_0^\infty(\mathbb{R} \times [0, \infty)) \), \( i = 1, 2 \),

\[
\begin{cases}
\int_0^\infty \int_{-\infty}^\infty v\phi(x,t)_{1t} - u\phi(x,t)_{1x}dxdt + \int_{-\infty}^\infty v_0(x)\phi_1(x,0)dx = 0, \\
\int_0^\infty \int_{-\infty}^\infty u\phi(x,t)_{2t} - c|v|^{s-1}v\phi(x,t)_{2x}dxdt + \int_{-\infty}^\infty u_0(x)\phi_2(x,0)dx = 0. \tag{1.6}
\end{cases}
\]

It is obvious that the generalized solutions of the Cauchy problem (1.3)-(1.4) are also the solutions of the Cauchy problem (1.1)-(1.2) if we specially choose \( \phi_1 = \phi_t, \phi_2 = \phi_x \) and \( v_1(x) = u_0'(x) \) in (1.6).

Two eigenvalues of (1.3) are

\[
\lambda_1 = -(cs)^{\frac{1}{2}}|v|^{\frac{s+1}{2}} = -\theta|v|^{\frac{s+1}{2}}, \quad \lambda_2 = (cs)^{\frac{1}{2}}|v|^{\frac{s+1}{2}} = \theta|v|^{\frac{s+1}{2}}, \tag{1.7}
\]

which coincide when \( v = 0 \) and so in which (1.3) is nonstrictly hyperbolic; two Riemann invariants of (1.3) are

\[
z = u + \int_0^v (cs)^{\frac{1}{2}}|v|^{\frac{s+1}{2}}dv, \quad w = u - \int_0^v (cs)^{\frac{1}{2}}|v|^{\frac{s+1}{2}}dv.
\]

Throughout this paper, we concentrate our study on the domain of \( v \geq 0 \), then the eigenvalues of (1.3) are \( \lambda_1 = -\theta v^{\frac{s+1}{2}}, \lambda_2 = \theta v^{\frac{s+1}{2}} \) and the Riemann invariants are

\[
z = u + \frac{2(cs)^{\frac{1}{2}}}{s+1}v^{\frac{s+1}{2}} = u + v^{\theta}, \quad w = u - \frac{2(cs)^{\frac{1}{2}}}{s+1}v^{\frac{s+1}{2}} = u - v^{\theta}.
\]

The Riemann invariants for (1.1) are given formally by

\[
w = u - v^{\theta}, \quad z = \tilde{u} + v^{\theta}, \tag{1.8}
\]

where

\[
u = \int_{-\infty}^x v_tdx, \quad \tilde{u} = -\int_x^\infty v_tdx.
\]

We denote \( w(x,0) \) and \( z(x,0) \) by \( w_0(x) \) and \( z_0(x) \) respectively. We have the following first main result in this paper.

**Theorem 1.** (I) Suppose that \( z_0(x) \) and \( w_0(x) \) are decreasing and satisfy

\[
c_1 \leq w_0(x) \leq c_0 \leq z_0(x) \leq c_2,
\]

where \( c_1, c_0, c_2 \) are three constants. Then the Cauchy problem (1.3) and (1.4) has a global weak solution satisfying (1.6).
(II). Let \( v_0(x) \geq 0 \) be bounded, \( v_1(x) \in L^1(\mathbb{R}) \) and the limits \( v_0^\theta(x)|_{x=\pm\infty} \) exist. Moreover suppose that
\[
\int_{-\infty}^{\infty} v_1(x)dx + v_0^\theta(x)|_{x=+\infty} + v_0^\theta(x)|_{x=-\infty} \geq 0
\] (1.10)
and
\[
v_1(x) \pm \theta v_0^\frac{x}{|x|} v_0(x) \leq 0.
\] (1.11)
Then the Cauchy problem (1.1) and (1.2) has a global weak solution satisfying (1.3).

Next we give the second main theorem of this paper, which implies the nonexistence of solutions satisfying \( v \geq 0 \). From the proof of Theorem 1, we can easily check that the global solutions of (1.1) and (1.3) constructed Theorem 1 satisfy that
\[
v(x,t) \geq 0 \text{ for a.a. } (x,t) \in \mathbb{R} \times \mathbb{R}^+
\] (1.12)
and \( w, z \) are decreasing with \( x \) for a.a. \( t \geq 0 \),
\[
w, z \in L^\infty(\mathbb{R} \times \mathbb{R}^+).
\] (1.14)
Furthermore, we can show that the following properties are also satisfied for the global solutions in Theorem 1, if we assume the additional regularity on initial data that \( w_0, z_0 \in W^{1,1}_{loc}(\mathbb{R}) \):
\[
w, z \in W^{1,1}_{loc}(\mathbb{R} \times \mathbb{R}^+),
\] (1.15)
\[
\|w_x(t)\|_{L^1} + \|z_x(t)\|_{L^1} \leq \|w_x(0)\|_{L^1} + \|z_x(0)\|_{L^1} \text{ for a.a. } t \geq 0
\] (1.16)
and
\[
\|w_t(t)\|_{L^1} + \|z_t(t)\|_{L^1} \leq C(\|w_x(0)\|_{L^1} + \|z_x(0)\|_{L^1}) \text{ for a.a. } t \geq 0.
\] (1.17)
Furthermore, the global solution of (1.1) satisfies that
\[
\partial_t v(t, \cdot) \in L^1(\mathbb{R}) \text{ for a.a. } t \geq 0.
\] (1.18)
We have the following second main result in this paper.

**Theorem 2.** (I). Suppose that \( w_0, z_0 \in W^{1,1}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and that \( w_0 \) and \( z_0 \) are decreasing. Furthermore, we assume that for some \( x, y \in \mathbb{R} \)
\[
w_0(x) > z_0(y).
\] (1.19)
Then the Cauchy problem (1.3) and (1.4) has no solutions satisfying the properties (1.12)-(1.17).

(II). Let \( v_0 \geq 0, w_0, z_0 \in W^{1,1}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( v_1 \in L^1(\mathbb{R}) \). Suppose that the limits \( v_0^\theta(x)|_{x=\pm\infty} \) exist. Furthermore, we assume that (1.11) is satisfied and
\[
\int_{-\infty}^{\infty} v_1(x)dx + v_0^\theta(x)|_{x=+\infty} + v_0^\theta(x)|_{x=-\infty} < 0.
\] (1.20)
Then the Cauchy problem (1.1) and (1.2) has no solutions satisfying the properties (1.12)-(1.18).

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This paper extends the results in $[Su]$. In $[Su]$, the second author obtained a threshold $-\frac{2}{a+1}$ of $\int_{-\infty}^{\infty} v_1(x)dx$ for the equation $v_{tt} = ((1 + v)^2 v_x)$. If the initial data (1.2) satisfy
\[ v_1(x) \pm (1 + v_0)^{\alpha}(x)\partial_x v_0(x) \leq 0 \] (1.21)
and
\[ (v_0, v_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}), \quad 1 + v_0(x) \geq c_0 > 0, \quad \int_{-\infty}^{\infty} v_1(x)dx > -\frac{2}{\alpha + 1}, \]
then the Cauchy problem (1.1) and (1.2) has a global unique solution $v \in C([0, \infty); H^2(\mathbb{R})) \cap C^1([0, \infty); H^1(\mathbb{R}))$ satisfying $v_0(x) \geq c_1 > 0$ for all $(x, t) \in [0, \infty) \times \mathbb{R}$, where $c_0, c_1$ are constants (The proof is classical and can be found in $[J, YN]$). If the initial data (1.2) satisfy (1.21) and
\[ (v_0, v_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}), \quad 1 + v_0(x) \geq c_0 > 0, \quad \int_{-\infty}^{\infty} v_1(x)dx < -\frac{2}{\alpha + 1}, \]
then equation (1.1) must degenerate at a finite time, namely, there exists $T^* > 0$ such that a local unique solution $v \in C([0, T^*); H^2(\mathbb{R})) \cap C^1([0, T^*); H^1(\mathbb{R}))$ of the Cauchy problem (1.1) and (1.2) exists, and
\[ \lim_{t \uparrow T^*} 1 + v_0(t, x_0) = 0 \quad \text{for some} \quad x_0 \in \mathbb{R}. \]
One can expect that classical solutions can not be extended after the degenerate occurs, since the strict hyperbolicity is lost. To avoid this difficulty, we treat our problem in the framework of the weak solution and construct a solutions with $v_0(x)$ having the degeneracy ($v_0(x_0) = 0$ for some $x_0 \in \mathbb{R}$). Under the assumption that $v \geq 0$, the decreasing property of $v_0$ and $z_0$ ensures the absence of shock waves. Our main theorems give a threshold separating the existence and nonexistence of solutions to (1.1) satisfying $v \geq v_0$, under the assumption that $w_0$ and $z_0$ are decreasing. In $[ZZ1, ZZ2]$, Zhang and Zheng proved the global existence of weak solutions to the 1D variational wave equation $v_{tt} = c(v)(c(v)v_x)_x$ with the non-degeneracy condition that $c(w) \geq c_0$ for some constant $c_0 > 0$.

This paper is organized as follows: In Section 2, we introduce a variant of the viscosity argument, and construct approximated solutions of the Cauchy problem (1.3) and (1.4) by using the solutions of the parabolic system (2.1) with the initial data (2.2). Under the conditions in the Part I of Theorem 1, we can easily obtain the necessary boundedness estimates (2.5) and (2.6) of $(w^\varepsilon\delta(x, t), z^\varepsilon\delta(x, t))$, where the bound $M(\delta)$ in (2.6) could tend to infinity as $\delta$ goes to zero. Based on the estimates (2.5) and (2.6) and the kinetic formulation by Lions, Perthame, Souganidis and Tadmor $[LPT, LPS]$, in Section 3, we prove the pointwise convergence of $(w^\varepsilon\delta(x, t), z^\varepsilon\delta(x, t))$ by using the theory of the compensated compactness, and its limit $(u, v)$ is a generalized solution of the Cauchy problem (1.3) and (1.4). Finally, in the last part of Section 3, we shall prove all conditions in the Part I of Theorem 1 are satisfied under the assumptions of the Part II of Theorem 1 and obtain the global existence of the generalized solutions of the Cauchy problem (1.1) and (1.2), which completes the proof of Theorem 1. In Section 4, we prove Theorem 2. The proof of Theorem 2 are based on similar, but refined and simpler method in $[Su]$. The key for the proof in $[Su]$ is the function $F(t)$:
\[ F(t) = \int_{-\infty}^{\infty} v(x, t) - v(x, 0)dx. \]
In the estimate for $F(t)$, we divide the integral region $(-\infty, \infty) \times (0, T]$ into three parts, using characteristic curves. However, for non-classical solutions, the characteristic curves would not be defined. Observing $0 \leq w_t(x,t)$ and $z_t(x,t) \leq 0$, we estimate $F$ more simply.

## 2 Viscosity Solutions

In this section we construct the approximated solutions of the Cauchy problem (1.3) and (1.4) by using the following parabolic systems

$$
\begin{align*}
\begin{cases}
    w_t + \lambda_2 w_x &= \varepsilon w_{xx} \\
    z_t + \lambda_1 z_x &= \varepsilon z_{xx}
\end{cases}
\end{align*}
$$

(2.1)

with initial data

$$
(w(x,0), z(x,0)) = \left(\phi_0(x), \phi_0(x)\right) * G_0,
$$

(2.2)

where $\varepsilon, \delta$ are small positive constants, $G_0$ is a mollifier,

$$
\begin{align*}
\begin{cases}
    \phi_0(x) = u_0(x) - (v_0(x))^\theta = u_0(x) - (v_0(x))^{\theta - \delta} \\
    \phi_0(x) = u_0(x) + (v_0(x))^\theta = u_0(x) + (v_0(x))^{\theta + \delta}
\end{cases}
\end{align*}
$$

and $(u_0(x), v_0(x))$ are given by (1.4). Thus $w(x,0)$ and $z(x,0)$ are smooth functions, and satisfy

$$
\begin{align*}
&c_1 - \delta \leq w(x,0) \leq c_0 - \delta, \quad c_0 + \delta \leq z(x,0) \leq c_2 + \delta, \\
&M(\delta) \leq w_x(x,0) \leq 0, \quad -M(\delta) \leq z_x(x,0) \leq 0,
\end{align*}
$$

(2.3), (2.4)

where $M(\delta)$ is a constant, and could tend to infinity as $\delta$ tends to zero.

First, we have the following Lemma.

**Lemma 3.** Let $w(x,0), z(x,0)$ be bounded in $C^1$ space and satisfy (2.3) and (2.4). Moreover, suppose that $(w^{\varepsilon, \delta}(x,t), z^{\varepsilon, \delta}(x,t))$ is a smooth solution of (2.1), (2.2) defined in a strip $(-\infty, \infty) \times [0, T]$ with $0 < T < \infty$. Then

$$
\begin{align*}
&c_1 - \delta \leq w^{\varepsilon, \delta}(x,t) \leq c_0 - \delta, \quad c_0 + \delta \leq z^{\varepsilon, \delta}(x,t) \leq c_2 + \delta,
\end{align*}
$$

(2.5)

and

$$
\begin{align*}
&M(\delta) \leq w^{\varepsilon, \delta}_x(x,t) \leq 0, \quad -M(\delta) \leq z^{\varepsilon, \delta}_x(x,t) \leq 0.
\end{align*}
$$

(2.6)

**Proof.** The estimates in (2.5) can be obtained by using the maximum principle to (2.1), (2.2) and the condition (2.3) directly.

We differentiate (2.1) with respect to $x$ and let $w_x = -R, z_x = -S$; then

$$
\begin{align*}
\begin{cases}
    R_t + \lambda_2 R_x - (\lambda_2 w R + \lambda_2 z S)R = \varepsilon R_{xx}, \\
    S_t + \lambda_1 S_x - (\lambda_1 w R + \lambda_1 z S)S = \varepsilon S_{xx}
\end{cases}
\end{align*}
$$

(2.7)

The nonnegativity $w^{\varepsilon, \delta}_x(x,t) \leq 0$ and $z^{\varepsilon, \delta}_x(x,t) \leq 0$ in (2.6) can be obtained by using the maximum principle to (2.7) and the condition $w_x(x,0) \leq 0$ and $z_x(x,0) \leq 0$ in (2.4).
A simple calculation yields
\[ u_w = \frac{1}{2}, \quad u_z = \frac{1}{2}, \quad v_w = -\frac{1}{2\theta v} - s, \quad v_z = \frac{1}{2\theta v} - \frac{s}{2} \]
and
\[ \lambda_{1w} = \lambda_{2z} = \frac{s - 1}{4v} > 0, \quad \lambda_{1z} = \lambda_{2w} = -\frac{s - 1}{4v} < 0. \] (2.8)
Thus the lower bound \(-M(\delta) \leq w_{\varepsilon,\delta}(x,t), -M(\delta) \leq z_{\varepsilon,\delta}(x,t)\) in (2.6) is a direct conclusion of Lemma 2.4 in [Lu1]. Lemma 3 is proved.

From the estimates in (2.5), we have the following estimates about the functions \(u_{\varepsilon,\delta}(x,t)\) and \(v_{\varepsilon,\delta}(x,t)\),
\[ \frac{c_1 + c_0}{2} \leq u_{\varepsilon,\delta}(x,t) \leq \frac{c_2 + c_0}{2}, \quad c_1 - c_0 + 2\delta \leq 2(v_{\varepsilon,\delta}(x,t))^\theta \leq c_2 - c_1 + 2\delta \] (2.9)
and
\[ -M(\delta) \leq w_{\varepsilon,\delta}(x,t) \leq 0, \quad -M(\delta) \leq 2\theta(v_{\varepsilon,\delta}(x,t))^\frac{\theta}{2} v_{\varepsilon,\delta}(x,t) \leq M(\delta) \] (2.10)
or
\[ |v_{\varepsilon,\delta}(x,t)| \leq M_1(\delta), \] (2.11)
where \(M_1(\delta)\) is a constant, and could tend to infinity as \(\delta\) tends to zero.

The positive, lower bound \(2(v_{\varepsilon,\delta}(x,t))^\theta \geq c_1 - c_0 + 2\delta\) in (2.9) ensures the regularity of \(\lambda_{1w}, \lambda_{2z}, i = 1, 2\) from (2.8) and the following global existence of the Cauchy problem \((2.1)-(2.2)\).

**Theorem 4.** Let \((w(x,0), z(x,0)) \in C^1\) satisfy (2.3) and (2.4), then the Cauchy problem \((2.1)\) with initial data \((2.2)\) has a unique global smooth solution satisfying (2.5) and (2.6).

The proof of Theorem 4 is standard, and the details can be found in [Sm] or Theorem 2.3 in [Lu1].

### 3 Proof of Theorem 1.

In this section, we prove Theorem 1.

A pair of smooth functions \((\eta(v,u), q(v,u))\) is called a pair of entropy-entropy flux of system \((1.3)\) in the region of \(v > 0\) if \((\eta(v,u), q(v,u))\) satisfies
\[ q_u = -\eta_v, \quad q_v = -\theta^2 v^{s-1} \eta_u. \] (3.1)
Eliminating the \(q\) from (3.1), we have the following entropy equation of system \((1.3)\),
\[ \eta_{vv} = \theta^2 v^{s-1} \eta_{uu}. \] (3.2)
Consider (3.2) in the region \(v > 0\) with the following initial conditions
\[ \eta(0,u) = d^0 f(u), \quad \eta_v(0,u) = 0, \] (3.3)

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where \( d^0 = \int_{-1}^{1} (1 - \tau^2)^\lambda d\tau \). Then from the results in [JPP, LPS, LPT], an entropy of (3.2) with (3.3) in the region \( v > 0 \) is

\[
\eta^0(v, u) = \int_{-\infty}^{\infty} f(\xi)G(v, \xi - u)d\xi,
\]

where the fundamental solution

\[
G(v, u - \xi) = v(v^{s+1} - (\xi - u)^2)^\lambda_+,
\]

the notation \( x_+ = \max(0, x) \) and \( \lambda = -\frac{s+3}{2(s+1)} \in (-1, 0) \). The entropy flux \( q^0(v, u) \) associated with \( \eta^0(v, u) \) in the region \( v > 0 \) is

\[
q^0(v, u) = -\int_{-\infty}^{\infty} f(\xi)\theta^\frac{\xi - u}{v}G(v, \xi - u)d\xi.
\]

Letting \( \xi = u + v^{s+1} \tau \), by simple calculations, we have on \( v > 0 \)

\[
\eta^0(v, u) = \int_{-\infty}^{\infty} f(\xi)G(v, \xi - u)d\xi = \int_{w}^{z} f(\xi)v(z - \xi)^\lambda(\xi - w)^\lambda d\xi
\]

\[
= \int_{-1}^{1} f(u + v^{s+1} \tau)\theta v^\frac{s+1}{2}(1 - \tau^2)^\lambda d\tau = \int_{-1}^{1} f(u + v^{s+1} \tau)(1 - \tau^2)^\lambda d\tau,
\]

and

\[
q^0(v, u) = -\int_{-\infty}^{\infty} f(\xi)\theta^\frac{\xi - u}{v}G(v, \xi - u)d\xi = -\int_{w}^{z} f(\xi)\theta(\xi - u)(z - \xi)^\lambda(\xi - w)^\lambda d\xi
\]

\[
= -\int_{-1}^{1} f(u + v^{s+1} \tau)\theta v^\frac{s+1}{2}(v^{s+1} \tau)(1 - \tau^2)^\lambda d\tau
\]

\[
= -\int_{-1}^{1} f(u + v^{s+1} \tau)\theta v^\frac{s+1}{2}(1 - \tau^2)^\lambda d\tau.
\]

We consider the matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w_v & w_u \\ z_v & z_u \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},
\]

and multiply it by the two sides in (3.4), then (3.4) can be rewritten as follows:

\[
v_t - u_x = \varepsilon(aw_{xx} + bz_{xx}) =\]

\[\varepsilon(aw_x + bz_x) - \varepsilon(a_x w_x + b_x z_x) = \varepsilon v_{xx} - \varepsilon(a_x w_x + b_x z_x)\]

and

\[
u_t - c(v^s)_x = \varepsilon(ew_{xx} + dz_{xx}) \]

\[= \varepsilon(ew_x + dz_x)_x - \varepsilon(e_x w_x + d_x z_x) = \varepsilon u_{xx}.
\]

First, we have the following Lemma:
Lemma 5.

\[ v^{\varepsilon, \delta}(x, t) - u^{\varepsilon, \delta}(x, t), \quad (3.6) \]
\[ u_t^{\varepsilon, \delta} - c((v^{\varepsilon, \delta})_x) \quad (3.7) \]
and
\[ \eta^0(v^{\varepsilon, \delta}(x, t), u^{\varepsilon, \delta}(x, t)) + q^0 (v^{\varepsilon, \delta}(x, t), u^{\varepsilon, \delta}(x, t)) \quad (3.8) \]
are compact in \( H^{-1}_0(\mathbb{R} \times \mathbb{R}^+) \), for any \( \eta^0 \in C^2(\mathbb{R}) \).

Proof. For simplicity, we omit the superscripts \( \varepsilon, \delta \). From the estimate in (2.11), for any function \( \phi \in H^1_0 \), we have that

\[ \left| \int_0^\infty \int_{-\infty}^\infty \varepsilon v_x \phi dxdt \right| = \left| \int_0^\infty \int_{-\infty}^\infty \varepsilon v_x \phi dxdt \right|, \]

is compact and the term \( \varepsilon (a_x w_x + b_x z_x) \) on the right-hand side of (3.6) is uniformly bounded, if we choose \( \varepsilon \) to be much smaller than \( \delta \). Thus (3.6) is compact in \( H^{-1}_0(\mathbb{R} \times \mathbb{R}^+) \). Similarly, we can prove from (2.9) that (3.7) is compact in \( H^{-1}_0(\mathbb{R} \times \mathbb{R}^+) \).

To prove (3.8) to be compact in \( H^{-1}_0(\mathbb{R} \times \mathbb{R}^+) \), we multiply (3.4) by \( \eta^0(v, u)_v \) and (3.5) by \( \eta^0(v, u)_u \) to obtain that

\[ \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_t + q^0 (v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \]
\[ = \varepsilon \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_{xx} - \varepsilon (a_x w_x + b_x z_x) \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_v \]
\[ - \varepsilon (\eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_v v + (v^{\varepsilon, \delta})^2 + 2 \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) u v^{\varepsilon, \delta} u^{\varepsilon, \delta} + \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) u u^{\varepsilon, \delta})^2. \]

Since the boundedness estimates given in (2.9)-(2.11), and for any \( \eta^0 \in C^2(\mathbb{R}) \), we know that the terms in (3.9) satisfy

\[ |(a_x w_x + b_x z_x) \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_v| \leq M(\delta) \]
and
\[ |\eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_v v + (v^{\varepsilon, \delta})^2 + 2 \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) u v^{\varepsilon, \delta} u^{\varepsilon, \delta} + \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) u u^{\varepsilon, \delta})^2| \leq M(\delta), \]
where \( M(\delta) \) is a positive constant, which could tend to infinity as \( \delta \) tends to zero.

Moreover, for any function \( \phi \in H^1_0 \), we have that

\[ |\int_0^\infty \int_{-\infty}^\infty \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_x \phi dxdt| = |\int_0^\infty \int_{-\infty}^\infty \eta^0(v^{\varepsilon, \delta}, u^{\varepsilon, \delta})_x \phi dxdt| \leq M(\delta), \]
thus the right-hand side of (3.9) is compact in \( H^{-1}_0(\mathbb{R} \times \mathbb{R}^+) \) if we choose \( \varepsilon \) to be much smaller than \( \delta \). Lemma 5 is proved.

Now we prove the Part I in Theorem 1. From the definition (1.6) of the generalized solutions of the Cauchy problem (1.3) and (1.4), and the equations (3.4) and (3.5), it is sufficient to prove the pointwise convergence of \( (v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \). By using the theory of compensated compactness [M], if \( \nu_{x,t} \) is the family of positive probability measures with respect to the viscosity solutions \( (v^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \), we only need to prove that the support...
set of the Young measure \( \nu_{x,t} \) is concentrated on one point \((x, t)\), or the Young measure \( \nu_{x,t} \) is a Dirac measure.

Suppose the support set of the Young measure \( \nu_{x,t} \) is concentrated on the line \( v = 0 \), or \( \text{supp} \nu_{x,t} = \{ v = 0 \} \), then since [3.6] and [3.7] given in Lemma 5 using the measure equation to the entropy-entropy flux pairs \((v, -u)\) and \((u, cv')\), we get

\[
< \nu_{x,t}, u >^2 = < \nu_{x,t}, u^2 >,
\]

which implies that \( \nu_{x,t} \) is a Dirac measure and the support set is one point \((0, \tilde{u})\).

Suppose, for fixed \((x, t)\), the support set of the Young measure \( \nu_{x,t} \) is concentrated on \( v \geq 0 \), but not only on \( v = 0 \). Then clearly \(< \nu_{x,t}, \eta^0 > \neq 0 \). As done in [LPS, LPT], using the measure equation in the theory of compensated compactness to the entropy-entropy flux pairs \((\eta^0(v, u), q^0(v, u))\), we get

\[
< \nu_{x,t}, \int_{-\infty}^{\infty} f_1(\xi_1) G(v, \xi_1 - u) d\xi_1 > < \nu_{x,t}, \int_{-\infty}^{\infty} f_2(\xi_2) \frac{\xi_2 - u}{v} G(v, \xi_2 - u) d\xi_2 >
- < \nu_{x,t}, \int_{-\infty}^{\infty} f_2(\xi_2) G(v, \xi_2 - u) d\xi_2 > < \nu_{x,t}, \int_{-\infty}^{\infty} f_1(\xi_1) \frac{\xi_1 - u}{v} G(v, \xi_1 - u) d\xi_1 >
= < \nu_{x,t}, \int_{-\infty}^{\infty} f_1(\xi_1) f_2(\xi_2) \frac{\xi_2 - \xi_1}{v} G(v, \xi_1 - u) G(v, \xi_2 - u) d\xi_1 d\xi_2 >
\]

or

\[
\int_{-\infty}^{\infty} f_1(\xi_1) < \nu_{x,t}, G(v, \xi_1 - u) > d\xi_1 \cdot \int_{-\infty}^{\infty} f_2(\xi_2) < \nu_{x,t}, \frac{\xi_2 - u}{v} G(v, \xi_2 - u) > d\xi_2
- \int_{-\infty}^{\infty} f_2(\xi_2) < \nu_{x,t}, G(v, \xi_2 - u) > d\xi_2 \cdot \int_{-\infty}^{\infty} f_1(\xi_1) < \nu_{x,t}, \frac{\xi_1 - u}{v} G(v, \xi_1 - u) > d\xi_1
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\xi_1) f_2(\xi_2) < \nu_{x,t}, \frac{\xi_2 - \xi_1}{v} G(v, \xi_1 - u) G(v, \xi_2 - u) > d\xi_1 d\xi_2.
\]  

(3.10) holds for arbitrary functions \( f_1, f_2 \), and this yields

\[
< vH(v, u, \xi_1) > < (\xi_2 - u)H(v, u, \xi_2) >
- < vH(v, u, \xi_2) > < (\xi_1 - u)H(v, u, \xi_1) >
= < (\xi_2 - \xi_1)vH(v, u, \xi_1)H(v, u, \xi_2) >,
\]

where we use the notation \(< H(v, u, \xi) > = < \nu_{x,t}, H(v, u, \xi) >\) and

\[
H(v, u, \xi) = (v^{s+1} - (\xi - u)^2)_+.
\]

Let \( I = [w, z] \) for each \((v, u) \in \text{supp} \nu_{x,t}\), where \( w = u - v^{s+1}, z = u + v^{s+1} \). Dividing (3.11) by \(< vH(v, u, \xi_1) > < vH(v, u, \xi_2) >\) and sending \( \xi_2 \) to \( \xi_1 \), we obtain

\[
\frac{\partial}{\partial \xi} \left( \frac{< (\xi - u)H(v, u, \xi) >}{< vH(v, u, \xi) >} \right) = \frac{< vH(v, u, \xi) >^2}{< vH(v, u, \xi) >^2}.
\]  

(3.12)

Again using the measure equation between \((\eta^0(v, u), q^0(v, u))\) and \((v, -u)\), we get

\[
-\theta < v > < (\xi - u)H(v, u, \xi) > + < u > < vH(v, u, \xi) >
= -\theta < (\xi - u)vH(v, u, \xi) > + < uvH(v, u, \xi) >,
\]

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Thus all conditions in Part I of Theorem 1 are satisfied. If we specially choose the Cauchy problem (1.1) and (1.2), Theorem 1 is proved.

We note that

$$\frac{\theta <v<(\xi-u)H(v,u,\xi)+w<vH(v,u,\xi)>}{<vH(v,u,\xi)>}$$

$$=-\theta \xi + (1+\theta)\frac{<uvH(v,u,\xi)>}{<vH(v,u,\xi)>}$$ \text{ in } I. \tag{3.13}

Differentiating (3.13) in $\xi$ and combining the outcome with (3.12), we also obtain

$$\frac{\partial}{\partial \xi} \left( <uvH(v,u,\xi)> \right) = -\frac{\theta}{1+\theta} <v> <vH(v,u,\xi)>^2 > -1 \leq 0.$$

Following the steps given in Proposition II.1 in [LPS] and Lemma 6 in [LPT], we can prove that the Young measure $\nu_{e,\delta}$ is a Dirac measure, which implies the pointwise convergence of the viscosity solutions $(v^{e,\delta}, u^{e,\delta})$. It is clear that the limit $(v, u)$ of $(v^{e,\delta}, u^{e,\delta})$ is a weak solution of the Cauchy problem (1.3) and (1.4) if we let $e, \delta$ in (2.2), (3.4), (3.5) go to zero. The Part I in Theorem 1 is proved.

Under the conditions given in the Part II in Theorem 1, we consider the Cauchy problem (1.3) and (1.4), where

$$u_0(x) = \int_{-\infty}^{x} v_1(\xi)d\xi.$$

Then from the condition (1.11), we know $z_0(\xi) \leq 0, w_0(\xi) \leq 0$ and so both $z_0(x)$ and $w_0(x)$ are decreasing. From the conditions (1.10) and (1.11), we have that

$$(u_0(+\infty) + v_0^\theta(+\infty)) - (u_0(x) + v_0^\theta(x)) = \int_{x}^{+\infty} u_0'(\xi) + (v_0^\theta(\xi))'d\xi \leq 0$$

and so

$$z_0(x) = u_0(x) + v_0^\theta(x) \geq u_0(+\infty) + v_0^\theta(+\infty) \tag{3.14}$$

where the constant $-v_0^\theta(-\infty)$ is corresponding to $c_0$ in (1.9). Similarly

$$(u_0(x) - v_0^\theta(x)) - (u_0(-\infty) - v_0^\theta(-\infty)) = \int_{-\infty}^{x} u_0'(\xi) - (v_0^\theta(\xi))'d\xi \leq 0,$$

and so

$$w_0(x) = u_0(x) - v_0^\theta(x) \leq u_0(-\infty) - v_0^\theta(-\infty) = -v_0^\theta(-\infty).$$

Thus all conditions in Part I of Theorem 1 are satisfied. If we specially choose $\phi_1 = \phi_\xi, \phi_2 = \phi_x$ and $v_1(x) = u_0'(x)$ in (1.6), the function $v$ given in Part I is a weak solution of the Cauchy problem (1.1) and (1.2). Theorem 1 is proved.

### 4 Proof of Theorem 2

First, we prove Part I in Theorem 2. We note that $w_0, z_0 \in C(\mathbb{R}) \subset W^{1,1}_{loc}(\mathbb{R})$. Since $w_0$ and $z_0$ are decreasing, (1.19) implies that

$$\lim_{x \to -\infty} w_0(x) \geq w_0(y) \geq \lim_{x \to -\infty} z(x).$$
Hence there exists a large constant $M_0 > 0$ such that if $M \geq M_0$, then

$$w_0(-M) > z_0(M).$$

Now we prepare some lemmas.

**Lemma 6.** Suppose that the assumptions (1.12)-(1.17) are satisfied. Then

$$\begin{cases}
w_t + \lambda_2 w_x = 0, \\
z_t + \lambda_1 z_x = 0
\end{cases} \quad (4.1)$$

is satisfied for a.a. $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. 

**Proof.** From (1.14) and (1.15), since $v \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ and $\theta = (s + 1)/2 > 1$, $\partial_x v^\theta$ can be defined in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$. From the second equation in (1.6) and the integration by parts,

$$\int_0^\infty \int_{-\infty}^\infty u_t \phi_2 - \theta v^{\theta-1} \partial_x (v^\theta) \phi_2 dx dt = 0. \quad (4.2)$$

Hence

$$u_t - \theta v^{\theta-1} \partial_x (v^\theta) = 0 \quad (4.3)$$

is satisfied for a.a. $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Similarly, we have that $v_t - u_x = 0$ is satisfied for a.a. $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. By multiplying this equation equality by $\theta v^{\theta-1}$, we have

$$\theta v^{\theta-1} v_t - \theta v^{\theta-1} \partial_x u = 0 \quad (4.4)$$

From (4.3) + (4.4) and (4.3) - (4.4), we have the first and the second equation in (4.1) respectively. 

From (1.13) and (1.16), $\lim_{x \to \pm \infty} (u_0(x), v_0(x))$ exists. We put

$$\lim_{x \to \pm \infty} (u_0(x), v_0(x)) = (u_\pm, v_\pm).$$

**Lemma 7.** Suppose that the assumptions (1.12)-(1.17) is satisfied, then we have

$$\lim_{x \to \pm \infty} u(x,t) = u_\pm \text{ and } \lim_{x \to \pm \infty} v(x,t) = v_\pm. \quad (4.5)$$

**Proof.** Since $w$ and $z$ are decreasing with $x$, from the definition of $w$ and $z$, we have $u_x \leq 0$. So, from $v_t - u_x = 0$, we have that $v(x,t)$ is decreasing with $t$ for a.a. $x \in \mathbb{R}$. Hence $0 \leq v(x,t) \leq v_0(x) \leq \|v_0\|_{L^\infty}$. We put $\lambda_M = \theta \|v_0\|_{L^\infty}$. Since $w_x, z_x \leq 0$ and $0 \leq v \leq C_M$, by (1.1), we have

$$0 \leq w_t \leq -\lambda_M w_x \text{ and } \lambda_M z_x \leq z_t \leq 0. \quad (4.6)$$

We set $\rho_{j,\varepsilon} = \varepsilon^{-1} \rho_j(\cdot/\varepsilon)$ as standard mollifiers with $x$ and $t$ for $j = 1$ and 2 respectively ($\rho_j \in C_0^\infty(\mathbb{R})$ and $\rho_j \geq 0$ and $\int_{-\infty}^\infty \rho_j(\cdot) dx = 1$ for $j = 1, 2$). We note that

$$\int_0^\infty \rho_{2,\varepsilon}(t-s) w_s(x,s) ds = \partial_t \int_0^\infty \rho_{2,\varepsilon}(t-s) w_s(x,s) ds + \rho_{2,\varepsilon}(t) w(x,0).$$
Hence, applying the mollifiers to the both side of the first inequality in (4.6), we have

\[ w_\varepsilon t - \lambda M w_\varepsilon x + \rho_{2,\varepsilon}(t) \rho_{1,\varepsilon} * w_0(x) \leq 0, \]

where \( w_\varepsilon = \int_0^\infty \rho_{2,\varepsilon}(t-s) \rho_{1,\varepsilon} * w(x, s)ds \). Noting \( w_\varepsilon(x + \lambda Mt, t) \) is differentiable with \( t \), we have

\[ \frac{d}{dt}w_\varepsilon(x + \lambda Mt, t) + \rho_{2,\varepsilon}(t) \rho_{1,\varepsilon} * w_0(x + \lambda Mt) \leq 0. \]

Integrating on \([0, t]\), we have

\[ w_\varepsilon(x + \lambda Mt, t) - w_\varepsilon(x, 0) + \int_0^t \rho_{2,\varepsilon}(s) \rho_{1,\varepsilon} * w_0(x + \lambda Ms)ds \leq 0. \]

Since \( w(x, \cdot) \) and \( w(\cdot, t) \) are continuous with a.a. fixed \( t \) and \( x \) respectively, we have

\[ \lim_{\varepsilon \to 0} w_\varepsilon(x, 0) \to \int_{-\infty}^0 \rho_2(t)dw_0(x) \]

and

\[ \int_0^t \rho_{2,\varepsilon}(s) \rho_{1,\varepsilon} * w(x + \lambda Ms, 0)ds \to \int_0^\infty \rho_2(t)dw_0(x). \]

Hence we have by taking \( \varepsilon \to 0, \)

\[ w(x + \lambda Mt, t) - w_0(x) \leq 0. \]

Therefore we have

\[ w_0(x) \leq w(x, t) \leq w(x - \lambda Mt, 0) \]

and

\[ z_0(x + \lambda Mt) \leq z(x, t) \leq z_0(x), \]

which implies that (4.5). \( \square \)

We put

\[ F(t) = - \int_{-\infty}^\infty v(x, t) - v_0(x)dx. \]

From the first equation (1.3) and Lemma 7, we have

\[ F(t) = (u_- - u_+)t. \quad (4.7) \]

We divide \( F(t) \) into the three parts as follows:

\[ F(t) = \left( \int_{-\infty}^{-M} + \int_{-M}^M + \int_{M}^\infty \right) v(x, t) - v_0(x)dx \]

\[ = F_1(t) + F_2(t) + F_3(t). \]

Now we estimate \( F_1(t) \). From the first equation in (1.3) and Lemma 7, we have

\[ \frac{d}{dt}F_1(t) = \int_{-\infty}^{-M} -u_x(x, t)dx = -u(t, -M) + u_- . \]
Since \( w_0(-M) \leq w(-M, t) \), from the definition of \( w \), we have
\[
-u(M, t) \leq -v^\theta(-M, t) - w_0(-M) \leq -w_0(-M)
\]
Hence we have
\[
F_1(t) \leq -t(u_+ - w_0(-M)).
\]
(4.8)
Since \( v \geq 0 \) under our contradiction argument and \( v \leq C \), we can estimate \( F_2(t) \) as
\[
F_2(t) \leq C_M,
\]
(4.9)
where \( C_M \) is a positive constant depending on \( M \). In the same way as in the above estimate of \( F_1 \), we have
\[
F_3(t) \leq t(-u_+ + z_0(M)).
\]
(4.10)
From (4.7), (4.8), (4.9) and (4.10), we have
\[
C_M + (z_0(M) - w_0(-M) + u_+ - u_+)t \geq (u_+ - u_+)t.
\]
Hence we have
\[
C_M > (w_0(-M) - z_0(M))t,
\]
which gives a contradiction for large \( t \), since \( w_0(-M) - z_0(M) > 0 \). Therefore we complete the proof of Part I in Theorem 2.

Next we prove Part II in Theorem 2. For the solutions of (1.1), we put \( u = \int_0^\infty v(t, y) dy \) and \( \tilde{u} = -\int_0^\infty v(t, y) dy \). We note \( u \) is well-defined, if we assume (1.18). We show the following Lemma.

**Lemma 4.1.** Let \( v \) be a weak solutions of the generalized Cauchy problem (1.5) and \( u \) and \( \tilde{u} \) be as above. Suppose that (1.12)-(1.18) are satisfied. Then \((v, u)\) and \((v, \tilde{u})\) are solutions of the generalized Cauchy problem (1.6).

**Proof.** We put \( \psi \in C_0^\infty(\mathbb{R}) \) such that \( \psi(0) = 1 \). Replacing \( \phi \) by \( \psi(\varepsilon x) \int_0^\infty \phi(t, y) dy \) in the generalized Cauchy problem (1.5), from (1.16), (1.17) and the integration by parts, we have that
\[
\int_0^\infty u(x, t)\psi(\varepsilon y)\partial_t\phi(t, y) - c(\varepsilon^\theta)\partial_x^2 \left( \psi(\varepsilon x) \int_0^\infty \phi(t, y) dy \right) dx dt - \int_{-\infty}^\infty u_0(x)\psi(\varepsilon y)\phi(0, y) dy = 0.
\]
Taking \( \varepsilon \to 0 \), from (1.16), we have
\[
\int_0^\infty \int_{-\infty}^\infty u(x, t)\partial_t\phi(t, y) - c(\varepsilon^\theta)\partial_x^2 \phi(t, y) dy dx dt - \int_{-\infty}^\infty u_0(x)\phi(0, y) dy = 0.
\]
Similarly, replacing \( \phi \) by \( \psi(\varepsilon x) \int_0^\infty \phi(t, y) dy \), we obtain that
\[
\int_0^\infty \int_{-\infty}^\infty \tilde{u}(x, t)\partial_t\phi(t, y) - c(\varepsilon^\theta)\partial_x^2 \phi(t, y) dy dx dt - \int_{-\infty}^\infty \tilde{u}_0(x)\phi(0, y) dy = 0.
\]
\hfill\Box
For the Riemann invariant (1.8), we obtain same results as in Lemmas 6 and 7. Therefore we can show Part II in Theorem 2 by using similar contradiction argument to in the proof of Part I.

**Remark 8.** Here we show that the solution constructed in Theorem 1 satisfies (1.15), (1.16) and (1.17), if \( w_0(x) \) and \( z_0(x) \) are bounded, decreasing and in \( W^{1,1}_{loc}(\mathbb{R}) \). Since the approximate solution \((R,S) = (w_x, z_x)\) to (2.7) satisfies that

\[
R_t + (\lambda_2 R)_x = R_{xx} \quad \text{and} \quad S_t + (\lambda_2 S)_x = S_{xx},
\]

we have from the negativity of \( R \) and \( S \)

\[
\|w_x(t)\|_{L^1} + \|z_x(t)\|_{L^1} = - \int_{-\infty}^{\infty} R(x,t) + S(x,t) \, dx
\]

\[
= - \int_{-\infty}^{\infty} R(x,0) + S(x,0) \, dx
\]

\[
= \lim_{x \to \infty} 2(u_0(-x) - u_0(x)).
\]

The \( \lim_{x \to \infty} 2(u(-x) - u(x)) \) exists and is finite, since \( u_0 = (w_0 + z_0)/2 \) is decreasing and bounded. Hence, for solutions of the non-viscous equations (1.3), we have that \( w(t,\cdot), z(t,\cdot) \in W^{1,1}_{loc}(\mathbb{R}) \) for a.a. \( t \geq 0 \) and that (1.16) is satisfied. Furthermore, in the similar way as to the proof of Lemma 6 we have that \( w, z \in W^{1,1}_{loc}(\mathbb{R} \times \mathbb{R}^+) \) and (4.1) are satisfied. Therefore we have by the boundedness of \( \lambda_1 \) and \( \lambda_2 \) that

\[
\|w_t(t)\|_{L^1} + \|z_t(t)\|_{L^1} \leq \|\lambda_2 w_x(t)\|_{L^1} + \|\lambda_2 z_x(t)\|_{L^1}
\]

\[
\leq C(\|w_x(0)\|_{L^1} + \|z_x(0)\|_{L^1}).
\]

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