Global Hopf bifurcation analysis of an susceptible-infective-removed epidemic model incorporating media coverage with time delay

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ABSTRACT
An susceptible-infective-removed epidemic model incorporating media coverage with time delay is proposed. The stability of the disease-free equilibrium and endemic equilibrium is studied. And then, the conditions which guarantee the existence of local Hopf bifurcation are given. Furthermore, we show that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of delay. The obtained results show that the time delay in media coverage can not affect the stability of the disease-free equilibrium when the basic reproduction number is less than unity. However, the time delay affects the stability of the endemic equilibrium and produces limit cycle oscillations while the basic reproduction number is greater than unity. Finally, some examples for numerical simulations are included to support the theoretical prediction.

1. Introduction
Recently, epidemic models with time delays have received much attention since time delays can change the qualitative behaviour of the models. For instance, it can change the stability of equilibrium and thus lead to periodic solutions by Hopf bifurcation [6,10–12,14,17,18,21,27–30].

In modelling of communicable diseases, the incidence rate is considered to play a key role which gives a reasonable qualitative description of the disease dynamics. In an epidemic model, the incidence rate \( g(I) \) (\( I \) denotes the number of infective individuals) may be affected by some factors, such as media coverage, density of population, and life style [4,5,7,13,20,22,24,26]. It is worthy to note that, media coverage plays an important role in helping both the government authority make interventions to contain the disease and people response to the disease [5,26]. The effect of media in the spread of infectious disease has long been under investigation, for example in the case of HIV/AIDS [3,15]. An other concrete example is a study conducted after the SARS outbreak [2]. The study shows that students from Ontario, Canada were aware that the risk of becoming infected by the SARS coronavirus was low, but they also predominantly had misconceptions about the virus.

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Thus, it is important for public health authorities to communicate accurate and timely information to the public about infectious disease outbreaks.

Several mathematical models have been formulated to describe the impact of media coverage on the transmission dynamics of infectious diseases. Especially, Cui et al. [5], Tchuenche et al. [22], and Sun et al. [20] incorporated a nonlinear function of the number of infective in their transmission term to investigate the effects of media coverage on the transmission dynamics:

\[ g(I) = \beta_1 - \frac{\beta_2 I}{m + I}, \]  

where \( \beta_1 > 0 \) is the contact rate before media alert; the term \( \beta_2 I/(m + I) \) measures the effect of reduction of the contact rate when infectious individuals are reported in the media. Because the coverage report cannot prevent disease from spreading completely we have \( \beta_1 \geq \beta_2 > 0 \). The half-saturation constant \( m > 0 \) reflects the impact of media coverage on the contact transmission. The function \( I/(m + I) \) is a continuous bounded function which takes into account disease saturation or psychological effects [7,20].

On the other hand, delays are ubiquitous in life, so it is in media coverage. There are two major routes for media coverage of an infectious outbreak [1,20]. The first route is that the media reports directly to the public on facts that they (the media) observe; the second is that the public health authorities use mass media or the internet to communicate about the outbreak. In fact, we need time to make sure an infectious outbreak and to get the statistical data, so the number of infections and the number of suspected infection reported by media today is often the statistical results of the day before. Hence, the effects of media coverage on the transmission dynamics (1) can be modified as follows:

\[ g(I) = \beta_1 - \frac{\beta_2 I(t - \tau)}{m + I(t - \tau)}, \]  

where \( \tau > 0 \) is the time delay representing the latent period of media coverage.

Based on the above assumption, we propose an susceptible–infective–removed (SIR) epidemic model incorporating media coverage with time delay which can be described as follows:

\[ \frac{dS(t)}{dt} = \Lambda - \left( \beta_1 - \frac{\beta_2 I(t - \tau)}{m + I(t - \tau)} \right) S(t)I(t) - dS(t), \]

\[ \frac{dI(t)}{dt} = \left( \beta_1 - \frac{\beta_2 I(t - \tau)}{m + I(t - \tau)} \right) S(t)I(t) - (d + \gamma)I(t), \]

\[ \frac{dR(t)}{dt} = \gamma I(t) - dR(t), \]

where \( S(t), I(t) \) and \( R(t) \) are the numbers of susceptible, infective and removed individuals at time \( t \), respectively. \( \Lambda > 0 \) is the recruitment rate of the population, \( d > 0 \) is the natural death rate of the population and \( \gamma > 0 \) is the natural recovery rate of the infective individuals.
In the following, we will investigate the effect of time delay on the dynamics of system (3). We suppose that the initial condition for system (3) takes the form

\[ S(\theta) = \phi_1(\theta), \quad I(\theta) = \phi_2(\theta), \quad R(\theta) = \phi_3(\theta), \]

with \( \phi_i(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1, 2, 3, \) (4)

where \((\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_+),\) which is the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \(\mathbb{R}^3_+,\) where \(\mathbb{R}^3_+ = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0\}.\)

By the fundamental theory of functional differential equations \([8, 9],\) system (3) has a unique solution \((S(t), I(t), R(t))\) satisfying the initial condition (4).

The rest of the paper is organized as follows. In Section 2, we first discuss the existence and stability of equilibria of system (3), and then we give the conditions at which Hopf bifurcation occurs. In Section 3, we consider the global existence of bifurcating periodic solutions. In Section 4, we will give some numerical simulations to support the theoretical prediction. In Section 5, a brief discussion is given.

2. Stability and Hopf bifurcation analysis

In this section, we first investigate the existence of equilibria, and then, by analysing the corresponding characteristic equations of each equilibria, we discuss the stability of equilibria and local Hopf bifurcation of the endemic equilibrium of system (3).

2.1. Equilibria and their existence

It’s easy to see that, irrespective of the parameter values, system (3) always possesses a disease-free equilibrium \(E_0(\Lambda/d, 0, 0).\)

Next, we calculate the basic reproduction number of model (3). By the next generation method in [23], we have

\[ \mathcal{F} = \begin{pmatrix} \beta_1 SI \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} \frac{\beta_2 I}{m+I} SI + (d + \gamma)I \\ dR - \gamma I \\ \left( \beta_1 - \frac{\beta_2 I}{m+I} \right) S(t)I(t) + dS(t) - \Lambda \end{pmatrix}. \]

It follows that

\[ F = \begin{pmatrix} \beta_1 \Lambda \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} d + \gamma \\ -\gamma \\ d \end{pmatrix}, \]

giving

\[ V^{-1} = \frac{1}{d(d + \gamma)} \begin{pmatrix} d & 0 \\ \gamma & d + \gamma \end{pmatrix}, \]

and the basic reproduction number for the model (3) is

\[ R_0 = \rho(FV^{-1}) = \frac{\beta_1 \Lambda}{d(d + \gamma)}. \] (5)
We next discuss the existence of endemic equilibrium.

**Lemma 2.1:** Solutions of system (3) with initial condition (4) are positive and ultimately bounded for all $t \geq 0$.

**Proof:** Assume $(S(t), I(t), R(t))$ is a solution of system (3) with initial condition (4). Let us consider $I(t)$ for $t \geq 0$. It follows from the second equation of system (3) that

$$I(t) = I(0) e^{\int_0^t (\beta_1 - \beta_2 I(s-\tau)/m + l(s-\tau)) S(s) - (d+\gamma)) \, ds}.$$

From the initial condition (4), we have $I(t) > 0$, for $t \geq 0$. Then, from the third equation of system (3), we have

$$\frac{dR(t)}{dt} = \gamma I(t) - dR(t) > -dR(t).$$

A comparison argument shows that

$$R(t) \geq R(0) e^{-dt}.$$

From the initial condition (4), we have $R(t) > 0$, for $t \geq 0$.

Next, we prove that $S(t)$ is positive. Assume the contrary, then let $t_1$ be the first time such that $S(t_1) = 0$. By the first equation of (3), we have

$$\frac{dS(t)}{dt} \bigg|_{t=t_1} = \Lambda > 0.$$

This means $S(t) < 0$ for $t \in (t_1 - \epsilon, t_1)$, where $\epsilon$ is an arbitrarily small positive constant. This leads to a contradiction. It follows that $S(t)$ is always positive for $t \geq 0$.

Let $N(t) = S(t) + I(t) + R(t)$. From Equation (3), we have

$$\frac{dN(t)}{dt} = \Lambda - dN(t).$$

Therefore, $0 < N(t) < \Lambda/d + \epsilon$ for all large $t$, where $\epsilon$ is an arbitrarily small positive constant. Thus, $S(t), I(t), R(t)$ are ultimately bounded. This ends the proof.  

**Theorem 2.2:** If $R_0 < 1$, system (3) has no endemic equilibria. If $R_0 > 1$, system (3) has an only endemic equilibrium.
\textbf{Proof:} Suppose $E^*(S^*, I^*, R^*)$ is an endemic equilibrium, then $(S^*, I^*, R^*)$ satisfies
\begin{align*}
\Lambda - g(I^*)S^* - dS^* &= 0, \\
g(I^*)S^* - (d + \gamma)I^* &= 0, \\
\gamma I^* - dR^* &= 0.
\end{align*}
(7)
It follows that
\begin{align*}
S^* &= \frac{\Lambda}{g(I^*)I^* + d}, \\
R^* &= \frac{\gamma I^*}{d},
\end{align*}
(8)
and $I^*$ determined by
\begin{equation}
\frac{\Lambda}{g(I^*)I^* + d} = \frac{d + \gamma}{g(I^*)},
\label{eq:9}
\end{equation}
where $g(I) = \beta_1 - \beta_2 I/(m + I)$. Equation (9) is equivalent to
\begin{equation}
\Lambda g(I^*) = (d + \gamma)(g(I^*)I^* + d).
\label{eq:10}
\end{equation}
Denote
\begin{align*}
G(I) &= \Lambda g(I), \\
H(I) &= (d + \gamma)(g(I)I + d).
\end{align*}
(11)
From Lemma 2.1 and Equation (6), we have $I^* \in [0, \Lambda/d]$. From Equation (11), we have
\begin{align*}
G(0) &= \beta_1 \Lambda, \\
H(0) &= (d + \gamma)d, \\
G \left( \frac{\Lambda}{d} \right) &= \Lambda g \left( \frac{\Lambda}{d} \right), \\
H \left( \frac{\Lambda}{d} \right) &= (d + \gamma)d + (d + \gamma)g \left( \frac{\Lambda}{d} \right) \frac{\Lambda}{d}.
\end{align*}
(12)
Hence, for $R_0 > 1$, we have
\begin{equation}
G(0) > H(0), \\
G \left( \frac{\Lambda}{d} \right) = \Lambda g \left( \frac{\Lambda}{d} \right) < (d + \gamma)g \left( \frac{\Lambda}{d} \right) \frac{\Lambda}{d} < H \left( \frac{\Lambda}{d} \right).
\label{eq:13}
\end{equation}
For $R_0 < 1$, we have
\begin{equation}
G(0) < H(0). 
\label{eq:14}
\end{equation}
Moreover, we can compute that
\begin{align*}
G'(I) &= -\frac{\Lambda \beta_2 m}{(m + I)^2} < 0, \\
H'(I) &= (d + \gamma) \left( \beta_1 - \frac{\beta_2 (2mI + I^2)}{(m + I)^2} \right) > 0.
\end{align*}
(15)
Note that $\beta_1 \geq \beta_2 > 0$. From Equation (15), one can easily prove that $G(I)$ is monotone decreasing and $H(I)$ is monotone increasing. Hence, from Equations (13)–(15), one can
verify that if \( R_0 > 1 \), the two curves \( G(I) \) and \( H(I) \) have only one positive intersection in \([0, \Lambda/d]\), which gives only one endemic equilibrium. However, if \( R_0 < 1 \), it follows that the two curves \( G(I) \) and \( H(I) \) has no intersection in \([0, +\infty)\), which implies that there is no endemic equilibria. This ends the proof.

2.2. Stability of disease-free equilibrium

Suppose that \( \bar{E}(\bar{S}, \bar{I}, \bar{R}) \) is an equilibrium of system (3), then the linearizing system (3) at the equilibrium \( \bar{E} \) is

\[
\begin{align*}
\frac{dS(t)}{dt} &= -\left( \beta_1 - \frac{\beta_2 \bar{I}}{m + I} \right) \bar{I}(t)S(t) - dS(t) - \beta_1 \bar{SI} + \beta_2 \bar{SI} \frac{mI(t - \tau)}{(m + I)^2}, \\
\frac{dI(t)}{dt} &= \left( \beta_1 - \frac{\beta_2 \bar{I}}{m + I} \right) \bar{I}(t)S(t) + (\beta_1 \bar{S} - d - \gamma)I - \beta_2 \bar{SI} \frac{mI(t - \tau)}{(m + I)^2}, \\
\frac{dR(t)}{dt} &= \gamma I(t) - dR(t),
\end{align*}
\]

(16)

It’s easy to see that the characteristic equation of system (3) at \( E_0 \) is

\[
\det \begin{bmatrix}
\lambda + d & \frac{\beta_1 \Lambda}{d} & 0 \\
0 & \lambda + d + \gamma - \frac{\beta_1 \Lambda}{d} & 0 \\
0 & -\gamma & \lambda + d
\end{bmatrix} = 0,
\]

(17)

which is equivalent to

\[
(\lambda + d)(\lambda + d + \gamma - \frac{\beta_1 \Lambda}{d}) = 0.
\]

(18)

It’s easy to see that, when \( R_0 < 1 \), Equation (18) have three negative roots, when \( R_0 > 1 \), Equation (18) has one positive root. Thus we have

**Theorem 2.3:**

(i) For any time delay \( \tau \geq 0 \), the disease-free equilibrium \( E_0 \) is locally asymptotically stable if \( R_0 < 1 \).

(ii) For any time delay \( \tau \geq 0 \), the disease-free equilibrium \( E_0 \) is always unstable if \( R_0 > 1 \).

2.3. Stability and Hopf bifurcation of endemic equilibrium

In this subsection, we suppose that \( R_0 > 1 \) and \( E^* \) is an endemic equilibrium satisfying Equations (7)–(9). We first investigate the stability of \( E^* \) with \( \tau = 0 \).

It is easy to prove the following Lemma.

**Lemma 2.4:** The plane \( S + I + R = \Lambda/d \) is an invariant manifold of system (3), which is globally attractive in \( R^3_+ \).

**Theorem 2.5:** If the condition \( R_0 > 1 \) holds, then when \( \tau = 0 \), the positive equilibrium \( E^* \) is globally stable in \( R^3_+ \).
Proof: When $\tau = 0$, system (3) becomes

$$
\frac{dS(t)}{dt} = \Lambda - \left( \beta_1 - \frac{\beta_2 I(t)}{m + I(t)} \right) S(t)I(t) - dS(t),
$$

$$
\frac{dI(t)}{dt} = \left( \beta_1 - \frac{\beta_2 I(t)}{m + I(t)} \right) S(t)I(t) - (d + \gamma)I(t),
$$

(19)

$$
\frac{dR(t)}{dt} = \gamma I(t) - dR(t).
$$

Noting that the variable $R$ only occurs in the third equation of system (19), from Lemma 2.4, it is enough to investigate the subsystem

$$
\frac{dS(t)}{dt} = \Lambda - \left( \beta_1 - \frac{\beta_2 I(t)}{m + I(t)} \right) S(t)I(t) - dS(t) = f_1(S,I),
$$

$$
\frac{dI(t)}{dt} = \left( \beta_1 - \frac{\beta_2 I(t)}{m + I(t)} \right) S(t)I(t) - (d + \gamma)I(t) = f_2(S,I).
$$

(20)

Taking a Dulac function $D = 1/S(t)I(t)$. Then, we have

$$
\frac{\partial (Df_1)}{\partial S} + \frac{\partial (Df_2)}{\partial I} = - \left( \frac{\Lambda}{S^2 I} + \frac{\beta_2 m}{(m + I)^2} \right) < 0.
$$

From Bendixson–Dulac theorem [31], we know that system (20) does not have a limit cycle in $\mathbb{R}^2_+$. Hence, system (19) has no limit cycle in $\mathbb{R}^3_+$. When $R_0 > 1$, by Theorem 2.3, $E_0$ is a hyperbolic unstable saddle point and repels solutions in its neighbourhood. Due to the hyperbolicity of $E_0$, it is not part of any cycle chain in $\mathbb{R}^3_+$. Thus, every bounded forward orbits of Equation (19) in $\mathbb{R}^3_+$ converges towards the unique endemic equilibrium $E^*$. Therefore, $E^*$ is global asymptotically stable. The proof is complete.

In what follows, using time delay as the bifurcation parameter, we investigate the Hopf bifurcation for system (3) and the stability of $E^*$ by using the method in [16,19].

The characteristic equation of system (3) at $E^*$ is

$$
\det \begin{bmatrix}
\lambda - a_1 & -a_2 - a_7 e^{-\lambda \tau} & 0 \\
-a_3 & \lambda - a_4 + a_7 e^{-\lambda \tau} & 0 \\
0 & -a_5 & \lambda - a_6
\end{bmatrix} = 0,
$$

(21)

where $a_1 = -d - \beta_1 I^* + \beta_2 I^2/(m + I^*)$, $a_2 = -\beta_1 S^* + \beta_2 S^* I^*/(m + I^*)$, $a_3 = \beta_1 I^* - \beta_2 I^2/(m + I^*)$, $a_4 = -(d + \gamma) + \beta_1 S^* - \beta_2 S^* I^*/(m + I^*)$, $a_5 = \gamma$, $a_6 = -d$, $a_7 = m\beta_2 S^* I^*/(m + I^*)^2$. Equation (21) is equivalent to

$$
(\lambda - a_6)[\lambda^2 + b_1 \lambda + b_2 + (b_3 \lambda + b_4) e^{-\lambda \tau}] = 0,
$$

(22)

where $b_1 = -(a_1 + a_4)$, $b_2 = a_1 a_4 - a_2 a_3$, $b_3 = a_7$, $b_4 = -a_7(a_1 + a_3)$. 
Because $\lambda = -d$ is always a negative root of Equation (22), thus we only need to investigate the roots of the second factor of Equation (22), which is

$$\lambda^2 + b_1 \lambda + b_2 + (b_3 \lambda + b_4) e^{-\lambda \tau} = 0, \quad (23)$$

Obviously, $i\omega$ is a root of Equation (23) if and only if $\omega$ satisfies

$$-\omega^2 + b_1 \omega + b_2 + (b_3 \omega + b_4) (\cos \omega \tau - i \sin \omega \tau) = 0.$$

Separating the real and imaginary parts, we have

$$\omega^2 - b_2 = b_4 \cos \omega \tau + b_3 \omega \sin \omega \tau,$$

$$b_1 \omega = b_4 \sin \omega \tau - b_3 \omega \cos \omega \tau,$$

which is equivalent to

$$\omega^4 + (b_1^2 - 2b_2 - b_3^2) \omega^2 + b_2^2 - b_4^2 = 0. \quad (25)$$

Let $z = \omega^2$ and denote $p = b_1^2 - 2b_2 - b_3^2, q = b_2^2 - b_4^2$. Then, Equation (25) becomes

$$z^2 + pz + q = 0. \quad (26)$$

**Lemma 2.6:** For the polynomial Equation (26), we have the following results.

(i) If $q < 0$ or $q = 0$ and $p < 0$, then Equation (26) has only one positive root.

(ii) If $q > 0$ and $p > 0$, or $\Delta = p^2 - 4q < 0$, then Equation (26) has no positive root.

(iii) If $q > 0$, $p < 0$ and $\Delta = p^2 - 4q \geq 0$, then Equation (26) has at least one positive root.

When $q > 0$, $p < 0$ and $\Delta = p^2 - 4q > 0$, then Equation (26) has two positive roots denoted by $z_1 = (-p - \sqrt{p^2 - 4q})/2$ and $z_2 = (-p + \sqrt{p^2 - 4q})/2$, respectively. Then, Equation (25) has two positive roots

$$\omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}.$$  

From Equation (24), we have

$$\cos \omega \tau = \frac{b_4 \omega^2 - b_2 b_4 - b_1 b_3 \omega^2}{(b_4^2 + b_3^2 \omega^2)}.$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k} \left\{ \arccos \left( \frac{b_4 \omega^2 - b_2 b_4 - b_1 b_3 \omega^2}{(b_4^2 + b_3^2 \omega^2)} \right) + 2j\pi \right\}, \quad (27)$$

where $k = 1, 2; \ j = 0, 1, 2, \ldots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of Equation (23) with $\tau = \tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k \in \{1, 2\}} \{ \tau_k^{(0)} \}, \quad \omega_0 = \omega_{k_0}. \quad (28)$$

Note that, when $\tau = 0, E^*$ is global asymptotically stable. Till now, we can employ a result from Ruan and Wei [16] to analyse Equation (23), which is stated as follows.
Lemma 2.7: Consider the exponential polynomial
\[
P(\lambda, e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) = \lambda^n + P_1^{(0)} \lambda^{n-1} + \cdots + P_n^{(0)} + \left( p_1^{(1)} \lambda^{n-1} + \cdots + P_{n-1}^{(1)} + p_n^{(1)} \right) e^{-\lambda \tau_1} + \cdots + \left( p_1^{(m)} \lambda^{n-1} + \cdots + P_{n-1}^{(m)} + p_n^{(m)} \right) e^{-\lambda \tau_m},
\]
where \( \tau_i \geq 0 (i = 1, 2, \ldots, m) \) and \( p_j^{(i)} (i = 0, 1, \ldots, m; j = 1, 2, \ldots, n) \) are constants. As \((\tau_1, \tau_2, \ldots, \tau_m) \) vary, the sum of the order of the zeros of \( P(\lambda, e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) \) on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Applying Lemmas 2.6 and 2.7 and the discussion above, we obtain the following Lemma.

Lemma 2.8: For the second degree transcendental Equation (23), we have

(i) if \( q > 0 \) and \( p > 0 \), or \( \Delta = p^2 - 4q < 0 \), then all roots of Equation (23) have negative real parts for all \( \tau \geq 0 \);

(ii) if either \( q < 0 \) or \( q = 0 \) and \( p < 0 \), or \( q > 0 \), \( p < 0 \) and \( \Delta = p^2 - 4q \geq 0 \), then all root of Equation (23) have negative real parts for \( \tau \in [0, \tau_0) \).

Let
\[
\lambda(\tau) = \alpha(\tau) + i\omega(\tau)
\]
be the root of Equation (23) near \( \tau = \tau_k^{(j)} \) satisfying \( \alpha(\tau_k^{(j)}) = 0 \) and \( \omega(\tau_k^{(j)}) = \omega_k \). Then, from Lemma 8 in [19], we have the following transversality condition.

Lemma 2.9: Suppose that \( z_k = \omega_k^2 \) and \( 2z_k + p \neq 0 \). Then
\[
\frac{d(\text{Re} \lambda(\tau_k^{(j)}))}{d\tau} \neq 0,
\]
and \( d(\text{Re} \lambda(\tau_k^{(j)})) / d\tau \) has the same sign with \( 2z_k + p \).

The proof of Lemma 2.9 is similar to that in the proof of Lemma 8 in [19], here we omit it. Then, from the above discussion and Lemmas 2.6–2.9, we have the following theorem.

Theorem 2.10: Suppose \( \tau_k^{(j)}, \omega_0, \tau_0 \) are defined by Equations (27) and (28), respectively. Then

(i) if \( q > 0 \) and \( p > 0 \), or \( \Delta = p^2 - 4q < 0 \), the endemic equilibrium \( E^* \) of system (3) is locally asymptotically stable for all \( \tau \geq 0 \);

(ii) if either \( q < 0 \) or \( q = 0 \) and \( p < 0 \), or \( q > 0 \), \( p < 0 \) and \( \Delta = p^2 - 4q \geq 0 \), the endemic equilibrium \( E^* \) of system (3) is locally asymptotically stable for \( \tau \in [0, \tau_0) \);

(iii) if the conditions of (ii) are satisfied and \( 2z_k + p \neq 0 \), then system (3) exhibits Hopf bifurcation at the endemic equilibrium \( E^* \) when \( \tau \) pass through \( \tau = \tau_k^{(j)} \).
3. Global continuation of local Hopf bifurcation

In this section, we study the global continuation of periodic solutions bifurcating from the positive equilibrium \( E^* \) of system (3).

Throughout this section, we follow closely the notations in [25]. For simplification of notations, setting \( z(t) = (z_1(t), z_2(t), z_3(t))^T = (S(t), I(t), R(t))^T \), we may rewrite system (3) as the following functional differential equation:

\[
\dot{z}(t) = \mathcal{F}(z(t), \tau, \rho),
\]

where \( z(t) = (z_1(t), z_2(t), z_3(t))^T = (z_1(t + \theta), z_2(t + \theta), z_3(t + \theta))^T \in C([-\tau, 0], \mathbb{R}^3). \) It is obvious that if \( R_0 > 1 \) holds, then system (3) has a semi-trivial equilibrium \( E_0(\Lambda/d, 0, 0) \) and a positive equilibrium \( E^*(S^*, I^*, R^*) \). Following, the work of Wu [25], we need to define

\[
X = C([-\tau, 0], \mathbb{R}^3),
\]

\[
\Gamma = Cl\{(z, \tau, \rho) \in X \times \mathbb{R} \times \mathbb{R}^+; \text{ } z \text{ is a nonconstant periodic solution of (29)}\},
\]

\[
\mathcal{N} = \{ (\bar{z}, \bar{\tau}, \bar{\rho}); \mathcal{F}(\bar{z}, \bar{\tau}, \bar{\rho}) = 0 \}.
\]

Let \( \ell(E^*, \tau_j, 2\pi/\omega_0) \) denote the connected component passing through \( (E^*, \tau_j, 2\pi/\omega_0) \) in \( \Gamma \), where \( \tau_j \) and \( \omega_0 \) are defined by Section 3. From Theorem 2.10, we know that \( \ell(E^*, \tau_j, 2\pi/\omega_0) \) is nonempty.

We first state the global Hopf bifurcation theory due to Wu [25] for functional differential equations.

**Lemma 3.1:** Assume that \((z_*, \tau, \rho)\) is an isolated centre satisfying the hypotheses \((A_1) - (A_4)\) in [25]. Denote by \( \ell(z_*, \tau, \rho) \) the connected component of \((z_*, \tau, \rho)\) in \( \Gamma \). Then either

(i) \( \ell(z_*, \tau, \rho) \) is unbounded, or

(ii) \( \ell(z_*, \tau, \rho) \) is bounded, \( \ell(z_*, \tau, \rho) \cap \Gamma \) is finite and

\[
\sum_{(z, \tau, \rho) \in \ell(z_*, \tau, \rho) \cap \mathcal{N}} \gamma_m(z_*, \tau, \rho) = 0,
\]

for all \( m = 1, 2, \ldots \), where \( \gamma_m(z_*, \tau, \rho) \) is the mth crossing number of \((z_*, \tau, \rho)\) if \( m \in J(z_*, \tau, \rho) \), or it is zero if otherwise.

Clearly, if (ii) in Lemma 3.1 is not true, then \( \ell(z_*, \tau, \rho) \) is unbounded. Thus, if the projections of \( \ell(z_*, \tau, \rho) \) onto \( z \)-space and onto \( \rho \)-space are bounded, then the projection onto \( \tau \)-space is unbounded. Further, if we can show that the projection of \( \ell(z_*, \tau, \rho) \) onto \( \tau \)-space is away from zero, then the projection of \( \ell(z_*, \tau, \rho) \) onto \( \tau \)-space must include interval \([\tau, +\infty)\). Following this idea, we can prove our results on the global continuation of local Hopf bifurcation.

From Lemma 2.1, it is easy to have

**Lemma 3.2:** If the condition \( R_0 > 1 \) holds, then all nonconstant periodic solutions of Equation (3) with initial condition (4) are uniformly bounded.
Lemma 3.3: If $R_0 > 1$, then system (3) has no nonconstant periodic solution with period $\tau$.

Proof: Suppose for a contradiction that system (3) has nonconstant periodic solution with period $\tau$. Then system (19) of ordinary differential equations has nonconstant periodic solution.

System (19) have the same equilibria to system (3), i.e. $E_0(\Lambda/d, 0, 0)$ and a positive equilibrium $E^*(S^*, I^*, R^*)$. Note that $I$-axis and $R$-axis are the invariable manifold of system (19) and the orbits of system (19) do not intersect each other. Thus, there is no solution crosses the coordinate axis.

On the other hand, note the fact that if system (19) has a periodic solution, then there must be the equilibrium in its interior and $E_0$ are located on the coordinate axis. Thus, we conclude that the periodic orbit of system (19) must lie in the first quadrant. From Theorem 2.5, the positive equilibrium is asymptotically stable and global stable in $\mathbb{R}^3_+$, thus, there is no periodic orbit in the first quadrant. This ends the proof.

Theorem 3.4: Let $\omega_0$ and $\tau_j (j = 0, 1, \ldots)$ be defined in (27) and (28). If $R_0 > 1$ and the conditions of (iii) in Theorem 2.10 holds, then system (3) have at least $j-1$ periodic solutions for every $\tau > \tau_j (j = 1, 2, \ldots)$.

Proof: It is sufficient to prove that the projection of $\ell(E^*, \tau_j, 2\pi/\omega_0)$ onto $\tau$-space is $[\bar{\tau}, +\infty)$ for each $j > 0$, where $\bar{\tau} \leq \tau_j$.

The characteristic matrix of (29) at an equilibrium $\bar{z} = (\bar{z}^{(1)}, \bar{z}^{(2)}, \bar{z}^{(3)}) \in \mathbb{R}^3$ takes the following form:

$$\Delta(\bar{z}, \tau, p)(\lambda) = \lambda \text{Id} - D\mathcal{F}(\bar{z}, \bar{\tau}, \bar{p})(e^{\lambda \text{Id}}),$$

(30)

where $(\bar{z}, \bar{\tau}, \bar{p})$ is called a centre if $\mathcal{F}(\bar{z}, \bar{\tau}, \bar{p}) = 0$ and $\det(\Delta(\bar{z}, \bar{\tau}, \bar{p})(2\pi/pi)) = 0$. A centre is said to be isolated if it is the only centre in some neighbourhood of $(\bar{z}, \bar{\tau}, \bar{p})$. It follows from Equation (30) that

$$\det(\Delta(E_0, \tau, p)(\lambda)) = (\lambda + d)^2\left(\lambda + d + \gamma - \frac{b_1 \Lambda}{d}\right) = 0,$$

(31)

and

$$\det(\Delta(E^*, \tau, p)(\lambda)) = (\lambda - d)(\lambda^2 + b_1 \lambda + b_2 + (b_3 \lambda + b_4) e^{-\lambda \tau}) = 0,$$

(32)

where $b_1, b_2, b_3,$ and $b_4$ are defined as in Section 3. From the discussion in Section 3, each of Equations (31) and (32) has no purely imaginary root provided that $R_0 < 1$. Thus, we conclude that Equation (29) has no the centre of the form as $(E_0, \tau, p)$ and $(E^*, \tau, p)$.

On the other hand, from the discussion in Section 3 about the local Hopf bifurcation, it is easy to verify that $(E^*, \tau_j, 2\pi/\omega_0)$ is an isolated centre. There exists $\epsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to \mathcal{C}$, such that $\det(\Delta(\lambda(\tau))) = 0$, $|\lambda(\tau) - \omega_0| < \epsilon$ for
all \( \tau \in [\tau_j - \delta, \tau_j + \delta] \) and
\[
\lambda(\tau_j) = i \omega_0, \quad \left. \frac{d \text{Re} \lambda(\tau)}{d \tau} \right|_{\tau = \tau_j} > 0
\]

Let
\[
\Omega_{\epsilon, 2\pi/\omega_0} = \left\{ (\eta, p); 0 < \eta < \epsilon, \left| p - \frac{2\pi}{\omega_0} \right| < \epsilon \right\}.
\]

It is easy to verify that on \([\tau_j - \delta, \tau_j + \delta] \times \partial \Omega_{\epsilon, 2\pi/\omega_0},\)
\[
\det \left( \Delta (E^*, \tau, p) \left( \eta + \frac{2\pi}{p} i \right) \right) = 0
\]
if and only if \( \eta = 0, \tau = \tau_j, p = 2\pi/\omega_0. \)

Therefore, the hypotheses \((A_1) - (A_4)\) in [25] are satisfied. Moreover, if we define
\[
H^\pm \left( E^*, \tau_j, \frac{2\pi}{\omega_0} \right) (\eta, p) = \det \left( \Delta (E^*, \tau_j \pm \delta, p) \left( \eta + \frac{2\pi}{p} i \right) \right),
\]
then we have the crossing number of isolated centre \((E^*, \tau_j, 2\pi/\omega_0)\) as follows:
\[
\gamma \left( E^*, \tau_j, \frac{2\pi}{\omega_0} \right) = \text{deg}_B \left( H^- \left( E^*, \tau_j, \frac{2\pi}{\omega_0} \right), \Omega_{\epsilon, 2\pi/\omega_0} \right) \\
- \text{deg}_B \left( H^+ \left( E^*, \tau_j, \frac{2\pi}{\omega_0} \right), \Omega_{\epsilon, 2\pi/\omega_0} \right) = -1.
\]

Thus, we have
\[
\sum_{(\bar{z}, \bar{\tau}, \bar{p}) \in C(E^*, \tau_j, 2\pi/\omega_0)} \gamma(\bar{z}, \bar{\tau}, \bar{p}) < 0,
\]
where \((\bar{z}, \bar{\tau}, \bar{p})\) has all or parts of the form \((E^*, \tau_k, 2\pi/\omega_0)(k = 0, 1, \ldots).\)

It follows from the Lemma 3.1 that the connected component \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) through \((E^*, \tau_j, 2\pi/\omega_0)\) in \(\Gamma\) is unbounded. From Equation (27), we know that if \(R_0 > 1\) holds, for \(j \geq 1,
\]
\[
\tau_j = \frac{1}{\omega_0} \left\{ \cos^{-1} \left( \frac{b_4 \omega_0^2 - b_2 b_4 - b_1 b_3 \omega_0^2}{b_4^2 + b_3^2 \omega_0^2} \right) + 2j\pi \right\} > \frac{2\pi}{\omega_0},
\]

Now we prove that the projection of \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) onto \(\tau\)-space is \([\bar{\tau}, +\infty),\) where \(\bar{\tau} \leq \tau_j.\) Clearly, it follows from the proof of Lemma 3.3 that system (3) with \(\tau = 0\) has no nontrivial periodic solution. Hence, the projection of \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) onto \(\tau\)-space is away from zero.

For a contradiction, we suppose that the projection of \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) onto \(\tau\)-space is bounded, this means that the projection of \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) onto \(\tau\)-space is included in an interval \((0, \tau^*).\) Noticing \(2\pi/\omega_0 < \tau_j\) and applying Lemma 3.3, we have \(0 < p < \tau^*\) for \((z(t), \tau, p)\) belonging to \(\ell(E^*, \tau_j, 2\pi/\omega_0).\) Applying Lemma 3.2, we know that the projection of \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) onto \(\tau\)-space is bounded. So the component of \(\ell(E^*, \tau_j, 2\pi/\omega_0)\) is bounded. It
contradicts our conclusion that \( \ell(E^*, \tau_j, 2\pi/\omega_0) \) is unbounded. The contradiction implies that the projection of \( \ell(E^*, \tau_j, 2\pi/\omega_0) \) onto \( \tau \)-space is unbounded above.

Hence, system (3) have at least \( j-1 \) periodic solutions for every \( \tau > \tau_j \), \( (j = 1, 2, \ldots) \). This completes the proof.

4. Numerical simulation

Example 4.1: In this case, we set \( \Lambda = 15, \beta_1 = 0.0008, \beta_2 = 0.0006, m = 30, d = 0.05, \gamma = 0.2 \), where ‘day’ is used as the unit of time. From Equation (5), we compute \( R_0 = 0.96 < 1 \). From Theorem 2.2, we know that system (3) has only a disease-free equilibrium \( E_0(300, 0, 0) \).

From Theorem 2.3, we know that the disease-free equilibrium \( E_0 \) is always asymptotically stable for any time delay \( \tau \geq 0 \). Figure 1 shows that \( E_0 \) is asymptotically stable, the trajectories of \( I(t) \) always converge to zero for \( \tau \) taking some different values.

This example means that, if the basic reproduction number \( R_0 < 1 \), then the disease-free equilibrium is always asymptotically stable for all \( \tau \geq 0 \). In the sense of infectious diseases, we conclude that the time delay in media coverage cannot influence the stability of the disease-free equilibrium when \( R_0 < 1 \).

Example 4.2: In this case, we set \( \Lambda = 15, \beta_1 = 0.002, \beta_2 = 0.0018, m = 30, d = 0.05, \gamma = 0.2 \), where ‘day’ is used as the unit of time. From Equation (5), we compute \( R_0 = 2.4 > 1 \). From Equation (7), we get a disease-free equilibrium \( E_0 = (300, 0, 0) \) and an endemic equilibrium \( E^* = (197.1728, 20.5654, 82.2617) \) of system (3). From the algorithm of Section 2.3, we can compute \( \tau_0 \approx 18.8644, \tau_1 \approx 143.5680, \) and \( 2z_k + p = 0.2438 > 0 \). Thus, from Theorems 2.3 and 2.10, we know that the disease-free equilibrium \( E_0 \) is unstable for all \( \tau \geq 0 \), the endemic equilibrium \( E^* \) is stable for \( \tau \in (0, 18.8644) \). When \( \tau \) crosses \( \tau_0 \), a family of periodic orbits bifurcate from \( E^* \).

![Figure 1](image.png)

**Figure 1.** The trajectories of \( I(t) \) with \( \tau = 5, 15, 25, 35, 45, 55 \), respectively.
Figure 2. The trajectories and phase graphs of system (3) with $\tau = 18$, $E^*$ is stable. Initial values: (200, 30, 20).

Figure 3. The trajectories and phase graphs of system (3) with $\tau = 20$, $E^*$ becomes unstable, and periodic orbits bifurcating from $E^*$. Initial values: (200, 30, 20).

Figure 2 shows that the endemic equilibrium $E^*$ is stable with $\tau = 18$. Figure 3 shows that the endemic equilibrium $E^*$ is unstable and a periodic orbit bifurcates from $E^*$ with $\tau = 20$. Figure 4 shows that the endemic equilibrium $E^*$ is still unstable and a periodic orbit bifurcates from $E^*$ with $\tau = 30$. We can see from Figures 3–4 that the period and amplitude of the oscillation is increasing with the increasing of time delay. Further more, Figure 5 shows that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of $\tau_1 \approx 143.5680$.

This example means that, if the basic reproduction number $R_0 > 1$, then the stability of the endemic equilibrium will be affected by the time delay in media coverage. In the sense of infectious diseases, we conclude that if the time delay in media coverage is less than the critical value $\tau_0$, then the epidemic will eventually become endemic disease. However, if
Figure 4. The trajectories and phase graphs of system (3) with $\tau = 50$, $E^*$ is unstable, and periodic orbits bifurcating from $E^*$. Initial values: (200, 30, 20).

Figure 5. The trajectories and phase graphs of system (3) with $\tau = 150$, $E^*$ is unstable, and periodic orbits bifurcating from $E^*$. Initial values: (200, 30, 20).

the time delay in media coverage is larger than the critical value $\tau_0$, then the epidemic will oscillate periodically.

5. Discussion

In this paper, we proposed an SIR epidemic model incorporating media coverage with time delay. We first investigated the positivity and boundedness of the solutions of system (3), we showed that the solution of system (3) with the initial condition (4) is positive and bounded. Our results also show that, when $R_0 < 1$, system (3) has only disease-free equilibrium, when $R_0 > 1$, system (3) has a disease-free equilibrium and an endemic equilibrium.

Second, we studied the stability of the disease-free and the endemic equilibrium. Our results show that the disease-free equilibrium is always asymptotically stable for all $\tau \geq 0$
when the basic reproduction number $R_0 < 1$, and is always unstable for all $\tau \geq 0$ when the basic reproduction number $R_0 > 1$. This is to say, the time delay in media coverage cannot influence the stability of the disease-free equilibrium. The simulation results also show that the effect of delay can be ignored for disease-free equilibrium.

However, the stability of the endemic equilibrium will be affected by the time delay in media coverage. We found that, when $R_0 > 1$ and the time delay $\tau$ equal to zero, the endemic equilibrium is globally stable. When the time delay $\tau > 0$, there exists a critical value of $\tau$, such that the stability of the endemic equilibrium changed and periodic oscillations occurred when the time delay pass through this critical value. Furthermore, we show that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of time delay.

These results mean that, when the time delay isn’t too large, the epidemic will eventually become endemic disease. However, if the delay of information about and appraisal of an epidemic on media coverage is large enough, it will lead to repeated episodes of epidemic, and then it is unfavourable for the containment of the epidemic. Based on these analysis, we suggest that it is helpful for controlling epidemic to communicate about the outbreak of epidemic with the public as soon as possible.

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