Incompatibility of Observables as State-Independent Bound of Uncertainty Relations

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For a pair of observables, they are called “incompatible”, if and only if the commutator between them does not vanish, which represents one of the key features in quantum mechanics. The question is, how can we characterize the incompatibility among three or more observables? Here we explore one possible route towards this goal through Heisenberg’s uncertainty relations, which impose fundamental constraints on the measurement precisions for incompatible observables. Specifically, we quantify the incompatibility by the optimal state-independent bounds of additive variance-based uncertainty relations. In this way, the degree of incompatibility becomes an intrinsic property among the operators, i.e., state independent. Specifically, we focus on the incompatibility of spin systems. For an arbitrary, including non-orthogonal, setting of a finite number Pauli-spin operators, the incompatibility is analytically solved; the spins are maximally incompatible if and only if they are orthogonal to each other. On the other hand, the measure of incompatibility represents a versatile tool for applications such as testing entanglement of bipartite states, and EPR-steering criteria.

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Introduction—A distinguished aspect of the quantum theory is that uncertainty relations [1] represent a fundamental limitation on measurement of physical systems; it is generally impossible to simultaneously measure two complementary observables of a physical system without an “uncertainty”. Furthermore, uncertainty relations manifest many intrinsic discrepancies between classical and quantum mechanics, leading to applications such as entanglement detection [2, 3], nonlocality of quantum systems [4], and EPR-steering criteria [5–8] etc.

One of the most well-known uncertainty relations, between a pair of observables \( M_1 \) and \( M_2 \), were formulated in terms of a commutator, \([M_1, M_2] \equiv M_1 M_2 - M_2 M_1\), by Robertson [9] in 1929,

\[
\Delta M_1 \Delta M_2 \geq \frac{1}{2} \left| \langle \psi \middle| [M_1, M_2] |\psi\rangle \right| ,
\]

where \( \Delta M_i \equiv \left( \langle M_i^2 \rangle - \langle M_i \rangle^2 \right)^{1/2} \) is the standard deviation in measuring the quantum state in state \( |\psi\rangle \). This uncertainty relation appears in almost every textbook of quantum mechanics, and is regarded as fundamental, connecting the physical concept of incompatibility of observables (IO) with quantum uncertainty.

However, Robertson’s inequality cannot be regarded as complete for describing the connection between incompatibility and uncertainty. What if the state \( |\psi\rangle \) is an eigenstate of \( M_i \) or \( M_2 \)? The left-hand side becomes zero, which makes no difference if \( M_1 \) and \( M_2 \) are incompatible or not. Another problem occurs when \( |\psi\rangle \) is an eigenstate of the commutator associated with an eigenvalue zero, which makes the inequality become trivial.

These problems point to the idea that incompatibility cannot be quantified properly when it depends on a quantum state [4, 10]. In particular, Deutsch [10] proposed that uncertainty relations should be expressed in a state-independent form:

\[
U(M_1, M_2, |\psi\rangle) \geq B(M_1, M_2) ,
\]

where the functional \( U \) denotes the total uncertainty, and \( B \) labels a state-independent bound, and its value can be taken as a quantitative measure of incompatibility between the observables \( M_1 \) and \( M_2 \).

Generally, we can show that there is a prevalent issue for state-dependent bounds in uncertainty relations of any form; it can always get improved. Let us suppose we have a state-dependent bound \( B_0 \) such that \( U \geq B_0 \), we can always define a convex combination, \( B_1 \equiv \lambda U - (1 - \lambda) B_0 \) to get an improved state-dependent bound: \( U \geq B_1 \geq B_0 \). This is the reason why we are interested in state-independent bounds.

On other hand, state-independent uncertainty relations have been investigated from the information-theoretic perspective [10–18], where \( U \) is taken to be the sum of entropies of different bases of measurements, e.g., \( \{|u_i\}\) and \( \{|v_j\}\), and the lower bound \( B \) is given by the overlap of the basis vectors

\[
c(i, j) \equiv |\langle u_i | v_j \rangle|^2 ,
\]

which is employed to quantify the incompatibility of measurement (IM). An immediate application of entropic uncertainty relations includes detecting entanglement [19, 20] and EPR-steering [7, 8].
For uncertainty relations of observables [21–28], let us consider a pair of Pauli spin operators \( S_{\vec{n}}, \) and \( S_{\vec{n}} \), pointing to different directions labeled by a unit vector \( \vec{n} \), where

\[
S_{\vec{n}} = \vec{n} \cdot \vec{\sigma}.
\]

It is known [28] that a state-independent bound, minimized over all possible states \( \rho \), for the additive total uncertainty, \( \Delta^2 S_{\vec{n}} + \Delta^2 S_{\vec{n}} \), is given by

\[
\min_{\rho} (\Delta^2 S_{\vec{n}} + \Delta^2 S_{\vec{n}}) = 1 - |\vec{n} \cdot \vec{n}| \equiv \mathcal{I}(\vec{n}, \vec{n}) ,
\]

which indicates that the incompatibility, \( \mathcal{I}(\vec{n}, \vec{n}) \), between the observables \( S_{\vec{n}} \) and \( S_{\vec{n}} \) depends on the geometric relation, \( |\vec{n} \cdot \vec{n}| \), which is the counterpart of \( c(i, j) \) (see Eq. (3)) in the entropic uncertainty relations.

The spin operators are compatible (\( \mathcal{I} = 0 \)), whenever \( \vec{n}_1 \cdot \vec{n}_2 = 1 \), which can be achieved by any eigenstate of \( S_{\vec{n}} \). They are maximally incompatible (\( \mathcal{I} = \) maximized), when the spin operators are orthogonal to each other \( \vec{n}_1 \cdot \vec{n}_2 = 0 \). A similar result is applicable with \( c(i, j) \). Therefore, the entropic uncertainty relations are revealing the incompatibility of different quantum measurements, and the variance-based uncertainty relations are for observables. In this sense, they can be regarded as the two sides of the same coin, i.e.,

\[
|\vec{n}_i \cdot \vec{n}_j| \Leftrightarrow c(i, j) .
\]

The question is, what about three or more observables? How does the lower bound look like? The point is that once we obtain an explicit form of a state-independent bound of a multi-observable uncertainty relation, we can then quantify the incompatibility \( \mathcal{I}(\{M_j\}) \) for multiple observables \( \{M_j\} \) by, \( \min_{\rho} \sum_j \Delta^2 M_j \), i.e.,

\[
\sum_j \Delta^2 M_j \geq \min_{\rho} \sum_j \Delta^2 M_j \equiv \mathcal{I}(\{M_j\}) .
\]

Setting the stage—In this work, we focus mainly on our results in determining state-independent bounds for 2-by-2 Hermitian observables. We shall later present some of the results associated with spin operators. First of all, any Hermitian operator \( M_j \) can be parametrized by a number and a 3D vector, denoted by \( a_j \) and \( \vec{b}_j \), i.e., \( M_j = a_j \hat{I} + \vec{b}_j \cdot \vec{\sigma} \). For any given density matrix parametrized by \( \rho = (I + \vec{r} \cdot \vec{\sigma})/2 \), we have

\[
\Delta^2 M = \text{tr} (M^2 \rho) - \text{tr}^2 (M \rho) = b_j^2 - (\vec{b}_j \cdot \vec{r})^2 \,,
\]

which means that the variance is independent of the value of the constant \( a \). In other words, we can instead consider the variances of a group of non-orthogonal spin operators (see Eq. (4)), i.e., \( \vec{b}_j \Leftrightarrow \vec{n}_j \) or

\[
\sum_j \Delta^2 M_j \leftrightarrow \sum_j \Delta^2 S_{\vec{n}_j} .
\]

Here we no longer require the vectors \( \vec{n}_j \) to be normalized. This result is consistent with the notion of characterizing compatibility with a commutator, \( [M_i, M_j] = [\vec{b}_i \cdot \vec{\sigma}, \vec{b}_j \cdot \vec{\sigma}] \), which is independent of the values of \( a_i \) and \( a_j \).

The next goal is to determine the incompatibility of the observables of non-orthogonal spins,

\[
\mathcal{I}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) \equiv \min_{\rho} \sum_{j=1}^N \Delta^2 S_{\vec{n}_j} \,,
\]

where the number of terms \( N > 1 \) is any finite integer larger than 1. For a special case of three spins \( N = 3 \), and all spin operators are orthogonal among one another, e.g., \( \{S_x, S_y, S_z\} \), it is known [2] that

\[
\Delta^2 S_x + \Delta^2 S_y + \Delta^2 S_z \geq 2 \,,
\]

which can be saturated by any pure state of a qubit. We shall see how to recover this result as a special case.

**Main results**—Let us consider again a general density matrix of a qubit, \( \rho = (I + \vec{r} \cdot \vec{\sigma})/2 \), where \( \vec{r} = (x, y, z) \), together with \( \text{tr} (S_{\vec{n}_i} \rho) = \vec{n}_i \cdot \vec{r} \) and \( \text{tr} (S_{\vec{n}_j} \rho) = n_i^2 \). First, in terms of the vector \( \vec{r} \) of \( \rho \), we have

\[
\min_{\rho} \sum_{i=1}^N \Delta^2 S_{\vec{n}_i} = \tau_1 - \max_{\vec{r}} \sum_{i=1}^N (\vec{n}_i \cdot \vec{r})^2 \,,
\]

where \( \tau_1 = \sum_{i=1}^N n_i^2 \) does not depend on \( \vec{r} \). To minimize the total uncertainty, our strategy is to find the eigenvalues of the following 3-by-3 matrix,

\[
A \equiv \sum_{i=1}^N |\vec{n}_i \rangle \langle \vec{n}_i| \,,
\]

where we have adopted the Dirac notation to denote vectors with three real elements, e.g., \( \langle \vec{r} \rangle \equiv \vec{r} = (x, y, z)^T \). In this way, we can write

\[
|\vec{n}_1 \rangle \langle \vec{n}_2| = |\vec{n}_2 \rangle \langle \vec{n}_1| = \vec{n}_1 \cdot \vec{n}_2 .
\]

Now, let us first consider the case with two non-orthogonal spins, \( \vec{n}_1 \), and \( \vec{n}_2 \). The eigenvalue equation is given by \( |\vec{n}_1 \rangle \langle \vec{n}_1| \vec{r} + |\vec{n}_2 \rangle \langle \vec{n}_2| \vec{r} = \lambda |\vec{r}| \), which yields the following matrix equation:

\[
\begin{bmatrix}
  n_1^2 & n_1 \cdot n_2 \\
  n_2 \cdot n_1 & n_2^2
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2
\end{bmatrix}
= \lambda
\begin{bmatrix}
s_1 \\
s_2
\end{bmatrix} .
\]
where \( s_i \equiv \langle \vec{n}_i | \vec{r} \rangle \) for \( i = 1, 2 \). The maximal eigenvalue \( \lambda_{\text{max}}(A) \) is therefore given by,

\[
\lambda_{\text{max}}(A) = \frac{1}{2} \left( \tau_1 + \sqrt{(n_1^2 - n_2^2)^2 + 4(n_1 \cdot n_2)^2} \right),
\]

where \( \tau_1 = n_1^2 + n_2^2 \). In the case of Pauli spins, where \( n_1 = n_2 = 1 \), we have, \( \lambda_{\text{max}}(A) = 1 + |\vec{n}_1 \cdot \vec{n}_2| \), which reduces to the result presented earlier in Eq. (5).

For three or more spins, let us consider the characteristic equation of \( T \), given by the determinant equation, \( \det(xI - A) = 0 \), or explicitly,

\[
x^3 - \tau_1 x^2 - x \left( \tau_2 - \tau_1^2 \right)/2 - \det(A) = 0,
\]

where \( \tau_k \equiv \text{tr}(A^k) \) for \( k = 1, 2, 3 \). In fact, we can express the determinant, \( \det(A) \), in terms of the \( \tau \)'s only, i.e., \( \det A = (\tau_1^3 + 2\tau_3 - 3\tau_1\tau_2)/6 \). Therefore, the characteristic equation can be completely determined by the values of the \( \tau \)'s. Explicitly, they are given by: (i) \( \tau_1 = \sum_{i=1}^{N} n_i^2 \), (ii) \( \tau_2 = \sum_{i,j=1}^{N} (\vec{n}_i \cdot \vec{n}_j)^2 \), and (iii) \( \tau_3 = \sum_{i,j,k=1}^{N} (\vec{n}_i \cdot \vec{n}_j)(\vec{n}_j \cdot \vec{n}_k)(\vec{n}_k \cdot \vec{n}_i) \). Therefore, we should expect that the solution of Eq. (17), and also the lower bound of the uncertainty relations, depends on the products of \( \vec{n}_i \cdot \vec{n}_j \), which is consistent with Eq. (5).

Let \( z \equiv x + \tau_1/3 \), we have equivalently,

\[
z^3 - 3\alpha^2 z - 2\beta = 0,
\]

where \( \alpha = \sqrt{(3\tau_2 - \tau_1^2)/18} \), and \( \beta \equiv \tau_1^3/27 + \tau_3/6 - \tau_1\tau_2/6 \).

In the following, we shall consider the cases where all vectors are normalized, i.e., \( \sum_{i=1}^{N} n_i^2 = 1 \) for all spin operators. Then, we have \( \tau_1 = N, \tau_2 = N + 2 \sum_{i,j} (\vec{n}_i \cdot \vec{n}_j)^2 \), and \( \tau_3 = 3\tau_2 - 2N + 6 \sum_{i,j,k} (\vec{n}_i \cdot \vec{n}_j)(\vec{n}_j \cdot \vec{n}_k)(\vec{n}_k \cdot \vec{n}_i) \).

For the case of three non-orthogonal spins (\( N = 3 \)), we have \( \tau_1 = 3, \tau_2 = 3 + 2(\vec{n}_1 \cdot \vec{n}_2 + \vec{n}_2 \cdot \vec{n}_3 + \vec{n}_1 \cdot \vec{n}_3) \), and \( \tau_3 = 3\tau_2 - 6 + 6(\vec{n}_1 \cdot \vec{n}_2)(\vec{n}_2 \cdot \vec{n}_3)(\vec{n}_1 \cdot \vec{n}_3) \). Furthermore, we found that

\[
\alpha^2 = ((\vec{n}_1 \cdot \vec{n}_2)^2 + (\vec{n}_2 \cdot \vec{n}_3)^2 + (\vec{n}_1 \cdot \vec{n}_3)^2)/3
\]

is the mean value of the products of \( (\vec{n}_i \cdot \vec{n}_j)^2 \), and

\[
\beta = (\vec{n}_1 \cdot \vec{n}_2)(\vec{n}_2 \cdot \vec{n}_3)(\vec{n}_1 \cdot \vec{n}_3).
\]

The transformed characteristic polynomial in Eq. (18) can be solved analytically, we found that the largest eigenvalue is given by, \( \lambda_{\text{max}}(A) = 1 + 2\alpha \cos \left( \cos^{-1} \left( \frac{2\beta}{\alpha^3} \right) \right) /3 \), which implies that the minimal uncertainty, or the incompatibility, for three non-orthogonal spins is given by,

\[
\min_{\rho} \sum_{j=1}^{3} \Delta^2 S_{n_j} = 2 - 2\alpha \cos \left( \cos^{-1} \left( \frac{2\beta}{\alpha^3} \right) \right).
\]

More generally, for \( N \) non-orthogonal spins, we have,

\[
\min_{\rho} \sum_{j=1}^{N} \Delta^2 S_{n_j} = \frac{2N}{3} - 2\alpha \cos \left( \cos^{-1} \left( \frac{2\beta}{\alpha^3} \right) \right).
\]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( J_3 \) & \( \theta_1 \) & \( \pi/9 \) & \( 2\pi/9 \) & \( 3\pi/9 \) & \( 4\pi/9 \) & \( 5\pi/9 \) \\
\hline
\( \pi/9 \) & 0.45108 & 0.58195 & 0.75235 & 0.89526 & 0.94979 \\
\( 2\pi/9 \) & 0.58195 & 0.71897 & 0.89872 & 1.04746 & 1.09190 \\
\( 3\pi/9 \) & 0.75235 & 0.89872 & 1.09498 & 1.25782 & 1.28681 \\
\( 4\pi/9 \) & 0.89526 & 1.04746 & 1.25782 & 1.43799 & 1.44804 \\
\( 5\pi/9 \) & 0.94979 & 1.09190 & 1.28681 & 1.44804 & 1.43799 \\
\hline
\end{tabular}
\caption{Numerical value of \( I \) with \( \theta_3 = \pi/3 \) for spin-1/2 system.}
\end{table}

\[ \alpha = \sqrt{\left( \sum_{1 \leq j < h \leq N} (\vec{n}_j \cdot \vec{n}_h)^2 - N(N-3)/6 \right)/3}, \]

and

\[ \beta = \sum_{h < j < t} (\vec{n}_h \cdot \vec{n}_j)(\vec{n}_t \cdot \vec{n}_h)(\vec{n}_j \cdot \vec{n}_t) + \frac{1}{3}(N-3) \sum_{j < h} (\vec{n}_j \cdot \vec{n}_h)^2 + \frac{1}{54}N(2N-3)(N-3). \]

\begin{example}
For spin-1/2 system, with the following four directions \( \vec{n}_1' = (1, 0, 0), \vec{n}_2' = (1/2, \sqrt{3}/2, 0), \vec{n}_3' = (1/2, 1/2, 1/2), \) and \( \vec{n}_4' = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \). Then

\[ I(\vec{n}_1', \vec{n}_2', \vec{n}_3', \vec{n}_4') = 0.94955. \]
\end{example}

\begin{example}
For spin-1/2 system, with observables \( S_{\vec{n}_i}, i = 1, 2, 3 \) such that \( \cos \theta_1 = \vec{n}_2 \cdot \vec{n}_3, \cos \theta_2 = \vec{n}_1 \cdot \vec{n}_3, \cos \theta_3 = \vec{n}_1 \cdot \vec{n}_2 \) while \( \theta_3 = \pi/3 \). We can immediately obtain a numerical table of \( I \) for arbitrary given directions \( \theta_1, \theta_2 \). It is shown in Table I.

Note that, for binary observables see also the work of J. Kaniewski et al. [29].

\section{III. APPLICATIONS}

We continue by discussing some instructive applications. First, the use of uncertainty arguments to study entanglement
is well known [2]. However, their arguments are based on a
unknown global minimum. Since in this section we will dis-
cuss a mathematical formulation of entanglement detection, it
will be important to review it in detail.

Consider incompatible observables \( S_{n_i} \), if there is no si-
multaneous eigenstate of all \( S_{n_i} \), there must be a nontrivial
lower limit \( \mathcal{U} \) for the sum of the uncertainties,
\[
\sum_i \Delta^2 S_{n_i} \geq \mathcal{U},
\]
while the bound \( \mathcal{U} \) is defined as the absolute minimum of the
uncertainty sum for any quantum state. It therefore represents
a universally valid limitation of the measurement statistics of
quantum systems.

In general, a bipartite quantum systems between Alice and
Bob can be characterized by the assemblages of incompati-
ble observables, \( \{S_{n_i}\}_i \) and \( \{S_{n_j}\}_j \), with the sum uncertainty
relations formulated by
\[
\sum_j \Delta^2 S_{n_j} \geq \mathcal{U}_j.
\]
Denote the index \( j \) as the result of some permutation \( \pi \), i.e.
\( j = \pi(i) \), then the measurement statistics of separable states
are limited by the following uncertainty relation
\[
\sum_i \Delta^2 (S_{n_i} \otimes I + I \otimes S_{n_{\pi(i)}}) \geq \mathcal{U}_i + \mathcal{U}_j.
\]

To derive a experimentally feasible criterion for entangle-
ment, \( \mathcal{U}_i \) and \( \mathcal{U}_j \) must have a specific expression. Here, we can
overcome this challenge easily. To show this, we consider IO
on three incompatible observables. Take measurements \( S_{n_i} \)
and \( S_{n_0} \), work on bipartite systems respectively, then for sep-
eparable states, the measurement values are uncorrelated and the
total uncertainties are limited by sum of the local uncertainties
\[
\sum_{i=1}^{3} \Delta^2 (S_{n_i} \otimes I + I \otimes S_{n_0}) \\
\geq \mathcal{I}(\bar{n}_1, \bar{n}_2, \bar{n}_3) + \mathcal{I}(\bar{m}_1, \bar{m}_2, \bar{m}_3).
\]

Any violation of (29) therefore proves that the measured
quantum state cannot be separated. However, entangled states
can overcome the limitation \( \mathcal{I}(\bar{n}_1, \bar{n}_2, \bar{n}_3) + \mathcal{I}(\bar{m}_1, \bar{m}_2, \bar{m}_3) \),
since entanglement describes quantum correlations that are
more precise than the ones represented by mixtures of product
states. Hence the sum of the incompatibility forms a sufficient
condition for the existence of entanglement directly. Note that
our method of witnessing entanglement does not involve an
(usually experimentally challenging) estimation of the \( d^2 \) ma-
trix elements of \( \rho_{AB} \), where \( d \) is the dimension of \( \rho_{AB} \).

Next, we consider the EPR-steering scenario [6, 30]: Alice
and Bob have local access to subsystems of a bipartite quan-
tum state \( \rho \). Alice chooses one of her measurements \( a \) with
outcomes \( A \), similar for Bob. Then a no-EPR-steering model
for Bob is
\[
p(A, B|a, b) = \sum_{\lambda} p(\lambda) p(A|a, \lambda)p_Q(B|b, \lambda),
\]
with probability distributions \( p(A|a, \lambda) \) and \( p(\lambda) \) under “past
factor” \( \lambda \) [31]. And \( p_Q(B|b, \lambda) \) represent probability distribu-
tions for outcomes \( B \) which are compatible with a quantum
state.

Following [6], if Alice tries to infer the outcomes of Bob’s
measurements through measurements on her subsystem. We
denote by \( B_{est}(A) \) Alice’s estimate of the value of Bob’s mea-
surement \( b \) as a function of the outcomes of her measurement
\( a \). The corresponding average inference variance of \( B \) given
estimate \( B_{est}(A) \) is defined by
\[
\Delta_{inf}^2(B) = \langle (B - B_{est}(A))^2 \rangle,
\]
and its minimum is
\[
\Delta_{min}^2(B) = \langle (B - \langle B \rangle_A)^2 \rangle,
\]
here the mean \( \langle B \rangle_A \) is over the conditional probability
\( p(B|A) \). Under the no-EPR-steering model, we can derive a
bound for \( \Delta_{inf}^2(B) \) [6]
\[
\Delta_{inf}^2(B) \geq \Delta_{min}^2(B) \geq \sum_{\lambda} p(\lambda) \Delta_Q^2(B|\lambda),
\]
where \( \Delta_Q^2(B|\lambda) \) represents the probability for \( B \) predicted by
a quantum state \( \rho_{\lambda} \). Consequently, we can derive the follow-
ing uncertainty relations for the no-EPR-steering model
\[
\sum_{j=1}^{N} \Delta_{inf}^2(S_{n_j}) \geq \mathcal{I}(\bar{n}_1, \cdots, \bar{n}_N).
\]

Here we have shown that it follows directly from the no-
EPR-steering model. Its experimental violation implies the
failure of the local hidden states (LHS) model to represent
the outcomes statistics, in another words, it is an experimen-
tal demonstration of EPR-steering. It is important to remark
that the spin \( S_{n_1}, \cdots, S_{n_{inf}} \) used by Alice to infer the val-
ues of the corresponding measurements of Bob are arbitrary.
Moreover, its lower bound only depends on the incompatibility $\mathcal{I}(\vec{n}_1, \ldots, \vec{n}_m)$.

Rather than trying to build EPR-steering criterion, our method shall show a deeper relation among the incompatibility, uncertainty and quantum correlations. If the shared state $\rho_{AB}$ is separable or satisfied the LHS model, by measuring its subsystems, we can derive a specific lower bound based on the incompatibility.

IV. CONCLUSION

To summarize, we proposed a general framework for detecting entanglement and characterizing steering based on variance-based uncertainty relations. By virtue of the incompatibility between observables (IO), we derived the state-independent lower bound for variance-based uncertainty relations. The relations between the incompatibility between observables (IO) and its geometrical representations are also given. As illustrations, we showed uncertainty relations are just the mathematical formulations of incompatibility between observables, and that is why both entropic uncertainty relations and variance-based uncertainty relations can be used to detect entanglement and characterize steering. Our work established intriguing connections among a number of fascinating subjects, including Lie theory, quantum foundations, uncertainty principle, quantum correlations and the geometry of quantum state space, which are of interest to researchers from diverse fields. In addition, our work has reveal a deeper theory behind uncertainty relations, the incompatibility between observables.

Note added. After this work was completed, independently, Schwonnek, Dammeier, and Werner [32] proposed a numerical method for locating a state-independent bound for the sum of the variances, and employed them for detecting entanglement. Here we provided an analytic form for the variances of spin observables, and utilize it for testing EPR-steering criteria, in addition to entanglement detection.

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[33] See Supplemental Material for our algorithm and theoretical details.

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