Involving copula functions in Conditional Tail Expectation

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Abstract

Our goal in this paper is to propose an alternative risk measure which takes into account the fluctuations of losses and possible correlations between random variables. This new notion of risk measures, that we call Copula Conditional Tail Expectation describes the expected amount of risk that can be experienced given that a potential bivariate risk exceeds a bivariate threshold value, and provides an important measure for right-tail risk. An application to real financial data is given.

Keywords: Conditional tail expectation; Positive quadrant dependence; Copulas; Dependence measure; Risk management; Market models.

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1. Introduction

In actuarial science literature a several risk measures have been proposed, namely: the Value-at-Risk (VaR), the expected shortfall or the conditional tail expectation (CTE), the distorted risk measures (DRM), and recently the copula distorted risk measure (CDRM) as risk measure which takes into account the fluctuations dependence between random variables (rv). See Brahimi et al. (2010).

The CTE in risk analysis represents the conditional expected loss given that the loss exceeds its VaR and provides an important measure for right-tail risk. In this paper we will always consider random variables with finite mean. For a real number s in (0, 1), the CTE of a risk X is given by

$$\text{CTE}(s) := \mathbb{E}[X | X > \text{VaR}_X(s)],$$

where \(\text{VaR}_X(s) := \inf\{x : F(x) \geq s\}\) is the quantile of order s pertaining to distribution function (df) \(F\).

One of the strategy of an Insurance companies is to set aside amounts of capital from which it can draw from in the event that premium revenues become insufficient to pay out claims. Of course, determining these amounts is not a simple calculation. It has to determine the...
The best risk measure that can be used to determine the amount of loss to cover with a high degree of confidence.

Suppose now that we deal with a couple of random losses \((X_1, X_2)\). It’s clear that the CTE of \(X_1\) is unrelated with \(X_2\). If we had to control the overflow of the two risks \(X_1\) and \(X_2\) at the same time, CTE does not answer the problem, then we need another formulation of CTE which takes into account the excess of the two risks \(X_1\) and \(X_2\). Then we deal with the amount

\[
E [X_1 | X_1 > VaR_{X_1}(s), X_2 > VaR_{X_2}(t)] .
\]

If the couple of random losses \((X_1, X_2)\) are independents rv’s then the amount \((1.2)\) defined only the CTE of \(X_1\). Therefore the case of independence is not important.

In the recent years dependence is beginning to play an important role in the world of risk. The increasing complexity of insurance and financial activities products has led to increased actuarial and financial interest in the modeling of dependent risks. While independence can be defined in only one way, but dependence can be formulated in an infinite ways. Therefore, the assumption of independence it makes the treatment easy. Nevertheless, in applications dependence is the rule and independence is the exception.

The copulas is a function that completely describes the dependence structure, it contains all the information to link the marginal distributions to their joint distribution. To obtain a valid multivariate distribution function, we combines several marginal distribution functions, or a different distributional families, with any copula function. Using Sklar’s theorem (Sklar, 1959), we can construct a bivariate distributions with arbitrary marginal distributions. Thus, for the purposes of statistical modeling, it is desirable to have a large collection of copulas at one’s disposal. A great many examples of copulas can be found in the literature, most are members of families with one or more real parameters. For a formal treatment of copulas and their properties, see the monographs by Hutchinson and Lai (1990), Dall’Aglio et al. (1991), Joe (1997), the conference proceedings edited by Beneš and Štěpán (1997), Cuadras et al. (2002), Dhaene et al. (2003) and the textbook of Nelsen (2006).

Recently in finance, insurance and risk management has emphasized the importance of positive or negative quadrant dependence notions (PQD or NQD) introduced by Lehmann (1966), in different areas of applied probability and statistics, as an example, see; Dhaene and Goovaerts (1997), Denuit et al. (2001). Two rv’s are said to be PQD when the probability that they are simultaneously large (or small) is at least as great as it would be were they are independent. In terms of copula, if their copula is greater than their product, i.e., \(C(u_1, u_2) > u_1u_2\) or, simply \(C > C^\perp\), where \(C^\perp\) denotes the product copula. For the sake of brevity, we will restrict ourselves to concepts of positive dependence.
The main idea of this paper is to use the information of dependence between PQD or NQD risks to quantifying insurance losses and measuring financial risk assessments, we propose a risk measure defined by:

$$\text{CCTE}_{X_1}(s; t) := \mathbb{E}[X_1 | X_1 > \text{VaR}_{X_1}(s), X_2 > \text{VaR}_{X_2}(t)].$$

We will call this new risk measure by the *Copula Conditional Tail Expectation* (CCTE), like a risk measure which measure the conditional expectation given the two dependents losses exceeds $\text{VaR}_{X_1}(s)$ and $\text{VaR}_{X_2}(t)$ for $0 < s, t < 1$ and usually with $s, t > 0.9$. Again, CCTE satisfies all the desirable properties of a coherent risk measure (Artzner et al., 1999). The notion of copula in risk measure filed has recently been considered by several authors, see for instance Embrechts et al. (2003a), Di Clemente and Romano (2004), Dalla Valle (2009), Brahimi et al. (2010) and the references therein.

This risk measures can give a good quantifying of losses when we have a combined dependents risks, this dependence can influence in the losses of interested risks. Therefore, quantify the riskiness of our position is useful to decide if it acceptable or not. For this reason we use the all informations about this interest risk. The dependence of our risk with other risks is one of important information that we must take it in consideration.

This paper is organized as follows. In section 2, we give an explicit formulations of the new notion copula conditional tail expectation risk measure in bivariate case. The relationship of this new concept and tail dependence measure, given in section 3. In section 4 we presents an illustration examples to explain how to use the new CCTE measure. Application in real financial data is given in section 5. Concluding notes are given in Section 6. Proofs are relegated to the Appendix.

2. Copula conditional tail expectation

A risk measure quantifies the risk exposure in a way that is meaningful for the problem at hand. The most commonly used risk measure in finance and insurance are: VaR and CTE (also known as Tail-VaR or expected shortfall). The risk measure is simply the loss size for which there is a small (e.g. 1%) probability of exceeding. For some time, it has been recognized that this measure suffers from serious deficiencies if losses are not normally distributed.

According to Artzner et al. (1999) and Wirch and Hardy (1999), the conditional tail expectation of a random variable $X_1$ at its $\text{VaR}_{X_1}(s)$ is defined by:

$$\text{CTE}_{X_1}(s) = \frac{1}{1 - F_{X_1}(\text{VaR}_{X_1}(s))} \int_{\text{VaR}_{X_1}(s)}^{\infty} x dF_{X_1}(x),$$

where $F_{X_1}$ is the df of $X_1$. 
Since \( X_1 \) is continuous, then \( F_{X_1}(VaR_{X_1}(s)) = s \), it follows that for all \( 0 < s < 1 \)
\[
\text{CTE}_{X_1}(s) = \frac{1}{1 - s} \int_s^1 VaR_{X_1}(u) \, du. \tag{2.3}
\]

The CTE can be larger than the VaR measure for the same value of level \( s \) described above since it can be thought of as the sum of the quantile \( VaR_{X_1}(s) \) and the expected excess loss. Tail-VaR is a coherent measure in the sense of Artzner et al. (1999). For the application of this kind of coherent risk measures we refer to the papers Artzner et al. (1999) and Wirch and Hardy (1999).

Thus the CTE is nothing, see Overbeck and Sokolova (2008), but the mathematical transcription of the concept of "average loss in the worst \( 100(1 - s)\% \) case", defining by \( \upsilon = VaR_{X_1}(s) \) a critical loss threshold corresponding to some confidence level \( s \), \( \text{CTE}_{X_1}(s) \) provides a cushion against the mean value of losses exceeding the critical threshold \( \upsilon \).

Now, assume that \( X_1 \) and \( X_2 \) are dependent with joint df \( H \) and continuous margins \( F_{X_i}, \ i = 1, 2 \), respectively. Through this paper we call \( X_1 \) the target risk and \( X_2 \) the associated risk. In this case, the problem becomes different and its resolution requires more than the usual background.

Our contribution is to introduce the copula notion to provide more flexibility to the CTE of risk of rv’s in terms of loss and dependence structure. For comprehensive details on copulas one may consult the textbook of Nelsen (2006).

According to Sklar’s Theorem Sklar (1959), there exists a unique copula \( C : [0, 1]^d \rightarrow [0, 1] \) such that
\[
H(x_1, x_2) = C(F_1(x_1), F_2(x_2)). \tag{2.4}
\]

The formula of CTE focuses only on the average of loss. For this you should think of an other formula to be more inclusive, this formula must take in consideration the dependence structure and the behavior of margin tails. These two aspects have an important influence when quantifying risks. On the other hand if the correlation factor is neglected, the calculation of the CTE follows from formula (2.3), which only focuses on the target risk.

Now by considering the correlation between the target and the associated risks, we define a new notion of CTE called \textit{Copula Conditional Tail Expectation} (CCTE) given in (1.2), this notion led to give a risk measurement focused in the target risk and the link between target and associated risk.

Let’s denote the survival functions by \( F_{i}(x_i) = \mathbb{P}(X_i > x_i), \ i = 1, 2 \), and the joint survival function by \( H(x_1, x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2) \). The function \( C \) which couples \( H \) to \( F_i, \ i = 1, 2 \)
via $\overline{H}(x_1, x_2) = \overline{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))$ is called the survival copula of $(X_1, X_2)$. Furthermore, $\overline{C}$ is a copula, and

$$\overline{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2), \quad (2.5)$$

where $C$ is the (ordinary) copula of $X_1$ and $X_2$. For more details on the survival copula function see, Section 2.6 in Nelsen (2006).

The CCTE of the target risk $X_1$ with respect to the associated risk $X_2$ is given in the following proposition.

**Proposition 2.1.** Let $(X_1, X_2)$ a bivariate rv with joint df represented by the copula $C$. Assume that $X_1$ have a finite mean and df $F_{X_1}$. Then for all $s$ and $t$ in $(0, 1)$ the copula conditional tail expected of $X_1$ with respect to the bivariate thresholds $(s, t)$ is given by

$$\text{CCTE}_{X_1}(s; t) = \int_0^1 \frac{F_{X_1}^{-1}(u) (1 - C_u(u, t))}{C(u, 1 - t)} du, \quad (2.6)$$

where $F_{X_1}^{-1}$ is the quantile function of $F_{X_1}$ and $C_u(u, v) := \partial C(u, v)/\partial u$.

This notion does not depend on the df of the associated risk, but it depend only by the copula function and the df of target risk.

By definition of PQD risks we have that $C(u, v) > uv$, then it easy to check that

$$\text{CCTE}_{X_1}(s; t) \leq \frac{\text{CTE}_{X_1}(s)}{(1 - t)} \text{ for } s, t < 1,$$

next, in Section 4, we will proved that the risk when we consider the correlation between PQD risks is greater than in the case of a single one. That means, for all $s \leq t$ and $s, t$ in $(0, 1)$ then

$$\text{CCTE}_{X_1}(s; t) \geq \text{CTE}_{X_1}(s). \quad (2.7)$$

Notice that in the NQD rv’s we have the reverse inequality of (2.7) and the CCTE coincide with CTE measures in the non-dependence case, i.e. the copula $C = C^\perp$.

3. CCTE and tail dependence

The concept of tail dependence is an asymptotic measure of the dependence between two random variables in the tail of their joint distribution function. Specifically, tail dependence is the probability that a random variable $X_1$ and $X_2$ takes a values in the extreme tail of its distributions simultaneously, for example we consider $X_1$ and $X_2$ which measure bankruptcy for two companies and both companies simultaneously go bankrupt.

We describes the joint upper tail dependence of the random variables $X_1$ and $X_2$:

$$\lim_{t \to 1^-} \lim_{s \to 1^-} \mathbb{P}(X_1 > F_{X_1}^{-1}(s) | X_2 > F_{X_2}^{-1}(t))$$
However, it can be seen as a good indicator of systemic risk (for \( s = t \)). If we considering the tail dependence as a dependence measure in the extreme tails of the joint distribution, it is possible for two rv’s to be dependent, but for there to be no dependence in the tail of the distributions, this is the case described for example by the Gaussian copula, hyperbolic copula or Farlie-Gumbel-Morgenstern copula (tail dependence is zero). Furthermore, the Clayton copula puts the entire tail dependence in the lower tail unlike Gumbel copula in the upper tail and the Student copula behave identically in the lower as in the upper tail. However, it is not suitable to model extreme negative outcomes similarly as with extreme positive outcomes.

**Remark 3.1.** Negative outcomes can be treated in the same way that the extremes positive outcomes by replacing their copula by the survival copula.

The tail dependence can be also expressed through copula

\[
\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \quad \text{and} \quad \lambda_L = \lim_{u \to 0^+} \frac{C(u, u)}{u}.
\]

Now, let’s denote by

\[
\tilde{\lambda}_U(u, v) := \frac{1 - u - v + C(u, v)}{1 - v} \quad \text{and} \quad \tilde{\lambda}_L(u, v) := \frac{C(u, v)}{v}.
\]

Note that \( \lim_{u, v \to 1^-} \tilde{\lambda}_U(u, v) = \lambda_U \) and \( \lim_{u, v \to 0^+} \tilde{\lambda}_L(u, v) = \lambda_L \). We can rewrite CCTE of according to \( \tilde{\lambda}_U \) as

\[
\text{CCTE}_{X_1}(s; t) = \frac{\int_s^1 F_{X_1}^{-1}(u) (1 - C_u(u, t)) \, du}{(1 - t) \tilde{\lambda}_U(s, t)},
\]

this has no impact on the limiting value at 0 for PQD risks. Then we have

\[
\lim_{s \to 1^- \atop t \to 1^-} (1 - t) \tilde{\lambda}_U(s, t) = 0.
\]

From Theorem 2.2.7 in (Nelsen, 2006, page 13) we have \( 0 \leq C_u(u, t) \leq 1 \) for such \( u \) and \( t \), then

\[
|\text{CCTE}_{X_1}(s; t)| \leq \left| \frac{\int_s^1 F_{X_1}^{-1}(u) \, du}{(1 - t) \tilde{\lambda}_U(s, t)} \right| \leq \frac{\mathbb{E}(X_1)}{(1 - t) \tilde{\lambda}_U(s, t)}.
\]

In the next section, we give an example to describe the impact of the upper tail dependence nearly 1 and the lower tail dependence near 0 in CCTE, and we discuss the relationship between the properties of the dependence of copula model with upper and lower tail dependence and how to derive the degree of risk in each case.
4. Illustration examples

4.1. CCTE via Farlie-Gumbel-Morgenstern Copulas. One of the most important parametric family of copulas is the Farlie-Gumbel-Morgenstern (FGM) family defined as

\[ C_{\theta}^{FGM}(u, v) = uv + \theta uv(1 - u)(1 - v), \quad u, v \in [0, 1], \]  

where \( \theta \in [-1, 1] \). The family was discussed by Morgenstern (1956), Gumbel (1958) and Farlie (1960).

The copula given in (4.8) is PQD for \( \theta \in (0, 1] \) and NQD for \( \theta \in [-1, 0) \). In practical applications this copula has been shown to be somewhat limited, for copula dependence parameter \( \theta \in [-1, 1] \), Spearman’s correlation \( \rho \in [-1/3, 1/3] \) and Kendall’s \( \tau \in [-2/9, 2/9] \), for more details on copulas see, for example, Nelsen (2006).

Members of the FGM family are symmetric, i.e., \( C_{\theta}^{FGM}(u, v) = C_{\theta}^{FGM}(v, u) \) for all \( (u, v) \) in \([0, 1]^2\) and have the lower and upper tail dependence coefficients equal to 0.

A pair \((X, Y)\) of rv’s is said to be exchangeable if the vectors \((X, Y)\) and \((Y, X)\) are identically distributed. Note that, in applications, exchangeability may not always be a realistic assumption. For identically distributed continuous random variables, exchangeability is equivalent to the symmetry of the FGM copula.

For practical purposes we consider a copula families with only positive dependence. Furthermore, risk models are often designed to model positive dependence, since in some sense it is the “dangerous” dependence: assets (or risks) move in the same direction in periods of extreme events, see Embrechts et al. (2003b).

Consider the bivariate loss PQD rv’s \((X_i, Y), i = 1, 2, 3\), having continuous marginal df’s \(F_{X_i}(x)\) and \(F_Y(y)\) and joint df \(H_{X_i,Y}(x, y)\) represented by FGM copula of parameters \(\theta_i\), respectively for \(i = 1, 2, 3\)

\[ H_{X_i,Y}(x, y) = C_{\theta_i}^{FGM}(F_{X_i}(x), F_Y(y)). \]

The marginal survival functions \(\overline{F}_{X_i}(x), i = 1, 2, 3\) and \(\overline{F}_Y(y)\) are given by

\[
\overline{F}_{X_i}(x) = \begin{cases} 
(1 + x)^{-\alpha}, & x \geq 0, \\
1, & x < 0,
\end{cases} \quad \text{and} \quad \overline{F}_Y(y) = \begin{cases} 
(1 + y)^{-\alpha}, & y \geq 0, \\
1, & y < 0,
\end{cases}
\]

(4.9)

where \(\alpha > 0\) called the Pareto index, the case \(\alpha \in (1, 2)\) means that \(X_i\) have a heavy-tailed distributions. So that \(X_i\) and \(Y\) have identical Pareto df’s.

For each couple \((X_i, Y), i = 1, 2, 3\), we propose \(\theta_1 = 0.01, \theta_2 = 0.5\) and \(\theta_3 = 1\), respectively. The choice of parameters \(\theta_i, i = 1, 2, 3\) correspond respectively to the weak, medium and the high dependence.
In this example, the target risks are $X_i$ and the associated risk is $Y$. The CTE’s and the VaR’s of $X_i$ are the same and are given respectively by

$$\text{CTE}_{X_i}(s) = \frac{\alpha (1-s)^{-1/\alpha}}{\alpha - 1}$$

and

$$\text{VaR}_{X_i}(s) = (1-s)^{-1/\alpha},$$

for $i = 1, 2, 3$.

We have that

$$C(1-s, 1-t) = (1-s)(1-t)(st\theta_i + 1).$$

Now, we calculate

$$\text{CCTE}_{X_i}(s,t) = \frac{1}{C(1-s, 1-t)} \int_s^1 (1-u)^{-1/\alpha} (1-t) (2tu\theta_i - t\theta_i + 1) du$$

by substitution (4.12) we get

$$\text{CCTE}_{X_i}(s,t) = \frac{\alpha (2\alpha + t\theta_i - 2st\theta_i + 2st\alpha\theta_i - 1)}{(2\alpha^2 - 3\alpha + 1)(st\theta_i + 1)} (1-s)^{-1/\alpha}.$$ 

We have in Table 4.1 and Figures 4.1 the comparison of the riskiness of $X_1$, $X_2$ and $X_3$. Recall that, the CTE’s risk measure of $X_i$ at level $s$ are the same in all cases. Note that CCTE coincide with CTE in the independence case ($\theta_1 = 0$). The CCTE of the loss $X_3$ is riskier than $X_2$ and $X_1$ but not very significant, in the 6th column of Table 4.1, the relative difference between 64.7946 and 64.633 is only about 0.025%. This is due to that FGM copula does not take into account the dependence in upper and lower tail ($\lambda_L = \lambda_U = 0$). In this case we can not clearly confirm which is the risk the more dangerous.

4.2. CCTE via Archimedean Copulas. A bivariate copula is said to be Archimedean (see, Genest and MacKay, 1986) if it can be expressed by

$$C(u_1, u_2) = \psi^{-1} \left( \psi(u_1) + \psi(u_2) \right),$$

where $\psi$, called the generator of $C$, is a continuous strictly decreasing convex function from $[0, 1]$ to $[0, \infty]$ such that $\psi(1) = 0$ with $\psi^{-1}$ denotes the pseudo-inverse of $\psi$, that is

$$\psi^{-1}(t) = \begin{cases} \psi^{-1}(t), & \text{for } t \in [0, \psi(0)] , \\ 0, & \text{for } t \geq \psi(0). \end{cases}$$

When $\psi(0) = \infty$, the generator $\psi$ and $C$ are said to be strict and therefore $\psi^{-1} = \psi^{-1}$. All notions of positive dependence that appeared in the literature, including the weakest one of PQD as defined by Lehmann (1966), require the generator to be strict.

Archimedean copulas are widely used in applications due to their simple form, a variety of dependence structures and other “nice” properties. For example, in the Actuarial field: the idea
Table 4.1. Risk measures of dependent pareto(1.5) rv’s with FGM copula.

| $s$  | 0.9000 | 0.9225 | 0.9450 | 0.9675 | 0.9900 |
|------|--------|--------|--------|--------|--------|
| $VaR_{X_i}(s)$ | 4.6415 | 5.5013 | 6.9144 | 9.8192 | 21.5443 |
| $CTE_{X_i}(s)$ | 13.9247 | 16.5039 | 20.7433 | 29.4577 | 64.6330 |

| $t$   | $\text{CCTE}_{X_1}(s,t)$, $\theta = 0.01$ |
|-------|------------------------------------------|
| 0.9000 | 13.9309 13.9311 13.9312 13.9314 13.9316  |
| 0.9225 | 16.5096 16.5097 16.5099 16.5100 16.5101  |
| 0.9450 | 20.7484 20.7485 20.7487 20.7488 20.7489  |
| 0.9675 | 29.4619 29.4620 29.4621 29.4623 29.4624  |
| 0.9900 | 64.6359 64.6359 64.6360 64.6361 64.6362  |

| $t$   | $\text{CCTE}_{X_2}(s,t)$, $\theta = 0.5$ |
|-------|------------------------------------------|
| 0.9000 | 14.1477 14.1517 14.1555 14.1594 14.1631  |
| 0.9225 | 16.7072 16.7108 16.7143 16.7178 16.7212  |
| 0.9450 | 20.9234 20.9266 20.9297 20.9327 20.9357  |
| 0.9675 | 29.6077 29.6103 29.6129 29.6154 29.6179  |
| 0.9900 | 64.7336 64.7353 64.7370 64.7387 64.7404  |

| $t$   | $\text{CCTE}_{X_3}(s,t)$, $\theta = 1$ |
|-------|------------------------------------------|
| 0.9000 | 14.2709 14.2756 14.2803 14.2848 14.2892  |
| 0.9225 | 16.8183 16.8226 16.8267 16.8308 16.8348  |
| 0.9450 | 21.0208 21.0245 21.0281 21.0316 21.0351  |
| 0.9675 | 29.6880 29.6910 29.6940 29.6969 29.6997  |
| 0.9900 | 64.7868 64.7888 64.7908 64.7927 64.7946  |

arose indirectly in Clayton (1978) and was developed in Oakes (1982), Cook and Johnson (1981). A survey of Actuarial applications is in Frees and Valdez (1998).

For an Archimedean copula, the Kendall’s tau can be evaluated directly from the generator of the copula, as shown in Genest and MacKay (1986)

$$
\tau = 4 \int_0^1 \frac{\psi(u)}{\psi'(u)} du + 1.
$$

where $\psi'(u)$ exists a.e., since the generator is convex. This is another “nice” feature of Archimedean copulas. As for tail dependency, as shown in (Joe, 1997, page 105) the coefficient of upper tail dependency is

$$
\lambda_U = 2 - 2 \lim_{s \to 0^+} \frac{\psi^{-1}(2s)}{\psi^{-1}(s)}.
$$
Figure 4.1. CCTE, CTE and VaR risks measures of PQD pareto (1,5) rv’s with FGM copula and $0.9 \leq s = t \leq 0.99$ and the coefficient of lower tail dependency is

$$
\lambda_L = 2 \lim_{s \to +\infty} \frac{\psi^{-1}(2s)}{\psi^{-1}(s)}.
$$

A collection of twenty-two one-parameter families of Archimedean copulas can be found in Table 4.1 of Nelsen (2006). Notice that in the case of Archimedean copula the copula conditional tail expectation has not an explicit formula, so we give by the following Proposition the expression of CCTE in terms of generator.

**Proposition 4.1.** Let $C$ be an Archimedean copula absolutely continuous with generator $\psi$, the CCTE of the target risk in terms of generator with respect to the bivariate thresholds $(s, t)$, $0 < s, t < 1$, is given by

$$
\text{CCTE}_{X_1}(s; t) = \frac{1}{C(1-s, 1-t)} \left( (1-s) \text{CTE}_{X_1}(s) - \int_s^1 \frac{\psi'(u)F_{X_1}^{-1}(u)}{\psi'(C(u, t))} du \right). \tag{4.15}
$$

Note that in practice we can easily fit copula-based models with the maximum likelihood method or with estimate the dependence parameter by the relationship between Kendall’s tau of the data and the generator of the Archimedean copula given in (4.14) under the specified copula model.

In the following section we give same examples to explain how to calculate and compare the CCTE with other risk measure such VaR and CTE.
4.2.1. **CCTE via Gumbel Copula.** The Gumbel family has been introduced by Gumbel (1960). Since it has been discussed in Hougaard (1986), it is also known as the Gumbel-Hougaard family. The Gumbel copula is an asymmetric Archimedean copula. This copula is given by

\[ C^G_\theta(u, v) = \exp \left\{ - \left[ (-\ln u)^\theta + (-\ln v)^\theta \right]^{1/\theta} \right\}, \]

its generator is

\[ \psi_\theta(t) = (-\ln t)^\theta. \]

The dependence parameter is restricted to the interval [1, \( \infty \)). It follows that the Gumbel family can represent independence and “positive” dependence only, since the lower and upper bound for its parameter correspond to the product copula and the upper Fréchet bound. The Gumbel copula families is often used for modeling heavy dependencies in right tail. It exhibits strong upper tail dependence \( \lambda_U = 2 - 2^{1/\theta} \) and relatively weak lower tail dependence \( \lambda_L = 0 \). If outcomes are known to be strongly correlated at high values but less correlated at low values, then the Gumbel copula will be an appropriate choice.

We give the CCTE of rv’s \( X_i, i = 1, 2, 3 \) in terms of Gumbel copula by

\[
\text{CCTE}_{X_i}(s; t) = \frac{1}{C^G_{\theta_i}(1 - s, 1 - t)} \left( \frac{\alpha (1 - s)^{1-1/\alpha}}{\alpha - 1} \right) - \int_s^1 u^{-1} (1 - u)^{-1/\alpha} \left(-\ln u\right)^{\theta_i - 1} C^G_{\theta_i}(u, t) \left(-\ln \left(C^G_{\theta_i}(u, t)\right)\right)^{1-\theta_i} du \right),
\]

(4.16)

where \( C^G_{\theta_i}(s, t) = s + t - 1 + C^G_{\theta_i}(1 - s, 1 - t) \).

The CTE’s and VaR’s of \( X_i \) is the same and it’s given respectively by (4.10) and (4.11), for \( i = 1, 2, 3 \).

By the relationship between Kendall’s tau \( \tau \) and the Gumbel copula parameter \( \theta_i \) given by:

\[ \tau = (\theta_i - 1) / \theta_i, \]

we select the values of \( \theta_i \) corresponding respectively to a weak, a moderate and a strong positive association witch summarized in Table 4.2.

Table 4.3 and Figure 4.2 shows that the loss \( X_1 \) is considerably riskier than \( X_2 \) and \( X_3 \), in the 6th column of Table 4.3, the relative difference between 112.1868 and 69.6017 is about 61.184%.

By definition of our risk measurement, the risks concern the study is necessarily comonotonic, then to have a good decision we must select a copula model with upper tail dependence, we
show in next example that the dependence models with no upper tails dependence do not helps us to take a decision.
4.2. **CCTE via Clayton Copula.** In the following example, we consider the bivariate Clayton copula which is a member of the class of Archimedean copula, with the dependence parameter $\theta$ in $[-1, \infty) \setminus \{0\}$.

The Clayton family was first proposed by Clayton (1978) and studied by Oakes, (1982; 1986), Cox and Oakes, (1981; 1986). The Clayton copula has been used to study correlated risks, it has the form

$$C_\theta^C(u, v) := \left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-1/\theta}. \quad (4.17)$$

For $\theta > 0$ the copulas are strict and the copula expression simplifies to

$$C_\theta^C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}. \quad (4.18)$$

Asymmetric tail dependence is prevalent if the probability of joint extreme negative realizations differs from that of joint extreme positive realizations. it can be seen that the Clayton copula assigns a higher probability to joint extreme negative events than to joint extreme positive events. The Clayton copula is said to display lower tail dependence $\lambda_L = 2^{-1/\theta}$, while it displays zero upper tail dependence $\lambda_U = 0$, for $\theta \geq 0$. The converse can be said about the Gumbel copula (displaying upper but zero lower tail dependence). The margins become independent as $\theta$ approaches to zero, while for $\theta \to \infty$, the Clayton copula arrives at the comonotonicity copula. For $\theta = -1$ we obtain the Fréchet-Hoeffding lower bound and the copula attains the Fréchet upper bound as $\theta$ approaches to infinity.
Clayton copula is the best suited for applications in which two outcomes are likely to experience low values together, since the dependence is strong in the lower tail and weak in the upper tail.

We take the same example as in the Subsection 4.1, we may now represents the joint df’s $H_i$, $i = 1, 2, 3$, respectively by the Clayton copulas $C^\theta_i$ given in (4.18) to have an idea about the effects of lower tail dependence on our risk measurement.

The relationship between Kendall’s tau $\tau$ and the Clayton copula is given by

$$
\tau = \frac{\theta_i}{\theta_i + 2},
$$

we select a different dependents parameters corresponding to several levels of positive dependency summarized in Table 4.4 for a weak, a moderate and a strong positive association, to calculate and compare the CCTE’s of $X_i$, $i = 1, 2, 3$.

| $\lambda_L$ | $\theta_i$ | $\tau$ |
|------------|----------|------|
| 0.250      | 0.5      | 0.200|
| 0.707      | 2        | 0.500|
| 0.943      | 12       | 0.857|

Table 4.4. Lower tail, Kendall’s tau and Clayton copula parameters used in calculate of risk measures.

The CCTE of the rv’s $X_i$ with respect to the bivariate thresholds $(s, t)$ is given by

$$
\text{CCTE}_{X_i}(s; t) = \frac{1}{C^\theta_i(1 - s, 1 - t)} \left( \frac{\alpha (1 - s)^{-1/\alpha + 1}}{(\alpha - 1)} - \int_s^1 \frac{(t^{\theta_i} + u^{\theta_i} - 1)^{-1-1/\theta_i}}{(1 - u)^{1/\alpha} u^{\theta_i+1}} du \right).
$$

The differences as reported in Table 4.5 and Figure 4.3 do not look very significant, in the 6th column of Table 4.5, the relative difference between 66.3802 and 64.6330 is only about 1.027%. The differences is not found also when $t$ is small compared to $s$, $\text{CCTE}_{X_1}(0.99; 0.01) = 64.6332$ and $\text{CCTE}_{X_3}(0.99; 0.01) = 64.6329$ the difference is about 1%.

5. Application

The relationship between the parameter of an Archimedean copula and Kendall’s tau has allowed us to calculate the value of this parameter assuming a well precise Archimedean copula e.g., Gumbel copula. Once endowed with the parameter value, we are able to compute any joint probability between the stock indices.
Table 4.5. Risk measures of dependent pareto (1.5) rv’s with Clayton copula.

For instance we analyzed 500 observations from four European stock indices return series calculated by \( \log \left( \frac{X_{t+1}}{X_t} \right) \) for the period July 1991 to June 1993 (see, Figure 5.4), available in ”QRM and datasets packages” of R software, it contains the daily closing prices of major European stock indices: Germany DAX (Ibis), Switzerland SMI, France CAC and UK FTSE. The data are sampled in business time, i.e., weekends and holidays are omitted. Table 5.6 summaries the Kendall’s tau between the four Market Index returns.

By assuming that Gumbel copula represents our four dependence structures, we obtain the fitted dependence parameters of the six bivariate joint df’s, presented in Table 5.7.

The \( \alpha \)-stable distribution offers a reasonable improvement to the alternative distributions, each stable distribution \( S_\alpha(\sigma; \beta; \mu) \) has the stability index \( \alpha \) that can be treated as the main parameter, when we make an investment decision, skewness parameter \( \beta \), in the range \([-1,1]\), scale parameter \( \sigma \) and shift parameter \( \mu \). In models that use financial data, it is
Figure 4.3. CCTE, CTE and VaR risks measures of PQD pareto (1,5) rv’s with Clayton copula and \(0.9 \leq s = t \leq 0.99\).

Figure 5.4. Scatterplots of 500 pseudo-observations drawn from a four European stock indices returns.

generally assumed that \(\alpha \in (1, 2]\). By using the "fBasics" package in R software, based on the maximum likelihood estimators to fit the parameters of a df’s of the four Market Index returns, the results are summarized in Table 5.8.
### Table 5.6. Kendall’s tau matrix estimates from four European stock indices returns.

| Variable | DAX   | SMI   | CAC   | FTSE  |
|----------|-------|-------|-------|-------|
| DAX      | 1     | 0.4052| 0.4374| 0.3706|
| SMI      | 0.4052| 1     | 0.3791| 0.3924|
| CAC      | 0.4374| 0.3791| 1     | 0.4076|
| FTSE     | 0.3706| 0.3924| 0.4076| 1     |

### Table 5.7. Fitted copula parameter corresponding to Kendall’s tau, Gumbel copula.

| Variable | DAX   | SMI   | CAC   | FTSE  |
|----------|-------|-------|-------|-------|
| DAX      | ∞     | 1.6815| 1.7777| 1.5888|
| SMI      | 1.6815| ∞     | 1.6106| 1.6459|
| CAC      | 1.7777| 1.6106| ∞     | 1.6880|
| FTSE     | 1.5888| 1.6459| 1.6880| ∞     |

### Table 5.8. Maximum likelihood fit of four-parameters stable distribution to four European stock indices returns data.

|         | DAX   | SMI   | CAC   | FTSE  |
|---------|-------|-------|-------|-------|
| $\alpha$ | 1.6420| 1.8480| 1.6930| 1.8740|
| $\beta$ | 0.1470| 0.1100| 0.0380| 0.9500|
| $\sigma$ | 0.0046| 0.0045| 0.0006| 0.0053|
| $\mu$   | −0.0001| 0.0006| −0.0001| −0.0005|

The $\alpha$-stable distribution has Pareto-type tails, it’s like a power function, i.e., $F$ is regularly varying (at infinity) with index $(-\alpha)$, meaning that $\overline{F}(x) = x^{-\alpha}L(x)$ as $x$ becomes large, where $L > 0$ is a slowly varying function, which can be interpreted as slower than any power function (see, Resnick; 1987 and Seneta; 1976 for a technical treatment of regular variation).

By using the Equations (4.16) for the Gumbel copula fitting, we calculate for a fixed levels $s = t = 0.95$ the CCTE’s risk measures for the all cases, the results are summarized in Table 5.9.

The smallest values in Table 5.9 gives the lowest risk. So, the less risky couples $(X, Y)$ are: (DAX, CAC), (SMI, DAX), (CAC, DAX) and (FTSE, CAC), where $X$ is the target risk and $Y$ is the associated risk.
| Variable | DAX   | SMI   | CAC   | FTSE  |
|---------|-------|-------|-------|-------|
| DAX     | −     | 21.509 | 21.0786 | 21.9731 |
| SMI     | 14.2812 | −     | 14.4703 | 14.3737 |
| CAC     | 18.8362 | 19.4915 | −     | 19.1671 |
| FTSE    | 13.9075 | 13.7593 | 13.6576 | −     |

Table 5.9. CCTE’s Risk measures for $s = 0.99$ and $t = 0.99$ with Gumbel copula (left panel) and Clayton copula (right panel).

6. Conclusion notes

This paper discussed a new risk measure called copula conditional tail expectation. This measure aid to understanding the relationships among multivariate assets and to help us significantly about how best to position our investments and improve our financial risk protection.

Tables 4.3 show that the copula conditional tail expectation measure become smaller as the dependency increase. However, CTE and VaR are neither increasing nor decreasing as the correlation increase. Therefore, the dependency information helps us to minimize the risk.

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7. Appendix

Proof of Proposition 2.1. By conditional probability is easily to obtain

$$P (X_1 \leq x \mid X_1 > VaR_{X_1} (s), X_2 > VaR_{X_2} (t)) = \frac{P (x \geq X_1 > VaR_{X_1} (s), X_2 > VaR_{X_2} (t))}{P (X_1 > VaR_{X_1} (s), X_2 > VaR_{X_2} (t))}$$

On the other hand, we have

$$P (x \geq X_1 > VaR_{X_1} (s), X_2 > VaR_{X_2} (t)) = 1 - P (F_1^{-1} (x) < F_1 (X_1) \leq s) - P (F_2 (X_2) \leq t) + P (F_1^{-1} (x) < F_1 (X_1) \leq s, F_2 (X_2) \leq t) ,$$

$$P (X_1 > VaR_{X_1} (s), X_2 > VaR_{X_2} (t)) = 1 - P (F_1 (X_1) \leq s) - P (F_2 (X_2) \leq t) + P (F_1 (X_1) \leq s, F_2 (X_2) \leq t)$$

$$= 1 - s - t + C (s, t)$$

$$= C (1 - s, 1 - t),$$

$$P (F_1^{-1} (x) < F_1 (X_1) \leq s) = s - VaR_{X_1} (x)$$
\[
\mathbb{P}(F_1^{-1}(x) < F_1(X_1) \leq s, F_2(X_2) \leq t) = C(s, t) - C(VaR_{X_1}(x), t).
\]

Then
\[
\mathbb{P}(X_1 \leq x \mid X_1 > VaR_{X_1}(s), X_2 > VaR_{X_2}(t)) = 1 + \frac{VaR_{X_1}(x) - C(VaR_{X_1}(x), t)}{C(1-s, 1-t)}.
\]

Then the CCTE is given by
\[
CCTE_{X_1}(s, t) = \int_{X_1 > VaR_{X_1}(s)} xd\mathbb{P}(X_1 \leq x \mid X_1 > VaR_{X_1}(s), X_2 > VaR_{X_2}(t))
\]
\[
= \frac{1}{C(1-s, 1-t)} \int_{VaR_{X_1}(s)}^{\infty} xd(VaR_{X_1}(x) - C(VaR_{X_1}(x), t)).
\]
\[
= \frac{1}{C(1-s, 1-t)} \int_s^1 F_{X_1}^{-1}(u) du - C(u, t)\]
\[
= \frac{1}{C(1-s, 1-t)} \left( \int_s^1 F_{X_1}^{-1}(u) du - \int_s^1 F_{X_1}^{-1}(u) dC(u, t) \right).
\]

This closes the proof of Proposition 2.1. \(\square\)

**Proof of Proposition 4.1.** Let’s denote by
\[
C_u(u, v) := \frac{\partial C(u, v)}{\partial u}
\]
then by (2.6), we have
\[
CCTE_{X_1}(s; t) = \frac{1}{C(1-s, 1-t)} \left( \int_s^1 F_{X_1}^{-1}(u) du - \int_s^1 F_{X_1}^{-1}(u) C_u(u, t) du \right).
\]
So, \(C\) is Archimedean copula, then
\[
C_u(u, v) = \frac{\psi'(u)}{\psi'(C(u, v))}, \quad (7.21)
\]
Finely, we get (4.15) by substitution of
\[
\int_s^1 F_{X_1}^{-1}(u) du = (1-s)CCTE_{X_1}(s)
\]
and (7.21) in (2.6). \(\square\)

**References**

Artzner, P. H., Delbaen, F., Eber, J. M., Heath, D., 1999. Coherent measures of risk. *Math. Finance* 9(3), 203-228.

Beneš, V., and Štěpán, J., 1997. Distributions with Given Marginals and Moment Problems. Proceedings of the conference held in Prague, September 1996. *Kluwer Academic Publishers, Dordrecht*.

Brahimi, B., Meraghni, D., Necir, A., 2010. Distortion risk measures for sums of dependent losses. *J. Afr. Stat.* 5, 260-267.
Clayton, D. G., 1978. A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*. **65**, 141-151.

Cook, R. D. and Johnson, M. E., 1981. A family of distributions for modeling non-elliptically symmetric multivariate data. *J. Roy. Statist. Soc.* Ser. B, **43**, 210-218.

Cook, R. D. and Johnson, M. E., 1986. Generalized Burr-Pareto-logistic distributions with applications to a uranium exploration data set. *Technometrics*. **28**, 123-131.

Cox, D. R. and Oakes, D., 1984. Analysis of Survival Data. Chapman & Hall, London.

Cuadras, C. M., Fortiana, J. and Rodriguez-Lallena, J. A., eds. 2002. Distributions with given marginals and statistical modelling. Kluwer, Dordrecht.

Dalla Valle, L., 2009. Bayesian copulae distributions, with application to operational risk management. *Methodol. Comput. Appl. Probab.* **11**(1), 95-115.

Dall’Aglio, G., Kotz, S., and Salinetti, G., 1991. Advances in probability distributions with given marginals. Mathematics and its Applications, 67. *Kluwer Academic Publishers Group, Dordrecht*.

Denuit, M., Dhaene, J. and Ribas, C., 2001. Does positive dependence between individual risks increase stop-loss premiums? *Insurance Math. Econom.* **28**(3), 305-308.

Dhaene, J. and Goovaerts, M. J. 1997. On the dependency of risks in the individual life model. *Insurance Math. Econom.* **19**(3), 243-253.

Dhaene, J., Kolev, N. and Morettin, P., 2003. Proceedings of the First Brazilian Conference on Statistical Modelling in Insurance and Finance. *Institute of Mathematics and Statistics, University of São Paulo*.

Di Clemente, A. and Romano, C., 2004. Measuring and optimizing portfolio credit risk: A copula-based approach. *Economic Notes*. **33**(3), 325-357.

Embrechts, P., Höing, A. and Juri, A., 2003. Using copulae to bound the value-at-risk for functions of dependent risks. *Finance Stoch.* **7**(2), 145-167.

Embrechts, P., Lindskog, F. and McNeil, A., 2003. Modelling dependence with copulas and applications to risk management. In: Rachev S (ed) Handbook of heavy tailed distributions in finance. Elsevier, New York, 329-384.

Farlie, D. J. G., 1960. The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*. **47**, 307-323.

Frees, E. W. and Valdez, E. A., 1998. Understanding relationships using copulas. *N. Am. Actuar. J.* **2**(1), 1-25.

Genest, C. and MacKay, R. J., 1986. Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. *Canad. J. Statist.* **14**, 145-159.
Gumbel, E. J., 1958. Distributions à plusieurs variables dont les marges sont données. *C. R. Acad. Sci. Paris* **246**, 2717-2719.

Gumbel, E. J., 1960. Bivariate exponential distributions. *J. Amer. Statist. Assoc.* **55**, 698-707.

Hougaard, P., 1986. A class of multivariate failure time distributions. *Biometrika*. **73**, 671-678.

Hutchinson, T. P. and Lai, C. D., 1990. Continuous bivariate distributions, emphasizing applications. Rumsby, Sydney, Australia.

Joe, H., 1997. Multivariate models and dependence concepts. Monographs on Statistics and Applied Probability, **73**. Chapman and Hall, London.

Lehmann, E. L., 1966. Some concepts of dependence. *Ann. Math. Statist.* **37**, 1137-1153.

Morgenstern, D., 1956. Einfache beispiele zweidimensionaler Verteilungen. (German) *Mitteilungsbl. Math. Statist.* **8**, 234-235.

Nelsen, R. B., 2006. *An Introduction to copulas*. second ed. Springer, New York.

Oakes, D., 1982. A model for association in bivariate survival data. *J. Roy. Statist. Soc. Ser. B*, **44**, 414-422.

Oakes, D., 1986. Semiparametric inference in a model for association in bivariate survival data. *Biometrika*. **73**, 353-361.

Overbeck, L. and Sokolova, M. 2008. Risk measurement with spectral capital allocation, applied quantitative finance, II, Springer, 139-159.

Resnick, S.I., 1987. Extreme values, regular variation and point processes. Applied Probability. A Series of the Applied Probability Trust, 4. *Springer-Verlag*, New York.

Seneta, E., 1976. Regularly varying functions. Lecture Notes in Mathematics, Vol. 508. *Springer-Verlag*, Berlin-New York.

Sklar, A., 1959. Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris*. **8**, 229-231.

Wirch, J. and Hardy, M., 1999. A Synthesis of Risk Measures for Capital Adequacy, *Insurance Math. Econom.*, **25**, 337-347.