On a new simple discriminant inequality

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Abstract

We prove a simple though nontrivial inequality involving a real-rooted polynomial, its successive derivatives and their discriminants. In particular we get the new inequality: \( \text{Dis}(p - tp') > \text{Dis}(p) \) for a polynomial \( p \), its derivative \( p' \) and \( t \) any non zero real. We also formulate a well-motivated conjecture generalizing our main result.

1 Introduction

In this paper, the polynomials we consider will be monic and real-rooted. The monic condition is no particular restriction, as we are only interested in polynomials through their roots and don’t care about the leading coefficient. Take such a polynomial of degree \( d \), \( p(x) = \prod_{i=1}^{d}(x - \lambda_i(p)) \) where all the roots are real. We denote in all the following by \( p' \) the polynomial derivative of \( p \). We also assume that the polynomial \( p \) has simple roots. This is again no particular restriction. Recall the following somewhat canonical definition of the discriminant of \( p \), which matches the canonical one for monic polynomials:

\[
\text{Dis}(p) = \prod_{i<j} (\lambda_i(p) - \lambda_j(p))^2
\]

Notice that defined this way, the discriminant is always positive for real-rooted polynomials with simple roots. Few inequalities involving discriminants are available that are true for all polynomials with real roots. Most of them are trivial re-writing of the quantity. The discriminant represents some sort of average pair-wise spacing between all the roots of the polynomial. Therefore, a high value discriminant amounts to a polynomial with well-spread roots over the real line, whereas a small quantity indicates roots close overall to one another - not well separated from each other (it suffices that two of them are very close to make the discriminant small). In this line of thought, recall that derivating increases in some sense the spreading of the roots. For instance, if a polynomial has a multiple root, derivating splits the multiple root into single distinct roots one by one. Other people have studied the effects of the derivative on the roots of a polynomial in different ways, see \([3]\) for an overview. In terms of discriminants, it is not possible to compare any normalized version of \( \text{Dis}(p) \) with \( \text{Dis}(p') \). However, and it constitutes the base case of our result, we can compare the discriminant of a polynomial to which we add (or subtract) a scalar multiple of its derivative to the original discriminant. Denote by \( \mathbb{R}^* \) the set of all real numbers except zero.

**Theorem 1.1.** For all \( \alpha \in \mathbb{R}^* \),

\[
\text{Dis}(p - \alpha p') > \text{Dis}(p)
\]
Corollary 1.1. Let \( r(x) = \prod_{i=1}^{d}(1 - \alpha_i x) \), then, if \( D \) denotes the derivative operator, and if at least one \( \alpha_i \) is nonzero,

\[
\text{Dis}(r(D)p) > \text{Dis}(p)
\]

To formulate the main form of our inequality, we will appeal to some recently discovered tools. We want a way to convolve polynomials that corresponds to "adding" the roots of \( p \) to the roots of \( q \). The problem with adding them pairwise is that this will depend on some ordering. So the obvious thing to do if you want something that does not depend on any ordering is to average over all possible pairings. That is, we take for \( p = \prod_{i=1}^{d}(x - a_i) \), \( q = \prod_{i=1}^{d}(x - b_i) \) and for \( \Sigma_d \) the set of all possible permutations of \( d \) elements

\[
p \boxplus_d q := \sum_{\pi \in \Sigma_d} \prod_{i=1}^{d}(x - a_i - b_{\pi(i)})
\]

This quantity called the finite free convolution of \( p \) and \( q \) depends only on the roots of \( p \) and \( q \), and in terms of derivatives we have the following.

Lemma 1.1 (see [1]).

\[
p \boxplus_d q = \sum_{k=0}^{d} p^{(k)}(-1)^k \frac{1}{k!} \sigma_k(q)
\]

where

\[
\sigma_k(q) = \frac{1}{(d)_k} \sum_{i_1 < i_2 < \ldots < i_k} \lambda_{i_1}(q) \lambda_{i_2}(q) \ldots \lambda_{i_k}(q).
\]

In particular we have that if \( q(x) = x^d - d\alpha x^{d-1} \),

\[
p \boxplus_d q = p - \alpha p'
\]
as \( \sigma_1(q) = \alpha \) and all the other symmetric sums are zero. Therefore, a natural generalization of Theorem 1.1 is

Theorem 1.2. For all polynomial \( q \) of degree \( d \) not equal to \((x - c)^d\) for some \( c \in \mathbb{R} \),

\[
\text{Dis}(p \boxplus_d q) > \text{Dis}(p)
\]

Remark 1.

\[
p \boxplus_d (x - c)^d = p(x - c)
\]

Therefore:

\[
\text{Dis}(p \boxplus_d (x - c)^d) = \text{Dis}(p)
\]

Remark 2.

\[
p \boxplus_d q = q \boxplus_d p
\]

As a consequence the inequality can be reversed, exchanging the roles of \( p \) and \( q \) (but we remind the reader that the assumption on \( p \) is that it is simple rooted, so the same should be expected from \( q \) if we exchange the roles).
So, in some sense, doing this finite convolution makes the distribution of the roots more "random", as they are more well-spread and less localized. It creates some sort of mixing of the roots. This operation is far from being exotic or anecdotic, as the following lemma from [2] shows (Proposition 1.2).

**Lemma 1.2.** A linear operator $T : \mathbb{R}^{\leq d}[x] \to \mathbb{R}^{\leq d}[x]$, where $\mathbb{R}[x]^{\leq d}$ is the real vector space of polynomials of degree at most $d$, is a differential operator which preserves real-rootedness if and only if it can be written in the form $T(p) = p \oplus_d q$ for some $q \in \mathbb{R}^{\leq d}[x]$.

As a result, we prove that applying any differential operator which preserves real-rootedness (a necessity to even be able to define the discriminant as a real quantity) and as long as the operator doesn’t decrease the degree (the condition on the polynomial $q$ being that it is of degree $d$), always increases the discriminant - i.e. the average spacing of the roots- which amounts to a sort of smoothness property. We will finally need to introduce the idea of time-dilation. For a polynomial $q(x) = \prod_{i=1}^{d}(x - \lambda_i(q))$, define the quadratic time-dilation:

$$q_t(x) = q(t, x) = \prod_{i=1}^{d}(x - \sqrt{t}\lambda_i(q))$$

Notice that $q_0(x) = x^d$ and $q_1(x) = q(x)$. So we interpolate between the zero polynomial and the original polynomial, using some quadratic parameter. Our choice of $\sqrt{t}$ over $t$ was motivated by connections with free probability (see Section 4), and will lead to nicer cancellations in the proof. We will in all the following denote by $r_t(x) = p \oplus_d q_t(x)$. We will denote by $\lambda_i(t)$ the roots of $r_t$, according to the context.

**Lemma 1.3.**

$$\frac{d\lambda_i(t)}{dt} = -\frac{\partial_t r_t}{r_t'}(\lambda_i(t))$$

**Proof.** We can easily prove using the implicit function theorem that the roots are differentiable functions of $t$. We have the following equality for all time: $r_t(\lambda_i(t)) = 0$. Therefore, derivating with respect to time we get

$$\frac{d\lambda_i(t)}{dt} r_t'(\lambda_i(t)) + \partial_t r_t(\lambda_i(t)) = 0$$

Also, we recall the definition of the Wronskian:

**Definition 1.** We define the Wronskian of two real-rooted polynomials as

$$W[p, q] = p'q - q'p$$

The following easy properties can be found in [6].

**Lemma 1.4.** If the roots of $p$ and $q$ interlace, then the Wronskian is of constant sign: $W[p, q] \geq 0$ or $W[p, q] \leq 0$.

**Lemma 1.5.** (Laguerre’s inequality) If $p$ has simple roots, then we have a strict interlacing between the first and second derivative; more precisely, for all $x$,

$$W[p', p''](x) > 0$$
Additionally, we recall the following important rewriting of the discriminant:

Lemma 1.6. For a (monic) polynomial \( r \) of degree \( d \),

\[
\text{Dis}(r) = (-1)^{\frac{d(d-1)}{2}} \prod_{i=1}^{d} r'[\lambda_i(r)]
\]

Proof.

\[
\text{Dis}(r) = \prod_{i=1}^{d} \prod_{j \neq i} |\lambda_i(r) - \lambda_j(r)| \quad (1)
\]

\[
= (-1)^{\frac{d(d-1)}{2}} \prod_{i=1}^{d} \prod_{j \neq i} (\lambda_i(r) - \lambda_j(r)) \quad (2)
\]

But

\[
r'(\lambda_i(r)) = \prod_{j \neq i} (\lambda_i(r) - \lambda_j(r))
\]

\[\blacksquare\]

2 The base case

This section is devoted to proving Theorem 1.1.

Proof. Let \( q_t(x) := x^d - \sqrt{t}dx^d - 1 \) and

\[ r_t(x) := p \oplus q_t = p - \sqrt{t} \alpha p' \]

It is sufficient to show that \( \text{Dis}(r_t) \) is strictly increasing as a function of time \( t \). We can use Lemma 1.6 to re-write:

\[
\text{Dis}(r_t)^2 = \prod_{i=1}^{d} r_t'[\lambda_i(t)]^2
\]

Now, it is sufficient to show that \( f_i(t) = r_t'[\lambda_i(t)]^2 \) is increasing in \( t \) for all \( i \). We square all the quantities not to bother about the sign but to consider only the absolute value increments.

We have

\[
\frac{df_i(t)}{dt} = 2r_t' (\lambda_i(t))\left( \frac{d\lambda_i(t)}{dt} r_t'' (\lambda_i(t)) + \partial_t r_t'[\lambda_i(t)] \right)
\]

On the other hand we have

\[
\partial_t r_t(x) = -\frac{\alpha}{2\sqrt{t}} p' \quad \quad \frac{d\lambda_i(t)}{dt} = -\frac{\partial_t r_t}{r_t'} (\lambda_i(t)) \quad (3)
\]

So that

\[
\frac{df_i(t)}{dt} = 2\left( -\partial_t r_t r_t'' + (\partial_t r_t') r_t' \right) (\lambda_i(t))
\]

\[
= 2\left( \frac{\alpha}{2\sqrt{t}} p' p'' \right) (p' - \sqrt{t} \alpha p''') (\lambda_i(t))
\]

\[
= \alpha^2 \left( p'^2 - p' p''\right) (\lambda_i(t)) \quad (4)
\]

\[
= \sigma_1(q)^2 W[p', p''\lambda_i(t)] > 0 \quad (5)
\]

The last inequality follows from Lemma 1.5 \[\blacksquare\]
3 The general case

We start with a few theorems on polynomial interlacings. They can be found in [6].

Theorem 3.1. (Hermite-Kakeya-Obreschkoff) Let \( f \) and \( g \) be real polynomials. Then the two following statement are equivalent:
1) \( f \) and \( g \) are real-rooted and their roots interlace (in particular \( W[f, g] \) is of constant sign)
2) \( af + bg \) is real-rooted for all \( a, b \in \mathbb{R} \)

Corollary 3.1. The finite free addition (convolution) preserves interlacing, or more precisely: if \( f \) and \( g \) are degree \( d \) polynomials that interlace then \( p \boxplus_d f \) and \( p \boxplus_d g \) interlace too.

Proof. Take \( a, b \in \mathbb{R} \). Assume \( f \) and \( g \) do interlace. It implies by Theorem 3.1 that \( af + bg \) is real rooted. But now, using the real-rootedness preservation (Theorem 3.1) and the linearity properties of \( p \boxplus_d \), we get that \( p \boxplus_d (af + bg) = a(p \boxplus_d f) + b(p \boxplus_d g) \) is real rooted. As \( a \) and \( b \) are arbitrary, we conclude using Theorem 3.1 the other way around that \( p \boxplus_d f \) and \( p \boxplus_d g \) do interlace. \( \square \)

More precisely we have the following result coming from [2] (Lemma 4.1):

Corollary 3.2. If \( W[f, g] \geq 0 \), then also \( W[f \boxplus_d p, g \boxplus_d p] \geq 0 \) (the convolution operation preserves the monotonicity of the interlacing and not merely the interlacing).

We will need at this point the notion of polar derivative.

Definition 2. If \( p \) is a polynomial of degree \( d \), we call polar derivative the polynomial of degree \( d - 1 \) defined by: \( \partial_x p = dp - xp' \).

Remark 3. Notice that the polar derivative can be considered as some sort of derivative with respect to all the roots at the same time, and as some sort of derivative dual to the usual derivative; for the following reason. If \( p(x) = \prod_{i=1}^{d} (x - \lambda_i) \), define \( p(x, t) = \prod_{i=1}^{d} (x - t\lambda_i) \). Now we can differentiate “with respect to the roots” by differentiating with respect to \( t \) and then making \( t \) disappear by evaluating at 1. And we precisely get the polar derivative: \( \partial_x p = \partial_t p(x, t) \big|_{t=1} \).

Lemma 3.1. For any real-rooted polynomial \( q \), \( \partial_t q_t(x) \) and \( q'_t := \partial_x q_t(x) \) interlace for all \( t > 0 \). In particular, \( \partial_t q_t(x) \) is always a real-rooted polynomial (this is equivalent as we will see to showing that \( \partial_x q \) and \( q' \) interlace).

Proof. We recall that: \( q_t(x) = \prod_{i=1}^{d} (x - \sqrt{t}\lambda_i(q)) \). We have the following:

\[ \partial_t q_t(x) = \frac{1}{2\sqrt{t}} \sum_{i=1}^{d} -\lambda_i(q) \cdot \frac{q_t(x)}{x - \sqrt{t}\lambda_i(q)}. \]

On the other hand we have that

\[ q'_t = \sum_{i=1}^{d} \frac{q_t(x)}{x - \sqrt{t}\lambda_i(q)}. \]

So we have the following relationship:

\[ 2t\partial_t q_t(x) + xq'_t(x) = dq_t(x) \]
Or isolating the scaled polar derivative,
\[
\partial_t q_t(x) = \frac{1}{2t} \left( dq_t(x) - x q'_t(x) \right) = \frac{1}{2t} \dot{x}_t q_t.
\]

Now notice that we can pull out the multiple roots factors out of \( r_i \) and \( r_i' \) and they will be roots of \( \partial_t q_t(x) \) with the same multiplicity as for \( q'_t(x) \). If we now consider the roots of \( q'_t \) which don’t come from roots of \( q_t \), denoting them by \( \alpha_1 < \alpha_2 < \cdots < \alpha_l \) we have that the sign of \( q_t \) will strictly change from one to another, that is \( \text{sign}(q_t(\alpha_i)) = (-1)^i \epsilon \). Therefore:
\[
\text{sign}(\partial_t q_t(\alpha_i)) = (-1)^i \epsilon
\]

This sign-interlacing gives us some roots \( \beta_1 < \ldots < \beta_l \) of \( \partial_t q_t(x) \) that interlace the \( \alpha_i \). Therefore we get (summing up) \( d \) real roots for the polar derivative, and whatever the multiplicity, the roots of the polar derivative and the normal derivative do interlace.

**Corollary 3.3.** In case \( q \) is not of the form \((x - c)^d\) for some \( c \in \mathbb{R} \), \( W[\partial_x q, q'] \geq 0 \), i.e. the interlacing is positive between the polar derivative and the derivative. In particular: \( W[\partial_t q_t, q'_t] \geq 0 \) (if \( q_t \) is a time-dilation of \( q \)). In case \( q \) is of the form \((x - c)^d\) we trivially have \( W[\partial_x q, q'] = 0 \) (as the two polynomials are collinear).

**Proof.** We already know by the previous lemma that the Wronskian has a constant sign, it is therefore sufficient to prove positivity, and positivity for large \( x \) is also sufficient (the sign being constant). We only need to isolate the leading term in \( W[\partial_x q, q'] \) and show that it is positive. Write:
\[
q = x^d + c_{d-1} x^{d-1} + c_{d-2} x^{d-2} + \ldots \text{(lower degrees)}
\]
\[
\dot{\partial}_x q = c_{d-1} x^{d-1} + 2 c_{d-2} x^{d-2} + \ldots
\]

After a few straightforward computations we get:
\[
W[\partial_x q, q'] = \dot{\partial}_x q' - \dot{\partial}_x q q'' = x^{2(d-2)} \left( c_{d-1}^2 (d - 1) - 2 dc_{d-2} \right) + \ldots
\]

Using the fact that \( c_{d-1} = - \sum_{i=1}^d \lambda_i(q) \) and \( c_{d-2} = \sum_{i < j} \lambda_i(q) \lambda_j(q) \) we get that:
\[
c_{d-1}^2 (d - 1) - 2 dc_{d-2} = \sum_{i \neq j} (\lambda_i(q) - \lambda_j(q))^2
\]

The positivity is then clear if all the roots are not equal to some constant \( c \).

We are now ready to prove Theorem 1.2.

**Proof.** The inequality amounts to proving that \( \text{Dis}(r_1(x)) > \text{Dis}(r_0(x)) \). We will prove first that \( \frac{\partial}{\partial t} \text{Dis}(r_t(x)) \geq 0 \), and then that for \( t \) close to zero \( \frac{\partial}{\partial t} \text{Dis}(r_t(x)) > 0 \). The result will immediately follow. We recall that
\[
\text{Dis}(r_t)^2 = \prod_{i=1}^d r_i^t \left[ \lambda_i(t) \right]^2
\]

Again we will prove that each factor in the former expression does indeed increase in time, it will follow that the squared discriminant will a fortiori be increasing in time, and by positivity of the
discriminant, the discriminant itself. As in the "base case", call \( f_i(t) = r'_i[\lambda_i(t)]^2 \), we want to show first that \( \frac{df_i(t)}{dt} \geq 0 \). But we have:

\[
\frac{df_i(t)}{dt} = 2r'_i[\lambda_i(t)]\left(\frac{d\lambda_i(t)}{dt}r''_i[\lambda_i(t)] + \partial_t r'_i[\lambda_i(t)]\right)
\]

(8)

\[
= 2r'_i[\lambda_i(t)]\left(- \frac{\partial_t r_t}{r_t}[\lambda_i(t)]r''_t[\lambda_i(t)] + \partial_t r'_t[\lambda_i(t)]\right)
\]

(9)

\[
= 2\left(- \partial_t r_t[\lambda_i(t)]r''_t[\lambda_i(t)] + r'_t[\lambda_i(t)]\partial_t r'_t[\lambda_i(t)]\right)
\]

(10)

\[
= 2W[\partial_q t, r'_t][\lambda_i(t)]
\]

(11)

\[
= 2W[\partial_q t \boxplus_d p, q'_t \boxplus_d p][\lambda_i(t)]
\]

(12)

\[
= 2W[\partial_q t \boxplus_d p, q'_t \boxplus_d p][(13)]
\]

(13)

The last equality follows from the fact that: \( \partial_t(q_t \boxplus_d p) = \partial_t(q_t \boxplus_d p)\) and \((q_t \boxplus_d p)' = q'_t \boxplus_d p\) (a proof of this fact can be found in [1]).

The next step is to notice that if the polynomial \( q \) is not of the form \((x - c)^d\), then its time-dilation \( q_t \) won’t be either. Using [3], we get that \( W[\partial_q q_t \boxplus_d p, q'_t \boxplus_d p](\lambda_i(t)) \geq 0 \), and using [3,2] we get that

\[
W[\partial_q q_t \boxplus_d p, q'_t \boxplus_d p](\lambda_i(t)) \geq 0
\]

What was needed to prove that it is nondecreasing. We will now prove that is in fact increasing for \( t \) close to zero, and therefore the strict inequality will follow. To this purpose, we can compute an asymptotic expansion of \( W[\partial_q q_t \boxplus_d p, q'_t \boxplus_d p] \) in time and we find that:

\[
W[\partial_t q_t \boxplus_d p, q'_t \boxplus_d p](x) \sim_{t \to 0} (\sigma_1(q)^2 - \sigma_2(q))W[p', p''](x)
\]

where

\[
\sigma_1(q)^2 - \sigma_2(q) = \sum_{i,j} (\lambda_i(q) - \lambda_j(q))^2 > 0
\]

If \( q \) is not of the form \((x - c)^d\). Using the strict interlacing lemma [1,5] again, we get that for \( t \) close to zero, and all \( x \),

\[
W[\partial_t q_t \boxplus_d p, q'_t \boxplus_d p](x) > 0
\]

what was needed.

\[\square\]

### 4 Conclusion and extension

The proof of the inequality is quite systematic and elementary, based purely on interlacing of polynomials and their derivatives. In fact, the convolution we are using comes from a newly developed theory which has linked operations on polynomials to operations in a field known as "free probability" (see [4]). Voiculescu (in [5]) introduced information theoretic notions in free probability that are analogous to Shannon’s theory. One can then use this link to define information theoretic notions on polynomials. We define the entropy of a polynomial \( p \) as:

\[
h(p) = \frac{1}{(d)} \sum_{i < j} \log |\lambda_i(p) - \lambda_j(p)|
\]
Therefore, if we take Voiculescu’s theory and link it back to polynomials, what we find is that the entropy function coincides with a log-discriminant! We can also define some Fisher information and other information theoretic quantities but we won’t push the analogy further in this paper. The main point is that the inequality we proved is a simple version of the more general Minkowski inequality:

**Conjecture 1.**

\[
\text{Dis}(p \boxplus_d q)^{\frac{1}{2}} \geq \text{Dis}(p)^{\frac{1}{2}} + \text{Dis}(q)^{\frac{1}{2}}
\]

*For which there is equality only for generalized Hermite polynomials.*

It is actually a finite free analogue of the entropy power inequality, as the previous inequality can be read as follows:

\[
e^{2h(p \boxplus_d q)} \geq e^{2h(p)} + e^{2h(q)}
\]

This conjecture seems to be true numerically.

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**References**

[1] Adam Marcus, Daniel A. Spielman, Nikhil Srivastava. *Finite free convolutions of polynomials*, arXiv:1504.00350

[2] Jonathan Leake, Nick Ryder. *On the Further Structure of the Finite Free Convolutions*, arXiv:1811.06382

[3] Qazi Ibadur Rahman, Gerhard Schmeisser. *Analytic Theory of Polynomials*

[4] J. A. Mingo, R. Speicher. *Free probability and random matrices*

[5] Dan Voiculescu. *The analogues of entropy and of Fisher’s information measure in free probability theory*, https://projecteuclid.org/euclid.cmp/1104253200

[6] David Wagner, *Multivariate stable polynomials: theory and applications*