UNBOUNDED KOBAYASHI HYPERBOLIC DOMAINS IN $\mathbb{C}^n$

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Abstract. We first give a sufficient condition, issued from pluripotential theory, for an unbounded domain in the complex Euclidean space $\mathbb{C}^n$ to be Kobayashi hyperbolic. Then, we construct an example of a rigid pseudoconvex domain in $\mathbb{C}^3$ that is Kobayashi hyperbolic and has a nonempty core. In particular, this domain is not biholomorphic to a bounded domain in $\mathbb{C}^3$ and the mentioned above sufficient condition for Kobayashi hyperbolicity is not necessary.

Introduction

According to the Riemann mapping theorem, every simply-connected domain in $\mathbb{C}$, different from $\mathbb{C}$, is biholomorphically equivalent to the unit disk $\Delta_1(0) := \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$. It is well known that this result has no direct generalization to higher dimension, since for instance every domain in $\mathbb{C}^n$ containing a nonconstant entire curve cannot be biholomorphic to a bounded domain in $\mathbb{C}^n$. There are different tools to distinguish domains, among which invariant metrics (under the action of biholomorphisms) play an important role. We recall that if $M$ is a complex manifold, $\Delta_r(0) := \{ \lambda \in \mathbb{C} : |\lambda| < r \}$ for every $r > 0$ and $\mathcal{H}(\Delta_r(0), M)$ denotes the set of holomorphic maps from $\Delta_r(0)$ to $M$, then the Kobayashi pseudometric $k_M$ is defined on $TM$ by

$$k_M(z; v) := \inf\{1/r > 0 : \exists f \in \mathcal{H}(\Delta_r(0), M), \ f(0) = z, \ f'(0) = v \}.$$ 

A complex manifold $M$ of complex dimension $n$ is Kobayashi hyperbolic if for every point $p \in M$, there is a neighbourhood $U$ of $p$ in $M$ and a constant $c > 0$ such that $k_M(z, v) \geq c\|v\|_g$ for every $z \in U$ and every $v \in T_zM$, where $\|\cdot\|_g$ is any Hermitian norm on $U$ induced from $\mathbb{C}^n$. If $K_M$ denotes the inner distance induced by $k_M$, then $M$ is Kobayashi hyperbolic if $K_M$ is a distance on $M$. Notice that the topology induced by $K_M$ on $M$ is then equivalent to the natural topology of $M$. From the definition of $k_M$ we see that every bounded domain in $\mathbb{C}^n$ is Kobayashi hyperbolic, whereas a complex manifold containing a nonconstant entire curve is not Kobayashi hyperbolic. Since the Kobayashi metric is a biholomorphic invariant, it follows that a complex manifold that is not Kobayashi hyperbolic does not admit any bounded representation, i.e., is not biholomorphic to any bounded domain in $\mathbb{C}^n$. The first purpose of the paper is to give a sufficient condition from pluripotential theory for an unbounded domain to be Kobayashi hyperbolic. For $r > 0$ and $z \in \mathbb{C}^n$, $n \geq 1$, we denote by $B_r^n(z)$ the Euclidean open ball centered at $z$ with radius $r$, i.e. $B_r^n(z) := \{ w \in \mathbb{C}^n : \|w - z\| < r \}$ where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^n$; in particular, $\Delta_r(z) := B_1(z)$. Finally, if $D$ is a domain in $\mathbb{C}^n$, we denote by $\partial D$ its Euclidean boundary.

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Definition 1. Let $\Omega$ be an unbounded domain in $\mathbb{C}^n$. A bounded continuous positive plurisubharmonic (for short, psh.) function $\varphi$ on $\Omega$ will be called strong antipeak at infinity for $\Omega$ if \[
abla \lim_{\|z\| \to \infty} \varphi(z) = 0.\]

The first result of the paper is the following

Theorem 1. Let $\Omega$ be an unbounded domain in $\mathbb{C}^n$. If $\Omega$ has a strong antipeak function at infinity, then $\Omega$ is Kobayashi hyperbolic.

Note that not every Kobayashi hyperbolic domain admits a bounded representation. One of the obstructions for the existence of such representations was introduced and studied by T.Harz, N.Shcherbina and G.Tomassini (see [5, 6, 7]) and later also by N.Shcherbina and E.Poletsky [9]. It was named the core of a domain and can be defined as follows.

Definition 2. Let $\Omega$ be an unbounded domain in $\mathbb{C}^n$. The core $c(\Omega)$ is defined by
\[
c(\Omega) := \{ z \in \Omega : \text{every bounded continuous plurisubharmonic function on } \Omega \text{ fails to be smooth and strictly plurisubharmonic near } z \}.
\]

Since the function $z \mapsto \|z\|^2$ is strictly plurisubharmonic in $\mathbb{C}^n$, every bounded domain in $\mathbb{C}^n$ has an empty core. It follows from the biholomorphic invariance of the core that an unbounded domain with a nonempty core will not admit any bounded representation. For instance, in [4, Theorem 1.2], the authors construct for every $n \geq 2$ an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that $c(\Omega)$ is not empty and contains no analytic variety of positive dimension. Another surprising example of a strictly pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary and nonempty core which is Kobayashi and Bergman complete, but has no nonconstant holomorphic functions, was constructed recently in [12]. It is also worth mentioning that S.Mongodi explained to us a construction of a domain in $\mathbb{CP}^2$ which is Kobayashi hyperbolic, but has no bounded representations. The second goal of the present paper is to construct an unbounded pseudoconvex domain in $\mathbb{C}^3$, whose boundary is globally defined by a graph, which is Kobayashi hyperbolic and has a nonempty core. More precisely, we say that a domain $\Omega \subset \mathbb{C}^n$ is rigid if there exists a function $\Psi$ defined in $\mathbb{C}^{n-1}$ such that
\[
\Omega = \{ (z, \zeta) \in \mathbb{C}^{n-1} \times \mathbb{C} : \Re(\zeta) > \Psi(z) \}.
\]

The domain $\Omega$ is pseudoconvex if and only if the function $\Psi$ is plurisubharmonic in $\mathbb{C}^{n-1}$. Rigid domains appear naturally as local models for pseudoconvex domains and reflect the geometry of such domains at some boundary points. For instance, the strictly pseudoconvex domain $\Omega := \{ (\zeta, z) \in \mathbb{C}^n : \Re(\zeta) > \|z\|^2 \}$, unbounded representation of the unit ball in $\mathbb{C}^n$, is a local model for domains in $\mathbb{C}^n$ near every strictly pseudoconvex boundary point. Likewise, if $D \subset \mathbb{C}^2$ is a bounded domain with smooth boundary of finite D’Angelo type $2m$ at $p \in \partial D$ (see [2] for the definition of the D’Angelo type), then there are a neighbourhood $U$ of $p$ in $\mathbb{C}^2$ and holomorphic coordinates $(\zeta, z)$ defined on $U$ such that
\[
D \cap U = \{ (z, \zeta) \in U : \Re(\zeta) > H(z) + \phi(z, \zeta) \},
\]
where $H$ is a subharmonic homogeneous polynomial of degree $2m$ which is not harmonic and $|\phi(z, \zeta)| \leq c(|\zeta|^2 + |\zeta||z| + |z|^{2m+1})$ on $U$. Notice that if $\Omega_H := \{ (z, \zeta) \in \mathbb{C}^2 : \Re(\zeta) > H(z) \}$,
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then the metric space $(\Omega_H, K_{\Omega_H})$ is complete. Indeed, since $H$ is homogeneous, there is a sequence of automorphisms of $\Omega_H$ that accumulates at the origin. Moreover, according to the Main Theorem in [1], there is a global holomorphic peak function at the origin for $\Omega_H$, i.e. a holomorphic function $f$ from $\Omega_H$ to $\Delta_1(0)$, continuous on $\overline{\Omega_H}$, such that $f(0) = 1$ and for every bounded open neighbourhood $U$ of the origin in $\mathbb{C}^2$, sup$_{U \setminus \{0\}} |f| < 1$. Notice that, by construction, $f(\Omega_H) \subset \Delta_1(0) \setminus \{0\}$ and lim$_{|p| \to \infty} f(p) = 0$. The completeness of the metric space $(\Omega_H, K_{\Omega_H})$ follows now from Proposition 3.1.4 in [3].

Observe, moreover, that $\Omega_H$ has an empty core. Indeed, assume to get a contradiction that $c(\Omega_H) \neq \emptyset$. Since $\Omega_H$ is a pseudoconvex domain of finite type in $\mathbb{C}^2$, it admits a local holomorphic peak function at each boundary point. It follows then that $c(\Omega_H) \cap \partial \Omega_H = \emptyset$. Moreover, we know by Theorem II in [9] (see also Theorem 3.3 in [14]) that the set $c(\Omega_H)$ is the disjoint union of some closed sets $E_j, j \in J$, that are 1-pseudoconcave in the sense of Rothstein and have the following Liouville property: every bounded continuous psh. function on $\Omega$ is constant on each of $E_j$. Let $E_{j_0}$ be one of the sets in the decomposition above. Then, in view of the 1-pseudoconcavity of $E_{j_0}$, $E_{j_0}$ is unbounded. Since $|f|^2$ is a bounded continuous psh. function on $\Omega_H$, the restriction of $|f|^2$ to $E_{j_0}$ is constant. Hence, it follows from the fact that lim$_{|p| \to \infty} f(p) = 0$ that $f$ vanishes identically on $E_{j_0}$. This is a contradiction since $f$ does not vanish on $\Omega_H$.

It was proved in a recent paper [11] that the existence of the Kobayashi and the Bergman metrics for pseudoconvex domains in $\mathbb{C}^2$ of more general form

$$\Omega_H := \{(z, \zeta) \in \mathbb{C}^2 : \text{Re}(\zeta) > H(z, \text{Im}(\zeta))\},$$

with $H$ being just a continuous function on $\mathbb{C} \times \mathbb{R}$, is equivalent to the fact that the core $c(\Omega_H)$ of $\Omega_H$ is empty.

The second result of the present paper shows that this kind of relations does not hold in the case of higher dimensions.

**Theorem 2.** There exists a nonnegative plurisubharmonic function $\Psi$ in $\mathbb{C}^2$ such that the rigid domain

$$\Omega_{\Psi} := \{(z, w, \zeta) \in \mathbb{C}^3 : \text{Re}(\zeta) > \Psi(z, w)\}$$

is Kobayashi hyperbolic and has a nonempty core. In particular, the domain $\Omega_{\Psi}$ is not biholomorphic to a bounded domain.

The following corollary shows that the existence of a strong antipeak function at infinity is not a necessary condition for an unbounded domain to be Kobayashi hyperbolic. Indeed, from the construction of $\Omega_{\Psi}$ in Theorem 2, we have

**Corollary 1.** The domain $\Omega_{\Psi}$ does not admit any strong antipeak function at infinity.

The paper is organized as follows. In Section 1 we prove Theorem 1. In Section 2 we construct explicitly the function $\Psi$ used in Theorem 2 and a Wermer type set contained in $\Omega_{\Psi}$. Finally, in Section 3 we prove Theorem 2 and Corollary 1.

1. **Proof of Theorem 1**

We first notice that a domain $\Omega \subset \mathbb{C}^n$ is Kobayashi hyperbolic if and only if it satisfies the following condition:
\[
\forall p \in \Omega, \exists r > 0, \exists c > 0, \forall q \in B^n_r(p), \forall v \in \mathbb{C}^n, k_{\Omega}(q, v) \geq c\|v\|,
\]

where \(\|v\|\) denotes the Euclidean norm in \(\mathbb{C}^n\).

Let now \(\Omega\) be a domain satisfying the assumptions of Theorem 1. Assume, to get a contradiction, that \(\Omega\) is not Kobayashi hyperbolic. It follows from (1.1) that there is a point \(p \in \Omega\) and for every positive integer \(k\) there is a holomorphic map \(f_k : \Delta_k(0) \to \Omega\) such that \(\|f'_k(0)\| = 1\) and the sequence \(\{f_k(0)\}\) converges to \(p\) when \(k\) goes to infinity. Moreover, we may assume that \(f_k\) is continuous up to \(\partial \Delta_k(0)\).

Denote by \(\varphi\) a psh. function on \(\Omega\) that is strong antipeak at infinity. Let \(C > 0\) be a constant which bounds the function \(\varphi\) from above, i.e. \(\varphi < C\) on \(\Omega\). Observe that, in view of the continuity of \(\varphi\), there is a positive constant \(\alpha\) such for sufficiently large \(k\) we have:

\(\varphi(f_k(0)) \geq \alpha.\)

For each \(R > 0\) we let \(c_R := \sup_{\mathbb{C}^n \setminus B^n_R(0)} \varphi\). Notice that, by Definition 1, we get: \(\lim_{R \to \infty} c_R = 0\). Since the Euclidean ball \(B^n_R(0)\) is Kobayashi hyperbolic, it follows that for every sufficiently large positive integer \(k\) we have:

\[ f_k(\partial \Delta_k(0)) \cap \left(\mathbb{C}^n \setminus B^n_R(0)\right) \neq \emptyset. \]

Let \(U_{k,R}\) denote the connected component of \(f_k^{-1}(f_k(\Delta_k(0)) \cap B^n_R(0))\) containing the origin.

**Claim 1.** For each \(k \in \mathbb{N}\) and \(R > 0\), the domain \(U_{k,R}\) is simply connected.

Indeed, if \(\partial U_{k,R}\) has at least two components, then, after maybe substituting \(R\) with a generic value \(\tilde{R} < R\), there is a disc \(V_{k,R}\), contained in \(\Delta_k(0)\), such that \(f_k(V_{k,R}) \subset \Omega \setminus B^n_{\tilde{R}}(0)\) and such that \(f_k(\partial V_{k,R}) \subset \partial B^n_{\tilde{R}}(0)\). Let \(R' = \sup_{V_{k,R}} |f_k|\). Then the complex disc \(f_k(V_{k,R})\) is tangent to \(\partial B^n_{\tilde{R}}(0)\) from inside which is not possible. This proves Claim 1.

Hence \(U_{k,R}\) is bounded by a piecewise smooth Jordan curve for generic values of \(R\). We denote by \(\Phi\) a Riemann map from \(\Delta_1(0)\) to \(U_{k,R}\) such that \(\Phi(0) = 0\). According to the Carathéodory Theorem, \(\Phi\) extends as a homeomorphism between \(\partial \Delta_1(0)\) and \(\partial U_{k,R}\).

**Claim 2.** There is \(R_0 > 0\) such that for each \(k \in \mathbb{N}\) and each \(R > R_0\) one has \(\overline{U_{k,R}} \cap \partial \Delta_k(0) \neq \emptyset\).

Indeed, assume to get a contradiction that \(\partial U_{k,R}\) is a closed curve contained in \(\Delta_k(0)\). Then \(f_k(\Phi(e^{i\theta})) \subset \partial B^n_{\tilde{R}}(0)\) for all \(\theta \in [0, 2\pi]\) and from the Mean Value Inequality

\[ (\varphi \circ f_k \circ \Phi)(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(f_k(\Phi(e^{i\theta}))) d\theta \]

we get the inequality

\[ \alpha \leq c_R. \]

If we choose \(R_0\) so large that \(c_{R_0} < \alpha\), then we get a contradiction. This proves Claim 2.

Let \(\omega(0, \overline{U_{k,R}} \cap \partial \Delta_k(0), U_{k,R})\) denote the harmonic measure of \(\overline{U_{k,R}} \cap \partial \Delta_k(0)\) at the point 0 with respect to the domain \(U_{k,R}\). We recall that if \(D\) is a bounded domain in \(\mathbb{C}\), \(p \in D\) and \(E\) is a Borel set in \(\partial D\), then \(\omega(p, E, D)\) denotes the harmonic measure given by the value at
p of the solution to the Dirichlet problem on $D$, whose boundary value on $\partial D$ is equal to the characteristic function of $E$. Then we have

$$\omega(0, \overline{U}_{k,R} \cap \partial \Delta_k(0), U_{k,R}) = \frac{l(\Phi^{-1}(\overline{U}_{k,R} \cap \partial \Delta_k(0)))}{2\pi},$$

where $l(\Phi^{-1}(\overline{U}_{k,R} \cap \partial \Delta_k(0)))$ denotes the length of the set $\Phi^{-1}(\overline{U}_{k,R} \cap \partial \Delta_k(0))$. Since $U_{k,R}$ is simply connected and contained in $\Delta_k(0)$, it follows from Lemma 3.4 of [10] that the Euclidean distance $\rho(0, \partial U_{k,R})$ from 0 to the boundary of $U_{k,R}$ satisfies:

$$\rho(0, \partial U_{k,R}) \geq \frac{\pi^2 k}{16} \left( \frac{\omega(0, \overline{U}_{k,R} \cap \partial \Delta_k(0), U_{k,R})}{\rho(0, \partial U_{k,R})} \right)^2.$$

To finish the proof of Theorem 1 we consider two cases.

**Case 1.** There exist $R > 0$ and $c > 0$ such that $\omega(0, \overline{U}_{k,R} \cap \partial \Delta_k(0), U_{k,R}) \geq c$ for some sequence of positive integers $k_p$ with $k_1 < k_2 < k_3 < \cdots$.

In this case we conclude from the previous inequality that for these numbers $k_p$ one has:

$$(1.3) \quad \rho(0, \partial U_{k_p,R}) \geq \frac{\pi^2 k_p}{16} c^2 = c^* k_p,$$

where $c^* = \frac{\pi^2 c^2}{16} > 0$.

Hence the ball $B_R^n(0)$ contains the set $f_{k_p}(\Delta_{c^*k_p})$ for arbitrarily large numbers $k_p$. This contradicts the Kobayashi hyperbolicity of $B_R^n(0)$.

**Case 2.** For each $R > 0$ one has $\lim_{k \to \infty} \omega(0, \overline{U}_{k,R} \cap \partial \Delta_k(0), U_{k,R}) = 0$.

We first observe that for each $\epsilon > 0$ we have $\omega(0, \partial U_{k,R} \cap \Delta_k(0), U_{k,R}) \geq 1 - \epsilon$ for every sufficiently large $k$. Since $f_k(\partial U_{k,R} \cap \Delta_k(0)) \subset B_R^n(0)$, we conclude that

$$\sup_{f_k(\partial U_{k,R} \cap \Delta_k(0))} \varphi \leq c_R.$$

It follows also from the choice of $C$ that

$$\sup_{f_k(\overline{U}_{k,R} \cap \partial \Delta_k(0))} \varphi \leq C.$$

Then the Mean Value Inequality (1.2) implies the inequality

$$\alpha \leq C\epsilon + c_R(1 - \epsilon).$$

If we choose now $\epsilon$ so small that $C\epsilon < \frac{a}{2}$ and then $R$ so large that $c_R(1 - \epsilon) < \frac{a}{2}$, then we get a contradiction. This concludes the proof of Theorem 1. $\square$

**Remark 1.** If the domain $\Omega$ is, moreover, assumed to be strictly pseudoconvex, then we can also give a completely different proof of Theorem 1 which uses recent nontrivial results on the structure of the core obtained in [6], [9] and [14]. Indeed, we first prove the following

**Claim.** The core $c(\Omega)$ is empty.

**Proof of the Claim.** The argument here is similar to the one used to prove that $c(\Omega_H) = \emptyset$ in the example considered in the introduction. Assume to get a contradiction that $c(\Omega) \neq \emptyset$.
and let $E_{j_0}$ be one of the closed 1-pseudoconcave sets in the decomposition of $\mathcal{E}(\Omega)$ granted by Theorem II in [9]. Then the restriction of the antipeak function $\varphi$ to $E_{j_0}$ is constant. In view of strict pseudoconvexity of $\Omega$, one has that $\mathcal{E}(\Omega) \cap \partial \Omega = \emptyset$. Then, since $E_{j_0}$ is 1-pseudoconcave in the sense of Rothstein, we conclude that the set $E_{j_0}$ has to be unbounded. It follows now from the requirement $\lim_{\|z\| \to \infty} \varphi(z) = 0$ on the strong antipeak function $\varphi$ that $\varphi \equiv 0$ on $E_{j_0}$, which is impossible by the definition of the antipeak function, since $\varphi$ is positive on $\Omega$. This proves the Claim.

Now, using the argument of Lemma 1 in [11], we get a bounded continuous strictly psh. function $\phi$ on $\Omega$. It follows then from Theorem 3 on p. 362 in [13] that $\Omega$ is Kobayashi hyperbolic.

We do not know if a similar argument can also be applied for general (not necessarily strictly pseudoconvex) unbounded domains in $\mathbb{C}^n$.

**Remark 2.** A weaker notion of an antipeak function was introduced and studied in [3]. That notion is not strong enough to guarantee the claim of Theorem 1 as it can be seen from the following example:

Let $\Omega := \{(z, w) \in \mathbb{C}^2 : \log |w| + (|z|^2 + |w|^2) < 0\} \subset \mathbb{C}^2$ and let $\varphi(z, w) := -\log |z|$. It is easy to see that $\varphi$ is an antipeak function for $\Omega$ in the sense of [3], but $\Omega$ is obviously not Kobayashi hyperbolic due to the fact that $\{w = 0\} \subset \Omega$.

We do not know if for an unbounded domain $\Omega$ in $\mathbb{C}^n$ (which we can assume in addition to be pseudoconvex or, even, strictly pseudoconvex) the fact that $\mathcal{E}(\Omega) = \emptyset$ implies that there is a strong antipeak function at infinity for $\Omega$.

Finally, we point out that in the paper [8], the authors considered (non necessarily continuous) bounded above psh. functions $\phi$ defined on some unbounded domains in $\mathbb{C}^n$ and having the property $\lim_{\|z\| \to \infty} \phi(z) = -\infty$ with the aim to study a Dirichlet type problem on some family of unbounded domains.

2. Construction of $\Psi$ and of a Wermer type set in $\Omega_\Psi$

Let $\{a_n, n \in \mathbb{N}\}$ be the sequence of points with entire coordinates in $\mathbb{C}$ such that $a_1 = 0$ and, for every $n \in \mathbb{N}$, the set $(\mathbb{Z} + i\mathbb{Z}) \cap \{\zeta \in \mathbb{C} : -n \leq \text{Re}(\zeta), \text{Im}(\zeta) \leq n\}$ consists of the points $a_1, \ldots, a_{(2n+1)^2}$. We may select the points to form a spiral turning anticlockwise, starting with $a_2 : (1,0)$, $a_3 : (1,1)$, ... (See Figure 1.)

2.1. Construction and properties of a Wermer type set in $\Omega_\Psi$. We consider the following Wermer type set, whose construction is similar to the one used in [4]. Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers, decreasing to zero. First conditions on the speed of convergence of $\{\varepsilon_n\}$ are provided by Lemma 2.2 in [4]. Then, for every $n \in \mathbb{N}$, let

$$E_n := \{(z, w) \in \mathbb{C}^2 : w = \sum_{j=1}^{n} \varepsilon_j \sqrt{z - a_j}\}.$$ 

Following [4], we define the Wermer type set
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$\mathcal{E} := \bigcup_{R>0} \left( \lim_{n \to \infty} \left( E_n \cap B^2_R(0) \right) \right) \subset \mathbb{C}^2,$

where the limit is understood with respect to the Hausdorff distance.

Moreover, the same argument as in Lemma 5.1 of [4] shows that there exists a psh. function $\phi : \mathbb{C}^2 \to [-\infty, +\infty)$ such that $\mathcal{E} = \{ \phi = -\infty \}$ and $\phi$ is pluriharmonic on $\mathbb{C}^2 \setminus \mathcal{E}$.

For $\rho : [0, +\infty) \to \mathbb{R}$, let $\Psi(\rho)$ be the function defined on $\mathbb{C}^2$ by

$$
\Psi(\rho)(z, w) := e^{\phi(z, w) + \rho(\Re(z)) + \rho(\Im(z))}.
$$

To prove Theorem 2 we will construct a convex function $\rho : [0, +\infty) \to [0, +\infty)$, with $\lim_{x \to +\infty} \rho(x) = +\infty$, such that the function $\Psi(\rho)$ satisfies the conclusion of Theorem 2.

But first we prove the following property of the domain $\Omega_{\Psi(\rho)}$ which is claimed in Theorem 2.

**Lemma 1.** The core $c(\Omega_{\Psi(\rho)})$ is not empty. In particular, $\Omega_{\Psi(\rho)}$ is not biholomorphic to a bounded domain in $\mathbb{C}^3$.

**Proof.** It follows from the same argument as in [6, Theorem 2.2] that the set $\mathcal{E} \times \{1\}$, which is contained in $\Omega_{\Psi(\rho)}$, satisfies the Liouville type property, meaning that every continuous psh. function defined in a neighbourhood of $\mathcal{E} \times \{1\}$ and bounded there from above is constant on $\mathcal{E} \times \{1\}$. Hence, $\mathcal{E} \times \{1\}$ is contained in $c(\Omega_{\Psi(\rho)})$. \qed

Let $\mathcal{F}_{\text{conv}}$ denote the set of convex functions $\rho : [0, +\infty) \to [0, +\infty)$, with $\lim_{x \to +\infty} \rho(x) = +\infty$. For $\rho \in \mathcal{F}_{\text{conv}}$, let $(z_0, w_0, \zeta_0) \in \Omega_{\Psi(\rho)}$ and let $U \ni (z_0, w_0, \zeta_0)$ be a neighbourhood of $(z_0, w_0, \zeta_0)$, relatively compact in $\Omega_{\Psi(\rho)}$. We first prove a localization lemma for a sequence of “large” holomorphic disks with center in $U$. Let $d := \sup_{(z, w, \zeta) \in U} \Re(\zeta)$ and let $\pi_{z, w} : \mathbb{C}^3_{z, w, \zeta} \to \mathbb{C}^2_{z, w}$ denote the canonical projection.

**Lemma 2.** Let $r > 0$ and $l : \Delta_r(0) \to \Omega_{\Psi(\rho)}$ be holomorphic. Assume that $f(0) \in U$ and $|f'(0)| = 1$. Then

$$
\pi_{z, w} \left( f(D_{r/7}(0)) \right) \subset F_d := \left\{ (z, w) \in \mathbb{C}^2 : \Psi(\rho)(z, w) < 2d \right\}.
$$

**Proof.** Set $f(\lambda) = (z(\lambda), w(\lambda), \zeta(\lambda))$ and observe that

$$
f(0) \in U \Rightarrow \Psi(\rho)(z(0), w(0)) < \Re(\zeta(0)) < d.
$$
Hence, we can consider the connected component $\Sigma^d$ of the set \( \{ \lambda \in \Delta_r(0) : \Psi(\rho)(z(\lambda), w(\lambda)) < 2d \} = \{ \lambda \in \Delta_r(0) : \pi_{z,w}(f(\lambda)) \in F_d \} \) which contains the origin. Then, in view of the plurisubharmonicity of $\Psi(\rho)$, $\Sigma^d$ is simply connected.

It follows now from [10, Lemma 3.4] that

\[
\omega(0, \partial \Delta_r(0) \cap \bar{\Sigma}^d, \Sigma^d) \leq \frac{4}{\pi \sqrt{r}} \sqrt{\rho(0, \partial \Sigma^d)},
\]

where $\rho(0, \partial \Sigma^d) := \inf \{ |\lambda| : \lambda \in \partial \Sigma^d \}$.

Using the harmonicity of the function $\text{Re}(\zeta)$, the Mean Value Inequality and the fact that $\Psi(z(\lambda), w(\lambda)) = 2d$, for every $k \geq 1$ and every $\lambda \in \Delta_r(0) \cap \partial \Sigma^d$, we get

\[
d \geq \text{Re}(\zeta(0)) \geq 2d \omega(0, \partial \Sigma^d \cap \Delta_r(0), \Sigma^d) + \omega(0, \partial \Delta_r(0) \cap \bar{\Sigma}^d, \Sigma^d) \inf_{\lambda \in \Sigma^d} \text{Re}(\zeta(\lambda)) \geq 2d \omega(0, \partial \Sigma^d \cap \Delta_r(0), \Sigma^d),
\]

since $\text{Re}(\zeta)$ is a positive function on $\Delta_r(0)$.

It follows that

\[
\omega(0, \partial \Sigma^d \cap \Delta_r(0), \Sigma^d) \leq \frac{1}{2},
\]

which implies

\[
\omega(0, \partial \Delta_r(0) \cap \bar{\Sigma}^d, \Sigma^d) \geq \frac{1}{2}.
\]

We finally conclude from (2.2) that

\[
\rho(0, \partial \Sigma^d) \geq \frac{\pi}{8} r > \frac{r}{7}.
\]

This completes the proof of Lemma 2.

\[\square\]

2.2. Construction of the convex function $\rho$. We now construct the function $\rho$ that will enter the definition of the function $\Psi$ in Theorem 2.

**Lemma 3.** Let \( \{ c(n), \ n \geq 0 \} \) be an arbitrary increasing sequence of positive numbers. Then there is a strictly convex function $\rho$ of class $C^\infty$ on $[0, +\infty)$ such that $\rho'(0) = 0$ and for every $n \geq 0$ and every $t \in (n, n + 1]$, one has

\[
\rho(t) > c(n).
\]

**Proof.** We first construct inductively an auxiliary convex function $\rho_1$ such that for every $n \geq 0$, $\rho_1$ is affine on the segment $[n, n + 1]$. For $n = 0$ we set $\rho_1|_{[0,1]} = c(1)$. Let $n \in \mathbb{N}$ and assume that $\rho_1$ is already constructed on $[0, n]$. In particular, there exist $a_n, b_n > 0$ such that for every $t \in (n - 1, n]$

\[
\rho_1(t) = a_n t + b_n.
\]

- If $a_n(n + 1) + b_n \geq c(n + 1)$, we set $\rho_1(t) = a_n t + b_n$ for every $t \in (n, n + 1]$,
- If $a_n(n + 1) + b_n < c(n + 1)$, we set $\rho_1(t) = (a_n n + b_n)(1 - (t - n)) + c(n + 1)(t - n)$ for every $t \in (n, n + 1]$. 
This defines the function $\rho_1$ on $[0, +\infty)$ by induction. Let now $\chi : \mathbb{R} \to \mathbb{R}$ be a nonnegative $C^\infty$-smooth function with support contained in $[-1/4, 1/4]$ and satisfying $\int_{\mathbb{R}} \chi = 1$. Then the restriction to $[0, +\infty)$ of the function $\rho$ defined on $\mathbb{R}$ by $\rho(t) := \tilde{\rho}_1 \star \chi(t) + t^2$, where $\tilde{\rho}_1(t) = \rho_1(|t|)$ for every $t \in \mathbb{R}$, will satisfy all the conditions of Lemma 3.

For every $n \in \mathbb{N}$, let

$$S_n := \left\{ z \in \mathbb{C} : -\left( n + \frac{3}{4} \right) \leq \text{Re}(z), \text{Im}(z) \leq n + \frac{3}{4} \right\} \setminus \left\{ z \in \mathbb{C} : -\left( n + \frac{1}{4} \right) < \text{Re}(z), \text{Im}(z) < n + \frac{1}{4} \right\}$$

and

$$T_n := \left\{ z \in \mathbb{C} : \text{Re}(z) = \pm \left( n + \frac{1}{2} \right), |\text{Im}(z)| \leq n + \frac{1}{2} \right\} \cup \left\{ z \in \mathbb{C} : |\text{Re}(z)| \leq n + \frac{1}{2}, \text{Im}(z) = \pm \left( n + \frac{1}{2} \right) \right\}.$$  

(See Figure 2.)

Since, for every $n \in \mathbb{N}$, the set $S_n$ does not contain any point with entire coordinates, then for every $n, m \in \mathbb{N}$ and for every $p \in S_n$, the restriction to $\Delta_{1/4}(p)$ of the defined above set $E_m$, denoted $E_m|_{\Delta_{1/4}(p)}$, is a union of holomorphic graphs of the form $\{w = f(z)\}$. More precisely, for every $p \in S_n$, there are holomorphic functions $f_1, \ldots, f_{2^m}$, defined on $\Delta_{1/4}(p)$, such that $E_m|_{\Delta_{1/4}(p)} = \bigcup_{1 \leq j \leq 2^m} \{w = f_j(z)\}$. Since $S_n$ is compact in $\mathbb{C}$, it follows from the Montel Theorem that for every $p \in S_n$, $E|_{\Delta_{1/4}(p)} = \bigcup_{\lambda \in \mathcal{A}} \{w = f_{\lambda}(z)\}$, where $\mathcal{A}$ denotes a Cantor set parametrising the branches of $E$ over $\Delta_{1/4}(p)$. Moreover, for every $\lambda \in \mathcal{A}$, $f_{\lambda}$ is holomorphic on $\Delta_{1/4}(p)$ and, hence, in view of the compactness of both the set $S_n$ and the family $\{w = f_{\lambda}(z)\}_{\lambda \in \mathcal{A}}$ with respect to the parameter $\lambda$, one can define

![Figure 2](Image)
that it is sufficient to prove Lemma 4 for these functions on ∆₁.

Lemma 4. We can choose the sequence \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) decreasing and converging to zero so fast that 

\[
\sum_{n} m > \varepsilon \quad \text{for every } m \in \mathbb{N}.
\]

Proof. We first point out that, since every map \( f_\lambda \) is the uniform limit on \( \Delta_{1/8}(p) \) of functions whose graph over \( \Delta_{1/8}(p) \) is a branch of the multivalued holomorphic function \( \sum_{k=1}^{m} \varepsilon_k \sqrt{z - a_k} \), it is sufficient to prove Lemma 4 for these functions on \( \Delta_{1/8}(p) \) uniformly with respect to \( m \in \mathbb{N} \).

Let \( \varepsilon > 0 \). Since \( \varepsilon_k \) decreases sufficiently fast to zero according to [4], there exists \( k_0 \geq 1 \) such that \( \sum_{k \geq (2k_0 + 1)^2 + 1} \varepsilon_k < \varepsilon/4 \). Moreover, for every \( n \in \mathbb{N} \) and every \( p \in S_n \) we have

\[
\inf_{z \in \Delta_{1/8}(p)} d(z, \mathbb{Z} + i\mathbb{Z}) \geq \frac{1}{8}.
\]

Hence, for every \( m > (2k_0 + 1)^2 \), for every \( n \in \mathbb{N} \) and every \( p \in S_n \), each branch of the multivalued holomorphic function \( \sum_{k=(2k_0+1)^2+1}^{m} \varepsilon_k \sqrt{z - a_k} \) is given by the graph of a holomorphic function such that the modulus of its derivative is bounded from above by \( \varepsilon/5 \) on \( \Delta_{1/8}(p) \).

Now, there exists \( n_0 > k_0 \) such that for every \( n \geq n_0 \),

\[
\frac{1}{\inf \{ \sqrt{d(a_j, S_n) : a_j \in \{ \lambda \in \mathbb{C} : -k_0 \leq \text{Re}(\lambda), \text{Im}(\lambda) \leq k_0 \} \} } < \sum_{j \geq 1} \varepsilon_j.
\]

This implies that for every \( n > n_0 \) and for every \( p \in S_n \), each branch of the multivalued holomorphic function \( \sum_{k=1}^{(2k_0 + 1)^2} \varepsilon_k \sqrt{z - a_k} \) is the graph of a holomorphic function such that the modulus of its derivative is bounded from above by \( \varepsilon/2 \) on \( \Delta_{1/8}(p) \).

We finally obtain that for every \( \varepsilon > 0 \), for every \( n \geq n_0 \), for every \( p \in S_n \) and for every \( m > (2k_0 + 1)^2 \), every branch of the multivalued holomorphic function \( \sum_{k=1}^{m} \varepsilon_k \sqrt{z - a_k} \) is the graph of a holomorphic function whose derivative is bounded from above by \( \varepsilon \) on \( \Delta_{1/8}(p) \). This completes the proof of Lemma 4.

\( \square \)

For every \( w \in \mathbb{C} \), \( \delta > 0 \), we consider the set

\[
\mathcal{E}^\delta := \cup_{(z,w) \in \mathcal{E}} \{ z \times \Delta_\delta(w) \}.
\]

In view of Lemma 4, we can define \( q_0 := \inf \{ k \in \mathbb{N} : \alpha(j) < \frac{1}{2} \text{ for all } j \geq k \} \). Now, for every \( n \geq q_0 \), let \( \mathcal{H}_n \) denote the set of holomorphic disks \( D = \{ (f(w), w) : w \in \Delta_1(w_0) \} \) such that

1. \( (f(w_0), w_0) \in \mathcal{E} \cap (T_n \times \mathbb{C}) \),
2. \( \text{for every } w \in \Delta_1(w_0), \ |f'(w)| < 1 \).

Let \( \pi_w : \mathbb{C}^2 \rightarrow \mathbb{C} \) denote the canonical projection and, for every \( D \in \mathcal{H}_n \), let

\[
\beta^\delta(D) := \sup \left\{ \text{diam}(c) : c \text{ connected component of the closure of } \pi_w \left( \frac{1}{4} D \cap \mathcal{E}^\delta \right) \right\},
\]

where for a holomorphic disk \( D = \{ (f(w), w) : w \in \Delta_1(w_0) \} \subset \mathcal{H}_n \) we denote by \( \frac{1}{4} D \) the set \( \frac{1}{4} D := \{ (f(w), w) : w \in \Delta_{1/4}(w_0) \} \). Here, in view of Conditions (1) and (2) above, we choose
the radius equal to 1/4 to insure that the disc $\frac{1}{4}D$ is contained in the set $S_n \times \mathbb{C}$. Moreover, it will also insure that the family of disks $\frac{1}{4}D_k$ considered in the proof of Lemma 5 will converge smoothly up to the boundary to $\frac{1}{4}D_\infty$.

Finally, define

$$\beta^\delta_n := \sup_{D \in \mathcal{H}_n} \beta^\delta(D).$$

**Lemma 5.** For every $n \geq q_0$ we have

$$\lim_{\delta \to 0} \beta^\delta_n = 0.$$

**Proof.** Assume, to get a contradiction, that there exists $n \geq q_0$, a sequence of positive real numbers $\delta_k$ decreasing to 0, a sequence of points $w_k \in \pi_w(\mathcal{E} \cap (T_n \times \mathbb{C}))$, a sequence of holomorphic disks $D_k = \{(f_k(w), w), \ w \in \Delta_1(w_k)\} \in \mathcal{H}_n$ and, for every $k \geq 1$, a connected component $c_k$ of the closure of $\pi_w(\frac{1}{4}D_k \cap \mathcal{E}^\delta_k)$, such that

$$\inf_{k \geq 1} \text{diam}(c_k) =: \alpha_\infty > 0.$$

We can assume that for every $k \geq 1$, $c_k$ is simply connected.

Since $\mathcal{E} \cap (T_n \times \mathbb{C})$ is compact, and since $D_k \in \mathcal{H}_n$ for every $k \geq 1$, it follows from Condition (2) and from the Montel Theorem that up to extracting a subsequence, $D_k$ will converge to a holomorphic disc $D_\infty := \{(f_\infty(w), w), \ w \in \Delta_1(w_\infty)\}$ for some point $w_\infty \in \pi_w(\mathcal{E} \cap (T_n \times \mathbb{C}))$, where $w_k \to w_\infty$ as $k \to \infty$, $f_\infty$ is holomorphic on $\Delta_1(w_\infty)$ and satisfies $\sup_{\Delta_1(w_\infty)} |f'_\infty| \leq 1$.

This implies, by the choice of $q_0$, that the disk $D_\infty$ is transversal to every branch of $\mathcal{E}$ and, hence, $\frac{1}{4}D_\infty$ intersects $\mathcal{E}$ on a Cantor set. However, up to extracting a subsequence, $c_k$ converges to a subset of some connected component $c_\infty$ of the closure of $\pi_w(\frac{1}{4}D_\infty \cap \mathcal{E})$ and then, by (2.7), $\text{diam}(c_\infty) \geq \alpha_\infty$. This is a contradiction. \(\square\)

For our next argument we need to define some notions. We will call a continuous curve $(z(t), w(t)) : [0, 1] \to \mathbb{C}^2$, a lifting to $\mathbb{C}^2$, of the curve $z(t) : [0, 1] \to \mathbb{C}$ (without restrictions of generality we can assume here that, up to a reparametrisation, if necessary, all the curves are parametrised by the segment $[0, 1]$). Let now $z(t) : [0, 1] \to \mathbb{C}$ be a closed (i.e. $z(0) = z(1)$) continuous curve. For a compact set $F$ in $\mathbb{C}$ which projects to the given curve $z(t)$ we consider the family $\{\gamma^F_\alpha(t)\}_{\alpha \in \mathcal{A}}$ of all liftings $\gamma^F_\alpha(t) = (z(t), w^F_\alpha(t))$ of the curve $z(t)$ which are contained in the set $F$ (i.e. such that $\gamma^F_\alpha(t) \in F$ for all $t \in [0, 1]$). Then we define the shift error $\theta(F)$ of the set $F$ as

$$\theta(F) := \inf_{\alpha \in \mathcal{A}} |w^F_\alpha(1) - w^F_\alpha(0)|.$$

Observe that for two sets $F_1 \subset F_2$ which project to the same curve $z(t)$ one obviously has that

$$\theta(F_2) \leq \theta(F_1).$$

Now we can finally make precise the construction of the Wermer type set $\mathcal{E}$, specifying conditions on the sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$. We first set $\varepsilon_1 = 1$ and set $E_1 := \{(z, w) \in \mathbb{C}^2 : w = \varepsilon_1 \sqrt{z} - a_1\}$. Then we will choose $\varepsilon_2$ as follows. Fix $0 < r_2 < 1/2$ such that the set $E_1 \cap (\Delta_{r_2}(a_2) \times \mathbb{C})$ is the
union of the graphs of holomorphic functions $f_1^k, f_2^k : \Delta_{r_2}(a_2) \to \mathbb{C}$ and such that, moreover, one has

$$\kappa_2 := \inf \{|f_1^k(z) - f_2^k(z)| : |z - a_2| = r_2| > 0.$$ 

Now, let us choose $\varepsilon_2$ such that $2\varepsilon_2 \sqrt{r_2} < \kappa_2/2$. Then for each $(z, w) \in E_2 := \{(z, w) \in \mathbb{C}^2 : w = \sum_{j=1}^{2} \varepsilon_j \sqrt{|z - a_j|}\}$ with $|z - a_2| = r_2$ we consider $w' \in \mathbb{C}$ such that $(a_2 + (z - a_2)e^{2i\pi}, w') \in E_2$ and observe that

$$|w - w'| = 2\varepsilon_2 \sqrt{r_2}.$$ 

Here we denote by $(a_2 + (z - a_2)e^{2i\pi}, w') \in E_2$ a point which is obtained from $(z, w)$ after one turn around $a_2$ starting at $z$ and keeping it, during this turn, on the set $E_2$.

We can continue the process inductively. Assume that for some $k \geq 3$, $r_2, \ldots, r_{k-1}$ and $\varepsilon_1, \ldots, \varepsilon_{k-1}$ are already constructed. We choose $0 < r_k < 1/2$ such that

$$E_{k-1} \cap (\Delta_{r_k}(a_k) \times \mathbb{C}) = \bigcup_{j=1}^{2k-1} \{(z, f_j^{k-1}(z)) : z \in \Delta_{r_k}(a_k)\},$$

where $f_1^{k-1}, \ldots, f_{2k-1}^{k-1}$ are functions holomorphic in a neighbourhood of $\overline{\Delta_{r_k}(a_k)}$ and such that for every $1 \leq j \neq l \leq 2^{k-1}$ and every $z \in \partial \Delta_{r_k}(a_k)$, $f_j^{k-1}(z) \neq f_l^{k-1}(z)$. Then we set

$$(2.8) \quad \kappa_k := \inf_{1 \leq j \neq l \leq 2^{k-1}} \{|f_j^{k-1}(z) - f_l^{k-1}(z)| : |z - a_k| = r_k\} > 0.$$ 

Let $\varepsilon_k > 0$ be such that

$$(2.9) \quad 2\varepsilon_k \sqrt{r_k} < \frac{\kappa_k}{2}$$

and, for every $2 \leq p \leq k - 1$,

$$(2.10) \quad 2\varepsilon_k \sqrt{|a_k - a_p| + r_p} < \frac{\varepsilon_p \sqrt{r_p}}{2^{k-p+1}}.$$ 

Condition (2.10) will insure that the set $E_k$, over the circle $\partial \Delta_{r_k}(a_p)$, will be a sufficiently small perturbation of the set $E_p$ and, hence, the shift error for liftings of the closed circle $\partial \Delta_{r_p}(a_p)$ to a sufficiently small neighbourhood of the set $E_k$ will be bounded from below by a positive constant independent of $k$. Let us make this argument more precise. We first introduce the following notations: for each $k \geq p$ we denote by $E_{k,p}$ the set

$$E_{k,p} := E_k \cap \{|z - a_p| = r_p\} \times \mathbb{C}_w,$$

for each $p \geq 1$ we denote by $E_p$ the set

$$E_p := E \cap \{|z - a_p| = r_p\} \times \mathbb{C}_w,$$

and for each compact set $F \subset \mathbb{C}^2_{z,w}$ and each $\delta > 0$ we denote by $F^\delta$ the set

$$F^\delta := \cup_{(z,w) \in F} \{|z - \Delta_{\delta}(w)\}.$$

Since, by Condition (2.10), we know that for each $k > p$ one has

$$|\varepsilon_k \sqrt{|z - a_k|} \leq \varepsilon_k \sqrt{|a_k - a_p| + r_p} < \frac{\varepsilon_p \sqrt{r_p}}{2^{k-p+2}}$$

for $z \in \partial \Delta_{r_p}(a_p)$, we conclude that

$$E_{k,p} \subset \frac{\varepsilon_k \sqrt{r_k}}{2^{k-p+2}} \subset \frac{\varepsilon_k \sqrt{r_k}}{2^{k-p+2} + \frac{1}{2^{k-p+1}}} \subset \frac{\varepsilon_k \sqrt{r_k}}{2^{k-p+2} + \frac{1}{2^{k-p+1}} + \ldots + \frac{1}{8}} \subset \frac{\varepsilon_k \sqrt{r_k}}{2^{k-p+2} + \frac{1}{8}} \subset E_{p,p}.$$
Then, after passing to the limit as $k \to \infty$, we will get that
\[ E_p \subseteq \text{cl} \left( E_{p,p}^{\epsilon_p \sqrt{r_p}} \right), \]
where for avoiding ambiguity we use the notation cl$(X)$ for the closure of $X$. Hence, we also have that
\[ \text{cl} \left( E_p^{\epsilon_p \sqrt{r_p}/4} \right) \subseteq \text{cl} \left( E_{p,p}^{\epsilon_p \sqrt{r_p}} \right). \]
By the construction of the Wermer type set $E$, it finally follows from the last inclusion, Property (2.7) and Conditions (2.8), (2.9) that for the shift error of the constructed above set $E$ we have the following

**Property (P):** For every $p \geq 1$, the inequality
\[ \theta \left( \text{cl} \left( E_p^{\epsilon_p \sqrt{r_p}/4} \right) \right) \geq \epsilon_p \sqrt{r_p} > 0 \]
holds.

We can now specify the choice of the sequence $\{c(n)\}_n$ and then, using Lemma 3, construct the function $\rho$.

For every $n \in \mathbb{N}$, let
\[ \tilde{S}_n := \{ z \in \mathbb{C} : -n - 2 \leq \text{Re}(z), \text{Im}(z) \leq n + 2 \} \setminus \{ z \in \mathbb{C} : -n + 1 < \text{Re}(z), \text{Im}(z) \leq n - 1 \}. \]
and let
\[ \kappa(n) := \inf_{\{ p \in \tilde{S}_n \}} \left\{ \frac{\epsilon_p \sqrt{r_p}}{4} \right\}. \]
The definition of $\tilde{S}_n$ insures that every disc of radius one contained in $\mathbb{C}$ will be contained in some $\tilde{S}_n$. This property will be used in Section 3 to prove the Kobayashi hyperbolicity of $\Omega \psi(\rho)$.

Now we choose $q(n)$ so large that
\[ (2.11) \quad \{ (z, w) \in \mathbb{C}^2 : e^{\phi(z,w) + q(n)} < 1 \} \cap (\tilde{S}_n \times \mathbb{C}) \subset E^{\kappa(n)} \cap (\tilde{S}_n \times \mathbb{C}), \]
the function $\phi$ being introduced at the beginning of Subsection 2.1.

Then, in view of Lemma 5, we can define for every $n > q_0$
\[ \delta(n) := \inf \{ \delta > 0 : \beta_k^\delta < \frac{1}{2}, \text{ for all } n - 1 \leq k \leq n + 2 \}. \]
It follows now that for every $n > q_0$ and every $n - 1 \leq k \leq n + 2$ the inequality
\[ (2.12) \quad \beta_k^{\delta(n)} \leq \frac{1}{2} \]
holds and then we choose $\tilde{q}(n) \geq q(n)$ such that
\[ (2.13) \quad \{ (z, w) \in \mathbb{C}^2 : e^{\phi(z,w) + \tilde{q}(n)} < 1 \} \cap (\tilde{S}_n \times \mathbb{C}) \subset E^{\delta(n)} \cap (\tilde{S}_n \times \mathbb{C}). \]
Hence, setting $c(n) = q(n) + n$ for $n < q_0$ and $c(n) = \tilde{q}(n) + n$ for $n \geq q_0$, and applying then Lemma 3, we obtain a strictly convex function $\rho : [0, +\infty) \to [0, +\infty)$ such that for each $n \in \mathbb{N}$
\[ (2.14) \quad \rho|_{[n-1,n+2]} \geq c(n). \]
Then the corresponding function $\Psi(\rho)$ defined by (2.1) is plurisubharmonic on $\mathbb{C}^2$ and the domain $\Omega_{\Psi(\rho)} := \{(z, w, \zeta) \in \mathbb{C}^3 : \text{Re}(\zeta) > \Psi(\rho)(z, w)\}$ is a rigid pseudoconvex domain in $\mathbb{C}^3$. In Section 3, we prove that $\Omega_{\Psi(\rho)}$ satisfies the conditions of Theorem 2.

3. Proofs of Theorem 2 and Corollary 1

3.1. Proof of Theorem 2. We prove that the domain $\Omega_{\Psi(\rho)}$ satisfies all the conclusions of Theorem 2. According to Lemma 1, the core of $\Omega_{\Psi(\rho)}$ is not empty. Hence, it remains to prove that $\Omega_{\Psi(\rho)}$ is Kobayashi hyperbolic. Assume, to get a contradiction, that there is a point $p \in \Omega_{\Psi(\rho)}$, a neighbourhood $U$ of $p$ in $\mathbb{C}^3$, $U$ relatively compact in $\Omega_{\Psi(\rho)}$, and for every $k \in \mathbb{N}$, a holomorphic function $f_k = (z_k, w_k, \zeta_k) : \Delta_k(0) \to \Omega_{\Psi(\rho)}$ such that $f_k(0) \in U$ and $||f_k'(0)||^2 := |z_k'(0)|^2 + |w_k'(0)|^2 + |\zeta_k'(0)|^2 = 1$. Since $\Psi(\rho)$ is nonnegative, it follows that the function $\text{Re}(\zeta_k)$ is positive on $\Delta_k(0)$ i.e., $\zeta_k(\Delta_k(0)) \subset \mathbb{H} := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$. For $v \in \mathbb{C}$ and $\zeta \in \mathbb{H}$ (resp. $\eta \in \Delta_k(0)$), let $||v||_{\zeta, \mathbb{H}}$ (respectively $||v||_{\eta, \Delta_k(0)}$) denote the hyperbolic norm of $v$ at $\zeta \in \mathbb{H}$ (respectively at $\eta \in \Delta_k(0)$). From the decreasing property of the hyperbolic metric (under the action of holomorphic maps) we get

$$\frac{|\zeta'_k(0)|}{\text{Re}(\zeta_k(0))} = ||\zeta'_k(0) \cdot 1||_{\zeta_k(0), \mathbb{H}} \leq ||1||_{0, \Delta_k(0)} = \frac{1}{k}.$$  

Hence, for every $k \geq 1$ \begin{equation} |\zeta_k'(0)| \leq \frac{|\zeta_k(0)|}{k}. \end{equation}

Since $U$ is relatively compact in $\Omega_{\Psi(\rho)}$, the set $\{|\zeta_k(0)|, \ k \in \mathbb{N}\}$ is bounded in $\mathbb{C}$ and, from (3.1), there exists $k_0 \geq 0$ such that $|z_k'(0)|^2 + |w_k'(0)|^2 > \frac{1}{2}$ for every $k \geq k_0$. If we set $r_k := \sqrt{|z_k'(0)|^2 + |w_k'(0)|^2}$, then the holomorphic map $g_k = (\tilde{z}_k, \tilde{w}_k, \tilde{\zeta}_k) : \lambda \in \Delta_k/r_k(0) \mapsto f_k(\lambda/r_k)$ satisfies

$$|\tilde{z}_k'(0)|^2 + |\tilde{w}_k'(0)|^2 = 1.$$  

It follows now from Lemma 2 that, setting $d := \sup_{(z, w, \zeta) \in U} \text{Re}(\zeta)$, we will have the following inclusion

\begin{equation} \left\{ (\tilde{z}_k(\lambda), \tilde{w}_k(\lambda)), \ \lambda \in \Delta_k/7r_k(0) \right\} \subset F_d := \{(z, w) \in \mathbb{C}^2 : \Psi(\rho)(z, w) < 2d\}. \end{equation}

Notice that Condition (3.2) is satisfied for every $\lambda \in \Delta_k/7r_k$ and hence, since $r_k > 1/\sqrt{2}$, for every $\lambda \in \Delta_k/7r_k$. We distinguish two cases.

Case 1. There exists an increasing sequence $\{k_m\}_{m \in \mathbb{N}}$ diverging to $+\infty$, such that for every $m \in \mathbb{N}$, $|z'_{k_m}(0)| > |\tilde{w}'_{k_m}(0)|$.

We will now need the following classical result.

The Bloch Theorem. There exists $0 < \theta < 1$ such that for every $r > 0$ and every holomorphic function $f : \Delta_r(0) \to \mathbb{C}$ with $|f'(0)| = 1$ there are $b \in \mathbb{C}$ and a holomorphic function $g : \Delta_{\theta r}(b) \to \mathbb{C}$ such that $f \circ g(\lambda) = \lambda$ for every $\lambda \in \Delta_{\theta r}(b)$. 
Since for every $m \in \mathbb{N}$ we have $\frac{1}{\sqrt{2}} < |\tilde{z}_{k_m}'(0)| \leq 1$, the function $\lambda \in \Delta_{k_m/14}(0) \mapsto \tilde{z}_{k_m}(\lambda/|\tilde{z}_{k_m}'(0)|)$ satisfies the assumptions of the Bloch Theorem and takes values in $\tilde{z}_{k_m}(\Delta_{k_m/14\sqrt{2}}(0))$. It follows that there exists $0 < \theta < 1$ such that for every $m \in \mathbb{N}$, there are $b_{k_m} \in \mathbb{C}$ and a holomorphic function $g_{k_m} : \Delta_{\theta k_m/14}(b_{k_m}) \to \mathbb{C}$ whose graph $\Gamma(g_{k_m}) := \{(z, g_{k_m}(z)) : z \in \Delta_{\theta k_m/14}(b_{k_m})\}$ will satisfy the condition

\[(3.3) \quad \Gamma(g_{k_m}) \subset F_d.\]

Now, since $\Psi(\rho)(z, w) := e^{\phi(z,w)+\rho(|Re(z)|)+\rho(|Im(z)|)}$, it follows from Condition (2.11) and from the definition of $c(n)$ in Condition (2.14) that for every positive integer $n$ such that $\frac{2d}{e^n} < 1$ we have

$$F_d \cap (\tilde{S}_n \times \mathbb{C}) = \{(z, w) \in \mathbb{C}^2 : e^{\phi(z,w)+\rho(|z|)+\rho(|y|)} < 2d\} \cap (\tilde{S}_n \times \mathbb{C})$$

$$\subset \{(z, w) \in \mathbb{C}^2 : e^{\phi(z,w)+q(n)} < \frac{2d}{e^n}\} \cap (\tilde{S}_n \times \mathbb{C})$$

$$\subset \{(z, w) \in \mathbb{C}^2 : e^{\phi(z,w)+q(n)} < 1\} \cap (\tilde{S}_n \times \mathbb{C})$$

$$\subset C^{\kappa(n)} \cap (\tilde{S}_n \times \mathbb{C}),$$

the last inclusion coming from Condition (2.11).

In particular, for every $m \in \mathbb{N}$,

\[(3.4) \quad \Gamma(g_{k_m}) \subset \bigcup_{n \geq n_0} \left(C^{\kappa(n)} \cap (\tilde{S}_n \times \mathbb{C})\right) \cup K,
\]

where $n_0$ satisfies $\frac{2d}{e^n_0} < 1$ and $K := F_d \cap (\{|z| \leq n_0\} \times \mathbb{C})$.

It follows now from the definition of $\kappa(n)$ and Property $(P)$ that the set

$$\bigcup_{n \geq n_0} \left(C^{\kappa(n)} \cap (\tilde{S}_n \times \mathbb{C})\right)$$

cannot contain large disks. This contradicts (3.4), since for sufficiently large $m$ the set $\Gamma(g_{k_m}) \setminus K$ will obviously contain an arbitrarily large disk.

**Case 2.** There exists $k_0 \geq 1$ such that for every $k \geq k_0$, $|\tilde{z}_{k}'(0)| \leq |\tilde{w}_{k}'(0)|$.

In particular, we have $|\tilde{w}_{k}'(0)| \geq \frac{1}{\sqrt{2}}$ for every $k \geq k_0$ and, as in the Case 1 above, according to the Bloch Theorem, there exists $0 < \theta < 1$ such that for every $k \geq k_0$, there are $b_{k}' \in \mathbb{C}$ and holomorphic functions $h_k : \Delta_{\theta k/14}(b_{k}') \to \mathbb{C}$ whose graph $\Gamma(h_k) := \{(h_k(\lambda), \lambda) : \lambda \in \Delta_{\theta k/14}(b_{k}')\}$ satisfies

\[(3.5) \quad \Gamma(h_k) \subset F_d.
\]

There are two subcases to consider.

**Subcase 2a.** There is an increasing sequence $(k_m)_{m \in \mathbb{N}}$ diverging to $+\infty$ and, for every $m \in \mathbb{N}$, a point $\lambda_{k_m} \in \Delta_{\theta k_m/2\delta}(b_{k_m})$ such that $|h_{k_m}'(\lambda_{k_m})| \geq 1$. 
In this case we can repeat the argument of Case 1, replacing $\tilde{z}_{km}$ with $h_{km}$, $\Delta_{km/7\sqrt{2}}(0)$ with $\Delta_{\theta_{km}/28}(\lambda_{km})$ (here we use a trivial observation that for $\lambda_{km} \in \Delta_{\theta_{km}/28}(b'_{km})$ one has $\Delta_{\theta_{km}/28}(\lambda_{km}) \subset \Delta_{\theta/14}(b'_{k})$) and then, using (3.5), we obtain the same contradiction as in Case 1.

Subcase 2b. For every $k \in \mathbb{N}$ large enough and every $\lambda \in \Delta_{\theta/28}(b'_{k})$ the inequality $|h_{k}(\lambda)| < 1$ holds.

It follows then from Condition (2.14), from the definition of $c(n)$ and from Condition (2.13) that for every $n \geq q_0$ such that $\frac{2d}{n^2} < 1$ one has

\begin{equation}
F_d \cap \left(\tilde{S}_n \times \mathbb{C}\right) \subset \mathcal{E}^{\delta(n)} \cap \left(\tilde{S}_n \times \mathbb{C}\right).
\end{equation}

Hence, for every $n \geq q_0$ there exists a compact set $K_n \subset \mathbb{C}^2$ such that

\[ F_d \cap \left\{ z \in \mathbb{C} : -n - \frac{1}{2} < \text{Re}(z) < n + \frac{1}{2}, -n - \frac{1}{2} < \text{Im}(z) < n + \frac{1}{2} \right\} \times \mathbb{C} \subset K_n. \]

(See Figure 3.)

Since \( \left\{ \Gamma(h_{k}^{\Delta_{\theta/28}(b'_{k})}) \right\}_{k \geq 1} \) forms a sequence of unbounded holomorphic disks, the set $\bigcup_{k \geq 1} \left( \Gamma(h_{k}^{\Delta_{\theta/18}(b'_{k})}) \cap (\mathbb{C}^2 \setminus K_n) \right)$ will also contain arbitrarily large discs. Hence, by (3.5), (3.6) and the definition of $\mathcal{E}^{\delta(n)}$, the set $\bigcup_{k \geq 1} \pi_z \left( \Gamma(h_{k}^{\Delta_{\theta/28}(b'_{k})}) \cap (\mathbb{C}^2 \setminus K_n) \right)$ is not bounded in $\mathbb{C}_z$.

In particular, we can choose $n \geq q_0$ such that (3.6) is satisfied, and $k(n) \geq 1$, $b''_{k(n)} \in \Delta_{(\theta(n)/28) - 1}(b'_{k(n)})$, such that $h_{k(n)}(b''_{k(n)}) \in T_n$, where $T_n$ is defined in (2.4). Notice that,
according to the assumption of Subcase 2b, the holomorphic disk \( \{(h_{k(n)}(\lambda), \lambda); \lambda \in \Delta_1(b''_{k(n)})\} \) belongs to \( \mathcal{H}_n \).

Since \( \Delta_1(b''_{k(n)}) \subset \tilde{S}_n \), it follows from (3.5) and (3.6) that

\[
\Gamma \left( h_{k(n)} \big|_{\Delta_1(b''_{k(n)})} \right) \subset \mathcal{E} \cap (\tilde{S}_n \times \mathbb{C}) .
\]

However, \( \text{diam} \left( \pi_w \left( \Gamma \left( h_{k(n)} \big|_{\Delta_1(b''_{k(n)})} \right) \right) \right) = 1 \), which contradicts Condition (2.12). This completes the proof of Theorem 2. \( \square \)

Remark 3. Observe, that the statement which is actually proved in the Case 1 and Case 2 above using the Bloch Theorem, can be formulated as the following property of the domain \( F_d \):

Property \((\mathcal{F})\): For each \( d > 0 \) there exists \( r = r(d) > 0 \) such that the domain \( F_d \) contains no holomorphic disks of radius \( r > r(d) \) (the last part of the statement means, more precisely, that for every holomorphic map \( h : \Delta_r(0) \to F_d \) such that \( \|h'(0)\| = 1 \) one has \( r \leq r(d) \)).

We give here an explicit formulation of this property because it will be needed in the forthcoming paper [12].

3.2. Proof of Corollary 1. Assume, to get a contradiction, that \( \varphi \) is a strong antipeak function at infinity for \( \Omega_{\psi(\rho)} \). Then \( \varphi \rvert_{\mathcal{E}} \) is continuous psh. and bounded from above in a neighbourhood of \( \mathcal{E} \). It follows now from the same argument as in [6, Theorem 2.2] that \( \varphi \equiv C \) on \( \mathcal{E} \) for some \( C \in \mathbb{R} \). Since, by the definition of a strong antipeak function, we know that \( \varphi(z) \to 0 \) for \( z \in \mathcal{E} \) as \( \|z\| \to \infty \), we conclude that \( \varphi \equiv 0 \) on \( \mathcal{E} \). This contradicts the definition of an antipeak function and, hence, completes the proof of Corollary 1. \( \square \)

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