HYPERSPACES OF CONVEX BODIES OF CONSTANT WIDTH

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Abstract. For every $n \geq 1$, let $cc(\mathbb{R}^n)$ denote the hyperspace of all non-empty compact convex subsets of the Euclidean space $\mathbb{R}^n$ endowed with the Hausdorff metric topology. For every non-empty convex subset $D$ of $[0, \infty)$ we denote by $cw_D(\mathbb{R}^n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex sets of constant width $d \in D$ and by $crw_D(\mathbb{R}^n)$ the subspace of the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ consisting of all pairs of compact convex sets of constant relative width $d \in D$. In this paper we prove that $cw_D(\mathbb{R}^n)$ and $crw_D(\mathbb{R}^n)$ are homeomorphic to $D \times \mathbb{R}^n \times Q$, whenever $D \neq \{0\}$ and $n \geq 2$, where $Q$ denotes the Hilbert cube. In particular, the hyperspace $cw(\mathbb{R}^n)$ of all compact convex bodies of constant width as well as the hyperspace $crw(\mathbb{R}^n)$ of all pairs of compact convex sets of constant relative positive width are homeomorphic to $\mathbb{R}^{n+1} \times Q$.

1. Introduction

Let $cc(\mathbb{R}^n)$, $n \geq 1$, denote the hyperspace of all non-empty compact convex subsets of $\mathbb{R}^n$ endowed with the topology induced by the Hausdorff metric:

$$\rho_H(Y, Z) = \max \left\{ \sup_{z \in Z} \| z - Y \|, \sup_{y \in Y} \| y - Z \| \right\}, \quad Y, Z \in cc(\mathbb{R}^n)$$

where $\| \cdot \|$ denotes the euclidean norm on $\mathbb{R}^n$, i.e.,

$$\| x \|^2 = \sum_{i=1}^{n} x_i^2, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

In case $n = 1$, it is easy to see that $cc(\mathbb{R})$ is homeomorphic to $\mathbb{R} \times [0, 1)$ and for every $n \geq 2$, it is well known that $cc(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube $Q_0 := Q \{0\}$, where $Q := [0, 1]^\infty$ is the Hilbert cube (see [13, Theorem 7.3]).

By a convex body we mean a compact convex subset of $\mathbb{R}^n$ with non-empty interior. A compact convex set in $\mathbb{R}^n$ is said to be of constant width $d \geq 0$, if the distance between any two of its parallel support hyperplanes is equal to $d$ (see e.g., [16, Chapter 7, §6]).

As a generalization of compact convex sets of constant width, H. Maehara introduced in [10] the concept of pairs of compact convex sets of constant relative width and showed that these pairs share certain properties of compact convex sets of constant width. A pair $(Y, Z) \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ is said to be of constant relative width $d \geq 0$, if

$$h_Y(u) + h_Z(-u) = d$$

for every $u \in S^{n-1}$, where $h_Y$ and $h_Z$ denote the support functions of $Y$ and $Z$, respectively (see formula (2.1) below) and

$$S^{n-1} = \{ x \in \mathbb{R}^n \mid \| x \| = 1 \}$$

is the unit sphere in $\mathbb{R}^n$.

For every non-empty convex subset $D$ of $[0, \infty)$ we denote by $cw_D(\mathbb{R}^n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex sets of constant width $d \in D$ and by $crw_D(\mathbb{R}^n)$ the subspace of the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ consisting of all pairs of compact convex

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sets of constant relative width \( d \in D \). We shall use \( \text{cw}(\mathbb{R}^n) \) and \( \text{crw}(\mathbb{R}^n) \) for \( \text{cw}(0, \infty)(\mathbb{R}^n) \) and \( \text{crw}(0, \infty)(\mathbb{R}^n) \), respectively.

Note that if \( D \subset (0, \infty) \), then every \( A \in \text{cw}_D(\mathbb{R}^n) \) is a convex body. This does not hold in the hyperspace \( \text{cw}_D(\mathbb{R}^n) \), i.e., there are pairs \((Y, Z)\) of compact convex sets of constant relative width \( d > 0 \), such that either \( Y \) or \( Z \) is not a convex body. For instance, the pair \((\mathbb{R}^n, \{0\})\) is of constant width 1, where

\[
\mathbb{B}^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}
\]

is the unit ball in \( \mathbb{R}^n \), while \( \{0\} \) is not a body.

Note also that if \( D = \{0\} \), then the hyperspaces \( \text{cw}_D(\mathbb{R}^n) = \{ \{x\} \mid x \in \mathbb{R}^n \} \) and \( \text{crw}_D(\mathbb{R}^n) = \{ \{\{x\},\{x\}\} \mid x \in \mathbb{R}^n \} \) are both homeomorphic to \( \mathbb{R}^n \).

In case \( n = 1 \), it is easy to see that \( \text{cw}_D(\mathbb{R}) \) is homeomorphic to \( D \times \mathbb{R} \) for every non-empty convex subset \( D \) of \([0, \infty)\) and that \( \text{crw}_D(\mathbb{R}) \) is homeomorphic to \( D \times \mathbb{R} \times [0, 1] \) for every non-empty convex subset \( D \neq \{0\} \) of \([0, \infty)\) (see Propositions 8.11 and 11.2).

However, for every \( n \geq 2 \), the topological structure of \( \text{cw}_D(\mathbb{R}^n) \) and \( \text{crw}_D(\mathbb{R}^n) \) had remained unknown, except for the cases of convex sets \( D \) of the form \([d_0, \infty)\) with \( d_0 \geq 0 \), for which it was proved in [1] Corollary 1.2 (relying on [1] Theorem 1.1) that \( \text{cw}_D(\mathbb{R}^n) \) is homeomorphic to the punctured Hilbert cube \( Q_0 \).

It is the purpose of this paper to give a complete description of the topological structure of the hyperspaces \( \text{cw}_D(\mathbb{R}^n) \) and \( \text{crw}_D(\mathbb{R}^n) \) for every \( n \geq 2 \) and every non-empty convex subset \( D \neq \{0\} \) of \([0, \infty)\). Namely, we prove in Theorem 3.9 that the hyperspace \( \text{cw}_D(\mathbb{R}^n) \) is homeomorphic to \( D \times \mathbb{R}^n \times Q \). In particular, the hyperspace \( \text{cw}(\mathbb{R}^n) \) of all convex bodies of constant width is homeomorphic to \( \mathbb{R}^{n+1} \times Q \) (Corollary 3.10). Besides, we prove in Theorem 4.1 that the hyperspace \( \text{crw}_D(\mathbb{R}^n) \) is homeomorphic to \( \text{cw}_D(\mathbb{R}^n) \) (in particular, \( \text{cw}(\mathbb{R}^n) \) is homeomorphic to \( \text{cw}_D(\mathbb{R}^n) \)).

Our argument is based, among other things, on [1] Theorem 1.1 which asserts that the hyperspace \( \text{cw}_D(\mathbb{R}^n) \), with \( D \neq \{0\} \), is a contractible Hilbert cube manifold. However, the proof of this result given in [1] contains a gap. Namely, it is claimed within the proof that for any \( n \geq 3 \) and any regular \( n \)-simplex \( \Delta \subset \mathbb{R}^n \) of side length \( d > 0 \), the intersection of all closed balls with centers at the vertices of \( \Delta \) and radius \( d \) is of constant width \( d \). But, this claim is not true. In fact, for every \( n \geq 3 \), no finite intersection of balls in \( \mathbb{R}^n \) is of constant width, unless it reduces to a single ball (see [1] Corollary 3.3). This shows a striking difference with the two-dimensional case, where the intersection of all closed discs in \( \mathbb{R}^2 \) of radius \( d > 0 \) and centers at the vertices of an equilateral triangle of side length \( d \), is a convex body of constant width \( d \), which is well known under the name of \textit{Reuleaux triangle} (see e.g., [16] Chapter 7, §6). Fortunately, this gap can be filled in using [1] Theorem 4.1, which describes a method for constructing convex bodies of constant width in arbitrary dimension \( n \), starting from a given projection in dimension \( n - 1 \). We decided to present below in Theorem 3.6 a detailed correct proof of [1] Theorem 1.1; this makes our presentation more complete.

2. Preliminaries

All maps between topological spaces are assumed to be continuous.

A metrizable space \( X \) is called an absolute neighborhood retract (denoted by \( X \in \text{ANR} \)) if for any metrizable space \( Z \) containing \( X \) as a closed subset, there exist a neighborhood \( U \) of \( X \) in \( Z \) and a retraction \( r : U \to X \).

Recall that a map \( f : X \to Y \) between topological spaces is called proper if for every compact subset \( K \) of \( Y \), the inverse image \( f^{-1}(K) \) is a compact subset of \( X \). A proper map \( f : X \to Y \) between ANR’s is called cell-like if it is onto and each point inverse \( f^{-1}(y) \) has the property \( UV \), i.e., for each neighborhood \( U \) of \( f^{-1}(y) \) there exists a neighborhood \( V \subset U \) of \( f^{-1}(y) \) such that the inclusion \( V \hookrightarrow U \) is homotopic to a constant map of \( V \) into \( U \). In particular, if \( f^{-1}(y) \) is contractible, then it has the property \( UV \) (see [6] Chapter XIII).
We refer the reader to [7, 11, 14] and [16] for the theory of convex sets. However, we recall here some notions of convexity that will be used throughout the paper. We begin with the Minkowski operations.

For any subsets $Y$ and $Z$ of $\mathbb{R}^n$ and $t \in \mathbb{R}$, the sets
\[ Y + Z = \{ y + z \mid y \in Y, z \in Z \} \quad \text{and} \quad tY = \{ ty \mid y \in \mathbb{R}^n \} \]
are called the Minkowski sum of $Y$ and $Z$ and the product of $Y$ by $t$, respectively. It is well known that these operations preserve compactness and convexity and are continuous with respect to the Hausdorff metric.

As usual, we denote by $C(S^{n-1})$ the Banach space of all maps from $S^{n-1}$ to $\mathbb{R}$ topologized by the supremum metric:
\[ g(f, g) = \sup \{ |f(u) - g(u)| \mid u \in S^{n-1} \}, \quad f, g \in C(S^{n-1}). \]

The support function of $Y \in cc(\mathbb{R}^n)$ is the map $h_Y \in C(S^{n-1})$ defined by
\[ h_Y(u) = \max \{ \langle y, u \rangle \mid y \in Y \}, \quad u \in S^{n-1} \]
where $\langle \ , \ \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

For every $Y, Z \in cc(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$, the support function of $\alpha Y + \beta Z$ satisfies the following equality:
\[ h_{\alpha Y + \beta Z} = \alpha h_Y + \beta h_Z. \]
(see e.g., [16 Theorem 5.6.2]).

It is well known that the map $\varphi : cc(\mathbb{R}^n) \rightarrow C(S^{n-1})$ defined by
\[ \varphi(Y) = h_Y, \quad Y \in cc(\mathbb{R}^n) \]
is an affine isometric embedding and the image $\varphi(cc(\mathbb{R}^n))$ is a locally compact closed convex subset of the Banach space $C(S^{n-1})$ (see e.g., [14] p. 57, Note 6). Here, the map $\varphi$ is affine with respect to the Minkowski operations, i.e., for any $Y, Z \in cc(\mathbb{R}^n)$ and $t \in [0, 1]$, equality (2.2) clearly implies that
\[ \varphi(tY + (1-t)Z) = t\varphi(Y) + (1-t)\varphi(Z). \]

The width function of $Y \in cc(\mathbb{R}^n)$ is the map $w_Y \in C(S^{n-1})$ defined by
\[ w_Y(u) = h_Y(u) + h_Y(-u), \quad u \in S^{n-1}. \]

Thus, a compact convex set $Y$ in $\mathbb{R}^n$ has constant width $d \geq 0$ if $w_Y$ is the constant map with value $d$. Equivalently, if
\[ Y - Y = d\mathbb{B}^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq d \} \]
where $-Y = \{-y \mid y \in Y\}$ (see e.g., [16] Chapter 7, §6).

In general, we denote by
\[ B(x, r) = x + r\mathbb{B}^n = \{ y \in \mathbb{R}^n \mid \|x - y\| \leq r \} \]
the closed ball with center $x \in \mathbb{R}^n$ and radius $r \geq 0$.

The diameter of $Y \in cc(\mathbb{R}^n)$ is denoted by $\operatorname{diam} Y$. It is well known that the function $\operatorname{diam} : cc(\mathbb{R}^n) \rightarrow [0, \infty)$ defined by
\[ Y \mapsto \operatorname{diam} Y, \quad Y \in cc(\mathbb{R}^n) \]
is continuous (see e.g., [16 Example 2.7.11]). Clearly, if $Y \in cw_{[0, \infty)}(\mathbb{R}^n)$, then $\operatorname{diam} Y$ is just the width of $Y$. Let $\omega$ denote the restriction to $cw_{[0, \infty)}(\mathbb{R}^n)$ of the diameter map. Then
\[ \omega : cw_{[0, \infty)}(\mathbb{R}^n) \rightarrow [0, \infty) \]
is obviously continuous. Furthermore, for every $Y, Z \in cw_{[0, \infty)}(\mathbb{R}^n)$ and $t \in [0, 1]$, one has
\[ \omega(tY + (1-t)Z) = t\omega(Y) + (1-t)\omega(Z). \]
Indeed,

\[(tY + (1 - t)Z) - (tY + (1 - t)Z) = t(Y - Y) + (1 - t)(Z - Z)\]
\[= t\omega(Y)B^n + (1 - t)\omega(Z)B^n\]
\[= (t\omega(Y) + (1 - t)\omega(Z))B^n.\]

Hence, according to equality (2.3), \(tY + (1 - t)Z\) is of constant width \(t\omega(Y) + (1 - t)\omega(Z)\).

The concept of a compact convex set of constant width was extended by H. Maehara [10] to that of pairs of compact convex sets of constant relative width. Namely, a pair \((Y, Z)\) of compact convex sets in \(\mathbb{R}^n\) is said to be of constant relative width \(d \geq 0\) if the map \(w_{(Y, Z)} \in C(S^{n-1})\) defined by

\[(2.8)\]
\[w_{(Y, Z)}(u) = h_Y(u) + h_Z(-u), \quad u \in S^{n-1}\]

is a constant map with value \(d\). Equivalently, if

\[Y - Z = dB^n.\]

Obviously, a compact convex set \(Y\) of \(\mathbb{R}^n\) is of constant width \(d \geq 0\) if and only if \((Y, Y)\) is a pair of constant relative width \(d \geq 0\). From this fact, one gets a natural embedding \(e : cw_D(\mathbb{R}^n) \to cw_D(\mathbb{R}^n)\) given by the rule:

\[(2.9)\]
\[e(Y) = (Y, Y), \quad Y \in cw_D(\mathbb{R}^n).\]

We consider the Minkowski operations in \(cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)\), i.e., for every \(t \in \mathbb{R}\) and \((Y, Z), (A, E) \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)\),

\[(Y, Z) + (A, E) = (Y + A, Z + E) \quad \text{and} \quad t(Y, Z) = (tY, tZ).\]

It follows from equality (2.3) and formula (2.2) that the map

\[\varphi \times \varphi : cc(\mathbb{R}^n) \times cc(\mathbb{R}^n) \rightarrow C(S^{n-1}) \times C(S^{n-1})\]

defined by

\[(2.10)\]
\[(Y, Z) \mapsto (h_Y, h_Z), \quad Y, Z \in cc(\mathbb{R}^n)\]

embeds \(cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)\) as a closed convex subset in the Banach space \(C(S^{n-1}) \times C(S^{n-1})\).

Next we recall that the group \(\text{Aff}(n)\) of all affine transformations of \(\mathbb{R}^n\) is defined to be the (internal) semidirect product:

\[\mathbb{R}^n \rtimes GL(n)\]

where \(GL(n)\) is the group of all non-singular linear transformations of \(\mathbb{R}^n\) endowed with the topology inherited from \(\mathbb{R}^{n^2}\) (see e.g. [11, p. 102]). As a semidirect product, \(\text{Aff}(n)\) is topologized by the product topology of \(\mathbb{R}^n \times GL(n)\), becoming a Lie group with two connected components. Each element \(g \in \text{Aff}(n)\) is usually represented by \(g = T_v + \sigma\), where \(\sigma \in GL(n)\) and \(T_v : \mathbb{R}^n \to \mathbb{R}^n\) is the translation by \(v \in \mathbb{R}^n\), i.e.,

\[T_v(x) = v + x, \quad x \in \mathbb{R}^n\]

and thus,

\[g(x) = v + \sigma(x), \quad x \in \mathbb{R}^n.\]

Note that for every \(g \in \text{Aff}(n)\), \(t \in \mathbb{R}\) and \(x, y \in \mathbb{R}^n\), we have that

\[(2.11)\]
\[g(tx + (1 - t)y) = tg(x) + (1 - t)g(y).\]

Indeed,

\[g(tx + (1 - t)y) = v + \sigma(tx + (1 - t)y)\]
\[= te + (1 - t)v + t\sigma(x) + (1 - t)\sigma(y)\]
\[= t(v + \sigma(x)) + (1 - t)(v + \sigma(y))\]
\[= tg(x) + (1 - t)g(y).\]
A map \( g : \mathbb{R}^n \to \mathbb{R}^n \) is called a similarity transformation of \( \mathbb{R}^n \), if there is a \( \lambda > 0 \), called the ratio of \( g \), such that
\[
\|g(x) - g(y)\| = \lambda \|x - y\|
\]
for every \( x, y \in \mathbb{R}^n \) (see e.g. [11, Chapter 4, §1]). Clearly, every similarity transformation \( g : \mathbb{R}^n \to \mathbb{R}^n \) with ratio \( \lambda \) is an affine transformation of \( \mathbb{R}^n \). Indeed, such a \( g \) is just the composition of the homothety with center at the origin and ratio \( \lambda \) and the isometry \( \frac{1}{\lambda} g \) (see [11, p. xii]). Let \( \text{Sim}(n) \) denote the closed subgroup of \( \text{Aff}(n) \) consisting of all similarity transformations of \( \mathbb{R}^n \). Clearly, the natural action of \( \text{Sim}(n) \) on \( \mathbb{R}^n \) given by the evaluation map
\[
(2.12) \quad (g, x) \mapsto gx := g(x), \quad g \in \text{Sim}(n), \quad x \in \mathbb{R}^n
\]
is continuous (see e.g., [8, Proposition 2.6.11 and Theorem 3.4.3]). This action induces a continuous action on the hyperspace \( cc(\mathbb{R}^n) \), which is given by the rule:
\[
(2.13) \quad (g, Y) \mapsto gY = \{gy \mid y \in Y\}, \quad g \in \text{Sim}(n), \quad Y \in cc(\mathbb{R}^n)
\]
(see [2, Proposition 3.1]).

It is a well known classical fact that for every \( Y \in cc(\mathbb{R}^n) \), there is a unique ball \( \mathcal{B}(Y) \subset \mathbb{R}^n \) of minimal radius \( \mathcal{B}(Y) \) containing \( Y \) (see, e.g., [11, Theorem 12.7.5]). The ball \( \mathcal{B}(Y) \) is known under the name of Chebyshev ball of \( Y \). In this case, the center of \( \mathcal{B}(Y) \) belongs to \( Y \), and we will denote it by \( \mathcal{C}(Y) \). By [11, Corollary 12.7.6], the function \( \mathcal{C} : cc(\mathbb{R}^n) \to \mathbb{R}^n \) defined by
\[
(2.14) \quad Y \mapsto \mathcal{C}(Y), \quad Y \in cc(\mathbb{R}^n)
\]
is continuous. Furthermore, if \( cc(\mathbb{R}^n) \) and \( \mathbb{R}^n \) are endowed with the actions (2.13) and (2.12), respectively, then clearly the map \( \mathcal{C} \) is also \( \text{Sim}(n) \)-equivariant, i.e.,
\[
\mathcal{C}(gY) = g \mathcal{C}(Y)
\]
for every \( g \in \text{Sim}(n) \) and \( Y \in cc(\mathbb{R}^n) \) (see e.g., [11, Exercise 12.20]).

A Hilbert cube manifold or a \( Q \)-manifold is a separable, metrizable space that admits an open cover each member of which is homeomorphic to an open subset of the Hilbert cube \( Q \). Every \( Q \)-manifold is stable, i.e., it is homeomorphic to its product with the Hilbert cube \( Q \) (see [6, Theorem 15.1]). We refer the reader to [6] and [15] for the theory of \( Q \)-manifolds.

Finally, we recall Edwards’ Theorem (see [6, Theorem 43.1]), which by the stability Theorem for \( Q \)-manifolds, can be stated in the following form:

**Theorem 2.1.** (R.D. Edwards) Let \( X \) be a \( Q \)-manifold and \( Y \) a locally compact ANR. If there exists a cell-like map \( f : X \to Y \) then \( X \) is homeomorphic to the product \( Y \times Q \).

3. The hyperspaces \( cw_D(\mathbb{R}^n) \)

In this section we describe for every \( n \geq 1 \), the topology of the hyperspaces \( cw_D(\mathbb{R}^n) \). We begin with the case \( n = 1 \).

**Proposition 3.1.** Let \( D \) be a non-empty convex subset of \([0, \infty)\). Then the hyperspace \( cw_D(\mathbb{R}) \) is homeomorphic to \( D \times \mathbb{R} \).

**Proof.** Define a map \( f : cw_D(\mathbb{R}) \to D \times \mathbb{R} \) by the rule:
\[
f((x, y)) = (y - x, (x + y)/2), \quad (x, y) \in cw_D(\mathbb{R}).
\]
A simple calculation shows that \( f \) is a homeomorphism. \( \square \)

Now we assume, for the rest of the section, that \( n \geq 2 \).

**Lemma 3.2.** Let \( Y \in cc(\mathbb{R}^n) \) be such that \( \mathcal{C}(Y) = 0 \). Then \( \mathcal{C}(Y + B) = 0 \) for every closed ball \( B = B(0, r) \).
Theorem 3.6. Proof. Let $\delta = R(Y)$ and $\varepsilon = R(Y+B)$ be the radii of $R(Y)$ and $R(Y+B)$, respectively. Then $R(Y) + B$ is just the closed ball $B(0, \delta + r)$. Since $Y \subset R(Y)$ and the Minkowski addition preserves inclusions, we get that $Y + B \subset R(Y) + B$. By minimality of the radius $\varepsilon = R(Y+B)$, we have that $\varepsilon \leq \delta + r$. Let
\[ O = \{x \in R(Y+B) \mid B(x,r) \subset R(Y+B)\}. \]
Clearly, $O = B(z, \varepsilon - r)$, where $z = C(Y + B)$. Since $Y \subset Y + B \subset R(Y + B)$ and since for every $y \in Y$,
\[ B(y,r) \subset \bigcup_{\gamma \in Y} B(\gamma, r) = \bigcup_{\gamma \in Y} \{\gamma + B(0,r)\} = Y + B \subset R(Y + B) \]
we infer that $Y \subset O$. By minimality of $R(Y)$, we have that $\delta \leq \varepsilon - r$. Consequently, $\varepsilon = \delta + r$. Uniqueness of $R(Y + B)$ yields that $z = C(Y + B) = 0$, as required. $\square$

Lemma 3.3. Let $B$ be a closed ball with center $y \in \mathbb{R}^n$ and let $Y \in cc(\mathbb{R}^n)$ be such that $C(Y) = y$. Then $C(tY + (1-t)B) = y$ for every $t \in \mathbb{R}$.

Proof. Let $r$ be the radius of $B$. Define $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by
\[ gx = x - y, \quad x \in \mathbb{R}^n. \]
Then clearly $g \in Sim(n)$ and
\[ C(tY + (1-t)B) = y \iff gC(tY + (1-t)B) = 0. \tag{3.1} \]
By the $Sim(n)$-equivariance of $C$, we have that $C(tgY) = tgC(Y) = tgy = t \cdot 0 = 0$. Note also that
\[ (1-t)gB = B(0, (1-t)r). \]
Then, again by the $Sim(n)$-equivariance of $C$, equality (3.1) and Lemma 3.2 one gets
\[ gC(tY + (1-t)B) = C(tgY + (1-t)gB) = 0. \]
Finally, by the equivalence (3.1), we get $C(tY + (1-t)B) = y$. $\square$

Lemma 3.4. For every non-empty closed subset $K$ of $[0, \infty)$, the hyperspace $cw_K(\mathbb{R}^n)$ of all compact convex sets of constant width $k \in K$ is closed in $cc(\mathbb{R}^n)$.

Proof. Let $(Y_i)_{i=1}^\infty$ be a sequence in $cw_K(\mathbb{R}^n)$ such that $Y_i \rightharpoonup Y \in cc(\mathbb{R}^n)$. Then, by [14, Theorem 1.8.11], we have that $h_{Y_i} \rightharpoonup h_Y$ and hence, for the width map $\mathcal{W}$ we get that $w_{Y_i} \rightharpoonup w_Y$. Since for every $i \geq 1$, $w_{Y_i}$ is a constant map with value in $K$ and $K$ is closed, $w_{Y_i}$ is also a constant map with value in $K$. Thus, $Y \in cw_K(\mathbb{R}^n)$, as required. $\square$

Lemma 3.5. If $D$ is a non-empty convex subset of $[0, \infty)$, then the hyperspace $cw_D(\mathbb{R}^n)$ is convex with respect to the Minkowski operations.

Proof. This follows directly from the convexity of $D$ and equality (2.7). $\square$

The proof of the following theorem is essentially the same as the one presented in [4, Theorem 1.1]. Here we just fill the gap in that proof.

Theorem 3.6. For every $d > 0$, the hyperspace $cw_d(\mathbb{R}^n)$ of all convex bodies of constant width $d$, is a contractible $Q$-manifold.

Proof. By formula (2.3) and Lemmas 3.3 and 3.5, the hyperspace $cw_d(\mathbb{R}^n)$ embeds as a locally compact closed convex subset in the Banach space $C(S^{n-1})$. Hence, according to [3, Theorem 7.1], $cw_d(\mathbb{R}^n)$ is homeomorphic to either $\mathbb{R}^m \times [0,1]^p$ or $[0,1] \times [0,1]^p$ for some $0 \leq m < \infty$ and $0 \leq p \leq \infty$.

Now, for the case $n = 2$, let $K$ denote the Reuleaux triangle in $\mathbb{R}^2$ that is the intersection of the closed discs of radius $d$, centered at the points $(0,0), (d,0)$ and $(d/2, d\sqrt{3}/2)$ of the plane $\mathbb{R}^2$. For any $\alpha \in [0, 2\pi]$, denote by $K_\alpha$ the image of $K$ under a counterclockwise rotation by an angle $\alpha$ around the origin. Note that
\[ \{K_\alpha \mid \alpha \in [0, 2\pi]\} \subset cw_d(\mathbb{R}^2). \]
Using formula (2.3), we identify the family \( \{K_{\alpha} \mid \alpha \in [0, 2\pi]\} \) with the family 
\[ \{h_{K_{\alpha}} \mid \alpha \in [0, 2\pi]\} \]
of the support functions of the sets \( K_{\alpha}, \alpha \in [0, 2\pi] \). Next, we show that the latter family contains linearly independent sets of arbitrary cardinality. Identify the circle \( S^1 \) with the subset \( \{e^{it} \mid t \in [0, 2\pi]\} \) of the complex plane. Since orthogonal transformations preserve the inner product, it follows from the definition of the support functions that the following equality holds for every \( \alpha \) and \( t \) in \( [0, 2\pi] \):
\[ h_{K_{\alpha}}(e^{it}) = h_{K}(e^{i(t-\alpha)}) . \]
Elementary geometric arguments show that
\[ h_{K}(e^{it}) = \begin{cases} 
  d & \text{if } t \in [0, \pi/3] \\
  0 & \text{if } t \in [\pi, 4\pi/3] \\
  x \in (0, d) & \text{if } t \in (\pi/3, \pi) \cup (4\pi/3, 2\pi) .
\end{cases} \]
Now, fixing \( l \in \mathbb{N} \), we define for every \( j = 0, 1, \ldots, l-1 \), the map
\[ h_j := h_{K_{\alpha}} \in \{h_{K_{\alpha}} \mid \alpha \in [0, 2\pi]\} . \]
To see that the set \( \{h_j \mid j = 0, 1, \ldots, l-1\} \) is linearly independent, let
\[ g = \sum_{j=0}^{l-1} \lambda_j h_j \]
be a linear combination such that \( g = 0 \). Using equalities (3.2) and (3.3), we get that
\[ h_j(e^{i\frac{\pi}{l}}) = h_{K}(e^{i(\frac{\pi}{l} - \frac{\pi}{l+1})}) = d \]
for every \( j = 0, 1, \ldots, l-1 \). Hence,
\[ 0 = g(e^{i\frac{\pi}{l}}) = \sum_{j=0}^{l-1} \lambda_j h_j(e^{i\frac{\pi}{l}}) = d \sum_{j=0}^{l-1} \lambda_j \]
and consequently, \( \sum_{j=0}^{l-1} \lambda_j = 0 \). Now, again by equalities (3.2) and (3.3) we get for every \( j = 1, 2, \ldots, l-1 \) that
\[ h_j(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})}) = h_{K}(e^{i(\frac{\pi}{l} - \frac{\pi}{l+1})}) = d \quad \text{and} \quad h_0(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})}) \in (0, d) \]
Hence,
\[ 0 = g(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})}) = \lambda_0 h_0(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})}) + \sum_{j=1}^{l-1} \lambda_j h_j(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})}) \]
\[ = \lambda_0 h_0(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})}) + d \sum_{j=1}^{l-1} \lambda_j = \left(d - h_0(e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})})\right) \sum_{j=1}^{l-1} \lambda_j \]
and consequently, \( \lambda_0 = -\sum_{j=1}^{l-1} \lambda_j = 0 \).
Repeating the argument but evaluating the map \( g \) at the points \( e^{i(\frac{\pi}{l} + \frac{\pi}{l+1})} \) for \( s = 2, 3, \ldots, l-1 \), we conclude that \( \lambda_j = 0 \) for every \( j = 0, 1, \ldots, l-1 \), and hence, the set \( \{h_j \mid j = 0, 1, \ldots, l-1\} \) is linearly independent. This yields that, \( \{K_{\alpha} \mid \alpha \in [0, 2\pi]\} \) is infinite-dimensional.
Now, for any \( n \geq 3 \), denote by \( p_2 : \mathbb{R}^n \to \mathbb{R}^2 \) the cartesian projection, i.e.,
\[ p_2((x_1, \ldots, x_n)) = (x_1, x_2), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n . \]
Denote also by \( R(2) \) the family of all Reuleaux triangles in \( \mathbb{R}^2 \) of constant width \( d \). Applying inductively the raising-dimension process described in [9] Theorem 4.1] to every \( Z \in R(2) \), we obtain the family \( R(n) \) of all convex bodies \( Y \subset \mathbb{R}^n \) of constant width \( d \), such that \( p_2(Y) \in R(2) \). Here we are considering
\[ \mathbb{R}^{n-1} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\} \]
as the affine hyperplane of \(\mathbb{R}^n\) in which the \((n-1)\)-dimensional convex body of constant width \(d\) is contained (see [3] Theorem 4.1)). Then clearly,
\[
\{ K_n \mid n \in [0,2\pi]\} \subset R(2) = \{ p_2(Y) \mid Y \in R(n) \} = p_2(R(n)).
\]
Consequently, the space \(R(n)\) is infinite-dimensional. Thus we conclude that the hyperspace \(cw_d(\mathbb{R}^n)\) is also infinite-dimensional for every \(n \geq 2\). Then, by virtue of [3] Theorem 7.1, \(cw_d(\mathbb{R}^n)\) is homeomorphic to either \(\mathbb{R}^m \times Q\) or \([0,1) \times Q\) for some \(0 \leq m < \infty\). In either case, it is a contractible \(Q\)-manifold. This completes the proof. 

\[\square\]

**Lemma 3.7** ([4] Theorem 1.1]). Let \(D \neq \{0\}\) be a non-empty convex subset of \([0,\infty)\). Then the hyperspace \(cw_d(\mathbb{R}^n)\) is a contractible \(Q\)-manifold.

**Proof.** It follows from Theorem 3.6 that the hyperspace \(cw_d(\mathbb{R}^n)\) is infinite-dimensional. By Lemma 3.5, it is convex and hence, contractible. It remains to show that \(cw_d(\mathbb{R}^n)\) is a \(Q\)-manifold.

If \(D\) is closed, then by Lemma 5.4, \(cw_d(\mathbb{R}^n)\) is closed in \(cc(\mathbb{R}^n)\), and therefore, it is locally compact. Then the map \(\varphi\) defined by formula (2.3) embeds \(cw_d(\mathbb{R}^n)\) as a locally compact closed convex subset in the Banach space \(C(S^{n-1})\). Thus, by [3] Theorem 7.1, \(cw_d(\mathbb{R}^n)\) is homeomorphic to either \(\mathbb{R}^m \times Q\) or \([0,1) \times Q\) for some \(0 \leq m < \infty\). In either case, \(cw_d(\mathbb{R}^n)\) is also a \(Q\)-manifold. Next, if \(D\) is open, then \(K_D := [0,\infty) \setminus D\) is closed. By Lemma 5.4, \(cw_{K_D}(\mathbb{R}^n)\) is closed in \(cc(\mathbb{R}^n)\), and hence, also in \(cw_{[0,\infty)}(\mathbb{R}^n)\). Equivalently, \(cw_d(\mathbb{R}^n)\) is open in \(cw_{[0,\infty)}(\mathbb{R}^n)\), which by the above paragraph is a \(Q\)-manifold. Thus, we infer that \(cw_d(\mathbb{R}^n)\) is also a \(Q\)-manifold.

Finally, let \(D\) be a half-open interval properly contained in \([0,\infty)\). Assume without loss of generality that \(D = [a,b)\) with \(b > a \geq 0\). Then \(D\) is open in \([a,b]\) and consequently, \(cw_d(\mathbb{R}^n)\) is open in \(cw_{[a,b]}(\mathbb{R}^n)\). Since \(cw_{[a,b]}(\mathbb{R}^n)\) is a \(Q\)-manifold, it then follows that \(cw_d(\mathbb{R}^n)\) is also a \(Q\)-manifold. This completes the proof. 

**Proposition 3.8.** The function \(\eta_D : cw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n\) defined by formula (3.3) is a cell-like map.

**Proof.** Continuity of \(\eta_D\) follows from the continuity of \(\omega\) and \(\mathcal{C}\). Let \((d,x) \in D \times \mathbb{R}^n\) and \(B = B(x,d/2) \in cw_d(\mathbb{R}^n)\). Then \(\eta_D(B) = (d,x)\), and hence, \(\eta_D\) is a surjective map. We claim that the inverse image \(\eta^{-1}_D((d,x))\) is contractible. Indeed, define a homotopy \(H : \eta^{-1}_D((d,x)) \times [0,1) \to \eta^{-1}_D((d,x))\) by the Minkowski sum:
\[
H(A,t) = tA + (1-t)B, \quad A \in \eta^{-1}_D((d,x)), \quad t \in [0,1].
\]

By equation (2.7) and Lemma 3.3 we get that
\[
\omega(H(A,t)) = d \quad \text{and} \quad \mathcal{C}(H(A,t)) = x
\]
for every \(t \in [0,1]\). Hence, \(H\) is a well-defined contraction to the point \(B \in \eta^{-1}_D((d,x))\).

It remains to show that \(\eta_D\) is proper. Let \(K\) be a compact subset of \(D \times \mathbb{R}^n\). Then the projections \(\pi_D(K)\) and \(\pi_{\mathbb{R}^n}(K)\) are compact subsets of \(D\) and \(\mathbb{R}^n\), respectively. Let \(\Gamma\) denote the compact set \(\pi_D(K) \times \pi_{\mathbb{R}^n}(K)\). Then \(\Gamma\) is a compact subset of \([0,\infty) \times \mathbb{R}^n\). By continuity of \(\eta_{[0,\infty)} : cw_{[0,\infty)}(\mathbb{R}^n) \to [0,\infty) \times \mathbb{R}^n\) and Lemma 3.3, we have that \(\eta^{-1}_{[0,\infty)}(\Gamma)\) is closed in \(cc(\mathbb{R}^n)\). We put
\[
\delta = \max \pi_D(K), \quad r = \max \{ ||y|| \mid y \in \pi_{\mathbb{R}^n}(K) \} \quad \text{and} \quad O = B(0,\delta + r).
\]
Then \(cc(O)\) is a compact subset of \(cc(\mathbb{R}^n)\) (see [12] p. 568) that contains \(\eta^{-1}_{[0,\infty)}(\Gamma)\). Indeed, if \(Y \in \eta^{-1}_{[0,\infty)}(\Gamma)\), then \(\omega(Y) \leq \delta\) and \(||\mathcal{C}(Y)|| \leq r\). By [3] Theorem 6, the radius \(\mathcal{R}(Y)\) of \(\mathcal{R}(Y)\) is less than \(\omega(Y)\). Hence, \(\mathcal{R}(Y) + ||\mathcal{C}(Y)|| \leq \delta + r\). Thus, we get that
\[
Y \subset \mathcal{R}(Y) \subset O.
\]
It follows that \( \eta_{[0,\infty)}^{-1}(\Gamma) \) is closed in \( cc(O) \), and therefore, it is compact. Finally, by continuity of \( \eta_D \), \( \eta_D^{-1}(K) \) is closed in \( \eta_D^{-1}(\Gamma) = \eta_{[0,\infty)}^{-1}(\Gamma) \), and thus, it is also compact. This completes the proof. \( \square \)

Next, we state the main result of the section.

**Theorem 3.9.** Let \( D \neq \{0\} \) be a convex subset of \([0, \infty)\). Then the hyperspace \( cw_D(\mathbb{R}^n) \) is homeomorphic to \( D \times \mathbb{R}^n \times Q \).

**Proof.** By Lemma 5.7, the hyperspace \( cw_D(\mathbb{R}^n) \) is a \( Q \)-manifold. Clearly, \( D \times \mathbb{R}^n \) is a locally compact ANR. By Proposition 7.8, the map \( \eta_D : cw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n \) defined by formula (3.4) is a cell-like map. Consequently, by Theorem 2.1, \( cw_D(\mathbb{R}^n) \) is homeomorphic to \( D \times \mathbb{R}^n \times Q \). \( \square \)

**Corollary 3.10.** Let \( D \neq \{0\} \) be a convex subset of \([0, \infty)\). Then

1. \( cw_D(\mathbb{R}^n) \) is homeomorphic to \( \mathbb{R}^n \times Q \), if \( D \) is compact,
2. \( cw_D(\mathbb{R}^n) \) is homeomorphic to \( \mathbb{R}^{n+1} \times Q \), if \( D \) is an open interval,
3. \( cw_D(\mathbb{R}^n) \) is homeomorphic to \( Q_0 := Q \setminus \{\ast\} \), if \( D \) is a half-open interval.

In particular, the hyperspace \( cw(\mathbb{R}^n) \) of all convex bodies of constant width is homeomorphic to \( \mathbb{R}^{n+1} \times Q \).

**Proof.** By Theorem 3.9, \( cw_D(\mathbb{R}^n) \) is homeomorphic to \( D \times \mathbb{R}^n \times Q \).

1. If \( D \) is compact, then \( D \) is either a point or a closed interval. In either case, \( D \times \mathbb{R}^n \) is homeomorphic to \( \mathbb{R}^n \times Q \) and thus, \( D \times \mathbb{R}^n \times Q \) is homeomorphic to \( \mathbb{R}^n \times Q \).
2. If \( D \) is an open interval, then \( D \) is homeomorphic to \( \mathbb{R} \) and thus, \( D \times \mathbb{R}^n \times Q \) is homeomorphic to \( \mathbb{R}^{n+1} \times Q \).
3. If \( D \) is a half-open interval. Since \( \mathbb{R}^n \times Q \) is a contractible \( Q \)-manifold, we have by [6, Corollary 21.4] that \( D \times \mathbb{R}^n \times Q \) is homeomorphic to \( Q \times [0,1) \), which in turn is homeomorphic to \( Q_0 \) (see [6, Theorem 12.2]). \( \square \)

**Corollary 3.11.** If a subspace \( U \) of \([0, \infty)\) can be represented as the topological sum \( \bigoplus_{i \in I} D_i \) of a family \( (D_i)_{i \in I} \) of pairwise disjoint non-empty convex subsets \( D_i \neq \{0\} \) of \([0, \infty)\) (e.g., if \( U \) is open in \([0, \infty)\)), then the hyperspace \( cw_U(\mathbb{R}^n) \) of all compact convex subsets of constant width \( u \in U \) is homeomorphic to \( U \times \mathbb{R}^n \times Q \).

**Proof.** Since the sets \( D_i, i \in I \), are pairwise disjoint open subsets of \( U \), the sets \( cw(D_i(\mathbb{R}^n)) \), \( i \in I \), are also pairwise disjoint open subsets of \( cw_U(\mathbb{R}^n) \). Moreover, since \( cw_U(\mathbb{R}^n) \) is the disjoint union of the hyperspaces \( cw(D_i(\mathbb{R}^n)) \), \( i \in I \), we have the homeomorphism:

\[
\text{cw}_U(\mathbb{R}^n) \cong \bigoplus_{i \in I} \text{cw}(D_i(\mathbb{R}^n))
\]

(see [8 Corollary 2.2.4]). Now, Theorem 3.9 implies that for every \( i \in I \), the hyperspace \( cw(D_i(\mathbb{R}^n)) \) is homeomorphic to \( D_i \times \mathbb{R}^n \times Q \) and, consequently,

\[
\bigoplus_{i \in I} \text{cw}(D_i(\mathbb{R}^n)) \cong \bigoplus_{i \in I} (D_i \times \mathbb{R}^n \times Q) \cong \left( \bigoplus_{i \in I} D_i \right) \times \mathbb{R}^n \times Q = U \times \mathbb{R}^n \times Q.
\]

This completes the proof. \( \square \)

4. **The Hyperspaces \( crw_D(\mathbb{R}^n) \)**

In this section we describe for every \( n \geq 1 \), the topology of the hyperspaces \( crw_D(\mathbb{R}^n) \). Here is the main result.

**Theorem 4.1.** Let \( D \neq \{0\} \) be a non-empty convex subset of \([0, \infty)\). Then for every \( n \geq 2 \), the hyperspace \( crw_D(\mathbb{R}^n) \) is homeomorphic to \( cw_D(\mathbb{R}^n) \).

The proof of this theorem is given at the end of the section; it is preceded by a series of auxiliary lemmas. However, for \( n = 1 \) we have the following simple result.
Proposition 4.2. Let $D \neq \{0\}$ be a non-empty convex subset of $[0, \infty)$. Then the hyperspace $\text{crw}_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R} \times [0, 1]$.

Proof. Let $\Delta_D = \{(d, a) \in D \times \mathbb{R} | a \leq 2d\}$ and define a map $f : \text{crw}_D(\mathbb{R}) \to \Delta_D \times \mathbb{R}$ by the rule:

$$f([x, y], [v, z]) = \left((z - x, y - x), \frac{x + y}{2}\right), \quad ([x, y], [v, z]) \in \text{crw}_D(\mathbb{R}).$$

By definition, $z - x = y - v \in D$ is just the width of the pair $([x, y], [v, z])$. Then $\frac{x + y}{2} = \frac{z + v}{2} \in \mathbb{R}$ is the middle point of $[x, y]$ and of $[v, z]$. Also, $y - x \leq 2(z - x)$. Indeed, if not, then

$$2z = z + z \geq z + v = y + x > 2z$$

which is a contradiction. Thus, $f$ is a well-defined map. We claim that $f$ is a homeomorphism. Indeed, define a map $g : \Delta_D \times \mathbb{R} \to \text{crw}_D(\mathbb{R})$ by the rule:

$$g((d, a), p) = \left([p - \frac{a}{2}, p + \frac{a}{2}], [p - (d - \frac{a}{2}), p + (d - \frac{a}{2})]\right), \quad (d, a) \in \Delta_D, \; p \in \mathbb{R}.$$

A simple calculation shows that $g$ is the inverse map of $f$. Thus, $f$ is a homeomorphism. Finally, it is easy to see that $\Delta_D$ is homeomorphic to $D \times [0, 1]$. This completes the proof. \hfill \Box

For the rest of the section, we assume that $n \geq 2$.

Lemma 4.3. For every non-empty closed subset $K$ of $[0, \infty)$, the hyperspace $\text{crw}_K(\mathbb{R}^n)$ of all pairs of compact convex sets of constant width $k \in K$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$.

Proof. Let $(Y_i, Z_i)_{i=1}^\infty$ be a sequence in $\text{crw}_K(\mathbb{R}^n)$ such that $(Y_i, Z_i) \rightharpoonup (Y, Z)$, where $Y, Z \in cc(\mathbb{R}^n)$. Then, by [14, Theorem 1.8.11], we have that $h_{Y_i} \rightharpoonup h_Y$ and $h_{Z_i} \rightharpoonup h_Z$. Hence, we get that $w_{(Y_i, Z_i)} \rightharpoonup w_{(Y, Z)}$ (see equation (2.3)). Since for every $i \in \mathbb{N}$, $w_{(Y_i, Z_i)}$ is a constant map with value in $K$ and $K$ is closed, we infer that $w_{(Y, Z)}$ is also a constant map with value in $K$. Thus, $(Y, Z) \in \text{crw}_K(\mathbb{R}^n)$. \hfill \Box

Lemma 4.4. If $D$ is a non-empty convex subset of $[0, \infty)$, then the hyperspace $\text{crw}_D(\mathbb{R}^n)$ is also convex with respect to the Minkowski operations.

Proof. Let $(Y, Z), (A, E) \in \text{crw}_D(\mathbb{R}^n)$, $t \in [0, 1]$ and $u \in S^{n-1}$. Then

$$t(Y, Z) + (1 - t)(A, E) = (tY + (1 - t)A, tZ + (1 - t)E)$$

and, by equality (2.2), we have

$$w_{(tY + (1 - t)A, tZ + (1 - t)E)}(u) = th_Y(u) + th_E(-u) + (1 - t)h_Z(u) + (1 - t)h_E(-u)$$

$$= t(h_Y(u) + h_Z(-u)) + (1 - t)(h_A(u) + h_E(-u))$$

$$= tw_{(Y, Z)}(u) + (1 - t)w_{(A, E)}(u).$$

Since $w_{(Y, Z)}$ and $w_{(A, E)}$ are constant maps with values in $D$, and since $D$ is convex, we get that $w_{(tY + (1 - t)A, tZ + (1 - t)E)}$ is also a constant map with value in $D$. Thus, the pair $t(Y, Z) + (1 - t)(A, E) \in \text{crw}_D(\mathbb{R}^n)$. \hfill \Box

Lemma 4.5 ([4, Theorem 3.1]). Let $D \neq \{0\}$ be a non-empty convex subset of $[0, \infty)$. Then the hyperspace $\text{crw}_D(\mathbb{R}^n)$ is a contractible $Q$-manifold.

Proof. First we note that by formula (2.9) and Theorem 3.3, the hyperspace $\text{crw}_D(\mathbb{R}^n)$ is infinite-dimensional. By Lemma 4.3, it is also convex, and hence, contractible. It remains to show that $\text{crw}_D(\mathbb{R}^n)$ is a $Q$-manifold.

If $D$ is closed, then by Lemma 4.3, $\text{crw}_D(\mathbb{R}^n)$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$, and therefore, it is locally compact. Then the map $\varphi \times \varphi$, defined by formula (2.10), embeds $\text{crw}_D(\mathbb{R}^n)$ as a locally compact closed convex subset in the Banach space $C(S^{n-1}) \times C(S^{n-1})$. Thus, by [3, Theorem 7.1], $\text{crw}_D(\mathbb{R}^n)$ is homeomorphic to either $Q_0$ or $\mathbb{R}^m \times Q$ for some $0 < m < \infty$. In either case, $\text{crw}_D(\mathbb{R}^n)$ is a $Q$-manifold.
Next, if $D$ is open, then $K_D := [0, \infty) \setminus D$ is closed. By Lemma 4.3 $\text{crw}_{K_D}(\mathbb{R}^n)$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$, and hence, also in $\text{crw}_{[0, \infty)}(\mathbb{R}^n)$. Equivalently, $\text{crw}_D(\mathbb{R}^n)$ is open in $\text{crw}_{[0, \infty)}(\mathbb{R}^n)$, which by the above paragraph is a $Q$-manifold. Thus, we infer that $\text{crw}_D(\mathbb{R}^n)$ is also a $Q$-manifold.

Finally, let $D$ be a half-open interval properly contained in $[0, \infty)$. Assume without loss of generality that $D = [a, b)$ with $b > a \geq 0$. Then $D$ is open in $[a, b]$ and consequently, $\text{crw}_D(\mathbb{R}^n)$ is open in $\text{crw}_{[a, b]}(\mathbb{R}^n)$. Since $\text{crw}_{[a, b]}(\mathbb{R}^n)$ is a $Q$-manifold, we infer that $\text{crw}_D(\mathbb{R}^n)$ is also a $Q$-manifold. This completes the proof.

It was proved in [10, Theorem 3] that if $(Y, Z)$ is a pair of constant relative width $d \geq 0$, then the Minkowski sum $Y + Z$ is a compact convex set of constant width $2d$. Moreover, we have the following proposition.

**Proposition 4.6.** For every non-empty convex subset $D$ of $[0, \infty)$, the map $\Phi_D : \text{crw}_D(\mathbb{R}^n) \to \text{crw}_D(\mathbb{R}^n)$ defined by

$$\Phi_D((Y, Z)) = \frac{1}{2}(Y + Z), \quad (Y, Z) \in \text{crw}_D(\mathbb{R}^n)$$

is a cell-like map.

**Proof.** It follows from formula (2.3), equality (2.2) and [10, Theorem 3] that the map $\Phi_D$ is well-defined and clearly, it is continuous. Let $Y \in \text{crw}_D(\mathbb{R}^n)$. Then the pair $(Y, Y) \in \text{crw}_D(\mathbb{R}^n)$ and $\Phi_D((Y, Y)) = Y$ (see [10, Theorem 2.1.7]). Hence, the map $\Phi_D$ is surjective. We claim that the inverse image $\Phi_D^{-1}(E)$ of every $E \in \text{crw}_D(\mathbb{R}^n)$ is convex and thus, contractible. Indeed, let $(A, B), (Y, Z) \in \Phi_D^{-1}(E)$ and $t \in [0, 1]$. Then

$$\frac{1}{2}(A + B) = \frac{1}{2}(Y + Z) = E.$$

By Lemma 1.4,

$$t(A, B) + (1 - t)(Y, Z) = (tA + (1 - t)Y, tB + (1 - t)Z) \in \text{crw}_D(\mathbb{R}^n).$$

Moreover,

$$\Phi_D\left((tA + (1 - t)Y, tB + (1 - t)Z)\right) = \frac{1}{2}(tA + (1 - t)Y + tB + (1 - t)Z) = \frac{1}{2}\left((A + B) + (1 - t)(Y + Z)\right) = tE + (1 - t)E = E.$$

Therefore, $t(A, B) + (1 - t)(Y, Z) \in \Phi_D^{-1}(E)$. Consequently, $\Phi_D^{-1}(E)$ is contractible. It remains to show that $\Phi_D$ is a proper map. Consider a compact subset

$$\Gamma \subset \text{crw}_D(\mathbb{R}^n) \subset \text{crw}_{[0, \infty)}(\mathbb{R}^n).$$

Observe that $\Phi_D$ is the restriction of $\Phi_{[0, \infty)}$ to $\text{crw}_D(\mathbb{R}^n)$ and $\Phi_{[0, \infty)}((Y, Z)) \in \text{crw}_D(\mathbb{R}^n)$ if and only if $(Y, Z) \in \text{crw}_D(\mathbb{R}^n)$. This implies that

$$\Phi_D^{-1}(\Gamma) = \Phi_{[0, \infty)}^{-1}(\Gamma)$$

is closed in $\text{crw}_{[0, \infty)}(\mathbb{R}^n)$ and according to Lemma 4.3 $\Phi_D^{-1}(\Gamma)$ is also closed in the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$. Now, since $\Gamma$ is compact and $\omega : \text{crw}_{[0, \infty)}(\mathbb{R}^n) \to [0, \infty)$ is continuous (see (2.6)), we can find a positive number $M \geq \max \omega(\Gamma) := d$ such that

$$A \subset B(0, M)$$

for every $A \in \Gamma$. Let $(Y, Z) \in \Phi_D^{-1}(\Gamma)$. Then $\Phi_D((Y, Z)) = \frac{1}{2}(Y + Z) \in \Gamma$. Consequently, the pair $(Y, Z)$ is of constant relative width $\omega\left(\frac{1}{2}(Y + Z)\right) \leq d$ and thus,

$$\|y - z\| \leq d < 2d$$
for every $y \in Y$ and $z \in Z$. On the other hand, by the choice of $M$, we infer that
\[
\|y + z\| \leq 2M
\]
for every $y \in Y$ and $z \in Z$. Finally, using the parallelogram law we get that
\[
\|y\|^2 + \|z\|^2 = \frac{1}{2}(\|y + z\|^2 + \|y - z\|^2) < \frac{4M^2 + 4d^2}{2} \leq 4M^2
\]
for each $y \in Y$ and $z \in Z$. This directly implies that $\|y\| < 2M$ and $\|z\| < 2M$, if $y \in Y$ and $z \in Z$. Therefore, we can infer that $(Y, Z) \in \text{cc}(B(0, 2M)) \times \text{cc}(B(0, 2M))$ and hence $\Phi^{-1}_D(\Gamma) \subset \text{cc}(B(0, 2M)) \times \text{cc}(B(0, 2M))$. Now, using the fact that $\Phi^{-1}_D(\Gamma)$ is closed in $\text{cc}(\mathbb{R}^n) \times \text{cc}(\mathbb{R}^n)$ and $\text{cc}(B(0, 2M)) \times \text{cc}(B(0, 2M))$ is compact (see [12] p. 568), we conclude that $\Phi^{-1}_D(\Gamma)$ is compact. This completes the proof.

Proof of Theorem 4.1. By Lemmas 3.5 and 3.7, the hyperspaces $\text{crw}_D(\mathbb{R}^n)$ and $\text{cw}_D(\mathbb{R}^n)$ are $Q$-manifolds. By Proposition 1.6, $\text{cw}_D(\mathbb{R}^n)$ is a cell-like image of $\text{crw}_D(\mathbb{R}^n)$. Consequently, Theorem 2.1 implies that $\text{crw}_D(\mathbb{R}^n)$ is homeomorphic to $\text{cw}_D(\mathbb{R}^n) \times Q$, which in turn, by the Stability Theorem for $Q$-manifolds ([6] Theorem 15.1), is homeomorphic to $\text{cw}_D(\mathbb{R}^n)$.

References

1. J.L. Alperin and R.B. Bell, Groups and Representations, Graduate texts in mathematics 162, Springer, New York, 1995.
2. S.A. Antonyan and N. Jonard-Pérez, Affine group acting on hyperspaces of compact convex subsets of $\mathbb{R}^n$, Fund. Math. 223 (2013) 99-136.
3. C. Bessaga and A. Pełczyński, Selected Topics in Infinite-Dimensional Topology, Polish Scientific Publishers, Warszawa, 1975.
4. L.E. Bazylevych and M.M. Zarichnyi, On convex bodies of constant width, Topol. Appl. 153 (2006) 1699-1704.
5. F. Bohnenblust, Convex regions and projections in Minkowski spaces, Annals of Mathematics, Second Series Vol. 39, No. 2 (1938) 301-308.
6. T.A. Chapman, Lectures on Hilbert Cube Manifolds, C. B. M. S. Regional Conference Series in Math., 28, Amer. Math. Soc., Providence, RI, 1975.
7. H.G. Eggleston, Convexity, Cambridge University Press, 1958.
8. R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
9. T. Lachand-Robert and E. Oudet, Bodies of constant width in arbitrary dimensions, Matematische Nachrichten 280 (2007) 740-750.
10. H. Maehara, Convex bodies forming pairs of constant width, Journal of Geometry Vol. 22 (1984) 101-107.
11. M. Moszyńska, Selected Topics in Convex Geometry, Birkhäuser Boston, 2006.
12. S.B. Nadler, Hyperspaces of Sets. A text with research questions, Marcel Dekker Inc., New York, 1978.
13. S.B. Nadler, Jr., J.E. Quinn and N.M. Stavrakas, Hyperspaces of compact convex sets, Pacific J. Math. 83 (1979) 441-462.
14. R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Encyclopedia of Mathematics, Cambridge University Press, 1993.
15. J. van Mill, Infinite-Dimensional Topology: Prerequisites and Introduction, North-Holland Math. Library 43, Amsterdam, 1989.
16. R. Webster, Convexity, Oxford Univ. Press, Oxford, 1994.

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