Fixed-Time Synchronization Analysis for Complex-Valued Neural Networks via a New Fixed-Time Stability Theorem

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ABSTRACT Based on variable substitution, calculating definite integral and solving the minimization problem, in this paper we establish a new fixed-time stability theorem, which can provide a novel upper bound estimate formula for the settling time. By dividing the considered complex-valued neural networks (CVNNs) into double-layer real-valued neural networks, the fixed-time synchronization of CVNNs is analyzed by means of the new fixed-time stability theorem. Both theoretical derivation and numerical simulation show the new upper bound estimate formula for the settling time in this paper is more accurate than that given in the classic fixed-time stability theorem. A numerical example is given to verify the effectiveness of the main results.

INDEX TERMS Complex-valued neural networks, fixed-time stability, fixed-time synchronization.

I. INTRODUCTION
The stability of nonlinear systems is an important issue. If stability time is considered, asymptotic stability will be unsatisfactory since the stability time is infinite. Finite-time stability means system can reach stability within some finite time (also known as the settling time), which changes with the initial value of the system. However, if the initial value of the system is unknown, the settling time can’t be calculated. In 2012, fixed-time stability was proposed [1]. Fixed-time stability means system can reach stability within the settling time, and the settling time has a fixed upper bound for any initial value of the system. Due to the above characteristics, fixed-time stability is indispensable in many practical applications, such as spacecraft attitude control [2], power system control [3], and so on.

Among the common fixed-time stability theorems [1], [4]–[6], the fixed-time stability theorem in [1] is the most popular one. However, for the fixed-time stability theorem in [1], its proof is not a strict one, and its upper bound estimate formula for the settling time is very inaccurate. In recent years, some fixed-time stability theorems have been used to analyze the fixed-time synchronization of neural networks [7]–[11].

The synchronization of neural networks can be used in many fields, such as secure communication [12], image encryption [13], associative memory [14] and pattern recognition [15]. In the above application fields, if synchronization time needs to be considered, fixed-time synchronization will be a good choice. Take secure communication for example, the encrypted signals can be decrypted only when the drive-response systems have achieved synchronization. If fixed-time synchronization is adopted, the encrypted signals can be decrypted in a fixed settling time, which can be calculated in advance and is independent on the initial synchronization errors of the drive-response systems. In this case, fixed-time synchronization has obvious advantages over asymptotic synchronization and finite-time synchronization.

Until now, most results about the dynamic analysis of neural networks have focused on real-valued neural networks [16]–[20]. Compared with real-valued neural networks, complex-valued neural networks (CVNNs) [21]–[27]
have advantages in some special application fields, such as exclusive or operation, the detection of symmetry, electromagnetic, quantum waves, optoelectronics, filtering, speech synthesis and remote sensing [28]–[38]. CVNNs have complex parameters, and they can present strange dynamic behaviors. Since the fixed-time synchronization of CVNNs can be applied in secure communication, image encryption, associative memory, pattern recognition and other potential areas, it is also necessary to analyze the fixed-time synchronization of CVNNs. In fact, there have been some works [39], [40] about the fixed-time synchronization of complex-valued neural networks, which are all based on the fixed-time stability theorem in [1]. However, the fixed-time controllers in [39], [40] are uncontinuous ones, which may cause chattering phenomenon in engineering applications. Furthermore, the activation functions in [39], [40] are required to be bounded, and the Lyapunov functions in [39], [40] are not derivable.

Motivated by the above analysis, in this paper a new fixed-time stability theorem is established based on strict mathematical derivations, and the new fixed-time stability theorem can provide a novel upper bound estimate formula for the settling time. We theoretically prove the new upper bound estimate formula for the settling time is more accurate than that in [1]. By dividing the considered CVNNs into double-layer real-valued neural networks, the fixed-time synchronization of CVNNs is analyzed by means of the new fixed-time stability theorem. In this paper, the fixed-time controller is continuous, the activation functions are not required to be bounded, and the Lyapunov function is derivable. Numerical simulation also illustrates that the upper bound estimate formula for the settling time in this paper is more accurate than that in [1].

The remainder of this paper is organized as follows. Some necessary preliminaries are included in Section 2, and the theoretical results are obtained in Section 3. In Section 4, a numerical example is given to verify the theoretical results. The conclusion is provided in Section 5.

Notations: Throughout this paper, \( \mathbb{R}, \mathbb{C}, \mathbb{R}^+ \) and \( \mathbb{R}^n \) denote the set of real numbers, the set of complex numbers, the set of positive real numbers, and the \( n \)-dimensional Euclidean space, respectively.

II. PRELIMINARIES

Consider a complex-valued neural network (CVNN):

\[
\frac{dz_j(t)}{dt} = -d_j(z_j(t)) + \sum_{k=1}^{n} a_{jk}(z_k(t)) + I_j, \quad (1)
\]

where \( j = 1, 2, \ldots, n \), and \( z_j(t) \in \mathbb{C} \) stands for the state of neuron; \( d_j \in \mathbb{C} \) represents the neuron self-inhibition; \( f_k(\cdot) : \mathbb{C} \to \mathbb{C} \) denotes the activation function; \( I_j \in \mathbb{C} \) expresses the external input; \( a_{jk} \in \mathbb{C} \) is the connection weight.

Assumption 1: \( z_j(t), d_j, a_{jk} \) and \( I_j \) can be written as

\[
z_j(t) = x_j(t) + iy_j(t), \quad d_j = d_j^R + id_j^I, \quad a_{jk} = a_{jk}^R + ia_{jk}^I, \quad I_j = I_j^R + iI_j^I, \quad j, k = 1, 2, \ldots, n,
\]

where \( i = \sqrt{-1} \) denotes the imaginary unit, \( x_j(t), y_j(t), d_j^R, d_j^I, a_{jk}^R, a_{jk}^I, I_j^R, I_j^I \in \mathbb{R} \). \( f_k(z_k(t)) \) can be written as

\[
f_k(z_k(t)) = f_k^R(x_k(t)) + if_k^I(y_k(t)), \quad k = 1, 2, \ldots, n.
\]

where \( f_k^R(\cdot) : \mathbb{R} \to \mathbb{R} \) and \( f_k^I(\cdot) : \mathbb{R} \to \mathbb{R} \) are real-valued functions.

According to Assumption 1, CVNN (1) can be rewritten as

\[
\begin{aligned}
\frac{dx_j(t)}{dt} &= -d_j^R x_j(t) + d_j^I y_j(t) + \sum_{k=1}^{n} a_{jk}^R f_k^R(x_k(t)) \\
&\quad - \sum_{k=1}^{n} d_{jk} f_k^R(y_k(t)) + I_j^R, \\
\frac{dy_j(t)}{dt} &= -d_j^R y_j(t) - d_j^I x_j(t) + \sum_{k=1}^{n} a_{jk}^R f_k^I(y_k(t)) \\
&\quad + \sum_{k=1}^{n} d_{jk} f_k^R(x_k(t)) + I_j^I, \\
&j = 1, 2, \ldots, n.
\end{aligned}
\]

In this paper, CVNN (1) is the drive system. The response system can be expressed as

\[
\begin{aligned}
\frac{dz_j^*(t)}{dt} &= -d_j^* z_j^*(t) + \sum_{k=1}^{n} a_{jk} f_k(z_k^*(t)) + I_j + u_j(t), \\
&j = 1, 2, \ldots, n, \quad \text{where } z_j^*(t) = x_j^*(t) + iy_j^*(t) \in \mathbb{C}, \quad u_j(t) = u_j^R(t) + iu_j^I(t) \in \mathbb{C} \text{ is the controller}, \quad x_j^*(t), y_j^*(t), u_j^R(t), u_j^I(t) \in \mathbb{R}.
\end{aligned}
\]

Similarly, CVNN (3) can be rewritten as

\[
\begin{aligned}
\frac{dx_j^*(t)}{dt} &= -d_j^R x_j^*(t) + d_j^I y_j^*(t) + \sum_{k=1}^{n} a_{jk}^R f_k^R(x_k^*(t)) \\
&\quad - \sum_{k=1}^{n} d_{jk} f_k^R(y_k^*(t)) + I_j^R + u_j^R(t), \\
\frac{dy_j^*(t)}{dt} &= -d_j^R y_j^*(t) - d_j^I x_j^*(t) + \sum_{k=1}^{n} a_{jk}^R f_k^I(y_k^*(t)) \\
&\quad + \sum_{k=1}^{n} d_{jk} f_k^R(x_k^*(t)) + I_j^I + u_j^I(t), \\
&j = 1, 2, \ldots, n.
\end{aligned}
\]

The synchronization errors between CVNNs (1) and (3) are defined as \( e_j^R(t) = x_j^*(t) - x_j(t), e_j^I(t) = y_j^*(t) - y_j(t), j = 1, 2, \ldots, n \). Then the error system is

\[
\begin{aligned}
\frac{de_j^R(t)}{dt} &= -d_j^R e_j^R(t) + d_j^I e_j^I(t) + F_j^R(t) + u_j^R(t), \\
\frac{de_j^I(t)}{dt} &= -d_j^R e_j^I(t) - d_j^I e_j^R(t) + F_j^I(t) + u_j^I(t), \\
&j = 1, 2, \ldots, n, \quad \text{where}
\end{aligned}
\]

\[
F_j^R(t) = \sum_{k=1}^{n} a_{jk}^R f_k^R(x_k^*(t)) - \sum_{k=1}^{n} a_{jk}^R f_k^R(x_k(t)) \\
- \sum_{k=1}^{n} d_{jk} f_k^R(y_k^*(t)) + \sum_{k=1}^{n} d_{jk} f_k^R(y_k(t)),
\]

\[
F_j^I(t) = \sum_{k=1}^{n} a_{jk}^R f_k^I(y_k^*(t)) - \sum_{k=1}^{n} a_{jk}^R f_k^I(y_k(t)) \\
- \sum_{k=1}^{n} d_{jk} f_k^I(x_k^*(t)) + \sum_{k=1}^{n} d_{jk} f_k^I(x_k(t)).
\]
\[ F_j(t) = \sum_{k=1}^{n} a_{jk}^R f_k^R(y_k^*(t)) - \sum_{k=1}^{n} a_{jk}^R f_k^R(y_k(t)) + \sum_{k=1}^{n} a_{jk}^R f_k^R(x_k^*(t)) - \sum_{k=1}^{n} a_{jk}^R f_k^R(x_k(t)). \]

**Assumption 2:** For all \( \mu, v \in \mathbb{R} \), there exist constants \( l_k^R > 0 \) and \( I_k^R > 0 \) such that
\[
\begin{align*}
|f_k^R(\mu) - f_k^R(v)| &\leq I_k^R |\mu - v|, \\
|f_k^R(\mu) - f_k^R(v)| &\leq l_k^R |\mu - v|, \\
k &\in 1, 2, \ldots, n.
\end{align*}
\]

**Lemma 1 [41]:** If \( x_1, x_2, \ldots, x_n \geq 0, 0 < p \leq 1 \) and \( q > 1 \), then
\[
\sum_{j=1}^{n} x_j^p \geq \left( \sum_{j=1}^{n} x_j \right)^p, \quad \sum_{j=1}^{n} x_j^q \geq n^{1-q} \left( \sum_{j=1}^{n} x_j \right)^q.
\]

Consider a nonlinear system:
\[
\dot{x}(t) = f(x(t)),
\]
where \( x(t) \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function. Assume the origin is an equilibrium point of system (6).

**Definition 1:** The origin of system (6) is finite-time stable, if there exists a constant \( T(0) > 0 \) satisfying \( \lim_{t \to T(0)} \|x(t)\| = 0 \) and \( \|x(t)\| \equiv 0 \) for \( t > T(0) \), where \( T(x(0)) \) is the settling time.

**Definition 2:** The origin of system (6) is fixed-time stable, if two conditions can be satisfied: (i) The origin of system (6) is finite-time stable; (ii) For any \( x(0) \), there exists a fixed constant \( T_{\text{max}} > 0 \) such that \( T(x(0)) \leq T_{\text{max}} \).

**Lemma 2 [1]:** If \( V(\cdot) : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\} \) is a continuous function, and it satisfies
(i) \( V(x(t)) = 0 \iff x(t) = 0 \);
(ii) For any nonzero solution \( x(t) \) of system (6), there exist \( \alpha, \beta > 0, 0 \leq \xi < 1 \) and \( \eta > 1 \) such that
\[
\dot{V}(x(t)) \leq -\alpha V^\xi(x(t)) - \beta V^\eta(x(t)),
\]
then the origin of system (6) is fixed-time stable, and
\[
T_{\text{max}}^1 = \frac{1}{\alpha(1-\xi)} + \frac{1}{\beta(\eta-1)}.
\]

**III. MAIN RESULTS**

Firstly, based on variable substitution, calculating definite integral and solving the minimization problem, we establish a new fixed-time stability theorem, which can provide a novel upper bound estimate formula for the settling time.

**Theorem 1:** If \( V(\cdot) : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\} \) is a continuous function, and it satisfies
(i) \( V(x(t)) = 0 \iff x(t) = 0 \);
(ii) For any nonzero solution \( x(t) \) of system (6), there exist \( \alpha, \beta > 0, 0 \leq \xi < 1 \) and \( \eta > 1 \) such that
\[
\dot{V}(x(t)) \leq -\alpha V^\xi(x(t)) - \beta V^\eta(x(t)),
\]
then the origin of system (6) is fixed-time stable, and
\[
T_{\text{max}}^2 = \frac{1}{\alpha(1-\xi)} \left( \frac{\alpha}{\beta} \right)^{1-\xi} + \frac{1}{\beta(\eta-1)} \left( \frac{\alpha}{\beta} \right)^{1-\eta}.
\]

**Proof:** Let \( W(s) = V^{1-\xi}(s) \), then it follows that
\[
\dot{W}(s) = \frac{1}{1-\xi} \dot{W}(s)V^\xi(s).
\]
Since \( \dot{W}(x(t)) \leq -\alpha V^\xi(x(t)) - \beta V^\eta(x(t)), \) we have
\[
\frac{1}{1-\xi} \dot{W}(x(t))V^\xi(x(t)) \leq -\alpha V^\xi(x(t)) - \beta V^\eta(x(t)).
\]

Then we can obtain that
\[
\dot{W}(x(t)) \leq (1-\xi) \left[ -\alpha - \beta V^\eta(x(t)) \right] \leq -\alpha(1-\xi).
\]

Since \( \dot{W}(x(t)) \leq -\alpha(1-\xi), \) there exists a constant \( T(0) = \frac{W(x(0))}{\alpha(1-\xi)} > 0 \) satisfying \( \lim_{t \to T(x(0))} \|x(t)\| = 0 \) and \( \|x(t)\| \equiv 0 \) for \( t > T(x(0)) \). Based on Definition 1, the origin of system (6) is finite-time stable.

Obviously, \( T(x(0)) = \int_0^{T(0)} dt \). Since \( dt > 0 \) and \( \dot{V}(x(t)) = \frac{dV(x(t))}{dt} \leq -\alpha V^\xi(x(t)) - \beta V^\eta(x(t)) < 0 \), we have
\[
dt \leq \frac{1}{\alpha(1-\xi)} \left( \frac{\alpha}{\beta} \right)^{1-\xi}.
\]

Let \( w = V(x(t)). \) As we know, if \( t = T(x(0)), \|x(t)\| = w \). Therefore,
\[
T(x(0)) = \int_0^{T(x(0))} dt \leq \int_0^{V(x(0))} \frac{1}{\alpha w^\xi + \beta w^\eta} dw.
\]

Next we discuss two cases separately:
(i) If \( 0 < V(x(0)) \leq \left( \frac{\alpha}{\beta} \right)^{1-\xi} \),
\[
T(x(0)) \leq \int_0^{V(x(0))} \frac{1}{\alpha w^\xi + \beta w^\eta} dw = \left( V(x(0)) \right)^{1-\xi} \frac{1}{\alpha(1-\xi)} \leq \int_0^{V(x(0))} \frac{1}{\alpha w^\xi + \beta w^\eta} dw.
\]

(ii) If \( V(x(0)) > \left( \frac{\alpha}{\beta} \right)^{1-\xi} \),
\[
T(x(0)) \leq s \frac{1}{\alpha s^\xi + \beta s^\eta} dw + \int_s^{T(x(0))} \frac{1}{\alpha w^\xi + \beta w^\eta} dw.
\]

\[
= \frac{s^{1-\xi}}{\alpha(1-\xi)} \left[ s^{1-\eta} + \frac{1}{\beta(\eta-1)} \right] \left( V(x(0)) \right)^{1-\eta} \leq \frac{s^{1-\xi}}{\alpha(1-\xi)} + \frac{1}{\beta(\eta-1)},
\]
where \( s > 0 \) is an undetermined variable.
Now let
\[ \Phi(s) = \frac{s^{1-\xi}}{\alpha(1-\xi)} + \frac{s^{1-\eta}}{\beta(\eta-1)}, \ s > 0, \]
then we get that
\[ \Phi(s) = \frac{s^{-\xi}}{\alpha} - \frac{s^{-\eta}}{\beta}, \ s > 0. \]

It is obvious that \( \Phi(s^*) = 0 \) implies \( s^* = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} \).

As we know, \( \Phi(s) > 0 \) for \( s \in (s^*, +\infty) \), and \( \Phi(s) < 0 \) for \( s \in (0, s^*) \). Therefore, function \( \Phi(s) \) obtains its minimum value at \( s^* = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} \), and the minimum value is
\[ \frac{1}{\alpha(1-\xi)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} + \frac{1}{\beta(\eta-1)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}}. \]
That means
\[ T(s(0)) \leq \int_{0}^{V(x(0))} \frac{1}{\alpha w^{\xi} + \beta w^{\eta}} dw + \int_{V(x(0))}^{V(\xi)} \frac{1}{\alpha w^{\xi} + \beta w^{\eta}} dw \]
\[ \leq \frac{1}{\alpha(1-\xi)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} + \frac{1}{\beta(\eta-1)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}}. \]

Based on Definition 2, the origin of system (6) is fixed-time stable, and
\[ T^2_{\max} = \frac{1}{\alpha(1-\xi)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} + \frac{1}{\beta(\eta-1)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}}. \]
This completes the proof.

The next theorem shows that, compared with Lemma 2, Theorem 1 can give a more accurate estimate.

**Theorem 2:** \( T^2_{\max} < T^1_{\max} \) for \( \alpha \neq \beta \), and \( T^2_{\max} = T^1_{\max} \) for \( \alpha = \beta \).

**Proof:** Without loss of generality, we set \( V(x(0)) > \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} \).
Since
\[ \Phi(s) = \frac{s^{1-\xi}}{\alpha(1-\xi)} + \frac{s^{1-\eta}}{\beta(\eta-1)}, \ s^* = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}}, \]
we have
\[ \Phi(s^*) = \frac{1}{\alpha(1-\xi)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} + \frac{1}{\beta(\eta-1)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} = T^2_{\max}, \]
\[ \Phi(1) = \frac{1}{\alpha(1-\xi)} + \frac{1}{\beta(\eta-1)} = T^1_{\max}. \]
If \( \alpha \neq \beta \), \( s^* = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} \neq 1 \). Since \( \Phi(s) \) can only obtain its minimum value at \( s^* \), we have \( \Phi(s^*) < \Phi(1) \), that means \( T^2_{\max} < T^1_{\max} \).
If \( \alpha = \beta \), \( s^* = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}} = 1 \). Therefore, \( \Phi(s^*) = \Phi(1) \), that means \( T^2_{\max} = T^1_{\max} \).

This completes the proof.

**Remark 1:** Obviously, the fixed-time stability criterion in Theorem 1 is the same as that in Lemma 2. However, the upper bound estimate formula for the settling time in Theorem 1 is more accurate than that in Lemma 2. In fact, deriving a more accurate upper bound estimate formula for the settling time isn’t an easy work due to the difficulty of theoretical analysis. That is why the fixed-time stability theorem in Lemma 2 has been the most popular one for so many years.

Let \( e(t) = (e_1^R(t), \ldots, e_n^R(t), e_1^l(t), \ldots, e_n^l(t))^T \). In order to analyze the fixed-time synchronization of CVNNs (1) and (3), we design the following controller:
\[
\begin{align*}
    u^R_j(t) &= -\delta_{j1} e^R_j(t) - \sigma_{j1} \text{sign}(e^R_j(t)) |e^R_j(t)|^\xi - \rho_{j1} \text{sign}(e^R_j(t)) |e^R_j(t)|^\eta, \\
    u^l_j(t) &= -\delta_{j2} e^l_j(t) - \sigma_{j2} \text{sign}(e^l_j(t)) |e^l_j(t)|^\xi - \rho_{j2} \text{sign}(e^l_j(t)) |e^l_j(t)|^\eta.
\end{align*}
\]
where \( \delta_{j1} \) and \( \delta_{j2} \) are undetermined constants, \( \sigma_{j1}, \rho_{j1}, \sigma_{j2} \) and \( \rho_{j2} \) are arbitrary positive constants, \( 0 \leq \xi < 1 \) and \( \eta > 1 \).

**Theorem 3:** Suppose Assumption 1 and Assumption 2 hold. If control gains \( \delta_{j1} \) and \( \delta_{j2} \) satisfy
\[
\begin{align*}
    \delta_{j1} &\geq -d_j^R + \left| d_j^l \right| + \frac{1}{2} \sum_{k=1}^{n} |d_{jk}^R|^2 l_k^R + \frac{1}{2} \sum_{k=1}^{n} |d_{jk}^l|^2 l_j^l, \\
    \delta_{j2} &\geq -d_j^l + \left| d_j^l \right| + \frac{1}{2} \sum_{k=1}^{n} |d_{jk}^R|^2 l_k^R + \frac{1}{2} \sum_{k=1}^{n} |d_{jk}^l|^2 l_j^l.
\end{align*}
\]
\( j = 1, 2, \ldots, n \), CVNNs (1) and (3) can reach fixed-time synchronization under controller (7). Additionally,
\[ T_{\max} = \frac{2}{\sigma(1-\xi)} + \frac{1}{\rho(\eta-1)} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\eta-\xi}}, \]
where \( \sigma = \min \{ \min_{j} |\sigma_{j1}|, \min_{j} |\sigma_{j2}| \} 2^{\frac{1+\xi}{\eta}} \), \( \rho = \min \{ \min_{j} |\rho_{j1}|, \min_{j} |\rho_{j2}| \} n^{\frac{1+\xi}{\eta}} \).

**Proof:** Choose a Lyapunov function:
\[ V(e(t)) = \frac{1}{2} e^T(t) e(t) = V_1(t) + V_2(t), \]
where
\[ V_1(t) = \frac{1}{2} \sum_{j=1}^{n} (e^R_j(t))^2, \ V_2(t) = \frac{1}{2} \sum_{j=1}^{n} (e^l_j(t))^2. \]
Calculate the derivative of $V_1(t)$:

$$
\dot{V}_1(t) = \sum_{j=1}^{n} e_j^R(t) \left[ -d_j^R e_j^R(t) + d_j^l e_j^l(t) + F_j^R(t) + u_j^R(t) \right]
$$

$$
= -\sum_{j=1}^{n} d_j^R (e_j^R(t))^2 + \sum_{j=1}^{n} d_j^l (e_j^l(t))^2 + \sum_{j=1}^{n} e_j^R(t) F_j^R(t) + \sum_{j=1}^{n} e_j^R(t) u_j^R(t).
$$

It can be obtained that

$$
\sum_{j=1}^{n} d_j^l e_j^l(t) e_j^l(t) \leq \sum_{j=1}^{n} d_j^l \cdot \frac{(e_j^R(t))^2 + (e_j^l(t))^2}{2}
$$

$$
= \sum_{j=1}^{n} \frac{d_j^l}{2} (e_j^R(t))^2 + \sum_{j=1}^{n} \frac{d_j^l}{2} (e_j^l(t))^2.
$$

Based on Assumption 2, we have

$$
|F_j^R(t)| \leq \sum_{k=1}^{n} |a_{jk}^R| \cdot l_k^R \cdot |e_k^R(t)| + \sum_{k=1}^{n} |a_{jk}^l| \cdot l_k^l \cdot |e_k^l(t)|,
$$

$$
|F_j^l(t)| \leq \sum_{k=1}^{n} |a_{jk}^R| \cdot l_k^l \cdot |e_k^l(t)| + \sum_{k=1}^{n} |a_{jk}^l| \cdot l_k^R \cdot |e_k^R(t)|,
$$

$j = 1, 2, \ldots, n$.

Then we can obtain that

$$
\sum_{j=1}^{n} e_j^R(t) F_j^R(t)
$$

$$
\leq \sum_{j=1}^{n} |e_j^R(t)| \sum_{k=1}^{n} |a_{jk}^R| \cdot l_k^R \cdot |e_k^R(t)|
$$

$$
+ \sum_{j=1}^{n} |e_j^R(t)| \sum_{k=1}^{n} |a_{jk}^l| \cdot l_k^l \cdot |e_k^l(t)|
$$

$$
\leq \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}^R| \cdot l_k^R \cdot \frac{(e_j^R(t))^2 + (e_k^R(t))^2}{2}
$$

$$
+ \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk}^l| \cdot l_k^l \cdot \frac{(e_j^R(t))^2 + (e_k^l(t))^2}{2}
$$

$$
= \sum_{j=1}^{n} \left( \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^R + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^l \right) (e_j^R(t))^2
$$

$$
+ \sum_{j=1}^{n} \left( \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^l \right) (e_j^l(t))^2
$$

$$
+ \sum_{j=1}^{n} \left( \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^R \right) (e_j^l(t))^2.
$$

On the other hand,

$$
\sum_{j=1}^{n} e_j^R(t) u_j^R(t) = -\sum_{j=1}^{n} \delta_j (e_j^R(t))^2
$$

$$
- \sum_{j=1}^{n} \sigma_j |e_j^R(t)|^{|\xi+1|} - \sum_{j=1}^{n} \rho_j |e_j^R(t)|^{|\eta+1|}.
$$

Therefore,

$$
\dot{V}_1(t) \leq \sum_{j=1}^{n} \left( -d_j^R + \frac{d_j^l}{2} \right) \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^R
$$

$$
+ \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^l + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^R - \delta_j (e_j^R(t))^2
$$

$$
+ \sum_{j=1}^{n} \left( \frac{d_j^l}{2} + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^R \right) (e_j^l(t))^2
$$

$$
- \sum_{j=1}^{n} \sigma_j |e_j^R(t)|^{|\xi+1|} - \sum_{j=1}^{n} \rho_j |e_j^R(t)|^{|\eta+1|}.
$$

Similarly, we have

$$
\dot{V}_2(t) = \sum_{j=1}^{n} e_j^l(t) \left[ -d_j^l e_j^l(t) - d_j^R e_j^R(t) + F_j^l(t) + u_j^l(t) \right]
$$

$$
\leq \sum_{j=1}^{n} \left( -d_j^l + \frac{d_j^R}{2} \right) \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^l + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^R
$$

$$
+ \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^R - \delta_j (e_j^l(t))^2
$$

$$
+ \sum_{j=1}^{n} \left( \frac{d_j^R}{2} + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^l \right) (e_j^R(t))^2
$$

$$
- \sum_{j=1}^{n} \sigma_j |e_j^l(t)|^{|\xi+1|} - \sum_{j=1}^{n} \rho_j |e_j^l(t)|^{|\eta+1|}.
$$

Then it follows that

$$
\dot{V}(e(t))
$$

$$
\leq \sum_{j=1}^{n} \left( -d_j^R + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^R + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^l \right)
$$

$$
+ \left| \frac{d_j^l}{2} + \frac{1}{2} \sum_{k=1}^{n} a_{jk}^l l_k^l \right| + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^R - \delta_j (e_j^R(t))^2
$$

$$
+ \sum_{j=1}^{n} \left( -d_j^R + \frac{d_j^l}{2} \right) \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^R| l_k^R + \frac{1}{2} \sum_{k=1}^{n} |a_{jk}^l| l_k^l
$$

$$
- \sum_{j=1}^{n} \sigma_j |e_j^R(t)|^{|\xi+1|} - \sum_{j=1}^{n} \rho_j |e_j^R(t)|^{|\eta+1|}.
$$

$$
- \sum_{j=1}^{n} \sigma_j |e_j^l(t)|^{|\xi+1|} - \sum_{j=1}^{n} \rho_j |e_j^l(t)|^{|\eta+1|}.
$$
Obviously, \(0 < \frac{\xi + 1}{2} < 1\) and \(\frac{\eta + 1}{2} > 1\). Let \(\sigma_1 = \min(\sigma_{ij})\), \(\sigma_2 = \min(\sigma_{ij})\), \(\rho_1 = \min(\rho_{ij})\) and \(\rho_2 = \min(\rho_{ij})\). If \(\delta_{ij}\) and \(\delta_{ij}\) satisfy the conditions given in Theorem 3, it can be derived from Lemma 1 that

\[
V(e(t)) \leq -\sigma_1 \sum_{j=1}^{n} \left( |e_j(t)|^2 \right)^{\frac{\xi + 1}{2}} - \sigma_2 \sum_{j=1}^{n} \left( |e_j(t)|^2 \right)^{\frac{\eta + 1}{2}} - \rho_1 \sum_{j=1}^{n} \left( |e_j(t)|^2 \right)^{\frac{\eta + 1}{2}} - \rho_2 \sum_{j=1}^{n} \left( |e_j(t)|^2 \right)^{\frac{\eta + 1}{2}}
\]

\[
= -\sigma_1 \left( |V_1(t)|^{\frac{\xi + 1}{2}} + |V_2(t)|^{\frac{\xi + 1}{2}} \right) - \sigma_2 \left( |V_1(t)|^{\frac{\eta + 1}{2}} + |V_2(t)|^{\frac{\eta + 1}{2}} \right) - \rho_1 \left( |V_1(t)|^{\frac{\eta + 1}{2}} + |V_2(t)|^{\frac{\eta + 1}{2}} \right) - \rho_2 \left( |V_1(t)|^{\frac{\eta + 1}{2}} + |V_2(t)|^{\frac{\eta + 1}{2}} \right)
\]

Based on Theorem 1, the origin of error system (5) is fixed-time stable, which means CVNNs (1) and (3) can reach fixed-time synchronization. Additionally,

\[
T_{\text{max}} = \frac{1}{\sigma(1 - \frac{\xi + 1}{2})} \left( \frac{\sigma}{2\rho} \right)^{\frac{1-\frac{\xi + 1}{2}}{1-\frac{\eta + 1}{2}}}
+ \frac{1}{2\rho(\frac{\eta + 1}{2} - 1)} \left( \frac{\sigma}{2\rho} \right)^{\frac{1-\frac{\xi + 1}{2}}{1-\frac{\eta + 1}{2}}}
= \frac{1}{\sigma(1 - \frac{\xi}{2})} \left( \frac{\sigma}{2\rho} \right)^{\frac{\xi}{\eta + 2}} + \frac{1}{\rho(\eta - 1)} \left( \frac{\sigma}{2\rho} \right)^{\frac{\xi}{\eta + 2}}.
\]

This completes the proof.

Remark 2: Theorem 3 shows, the fixed-time synchronization scheme in this paper is theoretically feasible if Assumption 1 and Assumption 2 hold. The fixed-time controller (7) is a concise continuous controller, which can be implemented in the engineering easily. In the proof of Theorem 3, we only use some basic operations, such as taking derivatives and inequality techniques. By simple algebraic operations, the fixed-time synchronization criteria can be verified, and \(T_{\text{max}}\) can be calculated. The computation complexity of verifying the synchronization criteria and calculating \(T_{\text{max}}\) can be \(O(n)\).

Based on Lemma 2, we can obtain the following corollary:

**Corollary 1:** Suppose Assumption 1 and Assumption 2 hold. If control gains \(\delta_{ij}\) and \(\delta_{ij}\) satisfy

\[
delta_{ij} \geq -d_{ij}^R + \left| d_j^R \right| + \frac{1}{2} \sum_{k=1}^{n} \left| a_{jk}^R \right| \theta_k^R + \frac{1}{2} \sum_{k=1}^{n} \left| a_{kj}^R \right| \theta_j^R,
\]

\[
\delta_{ij} \geq -d_{ij}^R + \left| d_j^R \right| + \frac{1}{2} \sum_{k=1}^{n} \left| a_{kj}^R \right| \theta_k^R + \frac{1}{2} \sum_{k=1}^{n} \left| a_{jk}^R \right| \theta_j^R,
\]

\[
f = 1, 2, \ldots, n, \text{ CVNNs} (1) \text{ and } (3) \text{ can reach fixed-time synchronization under controller (7). Additionally,}
\]

\[
T_{\text{max}} = \frac{2}{\sigma(1 - \frac{\xi}{2})} + \frac{1}{\rho(\eta - 1)},
\]

\[
\rho = \min(\min(\rho_{ij}), \min(\rho_{ij})) \frac{1}{n} \frac{\xi}{\eta + 2}.
\]

Remark 3: In fact, there is another fixed-time stability theorem which has been used frequently. In [6], the authors established a fixed-time stability theorem similar to Lemma 2, and the corresponding \(T_{\text{max}}\) is also smaller than that in Lemma 2. In Lemma 2 and Theorem 1, \(\xi\) and \(\eta\) should satisfy \(0 \leq \xi < 1\) and \(\eta > 1\), while \(\xi\) and \(\eta\) must satisfy \(\xi = 1 - \frac{1}{\eta}\) and \(\eta = 1 + \frac{1}{2\eta} (\mu > 1)\) in [6]. Obviously, the assumptions on \(\xi\) and \(\eta\) in [6] are stronger than that in Lemma 2 and Theorem 1, which limits the applicability of the fixed-time stability theorem in [6] in some sense.

**Remark 4:** It should be pointed out that, the upper bound estimate formula for the settling time in this paper isn’t the minimum one. In [42], by introducing beta function and gamma function the authors derived the minimum upper bound estimate formula for the settling time, and the corresponding fixed-time stability criterion was the same as that in Lemma 2 and Theorem 1. Undeniably, the result in Theorem 1 is inferior to that in [42]. However, the analysis techniques in Theorem 1 are new, and they can provide a new perspective for studying the relevant problems.

**Remark 5:** In [4], the authors proposed some novel fixed-time stability theorems with implicit Lyapunov functions. In [5], the authors proposed some novel fixed-time stability theorems, which required the Lyapunov functions satisfied switching conditions. However, in practical applications, it is difficult to choose suitable controllers and analysis techniques to match the above-mentioned fixed-time stability theorems.

**Remark 6:** There have been some works [39], [40] about the fixed-time synchronization of complex-valued neural networks, which are all based on the fixed-time stability theorem in [1]. However, the fixed-time controllers in [39], [40] are
uncontinuous ones, which may cause chattering phenomenon in engineering applications. Furthermore, the activation functions in [39], [40] are required to be bounded, and the Lyapunov functions in [39], [40] are not derivable. In this paper, we study the fixed-time synchronization of complex-valued neural networks based on the fixed-time stability theorem developed by ourselves. The new fixed-time stability theorem can provide a novel upper bound estimate formula for the settling time, which is tighter than that in [1]. Variable substitution, calculating definite integral and solving the minimization problem have been used in the proof of the new fixed-time stability theorem, which is very different from the proof of the fixed-time stability theorem in [1]. In this paper, the fixed-time controller is continuous, the activation functions are not required to be bounded, and the Lyapunov function is derivable.

Remark 7: In [43], the global asymptotical synchronization of fractional-order complex-valued neural networks with time delay was investigated. In [44], the global asymptotical synchronization of fractional-order quaternion-valued neural networks with leakage and discrete delays was studied. Undeniably, the neural network models in [43], [44] are more general than the neural network model in this paper. However, since there exists fractional-derivative in the neural network models in [43], [44], it is impossible to analyze their fixed-time synchronization.

IV. NUMERICAL SIMULATIONS

In this section, a specific numerical example is given to verify the effectiveness of the main results.

Example 1: Consider a CVNN model:

\[
\frac{dz_k(t)}{dt} = -d_3z_k(t) + \sum_{k=1}^{4}a_{jk}f_k(z_k(t)) + I_j, \quad (8)
\]

\[
j = 1, 2, 3, 4, \text{ where } d_1 = 2.5 + 1.9i, d_2 = 3 + 1.8i, d_3 = 0.3 + 2i, d_4 = 0.1 + 0.5i, a_{11} = 2 + 1.2i, a_{12} = -1 - 4.5i, a_{13} = i, a_{14} = -1.5 + i, a_{21} = 1 + 1.7i, a_{22} = 0.8 + 0.9i, a_{23} = 3 - 2i, a_{24} = 5, a_{31} = 0.6 - i, a_{32} = 2 + 0.5i, a_{33} = -0.1 + 3i, a_{34} = -1 - i, a_{41} = -0.3, a_{42} = 1.1i, a_{43} = 0.5 + i, a_{44} = 0, I_1 = \sin t + i\sin t, I_2 = 0, I_3 = \sin t + i\sin t, I_4 = 0, \]

The response system is

\[
\frac{dz^*_k(t)}{dt} = -d_3z^*_k(t) + \sum_{k=1}^{4}a_{jk}f_k(z^*_k(t)) + I_j + u_j(t). \quad (9)
\]

\[
j = 1, 2, 3, 4, \text{ where } z^*_j(t) = x^*_j(t) + iy^*_j(t), u_j(t) = \frac{d}{dt}I_j + iu^*_j(t) \text{ is the controller, } x^*_j(t), y^*_j(t), \frac{d}{dt}I_j, u^*_j(t) \in \mathbb{R}. \]

The evolutions of the synchronization errors are defined as \(e^{R(j)}_j(t) = x^{R(j)}_j(t) - x_j(t), e^{I(j)}_j(t) = y^{R(j)}_j(t) - y_j(t), j = 1, 2, 3, 4.\)

Fig.2 illustrates the evolutions of the synchronization errors without control.

For controller (7), we choose \(\delta_{11} = \delta_{21} = 9.4, \delta_{12} = \delta_{22} = 11.4, \delta_{13} = \delta_{23} = 11.6, \delta_{14} = \delta_{24} = 6.6, \sigma_{11} = \sigma_{12} = \sigma_{13} = \sigma_{14} = \sigma_{21} = \sigma_{22} = \sigma_{23} = \sigma_{24} = 0.5, \rho_{11} = \rho_{12} = \rho_{13} = \rho_{14} = \rho_{21} = \rho_{22} = \rho_{23} = \rho_{24} = 2, \xi = 0.3 \text{ and } \eta = 2.\)

Obviously, \(\sigma_{1j}, \sigma_{2j}, \rho_{1j}, \rho_{2j}, \xi \text{ and } \eta \) satisfy the conditions given in controller (7), \(j = 1, 2, 3, 4.\) It can also be verified that \(\delta_{1j} \text{ and } \delta_{2j} \) satisfy the conditions given in Theorem 3, \(j = 1, 2, 3, 4.\)

Based on Theorem 3, CVNNs (8) and (9) can reach fixed-time synchronization under controller (7), and \(T_{max} \approx 4.2112.\) Fig.3 and Fig.4 illustrate the evolutions of the synchronization errors under controller (7), which verifies that CVNNs (8) and (9) can reach synchronization within \(T_{max}.\)

Remark 8: Based on Corollary 1, CVNNs (8) and (9) can reach fixed-time synchronization under controller (7), and \(T_{max} \approx 4.6416 \approx 4.2112.\) As we know, Theorem 3 is based on Theorem 1, while Corollary 1 is based on Lemma 2. Example 1 shows that, compared with Lemma 2, Theorem 1 can give a more accurate estimate.

Remark 9: As we know, chaos synchronization can be used in secure communication, and the encrypted signals can be decrypted correctly only when synchronization has been reached. The fixed-time synchronization scheme in this
paper can also be used in secure communication, and the encrypted signals can be decrypted correctly irrespective of the initial synchronization errors of the drive-response CVNNs after the upper bound estimate for the settling time. Therefore, based on Example 1, the encrypted signals can be decrypted correctly since 4.2112 according to Theorem 3, while the encrypted signals can be decrypted correctly since 4.6416 according to Corollary 1.

V. CONCLUSION

In this paper, we establish a new fixed-time stability theorem based on variable substitution, calculating definite integral and solving the minimization problem. The new fixed-time stability theorem can provide a novel upper bound estimate formula for the settling time. By dividing the considered CVNNs into double-layer real-valued neural networks, the fixed-time synchronization of CVNNs is analyzed by means of the new fixed-time stability theorem. Both theoretical derivation and numerical simulation show that the upper bound estimate formula for the settling time in this paper is more accurate than that given in the classic fixed-time stability theorem. A specific numerical example is provided to verify the effectiveness of the main results.

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