Metric fluctuations and decoherence

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Abstract
Recently, a model of metric fluctuations has been proposed which yields an effective Schrödinger equation for a quantum particle with a modified inertial mass, leading to a violation of the weak equivalence principle. The renormalization of the inertial mass tensor results from a local space average over the fluctuations of the metric over a fixed background metric. Here, we demonstrate that the metric fluctuations of this model lead to a further physical effect, namely to an effective decoherence of the quantum particle. We derive a quantum master equation for the particle’s density matrix, discuss in detail its dissipation and decoherence properties, and estimate the corresponding decoherence timescales. By contrast to other models discussed in the literature, in the present approach the metric fluctuations give rise to a decay of the coherences in the energy representation, i.e., to a localization in energy space.

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1. Introduction

The search for a quantum theory of gravity is one of the main challenges of theoretical physics. Though until now there is no final version of a quantum gravity theory it is expected that one of the consequences of such a theory is the appearance of some kind of spacetime fluctuations, of a spacetime foam. These fluctuations may be given by fluctuations of the spacetime metric or of the connection if independent of the metric. This may include even changes in the topology. In the most simple version, one may think of a Minkowskian background with small metrical fluctuations. Within a semi-classical theory of quantum gravity, spacetime fluctuations are also expected to be a consequence of fluctuations of matter fields [1].

Since spacetime is the arena where all physical phenomena take place it is clear that all phenomena will be influenced by a fluctuating spacetime metric. At the first place, the propagation of light will be influenced by a fluctuating metric and will lead to fluctuating light cones and to a blurring of light signals, e.g., to angular and redshift blurring [2]. Since within general relativity lengths are defined through the time-of-flight of propagating light it is also clear that a fluctuating metric defines a fundamental length scale and will lead to a bound
on the sharpness of length measurements [3, 4] which also will add additional fundamental noise into gravitational wave detectors [5]. This already initiated an experimental search for a fundamental metrical noise in optical cavities [6]. Furthermore, metrical fluctuations have been shown to lead to a modified inertial mass in an effective Schrödinger equation derived from a Klein–Gordon equation minimally coupled to the spacetime metric [7]. Since fluctuations in space and time also will lead effectively to non-localities (in the sense of higher-order derivatives) in field equations it is also clear that spacetime fluctuations will also emerge in modified dispersion relations as it has been discussed first in [8, 9]. A number of master equations describing intrinsic decoherence in quantum mechanics are reviewed and discussed in a paper by Diósi [10]. Finally, in a recent paper by Wang and coworkers [11] it has been shown that spacetime fluctuations may be regarded as explanation for the cosmological constant.

In this paper, we further investigate the consequences of the model introduced in [7]. We show that beside a modification of the inertial mass followed by a violation of the weak equivalence principle, the spacetime fluctuations also will lead to an effective decoherence of a quantum system.

Quantum gravity induced decoherence of quantum systems has been considered in [12, 13]. Decoherence also appears in discretized quantum gravity scenarios [14]. The authors of [12] regard spacetime fluctuations as incoherent conformal waves which are produced by quantum-mechanical zero point fluctuations of a conformal field. The nonlinear contribution of this field causes a decoherence of quantum wave packets which yields a lower bound for a parameter which defines the borderline on which quantum to classical transition of gravity takes place. Generally, the value for the parameter is model dependent. In [13], the authors generalize this approach accounting for spin-2 gravitational waves yielding a more optimistic lower bound on the transition parameter which is well within an expected range for low-energy quantum gravity. They conclude that effects of quantum fluctuations of spacetime causing matter waves to lose coherence are worth to be explored with high-sensitivity matter wave interferometers.

2. Quantum field in a fluctuating spacetime metric

In [7], we considered a Klein–Gordon field minimally coupled to a spacetime metric $g_{\mu\nu}$. We assumed that this metric consists of a Minkowskian background $\eta_{\mu\nu} = \text{diag}(-+++)$ and a small fluctuating part $|h_{\mu\nu}(x,t)| \ll 1$ so that $g_{\mu\nu}(x,t) = \eta_{\mu\nu} + h_{\mu\nu}(x,t)$. We furthermore assumed that for the average over spacetime intervals $\langle h_{\mu\nu}(x,t) \rangle = f_{\mu\nu}$, where all $f_{\mu\nu} = 0$ except $f_{00}$ which we identify as a Newtonian potential.

Performing a non-relativistic limit of the Klein–Gordon equation we obtained an effective Schrödinger equation of the form

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} [H_0 + H_p(t)] |\psi(t)\rangle,$$

where $(i, j = 1, 2, 3)$

$$H_0 = \frac{p^2}{2m} - mU, \quad H_p(t) = \frac{1}{2m} a^{ij}(t)p_ip_j, \quad p_i = -i\hbar \partial_i.$$

Here, the metrical fluctuations are encoded in the tensorial function $a^{ij}(t)$ which is the spatial average of squares of the metrical fluctuations which in our approach are assumed to consist of wavelengths short compared with the Compton wavelength of the particle under consideration.

The tensorial function $a^{ij}(t)$ is then split into an average part and a fluctuating part

$$a^{ij}(t) = \bar{a}^{ij} + \gamma^{ij}(t) \quad \text{with} \quad \langle \gamma^{ij}(t) \rangle = 0,$$

with

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} \left( \frac{p^2}{2m} - mU + \frac{1}{2m} a^{ij}(t)p_ip_j \right) |\psi(t)\rangle.$$
where the average is denoted by angular brackets. In [7] the part $\tilde{\alpha}_{ij}(t)$ has been shown to lead to a redefinition of the inertial mass of the quantum field under consideration. This would imply a breakdown of the weak equivalence principle which may reach a level of $10^{-9}$ in terms of the Eötvös parameter.

In the following, we discuss the implications of the remaining fluctuating part $\gamma_{ij}(t)$. We show that this term leads to an effective decoherence of the quantum system.

3. Derivation of the quantum master equation

3.1. General form of the master equation

Having redefined the inertial mass of the particle as described in [7] we are left with an effective Schrödinger equation of the form (1) where, however, only the fluctuating part $\gamma_{ij}(t)$ enters the Hamiltonian $H_p(t)$,

$$H_p(t) = \frac{1}{2m} \gamma_{ij}(t) p_i p_j,$$

while $H_0$ is defined as in equation (2) with an appropriately renormalized inertial mass in the kinetic term. We start by transforming to the interaction picture,

$$|\tilde{\psi}(t)\rangle = e^{-iH_0 t/\hbar} |\tilde{\psi}(t)\rangle,$$

(5)

to obtain the Schrödinger equation

$$\frac{d}{dt} |\tilde{\psi}(t)\rangle = -\frac{i}{\hbar} \tilde{H}_p(t) |\tilde{\psi}(t)\rangle,$$

(6)

where the interaction picture Hamiltonian is given by

$$\tilde{H}_p(t) = e^{iH_0 t/\hbar} H_p(t) e^{-iH_0 t/\hbar}.$$

(7)

Formally, equation (6) can be regarded as a stochastic Schrödinger equation (SSE) involving a random Hamiltonian $\tilde{H}_p(t)$ with zero average, $\langle \tilde{H}_p(t) \rangle = 0$. For a given realization of the random process $\gamma_{ij}(t)$ the corresponding solution of the SSE represents a pure state with the density matrix $\tilde{R}(t) = |\tilde{\psi}(t)\rangle \langle \tilde{\psi}(t)|$,

satisfying the von Neumann equation

$$\frac{d}{dt} \tilde{R}(t) = -\frac{i}{\hbar} [\tilde{H}_p(t), \tilde{R}(t)] \equiv \mathcal{L}(t) \tilde{R}(t),$$

(9)

where $\mathcal{L}(t)$ denotes the Liouville superoperator. However, if we consider the average over the fluctuations of $\gamma_{ij}(t)$ the corresponding solution of the SSE represents a pure state with the density matrix

$$\tilde{\rho}(t) = \langle \tilde{\psi}(t) \rangle \langle \tilde{\psi}(t)|,$$

(10)

generally represents a mixed quantum state. Thus, when considering averages, the dynamics given by the SSE transforms pure states into mixtures and leads to dissipation and decoherence processes, i.e., a loss of quantum coherence. Consequently, the time evolution of $\tilde{\rho}(t)$ is no longer given by a unitary transformation, but must be described through a dissipative quantum-dynamical map that preserves the Hermiticity, the trace and the positivity of the density matrix [15].

An efficient way of describing a quantum-dynamical map consists in the formulation of an appropriate master equation for the density matrix $\tilde{\rho}(t)$. Thus, our goal is to derive from the linear stochastic differential equation (9) an equation of motion for the average given by equation (10). A standard approach to this problem is provided by the cumulant expansion.
method in which the equation of motion for $\tilde{\rho}(t)$ is represented by means of an expansion in terms of the ordered cumulants of the Liouville superoperator $\hat{L}(t)$ [16]. This method is widely used in the treatment of stochastic differential equations and in the theory of open quantum systems [15]. To second order in the strength of the fluctuations it yields the equation of motion

$$\frac{d}{dt}\tilde{\rho}(t) = \int_0^t dt_1 \langle \hat{L}(t)\hat{L}(t_1) \rangle \tilde{\rho}(t)$$

$$= -\frac{1}{\hbar^2} \int_0^t dt_1 \langle [\hat{H}_p(t), [\hat{H}_p(t_1), \tilde{\rho}(t_1)]] \rangle.$$  \hspace{1cm} (11)

This is the desired quantum master equation for the density matrix of the Schrödinger particle, representing a local first-order differential equation with time-dependent coefficients.

### 3.2. White-noise limit and Markovian master equation

To proceed further we have to specify the stochastic properties of the random quantities $\gamma^{ij}(t)$. We take the simplest ansatz assuming that the fluctuations are isotropic,

$$\gamma^{ij}(t) = \sigma \delta_{ij} \xi(t).$$  \hspace{1cm} (12)

It should be noted that this assumption singles out a certain reference frame, which can be identified with the frame in which the space averaging of [7] has been carried out. In equation (12), the function $\xi(t)$ is taken to be a Gaussian white-noise process with zero mean and a $\delta$-shaped auto-correlation function,

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t-t').$$  \hspace{1cm} (13)

Therefore, the quantity $\sigma^2$ has the dimension of time and we set

$$\sigma^2 = \tau_c.$$  \hspace{1cm} (14)

The assumption of a white-noise process means that the auto-correlation time of the metric fluctuations is small compared to the timescale of the free motion of the Schrödinger particle. The fluctuations thus appear as un-correlated on the timescale of the particle motion with a constant power spectrum. In the case of colored noise with a structured power spectrum one can determine systematic corrections to the above master equation by means of the cumulant expansion, which generally leads to a non-Markovian quantum master equation [17].

Within the white-noise limit the contributions from the higher-order cumulants vanish and the second-order master equation (11) becomes an exact equation [18]. Using equation (12) we find

$$\hat{H}_p(t) = \hbar \tilde{V}(t)\xi(t),$$  \hspace{1cm} (15)

where

$$\tilde{V}(t) = e^{iH_0t/\hbar} V e^{-iH_0t/\hbar}, \quad V = \frac{\sqrt{\tau_c} p^2}{\hbar} \frac{m}{2}.$$  \hspace{1cm} (16)

Substitution into the master equation (11) yields

$$\frac{d}{dt}\tilde{\rho}(t) = -\int_0^t dt_1 \langle \xi(t)\xi(t_1) \rangle [\tilde{V}(t), [\tilde{V}(t_1), \tilde{\rho}(t_1)]]].$$  \hspace{1cm} (17)

Employing equation (13) we therefore get

$$\frac{d}{dt}\tilde{\rho}(t) = -\frac{1}{2} [\tilde{V}(t), [\tilde{V}(t), \tilde{\rho}(t)]].$$  \hspace{1cm} (18)
Transforming back to the Schrödinger picture by means of
\[ \rho(t) = e^{-iH_0t/\hbar} \hat{\rho}(t) e^{iH_0t/\hbar} \]  
we finally arrive at the master equation
\[ \frac{d}{dt} \rho(t) = -\frac{i}{\hbar} [H_0, \rho(t)] + \mathcal{D}(\rho(t)), \]  
(20)

where
\[ \mathcal{D}(\rho(t)) = -\frac{1}{2} [V, [V, \rho(t)]] = V\rho(t)V - \frac{1}{2} \{V^2, \rho(t)\}. \]  
(21)

Equation (20) represents a Markovian quantum master equation for the Schrödinger particle. The commutator term involving the free Hamiltonian \( H_0 \) describes the contribution from the coherent motion, while the superoperator \( \mathcal{D}(\rho) \), known as dissipator, models all dissipative effects. We observe that the master equation is in Lindblad form and, thus, generates a completely positive quantum-dynamical semigroup [19, 20]. We remark further that the structure of the master equation (20) corresponds to the so-called singular coupling limit. Within a microscopic approach such master equations arise from the coupling of an open quantum system to a free quantum field [21].

4. Physical implications

4.1. Moment equations and increase of entropy

To discuss the physical implications of the master equation (20) we first investigate the dynamical behavior of the averages. The average of an arbitrary system observable \( A \) is defined by
\[ \langle A \rangle_t = \text{tr}[A \rho(t)], \]  
(22)

and the master equation (20) leads to the equation of motion
\[ \frac{d}{dt} \langle A \rangle_t = \frac{i}{\hbar} \langle [H_0, A] \rangle_t - \frac{1}{2} \langle [V, [V, A]] \rangle_t. \]  
(23)

Let us consider for simplicity the case of a vanishing Newtonian potential such that \( H_0 \) commutes with \( V \). An immediate consequence of equation (23) is then
\[ \frac{d}{dt} \langle H_0 \rangle_t = 0. \]  
(24)

This is an important property which shows that on average the particle neither gains nor loses energy from the fluctuating field, by contrast to other master equations proposed in the literature. Moreover, the equations of motion for the first moments of momentum \( p_i \) and position \( x_i \) are found to coincide with those of a free particle,
\[ \frac{d}{dt} \langle p_i \rangle_t = 0, \quad \frac{d}{dt} \langle x_i \rangle_t = \frac{1}{m} \langle p_i \rangle_t. \]  
(25)

The influence of the dissipator can however be seen in the dynamics of the second moments. Defining the spatial variance
\[ \sigma^2_x(t) = \langle x_i^2 \rangle_t - \langle x_i \rangle_t^2 \]  
(26)

we find with the help of the master equation
\[ \sigma^2_x(t) = \sigma^2_x(0) + \frac{\sigma_{px}(0)}{m} t + \frac{\sigma^2_p}{m^2} t^2 + \frac{\sigma^2_{px}}{m^3} t^3. \]  
(27)
Here, the momentum variance $\sigma_p^2(t) = \langle p_i^2 \rangle - \langle p_i \rangle^2$ is of course constant in time, and we have evaluated the above expression in the rest frame of the particle, assuming $\langle p_i \rangle = 0$. Moreover, we have defined the cross-correlation $\sigma_{px}(t) = \langle pxi \rangle \frac{t}{x} - \langle pi \rangle \frac{t}{x} \langle xi \rangle$. The first three terms on the right-hand side of equation (27) coincide with the corresponding expression that is obtained from the free Schrödinger equation. Thus, dissipative effects are described by the last term of equation (27). However, for large times $t \gg \tau_c$ this term is small compared to the quadratically increasing ballistic term. Thus we see that the influence of dissipative effects on the spreading of the wave packet is very small in the long-time limit.

The irreversible character of the dynamics can be quantified with the help of the dynamics of the entropy of the state $\rho(t)$. For technical simplicity we consider here the linear entropy which is defined by

$$ S(t) = \text{tr}[\rho(t) - \rho^2(t)], $$

i.e., by one minus the purity $\text{tr}\rho^2(t)$. Differentiating equation (28) and employing the master equation (20) we obtain

$$ \frac{d}{dt} S(t) = \text{tr}[W^\dagger(t)W(t)] \geq 0, \quad W(t) = [V, \rho(t)]. $$

Hence, the entropy increases monotonically because $W^\dagger(t)W(t)$ is a positive operator. We also conclude from this equation that $\dot{S}(t) = 0$ if and only if $\rho(t)$ commutes with $V$, which means that $\rho(t)$ represents an incoherent mixture of eigenstates of $V$. Since $V$ is proportional to the kinetic energy $\rho(t)$ must be a mixture of eigenstates of the kinetic energy, e.g. plane or spherical waves. In particular, a kinetic energy eigenstate is not affected by the dissipative term and behaves exactly as for the free Schrödinger equation.

### 4.2. Estimation of decoherence times

The quantum master equation (20) can easily be solved in the momentum representation. To this end, we define the density matrix in the momentum representation by

$$ \rho(p, p', t) = \langle p|\rho(t)|p'\rangle. $$

With the help of the master equation we then find

$$ \frac{d}{dt} \rho(p, p', t) = -\frac{i}{\hbar}[E(p) - E(p')]\rho(p, p', t) $$

$$ - \frac{\tau_c}{2\hbar^2}[E(p) - E(p')]^2 \rho(p, p', t), $$

where $E(p) = p^2/2m$. This equation is immediately solved to yield

$$ \rho(p, p', t) = \exp\left[-\frac{i}{\hbar} \Delta Et - \frac{(\Delta E)^2 \tau_c}{2\hbar^2} t\right] \rho(p, p', 0), $$

where $\Delta E = E(p) - E(p')$. We see that the matrix elements corresponding to $E(p) = E(p')$ stay constant in time. In particular, the diagonals of the density matrix in the momentum representation are constant. On the other hand, the coherences corresponding to different energies decay exponentially with the rate $(\Delta E)^2 \tau_c/2\hbar^2$. Thus we find an associated decoherence time $\tau_D$ which is given by

$$ \tau_D = \frac{2\hbar^2}{(\Delta E)^2 \tau_c} = 2 \left( \frac{\hbar}{\Delta E \cdot \tau_c} \right)^2 \tau_c. $$

Hence, the dissipator $D(\rho)$ of the master equation leads to a decay of the coherences of superpositions of energy eigenstates with different energies, resulting in an effective dynamical
localization in energy space. This feature of the master equation is due to the fact that the fluctuating quantities $\gamma^{ij}(t)$ couple to the components of the momentum operator.

In our model, the different possible realizations of the wavefunction due to the stochastic fluctuations of spacetime are accounted for by the ensemble average introduced in equation (10). In a matter wave experiment, this leads to a reduced contrast of the interference fringes when both atom beams are recombined. When the traveling time of both beams is of the order of the decoherence time $\tau_D$, the interference fringes get smeared out significantly indicating the partial destruction of quantum coherence.

Let us identify the timescale $\tau_c$ that characterizes the strength of the fluctuations (see equations (12) and (14)) with the Planck time $t_p$, i.e., we set $\tau_c = t_p = l_p/c$ with the Planck length $l_p$. Expression (33) then yields the estimate

$$\tau_D \approx \frac{10^{13}s}{(\Delta E/eV)^2}.$$  

(34)

The decoherence time depends strongly on the scale of the energy difference $\Delta E$. For example, $\Delta E = 1eV$ gives a decoherence time of the order of $10^{13}$ s, while $\Delta E = 1MeV$ leads to a decoherence time of the order of 10 s.

5. Summary and discussion

In [7] we showed that a fluctuating spacetime metric would modify the inertial mass of quantum particles and, thus, leads to an apparent violation of the equivalence principle which gave additional motivation to performing improved atom interferometric tests of the equivalence principle. Here we derived another, complementary, implication of such spacetime fluctuations, namely decoherence of quantum systems. In the case that the spacetime fluctuations are related to the Planck scale then the decoherence time corresponding to an energy difference of 1 eV would be of the order of 0.3 million years, far beyond any experimental relevance. Even if the relevant scale is given by the grand unification scale which is three orders of magnitude smaller than the Planck scale, the corresponding decoherence time of about 300 years still is too large to be detectable. However, this result does not rule out in general the experimental detection of dephasing effects caused by metric fluctuations if one considers, for example, composite quantum objects whose states can be extremely sensitive to environmental noise.

In the derivation of our result we made two specifications. First, we made the ansatz (12) characterizing the fluctuations. This could in principle be generalized by introducing off-diagonal terms in $\gamma^{ij}$. However, this is already excluded by the averaging scheme introduced in [7] since different components of the fluctuating metric have been assumed to be independent of each other. Therefore, the quadratic fluctuating tensorial quantity $\gamma^{ij}$ defined by (3) must be diagonal, too. It is still possible to have different diagonal elements describing anisotropic fluctuations leading to a replacement of (12) by $\gamma^{ij} = \sigma_i \delta_{ij} \xi_i$. However, since one expects that different diagonal elements of $\gamma^{ij}$ will differ only by a tiny amount (as one expects deviations from isotropy of space being minuscule), this should not modify the expression for the decoherence time $\tau_D$ significantly.

Second, the white-noise scenario characterized by the two moments (13) could be generalized to colored noise. This is indeed feasible by means of the technique indicated in section 3 and would lead to additional non-Markovian terms in the master equation (20). We expect however that such terms would manifest only as small corrections to the equation for the decoherence time $\tau_D$, which do not alter the order of magnitude. Thus, incorporating only
the simple white-noise scenario and isotropic fluctuations is sufficient for obtaining reasonable estimates for the decoherence time.

A master equation whose structure is similar to our master equation (20) has been derived in a paper by Milburn [22]. However, it is important to note that the methods of derivation and the underlying physical principles are fundamentally different. In the paper by Milburn, time $t$ is postulated to be a random variable following a certain distribution function $p(t)$. Decoherence in this model is then due to the associated uncertainty in the system’s state changes, leading to a discontinuous, stochastic sequence of unitary transformations rather than a continuous one. In fact, the crucial difference is that the dissipator of our master equation involves the Lindblad operator $V$ which is proportional to the kinetic energy operator and differs from the system Hamiltonian (see equation (16)). This is essentially due to the fact that in our model the metric fluctuations are minimally coupled to the kinetic energy of the quantum particle. By contrast, the Lindblad operator of the other master equations proposed in the literature is proportional to the total system Hamiltonian $H$, because the time uncertainty is connected to an uncertainty in the total system dynamics generated by $H$.

Finally, we emphasize that the structure of our master equation (20) differs significantly from the master equation which has been derived by Power and Percival [12] and investigated further by Wang et al [13]. Both master equations are in Lindblad form in accordance with general principles of the theory of open quantum systems, and describe decoherence effects that yield, for instance, a reduction of the visibility of interference fringes. However, the position space localization of the master equation of Power and Percival results in an unbounded increase of the average energy, the corresponding pointer states being given by position eigenstates. By contrast, the master equation (20) describes localization in energy space with energy eigenstates as pointer states, leading to a constant mean energy (see equation (24)).

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