A SURVEY OF CHARACTERISTIC CLASSES OF SINGULAR SPACES

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ABSTRACT. A theory of characteristic classes of vector bundles and smooth manifolds plays an important role in the theory of smooth manifolds. An investigation of reasonable notions of characteristic classes of singular spaces started since a systematic study of singular spaces such as singular algebraic varieties. We make a quick survey of characteristic classes of singular varieties, mainly focusing on the functorial aspects of some important ones such as the singular versions of the Chern class, the Todd class and the Thom–Hirzebruch’s L-class. Then we explain our recent “motivic” characteristic classes, which in a sense unify these three different theories of characteristic classes. We also discuss bivariant versions of them and characteristic classes of proalgebraic varieties, which are related to the motivic measures/integrations. Finally we explain some recent work on “stringy” versions of these theories, together with some references for “equivariant” counterparts.

Dedicated to Jean-Paul Brasselet on the occasion of his 60th birthday

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1. INTRODUCTION

Characteristic classes are usually certain kinds of cohomology classes for vector bundles over spaces and characteristic classes of smooth manifolds are defined via their tangent bundles. The most basic ones are Stiefel–Whitney, Euler and Pontrjagin classes in

(*) Partially supported by Grant-in-Aid for Scientific Research(No.17540088), the Japanese Ministry of Education, Science, Sports and Culture.
the real case, and Chern classes in the complex case. They were introduced in 1930’s and 1940’s and constructed in a topological manner, i.e., via the obstruction theory, and in a differential-geometrical manner, i.e., via the Chern–Weil theory. Various important characteristic classes of vector bundles and invariants of manifolds are expressed as polynomials of them. The theory of cohomological characteristic classes were used for classifying manifolds and the study of structures of manifolds.

In 1960’s a systematic study of singular spaces was started by R. Thom, H. Whitney, H. Hironaka, S. Łojasiewicz, et al.; they studied triangulations, stratifications, resolution of singularities (in characteristic zero) and so on. Already in 1958 R. Thom introduced in [Thom2] rational Pontrjagin and L-classes for rational PL-homology manifolds. In 1965 M.-H. Schwartz defined in [Schw1] certain characteristic classes using obstruction theory of the so-called radial vector fields; the Schwartz class is defined for a singular complex variety embedded in a complex manifold as a cohomology class of the manifold supported on the singular variety. In 1969, D. Sullivan [Sull] proved that a real analytic space is mod 2 Euler space, i.e., the Euler–Poincaré characteristic of the link of any point is even, which implies that the sum of simplices in the first barycentric subdivision of any triangulation is mod 2 cycle. This enabled Sullivan to define the “singular” Stiefel–Whitney class as a mod 2 homology class, which is equal to the Poincaré dual of the above cohomological Stiefel–Whitney class for a smooth variety.

P. Deligne and A. Grothendieck (cf. [Sull]) conjectured the unique existence of the Chern class version of the Sullivan’s Stiefel–Whitney class, and in 1974 R. MacPherson [Mac1] proved their conjecture affirmatively. Motivated by MacPherson’s proof of the conjecture, P. Baum, W. Fulton and R. MacPherson [BFM1] proved the so-called “singular Riemann–Roch theorem”, which is nothing but the Todd class transformation in the case of singular varieties.

M. Goresky and R. MacPherson ([GM1], [GM2]) have introduced Intersection Homology Theory, by using the notion of “perversity”. In [GM1] they extended the work of [Thom2] to stratified spaces with even (co)dimensional strata and introduced a homology L-class \( L^*_{GM}(X) \) such that if \( X \) is nonsingular it becomes the Poincaré dual of the original Thom–Hirzebruch L-class: \( L^*_{GM}(X) = L^*(T X) \cap [X] \). In [Si] this was further extended to so-called stratified “Witt-spaces”, whose intersection (co)homology complex (for the middle perversity) becomes self-dual (compare also with [Ban] for a more recent extension). Later, S. Cappell and J. Shaneson [CS1] (see also [CS2] and [Sh]) introduced a homology L-class transformation \( L_* \), which turns out to be a natural transformation from the abelian group \( \Omega(X) \) (see §7) of cobordism classes of selfdual constructible complexes to the rational homology group \([BSY2]\) (cf. [Y2]).

In the case of singular varieties, the characteristic cohomology classes have been individually extended to the corresponding characteristic homology classes without any unifying theory of characteristic classes of singular varieties, unlike the case of smooth manifolds and vector bundles. Only very recently such a unifying theory of “motivic characteristic classes” for singular spaces appeared in our work [BSY2]. The purpose of the present paper is to make a quick survey on the development of characteristic classes and the up date situation of characteristic classes of singular spaces. This includes our motivic characteristic classes, bivariant versions, characteristic classes of proalgebraic varieties and finally “stringy” versions of these theories, together with some references for “equivariant” counterparts.
The present survey is a kind of extended and up-dated version of MacPherson’s survey article [Mac2] of more than 30 years ago. There are other surveys, e.g., [Alu1], [Br2], [Pa], [Sch4], [Su] on characteristic classes of singular varieties written from different viewpoints.

Acknowledgements. It is a pleasure to thank P. Aluffi, J.-P. Brasselet, A. Libgober, P. Pragacz, J. Seade, T. Suwa, W. Veys and A. Weber for valuable conversations about different aspects of this subject.

This survey is a combined, modified and extended version of the author’s two talks at “Singularities in Geometry and Topology” (the 5th week of Ecole de la Formation Permanente du CNRS - Session résidentielle de la FRUMAM) held at Luminy, Marseille, during the period of 21 February – 25 February 2005. The authors would like to thank the organizers of the conference for inviting us to give these talks. The second named author (S.Y.) also would like to thank the staff of ESI (Erwin Schrödinger International Institute for Mathematical Physics, Vienna, Austria), where a part of the paper was written in August 2005, for providing a nice atmosphere in which to work.

2. Euler–Poincaré characteristic

The simplest, but most fundamental and most important topological invariant of a compact topological space is the Euler number or Euler–Poincaré characteristic. Its definition is quite simple; for a compact triangulable space or more generally for a cellular decomposable space \( X \), it is defined to be the alternating sum of the numbers of cells and denoted by \( \chi(X) \):

\[
\chi(X) = \sum_n (-1)^n \#(n \text{- cells}).
\]

By the homology theory, the Euler–Poincaré characteristic turns out to be equal to the alternating sum of Betti numbers, i.e.,

\[
\chi(X) = \sum_n (-1)^n \dim H_n(X; \mathbb{R}).
\]

With this fact, the Euler–Poincaré characteristic is defined for any topological space as long as the right-hand-side of (2.2) is defined, e.g. for locally compact semialgebraic sets. Note that taking the alternating sum is essential in the definition (2.1), but it is not the case in the definition (2.2). The following general form is called the Poincaré polynomial:

\[
P_t(X) := \sum_n \dim H_n(X; \mathbb{R}) t^n,
\]

which is also a topological invariant. The Euler–Poincaré characteristic has the following properties:

1. \( \chi(X) = \chi(X') \) if \( X \cong X' \),
2. \( \chi(X) = \chi(X, Y) + \chi(Y) \) for any closed subspace \( Y \subset X \), where the relative Euler–Poincaré characteristic \( \chi(X, Y) \) is defined by the relative homology groups \( H_*(X, Y) \).
3. \( \chi(X \times Y) = \chi(X) \cdot \chi(Y) \).

For a fiber bundle \( f : X \to Y \) we have \( \chi(X) = \chi(F) \cdot \chi(Y) \), if the Euler characteristic \( \chi(F) \) of all fibers \( F \) is constant, e.g. \( Y \) is connected. This generalizes the above property (3). The same properties also hold for the Euler characteristic with compact support

\[
\chi_c(X) := \sum_n (-1)^n \dim H_c^n(X; \mathbb{R}),
\]

together with the following additivity property

\[
\chi_c(X) - \chi_c(Y) = \chi_c(X, Y) = \chi_c(X \setminus Y)
\]
for any closed subspace \( Y \subset X \), where the relative Euler characteristic with compact support \( \chi_c(X, Y) \) is defined by the relative cohomology groups \( H^*_c(X, Y) = H^*_c(X \setminus Y) \).

Of course \( \chi(X) = \chi_c(X) \) for \( X \) compact.

**Remark 2.5.** For two topological spaces \( X, Y \), let \( X + Y \) denote the topological sum, which is the disjoint sum, we clearly have

\[
\chi(X + Y) = \chi(X) + \chi(Y).
\]

However, we should note that for a closed subspace \( Y \subset X \) the following additivity property does not hold in general:

\[
(2.6) \quad \chi(X) = \chi(X \setminus Y) + \chi(Y),
\]

although \( X = (X \setminus Y) + Y \) as a set, since the topological sum \( Y + (X \setminus Y) \) is not equal to the original topological space \( X \). In other words, \( \chi(X, Y) \neq \chi(X \setminus Y) \) in general.

However, in the category of complex algebraic varieties, the above formula (2.6) holds, i.e., for any closed subvariety \( Y \subset X \) we have that \( \chi(X) = \chi(X \setminus Y) + \chi(Y) \). The key geometric reason for the equality \( \chi(X) = \chi(X \setminus Y) + \chi(Y) \) is that a closed subvariety \( Y \) always has a neighborhood deformation retract \( N \) such that the Euler–Poincaré characteristic of the “link” \( \chi(N \setminus Y) \) vanishes due to a result of Sullivan (see [Fu2, Exercise, p.95, comments on p.141-142]). In other words \( \chi(X \setminus Y) = \chi_c(X \setminus Y) \) in the complex algebraic context, which also can be extended and proved in the language of complex algebraically constructible functions (see [Sch3 §6.0.6]).

### 3. Characteristic Classes of Vector Bundles

Very nice references for this section are the books [MiSt], [Hir], [Hus], [Sto]. A characteristic class of vector bundles over a topological space \( X \) is defined to be a map from the set of isomorphism classes of vector bundles over \( X \) to the cohomology group (ring) \( H^*(X; \Lambda) \) with a coefficient ring \( \Lambda \), which is supposed to be compatible with the pullback of vector bundle and cohomology group for a continuous map. Namely, it is an assignment \( c^\ell : \text{Vect}(X) \rightarrow H^*(X; \Lambda) \) such that the following diagram commutes for a continuous map \( f : X \rightarrow Y \):

\[
\begin{array}{ccc}
\text{Vect}(Y) & \xrightarrow{c^\ell} & H^*(Y; \Lambda) \\
\downarrow f^* & & \downarrow f^* \\
\text{Vect}(X) & \xrightarrow{c^\ell} & H^*(X; \Lambda).
\end{array}
\]

Here \( \text{Vect}(W) \) is the set of isomorphism classes of vector bundles over \( W \).

The theory of characteristic classes started in Stiefel’s paper [Sti], in which he considered the problem of the existence of tangential frames, i.e., linearly independent vector fields on a differentiable manifold. And at the same year H. Whitney defined such characteristic classes for sphere bundles over a simplicial complex [Wh1], and some time later he invented cohomology and proved his important “sum formula” [Wh2]. Then Pontrjagin [Pon] introduced other characteristic classes of real vector bundles, based on the study of the homology of real Grassmann manifolds. Finally Chern [Ch1], [Ch2] defined similar characteristic classes of complex vector bundles.

The most fundamental characteristic classes of a real vector bundle \( E \) over \( X \) are the Stiefel-Whitney classes \( w^i(E) \in H^i(X; \mathbb{Z}_2) \), Pontrjagin classes \( p^i(E) \in H^{4i}(X; \mathbb{Z}[1/2]) \), and for a complex vector bundle \( E \) the Chern classes \( c^i(E) \in H^{2i}(X; \mathbb{Z}) \). These characteristic classes \( c^\ell(E) \in H^*(X; \Lambda) \) are described axiomatically in a unified way:
Definition 3.1. The Stiefel Whitney resp. Pontrjagin classes of real vector bundles, resp. Chern classes of complex vector bundles, is the operator assigning to each real (resp. complex) vector bundle $E \to X$ cohomology classes

$$\begin{align*}
c\ell^i(E) &:= \begin{cases} w^i(E) &\in H^i(X; \mathbb{Z}_2) \\ p^i(E) &\in H^{2i}(X; \mathbb{Z}[1/2]) \\ c^i(E) &\in H^{2i}(X; \mathbb{Z}) \end{cases}
\end{align*}$$

of the base space $X$ such that the following four axioms are satisfied:

**Axiom-1:** (finiteness) For each vector bundle $E$ one has $c\ell^0(E) := 1$ and $c\ell^i(E) = 0$ for $i > \text{rank } E$ (in fact $p^i(E) = 0$ for $i > [\text{rank } E/2]$). $c\ell^*(E) := \sum_i c\ell^i(E)$ is called the corresponding total characteristic class. In particular $c\ell^*(0_X) = 1$ for the zero vector bundle $0_X$ of rank zero.

**Axiom-2:** (naturality) One has $c\ell^*(F) = c\ell^*(f^*E) = f^* c\ell^*(E)$ for any cartesian diagram

$$\begin{array}{ccc}
F & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X.
\end{array}$$

**Axiom-3:** (Whitney sum formula)

$$c\ell^*(E \oplus F) = c\ell^*(E) c\ell^*(F),$$

or more generally

$$c\ell^*(E) = c\ell^*(E') c\ell^*(E'')$$

for any short exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles.

**Axiom-4:** (normalization or the “projective space” condition) For the canonical (i.e., the dual of the tautological) line bundle $\gamma^1_n(K) := O_{P^n(K)}(1)$ over the projective space $P^n(K)$ (with $K = \mathbb{R}, \mathbb{C}$) one has:

- $(w^1)$: $w^1(\gamma^1_n(K))$ is non-zero.
- $(p^1)$: $p^1(\gamma^1_n(C)) = c^1(\gamma^1_n(C))^2$.
- $(c^1)$: $c^1(\gamma^1_n(C)) = [P^{n-1}(C)] \in H^2(P^n(C); \mathbb{Z})$ is the cohomology class represented by the hyperplane $P^{n-1}(C)$.

**Remark 3.2.** We use the superscript notation $c\ell^*$ for contravariant functorial characteristic classes of vector bundles in cohomology, to distinguish them from the subscript notation $c\ell_*$ for covariant functorial characteristic classes of singular spaces in homology, which we consider later on. Also note that in topology any short exact sequence of vector bundles over a reasonable (i.e. paracompact) space splits (by using a metric on $E$). But this is not the case in the algebraic or complex analytic context, where one should ask the “Whitney sum formula” for short exact sequences.

The existence of such a class for vector bundles of rank $n$ can be shown, for example, with the help of a classifying space, i.e., the infinite dimensional Grassmanian manifolds $G_n(K\infty)$ (with $K = \mathbb{R}, \mathbb{C}$), and the fact that the cohomology ring of this Grassmanian manifold is a polynomial ring

$$H^*(G_n(K\infty); \Lambda) = \begin{cases} \mathbb{Z}_2[w^1, w^2, \ldots, w^n] & \text{ for } K = \mathbb{R} \text{ and } \Lambda = \mathbb{Z}_2, \\ \mathbb{Z}[1/2][p^1, p^2, \ldots, p^{[n/2]}] & \text{ for } K = \mathbb{R} \text{ and } \Lambda = \mathbb{Z}[1/2], \\ \mathbb{Z}[c^1, c^2, \ldots, c^n] & \text{ for } K = \mathbb{C} \text{ and } \Lambda = \mathbb{Z}. \end{cases}$$

The most important axiom is Axiom-2 and the uniqueness of such a class follows then form Axiom-3 and Axiom-4. By the so-called “splitting principle” one can assume (after pulling back to a suitable bundle, whose pullback on the cohomology level is injective) that a given non-zero vector bundle $E$ splits into a sum of line (or 2-plane) bundles. These line (or 2-plane) bundles are then called the “Chern roots” of $E$. Then Axiom-3 reduces
the calculation of characteristic classes to the case of line bundles (for \(c_\ell = w, c\)) or real 2-plane bundles (for \(c_\ell = p\)). By naturality these are then fixed by Axiom-4, since

\[
G_1(K^\infty) = \lim_k P^k(K) \quad (\text{for } K = \mathbb{R}, \mathbb{C}),
\]

for the case \(c_\ell = w, c\), or from the fact that the canonical projection

\[
\lim_k P^k(C) \to G_2(\mathbb{R}^\infty)
\]

is the orientation double cover for the case \(c_\ell = p\).

From the axioms one gets in all cases \(w^1, p^1\) and \(c^1\) are nilpotent on finite dimensional spaces, and \(c_\ell^r(E) = 1\) for a trivial vector bundle \(E\). Note that a real oriented line bundle is always trivial so that a real line bundle \(L \to X\) has no interesting characteristic class \(c_\ell^j(L) = 0 \in H^j(X; \mathbb{Z}[1/2])\) for \(j > 0\). Just pullback to an orientation double cover \(\pi : \tilde{X} \to X\) so that \(\pi^*L\) is orientable with \(\pi^*: H^j(X; \mathbb{Z}[1/2]) \to H^j(\tilde{X}; \mathbb{Z}[1/2])\) injective (since \(2 \in \mathbb{Z}[1/2]\) is invertible). In particular a real vector bundle \(E\) of rank \(r\) is orientable if and only if \(w^1(E) = w^1(\Lambda^r E) = 0\).

If a characteristic class \(c_\ell^* : \text{Vect}(X) \to H^*(X; \Lambda)\) satisfies the Whitney sum condition

\[
c_\ell^*(E \oplus F) = c_\ell^*(E)c_\ell^*(F) \quad \text{with} \quad c_\ell^*(0_X) = 1,
\]

then \(c_\ell^*\) is called a multiplicative characteristic class. Another important multiplicative characteristic class of an oriented real vector bundle \(E \to X\) of rank \(r\) is the Euler class \(e(E) \in H^r(X; \mathbb{Z}),\) with \(e(E) \mod 2 = w^r(E),\) \(e(E)^2 = p_{r/2}(E)\) for \(r\) even and \(e(E) = c^r(E)\) in case \(E\) is given by a complex vector bundle \(E\) of rank \(r\). But the Euler class is not a normalized characteristic class with \(c_\ell^0(L) = 1\).

The Stiefel-Whitney, Pontrjagin and Chern classes are essential in the sense that any multiplicative characteristic class \(c_\ell^*\) over finite dimensional base spaces is uniquely expressed as a polynomial (or power series) in these classes, i.e. the “splitting principle” implies:

**Theorem 3.3.** Let \(\Lambda\) be a \(\mathbb{Z}_2\)-algebra (resp. a \(\mathbb{Z}[1/2]\)-algebra) for the case of real vector bundles, or a \(\mathbb{Z}\)-algebra for the case of complex vector bundles. Then there is a one-to-one correspondence between

1. multiplicative characteristic classes \(c_\ell^*\) over finite dimensional base spaces, and
2. formal power series \(f \in \Lambda[[z]]\)

such that \(c_\ell^*(L) = f(w^1(L))\) or \(c_\ell^*(L) = f(c^1(L))\) for any real or complex line bundle \(L\) (resp. \(c_\ell^*(L) = f(p^1(L))\) for any real 2-plane bundle \(L\)). In this case \(f\) is called the characteristic power series of the corresponding multiplicative characteristic class \(c_\ell^f\).

**Remark 3.4.** For the result above it is important that characteristic classes of vector bundles live in cohomology so that one can build new classes by multiplication (i.e. by the cup-product) of the basic ones. This is not possible in the case of characteristic classes of singular spaces, which live in homology (except in the case of homology manifolds where Poincaré duality is available).

Moreover \(c_\ell^f\) is invertible with inverse \(c_\ell^f_{/f}\), if \(f \in \Lambda[[z]]\) is invertible, i.e. if \(f(0) \in \Lambda\) is a unit (e.g. \(f\) is a normalized power series with \(f(0) = 1\)). Then the corresponding multiplicative characteristic class \(c_\ell^*\) extends over finite dimensional base spaces \(X\) to a natural transformation of groups

\[
c_\ell^* : (K(X), \oplus) \to (H^*(X; \Lambda), \cup)
\]

on the Grothendieck group \(K(X)\) of real or complex vector bundles over \(X\).
4. Characteristic classes of smooth manifolds

Let us now switch to smooth manifolds, which will be an important intermediate step on the way to characteristic classes of singular spaces. For a smooth (or almost complex) manifold \( M \) its real (or complex) tangent bundle \( TM \) is available and a characteristic class \( cl^*(TM) \) of the tangent bundle \( TM \) is called a characteristic cohomology class \( cl^*(M) \) of the manifold \( M \). We also use the notation

\[
cl_*(M) := cl^*(TM) \cap [M] \in H^B_M(M; \Lambda)
\]

for the corresponding characteristic homology class of the manifold \( M \), with \([M] \in H^B_M(M; \Lambda)\) the fundamental class in Borel-Moore homology of the (oriented) manifold \( M \). Note that \( H^B_M(X; \Lambda) = H_*(X; \Lambda) \) for \( X \) compact.

**Remark 4.1.** Using a relation to suitable cohomology operations, i.e. Steenrod squares, Thom \([\text{Thom}]\) has shown that the Stiefel-Whitney classes \( w^*(M) \) of a smooth manifold \( M \) are topological invariants. Later he introduced \([\text{Thom2}]\) rational Pontrjagin and \( L \)-classes for compact rational PL-homology manifolds so that the rational Pontrjagin classes \( p^*(M) \in H^*(M; \mathbb{Q}) \) of a closed smooth manifold \( M \) are combinatorial or piecewise linear invariants. A deep result of Novikov \([\text{Nov}]\) implies the topological invariance of these rational Pontrjagin classes \( p^*(M) \in H^*(M; \mathbb{Q}) \) of a smooth manifold \( M \).

For a closed oriented manifold \( M \) one has the interesting formula

\[
deg(c(M)) = \int_M e(TM) \cap [M] = \chi(M),
\]

which justifies the name “Euler class”. For a closed complex manifold \( M \) this formula becomes

\[
deg(c_*(M)) = \int_M c^*(TM) \cap [M] = \chi(M),
\]

which is called the Gauss–Bonnet–Chern Theorem (compare \([\text{Ch3}]\)). In this sense, the Chern class is a higher cohomology class version of the Euler–Poincaré characteristic. Similarly

\[
deg(w_*(M)) = \int_M w^*(TM) \cap [M] = \chi(M) \mod 2
\]

for any closed manifold \( M \).

More generally let \( \text{Iso}(n - \dim. \ G - mfd.) \) be the set of isomorphism classes of smooth closed (and oriented) pure \( n \)-dimensional manifolds \( M \) for \( G = O \) (or \( G = SO \)), or of pure \( n \)-dimensional weakly (“= stably”) almost complex manifolds \( M \) for \( G = U \), i.e. \( TM \oplus \mathbb{R}_M^N \) is a complex vector bundle (for suitable \( N \), with \( \mathbb{R}_M \) the trivial real line bundle over \( M \)). Then

\[
\text{Iso}(G - mfd.)_* := \bigoplus_n \text{Iso}(n - \dim. \ G - mfd.)
\]

becomes a commutative graded semiring with addition and multiplication given by disjoint union and exterior product, with \( 0 \) and \( 1 \) given by the classes of the empty set and one point space. Moreover any multiplicative characteristic class \( cl_f \) coming from the power series \( f \) in the variable \( z = w^1, p^1 \) or \( c^1 \) induces by

\[
M \mapsto \text{deg}(cl_{f*}(M)) := \int_M cl^*_f(TM) \cap [M]
\]

a semiring homomorphism

\[
\Phi_f : \text{Iso}(G - mfd.)_* \to \Lambda = \begin{cases}
\mathbb{Z}_2\text{-algebra for } G = O \text{ and } z = w^1, \\
\mathbb{Z}[1/2]\text{-algebra for } G = SO \text{ and } z = p^1, \\
\mathbb{Z}\text{-algebra for } G = U \text{ and } z = c^1.
\end{cases}
\]
Let $\Omega^G_\ast := Iso(G \setminus m.f.d.)/ \sim$ be the corresponding cobordism ring of closed ($G = O$) and oriented ($G = SO$) or weakly (“= stably”) almost complex manifolds ($G = U$), with $M \sim 0$ for a closed pure $n$-dimensional $G$-manifold $M$ if and only if there is a compact pure $n + 1$-dimensional $G$-manifold $B$ with boundary $\partial B \simeq M$. Note that this is indeed a ring with $-[M] = [M]$ for $G = O$ or $-[M] = [-M]$ for $G = SO, U$, where $-M$ has the opposite orientation of $M$. Moreover, for $B$ as above with $\partial B \simeq M$ one has

$$TB|\partial B \simeq TM \oplus \mathbb{R}_M$$

so that $\mathcal{e}_f^i(TM) = i^\ast \mathcal{e}_f^i(TB)$ for $i : M \simeq \partial B \to B$ the closed inclusion of the boundary. This also explains the use of the stable tangent bundle for the definition of a stably or weakly almost complex manifold. By a simple argument due to Pontrjagin one then gets

$$M \sim 0 \Rightarrow \deg(\mathcal{e}_f^i(TM)) = \int_M \mathcal{e}_f^i(TM) \cap [M] = 0$$

so that any multiplicative characteristic class $\mathcal{e}_f^i$ coming from the power series $f$ in the variable $z = w^1$, $p^1$ or $c^1$ induces a ring homomorphism called genus

$$\Phi_f : \Omega^G_\ast \to \Lambda = \begin{cases} \mathbb{Z}_2\text{-algebra for } G = O \text{ and } z = w^1, \\ \mathbb{Z}[1/2]\text{-algebra for } G = SO \text{ and } z = p^1, \\ \mathbb{Z}\text{-algebra for } G = U \text{ and } z = c^1. \end{cases}$$

In fact for $\Lambda$ a $\mathbb{Q}$-algebra this induces a one-to-one correspondence between

1. normalized power series $f$ in the variable $z = p^1$ (or $c^1$),
2. normalized and multiplicative characteristic classes $\mathcal{e}_f^i$ over finite dimensional base spaces, and
3. genera $\Phi : \Omega^G_\ast \to \Lambda$ for $G = SO$ (or $G = U$).

Here one uses the following structure theorem.

**Theorem 4.4.**

1. (Thom) $\Omega^SO_\ast \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{P}^{2n}(\mathbb{C})]|n \in \mathbb{N}$ is a polynomial algebra in the classes of the complex even dimensional projective spaces.
2. (Milnor) $\Omega^U_\ast \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{P}^n(\mathbb{C})]|n \in \mathbb{N}$ is a polynomial algebra in the classes of the complex projective spaces.

In particular, the corresponding genus $\Phi_f$ with values in a $\mathbb{Q}$-algebra $\Lambda$, or the corresponding normalized and multiplicative characteristic class $\mathcal{e}_f^i$, is uniquely fixed by the values $\Phi_f(M) = \int_M \mathcal{e}_f^i(TM) \cap [M]$ for all (complex even dimensional) complex projective spaces $M = \mathbb{P}^n(\mathbb{C})$. These are best codified by the *logarithm* $g \in \Lambda[\![t]\!]$ of $\Phi_f$:

$$g(t) := \sum_{i=0}^{\infty} \Phi_f(\mathbb{P}^i(\mathbb{C})) \cdot \frac{t^{i+1}}{i+1}.$$  

Moreover, a genus $\Phi_f : \Omega^U_\ast \otimes \mathbb{Q} \to \Lambda$ factorizes over the canonical map

$$\Omega^U_\ast \otimes \mathbb{Q} \to \Omega^SO_\ast \otimes \mathbb{Q}$$

if and only if $f(z)$ is an even power series in $z = c^1$, $f(z) = h(z^2)$ with $z^2 = (c^1)^2 = p^1$. Consider for example the *signature* $\sigma(M)$ of the cup-product pairing on the middle dimensional cohomology of the closed oriented manifold $M$ of real dimension $4n$, with $\sigma(M) := 0$ in all other dimensions. This defines a genus $\sigma : \Omega^SO_\ast \otimes \mathbb{Q} \to \mathbb{Q}$, as observed by Thom, with $\sigma(\mathbb{P}^{2n}(\mathbb{C})) = 1$ for all $n$. The signature genus comes from the normalized power series $h(z) = \sqrt{z}/\tanh(\sqrt{z})$ in the variable $z = p^1$ (or $f(z) = z/\tanh(z)$ in the variable $z = c^1$), whose corresponding characteristic class $\mathcal{e}_f^i = L^i$ is by definition
the Hirzebruch-Thom $L$-class. This is the content of the famous Hirzebruch's Signature Theorem (compare also with [Hir3]):

$$
\sigma(M) = \int_M L^*(TM) \cap [M].
$$

**Remark 4.6.** The first structure theorem about cobordism rings due to Thom is the description of $\Omega^*_C$ as a polynomial algebra $\mathbb{Z}_2[[M^n]]|n \in \mathbb{N}, n + 1 \neq 2^k|$ in the classes of suitable closed manifolds $M^n$ of dimension $n$, with one generator in each dimension $n$ with $n + 1$ not a power of 2. Then each genus $\Omega^*_C \to \Lambda$ to a $\mathbb{Z}_2$-algebra $\Lambda$ is coming form a normalized and multiplicative characteristic class $c^\ell_f$, but this correspondence is not injective.

The value $\Phi(M)$ of a genus $\Phi$ on the closed manifold $M$ is also called a characteristic number of $M$. All these numbers can be used to classify closed manifolds up to cobordism.

**Theorem 4.7.**

(1) (Pontrjagin–Thom) Two closed $C^\infty$-manifolds are cobordant (i.e., represent the same element in $\Omega^*_C$) if and only if all their Stiefel–Whitney numbers are the same.

(2) (Thom–Wall) Two closed oriented $C^\infty$-manifold are corbordant up to two-torsion (i.e., represent the same element in $\Omega^*_S \otimes \mathbb{Z}[1/2]$) if and only if all their Pontrjagin numbers are the same.

(3) (Milnor–Novikov) Two closed stably or weakly almost complex manifold are cobordant (i.e., represent the same element in $\Omega^*_U$) if and only if all their Chern numbers are the same.

5. **Hirzebruch–Riemann–Roch and Grothendieck–Riemann–Roch**

Let $X$ be a non-singular complex projective variety and $E$ a holomorphic vector bundle over $X$. Note that in this context we do not need to distinguish between holomorphic and algebraic vector bundles, and similarly for coherent sheaves, by the so-called “GAGA-principle” [Serre]. Then the Euler–Poincaré characteristic of $E$ is defined by

$$
\chi(X, E) = \sum_{i \geq 0} (-1)^i \dim \mathbb{C}H^i(X; \Omega(E)),
$$

where $\Omega(E)$ is the coherent sheaf of germs of sections of $E$. J.-P. Serre conjectured in his letter to Kodaira and Spencer (dated September 29, 1953) that there exists a polynomial $P(X, E)$ of Chern classes of the base variety $X$ and the vector bundle $E$ such that

$$
\chi(X, E) = \int_X P(X, E) \cap [X].
$$

Within three months (December 9, 1953) F. Hirzebruch solved this conjecture affirmatively: the above looked for polynomial $P(X, E)$ can be expressed as

$$
P(X, E) = ch^*(E)td^*(X)
$$

where $ch^*(E)$ is the total Chern character of $E$ and $td^*(TX)$ is the total Todd class of the tangent bundle $TX$ of $X$. Let us recall that the cohomology classes $ch^*(V)$ and $td^*(V)$ are defined as follows:

$$
ch^*(V) = \sum_{i=1}^{\text{rank } V} e^{\alpha_i} \in H^{2*}(X; \mathbb{Q})
$$

and

$$
td^*(V) = \prod_{i=1}^{\text{rank } V} \frac{\alpha_i}{1 - e^{-\alpha_i}} \in H^{2*}(X; \mathbb{Q})
$$

where $\alpha_i$’s are the Chern roots of $V$. So $td^*$ is just the normalized and multiplicative characteristic class corresponding to the normalized power series $f(z) = z/(1 - e^{-z})$ in
z = c^{1/2}. Similarly the Chern character defines a contravariant natural transformation of rings
\[ ch^* : (K(X), \oplus, \otimes) \to (H^{2*}(X; \mathbb{Q}), +, \cup) \]
on the Grothendieck group \( K(X) \) of complex vector bundles over \( X \). Then we have the following celebrated theorem of Hirzebruch:

**Theorem 5.1. (Hirzebruch–Riemann–Roch)**

\[ \chi(X, E) = T(X, E) := \int_X (ch^*(E) td^*(X)) \cap [X]. \]

\( T(X, E) \) is called the \( T \)-characteristic (\( \text{HRR} \)). For a more detailed historical aspect of HRR, see [Hir3].

**Remark 5.2.** The \( T \)-characteristic \( T(X, E) \) is a priori a rational number by the definitions of the Todd class and Chern character, but it has to be an integer as a consequence of the HRR. The \( T \)-characteristic \( T(X, E) \) of a complex vector bundle \( E \) can be defined for any almost complex manifold and Hirzebruch [Hir1] asked if the \( T \)-genus \( T(X) := T(X, 1) \) with \( 1 \) being a trivial line bundle is always an integer. Of course this follows from HRR and the later result of Quillen, that \( \Omega^n_X \otimes \mathbb{Q} \) is generated by complex projective algebraic manifolds. The identity
\[ z/(1 - e^{-z}) = e^{z/2} \cdot z/(2 \sinh(z/2)) \]
allows one to introduce the Todd class
\[ Td^*(X) := e^{c^{1/2}(TX)/2} \cdot \hat{A}^*(TX), \]
and therefore also the \( T \)-characteristic \( T(X, E) \), more generally for a so-called Spin\(^c\)-manifold \( X \). Here \( \hat{A} \) is the so-called \( A \) hat genus or characteristic class corresponding to the even normalized power series \( f(z) = z/(2 \sinh(z/2)) \) in the variable \( z = e^1 \) or to \( f(z) = \sqrt{z}/(2 \sinh(\sqrt{z}/2)) \) in the variable \( z = p^1 \). The \( T \)-characteristic \( T(X, E) \) of a complex vector bundle \( E \) is then an integer by an application of the Atiyah-Singer Index theorem [AS] for a suitable Dirac operator (compare [Hir1] p. 197, Theorem 26.1.1)).

\( A. \) Grothendieck (cf. [BoSe]) generalized HRR for non-singular quasi-projective algebraic varieties over any field and proper morphisms with Chow cohomology ring theory instead of ordinary cohomology theory (compare also with [Fu1] chapter 15)). For the complex case we can still take the ordinary cohomology theory (or the homology theory by the Poincaré duality). Here we stick ourselves to complex projective algebraic varieties for the sake of simplicity. For a variety \( X \), let \( G_0(X) \) denote the Grothendieck group of algebraic coherent sheaves on \( X \) and for a morphism \( f : X \to Y \) the pushforward \( f_! : G_0(X) \to G_0(Y) \) is defined by
\[ f_!(F) := \sum_{i \geq 0} (-1)^i R^if_* F, \]
where \( R^if_* F \) is (the class of) the higher direct image sheaf of \( F \). Then \( G_0 \) is a covariant functor with the above pushforward (see [Gro] and [Man]). Let similarly \( K^0(X) \) be the Grothendieck group of complex algebraic vector bundles over \( X \) so that one has a canonical contravariant transformation of rings \( K^0(\ ) \to K(\ ) \) to the Grothendieck group of complex vector bundles. Note that on a smooth algebraic manifold the canonical map \( K^0(\ ) \to G_0(\ ) \) is an isomorphism. With this isomorphism one can define characteristic classes of any algebraic coherent sheaf. Then Grothendieck showed the existence of a natural transformation from the covariant functor \( G_0 \) to the \( \mathbb{Q} \)-homology covariant functor \( H_{2*}(\ ; \mathbb{Q}) \) (see [BoSe]):
Theorem 5.3. (Grothendieck–Riemann–Roch) Let the transformation \( \tau_* : G_0(\ ) \to H_{2*}(\ ; Q) \) be defined by \( \tau_*(F) = td^\ast X \ ch^\ast (F) \cap [X] \) for any smooth variety \( X \). Then \( \tau_* \) is actually natural, i.e., for any morphism \( f : X \to Y \) the following diagram commutes:

\[
\begin{array}{ccc}
G_0(X) & \xrightarrow{\tau_*} & H_{2*}(X; Q) \\
\downarrow f & & \downarrow f_* \\
G_0(Y) & \xrightarrow{\tau_*} & H_{2*}(Y; Q)
\end{array}
\]

i.e.,

\[
(GRR) \quad td^\ast (T_Y) ch^\ast (f_!F) \cap [Y] = f_*(td^\ast X \ ch^\ast (F) \cap [X]).
\]

Clearly \( HRR \) is induced from \( GRR \) by considering a map from \( X \) to a point. Note that the target of the transformation of the original \( GRR \) is the cohomology \( H_{2*}(\ ; Q) \) with the Gysin homomorphism instead of the homology \( H_{2*}(\ ; Q) \), but, by the definition of the Gysin homomorphism the original \( GRR \) can be put in as above.

6. The Generalized Hirzebruch–Riemann–Roch

In Hirzebruch’s book [Hir2, §12.1 and §15.5] he has generalized the characteristics \( \chi(X, E) \) and \( T(X, E) \) to the so-called \( \chi_y \)-characteristic \( \chi_y(X, E) \) and \( T_y \)-characteristic \( T_y(X, E) \) as follows, using a parameter \( y \) (see also [HB] Chapter 5).

Definition 6.1.

\[
\chi_y(X, E) := \sum_{p \geq 0} \left( \sum_{q \geq 0} (-1)^q \dim \mathcal{C} H^q(X, \Omega(E) \otimes \Lambda^p T^\ast X) \right) y^p
\]

\[
= \sum_{p \geq 0} \chi(X, E \otimes \Lambda^p T^\ast X)) y^p
\]

where \( T^\ast X \) is the dual of the tangent bundle \( TX \), i.e., the cotangent bundle of \( X \).

\[
T_y(X, E) := \int_X \tilde{td}(y) (TX) ch(1+y) (E) \cap [X],
\]

\[
\tilde{td}(y) (TX) := \prod_{i=1}^{\dim X} \left( \frac{\alpha_i (1+y)}{1 - e^{-\alpha_i (1+y)}} - \alpha_i y \right),
\]

\[
ch(1+y) (E) := \sum_{j=1}^{\text{rank } E} e^{\beta_j (1+y)},
\]

where \( \alpha_i \)'s are the Chern roots of \( TX \) and \( \beta_j \)'s are the Chern roots of \( E \).

F. Hirzebruch [Hir2 §21.3] showed the following theorem:

Theorem 6.2. (The generalized Hirzebruch–Riemann–Roch)

\[
(g-HRR) \quad \chi_y(X, E) = T_y(X, E).
\]
The $g$-HRR can be shown as follows, using HRR:

$$
\chi_y(X, E) = \int_X \sum_{p \geq 0} \chi(X, E \otimes \Lambda^p T^* X)y^p \quad \text{(by the definition)}
$$

$$
= \int_X \sum_{p \geq 0} (ch^*(E \otimes \Lambda^p T^* X)td^*(X) \cap [X])y^p \quad \text{(by HRR)}
$$

$$
= \int_X \left( \sum_{p \geq 0} ch^*(E)td^*(X) \prod_{i=1}^{\dim X} (1 + ye^{-\alpha_i}) \right) \cap [X]
$$

However, the power series $(1 + ye^{-\alpha_i}) \frac{\alpha_i}{1 - e^{-\alpha_i}}$ is not a normalized power series because
the 0-degree part is $1 + y$, not 1. So, by dividing this non-normalized power series by $1 + y$
and furthermore by changing $\beta_j$ to $\beta_j(1 + y)$ and $\alpha_i$ to $\alpha_i(1 + y)$, which does not change
the value of $\chi_y(X, E)$ at all, and by noticing that

$$
\frac{1 + ye^{-\alpha_i(1+y)}}{1 + y} \frac{\alpha_i}{1 - e^{-\alpha_i(1+y)}} = \frac{\alpha_i(1 + y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y,
$$

we can see that the right hand side of the last equation is $T_y(X, E)$ (compare [HB] p.61-
62). In fact the same argument shows that a non-normalized power series $f(z)$ with
$a := f(0) \in \Lambda$ a unit induces the same genus as the normalized power series $f(az)/a$.

**Remark 6.3.** The generalized Hirzebruch Riemann-Roch theorem is also true for a holo-
morphic vector bundle $E$ over a compact complex manifold $X$, by an application of the
Atiyah-Singer Index theorem [AS].

The above modified Todd class $\widetilde{td}(y)$ is the normalized and multiplicative characteristic
class corresponding to the normalized power series (in $z = c^1$):

$$
f(z) = f_y(z) = \frac{z(1 + y)}{1 - e^{-z(1+y)}} - zy \in \mathbb{Q}[y][[z]].
$$

The associated genus $\chi_y : \Omega^U_{*} \to \mathbb{Q}[y]$ is called the Hirzebruch $\chi_y$-genus. A simple
residue calculation in [HR2] Lemma 1.8.1] implies for all $n \in \mathbb{N}$:

$$
\chi_y(P^n(\mathbb{C})) = \sum_{i=0}^{n} (-y)^i \in \mathbb{Z}[y] \subset \mathbb{Q}[y].
$$

So these values on $P^n(\mathbb{C})$ fix the $\chi_y$-genus and the modified Todd class $\widetilde{td}(y)$. Moreover,
the normalized power series $f_y(z)$ specializes to

$$
f_y(z) = \begin{cases} 
1 + z & \text{for } y = -1, \\
\frac{z}{(1 - e^{-z})} & \text{for } y = 0, \\
\frac{z}{\tanh(z)} & \text{for } y = 1.
\end{cases}
$$

So the modified Todd class $\widetilde{td}(y)$ defined above unifies the following three important character-
istic cohomology classes:
Therefore, when $E$ is the trivial line bundle, for these special values $y = -1, 0, 1$ the $g$-HRR reads as follows:

(y = -1) Gauss–Bonnet–Chern Theorem: 
$$\chi(X) = \int_X c^*(TX) \cap [X],$$

(y = 0) Riemann–Roch Theorem: denoting $\chi_a(X) := \chi(X, \mathcal{O}_X)$, called the arithmetic genus of $X$, to avoid the confusion with the above topological Euler–Poincaré characteristic $\chi(X)$,
$$\chi_a(X) = \int_X td^*(TX) \cap [X],$$

(y = 1) Hirzebruch’s Signature Theorem:
$$\sigma(X) = \int_X L^*(TX) \cap [X].$$

Remark 6.5. (Poincaré–Hopf Theorem) The above Gauss–Bonnet–Chern Theorem due to Chern [Ch3] is a generalization of the original Gauss–Bonnet theorem saying that the integration of the Guassian curvature is equal to $2\pi$ times the topological Euler–Poincaré characteristic. There is another well-known differential-topological formula concerning the topological Euler–Poincaré characteristic. That is the so-called Poincaré–Hopf theorem, saying that the index of a smooth vector field $V$ with only isolated singularities on a smooth compact manifold $M$ is equal to the topological Euler–Poincaré characteristic of the manifold $M$;
$$\text{Index}(V) = \chi(M),$$
where the index $\text{Index}(V)$ is defined to be the sum of the indices of the vector field at the isolated singularities. Compare with [Mi1] for a beautiful introduction to the Poincaré–Hopf theorem. Note that the Gauss–Bonnet–Chern Theorem follows from the Poincaré–Hopf theorem (cf. [Wi] and [Zh]).

7. CHARACTERISTIC CLASSES OF SINGULAR VARIETIES

In the following we consider for simplicity only compact spaces. For a singular algebraic or analytic variety $X$ its tangent bundle is not available any longer because of the existence of singularities, thus one cannot define its characteristic class $c_\ell(X)$ as in the previous case of manifolds, although a “tangent-like” bundle such as Zariski tangents is available. A main theme for defining reasonable characteristic classes for singular varieties is that reasonable ones should be interesting enough; for example, they should be geometrically or topologically interesting and quite well related to other well-known invariants of varieties and singularities (e.g., see [Mac2]).

The theory of characteristic classes of vector bundles is a natural transformation from the contravariant functor $\text{Vect}$ to the contravariant cohomology functor $H^*(\quad; \Lambda)$. This naturality is an important guide for developing various theories of characteristic classes for singular varieties. The known functorial characteristic classes for singular spaces are covariant functorial maps
$$cl_\ell : A(X) \rightarrow H_*(X; \Lambda)$$
from a suitable covariant theory $A$ depending on the choice of $c\ell_*$. Moreover, there is always a distinguished element $1_X \in A(X)$ such that the corresponding characteristic class of the singular space $X$ is defined as $c\ell_*(X) := c\ell_*(1_X)$. Finally one has the normalization
\[ c\ell_*(1_M) = c\ell^*(TM) \cap [M] \in H_*(X; \Lambda) \]
for $M$ a smooth manifold, with $c\ell^*(TM)$ the corresponding characteristic cohomology class of $M$. This justifies the notation $c\ell_*$ for this homology class transformation, which should be seen as a relative homology class version of the following characteristic number of the singular space $X$:
\[ \sharp(X) := c\ell_*(const_* 1_X) = const_*(c\ell_*(1_X)) \in H_*(\{pt\}; \Lambda) = \Lambda, \]
with $const : X \to \{pt\}$ a constant map. Note that the normalization implies for $M$ smooth:
\[ \sharp(M) = \deg(c\ell_*(M)) = \int_M c\ell^*(TM) \cap [M] \]
so that this is consistent with the notion of characteristic number of the smooth manifold $M$ as used before.

7.1. Stiefel-Whitney classes $w_*$. The first example of functorial characteristic classes is the theory of singular Stiefel–Whitney homology classes due to Dennis Sullivan [Sull] (also see [FM]). A crucial fact about the original Stiefel–Whitney class is the following fact: if $T$ is any triangulation of a manifold $X$, then the sum of all the simplices of the first barycentric subdivision is a mod 2 cycle and its homology class is equal to the Poincaré dual of the Stiefel–Whitney class. In [Sull] D. Sullivan observed that also a singular real algebraic variety $X$ is a mod 2 Euler space, i.e. the link of any point of $X$ has even Euler characteristic. And this condition implies that the sum of all the simplices of the first barycentric subdivision of any triangulation of $X$ is always a mod 2 cycle and he defined its homology class to be the singular Stiefel–Whitney class of the variety $X$. Then, with an insight of Deligne, Sullivan’s Stiefel–Whitney homology classes where enhanced as a natural transformation from a certain covariant functor to the mod 2 homology theory.

Let $X$ be a complex (or real) algebraic set and let $F(X)$ (or $F^{mod2}(X)$) be the abelian group of $\mathbb{Z}$- (or $\mathbb{Z}_2$-)valued complex (or real) algebraically constructible functions on a variety $X$. Then the assignment $F$ (or $F^{mod2}$) : $\mathcal{V} \to \mathcal{A}$ is a contravariant functor (from the category of algebraic varieties to the category of abelian groups) by the usual functional pullback for a morphism $f : X \to Y$: $f^*(\alpha) := \alpha \circ f$. For a constructible set $Z \subset X$, we define
\[ \chi(Z; \alpha) := \sum_{n \in \mathbb{Z}} n \cdot \chi_c(Z \cap \alpha^{-1}(n)) \quad (mod \ 2). \]
Then it turns out that the assignment $F$ (or $F^{mod2}$) : $\mathcal{V} \to \mathcal{A}$ also becomes a covariant functor by the following pushforward defined by
\[ f_*(\alpha)(y) := \chi(f^{-1}(y); \alpha) \quad for \ y \in Y. \]
To show that this is well-defined (i.e. $f_*(\alpha)$ is again constructible) and functorial requires, for example, stratification theory (see [Mac1]) or a suitable theory of constructible sheaves (see [Sch3]). For later use we also point out, that here in the (semi-)algebraic context we do not need the assumption that our spaces are compact or the morphism $f$ is proper for the definition of $f_*$. This properness of $f$ for the definition of $f_*$ is only needed in the corresponding (sub-)analytic context.

The above Sullivan’s Stiefel–Whitney class is now the special case of the following Stiefel–Whitney class transformation:
Theorem 7.1. On the category of compact real algebraic varieties there exists a unique natural transformation
\[ w_* : F^{mod_2}(\ ) \to H_*(\ ; \mathbb{Z}_2) \]
satisfying the normalization condition that for a nonsingular variety \( X \)
\[ w_*(\mathbb{I}_X) = w^*(TX) \cap [X]. \]
Here \( \mathbb{I}_X := 1_X \) is the characteristic function of \( X \).

Note that \( \sharp(X) = \deg(w_*(\mathbb{I}_X)) = \chi(X) \mod 2 \) is just the Euler characteristic \( \mod 2 \) of the singular space \( X \).

7.2. Chern classes \( c_* \). Based on Grothendieck’s ideas or modifying Grothendieck’s conjecture on a Riemann–Roch type formula concerning the constructible étale sheaves and Chow rings (see [Grot2, Part II, note (871), p.361 ff.]), Deligne made the following conjecture — this is usually simply phrased “Deligne and Grothendieck made the following conjecture” — and R. MacPherson [Mac1] proved it affirmatively:

Theorem 7.2. There exists a unique natural transformation
\[ c_* : F(\ ) \to H_{2*}(\ ; \mathbb{Z}) \]
from the constructible function covariant functor \( F \) to the integral homology covariant functor (in even degrees) \( H_{2*} \), satisfying the “normalization” that the value of the characteristic function \( \mathbb{I}_X := 1_X \) of a smooth complex algebraic variety \( X \) is the Poincaré dual of the total Chern cohomology class:
\[ c_*(\mathbb{I}_X) = c^*(TX) \cap [X]. \]

The main ingredients are Chern–Mather classes, local Euler obstruction and “graph construction”. The uniqueness follows from the above normalization condition and resolution of singularities. For an algebraic version of the Chern class transformation \( c_* \) over a base field of characteristic zero (taking values in Chow groups), compare with [Ken]. MacPherson’s approach [Mac1] also works in the complex analytic context, since the analyticity of the “graph construction” was solved by Kwiecinski in his thesis [Kw2].

Remark 7.3. (see [KMY]) The individual component \( c_1 : F(\ ) \to H_{2*}(\ ) \) of the transformation \( c_* : F(\ ) \to H_{2*}(\ ) \) is also a natural transformation and also any linear combination of these components is a natural transformation. Let us consider projective varieties. Then, modulo torsion, these linear combinations are the only natural transformations from the covariant functor \( F \) to the homology functor. In particular, the rationalized Chern–Schwartz–MacPherson class transformation \( c_* \otimes \mathbb{Q} \) is the only such natural transformation satisfying the weaker normalization condition that for each complex projective space \( P \) the top dimensional component of \( c_*(P) \) is the fundamental class \([P]\). A noteworthy feature of the proof of these statements is that one does not need to appeal to the resolution of singularities.

J.-P. Brasselet and M.-H. Schwartz [BrSc] showed that the distinguished value \( c_*(\mathbb{I}_X) \) of the characteristic function of a complex variety embedded into a complex manifold is isomorphic to the Schwartz class \( \text{Schw1}[Schw2] \) via the Alexander duality. Thus the above transformation \( c_* \) is usually called the Chern–Schwartz–MacPherson class transformation. For a complex algebraic variety \( X \), singular or nonsingular, we have the distinguished element \( \mathbb{I}_X := 1_X \) and \( c_*(X) := c_*(\mathbb{I}_X) \) is called the total Chern–Schwartz–MacPherson class of \( X \). By considering mapping \( X \) to a point, one gets
\[ \chi(X) = \deg(c_*(\mathbb{I}_X)) = \sharp(X), \]
which is a singular version of the Gauss–Bonnet–Chern theorem.
Remark 7.4. For a singular version of the Poincaré–Hopf theorem in terms of stratified vector fields see [BLSS], and for a version in terms of 1-forms and characteristic cycles of constructible functions, compare for example with [Sch3] §5.0.3 and [Sch5]. There are also other notions of Chern classes of a singular complex algebraic variety \(X\): Chern-Mather classes \(c^M_\alpha(X)\) (Mac1), Fulton- and Fulton-Johnson Chern classes \(c^F_\alpha(X), c^F_\alpha(X)\) (EM) and [Fu1] Ex. 4.2.6), and for “stringy and arc Chern classes” \(c^{str}_\alpha(X), c^{arc}_\alpha(X)\) see subsection 11.4. In many interesting cases these can be described as \(c_\alpha(\alpha_X)\) for a suitable constructible function \(\alpha_X\) related to some geometric properties of the singular space \(X\) (compare [Alu1, Br2] Pa [Sch1, Sch4, Sch5, Su]). Of course \(\alpha_X = 1_X\) for \(X\) smooth, but in general \(\alpha_X \neq 1_X\) so that the MacPherson Chern class transformation \(c_\alpha\) is the basic one, but in general \(\mathbb{I}_X = 1_X\) is not the only possible choice of a distinguished element \(\mathbb{I}_X\)!

7.3. Todd classes \(td_*\). Motivated by the formulation of the Chern–Schwartz–MacPherson class transformation, P. Baum, W. Fulton and R. MacPherson [BFM] have extended GRR to singular varieties, by introducing the so-called localized Chern character \(ch^M_\alpha(F)\) of a coherent sheaf \(F\) with \(X\) embedded into a non-singular quasi-projective variety \(M\), as a substitute of \(ch^*(F) \cap [X]\) in the above GRR. Note that if \(X\) is smooth \(ch^X_\alpha(F) = ch^*(F) \cap [X]\). In [BFM] they showed the following theorem:

Theorem 7.5. (Baum–Fulton–MacPherson’s Riemann–Roch)

(i) \(td_* (F) := td^*(i^*_M TM) \cap ch^M_\alpha(F)\) is independent of the embedding \(i_M : X \rightarrow M\).

(ii) Let the transformation \(td_* : G_0( ) \rightarrow H_{2*}( ; \mathbb{Q})\) be defined by

\[
\begin{align*}
\text{G}_0(X) & \xrightarrow{td_*} H_{2*}(X; \mathbb{Q}) \\
\downarrow f_* & \downarrow f_* \\
\text{G}_0(Y) & \xrightarrow{td_*} H_{2*}(Y; \mathbb{Q})
\end{align*}
\]

for any variety \(X\). Then \(td_*\) is actually natural, i.e., for any morphism \(f : X \rightarrow Y\) the following diagram commutes:

\(\text{G}_0(X) \xrightarrow{td_*} H_{2*}(X; \mathbb{Q})\)

\(f_*\)

\(\text{G}_0(Y) \xrightarrow{td_*} H_{2*}(Y; \mathbb{Q})\)

i.e., for any embeddings \(i_M : X \rightarrow M\) and \(i_N : Y \rightarrow N\)

(BFM-RR) \(td^*(i^*_N TN) \cap ch^N_\alpha(f_* F) = f_* (td^*(i^*_M TM) \cap ch^M_\alpha(F))\).

For a complex algebraic variety \(X\), singular or nonsingular, \(td_* (X) := td_* (\mathcal{O}_X)\) is called the Baum–Fulton–MacPherson’s Todd homology class of \(X\), i.e. the class of the structure sheaf is the distinguished element \(\mathbb{I}_X := [\mathcal{O}_X]\). And we get

\[\chi_\alpha(X) = \int_X td_* (X) = \sharp(X),\]

which is a singular version of the Riemann–Roch theorem. And in [BFM] this Todd class transformation is moreover factorized through complex K-homology, which maybe is the most natural formulation of this transformation. For the algebraic version of the Todd class transformation \(td_*\) over any base field compare with [Fu1] chapter 18.

Remark 7.6 (Euler homology class \(e_0\)). Even though the formulation of the BFM–RR was motivated by that of the Chern–Schwartz–MacPherson class, it was proved in a completely different way. And now there is available a similar proof of MacPherson’s theorem for the embedded context based on the theory of characteristic cycles \(CC\) of constructible functions, with the Segre class \(s_{CC}\) of these conic characteristic cycles playing the role of the localized Chern character in the proof of Baum–Fulton–MacPherson. Here these characteristic cycles are conic Lagrangian cycles in \(T^* M | X\), and the pullback

\[e_0 := k^* CC : F(X) \rightarrow H_0(X; \mathbb{Z})\]
by the zero section $k : X \to T^*M|X$ can be seen as a functorial Euler homology class transformation even in the context of real geometry. In particular
\[
\chi(X) = \deg(e(\mathbb{I}_X)) = z(X)
\]
also in this context. For more details of this, see [Sch4, Sch5]. Finally, this approach by characteristic cycles also gives a new approach to the Stiefel-Whitney class transformation $w_*$ of Sullivan as observed and explained in [FuMC].

7.4. L-classes $L_*$. Using the notion of “perversity”, M. Goresky and R. MacPherson ([GM1], [GM2]) have introduced Intersection Homology Theory. In [GM1] they introduced a homology $L$-class $L_\text{GM}^*(X)$ for stratified spaces $X$ with even (co)dimensional strata such that if $X$ is nonsingular it becomes the Poincaré dual of the original Thom-Hirzebruch $L$-class: $L_\text{GM}^*(X) = L^*(TX) \cap [X]$. And for rational PL-homology manifolds, their $L$-classes agree with the classes introduced by Thom long ago in [Thom2] as one of the first characteristic classes of suitable singular spaces.

Later, S. Cappell and J. Shaneson [CS1] (see also [CS2] and [Sh]) introduced a homology $L$-class transformation $L^*$, which turns out to be a natural transformation from the abelian group $\Omega(X)$ of cobordism classes of selfdual constructible complexes, whose definition we now explain, to the rational homology group $[BSY2]$ (cf. [Y2]).

Let $X$ be a compact complex analytic (algebraic) space with $D^b_c(X)$ the bounded derived category of complex analytically (algebraically) constructible complexes of sheaves of $\mathbb{Q}$-vector spaces (compare [KS] and [Sch3]). So we consider bounded sheaf complexes $\mathcal{F}$, which have locally constant cohomology sheaves with finite dimensional stalks along the strata of a complex analytic (algebraic) Whitney stratification of $X$. This is a triangulated category with translation functor $T = [1]$ given by shifting a complex one step to the left. It also has a duality in the sense of Youssin [You] induced by the Verdier duality functor (compare [Sch3, Chap.4] and [KS, Chap.VIII]):
\[
D_X := \text{Rhom}(\cdot, k^{\mathbb{Q}_{pt}}) : D^b_c(X) \to D^b_c(X),
\]
with $k : X \to \{\mathbb{pt}\}$ a constant map, together with its biduality isomorphism $\text{can} : id \sim D_X \circ D_X$. A constructible complex $\mathcal{F} \in \text{ob}(D^b_c(X))$ is called selfdual, if there is an isomorphism
\[
d : \mathcal{F} \sim D_X(\mathcal{F}).
\]
The pair $(\mathcal{F}, d)$ is called symmetric or skew-symmetric, if
\[
D_X(d) \circ \text{can} = d \quad \text{or} \quad D_X(d) \circ \text{can} = -d.
\]
Finally an isomorphism or isometry of selfdual objects $(\mathcal{F}, d)$ and $(\mathcal{F}', d')$ is an isomorphism $u$ such that the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{u} & \mathcal{F}' \\
\downarrow d & & \downarrow d' \\
D_X(\mathcal{F}) & \xleftarrow{D_X(u)} & D_X(\mathcal{F}')
\end{array}
\]

The isomorphism classes of such (skew-)symmetric selfdual complexes form a set, which becomes a monoid with addition induced by the direct sum. Using a definition of Youssin [You], the cobordism groups $\Omega_\pm(X)$ of (skew-)symmetric selfdual constructible complexes on $X$ are defined by introducing a suitable cobordism relation in terms of an octahedral diagram, i.e. a diagram (Oct) of the following form:
Here the morphism marked by [1] are of degree one, the triangles marked + are commutative, and the ones marked \( \tilde{d} \) are distinguished. Finally the two composite morphisms from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) (via \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \)) have to be the same, and similarly for the two composite morphisms from \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \) (via \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \)).

Application of the duality functor \( D := D_\mathcal{X} \) and a rotation by \( 180^\circ \) about the axis connecting upper-left and lower-right corner induces another octahedral diagram \((RD \cdot \text{Oct})\) such that \( RD \) applied to \((RD \cdot \text{Oct})\) gives the octahedral diagram \((D^2 \cdot \text{Oct})\) which one gets from \((\text{Oct})\) by application of \( D^2 \) (compare with [Yoo p.387/388]). Then the octahedral diagram \((\text{Oct})\) is called symmetric or skew-symmetric, if there is an isomorphism \( d : (\text{Oct}) \to (RD \cdot \text{Oct}) \) of octahedral diagrams such that

\[
RD(d) \circ \text{can} = d \quad \text{or} \quad RD(d) \circ \text{can} = -d
\]

as maps of octahedral diagrams \((\text{Oct}) \to (RD \cdot \text{Oct})\). Note that this induces in particular (skew-)symmetric dualities \( d_1 \) and \( d_2 \) of the corners \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), and \((\text{Oct}, d)\) is called an elementary cobordism between \((\mathcal{F}_1, d_1)\) and \((\mathcal{F}_2, d_2)\). This notion is a symmetric and reflexive relation. \((\mathcal{F}, d)\) and \((\mathcal{F}', d')\) are called cobordant, if there is a sequence

\[
(\mathcal{F}, d) = (\mathcal{F}_0, d_0), (\mathcal{F}_1, d_1), \ldots, (\mathcal{F}_m, d_m) = (\mathcal{F}', d')
\]

with \((\mathcal{F}_i, d_i)\) elementary cobordant to \((\mathcal{F}_{i+1}, d_{i+1})\) for \( i = 0, \ldots, m - 1 \). This cobordism relation is then an equivalence relation.

The cobordism group \( \Omega^\pm_*(X) \) of selfdual constructible complexes on \( X \) is the quotient of the monoid of isomorphism classes of (skew-)symmetric selfdual complexes by this cobordism relation. These are indeed abelian groups and not just monoids.

Consider now an algebraic (or holomorphic) map \( f : X \to Y \), with \( X, Y \) compact so that \( f \) is proper. Then \( Rf_* ) \simeq Rf_! \) maps \( D^b_\mathcal{C}(X) \) to \( D^b_\mathcal{C}(X) \). Moreover, the adjunction isomorphism

\[
Rf_* \text{Rhom}(\mathcal{F}, f^! k^! \mathbb{Q}_{\text{pt}}) \simeq \text{Rhom}(Rf_! \mathcal{F}, k^! \mathbb{Q}_{\text{pt}})
\]

induces the isomorphism

\[
(7.7) \quad Rf_* D_X \xrightarrow{\sim} D_Y Rf_! \simeq D_Y Rf_*
\]

so that \( Rf_* \) commutes with Verdier-duality. In particular \( Rf_* \) maps selfdual constructible complexes on \( X \) to selfdual constructible complexes on \( Y \) inducing group homomorphisms

\[
f_* : \Omega^\pm_*(X) \to \Omega^\pm_*(Y); \quad [(\mathcal{F}, d)] \mapsto [(Rf_* \mathcal{F}, Rf_*(d))].
\]
A simple example of a self-dual constructible complex is the shifted constant sheaf $\mathbb{Q}_Z[n]$ on a complex manifold $Z$ of pure dimension $n$, with the duality isomorphism induced from the complex orientation of $Z$ by Poincaré-Verdier duality:

$$k^!\mathbb{Q}_{pt} \cong \mathbb{Q}_Z[2n],$$

with $k : X \to \{pt\}$ a constant map.

This is (skew-)symmetric for $n$ even (or odd).

Then the results of Cappell-Shaneson [CS1, §5] can be reformulated as in [BSY2](cf. [Y2, Corollary 2.3]):

**Theorem 7.8** (Cappell-Shaneson). For a compact complex analytic (or algebraic) space $X$ there is a homology $L$-class transformation

$$L_* : \Omega(X) := \Omega_+(X) \oplus \Omega_-(X) \to H_\ast(X, \mathbb{Q}),$$

which is a group homomorphism functorial for the pushdown $f_*$ induced by a holomorphic (or algebraic) map. The degree of $L_0((\mathcal{F}, d))$ is the signature of the induced pairing

$$H^0(X, \mathcal{F}) \otimes_\mathbb{Q} \mathbb{R} \times H^0(X, \mathcal{F}) \otimes_\mathbb{Q} \mathbb{R} \to \mathbb{R}$$

(by definition this is 0 for a skew-symmetric pairing). Moreover, for $X$ smooth of pure dimension $n$ one has the normalization

$$L_*(([Q_X[n]], d)) = L^*(TX) \cap [X].$$

There is also a uniqueness statement in [CS1, §5] for such an $L$-class transformation, but for this one has to go outside the complex algebraic or analytic context.

For $X$ pure dimensional (otherwise one should only look at the top dimensional irreducible components of $X$) one has the distinguished self-dual constructible intersection cohomology complex $\mathbb{I}_X := IC_X$, whose global cohomology calculates the intersection (co)homology of Goresky-MacPherson. By definition one gets $L_*(X) := L_* IC_X = L_*^{GM}(X)$ so that

$$\int_X L_*(X) = \sharp(X)$$

is the signature of the global intersection (co)homology.

**Remark 7.9.** Thom used in [Thom2] his combinatorial $L$-classes for the definition of combinatorial Pontrjagin classes of rational PL-homology manifolds. Note that in the context of rational homology manifolds, rational $L$- and Pontrjagin classes carry the same information (i.e. can be deduced from each other). But this is not the case for more singular spaces, and only a corresponding $L$-class transformation exists for suitable singular spaces, but not a Pontrjagin class transformation.

So all these theories of characteristic homology class transformations for singular spaces have the same formalism, but their existence and construction is due to completely different underlying ideas: mod 2 Euler spaces for $w_\ast$, local Euler obstruction for $c_\ast$, localized Chern character for $td^\ast$ and duality for $L_\ast$. Nevertheless it is natural to ask for another theory of characteristic homology classes of singular spaces, which unifies these theories for complex algebraic varieties:

**Problem 7.10.** (cf. [Mac2] and [Y3]) Is there a "unifying and singular version" $\mathcal{L}_\ast$ of the generalized Hirzebruch–Riemann–Roch $g$-HRR such that

- $(y = -1)$ \(\mathcal{L}_\ast\) gives rise to the rationalized Chern–Schwartz–MacPherson’s class $c_\ast \otimes \mathbb{Q}$,
- $(y = 0)$ \(\mathcal{L}_\ast\) gives rise to the Baum–Fulton–MacPherson’s Todd class $td_\ast$, and
- $(y = 1)$ \(\mathcal{L}_\ast\) gives rise to the Cappell–Shaneson’s homology $L$-class $L_\ast$. 
An obvious obstacle for this problem is that the source covariant functors of these three natural transformations are all different. And even if such a theory is not known, its normalization condition for a smooth complex algebraic manifold $M$ has to be

$$cl_{\ell}(\mathbb{1}_M) = \overline{td}(TM) \cap [M]$$

by $g$-HRR so that this transformation has to be called a Hirzebruch $\overline{td}(g\ast)$ or $T_{g\ast}$-class transformation.

8. Relative Grothendieck rings of varieties and motivic characteristic classes

A “reasonable” answer for the above Problem [7.10] has been obtained in [BSY2] via the so-called relative Grothendieck ring of complex algebraic varieties over $X$, denoted by $K_0(V/X)$. This ring was introduced by E. Looijenga in [Lo] and further studied by F. Bittner in [Bit]. The relative Grothendieck group $K_0(V/X)$ (of morphisms over a variety $X$) is the quotient of the free abelian group of isomorphism classes of morphisms to $X$ (denoted by $[Y \to X]$ or $[Y \xrightarrow{h} X]$), modulo the following additivity relation:

$$[Y \xrightarrow{h} X] = [Z \hookrightarrow Y \xrightarrow{h} X] + [Y \setminus Z \xrightarrow{h} X]$$

for $Z \subset Y$ a closed subvariety of $Y$. The ring structure is given by the fiber square: for $[Y \xrightarrow{f} X], [W \xrightarrow{g} X] \in K_0(V/X)$

$$[Y \xrightarrow{f} X] \cdot [W \xrightarrow{g} X] := [Y \times_X W \xrightarrow{f \times g} X].$$

Here $Y \times_X W \xrightarrow{f \times g}$, $X$ is $g \circ g' = f \circ g'$ where $f'$ and $g'$ are as in the following diagram

$$\begin{array}{c}
Y \times_X W \\
g' \downarrow \quad \downarrow g
\end{array} \quad \begin{array}{c}
\xrightarrow{f'} W' \\
 X
\end{array}$$

The relative Grothendieck ring $K_0(V/X)$ has the unit $1_X := [X \xrightarrow{id_X} X]$, which later becomes the distinguished element $\mathbb{1}_X := [id_X]$. Similarly one gets an exterior product

$$\times : K_0(V/X) \times K_0(V/Y) \to K_0(V/X \times Y).$$

Note that when $X = \{pt\}$ is a point, then the relative Grothendieck ring $K_0(V/\{pt\})$ is nothing but the usual Grothendieck ring $K_0(V)$ of $V$, which is the free abelian group generated by the isomorphism classes of varieties modulo the subgroup generated by elements of the form $[V] - [V'] - [V \setminus V']$ for a subvariety $V' \subset V$, and the ring structure is given by the Cartesian product of varieties.

Remark 8.1. In some sense the Grothendieck ring $K_0(V)$ can be seen as an algebraic substitute for cobordism rings $\Omega_*$ of smooth manifolds, based on the additivity instead of a cobordism relation.

For a morphism $f : X' \to X$, the pushforward

$$f_* : K_0(V/X') \to K_0(V/X)$$

is defined by

$$f_*[Y \xrightarrow{h} X'] := [Y \xrightarrow{f \circ h} X].$$

With this pushforward, the assignment $X \mapsto K_0(V/X)$ is a covariant functor. The pullback

$$f^* : K_0(V/X) \to K_0(V/X')$$
is defined as follows: for a fiber square
\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & X' \\
\downarrow f' & & \downarrow f \\
Y & \xrightarrow{g} & X
\end{array}
\]
the pullback \( f^*[Y \xrightarrow{g} X] := [Y' \xrightarrow{g'} X'] \). With this pullback, the assignment \( X \mapsto - \) is a contravariant functor. Let \( \text{Iso}^{pr}(SV/X) \) be the free abelian groups on isomorphism classes of proper morphisms from smooth varieties to a given variety \( X \).

Then we get the canonical quotient homomorphism
\[
\text{quo} : \text{Iso}^{pr}(SV/X) \rightarrow K_0(V/X)
\]
which is surjective by the above additivity relation and Hironaka’s resolution of singularities \([\text{Hi}]\). And it turns out that the kernel of this surjection map is generated by the “blow-up relation”, more precisely we have the following theorem, which is due to F. Bittner \([\text{Bi}, \text{Theorem 5.1}]\), based on the very deep “weak factorization theorem” \([\text{AKMW}]\) and \([\text{W}]\):

**Theorem 8.2.** The relative Grothendieck group \( K_0(V/X) \) is isomorphic to the quotient of the free abelian group \( \text{Iso}^{pr}(SV/X) \) modulo the following “blow-up relation”
\[
[\emptyset \rightarrow X] := 0 \quad \text{and} \quad [\text{Bl}_Y X' \rightarrow X] - [E \rightarrow X] = [X' \rightarrow X] - [Y \rightarrow X]
\]
for any Cartesian “blow-up” diagram
\[
\begin{array}{ccc}
E & \xrightarrow{i} & \text{Bl}_Y X' \\
\downarrow \pi' & & \downarrow \pi \\
Y & \xrightarrow{i} & X' \xrightarrow{f} X,
\end{array}
\]
with \( i \) a closed embedding of smooth (pure dimensional) varieties and \( f : X' \rightarrow X \) proper.

Here \( \pi : \text{Bl}_Y X' \rightarrow X' \) is the blow-up of \( X' \) along \( Y \) with \( E \) denoting the exceptional divisor.

From this theorem we can get the following corollary:

**Theorem 8.3.** Let \( B_* : \mathcal{V}/k \rightarrow A \) be a functor from the category of reduced separated schemes of finite type over \( \mathbb{C} \) to the category of abelian groups such that
(i) \( B_*(\emptyset) := 0 \),
(ii) it is covariantly functorial for proper morphisms, and
(iii) for any smooth variety \( X \) there exists a distinguished element \( d_X \in B_*(X) \) such that
(iii-1) for any isomorphism \( h : X' \rightarrow X \), \( h_*(d_{X'}) = d_X \) and
(iii-2) for any Cartesian “blow-up” diagram as in the above Theorem 8.2 with \( f = \text{id}_X \),
\[
\pi_*(d_{\text{Bl}_Y X}) - i_*\pi'_*(d_E) = d_X - i_*(d_Y) \in B_*(X).
\]

Then we have by (iii-1) that there exists a unique natural transformation of covariant functors
\[
\Phi : \text{Iso}^{pr}(SV/X) \rightarrow B_*(X)
\]
satisfying the normalization condition that for smooth \( X \)
\[
\Phi([X \xrightarrow{\text{id}} X]) = d_X,
\]
and furthermore by (iii-2) there exists a unique natural transformation of covariant functors
\[
\Phi : K_0(V/X) \rightarrow B_*(X)
\]
satisfying the normalization condition that for smooth \( X \)
\[
\Phi([X \xrightarrow{\text{id}} X]) = d_X.
\]
Then, using results of [Gros IV.1.2.1] or [GNA] Proposition 3.3, we can get the following corollary about a motivic Chern class transformation $mC_*$. 

**Corollary 8.4.** There exists a unique natural transformation (with respect to proper maps) 

$$mC_* : K_0(V/\text{ } ) \to G_0(\text{ } ) \otimes \mathbb{Z}[y]$$

satisfying the normalization condition that for $X$ smooth 

$$mC_*([X \xrightarrow{id} X]) = \sum_{i=0}^{\dim X} [\Lambda^i T^*X] y^i := \Lambda_y([T^*X]) \cap [O_X].$$

Here $\Lambda_y(\text{ })$ is the so-called total $\Lambda$-class.

If we compose $mC_*|_{y=-1,0,1}$ with the natural transformation $G_0(\text{ }) \to K_0^{\text{top}}(\text{ })$ to topological K-homology constructed in [BFM2], then $mC_*(X)$ unifies for $X$ smooth the following K-theoretical homology classes:

(y=-1) the top-dimensional Chern class $c_{K}^{\text{top}}(TX) \cap [X]_K$ in K-theory 

$$mC_*|_{y=-1}(\text{[id}_X\text{])} = \Lambda_{-1}([T^*X]) \cap [X]_K,$$

(y=0) the fundamental class in K-homology of the complex manifold $X$ 

$$mC_*|_{y=0}(\text{[id}_X\text{])} = [X]_K,$$

(y=1) the class of the signature operator of the underlying spin$^c$ manifold of $X$ 

$$mC_*|_{y=1}(\text{[id}_X\text{])} = \Lambda_1([T^*X]) \cap [X]_K.$$

Consider the twisted BFM–RR transformation 

$$td_{(1+y)} : G_0(X) \otimes \mathbb{Z}[y] \to H_{2*}(X) \otimes \mathbb{Q}[y, (1 + y)^{-1}]$$

defined by 

$$td_{(1+y)}([F]) := \sum_{i \geq 0} td_i([F])(1 + y)^{-i}$$

and extending it linearly with respect to $\mathbb{Z}[y]$ (Y3). Using this twisted BFM–RR transformation $td_{(1+y)}$ and the above transformation $mC_*$, we define the Hirzebruch class transformation $T_{y*}$ as the composite $T_{y*} := td_{(1+y)} \circ mC_*$. Then we get the following theorem:

**Theorem 8.5.** Let $K_0(V/X)$ be the Grothendieck group of complex algebraic varieties over $X$. Then there exists a unique natural transformation (with respect to proper maps) 

$$T_{y*} : K_0(V/\text{ } ) \to H_{2*}^{BM}(\text{ } ) \otimes \mathbb{Q}[y] \subset H_{2*}^{BM}(\text{ } ) \otimes \mathbb{Q}[y, (1 + y)^{-1}]$$

such that for $X$ nonsingular 

$$T_{y*}([X \xrightarrow{id} X]) = \bar{td}_y(TX) \cap [X].$$

**Remark 8.6.** The transformations $mC_*$ and $T_{y*}$ can also be defined in the same way in the algebraic context over a base field of characteristic zero, using the algebraic version of the Todd transformation $td_*$ as in [F1] chapter 18, and in the compactifiable complex analytic context, using the analytic version of the Todd transformation $td_*$ constructed in [Levy] (compare with [BSY2] for more details).

For a later use, we observe that $T_{y*}$ commutes with the exterior product (and similarly for $mC_*$), i.e., the following diagram commutes: 

$$\begin{array}{ccc} K_0(V/X) \times K_0(V/Y) & \longrightarrow & K_0(V/X \times Y) \\
T_{y*} \times T_{y*} & \downarrow & \downarrow T_{y*} \\
H_{2*}(X) \otimes \mathbb{Q}[y] \times H_{2*}(Y) \otimes \mathbb{Q}[y] & \longrightarrow & H_{2*}(X \times Y) \otimes \mathbb{Q}[y]. \end{array}$$

And we have the following theorem for a compact complex algebraic variety $X$: 

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Theorem 8.7. \((y = -1)\) There exists a unique natural transformation \(\epsilon : K_0(V/\ ) \to F(\ )\) such that for \(X\) nonsingular \(\epsilon([X \xrightarrow{id} X]) = 1_X\). And the following diagram commutes

\[
\begin{array}{ccc}
K_0(V/X) & \xrightarrow{\epsilon} & F(X) \\
T_{-1} \downarrow & & \downarrow c_* \\
H_{2*}(X) \otimes \mathbb{Q} & & .
\end{array}
\]

\((y = 0)\) There exists a unique natural transformation \(\gamma : K_0(V/\ ) \to G_0(\ )\) such that for \(X\) nonsingular \(\gamma([X \xrightarrow{id} X]) = [O_X]\). And the following diagram commutes

\[
\begin{array}{ccc}
K_0(V/X) & \xrightarrow{\gamma} & G_0(X) \\
T_0 \downarrow & & \downarrow t_\ast \\
H_{2*}(X) \otimes \mathbb{Q} & & .
\end{array}
\]

\((y = 1)\) There exists a unique natural transformation \(\omega : K_0(V/\ ) \to \Omega(\ )\) such that for \(X\) nonsingular \(\omega([X \xrightarrow{id} X]) = \left[\mathbb{Q}X [\dim X]\right]\). And the following diagram commutes

\[
\begin{array}{ccc}
K_0(V/X) & \xrightarrow{\omega} & \Omega(X) \\
T_1 \downarrow & & \downarrow L_* \\
H_{\ast}(X) \otimes \mathbb{Q} & & .
\end{array}
\]

An original proof of the above Theorem 8.7 uses Saito’s theory of mixed Hodge modules [Sai] instead of the above Theorem 8.2. And an even more elementary proof can be given based on some classical results of [DuBo] about the so-called DuBois complex of a singular complex algebraic variety. Only the proof of the case \((y = 1)\) of the above Theorem 8.7 depends, up to now, on the Bittner’s theorem, i.e., the above Theorem 8.2 in other words, on the “weak factorization theorem” ([AKMW] and [W]). Also note that the transformation \(\epsilon\) is defined for any algebraic map of not necessarily compact algebraic varieties, and it also commutes with pullback and (exterior) products. For more details, see [BSY2].

Remark 8.8. The reader should be warned that the transformations \(\gamma\) and \(\omega\) above do not preserve the distinguished elements in general. For any compact singular complex algebraic variety \(X\) one has \(\epsilon([id_X]) = 1_X\) so that the Hirzebruch class \(T_{y\ast}(X) := T_{y\ast}([id_X])\) specializes to \(T_{-1\ast}(X) = c_*\) \((X) \in H_{2\ast}(X; \mathbb{Q})\). But in general

\[
\gamma([id_X]) \neq [O_X] \in G_0(X) \quad \text{and} \quad T_{0\ast}(X) \neq t_\ast(X).
\]

But \(T_{0\ast}(X) = t_\ast(X)\) if \(X\) has at most “Du Bois singularities”, e.g., “rational singularities”-like, for example, toric varieties. Similarly

\[
\omega([id_X]) \neq [IC_X] \in \Omega(X) \quad \text{and} \quad T_{1\ast}(X) \neq L_\ast(X)
\]

in general, but we conjecture that \(T_{1\ast}(X) = L_\ast(X)\) for \(X\) a rational homology manifold.

Moreover, the Hirzebruch characteristic class \(t_{\lambda\ast} = T_{\lambda\ast}\) is the most general normalized and multiplicative characteristic class of complex vector bundles

\[
cl_{\lambda\ast} : \text{Vect}(X) \to H^{2\ast}(X; \Lambda),
\]

with \(\Lambda\) a \(\mathbb{Q}\)-algebra, which satisfies the condition of Theorem 8.7 with

\[
d_X := cl_{\lambda\ast}(TX) \cap [X] \in H^{2\ast}_{BM}(X; \Lambda).
\]
for $X$ smooth. In fact, the corresponding genus $\Phi_f$ factorizes as

$$\text{Iso}^p(S\mathcal{V}/\{pt\}) \xrightarrow{\Omega^L \otimes \mathbb{Q}}$$

(8.9) $K_0(\mathcal{V}) \xrightarrow{\Phi_f} \Lambda = H_{2*}(\{pt\}; \Lambda).$

Moreover, the characteristic class $cL_f^*$ or its genus $\Phi_f$ is uniquely determined by

$$\Phi_f([P^n(\mathcal{C})]) = \int_{P^n(\mathcal{C})} (cL_f^*(TP^n(\mathcal{C}))) \cap [P^n(\mathcal{C})]$$

for all $n$. But if $\Phi_f$ also factorizes over $K_0(\mathcal{V})$ then we get from the decomposition

$$P^n(\mathcal{C}) = \{pt\} \cup \mathbb{C} \cup \cdots \cup \mathbb{C}^n$$

by “additivity” and “multiplicativity” (and compare with equation (6.4)):

(8.10) $\Phi_f([P^n(\mathcal{C})]) = 1 + (-y) + \cdots + (-y)^n$ with $y := 1 - \Phi_f([P^1(\mathcal{C})]).$

So $\Phi_f$ is a specialization of the Hirzebruch $\chi_0$-genus corresponding to the Hirzebruch characteristic class $T^*_f$. Of course here we use a decomposition into the non-compact manifolds $\mathbb{C}^n$, which is classically forbidden for a genus, with $y = -\Phi_f([\mathbb{C}])$.

**Remark 8.11.** So additivity is the underlying principle which “singles out” those normalized and multiplicative characteristic classes $cL^*_f$, which have (so far) a functorial extension to singular spaces. Also note that the specialization $y = 1$ corresponding to the signature genus $\text{sign} = \chi_1$ and the characteristic L-class transformation $L^* = T^*_1$ is the only one that factorizes by the canonical map $\Omega^L_1 \otimes \mathbb{Q} \to \Omega^SO_1 \otimes \mathbb{Q}$ over the cobordism ring $\Omega^SO_1$ of oriented manifolds, since $[P^1(\mathcal{C})] = 0 \in \Omega^SO_1$. In particular this “explains” why there is no functorial Pontrjagin class transformation for singular spaces.

For $X$ a compact complex algebraic variety one can also deduce from Theorem 8.3 the Chern class transformation

$$c_* : K_0(\mathcal{V}/X) \to H_{2*}(X; \mathbb{Z}),$$

on the relative Grothendieck group $K_0(\mathcal{V}/X)$ without appealing to MacPherson’s theorem, since the distinguished element

$$d_X := c^*(TX) \cap [X] \in H_{2*}(X; \mathbb{Z})$$

of a smooth space $X$ satisfies the corresponding conditions. Condition (iii-1) follows from the projection formula, and condition (iii-2) is an easy application (by pushing down to $X$) of the classical “blowing up formula for Chern classes” [Fu1, Theorem 15.4]. And recent work of Aluffi [Alu3] can be interpreted as showing that this transformation $c_*$ factorizes over $\epsilon : K_0(\mathcal{V}/X) \to F(\epsilon)$.

9. BIVARIANT CHARACTERISTIC CLASSES

In [FM] (also, see [Fu1]) W. Fulton and R. MacPherson introduced the notion of Bivariant Theory, which is a simultaneous generalization of a pair of covariant and contravariant functors. Most pairs of covariant and contravariant theories, e.g., such as homology theory, K-theory, etc., extend to bivariant theories. A bivariant theory $B$ on a suitable category $C$ (with a distinguished class of so-called “proper” or “confined” maps) with values in the category of abelian groups is an assignment to each morphism $X \overset{f}{\to} Y$ in the category $C$ an abelian group $B(X \overset{f}{\to} Y)$, which is equipped with the following three basic operations: (Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : B(X \overset{f}{\to} Y) \otimes B(Y \overset{g}{\to} Z) \to B(X \overset{gf}{\to} Z)$$
is defined.
(Pushforward operations): For morphisms \( f : X \to Y \) and \( g : Y \to Z \) with \( f \) proper, the pushforward operation
\[
 f_* : \mathcal{B}(X \xrightarrow{gf} Z) \to \mathcal{B}(Y \xrightarrow{g} Z)
\]
is defined.
(Pullback operations): For a fiber (or more generally a so-called independent) square
\[
\begin{array}{c}
X' \xrightarrow{g'} X \\
\downarrow f' \downarrow f \\
Y' \xrightarrow{g} Y
\end{array}
\]
the pullback operation
\[
g^* : \mathcal{B}(X \xrightarrow{f} Y) \to \mathcal{B}(X' \xrightarrow{f'} Y')
\]
is defined. And these three operations are required to satisfy seven compatibility axioms (see [FM, Part I, §2.2] for details). In particular, the class of “proper” maps has to be stable under composition and base change, and should contain all identity maps. Let \( \mathcal{B}, \mathcal{B}' \) be two bivariant theories on such a category \( \mathcal{C} \). Then a Grothendieck transformation from \( \mathcal{B} \) to \( \mathcal{B}' \)
\[
\gamma : \mathcal{B} \to \mathcal{B}'
\]
is a collection of homomorphisms
\[
\mathcal{B}(X \to Y) \to \mathcal{B}'(X \to Y)
\]
for a morphism \( X \to Y \) in the category \( \mathcal{C} \), which preserves the above three basic operations:
(i) \( \gamma(\alpha \bullet \beta) = \gamma(\alpha) \bullet_{\mathcal{B}'} \gamma(\beta) \),
(ii) \( \gamma(f_* \alpha) = f_* \gamma(\alpha) \), and
(iii) \( \gamma(g^* \alpha) = g^* \gamma(\alpha) \).

\( \mathcal{B}_*(X) := \mathcal{B}(X \to \text{pt}) \) and \( \mathcal{B}^*(X) := \mathcal{B}(X \xrightarrow{\text{id}} X) \) become a covariant functor for proper maps and a contravariant functor, respectively. And a Grothendieck transformation \( \gamma : \mathcal{B} \to \mathcal{B}' \) induces natural transformations \( \gamma_* : \mathcal{B}_* \to \mathcal{B}'_* \) and \( \gamma^* : \mathcal{B}^* \to \mathcal{B}'^* \) such that \( \gamma_* \) commutes with the (bivariant) exterior product, i.e. the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{B}_*(X) \times \mathcal{B}_*(Y) & \xrightarrow{\times} & \mathcal{B}_*(X \times Y) \\
\gamma_* \times \gamma_* \downarrow & & \downarrow \gamma_* \\
\mathcal{B}'_*(X) \times \mathcal{B}'_*(Y) & \xrightarrow{\times} & \mathcal{B}'_*(X \times Y)
\end{array}
\]
If we have a Grothendieck transformation \( \gamma : \mathcal{B} \to \mathcal{B}' \), then via a bivariant class \( b \in \mathcal{B}(X \xrightarrow{f} Y) \) we get the commutative diagram
\[
\begin{array}{c}
\mathcal{B}_*(Y) \xrightarrow{\gamma_*} \mathcal{B}'_*(Y) \\
\downarrow \gamma(b) \downarrow \\
\mathcal{B}_*(X) \xrightarrow{\gamma_*} \mathcal{B}'_*(X)
\end{array}
\]
This is called the Verdier-type Riemann–Roch formula associated to the bivariant class \( b \).

**Bivariant Todd class transformation** \( \tau \). The most important (and motivating) example of such a Grothendieck transformation of bivariant theories is the bivariant Riemann-Roch
transformation $\tau$ from the bivariant algebraic $K$-theory $\mathbb{K}_{alg}$ of perfect complexes to rational bivariant homology $\mathbb{H}_Q$

\[ \tau : \mathbb{K}_{alg} \to \mathbb{H}_Q \]

constructed in [FM] Part II in the complex quasi-projective context. Here $\mathbb{H}_Q$ is the bivariant homology theory corresponding to usual cohomology with rational coefficients constructed in [FM] §3.1 for more general homology theories. Then the associated contravariant theory $\mathbb{H}_Q(X) = H^*(X; \mathbb{Q})$ is the usual cohomology, and the associated covariant theory $\mathbb{H}_Q(X) = H_*^{BM}(X; \mathbb{Q})$ is the Borel-Moore homology. Similarly $\mathbb{K}_{alg}^* \simeq K^0$ is the Grothendieck group of algebraic coherent sheaves. Then the associated contravariant transformation $\tau^*$ is the Chern character

\[ ch^* : \mathbb{K}_{alg}(\ ) \simeq K^0(\ ) \to H^*(\ , \mathbb{Q}) \simeq H_*^Q(\ ) , \]

and the associated covariant transformation

\[ \tau_* : \mathbb{K}_{alg}^*(\ ) \simeq G_0(\ ) \to H_*^{BM}(\ , \mathbb{Q}) \simeq H_*^Q(\ ) \]

is just Baum–Fulton–MacPherson’s Todd class transformation $td_*$ constructed in [BFM]. And the bivariant transformation $\tau$ unifies many different known Riemann-Roch type theorems. In particular for a smooth morphism $f : X \to Y$ of possible singular varieties one has

\[ \Pi_f := [\mathcal{O}_X] \in \mathbb{K}_{alg}(X \xrightarrow{f} Y) , \]

with $\tau(\Pi_f) = td^*(T_f) \cdot [f]$. Here $T_f$ is the vector bundle of tangent spaces of fibers of $f$, and $[f] \in \mathbb{H}_Q(X \xrightarrow{f} Y)$ is the canonical orientation of the smooth morphism $f$. Then the Verdier-type Riemann–Roch formula (9.1) associated to $\Pi_f$ becomes the usual Verdier Riemann–Roch theorem for the Todd class transformation $td_* :$

\[ (9.2) \quad td_*(f^*\beta) = td^*(T_f) \cap f^!td_*(\beta) \quad \text{for } \beta \in G_0(Y) . \]

Here $f^! = [f] \cdot : H_*^{BM}(Y; \mathbb{Q}) \simeq \mathbb{H}_Q(Y) \to \mathbb{H}_Q(X) \simeq H_*^{BM}(X; \mathbb{Q})$ is the smooth pull-back in Borel-Moore homology. And for an algebraic version of this bivariant Riemann-Roch transformation $\tau$ compare with [Fu1] Ex. 18.3.16.

**Bivariant Stiefel-Whitney class transformation $\omega$.** In the context of real geometry (e.g. the piecewise linear, (semi-)algebraic or subanalytic context) one has the following interesting example of a bivariant theory (with “proper” the usual meaning). Here Fulton–MacPherson’s bivariant group $\mathbb{F}^{mod}_2(X \xrightarrow{f} Y)$ of $\mathbb{Z}_2$-valued constructible functions consists of all the constructible functions on $X$ which satisfy the local Euler condition with respect to $f$. Here a $\mathbb{Z}_2$-valued constructible function $\alpha \in F^{mod}_2(X)$ is said to satisfy the local Euler condition with respect to $f$, if for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbb{R}^N, 0)$ the equality

\[ \alpha(x) = \chi(B \cap f^{-1}(x); \alpha) \mod 2 \]

holds, where $B_\epsilon$ is a sufficiently small open ball of the origin 0 with radius $\epsilon$ and $z$ is any point close to $f(x)$ (cf. [Br1], [Sa]). In particular, if $\Pi_f := 1_{\mathbb{X}}$ belongs to the bivariant group $\mathbb{F}^{mod}_2(X \xrightarrow{f} Y)$, then the morphism $f : X \to Y$ is called an Euler morphism. For $f : X \to \{pt\}$ a constant map this just means (by the “local conic structure” of $X$), that $X$ is a mod 2 Euler space, i.e. the link $\partial B_\epsilon \cap X$ of any point $x \in X$ has vanishing Euler characteristic modulo 2:

\[ \chi(\partial B_\epsilon \cap X) = \chi_c(\partial B_\epsilon \cap X) = 1 - \chi_c(B_\epsilon \cap X) = 1 - \chi(B_\epsilon \cap X; 1_{\mathbb{X}}) = 0 \mod 2 \]
Also a smooth morphism, or a locally trivial fibration with fiber a mod 2 Euler space, is always an Euler morphism.

The three operations on $\mathbb{F}^{\text{mod}2}(X \xrightarrow{f} Y)$ are defined as follows:

(i) the product operation

\[
\bullet : \mathbb{F}^{\text{mod}2}(X \xrightarrow{f} Y) \otimes \mathbb{F}^{\text{mod}2}(Y \xrightarrow{g} Z) \to \mathbb{F}^{\text{mod}2}(X \xrightarrow{gf} Z)
\]

is defined by $\alpha \bullet \beta := \alpha \cdot f^* \beta$.

(ii) the pushforward operation $f_* : \mathbb{F}^{\text{mod}2}(X \xrightarrow{gf} Z) \to \mathbb{F}^{\text{mod}2}(Y \xrightarrow{g} Z)$ is the usual pushforward $f^*$, i.e.,

\[
f_*(\alpha)(y) := \chi(f^{-1}(\{y\}); \alpha) \mod 2.
\]

(iii) for a fiber square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

the pullback operation $g^* : \mathbb{F}^{\text{mod}2}(X \xrightarrow{f} Y) \to \mathbb{F}^{\text{mod}2}(X' \xrightarrow{f'} Y')$ is the functional pullback $g'^*$, i.e.,

\[
g^*(\alpha)(x') := \alpha(g'(x')).
\]

Note that for $f$ proper and any bivariant constructible function $\alpha \in \mathbb{F}^{\text{mod}2}(X \xrightarrow{f} Y)$, the Euler–Poincaré characteristic $\chi(f^{-1}(y); \alpha)$ of $\alpha$ restricted to each fiber $f^{-1}(y)$ is locally constant on $Y \mod 2$ (by the local Euler condition for $f_*(\alpha)$).

The correspondence $s \mathbb{F}^{\text{mod}2}(X \xrightarrow{f} Y) := F^{\text{mod}2}(X)$ assigning to a morphism $f : X \to Y$ the abelian group $F^{\text{mod}2}(X)$ of the source variety $X$, whatever the morphism $f$ is, becomes a bivariant theory with the same operations above. This bivariant theory is called the simple bivariant theory of constructible functions (see [Sch2] and [Y6]). In passing, what we then need to do for showing that the Fulton–MacPherson’s group of $\mathbb{Z}_2$-valued constructible functions satisfying the local Euler condition with respect to a morphism is a bivariant theory, is to show that the local Euler condition with respect to a morphism is preserved by each of these three operations.

For later use let us point out the abstract properties needed for the definition of a simple bivariant theory [Sch2 Definition, p.25-26]:

(SB1) We have a contravariant functor $G : \mathcal{C} \to \text{Rings}$ with values in the category of rings with unit.

(SB2) $G$ is also covariantly functorial with respect to proper maps (as a functor to the category of Abelian groups).

(SB3) $G$ satisfies the two-sided projection-formula, i.e. for $f : X \to Y$ proper and $\alpha \in G(Y)$ and $\beta \in G(X)$,

\[
f_*(f^* \alpha) = f^* \cup (f_\alpha),
\]

i.e., $f_*$ is a left $G(Y)$-module and

\[
f_*(\beta \cup (f^* \alpha)) = (f_\beta) \cup \alpha,
\]

i.e., $f_*$ is a right $G(Y)$-module. (Note that we do not assume $(G, \cup)$ is (graded) commutative so that both versions of the usual projection formula are needed.)
holds:

\[ g^* f_* = f'_* g'^* : G(X) \to G(Y') \]

for any fiber (or independent) square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

with \( f, f' \) proper.

Then one gets a (simple) bivariant theory \( sG \) by \( sG(X \xrightarrow{f} Y) := G(X) \), with the obvious push-down and pull-back transformations as above. Finally the bivariant product

\[ \bullet : sG(X \xrightarrow{f} Y) \times sG(Y \xrightarrow{g} Z) \to sG(X \xrightarrow{gf} Z) \]

is just given by \( \alpha \bullet \beta := \alpha \cup f^*(\beta) \), with \( \cup \) the given product of the ring-structure. Note that this construction does not only apply to constructible functions \( G(\ ) = F^{\text{mod2}}(\ ) \), but also to the relative Grothendieck group of complex algebraic varieties \( G(\ ) = K_0(V/\ ) \), even if we allow all algebraic morphisms as “proper” morphisms.

Let \( \mathbb{H}^{\text{mod2}}(X \xrightarrow{f} Y) \) be Fulton–MacPherson’s bivariant homology theory with \( \mathbb{Z}_2 \) coefficients, constructed from the corresponding cohomology theory in [FM] §3.1 so that \( \mathbb{H}^{\text{mod2},*}(X) = H^*(X; \mathbb{Z}_2) \) and \( \mathbb{H}^{\text{mod2}}_*(X) = H_*^{BM}(X; \mathbb{Z}_2) \). Then Fulton–MacPherson [FM] Theorem 6A] showed in the piecewise linear context the following theorem, which is a bivariant version of the singular Stiefel–Whitney class transformation \( w_* : F^{\text{mod2}}(\ ) \to H_*^{BM}(\ : \mathbb{Z}_2) \):

**Theorem 9.3.** There exists a unique Grothendieck transformation

\[ \omega : \mathbb{H}^{\text{mod2}} \to \mathbb{H}^{\text{mod2}} \]

satisfying the normalization condition that for a morphism from a smooth variety \( X \) to a point

\[ \omega(1_X) = w^*(T X) \cap [X] \in \mathbb{H}^{\text{mod2}}_*(X) = H_*^{BM}(X; \mathbb{Z}_2). \]

**Remark 9.4.** As to the bivariant mod 2 constructible functions, in the context of real geometry, the definition and the theory of them can be given in any of the following categories: the \( PL \)-category, the (semi-)algebraic category and the subanalytic category. Note that the above bivariant Stiefel–Whitney class transformation is only proved and known in the \( PL \)-category.

**Bivariant Chern class transformation \( \gamma \).** Instead of mod 2 constructible functions, in the complex analytic or algebraic context we certainly have similarly the bivariant group \( F(X \to Y) \) of \( \mathbb{Z} \)-valued constructible functions satisfying the local Euler condition with values in \( \mathbb{Z} \) (and not only in \( \mathbb{Z}_2 \)) and the bivariant homology theory \( \mathbb{H}(X \to Y) \) with integer coefficients, and W. Fulton and R. MacPherson conjectured or posed as a question the existence of a so-called bivariant Chern class transformation and J.–P. Brasselet [Br1] solved it:

**Theorem 9.5.** (J.–P. Brasselet) For the category of embeddable complex analytic varieties with cellular morphisms, there exists a Grothendieck transformation

\[ \gamma : F \to \mathbb{H} \]

such that for a morphism \( f : X \to \{pt\} \) from a nonsingular variety \( X \) to a point \( \{pt\} \) and the bivariant constructible function \( \mathbb{1}_f : 1_X \) the following normalization condition holds:

\[ \gamma(\mathbb{1}_f) = c^*(T X) \cap [X] \in \mathbb{H}_*(X) = H_*^{BM}(X; \mathbb{Z}). \]
Since then, the uniqueness of the Brasselet bivariant Chern class and the problem of whether "cellularity" of morphisms (which is not so easy to check) can be dropped or not have been unresolved. In [Sa], C. Sabbah constructed a bivariant Chern class transformation "micro-local analytically" in some cases. In [Z1], [Z2] J. Zhou showed that the bivariant Chern classes constructed by J.-P. Brasselet [Br1] and the ones constructed by C. Sabbah [Sa] in some cases are identical in the case when the target variety is a nonsingular curve. And in [YS] Theorem (3.7) we showed the following more general uniqueness theorem of bivariant Chern classes whose target varieties are nonsingular of any dimension:

**Theorem 9.6.** If there exists a bivariant Chern class transformation $\gamma : F \to H$, then it is unique when restricted to morphisms whose target varieties are nonsingular; explicitly, for a morphism $f : X \to Y$ with $Y$ nonsingular and for any bivariant constructible function $\alpha \in F(X \xrightarrow{f} Y)$ the bivariant Chern class $\gamma(\alpha)$ is expressed by

$$\gamma(\alpha) = f^* s(TY) \cap c_*(\alpha)$$

where $s(TY) := c^*(TY)^{-1}$ is the Segre class of the tangent bundle.

The twisted class $f^* s(TY) \cap c_*(\alpha)$ is called the Ginzburg–Chern class of $\alpha$ ([Gi1, Gi2] and [Y7, Y8]). Here, the above equality needs a bit of explanation. The left-hand-side $\gamma(\alpha)$ belongs to the bivariant homology group $\mathbb{H}(X \xrightarrow{f} Y)$ and the right-hand-side $f^* s(TY) \cap c_*(\alpha)$ belongs to the homology group $H_{BM}^*(X)$, and this equality is up to the isomorphism

$$\mathbb{H}(X \xrightarrow{f} Y) \otimes [Y] \xrightarrow{\cong} \mathbb{H}(X \to pt) \xrightarrow{A} H_{BM}^*(X),$$

where the first isomorphism is the bivariant product with the fundamental class $[Y]$ and the second isomorphism $A$ is the Alexander duality map. Since we usually identify $\mathbb{H}(X \to pt)$ as $H_{BM}^*(X)$ via this Alexander duality, we ignore this Alexander duality isomorphism, unless we have to mention it. Hence we have

$$\gamma(\alpha) \bullet [Y] = f^* s(TY) \cap c_*(\alpha).$$

We remark that this formula follows from the simple but crucial observation that

$$\gamma_f(\alpha) \bullet \gamma_{Y \to pt}(\mathbb{I}_Y) = \gamma_{X \to pt}(\alpha)$$

and the fact that $\gamma_{Y \to pt}$ is nothing but the Chern–Schwartz–MacPherson class transformation $c_*$. And in [BSY1] the above theorem is furthermore generalized to the case when the target variety can be singular but is "like a manifold".

**Definition 9.7.** (cf. [BM]) Let $A$ be a Noetherian ring. A complex variety $X$ is called an $A$-homology manifold (of dimension $2n$) or is said to be $A$-smooth if for all $x \in X$

$$H_i(X, X \setminus x; A) = \begin{cases} A & i = 2n \\ 0 & \text{otherwise.} \end{cases}$$

In this case $X$ has to be locally pure $n$-dimensional, where we consider $n$ as a locally constant function on $X$. Just look at the regular part of $X$, because a pure $n$-dimensional complex manifold is a homology manifold of dimension $2n$. Moreover the local orientation system $or_X$ with stalk $or_{X,x} = H_{2n}(X, X \setminus x; A) \simeq A_X$ is then already trivial (on each connected component of $X$) so that $X$ becomes an oriented $A$-homology manifold.

**Example 9.8.** If $A = \mathbb{Z}$, a $\mathbb{Z}$-homology manifold is called simply a homology manifold (cf. [MiSt]). There are singular complex varieties which are homology manifolds. Such examples are (products of) suitable singular hypersurfaces with isolated singularities (see [MT2]). If $A = \mathbb{Q}$, a $\mathbb{Q}$-manifold is called a rational homology manifold. As remarked in [BM] §1.4 Rational homology manifolds], examples of rational homology manifolds include surfaces with Kleinian singularities, the moduli space for curves of a given genus,
and more generally Satake’s $V$-manifolds or orbifolds. In particular, the quotient of a nonsingular variety by a finite group is a rational homology manifold.

**Theorem 9.9.** Let $Y$ be a complex analytic variety which is an oriented $A$-homology manifold for some commutative Noetherian ring $A$. If there exists a bivariant Chern class transformation $\gamma : \mathbb{F} \otimes A \to \mathbb{H} \otimes A$, then for any morphism $f : X \to Y$ the bivariant Chern class $\gamma_f : \mathbb{F}(X \xrightarrow{f} Y) \otimes A \to \mathbb{H}(X \xrightarrow{f} Y) \otimes A$ is uniquely determined and it is described by

$$\gamma_f(\alpha) = f^* c^*(Y)^{-1} \cap c_*(\alpha).$$

Here $c^*(Y)$ is the unique cohomology class such that $c_*(1_Y) = c^*(Y) \cap [Y]$. (Note that $c^*(Y)$ is invertible.)

When $Y$ is nonsingular, we see that the cohomology class $c^*(Y)$ is nothing but the total Chern class $c^*(TY)$ of the tangent bundle $TY$, hence the inverse $c^*(Y)^{-1}$ is the total Segre class $s(TY)$. Therefore the twisted class $f^* c^*(Y)^{-1} \cap c_*(\alpha)$ shall also be called the Ginzburg–Chern class of $\alpha$ and still denoted by $\gamma_{\text{Gin}}(\alpha)$. Note that we also have in this more general context the isomorphism

$$\mathbb{H}(X \xrightarrow{f} Y) \otimes A \xrightarrow{\gamma(Y)} \mathbb{H}(X \to pt) \otimes A \xrightarrow{A} H^B_{BM}(X) \otimes A,$$

since for an oriented $A$-homology manifold $Y$ the fundamental class $[Y] \in H^B_{BM}(X) \otimes A \cong \mathbb{H}(X \to pt) \otimes A$ is a strong orientation in the sense of bivariant theories (compare [BSY1]).

**Existence and uniqueness of bivariant characteristic classes.** Note that the proof of Theorem 9.9 also applies in the real (semi-)algebraic or subanalytic context to a bivariant Stiefel-Whitney class transformation $\gamma : \mathbb{F}^{\text{mod}} \to \mathbb{H}^{\text{mod}}$ (with the obvious modification of the notations from $c^*, c_*$ to $w^*, w_*$). In a similar manner, we can show the following theorem, which is an extended version of [YS] (Theorem (3.7)):

**Theorem 9.10.** The Grothendieck transformation from the bivariant algebraic $K$-theory $\mathbb{K}_{\text{alg}}$ of perfect complexes

$$\tau : \mathbb{K}_{\text{alg}} \to \mathbb{H}_{\mathbb{Q}}$$

constructed in [BM Part II] is unique on morphisms whose target varieties are rational homology manifolds. Explicitly, for a bivariant element $\alpha \in \mathbb{K}_{\text{alg}}(X \xrightarrow{f} Y)$ with $Y$ being a rational homology manifold

$$\tau(\alpha) = f^* \text{td}^*(Y)^{-1} \cap \text{td}_*(\alpha \bullet [O_Y]).$$

Here $[O_Y] \in \mathbb{K}_{\text{alg}}(Y) \simeq G_0(Y)$ is the class of the structure sheaf and the associated covariant transformation $\tau_* : \mathbb{K}_{\text{alg}}(\cdot) \simeq G_0(\cdot) \to H^B_{BM}(\cdot; \mathbb{Q})$ is Baum–Fulton–MacPherson’s Todd class transformation $\text{td}_*$ constructed in [BFM]. Moreover $\text{td}^*(Y) \in H^*(Y; \mathbb{Q})$ is the Poincaré dual of the Todd class $\text{td}_*(Y) := \text{td}_*([O_Y])$, which is invertible.

Conversely we ask ourselves whether the above Ginzburg–Chern class becomes a Grothendieck transformation for morphisms whose target varieties are oriented $A$-homology manifolds.

**Theorem 9.11.** For a morphism of complex analytic varieties $f : X \to Y$ with $Y$ an oriented $A$-homology manifold, we define $\mathfrak{F}(X \xrightarrow{f} Y)$ to be the set of all constructible functions $\alpha \in F(X)$ satisfying the following two conditions (i) and (ii) : for any fiber square

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}$$

(i) $g^* f_*(\alpha) = f'^* g'_*(\alpha)$

(ii) $g^* f_*(\alpha) = f'^* g'_*(\alpha)$

Using these functions, we define

$$\tau(\alpha) = f^* \text{td}^*(Y)^{-1} \cap \text{td}_*(\alpha \bullet [O_Y]).$$

This defines a homomorphism

$$\tau : \mathfrak{F}(X \xrightarrow{f} Y) \to \mathbb{H}_{\mathbb{Q}}(X \xrightarrow{f} Y)$$

which is unique on morphisms whose target varieties are rational homology manifolds.
with $Y'$ an oriented $A$-homology manifold the following equalities hold:

(1) for any constructible function $\beta' \in F(Y')$:

$$\gamma^{\text{Gin}}(g^*\alpha \cdot \beta') = \gamma^{\text{Gin}}(g^*\alpha) \cdot \gamma^{\text{Gin}}(\beta'),$$

(2)

$$\gamma^{\text{Gin}}(g^*\alpha) = g^*\gamma^{\text{Gin}}(\alpha).$$

Then $\mathcal{F}$ becomes a bivariant theory with the same operations as in $s\mathcal{F}$ and furthermore the transformation

$$\gamma^{\text{Gin}}: \mathcal{F} \to \mathbb{H}$$

is well-defined and becomes the unique Grothendieck transformation satisfying that $\gamma^{\text{Gin}}$ for morphisms to a point is the Chern–Schwartz–MacPherson class transformation $c_*: F \to H_*$. And also $\mathcal{F}(X \to \text{pt}) = F(X)$.

The proof of the theorem is the same as in [Y9], in which the case when the target variety $Y$ is nonsingular is treated. Note that to prove $\mathcal{F}(X \to \text{pt}) = F(X)$ we need the cross product formula or multiplicativity of the Chern–Schwartz–MacPherson class transformation $c_*$ due to Kwieciński [Kw1] (cf. [KY]), i.e. the commutativity of the following diagram:

$$\begin{array}{ccc}
F(X) \times F(Y) & \xrightarrow{x} & F(X \times Y) \\
\downarrow_{c_* \times c_*} & & \downarrow_{c_*}
\end{array}$$

$$\begin{array}{ccc}
H^B_*(X;\mathbb{Z}) \times H^B_*(Y;\mathbb{Z}) & \xrightarrow{x} & H^B_*(X \times Y;\mathbb{Z})
\end{array}$$

The cross product formula for Stiefel-Whitney classes in the real algebraic context can be shown similarly by using “resolution of singularities”, or the corresponding product formula for “characteristic cycles” of constructible functions so that a variant of this theorem also works in the real algebraic context.

And for a much more general version of Theorem 9.11 see [Sch2].

The above theorem led us to another uniqueness theorem, which in a sense gives a positive solution to the general uniqueness problem concerning Grothendieck transformations posed in [FM, §10 Open Problems].

**Theorem 9.12.** We define

$$\mathcal{F}(X \xrightarrow{f} Y)$$

with $\mathcal{F}$ the set consisting of all $\alpha \in s\mathcal{F}(X \xrightarrow{f} Y)$ satisfying the following condition: there exists a bivariant class $B_{\alpha} \in \mathbb{H}(X \xrightarrow{f} Y)$ such that for any base change $g: Y' \to Y$ (without any requirement) of an independent square

$$\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow^{f'} & & \downarrow^f \\
Y' & \xrightarrow{g} & Y,
\end{array}$$

and for any $\beta' \in F(Y')$ the following equality holds:

$$c_*(g^*\alpha \cdot \beta') = g^*B_{\alpha} \cdot c_*(\beta').$$

Then $\mathcal{F}$ is a bivariant theory. Furthermore $\mathcal{F}(X \to \text{pt}) = F(X)$.

The above bivariant class $B_{\alpha}$ should ideally be the unique bivariant Chern class of $\alpha$. However, so far we still do not know if it is the case or not. So, provisionally we call $B_{\alpha}$ a pseudo-bivariant Chern class of $\alpha$. 
Example 9.13 (VRR for smooth morphisms). Let $f : X \rightarrow Y$ be a smooth morphism of possible singular varieties. Then we have

$$
1_f := 1_X \in \overline{PH}(X \xrightarrow{f} Y)
$$

with $c^*(T_f) \bullet [f]$ being a pseudo-bivariant Chern class of $1_f$. Here $T_f$ is the vector bundle of tangent spaces of fibers of $f$, and $[f] \in H^*(X \xrightarrow{f} Y)$ is the canonical orientation of the smooth morphism $f$. Then as in Theorem 9.12 we have for $\beta' \in F(Y')$:

$$
c_*(g^*1_f \bullet \beta') = c_*(f^*\beta') = c^*(T_f) \cap f'^*c_*(\beta')
$$

Here $f'^* = [f'^*] : H^i_{BM}(Y') \simeq H^i_*(Y') \rightarrow H^i_*(X') \simeq H^i_{BM}(X')$ is the smooth pullback in Borel-Moore homology, and the equality

$$
c_*(f^*\beta') = c^*(T_f) \cap f'^*c_*(\beta')
$$

is the so-called Verdier-Riemann-Roch theorem for the smooth morphism $f'$ and the Chern class transformation $c_*$ (compare [FM, Sch1, Y4]).

In order to remedy this unpleasant possible non-uniqueness of the bivariant class $B_\alpha$ above, we set

$$
\overline{PH}(X \xrightarrow{f} Y) := \left\{ B \in H^*(X \xrightarrow{f} Y) | B \text{ is a pseudo-bivariant Chern class of some } \alpha \in \overline{PH}(X \xrightarrow{f} Y) \right\}
$$

to be the set of all pseudo-bivariant Chern classes for the morphism $f : X \rightarrow Y$. It is clear that $\overline{PH}$ is a bivariant subtheory of $H$, i.e., it is a subgroup stable under the three bivariant operations. Then we define

$$
\overline{H}(X \xrightarrow{f} Y) := \overline{PH}(X \xrightarrow{f} Y)/\sim
$$

where the relation $\sim$ is defined by

$$
B \sim B' \iff g^*B \bullet c_*(\beta') = g^*B' \bullet c_*(\beta')
$$

for all independent squares with $g : Y' \rightarrow Y$ and all $\beta' \in F(Y')$. Certainly the relation $\sim$ is an equivalence relation. In other words, with this identification we want to make possibly many pseudo-bivariant Chern classes into one unique bivariant Chern class. Indeed we have

**Theorem 9.15.** $\overline{H}(X \xrightarrow{f} Y)$ is an Abelian group and $\overline{H}$ is a bivariant theory with the canonical operations induced from those of $H$. Furthermore we have

$$
\overline{H}(X \rightarrow pt) = \text{Image}(c_* : F(X) \rightarrow H^*_{BM}(X)).
$$

And we have the following theorem

**Theorem 9.16.** There exists a unique Grothendieck transformation

$$
\overline{\gamma} : \overline{F} \rightarrow \overline{H}
$$

whose associated covariant transformation is $c_* : F \rightarrow \text{Im}(c_*)$, where

$$
\text{Im}(c_*)(X) := \text{Image}\left( c_* : F(X) \rightarrow H^*_{BM}(X) \right).
$$
Indeed, the bivariant theory for the “quotient” problem of whether or not there is a reasonable bivariant homology theory so that such a ring structure.

Here we assume that for the bivariant theory for and the divided by structible functions.

As mentioned above, a key for the above argument is the fact that $c_\ast(\alpha) = \gamma(\alpha) \ast c_\ast(\mathbb{I}_Y)$. So, very sloppy speaking, the bivariant class $\gamma(\alpha)$ is a kind of “$c_\ast(\alpha)$ divided by $c_\ast(\mathbb{I}_Y)$”, whatever it is meant to be. In our previous paper \cite{13} we posed the problem of whether or not there is a reasonable bivariant homology theory so that such a “quotient”

$$\frac{c_\ast(\alpha)}{c_\ast(\mathbb{I}_Y)}$$

is well-defined. The above theory $\tilde{\mathbb{H}}$ is in a sense a positive answer to this problem.

The above construction works for the following more general situation such that

1. there exists a natural transformation $\tau_\ast : F_\ast(X) \to H_\ast(X)$ between two covariant functors $F_\ast$ and $H_\ast$ (covariant with respect to proper maps) such that $F_\ast(pt)$ and $H_\ast(pt)$ are commutative rings with unit and such that $\tau_\ast$ maps the unit to the unit,

2. there are two bivariant theories $\mathbb{F}$ and $\mathbb{H}$ such that the associated covariant theories are

$$\mathbb{F}(X \to pt) = F_\ast(X) \quad \text{and} \quad \mathbb{H}(X \to pt) = H_\ast(X),$$

3. $\tau_\ast$ commutes with the bivariant exterior products, i.e., the following diagram commutes

$$F_\ast(X) \times F_\ast(Y) \xrightarrow{\times} F_\ast(X \times Y)$$

$$\tau_\ast \times \tau_\ast \downarrow \quad \downarrow \tau_\ast$$

$$H_\ast(X) \times H_\ast(Y) \xrightarrow{\times} H_\ast(X \times Y).$$

Here we assume that for $X = Y = \{pt\}$ a point this exterior product agrees with the given ring structure.

Certainly this construction works for the previous motivic Chern class transformation

$$mc_\ast : K_0(V/\_ ) \to G_0(\_ ) \otimes \mathbb{Z}[y]$$

and the motivic Hirzebruch class transformation

$$T_{y_\ast} : K_0(V/\_ ) \to H_\ast(\_ ) \otimes \mathbb{Q}[y].$$

Indeed, the bivariant theory for $K_0(V/\_ )$ is the simple bivariant theory

$$sK_0(X \to Y) := K_0(V/X),$$

the bivariant theory for $G_0(\_ ) \otimes \mathbb{Z}[y]$ is the Fulton–MacPherson’s bivariant algebraic K-theory $K_{alg}$ tensored with $\mathbb{Z}[y]$, and the bivariant theory for $H_\ast(\_ ) \otimes \mathbb{Q}[y]$ is of course the Fulton–MacPherson’s bivariant homology theory $\mathbb{H}$ tensored with $\mathbb{Q}[y]$. It also applies in the real algebraic context to the Stiefel-Whitney class transformation

$$w_\ast : F^{mod2}(\_ ) \to H_\ast(\_ ; \mathbb{Z}_2)$$

by using the simple bivariant theory $s\mathbb{H}^{mod2}$ of $\mathbb{Z}_2$-valued real algebraically constructible functions.

\textbf{Remark 9.17.} Let $f : X \to Y$ be a smooth morphism of possible singular varieties. Then also the example \cite{9,13} works in this context, with

$$\mathbb{I}_f := \mathbb{I}_X = [id_X] \in sK_0(X \xrightarrow{f} Y) \quad \text{or} \quad \mathbb{I}_f := 1_X \in s\mathbb{H}^{mod2}(X \xrightarrow{f} Y),$$

and $c_\ell(T_f) \bullet [f]$ being a pseudo-bivariant class of $\mathbb{I}_f$ for $c_\ell(T_f) = \lambda_y(T_f^\ast) \overline{td}_y(T_f)$ or $w_\ast(T_f)$. Here the corresponding Verdier-Riemann-Roch theorem for the smooth morphism $f'$ follows for the motivic characteristic classes $mc_\ast$ and $T_{y_\ast}$ from \cite{BSY2} Corollary 2.1 and Corollary 3.1. For the Stiefel-Whitney class transformation $w_\ast$ it can be shown as for Chern classes by using “resolution of singularities” or “characteristic cycles of constructible functions”.

\textbf{Remark 9.18.} Let $f : X \to Y$ be a smooth morphism of possible singular varieties. Then also the example \cite{9,13} works in this context, with

$$\mathbb{I}_f := \mathbb{I}_X = [id_X] \in sK_0(X \xrightarrow{f} Y) \quad \text{or} \quad \mathbb{I}_f := 1_X \in s\mathbb{H}^{mod2}(X \xrightarrow{f} Y),$$

and $c_\ell(T_f) \bullet [f]$ being a pseudo-bivariant class of $\mathbb{I}_f$ for $c_\ell(T_f) = \lambda_y(T_f^\ast) \overline{td}_y(T_f)$ or $w_\ast(T_f)$. Here the corresponding Verdier-Riemann-Roch theorem for the smooth morphism $f'$ follows for the motivic characteristic classes $mc_\ast$ and $T_{y_\ast}$ from \cite{BSY2} Corollary 2.1 and Corollary 3.1. For the Stiefel-Whitney class transformation $w_\ast$ it can be shown as for Chern classes by using “resolution of singularities” or “characteristic cycles of constructible functions”.

\textbf{Remark 9.19.} Let $f : X \to Y$ be a smooth morphism of possible singular varieties. Then also the example \cite{9,13} works in this context, with

$$\mathbb{I}_f := \mathbb{I}_X = [id_X] \in sK_0(X \xrightarrow{f} Y) \quad \text{or} \quad \mathbb{I}_f := 1_X \in s\mathbb{H}^{mod2}(X \xrightarrow{f} Y),$$

and $c_\ell(T_f) \bullet [f]$ being a pseudo-bivariant class of $\mathbb{I}_f$ for $c_\ell(T_f) = \lambda_y(T_f^\ast) \overline{td}_y(T_f)$ or $w_\ast(T_f)$. Here the corresponding Verdier-Riemann-Roch theorem for the smooth morphism $f'$ follows for the motivic characteristic classes $mc_\ast$ and $T_{y_\ast}$ from \cite{BSY2} Corollary 2.1 and Corollary 3.1. For the Stiefel-Whitney class transformation $w_\ast$ it can be shown as for Chern classes by using “resolution of singularities” or “characteristic cycles of constructible functions”.
This Verdier-Riemann-Roch theorem for smooth morphisms is also very important for the definition of $G$-equivariant characteristic class transformations in the equivariant algebraic context with $G$ a reductive linear algebraic group. Here we refer to [EG1, EG2, BZ] for the equivariant Todd class transformation $td^G_*$, and to [Oh] for the equivariant Chern class transformation $c^G_∗$. In fact, in future work we will construct in this equivariant algebraic context equivariant versions $mC^G_*$ and $T^G_*$ of our motivic characteristic classes, together with the equivariant version of Theorem 8.5 relating $T^G_{−1}$ with $c^G_*$ and $T^G_0$ with $td^G_*$.

Bivariant L-classes. At the moment we have no bivariant version with values in bivariant homology of the L-class transformation $L_* : Ω(X) → H_*(X, Q)$, since we do not know a suitable bivariant theory, whose associated covariant theory reduces to the cobordism group $Ω( )$ of selfdual constructible sheaf complexes. Note that in this case we cannot define a simple bivariant theory $sΩ$. Of course the Grothendieck group of constructible sheaf complexes $K_∗( )$ satisfies the properties (SB1-4) with respect to the induced proper push down $f_∗$, pullback $f^*$ and tensor product $⊗$ so that one gets a simple bivariant theory $sK_∗$. But the problem is that $f^*$ and $⊗$ do not commute with duality in general so that this approach doesn’t apply to $Ω( )$.

A similar problem appears in the context of real semialgebraic and subanalytic geometry for the group $F^{mod2}_{Eu}( )$ of $Z_2$-valued constructible functions satisfying the mod 2 local Euler condition (for a constant map), which also can be interpreted as a “duality” condition (compare [Sch3, p.135 and Remark 5.4.4, p.367]). This group (or condition) is also not stable under general pullback or product so that one cannot define a simple bivariant theory $sF^{mod2}_{Eu}$ in this context (comparable to $sF^{mod2}$ in the real algebraic context). Nevertheless one can define a Stiefel-Whitney class transformation $w_* : F^{mod2}_{Eu}( ) → H^BM_{∗}( ; Z_2)$ with the help of “characteristic cycles of constructible functions” (compare [FuMC]), which is multiplicative for exterior products and satisfies the Verdier Riemann-Roch theorem for smooth morphisms.

Similarly one can define in the complex algebraic or analytic context an exterior product and smooth pullback for the cobordism group $Ω( )$ of selfdual constructible sheaf complexes (compare [BSY2]), and the L-class transformation $L_*$ is also multiplicative by an argument similarly as in the recent paper [Wo, p.26, Proposition 5.16]. Also the corresponding Verdier Riemann-Roch theorem for smooth morphisms seems reasonable, but at the moment we have no proof or reference for this. Of course this VRR theorem holds on the image of the transformation $ω : K_0(V/ ) → Ω( )$ from Theorem 8.5 (compare [BSY2]).

Then in both these cases, L-class and Stiefel-Whitney class transformations, we can apply the results of [Y6] to get at least bivariant versions of these theories for the corresponding operational bivariant theories.

10. CHARACTERISTIC CLASSES OF PROALGEBRAIC VARIETIES

A pro-algebraic variety is defined to be a projective system of complex algebraic varieties and a proalgebraic variety is defined to be the projective limit of a pro-algebraic variety. Proalgebraic varieties are the main objects in [Grom]. A pro-category was first introduced by A. Grothendieck [Grot1] and it was used to develop the Etale Homotopy Theory [AM] and Shape Theory (e.g., see [Bor], [MaSe], etc.) and so on. In [Grom 1]
M. Gromov investigated the surjectivity, i.e. being either surjective or non-injective, in the category of proalgebraic varieties. The original or classical surjectivity theorem is the so-called Ax’ Theorem [Ax], saying that every regular selfmapping of a complex algebraic variety is surjective; thus if it is injective then it has to be surjective.

A very simple example of a proalgebraic variety is the Cartesian product $X^N$ of countable infinitely many copies of a complex algebraic variety $X$, which is one of the main objects treated in [Grom]. Then, what would be the “Chern–Schwartz–MacPherson class” of $X^N$? In particular, what would be the “Euler–Poincaré characteristic” of $X^N$? This simple question led us to a study of characteristic classes of proalgebraic varieties and it naturally led us to the so-called motivic measures (see [Y10, Y11]). The motivic measures/integrations have been actively studied by many people (e.g., see [Cr], [DL1], [DL2], [Kon], [Lo], [Ve2] etc.).

In a general set-up one can deal with the so-called bifunctors. The bifunctors which we consider are bifunctors $\mathcal{F} : C \to A$ from a category $C$ to the category $A$ of abelian groups, i.e., $\mathcal{F}$ is a pair $(\mathcal{F}_*, \mathcal{F}^*)$ of a covariant functor $\mathcal{F}_*$ and a contravariant functor $\mathcal{F}^*$ such that $\mathcal{F}_*(X) = \mathcal{F}^*(X)$ for any object $X$. Unless some confusion occurs, we just denote $\mathcal{F}(X)$ for $\mathcal{F}_*(X) = \mathcal{F}^*(X)$. A typical example is the constructible function functor $F(X)$. Furthermore we assume that for a final object $pt \in Obj(C)$, $F(pt)$ is a commutative ring $\mathcal{R}$ with a unit. The morphism from an object $X$ to a final object $pt$ shall be denoted by $\pi_X : X \to pt$. Then the covariance of the bifunctor $\mathcal{F}$ induces the homomorphism $\pi^\ast_X := \mathcal{F}(\pi_X) : \mathcal{F}(X) \to \mathcal{F}(pt) = \mathcal{R}$, which shall be denoted by $\chi_\mathcal{F} : \mathcal{F}(X) \to \mathcal{R}$ and called the $\mathcal{F}$-characteristic, just mimicking the Euler–Poincaré characteristic (with compact support) $\chi : F(X) \to \mathbb{Z}$ in the case when $\mathcal{F} = F$.

Let $X_\infty = \lim_{\lambda \in \Lambda} \left\{ X_\lambda, \pi_{\lambda \mu} : X_\mu \to X_\lambda \right\}$ be a proalgebraic variety. Then we define

$$\mathcal{F}^{\text{ind}}(X_\infty) := \lim_{\lambda \in \Lambda} \left\{ \mathcal{F}(X_\lambda), \pi^\ast_{\lambda \mu} : \mathcal{F}(X_\lambda) \to \mathcal{F}(X_\mu)(\lambda < \mu) \right\},$$

which may not belong to the category $A$. Another finer one can be defined as follows. Let $P = \{ p_{\lambda \mu} \}$ be a projective system of elements of $\mathcal{R}$ by the directed set $\Lambda$, i.e., a set such that $p_{\lambda \lambda} = 1$ (the unit) and $p_{\lambda \mu} \cdot p_{\mu \nu} = p_{\lambda \nu}$ ($\lambda < \mu < \nu$). For each $\lambda \in \Lambda$ the subobject $\mathcal{F}_P^{\text{ind}}(X_\lambda)$ of $\chi_\mathcal{F}$-stable elements in $\mathcal{F}(X_\lambda)$ is defined to be

$$\mathcal{F}_P^{\text{st}}(X_\lambda) := \left\{ \alpha_\lambda \in \mathcal{F}(X_\lambda) | \chi_\mathcal{F}(\pi^\ast_{\lambda \mu} \alpha_\lambda) = p_{\lambda \mu} \cdot \chi_\mathcal{F}(\alpha_\lambda) \text{ for any } \mu \text{ such that } \lambda < \mu \right\}.$$

The inductive limit

$$\lim_{\Lambda} \left\{ \mathcal{F}_P^{\text{st}}(X_\lambda), \pi^\ast_{\lambda \mu} : \mathcal{F}_P^{\text{st}}(X_\lambda) \to \mathcal{F}_P^{\text{st}}(X_\mu)(\lambda < \mu) \right\}$$

considered for a proalgebraic variety $X_\infty = \lim_{\lambda \in \Lambda} X_\lambda$ is denoted by

$$\mathcal{F}_P^{\text{st, ind}}(X_\infty).$$

Of course this definition is not intrinsic to the proalgebraic variety $X_\infty$, but depends on the given projective system $\{ X_\lambda, \pi_{\lambda \mu} : X_\mu \to X_\lambda \}$. But for simplicity we use this notation. Our key observation, which is an application of standard facts on inductive systems and limits, is the following:
Theorem 10.1. (i) For a proalgebraic variety $X_\infty = \lim_{\xrightarrow{\lambda \in \Lambda}} \{X_\lambda, \pi_{\lambda\mu} : X_\mu \to X_\lambda\}$ and a projective system $P = \{p_{\lambda\mu}\}$ of elements of $R$, we have the homomorphism
\[
\chi_{F}^{\text{ind}} : F_{P}^{\text{st.ind}}(X_\infty) \to \lim_{\xrightarrow{\lambda \in \Lambda}} \left\{ \times p_{\lambda\mu} : R \to R \right\},
\]
which is called the proalgebraic $F$-characteristic homomorphism.

(ii) Assume $\Lambda = \mathbb{N}$. For a proalgebraic variety $X_\infty = \lim_{\xrightarrow{n \in \mathbb{N}}} \{X_n, \pi_{nm} : X_m \to X_n\}$ and a projective system $P = \{p_{nm}\}$ of elements of $R$, the proalgebraic $F$-characteristic homomorphism $\chi_{F}^{\text{ind}} : F_{P}^{\text{st.ind}}(X_\infty) \to \lim_{\xrightarrow{n}} \left\{ \times p_{nm} : R \to R \right\}$ is realized as the homomorphism
\[
\chi_{F}^{\text{ind}} : F_{P}^{\text{st.ind}}(X_\infty) \to R_P
\]
defined by
\[
\chi_{F}^{\text{ind}}([\alpha_n]) := \frac{\chi_{F}(\alpha_n)}{p_{01} \cdot p_{12} \cdot p_{23} \cdots p_{(n-1)n}}.
\]
Here $p_{01} := 1$ and $R_P$ is the ring $R_S$ of fractions of $R$ with respect to the multiplicatively closed set $S$ consisting of all the finite products of powers of elements in $P$.

(iii) In particular, in the case when the above projective system $P = \{p^s\}$ consists of powers of an element $p$, we get the homomorphism
\[
\chi_{F}^{\text{ind}} : F_{P}^{\text{st.ind}}(X_\infty) \to R_{\left[\frac{1}{p}\right]}
\]
defined by
\[
\chi_{F}^{\text{ind}}([\alpha_n]) := \frac{\chi_{F}(\alpha_n)}{p^{n-1}}.
\]
Here $R_{\left[\frac{1}{p}\right]}$ is the localization by the multiplicatively closed set $S := \{p^s | s \in \mathbb{N}_0\}$.

Note that $R_S$ or $R_{\left[\frac{1}{p}\right]}$ is the zero ring in the case when $0 \in S$ for the corresponding multiplicatively closed set $S$. A typical example for the above theorem is the following.

Example 10.2. Let $X_\infty = \lim_{\xrightarrow{n \in \mathbb{N}}} \{X_n, \pi_{nm} : X_m \to X_n\}$ be a proalgebraic variety such that for each $n$ the structure morphism $\pi_{n,n+1} : X_{n+1} \to X_n$ satisfies the condition that the Euler–Poincaré characteristics of the fibers of $\pi_{n,n+1}$ are non-zero (which implies the surjectivity of the morphism $\pi_{n,n+1}$) and constant; for example, $\pi_{n,n+1} : X_{n+1} \to X_n$ is a locally trivial fiber bundle with fiber variety being $F_n$ and $\chi(F_n) \neq 0$. Let us denote the constant Euler–Poincaré characteristic of the fibers of the morphism $\pi_{n,n+1} : X_{n+1} \to X_n$ by $\epsilon_n$ and we set $e_0 := 1$. Then we get the canonical proalgebraic Euler–Poincaré characteristic homomorphism
\[
\chi_{F}^{\text{ind}} : F^{\text{ind}}(X_\infty) \to \mathbb{Q}
\]
described by
\[
\chi_{F}^{\text{ind}}([\alpha_n]) = \frac{\chi(\alpha_n)}{e_0 \cdot e_1 \cdot e_2 \cdots e_{n-1}}.
\]
In particular, if the Euler–Poincaré characteristics $\epsilon_n$ are all the same, say $\epsilon_n = e$ for any $n$, then the canonical proalgebraic Euler–Poincaré characteristic homomorphism $\chi_{F}^{\text{ind}} : F^{\text{ind}}(X_\infty) \to \mathbb{Q}$ is described by $\chi_{F}^{\text{ind}}([\alpha_n]) = \frac{\chi(\alpha_n)}{e^{n-1}}$, and furthermore the target ring $\mathbb{Q}$ can be replaced by the ring $\mathbb{Z}_{\left[\frac{1}{e}\right]}$.

Note that this example applies especially to the Cartesian product $X^N$ of countable infinitely many copies of a complex algebraic variety $X$ with $\chi(X) \neq 0$. In fact this example of Cartesian products is a special case of the following more general example:
Example 10.3. We make the following additional assumptions for our bifunctor:

(1) The contravariant functor $F^*$ takes values in the category of commutative rings with unit. The corresponding unit in $F(X)$ is denoted by $\mathbb{I}_X$, and $F(X)$ becomes an $\mathcal{R} := F(pt)$-algebra by the pullback for $\pi_X : X \to pt$.

(2) $F^*$ and $F_*$ are related for a morphism $f : X \to Y$ by the projection formula

$$ f_* (\alpha \cdot f^* \beta) = f_*(\alpha) \cdot \beta \quad \text{for all } \alpha \in F(X) \text{ and } \beta \in F(Y) $$

so that $f_* : F(X) \to F(Y)$ is $F(Y)$- and $\mathcal{R}$-linear. (This is just a special case of our simple bivariant theories, where all morphisms are “proper” and only the “trivial fiber squares” are “independent”.)

Consider a proalgebraic variety $X_\infty = \varprojlim_{n \in \mathbb{N}} \{ X_n, \pi_{nm} : X_n \to X_m \}$ such that for each $n$ the structure morphism $\pi_{n,n+1} : X_{n+1} \to X_n$ satisfies the condition

$$ \pi_{n,n+1}(\mathbb{I}_{X_{n+1}}) = e_n \cdot \mathbb{I}_{X_n} \in F(X_n) \quad \text{for some } e_n \in \mathcal{R}, \text{ with } e_0 := \mathbb{I}_{pt}.$$ 

Then we get the canonical proalgebraic $F$-characteristic homomorphisms

$$ \chi_{F,X_1}^{\text{ind}} : F^\text{ind}(X_\infty) \to F(X_1)_E \quad \text{and} \quad \chi_F^{\text{ind}} : F^\text{ind}(X_\infty) \to \mathcal{R}_E $$
described by

$$ \chi_{F,X_1}^{\text{ind}}([\alpha_n]) = \frac{\pi_{1,n} \cdot (\alpha_n)}{e_0 \cdot e_1 \cdot e_2 \cdots e_{n-1}} \quad \text{and} \quad \chi_F^{\text{ind}}([\alpha_n]) = \frac{\chi(\alpha_n)}{e_0 \cdot e_1 \cdot e_2 \cdots e_{n-1}}. $$

Here $\mathcal{R}_E$ (or $F(X_1)_E$) is the ring of fractions of $\mathcal{R}$ with respect to the multiplicatively closed set consisting of all the finite products of powers of the elements $e_i$ (or their pullbacks to $X_1$).

Consider a bifunctor as in example [10.3] with $f : X \to Y$ being a morphism such $f_*(\mathbb{I}_X) = e_f \cdot \mathbb{I}_Y$ for some $e_f \in \mathcal{R}$. Then one gets any $\alpha \in F(Y)$:

$$ f_* f^* \alpha = f_* (\mathbb{I}_X \cdot f^* \alpha) = e_f \cdot \alpha $$

so that for any morphism $g : Y \to Z$ (e.g. $g = \pi_Y : Y \to pt$):

$$ (g \circ f)_* (f^* \alpha) = g_* (f_* f^* \alpha) = g_*(e_f \cdot \alpha) = e_f \cdot g_*(\alpha). $$

Hence if we set in the context of the example

$$ p_{nm} = \begin{cases} 1 & n = m \\ e_n \cdot e_{n+1} \cdots e_{m-1} & n < m, \end{cases} $$

then $P := \{ p_{nm} \}$ is a projective system and $F^{\text{st, ind}}(X_\infty) = F^\text{ind}(X_\infty)$ for both notions of Euler characteristics working over the base space $X_1$ or over $pt$. Thus the above description of $\chi_{F,X_1}^{\text{ind}}$ and $\chi_F^{\text{ind}}$ follows from Theorem [10.1].

A “motivic” version of the Euler–Poincaré characteristic $\chi : F(X) \to \mathbb{Z}$ is the homomorphism $\Gamma_X : F(X) \to K_0(\mathcal{V}/X)$ “tautologically” defined by

$$ \Gamma_X(\sum_W a_W \mathbb{I}_W) := \sum_W a_W [W \subset X], $$
or better is the composite $\Gamma := \pi_X \cdot \Gamma_X : F(X) \to K_0(\mathcal{V})$. Note that $\Gamma_X$ commutes with pullback $f^*$ (but not with push down $f_*$). Then we get the following theorem, which is a generalization of the (naive) motivic measure:
Theorem 10.4. (i) For a proalgebraic variety \( X_\infty = \lim_{\lambda \in \Lambda} \{ X_\lambda, \pi_{\lambda \mu} : X_\mu \to X_\lambda \} \) and a projective system \( G = \{ \gamma_{\lambda \mu} \} \) of Grothendieck classes, we get the proalgebraic Grothendieck class homomorphism

\[
\Gamma_{\text{ind}} : F^\text{st. ind}_{\Gamma} (X_\infty) \to \lim_{\lambda \in \Lambda} \{ \times \gamma_{\lambda \mu} : K_0(V) \to K_0(V) \}.
\]

(ii) Assume \( \Lambda = \mathbb{N} \). For a proalgebraic variety \( X_\infty = \lim_{n \in \mathbb{N}} \{ X_n, \pi_{nm} : X_m \to X_n \} \) and a projective system \( G = \{ \gamma_{n,m} \} \) of Grothendieck classes, we have the following canonical proalgebraic Grothendieck class homomorphism

\[
\tilde{\Gamma}_{\text{ind}} : F^\text{st. ind}_{\Gamma} (X_\infty) \to K_0(V)_G
\]

which is defined by

\[
\tilde{\Gamma}_{\text{ind}} ([\alpha_n]) := \frac{\Gamma(\alpha_n)}{\gamma_{01} \cdot \gamma_{12} \cdot \gamma_{23} \cdots \gamma_{(n-1)n}}.
\]

Here we set \( \gamma_{01} := 1 \) and \( K_0(V)_G \) is the ring of fractions of \( K_0(V) \) with respect to the multiplicatively closed set consisting of finite products of powers of elements of \( G \).

(iii) Let \( X_\infty = \lim_{n \in \mathbb{N}} \{ X_n, \pi_{nm} : X_m \to X_n \} \) be a proalgebraic variety such that each structure morphism \( \pi_{n,n+1} : X_{n+1} \to X_n \) satisfies the condition:

\[
\pi_{n,n+1}([id_{X_{n+1}}]) = \gamma_n \cdot [id_{X_n}] \in K_0(V/X_n) \text{ for some } \gamma_n \in K_0(V); \]

for example \( \pi_{n,n+1} : X_{n+1} \to X_n \) is a Zariski locally trivial fiber bundle with fiber variety being \( F_n \) (in which case one can take \( \gamma_n := [F_n] \in K_0(V) \)). Then the canonical proalgebraic Grothendieck class homomorphisms

\[
\Gamma_{\text{ind}} : F^\text{ind}(X_\infty) \to K_0(V/X_1)_G \text{ and } \Gamma_{\text{ind}} : F^\text{ind}(X_\infty) \to K_0(V)_G
\]

are described by

\[
\Gamma_{\text{ind}} ([\alpha_n]) = \frac{\pi_{1,n+1}^*(\Gamma_{X_n}(\alpha_n))}{\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{n-1}} \text{ and } \Gamma_{\text{ind}} ([\alpha_n]) = \frac{\Gamma(\alpha_n)}{\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{n-1}}.
\]

Here \( \gamma_0 := 1 \) and \( K_0(V)_G \) (or \( K_0(V/X_1)_G \)) is the ring of fractions of \( K_0(V) \) with respect to the multiplicatively closed set consisting of finite products of powers of \( \gamma_m \) (\( m = 1, 2, 3 \cdots \)) (or their pullbacks to \( X_1 \)).

(iv) In particular, if \( \gamma_n = \gamma \) for all \( n \), then the canonical proalgebraic Grothendieck class homomorphisms

\[
\Gamma_{X_1} : F^\text{ind}(X_\infty) \to K_0(V/X_1)_G \text{ and } \Gamma_{\text{ind}} : F^\text{ind}(X_\infty) \to K_0(V)_G
\]

are described by

\[
\Gamma_{X_1} ([\alpha_n]) = \frac{\pi_{1,n+1}^*(\Gamma_{X_n}(\alpha_n))}{\gamma_{n-1}} \text{ and } \Gamma_{\text{ind}} ([\alpha_n]) = \frac{\Gamma(\alpha_n)}{\gamma_{n-1}}.
\]

In this special case the quotient ring \( K_0(V)_G \) (or \( K_0(V/X_1)_G \)) shall be simply denoted by \( K_0(V)_{\gamma} \) (or \( K_0(V/X_1)_{\gamma} \)).

Example 10.5. The arc space \( \mathcal{L}(X) \) of an algebraic variety \( X \) is defined to be the projective limit of the projective system consisting of the truncated arc varieties \( \mathcal{L}_n(X) \) of jets of order \( n \) together with the canonical projections \( \pi_{n,n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X) \). Note that \( \mathcal{L}_0(X) = X \) so that this time we use \( \Lambda = \mathbb{N}_0 \). Thus the arc space is a nontrivial example of a proalgebraic variety. If \( X \) is nonsingular and of complex dimension \( d \), then the projection \( \pi_{n,n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X) \) is a Zariski locally trivial fiber bundle with fiber being \( \mathbb{C}^d \). Thus in this case, in (iv) of Theorem 10.4 the Grothendieck class \( \gamma \) is \( \mathbb{L}^d \), with \( \mathbb{L} := [\mathbb{C}] \).
An element of $F^\text{ind}(X_\infty) = \lim_{\lambda \in \Lambda} F(X_\lambda)$ is called and indconstructible function and up to now we have not discussed the role of functions, even though it is called “function”. In fact, the indconstructible function can be considered in a natural way as a function on the proalgebraic variety simply as follows: for $[\alpha_\lambda] \in F^\text{ind}(X_\infty) = \lim_{\lambda \in \Lambda} F(X_\lambda)$ the value of $[\alpha_\lambda]$ at a point $(x_\mu) \in X_\infty = \lim_{\lambda \in \Lambda} X_\lambda$ is defined by

$$[\alpha_\lambda](x_\mu) := \alpha_\lambda(x_\lambda)$$

which is well-defined. So, if we let $\text{Fun}(X_\infty, \mathbb{Z})$ be the abelian group of $\mathbb{Z}$-valued functions on $X_\infty$, then the homomorphism $\Psi : \lim_{\lambda \in \Lambda} F(X_\lambda) \to \text{Fun}(X_\infty, \mathbb{Z})$ defined by $\Psi([\alpha_\lambda])((x_\mu)) := \alpha_\lambda(x_\lambda)$ shall be called the “functionization” homomorphism.

One can describe this in a fancier way as follows. Let $\pi_\lambda : X_\infty \to X_\lambda$ denote the canonical projection. Consider the following commutative diagram (which follows from $\pi_\lambda = \pi_{\lambda\mu} \circ \pi_\mu (\lambda < \mu)$):

$$\begin{array}{ccc}
F(X_\lambda) & \xrightarrow{\pi_\lambda^*} & \text{Fun}(X_\infty, \mathbb{Z}) \\
\pi_{\lambda\mu} \downarrow & & \downarrow \pi_\mu^* \\
F(X_\mu) & \xrightarrow{\pi_\mu^*} & \\
\end{array}$$

Then the “functionization” homomorphism $\Psi : \lim_{\lambda \in \Lambda} F(X_\lambda) \to \text{Fun}(X_\infty, \mathbb{Z})$ is the unique homomorphism such that the following diagram commutes:

$$\begin{array}{ccc}
F^\text{ind}(X_\infty) & \xrightarrow{\rho^\lambda} & F(X_\lambda) \\
\downarrow \Psi & & \pi_\lambda^* \downarrow \\
\text{Fun}(X_\infty, \mathbb{Z}). & \xrightarrow{\pi_\mu^*} & \\
\end{array}$$

To avoid some possible confusion, the image $\Psi([\alpha_\lambda]) = \pi_\lambda^* \alpha_\lambda$ shall be denoted by $[\alpha_\lambda]_\infty$. For a constructible set $W_\lambda \in X_\lambda$, by the definition we have

$$[\mathbb{I}_{W_\lambda}]_\infty = \mathbb{I}_{\pi_\lambda^{-1}(W_\lambda)}.$$ 

$\pi_\lambda^{-1}(W_\lambda)$ is called a proconstructible or a cylinder set, mimicking [Cr]. And the characteristic function supported on a proconstructible set is called a procharacteristic function and a finite linear combination of procharacteristic functions is called a proconstructible function. Let $F^{\text{pro}}(X_\infty)$ denote the abelian group of all proconstructible functions on the proalgebraic variety $X_\infty = \lim_{\lambda \in \Lambda} \{X_\lambda, \pi_{\lambda\mu} : X_\mu \to X_\lambda\}$. Thus we have the following

**Proposition 10.6.** For a proalgebraic variety $X_\infty = \lim_{\lambda \in \Lambda} \{X_\lambda, \pi_{\lambda\mu} : X_\mu \to X_\lambda\}$

$$F^{\text{pro}}(X_\infty) = \text{Image}(\Psi : F^\text{ind}(X_\infty) \to \text{Fun}(X_\infty, \mathbb{Z})) = \bigcup_{\mu} \pi_\mu^*(F(X_\mu)).$$

If the structure morphisms $\pi_{\lambda\mu} : X_\mu \to X_\lambda (\lambda < \mu)$ are all surjective, then we have

$$F^\text{ind}(X_\infty) \cong F^{\text{pro}}(X_\infty).$$

In the case of the arc space $L(X)$ of a nonsingular variety $X$, since each structure morphism $\pi_{n,n+1} : L_{n+1}(X) \to L_n(X)$ is always surjective, we get the following
Corollary 10.7. Assume $X$ is a nonsingular variety of dimension $d$. Then we have for the arc space $\mathcal{L}(X)$ the canonical isomorphism

$$F^\text{ind}(\mathcal{L}(X)) \cong F^\text{pro}(\mathcal{L}(X)),$$

together with the following canonical Grothendieck class homomorphisms

$$\Gamma^\text{ind}_X: F^\text{pro}(\mathcal{L}(X)) \to K_0(\mathcal{V}/X)[L^\ell] \quad \text{and} \quad \Gamma^\text{ind}: F^\text{pro}(\mathcal{L}(X)) \to K_0(\mathcal{V})[L^\ell]$$
described by

$$\Gamma^\text{ind}_X([\alpha_n]_\infty) = \frac{\pi_{0,n*}(\mathcal{L}_n(X)(\alpha_n))}{[L]^\text{ind}} \quad \text{and} \quad \Gamma^\text{ind}([\alpha_n]_\infty) = \frac{\Gamma(\alpha_n)}{[L]^\text{ind}}.$$

In particular, we get that $\Gamma^\text{ind}_X(\mathbb{I}_{\mathcal{L}(X)}) = [id_X]$ and $\Gamma^\text{ind}(\mathbb{I}_{\mathcal{L}(X)}) = [X]$.

So $\Gamma^\text{ind}_X$ and $\Gamma^\text{ind}$ define finitely additive measures $\mu_X$ and $\mu$ on the algebra of cylinder sets in the arc space $\mathcal{L}(X)$ of a nonsingular variety $X$, which are called naive motivic measures. So we can rewrite $\Gamma^\text{ind}_X(\alpha)$ and $\Gamma^\text{ind}(\alpha)$ for $\alpha \in F^\text{pro}(\mathcal{L}(X))$ as motivic integrals

$$\Gamma^\text{ind}_X(\alpha) = \int_{\mathcal{L}(X)} \alpha \, d\mu_X \quad \text{and} \quad \Gamma^\text{ind}(\alpha) = \int_{\mathcal{L}(X)} \alpha \, d\mu .$$

Therefore we see that our proalgebraic Grothendieck class homomorphisms of Theorem 10.4 are a generalization of these naive motivic measures. Here for “naive” we point out that for the applications of a good motivic integration theory (e.g., as described in the next section) one needs to take values in a suitable completion of $K_0(\mathcal{V}/X)[L^\ell]$ or $K_0(\mathcal{V})[L^\ell]$ so that more general sets than just cylinder sets become “measurable”. Also the use of the “relative measure” $\Gamma^\text{ind}_X$ over the base space $X$ due to Looijenga [Lo] is more recent, and the use of the “limit surely depends on the choice of it. So we make the covariant functor $\mathbb{B}$ into a bifunctor using the functorial “Gysin homomorphisms” $b_{\lambda\mu} \bullet : \mathbb{B}_*(X_\lambda) \to \mathbb{B}_*(X_\mu)$ induced
by the projective system \( \{ b_{\lambda \mu} \} \). For example, in the above Example 10.2, we have that
\[
F^{\text{ind}}(X_\infty) = \mathbb{B}^t \{ X_\infty; \{ \mathbb{B} \mu \} \}.
\]

Definition 10.8. Let \( \{ f_\lambda : X_\lambda \rightarrow Y_\lambda \}_{\lambda \in \Lambda} \) be a pro-morphism of pro-algebraic varieties \( \{ X_\lambda, \pi_{\lambda \mu} : X_\mu \rightarrow X_\lambda \} \) and \( \{ Y_\lambda, \rho_{\lambda \mu} : Y_\mu \rightarrow Y_\lambda \} \). If the following commutative diagram for \( \lambda < \mu \)
\[
\begin{array}{ccc}
X_\mu & \xrightarrow{f_\mu} & Y_\mu \\
\pi_{\lambda \mu} \downarrow & & \downarrow \rho_{\lambda \mu} \\
X_\lambda & \xrightarrow{f_\lambda} & Y_\lambda
\end{array}
\]
is a fiber square, then we call the pro-morphism \( \{ f_\lambda : X_\lambda \rightarrow Y_\lambda \}_{\lambda \in \Lambda} \) a fiber-square pro-morphism, abusing words.

With these definitions we have the following theorem:

Theorem 10.9. (i) Let \( \gamma : \mathbb{B} \rightarrow \mathbb{B}' \) be a Grothendieck transformation between two bivariant theories \( \mathbb{B}, \mathbb{B}' : \mathcal{C} \rightarrow \mathcal{A} \) and let \( \{ (\pi_{\lambda \mu}; b_{\lambda \mu}) : X_\mu \rightarrow X_\lambda \} \) be a projective system of bivariant-class-equipped morphisms. Then we get the following pro-version of the natural transformation \( \gamma_* : \mathbb{B}_* \rightarrow \mathbb{B}'_* \):
\[
\gamma_*^{\text{ind}} : \mathbb{B}_*^{\text{ind}} \{ X_\infty; \{ b_{\lambda \mu} \} \} \rightarrow \mathbb{B}'_*^{\text{ind}} \{ X_\infty; \{ \gamma(b_{\lambda \mu}) \} \}.
\]

(ii) Let \( \{ f_\lambda : X_\lambda \rightarrow Y_\lambda \} \) be a fiber-square pro-morphism between two projective systems \( \{ (\rho_{\lambda \mu}; d_{\lambda \mu}) : Y_\mu \rightarrow Y_\lambda \} \) and \( \{ (\pi_{\lambda \mu}; b_{\lambda \mu}) : X_\mu \rightarrow X_\lambda \} \) of bivariant-class-equipped morphisms such that \( d_{\lambda \mu} = f_\lambda^* b_{\lambda \mu} \). Then we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{B}_*^{\text{ind}}(Y_\infty; \{ d_{\lambda \mu} \}) & \xrightarrow{\gamma_*^{\text{ind}}} & \mathbb{B}'_*^{\text{ind}}(X_\infty; \{ \gamma(d_{\lambda \mu}) \}) \\
\downarrow f_\infty & & \downarrow f_\infty \\
\mathbb{B}_*^{\text{ind}}(X_\infty; \{ b_{\lambda \mu} \}) & \xrightarrow{\gamma_*^{\text{ind}}} & \mathbb{B}'_*^{\text{ind}}(X_\infty; \{ \gamma(b_{\lambda \mu}) \}).
\end{array}
\]

(iii) Let \( \mathbb{B}_*(pt) = \mathbb{B}'_*(pt) \) be a commutative ring \( \mathcal{R} \) with a unit and we assume that the homomorphism \( \gamma : \mathbb{B}_*(pt) \rightarrow \mathbb{B}'_*(pt) \) is the identity. Let \( P = \{ p_{\lambda \mu} \} \) be a projective system of elements \( p_{\lambda \mu} \in \mathcal{R} \). Then we get the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{B}'_*^{\text{ind}}(X_\infty; \{ b_{\lambda \mu} \}) & \xrightarrow{\gamma_*^{\text{ind}}} & \mathbb{B}'_*^{\text{ind}}(X_\infty; \{ \gamma(b_{\lambda \mu}) \}) \\
\downarrow \chi_{\pi_{\lambda \mu}}^{\text{ind}} & & \downarrow \chi_{\pi_{\lambda \mu}}^{\text{ind}} \\
\lim_{\lambda \in \Lambda} \times p_{\lambda \mu} : \mathcal{R} \rightarrow \mathcal{R}.
\end{array}
\]

If we apply this theorem to the Brasselet’s bivariant Chern class [Br] or to the one of [BSY], we get a proalgebraic version \( c_\lambda^{\text{ind}} \) of the Chern–Schwartz–MacPherson class transformation \( c_\lambda : F \rightarrow H_\lambda \). But of course we also can apply it to the bivariant versions of our motivic characteristic class transformations \( mC_\lambda \) and \( T_{b\lambda} \).

As a very simple example, consider a proalgebraic variety \( X_\infty = \lim_{\lambda \in \Lambda} \{ X_\lambda, \pi_{\lambda \mu} : X_\mu \rightarrow X_\lambda \} \), whose structure maps \( \pi_{\lambda \mu} \) are smooth (and therefore “Euler morphisms”) and proper. Then we can apply the proalgebraic Chern–Schwartz–MacPherson class transformation \( c_\lambda^{\text{ind}} \) to
\[
F^{\text{ind}}(X_\infty) = \mathbb{B}^t \{ X_\infty; \{ \mathbb{B} \mu \} \}.
\]
Note that in this case \( \gamma(\pi_{i\lambda}) = c^*(T_{\pi_{i\lambda}}) \bullet [\pi_{i\lambda}] \) by the Verdier Riemann-Roch theorem for a smooth morphism, so that \( H_*^{\text{ind}} \left( X_\infty; \{ \gamma(\pi_{i\lambda}) \} \right) \) is just the inductive limit of the following system of “twisted” smooth pullbacks in homology:

\[
\pi_{i\lambda}^{\Pi} := c^*(T_{\pi_{i\lambda}}) \cap \pi_{i\lambda}^1 : H_* (X_\lambda; \mathbb{Z}) \to H_* (X_{\mu}; \mathbb{Z})
\]

Suitable modifications of such “inductive limits of twisted smooth pullback morphisms” are closely related to the construction of equivariant characteristic classes (compare for example with [Oh] §3.3, p.12-13).

11. Stringy and arc characteristic classes of singular spaces

In this last section we explain another and more recent extension of characteristic classes to singular spaces. These are not functorial theories as before, but have a better “birational invariance”, in particular for \( K \)-equivalent manifolds, i.e. \( M_i (i = 1, 2) \) are irreducible (or pure dimensional) complex algebraic manifolds dominated by a third such manifold \( M \), with \( \pi_i : M \to M_i \) proper birational \( (i = 1, 2) \) such that the pullbacks of their canonical bundles (or divisors) \( \pi_i^* K_{M_i} \simeq \pi_{i\lambda}^* K_M \) are isomorphic (or linearly equivalent). For example \( M_1 \) and \( M_2 \) are both Calabi-Yau manifolds in the sense that their canonical bundle is trivial. In fact the origin of these classes and invariants goes back to two different generalizations of Hirzebruch’s \( \chi_y \)-genus (which was related to our motivic characteristic classes \( mC_* \) and \( T_{\mu*} \)).

The first one is the \( E \)-polynomial or Hodge characteristic \( E(X)(u, v) \in \mathbb{Z}[u, v] \) defined in terms of Deligne’s mixed Hodge structure [Dc1, Dc2] for the cohomology with compact support \( H^*_c (X, \mathbb{Q}) \) of a complex algebraic variety. We have that \( E(X)(1, 1) = \chi(X) \) for any variety \( X \) and \( E(X)(-y, 1) = \chi_y(X) \) for \( X \) smooth and compact. In the 90’s V. Batyrev [Bat] extended this \( E \)-polynomial to a stringy \( E \)-function \( E_{\text{str}} \) and stringy Euler numbers \( \chi_{\text{str}} \) of “log-terminal pairs” \( (X, D) \) relating them in some cases known as the “McKay correspondence” to orbifold invariants of suitable quotient varieties. He also used in [Bat2] methods from \( p \)-adic integration theory to prove that different “crepant resolutions” of a given singular space, and also birationally equivalent Calabi-Yau manifolds, have equal Betti numbers. Later on M. Kontsevich [Kon] invented “motivic integration” (with some analogy to \( p \)-adic integration) for extending these results from Betti numbers to Hodge numbers.

The other generalization of the \( \chi_y \)-genus is the (complex) elliptic genus \( \ell_k \) studied by L. Krichever [Kric] and G. Höhn [Höhn]. As observed by Totaro [Tt] (and compare with [BR]), this is the most general genus on the complex cobordism ring \( \Omega^F \otimes \mathbb{Q} \), which can be invariant under a suitable notion of “flips”. Later on this was extended by L. Borisov and A. Libgober [BL1] and C.-L. Wang [Wang] for showing the invariance of this elliptic genus \( \ell_k \) for \( K \) – equivalent complex algebraic manifolds, a notion coming from “minimal model theory”. Both works use the very deep “weak factorization theorem” ([AKMW] and [W]) for the comparison of different resolution spaces. They also introduced in this way the elliptic homology class \( \mathcal{E} \) of a \( \mathbb{Q} \)-Gorenstein log-terminal singular complex algebraic variety \( X \) ([BL2] [Wang]). Here \( \mathbb{Q} \)-Gorenstein for a normal irreducible (or pure dimensional) variety \( X \) just means that some multiple \( r \cdot K_X (r \in \mathbb{N}) \) of the canonical Weil divisor \( K_X \) is already a Cartier divisor, with \( r = 1 \) corresponding to a \( Gorenstein \) variety (e.g. \( X \) is smooth). Here \( K_X \) is just the closure of a canonical divisor on the regular part. In fact, Borisov-Libgober proved in [BL] for this elliptic homology class a very general version of the “McKay correspondence”.

More recently simpler stringy Chern classes \( c_{\text{str}}^*(X) \) were introduced by Aluffi [Al1], based on the “weak factorization theorem”, and independently by de Fernex, Lupercio,
Nevins and Uribe [FLNU], based on “motivic integration” and MacPherson’s functorial Chern class transformation $c_s$. In fact Aluffi pointed out that there are two possible notions of such classes, depending on two different choices of a system of “relative canonical divisors” $K_{\pi}$ for a suitable resolution of singularities $\pi : M \to X$ (i.e. $\pi$ is proper and $M$ smooth), which he calls the “$\Omega$-flavor” and “$\omega$-flavor”.

The “$\omega$-flavor” is related to “stringy invariants and characteristic classes” (like $E_{str}$, $E_{ill}$, and $c^{(str)}_s$). Here one assumes $X$ is irreducible and $\mathbb{Q}$-Gorenstein so that the relative canonical divisor $K_{\pi} := K_M - \pi^* K_X$ is at least a $\mathbb{Q}$-Cartier divisor (class). Moreover it is supported on the exceptional locus $E$ of the resolution, which is supposed to be (contained in) a normal crossing divisor with smooth irreducible components $E_i$. Then $K_{\pi} \simeq \sum_i a_i \cdot E_i$ for some fixed $a_i \in \mathbb{Q}$ (depending on the resolution). And for the definition of all these “stringy invariants” one needs the condition $a_i > -1$ for all $i$, which exactly means that $X$ has only log-terminal singularities. If this condition holds for one such resolution, then it is true for any resolutions of this type. A resolution $\pi$ is called crepant, if $K_M \simeq \pi^* K_X$, e.g. all $a_i = 0$ for $E$ a normal crossing divisor as before.

The “$\Omega$-flavor” is related to what we call “arc invariants and arc characteristic classes”, because these generalize corresponding “arc invariants” of Denef and Loeser [DL1, §6] and [DL2, §4.1], which they introduced already before by their work on “motivic integration”. In this case $X$ is only assumed to be pure $d$-dimensional and $K_{\pi}$ is defined for all resolutions $\pi$ such that the canonical map $\pi^* \Omega^d_X \to \Omega^d_M$ of Kähler differentials has an image $\mathcal{I} \otimes \Omega^d_M$ with $\mathcal{I}$ a principal ideal in $\mathcal{O}_M$ (this can always be achieved by Hironaka [HI]). Then $K_{\pi}$ is defined by $\mathcal{I} = \mathcal{O}_M(-K_{\pi})$. The effective Cartier divisor $K_{\pi}$ is again supported on the exceptional locus $E$ of the resolution, which can also be supposed to be (contained in) a normal crossing divisor with smooth irreducible components $E_i$. Then one can introduce the $a_i \in \mathbb{N}_0$ as before.

For $X$ already smooth, both notions of a relative canonical divisor $K_{\pi}$ agree with the divisor of the Jacobian of $\pi$ defined by the section $s$ of $K_M \otimes \pi^* K_X^*$ corresponding to the canonical map $\pi^* \Omega^d_X \to \Omega^d_M$. Note that in both cases the corresponding resolutions $\pi : M \to X$ as above form a directed set, i.e. two of them can be dominated by a third one of this type (and taking suitable limits over this directed set corresponds to the view point of Aluffi [Alu4]). If $\pi' : M' \to M$ is a proper birational map with $\pi$ and $\pi \circ \pi'$ as above, then the relative canonical divisors have (in both cases) the following crucial transitivity property:

\begin{equation}
K_{\pi \circ \pi'} \simeq K_{\pi'} + \pi'^* K_{\pi}.
\end{equation}

Then all these new invariants $I(X)$ for a singular space $X$ as above are defined as

\[ I(X) := \pi_*(I(M) \cdot J(\{E_i, a_i\})) \in B_*(X) \]

for such a special resolution $\pi : M \to X$, with $E$ a normal crossing divisor with smooth irreducible components $E_i$, where $I(M) \in B_*(M)$ is the corresponding invariant of the smooth space $M$, together with some “correction term” $J(\{E_i, a_i\}) \in \mathbb{B}^*(M)$ depending on the exceptional divisor $E$ and the multiplicities $a_i$ defined by the relative canonical divisor $K_{\pi}$. Here $B_*$ and $\mathbb{B}^*$ are suitable covariant and contravariant theories taking values in the category of Abelian groups and commutative rings with unit, related by the projection formula as in Example [10.3]. Typical examples are

1. $B_*(X) = \mathbb{B}^*(X) = \Lambda$ is a commutative ring with unit (with all pullbacks and push downs the identity transformation $id_{R}$) so that $I(M) \in R$ corresponds to a suitable generalized “Euler characteristic type invariant”.

2. $B_*$ and $\mathbb{B}^*$ correspond to a suitable (co)homology theory like $(B_*(X), \mathbb{B}^*(X)) = (H^*(X) \otimes \Lambda, H^*(X) \otimes \Lambda)$ or $(B_*(X), \mathbb{B}^*(X)) = (G^0(X) \otimes \Lambda, K^0(X) \otimes \Lambda)$ so that $I(M) \in B_*(M)$ is a suitable characteristic class of $M$. 

(3) $\mathbb{B}(X) := \mathbb{B}_s(X) = \mathbb{B}^*(X)$ is a bifunctor as in Example JUS e.g. like constructible functions $\mathbb{B}(X) = F(X) \otimes \Lambda$ or relative Grothendieck rings of varieties $K_0(V/X) \otimes \Lambda$ coming up from “motivic integrals”.

If $I(X) \in \mathbb{B}_s(X)$ is such an invariant not depending on the choice of the resolution $\pi$, then the same is true for $\gamma_*(I(X)) \in \mathbb{B}'(X)$ for any natural transformation of covariant theories $\gamma_* : \mathbb{B}_s \rightarrow \mathbb{B}'$. For example $I(X) \in H_*(X) \otimes \Lambda$ is a characteristic homology class with $X$ compact, and $\text{deg} := \gamma_* : H_*(X) \otimes \Lambda \rightarrow H_*(\{pt\}) \otimes \Lambda = \Lambda$ is just its degree (or push down to a point). Or we apply suitable “completions” of our motivic characteristic class transformations $mC_*$ and $T_{\nu_*}$ to invariants $I(X)$ coming from motivic integration!

For showing that the final result $I(X)$ does not depend on the choice of the resolution, either “motivic integration with its transformation rule” related to the “Jacobian factor” $J(\{E_i, a_i\})$ is used:

$$\int_{\mathcal{M}} L^{-\alpha} d\mu = \pi_* \int_{\mathcal{M}'} L^{-(\pi^{-1}\alpha + K_\pi)} d\mu$$

for $\pi' : M' \rightarrow M$ a proper birational map of manifolds and $L := [\mathbb{C}] \in K_0(V)$. This suggests to think of $I(X)$ as the push down of an “integral with respect to the invariant $I(M)$”:

$$I(X) = \pi_* \int_M L^{-K_\pi} dI(M).$$

Or the “weak factorization theorem” is used, in which case only the invariance under suitable “blowing ups” has to be checked.

Moreover $J(\{E_i, a_i\}) = 1$ in case all $a_i = 0$, so that $I(X) = \pi_*(I(M))$ in case of a crepant resolution. In particular $\pi_* (I(M))$ does not depend on the choice of this crepant resolution. Suppose two maybe singular spaces $X_i (i = 1, 2)$ are $K$-equivalent in the sense that they are dominated by a manifold $M$, with $\pi_i : M \rightarrow X_i$ a resolution of singularities such that the relative canonical divisors $K_{\pi_i}$ are defined ($i = 1, 2$) and equal. After taking another resolution of $M$, we can even assume that the exceptional locus of both maps is contained in a normal crossing divisor $E$ with smooth irreducible components $E_i$ (here we use the transitivity property of the relative canonical divisors). But then the correction factor $J(\{E_i, a_i\})$ for both maps is the same, so that

$$I(X_1) = \pi_1 (I(M) \cdot J(\{E_i, a_i\})) \quad \text{and} \quad I(X_2) = \pi_2 (I(M) \cdot J(\{E_i, a_i\})),$$

i.e. both invariants $I(X_1)$ and $I(X_2)$ are “dominated” by the same element coming from $M$. In particular

$$I(X_1) = I(X_2)$$

in case of “Euler characteristic type invariants”, and

$$\text{deg}(I(X_1)) = \text{deg}(I(X_2))$$

in case of “characteristic homology classes” for compact spaces $X_i$. If we are working in the “$\omega$-flavor” of stringy homology classes $I(X_i) \in H^{BM}_s(X_i) \otimes \Lambda$ for $\mathbb{Q}$-Gorenstein varieties $X_i$, we can use the first Chern class $c^1(K_{X_i}) := c^1(r \cdot K_{X_i})/r \in H^2(X_i; \mathbb{Q})$ (for $\Lambda$ a $\mathbb{Q}$-algebra) to modify $I(X_i)$ into

$$I'(X_i) := f(c^1(K_{X_i})) \cdot I(X_i) \in H^{BM}_s(X_i) \otimes \Lambda.$$

By the projection formula also these new invariants $I'(X_1)$ and $I'(X_2)$ are “dominated” by the same element coming from $M$, where $f \in \Lambda[[z]]$ can be any power series. If $X_i$ are both Gorenstein, we can do the same thing for corresponding invariants $I(X_i) \in G_0(X) \otimes \Lambda$ by using polynomials in the (inverse) classes $[K_{X_i}^{\pm 1}] \in K^0(X_i)$ of the canonical Cartier divisors (instead of their first Chern classes).
11.1. **Elliptic classes.** Let us start with the definition of the (*complex*) *elliptic class* $E LL(E)$ of a complex vector bundle $E \to X$. Consider the formal power series

$$\Lambda_t(E) := \sum_{n \geq 0} t^n \Lambda^n E \quad \text{and} \quad S_t(E) := \sum_{n \geq 0} t^n S^n E,$$

with $\Lambda^n E$ and $S^n E$ the corresponding exterior and symmetric power of $E$ (so $\Lambda^n E = 0$ for $n > \text{rank} \ E$, with $\Lambda_t$ the total Lambda class coming up in our definition of the motivic Chern class transformation $mC_*$ in Corollary 8.4). Then one has

$$\Lambda_t(E \oplus F) = \Lambda_t(E) \Lambda_t(F), \quad S_t(E \oplus F) = S_t(E) S_t(F), \quad \text{and} \quad \Lambda_t(E) S_{-t}(E) = 1.$$  

So these operations extend to the Grothendieck group of complex vector bundles (and similarly in the algebraic context):

$$\Lambda_t, S_t : (K(X), \oplus) \to (1 + K(X)[[t]], \otimes) \subset (K(X)[[t]], \otimes).$$

Then we define the *complex elliptic class* $E LL(E) = E LL(y, q)(E) \in K(X)[[q]][y^{\pm 1}]$ of a complex vector bundle $E \to X$ as $E LL(E) := \Lambda_y(E^*) \otimes W(E)$, with

$$W(E) := \bigotimes_{n \geq 1} \left( \Lambda_{y^n}(E^*) \otimes \Lambda_{y^{-1}q^n}(E) \otimes S_{q^n}(E^*) \otimes S_{q^n}(E) \right).$$

More generally the *elliptic class* of order $k$

$$E LL_k(E) = E LL_k(y, q)(E) \in K(X)[[q]][y^{\pm 1}]$$

with $k \in \mathbb{Z}$ of a complex vector bundle $E \to X$ is defined as the twisted class

$$E LL_k(E) := \text{det}(E)^{\otimes -k} \otimes E LL(E),$$

with $\text{det}(E) := \Lambda^\text{rank} E(E)$ being the determinant line bundle of $E$. So $E LL(E)$ (or $E LL_k(E)$) is a one (or two) parameter deformation of the total Lambda class $\Lambda_y(E^*)$, with

$$E LL_0(E) = E LL(E) \quad \text{and} \quad E LL(E)|_{q=0} = \Lambda_y(E^*).$$

For $M$ a complex projective algebraic manifold (or a compact almost complex manifold) one can introduce as in [5] the $\chi = T$-characteristic

$$\chi(E LL_k(E)) \in \mathbb{Q}[[k, q]][y^{\pm 1}]$$

of $E LL_k(E)$ as

$$\chi(E LL_k(E)) := \int_M \text{ch}^*(E LL_k(E)) \cdot \text{td}^*(TM) \cap [M]$$

$$= \int_M e^{-k \cdot c^1(TM)} \cdot \text{ch}^*(E LL(E)) \cdot \text{td}^*(TM) \cap [M].$$

Note that in the last term one can introduce $k$ as a formal parameter. $\text{ch}^*(E LL_k(E))$ and $\text{ch}^*(E LL(E))$ are multiplicative (but not normalized) characteristic classes so that we get the induced Krichever–Höhn *elliptic genus*

$$ell_k : \Omega_*^U \otimes \mathbb{Q} \to \mathbb{Q}[[k, q]][y^{\pm 1}],$$

with

$$(\text{11.5}) \quad ell_k(M) := \int_M e^{-k \cdot c^1(TM)} \cdot \text{ch}^*(E LL(TM)) \cdot \text{td}^*(TM) \cap [M].$$
The corresponding complex elliptic genus \( \text{ell} := \text{ell}_0 : \Omega^* \otimes \mathbb{Q} \to \mathbb{Q}[[q]][y^{\pm 1}] \) given by

\[
\text{ell}_0(M) = \int_M \text{ch}^*(\mathcal{W}(E)) \cdot \text{ch}^*(\Lambda_y T^*M) \cdot td^*(TM) \cap [M]
\]

\[
= \chi_y(M, \mathcal{W}(E)) = \chi_y(M, \bigotimes_{n \geq 1} (\Lambda_y q^n (T^*M) \otimes \Lambda_y^{-1} q^n (TM) \otimes S_q^n (TM^*))
\]

was formally interpreted by Witten as the \( S^1 \)-equivariant \( \chi_y \)-genus \( \chi_y(S^1, LM) \) of the free loop space \( LM = \{ f : S^1 \to M \mid f \text{ smooth} \} \) of \( M \) (compare [HBJ] Appendix III) and [BR].

\[
\chi_{k,y}(M) := \text{ell}_k(M)|_{q=0} \in \mathbb{Q}[y][[k]]
\]

is called the twisted \( \chi_y \)-genus of \( M \):

\[
(11.6) \quad \chi_{k,y}(M) = \int_M e^{-k \cdot \text{ch}^1(TM)} \cdot \text{ch}^*(\Lambda_y (T^*M)) \cdot td^*(TM) \cap [M].
\]

Another specialization is the real elliptic genus \( \text{ell} |_{y=1} \), which factorizes over the oriented cobordism ring

\[
\text{ell}|_{y=1} : \Omega^*_S \otimes \mathbb{Q} \to \mathbb{Q}[[q]].
\]

This one parameter genus interpolates between the signature genus (for \( q \to 0 \)) and the \( \tilde{A} \)-genus (for \( q \to \infty \)), and was formally interpreted by Witten as the \( S^1 \)-equivariant signature \( \sigma(S^1, LM) \) of the free loop space \( LM \) of the oriented manifold \( M \) (compare [HBJ] §6 and [BR]).

**Remark 11.7.** We should point out that there are many different normalizations of the elliptic genus and classes in the literature. First of all many authors (like [BL1] [BL2] [W1] [W2]) use \( -y \) instead of \( y \) so that their elliptic genus is related to the \( \chi_y \)-genus. But what is maybe more important, we do not work with “normalized characteristic classes”, i.e. the power series \( f(z) \in \mathbb{Q}[[k,q]][y^{\pm 1}][[z]] \) in the variable \( z = c^1 \) corresponding to the multiplicative characteristic class \( \text{ch}^*(\mathcal{E}LL_k(\_)) \) has a constant coefficient \( a := f(0) \neq 1 \), since \( \text{ch}^*(\mathcal{E}LL(E))|_{q=0} = \text{ch}^*(\Lambda_y (E^*)) \) implies \( a(k = 0, q = 0) = 1 + y \in \mathbb{Q}[y^{\pm 1}] \). So twisting \( f(z) \) to a normalized power series \( f(z)/a \) (as used in [BR] [W]) would change the elliptic genus only to \( \text{ell}_k(M)/a^\sigma \) for \( M \) an (almost) complex manifold of complex dimension \( n \), and similarly a characteristic homology class \( e_i(\_ \in H^{BM}_{2i} (\_)) \otimes \Lambda \) would just be multiplied by \( a^{-1} \). For example in theorem 5.3 we could have started with the natural transformation (with respect to proper maps):

\[
\tilde{T}_{y*,} := td_* \circ MC_* : K_0(V/\_ \to H^{BM}_{2i}(\_ \otimes \mathbb{Q}[y],
\]

satisfying for \( M \) nonsingular the normalization

\[
\tilde{T}_{y*}([M \xrightarrow{\text{id}} M]) = \text{ch}^*(\Lambda_y T^*M) \cdot td^*(TM) \cap [M].
\]

And “twisting” by \( 1 + y \) would then give our motivic characteristic class transformation \( T_{y*} \) with

\[
(11.8) \quad T_{y,i}(\_ \in H^{BM}_{2i}(\_ \otimes \mathbb{Q}[y], (1 + y)^{-1}).
\]

But since we work in this section only with pure dimensional spaces, this “twisting” does not matter for the question of getting invariants of pure dimensional singular complex algebraic varieties. Similarly it will be enough to consider only the complex elliptic genus and classes corresponding to \( k = 0 \) (as in [BL1] [BL2]), since the case of general \( k \) follows then from the projection formula (as already explained before). So the elliptic classes \( \mathcal{E}ll^*(z, \tau) \) used in [BL1] [BL2] correspond in our notation to

\[
\mathcal{E}ll^*(z, \tau)(TM) := y^{-\text{dim}(M)/2} \cdot \text{td}^*(TM) \cdot \text{ch}^*(\mathcal{E}LL(TM))(-y, q),
\]

with \( y = e^{2\pi i z} \) and \( q = e^{2\pi i \tau} \).
With these notations, we can now explain the definition of Libgober and Borisov (BL2, Definition 3.2) with $G := \{id\}$ for their elliptic class $Ell_\ast((X, D))$ of a “Kawamata log terminal pair $(X, D)$”, i.e. $X$ is a normal irreducible complex algebraic variety, with $D$ a $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + D$ is a $\mathbb{Q}$-Cartier divisor satisfying the following condition: There is a resolution of singularities $\pi : M \to X$ with the exceptional locus $E$ and the support of $K_\pi(E) := K_M - \pi^*(K_X + D)$ contained in a normal crossing divisor with smooth irreducible components $E_i$ ($i \in I$) such that $K_\pi(E) \simeq \sum_i a_i \cdot E_i$, with all $a_i \in \mathbb{Q}$ satisfying the inequality $a_i > -1$. Note that the last condition is then independent of the choice of such a resolution (compare [KM] Definition 2.34, Corollary 2.31), with the case $D = 0$ corresponding to the case “$X$ is $\mathbb{Q}$-Gorenstein with only log-terminal singularities”. Moreover, the “relative canonical divisor $K_\pi(D)$ of $D$” also satisfies the transitivity property

$$K_{\pi_0 \pi}(D) \simeq K_{\pi_0} + \pi_0^* K_\pi(D)$$

for $\pi' : M' \to M$ a proper birational map with $\pi$ and $\pi \circ \pi'$ as before. Then

$$Ell_\ast((X, D)) := \pi_\ast (Ell_\ast(TM) \cap [M]) \cap \prod_i J(E_i, a_i),$$

with

$$J(E_i, a_i)(z, \tau) := \frac{\theta(\frac{e_i}{2\pi i}, (a_i + 1)z, \tau)\theta(-z, \tau)}{\theta(\frac{e_i}{2\pi i} - z, \tau)\theta(-(a_i + 1)z, \tau)} \in H^\ast(M; \mathbb{Q})[[y, q]].$$

Here $\theta(z, \tau)$ is the Jacobi theta function in $y = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$, with $e_i = e_i^1(E_i) \in H^2(M, \mathbb{Z})$ the first Chern class of the smooth divisor $E_i$.

The proof of the independence of the resolution $\pi$ uses the “weak factorization theorem” for reducing it to the comparison with a suitable blowing up along a smooth center. Using some modularity properties of the $\theta$-function, this is finally reduced to the vanishing of a suitable residue (of an elliptic function with exactly one pole, compare [BL2, p.11] and [Wang, §4]). If $X$ is compact, then

$$\text{ell}((X, D)) := \text{deg}(Ell_\ast((X, D)))$$

is just the singular elliptic genus of the Kawamata log terminal pair $(X, D)$ as defined in [BL1, Definition 3.1] (up to a normalization factor).

Later on we only need the following limit formula (with $y = e^{2\pi iz}$):

$$\lim_{\tau \to \infty} J(E_i, a_i)(z, \tau) = \frac{(y - 1)(1 - ye^{1 + e_i})}{(y^{e_i + 1} - 1)(1 - ye^{1 - e_i})} = 1 + \frac{(y - ye^{1 + e_i})(1 - e_i)}{(y^{e_i + 1} - 1)(1 - ye^{1 - e_i})}.$$

Note that the multiplicative characteristic class

$$\tilde T^\ast_y(E) := ch^\ast(E LL(E))|_{y = 0} \cdot td^\ast(E) = ch^\ast(\Lambda_y(E^\ast)) \cdot td^\ast(E)$$

exactly corresponds to the power series $f(z) = \frac{z(1 + ye^{-z})}{1 + ze^{-z}}$ in the variable $z = e^1$ (compare with section 6). If we denote for $J \subset I$ the closed embedding $i_J : E_J := \bigcap_{i \in J} E_i \to M$ of the submanifold $E_J$ (with $E_0 := M$), then one has by the “adjunction formula” $i_J \ast i^\ast_J = \prod_{i \in J} e_i \cap \cdot$, with $TE_J = i^\ast_J(TM - \sum_{i \in J} \mathcal{O}(E_i))$ (compare [HBJ] p.36):

$$i_J \ast (\tilde T^\ast_y(TE_J) \cap [E_J]) = (\tilde T^\ast_y(TM) \cap [M]) \cap \prod_{i \in J} \frac{1 - e^{-e_i}}{1 + ye^{-e_i}}.$$
So altogether we get the following “limit formula” (with $y = e^{2\pi i z}$):

$$\lim_{\tau \to \infty} y^{\dim(X)/2} \cdot \mathcal{E}H_*(X, D) = \pi_* \left( \sum_{j \in I} i_{j*}(\mathcal{T}_j - y_\tau(E_j)) \cdot \prod_{i \in J} y_i - y^{\mu_i+1} - 1 \right).$$

Recall that we use the notation $cl_*(E_j) = cl^*(TE_j) \cap [E_j]$ for the characteristic homology class of a manifold (corresponding to a characteristic class $cl^*$ of vector bundles).

11.2. Motivic integration. Motivic integration was invented by Kontsevich [Kon] for showing that birational equivalent Calabi-Yau manifolds have equal Hodge numbers. In all details with many different applications it was developed by Denef-Loeser (e.g. [DL1, DL2, DL3]), with some improvements by Looijenga [Lo], who in particular introduced the calculus of relative Grothendieck rings $K_0(V/X)$ of algebraic varieties. For a nice introduction to “stringy invariants of singular spaces” we recommend [Ve1, Ve2]. Even though motivic integration can be directly studied on singular spaces, we restrict ourselves to the simpler case of smooth spaces, which will be enough for our applications. Also in this way it can easily be compared to results coming from the use of the “weak factorization theorem”. For a quick introduction to “motivic integration on smooth spaces” compare with [Cr] (where by Corollary 10.7 all arguments of [Cr] extend to the framework of “relative motivic measures).

Let $M$ be a pure $d$-dimensional complex algebraic manifold and $E = \sum_{i=1}^{k} a_i E_i$ be an effective normal crossing divisor (e.g. $a_i \in \mathbb{N}_0$) on $M$, with smooth irreducible components $E_i$. Then one can introduce on the arc space $\mathcal{L}(M) = \{\gamma_u | u \in M\}$ the order function along $E$:

$$ord(E) := \sum_i a_i \cdot ord(E_i) : \mathcal{L}(M) \to \mathbb{N}_0 \cup \infty,$$

with $ord(E_i)(\gamma_u) := ord_0 f_i \circ \gamma_u(t)$ the zero order of $f_i \circ \gamma_u(t) \in \mathbb{C}[[t]]$, if $f_i$ is a local defining equation of $E_i$ near the point $u \in M$. In particular

$$ord(D_i)(\gamma_u) = 0 \iff u \notin D_i \quad \text{and} \quad ord(D_i)(\gamma_u) = \infty \iff \gamma_u \subset D_i.$$

Then $\{ord(E) = n\} \subset \mathcal{L}(M)$ is for all $n \in \mathbb{N}_0$ a proconstructible or cylinder set in the sense of [10]. Then one would like to introduce the following motivic integral:

$$\int_{\mathcal{L}(M)} L^{-ord(E)} d\mu_M := \sum_{p \in \mathbb{N}_0} \mu_M(\{ord(E) = p\}) \cdot L^{-p}$$

with values in the localized ring $K_0(V/M)[L, L^{-1}]$ as in Corollary [10.7] Recall that we normalized the (naive) motivic measure $\mu_M$ in such a way that we get for $E = 0$:

$$\int_{\mathcal{L}(M)} 1 d\mu_M = [M] \in K_0(V/M)[L, L^{-1}].$$

But the problem with the definition (11.14) is that this is not a finite series, and that $\{ord(E) = \infty\}$ is not a cylinder set in $\mathcal{L}(M)$. Both problems are solved by taking a suitable completion of $K_0(V/M)[L, L^{-1}]$.

Let $\bar{\mathcal{M}}(V/X)$ be the completion of $K_0(V/X)[L^{-1}]$ with respect to the following dimension filtration (for $k \to \infty$):

$$F_k(K_0(V/X)[L^{-1}]) \text{ is generated by } [X' \to X]L^{-n} \text{ with } \text{dim}(X') - n \leq k.$$
Moreover it can easily be computed:

\[(11.16) \quad \int_{\mathcal{L}(M)} L^{-ord(E)} d\hat{\mu}_M := \sum_{p \in \mathbb{N}_0} \hat{\mu}_M(\{ord(E) = p\}) \cdot L^{-p} \in \hat{\mathcal{M}}(\mathcal{V}/M).
\]

Moreover it can easily be computed:

\[(11.17) \quad \int_{\mathcal{L}(M)} L^{-ord(E)} d\hat{\mu}_M = \sum_{I \subset \{1, \ldots, k\}} \left[ E_I^p \rightarrow M \right] \cdot \prod_{i \in I} \frac{L - 1}{L^{a_i+1} - 1}.
\]

Here we use the notation:

\[E_I := \bigcap_{i \in I} E_i \quad (\text{with } E_\emptyset := M), \quad E_I^p := E_I \setminus \bigcup_{i \in \{1, \ldots, k\} \setminus I} E_i,
\]

and the factor \((L^{a_i+1} - 1)^{-1} = L^{-(a_i+1)} \cdot (1 - L^{-(a_i+1)})^{-1}\) has to be developed as the corresponding geometric series in \(\hat{\mathcal{M}}(\mathcal{V})\). Moreover one gets the last equality in (11.17) by multiplying out the following products:

\[(11.18) \quad \prod_{i=1}^k \left( b_i \cdot [E_i \rightarrow M] + [M \setminus E_i \rightarrow M] \right) = \prod_{i=1}^k \left( (b_i - 1) \cdot [E_i \rightarrow M] + [id_M] \right) \in \hat{\mathcal{M}}(\mathcal{V}/M),
\]

with \(b_i := (L - 1)(L^{a_i+1} - 1)^{-1} \in \hat{\mathcal{M}}(\mathcal{V})\). Recall that multiplication in \(\hat{\mathcal{M}}(\mathcal{V}/M)\) is induced from taking the fiber product over \(M\).

The other piece of information that we need is the transformation rule

\[(11.19) \quad \int_{\mathcal{L}(M)} L^{-ord(E)} d\hat{\mu}_M = \pi_* \int_{\mathcal{L}(M')} L^{-ord(\pi'^*E + K\pi')} \ d\hat{\mu}_{M'},
\]

for \(\pi' : M' \rightarrow M\) a proper birational map of pure dimensional complex algebraic manifolds such that \(\pi'^*E + K\pi'\) is a normal crossing divisor with smooth irreducible components.

Assume now that we have a proper birational map \(\pi : M \rightarrow X\), with \(X\) pure dimensional but maybe singular, together with a Cartier divisor \(D\) on \(M\) such that \(D\) and the exceptional locus of \(\pi\) are contained in (the support of) \(E\). Finally we assume

\[K_\pi(D) := K_\pi - D \simeq \sum_i a_i \cdot E_i,
\]

\(X\), and are denoted by \(K_0(\mathcal{V}/X)_S\). In case \(S = \{L^n | n \in \mathbb{N}_0\}\), with \(L := [C] \in K_0(\mathcal{V})\), we also use the notation \(K_0(\mathcal{V}/X)[L^{-1}]\) above.

Also note that the filtration and completion as above are compatible with push down \(f_*\) and exterior product \(\times\) so that in particular \(\hat{\mathcal{M}}(\mathcal{V}/X)\) is a \(\hat{\mathcal{M}}(\mathcal{V}/\{pt\})\)-module, with an induced \(\hat{\mathcal{M}}(\mathcal{V})\)-linear push down \(f_* : \hat{\mathcal{M}}(\mathcal{V}/X) \rightarrow \hat{\mathcal{M}}(\mathcal{V}/Y)\) for \(f : X \rightarrow Y\) an algebraic morphism.

Let us come back to our motivic integral \((11.14)\) on the manifold \(M\). The composed relative motivic measure

\[\hat{\mu}_M : F^{pro}(\mathcal{L}(M)) \rightarrow \hat{\mathcal{M}}(\mathcal{V}/M)
\]

can now be extended from cylinder sets to a more general class of “measureable subsets” of the arc space \(\mathcal{L}(M)\) in such a way that \(\{ord(E) = \infty\}\) becomes measureable with measure 0, and the series \((11.14)\) above converges in \(\hat{\mathcal{M}}(\mathcal{V}/M)\). So now one can define

\[(11.16) \quad \int_{\mathcal{L}(M)} L^{-ord(E)} d\hat{\mu}_M := \sum_{p \in \mathbb{N}_0} \hat{\mu}_M(\{ord(E) = p\}) \cdot L^{-p} \in \hat{\mathcal{M}}(\mathcal{V}/M).
\]

The other piece of information that we need is the transformation rule

\[(11.19) \quad \int_{\mathcal{L}(M)} L^{-ord(E)} d\hat{\mu}_M = \pi_* \int_{\mathcal{L}(M')} L^{-ord(\pi'^*E + K\pi')} \ d\hat{\mu}_{M'},
\]

for \(\pi' : M' \rightarrow M\) a proper birational map of pure dimensional complex algebraic manifolds such that \(\pi'^*E + K\pi'\) is a normal crossing divisor with smooth irreducible components.
with all $a_i \in \mathbb{Z}$ satisfying the inequality $a_i > -1$ (i.e. $a_i \in \mathbb{N}_0$). Here we use of course the relative canonical divisor $K_\pi$ in the “$\omega$-flavor”. Then we define the following motivic arc invariant
\[ E^{arc}((X, D)) \in \widehat{\mathcal{M}}(\mathcal{V}/X) \]
of the pair $(X, D)$:
\begin{equation}
E^{arc}((X, D)) := \pi_\ast \left( \int_{\mathcal{L}(M)} L^{-ord(K_\pi D)} d\tilde{\mu}_M \right),
\end{equation}
which more explicitly can be calculated as in (11.17). This invariant is “independent” of the choice of $\pi$ in the following sense. Let $\pi' : M' \to M$ be a proper birational map of pure dimensional complex algebraic manifolds such that $\pi'^*D$ and the exceptional locus of $\pi \circ \pi' : M' \to X$ is contained in a normal crossing divisor with smooth irreducible components. Then
\[ K_{\pi \circ \pi'}(\pi'^*D) = K_{\pi \circ \pi'} - \pi'^*D = \pi'^*K_\pi(D) + K_{\pi'} \]
is also an effective Cartier divisor with
\[ E^{arc}((X, D)) = E^{arc}((X, \pi'^*D)) \]
by the transformation rule. So this is an invariant of the pair $(X, D)$, if we consider $D$ as a Cartier divisor (in the sense of Aluffi [Alu]) on the directed set of all such resolutions $\pi : M \to X$. In particular $E^{arc}(X) := E^{arc}((X, 0))$ is an invariant of the singular space $X$. In fact in the language of [DL1 sec.6] and [DL2 sec.4.4] it is just the “motivic volume of the arc space $\mathcal{L}(X)$” of the singular space $X$:
\[ E^{arc}(X) = \int_{\mathcal{L}(X)} 1 \ d\tilde{\mu}_X. \]
And this fits with our general description in the introduction of this section, if we set
\[ I(M) := [id_M] \in \widehat{\mathcal{M}}(\mathcal{V}/M), \quad \text{with} \quad J(E_i, a_i) := \int_{\mathcal{L}(M)} L^{-ord(K_\pi)} d\tilde{\mu}_M. \]

For the corresponding “stringy invariant” in the “$\omega$-flavor”, one has first to extend these motivic integrals to $\mathbb{Q}$-Cartier divisors supported on a normal crossing divisor with smooth irreducible components $E_i$, i.e. we start with a strict normal crossing divisor $E = \sum_{i=1}^k a_i E_i$ on the smooth manifold $M$, with $a_i \in \mathbb{Q}$ such that $r \cdot E$ is a Cartier divisor for some $r \in \mathbb{N}$, i.e. $r \cdot a_i \in \mathbb{Z}$ for all $i$. Add a formal variable $L^{1/r}$ to $\widehat{\mathcal{M}}(\mathcal{V})$ and $const^r L^{1/r}$ to $\widehat{\mathcal{M}}(\mathcal{V}/X)$, with $(L^{1/r})^r = L$. Then one can introduce and evaluate the integral
\begin{equation}
\int_{\mathcal{L}(M)} L^{-ord(E)} d\tilde{\mu}_M := \sum_{p \in \mathbb{Z}} \tilde{\mu}_M(\{ord(rE) = p\}) \cdot (L^{1/r})^{-p},
\end{equation}
with value in $\widehat{\mathcal{M}}(\mathcal{V}/M)[L^{1/r}]$, if $a_i > -1$ for all $i$. Moreover the corresponding formula (11.17) with $L^{a_i+1} := (L^{1/r})^{-r(a_i+1)}$ and transformation rule (11.19) are also true in this more general context (compare with [Ve1 Appendix] for more details).

With these improvements, one can introduce for a “Kawamata log terminal pair $(X, D)$” the corresponding motivic stringy invariant (for a suitable $r \in \mathbb{N}$):
\[ E^{str}((X, D)) \in \widehat{\mathcal{M}}(\mathcal{V}/X)[L^{1/r}]. \]
Let $D$ be a $\mathbb{Q}$-Weil divisor on the normal and irreducible complex variety $X$ such that $K_X + D$ is a $\mathbb{Q}$-Cartier divisor (with $r \cdot (K_X + D)$ a Cartier divisor) satisfying the following condition: There is a resolution of singularities $\pi : M \to X$ with the exceptional locus $E$ and the support of $K_\pi(D) := K_M - \pi^*(K_X + D)$ contained in a normal crossing divisor
with smooth irreducible components \( E_i (i \in I) \) such that \( K_\pi(D) \simeq \sum_i a_i \cdot E_i \), with all \( a_i \in \mathbb{Q} \) satisfying the inequality \( a_i > -1 \). Then we set

\[
\mathcal{E}^{\text{str}}((X,D)) := \pi_* \left( \int_{\mathcal{L}(M)} L^{-\text{ord}(K_\pi(D))} \, d\mu_M \right),
\]

which more explicitly can be calculated as in (11.17). Once more this is an invariant of the pair \((X,D)\), not depending on the resolution \(\pi\) by the transformation rule! In the language of [DL1, DL2, DL3] it is for \(D = 0\) just the “motivic Gorenstein volume of the arc space \(\mathcal{L}(X)\)” of the singular space \(X\), i.e. the following “motivic integral” on the singular space \(X\):

\[
\mathcal{E}^{\text{str}}((X)) = \int_{\mathcal{L}(X)} L^{-\text{ord}(K_X)} \, d\mu_X.
\]

Note that by our conventions \(\mathcal{E}^{\text{str}}((X,D)) = \mathcal{E}^{\text{arc}}((X,D))\) in case \(D\) a Cartier divisor (with strict normal crossing) on a smooth manifold \(X = M\).

11.3. Stringy/arc \(E\)-function and Euler characteristic. By application of suitable transformations, one can build from the motivic invariants \(\mathcal{E}^{\text{str}}((X,D))\) and \(\mathcal{E}^{\text{arc}}((X,D))\) other invariants. For example by pushing down by a constant map:

\[
\text{const}_*: \hat{M}(\mathcal{V}/X)[L^{1/r}] \to \hat{M}(\mathcal{V})[L^{1/r}],
\]

one can transform these “relative invariants over \(X\)” to ”absolute invariants” (with \(r = 1\) in the case of “arc invariants”). And then one can apply for example the “\(E\)-function characteristic”

\[
E: \hat{M}(\mathcal{V})[L^{1/r}] \to \mathbb{Z}[u,v][[(uv)^{-1}]][(uv)^{1/r}],
\]

which is defined with the help of Deligne’s mixed Hodge theory. Then

\[
E^{\text{str}}((X,D)) := E(\mathcal{E}^{\text{str}}((X,D)))
\]

becomes Batyrev’s stringy \(E\)-function of the Kawamata log terminal pair \((X,D)\) (as in [Baty1]). Similarly

\[
E^{\text{arc}}(X) := E(\mathcal{E}^{\text{arc}}(X))
\]

is the “Hodge-arc invariant” of \(X\) in the sense of [DL1 §6] and [DL2 §4.4.1] (up to a normalization factor \((uv)^{\text{dim}(X)}\) coming from a different normalization of the motivic measure).

Here \(E: K_0(\mathcal{V}) \rightarrow \mathbb{Z}[u,v]\) is induced from

\[
X \mapsto E(X) := \sum_{i,p,q \geq 0} (-1)^i \cdot \text{dim}_C \left( gr^p_F gr^q_W H^i_c(X^{an},\mathbb{C}) \right) u^p v^q,
\]

with \(F\) the decreasing Hodge filtration and \(W\) the increasing weight filtration of Deligne’s canonical and functorial mixed Hodge structure on \(H^*_c(X^{an},\mathbb{Q})\). Here \(X^{an}\) means the complex algebraic variety \(X\) with its classical (and not the Zariski) topology. This \(E\)-polynomial satisfies the defining “additivity” relation of \(K_0(\mathcal{V})\), because the corresponding long exact cohomology sequence is strictly compatible with the filtrations \(F\) and \(W\) (i.e. the sequence remains exact after application of \(gr^p_F gr^q_W\)).

In particular, \(E(-1,-1)(X) = \chi(X)\) is the topological Euler characteristic of \(X\). Finally classical Hodge theory implies, for \(X\) smooth and compact, the “purity result”

\[
\text{dim}_C \left( gr^p_F H^q(X^{an},\mathbb{C}) \right) = \text{dim}_C \left( H^q(X^{an},\Lambda^p T^* X^{an}) \right)
\]

with \(F\) the decreasing Hodge filtration and \(W\) the increasing weight filtration of Deligne’s canonical and functorial mixed Hodge structure on \(H^*_c(X^{an},\mathbb{Q})\). Here \(X^{an}\) means the complex algebraic variety \(X\) with its classical (and not the Zariski) topology. This \(E\)-polynomial satisfies the defining “additivity” relation of \(K_0(\mathcal{V})\), because the corresponding long exact cohomology sequence is strictly compatible with the filtrations \(F\) and \(W\) (i.e. the sequence remains exact after application of \(gr^p_F gr^q_W\)).
Remark 11.24. One can get the transformation $E : K_0(V) \to \mathbb{Z}[u, v]$ also as an application of Theorem 8.3 (but in a less explicit way), since the invariant

$$d_X := E(X) = \sum_{p, q \geq 0} (-1)^{p+q} \cdot \dim C H^q(X, \Lambda^p T^* X) u^p v^q$$

for $X$ compact and smooth satisfies the corresponding properties (iii-1) and (iii-2).

In particular, $\chi_y(X) = E(-y, 1)(X)$ for $X$ smooth and compact by (g-HRR), so that this E-function is another generalization of the $\chi_y$-genus. But the classes $[X]$ for $X$ smooth and compact generate $K_0(V)$ so that we get the following Hodge theoretic description for any $X$ (with $T_{ys}$ our Hirzebruch class transformation of Theorem 8.3):

$$T_{ys}([X]) = \sum_{i, p \geq 0} (-1)^i \dim C \left( y^p \cdot H^i(X^{an}, \mathbb{C}) \right) (-y)^p = E(-y, 1)(X) .$$

Moreover $\chi_y(X) := E(-y, 1)(X)$ is for $X \neq \emptyset$ of dimension $d$ a polynomial of degree $d$, with $E(L) = E(\mathbb{C}) = uv \in \mathbb{Z}[u, v]$ so that one gets an induced map

$$E : \tilde{\mathcal{M}}(V)[L^{1/\tau}] \to \mathbb{Z}[u, v][[(uv)^{-1}]](uv)^{1/\tau} .$$

By (11.17) we get the following explicit description of $E_{str}((X, D))$, with $\pi : M \to X$ a resolution of singularities such that $K_{\pi}(D) \simeq \sum a_i \cdot E_i$ is a strict normal crossing divisor with $a_i > -1$ for all $i$ as before (and similarly for $E_{arc}((X))$):

$$E_{str}((X, D)) = \sum_{I \subset \{1, \ldots, k\}} E(E_0^I) \cdot \prod_{i \in I} \frac{uv - 1}{(uv)^{a_i+1} - 1}$$

Putting $(u, v) = (-y, 1)$ gives a similar formula for (or defines) the “stringy $\chi_y$-characteristic” $\chi_y^{str}((X, D))$ (or the “arc $\chi_y$-characteristic” $\chi_y^{arc}((X))$), and also the limit $u, v \to 1$ exists with

$$\chi^{str}((X, D)) := \lim_{u, v \to 1} E_{str}((X; D))$$

$$= \sum_{I \subset \{1, \ldots, k\}} \chi(E_0^I) \cdot \prod_{i \in I} \frac{1}{a_i + 1}$$

This $\chi^{str}((X, D))$ is just Batyrev’s stringy Euler number of the log-terminal pair $(X, D)$ (as defined in [Bat]). Similarly $\chi^{arc}(X)$ is just the arc Euler characteristic of $X$ in the sense of [DL1] §6 and [DL2] §4.4.1. Finally note that (11.26) and the “limit formula” (11.13) for the elliptic class $Ell((X, D))$ of the pair $(X, D)$ imply for $X$ compact (with $y = e^{2\pi i z}$):

$$\lim_{y \to \infty} y^{\dim(X)/2} \cdot ell((X, D)) = \chi^{str}_{-y}((X, D)) = E_{str}((X, D))(y, 1) .$$

11.4. Stringy and arc characteristic classes. Recall our motivic characteristic class transformations $m_C \ast$ from Corollary 8.4, $T_{ys}$ from Theorem 8.3 and $\hat{T}_{ys}$, from Remark 11.7. Here $T_{ys}(\cdot) = (1 + y)^{-1} \cdot \hat{T}_{ys}(\cdot)$ for all $i$, so that both classes carry the same information. These classes all satisfy $cl_\ast([\mathbb{C}]) = -y$, so that they induce similar transformations on $K_0(V/X)[L^{-1}]$:

$$m_C : K_0(V/X)[L^{-1}] \to G_0(X) \otimes \mathbb{Z}[y, y^{-1}] ,$$

$$T_{ys}, \hat{T}_{ys} : K_0(V/X)[L^{-1}] \to H^B_{arc}(X) \otimes \mathbb{Q}[y, y^{-1}] .$$
And these extend by [BSY2] Corollary 2.1.1, Corollary 3.1.1] to the completions

\[
\begin{align*}
    mC^\wedge_* & : \hat{M}(V/X)[L_{1/r}] \to G_0(X) \otimes \mathbb{Z}[y][y^{-1}][(-y)^{1/r}] , \\
    T^\wedge_{ys}, T^\wedge_{ys} & : \hat{M}(V/X)[L_{1/r}] \to H^BM_*(X) \otimes \mathbb{Q}[y][y^{-1}][(-y)^{1/r}] .
\end{align*}
\]

(11.29)

So we can introduce for \( c_* = mC_*, T_{ys}, T_{ys} \) the corresponding stringy characteristic homology class \( cl^{str}_*(X, D) \) of the Kawamata log terminal pair \((X, D)\) by

\[
    cl^{str}_*(X, D) = cl^\wedge_*(E^{str}((X, D))) .
\]

(11.30)

Moreover these transformations \( cl^\wedge_* \) commute with proper push down and exterior products so that

\[
    cl^\wedge_*(f_*(\alpha \cdot const^* \beta)) = (f_*(cl^\wedge_*(\alpha))) \cdot const^* \beta
\]

for \( f : X \to Y \) proper, with \( \alpha \in \hat{M}(V/X) \) and \( \beta \in \hat{M}(V) \). By (11.17) we get the following explicit description of \( cl^{str}_*(X, D) \), with \( \pi : M \to X \) a resolution of singularities such that \( K_\pi(D) \simeq \sum a_i \cdot E_i \) is a strict normal crossing divisor with \( a_i > -1 \) for all \( i \) as before:

\[
    cl^{str}_*(X, D) = \sum_{I \subset \{1, \ldots, k\}} cl_*(|E_I^0 \to X|) \cdot \prod_{i \in I} \frac{(-y) - 1}{(-y)^{a_i+1} - 1} .
\]

(11.31)

But \( E_I \) is a closed smooth submanifold of \( M \) so that \( cl_*(|E_I \to X|) \) is just the proper pushforward to \( X \) of the corresponding characteristic (homology) class

\[
    cl_*(E_I) = cl^*(T_{E_I}) \cap [E_I] \quad \text{for} \quad cl_* = mC_*, T_{ys}, T_{ys} .
\]

The stringy Hirzebruch classes \( T^{str}_{ys}((X, D)) \) and \( T^{str}_{ys}((X, D)) \) interpolate by (11.13) and (11.31) in the following sense between the elliptic class \( ELL_*((X, D)) \) of Borisov-Libgober defined in [BSY2]:

\[
    \lim_{\tau \to i \infty} y^{dim(X)/2} \cdot ELL((X, D))(z, \tau) = \hat{T}^{str}_{ys}((X, D)) \quad \text{for} \quad y = e^{2\pi i z},
\]

and for compact \( X \) the stringy \( E \)-function \( E^{str}_{ys}((X, D)) \) of Batyrev as in (11.26):

\[
    \chi^{str}_{ys}(X, D) : = \deg(T^{str}_{ys}((X, D)))
\]

(11.32)

\[
    = \deg(\hat{T}^{str}_{ys}((X, D))) = E^{str}_{ys}((X, D))(y, 1) .
\]

(11.33)

So these stringy Hirzebruch classes are “in between” the elliptic class and the stringy \( E \)-function, and as suitable limits they are “weaker” than these more general invariants. But they have the following good properties of both of them:

- The stringy Hirzebruch classes come from a functorial “additive” characteristic homology class.
- The stringy \( E \)-function comes from the “additive” \( E \)-polynomial defined by Hodge theory, which does not have a homology class version (compare with [BSY2] [5]).
- The elliptic class is a homology class, which does not come from an “additive” characteristic class (of vector bundles), since the corresponding elliptic genus is more general than the Hirzebruch \( \chi^y \)-genus, which is the most general “additive” genus of such a class.

Finally the stringy Hirzebruch class \( T^{str}_{ys}((X, D)) \) specializes for \( y = -1 \) in the following way to the stringy Chern class \( c^{str}_{ys}((X, D)) \) of \((X, D)\) as introduced in [Alu-1] [F-LNU]:

\[
    \lim_{y \to -1} T^{str}_{ys}((X, D)) = c^{str}_{ys}((X, D)) \in H^BM_*(X) \otimes \mathbb{Q} .
\]

(11.34)
In fact
\[
\lim_{y \to 1} T^\text{str}_{y^*} ((X, D)) = \sum_{I \subset \{1, \ldots, k\}} T_{-1*}([E_I^y \to X]) : \prod_{i \in I} \left( \frac{1}{a_i + 1} \right) \\
= \sum_{I \subset \{1, \ldots, k\}} T_{-1*}([E_I \to X]) : (-1)^{|I|} \cdot \prod_{i \in I} \left( \frac{a_i}{a_i + 1} \right).
\]
(11.35)

So by Theorem 8.7 (for \(y = -1\)) we get:
\[
\lim_{y \to -1} T^\text{str}_{y^*} ((X, D)) = c_* \left( \sum_{I \subset \{1, \ldots, k\}} \pi_* (1_{E_I^y}) : \prod_{i \in I} \left( \frac{1}{a_i + 1} \right) \right)
\]
\[= \sum_{I \subset \{1, \ldots, k\}} (-1)^{|I|} \cdot \prod_{i \in I} \left( \frac{a_i}{a_i + 1} \right) \cdot \pi_* (c_*(E_I)).
\]
(11.36)

And the right hand side is just \(c^\text{str}_*(X, D)\) by [Alu1] \(\S\S 3.4, 5.5, 6.5\] and [FLNU] Corollary 2.5, §4. In a similar way one gets for \(cl_* = mC_*, T_y, \tilde{T}_y\) the arc characteristic classes
\[
cl^\text{arc}_* ((X, D)) = cl^\text{arc}_* (\mathcal{E}^\text{arc}((X, D))
\]
with
\[
\lim_{y \to -1} T^\text{arc}_{y^*} ((X, D)) = c^\text{arc}_* ((X, D)) \in H^BM_*(X) \otimes \mathbb{Q}
\]
the Chern class \(\int_X \mathbb{L}(-D) d\chi\) of the pair \((X, -D)\) as introduced and studied in [Alu1] \(\S\S 3.3, 5.5\], with \(\mathbb{L}^{-ord(K_*(D))}\) corresponding to \(\mathbb{L}(-D)\) for \(L \to y \to 1\).

Of course it is also natural to look at the other specializations \(y \to 0\) and \(y \to 1\) of the stringy and arc characteristic classes \(cl^\text{str/arc}_* ((X, D))\) for \(cl_* = mC_*, T_y, \tilde{T}_y\). But the limit \(y \to 1\) doesn’t exist in general so that one can not introduce “stringy or arc L-classes and signature” in this generality. But if we specialize in (11.31) for \(D = 0\) to \(y = 0\), then we get by “additivity”:
\[
\lim_{y \to 0} mC^\text{str}_* (X) = \pi_* ([\mathcal{O}_M]) = \lim_{y \to 0} mC^\text{arc}_* (X)
\]
and
\[
\lim_{y \to 0} T^\text{str}_{y^*} (X) = \pi_* (T^d (TM) \cap [M]) = \lim_{y \to 0} T^\text{arc}_{y^*} (X).
\]
In particular the middle terms are independent of a resolution \(\pi : M \to X\), whose exceptional locus is contained in a strictly normal crossing divisor. And by the “weak factorization theorem” one can even conclude (compare [RSY2] Corollary 3.2):

**Proposition 11.39.** Let \(\pi : M \to X\) be a resolution of singularities of the pure dimensional complex algebraic variety \(X\). Then the classes
\[
\pi_* ([\mathcal{O}_M]) \in G_0(X) \quad \text{and} \quad \pi_* (T^d (TM) \cap [M]) \in H^BM_*(X) \otimes \mathbb{Q}
\]
are independent of \(\pi\).

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