All-Orders Renormalon Resummations for some QCD Observables

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Exact large-$N_f$ results for the QCD Adler $D$-function and Deep Inelastic Scattering sum rules are used to resum to all orders the portion of QCD perturbative coefficients containing the highest power of $b=\frac{1}{6}(11N-2N_f)$, for SU($N$) QCD with $N_f$ quark flavours. These terms correspond to renormalon singularities in the Borel plane and are expected asymptotically to dominate the coefficients to all orders in the $1/N_f$ expansion. Remarkably, we note that this is already apparent in comparisons with the exact next-to-leading order (NLO) and next-to-NLO (NNLO) perturbative coefficients. The ultra-violet (UV) and infra-red (IR) renormalon singularities in the Borel transform are isolated and the Borel sum (principal value regulated for IR) performed. Resummed results are also obtained for the Minkowski quantities related to the $D$-function, the $e^+e^-$ $R$-ratio and the analogous $\tau$-lepton decay ratio, $R_\tau$. The renormalization scheme dependence of these partial resummations is discussed and they are compared with the results from other groups [1–3] and with exact fixed order perturbation theory at NNLO. Prospects for improving the resummation by including more exact details of the Borel transform are considered.
1 Introduction

In several recent papers [1–3] the possibility of resummation to all orders of the part of perturbative corrections contributed by QCD renormalons has been explored.

The QCD perturbative corrections to some generic QCD Green’s function or current correlator (to the Adler $D$-function of QCD vacuum polarization for instance) can be written:

$$D = a + d_1 a^2 + d_2 a^3 + \cdots + d_k a^{k+1} + \cdots ,$$  

(1)

where $a \equiv \alpha_s / \pi$ is the renormalization group (RG) improved coupling; and the perturbative coefficients $d_k$ can themselves be written as polynomials of degree $k$ in the number of quark flavours, $N_f$; we shall assume massless quarks.

$$d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} + \cdots + d_k^{[0]} .$$  

(2)

The “$N_f$-expansion” coefficients, $d_k^{[k-r]}$, will consist of sums of multinomials in the adjoint and fundamental Casimirs, $C_A = N_c$, $C_F = (N_c^2 - 1)/2 N_c$, of SU($N_c$) QCD; and will have the structure $C_A^{k-r-s} C_F^s$. The terms in this “$N_f$-expansion” will correspond to Feynman diagrams with differing numbers of vacuum polarization loops. By explicit evaluation of diagrams with chains of such loops inserted, it has been possible to obtain the leading $d_k^{[k]}$ coefficient exactly to all orders for the Adler $D$-function [4–6] (and hence its Minkowski continuations, the $e^+ e^-$ QCD $R$-ratio and the $\tau$-decay ratio, $R_{\tau}$); the Gross Llewellyn-Smith (GLS) sum rule corrections [3]; and heavy quark decay widths and pole masses [4]. A general procedure enabling $d_k^{[k]}$ to be obtained from knowledge of the one-loop correction with a fictitious gluon mass has been developed [1, 2].

In a recent paper [8] we pointed out that the large order behaviour of perturbative coefficients is most transparently discussed in terms of an expansion of the perturbative coefficients in powers of $b = (11 C_A - 2 N_f)/6$, the first QCD beta-function coefficient:

$$d_k = d_k^{(k)} b^k + d_k^{(k-1)} b^{k-1} + \cdots + d_k^{(0)} .$$  

(3)

This “$b$-expansion” is uniquely obtained by substituting $N_f = (11 C_A - 3 b)$ in equation (2). $d_k^{[k]} = (-1/3)^k d_k^{(k)}$ and so exact knowledge of the leading-$N_f$ $d_k^{[k]}$ to all orders implies exact knowledge of the $d_k^{[k]}$.

In QCD one expects the large-order growth of perturbative coefficients to be driven by Borel plane singularities at $z = z_r = 2 \ell / b$, with $\ell = \pm 1, \pm 2, \pm 3, \ldots$. The singularities on the negative real axis are the so-called ultra-violet renormalons, $UV_r$, and those on the positive real axis are the infra-red renormalons, $IR_r$. These singularities result in the large-order behaviour of the coefficients $d_k \sim b^k k!$. Indeed we showed in reference [8] that, given a set of renormalon singularities at the expected positions in the Borel plane, the leading terms in the $b$-expansion, $d_k^{(k)} b^k$, should, if expanded in powers of $N_f$, asymptotically reproduce the $d_k^{[k-r]}$ coefficients of equation (2) up to $O(1/k)$ accuracy. We conversely checked that the exact $d_k^{(k)} b^k$ results corresponded to a set of renormalon singularities at the expected positions. We shall demonstrate in section 2 of the present paper that, for the Adler $D$-function and the GLS sum rule, the $N_f$-expansion coefficients obtained by expanding $d_1^{(1)} b$ and $d_2^{(2)} b^2$ are in good (10–20% level) agreement with those of the exact $O(a^3)$ next-to-next-to-leading order (NNLO) perturbative calculations for these quantities, so that
the anticipated asymptotic dominance of the leading-$b$ term is already apparent in low orders.

Given the dominance of the leading-$b$ terms, an obvious proposal is to sum them to all orders. That is to split $D$ into two components:

$$D = D^{(L)} + D^{(NL)}$$

(4)

where ‘$L$’ and ‘$NL$’ superscripts refer to leading and non-leading terms in the $b$-expansion.

$$D^{(L)} \equiv \sum_{k=1}^{\infty} d_k^{(L)} b^k a^{k+1}$$

(5)

and

$$D^{(NL)} \equiv \sum_{k=1}^{\infty} a^{k+1} + \sum_{\ell=0}^{k-1} d_k^{(NL)} b^\ell .$$

(6)

The summation of terms can be achieved by using the Borel sum. The Borel integral can itself be split into two components and is well defined for the UV $\ell$ singularities on the negative axis, which contribute poles to the Borel transform of $D^{(L)}$. The integral can be performed explicitly in terms of exponential integral functions and other elementary functions. The piece of the Borel integral for $D^{(L)}$ involving the IR $\ell$ singularities on the positive real axis is formally divergent; but a principal value or other prescription can be used to go around the poles. The specification of this prescription is intimately linked to the procedure needed to combine the non-perturbative vacuum condensates in the operator product expansion (OPE) with the perturbation theory in order to arrive at a well-defined result for $D$.

In recent papers by Neubert [3] and by Ball, Beneke and Braun [1, 2] a summation of the leading-$b$ terms has also been considered. In these papers it has been motivated as a generalisation of the BLM scale fixing prescription [10] and termed “naïve non-abelianization” [11]. The Neubert procedure uses weighted integrals over a running coupling. For the Euclidean Adler $D$-function this representation is equivalent to splitting the Borel integral into ultra-violet renormalon and infra-red renormalon singularities and principal value regulating the latter. When one continues to Minkowski space to obtain the $e^+e^- R$-ratio and the $\tau$-decay ratio, $R_\tau$, there are several inequivalent ways to perform the continuation of the running coupling representation; and hence apparent additional non-perturbative ambiguities are claimed. In reference [2] the resummation is defined by using the principal value regulated Borel integral, as we shall do. They concentrate on $R_\tau$ and heavy quark pole masses. We agree with reference [2] that only a consideration of the singularities in the Borel integral provides a satisfactory way of combining perturbative effects with non-perturbative condensates, along the lines discussed in reference [9]; and that the extra uncertainties claimed in reference [3] are spurious.

Our intention in this paper is to focus on the Adler $D$-function, the $e^+e^- R$-ratio, $R_\tau$ and the GLS sum rule (the latter was not considered in references [1–3]). For all of these quantities there exist exact NNLO fixed order perturbative calculations and our interest is in comparing the leading-$b$ resummation with these exact fixed order results. The large-$b$ results provide partial information about the Borel transform and the question is how this can best be utilised. We discuss the renormalization scheme (RS) dependence of
the split between $D^{(L)}$ and $D^{(NL)}$ in equation (4), the relative contribution of $D^{(L)}$ being RS-dependent. This RS uncertainty needs to be kept in mind and is carefully discussed.

The organisation of the paper is as follows. In section 2 we shall discuss the $b$-expansion for the Adler $D$-function and the GLS sum rule and will consider the extent to which the dominance of the leading-$b$ term is RS-dependent. In section 3 the exact large-$b$ results are used to determine partially the Borel transforms for these quantities; and, having split them into $UV$ and $IR$ renormalon pieces, a resummation along the lines discussed above is performed.

In section 4 the RS dependence of the resummed results is considered, the numerical results are presented and a comparison with the results of references [2, 3] and with the exact NNLO perturbative results is made. A discussion of the uncertainties and the prospects for improving the resummation by including more of the exact structure of the Borel transform is then undertaken. Section 5 contains overall conclusions.

2 The $b$-expansions for $\tilde{D}$ and $\tilde{K}$

We begin by defining the Adler $D$-function and the GLS sum rule, the quantities with which we shall be concerned in this paper.

$D(Q^2)$ is related to the vacuum polarization function, the correlator, $\Pi(Q^2)$, of two vector currents in the Euclidean region,

$$(q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(Q^2) = 16\pi^2 i \int d^4x e^{i q\cdot x} \langle 0 | T \{ J_\mu(x) J_\nu(0) \} | 0 \rangle,$$

by

$$D(Q^2) = -\frac{3}{4} Q^2 \frac{d}{dQ^2} \Pi(Q^2),$$

with $Q^2 = -q^2 > 0$.

In perturbation theory one has

$$D(Q^2) = d(R) \sum_f Q_f^2 \left( 1 + \frac{3}{4} C_F \tilde{D} \right) + \left( \sum_f Q_f \right)^2 \tilde{D}.$$

Here $d(R)$ is the dimension of the quark representation of the colour group, $d(R)=N$ for SU($N$) QCD, and $Q_f$ denotes the quark charges, summed over the accessible flavours at a given energy. $\tilde{D}$ denotes corrections of the “light-by-light” type which first enter at $O(a^3)$ and will be subleading in $N_f$; they are, it is to be hoped, small. $\tilde{D}$ represents the QCD corrections to the zeroth order parton model result and has the form

$$\tilde{D} = a + d_1 a^2 + d_2 a^3 + \cdots + d_k a^{k+1} + \cdots.$$

We shall next consider two QCD deep inelastic scattering sum rules. The first is the polarized Bjorken sum rule (PBjSR):

$$K_{PBj} \equiv \int_0^1 g_1^{en}(x, Q^2) dx = \frac{1}{3} \frac{g_A}{g_V} \left( 1 - \frac{3}{4} C_F \tilde{K} \right).$$
Here $\tilde{K}$ denotes the perturbative corrections to the zeroth order parton model sum rule,

$$\tilde{K} = a + K_1 a^2 + K_2 a^3 + \cdots + K_k a^{k+1} + \cdots. \quad (12)$$

We can also consider the GLS sum rule,

$$K_{GLS} \equiv \frac{1}{6} \int_0^1 \frac{F_3^{p+p}(x,Q^2)}{x} dx = \left(1 - \frac{3}{4} C_F \tilde{K} + \tilde{K} \right). \quad (13)$$

The perturbative corrections, $\tilde{K}$, are the same as for the PBjSR; but there are additional corrections of “light-by-light” type, $\tilde{K}$, analogous to $\tilde{D}$ of equation (9). These will similarly enter at $O(a^3)$ and be subleading in $N_f$ and we shall assume once again that they are small.

For both $\tilde{D}$ and $\tilde{K}$ the first two perturbative coefficients, $d_1$, $d_2$ and $K_1$, $K_2$, are known from the exact perturbative calculations [12–15]. We shall assume $\overline{MS}$ renormalization with renormalization scale $\mu=Q$ for the present; but will later discuss RS dependence more generally.

Writing $d_1$ and $d_2$ expanded in $N_f$ as in equation (2), the exact calculations give:

$$d_1 = \left(\frac{11}{12} + \frac{2}{3} \zeta_3\right) N_f + C_A \left(\frac{41}{8} - \frac{11}{3} \zeta_3\right) - \frac{1}{8} C_F,$$  

$$d_2 = \left(\frac{151}{162} - \frac{19}{27} \zeta_3\right) N_f^2 + C_A \left(-\frac{970}{81} + \frac{224}{27} \zeta_3 + \frac{5}{9} \zeta_5\right) N_f$$

$$+ C_F \left(-\frac{29}{96} + \frac{19}{6} \zeta_3 - \frac{10}{3} \zeta_5\right) N_f + C_A \left(90445/2932 - \frac{2737}{108} \zeta_3 - \frac{55}{18} \zeta_5\right)$$

$$+ C_A C_F \left(-\frac{127}{48} - \frac{143}{12} \zeta_3 + \frac{55}{3} \zeta_5\right) + C_F^2 \left(-\frac{23}{32}\right). \quad (15)$$

Here $\zeta_3$ and $\zeta_5$ are Riemann $\zeta$-functions. For later comparisons it will be useful to write these results numerically for $SU(N)$ QCD.

$$d_1 = -0.115 N_f + \left(0.655 N + \frac{0.063}{N}\right),$$  

$$d_2 = 0.086 N_f^2 + N_f \left(-1.40 N - \frac{0.024}{N}\right) + \left(2.10 N^2 - 0.661 - \frac{0.180}{N^2}\right). \quad (17)$$

Expanding $d_1$ and $d_2$ in $b$ as in equation (3) gives

$$d_1 = \left(\frac{11}{4} - 2 \zeta_3\right) b + \frac{C_A}{12} b - \frac{C_F}{8},$$  

$$d_2 = \left(\frac{151}{18} - \frac{19}{3} \zeta_3\right) b^2 + C_A \left(\frac{31}{6} - \frac{5}{3} \zeta_3 - \frac{5}{3} \zeta_5\right) b$$

$$+ C_F \left(\frac{29}{32} - \frac{19}{2} \zeta_3 + 10 \zeta_5\right) b + C_A^2 \left(-\frac{799}{288} - \zeta_3\right)$$

$$+ C_A C_F \left(-\frac{827}{192} + \frac{11}{2} \zeta_3\right) + C_F^2 \left(-\frac{23}{32}\right). \quad (19)$$

As observed in reference [8], the $b$-expansion exhibits certain simplifications relative to that in $N_f$. In particular the $\zeta_3$, present in all orders of the $N_f$-expansion for $d_1$, is present
only in the leading term in the $b$-expansion. For $d_2$ the $\zeta_5$, present in all but the leading term in the $N_f$-expansion, is now present only in $d_2^{(1)}$. In both cases the highest $\zeta$-function present cancels and is absent in the ‘conformal’ $b \to 0$ limit \[10\]; so $d_1^{(0)}$ does not involve $\zeta_3$ and $d_2^{(0)}$ does not involve $\zeta_5$. Since the $\overline{\text{MS}}$ beta-function coefficients do not involve $\zeta$-functions, this is presumably not an artefact of the particular RS chosen but may well be of more fundamental significance. It may ultimately be connected with the fact that in the $b \to 0$ limit the $z=2\ell/b$ UV and $IR_\ell$ singularities in the Borel plane move off to infinity, leaving only instanton singularities. As discussed in reference \[8\] the $\zeta$-functions are intimately linked with the presence of renormalon singularities. It is amusing to notice that in the original NNLO result for $d_2$ \[11\], which was subsequently found to be in error \[13\], $d_2^{(0)}$ does contain a non-vanishing $-\frac{2}{9}\zeta_5$ term for SU(3) QCD. If a fundamental result about the absence of $\zeta$-functions in the conformal limit could be established it would have enabled the incorrect result to have been dismissed at once.

The corresponding results for the $N_f$-expansion for the deep inelastic sum rules are:

\[
K_1 = -\frac{1}{3} N_f + \left(\frac{23}{12} C_A - \frac{7}{8} C_F\right),
\]
\[
K_2 = N_f^2 \left(\frac{115}{648}\right) + N_f \left(\frac{3535}{1296} - \frac{\zeta_3}{2} + \frac{5}{9} \zeta_5\right) C_A + N_f \left(\frac{133}{864} + \frac{5}{18} \zeta_3\right) C_F
+ C_A^2 \left(\frac{5437}{648} - \frac{55}{18} \zeta_5\right) + C_A C_F \left(\frac{-1241}{432} + \frac{11}{9} \zeta_3\right) + C_F^2 \left(\frac{1}{32}\right).
\]

In numerical form, for SU($N$) QCD,

\[
K_1 = -.333 N_f + \left(1.48 N + \frac{.438}{N}\right),
\]
\[
K_2 = .177 N_f^2 + N_f \left(-2.51 N - \frac{.244}{N}\right) + \left(4.53 N^2 + .686 + \frac{.008}{N^2}\right).
\]

Expanding in powers of $b$ as in equation (3),

\[
K_1 = b + \left(\frac{C_A}{12} - \frac{7}{8} C_F\right),
\]
\[
K_2 = b^2 \left(\frac{115}{72}\right) + b \left(\frac{335}{144} + \frac{3}{2} \zeta_3 - \frac{15}{9} \zeta_5\right) C_A + b \left(-\frac{133}{288} - \frac{5}{6} \zeta_3\right) C_F
+ C_A^2 \left(-\frac{179}{144} - \frac{11}{4} \zeta_3\right) + C_A C_F \left(-\frac{389}{192} + \frac{11}{4} \zeta_3\right) + C_F^2 \left(\frac{1}{32}\right).
\]

Notice that $K_1$ does not contain $\zeta_3$; but a similar remark about the absence of $\zeta_5$ in $K_2^{(0)}$ holds.

We now wish to demonstrate that the leading term in the $b$-expansion, when expanded in $N_f$, approximates the $N_f$-expansion coefficients well, even in rather low orders.

For $d_1$ and $d_2$ we have

\[
d_1^{(1)} b = .345 b = -.115 N_f + .634 N,
\]
\[
d_2^{(2)} b^2 = .776 b^2 = .086 N_f^2 - .948 N_f N + 2.61 N^2.
\]

The subleading, $N$, $N_f N$ and $N^2$, coefficients approximate well in sign and magnitude those in the exact expressions in equations (16) and (17). The leading, $N_f$ and $N_f^2$, coefficients of course agree exactly.
For $K_1$ and $K_2$ we have

\begin{align}
K_1^{(1)} b &= b = -0.333 N_f + 1.83 N, \\
K_2^{(2)} b^2 &= 1.59 b^2 = 0.177 N_f^2 - 1.95 N_f N + 5.37 N^2.
\end{align}

The agreement with the exact $N$, $N_f N$ and $N^2$ coefficients in equations (22) and (23) is again rather good.

We now turn to a consideration of the RS dependence of the $N_f$ and $b$ expansions. In variants of minimal subtraction, where the $1/\epsilon$ pole in dimensional regularization is subtracted along with an $N_f$-independent finite part, $K$, the QCD perturbative coefficients will have the form of polynomials in $N_f$ as in equation (2). Modified minimal subtraction (\textit{MS}), corresponding to $K = (\ln 4\pi - \gamma_E)$, with $\gamma_E = 0.5722 \ldots$, Euler’s constant, is most commonly employed. We can consider \textit{MS} with renormalization scale $\mu = e^u Q$, where $u$ is an $N_f$-independent number. The most general subtraction procedure which will result in perturbative coefficients polynomial in $N_f$, however, can be regarded as \textit{MS} with scale $\mu = e^{u + v/b} Q$, where $v$ is again $N_f$-independent. We shall refer to such renormalization schemes as ‘regular’ schemes. Of course the renormalization scheme is not specified by the scale and subtraction procedure alone but by higher order beta-function coefficients as well. Any variant of minimal subtraction with an $N_f$-independent renormalization scale will have $v = 0$. Momentum space subtraction (MOM) based on the $ggg$ vertex at a symmetric subtraction point $\mu^2 = Q^2$ corresponds to $u = 2.56$ and $v = C_A f(\xi)$, where $f$ is a cubic polynomial in the gauge parameter $\xi$. For the Landau gauge, $\xi = 0$, $v = -2.49 C_A$. For other versions of MOM based on the $qgg$ or ghost vertices, $v$ will involve $C_A$ and $C_F$. Let us denote the perturbative coefficients in the \textit{MS} scheme with $\mu = Q$ ($u = v = 0$) by $d_k$; and those with general $u$ and $v$ by $d'_k$. Then

\begin{equation}
\begin{aligned}
d_1' &= (d_1^{(1)} + u) b + (d_1^{(0)} + v) \\
&= d_1 + bu + v.
\end{aligned}
\end{equation}

Changing $v$, one can make the $d_1^{(0)}$ coefficient as large as one pleases and hence destroy the dominance of the leading-$b$ term noted above for the $D$-function and the sum rules in low orders; although the leading-$b$ term should still reproduce asymptotically the $d_k^{[k-r]}$ coefficients to $O(1/k)$ accuracy.

For $d_2$ and higher coefficients the specification of the RS will involve higher beta-function coefficients as well as the scale and subtraction procedure. The RG-improved coupling, $a(\mu^2)$, will evolve with renormalization scale according to the beta-function equation

\begin{equation}
\frac{da}{d\ln \mu} = -ba^2(1 + ca + c_2 a^2 + \cdots + c_k a^k + \cdots).
\end{equation}

Here $b$ and $c$ are universal with

\begin{equation}
\begin{aligned}
b &= \frac{1}{6} (11 C_A - 2 N_f), \\
c &= \left[-\frac{7 C_A^2}{8 b} - \frac{11}{8} \frac{C_A C_F}{b} + \frac{5}{4} C_A + \frac{3}{4} C_F \right].
\end{aligned}
\end{equation}
Integrating equation (31) with a suitable choice of boundary condition \[20\], one obtains a transcendental equation for \(a\):

\[
\frac{b \ln \mu}{\Lambda} = \frac{1}{a} + c \ln \frac{ca}{1 + ca} + \int_0^a \frac{dx}{x^2 B(x) + \frac{1}{x^2 (1 + cx)}} ,
\]

(33)

where \(B(x) = (1 + cx + cx^2 + \cdots + c_k x^k + \cdots)\). The beta-function coefficients, \(c_2, c_3, \ldots\) together with \(b \ln \frac{\mu}{\Lambda}\) label the RS. In a fixed order perturbative calculation one would truncate the beta-function. For the all-orders resummations of the next section, however, one requires an all-orders definition of the coupling. In the \(\overline{\text{MS}}\) scheme the higher beta-function coefficients, \(c_2^{\overline{\text{MS}}}, c_3^{\overline{\text{MS}}}, \ldots\), presumably exhibit factorial growth, \(c_k^{\overline{\text{MS}}} \sim k!\); and the ‘\(a\)’ coupling in the Borel integral would not be defined, since \(B(x)\) would itself need to be defined by a Borel integral or other summation. One therefore needs to use a finite scheme \[21\] where \(B(x)\) has a finite radius of convergence and can be summed. An extreme example is the so-called ’t Hooft scheme \[21\] where \(c_2 = c_3 = \cdots = c_k = \cdots = 0\), \(B(x) = 1 + cx\). This results in the all-orders definition of the coupling,

\[
\frac{b \ln \mu}{\Lambda} = \frac{1}{a} + c \ln \frac{ca}{1 + ca} .
\]

(34)

In such a finite scheme, where \(c_2, c_3, \ldots\) are \(N_f\)-independent, the \(b\)-expansion of equation (3) will contain an extra \(d_k^{(-1)}/b\) term (for \(k > 1\)) and the \(d_k\) will strictly no longer be polynomials in \(N_f\). \(bd_k\), however, is a polynomial in \(N_f\) of degree \(k + 1\). The leading-\(b\) coefficients, \(d_k^{(k)}\), in regular schemes are independent of \(c_2, c_3, \ldots\), since these beta-function coefficients, \(c_k\), are \(O(1/N_f)\) relative to \(d_k\).

3 Leading-\(b\) Resummations

Let us begin by recalling the definition of the Borel transform. If \(D\) has a series expansion in ‘\(a\)’ as in equation (1), we can write

\[
D = \int_0^\infty dz \ e^{-z/a} B[D](z) ,
\]

(35)

where \(B[D](z)\) is the Borel transform of \(D\), defined by \((d_0 = 1)\)

\[
B[D](z) = \sum_{m=0}^\infty \frac{z^m d_m}{m!} .
\]

(36)

For the Euclidean quantities, \(\tilde{D}\) and \(\tilde{K}\), defined earlier we shall deduce from the exact large-\(N_f\) results that, in the \(\overline{\text{MS}}\) scheme with \(\mu = e^{-5/6}Q\), \(B[D](z)\) is of the form

\[
B[D](z) = \sum_{\ell=1}^\infty \frac{A_0(\ell) + A_1(\ell) z + \overline{A}_1(\ell) z + \overline{A}_2(\ell) z^2 + \cdots}{(1 + \frac{z}{\ell})^{\alpha_\ell + \overline{\alpha}_\ell}}
\]

\[
\cdot \left(1 - \frac{z}{\ell} \right)^{\beta_\ell + \overline{\beta}_\ell} + \cdots ,
\]

(37)

\[
+ \sum_{\ell=1}^\infty \frac{B_0(\ell) + B_1(\ell) z + \overline{B}_1(\ell) z + \overline{B}_2(\ell) z^2 + \cdots}{(1 - \frac{z}{\ell})^{\gamma_\ell + \overline{\gamma}_\ell}} + \cdots ,
\]
where $z_\ell = 2\ell/b$. The two terms correspond to a summation over the ultra-violet renormalons, $UV_\ell$, and infra-red renormalons, $IR_\ell$, respectively. $A_0(\ell)$, $A_1(\ell)$, $\alpha_\ell$ and $B_0(\ell)$, $B_1(\ell)$, $\beta_\ell$ will be obtained from the large-$N_f$ results. The barred terms are sub-leading in $N_f$ and remain unknown. The use of the so-called ‘V-scheme’ $\overline{\text{MS}}$ with $\mu = e^{-5/6}Q$, means that only the constant and $O(z)$ terms in the numerator polynomials are leading in $N_f$. For a general $\overline{\text{MS}}$ scale, $\mu = e^Q$, an overall factor $e^{b_0(u + 5/6)}$ should multiply the unbarred leading-$N_f$ terms in the numerator. The presence of this exponential factor, when it is expanded in powers of $z$, can mask the presence of the $UV$ and $IR$ renormalons in low orders of perturbation theory.

Notice that, whilst the residue at each renormalon singularity is only known to leading order in $N_f$, the $A_0(\ell)$ and $B_0(\ell)$ constant terms in the numerator polynomials are known exactly; indeed $\sum_{\ell = 1}^{\infty} (A_0(\ell) + B_0(\ell)) = 1$, as is required to reproduce the unit coefficient of the $O(a)$ term in $\hat{D}$ in equation (1). We now turn to the explicit determination of the coefficients and exponents for $\tilde{D}$ and $\tilde{K}$.

For $\tilde{D}$ the leading-$N_f$ terms in the QED Gell-Mann–Low function are generated by

$$
\psi[n]_n = \frac{3^{2-n}}{2} \left( \frac{d}{dx} \right)^{n-2} P(x) \bigg|_{x=1},
$$

where

$$
P(x) = \frac{32}{3(1 + x)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - x^2)^2}.
$$

(38)

In the V-scheme, $\overline{\text{MS}}$ with $\mu = e^{-5/6}Q$, one then has

$$
d^{(n)}_n = \left( -\frac{3}{2} \right)^n 2\psi[n+2]_n.
$$

(39)

It is then straightforward to deduce that the coefficients and exponents in equation (37) for $B[\tilde{D}](z)$ are

$$
A_0(\ell) = \frac{8}{3} \frac{(-1)^{\ell+1}(3\ell^2 + 6\ell + 2)}{\ell^2(\ell + 1)^2(\ell + 2)^2}, \quad A_1(\ell) = \frac{8}{3} \frac{b(-1)^{\ell+1}(\ell + \frac{3}{2})}{\ell^2(\ell + 1)^2(\ell + 2)^2},
$$

$$\ell = 1, 2, 3, \ldots.
$$

$$
B_0(1) = 0, \quad B_0(2) = 1, \quad B_0(\ell) = -A_0(-\ell) \quad \ell \geq 3
$$

$$
B_1(1) = 0, \quad B_1(2) = 0, \quad B_1(\ell) = -A_1(-\ell) \quad \ell \geq 3
$$

$$
\alpha_\ell = 2 \quad \ell = 1, 2, 3, \ldots, \quad \beta_2 = 1, \quad \beta_\ell = 2 \quad \ell \geq 3.
$$

(40)

So $IR_1$ is absent, as required from the absence of a dimension two condensate in the OPE. $IR_2$ is a single pole. All the other singularities are double poles. Not only are the coefficients for the $UV_\ell$ and $IR_\ell$ singularities related by the curious symmetry $B_{0,1}(\ell) = -A_{0,1}(-\ell)$; but the form of $A_0(\ell)$ means that there is an additional relation, $A_0(\ell) = -B_0(\ell + 2)$, so that the constant term in the numerator polynomial for $UV_\ell$ exactly cancels that for $IR_{\ell+2}$. This ensures that

$$
\sum_{\ell=1}^{\infty} (A_0(\ell) + B_0(\ell)) = B_0(2) = 1,
$$
Table 1: Leading-$b$ coefficients, $d_n^{(n)}$, for the Adler $D$-function, $\tilde{D}$, compared with the contribution of the first $UV$ renormalon, $'UV_1'$, (equation (41)). The V-scheme, $\overline{\text{MS}}$ with $\mu=e^{-5/6}Q$, is assumed. ‘$UV$’ and ‘$IR$’ denote the separate sums over the $UV_\ell$ and $IR_\ell$ singularities.

| $n$ | $d_n^{(n)}$ | $UV_1$ | $UV$ | $IR$ |
|-----|-------------|--------|------|------|
| 0   | 1           | .8148148 | .7198242 | .28018 |
| 1   | -.4874471   | -.6296296 | -.5921448 | .10470 |
| 2   | .8938293    | .8518519  | .8258924  | .06794 |
| 3   | -1.525257   | -1.611111 | -1.586113 | .09405 |
| 4   | 3.927235    | 3.888889  | 3.858335  | .06086 |
| 5   | -11.24973   | -11.38889 | -11.34378 | .10470 |
| 6   | 39.23893    | 39.16667  | 39.08871  | .06794 |
| 7   | -3410.339   | -3412.500 | -3411.647 | .06794 |
| 8   | 688.5574    | 688.3333  | 687.9894  | .06794 |
| 9   | -154.1541   | -154.5833 | -154.4291 | .06794 |
| 10  | 18638.50    | 18637.50  | 18635.17  | .06794 |

which, as noted earlier, is required to reproduce the unit coefficient of the $O(a)$ term in the perturbative expansion. The precise origin of these relations between $UV$ and $IR$ renormalons remains unclear and deserves further study. They have also been noted and discussed in reference [22].

For the $D$-function the singularity nearest the origin is $UV_1$ and from the $A_0(1)$, $A_1(1)$ in equation (40) this should correspond to

$$d_n^{(n)}|_{UV_1} = \frac{12n + 22}{27} n! \left( \frac{1}{2} \right)^n. \quad (41)$$

In Table 1 we compare the exact leading-$b$, $d_n^{(n)}$, coefficients with the contribution from $UV_1$ of equation (41). $UV$ and $IR$ denote the separate sums over the $UV_\ell$ and $IR_\ell$ singularities. The $\overline{\text{MS}}$ scheme with $\mu=e^{-5/6}Q$ (V-scheme) is assumed. With this choice of scheme $UV_1$ dominates even in low orders and the alternating factorial behaviour is apparent.

We now consider the coefficients and exponents in equation (37) for $B[\tilde{K}](z)$. The generating function for the leading-$b$ coefficient in the V-scheme is

$$K_n^{(n)} = \frac{1}{3} \left( \frac{1}{2} \right)^n \left. \frac{d^n}{dx^n} \frac{(3 + x)}{(1 - x^2)(1 - \frac{x^2}{4})} \right|_{x=0}. \quad (42)$$

This results in

$$B[\tilde{K}](z) = \frac{\frac{4}{9}}{(1 + \frac{5z}{2})} - \frac{\frac{18}{15}}{(1 + \frac{5z}{4})} + \frac{\frac{8}{9}}{(1 - \frac{5z}{2})} - \frac{\frac{5}{15}}{(1 - \frac{5z}{4})}. \quad (43)$$

The terms correspond to $UV_1$, $UV_2$, $IR_1$, $IR_2$ respectively. Each numerator and exponent will contain in addition $O(1/N_f)$ corrections corresponding to the barred terms.
\[
\begin{array}{|c|c|c|c|c|}
\hline
n & K_n^{(n)} & UV_1 + IR_1 & UV & IR \\
\hline
0 & 1 & 1.333333 & .3888889 & .6111111 \\
1 & .1666667 & .2222222 & -.2083333 & .3750000 \\
2 & .6250000 & .6666667 & .2152778 & .4097222 \\
3 & .3125000 & .3333333 & -.3281250 & .6406250 \\
4 & 1.968750 & 2.000000 & .6614583 & 1.307292 \\
5 & 1.640625 & 1.666667 & -1.660156 & 3.300781 \\
6 & 14.94141 & 15.00000 & 4.990234 & 9.951172 \\
7 & 17.43164 & 17.50000 & -17.48291 & 34.91455 \\
8 & 209.7949 & 210.0000 & 69.96582 & 139.8291 \\
9 & 314.6924 & 315.0000 & -314.9231 & 629.6155 \\
10 & 4723.846 & 4725.000 & 1574.808 & 3149.039 \\
\hline
\end{array}
\]

Table 2: As for Table 1 but for the Deep Inelastic Scattering sum rules, $\tilde{K}$, $K_n^{(n)}$. ‘$UV_1 + IR_1$’ denotes the contribution of the singularities nearest the origin, the first two terms of equation (44).

in equation (37). The constant terms in the numerators sum to 1, again ensuring a unit $O(a)$ coefficient in $\tilde{K}$. It would be interesting to try to understand the fact that only the first two $UV$ and $IR$ renormalons are leading in $N_f$ in the context of the OPE for the deep inelastic sum rules, a topic discussed in reference [23].

$K_n^{(n)}$ is then given by (in the V-scheme)

\[
K_n^{(n)} = \frac{8}{9} n! \left(\frac{1}{2}\right)^n + \frac{4}{9} n! \left(-\frac{1}{2}\right)^n - \frac{5}{18} n! \left(\frac{1}{4}\right)^n - \frac{1}{18} n! \left(-\frac{1}{4}\right)^n .
\]

(44)

Table 2 shows that $K_n^{(n)}$ is dominated, even in low orders, by the combined $UV_1 + IR_1$ contributions of the two singularities nearest the origin, the first two terms of equation (44).

We now wish to use the Borel integrals to perform the leading-$b$ resummation defined in equations (5,6). For the Adler $D$-function we have

\[
\bar{D}^{(L)}(a) = \int_0^\infty dz e^{-F(a)z} \sum_{\ell=1}^{\infty} \frac{A_0(\ell) + A_1(\ell)z}{(1 + \frac{z}{2})^2} + \int_0^\infty dz e^{-F(a)z} \left( \frac{B_0(2)}{(1 - \frac{z}{2})} + \sum_{\ell=3}^{\infty} \frac{B_0(\ell) + B_1(\ell)z}{(1 - \frac{z}{2})^2} \right) .
\]

(45)

The coefficients $A_0$, $A_1$, $B_0$, $B_1$ are summarised in equation (40). We assume that the resummation has been performed in a finite RS corresponding to $\overline{MS}$ subtraction with $\mu=e^{a+\nu/b}Q$. ‘$a$’, the coupling, is then defined by the integrated beta-function equation of equation (33). One then finds that the exponent in equation (45) is

\[
F(a) = b \ln \frac{Q}{\Lambda_{\overline{MS}}} - \frac{5}{6} b - c \ln \frac{ca}{1 + ca} + v - \int_0^a dx \left[ -\frac{1}{x^2B(x)} + \frac{1}{x^2(1 + cx)} \right] .
\]

(46)
The RS dependence of $\tilde{D}^{(L)}$ reflects the fact that only a subset of the perturbation series has been resummed, hence violating the exact RS-invariance which would apply to the full series. We shall return to this RS dependence in a moment.

The first, $UV$ renormalon, term in equation (45) is a completely well-defined integral. It may be performed in terms of the exponential integral function (with negative argument),

$$\text{Ei}(x) = -\int_{-x}^{\infty} dt \frac{e^{-t}}{t}.$$  \hfill (47)

The first term yields

$$\tilde{D}^{(L)}(a)|_{UV} = \sum_{\ell=1}^{\infty} z_\ell \left\{ e^{F(a)z_\ell} \text{Ei}(-F(a)z_\ell) \left[ F(a)z_\ell(A_0(\ell) - z_\ell A_1(\ell)) - z_\ell A_1(\ell) \right] + (A_0(\ell) - z_\ell A_1(\ell)) \right\}. \hfill (48)$$

To evaluate the second, $IR$ renormalon, term we shall use a principal value prescription; correspondingly we need to define $\text{Ei}(x)$ with a positive argument as a principal value. We find

$$\tilde{D}^{(L)}(a)|_{IR} = e^{-F(a)z_2} z_2 B_0(2) \text{Ei}(F(a)z_2) + \sum_{\ell=3}^{\infty} z_\ell \left\{ e^{-F(a)z_\ell} \text{Ei}(F(a)z_\ell) \left[ F(a)z_\ell(B_0(\ell) + z_\ell B_1(\ell)) - z_\ell B_1(\ell) \right] - (B_0(\ell) + z_\ell B_1(\ell)) \right\}. \hfill (49)$$

Finally

$$\tilde{D}^{(L)}(a) = \tilde{D}^{(L)}(a)|_{UV} + \tilde{D}^{(L)}(a)|_{IR}. \hfill (50)$$

For the Deep Inelastic Scattering sum rules we will have, analogously, using equation (43),

$$\tilde{K}^{(L)}(a) = \int_0^\infty dz \ e^{-F(a)z} \left[ \frac{4}{9} \left(\frac{1}{1 + \frac{a}{z_1}} - \frac{1}{18} \right) \right] + \int_0^\infty dz \ e^{-F(a)z} \left[ \frac{8}{9} \left(\frac{1}{1 - \frac{a}{z_1}} - \frac{5}{18} \right) \right]. \hfill (51)$$

These integrals may be expressed once again in terms of $\text{Ei}(x)$:

$$\tilde{K}^{(L)}(a)|_{UV} = \left[ -\frac{4}{9} e^{F(a)z_1} z_1 \text{Ei}(-F(a)z_1) + \frac{1}{18} e^{F(a)z_2} z_2 \text{Ei}(-F(a)z_2) \right] \hfill (52)$$

and

$$\tilde{K}^{(L)}(a)|_{IR} = \left[ \frac{8}{9} e^{-F(a)z_1} z_1 \text{Ei}(F(a)z_1) - \frac{5}{18} e^{-F(a)z_2} z_2 \text{Ei}(F(a)z_2) \right]. \hfill (53)$$

Similarly

$$\tilde{K}^{(L)}(a) = \tilde{K}^{(L)}(a)|_{UV} + \tilde{K}^{(L)}(a)|_{IR}. \hfill (54)$$

Before we numerically evaluate these results and comment further on RS dependence, we shall derive the analogous resummations for the Minkowski continuations of the $D$-function, the $e^+e^-$ annihilation $R$-ratio and the analogous quantity in $\tau$-decay, $R_\tau$. The $R$-ratio is related to $D$ by a dispersion relation,

$$R(s) = \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} dQ^2 \frac{D(Q^2)}{Q^2}. \hfill (55)$$
Here \( s \) is the physical timelike Minkowski squared momentum transfer. A perturbative result for \( R \) of the form of equation (9) can be written down involving a quantity \( \tilde{R} \) with perturbative coefficients \( r_k \). The \( r_k \) are directly related to the \( d_k \) via the dispersion relation (55): \( r_1=d_1, r_2=d_2-\pi^2 b^2/12 \). The \( \pi^2 \) term arises due to analytical continuation.

In the Borel plane one finds (to leading order in \( N_f \))

\[
B[\tilde{R}](z) = \frac{\sin(\pi bz/2)}{\pi bz/2} B[\tilde{D}](z). 
\]  

(56)

The leading-\( b \) resummation is then obtained from equation (45) simply by adding an extra \( \frac{\sin(\pi bz/2)}{\pi bz/2} \) factor in the integrand.

\[
\tilde{R}^{(L)}(a) = \int_0^\infty dz \ e^{-F(a)z} \frac{\sin(\pi bz/2)}{\pi bz/2} \sum_{\ell=1}^\infty \frac{A_0(\ell) + A_1(\ell)z}{(1 + \frac{z}{z_\ell})^2} 
+ \int_0^\infty dz \ e^{-F(a)z} \frac{\sin(\pi bz/2)}{\pi bz/2} \left( \frac{B_0(2)}{(1 - \frac{z}{z_2})} + \sum_{\ell=3}^\infty \frac{B_0(\ell) + B_1(\ell)z}{(1 - \frac{z}{z_\ell})^2} \right). 
\]  

(57)

Writing the ‘sin’ as a sum of complex exponentials and using partial fractions, the \( UV \) integrals can be explicitly performed in terms of the generalised exponential integral functions \( Ei(n, w) \), with complex argument \( w \), defined for \( \text{Re} w > 0 \) by

\[
Ei(n, w) = \int_1^\infty dt \frac{e^{-wt}}{t^n}. 
\]  

(58)

One also needs

\[
\int_0^\infty dz \ e^{-F(a)z} \frac{\sin(\pi bz/2)}{z} = \arctan \left( \frac{\pi b}{2F(a)} \right). 
\]  

(59)

One finds

\[
\tilde{R}^{(L)}(a)|_{UV} = \frac{2}{\pi b} \left( \frac{8\zeta_2}{3} - \frac{11}{3} \right) \arctan \left( \frac{\pi b}{2F(a)} \right) 
+ \frac{2}{\pi b} \sum_{\ell=1}^\infty \left\{ A_0(\ell)\phi_+(1, \ell) + (A_0(\ell) - A_1(\ell)z_\ell)\phi_+(2, \ell) \right\}, 
\]  

(60)

where

\[
\phi_+(p, q) = e^{F(a)z_q}(-1)^q \text{Im}[Ei(p, F_+z_q)]. 
\]  

(61)

with \( F_\pm = F(a) \pm \frac{inb}{2} \).

To evaluate the principal value of the \( IR \) contribution in equation (57) one needs to continue \( Ei(n, w) \), defined by equation (58) for \( \text{Re} w > 0 \), to \( \text{Re} w < 0 \). With the standard continuation one then arrives at a function analytic everywhere in the cut complex \( w \)-plane, except at \( w=0 \); and with a branch cut running along the negative real axis. Explicitly [32]

\[
Ei(n, w) = \frac{(-w)^{n-1}}{(n-1)!} \left[ -\ln w - \gamma_E + \sum_{m=1}^{n-1} \frac{1}{m} \right] - \sum_{m=0}^{\infty} \frac{(-w)^m}{(m-n+1)m!}, 
\]  

(62)
with $\gamma_E = 0.572 \ldots$, Euler's constant. The $\ln w$ term in equation (62) means that $\text{Ei}(n, w)$ is not a real function. For instance, for negative real $w$ one has $\text{Ei}(1, -x + i\epsilon) = \text{Ei}(x) \mp i\pi$, where $\text{Ei}(x)$ is the principal value of equation (47) used to define the $IR$ renormalon contribution for the Euclidean quantities.

In order to evaluate the $IR$ renormalon contribution correctly, one in fact needs to continue $\text{Ei}(n, w)$ as a real function, so that for $\text{Re} w < 0$ one makes the replacement $\ln w \rightarrow \ln w + i\pi \text{sign}(\text{Im} w)$. Correspondingly, one should define the $IR$ analogue of equation (61),

$$
\phi_-(p, q) = e^{-F(a)z_q}(-1)^q \text{Im}[\text{Ei}(p, -F_+ z_q)] - \frac{e^{-F(a)z_q}(-1)^q z_q^{p-1}}{(p-1)!} \pi \text{Re}[(F_+)^{p-1}],
$$

where $\text{Ei}(p, -F_+ z_q)$ is defined by equation (62). The principal value of the $IR$ renormalon contribution is then given by

$$
\tilde{R}^{(L)}(a)|_{IR} = \frac{2}{\pi b} \left( \frac{14}{3} - \frac{8\zeta_2}{3} \right) \arctan \left( \frac{\pi b}{2F(a)} \right) + \frac{2B_0(2)}{\pi b} \phi_-(1, 2)
$$

$$
+ \frac{2}{\pi b} \sum_{\ell=3}^{\infty} \left\{ B_0(\ell) \phi_-(1, \ell) + (B_0(\ell) + B_1(\ell)z_\ell) \phi_-(2, \ell) \right\}.
$$

Then

$$
\tilde{R}^{(L)}(a) = \tilde{R}^{(L)}(a)|_{UV} + \tilde{R}^{(L)}(a)|_{IR}.
$$

The $\tau$-decay analogue of the $R$-ratio, $R_\tau$, can be defined in terms of the $R$-ratio by the integral representation [24]

$$
R_\tau = 2 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}(1 - s/M_\tau^2)^2(1 + 2s/M_\tau^2) \tilde{R}(s)
$$

$$
= d(R)(|V_{ud}|^2 + |V_{us}|^2) \left[ 1 + \frac{3}{4} C_F \tilde{R}_\tau \right].
$$

Here $\tilde{R}(s)$ denotes $R(s)$ with the $\sum_f Q_f^2$ replaced by $|V_{ud}|^2 + |V_{us}|^2 \approx 1$, where the $V$’s are KM mixing matrix elements. $\tilde{R}_\tau$ has the form

$$
\tilde{R}_\tau = a + r_1^2 a^2 + r_2^2 a^3 + \cdots + r_k^2 a^{k+1} + \cdots
$$

It is then straightforward to show that

$$
B[\tilde{R}_\tau](z) = \frac{\sin(\pi bz/2)}{\pi bz/2} \left[ \frac{2}{(1 - \frac{bz}{2})} - \frac{2}{(1 - \frac{bz}{6})} + \frac{1}{(1 - \frac{bz}{8})} \right] B[D](z). 
$$

Proceeding in a manner analogous to that for the $R$-ratio, we find

$$
\tilde{R}^{(L)}(a)|_{UV} = \frac{2}{\pi b} \left( \frac{8\zeta_2}{3} - \frac{11}{3} \right) \arctan \left( \frac{\pi b}{2F(a)} \right)
$$

$$
+ \frac{4}{\pi b} \sum_{\ell=1}^{\infty} \left[ (A_0(\ell)(G(\ell) + H(\ell)) - z_\ell A_1(\ell)G(\ell)) \phi_+(1, \ell)
$$

$$
+ H(\ell)(A_0(\ell) - z_\ell A_1(\ell)) \phi_+(2, \ell) \right].
$$
\[ \tilde{R}_\tau(L)(a)_{IR} = \frac{2}{\pi b} \left( \frac{14}{3} - \frac{8\zeta_2}{3} \right) \arctan \left( \frac{\pi b}{2F(a)} \right) \]

\[ + \frac{4}{\pi b} \left( -\frac{14}{3} + \frac{64}{3} \ln 2 - 8\zeta_3 + bz_1 \left( \frac{23}{3} - \frac{32}{3} \ln 2 \right) \right) \phi_-(1, 1) \]

\[ - \frac{12}{\pi b} \phi_-(1, 2) + \frac{4}{\pi b} \left( -703 + 64 \ln 2 + bz_3 \left( \frac{245}{36} - \frac{32}{3} \ln 2 \right) \right) \phi_-(1, 3) \]

\[ + \frac{4}{\pi b} \left( -1627 - \frac{128}{18} + bz_4 \left( \frac{2035}{7776} + \frac{16}{81} \ln 2 \right) \right) \phi_-(1, 4) \]

\[ + \frac{4}{\pi b} \left( -\frac{11}{27} + \frac{b_3}{6} \right) \phi_-(2, 3) + \frac{4}{\pi b} \left( -\frac{247}{648} - \frac{5bz_4}{162} \right) \phi_-(2, 4) \]

\[ - \frac{8}{\pi b} \left( -\frac{11}{27} + \frac{b_3}{18} \right) \phi_-(3, 3) + \frac{8}{\pi b} \left( \frac{13}{432} - \frac{5bz_4}{1728} \right) \phi_-(3, 4) \]

\[ + \frac{4}{\pi b} \sum_{\ell=5}^\infty \left[ (B_0(\ell)(G(-\ell) + H(-\ell)) + z_\ell B_1(\ell)G(-\ell)) \phi_-(1, \ell) \right. \]

\[ + H(-\ell)(B_0(\ell) + z_\ell B_1(\ell)) \phi_-(2, \ell) \right], \quad (70) \]

where

\[ G(\ell) = \frac{6\ell(3\ell^2 + 16\ell + 19)}{(\ell + 1)^2(\ell + 3)^2}, \quad H(\ell) = \frac{6}{(\ell + 1)(\ell + 3)(\ell + 4)}. \]

Then, as before,

\[ \tilde{R}_\tau(L)(a) = \tilde{R}_\tau(L)(a)_{UV} + \tilde{R}_\tau(L)(a)_{IR}. \quad (71) \]

Before we proceed to discuss RS dependence further and to evaluate numerically these resummed expressions, we would like to make some remarks. The first concerns the ease of evaluation of both the Euclidean resummed expressions, equations (48), (49) and (52), (53), and those for the Minkowski quantities, equations (60), (64) and (69), (70). Even though these expressions contain infinite summations over the contributions for the UV and IR singularities, successive terms in the sums are strongly damped, with the result that, in order to obtain the three significant figure accuracy of the resummed results to be tabulated in the next section, it is only necessary to retain terms up to and including \( \ell = 7 \) in each sum. The resummations can then be straightforwardly and rapidly evaluated.

The second remark concerns the connection between these explicit expressions for the principal value of the Borel sum and the inequivalent continuations of the running coupling representation to the Minkowski region in reference [3]. It is straightforward to show that procedure ‘1’ of reference [3] for \( \tilde{R} \) and \( \tilde{R}_\tau \) corresponds exactly to evaluating equations (60), (64) and (69), (70) using \( \phi_{-(p, q)} \) defined by equation (63) with the second term omitted, i.e. using the standard continuation of \( \text{Ei}(n, w) \) defined in equation (62). This does not produce the principal value of the Borel sum. Worse still, the Borel sum contains pieces involving single IR renormalon poles together with a \( \sin \frac{\pi b z_2}{2} \) factor, which are well-defined and finite due to the compensating zero contained in the ‘sin’. These contributions, which do not require regulation, are evaluated incorrectly with the standard continuation. Procedure ‘2’ of reference [3] corresponds to evaluating the Borel sum, incorrectly omitting the second term in equation (63) for some of the IR singularities and, correctly, retaining it for others. In our view the inequivalent continuations of the
running coupling representation of reference \[3\] to the Minkowski region correspond to various ways of wrongly evaluating the Borel sum. We see no reason to believe that these discrepancies have a physical relevance, or that they reflect inadequacies in the definition of the OPE in the Minkowski region, as suggested in reference \[3\]. We agree with reference \[2\] that, with our present state of knowledge, the regulated Borel sum provides a satisfactory framework for combining IR renormalon ambiguities with the vacuum condensate ambiguities in the OPE.

4 RS-dependence of the resummed results

Armed with these resummed expressions, we now return to the question of the RS dependence of $D^{(L)}(a)$ before presenting the numerical results.

For a generic quantity we can write

$$D = D^{(L)}(a) + D^{(NL)}(a).$$

(72)

‘$D$’ on the left of the equation denotes the full Borel sum, i.e. equation (37) with barred and unbarred terms included and the IR singularities principal value regulated; and we assume that this exists. Crucially, the full Borel sum is RS independent and so will not depend on ‘$a$’. The L and NL components do depend on ‘$a$’, however. We can consider ‘$a$’ varying between $a=0$ and $a=+\infty$, labelling possible RS’s. From equation (46), as $a \to 0$ so $F(a) \to +\infty$, resulting from the $-c \ln \frac{ca}{1+ca}$ term, and we have assumed $c > 0$, which is true for $N_f \leq 8$ in SU(3) QCD. One then has $D^{(L)}(0)=0$. Correspondingly, from equation (72), $D^{(NL)}(0)=D$ and so, as $a \to 0$, the NL component contributes the whole resummed $D$. As ‘$a$’ increases the $-c \ln \frac{ca}{1+ca}$ term in equation (46) decreases and as $a \to \infty$ it vanishes, resulting in a finite limit $F(\infty)$:

$$F(\infty) = b \ln \frac{Q}{\Lambda_{\text{MS}}} - \frac{5}{6} b + v - \int_0^\infty dx \left[ -\frac{1}{x^2 B(x)} + \frac{1}{x^2 (1 + cx)} \right].$$

(73)

We assume that $B(x)$ is such that the integral exists. Thus $D^{(L)}(a)$ increases from $D^{(L)}(0)=0$ to a finite maximum value, $D^{(L)}(\infty)$, as ‘$a$’ increases; correspondingly, $D^{(NL)}(a)$, which provides the whole resummed $D$ at $a=0$, decreases as ‘$a$’ increases.

This RS dependence of $D^{(L)}(a)$ is clearly problematic. It is monotonic and hence there is no basis for choosing a particular scheme. There is also a dependence on the particular finite scheme, characterised by the choice of $B(x)$, and on the parameter $v$. The maximum value, $D^{(L)}(\infty)$, does perhaps minimize the relative contribution of the unknown $D^{(NL)}$ component but there is no guarantee that $D^{(NL)}(\infty)$ is positive; and it is entirely possible that $D^{(L)}(\infty)$ overestimates $D$.

We shall choose $v=0$, corresponding to a variant of minimal subtraction, a choice which one can motivate by the observed dominance of the leading-$b$ term in $\overline{\text{MS}}$ noted in section 2. For reasons of simplicity we shall choose the ’t Hooft scheme, corresponding to $B(x)=1+cx$. With these choices one has

$$F(\infty) = b \ln \frac{Q}{\Lambda_{\text{MS}}} - \frac{5}{6} b;$$

(74)
and we shall use this in the resummations. It corresponds simply to taking ‘a’ as the one-loop coupling in the \( \overline{\text{MS}} \) scheme with \( \mu = e^{-5/6} Q \) (the V-scheme):

\[
a_{\text{1-loop}} = \frac{1}{b \ln \frac{Q}{\Lambda_V}},
\]

where \( \Lambda_V = e^{5/6} \Lambda_{\overline{\text{MS}}} \). This is in fact the same choice for \( a \) as in references [1–3], where it is motivated by noting that using the one-loop form for \( a(\mu^2) \) makes the leading-\( b \) summation \( \mu \)-independent. We stress once again that in our view its significance is that it maximizes \( D(\mu) \) for a given choice of finite scheme, \( B(x) \), and parameter \( v \). In Figure 1 we show \( \tilde{D}(\mu) \) plotted versus ‘a’ (’t Hooft scheme with \( v=0 \)). In the figures we have plotted versus \( \tilde{a} = 1 - e^{-\alpha} \), so that the full RS variation can be fitted in a unit interval in \( \tilde{a} \). We have taken \( Q = M_Z = 91 \text{GeV} \) and \( \Lambda_{\overline{\text{MS}}} (N_f = 5) = 111 \text{MeV} \). The solid curve gives the overall \( \tilde{D}(\mu) \), split into \( \tilde{D}(\mu)|_{\text{UV}} \) (dashed) and \( \tilde{D}(\mu)|_{\text{IR}} \) (dashed-dot) contributions. Similar curves for \( \tilde{K}(\mu) \) with \( Q^2 = 2.5 \text{GeV}^2 \) and \( \Lambda_{\overline{\text{MS}}} (N_f = 3) = 201 \text{MeV} \) are given in Figure 2; and for \( \tilde{R}(\mu) \) with \( Q = 91 \text{GeV} \) in Figure 3. The corresponding curve for \( \tilde{R}(\tau) \) is given in Figure 4, \( Q = M_\tau = 1.78 \text{GeV} \) and \( \Lambda_{\overline{\text{MS}}} (N_f = 3) = 201 \text{MeV} \). The qualitative behaviour is as we described earlier. The relative sizes of the \( \text{UV} \) and \( \text{IR} \) contributions reflect the disposition of the \( \text{UV} \) and \( \text{IR} \) singularities described above for the different quantities.
Figure 1: The leading-$b$ resummation, $\tilde{D}^{(L)}(a)$, plotted versus $\tilde{a}=1-e^{-a}$ ('t Hooft scheme $\nu=0$), $Q=M_Z=91\text{GeV}$ and $\Lambda_{\text{MS}}(N_f=5)$ is as in the text. The solid curve is the overall result split into $\tilde{D}^{(L)}(a)|_{UV}$ (dashed) and $\tilde{D}^{(L)}(a)|_{IR}$ (dashed-dot) contributions.
Figure 2: As for Figure 1 but for $\tilde{K}^{(L)}(a)$ with $Q^2=2.5\text{GeV}^2$. 
Figure 3: As for Figure 1 but for $\hat{R}^{(L)}(a)$ with $Q=\hat{M}_Z=91\text{GeV}$.

The $a \to \infty$ limits obtained using $F(\infty)$ as in equation (74) are tabulated in Table 3. Specifically, the ‘Resummed’ column contains $1 + \hat{D}^{(L)}(\infty)$, $1 + \hat{R}^{(L)}(\infty)$, $1 + \hat{R}_\tau^{(L)}(\infty)$ and $1 - \hat{K}^{(L)}(\infty)$ for each observable and we have added some extra energies, $Q_0=20\text{GeV}$ and $Q^2=2.5\text{GeV}^2$. The values of $\Lambda_{\overline{\text{MS}}}$ for $N_f=5,3$ are as noted above. In each case we have taken care to include sufficient terms in the summation over $\text{UV}_\ell$ and $\text{IR}_\ell$ singularities to guarantee accuracy to the quoted number of significant figures.

As noted, using $F(\infty)$ is equivalent to the one-loop definition of the coupling used in references [1–3]; and the same value of $\Lambda_{\overline{\text{MS}}}(N_f=3)$ has been used. Our resummed results for $D(m_\tau^2)$, $R(m_\tau^2)$ agree with the principal value of the Borel sum for these quantities quoted in reference [3]; and the resummed result for $R_\tau$ agrees with the results quoted in references [3, 3]. As discussed at the end of section 3 we can reproduce the results of procedures ‘1’ and ‘2’ for $R$ and $R_\tau$ in reference [3] by incorrectly omitting the second term in equation (63) for some of the $\text{IR}$ renormalon singularities.
Figure 4: As for Figure 1 but for $\tilde{R}^{(L)}(a)$ with $Q=M_\tau=1.78\text{GeV}$.
| Observable | Energy | Resummed | FOPT | Expt |
|------------|--------|----------|------|------|
| $D$        | $m_\tau$ | 1.151    | 1.087| —    |
|            | $Q_0$   | 1.055    | 1.045| —    |
|            | $m_Z$   | 1.042    | 1.035| —    |
| $R$        | $m_\tau$ | 1.105    | 1.080| —    |
|            | $Q_0$   | 1.053    | 1.044| —    |
|            | $m_Z$   | 1.041    | 1.035| 1.040±0.004 |
| $R_\tau$   | $m_\tau$ | 1.228    | 1.115| 1.183±0.010 |
| $K$        | $Q_1$   | .784     | .889 | .768±0.09 |

Table 3: Comparison of resummed results (‘Resummed’) of section 3 using $F(\infty)$, equation (74) (see text). $Q_0^2=(20\text{GeV})^2$, $Q_1^2=2.5\text{GeV}^2$. ‘FOPT’ gives the exact NNLO perturbative results with $\mu=Q\ MS$ scheme and the $\Lambda_{\overline{\text{MS}}}$ values noted in the text. ‘Expt’ gives the experimentally deduced values for some of these quantities [25–27].

There is clearly an ambiguity associated with the IR renormalons or, correspondingly, with vacuum condensates in the OPE. For a leading IR singularity in which there is a single pole one would expect an ambiguity in the principal value $\sim \tilde{B} e^{-z/a}$, where $\tilde{B}$ is the residue of the renormalon. For $\tilde{D}$ the leading IR singularity is $IR_2$. For the Minkowski quantities, $\tilde{R}$ and $\tilde{R}_\tau$, the $\frac{\sin(\pi b z/2)}{\pi b z/2}$ factor apparently removes the single pole at $z=z_2$ but there is presumably still a branch point singularity at $z=z_2$ beyond the leading-$N_f$ approximation [4]. The determination of the residue $\tilde{B}$ would require a resummation of the numerator polynomial and so it is only known to leading-$N_f$ (for $\tilde{D} B_0(2)=1$). Taking $\tilde{B}=1$ and putting $N_f=3$, 5 values for $b$ and ‘$a$’ values corresponding to the energies and $\Lambda_{\overline{\text{MS}}}$ considered in Table 3 yields an ambiguity $\lesssim 10^{-4}$ for $\tilde{D}$, $\tilde{R}$, $\tilde{R}_\tau$, so the significant figures quoted in the resummed result do not change. For $\tilde{K}$, however, the leading singularity is $IR_1$ and one would estimate the IR ambiguity $\sim 10^{-2}$ for $N_f=3$ and $Q^2=2.5\text{GeV}^2$, which is clearly significant.

The column labelled ‘FOPT’ gives the fixed order perturbation theory results obtained at NNLO (up to $O(a^3)$) using the exact perturbative coefficients in the $\overline{\text{MS}}$ scheme with $\mu=Q$ [12–15]. The coupling ‘$a$’ is defined using the NNLO truncated beta-function, $B(x)=1+cx+c_{\overline{\text{MS}}}^2 x^2$, in equation (33), with the values of $\Lambda_{\overline{\text{MS}}}$ as above.

The column labelled ‘Expt’ gives the values determined from experimental data for $(1+\tilde{R})$ [25], $(1+\tilde{R}_\tau)$ [26] and $(1-\tilde{K})$ [27]. The results of adjusting $\Lambda_{\overline{\text{MS}}}$ to fit the FOPT and resummed predictions to these experimental values are summarised in Table 4. Both fixed order perturbation theory and the leading-$b$ resummed results exhibit RS-dependence and it is not at all obvious which procedure gives the closest approximation to the all-orders sum. We will defer a more detailed discussion of this question, including a consideration of how the resummed results compare with use of the effective charge formalism [28, 29], until a future work.

To conclude this section let us consider how the leading-$b$ resummation might be improved. An obvious improvement would be to include the full branch point structure of the renormalon singularities by incorporating the subleading in $N_f$, $\alpha_\ell$ and $\beta_\ell$, pieces...
Table 4: Values of $\Lambda_{\text{MS}}$ adjusted to fit the predictions of NNLO FOPT and the resummed results to the experimental data for $(1+\bar{R})$ [24], $(1+\bar{R}_\tau)$ [26] and $(1-\bar{K})$ [27].

| Observable | $N_f$ | $\Lambda_{\text{MS}}$/MeV fitted to experimental data |
|------------|-------|--------------------------------------------------------|
|            |       | Resummed                                               |
| $R$        | 5     | $98^{+93}_{-55}$                                       |
| $R_\tau$   | 3     | $159^{+9}_{-10}$                                       |
| $K$        | 3     | $375^{+20}_{-33}$                                       |
|            |       | NNLO FOPT                                              |
| $R$        | 5     | $287^{+226}_{-145}$                                    |
| $R_\tau$   | 3     | $386^{+21}_{-26}$                                      |
| $K$        | 3     | $495^{+15}_{-16}$                                      |

of the exponents in equation (37) into the resummations. One expects

$$\overline{\alpha}_\ell = -cz_\ell + \gamma_\ell,$$

$$\overline{\beta}_\ell = cz_\ell + \gamma'_\ell.$$ (76)

The first term can be deduced from RG considerations but the $\gamma_\ell$ and $\gamma'_\ell$ are the one-loop anomalous dimensions of the relevant operators [8, 30]. For $\bar{D}$ it is known that $IR_2$ has a corresponding OPE operator with vanishing one-loop anomalous dimension, $\gamma_2'=0$ [31], so $\overline{\beta}_2'=c_2$ is known. To the best of our knowledge the remaining $\gamma_\ell$ and $\gamma'_\ell$’s are not known. One could nonetheless include the first RG-predictable terms in equation (76) in the resummations and see by how much the results change. The problematic RS-dependence of $D^{(L)}(a)$ would be qualitatively unchanged, however.

Further improvement could be achieved by including some of the subleading in $N_f$, barred, coefficients in the numerator polynomials in equation (37). A complete fixed order perturbative calculation for $D$ up to $O(a^n)$ would enable the series coefficients of $B[D](z)$ up to $O(z^n)$ to be determined exactly; but to obtain the coefficients of the numerator polynomials for each singularity up to $O(z^n)$, even given knowledge of the full branch point exponents discussed above, would still be very difficult. The improvement of perturbation series by developing a representation of the form of equation (37) with truncated numerator polynomials was suggested in reference [31].

One further caveat concerns the structure of the $UV_\ell$ singularities. It has been suggested in reference [30] that diagrams with more than one renormalon chain will modify the form of the ultraviolet renormalon singularities [8]; this would lead to additional $UV$ terms in equation (37) with $O(z)$ or higher, subleading in $N_f$, numerator coefficients but with increasing leading-$N_f$ exponents $\alpha_\ell$. The presence of such terms would destroy the asymptotic dominance of the leading-$b$ coefficient to all-orders in $N_f$ [8], which seemed to be already evident in the comparisons with the exact NLO and NNLO coefficients.
discussed in section 2. The motivation for leading-$b$ resummation would hence disappear if this UV structure is correct. Further clarification of this point is obviously required.

5 Conclusions

In this paper we have investigated the possibility of resummation to all orders of the leading term in the ‘$b$-expansion’ of QCD perturbative coefficients in equation (3). This expansion was introduced and motivated in reference [8] by a consideration of renormalon singularities in the Borel plane; and, if such singularities are present, then the $d^{(k)}_k b^k$ term should, when expanded in $N_f$, reproduce the $N_f$-expansion coefficients of equation (2) to all orders in $N_f$ with asymptotic accuracy $O(1/k)$. We checked explicitly in section 2 that for the QCD Adler $D$-function ($\tilde{D}$) and Deep Inelastic Scattering sum rules ($\tilde{K}$) this asymptotic dominance of the leading-$b$ terms was already evident in comparisons with the exact NLO and NNLO perturbative coefficients for those quantities. The interesting absence of $\zeta$-functions from the $d^{(0)}_n$ coefficient which survives in the $b \to 0$ limit was also noted. The RS dependence of the leading-$b$ coefficient and the need to give an all-orders definition of the coupling in order to perform resummations was discussed.

In section 3 we used exact large-$N_f$ results [4–6] to obtain partial information about the Borel transforms $B[\tilde{D}](z)$ and $B[\tilde{K}](z)$. Ultra-violet and infra-red renormalon singularities are present and we obtained the constant coefficients in the numerator polynomials exactly; and the exponents, single and double poles, to leading-$N_f$ (equations (40), (43)). We showed that in the V-scheme ($\overline{\text{MS}}$ with $\mu=e^{-5/6}Q$) the leading-$b$ coefficients, $d^{(0)}_n$ and $K^{(n)}_n$, are dominated, even in low orders, by the renormalon singularities nearest the origin, respectively $UV_1$ and $UV_1+IR_1$ combined (see Tables 1 & 2).

For each quantity we split the leading-$b$ Borel sum into UV and IR poles. The first contribution could be evaluated exactly in terms of the exponential integral function $\text{Ei}(x)$ (equation (47)) and elementary functions; and the second, IR, contribution could be obtained as a principal value, in terms of a principal value of $\text{Ei}(x)$. We showed how to modify the Borel transform for the Minkowski continuations of $\tilde{D}$, the $e^+e^- R$-ratio ($\tilde{R}$) and the $R$-ratio for $\tau$-decay ($\tilde{R}_\tau$), and a similar resummation was performed for these Minkowski quantities in terms of a generalised $\text{Ei}(n,w)$ function (equations (58) and (62)).

In this way we obtained the $D^{(L)}$ component of the split defined in equation (4) for the above quantities. Unfortunately the result obtained by summing the leading-$b$ terms is RS-dependent, $D^{(L)}(a)$. In section 4 we showed that one can maximize $D^{(L)}(a)$ and hence perhaps minimize $D^{(NL)}(a)$ for any given choice of finite scheme and subtraction procedure. Maximizing $D^{(L)}(a)$ whilst choosing the ’t Hooft scheme and a variant of minimal subtraction ($v=0$) was equivalent to using the one-loop coupling in the V-scheme, the choice also made in references [1–3].

We compared our resummed results with the principal values of the Borel sum for $D(m^2_\tau)$ and $R(m^2_\tau)$ quoted in reference [8] and with that for $R_\tau$ in references [2] and found agreement. They are tabulated in Table 3. The procedures ‘1’ and ‘2’ for continuing the running coupling representation of reference [5], for $R$ and $R_\tau$, to the Minkowski region were shown to correspond to using different continuations of the $\text{Ei}(n,w)$ function from Re $w > 0$ to Re $w < 0$. Only a continuation of $\text{Ei}(n,w)$ as a real function enables
one to evaluate correctly the well-defined pieces of the Borel sum involving single $IR$ renormalon poles with a $\sin\frac{\pi b_5}{2}$ factor. The inequivalent procedures of reference [3] are seen to correspond to various ways of wrongly evaluating the Borel sum and are therefore spurious. We stress that the Borel sum with $IR$ singularities identified and principal value regulated provides a unique result and, in our view, a firm foundation for combining vacuum condensates in the OPE with $IR$ renormalons to achieve a well-defined overall result (see reference [9]). There is, of course, ambiguity due to the $IR$ poles and we estimated this to be $\lesssim 10^{-4}$ for the $\tilde{D}, \tilde{R}, \tilde{R}_\tau$ resummed results in Table 3, so the displayed significant figures should be valid; but much larger, $\sim 10^{-2}$, for $\tilde{K}$ due to the presence of an $IR_1$ singularity.

We also compared with fixed order perturbation theory up to NNLO for these quantities. Since both $D^{(L)}(a)$ and the NNLO fixed order result suffer from RS dependence it is unclear which is the more reliable and further investigation of this question is required.

We finally considered how the leading-$b$ resummation might be improved by including more exact information about the Borel transform.

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