Reconstruction of the number and positions of dipoles and quadrupoles using an algebraic method

Takaaki Nara
The University of Electro-Communications, 1-5-1, Chofugaoka, Chofu-city, Tokyo, 182-8585, Japan
E-mail: nara@mce.uec.ac.jp

Abstract. Localization of dipoles and quadrupoles is important in inverse potential analysis, since they can effectively express spatially extended sources with a small number of parameters. This paper proposes an algebraic method for reconstruction of pole positions as well as the number of dipole-quadrupoles without providing an initial parameter guess or iterative computing forward solutions. It is also shown that a magnetoencephalography inverse problem with a source model of dipole-quadrupoles in 3D space is reduced into the same problem as in 2D space.

1. Introduction
Pole identification is an important inverse source problem in various fields. Especially, the dipole-quadrupole identification has recently attracted attention in magnetoencephalography and electroencephalography inverse problems [17, 12, 18, 5, 6], since spatially extended dipoles can be equivalently and efficiently expressed with a small number of multipole parameters. The essence of this problem is formulated using holomorphic functions as follows: let Ω and Σ be the unit circle and a circle with a radius of $R > 1$, respectively, in $\mathbb{C}$. Let $f(z)$ be a holomorphic function in Σ, except for $N$ positions $z_k \in \Omega$, given by

$$f(z) = \sum_{k=1}^{N} \frac{\mu_k}{z - z_k} + \sum_{k=1}^{N} \frac{\nu_k}{(z - z_k)^2} + g(z), \quad (1)$$

where $\nu_k \neq 0$, and $g(z)$ is a holomorphic function in Σ. Given $f(z)$ on the boundary of the domain $\Gamma = \partial \Omega$, reconstruct $z_k$, $\mu_k$ and $\nu_k$, and $N$.

Physically, the first two terms in the right side of Eq. (1) represent potential generated by dipoles and quadrupoles whose positions, dipole moments, and quadrupole moments are $z_k$, $\mu_k$, and $\nu_k$, respectively. $g(z)$ is added to satisfy boundary conditions specified in each application. For example, in a simplified, 2D model of electroencephalography inverse problems, the electric potential, $\phi$, which is a harmonic function except the source points in $\Omega$ and is measured on $\partial \Omega$, should satisfy the Neumann boundary condition, $\partial \phi / \partial n = 0$ on $\partial \Omega$. In this case, the complex potential $f(z)$ is created such that the real and imaginary parts of $f(z)$ are $\phi$ and its conjugate harmonic function, respectively. The Neumann boundary condition is then expressed as $\text{Im} f(z) = \text{constant}$. From some calculations using the method of images, it is shown that $g(z)$ to
satisfy this boundary condition is given by

\[ g(z) = \sum_{k=1}^{N} \frac{\mu_k z^k}{z - z_k'} + \sum_{k=1}^{N} \frac{\nu_k z^k}{(z - z_k')^2}, \quad (2) \]

where \( z_k' \) is the reflection of \( z_k \) in the unit circle given by \( 1/\bar{z}_k \), and

\[ \mu_k' = -\overline{\mu}_k z_k^2 + \frac{2\overline{\nu}_k}{z_k + 1} z_k - \frac{2\overline{\nu}_k}{z_k + 1} \bar{z}_k, \quad \nu_k' = \nu_k z_k^4, \quad (3) \]

where \( i \) is the imaginary unit. Since \( |z_k'| > 1 \), \( g(z) \) is holomorphic in \( \Sigma \), where \( R = 1/\max_k \{|z_k|\} \). Furthermore, since the constant of \( \text{Im} f(z) \) does not affect the contour integral \( \int_{\partial\Omega} f(z)z^n dz \) used in section 2, it can be set to be zero. Thus in this application, \( f(z) \) on \( \partial\Omega \), which are the data of our inverse problem, is the measured electric potential, \( \phi \) on \( \partial\Omega \).

A problem with \( \nu_k = 0 \) is called dipole identification and has been extensively studied so far in various applications such as bio-electromagnetic inverse problems\[3, 13, 2, 14\], in leakage magnetic flux methods for crack detection\[11, 15\], in tomographic reconstruction\[9, 4\], and in electrical impedance tomography\[7, 2\]. Although a typical solution for dipole identification requires iterative computations\[19, 10\], algebraic methods which reconstruct dipoles, without resorting to an initial parameter guess or iterative computing forward solutions, have been proposed\[3, 4, 7, 8, 13, 20\].

We call in this paper a problem with \( \nu_k \neq 0 \) dipole-quadrupole identification. In connection with the bio-electro-magnetic inverse problem, a nonlinear least-squares method was applied to dipole-quadrupole identification\[6\]. We proposed an algebraic method for this problem\[16\], in which the elementary symmetric polynomials of the dipole-quadrupole positions are obtained by linear equations. However, the number of dipole-quadrupoles was assumed to be known. In this paper, we propose a method to estimate \( N \) and show numerical simulations. To show that a 3D problem can be reduced into Eq. (1) is another aim of this paper.

This paper is organized as follows. In section 2, the algebraic method proposed in \[16\] is summarized. In section 3, a method for estimation of \( N \) and its numerical verification are given. Section 5 is devoted to extention of the method to a 3D problem.

**2. Summary of the algebraic method proposed in [16]**

Applying the residue theorem to Eq. (1) leads to

\[ \sum_{k=1}^{N} \mu_k z_k^n + n \sum_{k=1}^{N} \nu_k z_k^{n-1} = c_n, \quad (n = 0, 1, \cdots). \quad (4) \]

where

\[ c_n = \frac{1}{2\pi i} \int_{\Gamma} f(z)z^n dz. \quad (5) \]

Our problem is to estimate \( z_k, \mu_k, \nu_k \) and \( N \) from \( c_n \) \((n = 0, 1, \cdots)\). In \[16\], assuming that \( N \) is known, we proposed a method to obtain the other parameters algebraically summarized in the following. We will propose how to estimate \( N \) and examine it numerically in the next section.

For the \( N \) positions, \( z_1, z_2, \cdots, z_N \), of dipole-quadrupole sources, let \( K_1 \) through \( K_N \) denote their elementary symmetric polynomials:

\[ K_1 = z_1 + z_2 + \cdots + z_N, \quad (6) \]

\[ K_2 = -(z_1 z_2 + z_1 z_3 + \cdots + z_{N-1} z_N), \quad (7) \]

\[ \vdots \]

\[ K_N = (-1)^{N-1} z_1 z_2 \cdots z_N. \quad (8) \]
Also, we call the \((N + 1)\) dimensional vector
\[
k = (K_N, K_{N-1}, \ldots, K_1, -1)^T \in \mathbb{C}^{N+1}
\] (9)
the elementary symmetric polynomial vector.

Define the Hankel matrix
\[
H_{N+1,n} \equiv \begin{pmatrix}
    c_n & c_{n+1} & \cdots & c_{n+N} \\
    c_{n+1} & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    c_{n+N} & \cdots & c_{n+2N} & \cdots
\end{pmatrix},
\] (10)
where the suffix \(N + 1\) represents \((N + 1) \times (N + 1)\) matrix, and the suffix \(n\) represents that the matrix consists of the Laurent coefficients of order \(n\) through \(n + 2N\). We call the anti-diagonal components of the Hankel matrix \(H = (h_{ij})\) where \(i + j = k + 1\) ‘the \(k\)-th anti-diagonal components (ADCs)’, and denote them by \((H)_k\). For example in Eq. (10), \(c_n, c_{n+1}, \ldots, c_{n+2N}\) are called the first, second, \(\cdots\), \((2N + 1)\)-th ADCs, and denoted by \((H_{N+1,n})_1, (H_{N+1,n})_2, \cdots, (H_{N+1,n})_{2N+1}\), respectively.

Then, it was shown (Theorem 1 in [16]) that the elementary symmetric polynomial vector, \(k\), satisfies the system of second order equations
\[
k^TH_{N+1,m}k = 0,
\] (11)
for non-negative integers, \(m\). We used 2\(N\) equations (11) for \(m = 0, 1, \cdots, 2N - 1\) and showed that they can be ‘triangulated’ for \(K_1, K_2, \cdots, K_N\) by using a property of the Hankel matrices. For simplicity an algorithm for the case when \(N = 2\) is explained here. In this case, we start with a set of four Hankel matrices
\[
\mathcal{H}_4 = \{H_{3,0}, H_{3,1}, H_{3,2}, H_{3,3}\}
\]
\[
= \left\{ \begin{pmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix}, \begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{pmatrix}, \begin{pmatrix} c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \\ c_4 & c_5 & c_6 \end{pmatrix}, \begin{pmatrix} c_3 & c_4 & c_5 \\ c_4 & c_5 & c_6 \\ c_5 & c_6 & c_7 \end{pmatrix} \right\},
\] (12)
and corresponding four equations
\[
k^TH_{3,0}k = 0, \quad k^TH_{3,1}k = 0, \quad k^TH_{3,2}k = 0, \quad k^TH_{3,3}k = 0.
\] (13)

First, define the elimination operator, \(e_l(G_1, G_2)\), mapping from two Hankel matrices, \(G_1\) and \(G_2\), to a single Hankel matrix whose \(l\)-th ADCs are zero as
\[
e_l(G_1, G_2) \equiv \begin{cases} \begin{pmatrix} (G_2)_l G_1 - (G_1)_l G_2 \\ G_1 \end{pmatrix} & \text{when } (G_1)_l \neq 0 \text{ or } (G_2)_l \neq 0, \\ \begin{pmatrix} (G_1)_l \end{pmatrix} & \text{when } (G_1)_l = (G_2)_l = 0. \end{cases}
\] (14)

Also, define the elimination operator, \(E_l(\{G_1, G_2, \cdots, G_n\})\), mapping from a set of \(n\) Hankel matrices, \(\{G_1, G_2, \cdots, G_n\}\), to a set of \(n - 1\) Hankel matrices whose \(l\)-th ADCs are zero as
\[
E_l(\{G_1, G_2, \cdots, G_n\}) \equiv \left\{ e_l(G_1, G_2), e_l(G_2, G_3), \cdots, e_l(G_{n-1}, G_n) \right\}.
\] (15)

Then, consider \(E_3(\{E_2(E_1(\mathcal{H}_4))\})\), which we hereafter denote by \(E_3E_2E_1(\mathcal{H}_4)\) for brevity. By definition of the elimination operators, what is obtained by this successive elimination is a set of single Hankel matrix, denoted by \(G_{1,1}\), whose first through third ADCs are zero such as
\[
Y_1 \equiv E_3E_2E_1(\mathcal{H}_4) = \{G_{1,1}\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (G_{1,1})_4 \\ 0 & (G_{1,1})_4 & (G_{1,1})_5 \end{pmatrix},
\] (16)
where \((G_{1,1})_4\) and \((G_{1,1})_5\) are the fourth and fifth ADCs of \(G_{1,1}\), respectively. Since the elimination procedure is the computation of a linear combination of \(H_{3,0}\) through \(H_{3,3}\), it holds that

\[
k^T G_{1,1} k = 0 \quad (17)
\]

which is written as

\[
-2(G_{1,1})_4 K_1 + (G_{1,1})_5 = 0. \quad (18)
\]

Since it was shown that \((G_{1,1})_4 \neq 0\) (Theorem 4 in [16]), we have a linear equation (18) for \(K_1\).

Consider next \(E_2 E_1(\mathcal{H}_4)\), which gives us a set of two Hankel matrices, denoted by \(G_{2,1}\) and \(G_{2,2}\), whose first and second ADCs are zero:

\[
Y_2 \equiv E_2 E_1(\mathcal{H}_4) = \{G_{2,1}, G_{2,2}\}
\]

\[
= \left\{ \begin{pmatrix} 0 & (G_{2,1})_3 \\ 0 & (G_{2,1})_4 \\ (G_{2,1})_3 & (G_{2,1})_4 \end{pmatrix}, \begin{pmatrix} 0 & (G_{2,2})_3 & (G_{2,2})_4 \\ 0 & (G_{2,2})_3 \\ (G_{2,2})_3 & (G_{2,2})_4 \end{pmatrix} \right\}. \quad (19)
\]

Correspondingly, it holds that (see Corollary 2 in [16])

\[
k^T G_{2,1} k = 0, \quad k^T G_{2,2} k = 0, \quad (20)
\]

which are rewritten as

\[
-2(G_{2,1})_3 K_2 + (G_{2,1})_3 K_1^2 - 2(G_{2,1})_4 K_1 + (G_{2,1})_5 = 0, \quad (21)
\]

\[
-2(G_{2,2})_3 K_2 + (G_{2,2})_3 K_1^2 - 2(G_{2,2})_4 K_1 + (G_{2,2})_5 = 0. \quad (22)
\]

Since again it was shown that \((G_{2,1})_3\) and \((G_{2,2})_3\) do not become zero simultaneously, either Eq. (21) or (22) provides us with a linear equation for \(K_2\), since \(K_1\) has been already obtained. When both \((G_{2,1})_3\) and \((G_{2,2})_3\) are not zero, Eqs. (21) and (22) are essentially the same equation.

In this way, the system of algebraic equations (13) for \(N = 2\) is reduced to a triangular form. After \(K_1\) and \(K_2\) are obtained, \(z_1\) and \(z_2\) are obtained by solving a single, second degree equation. Once \(z_1\) and \(z_2\) are obtained, the dipole moments, \(\mu_k\), and the quadrupole moments, \(\nu_k\), are obtained by a linear least-squares method using Eq. (1).

The above procedure is readily generalized for the case when there are \(N\) dipole-quadrupoles. See [16] for the detail.

3. Estimation of the number of dipole-quadrupoles

Although we assumed that \(N\) is known in [16], we showed that it holds that

\[
\det H_{n,0} = \begin{cases} (-1)^N \prod_{k=1}^N \nu_k^2 \prod_{i>j} (z_i - z_j)^8 & (n = 2N), \\
0 & (n \geq 2N + 1). \end{cases} \quad (23)
\]

when there are \(N\) dipole-quadrupoles (Theorem 3 in [16]). Using this property, we propose to estimate \(N\) in this paper.

Let us define the dimensionless quantity

\[
\gamma_{N'} \equiv \frac{|\det H_{2N',0}|}{\prod_{k=1}^{N'} \nu_k^2 \prod_{i>j} (z_i - z_j)^8},
\]

where \(N'\) is the number of dipole-quadrupoles.
where $N'$ is the assumed source number.

When $N' = N$, $\gamma_{N'}$ is theoretically equal to 1, from Eq. (23), and is expected to be close to 1, if the noise level is not so high. When $N' > N$, $\det H_{2N',0}$ is theoretically zero. Thus, it is expected that $\gamma_{N'}$ is much smaller than 1 when $N' > N$.

We examined this numerically. $N = 2$ dipole-quadrupole sources were set at $z = 0.5 + 0.3\, i$ and $-0.4 - 0.4\, i$. The noise level was 5%. 10 data sets were used to compute $\gamma_{N'}$ for $N' = 1, 2, 3$. Four cases were examined:

- case (i) $|\nu_k/\mu_k| = 1$ : the absolute values of the quadrupole moments $|\nu_k|$ were not negligible compared to the absolute values of dipole moments $|\mu_k|$.
- case (ii) $|\nu_k/\mu_k| = 0.5$
- case (iii) $|\nu_k/\mu_k| = 0.1$
- case (iv) $|\nu_k/\mu_k| = 0.05$: $|\nu_k|$ were relatively small so that the sources were almost dipolar.

The left- and right- top figures in Fig. 1 show the geometric mean of $\gamma_{N'}$ for $N' = 1, 2, 3$ using 10 data sets in cases (i) and (ii), respectively. It was observed that the method worked well: $\gamma_2$ was almost 1 while $\gamma_3$ was much smaller than 1. However in cases (iii) and (iv), the method did not work well as shown in the left- and right- bottom figures in Fig. 1. The threshold above which the method using $\gamma_{N'}$ succeeded was about $|\nu_k/\mu_k| = 0.2$. From these results, the method using $\gamma_{N'}$ would be used if the quadrupole moments were not too small and the sources were not regarded to be purely dipolar.

On the other hand, Fig. 2 shows the result for the same sources as above using the method for estimation of $N$, which we used in [12], with the dipole-only source model. The left- and right- top figures show the results in cases (i) and (ii), respectively, and the left- and right- bottom figures show the results in cases (iii) and (iv), respectively. We assumed $N' = 3$ dipole-only sources, and estimated the dipole moments using the algebraic method. From the results, the number of sources was well judged when the sources were well approximated by the dipole-only source model (when $|\nu_k/\mu_k| = 0.1, 0.05$), while it was not when the quadrupole moments were not neglected (when $|\nu_k/\mu_k| = 1, 0.5$).

From these observations, the method using $\gamma_{N'}$ and the method using the dipole-only source model, would be complementarily used.

4. 3D problem in magnetoencephalography

In this section, we show that a magnetoencephalography inverse problem with a source model consisting of dipoles and quadrupoles in 3D space can be reduced into the same problem as in 2D space.

Let $\Omega$ be three, concentric spheres, centered at the origin in 3D space, representing the brain, skull, and scalp. Let $J_p$ represent neural currents in the brain. Then, it is known [5] that the radial component of the magnetic field is expressed by the multipole expansion

$$
\hat{\mathbf{B}}(\mathbf{r}) \cdot \mathbf{r} = \mu_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l + 1}{2l + 1} M_{lm} \hat{Y}_{lm}^*(\theta, \phi) \frac{r^{l+1}}{r^{l+1}},
$$

where

$$
\hat{Y}_{lm}(\theta, \phi) = \sqrt{\frac{(l-m)!}{(l+m)!}} Y_{lm}(\theta, \phi), \quad Y_{lm}(\theta, \phi) = P_l^m(\cos \theta) e^{im\phi},
$$

are the normalized spherical harmonics, and the spherical harmonics, respectively.
Figure 1. Estimation of $N$ using $\gamma_{N'}$. (left-top) case (i): $\gamma_{N'}$ for $N' = 1, 2, 3$ when $|\nu_k/\mu_k| = 1$ ($\mu_k = \nu_k = 1$). (right-top) case (ii) when $|\nu_k/\mu_k| = 0.5$ ($\mu_k = 1, \nu_k = 0.5$). In cases (i) and (ii), it is observed that $\gamma_2$ is close to 1 and $\gamma_3 \ll 1$, from which $N$ is judged to be 2. (left-bottom) case (iii) when $|\nu_k/\mu_k| = 0.05$ ($\mu_k = 1, \nu_k = 0.05$). (right-bottom) case (iv) when $|\nu_k/\mu_k| = 0.01$ ($\mu_k = 1, \nu_k = 0.01$). In cases (iii) and (iv) where the quadrupole moments are small and the sources are regarded to be almost dipolar, the method using $\gamma_{N'}$ does not work well.

Figure 2. Estimation of $N$ using the dipole-only source model. Assuming $N' = 3$, the estimated dipole moments were plotted. In contrast to the method using $\gamma_{N'}$, it is difficult to judge $N$ when $|\nu_k/\mu_k| = 1$ (left-top) and when $|\nu_k/\mu_k| = 0.5$ (right-top), while $N$ can be judged to be two when $|\nu_k/\mu_k| = 0.1$ (left-bottom) and when $|\nu_k/\mu_k| = 0.05$ (right-bottom) where the sources are regarded to be purely dipolar.
$P_l^m(\cos \theta)$ are the associated Legendre polynomials, and $\mu_0$ is the magnetic permeability. The multipole moment has a relationship with $J_p$

$$M_{lm} = \frac{1}{l+1} \int \nabla'[r'^d \dot{Y}_{lm}(\theta', \phi')] \cdot (r' \times J_p(r')) \, \text{d}^3 r' .$$  \hspace{1cm} (27)

On the other hand, using the orthogonality of the spherical harmonics, the multipole moment is shown to have another relationship with the radial magnetic components on a spherical surface, denoted by $\partial \Sigma$, with a radius of $R$ which encloses $\Omega[1]$:

$$M_{lm} = \frac{2l + 1}{\mu_0 (l + 1)} R^l \int_0^{2\pi} \int_0^{\pi} n' \cdot B(r') \dot{Y}_{lm}(\theta', \phi') R^2 \sin \theta' \, \text{d} \theta' \, \text{d} \phi' .$$  \hspace{1cm} (28)

Equating Eqs. (27) and (28) for $l \geq m \geq 0$ leads to algebraic equations relating the neural current to the radial MEG data, $n \cdot B(r)$, on $\partial \Sigma$.

Let us assume that the neural current is expressed by equivalent current dipoles and quadrupoles as

$$J_p = \sum_{k=1}^N p_k \delta(\mathbf{r} - \mathbf{r}_k) + \sum_{k=1}^N Q_k \nabla \delta(\mathbf{r} - \mathbf{r}_k) ,$$  \hspace{1cm} (29)

where $\mathbf{r}_k$ is the position of the $k$-th dipole-quadrupole, $p_k$ is the dipole moment, and $Q_k$ is the quadrupole moment tensor. We consider the case when the neural current ‘patches’ are parallel to the $xy$-plane so that $Q_k$ is written as

$$Q_k = \begin{pmatrix} a_k & b_k & 0 \\ c_k & d_k & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$  \hspace{1cm} (30)

Substituting Eq. (29) into Eq. (27) provides us with

$$M_{lm} = \frac{1}{l+1} \sum_{k=1}^N \left( \nabla'[r'^d \dot{Y}_{lm}(\theta', \phi')] |_{\mathbf{r} = \mathbf{r}_k} \cdot (\mathbf{r}_k \times p_k) + Q_k : [\nabla'(r'^d \dot{Y}_{lm}(\theta', \phi') \times \mathbf{r}') |_{\mathbf{r} = \mathbf{r}_k} \right) ,$$  \hspace{1cm} (31)

where ‘:’ represents the tensor contraction. Put now $l = m = n$. Since it holds that $r^n Y_{mn}(\theta, \phi) = (2n - 1)!!(x + iy)^n$ and

$$\nabla (x + iy)^n = n \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (x + iy)^{n-1}$$  \hspace{1cm} (32)

and

$$\nabla(\nabla(x + iy)^n) \times \mathbf{r} = n \begin{pmatrix} i(n-1)z(x+iy)^{n-2} \\ -(n-1)z(x+iy)^{n-2} \\ -i(n-1)z(x+iy)^{n-2} \\ n(x+iy)^{n-1} \end{pmatrix} ,$$  \hspace{1cm} (33)

Eqs. (28) and (31) turn to

$$\sum_{k=1}^N \mu_k (x_k + iy_k)^n + n \sum_{k=1}^N \nu_k (x_k + iy_k)^{n-1} = c'_n ,$$  \hspace{1cm} (34)
where

\[ \mu'_k = [r_k \times p_k]_{x+iy}, \]  
\[ \nu'_k = iz_k((a_k - d_k) + i(b_k + c_k)), \]

\[ c'_n = 2n + 3 \frac{\mu_0}{(n+1)} R^{n+1} \int_0^{2\pi} \int_0^{\pi} n' \cdot B(r'(x + iy)) R^2 \sin \theta' d\theta' d\phi'. \]  

Since Eq. (34) has the same form as that of Eq. (1), the dipole-quadrupole position projected on the xy-plane, \( x_k + iy_k \), and \( \mu'_k, \nu'_k \) as well as \( N \) can be reconstructed from \( c'_n \), which are computed from radial MEG. By orthogonal projections, the 3D positions are also reconstructed.

5. Conclusion

This paper has proposed an algorithm to estimate the number of dipole-quadrupoles directly from data. It was also shown that a magnetoencephalography inverse problem with a source model of dipole-quadrupoles in 3D space is reduced into the same problem as in 2D space. Extension of the analysis to the case when the patch source has general orientation is an important aspect of future work.

References

[1] Alvarez R E 1991 IEEE Trans. Med. Imag. 10 375
[2] Baratchart L, Ben Abda A, Ben Hassen F, and Leblond J, 2005 Inverse Problems 21 51
[3] El-Badia A and Ha-Duong T 2000 Inverse Problems 16 651
[4] Golub G H, Milanfar P, and Varah J 1999 SIAM Journal of scientific computing, 21 1222
[5] Jerbi K, Mosher J C, Bailet S and Leahy R M 2002 Physics in Medicine and Biology 47 523
[6] Jerbi K, Baillet S, Mosher J C, Nolte G, Garnero L, and Leahy R M 2004 NeuroImage 22 779
[7] Kang H and Lee H 2004 Inverse Problems, 20 1853
[8] Kravanja P, Sakurai T, and Barel M V 1998 BIT 39 101
[9] Milanfar P, Verghese G C, Karl W C and Willsky A S 1995 IEEE Trans. Signal Processing 43 432
[10] Miller K 1970 SIAM J. Appl. Math 18 346
[11] Minkov D et al., Applied Physics A 74 169
[12] Mosher J C, Leahy R M, Shattuck D W, and Baillet S 1999 Lecture notes in Computer Science 98
[13] Nara T and Ando S 2003 Inverse Problems 19 355
[14] Nara T, Oohama J, Hashimoto M, Takeda T, and Ando S 2007 Physics in Medicine and Biology, 52 3859
[15] Nara T and Ando S 2007 Journal of inverse and ill-posed problems 15 403
[16] Nara T 2008 Inverse Problems 24 025010
[17] Nolte G and Curio G Biophysical Journal 73 1253
[18] Nolte G and Curio G 2000 IEEE Trans. Biomedical Engineering 47 1347
[19] Ohnaka K and Uosaki K 1989 Int. J. Control, 49 119
[20] Ohe T and Ohnaka K 1994 Appl. Math. Modeling 18 446

8