On the Abel-Jacobi map for bisections of rational elliptic surfaces 
and 
Zariski $N$-plets for conic arrangements

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Abstract

We study the Abel-Jacobi map for bisections of a certain rational elliptic surface. As an application, we construct examples of Zariski $N$-plets for conic arrangements.

Introduction

In this article, we study the geometry of bisections and the Abel-Jacobi map for them on rational elliptic surfaces. We apply our result to construct certain conic arrangements and to study the topology of the conic arrangement via Galois covers. As an application, we give examples of Zariski $N$-plets.

Let $\varphi : S \to C$ be an elliptic surface over a smooth projective curve $C$ with a section $O$ such that no $(-1)$ curve is contained in any fiber and $\varphi$ has at least one degenerate fiber (see §1 for terminologies on elliptic surfaces). We denote the set of sections of $\varphi$ by $\text{MW}(S)$. The following facts are well-known:

(I) By regarding $O$ as the zero element and considering fiberwise addition, $\text{MW}(S)$ becomes an abelian group. We denote its addition and the multiplication-by-$m$-map by $+ \text{ and } [m]$, respectively.

(II) Abel’s theorem. Let $D$ be a divisor on $S$ and put $d = DF$, where $F$ is a fiber of $\varphi$. There exists a unique section $s(D)$ (the image of the Abel-Jacobi map) such that $D - s(D) - (d - 1)O$ is algebraically equivalent to a divisor whose irreducible components are all in fibers of $\varphi$.

From these facts, by identifying a section and its image, we immediately observe the following:

(a) Given $s_1, s_2 \in \text{MW}(S)$, then we have new curves $s_1 + s_2$ and $[m]s_i$ ($i = 1, 2$) on $S$.

(b) Given a divisor $D$, then we have a new curve $s(D)$ on $S$.

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In [14], we make use of observation (a) in order to study Zariski pairs for conic-line arrangements. In this article, however, we study the relation between $D$ and $s(D)$ in observation (b) in the case where $S$ is a rational elliptic surface with an $I_2$ fiber and $D$ is a bi-section in order to give conic arrangements with prescribed conditions.

We now explain our main result. Let $Q$ be a reduced plane quartic curve. $Q$ can be reducible, but we will assume that $Q$ has a component of degree greater than or equal to 2. Let $S''$ be the double cover of $\mathbb{P}^2$ branched along $Q$ and $\bar{S}$ the canonical resolution of the singularities of $S''$ (see [4] for the canonical resolution). Let $p_1$ and $p_2$ be general points on a component of $Q$ with degree $\geq 2$. We will denote the inverse images of $p_1$ and $p_2$ in $\bar{S}$ by the same symbol. The pencils of lines through $p_i$ ($i = 1, 2$) give rise to pencils of elliptic curves through $p_i$ on $\bar{S}$. By resolving the base points of each pencil we obtain rational elliptic surfaces $\varphi_{p_1}: S_{p_1} \to \mathbb{P}^1$ and $\varphi_{p_2}: S_{p_2} \to \mathbb{P}^1$. The resolution maps will be denoted by $\mu_i: S_{p_i} \to \bar{S}$. Each $\mu_i$ is a composition of two blowing ups.

The exceptional divisor of the second blowing-up of $\mu_1$ (resp. $\mu_2$) gives rise to a section of $S_{p_1}$ (resp. $S_{p_2}$). We will regard this section as the zero section and denote it by $O_1$ (resp. $O_2$). By construction, $S_{p_1}$ and $S_{p_2}$ are rational elliptic surfaces that have the same configuration of singular fibers.

Let $D_1, \ldots, D_m$ be divisors on $\bar{S}$ such that they do not pass through $p_1, p_2$ and their strict transforms under $\mu_1$ (resp. $\mu_2$) give rise to sections of $S_{p_1}$ (resp. $S_{p_2}$). Let $s_i(D_j)$ denote the section corresponding to $D_j$ on $S_{p_i}$. Let $C_i = s_i(D_1) + \cdots + s_i(D_m)$. Put $\hat{C}_i = \mu_i(C_i)$ and let $\hat{C}_i$ be the strict transform of $C_i$ under $\mu_i^{-1}$ ($i \neq j$). For general $p_1$ and $p_2$, $\hat{C}_2$ becomes a multi-section of $S_{p_1}$. Under this setting, we have:

**Theorem 1** Suppose that $C_2 \neq O_2$ and $p_1 \notin \hat{C}_2$. Then

\[ s(\hat{C}_2) = C_1. \]

We apply Theorem 1 to study dihedral covers of $\mathbb{P}^2$ whose branch locus is a conic arrangement and give some examples of Zariski $N$-plets for conic arrangements.

Let us first recall the definition of a Zariski $N$-plet.

**Definition 2** An $N$-plet of reduced plane curves $(\mathcal{B}_1, \ldots, \mathcal{B}_N)$ in $\mathbb{P}^2$ is said to be a Zariski $N$-plet if it satisfies the following conditions:

(i) For each $i$, there exists a tubular neighborhood $T(\mathcal{B}_i)$ of $\mathcal{B}_i$ such that $(T(\mathcal{B}_i), \mathcal{B}_i)$ is homeomorphic to $(T(\mathcal{B}_j), \mathcal{B}_j)$ for any $1 \leq i < j \leq N$. 
(ii) For any $1 \leq i < j \leq N$, there exists no homeomorphism from $(\mathbb{P}^2, B_i)$ to $(\mathbb{P}^2, B_j)$.

The first condition can be replaced by the combinatorial type of each curve. If we denote the irreducible decomposition of $B_i$ by $B_i = C_{i,1} + \ldots + C_{i,r_i}$, the combinatorial type of $B_i$ is, roughly speaking, determined by $\deg C_{i,j}$, the set of topological types of the singularities of $C_{i,j}$ and how the irreducible components meet each other (For details, see [1]).

The combinatorial type of the conic arrangement which we consider in this article is as follows:

**Definition 3** A reduced plane curve $B$ consisting of $(k + 2)$ irreducible conics is called a Namba-Tsuchihashi conic arrangement of type $k$ ($k$-NT arrangement, for short) if $B$ is of the form $Q + C_1 + \ldots + C_k$ satisfying the following conditions:

(i) $Q$ is a quartic consisting of 2 irreducible conics $C', C''$ intersecting transversely.

(ii) $C', C''$ are tangent to $C_j$ ($j = 1, \ldots, k$).

(iii) $C_i$ and $C_j$ ($1 \leq i < j \leq k$) intersect transversally.

(iv) The singularities of $B$ are only nodes and tacnodes, i.e, each component of $Q$ is tangent to $C_j$ ($j = 1, \ldots, k$) at two distinct points and no three conics meet at one point.

For $B$ satisfying only the first two conditions, we call $B$ a weak Namba-Tsuchihashi arrangement of type $k$ (a weak $k$-NT arrangement, for short).

In [8], Namba and Tsuchihashi give an example of a Zariski pair for Namba-Tsuchihashi arrangements of type $2$. In this article, as an application of Theorem 1 and [14, Theorems 3.1 and 4.1] we generalize Namba-Tsuchihashi’s example and prove the following:

**Theorem 4** Let $y(k,3)$ be the number of Young diagrams with $k$ boxes and at most 3 rows. There exist $y(k,3)$ $k$-NT arrangements $B_1, \ldots, B_{y(k,3)}$ such that no homeomorphism $h : (\mathbb{P}^2, B_i) \to (\mathbb{P}^2, B_j)$ with $h(Q) = Q$ exists for any $i < j$. In particular, they form a Zariski $y(k,3)$-plet for $k \geq 3$.

**Remark 5** Professor Miles Reid told the second author that $y(k,3)$ can be computed by the orbifold Riemann-Roch formula. In fact, since $y(k,3)$ satisfies the recursive formula

$$y(k,3) = 1 + \left[ \frac{k}{2} \right] + y(k-3,3)$$

for $k \geq 4$, where $\left[ \bullet \right]$ denotes the maximum integer not exceeding $\bullet$, one can compute explicit formulas for $y(k,3)$ as follows:

$$y(k,3) = \begin{cases} \frac{1}{12}(k^2 + 6k + 12) & k \equiv 0 \mod 6 \\
\frac{1}{12}(k + 1)(k + 5) & k \equiv \pm1 \mod 6 \\
\frac{1}{12}(k + 2)(k + 4) & k \equiv \pm2 \mod 6 \\
\frac{1}{12}(k + 3)^2 & k \equiv 3 \mod 6 \end{cases}$$

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This article consists of 4 sections. We summarize some necessary facts on the theory of elliptic surfaces and dihedral covers in §1 and prove Theorem 1 in §2. In §3, we construct $y(k,3)$ weak $k$-NT arrangement and prove Theorem 4 in §4.
1 Preliminaries

1.1 Elliptic surfaces

1.1.1 General Facts

We first summarize some facts from the theory of elliptic surfaces. As for details, we refer to [5], [6], [7] and [10].

In this article, by an elliptic surface, we always mean a smooth projective surface $S$ with a fibration $\varphi : S \to C$ over a smooth projective curve, $C$, as follows:

(i) There exists a non-empty finite subset, $\text{Sing}(\varphi)$, of $C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 (resp. a singular curve) for $v \in C \setminus \text{Sing}(\varphi)$ (resp. $v \in \text{Sing}(\varphi)$)

(ii) $\varphi$ has a section $O : C \to S$ (we identify $O$ with its image).

(iii) $\varphi$ is minimal, i.e., there is no exceptional curve of the first kind in any fiber.

For $v \in \text{Sing}(\varphi)$, we put $F_v = \varphi^{-1}(v)$. We denote its irreducible decomposition by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i} \Theta_{v,i},$$

where $m_v$ is the number of irreducible components of $F_v$ and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ the identity component.

The classification of singular fibers is well known ([5]). Note that every smooth irreducible component of reducible singular fibers is a rational curve with self-intersection number $-2$.

We also define a subset of $\text{Sing}(\varphi)$ by $\text{Red}(\varphi) := \{ v \in \text{Sing}(\varphi) \mid F_v \text{ is reducible} \}$. Let $\text{MW}(S)$ be the set of sections of $\varphi : S \to C$. From our assumption, $\text{MW}(S) \neq \emptyset$. By regarding $O$ as the zero element of $\text{MW}(S)$ and considering fiberwise addition (see [5] §9 or [13] §1 for the addition on singular fibers), $\text{MW}(S)$ becomes an abelian group. We denote its addition by $\dot{+}$. Note that the ordinary $+$ is used for the sum of divisors, and the two operations should not be confused.

Let $\text{NS}(S)$ be the Néron-Severi group of $S$ and let $T_\varphi$ be the subgroup of $\text{NS}(S)$ generated by $O, F$ and $\Theta_{v,i}$ ($v \in \text{Red}(\varphi)$, $1 \leq i \leq m_v - 1$). Then we have the following theorems:

**Theorem 1.1** ([10] Theorem 1.2) Under our assumption, $\text{NS}(S)$ is torsion free.

**Theorem 1.2** ([10] Theorem 1.3) Under our assumption, there is a natural map $\tilde{\psi} : \text{NS}(S) \to \text{MW}(S)$ which induces an isomorphism of groups

$$\psi : \text{NS}(S)/T_\varphi \cong \text{MW}(S).$$

In particular, $\text{MW}(S)$ is a finitely generated abelian group.

In the following, by the rank of $\text{MW}(S)$, denoted by $\text{rank MW}(S)$, we mean that of the free part of $\text{MW}(S)$. For a divisor on $S$, we put $s(D) = \psi(D)$. Then we have
Lemma 1.3 (Lemma 5.1) $D$ is uniquely written in the form:

$$D ≈ s(D) + (d - 1)O + nF + \sum_{v \in \text{Red}(\varphi)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i},$$

where $≈$ denotes the algebraic equivalence of divisors, and $d, n$ and $b_{v,i}$ are integers defined as follows:

$$d = DF, \quad n = (d - 1)\chi(O_S) + OD - s(D)O,$$

and

$$\begin{pmatrix}
b_{v,1} \\
\vdots \\
b_{v,m_v-1}
\end{pmatrix} = A_v^{-1} \begin{pmatrix}
D\Theta_{v,1} - s(D)\Theta_{v,1} \\
\vdots \\
D\Theta_{v,m_v-1} - s(D)\Theta_{v,m_v-1}
\end{pmatrix}.$$

Here $A_v$ is the intersection matrix $(\Theta_{v,i} \Theta_{v,j})_{1 \leq i,j \leq m_v - 1}$.

For a proof, see [10].

Also, in [10], a $\mathbb{Q}$-valued bilinear form $\langle \cdot, \cdot \rangle$ on $\text{MW}(S)$ is defined by using the intersection pairing on NS. Here are two basic properties of $\langle \cdot, \cdot \rangle$:

- $\langle s, s \rangle \geq 0$ for $\forall s \in \text{MW}(S)$ and the equality holds if and only if $s$ is an element of finite order in $\text{MW}(S)$.

- An explicit formula for $\langle s_1, s_2 \rangle$ ($s_1, s_2 \in \text{MW}(S)$) is given as follows:

$$\langle s_1, s_2 \rangle = \chi(O_S) + s_1 O + s_2 O - s_1 s_2 - \sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_1, s_2),$$

where $\text{Contr}_v(s_1, s_2)$ is given by

$$\text{Contr}_v(s_1, s_2) = (s_1 \Theta_{v,1}, \ldots, s_1 \Theta_{v,m_v-1})(-A_v)^{-1} \begin{pmatrix}
s_2 \Theta_{v,1} \\
\vdots \\
s_2 \Theta_{v,m_v-1}
\end{pmatrix}.$$

As for explicit values of $\text{Contr}_v(s_1, s_2)$, we refer to [10 (8.16)].

- Let $\text{MW}(S)^0$ be the subgroup of $\text{MW}(S)$ given by

$$\text{MW}(S)^0 := \{ s \in \text{MW}(S) \mid s \text{ meets } \Theta_{v,0} \text{ for } \forall v \in \text{Red}(\varphi) \}.$$

$\text{MW}(S)^0$ is called the narrow part of $\text{MW}(S)$. By the explicit formula as above, $(\text{MW}(S)^0, \langle \cdot, \cdot \rangle)$ is a positive definite even integral lattice.
1.1.2 Double cover construction of an elliptic surface

We refer to [6, Lectures III and IV] for details. Let \( \varphi : S \to C \) be an elliptic surface. We can regard the generic fiber \( S_\eta \) of \( \varphi \) as an elliptic curve over \( \mathbb{C} \), the rational function field of \( C \), under our assumption. The inverse morphism with respect to the group law on \( S_\eta \) induces an involution \( [-1]_\varphi \) on \( S \). Let \( S/\langle [-1]_\varphi \rangle \) be the quotient by \( [-1]_\varphi \). It is known that \( S/\langle [-1]_\varphi \rangle \) is smooth and we can blow down \( S/\langle [-1]_\varphi \rangle \) to its relatively minimal model \( W \) over \( C \) in the following way:

Let us denote

- \( f : S \to S/\langle [-1]_\varphi \rangle \): the quotient morphism,
- \( q' : S/\langle [-1]_\varphi \rangle \to W \): the blow down, and
- \( S \xrightarrow{\mu'} S' \xrightarrow{f'} W \): the Stein factorization of \( q' \circ f \).

Then we have:

1. The branch locus \( \Delta_{f'} \) of \( f' \) consists of a section \( \Delta_0 \) and a trisection \( T \) such that its singularities are at most simple singularities (see [2, Chapter II, §8] for simple singularities and their notation) and \( \Delta_0 \cap T = \emptyset \).

2. \( \Delta_0 + T \) is 2-divisible in \( \text{Pic}(W) \).

3. The morphism \( \mu \) is obtained by contracting all the irreducible components of singular fibers not meeting \( O \).

Conversely, if \( \Delta_0 \) and \( T \) on \( W \) satisfy the above conditions, we obtain an elliptic surface \( \varphi : S \to C \), as the canonical resolution of a double cover \( f' : S' \to W \) with \( \Delta_{f'} = \Delta_0 + T \), and the diagram (see [3] for the canonical resolution):

\[
\begin{array}{ccc}
S' & \xleftarrow{\mu'} & S \\
\downarrow f' & & \downarrow f \\
W & \xleftarrow{q'} & \hat{W}.
\end{array}
\]

Here \( q \) is a composition of blowing-ups so that \( \hat{W} = S/\langle [-1]_\varphi \rangle \). Hence any elliptic surface is obtained in this way. In the following, we call the diagram above the double cover diagram for \( S \).

In the case when \( S \) is a rational elliptic surface, \( W \) is the Hirzebruch surface, \( \Sigma_2 \), of degree \( d = 2 \) and \( \Delta_{f'} \) is of the form \( \Delta_0 + T \), where \( \Delta_0 \) is a section with \( \Delta_0^2 = -2 \) and \( T \sim 3(\Delta_0 + 2f) \), \( f \) being a fiber of the ruling \( \Sigma_2 \to \mathbb{P}^1 \).

**Remark 1.4**

- For each \( v \in \text{Sing}(\varphi) \), the type of \( \varphi^{-1}(v) \) is determined by the type of singularity of \( T \) on \( f_v \) and the relative position between \( f_v \) and \( T \) (see [7, Table 6.2]).

- Let \( \sigma_f \) be the covering transformation of \( f \). We can check that \( \sigma_f \) coincides with \( [-1]_\varphi \) and how \( [-1]_\varphi \) acts on irreducible components of singular fibers (see [14] Remark 3.1 (i)).
1.2 Dihedral Covers.

1.2.1 Branched Galois covers

We first explain our terminology for Galois covers. For normal projective varieties $X$ and $Y$ with finite morphism $\pi : X \to Y$, we say that $X$ is a Galois cover of $Y$ if the induced field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois, where $\mathbb{C}(\bullet)$ denotes the rational function field of $\bullet$. Recall that the Galois group $\text{Gal}(\mathbb{C}(X)/\mathbb{C}(Y))$ acts on $X$ such that $Y$ is obtained as the quotient space with respect to this action (cf. [11 §1]). If the Galois group $\text{Gal}(\mathbb{C}(X)/\mathbb{C}(Y))$ is isomorphic to a finite group $G$, we simply call $X$ a $G$-cover of $Y$. The branch locus of $\pi : X \to Y$, which is denoted by $\Delta_\pi$ or $\Delta(X/Y)$, is a subset of $Y$ consisting of points $y$ of $Y$, over which $\pi$ is not locally isomorphic. It is well-known that $\Delta_\pi$ is an algebraic subset of pure codimension 1 if $Y$ is smooth ([15]).

Now assume that $Y$ is smooth. Let $B$ be a reduced divisor on $Y$, and denote its irreducible decomposition by $B = \sum_{i=1}^r B_i$. We say that a $G$-cover $\pi : X \to Y$ is branched at $\sum_{i=1}^r e_i B_i$ if (i) $\Delta_\pi = B$ (here we identify $B$ with its support) and (ii) the ramification index along $B_i$ is $e_i$ for each $i$, where the ramification index means the one along the smooth part of $B_i$ for each $i$. Note that the study of $G$-covers is related to that of the topology of the complement of $B$, as the proposition below holds:

**Proposition 1.5 ([11 Proposition 3.6])** Under the notation as above, let $\gamma_i$ be a meridian around $B_i$, and $[\gamma_i]$ denote its class in the topological fundamental group $\pi_1(Y \setminus B, p_0)$. If there exists a $G$-cover $\pi : X \to Y$ branched at $e_1 B_1 + \cdots + e_r B_r$, then there exists a normal subgroup $H_\pi$ of $\pi_1(Y \setminus B, p_0)$ such that:

(i) $[\gamma_i]^{e_i} \in H_\pi, [\gamma_i]^k \not\in H_\pi, (1 \leq k \leq e_i - 1)$, and

(ii) $\pi_1(Y \setminus B, p_0)/H_\pi \cong G$.

Conversely, if there exists a normal subgroup $H$ of $\pi_1(Y \setminus B, p_0)$ satisfying the above two conditions for $H_\pi$, then there exists a $G$-cover $\pi_H : X_H \to Y$ branched at $e_1 B_1 + \cdots + e_r B_r$.

For $G$-covers $\pi_1 : X_1 \to Y$ and $\pi_2 : X_2 \to Y$, we identify them if there exists an isomorphism $\Phi : X_1 \to X_2$ such that $\pi_1 = \Phi \circ \pi_2$. Under the same notation as in Proposition 1.5, put

- $\text{Cov}(Y, B, G) :=$ the set of isomorphism classes of $G$-covers $\pi : X \to Y$ such that $\Delta_\pi \subseteq B$, and

- $\text{Cov}_b(Y, e_i B_i + \cdots + e_i B_{i_r}, G) :=$ the set of isomorphism classes of $G$-covers $\pi : X \to Y$ branched at $\sum_{j} e_i B_{i_j}$, Here we only assume that $\text{Supp}(\sum_{j} e_i B_{i_j}) \subseteq B$.

Note that if $\text{Supp}(\sum_{j} e_i B_{i_j}) \subseteq B$, then $\text{Cov}_b(Y, e_i B_{i_1} + \cdots + e_i B_{i_r}, G)$ is a subset of $\text{Cov}(Y, B, G)$. By Proposition 1.5 we have the following:

**Proposition 1.6** We keep the notation in Proposition 1.5. Let $B_i$ ($i = 1, 2$) be reduced divisors on $Y$. Let their irreducible decompositions be denoted by $B_{i_1} + \cdots + B_{i_{r_i}}$ ($i = 1, 2$). If there exists a homeomorphism $h$ from $(Y, B_1)$ to $(Y, B_2)$ such that $h(B_{1,j}) = B_{2,j}$ ($j = 1, \ldots, r_1 (= r_2)$), then there exists a one-to-one correspondence between $\text{Cov}_b(Y, e_i B_{1,i_1} + \cdots + e_i B_{1,i_r}, G)$ and $\text{Cov}_b(Y, e_i B_{2,i_1} + \cdots + e_i B_{2,i_r}, G)$.
1.2.2 $D_{2n}$-covers

We here introduce notation for dihedral covers. Let $D_{2n}$ be the dihedral group of order $2n$. In order to present $D_{2n}$, we use the notation

$$D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma \tau)^2 = 1 \rangle.$$ 

Given a $D_{2n}$-cover, we obtain a double cover, $D(X/Y)$, of $Y$ canonically by considering the $\mathbb{C}(X)$-$n$-normalization of $Y$, where $\mathbb{C}(X)$ denotes the fixed field of the subgroup generated by $\tau$. $X$ is an $n$-fold cyclic cover of $D(X/Y)$ and we denote these covering morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$, respectively.

1.2.3 Elliptic surfaces and $D_{2p}$-covers of $\Sigma_d$

Let $\varphi : S \to \mathbb{P}^1$ be an elliptic surface. As for the double cover diagram for $S$, we have the following:

- $W$ is the Hirzebruch surface, $\Sigma_d$, of degree $d = 2\chi(O_S)$. Hence $d$ is always even. We denote $\tilde{W}$ by $\tilde{\Sigma}_d$.
- $\Delta_f$ is of the form $\Delta_0 + T$, where $\Delta_0$ is a section with $\Delta_0^2 = -d$ and $T \sim 3(\Delta_0 + d)\, f$, $f$ being a fiber of the ruling $\Sigma_d \to \mathbb{P}^1$.

In previous articles, [12, 14], we studied $p$-cyclic covers ($p$: odd prime) $g : X \to S$ such that $f \circ g : X \to \tilde{\Sigma}_d$ gives rise to a $D_{2p}$-cover of $\tilde{\Sigma}_d$. One of the main results is as follows:

**Theorem 1.7** [14, Theorem 3.1] Let $C$ be a reduced divisor on $S$ such that

- all irreducible components $C_i$ of $C = \sum_i C_i$ are horizontal with respect to the elliptic fibration (i.e., any irreducible component is not contained in any fiber), and
- $C$ and $\sigma_f^*C$ have no common component.

Then there exists a $p$-cyclic cover $g : X \to S$ such that

(i) $\Delta_g = C + \sigma_f^*C + \Xi + \sigma_f^*\Xi$, where $\Xi$ is effective divisor on $S$ such that irreducible components of $\Xi$ are all vertical and there is no common component between $\Xi$ and $\sigma_f^*\Xi$.

(ii) $f \circ g : X \to \tilde{\Sigma}_d$ is a $D_{2p}$-cover of $\tilde{\Sigma}_d$ such that $D(X/\tilde{\Sigma}_d) = S$, $\beta_1(\pi) = f$ and $\beta_2(\pi) = g$, if and only if the following condition holds:

Let $s(C_i) = \psi(C_i)$ ($i = 1, \ldots, r$). There exist integers $a_i$ ($i = 1, \ldots, r$) such that

- $1 \leq a_i < p$ ($i = 1, \ldots, r$) and
- $\sum_{i=1}^r |a_i| s(C_i)$ is $p$-divisible in $\text{MW}(S)$, i.e.,

$$\sum_{i=1}^r |a_i| s(C_i) \in [p] \text{MW}(S) := \{[p]s \mid s \in \text{MW}(S)\}.$$
Corollary 1.8 Suppose that $p \not| \text{MW}(S)_{\text{tor}}$ and $C$ consists of two components $C_1$ and $C_2$. Then there exists a $p$-cyclic cover as in Theorem 1.7 if and only if the images of $s(C_1)$ and $s(C_2)$ are linearly dependent in $\text{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$.

Corollary 1.9 Suppose that $p \not| \text{MW}(S)_{\text{tor}}$ and $C$ consists of two components $C_1$ and $C_2$. Then there exists a $D_2$-cover of $\Sigma_d$ branched at $2\Delta_f + f(C)$ if and only if the images of $s(C_1)$ and $s(C_2)$ are linearly dependent in $\text{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$.

2 Proof of Theorem 1

We keep our notation from the Introduction. Note that each surface $S_{p_i}$ has a singular fiber of type $I_0$ whose components arise from the exceptional divisor of the first blow up in $\mu_i$ which meets $O_i$ and the strict transform of the tangent line $l_{p_i}$ of $Q$ at $p_i$. We will denote these components by $\Theta_{p_i,0}$ and $\Theta_{p_i,1}$. All the other reducible singular fibers arise from the exceptional sets of the resolution $\bar{S} \to S''$, hence they are in 1 to 1 correspondence with the singularities of $Q$. We will denote their components by $\Theta_{v,i}$ where $v \in \text{Sing}(Q)$. We will use the same symbol $\Theta_{v,i}$ for these components on both $S_{p_i}$.

Let $D' = s_1(D_1) + \cdots + s_2(D_n)$. Note that the sum taken here is regarded as a sum of divisors on $S_{p_1}$. Then since the Abel-Jacobi map $\tilde{\psi}$ is a homomorphism, $\tilde{\psi}(D') = s_1(D_1) + \cdots + s_1(D_n) = C_1$. Hence by Lemma 1.3 we have the equivalence

$$D' \sim_{S_{p_1}} (C_1) + d(O_1) + nF_1 + a\Theta_{p_1,1} + \sum_{v \in \text{Sing}(Q)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}$$

Where $\sim$ denotes linear equivalence of divisors on $S_{p_1}$.

Similarly for $D'' = s_2(D_1) + \cdots + s_2(D_n)$, we have the equivalence

$$D'' \sim_{S_{p_2}} (C_2) + d(O_2) + nF_2 + a\Theta_{p_2,1} + \sum_{v \in \text{Sing}(Q)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}.$$

Note that by construction of $D', D''$, the coefficients $d, n, a, b_{v,i}$ are the same in both cases. Since $\mu_{s_1}(D') = D_1 + \cdots + D_m = \mu_{s_2}(D'')$ and since linear equivalence behaves well under push-forward, we have

$$\overline{C}_1 + nF_1 + a\Theta_{p_1,1} + \sum_{v \in \text{Sing}(Q)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i} \sim_{\overline{S}} \overline{C}_2 + nF_2 + a\Theta_{p_2,1} + \sum_{v \in \text{Sing}(Q)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}$$

where we denote the images of $F_1$ and $\Theta_{p_1,1}$ by the same symbols. Then since $F_1 \sim F_2$ and $\Theta_{p_1,1} \sim \Theta_{p_2,1}$, because they are inverse images of lines of $\mathbb{P}^2$, we obtain the equivalence

$$\overline{C}_1 \sim_{\overline{S}} \overline{C}_2.$$
By pulling this equivalence back by $\mu_1$, we obtain
\[
\hat{C}_2 \sim_{S_{p_1}} C_1 + \alpha O_1 + \beta \Theta_{p_{1,0}}
\]
\[
\sim_{S_{p_1}} C_1 + \alpha O_1 + \beta F - \beta \Theta_{p_{1,1}},
\]
for some integers $\alpha$ and $\beta$. Hence by Theorem 1.2 we have $\tilde{\psi}(\hat{C}_2) = C_1$. □

3 Construction for weak Namba-Tsuchihashi arrangements of type $k$ and $D_{2p}$-covers

Let $\mathcal{Q}$ be a quartic consisting of 2 irreducible conics intersecting at 4 distinct points. Let $f'' : S'' \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ with $\Delta f'' = \mathcal{Q}$ and let

![Diagram](image)

be the diagram for the canonical resolution of $S''$. Note that $q_o : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ is a composition of blowing ups at the 4 nodes Sing($\mathcal{Q}$) = \{P_0, P_1, P_2, P_3\}. We put $f_o = \mu_o \circ f'' = q_o \circ f_o$. We choose a point $x \in \mathcal{Q}$ such that

**Assumption.**

(i) the tangent line $l_x$ at $x$ meets $\mathcal{Q}$ at two other distinct points and

(ii) no tangent line at Sing($\mathcal{Q}$) passes through $x$.

Note that these conditions imply that $x$ is on a component of $\mathcal{Q}$ having degree greater than or equal to 2.

The pencil of lines through $x$ gives rise to a pencil of curves of genus 1, $\Lambda_x$, on $\bar{S}$. By resolving the base points, $\mu_x : S_x \to \bar{S}$, of $\Lambda_x$, we have a rational elliptic surface $\varphi_x : S_x \to \mathbb{P}^1$.

By our choice of $x$, $\varphi_x$ has 5 singular fibers of type $I_2$. Let $L_1, L_2$ and $L_3$ be lines through \{P_0, P_1\}, \{P_0, P_2\} and \{P_0, P_3\}. For each $L_i$, $\mu_x^* L_i$ is of the form $L_i^+ + L_i^-$. By our construction, $s_{L_i} := \mu_x^* L_i^+$ is a section with $\langle s_{L_i}, s_{L_i} \rangle = 1/2$ for each $i$. Thus, by labeling the singular fibers suitably, we may assume that $s_{L_i}$ ($i = 1, 2, 3$) meet the singular fibers as follows:
Here $\Theta_{1,1}$ is the component arising from $l_x$. Put $s_i := [2]s_{L_i}$. By \[\text{Theorem 9.1},\] we infer that $s_iO = 0$ ($i = 1, 2, 3$). Hence we have

\[(s_i, s_i) = 2 \quad (i = 1, 2, 3) \quad (s_i, s_j) = 0 \quad (1 \leq i < j \leq 3).\]

Hence $\text{MW}(S)^0 \cong A_1^{\oplus 3} = \mathbb{Z}s_1 \oplus \mathbb{Z}s_2 \oplus \mathbb{Z}s_3$, where $\text{MW}(S)^0$ denote the narrow part of $\text{MW}(S)$.

**Lemma 3.1** Let $Q$ and $x \in Q$ be as above. There exists exactly 3 conics $C_{1,x}, C_{2,x}$ and $C_{3,x}$ through $x$ such that (i) $C_{i,x}$ does not pass through $P_0, P_1, P_2$ and $P_3$ for each $i$ and (ii) for each $z_o \in C_{i,x} \cap Q$, the intersection multiplicity at $z_o$, $I_{z_o}(C_{i,x}, Q)$ is even.

**Proof.** Put $C_{i,x} := \tilde{f}_o \circ \mu_x(s_i) = \tilde{f}_o \circ \mu_x([-1]s_i)$ ($i = 1, 2, 3$). Since $s_i \in \text{MW}(S)^0$, $s_i$ always meets the identity component of a fiber for each $i$. For each $i$, $C_{i,x}$ passes $x$ and meets a general line through $x$ at another point distinct from $x$. Hence $C_{i,x}$ is a smooth conic. Since $s_i \neq [-1]s_i$, for each $z_o \in C_{i,x} \cap Q$, the intersection multiplicity at $z_o$, $I_{z_o}(C_{i,x}, Q)$ is even.

Conversely, any conic $C$ satisfying the conditions gives rise to sections $s_C, [-1]s_C \in \text{MW}(S)^0$ with $(s_C, s_C) = ([−1]s_C, [−1]s_C) = 2$. Thus $C$ is one of $C_{i,x}$ ($i = 1, 2, 3$). \[\square\]

**Lemma 3.2** Choose $x, y \in Q$ satisfying the Assumption. Put $s_{i,x} = [2]s_{L_{i,x}}$ and $s_{i,y} = [2]s_{L_{i,y}}$ ($i = 1, 2, 3$), where $s_{L_{i,x}}$ and $s_{L_{i,y}}$ are the sections arising from $(\tilde{f}_o \circ \mu_x)^\ast(L_i)$ and $(\tilde{f}_o \circ \mu_y)^\ast(L_i)$, respectively. Put $C_{i,x} = f_0 \circ \mu_x(s_{i,x})$ and $C_{i,y} = f_0 \circ \mu_y(s_{i,y})$ Let $C_{i,x}^\pm$ (resp. $C_{i,y}^\pm$) be bisections on $S_y$ (resp. $S_x$) arising $C_{i,x}$ (resp. $C_{i,y}$). Then $s(C_{1,y}^+) = s_{i,x}$ or $s(C_{1,x}^+) = [−1]s_{i,x}$ on $\text{MW}(S_x)$ and $s(C_{i,x}^+) = s_{i,y}$ or $s(C_{i,x}^+) = [−1]s_{i,x}$ on $\text{MW}(S_y)$.

**Proof.** Our statement is immediate by Theorem [1]. \[\square\]

Fix a positive integer $k \geq 2$ and take $(k_1, k_2, k_3) \in \mathbb{Z}_{>0}^3$ with $k_1 \geq k_2 \geq k_3$ and $k = k_1 + k_2 + k_3$. Choose $x_i$ ($i = 1, \ldots, k$) $\in Q$ satisfying the Assumption. We consider $k$ conics
Let $p$ be an odd prime. For $\mathcal{C}_i$ and $\mathcal{C}_j$ as above, there exists a $D_{2p}$-cover of $\mathbb{P}^2$ branched at $2\mathcal{Q} + p(\mathcal{C}_i + \mathcal{C}_j)$ if and only if $i, j \in \{1, \ldots, k\}$, $\{k_1 + 1, \ldots, k_1 + k_2\}$ or $\{k_1 + k_2 + 1, \ldots, k\}$.

Proof. Choose a general point $x \in \mathcal{Q} \setminus \{x_1, \ldots, x_k\}$ satisfying the Assumption. Let $\varphi_x : S_x \to \mathbb{P}^1$ be a rational elliptic surface as before and let

$$
S_x' \xleftarrow{\; \mu_s \;} S_x
$$

be its double cover diagram. By our construction of $S_x$, we infer that there exists a composition of blowing ups $q_x : \tilde{\Sigma}_2 \to \tilde{\mathbb{P}^2}$ such that

$$
\begin{array}{c}
S'' \xleftarrow{\; \mu_s \;} \tilde{S} \xleftarrow{\; \mu_s \;} S_x \\
\downarrow f' \quad \downarrow f \\
\tilde{\mathbb{P}^2} \xleftarrow{\; q_x \;} \mathbb{P}^2
\end{array}
$$

Let $\tilde{\mathcal{C}}_i$ be the strict transform of $\mathcal{C}_i$ with respect to $q_x \circ q_x$. By considering the Stein factorization, there exists a $D_{2p}$-cover of $\mathbb{P}^2$ branched at $2\mathcal{Q} + p(\mathcal{C}_i + \mathcal{C}_j)$ if and only if there exists a $D_{2p}$-cover of $\tilde{\Sigma}_2$ branched at $2\Delta_{f_x} + p(\tilde{\mathcal{C}}_i + \tilde{\mathcal{C}}_j)$. Put $f_x \tilde{\mathcal{C}}_i = \tilde{\mathcal{C}}_i' + \tilde{\mathcal{C}}_j'$. By our choice of $\mathcal{C}_i$, we have

$$
s(\mathcal{C}_j') = \begin{cases} 
    s_1.x \text{ or } [-1]s_1.x & \text{if } 1 \leq j \leq k_1 \\
    s_2.x \text{ or } [-1]s_2.x & \text{if } k_1 + 1 \leq j \leq k_1 + k_2 \\
    s_3.x \text{ or } [-1]s_3.x & \text{if } k_1 + k_2 + 1 \leq j \leq k
\end{cases}
$$

Hence by Corollary 3.2 our statement follows.

Let $\succ \text{lex}$ be the lexicographic order on $\mathbb{Z}_{\geq 0}^3$ (see [3]). Take $(k_1, k_2, k_3), (k'_1, k'_2, k'_3) \in \mathbb{Z}_{\geq 0}^3$ as in Lemma 3.3 such that $(k_1, k_2, k_3) \succ \text{lex} (k'_1, k'_2, k'_3)$. Choose

- $\mathcal{C}^{(1)}_1, \ldots, \mathcal{C}^{(1)}_k$ for $(k_1, k_2, k_3)$, and
- $\mathcal{C}^{(2)}_1, \ldots, \mathcal{C}^{(2)}_k$ for $(k'_1, k'_2, k'_3)$
as above. Put $B_1 = \mathcal{Q} + \sum_{i=1}^{k} C_i^{(1)}$ and $B_2 = \mathcal{Q} + \sum_{i=1}^{k} C_i^{(2)}$. Note that both $B_1$ and $B_2$ are weak $k$-Namba-Tsuchihashi arrangements.

**Proposition 3.4** There exists no homeomorphism $h : (\mathbb{P}^2, B_1) \to (\mathbb{P}^2, B_2)$ such that $h(\mathcal{Q}) = \mathcal{Q}$.

**Proof.** Suppose that a homeomorphism $h : (\mathbb{P}^2, B_1) \to (\mathbb{P}^2, B_2)$ exists. Put

$$h(c_i^{(1)}) = c_i^{(2)}, \quad h(c_{i+1}^{(1)}) = c_{i+1}^{(2)}.$$ 

Since Cov$_b(\mathbb{P}^2, 2Q + p(c_i^{(1)} + C_j^{(1)}), D_{2p}) \neq \emptyset$ $(j = 2, \ldots, k_1)$, by Proposition 1.6 there exists at least $(k_1 - 1) C_j^{(2)}$ different form $C_i^{(2)}$ such that Cov$_b(\mathbb{P}^2, 2Q + p(C_i^{(2)} + C_j^{(2)}), D_{2p}) \neq \emptyset$. As $k_1 \geq k_1'$, we infer that $k_1 = k_1'$ by Lemma 3.3. Similarly, since Cov$_b(\mathbb{P}^2, 2Q + p(C_{k_1+1}^{(1)} + C_j^{(1)}), D_{2p}) \neq \emptyset$ $(j = k_1 + 1, \ldots, k_1 + k_2)$, there exists at least $(k_2 - 1) C_j^{(2)}$ different form $C_i^{(2)}$ such that Cov$_b(\mathbb{P}^2, 2Q + p(C_{i_2}^{(2)} + C_j^{(2)}), D_{2p}) \neq \emptyset$ and we infer that $k_2 = k_2'$. This contradicts $(k_1, k_2, k_3) \gtrdot (k_1', k_2', k_3')$. $\square$

**Corollary 3.5** If both $B_1$ and $B_2$ are $k$-NT arrangements for $k \geq 3$, then $(B_1, B_2)$ is a Zariski pair.

**Proof.** If both $B_1$ and $B_2$ are $k$-NT arrangements and $k \geq 3$, $h(\mathcal{Q}) = \mathcal{Q}$ holds for any homeomorphism $(\mathbb{P}^2, B_1) \to (\mathbb{P}^2, B_2)$.

### 4 Proof of Theorem 4

In this section we will construct $y(k, 3)$ Namba-Tsuchihashi arrangements of type $k$ which form Zariski $y(k, 3)$-plets for conic arrangements. The main ingredient of the proof is an explicit method to calculate the equations of the conics that appeared in Proposition 3.4. We keep the notation in §3.

#### 4.1 Explicit method in finding equations of bisections

Let $S$ be an elliptic surface over $\mathbb{P}^1$, whose generic fiber is given by a Weirstrass form $y^2 = f(t, x)$. Note that the degree of $f$ with respect to $x$, $\deg_x f(t, x)$, is equal to 3. Consider a rational point $P$ of the generic fiber of $S$ with coordinates $(x(t), y(t))$. Let $L$ be the line passing through $P$ in $\mathbb{A}^2_{\mathbb{C}(t)}$ defined by

$$L : y = r(t)(x - x(t)) + y(t)$$

for some $r(t) \in \mathbb{C}(t)$. Then, since $(x(t), y(t))$ is a rational point of $S$ and since $L$ passes through $(x(t), y(t))$, the equation

$$\{r(t)(x - x(t)) + y(t))\}^2 - f(t, x)$$

factors into the form

$$\{r(t)(x - x(t)) + y(t))\}^2 - f(x, t) = (x - x(t))g(t, x).$$
Since \( \deg_x g(t, x) = 2 \), the support of the intersection of \( L \) and \( g(t, x) = 0 \), viewed as rational functions of \( S \) over the field \( \mathbb{C} \), defines a bisection \( D \) on \( S \), and by the definition of the Abel-Jacobi map, \( s(D) = -P \). By varying \( r(t) \), we get a family of bisections \( \mathcal{D} \) such that any \( D \in \mathcal{D} \) has the same image \( s(D) = -P \) under the Abel-Jacobi map.

In general, \( r(t) \) can be any rational function in \( \mathbb{C}(t) \) and hence \( \mathcal{D} \) becomes an enormously large family. We will find various useful strata of \( \mathcal{D} \) by restricting \( r(t) \).

Note that the images of the bisections, obtained by the method above, in \( \mathbb{P}^2 \) is defined by \( g(t, x) = 0 \).

### 4.2 Construction of \( y(k, 3) \) Namba-Tsuchihashi arrangements of type \( k \)

Let \( [T : X : Z] \) be homogeneous coordinates of \( \mathbb{P}^2 \) and let \( t = T/Z \) and \( x = X/Z \). Let \( Q : (XZ - T^2 + Z^2)(X^2 - 2XZ + T^2 - 4Z^2) = 0 \), and \( z_o = [0 : 1 : 0] \). We will consider the elliptic surface \( S \), as in §3, branched along \( Q \) with blow-up center \( z_o = [0 : 1 : 0] \). The Weierstrass equation of \( S \) is

\[
y^2 = (x - t^2 + 2)(x^2 - 2x + t^2 - 4).
\]

The singularities of \( Q \) consists of four nodes at \( \{[\pm 1, -1, 1], [\pm 2, 1, 1] \} \) and we put \( P_0 = [2, 2, 1], P_1 = [-1, -1, 1], P_2 = [1, -1, 1], P_3 = [-2, 2, 1] \). Then the lines \( L_1, L_2 \) and \( L_3 \) in §3 are given by \( L_1 : X - T = 0, L_2 : X - 3T + 4Z = 0 \) and \( L_3 : X - 2Z = 0 \), respectively. These lines gives rise to sections \( s_{L_1}, s_{L_2} \) and \( s_{L_3} \) as in §3. Also the conic \( XZ - T^2 + Z^2 = 0 \) gives rise to the two-torsion section, which we will denote by \( s_t \). These sections are given explicitly as follows:

\[
\begin{align*}
   s_{L_1} &= \left( t, \sqrt{-2}(t + 1)(t - 2) \right) \\
   s_{L_2} &= \left( 3t - 4, \sqrt{-10}(t - 1)(t + 2) \right) \\
   s_{L_3} &= \left( 2, \sqrt{-1}(t + 2)(t - 2) \right) \\
   s_t &= \left( t^2 + 2, 0 \right)
\end{align*}
\]

Put \( s_i = [2]s_{L_i} \). Explicitly, \( s_1, s_2 \) and \( s_3 \) are given as follows:

\[
\begin{align*}
   s_1 &= \left( \frac{1}{2} t^2 - 2, -\frac{\sqrt{-2}}{4} t (t^2 - 4) \right) \\
   s_2 &= \left( \frac{1}{10} t^2 - 2, -\frac{3\sqrt{-10}}{100} t (t^2 + 20) \right) \\
   s_3 &= \left( t^2 - \frac{17}{4}, -\frac{3\sqrt{-1}}{8} (4t^2 - 19) \right)
\end{align*}
\]

As we see in §3, each \( s_i \) gives rise to a conic in \( \mathbb{P}^2 \) that is tangent to \( Q \) at \( z_o \) and three other points. The defining equation of the conic \( C_1 \) (resp. \( C_2, C_3 \)) corresponding to \( s_1 \) (resp. \( s_2, s_3 \)) is \( C_1 : XZ - \frac{1}{2} T^2 + 2Z^2 = 0 \) (resp. \( C_2 : XZ - \frac{1}{10} T^2 + 2Z^2 = 0, C_3 : XZ - T^2 + 2Z^2 = 0 \)).

By applying the method of finding the explicit equations of bisections whose image of the Abel-Jacobi map is \([-1]s_1\) in 4.11 we obtain a family of bisections \( \mathcal{D}_1 \) whose images in \( \mathbb{P}^2 \)
under \( f_o \circ \mu_o \), are curves given by \( g_{1,r}(t,x) = 0 \), \( r \in \mathbb{C}(t) \) where

\[
g_{1,r}(t,x) = \frac{t^4}{4} + \frac{r\sqrt{-2}}{2} t^3 + \frac{1}{2} t^2 x - \left(3 + \frac{r^2}{2}\right) t^2 - x^2 - 2r\sqrt{-2} t + (r^2 + 2) x + 2r^2 + 4
\]

By specializing \( r \) to \( r = \frac{1}{2}\sqrt{-2}t - a_1 \) \( (a_1 \in \mathbb{C}) \), we obtain a sub-family \( D_1 \subset \mathcal{D}_1 \) of bisectons. The defining equations of the images under \( \bar{q} \circ f \) specialize to \( g_{1,a_1}(t,x) \), \( a_1 \in \mathbb{C} \) where \( g_{1,a_1}(t,x) \) is given by

\[
g_{1,a_1}(t,x) = \left( -2 - \frac{a_1^2}{2} \right) t^2 - xt\sqrt{-2}a_1 - x^2 + (a_1^2 + 2) x + 2a_1^2 + 4.
\]

By construction, any \( D_{1,a} \in D_1 \) satisfies \( s(D_{a_1}) = s_1 \).

Similarly, by applying the same method to \([-1]s_2 \) and \([-1]s_3 \), we obtain families of bisectons \( D_2 \) (resp. \( D_3 \)) parametrized by \( a_2 \in \mathbb{C} \) (resp. \( a_3 \in \mathbb{C} \)) such that any \( D_{1,a} \in \mathcal{D}_1 \) satisfies \( s(D_{a,i}) = s_i \) \( (i = 2,3) \) and the defining equations of their images in \( \mathbb{P}^2 \) are given by

\[
g_{2,a_2}(t,x) = \frac{1}{10} a_2^2 t^2 - 10 t^2 - \frac{12}{5} \sqrt{-10} a_2 t + a_2^2 + \frac{3}{5} \sqrt{-10} a_2 t^2 + a_2^2 x + 4 + 2x - x^2,
\]

\[
g_{3,a_3}(t,x) = -a_2^2 t^2 + \frac{17}{4} a_2^2 + a_2^3 x - 3\sqrt{-1} a_3 t^2 + \frac{57}{4} \sqrt{-1} a_3 + \frac{5}{4} t^2 - \frac{161}{16} + \frac{17}{4} x - x^2.
\]

We will denote the image of \( D_{1,a} \) in \( \mathbb{P}^2 \) by \( C_{i,a} \), which is defined by \( g_{1,a}(t,x) = 0 \).

**Lemma 4.1** Under the notation above, the following statements hold:

1. For \( i = 1, 2, 3 \) and general \( a_i \), \( C_{i,a} \) is a smooth conic. In particular there are only a finite number of non-reduced members of \( \mathcal{D}_i \).

2. For \( i = 1, 2, 3 \), there exist only a finite number of \( C_{i,a} \) passing through any given point \( p \in \mathbb{P}^2 \).

3. For \( i = 1, 2, 3 \), there exist only a finite number of \( a_i \) such that \( C_{i,a} \) and \( Q \) have an intersection point with multiplicity \( \geq 4 \).

4. For all \( \{i, j\} \subset \{1, 2, 3\} \) and any given \( a_i \in \mathbb{C} \) such that \( C_{i,a} \) is reduced, there exist only a finite number of \( a_j \) such that \( C_{j,a} \) and \( C_{i,a} \) do not intersect transversally.

**Proof** Each statement can be proved by direct calculations. The computer algebra software MAPLE was used in the calculations.

**Lemma 4.2** Let \( k \) be an integer \( \geq 3 \) and \( k_1, k_2, k_3 \) be non-negative integers such that \( k = k_1 + k_2 + k_3 \). Then there exists \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \) such that the configuration

\[
Q + \sum_{i=1}^{k_1} C_{1,\alpha_i} + \sum_{i=k_1+1}^{k_1+k_2} C_{2,\alpha_i} + \sum_{i=k_1+k_2+1}^{k} C_{3,\alpha_i}
\]

becomes a Namba-Tsuchihashi configuration of type \( k \) satisfying the properties described before Proposition 3.4.

**Proof** This follows directly from the previous lemma.

By combining Theorem 1.1, Lemma 4.2 and Proposition 3.4, we obtain Theorem 4.4.
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