I. INTRODUCTION

Exact solutions of the Einstein equations are difficult to obtain due to their nonlinear nature. In this paper we show that the system of equations for homogeneous, isotropic cosmological models with a variety of matter sources reduce to particular cases of the ordinary differential equation

\[ \ddot{y} + y^n \dot{y} + \beta y^{2n+1} + cy^n = 0 \] (1)

where \( \beta \), \( c \) and \( n \) are constants.

The problem of a causal viscous fluid with the bulk viscosity coefficient \( \zeta \) proportional to \( \rho^{1/2} \) corresponds to \( n = 1 \), \( c = 0 \), \( y \propto H \) in the truncated Extended Irreversible Thermodynamics theory,\(^1\) and \( n = -1/r, c = 0, y \propto H^{-r} \) in the full theory, where the relation between temperature and energy density is assumed to be of the form \( T \sim \rho^r.\(^2\)\)

Also, the behavior near the singularity, when the relaxation term is much more important than the viscous term in the transport equation, corresponds to \( n = 1, c = 0, y \propto H \) for generic power-law relation \( \zeta = \alpha \rho^m.\(^3\)\) For a perfect fluid source, with an equation of state \( p = (\gamma - 1) \rho \), and a cosmological constant \( \Lambda \), we recover Eq.(1) with \( n = 1, c = -\gamma \Lambda \) provided we derive twice the 00 Einstein equation

\[ H^2 = \frac{1}{3} \frac{\rho_0}{a^{3\gamma}} - \frac{k}{a^2} + \frac{\Lambda}{3} \] (2)

We also find Eq.(1) with \( n = 1, c = \Lambda, y \propto H \) for a fluid with equation of state \( 3p = -\rho - C/a^2, \rho = C \log a/a^2, C \) a constant. For two scalar fields, one free and the other self–interacting with a potential \( V(\phi) = V_0 \exp(-A\phi) \), the change \( y = a^{A\phi/2} \) in the Einstein equation leads to Eq.(1) with \( n = -6/A^2 \) and \( c = 0 \), as can be seen in the paper "Two-scalar field cosmologies" appearing in this volume. For a time decaying cosmological "constant", \( \dot{\Lambda} \sim -H^3 \Lambda \) with \( y \propto H \), the case with \( n = 1, c = 0 \) also arises.\(^4\)

Thus, it turns out to be of great interest to analyze Eq.(1) from the mathematical point of view. Its general solution will be studied elsewhere, and we concentrate here on the families of real solutions of the case \( n = 1, c = 0 \).

II. ANALYSIS OF THE SOLUTIONS FOR \( n = 1, c = 0 \)

Unless \( \beta = 1/9 \), Eq.(1), for \( n = 1 \) and \( c = 0 \) has only two point Lie symmetries and it is not equivalent to a second order linear equation under a point transformation.\(^5\) So, we consider the nonlocal transformation

\[ z = y^2, \quad \eta = \int y dt \] (3)

which turns Eq.(1) into the equation of a damped linear oscillator

\[ \frac{d^2z}{d\eta^2} + \frac{dz}{d\eta} + 2\beta z = 0 \] (4)

and we obtain the general solution of in a parametrized form \( (t(\eta), y(\eta)) \). The real solutions of Eq.(1) can be classified as follows:
a. $\beta < 1/8$ (strong damping).

$$z(\eta) = C \exp(\lambda_\eta \eta)$$  \hfill (5)

$$z(\eta) = 2C \exp(-\eta/2) \cosh(\delta \eta/2 + \phi)$$  \hfill (6)

$$z(\eta) = 2C \exp(-\eta/2) \sinh(\delta \eta/2 + \phi)$$  \hfill (7)

b. $\beta = 1/8$ (critical damping).

$$z(\eta) = C \exp(-\eta/2)$$  \hfill (8)

$$z(\eta) = C(\eta + \phi) \exp(-\eta/2)$$  \hfill (9)

c. $\beta > 1/8$ (weak damping)

$$z(\eta) = 2C \exp(-\eta/2) \sin(\delta \eta/2 + \phi)$$  \hfill (10)

where $\lambda_\pm$ are the roots of the characteristic polynomial, $\delta = |1 - 8\beta|^{1/2}$, and $C, \phi$ are arbitrary integration constants. Through the transformation (3), both Lie point symmetries of (1) have a simple equivalent: $t(\eta)$ is defined up to an arbitrary integration constant $t_0$, and this reflects the invariance of (1) under $t \to t + t_0$. Also, the invariance $z \to z/A^2 (A \neq 0)$, of (1) is equivalent to the symmetry transformation $t \to |A|t, y \to y/|A|$. Besides, the permutation between the two branches of $z$ leads to the discrete symmetry transformation $y \to -y, t \to -t$.

Whenever $z(\eta)$ has a zero, extremum or inflexion point at $\eta_1$, $y(t)$ has a zero, extremum or inflexion point at $t_1 = t(\eta_1)$. Besides, it can be seen that $\dot{y}$ is finite at any zero point, and so $y(t)$ is odd in a neighborhood of all finite zero points (see below).

There are two groups of solutions $z(\eta)$:

i) Those that never vanish, i.e., (5), (6), and (8), so that we may choose $z(\eta) > 0$ for all $\eta$. For these solutions $y(t)$ is nonvanishing, and it is obtained from any of the two branches of $\sqrt{x}$ (depending on sign $y$).

ii) Those that have (at least) one zero point, i.e., (7), (9), and (10). The requirement that $z(\eta) > 0$ cannot be satisfied on both sides of the zero point by the same solution (with a given value of $C$). Therefore, $z(\eta)$ gives rise to two solutions $y(t)$, one for each sign of $C$. Since $y(t)$ is odd (see below), these solutions are obtained by joining at the zero point both branches of $\sqrt{x}$.

From (3), we see that $\eta(t)$ is even for odd $y(t)$ and has extrema at the zero points of $y(t)$. Then, for the group (i), $\eta(t)$ is monotonic, that is, there is a one to one mapping between the real axis $\eta$ and some interval of the axis $t$. However, for the group (ii), the pair of branches at each side of the zero point correspond to different mappings between $\eta$ and $t$. Besides, each singularity of $y(t)$ (where $\eta(t)$ diverges logarithmically) marks a boundary for the mapping $\eta \to t$. For solutions (6), (7), (8), and (10), $t(\eta)$ can be expressed in terms of a hypergeometric function. Only for $\beta = 1/9, 0$ or $-1, t(\eta)$ can be inverted in closed form.

Due to the symmetries of (1), if $y(t)$ is a solution, $Ay(\Delta t)$ is also a solution, where $\Delta t = t - t_0$. In particular, if $y(\Delta t)$ satisfies $y = 0$ and $\dot{y} \neq 0$ at $t = t_0$, $-y(-\Delta t)$ is also a solution with the same initial data. However, as (1) satisfies a Lipschitz condition, given these initial data the solution is unique. Thus, we conclude that $y(\Delta t)$ is odd. Further, it is easy to see that $y(t)$ is analytic at $t_0$, so that there is an interval where its Taylor series converges. As $y(t)$ must be odd also about any further zero point within the interval of convergence of the series, it comes out that there are only two possibilities for an interval with a zero point:

a) The interval contains only one zero point.
b) There are infinitely many equispaced zero points; that is, $y(t)$ is an oscillatory periodic function (the radius of convergence is infinite).

Solutions which exhibit behavior (a) occur only for $\beta < 1/8$, while the solutions for $\beta > 1/8$ have behavior (b). So, for $\beta < 1/8$, $y(t)$ has either one or no zero point in the interval where it is bounded.

The solutions (10) lead to the two one-parameter families of solutions for $\beta < 1/8$:

$$y_{\pm}(t) = \alpha/\Delta t, \quad \alpha_{\pm} = -2/\lambda_{\pm}, \quad \lambda_+ \neq 0; \quad y_+ = K, \quad \lambda_+ = 0$$  \hfill (11)

and we wish to investigate small departures from them. Let us consider first the case when $\exp[(\lambda_- - \lambda_+)\eta - 2\phi] \ll 1$. As $\lambda_+ > \lambda_-$, this occurs for any $\phi$ if $\eta$ is big enough. Then to first order we get the approximated solution

$$y(t) = \frac{\alpha}{\Delta t} (1 + \gamma \Delta t^r), \quad \lambda_+ \neq 0; \quad y(t) = K (1 + \gamma \exp(-Kt)), \quad \lambda_+ = 0$$  \hfill (12)
where $\gamma \propto \exp(-2\phi)$, $r = 4 - \alpha$ is the Kowalevski exponent and $\alpha = \alpha_+$ in this case.\[3\] In the opposite case, that is for $\eta \to -\infty$, we get also (12b), but now $\alpha = \alpha_-$ and $\gamma \propto \exp(2\phi)$. Whenever $y(t)$ has a singularity, $z(\eta)$ diverges, and this occurs for $\eta \to \infty$ as well as for $\eta \to -\infty$ if $\beta < 0$. Hence, using (3), we find that any singularity is located at a finite time, and in a neighborhood of it, $y(t)$ has the asymptotic form (12b) with $r > 0$. When $\beta > 0$, $z(\eta) \to 0$ and $|t(\eta)| \to \infty$ for $\eta \to \infty$. Then, $y(t)$ vanishes at infinity with the asymptotic behavior (12b), where $\alpha = \alpha_+$ and $r < 0$.

The two-parameter families of solutions arise from (6), (7), (9), and (10). We classify them in two groups: those which have a singularity at a finite time, and those which are regular for all time; and we give the main features of their behavior.

a. Singular solutions.

0 $\leq \beta < 1/8$, $C > 0$.

a1. They have a singularity, where the leading behavior is (11+) decrease monotonically and vanish at infinity with leading behavior (11-) unless $\beta = 0$, when they have a nonvanishing limit.

$\beta < 0$, $C > 0$.

a2. They have two singularities, one with leading behavior (11+) and the other with leading behavior (11-). There is a minimum (maximum) between them.

a3. They have a zero point between two singularities, where the leading behavior is (11+). They increase monotonically and have three inflexion points.

$\beta \leq 1/8$, $C < 0$.

a4. They have a zero point between two singularities, where the leading behavior is (11-) for $\beta < 1/8$ and

$$y(t) \sim \frac{A}{\Delta t} \left[ 1 + \frac{1}{2 \ln |\Delta t|} + \frac{A - \ln \ln(1/|\Delta t|)}{2 \ln |\Delta t|^2} \right]$$

for $\beta = 1/8$. They decrease monotonically.

b. Regular solutions.

0 $< \beta \leq 1/8$, $C > 0$.

b1. They have a zero point between two extrema. They have three inflexion points and they vanish at infinity, with leading behavior (11+) for $\beta < 1/8$ and like a4., with the replacement $\ln(1/|\Delta t|) \to \ln |\Delta t|$, for $\beta = 1/8$.

$\beta = 0$, $C > 0$.

b2. They have a zero point at $t_0$ and increase monotonically with a nonvanishing limit at infinity.

$\beta > 1/8$.

b3. They are oscillatory periodic, and its period and amplitude have a relation of the form $AT^2 = f(\delta)$. The period diverges as $\beta \to 1/8^+$ and has the limit $T \to 2\pi^{3/2} / (\sqrt{\Gamma(3/4)^2})$, $\beta \to \infty$. These results agree with the phase space analysis.\[5\] and confirm the numerical simulations.\[8\]

### III. CONCLUSIONS

We have obtained the general solution of Eq. (1) for $n = 1$ and $c = 0$ in a parametrized form by means of Eq. (3). The solutions have moving singularities and depending on whether these points are real or not, two groups of real solutions arise: the singular and the regular ones.\[6\] Both one-parameter families of solutions for $\beta < 1/8$ are singular, unless $\beta = 0$, when one of them turns into a constant. They coalesce for $\beta = 1/8$, and there is no real one-parameter family of solutions for $\beta > 1/8$. These one-parameter solutions give the leading behavior of the solutions about a singularity. Only for $\beta = 1/9, 0, -1$ two-parameter solutions are functions on the complex plane and real solutions can be expressed in closed form.

In general, the problem of the construction of explicit solutions of a given integrable nonlinear differential equation remains open. One direction along which one can attempt to proceed is linearization, i.e. the reduction of the equation to a linear ordinary differential equation, which is, by definition, integrable. Only for $\beta = 1/9$ Eq. (1) possesses eight symmetries and is linearizable by a point transformation. On the other hand, the transformation (3) linearizes it for any value of $\beta$. Thus, although it has only two Lie point symmetries, it possesses eight nonlocal symmetries. We think that it is of utmost importance to study this kind of linearizing transformations, which have received up to now little attention.
IV. REFERENCES

[1] L. P. Chimento and A. S. Jakubi, Class. Quantum Grav. 10 (1993) 2047.
[2] R. Maartens, Class. Quantum Grav. 12 (1995) 1455.
[3] M. Zakari and D. Jou, Phys. Lett. A 175 (1993) 395.
[4] M. Reuter and C. Wetterich Phys. Lett. B 188 (1987) 38.
[5] F. M. Mahomed and P. G. L. Leach, Quaest. Math. 48 (1985) 241.
[6] H. Yoshida, Celest. Mech. 31 (1983) 363.
[7] N. Minorsky, Nonlinear Oscillations, (Van Nostrand, New York, 1962).
[8] P. G. L. Leach, M. R. Feix and S. Bouquet, J.Math.Phys., 429 (1988) 2563.
[9] E. L. Ince, Ordinary Differential Equations (Dover, New York, 1956)