Local Additivity Revisited

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Abstract
We make a number of simplifications in Gour and Friedland’s proof of local additivity of minimum output entropy of a quantum channel. We follow them in reframing the question as one about entanglement entropy of bipartite states associated with a $d_B \times d_E$ matrix. We use a different approach to reduce the general case to that of a square positive definite matrix. We use the integral representation of the log to obtain expressions for the first and second derivatives of the entropy, and then exploit the modular operator and functional calculus to streamline the proof following their underlying strategy. We also extend this result to the maximum relative entropy with respect to a fixed reference state which has important implications for studying the superadditivity of the capacity of a quantum channel to transmit classical information.
1 Introduction

It is now well established that the availability of entangled states for quantum systems implies that many information theoretic quantities, such as the capacity of a quantum channel to transmit information, need not be additive under tensor products. In this paper we consider the minimal output entropy $S_{\text{min}}(\Phi)$ associated with a quantum channel $\Phi$ which acts on $M_d$ the set of $d \times d$ matrices over $\mathbb{C}$. A quantum state is described by a density matrix $\rho$, i.e., a positive semi-definite matrix with $\text{Tr} \rho = 1$. The quantum entropy of the state is $S(\rho) \equiv -\text{Tr} \rho \log \rho$. Then

$$S_{\text{min}}(\Phi) \equiv \inf \{ S[\Phi(\rho)] : \rho \geq 0, \text{Tr} \rho = 1 \}. \quad (1)$$

In 2009, Hastings [4] showed that $S_{\text{min}}(\Phi)$ is not additive, i.e., there are quantum channels $\Phi$ for which $S_{\text{min}}(\Phi \otimes \Phi) < 2S_{\text{min}}(\Phi)$. (See [1, Section 8.2.3] and endnotes to Chapter 8 of [1] for a discussion of later developments.)

Nevertheless, Gour and Friedland [3] proved that the minimal output entropy of a quantum channel is locally additive. $S[\Phi(\rho)]$ has a local minimum at $\rho_c$ if there is a neighborhood $\mathcal{N}(\rho_c)$ such that $S[\Phi(\rho)] \geq S[\Phi(\rho_c)]$ for all $\rho \in \mathcal{N}(\rho_c)$. If the inequality is strict for all $\rho \neq \rho_c$ the minimum is non-degenerate. Because the entropy $S(\rho)$ is a concave function and the set of density matrices is compact and convex, any minimum, local or global, must be achieved at an extreme point so that $\rho_c = |\psi\rangle\langle\psi|$ is a pure state.

The main result of [3] can be stated as follows. If a pair of quantum channels $\Phi_A, \Phi_C$ have non-degenerate local minima of $S[\Phi_A(\rho_A)]$ and $S[\Phi_C(\rho_C)]$ at $\rho_A = |\psi_A\rangle\langle\psi_A|$ and $\rho_C = |\psi_C\rangle\langle\psi_C|$ respectively, then $S[(\Phi_A \otimes \Phi_C)(\rho_{AC})]$ has a local minimum at $\rho_{AC} = |\psi_A \otimes \psi_C\rangle\langle\psi_A \otimes \psi_C|$. In this manuscript, we present this result with a number of simplifications to their proof. We also extend it to the maximum output for relative entropy with respect to a fixed reference state.

In Section 2.1 we follow [3] in using the Stinespring representation to reformulate the problem in terms of the entanglement entropy of a bipartite state and then as the entropy $S(XX^\dagger)$ when $X$ is a $d_B \times d_E$ matrix restricted to a subspace of such matrices. After doing this, we restate the main result as Theorem 2.

However, we use a completely different approach to reduce the general case to that of $X$ a square positive definite matrix. This reduction is based on the the fact that $S(X^\dagger X) = S(XX^\dagger)$ and a simple majorization argument as described in Section 6. The further reduction to a positive definite matrix is given in Section 2.2 using the polar decomposition theorem; this section also presents the framework needed later in Section 6.

When $d_B = d_E$ and $X$ is non-singular, we follow the underlying strategy of [3] in Sections 3 to 5. However, we make significant simplifications to their arguments. In Section 3 we use the integral representation of the log to easily obtain the relevant derivatives. We then rewrite these derivatives using the elegant modular operator theory of Tomita-Takesaki as advocated by D. Petz [7]. This approach avoids the need to write matrix elements explicitly. It also allows us to associate each critical point $X_c$ with an operator $H(X_c)$ which acts on a particular self-adjoint subspace of $M_d \oplus M_d$. The condition that this critical point corresponds to a non-degenerate local minimum...
of $S(XX^\dagger)$ is that this operator is positive definite on that subspace. We complete the proof in Section 5 with a cleaner presentation using the relative modular operator.

In Section 7 we extend our results in a straightforward way to the local maxima of the relative entropy $H[\Phi(\rho), \Phi(\omega)]$ with $\omega$ fixed and discuss the implications for studying superadditivity of the capacity of a quantum channel to transmit classical information.

In Appendix A we give two proofs of an elementary key inequality from [3]. In Appendix B we use an integral representation of $\log \rho(t)$ to find useful formulas for its first derivative.

## 2 Reformulations

### 2.1 Matrix reformulation

Recall that a quantum channel $\Phi : M_{d_A} \mapsto M_{d_B}$ is a completely positive trace-preserving map which can be represented using an auxiliary space $C_{d_E}$ of dim $d_E = d_A d_B$ via the Stinespring representation in the form

$$\Phi(\rho) = \text{Tr}_E K \rho K^\dagger,$$

where $K : C_{d_A} \mapsto C_{d_B} \otimes C_{d_E}$ satisfies $K^\dagger K = I_A$. Since the minimal output entropy is the optimization of a concave function over a convex set, it suffices to consider pure inputs $|\psi \rangle \langle \psi|$. Then $K|\psi\rangle \in C_{d_B} \otimes C_{d_E}$. If we interpret $u, v$ as column vectors then $\Omega[u \otimes v] = uu^T$ extended by linearity takes vectors in $C_{d_B} \otimes C_{d_E}$ to $d_B \times d_E$ matrices. If $|\psi\rangle \in C_{d_A}$ and $X = \Omega(K|\psi\rangle)$ then $X : C_{d_E} \mapsto C_{d_B}$ so that $X$ is a $d_B \times d_E$ matrix and

$$XX^\dagger = \Phi(|\psi\rangle \langle \psi|) = \text{Tr}_E K|\psi\rangle \langle \psi| K^\dagger.$$

The entropy $S(XX^\dagger)$ is the entanglement entropy of the bipartite state $K|\psi\rangle$. Thus the problem of finding the minimal output entropy of a quantum channel is equivalent to finding the minimal entanglement entropy of a bipartite state $|\psi_{BE}\rangle$ in the subspace $KK^\dagger(C_{d_B} \otimes C_{d_E})$ which is the range of $K$. Moreover, if $K \equiv \Omega[KK^\dagger(C_{d_B} \otimes C_{d_E})] = \Omega[KK^\dagger]$ is the corresponding subspace of $d_B \times d_E$ matrices this is equivalent to finding $\inf_{X \in \mathcal{K}} S(XX^\dagger)$ where $\mathcal{K} = \{X \in \mathcal{K} : \text{Tr} XX^\dagger = 1\}$ is the unit sphere in $\mathcal{K}$. We have thus proved the following

**Theorem 1** Let $\Phi : M_{d_A} \mapsto M_{d_B}$ be a quantum channel and $\mathcal{K}$ the subspace of $d_B \times d_E$ matrices defined via the Stinespring representation as above. Then the entropy $S[\Phi(\rho)]$ has a local minimum at $\rho = |\psi\rangle \langle \psi|$ if and only if $S(XX^\dagger)$ with $X \in \mathcal{K}$ has a local minimum at $X = \Omega K|\psi\rangle$.

In this framework $X^T \overline{X} = \Phi^C(|\psi\rangle \langle \psi|)$ is the output of the complementary channel. Since $XX^\dagger$ and $X^\dagger X$ have the same non-zero eigenvalues, $S(XX^\dagger) = S(X^\dagger X) = S(X^T \overline{X})$. Note that $X^\dagger X = \text{Tr}_B \overline{\rho} \langle \overline{\psi}| K^T$ is an output of the complementary channel associated with $\overline{K}$.

It follows from Theorem 1 that the local additivity of the minimal output entropy of a pair of quantum channels when both minima are non-degenerate is equivalent to the following
Theorem 2 If \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o} \) and \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o_C} \) have non-degenerate local minima at \( X_B \) and \( X_C \) respectively, then \( S(XX^\dagger) \big|_{X \in (\mathcal{K}_B \otimes \mathcal{K}_C)^o} \) has a non-degenerate local minimum at \( X_B \otimes X_C \).

This result can be extended to some cases of degenerate local minima as discussed at the end of Section 5. In Sections 3 to 5, we prove Theorem 2 under the additional assumption that \( X_B \) and \( X_C \) are square positive definite matrices. In the next subsection we carefully describe the framework needed to consider the general case. In Section 6 we complete the proof by showing that the general case can be reduced to that of a square non-singular matrix.

2.2 Reduction to square non-singular and positive definite forms

Whether or not \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o} \) has a local minimum at \( X_c \) is determined by directional derivatives associated with perturbations of the form \( X(t) \equiv \sqrt{1-t^2}X_c + tY \) with \( X_c \in \mathcal{K}^o, Y \in \mathcal{K}^o \) and \( \text{Tr} X_c Y = 0 \). Then \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o} \) has a local minimum at \( X_c \) if and only if \( g(t) = S[X(t)X(t)^\dagger] \) has a local minimum at \( t = 0 \) for all \( Y \in \mathcal{K}^o \cap \mathcal{X}^\perp \).

For \( X_c \in \mathcal{K}^o \), let \( P_B \) be the projection onto \( (\ker X_c X_c^\dagger + \ker X_c^\dagger X_c)^\perp \) and \( P_E \) the projection onto \( (\ker X_c X_c^\dagger + \ker X_c^\dagger X_c)^\perp \) so that \( X_c = P_B X_c P_E \). The following theorem, which is proved in Section 6, implies that to determine whether or not \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o} \) has a local minimum at \( X_c \) it suffices to consider directional derivatives \( X(t) \equiv \sqrt{1-t^2}X_c + tY \) with \( Y \in (P_B KP_E)^o \).

Theorem 3 Let \( X_c = P_B X_c P_E \) as above. Then \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o} \) has a local minimum at \( X_c \) if and only if \( S(XX^\dagger) \big|_{X \in (P_B KP_E)^o} \) has a local minimum at \( X_c \).

Since \( \text{Tr} P_B = \text{Tr} P_E \equiv d \), this essentially reduces the problem to \( d \times d \) matrices on the restriction to \( P_B KP_E \) with \( X \) having full rank \( d \). Using a variant of the polar decomposition, we can write \( X = \sqrt{XX^\dagger} V \) with \( V \in M_d \) unitary. Since \( VV^\dagger = I_d \), we can replace

\[
\sqrt{1-t^2}X + tY = (\sqrt{1-t^2\sqrt{XX^\dagger}} + tYV^\dagger)V
\]

by \( (\sqrt{1-t^2\sqrt{XX^\dagger}} + tYV^\dagger) \). Thus, we can assume that \( X = \sqrt{XX^\dagger} \) is positive definite by changing the perturbation from \( Y \) to \( YV^\dagger \).

In terms of our original formulation, we can identify \( V \) with a partial isometry of rank \( d \) with \( VV^\dagger = P_B \) and \( V^\dagger V = P_E \). Then \( Y \in P_B KP_E = P_B KV^\dagger V \) if and only if \( YV^\dagger \in P_B KV^\dagger \). Note that elements of \( P_B KV^\dagger \) map \( C_d \rightarrow C_d \) and \( \mathcal{K}_B \equiv P_B KV^\dagger \) can be identified with a subspace of \( M_{d_B} \). Thus, we can replace \( X \in \mathcal{K} \) by the positive definite matrix \( X = \sqrt{XX^\dagger} \in \mathcal{K}_B \) by considering perturbations \( Y \in \mathcal{K}_B = P_B KV^\dagger \).

Remark 4 Thus \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o} \) has a local minimum at \( X_c = \sqrt{X_c X_c^\dagger} V \in \mathcal{K}^o \) if and only if

a) \( S(XX^\dagger) \big|_{X \in \mathcal{K}^o_B} \) has a local minimum at \( X_c^\dagger \equiv \sqrt{X_c X_c^\dagger} \in \mathcal{K}^o_B \) or, equivalently,
b) $g(t) = S[X(t)X(t)\dagger]$ with $X(t) = \sqrt{1-t^2}X_cX_c\dagger + tY$ has a local minimum at $t = 0$ for all $Y \in \mathcal{K}_B \cap X^\perp$.

Thus, we can assume that $X$ is positive definite and consider $X(t) = \sqrt{1-t^2}X + tY$ with $X, Y$ in the unit sphere $\mathcal{K}^\circ$ of some fixed subspace of $M_q$ and $\text{Tr} XY\dagger = 0$.

If we write $Y = W + iZ$ with $W = W\dagger$ and $Z = Z\dagger$, then our assumption that $X > 0$ implies that if $\text{Tr} XY\dagger = 0$ then $\text{Tr} XW = \text{Tr} XZ = \text{Tr} XY = 0$. As observed after (15), the condition which determines the critical points, requires only perturbations $W = W\dagger$ in $X^\perp$. Whether or not these critical points correspond to non-degenerate local maxima requires only perturbations $W$ and $iZ$ with $W = W\dagger \in X^\perp$ and $Z = Z\dagger \in X^\perp$ as can be seen from (27).

3 Derivatives of entropy expressions

3.1 Derivatives of the entropy

Let $\rho(t)$ be a one-parameter family of density matrices which is twice differentiable and has full rank in a neighborhood of $t = 0$. The derivatives of $S[\rho(t)] = -\text{Tr} \rho(t) \log \rho(t)$ can be easily obtained from those for $\log \rho(t)$. For completeness, these are derived using an integral representation for $\log \rho$ in Appendix B.

It is well known [7] and shown in Appendix B that

$$\left. \frac{d}{dt} \log \rho(t) \right|_{t=t_1} = \int_0^\infty \frac{1}{\rho(t_1) + uI} \rho'(t_1) \frac{1}{\rho(t_1) + uI} du. \quad (5)$$

Next, observe that the derivative of $\log \rho(t)$ is always orthogonal to $\rho(t)$, i.e.,

$$\text{Tr} \rho(t) \int_0^\infty \frac{1}{\rho(t) + uI} \rho'(t) \frac{1}{\rho(t) + uI} du = \text{Tr} \int_0^\infty \frac{\rho'(t)}{\rho(t) + uI} \rho(t) \frac{1}{\rho(t) + uI} du$$

$$= \text{Tr} \rho'(t) \rho(t) \rho(t)^{-1}$$

$$= \text{Tr} \rho'(t) = 0 \quad (6)$$

where we used the cyclicity of the trace.

It follows immediately from (6) that

$$\frac{d}{dt} S[\rho(t)] = -\text{Tr} \rho'(t) \log \rho(t) - \text{Tr} \rho(t) \frac{d}{dt} \log \rho(t) = -\text{Tr} \rho'(t) \log \rho(t) \quad (7)$$

$$\frac{d^2}{dt^2} S[\rho(t)] = -\text{Tr} \rho''(t) \log \rho(t) - \text{Tr} \rho'(t) \int_0^\infty \frac{1}{\rho(t) + uI} \rho'(t) \frac{1}{\rho(t) + uI} du. \quad (8)$$
### 3.2 Derivatives in matrix form

We now consider \( X, Y \in K^o \) the unit sphere on a subspace of \( M_d \) as described in Section 2.2 with \( \text{Tr} XY^\dagger = 0 \) and define \( X(t) = \sqrt{1 - t^2} X + t Y \), so that \( \text{Tr} X(t)X(t)^\dagger = 1 \). We also assume that \( XX^\dagger \) is non-singular so that we can apply (7) and (8) to

\[
\rho(t) = (\sqrt{1 - t^2} X + t Y)(\sqrt{1 - t^2} X + t Y)^\dagger
\]

\[
= (1 - t^2)XX^\dagger + t\sqrt{1 - t^2}(YX^\dagger + XY^\dagger) + t^2 YY^\dagger
\]

(9)



to obtain

\[
D_1[X,Y] \equiv \left. \frac{d}{dt} S[\rho(t)] \right|_{t=0} = -\text{Tr} (YX^\dagger + XY^\dagger) \log XX^\dagger = -\text{Tr} \Gamma_1 \log XX^\dagger
\]

(10)

\[
D_2[X,Y] \equiv \left. \frac{d^2}{dt^2} S[\rho(t)] \right|_{t=0} = -2 \text{Tr} \Gamma_2 \log XX^\dagger - \text{Tr} \Gamma_1 \int_0^\infty \frac{1}{XX^\dagger + uI} \Gamma_1 \frac{1}{XX^\dagger + uI} du
\]

(11)

where \( \Gamma_1 = YY^\dagger + YX^\dagger = \rho'(0) \) and \( \Gamma_2 = YY^\dagger - XX^\dagger = \frac{1}{2}\rho''(0) \).

With the additional assumption that \( X > 0 \) and \( X, Y \in K^o_B \) as discussed above Remark 4, it is convenient to write

\[
D_2[X,Y] = R[X,Y] - Q[X,Y]
\]

(12)

where

\[
R[X,Y] \equiv -2 \text{Tr} (YY^\dagger \log X^2) - 2 S(X^2)
\]

(13)

\[
Q[X,Y] = \text{Tr} (YX + XY^\dagger) \int_0^\infty \frac{1}{X^2 + uI} (YX + XY^\dagger) \frac{1}{X^2 + uI} du
\]

(14)

Observe that \(-2 \text{Tr} (YY^\dagger \log X^2)\) is the only positive term in the second derivative. Therefore, if \( S(X^2) \) is a local minimum, this term must dominate.

We can easily extend all of the expressions above to situations when \( \text{Tr} YY^\dagger = \lambda^2 \neq 1 \) by observing that \( D_1[X,\lambda Y] = \lambda D_1[X,Y] \) and will occasionally find it useful to do so.

### 3.3 Modular operator expression for second derivative

For simplicity, we now assume that \( X = X^\dagger > 0 \) is positive definite as discussed above Remark 4.

We can rewrite (14) in an elegant and useful way using the modular operator formalism. Let \( L_A(W) = AW \) and \( R_B(W) = WB \) denote the operations of left and right multiplication on \( M_d \). Since \( L_A R_B(W) = AWB = R_B L_A(W) \), the operators \( L_A, R_B \) commute even when \( AB \neq BA \). When \( A, B \) are Hermitian, positive semi-definite, or positive definite, then \( L_A \) and \( R_B \) are
correspondingly Hermitian, positive semi-definite, or positive definite with respect to the Hilbert Schmidt inner product. Moreover $f(L_A) = L_{f(A)}$ etc.

Now write $Y = W + iZ$ with
\[ W = \frac{1}{2} (Y^† + Y) = W^† \] and $Z = i\frac{1}{2} (Y^† - Y) = Z^†$.

Then
\[ \Gamma_1 = (W + iZ)X + X(W - iZ) \]
\[ = (L_X + R_X)(W) - i(L_X - R_X)(Z) \]
\[ = M_+(W) - iM_-(Z) \quad \text{with} \quad M_\pm = L_X \pm R_X. \]

It follows from (10) that $D_1[X, Y] = D_1[X, W] = -2\Tr WX \log X^2$ when $X > 0$.

Next, observe that
\[ R[X, Y] = -2\Tr (W^2 + Z^2) \log X^2 - 2S(X^2) + 2i\Tr (WZ - ZW) \log X^2. \]

To treat $Q[X, Y]$ we will use the following well-known formula [7, Appendix A]
\[ \frac{\log L_\rho - \log R_\rho}{L_\rho - R_\rho} (\Gamma) = \frac{\log \Delta_\rho}{L_\rho - R_\rho} (\Gamma) = \int_0^\infty \frac{1}{\rho + tI} \frac{1}{tI} \, dt \]

where $\Delta_\rho = L_\rho R_\rho^{-1}$ is the modular operator. This follows from (B.5) in Appendix B and holds whenever $\rho > 0$ is positive definite. Thus, (14) can be written as
\[ Q[X, Y] = \Tr \Gamma_1 \frac{\log \Delta X^2}{LX^2 - RX^2} (\Gamma_1) = \Tr \Gamma_1 \frac{\log \Delta X^2}{M_+ M_-} (\Gamma_1) \]

and the fact that $M_\pm$ and $\Delta X^2$ all commute implies that (18) can be written as
\[ Q[X, Y] = \Tr \left[ M_+(W) - iM_-(Z) \right] \frac{\log \Delta X^2}{LX^2 - RX^2} \left[ M_+(W) - iM_-(Z) \right] \]
\[ = \Tr M_+(W) \frac{\log \Delta X^2}{M_+ M_-} M_+(W) - \Tr M_-(Z) \frac{\log \Delta X^2}{M_+ M_-} M_-(Z) \]
\[ - i \Tr M_+(W) \frac{\log \Delta X^2}{M_+ M_-} M_-(Z) - i \Tr M_-(Z) \frac{\log \Delta X^2}{M_+ M_-} M_+(W). \]

Since the cyclicity of the trace implies $\Tr M_\pm(A)B = \pm \Tr AM_\pm(B)$, this gives
\[ Q[X, W + iZ] = Q[X, Y] - 2i\Tr (WZ - ZW) \log X^2 \]
\[ = \Tr W M_+ \frac{\log \Delta X^2}{M_+ M_-} M_+(W) + \Tr Z M_- \frac{\log \Delta X^2}{M_+ M_-} M_-(Z) \]
\[ = \Tr W M_+ \frac{\log \Delta X^2}{M_+ M_-} W + \Tr Z M_- \frac{\log \Delta X^2}{M_+ M_-} Z \]
\[ = \Tr W \frac{\Delta X + I}{\Delta X - I} \log \Delta X^2 W + \Tr Z \frac{\Delta X - I}{\Delta X + I} \log \Delta X^2 Z \]
\[ = \Tr W \phi(\Delta X)(W) + \Tr Z \phi(-\Delta X)(Z). \]

\[ \text{If we had not assumed that } X > 0, \text{ we could obtain an equivalent result by writing } X = \sqrt{X}X^†V \text{ and letting } V Y^† = W - iZ \text{ with } W = \frac{1}{2}(V Y^† + YV^†) \text{ and } Z = i\frac{1}{2}(V Y^† - YV^†). \]
where we used
\[
\frac{M_+}{M_-} = \frac{L_+ + R_X}{L_- + R_X} = \frac{\Delta_X + 1}{\Delta_X - 1}
\] (23)
and define
\[
\phi(a) = \frac{a + 1}{a - 1} \log a^2 .
\] (24)

The properties of \( \phi \), which plays a key role, are summarized in Appendix A where we also include two proofs of the following inequality of Gour and Friedland [3, Lemma 6].
\[
2\phi(bc) \leq \phi(b) + \phi(-b) + \phi(c) + \phi(-c).
\] (25)

Since \( \phi(a) + \phi(-a) = \phi(a^2) \), this can be rewritten as
\[
2\phi(bc) \leq \phi(b^2) + \phi(c^2).
\] (26)

Combining (16), (20) and (22) gives the following

**Theorem 5** When \( X > 0 \) is positive definite, and \( Y = W + iZ \)
\[
D_2[X, Y] \equiv \frac{d^2}{dt^2} S \left( (\sqrt{1 - t^2} X + tY)(\sqrt{1 - t^2} X + tY)^\dagger \right) \bigg|_{t=0} = -2S(X^2) - 2\text{Tr} (W^2 + Z^2) \log X^2 - \bar{Q}[X, Y]
\] (27)
\[
= -2S(X^2) - 2\text{Tr} (W^2 + Z^2) \log X^2 - \text{Tr} W \phi(\Delta_X)(W) - \text{Tr} Z \phi(-\Delta_X)(Z)
\]
with \( W, Z \) as defined above (15) and \( \phi(a) \) in (24).

**Remark 6** We now make a number of useful observations with \( Y = W + iZ \).

a) \( 2(W^2 + Z^2) = YY^\dagger + Y^\dagger Y \).

b) Since \( \phi(\pm \Delta_X) \) is self-adjoint with respect to the Hilbert Schmidt inner product,
\[
\text{Tr} W \phi(\pm \Delta_X)(W) + \text{Tr} Z \phi(\pm \Delta_X)(Z) = \text{Tr} (W + iZ)\phi(\pm \Delta_X)(W - iZ)
\]
\[
= \text{Tr} Y \phi(\pm \Delta_X)(Y^\dagger)
\]
which implies \( \text{Tr} W \phi(\Delta_X^2)(W) + \text{Tr} Z \phi(\Delta_X^2)(Z) = \text{Tr} Y \phi(\Delta_X^2)(Y^\dagger) \).

c) We can also write
\[
\frac{1}{2}D_2[X, Y] + \frac{1}{2}D_2[X, iY] = -2S(X^2) - \text{Tr} (YY^\dagger + Y^\dagger Y) \log X^2 - \frac{1}{2} \text{Tr} Y[\phi(\Delta_X) + \phi(-\Delta_X)](Y^\dagger)
\]
\[
= -2S(X^2) - \text{Tr} (YY^\dagger + Y^\dagger Y) \log X^2 - \frac{1}{2} \text{Tr} Y \phi(\Delta_X^2)Y^\dagger.
\] (28)
d) Let $H_\pm(X) \equiv -2S(X^2)T_d - 2R_{\log X^2} - \phi(\pm \Delta X)$ with $T_d$ the identity acting on $M_d$. Then when $\text{Tr} W^2 + \text{Tr} Z^2 = 1$

$$D_2[X, W + iZ] = \text{Tr}(W, Z) \begin{pmatrix} H_+(X) & 0 \\ 0 & H_-(X) \end{pmatrix} \begin{pmatrix} W \\ Z \end{pmatrix}.$$  

(29)

As discussed in [3], the minimal output entropy is not locally additive if we restrict to vector space of matrices whose entries are real. In [29], we replace $\mathcal{K}$ by the real subspace of self-adjoint matrices in $\mathcal{K} \oplus \mathcal{K}$. The complex structure is implicit in our use of $\phi(\pm \Delta X)$ in the first block and $\phi(-\Delta X)$ in the second block of $H(X) = \begin{pmatrix} H_+(X) & 0 \\ 0 & H_-(X) \end{pmatrix}$. Then $D_2[X_c, Y] > 0$ for all $Y \in X_C^\perp$ is equivalent to the strict positivity of $H(X_c)$ on the subspace of self-adjoint matrices in $\mathcal{K} \oplus \mathcal{K} \cap \{(X_c, 0) \cup (0, X_c)\}^\perp = \{X_c^\perp \oplus X_c^\perp\}$. Let $X_c > 0$ and $D_1[X_c, W] = 0$ for all $W = W^\dagger \in \mathcal{K}^0$ so that $X_c$ is a critical point of $S(XX^\dagger)$. The strict positivity of $H(X_c)$ on the subspace of self-adjoint matrices in $X_c^\perp \oplus X_c^\perp$ means that there is a $\nu > 0$ such that $H(X_c) > \nu I_{2d-2}$ on this subspace which implies that $D_2[X_c, Y] > \nu$ for all $Y \in \mathcal{K}^0 \cap X_c^\perp$. Thus, with $X(t) = \sqrt{1-t^2}X_c + tY$ it follows from Taylor’s theorem with remainder that

$$S(X(t)X(t)^\dagger) > S(X_c^2) + \frac{1}{2} \nu t^2 - \frac{R}{6} |t|^3 \quad \forall Y \in \mathcal{K}^0 \cap X_c^\perp. \quad (30)$$

if $|\frac{d^2}{dt^2} S(X(t)X(t)^\dagger)| < R$. We show in Appendix B.3 that there is an $R > 0$ and $\tau \in (0, 1)$ such that this holds for all $t \in (-\tau, \tau)$. Thus, $S(X(t)X(t)^\dagger) > S(X_c^2)$ when $0 < |t| < \min\left\{\tau, \frac{3\nu}{R}\right\}$. Since $S(X(0)X(0)^\dagger) = S(X_c^2)$, this implies that $S(XX^\dagger)$ has a non-degenerate local minimum at $X_c$.

4 First derivative essentially additive

We now assume that $S(X^2)|_{X \in \mathcal{K}^0_B}$ and $S(X^2)|_{X \in \mathcal{K}^0_C}$ have local minima at $X_B > 0$ and $X_C > 0$ so that $D_1[X_B, Y_B] = D_1[X_C, Y_C] = 0$ for all $Y_B \in \mathcal{K}^0_B \cap X_B^\perp$ and $Y_C \in \mathcal{K}^0_C \cap X_C^\perp$. Then for all $Y_{BC} \in (\mathcal{K}_B \otimes \mathcal{K}_C)^0 \cap (X_B \otimes X_C)^\perp$

$$D_1[X_B \otimes X_C, Y_{BC}] = -\text{Tr}_{BC} \left[ (X_B \otimes X_C)Y_{BC}^\dagger + Y_{BC}(X_B \otimes X_C) \right] \log(X_B^2 \otimes X_C^2)$$

$$= -\text{Tr}_{B} \left[ X_B\tilde{Y}_B^\dagger + \tilde{Y}_B X_B \right] \log X_B^2 - \text{Tr}_{C} \left[ X_C \tilde{Y}_C^\dagger + \tilde{Y}_C X_C \right] \log X_C^2$$

$$= D_1[X_B, \tilde{Y}_B] + D_1[X_C, \tilde{Y}_C] = 0 + 0 = 0 \quad (31)$$

where $\tilde{Y}_B = \text{Tr}_{C} X_C Y_{BC}$ and $\tilde{Y}_C = \text{Tr}_{B} X_B Y_{BC}$ and, e.g., $\text{Tr}_{B} X_B \tilde{Y}_B^\dagger = \text{Tr}_{BC} (X_B \otimes X_C)Y_{BC}^\dagger = 0$.

Note that we did not find $D_1[X_B \otimes X_C, Y_{BC}] = D_1[X_B, Y_B] + D_1[X_C, Y_C]$. Instead we used the fact that the directional derivatives with respect to all allowed perturbations are zero. We will use a similar strategy to treat the second derivative.
5 Second derivative increases

5.1 Key inequality of Gour-Friedland

To simplify the notation we write $\Delta_B$ for $\Delta_{X_B}$ etc. and use $I$ to denote the identity operator acting on $d \times d$ matrices. The key result needed to treat $Q[X,Y]$ is

$$2\phi(\pm\Delta_B \otimes \Delta_C) \leq \phi(\Delta_B^2)\otimes I_C + I_B \otimes \phi(\Delta_C^2).$$

This follows immediately from (26) and the spectral theorem applied to $\Delta_B$ and $\Delta_C$. This inequality is a substitute for additivity. It implies subadditivity of $D_2[X,Y]$. The presence of either $I_B$ or $I_C$ also allows us to eliminate many cross terms in $\tilde{Q}[X,Y]$.

In what follows we will make frequent use of the fact that for $X > 0$ both $\Delta_X$ and $\phi(\Delta_X)$ are self-adjoint with respect to the Hilbert-Schmidt inner product. Moreover, $\Delta_X(X) = X$ implies that $\phi(\Delta_B \otimes \Delta_C)(X_B \otimes Y_C) = X_B \otimes \phi(\Delta_C)(Y_C)$. We will frequently combine this with self-adjointness to conclude that when $W = W^\dagger$

$$\text{Tr}_{BC} (X_B \otimes W_C)\{\phi(\Delta_B \otimes \Delta_C)(F_{BC})\}^\dagger = \text{Tr}_{BC} X_B \otimes \phi(\Delta_C)(W_C) F_{BC}^\dagger.$$

As previously observed, since $X$ is self-adjoint $\text{Tr} XY^\dagger = 0 \iff \text{Tr} XY = 0$ and similarly for $W, Z$.

We will also use the following result. When $D \in \mathcal{A} \otimes \mathcal{B}$ this is essentially the singular value decomposition of the operator that maps $A \in \mathcal{A}$ to $\text{Tr}_A A^\dagger D \in \mathcal{B}$.

**Remark 7** Let $\mathcal{A}$ and $\mathcal{B}$ be vector spaces of matrices and $D \in \mathcal{A} \otimes \mathcal{B}$. Then one can always write $D = \sum_k \omega_k A_k \otimes B_k$ with $\omega_k \in \mathbb{C}$, $A_k \in \mathcal{A}$, $B_k \in \mathcal{B}$ and $\text{Tr} A_j^\dagger A_k = \text{Tr} B_j^\dagger B_k = 0$ when $j \neq k$.

5.2 Proof

We assume that $S(XX^\dagger)|_{X \in K_B^o}$ and $S(XX^\dagger)|_{X \in K_C^o}$ have non-degenerate local minima at $X_B$ and $X_C$ respectively.

First, observe that for $X_B \in K_B^o$ and $X_C \in K_C^o$,

$$(X_B \otimes X_C)^\perp = \text{span}\{(X_B \otimes X_C^\perp) \cup (X_B^\perp \otimes X_C) \cup (X_B^\perp \otimes X_C^\perp)\}$$

where it is understood that $X_B^\perp$ and $X_C^\perp$ are subspaces of $K_B$ and $K_C$ respectively. Remark 7 implies that any matrix in $(X_B^\perp \otimes X_C^\perp)^o$ can be written in the form

$$T_{BC} = \sum_{j>0} \xi_j Y_B^j \otimes Y_C^j \quad \text{with} \quad \text{Tr} Y_B^j (Y_B^k)^\dagger = \text{Tr} Y_C^j (Y_C^k)^\dagger = \delta_{jk}.$$ 

Therefore, the most general perturbation we need to consider can be written in the form

$$Y_{BC} = u_1 X_B \otimes Y_C^0 + u_2 Y_B^0 \otimes X_C + \eta T_{BC} = u_1 X_B \otimes Y_C^0 + u_2 Y_B^0 \otimes X_C + \eta \sum_{j>0} \xi_j Y_B^j \otimes Y_C^j$$

(36)
where \( Y_B^0 \in (X_B^\perp)'^o \) and \( Y_C^0 \in (X_C^\perp)'^o \). Note, however, that \( \text{Tr} Y_B^0 (Y_B^j)^\dagger \) and \( \text{Tr} Y_B^0 (Y_B^j)^\dagger \) are not zero in general. Because \( Y_B^0 \in X_B^\perp \) and \( X > 0 \), \( \text{Tr} X_B (Y_B^j)^\dagger = \text{Tr} X_B Y_B^j = 0 \) and similarly for \( Y_C^j \). We assume that \(|u_1|^2 + |u_2|^2 + |\eta|^2 = 1\) and \( \sum_{j > 0} |\xi_j|^2 = 1 \) so that \( \text{Tr}_{BC} (Y_{BC}^j)^\dagger Y_{BC} = 1 \).

**Lemma 8** With \( Y_{BC} \) defined as in (36),

\[
D_2 [X_B \otimes X_C, Y_{BC}] = |u_1|^2 D_2 [X_B \otimes X_C, X_B \otimes Y_C^0] + |u_2|^2 D_2 [X_B \otimes X_C, Y_C^0 \otimes X_C] \\
+ |\eta|^2 D_2 [X_B \otimes X_C, T_{BC}] \\
= |u_1|^2 D_2 [X_C, Y_C^0] + |u_2|^2 D_2 [X_B, Y_B^0] + |\eta|^2 D_2 [X_B \otimes X_C, T_{BC}] 
\]

(37)

(38)

**Proof:** We first consider the cross terms in \( \bar{Q}[X_B \otimes X_C, Y_{BC}] \). Observe that

\[
W_{BC} = \frac{1}{2} (Y_{BC} + Y_{BC}^\dagger) = u_1 X_B \otimes W_C^0 + u_2 W_B^0 \otimes X_C + \frac{1}{2} (T_{BC} + T_{BC}^\dagger)
\]

The cross-term with coefficient \( u_1 \bar{\eta}_2 \) is

\[
\text{Tr}_{BC} X_B \otimes W_C^0 \phi(\Delta_B \otimes \Delta_C) W_B^0 \otimes X_C = \text{Tr}_{BC} X_B W_B^0 \otimes \phi(\Delta_C)(W_C^0) X_C = 0
\]

where we used (33) and \( \text{Tr} B X_B W_B^0 = 0 \). Similarly, we find

\[
\text{Tr}_{BC} (X_B \otimes W_C^0)[\phi(\Delta_B \otimes \Delta_C)(Y_B^j \otimes Y_C^j)]^\dagger = \text{Tr}_{BC} X_B (Y_B^j)^\dagger \otimes \phi(\Delta_C)(Y_C^j)^\dagger = 0
\]

(39)

which is easily seen to imply that \( \text{Tr}_{BC} (X_B \otimes W_C^0) \phi(\Delta_B \otimes \Delta_C) W_{BC}^j = 0 \). Similar arguments hold with \( Z_{BC} \). Thus we conclude that all cross terms in \( Q(X_B \otimes X_C, Y_{BC}) \) with coefficient \( u_1 \bar{\eta}_2, u_1 \bar{\eta}, u_2 \bar{\eta} \) and their conjugates are zero.

Next we consider cross terms arising from \( \text{Tr} (Y_{BC} Y_{BC}^\dagger + Y_{BC}^\dagger Y_{BC}) \log(X_B^2 \otimes X_C^2) \). Those with coefficient \( u_1 \bar{\eta}_2 \) are easily seen to be zero. Those involving \( u_1 \bar{\eta}_2, u_1 \bar{\eta}, u_2 \bar{\eta} \) are trickier. Let

\[
Y_1 = u_1 \bar{\eta} (X_B \otimes Y_C^0)(\xi_j Y_B^j \otimes Y_C^j)^\dagger + \bar{\eta}_1 \bar{\eta} (\xi_j Y_B^j \otimes Y_C^j)(X_B \otimes Y_C^0)^\dagger \\
Y_2 = u_1 \bar{\eta} (\xi_j Y_B^j \otimes Y_C^j)^\dagger (X_B \otimes Y_C^0)^\dagger + \bar{\eta}_1 \bar{\eta} (X_B \otimes Y_C^0)^\dagger (\xi_j Y_B^j \otimes Y_C^j)^\dagger
\]

Then we find

\[
\text{Tr}_{BC} Y_1 \log(X_B^2 \otimes X_C^2) \\
= \text{Tr}_{BC} [u_1 \bar{\eta}(X_B \otimes Y_C^0)(\xi_j Y_B^j \otimes Y_C^j)^\dagger + \bar{\eta}_1 \bar{\eta}(\xi_j Y_B^j \otimes Y_C^j)(X_B \otimes Y_C^0)^\dagger] \log X_B^2 \\
+ \text{Tr}_{C} ((\text{Tr}_{BC} X_B Y_B^j)u_1 \bar{\eta} Y_C^0)(Y_C^j)^\dagger + (\text{Tr}_{BC} Y_B^j X_B) \bar{\eta}_1 \bar{\eta} Y_C^0 (Y_C^j)^\dagger \log X_C^2
\]

(40a)

(40b)

The last term (40b) is zero because \( \text{Tr}_{BC} X_B Y_B^j = \text{Tr}_{BC} X_B (Y_B^j)^\dagger = 0 \). If we let \( \omega = \text{Tr}_{BC} \bar{\xi}_j Y_C^0 (Y_C^j)^\dagger \) and \( \bar{Y}_B = \bar{\eta}_1 \bar{\eta} \bar{\omega} Y_B^j \), the first term (40a) can be rewritten as

\[
\text{Tr}_B (X_B u_1 \bar{\eta} \omega (Y_B^j)^\dagger + \bar{\eta}_1 \bar{\eta} \bar{\omega} Y_B^j X_B) \log X_B^2 = D_1 [X_B, \bar{Y}_B] = 0
\]
since $X_B$ is a critical point. Because, e.g., $\text{Tr}_C Y_C^0(Y_C^j)\dagger = \text{Tr}_C (Y_C^j)\dagger Y_C^0$, one similarly finds $\text{Tr}_{BC} Y_2 \log (X_B^2 \otimes X_C^2) = D_1[X_B, Y_B] = 0$ so that the total contribution from these cross-terms is $\text{Tr}_{BC}(Y_1 + Y_2) \log (X_B^2 \otimes X_C^2) = 0$. Note that the cross-terms with coefficient $u_1 \eta$ need not be zero. They must be combined with those with coefficient $\overline{\eta} u_1$ and, moreover, $u_1$ and $\eta$ are needed in the definition of $\overline{Y}_B$. The corresponding cross terms involving the term with coefficient $u_2$ can be handled in the same way. Thus, all cross terms arising from $\text{Tr} (Y_B BC Y_{BC}^0 + Y_{BC}^0 Y_B) \log (X_B^2 \otimes X_C^2)$ are zero.

To complete the proof let $C(t) = \sqrt{1 - t^2}X_C + tY_0^0$ and observe that the entropy associated with the first term is $S(X_B^2 \otimes C(t)^\dagger C(t)) = S(X_B^2) + S(C(t)^\dagger C(t))$ so that $D_2[X_B \otimes X_C, X_B \otimes Y_0^0] = D_2[X_C, Y_0^0]$. A similar result holds for the second term which gives (38). QED.

We now consider $D_2[X_B \otimes X_C, T_{BC}]$ and write $T_{BC} = W_{BC} + iZ_{BC}$ as before. Then (32) implies

$$2 \text{Q}(\Delta_B \otimes \Delta_C, T_{BC}) = 2 \text{Tr}_{BC} W_{BC} \phi(\Delta_B \otimes \Delta_C) W_{BC} + 2 \text{Tr}_{BC} Z_{BC} \phi(-\Delta_B \otimes \Delta_C) Z_{BC} \leq \text{Tr}_{BC} W_{BC} \phi(\Delta_B^2 \otimes I_C) W_{BC} + \text{Tr}_{BC} W_{BC} (I_B \otimes \phi(\Delta_C^2)) W_{BC} + \text{Tr}_{BC} Z_{BC} (I_B \otimes \phi(\Delta_C^2)) Z_{BC} = \text{Tr}_{BC} \text{Tr}_C (\phi(\Delta_B^2 \otimes I_C) T_{BC} + \text{Tr}_{BC} T_{BC} (I_B \otimes \phi(\Delta_C^2)) T_{BC} = \sum_{j>0} |\xi_j|^2 \left( \text{Tr}_B Y_B^j \phi(\Delta_B^2)(Y_B^j)^\dagger + \text{Tr}_C Y_C^j \phi(\Delta_C^2)(Y_C^j)^\dagger \right) \right)$$

(41)

where the last two lines follow from Remark 6b and

$$\text{Tr}_{BC} \text{Tr}_C (\phi(\Delta_B^2 \otimes I_C) T_{BC}^\dagger = \sum_{j>0} \sum_{k>0} \xi_j \xi_k \text{Tr}_{BC} Y_B^j \otimes Y_C^k (\phi(\Delta_B^2 \otimes I_C))(Y_B^k \otimes Y_C^j)^\dagger$$

$$= \sum_{j>0} |\xi_j|^2 \text{Tr}_B Y_B^j \phi(\Delta_B^2)(Y_B^j)^\dagger$$

since $\text{Tr}_C Y_C^j (Y_C^k)^\dagger = \delta_{jk}$. A similar argument holds for the term with $I_B \otimes \phi(\Delta_C^2)$.

We similarly find that since $\log (X_B^2 \otimes X_C^2) = \log (X_B^2) \otimes I_C + I_B \otimes \log(X_C^2)$

$$\text{Tr}_{BC} (Y_B BC Y_{BC}^+ + Y_{BC}^+ Y_B) \log (X_B^2 \otimes X_C^2) = \sum_{j>0} |\xi_j|^2 \text{Tr}_B (Y_B^j(Y_B^j)^\dagger + (Y_B^j)^\dagger Y_B^j) \log X_B^2 + \sum_{j>0} |\xi_j|^2 \text{Tr}_C (Y_C^j(Y_C^j)^\dagger + (Y_C^j)^\dagger Y_C^j) \log X_C^2$$

(42)

Combining (41) and (42) and noting $S(X_B^2 \otimes X_C^2) = \sum_{j>0} |\xi_j|^2 [S(X_B^2) + S(X_C^2)]$ we find

$$D_2[X_B \otimes X_C, T_{BC}] \geq$$

$$\sum_{j>0} |\xi_j|^2 \left( -2S(X_B^2) + \text{Tr}_B [Y_B^j(Y_B^j)^\dagger + (Y_B^j)^\dagger Y_B^j] \log X_B^2 - \frac{1}{2} \text{Tr}_B Y_B^j \phi(X_B^2)(Y_B^j)^\dagger \right)$$

$$+ \sum_{j>0} |\xi_j|^2 \left( -2S(X_C^2) + \text{Tr}_C [Y_C^j(Y_C^j)^\dagger + (Y_C^j)^\dagger Y_C^j] \log X_C^2 - \frac{1}{2} \text{Tr}_C Y_C^j \phi(X_C^2)(Y_C^j)^\dagger \right)$$

$$= \frac{1}{2} \sum_{j>0} |\xi_j|^2 \left( D_2[X_B, Y_B^j] + D_2[X_B, iY_B^j] + D_2[X_C, Y_C^j] + D_2[X_C, iY_C^j] \right)$$

(43)
where we used (28). Combining (38) and (44) we find that

\[
D_2[X_B \otimes X_C, Y_{BC}] \geq |u_1|^2 D_2[X_C, Y_C^0] + |u_2|^2 D_2[X_B, Y_B^0] + |\eta|_2 \sum_{j>0} |\xi_j|^2 D_2[X_B, iY_B^j] + D_2[X_C, Y_C^j] + D_2[X, Y_C^j] \geq 0
\]  
(45)

with strict inequality if both minima are non-degenerate.

This completes the proof of Theorem 2 when \(X\) is square and nonsingular. As observed at the end of Section III of [3], this argument also shows that if one of the local minima is non-degenerate, and the other degenerate, then the product is a degenerate local minimum.

We emphasize again that we do not compare \(D_2[X_B \otimes X_C, Y_{BC}]\) to \(D_2[X_B, Y_B] + D_2[X_C, Y_C]\). Instead we use the fact that the second derivative is positive for arbitrary perturbations in \(K_B\) and \(K_C\) so that we can bound the second derivative by a sum of positive terms.

6 Reduction to square non-singular form

6.1 \(XX^\dagger\) or \(X^\dagger X\) non-singular

As in Section 2.2, \(P_B\) and \(P_E\) are the projections onto \((\ker XX^\dagger)^\perp\) and \((\ker X^\dagger X)^\perp\) respectively so that \(X = P_B XP_E\). We can write \(X\) and \(Y\) in block form as

\[
\begin{pmatrix}
X_{11} & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{pmatrix}
\]

where we identify \(X_{11} \simeq P_B XP_E, Y_{11} \simeq P_B Y P_E, Y_{12} \simeq P_B Y P_{E^\perp}, Y_{21} \simeq P_{B^\perp} Y P_E, Y_{22} \simeq P_{B^\perp} Y P_{E^\perp}\). If \(XX^\dagger\) is non-singular so that \(P_B = I\), we can write \(X = (X_{11} \ 0), Y = (Y_{11} \ Y_{12})\) and observe that \(\Gamma_1 = X_{11} Y_{11}^\dagger + Y_{11} X_{11}^\dagger\). Thus, \(Y_{12}\) has no effect on \(D_1[X, Y]\) and its only contribution to the second derivative is \(-\text{Tr} Y_{12} Y_{12}^\dagger \log XX^\dagger \geq 0\). Since non-zero \(Y_{12}\) can only increase the second derivative and, hence, the entropy, there is no loss of generality in assuming that \(Y_{12} = 0\), i.e., it suffices to consider \(Y \in P_B K P_E\).

If \(X^\dagger X\) is non-singular, then we can exploit the fact that \(S(XX^\dagger) = S(X^\dagger X)\) and use the same argument with \(X^\dagger = (X_{11}^\dagger \ 0), Y^\dagger = (Y_{11}^\dagger \ Y_{12}^\dagger)\) to show that there is no loss of generality in assuming that \(Y_{21} = 0\). Thus, it again suffices to consider \(Y \in P_B K P_E\).

6.2 General case

To deal with the general case, we assume \(d_E \geq d_B\) and replace \(X\) by \(X_\epsilon = (1 - \epsilon^2 P_{B^\perp})^{1/2} X_{11}\). As discussed in Section 2.2, there is no loss of
generality in assuming that $X_{11}$ is positive definite.

$$XX^\dagger = \begin{pmatrix} X_{11}^2 & 0 \\ 0 & \epsilon^2 P_B \end{pmatrix}$$

$$\log XX^\dagger = \begin{pmatrix} \log X_{11}^2 & 0 \\ 0 & P_B \log \epsilon^2 \end{pmatrix}$$

$$\Gamma_1 = \begin{pmatrix} X_{11}Y_{11}^\dagger + Y_{11}X_{11} & X_{11}Y_{21}^\dagger + \epsilon Y_{12}F^\dagger \\ Y_{21}X_{11} + \epsilon FY_{12}^\dagger & \epsilon (FY_{22}^\dagger + 2Y_{22}F^\dagger) \end{pmatrix}$$

It is straightforward to see that as $\epsilon \to 0$

$$D_1[X,Y] = - \text{Tr} (X_{11}Y_{11}^\dagger + Y_{11}X_{11}) \log X_{11}^2 + \text{Tr} FY_{22}^\dagger + Y_{22}F^\dagger \epsilon \log \epsilon^2$$

$$\to \text{Tr} (X_{11}Y_{11}^\dagger + Y_{11}X_{11}) \log X_{11}^2 = D[X_{11}, Y_{11}]$$

Thus, only $Y_{11}$ affects the first derivative.

The treatment of the second derivative is more complex. The terms $R[X,Y]$ become

$$R[X,Y] = -2 \text{Tr} (Y_{11}Y_{11}^\dagger + Y_{12}Y_{12}^\dagger) \log X_{11}^2 - 2 \text{Tr} (Y_{21}Y_{21}^\dagger + Y_{22}Y_{22}^\dagger) \log \epsilon^2$$

$$- 2S(X_{11}^2) + 2 \epsilon^2 \log \epsilon^2 \text{Tr} P_B^\perp \tag{46}$$

and

$$Q[X,Y] = \text{Tr} \int_0^\infty (X_{11}Y_{11}^\dagger + Y_{11}X_{11}) \frac{P_B}{X_{11}^2 + uP_B} (X_{11}Y_{11}^\dagger + Y_{11}X_{11}) \frac{P_B}{X_{11}^2 + uP_B} du \tag{47a}$$

$$+ 2 \text{Tr} \int_0^\infty (X_{21}Y_{21}^\dagger + \epsilon Y_{12}F^\dagger) \frac{P_B}{P_B^\perp (\epsilon^2 + u)} (Y_{21}X_{11} + \epsilon FY_{12}^\dagger) \frac{P_B}{X_{11}^2 + uP_B} du \tag{47b}$$

$$+ \text{Tr} \int_0^\infty (\epsilon (FY_{22}^\dagger + 2Y_{22}F^\dagger) \frac{P_B}{\epsilon^2 + u} (FY_{22}^\dagger + 2Y_{22}F^\dagger) \frac{P_B}{\epsilon^2 + u} du \tag{47c}$$

One can omit $P_B$ and $P_B^\perp$ when they are sandwiched between matrices which are invariant under $P_B$ and $P_B^\perp$ respectively.

The terms in (46) and (47a) involving $Y_{11}$ converge to $D_2[X_{11}, Y_{11}]$ when $\epsilon \to 0$.

To analyze (47b) observe that

$$\int_0^\infty \frac{1}{\epsilon^2 + u \frac{X_{11}^2}{X_{11}^2 + uP_B}} du = \frac{P_B}{X_{11}^2} \left( \log X_{11}^2 - \log \epsilon^2 \right). \tag{48}$$

Therefore the terms with $\epsilon Y_{12} \to 0$ as $\epsilon \to 0$ so that the only contribution of $Y_{12}$ to $D_2[X,Y]$ is $- \text{Tr} Y_{12}Y_{12}^\dagger \log X^2$ which is positive. Although the terms with $Y_{21}$ in (46) and (47b) are not well-behaved, we can exploit the fact that $S(XX^\dagger) = S(X^\dagger X)$. Since the exchange $X \leftrightarrow X^\dagger$ gives $Y_{12} \leftrightarrow Y_{21}^\dagger$ we find that, as above, the only contribution of $Y_{21}$ to $D_2[X,Y]$ is $- \text{Tr} Y_{21}^\dagger Y_{21} \log X^2$ which is positive. Hence, there is no loss of generality in assuming that $Y_{12}$ and $Y_{21}$ are both zero.

To treat $Y_{22}$, observe that the eigenvalues of $\begin{pmatrix} X_{11}^2 & 0 \\ 0 & 0 \end{pmatrix}$ majorize those of $\begin{pmatrix} \xi X_{11}^2 & 0 \\ 0 & \epsilon^2 Y_{22}Y_{22}^\dagger \end{pmatrix}$ when $\epsilon$ is sufficiently small and $\xi = (1 - \epsilon^2 \text{Tr} Y_{22}^\dagger Y_{22})^{1/2}$. Thus perturbations in which $Y_{22}$ is the
only non-zero block always increase the entropy. Since we showed above that we can assume that \( Y \) is block diagonal, this suffices to conclude that non-zero \( Y_{22} \) can only increase the entropy. (We could observe that the contribution of \( Y_{22} \) to \( Q[X,Y] \) above is completely decoupled from those of \( Y_{12} \) and \( Y_{21} \) so that there is no loss of generality in assuming that \( Y \) is block diagonal.)

Thus, we have shown that if \( D_2[X_{11}, Y_{11}] \) is positive for all \( Y_{11} \in P_BKP_E \), then \( D_2[X,Y] \) is positive for all \( Y \in K \). This completes the proof of Theorem 3.

6.3 Refined treatment of (47b) and (47c)

Although not needed for the proof, we can say more about the divergent terms in \( D_2[X,Y] \).

First, observe that using (48) in (47b) with \( Y_{12} = 0 \) and ignoring terms of order \( \epsilon^2 \) implies that the contribution of \( Y_{21} \) to \( Q[X,Y] \) is

\[
2 \text{Tr} X \epsilon Y_{21}^\dagger Y_{21} X (X_\epsilon^{-1} \log X_\epsilon^2 - \log \epsilon^2) = -2 \text{Tr} Y_{21}^\dagger Y_{21} \log \epsilon^2 + 2 \text{Tr} Y_{21}^\dagger Y_{21} \log X_\epsilon^2 \tag{49}
\]

so that the net contribution of \( Y_{21} \) to \( D_2[X,Y] = R[X,Y] - Q[X,Y] \) is

\[
-2 \text{Tr} Y_{21}^\dagger Y_{21} \log \epsilon^2 + 2 \text{Tr} Y_{21}^\dagger Y_{21} \log \epsilon^2 - 2 \text{Tr} Y_{21}^\dagger Y_{21} \log X_\epsilon^2 \quad \rightarrow \quad -2 \text{Tr} Y_{21}^\dagger Y_{21} \log X_\epsilon^2 , \tag{50}
\]

which is exactly what we found by considering \( S(X_\epsilon^1 X) \).

Since \( F^\dagger P_B \perp F = F^\dagger F \), the term (47c) can be written as

\[
\text{Tr} (FY_{22}^\dagger + Y_{22}F^\dagger)^2 \int_0^\infty \frac{\epsilon^2}{(\epsilon^2 + u)^2} du = \text{Tr} (FY_{22}^\dagger + Y_{22}F^\dagger)^2
\]

where we used \( \int_0^\infty \frac{\epsilon^2}{(\epsilon^2 + u)^2} du = -\epsilon^2 \frac{1}{\epsilon^2 + u} \bigg|_0^\infty = 1 \). This term does not \( \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). However, since the term (46) contains \( -2 \text{Tr} Y_{22}^\dagger Y_{22}^\dagger \log \epsilon^2 \) which diverges to \( +\infty \) we again conclude that non-zero \( Y_{22} \) must increase the second derivative (and, hence, the entropy). This also proves Theorem 7 of [3] which says that \( D_2[X,Y] \) diverges to \( +\infty \) if and only if \( Y_{22} \) is non-zero.

7 Extension to relative entropy

7.1 Local additivity

The relative entropy is defined (when ker \( \omega \subseteq \text{ker} \rho \)) as

\[
H(\rho, \omega) = \text{Tr} \rho (\log \rho - \log \omega) = -S(\rho) - \text{Tr} \rho \log \omega \tag{51}
\]

The arguments above are easily generalized to show that the maximization of relative entropy with respect to a fixed reference state is locally additive because

\[
\frac{d}{dt} H(\rho(t), \omega) = -\frac{d}{dt} S[\rho(t)] - \text{Tr} \rho'(t) \log \omega \tag{52}
\]
and
\[
\frac{d^2}{dt^2} H(\rho(t), \omega) = \frac{d^2}{dt^2} S[\rho(t)] - \text{Tr} \rho''(t) \log \omega.
\] (53)

Since the terms involving \( \log \omega \) are additive and the others come from differentiating \( S(\rho) \), the following result is a straightforward corollary to our results for the minimal output entropy.

**Theorem 9** Let \( \Phi_A : M_{da} \mapsto M_{db} \) and \( \Phi_C : M_{dc} \mapsto M_{dC} \) be quantum channels, and let the states \( \omega_A, \omega_C \) be fixed. If \( H[\Phi_A(\rho_A), \Phi_A(\omega_A)] \) and \( H[\Phi_C(\rho_C), \Phi_C(\omega_C)] \) have non-degenerate local maxima at \( \rho_A = |\psi_A \rangle \langle \psi_A| \) and \( \rho_C = |\psi_C \rangle \langle \psi_C| \), then \( H[\Phi_A \otimes \Phi_C(\rho_{AC}), (\Phi_A \otimes \Phi_C)(\omega_A \otimes \omega_C)] \) has a local maximum at \( \rho_{AC} = |\psi_A \otimes \psi_C \rangle \langle \psi_A \otimes \psi_C| \).

### 7.2 Capacity of a quantum channel

This result is of particular interest in studying the additivity of Holevo capacity which describes the capacity of a quantum channel to transmit classical information using product inputs. This is defined as

\[
C_\text{Holv}(\Phi) = \sup_{\rho_j, \rho_A} \left( S(\Phi(\rho_A)) - \sum_j \pi_j S(\Phi(\rho_j)) \right) = \sup_{\pi_j, \rho_j} \chi(\{\pi_j, \Phi(\rho_j)\})
\] (54)

where \( \rho_A = \sum_j \pi_j \rho_j \) and \( \chi(\{\pi_j, \rho_j\}) = S(\rho_A) - \sum_j \pi_j S(\rho_j) \) and \( \rho_j = |\psi_j \rangle \langle \psi_j| \) is a pure state. The strict concavity of \( S(\rho) \) implies that the optimal output average \( \Phi(\rho_A) \) is unique.

The optimization over ensembles can be replaced by the following max-min expression [6, 8]

\[
C_\text{Holv}(\Phi) = \min_\gamma \max_\rho H[\Phi(\rho), \Phi(\gamma)]
\] (55)

\[
= \max_\rho H[\Phi(\rho), \Phi(\rho_A)]
\] (56)

where \( \rho, \gamma \) are density matrices. It follows from (56) that the outputs \( \Phi(\rho_j) \) in any ensemble that optimizes (54) are “equi-distant” from the optimal average output \( \Phi(\rho_A) \) since they satisfy \( H[\Phi(|\psi_j \rangle \langle \psi_j|), \Phi(\rho_A)] = C_\text{Holv}(\Phi) \). It also follows from (56) that \( H[\Phi(\rho), \Phi(\rho_A)] \) has a (possibly degenerate) local maximum at each input \( \rho_j = |\psi_j \rangle \langle \psi_j| \) in an ensemble that maximizes (54).

Moreover, this max-min expression gives an important criterion for superadditivity which we state only in the simplest case.

**Theorem 10** A channel \( \Phi : M_d \mapsto M_d \) satisfies \( C_\text{Holv}(\Phi \otimes \Phi) > 2C_\text{Holv}(\Phi) \) if and only if there exists a \( |\Psi\rangle \in \mathcal{C}_d \otimes \mathcal{C}_d \) such that

\[
H[(\Phi \otimes \Phi)(|\Psi \rangle \langle \Psi|), (\Phi \otimes \Phi)(\rho_A \otimes \rho_A)] > 2C_\text{Holv}(\Phi).
\] (57)

**Proof:** First, observe that superadditivity and (55) imply

\[
2C_\text{Holv}(\Phi) < C_\text{Holv}(\Phi \otimes \Phi) = \max_\Psi H[(\Phi \otimes \Phi)(|\Psi \rangle \langle \Psi|), (\Phi \otimes \Phi)(\Gamma_{AV})] \leq \max_\Psi H[(\Phi \otimes \Phi)(|\Psi \rangle \langle \Psi|), (\Phi \otimes \Phi)(\rho_A \otimes \rho_A)]
\]
where the max is taken over $|\Psi\rangle \in C_d \otimes C_d$ and $\Gamma_{\text{Av}}$ is the true optimal average input for $\Phi \otimes \Phi$.

To prove the converse recall that the optimal average output $(\Phi \otimes \Phi)(\Gamma_{\text{Av}})$ is unique and observe that one can always achieve $2C_{\text{Holv}}(\Phi)$ with a “product ensemble” $\chi(\{\pi_j \pi_k, (\Phi \otimes \Phi)(\rho_j \otimes \rho_k)\}) = 2C_{\text{Holv}}(\Phi)$ for which the optimal average input is $\rho_{\text{Av}} \otimes \rho_{\text{Av}}$. If this is not the true $\Gamma_{\text{Av}}$ then the supremum in (54) must be $> 2C_{\text{Holv}}(\Phi)$. On the other hand if $\rho_{\text{Av}} \otimes \rho_{\text{Av}} = \Gamma_{\text{Av}}$ then (56) and (57) imply superadditivity. QED

7.3 Implications for numerical work

Theorem 10 implies that one can demonstrate superadditivity of the capacity without the need to find either the capacity $C_{\text{Holv}}(\Phi \otimes \Phi)$ or the optimal average input $\Gamma_{\text{Av}}$. It suffices to show that (57) holds. If $\rho_{\text{Av}}$ and $C_{\text{Holv}}(\Phi)$ for a single use of the channel have been determined, the additional numerical effort is comparable to showing that $S_{\text{min}}(\Phi)$ is not additive.

Before Hastings’ work, Shor [9] had shown the equivalence of several additivity conjectures including those for both $C_{\text{Holv}}(\Phi)$ and $S_{\text{min}}(\Phi)$. Thus, Hastings’ breakthrough also implies superadditivity of $C_{\text{Holv}}(\Phi)$. To find explicit examples of channels which violate additivity, attention focussed on $S_{\text{min}}(\Phi)$ as the simplest. But subsequent work suggests that violations sufficiently large to be observed requires very large dimensions. (See [2] and endnotes to [1, Chapter 8].) It is, however, plausible that examples for superadditivity of $C_{\text{Holv}}(\Phi)$ can be found in lower dimensions.

If $|\psi_j\rangle\langle\psi_j|$ are optimal inputs for $H[\Phi(\rho), \Phi(\rho_{\text{Av}})]$, then $H[(\Phi \otimes \Phi)(\Gamma), (\Phi \otimes \Phi)(\rho_{\text{Av}} \otimes \rho_{\text{Av}})]$ will always have local maxima at $\Gamma = |\psi_j \otimes \psi_k\rangle\langle\psi_j \otimes \psi_k|$. If $C_{\text{Holv}}(\Phi)$ is superadditive, There will also be at least one entangled input $|\Psi_{\text{max}}\rangle$ for which there is a local maximum satisfying (57). Although finding this $|\Psi_{\text{max}}\rangle$ might seem daunting in view of the many local maxima at products $|\psi_j \otimes \psi_k\rangle$, that knowledge might also help to begin the search with highly entangled inputs designed to avoid those products.

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A Proof of key inequality

We present two proofs of the key inequality (26), i.e., $2\phi(ab) \leq \phi(a^2) + \phi(b^2)$ with $\phi(a) = \frac{a+1}{a-1} \log a^2$ as in (24). Before doing so, we make some observations useful in both arguments.

a) $a > 0$ implies $\phi(-a) \leq \phi(a)$ so that it suffices to prove (26) for $a, b > 0$.

b) It is straightforward to verify that $\phi(x)$ is continuous on $(0, \infty)$ with a minimum at $x = 1$ satisfying $\phi(1) = 4$.

c) $\phi(a) = \phi(a^{-1})$ and equality holds in (26), when $a = b$.

Elementary proof: We now assume $a, b > 0$ and observe that (26) is equivalent to

$$\frac{ab + 1}{ab - 1} [\log a + \log b] \leq \frac{a^2 + 1}{a^2 - 1} \log a + \frac{b^2 + 1}{b^2 - 1} \log b$$

which is equivalent to

$$\frac{b - a}{ab - 1} [\zeta(a) - \zeta(b)] \geq 0 \quad \text{with} \quad \zeta(x) = \frac{x \log x}{x^2 - 1}.$$  \hspace{1cm} (A.1)

Now observe that $\zeta(x) = \zeta(x^{-1})$ and $\zeta'(x) = \frac{1}{x^2-1} \left(1 - \frac{1}{2} \phi(x^2)\right)$. Thus, when $x > 1$, $\zeta'(x) < 0$ and $\zeta$ is strictly decreasing for $x > 1$. We can assume $b > a$ so that it suffices to consider three cases:

- When $1 < a < b$, this implies $\zeta(b) < \zeta(a)$ so that (A.1) holds.
- When $a < 1 < b$ and $ab > 1$, then $1 < a^{-1} < b$ and $\zeta(b) < \zeta(a^{-1}) = \zeta(a)$ so that (A.1) holds.
- When $a < 1 < b$ and $ab < 1$, then $1 < b < a^{-1}$ so that $\zeta(b) > \zeta(a^{-1}) = \zeta(a)$ which again implies (A.1).

Combining this with the continuity of $\phi(a)$ at $a = 1$ proves (26).

QED

Convexity Proof: Let $\chi(x) = \phi(e^x) = 2x \frac{e^{x+1}}{e^x - 1}$. Then $\chi(\log a) = \phi(a)$ and (26) is equivalent to

$$\chi(\log ab) = \chi\left(\frac{x}{2} \log a^2 + \frac{x}{2} \log b^2\right) \leq \frac{1}{2} \chi(\log a^2) + \frac{1}{2} \chi(\log b^2)$$  \hspace{1cm} (A.2)

which holds if $\chi(x)$ is convex. One can verify that

$$\chi'(x) = \frac{\sinh(x) - x}{\sinh^2(x/2)}$$

$$\chi''(x) = \frac{x \coth(x/2) - 2}{\sinh^2(x/2)} = \frac{\chi(x) - 4}{2 \sinh^2(x/2)}.$$

Observation (b) above implies $\chi(x) \geq \chi(0) = 4$ on $\mathbb{R}$ which implies that $\chi''(x) \geq 0$. Thus $\chi(x)$ is convex and (A.2) holds.

QED
B Derivative formulas involving \( \log \rho(t) \)

B.1 Derivative of \( \log \rho(t) \)

Let \( \rho(t) \) be a one parameter family of density matrices twice differentiable in some neighborhood \( \mathcal{N}(0) \) of \( t = 0 \) so that for \( t_1 \in \mathcal{N}(0) \) there is a neighborhood \( \mathcal{N}(t_1) \subset \mathcal{N}(0) \) in which

\[
\rho(t) = \rho(t_1) + (t - t_1) \rho'(t_1) + O(t - t_1)^2 \tag{B.1}
\]

and \( \rho(t) \) has full rank. To find the derivative of \( \log \rho(t) \) we begin with the integral representation

\[
\log \rho = \lim_{M \to \infty} \int_0^M \left( \frac{1}{1 + u} - \frac{1}{\rho + u} \right) du \tag{B.2}
\]

which is valid when \( \rho \) has full rank. Then

\[
\log \rho(t_2) - \log \rho(t_1) = \int_0^\infty \left( \frac{1}{\rho(t_1) + uI} - \frac{1}{\rho(t_2) + uI} \right) du
\]

\[
= \int_0^\infty \frac{1}{\rho(t_1) + uI} \left[ \rho(t_2) - \rho(t_1) \right] \frac{1}{\rho(t_2) + uI} du
\]

\[
= (t_2 - t_1) \int_0^\infty \frac{1}{\rho(t_1) + uI} \rho'(t_1) \frac{1}{\rho(t_2) + uI} du + O(t_2 - t_1)^2
\]

\[
= (t_2 - t_1) \int_0^\infty \frac{1}{\rho(t_1) + uI} \rho'(t_1) \frac{1}{\rho(t_1) + uI} du + O(t_2 - t_1)^2 \tag{B.3}
\]

where we used \( A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \). Then

\[
\frac{d}{dt} \log \rho \bigg|_{t=t_1} \equiv \lim_{t_2 \to t_1} \frac{\log \rho(t_2) - \log \rho(t_1)}{t_2 - t_1}
\]

\[
= \int_0^\infty \frac{1}{\rho(t_1) + uI} \rho'(t_1) \frac{1}{\rho(t_1) + uI} du \tag{B.4}
\]

B.2 Identity (17) for the modular operator \( \Delta_\rho = L_\rho R_\rho^{-1} \)

Next, observe that we can use (B.2) to show that

\[
(L_{\log \rho} - R_{\log \rho})(\Gamma) = \int_0^\infty \left[ \frac{\Gamma}{\rho + uI} - \frac{1}{\rho + uI} \Gamma \right] du
\]

\[
= \int_0^\infty \frac{1}{\rho + uI} \left[ \rho \Gamma - \Gamma \rho \right] \frac{1}{\rho + uI} du
\]

\[
= (L_\rho - R_\rho) \int_0^\infty \frac{1}{\rho + uI} \Gamma \frac{1}{\rho + uI} du . \tag{B.5}
\]

Then dividing by \( L_\rho - R_\rho \) and using \( L_{\log \rho} = \log L_\rho \) gives (17) with \( \Delta_\rho = L_\rho R_\rho^{-1} \).

Since \( \frac{\log \rho}{L_\rho - R_\rho} \) can be defined by continuity at \( x = 1 \), \( \frac{\log \Delta}{L_\rho - R_\rho}(W) = R_\rho^{-1} \frac{\log \Delta}{L_\rho - R_\rho}(W) = \rho^{-1}W \) when \( L_\rho W = R_\rho W \).
B.3 Bound on third derivative

Let \( L(t) = \log \rho(t) \). Then it follows immediately from Eqs. (5) and (8) in Section 3.1 that

\[
- \frac{d^3}{dt^3} S[\rho(t)] = \text{Tr} \rho'''(t) L(t) + 2 \text{Tr} \rho''(t) L'(t) + \text{Tr} \rho'(t) L''(t) \tag{B.6}
\]

\[
= \text{Tr} \rho'''(t) \log \rho(t) + 2 \text{Tr} \rho''(t) \int_0^\infty \frac{1}{\rho(t) + uI} \rho'(t) \frac{1}{\rho(t) + uI} du \\
+ \text{Tr} \rho'(t) \frac{d}{dt} \int_0^\infty \frac{1}{\rho(t) + uI} \rho'(t) \frac{1}{\rho(t) + uI} du. \tag{B.7}
\]

When \( \rho(t) \) is given by (9), \( \rho(0) = X^2 \) is strictly positive definite. Therefore, one can find an \( \alpha > 0 \) and a \( \tau \in (0,1) \) such that \( t \in (-\tau, \tau) \) implies that \( I_d > \rho(t) > \alpha_0 \), which implies that one can find \( \{A_k : k = 0, 1, 2\} \) such that

\[
|\text{Tr} \log \rho(t)| < A_0, \quad \text{Tr} \rho^{-1}(t) < A_1, \quad \text{Tr} \rho^{-2}(t) < A_2 \tag{B.8}
\]

for all \( t \in (-\tau, \tau) \).

Next, observe that \( \text{Tr} X^2 = \text{Tr} YY^\dagger = 1 \), implies \( \|X\| = \text{Tr} |X| \leq 1 \) and \( \|Y\| = \text{Tr} |Y| \leq 1 \). Since the derivatives are given by

\[
\rho^{(1)}(t) = \rho'(t) = 2t(Y Y^\dagger - X^2) + \frac{1 - 2t^2}{\sqrt{1 - t^2}}(XY^\dagger + YX)
\]

\[
\rho^{(2)}(t) = \rho''(t) = 2(Y Y^\dagger - X^2) + \frac{2(2t^3 - 3)}{(1 - t^2)^{3/2}}(XY^\dagger + YX)
\]

\[
\rho^{(3)}(t) = \rho'''(t) = \frac{-3}{(1 - t^2)^{5/2}}(XY^\dagger + YX)
\]

and \( 0 < \tau < 1 \), one can find \( \{B_k > 0 : k = 1, 2, 3\} \) such that \( |\rho^{(k)}(t)| < B_k \) on \( (-\tau, \tau) \).

Then it is straightforward to show that there is an \( R > 0 \) such that

\[
\left| \frac{d^3}{dt^3} S[\rho(t)] \right| < R \quad \forall \ t \in (-\tau, \tau). \tag{B.10}
\]

In particular, the magnitudes of the first and second terms in the third derivative are bounded by \( B_3A_0 \) and \( 2B_1B_2A_1 \) respectively, while that for the third term is bounded by \( B_1B_2A_1 + 2B_3^2A_2 \).
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