The glueball Regge trajectory from the string - inspired theory

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The special case of $4D$ string-like theory proposed early is investigated. Regge trajectories in the developed model are non-linear for the small masses and the values of spin and have the different asymptotical slopes $\alpha_p$. The value $\alpha_p$ just as the form of the trajectory depends from the quantum state of some "internal" (relativistic invariant) variables. It is also shown that some trajectories have the asymptotical slope $\alpha_g \simeq 0.21 Gev^{-2}$.

I. Introduction

The conventional approaches in a string theory (see, for example, [1, 2]) lead as is well known to the linear Regge trajectories for free string such that the slope $\alpha'$ is in-put parameter in a theory. But the trajectories $s = \alpha' M^2 + c$, where the value $\alpha' \simeq 0.9 Gev^{-2}$ is the universal constant, describe the spectrum of real particles well but only approximately. Indeed, the linearity means that the width of any resonance is equal to zero; the universality of the slope $\alpha'$ is connected with the absence of exotic particles [3]. In the meantime we have the stable evidence on hadronic exotics now [4]. These data give the true information about Regge trajectories with slopes $\alpha_g \neq 0.9 Gev^{-2}$ [5, 6, 7]. As regards the form of the trajectory, the linear dependence gives a good approximation for light-flavoured mesons and baryons only (see, for example [8]). Therefore we have to modify standard string theory to describe the real (extended) particles. Of course, such modification must solve in some way the problem of the extra space - time dimensions which is usual for conventional approaches. To solve this problem the various (super)string models have been constructed by many authors both in the physical (4D) and in the arbitrary (non-critical) dimensions (see, for example, [9, 10, 11, 12, 13, 14, 15]). Most of them lead to the linear Regge trajectories too.

Recently new approach was suggested to describe the classical and quantum dynamics of $4D$ open spinning string [16]. Proposed theory differs, in our knowledges, from other because it founded on the new conception of "adjunct phase space". Generally speaking, we describe the classical dynamics of the string with canonical

\[ ^1 \text{This list can be continued, of course.} \]
phase space $\mathcal{H}$, first type constraints $F_i$ and some additional conditions $G_i$ in terms of another dynamical system with phase space $\mathcal{H}^{ad}$. These phase spaces connected in accordance with diagram

$$\mathcal{H} \supset \mathcal{V} \approx \mathcal{W} \subset \mathcal{H}^{ad},$$

where set $\mathcal{V}$ is the surface of the constraints and additional conditions in the space $\mathcal{H}$, set $\mathcal{W}$ is the surface of some (first type) constraints $\Phi_i$, $(i = 1, \ldots, n < \infty)$ in the space $\mathcal{H}^{ad}$. The symbol $\approx$ means the diffeomorphism conserved in the dynamics. From the classical viewpoint, the manifold $\mathcal{W}$ is equivalent to the manifold $\mathcal{V}$, because it contains same information about the physical degrees of the freedom. It should be stressed that $\mathcal{H} \not\approx \mathcal{H}^{ad}$ as the Poisson manifolds: there is no any canonical transformation $\mathcal{H} \rightarrow \mathcal{H}^{ad}$.

We fulfill the subsequent quantization of the string theory in terms of the space $\mathcal{H}^{ad}$. Rigorously speaking, we quantize another dynamical system and select the "string sector" which corresponds to the set $\mathcal{W}$. Relativistic invariance does not broken in quantum case because special selection of the phase space $\mathcal{H}^{ad}$ which will be quite natural in our opinion. Finally, as a quantum theory we have the operator representation of the classical algebra

$$\mathcal{A}_{cl} = \mathcal{P} \oplus \mathcal{A}_{int},$$

where $\mathcal{P}$ is the Poincaré algebra and $\mathcal{A}_{int}$ the Poisson brackets algebra of some two-dimensional fields in the "box" (there are "internal" variables). Thus our approach is a natural generalization of the Wigner's conception of "elementary" particle as the representation of Poincaré algebra. Non-trivial spectrum appears here because existence of constraints depending from the Casimir functions $P^2$ and $w^2$. We interpret such quantum theory as the model of extended relativistic particle.

II. Basic points of a classical theory.

As the model of spinning string we consider the following theory. Let the fields $X_{\mu}(\xi^{0}, \xi^{1})$ and $\Psi^A_{\pm}(\xi^{0}, \xi^{1})$ interact with two-dimensional gravity $h_{ij}(\xi^{0}, \xi^{1})$, where $\xi^{1} \in [0, \pi]$ and $\xi^{0} \in (-\infty, \infty)$, such that dynamics is defined by the action constructed in accordance with the well-investigated manner [1]:

$$S = -\frac{1}{4\pi\alpha} \int d\xi^{0} d\xi^{1} \sqrt{-h} \left[ h^{ij} \partial_{i} X^{\mu} \partial_{j} X_{\mu} - i \partial_{0} \Psi^A_{\pm} (\Gamma^{0})_{\alpha \beta} \Psi^A_{\pm} \partial_{0} \Psi^A_{\pm} \right]. \quad (2)$$

The notations are following: $h = \det(h^{ij})$, the vectors $e^I_j(\xi^{0}, \xi^{1})$ are the vectors of two-dimensional basis such that the equalities $h^{ij} = e^{I i} e^{J j}$ take place and the matrices $\Gamma^{\mu}$ and $\gamma_i$ are the Dirac matrices in the four- and two-dimensional space-time respectively. The field $X_{\mu}$ takes the values in Minkowski space-time $E_{1,3}$; the fields $\Psi^A_{\pm}$ with components $\Psi^A_{\pm}$ are the spinor fields in two-dimensional space; index $A$ is the spinor index in the space $E_{1,3}$ such that the fields $\Psi^A_{\pm}$ are the Majorana spinor fields in four-dimensional space-time. The numbers $\Psi^A_{\pm}$ are the complex numbers, thus there are no classical Grassmann variables in our action. The consideration of the spinning string without the Grassmann variables is not new (see, for example, [17]).

To fix the gauge arbitrariness we demand, as usually, $e^I_j = \delta^I_j$ so that $h_{ij} = \text{diag}(1, -1)$ and the equations of motion can be written in simplest form. For fields $X$ and $\Psi$ we have $\partial_{\xi} \partial_{\xi} X_{\mu} = 0$, $\partial_{\xi} \Psi_{\pm} = 0$; the equations of motion $\delta S / \delta h^{ij} = 0$ for gravity $h$ lead as well-known to the equalities

$$F_{\pm}(\xi) \equiv (\partial_{\xi} X)^{2} \pm \frac{i}{2} \Psi^A_{\pm} \partial_{\xi} \Psi^A_{\pm} = 0, \quad (3)$$

where $\partial_{\xi}$ are derivatives with respect to cone parameters $\xi_{\pm} = \xi^{1} \pm \xi^{0}$.

We still have the remained gauge freedom [16]

$$\xi_{\pm} \rightarrow \tilde{\xi}_{\pm} = \pm A(\pm \xi_{\pm}). \quad (4)$$

The function $A(\xi)$ must satisfy the property $A(\xi + 2\pi) = A(\xi) + 2\pi$ ($A' \neq 0$) in accordance with the standard boundary conditions for original variables $X$ and $\Psi$: $X'_{\mu}(\xi^{0}, 0) = X'_{\mu}(\xi^{0}, \pi) = 0$, $\Psi_{+}(\xi^{0}, 0) = \Psi_{+}(\xi^{0}, \pi)$ and $\Psi_{+}(\xi^{0}, \pi) = e^{\Psi^{-}}(\xi^{0}, \pi)$, where $\epsilon = \pm$. Our subsequent studies are founded on two conjectures. Firstly, we restrict the set of all string configurations by requirement

$$\pm \Psi^A_{\pm} \Gamma^{\mu} \Psi_{\pm} \partial_{\xi} X_{\mu} > 0. \quad (5)$$
Because the boundary conditions on original string variables:

\[ \partial_\pm \left[ \mathbb{V}_\pm \Gamma^\mu \Psi_\pm \partial_\pm X_\mu \right] = 0. \tag{6} \]

hold. It is more convenient to integrate eq. (3) so that

\[ G_\pm(\xi) \equiv \mathbb{V}_\pm \Gamma^\mu \Psi_\pm \partial_\pm X_\mu = \pm \frac{\kappa^2}{2} \tag{7} \]

for any positive constant \( \kappa \). These equalities are equivalent to the original conditions (3) and (2). Note that the conditions (3) are invariant both under Poincaré and under scale transformations of the space-time \( E_{1,3} \). Such invariance is first reason of the motivation for the conditions (3). Second reason is to the gauge (3) generalizes naturally the well-known light-cone gauge in a string theory (this fact is discussed firstly in the work \( 38 \) in connection with geometrical description 3D spinning string). We define the set \( V \) as a set of the string configurations which satisfy the constraints (3) and the additional conditions (5) and (6).

The original phase space \( \mathcal{H} \) has the coordinates \( \dot{X}_\mu \equiv \partial_0 X_\mu, X_\mu, \Psi_\pm^A \) and \( \Psi_\pm^A \). As usually, canonical Poisson bracket structure is following:

\[ \{ \dot{X}_\mu(\xi), X_\nu(\eta) \} = -4\pi \alpha' g_{\mu\nu} \delta(\xi - \eta), \]
\[ \{ \Psi_\pm^A(\xi), \Psi_\pm^B(\eta) \} = \frac{8\pi i \alpha'}{g_0} (\Gamma^0)^{AB} \delta(\xi - \eta). \]

Let us define two functions \( F \) and \( G \) which will be continuous and 2\pi-periodical in accordance with boundary conditions on original string variables:

\[ F(\xi) = \begin{cases} F_+(\xi), & \xi \in [0, \pi), \\ F_-(\xi), & \xi \in [-\pi, 0), \end{cases} \quad G(\xi) = \begin{cases} G_+(\xi), & \xi \in [0, \pi), \\ G_-(\xi), & \xi \in [-\pi, 0). \end{cases} \]

In terms of Fourier modes \( F_n \) and \( G_n \) of the introduced functions \( F \) and \( G \), the set \( V \) is defined by the equalities

\[ F_n = 0 \quad (n = 0, \pm 1, \ldots), \quad G_n = 0 \quad (n \neq 0). \]

In accordance with canonical Poisson bracket structure we have the equalities

\[ \{ F_n, G_m \} = 4\pi i \alpha'(n + m) G_{n-m}. \]

Because \( G_0 = \kappa^2/2 > 0 \) in our theory, the system of ”constraints”

\[ F_n = 0, \quad (n \neq 0), \quad G_n = 0 \quad (n \neq 0) \tag{8} \]

will be the second type system in Dirac terminology. We can introduce correspondent Dirac brackets and consider the reduced phase space \( \mathcal{H}_1 \) (defined by the equalities (8)). In this space the set \( V \subset \mathcal{H}_1 \) will be the surface of single constraint \( F_0 = 0 \). It is clear that such constraint generates the transformations (4) such that \( \Lambda(\xi) \equiv \xi + c \), where \( c = \text{const} \). Obviously, they give the shifts \( \xi^0 \rightarrow \xi^0 + c \), which correspond to \( \xi^0 \)-dynamics.

As a next step of our classical theory we consider another dynamical hamiltonian system (”extended particle”). To do it we must define the objects: a phase space \( \mathcal{H}^{ad} \), a Poisson brackets \( \{ \cdot, \cdot \}^0 \), a hamiltonian \( h_0 \), and, may be, some constraints \( \Phi_i \). The definitions will be following. The space \( \mathcal{H}^{ad} \) has a structure

\[ \mathcal{H}^{ad} = \mathcal{H}_p \times \mathcal{H}_b \times \mathcal{H}_j \times \mathcal{H}_0, \]

where: \( \mathcal{H}_p \) is the algebraic space of the Poincaré algebra with coordinates \( P_\mu \) and \( M_{\mu\nu} \); \( \mathcal{H}_b \) – the phase space of the complex D’Alembert field \( b(\xi^0, \xi^1) \) which is defined for \( \xi^1 \in [0, \pi] \) and satisfyed to the boundary conditions

\[ b(\xi^0, 0) = b(\xi^0, \pi) = 0; \]

the space \( \mathcal{H}_j = \mathcal{H}_{j0} \times \mathcal{H}_U \) be composed from the space \( \mathcal{H}_{j0} \) which is the phase space of the real D’Alembert field \( j(\xi^0, \xi^1) \) in same ”box” \( \xi^1 \in [0, \pi] \) and satisfyed to the boundary conditions

\[ j(\xi^0, 0) = 0, \quad j(\xi^0, \pi) = 2\pi j^0_0 \]
and from the space $\mathcal{H}_U$. Last one is the phase space of $SU(2)$ - valued WZNW - field $U(\xi^0, \xi^1)$ defined in the "box" $\xi^1 \in [0, \pi]$ and satisfied to the boundary conditions

$$U(\xi^0, 0) = U(\xi^0, \pi) = 1_2.$$  

The space $\mathcal{H}_0$ is some auxiliary (finite - dimensional) space; it is not important here and will be dropped out from subsequent consideration. We use following representations for introduced fields:

$$b(\xi^0, \xi^1) = \int_{-\xi^+}^{\xi^+} f(\eta) d\eta,$$

where $f(\xi)$ - complex $2\pi$ - periodical function without zero mode;

$$j(\xi^0, \xi^1) = \int_{-\xi^+}^{\xi^+} j_0(\eta) d\eta,$$

where $j_0(\xi)$ - real $2\pi$ - periodical function such that the zero mode of this function is the constant $j_0^0$.

The space $\mathcal{H}_U$ is coordinatized by three real functions $j_0(\xi)$ (a = 1, 2, 3) (see [19]). $2\pi$ - periodicity of these functions is the consequence of the boundary conditions for the WZNW-field $U$. We assume that all coordinates of spaces $\mathcal{H}_b$, $\mathcal{H}_f$ and $\mathcal{H}_0$ are Poincaré - scalars.

Natural Poisson brackets for the coordinates of the phase space $\mathcal{H}^{ad}$ will be following:

$$\{M_{\alpha\beta}, M_{\gamma\delta}\}^0 = g_{\alpha\delta} M_{\beta\gamma} + g_{\beta\gamma} M_{\alpha\delta} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma},$$

$$\{M_{\alpha\beta}, P_{\gamma}\}^0 = g_{\beta\gamma} P_{\alpha} - g_{\alpha\gamma} P_{\beta}, \quad \{P_{\alpha}, P_{\beta}\}^0 = 0,$$

$$\{f(\xi), \bar{f}(\eta)\}^0 = S_0^{-1} \delta(\xi - \eta),$$

$$\{j_0(\xi), j_0(\eta)\}^0 = -2S_0^{-1} \delta(\xi - \eta),$$

$$\{j_0(\xi), j_0(\eta)\}^0 = 2S_0^{-1} \left(-\delta_{ab}\delta(\xi - \eta) + \varepsilon_{abc} j_0(\xi) \delta(\xi - \eta)\right).$$

(a, b, c = 1, 2, 3 and $\delta(\xi) = \sum_n e^{in\xi}$ here.) As hamiltonian we postulate the function

$$h_0 = \frac{S_0}{2\pi} \left( \int_0^{2\pi} |f(\xi)|^2 d\xi + \frac{1}{4} \sum_{a=0}^3 \int_0^{2\pi} j_0^2(\xi) d\xi \right),$$

so that the following non-trivial equations of motion hold:

$$\{h_0, j_a\}^0 = j_a^0, \quad \{h_0, f\}^0 = f'.$$

(10)

**Main result** of the work [16] is formulated as the following statement.

The equality

$$\Phi(P^2, w^2; f, j_0, \ldots, j_3) = 0,$$

(11)

where $w^2$ is the square of the pseudo-vector $w_\mu = -(1/2)\varepsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} P^\sigma$, exists on the phase space $\mathcal{H}^{ad}$ such the surface $W \subset \mathcal{H}^{ad}$, corresponded to "constraint" [17] will be diffeomorphic to the surface $V$ in the string phase space $\mathcal{H}$. Coordinates $P_\mu$ and $M_{\mu\nu}$ of the space $\mathcal{H}^{ad}$ corresponded any string configuration $(X_\mu, \Psi_{\pm})$ for such diffeormorphism, will coincide with the string energy-momentum and moment calculated for this configuration in accordance with the Noether theorem. Constraint [17] is stable, i.e. the equality

$$\{h_0, \Phi\}^0 = 0$$



\(^2\)Because the functions $f$ and $j_\alpha$ are dimensionless we must use some constant $S_0$ which has a dimension of action. For the subsequent calculations $S_0 = \sqrt{\mu'/\alpha}$ and $c = \hbar = 1$. 

4
fulfill the quantization of the variable $f$

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The following proposition clarifies this question \[20\]
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The "constraint" \[11\] was deduced in the work \[16\] in the
κ
domain does not correspond any string, it may be interesting too.
The equalities \(2\) and \[3\] gives the closed form (11). The evident expression o f the function $Φ(\ldots)$ as its domain must
described more detail in general. We will consider reduced case here and prefer the form \[3\] and \[4\].
Let us note that the continuation of these equalities on the domain $κ \leq 0$ is quite possible. Although such
domain does not correspond any string, it may be interesting too.

III. Reduction and quantization

The variables of the phase space $H^{ad}$ are complicated functions from the string fields $X$ and $Ψ$, that is
why the correct selection of quantum statistics for any "internal" variables $f$ and $j_a$ is not so obvious here.
The following proposition clarifies this question \[20\]
The equalities $Ψ^a(ξ) \equiv \text{const}$ hold if and only if the equalities $j_a(ξ) \equiv 0$ for $a = 0, \ldots, 3$ take place.
This statement means that, in spite of the complicated dependence of the variables $f$ and $j_a$ from the
variables $X_µ$ and $Ψ$, the bosonic and fermionic degrees of the freedom are still non-mixed. It is natural to
fulfill the quantization of the variable $f$ in terms of some bosonic field but the variables $j_a$ in terms of the fermionic fields with help of the bosonization procedure \[23\].
Thus the natural Hilbert space of the quantum states in our theory will be the following:

$$H = \bigoplus_{l,i,s} (H_b \otimes H_j \otimes H_{µ^2,s}),$$

where the spaces $H_{µ^2,s}$ are the spaces of irreducible representations of Poincaré algebra $P$, corresponded to
the eigenvalues $µ^2$ and $s(s+1)$ of the Casimir operators $α'^2 P^2$ and $S^2$; $H_b$ – the Fock space of two-dimensional
complex bosonic field in the "box" and $H_j$ – the Fock space of some two-dimensional fermionic field in the
"box". The corresponding physical states $|\psi_{\text{phys}}\rangle$ must being selected with help of the "Shrödinger equation"

$$Φ(P^2, w^2, \ldots) |\psi_{\text{phys}}\rangle = 0,$$

where $Φ_i$ are the quantum expressions for constraint \[11\].
Thus the reduction $j_a(ξ) \equiv 0$ corresponds to open 4D string with additional spinor field which has
a constant non-zero components on the world-sheet. We consider in detail the quantum version of the
dynamical system "extended particle" for such reduction in this work. Let us introduce the following objects.
• bosonic creation and annihilation operators \( b^+_n, b_n \) \((n = \pm 1, \pm 2, \ldots)\) acting in the corresponding Fock space \( H_b \) such that the canonical commutation relations \([b_n, b^+_m] = \delta_{mn}\) hold;

• some sequence of non-negative numbers 

\[
\{\mu_i^2(s)\} \quad (s = 0, 1/2, 1, 3/2, \ldots, i = 0, 1, \ldots, \infty).
\]

All quantum states in our theory are the vectors of space

\[
H = \bigoplus_{i,s} \left( H_b \otimes H_{\mu_i^2(s),s} \right).
\]

It should be stressed that we can drop out the summands which correspond to integer or half-integer values of the number \( s \). Therefore we can construct the models both bosons and fermions which have the additional bosonic internal degrees of the freedom. It is interesting to note that we can include in the definition of the space \( H \) the summands \( H_b \otimes H_{\mu_i^2} \) which correspond to the irreducible representations of the Poincaré algebra with \( \mu_i^2 < 0 \). This case corresponds to the theory with the tachyon states. In accordance with our definition \( \mu_i^2 \geq 0 \) so there are no tachyons in the model.

We find the sequence \( \mu_i^2(s) \) such that non-zero solutions \( |\psi_{\text{phys}}\rangle \in H \) of the "Schrödinger equation" (15) exist and have the finite norm.

Let us define the operator-valued function

\[
f(\xi) = \sum_{n>0} \sqrt{n} \left( b_{-n} e^{-in \xi} + b^+_n e^{in \xi} \right).
\]

It is clear that the equality

\[
[f(\xi), f^+(\eta)] = i\delta'(\xi - \eta)
\]

takes place.

To construct the quantum theory correctly, we must describe evidently the algebra of the classical dynamical variables ("observables"). By definition the observables will be the functions:

\[
\mathcal{D} = \mathcal{D}[f(\xi); P_{\mu}, M_{\mu\nu}] \equiv \mathcal{D}^m[A_1, A_2, \ldots, A_n],
\]

where \( \mathcal{D}^m \) some m-th power polynomial of the variables \( A_i \). Last ones can be the following quantities:

1. the component of the energy-momentum \( P_\mu \);

2. the component of the moment \( M_{\mu\nu} \);

3. some integral

\[
\int_0^{2\pi} \cdots \int_0^{2\pi} \varphi(\xi_1, \ldots, \xi_k) f(\xi_1) \cdots f'(\xi_i) \cdots f^+(\xi_k) d\xi,
\]

such that the kernel \( \varphi \in C^\infty \) non-degenerated.

The polynomial coefficients are arbitrary analytical functions from the values \( P^2 \) and \( w^2 \).

The quantization \( \Upsilon \) for considered reduced dynamical system will be the correspondence

\[
\mathcal{D} \rightarrow \Upsilon(\mathcal{D}) \equiv D[\sqrt{\gamma} f(\xi); P_\mu, M_{\mu\nu}],
\]

where the number \( \gamma' = \alpha' \hbar / 6\hbar^3 \) is the natural dimensionless parameter in our theory and bold-faced symbols \( P_\mu \) and \( M_{\mu\nu} \) mean the correspondent operators acting in the spaces \( H_{\mu_i^2,s} \). Ordering rules are following:

1. in the integrals

\[
\Upsilon \left( f(\xi) \cdots f'(\xi) \cdots f^+(\xi) \right) =: f(\xi) \cdots f'(\xi) \cdots f^+(\xi). \]
2. in the polynomial $D^m$

$$\Upsilon(A_1 \ldots A_k) = \frac{1}{k!} \sum_{i_1, \ldots, i_k} \Upsilon(A_{i_1}) \ldots \Upsilon(A_{i_k}),$$

where the summands correspond to the rearrangements of the numbers 1, \ldots, $k$.

Thus all observables in our model are well-defined operators in the space $H$ (the operator $b_n$ has to be defined as $\oplus_{i,s}(b_n \otimes 1)$ and so on). The constraint $\Phi = 0$ leads to the equation (15); solutions of this equation define the physical subspace $H_{phys} \subset H$.

Note that the different classical forms same constraints can lead to the different operator equations. Indeed, as the general situation we have

$$\Upsilon(D_1) \Upsilon(D_2) \neq \Upsilon(D_1 D_2)$$

for arbitrary observables $D_1$ and $D_2$ so that the constraints $\Phi = 0$ and $\Phi^2 = 0$, for example, lead to the different quantum theories. As the standard situation, we have some (singular) Lagrangian $L(q \ldots \dot{q} \ldots)$ and the natural form of the constraints which are consequence of the definitions $p_i = \partial L/\partial \dot{q}_i$ for the canonical momenta. In our case we construct the hamiltonian system without lagrangian formalism (in accordance with general Dirac’s ideas); moreover the equality $\Phi = 0$ appears as some external condition. But the existence of the lagrangian scheme for any constraint hamiltonian dynamics is non-trivial problem. Moreover there is no such scheme for the arbitrary constraints [22]. All these arguments mean that we must postulate strongly the form of the constraint (11) before quantization.

Let us introduce the parameter $\lambda$ instead the parameter $\kappa$ such that

$$\lambda = \frac{\alpha' P^2}{\kappa} \left( P = \sqrt{P^2} \right).$$

Because $\kappa \neq 0$ in our theory such redefinition of the parameter will be correct. Note that the point $\lambda = 0$ corresponds to the massless particles. As the consequence we have the following $\lambda$ - parametric representation of the constraint $\Phi = 0$:

$$h_0 - \lambda^2 = 0, \quad \Phi_1(\lambda | B_2, B_3, P^2, S^2) = -S^2 \lambda^6 + \frac{1}{4} \varepsilon_0^2 \lambda^2 P^2 -$$

$$-\frac{1}{2} \alpha' P^3 \varepsilon_0 \lambda \left( B_3 + B_3^+ \right) + \alpha^2 P^4 \left( B_2^2 \lambda^2 + B_3 B_3^+ \right) = 0, \quad (17)$$

where the following notations are stated:

$$B_2 = \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \int_0^{\xi} \mathcal{F}(\zeta) d\zeta d\xi, \quad B_3 = \frac{1}{2\pi} \int_0^{2\pi} |f(\xi)|^2 \int_0^{\xi} \mathcal{F}(\zeta) d\zeta d\xi.$$

Symbol $\mathcal{F} f$ means such antiderivative of the function $f$ that has not zero mode in Fourier expansion. Let us note that the Poisson brackets of l.h.s. of the equalities (16) and (17) are vanished strongly that is why such representation of our constraint $\Phi = 0$ will be correct. This form of the constraint $\Phi = 0$ – equalities (16) and (17) – is postulated before the quantization.

Let us introduce the notations

$$h = \gamma'^{-1} \Upsilon \left( \frac{1}{2\pi} \int |f(\xi)|^2 d\xi \right) = \sum_n |n |b_n^+ b_n.$$

\[3\text{Let us remind that all evident expressions here are deduced from the general theory constructed in the work [16].} \]
\[
\begin{align*}
\mathbf{w}_2 &= \gamma'^{-1} \int_0^{2\pi} f(\xi) d\eta d\xi = \sum_{n \neq 0} \text{sgn}(n) b_n^+ b_n, \\
\mathbf{w}_3 &= \gamma'^{-3/2} \int_0^{2\pi} |f(\xi)|^2 f(\eta) d\eta d\xi = \\
&= i \sum_{n,k>0} \frac{n}{k(n+k)} \left[ (n+k)(b_{n-k}^+ b_{n-k} - b_n^+ b_{n+k}^+) + \\
&+ n(b_{n+k}^+ b_{n-k} - b_{-n+k}^+ b_{-n-k}) \right].
\end{align*}
\]

In accordance with the classical expressions \((16)\) and \((17)\), the quantum equations for the physical states vectors \(|\psi_{\text{phys}}\rangle \in \mathcal{H}\) have the form:

\[
\left( \gamma' \mathbf{h} - \lambda^2 \right) |\psi_{\text{phys}}\rangle = 0, \quad (18)
\]

\[
\Phi_1(\lambda |\mathbf{w}_2, \mathbf{w}_3, P^2, S^2) |\psi_{\text{phys}}\rangle = 0. \quad (19)
\]

Operator \(\Phi_1\) must be constructed in accordance with the ordering rules formulated above.

It can be verified directly that

\[
[\mathbf{h}, \mathbf{w}_2] = [\mathbf{h}, \mathbf{w}_3] = 0,
\]

so that

\[
[\mathbf{h}, \Phi_1(\lambda |\mathbf{w}_2, \mathbf{w}_3, P^2, S^2)] = 0.
\]

The structure of the space \(\mathcal{H}\) means that any special solution of the system \((18) - (19)\) has a form

\[
|\psi_{\text{phys}}\rangle = |\mu_{\phi}, s \rangle |\phi\rangle,
\]

where \(|\mu_{\phi}, s\rangle \in \mathcal{H}_{\mu^2, s}, |\phi\rangle \in \mathcal{H}_{\phi}\) and \(\mu_{\phi}^2 = \mu^2(|\phi\rangle)\) in general. Such representation for the vector \(|\psi_{\text{phys}}\rangle\) leads to following spectral tasks in the space \(\mathcal{H}_{\phi}\) (\(\lambda^2 = \gamma'N\)):

\[
(\mathbf{h} - N) |\phi\rangle = 0, \quad (20)
\]

\[
\left[ \mu^4 \left( N\mathbf{w}_2^2 + \frac{1}{2}(\mathbf{w}_3^+ \mathbf{w}_3 + \mathbf{w}_3 \mathbf{w}_3^+) \right) + \\
+ \epsilon \mu^3 \sqrt{\frac{N}{4\gamma'\tau^3}} \left( \mathbf{w}_3 + \mathbf{w}_3^+ \right) + \frac{\mu^2 N}{4\gamma'\tau^3} - s(s+1)N^3 \right] |\phi\rangle = 0, \quad (21)
\]

where \(\epsilon = \text{sgn} \lambda = \frac{P_0}{|P_0|}\).

In our case \(P^2 > 0\) that is why the value \(\epsilon\) will be additional Cazimir function (for example we must choose \(\epsilon = 1\) for string sector \(\kappa > 0\)).

The eigenvalues \(N\) of the spectral task \((20)\) will be integer: \(N = 0, 1, 2, \ldots\). Correspondent eigenspaces \(\mathcal{H}_N \in \mathcal{H}_f\) are spanned on the vectors

\[
|l_1, \ldots, l_m\rangle = c_0 \prod_{i=1}^{m} b_{l_i}^+ |0\rangle \quad m = 1, \ldots, N
\]

such that \(\sum_{i=1}^{m} |l_i| = N\) (the factor \(c_0\) – is the normalization factor).
The following decomposition takes place:

\[ H_N = \bigoplus_{N=0}^{\infty} H_N, \quad \text{where} \quad H_0 = \{ c \mid 0 \}. \]

As regards the equality (21) it must be considered as the spectral task (\( \mu \) will be spectral parameter) for each value \( s = 0, 1/2, 1, 3/2, \ldots \).

The evident expressions for the operators \( w_2 \) and \( w_3 \) prove that the following inclusions are true:

\[ w_2 H_N \subset H_N, \quad w_3 H_N \subset H_N. \]

This fact means, firstly, that the expression \( \Phi_1(w_2, w_3, \ldots) \) defines correctly some operator in the each space \( H_N \) and, secondly, the eigenvectors \( | \phi_\mu \rangle \) of the task (21) must be searched as

\[ | \phi_\mu \rangle = | \phi_{N, \mu} \rangle \in H_N. \]

It is clear that the number \( d_N = \dim H_N \) is the finite number so that the operator \( \Phi_1(w_2, w_3, \ldots) \) defines some matrix \( d_N \times d_N \) in the each space \( H_N \). Thus the spectral task (21) is reduced to the series of the matrix tasks marked by number \( N = 1, 2, \ldots \).

Let \( \{ | N, l \rangle; l = 0, 1, \ldots, d_N \} \) denote the set of the vectors \( | l_1, \ldots, l_m \rangle \) introduced above and numbered in some way. Then the eigenvalues \( \mu = \mu_l(s; N, \gamma', \epsilon) \) will be the roots of the algebraic equations \( (N = 1, 2, \ldots) \):

\[
\det \left[ \mu^4 A + \epsilon \mu^3 \sqrt{N \over 4 \gamma'^2} B + \left( {\mu^2 N \over 4 \gamma'^2} - s(s + 1)N^2 \right) 1_{d_N} \right] = 0, \tag{22}
\]

where \( 1_{d_N} \) is the unit matrix \( d_N \times d_N \) and the symbols \( A \) and \( B \) denote the matrices

\[
A = \langle l', N | \left( N w_2^2 + \frac{1}{2} (w_3^+ w_3 + w_3 w_3^+) \right) \mid N, l \rangle, \\
B = \langle l', N | (w_3 + w_3^+) \mid N, l \rangle.
\]

The following statement is true.

For each numbers \( N > 0, s > 0 \) equation (22) has the real positive roots, at least in some neighbourhood \( \mathcal{O} \) of the point \( \Phi_0 = 0 \).

We drop out detail proof here and restrict himself by the concrete examples. Note that the degenerated case of our theory \( \Phi_0 = 0 \) leads to the quasi-linear (in the plane (\( \mu^2, s \)) Regge trajectories

\[ s(s + 1) = \alpha_k^2 \mu^4, \quad \mu^2 \geq 0, \quad s = 0, 1/2, 1, 3/2, \ldots. \]

Corresponding asymptotics have a different slopes \( \alpha_k \) and same intercepts \( s = -1/2 \). It is interesting that the analogous behaviour – the interception of the infinitely many trajectories in this point – was founded many years ago in some realistical model of the potential scattering [23].

Let the symbol \( \mu_l(s; N, k) \) denotes \( k \)-th real root of the equation (22) for some fixed numbers \( N \) and \( s \). It is clear that the total set of the roots will be enumerable thus we have the sequence \( \mu_l^2(s) \) which is need for the construction of the total space of quantum states \( H \). The "physical" states \( | \psi_{\text{phys}} \rangle \in H_{\text{phys}} \) have a form

\[
| \psi_{\text{phys}} \rangle = \sum_{n,s} c_{n,s} \mid \mu_n^2(s), s \rangle \mid \phi_{(n)} \rangle,
\]

where \( | \phi_0 \rangle = | 0 \rangle, \quad | \phi_{(n)} \rangle (n = 1, \ldots, \infty) \) – the eigenvector of spectral task (22) which corresponds to the eigenvalue \( \mu_n(s) \) and the arbitrary constants \( c_{n,s} \) must be subjected to the condition \( \sum_{n,s} | c_{n,s} |^2 < \infty \).
IV. Examples.

1. $N = 0$. The equation (21) has a unique solution $|\phi\rangle = |0\rangle$. In accordance with definition of the number $N$ we have $\mu = 0$. Spin $s \neq 0$ in general. We can include or not include in the definition of the Hilbert space $H$ corresponding subspaces $H_{\alpha,s}, s \neq 0$.

2. $N = 1$. The space $H_1$ will be two-dimensional; it is spanned on the vectors $b_{1-}^+|0\rangle$ and $b_{1+}^-|0\rangle$. As regards to the equation (21), the operator $w_2^2$ will be the unit operator (other operators will be zero). As the consequence we have the equation

$$\mu^4 + \frac{\mu^2}{4\gamma'} = s(s + 1).$$

Thus we have the non-linear Regge trajectory in the plane $(\mu^2, s)$. Correspondent asymptotic line is following:

$$s = \mu^2 + \frac{1}{2} \left( \frac{1}{4\gamma'^3} - 1 \right).$$

The results of the numerical calculations for $\alpha' = 0.9\text{ Gev}^{-2}$ and some values of the constant $\gamma'$ are represented in the table 1. We can compare these masses with the masses of some neutral mesons with zero isospin.

| spin $s$ | $M$ (Gev) $(\gamma' = 0.44)$ | $M$ (Gev) $(\gamma' = 15)$ | Some mesons (isospin $I = 0$) |
|---------|-----------------|-----------------|-------------------|
| 1       | 0.796           | 1.236           | $\omega(0.783)$; $f_1(1.285)$ |
| 2       | 1.242           | 1.650           | $f_2(1.270)$; $f_2(1.640)$ |
| 3       | 1.600           | 1.962           | $\omega_3(1.670)$; $\phi_3(1.850)$ |
| 4       | 1.900           | 2.230           | $f_4(2.220)$ |

Table 1: Mass spectrum in the sector $N = 1$ ($I = 0$, $Q = 0$)

3. $N = 2$. Corresponding subspace $H_2$ will be the linear combination of the vectors

$$b_{1-2}^+|0\rangle, \ b_{1-1}^+b_{1+}^-|0\rangle, \ b_{1+}^+b_{1-}^-|0\rangle, \ b_{1-2}^+|0\rangle, \ b_{1+}^+|0\rangle,$$

so that $\dim H_2 = 5$. The numerical results of the solution of the equation (22) are represented in the table 2 (the values of the fundamental constants are $\alpha' = 0.9\text{ Gev}^{-2}$ and $\gamma' = 15$).

Thus we have three quasi-linear Regge trajectories in the Plane $(M^2, s)$:

$$s \simeq \alpha_i M^2 - 0.5, \quad i = 1, 2, 3,$$

where $\alpha_1 = 1.1\alpha'$, $\alpha_2 = 0.72\alpha'$ and $\alpha_3 = 0.24\alpha'$. Trajectories corresponded to the index $i = 1, 2$ are quite appropriate for the description of some standard meson states. As regards of the trajectory corresponded to the index $i = 3$, it has the slope which differs essentially from the standard one. In our opinion the corresponding states can be interpreted as some glueball states. Indeed, for example, the phenomenology of the scalar and tensor glueballs was investigated in the work [23]; it was supposed that the slope for glueball Regge trajectory $\alpha_{\text{glue}} \simeq 0.2\text{ Gev}^{-2}$.

It is interesting to see the mass spectrum deformation if we decrease the dimensionless constant $\gamma'$ (remember this procedure means increasing the constant $\theta_0$ which is in-put constant for the action (2)). Corresponding results for $\gamma' = 0.44$ are represented in the table 3. We can see two effects here: firstly, the trajectories become more non-linear for small values masses and spin and, secondly, some trajectories are splitting. Note that the existence of the mesons which have same quantum numbers and the near-by masses is considered as a problem in some works (see, for example, [26]).

The operator $h$ is the energy for the quantized bosonic field $b(\xi, \xi^\dagger)$. Therefore, the quantum number $N$ plays the role of internal energy level. Note that the non-linearity of the Regge trajectories is increased for large $N$, just as a calculation difficulties, unfortunately.

The string models of the glueballs are investigated too. So, in the work [25] the glueballs was considered as some states of the (elliptic) Nambu-Goto string.
Table 2: Mass spectrum in the sector $N = 2$ for $\gamma' = 15$

| Spin s | $M_1$ (Gev) | $M_2$ (Gev) | $M_3$ (Gev) | Some mesons               |
|--------|-------------|-------------|-------------|---------------------------|
| 1      | 1.17        | 1.47        | 2.51        | $h_1(1.17)$; $f_1(1.51)$  |
| 2      | 1.56        | 1.93        | 3.30        | $f_2(1.64)$; $\eta_2(1.87)$ |
| 3      | 1.86        | 2.30        | 3.92        | $\phi_3(1.85)$            |
| 4      | 2.11        | 2.61        | 4.49        | $f_4(2.20)$               |

Table 3: Mass spectrum in the sector $N = 2$ for $\gamma' = 0.44$

| Spin s | $M_1$ (Gev) | $M_2$ (Gev) | $M_3$ (Gev) | $M_4$ (Gev) | $M_5$ (Gev) |
|--------|-------------|-------------|-------------|-------------|-------------|
| 1      | 1.037       | 1.042       | 1.31        | 1.38        | 1.78        |
| 2      | 1.441       | 1.443       | 1.81        | 1.86        | 2.67        |
| 3      | 1.753       | 1.755       | 2.20        | 2.24        | 3.37        |
| 4      | 2.016       | 2.017       | 2.52        | 2.56        | 3.95        |

V. Concluding remarks.

It is our belief that further studies of suggested theory may well help to guide the constructing of the realistic models of particles. Indeed, the case considered in this work connects the mass and the spin of extended particle with two annihilators of the Poisson structure $\{\cdot,\cdot\}^0$ – the Casimir operators $P^2$ and $w^2$.

In general case of our theory, when the variables $j_a(\xi) \neq 0$, the Poisson structure $\{\cdot,\cdot\}^0$ has four annihilators. Additional annihilators here the value $j_0^0 = \int_0^{2\pi} j_0(\xi) d\xi$ and the annihilator for the current algebra $[\xi]$ (see [19]). Last one is the value $n = (1/\pi) \arccos[\text{Tr} M/2]$, were the matrix $M$ is the monodromy matrix of the auxiliary linear system

$$U'(\xi) + \frac{i}{2} \left( \sum_{a=1}^{3} j_a(\xi) \sigma_a \right) U(\xi) = 0.$$ 

Note that the standard boundary conditions on the original string variables $X_\mu$ and $\Psi_\pm$ mean that the number $n$ will be integer in the classical theory. The quantization of the variables $j_a$ must be fulfilled in terms of the massless fermionic fields [27] by means of the bosonization procedure [21]. In our opinion we can interpret the value $j^0$ as some (“generalized”) charge and the value $n/2$ as the isotopical spin after quantization. The subsequent works will be devoted to the corresponded models. The author hope, in particular, that the consideration of more general case allows to fix the input constants $\alpha'$ and $\varrho_0$.

We construct the model of the free extended particle here. The interaction of such particles is separate problem. As regards to the possible approaches it must be remembered the old point of view [28] which means that the Regge trajectories determine the scattering amplitude. Note also the work [29], where so-called $q$-deformed dual string theory was studied; it was shown that the $q$-deformation of the amplitude leads to the non-linear (square-root) trajectories.

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