Characterizing $S$-flat modules and $S$-von Neumann regular rings by uniformity

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Abstract

Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $T$ is called $u$-$S$-torsion ($u$- always abbreviates uniformly) provided that $sT = 0$ for some $s \in S$. The notion of $u$-$S$-exact sequences is also introduced from the viewpoint of uniformity. An $R$-module $F$ is called $u$-$S$-flat provided that the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$. A ring $R$ is called $u$-$S$-von Neumann regular provided there exists an element $s \in S$ satisfying that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$. We obtain that a ring $R$ is a $u$-$S$-von Neumann regular ring if and only if any $R$-module is $u$-$S$-flat. Several properties of $u$-$S$-flat modules and $u$-$S$-von Neumann regular rings are obtained.

Key Words: $u$-$S$-torsion module, $u$-$S$-exact sequence, $u$-$S$-flat module, $u$-$S$-von Neumann regular ring.

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1. Introduction

Throughout this article, $R$ is always a commutative ring with identity and $S$ is always a multiplicative subset of $R$, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S, s_2 \in S$. Let $S$ be a multiplicative subset of $R$. Recall from [12, Definition 1.6.10] that an $R$-module $M$ is called an $S$-torsion module if for any $m \in M$, there is an $s \in S$ such that $sm = 0$. $S$-torsion-free modules can be defined as the right part of the hereditary torsion theory $\tau_S$ generated by $S$-torsion modules (see [11]). Early in 1965, Năstăsescu et al. [10] defined $\tau_S$-Noetherian rings as rings $R$ satisfying that for any ideal $I$ of $R$ there is a finitely generated sub-ideal $J$ of $I$ such that $I/J$ is $S$-torsion. However, to tie together some Noetherian properties of commutative rings and their polynomial rings or formal power series rings, Anderson and Dumitrescu [11] defined $S$-Noetherian rings $R$, that is, any ideal of $R$ is $S$-finite in 2002. Recall from [11] that an $R$-module $M$ is called $S$-finite provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule $F$ of $M$. One can see that there is
some uniformity is hidden in the definition of $S$-finite modules. In fact, an $R$-module $M$ is $S$-finite if and only if $s(M/F) = 0$ for some $s \in S$ and some finitely generated submodule $F$ of $M$. In this article, we introduce the notion of $u$-S-torsion modules $T$ for which there exists $s \in S$ such that $sT = 0$. The notion of $u$-S-torsion modules is different from that of S-torsion modules (see Example 2.2). In the past few years, the notions of $S$-analogues of Noetherian rings, coherent rings, almost perfect rings and strong Mori domains are introduced and studied extensively in [1, 2, 3, 8, 9, 7].

In this article, we introduce the notions of $u$-S-monomorphisms, $u$-S-epimorphisms, $u$-S-isomorphisms and $u$-S-exact sequences according to the idea of uniformity (see Definition 2.7). Some properties of $u$-S-torsion modules and $S$-finite modules with respect to $u$-S-exact sequences are given in Proposition 2.8 and Proposition 2.9. We say an $R$-module $F$ is $u$-S-flat provided that the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is $u$-S-exact for any $u$-S-exact sequence $0 \to A \to B \to C \to 0$ (see Definition 3.1). Some basic characterizations of $u$-S-flat modules are given (see Theorem 3.2). It is well known that an $R$-module $F$ is flat if and only if $\text{Tor}_1^R(R/I, F) = 0$ for any ideal $I$ of $R$. However, the $S$-analogue of this result is not true (see Example 3.3). It is also worth remarking that the class of $u$-S-flat modules is not closed under direct limits and direct sums (see Remark 3.5). If an $R$-module $F$ is $u$-S-flat, then $F_S$ is flat over $R_S$ (see Corollary 3.6). However, the converse does not hold (see Remark 3.7). A new local characterization of flat modules is given in Proposition 3.9. A ring $R$ is called a $u$-S-von Neumann regular ring if there exists an element $s \in S$ satisfies that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$ (see Definition 3.12). A ring $R$ is $u$-S-von Neumann regular if and only if any $R$-module is $u$-S-flat (see Theorem 3.13). Every $u$-S-von Neumann regular ring is locally von Neumann regular at $S$ (see Corollary 3.14). However, the converse is also not true in general (see Example 3.15). We also give a non-trivial example of $u$-S-von Neumann regular which is not von Neumann regular (see Example 3.18). Finally, we give a new local characterization of von Neumann regular rings in Proposition 3.19.

2. $u$-S-TORSION MODULES

Recall from [12, Definition 1.6.10] that an $R$-module $T$ is said to be an $S$-torsion module if for any $t \in T$ there is an element $s \in S$ such that $st = 0$. Note that the choice of $s$ is decided by the element $t$. In this article, we care more about the uniformity of $s$ on $T$. 

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**Definition 2.1.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $T$ is called a $u$-$S$-torsion (abbreviates uniformly $S$-torsion) module provided that there exists an element $s \in S$ such that $sT = 0$.

Obviously, the submodules and quotients of $u$-$S$-torsion modules are also $u$-$S$-torsion. Note that finitely generated $S$-torsion modules are $u$-$S$-torsion and any $u$-$S$-torsion modules are $S$-torsion. However, $S$-torsion modules are not necessary $u$-$S$-torsion. We also note that every $R$-module does not have a maximal $u$-$S$-torsion submodule.

**Example 2.2.** Let $\mathbb{Z}$ be the ring of integers, $p$ a prime in $\mathbb{Z}$ and $S = \{ p^n \mid n \geq 0 \}$. Let $M = \mathbb{Z}_{(p)}/\mathbb{Z}$ be a $\mathbb{Z}$-module where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at $S$. Then

1. $M$ is $S$-torsion but not $u$-$S$-torsion.
2. $M$ has no maximal $u$-$S$-torsion submodule.

**Proof.** (1) Obviously, $M$ is an $S$-torsion module. Suppose there is an $p^n$ such that $p^nM = 0$. However, $p^n(\frac{1}{p^n} + \mathbb{Z}) = \frac{1}{p} + \mathbb{Z} \neq 0 + \mathbb{Z}$ in $M$. Thus $M$ is not $u$-$S$-torsion.

(2) Suppose $N$ is a maximal $u$-$S$-torsion submodule of $M$. Then there is an element $p^n \in S$ such that $p^nN = 0$. Note $N$ is a submodule of $M_n := \{ \frac{a}{p^n} + \mathbb{Z} \in M \mid a \in \mathbb{Z} \}$. Since $M_{n+1} := \{ \frac{a}{p^n} + \mathbb{Z} \in M \mid a \in \mathbb{Z} \}$ is a $u$-$S$-torsion submodule of $M$ and $N$ is a proper submodule of $M_{n+1}$. It is a contradiction. \hfill $\square$

**Proposition 2.3.** Let $R$ be a ring and $M$ an $R$-module. Let $S$ be a multiplicative subset of $R$ consisting of finite elements. Then $M$ is $S$-torsion if and only if $M$ is $u$-$S$-torsion.

**Proof.** If $M$ is $u$-$S$-torsion, then $M$ is trivially $S$-torsion. Let $S = \{ s_1, ..., s_n \}$ and $s = s_1...s_n$. Suppose $M$ is an $S$-torsion module. Then for any $m \in M$, there is an element $s_i \in S$ such that $s_i m = 0$. Thus $sm = 0$ for any $m \in M$. So $sM = 0$. \hfill $\square$

**Proposition 2.4.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. If an $R$-module $M$ has a maximal $u$-$S$-torsion submodule, then $M$ has only one maximal $u$-$S$-torsion submodule.

**Proof.** Let $M_1$ and $M_2$ be maximal $u$-$S$-torsion submodules of $M$ such that $s_1M_1 = 0$ and $s_2M_2 = 0$ for some $s_1, s_2 \in S$. We claim that $M_1 = M_2$. Indeed, otherwise we may assume there is an $m \in M_2 - M_1$. Let $M_3$ be a submodule of $M$ generated by $M_1$ and $m$. Then $s_1s_2M_3 = 0$. Thus $M_3$ is a $u$-$S$-torsion submodule properly containing $M_1$, which is a contradiction. \hfill $\square$

Recall from [12, Definition 1.6.10] that an $R$-module $M$ is said to be an $S$-torsion-free module if $sm = 0$ for some $s \in S$ and $m \in M$ implies that $m = 0$. The classes of
S-torsion modules and S-torsion-free modules constitute a hereditary torsion theory (see [11]). From this result it follows immediately the next result (see [12, Theorem 6.1.6]). However we give direct proof for completeness.

**Proposition 2.5.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then an $R$-module $F$ is S-torsion-free if and only if $\text{Hom}_R(T, F) = 0$ for any u-$S$-torsion module $T$.

**Proof.** Assume that $F$ is an S-torsion-free module and let $T$ be a u-$S$-torsion module and $f \in \text{Hom}_R(T, F)$. Then there exists $s \in S$ such that $st = 0$. Thus for any $t \in T$, $sf(t) = f(st) = 0 \in F$. Thus $f(t) = 0$ for any $t \in T$. Conversely suppose that $sm = 0$ for some $s \in S$ and $m \in F$. Set $F_s = \{x \in F \mid sx = 0\}$. Then $F_s$ is a u-$S$-torsion submodule of $F$. Thus $\text{Hom}_R(F_s, F) = 0$. It follows that $F_s = 0$ and thus $m=0$. So $F$ is S-torsion-free.

**Corollary 2.6.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $T$ a u-$S$-torsion module. Then $\text{Tor}^R_n(M, T)$ is u-$S$-torsion for any $R$-module $M$ and $n \geq 0$.

**Proof.** Let $T$ be a u-$S$-torsion module with $sT = 0$. If $n = 0$, then for any $\sum a \otimes b \in M \otimes_R T$, we have $s \sum a \otimes b = \sum a \otimes sb = 0$. Thus $s(M \otimes_R T) = 0$. Let $0 \to \Omega(M) \to P \to M \to 0$ be a short exact sequence with $P$ projective. Then $\text{Tor}^R_1(M, T)$ is a submodule of $\Omega(M) \otimes_R T$ which is u-$S$-torsion. Thus $\text{Tor}^R_1(M, T)$ is u-$S$-torsion. For $n \geq 2$, we have an isomorphism $\text{Tor}^R_n(M, T) \cong \text{Tor}^R_1(\Omega^{n-1}(M), T)$ where $\Omega^{n-1}(M)$ is the $(n-1)$-th syzygy of $M$. Since $\text{Tor}^R_1(\Omega^{n-1}(M), T)$ is u-$S$-torsion by induction, $\text{Tor}^R_n(M, T)$ is u-$S$-torsion.

**Definition 2.7.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $M$, $N$ and $L$ be $R$-modules.

1. An $R$-homomorphism $f : M \to N$ is called a u-$S$-monomorphism (resp., u-$S$-epimorphism) provided that $\text{Ker}(f)$ (resp., $\text{Coker}(f)$) is a u-$S$-torsion module.
2. An $R$-homomorphism $f : M \to N$ is called a u-$S$-isomorphism provided that $f$ is both a u-$S$-monomorphism and a u-$S$-epimorphism.
3. An $R$-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called u-$S$-exact provided that there is an element $s \in S$ such that $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$.

It is easy to verify that $f : M \to N$ is a u-$S$-monomorphism (resp., u-$S$-epimorphism) if and only if $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$) is u-$S$-exact.

**Proposition 2.8.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. Then the following assertions hold.
Proposition 2.9. Let \( S \subseteq sM \) be finitely generated submodules of \( M \). Then the following assertions hold.

(1) Suppose \( M \) is \( u\)-\( S \)-torsion and \( f : L \to M \) is a \( u\)-\( S \)-monomorphism. Then \( L \) is \( u\)-\( S \)-torsion.

(2) Suppose \( M \) is \( u\)-\( S \)-torsion and \( g : M \to N \) is a \( u\)-\( S \)-epimorphism. Then \( N \) is \( u\)-\( S \)-torsion.

(3) Let \( f : M \to N \) be a \( u\)-\( S \)-isomorphism. If one of \( M \) and \( N \) is \( u\)-\( S \)-torsion, so is the other.

(4) Let \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) be a \( u\)-\( S \)-exact sequence. Then \( M \) is \( u\)-\( S \)-torsion if and only if \( L \) and \( N \) are \( u\)-\( S \)-torsion.

Proof. We only prove (4) since (1), (2) and (3) are the consequences of (4).

Suppose \( M \) is \( u\)-\( S \)-torsion with \( sM = 0 \). Since \( \text{Ker}(f) \) (resp., \( \text{Coker}(g) \)) is \( u\)-\( S \)-torsion with \( s_1\text{Ker}(f) = 0 \) (resp., \( s_2\text{Coker}(g) = 0 \)) for some \( s_1 \in S \) (resp., \( s_2 \in S \)), it follows that \( ss_1L = 0 \) (resp., \( ss_2N = 0 \)). Consequently, \( L \) (resp., \( N \)) is \( u\)-\( S \)-torsion. Now suppose \( L \) and \( N \) are \( u\)-\( S \)-torsion with \( s_1L = s_2N = 0 \) for some \( s_1, s_2 \in S \). Since the \( u\)-\( S \)-exact sequence is exact at \( M \), there exists \( s \in S \) such that \( s\text{Ker}(g) \subseteq \text{Im}(f) \) and \( s\text{Im}(f) \subseteq \text{Ker}(g) \). Let \( m \in M \). Then \( s_2g(m) = g(s_2m) = 0 \). Thus there exists \( l \in L \) such that \( ss_2m = f(l) \). So \( s_1ss_2m = s_1f(l) = f(s_1l) = 0 \). So \( M \) is \( u\)-\( S \)-torsion.

Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Recall from [1] that an \( R \)-module \( M \) is called \( S \)-finite provided that there exists \( s \in S \) such that \( sM \subseteq N \subseteq M \), where \( N \) is a finitely generated \( R \)-module. Let \( M \) be an \( R \)-module, \( \{m_i\}_{i \in \Lambda} \subseteq M \) and \( N = \langle m_i \rangle_{i \in \Lambda} \). We say an \( R \)-module \( M \) is \( S \)-generated by \( \{m_i\}_{i \in \Lambda} \) provided that \( sM \subseteq N \) for some \( s \in S \). Thus an \( R \)-module \( M \) is \( S \)-finite provided that \( M \) can be \( S \)-generated by finite elements.

Proposition 2.9. Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( M \) an \( R \)-module. Then the following assertions hold.

(1) Let \( M \) be an \( S \)-finite \( R \)-module and \( f : M \to N \) a \( u\)-\( S \)-epimorphism. Then \( N \) is \( S \)-finite.

(2) Let \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) be a \( u\)-\( S \)-exact sequence. If \( L \) and \( N \) are \( S \)-finite, so is \( M \).

(3) Let \( f : M \to N \) be a \( u\)-\( S \)-isomorphism. If one of \( M \) and \( N \) is \( S \)-finite, so is the other.

Proof. (1) Consider the exact sequence \( M \xrightarrow{f} N \to T \to 0 \) with \( sT = 0 \) for some \( s \in S \). Let \( F \) be a finitely generated submodule of \( M \) such that \( s'\)\( M \subseteq F \) for some \( s' \in S \). Then \( f(F) \) is a finitely generated submodule of \( N \) such that \( ss'N \subseteq f(F) \).

(2) Suppose \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) is a \( u\)-\( S \)-exact sequence. Let \( L_1 \) and \( N_1 \) be finitely generated submodules of \( L \) and \( N \) such that \( s_1L \subseteq L_1 \) and \( s_1N \subseteq N_1 \)...
for some $s_L, s_N \in S$ respectively. Let $M_1$ be a finitely generated submodule of $M$ generated by the finite images of generators of $L_1$ and the finite pre-images of finite generators of $N_1$. Then for any $m \in M$, $s_N g(m) \in N_1$. Thus there exists $m_1 \in M_1$ such that $s_N g(m) = g(m_1)$. We have $s_N m - m_1 \in \text{Ker}(g)$. Since there exists $s \in S$ such that $s \text{Ker}(g) \subseteq \text{Im}(f)$. So there exists $l \in L$ such that $s(s_N m - m_1) = f(l)$. Then there exists $l_1 \in L_1$ such that $s_L l = l_1$. Thus $s_L s(s_N m - m_1) = s_L f(l) = f(s_L l) = f(l_1)$. Consequently, $s_L s s_N m = s_L s m_1 + s f(l_1) \in M_1$. So $s_L s s_N M \subseteq M_1$. Since $M_1$ is finitely generated, we have $M$ is $S$-finite.

(3) It is a consequence of (2). 

□

3. $u$-$S$-FLAT MODULES AND $u$-$S$-VON NEUMANN REGULAR RINGS

Recall from [12] that an $R$-module $F$ is called flat provided that for any short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is exact. Now, we give an $S$-analogue of flat modules.

**Definition 3.1.** Let $R$ be a ring, $S$ a multiplicative subset of $R$. An $R$-module $F$ is called $u$-$S$-flat (abbreviates uniformly $S$-flat) provided that for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$, the induced sequence $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is $u$-$S$-exact.

Recall from [12] that an $R$-module $F$ is flat if and only if $\text{Tor}_1^R(M, F) = 0$ for any $R$-module $M$, if and only if $\text{Tor}_n^R(M, F) = 0$ for any $R$-module $M$ and $n \geq 1$. We give an $S$-analogue of this result.

**Theorem 3.2.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $F$ an $R$-module. The following statements are equivalent:

1. $F$ is $u$-$S$-flat;
2. for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to A \otimes_R F \xrightarrow{f \otimes_R F} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F \to 0$ is $u$-$S$-exact;
3. $\text{Tor}_1^R(M, F)$ is $u$-$S$-torsion for any $R$-module $M$;
4. $\text{Tor}_n^R(M, F)$ is $u$-$S$-torsion for any $R$-module $M$ and $n \geq 1$.

**Proof.** (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3): Trivial.

(2) $\Rightarrow$ (3): Let $0 \to L \to P \to M \to 0$ be a short exact sequence with $P$ projective. Then there exists a long exact sequence

$$0 \to \text{Tor}_1^R(M, F) \to F \otimes L \to P \otimes F \to M \otimes F \to 0.$$ 

Thus $\text{Tor}_1^R(M, F)$ is $u$-$S$-torsion by (2).
(3) ⇒ (4): Let $M$ be an $R$-module. Denote by $\Omega^{n-1}(M)$ the $(n-1)$-th syzygy of $M$. Then $\text{Tor}_n^n(M, F) \cong \text{Tor}_1^1(\Omega^{n-1}(M), F)$ is $u$-$S$-torsion by (3).

(2) ⇒ (1): Let $F$ be an $R$-module satisfies (2). Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a $u$-$S$-exact sequence. Then there is an exact sequence $B \xrightarrow{g} C \to T \to 0$ where $T = \text{Coker}(g)$ is $u$-$S$-torsion. Tensoring $F$ over $R$, we have an exact sequence

$$B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F \to T \otimes_R F \to 0.$$ 

Then $T \otimes_R F$ is $u$-$S$-torsion by Corollary 2.6. Thus $0 \to A \otimes_R F \xrightarrow{f \otimes_R F} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F \to 0$ is $u$-$S$-exact at $C \otimes_R F$.

There are naturally two short exact sequences: $0 \to \text{Ker}(f) \to A \to \text{Im}(f) \to 0$, $0 \to \text{Im}(f) \to B \to \text{Coker}(f) \to 0$, where $\text{Ker}(f)$ is $u$-$S$-torsion. Consider the induced exact sequence

$$0 \to \text{Ker}(f) \otimes_R F \xrightarrow{i_{\text{Ker}(f)} \otimes_R F} A \otimes_R F \to \text{Im}(f) \otimes_R F \to 0,$$

$$0 \to \text{Im}(f) \otimes_R F \xrightarrow{i_{\text{im}(f)} \otimes_R F} B \otimes_R F \to \text{Coker}(f) \otimes_R F \to 0,$$

where $\text{Ker}(i_{\text{im}(f)} \otimes_R F)$ and $\text{Ker}(i_{\text{Ker}(f)} \otimes_R F)$ are $u$-$S$-torsion. We have the following pull-back diagram:

\[
\begin{array}{ccccccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\text{Im}(i_{\text{Ker}(f)} \otimes_R F) & \to & \text{Im}(i_{\text{Ker}(f)} \otimes_R F) \\
\downarrow & & \downarrow \\
Y & \to & A \otimes_R F & \to & \text{Im}(i_{\text{im}(f)} \otimes_R F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Ker}(i_{\text{im}(f)} \otimes_R F) & \to & \text{Im}(f) \otimes_R F & \to & \text{Im}(i_{\text{im}(f)} \otimes_R F) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

Since $\text{Ker}(f)$ is $u$-$S$-torsion, so is $\text{Ker}(f) \otimes_R F$ by Corollary 2.6. Hence $\text{Im}(i_{\text{Ker}(f)} \otimes_R F)$ is $u$-$S$-torsion, and thus $Y$ is also $u$-$S$-torsion by Proposition 2.8. So the composition $f \otimes_R F : A \otimes_R F \to \text{Im}(i_{\text{im}(f)} \otimes_R F) \to B \otimes_R F$ is a $u$-$S$-monomorphism.

Thus $0 \to A \otimes_R F \xrightarrow{f \otimes_R F} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F \to 0$ is $u$-$S$-exact at $A \otimes_R F$.

Since the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $u$-$S$-exact at $B$, there exists $s_1 \in S$ such that $s_1 \text{Ker}(g) \subseteq \text{Im}(f)$ and $s_1 \text{Im}(f) \subseteq \text{Ker}(g)$. By (2), there are two exact sequences $0 \to T_1 \to s_1 \text{Ker}(g) \otimes_R F \to \text{Im}(f) \otimes_R F$ with $s_2 T_1 = 0$ for some $s_2 \in S$, and $0 \to T_2 \to s_1 \text{Im}(f) \otimes_R F \to \text{Ker}(g) \otimes_R F$ with $s_3 T_2 = 0$ for some $s_3 \in S$. Consider
the induced sequence $0 \to T \to \text{Ker}(g) \otimes_R F \to B \otimes_R F \to \text{Coker}(g) \otimes_R F \to 0$ with $s_4 T = 0$ for some $s_4 \in S$. Set $s = s_1 s_2 s_3 s_4$, we will show $s \text{Ker}(g \otimes_R F) \subseteq \text{Im}(f \otimes_R F)$ and $s \text{Im}(f \otimes_R F) \subseteq \text{Ker}(g \otimes_R F)$. Consider the following exact sequence

$$0 \to T \to \text{Ker}(g) \otimes_R F \xrightarrow{i_{\text{Ker}(g) \otimes_R F}} B \otimes_R F \xrightarrow{g \otimes_R F} C \otimes_R F.$$ 

Then $\text{Im}(i_{\text{Ker}(g) \otimes_R F}) = \text{Ker}(g \otimes_R F)$. Thus $s \text{Ker}(g \otimes_R F) = s_1 s_2 s_3 s_4 \text{Ker}(g \otimes_R F) = s_1 s_2 s_3 s_4 i_{\text{Ker}(g) \otimes_R F} \subseteq s_1 s_2 s_3 \text{Ker}(g) \otimes_R F \subseteq s_3 \text{Im}(f) \otimes_R F = s_3 \text{Im}(f \otimes_R F) \subseteq \text{Im}(f \otimes_R F)$, and $s \text{Im}(f \otimes_R F) = s_1 s_2 s_3 s_4 \text{Im}(f) \otimes_R F \subseteq s_2 s_4 \text{Ker}(g) \otimes_R F \subseteq s_2 \text{Im}(i_{\text{Ker}(g) \otimes_R F}) = s_2 \text{Ker}(g \otimes_R F) \subseteq \text{Ker}(g \otimes_R F)$. Thus $0 \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0$ is $u$-$S$-exact at $B \otimes_R F$.

By Corollary 2.6 and Theorem 3.2, flat modules and $u$-$S$-torsion modules are $u$-$S$-flat. And $u$-$S$-flat modules are flat provided that any element in $S$ is a unit. Moreover, if any element in $S$ is regular and all $u$-$S$-flat modules are flat, then any element in $S$ is a unit. Indeed, for any $s \in S$, we have $R/\langle s \rangle$ is $u$-$S$-flat and thus flat. So $\langle s \rangle$ is a pure ideal of $R$. By [2] Theorem 1.2.15], there exists $r \in R$ such that $s(1 - rs) = 0$. Since $s$ is regular, $s$ is a unit.

The following example shows that the condition “$\text{Tor}^R_1(M, F)$ is $u$-$S$-torsion for any $R$-module $M$” in Theorem 3.2 can not be replaced by “$\text{Tor}^R_1(R/I, F)$ is $u$-$S$-torsion for any ideal $I$ of $R$”.

**Example 3.3.** Let $\mathbb{Z}$ be the ring of integers, $p$ a prime in $\mathbb{Z}$ and $S = \{p^n \mid n \geq 0\}$ as in Example 2.2. Let $M = \mathbb{Z}(p)/\mathbb{Z}$. Then $\text{Tor}^R_1(R/I, M)$ is $u$-$S$-torsion for any ideal $I$ of $R$. However, $M$ is not $u$-$S$-flat.

**Proof.** Let $\langle n \rangle$ be an ideal of $\mathbb{Z}$. It follows from [1] Chapter I, Lemma 6.2(a)] that $\text{Tor}^\mathbb{Z}_1(\mathbb{Z}/\langle n \rangle, M) \cong \{m \in M \mid nm = 0\} = \{\frac{a}{p^b} + \mathbb{Z} \in \mathbb{Z}(p)/\mathbb{Z} \mid a, b \text{ satisfies } p^b|nb\}$. Write $n = p^k m$ where $(p, m) = 1$. If $k = 0$, then $\text{Tor}^\mathbb{Z}_1(\mathbb{Z}/\langle n \rangle, M) = 0$. If $k \geq 1$, then $\text{Tor}^\mathbb{Z}_1(\mathbb{Z}/\langle n \rangle, M) = \{\frac{b}{p^k} + \mathbb{Z} \in \mathbb{Z}(p)/\mathbb{Z} \mid a, b \in \mathbb{Z}\}$. Thus $p^k \cdot \text{Tor}^\mathbb{Z}_1(\mathbb{Z}/\langle n \rangle, M) = 0$. So $\text{Tor}^\mathbb{Z}_1(\mathbb{Z}/\langle n \rangle, M)$ is $u$-$S$-torsion for any ideal $\langle n \rangle$ of $\mathbb{Z}$. However, $\text{Tor}^\mathbb{Z}_1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}(p)/\mathbb{Z}) \cong t(\mathbb{Z}(p)/\mathbb{Z}) = \mathbb{Z}(p)/\mathbb{Z}$ by [1] Chapter I, Lemma 6.2(b)]. Since $\mathbb{Z}(p)/\mathbb{Z}$ is not $u$-$S$-torsion by Example 2.2, $M = \mathbb{Z}(p)/\mathbb{Z}$ is not $u$-$S$-flat. 

**Proposition 3.4.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then the following statements hold.

1. Any pure quotient of $u$-$S$-flat modules is $u$-$S$-flat.
2. Any finite direct sum of $u$-$S$-flat modules is $u$-$S$-flat.
3. Let $0 \to A \xrightarrow{i} B \xrightarrow{f} C \to 0$ be a $u$-$S$-exact sequence. If $A$ and $C$ are $u$-$S$-flat modules, so is $B$. 

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(4) Let $A \to B$ be a $u$-$S$-isomorphism. If one of $A$ and $B$ is $u$-$S$-flat, so is the other.

(5) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence. If $B$ and $C$ are $u$-$S$-flat, then $A$ is $u$-$S$-flat.

Proof. (1) Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with $B$ $u$-$S$-flat. Let $M$ be an $R$-module. Then there is an exact sequence $\text{Tor}_1^R(M, B) \to \text{Tor}_1^R(M, C) \to 0$. Since $\text{Tor}_1^R(M, B)$ is $u$-$S$-torsion, $\text{Tor}_1^R(M, C)$ also is $u$-$S$-torsion. Thus $C$ is $u$-$S$-flat.

(2) Let $F_1, \ldots, F_n$ be $u$-$S$-flat modules. Let $M$ be an $R$-module. Then there exists $s_i \in S$ such that $s_i \text{Tor}_1^R(M, F_i) = 0$. Set $s = s_1 \ldots s_n$. Then $s \text{Tor}_1^R(M, \bigoplus_{i=1}^n F_i) \cong \bigoplus_{i=1}^n s \text{Tor}_1^R(M, F_i) = 0$. Thus $\bigoplus_{i=1}^n F_i$ is $u$-$S$-flat.

(3) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence. Then there are three short exact sequences: $0 \to \text{Ker}(f) \to A \to \text{Im}(f) \to 0$, $0 \to \text{Ker}(g) \to B \to \text{Im}(g) \to 0$ and $0 \to \text{Im}(g) \to C \to \text{Coker}(g) \to 0$. Then Ker($f$) and Coker($g$) are all $u$-$S$-torsion and $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$ for some $s \in S$. Let $M$ be an $R$-module. Suppose $A$ and $C$ are $u$-$S$-flat. Then

$$\text{Tor}_1^R(M, A) \to \text{Tor}_1^R(M, \text{Im}(f)) \to M \otimes_R \text{Ker}(f)$$

is exact. Since $\text{Ker}(f)$ is $u$-$S$-torsion and $A$ is $u$-$S$-flat, it follows that $\text{Tor}_1^R(M, \text{Im}(f))$ is $u$-$S$-torsion. Note

$$\text{Tor}_2^R(M, \text{Coker}(g)) \to \text{Tor}_1^R(M, \text{Im}(g)) \to \text{Tor}_1^R(M, C)$$

is exact. Since $\text{Coker}(g)$ is $u$-$S$-torsion, then $\text{Tor}_2^R(M, \text{Coker}(g))$ is $u$-$S$-torsion by Corollary 2.6. Thus $\text{Tor}_1^R(M, \text{Im}(g))$ is $u$-$S$-torsion as $\text{Tor}_1^R(M, C)$ is $u$-$S$-torsion. We also note that

$$\text{Tor}_1^R(M, \text{Ker}(g)) \to \text{Tor}_1^R(M, B) \to \text{Tor}_1^R(M, \text{Im}(g))$$

is exact. Thus to verify $\text{Tor}_1^R(M, B)$ is $u$-$S$-torsion, we just need to show $\text{Tor}_1^R(M, \text{Ker}(g))$ is $u$-$S$-torsion. Set $N = \text{Ker}(g) + \text{Im}(f)$. Consider the following two exact sequences

$$0 \to \text{Ker}(g) \to N \to N/\text{Ker}(g) \to 0 \text{ and } 0 \to \text{Im}(f) \to N \to N/\text{Im}(f) \to 0.$$ 

Then it is easy to verify $N/\text{Ker}(g)$ and $N/\text{Im}(f)$ are all $u$-$S$-torsion. Consider the following induced two exact sequences

$$\text{Tor}_2^R(M, N/\text{Im}(f)) \to \text{Tor}_1^R(M, \text{Ker}(g)) \to \text{Tor}_1^R(M, N) \to \text{Tor}_1^R(M, N/\text{Im}(f)),$$

$$\text{Tor}_2^R(M, N/\text{Ker}(g)) \to \text{Tor}_1^R(M, \text{Im}(f)) \to \text{Tor}_1^R(M, N) \to \text{Tor}_1^R(M, N/\text{Ker}(g)).$$
Thus $\text{Tor}_R^1(M, \text{Ker}(g))$ is $u$-$S$-torsion if and only if $\text{Tor}_R^1(M, \text{Im}(f))$ is $u$-$S$-torsion. Consequently, $B$ is $u$-$S$-flat since $\text{Tor}_R^1(M, \text{Im}(f))$ is proved to be $u$-$S$-torsion as above.

(4) It can be certainly deduced from (3).

(5) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence. Then, as in the proof of (3), there are three short exact sequences: $0 \to \text{Ker}(f) \to A \to \text{Im}(f) \to 0$, $0 \to \text{Ker}(g) \to B \to \text{Im}(g) \to 0$ and $0 \to \text{Im}(g) \to C \to \text{Coker}(g) \to 0$. Then $\text{Ker}(f)$ and $\text{Coker}(g)$ are all $u$-$S$-torsion and $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$ for some $s \in S$. Let $M$ be an $R$-module. Note that

$$\text{Tor}_R^1(M, \text{Ker}(f)) \to \text{Tor}_R^1(M, A) \to \text{Tor}_R^1(M, \text{Im}(f)) \to M \otimes_R \text{Ker}(f)$$

is exact. Since $\text{Ker}(f)$ is $u$-$S$-torsion, then $\text{Tor}_R^1(M, \text{Ker}(f))$ and $M \otimes_R \text{Ker}(f)$ are $u$-$S$-torsion by Corollary [2.6]. It just need to verify $\text{Tor}_R^1(M, \text{Im}(f))$ is $u$-$S$-torsion. By the proof of (3), we just need to show $\text{Tor}_R^1(M, \text{Ker}(g))$ is $u$-$S$-torsion. Since

$$\text{Tor}_R^2(M, \text{Im}(g)) \to \text{Tor}_R^1(M, \text{Ker}(g)) \to \text{Tor}_R^1(M, B)$$

is exact and $\text{Tor}_R^1(M, B)$ is $u$-$S$-torsion, we just need to show $\text{Tor}_R^2(M, \text{Im}(g))$ is $u$-$S$-torsion. Note that

$$\text{Tor}_R^3(M, \text{Coker}(g)) \to \text{Tor}_R^2(M, \text{Im}(g)) \to \text{Tor}_R^2(M, C)$$

is exact. Since $\text{Coker}(g)$ is $u$-$S$-torsion and $C$ is $u$-$S$-flat, we have $\text{Tor}_R^3(M, \text{Coker}(g))$ and $\text{Tor}_R^2(M, C)$ are $u$-$S$-torsion. So $\text{Tor}_R^2(M, \text{Im}(g))$ is $u$-$S$-torsion. \hfill \Box

Remark 3.5. It is well known that any direct limit of flat modules is flat. However, every direct limit of $u$-$S$-flat modules is not $u$-$S$-flat. Let $Z$ be the ring of integers, $p$ a prime in $Z$ and $S = \{p^n \mid n \geq 0\}$ as in Example [3.3]. Let $F_n = Z/\langle p^n \rangle$ be a $Z$-module. Then $F_n$ is $u$-$S$-torsion, and thus $u$-$S$-flat. Note that each $F_n$ is isomorphic to $M_n = \{\frac{a}{p^n} + Z \in \mathbb{Z}_{(p)}/\mathbb{Z} \mid a \in \mathbb{Z}\}$. It is easy to verify $\mathbb{Z}_{(p)}/\mathbb{Z} = \bigcup_{i=1}^{\infty} M_n \cong \lim_{\longrightarrow} F_n$.

However, $\mathbb{Z}_{(p)}/\mathbb{Z}$ is not $u$-$S$-flat (see Example 3.3).

It is also worth noting infinite direct sums of $u$-$S$-flat modules need not be $u$-$S$-flat. Let $M_n = \{\frac{a}{p^n} + Z \in \mathbb{Z}_{(p)}/\mathbb{Z} \mid a \in \mathbb{Z}\}$ as above. Then $M_n$ is $u$-$S$-flat. Set $N = \bigoplus_{n=1}^{\infty} M_n$. Then $N$ is a torsion module. Thus $\text{Tor}_R^1(\mathbb{Q}/\mathbb{Z}, N) = N$ by [4, Chapter I, Lemma 6.2(b)]. It can similarly be deduced from the proof of Example 2.2 that $N$ is not $u$-$S$-torsion. Thus $N$ is not $u$-$S$-flat.

**Corollary 3.6.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. If $F$ is $u$-$S$-flat over a ring $R$, then $F_S$ is flat over $R_S$. 

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Proof. Let \( I_S \) be a finitely generated ideal of \( R_S \), where \( I \) is a finitely generated ideal of \( R \). Then there exists \( s \in S \) such that \( s\text{Tor}^R_1(R/I, F) = 0 \). Thus \( 0 = \text{Tor}^R_1(R/I, F)_S \cong \text{Tor}^{R_S}_1(R_S/I_S, F_S) \). So \( F_S \) is flat over \( R_S \). \qed

Remark 3.7. Note that the converse of Corollary 3.6 does not hold. Consider \( \mathbb{Z} \)-module \( M = \mathbb{Z}_{(p)}/\mathbb{Z} \) in Example 2.2. Let \( S = \{p^n \mid n \geq 0\} \). Then \( M_S = 0 \) and thus is flat over \( \mathbb{Z}_S \). However, \( M \) is not u-S-flat over \( \mathbb{Z} \) (see Example 3.3).

Proposition 3.8. Let \( R \) be a ring and \( F \) an \( R \)-module. Let \( S \) be a multiplicative subset of \( R \) consisting of finite elements. Then \( F \) is u-S-flat over a ring \( R \) if and only if \( F_S \) is flat over \( R_S \).

Proof. We just need to show that if \( F_S \) is flat over \( R_S \), then \( F \) is u-S-flat over a ring \( R \). Let \( 0 \to A \xrightarrow{f} B \to C \to 0 \) be a short exact sequence over \( R \). By tensoring \( F \), we have an exact sequence \( 0 \to T \to A \otimes_R F \xrightarrow{f \otimes R F} B \otimes_R F \to C \otimes_R F \to 0 \) where \( T \) is the kernel of \( f \otimes R F \). By tensoring \( R_S \), we have an exact sequence \( 0 \to T_S \to A_S \otimes_{R_S} F_S \to B_S \otimes_{R_S} F_S \to C_S \otimes_{R_S} F_S \to 0 \) over \( R_S \). Since \( F_S \) is flat over \( R_S \), \( T_S = 0 \). Thus \( T \) is \( S \)-torsion. By Proposition 2.3, \( T \) is u-S-torsion. So \( F \) is u-S-flat over a ring \( R \). \qed

Let \( p \) be a prime ideal of \( R \). We say an \( R \)-module \( F \) is \( u-p \)-flat shortly provided that \( F \) is u-(\( R \setminus p \))-flat.

Proposition 3.9. Let \( R \) be a ring and \( F \) an \( R \)-module. Then the following statements are equivalent:

1. \( F \) is flat;
2. \( F \) is u-p-flat for any \( p \in \text{Spec}(R) \);
3. \( F \) is u-m-flat for any \( m \in \text{Max}(R) \).

Proof. (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) : Trivial.

(3) \( \Rightarrow \) (1) : Let \( M \) be an \( R \)-module. Then \( \text{Tor}^R_1(M, F) \) is (\( R \setminus m \))-torsion. Thus for any \( m \in \text{Max}(R) \), there exists \( s_m \in S \) such that \( s_m \text{Tor}^R_1(M, F) = 0 \). Since the ideal generated by all \( s_m \) is \( R \), \( \text{Tor}^R_1(M, F) = 0 \). So \( F \) is flat. \qed

Recall that a ring \( R \) is called von Neumann regular provided that for any \( a \in R \), there exists \( r \in R \) such that \( a = ra^2 \). One of the main topics is the \( S \)-analogue of von Neumann regular rings. In order to study further, we will characterize when a ring \( R_S \) is von Neumann regular in the next result.

Proposition 3.10. Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following statements are equivalent:

1. \( R_S \) is a von Neumann regular ring;
(2) any principal ideal of \( R \) is \( S \)-generated by an idempotent;
(3) any \( S \)-finite ideal of \( R \) is \( S \)-generated by an idempotent;
(4) for any \( a \in R \), there exist \( s \in S \) and \( r \in R \) such that \( sa = ra^2 \);
(5) any \( R_S \)-module is flat over \( R_S \).

**Proof.** (1) \( \Leftrightarrow \) (5) : It is well known. (3) \( \Rightarrow \) (2) : Trivial.

(1) \( \Rightarrow \) (4) : Let \( a \in R \). Then there exists \( \frac{a}{s} \in R_S \) such that \( \frac{a}{s} = \frac{r}{s_1} \frac{a^2}{1} \). Thus there exists \( s_2 \in S \) such that \( s_1 s_2 a = s_2 r_1 a^2 \). Set \( s = s_1 s_2 \) and \( r = s_2 r_1 \), (4) holds naturally.

(4) \( \Rightarrow \) (1) : Let \( \frac{a}{s} \) be an element in \( R_S \). Then there are \( s' \in S \) and \( x \in R \) such that \( s'a = xa^2 \). Thus \( \frac{a}{s} = \frac{x}{s_1} \frac{a^2}{1} \). So \( R_S \) is a von Neumann regular ring.

(4) \( \Rightarrow \) (2) : Let \( \langle a \rangle \) be a principal ideal of \( R \). Then there exists \( s \in S \) such that \( sa = ra^2 \) for some \( r \in R \). Set \( e = ra \). Then \( se = e^2 \) and \( e \in \langle a \rangle \). Since \( sa = ea \in \langle e \rangle \), we have \( s \langle a \rangle \subseteq \langle e \rangle \subseteq \langle a \rangle \).

(2) \( \Rightarrow \) (3) : Let \( K \) be an \( S \)-finite ideal and \( I = Ra_1 + \cdots + Ra_n \) be a finitely generated sub-ideal of \( I \) such that \( s'K \subseteq I \) for some \( s' \in S \). By (2), for each \( i \) there is an idempotent \( e_i \in Ra_i \) such that \( s_i \langle a_i \rangle \subseteq \langle e_i \rangle \) for some \( s_i \in S \) \((i = 1, \cdots, n)\). Set \( s = s_1 s_2 \cdots s_n \). Then \( s \langle a_i \rangle \subseteq \langle e_i \rangle \). Set \( J = Re_1 + \cdots + Re_n \). Then \( J \) is a sub-ideal of \( I \) (thus of \( K \)) such that \( sK \subseteq s_1 \cdots s_n I \subseteq J \). Claim that \( J \) is generated by an idempotent. Indeed, for any \( x \in J \), we have \( x = r_1 e_1 + \cdots + r_n e_n = r_1 e_1^2 + \cdots + r_n e_n^2 \in J^2 \). Thus \( J^2 = J \). Since \( J \) is finitely generated, \( J = \langle e \rangle \) for some idempotent \( e \in I \) by [12, Theorem 1.8.22].

(2) \( \Rightarrow \) (4) : Let \( a \in R \). Then there is an idempotent \( e \) such that \( s \langle a \rangle \subseteq \langle e \rangle \subseteq \langle a \rangle \). If \( e = ba \) for some \( b \in R \), then \( e = e^2 = b^2 a^2 \). Thus \( sa = ce = cb^2 a^2 \) for some \( cb^2 \in R \). So (4) holds. \( \square \)

Recall from [3] that a ring \( R \) is called \( c \)-\( S \)-coherent if any \( S \)-finite ideal \( I \) is \( c \)-\( S \) finitely presented, that is, there exists a finitely presented sub-ideal \( J \) of \( I \) such that \( sI \subseteq J \subseteq I \). By Proposition 3.10, the following result holds since any ideal generated by an idempotent is projective, and thus is finitely presented.

**Corollary 3.11.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). If \( R_S \) is a von Neumann regular ring, then \( R \) is \( c \)-\( S \)-coherent.

It is certain that for a ring \( R \) such that \( R_S \) is von Neumann regular, the element \( s \in S \) such that \( sa = ra^2 \) for some \( r \in R \) depends on \( a \in R \) by Proposition 3.10. Now we give the definition of \( u \)-\( S \)-von Neumann regular ring for which the element \( s \in S \) is uniform on any element \( a \in R \).

**Definition 3.12.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). \( R \) is called a \( u \)-\( S \)-von Neumann regular ring (abbreviates uniformly \( S \)-von Neumann regular ring)
provided there exists an element \( s \in S \) satisfying that for any \( a \in R \) there exists \( r \in R \) such that \( sa = ra^2 \).

Let \( \{M_j\}_{j \in \Gamma} \) be a family of \( R \)-modules. Let \( \{m_{i,j}\}_{i \in \Lambda_j} \subseteq M_j \) for each \( j \in \Gamma \) and \( N_j = \langle m_{i,j}\rangle_{i \in \Lambda_j} \). We say a family of \( R \)-modules \( \{M_j\}_{j \in \Gamma} \) is \( u \)-\( S \)-generated by \( \{\{m_{i,j}\}_{i \in \Lambda_j}\}_{j \in \Gamma} \) provided that there exists an element \( s \in S \) such that \( sM_j \subseteq N_j \) for each \( j \in \Gamma \). It is well known that a ring \( R \) is a von Neumann regular ring if and only if every \( R \)-module is flat, if and only if any principal (finitely generated) ideal is generated by an idempotent (see [12, Theorem 3.6.3]). Now we give an \( S \)-analogue of this result.

**Theorem 3.13.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following statements are equivalent:

1. \( R \) is a \( u \)-\( S \)-von Neumann regular ring;
2. for any \( R \)-module \( M \) and \( N \), there exists \( s \in S \) such that \( s\text{Tor}_1^R(M, N) = 0 \);
3. there exists \( s \in S \) such that \( s\text{Tor}_1^R(R/I, R/J) = 0 \) for any ideals \( I \) and \( J \) of \( R \);
4. there exists \( s \in S \) such that \( s\text{Tor}_1^R(R/I, R/J) = 0 \) for any \( S \)-finite ideals \( I \) and \( J \) of \( R \);
5. there exists \( s \in S \) such that \( s\text{Tor}_1^R(R/\langle a \rangle, R/\langle a \rangle) = 0 \) for any element \( a \in R \);
6. any \( R \)-module is \( u \)-\( S \)-flat;
7. the class of all principal ideals of \( R \) is \( u \)-\( S \)-generated by idempotents;
8. the class of all finitely generated ideals of \( R \) is \( u \)-\( S \)-generated by idempotents.

**Proof.** (1) \( \iff \) (5): It follows from the equivalences: \( s\text{Tor}_1^R(R/\langle a \rangle, R/\langle a \rangle) = 0 \) if and only if \( \frac{s(a)}{[s]} = 0 \), if and only if there exists \( r \in R \) such that \( sa = ra^2 \).

(2) \( \iff \) (6), (8) \( \Rightarrow \) (7) and (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5): Trivial.

(2) \( \Rightarrow \) (3): Set \( M = N = \bigoplus_{I \in R} R/I \). Then (3) holds naturally.

(3) \( \Rightarrow \) (2): Suppose \( M \) is generated by \( \{m_i \mid i \in \Gamma \} \) and \( N \) is generated by \( \{n_i \mid i \in \Lambda \} \). Well-order \( \Gamma \) and \( \Lambda \). Set \( M_0 = 0 \) and \( M_{\alpha} = \langle m_i \mid i < \alpha \rangle \) for each \( \alpha \leq \Gamma \). Then \( M \) has a continuous filtration \( \{M_\alpha \mid \alpha \leq \Gamma \} \) with \( M_{\alpha+1}/M_{\alpha} \cong R/I_{\alpha+1} \) and \( I_\alpha = \text{Ann}_R(m_\alpha + M_\alpha \cap Rm_\alpha) \). Similarly \( N \) has a continuous filtration \( \{N_\beta \mid \beta \leq \Lambda \} \) with \( N_{\beta+1}/N_\beta \cong R/J_{\beta+1} \) and \( J_\beta = \text{Ann}_R(n_\beta + N_\beta \cap Rn_\beta) \). Since \( s\text{Tor}_1^R(R/I_\alpha, R/J_\beta) = 0 \) for each \( \alpha \leq \Gamma \) and \( \beta \leq \Lambda \), it is easy to verify \( s\text{Tor}_1^R(M, N) = 0 \) by transfinite induction on both positions of \( M \) and \( N \).

(5) \( \Rightarrow \) (3): By [12, Exercise 3.20], we have \( s\text{Tor}_1^R(R/I, R/J) = \frac{s(I \cap J)}{[s]} \) for any ideals \( I \) and \( J \) of \( R \). So we just need to show \( s(I \cap J) \subseteq IJ \). Let \( a \in I \cap J \). Since \( s\text{Tor}_1^R(R/\langle a \rangle, R/\langle a \rangle) = \frac{s(a)}{[s]} = 0 \), it follows that \( sa \in s\langle a \rangle \subseteq \langle a^2 \rangle \subseteq IJ \). Thus \( s\text{Tor}_1^R(R/I, R/J) = 0 \).
Claim that generated, \( x = \{ s \text{ generated ideals of } R \} \) for any \( s \in S \) such that \( sa = e^2 \) and \( e \in \langle a \rangle \). Since \( sa = ea \in \langle e \rangle \), we have \( s\langle a \rangle \subseteq \langle e \rangle \subseteq \langle a \rangle \) for any \( a \in R \).

(7) \( \Rightarrow \) (8) : Let \( \{ I_j = Ra_{i,j} + \cdots + Ra_{n,j} \mid j \in \Gamma \} \) be the family of all finitely generated ideals of \( R \). By (3), there exists an element \( s \in S \) such that for each \( j \in \Gamma \) and \( i = 1,\ldots,n \), there is an idempotent \( e_{i,j} \in Ra_{i,j} \) such that \( s\langle a_{i,j} \rangle \subseteq \langle e_{i,j} \rangle \). Set \( J_j = Re_{1,j} + \cdots + Re_{n,j} \). Then \( J_j \) is a sub-ideal of \( I_j \) such that \( sJ_j \subseteq I_j \subseteq J_j \). Claim that \( J_j \) is generated by an idempotent. Indeed, for any \( x \in J_j \), we have \( x = r_1e_1 + \cdots + r_ne_n = r_1e_1^2 + \cdots + r_ne_n^2 \in J_j^2 \). Thus \( J_j^2 = J_j \). Since \( J_j \) is finitely generated, \( J_j = \langle e_j \rangle \) for some idempotent \( e_j \in I_j \) by [12, Theorem 1.8.22]. So \( \{ I_j \mid j \in \Gamma \} \) is \( u\)-\( S \)-generated by \( \{ \langle e_j \rangle \mid j \in \Gamma \} \).

(7) \( \Rightarrow \) (1) : There are an element \( s \in S \) and a family of idempotents \( \{ e_a \mid a \in R \} \) such that \( s\langle a \rangle \subseteq \langle e_a \rangle \subseteq \langle a \rangle \) for any \( a \in R \). Write \( e_a = ba \) for some \( b \in R \). Then \( e_a = e_a^2 = b^2a^2 \). Thus \( sa = ce_a = cb^2a^2 \) for some \( cb^2 \in R \). So \( R \) is \( u\)-\( S \)-von Neumann regular.

\[\square\]

**Corollary 3.14.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). If \( R \) is a \( u\)-\( S \)-von Neumann regular ring, then \( R_S \) is a von Neumann regular ring. Consequently, any \( u\)-\( S \)-von Neumann regular ring is \( c\)-\( S \)-coherent.

**Proof.** It follows from Proposition 3.10, Corollary 3.11 and Theorem 3.13. \[\square\]

Note that a ring \( R \) such that \( R_S \) is von Neumann regular is not necessarily \( u\)-\( S \)-von Neumann regular.

**Example 3.15.** Let \( \mathbb{Z} \) be the ring of all integers, \( S = \mathbb{Z} \setminus \{0\} \). Then \( \mathbb{Z}_S = \mathbb{Q} \) is a von Neumann regular ring. Let \( p \) be a prime in \( \mathbb{Z} \) and \( M = \mathbb{Z}_{(p)}/\mathbb{Z} \). Then \( \text{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_{(p)}/\mathbb{Z}) \cong \mathbb{Z}_{(p)}/\mathbb{Z} \) by [4, Chapter I, Lemma 6.2(b)]. It is easy to verify that \( n\mathbb{Z}_{(p)}/\mathbb{Z} \neq 0 \) for any \( n \in S \). Thus \( M \) is not \( u\)-\( S \)-torsion, and so \( \mathbb{Z} \) is not a \( u\)-\( S \)-von Neumann regular ring.

**Corollary 3.16.** Let \( R \) be a ring. Let \( S \) be a multiplicative subset of \( R \) consisting of finite elements. Then \( R \) is a \( u\)-\( S \)-von Neumann regular ring if and only if \( R_S \) is a von Neumann regular ring.

**Proof.** We just need to show that if \( R_S \) is a von Neumann regular ring then \( R \) is a \( u\)-\( S \)-von Neumann regular ring. Let \( S = \{ s_1, \cdots, s_n \} \). Set \( s = s_1 \cdots s_n \). By Proposition 3.10 for any \( a \in R \), there exists \( s_i \in S \) and \( r_a \in R \) such that \( s_i a = r_a a^2 \). Thus \( sa = ra^2 \) for any \( a \in R \) and some \( r \in R \).

\[\square\]

Since every flat module is \( u\)-\( S \)-flat, von Neumann regular rings are \( u\)-\( S \)-von Neumann regular. The following result shows \( u\)-\( S \)-von Neumann regular rings are always...
von Neumann regular provided $S$ is a regular multiplicative set, i.e., the multiplicative set $S$ is composed of non-zero-divisors.

**Proposition 3.17.** Let $R$ be a ring and $S$ a regular multiplicative subset of $R$. Then $R$ is $u$-$S$-von Neumann regular if and only if $R$ is von Neumann regular.

**Proof.** We just need to show if $R$ is $u$-$S$-von Neumann regular, then $R$ is von Neumann regular. Suppose $R$ is a $u$-$S$-von Neumann regular ring. Then there exists $s \in S$ such that for any $a \in R$ there exists $r \in R$ satisfying $sa = ra^2$. Taking $a = s^2$, we have $s^3 = rs^4$. Since $s$ is a non-zero-divisor of $R$, we have $1 = sr$. Thus $s$ is a unit. So for any $a \in R$ there exists $r \in R$ such that $a = (s^{-1}r)a^2$. It follows that $R$ is a von Neumann regular ring. \( \square \)

However, the condition that “any element in $S$ is a non-zero-divisor” in Proposition 3.17 cannot be removed. Let $R$ be any ring and $S$ a multiplicative subset of $R$ containing a nilpotent element. Then $R$ is a $u$-$S$-von Neumann regular ring. Indeed, let $s$ be a nilpotent element in $R$ with nilpotent index $n$. Then $0 = s^n \in S$. Thus for any $a \in R$, we have $0 = 0a = 0a^2 = 0$. So $R$ is $u$-$S$-von Neumann regular. If the multiplicative subset $S$ of $R$ does not contain 0, the condition that “any element in $S$ is a non-zero-divisor” in Corollary 3.17 also cannot be removed.

**Example 3.18.** Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ be a semi-simple ring and $s = (1, 0) \in T$. Then any element $a \in T$ satisfies $a^2 = a$ and $2a = 0$. Let $R = T[x]/(sx, x^2)$ with $x$ the indeterminate and $S = \{1, s\}$ be a multiplicative subset of $R$. Then $R$ is a $u$-$S$-von Neumann regular ring, but $R$ is not von Neumann regular. Indeed, let $r = a + b\overline{x}$ be any element in $R$, where $\overline{x}$ is the residual element of $x$ in $R$ and $a, b \in T$. Then $sr = s(a + b\overline{x}) = sa = sa^2 = s(a^2 + 2ab\overline{x} + b^2\overline{x}^2) = s(a + b\overline{x})^2 = sr^2$. Thus $R$ is $u$-$S$-von Neumann regular. However, since $R$ is not reduced, $R$ is not von Neumann regular by [12, Theorem 3.6.16(2), Exercise 3.48].

Let $\mathfrak{p}$ be a prime ideal of $R$. We say a ring $R$ is a $u$-$\mathfrak{p}$-von Neumann regular ring shortly provided $R$ is an $u$-$(R \setminus \mathfrak{p})$-von Neumann regular ring. The final result gives a new local characterization of von Neumann regular rings.

**Proposition 3.19.** Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ is a von Neumann regular ring;
2. $R$ is a $u$-$\mathfrak{p}$-von Neumann regular ring for any $\mathfrak{p} \in \text{Spec}(R)$;
3. $R$ is a $u$-$\mathfrak{m}$-von Neumann regular ring for any $\mathfrak{m} \in \text{Max}(R)$.

**Proof.** (1) $\Rightarrow$ (2) : Let $F$ be an $R$-module and $\mathfrak{m} \in \text{Max}(R)$. Then $F$ is flat, and thus $u$-$\mathfrak{m}$-flat. So $R$ is an $u$-$\mathfrak{m}$-von Neumann regular ring.
(2) ⇒ (3) : Trivial.
(3) ⇒ (1) : Let $M$ be an $R$-module. Then $M$ is $m$-flat for any $m \in \text{Max}(R)$. Thus $M$ is flat by Proposition 3.9. So $R$ is a von Neumann regular ring. □

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