Geodesics in the Space of Kähler Cone Metrics II: Uniqueness of Constant Scalar Curvature Kähler Cone Metrics

KAI ZHENG
University of Warwick

Abstract
In this article, we give a complete construction of geodesics in the space of Kähler cone metrics (cone geodesics), and we address the problem on the uniqueness of constant scalar curvature Kähler (cscK) cone metrics when the cone angle \( \beta \) stays in the whole interval \((0, 1]\). The part \( \beta \in \left[\frac{1}{2}, 1\right) \) requires new weighted function spaces and new analytic techniques. We determine the asymptotic behavior of both cone geodesics and cscK cone metrics, prove the reductivity of the automorphism group, and establish the linear theory for the Lichnerowicz operator, which immediately implies the openness of the path deforming the cone angles of cscK cone metrics. © 2019 Wiley Periodicals, Inc.

Contents

1. Introduction 2621
2. Cone Geodesics 2626
3. Uniqueness of cscK Cone Metrics 2650
4. Regularity of cscK Cone Metrics 2668
5. Linear Theory of Lichnerowicz Operators 2680
6. The Automorphism Group Is Reductive If a cscK Cone Metric Exists 2691
7. Bifurcation of the \( J \)-Twisted Path 2693
Bibliography 2698

1 Introduction

Geodesics in the space of Kähler metrics, first established in Chen [20], are fundamental geometric objects in Kähler geometry and link constant scalar curvature Kähler (cscK) metrics in differential geometry to geometric invariant theory; see the foundational articles by Calabi-Chen [15] and also Donaldson [32, 33]. One main aim of this article is to construct geodesics in the space of Kähler metrics with cone singularities (cone geodesics), Theorem 1.1 below. To a large extent,
there is a continuously growing interest in studying constant scalar curvature Kähler metrics with cone singularities (see Chen [21]) due to the recent achievements in Kähler geometry. Another aim of this article is to utilize cone geodesics to study constant scalar curvature Kähler metrics with cone singularities (cscK cone metrics, Definition 3.1), Theorem 1.10 in the general conjectural picture. We refer to the excellent review articles by Chen [21] and Donaldson [35] for many more references and a more detailed and updated account of the development of the cscK metric problem.

The geometric objects we study in the present article are cone geodesics and cscK cone metrics. Let \( D \) be a smooth positive divisor in a smooth closed \( n \)-dimensional Kähler manifold \( X \), and \( \omega_0 \) be a smooth Kähler metric in \( X \). In this article, the cone angle is assumed to be

\[
0 < \beta \leq 1,
\]

and we are mainly concerned with the case when the cone angle \( \frac{1}{2} \leq \beta < 1 \), which requires more new ideas and is technically more involved than the complement half-angle case \( 0 < \beta < \frac{1}{2} \), which has been done in our series of articles [16, 45, 49].

The Kähler cone metrics of cone angle \( \beta \) is a Kähler metric in the regular part \( M := X \setminus D \) and has cone singularities of angle \( \beta \) along the divisor \( D \); see Definition 2.1. We let \( \mathcal{H}_\beta \) be the space of Kähler cone metrics of cone angle \( \beta \). The geodesic segment \( \{ \varphi(t) : 0 \leq t \leq 1 \} \) connecting two Kähler cone metrics \( \{ \varphi_i \in \mathcal{H}_\beta, i = 0, 1 \} \) turns out to be a homogeneous complex Monge-Ampère equation on the product manifold \( M \times R \),

\[
(\Omega_0 + i\partial\overline{\partial}\Psi)^{n+1} = 0.
\] (1.1)

Here \( R = S^1 \times [0, 1] \) is a cylinder with boundary, and \( \Omega_0 \) and \( \Psi \) are the lifting of \( \omega_0 \) and \( \varphi \), respectively, to the product manifold; see (2.3).

There is extensive literature on the Dirichlet problem for the homogeneous complex Monge-Ampère equation. We will extend the method developed in Chen [20] and He [43] to our cone geodesics. In our case, the source of difficulties are twofold: not only is the right-hand side degenerate, but the boundary values are also singular. They are the main technical difficulties, in particular for the regularity theory. We first construct the generalized cone geodesic, which has bounded spatial Laplacian; see Definition 2.3.

**Theorem 1.1 (Cone geodesic).** Suppose that \( 0 < \beta \leq 1 \) and \( \omega_1, \omega_2 \) are two Kähler cone metrics in \( \mathcal{H}_\beta \). Then there exists a unique generalized cone geodesic connecting them.

A more precise statement of this theorem is Theorem 2.4. We then consider more regularity of the cone geodesic, i.e., the full Laplacian. In [16], we restricted the cone angle to less than \( \frac{1}{2} \) and constructed the \( C^{1,1,\beta} \) cone geodesic between Kähler cone metrics with some bounded geometry, i.e., the subspace \( \mathcal{H}_C \) in \( \mathcal{H}_\beta \).
The boundary values in $\mathcal{H}_C$ were used to construct the background metric with bounded Christoffel symbols and Ricci upper bound.

The background metric has two functions in the construction of the $C^{1,1,\beta}$ cone geodesic; it served as an initial solution for the continuity method, and the a priori estimates required the geometric conditions of the background metric. But we will show a different approach in this paper. In Section 2, we will use an approximation method and choose different background metrics. As a result, we relax the geometric conditions of the background metric to be (2.8), (2.9), and (2.10), improve the a priori estimates, and construct the $C^{1,1,\beta}_w$ cone geodesics with appropriate weights. The precise statements are Theorem 2.4 and Theorem 2.6.

We will introduce the definitions of cscK cone metrics and $J$-twisted cscK cone metrics in Definition 3.1. Their approximation will also be illustrated in Section 3 for further applications. In Section 4, we describe the regularity of constant scalar curvature Kähler cone metrics. The following regularity theorem is a loose statement of this result; a full description of the higher-order asymptotic of cscK cone metrics are presented in Section 4. We refer to Theorem 4.8 for the complete statement.

**Theorem 1.2 (Regularity of cscK cone metrics).** Let $\omega = \omega_0 + i \partial \bar{\partial} \varphi$ be a constant scalar curvature Kähler cone metric. Suppose that $0 < \beta \leq 1$ and the Hölder exponent $\alpha$ satisfies

$$\alpha \beta < 1 - \beta.$$

Then $\varphi$ is $C^{3,\alpha,\beta}_w$.

**Remark 1.3.** When $\beta = 1$, $\alpha$ can be chosen arbitrarily in $(0, 1)$.

**Remark 1.4.** The “$w$” in $C^{3,\alpha,\beta}_w$ stands for “weak.” It will become clear in Section 4.1 why we chose this terminology.

**Remark 1.5.** Theorem 1.2 is a simple corollary of our main regularity Theorem 4.8.

The strategy pursued in this work is to scale the Kähler potential $\varphi$ and then to derive the estimates by using techniques inspired by our previous paper [61] on an asymptotic analysis of the complex Monge-Ampère equation with cone singularities. The new difficulty is that we need to deal with an elliptic system (see (4.4)). In [61], the expansion formula has more detailed information than [44,51], and the method is very different. We will see in Section 4 that this new method can be adapted for the cscK cone metrics as (4.4).

In Section 5, we investigate the linearized operator of the cscK cone metric, i.e., the Lichnerowicz operator (5.2). We introduce a new space $C^{4,\alpha,\beta}_w(\omega)$ (Definition 5.1). We show that this is the right space to prove the Fredholm alternative theorem for the Lichnerowicz operator.
Theorem 1.6 (Linear theory). Suppose that $0 < \beta \leq 1$ and $0 < \alpha \beta < 1 - \beta$. Let $\omega$ be a constant scalar curvature Kähler cone metric. Assume that $f \in C^{0, \alpha, \beta} \cap C^{2}$ with normalization condition $\int_X f \omega^n = 0$. Then one of the following holds:

- The Lichnerowicz equation $\Box_{\omega} u = f$ has a unique $C^{4, \alpha, \beta}(\omega)$ solution, or
- the kernel of $\Box_{\omega} (u)$ generates a holomorphic vector field tangent to $D$.

Remark 1.7. The linear theory implies the openness of the cscK cone path, which deforms the cone angle of cscK cone metrics by following the implicit function theorem argument in section 4.4 in Donaldson’s paper [34].

This theorem was proved in our paper [45] under the half-angle restriction $0 < \beta < \frac{1}{2}$ and in the space $C^{4, \alpha, \beta}(\omega)$, which has better regularities.

The key steps of the proof of Theorem 1.6 can be summarized as follows:

- solving the perturbed bi-Laplacian equation (5.5) in $C^{4, \alpha, \beta}(\omega)$ by developing a $L^2$-theory for the fourth-order elliptic PDEs with cone leading coefficients, and
- building the continuity path connecting the perturbed Lichnerowicz equation with the perturbed bi-Laplacian equation and showing that along the continuity path, the $C^{4, \alpha, \beta}(\omega)$ estimates hold uniformly.

In Section 6, we generalize the Matsushima-Lichnerowicz reductivity theorem of the automorphism group for the cscK metric to the cscK metric with cone singularities with full angle $0 < \beta \leq 1$. In our previous paper [49], we proved this theorem under the half-angle condition $0 < \beta < \frac{1}{2}$ after proving the regularity of the half-angle cscK cone metric. New ingredients in this part are the asymptotic of Christoffel symbols of the cscK cone metric in Section 4.2 and the asymptotic of functions in $C^{4, \alpha, \beta}(\omega)$ in Section 5.3.

Our precise result is the content of Theorem 6.1. Let us state it loosely as follows. Let $\mathfrak{h}(X; D)$ be the space of all holomorphic vector fields tangential to the divisor.

Theorem 1.8 (Reductivity). Suppose $\omega$ is a constant scalar curvature Kähler cone metric. Then the Lie algebra $\mathfrak{h}(X; D)$ has a direct sum decomposition:

\[ \mathfrak{h}(X; D) = \mathfrak{a}(X; D) \oplus \mathfrak{b}(X; D), \]

in which $\mathfrak{a}(X; D)$ is the complex, abelian Lie subalgebra of $\mathfrak{h}(X; D)$ consisting of all parallel holomorphic vector fields tangential to $D$. $\mathfrak{b}(X; D)$ is the Lie algebra ideal of $\mathfrak{h}(X; D)$ consisting of the image under the $\text{grad}_X$ of the kernel of the $\Box_{\omega}$ operator, and it is also the complexification of the Lie algebra consisting of Killing vector fields of $X$ tangential to $D$ with nonempty zero set. Moreover, the Lie algebra $\mathfrak{h}(X; D)$ is reductive.
In Section 7, we prove the bifurcation of the $J$-twisted path at a cscK cone metric ($t = 1$),

\begin{equation}
\Phi(t, \varphi(t)) := S(\omega_\varphi(t)) - S_\beta - (1 - t) \left( \frac{\omega^n_0}{\omega^n_\varphi(t)} - 1 \right) = 0.
\end{equation}

The bifurcation argument of Bando-Mabuchi [3] for the Aubin-Yau path at a Kähler-Einstein metric was extended to the cone path in our previous paper [50] and to Chen’s path [21] at the extremal metric in [27]. We use the word “twisted” to describe the term $(1 - t)(\omega^n_0 / \omega^n_\varphi(t) - 1)$, which is different from Chen’s continuity path [21]; see Remark 3.8 for further information.

We will prove the bifurcation at a cscK cone metric by utilizing the linear theory in Section 5 and the reductivity theorem in Section 6.

**Theorem 1.9 (Bifurcation).** Suppose that $\omega = \omega_0 + i \partial \bar{\partial} \varphi$ is a constant scalar curvature Kähler cone metric. Then there exists a parameter $\tau > 0$ such that $\varphi(t)$ with $\varphi(1) = \varphi$ is extended uniquely to a smooth one-parameter family of solutions of the $J$-twisted path (1.3) on $(1 - \tau, 1]$.

Let us conclude this introductory section with the uniqueness of constant scalar curvature Kähler cone metrics. This article addresses the following fundamental question: *Is the constant scalar curvature Kähler cone metric in $\mathcal{H}_\beta$ unique up to automorphisms?*

**Theorem 1.10 (Uniqueness of cscK cone metrics).** The constant scalar curvature Kähler cone metric along $D$ is unique up to automorphisms.

**Remark 1.11.** A by-product of the uniqueness theorem is that the log-$K$-energy is bounded from below in $\mathcal{H}_\beta$ if we assume the existence of a cscK cone metric in $\mathcal{H}_\beta$. The lower bound implies log-$K$-semistability immediately, which extends the result in [53] for cscK orbitfold metrics.

**Remark 1.12.** On a Riemannian surface, the uniqueness of the singular Gauss curvature metric is a classical outstanding problem in conformal geometry and geometric topology; see [18, 52, 57, 58] for references and related uniqueness results. About the nonuniqueness result, there are examples of constant Gauss curvature conical metrics with angle larger than 1 that are not unique. One may ask whether a similar phenomenon occurs for Calabi’s extremal metrics (a generalization notion of cscK metrics). In general, when the cone angle $\beta > 1$, the uniqueness of such critical metrics also fails, e.g., Chen’s examples for HCMU (the Hessian of the curvature of the metric is umbilical) metrics in [19].

**Remark 1.13.** In a recent article [12], the uniqueness of Kähler-Einstein cone metrics is proved as a result of the convexity of the Ding functional along bounded geodesics. We provide a different approach of this result in [50] by extending Bando-Mabuchi’s bifurcation argument [3] to the conical case. The results are extended to more general singularities, i.e., the Kawamata log terminal pairs in [9].
The method in [12] is also applied to Kähler-Ricci solitons, whose uniqueness was proved in [56] by the bifurcation argument.

As shown in a series of previous papers [45, 49], we observe the loss of regularity phenomena along the singular direction of cscK cone metrics when the angle is larger than half. This is the main difficulty of the proof of Theorem 1.10 which causes the technical complexity. Approaches to address this problem will be presented in Section 3, as well as Sections 4, 5, 6, and 7.

In Section 3, we will show that the log-\(K\)-energy is continuous and convex along the cone geodesics (Theorem 3.19). Recall that the key ingredient to show the uniqueness of cscK metrics, as well as extremal Kähler metrics, is the convexity of the \(K\)-energy along Chen’s geodesic in the recent papers by Berman-Berndtsson [8], Chen-Li-Păun [26], and Chen-Păun-Zeng [27]. It is also shown in Li [47] that the log-\(K\)-energy is convex along the \(C^{1,1}\) geodesic, which is constructed in our previous paper [16]. In Section 3.3, we will extend their methods to prove the convexity of the log-\(K\)-energy along the cone geodesics for any cone angle \(\beta \in (0, 1]\) constructed in this article. That is used to prove the uniqueness of \(J\)-twisted cscK cone metrics. The new difficulty is again the loss of regularity of the cone geodesic, which is different than in the nonsingular case.

Utilizing Bando-Mabuchi’s bifurcation method, the proof of Theorem 1.10 is now reduced to showing the deformation of the \(J\)-twisted path (1.3) from a cscK cone metric to a \(J\)-twisted cscK cone metric. In order to realize such deformation, we perturb the cscK cone metric along the \(J\)-twisted path and assemble all parts of the proof including the following ingredients, which are organized in this article as follows:

- the regularity of cscK cone metrics (Section 4),
- the Fredholm alternative theorem for the Lichnerowicz operator (Section 5),
- the reductivity of the automorphism group (Section 6), and
- the bifurcation argument (Section 7).

The methods and results in this paper have potential application to other singular canonical models, which could be with Poincaré singularities [2, 45, 55], on singular varieties [9, 24, 25], in degenerate situations [4-7, 10, 14, 60], etc. We raise the following question:

**Question 1.14.** Could we extend the uniqueness theorem (Theorem 1.10) to more general situations, when both the metric \(\omega\) and the space \(X\) are singular? If so, could we completely determine all possible singularities?

## 2 Cone Geodesics

In this section, we solve the boundary value problem for geodesics in the space of Kähler cone metrics. Recall that \(X\) is a smooth compact \(n\)-dimensional Kähler manifold without boundary and \(\omega_0\) is a smooth Kähler metric. We are also given
a smooth hypersurface $D$ in $X$ and a cone angle $0 < \beta \leq 1$. Let $s$ be a global section of the associated bundle $L_D$ of $D$ and $h$ be a Hermitian metric on $L_D$.

**Definition 2.1.** A Kähler cone metric $\omega$ of cone angle $\beta$ along $D$ is a smooth Kähler metric on the regular part $M := X \setminus D$, and quasi-isometric to the flat cone metric,

$$\omega_{\text{cone}} := \beta^2 |z^1|^2(\beta-1) i \, dz^1 \wedge d\bar{z}^1 + \sum_{2 \leq j \leq n} i \, dz^j \wedge d\bar{z}^j,$$

under the cone chart $\{U_p; z^1, \ldots, z^n\}$ near $p \in D$, where $z^1$ is the local defining function of $D$.

**Definition 2.2.** We let $\mathcal{H}_\beta$ be the space of all Kähler potentials such that the associated Kähler metrics are Kähler cone metrics in $[\omega_0]$, i.e.,

$$\mathcal{H}_\beta = \{ \phi \mid \omega_\phi = \omega_0 + i \, \delta \bar{\delta} \phi \text{ is a Kähler cone metric in } [\omega_0] \}.$$

The space $\mathcal{H}_\beta$ contains the model metric

$$\omega_D = \omega_0 + \delta i \, \delta \bar{\delta} |s|^2,$$

provided with a sufficiently small constant $\delta > 0$. We will also use $\omega$ without specification to denote $\omega_D$ for short in this paper.

**Cone Geodesic Equation**

We denote the cylinder $R = [0, 1] \times S^1$ and $z^{n+1} = x^{n+1} + \sqrt{-1} \, y^{n+1}$ with $x^{n+1} = t$. We also denote the $(n+1)$-dimensional product manifold with boundary and its regular part by

$$\mathcal{X} = X \times R, \quad \mathcal{M} = M \times R.$$

We extend the segment $\{ \varphi(z^1, \ldots, z^n, t) : 0 \leq t \leq 1 \}$ to the product manifold $\mathcal{X}$ as

$$\varphi(z', z^{n+1}) := \varphi(z^1, \ldots, z^n, x^{n+1}) = \varphi(z^1, \ldots, z^n, t).$$

We let $\pi$ be the natural projection from $\mathcal{X}$ to $X$ and also denote, after lifting up, the new coordinates, Kähler metric $\omega_0$ and Kähler potential $\varphi$ on $\mathcal{X}$, by

$$\begin{aligned}
\begin{cases}
z & = (z', z^{n+1}) = (z^1, \ldots, z^n, z^{n+1}) \\
\Omega_0 & = \pi^* \omega_0 + d z^{n+1} \wedge d z^{n+1} \\
\Omega_{\text{cone}} & = \pi^* \omega_{\text{cone}} + d z^{n+1} \wedge d z^{n+1} \\
\Omega & = \pi^* \omega_D + d z^{n+1} \wedge d z^{n+1} = \Omega_0 + i \, \delta \bar{\delta} \psi_D, \\
\psi(z) & = \varphi(z) - |z^{n+1}|^2, \\
\Omega_\psi & = \Omega_0 + i \, \delta \bar{\delta} \psi.
\end{cases}
\end{aligned}$$
With the notations above, a cone geodesic connecting \( \varphi_0 \) and \( \varphi_1 \) is a curve segment \( \{ \varphi(t) : 0 \leq t \leq 1 \} \subset \mathcal{H}_B \) and satisfies a homogeneous complex Monge-Ampère equation
\[
[\varphi'' - (\partial \varphi', \partial \varphi')_{g_\varphi}] \det(\omega_\varphi) = \det(\Omega_\varphi) = 0 \quad \text{in } \mathcal{M},
\]
in which
\[
\varphi' = \frac{\partial \varphi}{\partial t} \quad \text{and} \quad (\partial \varphi', \partial \varphi')_{g_\varphi} = \sum_{1 \leq i, j \leq n} g^{ij}_{\varphi} \frac{\partial \varphi'}{\partial z^i} \frac{\partial \varphi'}{\partial \overline{z}^j}.
\]
Along the cone geodesic segment, \( \omega_\varphi(t) \) is a possibly degenerate Kähler cone metric for any \( 0 < t < 1 \).

The Dirichlet boundary conditions \( \Omega_0 \) are on two disjoint copies of \( X \),
\[
\begin{aligned}
\Psi_0(z', 0) &= \Psi(z', \sqrt{-1}y^{n+1}) \\
&= \varphi_0(z') - (y^{n+1})^2 \quad \text{on } X \times \{0\} \times S^1, \\
\Psi_0(z', 1) &= \Psi(z', 1 + \sqrt{-1}y^{n+1}) \\
&= \varphi_1(z') - 1 - (y^{n+1})^2 \quad \text{on } X \times \{1\} \times S^1.
\end{aligned}
\]
We let the Hölder exponent satisfy
\[
\alpha \beta < 1 - \beta.
\]
The Hölder spaces \( C^{0, \alpha, \beta}, C^{2, \alpha, \beta} \) on \( \mathcal{X} \) are defined in section 2 in [16].

Assume that \( \Omega_b \) is a Kähler cone metric in the product manifold \( \mathcal{X} \). We will show in the following sections, how a priori estimates depend on \( \Omega_b \), and leave the construction of \( \Omega_b \) in Section 2.

We approximate the geodesic equation by a family of equations,
\[
\begin{aligned}
\Omega^{n+1}_{\Psi} &= \tau \cdot \Omega_0^{n+1} \quad \text{in } \mathcal{M}, \\
\Psi &= \Psi_0 \quad \text{on } \partial \mathcal{X}.
\end{aligned}
\]
We let \( \Psi_{0, \epsilon} \) be the smooth approximation of the boundary value by Richberg’s regularization (see [13, 31]), since our boundary Kähler potentials are Hölder continuous. However, when the boundary values are cscK cone metrics, we will present alternative approximations of them in Section 3.2 for future application. We also need the background metric \( \Omega_b \) to have a smooth approximation \( \Omega_{b_\epsilon} \); see Section 2, that is,
\[
\Omega^{n+1}_{b_\epsilon} = \frac{e^{-f_0} \cdot \Omega_0^{n+1}}{(|s|^2 + \epsilon^2)^{1-\beta}}.
\]
Then we consider the smooth approximation equation
\[
\begin{aligned}
\Omega^{n+1}_{\Psi_\epsilon} &= \tau \cdot \Omega^{n+1}_{b_\epsilon} \quad \text{in } \mathcal{X}, \\
\Psi_\epsilon &= \Psi_{0, \epsilon} \quad \text{on } \partial \mathcal{X}.
\end{aligned}
\]
The smooth approximation equation (2.7) has a unique smooth solution $\Psi$. Moreover, according to Chen [20], the $C^{1,1}$-norm of $\Psi$ is uniformly bounded, independent of $\tau$, i.e.,

$$\|\Psi\|_{C^{1,1}} = \sup_{\mathcal{X}} \{ |\Psi| + |\partial^1 \Psi|_{\Omega_0} + |\partial^2 \Psi|_{\Omega_0} \}.$$ 

**Geometric Conditions on the Background Metric $\omega_b$ and $\Omega_b$**

In [16], we reduce the $C^{1,1,\beta}$ estimate to the geometric conditions of the background metric $\Omega_1$, i.e.,

- the cone angle less than $\frac{1}{2}$,
- bounded Christoffel symbols, and
- the bounded Ricci upper bound.

But in this paper, we will use $\Omega_1$ in a different way.

We recall the construction of the background metric $\Omega_1$. It is constructed by using the boundary values. Let $\Psi_0$ be the line segment between the boundary Kähler cone potentials $\Psi_0 = t \varphi_1 + (1 - t) \varphi_0$, and $\Phi_{\text{convex}}$ be a convex function on $\mathcal{X}^{n+1}$ that vanishes on the boundary. Letting $\Psi_1 := \Psi_0 + m \Phi_{\text{convex}}$, we know from proposition 2.5 in [16] that

$$\Omega_1 := \Omega_0 + \sum_{1 \leq i, j \leq n+1} i \partial^1_i \partial^1_j \Psi_1$$

is a Kähler cone metric on $(\mathcal{X}, \mathcal{D})$. The $\Psi_1$ inherits regularities from the boundary values. When $\varphi_0, \varphi_1 \in C_w^{3,\alpha,\beta}$, we have $\Psi_1 \in C_w^{3,\alpha,\beta}$ by using the computation from corollary 4.2 in [16]. We denote the weight $\kappa = \beta - \alpha \beta$.

This weight comes from the computation of the Christoffel symbols of cscK cone metrics; see Section 4.2. We obtain that under the normal coordinate, the Christoffel symbols satisfy

$$\text{sup} \left( \sum_{2 \leq i \leq n+1} |\nabla^1_i \Omega_1|_\Omega + |z^1|_\Omega \cdot |\nabla^1 \Omega_1|_\Omega \right) \leq C,$$ 

according to Corollary 4.3 and 4.4.

We will use background metrics $\Omega_b$ and $\omega_b$, whose geometry can be unbounded. The background metric $\omega_b$ stays on $\mathcal{X}$, and $\Omega_b$ is the one on the product manifold $\mathcal{X}$, i.e.,

$$\Omega_b = \pi^* \omega_b + d z^{n+1} \wedge d \bar{z}^{n+1}.$$ 

We present the construction of the background metrics $\Omega_b$ and $\omega_b$ in order to fulfill the requirement of a priori estimates in later sections.

The following geometric conditions of $\omega_b$ (same for $\Omega_b$ on $\mathcal{X}$) will be used in obtaining the a priori estimates.
(1) Christoffel symbols:

\[
\sup_X \sum_{2 \leq i \leq n} |\nabla^\text{cone}_i \omega_b|_{\omega_b} + |z^{1/2} \cdot |\nabla^\text{cone}_i \omega_b|_{\omega_b}| \leq C_1.
\]

(2) Riemannian curvature for some fixed constants $C_2, C_3$:

\[
\begin{cases}
R_{ijkl}(\omega_b) \geq -(\tilde{g}_b)_{ij} \cdot (\tilde{g}_b)_{kl} \\
\tilde{\omega}_b = C_2 \cdot \omega_b + i \tilde{\partial} \Phi_b \geq 0, \\
|\Phi_b|_{\infty} \leq C_3.
\end{cases}
\]

A simple way to construct $\omega_b$ and its approximation is to use $\omega_b = \omega_D$ and $\omega_{\mathbf{b}e} = \omega_0 + \delta i \tilde{\partial}(|s|^2 + e)^{\beta}$ with small $\delta$ and $\tilde{\omega}_{\mathbf{b}e} = \omega_0 + \delta i \tilde{\partial}(|s|^2 + e)^{\gamma}$ with $0 < \gamma < \min\{\beta, 1 - \beta\}$. Then $\Phi^{\mathbf{b}e} = (|s|^2 + e)^{\gamma}$ is bounded.

Another way to construct $\omega_{\mathbf{b}e}$ (and also $\omega_b$) uses the weight function

\[
\lambda_{\mathbf{b}e}(e^2 + t) = \gamma^{-1} \int_0^t \left( e^2 + r \right)^{\gamma} - e^{2\gamma} \frac{d r}{r}
\]

in [41]; then

\[
\omega_{\mathbf{b}e} = \omega_0 + \delta i \tilde{\partial} \lambda_{\mathbf{b}e}(|s|^2 + e), \quad \Phi^{\mathbf{b}e} = \lambda_{\mathbf{b}e}(|s|^2 + e), \quad \tilde{\omega}_{\mathbf{b}e} = \omega_{\mathbf{b}e} + i \tilde{\partial} \Phi^{\mathbf{b}e}.
\]

In this construction, $\Phi^{\mathbf{b}e}$ is also bounded. Meanwhile, both $\omega_b$ satisfy property (2.10), and the second construction has an additional property of

\[
F_{\epsilon} = \log \frac{(|s|^2 + e)^{\beta-1} \omega_{\mathbf{b}e}^{\beta}}{\omega_{\mathbf{b}0}^{\beta-1}},
\]

that is,

\[
i \tilde{\partial} F_{\epsilon} \geq -(C \omega_{\mathbf{b}e} + i \tilde{\partial} \Phi^{\mathbf{b}e}).
\]

**Generalized Cone Geodesic**

After taking $\epsilon \to 0$, the solutions of the smooth approximation equation (2.7) converge to a solution of the approximation equation (2.6). Then taking $\tau \to 0$, we obtain a solution of the geodesic equation (2.4).

**Definition 2.3.** We say $\varphi(t) : 0 \leq t \leq 1$ is a generalized cone geodesic if it is the limit of solutions to the approximation equation (2.6) as $\tau \to 0$ under the following norm:

\[
\|\varphi\|_{C_\Delta} = \sup_{(t', z')^n+1} \{\|\varphi\| + |\partial_t \varphi| + |\partial_z \varphi|_{\omega} + |\partial_{z'} \partial_{z'} \varphi|_{\omega}\}.
\]

The estimate of $|\varphi| + |\partial_t \varphi| + |\partial_z \varphi|_{\omega}$ follows from Lemma 2.10, Lemma 2.11, Lemma 2.12, and the interior spatial Laplacian estimate (Proposition 2.23). The interior spatial gradient estimate $|\partial_z \varphi|_{\omega}$ follows from Proposition 2.24.
**Theorem 2.4.** Assume \( \{q_0, \sigma = 0, 1\} \) are two Kähler cone metrics. Let \( \Omega_0 \) be the background metric constructed in Section 2 and satisfying curvature condition (2.10). Then there exists a unique \( C_4^n \) generalized cone geodesic connecting \( q_0 \), and there is a constant \( C \) independent of \( \tau \) such that

\[
\|q_0\|_{C_4^n} \leq C.
\]

The constant \( C \) depends on constants in (2.10) and

\[
\sup_{x} \text{Tr}_{\omega} \omega q_0, \quad \sup_{x} |q_0|.
\]

**Cone Geodesic**

**Definition 2.5.** We say \( \{q(t) : 0 \leq t \leq 1\} \) is a cone geodesic if it is the limit of solutions to the approximation equation (2.6) as \( \tau \to 0 \) under the following norm:

\[
\|q\|_{C_{\alpha}^{1.1.\beta}} = \|q\|_{C_4^n} + \sup_{x} \left\{ \sum_{2 \leq i \neq n} |z^1|^{\kappa+1} \left| \frac{\partial^2 q}{\partial z^i \partial t} \right| + |z^1|^{\kappa+1-\beta} \left| \frac{\partial^2 q}{\partial z^i \partial t} \right| + |z^1|^{2\kappa} \left| \frac{\partial^2 q}{\partial t^2} \right| \right\}
\]

where \( \kappa = \beta - \alpha \beta \).

The second-order estimates follow from the boundary Hessian estimates (Proposition 2.13) and the interior Laplacian estimates (Proposition 2.21).

When the points are far away from the divisor, all estimates are the same as in the smooth case. So we always focus on the cone chart, which intersects with the divisor. In the definition of the norm \( C_{\alpha}^{1.1.\beta} \), we write down the formulas in the cone chart but omit the smooth ones for convenience. We will always use this convention in the following estimates.

**Theorem 2.6.** Assume \( \{q_0 : \sigma = 0, 1\} \) are two \( C_{\alpha}^{3.\beta} \) Kähler cone metrics. Let \( \Omega_0 \) be the background metric constructed in Section 2 and satisfying curvature conditions (2.9) and (2.10). Then there exists a unique \( C_{\alpha}^{1.1.\beta} \) cone geodesic connecting \( q_0 \) and satisfying (2.6). Furthermore, there is a constant \( C \) independent of \( \tau \) such that

\[
\|q_0\|_{C_{\alpha}^{1.1.\beta}} \leq C.
\]

The constant \( C \) depends on the constants in (2.8), (2.9), and (2.10), and

\[
\sup_{x} |\Psi|, \quad \sup_{x} |\Psi_0|, \quad \sup_{x} |\partial \Psi_0|_{\Omega}, \quad \sup_{x} |\partial \Psi_1|_{\Omega}^2, \quad \sup_{x} |\Omega_0|_{\Omega}, \quad \sup_{x} \text{Tr}_{\omega} \omega q_0, \quad \sup_{x} |z^1|^{\kappa+1-\beta} \left| \frac{\partial^2 \Psi_1}{\partial z^i \partial t} \right| \quad \sup_{x} \left| \frac{\partial^2 \Psi_1}{\partial z^i \partial t} \right| \quad 2 \leq i \leq n.
\]

**Remark 2.7.** The \( C_{\alpha}^{3.\beta} \)-norm of \( q_0 \) controls the terms involving \( \Psi \) and \( \Psi_1 \).

The \( C_{\alpha}^{1.1.\beta} \) cone geodesic in Theorem 2.6 has more regularity than the \( C_{\alpha}^{3.\beta} \) generalized cone geodesic in Theorem 2.4. The (twisted) cscK cone metrics satisfy...
the boundary conditions in Theorem 2.6 and thus produce a better cone geodesic connecting them. This observation will be used in the later sections.

Remark 2.8. The ideas and techniques developed in [61] could be applied to cone geodesics; we leave this discussion for future development.

Remark 2.9. The construction of bounded geodesics (weak solutions) has been elucidated in well-known literature; see, for example, [11, 12, 39, 40]. However, it is necessary to establish a self-contained asymptotic analysis theory to understand smoothness/regularity of the cone geodesics precisely for further application.

Some Instant Estimates

We collect basic estimates of (2.6). It is sufficient to consider the case when \( \tau \) is sufficiently small. Then

\[
\begin{align*}
\Omega^{n+1} = \tau & \frac{\Omega_1^{n+1}}{\Omega_1^{n+1}} \Omega_1^{n+1} \leq \Omega_1^{n+1} \quad \text{in } \mathcal{M}, \\
\psi = \psi_1 = \psi_0 & \quad \text{on } \partial \mathcal{X}.
\end{align*}
\]

Let \( h \) satisfy the linear equation

\[
\begin{align*}
\Delta \Omega_0 h &= -n - 1 \quad \text{in } \mathcal{M}, \\
h &= \psi_0 & \quad \text{on } \partial \mathcal{X}.
\end{align*}
\]

Lemma 2.10 (\( L^\infty \)-estimate). For any point \( x \in \mathcal{X} \), it holds that

\[
\psi_1(x) \leq \psi(x) \leq h(x).
\]

Proof. For all \( 0 < \beta \leq 1 \), the maximum principle implies the \( L^\infty \)-estimate. The lower bound follows from Jeffreys’ method to avoid the maximum achieved on the boundary (lemma 2.4 and proposition 3.1 in [16]), and the upper bound follows from applying the weak maximum principle for the conical Laplacian equation (lemma 5.5 and proposition 3.2 in [16]).

The \( L^\infty \)-estimate yields the boundary gradient estimate. Since all \( \psi_1, \psi, \) and \( h \) have the same values on the boundary, we conclude that the gradient of \( \psi \) is controlled by the gradient of \( \psi_1 \) and \( h \).

Lemma 2.11 (Boundary gradient estimate).

\[
\sup_{\partial \mathcal{X}} |\nabla \psi| \leq \sup_{\mathcal{X}} |\nabla \psi_1| + \sup_{\mathcal{X}} |\nabla h|.
\]

Lemma 2.12. \( \frac{\partial \psi}{\partial t} \) is bounded uniformly along the whole path \( 0 \leq t \leq 1 \).

Proof. The boundary gradient estimate tells us that \( \frac{\partial \psi}{\partial x} \) is uniformly bounded at the boundary. By making use of (2.3), we see that \( \frac{\partial \psi}{\partial t} \) is also uniformly bounded at the boundary. By (2.6), \( \frac{\partial^2 \psi}{\partial t^2} \geq 0 \). Thus the lemma is proved.
2.1 Boundary Hessian Estimates

In this section, we improve the boundary Hessian estimates, section 3.3 in [16]. As shown in [16], there is an obstruction to directly obtain the mixed singular tangent-normal estimates on the boundary

\[ |z^{1}|^{1-\beta} \left| \frac{\partial^{2}\Psi}{\partial z^{1} \partial t} \right|, \]

since the term \(|\nabla_{1}^{\text{cone}} \Omega_{b}|_{\Omega_{\text{cone}}}^{\text{cscK}}| \) is generally not bounded for large angles. For example, we know from [16] that the model cone metric \(\omega_{D}\) only has bounded \(|\nabla_{1}^{\text{cone}} \omega_{D}|_{\Omega_{\text{cone}}} \) when the angle is less than \(\frac{\pi}{2}\). From Corollary 4.1 we will see that the KE cone metric \(\omega_{\text{KE}}\) or, more generally, the cscK cone metric, has bounded \(|\nabla_{1}^{\text{cone}} \omega_{\text{cscK}}|_{\Omega_{\text{cone}}}^{\text{cscK}}| \) when the angle is less than \(\frac{1}{2}\). Actually, it has growth rate \(|z^{1}|^{-\kappa}\) with \(\kappa = \beta - \alpha\beta\) for any \(\beta \in (0, 1)\).

The boundary Hessian estimates are divided into four parts:

1. the tangent-tangent estimates on the boundary, which follow directly from the boundary values,

\[ \frac{\partial^{2}(\Psi - \Psi_{1})}{\partial z^{i} \partial \bar{z}^{j}} = \frac{\partial^{2}(\Psi - \Psi_{1})}{\partial z^{i} \partial \bar{z}^{j}} = 0 \quad \forall \ 1 \leq i, j \leq n; \]

2. the mixed regular tangent-normal estimates on the boundary (Proposition 2.18),

\[ \sup_{\partial z} \left| \frac{\partial^{2}\Psi}{\partial z^{i} \partial \bar{z}^{n+1}} \right|, \quad \sup_{\partial z} \left| \frac{\partial^{2}\Psi}{\partial z^{i} \partial \bar{z}^{n+1}} \right|, \quad \forall \ 2 \leq i \leq n; \]

3. the mixed singular tangent-normal estimates on the boundary (Proposition 2.19),

\[ \sup_{\partial z} |z^{1}|^{1-\beta} \left| \frac{\partial^{2}\Psi}{\partial z^{1} \partial \bar{z}^{n+1}} \right|, \quad \sup_{\partial z} |z^{1}|^{1-\beta} \left| \frac{\partial^{2}\Psi}{\partial z^{1} \partial \bar{z}^{n+1}} \right|; \]

4. the normal-normal estimates on the boundary (Proposition 2.20),

\[ \sup_{\partial z} |z^{1}|^{2\alpha} \left| \frac{\partial^{2}\Psi}{\partial z^{n+1} \partial \bar{z}^{n+1}} \Psi \right|, \quad \sup_{\partial z} |z^{1}|^{2\alpha} \left| \frac{\partial^{2}\Psi}{\partial z^{n+1} \partial \bar{z}^{n+1}} \right|. \]

**Proposition 2.13** (Boundary Hessian estimates). Assume that the Christoffel symbols of the background metrics \(\Omega_{1}\) and \(\Omega_{b}\) satisfy conditions (2.8) and (2.9), respectively. Then the terms in (2), (3), and (4) are bounded by

\[ C \left( \sup_{x} |\partial \Psi|^{2}_{\Omega} + 1 \right). \]
The constant $C$ depends on $X$, $\inf_X Tr_{\Omega_1}$, the constants in (2.8) and (2.9), and
\[
\sup_x |\Omega_h|_{\Omega_1}, \sup_x |\partial^2 \Psi_1|^2, \sup_x \left| \frac{\partial^2 \Psi_1}{\partial z^i \partial z^{n+1}} \right|, 2 \leq i \leq n.
\]

Remark 2.14. In general, a similar argument can be carried on with a different weight from $|z^1|^k$, which depends on the boundary values.

Boundary Mixed Regular Tangent-Normal Estimates

We will apply the barrier argument [37]. We are given a point $p$ on the boundary of $X$, i.e., $\partial X$. We assume the point $p$ is on the divisor; otherwise the estimates are simpler. We use the cone chart $\{U; z^1, \ldots, z^{n+1}\}$ centered at $p$ as before, i.e., $p = 0$, $z^1$ is the normal direction to $D$, and $z^{n+1} := x + \sqrt{-1} y$ parametrizes the cylinder $R$. We fix a half-ball of $p$; i.e., $B_i = x + \sqrt{-1} y$ parametrizes the cylinder $R$. We consider in $B_0^+$
\[
(2.13) \quad v := (\Psi - \Psi_1) + sx - Nx^2,
\]
where $s = 2N$ and $N$ is determined in (2.15) by $\epsilon_0$ and $\inf_X \Omega_{\text{cone}}^{n+1}/\Omega_h^{n+1}$.

Lemma 2.15. Let $\Delta_\Psi$ be the Laplacian with respect to $g_\Psi$. Let $\epsilon_0$ be the constant such that
\[
(2.14) \quad \Omega_1 \geq \epsilon_0 \cdot \Omega_{\text{cone}}^{n+1}/\Omega_h^{n+1}.
\]
Then there exists a constant $N$ depending on $\epsilon_0$ and $\inf_X \Omega_{\text{cone}}^{n+1}/\Omega_h^{n+1}$ such that the following inequalities hold:
\[
\begin{align*}
\Delta_\Psi v &\leq -\frac{\epsilon_0}{2} (1 + Tr_{\Omega_1} \Omega_{\text{cone}}) \quad \text{in } B_0^+, \\
v &\geq 0 \quad \text{on } \partial B_0^+, \\
v &\equiv 0 \quad \text{on } \partial B_0^+ \cap \partial X.
\end{align*}
\]

Proof. Recall that $\partial z^i = \frac{1}{2}(\partial x^i - i \partial y^i)$. Then direct computation shows that
\[
\Delta_\Psi v = n + 1 - Tr_{\Omega_1} \Omega_1 - N \cdot g_\Psi^{n+1, n+1}
\]
\[
\leq n + 1 - \frac{\epsilon_0}{2} Tr_{\Omega_1} \Omega_{\text{cone}} - \left[ \frac{\epsilon_0}{2} Tr_{\Omega_1} \Omega_{\text{cone}} + N \cdot g_\Psi^{n+1, n+1} \right].
\]
We then have, by the inequality of arithmetic and geometric means, that
\[
\frac{\epsilon_0}{2} Tr_{\Omega_1} \Omega_{\text{cone}} + N \cdot g_\Psi^{n+1, n+1} \geq (n + 1) \left[ \left( N + \frac{\epsilon_0}{2} \right) \frac{1}{2\pi} \Omega_{\text{cone}}^{n+1}/\Omega_{\text{cone}}^{n+1} \right]^\frac{1}{n+1}.
\]
Since $\Omega^{n+1}_\Psi = \tau \cdot \Omega^{n+1}_b$ and $\inf_X \Omega^{n+1}_{\text{cone}} / \Omega^{n+1}_b > 0$, we choose large $N$ such that

\begin{equation}
(n + 1) - (n + 1) \left[ \left( N + \frac{\epsilon_0}{2} \right)^{\frac{1}{2n}} \inf_X \Omega^{n+1}_{\text{cone}} \right]^{\frac{1}{n+1}} \leq -\frac{\epsilon_0}{2}.
\end{equation}

So the first inequality is proved. Moreover, the boundary inequality follows from $\Psi_1 \leq \Psi$ and the choice of $s$. \hfill \Box

**Lemma 2.16.** There exists a function $u$ such that

- $u$ vanishes at the center $p$ of $B_\rho^+$ and is nonnegative on $\partial B_\rho^+ \cap \partial X$;
- for $q \in \Gamma_\rho^+ := \partial B_\rho^+ \cap \text{Int}(X)$, $u(q) \geq \epsilon_1(\rho_0)$ for some constant $\epsilon_1(\rho_0)$ depending only on the fixed $\rho_0$;
- for some fixed constant $\epsilon_2 \geq 1$, it holds in $B_\rho^+$ that

$$\Delta_g u \leq \epsilon_2 \cdot \text{Tr} \Omega_g \Omega_{\text{cone}}.$$  

**Proof.** We could choose nonnegative $u$ to be the auxiliary globally bounded function in $M \times R$ constructed on page 1173 in [16] or the local function $|z|^2 + \sum_{2 \leq j \leq n+1} |z_j|^2$. In the latter case,

\begin{equation}
\epsilon_1(\rho_0) = \rho_0 \quad \text{and} \quad \epsilon_2 = 1. \hfill \Box
\end{equation}

We define the real operator

$$D_i := \partial_{z^i} + \partial_{\bar{z}^i} \log \sqrt{-1} \left( \partial_{z^i} - \partial_{\bar{z}^i} \right) \quad \forall \ 1 \leq i \leq n + 1,$$

and we consider $D_i(\Psi - \Psi_1)$.

**Lemma 2.17.** Suppose $1 \leq i \leq n + 1$. Then there exists a constant

\begin{equation}
\epsilon_3 = \sup \text{Tr} \Omega_g \Omega_{\text{cone}}
\end{equation}

such that on $M$,

$$|\Delta_g [D_i(\Psi - \Psi_1)]| \leq \epsilon_3 \cdot F_i \cdot (1 + \text{Tr} \Omega_g \Omega_{\text{cone}}),$$

where $F_i = |\nabla^\text{cone}_i \Omega_g|\Omega_{\text{cone}} + |\nabla^\text{cone}_i \Omega_1|\Omega_{\text{cone}}$.

**Proof.** We denote $g$ the local Kähler potential of $\omega$ and note that

$$\partial_{\bar{z}^a} \partial_{\bar{z}^b} g = g_{a\bar{b}}, \quad 1 \leq a, b \leq n + 1.$$

From $\Delta_g = \partial_{\bar{z}^a} \partial_{z^a} \partial_{\bar{z}^a} \partial_{z^a}$, we have

$$\Delta_g \partial_{\bar{z}^i} (\Psi - \Psi_1) = \partial_{\bar{z}^i} \log \Omega^{n+1}_\Psi - g_{\bar{a}a} \partial_{\bar{z}^i} (g_{\Psi_1})_{a\bar{b}}.$$

Using the approximation equation (2.6), we have

$$\partial_{\bar{z}^i} \log \Omega^{n+1}_\Psi = \partial_{\bar{z}^i} \log \Omega^{n+1}_b,$$

and then

\begin{equation}
\Delta_g \partial_{\bar{z}^i} (\Psi - \Psi_1) = \partial_{\bar{z}^i} \log \Omega^{n+1}_b - g_{\bar{a}a} \partial_{\bar{z}^i} (g_{\Psi_1})_{a\bar{b}}.
\end{equation}
Furthermore, we get
\[
|\Delta \psi [\partial_x^i (\Psi - \Psi_1)] |
\]
\[
= \sum_{1 \leq a, b \leq n+1} \left[ g^{ab}_{\psi \psi} \nabla^\text{cone}_i (\Omega_b)_{ab} - g^{ab}_{\psi \psi} \nabla^\text{cone}_i (\Omega_1)_{ab} \right]
\]
\[
\leq \epsilon_3 \cdot (|\nabla^\text{cone}_i \Omega_b|_{\Omega_{\text{cone}}} + |\nabla^\text{cone}_i \Omega_1|_{\Omega_{\text{cone}}}) \cdot (1 + \text{Tr}_{\Omega_{\psi} \Omega_{\text{cone}}}).
\]

The lemma follows from the definition of $D_i$.

\[\square\]

The mixed regular tangent-normal estimates on the boundary are already in the proof of proposition 3.7 in [16]. We present the proof here for later use.

**Proposition 2.18 (Boundary mixed regular tangent-normal estimates). For $2 \leq i \leq n$, there exists a constant $C$ such that**

\[
(2.19)
\]
\[
\sup_{\partial x} \left[ \left| \frac{\partial^2 (\Psi - \Psi_1)}{\partial z^i \partial z^{n+1}} \right| + \left| \frac{\partial^2 (\Psi - \Psi_1)}{\partial z^i \partial z^{n+1}} \right| \right]
\]
\[
\leq C \left[ 1 + \sup_{\partial x} |\partial_{n+1} (\Psi - \Psi_1)|_{\Omega} \right] \cdot \left[ 1 + \sup_{\partial x} |\partial_i (\Psi - \Psi_1)|_{\Omega} \right].
\]

**The constant $C$ depends on $X$, $\epsilon_0$ in (2.14) and**

\[
\sup_X |\Omega_b|_{\Omega_{\text{cone}}}, \quad \sup_X |\nabla^\text{cone}_i \Omega_b|_{\Omega_{\text{cone}}}, \quad \sup_X |\nabla^\text{cone}_i \Omega_1|_{\Omega_{\text{cone}}}, \quad 1 \leq a, b \leq n+1.
\]

**Proof. We consider**

\[
h := \lambda_1 v + \lambda_2 u + D_i (\Psi - \Psi_1),
\]

with two constants:

\[
(2.20) \quad \lambda_2 = \frac{1}{\epsilon_2} \cdot \left[ 1 + \sup_{\partial x} |\partial_i (\Psi - \Psi_1)|_{\Omega} \right],
\]

\[
(2.21) \quad \lambda_1 = \frac{4}{\epsilon_0} \cdot \left[ \lambda_2 \cdot \epsilon_2 + \epsilon_3 \cdot \sup_{\partial x} F_i \right].
\]

We check the boundary value of $h$. On $\partial B^{+}_{\rho_0} \cap \partial X$, $h \geq 0$. Using Lemma 2.15 and (2.20), we have the estimates of $h$ on the upper boundary, that is, for $q \in \partial B^{+}_{\rho_0} \cap \text{Int}(X)$,

\[
h(q) \geq \lambda_2 u(q) - |D_i (\Psi - \Psi_1)(q)|
\]
\[
\geq \lambda_2 \epsilon_1 - \sup_{\partial x} |\partial_i (\Psi - \Psi_1)|_{\Omega} \geq 0.
\]

Applying Lemma 2.15, Lemma 2.16, Lemma 2.17 and (2.21), we have the differential inequality that $h$ satisfies,

\[
\Delta \psi h \leq \left[ -\frac{\epsilon_0}{2} \lambda_1 + \lambda_2 \epsilon_2 + \epsilon_3 F_i \right] \cdot (1 + \text{Tr}_{\Omega_{\psi} \Omega_{\text{cone}}}) < 0.
\]
Then using the maximum principle and \( h(p) = 0 \), we have (recall \( z^{n+1} = x + \sqrt{-1} y \))

\[
\frac{\partial h}{\partial x}(p) \geq 0.
\]

Thus we have

\[
\frac{\partial}{\partial x} D_i (\Psi - \Psi_1)(p) \geq -\lambda_1 \left[ \frac{\partial (\Psi - \Psi_1)}{\partial x} + 2N - 2N x \right] - \lambda_2 \frac{\partial u}{\partial x}(p) := f.
\]

The same argument holds for \(-D_i\),

\[-\frac{\partial}{\partial x} D_i (\Psi - \Psi_1)(p) \geq f.\]

But along the tangent direction

\[
\frac{\partial}{\partial y} D_i (\Psi - \Psi_1) = 0.
\]

Recall that \( D_i = \frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial y^i} \). Combining these estimates for \( \frac{\partial}{\partial x} D_i (\Psi - \Psi_1)(p) \) with the identity along the tangent direction, we get that for any \( 1 \leq i \leq n \),

\[
\left| \frac{\partial^2 (\Psi - \Psi_1)}{\partial z^i \partial \bar{z}^{n+1}}(p) \right| \leq |f|, \quad \left| \frac{\partial^2 (\Psi - \Psi_1)}{\partial \bar{z}^i \partial z^{n+1}}(p) \right| \leq |f|.
\]

From the choice of \( u \), we know that \( \sup_{\partial X} |\partial_{n+1} u|_\Omega \leq 1 \). Therefore,

\[ |f| \leq \lambda_1 \left[ \sup_{\partial X} |\partial_{n+1} (\Psi - \Psi_1)|_\Omega + 4N \right] + \lambda_2. \]

The conclusion is proved by putting the formulas of \( \lambda_1 \) and \( \lambda_2 \) into the estimate of \( |f| \). \( \square \)

**Boundary Mixed Singular Tangent-Normal Estimates**

When we directly estimate the mixed singular tangent-normal derivative on the boundary, that is, \( i = 1 \), the term \( |\nabla_1^{\text{cone}} \Omega_1 |_{\Omega_\text{cone}} \) is not finite in general when the angle is larger than \( \frac{2}{3} \), as shown in section 3.3 of [16]. However, according to Corollary 4.5 this term has the growth rate \( O(|z|^{1-\beta}) \) when the boundary values are cscK cone metrics. We present a direct adaption of Section 2.1 which gives us the boundary mixed singular tangent-normal estimates.

**Proposition 2.19 (Boundary mixed singular tangent-normal estimates).** There exists a constant \( C \) such that

\[
\sup_{\partial X} \left[ |z|^{1+1-\beta} \left( \left| \frac{\partial^2 (\Psi - \Psi_1)}{\partial z^i \partial \bar{z}^{n+1}} \right| + \left| \frac{\partial^2 (\Psi - \Psi_1)}{\partial \bar{z}^i \partial z^{n+1}} \right| \right) \right] \\
\leq C \left[ 1 + \sup_{\partial X} |\partial_{n+1} (\Psi - \Psi_1)|_\Omega \right] \left[ 1 + \sup_{\partial X} |\partial_1 (\Psi - \Psi_1)|_\Omega \right].
\]

The constant \( C \) depends on \( X \), \( \epsilon_0 \) in (2.14),

\[
\sup |\Omega_1|_{\Omega_\text{cone}}, \quad \sup |z|^{1+1-\beta} |\nabla_1^{\text{cone}} \Omega_1|_{\Omega_\text{cone}},
\]
\[
\sup |z^1|^{\kappa} |\nabla_1 \Omega_1| \Omega_{\text{cone}}, \quad 1 \leq a, b \leq n + 1.
\]

**Proof.** We choose \(0 < \kappa_s < 1\) to be a sequence of rational numbers decreasing to \(\kappa = \beta - \alpha \beta\). We write \(\kappa_s\) as a reduced fraction \(\frac{a_s}{b_s}\) with two integers \(a_s < b_s\). We set \(z_s^1 = (z^1)^{1/b_s}/(-\kappa_s + 1)\) and choose one of the \(b_s\) branched coverings. Outside the divisor, we could further set \(z_s^1 = (-\kappa_s + 1)^{b_s - a_s - 1}(z_s^1)^{b_s - a_s}\) and use \(\bar{z}\) to denote the pullback from \(z^1\) to \(\bar{z^1}\). We write \(\tilde{D}_1 = (z^1)^{\kappa_s} D_1\) and notice that

\[
\Delta_{\bar{z}} \tilde{D}_1(\Psi - \Psi_1) = (z^1)^{\kappa_s} \cdot \Delta_{\bar{z}}[D_1(\Psi - \Psi_1)].
\]

By Lemma 2.17 we have

(2.22) \[
\Delta_{\bar{z}} \tilde{D}_1(\Psi - \Psi_1) \leq \epsilon_3 \cdot |z^1|^{\kappa_s} \cdot F_1 \cdot (1 + \text{Tr}_{\bar{z}} \Omega_{\text{cone}}).
\]

We consider \(\tilde{h} := \lambda_1 \tilde{u} + \lambda_2 \tilde{u} + \tilde{D}_1(\tilde{\Psi} - \tilde{\Psi}_1)\), with two constants determined by

(2.23) \[
\lambda_2 = \frac{1}{\epsilon_2} \cdot \left[1 + \sup_{\bar{z}} |\partial_1(\Psi - \Psi_1)| \Omega_{\text{cone}}\right],
\]

(2.24) \[
\lambda_1 = \frac{4}{\epsilon_0} \cdot \left[\lambda_2 \cdot \epsilon_2 + \epsilon_3 \cdot \sup_{\bar{z}} |z^1|^{\kappa_s} \cdot F_1\right].
\]

Here, \(F_1 = |\nabla_1 \Omega_{\text{cone}}| \Omega_{\text{cone}} + |\nabla_1 \bar{\Omega}_{\text{cone}}| \Omega_{\text{cone}}\).

By Lemma 2.15, Lemma 2.16, and (2.22), \(\tilde{h}\) satisfies the differential inequality

\[
\Delta_{\bar{z}} \tilde{h} \leq \left[\frac{-\epsilon_0}{2} \lambda_2 \cdot \epsilon_2 + \epsilon_3 \cdot |z^1|^{\kappa_s} \cdot F_1\right] \cdot (1 + \text{Tr}_{\bar{z}} \Omega_{\text{cone}}) < 0.
\]

We then check the boundary conditions by using Lemma 2.15 and Lemma 2.16. On \(\partial \bar{B}_{\delta_0} \cap \bar{\Omega}\), we have \(\tilde{h} \geq 0\). Then we check the boundary point \(q \in \partial \bar{B}_{\delta_0} \cap \text{Int} \bar{\Omega}\). Note that

\[
|\tilde{D}_1(\tilde{\Psi} - \tilde{\Psi}_1)| \Omega = |\partial_1(\Psi - \Psi_1)| \Omega
\]

is bounded. The boundary value of \(h(q)\) satisfies, by Lemma 2.15 and (2.23),

\[
\tilde{h}(q) \geq \lambda_2 \tilde{u}(q) - |\tilde{D}_1(\tilde{\Psi} - \tilde{\Psi}_1)(q)| \Omega \\
\geq \lambda_2 \epsilon_1 - |\partial_1(\Psi - \Psi_1)(q)| \Omega \geq 0.
\]

Since \(\tilde{h}(p) = 0\), we have by the strong maximum principle

\[
\frac{\partial \tilde{h}}{\partial x}(p) \geq 0.
\]

That is,

\[
\frac{\partial}{\partial x} \tilde{D}_1(\tilde{\Psi} - \tilde{\Psi}_1)(p) \geq -\lambda_1 \left(\frac{\partial(\tilde{\Psi} - \tilde{\Psi}_1)}{\partial x} + s - 2N x\right) - \lambda_2 \frac{\partial \tilde{u}}{\partial x}(p) := f.
\]
The same inequality holds for $-\tilde{D}_1$. After transforming back from $\tilde{z}$ to $z$, we have that the mixed singular tangent-normal derivative is bounded by $|f|_\infty$, that is,

$$|z|^{1+\varepsilon}|\frac{\partial^2(\Psi - \Psi_1)}{\partial x \partial x_1}| \leq |f|_\infty.$$

Using the same argument for $\pm \partial_{y_1}(\Psi - \Psi_1)$, combined with the tangent direction $\frac{\partial}{\partial y_1}D_t(\Psi - \Psi_1) = 0$, and letting $s \to \infty$, we complete the proof of the proposition by estimating $|f|_\infty$ with (2.23) and (2.24).

Applying the approximation geodesic equation (2.6), we obtain the normal-normal estimates on the boundary immediately.

**Proposition 2.20 (Boundary normal-normal estimates).** There exists a constant $C$ such that

$$\sup_{\partial X} \left| z^{1+\varepsilon} \left( \left| \frac{\partial^2 \Psi}{\partial z_1^{n+1} \partial z_n^{n+1}} \right| \right) \right| \leq C \left( \sup_{\partial X} |\partial \Psi_1 |_{L_1}^2 + 1 \right).$$

The constant $C$ depends on the constants in Proposition 2.18 and Proposition 2.19.

**Proof.** From the approximate geodesic equation (2.6), we have

$$\left[ \frac{\partial^2 \Psi}{\partial z_1^{n+1} \partial z_n^{n+1}} \right] \left[ \frac{\partial^2 \Psi_0}{\partial z_1^{n+1} \partial z_n^{n+1}} \right] \left( \frac{\partial^2 \Psi}{\partial z_1^{n+1} \partial z_n^{n+1}} \right) \left( \frac{\partial^2 \Psi_0}{\partial z_1^{n+1} \partial z_n^{n+1}} \right) \leq \tau \left( \frac{\det(\Omega_{bij})}{\det \omega_\phi} \right).$$

The normal-normal term on the boundary follows as

$$\psi_{tt} \leq (\partial \psi', \partial \psi')_{\omega_\phi} + \tau \left( \frac{\det(\Omega_{bij})}{\det \omega_\phi} \right).$$

We multiply this inequality with $|z|^{1+\varepsilon}$ on the both sides. Then the conclusion follows from applying

$$\sup_{\partial X} \left[ \left| \frac{\partial^2 \Psi_1}{\partial z_1^{n+1} \partial z_n^{n+1}} \right| \right], \quad 2 \leq i \leq n,$

in Proposition 2.18 and

$$\sup_{\partial X} \left[ \left| z^{1+\varepsilon} \left( \left| \frac{\partial^2 \Psi_1}{\partial z_1^{n+1} \partial z_n^{n+1}} \right| \right) \right| \right]$$

in Proposition 2.19. □
2.2 Interior Laplacian Estimates

In this section, we will prove the following interior Laplacian estimate, which includes four parts:

1. the tangent-tangent estimates in the interior (Proposition 2.23),
   \[ \sup_x |\frac{\partial^2 \Psi}{\partial \xi_i \partial \xi_j}|, \sup_x |\frac{\partial^2 \Psi}{\partial \xi_i \partial \eta_j}| \forall 1 \leq i, j \leq n; \]

2. the mixed regular tangent-normal estimates in the interior (Proposition 2.22),
   \[ \sup_x |z^{1|1}| \frac{\partial^2 \Psi}{\partial \xi_i \partial \eta_1}|, \sup_x |z^{1|1}| \frac{\partial^2 \Psi}{\partial \xi_i \partial \eta_{n+1}}| \forall 2 \leq i \leq n; \]

3. the mixed singular tangent-normal estimates in the interior (Proposition 2.22),
   \[ \sup_x |z^{1|1}| \frac{\partial^2 \Psi}{\partial \xi_1 \partial \eta_1}|, \sup_x |z^{1|1}| \frac{\partial^2 \Psi}{\partial \xi_1 \partial \eta_{n+1}}| \]

4. the normal-normal estimates in the interior (Proposition 2.22),
   \[ \sup_x |z^{1|2}| \frac{\partial^2 \Psi}{\partial \eta_1 \partial \eta_1}|, \sup_x |z^{1|2}| \frac{\partial^2 \Psi}{\partial \eta_1 \partial \eta_{n+1}}| \]

We will prove the following proposition in this section.

**Proposition 2.21 (Interior Laplacian estimates).** Assume that the background metric \( \Omega \) satisfies curvature condition (2.10). Then there exists a constant \( C \) (independent of \( \tau \)), which depends on the constants in (2.10) and \( \sup_x |\Omega_{\text{cone}}|, \sup_x |\Psi| \) such that (1), (2), (3), and (4) are bounded by

\[ C \left[ \sup_{x} \frac{\text{Tr} \omega_{\theta}}{\text{det} \omega_{\theta}} + \sup_{x} |s| \frac{2^x \cdot \text{Tr} \Omega \Psi}{} + 1 \right]. \]

**Proof.** We obtain (1) from Proposition 2.23 and (4) from Proposition 2.22 directly. Then we apply the approximate geodesic equation (2.6).

\[ (\partial \varphi', \partial \varphi')_{g_{\varphi}} = \varphi_{tt} - \tau \frac{\text{det}(\Omega \omega_{\theta})}{\text{det} \omega_{\theta}}. \]

Multiplying \(|z^{1|2}| \) on both sides, the RHS is bounded by (1) and (4). Therefore, we obtain (2) and (3). \( \square \)
**Interior Laplacian Estimate**

This section improves upon proposition 3.3 in [16], where we get three estimates, \( C_1, C_2, C_3 \). It is sufficient to consider the case when \( \frac{1}{2} \leq \beta < 1 \). When the angle is large, \( \inf Riem(\Omega) \) in \( C_1 \) and \( |Riem(\Omega_1)|_{\infty} \) in \( C_2 \) can be unbounded. The constant \( C_3 \) relies on the boundary estimates \( \sup_{\partial X} \text{Tr}_{\Omega} \psi \), which may not be bounded, also. However, according to Proposition 2.13 in Section 2.1, we have a bound on the boundary \( \sup_{\partial X} (|s|^{2\beta} \cdot \text{Tr}_{\Omega} \Omega_{\psi}(z)) \). As a result, we could extend boundary estimates to the interior to obtain the interior Laplacian estimate.

**Proposition 2.22 (Interior Laplacian estimate).** Assume that the background metric \( \Omega_b \) satisfies curvature condition \( (2.10) \). Then there exists a constant \( C \) such that

\[
\sup_{x} |s|^{2\beta} \cdot \text{Tr}_{\Omega} \Omega_{\psi} \leq C \left[ \sup_{\partial X} (|s|^{2\beta} \cdot \text{Tr}_{\Omega} \Omega_{\psi}) + 1 \right].
\]

The constant \( C \) depends on the constants in \( (2.10) \) and

\[
\sup_{x} |\Omega_b|_{\Omega_{\psi}}, \sup_{x} |\Psi_b|, \sup_{x} |\Psi|.
\]

**Proof.** We apply the inequality from Yau’s second-order estimate [60],

\[
\Delta_\psi \log \text{Tr}_{\Omega_b} \Omega_{\psi} \geq -g^{i\bar{j}}_{\Omega_b} R_{i\bar{j}}(\Omega_{\psi}) + g^{k\bar{\ell}}_{\psi} R^{k\bar{\ell}}_{\Omega_b} g_{\psi \bar{\ell}}.
\]

From the approximation geodesic equation \( (2.6) \), we have

\[
\text{Ric}(\Omega_{\psi}) = \text{Ric}(\Omega_b)
\]

and

\[
\text{RHS} = -S(\Omega_b) + g^{k\bar{\ell}}_{\psi} R^{k\bar{\ell}}_{\Omega_b} g_{\psi \bar{\ell}}.
\]

Under the local normal coordinate of \( \Omega_b \) (understood in the approximation sense), we denote by \( \lambda_a, 1 \leq a \leq n + 1 \), the eigenvalue of \( \Omega_\psi \). We then use an inequality in proposition 2.1 in [41] to get

\[
\text{RHS} \geq \sum_{a \neq b} \left( \frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2 \right) R_{a\bar{a}b\bar{b}}(\Omega_b) \cdot \sum_{c} \lambda_c.
\]

We use the curvature condition \( (2.10) \), \( R_{a\bar{a}b\bar{b}}(\Omega_b) \geq -(\tilde{g}_b)_{a\bar{a}} \cdot (g_b)_{b\bar{b}} \). Note that \( \frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2 \geq 0 \). Then

\[
\text{RHS} \geq -C \sum_{a \neq b} \left( \frac{\lambda_a}{\lambda_b} (\tilde{g}_b)_{b\bar{b}} + \frac{\lambda_b}{\lambda_a} (\tilde{g}_b)_{a\bar{a}} \right) \cdot \sum_{c} \lambda_c.
\]

Here \( \Phi \) is defined in \( (2.10) \).
Recall that $L_D$ is the line bundle associated to the divisor $D$ and $s$ is the holomorphic section. We choose $h$ to be a smooth Hermitian metric on $L_D$ such that for some constant $C$,

$$i \partial \overline{\partial} \log |s|_h \geq -C \cdot \Omega_b.$$ 

Combining this with (2.28), we have

$$\Delta_\psi \log (|s|^{2^x}_{h} \cdot \text{Tr}_\Omega \Omega_b \cdot \psi) \geq -C(\text{Tr}_\Omega \Omega_b \cdot \psi + \Delta_\psi \psi).$$

Since $\Delta_\psi (\psi - \psi_b) = n + 1 - \text{Tr}_\Omega \Omega_b \cdot \psi_b$, we obtain the differential inequality of

$$Z = \log (|s|^{2^x}_{h} \cdot \text{Tr}_\Omega \Omega_b \cdot \psi) + C \Phi - (C + 1)(\psi - \psi_b),$$

which reads

$$\Delta_\psi Z \geq \text{Tr}_\Omega \Omega_b \cdot (n + 1)(C + 1).$$

If the maximum of $Z$ is achieved on the boundary, then

$$|s|^{2^x}_{h} \cdot \text{Tr}_\Omega \Omega_b \cdot e^{C \Phi - (C + 1)(\psi - \psi_b)}$$

is controlled by the boundary maximum and the proof is finished. When the maximal point $p$ is an interior point, we have

$$\text{Tr}_\Omega \Omega_b (p) \leq (n + 1)(C + 1).$$

The inequality of arithmetic and geometric means implies that at $p$,

$$\text{Tr}_\Omega \Omega_b \cdot \psi \leq \frac{n + 1}{n} (\text{Tr}_\Omega \Omega_b \cdot \psi_b)^n \cdot \frac{\Omega_b^{n+1}}{\Omega_b^{n+1}} = \frac{n + 1}{n} (\text{Tr}_\Omega \Omega_b \cdot \psi_b)^n.$$

Letting $L = C \Phi - (C + 1)(\psi - \psi_b)$, we have for any $z \in \mathcal{X}$,

$$\log (|s|^{2^x}_{h} \cdot \text{Tr}_\Omega \Omega_b \cdot \psi)(z) \leq Z(p) - L(z).$$

The proof is complete, since $L$ is bounded. \hfill \Box

**Interior Spatial Laplacian Estimate**

The interior spatial Laplacian estimate [43] was extended to a singular case in [1]. This method, as a generalization of Yau’s second-order estimate, relies on both the lower bound of the bisectional curvature and the upper bound of the scalar curvature of the background metric, which are bounded in general when the cone angle is less than $\frac{1}{2}$. However, we will see that the estimate is still true when the curvature is not bounded. We use the approximate geodesic equation (2.6) in $\mathcal{M}$,

$$\det(\Omega_\psi) = [\varphi'' - (\partial \varphi', \partial \varphi')]_{g_\varphi} \det(\omega_\varphi) = \tau \cdot \det(\Omega_b).$$

Recall the notation $\psi(z) = \varphi(z) - |z|^{n+1}$. The improvement from Proposition [2.22] to the following one is to remove the weight $|s|^{2^x}$. We will see that if we consider the interior Laplacian estimate along the spatial directions, not the full directions, the weight $|s|^{2^x}$ can be removed eventually.
PROPOSITION 2.23 (Interior spatial Laplacian estimate). Assume that the background metric $\Omega_b$ satisfies curvature condition (2.10). Then there exists a constant $C$ such that

$$\sup_X \text{Tr}_\omega \phi \leq C \left( \sup_{\partial X} \text{Tr}_\omega \phi + 1 \right).$$

The constant $C$ depends on the constants in (2.10) and $\sup_X |\omega_b|_\omega, \text{osc}_X \phi$.

PROOF. If we let $G = \phi'' - (\partial \phi', \partial \phi')_{g_b}$, its linearized operator with variation $\delta \phi = u$ at $\phi$ is

$$L(u) = \frac{u'' + g_{i\bar{j}}^{\ell} g_{j\ell}^{\bar{k}} u_{j\bar{k}} \phi'_{i\bar{j}} - (\partial u', \partial \phi')_{g_b} - (\partial \phi', \partial u')_{g_b}}{G} + \Delta \phi u.$$

We denote $h = \text{Tr}_{\omega_b} \omega_b$. Then we use the equation (2.11) in [43]. In our case, we have $f = 0$, according to (2.29). So we get

$$L(h) = |\partial \log G|_{g_b}^2 + I + \frac{\text{II} + \text{III}}{G} + \text{IV}.$$

Here the third term

$$\text{III} \geq A G h,$$

with the notation

$$A = \frac{[h' - (\partial \phi', \partial h)_{g_b}] [h' - (\partial \phi', \partial h)]_{g_b}}{h^2 G};$$

the fourth term

$$\text{IV} = g_{i\bar{j}}^{\ell} g_{j\ell}^{\bar{k}} R_k \phi_{i\bar{j}} \phi_{\bar{k} \ell} \geq h |\partial \log h|_{g_b}^2.$$

We need to handle terms I and II, which contain the curvature. We use the curvature condition (2.10). Similarly to the proof of Proposition 2.22, i.e., (2.28) and (2.27), we control the first term by

$$I = -S(\omega_b) + g_{i\bar{j}}^{\ell} R_k \phi_{i\bar{j}} (\omega_b) g_{\ell k} \phi_{i\bar{j}} \geq -C (\text{Tr}_{\omega_b} \omega_b + \Delta \phi \Phi) \text{Tr}_{\omega_b} \omega_b,$$

and the second term by

$$\text{II} = g_{i\bar{j}}^{\ell} g_{\ell k} \phi_{i\bar{j}} R_k \phi_{\ell j} \geq -2 C h \sum_i \frac{(1 + \Phi_{i\bar{j}}) \phi_{i\bar{j}}'}{(1 + \phi_{i\bar{j}})^2}.$$
Letting $Z = \log \text{Tr}_{\omega_b} \omega_{\varphi} - B_1 \varphi + \frac{t^2}{2} + B_2 \Phi$, we have the identity

$$L(Z) = \frac{L(h)}{h} - |\vartheta \log h|_{g_\omega}^2 - A - (n + 1) B_1 + B_1 \text{Tr}_{\omega_b} \omega_b$$

$$+ \frac{B_1}{G} \sum_i \frac{\varphi_i \varphi_i^l}{(1 + \varphi_i T)^2} + \frac{1}{G} \sum_i \frac{\Phi_{iT} \varphi_i \varphi_i^l}{(1 + \varphi_i T)^2} + B_2 \Delta \varphi \Phi.$$ 

Inserting $L(h)$ into the formula of $L(Z)$, we have

$$L(Z) \geq -(n + 1) B_1 + (B_1 - C) \text{Tr}_{\omega_b} \omega_b + \frac{B_1 - 2C}{G} \sum_i \frac{\varphi_i \varphi_i^l}{(1 + \varphi_i T)^2} + \frac{1}{G}$$

$$+ \frac{B_2 - 2C}{G} \sum_i \frac{\Phi_{iT} \varphi_i \varphi_i^l}{(1 + \varphi_i T)^2} + (B_2 - C) \Delta \varphi \Phi.$$ 

We now deal with the terms involved with $\Phi$. By using $\text{Tr}_{\omega_b} \tilde{\omega}_b = \text{Tr}_{\omega_b} \omega_b + \Delta \varphi \Phi$, we have

$$\mathcal{G}_1 := (B_1 - C) \text{Tr}_{\omega_b} \omega_b + (B_2 - C) \Delta \varphi \Phi$$

$$= (B_1 - B_2) \text{Tr}_{\omega_b} \omega_b + (B_2 - C)(\text{Tr}_{\omega_b} \tilde{\omega}_b)$$

and

$$\mathcal{G}_2 := \frac{B_1 - 2C}{G} \sum_i \frac{\varphi_i \varphi_i^l}{(1 + \varphi_i T)^2} + \frac{B_2 - 2C}{G} \sum_i \frac{\Phi_{iT} \varphi_i \varphi_i^l}{(1 + \varphi_i T)^2}$$

$$= \frac{B_1 - B_2}{G} \sum_i \frac{\varphi_i \varphi_i^l}{(1 + \varphi_i T)^2} + \frac{B_2 - 2C}{G} \sum_i \frac{(1 + \Phi_{iT}) \varphi_i \varphi_i^l}{(1 + \varphi_i T)^2}.$$ 

We choose $B_1 = B_2 + 1 = 2C + 2$; then

$$\mathcal{G}_1 \geq \text{Tr}_{\omega_b} \omega_b, \quad \mathcal{G}_2 \geq 0.$$ 

So we have

$$L(Z) \geq -(n + 1) B_1 + \text{Tr}_{\omega_b} \omega_b + \frac{1}{G}.$$ 

We denote the addendum $E = -B_1 \varphi + \frac{t^2}{2} + B_2 \Phi$. It is bounded. When the maximum of $L(Z)$ is on the boundary, the proof is finished. If the maximal point $p$ appears in the interior of $X$, $\text{Tr}_{\omega_b} \omega_b + \frac{1}{G}$ at $p$ is bounded. We apply the inequality between two positive matrices $M$ and $N$, i.e., $\text{Tr}_M N \leq \frac{n + 1}{n} (\text{Tr}_N M)^{\det N \det M}$, to see at $p$ that

$$\text{Tr}_{\omega_b} \omega_{\varphi} + G \leq \frac{n + 1}{n} \left( \text{Tr}_{\omega_b} \omega_b + \frac{1}{G} \right)^{\frac{n \Omega_{\omega_{\varphi}} + 1}{\Omega_{\omega_b} + 1}} = \frac{n + 1}{n} \left( \text{Tr}_{\omega_b} \omega_b + \frac{1}{G} \right)^{\frac{n}{\Omega_{\omega_b} + 1}}.$$ 

Since $G \geq 0$, we have

$$\log \text{Tr}_{\omega_b} \omega_{\varphi}(z) \leq \log \text{Tr}_{\omega_b} \omega_{\varphi}(p) + E(p) - E(z) \quad \forall z \in X.$$ 

The proof is done.
2.3 Interior Gradient Estimate

According to Proposition 2.13 and Proposition 2.22 in order to complete the proofs of Theorem 2.4 and Theorem 2.6 we need to prove the interior gradient estimate.

The interior gradient estimate will be obtained from the following interior spatial gradient estimate (Proposition 2.24 in Section 2.3) and the $t$-derivative estimate (Lemma 2.12 in Section 2).

We also present a direct proof of the interior gradient estimate in Section 2.3 which is of independent interest.

Interior Spatial Gradient Estimate

We use the interior spatial Laplacian estimate (Proposition 2.23) to prove the interior spatial gradient estimate.

Proposition 2.24 (Interior spatial gradient estimate). There is a constant $C$ depending on the constants in Lemma 2.10 and Proposition 2.23 such that

\[ \sup_X |\partial_z^j \varphi|_\infty \leq C. \]

Proof. From the interior spatial Laplacian estimate (Proposition 2.23), we already know that $\Delta \varphi$ is bounded. The potential $\varphi$ is also bounded, due to Lemma 2.10.

In order to get $|\partial_z^j \varphi|_\infty$, we apply the global $L^p$-estimate for the cone metric to $\Delta \varphi$, which is glued together from the local ones. The local $L^p$-estimate is proved by applying the cone Green function (see [29]). The statement is that there exists a constant $C$ depending on $n, p, \beta$ such that

\[ \|u\|_{C^{1,\alpha,\beta}(\omega; B_r)} \leq C (\|\Delta \varphi\|_{L^p(\omega; B_{2r})} + \|u\|_{W^{1,2}(\omega; B_{2r})}). \]

We will also need an interpolation inequality.

Lemma 2.25 (Interpolation inequality). Suppose that $\epsilon > 0$ and $1 < p < \infty$. There exists a constant $C$ such that for all $u \in C^{1,\alpha,\beta}(\omega)$, we have

\[ \|u\|_{W^{1,\frac{p}{\epsilon}+(\omega)}} \leq \epsilon \|u\|_{C^{1,\alpha,\beta}(\omega)} + C \|u\|_{L^p(\omega)}. \]

The proof of this lemma is similar to that of proposition 3.1 in [45]. We could replace $W^{2,\beta}$ with $C^{1,\alpha,\beta}$ and use the compactness argument of $C^{1,\alpha,\beta}$ in the same way.

We continue back to our proof and patch the local estimates together. We let the manifold $X$ be covered by a finite number of coordinates charts $\{U_i, \psi; 1 \leq i \leq N\}$. We let $\rho_i$ be the smooth partition of unity subordinate to $\{U_i\}$ and be supported in $B_r \subset B_{3r} \subset U_i$ for each $i$. From

\[ \Delta (\rho_i \varphi) = \Delta \rho_i \varphi + \rho_i \Delta \varphi + 2(\varphi, \partial \rho_i \varphi) \omega. \]
using the equation and putting all estimates in each $U_i$ together, we obtain from (2.33) that
\[
\|\varphi\|_{C^{1,\alpha}}(\omega) = \|\sum_i \rho_i \varphi\|_{C^{1,\alpha}}(\omega) \leq C \sum_i \|\rho_i \varphi\|_{C^{1,\alpha}}(\omega) \\
\leq C \sum_i (\|\rho_i \varphi\|_{W^{1,2}}(\omega) + \|\Delta_{\omega}(\rho_i \varphi)\|_{L^p}(\omega)).
\]
The RHS is bounded by
\[
\leq C(\|\varphi\|_{L^\infty} + \|\Delta_{\omega}(\varphi)\|_{L^\infty} + \|\varphi\|_{W^{1,p}}(\omega)).
\]
Applying the interpolation inequality (Lemma 2.25), we obtain that
\[
\|\varphi\|_{C^{1,\alpha}}(\omega) \leq C(\|\varphi\|_{L^\infty} + \|\Delta_{\omega}(\varphi)\|_{L^\infty}).
\]
Thus the interior spatial gradient estimate is proved.

**Contradiction Method**

We now present an alternative proof by the contradiction method in this section. There is another direct method to get the interior gradient estimate; see section 3.4 in [16].

The gradient estimate in the next proposition contains all the directions, while the one in Proposition 2.24 concerns the spatial directions only. This section has its own independent interest, especially when the extra weight $|s|^k$ in the Laplacian estimates can be successfully removed, such as in the half-angle case.

**Proposition 2.26.** Assume the following:

- *interior Laplacian estimate:* $\sup_X \Omega \varphi \leq C_1$;
- *boundary Laplacian estimate:* $\sup_{\partial \Omega} \Omega \varphi \leq C_2(\sup_X |\partial \varphi|_{\Omega}^2 + 1)$.

Then there exists a constant $C$ depending on the constants $C_1$ and $C_2$ such that
\[
\sup_X |\partial \varphi|_{\Omega} \leq C.
\]

**Proof.** We prove it with an argument by contradiction. Assume that we have a sequence of $\{\Psi_s\}$ and points $\{p_s\}$ such that
\[
|\partial \Psi_s|_{\Omega}(p_s) = \sup_X |\partial \Psi_s|_{\Omega} \to \infty \quad \text{as} \quad s \to \infty.
\]
We take the limit of this sequence in the underlying topology on $\mathcal{X}$ (not in the topology induced by the cone metrics, which are incomplete), and we denote the limit point $p$. We need to consider where the limit point $p$ is located.

When $p$ is located in $\mathcal{X} \setminus \mathcal{D}$, the argument follows directly from [20]. The new situation is $p \in \mathcal{D}$. We will use a different scaling method.

There are two subcases,

- $p$ is in $\mathcal{D}$ but not in the boundary $\partial \mathcal{X}$, and
- $p$ lies on the boundary $\mathcal{D} \cap \partial \mathcal{X}$.
Choose a small ball $B_{\rho_0}(p)$ centered at $p$ with fixed radius $\rho_0$ and holomorphic normal coordinate chart $z$ in $B_{\rho_0}(p)$ and $p = 0$. Because this is a local argument, we choose $\Omega_{\text{cone}} = \beta^2 |z|^2 (\beta-1) i dz \wedge d\bar{z} + \sum_{2 \leq j \leq n+1} i dz^j \wedge d\bar{z}^j$ as the background metric. When $p \in \mathcal{D} \cap \partial X$, we choose a half-ball $B^+_{\rho_0}(p)$ instead.

We assume that $\Psi_s$ achieves its supremum at points $p_s \in X$ with

\begin{equation}
|\partial_z \Psi_s|_{\Omega_{\text{cone}}}(p_s) = \sup_{X} |\partial_z \Psi|_{\Omega_{\text{cone}}} = \frac{1}{m_s}
\end{equation}

such that $m_s \to 0$ as $s \to \infty$.

**Step 1.** For any $q \in B_{\rho_0}(p)$,

\begin{equation}
|\partial_z \Psi_s|_{\Omega_{\text{cone}}}(q) \leq \frac{1}{m_s}.
\end{equation}

Then the interior Laplacian estimate and the boundary Laplacian estimate imply that

\begin{equation}
(g_{\text{cone}})_{j\bar{j}}(q) + \partial_{z_j} \partial_{\bar{z}_j} \Psi_s(q) \leq \frac{C}{m_s^2} (g_{\text{cone}})_{j\bar{j}}(q),
\end{equation}

\begin{equation}
|\partial_z \partial_{\bar{z}} \Psi_s|_{\Omega_{\text{cone}}}(q) \leq \frac{C}{m_s^2}.
\end{equation}

The constant $C$ depends on $C_1, C_2$ in the assumption of the proposition.

**Step 2.** We define $\lambda = (\lambda_1, \ldots, \lambda_j, \ldots)$,

\[ \lambda_1 = m_s^{1/\beta}, \quad \lambda_j = m_s \quad \forall 2 \leq j \leq n+1. \]

We use the transform $T: B_{\rho_0}(p) \to B_{\frac{\rho_0}{m_s}}(0) \subset \mathbb{C}^n, z \mapsto \tilde{z}$,

\[ \tilde{z}_j = \frac{z_j}{\lambda_j} \quad \forall 1 \leq j \leq n+1. \]

Now we define the following rescaled sequences of functions:

\[ \tilde{\Psi}_s(\tilde{z}) = \Psi_s(\lambda \cdot \tilde{z}), \quad \tilde{\Psi}_b(\tilde{z}) = \Psi_b(\lambda \cdot \tilde{z}), \quad \tilde{h}(\tilde{z}) = h(\lambda \cdot \tilde{z}), \]

for any $\tilde{z} \in B_{\frac{\rho_0}{m_s}}(0)$. We also denote $\tilde{\Omega}(\tilde{z}) = \Omega(\lambda \cdot \tilde{z})$ and

\[ \tilde{p}_s = \left( \frac{z_1(p_s)}{\lambda_1}, \ldots, \frac{z_j(p_s)}{\lambda_j}, \ldots, \frac{z_{n+1}(p_s)}{\lambda_1} \right). \]

It is direct to compute that $\tilde{\Omega}_{\text{cone}} = m_s^2 \Omega_{\text{cone}}.$
Then (2.34), (2.35), and (2.37) are rescaled to be

\[
|z \tilde{\Psi}_s|_{\Omega_{\text{cone}}} (\tilde{\rho}_s) = 1,
\]

(2.38)

\[
\sup_{B(\rho_0/m_s)(0)} |z \tilde{\Psi}_s|_{\Omega_{\text{cone}}} = m_s \sup_{B(\rho_0)(p)} |z \Psi_s|_{\Omega_{\text{cone}}} \leq 1,
\]

(2.39)

\[
\sup_{B(\rho_0/m_s)(0)} |z z \tilde{\Psi}_s|_{\Omega_{\text{cone}}} = m_s^2 \sup_{B(\rho_0)(p)} |z z \tilde{z} \Psi_s|_{\Omega_{\text{cone}}} \leq C.
\]

(2.40)

**Step 3.** Any closed set \( K \subset \mathbb{C}^{n+1} \) stays in \( B(\rho_0/m_s)(0) \) for sufficiently large \( i \).

According to (2.39) and (2.40), after taking the standard diagonal sequence and the Arzelà-Ascoli theorem, we can extract a subsequence of \( \tilde{\Psi}_s \) that converges to \( \tilde{\Psi}_s \) in \( C^{1,\alpha,\beta} (\mathbb{C}, \Omega_{\text{cone}}) \), and from (2.38) the limiting function \( \tilde{\Psi}_s \) is not a constant.

**Step 4.** When \( p \) lies on the boundary \( \partial X \), we always choose half-balls in the argument above. From the \( L^\infty \)-estimate (2.10), i.e., \( \Psi_b(z) \leq \Psi_s(z) \leq h(z) \forall s \), we have

\[
\tilde{\Psi}_b(z) \leq \tilde{\Psi}_s(z) \leq \tilde{h}(z) \quad \forall s.
\]

So after taking \( s \to \infty \), the rescaled sequence converges as

\[
\Psi_b(0) \leq \tilde{\Psi}_\infty(z) \leq h(0).
\]

On the boundary \( \partial X \), \( \Psi_b(0) = h(0) \); thus \( \tilde{\Psi}_\infty \) has to be constant; contradiction!

**Step 5.** When \( p \) is in \( D \) but not in the boundary \( \partial X \), we apply (2.36) on the complex plane \( \mathbb{C} \) containing \( \{z_1, z_2\} \) in \( B(\rho_0)(\rho_s) \). It suffices to consider \( j = 1 \); otherwise we just take \( \beta = 1 \) for \( 2 \leq j \leq n + 1 \). We compute

\[
0 < z^{1+2\beta - 2j} \partial z_1 \partial z_1 \Psi_s \leq C\frac{m_s}{\rho_s} z^{1+2\beta - 2j};
\]

then by scaling we have on \( B(\rho_0/m_s)(0) \)

\[
0 < \lambda_1 z^{1+2\beta - 2j} \partial z_1 \partial z_1 \tilde{\Psi}_s \leq C m^{-2j} \lambda_1 z^{1+2\beta - 2j}.
\]

Then

\[
0 < \lambda_1 z^{1+2\beta - 2j} \partial z_1 \partial z_1 \tilde{\Psi}_s \leq C m^{-2j} \lambda_1 z^{1+2\beta - 2j}.
\]

After taking \( s \to \infty \), we have on \( \mathbb{C} \)

\[
0 < \partial z_1 \partial z_1 \tilde{\Psi}_\infty \leq C z^{1+2\beta - 2j} \quad \text{in } W^{2,p}.
\]

Due to the following Liouville theorem (Proposition (2.27)), \( \Psi_\infty \) is a constant. This is a contradiction. Since

\[
\tilde{\Psi}_s \to \tilde{\Psi}_\infty \quad \text{in } C^{1,\alpha,\beta} (\mathbb{C}, \Omega_{\text{cone}}),
\]

we obtain, from (2.38), that

\[
|z \tilde{\Psi}_s|_{\Omega_{\text{cone}}} (\tilde{\rho}_s) > 0.5,
\]

when \( s \) is sufficiently large. □
Proposition 2.27 (Liouville theorem). We denote the cone metric
\[ \tilde{\omega} = |z|^{2\beta - 2} i dz \wedge d\overline{z} \quad \text{in } \mathbb{C}. \]
Suppose \( u \in C^{1,\alpha,\beta} \cap W^{2,p}(\mathbb{C}) \) has bounded \( C^1(\tilde{\omega}) \)-norm, i.e.,
\[ |u|_{C^1(\tilde{\omega})} = \sup_{z \in \mathbb{C}} \{|u(z)| + |z|^{1-\beta} |\partial u(z)|\} < \infty. \]
Suppose \( u \) satisfies in the distribution sense
\[ |z|^{2-2\beta} \partial_z \partial_{\overline{z}} u \geq 0. \]
Then \( u \) must be a constant.

Proof. Since \( u \) is bounded, we could assume that \( u \) is positive. We denote by \( \Delta \) the associated Laplacian with regard to \( \tilde{\omega} \). We see that for \( p \geq 2 \)
\[ \Delta u^p = p(p - 1)|\partial u|^2 u^{p-2} + p u^{p-1} \Delta u \geq p(p - 1)|\partial u|^2 u^{p-2}. \]
Choose \( 0 < r_0 < s < r \) and a cutoff function \( \eta \) with \( \eta = 1 \) in \( B_s \) and \( \eta = 0 \) outside \( B_r \). Let \( \chi_\epsilon \) be the smooth cutoff function supported outside the \( r_0 \)-tubular neighborhood of the divisor with the properties that such that
\[ |\nabla \chi_\epsilon| = \epsilon \cdot O(r^{-1}). \]
Then we have
\[ \int_{B_r} \chi_\epsilon \text{div}(\eta^2 \partial u^p) \tilde{\omega} = \int_{B_r} - (\partial \chi_\epsilon, \partial u^p) \eta^2 \tilde{\omega}. \]
Since \( u \) and \( |\partial u|_g \) are both bounded, we have, as \( \epsilon \to 0 \),
\[ \int_{B_r} \text{div}_g (\eta^2 \partial u^p) \tilde{\omega} = 0; \]
that is,
\[ 0 \geq \int_{B_r} 2p \eta(\partial \eta, \partial u) u^{p-1} \tilde{\omega} + \int_{B_r} p(p - 1) \eta^2 |\partial u|_g^2 u^{p-2} \tilde{\omega}. \]
Applying the Schwarz inequality, we have
\[ \left[ \int_{B_r} \eta^2 |\partial u|^2_g u^{p-2} \tilde{\omega} \right]^2 \leq \frac{2}{p - 1} \int_{B_r \setminus B_s} |\partial \eta|^2_g u^p \tilde{\omega} \int_{B_r \setminus B_s} \eta^2 |\partial u|_g^2 u^{p-2} \tilde{\omega}. \]
Since \( u \) is bounded and the real dimension is 2, we have
\[ \int_{B_s} \eta^2 |\partial u|^2_g u^{p-2} \tilde{\omega} \leq \int_{B_r} \eta^2 |\partial u|^2_g u^{p-2} \tilde{\omega} \leq C \frac{1}{r - s}. \]
Fixing \( s \) and taking \( r \to \infty \), we obtain that \( u \) is a constant. \( \square \)
3 Uniqueness of cscK Cone Metrics

Proof of Theorem 1.10. We are given two cscK cone metrics $\omega_1, \omega_2$. We assume that there are two different orbits $O_1$ and $O_2$ containing them. We minimize the functional $J$ (see (3.10)) in each orbit, i.e., in $O_1$ to get a cscK cone metric $\theta_1$ and in $O_2$ to get $\theta_2$ (Proposition 7.1).

According to the bifurcation theorem (Theorem 7.5 in Section 7), at each cscK cone metric $\theta_i, i = 1, 2$, we are able to perturb $\theta_i$ to two continuity paths of $J$-twisted cscK cone metrics $\theta_i(t)$ for $1 - \tau < t \leq 1$. The bifurcation construction requires the linear theory for Lichnerowicz operator (Section 5) and the reductivity of the automorphism group (Section 6).

The regularity theorem of the $J$-twisted cscK cone metric (Corollary 4.5 in Section 4) tells us that its Kähler potential is $C^{3,\alpha,\beta}$. Then fixing $t_2 = 1$, we can connect $\theta_1(t), \theta_2(t)$ with the cone geodesic, according to Theorem 2.6 in Section 2. The $J$-twisted cscK cone metric is the critical point of the $J$-twisted log-$K$-energy $\mathcal{E}$, while the $J$-twisted log-$K$-energy is strictly convex along the cone geodesic (Proposition 3.21). Thus the $J$-twisted cscK cone metric is unique.

The uniqueness of the $J$-twisted cscK cone metrics implies two paths have to coincide with each other when $1 - \tau < t < 1$. As a consequence, $O_1 = O_2$. Therefore the proof of the main theorem (Theorem 1.10) is complete. □

3.1 cscK Cone Metrics and $J$-Twisted cscK Cone Metrics

Here we define the cscK cone metric. We need to start with a reference metric.

Reference Metrics

There are two ways to construct reference metric $\omega_0$.

**Solving cone Calabi’s conjecture.** In any Kähler class $[\omega_0]$, given a smooth $(1, 1)$-form

$$\theta \in C_1(X) - (1 - \beta)C_1(L_D),$$

there exists a unique $\omega_\theta := \omega_{\varphi_\theta} = \omega_0 + i \partial \bar{\partial} \varphi_\theta$ with $\varphi_\theta \in C^{2,\alpha,\beta}$ such that

$$\text{Ric}(\omega_\theta) = \theta + 2\pi (1 - \beta) [D].$$

The potential $\varphi_\theta$ satisfies the equation

$$\frac{\omega_\theta^n}{\omega_0^n} = e^{h_0} |s|^{2-2\beta},$$

where $h_0$ is a smooth function and $h$ is a smooth Hermitian metric on $L_D$. The construction of a weak solution can be found in [59], and the weak solution is actually $C^{2,\alpha,\beta}$ in [41]. According to our expansion formula of the complex Monge-Ampère equation in our article [61], $\varphi_\theta$ has higher-order estimates and the expansion formula.

**Perturbation method.** We could perturb the model metric $\omega_D$ a little bit to have a Kähler cone metric with bounded Ricci curvature. This method is presented
in [30]. The function
\[ f := \log \frac{|s|^{2\beta-2\omega_0^n}}{\omega_0^n} \]
is \( C^0, \alpha, \beta \) and can be approximated by a smooth function \( f_0 \) in \( C^0, \alpha, \beta \) with smaller \( \alpha' < \alpha \). We use the same \( \alpha \), but keep in mind that \( \alpha \) can be adjusted to be smaller. By the implicit function theorem and the resolvability of the linear equation with cone coefficients, we can solve \( \omega_\theta := \omega + i \partial \bar{\partial} \varphi_\theta \) with \( \varphi_\theta \) in \( C^2, \alpha, \beta \) such that
\[ \frac{\omega_\theta^n}{\omega_0^n} = e^f - f_0 \text{ in } M. \]

Then direct computation shows that
\[ \text{Ric}(\omega_\theta) = \text{Ric}(\omega_0) + i \partial \bar{\partial} f_0 + i \partial \bar{\partial} \log |s|^{2\beta-2} := \theta + 2\pi(1-\beta)[D]. \]

**Constant Scalar Curvature Kähler Cone Metrics**

Motivated by the definition of the Kähler-Einstein cone metrics, we define the constant scalar curvature Kähler cone metrics as follows. Some partial progress in this direction has already been made in some very recent papers [21, 42, 45, 47, 63].

**DEFINITION 3.1.** We say \( \omega_{cscK} = \omega_0 + i \partial \bar{\partial} \varphi \) is a constant scalar curvature Kähler cone metric if the following hold:

1. \( \omega_{cscK} \) is a Kähler cone metric with angle \( \beta \).
2. \( \omega_{cscK} \) satisfies the equations
\[ \begin{cases} \frac{\omega_{cscK}^n}{\omega_0^n} = e^P, \\ \Delta_{\omega_{cscK}} P = \text{Tr}_{\omega_{cscK}} \theta \ominus S_{\beta}. \end{cases} \]
3. Its Kähler potential \( \varphi \in C^{2, \alpha, \beta} \) and the Hölder exponent \( \alpha \) satisfies
\[ \alpha \beta < 1 - \beta. \]

In general, the Ricci curvature of a Kähler cone metric is singular and it is difficult to make sense of the RHS of (3.5). In the half-angle case, it is simpler, because the model cone metric (2.2) has bounded curvature. But when the angle is not less than one-half, we need to introduce an appropriate reference metric, that is, the metric \( \omega_\theta \) in Section 3.1.

One advantage in choosing the smooth \( \theta \) is to make sure the RHS of (3.5) is well-defined. Another advantage of choosing such a \( \omega_\theta \), which is a Kähler cone metric rather than a smooth metric, is to force \( P \) to be a bounded function instead of a singular one.

**Remark 3.2.** Condition (3) can be removed. When \( \omega_{cscK} \) is a Kähler cone metric by condition (1), we have \( C^{-1} \omega_D \leq \omega_{cscK} \leq C \omega_D \) for a positive constant \( C \). Then we apply Moser iteration (see [50]) to (3.5) to see that \( P \in C^{0, \alpha, \beta} \). Moreover, we have \( \varphi \in C^{2, \alpha, \beta} \) by applying a conical Evans-Krylov estimate [28] to (3.4).
Lemma 3.3. Assume that $\omega$ is a cscK cone metric. Then

$$P \in C^{2, \alpha, \beta}.$$  

Proof. Since the cscK cone potential $\varphi_{cscK}$ is in $C^{2, \alpha, \beta}$, we are able to use (3.5) to obtain that $P$ is $C^{2, \alpha, \beta}$, according to the second-order linear Schauder theory for Kähler cone metrics.

Assume that the angle $0 < \beta < \frac{1}{2}$. Then the $C^{2, \alpha, \beta}$ potential function $\varphi_{cscK}$ of a cscK cone metric is actually in $C^{4, \alpha, \beta}$ in [49]. We are going to prove the higher regularity of $\varphi_{cscK}$ for any $0 < \beta \leq 1$ in Section 4.3 and the following sections, i.e., Theorem 4.8. The theorem implies immediately that $\varphi \in C^{3, \alpha, \beta}$, according to Corollary 4.5 and Corollary 4.6.

By substitution into equation (3.4) in Definition 3.1 with equation (3.2) of $\omega$, we have the potential equation

$$\frac{\omega^n_{cscK}}{\omega^n_0} = \frac{e^{P + h_0}}{|s|^{2-2\beta}}.$$  

We denote the tensor

$$T = -i \bar{\partial} P + \theta.$$  

Lemma 3.4. The tensor $T$ is $C^{0, \alpha, \beta}$, and $\text{Ric}(\omega_{cscK}) = T + 2\pi(1-\beta)[D]$. Moreover, the volume on the averaged scalar curvature $S_\beta$ is a well-defined topological invariant,

$$S_\beta = \frac{(C_1(X) - (1-\beta)C_1(L_D))[\omega_0]^n-1}{[\omega_0]^n}.$$  

Proof. The first conclusion holds, since $P \in C^{2, \alpha, \beta}$ (Lemma 3.3) and $\theta$ is smooth. The second conclusion follows from (3.4).

$$\text{Ric}(\omega_{cscK}) = \text{Ric}(\omega_\theta) - i \bar{\partial} P = \theta + 2\pi(1-\beta)[D] - i \bar{\partial} P.$$  

Thus $\text{Ric}(\omega_{cscK})$ has a lower bound, i.e., $\text{Ric}(\omega_{cscK}) \geq -C \cdot \omega_{cscK}$.

From equation (3.5) of $P$, we see that the scalar curvature of $\omega_{cscK}$ is equal to the averaged scalar curvature $S(\omega_{cscK})$ over the regular part $M$,

$$S(\omega_{cscK}) = \frac{\int_M \text{Ric}(\omega_{cscK}) \wedge \omega^{-1}_{cscK}}{\int_M \omega^{-1}_{cscK}} = S_\beta.$$  

In the remainder of the proof, we will show that it is a topological invariant.

The volumes of the $\omega_i$ are topological invariants, since their Kähler potential is $C^{2, \alpha, \beta}$ and the integration-by-parts formula works. It is sufficient to prove that

$$\int_M \text{Ric}(\omega_1) \wedge \omega_1^{-1} = \int_M \text{Ric}(\omega_2) \wedge \omega_2^{-1}$$  

for two cscK cone metrics $\omega_i$, $i = 1, 2$. 


We can choose a large constant $C$ such that they are both controlled by the model cone metric multiplying the constant $C$, that is,

$$\omega_i \leq C\omega_D.$$ 

Thus the lower bound of the Ricci curvature implies that the twisted Ricci curvature $\mathcal{T}_i = \text{Ric}(\omega_i) + C\omega_D$ is nonnegative. By

$$\mathcal{T}_1 - \mathcal{T}_2 = i\partial\bar{\partial} \log \frac{\omega^n_2}{\omega^n_1},$$

the Ricci potential $\log(\omega^n_2/\omega^n_1)$ is globally bounded. As also observed in proposition 2.8 in [47], we are able to apply the integration-by-parts formula in theorem 1.14 in [14] with $u = 1$ and $v = \log(\omega^n_2/\omega^n_1)$ to obtain that

$$\int_M (\text{Ric}(\omega_1) - \text{Ric}(\omega_2)) \wedge \omega_{1}^{n-1} = 0.$$ 

We also have

$$\int_M (\omega_1 - \omega_2) \wedge \text{Ric}(\omega_i) \wedge \omega_i^{n-2} = \int_M (\omega_1 - \omega_2) \wedge (\text{Ric}(\omega_i) + C\omega_D) \wedge \omega_i^{n-2} = 0,$$

and then similarly have

$$\int_M (\omega_1^{n-1} - \omega_2^{n-1}) \wedge \text{Ric}(\omega_i) = 0.$$ 

By taking $i = 2$ and adding $\int_M (\text{Ric}(\omega_1) - \text{Ric}(\omega_2)) \wedge \omega_{1}^{n-1} = 0$, we have proved (3.8).

**Energy Functionals**

We recall that $\omega_0$ is a smooth Kähler metric and $h$ is a smooth Hermitian metric on $L_D$. The following functionals are defined for any $\varphi \in \mathcal{H}_\beta$:

$$E_\beta(\varphi) = \frac{1}{V} \int_M \log \frac{\omega^n_\varphi}{\omega^n_0} \frac{\omega^n_0}{s^{2\beta-2} e^{h_0}} \omega^n_\varphi,$$

$$D(\varphi) = \frac{1}{V} \frac{1}{n+1} \sum_{j=0}^n \int_M \varphi \omega_0^j \wedge \omega^{n-j}_\varphi,$$

$$j_{-\theta}(\varphi) = \frac{1}{V} \sum_{j=0}^{n-1} \int_M \varphi \omega_0^j \wedge \omega^{n-1-j}_\varphi \wedge \theta.$$ 

Denoting $\mathfrak{h} := -(1 - \beta) \log |s|^2_{h}$. The log-$K$-energy defined over the space of Kähler cone metrics $\mathcal{H}_\beta$ is

$$(3.9) \quad \nu_\beta(\varphi) = E_\beta(\varphi) + S_\beta \cdot D(\varphi) + j_{-\theta}(\varphi) + \frac{1}{V} \int_M (\mathfrak{h} + h_0) \omega^n.$$
The $J$-functional is defined on $\mathcal{H}_\beta$ as follows:

\begin{equation}
J(\varphi) = -D(\varphi) + \frac{1}{V} \int_M \varphi \omega^n_0.
\end{equation}

**Lemma 3.5.** The cscK cone metrics (Definition 3.1) are the critical points of the log-$K$-energy.

**Proof.** We compute the first variation of the log-$K$-energy with the variation of $\varphi$, both in $C^2_{0, \beta}$,

$$\delta v_\beta(\varphi) = \delta E_\beta(\varphi) + \frac{1}{V} \int_M \varphi (S_\beta - \text{Tr}_{\omega_\varphi} \theta) \omega^n_\varphi$$

and

$$\delta E_\beta(\varphi) = \frac{1}{V} \int_M \Delta_\varphi \varphi \omega^n_\varphi + \frac{1}{V} \int_M \log \frac{\omega^n_\varphi}{\omega^n_\theta} \cdot \Delta_\varphi \varphi \cdot \omega^n_\varphi.$$

If $\omega_\varphi = \omega_{\text{cscK}}$, we insert the cscK equations (3.4) and (3.5) into the identities above to obtain

$$\delta v_\beta(\varphi) = \frac{1}{V} \int_M P \cdot \Delta_\varphi \varphi \cdot \omega^n_\varphi - \frac{1}{V} \int_M \varphi \cdot \Delta_{\text{cscK}} P \cdot \omega^n_\varphi.$$

This is equal to 0, since both $P$ and $\varphi$ are $C^2_{0, \beta}$, and we can apply the integration-by-parts formula. \qed

**Remark 3.6.** We could also use the Kähler cone metric $\omega_\theta$ constructed in (3.2) as the reference metric to rewrite the functionals above.

**J-Twisted Constant Scalar Curvature Kähler Cone Metrics**

We consider the $J$-twisted log-$K$-energy with $0 \leq \sigma < 1$ over $\mathcal{H}_\beta$,

\begin{equation}
\mathcal{E}_\beta(\varphi) = v_\beta(\varphi) + (1 - \sigma)J(\varphi).
\end{equation}

In this paper, we remove the lower index $\beta$ without confusion when we use these functionals.

**Definition 3.7.** We call the critical points of (3.11) the $J$-twisted cscK cone metrics. Letting

$$\gamma := S_\beta + (1 - \sigma)\left(\frac{\omega^n_\theta}{\omega^n_\varphi} - 1\right),$$

we see that the $J$-twisted cscK cone metric $\omega_\varphi$ solves the following equations:

\begin{align*}
\begin{cases}
\frac{\omega^n_\varphi}{\omega^n_\theta} = e^P, \\
\Delta_{\omega_\varphi} P = \text{Tr}_{\omega_\varphi} \theta - \gamma.
\end{cases}
\end{align*}
It can also be written as the following fourth-order equation outside the divisor,
\[ S(\omega_\varphi) - S_\theta - (1 - \sigma) \left( \frac{\omega_\varphi^n}{\omega_0^n} - 1 \right) = 0. \]

Furthermore, using the equation of the reference metric \( \theta \) in (3.2), we rewrite the \( J \)-twisted cscK cone equation with respect to the smooth metric \( \omega_0 \),
\[ \frac{\omega_\varphi^n}{\omega_0^n} = \frac{e^{p + h_0}}{|\beta|^{2 - 2\beta}}. \]

**Remark 3.8.** (Compare with Chen’s path) The \( J \)-twisted log-\( K \)-energy \( E_\beta,1 \) and the \( J \)-twisted path (1.3) are different from Chen’s definitions [21]. Recall that Chen’s \( \chi \)-twisted \( K \)-energy is defined by
\[ \sigma v_\beta(\varphi) + (1 - \sigma) J_\chi(\varphi), \quad 0 \leq \sigma \leq 1, \]
with the \( \chi \)-twisted term
\[ J_\chi(\varphi) = j_\chi(\varphi) - \chi D(\varphi) \quad \text{and} \quad j_\chi(\varphi) = \frac{1}{V} \sum_{j=0}^{n-1} \int_M \varphi \omega_0^j \omega_\varphi^{n-1-j} \wedge \chi, \]
in which \( \chi \) is a positive, closed \((1,1)\)-form and \( \chi = \frac{[\chi_1^1]}{[\omega]}\frac{[\varphi]}{[\omega]} - 1 \). Chen’s \( \chi \)-twisted path is then
\[ \sigma (S - S) = (1 - \sigma) (\text{Tr}_\varphi \chi - \chi). \]

Nevertheless, Chen’s path serves as another candidate for the \( J \)-twisted path (3.11), after adjusting the appropriate parameter \( \sigma \).

We further remark that we could prove the convexity of the \( \chi \)-twisted energy (3.14) for Chen’s path with strictly positive closed \((1,1)\)-form \( \chi \) in the conical case, making use of Lemma 3.11. As a result, we have the uniqueness of the \( \chi \)-twisted cscK cone metrics satisfying (3.15) for \( \sigma < 1 \). In this case, we do not need Lemma 3.12. This is different from the proof of the convexity of (3.11) and the uniqueness of the \( J \)-twisted cscK cone metric (3.12).

Moreover, in the bifurcation argument (Section 7), we need to replace Lemma 7.6 with the convexity of the \( j_-\chi \) functional, which is essentially the same as Lemma 3.11. Therefore, Chen’s path is another way to prove Theorem 1.10 instead of the \( J \)-twisted path (1.3).

**Convexity and Continuity of Energy Functionals along Cone Geodesics**

The following lemma is obvious from the explicit formulas of the functionals.

**Lemma 3.9.** The log-\( K \)-energy \( v_\beta \), the log energy functional \( E_\beta \), as well as \( D \) and \( j_-\varphi \) functionals, are well-defined on the space
\[ \mathcal{H}_\Delta := \{ \varphi \in \mathcal{H}_\beta \mid \sup_{\mathcal{H}} \{ |\varphi| + |\partial_\varphi \varphi|_\omega + |\partial_\varphi \partial_\varphi \varphi|_\omega \} < \infty \}. \]

In particular, they are well-defined along the \( C^\beta_\Delta \) generalized cone geodesic.
We denote by $dV$ the standard volume form on the Kähler manifold $X$.

**Lemma 3.10.** Assume $\{\varphi(t) : 0 \leq t \leq 1\}$ is a $C^{1,\beta}_w$ cone geodesic. Then the second derivative of the $D$-functional is computed in the distribution sense; i.e., letting $\eta$ be a smooth nonnegative cutoff function supported in the interior of $[0, 1]$, we have

$$\int_0^1 \partial_t^2 \eta \cdot D(\varphi) \cdot dt = \frac{1}{V} \int_0^1 \eta \cdot \int_M \det(\Omega_{\varphi})dV \cdot dt.$$  

**Proof.** Along the smooth geodesic, we compute that

$$\partial_t D = \frac{1}{V} \int_M \varphi' \omega^n_{\varphi},$$

and then

$$\partial_t^2 D = \frac{1}{V} \int_M \left[ \varphi'' + \varphi' \Delta_{\varphi} \varphi \right] \omega^n_{\varphi} = \frac{1}{V} \int_M \left[ \varphi'' - (\partial \varphi', \partial \varphi')_{g_\varphi} \right] \omega^n_{\varphi}.$$  

After lifting up to the product manifold $X \times [0, 1]$ and using the notations in (2.3), we have

(3.16) $$\partial_t^2 D = \frac{1}{V} \int_M \det(\Omega_{\varphi})dV.$$  

So we see that $D$ is linear along the smooth geodesic by using the geodesic equation (2.4).

In order to prove this identity in the distribution sense along the cone geodesic, we use the approximation of the plurisubharmonic function. Since $\varphi$ is Hölder continuous, there exists a smooth sequence $\varphi_s$ decreasing to $\varphi$ in any open subset of $[0, 1] \times X$ (see, e.g., [13, 31]). Thus along the smooth approximation, we have from (3.16) that

$$\int_0^1 \partial_t^2 \eta \cdot D(\varphi_s) \cdot dt = \int_0^1 \eta \cdot \frac{1}{V} \int_M \det(\Omega_{\varphi_s})dV \cdot dt.$$  

From the $C^{1,\beta}_w$ estimates of $\Psi$ and also $\Psi_s$, we know that

$$\det(\Omega_{\varphi_s}) = O(|z|^2 \beta - 2 - 2 \kappa).$$

It is integrable, since

$$2\beta - 2 - 2\kappa = -2 + 2\alpha \beta > -2.$$  

The Monge-Ampère operator is continuous under deceasing sequence $\Psi_s$ by Bedford-Taylor’s monotonic continuity theorem, (see, e.g., [38]), thus the RHS converges to

$$\int_0^1 \eta \cdot \frac{1}{V} \int_M \det(\Omega_{\varphi})dV \cdot dt$$.
by Lebesgue’s dominated convergence theorem. Since the $D$-functional is well-defined along the $C^2_\Delta$ generalized cone geodesic, the LHS converges by the dominated convergence theorem too. Therefore, we have obtained the conclusion of the lemma. □

Recall that $R = [0, 1] \times S^1$ and $z^{n+1} = x^{n+1} + \sqrt{-1} y^{n+1}$ with $x^{n+1} = t$.

**Lemma 3.11.** Assume $\{\varphi(t) : 0 \leq t \leq 1\}$ is a $C^{1,1,1}_W$ cone geodesic. Then the second derivative of the $j_{-\theta}$-functional is computed in the distribution sense; i.e., letting the test function $\eta$ be as above, we have

$$
\int_R \partial_{\tau}^2 \eta \cdot j_{-\theta}(\varphi) \cdot dz^{n+1} \wedge dz^{n+1} = -\frac{1}{V} \int_{M} \eta \cdot \Omega^n_{\Psi} \wedge \pi^* \theta.
$$

**Proof.** We first compute along the smooth geodesic (cf. section 2 in [62] and (3.5) in [8]):

$$
\partial_{\tau} j_{-\theta} = -\frac{1}{V} \int_{M} \varphi' \text{Tr}_{\varphi} \theta \omega^n_{\varphi}
$$

and

$$
\partial_{\tau}^2 j_{-\theta} = -\frac{1}{V} \int_{M} \{\varphi'' \text{Tr}_{\varphi} \theta + \varphi'[-(\varphi')^j_{i,j} \theta_{ij} + \text{Tr}_{\varphi} \theta \Delta \varphi']\} \omega^n_{\varphi}
$$

$$
= -\frac{1}{V} \int_{M} \{[\varphi'' - (\partial \varphi', \partial \varphi')_g] \text{Tr}_{\varphi} \theta + (\varphi')^j (\varphi')^i \theta_{ij} \} \omega^n_{\varphi}
$$

$$
= -\frac{1}{V} \int_{M} \{\text{Tr}_{\varphi} \theta \det(\Omega_{\Psi}) + (\varphi')^j (\varphi')^i \theta_{ij} \det(\omega_{\varphi})\} dV
$$

$$
= -\frac{1}{V} \int_{M} \frac{\Omega^n_{\Psi} \wedge \pi^* \theta}{\Omega^n_{\Psi} + 1} \det(\Omega_{\Psi}) dV.
$$

The rest of the proof is similar to that for Lemma 3.10. We could approximate $\varphi$ by smooth sequence and apply the convergence theorems to prove this lemma. □

Then, we move to the following strict convexity of the $J$-functional.

**Lemma 3.12.** Assume $\{\varphi(t) : 0 \leq t \leq 1\}$ is a $C^{1,1,1}_W$ cone geodesic. Then the $J$-functional is convex in the distribution sense; i.e., letting the test function $\eta$ be as above, we have

$$
\int_0^1 \partial_{\tau}^2 \eta \cdot J(\varphi) \cdot dt = -\int_0^1 \eta \cdot (\int_M |\partial \varphi|^2_{\varphi} \omega^n_{\varphi}) dt \geq 0.
$$

The equality holds if and only if $\varphi(t)$ is a constant geodesic.

**Proof.** We compute the second derivative of $\frac{1}{V} \int_M \varphi \omega^n_{\varphi}$ to see that

$$
\partial_{\tau}^2 \left[ \frac{1}{V} \int_M \varphi \omega^n_{\varphi} \right] = -\frac{1}{V} \int_M \varphi'' \omega^n_{\varphi}.
$$

(3.17)
Then we apply the approximation geodesic (2.6) to prove that (3.17) holds for the $C^{1,1}_w$ cone geodesic. With the $C^{1,1}_w$ estimate, we have $q'' = O(|z|^{-2\kappa})$. It is integrable, since

$$-2\kappa = -2\beta + 2\alpha\beta > -2\beta > -2.$$  

So we can apply the dominated convergence theorem to both sides. The conclusion then follows from applying Lemma 3.10 to

$$J(\varphi) = -D(\varphi) + \frac{1}{V} \int_M \varphi \omega_0^n.$$  

**Lemma 3.13.** The $D$-functional, $J$-functional, and $j_{-\theta}$-functional are continuous along the $C^{1,1}_w$ cone geodesic.

**Proof.** Given two points $\varphi_i = \varphi(t_i), i = 1, 2$, in the $C^{1,1}_w$ cone geodesic, we have their decreasing approximation $\varphi_{i\epsilon}$ as [13][20]. The cocycle condition of $D$, i.e.,

$$D(\varphi_{1\epsilon}) - D(\varphi_{2\epsilon}) = \frac{1}{V} \frac{1}{n+1} \sum_{j=0}^{n} \int_M (\varphi_{1\epsilon} - \varphi_{2\epsilon}) \omega_{\varphi_{1\epsilon}}^j \wedge \omega_{\varphi_{2\epsilon}}^{n-j},$$

implies that $D$ is bounded by $|\varphi_{1\epsilon} - \varphi_{2\epsilon}|_\infty$ multiplied by a constant depending on the spatial Laplacian estimate of $\varphi_i$. The approximation $D_{i\epsilon}$ converges to $D_i$ by the continuity of the weighted Monge-Ampere operator [38]. Thus $D$ is continuous along the $C^{1,1}_w$ cone geodesic.

The continuity of $j_{-\theta}$ follows in the same way, but using the cocycle condition that

$$j_{-\theta}(\varphi_1) - j_{-\theta}(\varphi_2) = -\frac{1}{V} \sum_{j=0}^{n-1} \int_M (\varphi_1 - \varphi_2) \omega_{\varphi_1}^j \wedge \omega_{\varphi_2}^{n-1-j} \wedge \theta.$$  

Since $\int_M \varphi_0^n$ is continuous when $\varphi(t)$ is continuous on $t$, we have the continuity of $J$. \hfill \Box

Combining Lemma 3.12 and Lemma 3.13 above, we have the following:

**Proposition 3.14.** Assume $\{\varphi(t) : 0 \leq t \leq 1\}$ is a $C^{1,1}_w$ cone geodesic. Then the $J$-functional is convex along $\varphi(t)$. The equality holds if and only if $\varphi(t)$ is a constant geodesic.

We then consider the entropy along the cone geodesic.

**Lemma 3.15.** The log-entropy $E(\varphi)$ and also the log-$K$-energy are lower-semicontinuous along the $C^{1,1}_w$ cone geodesic.

**Proof.** Since $\varphi(t) \in C^\beta_\Delta$, the volume ratio $h(\varphi) = \omega_n^\varphi / \omega_0^n$ is uniformly bounded and nonnegative. The sequence $h_i = \omega_n^\varphi_i / \omega_0^n$ is then uniformly bounded in $L^1(\omega_0)$ and $L^p(\omega_0)$ for some $p > 1$. So it has a weakly $L^1(\omega_0)$-convergent
subsequence. The subsequence also converges weak-star to a limit $f$, which is $L^\infty$. Therefore the assumption of lemma 4.7 in [26] is verified and $E$ is lower-semicontinuous. Then the log-$K$-energy is lower semicontinuous, according to the continuity of $D$ and $j_{\varphi_0}$ (Lemma [3.13]). \hfill \square

3.2 Approximation of $J$-Twisted cscK Cone Metrics

In this section, we illustrate a method to construct a smooth approximation sequence of the $J$-twisted cscK cone metric. Related works of Kähler-Einstein cone metrics have been seen (for example, [23]).

One possible application is in constructing generalized cone geodesics between $J$-twisted cscK cone metrics, where we need the smooth approximation equation (2.7) with smooth boundary values. But in our situation, the boundary Kähler potentials are Hölder-continuous, and Richberg’s regularization is sufficient for our cone geodesic problem (see Section 2).

Further application of the following approximation of $J$-twisted cscK cone metrics will be seen in the subsequent development of the existence problem [64].

Approximation of the Reference Metric $\omega_\theta$

We solve smooth $\varphi_{\theta_k}$ from the following approximation equation,

$$\frac{\omega^n_{\theta_k}}{\omega^n_0} = \frac{e^{h_0+c}}{(|s|^2_{h} + \epsilon)^{1-\beta}}.$$  

The volume is normalized to be

$$\int_M \omega^n_{\theta_k} = \int_M \frac{e^{h_0+c}}{(|s|^2_{h} + \epsilon)^{1-\beta}} \omega^n_0 = V.$$

**Lemma 3.16.** We denote by $\Theta_D = -i\partial\bar\partial \log |s|^2_{h}$ the curvature form of the Hermitian metric $h$ on the line bundle $L_D$. Let $\omega_{\theta_k}$ be the approximate metrics defined above by (3.18). Then, for any $\epsilon > 0$, we have

$$\text{Ric}(\omega_{\theta_k}) \geq \tilde{\theta} := \theta + \min\{(1-\beta)\Theta_D, 0\}.$$

In particular, when $C_1(L_D)$ is nonnegative, we have

$$\text{Ric}(\omega_{\theta_k}) \geq \theta.$$

**Proof.** Direct computation shows that

$$(3.19) \quad \text{Ric}(\omega_{\theta_k}) = \text{Ric}(\omega_0) - i\partial\bar\partial h_0 + (1-\beta)i\partial\bar\partial \log (|s|^2_{h} + \epsilon).

By (3.2), the right-hand side becomes

$$= \text{Ric}(\omega_0) - (1-\beta)i\partial\bar\partial \log |s|^2_{h} + (1-\beta)i\partial\bar\partial \log (|s|^2_{h} + \epsilon)$$

$$= \theta + (1-\beta)\Theta_D + (1-\beta)i\partial\bar\partial \log (|s|^2_{h} + \epsilon).$$
Then we use
\[
i\partial\bar{\partial} \log (|\nu|^2 + \epsilon) \geq \frac{|\nu|^2}{|\nu|^2 + \epsilon} i\partial\bar{\partial} \log |\nu|^2 = \frac{|\nu|^2}{|\nu|^2 + \epsilon} \Theta_D
\]
to see
\[
\text{Ric}(\omega_{\varphi_e}) \geq \theta + (1 - \beta) \frac{\epsilon}{|\nu|^2 + \epsilon} \Theta_D.
\]
Thus we have proved the lemma. 

**Approximation of \( P \)**

**Lemma 3.17.** We construct a smooth sequence \((\varphi_e, P_e)\) of \((\varphi, P)\) satisfying equation (3.12) of the \( J \)-twisted cscK cone metric.

**Proof.** Let \( \eta_e \) be a smooth sequence approximating the volume ratio
\[
\frac{\omega_0^n}{|\nu|^{2\beta - 2}} \frac{\omega_{\varphi_e}^n}{\omega_0^n} = \frac{\eta_e + c_p}{(|\nu|^2 + \epsilon)^{1-\beta}}.
\]
Here, we normalize
\[
\int_M \omega_{\varphi_e}^n = \int_M \frac{\eta_e + c_p}{(|\nu|^2 + \epsilon)^{1-\beta}} \omega_0^n = V,
\]
while we solve \( P_e \) by solving
\[
\triangle_{\omega_{\varphi_e}} P_e = \text{Tr}_{\omega_{\varphi_e}} \Theta - \gamma.
\]
Since \( \varphi_e \) has a uniform \( C^{2, \alpha, \beta} \) bound and \( \theta, \gamma \) have a uniform \( C^{0, \alpha, \beta} \) bound, we obtain the uniform \( C^{2, \alpha, \beta} \) estimate of \( P_e \). 

**Approximation of \( \varphi \)**

**Lemma 3.18.** There exists a smooth sequence \( \psi_e \) of \( \varphi \) in equation (3.12) of the \( J \)-twisted cscK cone metric such that \( \lim_{e \to 0} \psi_e = \varphi + \text{const.} \) Furthermore, the Ricci curvature satisfies
\[
\text{Ric}(\omega_{\psi_e}) \geq T_e := \theta - i\partial\bar{\partial} P_e,
\]
and the scalar curvature is
\[
S(\omega_{\psi_e}) = \text{Tr}_{\omega_{\psi_e}} \text{Ric}(\omega_{\psi_e}) - \triangle_{\psi_e} P_e.
\]
PROOF. We approximate $\varphi$ by the smooth sequence $\psi_\epsilon$ solving
\begin{equation}
\frac{\omega^n_{\psi_\epsilon}}{\omega^n_0} = e^{P_\epsilon + h_0 + c_\varphi} \left( |\xi|^2_h + \epsilon \right)^{1-\beta}
\end{equation}
with
\[ \int_M \omega^n_{\psi_\epsilon} = \int_M e^{P_\epsilon + h_0 + c_\varphi} \left( |\xi|^2_h + \epsilon \right)^{1-\beta} \omega^n_0 = V. \]
Since $P_\epsilon$ is uniformly bounded, $\psi_\epsilon$ has a uniform $C^{0,\alpha'}$ bound and converges to $\psi_0$ in $C^{0,\alpha'}$ for any $\alpha' < \alpha$ as $\epsilon \to 0$. Because both $\omega^n_{\psi_\epsilon}$ and $\omega^n_{\psi_\epsilon}$ converge to $\omega^n_\varphi$ in the $L^p$-norm, we see that
\[ \lim_{\epsilon \to 0} \psi_\epsilon = \varphi + \text{const}. \]
Furthermore, the smooth sequence $\psi_\epsilon$ converges to $\varphi$ smoothly outside the divisor.

We compute the Ricci curvature of $\omega_{\psi_\epsilon}$. By (3.22),
\[ \text{Ric}(\omega_{\psi_\epsilon}) = \text{Ric}(\omega_0) - i \bar{\partial} \partial h_0 + (1-\beta) i \bar{\partial} \partial \log \left( |\xi|^2_h + \epsilon \right) - i \bar{\partial} \partial P_\epsilon. \]
Then by (3.18),
\[ \text{RHS} = \text{Ric}(\omega_{\theta_\epsilon}) - i \bar{\partial} \partial P_\epsilon. \]
Using Lemma 3.16, we have
\[ \text{Ric}(\omega_{\psi_\epsilon}) \geq T_\epsilon := \theta - i \bar{\partial} \partial P_\epsilon. \]
Furthermore, we can see that the scalar curvature of $\omega_{\psi_\epsilon}$ is
\[ S(\omega_{\psi_\epsilon}) = T_{\omega_{\psi_\epsilon}} \left( \text{Ric}(\omega_{\psi_\epsilon}) - i \bar{\partial} \partial P_\epsilon \right). \]

Additional properties and improvement of the approximation of the twisted cscK cone metrics are obtained in proposition 3.14 in [64], as well as the uniform a priori estimates of the approximation sequence $\psi_\epsilon$.

3.3 Convexity along Cone Geodesics

The convexity of the $K$-energy along the $C^{1,1}$-geodesic is proved in [8, 26] and extended to the $C^{1,1, \beta}$ cone geodesic (half-angle cone geodesic) in [47]. We adapt the convexity to the $C^{1,1, \beta}$ cone geodesic $\Omega_\psi$ in this section. The delicate issue is that we lose regularity along the directions
\[ \begin{bmatrix} \frac{\partial^2 \Psi}{\partial z^1 \partial \tau} \\ \frac{\partial^2 \Psi}{\partial z^1 \partial r} \end{bmatrix}_\Omega \quad \text{and} \quad \begin{bmatrix} \frac{\partial^2 \Psi}{\partial \tau^2} \\ \frac{\partial^2 \Psi}{\partial r^2} \end{bmatrix}_\Omega, \]
but they can be bounded with proper weights, i.e.,
\[ \sum_{2 \leq j < n} |z^j|^\kappa \begin{bmatrix} \frac{\partial^2 \Psi}{\partial z^j \partial \tau} \\ \frac{\partial^2 \Psi}{\partial z^j \partial r} \end{bmatrix}_\Omega, \quad |z^1|^{\kappa+1-\beta} \begin{bmatrix} \frac{\partial^2 \Psi}{\partial z^1 \partial \tau} \\ \frac{\partial^2 \Psi}{\partial z^1 \partial r} \end{bmatrix}_\Omega, \quad |z^1|^{2\kappa} \begin{bmatrix} \frac{\partial^2 \Psi}{\partial \tau^2} \\ \frac{\partial^2 \Psi}{\partial r^2} \end{bmatrix}_\Omega. \]
We observe that the convexity still survives.
THEOREM 3.19. The log-K-energy is continuous and convex along the $C^{1,1,\beta}_{w}$ cone geodesic.

Remark 3.20. The proof below could be further improved to the $C^{\beta}_{\Delta}$ generalized cone geodesic.

Before we start to prove this theorem, we obtain as follows its main application to the $J$-twisted log-K-energy $E$ and the $J$-twisted $\text{cscK}$ cone metric.

PROPOSITION 3.21. The $J$-twisted log-K-energy $E$ is strictly convex along the $C^{1,1,\beta}_{w}$ cone geodesic, and the $J$-twisted $\text{cscK}$ cone metric is unique.

PROOF. Combining the strict convexity of the $J$-functional (Proposition 3.14) and the convexity of the log-K-energy (Theorem 3.19), we have proved the strict convexity of the $J$-twisted log-K-energy $E$. The uniqueness follows directly from the strict convexity along the cone geodesic. □

PROOF OF THEOREM 3.19. It is sufficient to prove that the log-K-energy is convex. As long as we have the convexity of $\nu_{\beta}$, the upper semicontinuity is obtained immediately as a corollary, while Lemma 3.15 provides the lower semicontinuity. So the log-K-energy is continuous. We follow the proof in the spirit of [8, 26, 47], which contains four steps. Let $\epsilon, \delta$ be small constants.

Step 1. Regularization in the interior of $X$. Recall that $\Omega_{\Psi}$ is a $C^{1,1,\beta}_{w}$ cone geodesic, that is,

$$\Omega_{\Psi} = \Omega_{0} + i \bar{\partial} \Psi, \quad \Psi \in C^{1,1,\beta}_{w}, \quad \Omega_{0} = \pi^{\ast} \omega_{0} + d \omega^{n+1} \wedge d \bar{\omega}^{n+1}.$$ 

Since the potential $\Psi$ of the cone geodesic $\Omega_{\Psi}$ is Hölder-continuous, Richberg’s convolution regularization process implies that there exists a smooth approximation $\Omega_{\delta} = \Omega_{0} + i \bar{\partial} \Psi_{\delta}$ in the interior of the product manifold $R_{0} \times X \subset X$ [13, 31] and $\Omega_{\delta}$ satisfies

$$C_{\delta} \Omega_{0} + \Omega_{\delta} \geq 0 \quad \text{for some constant } C.$$ 

We use $\varphi_{\delta}$ to denote the potential of the restriction of $\Omega_{\delta}$ on each fiber $\{t\} \times X$, i.e.,

$$\Omega_{\delta} \big|_{\{t\} \times X} := \omega_{0} + i \bar{\partial} \varphi_{\delta}.$$ 

We set the perturbed approximate geodesic to be

$$\tilde{\Omega}_{\delta} := 2C_{\delta} \Omega_{0} + \Omega_{\delta}$$

and its restriction to each fiber to be

$$\tilde{\omega}_{\delta} := 2C_{\delta} \omega_{0} + \Omega_{\delta} \big|_{\{t\} \times X} > 0 \quad \text{on } \{t\} \times X.$$ 

Note that these are strictly positive forms.

We also set $\varphi(t)$, the restriction of $\Omega_{\Psi}$ on $\{t\} \times X$. Note that this is slightly different from the notion (up to a $t^{2}$) we used in the previous sections. Additional properties of Richberg’s regularization $\varphi_{\delta}$ hold as follows:
\[ \sup_{t \times X} |\varphi_\delta - \varphi|_{C^0} \to 0 \text{ as } \delta \to 0. \]
\[ |i \bar{\partial} \varphi_\delta|_{C^{1,0}} \text{ and also } \|\Psi\|_{C^{1,1}} \text{ are uniformly bounded.} \]
\[ g_{\varphi_\delta} \text{ converges to } g_{\varphi} \text{ in the } L^p \text{-norm for some } p > 1 \text{ on each fiber } \{t\} \times X. \]

Since \( \varphi \in C^{1,1}_{\text{per}} \), the range of \( p \) can be determined as
\[ 1 < p < \min \left\{ \frac{1}{k}, \frac{1}{1 - \beta}, \frac{2}{k + 1 - \beta} \right\}. \]

**Step 2. Improved regularization on each fiber \( \{t\} \times X. \)**

In order to improve the regularization on each fiber, we solve the Monge-Ampère equation of \( \varphi_\theta \) with the parameters \( \theta := (\epsilon, \delta, \mu) \):
\[ (-\epsilon \theta + \tilde{\omega}_\delta + \epsilon i \bar{\partial} \varphi_\theta)^n = e^{\theta_\mu} \omega^n_{\theta_{\mu}}. \]

Here \( \omega^n_{\theta_{\mu}} \) is the approximate reference metric constructed in (3.18), i.e.,
\[ \omega_0^n = \frac{e^{\rho_0 + c}}{(|s|^2 + \mu)^{1-\beta}}. \]

The parameter \( \mu \) depends on the parameter \( \delta \) and will be determined later.

By the construction of \( \tilde{\omega}_\delta \) in (3.23), we can choose a small constant \( \epsilon \) such that
\[ \epsilon [\mathcal{C}_1(X) + (1 - \beta)\mathcal{C}_1(L_D)] + [\tilde{\omega}_\delta] \]
is Kähler. Recall in (3.1) that
\[ \theta \in \mathcal{C}_1(X) - (1 - \beta)\mathcal{C}_1(L_D). \]

Then we have the form
\[ -\epsilon \theta + \tilde{\omega}_\delta + \epsilon i \bar{\partial} \varphi_\theta > 0 \text{ on } \{t\} \times X. \]

The solution of (3.24) satisfies the following estimates.

**Lemma 3.22 (\( L^\infty \)-upper bound).** There exists a constant \( C \) independent of \( \epsilon, \delta \) such that
\[ \sup_{x} \varphi_\theta \leq C. \]

**Proof.** At the maximum point \( q \) of \( \varphi_\theta, i \bar{\partial} \varphi_\theta(q) \leq 0. \) By the Monge-Ampère equation (3.24), we have
\[ e^{\theta_\mu} \leq \frac{(-\epsilon \theta + \tilde{\omega}_\delta)^n}{\omega^n_{\theta_{\mu}}}(q). \]

Since the numerator satisfies (3.25), we could choose \( \mu \) depending on \( \delta \) such that the fraction is bounded. \( \square \)

**Lemma 3.23 (\( L^\infty \)-lower bound).** There exists a constant \( C \) independent of \( \epsilon, \delta \) such that
\[ -\epsilon \inf_{x} \varphi_\theta \leq C. \]
The Monge-Ampère equation (3.24) can be written as
\[ (\omega_{e,\delta} + i \partial \overline{\partial} \psi_\theta)^n = e^{\frac{\overline{\omega}_0 - \psi_\delta + \psi_{D\delta}}{\varepsilon} - \omega_0^n} \]
with the background Kähler cone metrics
\[ \omega_{e,\delta} = -\varepsilon \theta + C \delta \omega_0 + \omega_{D\delta} \quad \text{and} \quad \overline{\psi}_\theta = \psi_\delta - \psi_{D\delta} + \varepsilon \psi_\theta. \]
At the minimum point of \( \overline{\psi}_\theta \), we have \( i \partial \overline{\partial} \overline{\psi}_\theta \geq 0 \). Inserting this into (3.27), we have
\[ e^{\frac{\overline{\omega}_0 (\omega_0^n - \psi_\delta (t_1) + \psi_{D\delta} (t_1))}{\varepsilon} - \omega_0^n} \geq \frac{\omega_{e,\delta}^n}{\omega_0^n} (q). \]
Since \( \omega_{e,\delta}^n \) is \( O((|s|^2 + \delta)^{\beta - 1}) \), we have that the lower bound of the RHS side is strictly positive, denoted by \( C \). Thus,
\[ \epsilon \psi_\theta \geq -\psi_\delta + \psi_{D\delta} + \epsilon \log C + \psi_\delta (q) - \psi_{D\delta} (q), \]
which is bounded below, by the construction of \( \psi_\delta \) in Step 1.

**Lemma 3.24 (Equicontinuous).** The solution \( \psi_\theta (t) \) is equicontinuous.

**Proof.** Let \( \overline{\omega}_{\psi_\theta (t)} = \omega_{e,\delta} + i \partial \overline{\partial} \overline{\psi}_\theta (t) \). Then we compare two solutions of (3.27) at \( t_1, t_2 \),
\[ (\omega_{\psi_\theta (t_2)} + i \partial \overline{\partial} (\overline{\psi}_\theta (t_1) - \overline{\psi}_\theta (t_2)))^n = e^{\frac{\overline{\omega}_0 (\omega_0^n - \psi_\delta (t_1) + \psi_{D\delta} (t_1)) - \psi_\delta (t_2) + \psi_{D\delta} (t_2)}{\varepsilon} - \omega_0^n}. \]
Since \( \partial_t \psi \) is bounded, we have \( \partial_t \psi_\delta \) is uniformly bounded. The cone maximum principle implies the equicontinuity of \( \psi_\theta (t) \) on \( t \). Similarly, using the estimate that \( |\partial_z \psi|_{\omega_{D\delta}} \) is uniformly bounded, we obtain the conclusion.

**Lemma 3.25 (Laplacian estimates).** There exists a constant \( C \) independent of \( \varepsilon, \delta \) such that
\[ \epsilon \sup_x |i \partial \overline{\partial} \psi_\theta|_{\omega_{D\delta}} \leq C. \]

**Proof.** The proof is similar to that of Proposition 2.22. In our case the smooth background metric is \( \overline{\omega}_0 = -\varepsilon \theta + C \delta \omega_0 + \omega_0 \). We choose the background approximate cone metric as in Section 2, that is, \( \omega_b = \overline{\omega}_0 + i \partial \overline{\partial} \overline{\psi}_b \). We denote \( \overline{\omega} = \omega_b + i \partial \overline{\partial} \overline{\psi}_b \). \( \overline{\psi}_b = \psi_\delta + \epsilon \psi_\theta - \psi_{\text{hc}} \),
and rewrite (3.27) as
\[ \overline{\omega}^n = e^{\psi_\theta + F} \omega_b^n \text{ with } F = \log \frac{\omega_0^n}{\omega_b^n}, \overline{F} = F + \psi_\theta. \]
Yau’s computation shows that
\[ \bar{\Delta} \log \text{Tr}_{\omega_b} \omega_b \geq -g^{i\bar{j}} R_{i\bar{j}}(\omega_b) + 2k^T R^k R_{i\bar{j}}(\omega_b) \bar{g}_{i\bar{j}}. \]

Using (3.28), we have
\[ \text{RHS} \geq \frac{\Delta_{\omega_b} \bar{F} - S(\omega_b) + 2k^T R^k R_{i\bar{j}}(\omega_b) \bar{g}_{i\bar{j}}}{\text{Tr}_{\omega_b} \omega_b}. \]

Then the last two terms are controlled after introducing the extra weight \( \Delta \Phi \). Here we apply the same method as we used to derive a similar estimate, (2.28).

Now we compute
\[ i \partial \bar{\partial} F = i \partial \bar{\partial} \log \frac{1|\omega_{\mu_1}^{\omega_{\mu_2}} e^{\omega_{\mu_1} + \epsilon \omega_b}}{\omega_b} \]
and make use of (2.11) to get
\[ i \partial \bar{\partial} F \geq -(C \omega_b + i \partial \bar{\partial} \Phi). \]

Moreover, by \( C \partial \bar{\partial} \Phi \geq 0 \) from (2.10), we get
\[ \Delta_{\omega_b} F \geq - \text{Tr}_{\omega_b} \omega_b (C \omega_b + i \partial \bar{\partial} \Phi). \]

Putting the inequalities above together, we obtain that
\[ \bar{\Delta} \log \text{Tr}_{\omega_b} \omega_b \geq -C \text{Tr}_{\omega_b} \omega_b - C \Phi + \frac{\Delta_{\omega_b} \phi_{\lambda}}{\text{Tr}_{\omega_b} \omega_b}. \]

Then we use the auxiliary function \( Z = -(C + 1) \phi_{\lambda} + C \Phi \) to cancel the middle terms. Since \( \tilde{\Delta} \phi_{\lambda} = n - \text{Tr}_{\omega_b} \omega_b \), we see that
\[ \bar{\Delta} (\log \text{Tr}_{\omega_b} \omega_b + Z) \geq -(C + 1)n + \frac{\Delta_{\omega_b} \phi_{\lambda}}{\text{Tr}_{\omega_b} \omega_b}. \]

At the maximum point \( p \) of \( \log \text{Tr}_{\omega_b} \omega_b + Z \), it holds that
\[ \Delta_{\omega_b} \phi_{\lambda} \leq C \text{Tr}_{\omega_b} \omega_b. \]

Now we make use of \( \text{Tr}_{\omega_b} \omega_b = n + \Delta_{\omega_b} \phi_{\lambda} \) and also compute
\[ \Delta_{\omega_b} \phi_{\lambda} = \Delta_{\omega_b} (\phi_{\lambda} + \epsilon \phi_0 - \phi_b) \leq \epsilon \Delta_{\omega_b} \phi_0 + C. \]

Rearranging these inequalities and direct computation shows that
\[ \text{Tr}_{\omega_b} \omega_b(p) \leq \frac{n + C}{1 - C \epsilon} \]
is bounded, as long as \( \epsilon \leq (2C)^{-1} \). At arbitrary point \( z \), we have the estimate of \( \log \text{Tr}_{\omega_b} \omega_b(z) \) as follows:
\[ \log \text{Tr}_{\omega_b} \omega_b(z) \leq \log \text{Tr}_{\omega_b} \omega_b(p) + Z(p) - Z(z). \]
According to the zero estimates Lemma 3.22 and Lemma 3.23 and the weight $\hat{\omega}$ in (2.10), we obtain that the auxiliary function $Z$ is bounded. Therefore the Laplacian estimate is proved.

Step 3. Passing to limits. Once we have the estimates from Steps 1 and 2, we are able to take the limit of the sequence $\varphi_{\theta}$. We first fix $\epsilon$ and take $\delta \to 0$. Then

$$\varphi_{\theta} \to \varphi_{\epsilon} \quad \text{in } C^0$$

as a consequence of the equicontinuity of $\varphi_{\theta}$ (Lemma 3.24). Furthermore, with the help of the $L^\infty$ estimate (Lemma 3.22 and Lemma 3.23), the Laplacian estimate of $\epsilon \varphi_{\theta}$ (Lemma 3.25), and the corresponding estimates of the interior approximation of the cone geodesic $\varphi_{\delta}$, the limit $\varphi_{\epsilon}$ also satisfies the limiting equation of (3.24) as $\delta \to 0$, i.e.,

$$(-\epsilon \theta + \omega_0 + i \tilde{\partial}(\varphi + \epsilon \varphi_{\epsilon}))^n = e^{\varphi_{\epsilon}} \cdot \omega_0^n.$$

The positivity is also passing from the fiber to the family in $R_0 \times X$.

**Lemma 3.26.**

$$-\epsilon \pi^* \theta + \tilde{\Omega}_\psi + \epsilon i \tilde{\partial} \varphi_{\epsilon} \geq 0 \quad \text{in } R_0 \times X.$$

**Proof.** According to theorem 4.1 in [26], the fiberwise inequality (3.26) could be extended to the whole interior,

$$-\epsilon \pi^* \theta + \tilde{\Omega}_\delta + \epsilon i \tilde{\partial} \varphi_{\theta} > 0 \quad \text{in } R_0 \times X.$$

Here the $i \tilde{\partial}$ operator is also extended to the product manifold. Since the perturbed approximate geodesic $\tilde{\Omega}_\delta$ converges to the cone geodesic $\Omega_\psi$ as $\delta \to 0$, the lemma follows from (3.30).

Second, we take $\epsilon \to 0$.

**Lemma 3.27.** The volume form

$$e^{\varphi_{\epsilon}} \cdot \omega_0^n \to \omega_\varphi^n$$

weakly in $L^p$ for some $p > 1$ as $\epsilon \to 0$.

**Proof.** Applying the estimates Lemma 3.22, Lemma 3.23, and Lemma 3.25 again, we have that the LHS of (3.29) converges:

$$(\omega_\varphi + i \tilde{\partial} \varphi)^n \to (\omega_\varphi + i \tilde{\partial} \phi)^n$$

weakly in $L^p$ for some $p > 1$ and the potential $\epsilon \varphi_{\epsilon}$ converges to $\phi$ in $C^{1, \alpha, \beta}$ as $\epsilon \to 0$.

Actually, we will see that $\phi = 0$. It is direct to find that $\phi \leq 0$; otherwise $\frac{\epsilon \varphi_{\epsilon}}{\epsilon}$ diverges to $+\infty$, which contradicts the convergence of the LHS of (3.29). Let $a$ be a small nonnegative constant. We have that $e^{\varphi_{\epsilon}} \cdot \omega_0^n = e^{\varphi_{\epsilon}} \cdot \omega_\varphi^n$ is forced to converge to 0 in the domain $\{\phi < -a\}$. By (3.29) and the convergence (3.32), we
have $\int_{\phi < -a} (\omega_{\phi} + i \partial \bar{\partial} \phi)^n = 0$. From the comparison theorem (theorem 3 in [13]), we have $\int_{\phi < -a} \omega^n_{\phi} \leq \int_{\phi < -a} (\omega_{\phi} + i \partial \bar{\partial} \phi)^n$. Then the following equality holds:

$$\int_{\phi < -a} \omega^n_{\phi} = \int_{\phi < -a} (\omega_{\phi} + i \partial \bar{\partial} \phi)^n.$$  

(3.33)

Since $\omega^n_{\phi}$ is integrable, we get (3.33) in the closure of the domain $\{ \phi < 0 \}$ after taking $a \to 0$. In $\{ \phi = 0 \}$, (3.33) is clear. In conclusion, $\omega^n_{\phi} = (\omega_{\phi} + i \partial \bar{\partial} \phi)^n$ holds as currents in $M$.

Since both $(\omega_{\phi} + i \partial \bar{\partial} \phi)^n$ and $\omega^n_{\phi}$ are $L^p$, the stability of the Monge-Ampère equation (theorem 1.1 in [46]) implies that $\phi$ has to vanish. Thus the convergence of the approximation volume form is obtained. □

**Step 4. Convexity.** We will prove the convexity of the log-$K$-energy $v_\beta$ by comparing its values at three arbitrary points $t_1 < t_2 < t_3 \in [0, 1]$ on the cone geodesic. That is, to show the value at $t_2$ lies below the line segment between the endpoints $t_1$ and $t_3$. We will construct an approximation of $v_\beta$ and prove this convexity property for the approximate sequence. As a result, the convexity of $v_\beta$ follows after taking the limit.

**Construct an approximation of $v_\beta$:** Repeating the Banach-Saks theorem at these three points and picking subsequences of the Cesàro mean, we have by Lemma 3.27 that

$$\sum_{j=1}^{k} d_j^k e^{\phi_{j}(t_m)} \omega^n_\theta \to \omega^n_\phi(t_m)$$

in $L^p(\{t_m\} \times X)$ for some $p > 1$ as $k \to \infty$ for each $m = 1, 2, 3$. We define

$$V_k(t) := \sum_{j=1}^{k} d_j^k e^{\phi_{j}(t)} \omega^n_\theta$$

for any $t \in [0, 1]$.

The approximation of the log-$K$-energy (3.9) is defined to be

$$v_k^\beta(\phi) := E_k^\beta(\phi) + S_\beta \cdot D(\phi) + j_{-\phi}(\phi) + \frac{1}{V} \int_M (\theta + h_0) \omega^n.$$  

$$E_k^\beta(\phi) := \frac{1}{V} \int_M \log \frac{V_k(t)}{\omega^n_\theta} \omega^n_\phi,$$

$$\omega^n_\theta = \omega^n_\phi |\xi|^2 h^{-2} e^{h_0}.$$  

**Convexity of the approximation $v_k^\beta$:** Thanks to Lemma 3.13, the approximation $K$-energy $v_k^\beta(\phi)$ is continuous in $t \in [0, 1]$. In order to prove the convexity of $v_k^\beta$ along the cone geodesic, it remains to prove it is weakly convex, i.e., is convex in the distribution sense.

**Lemma 3.28.** The approximate log-$K$-energy $v_k^\beta$ is convex in the distribution sense along the cone geodesic.
PROOF. According to the second derivatives of \( D \) and \( j_\theta \) in Lemma 3.10 and Lemma 3.11, it is sufficient to prove that

\[
\int_R \partial^2 \eta \cdot E^k \cdot dz^{n+1} \wedge \partial z^{n+1} = \frac{1}{V} \int_{\Omega^n} \eta \cdot \Omega^n \wedge \pi^* \theta
\]

along the cone geodesic. Here \( \eta \) is a smooth nonnegative cutoff function supported in the interior of \([0, 1]\). As shown in theorem 3.3 in [8], it is reduced to show that

\[
i \partial \bar{\partial} \log \frac{V_k(t)}{\omega_0^{n+1}} \geq \pi^* \theta \wedge \Omega^n \Psi.
\]

That follows immediately from computing

\[
i \partial \bar{\partial} \log \frac{V_k(t)}{\omega_0^{n+1}} = \partial \bar{\partial} \log \sum_{j=1}^k d^k e^{\phi_{ej}} = \sum_{j=1}^k a^k_{j-1} d^k e^{\phi_{ej}} i \partial \bar{\partial} \phi_{ej}
\]

and then using \(-\epsilon_j \pi^* \theta + \Omega^n \Psi \geq 0\) (Lemma 3.26) and the geodesic equation \( \Omega^{n+1} = 0 \). So the lemma is proved.

Having taken the limit of the convexity inequality of the approximation log-\( K \)-energy at three points, we have completed the proof of Theorem 3.19 that the log-\( K \)-energy \( v_\eta \) is also convex in \([0, 1]\).

4 Regularity of cscK Cone Metrics

We prove the geometric polyhomogeneity of cscK cone metrics in this section. The main theorem is Theorem 4.8.

4.1 Hölder Spaces \( C^{3, \alpha, \beta}_{pw} \) and \( C^{3, \alpha, \beta}_w \)

Before we prove the regularity theorem, we introduce some new function spaces, which are designed for the regularity of cscK cone metrics.

DEFINITION 4.1. A function \( \varphi \) belongs to \( C^{3, \alpha, \beta}_{pw} \) with the Hölder exponent \( \alpha \) satisfying \( \alpha \beta < 1 - \beta \) if the following hold:

- \( \varphi \in C^{2, \alpha, \beta} \),
- In the coordinate chart intersecting the divisor, for any \( 2 \leq i, k, l \leq n \), the following items are \( C^{0, \alpha, \beta} \):

\[
\frac{\partial g_{k\bar{l}}}{\partial z^i}, \quad \left| z^1 \right|^{1-\beta} \frac{\partial g_{k\bar{l}}}{\partial \bar{z}^i}, \quad \left| z^1 \right|^{1-\beta} \frac{\partial g_{k\bar{l}}}{\partial z^i}, \quad \left| z^1 \right|^{2-2\beta} \frac{\partial g_{k\bar{l}}}{\partial \bar{z}^i}.
\]

Here \( (z^1, \ldots, z^n) \) is the local coordinate and \( z^1 \) is the local defining function of \( D \). We also set the corresponding metric \( g \) to be the Kähler cone metric given by \( \varphi \), satisfying \( g_{k\bar{l}} = g_{0k\bar{l}} + \varphi_{k\bar{l}} \).

When the coordinate chart does not intersect the divisor, all definitions are in the classical way by using the locally defined smooth metric in this coordinate chat.

We define another space \( C^{3, \alpha, \beta}_w \), with information along the singular directions.
DEFINITION 4.2. A function $\varphi$ is in $C^{3,\alpha,\beta}_w$ with the Hölder exponent $\alpha$ satisfying $\alpha \beta < 1 - \beta$ if $\varphi \in C^{3,\alpha,\beta}_{pw}$. In addition, in the normal cone chart, the following terms are $O(\rho^{1-\beta})$:

$$|z|^{1-\beta} \frac{\partial g_{k\ell}}{\partial z^i}, \quad |z|^{1-\beta} \frac{\partial g_{k\ell}}{\partial z^1}, \quad |z|^{1-\beta} \frac{\partial g_{1\ell}}{\partial z^1}.$$

The spaces above could be generalized to the function $\Psi$ on the product manifold $X$, adding the new direction $\ell$ to the regular directions $n$, i.e., changing $n$ to $n + 1$ in the definition. We list the terms $|\nabla_i^{\text{cone}}(g_\psi)_\alpha\beta|_{\Omega_{\text{cone}}}$ as a corollary of the definitions. The computation will be given in the next section.

COROLLARY 4.3. Suppose $\Psi \in C^{3,\alpha,\beta}_{pw}$. Then it holds that $|\nabla_i^{\text{cone}}(g_\psi)_\alpha\beta|_{\Omega_{\text{cone}}}$ are in $C^{0,\alpha,\beta}$ for all $2 \leq i \leq n + 1$ and $1 \leq a, b \leq n + 1$.

PROOF. When $2 \leq i, k, l \leq n + 1$, we have the formulas

$$\begin{align*}
\nabla_i^{\text{cone}}(g_\psi)_k\ell = & \frac{\partial (g_\psi)_k\ell}{\partial z^i}, & \nabla_i^{\text{cone}}(g_\psi)_1\ell = & \frac{\partial (g_\psi)_1\ell}{\partial z^i}, \\
\nabla_i^{\text{cone}}(g_\psi)_1\ell = & \frac{\partial (g_\psi)_1\ell}{\partial z^i}, & \nabla_i^{\text{cone}}(g_\psi)_1\ell = & \frac{\partial (g_\psi)_1\ell}{\partial z^i}.
\end{align*}$$

Then the conclusions follow from Definition 4.1. $\square$

COROLLARY 4.4. Let $\Psi \in C^{3,\alpha,\beta}_w$. Then the following terms are $O(\rho^{1-\beta})$:

$$\begin{align*}
|\nabla_i^{\text{cone}}(g_\psi)_k\ell|_{\Omega_{\text{cone}}}, & \quad |\nabla_i^{\text{cone}}(g_\psi)_k\ell|_{\Omega_{\text{cone}}}, \\
|\nabla_i^{\text{cone}}(g_\psi)_1\ell|_{\Omega_{\text{cone}}}, & \quad |\nabla_i^{\text{cone}}(g_\psi)_1\ell|_{\Omega_{\text{cone}}},
\end{align*}$$

for all $2 \leq k, l \leq n + 1$.

PROOF. The computation will be given in the proof of Corollary 4.5. $\square$

4.2 Christoffel Symbols of cscK Cone Metrics

Before we prove Proposition 4.13 and Proposition 4.15, we make note of some applications where they are applied to obtain the following asymptotic behavior of the first derivatives of the cscK cone metric $g$ and its Christoffel symbols for further application.

We are given a point $z$ outside the divisor, and in the cone chart we denote

$$\rho_0 = |z|^\beta.$$ 

COROLLARY 4.5. Let $\varphi$ be a $C^{2,\alpha,\beta}_w \cap \mathcal{H}_\beta$ solution of (4.4) of a cscK cone metric (or $J$-twisted cscK cone metric). Then $\varphi \in C^{3,\alpha,\beta}_w \cap \mathcal{H}_\beta$, i.e., the first derivatives

$$\frac{\partial \varphi}{\partial z^i}, \quad \frac{\partial \varphi}{\partial z^1}, \quad \frac{\partial \varphi}{\partial z^{1-1}},$$

are in $C^{0,\alpha,\beta}_w \cap \mathcal{H}_\beta$. The computation will be given in the proof of Corollary 4.5.
of the corresponding metric \( g \) satisfy, for any \( 2 \leq i, k, l \leq n, \)

\[
\begin{align*}
\left\{ \begin{array}{ll}
|z|^{1-\beta} \frac{\partial g_{k\bar{I}}}{\partial z^i} & \in C^{0, \alpha, \beta}, \\
|z|^{1-\beta} \frac{\partial g_{\bar{I}I}}{\partial z^j} & \in C^{0, \alpha, \beta}, \\
|z|^{1-\beta} \frac{\partial g_{k\bar{I}}}{\partial z^l} & \in C^{0, \alpha, \beta}, \\
|z|^{2-2\beta} \frac{\partial g_{I\bar{I}}}{\partial z^i} & \in C^{0, \alpha, \beta}, \\
|z|^{1-\beta} \frac{\partial g_{k\bar{I}}}{\partial z^l} & = O(\rho_0^{g-1}), \\
|z|^{1-\beta} \frac{\partial g_{\bar{I}I}}{\partial z^l} & = O(\rho_0^{g-1}), \\
|z|^{1-\beta} \frac{\partial g_{k\bar{I}}}{\partial z^l} & = O(\rho_0^{g-1}).
\end{array} \right.
\end{align*}
\]

Moreover,

\[
\begin{align*}
\left\{ \begin{array}{ll}
|z|^{1-\beta} \nabla_1^{\text{cone}} g_{k\bar{I}} & = O(\rho_0^{g-1}), \\
|z|^{1-\beta} \nabla_1^{\text{cone}} g_{\bar{I}I} & = O(\rho_0^{g-1}), \\
|z|^{1-\beta} \nabla_1^{\text{cone}} g_{k\bar{I}} & = O(\rho_0^{g-1}), \\
|z|^{1-\beta} \nabla_1^{\text{cone}} g_{\bar{I}I} & = O(\rho_0^{g-1}).
\end{array} \right.
\end{align*}
\]

**Proof.** Using Proposition 4.13, we have in \( V \)-coordinates that the metric \( g \) satisfies

\[
\|g_{ij}\| \leq C(1).
\]

We choose the normal coordinate such that at \( \bar{v} = 0 \), \( g \) is the flat cone metric and \( \partial \bar{v} g_{k\bar{I}} = 0. \)

Then we have

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial \bar{v} \ g_{k\bar{I}} & \in C^{0, \alpha, \beta}, \\
\partial \bar{v} \ g_{\bar{I}I} & \in C^{0, \alpha, \beta}, \\
\partial \bar{v} \ g_{k\bar{I}} & \in C^{0, \alpha, \beta}, \\
\partial \bar{v} \ g_{\bar{I}I} & \in C^{0, \alpha, \beta}, \\
\partial \bar{v} \ g_{k\bar{I}} & = O(\rho_0^{g-1}), \\
\partial \bar{v} \ g_{\bar{I}I} & = O(\rho_0^{g-1}), \\
\partial \bar{v} \ g_{k\bar{I}} & = O(\rho_0^{g-1}).
\end{array} \right.
\end{align*}
\]

Then transforming back to the \( z \)-coordinates with

\[
v^1 = (z^1)^\beta \quad \text{and} \quad \bar{g}_{k\bar{I}} = \beta^{-1}(z^1)^{1-\beta} \ g_{k\bar{I}}, \quad \bar{g}_{\bar{I}I} = \beta^{-2} |z|^2 |z|^{-2\beta} g_{\bar{I}I},
\]

we obtain (4.1).

In order to prove (4.2), we first compute the Christoffel symbols of \( \omega_{\text{cone}} \). We use the standard formula

\[
\Gamma^{k}_{ij}(\omega_{\text{cone}}) = g_{k\bar{I}} \frac{\partial (g_{\text{cone}})}{\partial z^j}.
\]

and compute

\[
\frac{\partial}{\partial z^i} |z|^{2\beta - 2} = (\beta - 1) |z|^{1 - 2\beta} \frac{\partial}{\partial z^i} |z|^2.
\]

to see that the Christoffel symbols of \( \omega_{\text{cone}} \) in a cone chart all vanish, that is, for all \( 2 \leq i, j, k \leq n, \)

\[
\Gamma^{1}_{1k}(\omega_{\text{cone}}) = \Gamma^{1}_{11}(\omega_{\text{cone}}) = \Gamma^{1}_{jk}(\omega_{\text{cone}}) = \Gamma^{i}_{1k}(\omega_{\text{cone}}) = \Gamma^{i}_{jk}(\omega_{\text{cone}}) = 0.
\]
The only exception is
\[ \Gamma^{1}_{11}(\omega_{\text{cone}}) = -\frac{1 - \beta}{z^{1}}. \]

Second, we compute \( \nabla_{1}^{\text{cone}} g_{1\bar{1}}^{TI} \) and \( \nabla_{1}^{\text{cone}} g_{11}^{T} \). We use the following formulas. One is the first covariant derivative w.r.t. \( \omega_{\text{cone}} \):

\[
\nabla_{1}^{\text{cone}} \varphi_1 = \frac{\partial \varphi_1}{\partial z^{1}} - \sum_{p=1}^{n} \Gamma^{p}_{11} \varphi_{p} = \frac{\partial \varphi_1}{\partial z^{1}} + \frac{1 - \beta}{z^{1}} \varphi_1.
\]

The other one is the first covariant derivative of a \((1, 1)\)-form \( a_{i\bar{j}}dz^{i} \wedge d\bar{z}^{j} \) w.r.t. \( \omega_{\text{cone}} \), i.e.,

\[
\nabla_{1}^{\text{cone}} a_{i\bar{j}} = \frac{\partial a_{i\bar{j}}}{\partial z^{1}} - \sum_{p=1}^{n} \Gamma^{p}_{1i} a_{pj} - \sum_{p=1}^{n} \Gamma^{\bar{p}}_{1\bar{j}} a_{pi\bar{p}} = \frac{\partial a_{i\bar{j}}}{\partial z^{1}} + \frac{1 - \beta}{z^{1}} a_{i\bar{j}}.
\]

By using these formulas, we have

\[
|z|^{1-\beta} \nabla_{1}^{\text{cone}} g_{k\bar{1}} := |z|^{1-\beta} \frac{\partial g_{k\bar{1}}}{\partial z^{1}},
\]

\[
|z|^{2-2\beta} \nabla_{1}^{\text{cone}} g_{k\bar{1}} := |z|^{2-2\beta} \left[ \frac{\partial g_{k\bar{1}}}{\partial z^{1}} + \frac{1 - \beta}{z^{1}} g_{k\bar{1}} \right],
\]

\[
|z|^{2-2\beta} \nabla_{1}^{\text{cone}} g_{1\bar{1}} := |z|^{2-2\beta} \left[ \frac{\partial g_{1\bar{1}}}{\partial z^{1}} + \frac{1 - \beta}{z^{1}} g_{1\bar{1}} \right].
\]

Third, since \( g_{k\bar{1}}, g_{1\bar{1}} \to 0 \), when points approach the point on the divisor, we have the following estimates:

\[
|z|^{1} [2-2\beta] \frac{1 - \beta}{z^{1}} g_{k\bar{1}} = O(\rho_0^{\alpha-1}), \quad |z|^{2-2\beta} \frac{1 - \beta}{z^{1}} g_{1\bar{1}} = O(\rho_0^{\alpha-1}).
\]

Together with (4.1) and the formulas above, we see that the first three identities in the conclusion (4.2) follow directly.

Finally, the fourth identity in the conclusion is proved by using \( \nabla_{1}^{\text{cone}} (g_{\text{cone}})_{1\bar{1}} = 0, \) (4.1), and

\[
|z|^{1} [3-3\beta] \frac{1 - \beta}{z^{1}} (g_{1\bar{1}} - (g_{\text{cone}})_{1\bar{1}}) = O(\rho_0^{\alpha-1}).
\]

Putting these facts together, we have

\[
|z|^{1} [3-3\beta] \nabla_{1}^{\text{cone}} g_{1\bar{1}} := |z|^{1} [3-3\beta] \left[ \frac{\partial (g_{1\bar{1}} - (g_{\text{cone}})_{1\bar{1}})}{\partial z^{1}} + \frac{1 - \beta}{z^{1}} (g_{1\bar{1}} - (g_{\text{cone}})_{1\bar{1}}) \right] = O(\rho_0^{\alpha-1})
\]

and complete the proof.
We are now ready to determine the Christoffel symbols of cscK cone metrics.

**Corollary 4.6.** Let \( \varphi \) be a \( C^{2,\alpha,\beta} \) solution of (4.4) of a cscK cone metric (or \( J \)-twisted cscK cone metric). Then the Christoffel symbols of the corresponding metric \( g \) satisfy, for any \( 2 \leq i, j, k \leq n \),

\[
\begin{align*}
\Gamma_{jk}^i &\in C^{0,\alpha,\beta}, \quad |z|^\beta - 1 \Gamma_{jk}^i \in C^{0,\alpha,\beta}, \\
|z|^1 \Gamma_{jk}^i &\in C^{0,\alpha,\beta}, \quad |z|^1 \Gamma_{jk}^i = O(\rho_0^{-1}), \\
\Gamma_{j1}^1 &= O(\rho_0^{\alpha-1}), \quad \Gamma_{1k}^1 = O(\rho_0^{\alpha-1}), \\
|z|^2 \Gamma_{11}^1 &= O(\rho_0^{\alpha-1}), \quad |z|^1 \Gamma_{11}^1 = O(\rho_0^{\alpha-1}).
\end{align*}
\]

**Proof.** The Christoffel symbols of \( \omega_\varphi \) are given by the following formula and can be checked one by one by using Corollary 4.5.

\[
\Gamma_{jk}^i (\omega) = \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^k} = \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^k} + g^{i1} \frac{\partial g_{1j}}{\partial z^k}.
\]

Also

\[
|z|^1 \Gamma_{jk}^i (\omega) = |z|^1 \Gamma_{jk}^i \left[ \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^k} + g^{i1} \frac{\partial g_{1j}}{\partial z^k} \right].
\]

Then

\[
\begin{align*}
|z|^1 \Gamma_{jk}^i (\omega) &= |z|^1 \left[ \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^k} + g^{i1} \frac{\partial g_{1j}}{\partial z^k} \right], \\
|z|^1 \Gamma_{jk}^i (\omega) &= |z|^1 \left[ \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^k} + g^{i1} \frac{\partial g_{1j}}{\partial z^k} \right].
\end{align*}
\]

Furthermore,

\[
\Gamma_{j1}^1 (\omega) = \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^1} + g^{i1} \frac{\partial g_{j1}}{\partial z^1}, \quad \Gamma_{1k}^1 (\omega) = \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{1\ell}}{\partial z^k} + g^{i1} \frac{\partial g_{11}}{\partial z^k}.
\]

The next two terms involve \( \frac{\partial g_{j1}}{\partial z^1} \) and \( \frac{\partial g_{j1}}{\partial z^1} \):

\[
\begin{align*}
|z|^1 |^2 \Gamma_{11}^1 (\omega) &= |z|^1 |^2 \left[ \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^1} + g^{i1} \frac{\partial g_{1j}}{\partial z^1} \right], \\
|z|^1 |^1 \Gamma_{11}^1 (\omega) &= |z|^1 |^1 \left[ \sum_{2 \leq l \leq n} g^{i\ell} \frac{\partial g_{j\ell}}{\partial z^1} + g^{i1} \frac{\partial g_{1j}}{\partial z^1} \right].
\end{align*}
\]

Thus the conclusion is proved. \( \square \)
4.3 Higher-Order Estimates: Geometric Polyhomogeneity

We use the same terminology from [61]. We consider asymptotic for cscK cone metrics in the unit ball $B_1 \subseteq B^* := \mathbb{C} \times \mathbb{C}^{n-1}$ centered at the origin,

\begin{equation}
\begin{cases}
\det(\varphi_{i\overline{j}}) = e^K, \\
\Delta \varphi K = S.
\end{cases}
\end{equation}

Assume that

- $S$ is a smooth function;
- $\varphi \in C^{2,\alpha,\beta}$ (then from the second equation $K$ is also in $C^{2,\alpha,\beta}$);
- there is some constant $c > 1$ such that

\begin{equation}
\frac{1}{c} \omega_{\text{cone}} \leq \sqrt{-1} \partial \overline{\partial} \varphi \leq c \omega_{\text{cone}}.
\end{equation}

We also consider the $J$-twisted cscK cone metrics with

$$S = \gamma(\varphi_{i\overline{j}}) := S_{\beta} + (1 - t) \left( \frac{\omega_0^n}{\omega^m_\varphi} - 1 \right).$$

We will also obtain geometric polyhomogeneity of the solution $\varphi$ for this case. The proof is slightly different from the case when $S$ is a smooth function. We will specify necessary changes and put them in remarks for the reader’s convenience.

Given a point

$$Z_0 = (r_0, \theta_0, \xi_0) \in B_1 \setminus \{0\},$$

we consider a neighborhood $\Omega$ of $Z_0$ that is contained in the regular part $M$. Note that $\Omega$ does not contain $0$.

**Definition 4.7 (Polar coordinates).** We use the polar coordinates $(\rho, \theta, \xi)$ in $B_1$, defined by

$$\rho = |z_1|^\beta, \quad z_1 = |z_1| e^{i\theta}, \quad \xi = (z_2, \ldots, z_n).$$

With this terminology in place, our result on the higher-order estimates of the cscK cone metric and $J$-twisted cscK cone metric can be stated as follows.

**Theorem 4.8 (Geometric polyhomogeneity).** Let $\varphi$ be a $C^{2,\alpha,\beta}(B_1)$ solution of (4.4). Then given any $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}$, there exists a constant $C(k_1, k_2, k_3)$ such that

\begin{equation}
\left| (\rho \partial_\rho)^{k_1} (\partial_\theta)^{k_2} (\nabla_\xi)^{k_3} \varphi \right| \leq C(k_1, k_2, k_3)
\end{equation}

for any $\rho \in (0, \frac{1}{2})$ and $|\xi| < \frac{1}{2}$.

Theorem 4.8 is very similar to our previous work [61], where the asymptotic for the Kähler-Einstein cone metrics is proved. The new ingredient is that the equation (4.4) for the cscK cone metric is an elliptic system. We need to handle the equation
of \( K \) at the same time when we apply the method in [61] to estimate \( \varphi \). The argument below can be extended to more general equations with the appropriate \( K \) in (4.4).

**The Scaling Coordinate Systems**

We recall the scaling coordinate systems.

**Definition 4.9 (Lifted holomorphic coordinates).** For points in \( \Omega \), we write
\[
v_1 := z_1^\beta
\]
and define for \( z \in \Omega \),
\[
\text{LH: } z \mapsto v = (v_1, \xi) = (v_1, \ldots, v_n) \in \mathbb{C}^n.
\]
Then the center of the domain \( Z_0 \) becomes
\[
\text{LH}(z_0) = v_0 = (\rho_0 e^{i\beta \theta_0}, \xi_0)
\]
and the image LH(\( \Omega \)) is contained in
\[
B_{c_\beta \rho_0}(v_0) \subset \mathbb{C}^n \setminus \{0\}.
\]
We denote the partial derivatives by
\[
\varphi_{V,i}, \quad \varphi_{V,j}, \quad \varphi_{V,i_j}, \quad \text{etc.}
\]
and tangent derivatives by
\[
\nabla_{V,k} \in \{\partial_{v_i}, \partial_{\xi_i} \forall i = 2, \ldots, n\}.
\]

**Definition 4.10 (Rescaled lifted holomorphic coordinates).** Now we define the rescaled lifted holomorphic coordinates \( \bar{v} \) by
\[
\text{SLH: } v \mapsto \bar{v} = \frac{v - v_0}{\rho_0}.
\]
Then after scaling by SLH, the domain \( \Omega \) is contained in
\[
B_{c_\beta} := B_{c_\beta}(0) \subset \mathbb{C}^n
\]
for a universal constant \( c_\beta \), which is a ball in the \( \bar{v} \)-coordinates.

We now denote the partial derivatives by
\[
\varphi_{\bar{v},i}, \quad \varphi_{\bar{v},j}, \quad \varphi_{\bar{v},i_j}, \quad \text{etc.}
\]
and the tangent derivatives, as before, by
\[
\nabla_{\bar{v},k}.
\]

The rescaled lifted holomorphic coordinates are the right coordinates to carry on the interior Schauder estimates and to insure that the constants appearing from the estimates are independent of the position of the point \( Z_0 \).

The Hölder functions are defined in the usual sense with respect to the distance given by the \( v \)-coordinates. The norm
\[
\|\cdot\|_{C^{k,\alpha}_v(B_{c_\beta \rho_0}(v_0))}
\]
is the usual Hölder norm. Here we use the lower subscript to emphasize the use of the \( V \)-coordinates around \( v_0 \). A similar convention holds for \( C^k_V \).

We always require that the constant \( C(k) \) increases on \( k \) and may be different from line to line.

**Fundamental Estimates: Startup**

From lemma 2.3 in [61], the following estimates hold by coordinate transformation, since \( \varphi \) and \( K \) are both in \( C^{2,\alpha,\beta}(B_1) \).

**Lemma 4.11.** Suppose \( u \in C^{2,\alpha,\beta}(B_1) \). There is a constant \( C > 0 \) independent of \( Z_0 \) and \( u \) such that

\[
\|u \|_{C^{0,\alpha}(B_{C^2 \rho^0(v_0)})} + \|u_{V,i} \|_{C^{0,\alpha}(B_{C^2 \rho^0(v_0)})} + \|u_{V,j} \|_{C^{0,\alpha}(B_{C^2 \rho^0(v_0)})} \leq C \|u\|_{C^{2,\alpha,\beta}(B_1)}.
\]

All tangent derivatives of \( \varphi \) and \( K \) are bounded in \( C^{2,\alpha,\beta} \) space, because along the tangent direction, no singularities occur.

**Lemma 4.12.** For \( l = 0, 1, \ldots \), there exists a constant \( C(l) \) such that

\[
\|\nabla^l \varphi \|_{C^0(V,B_{C^2 \rho^0(v_0)})} \leq C(l),
\]

\[
\|\nabla^l K \|_{C^0(V,B_{C^2 \rho^0(v_0)})} \leq C(l).
\]

**Proof.** When \( l = 0 \), this lemma follows from Lemma 4.11. For \( l = 1 \), we get the estimates along the tangent direction \( \xi \) by differentiating (4.4) and applying the Schauder estimate. The higher-order estimates in the lemma are then obtained by bootstrapping.

**Rescaled Estimates on \( g \) and \( K \)**

We rescale the Kähler potential,

\[
\tilde{\varphi}(\tilde{v}) = \rho_0^{-2} \varphi(v_0 + \tilde{v}\rho_0).
\]

Then the rescaled metric

\[
g_{\tilde{V},\tilde{J}} := \tilde{\varphi} V_{\tilde{V},\tilde{J}} \quad \text{for } i, j = 1, \ldots, n
\]

is equivalent to the standard euclidean metric, and also the \( C^0_V \) norm is bounded,

\[
\|g_{\tilde{V},\tilde{J}}\|_{C^0_V(B_{C\rho})} \leq C.
\]

We will prove the higher-order interior estimates of both \( g_{\tilde{V},\tilde{J}} \) and \( K \). We denote the strictly increasing sequence \( \eta_k \) for \( k = 0, 1, \ldots \) such that \( \lim_{k \to \infty} \eta_k = 0.25c_\beta \). This is a technical arrangement to shrink the radius of balls in each interior estimate.
Recall the first equation in (4.4), i.e.,
\[
\text{det}(\varphi^I) = \frac{e^K}{|\xi|^12.3\beta}.
\]
Rewriting it in the rescaled lifted holomorphic coordinates \(\tilde{\nu}\), we have
\[
(4.11) \log \text{det}(\tilde{\varphi}^I, \tilde{J}) = K.
\]
Taking \(\partial_{\tilde{\nu}_i} \partial_{\tilde{\nu}_j}\), we get
\[
\Delta_g g^I_j - g^k \tilde{m} g^n \frac{\partial g^I_k}{\partial \tilde{\nu}_n} \frac{\partial g^L_j}{\partial \tilde{\nu}_l} = K^I_j.
\]
where \(\Delta_g = g^i_j \frac{\partial^2}{\partial \tilde{\nu}_i \partial \tilde{\nu}_j}\).

**PROPOSITION 4.13.** There are constants \(C(k)\) such that for \(k = 0, 1, \ldots\),
\[
\|g^I_j\|_{C_{\tilde{\nu}}^{1,\alpha'}(B_{c\nu,\beta/4})} \leq C(k), \quad \|K\|_{C_{\tilde{\nu}}^{4,\alpha}(B_{c\nu,\beta/4})} \leq C(k).
\]

**PROOF.** From Lemma 4.11, the RHS of (4.12) is bounded by
\[
\|K^I_j\|_{C_{\tilde{\nu}}^{4,\alpha}(B_{c\nu,\beta})} \leq C \|K\|_{C_{\tilde{\nu}}^{2,\alpha}(B_{\nu})} \cdot c^{2+\alpha}.
\]
Then using the same technique used in [61] (see the appendix therein) to obtain the interior estimate of (4.12), we obtain that there exists some \(\alpha' > 0\) such that
\[
(4.13) \quad \|g^I_j\|_{C_{\tilde{\nu}}^{1,\alpha'}(B_{c\nu,\beta/4})} \leq C(1)
\]
for some constant \(C(1)\) depending on \(\eta_1, \alpha', c\) in (4.5) and the constant in (4.10).

Applying the classical Schauder estimate to (4.12), we get that there is a constant \(C(2)\) independent of \(Z_0\) such that
\[
(4.14) \quad \|g^I_j\|_{C_{\tilde{\nu}}^{2,\alpha}(B_{c\nu,\beta/2})} \leq C(2).
\]
Under the rescaled lifted holomorphic coordinates \(\tilde{\nu}\), the second equation in (4.4) becomes
\[
(4.15) \quad S = \Delta_g K = g^I_j K^I_j.
\]
Since \(S\) is a smooth function,
\[
(4.16) \quad \|S\|_{C_{\tilde{\nu}}^{1,\alpha}(B_{c\nu,\beta})} \leq C(k) \quad \forall k \geq 0.
\]
Then applying the Schauder estimate to (4.15) with (4.14), we have the fourth-order estimates of \(K\),
\[
\|K\|_{C_{\tilde{\nu}}^{4,\alpha}(B_{c\nu,\beta/4})} \leq C(3).
\]
Returning to (4.12), the third- and the fourth-order estimates of \(g^I_j\) are also obtained by the bootstrapping method:
\[
(4.17) \quad \|g^I_j\|_{C_{\tilde{\nu}}^{4,\alpha}(B_{c\nu,\beta/4})} \leq C(4).
\]
Repeating the argument above, we arrive at the estimates for $g_{i\bar{j}}$ and $K$. □

Remark 4.14 (The $J$-twisted case $S = \gamma(\varphi_{i\bar{j}})$). We only need to change (4.16) to be
\[ \| S \| C^2_{\bar{\psi}}(B_{r_0}) \leq C(2), \]
which follows from (4.14).

Refined Rescaled Tangent Estimates on $g$ and $K$

We rewrite Lemma 4.12 in the rescaled lifted holomorphic coordinates. The tangent estimates on metric $g_{i\bar{j}}$ are rescaled as
\[ (4.18) \quad \| \nabla_{\bar{\psi}, \xi}^l g_{i\bar{j}} \| C^0_{\bar{\psi}}(B_{r_0}) \leq C(l) \cdot \rho_0^l \quad \forall l = 0, 1, \ldots. \]
We will further improve the tangent estimates from $C^0_{\bar{\psi}}$ to $C^k_{\bar{\psi}}$ in shrinking balls.

We take tangent derivatives $\nabla_{\bar{\psi}, \xi}$ of (4.12) and (4.15) with respect to the rescaled coordinates
\[ (4.19) \quad \Delta_g (\nabla_{\bar{\psi}, \xi} g_{i\bar{j}}) = (\nabla_{\bar{\psi}, \xi} K) \bar{\psi}, \bar{j} \]
\[ = G_1(g, \nabla_{\bar{\psi}, \xi} g, \nabla^2_{\bar{\psi}, \xi} g) \# \nabla_{\bar{\psi}, \xi} g + G_2(g, \nabla_{\bar{\psi}, \xi} g) \# \nabla_{\bar{\psi}, \xi} (\nabla_{\bar{\psi}, \xi} g), \]
\[ (4.20) \quad g^{i\bar{j}} (\nabla_{\bar{\psi}, \xi} K) \bar{\psi}, \bar{j} = \nabla_{\bar{\psi}, \xi} S - \nabla_{\bar{\psi}, \xi} g^i \bar{j} K_{\bar{\psi}, \bar{j}}. \]
In (4.19), $G_1, G_2$ are smooth functions of their arguments and $a \# b$ is a sum of products of $a$ and $b$.

In order to obtain the interior higher-order estimates, it remains to examine the coefficients and the nonhomogeneous terms.

**Proposition 4.15.** For any $k, l = 0, 1, \ldots$, there are constants $C(k, l)$ such that
\[ (4.21) \quad \| (\nabla_{\bar{\psi}, \xi}^l g_{i\bar{j}}) C^0_{\bar{\psi}}(B_{3r_0/4}) \| \leq C(k, l) \rho_0^l, \]
\[ (4.22) \quad \| (\nabla_{\bar{\psi}, \xi}^l K) C^0_{\bar{\psi}}(B_{3r_0/4}) \| \leq C(k, l) \rho_0^l. \]

**Proof.** The proof is by induction. We first prove the lemma with $l = 1$. Applying the Schauder estimate to (4.20), we derive the tangent estimates on $\nabla_{\bar{\psi}, \xi} K$, i.e.,
\[ (4.23) \quad \| \nabla_{\bar{\psi}, \xi} K \| C^2_{\bar{\psi}}(B_{3r_0/4}) \| \leq C \left( \| \nabla_{\bar{\psi}, \xi} S \| C^0_{\bar{\psi}}(B_{r_0}) + \| \nabla_{\bar{\psi}, \xi} S \| C^0_{\bar{\psi}}(B_{r_0}) \right). \]
Since $S$ is smooth,
\[ (4.24) \quad \| \nabla_{\bar{\psi}, \xi} S \| C^0_{\bar{\psi}}(B_{r_0}) \| \leq C r_0. \]
From Lemma 4.11
\[ (4.25) \quad \| \nabla_{\bar{\psi}, \xi} K \| C^0_{\bar{\psi}}(B_{r_0}) \| \leq C r_0. \]
Putting (4.24) and (4.25) into the RHS of (4.23), we obtain the second inequality (4.22) with \( l \) in the conclusion,

\[
\| \nabla \bar{\varphi} \|_{C^{k+2,0}(B_{3c/4-\eta_1})} \leq C(k, 1)\rho_0.
\]

We apply the Schauder estimate to (4.19) with coefficients bounded from Proposition 4.13 to obtain

\[
\| \nabla \bar{\varphi} \|_{C^{k+2,0}(B_{3c/4-\eta_1})} \leq C \left( \| \nabla \bar{\varphi} \|_{C^0(B_{c\rho})} + \| \nabla \bar{\varphi} \|_{C^{k+2,0}(B_{c\rho})} \right).
\]

The RHS of (4.27) is bounded because of (4.18) and (4.23), so we prove the first inequality (4.21) with \( l = 1 \), i.e.,

\[
\| g_{ij}^{\bar{\varphi}} \|_{C^{k+2,0}(B_{3c/4-\eta_1})} \leq C(k, 1)\rho_0.
\]

If (4.21) holds for \( l = l_0 \), we take \( l_0 \) times the tangent derivatives on both the \( g \)-equation (4.19) and the \( K \)-equation (4.20), and the argument is similar to that above. We leave the proof to the interested reader.

**Remark 4.16.** We notice that (4.24) follows from (4.18).  

**Final Step of the Proof of Higher-Order Estimates (Theorem 4.8)**

Now we are ready for the final step to proving the higher-order interior estimates for the cscK cone potential \( \varphi \). We will carry on the proof by induction on \( k_3 \) in the estimate (4.6).

Recall the definition of \( \bar{\varphi} \), i.e., \( \bar{\varphi}(\bar{v}) = \rho_0^{-2} \varphi(v_0 + \bar{v}\rho_0) \). Since \( \varphi \in C^{2,0,\beta} \) (Lemma 4.11), we see that

\[
\| \bar{\varphi}(\bar{v}) \|_{C^0(B_{c\rho})} \leq C\rho_0^{-2}
\]

and the \( C^0 \) norm of the first-order derivative of \( \bar{\varphi} \) becomes

\[
\| \nabla \bar{\varphi}(\bar{v}) \|_{C^0(B_{c\rho})} \leq C\rho_0^{-1}.
\]

**The case** \( k_3 = 0 \). We take the derivative \( \nabla \bar{v} \) of the rescaled equation (4.11) of \( \bar{\varphi} \) to obtain

\[
\Delta g_{ij} \bar{\varphi} = \nabla \bar{\varphi} \cdot K.
\]

The coefficients and the right-hand side are estimated in Proposition 4.13 as

\[
\| g_{ij} \|_{C^{k,0}(B_{3c/4})} \leq C(k), \quad \| K \|_{C^{k,0}(B_{3c/4})} \leq C(k).
\]

The Schauder estimate applied to (4.31) gives for any \( k \geq 0 \),

\[
\| \nabla \bar{v} \|_{C^{k+2,0}(B_{c\rho/2})} \leq C \left( \| \nabla \bar{v} \|_{C^0(B_{c\rho/2})} + \| \nabla \bar{\varphi} \|_{C^{k,0}(B_{c\rho/2})} \right)
\]

\[
\leq C\rho_0^{-1}.
\]
This implies that
\[ \| \tilde{\varphi} \|_{C_{\tilde{r}}^{k+3,\alpha}(B_{c/4})} \leq C \rho_0^{-1}. \]
Rescaled back to \( \varphi \), this inequality becomes
\[ (4.33) \quad \| \varphi \|_{C_{\tilde{r}}^{k}(B_{c/4})} \leq C \quad \forall k \geq 3. \]
Thus Theorem 4.8 is proved with \( k_3 = 0 \), combined with the lower-order estimates in Lemma 4.11.

The case \( k_3 = 1 \). Rewrite (4.32) in \( \varphi \),
\[ (4.34) \quad \| \nabla \tilde{V} \tilde{\varphi} \|_{C_{\tilde{r}}^{k+2,\alpha}(B_{c/4})} \leq C \quad \forall k \geq 0. \]
In particular,
\[ \| \nabla \tilde{V} \tilde{\varphi} \|_{C_{\tilde{r}}^{k+2,\alpha}(B_{c/4})} \leq C \quad \forall k \geq 0. \]
The lower-order estimates, \( \| \nabla \tilde{V} \tilde{\varphi} \|_{C_{\tilde{r}}^{0,\alpha}(B_{c/4})} \) and \( \| \nabla \tilde{V} \tilde{\varphi} \|_{C_{\tilde{r}}^{1,\alpha}(B_{c/4})} \), follow from Lemma 4.11. So we have proved Theorem 4.8 with \( k_3 = 1 \).

The case \( k_3 \geq 2 \). From the boundedness of the tangential derivatives of \( \varphi \) and the definition of \( \tilde{\varphi} \), we get
\[ (4.35) \quad \| (\nabla_{\tilde{V}, \tilde{g}})_{k_3} \tilde{\varphi} \|_{C_{\tilde{r}}^{l,\alpha}(B_{c/4})} \leq C (l) \rho_0^{k_3-2} \quad \forall l \geq 0. \]
The rest of this section is devoted to the proof of (4.35), which is an induction on \( k_3 \). For \( k_3 = 2 \), we take one more \( \nabla_{\tilde{V}, \tilde{g}} \) of the equation from (4.31),
\[ \Delta_{\tilde{g}} (\nabla_{\tilde{V}, \tilde{g}} \tilde{\varphi}) = \nabla_{\tilde{V}, \tilde{g}} K \]
to get
\[ (4.36) \quad \Delta_{\tilde{g}} (\nabla_{\tilde{V}, \tilde{g}}^2 \tilde{\varphi}) = (\nabla_{\tilde{V}, \tilde{g}} \tilde{\varphi})_{V, i, j} + \nabla_{\tilde{V}, \tilde{g}}^2 \tilde{\varphi} + \nabla_{\tilde{V}, \tilde{g}}^2 \tilde{\varphi} K. \]
In order to apply the Schauder estimate, we check the following:
- Proposition 4.13 tells us that the coefficient \( g \) and the term \( K \) are both in \( C_{\tilde{r}}^{l,\alpha}(B_{3c/4}) \) for \( l = 1, 2, \ldots \).
- (4.34) gives us that the \( C_{\tilde{r}}^{0}(B_{c/4}) \) norm of \( \nabla_{\tilde{V}, \tilde{g}}^2 \tilde{\varphi} \) is bounded by a constant independent of \( \rho_0 \).
- The nonhomogeneous term is (by switching the order of derivatives)
\[ (\nabla_{\tilde{V}, \tilde{g}} \tilde{\varphi})_{V, i, j} + \nabla_{\tilde{V}, \tilde{g}}^2 \tilde{\varphi} + \nabla_{\tilde{V}, \tilde{g}}^2 \tilde{\varphi} K = \nabla_{\tilde{V}, \tilde{g}} g_{ij} + \nabla_{\tilde{V}, \tilde{g}}^2 g_{ij} + \nabla_{\tilde{V}, \tilde{g}}^2 K, \]
where the first term is estimated by Proposition 4.15 for \( l = 1, 2, \ldots \):
\[ (4.37) \quad \| (\nabla_{\tilde{V}, \tilde{g}} g_{ij} + \nabla_{\tilde{V}, \tilde{g}}^2 g_{ij} \|_{C_{\tilde{r}}^{l,\alpha}(B_{c/4})} \leq C \rho_0^2 \leq C \]
and the second term is controlled by Proposition 4.13 (applied to $\nabla^2 K$):

\[(4.38) \quad \| \nabla^2_{\omega, K} K \|_{C^1(B_{1/2})} \leq C \rho_0^2 \leq C.\]

Applying the Schauder estimate to (4.36), we conclude the proof of (4.35) for $k_3 = 2$. We repeat this argument to get (4.35) for any $k_3$ holds. Hence Theorem 4.8 is completely proved.

5 Linear Theory of Lichnerowicz Operators

We now denote the cscK cone metric with $C^{2,\alpha,\beta}$ cscK potential $\varphi$ by

$$\omega := \omega_{\text{cscK}} = \omega_{D} + i \partial \bar{\partial} \varphi.$$ 

Recall that $\theta$ is a smooth closed $(1, 1)$-form and the cscK cone equations (3.5) and (3.7) in Section 3.1, with $P \in C^{2,\alpha,\beta}$ satisfying

\[
\begin{aligned}
\Delta_{\omega} P &= \text{Tr}_{\omega} \theta - S_{\beta}, \\
\omega^{\alpha} \omega_\beta &= e^{P + h_0} |z|^{2-2\beta}.
\end{aligned}
\]

Let $Q$ be the variation of $P$, and $u$ the variation of $\varphi$. Recall that $T = -i \partial \bar{\partial} P + \theta$. The linearized equation of the cscK cone equation at the cscK cone metric $\omega$ on functions $u$ and $Q$ is

\[
\begin{aligned}
\Delta_{\omega} Q &= -u^I \bar{J} T_I \bar{J}, \\
\Delta_{\omega} u &= Q.
\end{aligned}
\]

or, as a fourth-order Lichnerowicz operator,

\[
\Delta_{\omega}(u) = \Delta_{\omega}^2 u = u^{I} \bar{J} T_{I} \bar{J}.
\]

We introduce the following Hölder spaces $C^{4,\alpha,\beta}_w(\omega)$ to be our solution space for (5.1).

DEFINITION 5.1 (Hölder spaces $C^{4,\alpha,\beta}_w(\omega)$). We define a “weak” fourth-order Hölder space with respect to a given Kähler cone metric $\omega$,

\[
C^{4,\alpha,\beta}_w(\omega) = \{ u \in C^{2,\alpha,\beta} : \Delta_{\omega} u \in C^{2,\alpha,\beta} \}.
\]

We also use an appropriate normalization condition later. In this definition, not all fourth-order derivatives of $u$ are $C^{0,\alpha,\beta}$. Apparently, if the reference Kähler cone $\omega$ is classically $C^{2,\alpha}$ outside the divisor, on the regular part $M$, then the function in this space is $C^{4,\alpha}$ in the usual sense, according to the interior Schauder estimate. The advantage of Definition 5.1 is that we don’t need geometric conditions of the background metric $\omega$.

We defined a different space $C^{4,\alpha,\beta}_w(\omega)$ in section 2 in [50] and section 2 in [45]. The function in such space $C^{4,\alpha,\beta}_w(\omega)$ is required to have all its fourth-order derivatives in $C^{0,\alpha,\beta}$ space. The idea is first to define a local fourth-order Hölder space with respect to the flat cone metric $\omega_{\beta}$ near the cone points. Then we glue
the local fourth-order Hölder spaces together to the global spaces \( C^{4,\alpha,\beta}(\omega) \) with respect to a background Kähler cone metric \( \omega \) as proposition 2.2 in [45]. The background metric \( \omega \) is required to have nice geometric properties, and with such a nice \( \omega \), we can prove that \( C^{4,\alpha,\beta}(\omega) \) coincides with \( C^{4,\alpha,\beta}(\omega) \). For example, the background metric \( \omega \) can be chosen to be the model cone metric \( \omega_D \) as in proposition 5.5 in [50], when the cone angle and the Hölder exponent satisfy the half-angle condition

\[
0 < \beta < \frac{1}{2}, \quad \alpha\beta < 1 - 2\beta.
\]

When \( \beta = 1 \), it is simply the smooth case. As a result, the unsolved part is the case when \( \frac{1}{2} \leq \beta < 1 \). In this section, we focus on this case and prove the Fredholm alternative theorem in \( C^{4,\alpha,\beta}(\omega) \). The method also works for the half-angle case and the smooth case.

**Theorem 5.2 (Linear theory; \( 0 < \beta \leq 1 \)).** Let \( X \) be a compact Kähler manifold, \( D \subset X \) a smooth divisor, and \( \omega \) a cscK cone metric with \( C^{2,\alpha,\beta} \) potential such that the cone angle \( 2\pi\beta \) and the Hölder exponent \( \alpha \) satisfy

\[
0 < \beta \leq 1, \quad \alpha\beta < 1 - \beta.
\]

Assume that \( f \in C^{0,\alpha,\beta} \) with normalization condition \( \int_X f \omega^n = 0 \). Then one of the following holds:

- The Lichnerowicz equation \( \mathbb{L}ic_{\omega}(u) = f \) has a unique \( C^{4,\alpha,\beta}(\omega) \) solution, or
- the kernel of \( \mathbb{L}ic_{\omega}(u) \) generates a holomorphic vector field tangent to \( D \).

Actually, we will solve the general linear problem.

**Theorem 5.3.** Let \( X \) be a compact Kähler manifold, \( D \subset X \) a smooth divisor, and \( \omega \) a Kähler cone metric with \( C^{2,\alpha,\beta} \) potential such that the cone angle \( 2\pi\beta \) and the Hölder exponent \( \alpha \) satisfy

\[
0 < \beta \leq 1, \quad \alpha\beta < 1 - \beta.
\]

We are given a \((1,1)\)-form \( T \in C^{0,\alpha,\beta} \) with \( \nabla T = 0 \) and a function \( f \in C^{0,\alpha,\beta} \) with normalization condition \( \int_X f \omega^n = 0 \). Then one of the following holds:

- The equation \( \mathbb{L}ic_{\omega}(u) = \Delta^2 \omega u + u^{ij}T_{ij} = f \) has a unique \( C^{4,\alpha,\beta}(\omega) \) solution, or
- the kernel space of \( \mathbb{L}ic_{\omega}(u) \) is finite dimensional.

With the half-angle restriction [54], the Fredholm alternative theorem above is proved in [45] in the space \( C^{4,\alpha,\beta}(\omega) \) with better regularity. When the angle is larger than half, we use the weaker space \( C^{4,\alpha,\beta}(\omega) \). The new observation is that we can prove appropriate a priori estimates for partial fourth-order derivatives but
not the full control of all fourth-order derivatives. In Section 5.3, we prove an asymptotic of functions in $C^4_{w,\omega}(\omega)$ for further use.

We first consider the $K$-bi-Laplacian operator

$$(\triangle_{\omega} - K)\triangle_{\omega} u$$

and prove the existence and uniqueness of weak solutions in the Sobolev space $H^{2,\beta}_{w,0}(\omega)$. The constant $K$ is determined in Theorem 5.12. We next improve the regularity of the weak solution to the Hölder spaces $C^4_{w,\omega}(\omega)$ using the second Schauder estimate. These two steps will be done in Section 5.1.

Then we define a continuity path

$$L^K_{t} u := (\triangle_{\omega} - K)\triangle_{\omega} u + tu^i\partial_i T_t(\omega)$$

connecting the $K$-bi-Laplacian operator ($t = 0$) and the $K$-Lichnerowicz operator ($t = 1$)

$$\mathbb{Li}_{\omega}(u) - K\triangle_{\omega} u,$$

and prove uniform a priori $C^4_{w,\omega}(\omega)$ estimates along the path in Section 5.2.

Finally, the theorem follows from the functional analysis theorems for the continuity method.

The difficulties come from the Hölder space $C^{4,\alpha,\beta}_{w}(\omega)$, which is the most natural one for the solution to stay in, but the function $u$ in this space has no a priori boundedness on the pure second derivatives, $\partial_1^2 u$. We overcome this problem by using $\int_M |\partial \partial u|_{\omega}^2 \omega^n$ to define a bilinear form, instead of $\int_M |\partial \nabla_1 u|_{\omega}^2 \omega^n + \int_M |\partial \partial u|_{\omega}^2 \omega^n$. Fortunately, it also leads to weak Sobolev spaces $H^{2,\beta}_{w,0}(\omega)$ and global $L^2$-estimates. Then the second Schauder estimate and compactness of the second Hölder space $C^{2,\alpha,\beta}$ are applied to obtain the $C^{4,\alpha,\beta}_{w}(\omega)$ estimate.

We close this section by showing that our Fredholm alternative for the Lichnerowicz operator (Theorem 5.2) is reduced to Theorem 5.12.

PROOF OF THEOREM 5.2 AND THEOREM 5.3 Due to Theorem 5.12, the $K$-Lichnerowicz operator

$$\mathbb{Li}_{\omega}^K(u) = \mathbb{Li}_{\omega}(u) - K\triangle_{\omega} u$$

is invertible and the inverse map

$$\left(\mathbb{Li}_{\omega}^K\right)^{-1} : C^{0,\alpha,\beta}_{w}(\omega) \rightarrow C^{4,\alpha,\beta}_{w}(\omega)$$

is compact. Now, we consider the Lichnerowicz equation, i.e.,

$$\mathbb{Li}_{\omega}(u) = \mathbb{Li}_{\omega}^K(u) + K\triangle_{\omega} u = f.$$

Actually, this is equivalent to, after taking $\left(\mathbb{Li}_{\omega}^K\right)^{-1}$,

$$u + K\left(\mathbb{Li}_{\omega}^K\right)^{-1} \triangle_{\omega} u = \left(\mathbb{Li}_{\omega}^K\right)^{-1} f.$$
According to Theorem 5.12, the mapping
\[ T : = - K ( \mathbb{L} \cdot \text{ic}^{\text{K}} )^{-1} \Delta_\omega : C^4_w,^{\alpha, \beta} (\omega) \to C^4_w,^{\alpha, \beta} (\omega) \]
is compact, since
\[ \Delta_\omega : C^4_w,^{\alpha, \beta} (\omega) \to C^{2,\alpha, \beta} (\omega) \]
and the identity map
\[ C^{2,\alpha, \beta} (\omega) \to C^{0,\alpha, \beta} (\omega) \]
is compact. Thus Theorem 5.3 follows from the Fredholm alternative in functional analysis theory.

The second statement in Theorem 5.2 follows from the integral identity (Lemma 6.5) proved in Section 6. □

5.1 Bi-Laplacian Equations: Existence and Regularity

We will solve the following $K$-bi-Laplacian equation in $C^4_w,^{\alpha, \beta} (\omega)$,

(5.5) \[ (\Delta_\omega - K) \Delta_\omega u = f. \]

We recall the Poincaré inequality regarding $\omega$. We denote the Poincaré constant by $C_P$.

**Lemma 5.4 (Poincaré inequality).** There is a constant $C_P$ such that for any $u \in H_0^{1,\beta} = \{ u \in H^{1,\beta} \mid \int_M u \omega^n = 0 \}$,

\[ \| u \|_{L^2(\omega)} \leq C_P \| \partial u \|_{L^2(\omega)}. \]

The positive constant $K$ will be determined later in Proposition 5.10.

**Theorem 5.5 ($K$-bi-Laplacian equation).** Assume that $\omega = \omega_D + i \partial \bar{\partial} \varphi$ is a Kähler cone metric with

\[ \varphi \in C^{2,\alpha, \beta}, \quad f \in C^{0,\alpha, \beta}, \quad 0 < \beta \leq 1, \]

and the Hölder exponent satisfies $\alpha \beta < 1 - \beta$. Suppose that

\[ K > C_P + 1. \]

Then there exists a unique $C^4_w,^{\alpha, \beta} (\omega)$ solution to equation (5.5).

**Proof.** The proof is divided into the existence part (Proposition 5.10) and the regularity part (Proposition 5.11). □

K-bi-Laplacian Equations: Sobolev Spaces and Weak Solutions

We first need to define a proper notion of weak solution for equation (5.5). Given a Kähler cone metric, its volume element is an $L^p$ function (for some $p \geq 1$) with respect to a smooth metric and gives rise to a measure $\omega^n$ on $M$. We will use the following Sobolev spaces and their embedding theorems with respect to $\omega$. 

DEFINITION 5.6 (Sobolev spaces $W^{1,p,\beta}(\omega)$). For a Kähler cone metric $\omega$, the Sobolev spaces $W^{1,p,\beta}(\omega)$ for $p \geq 1$ are defined with respect to $\omega$. The $W^{1,p,\beta}(\omega)$ norm is

$$
\|u\|_{W^{1,p,\beta}(\omega)} = \left( \int_M |u|^p + |\partial u|^p_{\omega^\beta} \right)^{1/p}.
$$

DEFINITION 5.7. We define the Sobolev spaces $H^{1,\beta} := W^{1,2,\beta}(\omega)$ and $H_0^{1,\beta} = \{u \in H^{1,\beta} | \int_M u \omega^n = 0 \}$. The Sobolev norm remains the same.

LEMMA 5.8 (Sobolev inequality). Assume that $u \in W^{1,p,\beta}(\omega)$. If $p < 2n$, then there exists a constant $C$ such that

$$
\|u\|_{L^q(\omega)} \leq C \|u\|_{W^{1,p,\beta}(\omega)} \quad \text{for any } q \leq \frac{2np}{2n - p}.
$$

DEFINITION 5.9 (Sobolev spaces $H^{2,\beta}_w(\omega)$). We define the second Sobolev space $H^{2,\beta}_w$ with seminorm $\partial_a \partial_{\overline{B}} u = \frac{\partial^2 u}{\partial z^a \partial \overline{z}^B} i dz^a \wedge d\overline{z}^B$ as

$$
[u]_{H^{2,\beta}_w(\omega)} = \sum_{1 \leq \alpha, \beta \leq n} \|\partial_a \partial_{\overline{B}} u\|_{L^2(\omega)}
$$

and the norm by

$$
\|u\|_{H^{2,\beta}_w(\omega)} = \|u\|_{H^{1,\rho}(\omega)} + [u]_{H^{2,\beta}_w(\omega)}.
$$

We define the bilinear form on $H^{2,\beta}_{w,0}(\omega)$ by

$$
\mathcal{B}^K(u, \eta) := \int_M [(i \partial \overline{\partial} u, i \partial \overline{\partial} \eta)_{\omega} + K(\partial u, \partial \eta)_{\omega}] \omega^n
$$

for all $u, \eta \in H^{2,\beta}_{w,0}(\omega)$, and define $u$ to be the $H^{2,\beta}_{w,0}(\omega)$-weak solution of the $K$-bi-Laplacian equation (5.5) if it satisfies the following identity for all $\eta \in H^{2,\beta}_{w,0}(\omega)$:

$$
\mathcal{B}^K(u, \eta) = \int_M f \eta \omega^n.
$$

PROPOSITION 5.10 ($H^{2,\beta}_{w,0}(\omega)$-weak solution). Assume that $\omega$ is a Kähler cone metric. Suppose that $K > C_K + 1$ and $f$ is in the dual space $(H^{2,\beta}_{w,0}(\omega))^*$. Then the $K$-bi-Laplacian equation (5.5) has a unique weak solution $u \in H^{2,\beta}_{w,0}(\omega)$.

PROOF. By using the Cauchy-Schwarz inequality, the boundedness of $\mathcal{B}^K$ follows from its definition:

$$
\mathcal{B}^K(u, \eta) \leq \|u\|_{H^{2,\beta}_{w,0}(\omega)} \|\eta\|_{H^{2,\rho}_w(\omega)} + K \|u\|_{H^{1,\rho}(\omega)} \|\eta\|_{H^{1,\beta}(\omega)}.
$$
Then we prove the coercivity. The definition of the bilinear form $B^K$ implies that

$$
\|u\|^2_{H^{2,\beta}_w(\omega)} = \int_M \left( |\overline{\partial} \partial u|^2 + |\partial u|^2 + |u|^2 \right) \omega^n
$$

$$
= B^K(u, u) + \int_M [(1 - K)|\partial u|^2 + |u|^2] \omega^n.
$$

With the Poincaré inequality (Lemma 5.4), the RHS above is controlled by

$$
\leq B^K(u, u) + \int_M [(1 - K + C_P)|\partial u|^2] \omega^n.
$$

Thus choosing $K > C_P + 1$, we have

$$
\|u\|^2_{H^{2,\beta}_w(\omega)} \leq B^K(u, u).
$$

Thus the Lax-Milgram theorem applies and there is a unique weak solution $u \in H^{2,\beta}_{w,0}$ to the equation (5.5). □

**K-bi-Laplacian Equations: Regularity**

We now improve the regularity of the $H^{2,\beta}_{w,0}(\omega)$-weak solution $u$.

**Proposition 5.11 (Schauder estimate).** Assume that $\omega = \omega_D + i \overline{\partial} \phi$ is a Kähler cone metric with

$$
\varphi \in C^{2,\alpha,\beta}, \quad f \in C^{0,\alpha,\beta}, \quad 0 < \beta \leq 1,
$$

and the Hölder exponent satisfies $\alpha \beta < 1 - \beta$. Then the weak solution $u \in H^{2,\beta}_{w,0}$ to the equation (5.5) is actually $C^{4,\alpha,\beta}(\omega)$.

**Proof.** Applied to the equation $(\Delta_{\omega} - K)v = f$, the second-order linear elliptic theory [34] tells us that this equation has a unique $C^{2,\alpha,\beta}$ solution $v$, and also

(5.6) \[
\int_M v(\Delta_{\omega} - K) \eta \omega^n = \int_M f \eta \omega^n
\]

with $\eta$ a smooth function with vanishing average.

We claim that this unique solution $v$ has to be $\Delta_{\omega} u$ in Proposition 5.10. We then prove this claim. Choosing $\eta$ as above, we have, from the definition of the $H^{2,\beta}_{w,0}$ weak solution,

$$
\int_M [(i \overline{\partial} u, i \overline{\partial} \eta)_{\omega} + K(\partial u, \partial \eta)_{\omega}] \omega^n = \int_M f \eta \omega^n.
$$

Since $\eta$ is smooth, we are able to use integration by parts on the left-hand side

$$
\int_M (i \overline{\partial} u, i \overline{\partial} \eta)_{\omega} \omega^n = \int_M u_i \eta_{j \overline{\partial}} \omega^n = \int_M \Delta_{\omega} u \Delta_{\omega} \eta \omega^n,
$$

$$
\int_M (\partial u, \partial \eta)_{\omega} \omega^n = \int_M \Delta_{\omega} u \eta \omega^n.
$$
and then obtain
\[ \int_M \triangle_{\omega} u (\triangle_{\omega} - K) \eta \omega^n = \int_M f \eta \omega^n. \]
Comparing this with (5.6) and using the uniqueness of \( v \), we see that \( \triangle_{\omega} u \) is the same as \( v \), and thus in \( C^{2,\alpha,\beta} \). Applying linear theory again, we obtain that \( u \in C^{2,\alpha,\beta} \).

5.2 Fredholm Alternative for Lichnerowicz Operators

In order to prove the Fredholm alternative Theorem 5.2, it suffices to consider the \( K \)-Lichnerowicz equation
\[ \mathbb{L}ic^K_{\omega}(u) = \mathbb{L}ic_\omega(u) - K \triangle_{\omega} u = f. \]

**Theorem 5.12 (K-Lichnerowicz equation).** Assume that \( \omega = \omega_D + i \partial \bar{\partial} \varphi \) is a Kähler cone metric with
\[ \varphi \in C^{2,\alpha,\beta}, \quad f \in C^{0,\alpha,\beta}, \quad 0 < \beta \leq 1, \]
and the Hölder exponent satisfies \( \alpha \beta < 1 - \beta \). Suppose that
\[ K > 1 + \| T \|_{L^\infty} + 2C_P. \]
Then there exists a unique \( C^{4,\alpha,\beta}_w(\omega) \) solution to equation (5.7).

**Proof.** We define the continuity path
\[ L^K_t u := (\triangle_{\omega} - K) \triangle_{\omega} u + t u \overline{i} T_i \overline{T}_j(\omega) \]
with
\[ L^K_t : C^{4,\alpha,\beta}_w(\omega) \rightarrow C^{0,\alpha,\beta}_w \]
for any \( 0 \leq t \leq 1 \).

When \( t = 0 \), we already solved
\[ (\triangle_{\omega} - K) \triangle_{\omega} u = f \]
for any \( f \in C^{0,\alpha,\beta}_w \) and obtained a unique solution \( u \in C^{4,\alpha,\beta}_w(\omega) \) thanks to Theorem 5.5.

In order to use the continuity method in the Hölder space \( C^{4,\alpha,\beta}_w(\omega) \) to prove the existence and uniqueness of the \( C^{4,\alpha,\beta}_w(\omega) \) solution for the \( K \)-Lichnerowicz equation, it is sufficient to obtain the a priori estimates (Theorem 5.17). We will prove these estimates in the rest of this section.

We will need the following lemmas to do integration by parts.

**Lemma 5.13.** Assume that \( u \in C^{2,\alpha,\beta}_w \), the tensor \( T \) is bounded, and \( \nabla T = 0 \) on \( M \). Then it holds that
\[ \int_M u \overline{i} T_i \overline{T}_j(\omega) u \omega^n = - \int_M u^i T_i(\omega) u^j \omega^n. \]
PROOF. We apply the cutoff function $\chi_\epsilon$, which is fully discussed in \cite{50}. Then the argument is essentially lemma 4.10 in \cite{49}. By the dominated convergence theorem, we have
\[
\lim_{\epsilon \to 0} \int_M u^i \overline{J} T^i_{\overline{J}}(\omega) u \chi_\epsilon \omega^n = \int_M u^i \overline{J} T^i_{\overline{J}}(\omega) u \omega^n.
\]
Using $\nabla T = 0$ on $M$, we have
\[
\int_M u^i \overline{J} T^i_{\overline{J}}(\omega) u \chi_\epsilon \omega^n = -\int_M u^i T^i_{\overline{J}}(\omega) (u \chi_\epsilon) \overline{J} \omega^n
\]
\[
= -\int_M u^i T^i_{\overline{J}}(\omega) u \chi_\epsilon \omega^n - \int_M u^i T^i_{\overline{J}}(\omega) \chi_\epsilon \overline{J} u \omega^n.
\]
The first term converges under the assumptions on $u$ and $T$. The second term also converges, since for $2 \leq i, j \leq n$, $u^1 T^i_{\overline{J}}(\omega) \chi_\epsilon = \epsilon \cdot O(1)$, $u^i T^i_{\overline{J}}(\omega) \chi_\epsilon = \epsilon \cdot O(1)$.
\[\square\]

**Lemma 5.14.** Assume that $u \in C^4_{w, \alpha, \beta}(\omega)$. Then the following identity holds:
\[
\mathcal{B}^K_i(u, u) := \int_M u L^K_i u \omega^n = \int_M \left[ |\Delta_\omega u|^2 - t u^i T^i_{\overline{J}}(\omega) u \overline{J} + K |\partial u|^2_\omega \right] \omega^n.
\]

**Proof.** We multiply $L^K_i u$ with $u$ and integrate over the manifold $M$:
\[
\int_M u L^K_i u \omega^n = \int_M u \left[ (\Delta_\omega - K) \Delta_\omega u + t u^i T^i_{\overline{J}}(\omega) \right] \omega^n.
\]
Since both $u$ and $\Delta_\omega u$ are $C^{2, \alpha, \beta}(\omega)$, we have
\[
\int_M u \Delta_\omega^2 u \omega^n = \int_M |\Delta_\omega u|^2 \omega^n \quad \text{and} \quad \int_M u \Delta_\omega u \omega^n = -\int_M |\partial u|^2_\omega \omega^n.
\]
Thus the identity follows from Lemma 5.13. \[\square\]

We need a global $L^2$-estimate for the linear equation with Kähler cone coefficients.

**Lemma 5.15 (Global $L^2$-estimate).** There exists a constant $C > 0$ depending on $n, \omega, M$ such that
\[
\|u\|_{H^2, \beta,(\omega)} \leq C \left( \|\Delta_\omega u\|_{L^2(\omega)} + \|u\|_{H^1, \beta,(\omega)} \right).
\]

**Proof.** The proof of this lemma is similar to the proof of proposition 3.10 in section 3.2 in \cite{45} by patching local estimates together. The difference is that we do not need to control $\partial_i \partial_j u$, so we do not require bounds on the Christoffel symbols of $\omega$.

We let $\{U_i, \psi_i : 1 \leq i \leq N\}$ be the finite cove of $M$ and $\rho_i$ be the corresponding partition of unity, supported on $\mathbb{B}_1 \subset \mathbb{B}_3 \subset U_i$ for each $i$. In the cone charts, where
the charts have nonempty intersections with the divisor $D$, we choose $\psi_i$ such that $|\omega - \omega_{\text{cone}}|_{L^\infty(\mathbb{H}_3)}$ is sufficiently small, which can be done since $\omega$ is $C^{0,\alpha,\beta}$. Then freezing the leading coefficients and applying the local $L^2$-estimate to the cutoff equation

$$\Delta_{\omega_{\text{cone}}} (\rho_t u) = (\Delta_{\omega_{\text{cone}}} (\rho_t u) - \Delta_\omega(\rho_t u)) + \Delta_\omega \rho_t u + \rho_t \Delta_\omega u + 2(\partial \rho_t \cdot \bar{\partial} u)_\omega := f,$$

we obtain the estimate on $\rho_t u$,

$$[\rho_t u]_{H^{2,\beta}_0(\mathbb{H}_3; \omega_{\text{cone}})} \leq C \| f \|_{L^2(\mathbb{H}_3; \omega_{\text{cone}})}.$$

Then the conclusion follows from adding $\sum_i \rho_t u$ and the Cauchy-Schwarz inequality.

We also need a Hölder interpolation inequality, which follows from the compactness of $C^{2,\alpha,\beta}$ to $C^{0,\alpha,\beta}$.

**Lemma 5.16 (Hölder interpolation inequality).** For all $u \in C^{2,\alpha,\beta}$ and any $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that

$$|u|_{C^{0,\alpha,\beta}} \leq \epsilon |u|_{C^{2,\alpha,\beta}} + C(\epsilon) \| u \|_{L^2(\omega)}.$$

**Proof.** We prove by contradiction. Assume that there are $\epsilon_0$ and $u_k$ such that for any $k \to \infty$, we have

$$|u_k|_{C^{0,\alpha,\beta}} > \epsilon_0 |u_k|_{C^{2,\alpha,\beta}} + k \| u_k \|_{L^2(\omega)}.$$

Normalize $u_k$ such that $|u_k|_{C^{2,\alpha,\beta}} = 1$. The compactness of $C^{2,\alpha,\beta}$ to $C^{0,\alpha,\beta}$ gives a limit of $u_k$ to $u_\infty$ in $C^0$ by Hölder. Inequality (5.9) implies that $\| u_\infty \|_{L^2(\omega)} = 0$, which contradicts the normalization condition $|u_\infty|_{C^{2,\alpha,\beta}} = 1$.

Now we are ready to prove the key a priori estimates.

**Theorem 5.17.** Assume that $\omega = \omega_D + i \partial \bar{\partial} \varphi$ is a Kähler cone metric with

$$\varphi \in C^{2,\alpha,\beta}, \quad f \in C^{0,\alpha,\beta}, \quad T \in C^{0,\alpha,\beta}, \quad 0 < \beta \leq 1,$$

and the Hölder exponent satisfies $\alpha \beta < 1 - \beta$. Assume that

$$K > 1 + \| T \|_{L^\infty} + 2C_p.$$

There is constant $C_1$ such that for any $u \in C^{4,\alpha,\beta}_w(\omega)$ along the continuity path (5.8) with $0 \leq t \leq 1$, we have

$$|u|_{C^{4,\alpha,\beta}_w(\omega)} \leq C_1 |L^K_t u|_{C^{0,\alpha,\beta}}.$$
As a direct corollary, we have proved the following:

**Theorem 5.18.** Assume $\omega$ is a cscK cone metric, and that the Hölder exponent satisfies $\alpha \beta < 1 - \beta$. Assume that

$$K > 1 + \|T\|_{L^\infty} + 2C_p.$$ 

There is constant $C_1$ such that for any $u \in C^{4, \alpha, \beta}_w(\omega)$ along the continuity path (5.8) with $0 \leq t \leq 1$, we have

$$|u|_{C^{4, \alpha, \beta}_w(\omega)} \leq C_1 |L_t^K u|_{C^{0, \alpha, \beta}}.$$

**Proof of Theorem 5.17.** Applying the Schauder estimate to the equation

$$\Delta_\omega^2 u = L_t^K u - t u^j T_{i j}^T(\omega) + K \Delta_\omega u,$$

we have

$$|\Delta_\omega u|_{C^{2, \alpha, \beta}} \leq C_2 \left( |L_t^K u - t u^j T_{i j}^T(\omega) + K \Delta_\omega u|_{C^{0, \alpha, \beta}} + \|\Delta_\omega u\|_{L^2(\omega)} \right).$$

Note that the tensor $T$ is $C^{0, \alpha, \beta}$, the RHS is bounded by

$$\leq C_3 \left( |L_t^K u|_{C^{0, \alpha, \beta}} + |u|_{C^{2, \alpha, \beta}} + |\Delta_\omega u|_{C^{0, \alpha, \beta}} \right).$$

Applying the Schauder estimate to the middle term, we get

$$|u|_{C^{2, \alpha, \beta}} \leq C_4 (|\Delta_\omega u|_{C^{0, \alpha, \beta}} + \|u\|_{L^2(\omega)}).$$

Combining these inequalities and applying the Schauder estimate to $\Delta_\omega u$ again, we have

$$|u|_{C^{4, \alpha, \beta}_w(\omega)} \leq C_5 \left( |L_t^K u|_{C^{0, \alpha, \beta}} + |\Delta_\omega u|_{C^{0, \alpha, \beta}} + \|u\|_{L^2(\omega)} \right).$$

We use the $\epsilon$-interpolation inequality of the Hölder spaces, Lemma 5.16 with sufficiently small $\epsilon$, to the second term on the right-hand side:

$$|\Delta_\omega u|_{C^{0, \alpha, \beta}} \leq \epsilon |u|_{C^{2, \alpha, \beta}} + C(\epsilon) \|u\|_{L^2(\omega)}.$$

Thus we have

$$|u|_{C^{4, \alpha, \beta}_w(\omega)} \leq C_6 \left( |L_t^K u|_{C^{0, \alpha, \beta}} + \|u\|_{L^2(\omega)} \right).$$

We now estimate $\|u\|^2_{H^2_{w, \beta}(\omega)}$ to control $\|u\|_{L^2(\omega)}$. We first use the $L^2$-estimate of the cone metrics to the standard linear operator $\Delta_\omega u$ (Lemma 5.15); i.e., there exists a constant $C_7 > 0$ such that

$$\|u\|^2_{H^2_{w, \beta}(\omega)} \leq C_7 \int_M \left( |\Delta_\omega u|^2 + |\nabla u|^2_\omega + |u|^2 \right) \omega^n.$$

With Lemma 5.14 the RHS of the previous inequality becomes

$$= C_8 \left( B^K_t(u, u) + \int_M \left[ t \nabla u \cdot T_{i j}^T(\omega) u^j + (1 - K) |\nabla u|^2_\omega + |u|^2 \right] \omega^n \right).$$
Then we use the Cauchy-Schwarz inequality to see
\[ B^K_L(u, u) = \int_M u L^K_L u \omega^n \leq \|u\|^2_{L^2(\omega)} + \|L^K_L u\|^2_{L^2(\omega)}, \]
and apply the Poincaré inequality (Lemma 5.4) to the fourth term \( \int_M |u|^2 \omega^n \) to get
\[ \|u\|^2_{L^2(\omega)} \leq C^2_P \|\partial u\|^2_{L^2(\omega)}. \]
We set
\[ K_0 = 1 - K + \|T(\omega)\|_{L^\infty} + 2C^2_P \]
and obtain that
\[ \|u\|^2_{H^{2, \beta}(\omega)} \leq C_9 (\|L^K_L u\|^2_{L^2(\omega)} + \|\nabla u\|^2_{L^2(\omega)}). \]
We further choose an appropriate \( K \) such that \( K_0 < 0 \); then
\[ \|u\|^2_{L^2(\omega)} \leq \|u\|^2_{H^{2, \beta}(\omega)} \leq C_0 \|L^K_L u\|^2_{L^2(\omega)} \leq C_{10} \|L^K_L u\|^2_{C^{0, \beta}}. \]
By substitution into (5.10), we conclude the required estimate, i.e.,
\[ |u|_{C^{0, \alpha, \beta}(\omega)} \leq C_1 |L^K_L u|_{C^{0, \alpha, \beta}}. \]

5.3 Asymptotic of Functions in \( C^{4, \alpha, \beta}(\omega) \)

In local coordinates,
\[ \bar{\partial} \nabla^{1,0} u = g^{zi} \left[ \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} - \Gamma^j_{ki} \frac{\partial u}{\partial z^j} \right] \frac{\partial}{\partial \bar{z}^j} \otimes \frac{\partial}{\partial z^i}. \]
We prove the growth rate of
\[ \nabla_k \nabla_i u := \frac{\partial^2 u}{\partial z^k \partial \bar{z}^i} - \Gamma^j_{ki} (g) \frac{\partial u}{\partial z^j} \]
near the divisor. Here \( \Gamma^j_{kl} (g) \) are the Christoffel symbols of the cscK cone metric \( g \).

We use the notations in Section 4 under the rescaled lifted holomorphic coordinates \( \bar{v} \); see Definition 4.9 and Definition 4.10. We are given a point \( v_0 \) and a small neighborhood \( B_{c_{\beta}}(v_0) \) outside the divisor. After rescaling \( v \) to \( \bar{v} \), the neighborhood becomes a fixed set \( B_{c_{\beta}}(v_0) \).

**Proposition 5.19.** Suppose \( u \in C^{4, \alpha, \beta}(\omega) \) and \( \omega \) is a cscK cone metric. Then
\[ |\bar{\partial} \nabla^{1,0} u|_{g} = O(r_0^{\alpha \beta - \beta}). \]

**Proof.** We denote
\[ \triangle_{\omega, \bar{v}} := g^{i\bar{j}} \frac{\partial^2}{\partial \bar{v}^i \partial v^j} = \rho^{2g^{i\bar{j}}} \frac{\partial^2}{\partial v^i \partial v^j}, \]
the Laplacian with respect to \( \omega \) in the coordinate \( \bar{v} \). We let
\[ f := \triangle_{\omega, \bar{v}} u, \]
and then modify $u$ by a polynomial function such that the modified functions and their gradient vanish at the point $\tilde{v} = 0$; i.e., let

$$u_0 = \sum_{2 \leq i \leq n} \frac{\partial u(0)}{\partial \tilde{v}^i}(\tilde{v}^i) + \sum_{2 \leq i \leq n} \frac{\partial u(0)}{\partial \tilde{v}^i}(\tilde{v}^i)$$

and

$$w := u - u_0.$$ 

Then with the definition $\rho_0 = r_0^\beta$, we have

$$w = O(\rho_0^{g+1}).$$

Since $u_0$ is a linear function, we get $\triangle_\omega, \tilde{v} u_0 = 0$. Then we have

$$\triangle_\omega, \tilde{v} w = f.$$ 

The metric $\omega$ is equivalent to the flat cone metric and $\|g_i^j\|_{C^{0,\alpha}(B_{c^{\beta}})}$ is bounded by Proposition 4.13. We apply the interior Schauder estimate to this equation in $B_{c^{\beta}}$ and rescaling back (or see theorem 4.6 in [36]), we have for any $1 \leq i, j \leq n$,

$$\frac{\partial^2 w}{\partial \tilde{v}^i \partial \tilde{v}^j} = O(\rho_0^{g-1}).$$

The function $u$ satisfies the same identity as $w$, because $u_0$ is a linear function. Since $u \in C^{1,\alpha,\beta}$ and $\omega$ is a cscK cone metric, from Corollary 4.6

$$|\Gamma^j_{ki}(g) \partial_j u|_g = O(\rho_0^{g-1}).$$

The conclusion follows from $\rho_0 = r_0^\beta$. 

\[\Box\]

### 6 The Automorphism Group Is Reductive

**If a cscK Cone Metric Exists**

We need three notations:

- $\text{Aut}(X; D)$ is the identity component of the group of holomorphic automorphisms of $X$ that fix the divisor $D$;
- $\mathfrak{h}(X; D)$ is the space of all holomorphic vector fields tangential to the divisor;
- $\mathfrak{h}'(X; D)$ is the complexification of a Lie algebra consisting of Killing vector fields of $X$ tangential to $D$.

**Theorem 6.1.** Suppose $\omega$ is a cscK cone metric. Then there exists a one-to-one correspondence between $\mathfrak{h}'(X; D)$ and the kernel of $\text{Lie}_\omega$.

Specifically, the Lie algebra $\mathfrak{h}(X; D)$ has a direct sum decomposition,

$$\mathfrak{h}(X; D) = \mathfrak{a}(X; D) \oplus \mathfrak{h}'(X; D),$$

where $\mathfrak{a}(X; D)$ is the complex Lie subalgebra of $\mathfrak{h}(X; D)$ consisting of all parallel holomorphic vector fields tangential to $D$, and $\mathfrak{h}'(X; D)$ is the ideal of $\mathfrak{h}(X; D)$.
Combining these two identities together, we prove the lemma. □

From Lemma 3.4 and Lemma 5.13, we have the help of the asymptotic behavior of functions in Proposition 5.19. Inserting this into the right-hand side of the identity above, we have of 

In the local normal coordinate on the regular part $M$, consisting of the image under $\text{grad}_g$ of the kernel of $\mathcal{D}$ operator. The operator $\text{grad}_g$ is defined to be $\text{grad}_g (u) = \partial^\omega \tilde{\delta}u = g^{ij} \partial^\theta \partial_\theta \partial_\sigma \partial_\sigma^j$. 

Furthermore $h'(X; D)$ is the complexification of a Lie algebra consisting of Killing vector fields of $X$ tangential to $D$. In particular, $h'(X; D)$ is reductive. Moreover, $h(X; D)$ is reductive.

PROOF. The proof follows along the same lines as that of theorem 4.1 in [49], with the help of the following lemmas. The new gradients are the use of the space $C^4_{\alpha, \beta}(\omega)$ and obtaining the growth rate of $\tilde{\partial} \nabla^{1,0} u$, which follows from the asymptotic of the cscK cone metrics in Section 4 and also the asymptotic of the functions in the space $C^4_{\alpha, \beta}(\omega)$ in Section 5.3. □

LEMMA 6.2. Suppose $u \in C^4_{\alpha, \beta}(\omega)$ and $\omega$ is a cscK cone metric. Then we have 

$$
\int_M u \cdot \|\mathcal{L}c_\omega(u)\omega^n = - \int_M g^{ij} g^{kT} \cdot \nabla_j \nabla_k \nabla_i \nabla_i u \cdot \omega^n.
$$

PROOF. We will apply the integration by parts directly to the Lichnerowicz operator 

$$
\mathcal{L}c_\omega(u) = \Delta_\omega^2 u + u^{ij} T_{ij}.
$$

First, since $u, \Delta_\omega u \in C^2_{\alpha, \beta}$, we have 

$$
\int_M u \cdot \Delta_\omega^2 u \cdot \omega^n = - \int_M g^{ij} \nabla_j \nabla_i (\Delta_\omega u)\omega^n.
$$

In the local normal coordinate on the regular part $M$, we use the Kähler condition of $g$ and the Ricci formula to prove 

$$
\nabla_i (\Delta_\omega u) = g^{kT} \nabla_i \nabla_k \nabla_i u = g^{kT} \nabla_k \nabla_i \nabla_i u = g^{kT} \nabla_k \nabla_i \nabla_i u - g^{pq} R_{kli}^p \nabla_p u.
$$

Inserting this into the right-hand side of the identity above, we have 

$$
\int_M u \cdot \Delta_\omega^2 u \cdot \omega^n = - \int_M [g^{ij} g^{kT} \nabla_j \nabla_k \nabla_i \nabla_i u - g^{pqk} R_{kli}^p \nabla_p u] \omega^n.
$$

From Lemma 3.4 and Lemma 5.13, we have 

$$
\int_M u^{ij} T_{ij}(\omega) \omega^n = - \int_M T_{ij}(\omega) \nabla_j \nabla_i u \omega^n = - \int_M R_{ij}(\omega) \nabla_j \nabla_i u \omega^n.
$$

Combining these two identities together, we prove the lemma. □

We then continue from the previous lemma and apply integration by parts, with the help of the asymptotic behavior of functions in Proposition 5.19.

LEMMA 6.3. Suppose $u \in C^4_{\alpha, \beta}(\omega)$, $\omega$ is a Kähler cone metric, and $|\tilde{\partial} \nabla^{1,0} u|_g = O(r^{-\gamma})$ with $-\gamma + \beta > 0$. Then we have 

$$
- \int_M g^{ij} g^{kT} \cdot \nabla_j \nabla_k \nabla_i u \cdot \omega^n = \int_M |\tilde{\partial} \nabla^{1,0} u|^2_g \omega^n.
$$
We let \( \chi_\epsilon \) be the smooth cutoff function supported outside the \( \rho \)-tubular neighborhood of the divisor with the properties such that
\[
| \nabla \chi_\epsilon | = \epsilon \cdot O(r^{-1}).
\]

Multiplying the integrand in (6.2) with the cutoff function \( \chi_\epsilon \), we compute by integration by parts
\[
\int_M g^{ij} g^{kT} \cdot \nabla_j u \cdot \nabla_T \nabla_k \nabla_i u \cdot \chi_\epsilon \cdot \omega^n =
\int_M |\bar{\partial} \nabla^{1,0} u|_g^2 u \cdot \chi_\epsilon \cdot \omega^n + \int_M g^{ij} g^{kT} \cdot \nabla_j u \cdot \nabla_k \nabla_i u \cdot \nabla_T \chi_\epsilon \cdot \omega^n.
\]

We denote
\[
I_\epsilon = \int_M g^{ij} g^{kT} \cdot \nabla_j u \cdot \nabla_k \nabla_i u \cdot \nabla_T \chi_\epsilon \cdot \omega^n.
\]

We know from \( u \in C^{1,\alpha,\beta}_0 \) and \( g \) being \( C^{0,\alpha,\beta} \) that the growth rate of the following terms near the divisor satisfies
\[
| \partial \bar{z}^i u |_g = O(r^{\alpha \beta}), \quad | \partial z^i u |_g = O(1) \quad \forall 2 \leq i \leq n, \\
| \nabla \chi_\epsilon |_g = \epsilon \cdot O(r^{-\beta}), \quad \omega^n = O(r^{2\beta-2}).
\]

Then from the assumption the growth order of the integrand is \(-\kappa + \beta - 2 > -2\) and thus \( I_\epsilon \to 0 \) as \( \epsilon \to 0 \). □

**Corollary 6.4.** Suppose \( u \in C^{4,\alpha,\beta}_w(\omega) \) is in the kernel of \( \mathbb{L} \mathfrak{c} \omega \) and \( \omega \) is a cscK cone metric. Then we have
\[
|\bar{\partial} \nabla^{1,0} u|_g = 0 \quad \text{on } M.
\]

**Proof.** According to Proposition 5.19, the growth rate of \( |\bar{\partial} \nabla^{1,0} u|_g \) is \(-\gamma = \alpha \beta - \beta \). Thus the conclusion follows from Lemma 6.2 and Lemma 6.3 □

Next we are going to prove that the kernel of \( \mathbb{L} \mathfrak{c} \omega \) generates a holomorphic vector field on \( X \) tangential to the divisor.

**Lemma 6.5.** Suppose \( u \in C^{4,\alpha,\beta}_w(\omega) \) is in the kernel of \( \mathbb{L} \mathfrak{c} \omega \) and \( \omega \) is a cscK cone metric. Then the lifting of \( u \) by the metric \( \omega \) is a holomorphic vector field on \( X \) tangential to \( D \).

**Proof.** According to Corollary 6.4 when \( \bar{\partial} \nabla^{1,0} u = 0 \) on the regular part \( M \), then \( X = g^{ij} \frac{\partial u}{\partial z^j} \frac{\partial \bar{z}^i}{\partial z^j} \) is a holomorphic vector field on \( M \). Since \( u \in C^{2,\alpha,\beta} \), \( X \) vanishes along \( \bar{D} \) and the tangent directions can be extended to \( D \), which extends \( X \) to a holomorphic vector field on the whole of \( X \). □
7 Bifurcation of the $J$-Twisted Path

We now introduce a new continuity path towards a cscK metric:

$$
\Phi(t, \varphi) = t (S(\omega_\varphi) - S) - (1 - t) \left( \frac{\omega_n^\varphi}{\omega_n^0} - 1 \right) = 0. \tag{7.1}
$$

It is clear that this path is the critical point of the perturbed functional

$$
E(\varphi(t)) = t \nu(\varphi(t)) + (1 - t) J(\varphi(t)). \tag{7.2}
$$

The $J$-twisted term also appears in [17, 62]. We refer readers to Remark 3.8 for comparing this with Chen’s path [21].

We consider the endpoint at $t = 1$, and we reparametrize the continuity path

$$
\Phi(t, \varphi(t)) = S(\omega_\varphi) - S_\beta - (1 - t) \left( \frac{\omega_n^\varphi}{\omega_n^0} - 1 \right) = 0. \tag{7.3}
$$

When $\varphi, P \in C^{2,\alpha,\beta}$, we rewrite the equation of $\Phi(t, \varphi(t))$ as the fully nonlinear equations

$$
\Phi(t, \varphi(t)) = -\Delta_{\varphi(t)} P(t) + g^{ij}_{\varphi(t)} R_{ij}(\omega_\theta) - S_\beta - (1 - t) \left( \frac{\omega_n^\varphi}{\omega_n^0} - 1 \right)
= 0. \tag{7.4}
$$

and

$$
\log \frac{\omega_n^\varphi(t)}{\omega_n^\theta} = P. \tag{7.4}
$$

We denote by $O$ an orbit of the cscK cone metrics. We minimize the functional $J$ over $O$.

**Proposition 7.1.** The functional $J$ has a unique minimizer on $O$.

**Proof.** It follows from the reductivity of the automorphism group (Theorem 6.1) that $O$ is finite dimensional and compact. So $J$ is a proper function on $O$, and so a minimizing sequence converges to some point $\theta$ on $O$ under the norm on the Lie group $O$. Since all norms are equivalent in the finite-dimensional space, $\theta$ is actually the minimizer. We then show the uniqueness. By Theorem 6.1 again, any cscK cone metric on $O$ can be connected to $\theta$ by the cone geodesic $\exp(t \mathfrak{g} X)$ for some holomorphic vector field $X$. From Proposition 3.14, i.e., $J$ is convex along the $C^{1,\alpha,\beta}$ cone geodesic, we see that the minimizer is unique over $O$. \qed

**Remark 7.2.** We remark that in general the minimizer of $J$ cannot be obtained directly.

**Remark 7.3.** There is another way to prove $J$ is proper by using the a priori estimates of the cscK cone equation, since any point on $O$ satisfies this equation. We refer readers to the most recent development in [22, 48, 64] and references therein.
We denote by $H_\theta$ the kernel space of the Lichnerowicz operator,

$$L_{ic\theta}(u) = \Delta_\theta^2 u + u^j T_{ij}(\theta).$$

The following lemma is direct from Proposition 7.1.

**Lemma 7.4.** At $\theta$, for any $u \in H_\theta$,

$$\frac{1}{V} \int_M u \cdot \left( \omega_0^n \frac{\partial}{\partial \omega^n} - 1 \right) \theta^n = 0.$$

Now we prove Theorem 1.9 at the cscK cone metric $\theta$. We denote by $\lambda_\theta$ the Kähler cone potential of the minimizer $\theta$ such that

$$\theta := \omega_\theta = \omega + i \partial \overline{\partial} \lambda_\theta.$$

We consider the continuity path (7.3) with $\varphi(1) = \lambda_\theta$ and construct the bifurcation at $\varphi(1)$.

**Theorem 7.5.** Assume the notations above. There exists a parameter $\tau > 0$ such that $\varphi(t)$ with $\varphi(1) = \lambda_\theta$ is extended uniquely to a smooth one-parameter family of solutions of the $J$-twisted path (7.3) on $(1 - \tau, 1]$.

**Proof.** The proof is divided into six steps.

**Step 1.** The linearized operator at $t = 1$ is the Lichnerowicz operator. Then we decompose the whole space $C_w^{A,\alpha,\beta}(\theta)$ into the direct sum of $H_\theta$ and its orthogonal space $H_\theta^\perp$, i.e.,

$$C_w^{A,\alpha,\beta}(\theta) = H_\theta \oplus H_\theta^\perp.$$

The path $\varphi(t) - \lambda_\theta$ is then decomposed into

$$\varphi(t) - \lambda_\theta = \varphi^1 + \varphi^\perp.$$

$\Phi(t, \varphi(t)$ vanishes at

$$(t, \varphi^1, \varphi^\perp) = (1, 0, 0),$$

since $\theta$ is a cscK cone metric. We let $P$ denote the projection from $C_w^{A,\alpha,\beta}(\theta)$ to $H_\theta$ and decompose the linear operator $\Phi$ also into two parts $\Phi^1$ and $\Phi^\perp$.

**Step 2.** We first consider the vertical part,

$$\Phi^\perp(t, \varphi^1, \varphi^\perp) = (1 - P)[S(\varphi(t)) - S_\theta] - (1 - t) \left( \frac{\omega_0^n}{\omega_0^n} - 1 \right).$$

Meanwhile, its derivative on $\varphi^\perp$ at $(1, 0, 0)$ is

$$\delta_{\varphi^\perp} \Phi^\perp|_{(1,0,0)}(\xi) = (1 - P) \ll_{ic\theta} \xi \quad \text{for any } \xi \in H_\theta^\perp,$$

which is invertible according to Theorem 5.2. Therefore, we are able to use the implicit function theorem on $C_w^{A,\alpha,\beta}(\theta)$-space to conclude that there is a small neighborhood $U$ near $(t, \varphi^1) = (1, 0)$ such that when $(t, \varphi^1) \in U$,

$$\varphi^\perp: U \subset (1 - \tau, 1) \times H_\theta \to H_\theta^\perp,$$

$$(t, \varphi^1) \mapsto \varphi^\perp(t, \varphi^1).$$
solves
\( (7.7) \quad \Phi^\perp(t, \varphi^1, \varphi^\perp(t, \varphi^1)) = 0, \)
with \( \varphi^\perp(1, 0) = 0. \)

**Step 3.** We differentiate (7.7) along the kernel direction, at \( (t, \varphi^1) = (1, 0), \) for any \( u \in H^\perp_\Theta, \)
\[
0 = \delta_{\varphi^1} \Phi^\perp|_{(1,0)}(u) = (1 - P)[- \mathbb{L} \text{ic}_\Theta(\delta_{\varphi^1} \varphi^\perp|_{(1,0)}(u))].
\]
Since both the image of \( 1 - P \) and \( \mathbb{L} \text{ic}_\Theta \) are in \( H^\perp_\Theta, \) we conclude that
\( (7.8) \quad \delta_{\varphi^1} \varphi^\perp|_{(1,0)}(u) = 0 \quad \forall u \in H_\Theta. \)
Furthermore, taking the \( t \)-derivative of the path \( \Phi^\perp = 0, \) we have
\[
0 = \frac{\partial \Phi^\perp}{\partial t} = (1 - P) \left\{ - \Delta \varphi \left[ \frac{\partial \varphi^1}{\partial t} + \frac{\partial \varphi^\perp}{\partial t} + \delta_{\varphi^1} \varphi^\perp \left( \frac{\partial \varphi^1}{\partial t} \right) \right] 
+ \left( \frac{\omega^\Theta_0}{\omega^\Theta} - 1 \right) + (1 - t) \frac{\omega^\Theta_0}{\omega^\Theta} \Delta \varphi \left[ \frac{\partial \varphi^1}{\partial t} + \frac{\partial \varphi^\perp}{\partial t} + \delta_{\varphi^1} \varphi^\perp \left( \frac{\partial \varphi^1}{\partial t} \right) \right] \right\}.
\]
At \( (t, \varphi^1) = (1, 0), \) the equation reads
\[
0 = \left. \frac{\partial \Phi^\perp}{\partial t} \right|_{(1,0)} = (1 - P) \left\{ - \mathbb{L} \text{ic}_\Theta \left( \frac{\partial \varphi^\perp}{\partial t} \right)_{(1,0)} + \frac{\omega^\Theta_0}{\omega^\Theta} - 1 \right\}.
\]
From Lemma 7.4, \( \frac{\omega^\Theta_0}{\omega^\Theta} - 1 \in H^\perp_\Theta, \) we have
\( (7.9) \quad - \mathbb{L} \text{ic}_\Theta \left( \frac{\partial \varphi^\perp}{\partial t} \right)_{(1,0)} + \frac{\omega^\Theta_0}{\omega^\Theta} - 1 = 0. \)

**Step 4.** We next consider the horizontal operator on the finite-dimensional space \( H_\Theta, \)
\[
\Phi^1(t, \varphi^1) = P \left[ S(\varphi(t)) - S_\beta - (1 - t) \left( \frac{\omega^\Theta_0}{\omega^\Theta} - 1 \right) \right].
\]
Note that \( \Phi^1(t, \varphi^1) = 0 \) at \( (1, 0). \) From (7.9), we see that
\[
\left. \frac{\partial \Phi^1}{\partial t} \right|_{(1,0)} = 0.
\]
Then we consider the modified functional
\[
\tilde{\Phi}^1(t, \varphi^1) = \frac{\Phi^1(t, \varphi^1)}{t - 1}.
\]
Again, we let $\xi = \frac{\partial \mathcal{g}^{-1}}{\partial t}(t-1, x^1 - 0)$, and then compute for all $u \in H_\theta$,
\[
\delta_{\psi} \tilde{\Phi}^1|_{(1,0)}(u) = \delta_{\psi} \frac{\partial}{\partial t} \Phi^1|_{(1,0)}(u)
\]
\[
= P \left[ \begin{array}{c} \operatorname{div} \Delta_{\theta} \Phi^1 \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) + \Delta_{\theta} \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) + \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \right] \\
+ u^i \phi \kappa \mathcal{g} (\mathcal{g}^i j \kappa \mathcal{g} + \phi \mathcal{g}^i j \kappa \mathcal{g}^j \mathcal{g}^k \kappa \mathcal{g}^l \mathcal{g}^m \mathcal{g}^n \mathcal{g} \Delta_{\theta} \Delta_{\theta} \Phi^1 \right].
\]

**Step 5.** We will carry on the detailed computation in Lemma 7.6 below to see that
\[
(\delta_{\psi} \tilde{\Phi}^1|_{(1,0)}(u), v)_{L^2(\theta)}
\]
is positive definite, and the equality holds if and only if $u = 0$.

**Lemma 7.6.** The following identity holds:
\[
\int_M \delta_{\psi} \tilde{\Phi}^1|_{(1,0)}(u) \cdot v \cdot \theta^n = \int_M (\partial \mathcal{g}^i j \kappa \mathcal{g}^j \mathcal{g}^k \mathcal{g}^l \mathcal{g}^m \mathcal{g}^n \mathcal{g} \Delta_{\theta} \Delta_{\theta} \Phi^1 \right].
\]

**Proof.** On the regular part $M$, we use the direct computation (see lemma 2.4 in [27]) to see that
\[
\delta_{\psi} \tilde{\Phi}^1|_{(1,0)}(u) = P \left[ \begin{array}{c} \operatorname{div} \Delta_{\theta} \Phi^1 \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) + \Delta_{\theta} \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) + \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \right] \\
+ u^i \phi \kappa \mathcal{g} (\mathcal{g}^i j \kappa \mathcal{g} + \phi \mathcal{g}^i j \kappa \mathcal{g}^j \mathcal{g}^k \kappa \mathcal{g}^l \mathcal{g}^m \mathcal{g}^n \mathcal{g} \Delta_{\theta} \Delta_{\theta} \Phi^1 \right].
\]

We multiply this identity with $v$ and the cutoff function $\chi_{\varepsilon}$ (see Lemma 6.3) and then integrate over $X$ to have
\[
\int_X \delta_{\psi} \tilde{\Phi}^1|_{(1,0)}(u) \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n = - \int_M \left[ \begin{array}{c} \operatorname{div} \Delta_{\theta} \Phi^1 \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) + \Delta_{\theta} \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) + \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \right] \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n.
\]

By the formula for integration by parts, the second term becomes
\[
- \int_M \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n
\]
\[
= \int_M \left[ \operatorname{div} \Delta_{\theta} \Phi^1 \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \right] \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n + \int_M \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n.
\]

Then we have
\[
\int_X \delta_{\psi} \tilde{\Phi}^1|_{(1,0)}(u) \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n = \int_M \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n
\]
\[
+ \int_M \left( \frac{\partial}{\partial \theta} \Delta_{\theta} \Phi^1 \right) \cdot v \cdot \chi_{\varepsilon} \cdot \theta^n.
\]
Thus the lemma is proved as \( \epsilon \rightarrow 0 \), since \( u, v, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial \theta} \in C^{2,\alpha,\beta} \).

**Step 6.** At last, we are able to apply the implicit function theorem to \( \Phi^1(t, \varphi^1) \) over \( C^{4,\alpha,\beta}(\theta) \) to construct a solution \( \varphi^1(t) \in C^{4,\alpha,\beta}(\theta) \) with \( t \in (1 - \tau, 1] \) such that

\[
\Phi^1(t, \varphi^1(t)) = 0 \quad \text{and} \quad \varphi^1(1) = 0.
\]

Then the original nonlinear equation is solved as

\[
\Phi(t, \varphi^1(t), \varphi^1(t), \varphi^1(t)) = 0.
\]

Moreover,

\[
\varphi(t) = \lambda_0 + \varphi^1(t) + \varphi^1(t, \varphi^1(t))
\]

is the solution to the continuity path on \( t \in (1 - \tau, 1] \) with \( \varphi(1) = \lambda_0 \).

The proof of Theorem 7.5 is complete.

**Acknowledgments.** The work of K. Zheng received funding from the European Union’s Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement no. 703949, and was also partially supported by EPSRC grant number EP/K00865X/1.

**Bibliography**

[1] Aleyasin, S. A.; Chen, X. On the geodesics in the space of Kähler metrics with prescribed singularities. Preprint, 2013. arXiv:1306.1867 [math.DG]

[2] Auvray, H. The space of Poincaré type Kähler metrics on the complement of a divisor. J. Reine Angew. Math. **722** (2017), 1–64. doi:10.1515/crelle-2014-0058

[3] Bando, S.; Mabuchi, T. Uniqueness of Einstein Kähler metrics modulo connected group actions. Algebraic geometry, Sendai, 1985, 11–40. Advanced Studies in Pure Mathematics, 10. North-Holland, Amsterdam, 1987.

[4] Bedford, E.; Taylor, B. A. The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math. **37** (1976), no. 1, 1–44. doi:10.1007/BF01418826

[5] Bedford, E.; Taylor, B. A. Variational properties of the complex Monge-Ampère equation. I. Dirichlet principle. Duke Math. J. **45** (1978), no. 2, 375–403.

[6] Bedford, E.; Taylor, B. A. Variational properties of the complex Monge-Ampère equation. II. Intrinsic norms. Amer. J. Math. **101** (1979), no. 5, 1131–1166. doi:10.2307/2374130

[7] Bedford, E.; Taylor, B. A. A new capacity for plurisubharmonic functions. Acta Math. **149** (1982), no. 1-2, 1–40. doi:10.1007/BF02392348

[8] Berman, R. J.; Berndtsson, B. Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics. J. Amer. Math. Soc. **30** (2017), no. 4, 1165–1196. doi:10.1090/jams/880

[9] Berman, R. J.; Boucksom, S.; Eyssidieux, P.; Guedj, V.; Zeriahi, A. Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties. J. Reine Angew. Math. (2016). doi:10.1515/crelle-2016-0033

[10] Berman, R. J.; Boucksom, S.; Guedj, V.; Zeriahi, A. A variational approach to complex Monge-Ampère equations. Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245. doi:10.1007/s10240-012-0046-6
[11] Berman, R.; Demailly, J.-P. Regularity of plurisubharmonic upper envelopes in big cohomology classes. Perspectives in analysis, geometry, and topology, 39–66. Progress in Mathematics, 296. Birkhäuser/Springer, New York, 2012. doi:10.1007/978-0-8176-8277-4_3

[12] Berndtsson, B. A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. Invent. Math. 200 (2015), no. 1, 149–200. doi:10.1007/s00222-014-0532-1

[13] Błocki, Z.; Kołodziej, S. On regularization of plurisubharmonic functions on manifolds. Proc. Amer. Math. Soc. 135 (2007), no. 7, 2089–2093. doi:10.1090/S0002-9939-07-08858-2

[14] Boucksom, S.; Eyssidieux, P.; Guedj, V.; Zeriahi, A. Monge-Ampère equations in big cohomology classes. Acta Math. 205 (2010), no. 2, 199–262. doi:10.1007/s11511-010-0054-7

[15] Calabi, E.; Chen, X. X. The space of Kähler metrics. II. J. Differential Geom. 61 (2002), no. 2, 173–193.

[16] Calamai, S.; Zheng, K. Geodesics in the space of Kähler cone metrics, I. Amer. J. Math. 137 (2015), no. 5, 1149–1208. doi:10.1353/ajm.2015.0036

[17] Cao, H.-D.; Keller, J. On the Calabi problem: a finite-dimensional approach. J. Eur. Math. Soc. (JEMS) 15 (2013), no. 3, 1033–1065. doi:10.4171/JEMS/385

[18] Chen, W. X.; Li, C. What kinds of singular surfaces can admit constant curvature? Duke Math. J. 78 (1995), no. 2, 437–451. doi:10.1215/S0012-7094-95-07821-1

[19] Chen, X. Obstruction to the existence of metric whose curvature has umbilical Hessian in a K-surface. Comm. Anal. Geom. 8 (2000), no. 2, 267–299. doi:10.4310/CAG.2000.v8.n2.a2

[20] Chen, X.; Donaldson, S.; Sun, S. Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. J. Amer. Math. Soc. 28 (2015), no. 1, 183–197. doi:10.1090/S0894-0347-2014-00799-2

[21] Chen, X.; Donaldson, S.; Sun, S. Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π. J. Amer. Math. Soc. 28 (2015), no. 1, 199–234. doi:10.1090/S0894-0347-2014-00800-6

[22] Chen, X.; Donaldson, S.; Sun, S. Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof. J. Amer. Math. Soc. 28 (2015), no. 1, 235–278. doi:10.1090/S0894-0347-2014-00801-8

[23] Chen, X.; Li, L.; Păuni, M. Approximation of weak geodesics and subharmonicity of Mabuchi energy. Ann. Fac. Sci. Toulouse Math. (6) 25 (2016), no. 5, 935–957. doi:10.5802/afst.1516

[24] Chen, X.; Păun, M.; Zeng, Y. On deformation of extremal metrics. Preprint, 2015. arXiv:1506.01290 [math.DG]

[25] Chen, X.; Wang, Y. C^{2,\alpha} estimate for Monge-Ampère equations with Hölder-continuous right hand side. Ann. Global Anal. Geom. 49 (2016), no. 2, 195–204. doi:10.1007/s10455-015-9487-8

[26] Chen, X.; Wang, Y. On the regularity problem of complex Monge-Ampere equations with conical singularities. Ann. Inst. Fourier (Grenoble) 67 (2017), no. 3, 969–1003.

[27] de Borbon, M. Kähler metrics with cone singularities along a divisor of bounded Ricci curvature. Ann. Global Anal. Geom. 52 (2017), no. 4, 457–464. doi:10.1007/s10455-017-9565-1

[28] Demailly, J.-P. Complex analytic and differential geometry, 1997. Available at: https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbg.pdf

[29] Donaldson, S. K. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. Northern California Symplectic Geometry Seminar, 13–33. American Mathematical Society Translations, Series 2, 196. Adv. Math. Sci., 45. American Mathematical Society, Providence, R.I., 1999. doi:10.1090/trans2/196/02
[33] Donaldson, S. K. Conjectures in Kähler geometry. *Strings and geometry*, 71–78. Clay Mathematics Proceedings, 3. American Mathematical Society, Providence, R.I., 2004.

[34] Donaldson, S. K. Kähler metrics with cone singularities along a divisor. *Essays in mathematics and its applications*, 49–79. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-34446-4_4

[35] Donaldson, S. Algebraic families of constant scalar curvature Kähler metrics. *Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations*, 111–137. Surveys in Differential Geometry, 19. International Press, Somerville, Mass, 2015. doi:10.4310/SDG.2014.v19.n1.a5

[36] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, Berlin, 2001.

[37] Guan, B.; Spruck, J. Boundary-value problems on $S^n$ for surfaces of constant Gauss curvature. *Ann. of Math. (2)* **138** (1993), no. 3, 601–624. doi:10.2307/2946558

[38] Guedj, V.; Zeriahi, A. The weighted Monge-Ampère energy of quasiplusher subharmonic functions. *J. Funct. Anal.* **250** (2007), no. 2, 442–482. doi:10.1016/j.jfa.2007.04.018

[39] Guedj, V.; Zeriahi, A. Dirichlet problem in domains of $\mathbb{C}^n$. *Complex Monge-Ampère equations and geodesics in the space of Kähler metrics*, 13–32. Lecture Notes in Mathematics, 2038. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-23669-3_2

[40] Guedj, V.; Zeriahi, A. Degenerate complex Monge-Ampère equations. EMS Tracts in Mathematics, 26. European Mathematical Society (EMS), Zürich, 2017. doi:10.4171/167

[41] Guenancia, H.; Păun, M. Conic singularities metrics with prescribed Ricci curvature: general cone angles along normal crossing divisors. *J. Differential Geom.* **103** (2016), no. 1, 15–57.

[42] Hashimoto, Y. Scalar curvature and Futaki invariant of Kähler metrics with cone singularities along a divisor. *Ann. Inst. Fourier (Grenoble)* **69** (2019), no. 2, 591–652. doi:10.5802/aif.3252

[43] He, W. On the space of Kähler potentials. *Comm. Pure Appl. Math.* **68** (2015), no. 2, 332–343. doi:10.1002/cpa.21515

[44] Jeffres, T.; Mazzeo, R.; Rubinstein, Y. A. Kähler-Einstein metrics with edge singularities. *Ann. of Math. (2)* **183** (2016), no. 1, 95–176. doi:10.4007/annals.2016.183.1.3

[45] Keller, J.; Zheng, K. Construction of constant scalar curvature Kähler cone metrics. *Proc. Lond. Math. Soc. (3)* **117** (2018), no. 3, 527–573. doi:10.1112/plms.12132

[46] Kolodziej, S. Hölder continuity of solutions to the complex Monge-Ampère equation with the right-hand side in $L^p$: the case of compact Kähler manifolds. *Math. Ann.* **342** (2008), no. 2, 379–386. doi:10.1007/s00208-008-0239-7

[47] Li, L. Subharmonicity of conic Mabuchi’s functional, I. *Ann. Inst. Fourier (Grenoble)* **68** (2018), no. 2, 805–845.

[48] Li, L.; Wang, J.; Zheng, K. Conic singularities metrics with prescribed scalar curvature: a priori estimates for normal crossing divisors. *Ann. of Math. (2)* **183** (2016), no. 1, 95–176. doi:10.4007/annals.2016.183.1.3

[49] Li, L.; Zheng, K. Uniqueness of constant scalar curvature Kähler metrics with cone singularities, I: Reductivity. *Math. Ann.* **373** (2019), no. 1-2, 679–718. doi:10.1007/s00208-017-1626-z

[50] Li, L.; Zheng, K. Generalized Matsuhashi’s theorem and Kähler-Einstein cone metrics. *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 31, 43 pp. doi:10.1007/s00526-018-1313-z

[51] Mazzeo, R. Kähler-Einstein metrics singular along a smooth divisor. *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1999)*, Exp. No. VI, 10 pp. Univ. Nantes, Nantes, 1999.

[52] Mondello, G.; Panov, D. Spherical metrics with conical singularities on a 2-sphere: angle constraints. *Int. Math. Res. Not. IMRN* (2016), no. 16, 4937–4995. doi:10.1093/imrn/rnv300

[53] Ross, J.; Thomas, R. Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics. *J. Differential Geom.* **88** (2011), no. 1, 109–159.

[54] Tian, G.; Yau, S.-T. Complete Kähler manifolds with zero Ricci curvature. I. *J. Amer. Math. Soc.* **3** (1990), no. 3, 579–609. doi:10.2307/1990928
[55] Tian, G.; Yau, S.-T. Complete Kähler manifolds with zero Ricci curvature. II. \emph{Invent. Math.} \textbf{106} (1991), no. 1, 27–60. [doi:10.1007/BF01243902]

[56] Tian, G.; Zhu, X. Uniqueness of Kähler-Ricci solitons. \emph{Acta Math.} \textbf{184} (2000), no. 2, 271–305. [doi:10.1007/BF02392630]

[57] Troyanov, M. Prescribing curvature on compact surfaces with conical singularities. \emph{Trans. Amer. Math. Soc.} \textbf{324} (1991), no. 2, 793–821. [doi:10.2307/2001742]

[58] Wang, G.; Zhu, X. Extremal Hermitian metrics on Riemann surfaces with singularities. \emph{Duke Math. J.} \textbf{104} (2000), no. 2, 181–210. [doi:10.1215/S0012-7094-00-10421-8]

[59] Yao, C. Existence of weak conical Kähler-Einstein metrics along smooth hypersurfaces. \emph{Math. Ann.} \textbf{362} (2015), no. 3-4, 1287–1304. [doi:10.1007/s00208-014-1140-5]

[60] Yau, S. T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. \emph{Comm. Pure Appl. Math.} \textbf{31} (1978), no. 3, 339–411. [doi:10.1002/cpa.3160310304]

[61] Yin, H.; Zheng, K. Expansion formula for complex Monge-Ampère equation along cone singularities. \emph{Calc. Var. Partial Differential Equations} \textbf{58} (2019), no. 2, Art. 50, 32 pp. [doi:10.1007/s00526-019-1498-z]

[62] Zheng, K. \textit{I}-properness of Mabuchi’s $K$-energy. \emph{Calc. Var. Partial Differential Equations} \textbf{54} (2019), no. 3, 2807–2830. [doi:10.1007/s00526-015-0884-4]

[63] Zheng, K. Kähler metrics with cone singularities and uniqueness problem. \textit{Current trends in analysis and its applications}, 395–408. Trends in Mathematics. Birkhäuser/Springer, Cham, 2015.

[64] Zheng, K. Existence of constant scalar curvature Kähler cone metrics, properness and geodesic stability. Preprint, 2018; [arXiv:1803.09506] [math.DG]

KAI ZHENG
Tongji University
Shanghai 200092
P.R. CHINA
E-mail: [kaizheng@amss.ac.cn]
Received August 2018.