Roman domination excellent graphs: trees

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Abstract: A Roman dominating function (RDF) on a graph $G = (V, E)$ is a labeling $f : V \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The weight of $f$ is the value $f(V) = \sum_{v \in V} f(v)$ The Roman domination number, $\gamma_R(G)$, of $G$ is the minimum weight of an RDF on $G$. An RDF of minimum weight is called a $\gamma_R$-function. A graph $G$ is said to be $\gamma_R$-excellent if for each vertex $x \in V$ there is a $\gamma_R$-function $h_x$ on $G$ with $h_x(x) \neq 0$. We present a constructive characterization of $\gamma_R$-excellent trees using labelings. A graph $G$ is said to be in class $UVR$ if $\gamma(G - v) = \gamma(G)$ for each $v \in V$, where $\gamma(G)$ is the domination number of $G$. We show that each tree in $UVR$ is $\gamma_R$-excellent.

Keywords: Roman domination number, excellent tree, coalescence

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1. Introduction and preliminaries

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes et al. [9]. Specifically, let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A spanning subgraph for $G$ is a subgraph of $G$ which contains every vertex of $G$. In a graph $G$, for a subset $S \subseteq V(G)$ the subgraph induced by $S$ is the graph $\langle S \rangle$ with vertex set $S$ and edge set $\{xy \in E(G) \mid x, y \in S\}$. The complement $\overline{G}$ of $G$ is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of $G$. We write $K_n$ for the complete graph of order $n$ and $P_n$ for the path on $n$ vertices. Let $C_m$ denote the cycle of length $m$. For any vertex $x$ of a graph $G$, $N_G(x)$ denotes the set of all neighbors of $x$ in $G$, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of $x$ is $\deg_G(x) = |N_G(x)|$. The minimum and maximum degrees of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S$ of vertices, let
Denote by $P = \{v \in V(G) - S \mid N_G(u) \cap S = \{v\}\}$. The external private neighborhood epn$(v, S)$ of $v \in S$ is defined by $\text{epn}(v, S) = \{u \in V(G) - S \mid N_G(u) \cap S = \{v\}\}$. A leaf is a vertex of degree one and a support vertex by $N = \{v \in V(G) - S \mid \deg_G(v) = 1\}$. Let a $\gamma$-good graph be a finite set of integers which has positive as well as non-positive elements. If $F$ and $H$ are disjoint graphs, $v_F \in V(F)$ and $v_H \in V(H)$, then the coalescence $(F \cdot H)(v_F, v_H : v)$ of $F$ and $H$ via $v_F$ and $v_H$, is the graph obtained from the union of $F$ and $H$ by identifying $v_F$ and $v_H$ in a vertex labeled $v$. If $F$ and $H$ are graphs with exactly one vertex in common, say $x$, then the coalescence $(F \cdot H)(x)$ of $F$ and $H$ via $x$ is the union of $F$ and $H$.

Let $Y$ be a finite set of integers which has positive as well as non-positive elements. Denote by $P(Y)$ the collection of all subsets of $Y$. Given a graph $G$, for a $Y$-valued function $f : V(G) \rightarrow Y$ and a subset $S \subseteq V(G)$ we define $f(S) = \sum_{v \in S} f(v)$. The weight of $f$ is $f(V(G))$. A $Y$-valued Roman dominating function on a graph $G$ is a function $f : V(G) \rightarrow Y$ satisfying the conditions: (a) $f(N_G[v]) \geq 1$ for each $v \in V(G)$, and (b) if $v \in V(G)$ and $f(v) \leq 0$, then there is $u_v \in N_G(v)$ with $f(u_v) = \max\{k \mid k \in Y\}$. For a $Y$-valued Roman dominating function $f$ on a graph $G$, where $Y = \{r_1, r_2, \ldots, r_k\}$ and $r_1 < r_2 < \cdots < r_k$, let $V_f^{j} = \{v \in V(G) \mid f(v) = r_j\}$ for $i = 1, \ldots, k$. Since these $k$ sets determine $f$, we can equivalently write $f = (V_f^{r_1}; V_f^{r_2}; \ldots; V_f^{r_k})$. If $f$ is $Y$-valued Roman dominating function on a graph $G$ and $H$ is a subgraph of $G$, then we denote the restriction of $f$ on $H$ by $f|_H$. The $Y$-Roman domination number of a graph $G$, denoted $\gamma_{YR}(G)$, is defined to be the minimum weight of a $Y$-valued dominating function on $G$. As examples, let us mention: (a) the domination number $\gamma(G) \equiv \gamma_{\{0,1\}}(G)$, (b) the minus domination number [6], where $Y = \{-1, 0, 1\}$, (c) the signed domination number [5], where $Y = \{-1, 1\}$, (d) the Roman domination number $\gamma_R(G) \equiv \gamma_{\{0,1,2\}}(G)$ [4], and (e) the signed Roman domination number [1], where $Y = \{-1, 1, 2\}$. A $Y$-valued Roman dominating function $f$ on $G$ with weight $\gamma_{YR}(G)$ is called a $\gamma_{YR}$-function on $G$.

Now we introduce a new partition of a vertex set of a graph, which plays a key role in the paper. In determining this partition, all $\gamma_{YR}$-functions of a graph are necessary. For each $X \in P(Y)$ we define the set $V^X(G)$ as consisting of all $v \in V(G)$ with $\{f(v) \mid f \text{ is a } \gamma_{YR}\text{-function on } G\} = X$. Then all members of the family $(V^X(G))_{X \in P(Y)}$ clearly form a partition of $V(G)$. We call this partition the $\gamma_{YR}$-partition of $G$.

Fricke et al. [7] in 2002 began the study of graphs, which are excellent with respect to various graph parameters. Let us concentrate here on the parameter $\gamma_{YR}$. A vertex $v \in V(G)$ is said to be (a) $\gamma_{YR}$-good, if $h(v) \geq 1$ for some $\gamma_{YR}$-function $h$ on $G$, and (b) $\gamma_{YR}$-bad otherwise. A graph $G$ is said to be $\gamma_{YR}$-excellent if all vertices of $G$ are $\gamma_{YR}$-good. Any vertex-transitive graph is $\gamma_{YR}$-excellent. Note that when $\gamma_{YR} \equiv \gamma$, the set of all $\gamma$-good and the set of all $\gamma$-bad vertices of a graph $G$ form the $\gamma$-partition of $G$. For further results on this topic see e.g. [2, 10–15].

In this paper we begin an investigation of $\gamma_{YR}$-excellent graphs in the case when $Y = \{0, 1, 2\}$. In what follows we shall write $\gamma_R$ instead of $\gamma_{\{0,1,2\}}$, and we shall abbreviate a $\{0,1,2\}$-valued Roman dominating function to an RD-function. Let us describe all members of the $\gamma_R$-partition of any graph $G$ (we write $V^i(G)$, $V^{ij}(G)$ and $V^{ijk}(G)$ instead of $V^{\{i\}}(G), V^{\{i,j\}}(G)$ and $V^{\{i,j,k\}}(G)$, respectively).

(i) $V^{i}(G) = \{x \in V(G) \mid f(x) = i \}$ for each $\gamma_R$-function $f$ on $G$, $i = 1, 2, 3;$
Denote by $R_{n,k}$ the family of all mutually non-isomorphic $n$-order $\gamma_R$-excellent connected graphs having the Roman domination number equal to $k$. With the family $G_{n,k}$, we associate the poset $\mathbb{R}E_{n,k} = (G_{n,k}, \prec)$ with the order $\prec$ given by $H_1 \prec H_2$ if and only if $H_2$ has a spanning subgraph which is isomorphic to $H_1$ (see [16] for terminology on posets). Remark 1 shows that all maximal elements of $\mathbb{R}E_{n,k}$ are in $R_{CEA}$. Here we concentrate on the set of all minimal elements of $\mathbb{R}E_{n,k}$. Clearly a graph $H \in G_{n,k}$ is a minimal element of $\mathbb{R}E_{n,k}$ if and only if for each $e \in E(H)$ at
least one of the following holds: (a) $H - e$ is not connected, (b) $\gamma_R(H) \neq \gamma_R(H - e)$, and (c) $H - e$ is not $\gamma_R$-excellent. All trees in $G_{n,k}$ are obviously minimal elements of $\mathbb{R}E_{n,k}$.

The remainder of this paper is organized as follows. In Section 2, we formulate our main result, namely, a constructive characterization of $\gamma_R$-excellent trees. We present a proof of this result in Sections 3 and 4. Applications of our main result are given in Sections 5 and 6. We conclude in Section 7 with some open problems.

We end this section with the following useful result.

Lemma 3. ([4]) Let $f = (V_0^f;V_1^f;V_2^f)$ be any $\gamma_R$-function on a graph $G$. Then each component of $\langle V_0^f \rangle$ has order at most 2 and no edge of $G$ joins $V_1^f$ and $V_2^f$.

In most cases Lemmas 1, 2 and 3 will be used in the sequel without specific reference.

2. The main result

In this section, we present a constructive characterization of $\gamma_R$-excellent trees using labelings. We define a labeling of a tree $T$ as a function $S : V(T) \to \{A, B, C, D\}$. A labeled tree is denoted by a pair $(T, S)$. The label of a vertex $v$ is also called its status, denoted $sta_T(v : S)$ or $sta_T(v)$ if the labeling $S$ is clear from context. We denote the sets of vertices of status $A, B, C$ and $D$ by $S_A(T), S_B(T), S_C(T)$ and $S_D(T)$, respectively. In all figures in this paper we use $\bullet$ for a vertex of status $A$, $\bullet$ for a vertex of status $B$, $\bullet$ for a vertex of status $C$, and $\circ$ for a vertex of status $D$. If $H$ is a subgraph of $T$, then we denote the restriction of $S$ on $H$ by $S|_H$.

Figure 1. All trees with $|L_B \cup L_C| \leq 2$.

To state a characterization of $\gamma_R$-excellent trees, we introduce four types of operations. Let $\mathcal{T}$ be the family of labeled trees $(T, S)$ that can be obtained from a
sequence of labeled trees $\tau : (T^1, S^1), \ldots, (T^j, S^j), (j \geq 1)$, such that $(T^1, S^1)$ is in $\{(H_1, I^1), \ldots, (H_5, I^5)\}$ (see Figure 1) and $(T, S) = (T^i, S^i)$, and, if $j \geq 2$, $(T^{i+1}, S^{i+1})$ can be obtained recursively from $(T^i, S^i)$ by one of the operations $O_1, O_2, O_3$ and $O_4$ listed below; in this case $\tau$ is said to be a $\mathcal{T}$-sequence of $T$. When the context is clear we shall write $T \in \mathcal{T}$ instead of $(T, S) \in \mathcal{T}$.

\begin{figure}[h]
\centering
\begin{tikzpicture}

\begin{scope}[every node/.style={draw, circle, fill=black, inner sep=1pt}]
\node (v1) at (0,0) {$x$};
\node (v2) at (1,1) {$x$};
\node (v3) at (2,0) {$x$};
\node (v4) at (3,1) {$x$};
\end{scope}

\begin{scope}[every edge/.style={thick}]
\path (v1) edge (v2);
\path (v2) edge (v3);
\path (v3) edge (v4);
\end{scope}

\node at (-1.5,0) {$(F_1, J^1)$};
\node at (1.5,0) {$(F_2, J^2)$};
\node at (4.5,0) {$(F_3, J^3)$};
\node at (2.5,1) {$(F_4, J^4)$};
\end{tikzpicture}
\caption{$(F, J)$-graphs}
\end{figure}

**Operation $O_1$.** The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F, J) \in \{(F_1, J^1), (F_2, J^2), (F_3, J^3)\}$ (see Figure 2) by adding the edge $ux$, where $u \in V(T_i)$, $x \in V(F)$ and $\text{stat}_{T_i}(u) = \text{stat}_F(x) = C$.

**Operation $O_2$.** The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F_4, J^4)$ (see Figure 2) by adding the edge $ux$, where $u \in V(T_i)$, $x \in V(F_4)$, $\text{stat}_{T_i}(u) = D$, and $\text{stat}_{F_4}(x) = C$.

**Operation $O_3$.** The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(H_k, I^k)$, $k \in \{2, 3, \ldots, 7\}$ (see Figure 1), in such a way that $T^{i+1} = (T^i \cdot H_k)(u, v : u)$, where $\text{stat}_{T^i}(u) = \text{stat}_{H_k}(v) = A$, and $\text{stat}_{T^{i+1}}(u) = A$.

**Operation $O_4$.** The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(H_k, I^k)$, $k \in \{3, 4, 6\}$ (see Figure 1), in such a way that $T^{i+1} = (T^i \cdot H_k)(u, v : u)$, where $\text{stat}_{T^i}(u) = D$, $\text{stat}_{H_k}(v) = A$, and $\text{stat}_{T^{i+1}}(u) = D$.

Remark that if $y \in V(T^i)$ and $i \leq k \leq j$, then $\text{stat}_{T^i}(y) = \text{stat}_{T^j}(y)$. Now we are prepared to state the main result.

**Theorem 1.** Let $T$ be a tree of order at least 2. Then $T$ is $\gamma_R$-excellent if and only if there is a labeling $S : V(T) \to \{A, B, C, D\}$ such that $(T, S)$ is in $\mathcal{T}$. Moreover, if $(T, S) \in \mathcal{T}$ then

\begin{align*}
(\mathcal{P}_1) & \quad S_B(T) = \{x \in V^{02}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{02}(T)| = 1\}, \quad S_A(T) = V^{01}(T), \\
S_D(T) & \quad = V^{012}(T), \quad \text{and } S_C(T) = V^{02}(T) - S_B(T).
\end{align*}

3. Preparation for the proof of Theorem 1

3.1. Coalescence

We shall concentrate on the coalescence of two graphs via a vertex in $V^{01}$ and derive the properties which will be needed for the proof of our main result.
Proposition 1. Let $G = (G_1 \cdot G_2)(x)$ be a connected graph and $x \in V^{01}(G)$. Then the following holds.

(i) If $f$ is a $\gamma_R$-function on $G$ and $f(x) = 1$, then $f|_{G_i}$ is a $\gamma_R$-function on $G_i$, and $f|_{G_i-x}$ is a $\gamma_R$-function on $G_i - x$, $i = 1, 2$.

(ii) $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) If $h$ is a $\gamma_R$-function on $G$ and $h(x) = 0$, then exactly one of the following holds:

(iii.1) $h|_{G_1}$ is a $\gamma_R$-function on $G_1$, $h|_{G_2-x}$ is a $\gamma_R$-function on $G_2 - x$, and $h|_{G_2}$ is no RD-function on $G_2$.

(iii.2) $h|_{G_1-x}$ is a $\gamma_R$-function on $G_1 - x$, $h|_{G_1}$ is no RD-function on $G_1$, and $h|_{G_2}$ is a $\gamma_R$-function on $G_2$.

(iv) Either $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$ or $\{x\} = V^{01}(G_i) \cap V^1(G_j)$, where $\{i, j\} = \{1, 2\}$.

Proof. (i) and (ii): Since $f(x) = 1$, $f|_{G_i}$ is an RD-function on $G_i$, and $f|_{G_i-x}$ is an RD-function on $G_i - x$, $i = 1, 2$. Assume $g_1$ is a $\gamma_R$-function on $G_1$ with $g_1(V(G_1)) < f|_{G_1}(V(G_1))$. Define an RD-function $f'$ as follows: $f'(u) = g_1(u)$ for all $u \in V(G_1)$ and $f'(u) = f(u)$ when $u \in V(G_2 - x)$. Then $f'(V(G)) = g_1(V(G_1)) + f|_{G_2-x}(V(G_2 - x)) < f(V(G))$, a contradiction. Thus, $f|_{G_i}$ is a $\gamma_R$-function on $G_i$, $i = 1, 2$. Now, Lemma 1 implies that $f|_{G_i-x}$ is a $\gamma_R$-function on $G_i - x$, $i = 1, 2$. Hence $\gamma_R(G) = f|_{G_1}(V(G_1)) + f|_{G_2}(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) First note that $h(x) = 0$ implies $h|_{G_i}$ is an RD-function on $G_i$ for some $i \in \{1, 2\}$, say $i = 1$. If $h|_{G_2}$ is an RD-function on $G_2$ then $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction with (ii). Thus, $h|_{G_2-x}$ is an RD-function on $G_2 - x$. Now we have $\gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G) = h(V(G)) = h|_{G_1}(V(G_1)) + h|_{G_2-x}(V(G_2 - x)) \geq \gamma_R(G_1) + (\gamma_R(G_2) - 1)$. Hence $h|_{G_1}$ is a $\gamma_R$-function on $G_1$ and $h|_{G_2-x}$ is a $\gamma_R$-function on $G_2 - x$.

(iv) Let $f_1$ be a $\gamma_R$-function on $G_1$. Assume first that $f_1(x) = 2$. Define an RD-function $g$ on $G$ as follows: $g(u) = f_1(u)$ when $u \in V(G_1)$ and $g(u) = f(u)$ when $u \in V(G_2 - x)$, where $f$ is defined as in (i). The weight of $g$ is $\gamma_R(G_1) + (\gamma_R(G_2) + 1) - 2 = \gamma_R(G)$. But $g(x) = 2$ and $x \in V^{01}(G)$, a contradiction. Thus $f_1(x) \neq 2$. Now by (i) we have $x \in V^1(G_i) \cup V^{01}(G_i)$, $i = 1, 2$, and by (iii), $x \in V^{01}(G_j)$ for some $j \in \{1, 2\}$.

Proposition 2. Let $G = (G_1 \cdot G_2)(x)$, where $G_1$ and $G_2$ are connected graphs and $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$.

(i) If $f_i$ is a $\gamma_R$-function on $G_i$ with $f_i(x) = 1$, $i = 1, 2$, then the function $f : V(G) \rightarrow \{0, 1, 2\}$ with $f|_{G_i} = f_i$, $i = 1, 2$, is a $\gamma_R$-function on $G$.

(ii) $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) Let $V_R = \{V^0, V^1, V^2, V^{01}, V^{02}, V^{12}, V^{012}\}$. Then for any $A \in V_R$, $A(G_1) \cup A(G_2) = A(G)$. 

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Proof. (i) and (ii): Note that $f$ is an RD-function on $G$ and $\gamma_R(G) \leq f(V(G)) = f_1(V(G_1)) + f_2(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$. Now let $h$ be any $\gamma_R$-function on $G$.

Case 1: $h(x) \geq 1$. Then $h|_{G_i}$ is an RD-function on $G_i$, $i = 1, 2$. If $h(x) = 2$ then since $x \in V^{01}(G_1) \cap V^{01}(G_2)$, $h|_{G_i}$ is no $\gamma_R$-function on $G_i$, $i = 1, 2$. Hence $\gamma_R(G) \geq (\gamma_R(G_1) + 1) + (\gamma_R(G_2) + 1) - h(x) = \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. If $h(x) = 1$ then $\gamma_R(G) = h(V(G)) = h(V(G_1)) + h(V(G_2)) - h(x) \geq \gamma_R(G_1) + \gamma_R(G_2) - 1$. Thus $h(x) = 1$, $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and $f$ is a $\gamma_R$-function on $G$.

Case 2: $h(x) = 0$. Then at least one of $h|_{G_1}$ and $h|_{G_2}$ is an RD-function, say the first. If $h|_{G_2}$ is an RD-function on $G_2$ then $h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. Hence $h|_{G_2-x}$ is a $\gamma_R$-function on $G_2 - x$. But then $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2) - x \geq \gamma_R(G_1) + \gamma_R(G_2) - 1 \geq \gamma_R(G)$.

Thus, (i) and (ii) hold.

(iii): Let $g_1$ be a $\gamma_R$-function on $G_1$ with $g_1(x) = 0$, and $g_2$ a $\gamma_R$-function on $G_2 - x$. Then the RD-function $g$ on $G$ for which $g|_{G_1} = g_1$ and $g|_{G_2-x} = g_2$ has weight $g_1(V(G_1)) + g_2(V(G_2 - x)) = \gamma_R(G_1) + \gamma_R(G_2 - x) = \gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G)$. Hence by (i), $x \in V^{01}(G) \cup V^{012}(G)$. However, by Case 1 it follows that $h(x) \neq 2$ for any $\gamma_R$-function $h$ on $G$. Thus $x \in V^{01}(G)$.

Let $g \in V(G_1 - x)$, $l_1$ a $\gamma_R$-function on $G_1$, and $h$ a $\gamma_R$-function on $G$. We shall prove that the following holds.

Claim 4.1 There are a $\gamma_R$-function $l$ on $G$, and a $\gamma_R$-function $h_1$ on $G_1$ such that $l(y) = l_1(y)$ and $h_1(y) = h(y)$.

Define an RD-function $l$ on $G$ as $l|_{G_1} = l_1$ and $l|_{G_2-x} = l_2$, where $l_2$ is a $\gamma_R$-function on $G_2 - x$. Since $l(V(G)) = \gamma_R(G_1) + \gamma_R(G_2 - x) = \gamma_R(G)$, $l$ is a $\gamma_R$-function on $G$ and $l(y) = l_1(y)$.

Assume now that there is no $\gamma_R$-function $h_1$ on $G_1$ with $h_1(y) = h(y)$. Proposition 1 implies that, $h|_{G_1-x}$ is a $\gamma_R$-function on $G_1 - x$. But then the function $h': V(G_1) \rightarrow \{0, 1, 2\}$ defined as $h'(u) = 1$ when $u = x$ and $h'(u) = h|_{G_1}(u)$ otherwise, is a $\gamma_R$-function on $G_1$ with $h'(y) = h|_{G_1}(y)$, a contradiction.

By Claim 4.1 and since $x \in V^{01}(G)$, $A(G_1) = A(G) \cap V(G_1)$ for any $A \in V_R$. By symmetry, $A(G_2) = A(G) \cap V(G_2)$. Therefore $A(G_1) \cup A(G_2) = A(G)$ for any $A \in V_R$. □

Lemma 4. Let $G = (G_1 \cdot G_2)(x)$, where $G_1$ and $G_2$ are connected graphs and $\{x\} = V^{012}(G_1) \cap V^{01}(G_2)$. Then $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and $x \in V^{012}(G)$.

Proof. Let $f_i$ be a $\gamma_R$-function on $G_i$ with $f_i(x) = 1$, $i = 1, 2$. Then the function $f$ defined as $f|_{G_i} = f_i$ is an RD-function on $G_i$, $i = 1, 2$. Hence $\gamma_R(G) \leq f(V(G)) = \gamma_R(G_1) + \gamma_R(G_2) - 1$. Let now $h$ be any $\gamma_R$-function on $G$.

Case 1: $h(x) = 2$. 

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Since \( x \in V^{012}(G_1) \cap V^{01}(G_2) \), \( h|_{G_1} \) is a \( \gamma_{R}\)-function on \( G_1 \) and \( h|_{G_2} \) is an RD-function on \( G_2 \) of weight more than \( \gamma_R(G_2) \). Hence \( \gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + (\gamma_R(G_2) + 1) - h(x) \). Thus \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \).

**Case 2:** \( h(x) = 1 \).
Then obviously \( h|_{G_1} \) and \( h|_{G_2} \) are \( \gamma_{R}\)-functions. Hence \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \).

**Case 3:** \( h(x) = 0 \).
Hence at least one of \( h|_{G_1} \) and \( h|_{G_2} \) is a \( \gamma_{R}\)-function. If both \( h|_{G_1} \) and \( h|_{G_2} \) are \( \gamma_{R}\)-functions, then \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) \), a contradiction. Hence either \( h|_{G_1} \) and \( h|_{G_2-x} \) are \( \gamma_{R}\)-functions, or \( h|_{G_1-x} \) and \( h|_{G_2} \) are \( \gamma_{R}\)-functions. Since \( \{x\} = V^{012}(G_1) \cap V^{01}(G_2) \), in both cases we have \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \).
Thus, \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \) and \( x \in V^{012}(G) \).

\[ \square \]

### 3.2. Three lemmas for trees

**Lemma 5.** Let \( T \) be a \( \gamma_{R}\)-excellent tree of order at least 2. Then \( V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T) \).

**Proof.** Let \( x \in V(T) \), \( y \in N(x) \) and \( f \) a \( \gamma_{R}\)-function on \( T \). Suppose \( x \in V^1(T) \). If \( f(y) = 1 \), then the RD-function \( g \) on \( T \) defined as \( g(x) = 2 \), \( g(y) = 0 \) and \( g(u) = f(u) \) for all \( u \in V(T) \) - \( \{x, y\} \) is a \( \gamma_{R}\)-function on \( T \), a contradiction. But then \( N(x) \subseteq V^0(T) \), which is impossible.

Suppose now \( x \in V^2(T) \cup V^{12}(T) \). Hence \( x \) is not a leaf. Choose a \( \gamma_{R}\)-function \( h \) on \( T \) such that (a) \( h(x) = 2 \), and (b) \( k = |epn[x, V^k_2]| \) to be as small as possible. Let \( epn[x, V^2_2] = \{y_1, y_2, \ldots, y_k\} \) and denote by \( T_i \) the connected component of \( T - x \), which contains \( y_i \). Hence \( h(y_i) = 0 \) for all \( i \leq k \). Since \( T \) is \( \gamma_{R}\)-excellent, there is a \( \gamma_{R}\)-function \( f_k \) on \( T \) with \( f_k(y_k) \neq 0 \). Since \( x \in V^2(T) \cup V^{12}(T) \), \( f_k(x) \neq 0 \). If \( f_k(y_k) = 1 \) then \( f_k(x) = 1 \), which easily implies \( x \in V^{012}(T) \), a contradiction. Hence \( f_k(y_k) = f_k(x) = 2 \). Define a \( \gamma_{R}\)-function \( l \) on \( T \) as \( l|_{T_k} = f_k|_{T_k} \) and \( l(u) = h(u) \) for all \( u \in V(T) - V(T_k) \). But \( |epn[x, V^k_2]| < k \), a contradiction with the choice of \( h \). Thus \( V^1(T) \cup V^2(T) \cup V^{12}(T) \) is empty, and the required follows. \[ \square \]

**Lemma 6.** Let \( T \) be a tree and \( V^-(T) \) is not empty. Then each component of \( \langle V^-(T) \rangle \) is either \( K_1 \) or \( K_2 \).

**Proof.** Assume that \( P : x_1, x_2, x_3 \) is a path in \( T \) and \( x_1, x_2, x_3 \in V^-(T) \). Then there is a \( \gamma_{R}\)-function \( f_i \) on \( T \) with \( f_i(x_i) = 1 \), \( i = 1, 2, 3 \) (by Lemma 1). Denote by \( T_j \) the connected component of \( T - x_2 \cup x_j \) that contains \( x_j \), \( j = 1, 3 \). Then \( f_2|_{T_j} \) and \( f_3|_{T_j} \) are \( \gamma_{R}\)-functions on \( T_j \), \( j = 1, 3 \). Now define a \( \gamma_{R}\)-function \( h \) on \( T \) such that \( h|_{T_j} = f_j|_{T_j} \), \( j = 1, 3 \), and \( h(u) = f_2(u) \) when \( u \in V(T) - (V(T_1) \cup V(T_3)) \). But \( h(x_1) = h(x_2) = h(x_3) = 1 \), a contradiction. \[ \square \]

**Lemma 7.** Let \( T \) be a \( \gamma_{R}\)-excellent tree of order at least 2.
(i) If $x \in V^{012}(T)$, then $x$ is adjacent to exactly one vertex in $V^-(T)$, say $y_1$, and $y_1 \in V^{012}(T)$.

(ii) Let $x \in V^{02}(T)$. If $\deg(x) \geq 3$ then $x$ has exactly 2 neighbors in $V^-(T)$. If $\deg(x) = 2$ then either $N_T(x) \subseteq V^{012}(T)$ or there is a path $u, x, y, z$ in $T$ such that $u, z \in V^{01}(T)$, $y \in V^{02}(T)$ and $\deg(y) = 2$.

(iii) $V^{01}(T)$ is either empty or independent.

Proof. Let $x \in V^{012}(T) \cup V^{02}(T)$ and $N(x) = \{y_1, y_2, \ldots, y_k\}$. If $x$ is a leaf, then clearly $x, y_1 \in V^{012}(T)$. So, let $r \geq 2$. Denote by $T_i$ the connected component of $T - x$ which contains $y_1, i \geq 1$. Choose a $\gamma_R$-function $h$ on $T$ such that (a) $h(x) = 2$, and (b) $k = |epm[x, V^2_1]|$ to be as small as possible. Let without loss of generality $epm[x, V^2_1] = \{y_1, y_2, \ldots, y_k\}$. By the definition of $h$ it immediately follows that (c) $h|_{T_j}$ is a $\gamma_R$-function on $T_j$ for all $j \geq k + 1$, (d) for each $i \in \{1, \ldots, k\}$, $h|_{T_i}$ is no RD-function on $T_i$, and (e) $h|_{T_i \setminus y_i}$ is a $\gamma_R$-function on $T_i - y_i$, $i \in \{1, \ldots, k\}$. Hence $\gamma_R(T_i) \leq \gamma_R(T_i - y_i) + 1$ for all $i \in \{1, \ldots, k\}$. Assume that the equality does not hold for some $i \leq k$. Define an RD-function $h_i$ on $T$ as follows: $h_i(u) = h(u)$ when $u \in V(T) - V(T_i)$ and $h_i|_{T_i} = h'_i$, where $h'_i$ is some $\gamma_R$-function on $T_i$. But then either $h_i$ has weight less than $\gamma_R(T)$ or $h_i$ is a $\gamma_R$-function on $T$ with $epm[x, V^2_i] = epm[x, V^2_i] - \{y_i\}$. In both cases we have a contradiction. Thus $\gamma_R(T_i) = \gamma_R(T_i - y_i) + 1$ for all $i \in \{1, \ldots, k\}$. Therefore $\gamma_R(T) = h(V(T)) = 2 + \sum_{i=1}^k(\gamma_R(T_i) - 1) + \sum_{j=k+1}^{r} \gamma_R(T_j) = 2 - k + \sum_{i=1}^k \gamma_R(T_i) = 2 - k + \gamma_R(T - x)$. Thus $\gamma_R(T) = 2 - k + \gamma_R(T - x)$.

(i) Since $\gamma_R(T - x) + 1 = \gamma_R(T), k = 1$. We already know that $h|_{T_j}$ is a $\gamma_R$-function on $T_j$, $j \geq 2$. Assume that $y_j \in V^{012}(T) \cup V^{01}(T)$ for some $j \geq 2$. Then there is a $\gamma_R$-function $l$ on $T$ with $l(y_j) = 1$. Clearly $l|_{T_j}$ is a $\gamma_R$-function on $T_j$. Now define a $\gamma_R$-function $h''$ on $T$ as follows: $h''(u) = h(u)$ when $u \in V(T) - V(T_j)$ and $h''|_{T_j} = l|_{T_j}$. But then $h''(x) = h''(y_j) = 1$ and $xy_j \in E(G)$, which is impossible. Thus, $y_2, y_3, \ldots, y_r \in V^{02}(T)$. Define now $\gamma_R$-functions $h_1$ and $h_2$ on $T$ as follows: $h_1(u) = h_2(u) = h(u)$ for all $u \in V(T) - \{x, y_1\}$, $h_1(x) = h_1(y_1) = 1$, $h_2(x) = 0$ and $h_2(y_1) = 2$. Thus $y_1 \in V^{01}(T)$.

(ii) Since $\gamma_R(T - x) = \gamma_R(T), k = 2$. Recall that $h|_{T_j}$ is a $\gamma_R$-function on $T_j$, $j \geq 3$, and $\gamma_R(T_i - y_i) = \gamma_R(T_i) - 1$ for $i = 1, 2$. Hence there is a $\gamma_R$-function $f_i$ on $T_i$ with $f_i(y_1) = 1$, $i = 1, 2$. Suppose first that $r \geq 3$. As in the proof of (i), we obtain $y_3, \ldots, y_r \in V^{02}(T)$. Hence there is a $\gamma_R$-function $g$ on $T$ such that $g(y_3) = 2$. By the choice of $h$, $g(x) = 0$. Then $g|_{T_1}$ is a $\gamma_R$-function on $T_i$, $i = 1, 2$. Define now a $\gamma_R$-function $g'$ on $T$ as $g'|_{T_i} = f_i$, $i = 1, 2$, and $g'(u) = g(u)$ when $u \in V(T) - (V(T_1) \cup V(T_2))$. Since $g'(y_1) = g'(y_2) = 1$, $y_1, y_2 \in V^-(T)$.

So, let $r = 2$ and let $f$ be a $\gamma_R$-function on $T$ with $f(x) = 0$. Then there is $y_s$ such that $f(y_s) = 2$, say $s = 2$. Hence $y_2 \in V^{02}(T) \cup V^{012}(T)$ and $f|_{T_1}$ is a $\gamma_R$-function on $T_1$. Define the $\gamma_R$-function $l$ on $T$ as $l|_{T_1} = f_1$ and $l(u) = f(u)$ when $u \in V(T) - V(T_1)$. Since $l(y_1) = 1, y_1 \in V^{01}(T) \cup V^{012}(T)$. 

Assume first that $y_1 \in V^{012}(T)$. Then there is a $\gamma_R$-function $f'$ on $T$ with $f'(y_1) = 2$. Since $x \in V^{02}(T)$ and $deg(x) = 2$, $f'(x) = 0$. Hence $f'|_{T_2}$ is a $\gamma_R$-function on $T_2$. But then we can choose $f'$ so that $f'|_{T_2} = f_2$. Thus $y_2 \in V^{012}(T)$. So let $y_1 \in V^{01}(T)$ and suppose $y_2 \in V^{012}(T)$. Then there is a $\gamma_R$-function $f''$ on $T$ with $f''(y_2) = 1$. Since $x \in V^{02}(T)$, $f''(x) = 0$ and $f''(y_1) = 2$, a contradiction. Thus, if $y_1 \in V^{01}(T)$ then $y_2 \in V^{02}(T)$.

Finally, let us consider a path $y_1, x, y_2, z$ in $T$, where $y_1 \in V^{01}(T)$, $x, y_2 \in V^{02}(T)$ and $deg(x) = 2$. Assume to the contrary that $N(y_2) = \{z_1, z_2, \ldots, z_s \in x\}$ with $s \geq 3$. Denote by $T_{y}$ the connected component of $T - y_2$ that contains $z_p$, $p = 1, 2, \ldots, s$. By applying results proved above for $x \in V^{02}(T)$ with $deg(x) \geq 3$ to $y_2$, we obtain that (a) $y_2$ has exactly 2 neighbors in $V^-(T)$, say, without loss of generality, $z_1, z_2 \in V^-(T)$, and (b) $\gamma_R(T_{z_1} - z_1) = \gamma_R(T_{z_2} - z_1) - 1$, where $i = 1, 2$. Recall now that: $h(x) = 2$, $h|_{T_i}$ is no RD-function on $T_i$ and $h|_{T_i - y_i}$ is a $\gamma_R$-function on $T_i - y_i$, $i = 1, 2$. Hence $h(y_1) = h(y_2) = 0$ and $h|_{T_{y}}$ is a $\gamma_R$-function on $T_{y}$, $j \leq s - 1$. Since $\gamma_R(T_{z_i} - z_i) = \gamma_R(T_{z_i}) - 1$, $i = 1, 2$, additionally we can choose $h$ so that $h(z_1) = h(z_2) = 1$. But then the function $h_1$ defined as $h_1(u) = h(u)$ when $u \in V(T) - \{y_1, x, y_2, z_1, z_2\}$ and $h_1(y_1) = h_1(x) = 1$, $h_1(y_2) = 2$, $h_1(z_1) = h(z_2) = 0$ is a $\gamma_R$-function on $T$. Now $h_1(x) = 1$, $h_1(y_2) = 2$ and $xy_2 \in E(G)$ lead to a contradiction. Thus, $N(y_2) = \{x, z\}$. Suppose $z \notin V^{01}(T)$. Then there is a $\gamma_R$-function $h_4$ on $T$ with $h_4(z) = 2$. If $h_4(y_2) = 2$, then $h_4(x) = 0$ and the function $h_5$ on $T$ defined as $h_5(x) = h_5(y_2) = 1$ and $h_5(u) = h_4(u)$ otherwise, is a $\gamma_R$-function on $T$, a contradiction. Hence $h_4(y_2) = 0$ and since $y_1 \in V^{01}(T)$, $h_4(x) = 2$ and $h_4(y_1) = 0$. But then the function $h_6$ on $T$ defined as $h_6(x) = h_6(y_1) = 1$ and $h_6(u) = h_4(u)$ otherwise, is a $\gamma_R$-function on $T$, a contradiction. Therefore $z \in V^{01}(T)$, and we are done.

(iii) Assume that $u_1, u_2 \in V^{01}(T)$ are adjacent. Let $T_{u_{i}}$ be the component of $T - u_1u_2$ that contains $u_{i}$, $i = 1, 2$. Let $g_i$ be a $\gamma_R$-function on $T$ with $g_i(u_i) = 1$, $i = 1, 2$. Hence $g_i(T_{u_{i}})$ is a $\gamma_R$-function on $T_{u_{i}}$, $i, j = 1, 2$. Thus $\gamma_R(T) = \gamma_R(T_{u_{i}}) + \gamma_R(T_{u_{j}})$.

Define now a $\gamma_R$-function $g_3$ on $T$ as $g_3|_{T_{i}} = g_1|_{T_{i}}$, $i = 1, 2$. But then a function $g_4$ defined as $g_4(u) = g_3(u)$ when $u \in V(T) - \{u_1, u_2\}$, $g_4(u_1) = 2$ and $g_4(u_2) = 0$ is a $\gamma_R$-function on $T$, contradicting $u_1 \in V^{01}(T)$. Thus $V^{01}(T)$ is independent. □

4. Proof of the main result

Proof of Theorem 1. Let $T$ be a $\gamma_R$-excellent tree. First, we shall prove the following statement.

$\mathcal{P}_2$. There is a labeling $L : V(T) \rightarrow \{A, B, C, D\}$ such that (a) $L_A(T)$ is either empty or independent, (b) each component of $\langle L_B(T) \rangle$ and $\langle L_D(T) \rangle$ is isomorphic to $K_2$, (c) each element of $L_B(T)$ has degree 2 and it is adjacent to exactly one vertex in $L_A(T)$, (d) each vertex $v$ in $L_C(T)$ has exactly 2 neighbors in $L_A(T) \cup L_D(T)$, and if $deg(v) = 2$ then both neighbors of $v$ are in $L_D(T)$.

By Lemma 5 we know that $V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T)$. Define a labeling $L : V(T) \rightarrow \{A, B, C, D\}$ by $L_A(T) = V^{01}(T)$, $L_D(T) = V^{012}(T)$, $L_B(T) = \{x \in$
\( V^{02}(T) \mid \deg(x) = 2 \) and \(|N(x) \cap V^{02}(T)| = 1\), and \( L_C(T) = V^{02}(T) - L_B(T) \). The validity of \((P_2)\) immediately follows by Lemma 7.

Denote by \( \mathcal{T} \) the family of all labeled, as in \((P_2)\), trees \( T \). We shall show that if \( (T, L) \in \mathcal{T} \) then \( (T, L) \in \mathcal{T} \).

(I) Proof of \( (T, L) \in \mathcal{T} \Rightarrow (T, L) \in \mathcal{T} \).

Let \( (T, L) \in \mathcal{T} \). The following claim is immediate.

**Claim 1.1**

(i) Each leaf of \( T \) is in \( L_A(T) \cup L_D(T) \).

(ii) If \( v \) is a support vertex of \( T \), then \( v \) is adjacent to at most 2 leaves.

(iii) If \( u_1 \) and \( u_2 \) are leaves adjacent to the same support vertex, then \( u_1, u_2 \in L_A(T) \).

We now proceed by induction on \( k = |L_B \cup L_C| \). The base case, \( k \leq 2 \), is an immediate consequence of the following easy claim, the proof of which is omitted.

**Claim 1.2** (see Fig.1)

(i) If \( k = 0 \) then \( (T, L) = (H_1, I^1) \).

(ii) If \( k = 1 \) then \( (T, L) \) is obtained from \((H_1, I_1)\) by operation \( O_2 \), i.e. \( (T, L) = (H_{11}, I^{11}) \).

(iii) If \( k = 2 \) then either \( (T, L) \) is \((H_r, I^r)\) with \( r \in \{2, 3, 4, 5\} \), or \( (T, L) \) is obtained from \((H_{11}, I^{11})\) by operation \( O_1 \) or by operation \( O_2 \) (see the graphs \((H_s, I^s)\) where \( s \in \{6, 7, 8, 9, 10\} \).

Let \( k \geq 3 \) and suppose that each tree \((H, L') \in \mathcal{T} \) with \( |L_B(H) \cup L_C(H)| < k \) is in \( \mathcal{T} \). Let now \((T, L) \in \mathcal{T} \) and \( k = |L_B(T) \cup L_C(T)| \). To prove the required result, it suffices to show that \( T \) has a subtree, say \( U \), such that \((U, L|_U) \in \mathcal{T} \), and \((T, L) \) is obtained from \((U, L|_U)\) by one of operations \( O_1 \), \( O_2 \), \( O_3 \) and \( O_4 \). Consider any diametral path \( P : x_1, x_2, \ldots, x_n \) in \( T \). Clearly \( x_1 \) is a leaf. Denote by \( x_1^1, x_1^2, \ldots \) all neighbors of \( x_i \), which do not belong to \( P \), \( 2 \leq i \leq n - 1 \).

**Case 1:** \( sta(x_1) = A \) and \( sta(x_2) = B \).

Then \( deg(x_1) = 1 \), \( deg(x_2) = deg(x_3) = 2 \), \( sta(x_3) = B \) and \( sta(x_4) = A \). Thus \( T \) is obtained from \( T - \{x_1, x_2, x_3\} \in \mathcal{T} \) and a copy of \( H_2 \) by operation \( O_3 \) (via \( x_4 \)).

**Case 2:** \( sta(x_1) = A \) and \( sta(x_2) = C \).

Hence \( deg(x_2) \geq 3 \). By the choice of \( P \), \( deg(x_2) = 3 \), \( x_2^1 \) is a leaf, \( sta(x_2^1) = A \) and \( sta(x_3) = C \). If \( deg(x_3) \geq 4 \) then \( T \) is obtained from \( T - \{x_2^1, x_1, x_2\} \in \mathcal{T} \) and a copy of \( F_1 \) by operation \( O_1 \). So, let \( deg(x_3) = 3 \). Assume first that \( sta(x_4) = A \). Then either \( x_3^1 \) is a leaf of status \( A \) or \( x_3^1 \) is a support vertex, \( deg(x_3^1) = 2 \), and both \( x_3^1 \) and its leaf-neighbor have status \( D \). Thus, \( T \) is obtained from \( T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T} \) and a copy of \( H_3 \) or \( H_4 \), respectively, by operation \( O_3 \) (via \( x_4 \)). Finally let \( sta(x_4) = D \). By the choice of \( P \), either \( x_3^1 \) is a leaf of status \( A \) and then \( T \) is obtained from
$T - (N[x_2] \cup \{x_3^1\}) \in \mathcal{T}_1$ and a copy of $H_3$ by operation $O_4$, or $x_3^1$ is a support vertex of degree 2 and both $x_3^1$ and its leaf-neighbor have status $D$, and then $T$ is obtained from $T - \{x_2^1, x_1, x_2\} \in \mathcal{T}_1$ and a copy of $F_1$ by operation $O_1$.

In what follows, let $sta(x_1) = D$. Hence $de(x_2) = 2$, $sta(x_2) = D$ and $sta(x_3) = C$. If $de(x_3) = 2$ then $T$ is obtained from $T - N[x_2] \in \mathcal{T}_1$ and a copy of $F_4$ by operation $O_2$.

**Case 3:** $de(x_3) = 3$ and $sta(x_4) \in \{A, D\}$.
In this case $sta(x_3^1) = C$, $x_3^1$ is a support vertex, $de(x_3^1) = 3$, and the leaf neighbors of $x_3^1$ have status $A$. Now (a) if $sta(x_4) = A$ then $T$ is obtained from $T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1$ and a copy of $H_4$ by operation $O_3$ (via $x_4$), and (b) if $sta(x_4) = D$ then $T$ is obtained from $T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1$ and a copy of $H_4$ by operation $O_4$ (via $x_4$).

**Case 4:** $de(x_3) = 3$, $sta(x_4) = C$ and $sta(x_3^1) = A$.
Hence $x_3^1$ is a leaf. If $de(x_4) = 3$ and $sta(x_5) = sta(x_4^1) = D$, or $de(x_4) \geq 4$, then $T$ is obtained from $T - \{x_1, x_2, x_3, x_3^1\} \in \mathcal{T}_1$ and a copy of $F_2$ by operation $O_1$. So, let $de(x_4) = 3$ and the status of at least one of $x_5$ and $x_4^1$ is $A$. Assume first that $sta(x_4^1) = A$. Hence $x_4^1$ is a leaf (by the choice of $P$). If $sta(x_5) = A$ then $T$ is obtained from a copy of $H_4$ and a tree in $\mathcal{T}_1$ by operation $O_3$ (via $x_5$). If $sta(x_5) = D$ then $T$ is obtained from a copy of $H_4$ and a tree in $\mathcal{T}_1$ by operation $O_4$ (via $x_5$). Second, let $sta(x_4^1) = D$. Hence $sta(x_5) = A$, $de(x_4^1) = 2$ and the status of the leaf-neighbor of $x_4^1$ is $D$. But then $T$ is obtained from a copy of $H_5$ and a tree in $\mathcal{T}_1$ by operation $O_3$ (via $x_5$).

**Case 5:** $de(x_3) = 3$, $sta(x_4) = C$ and $sta(x_3^1) = D$.
Hence $de(x_3^1) = 2$, $x_3^1$ is a support vertex, and the leaf-neighbor of $x_3^1$ has status $D$. If $de(x_4) \geq 4$ or $sta(x_5) = sta(x_4^1) = D$, then $T$ is obtained from $T - N[\{x_2, x_3^1\}] \in \mathcal{T}_1$ and a copy of $F_3$ by operation $O_1$. So, let $de(x_4) = 3$ and at least one of $x_5$ and $x_4^1$ has status $A$. Assume $sta(x_4^1) = A$. Hence $x_4^1$ is a leaf. If $sta(x_5) = A$ then $T$ is obtained from $T - N[\{x_2, x_3^1, x_4^1\}] \in \mathcal{T}_1$ and a copy of $H_6$ by operation $O_3$ (via $x_5$). If $sta(x_5) = D$ then $T$ is obtained from $T - N[\{x_2, x_3, x_4^1\}] \in \mathcal{T}_1$ and a copy of $H_6$ by operation $O_4$ (via $x_5$). Now let $sta(x_4^1) = D$. Hence $sta(x_5) = A$ and then $T$ is obtained from a copy of $H_7$ and a tree in $\mathcal{T}_1$ by operation $O_3$ (via $x_5$).

**Case 6:** $de(x_3) \geq 4$.
Hence $x_3$ has a neighbor, say $y$, such that $y \neq x_4$ and $sta(y) = C$. By the choice of $P$, $y$ is a support vertex which is adjacent to exactly 2 leaves, say $z_1$ and $z_2$, and $sta(z_1) = sta(z_2) = A$. But then $T$ is obtained from $T - \{y, z_1, z_2\} \in \mathcal{T}_1$ and a copy of $F_1$ by operation $O_1$.

By Claim 2.1, there are no other possibilities.

(II) $(T, S) \in \mathcal{T} \Rightarrow (T, S) \in \mathcal{T}_1$. Obvious.

It remains the following.

(III) Proof of $(T, S) \in \mathcal{T} \Rightarrow T$ is $\gamma_R$-excellent and $(P_1)$ holds.
Let \((T, S) \in \mathcal{T}\). We know that \((T, S) \in \mathcal{T}_1\). We now proceed by induction on \(k = |S_B \cup S_C|\). First let \(k \leq 2\). By Claim 1.2, \(T \in \mathcal{H} = \{H_1, \ldots, H_{11}\}\). It is easy to see that all elements of \(\mathcal{H}\) are \(\gamma_R\)-excellent graphs and \((\mathcal{P}_1)\) holds for each \(T \in \mathcal{H}\).

Let \(k \geq 3\) and suppose that if \((H, S') \in \mathcal{T}\) and \(|S_B'(H) \cup S_C'(H)| < k\), then \(H\) is \(\gamma_R\)-excellent and \((\mathcal{P}_1)\) holds with \((T, S)\) replaced by \((H, S')\). So, let \((T, S) \in \mathcal{T}\) and \(k = |S_B(T) \cup S_C(T)|\). Then there is a \(\mathcal{T}\)-sequence \(\tau : (T^1, S^1), \ldots, (T^{j-1}, S^{j-1}), (T, S)\). By induction hypothesis, \(T^{j-1}\) is \(\gamma_R\)-excellent and \((\mathcal{P}_1)\) holds with \((T, S)\) replaced by \((T^{j-1}, S^{j-1})\). We consider four possibilities depending on whether \(T\) is obtained from \(T^{j-1}\) by operation \(O_1, O_2, O_3\) or \(O_4\).

**Case 7:** \(T\) is obtained from \(T^{j-1} \in \mathcal{T}\) and \(F_a\) by operation \(O_1, a \in \{1, 2, 3\}\). Hence \(T\) is obtained after adding the edge \(ux\) to the union of \(T^{j-1}\) and \(F_a\), where \(\text{sta}_{T^{j-1}}(u) = \text{sta}_{F_a}(x) = C\) (see Fig. 2). First note that \(\gamma_R(F_a) = a + 1\), and \(F_2\) and \(F_3\) are \(\gamma_R\)-excellent graphs. Since \(\gamma_R(F_a - x) = \gamma_R(F_a)\) and \(u \in V^{02}(T^{j-1})\), Lemma 2 implies \(\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(F_a)\). Hence for any \(\gamma_R\)-function \(g\) on \(T\), the weight of \(g|_{F_a}\) is not more than \(\gamma_R(F_a)\). But then \(g(x) \neq 1\) and either \(g|_{F_a-x}\) is a \(\gamma_R\)-function on \(F_a\) or \(g|_{F_a-x}\) is a \(\gamma_R\)-function on \(F_a - x\). By inspection of all \(\gamma_R\)-functions on \(F_a\) and \(F_a - x\), we obtain

\[
(\alpha_1) \quad S_A(T) \cap V(F_a) = V^{01}(T) \cap V(F_a), \quad S_B(T) \cap V(F_a) = \emptyset, \quad \{x\} = S_C(T) \cap V(F_a) = V^{02}(T) \cap V(F_a), \quad \text{and} \quad S_D(T) \cap V(F_a) = V^{012}(T) \cap V(F_a).
\]

By the definition of operation \(O_1\) it immediately follows

\[
(\alpha_2) \quad S_X(T) \cap V(T^{j-1}) = S^{-1}_X(T^{j-1}), \quad \text{for all} \quad X \in \{A, B, C, D\}.
\]

Let \(f_1\) be a \(\gamma_R\)-function on \(T^{j-1}\) and \(f_2\) a \(\gamma_R\)-function on \(F_a\). Then the RD-function \(f\) on \(T\) defined as \(f|_{T^{j-1}} = f_1\) and \(f|_{F_a} = f_2\) is a \(\gamma_R\)-function on \(T\). Since \(f_1\) was chosen arbitrarily, we have

\[
(\alpha_3) \quad V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T), \quad V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T), \quad \text{and} \quad V^{012}(T^{j-1}) \subseteq V^{012}(T).
\]

By \((\alpha_1)\) and \((\alpha_3)\) we conclude that \(T\) is \(\gamma_R\)-excellent.

Now we shall prove that

\[
(\alpha_4) \quad V^{01}(T) \cap V(T^{j-1}) = V^{01}(T^{j-1}), \quad V^{02}(T) \cap V(T^{j-1}) = V^{02}(T^{j-1}), \quad \text{and} \quad V^{012}(T) \cap V(T^{j-1}) = V^{012}(T^{j-1}).
\]

Assume there is a vertex \(z \in V^{02}(T^{j-1}) \cap V^{012}(T)\). By Lemma 7, \(z\) is adjacent to at most 2 elements of \(V^{-}(T^{j-1})\). Now by \((\alpha_3)\) and since \(\Delta((V^{-}(T))) \leq 1\) (by Lemma 6), \(z\) is adjacent to exactly one element of \(V^{-}(T^{j-1})\). But then Lemma 7 implies that there is a path \(z_1, z, z_2, z_3\) in \(T^{j-1}\) such that \(\text{deg}_{T^{j-1}}(z) = \text{deg}_{T^{j-1}}(z_2) = 2\), \(z, z_2 \in V^{02}(T^{j-1})\) and \(z_1, z_3 \in V^{01}(T^{j-1})\). Since \((\mathcal{P}_1)\) is true for \(T^{j-1}\), \(\text{sta}_{T^{j-1}}(z_1) = \text{sta}_{T^{j-1}}(z_3) = A\), and \(\text{sta}_{T^{j-1}}(z) = \text{sta}_{T^{j-1}}(z_2) = B\). Clearly, at least one of \(z_1\) and \(z_3\) is a cut-vertex. Denote by \(Q\) the graph \(\langle\{z_1, z, z_2, z_3\}\rangle\) and let the vertices of \(Q\)
are labeled as in $T^{j-1}$. Let $U_s$ be the connected component of $T - \{z,z_2\}$, which contains $z_s$, $s=1,3$.

Assume first that $T^1$ is a subtree of $U \in \{U_1,U_3\}$. Then there is $i$ such that $T^i$ is obtained from $T^{i-1}$ and $Q$ by operation $O_3$. Hence $T^{i-1}$ is a subtree of $U$. Recall that if $y \in V(T^r)$ and $r \leq s \leq j-1$, then $sta_{T^r}(y) = sta_{T^s}(y)$. Using this fact, we can choose $\tau$ so that $T^{j-1} = U$. Therefore $U$ is in $\mathcal{F}$. Suppose that neither $z_1$ nor $z_3$ is a leaf of $T^{j-1}$. Define $R^s = T^{i+s} - (V(T^{i-1}) \cup \{z,z_2\})$, $s = 1,2,\ldots,j-1-i$. Since clearly $R^1$ is in $\{H_2,H_3,\ldots,H_7\}$, the sequence $R^1,R^2,\ldots,R^{j-1-i}$ is a $\mathcal{F}$-sequence of $U'$, where $\{U,U'\} = \{U_1,U_3\}$. Thus, both $U_1$ and $U_3$ are in $\mathcal{F}$, and $sta_{U_1}(z_1) = A$. By the induction hypothesis, $z_1 \in V^{01}(U_1)$.

Suppose now that $u \in V(U_3)$. Consider the sequence of trees $U_3,U_4,U_5$, where $U_4$ is obtained from $U_3$ and $Q$ by operation $O_4$ (via $z_3$), and $U_5$ is obtained from $U_4$ and $F_a$ by operation $O_1$. Clearly $U_5$ is in $\mathcal{F}$, $sta_{U_5}(z_1) = A$ and by the induction hypothesis, $z_1 \in V^{01}(U_5)$. Since $T = (U_5 \cdot U_1)(z_1)$ and $\{z_1\} = V^{01}(U_1) \cap V^{01}(U_5)$, by Proposition 2 we have $z_1 \in V^{01}(T)$. But then Lemma 7 implies $z_2 \in V^{02}(T)$, a contradiction.

Now let $u \in V(U_1)$. Denote by $U_2$ the graph obtained from $U_1$ and $F_a$ by operation $O_3$. Then $U_2$ is in $\mathcal{F}$, $sta_{U_2}(z_1) = A$, and by induction hypothesis, $z_1 \in V^{01}(U_2)$. Define also the graph $U_6$ as obtained from $U_3$ and $Q$ by operation $O_3$, i.e. $U_6 = (U_3 \cdot Q)(z_3)$. Then $U_6$ is in $\mathcal{F}$, $sta_{U_6}(z_1) = A$ and by induction hypothesis, $z_1 \in V^{01}(U_6)$. Now by Proposition 2, $z_1 \in V^{01}(T)$, which leads to $z_2 \in V^{02}(T)$ (by Lemma 7), a contradiction.

Thus, in all cases we have a contradiction. Therefore $V^{02}(T^{j-1}) \subseteq V^{02}(T)$ when both $z_1$ and $z_3$ are cut-vertices. If $z_1$ or $z_3$ is a leaf, then, by similar arguments, we can obtain the same result.

Let now $T^1 = Q$. Then $T^2$ is obtained from $T^1$ and $H_k$ by operation $O_3$. Consider the sequence of trees $\tau_1 : T^1_k = H_k,T^2,T^3,\ldots,T^{j-1}$. Clearly $\tau_1$ is a $\mathcal{F}$-sequence of $T^{j-1}$ and $T^1_k \neq Q$. Therefore we are in the previous case. Thus, $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$.

Assume now that there is a vertex $w \in V^{01}(T^{j-1}) \cap V^{012}(T)$. By Lemma 7(i) $w$ has a neighbor in $T$, say $w'$, such that $w' \in V^{012}(T)$. Since $w \neq u$, $w' \in V(T^{j-1})$. But all neighbors of $w$ in $T^{j-1}$ are in $V^{02}(T^{j-1})$ (by Lemma 7 applied to $T^{j-1}$ and $w$). Since $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$, we obtain a contradiction.

Thus $(\alpha_4)$ is true.

Now we are prepared to prove that $(\mathcal{P}_1)$ is valid. Using, in the chain of equalities below, consecutively $(\alpha_2)$, the induction hypothesis, $(\alpha_1)$ and $(\alpha_4)$, we obtain

$$S_A(T) = S_A^{j-1}(T^{j-1}) \cup (S_A(T) \cap V(F_a)) = V^{01}(T^{j-1}) \cup (V^{01}(T) \cap V(F_a)) = V^{01}(T),$$
and similarly, $S_D(T) = V^{012}(T)$. Since $u \notin S_B(T)$ and $S_B(T) \cap V(F_a) = \emptyset$, we have

$$
S_B(T) = S_B(T) \cap V(T^{j-1}) \overset{(\alpha_2)}{=} S_B^{-1}(T^{j-1})
$$

$$
= \{ t \in V^{02}(T^{j-1}) | \deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1 \}
$$

$$
\overset{(\alpha_4)}{=} \{ t \in V^{02}(T) \cap V(T^{j-1}) | \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1 \}
$$

$$
= \{ t \in V^{02}(T) | \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1 \}.
$$

The last equality follows from $\deg_T(x) > 2$ and $\{ x \} = V^{02}(T) \cap V(F_a)$ (see ($\alpha_1$)). Now the equality $S_C(T) = V^{02}(T) - S_B(T)$ is obvious. Thus, ($P_1$) holds and we are done.

**Case 8:** $T$ is obtained from $T^{j-1} \in \mathcal{F}$ by operation $O_2$.

Clearly, $\gamma_R(F_4) = \gamma_R(F_4 - x) = 2$. By Lemma 2, $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_4)$. Let $f_1$ be a $\gamma_R$-function on $T^{j-1}$ and $f_2$ a $\gamma_R$-function on $F_4$. Then the function $f$ defined as $f|_{T^{j-1}} = f_1$ and $f|_{F_4} = f_2$ is a $\gamma_R$-function on $T$. Therefore $V^{012}(T^{j-1}) \subseteq V^{012}(T)$, $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{02}(T)$, and $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$.

Assume that there is $y \in V^{0s}(T^{j-1}) \cap V^{012}(T)$, $s \in \{1, 2\}$, and let $f'$ be a $\gamma_R$-function on $T$ with $f'(y) = r \notin \{0, s\}$. If $f'|_{T^{j-1}}$ is an RD-function on $T^{j-1}$, then $f'|_{T^{j-1}}(V(T^{j-1})) > \gamma_R(T^{j-1})$ and $f'|_{F_4}(V(F_4)) \geq 2$. This leads to $f'(V(T)) > \gamma_R(T)$, a contradiction. Hence $f'|_{T^{j-1}}$ does not have a RD-function on $T^{j-1}$ and $f'|_{T^{j-1} - u}$ is a $\gamma_R$-function on $T^{j-1} - u$. Define now an RD-function $f''$ on $T^{j-1}$ as $f''|_{T^{j-1} - u} = f'|_{T^{j-1} - u}$ and $f''(y) = 1$. Since $u \in V^-(T^{j-1})$, $f''$ is a $\gamma_R$-function on $T^{j-1}$ with $f''(y) = r \notin \{0, s\}$, a contradiction with $y \in V^{0s}(T^{j-1})$.

Thus

$$
(\alpha_5) \quad V^{012}(T^{j-1}) = V^{012}(T) \cap V(T^{j-1}), \quad V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1}), \quad V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1}).
$$

Let $x, x_1, x_2$ be a path in $F_4$, $h_1$ a $\gamma_R$-function on $T^{j-1}$ with $h_1(u) = 2$, and $h_2$ a $\gamma_R$-function on $T^{j-1} - u$. Define $\gamma_R$-functions $g_1, \ldots, g_4$ on $T$ as follows:

- $g_1|_{T^{j-1}} = h_1$, $g_1(x) = g_1(x_2) = 0$ and $g_1(x_1) = 2$;
- $g_2|_{T^{j-1}} = h_1$, $g_2(x) = 0$ and $g_2(x_1) = g_2(x_2) = 1$;
- $g_3|_{T^{j-1}} = h_1$, $g_3(x) = g_3(x_1) = 0$ and $g_3(x_2) = 2$;
- $g_4|_{T^{j-1} - u} = h_2$, $g_4(u) = g_4(x_1) = 0$, $g(x) = 2$ and $g_4(x_2) = 1$.

This, ($\alpha_5$) and Lemma 6 allows us to conclude that $T$ is $\gamma_R$-excellent, $x_1, x_2 \in V^{012}(T)$ and $x \in V^{02}(T)$.

By induction hypothesis, ($P_1$) holds with $(T, S)$ replaced by $(T^{j-1}, S^{j-1})$. Then Since $u \notin S_B(T)$ and $S_B(T) \cap V(F_4) = \emptyset$, we have

$$
S_B(T) = S_B^{-1}(T^{j-1})
$$

$$
= \{ t \in V^{02}(T^{j-1}) | \deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1 \}
$$

$$
= \{ t \in V^{02}(T) | \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1 \}.
$$
The last equality follows from $\deg_T(x) > 2$ and $\{x\} = V^{02}(T) \cap V(F_4)$. Now the equality $S_C(T) = V^{02}(T) - S_B(T)$ is obvious. Thus, $(P_1)$ is true.

**Case 9:** $T$ is obtained from $T^{j-1} \in \mathcal{F}$ by operation $O_3$.

Let $T = (T^{j-1} \cdot H_k)(u, v : u)$, where $\text{stat}_{T^{j-1}}(u) = \text{stat}_{H_k}(v) = \text{stat}_T(u) = A$ and $k \in \{2, \ldots, 7\}$. Hence $S_X(T) = S_X^{j-1}(T^{j-1}) \cup I_X^k(H_k)$, for any $X \in \{A, B, C, D\}$. We know that $(P_1)$ holds with $(T, S)$ replaced by any of $(T^{j-1}, S^{j-1})$ and $(H_k, I^k)$. Hence $S_A(T) = S_A^{j-1}(T^{j-1}) \cup I_A^k(H_k) = V^{01}(T^{j-1}) \cup V^{01}(H_k)$. Now, by Proposition 2, applied to $T^{j-1}$ and $H_k$, $S_A(T) = V^{01}(T)$. Similarly we obtain $S_D(T) = V^{012}(T)$.

We also have

$$S_B(T) = S_B^{j-1}(T^{j-1}) \cup I_B^k(H_k) = \{t \in V^{02}(T^{j-1}) \mid \deg_T(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \cup \{t \in V^{02}(H_k) \mid \deg_{H_k}(t) = 2 \text{ and } |N_{H_k}(t) \cap V^{02}(H_k)| = 1\} = \{t \in V^{02}(T^{j-1}) \cup V^{02}(H_k) \mid \deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\},$$

as required, because $V^{02}(T^{j-1}) \cup V^{02}(H_k) = V^{02}(T)$ (by Proposition 2). Now the equality $S_C(T) = V^{02}(T) - S_B(T)$ is obvious.

**Case 10:** $T$ is obtained from $T^{j-1} \in \mathcal{F}$ and $H_k \in \mathcal{F}$, $k \in \{3, 4, 6\}$, by operation $O_4$.

By induction hypothesis and Lemma 4, we have $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_k) - 1$ and $u \in V^{012}(T)$. Let $f_1$ be a $\gamma_R$-function on $T^{j-1}$ and $f_2$ a $\gamma_R$-function on $H_k - v$. Then the function $f$ defined as $f|_{T^{j-1}} = f_1$ and $f|_{H_k-v} = f_2$ is a $\gamma_R$-function on $T$. Therefore $V^{012}(T^{j-1}) \subseteq V^{012}(T)$, $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T)$, and $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$. Assume that there is $y \in V^{08}(T^{j-1}) \cap V^{012}(T)$, $s \in \{1, 2\}$, and let $f'$ be a $\gamma_R$-function on $T$ with $f'(y) = r \notin \{0, s\}$. But then $f'|_{T^{j-1}}$ is no RD-function on $T^{j-1}$, $f'(u) = 0$, $f'|_{T^{j-1}-u}$ is a $\gamma_R$-function on $T^{j-1} - u$ and $f'|_{H_k}$ is a $\gamma_R$-function on $H_k$. Define now an RD-function $g_1$ on $T^{j-1}$ as $g_1|_{T^{j-1}-u} = f'|_{T^{j-1}-u}$ and $g_1(u) = 1$. Since $g_1(V(T^{j-1})) = \gamma_R(T^{j-1} - u) + 1 = \gamma_R(T^{j-1})$, $g_1$ is a $\gamma_R$-function on $T^{j-1}$. But $g_1(y) = r \notin \{0, s\}$, a contradiction. Thus

$$(\alpha_6) \quad V^{012}(T^{j-1}) = V^{012}(T) \cap V(T^{j-1}), \quad V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1}), \quad V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1}).$$

The next claim is obvious.

**Claim 1.3** Let $x$ be the neighbor of $v$ in $H_k$, $k \in \{3, 4, 6\}$. Then $\gamma_R(H_3) = 4, \gamma_R(H_4) = 5, \gamma_R(H_6) = 6, \gamma_R(H_k - v) = \gamma_R(H_k - \{v, x\}) = \gamma_R(H_k)$, and $l(x) = 0$ for any $\gamma_R$-function $l$ on $H_k - v$.

Let $h$ be a $\gamma_R$-function on $T$. We know that $u \in V^{012}(T), u \in V^{012}(T^{j-1}), v \in V^{01}(H_k)$, and $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_k) - 1$. Then by Claim 1.3 we clearly have:

(a1) If $h(u) = 2$ then at least one of the following holds:

(a1.1) $h|_{H_k-v}$ is a $\gamma_R$-function on $H_k - v$, and

(a1.2) $h|_{H_k-\{v, x\}}$ is a $\gamma_R$-function on $H_k - \{v, x\}$.  


(a2) If $h(u) = 1$ then $h|_{H_k - v}$ is a $\gamma_R$-function on $H_k - v$.

(a3) If $h(u) = 0$ then either $h|_{H_k}$ is a $\gamma_R$-function on $H_k$, or $h|_{H_k - v}$ is a $\gamma_R$-function on $H_k - v$.

Let $l_1, l_2, l_3, l_4$ and $l_5$ be $\gamma_R$-functions on $H_k, H_k - v, H_k - \{v, x\}, T^{j-1} - u$ and $T^{j-1}$, respectively, and let $l_5(u) = 2$. Define the functions $h_1, h_2,$ and $h_3$ on $T$ as follows: (i) $h_1|_{T^{j-1}} = l_5, h_1(x) = 0$ and $h_1|_{H_k - \{v, x\}} = l_3$, (ii) $h_2|_{T^{j-1}} = l_5$ and $h_1|_{H_k - v} = l_2$, and (iii) $h_3|_{T^{j-1} - u} = l_4$ and $h_3|_{H_k} = l_1$. Clearly $h_1, h_2,$ and $h_3$ are $\gamma_R$-functions on $T$. After inspection of all $\gamma_R$-functions of $H_k, H_k - v$ and $H_k - \{v, x\}$, we conclude that $V^{01}(H_k) - \{v\} \subseteq V^{01}(T), V^{02}(H_k) \subseteq V^{02}(T)$, and $V^{012}(H_k) \subseteq V^{012}(T)$. This and $(\alpha_7)$ imply

$$V^{012}(T) = V^{012}(T^{j-1}) \cup V^{012}(H_k), \ V^{02}(T) = V^{02}(T^{j-1}) \cup V^{02}(H_k), \text{ and} \ V^{01}(T) = V^{01}(T^{j-1}) \cup V^{01}(H_k) - \{v\}.$$  

Since $(\mathcal{P}_3)$ holds with $T$ replaced by $H_k$ or by $T^{j-1}$ (by induction hypothesis), using $(\alpha_7)$ we obtain that $(\mathcal{P}_1)$ is satisfied. \qed

5. Corollaries

The next three results immediately follow by Theorem 1.

**Corollary 1.** If $(T, S_1), (T, S_2) \in \mathcal{T}$ then $S_1 \equiv S_2$.

If $(T, S) \in \mathcal{T}$ then we call $S$ the $\mathcal{T}$-labeling of $T$.

**Corollary 2.** Let $T$ be a $\gamma_R$-excellent tree of order $n \geq 5$, and $S$ the $\mathcal{T}$-labeling of $T$. Then $\frac{3}{2} \leq |V^{02}(T)| \leq \frac{3}{2}(n - 1)$ and $\frac{1}{2}n \geq |V^{-}(T)| \geq \frac{1}{2}(n + 2)$. Moreover,

1. $\frac{3}{2} = |V^{02}(T)|$ if and only if $(T, S)$ has a $\mathcal{T}$-sequence $\tau : (T_1, S_1), \ldots, (T^j, S^j)$, such that $(T_1, S_1) = (F_3, J^3)$ and if $j \geq 2$, $(T^{j+1}, S^{j+1})$ can be obtained recursively from $(T^j, S^j)$ and $(F_3, J^3)$ by operation $O_1$.

2. $|V^{02}(T)| \leq \frac{3}{2}(n - 1)$ if and only if $(T, S)$ has a $\mathcal{T}$-sequence $\tau : (T_1, S_1), \ldots, (T^j, S^j)$, such that $(T_1, S_1) = (H_2, I^2)$ and if $j \geq 2$, $(T^{j+1}, S^{j+1})$ can be obtained recursively from $(T^j, S^j)$ and $(H_2, I^2)$ by operation $O_3$.

**Corollary 3.** Let $G$ be an $n$-order $\gamma_R$-excellent connected graph of minimum size. Then either $G = K_3$ or $n \neq 3$ and $G$ is a tree.

6. Special cases

Let $G$ be a graph and $\{a_1, \ldots, a_k\} \subseteq \{0, 1, 2, 01, 02, 12, 012\}$. We say that $G$ is a $\mathcal{R}_{a_1, \ldots, a_k}$-graph if $V(G) = \bigcup_{i=1}^{k} V^{a_i}(G)$ and all $V^{a_1}(G), \ldots, V^{a_k}(G)$ are nonempty. Now let $T$ be a $\gamma_R$-excellent tree of order at least 2. By Theorem 1, we immediately conclude that $T \in \mathcal{R}_{012} \cup \mathcal{R}_{01,02} \cup \mathcal{R}_{02,012} \cup \mathcal{R}_{01,02,012}$. Moreover,
(i) \( T \in \mathcal{R}_{012} \) if and only if \( T = K_2 \), and

(ii) \( T \in \mathcal{R}_{01,02,012} \) if and only if none of \( S_A(T), S_C(T) \) and \( S_D(T) \) is empty, where \( S \) is the \( \mathcal{F} \)-labeling of \( T \).

In this section, we turn our attention to the classes \( \mathcal{R}_{01,02} \) and \( \mathcal{R}_{02,012} \).

### 6.1. \( \mathcal{R}_{01,02} \)-graphs.

Here we give necessary and sufficient conditions for a tree to be in \( \mathcal{R}_{01,02} \). We define a subfamily \( \mathcal{F}_{01,02} \) of \( \mathcal{F} \) as follows. A labeled tree \( (T, S) \in \mathcal{F}_{01,02} \) if and only if \( (T, S) \) can be obtained from a sequence of labeled trees \( \tau : (T^1, S^1), \ldots, (T^j, S^j) \), \((j \geq 1)\), such that \((T^1, S^1)\) is in \( \{(H_2, I^2), (H_3, I^3)\}\) (see Figure 1) and \((T, S) = (T^j, S^j)\), and, if \( j \geq 2 \), \((T^{i+1}, S^{i+1})\) can be obtained recursively from \((T^i, S^i)\) by one of the operations \( O_5 \) and \( O_6 \) listed below; in this case \( \tau \) is said to be a \( \mathcal{F}_{01,02} \)-sequence of \( T \).

**Operation** \( O_5 \). The labeled tree \((T^{i+1}, S^{i+1})\) is obtained from \((T^i, S^i)\) and \((F_1, I^1)\) (see Figure 2) by adding the edge \( u \cdot x \), where \( u \in V(T_i) \), \( x \in V(F_1) \) and \( sta_{T_i}(u) = sta_{F_i}(x) = C \).

**Operation** \( O_6 \). The labeled tree \((T^{i+1}, S^{i+1})\) is obtained from \((T^i, S^i)\) and \((H_k, I^k)\), \(k \in \{2, 3\}\) (see Figure 1), in such a way that \( T^{i+1} = (T^i \cdot H_k)(u, v : u) \), where \( sta_{T_i}(u) = sta_{H_k}(v) = A \), and \( sta_{T^{i+1}}(u) = A \).

Remark that once a vertex is assigned a status, this status remains unchanged as the labeled tree \((T, S)\) is recursively constructed. By the above definitions we see that \( S_D(T) \) is empty when \((T, S) \in \mathcal{F}_{01,02} \). So, in this case, it is naturally to consider a labeling \( S \) as \( S : V(T) \to \{A, B, C\} \). From Theorem 1 we immediately obtain the following result.

**Corollary 4.** Let \( T \) be a tree of order at least 2. Then \( T \in \mathcal{R}_{01,02} \) if and only if there is a labeling \( S : V(T) \to \{A, B, C\} \) such that \((T, S) \) is in \( \mathcal{F}_{01,02} \). Moreover, if \((T, S) \in \mathcal{F}_{01,02} \) then

\[
(P_3) \quad S_B(T) = \{x \in V^{02}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{02}(T)| = 1\}, \quad S_A(T) = V^{01}(T), \quad \text{and} \quad S_C(T) = V^{02}(T) - S_B(T).
\]

As an immediate consequence of Corollary 1 we obtain:

**Corollary 5.** If \((T, S_1), (T, S_2) \in \mathcal{F}_{01,02} \) then \( S_1 \equiv S_2 \).

A graph \( G \) is called a 2-corona if each vertex of \( G \) is either a support vertex or a leaf, and each support vertex of \( G \) is adjacent to exactly 2 leaves. In a labeled 2-corona all leaves have status \( A \) and all support vertices have status \( C \).
Proposition 3. Every connected $n$-order graph $H$, $n \geq 2$, is an induced subgraph of a $\mathcal{R}_{01,02}$-graph with the domination number equals to $2|V(H)|$.

Proof. Let a graph $G$ be a 2-corona such that the induced subgraph by the set of all support vertices of $G$ is isomorphic to $H$. Let $x$ be a support vertex of $G$ and $y, z$ the leaf neighbors of $x$ in $G$. Then clearly for any $\gamma_R$-function $f$ on $G$, $f(x) + f(y) + f(z) \geq 2$, $f(y) \neq 2 \neq f(z)$ and $f(x) \neq 1$. Define RD-functions $h$ and $g$ on $G$ as follows: (a) $h(u) = 2$ when $u$ is a support vertex of $G$ and $h(u) = 0$, otherwise, and (b) $g(v) = h(v)$ when $v \notin \{x, y, z\}$, and $g(x) = 0$, $g(y) = g(z) = 1$. Therefore $\gamma_R(G) = 2|V(H)|$ and $G$ is in $\mathcal{R}_{01,02}$. □

Corollary 6. There does not exist a forbidden subgraph characterization of the class of $\mathcal{R}_{01,02}$-graphs. There does not exist a forbidden subgraph characterization of the class of $\gamma_R$-excellent graphs.

Let $\mathcal{S}_{01,02}$ be the family of all labeled trees $(T, L)$ that can be obtained from a sequence of labeled trees $\lambda: (T^1, L^1), \ldots, (T^j, L^j), (j \geq 1)$, such that $(T, L) = (T^j, L^j)$, $(T^1, L^1)$ is either $(H_2, I^2)$ (see Figure 1) or a labeled 2-corona tree, and, if $j \geq 2$, $(T^{i+1}, L^{i+1})$ can be obtained recursively from $(T^i, L^i)$ by one of the operations $O_7$ and $O_8$ listed below; in this case $\lambda$ is said to be a $\mathcal{S}'_{01,02}$-sequence of $T$.

Operation $O_7$. The labeled tree $(T^{i+1}, L^{i+1})$ is obtained from $(T^i, L^i)$ and $(H_2, I^2)$, in such a way that $T^{i+1} = (T^i \cdot H_2)(u, v : u)$, where $sta_{T^i}(u) = sta_{H_2}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Operation $O_8$. The labeled tree $(T^{i+1}, L^{i+1})$ is obtained from $(T^i, L^i)$ and a labeled 2-corona tree, say $U_i$, in such a way that $T^{i+1} = (T^i \cdot U_i)(u, v : u)$, where $sta_{T^i}(u) = sta_{U_i}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Again, once a vertex is assigned a status, this status remains unchanged as the 2-labeled tree $T$ is recursively constructed.

Theorem 2. For any tree $T$ the following are equivalent.

$(A_1) \; T$ is in $\mathcal{R}_{01,02}$.

$(A_2) \; There is a labeling $S: V(T) \rightarrow \{A, B, C\}$ such that $(T, S)$ is in $\mathcal{S}_{01,02}$.

$(A_3) \; There is a labeling $L: V(T) \rightarrow \{A, B, C\}$ such that $(T, L)$ is in $\mathcal{S}'_{01,02}$.

Proof. $(A_1) \Leftrightarrow (A_2)$: By Corollary 4.

$(A_3) \Rightarrow (A_2)$:
Let a tree $(T, L) \in \mathcal{S}'_{01,02}$. It is clear that all $\mathcal{S}'_{01,02}$-sequences of $(T, L)$ have the same number of elements. Denote this number by $r(T)$. We shall prove that $(T, L) \in \mathcal{S}'_{01,02} \Rightarrow (T, L) \in \mathcal{S}_{01,02}$. We proceed by induction on $r(T)$. If $r(T) = 1$ then either
(T, L) is a labeled 2-corona tree, or (T, L) = (H_2, I^2). In both cases (T, L) ∈ ℰ_{01,02}.

We need the following obvious claim.

**Claim 2.1** If (T', L') is a labeled 2-corona tree, w ∈ V(T') and sta(w) = A, then either (T', L') is (H_3, I^3) or there is a ℰ-sequence τ : (T^1, S^1), . . . , (T^i, S^i), (l ≥ 2), such that (T^1, S^1) = (H_3, I^3), w ∈ V(T^1), (T^i, S^i) = (T', L'), and (T^{i+1}, S^{i+1}) can be obtained recursively from (T^i, S^i) and (F_1, J^1) by operation O_5.

Suppose now that each tree (H, L_H) ∈ ℰ_{01,02} with r(H) < k is in ℰ_{01,02}, where k ≥ 2. Let λ : (T^1, L^1), . . . , (T^l, L^l), be a ℰ_{01,02}-sequence of a labeled tree (T, L) ∈ ℰ'_{01,02}. By the induction hypothesis, (T^{l-1}, L^{l-1}) is in ℰ_{01,02}. Let τ : (U^1, S^1), . . . , (U^m, S^m) be a ℰ-sequence of (T^{l-1}, L^{l-1}). Hence U^m = T^{l-1} and S^m = L^{l-1}. If (T^{l-1}, L^{l-1}) is obtained from (T^{l-1}, L^{l-1}) and (H_2, I^2) by operation O_7, then (U^1, S^1), . . . , (U^m, S^m), (T^{l-1}, L^{l-1}) = (T, L) is a ℰ-sequence of (T, L). So, let (T^{l-1}, L^{l-1}) be obtained from (T^{l-1}, L^{l-1}) and a labeled 2-corona tree, say (Q, L_q) by operation O_8. Hence (T^{l-1}, L^{l-1}) and Q have exactly one vertex in common, say w, and sta_{T^{l-1}}(w) = sta_Q(w) = sta_{T^l}(w) = A. By Claim 2.1, (Q, L_q) ∈ ℰ_{01,02} and it has a ℰ_{01,02}-sequence, say (Q^1, L_q^1), . . . , (Q^s, L_q^s) such that Q^s = Q, L_q = L_q^s, and w ∈ V(Q^1). Denote W^{m+i} = (V(U^m) ∪ V(Q^i)), and let a labeling S^{m+i} be such that S^{m+i}|_{U^m} = S^m and S^{m+i}|_{Q^i} = L_q^s. Then the sequence of labeled trees (U^1, S^1), . . . , (U^m, S^m), (W^{m+1}, S^{m+1}), . . . , (W^{m+s}, S^{m+s}) = (T, L) is a ℰ_{01,02}-sequence of (T, L).

(A_2) ⇒ (A_3):

Let a labeled tree (T, S) ∈ ℰ_{01,02}. Then (T, S) has a ℰ-sequence τ : (T^1, S^1), . . . , (T^i, S^i) = (T, S), where (T^1, S^1) ∈ {(H_2, I^2), (H_3, I^3)} ∈ ℰ'_{01,02}. We proceed by induction on p(T) = ∑_{z ∈ ℰ(T)} deg_T(z), where ℰ(T) is the set of all cut-vertices of T that belong to S_A(T). Assume first p(T) = 0. If j = 1 then we are done. If j ≥ 2 then (T^j, S^j) = (H_3, I^3) and (T^{j+1}, S^{j+1}) is obtained from (F_1, J^1) and (T^j, S^j) by operation O_5. Thus, (T, S) is a labeled 2-corona tree, which allow us to conclude that (T, S) is in ℰ'_{01,02}.

Suppose now that p(T) = k ≥ 1 and for each labeled tree (H, S_H) ∈ ℰ_{01,02} with p(H) < k is fulfilled (H, S_H) ∈ ℰ'_{01,02}. Then there is a cut-vertex, say z, such that (a) z ∈ S_A(T), (b) (T, S) is a coalescence of 2 graphs, say (T', S|_{T'}) and (T'', S|_{T''}), via z, and (c) no vertex in S_A(T) ∩ V(T'') is a cut-vertex of T''. Hence (T', S|_{T'}) ∈ ℰ'_{01,02} (by induction hypothesis) and (T'', S|_{T''}) is either a labeled 2-corona tree or H_2. Thus (T, S) is in ℰ'_{01,02}.

### 6.2. ℰ_{02,012}-trees.

Our aim in this section is to present a characterization of ℰ_{02,012}-trees. For this purpose, we need the following definitions. Let ℰ_{02,012} ⊂ ℰ be such that (T, S) ∈ ℰ_{02,012} if and only if (T, S) can be obtained from a sequence of labeled trees τ : (T^1, S^1), . . . , (T^j, S^j), (j ≥ 1), such that (T^1, S^1) = (F_3, J^3) (see Figure 2) and (T, S) = (T^j, S^j), and, if j ≥ 2, (T^{j+1}, S^{j+1}) can be obtained recursively from (T^j, S^j) by one of the operations O_9 and O_{10} listed below.
As a consequence of Theorem 3 and Corollary 7 we have:

**Operation** $O_9$. The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F_3, J^3)$ by adding the edge $ux$, where $u \in V(T^i)$, $x \in V(F_3)$, and $sta_{T^i}(u) = sta_{F_3}(x) = C$.

**Operation** $O_{10}$. The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F_4, J^4)$ (see Figure 2) by adding the edge $ux$, where $u \in V(T^i)$, $x \in V(F_4)$, $sta_{T^i}(u) = D$, and $sta_{F_4}(x) = C$.

Note that once a vertex is assigned a status, this status remains unchanged as the labeled tree $(T, S)$ is recursively constructed. By the above definitions we see that if $(T, S) \in \mathcal{R}_{01,02}$, then $S_A(T) = S_B(T) = \emptyset$. Therefore it is naturally to consider a labeling $S$ as $S : V(T) \rightarrow \{C, D\}$.

From Theorem 1 we immediately obtain the following result.

**Corollary 7.** A tree $T$ is in $\mathcal{R}_{02,012}$ if and only if there is a labeling $S : V(T) \rightarrow \{C, D\}$ such that $(T, S)$ is in $\mathcal{R}_{02,012}$. Moreover, if $(T, S) \in \mathcal{R}_{02,012}$ then $S_C(T) = V^{02}(T)$ and $S_D(T) = V^{012}(T)$.

As an immediate consequence of Corollary 1 we obtain:

**Corollary 8.** If $(T, S_1), (T, S_2) \in \mathcal{R}_{02,012}$ then $S_1 \equiv S_2$.

**Theorem 3.** [3] If $G$ is a connected graph of order $n \geq 3$, then $\gamma_R(G) \leq 4n/5$. The equality holds if and only if $G$ is $C_5$ or is obtained from $\frac{n}{2}P_5$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2}P_5$.

As a consequence of Theorem 3 and Corollary 7 we have:

**Corollary 9.** Let $G$ be a connected $n$-vertex graph with $n \geq 6$ and $\gamma_R(G) = 4n/5$. Then $G$ is in $\mathcal{R}_{02,012}$ and $V^{012}(G)$ consists of all leaves and all support vertices. Moreover, if $G$ is a tree, then $G$ has a $\mathcal{T}$-sequence $\tau : (G^1, S^1), \ldots, (G^j, S^j)$, $(j \geq 1)$, such that $(G^1, S^1) = (F_3, J^3)$ (see Figure 2) and if $j \geq 2$, then $(G^{i+1}, S^{i+1})$ can be obtained recursively from $(G^i, S^i)$ by operation $O_9$.

A graph $G$ is said to be in class $UVR$ if $\gamma(G - v) = \gamma(G)$ for each $v \in V(G)$. Constructive characterizations of trees belonging to $UVR$ are given in [14] by Samodivkin, and independently in [11] by Haynes and Henning. We need the following result in [14] (reformulated in our present terminology).

**Theorem 4.** [14] A tree $T$ of order at least 5 is in $UVR$ if and only if there is a labeling $S : V(T) \rightarrow \{C, D\}$ such that $(T, S)$ is in $\mathcal{R}_{02,012}$. Moreover, if $(T, S) \in \mathcal{R}_{02,012}$ then $S_C(T)$ and $S_D(T)$ are the sets of all $\gamma$-bad and all $\gamma$-good vertices of $T$, respectively.

We end with our main result in this subsection.
Theorem 5. For any tree $T$ the following are equivalent:

$(A_1)$ $T$ is in $R_{02,012}$, \hspace{1cm} $(A_5)$ $T$ is in $R_{02,012}$, \hspace{1cm} $(A_6)$ $T$ is in $UVR$.

Proof. Corollary 7 and Theorem 4 together imply the required result. \hfill \square

7. Open problems and questions

We conclude the paper by listing some interesting problems and directions for further research. Let first note that if $n \geq 3$ and $G_{n,k}$ is not empty, then $k \leq 4n/5$ (Theorem 3).

An element of $RE_{n,k}$ is said to be isolated, whenever it is both maximal and minimal. In other words, a graph $H \in G_{n,k}$ is isolated in $RE_{n,k}$ if and only if $H \in R_{CEA}$ and for each $e \in E(H)$ at least one of the following holds: (a) $H - e$ is not connected, (b) $\gamma_R(H) \neq \gamma_R(H - e)$, (c) $H - e$ is not $\gamma_R$-excellent.

Example 1. (i) All $\gamma_R$-excellent graphs with the Roman domination number equals to 2 are $K_2$ and $K_n$, $n \geq 2$. If a graph $G \in R_{CEA}$ and $\gamma_R(G) = 2$, then $G$ is complete. $K_n$ is isolated in $RE_{n,2}$, $n \geq 2$.

(ii) [8] $K_2$, $H_7$ and $H_8$ (see Fig. 1) are the only trees in $R_{CEA}$.

(iii) If $RE_{n,k}$ has a tree $T$ as an isolated element, then either $(n,k) = (2,2)$ and $T = K_2$, or $(n,k) = (9,7)$ and $T = H_7$, or $(n,k) = (10,8)$ and $T = H_8$.

- Find results on the isolated elements of $RE_{n,k}$.

- What is the maximum number of edges $m(G_{n,k})$ of a graph in $G_{n,k}$? Note that (a) $m(G_{n,2}) = n(n - 1)/2$, (b) $m(G_{n,3}) = n(n - 1)/2 - \lceil n/2 \rceil$.

- Find results on those minimal elements of $RE_{n,k}$ that are not trees.

Example 2. (a) A cycle $C_n$ is a minimal element of $RE_{n,k}$ if and only if $n \equiv 0 \pmod{3}$ and $k = 2n/3$. (b) A graph $G$ obtained from the complete bipartite graph $K_{p,q}$, $p \geq q \geq 3$, by deleting an edge is a minimal element of $RE_{p+q,4}$.

The height of a poset is the maximal number of elements of a chain.

- Find the height of $RE_{n,k}$.

Example 3. (a) It is easy to check that any longest chain in $RE_{6,4}$ has as the first element $H_3$ (see Fig 1) and as the last element one of the two 3-regular 6-vertex graphs. Therefore the height of $RE_{6,4}$ is 5.
(b) Let us consider the poset $\mathcal{RE}_{5r,4r}$, $r \geq 2$. All its minimal elements are $\gamma_R$-excellent trees (by Theorem 3 and Corollary 9), which are characterized in Corollary 9. Moreover, the graph obtained from $rP_5$ by adding a complete graph on the set of centers of the components of $rP_5$ is the largest element of $\mathcal{RE}_{5r,4r}$. Therefore the height of $\mathcal{RE}_{5r,4r}$ is $(r - 1)(r - 2)/2 + 1$.

- Find results on $\gamma_{YR}$-excellent graphs at least when $Y$ is one of $\{-1, 0, 1\}$, $\{-1, 1\}$ and $\{-1, 1, 2\}$.

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