Momentum Spaces of Arbitrary Signature for $\kappa$-Minkowski

Fedele Lizzi\textsuperscript{a,b,c,*}, Flavio Mercati\textsuperscript{a,b†}, Mattia Manfredonia\textsuperscript{a,b‡}

\textsuperscript{a} Dipartimento di Fisica “Ettore Pancini”, Università di Napoli Federico II, Napoli, Italy; 
\textsuperscript{b} INFN, Sezione di Napoli, 
\textsuperscript{c} Departament de Física Quànctica i Astrofísica and Institut de Ciencies del Cosmos (ICCUB), Universitat de Barcelona, Barcelona, Spain.

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Abstract

The $\kappa$-Minkowski noncommutative space requires a curved momentum space. We find that this is not unique, and it can have any signature, Euclidean, Lorentzian, and (+,+,-,-), as well as degenerate (with zero eigenvalues) cases. An interesting feature is the presence of two Lorentzian-signature momentum spaces. One, well discussed in the literature, is the known momentum space with de Sitter geometry. The other, with anti de Sitter geometry, is new. It is based on a previously-unnoticed five dimensional matrix representation of the $\kappa$-Minkowski commutation relations. Furthermore, for any choice of a four dimensional metric there is a quantum group of symmetries of $\kappa$-Minkowski preserving it. We associate a momentum space to each nondegenerate choice of such metric. These momentum spaces are all maximally symmetric, and the isotropy subgroup of their isometries coincides with the homogeneous part of the quantum group. We also discuss the degenerate cases.

\textsuperscript{*}fedele.lizzi@na.infn.it  
\textsuperscript{†}flavio.mercati@gmail.com  
\textsuperscript{‡}mattia.manfredonia91@gmail.com
1 Introduction

The $\kappa$-Minkowski noncommutative spacetime [1–3] is defined by the commutation relations:

$$[x^0, x^i] = \frac{i}{\kappa} x^i, \quad [x^i, x^j] = 0, \quad i, j = 1, \ldots, 3, \quad (1)$$

where $\kappa$ is a constant with the dimensions of an energy (in $\hbar = 1$ units). The above relations close a Lie algebra, known as $\mathfrak{an}(3)$. The commutation relations (1) can be generalized to $[x^\mu, x^\nu] = i(\nu^\mu x^\nu - \nu^\nu x^\mu)$, $\mu = 0, \ldots, 3$, where $x^\mu$ is any set of four real numbers. However, all these algebras are isomorphic and can be put in the form (1) by a linear redefinition of generators. The generator $x^0$ is usually interpreted as a time coordinate, and $x^i$ as a spatial one.

This interpretation derives from the fact that the above algebra can be derived as the “quantum homogeneous space” of a quantum-group deformation of the Poincaré group known as $\kappa$-Poincaré [1–9]. This group is generated by the elements $a^\mu$ and $\Lambda^\mu\nu$, satisfying the following commutation and cocommutation rules:

$$\Delta[\Lambda^\mu\nu] = \Lambda^\mu\alpha \otimes \Lambda_\alpha^\nu, \quad [\Lambda^\mu\nu, \Lambda_\alpha^\beta] = 0,$$

$$\Delta[a^\mu] = a^\nu \otimes a^\mu + a^\mu \otimes 1, \quad [a^\mu, a^\nu] = \frac{i}{\kappa} \left[ (\Lambda^\mu_\alpha \delta^\alpha^0 - \delta^\mu^0) \Lambda_\nu^\gamma + (\Lambda_\alpha^\mu \delta^0_\alpha - \delta^0_\nu) \eta^\gamma^\nu \right],$$

$$S[\Lambda] = \Lambda^{-1}, \quad S[a^\mu] = -a^\mu, \quad [a^0, a^i] = \frac{i}{\kappa} a^i, \quad [a^i, a^j] = 0,$$

$$\varepsilon[\Lambda^\mu\nu] = \delta^\mu_\nu, \quad \varepsilon[a^\mu] = 0, \quad \Lambda_\alpha^\mu \Lambda_\beta^\nu \eta^{\alpha\beta} = \eta^\mu^\nu, \quad \Lambda_\mu^\rho \Lambda_\sigma^\nu \eta_{\rho\sigma} = \eta_{\mu\nu}, \quad (2)$$

where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the usual Minkowski flat metric. Note that the commutation and cocommutation rules involving the Lorentz sector are undeformed, the deformation being limited to the translation sector, and the mixed part. The very last relation gives the appropriate number of constraints so that the independent components of $\Lambda$ are six. The $\kappa$-Minkowski commutation relations (1) are left invariant by the following left co-action of $\kappa$-Poincaré:

$$\Delta_L[x^\mu] = \Lambda^\mu_\nu \otimes x^\nu + a^\mu \otimes 1, \quad (3)$$

which is an algebra homomorphism for the relations (1). This is the sense in which $\kappa$-Minkowski is the quantum homogeneous space associated to $\kappa$-Poincaré, and, in this light, it is legitimate to interpret $x^0$ as temporal and $x^i$ as spatial coordinates, because they transform as such, and the separation between time and space indices in the generators of the $\kappa$-Poincaré group is determined by the form of the metric $\eta^{\mu\nu}$ appearing in its relations.

One unusual property of the $\kappa$-Minkowski spacetime is that it is associated to a curved momentum space, a momentum space that generalizes the vector momentum space of Special/General Relativity into a pseudo-Riemannian geometry. This was first noticed in [10], and then further studied in a variety of works [10–16]. The simplest way to see in what sense one should think of momenta as living in a pseudo-Riemannian geometry is to consider the ordered plane waves built from the noncommutative coordinates (1):

$$\exp(ik_\mu x^\mu), \quad k_\mu \in \mathbb{R}^4. \quad (4)$$

these are useful because they provide a basis in which we can expand functions, in order to discuss field theories on $\kappa$-Minkowski (1) [17–23]. What is unusual is the fact that, because the product (1) is noncommutative, these plane waves do not combine in a linear way:

$$\exp(ik_\mu x^\mu) \exp(ig_\mu x^\mu) = \exp\left( i \frac{(k_0 + q_0) / \kappa}{e^{(k_0 + q_0) / \kappa} - 1} \left[ e^{k_0 / \kappa} - 1 \right] k_i + e^{-k_0 / \kappa} \left( e^{q_0 / \kappa} - 1 \right) q_i \right) x^i + i(k_0 + q_0)x^0. \quad (5)$$
This can be proven explicitly using only the commutation relations [19, 21]. We see that, since the commutation rules are those of a Lie algebra, the exponentials are closed under product, they form a subalgebra of the universal enveloping algebra of \( \mathfrak{an}(3) \). The law:

\[
(k, q) \rightarrow p, \quad p_i = \left( e^{k_0/\kappa - 1} \right)^{-1} k_i + e^{-k_0/\kappa} \left( e^{q_0/\kappa - 1} \right) q_i, \quad p_0 = k_0 + q_0,
\]

generalizes in a nonlinear way the familiar composition law of “wave vectors” (or Fourier parameters) \((k, q) \rightarrow k_\mu + q_\mu\), and reduces to it in the limit \(\kappa \rightarrow \infty\). It can be seen as a small deformation of it, when the wave vectors are much smaller than \(\kappa\) [24,25]. There is a consensus in the literature on the fact that this nonlinearity is a manifestation of the fact that the Fourier parameters are coordinates on a nonlinear manifold. In fact, exponentiating the generators of a Lie algebra like \( \mathfrak{an}(3) \), one obtains elements of the associated Lie group, which in our case is \( \text{AN}(3) \) [14]. Then, since the algebra is not Abelian, the composition law between the parameters in the exponentials is not linear, and they just codify the group product. As the theory of Lie groups prescribes, these parameters are coordinate systems on the group manifold.

The expression of plane waves in (4) is not the only possible choice to represent a plane wave, we had implicitly chosen an ordering prescription. There are other ordering choices, which give rise to other factorizations of the group elements. For example, the time generator can be ordered to the right, \( \exp(iq_i x^i) \exp(iq_0 x^0) \). Different ordering are related through nonlinear relations between the real parameters appearing in the exponentials. For the two examples above:

\[
\exp (i k_\mu x^\mu) = \exp \left( i \left( \frac{e^{k_0/\kappa - 1}}{k_0/\kappa} \right) k_i x^i \right) \exp \left( i k_0 x^0 \right),
\]

this transformation, \((k_0, k_i) \rightarrow \left( k_0, \frac{e^{k_0/\kappa - 1}}{k_0/\kappa} k_i \right)\) is a general coordinate change, \(i.e.,\) a diffeomorphism on the group manifold.

We interpret the group manifold associated to the Lie group \( \text{AN}(3) \) as the momentum space of theories on \( \kappa \)-Minkowski that make use of noncommutative plane waves, \(e.g.,\) (quantum) field theories, in which ordered plane waves are a basis for scalar fields and solutions of the equations of motion. Here we are interested in the geometry of this momentum space. In Lie group theory, there is a natural way to define a metric on the group manifold: if there is a nondegenerate Killing form, one can immediately define a bi-invariant metric. However, since the group \( \text{AN}(3) \) is not semi-simple, the Killing form is degenerate and there is no bi-invariant metric. There is, however, a basis of left-invariant forms and another one of right-invariant forms. As observed in [26], any quadratic form built from the symmetrized tensor product of right-invariant forms will give a right-invariant metric, the same for left-invariant metrics. All right(left)-invariant metrics with the same signature (and same rank) are equivalent modulo diffeomorphisms,\(^1\) but no right- and left-invariant metric are equivalent to each other. There is a certain freedom in choosing right- or left-invariant metrics, and we need the conditions which identify them.

The first paper which introduced a curved geometry for the momentum space of \( \kappa \)-Minkowski was [10], using a matrix representation of \( \mathfrak{an}(3) \) (see also [27–29]). The algebra (1) can be seen as a subalgebra of the five-dimensional Lorentz algebra \( \mathfrak{so}(4,1) \) via the isomorphism:

\[
x^\mu \sim M_0\mu + M_4\mu,
\]

\(^1\)In [26] it is stated that there is a unique right-invariant metric and a unique left-invariant one, but the author is implicitly assuming that the signature is \((+,-,-,-)\), and the rank is maximal (no zero eigenvalues).
where $M_{AB}$ ($A, B = 0, \ldots, 4$) are the Lorentz generators in the standard antisymmetric $5 \times 5$ matrix representation. This isomorphism induces the following five-dimensional representation of the commutation relations (1):

$$
\rho(x^0) = -\frac{i}{\kappa} \begin{pmatrix}
0 & 0 & 1 \\
0 & \hat{0} & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \rho(x^i) = -\frac{i}{\kappa} \begin{pmatrix}
0 & e_i & 0 \\
e_i & \hat{0} & e_i \\
0 & -e_i & 0
\end{pmatrix},
$$

(9)

where $e^a_i = \delta^a_i$, three-dimensional vector quantities are in boldface (we do not distinguish between rows and columns, it should be clear from the position in the matrix), $\hat{0}$ is the zero $3 \times 3$ matrix. This is a $*$-representation under the involution compatible with the Lorentz group $(\rho^*_\beta)^\alpha = \eta^{\alpha\lambda} \eta_{\beta\gamma} \rho^\gamma_\lambda$ (i.e. rising an index, flipping indices, complex conjugating and lowering back the index), which leaves all generators $\rho(x^\mu)$ invariant. Under this representation, the plane waves/group elements are represented as the matrices (in order to get simpler formulas we use the “time-to-the-right” ordering):

$$
G^*(p_\mu) = e^{ip_\mu \rho(x^i)} e^{ip_0 \rho(x^0)} = \begin{pmatrix}
cosh \frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa} \frac{\|p\|^2}{2\kappa^2}} & \frac{p_0}{\kappa} & \sinh \frac{p_0}{\kappa} + e^{\frac{p_0}{\kappa} \frac{\|p\|^2}{2\kappa^2}} \\
e^{-\frac{p_0}{\kappa} \frac{\|p\|^2}{2\kappa^2}} & 1 & e^{\frac{p_0}{\kappa} \frac{\|p\|^2}{2\kappa^2}} \\
\sinh \frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa} \frac{\|p\|^2}{2\kappa^2}} & -\frac{p_0}{\kappa} & \cosh \frac{p_0}{\kappa} - e^{\frac{p_0}{\kappa} \frac{\|p\|^2}{2\kappa^2}}
\end{pmatrix}.
$$

(10)

The idea is that, since the above representation is free, and the nondegenerate orbits have the dimension of the group, these will be diffeomorphic to the group manifold. The group orbits can be obtained by taking a fiducial vector $u^A$ in the five-dimensional vector space on which (10) acts, and considering the points obtained by acting upon $u$ with $G^*(p_\mu)$ for all choices of $p^\mu$:

$$
X^A = X^A(p_\mu) = G^*(p_\mu) A B u^B.
$$

(11)

$X^A(p_\mu)$ are the parametric representation of a four-dimensional submanifold embedded in a five-dimensional Minkowski space. This submanifold is diffeomorphic to the group manifold of $AN(3)$, and to momentum space. Since all $G^*(p_\mu)$ are elements of $SO(4, 1)$, we have that

$$
X^A X_A = X^A(p) X^B(p) \eta_{AB} = u^A u^B \eta_{AB}, \quad \eta_{AB} = \text{diag}(1, -1, -1, -1, -1),
$$

(12)

for all $p^\mu \in \mathbb{R}^4$. Choosing $u^A = (0, 0, 0, 0, 1)$, the above equation is that of de Sitter spacetime. The conclusion in [10] or [27] is that the geometry of momentum space is de Sitter. And indeed the metric on the orbit that is induced by the embedding $X^A(p)$:

$$
d s^2 = -\frac{\partial X^A}{\partial p_\mu} \frac{\partial X^B}{\partial p_\nu} \eta_{AB} dp_\mu dp_\nu = \frac{1}{\kappa^2} \left( -dp_0^2 + e^{2p_0/\kappa} \sum_{i=1}^3 dp_i^2 \right),
$$

(13)

is the same right-invariant metric found by [26]. Moreover, one can verify that for $u^A = (0, 0, 0, 0, 1)$ the relation $X^0 + X^4 > 0$ is verified for all choices of $p_\mu$, and therefore we are actually dealing with half of de Sitter spacetime, the half covered by the flat slicing (the coordinates $p_\mu$ corresponding to time-to-the-right ordering of plane waves are what cosmologists call comoving coordinates for de Sitter spacetime). This constraint makes the portion of momentum space covered by the $p_\mu$ coordinates non-Lorentz-invariant [12], and

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2 For example, exponentiating the standard representation of $su(2)$ as $2 \times 2$ complex matrices acting on the vector space of 2D spinors $\mathbb{C}^2$, one can prove that the nondegenerate orbits of the group are all 3-spheres embedded in $\mathbb{R}^4$ (under the canonical identification $\mathbb{R}^4 \sim \mathbb{C}^2$), and indeed the group manifold of $su(2)$ is, topologically, a 3-sphere.
one has to choose a different global topology for the ambient space (the elliptic topology) in order to restore Lorentz invariance \([23,30]\).

In [15] it was noticed that a different fiducial vector (in particular a time-like one \(u^A = (1,0,0,0,0)\)) gives rise to a different momentum space. Since the Lorentz group has disconnected different orbits, corresponding to different fiducial vectors, their choice is not inconsequential. The purpose of the present paper is to study the other momentum has disconnected different orbits, corresponding to different fiducial vectors, their choice is not inconsequential. The purpose of the present paper is to study the other momentum spaces with any (degenerate or not) choice of signature. We will show, in conclusion of the paper, a unifying picture that clarifies the role of these momentum spaces.

## 2 Geometries of momentum space

Consider a generic fiducial vector \(u^A\). The orbit of \(AN(3)\) acting on such a vector is:

\[
X^A(p_\mu) = G^*(p_\mu)^A Bu^B = \begin{pmatrix}
\frac{p \cdot u}{\kappa} + (u^0 + u^4)e^{\frac{p_\mu}{2\kappa}} + u^\mu \cos \frac{p_\mu}{\kappa} + u^4 \sinh \frac{p_\mu}{\kappa} \\
\kappa u + (u^0 + u^4)e^{\frac{p_\mu}{2\kappa}} \\
-\frac{p \cdot u}{\kappa} - (u^0 + u^4)e^{\frac{p_\mu}{2\kappa}} + u^\mu \sinh \frac{p_\mu}{\kappa} + u^4 \cosh \frac{p_\mu}{\kappa}
\end{pmatrix}.
\]  

(14)

Given a choice of fiducial vector, one can always realign the axes of the embedding space through a Lorentz transformation: \(X'^A = \Lambda^A_B X^B\), so that the vector \(\Lambda^A_B u^B\) is aligned along one (or two, in the lightlike case) of the \(X'^A\) axes. Then we identify three equivalence classes of choices of fiducial vectors, whose element all give rise to the same geometry: the spacelike, lightlike and timelike class. We choose to align the spacelike choice along the 4 axis, the lightlike case along the 0 − 4 plane, and the timelike choice along the 0 axis.

For the spacelike choice \(u^A = \delta^A_4\), we reproduce the known result of an embedding in Minkowski space of the patch of de Sitter space that is covered by comoving coordinates/flat slicing. If the fiducial vector is lightlike \(u^0 = u^4\), \(u^i = 0\), we simply obtain the limit of vanishing cosmological constant of the manifold above. This is the future-oriented light cone of the ambient Minkowski space, with the whole \(X^0 = -X^4\), \(X^i = 0\) line cut off (including the origin \(X^A = 0\)), as the \(p^\mu\) coordinates do not reach that line for any finite value of \(p^\mu\).

Finally, for \(u^A = \delta^A_0\), we get one of the two sheets of a Riemannian hyperbolic space, i.e. the positive-frequency mass-shell hyperboloid of a massive particle. The coordinates \(p_\mu\) in this case cover an entire sheet of the hyperboloid, because the whole sheet lies above the plane \(X^0 = -X^4\). The three manifolds we found are diffeomorphic to each other, i.e. they have the same topology, that of a plane. This is to be expected, because they are all diffeomorphic to the group manifold of of \(AN(3)\).

In addition to the three four-dimensional momentum spaces just described, there is a family of degenerate cases: if the fiducial vector is such that \(u^0 = -u^4\), then \(X^0(p) = -X^4(p)\) for all values of \(p\), and it generates a one-parameter family of points (a straight line). There is a three-parameter family of such straight lines, parametrized by the \(u^i\) components of \(u^A\). Each of these straight lines is lightlike, so the induced metric vanishes. In Table 2 we illustrate these results.

### 2.1 Isometries of the three momentum spaces

The representation of \(AN(3)\) we used above exploits the isomorphism with a subgroup of \(SO(4,1)\), the momentum spaces we found are the orbits of this subgroup in the ambient
Table 1: First column: norm of the fiducial vector. Second column: components of fiducial vector of choice. Third column: embedding coordinates for the corresponding momentum space. Fourth column: induced metric on momentum space. Last column: plot of the momentum space manifold embedded in the ambient Minkowski space (with coordinates \( X^2 \) and \( X^3 \) suppressed, one should imagine that each point on the manifold really represents a sphere of radius \( |X^1| \)).

The induced metric on these spaces, Eq. (13), is, by construction, explicitly invariant under \( SO(4,1) \) transformations of the ambient space (it depends on the ambient coordinates only through the invariant combination \( \partial X^A / \partial p^\mu \partial X^B / \partial p^\nu \eta_{AB} \)), so the isometry group of our momentum spaces are always \( SO(4,1) \). To find how this
group acts on the coordinates $p_\mu$ we can calculate the pull-back of the adjoint action of a
generic $SO(4, 1)$ element on the representation (10), and deduce a transformation law for the
momentum-space coordinates $p_\mu$. This is, in general, a rather complicated expression (check,
for example, the spacelike-$u$ case in the Appendix of [30]), however a formal argument allows
us to make some deductions. Call $\Lambda^{AB}$ a generic $SO(4, 1)$ element, then the transformation
laws of the point of coordinate $p_\mu$ is given by the formula:

$$X^A(p') = \Lambda^{AB}X^A(p) = \Lambda^{AB}G^*(p_\mu)^B_C u^C,$$

and, if the point is at the origin of the coordinate system, $p_\mu = 0$, then $G^*(p_\mu)^A_B = \delta^A_B,$
and $X^A(p') = \Lambda^{AB}u^B$. From this we can say that the stabilizer of $u^A$, $\Lambda^{AB}u^B = u^A$, leaves
the origin invariant. This allows us to identify the isotropy subgrop, i.e. the “Lorentz” part
of the isometries of our momentum space: they are the subgroups of $SO(4, 1)$ that stabilize
$u^A$.

The isotropy subgroup generalizes the Lorentz transformations of momentum space. In
the lightlike-$u^A$ (de Sitter hyperboloid) case, this is just the undeformed four-dimensional
Lorentz group, first noticed in [3]. For $u^A$ lightlike, the subgroup is ISO$(3)$, i.e. the Eu-
clidean group of rigid motions of $\mathbb{R}^3$. We will see that this has to do with the degenerate
geometry of the light-cone. Finally, in the timelike-$u^A$ case (two-sheeted Riemannian hy-
perboloid), the isotropy subgroup is $SO(4)$, the group of rotations of the three-dimensional
sphere. This is the appropriate isotropy group for a Riemannian space, whose metric is
positive-definite.

2.2 Embedding of $AN(3)$ into $SO(3, 2)$

Eq. (9) is not the only representation of $\mathfrak{an}(3)$. Consider this:

$$\rho'(x^0) = -\frac{i}{\kappa} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho'(x^1) = \frac{i}{\kappa} \begin{pmatrix} 0 & -e_1 & 0 \\ e_1 & 0 & e_1 \\ 0 & e_1 & 0 \end{pmatrix},$$

$$\rho'(x^2) = \frac{i}{\kappa} \begin{pmatrix} 0 & e_2 & 0 \\ e_2 & 0 & e_2 \\ 0 & -e_2 & 0 \end{pmatrix}, \quad \rho'(x^3) = \frac{i}{\kappa} \begin{pmatrix} 0 & e_3 & 0 \\ e_3 & 0 & e_3 \\ 0 & -e_3 & 0 \end{pmatrix},$$

(16)

The matrices above are all $\mathfrak{so}(3, 2)$ matrices, as in:

$$x^0 \sim J_{0,4}, \quad x^1 \sim J_{0,1} + J_{4,1}, \quad x^2 \sim J_{0,2} + J_{4,2}, \quad x^3 \sim J_{0,3} + J_{4,3}, \quad J_{AB} \in \mathfrak{so}(3, 2),$$

(17)

where the coordinates 0 and 1 have the same signature, opposite to that of coordinates 2,
3 and 4. The difference between Eq. (9) and (16) is merely the form of $\rho(x^1)$. In (9) which
the antisymmetric components are the 4-1 and the 0-1 are symmetric, indicating that the
coordinate 1 has the same nature of coordinate 4 and opposite signature with respect to
coordinate 0. In Eq. (16) this is inverted, and so the axis 1 has the same signature as 0. We
can play this game of choosing the signature of the 1, 2 and 3 axes only in two ways: either
they have all the same signature, which will be the same of either axis 0 or 4 (they are forced
to have opposite signatures by the fact that $\rho(x^0)$ is symmetric), and we fall into case (9).
Otherwise, one of the three coordinates can have a different signature from the others, and
then, exchanging the axes, we can always cast our matrices in the form (16).

The different embeddings have consequences for the corresponding momentum spaces.
Consider the exponentiation
\[ G^\gamma(p_\mu) = e^{ip_\rho(x')} e^{ip_\rho'(x'')} = \left( \begin{array}{cc} \cosh \frac{p_0}{\kappa} + e^{p_0/\kappa} \left( \frac{p_1^2 + p_2^2 + p_3^2}{2\kappa^2} \right) & \frac{p_1}{\kappa} - \frac{p_2}{\kappa} - \frac{p_3}{\kappa} \
- e^{p_0/\kappa} p_1 & 1 \\
- e^{p_0/\kappa} p_2 & 0 \\
- e^{p_0/\kappa} p_3 & 0 \end{array} \right) \sinh \frac{p_0}{\kappa} + e^{p_0/\kappa} \left( \frac{p_1^2 + p_2^2 + p_3^2}{2\kappa^2} \right). \] (18)

The action of the group element on a generic fiducial vector is
\[ X'^A(p_\mu) = G^\gamma(p_\mu)^A_B u^B = \left( \begin{array}{c} u^0 \cosh \left( \frac{p_0}{\kappa} \right) + \frac{(p_1^2 + p_2^2 + p_3^2)}{2\kappa^2}(u^0 + u^4) e^{p_0/\kappa} + u^4 \sinh \left( \frac{p_0}{\kappa} \right) + \frac{-p_3 u^3 - p_2 u^2 + p_1 u^4}{\kappa} \\
u^1 = \frac{p_1 (u^0 + u^4) e^{p_0/\kappa}}{\kappa} \\
u^2 = \frac{p_2 (u^0 + u^4) e^{p_0/\kappa}}{\kappa} \\
u^3 = \frac{p_3 (u^0 + u^4) e^{p_0/\kappa}}{\kappa} \end{array} \right). \] (19)

We see that
\[ X'^A X'_A = X'^A(p) X'^B(p) \eta'_{AB}(p) = u^A u^B \eta'_{AB}, \quad \eta'_{AB} = \text{diag}(-1, -1, +1, +1, +1). \] (20)

Therefore, if \( u^A u^B \eta'_{AB} < 0, X'^A \) are embedding coordinates for anti-de Sitter space, which is a one-sheeted hyperboloid with axis aligned along the spacelike coordinates. If \( u^A u^B \eta'_{AB} = 0 \) we have the light cone that the AdS hyperboloid tends to in the limit of vanishing cosmological constant (unless \( u^0 = -u^4 \), in which case we have a degenerate geometry). Finally if \( u^A u^B \eta'_{AB} > 0 \) we have a two-sheeted hyperboloid with signature \((+, +, -, -)\).

We also have to take into account the fact that:
\[ X'^0 + X'^4 = e^{p_0/\kappa} (u^0 + u^4), \] (21)
which implies that the sign of \( X'^0 + X'^4 \) is fixed and equal to the sign of \( u^0 + u^4 \). Assume without loss of generality that \( u^0 + u^4 > 0 \) (the other case mirrors this one). In the anti-de Sitter case, \( i.e. \) \( u^A u^B \eta'_{AB} < 0 \), this bound implies that the coordinates \( p_\mu \) cover the half-space coordinatization of anti-de Sitter. In fact the induced metric on the orbit is, for example if \( u^A = (1, 0, 0, 0, 0) \):
\[ ds^2 = \eta'_{AB} \frac{\partial X'^A}{\partial p_\mu} \frac{\partial X'^B}{\partial p_\nu} dp_\mu dp_\nu = \frac{1}{\kappa^2} \frac{e^{2p_0/\kappa}}{\kappa^2} (dp_1^2 - dp_2^2 - dp_3^2), \] (22)
and by transforming \( p_0 = -\kappa \log(y/\kappa) \)
\[ ds^2 = \frac{1}{y^2} \left( dy^2 + dp_2^2 + dp_3^2 - dp_1^2 \right), \] (23)
which we recognize as the coordinate patch covering half of AdS spacetime [31].

The \( u^A u^B \eta'_{AB} = 0 \) case will be again a cone, but its intersection with the half-space \( X'^0 + X'^4 > 0 \) this time will not leave out simply a line, unless we are in the 1+1-dimensional case. In fact Eq. (12) implied that
\[ (X'^0)^2 = (X'^4)^2 + \sum_{i=1}^{3} (X'^i)^2 \geq (X'^4)^2, \] (24)
and so $X^0 = -X^4$ only on the line $X^i = 0$. On the other hand, the embedding of $\mathfrak{an}(3)$ into $\mathfrak{so}(3,2)$ gives

$$(X^0)^2 = (X^4)^2 - (X^i)^2 + (X^3)^2 + (X^1)^2,$$  \hspace{1cm} (25)

and the intersection of this submanifold with $X^0 + X^4 > 0$ leaves out a non-zero measure portion of the cone. In the 1+1-dimensional case, however, this difference disappears, because $(X^0)^2 + (X^4)^2 = (X^4)^2$ implies that $(X^4)^2 \geq (X^0)^2$, and only the line $X^1 = 0$ is left out (see Table 2.4).

The $u^A u^B \eta_{AB} > 0$ case too is greatly simplified by going to 1+1 dimensions: in general one has, in fact:

$$(X^4)^2 + (X^2)^2 + (X^3)^2 > (X^0)^2 + (X^1)^2,$$  \hspace{1cm} (26)

which has a quite complicated intersection with $X^0 + X^4 > 0$. But if we suppress $X^2$ and $X^3$ we get

$$(X^4)^2 > (X^0)^2 + (X^1)^2,$$  \hspace{1cm} (27)

which never intersects the plane $X^0 = -X^4$.

2.3 Isometries of the three new momentum spaces

Just like in the $SO(4,1)$ case, the isotropy subgroups have to be identified with the subgroups of $SO(3,2)$ that stabilize $u^A$. In the $u^A u^B \eta_{AB} < 0$ case, this is the subgroup that stabilizes a timelike vector, and so it is the Lorentz group $SO(3,1)$. For $u^A$ lightlike, the subgroup is $ISO(2,1)$, i.e. the Poincaré group in 2+1 dimensions. Finally, in the $u^A u^B \eta'_{AB} > 0$ case, the group is $SO(2,2)$.

The action of these groups on the corresponding momentum spaces are such that a finite transformation can bring a point outside of the coordinate patch covered by the $p_\mu$ coordinates, just like in the previous Section for space- and light-like fiducial vectors. This time, however, this phenomenon happens for all choices of fiducial vector - there is no momentum space that is exempt from it.

2.4 The right-invariant metrics on $AN(3)$

As we remarked in the introduction, the paper [26] observed how all the quadratic form built from the right-invariant forms on $AN(3)$ is a right-invariant metric, and all such metrics are equivalent modulo diffeomorphisms. However, [26] implicitly assumed a Lorentzian signature for these metrics, effectively eliminating the only freedom that one has in choosing the metric: its signature, which cannot be changed by a diffeomorphism. One has therefore the following four inequivalent choices (if we limit ourselves to nondegenerate metrics):

$$ds^2 = dp_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2} \left( dp_1^2 + dp_2^2 + dp_3^2 \right),$$  \hspace{1cm} (28)

$$ds^2 = dp_0^2 - \frac{e^{2p_0/\kappa}}{\kappa^2} \left( dp_1^2 + dp_2^2 + dp_3^2 \right),$$  \hspace{1cm} (29)

$$ds^2 = dp_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2} \left( -dp_1^2 + dp_2^2 + dp_3^2 \right),$$  \hspace{1cm} (30)

$$ds^2 = -dp_0^2 + \frac{e^{2p_0/\kappa}}{\kappa^2} \left( dp_1^2 + dp_2^2 + dp_3^2 \right).$$  \hspace{1cm} (31)

We have encountered all these metrics with our embeddings of $AN(3)$ into $SO(4,1)$ and $SO(3,2)$: the first is the Riemannian metric of the two-sheeted hyperboloid of $SO(4,1)$, the
second is the dS metric of $SO(4,1)$, the third is the AdS metric of $SO(3,2)$ and the last is the signature $(+,+,−,−)$ hyperboloid of $SO(3,2)$.

| $u^A$ | $X'^A$ | $X'^A X'^A$ | $ds^2$ | plot |
|-------|---------|-------------|--------|------|
| $(1\ 0\ 0)$ | $\begin{pmatrix} \cosh (\frac{p_0}{\kappa}) + \frac{(p \cdot p)e^{\frac{p_0}{2\kappa^2}}}{2e^2} \\ -\frac{p_1 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ -\frac{p_2 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ -\frac{p_3 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ \sinh (\frac{p_0}{\kappa}) - \frac{(p \cdot p)e^{\frac{p_0}{2\kappa^2}}}{2e^2} \end{pmatrix}$ | $-1$ | $\frac{1}{\kappa^2} dp_0^2 + \frac{e^{2\kappa^2/\kappa^2}}{\kappa^2} dp \cdot dp$ |
| $(\frac{1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}})$ | $\begin{pmatrix} \frac{(\kappa^2+p \cdot p)e^{\frac{p_0}{2\kappa^2}}}{\sqrt{2}e^2} \\ -\frac{\kappa^2 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ -\frac{\kappa^2 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ -\frac{\kappa^2 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ \frac{(\kappa^2 - p^2)e^{\frac{p_0}{2\kappa^2}}}{\sqrt{2}e^2} \end{pmatrix}$ | $0$ | $\frac{2}{\kappa^2} e^{\frac{2\kappa^2}{\kappa^2}} dp \cdot dp$ |
| $(0\ 0\ 1)$ | $\begin{pmatrix} \sinh (\frac{p_0}{\kappa}) + \frac{(p \cdot p)e^{\frac{p_0}{2\kappa^2}}}{2e^2} \\ -\frac{p_1 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ -\frac{p_2 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ -\frac{p_3 e^{\frac{p_0}{\kappa}}}{\sqrt{2}e^2} \\ \cosh (\frac{p_0}{\kappa}) - \frac{(p \cdot p)e^{\frac{p_0}{2\kappa^2}}}{2e^2} \end{pmatrix}$ | $1$ | $\frac{1}{\kappa^2} e^{\frac{2\kappa^2}{\kappa^2}} dp \cdot dp - \frac{1}{\kappa^2} dp_0^2$ |
| $(u^0\ u\ -u^0)$ | $\begin{pmatrix} u^0 e^{-\frac{p_0}{\kappa}} - \frac{p \cdot u}{\kappa} \\ u^1 \\ u^2 \\ u^3 \\ \frac{p \cdot u}{\kappa} - u^0 e^{-\frac{p_0}{\kappa}} \end{pmatrix}$ | $u \cdot u$ | $0$ |

Table 2: First column: norm of the fiducial vector. Second column: components of fiducial vector of choice, where $p \cdot p = p_1^2 + p_2^2 + p_3^2$, $p \cdot u = -p_1 u^1 + p_2 u^2 + p_3 u^3$ and $dp \cdot dp = -dp_1^2 + dp_2^2 + dp_3^2$. Third column: embedding coordinates for the corresponding momentum space. Fourth column: induced metric on momentum space. Last column: plot of the momentum space manifold embedded in the ambient Minkowski space in the 1+1-dimensional case. The higher-dimensional cases are impossible to represent on paper, and are significantly more complicated, in that the regions of the submanifolds that are excluded from the coordinate patch, in the second and third lines, are not measure-zero and are rather complicated.
3 κ–Minkowski: a noncommutative space with arbitrary signature

In this section we show that one can start with an arbitrary signature Poincaré-like Lie algebra, and show that with a proper (and unique) $r$-matrix its deformation is compatible with a κ-Minkowski space.

Consider the $ISO(p, 4 - p)$ algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\nu\sigma}M_{\mu\rho},$$  
$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu,$$  

(32)

where now $g_{\mu\nu} = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, and $\lambda_\mu = \pm 1$ or 0. Let us also call $K_i = M_{0i}$ and $M_1 = R_3, M_2 = R_1, M_3 = R_2$.

According to Zakrzewski, [32, 33] all Lie bialgebras built upon $\text{iso}(p, q)$ are coboundaries. Then we can write the most generic ansatz for the $r$-matrix:

$$r = a^\mu_\nu P_\mu \wedge P_\nu + b^{\mu\rho\sigma}M_{\mu\nu} \wedge P_\rho + c^{\mu\nu\rho\sigma}M_{\mu\nu} \wedge M_{\rho\sigma}.$$  

(33)

moreover, imposing that the corresponding cocommutators are such that:

$$\delta(P_0) = 0, \quad \delta(P_1) \propto P_1 \wedge P_0, \quad \delta(M) = M \wedge P + M \wedge M,$$  

(34)

where the last equation is a formal expression indicating that terms of the type $P_\mu \wedge P_\nu$ cannot appear in the cocommutator of $M_{\mu\nu}$.

This leaves an $r$-matrix of the form:

$$r = \lambda_1 K_1 \wedge P_1 + \lambda_2 K_2 \wedge P_2 + \lambda_3 K_2 \wedge P_3 + \alpha R_1 \wedge R_2 + \beta R_1 \wedge R_3 + \gamma R_2 \wedge R_3,$$  

(35)

Imposing the co-Jacobi equations (or, equivalently, the YBE), we get:

$$r = \lambda_1 K_1 \wedge P_1 + \lambda_2 K_2 \wedge P_2 + \lambda_3 K_3 \wedge P_3 + \alpha R_1 \wedge R_2 + \beta R_1 \wedge R_3 + \gamma R_2 \wedge R_3,$$  

(36)

where $\lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2 = 0$. The $\alpha, \beta, \gamma$ terms generalize to arbitrary signature the twist described by Ballesteros et al. [29], while the other term generates the κ-Minkowski cocommutators (34) for any choice of signature.

We conclude that, for any choice of $g_{\mu\nu} = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, the algebra of isometries of $g_{\mu\nu}$ admits a quantum deformation which preserves the κ-Minkowski commutation relation in the sense that Eqs. (34) are satisfied.

We can conclude that there are momentum spaces associated to the κ–Minkowski noncommutative space-time with all possible (degenerate or not) signatures: κ–Minkowski, defined by (1) is left invariant by the left co-action$^3$:

$$\Delta[x_\mu] = \Lambda_\mu^\nu \otimes x^\nu + a_\mu \otimes 1,$$  

(37)

of the following Quantum Group(s):

$$\Delta_R[\Lambda_\mu^\nu] = \Lambda_\mu^\alpha \otimes \Lambda_\nu^\alpha, \quad [\Lambda_\mu^\nu, \Lambda_\rho^\gamma] = 0, \quad [\Lambda_\mu^\nu, \Lambda_\beta^\sigma] = 0,$$  
$$\Delta[a^\mu] = a^\mu \otimes 1 + a^\mu \otimes \Lambda_\nu^\mu, \quad [a^\mu, a^\nu] = \frac{1}{\kappa} [\Lambda_\alpha^\mu \delta_\alpha^\nu - \delta_\nu^\alpha] \Lambda^\gamma_\nu + (\Lambda_\nu^\alpha \delta_\alpha^\mu - \delta_\mu^\alpha) a^\gamma]$$,  
$$\Delta[\Lambda] = \Lambda^{-1} \otimes 1, \quad [\Lambda^\mu_\nu, a^\gamma] = \frac{1}{\kappa} a^\gamma, \quad [a^\mu, a^\nu] = 0$$  

(38)

$^3$The action $\Delta_R$ is a homomorphism for (1).
for $\Lambda_\mu^\nu$ satisfying the following algebraic rules

\begin{align}
\Lambda_\nu^\mu \Lambda_\beta^\alpha g^{\alpha\beta} &= g^{\mu\nu}, \\
\Lambda_\rho^\mu \Lambda_\sigma^\nu g_{\rho\sigma} &= g_{\mu\nu},
\end{align}

(39)

for any choice of matrices $g^{\mu\nu}$ and $g_{\rho\sigma}$. In the nondegenerate cases $\det g \neq 0$ the metrics with upper and lower indices are the inverse of each other. In the degenerate cases $\det g = 0$, the matrices are not invertible, but they are complementary, in the sense that their null eigenspaces are orthogonal. This allows us to describe the known Carroll and Galilei groups [34] as, respectively, $g_{\mu\nu} = \text{diag}(0,1,1,1)$, $g^{\mu\nu} = \text{diag}(1,0,0,0)$, and $g_{\mu\nu} = \text{diag}(1,0,0,0)$, $g^{\mu\nu} = \text{diag}(0,1,1,1)$.

4 Conclusions and Outlook

We found that it is possible to put any constant (diagonal) metric on momentum spaces of a $\kappa$-Minkowski spacetime, finding always a maximal symmetry group which respects the commutation relations (1), leaving the metric invariant. In particular the nondegenerate cases are compatible with a de Sitter case, which was well studied, but also with an anti de Sitter case, the Euclidean metric, and with a “two times” metric ($-,+,+,+)$). Keeping with the usual metric, we found in Sect. 2.2 a novel five dimensional representation of the Lie algebra of the coordinates of $\kappa$-Minkowski. This gives rise to a new bona fide Lorentzian case. It will be important in the future to understand if this new case enables a genuine new different physical model. The topology of all these momentum spaces is that of a plane, but the way the metric deals with the time and space components is very different. In particular, the two Lorentzian cases attribute a timelike nature to different coordinates: the known de Sitter momentum space makes $x^0$ timelike and $x^i$ spacelike, while the news anti-de Sitter one makes one of the $x^i$ (or a linear combination thereof) timelike, and the others spacelike. This issue deserves further scrutiny.

Likewise there are things to understand for the degenerate cases as well. From a physical point of view it is too early to dismiss them as “unphysical”, since they might have applications in extreme situations, like near the horizon of black holes or near Big Bang singularities, where the local symmetries of spacetime become Carrollian [35–37]. Moreover, these kinds of metrics are limits obtainable as group contractions, for example by sending $c$ to 0 (Carrollian limit) or $\infty$ (Galilean limit). In this respect it is intriguing the fact that we did not find the momentum spaces with metric $g^{\mu\nu} = \text{diag}(1,0,0,0)$ or any case doubly degenerate (with two zero and two nonzero diagonal elements). For the first there can be connections with the $\kappa$-Galileo Hopf algebra [34,38–41].

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