A WELL-POSEDNESS THEORY IN SOBOLEV SPACES FOR THE STOCHASTIC MAGNETOHYDRODYNAMIC EQUATIONS IN THE WHOLE SPACE

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ABSTRACT. We prove the existence of a mild solution to the three dimensional incompressible stochastic magnetohydrodynamic equations in the whole space with the initial data which belong to the Sobolev spaces. Keywords: stochastic partial differential equations; magnetohydrodynamic equations; mild solution
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1. INTRODUCTION

Magnetohydrodynamics is the branch of physics that studies the dynamics for electrically conducting fluids influenced by a magnetic field. They are frequently generated in nature, for example, the sun, beneath the Earth’s mantle, plasmas in space, liquid metals, and so on. For the background knowledge we refer the reader to Davidson’s monograph [1]. The aim of this paper is to establish the existence of a stochastic mild solution to the three dimensional incompressible magnetohydrodynamic (MHD) equations driven by stochastic external forces, which can be described as the following system of stochastic partial differential equations

\[
\begin{aligned}
(\partial_t - \Delta) u + \nabla \cdot (u \otimes u - b \otimes b) + \nabla \pi &= g_1 \frac{dW^1_k}{dt} \\
(\partial_t - \Delta) b + \nabla \cdot (u \otimes b - b \otimes u) &= g_2 \frac{dW^2_k}{dt} \\
\nabla \cdot u &= \nabla \cdot b = 0
\end{aligned}
\]

in \((0, T) \times \mathbb{R}^3\) with divergence-free initial vector fields \(u_0\) and \(b_0\), where \(u, b, \pi\), and \(\frac{dW_k}{dt}\) denote the velocity field, the magnetic field, the pressure of fluid, and independent one-dimensional white noises \((k = 1, 2, \ldots)\), respectively. We note that Einstein’s summation convention is used throughout the paper.

In the absence of magnetic field, the MHD equations reduce to the Navier–Stokes equations. There are huge literature about the theory of the Navier–Stokes equations. For the deterministic Navier–Stokes equations, Fujita and Kato [2] initiated the study for the existence of the mild solution with initial data in the critical Sobolev space \(H^{1/2}_2\). Many mathematicians have been interested in its stochastic versions due to the complicate dynamics of fluid motions. Naturally, there are many articles handling stochastic Navier–Stokes equations extend the deterministic results. However, there are only a few results for the
stochastic MHD equations. We refer the reader for background and history of these results to the introduction of [9], where the authors also considered well-posedness of three-dimensional incompressible MHD equations with stochastic external forces.

Before stating our main results, we motivate the definition of a mild solution and simplify the settings by introducing a few notations. We denote by \( P \) the Leray projection operator onto divergence-free vector fields. In \( \mathbb{R}^3 \) it can be expressed as

\[
P = I + R \otimes R,
\]

where \( R = (R_1, R_2, R_3) \) denotes the Riesz transforms and \( ((R \otimes R)u) = \sum R_i R^j u^j \). By applying the Leray projection \( P \) to (1) formally, one can remove the pressure term since \( P \nabla \pi = 0 \). By introducing new variables

\[
v = u + b, \quad w = u - b,
\]

one can rewrite the equations (1) as

\[
\begin{align*}
(\partial_t - \Delta)v &= -P \nabla \cdot (w \otimes v) + G^1_k \frac{dw^k}{dt} \\
(\partial_t - \Delta)w &= -P \nabla \cdot (v \otimes w) + G^2_k \frac{dw^k}{dt} \\
\nabla \cdot v &= \nabla \cdot w = 0
\end{align*}
\]

with the corresponding divergence-free initial data \( v_0 = u_0 + b_0 \) and \( w_0 = u_0 - b_0 \) and the stochastic external forces \( G^1_k = P g^1_k + P g^2_k \) and \( G^2_k = P g^1_k - P g^2_k \).

In order to neatly write the equations, we denote the heat semi-group by \( S(t) = e^{t\Delta} \) and define the following bilinear map

\[
B(u,v)(t) = \int_0^t S(t-s)P \nabla \cdot (u \otimes v)(s)ds.
\]

Solving the heat equation by Duhamel’s formula motivates the definition of a mild solution of the stochastic MHD equations. We say that \( (v, w) \) is a (mild) solution to (3) on \( (0,T) \) if it solves for \( 0 \leq t < T \) the integral equations

\[
\begin{align*}
v &= S(t)v_0 - B(w, v) + \int_0^t S(t-s)G^1_k(s)dw^k_s \\
w &= S(t)w_0 - B(v, w) + \int_0^t S(t-s)G^2_k(s)dw^k_s,
\end{align*}
\]

where \( w^k_s \) is the Brownian motion related to the white noisy \( \frac{dw^k}{dt} \).

In [9], well-posedness for the stochastic MHD equations was studied for the initial data

\[
v_0, w_0 \in L_2(\Omega, \mathcal{F}_0; H^{1/2+\alpha}(\mathbb{R}^3))
\]

with \( 0 < \alpha < 1/2 \). We extend the well-posedness result for wider class of initial data

\[
v_0, w_0 \in L_5(\Omega, \mathcal{F}_0; H^{-2/5}_P),
\]
but we admit that this paper does not cover the multiplicative noise case handled in [9] with extra regularity condition on initial value. The precise statements of our main results is the following two theorems. The exact notations and definitions for function spaces are presented in the next section.

**Theorem 1.** Let \( v_0, w_0 \in L_2 \left( \Omega, \mathcal{F}_0; \dot{H}^{1/2}_{4,\sigma}(\mathbb{R}^3) \right) \) and \( G_1, G_2 \in \dot{H}^{1/2}_{3,\sigma}(\infty, l_2) \).

(i) There exists a positive number \( T_0 \) such that the equation (3) with \( T = T_0 \) has a solution

\[
v, w \in L_2 \left( \Omega, \mathcal{F}; C \left( [0, T_0]; \dot{H}^{1/2}_2 \right) \right) \cap L_2 \left( \Omega \times (0, T_0), \mathcal{P}; \dot{H}^{3/2}_2 \right).
\]

(ii) There exists a positive number \( \epsilon \) such that if

\[
\| (v_0, w_0) \|_{L_2(\Omega, \mathcal{F}; \dot{H}^{1/2}_2)} + \| (G_1, G_2) \|_{\dot{H}^{1/2}_{3,\sigma}(\infty, l_2)} < \epsilon,
\]

then there exists a global in time solution \( (v, w) \) to the equation (3) with

\[
v, w \in L_2 \left( \Omega, \mathcal{F}; C \left( [0, \infty); \dot{H}^{1/2}_2 \right) \right) \cap L_2 \left( \Omega \times (0, \infty), \mathcal{P}; \dot{H}^{3/2}_2 \right).
\]

**Theorem 2.** Let \( v_0, w_0 \in L_5 \left( \Omega, \mathcal{F}_0; \dot{H}^{-2/5}_{3,\sigma} \right) \) and \( G_1, G_2 \in \dot{H}^{-1}_{3,\sigma}(\infty, l_2) \).

(i) There exists a positive number \( T_0 \) such that the equation (3) with \( T = T_0 \) has a solution \( (v, w) \in L_5(T_0) \times L_5(T_0) \).

(ii) There exists a positive number \( \epsilon \) such that if

\[
\| (v_0, w_0) \|_{L_5(\Omega, \mathcal{F}; \dot{H}^{-2/5}_3)} + \| (G_1, G_2) \|_{\dot{H}^{-1}_{3,\sigma}(\infty, l_2)} < \epsilon,
\]

then there exists a global in time solution \( (v, w) \) to the equation (3) with

\[
v, w \in L_5(\infty).
\]

**Remark 1.** If \( X = L_5(\infty) \) or

\[
X = L_4 \left( \Omega, \mathcal{F}; L_4 \left( [0, \infty); \dot{H}^{1/2}_2(\mathbb{R}) \right) \right),
\]

then there is a positive constant \( C \) such that

\[
\| B(v, w) \|_{X \times X} \leq C \| v \|_X \| w \|_X \quad \forall v, w \in X,
\]

which is proved in Proposition 2 and Proposition 3. We can obtain an upper bound \( \epsilon < \frac{1}{4C_i} \) \((i = 5, 6)\) in the second parts of main theorems, where constants \( C_5 \) and \( C_6 \) appear in Corollary 4 and Corollary 2 respectively. Moreover, in this case, by the uniqueness of the fixed point theorem, the solution is unique in the closed subspace \( \{(u, v) \in X \times X : \|(u, v)\|_{X \times X} \leq 2\epsilon\} \) (see Lemma 7).
Remark 2. Although we took physical constants to be 1, it is possible to consider the general physical constants in the MHD equations, that is,
\[
\begin{align*}
(\partial_t - \nu_1 \Delta)v &= -\mathbb{P} \nabla \cdot (w \otimes v) + G^k_1 \frac{dw^k}{dt} \\
(\partial_t - \nu_2 \Delta)w &= -\mathbb{P} \nabla \cdot (v \otimes w) + G^k_2 \frac{dw^k}{dt} \\
\nabla \cdot v &= \nabla \cdot w = 0,
\end{align*}
\]
where the kinematic viscosity \( \nu_1 \) and the magnetic resistivity \( \nu_2 \) are positive constants. In this paper, we focus on studying (3) for simplicity.

The organization of this paper is as follows. In Section 2, we introduce notations and definitions used throughout this paper. In Section 3, we survey linear theories for stochastic partial differential equations. In Section 4, we prove bilinear estimates which play a crucial role. In Section 5, we complete the proofs of main theorems.

2. Notations and definitions

The purpose of this section is to introduce notations and definitions which will be used throughout this paper.

- Let \( \mathbb{N} \) and \( \mathbb{Z} \) denote the natural number system and the integer number system, respectively. As usual \( \mathbb{R}^d \), \( d \in \mathbb{N} \), stands for the Euclidean space of points \( x = (x^1, \ldots, x^d) \).

- The gradient of a function \( f \) is denoted by \( \nabla f = (D_1 f, D_2 f, \ldots, D_d f) \).

  where \( D_i f = \frac{\partial f}{\partial x_i} \) for \( i = 1, \ldots, d \) and the divergence of a vector field \( v = (v^1, \ldots, v^d) \) is denoted by

  \[
  \nabla \cdot v := \sum_{i=1}^d D_i v^i.
  \]

- Let \( C^\infty(\mathbb{R}^d) \) denote the space of infinitely differentiable functions on \( \mathbb{R}^d \). Let \( C_c^\infty(\mathbb{R}^d) \) denote the subspace of \( C^\infty(\mathbb{R}^d) \) with the compact support. Let \( C_c^\infty,\sigma(\mathbb{R}^d) \) denote the subspace of \( C_c^\infty(\mathbb{R}^d) \) with divergence free. Let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space consisting of infinitely differentiable and rapidly decreasing functions on \( \mathbb{R}^d \). We simply write \( C^\infty, C_c^\infty, C_c^\infty,\sigma, \mathcal{S} \) by omitting \( (\mathbb{R}^d) \).

- For \( \mathcal{O} \subset \mathbb{R}^d \) and a normed space \( F \), we denote by \( C(\mathcal{O}; F) \) the space of all \( F \)-valued continuous functions \( u : \mathcal{O} \to F \) with the norm

  \[
  |u|_C := \sup_{x \in \mathcal{O}} |u(x)|_F < \infty.
  \]
• For \( p \in [1, \infty) \), a normed space \( F \), and a measure space \((X, \mathcal{M}, \mu)\), we denote by \( L_p(X, \mathcal{M}, \mu; F) \) the space of all \( \mathcal{M}^\mu \)-measurable functions \( u : X \to F \) with the norm
\[
\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|^p \mu(dx) \right)^{1/p} < \infty
\]
where \( \mathcal{M}^\mu \) denotes the completion of \( \mathcal{M} \) with respect to the measure \( \mu \). If there is no confusion for the given measure and \( \sigma \)-algebra, we usually omit them.

• We denote by \(|\theta|\) the Lebesgue measure of a measurable set \( \theta \subset R^d \).

• Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \( u(\omega, t, x) \) and \( v(\omega, t, x) \) be stochastic processes on \( \Omega \times (0, \infty) \times R^d \). We say that with probability one, for all \( t \in (0, \infty) \)
\[
u(\omega, t, x) = v(\omega, t, x) \quad (x-a.e.)
\]
if there exists \( \Omega' \subset \Omega \) such that \( P(\Omega') = 1 \) and for all \( (\omega', t) \in \Omega' \times (0, \infty) \),
\[
u(\omega', t, x) = v(\omega', t, x)
\]
holds for almost every \( x \in R^d \). For the notational convenience, the random parameter \( \omega \) will be usually omitted.

• We denote the \( d \)-dimensional Fourier transform of \( f \) by
\[
\mathcal{F}[f](\xi) := \int_{R^d} e^{-2\pi i \xi \cdot x} f(x)dx
\]
and the \( d \)-dimensional inverse Fourier transform of \( f \) by
\[
\mathcal{F}^{-1}[f](x) := \int_{R^d} e^{2\pi i \xi \cdot x} f(\xi)d\xi.
\]

• If we write \( C = C(a, b, \cdots) \), this means that the constant \( C \) depends only on \( a, b, \cdots \).

• We shall write \( A \lesssim B \) if there is a positive generic constant \( C \) such that \( |A| \leq C|B| \).

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, and \( \{\mathcal{F}_t, t \geq 0\} \) be a filtration satisfying the usual condition, i.e. \( \{\mathcal{F}_t\} \) is increasing, right continuous, and each \( \mathcal{F}_t \) contains all \( (\mathcal{F}, P) \)-null sets. In other words, \( \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \) if \( t_1 \leq t_2 \), \( \bigcap_{t < \infty} \mathcal{F}_t = \mathcal{F}_\infty \), and \( A \subset \mathcal{F}_t \) for all \( t \geq 0 \) if there exists a \( B \in \mathcal{F} \) such that \( A \subset B \) and \( P(B) = 0 \). We denote by \( \mathcal{P} \) the predictable \( \sigma \)-field generated by \( \{\mathcal{F}_t, t \geq 0\} \). We assume that \( w^k_t \) are independent one-dimensional Brownian motions (Wiener processes) on \((\Omega, \mathcal{F}, P)\) for \( (k = 1, 2, \cdots) \) and they are relative to \( \{\mathcal{F}_t, t \geq 0\} \).

We end this section by introducing inhomogeneous and homogeneous Sobolev spaces and related function spaces used in this article.

**Definition 1** (Inhomogeneous Sobolev spaces). (1) For \( n \in R \) and \( p \in (1, \infty) \), define the space
\[
H^n_p(R^d) = (1 - \Delta)^{-n/2} L_p(R^d)
\]
(called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the set of all tempered distributions \( u \) such that
\[
(1 - \Delta)^{n/2} u := \mathcal{F}^{-1} \left[ (1 + |2\pi \xi|^2)^{n/2} \mathcal{F}(u)(\xi) \right] \in L_p(R^d)
\]
with the norm
\[ \|u\|_{H_p^n(R^3)} := \|(1 - \Delta)^{n/2} u\|_{L_p(R^3)} < \infty. \]

(2) The set of all \( u = (u^1, u^2, u^3) \) such that \( u^1, u^2, u^3 \in H_p^n(R^3) \) is denoted by \( H_p^n(R^3; R^3) \) and the norm is given by
\[ \|u\|_{H_p^n(R^3; R^3)} := \sum_{i=1}^{3} \| (1 - \Delta)^{n/2} u^i \|_{L_p(R^3)} < \infty. \]

\( H_p^n(R^3; R^3) \) denotes the closure of \( C_c^\infty(R^3; R^3) \) in \( H_p^n(R^3; R^3) \).

For a sequence
\[ a = (a^1, a^2, \ldots) = (a^k)_{k \in \mathbb{N}}, \]
we define \( \|a\|_{l_2}^2 := \sum_{k=1}^{\infty} |a^k|^2 \) and denote by \( l_2 \) the space of all sequences \( a \) so that \( \|a\|_{l_2} < \infty \). For \( u = (u^k)_{k \in \mathbb{N}} \) (\( u^k \in H_p^n(R^3) \)), we define the space \( H_p^n(R^3; l_2) \) as the set of all \( l_2 \)-valued tempered distributions such that
\[ \|u\|_{H_p^n(R^3; l_2)} := \|(1 - \Delta)^{n/2} u\|_{L_p(l_2)} \] is finite.

The set of all \( u = (u^1, u^2, u^3) \) such that \( u^1, u^2, u^3 \in H_p^n(R^3; l_2) \) is denoted by \( H_p^n(R^3; R^3 \times l_2) \) with the norm
\[ \|u\|_{H_p^n(R^3; R^3 \times l_2)} := \sum_{i=1}^{3} \|u^i\|_{H_p^n(l_2)} \]
\( H_p^n(R^3; R^3 \times l_2) \) denotes the subspace of \( H_p^n(R^3; R^3 \times l_2) \) in which every component is divergence-free, i.e.
\[ u = (u^i \ (i = 1, 2, 3, \ k \in \mathbb{N})) \in H_p^n(R^3; R^3 \times l_2) \]
if and only if \( u^i \in H_p^n(R^3; R^3) \) for all \( i = 1, 2, 3 \) and \( k \in \mathbb{N} \). In particular, we put \( L_{p,\sigma} := H_p^0; \sigma \). For the notational convenience, we set
\[ H_p^n = H_p^n(R^3; R^3), \]
\[ H_p^n;\sigma = H_p^n(R^3; R^3), \]
\[ H_p^n(l_2) = H_p^n(R^3; R^3 \times l_2), \]
\[ H_p^n(l_2) = H_p^n(R^3; R^3 \times l_2). \]

We write \( u \in \mathbb{H}_p^n(T) \) if \( u \) is an \( H_p^n \)-valued \( \mathbb{F} \)-measurable process satisfying
\[ \|u\|_{\mathbb{H}_p^n(T)} := \left( \mathbb{E} \left[ \int_0^T \|u\|_{H_p^n}^p \ dt \right] \right)^{1/p} < \infty. \]
For the notational convenience, we set

\[ H^n_p(T) := L_p(\Omega \times (0, T), \mathcal{F} ; H^n_p), \]
\[ H^n_{p, \sigma}(T) := L_p(\Omega \times (0, T), \mathcal{F} ; H^n_{p, \sigma}), \]
\[ H^n_p(T, l_2) := L_p(\Omega \times (0, T), \mathcal{F} ; H^n_p(l_2)), \]
\[ H^n_{p, \sigma}(T, l_2) := L_p(\Omega \times (0, T), \mathcal{F} ; H^n_{p, \sigma}(l_2)). \]

Similarly, we write

\[ U^n_p := L_p(\Omega, \mathcal{F} ; H^n_p), \]
\[ U^n_{p, \sigma} := L_p(\Omega, \mathcal{F} ; H^n_{p, \sigma}), \]
and simply

\[ L_p(T) := H^0_p(T), \]
\[ L_{p, \sigma}(T) := H^0_{p, \sigma}(T), \]
\[ L_p(T, l_2) := H^0_p(T, l_2), \]
\[ L_{p, \sigma}(T, l_2) := H^0_{p, \sigma}(T, l_2). \]

**Definition 2** (Homogeneous Sobolev spaces). (1) We denote by \( H^n_p(\mathbb{R}^3 ; \mathbb{R}^3) \) the space of all \( \mathbb{R}^3 \)-valued tempered distribution \( u = (u^1, u^2, u^3) \) modulo by polynomials such that

\[ (-\Delta)^{n/2} u^i := \mathcal{F}^{-1} \left[ |2\pi\xi|^n \mathcal{F}(u^i)(\xi) \right] \in L_p(\mathbb{R}^3) \]

with the norm

\[ \|u\|_{H^n_p(\mathbb{R}^3 ; \mathbb{R}^3)} := \sum_{i=1}^3 \left\| (-\Delta)^{n/2} u^i \right\|_{L_p(\mathbb{R}^3)} < \infty. \]

We denote by \( H^n_p(\mathbb{R}^3 ; \mathbb{R}^3 \times l_2) \) the space of all sequence \( u = (u^i)_{i=1,2,3} \), \( k \in \mathbb{N} \) such that \( u^k \in H^n_p(\mathbb{R}^3 ; \mathbb{R}^3) \) for all \( k \in \mathbb{N} \) and

\[ \|u\|_{H^n_p(\mathbb{R}^3 ; \mathbb{R}^3 \times l_2)} := \sum_{i=1}^3 \left( \sum_{k=1}^\infty \left\| (-\Delta)^{n/2} u^i \right\|_{L_p(\mathbb{R}^3)} \right)^{1/2} < \infty. \]

\( H^n_{p, \sigma}(\mathbb{R}^3 ; \mathbb{R}^3) \) and \( H^n_{p, \sigma}(\mathbb{R}^3 ; \mathbb{R}^3 \times l_2) \) denote the subspace of \( H^n_p(\mathbb{R}^3 ; \mathbb{R}^3) \) and \( H^n_p(\mathbb{R}^3 ; \mathbb{R}^3 \times l_2) \) whose elements are divergence free, respectively, i.e. \( H^n_{p, \sigma}(\mathbb{R}^3 ; \mathbb{R}^3) \) and \( H^n_{p, \sigma}(\mathbb{R}^3 ; \mathbb{R}^3 \times l_2) \) are closures of \( C_c^{\infty} \) with respect the norms in \( H^n_p(\mathbb{R}^3 ; \mathbb{R}^3) \) and \( H^n_p(\mathbb{R}^3 ; \mathbb{R}^3 \times l_2) \), respectively.
(2) Similar to inhomogeneous function spaces, we set deterministic spaces

\[ \dot{H}^n_p = H^n_p(\mathbb{R}^3; \mathbb{R}^3) \]
\[ \dot{H}^n_{p,\sigma} = H^n_{p,\sigma}(\mathbb{R}^3; \mathbb{R}^3) \]
\[ \dot{H}^n_p(l_2) = H^n_p(\mathbb{R}^3; \mathbb{R}^3 \times l_2) \]
\[ \dot{H}^n_{p,\sigma}(l_2) = H^n_{p,\sigma}(\mathbb{R}^3; \mathbb{R}^3 \times l_2) \]

and stochastic spaces

\[ \mathcal{H}^n_p(T) := L^p(\Omega \times (0, T), \mathcal{F}; \dot{H}^n_p) \]
\[ \mathcal{H}^n_{p,\sigma}(T) := L^p(\Omega \times (0, T), \mathcal{F}; \dot{H}^n_{p,\sigma}) \]
\[ \mathcal{H}^n_p(T, l_2) := L^p(\Omega \times (0, T), \mathcal{F}; \dot{H}^n_p(l_2)) \]
\[ \mathcal{H}^n_{p,\sigma}(T, l_2) := L^p(\Omega \times (0, T), \mathcal{F}; \dot{H}^n_{p,\sigma}(l_2)) \]

and

\[ U^n_p := L_p(\Omega, \mathcal{F}_0; \dot{H}^n_p) \]
\[ U^n_{p,\sigma} := L_p(\Omega, \mathcal{F}_0; \dot{H}^n_{p,\sigma}) \].

Remark 3. It is well-known that two norms \( \| \cdot \|_{H^n_p} \) and \( \| \cdot \|_{\dot{H}^n_p} + \| \cdot \|_{L^p} \) are equivalent if \( 1 < p < \infty \) and \( n > 0 \) (cf. [4] Theorem 6.3.2). Thus, \( H^n_p = \dot{H}^n_p \cap L^p \) and \( \| \cdot \|_{H^n_p} \preceq \| \cdot \|_{\dot{H}^n_p} \) if \( 1 < p < \infty \) and \( n > 0 \).

3. Linear theories for stochastic heat equations

In this section, we introduce linear theories for stochastic PDEs. Recently, analytic regularity theories for stochastic PDEs have been well developed. We refer the reader to [5], which is considered as one of bibles in this area. However, most of the estimates and theories are handled in inhomogeneous Sobolev space setting. To prove our main theorems, we need to obtain homogeneous type estimates for linear stochastic PDEs. Since we could not find an appropriate reference to show homogeneous type estimates for stochastic PDEs, we give detailed proofs. In addition, we note that all functions in the following theorem are \( \mathbb{R}^3 \)-valued, but the results in [5] are for scalar-valued functions. Since our leading operator is Laplacian, those are easily extended to \( \mathbb{R}^3 \)-valued functions without any difficulty.

Proposition 1. Let \( n \in \mathbb{R} \) and \( T \in (0, \infty) \). If \( u_0 \in U^{n+2-2/p}_p \), \( f \in \mathcal{H}^n_p(T) \), and \( g \in \mathcal{H}^{n+1}_p(T, l_2) \), then there exists a unique solution \( u \in \mathcal{H}^{n+2}_p(T) \) to

\[
\begin{aligned}
    u_t(t, x) &= \Delta u(t, x) + f(t, x) + g^k(t, x) \frac{dw^k}{dt} \\
    u(0, x) &= u_0(x)
\end{aligned}
\]
Proof. This proposition with the estimates (7) and (9) was proved in [5] Lemma 4.1 and Theorem 4.10. We only prove (8) since the proof of (10) is similar. We may prove it with assuming \( T \) is also a Brownian motion. If the proposition is proved for \( T \) where \( u \) satisfy (6) so that \( \tilde{u} \) is a solution to
\[
\tilde{u}_t(t,x) = \Delta \tilde{u}(t,x) + \tilde{f}(t,x) + \tilde{g}^k(t,x) \frac{d\tilde{w}^k}{dt} \quad \text{in } (0,T) \times \mathbb{R}^3
\]
in \( (0,T) \times \mathbb{R}^3 \) in the sense that for any \( \phi \in C_c^\infty \), with probability one, for all \( t \in (0,T) \),
\[
(u(t,\cdot), \phi) = (u_0, \phi) + \int_0^t [(u(s,\cdot), \Delta \phi) + (f(s,\cdot), \phi)] \, ds + \int_0^t (g^k(s,\cdot), \phi) \, dw^k.
\]
Moreover, there exist positive constants \( C_1(T,p) \), \( C_2(p) \), \( C_3(T,p) \), and \( C_4(p) \) such that
\[
\|u\|_{L^{p+2-2/p}(T)} \leq C_1 \left( \|u_0\|_{L_p^{\alpha+1-2/p}(T)} + \|f\|_{L_p^{\beta+2-2/p}(T)} + \|g\|_{L_p^{\gamma+1-2/p}(T)} \right) \quad (7)
\]
\[
\|u\|_{L^{p+2-2/p}(T)} \leq C_2 \left( \|u_0\|_{L_p^{\alpha+2-2/p}(T)} + \|f\|_{L_p^{\beta+2-2/p}(T)} + \|g\|_{L_p^{\gamma+1-2/p}(T)} \right) \quad (8)
\]
and
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|u(t,\cdot)\|^2_{L^{p+2-2/p}} \leq C_3 \left( \|u_0\|^2_{L_p^{\alpha+1-2/p}(T)} + \|f\|^2_{L_p^{\beta+2-2/p}(T)} + \|g\|^2_{L_p^{\gamma+1-2/p}(T)} \right) \quad (9)
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|u(t,\cdot)\|_{p+2-2/p} \leq C_4 \left( \|u_0\|_{L_p^{p+2-2/p}(T)} + \|f\|_{L_p^{p+2-2/p}(T)} + \|g\|_{L_p^{p+2-2/p}(T)} \right). \quad (10)
\]

Proof. This proposition with the estimates (7) and (9) was proved in [5] Lemma 4.1 and Theorem 4.10. We only prove (8) since the proof of (10) is similar. We may prove it with assuming \( T = 1 \) due to scaling. Indeed, Brownian motions have the self-similarity, that is,
\[
\tilde{w}_t := \frac{1}{\sqrt{T}} w_{tT}.
\]
is also a Brownian motion. If the proposition is proved for \( T = 1 \), then we define
\[
\tilde{u}(t,x) = u(Tt, \sqrt{T}x)
\]
\[
\tilde{u}_0(x) = u_0(\sqrt{T}x)
\]
\[
\tilde{f}(t,x) = Tf(Tt, \sqrt{T}x)
\]
\[
\tilde{g}(t,x) = \sqrt{T}g(Tt, \sqrt{T}x),
\]
where \( u, u_0, f, g \) satisfy (6) so that \( \tilde{u} \) is a solution to
\[
\tilde{u}_t(t,x) = \Delta \tilde{u}(t,x) + \tilde{f}(t,x) + \tilde{g}^k(t,x) \frac{d\tilde{w}^k}{dt}
\]
in \( (0,1) \times \mathbb{R}^3 \). Due to the estimate for \( \tilde{u} \), we have
\[
\|u\|_{L^{p+2}(T)} = T^{1/p} \|T^{-\alpha/2/p} T^{-\gamma/2/p} \|\tilde{u}\|_{L^{p+2}(1)}
\]
\[
\leq T^{1/p} \|T^{-\alpha/2/p} T^{-\gamma/2/p} \|C_2 \left( \|u_0\|_{L_p^{\alpha+2-2/p}(T)} + \|f\|_{L_p^{\beta+2-2/p}(T)} + \|g\|_{L_p^{\gamma+1-2/p}(T)} \right)
\]
\[
= C_2 \left( \|u_0\|_{L_p^{\alpha+2-2/p}(T)} + \|f\|_{L_p^{\beta+2-2/p}(T)} + \|g\|_{L_p^{\gamma+1-2/p}(T)} \right).
\]

Now, we assume \( T = 1 \) and split the proof into four cases.

• **Case 1.** \((f = 0 \text{ and } g = 0)\). Then taking \((-\Delta)^{\alpha/2-1/p}\) to both sides of (6) and applying (7) with \( n = -2 + 2/p \), we have
\[
\|u\|_{L^{p+2}} \lesssim \|\Delta^{(n+2)/2-1/p} u\|_{L_p^{p}} \lesssim \|\Delta^{(n+2)/2-1/p} u_0\|_{L_p^{p}} = \|u_0\|_{L_p^{\alpha+2-2/p}}. \quad (11)
\]
Remark 4. The constants $C_1$ and $C_3$ depend on $T$. However, one can take $C_1$ and $C_3$ uniformly for all $T \in [0, \hat{T}]$, i.e. for all $T \in [0, \hat{T}]$,

$$
\|u\|_{H^{n+2}(T)} \leq C_1 \left( \|u_0\|_{H^{n+2-2/p}} + \|f\|_{H^p(T)} + \|g\|_{H^{n+1}(T, L_2)} \right),
$$

and

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^{p+2-2/p}} \leq C_2 \left( \|u_0\|_{L^{p+2-2/p}} + \|f\|_{H^{p+1-2/p}(T)} + \|g\|_{H^{p+2-2/p}(T, L_2)} \right),
$$

where constants $C_1$ and $C_3$ depend only on $\hat{T}$ and $p$.

We denote by $H^\infty_t(T, L_2)$ the space of stochastic processes $g = (g^1, g^2, \ldots)$ such that $g^k = 0$ for all large $k$ and each $g^k$ is of the type

$$
g^k(t, x) = \sum_{i=1}^{\lceil k\rceil} 1_{(\tau_{i-1}, \tau_i]}(t) g_i^k(x),
$$
where \(j(k) \in \mathbb{N}, g^i \in C_c^\infty\), and \(\tau_i\) are stopping times with \(\tau_i \leq T\). Similarly, we denote by \(\mathbb{D}^\infty_c(T)\) the space of the processes \(g\) such that
\[
g(t, x) = \sum_{i=1}^{j} 1_{(\tau_{i-1}, \tau_i]}(t)g^i(x),
\]
where \(j \in \mathbb{N}\), \(g^i \in C_c^\infty\), and \(\tau_i\) are stopping times with \(\tau_i \leq T\). Lastly, we denote by \(\mathbb{D}^\infty_c(\mathbb{R}^3)\) the space of the random variables \(g_0\) of the type
\[
g_0(\omega, x) = 1_A(\omega)g(x)
\]
where \(g \in C_c^\infty\), and \(A \in \mathcal{F}_0\).

**Remark 5.**

(i) It is known that \(\mathbb{D}^\infty_c(T, l_2)\) is dense in \(\mathbb{H}^p_c(T, l_2)\) for all \(p \in (1, \infty)\) and \(n \in \mathbb{R}\) (for instance, see [5, Theorem 3.10]). In particular, \(\mathbb{D}^\infty_c(T)\) is dense in \(\mathbb{H}^p_c(T)\) for all \(p \in (1, \infty)\) and \(n \in \mathbb{R}\). Following the idea of [5, Theorem 3.10], one can also easily check that \(\mathbb{D}^\infty_c(\mathbb{R}^3)\) is dense in \(U^p_c\) for all \(p \in (1, \infty)\) and \(n \in \mathbb{R}\).

(ii) If \(g \in \mathbb{H}^{n+1}_p(T, l_2)\), then the stochastic integral
\[
\int_0^t (g^k(s, \cdot), \phi) \, dw^k_s
\]
is well-defined in the classical Itô-sense (cf. [5, Remark 3.2]). Moreover, the \(H^{n+1}_p(l_2)\)-valued stochastic integral
\[
\int_0^t g(s, x) \, dw_s
\]
is defined by recently developed UMD-space valued stochastic integral theory (cf. [6]).

(iii) The dual space of \(H^n_p\) is \(H_q^{-n}\), where \(q = \frac{p}{p-n}\), and one can find a countable subset of \(C_c^\infty\) which is dense in \(H_q^{-n}\). Thus, (7) can be interpreted in strong sense. i.e. (6) holds if and only if with probability one, for all \(t \in (0, T)\),
\[
u(t, x) = u_0 + \int_0^t (\Delta u(s, x) + f(s, x)) \, ds + \int_0^t g^k(s, x) \, dw^k_s
\]
where the equality holds as an element of \(H^n_p\)
\[
\int_0^t (\Delta u(s, x) + f(s, x)) \, ds = H^n_p\text{-valued classical Bochner's integral,}
\]
and \(\int_0^t g(s, x) \, dw_s\) is \(H^n_p(l_2)\)-valued stochastic integral.

(iv) If \(u_0 \in \mathbb{H}_c^\infty(\mathbb{R}^3)\), \(f \in \mathbb{H}^\infty_c(T)\), and \(g \in \mathbb{H}^\infty_c(T, l_2)\), then the solution \(u\) is given by (cf. [5, proof of Theorem 4.2])
\[
u(t, x) = S(t)u_0(x) + \int_0^t S(t-s)f(s, \cdot)(x) \, ds + \int_0^t S(t-s)g(s, \cdot) \, dw^k_s,
\]
where
\[
p(t, x) = (4\pi t)^{-3/2} \exp\left(-|x|^2/(4t)\right)
\]
\[
S(t)u_0(x) = \int_{\mathbb{R}^3} p(t, x-y)u_0(y) \, dy.
\]
Due to (7), the standard approximation, and UMD space-valued stochastic integration theories, (14) holds even for general \( u_0 \in \mathbb{H}_p^{n+2}(\mathbb{R}^3), f \in \mathbb{H}_p^n(T), \) and \( g \in \mathbb{H}_p^n(T,L_2). \)

(v) In Proposition 1, we assumed that \( u_0 \in \dot{U}_p^{n+2-\frac{d}{p}}, f \in \mathbb{H}_p^n(T), \) and \( g \in \mathbb{H}_p^{n+1}(T,L_2). \) However by using approximations in \( u_0^N \in \dot{U}_p^{n+2-\frac{d}{p}} \cap \dot{L}_p^{n+2-\frac{d}{p}}, f^N \in \mathbb{H}_p^n(T) \cap \mathbb{H}_p^n(T), \) \( g^N \in \mathbb{H}_p^{n+1}(T,L_2) \cap \mathbb{H}_p^{n+1}(T,L_2) \) (\( N = 1, 2, \ldots \)) or applying a UMD-space valued stochastic maximal \( L_p \)-inequality ([7] Theorem 1.1), one can easily check that (3) and (10) hold for all \( u_0 \in \dot{U}_p^{n+2-\frac{d}{p}}, f \in \mathbb{H}_p^n(T), \) and \( u \) defined as in (14).

We state a few corollaries, which is easily deduced by Proposition 1 and the representation (14).

**Corollary 1.** There exists a positive constant \( C_5 \) such that for all \( T \in (0, \infty), u_0 \in \dot{U}_p^{1/2}, \) and \( g \in \mathbb{H}_p^{1/2}(T,L_2), \)

\[
\left\| S(t)u_0(x) + \int_0^t S(t-s)g(s,)dw_s^k \right\|_{\mathbb{H}_p^{3/2}(T)} + \mathbb{E} \sup_{0 \leq s \leq T} \left\| S(t)u_0(x) + \int_0^t S(t-s)g(s,)dw_s^k \right\|_{\mathbb{H}_p^{3/2}} \leq C_5 \left( \|u_0\|_{\mathbb{H}_p^{1/2}} + \|g\|_{\mathbb{H}_p^{1/2}(T,L_2)} \right).
\]

**Corollary 2.** There exists a positive constant \( C_6 \) such that for all \( T \in (0, \infty), u_0 \in \dot{U}_p^{-2/5}, \) and \( g \in \mathbb{H}_p^{-1}(T,L_2), \)

\[
\left\| S(t)u_0(x) + \int_0^t S(t-s)g(s,)dw_s^k \right\|_{L_1(T)} \leq C_6 \left( \|u_0\|_{\mathbb{H}_p^{-2/5}} + \|g\|_{\mathbb{H}_p^{-1}(T,L_2)} \right).
\]

**Corollary 3.** (i) There exists a positive constant \( C_7 \) such that for all \( T \in (0, \infty) \) and \( u_0 \in \mathbb{H}_p^{1/2}, \)

\[
\int_0^T \|S(t)u_0\|_{\mathbb{H}_p^{1/2}}^2 \, dt \leq C_7 \|u_0\|_{\mathbb{H}_p^{1/2}}.
\]

(ii) There exists a positive constant \( C_8 \) such that for all \( T \in (0, \infty) \) and \( f \in L^2((0,T);\mathbb{H}_p^{-1/2}), \)

\[
\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)f(s,)ds \right\|_{\mathbb{H}_p^{1/2}}^2 + \int_0^T \left\| \int_0^t S(t-s)f(s,)ds \right\|_{\mathbb{H}_p^{1/2}}^2 \, dt \leq C_8 \int_0^T \| f(t,) \|^2_{\mathbb{H}_p^{-1/2}} dt.
\]

(iii) There exists a positive constant \( C_9 \) such that for all \( T \in (0, \infty) \) and \( u_0 \in \mathbb{H}_p^{-2/5}, \)

\[
\|S(t)u_0\|_{L_1((0,T)\times \mathbb{R}^3)} \leq C_9 \|u_0\|_{\mathbb{H}_p^{-2/5}}.
\]

4. **Bilinear Estimates**

The goal of this section is to prove bilinear estimates, Proposition 2 and Proposition 3 which play key roles to obtain our existence results for stochastic MHD equations.
Before going further, we recall that if
\[ \mathbf{v} \in C([0, T]; \dot{H}^{1/2}_2) \cap L^2((0, T); \dot{H}^{3/2}_2), \]
then from an interpolation inequality in the Sobolev space,
\[ \| \mathbf{v} \|_{L^4((0, T); \dot{H}^{1/2}_2)} \lesssim \| \mathbf{v} \|_{C([0, T]; \dot{H}^{1/2}_2)}^{1/2} \| \mathbf{v} \|_{L^2((0, T); \dot{H}^{3/2}_2)}^{1/2} \] (15)
and hence
\[ C([0, T]; \dot{H}^{1/2}_2) \cap L_2((0, T); \dot{H}^{3/2}_2) \subset L_4((0, T); \dot{H}^{1}_2). \]
For the notational convenience, we set
\[ X_1 = C([0, T]; \dot{H}^{1/2}_2) \cap L^2((0, T); \dot{H}^{3/2}_2) \]
with the norm
\[ \| \cdot \|_{X_1} = \| \cdot \|_{C([0, T]; \dot{H}^{1/2}_2)} + \| \cdot \|_{L_2((0, T); \dot{H}^{3/2}_2)}. \]
For the corresponding stochastic function spaces, we set
\[ X_{1, T} := L_2\left( \Omega, \mathcal{F}; C([0, T]; \dot{H}^{1/2}_2(R)) \right) \cap \dot{H}^{3/2}_2(T), \]
\[ X_{2, T} := L_4\left( \Omega, \mathcal{F}; L_4([0, T]; \dot{H}^{1/2}_2(R)) \right) \]
with the norms
\[ \| \cdot \|_{X_{1, T}} = \| \cdot \|_{L_2(\Omega, \mathcal{F}; C([0, T]; \dot{H}^{1/2}_2(R)))} + \| \cdot \|_{\dot{H}^{3/2}_2(T)}, \]
\[ \| \cdot \|_{X_{2, T}} = \left( \mathbb{E} \int_0^T \| \cdot \|_{\dot{H}^{3/2}_2}^4 \ dt \right)^{1/4}. \]
Due to (15), we have \( X_{1, T} \subset X_{2, T} \).

**Proposition 2.** There exists a constant \( C_{10} > 0 \) such that for all \( T > 0 \) and \( \mathbf{v}, \mathbf{w} \in X_{1, T} \),
\[ \| B(\mathbf{w}, \mathbf{v}) \|_{X_{1, T}} \leq C_{10} \| \mathbf{v} \|_{X_{2, T}} \| \mathbf{w} \|_{X_{2, T}}. \]

**Proof.** We begin by recalling
\[ B(\mathbf{u}, \mathbf{v})(t) = \int_0^t S(t-s) \mathbb{P} \cdot (\mathbf{u} \otimes \mathbf{v})(s) ds. \]
Since the embedding \( L_{3/2} \subset \dot{H}^{-1/2}_2 \) is continuous and the projection operator \( \mathbb{P} \) is continuous on \( L_{3/2} \), we have
\[ \| \mathbb{P} \cdot (\mathbf{v} \otimes \mathbf{w}) \|_{\dot{H}^{-1/2}_2} \lesssim \| \mathbb{P} \cdot (\mathbf{v} \otimes \mathbf{w}) \|_{L_{3/2}} \lesssim \| \mathbf{v} \cdot (\mathbf{v} \otimes \mathbf{w}) \|_{L_{3/2}}. \]
Since \( \nabla \cdot \mathbf{v} = 0 \), we use the Hölder inequality and the Sobolev inequality to get
\[ \| \mathbf{v} \cdot (\mathbf{v} \otimes \mathbf{w}) \|_{L_{3/2}} \lesssim \| \mathbf{v} \cdot \nabla \mathbf{w} \|_{L_{3/2}} \lesssim \| \nabla \mathbf{w} \|_{L_2} \cdot \lesssim \| \mathbf{v} \|_{L_6} \| \nabla \mathbf{w} \|_{L_2} \lesssim \| \mathbf{v} \|_{L_6} \| \mathbf{w} \|_{H^1_2}. \]
Thus, we use the Cauchy–Schwarz inequality to get
\[
\int_0^T \|\mathbb{P} \nabla \cdot (v \otimes w)\|^2_{H^{-1/2}} dt \\
\leq \int_0^T \|v\|^2_{H^1} \|w\|^2_{H^1} dt \\
\leq \left( \int_0^T \|v\|^4_{H^1} dt \right)^{1/2} \left( \int_0^T \|w\|^4_{H^1} dt \right)^{1/2}.
\]
(16)

If we put
\[
f := \mathbb{P} \nabla \cdot (v \otimes w) \in L^2((0, T) : H^{-1/2}_w(\mathbb{R}))
\]
then from Corollary 3(ii),
\[
\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)f(s, \cdot) ds \right\|^2_{H^{-1/2}} + \int_0^T \left\| \int_0^t S(t-s)f(s, \cdot) ds \right\|^2_{H^{-1/2}} dt \\
\leq \int_0^T \|f(t, \cdot)\|^2_{H^{-1/2}} dt.
\]
(17)

Combining (16) and (17), we obtain for all \( T > 0 \) and \( v, w \in L^4((0, T); H^1_w) \),
\[
\|B(v, w)\|_{X_1}^2 \leq \left( \int_0^T \|v\|^4_{H^1} dt \right)^{1/2} \left( \int_0^T \|w\|^4_{H^1} dt \right)^{1/2}.
\]

Thus, we use the Cauchy–Schwarz inequality to obtain that
\[
\|B(v, w)\|_{X_1}^2 \leq \left( \int_0^T \|v\|^4_{H^1} dt \right)^{1/2} \left( \int_0^T \|w\|^4_{H^1} dt \right)^{1/2} \\
\leq \|v\|_{X_1}^2 \|w\|_{X_1}^2.
\]

We now prove that the bilinear operator is jointly continuous on \( \mathbb{L}_5(t) \times \mathbb{L}_5(t) \). We divide its proof into a few steps.

**Proposition 3.** There exists a constant \( C_{11} > 0 \) such that for all \( T > 0 \) and \( v, w \in \mathbb{L}_5(t) \),
\[
\|B(v, w)\|_{\mathbb{L}_5(T)} \leq C_{11} \|v\|_{\mathbb{L}_5(T)} \|w\|_{\mathbb{L}_5(T)}.
\]

**Proof.** \( \text{Step 1) } \) To prove this proposition, it is useful to have an integral representation
\[
B(v, w)(t) = \int_0^t S(t-s)\mathbb{P} \nabla \cdot (v \otimes w)(s) ds \\
= \int_0^t K(t-s) * (v \otimes w)(s) ds
\]
with a kernel \( K \).
Let $h$ be a $L_p(\mathbb{R}^3)$-valued function defined on $(0, T)$ with $T > 0$ and $p \in (1, \infty)$. For each $s \in (0, T)$, $h(s) \in L_p(\mathbb{R}^3)$. That is, fixing $s$, we can regard $h(s)$ as a function defined on $\mathbb{R}^3$ and by $h(s, y)$ we denote the value of this function at $y \in \mathbb{R}^3$. The kernel $K$ should satisfy

$$K(t-s, \cdot) * h(s, \cdot) = S(t-s)\mathcal{P}h(s) = S(t-s)(I + \mathcal{R} \otimes \mathcal{R})\nabla h(s)$$

so that we can write its $j$-th component of $K$ as an inverse Fourier transform of the multiplier

$$(K(t, x))^j = -i \sum_{k, m \in \{1, 2, 3\}} \mathcal{F}^{-1} \left[ e^{-\epsilon |\xi|^2} |\xi|^{-2} \xi_j \xi_k \xi_m \right](x).$$

The key points of this kernel representation are the following scaling property

$$K(t, x) = \tau^{-2} K(1, \tau^{-1/2} x)$$

(18)

and the pointwise decay property

$$\sup_{x \in \mathbb{R}^3} (1 + |x|)^4 |K(1, x)| < \infty. \quad \text{(19)}$$

Oseen studied this kind of kernel and now its properties are regarded as well-known facts. We refer the reader to [8] for its generalization and other fine properties. For reader’s convenience we provide a proof of these two basic facts at Step 3.

**Step 2** We use the Young inequality and the scaling property (18) to obtain that

$$\|B(v, w)(t)\|_{L^1(\mathbb{R}^3)}$$

$$\lesssim \int_0^t \|K(t-s, \cdot) * (v \otimes w)(s)\|_{L_4(\mathbb{R}^3)} \, ds$$

$$\lesssim \int_0^t \|K(t-s, \cdot)\|_{L_{4/3}(\mathbb{R}^3)} \|v \otimes w(s)\|_{L_{4/3}(\mathbb{R}^3)} \, ds$$

$$\lesssim \int_0^t (t-s)^{-4/3} \|K(1, \cdot)\|_{L_{4/3}(\mathbb{R}^3)} \|v \otimes w(s)\|_{L_{4/3}(\mathbb{R}^3)} \, ds.$$

Since $\|K(1, \cdot)\|_{L_{4/3}(\mathbb{R}^3)} < \infty$ from the pointwise decay property (19), we use the Hardy–Littlewood–Sobolev theorem on fractional integration (cf. [3 Theorem 6.1.3]) to obtain that

$$\int_0^t \|B(v, w)(t)\|_{L_4(\mathbb{R}^3)}^5 \, dt$$

$$\lesssim \left( \int_0^T \left( \int_0^t (t-s)^{-4/3} \|v \otimes w(s)\|_{L_{4/3}(\mathbb{R}^3)} \, ds \right)^5 \, dt \right)^{2/5}$$

$$\lesssim \left( \int_0^T \|v \otimes w\|_{L_{4/3}(\mathbb{R}^3)}^{5/2} \, dt \right)^{2}.$$

Since $\|v \otimes w\|_{L_{4/3}(\mathbb{R}^3)} \leq \|v\|_{L_{4}(\mathbb{R}^3)} \|w\|_{L_{4}(\mathbb{R}^3)}$, we use the Cauchy–Schwarz inequality inequality to get the result.

**Step 3** Finally, we prove (18) and (19). We denote

$$K_{j,k,m}(t, x) = \mathcal{F}^{-1} \left[ e^{-\epsilon |\xi|^2} |\xi|^{-2} \xi_j \xi_k \xi_m \right].$$
Since we have

\[(K(t, x))^j = -i \sum_{k,m} K_{j,k,m}(t, x),\]

it suffices to prove that \(K_{j,k,m}(t, x)\) satisfies the same properties as (18) and (19). We notice that

\[\partial_t K_{j,k,m}(t, x) = -\int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-t|\xi|^2} \xi_j \xi_k \xi_m d\xi\]

and \(K_{j,k,m}(t, x)\) goes to 0 as \(t \to \infty\). Thus, by the Fundamental theorem of calculus, we have

\[K_{j,k,m}(t, x) = i \int_t^\infty D_x^j D_x^k D_x^m (\pi/s)^{3/2} e^{-t|x|^2/s} ds\]

By the change of variables \(\pi^2|x|^2/s = a\), we get

\[K_{j,k,m}(t, x) = i \pi^{15/2} x_j x_k x_m \int_t^\infty s^{-9/2} e^{-t|x|^2/s} ds\]

From this identity it is easy to check the scaling

\[K_{j,k,m}(t, x) = t^{-2} K_{j,k,m}(1, t^{-1/2} x),\]

and the decay

\[|K_{j,k,m}(1, x)| \lesssim (1 + |x|)^{-4}.\]

5. Proof of main theorems

The proofs of Theorem 2 and Theorem 1 go along the same series of steps. In order to write the procedures in a tidy manner, we shall prove the following abstract fixed point lemma.

We recall that if \(X\) is a Banach space with the norm \(\| \cdot \|_X\), then the product space \(X \times X\) is a Banach space with the norm

\[\|(v, w)\|_{X \times X} = \|v\|_X + \|w\|_X.\]

We shall use the Banach fixed point theorem to the product space \(X \times X\).

Lemma 1. Let \(X\) be a Banach space with the norm \(\| \cdot \|_X\). Assume that \(B(v, w)\) be a jointly continuous bilinear operator on \(X \times X\), that is, there exists a positive constant \(C_X\) such that for all \(v, w \in X\),

\[\|B(v, w)\|_X \leq C_X \|v\|_X \|w\|_X.\]  \hspace{1cm} (20)
If $\varepsilon < (4C_X)^{-1}$,
\begin{equation}
\|v^1\|_{X} + \|w^1\|_{X} \leq \varepsilon,
\end{equation}
and $(v^n, w^n)$ is a sequence in $X \times X$ satisfying
\begin{align*}
v^{n+1} &= v^1 - B(w^n, v^n) \\
w^{n+1} &= w^1 - B(v^n, w^n),
\end{align*}
then there exists a unique $(v, w) \in X \times X$ satisfying
\begin{equation}
\|v\|_{X} + \|w\|_{X} \leq 2\varepsilon.
\end{equation}
and
\begin{align*}
v &= v^1 - B(w, v) \\
w &= w^1 - B(v, w).
\end{align*}

Proof. We note first that for all $n \in \mathbb{N}$,
\begin{equation}
\|v^n\|_{X} + \|w^n\|_{X} \leq 2\varepsilon.
\end{equation}
Indeed, mathematical induction yields
\begin{align*}
\|v^{n+1}\|_{X} + \|w^{n+1}\|_{X} \\
&\leq \|v^n\|_{X} + \|w^n\|_{X} + \|B(w^n, v^n)\|_{X} + \|B(v^n, w^n)\|_{X} \\
&\leq \varepsilon + 2C_X\|v^n\|_{X}\|w^n\|_{X} \\
&\leq \varepsilon + 2C_X\varepsilon^2 \\
&\leq 2\varepsilon.
\end{align*}
Since $B$ is bilinear, we may write for $n \geq 2$,
\begin{align*}
v^{n+1} - v^n &= -B(w^n, v^n) + B(w^{n-1}, v^{n-1}) \\
&= -B(w^n, v^n - v^{n-1}) - B(w^n - w^{n-1}, v^{n-1})
\end{align*}
and
\begin{align*}
w^{n+1} - w^n &= -B(v^n, w^n) + B(v^{n-1}, w^{n-1}) \\
&= -B(v^n, w^n - w^{n-1}) - B(v^n - v^{n-1}, w^{n-1}).
\end{align*}
Thus, we have
\begin{align*}
\|(v^{n+1}, w^{n+1}) - (v^n, w^n)\|_{X \times X} \\
&= \|v^{n+1} - v^n\|_{X} + \|w^{n+1} - w^n\|_{X} \\
&\leq 4\varepsilon C_X(\|v^n - v^{n-1}\|_{X} + \|w^n - w^{n-1}\|_{X}) \\
&< \delta \|(v^n, w^n) - (v^{n-1}, w^{n-1})\|_{X \times X}
\end{align*}
by (24), where $\delta$ is a constant satisfying $4\varepsilon C_X < \delta < 1.$
Therefore, the Banach fixed-point theorem implies that \((v^1, w^0)\) converges to a unique \((v, w) \in X \times X\) satisfying (23), which follows by taking the limit to (22) and (24).

\[\square\]

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1.** Recall

\[X_{1,T} := L_2\left(\Omega, \mathcal{F}[[0, T] : H^{1/2}(\mathbb{R})]\right) \cap L_{2,2}^{3/2}(T),\]

\[X_{2,T} := L_4\left(\Omega, \mathcal{F}[[0, T] : H^1(\mathbb{R})]\right).\]

From (5), Corollary 1 and Proposition 2, we have

\[\| (v, w) \|_{X_{1,T} \times X_{1,T}} \lesssim \| (v_0, w_0) \|_{L_2^1 \times L_4^1} + \| (G_1, G_2) \|_{L_2^1(T_L) \times L_4^1(T_L)} + \| w \|_{X_{2,T}}.\]

Thus, it suffices to find a solution \((v, w) \in X_{2,T} \times X_{2,T}\).

We set

\[v^1(t, x) = S(t)v_0(x) + \int_0^t S(t-s)G_1(s, \cdot)(x)dw_s^k,\]

\[w^1(t, x) = S(t)w_0(x) + \int_0^t S(t-s)G_2(s, \cdot)(x)dw_s^k,\]

and inductively define \((v^n, w^n)\) for \(n \geq 2\) by

\[v^n(t, x) = S(t)w_0(x) - B(w^{n-1}, v^{n-1}) + \int_0^t S(t-s)G_1(s, \cdot)dw_s^k,\]

\[w^n(t, x) = S(t)v_0(x) - B(v^{n-1}, w^{n-1}) + \int_0^t S(t-s)G_2(s, \cdot)dw_s^k.\]

By mathematical induction, we have for all \(n \in \mathbb{N}\),

\[v^n, w^n \in X_{2,T}. \quad (25)\]

Indeed, due to (15) and Corollary 1 we have \(v^1, w^1 \in X_{2,T}\). If we assume that (25) holds with \(n = k\), then by (15), Corollary 1 and Proposition 2

\[\| (v^{k+1}, w^{k+1}) \|_{X_{2,T} \times X_{2,T}} \lesssim \| (v_0, w_0) \|_{L_2^1 \times L_4^1} + \| (G_1, G_2) \|_{L_2^1(T_L) \times L_4^1(T_L)} + \| v^k \|_{X_{2,T}} \| w^k \|_{X_{2,T}} < \infty.\]

This proves (25).

Finally, we can apply Lemma 1 due to (15), Corollary 1 and Proposition 2 so that we get the second part, Theorem 1(ii).
To obtain the first part, Theorem 1(i), it suffices to show that there exists a positive constant $T_0 \in (0, \infty)$ such that

$$\|v^1\|_{X^{2,T_0}} + \|w^1\|_{X^{2,T_0}} < \frac{1}{4C_{10}},$$

which is an immediate consequence of the fact that

$$\lim_{T \to 0} \|v^1\|_{X^{2,T}} + \|w^1\|_{X^{2,T}} = 0.$$

This completes the proof of Theorem 1.\hfill \square

The proof of Theorem 2 is almost identical to the proof of Theorem 1 except that Corollary 2 and Proposition 3 are used in place of Corollary 1 and Proposition 2.

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