GROUPS WITH A BASE PROPERTY ANALOGOUS TO THAT OF VECTOR SPACES

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Abstract. A $B$-group is a group such that all its minimal generating sets (with respect to inclusion) have the same size. We prove that the class of finite $B$-groups is closed under taking quotients and that every finite $B$-group is solvable. Via a complete classification of Frattini-free finite $B$-groups we obtain a general structure theorem for finite $B$-groups. Applications include new proofs for the characterization of finite matroid groups and the classification of finite groups with the basis property.

1. Introduction

Let $G$ be a finite group. A generating set $X$ of $G$ is said to be minimal if no proper subset of $X$ generates $G$. We denote by $d(G)$ the minimal number of generators of $G$, i.e., the smallest size of a minimal generating set of $G$, and we write $m(G)$ for the largest size of a minimal generating set of $G$.

Whereas the invariant $d(G)$ has been well studied for many groups $G$, its counterpart $m(G)$ has not received a similar degree of attention. First steps toward investigating the latter have been taken in the context of permutation groups. For instance, in [10] Whiston proved that $m(\text{Sym}(n)) = n - 1$ for the finite symmetric group of degree $n$. Furthermore, Cameron and Cara gave in [1] a complete description of the maximal independent generating sets of $\text{Sym}(n)$; these are precisely the minimal generating sets of maximal size. Clearly, Whiston’s result implies that

$$m(\text{Sym}(n)) - d(\text{Sym}(n)) \to \infty \quad \text{as } n \to \infty.$$ 

This suggests a natural ‘classification problem’: given a non-negative integer $c$, characterize all finite groups $G$ such that $m(G) - d(G) \leq c$. The results of Saxl and Whiston in [10] show that for projective special linear groups $G = \text{PSL}_2(p^f)$ the difference $m(G) - d(G)$ depends on the number of prime divisors of $r$. In particular, $m(G) - d(G) = 1$ for all $G = \text{PSL}_2(p)$ with $p$ not congruent to $\pm 1$ modulo 8 or 10. Since the Frattini subgroup $\Phi(G)$ consists of all ‘non-generators’ of $G$, we have $d(G) = d(G/\Phi(G))$ and $m(G) = m(G/\Phi(G))$. Hence one may initially focus on groups $G$ which are Frattini-free, i.e., for which $\Phi(G) = 1$.

In the present article we solve the above stated problem for $c = 0$. We say that the group $G$ has property $B$, or the weak basis property, if all its minimal generating sets have the same size, equivalently if $m(G) = d(G)$. Groups with property $B$ are called $B$-groups for short. A group is said to have the basis property if all its subgroups have property $B$. The Burnside basis theorem states that all finite $p$-groups are $B$-groups and, consequently, have the basis property.
Groups with the basis property as well as variants, such as matroid groups, have been considered by a number of authors; e.g., see [5] and references therein. Indeed, McDougall-Bagnall and Quick initiated in [5] the systematic study of finite $B$-groups and used this to classify groups with the basis property. In this context they raised the following fundamental questions. Is it true that property $B$ is inherited by quotient groups? Is it possibly true that every finite $B$-group is solvable? We answer both questions positively.

**Proposition 1.1.** Every quotient of a finite $B$-group is again a $B$-group.

**Theorem 1.2.** Every finite $B$-group is solvable.

From these structural results we obtain a characterization of finite $B$-groups, based on a complete classification of Frattini-free finite $B$-groups. For any prime $p$ we denote by $\mathbb{F}_p$ the field with $p$ elements.

**Theorem 1.3.** Let $G$ be a finite group. Then $G$ is a Frattini-free $B$-group if and only if one of the following holds:

1. $G$ is an elementary abelian $p$-group for some prime $p$;
2. $G = P \rtimes Q$, where $P$ is an elementary abelian $p$-group and $Q$ is a non-trivial cyclic $q$-group, for distinct primes $p \neq q$, such that $Q$ acts faithfully on $P$ and the $\mathbb{F}_pQ$-module $P$ is a direct sum of isomorphic copies of one simple module.

**Remark 1.4.** This means that there are no Frattini-free finite $B$-groups beyond the examples constructed in [5, §3]. Indeed, the groups listed in (2) of Theorem 1.3 can be concretely realized as ‘semidirect products via multiplication in finite fields of characteristic $p$’: the simple module in question is of the form $\mathbb{F}_p(\zeta)$, the additive group of a finite field generated by a $q^k$th root of unity $\zeta$ over $\mathbb{F}_p$, with a generator $z$ of $Q$ acting on $\mathbb{F}_p(\zeta)$ as multiplication by $\zeta$.

Using the explicit description of Frattini-free finite $B$-groups, we determine the automorphism groups of such groups; see Theorem 1.4. From McDougall-Bagnall and Quick’s results in [5] we obtain a characterization of finite $B$-groups.

**Theorem 1.5.** Let $G$ be a finite group. Then $G$ is a $B$-group if and only if one of the following holds:

1. $G$ is a $p$-group for some prime $p$;
2. $G = P \rtimes Q$, where $P$ is a $p$-group and $Q$ is a cyclic $q$-group for distinct primes $p \neq q$, such that $C_Q(P) \neq Q$ and every non-trivial element of $Q/C_Q(P)$ acts fixed-point-freely on $P/\Phi(P)$.

Moreover, in case (2) one has $\Phi(G) = \Phi(P) \times C_Q(P)$.

As applications of Theorems 1.3 and 1.3 we provide new, streamlined proofs for the characterization of finite matroid groups (cf. [8]) and the classification of finite groups with the basis property (cf. [5]). Furthermore, we record as Corollary 3.2 a description of finite groups $G$ with $m(G) \leq 2$.

The proofs of our main results rely ultimately on consequences of the Classification of Finite Simple Groups. These enter our proofs directly as well as indirectly,
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namely via results of Lucchini and Menegazzo on generation properties of finite groups with a unique minimal normal subgroup; see [3] and [4].

An outline of the paper is as follows. Proposition 1.1 and Theorem 1.2 are proved in Section 2. Theorems 1.3 and 1.5 are proved in Section 3. Theorem 4.1 in Section 4 describes the automorphism group of a Frattini-free \( \mathcal{B} \)-group. In Sections 5 and 6 we use our main results to derive a characterization of finite matroid groups and a classification of finite groups with the basis property.

2. Quotients and solvability of \( \mathcal{B} \)-groups

Proof of Proposition 1.1. Let \( G \) be a finite \( \mathcal{B} \)-group with normal subgroup \( N \), and let \( \pi: G \to G/N \) denote the projection homomorphism. Writing \( d = d(G/N) \), we choose \( x_1, \ldots, x_d \in G \) such that \( x_1^\pi, \ldots, x_d^\pi \) is a minimal generating sequence of \( G/N \).

For a contradiction, assume that \( G/N \) does not have property \( \mathcal{B} \): let \( \bar{y}_1, \ldots, \bar{y}_e \) be a minimal generating sequence of \( G/N \) with \( e > d \). Express each element \( \bar{y}_i \) as a word in \( x_1^\pi, \ldots, x_d^\pi \) and then let \( y_i \) denote the same word in \( x_1, \ldots, x_d \) so that \( y_i^\pi = \bar{y}_i \). Since \( \langle y_1^\pi, \ldots, y_e^\pi \rangle = G/N \), there is a sequence \( z_1, \ldots, z_f \) in \( N \) such that \( y_1, \ldots, y_e, z_1, \ldots, z_f \) minimally generates \( G \). However, the strictly shorter sequence \( x_1, \ldots, x_d, z_1, \ldots, z_f \) also generates \( G \), contradicting the fact that \( G \) is a \( \mathcal{B} \)-group. \( \Box \)

Proposition 2.1. Let \( G \) be a finite group.

(1) Suppose that \( G \) is simple. Then \( G \) has property \( \mathcal{B} \) if and only if \( G \) is cyclic.

(2) Suppose that \( G \) is cyclic. Then \( G \) has property \( \mathcal{B} \) if and only if \( G \) has prime-power order.

Proof. (1) Suppose that \( G \) is a non-abelian simple group. Then the classification of finite simple groups implies that \( d(G) = 2 \) whereas \( m(G) \geq 3 \). The latter follows, for instance, from the fact that \( G \) is generated by involutions.

(2) As \( G \) is cyclic, \( d(G) = 1 \) and the primary decomposition of \( G \) shows that \( m(G) \) is equal to the number of primes dividing \( |G| \). \( \Box \)

The next three lemmas follow from results of Lucchini and Menegazzo in [3] and [4]. Their work relies on the Classification of Finite Simple Groups.

Lemma 2.2. Let \( G \) be a finite \( \mathcal{B} \)-group with a minimal normal subgroup \( N \). Then

\[
d(G) = \begin{cases} 
    d(G/N) & \text{if } N \leq \Phi(G), \\
    d(G/N) + 1 & \text{otherwise}.
\end{cases}
\]

Proof. If \( d(G) = d(G/N) \) then every minimal generating sequence of \( G \) projects to a minimal generating sequence of \( G/N \); so elements of \( N \) never appear in a minimal generating sequence of \( G \), that is \( N \leq \Phi(G) \). Conversely, if \( N \leq \Phi(G) \), then \( d(G) = d(G/N) \). On the other hand, if \( d(G) > d(G/N) \), then \( d(G) = d(G/N) + 1 \) because \( d(G) \leq d(G/N) + 1 \) from [3]. \( \Box \)

Lemma 2.3. Let \( G \) be a non-cyclic, Frattini-free finite \( \mathcal{B} \)-group with a unique minimal normal subgroup \( N \). Then \( d(G) = 2 \) and \( G/N \) is cyclic of prime-power order.
Proof. From [4] we have $d(G) = \max\{2, d(G/N)\}$ and Lemma 2.2 yields $d(G) = d(G/N) + 1$. Thus $d(G) = 2$ and $d(G/N) = 1$ so that $G/N$ is cyclic. Moreover, Propositions [3] and [2.1] imply that $G/N$ has prime-power order.

Lemma 2.4. Let $G$ be a Frattini-free finite $\mathcal{B}$-group with a non-abelian minimal normal subgroup $N$. If $G/N$ is cyclic, then $m(G) \geq 3$.

Proof. Suppose that $G/N$ is cyclic. Then, by Propositions [1.1] and [2.1] the quotient $G/N$ is cyclic of $p$-power order for some prime $p$. Let $P$ be a Sylow-$p$ subgroup of $G$. Since $G$ cannot be a $p$-group, we find a maximal subgroup $H$ of $G$ which contains $P$. Let $q$ be a prime dividing $[G : H]$.

Let $Q$ be a Sylow-$q$ subgroup contained in $N$, and observe that $Q$ is also a Sylow-$q$ subgroup of $G$. From $Q \neq N$ we conclude that $N_G(Q) \neq G$. Furthermore, the Frattini argument yields $G = N_G(Q)N$. Since $G/N$ is cyclic of $p$-power order, we find an element $g \in N_G(Q)$ of $p$-power order such that $(g)N = G$. Moreover, replacing $g$ and $Q$ by conjugates $g^x$ and $Q^x$ for a suitable $x \in G$, we may assume without loss of generality that $g \in P \leq H$.

Thus $\langle Q \cup H \rangle = G$ and $\langle (H \cap N) \cup \{g\} \rangle = H$ so that $\langle Q \cup (H \cap N) \cup \{g\} \rangle = G$.

We choose a minimal generating set $X$ of $G$ with $X \subseteq Q \cup (H \cap N) \cup \{g\}$. Since $\langle Q \cup (H \cap N) \rangle \subseteq N$, $\langle Q \cup \{g\} \rangle \subseteq N_G(Q)$, and $\langle (H \cap N) \cup \{g\} \rangle \subseteq H$ are all properly contained in $G$, we conclude that $m(G) \geq |X| \geq 3$. \qed

Proof of Theorem 1.2. Let $G$ be a finite $\mathcal{B}$-group. By Proposition 1.1 every quotient of $G$ has property $\mathcal{B}$ and thus, by induction on the order, every proper quotient of $G$ is solvable.

For a contradiction, assume that $G$ is not solvable and consequently has no non-trivial solvable normal subgroups. In particular, since $\Phi(G)$ is nilpotent, this implies that $G$ is Frattini-free. Let $M$ be a minimal normal subgroup of $G$. Then $M$ is non-abelian and $G$ has no other minimal normal subgroup $M$ besides $M$; otherwise $M$ would embed into the solvable group $G/M$.

Hence Lemmas 2.3 and 2.4 yield the contradiction $2 = d(G) = m(G) \geq 3$. \qed

3. The classification of $\mathcal{B}$-groups

Recall that the socle $\text{Soc}(G)$ of a finite group $G$ is the subgroup generated by all minimal normal subgroups.

Lemma 3.1. Let $G$ be a Frattini-free finite $\mathcal{B}$-group. Then $G = S \rtimes K$, where $S = \text{Soc}(G)$ is elementary abelian and $C_K(S)$ is trivial.

Proof. By Theorem 1.2 the group $G$ is solvable. Hence $S$ is abelian. We recall that every abelian normal subgroup of a Frattini-free finite group admits a complement; e.g., see [6, Proposition 5.2.13]. Hence $G = S \rtimes K$ for a suitable subgroup $K$.

Now assume for a contradiction that $S$ is not elementary abelian. Let $P$ be a nontrivial Sylow-$p$ subgroup of $S$ and $Q$ a nontrivial Sylow-$q$ subgroup of $S$, for distinct primes $p \neq q$. Let $L$ be a complement for $P \times Q$ in $G$ so that $G = \langle P \times Q \rangle \rtimes L$.  

Choose a minimal generating sequence $z_1, \ldots, z_f$ of $L$ and extend this to a minimal generating sequence

$$x_1, \ldots, x_d, y_1, \ldots, y_e, z_1, \ldots, z_f$$

of $G$ of length $d + e + f$ by choosing a minimal generating sequence $x_1, \ldots, x_d$ of $P$ as a $\mathbb{F}_pL$-module and a minimal generating sequence $y_1, \ldots, y_e$ of $Q$ as a $\mathbb{F}_pL$-module. Since $P$ and $Q$ are non-trivial, the parameters $d$ and $e$ are positive. But then

$$x_1y_1, x_2 \ldots, x_d, y_2, \ldots, y_e, z_1, \ldots, z_f$$

is a generating sequence of $G$ of shorter length $d + e + f - 1$, contradicting the fact that $G$ is a $\mathcal{B}$-group.

Finally, $C_K(S)$ is invariant under conjugation by $S$ and by $K$. Hence it is a normal subgroup of $G$ which intersects the socle $S$ trivially, and $C_K(S) = 1$. \hfill \Box

**Proof of Theorem 1.3.** It is shown in [5, §3] that groups of the form specified in the theorem are Frattini-free and have property $\mathcal{B}$.

Now suppose that $G$ is a Frattini-free $\mathcal{B}$-group. By Lemma 3.1, the group $G$ is abelian if and only if $G$ is elementary abelian. Thus it suffices to analyse the situation where $G$ is non-abelian. By Lemma 3.1, we have $G = P \times Q$, where $P = \text{Soc}(G)$ is an elementary abelian $p$-group for a prime $p$ and the complement $Q \neq 1$ acts faithfully on $P$. We decompose $P$ as a direct product $P = M_1 \times \ldots \times M_d$, where each $M_i$ is a minimal normal subgroup of $G$, that is each $M_i$ is a simple $\mathbb{F}_pQ$-module.

Fix a minimal generating sequence $z_1, \ldots, z_e$ of $Q$. By choosing in each $M_i$ an element $x_i \neq 1$, we obtain a minimal generating sequence $x_1, \ldots, x_d, z_1, \ldots, z_e$ of $G$ of length $d + e$. If $M_i$ and $M_j$ were not isomorphic as $\mathbb{F}_pQ$-modules for some $i \neq j$ we could replace $x_i$ and $x_j$ by a single element $x_ix_j$ to obtain a minimal generating sequence of $G$ of length $d + e - 1$. Since $G$ has property $\mathcal{B}$ this cannot happen and thus all the $M_i$ are isomorphic to one another as $\mathbb{F}_pQ$-modules. In particular, this implies that $Q$ acts faithfully on each $M_i$.

Finally we show that $Q$ is a cyclic $q$-group for some prime $q \neq p$. For this we consider the quotient group

$$\bar{G} = G/(M_2 \times \ldots \times M_d) = \bar{M}_1 \times \bar{Q}$$

which has property $\mathcal{B}$ by Proposition 1.1. Clearly, $\bar{M}_1$ is an abelian minimal normal subgroup of $\bar{G}$ and thus $\bar{Q}$ a maximal subgroup of $\bar{G}$. We claim that $\bar{M}_1$ is the unique minimal normal subgroup of $\bar{G}$. Indeed, if $\bar{g} \in \bar{G} \setminus \bar{M}_1$ then $g \in G \setminus P$ and $\langle g \rangle^G \supseteq [M_1, g]^G = \bar{M}_1$; consequently, every normal subgroup of $\bar{G}$ contains $\bar{M}_1$. Moreover, from $\bar{M}_1 \cap \bar{Q} = 1$ we conclude that $\bar{G}$ is Frattini-free. Lemma 2.3 shows that $Q \cong \bar{Q}$ is a cyclic $q$-group for some prime $q$. Since $G$ is non-abelian and Frattini-free, it cannot be a $p$-group, and hence $q \neq p$. \hfill \Box

**Proof of Theorem 1.5.** First suppose that $G$ has property $\mathcal{B}$. From Proposition 1.1 we deduce that $H = G/\Phi(G)$ is a Frattini-free $\mathcal{B}$-group so that Theorem 1.3 gives a detailed description of $H$. We observe that if $H$ is a $p$-group for some prime $p$ then $G$ is a $p$-group, because $H$ is the image of any Sylow-$p$ subgroup of $G$ modulo $\Phi(G)$ and $\Phi(G)$ consists of the ‘non-generators’ of $G$. Now suppose that $H = P \times Q$ with $P$ and $Q$ as in (2) of Theorem 1.3.
We claim that every non-trivial element of $Q$ acts fixed-point-freely on $P$, i.e., $C_P(y) = 1$ for $y \in Q \setminus \{1\}$. Recall that $P$ is a direct sum of isomorphic copies of one simple $\mathbb{F}_pQ$-module $M$. Thus $Q$ acts faithfully on $M$ and, since $Q$ is abelian, this implies that every non-trivial element of $Q$ acts fixed-point-freely on $M$ and therefore also on $P$. From [5] Proposition 3.3 and Theorem 3.2] we deduce that $G$ is of the shape described in the theorem.

Conversely, if $G$ has prime-power order then $G$ is a $\mathcal{B}$-group by the Burnside basis theorem, and it suffices to consider the remaining case: $G \cong P \times Q$, where $P$ is a $p$-group and $Q$ is a cyclic $q$-group, for distinct primes $p \neq q$, such that $C_Q(P) \neq Q$ and every non-trivial element of $Q/C_Q(P)$ acts fixed-point-freely on $P/\Phi(P)$. Since $P$ is a $p$-group and since $C_Q(P)$ is a proper subgroup of the cyclic $q$-group $Q$, each subset of $G$ that generates $G$ modulo the normal subgroup $\Phi(P) \times C_Q(P)$ already generates $G$. Thus $\Phi(P) \times C_Q(P) \subseteq \Phi(G)$ and in order to show that $G$ has property $\mathcal{B}$ we may assume that $\Phi(P) \times C_Q(P) = 1$. Then $P$ is elementary abelian and every element of $Q$ acts fixed-point-freely on $P$. It follows from [5] Proposition 3.3 and Theorem 3.2] that $G$ has property $\mathcal{B}$. \hfill \square

As a consequence of Theorem 1.3, Remark 1.4 and Theorem 1.5 we record the following corollary.

**Corollary 3.2.** Let $G$ be a non-trivial finite group with $m(G) \leq 2$. Then precisely one of the following holds:

1. $G$ is cyclic of prime-power order so that $d(G) = m(G) = 1$;
2. $G$ is cyclic of order divisible by exactly two distinct primes so that $d(G) = 1$ and $m(G) = 2$;
3. $G$ is a group of prime-power order with $d(G) = m(G) = 2$;
4. $G$ is a $\mathcal{B}$-group of order divisible by exactly two distinct primes so that $d(G) = m(G) = 2$.

In the last case $G/\Phi(G) \cong P \times Q$, where the elementary abelian $p$-group $P$ is isomorphic to the additive group of a finite field $F = \mathbb{F}_p(\zeta)$, the cyclic $q$-group $Q = \langle \zeta \rangle$ embeds as $\langle \zeta \rangle$ into $F^\times$ and $Q$ acts on $P$ via multiplication in $F$.

## 4. Automorphisms of Frattini-free $\mathcal{B}$-groups

Let $G$ be a Frattini-free finite $\mathcal{B}$-group. By Theorem 1.3 we have $|G| = p^{rd}q^k$, where $p \neq q$ are distinct primes and the socle $\text{Soc}(G)$ is a direct product of $d$ minimal normal subgroups, each elementary abelian of size $p^r$. Moreover, as indicated in Remark 1.4, the group $G$ can be embedded into the general semi-affine group $\Gamma GL(d, F)$ of degree $d$ over the finite field $F = \mathbb{F}_p(\zeta)$, where $\zeta$ denotes a primitive $q^k$th root of unity over $\mathbb{F}_p$ and $[F : \mathbb{F}_p] = r$: writing

$$\Gamma GL(d, F) = F^d \rtimes GL(d, F) \rtimes \text{Aut}(F)$$

we can realize $G$ as the subgroup consisting of all elements of the form

$$(v, \zeta^n I, \text{id}_F), \quad \text{where } v \in F^d \text{ and } \zeta^n I \in GL(d, F) \text{ is scalar for } 0 \leq n < q^k.$$
the full automorphism group of $G$. Moreover, the action of $\text{AGL}(d, F)$ on $G$ is faithful, unless $G$ is abelian.

Proof. Clearly, $G$ is a normal subgroup of $\text{AGL}(d, F)$. If $G$ is abelian, then $r = 1$, $k = 0$ and $F = \mathbb{F}_p$, so that $G = \mathbb{F}^d_p$ and $\text{AGL}(d, F) = \mathbb{F}^d_p \rtimes \text{GL}(d, \mathbb{F}_p)$. In this case $\text{Aut}(G) \cong \text{GL}(d, \mathbb{F}_p)$ is realized by the action of $\text{AGL}(d, \mathbb{F}_p)$ modulo $\mathbb{F}^d_p$.

Now suppose that $G$ is non-abelian. Since $F^d \subseteq G$, the centralizer of $G$ in $\text{AGL}(d, F)$ is contained in $F^d$. Since $\zeta I \in G$ acts fixed-point-freely on $F^d$, the centralizer of $G$ in $\text{AGL}(d, F)$ is trivial. Hence the action of $\text{AGL}(d, F)$ on $G$ by conjugation is faithful. It remains to show that every automorphism of $G$ can be realized as conjugation by a suitable element of $\text{AGL}(d, F)$.

Let $\varphi \in \text{Aut}(G)$. We observe that $\text{Soc}(G) = F^d$ and that the action of $G$ on $\text{Soc}(G)$ by conjugation induces an embedding of $G/\text{Soc}(G)$ into $F^\times$. The element $\zeta g \in F^\times$ corresponding to the action of $(\zeta I)^{\varphi}$ on $F^d$ satisfies the same minimal polynomial over $\mathbb{F}_p$ as $\zeta$. Thus the action of $\varphi$ on $G/\text{Soc}(G)$ can also be realized by an element of $\text{Aut}(F) \leq \text{AGL}(d, F)$. Without loss of generality we may therefore assume that $\varphi$ acts as the identity on $G/\text{Soc}(G)$. Then $\varphi$ acts on $\text{Soc}(G) = F^d$ as an $F$-linear isomorphism. A suitable element of $G \rtimes \text{GL}(d, F) \leq \text{AGL}(d, F)$ realizes the same action and we may further assume that $\varphi$ restricts to the identity on $\text{Soc}(G)$. This means that $(\zeta I)^{\varphi} = v(\zeta I)$ for some $v \in F^d$. Finally, we notice that conjugation by $(\zeta - 1)^{-1}v \in F^d \leq \text{AGL}(d, F)$ induces the automorphism $\varphi$. \qed

5. The characterization of matroid groups

A subset $X$ of a finite group $G$ is called independent, respectively Frattini-independent, if there is no proper subset $Y \subset X$ such that $\langle X \rangle = \langle Y \rangle$, respectively $\langle X \cup \Phi(G) \rangle = \langle Y \cup \Phi(G) \rangle$. The group $G$ is called a matroid group if $G$ has property $\mathcal{B}$ and every Frattini-independent subset of $G$ can be extended to a minimal generating set of $G$. Alternatively, $G$ is a matroid group if $H = G/\Phi(G)$ is a Frattini-free $\mathcal{B}$-group and every independent subset of $H$ can be extended to an minimal generating set. The definition of a matroid group given here is the one used in [8, 9].

We obtain a small variation of the characterization of matroid groups in [8].

**Theorem 5.1** (Scapellato and Verardi [8]). Let $G$ be a finite group and let $H = G/\Phi(G)$. The group $G$ is a matroid group if and only if one of the following holds:

1. $G$ is a $p$-group for some prime $p$,
2. $H = P \rtimes Q$, where $P \cong \mathbb{F}^d_p$ and $Q$ is cyclic of order $q$, for primes $p, q$ such that $q \mid p - 1$, and $Q \hookrightarrow \mathbb{F}^\times_p$ acts on $P$ via field multiplication.

Proof. By the Burnside basis theorem every finite group of prime-power order is a matroid group. From now suppose that $G$ does not have prime-power order.

First suppose that $G$ is a matroid group. Then, by Theorem [1.3] and Remark [1.4] the Frattini quotient $H$ is a matroid group of the form $H = P \rtimes Q$, where $P$ is an elementary abelian $p$-group and $Q$ is a non-trivial cyclic group of order $q^k$, for distinct primes $p \neq q$, such that $Q \hookrightarrow F^\times$ acts faithfully on $P \cong F^d$ via multiplication in a finite field $F$. Here $F$ is obtained from $\mathbb{F}_p$ by adjoining a
primitive $q^k$th root of unity and we set $r = [F : \mathbb{F}_p]$. We observe that the common size of all minimal generating sets of $G$ is $d + 1$.

Being isomorphic to an $\mathbb{F}_p$-vector space of dimension $rd$, the subgroup $P$ contains an independent subset of size $rd$. This subset extends to a minimal generating set of $H$. We deduce that $rd \leq d$, thus $r = 1$. Let $z$ be a generator of $Q$ and assume for a contradiction that $k \geq 2$. Choose a minimal generating set $X$ for $P$ as an $\mathbb{F}_pQ$-module. Then $X \cup \{z^0\}$ is an independent subset of size $d + 1$ that does not generate $H$ and does not extend to a minimal generating set of $H$. This implies that $H$ is not a matroid group in contradiction to our assumptions. Hence, $k = 1$, i.e., $Q$ is cyclic of order $q$. From $Q \hookrightarrow \mathbb{F}_p^\times$ we obtain $q \mid p - 1$.

Conversely, suppose that $H = P \rtimes Q$, where $P \cong \mathbb{F}_p^d$ and $Q = \langle z \rangle$ is cyclic of order $q$, for primes $p, q$ such that $q \mid p - 1$, and $Q \hookrightarrow \mathbb{F}_p^\times$ acts on $P$ via field multiplication. By Theorem [13] the group $H$ has property $\mathcal{B}$ and it suffices to show that every independent subset of $H$ extends to a minimal generating set. Let $X = \{x_1, \ldots, x_m\} \subseteq H$ be an independent subset of size $m$. If $X \subseteq P$ then, regarding $P$ as an $\mathbb{F}_p$-vector space, we extend $X$ to a minimal generating set of $P$ and add the generator $z$ of $Q$ to obtain a minimal generating set of $H$. Now suppose that $X \not\subseteq P$. Since $H$ does not contain any element of order $pq$, we may assume without loss of generality that $x_1 = z$. Then $X = \{z, v_2z^{j_2}, \ldots, v_mz^{j_m}\}$ where $\{v_2, \ldots, v_m\} \subseteq P$ is an independent subset of size $m - 1$ and $j_2, \ldots, j_m$ are integers. We extend $\{v_2, \ldots, v_m\}$ to a minimal generating set $\{v_2, \ldots, v_d\}$ of $P$. Then $X \cup \{v_{m+1}, \ldots, v_d\}$ is a minimal generating set of $H$. \qed

Using Theorem [1.3] we obtain the following consequence.

**Corollary 5.2.** Let $G$ be a finite group. Then $G$ is a matroid group if and only if one of the following holds:

1. $G$ is a $p$-group for some prime $p$,
2. $G = P \rtimes Q$, where $P$ is a $p$-group, $Q$ is a cyclic $q$-group for primes $p, q$ such that $q \mid p - 1$, $Q/C_Q(P)$ has order $q$ and acts on $P/\Phi(P)$ fixed-point-freely.

### 6. The classification of groups with the basis property

**Lemma 6.1.** Let $G = P \rtimes Q$, where $P$ is a $p$-group and $Q$ a cyclic $q$-group, for distinct primes $p \neq q$, such that every non-trivial element of $Q$ acts fixed-point-freely on $P$. Then $G$ has property $\mathcal{B}$.

**Proof.** If $Q = 1$ then the claim follows from the Burnside basis theorem. From now assume that $Q$ is non-trivial. Clearly, $C_Q(P) = 1$ and by Theorem [1.3] it suffices to show that every non-trivial element of $Q$ acts fixed-point-freely on $P/\Phi(P)$. Let $y \in Q \setminus \{1\}$ and assume for a contradiction that $y$ has a non-zero set of fixed points $U$ in $P/\Phi(P)$. By Maschke’s theorem $P/\Phi(P)$ is a semisimple $\mathbb{F}_pQ$-module. Hence there is a submodule $W$ such that $P/\Phi(P) = U \oplus W$. Then $[P, \langle y \rangle]$ is strictly smaller than $P$ because $[P/\Phi(P), \langle y \rangle] \leq W$. By [2. Theorem 5.3.5] we have $P = C_P(\langle y \rangle)[P, \langle y \rangle]$, hence $C_P(\langle y \rangle) \neq 1$. This contradicts the fact that $y$ acts on $P$ fixed-point-freely. \qed
Lemma 6.2. Let $G = P \rtimes Q$, where $P$ is a $p$-group and $Q$ is a cyclic $q$-group, for distinct primes $p \neq q$, such that every non-trivial element of $Q$ acts fixed-point-freely on $P$. Let $H \leq G$. Then one of the following holds:

1. $H$ is a $p$-group or a $q$-group,
2. $H$ is conjugate in $G$ to a group of the form $R \rtimes S$ with $R \leq P$ and $S \leq Q$.

Proof. Suppose that $H$ is a subgroup of $G$ which is not of prime-power order. Then $H \cap P$ and $H/(H \cap P)$ are non-trivial. Choose an element $h \in H$ such that $H = (H \cap P)\langle h \rangle$ and let $y \in Q$ such that $h \equiv y$ modulo $P$. Since $y$ acts fixed-point-freely on $P$, we have $yP = \{ y^x \mid x \in P \}$ so that $h = y^x$ for some $x \in P$. Consequently, $H = (R \rtimes S)^x$, where $R = (H \cap P)^{x^{-1}}$ and $S = \langle y \rangle$. □

Recall that a finite group $G$ has the basis property if all its subgroups are $B$-groups.

Theorem 6.3 (McDougall-Bagnall and Quick [5]). Let $G$ be a finite group. Then $G$ has the basis property if and only if one of the following holds:

1. $G$ is a $p$-group for some prime $p$,
2. $G \cong P \rtimes Q$, where $P$ is a $p$-group and $Q$ is a non-trivial cyclic $q$-group, for distinct primes $p \neq q$, such that every non-trivial element of $Q$ acts fixed-point-freely on $P$.

Proof. Lemmas 6.1 and 6.2 imply that every group of the described form has the basis property. Conversely, suppose that $G$ has the basis property and is not of prime-power order. From Theorem 1.5 we deduce that $G \cong P \rtimes Q$, where $P$ is a $p$-group and $Q$ is a non-trivial cyclic $q$-group. If there were non-trivial, commuting elements $x \in P$ and $z \in Q$ then the cyclic group $\langle xz \rangle = \langle x \rangle \rtimes \langle z \rangle$ would not have property $B$, contradicting the basis property. Hence every non-trivial element of $Q$ acts fixed-point-freely on $P$. □

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REFERENCES

[1] P. J. Cameron, P. Cara, Independent generating sets and geometries for symmetric groups, J. Algebra 258 (2002), 641–650.
[2] D. Gorenstein, Finite Groups, Harper & Row, Publishers, New York-London, 1968.
[3] A. Lucchini, Generators and minimal normal Subgroups Arch. Math. (Basel) 64 (1995), 273–276.
[4] A. Lucchini, F. Menegazzo, Generators for finite groups with a unique minimal normal subgroup, Rend. Sem. Mat. Univ. Padova 98 (1997), 173–191.
[5] J. McDougall-Bagnall, M. Quick, Groups with the basis property, J. Algebra 346 (2011), 332–339.
[6] D. J. S. Robinson, A course in the theory of groups, Springer-Verlag, New York, 1982.
[7] J. Saxl, J. Whiston, On the maximal size of independent generating sets of $PSL_2(q)$, J. Algebra 258 (2002), 651–657.
[8] R. Scapellato, L. Verardi, *Groupes finis quis jouissent d’une propriété analogue au théorème des bases de Burnside*, Boll. Un. Mat. Ital. A 5 (1991), 187–194.

[9] R. Scapellato, L. Verardi, *Bases of certain finite groups*, Annales mathématiques Blaise Pascal 1 (1994), 85–93.

[10] J. Whiston, *Maximal independent generating sets of the symmetric group*, J. Algebra 232 (2000), 255–268.

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