Distorted charged dilaton black holes

Stoytcho S. Yazadjiev *
Department of Theoretical Physics, Faculty of Physics, Sofia University,
5 James Bourchier Boulevard, Sofia 1164, Bulgaria

Abstract

We construct exact static, axisymmetric solutions of Einstein-Maxwell-dilaton gravity presenting distorted charged dilaton black holes. The thermodynamics of such distorted black holes is also discussed.

1 Introduction

The notion of distorted black holes is very natural from physical point of view. The distorted black hole can be viewed as an isolated system with a black hole in the center and a matter distribution at a finite distance from the black hole. The surrounding matter influences the black hole and thus the latter will be distorted.

Distorted black holes within the framework of general relativity have been considered by many authors (see, for example, [1] - [4]). Geroch and Hartle [1] have shown how to construct general relativistic solutions presenting distorted black holes and have established their global structure, thermodynamic behavior and their evolution with the emission of Hawking radiation. It turns out that the spherical and toroidal black holes are the only possibilities for the topology of horizon cross sections [1].

Very recently, charged distorted black holes in Einstein-Maxwell gravity have been discussed in [5]. The authors have presented new static, axisymmetric solutions to Einstein-Maxwell gravity generalizing the uncharged solutions studied in [1]. The zeroth and the first laws of charged distorted black holes have also been discussed.

The aim of the present work is to construct exact solutions presenting distorted charged dilaton black holes within the framework of a low energy string model - the so called Einstein-Maxwell-Dilaton (EMD) gravity. The thermodynamics of distorted dilaton black holes will also be discussed.

In the case of vacuum, static and axisymmetric space-times, Einstein equations reduce in practice to the Laplace equation on flat space. Making use of the linearity of the Laplace equation, distorted black holes solutions can be obtained in a simple way by adding an appropriate distortion function to the solution presenting an isolated

*E-mail: yazad@phys.uni-sofia.bg
Schwarzschild black hole. This method, however, can’t be applied to the EMD gravity because even for static, axisymmetric space-times the EMD equations are a highly nonlinear set of coupled partial differential equations. Fortunately, after a dimensional reduction, the static EMD equations possess a large group of an internal symmetry which can be employed to construct exact solutions \([6], [7]\). The group action transforms the distorted Schwarzschild solutions into new static, axisymmetric solutions of EMD equations. These new solutions present distorted charged dilaton black holes and depend on three different parameters. One of the parameters comes directly from the distorted Schwarzschild solution, while the remaining are group parameters. From a physical point of view the free parameters describe how different kinds of external matter affect the black hole.

2 Static axisymmetric EMD equations

In the so called Einstein frame the EMD gravity is described by the following system of equations:

\[
R_{ab} = 2\partial_a\varphi \partial_b\varphi + 2e^{-2\varphi} \left( F_{ac}F^c_b - \frac{1}{4}g_{ab}F_{cd}F^{cd} \right) \tag{1}
\]

\[
\Box \varphi = -\frac{1}{2}e^{-2\varphi}F_{ab}F^{ab}
\]

\[
\nabla_b \left( e^{-2\varphi}F^{ab} \right) = 0.
\]

Here \(R_{ab}\) is the Ricci tensor with respect to the space time metric \(g_{ab}\), \(F_{ab} = (dA)_{ab}\) is the Maxwell 2-form and \(\varphi\) is the dilaton field.

The metric of a static and axisymmetric space-time has the Weyl form

\[
ds^2 = -e^{2u}dt^2 + e^{2(h-u)} \left( d\rho^2 + dz^2 \right) + e^{-2u}\rho^2 d\phi^2 \tag{2}
\]

where \(u\) and \(h\) are functions of \(\rho\) and \(z\) only. The metric admits two Killing vectors \(\xi = \frac{\partial}{\partial t}\) and \(\eta = \frac{\partial}{\partial \phi}\).

In what follows we consider only the electrically charged case. Then the Maxwell 2-form can be cast in the form

\[
F = e^{-2u}d\Phi \wedge \xi. \tag{3}
\]

Here we have denoted the electric potential by \(\Phi\).

Pure magnetic solutions can be obtained by performing the discrete duality transformation

\[
e^{-2\varphi}F \rightarrow *F, \ varphi \rightarrow -\varphi, \ g_{ab} \rightarrow g_{ab} \tag{4}
\]
which is a symmetry of (1).

In the static, axisymmetric case the EMD gravity equations are expressed in terms of the metric functions $u$ and $h$, the electric potential $\Phi$ and the dilaton $\varphi$. The resulting equations are:

\[
\begin{align*}
\frac{\partial^2 \rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \rho}{\partial \rho} + \frac{\partial^2 \rho}{\partial z^2} &= e^{-2u-2\varphi} \left( (\rho \partial_{\rho} \Phi)^2 + (\partial_z \Phi)^2 \right) \\
\frac{\partial^2 \rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \rho}{\partial \rho} + \frac{\partial^2 \rho}{\partial z^2} &= e^{-2u-2\varphi} \left( (\rho \partial_{\rho} \Phi)^2 + (\partial_z \Phi)^2 \right) \\
\partial_{\rho} \left( \rho \rho \partial_{\rho} \Phi e^{-2u-2\varphi} \right) + \partial_z \left( \rho \partial_z \Phi e^{-2u-2\varphi} \right) &= 0 \\
\frac{1}{\rho} \partial_{\rho} h &= (\partial_{\rho} u)^2 - (\partial_z u)^2 + (\partial_{\rho} \varphi)^2 - (\partial_z \varphi)^2 - e^{-2u-2\varphi} \left( (\rho \partial_{\rho} \Phi)^2 - (\partial_z \Phi)^2 \right) \\
\frac{1}{\rho} \partial_z h &= 2 \left( \partial_{\rho} u \partial_z u + \partial_{\rho} \varphi \partial_z \varphi - e^{-2u-2\varphi} \partial_{\rho} \Phi \partial_z \Phi \right).
\end{align*}
\]

As it has been shown in [6], [7] the static EMD equation have $GL(2, R)$ as a group of a global internal symmetry. This symmetry becomes transparent if we introduce the following symmetric matrix:

\[
S = \begin{pmatrix}
e^{2u} - 2\Phi^2 e^{-2\varphi} & -\sqrt{2} \Phi e^{-2\varphi} \\
-\sqrt{2} \Phi e^{-2\varphi} & -e^{-2\varphi}
\end{pmatrix}.
\]

Because the group $GL(2, R)$ is not connected we may restrict ourselves to the one of the connected components, say $GL(+) (2, R)$.

In terms of the matrix $S$ the static, axisymmetric EMD equations are written in the following compact form:

\[
\begin{align*}
\partial_{\rho} \left( \rho S^{-1} \partial_{\rho} S \right) + \partial_z \left( \rho S^{-1} \partial_z S \right) &= 0 \\
\frac{1}{\rho} \partial_{\rho} h &= \frac{1}{4} Tr \left( \partial_z S^{-1} \partial_z S - \partial_{\rho} S^{-1} \partial_{\rho} S \right) \\
\frac{1}{\rho} \partial_z h &= -\frac{1}{2} Tr \left( \partial_{\rho} S \partial_z S^{-1} \right).
\end{align*}
\]

The equation for the matrix $S$ is just the chiral equation. It is, therefore, obvious that the static, axisymmetric EMD equations form a completely integrable system. There are powerful soliton methods for solving the chiral equation. In the present work, however we shall not solve this equation directly. Instead, we shall apply the symmetry group to construct exact solutions presenting distorted charged dilaton black holes from uncharged general relativistic ones.

First, we shall demonstrate how the solution describing an isolated charged dilaton black hole can be generated from a Schwarzschild one.

The group $GL(2, R)$ acts as follows:

\[
S \to ASA^T
\]
where \( A \in GL(2, R) \).

The action of the symmetry group does not, in general, preserve the asymptotic flatness. That is why when we are concerned with isolated black holes, we will need the \( GL(^+)(2, R) \)-subgroup preserving asymptotic flatness. The desired subgroup can be easily found. In order for \( S \) to present an asymptotically flat spacetime it is necessary\(^1\) that \( S(\infty) = \sigma_3 \) at spatial infinity. Here \( \sigma_3 \) is the third Pauli matrix. The elements of the subgroup should then satisfy

\[
B\sigma_3B^T = \sigma_3.
\]

Therefore, the subgroup preserving the asymptotic flatness is \( SO(1, 1) \).

The parameterization of \( SO(1, 1) \) is taken to be the standard one:

\[
B = \begin{pmatrix}
\frac{1}{\sqrt{1-\beta^2}} & \frac{\beta}{\sqrt{1-\beta^2}} \\
\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}}
\end{pmatrix} \quad (9).
\]

In order to generate the charged dilaton black hole solution we start with the Schwarzschild solution written in the standard spherical coordinates,

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2 \quad (10)
\]

where \( d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \). The relationship between the coordinates \( r \) and \( \theta \) and the Weyl coordinates is given by equations (19) - (21) below.

The matrix \( S \) which corresponds to the Schwarzschild solution is

\[
S_S = \begin{pmatrix}
e^{2\lambda_S} & 0 \\
0 & -1
\end{pmatrix} \quad (11)
\]

where \( e^{2\lambda_S} = 1 - \frac{2M}{r} \).

The desired solution is presented by the matrix

\[
S = BS_SB^T = \frac{1}{1 - \beta^2} \begin{pmatrix}
e^{2\lambda_S} - \beta^2 & \beta \left( e^{2\lambda_S} - 1 \right) \\
\beta \left( e^{2\lambda_S} - 1 \right) & \beta^2 e^{2\lambda_S} - 1
\end{pmatrix}. \quad (12)
\]

Recovering the four-metric we have

\[
ds^2 = \frac{1}{1 - \frac{2M}{r}} - \frac{2M}{r} dt^2 + e^{-2\phi} \left( \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2 \right) \quad (13)
\]

\(^1\)Without loss of generality we take \( \phi(\infty) = 0 \).
where
\[ e^{-2\varphi} = 1 + \frac{2\beta^2 M}{(1 - \beta^2)r}. \] (14)

The electric potential is, respectively,
\[ \Phi = \frac{\sqrt{2}\beta M}{1 - \beta^2} \left( r + \frac{1}{1 - \beta^2} \right). \] (15)

The obtained solution describes an isolated charged dilaton black hole with mass \( M = \frac{M}{1 - \beta^2} \) and charge \( Q = \frac{\sqrt{2}\beta M}{1 - \beta^2} \). It may be cast in more familiar form by performing the coordinate shift \( R = r + \frac{Q^2}{M} \). Then we obtain the black hole solution in Garfinkle-Horowitz-Strominger coordinates:
\[
\begin{align*}
ds^2 &= -(1 - \frac{2M}{R})dt^2 + (1 - \frac{2M}{R})^{-1}dR^2 + R(R - \frac{Q^2}{M})d\Omega^2 \\
e^{-2\varphi} &= 1 - \frac{Q^2}{MR} \\
\Phi &= \frac{Q}{R}.
\end{align*}
\] (16)

We shall follow a similar line of thoughts in the next section to construct distorted charged dilaton black hole solutions from general relativistic ones.

The matrix \( B \) transforming the Schwarzschild solution into the solution presenting charged dilaton black holes with mass \( M \) and charge \( Q \) can be written as
\[ B_{(M,Q)} = \frac{1}{\sqrt{1 - \frac{Q^2}{2M^2}}} \left( \begin{array}{cc}
\frac{Q}{\sqrt{2M}} & \frac{Q}{\sqrt{2M}} \\
\frac{Q}{\sqrt{2M}} & 1
\end{array} \right). \] (17)

3 Distorted black holes

This section is devoted to the construction of charged dilaton solutions presenting distorted black holes. For the reader’s convenience we begin with a brief review of the uncharged general relativistic distorted black holes following [5] closely. Finally, some interesting properties of the distorted charged black holes will be discussed.

3.1 Distorted Schwarzschild black hole

The vacuum, static and axisymmetric Einstein equations reduce to the following system of partial differential equations for the metric functions \( \lambda \) and \( h $2$In this subsection we write \( \lambda \) instead of \( u \).
\[ \partial_\rho^2 \lambda + \frac{1}{\rho} \partial_\rho \lambda + \partial_z^2 \lambda = 0 \]
\[ \frac{1}{\rho} \partial_\rho h = (\partial_\rho \lambda)^2 - (\partial_z \lambda)^2 \]  
(18)
\[ \frac{1}{\rho} \partial_z h = 2 \partial_\rho \lambda \partial_z \lambda . \]

If a solution for \( \lambda \) has been found then the second metric function \( h \) is obtained by integrating the remaining equations. The function \( h \) is determined up to a constant. The constant can be fixed by imposing the boundary condition: \( h = 0 \) on the \( \rho = 0 \) axis at all points where \( \lambda \) is regular. This condition also ensures that, near the axis, the orbits of the Killing vector \( \eta \) are closed curves of period \( 2\pi \).

The exterior Schwarzschild solution is a vacuum Weyl solution. The explicit form of the Schwarzschild solution presenting black hole with mass \( M \) is

\[ 2\lambda_S = \ln \left( \frac{L-M}{L+M} \right) \quad \text{and} \quad 2h_S = \ln \left( \frac{L^2 - M^2}{L^2 - \Delta^2} \right) \]  
(19)

where \( L \) and \( \Delta \) are functions of \( \rho \) and \( z \) given by

\[ L = \frac{1}{2} \left( \sqrt{\rho^2 + (z+M)^2} + \sqrt{\rho^2 + (z-M)^2} \right) , \]
\[ \Delta = \frac{1}{2} \left( \sqrt{\rho^2 + (z+M)^2} - \sqrt{\rho^2 + (z-M)^2} \right) . \]  
(20)

In Weyl coordinates, the horizon \( \mathcal{H} \) is placed in the segment \( |z| \leq M \) on the axis \( \rho = 0 \). As it can be seen the metric functions \( \lambda_S \) and \( h_S \) are well behaved everywhere except in the limit \( \rho \to 0 \) for the segment \( |z| \leq M \) where they diverge logarithmically. There is, however, nothing dangerous because the divergence is just the well-known coordinate singularity. This can be seen by performing the following coordinate transformation:

\[ r = L + M, \quad z = L \cos(\theta) \]
\[ \cos(\theta) = \frac{\Delta}{M}, \quad \rho^2 = (L^2 - M^2) \sin^2(\theta) \]  
(21)

Under this transformation the solution takes the standard Schwarzschild form with an event horizon \( \mathcal{H} \) placed at \( r = 2M \).

Distorted black holes solutions are constructed as follows. The vacuum, static and axisymmetric Einstein equations (18) are linear in the function \( \lambda \). This fact allows us to construct new solutions simply by adding any harmonic potential \( \lambda_D \) to the Schwarzschild potential \( \lambda_S \). The solutions obtained in this way can be viewed as distorted analogs of the Schwarzschild solution. When the potential \( \lambda_D \) is everywhere
regular the location of the horizon $\mathcal{H}$ will be unchanged. It should be noted, however, that since $\lambda_D$ is a harmonic function and is regular on the horizon it cannot tend to zero at infinity. Moreover, $\lambda_D$ must diverge at infinity and therefore the solution will not be asymptotically flat.

Having the new potential $\lambda = \lambda_S + \lambda_D$, the other metric function $h$ is obtained by integrating the remaining equations of (18) with the boundary condition that has been imposed. It turns out that the boundary condition places a restriction on the distorting potential $\lambda_D$, too. In order for $h$ to vanish on both disconnected sections of the axis, it is required that $\lambda_D$ takes the same value $u_*$ at both ends of the segment $\mathcal{H}$. Moreover, it turns out that

$$h \mid_{\mathcal{H}} = 2 \lambda_D \mid_{\mathcal{H}} - 2u_*.$$  \hspace{1cm} (22)

This important relation allows us to show that the metric is regular on the horizon.

Finally, the distorted metric can be presented in Schwarzschild coordinates as follows:

$$ds^2 = -e^{2\lambda_D} \left(1 - \frac{2M}{r}\right) dt^2 + e^{2(h_D - \lambda_D)} \left(\frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2\right) + e^{-2\lambda_D} r^2 \sin^2(\theta) d\phi^2$$ \hspace{1cm} (23)

where we have set $h_D = h - h_S$.

As has been shown in [1], this metric can be analytically continued through the horizon.

As we have already mentioned the distorted solutions are not asymptotically flat provided the function $\lambda_D$ is everywhere harmonic and is not a constant. As shown in [1] it is possible to find asymptotically flat extensions if we require that $\lambda_D$ is harmonic only in a neighborhood of the horizon and to extend $\lambda_D$ and $h_D$ so that they vanish at infinity. Therefore, we assume that in the intervening region the vacuum Einstein equations are not satisfied, i.e. there is some kind of matter present in this region. The distortion of the black hole is caused by this matter. It was shown that if the matter satisfies the strong energy condition the function $\lambda_D$ must be non-positive everywhere in space-time and therefore $u_* \leq 0$ (see [1]).

For more details concerning distorted Schwarzschild black holes we refer the reader to [1] and [3].

### 3.2 Distorted charged dilaton black holes

The existence of a group of an internal symmetry for the dimensionally reduced EMD equations allows us to transform any given static solution of vacuum Einstein equations to a static solution of EMD equations. Therefore, the vacuum Weyl solutions describing distorted Schwarzschild black holes can be transformed into Weyl EMD solutions. These new solutions, as it is natural to expect, should present distorted charged dilaton black holes.

The group action transforming a distorted Schwarzschild solution into a static, axisymmetric solution of EMD equations involve arbitrary $GL(\mathbb{R})$ matrices and we need to specify them. The $GL(\mathbb{R})$ matrices should be of the form
\[ A = \begin{pmatrix} e^{\upsilon_1} & 0 \\ 0 & e^{\upsilon_2} \end{pmatrix} B(M,Q) \]  

(24)

where \( \upsilon_1 \) and \( \upsilon_2 \) are arbitrary constants. This is the most general form of the \( GL^+(2,R) \) matrices which are physically interesting in the present context because the remaining transformation, which is not included in (24), is a pure electromagnetic gauge.

Starting with a distorted Schwarzschild solution presented by the matrix

\[ S_{DS} = \begin{pmatrix} e^{2\lambda} & 0 \\ 0 & -1 \end{pmatrix} \]  

(25)

where \( \lambda = \lambda_S + \lambda_D \), we obtain a new static, axisymmetric EMD solution given by the matrix

\[ S = AS_{DS}A^T. \]  

(26)

It should be noted that the new EMD solution has the same metric function \( h \) as the original distorted Schwarzschild solution \( (h = h_{DS}) \).

The space-time metric can be recovered and its form is

\[
ds^2 = -e^{2u_D} \left( 1 - \frac{2M}{R} \right) dt^2 + e^{2(h_D - u_D)} \left[ \frac{dR^2}{1 - \frac{2M}{R}} + R \left( R - \frac{Q^2}{M} \right) d\theta^2 \right] + e^{-2u_D} R \left( R - \frac{Q^2}{M} \right) \sin^2(\theta) d\phi^2
\]  

(27)

where the distorting potential \( u_D \) is given by

\[
e^{2u_D} = e^{2\upsilon_1} \frac{\left( 1 - \frac{Q^2}{2MR^2} e^{2\lambda_S} \right) e^{2\lambda_D}}{1 - \frac{Q^2}{2MR^2} e^{2\lambda_S} e^{2\lambda_D}}. \]  

(28)

To specify the solution completely we should also give the dilaton and electromagnetic fields. They are given, respectively, by the following expressions:

\[
e^{2\varphi} = \left( 1 - \frac{Q^2}{MR} \right) e^{2\varphi_D} = \left( 1 - \frac{Q^2}{MR} \right) e^{-2\upsilon_2} \frac{\left( 1 - \frac{Q^2}{2MR^2} e^{2\lambda_S} \right)}{1 - \frac{Q^2}{2MR^2} e^{2\lambda_S} e^{2\lambda_D}},
\]  

(29)

\[
\Phi = \frac{Q}{2M} e^{\upsilon_1 - \upsilon_2} \frac{1 - e^{2\lambda_S} e^{2\lambda_D}}{1 - \frac{Q^2}{2MR^2} e^{2\lambda_S} e^{2\lambda_D}}.
\]  

(30)
It is useful to express the electric potential $\Phi$ as the standard potential $\frac{Q}{R}$ plus a distortion part $\Phi_D$:

$$\Phi = \frac{Q}{R} + \Phi_D = \frac{Q}{R} + \frac{Q}{2M} \left[ e^{v_1-v_2} \frac{1 - e^{2\lambda s}e^{2\lambda_D}}{1 - \frac{Q^2}{2M}e^{2\lambda s}e^{2\lambda_D}} - \frac{1 - e^{2\lambda s}}{1 - \frac{Q^2}{2M}e^{2\lambda s}} \right].$$ (31)

It is easy to see that one has

$$u_D|_H = \lambda_D|_H + v_1.$$ (32)

on the horizon.

Therefore, we obtain that the following is satisfied on the horizon:

$$(h_D - 2u_D)|_H = -2(\mu + v_1).$$ (33)

Since $\lambda_D$ is a regular harmonic function the metric (27) is non-singular in a neighbourhood of the horizon. In the case when $\lambda_D$ is non-positive the metric is non-singular everywhere outside the horizon.

It remains to show that the metric (27) is regular on the horizon. This can be demonstrated by standard techniques. For this purpose we introduce the new Edington-Finkelstein-like coordinate $\omega$, given by $d\omega = dt + e^{-2(u_* + v_1)} \frac{dR}{1 - \frac{Q^2}{MR}}$. In terms of the coordinate $\omega$ the metric (27) is written in the form

$$ds^2 = -e^{2u_D} \left( 1 - \frac{2M}{R} \right) d\omega^2 + 2e^{-2(u_* + v_1)} e^{2u_D} dR d\omega + e^{2u_D} \left( \frac{e^{2(h_D - 2u_D)} - e^{-4(u_* + v_1)}}{1 - \frac{Q^2}{MR}} \right) dR^2 + R \left( R - \frac{Q^2}{MR} \right) [e^{2(h_D - u_D)} d\theta^2 + e^{-2u_D} \sin^2(\theta) d\phi^2].$$ (34)

It is not difficult to show that the coefficient of the $dR^2$ term remains finite on the horizon. Therefore, we can conclude that the metric (27) is regular on the horizon.

As it is clear our space-time is not asymptotically flat. Our solution, however, can be extended as in [1] to be asymptotically flat. This can be done as follows. We assume that the solution is valid only in a neighbourhood of the horizon. Outside this neighbourhood the solution can be extended arbitrary to present asymptotically flat space-time. In the intervening region the EMD equations will not be satisfied, which shows the presence of external matter causing the black hole distortion.

### 3.3 Properties of distorted charged dilaton black holes

The horizon geometry is presented by the metric

$$ds^2_{\mathcal{H}} = R^2_{\mathcal{H}} \left( 1 - \frac{Q^2}{MR_{\mathcal{H}}} \right) [e^{2h_D(\theta) - 2u_D(\theta)} d\theta^2 + e^{-2u_D(\theta)} \sin^2(\theta) d\phi^2]$$ (35)
where \( h_D(\theta) = h_D(R_H, \theta) \), \( u_D(\theta) = u_D(R_H, \theta) \) and \( R_H = 2 \mathcal{M} \). This is an axisymmetric but not in general spherically symmetric metric on a topological 2-sphere. Therefore, the geometry of the horizon is distorted. In order to see this in more explicit form we write the curvature of the horizon 2-sphere

\[
2R = 2e^{4(u_* + v_1)} \frac{e^{-2u_D(\theta)}}{R_H^2 \left( 1 - \frac{Q^2}{\mathcal{M}R_H} \right)} \left( 1 - 2(\partial_\theta u_D(\theta))^2 + 3 \cot(\theta) \partial_\theta u_D(\theta) + \partial^2_{\theta\theta} u_D(\theta) \right) . \tag{36}
\]

As is obvious the curvature of the two-sphere is not constant. Only in the particular case \( u_D = 0 \) we obtain a constant curvature.

The area of the horizon can be easily calculated and the result is

\[
\mathcal{A}_H = 4\pi R_H^2 \left( 1 - \frac{Q^2}{\mathcal{M}R_H} \right) e^{-2(u_* + v_1)} . \tag{37}
\]

The dual of the Maxwell tensor evaluated on the black hole horizon is

\[
\star F|_H = Qe^{-v_2 - v_1} \left( 1 - \frac{Q}{\mathcal{M}R_H} \right) \sin(\theta) d\theta \wedge d\phi . \tag{38}
\]

The charge of the distorted black hole is given by

\[
Q_H = \frac{1}{4\pi} \oint_H e^{-2\varphi} \star F . \tag{39}
\]

Performing integration on the horizon we obtain the explicit expression

\[
Q_H = Qe^{v_2 - v_1} . \tag{40}
\]

4 Thermodynamics of distorted charged dilaton black holes

In this section we discuss the zeroth and the first laws of distorted charged dilaton black hole mechanics.

4.1 Zeroth law

The zeroth law states that the surface gravity is constant over the horizon. The surface gravity is defined as

\[
\xi^a \nabla_a \xi^b = \kappa \xi^b \tag{41}
\]
where $\xi$ is the horizon-generating Killing field. It should be mentioned that the above definition gives the surface gravity up to a constant because there is a freedom to rescale $\xi$ by a constant. Although, this rescaling does not influence the zeroth law, for simplicity we assume from now on that our solutions have been extended to be asymptotically flat. Then the freedom to rescale the Killing vector can be fixed, requiring that the Killing vector becomes time translation at infinity.

The explicit form of the surface gravity can be easily calculated and the result is

$$
\kappa = \frac{1}{4M}e^{2(u_\ast + \nu_1)}.
$$

(42)

The above result shows that the surface gravity is indeed constant over the horizon and therefore the zeroth law is satisfied.

Moreover, as one may expect, the electric potential and the dilaton turn out to be constant over the black hole horizon. Indeed, it is not difficult to show that

$$
\Phi_H = \frac{Q}{2M}e^{\nu_1 - \nu_2}, \quad e^{2\varphi_H} = e^{-2\nu_2} \left(1 - \frac{Q^2}{2M^2}\right).
$$

(43)

4.2 First law

Variation of the black hole mass between neighboring equilibrium states yields the first law of black hole physics. We want to write down the first law for distorted charged dilaton black holes regarded as single systems acted on by external matter. The form of the first law, of course, depends on how the black hole mass is defined \[1, 5\]. We have three possible choices for black hole mass in our case. The first natural choice is the ADM mass $M_{\text{ADM}}$ of space-time. The second choice is the mass parameter $M$ appearing in the black hole solution and the third choice is the Komar mass of the horizon $M_H = \frac{1}{4\pi} \kappa A_H$.

The ADM mass $M_{\text{ADM}}$ contains a contribution from the distorting external matter. While we consider the distorted charged dilaton black holes as single systems the ADM mass is not appropriate for our purpose. The Komar mass $M_H$ is just the gravitational mass of the black hole and does not contain a contribution from the electromagnetic and dilatonic hair of the black hole. Therefore, it is not appropriate, too. Only the mass parameter $M$ is appropriate in our case. The parameter $M$ can be interpreted physically as "the mass of the black hole alone as measured at infinity". This interpretation is in agreement with the fact that $M$ satisfies a Smarr formula:

$$
M = \frac{\nu_1}{4\pi} A_H + \Phi_H Q_H.
$$

Varying the mass $M$ we obtain

$$
\delta M = \frac{1}{8\pi} \kappa \delta A_H + \Phi_H \delta Q_H + M_H \delta u_\ast + M \delta \nu_1 + (M_H - M) \delta \nu_2
$$

(44)

This algebraic identity is the black hole first law for an observer at infinity. The first two terms on the right-hand side of (44) are the standard terms describing the
change of the mass due to the change in the area and the charge of the black hole. There are, however, three new terms related to the parameters $u_*, \upsilon_1$ and $\upsilon_2$.

The parameter $u_*$ comes from the Schwarzschild solution and thus it may be viewed as a quantity describing how the uncharged and non-dilatonic external matter affects the black hole. Moreover, $u_*$ couples to the pure gravitational energy $\mathcal{M}_H$ of the black hole and therefore the term $\mathcal{M}_H \delta u_*$ (as in [1], [2]) can be interpreted as the gravitational work done by uncharged and non-dilatonic matter on the black hole.

The parameter $\upsilon_1$ can be considered as a quantity describing the influence of the charged external matter on the black hole. As can be seen from eqs. (28) - (30), $\upsilon_1$ affects both the gravitational and electric potentials and therefore the gravitational and electric parameters on the horizon. That is why it sounds naturally that $\upsilon_1$ couples to the total mass of the black hole which contains contributions from gravitational, electromagnetic and dilaton fields. Hence we can interpreted the term $\mathcal{M} \delta \upsilon_1$ as the work done on the black hole by variations in the charged external matter.

The parameter $\upsilon_2$ can be considered as a quantity describing how the external dilatonic matter affects the black hole. As can be seen from eqs. (28) - (30), the parameter $\upsilon_2$ affects the electric potential and dilaton field but not the gravitational potential. That is why it is reasonable that the parameter $\upsilon_2$ couples only to the mass of the dilaton-electromagnetic hair of the black hole but not to the gravitational part of mass. This is a consequence of the well-known fact the black hole can not support pure scalar hair. So, the term $(\mathcal{M}_H - \mathcal{M}) \delta \upsilon_2$ can be viewed as the work done on the black hole by the external dilatonic matter.

The first law for distorted dilaton black holes obtained in the present work can be considered as a natural generalization of the first law for undistorted dilaton black holes found by Ashtekar and Corichi in the isolated horizon context [8]. Here we have considered the first law from the standard classical point of view. The general treatment of the first law in the isolated horizon framework can be found in [9] where some particular examples as Einstein-Maxwell theory and EMD gravity are considered. It should be noted, however, that no exact distorted dilaton black hole solutions are presented in [9].

5 Conclusion

In the present work we have constructed exact static, axisymmetric solutions of the Einstein-Maxwell-dilaton gravity presenting distorted charged dilaton black holes. The method for constructing EMD solution is based on the global symmetry group of the dimensionally reduced EMD equations. Making use of the group action we can transform any given distorted black solution of the pure Einstein equations into a static, axisymmetric solution of EMD equations. The new solution presents a distorted charged dilaton black hole and depends on three arbitrary parameters. One of the parameters comes directly from the seed distorted Schwarzschild solution and describes how the uncharged and non-dilatonic external matter affects the black hole. Two remaining parameters are group parameters. They can be physically interpreted as quantities describing correspondingly how the charged and dilatonic external matter affects the
black hole.

The zeroth and the first law of black hole thermodynamics have been discussed, too.

It is worth noting that the method presented in this work can also be applied within the framework of Einstein-Maxwell gravity to generate distorted charged black holes. Indeed, after a dimensional reduction the static, axisymmetric Einstein-Maxwell equations have $SL(2, R)$ as a group of global symmetry. The corresponding subgroup, preserving asymptotic flatness is again $SO(1, 1)$. Moreover, the method and the results of this work can be generalized for EMD gravity with arbitrary dilaton coupling parameter including as particular cases the model considered here and Einstein-Maxwell equations.

We would also like to discuss briefly the following question. We have generated the distorted charged dilaton black hole solutions starting with the distorted Schwarzschild solutions. One may, however, wonder whether the distorted charged dilaton black hole solutions can be generated from the distorted black hole solutions of Einstein-Maxwell gravity. We shall demonstrate that this is possible without going into the details of calculation.

Let us return to the static, axisymmetric EMD equations (5) and to introduce the new potentials $U = u + \varphi$, $\Psi = u - \varphi$, $\Lambda = \sqrt{2}\Phi$ and the new metric function $H = 2h$. The static, axisymmetric EMD equations can be rewritten in the form

$$
\partial^2_{\rho}\psi + \frac{1}{\rho}\partial_{\rho}\psi + \partial^2_{\rho}\psi = 0
$$

$$
\partial^2_{\rho}U + \frac{1}{\rho}\partial_{\rho}U + \partial^2_{\rho}U = e^{-2U}\left((\partial_{\rho}\Lambda)^2 + (\partial_{\rho}\Lambda)^2\right)
$$

$$
\partial_{\rho}\left(\rho\partial_{\rho}e^{-2u-2\varphi}\right) + \partial_{\rho}\left(\rho\partial_{\rho}e^{-2u-2\varphi}\right) = 0
$$

(45)

$$
\frac{1}{\rho}\partial_{\rho}H = (\partial_{\rho}U)^2 - (\partial_{\rho}U)^2 + (\partial_{\rho}\Psi)^2 - (\partial_{\rho}\Psi)^2 - e^{-2U}\left((\partial_{\rho}\Lambda)^2 - (\partial_{\rho}\Lambda)^2\right)
$$

$$
\frac{1}{\rho}\partial_{\rho}H = 2\partial_{\rho}U\partial_{\rho}U + 2\partial_{\rho}\Psi\partial_{\rho}\Psi - 2e^{-2U}\partial_{\rho}\Lambda\partial_{\rho}\Lambda
$$

It is not difficult to recognize that the above system is just the static, axisymmetric Einstein-Maxwell equations along with minimally coupled scalar field $\Psi$. Let us denote by $U_0$ and $\Psi_0$ the potentials presenting the isolated charged dilaton black hole. Note that $U_0$ is just the Reissner-Nordström potential. Having once the potential $U = U_0 + U_D$ describing a distorted Reissner-Nordström black hole we may construct immediately a distorted charged dilaton black hole solution taking the pair of potentials $U$ and $\Psi_0$. This solution, however, is not the most general one. The most general charged dilaton black hole solution can be obtained by adding to $\Psi_0$ an arbitrary regular distortion harmonic function $\Psi_D$.

In this paper we have consider only pure electrically (magnetically) charged distorted dilaton black holes. Distorted dilaton black holes with both electric and magnetic charges, however, are not a trivial generalization of the above considerations. They form more complicated systems because they can support additional axion hair.
Acknowledgments

The author would like to thank P. Fiziev and V. Rizov for stimulating discussions.

References

[1] R. Geroch, J. Hartle, J. Math. Phys. 23, 680 (1981)
[2] L. Mysak, G. Szekeres, Can. J. Phys. 44, 617 (1966)
[3] P. Peters, J. Math. Phys. 20, 1481 (1979)
[4] B. Xanthopolous, Proc. Roy. Soc. Ser. A388, 117 (1983)
[5] S. Fairhurst, B. Krishnan, Distorted black holes with charge, Preprint gr-qc/0010088
[6] S. Yazadjiev, Int. J. Mod. Phys. D8, 635 (1999)
[7] S. Yazadjiev, Exact static solutions in Einstein-Maxwell-dilaton gravity with arbitrary dilaton coupling parameter, gr-qc/0101078
[8] A. Ashtekar, A. Corichi, Class. Quantum Grav. 17, 1317 (2000)
[9] A. Ashtekar, S. Fairhurst, B. Krishnan, Phys. Rev. D62, 104025-1 (2000)