TWISTED ALEXANDER INVARIANT AND NON–ABELIAN
REIDEMEISTER TORSION FOR HYPERBOLIC
THREE–DIMENSIONAL MANIFOLDS WITH CUSPS

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Abstract. We study a computational method of the hyperbolic Reidemeister torsion
(also called in the literature the non–abelian Reidemeister torsion) considered by J.
Porti for complete hyperbolic three–dimensional manifolds with cusps. The derivative
of the twisted Alexander invariant for a hyperbolic knot exterior gives the hyperbolic
torsion. We prove such a derivative formula of the twisted Alexander invariant for
hyperbolic link exteriors like the Whitehead link exterior. We provide the framework
for the derivative formula to work, which consists of assumptions on the topology
of the manifold and on the representations involved in the definition of the twisted
Alexander invariant, and prove derivative formula in that context. We also explore
the symmetry properties (with sign) of the twisted Alexander invariant and prove
that it is in fact a polynomial invariant, like the usual Alexander polynomial.

1. Introduction

Hyperbolic Reidemeister torsion, i.e. the Reidemeister torsion twisted by the ad-
joint representation associated to an irreducible representation into $\text{SL}_2(\mathbb{C})$, was first
introduced by D. Johnson in unpublished notes [Joh]. Later it was considered by E. Witten
[Wit91] as a symplectic volume form on the moduli space of $\text{SU}(2)$–flat connections
over two–dimensional surfaces and developed to the case for closed three–manifolds by
L. Jeffrey and J. Weitsman [JW93] and J. Park [Par97]. Furthermore, J. Porti [Por97]
applied the Reidemeister torsion twisted by the adjoint representation to the study of
three–dimensional hyperbolic manifolds eventually with cusps via $\text{SL}_2(\mathbb{C})$–character va-
rieties.

Let $M$ be a hyperbolic three–dimensional manifold with cusps. We consider the char-
acter variety $X(M)$ of $M$, which is in a sense the “algebraic quotient” of the represen-
tation space $\text{Hom}(\pi_1(M),\text{SL}_2(\mathbb{C}))$ under the action by conjugation. This set has the
structure of a complex algebraic affine set as it is proved in [CS83, LM85]). The geo-
metric component of $X(M)$ is the connected component which contains the discrete and
faithful representation of the complete hyperbolic structure.

The construction of the hyperbolic torsion uses $\text{SL}_2(\mathbb{C})$–representations whose con-
jugacy classes lie in the geometric component. In general, it is not easy to compute
the hyperbolic torsion directly from the definition because the twisted complex involved
for the computation is not acyclic is that case. But for hyperbolic knot exteriors, we
can carry out the computation of the hyperbolic torsion by using the derivative of the
twisted Alexander invariant given by second author’s work [Yam08]. Such formula gives
a link between the hyperbolic torsion, which is a “non–acyclic” Reidemeister torsion,
and another one which has the advantage to be computed using an acyclic complex. We
call this procedure the derivative formula of the twisted Alexander invariant and we use
the terminology non–abelian Reidemeister torsion instead of the hyperbolic Reidemeister torsion following author’s previous works [Dub05, DHY09].

This paper provides the generalized framework to compute the non–abelian Reidemeister torsion of a hyperbolic three–dimensional manifold with cusps by using the derivative of the twisted Alexander invariant with multivariables. We proceed from the technical conditions required by our framework to the computation procedure of the non–abelian Reidemeister torsion. Our framework requires the following three kinds of assumptions:

- topological conditions on the hyperbolic three–dimensional manifold $M$ whose boundary consists in the disjoint union of $b$ two–dimensional tori:
  \[ \partial M = \bigcup_{\ell=1}^{b} T^2_{\ell}; \]
- conditions on the surjective homomorphism
  \[ \varphi: \pi_1(M) \to \mathbb{Z}^n = \langle t_1, \ldots, t_n | t_it_j = t_jt_i \ (\forall \ i, j) \rangle \]
  which gives the variables of the twisted Alexander invariant and;
- conditions on an irreducible $SL_2(\mathbb{C})$–character $\rho: \pi_1(M) \to SL_2(\mathbb{C})$ which lies in the geometric component of the character variety.

These assumptions are referred to as the symbols $(A_M)$, $(A_{\varphi})$ and $(A_{\rho})$. Under these assumptions, we show the following four results:

- the non–abelian Reidemeister torsion $T^M_{\Lambda}(\rho)$ and the twisted Alexander invariant $\Delta^\otimes_{M, \rho}(t_1, \ldots, t_n)$ are well–defined;
- the derivative of the twisted Alexander invariant with multivariables gives the non–abelian Reidemeister torsion. More precisely, we have:
  \[ \lim_{\ell_1, \ldots, \ell_n \to 1} \frac{\Delta^\otimes_{M, \rho}(t_1, \ldots, t_n)}{\prod_{\ell=1}^{b} (t_1^{\otimes\ell_1} \cdots t_n^{\otimes\ell_n} - 1)} = (-1)^b \cdot T^M_{\Lambda}(\rho). \]

Here $b$ denotes the number of tori components of the boundary $\partial M$ of $M$ and
  \[ \varphi(\pi_1(T^2_{\ell})) = \langle t_1^{\otimes\ell_1} \cdots t_n^{\otimes\ell_n} \rangle \]
  \[ \forall \ (t_1^{\otimes\ell_1} \cdots t_n^{\otimes\ell_n}); \]
- under some technical conditions on the representations $\varphi$ and $\rho$, we prove that the twisted Alexander invariant $\Delta^\otimes_{M, \rho}(t_1, \ldots, t_n)$, which a priori lies in the fraction field $\mathbb{C}(t_1, \ldots, t_n)$, is in fact contained in $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ (up to a factor $t_1^{m_1} \cdots t_n^{m_n}$ for some integers $m_1, \ldots, m_n$), and;
- we give the computation example of the non–abelian Reidemeister torsion for the Whitehead link exterior by using this generalized formula.

The advantage of our generalization of the derivative formula lies in the fact that we can obtain a sequences of hyperbolic knot exteriors by Dehn filling a hyperbolic link exterior with some solid tori. Since twist knots are obtained by Dehn filling the Whitehead link exterior, the computations result of the non–abelian Reidemeister torsion for the Whitehead link exterior is useful to observe the asymptotic behavior of the non–abelian Reidemeister torsions for the twist knot exteriors, as observed in [DHY09] by V. Huynh and the authors.

It is expected that the twisted Alexander invariant in our framework has other properties than the derivative formula. We also prove a duality formula (with sign) for the twisted Alexander invariant which can be compared with the well–known symmetry property of the usual Alexander polynomial. More precisely, if $M$ is a link exterior $M = E_L = S^3 \setminus N(L)$, then the polynomial torsion of $E_L$ satisfies the following formula:

\[ \Delta^\otimes_{E_L}(t^{-1}) = (-1)^b \Delta^\otimes_{E_L}(t). \]
Organization

The outline of the paper is as follows. Section 2 deals with some reviews on the sign–determined Reidemeister torsion for a manifold and on the multiplicative property of Reidemeister torsions (the Multiplicativity Lemma) which is the main tool for computing Reidemeister torsions by using a cut and past argument. In Section 3, we set down technical assumptions used in the whole of this paper, prove that a certain twisted homology vanishes and then give the definition of the twisted Alexander invariant for a hyperbolic manifold with boundary. Section 4 presents some examples of computations of the twisted Alexander invariant for the figure eight knot exterior and for the Whitehead link exterior. Section 5 gives a reduction of the number of variables in the twisted Alexander invariant when we change coefficients in the twisted chain complex in the definition of the torsion. Section 6 is devoted to the derivative formula which gives a bridge joining the twisted Alexander invariant and the non–abelian Reidemeister torsion in the adjoint representation associated with an $SL_2(\mathbb{C})$-representation in the geometric component. In Section 7, we prove that the twisted Alexander invariant is generically a polynomial by using a cut and paste argument and the multiplicative property of Reidemeister torsions (see Theorem 5). The symmetry properties (with sign) of this polynomial torsion are investigated in Section 8 (see Theorem 14).

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2. Preliminaries

2.1. The Reidemeister torsion. We review the basic notions and results about the sign–determined Reidemeister torsion introduced by V. Turaev which are needed in this paper. Details can be found in Milnor’s survey [Mil66] and in Turaev’s monograph [Tur02].

Torsion of a chain complex. Let $C_\ast = (0 \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to d_1 : C_1 \to C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over a field $F$. Choose a basis $c_1^{(i)}$ of $C_i$ and a basis $h^i$ of the $i$-th homology group $H_i(C_\ast)$. The torsion of $C_\ast$ with respect to these choices of bases is defined as follows.

For each $i$, let $b^i$ be a set of vectors in $C_i$ such that $d_i(b^i)$ is a basis of $B_{i-1} = \text{im}(d_i : C_i \to C_{i-1})$ and let $\tilde{h}^i$ denote a lift of $h^i$ in $Z_i = \ker(d_i : C_i \to C_{i-1})$. The set of vectors $d_{i+1}(b^{i+1})\tilde{h}^i b^i$ is a basis of $C_i$. Let $[d_{i+1}(b^{i+1})\tilde{h}^i b^i/c^i] \in F^\ast$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(b^{i+1})\tilde{h}^i b^i$ with respect to $c^i$). The sign-determined Reidemeister torsion of $C_\ast$ (with respect to the bases $c^i$ and $h^i$) is the following alternating product (see [Tur01, Definition 3.1]):

\begin{equation}
\text{Tor}(C_\ast, c^\ast, h^\ast) = (-1)^{|C_1|} \prod_{i=0}^n [d_{i+1}(b^{i+1})\tilde{h}^i b^i/c^i]^{-1}\cdots(-1)^{i+1} \in F^\ast.
\end{equation}
Here

$$|C_*| = \sum_{k \geq 0} \alpha_k(C_*) \beta_k(C_*),$$

where $\alpha_k(C_*) = \sum_{i=0}^k \dim C_i$ and $\beta_k(C_*) = \sum_{i=0}^k \dim H_i(C_*)$.

The torsion $\text{Tor}(C_*, c^*, h^*)$ does not depend on the choices of $h^*$ nor on the lifts $\tilde{h}^i$. Note that if $C_*$ is acyclic (i.e. if $H = 0$ for all $i$), then $|C_*| = 0$.

**Torsion of a CW-complex.** Let $W$ be a finite CW-complex and $(V, \rho)$ be a pair of a vector space with an inner product over $\mathbb{F}$ and a homomorphism of $\pi_1(W)$ into $\text{Aut}(V)$. The vector space $V$ turns into a right $\mathbb{Z}[\pi_1(W)]$-module denoted $V_\rho$ by using the right action of $\pi_1(W)$ on $V$ given by $v \cdot \gamma = \rho(\gamma)^{-1}(v)$, for $v \in V$ and $\gamma \in \pi_1(W)$. The complex of the universal cover with integer coefficients $C_n(\tilde{W}; \mathbb{Z})$ also inherits a left $\mathbb{Z}[\pi_1(W)]$-module via the action of $\pi_1(W)$ on $\tilde{W}$ as the covering group. We define the $V_\rho$-twisted chain complex of $W$ to be

$$C_*(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\tilde{W}; \mathbb{Z}).$$

The complex $C_*(W; V_\rho)$ computes the $V_\rho$-twisted homology of $W$ which we denote as $H_*(W; V_\rho)$.

Let $\{e_1, \ldots, e_{n_1}\}$ be the set of $i$-dimensional cells of $W$. We lift them to the universal cover and we choose an arbitrary order and an arbitrary orientation for the cells $\{\tilde{e}_1, \ldots, \tilde{e}_{n_1}\}$. If we let $\{v_1, \ldots, v_{n_1}\}$ be an orthonormal basis of $V$, then we consider the corresponding basis

$$c^i = \{v_1 \otimes \tilde{e}^i_1, \ldots, v_m \otimes \tilde{e}^i_1, \ldots, v_1 \otimes \tilde{e}^i_{n_1}, \ldots, v_m \otimes \tilde{e}^i_{n_1}\}$$

of $C_i(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_i(\tilde{W}; \mathbb{Z})$. We call the basis $c^i = \oplus c^i$ a geometric basis of $C_*(W; V_\rho)$. Now choosing for each $i$ a basis $h^i$ of the $V_\rho$-twisted homology $H_i(W; V_\rho)$, we can compute the torsion

$$\text{Tor}(C_*(W; V_\rho), c^*, h^*) \in \mathbb{F}^*.$$

The cells $\{\tilde{e}^i_j\mid 0 \leq i \leq \dim W, 1 \leq j \leq n_i\}$ are in one–to–one correspondence with the cells of $W$, their order and orientation induce an order and an orientation for the cells $\{e^i_j\mid 0 \leq i \leq \dim W, 1 \leq j \leq n_i\}$. Again, corresponding to these choices, we get a basis $c^i_j$ over $\mathbb{R}$ of $C_i(W; \mathbb{R})$.

Choose an homology orientation of $W$, which is an orientation of the real vector space $H_i(W; \mathbb{R}) \cong \bigoplus_{j=0}^{\dim W} H_i(W; \mathbb{R})$. Let $\sigma$ denote this chosen orientation. Provide each vector space $H_i(W; \mathbb{R})$ with a reference basis $h^i_j$ such that the basis $\{h^0_{\tilde{e}^i_j}, \ldots, h^\dim W_{\tilde{e}^i_j}\}$ of $H_i(W; \mathbb{R})$ is positively oriented with respect to $\sigma$. Compute the sign–determined Reidemeister torsion $\text{Tor}(C_*(W; \mathbb{R}), c^i_j, h^i_j) \in \mathbb{R}^*$ of the resulting based and homology based chain complex and consider its sign

$$\tau_0 = \text{sgn} \left(\text{Tor}(C_*(W; \mathbb{R}), c^i_j, h^i_j)\right) \in \{\pm 1\}.$$

We define the **sign–refined twisted Reidemeister torsion** of $W$ (with respect to $h^*$ and $\sigma$) to be

$$\tau_0 \cdot \text{Tor}(C_*(W; V_\rho), c^*, h^*) \in \mathbb{F}^*.$$  

This definition only depends on the combinatorial class of $W$, the conjugacy class of $\rho$, the choice of $h^*$ and the homology orientation $\sigma$. It is independent of the orthonormal basis of $V$, of the choice of the lifts $\tilde{e}^i_j$, and of the choice of the positively oriented basis of $H_i(W; \mathbb{R})$. Moreover, it is independent of the order and orientation of the cells (because they appear twice).

**Remark 1.** In particular, if the Euler characteristic $\chi(W)$ is zero, then we can use any basis of $V$. If we change the basis of $V$ by another one, then the torsion is multiplicatively by the determinant of the bases change matrix to the power $\chi(W)$.
One can prove that the sign–refined Reidemeister torsion is invariant under cellular subdivision, homeomorphisms and simple homotopy equivalences. In fact, it is precisely the sign \((-1)^{C_1}\) in Equation (1) which ensures all these important invariance properties to hold (see [Tur02]).

2.2. The Multiplicativity Lemma for torsions. In this section, we briefly review the Multiplicativity Lemma for Reidemeister torsions (with sign).

First, we review the notion of compatible bases. Let \(0 \rightarrow E' \rightarrow E \xrightarrow{\partial} E'' \rightarrow 0\) be a short exact sequence of finite dimensional vector spaces and let \(s\) denotes a section of \(\partial\). Thus, \(i \oplus s : E' \oplus E'' \rightarrow E\) is an isomorphism. We equip the three vector spaces \(E', E\) and \(E''\) respectively with the following three bases: \(b' = (b_1', \ldots, b_n')\), \(b = (b_1, \ldots, b_n)\), and \(b'' = (b_1'', \ldots, b_n'')\). With such notation, one has \(n = p + q\), and we say that the bases \(b', b\) and \(b''\) are compatible if the isomorphism \(i \oplus s : E' \oplus E'' \rightarrow E\) has determinant 1 in the bases \(b' \cup b'' = (b_1', \ldots, b_p', b_1'', \ldots, b_q'')\) of \(E' \oplus E''\) and \(b\) of \(E\). If it is the case, we write \(b \sim b' \cup b''\).

Let us now review the multiplicativity property of the Reidemeister torsion (with sign).

**Multiplicativity Lemma** (Lemma 3.4.2 in [Tur86]). Let

\[
0 \rightarrow C_+^i \rightarrow C_+ \rightarrow C_+^q \rightarrow 0
\]

be an exact sequence of chain complexes. Assume that \(C_+^i\), \(C_+\) and \(C_+^q\) are based and homology based. For all \(i\), let \(c^i\), \(c^i_+\) and \(c^q_+\) denote the reference bases of \(C_+^i\), \(C_+\) and \(C_+^q\) respectively. Associated to (3) is the long sequence in homology

\[
\cdots \rightarrow H_i(C_+^i) \rightarrow H_i(C_+) \rightarrow H_i(C_+^q) \rightarrow H_{i-1}(C_+^i) \rightarrow \cdots
\]

Let \(\mathcal{H}_i\) denote this acyclic chain complex and base \(\mathcal{H}_{3+2} = H_2(C_+^i), \mathcal{H}_{3+1} = H_1(C_+)\) and \(\mathcal{H}_{3} = H_0(C_+^q)\) with the reference bases of \(H_i(C_+^i), H_i(C_+)\) and \(H_i(C_+^q)\) respectively. If for all \(i\), the bases \(c^i\), \(c^i_+\) and \(c^q_+\) are compatible, i.e. \(c^i \sim c^i_+ \cup c^q_+\), then

\[
\text{Tor}(C_+, c^+, h^+) = (-1)^{\alpha(C_+^i, C_+^q) + \varepsilon(C_+, C_+^i, C_+^q)} \text{Tor}(C_+^i, c^i_+, h^+) \text{Tor}(C_+^q, c^q_+, h^q) \text{Tor}(\mathcal{H}_+, \{h^+, h^*, h^{q+}\}, \emptyset)
\]

where

\[
\alpha(C_+^i, C_+^q) = \sum_{i \geq 0} \alpha_i(C_+^i) \alpha_i(C_+^q) \in \mathbb{Z}/2\mathbb{Z}
\]

and

\[
\varepsilon(C_+, C_+^i, C_+^q) = \sum_{i \geq 0} \{\beta_i(C_+) + 1)(\beta_i(C_+^i) + \beta_i(C_+^q)) + \beta_{i-1}(C_+^i) \beta_i(C_+^q)\} \in \mathbb{Z}/2\mathbb{Z}
\]

The proof is a careful computation based on linear algebra, see [Tur86, Lemma 3.4.2] and [Mil66, Theorem 3.2]. This lemma appears to be a very powerful tool for computing Reidemeister torsions. It will be used all over this paper.

3. Definition of the polynomial torsion

In this section, we define the twisted Alexander invariant and called it for short the polynomial torsion. This invariant is the twisted Alexander invariant with coefficients in the adjoint representation associated to a character which lies in the geometric component of the character variety of the three–manifold. We define it following the presentation given by Friedl and Vidussi in their survey [FV09] using Reidemeister torsions theory.

Hereafter \(M\) denotes a compact and connected hyperbolic three–dimensional manifold such that its boundary \(\partial M\) consists in a disjoint union of \(b\) two–dimensional tori:

\[
\partial M = T^2_1 \cup \ldots \cup T^2_b.
\]
In the sequel, $\rho$ denotes a representation of $\pi_1(M)$ into $\text{SL}_2(\mathbb{C})$. The composition of $\rho$ with the adjoint action $Ad$ of $\text{SL}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})$ gives us the following representation:

$$Ad \circ \rho: \pi_1(M) \to \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$$

$$\gamma \mapsto (v \mapsto \rho(\gamma) v \rho(\gamma)^{-1})$$

We let $\mathfrak{sl}_2(\mathbb{C})_\rho$ denote the right $\mathbb{Z}[\pi_1(M)]$-module $\mathfrak{sl}_2(\mathbb{C})$ via the action $Ad \circ \rho^{-1}$.

Now we introduce the two different twisted bases which will be considered throughout this paper.

The first twisted complex under consideration is the complex $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ defined by:

$$C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho) = \mathfrak{sl}_2(\mathbb{C})_\rho \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\tilde{M}; \mathbb{Z}).$$

This chain complex is called the $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted chain complex of $M$. The twisted chain complex $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ computes the so-called $\mathfrak{sl}_2(\mathbb{C})_\rho$-twisted homology denoted by $H_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$. It is well-known that for a three-manifold $M$ with non-empty boundary one has $\dim H_2(M) \geq b$. Thus $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is never acyclic for three-manifolds with non-empty boundary. We will use the symbol $\mathbb{T}^M$ to denote the sign-refined Reidemeister torsion of $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$.

Next we introduce a twisted chain complex with some variables. It will be done by using a $\mathbb{Z}[\pi_1(M)]$-module with variables to define a new twisted chain complex. We regard $\mathbb{Z}^n$ as the multiplicative group generated by $n$ variables $t_1, \ldots, t_n$, i.e.,

$$\mathbb{Z}^n = \langle t_1, \ldots, t_n | t_it_j = t_jt_i [\forall i, j] \rangle$$

and consider a surjective homomorphism $\varphi: \pi_1(W) \to \mathbb{Z}^n$. We often abbreviate the $n$ variables $(t_1, \ldots, t_n)$ to $t$ and the rational functions $\mathbb{C}(t_1, \ldots, t_n)$ to $\mathbb{C}(t)$. Moreover, we write $\mathfrak{sl}_2(t_1, \ldots, t_n) = \mathfrak{sl}_2(t)$ for $\mathbb{C}(t) \otimes_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C})$ for brevity. Note that $\mathfrak{sl}_2(t)$ is naturally identified with $\mathfrak{sl}_2(\mathbb{C}(t))$ which is the vector space of trace free matrices whose components are rational functions in $\mathbb{C}(t) = \mathbb{C}(t_1, \ldots, t_n)$. The group $\pi_1(M)$ acts on $\mathfrak{sl}_2(t)$ via the following action:

$$\varphi \otimes Ad \circ \rho: \pi_1(M) \to \text{Aut}(\mathbb{C}(t) \otimes \mathfrak{sl}_2(\mathbb{C})) = \text{Aut}(\mathfrak{sl}_2(t)).$$

Thus, $\mathfrak{sl}_2(t)$ inherits the structure of a right $\mathbb{Z}[\pi_1(M)]$-module, $\mathfrak{sl}_2(t)_\rho$, and we consider the associated twisted chain complex $C_*(M; \mathfrak{sl}_2(t)_\rho)$ given by:

$$C_*(M; \mathfrak{sl}_2(t)_\rho) = \mathfrak{sl}_2(t) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\tilde{M}; \mathbb{Z}).$$

where $f \otimes v \otimes \gamma \cdot \sigma$ is identified with $f \varphi(\gamma) \otimes Ad_{\rho(\gamma)^{-1}}(v) \otimes \sigma$ for any $\gamma \in \pi_1(M)$, $\sigma \in C_*(\tilde{M}; \mathbb{Z})$, $v \in \mathfrak{sl}_2(\mathbb{C})$ and $f \in \mathbb{C}(t)$. We call this complex the $\mathfrak{sl}_2(t)_\rho$-twisted chain complex of $M$, and its homology is denoted $H_*(M; \mathfrak{sl}_2(t)_\rho)$.

Now we define geometric bases. We choose a basis of $\mathfrak{sl}_2(\mathbb{C})$, for example,

$$\{E, H, F\} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

a geometric basis $c^*$ of $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is obtained from the CW-structure of $M$. The geometric basis $c^*$ automatically gives us the geometric basis $1 \otimes c^*$ of $C_*(M; \mathfrak{sl}_2(t)_\rho)$.

In this paper these two bases will be abusively denoted with the same notation.

**Definition 1.** Fix a homology orientation on $M$. If $C_*(M; \mathfrak{sl}_2(t)_\rho)$ is acyclic, then the sign-refined Reidemeister torsion of $C_*(M; \mathfrak{sl}_2(t)_\rho)$:

$$\Delta_M^{\varphi \otimes Ad_{\rho}}(t_1, \ldots, t_n) = \tau_0 \cdot \text{Tor}(C_*(M; \mathfrak{sl}_2(t)_\rho), c^*, \emptyset) \in \mathbb{C}(t_1, \ldots, t_n) \setminus \{0\}$$

is called the twisted Alexander invariant (or the polynomial torsion for short) of $M$.

Note that the sign-refined Reidemeister torsion $\Delta_M^{\varphi \otimes Ad_{\rho}}$ is determined up to a factor $t_1^{n_1} \cdots t_n^{n_n}$ such as the classical Alexander polynomial.
Example 1. Suppose that $M$ is the knot exterior $E_K = S^3 \setminus N(K)$ of a knot $K$ in $S^3$ where $N(K)$ is an open tubular neighbourhood of $K$. If the representation $\rho \in \text{Hom}(\pi_1(E_K); \mathbb{Q})$ is the trivial homomorphism and $\varphi$ is the abelianization of $\pi_1(E_K)$, i.e., $\varphi : \pi_1(E_K) \to H_1(E_K; \mathbb{Z}) \cong \langle t \rangle$, then the twisted chain complex $C_\star(E_K; \mathbb{Q}(t) \rho)$ is acyclic and the torsion $\Delta_{E_K} \rho(t)$ is the Alexander polynomial divided by $(t - 1)$ (see [Mil62] and [Tur02]).

3.1. Technical assumptions. In this subsection, we give some sufficient conditions on the compact hyperbolic three–manifold $M$ whose boundary consists of a disjoint union of tori and on the representations $\varphi$ and $\rho$ which assure the acyclicity of the twisted chain complex $C_\star(M; \mathbb{sl}_2(t) \rho)$.

We require assumptions on the topology of $M$, on the surjective homomorphism $\varphi$ of $\pi_1(M)$ onto $\mathbb{Z}^n$ and on the $\text{SL}_2(\mathbb{C})$–representation $\rho$. The assumption on $\varphi$ is also related to the topology of $M$, we will show this in the second half of this subsection.

We also prove that if $M$ is a hyperbolic knot exterior, $\varphi : \pi_1(M) \to \mathbb{Z}$ the abelianization and $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ the holonomy (i.e. the discrete and faithful representation), then all our assumptions are satisfied.

3.1.1. Topological assumption for $M$. Let $b$ be the number of components of $\partial M$. We let $T^2_\ell$ denote the $\ell$–component of $\partial M$. The usual inclusion $i : \partial M \to M$ induces an homomorphism $i_* : H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$. First we assume the following condition on the homology group $H_1(M; \mathbb{Z})$:

$$(A_M) \text{ the homomorphism } i_* : H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \text{ is onto and its restriction } (i|_{T^2_\ell})_*$$

to the $\ell$–th component of $\partial M$ has rank one for all $\ell$.

Thus we can choose two closed loops $\mu_\ell$ and $\lambda_\ell$ on $T^2_\ell$ such that the homology classes $[\mu_\ell]$ and $[\lambda_\ell]$ form a basis of $H_1(T^2_\ell; \mathbb{Z})$ and the image $i_*([\mu_\ell])$ generates the subgroup $\text{im}(i|_{T^2_\ell})_*$ and $[\lambda_\ell]$ generates the kernel of $(i|_{T^2_\ell})_*$. We call $\mu_\ell$ a meridian and $\lambda_\ell$ a longitude.

We let $\lambda$ denote the set $(\lambda_1, \ldots, \lambda_b) \subset \partial M$ of such generators of $\ker \varphi|_{\pi_1(T^2_\ell)}$ and call $\lambda$ the multi–longitude curve.

Remark 2. From the homology long exact sequence of $(M, \partial M)$, it follows that $b_1(M) = b$ and $H_1(M; \mathbb{Z})$ has no–torsion.

Example 2 (Knot exteriors). Suppose that $M$ is the exterior $E_K$ of a hyperbolic knot $K$ in $S^3$. Here $E_K = S^3 \setminus N(K)$ where $N(K)$ is an open tubular neighbourhood of $K$. Then $M = E_K$ satisfies the condition $(A_M)$. This is due to the existence of a Seifert surface of the knot.

Remark 3 (Link exteriors). Let $L$ be a hyperbolic link such that each component of the link bounds a Seifert surface missing the other components. This condition for a link $L = K_1 \cup \ldots \cup K_b$ is equivalent to that the linking numbers $lk(K_i, K_j)$ are zero for all $i, j$. It is the reason that we can obtain the required Seifert surface by first choosing an arbitrary Seifert surface for $K_i$ and then getting rid of the intersections by adding tubes as in Fig 1. The intersections of such Seifert surfaces and the boundary $\partial E_L$ form a set of longitudes.

Note that it is not necessarily the case that Seifert surfaces are disjoint if each component of a link $L = K_1 \cup \ldots \cup K_b$ bounds a Seifert surface missing the other components. Links whose components do bound disjoint Seifert surfaces are called boundary links. For example, the Whitehead link $L = K_1 \cup K_2$ has the linking number $lk(K_1, K_2) = 0$. Hence there exist two Seifert surfaces $F_i$ ($i = 1, 2$) such that $\partial F_i = K_i$ and $F_i \cap K_j = \emptyset$ ($i \neq j$). But in fact the Whitehead link does not bound disjoint Seifert surfaces since the Whitehead link is not a boundary link, for more details see [Rol90, Chapter 5 E].

In this paper, three–manifold under considerations has the same properties about the homology group like as those of a boundary link exterior.
abelianization

by containing a lift of the holonomy representation

components related to the complete hyperbolic structure. These components are defined

Remark 4. Every homomorphism from \( \pi_1(M) \) to an abelian group factors through the
abelianization \( H_1(M; \mathbb{Z}) \) of \( \pi_1(M) \), i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
\pi_1(M) & \longrightarrow & H_1(M; \mathbb{Z}) \\
\varphi \downarrow & & \downarrow h_n \\
\mathbb{Z}^n & \longrightarrow & \\
\end{array}
\]

When we consider a surjective homomorphism \( \varphi \) onto \( \mathbb{Z}^n \), the induced homomorphism
\( \varphi_h \) is also surjective. Together with the condition \((A_M)\), this means that \( n \) must be not
greater than \( b_1(M) = b \). Moreover we have the null–homologous closed curve \( \lambda_\ell \) on each
\( T_\ell^2 \), every homomorphism \( \varphi \) has the multi–longitude consisting of these \( \lambda_\ell \).

Example 3 (Abelianization representation). Suppose that \( M \) is the exterior \( E_K \) of a
knot \( K \subset S^3 \) or a link exterior \( E_L \) of a link \( L \subset S^3 \) whose linking number between
arbitrary two components is zero. The abelianization \( \varphi: \pi_1(M) \rightarrow H_1(M; \mathbb{Z}) \cong \mathbb{Z}^b \) satisfies
Assumption \((A_\varphi)\). From Assumption \((A_M)\), it follows that \( H_1(M; \mathbb{Z}) = (\mu_1, \ldots, \mu_b) \).
Assumption \((A_\varphi)\) means that each entry of any representative matrix of \( \varphi_h \) is non-zero.

3.1.3. Assumption on \( \text{SL}_2(\mathbb{C}) \)-representations. It is required for an \( \text{SL}_2(\mathbb{C}) \)-representation
\( \rho \) of \( \pi_1(M) \) to be a “generic” representation, which essentially means that \( \rho \) lies in the
geometric components of the character variety.

The character variety of \( \pi_1(M) \) is the set of characters of \( \text{SL}_2(\mathbb{C}) \)-representations. Here
the character of an \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) is a map \( \pi_1(M) \rightarrow \mathbb{C} \) given by the assignment
\( \gamma \mapsto \text{tr}(\rho(\gamma)) \) for all \( \gamma \in \pi_1(M) \), where \( \text{tr} \) denotes the usual trace of square matrices.
This set has a structure of an affine algebraic variety (refer to [CS83]) denoted by \( X(M) \). For
a complete hyperbolic manifold, the character variety \( X(M) \) contains the distinguished
components related to the complete hyperbolic structure. These components are defined
by containing a lift of the holonomy representation \( \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{C}) \) (i.e., the discrete
and faithful representation) determined by the complete hyperbolic structure. We call
these components the geometric components.

Here \( \lambda \) is the multi–longitude corresponding to \( \varphi \). The precise definition is given as
follows.

Notation. For each boundary component \( T_\ell^2 \) of \( M \), we fix a generator \( P_\ell^g \) of the
homology group \( H_0(T_\ell^2; \mathfrak{sl}_2(\mathbb{C})_\rho) \), i.e. a non–trivial vector \( P_\ell^g \in \mathfrak{sl}_2(\mathbb{C}) \) which satisfies
\( \text{Ad}_{\rho(g)}(P_\ell^g) = P_\ell^g \) for all \( g \in \pi_1(T_\ell^2) \) (for more details, see [Por97, Section 3.3.1]).

Definition 2. For the multi–longitude \( \lambda = \{\lambda_1, \ldots, \lambda_b\} \), an irreducible representation
\( \rho \) is called \( \lambda \)-regular if \( \rho \) satisfies the following conditions:

1. for each boundary component \( T_\ell^2 \) of \( \partial M \), the restriction of \( \rho \) to \( \pi_1(T_\ell^2) \) is non–trivial;
(2) the following homomorphism, induced from all inclusions \( \lambda_\ell \hookrightarrow M \) (1 \( \leq \ell \leq b \)),

\[
\bigoplus_{\ell=1}^{b} H_1(\lambda_\ell; \mathfrak{sl}_2(\mathbb{C})) \to H_1(M; \mathfrak{sl}_2(\mathbb{C}))
\]

is surjective and;

(3) if \( \text{tr} \rho(\pi_1(T_2^\ell)) \subset \{\pm 2\} \), then \( \rho(\lambda_\ell) \neq \pm 1 \).

Remark 5. The chain \( P^\rho_\ell \otimes \lambda_\ell \) becomes a cycle i.e., it defines a homology class in \( H_1(\lambda; \mathfrak{sl}_2(\mathbb{C})) \). By [Por97, Proposition 3.22 and Corollaire 3.21], for a \( \lambda \)-regular representation \( \rho \), we have the following bases of the twisted homology groups:

- the homology group \( H_1(M; \mathfrak{sl}_2(\mathbb{C})) \) has a basis \( \{[P^\rho_1 \otimes \lambda_1], \ldots, [P^\rho_n \otimes \lambda_n]\} \),
- the homology group \( H_2(M; \mathfrak{sl}_2(\mathbb{C})) \) has a basis \( \{[P^\rho_1 \otimes T^\rho_1], \ldots, [P^\rho_2 \otimes T^\rho_n]\} \).

In [Por97, Definition 3.21], irreducibility is not required and the second condition is written by using twisted cohomology groups. Since we consider representations near the holonomy representation of \( \pi_1(M) \), we focus on \( \lambda \)-regularity of irreducible representations in the present article.

Here we use the same symbol \( \lambda_\ell \) and \( T_2^\ell \) for lifts of \( \lambda_\ell \) and \( T_2^\ell \) to the universal cover.

Remark 6. For generic points on the geometric component of \( X(M) \), the corresponding \( \text{SL}_2(\mathbb{C}) \)-representations satisfy \( \lambda \)-regularity.

We assume the following assumption for \( \text{SL}_2(\mathbb{C}) \)-representation of \( \pi_1(M) \)

(\( \rho \)) the representation \( \rho: \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) is \( \lambda \)-regular for the multi–longitude \( \lambda \) determined by \( \varphi: \pi_1(M) \to \mathbb{Z}^n \).

Example 4 (Holonomy representation). Suppose that \( M \) is a hyperbolic three–dimensional manifold. Let \( \rho_0: \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) be a lift of the discrete and faithful representation of \( \pi_1(M) \) to \( \text{PSL}_2(\mathbb{C}) \) given by the hyperbolic structure. Porti proves [Por97] that \( \rho_0 \) satisfies Assumption (\( \rho \)) for any system of homotopically non–trivial curves (\( \gamma_1, \ldots, \gamma_6 \)).

We mention a relation between the character variety \( X(M) \) and the twisted homology group \( H_1(M; \mathfrak{sl}_2(\mathbb{C})) \) for \( \lambda \)-regular representation.

Remark 7. The twisted cohomology group \( H^1(M; \mathfrak{sl}_2(\mathbb{C})) \) is the dual space of the twisted homology group \( H_1(M; \mathfrak{sl}_2(\mathbb{C})) \) by the Universal Coefficient Theorem. Following [Thu02] and [CS83, Proposition 3.2.1] together with \( \chi(M) = 0 \), it is known that, near the discrete and faithful representation, the character variety \( X(M) \) is a complex affine variety with dimension \( b \) where \( b \) is the number of torus boundary components. In affine varieties, the dimension of Zariski tangent space is not less that the one of the variety. The Zariski tangent space of the character variety can be injectively mapped into \( H^1(M; \mathfrak{sl}_2(\mathbb{C})) \). We also have dimc \( H^1(M; \mathfrak{sl}_2(\mathbb{C})) = b \) near the discrete and faithful representation, thus the spaces \( X(M), T^\rho_{\pi_1(M)}X(M) \) and \( H^1(M; \mathfrak{sl}_2(\mathbb{C})) \) have the same dimension \( b \). This means that characters near the discrete and faithful representation are smooth points and \( H^1(M; \mathfrak{sl}_2(\mathbb{C})) \) is identified with the tangent space \( T_{\rho_0}X(M) \) (see also [Por97, Chapter 3] for such identifications).

3.2. Acyclicity. This subsection is devoted to prove the acyclicity of \( C_*(M; \mathfrak{sl}_2(t)) \), i.e., \( H_*(M; \mathfrak{sl}_2(t)) = 0 \), from the assumptions referred to as (\( A_M \)), (\( A_c \)) and (\( A_\rho \)):

(\( A_M \)): the canonical inclusion \( i: \partial M \to M \) of \( \partial M \) into \( M \) induces an homomorphism \( i_*: H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \) which is onto and such that \( \text{rank}(i_*|_{T^\rho_1}) = 1 \) for all boundary component \( T^\rho_1 \) of \( \partial M = \bigcup_{i=1}^b T^\rho_i \);

(\( A_c \)): for all boundary component \( T^\rho_i \) of \( \partial M \), there exist non negative integers \( (a_1(t), \ldots, a_n(t)) \in \mathbb{N}^n \setminus \{0\} \) such that \( \varphi(\pi_1(T^\rho_i)) = (t_1^{a_1(t)} \cdots t_n^{a_n(t)}) \);
Proposition 1. \( \text{We have} \ H_\ast(M; \mathfrak{sl}_2(t)_\rho) = 0. \)

The proof of Proposition 1 is based on Milnor's construction \cite{Mil68}, but the techniques are rather different, actually we use the restriction map induces by the inclusion \( \partial M \hookrightarrow M. \)

Let \( \overline{M} \) denote the infinite cyclic covering of \( M. \) We have ker\( (pr_0 \circ \varphi) = \pi_1(\overline{M}) \)
where \( pr_0: \mathbb{Z}^n \to \mathbb{Z} \) denotes the projection by substituting \( t_1 = \cdots = t_{n-1} = 1. \) We use the symbol \( F \) for the fraction field \( \mathbb{C}(t_1, \ldots, t_{n-1}). \) The action of \( \varphi \otimes Ad \circ \rho^{-1} \) of \( \mathbb{C}[t_n, t_n^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(F)_\rho \) is given by the tensor product of \( pr_0 \circ \varphi \) and \( (pr_1, \ldots, n-1) \circ \varphi \otimes Ad \circ \rho^{-1} \)
where \( pr_1, \ldots, n-1 \) denotes the projection by substituting \( t_n = 1. \) Moreover, one can observe that under the inclusion \( \mathbb{Z}[t_n, t_n^{-1}] \to \mathbb{C}[t_n, t_n^{-1}], \)
\[
C_\ast(\overline{M}; \mathfrak{sl}_2(F)_\rho) \simeq C_\ast(M; \mathbb{C}[t_n, t_n^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(F)_\rho).
\]

The Milnor sequence:

\[
0 \to C_\ast(\overline{M}; \mathfrak{sl}_2(F)_\rho) \xrightarrow{t_n-1} C_\ast(\overline{M}; \mathfrak{sl}_2(F)_\rho) \xrightarrow{t_n=1} C_\ast(M; \mathfrak{sl}_2(F)_\rho) \to 0
\]
induces the long exact sequence in twisted homology:

\[
0 \to H_\ast(\overline{M}; \mathfrak{sl}_2(F)_\rho) \xrightarrow{t_n-1} H_\ast(\overline{M}; \mathfrak{sl}_2(F)_\rho) \xrightarrow{t_n=1} H_\ast(M; \mathfrak{sl}_2(F)_\rho) \\
\xrightarrow{\delta} H_1(\overline{M}; \mathfrak{sl}_2(F)_\rho) \xrightarrow{t_n-1} H_1(\overline{M}; \mathfrak{sl}_2(F)_\rho) \xrightarrow{t_n=1} H_1(M; \mathfrak{sl}_2(F)_\rho) \to 0.
\]

Proposition 1 is a consequence of the following lemma.

Lemma 2. Let \( F \) be the fraction field \( \mathbb{C}(t_1, \ldots, t_{n-1}). \) The homology group

\[
H_\ast(M; \mathbb{C}[t_n, t_n^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(F)_\rho)
\]
has no free part.

Proof. The proof is by induction on the number \( n \) of variables \( t_1, \ldots, t_n. \) The first step is to prove the lemma in the case of a single variable \( t. \)

(i). We prove that \( H_\ast(M; \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(C)_\rho) \) has no free part.

Let \( \Lambda_r \) be the annulus in \( \overline{M} \) over \( T^2 \) and \( b \) the number of the components of \( \partial M. \) The Milnor sequence for the boundary \( \partial M \) induces the first line of the following commutative diagram, the second one is the exact sequence of Equation (7), and the diagram is commutative because all the constructions are natural:

\[
0 \xrightarrow{b} \bigoplus_{\ell=1}^b H_2(T^2; \mathfrak{sl}_2(C)_\rho) \xrightarrow{\delta} \bigoplus_{\ell=1}^b H_1(\Lambda_r; \mathfrak{sl}_2(C)_\rho) \xrightarrow{b} \bigoplus_{\ell=1}^b H_0(\Lambda_r; \mathfrak{sl}_2(C)_\rho) \xrightarrow{t_n=1} \cdots \\
\cdots \xrightarrow{t_n=1} \bigoplus_{\ell=1}^b H_2(M; \mathfrak{sl}_2(C)_\rho) \xrightarrow{\delta} \bigoplus_{\ell=1}^b H_1(M; \mathfrak{sl}_2(C)_\rho) \xrightarrow{t_n=1} H_1(M; \mathfrak{sl}_2(C)_\rho) \xrightarrow{t_n=1} \cdots
\]

The proof is by contradiction, so we make the following hypothesis:

\[
(\mathcal{H}) \quad H_1(\overline{M}; \mathfrak{sl}_2(C)_\rho) \simeq H_1(M; \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(C)_\rho) \text{ has a free part of rank } r > 0.
\]

First observe that (because \( \chi(M) = 0 \) and \( H_0(\overline{M}; \mathfrak{sl}_2(C)_\rho) = 0\)):

\[
\text{rk} \ H_2(\overline{M}; \mathfrak{sl}_2(C)_\rho) = \text{rk} \ H_1(\overline{M}; \mathfrak{sl}_2(C)_\rho) = r.
\]

Our proof is as follows and based on the following technical claim.
Claim 3. The map

$$\delta_0 : \bigoplus_{\ell=1}^{b} H_2(T^2_\ell; sl_2(\mathbb{C})_\rho) \to \bigoplus_{\ell=1}^{b} H_1(A_\ell; sl_2(\mathbb{C})_\rho)$$

in the previous diagram is non-trivial, and moreover for all $\ell$, $\delta_0([P^\rho_\ell \otimes T^2_\ell])$ is non zero.

Proof of the Claim. Let $P^\rho_\ell$ ($1 \leq \ell \leq b$) be the chosen invariant vector in $sl_2(\mathbb{C})$. Since $\partial M = \bigcup_{\ell=1}^{b} T^2_\ell$, we have $H_0(\partial M; sl_2(\mathbb{C})_\rho) = \bigoplus_{\ell=1}^{b} H_0(T^2_\ell; sl_2(\mathbb{C})_\rho)$. It follows from our assumptions that $H_2(M; sl_2(\mathbb{C})_\rho)$ is generated by the vectors $[[P^\rho_\ell \otimes T^2_\ell]]$ ($1 \leq \ell \leq b$) and $H_1(M; sl_2(\mathbb{C})_\rho)$ is also generated by the vectors $[[P^\rho_\ell \otimes \lambda_\ell]]$ ($1 \leq \ell \leq b$). The space $H_2(T^2_\ell; sl_2(\mathbb{C})_\rho)$ is generated by $[P^\rho_\ell \otimes T^2_\ell]$ and we have:

$$\delta_0([P^\rho_\ell \otimes T^2_\ell]) = (1 + t + \cdots + t^{a_\ell - 1})[[P^\rho_\ell \otimes \lambda_\ell]],$$

where $\varphi(\mu_\ell) = t^{a_\ell}$, $a_\ell > 0$. Next it is easy to observe that each $(\sum_{k=1}^{a_\ell - 1} t^k)[P^\rho_\ell \otimes \lambda_\ell]$ is a non zero element in $H_1(M; sl_2(\mathbb{C})_\rho)$, because its image by the map $(t = 1)$ is $a_\ell[[P^\rho_\ell \otimes \lambda_\ell]]$, which is non zero in $H_1(M; sl_2(\mathbb{C})_\rho)$. This proves that $\delta_0 M \neq 0$.

Using the exactness of the Milnor sequence (7) and Equation (8), we know that

$$\text{im}(t = 1) \simeq H_2(M; sl_2(\mathbb{C})_\rho)/(t - 1)H_2(M; sl_2(\mathbb{C})_\rho) \neq 0.$$  

Since $\text{im}(t = 1) = \ker \delta$, we deduce that $\ker \delta \neq 0$. Let $\xi$ be a non zero element in $\ker \delta$ and write it in $H_2(M; sl_2(\mathbb{C})_\rho)$ as follows:

$$\xi = \sum_{1}^{b} b_\ell [P^\rho_\ell \otimes T^2_\ell].$$

One has (see Equation (9)):

$$\delta_0(\xi) = \sum_{1}^{b} b_\ell (1 + t + \cdots + t^{a_\ell - 1})[[P^\rho_\ell \otimes \lambda_\ell]]$$

and thus,

$$(t = 1) \circ \delta_0(\xi) = \sum_{1}^{b} a_\ell b_\ell[[P^\rho_\ell \otimes \lambda_\ell]].$$

Now we prove by contradiction that $\delta_0(\xi)$ is non zero. If $\delta_0(\xi) = 0$, then its image by $(t = 1)$ is also zero, so $a_\ell = 0$ or $b_\ell = 0$, for all $\ell$. Since $a_\ell > 0$, for all $\ell$, we deduce that $b_\ell = 0$, for all $\ell$. So that, $\xi = 0$, which is a contradiction and thus $\delta_0(\xi) \neq 0$.

With our assumption we have $(t = 1) \circ \delta_0(\xi) = \sum_{1}^{b} a_\ell b_\ell[[P^\rho_\ell \otimes \lambda_\ell]]$, and this element is non zero in $H_1(M; sl_2(\mathbb{C})_\rho)$ as we seen. But this is in contradiction with the fact that $\xi$ is chosen in $\ker \delta_0$, so that the hypothesis (H) on the free part of $H_2(M; sl_2(\mathbb{C})_\rho)$ is absurd and proves Lemma 2 in the case of a single variable.

(ii). Now we finish the proof by induction on the number of variables, and suppose that

$$H_*(M; sl_2(F)_{\rho})$$

has no free part. Thus, $H_*(M; sl_2(F)_{\rho})$ vanishes and the long exact sequence (7) induces the isomorphism:

$$0 \to H_*(\overline{M}; sl_2(F)_{\rho}) \xrightarrow{\overline{t}_{n-1}} H_*(\overline{M}; sl_2(F)_{\rho}) \to 0.$$ 

As a conclusion, $C_*(M; sl_2(F)_{\rho})$ is acyclic which proves Lemma 2. 

$\square$
Thus, the twisted complex $C_*(M; \mathfrak{sl}_2(t))$ is acyclic and the torsion is well-defined (even its sign if we provide $M$ with its natural homology orientation, see e.g. [Tur02]):

$$\Delta_{M}^{\varphi \otimes Ad \circ \rho}(t_1, \ldots, t_n) = \tau_0 \cdot \text{Tor}(C_*(M; \mathfrak{sl}_2(t)), \mathcal{C}_*^+, \emptyset) \in \mathbb{C}(t_1, \ldots, t_n).$$

Here $\tau_0 = \text{sgn}(\text{Tor}(C_*(M; \mathbb{R}), \mathcal{C}_R^+, h_R^{\infty})).$

4. Examples of computations

We compute the polynomial torsions for the figure knot exterior and the Whitehead link exterior by using Fox differential calculus as shown in [Mil68, Kit96, KL99].

First, we consider the Jacobian matrix using Fox free differential calculus associated to a Wirtinger presentations of a link group. To express the polynomial torsion, we need a square minor in the Jacobian matrix. Since the number of relations in a Wirtinger presentation is one less than that of generators, we have a square minor in the Jacobian matrix by dropping one column. For an SL$_2(\mathbb{C})$-representation $\rho$ of the link group, when we replace each element of the link group in the square minor of the Jacobian matrix by the $3 \times 3$ matrix derived from the action of $\varphi \otimes Ad \circ \rho$, we obtain a large matrix whose entries are Laurent polynomials with coefficients in $\mathbb{C}$. Then we can express the polynomial torsion for the link exterior as the rational function whose numerator is the determinant of the square minor replaced each component with $\varphi \otimes Ad \circ \rho$. The denominator of the polynomial torsion is the characteristic polynomial of the SL$_3(\mathbb{C})$-element given by the generator corresponding to the dropped column from the Jacobian matrix. It remains a problem to construct SL$_2(\mathbb{C})$-representations of link group. However we can find explicit constructions for the figure eight knot in [KK90] and for the Whitehead link in [HLMA92].

4.1. The figure eight knot exterior. We consider the figure eight knot $K$ as in Figure 2. The knot group $\pi_1(E_K)$ is expressed as

$$E_K = \langle x, y \mid [x^{-1}, y] x = y[x^{-1}, y] \rangle$$

where $E_K$ is the complement an open tubular neighbourhood $N(K)$ of $K$ in $S^3$. The

![Figure 2. The figure eight knot](image)

following correspondences give an SL$_2(\mathbb{C})$-representation $\rho_{\sqrt{s},u}$ of $\pi_1(E_K)$

$$x \mapsto \left( \frac{\sqrt{s}}{0}, \frac{1}{\sqrt{s}} \right) \quad y \mapsto \left( \frac{\sqrt{s}}{0}, \frac{-u\sqrt{s}}{1/\sqrt{s}} \right)$$

when the pair $(s, u)$ is a root of $\phi(s, u) = u^2 + (3 - (s + 1/s))(u + 1)$.

By using formula in [Mil68, Kit96, KL99], the polynomial torsion is expressed as

$$\Delta_{E_K}^{\varphi \otimes Ad \circ \rho_{\sqrt{s},u}}(t) = \tau_0 \cdot \frac{\text{det } \Phi(x^{-1}, y[x, x^{-1}]y^{-1})}{\text{det } \Phi(x - 1)}$$

where $\Phi$ is the linear extension of $\varphi \otimes Ad \circ \rho_{\sqrt{s},u}$ on $\mathbb{Z}[\pi_1(E_L)]$. The Fox differential turns into

$$\frac{\partial}{\partial b} \left( [x^{-1}, y][x[y, x^{-1}]y^{-1}] \right) = x^{-1} - x^{-1} yxy^{-1} + x^{-1} yxy^{-1} x - yx^{-1} - 1.$$
Therefore the numerator of $\Delta_{E K}^{x \otimes \text{Ad}_\rho \otimes \tau_\nu}(t)$ turns out
\[
\tau_0 \cdot \det (x^{-1} - x^{-1} yx^{-1} + x^{-1} yxy^{-1} x - yx^{-1} - 1).
\]

When we choose the basis $\{E, H, F\}$ in $\mathfrak{sl}_2(\mathbb{C})$, the adjoint actions $\text{Ad}_{\rho(x)^{-1}}$ and $\text{Ad}_{\rho(y)^{-1}}$ are represented by the following upper and lower triangular matrices:
\[
\text{Ad}_{\rho(x)^{-1}} = \begin{pmatrix} 1/s & 2/s & -1/s \\ 0 & 1 & -1 \\ 0 & 0 & s \end{pmatrix} \quad \text{Ad}_{\rho(y)^{-1}} = \begin{pmatrix} 1/s & 0 & 0 \\ -u & 1 & 0 \\ -su^2 & 2su & s \end{pmatrix}.
\]

Calculating the determinant and reducing with the equation $\phi(s, u) = 0$, we can obtain the polynomial as
\[
\Delta_{E K}^{x \otimes \text{Ad}_\rho \otimes \tau_\nu}(t) = \frac{\tau_0 \cdot \frac{1}{t^2} \cdot (t - 1)^2 (t - s)(t - 1/s)(t^2 - (2s + 2/s + 1)t + 1)}{(t - s)(t - 1)(t - 1/s)}
\]
(12)
\[
= \tau_0 \cdot \frac{1}{t^2} (t - 1) (t^2 - (2I_x^2 - 3)t + 1)
\]
where $I_x = \sqrt{s} + 1/\sqrt{s}$ is the trace function of the meridian $x$. Note that we can use $I_y$ instead of $I_x$ since all generators in a Wirtinger presentation are conjugate.

4.2. The Whitehead link exterior. Let $L$ be the Whitehead link and choose the following Wirtinger presentation of the Whitehead link group:
\[
\pi_1(E_L) = \langle a, b | awa^{-1}b^{-1}w^{-1} \rangle \text{ where } w = bab^{-1}a^{-1}b^{-1}ab.
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{whitehead_link}
\caption{The Whitehead link with meridians}
\end{figure}

Hilden, Lozano and Montesinos has shown an explicit description of the character variety of the Whitehead link group in [HLMA92]. Trace functions play the role of local coordinates in this description. We set $x, y$ and $v$ as $x = I_a$, $y = I_b$ and $v = I_{ab}$, where $I_{\gamma}: X(E_L) \to \mathbb{C}$ is again given by $I_{\gamma}(\chi) = \chi(\gamma)$. Then the character variety of $X(E_L)$ is expressed as
\[
X(E_L) = \{ (x, y, v) \in \mathbb{C}^3 | p(x, y, v)q(x, y, v) = 0 \},
\]
where
\[
\begin{align*}
p(x, y, v) &= xy - (x^2 + y^2 - 2)v + xyv^2 - v^3, \\
q(x, y, v) &= x^2 + y^2 + v^3 - xyv - 4.
\end{align*}
\]

The component of irreducible characters is given by
\[
X^{\text{irr}}(E_L) = \{ (x, y, v) \in \mathbb{C}^3 | p(x, y, v) = 0 \} \setminus R
\]
where $R = \{ x = \pm 2, v = \pm y \} \cup \{ y = \pm 2, v = \pm x \}$. We consider an $\mathfrak{sl}_2(\mathbb{C})$-representation $\rho$ of $\pi_1(E_L)$ whose character is contained in $X^{\text{irr}}(E_L)$.

Let $\rho: \pi_1(E_L) \to \text{SL}_2(\mathbb{C})$ be an irreducible representation and consider the abelianization $\varphi$ of $\pi_1(E_L)$: $\varphi: \pi_1(E_L) \to \mathbb{Z}^2 = \langle t_1, t_2 | t_1t_2 = t_2t_1 \rangle$ defined by $\varphi(a) = t_1$ and $\varphi(b) = t_2$.

Using irreducibility of $\rho$, we can suppose that the pair of $\rho(a)$ and $\rho(b)$ are expressed as (after taking conjugation with eigenvectors of $\rho(a)$ and $\rho(b)$ if necessary):
\[
\rho(a) = \begin{pmatrix} \alpha & 0 \\
0 & 1/\alpha \end{pmatrix} \quad \rho(b) = \begin{pmatrix} \beta & 0 \\
\gamma & 1/\beta \end{pmatrix},
\]

where $\alpha$, $\beta$, $\gamma$ and $\gamma$ are roots of
\[
\begin{align*}
\frac{1}{t} &= (2s + 2/s + 1) \\
&= (2I^2 - 3) t + 1
\end{align*}
\]
Note that we have set the local coordinates \((x, y, v)\) as \(x = \alpha + \alpha^{-1}, y = \beta + \beta^{-1}\) and \(v = \gamma + \alpha\beta + \alpha^{-1}\beta^{-1}\).

By using formula in [Mil68, Kit96, KL99], the polynomial torsion is expressed as

\[
\Delta_{E_L}^{\varphi \otimes \text{Ad}_p}(t_1, t_2) = \tau_0 \cdot \frac{\det \Phi \left( \frac{\partial}{\partial a} aw^{-1}w^{-1} \right)}{\det \Phi(a - 1)}
\]

where \(\Phi\) is the linear extension of \(\varphi \otimes \text{Ad} \circ \rho^{-1}\) on \(\mathbb{Z}[\pi_1(E_L)]\). The Fox differential turns into

\[
\frac{\partial}{\partial b} (aw^{-1}w^{-1}) = (a - 1)(1 - bab^{-1} - bab^{-1}a^{-1}b^{-1} + bab^{-1}a^{-1}b^{-1}a).
\]

Therefore the polynomial torsion \(\Delta_{E_L}^{\varphi \otimes \text{Ad}_p}(t_1, t_2)\) turns out

\[
\Delta_{E_L}^{\varphi \otimes \text{Ad}_p}(t_1, t_2) = \tau_0 \cdot \det \Phi(1 - bab^{-1} - bab^{-1}a^{-1}b^{-1} + bab^{-1}a^{-1}b^{-1}a).
\]

So that, with \(\det \text{Ad}_{p(a)^{-1}} = \det \text{Ad}_{p(b)^{-1}} = 1\) in mind,

\[
\Delta_{E_L}^{\varphi \otimes \text{Ad}_p}(t_1, t_2) = \tau_0 \cdot \det \Phi(1 - bab^{-1} - bab^{-1}a^{-1}b^{-1} + bab^{-1}a^{-1}b^{-1}a) = \tau_0 \cdot \det \Phi((bab^{-1}a^{-1})(bab^{-1}b^{-1}a - a^{-1}b^{-1} + b^{-1}a))
\]

When we choose the basis \(\{E, H, F\}\) in \(\mathfrak{s}\mathfrak{l}_2(\mathbb{C})\), the adjoint actions \(\text{Ad}_{p(a)^{-1}}\) and \(\text{Ad}_{p(b)^{-1}}\) are represented by the following upper and lower triangular matrices

\[
\text{Ad}_{p(a)^{-1}} = \begin{pmatrix}
1/\alpha^2 & 2/\alpha & -1 \\
0 & 1 & -\alpha \\
0 & 0 & \alpha^2
\end{pmatrix}
\quad\text{and}\quad
\text{Ad}_{p(b)^{-1}} = \begin{pmatrix}
1/\beta^2 & 0 & 0 \\
\gamma/\beta & 1 & 0 \\
-\gamma^2 & -2\beta\gamma & \beta^2
\end{pmatrix}.
\]

Calculating the determinant and reducing the degree of \(\gamma\) by using the following identity \(\alpha + \alpha^{-1}, \beta + \beta^{-1}, \gamma + \alpha\beta + \alpha^{-1}\beta^{-1} = 0\), we can see that the polynomial torsion is expressed as

\[
(13) \quad \Delta_{E_L}^{\varphi \otimes \text{Ad}_p}(t_1, t_2) = \tau_0 \cdot \frac{(t_1 - 1)(t_2 - 1)}{t_1^2} \left(-2xyvt_1t_2 + x^2t_1(t_2 + 1)^2 + y^2(t_1 + 1)^2t_2 - (t_1 + 1)^2(t_2 + 1)^2\right).
\]

5. Change of Coefficients

The aim of this section is to give some notation and explanation about the reduction of variables (for more details, see [Mil66]).

Let \(\varphi\) be a surjective homomorphism of \(\pi_1(M)\) onto \(\mathbb{Z}^n = \langle t_1, \ldots, t_n | t_it_j = t_jt_i \rangle\). We let \(h_{(a_1, \ldots, a_n)}\) be the homomorphism of \(\mathbb{Z}^n\) into \(\mathbb{Z} = \langle t \rangle\), given by

\[
h_{(a_1, \ldots, a_n)}(t_1, \ldots, t_n) = (t^{a_1}, \ldots, t^{a_n})
\]

where each \(a_\ell\) is a positive integer. We use the notation \(\varphi_{(a_1, \ldots, a_n)}\) for the composition of \(\varphi\) and \(h_{(a_1, \ldots, a_n)}\):

\[
\pi_1(M) \xrightarrow{\varphi} \mathbb{Z}^n \xrightarrow{h_{(a_1, \ldots, a_n)}} \mathbb{Z}
\]

Observe that \(\varphi_{(a_1, \ldots, a_n)}\) is onto if and only if \(G.C.D (a_1, \ldots, a_n) = 1\). Moreover \(\varphi_{(a_1, \ldots, a_n)}\) satisfies the condition \((A_\varphi)\) since each \(a_\ell\) is positive.

Later in this paper, we often make a reduction of several variables into one variable. We let \(\varphi\) be a surjective homomorphism of \(\pi_1(M)\) onto \(\mathbb{Z}^n\) satisfying \((A_\varphi)\) and \(\rho\) be an \(\text{SL}_2(\mathbb{C})\)-representation satisfying \((A_\rho)\). We choose relatively prime positive integers
(a_1, \ldots, a_n) and let \( \varphi(a_1, \ldots, a_n) \) be the composition of \( \varphi \) and \( h_{(a_1, \ldots, a_n)} \). From Section 3.2, the Reidemeister torsion \( \Delta_M^{\varphi \otimes \text{Ad} \rho}(t_1, \ldots, t_n) \) and \( \Delta_M^{\varphi_{(a_1, \ldots, a_n)} \otimes \text{Ad} \rho}(t) \) are defined for both the abelian homomorphisms \( \varphi \) and \( \varphi_{(a_1, \ldots, a_n)} \). The following result (with sign) is a consequence of the definitions.

**Proposition 4** ([Mil66]). One has the following formula:

\[
\Delta_M^{\varphi \otimes \text{Ad} \rho}(t_1, \ldots, t_n) = \Delta_M^{\varphi_{(a_1, \ldots, a_n)} \otimes \text{Ad} \rho}(t).
\]

6. A DERIVATIVE FORMULA

We prove a relation between the polynomial torsion and the non–abelian Reidemeister torsion. More precisely, we prove that the non–abelian Reidemeister is a sort of “differential coefficient” associated to the polynomial torsion.

We review the definition of the non–abelian Reidemeister torsion (for more details, we refer to [Por97, Chap. 3]).

**Definition 3.** Let \( M \) be a compact hyperbolic three–dimensional manifold whose boundary is the disjoint union of \( b \) tori \( \partial M = \bigcup_{\ell=1}^b T_\ell^2 \). Consider an \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho: \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) which is \( \lambda \)-regular for a set of closed loops \( \Lambda = \{ \lambda_\ell \subset T_\ell^2 \ | \ 1 \leq \ell \leq b \} \). The non–abelian Reidemeister torsion \( \mathbb{T}_M^\lambda(\rho) \) is defined to be the sign–refined Reidemeister torsion for \( C_*(M; \text{sl}_2(\mathbb{C})_\rho) \) and the basis \( h_\lambda^* = \{ [P_{\ell}^i \otimes T_{\ell}^2], \ldots, [P_{\ell}^n \otimes T_{\ell}^2], [P_{\ell}^i \otimes \lambda_1], \ldots, [P_{\ell}^n \otimes \lambda_0] \} \) as

\[
\mathbb{T}_M^\lambda(\rho) = \tau_0 \cdot \text{Tor}(C_*(M; \text{sl}_2(\mathbb{C})_\rho), e^*, h_\lambda^*).
\]

6.1. Bridge from the polynomial torsion to the non–abelian Reidemeister torsion. The following theorem proves that the non–abelian Reidemeister torsion can be deduced from the polynomial torsion \( \Delta_M^{\varphi \otimes \text{Ad} \rho}(t) \) (see [Yam08] for the case of knots).

In this section, we suppose that \( M \) is a compact hyperbolic three–dimensional manifold satisfying assumption (A_M), \( \varphi: \pi_1(M) \to \mathbb{Z} \langle \ell \rangle \) is a surjective homomorphism which satisfies assumption (A_2) and that \( \rho \) satisfies assumption (A_\rho). We equip the three–manifold \( M \) with a distinguished homology orientation.

**Theorem 5.** The following equality holds:

\[
\lim_{t \to 1} \frac{\Delta_M^{\varphi \otimes \text{Ad} \rho}(t)}{\prod_{\ell=1}^b (t^{2n_{\ell}} - 1)} = (-1)^b \cdot \mathbb{T}_M^\lambda(\rho) \quad (\lambda = (\lambda_1, \ldots, \lambda_0)),
\]

where \( \varphi(\pi_1(T_\ell^2)) = \ell n_\ell, n_\ell \in \mathbb{Z}_{>0} \), and \( b \) is the number of components of \( \partial M \).

Before proving this result, we give a couple of remarks.

**Remark 8.** Using Theorems 12 & 5 one can observe that if \( \varphi: \pi_1(M) \to \mathbb{Z} \langle \ell \rangle \) satisfies assumption (A_2), then \( (t-1)^b \) divides \( \Delta_M^{\varphi \otimes \text{Ad} \rho}(t) \).

**Remark 9** (The multivariable case). Here we suppose that \( \varphi: \pi_1(M) \to \mathbb{Z}^n \) where \( \mathbb{Z}^n = \langle t_1, \ldots, t_n \ | \ t_i t_j = t_j t_i, \forall i, j \rangle \).

**Corollary 6.** We have the following identity:

\[
\lim_{t_1, \ldots, t_n \to 1} \frac{\Delta_M^{\varphi \otimes \text{Ad} \rho}(t_1, \ldots, t_n)}{\prod_{\ell=1}^b (t_1^{a_{i,\ell}} \cdots t_n^{a_{i,\ell}} - 1)} = (-1)^b \cdot \mathbb{T}_M^\lambda(\rho),
\]

where \( \varphi(\pi_1(T_\ell^2)) = \langle t_1^{a_{1,\ell}}, \ldots, t_n^{a_{n,\ell}} \rangle, a_{1,\ell}, \ldots, a_{n,\ell} \in \mathbb{Z}_{>0} \).
6.2. Proof of Theorem 5. We begin by introducing the complexes needed in the proof and some notation.

Let \( C_s = C_s(M; \mathfrak{sl}_2(\mathbb{C})) \) and \( C_s(t) = C_s(M; \mathfrak{sl}_2(\mathbb{C}(t))) \). We define a pair of complexes \((C'_s, C''_s(t))\) as follows. The complex \( C'_s \) is defined as a subchain complex of \( C_s \) which is a lift of the homology group \( H^i_s(M) = H^i_s(M; \mathfrak{sl}_2(\mathbb{C})) \). This is a “degenerated complex” in the sense that the boundary operators are all zero. More precisely, \( C'_s = C''_s = 0 \) for conventions, \( C'_s \) is spanned (over \( \mathbb{C} \)) by \( \{P^t \otimes \lambda_t \mid 1 \leq t \leq b\} \), and \( C'_s \) is spanned (over \( \mathbb{C} \)) by \( \{P^t \otimes \lambda_t \mid 1 \leq t \leq b\} \). Similarly as \( C_s(t) \), we define \( C'_s(t) \) to be \( C(t) \otimes_{\mathbb{C}} C'_s \), in particular \( C'_s(t) \) is spanned by \( \{1 \otimes P^t \otimes \lambda_t \mid 1 \leq t \leq b\} \) and \( C'_s(t) \) is spanned by \( \{1 \otimes P^t \otimes T^t \mid 1 \leq t \leq b\} \). Observe that \( C'_s(t) \) is a subchain complex of \( C_s(t) \).

More precisely, one has:

\[
\begin{align*}
C'_s(t) &= 0 \rightarrow C'_s(t) \rightarrow C''_s(t) \xrightarrow{\partial'} C_s(t) \rightarrow C'_s(t) \rightarrow 0
\end{align*}
\]

where the boundary operator \( \partial' \) works as follows:

\[
\partial': (1 \otimes P^t) \otimes T^t \rightarrow (t^a - 1) \cdot (1 \otimes P^t) \otimes \lambda_t.
\]

Finally, we define \( C''_s \) as the quotient complex \( C_s/C'_s \) and \( C''_s(t) = C_s(t)/C'_s(t) \). Hence we have the two following exact sequence of complexes:

\[
\begin{align*}
0 &\rightarrow C'_s \rightarrow C_s \rightarrow C''_s \rightarrow 0. \\
0 &\rightarrow C'_s(t) \rightarrow C_s(t) \rightarrow C''_s(t) \rightarrow 0.
\end{align*}
\]

As we already observe, the complexes \( C'_s, C_s \) are not acyclic and we completely know their homology groups. The homology of \( C''_s \) is given in the following claim.

**Lemma 7.** The complex \( C''_s \) is acyclic.

**Proof of the claim.** Write down the long exact sequence in homology associated to the short exact sequence (18):

\[
\cdots \rightarrow H_i(C'_s) \rightarrow H_i(C_s) \rightarrow H_i(C''_s) \rightarrow H_{i-1}(C'_s) \rightarrow H_{i-1}(C_s) \rightarrow \cdots.
\]

Observe that \( H_i(C'_s) \simeq H_i(C_s) \) by definition of \( C'_s \). Thus \( H_i(C''_s) = 0 \).

**Remark 10.** We think of \( C''_s \) as the original complex \( C_s \) in which we have “killed” the homology.

The homology groups of the complexes \( C'_s(t), C_s(t) \) and \( C''_s(t) \) are given in the following claim.

**Lemma 8.** The complexes \( C'_s(t), C_s(t) \) and \( C''_s(t) \) are acyclic.

**Proof of the claim.** According to Proposition 1, \( C_s(t) = C_s(M; \mathfrak{sl}_2(\mathbb{C}(t))) \) is acyclic. One can observe that the map \( \partial' \) is invertible, thus \( C'_s(t) \) is acyclic. And finally \( C''_s(t) = C_s(t)/C'_s(t) \) is also acyclic (as a quotient of two acyclic complexes).

We endow the complexes in Sequence (19) with compatible bases in order to compute the torsions. From the definition, \( C'_s \) is endowed with a distinguished basis \( e^* \) given by

\[
\{P^t \otimes T^t, \ldots, P^t \otimes T^t, P^t \otimes \lambda_1, \ldots, P^t \otimes \lambda_b\}
\]

and we equip \( C''_s(t) \) with the corresponding distinguished basis \( 1 \otimes e^{**} \) improperly denoted again for simplicity \( e^{**} \). Similarly, we endowed the quotient \( C''_s = C_s/C'_s \) with a distinguished basis \( e^{***} \), and the same for \( C''_s(t) \). Using the exact sequence (19), we finally endowed \( C_s(t) \) with the compatible basis \( e^{**} \cup e^{***} \) obtained by lifting and concatenation (here again our notation is improper). Note that this last basis is different from the distinguished geometric basis \( e^* \) of \( C_s(t) = C_s(M; \mathfrak{sl}_2(\mathbb{C}(t))) \) described in Subsection 3.

From now on, we write \( \text{Tor}(C'_s(t), e^*, \emptyset) \) (resp. \( \text{Tor}(C''_s(t), e^{**}, \emptyset) \), \( \text{Tor}(C''_s(t), e^{***}, \emptyset) \)) for the Reidemeister torsion of \( C'_s(t) \) (resp. \( C''_s(t), C''_s(t) \)) computed in the basis \( e^* \cup e^{**} \cup e^{***} \).
\(c^{\prime\prime\prime}\) (resp. \(c^{\prime\prime}\), \(c^{\prime\prime\prime}\)); whereas we write \(\text{Tor}(C_*(M; \mathfrak{sl}_2(\mathbb{C}(t)))_\rho, c^*, \emptyset)\) for the torsion of \(C_*(t) = C_*(M; \mathfrak{sl}_2(\mathbb{C}(t)))_\rho\) but computed in the geometric basis \(c^*\). Using the basis change formula (see [Por97, Proposition 0.2]), we have:

\[
\text{Tor}(C_*(M; \mathfrak{sl}_2(\mathbb{C}(t)))_\rho, c^*, \emptyset) = \text{Tor}(C_*(t), c^{\prime\prime\prime} \cup c^{\prime\prime\prime}, \emptyset) \cdot \prod_i (c^i / c^{\prime\prime\prime}_i \cup c^{\prime\prime\prime}_i)^{(-1)^i}.
\]

Hence from the definition of the polynomial torsion, we have:

\[
\Delta_{c^{\prime\prime\prime}/c^{\prime\prime\prime}}(t) = \tau_0 \cdot \text{Tor}(C_*(M; \mathfrak{sl}_2(\mathbb{C}(t)))_\rho, c^*, \emptyset) = \prod_i (c^i / c^{\prime\prime\prime}_i \cup c^{\prime\prime\prime}_i)^{(-1)^i} \cdot \tau_0 \cdot \text{Tor}(C_*(t), c^{\prime\prime\prime} \cup c^{\prime\prime\prime}, \emptyset)
\]

where \(\tau_0 = \text{sgn}(\text{Tor}(C_*(M; \mathbb{R}), c^\prime, \mathfrak{h}_\mathbb{R}))\), see Section 3.

Applying the Multiplicativity Lemma to the exact sequence in Equation (19) we get:

\[
\text{Tor}(C_*(t), c^{\prime\prime\prime} \cup c^{\prime\prime\prime}, \emptyset) = (-1)^{\alpha} \cdot \text{Tor}(C_*(t), c^{\prime\prime\prime}, \emptyset) \cdot \text{Tor}(C_*(t), c^{\prime\prime\prime}, \emptyset)
\]

where \(\alpha \equiv \sum_j \alpha_{j-1}(C_*(t)) \alpha_j(C_*(t)) \mod 2\). We first compute the sign in Equation (21):

**Lemma 9.** The sign \((-1)^{\alpha}\) in Equation (21) is given by \(\alpha \equiv b \cdot \dim C_3 \mod 2\).

*Proof of Lemma 9.* It is easy to see from the definition that:

\[
\dim C_3(t) = \dim C_*(t) - \dim C_3(t).
\]

Moreover, \(\dim C_0(t) = 0 = \dim C_1(t)\) and \(\dim C_2(t) = 3b\), where \(b\) is the number of boundary components of \(M\). Thus, reduced modulo 2, \(\alpha_j(C_*(t))\) are all zero except \(\alpha_1(C_3(t)) \equiv b \mod 2\). As a consequence,

\[
\alpha \equiv \sum_j \alpha_{j-1}(C_*(t)) \alpha_j(C_3(t)) \\
\equiv \alpha_1(C_3(t)) \alpha_2(C_3(t)) \\
\equiv b \cdot (\dim C_0 + \dim C_1 + \dim C_2) \mod 2.
\]

Since the Euler characteristic of \(M\) is equal to zero, we have that \(\alpha \equiv b \cdot \dim C_3 \mod 2\).

\[\square\]

Next we compute the torsion of \(C_*(t)\) (with respect to the bases \(c_\rho\)).

**Lemma 10.** We have:

\[
\text{Tor}(C_*(t), c^{\prime\prime\prime}, \emptyset) = \prod_{\ell=1}^{b} (t^{a_\ell} - 1).
\]

*Proof of Lemma 10.* It is easy to observe that (see Complex (17)):

\[
\text{Tor}(C_*(t), c^{\prime\prime\prime}, \emptyset) = \det \partial' = \prod_{\ell=1}^{b} (t^{a_\ell} - 1).
\]

\[\square\]

If we substitute Equation (22) into Equation (21), we obtain:

\[
\text{Tor}(C_3(t), c^{\prime\prime\prime}, \emptyset) = (-1)^{\alpha} \frac{\text{Tor}(C_*(t), c^{\prime\prime\prime} \cup c^{\prime\prime\prime}, \emptyset)}{\prod_{\ell=1}^{b} (t^{a_\ell} - 1)}.
\]

Now we consider the limit of Equation (23) as \(t\) goes to 1 and prove the following lemma which gives a relation between the torsion of \(C_3(t)\) and the non-abelian torsion of \(M\) in the adjoint representation.
Lemma 11. Let $\delta_0 = \tau_0 \prod_{i \geq 0} [c^i / c^{i+1} \cup c^{i+1}]^{-1}$. We have the following identity

$$\lim_{t \to 1} \text{Tor}(C^*(t), c^{\omega t}, \emptyset) = (-1)^{b+\alpha} \cdot \delta_0 \cdot \mathcal{T}_\mathfrak{M}(\rho).$$

Proof of Lemma 11. We begin the proof by some considerations on the complexes $C^*(t)$ and $C_*(t)$ and their respective “limits” $C^\omega$ and $C_\omega$ when $t$ goes to 1.

Return to the definition of the complex $C^\omega_*(t) = C_*(t) / C'_*(t)$. If $t$ goes to 1, then the acyclic complex $C'_*(t)$ changes into the “degenerated” complex $C'_\omega$ in the sense that the map $\partial'$ becomes the zero map. Hence $C'_1 \simeq H_1(C_*)$ and $C'_2 \simeq H_2(C_*)$. Like $C^\omega_*(t)$, the complex $C'_*(t)$ is acyclic (see Lemma 8); more precisely if $t$ goes to 1, then we get in fact a complex related to the complex $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ but without homology, together with a distinguished basis different from the geometric one. Repeat again that the twisted homology groups $H^\omega_*(M) = H_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$ are endowed with the following distinguished bases (coming from the ones of $C'_\omega_*$, in fact it is not exactly a basis but a lift of the basis into $C_\omega$):

1. $H^1_\omega(M)$ is endowed with $b^1 = c^1 = \{[P_1^* \otimes \lambda_1], \ldots, [P_b^* \otimes \lambda_b]\}$,
2. $H^2_\omega(M)$ is endowed with $b^2 = c^2 = \{[P_1^* \otimes T^1], \ldots, [P_b^* \otimes T^b]\}$.

With obvious notation, choose a set of vectors $b^{\omega i+1}$ in $C^\omega_{i+1}(t)$ such that $\partial_{i+1}^{\omega}(b^{\omega i+1})$ is a basis of $B'_{i+1} = \text{im}(\partial_{i+1}'; C^\omega_{i+1} \to C^\omega_i)$.

Observe that the set of vectors $1 \otimes b^{\omega i+1}$ in $C^\omega_{i+1}(t)$ generates a subspace on which the boundary operator $\partial_{i+1} : C_{i+1}(t) \to C_i(t)$ is injective.

With $b^{\omega 3} = c^{\omega 3}$ in mind, the torsion of $C^\omega_*(t)$ (with respect to the basis $c^{\omega t}$) can be computed as follows:

$$\text{Tor}(C^\omega_*(t), c^{\omega t}, \emptyset) = \prod_{i=0}^2 \left[ \partial_{i+1}^{\omega}(1 \otimes b^{\omega i+1}) 1 \otimes b^{\omega t} / c^{\omega t} \right]^{(-1)^{i+1}}$$

$$= \prod_{i=0}^2 \left[ c^i \cup \partial_{i+1}(b^{i+1}) b^i / c^{\omega t} \right]^{(-1)^{i+1}}.$$

Here $b^i$ denotes a lift of $1 \otimes b^{\omega i}$ to $C_*(t)$. As a result, we can rewrite

$$\text{Tor}(C^\omega_*(t), c^{\omega t}, \emptyset) = \left[ c^2 \partial_3(b^3) b^2 / c^{\omega 2} \cup c^{\omega 2} \right]^{-1} \cdot \left[ c^1 \partial_2(b^2) b^1 / c^{\omega 1} \cup c^{\omega 1} \right] \cdot \left[ \partial_1(b^1) / c^{\omega 0} \right]^{-1}.$$

We want now to relate Equation (25) to an expression closer to the torsion of the twisted complex $C_*(M; \mathfrak{sl}_2(\mathbb{C})_\rho)$. For this we permute the vectors of $\partial_2(b^2)$ and $\partial_1(b^1)$ with the ones of $c^{\omega 2}$ and $c^{\omega 1}$ in one of the determinants in Equation (25). Each set of $b^i$ and $c^i$ consists of $b$ vectors and it is easy to observe that $\partial_3(b^3)$ consists of $\dim C_3(t) (= \dim C_3)$ vectors and $\partial_2(b^2)$ consists of $\text{rk} \partial_2 = \dim C_2(t) - b - \dim C_0(t) (= \dim C_0 + \dim C_1 + b \mod 2)$ vectors. Hence the sign arises from the permutation, whose exponent is given by $b(\dim C_0 + \dim C_1 + b) + b \dim C_3$. When we write $\varepsilon = (-1)^b(\dim C_0 + \dim C_1 + b)$ and $(-1)^\alpha$ for $(-1)^{b(\dim C_3)}$ as in Lemma 9, thus we have:

$$\text{Tor}(C^\omega_*(t), c^{\omega t}, \emptyset) = (-1)^{b+\alpha} \cdot \varepsilon \cdot \left[ \partial_3(b^3) c^2 b^2 / c^{\omega 2} \cup c^{\omega 2} \right]^{-1} \cdot \left[ \partial_2(b^2) c^1 b^1 / c^{\omega 1} \cup c^{\omega 1} \right] \cdot \left[ \partial_1(b^1) / c^{\omega 0} \right]^{-1}.$$
By making a change of basis in Equation (26), we obtain the following expression:

\[
\text{Tor}(\mathcal{C}_n''(t), \mathcal{C}''', \emptyset) = (-1)^{b+\alpha} \cdot \left( \prod_{i \geq 0} \left[ \mathcal{C}_i / \mathcal{C}_i^o \right]^{(-1)^i+1} \right)
\]

(27)

\[
\cdot \varepsilon \cdot \left[ \partial_1(b_3) \mathcal{C}_1^2 b_2 / \mathcal{C}_2 \right]^{-1} \cdot \left[ \partial_2(b_2) \mathcal{C}_1^1 b_1 / \mathcal{C}_1 \right] \cdot \left[ \partial_3(b_1) / \mathcal{C}_0 \right]^{-1}.
\]

Moreover, using the definition of the bases \( c^1 \) and \( c^2 \), it is easy to observe that

\[
\lim_{t \to 1} \varepsilon \cdot \left[ \partial_1(b_3) \mathcal{C}_1^2 b_2 / \mathcal{C}_2 \right]^{-1} \cdot \left[ \partial_2(b_2) \mathcal{C}_1^1 b_1 / \mathcal{C}_1 \right] \cdot \left[ \partial_3(b_1) / \mathcal{C}_0 \right]^{-1}
\]

(28)

\[
= \varepsilon \cdot \left[ \partial_1(b_3) \mathcal{C}_1^2 b_2 / \mathcal{C}_2 \right]^{-1} \cdot \left[ \partial_2(b_2) \mathcal{C}_1^1 b_1 / \mathcal{C}_1 \right] \cdot \left[ \partial_3(b_1) / \mathcal{C}_0 \right]^{-1}
\]

\[
= \text{Tor}(\mathcal{C}_*, \mathcal{C}^*, \mathcal{h}^*)
\]

The last step in Equations (28) is due to the fact that

\[
(-1)^{[C]} = (-1)^{[\alpha(C)]} = (-1)^{(\dim C_1 + \dim C_0)b} = \varepsilon.
\]

Hence, combining Equations (27) and (28), we obtain

\[
\lim_{t \to 1} \text{Tor}(\mathcal{C}_n''(t), \mathcal{C}''', \emptyset) = (-1)^{b+\alpha} \left( \prod_{i \geq 0} \left[ \mathcal{C}_i / \mathcal{C}_i^o \right]^{(-1)^i+1} \right) \cdot \text{Tor}(\mathcal{C}_*, \mathcal{C}^*, \mathcal{h}^*)
\]

which is exactly the desired equality because \( T^M_\lambda(\rho) = \tau_0 \cdot \text{Tor}(\mathcal{C}_*, \mathcal{C}^*, \mathcal{h}^*) ).\]

We finish the proof by combining Equation (23) and (24) and using the definition of the polynomial torsion \( \Delta_{M}^{\varphi \odot \text{Adop}}(t) \) given in Equation (20).

6.3. Example: the non-abelian Reidemeister torsion of the Whitehead link exterior. We apply Corollary 6 to the polynomial torsion \( \Delta_{E_n}^{\varphi \odot \text{Adop}}(t_1, t_2) \) of the Whitehead link exterior \( E_L \) (see Fig. 3). First we substitute \( t \) into both variables \( t_1 \) and \( t_2 \) in Equation (13). The resulting homomorphism \( \varphi_{(1,1)} \) corresponds to the induced homomorphism \( \pi_1(E_L) \to \pi_1(S^1) \) by the fibered structure of \( E_L \). The dual surface is the fiber and is also a Seifert surface for \( L \). Thus:

\[
\Delta_{E_L}^{\varphi \odot \text{Adop}}(t, t) = \tau_0 \cdot \left( \frac{t-1}{t^3} \right) (-2xvty^2 + x^2(t+1)^2 + y^2t(t+1)^2 - (t+1)^4).
\]

Multiplying \( (t-1)^{-2} \) and taking the limit for \( t \) goes to 1, we obtain the hyperbolic torsion (the non-abelian Reidemeister torsion):

\[
T^{E_L}_\lambda(\rho_{x,y,v}) = \tau_0 \cdot \left( 4(x^2 + y^2) - 16 - 2xyv \right).
\]

(29)

Since a lift of the holonomy representation is irreducible and the traces of meridians are \( \pm 2 \), Equation (29) is also valid at \( x = \pm 2 \). Moreover the points \( \pm 2, \pm 2 + 1 + \sqrt{-1} \) and \( \pm 2, \pm 2 + 1 - \sqrt{-1} \) correspond to lifts of the holonomy representation and its complex conjugate. When we substitute \( \pm 2, \pm 2 + 1 + \sqrt{-1} \) and \( \pm 2, \pm 2 + 1 - \sqrt{-1} \) into \( (x, y, v) \), we have the following values of the hyperbolic torsion (the non-abelian Reidemeister torsion) for the Whitehead link exterior and its holonomy representation \( \rho_0 \) and the complex conjugate \( \bar{\rho}_0 \)

\[
\left\{ T^{E_L}_\lambda(\rho_0), T^{E_L}_\lambda(\bar{\rho}_0) \right\} = \{ \tau_0 \cdot 8(1 + \sqrt{1}) \}.
\]
The torsion $\Delta_M^{\varphi \otimes \Ad\rho}$ is an element in the fraction field $\mathbb{C}(t_1, \ldots, t_n)$ by definition of the Reidemeister torsion. But actually, under a technical condition on the representations $\rho: \pi_1(M) \to \SL_2(\mathbb{C})$ and $\varphi: \pi_1(M) \to \mathbb{Z}^n$, this torsion is in fact contained in $\mathbb{C}[t_1, \ldots, t_n]$, up to a factor $t_1^{m_1} \cdots t_n^{m_n}$ for some integers $m_1, \ldots, m_n$. This result is obtained by a cut and paste argument by using the Multiplicativity Lemma for torsions. This divisibility problem for link exteriors in $S^3$ has been also investigated by Kitano and Morifuji [KM05] and Wada [Wad94].

7.1. Dual surfaces of $\varphi$ with rank one. Using the universal coefficient theorem, a homomorphism $\varphi: \pi_1(M) \to \mathbb{Z}$ can be regarded as a cohomology class in $H^1(M, \mathbb{Z})$, and its Poincaré dual $PD(\varphi)$ lies in $H_2(M, \mathbb{Z})$. Each representative of $PD(\varphi)$ consists of proper embedded surfaces, i.e., embedded surfaces whose boundary is contained in $\partial M$. By Turaev [Tur], we can choose proper embedded surfaces satisfying that the complement $M \setminus S$ of $S$ in $M$ is connected as a representative of $PD(\varphi)$. We let $S_\varphi$ denote such representative surfaces.

In the case of the Reidemeister torsion with multivariable $(t_1, \ldots, t_n)$, if we substitute $t_\ell = t_\ell^n$ for all $\ell$, then we have the homomorphism $\varphi(a_1, \ldots, a_n): \pi_1(M) \to \mathbb{Z}$. Moreover if $(a_1, \ldots, a_n)$ are relatively prime positive integers then the composition $\varphi(a_1, \ldots, a_n)$ satisfies assumption $(A_\varphi)$. When we regard $\varphi(a_1, \ldots, a_n)$ as an element in $H^1(M; \mathbb{Z})$, we let $S(a_1, \ldots, a_n)$ denote a representative of $PD(\varphi)$ such that $M \setminus S(a_1, \ldots, a_n)$ is connected.

7.2. A sufficient condition to be a polynomial. Under some additional assumptions for the $\SL_2(\mathbb{C})$-representation $\rho$ and dual surfaces determined by $\varphi$, we prove that the Reidemeister torsion $\Delta_M^{\varphi \otimes \Ad\rho}$ is a polynomial.

**Theorem 12.** Let $M$ be a hyperbolic three-manifold with tori boundary which satisfy $(A_M)$. Let $\varphi$ be a surjective homomorphism $\pi_1(M) \to \mathbb{Z}^n$ which satisfies $(A_\varphi)$ and $\rho$ be an $\SL_2(\mathbb{C})$-representation of $\pi_1(M)$ satisfying $(A_\rho)$. Let $S_\varphi = \cup S_\ell$ be the dual surfaces corresponding to $\varphi$.

1. Suppose that $n = 1$. If the restriction $\rho|_{\pi_1(S_\ell)}$ of $\rho$ is non-abelian for all $\ell$, then $\Delta_M^{\varphi \otimes \Ad\rho}$ is a polynomial in $t$, up to a factor $t^m$, $m \in \mathbb{Z}$.

2. Suppose that $n \geq 2$. If for any natural number $N$ there exists relatively prime integers $(a_1, \ldots, a_n)$ such that $|a_i - a_j| > N$ for all distinct $i, j$ and the restriction $\rho|_{\pi_1(S)}$ on every component $S$ of the dual surfaces $S(a_1, \ldots, a_n)$ corresponding to $\varphi(a_1, \ldots, a_n)$ is non-abelian, then $\Delta_M^{\varphi \otimes \Ad\rho}$ is a polynomial in $t_1, \ldots, t_n$, up to a factor $t_1^{m_1} \cdots t_n^{m_n}$ for $m_1, \ldots, m_n \in \mathbb{Z}$.

Before proving this theorem, we give some explanations, examples and counterexamples.

**Remark 11.** Let $K$ be a hyperbolic knot in $S^3$. Consider $\rho_0: \pi_1(E_K) \to \SL_2(\mathbb{C})$ (a lift of) the holonomy representation and $\varphi: \pi_1(E_K) \to \mathbb{Z}$ the abelianization. The dual surface $S_\varphi$ corresponding to $\varphi$ is a Seifert surface of $K$. One can observe that the longitude lies in the second commutator subgroup of $\pi_1(E_K)$. Thus, the restriction $\rho_0|_{\pi_1(S)}$ sends the longitude to a parabolic element which is not $\pm 1$. This means that the restriction of $\rho_0$ on $\pi_1(S_\varphi)$ is non-abelian.

**Remark 12.** Let $K$ be a fibered knot in $S^3$. Consider an irreducible, non-metabelian representation $\rho: \pi_1(E_K) \to \SL_2(\mathbb{C})$ and the homomorphism $\varphi: \pi_1(E_K) \to \pi_1(S) = \mathbb{Z}$ induced by the fibration $E_K \to S^1$. The dual surface $S_\varphi$ corresponding to $\varphi$ is the fiber of $K$. One can easily observe that $\rho|_{\pi_1(S)}$ is non-abelian, because the commutator subgroup $[\pi_1(E_K), \pi_1(E_K)]$ of $\pi_1(E_K)$ is $\pi_1(S_\varphi)$. 

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Thus, since the torsions of \( W \) to the manifold \( S \) up to sign we will work to the Mayer–Vietoris sequence associated to this splitting to compute the Reidemeister variable case.

Remark 14. In what follows the sign in the Reidemeister torsion will be not relevant; so, we will work up to sign. We let \( \text{Tor}(W, c^n_M, \emptyset) \) denote the (acyclic) Reidemeister torsion of the manifold \( W \) with coefficients in \( \mathfrak{s}_2(\mathbb{C}(t)) \) and computed in the appropriate geometric basis \( c^n_M \).

7.3. Proof of Theorem 12. The proof is divided into two main steps: we first consider the case of a single variable \( t \) and next use the naturality property of \( \Delta_M^{\phi_2} \) (see Section 5) to deduce by an algebraic argument the multivariable case from the one variable case.

Remark 13. For every \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) and \( \gamma \in \pi_1(M) \), the linear map \( \text{Ad} \circ \rho(\gamma) \) always has eigenvalue 1. In the one variable case with the \( \text{SL}_2(\mathbb{C}) \)-representation \( \text{Ad} \circ \rho \), we can not use Wada’s criterion [Wad94, Proposition 8] directly.

7.3.1. Proof for one variable. We cut the manifold \( M \) along \( S = S_\phi \) and obtain the following splitting: \( M = N \cup (S \times I) \), where \( I = [0,1] \) is the closed unit interval. The boundaries of \( N \) and \( S \times I \) are equal and consist in the disjoint union of two copies of \( S \), denoted \( S^- = S \times \{0\} \) and \( S^+ = S \times \{1\} \). We apply the Multiplicativity Lemma to the Mayer–Vietoris sequence associated to this splitting to compute the Reidemeister torsion of \( M \). Our assumptions on \( \rho: \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) and \( \phi: \pi_1(M) \to \mathbb{Z}^n \) say that every restrictions of \( \rho \) to \( \pi_1(S_t) \) are non–abelian, so we have:

**Lemma 13.** The twisted homology groups of \( N \) and \( S \times I \) are given by:

\[
H_*(S \times I; \mathfrak{s}_2(\mathbb{C}(t))) = \mathbb{C}(t) \otimes \mathbb{C} H_*(S \times I; \mathfrak{s}_2(\mathbb{C})) \simeq \mathbb{C}(t) \otimes H_*(S; \mathfrak{s}_2(\mathbb{C})),
\]

\[
H_*(N; \mathfrak{s}_2(\mathbb{C}(t))) = \mathbb{C}(t) \otimes H_*(N; \mathfrak{s}_2(\mathbb{C})).
\]

Moreover, \( H_0(S \times I; \mathfrak{s}_2(\mathbb{C}(t))) = H_2(S \times I; \mathfrak{s}_2(\mathbb{C}(t))) = 0 \).

**Proof of Lemma 13.** One has \( M = N \cup (S \times I) \), where \( N \) is a three–dimensional connected manifold whose boundary consists in \( S^- \cup S^+ \). One can observe that the actions of the restrictions \( \varphi_\pi(N) \) and \( \varphi_\pi(S_t) \) of the representation \( \varphi \) are trivial. So the first two equalities of the lemma hold.

As \( \rho|_{\pi_1(S_t)} \) is non–abelian, then \( H_0(S_t \times I; \mathfrak{s}_2(\mathbb{C}(t))) = 0 \). In the case of a closed surface, last equality \( H_2(S_t \times I; \mathfrak{s}_2(\mathbb{C}(t))) = 0 \) follows from Poincaré duality. For (compact) surface with boundary, last equality \( H_2(S_t \times I; \mathfrak{s}_2(\mathbb{C}(t))) = 0 \) follows from the fact that \( S \times I \) has the same homotopy type as a one–dimensional complex. So that \( H_0(S \times I; \mathfrak{s}_2(\mathbb{C}(t))) = H_2(S \times I; \mathfrak{s}_2(\mathbb{C}(t))) = 0 \).

As \( H_*(M; \mathfrak{s}_2(\mathbb{C}(t))) = 0 \), see Proposition 1, the Mayer–Vietoris sequence with coefficients in \( \mathfrak{s}_2(\mathbb{C}(t)) \), denoted \( V \), reduces to a single isomorphism:

\[
V: H_1(S^-; \mathfrak{s}_2(\mathbb{C}(t))) \oplus H_1(S^+; \mathfrak{s}_2(\mathbb{C}(t))) \simeq H_1(N; \mathfrak{s}_2(\mathbb{C}(t))) \oplus H_1(S \times I; \mathfrak{s}_2(\mathbb{C}(t)));
\]

where \( H_1(S^\pm; \mathfrak{s}_2(\mathbb{C}(t))) \simeq H_1(S; \mathfrak{s}_2(\mathbb{C}(t))) \simeq H_1(S \times I; \mathfrak{s}_2(\mathbb{C}(t))). \) The isomorphism in sequence (30) is represented by the following matrix:

\[
\begin{pmatrix}
 1 & \cdot \\
 -1 & 1
\end{pmatrix},
\]

here \( i^\pm: S_t \to N \) is the inclusion and \( 1 \) is the identity matrix. The Multiplicativity Lemma for Reidemeister torsion gives us the identity below, because the common boundary of \( N \) and \( S \times I \) is the disjoint union of two copies of \( S \):

\[
\pm \Delta_M^{\phi_2}(t) \cdot \text{Tor}(S, c^n_M, h^n_M)^{-1} \cdot \text{Tor}(V, \{ h^n_M, h^n_M \}, \emptyset) = \text{Tor}(N, c^n_M, h^n_M) \cdot \text{Tor}(S \times I, c^n_M, h^n_M).
\]

Thus, since the torsions of \( S \) and \( S \times I \) are the same,

\[
\pm \Delta_M^{\phi_2}(t) = \text{Tor}(V, \{ h^n_M, h^n_M \}, \emptyset)^{-1} \cdot \text{Tor}(N, c^n_M, h^n_M) = \det \begin{pmatrix}
 1 & -1 \\
 -1 & 1
\end{pmatrix} \frac{\text{Tor}(N, c^n_M, h^n_M)}{\text{Tor}(S, c^n_M, h^n_M)}.
\]
The fraction of torsions $\text{Tor}(N, c^*_N, b^*_N)/\text{Tor}(S, c^*_S, b^*_S)$ is independent of $t$ (because as we have already observed in the proof of Lemma 13, the actions of the restrictions $\varphi|_{\pi_1(N)}$ and $\varphi|_{\pi_1(S)}$ are trivial). This proves that, up to sign,

$$\pm \Delta^*_{\text{Adop}}(t) = \det(t_i^* - i^*_i) \frac{\text{Tor}(N, c^*_N, b^*_N)}{\text{Tor}(S, c^*_S, b^*_S)}$$

is a polynomial in $t$. The one variable case in Theorem 12 is proved.

**Remark 15.** A result similar to Lemma 13 can be found in [KL99, Proposition 3.6].

7.3.2. *Proof from one variable to two variables.* Suppose that $\varphi: \pi_1(M) \to \mathbb{Z} \oplus \mathbb{Z}$. A priori, up to multiplications by $t_1^* t_2^*$, the torsion $\Delta^*_{\text{Adop}}(t_1, t_2)$ is a rational function $P(t_1, t_2)/Q(t_1, t_2)$, where $P(t_1, t_2)$ and $Q(t_1, t_2) \neq 0$ are coprime in $\mathbb{C}[t_1, t_2]$. We will prove in fact, reducing the situation to one variable, that the polynomial $Q(t_1, t_2)$ is constant.

To this end, suppose that $Q(t_1, t_2)$ is a non–constant polynomial. Without loss of generality we assume that $Q(t_1, t_2)$ is a non–constant in $t_2$. Applying Euclidean algorithm to $P(t_1, t_2)$ and $Q(t_1, t_2)$ in $\mathbb{C}(t_1)[t_2]$, we obtain the following equality in the polynomial ring $\mathbb{C}(t_1)[t_2]$ over the rational function field $\mathbb{C}(t_1)$:

$$P(t_1, t_2)\tilde{u}(t_1, t_2) + Q(t_1, t_2)\tilde{v}(t_1, t_2) = 1.$$  

Here the coefficients of the polynomials in $t_2$ $\tilde{u}(t_1, t_2)$ and $\tilde{v}(t_1, t_2)$ are rational functions in $\mathbb{C}(t_1)$. By taking product with some polynomial $w(t_1) \in \mathbb{C}[t_1]$, the following equality holds in $\mathbb{C}[t_1, t_2]$:

$$P(t_1, t_2)w(t_1, t_2) + Q(t_1, t_2)v(t_1, t_2) = w(t_1).$$  

To each pair of coprime integers $(a_1, a_2)$, consider the homomorphism $h_{(a_1, a_2)}: \mathbb{Z}^2 \to \mathbb{Z}$ defined by $h_{(a_1, a_2)}(t_1) = t^{a_1}$ and $h_{(a_1, a_2)}(t_2) = t^{a_2}$ and let $\varphi_{(a_1, a_2)} = h_{(a_1, a_2)} \circ \varphi$. Using Proposition 4 and by the first part of the proof (for one variable), we know that:

$$\Delta^*_{\text{Adop}}(t) = \Delta^*_{\text{Adop}}(t^{a_1}, t^{a_2}) = \frac{P(t^{a_1}, t^{a_2})}{Q(t^{a_1}, t^{a_2})} \in \mathbb{C}[t].$$

Thus there exists a polynomial $R(t) \in \mathbb{C}[t]$ such that $P(t^{a_1}, t^{a_2}) = R(t)Q(t^{a_1}, t^{a_2})$. Hence by substituting $t^{a_1}$ and $t^{a_2}$ to $t_1$ and $t_2$ respectively, Equation (31) turns into

$$Q(t^{a_1}, t^{a_2})\{R(t)u(t^{a_1}, t^{a_2}) + v(t^{a_1}, t^{a_2})\} = w(t^{a_1}).$$

From our assumption $a_2$ can be chosen sufficiently large. Since we suppose that $Q(t_1, t_2)$ is not constant in $t_2$, by changing $a_2$ into a sufficient large integer, we obtain an arbitrary large degree polynomial $Q(t^{a_1}, t^{a_2})$. This contradicts the fact that the degree of $Q(t^{a_1}, t^{a_2})$ must be less or equal to that of $w(t^{a_1})$. Therefore $Q(t_1, t_2)$ is constant in $t_2$.

Similarly, we can also conclude that $Q(t_1, t_2)$ is constant in $t_1$. As a consequence, the polynomial $Q(t_1, t_2)$ is constant which proves that $\Delta^*_{\text{Adop}}(t_1, t_2)$ is a polynomial in two variables.

By an inductive argument (in descending order of $a_i$), the general case works in the same way: $\Delta^*_{\text{Adop}}(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$. The multivariable case in Theorem 12 is also proved.

**Remark 16.** Actually, in the multivariable case of Theorem 12, it is sufficient to assume that there exists some positive integer $N_0$ such that for any $N \geq N_0$ we have a dual surfaces $S_{(a_1, \ldots, a_n)}$ satisfying that $|a_i - a_j| > N$ for all distinct $i, j$. 

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### 8. Reciprocity of the polynomial torsion (with sign)

#### 8.1. Reciprocity properties

The Alexander polynomial $\Delta_K(t)$ of a knot $K \subset S^3$ is known to be reciprocal in the sense that $\Delta_K(t^{-1}) = \pm t^\ell \Delta_K(t)$ from a long time. This property was first observed by Seifert [Sei34] and is a consequence of Poincaré duality. Milnor [Mil62] proved it again using the interpretation of the Alexander polynomial as an abelian Reidemeister torsion. Kitano [Kit96], Kirk and Livingston [KL99] observe that Milnor’s argument work fine in the context of twisted Alexander invariants for representations into $\text{SL}_n(C)$. All the known reciprocity formulas are sign-less. In our situation Milnor’s argument also work, and using the fact that the torsion of $\partial M = \bigcup_{f=1}^b T_f^2$ is trivial, we have the following result, in which the sign is analyzed in details (cf. [Mil62, Theorem 2] and [KL99, Theorem 5.1]). We also refer to [FKK] for duality formulas for the polynomial torsion in more general situations.

**Theorem 14.** Let $\tau: \mathbb{Z} \to \mathbb{Z}$ be the involution defined by $\bar{t} = t^{-1}$. If we consider a homomorphism $\varphi: \pi_1(M) \to \mathbb{Z}$ and its composition $\bar{\varphi}$ with the involution $\tau$, then the polynomial torsion satisfies the following identity up to a factor $t^k (k \in \mathbb{Z})$:

$$
\Delta^{\varphi \circ \text{Ad}_{E_L}}_M(t^{-1}) = \Delta^{\varphi \circ \text{Ad}_{E_L}}_M(t) = \epsilon(-1)^{kb(1)/2} \Delta^{\varphi \circ \text{Ad}_{E_L}}_M(t).
$$

Here $\epsilon$ is the sign of the Reidemeister torsion of the long exact sequence in homology associated to the pair $(M, \partial M)$ with the basis given by the bases of $H_*(\partial M; \mathbb{R})$, $H_*(M; \mathbb{R})$ and the Poincaré dual bases of $H_*(M; \mathbb{R})$ in $H_*(M, \partial M; \mathbb{R})$ as in Equation (39).

**Remark 17** (A duality property for link exteriors). As a special case, if $M$ is a link exterior $E_L = S^3 \setminus N(L)$, then we can explicitly compute the sign–term in Theorem 14. Proposition 16 in Appendix gives us the sign $\epsilon = (-1)^{b(b-1)/2}$, where $b$ is the number of boundary components of $M$. Using naturality of the polynomial torsion (see Section 5) then the following duality for the polynomial torsion $\Delta^{\varphi \circ \text{Ad}_{E_L}}_M$ holds:

$$
\Delta^{\varphi \circ \text{Ad}_{E_L}}_M(t^{-1}) = (-1)^{kb(1)/2} \cdot (-1)^{kb(1)/2} \Delta^{\varphi \circ \text{Ad}_{E_L}}_M(t)
$$

(32)

In particular, if $L$ is a knot $K$ in $S^3$, then the following duality holds:

$$
\Delta^{\varphi \circ \text{Ad}_{E_K}}_K(t^{-1}) = -\Delta^{\varphi \circ \text{Ad}_{E_K}}_K(t).
$$

Equation (33), and more generally Equation (32), can be considered as a sign–refined version of Milnor’s duality Theorem [Mil62] for Reidemeister torsion.

Our attention is restricted to the composition of an $\text{SL}_2(C)$-representation and the adjoint action. We refer to [FK06, FV09, HSW] for reciprocity formulas for other types of representations.

#### 8.2. Proof of Theorem 14

The proof is essentially based on the Multiplicativity Lemma for torsions (with sign). We apply the Multiplicativity Lemma in Section 2.2 for short exact sequences with the coefficient $\mathbb{R}$ and $\text{sl}_2(C(t))$ to observe the relation between the signed torsion of the original chain complex and that of the dual chain complex. With the fact that $\alpha(C', C'')$ for the coefficients in $\mathbb{R}$ are same as $\alpha(C', C'')$ for the coefficients in $\text{sl}_2(C(t))$ in mind, the Multiplicativity Lemma for the pair $(M, \partial M)$ yields the following equation for the sign–refined torsions:

$$
\Delta^{\varphi \circ \text{Ad}_{E_L}}_M(t) = \epsilon(-1)^{\nu} \cdot \Delta^{\varphi \circ \text{Ad}_{E_L}}_{\partial M}(t) \cdot \Delta^{\varphi \circ \text{Ad}_{E_L}}_{(M, \partial M)}(t).
$$

(34)
Here $\nu \in \mathbb{Z}/2\mathbb{Z}$ is the sign given by the sum $\nu = \sum_{i=0}^{3} (\beta_i + 1)(\beta'_i + \beta''_i) + \beta'_1 \beta''_1 \in \mathbb{Z}/2\mathbb{Z}$ where

$$\beta_i = \sum_{r=0}^{i} \dim_{\mathbb{R}} H_r(M; \mathbb{R}), \quad \beta'_i = \sum_{r=0}^{i} \dim_{\mathbb{R}} H_r(\partial M; \mathbb{R}) \text{ and } \beta''_i = \sum_{r=0}^{i} \dim_{\mathbb{R}} H_r(M, \partial M; \mathbb{R}).$$

We compute each terms appearing in Equation (34).

8.2.1. Computation of the sign $\nu$. By our assumption on $M$ and $\partial M$, we observe that $\nu = 1$ in $\mathbb{Z}/2\mathbb{Z}$.

8.2.2. Computation of $\Delta_{\partial M}^{\varphi \otimes Ad \rho}(t)$. By using the Multiplicativity Lemma, we have

$$\Delta_{\partial M}^{\varphi \otimes Ad \rho}(t) = (-1)^{b-1} \Delta_{T_E}^{\varphi \otimes Ad \rho}(t) \cdot \Delta_{\partial M}^{\varphi \otimes Ad \rho}(t).$$

Hence we have

$$\Delta_{\partial M}^{\varphi \otimes Ad \rho}(t) = (-1)^{(b-1)/2} \prod_{\ell=1}^{b} \Delta_{T_E}^{\varphi \otimes Ad \rho}(t).$$

Furthermore a direct computation provides the following lemma:

**Lemma 15.** The torsion $\Delta_{T_E}^{\varphi \otimes Ad \rho}(t)$ of the $\ell$-th component of $\partial M$ with the homology orientation given by the ordered basis $\{[T_E^2], [\lambda_1], [\mu_1], [p_2]\}$ is equal to $+1$.

As a consequence, the twisted Alexander invariant of $\partial M$ is given by:

$$\Delta_{\partial M}^{\varphi \otimes Ad \rho}(t) = (-1)^{(b-1)/2}. \tag{35}$$

8.2.3. Computation of $\Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t)$. Here we use the duality properties of torsion proved by M. Farber and V. Turaev in [FT00]. We write the torsion $\Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t)$ in the right hand side of Equation (34) as the torsion in the left hand side with a sign term. Let $(M', \partial M')$ denotes the dual cell decomposition of $(M, \partial M)$. Using the invariance of the Reidemeister torsion under subdivisions of CW–pairs we obtain:

$$\Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t) = \Delta_{(M',\partial M')}^{\varphi \otimes Ad \rho}(t).$$

Let $\rho^* : \pi_1(M) \to SL_2(\mathbb{C})$ be the following representation

$$\rho^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}.$$

Using the invariance of the Reidemeister torsion under conjugation of representations we thus have

$$\Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t) = \Delta_{(M',\partial M')}^{\varphi \otimes Ad \rho}(t) = \Delta_{(M',\partial M')}^{\varphi \otimes Ad \rho^*}(t).$$

Observe that the pair $(M', \partial M')$ and the representation $\varphi \otimes Ad \circ \rho^*$ give the dual chain complex of $M$ twisted by the representation $\varphi \otimes Ad \rho$, i.e., the $\mathfrak{sl}_2(\mathbb{C}(t))_{\rho^*}$-twisted chain complex $C_*(M', \partial M'; \mathfrak{sl}_2(\mathbb{C}(t))_{\rho^*})$ can be identified with the dual chain complex of $C_*(M; \mathfrak{sl}_2(\mathbb{C}(t))_{\rho})$. By using this identification and the duality of torsion in [FT00], the torsion $\Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t)$ is expressed as

$$\Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t) = \Delta_{(M',\partial M')}^{\varphi \otimes Ad \rho^*}(t) = (-1)^{(b-1)/2} \cdot \Delta_{(M,\partial M)}^{\varphi \otimes Ad \rho}(t). \tag{36}$$

Here $s(C_*) = \sum_{q=0}^{m} \alpha_{q-1}(C_*) \alpha_q(C_*)$ for a chain complex $C_* = C_m \oplus \cdots \oplus C_0$, with $\alpha_q(C_*) = \sum_{j=0}^{q} \dim C_j$. Further observe the sign term difference between Farber–Turaev’s formula and Equation (36): here the sign term $\sum_{q=0}^{m-1/2} \alpha_{2q}(C_*)$ used in [FT00] is omitted from $s(C_*)$ because we use the complex $C_*(M', \partial M'; \mathfrak{sl}_2(\mathbb{C}(t))_{\rho})$ instead of the dual complex of $C_*(M; \mathfrak{sl}_2(\mathbb{C}(t))_{\rho})$. 

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The first sign \((-1)^{(H_*(M;\mathbb{R})+s(C, M; \mathbb{R}))}\) in Equation (36) comes from the duality of the sign terms \(\tau_0\) in \(\Delta_{M,\partial M}^{s, Ad\rho}(t)\) and the second one \((-1)^{(s(C, M; \mathbb{R}))}\) in Equation (36) comes from the duality of torsions for \(C_*(M; \mathbb{Z}_2(C(t)))\). Since \(s(C_*(M; \mathbb{R}))\) and \(s(C_*(M; \mathbb{Z}_2(C(t))))\) are equal in \(\mathbb{Z}/2\mathbb{Z}\), it turns out that

\[
\Delta_{M,\partial M}^{s, Ad\rho}(t) = (-1)^{(H_*(M;\mathbb{R}))} \Delta_{M}^{-s, Ad\rho}(t).
\]

**8.2.4. Conclusion.** Substituting \(\nu = 1\), \(s(H_*(M;\mathbb{R})) = b + 1\) and Equation (37) into Equation (34) we obtain that

\[
\Delta_{M}^{s, Ad\rho}(t) = \epsilon(-1)^b \Delta_{M}^{s, Ad\rho}(t) \Delta_{M}^{-s, Ad\rho}(t).
\]

Finally, using Equation (35), we can see that Equation (38) turns into

\[
\Delta_{M}^{s, Ad\rho}(t) = (-1)^{b+b(b-1)/2} \Delta_{M}^{-s, Ad\rho}(t) = (-1)^{b(b+1)/2} \Delta_{M}^{-s, Ad\rho}(t)
\]

which achieves the proof of Theorem 14.

**Appendix A. A natural homology orientation for link exterior**

In this section, we exhibit a natural, and in a sense compatible, homology orientation in the case where \(M\) is the exterior \(E_L = S^3 \setminus N(L)\) of a link \(L\) in \(S^3\). Here \(N(L)\) denotes a tubular neighborhood of \(L\). Observe that the boundary \(\partial M\) of \(M\) consists in the disjoint union of \(b\) tori \(T^2_1, \ldots, T^2_b\). For more details on homology orientations the reader is invited to refer to Turaev’s monograph [Tur02].

The definition of the homology orientation needs some orientation conventions. We suppose that \(S^3\) and \(M\) are oriented. Thus \(E_L = S^3 \setminus N(L)\) and each torus \(T^2_\ell\) inherits the orientation induced by the one of \(S^3\). Moreover each torus \(T^2_\ell\) is given together with its peripheral-system \((\lambda_\ell, \mu_\ell)\), where \(\lambda_\ell\) the longitude, and \(\mu_\ell\) the meridian, generates \(H_1(T^2_\ell; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). These two curves are oriented using the following rules: \(\mu_\ell\) is oriented using the convention \(\ell k(\mu_\ell, L) = +1\), and \(\lambda_\ell\) is oriented using the convention \(\text{int}(\mu_\ell, \lambda_\ell) = +1\).

In what follows, we construct natural homology orientations for the link exterior \(E_L\), for its boundary \(\partial E_L = \biguplus_{\ell=1}^b T^2_\ell\) and for the pair \((E_L, \partial E_L)\).

**A.1. Homology groups and homology orientations.** We begin our investigations by describing in details the homology groups of the link exterior \(E_L\), of its boundary \(\partial E_L = \biguplus_{\ell=1}^b T^2_\ell\) and of the pair \((E_L, \partial E_L)\).

The homology groups of the torus \(T^2_\ell\) are given as follows:

\[
H_2(T^2_\ell; \mathbb{R}) = \mathbb{R}[T^2_\ell], \quad H_1(T^2_\ell; \mathbb{R}) = \mathbb{R}[\lambda_\ell] \oplus \mathbb{R}[\mu_\ell], \quad H_0(T^2_\ell; \mathbb{R}) = \mathbb{R}[p_{\ell}].
\]

Here \([T^2_\ell]\) is the fundamental class induced by the orientation of \(T^2_\ell\), \([\mu_\ell]\) and \([\lambda_\ell]\) are given by the oriented meridian and the oriented longitude (see above) and \([p_{\ell}]\) is the homology class of the base point.

An application of Mayer-Vietoris sequence associated to the following decomposition of the three–sphere: \(S^3 = E_L \cup (\biguplus_{\ell=1}^b S^1 \times D^2_\ell)\) gives us the following generators for the homology groups of \(E_L\):

\[
H_2(E_L; \mathbb{R}) = \bigoplus_{\ell=1}^b \mathbb{R}[T^2_\ell], \quad H_1(E_L; \mathbb{R}) = \bigoplus_{\ell=1}^b \mathbb{R}[\mu_\ell], \quad H_0(E_L; \mathbb{R}) = \mathbb{R}[p_{b}].
\]

To describe the homology groups of the pair \((E_L, \partial E_L)\) we use Poincaré duality and get the following generators:

\[
H_3(E_L, \partial E_L; \mathbb{R}) = \mathbb{R}[E_L, \partial E_L],
\]

\[
H_2(E_L, \partial E_L; \mathbb{R}) = \mathbb{R}[S_1] \oplus \cdots \oplus \mathbb{R}[S_b],
\]

\[
H_1(E_L, \partial E_L; \mathbb{R}) = \mathbb{R}[\gamma_1] \oplus \cdots \oplus \mathbb{R}[\gamma_{b-1}],
\]

\[
H_0(E_L, \partial E_L; \mathbb{R}) = \mathbb{R}[\mu].
\]
Here \([E_L, \partial E_L] = [p_0]^*\) is the fundamental class induced by orientations. If \(S_t\) denotes the restriction of a Seifert surface of the component \(K_t\) in \(L \rightarrow E_L\), then its class \([S_t]\) = \([\mu_1]^*\). And finally, if \(\gamma_t\) denotes a path connecting the point \(p_t\) to \(p_0\), then its class \([\gamma_t] = [T^2_T]\).

The homology orientations we fix on \(\partial E_L\), \(E_L\) and \((E_L, \partial E_L)\) are respectively induced by the following ordered bases:

\[
\begin{align*}
H_*(\partial E_L; \mathbb{R}) &= \langle [T^2_1], \ldots, [T^2_b], [\lambda_1], [\mu_1], \ldots, [\lambda_b], [\mu_b], [p_1], \ldots, [p_b]\rangle, \\
H_*(E_L; \mathbb{R}) &= \langle [T^2_1], \ldots, [T^2_b], [\lambda_1], [\mu_1], \ldots, [\lambda_b], [\mu_b], [p_1], \ldots, [p_b]\rangle, \\
H_*(E_L, \partial E_L; \mathbb{R}) &= \langle [E_L, \partial E_L], [S_1], \ldots, [S_b], [\gamma_1], \ldots, [\gamma_{b-1}]\rangle.
\end{align*}
\]

### A.2. A sign term.

This combination of bases for the pair \((E_L, \partial E_L)\) are natural in the sense that they are given by Poincaré duality. In the case of link exteriors, using such homology orientations we will fix numbers of sign indeterminacy in our formulas. However, to be exhaustive it remains to us to explicitly compute the Reidemeister torsion of the long exact sequence in homology associated to the pair \((E_L, \partial E_L)\). This torsion is a new sign term, given in the following proposition, which will give us the sign–term in the symmetry formula for the polynomial torsion (see Section 8, in particular Theorem 14 and Equation (32)).

**Proposition 16.** The torsion of the long exact sequence in homology associated to the pair \((E_L, \partial E_L)\), in which homology groups are endowed with the distinguished bases given in Equation (39), is equal to \((-1)^{b(b-1)/2}\).

**Proof.** Let \(\mathcal{H}_*\) denotes the long exact sequence in homology associated to the pair \((E_L, \partial E_L)\). Counting dimensions, it is easy to observe that \(\mathcal{H}_*\) is decomposed into the following three short exact sequences:

\[
(40) \quad \mathcal{H}_*^{(i)} : 0 \rightarrow H_{i+1}(E_L, \partial E_L; \mathbb{R}) \xrightarrow{\delta_{i}^{(i)}} H_i(\partial E_L; \mathbb{R}) \xrightarrow{-j_{i}^{(i)}} H_i(E_L; \mathbb{R}) \rightarrow 0
\]

for \(i = 0, 1, 2\). Here \(\delta_{i}^{(i)}\) denotes the connecting homomorphism and \(j_{i}^{(i)}\) is induced by the usual inclusion \(\partial E_L \hookrightarrow E_L\). Thus, the torsion of \(\mathcal{H}_*\) is equal to the alternative product of the three torsions of the above short exact sequences (40).

The torsions are computed with respect to the bases given in Equation (39) and we have:

**Claim 17.** The Reidemeister torsions of the short exact sequences \(\mathcal{H}_*^{(i)}\) are given by:

\[
\begin{align*}
\text{Tor}(\mathcal{H}_*^{(2)}, \{h_{E_L}^*, h_{\partial E_L}^*, h_{[E_L, \partial E_L]}^*\}, \emptyset) &= (-1)^{b-1}, \\
\text{Tor}(\mathcal{H}_*^{(1)}, \{h_{E_L}^*, h_{\partial E_L}^*, h_{[E_L, \partial E_L]}^*\}, \emptyset) &= (-1)^{b(b-1)/2}, \\
\text{Tor}(\mathcal{H}_*^{(0)}, \{h_{E_L}^*, h_{\partial E_L}^*, h_{[E_L, \partial E_L]}^*\}, \emptyset) &= (-1)^{b-1}.
\end{align*}
\]

**Proof of the claim.** Each torsion of the short exact sequences \(\mathcal{H}_*^{(i)}\) can be calculated as follows.

- **Computation of** \(\text{Tor}(\mathcal{H}_*^{(2)}, \{h_{E_L}^*, h_{\partial E_L}^*, h_{[E_L, \partial E_L]}^*\}, \emptyset)\).
  
  The connecting homomorphism \(\delta^{(2)}\) maps \([E_L, \partial E_L]\) to \([T^2_1] + \cdots + [T^2_b]\).
  
  Moreover one has

  \[
j_{*}^{(2)}([T^2_T]) = [T^2_T] \quad \text{for} \quad \ell = 1, \ldots, b - 1.
\]

  Thus the set \(\{[T^2_1], \ldots, [T^2_{b-1}]\}\) of vectors in \(H_2(\partial E_L; \mathbb{R})\) can be chosen as lifts of the distinguished basis of \(H_2(E_L; \mathbb{R})\). As a result, the torsion of this short exact
sequence $\mathcal{H}_c^{(2)}$ is given by the following base change determinant:

$$
\text{Tor}(\mathcal{H}_c^{(2)}, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset)
= \left[\begin{array}{cccc}
\delta([E_L, \partial E_L]), [T_1^2], \ldots, [T_{b-1}^2]/\{[T_1^2], [T_2^2], \ldots, [T_b^2]\}
\end{array}\right]
= (-1)^{b-1}.
$$

- **Computation of Tor($\mathcal{H}_c^{(1)}$, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset).**

The connecting homomorphism $\delta^{(1)}$ maps $[S_i]$ to $[\lambda_i] + \sum_{k \neq \ell} \ell k(K_i, K_k)[\mu_k]$ where $\ell k$ denotes the linking number. Moreover, $j^{(1)}([\mu_k]) = [\mu_k]$, for $\ell = 1, \ldots, b$. Thus, the set $\{[\mu_1], \ldots, [\mu_b]\}$ of vectors in $H_L(\partial E_L; \mathbb{R})$ can be chosen as lifts of the distinguished basis of $H_L(E_L; \mathbb{R})$. It follows that the torsion of the short exact sequence $\mathcal{H}_c^{(1)}$ is given by the ratio of the following two determinants of bases change matrices:

$$
\text{Tor}(\mathcal{H}_c^{(1)}, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset)
\quad (41)
= \left[\begin{array}{cccc}
\delta([S_1]), \ldots, [S_b], [\mu_1], \ldots, [\mu_b]/\{[\lambda_1], [\lambda_2], \ldots, [\lambda_b]\}
\end{array}\right].
$$

Observe that the determinant in the right-hand side of Equation (41) can be written again as the product of the following two base change determinants:

$$
D_1 = \left[\begin{array}{cccc}
[\delta([S_1]), \ldots, [S_b], [\lambda_1], [\mu_1], \ldots, [\mu_b]/\{[\lambda_1], [\lambda_2], \ldots, [\lambda_b]\}
\end{array}\right]
\quad D_2 = \left[\begin{array}{cccc}
[\lambda_1], \ldots, [\lambda_b], [\lambda_1], [\mu_1], \ldots, [\mu_b]/\{[\mu_1], [\mu_2], \ldots, [\mu_b]\}
\end{array}\right].
$$

It is easy to observe that

$$
D_1 = \left[\begin{array}{cccc}
0 & \cdots & 1
\end{array}\right] = (-1)^{b(b-1)/2},
$$

and that $D_2 = 1$, because it is the signature of a product of even permutations.

As a conclusion, we have that

$$
\text{Tor}(\mathcal{H}_c^{(1)}, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset) = (-1)^{b(b-1)/2}.
$$

- **Computation of Tor($\mathcal{H}_c^{(0)}$, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset).**

The connecting homomorphism $\delta^{(0)}$ maps $[\gamma_\ell]$ to $[p_\ell] - [p_\ell]$, for all $\ell = 1, \ldots, b - 1$. Also observe that $j^{(0)}([p_b]) = [p_b]$. Thus, the torsion of the short exact sequence $\mathcal{H}_c^{(0)}$ is given by the following base change determinant:

$$
\text{Tor}(\mathcal{H}_c^{(0)}, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset)
= \left[\begin{array}{cccc}
\delta([\gamma_1]), \ldots, [\gamma_{b-1}], [p_1], \ldots, [p_b]
\end{array}\right]
= (-1)^{b-1}.
$$

Using this result, we conclude that the torsion $\text{Tor}(\mathcal{H}_c, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset)$ of $\mathcal{H}_c$ is given by $\prod_{i=0}^2 \text{Tor}(\mathcal{H}_c^{(i)}, \{h^*_{E_L}, h^*_{\partial E_L}, h^*_{(E_L, \partial E_L)}\}, \emptyset)^{(-1)^i} = (-1)^{b(b-1)/2}$.  

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