Elicitation, measuring bias, checking for prior-data conflict and inference with a Dirichlet prior

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Abstract: Methods are developed for eliciting a Dirichlet prior based upon bounds on the individual probabilities that hold with virtual certainty. This approach to selecting a prior is applied to a contingency table problem where it is demonstrated how to assess the bias in the prior as well as how to check for prior-data conflict. It is shown that the assessment of a hypothesis via relative belief can easily take into account what it means for the falsity of the hypothesis to correspond to a difference of practical importance and provide evidence in favor of a hypothesis.

Key words and phrases: elicitation, bias, relative belief inferences.

1 Introduction

Perhaps the most basic statistical model is the multinomial\((n, p_1, \ldots, p_k)\) where \(n \in \mathbb{N}, \langle p_1, \ldots, p_k \rangle \in S_k = \{(x_1, \ldots, x_k) : x_i \geq 0 \text{ and } x_1 + \cdots + x_k = 1\}, S_k\) is the \((k - 1)\)-dimensional simplex and \((p_1, \ldots, p_k)\) is unknown. This arises from an i.i.d. sample from the multinomial\((1, p_1, \ldots, p_k)\) distribution. The goal is then inference about the unknown value of \((p_1, \ldots, p_k)\).

Bayesian inference requires a prior and the Dirichlet\((\alpha_1, \ldots, \alpha_k)\), for some choice of hyperparameters \(\alpha_1, \ldots, \alpha_k\), is a convenient choice due to its conjugacy. To employ such a prior it is necessary to have an easy to use elicitation algorithm. The purpose of this paper is to develop such an algorithm, to show how the chosen prior can be assessed with respect to the bias that it induces, to check whether the prior conflicts with the data, to show how to modify the prior when such a conflict is encountered and to implement inferences using the prior based on a measure of statistical evidence.

In Section 2 an elicitation algorithm is developed for the Dirichlet. In Section 3 the bias in the prior is discussed and in Section 4 the issue of prior-data conflict and possible modification of the prior is addressed. Section 5 deals with inference for the multinomial based on the relative belief ratio as a measure of
Table 1: The data in Example 1.

|       | Y = O | Y = A | Y = B | Total |
|-------|------|------|------|-------|
| X = P | 983  | 679  | 134  | 1796  |
| X = G | 383  | 416  | 84   | 883   |
| X = C | 2892 | 2625 | 570  | 6087  |
| Total | 4258 | 3720 | 788  | 8766  |

evidence. This presents a full treatment of a statistical analysis for the multinomial although it is assumed that the multinomial model is correct. Strictly speaking, provided the data is available, it should also be checked that the initial sample is i.i.d. from a multinomial(1, p₁, ..., pₖ) distribution, perhaps using a multivariate version of a runs test, but this is not addressed here.

Throughout the paper the following example, taken from Snedecor and Cochran (1967), is considered as a practical application of the methodology.

**Example 1. Assessing independence**

Individuals were classified according to their blood type Y (O, A, B, and AB, although the AB individuals were eliminated, as they were small in number) and also classified according to X, their disease status (peptic ulcer = P, gastric cancer = G, or control = C). So there are three populations; namely, those suffering from a peptic ulcer, those suffering from gastric cancer, and those suffering from neither and it is assumed that the individuals involved in the study can be considered as random samples from the respective populations. The data are in Table 1 and the goal is to determine whether or not X and Y are independent. So the counts are assumed to be multinomial(8766, p₁₁, p₁₂, p₁₃, p₂₁, p₂₂, p₂₃, p₃₁, p₃₂, p₃₃) where the first index refers to X and the second to Y and with a relabelling of the categories, e.g. X = G is relabeled as X = 2.

Using the chi-squared test, the null hypothesis of no relationship is rejected with a value of the chi-squared statistic of 40.54 and a p-value of 0.0000. Table 2 gives the estimated cell probabilities based on the full multinomial as well as the estimated cell probabilities based on independence between the X and Y. The difference between the two tables is very small and of questionable practical significance. For example, the largest difference between corresponding cells is 0.012 and, as a natural measure of difference between two distributions, the estimated Kullback-Leibler divergence, based on the raw data, is estimated as 0.002. This suggests that in reality the deviation from independence is not meaningful. The cure for this is that, in assessing any hypothesis, it is necessary to say what size of deviation δ from the null is of practical significance and take this into account when performing the test. This arises as a natural aspect of the relative belief approach to this problem and will be discussed in Section 3, where a very different conclusion is reached in this example.
that the restriction $1$. If $u$ be chosen.

The value $\tau$ is completely determined by $\Pi_{\alpha_1,\alpha_2}([l_1,u_1]) = \gamma$ provided that $u_1 - l_1 \leq \gamma$ as it is easy to see that $\Pi_{1+\tau\xi,1+\tau(1-\xi)}([l_1,u_1]) \uparrow 1$ as $\tau \uparrow \infty$. Note that the restriction $\alpha_1,\alpha_2 \geq 1$ is natural as this avoids singularities at 0 or 1. If $u_1 - l_1 > \gamma$, then the requirement can be relaxed to requiring $(\alpha_1,\alpha_2)$ satisfy $\Pi_{\alpha_1,\alpha_2}([l_1,u_1]) \geq \gamma$, so the beta$(1,1)$ suffices or a larger value of $\gamma$ can be chosen.

Supposing $u_1 - l_1 \leq \gamma$, it is then straightforward to solve for $\tau$ via an iterative algorithm. To start set $\tau_0 = 0$, which implies $(\alpha_1,\alpha_2) = (1,1)$ and $\Pi_{1+\tau_0\xi,1+\tau_0(1-\xi)}([l_1,u_1]) = u_1 - l_1$, find $\tau_1$ such that $\Pi_{1+\tau_1\xi,1+\tau_1(1-\xi)}([l_1,u_1]) >$

| Full   | $Y = O$ | $Y = A$ | $Y = B$ | Ind.   | $Y = O$ | $Y = A$ | $Y = B$ |
|--------|---------|---------|---------|--------|---------|---------|---------|
| $X = P$| 0.112   | 0.077   | 0.015   | $X = P$| 0.100   | 0.087   | 0.018   |
| $X = G$| 0.043   | 0.047   | 0.009   | $X = G$| 0.049   | 0.043   | 0.009   |
| $X = C$| 0.330   | 0.299   | 0.065   | $X = C$| 0.337   | 0.295   | 0.062   |

Table 2: The estimated cell probabilities in Example 1 based on the full and independence models.

2 Elicitation

A key component of a Bayesian statistical analysis is the choice of the prior. For this it is recommended that an elicitation algorithm be used so that the selection of the prior be based upon what is known about problem under study. Typically this will involve some knowledge of what kind of values are expected for the data as these arise via some measurement process. In the context of the Dirichlet this knowledge will take the form of how likely a success is expected on each of the $k$ categories being counted. Of course, there can be a variety of elicitation algorithms that are appropriate. Our approach here is to develop one that is simple to use and results in an appropriate expression of belief. Discussions about the process of elicitation for general problems can be found in Gathwaite et al. (2005) and O’Hagan et al. (2006).

Consider first the situation where $k = 2$ and the prior $\Pi_{\alpha_1,\alpha_2}$ on $p_1$ is beta$(\alpha_1,\alpha_2)$. Suppose it is known with ‘virtual certainty’ that $l_1 \leq p_1 \leq u_1$ where $l_1, u_1 \in [0,1]$ are known. This immediately implies that $1 - u_1 \leq p_2 = 1 - p_1 \leq 1 - l_1$ with virtual certainty. Here ‘virtual certainty’ is interpreted to mean that the true value of $p_1$ is in the interval $[l_1,u_1]$ with high prior probability $\gamma$, say $\gamma = 0.99$. So this restricts the prior to those values of $(\alpha_1,\alpha_2)$ satisfying $\Pi_{\alpha_1,\alpha_2}([l_1,u_1]) = \gamma$. To completely determine $(\alpha_1,\alpha_2)$ another condition is added, namely, it is required that the mode of the prior be at the point $\xi \in [l_1,u_1]$ as this allows the placement of the primary amount of the prior mass at an appropriate place within $[l_1,u_1]$. For example, a natural choice of the mode in this context is $\xi = (l_1 + u_1)/2$, namely, the midpoint of the interval. When $\alpha_1,\alpha_2 \geq 1$ the mode of the beta$(\alpha_1,\alpha_2)$ occurs at $\xi = (\alpha_1 - 1)/\tau$ where $\tau = \alpha_1 + \alpha_2 - 2$. There is thus a 1-1 correspondence between the values $(\alpha_1,\alpha_2)$ and $(\xi,\tau)$ given by $\alpha_1 = 1 + \tau\xi,\alpha_2 = 1 + \tau(1-\xi)$. Therefore, after specifying the mode, only the scaling of the beta prior is required through the choice of $\tau$. The value $\tau$ is completely determined by $\Pi_{\alpha_1,\alpha_2}([l_1,u_1]) = \gamma$ provided that $u_1 - l_1 \leq \gamma$ as it is easy to see that $\Pi_{1+\tau\xi,1+\tau(1-\xi)}([l_1,u_1]) \uparrow 1$ as $\tau \uparrow \infty$. Note that the restriction $\alpha_1,\alpha_2 \geq 1$ is natural as this avoids singularities at 0 or 1. If $u_1 - l_1 > \gamma$, then the requirement can be relaxed to requiring $(\alpha_1,\alpha_2)$ satisfy $\Pi_{\alpha_1,\alpha_2}([l_1,u_1]) \geq \gamma$, so the beta$(1,1)$ suffices or a larger value of $\gamma$ can be chosen.
\( \gamma \) and then proceed iteratively via the bisection root finding algorithm.

**Example 2. Determining a beta prior.**

Suppose that \([l_1, u_1] = (0.25, 0.75), \xi = 0.5 \) and \( \gamma = 0.99 \). The solution obtained via the iterative algorithm is then \( \tau = 22.0 \) where the iteration is stopped when \( |\Pi_{1+\tau,1+\tau,(1-\xi)}([l_1, u_1]) - \gamma| \leq 0.005 \). This took 7 iterations and the prior is given by \( (\alpha_1, \alpha_2) = (12.0, 12.0) \) and \([l_1, u_1]\) contains 0.993 of the prior probability. If instead of 0.005 the error tolerance for stopping was set equal to 0.001, then the solution \( \tau = 22.04 \) and \( (\alpha_1, \alpha_2) = (12.02, 12.02) \) was obtained after 20 iterations with \([l_1, u_1]\) containing 0.990 of the prior probability.

The approach to eliciting a beta prior seems very natural and allows for a great deal of flexibility in where the prior allocates the bulk of its mass in \([0, 1]\). The question, however, is how to generalize this to the Dirichlet(\(\alpha_1, \ldots, \alpha_k\)) prior. As will be seen, it is necessary to be careful about how \( (\alpha_1, \ldots, \alpha_k) \) is elicited. Again we make the restriction that each \( \alpha_i \geq 1 \) to avoid singularities for the prior on the boundary.

It seems quite natural to think about putting probabilistic bounds on the \( p_i \) such as requiring \( l_i \leq p_i \leq u_i \) with high probability, for fixed constants \( l_i, u_i \), to reflect what is known with ‘virtual certainty’ about \( p_i \). For example, it may be known that \( p_i \) is very small and so we put \( l_i = 0 \), choose \( u_i \) small and require that \( p_i \leq u_i \) with prior probability at least \( \gamma \). While placing bounds like this on the \( p_i \) is reasonable, such an approach can result in a complicated shape for the region that is to contain the true value of \((p_1, \ldots, p_k)\) with virtual certainty. This complexity can make the computations associated with inference very difficult. In fact it can be hard to determine exactly what the full region is. As such, it seems better to use an elicitation method that fits well with the geometry of the Dirichlet family. If it is felt that more is known a priori than a Dirichlet prior can express, then it is appropriate to contemplate using some other family of priors. Given the conjugacy property of Dirichlet priors, which vastly simplifies many computations, the focus here is on devising elicitation algorithms that work well with this family. First we consider elicitation approaches for this problem that have been presented in the literature.

Chaloner and Duncan (1987) discuss an iterative elicitation algorithm based on specifying characteristics of the prior predictive distribution of the data which is Dirichlet-multinomial. Regazzini and Sazonov (1999) discuss an elicitation algorithm which entails partitioning the simplex, prescribing prior probabilities for each element of the partition and then selecting a mixture of Dirichlet distributions as the prior such that this prior has Prohorov distance less than some \( \epsilon > 0 \) from the true prior associated with de Finetti’s representation theorem. Both of these approaches are complicated to implement. Closest to the method presented here is that discussed in Dorp and Mazzuchi (2003) where \( (\alpha_1, \ldots, \alpha_k) \) is specified by choosing \( i \in \{1, \ldots, k\} \), stating two prior quantiles \( (p_{\gamma_1}, p_{\gamma_2}) \) where \( 0 < \gamma_1 < \gamma_2 < 1 \) for \( p_i \) and specifying prior quantile \( p_{\gamma_j} \) for \( p_j \) for each \( j \neq i, k \). So there are \( k \) constraints that the Dirichlet(\(\alpha_1, \ldots, \alpha_k\)) has to satisfy and an algorithm is provided for computing \( (\alpha_1, \ldots, \alpha_k) \). Drawbacks include the fact that the \( p_i \) are not treated symmetrically as there is a need to place two
The edges of $S$ is Theorem 1. Following result then holds.

For this we ask for a set of lower bounds $\{l_i\}$, $i = 1, \ldots, k$. Generally the elicitation process allows for a single lower or upper bound to be specified for each $p_i$. These bounds specify a subsimplex of the simplex $S_k$ with all edges of the same length. As will be seen, this implicitly takes into account the dependencies among the $p_i$. With such a region determined, it is straightforward to determine $(\alpha_1, \ldots, \alpha_k)$ such that the subsimplex contains $\gamma$ of the prior probability for $(p_1, \ldots, p_k)$.

Note that a $(k-1)$-dimensional simplex can be specified by specifying $k$ distinct points in $R^k$, say $a_1, \ldots, a_k$, and then taking all convex combinations of these points. This simplex will be denoted as $S(a_1, \ldots, a_k) = \{\sum_{i=1}^k c_i a_i : c_i \geq 0$ with $c_1 + \cdots + c_k = 1\}$. So $S_k = S(e_1, \ldots, e_k)$ and it is clear that $S(a_1, \ldots, a_k) \subset S(e_1, \ldots, e_k)$ whenever $a_1, \ldots, a_k \in S(e_1, \ldots, e_k)$. The centroid of $S(a_1, \ldots, a_k)$ is equal to $CS(a_1, \ldots, a_k) = \sum_{i=1}^k a_i/k$.

### 2.1 Lower bounds on the probabilities

For this we ask for a set of lower bounds $l_1, \ldots, l_k \in [0,1]$ such that $l_i \leq p_i$ for $i = 1, \ldots, k$. To make sense there is only one additional constraint that the $l_i$ must satisfy, namely, $L_{1:k} = l_1 + \cdots + l_k \leq 1$. If $L_{1:k} = 1$, then it is immediate that $p_i = l_i$, otherwise $p_1 + \cdots + p_k > 1$. So the $p_i$ are completely determined when $L_{1:k} = 1$. Attention is thus restricted to the case where $L_{1:k} < 1$. The following result then holds.

**Theorem 1.** Specifying the lower bounds $l_1, \ldots, l_k \in [0,1]$ such that $l_i \leq p_i$ for $i = 1, \ldots, k$ and

$$ L_{1:k} < 1, $$

prescribes $S(a_1, \ldots, a_k) \subset S_k$ where $a_i = (l_1, \ldots, l_{i-1}, u_i, l_{i+1}, \ldots, l_k)$ and

$$ u_i = 1 - \sum_{j \neq i} l_j. $$

The edges of $S(a_1, \ldots, a_k)$ each have length $\sqrt{2(1 - L_{1:k})}$ and $S(a_1, \ldots, a_k) = \{(p_1, \ldots, p_k) : p_1 + \cdots + p_k = 1, l_i \leq p_i \leq u_i, i = 1, \ldots, k\}$.

**Proof:** Note that (1) implies that $p_i = 1 - \sum_{j \neq i} p_j \leq 1 - \sum_{j \neq i} l_j = u_i$, and so stating the lower bounds implies a set of upper bounds, and also $l_i < u_i$. Consider now the set $S = \{(p_1, \ldots, p_k) : p_1 + \cdots + p_k = 1, l_i \leq p_i \leq u_i, i = 1, \ldots, k\}$ and note that $a_i \in S$ for $i = 1, \ldots, k$. For $e_i \geq 0$ with $e_1 + \cdots + e_k = 1$, then $(p_1, \ldots, p_k) = \sum_{i=1}^k e_i a_i \in S$ since, for example, the first coordinate

5
satisfies $p_1 = c_1 u_1 + (\sum_{i=2}^k c_i) l_1 = c_1 u_1 + (1 - c_1) l_1$ so $l_1 \leq p_1 \leq u_1$. Therefore $S(a_1, \ldots, a_k) \subset S$.

If $(p_1, \ldots, p_k) \in S$, then $p_i = c_i^* l_i + (1 - c_i^*) u_i$ where $c_i^* \in [0, 1]$. Now $1 = p_1 + \cdots + p_k = \sum_{i=1}^k c_i^* l_i + \sum_{i=1}^k (1 - c_i^*) u_i = \sum_{i=1}^k c_i^* l_i + \sum_{i=1}^k (1 - c_i^*) (l_i + 1 - L_{1:k}) = L_{1:k} + (\sum_{i=1}^k (1 - c_i^*)) (1 - L_{1:k})$ and so $\sum_{i=1}^k (1 - c_i^*) = 1$. For $(p_1, \ldots, p_k) = \sum_{j=1}^k (1 - c_j^*) a_j$ we have $p_i = (\sum_{j \neq i} (1 - c_j^*)) l_i + (1 - c_i^*) u_i = c_i^* l_i + (1 - c_i^*) u_i$. This proves that $S \subset S(a_1, \ldots, a_k)$ and so we have $S(a_1, \ldots, a_k) = S$.

Finally note that $||a_i - a_j||^2 = (u_i - l_i)^2 + (u_i - l_j)^2 = 2(1 - L_{1:k})^2$ and so $S(a_1, \ldots, a_k)$ has edges all of the same length. This completes the proof.

### 2.2 Upper bounds on the probabilities

Of course, it may be that prior beliefs are instead expressed via upper bounds on the probabilities or a mixture of upper and lower bounds. The case of all upper bounds is considered first. Our goal is to specify the upper bounds in such a way that these lead unambiguously to lower bounds $l_1, \ldots, l_k \in [0, 1]$ satisfying (1) and so to the simplex $S(a_1, \ldots, a_k)$.

Suppose that we have the upper bounds $u_1, \ldots, u_k \in [0, 1]$ such that $p_i \leq u_i$. It is clear then that $l_1, \ldots, l_k$ must satisfy the system of linear equations given by (2) as well as $0 \leq l_i \leq u_i$ for $i = 1, \ldots, k$ and (1). So the $l_i$ must satisfy

$$\mathbf{u} = \mathbf{1}_k - \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} l = \mathbf{1}_k + (I_k - \mathbf{1}_k \mathbf{1}_k') l$$

where $\mathbf{1}_k$ is the $k$-dimensional vector of 1’s and $I_k$ is the $k \times k$ identity. Noting that $(I_k - \mathbf{1}_k \mathbf{1}_k')^{-1} = I_k - (k-1)^{-1} \mathbf{1}_k \mathbf{1}_k'$, it is immediate that

$$l = (I_k - (k-1)^{-1} \mathbf{1}_k \mathbf{1}_k')(\mathbf{u} - \mathbf{1}_k).$$

Note that this requires that $k \geq 2$ as is always the case.

Putting $U_{1:k} = \sum_{j=1}^k u_j$, then (2) implies $L_{1:k} = (k - U_{1:k})/(k-1)$ and so $0 \leq L_{1:k} < 1$ provided $U_{1:k}$ satisfies

$$1 < U_{1:k} \leq k.$$  

From (2)

$$l_i = (u_i - 1) - \frac{U_{1:k} - k}{k - 1} = u_i + \frac{1 - U_{1:k}}{k - 1}$$

and, for $i = 1, \ldots, k$, this implies that $l_i \geq 0$ iff

$$u_i \geq \frac{U_{1:k} - 1}{k - 1}.$$  

Also, when (5) is satisfied, then $l_i < u_i$ for $i = 1, \ldots, k$. This completes the proof of the following result.
Theorem 2. Specifying upper bounds \( u_1, \ldots, u_k \in [0, 1] \), such that \( p_i \leq u_i \) for \( i = 1, \ldots, k \), satisfying inequalities (5) and (7), determines the lower bounds \( l_1, \ldots, l_k \), given by (6), which determine the simplex \( S(a_1, \ldots, a_k) \) defined in Theorem 1.

The difficult aspect of this approach to elicitation is to make sure the upper bounds satisfy (5) and (7). If we take \( u_1 = \cdots = u_k = u \geq 1/k \), then (5) is satisfied and \((k - 1)u \geq ku - 1\) implies that (7) is satisfied as well.

2.3 Upper and lower bounds on the probabilities

Now, perhaps after relabelling the probabilities, suppose that lower bounds \( 0 \leq l_i \leq p_i \) for \( i = 1, \ldots, m \) as well as upper bounds \( p_i \leq u_i \leq 1 \) for \( i = m + 1, \ldots, k \), have been provided. Again it is required that \( L_{1:m} = l_1 + \cdots + l_m < 1 \) and we search for conditions on the \( u_i \) that complete the prescription of a full set of lower bounds \( l_1, \ldots, l_k \) so that Theorem 1 applies. Again the \( l \) and \( u \) vectors must satisfy (3). Let \( x_{r:s} \) denote the subvector of \( x \) given by its consecutive \( r \)-th through \( s \)-th coordinates and \( X_{r:s} \) the sum of these coordinates provided \( r \leq s \) and be null otherwise. The following equations hold

\[
\begin{align*}
u_{1:m} &= l_m + L_{1:m} 1_m - L_{m+1:k} 1_m \\
u_{m+1:k} &= 1_{k-m} - L_{1:m} 1_{k-m} + (I_{k-m} - 1_{k-m} I_{k-m}) l_{m+1:k}.
\end{align*}
\]

Rearranging these equations so the knowns are on the left and the unknowns are on the right gives

\[
\begin{align*}
l_{1:m} + (1 - L_{1:m}) 1_m &= u_{1:m} + L_{m+1:k} 1_m \\
u_{m+1:k} - (1 - L_{1:m}) 1_{k-m} &= (I_{k-m} - 1_{k-m} I_{k-m}) l_{m+1:k}.
\end{align*}
\]

It follows from (9) that

\[
l_{m+1:k} = (I_{k-m} - 1_{k-m} I_{k-m})^{-1} [u_{m+1:k} - (1 - L_{1:m}) 1_{k-m}] = (I_{k-m} - (k - m - 1)^{-1} 1_{k-m} I_{k-m}) [u_{m+1:k} - (1 - L_{1:m}) 1_{k-m}]
\]

and substituting this into (8) gives the solution for \( u_{1:m} \) as well.

So it is only necessary to determine what additional conditions have to be imposed on the \( l_1, \ldots, l_m, u_m, \ldots, u_k \) so that Theorem 1 applies. Note that it follows from (8) that \( u_{1:m} \) takes the correct form, as given by (2), so it is really only necessary to check that \( l \) is appropriate.

First it is noted that it is necessary that \( k - m > 1 \). The case \( k - m = 1 \) only occurs when \( m = k - 1 \) and then \( p_k = 1 - p_1 - \cdots - p_{k-1} \leq 1 - l_1 - \cdots - l_{k-1} \) which is the required value for \( u_k \) for Theorem 1 to apply. So when \( k - m = 1 \) there is no choice but to put \( u_k = 1 - l_1 - \cdots - l_{k-1} \) and choose a lower bound for \( p_k \), which of course could be 0, which means that Theorem 1 applies. It is assumed hereafter that \( k - m > 1 \).
Now $L_{1:k} = L_{1:m} + L_{m+1:k}$ and the requirement $0 \leq L_{1:k} < 1$ imposes the requirement $0 \leq L_{m+1:k} < 1 - L_{1:m}$. Using (10) gives

$$L_{m+1:k} = 1_{k-m}^t L_{m+1:k} = \left(1 - \frac{k-m}{k-m-1}\right)(U_{m+1:k} - (k-m)(1-L_{1:m}))$$

and therefore $0 \leq L_{m+1:k} < 1 - L_{1:m}$ iff

$$1 - L_{1:m} < U_{m+1:k} \leq (k-m)(1-L_{1:m}).$$

(11)

It is seen that (11) generalizes (5) on taking $m = 0$. Now for $i > m$

$$l_i = u_i - (1 - L_{1:m}) - \frac{U_{m+1:k}}{k-m-1} + \frac{(k-m)(1-L_{1:m})}{k-m-1}$$

$$= u_i + \frac{(1-L_{1:m}) - U_{m+1:k}}{k-m-1}$$

(12)

and so, for $i = m+1, \ldots, k$, this implies that $l_i \geq 0$ iff

$$u_i \geq \frac{U_{m+1:k} - (1-L_{1:m})}{k-m-1}.$$  

(13)

So (13) generalizes (5) on taking $m = 0$. Also, if (11) is satisfied, then $l_i \leq u_i$ for $i = m+1, \ldots, k$.

The above argument establishes the following result.

**Theorem 3.** For $m$ satisfying $1 \leq m \leq k-2$, specifying the bounds
(i) $l_i \leq p_i$ with $l_i \in [0,1]$ for $i = 1, \ldots, m$, satisfying $L_{1:m} < 1$ and
(ii) $u_i \geq p_i$ with $u_i \in [0,1]$ for $i = m+1, \ldots, k$, satisfying (11) and (13),
determines the lower bounds $l_{m+1}, \ldots, l_k$, given by (12), which, together with
$l_1, \ldots, l_m$, determine the simplex $S(a_1, \ldots, a_k)$ defined in Theorem 1.

### 2.4 Determining the Elicited Prior

So now suppose there is an elicited set of bounds that lead to the simplex specified by Theorem 1 and it is necessary to determine the Dirichlet($\alpha_1, \ldots, \alpha_k$) prior, denoted $\Pi(\alpha_1, \ldots, \alpha_k)$, such that $\Pi(\alpha_1, \ldots, \alpha_k)(S(a_1, \ldots, a_k)) = \gamma$. Again we pick a point $\xi = (\xi_1, \ldots, \xi_k) \in S(a_1, \ldots, a_k)$ and place the mode at $\xi_i = (a_i - 1)/\tau$ for $i = 1, \ldots, k$ with $\tau = \alpha_1 + \cdots + \alpha_k - k$. For example, $\xi = CS(a_1, \ldots, a_k)$ would often seem like a sensible choice and then only $\tau$ needs to be determined. There is a 1-1 correspondence between $(\alpha_1, \ldots, \alpha_k)$ and $(\xi_1, \ldots, \xi_k, \tau)$ given by $\alpha_i = 1 + \tau \xi_i$.

Again it makes sense to proceed via an iterative algorithm to determine $\tau$. Provided $\Pi(\alpha, \ldots, \alpha)(S(a_1, \ldots, a_k)) \leq \gamma$, set $\tau = 0$ and find $\tau_0$ such that $\Pi(1+\tau_0, \ldots, 1+\tau_0)(S(a_1, \ldots, a_k)) \geq \gamma$. As before set $\tau_1 = (\tau_0+\tau_0)/2$ and then the algorithm proceeds via bisection. Determining $\Pi(1+\tau, \xi_1, \ldots, 1+\tau, \xi_k)(S(a_1, \ldots, a_k))$
Figure 1: Plots of the marginal densities determined when specifying the lower bounds \(l_1 = 0.2, l_2 = 0.2, l_3 = 0.3, l_4 = 0.2\) in Example 3.

at each step becomes problematical even for \(k = 3\). In the approach adopted here this probability content was estimated via a Monte Carlo sample from the relevant Dirichlet. This is seen to work quite well as, in the case of determining a prior, high accuracy for the computations is not required.

Consider an example. **Example 3. Determining a Dirichlet\((\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) prior.**

Suppose that \(k = 4\) and the lower bounds \(l_1 = 0.2, l_2 = 0.2, l_3 = 0.3, l_4 = 0.2\) are placed on the probabilities. This results in the bounds \(0.2 \leq p_1 \leq 0.3, 0.2 \leq p_2 \leq 0.3, 0.3 \leq p_3 \leq 0.4,\) and \(0.2 \leq p_4 \leq 0.3\) which are reasonably tight. The mode was placed at the centroid \(\xi = (0.22, 0.22, 0.32, 0.22)\). For \(\gamma = 0.99\), an error tolerance of \(\epsilon = 0.005\) and a Monte Carlo sample of size of \(N = 10^3\) at each step, the values \(\tau = 2560\) and \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (577.0, 577.0, 833.0, 577.0)\) were obtained after 13 iterations. The prior content of \(S(a_1, a_2, a_3, a_4)\) was estimated to be 0.989. If greater accuracy is required then \(N\) can be increased and/or \(\epsilon\) decreased.

This choice of lower bounds results in a fairly concentrated prior as is reflected in the plots of the marginals in Figure 1. This is reflected also in Figure 2 where scatter plots are provided of a sample of 300 from the joint distribution for the pairs of probabilities \((p_1, p_2), (p_2, p_3)\) and \((p_3, p_4)\). This concentration is not a defect of the elicitation as (2) indicates that it must occur when the sum of the bounds is close to 1. So the concentration is forced by the dependencies among the probabilities.

Consider now another example.
probabilities. This leads to the following bounds for the probabilities.

\[ p_{ij} \]

Example 4. Determining a Dirichlet(\(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\)) prior.

Suppose that \(k = 9\) and the lower bounds \(l_1 = 0.02, l_2 = 0.02, l_3 = 0.0, l_4 = 0.00, l_5 = 0.00, l_6 = 0.00, l_7 = 0.10, l_8 = 0.10, l_9 = 0.00\) are placed on the probabilities. This leads to the following bounds for the probabilities.

\[
\begin{align*}
0.02 \leq p_1 & \leq 0.78 \\
0.00 \leq p_4 & \leq 0.76 \\
0.10 \leq p_7 & \leq 0.86 \\
0.00 \leq p_3 & \leq 0.76 \\
0.00 \leq p_5 & \leq 0.76 \\
0.00 \leq p_6 & \leq 0.76 \\
0.10 \leq p_8 & \leq 0.86 \\
0.00 \leq p_9 & \leq 0.76 \\
\end{align*}
\]

The mode was placed at the centroid \(\xi = (0.1, 0.1, 0.08, 0.08, 0.08, 0.08, 0.18, 0.18, 0.08)\). For \(\gamma = 0.99\), an error tolerance of \(\epsilon = 0.005\) and a Monte Carlo sample of size of \(N = 10^5\) at each step, the values \(\tau = 96\) and \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9) = (11.03, 11.03, 9.11, 9.11, 9.11, 18.71, 18.71, 18.71, 9.11)\) were obtained after 7 iterations. The prior content of \(S(a_1, \ldots, a_9)\) was estimated to be 0.987. Figure 3 is a plot of the 9 marginal priors for the \(p_i\). Again the dependencies among the \(p_i\) make the marginal priors quite concentrated.

Example 1. (continued) Choosing the prior.

Given that we wish to assess independence, it is necessary that any elicited prior include independence as a possibility so this is not ruled out a prior. A natural elicitation is to specify valid bounds (namely, bounds that satisfy our theorems) on the \(p_i\) and the \(p_j\) and then use these to obtain bounds on the \(p_{ij}\) which in turn leads to the prior. So suppose valid bounds have been specified that lead to the lower bounds \(a_i \leq p_i, b_j \leq p_j\). Then it is necessary that \(l_{ij} = a_ib_j\) is the lower bound on \(p_{ij}\). Note that it is immediate that the \(l_{ij}\) satisfy the conditions of Theorem 1 and from \(\sum_{s} p_{rs} + l_{ij} = 1 - \sum_{r} a_r \sum_{s} b_s + a_i b_j\) which is greater than \(l_{ij} = a_i b_j\) since \(0 \leq \sum_{r} a_r < 1\) and \(0 \leq \sum_{s} b_s < 1\). As such the region for the \(p_{ij}\) contains elements of \(H_0\).

For this example, the lower bounds \(a_1 = 0.1, a_2 = 0.0, a_3 = 0.5, b_1 = 0.0, b_2 = 0.0, b_3 = 0.0\) and the upper bounds \(\sum_{s} b_s + a_i b_j = 0.99\).
0.2, \(b_2 = 0.2, b_3 = 0.0\) were chosen which leads to the lower bounds

\[
L = \begin{pmatrix}
0.02 & 0.02 & 0.00 \\
0.00 & 0.00 & 0.00 \\
0.10 & 0.10 & 0.00
\end{pmatrix}
\]

on the \(p_{ij}\). Note that these are precisely the bounds used in Example 4 so the prior is as determined in that example where the indexing is row-wise.

3 Measuring Bias in the Prior

Here we specialize the developments discussed in Evans (2015) to the multinomial problem with a Dirichlet prior. Suppose a quantity \(\psi = \Psi(p_1, \ldots, p_k)\) is of interest and there is a need to assess the hypothesis \(H_0 : \Psi(p_1, \ldots, p_k) = \psi_0\). Let \(\pi_{\Psi}\) denote the prior density and \(\pi_{\Psi}(\cdot | f_1, \ldots, f_k)\) denote the posterior density of \(\Psi\), where \((f_1, \ldots, f_k)\) gives the observed cell counts. When \(\Psi(p_1, \ldots, p_k) = (p_1, \ldots, p_k)\), then \(\pi_{\Psi}\) is the Dirichlet(\(\alpha_1, \ldots, \alpha_k\)) density and \(\pi_{\Psi}(\cdot | f_1, \ldots, f_k)\) is the Dirichlet(\(\alpha_1 + f_1, \ldots, \alpha_k + f_k\)) density. The relative belief ratio \(RB_{\Psi}(\psi_0 | f_1, \ldots, f_k)\) is defined as the limiting ratio of the posterior probability of a set containing \(\psi_0\) to the prior probability of this set where the limit is taken as the set converges (nicely) to the point \(\psi_0\). Whenever \(\pi_{\Psi}(\psi_0) > 0\) and \(\pi_{\Psi}\) is continuous at \(\psi_0\), then \(RB_{\Psi}(\psi_0 | f_1, \ldots, f_k) = \pi_{\Psi}(\psi_0 | f_1, \ldots, f_k)/\pi_{\Psi}(\psi_0)\).
As such $RB_{\Psi}(\psi_0 | f_1, \ldots, f_k)$ is measuring how beliefs about $\psi_0$ have changed from a priori to a posteriori and is a measure of evidence concerning $H_0$. If $RB_{\Psi}(\psi_0 | f_1, \ldots, f_k) > 1$, then there is evidence that $H_0$ is true, as belief in the truth of $H_0$ has increased, if $RB_{\Psi}(\psi_0 | f_1, \ldots, f_k) < 1$, then there is evidence that $H_0$ is false, as belief in the truth of $H_0$ has decreased and if $RB_{\Psi}(\psi_0 | f_1, \ldots, f_k) = 1$, then there is no evidence either way.

Given that there is a measure of evidence for $H_0$, it is possible to assess the bias in the prior with respect to $H_0$. For this let $M(\cdot | \psi_0)$ denote the prior predictive distribution of $(f_1, \ldots, f_k)$ given that $\Psi(p_1, \ldots, p_k) = \psi_0$. The bias against $H_0$ is assessed by

$$M(RB_{\Psi}(\psi_0 | f_1, \ldots, f_k) \leq 1 | \psi_0),$$

the prior probability that evidence in favor of $H_0$ will not be obtained when $H_0$ is true. If (14) is large, then there is bias in the prior against $H_0$ and, as such, if evidence against $H_0$ is obtained after seeing the data, then this should have little impact. In essence the ingredients of the study are such that it is not meaningful to find evidence against $H_0$. To measure bias in favor of $H_0$, let $\psi_*$ be a value of $\Psi$ that is just meaningfully different than $\psi_0$. In other words values $\psi$ that differ from $\psi_0$ less than $\psi_*$ does, are not considered as practically different than $\psi_0$. Then the bias in favor of $H_0$ is measured by

$$M(RB_{\Psi}(\psi_0 | f_1, \ldots, f_k) \geq 1 | \psi_*).$$

If (15) is large, then there is bias in favor of $H_0$ and if evidence in favor of $H_0$ is obtained after seeing the data, then this should have little impact. It is shown in Evans (2015) that both (14) and (15) converge to 0 as $n \to \infty$. So bias can be controlled by sample size.

The computation of (14) and (15) can be difficult in certain contexts with the primary issue being the need to generate from the conditional prior predictives of the data. As in the following example, however, great accuracy is typically not required for these computations and so effective methods are available.

**Example 1. (continued) Measuring bias and choosing $\delta$.**

To assess independence between $X$ and $Y$, the marginal parameter

$$\psi = \Psi(p_{11}, p_{12}, \ldots, p_{kk}) = \sum_{i,j} p_{ij} \ln(p_{ij}/p_i p_{-j})$$

is used. Note that (16) is the minimum Kullback-Leibler distance between the $p_{ij}$ values and an element of $H_0$. Furthermore, $\psi = 0$ iff independence holds.

As discussed previously, it is necessary to specify a $\delta > 0$ such that a practically meaningful lack of independence occurs iff the true value $\psi \geq \delta$. One approach is to specify a $\delta$ such that, if $-\delta \leq (p_{ij} - p_i p_{-j})/p_{ij} < \delta$ for all $i$ and $j$, then any such deviation is practically insignificant, as the relative errors are all bounded by $\delta$. Using $\ln(1 + x) \approx x$ for small $x$, this condition implies that $-\delta \leq \psi < \delta$. The range of $\psi$ is then discretized using this $\delta$ and the hypothesis to be assessed is now, because $\psi \geq 0$ always, $H_0 : 0 \leq \psi < \delta$. This assessment is carried out using the relative belief ratios based on the discretized prior
and posterior of $\Psi$ as discussed in Section 5. For the data in this problem we take $\delta = 0.01$ which corresponds to a 1% relative error. So this says that we do not consider independence as failing when the true probabilities differ from probabilities based on independence with a relative error of less than 1%.

With this choice of $\delta$ the issue of bias is now addressed. The prior distribution of the discretized $\Psi$ is determined by simulation. For this, generate the $p_{ij}$ from the elicited prior and compute $\psi$ and the prior probability contents of the intervals for $\psi$ given by $[0, \delta), [\delta, 2\delta), \ldots, ([k - 1]\delta, k\delta)$ where $k$ is determined so as to cover the full range of observed generated values of $\psi$. The plot of the prior density histogram for $\psi$ is provided in Figure 4. For inference the posterior contents of these intervals are also determined via simulating from the posterior based on the observed data. For measuring bias, however, we proceed as follows. Each time a generated $\psi$ satisfies $[0, \delta)$ the corresponding $p_{ij}$ are used to generate a new data set $F_{ij}$ and $RB_{\Psi}([0, \delta) \mid F_{11}, \ldots, F_{kl})$ is determined and note that this requires generating from the posterior based on the $F_{ij}$. The probability $M(RB_{\Psi}([0, \delta) \mid F_{11}, \ldots, F_{kl}) \leq 1 \mid [0, \delta))$ is then estimated by the proportion of these relative belief ratios that are less than or equal to 1. This gives an estimate of the bias against $H_0$. Estimating the bias in favor of $H_0$ proceeds similarly, but now the $F_{ij}$ are generated whenever $\psi \in [\delta, 2\delta)$ is satisfied, as these represent values that correspond to just differing from independence meaningfully.

Clearly this procedure could be computationally quite demanding if highly accurate estimates of the biases are required. In general, however, high accuracy is not necessary. Even accuracy to one decimal place will provide a clear indication of whether or not there is serious bias. In this problem the biases for the elicited prior are estimated to be 0.12 for bias for and 0.02 for bias against.
So while there is some bias in favor of $H_0$, it is not serious and there is virtually no bias against $H_0$. These values depend on the chosen value of $\delta$ but in fact are reasonably robust to this choice. The prior probability content of the interval $[0, 0.01]$ is 0.14 while $[0.01, 0.02]$ contains 0.25 of the prior probability. So there is a reasonable amount of prior probability allocated to effective independence and also to the smallest nonindependence of interest.

4 Checking for Prior-Data Conflict

Anytime a prior is used it is reasonable to question whether or not the prior is contradicted by the data. For the elicitation could be in error, namely, what if the true probabilities lie well outside the intervals obtained. If the data demonstrate this in a reasonably conclusive way, then it would seem incorrect to proceed with an analysis based on this prior unless there was an absolute conviction that the amount of data was sufficient to overwhelm the influence of the prior. Such a situation is referred to as a prior-data conflict and methods exist to check whether or not this exists as well as methods to deal with it.

To check for prior-data conflict we follow Evans and Moshonov (2006) and compute the tail probability

$$M(m(F_1, \ldots, F_k) \leq m(f_1, \ldots, f_k))$$

where $(f_1, \ldots, f_k)$ is the observed value of the minimal sufficient statistic and $M$ is the prior predictive distribution of this statistic with density $m$. Evans and Jang (2011a) prove that quite generally (17) converges to $\Pi(\pi(p_1, \ldots, p_k) \leq \pi(p_{1, true}, \ldots, p_{k, true}))$ as $n \rightarrow \infty$, where $\Pi$ is the prior on $(p_1, \ldots, p_k)$. So (17) is indeed a valid check on the prior.

When the prior is given by the uniform, then a simple computation shows that (17) is equal to 1 and so there is no prior-data conflict. Intuitively, the closer $\tau$ is to 0, then the less information the prior is putting into the analysis. This idea can be made precise in terms of the weak informativity of one prior with respect to another as developed in Evans and Jang (2011b). As such, if prior-data conflict is obtained with the prior specified by a value of $(\xi_1, \ldots, \xi_k, \tau)$, then this prior can be replaced by a prior that is weakly informative with respect to it so that the conflict can be avoided and this entails choosing a value $\tau' < \tau$.

Example 1. (continued) Checking the elicited prior.

For the elicited Dirichlet prior the value of (17) is approximately equal to 1 (to the accuracy of the computations) and so there is definitely no prior-data conflict.

5 Inference

For data $(f_1, \ldots, f_k)$ and Dirichlet($\alpha_1, \ldots, \alpha_k$) prior the posterior, of $(p_1, \ldots, p_k)$ is Dirichlet($\alpha_1 + f_1, \ldots, \alpha_k + f_k$). As such it is easy to generate from the posterior of $\psi$, estimate the posterior contents of the intervals $[(i - 1)\delta, i\delta)$ and then
estimate the relative belief ratios $RB_\Phi(\{ (i-1)\delta, i\delta \} | f_1, \ldots, f_k)$. From this a relative belief estimate of the discretized $\psi$ can be obtained and various hypotheses assessed for this quantity.

As discussed in Evans (2015) the strength of the evidence provided by $RB_\Phi(\psi_0 | f_1, \ldots, f_k)$ is measured by

$$\Pi_\Phi( RB_\Phi(\psi | f_1, \ldots, f_k) \leq RB_\Phi(\psi_0 | f_1, \ldots, f_k) | f_1, \ldots, f_k),$$

namely, the posterior probability that the true value of $\psi$ has a relative belief ratio no greater than the hypothesized value. When $RB_\Phi(\psi_0 | f_1, \ldots, f_k) < 1$, so there is evidence against $\psi_0$, a small value for (18) implies there is strong evidence against $\psi_0$ since there is a large posterior probability that the true value has a larger relative belief ratio than $\psi_0$. When $RB_\Phi(\psi_0 | f_1, \ldots, f_k) > 1$, so there is evidence in favor of $\psi_0$, a large value for (18) indicates there is strong evidence in favor of $\psi_0$ since there is a small posterior probability that the true value has a larger relative belief ratio than $\psi_0$.

Note that when $RB_\Phi(\psi_0 | f_1, \ldots, f_k) > 1$, then the best estimate of $\psi$ in the set $\{ \psi : RB_\Phi(\psi | f_1, \ldots, f_k) \leq RB_\Phi(\psi_0 | f_1, \ldots, f_k) \}$ is $\psi_0$ as it has the most evidence in its favor. Note that while the measure of strength looks like a $p$-value, it has a very different interpretation and it is not measuring evidence.

Given that there is no prior-data conflict with the elicited prior and little or no bias in this prior relative to the hypothesis $H_0$ of independence, we can proceed to inference.

**Example 1. (continued) Inference.**

The posterior of the $p_{ij}$ is the Dirichlet($998.2, 694.2, 146.48, 395.48, 428.48, 96.48, 2918.1, 2651.1, 582.48$) distribution. For the hypothesis $H_0$ of independence between the variables, and using the discretized Kullback-Leibler divergence with $\delta = 0.01$, the value $RB_\Phi(0, \delta) | f_1, \ldots, f_k) = 7.13$ was obtained so there is evidence in favor of $H_0$. For the strength of this evidence the value of (18) equals 1. So the evidence in favor of $H_0$ is of the maximum possible strength. Of course, this is due to the large sample size and the fact that the posterior distribution concentrates entirely in $[0, \delta)$. Note that is a very different conclusion than that obtained by the $p$-value based on the chi-squared test.

### 6 Conclusions

A very natural and easy to use method has been developed for eliciting Dirichlet priors based upon placing single bounds on the individual probabilities that takes into account the dependencies among the probabilities. Of course, there may be more information available, such as upper and lower bounds on many of the probabilities. The price paid for this, however, is a much more complicated region where the bulk of the prior mass is located and even difficulties in determining what that region is. So indeed further research into the development of elicitation algorithms for this family of priors is warranted.

The application of this prior to an inference problem has also been illustrated using a measure of statistical evidence, the relative belief ratio, as a basis for the
inferences. Given that a measure of evidence has been identified, it is possible to assess the bias in the prior before proceeding to inference. Also, the prior has been checked to see if it is contradicted by the data. Finally, it is seen that the assessment of a hypothesis can be different than that obtained by a standard \( p \)-value and, in particular, provide evidence in favor of a hypothesis. Of course, this is based on a well-known defect in \( p \)-values, namely, with a large enough sample a failure of the hypothesis of no practical importance can be detected. The solution to this problem is to say what difference matters and use an approach that incorporates this. Relative belief inferences are seen to do this in a very natural way. The choice of \( \delta \) is not arbitrary but is rather a fundamental characteristic of the application. When such a \( \delta \) can’t be determined it is not a failure of the inference methodology, but rather reflects a failure of the analyst to understand an aspect of the application that is necessary for a more refined analysis to take place.

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