Irreducible $SL_{n+1}$–Representations remain Indecomposable restricted to some Abelian Subalgebras

Paolo Casati

Dipartimento di Matematica e applicazioni
Università di Milano-Bicocca
Via degli Arcimboldi 8, I-20125 Milano, Italy

E–mail: casati@matapp.unimib.it

Abstract

In this paper we show that any irreducible finite dimensional representation of $SL_{n+1}$ remains indecomposable if restricted to $n$–dimensional abelian subalgebras spanned by simple root vectors.

Mathematics Subjects Classification: Primary 22E47 Secondary 17B10
1 Introduction

Surely the finite dimensional irreducible representations of complex simple Lie algebras are one of the most fascinating and studied subjects in the theory of representations. Their beautiful and complicate structure still presents unknown aspects worth to be studied (see [1] and [4] for recent examples). This paper concerns with one of these, namely the restriction of such representations to some subalgebras. More precisely we shall show that any finite dimensional irreducible representation of a complex simple Lie algebra of type $A_n$ remains indecomposable if restricted to some abelian subalgebras of maximal dimension (Theorem 3.9). Such abelian subalgebra $\mathfrak{a}$ can be constructed as follows. Let $\mathfrak{g}$ be the complex simple Lie algebra $A_n$, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the corresponding set of roots. Further for any $\alpha \in \Delta$ let $X_\alpha$ be a basis of $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \forall H \in \mathfrak{h} \}$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a set of simple roots in $\Delta$ and set $Y_{\alpha_i} = X_{-\alpha_i}$, then $\mathfrak{a}$ is the abelian subalgebra of $\mathfrak{g}$ spanned by the vectors $\{Y_{\alpha_{2i+1}}\} (i = 0, \ldots, [n/2])$ and $\{X_{\alpha_j}\} (j = 1, \ldots, [n/2])$, where $[x]$ denotes the integer part of $x$.

Theorem 3.9 is almost trivial for the Lie algebra $A_1$, while for the Lie algebra $A_2$ was proved by Douglas and Premat in [5], and for the remaining simple Lie algebras of rank two $B_2$ and $G_2$ by Premat in [12]. These two papers have played an inspiring role in our work. As far as we know Theorem 3.9 is still unknown for $A_n$ with $n \geq 3$.

The paper is organized as follows. In section 2 we recall some known facts about the simple Lie algebras of type $A$ and their finite dimensional modules, and describe the maximal dimensional abelian Lie algebra $\mathfrak{a}$. This section also devoted to present basis of the finite dimensional irreducible $A_n$--modules found by Littelmann in [10]. In section 3 we find a minimal set of generators for the restriction to the abelian subalgebra $\mathfrak{a}_n$ of the finite dimensional representations of the Lie algebra $A_n$, and prove the main result of this paper: the indecomposableness of such restricted representations.

The author wishes to thank Alejandra Premat for sending her preprint [12], which plays a crucial role in the present work, and Veronica Magenes for discussions about the case concerning the Lie algebra $\mathfrak{sl}(4, \mathbb{C})$.

2 Irreducible finite dimensional $\mathfrak{sl}(n + 1, \mathbb{C})$--modules

In this section we recall some basic facts on $\mathfrak{sl}(n + 1, \mathbb{C})$ and its irreducible finite dimensional representations, and describe the basis of such representations first constructed by Littelmann in [10]. Good references on the structure and representation theory of the complex simple Lie groups and Lie algebras are, for instance, the books [6] [7].

Let $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{C})$ be the simple Lie algebra of all $(n + 1) \times (n + 1)$ complex matrices of zero trace, let $\mathfrak{h}$ be its Cartan subalgebra given by all diagonal matrices in $\mathfrak{sl}(n + 1, \mathbb{C})$, $\mathfrak{h}^*$ its complex dual, and $\Delta = \Delta(\mathfrak{sl}(n + 1, \mathbb{C}), \mathfrak{h}) \subset \mathfrak{h}^*$ the corresponding set of roots. Let $\mathfrak{g} = n^+ \oplus \mathfrak{h} \oplus n^-$ be its decomposition into the direct sum of strictly upper triangular, diagonal, and strictly lower triangular matrices, and $\Delta = \Delta^+ \cup -\Delta^+$ the decomposition of
the set of root such that
\[ n^+ = \sum_{\beta \in \Delta^+} g_\beta, \quad n^- = \sum_{\beta \in -\Delta^+} g_\beta \]
where \( g_\beta = \{ X \in \mathfrak{g} \mid [H, X] = \beta(H)X \forall H \in \mathfrak{h} \} \). We denote by \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) the corresponding set of simple roots and accordingly we fix a Chevalley basis of \( \mathfrak{g} \): \( X_\beta \in \mathfrak{g}_\beta \) and \( Y_\beta \in \mathfrak{g}_{-\beta} \) for \( \beta \in \Delta^+ \), and \( H_\alpha \in \mathfrak{h} \) for \( \alpha \) simple, in such a way that \([X_\alpha, Y_\alpha] = H_\alpha \). The Weyl group of \( \mathfrak{sl}(n + 1, \mathbb{C}) \) is denoted by \( W \), as subgroup of \( GL(\mathfrak{h}^*) \) it is generated by the hyperplane reflections \( s_\lambda : \lambda \mapsto \lambda - \lambda(H_\alpha)\alpha \) for any \( \lambda \in \mathfrak{h}^* \) and \( \alpha \in \Delta \).

Denote by \( \mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{n}^+), \mathcal{U}(\mathfrak{n}^-) \) the universal enveloping algebras of \( \mathfrak{g}, \mathfrak{n}^+, \mathfrak{n}^- \) respectively. (More in general \( \mathcal{U}(\mathfrak{a}) \) will denote the universal enveloping algebra of a given subalgebra \( \mathfrak{a} \) of \( \mathfrak{g} \).) Following Littelmann [10] we use the following abbreviations:

\[
Y^{(k)}_\beta := \frac{Y^k_\beta}{k!} \quad X^{(k)}_\beta := \frac{X^k_\beta}{k!} \quad \left( \frac{H_a}{k} \right) := \frac{H_a(H_a - 1) \cdots (H_a - k + 1)}{k!}.
\]

Fix an ordering \( \{ \gamma_1, \ldots, \gamma_N \} \) of the positive roots \( (N = n(n + 1)/2) \). For \( (n) \in \mathbb{N}^N \) we set:

\[
X^{(n)} := X^{(n_1)}_{\gamma_1} \cdots X^{(n_N)}_{\gamma_N}, \quad Y^{(n)} := Y^{(n_1)}_{\gamma_1} \cdots Y^{(n_N)}_{\gamma_N}.
\]

Fix an ordering \( \{ \alpha_1, \ldots, \alpha_n \} \) of the simple roots. For \( (k) \in \mathbb{N}^n \) we set:

\[
H^{(k)} := \left( \frac{H_{\alpha_1}}{k_1} \right) \cdots \left( \frac{H_{\alpha_n}}{k_n} \right).
\]

(we shall sometime write \( X_\beta, Y_\beta, H_\alpha \) respectively for \( X_{\alpha_1}, Y_{\alpha_1}, H_{\alpha_1} \) for a simple root \( \alpha_1 \)).

Recall that the monomials \( Y^{(m)} H^{(k)} X^{(n)} \) form a Poincaré–Birkhoff–Witt basis of the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \), and the monomials \( X^{(n)} \) and \( Y^{(m)} \) form a P–B–W basis of \( \mathcal{U}^+ = \mathcal{U}(\mathfrak{n}^+) \) respectively \( \mathcal{U}^- = \mathcal{U}(\mathfrak{n}^-) \).

An element of \( \mathfrak{h}^* \) is called a weight. The set \( P = \{ \lambda \in \mathfrak{h}^* \mid \lambda(h_a) \in \mathbb{Z}, \forall \alpha \in \Delta \} \) is said the set of integral weights of \( \mathfrak{g} \). A weight \( \lambda \) of \( P \) is said dominant if \( \lambda(H_\alpha) \geq 0 \) for any simple root \( \alpha \). The complex finite dimensional irreducible representations of \( \mathfrak{sl}(n + 1, \mathbb{C}) \) are parameterized by the dominant integral weights. We denote by \( V(\lambda) \) the finite dimensional irreducible \( \mathfrak{sl}(n + 1, \mathbb{C}) \)–module corresponding to the integral dominant weight \( \lambda \). An element \( \mu \) of \( \mathfrak{h}^* \) is said a weight of an irreducible finite dimensional module \( V(\lambda) \) if the weight space \( V_\mu = \{ v \in V(\lambda) \mid Hv = \mu(H)v \forall H \in \mathfrak{h} \} \) is different from zero. Denote by \( P(\lambda) \) the set of all weights of \( V(\lambda) \) then \( V(\lambda) \) may be decomposed as the direct sum of its weight spaces:

\[
V(\lambda) = \bigoplus_{\mu \in P(\lambda)} V_\mu. \tag{2.1}
\]

Let \( \Pi = \Pi_Y \cup \Pi_X \) a decomposition of the set of simple roots \( \Pi \) such that the \( n \)–dimensional subalgebra spanned by the elements \( \{ X_\alpha, Y_\beta \}_{\alpha \in \Pi_X, \beta \in \Pi_Y} \) is an abelian subalgebra. If \( \alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n - e_{n+1} \) is the usual ordering of the simple roots of \( \mathfrak{sl}(n + 1, \mathbb{C}) \), where \( e_i : \mathfrak{h} \to \mathbb{C} \) denotes the projection of a diagonal matrix onto its \( i \)–th entry, then it easy to see that this decomposition of the set of simple roots \( \Pi \) can be achieved in two ways:
either \( \Pi_\gamma = \{ \alpha_{2i+1} \}_{i=0}^{\infty} \) and \( \Pi_X = \{ \alpha_{2i} \}_{i=1}^{\infty} \}, \) where \([x]\) denote the integer part of \(x\) or the converse case. Since the two choices are equivalent, let us for the sake of concreteness choose in this paper the first one and give the

**Definition 2.1** Let \(a_n\) be the abelian subalgebra of \(sl(n+1, \mathbb{C})\) spanned by the simple root–vectors \(\{X_{\alpha_{2i}}, Y_{\alpha_{2i+1}}\} \), \(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor\).

The aim of this paper is to show how any irreducible \(sl(n+1, \mathbb{C})\)–module \(V(\lambda)\) restricted to the maximal abelian subalgebra \(a_n\) remains indecomposable.

Further since any of such abelian algebra \(a_n\) may be imbedded in a solvable Lie algebra endowed with a non singular \(ad\)–invariant bilinear form \([11\)[3] which is still a subalgebra of \(sl(n+1, \mathbb{C})\), this result provides a way to construct a fairly wide class of indecomposable (and therefore not trivial) finite dimensional modules of solvable quadratic Lie algebras \([2\]. In order to achieve such result we need to consider the basis of the irreducible \((\text{and therefore not trivial})\) finite dimensional modules of solvable quadratic Lie algebras \([1\] and \(2\)). First we introduce the following concepts

**Definition 2.2** A monomial in the \(Y_{\alpha_i}\) is called semi-standard if it is of the form:

\[Y^{(a)} = Y^{(a_1)}_1 \left(Y^{(a_2)}_2 Y^{(a_3)}_3 \right) \cdots \left(Y^{(a_{i-1})}_{i-1} Y^{(a_i)}_i \right)\left(\cdots \left(Y^{(a_{n-1})}_{n-1} \right) Y^{(a_n)}_n \right)\]

where \(a = (a_1, a_2, a_3, \ldots, a_n, \ldots) \in \mathbb{N}^n\). The tuple \(a\) and the monomial \(Y^{(a)}\) are called standard if:

\[a \in S = \{(a) \in \mathbb{N}^n | a_1^2 \geq a_1^1, a_2^2 \geq a_2^1 \geq a_2^3, \ldots, a_i^j \geq a_i^{j-1} \geq \cdots \geq a_i^1, \ldots, a_n^j \geq a_{n-1}^j \geq \cdots \geq a_1^j\}\]

Then we can formulate the following important result due to Littelman.

**Theorem 2.3** \([10\] For a dominant weight \(\lambda\) of \(\mathfrak{g}\), let \(V(\lambda)\) be the corresponding irreducible finite dimensional \(\mathfrak{g}\)–module of highest weight \(\lambda\) and \(u_\lambda \in V(\lambda)\) be a highest weight vector.

Denote by \(\lambda^j_0\) the weight of

\[\left(\left(Y^{(a_1)}_1 \cdots Y^{(a_{i-1})}_{i-1} Y^{(a_i)}_i \right) \cdots \left(Y^{(a_{n-1})}_{n-1} Y^{(a_n)}_n \right)\right) u_\lambda\]

and set

\(\lambda_0^0 := \lambda\), and \(\lambda_0^{i-1} := \lambda^j_0\) for \(1 \leq j \leq n\). Then the elements of \(V(\lambda)\)

\[Y^{(a)} u_\lambda = Y^{(a_1)}_1 \left(Y^{(a_2)}_2 Y^{(a_3)}_3 \right) \cdots \left(Y^{(a_{i-1})}_{i-1} Y^{(a_i)}_i \right)\left(\cdots \left(Y^{(a_{n-1})}_{n-1} \right) Y^{(a_n)}_n \right)\]
with \(a \in S\) such that

\[
\lambda_0^a(H_1) \geq a_1^a \quad \lambda_1^a(H_2) \geq a_2^a \quad \lambda_2^a(H_3) \geq a_3^a \quad \cdots \quad \lambda_{i-1}^a(H_i) \geq a_i^a \quad \cdots \quad \lambda_{n-1}^a(H_n) \geq a_n^a
\]

\[
\lambda_0^1(H_1) \geq a_1^1 \quad \lambda_1^1(H_2) \geq a_2^1 \quad \lambda_2^1(H_3) \geq a_3^1 \quad \cdots \quad \lambda_{j-1}^1(H_j) \geq a_j^1 \quad \cdots \quad \lambda_{k-1}^a(H_k) \geq a_k^a
\]

form a basis \(\Omega_\lambda\) of \(V(\lambda)\).

**Remark 2.4** Let \(\lambda, i = 1, \ldots, n\) be the elements of \(\mathfrak{h}^*\) defined by the relations \(\Lambda_i(\alpha_j) = \delta_{ij}\) where \(\delta_{ij}\) is the usual Kronecker delta. Then if we write the dominant weight \(\lambda\) in the form: \(\lambda = \sum_{i=1}^n m_i \lambda_i\) (with \(m_i \in \mathbb{N}, \ i = 1, \ldots, n\), the conditions (2.3) become:

\[
0 \leq a_i^1 \leq m_1 - 2 \sum_{j=i+1}^n a_i^1 + \sum_{j=i+1}^n a_j^1 \quad i = 1, \ldots, n
\]

\[
a_{k-1}^i \leq a_k^i \leq m_k - 2 \sum_{j=i+1}^n a_k^j + \sum_{j=i+1}^n a_{k-1}^j + \sum_{j=i+1}^n a_{k-1}^1 \quad i = 1, \ldots, n - k + 1 \quad 2 \leq k \leq n - 1
\]

\[
a_{n-1}^n \leq a_n^i \leq m_n + a_{n-1}^n.
\]

Finally observe that in general the complex simple Lie algebras \(\mathfrak{sl}(n+1, \mathbb{C})\) does not have any subalgebra of dimension strictly less than \(n\) with the same property of the subalgebra \(a_0\) discussed above.

Let us indeed consider the first non trivial case, namely the Lie algebra \(\mathfrak{sl}(3, \mathbb{C})\). In this case it is easy to show that there is not an one dimensional subalgebra such that the restriction on it of any irreducible finite dimensional representation of \(\mathfrak{sl}(3, \mathbb{C})\) remains indecomposable. Let \(X\) be indeed a basis for such latter algebra. Then \(X\) must act as a single Jordan block in any irreducible finite dimensional representation of \(\mathfrak{sl}(3, \mathbb{C})\). In particular if \(\pi : \mathfrak{sl}(3, \mathbb{C}) \to \text{End}(\mathbb{C}^3)\) is the irreducible representation with \(V(\lambda) = V(\lambda_1)\) (so that \(\dim_{\mathbb{C}}(V(\lambda) = 3)\) then, since the trace of \(\pi(X)\) is zero, it must exist a \(\xi \in \text{Aut}(\mathbb{C}^3)\) such that

\[
\xi \pi(X) \xi^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or eq.} \quad \xi \pi(X) \xi^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

I.e., \(X\) can be take equal to \(Y_1 + Y_2\) (or eq. to \(X_1 + X_2\)). But the restriction to these one-dimensional sub–algebras of the module \(V(\lambda) = V(\lambda_1 + \lambda_2)\) is not indecomposable because on it in both cases \(X^3 = 0\) while \(\dim(V(\lambda)) = 8\).
3 \( V(\lambda) \) as indecomposable \( \alpha_n \)-module

Let us fix a dominant integral weight \( \lambda \) of \( \mathfrak{sl}(n+1, \mathbb{C}) \). We shall show in this section that the \( \mathfrak{sl}(n+1, \mathbb{C}) \)-module \( V(\lambda) \) viewed as \( \alpha_n \)-modules is indecomposable.

We first need to find a (minimal) set of generators for the \( \alpha_n \)-modules \( V(\lambda) \).

**Definition 3.1** Let \( V \) be a \( \alpha_n \)-modules, a subset of elements \( \{v_1, \ldots, v_m\} \) in \( V \) is said to be a set of generators of \( V \) if \( V = \mathcal{U}(\alpha_n)\{v_1, \ldots, v_m\} \). The set is called a minimal set of generators if fewer than \( m \) vectors will not generate \( V \). In the case of the \( \alpha_n \)-modules \( V(\lambda) \) a set of generators \( \mathfrak{B} \) is a set of homogeneous generators if any element in \( \mathfrak{B} \) is a \( \mathfrak{sl}(n+1, \mathbb{C}) \)-weight vector.

**Theorem 3.2** Let \( \mathcal{G}_\lambda \) be the subset of \( \mathcal{L}_\lambda = \{a \in S\} Y(\alpha)u_1 \in \mathcal{L}_\lambda \) given by:

\[
\mathcal{G}_\lambda = \left\{ g \in \mathcal{L}_\lambda \mid \begin{array}{l}
a^{2j} = \lambda^{2j}_{2j-1}(H_{2j}) \\
a^{2j+1}_{2j} \neq 0 \Rightarrow a^{2j}_{2j-1} \neq 0 \\
a^{2j+1}_{2j} = 0 \\
a^{2j+1}_{2j} \neq 0 \Rightarrow a^{2j}_{2j-1} \neq 0 \\
\lambda^{2j+1}_{2j-1}(H_{2j}) \neq 0 \text{ and } \lambda^{2j+1}_{0}(H_{1}) = 0, \lambda^{2j+1}_{2j-r-1}(H_{2r}) = a^{2j+1}_{2r}, \quad 1 \leq r < i \\
\Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \ldots 2j-1 \\
\end{array} \right\}
\]

then the corresponding subset \( \mathcal{G}_\lambda \) of the Littelmann basis \( \mathcal{L}_\lambda \) is a set of homogeneous \( \alpha_n \)-generators of \( V(\lambda) \).

**Proof** Set \( \mathcal{L}_\lambda^0 = \{a \in \mathcal{L}_\lambda \mid a^1 = 0\} \). Since \( X_1 \) belongs to \( \alpha_n \), we have of course only to prove that acting with \( \alpha_n \) we may construct the subset \( \mathcal{L}_\lambda^1 = \{Y(\alpha)u_1|a \in \mathcal{L}_\lambda^0\} \) of \( \mathcal{L}_\lambda \). We divide the proof in four steps.

1. First, if we define

\[
\mathcal{L}_\lambda^1 = \left\{ a \in \mathcal{L}_\lambda^0 \mid \begin{array}{l}
a^{2h+1} = 0, \quad a^{2h+1} \neq 0 \Rightarrow a^{2h} \neq 0 \\
\lambda^{2j+1}_{2j-1}(H_{2j}) \neq 0 \text{ and } \lambda^{2j+1}_{0}(H_{1}) = 0, \lambda^{2j+1}_{2j-r-1}(H_{2r}) = a^{2j+1}_{2r}, \quad 1 \leq r < i \\
\Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \ldots 2j-1 \\
\end{array} \right\}
\]

and set \( \mathcal{L}_\lambda^1 = \{Y(\alpha)|a \in \mathcal{L}_\lambda^1\} \), then \( \mathcal{L}_\lambda^1 \subset \mathcal{U}(\alpha_n)(\mathcal{G}_\lambda) \).

Let us consider indeed for any \( 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \) the subsets \( \mathcal{L}_\lambda^1 \) of \( \mathcal{L}_\lambda^1 \) given by:

\[
\mathcal{L}_\lambda^1 = \left\{ \begin{array}{l}
Y(\alpha)u_1 \in \mathcal{L}_\lambda^1 \mid \exists Y(\alpha^0)u_1 \in \mathcal{G}_\lambda \\
(a^h_{2l+1} = a^{h}_{2l+1}, \quad l = 0, \ldots \left\lfloor \frac{n-1}{2} \right\rfloor, \quad 2l + 1 \leq h \leq n \\
(a^h_{2k} = a^h_{2k}, \quad k = j, \ldots \left\lfloor \frac{n}{2} \right\rfloor, \quad 2l \leq h \leq n.
\end{array} \right\}
\]
and the corresponding filtration of $\mathbb{U}_\lambda^1$: 

$$\mathcal{O}_\lambda = \mathbb{U}_2^1 \subset \cdots \subset \mathbb{U}_2^{1} \subset \mathbb{U}_2^{[\frac{1}{2}]} \subset \mathbb{U}_2^{1} = \mathbb{U}_\lambda^1.$$ 

Obviously it suffices to show that $\mathbb{U}_2^{1} \subset \mathcal{U}(a_n)(\mathbb{U}_2^{1})$ for any $1 \leq j \leq \left[ \frac{n}{2} \right]$. We shall do it (for a fixed index $j$) by induction over the partial ordering “$\leq_j$” of $\mathcal{L}_\lambda^0$ (and of $\mathcal{O}_\lambda^0$ as well) given by the relations

$$a \leq_j b \Leftrightarrow a^j_2 - a^j_{2j+1} \leq b^j_2 - b^j_{2j+1} \quad i = 2j + 1, \ldots, n.$$ 

With respect to this ordering the minimal elements in $\mathbb{U}_2^{1}$ are those $Y^{(a)}u_{\lambda}$ with $a^j_2 - a^j_{2j+1} = -\lambda^j_2(H_{2j+1})$, $2j + 1 \leq i \leq n$. For any of this element there exists a positive integer number $k$ (namely $k = a^j_2 - a^j_{2j+1}(H_{2j})$) such that the element $Y^{(g(a))}u_{\lambda}$ with $g(a)^j_2 = a^j_2 + k$ and $g(a)^j_{2j+1} = a_{2j+1}^j$ if $(i, l) \neq (2j, 2j)$ belongs to $\mathcal{O}_\lambda$ and (using the $\mathfrak{sl}(2, \mathbb{C})$–representation theory and $[10]$)

$$X^k_{2j} \cdot Y^{(g(a))}u_{\lambda} = c^j_k(a) Y^{(a)}u_{\lambda} = \prod_{i=1}^{k-1} \left( \sum_{h=1}^{n-2j} a^{j+h}_{2j+1} - 2 \sum_{h=1}^{n-2j} a^{j+h}_{2j} + \sum_{h=0}^{n-2j} a^{j+h}_{2j+1} - a^{j+h}_{2j} + k + i \right) Y^{(a)}u_{\lambda}$$

hence, since in our Hypothesis the coefficients $c^j_k(a)$ are always different from zero, any minimal element of $\mathbb{U}_2^{1}$ belongs to $\mathcal{U}(a_n)(\mathbb{U}_2^{1})$. Suppose now by induction hypothesis that we have constructed any elements $Y^{(b)}u_{\lambda} \in \mathbb{U}_2^{1}$ for any $b < a$. Since there exists a tuple $a_{\mathbb{U}_2^{1}}$ such that $(a_{\mathbb{U}_2^{1}})^j_2 = a^j_2$, if $(i, l) \neq (2j, 2j)$ and $Y^{(a_{\mathbb{U}_2^{1}})}u_{\lambda} \in \mathbb{U}_2^{1}$, we have

$$X^k_{2j} \cdot Y^{(a_{\mathbb{U}_2^{1}})}u_{\lambda} = c^j_k(a_{\mathbb{U}_2^{1}}) Y^{(a)}u_{\lambda} + \sum_{b < a} c_b Y^{(b)}u_{\lambda}$$

which shows (being again $c^j_k(a_{\mathbb{U}_2^{1}}) \neq 0$) that also $Y^{(a)}u_{\lambda}$ belongs to $\mathcal{U}(a_n)(\mathbb{U}_2^{1})$.

2. Define now

$$\mathcal{L}_\lambda^2 = \left\{ a \in \mathcal{L}_\lambda^0 \mid \begin{array}{l}
\lambda^j_{2j+1}(H_{2j}) \neq 0 \text{ and } \lambda^j_{2j+1}(H_{2j}) = 0, \lambda^j_{2j+1}(H_{2j}) = a^j_{2j+1}, \quad 1 \leq r < i \\
\Rightarrow a^j_{2i-1} = \lambda^j_{2i-1}(H_{2i-1}) \quad i = 1, \ldots, 2j - 1 \\
\end{array} \right\}$$

and set $\mathbb{U}_\lambda^2 = \left\{ Y^{(a)}u_{\lambda} \mid a \in \mathcal{L}_\lambda^2 \right\}$ then $\mathbb{U}_\lambda^2 \subset \mathcal{U}(a_n)(\mathbb{U}_\lambda^2)$.

For any $1 \leq s \leq \left[ \frac{n-1}{2} \right]$ let $\mathcal{O}_s^2$ be the set

$$\mathcal{O}_s^2 = \left\{ Y^{(a)}u_{\lambda} \in \mathbb{U}_\lambda^2 \mid a^j_{2k} \neq 0 \Rightarrow a^j_{2k-1} \neq 0, \quad k = s, \ldots, \left[ \frac{n-1}{2} \right] \right\}$$
and consider the corresponding filtration of $\mathfrak{U}^2_{s,t}$:

$$\mathfrak{U}^1_{s,t} \subset \mathfrak{U}^2_{s,t} \subset \cdots \subset \mathfrak{U}^2_{s-1} \subset \mathfrak{U}^2_{s} \subset \mathfrak{U}^2_{s+1} = \mathfrak{U}^2_{s,t}.$$ 

Again it will suffice to show that $\mathfrak{U}^2_{s} \subset \mathcal{U}(x_0)^{(2)}(\mathfrak{U}_{s-1}^1)$ for any $2 \leq s \leq \frac{n+1}{2} + 1$. We shall still do it by induction. Indeed consider first an element in $\mathfrak{U}_s^2$ of the form $Y^{(a)}u_{\lambda} = (Y^{(a_2^1)})(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda}$ then the element $Y^{(b(a))}u_{\lambda}$ with the tuple $b(a)$ given by the relations $b(a)_{2s} = a_{2s}^2 + 1$, $b(a)_{2s+1} = a_{2s+1}^2 - 1$, $b(a)_{2s+1}^2 = 0$ and $b(a)_{j} = a_{j}$ otherwise, belongs to $\mathfrak{U}^2_{s-1}$. Further from the relation \([10]\)

$$Y_{2s+1} \cdot Y^{(b(a))}u_{\lambda}$$

$$= p(1, a_{2s}^2 + 1, a_{2s+1}^2, 0)(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda}$$

$$+ p(1, a_{2s}^2 + 1, a_{2s+1}^2, 1)(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda},$$

where

$$p(a, b, c, d) = \binom{a + c - b}{a - d} \quad a, b, c, d \in \mathbb{N} \quad d \leq b$$

and to have binomial coefficients also available for negative integers, following Littelmann we used the definition:

$$\binom{a}{b} = \lim_{t \to 0} \frac{\Gamma(a + 1 + t)}{\Gamma(b - a + 1 + t) \Gamma(b + 1 + t)};$$

it follows that $Y^{(a)}u_{\lambda}$ belongs to $\mathcal{U}(x_0)^{(2)}(\mathfrak{U}_{s-1}^1)$ because $p(1, a_{2s}^2 + 1, a_{2s+1}^2, 1) = 1$ (but also see \([10]\) remark 7) and both $Y^{(b(a))}u_{\lambda}$ and

$$(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda}$$

are in $\mathcal{U}(x_0)^{(2)}(\mathfrak{U}_{s-1}^1)$. Let us now consider an element in $\mathcal{L}^2_{s}$ of the type

$$Y^{(a_2^s)} Y^{(a_2^{s+1})} Y^{(a_2^{s+2})}(\cdots)u_{\lambda},$$

since by induction Hypothesis $Y^{(b)}u_{\lambda}$, with $b_{2s}^2 = a_{2s}^2 + 1$, $b_{2s+1}^2 = a_{2s+1}^2 - 1$, $b_{2s+1}^2 = k$ and $b_j = a_j$ otherwise, belongs to $\mathcal{U}(x_0)^{(2)}(\mathfrak{L}_{s-1}^2)$ from

$$Y_{2s+1} \cdot Y^{(b)}u_{\lambda}$$

$$= p(1, a_{2s}^2 + 1, a_{2s+1}^2, 0)(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda}$$

$$+ p(1, a_{2s}^2 + 1, a_{2s+1}^2, 1)(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda},$$

it follows that also

$$(\cdots)(Y^{(a_2^s)} Y^{(a_2^{s+1})})Y^{(a_2^{s+2})}(\cdots)u_{\lambda}$$

belongs to $\mathcal{U}(x_0)^{(2)}(\mathfrak{L}_{s-1}^2)$.
3. Let us now define \( L^3_3 \) as:

\[
L^3_3 = \left\{ a \in L_0 \mid \begin{array}{l}
\lambda^{2j+1}_{2i-1}(H_{2i}) \neq 0 \text{ and } \lambda^{2j+1}_{0}(H_1) = 0, \lambda^{2j+1}_{2i-1}(H_{2r}) = a^{2j+1}_{2r}, \quad 1 \leq r < i \\
\Rightarrow a^{2j}_{2i-1} = \lambda^{2j}_{2i-2}(H_{2i-1}) \quad i = 1, \ldots, 2j - 1 \quad j = 1, \ldots, \left[ \frac{n+1}{2} \right]
\end{array} \right\}
\]

and set \( \Omega^3 = \{ Y^{(a)} u_1 \mid a \in L^3_2 \} \) then \( \Omega^3 \subseteq \mathcal{U}(a_0)(\Omega^2) \).

Defining for any \( 1 \leq s \leq \left[ \frac{n-1}{2} \right] \) the sets

\[
\Omega^s_j = \{ Y^{(a)} u_j \in \Omega^3 \mid a^{s+1}_{j} = 0 \quad j = s, \ldots, \left[ \frac{n-1}{2} \right]\}
\]

we have the filtration of \( L^3 \):

\[
\Omega^2 = \Omega^3 \subseteq \cdots \subseteq \Omega^3 \subseteq \cdots \subseteq \Omega^3 \left[ \frac{n-1}{2} \right] \subseteq \Omega^3 \left[ \frac{n-1}{2} \right] + 1 = \Omega^3.
\]

Once again it suffices to prove that \( \Omega^3 \subseteq \mathcal{U}(a_0)(\Omega^3_{s-1}) \) for any fixed \( s \). We proceed by induction. Let us first consider an element of \( \Omega^3 \) of the type

\[
Y^{(a)} u_1 = \left( Y^{(a)}_{2s} \right) \left( Y^{(a)}_{2s+1} \right) \cdots \left( Y^{(a)}_{2s+1} \right) Y_{2s+1}^{(a^{s+1}_{s+1})} Y_{2s+1}^{(a^{s+1}_{s+1})} Y_{2s+1}^{(a^{s+1}_{s+1})} Y_{2s+1}^{(a^{s+1}_{s+1})} Y_{2s+1}^{(a^{s+1}_{s+1})} Y_{2s+1}^{(a^{s+1}_{s+1})} \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) u_1
\]

then the element

\[
Y^{(b)} u_1 \text{ with } b(a)_{2s-l} = a_{2s-l}^2 + 1, 0 \leq l \leq 2s - 1, \text{ and } b(a)_{2s-l} = a_{2s-l}^2 + 1, -1 \leq l \leq 2s - 1, \text{ and } b(a)_{2s-l}^2 = a_{2s-l}^2 \text{ otherwise, belongs to } \Omega^s_{s-1}, \text{ and we have}
\]

\[
Y_{2s+1} \cdot Y^{(b)} u_1 = p(1, a_{2s+1}^2 + 1, a_{2s+1}^2 - 1, 0) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) u_1
\]

\[
+ p(1, a_{2s+1}^2 + 1, a_{2s+1}^2 - 1, 1) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) u_1
\]

\[
= \ldots
\]

\[
= \sum_{k=1}^{h} p_k(a) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) u_1
\]

\[
+ q_h(a) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) u_1
\]

\[
= \sum_{k=1}^{h} p_k(a) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) \left( \cdots \right) u_1
\]

where

\[
p_k(a) = p(1, a_{2s+1-k}^2 + 1, a_{2s-k+2}^2 - 1, 0) \prod_{l=1}^{k-1} p(1, a_{2s+1-l}^2 + 1, a_{2s-l+2}^2 - 1, 1)
\]

\[
q_h(a) = \prod_{l=1}^{k} p(1, a_{2s+1-l}^2 + 1, a_{2s-l+2}^2 - 1, 1)
\]

which show that \( Y^{(a)} u_1 \in \mathcal{U}(a_0)(\Omega^3_{s-1}) \) because \( q_2(a) \neq 0 \) and all other elements belong to \( \mathcal{U}(a_0)(\Omega^3_{s-1}) \). If we now consider an element \( Y^{(a)} u_1 \) of the type
Finally we can show that we have already constructed all the elements of computations done before we have
\[ Y^{(a)}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \]
then by induction Hypothesis the element \( Y^{(b)(a)}_{2s+1} \) with \( b(a)^{2s}_{2s-l} = a^{2s}_{2s-l} + 1, 0 \leq l \leq 2s-1 \), \( b(a)^{2s+1}_{2s-l} = a^{2s+1}_{2s-l} - 1 \), \( -1 \leq l \leq 2s-1 \), \( b(a)^{b}_{l} = a^{b}_{l} \) otherwise, belongs to \( \mathcal{U}_{s-1} \), and with the same computations done before we have
\[
Y^{(b)(a)}_{2s+1} : Y^{(b)(a)}_{2s+1} \cup \mathcal{U}^{(a)}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \]
which implies (by induction) \( Y^{(a)} \cup \mathcal{U}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \).

4. Finally we can show that \( \mathcal{U}^{0}_{s} \subset \mathcal{U}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \). The computations are similar to those done in the previous step. We define indeed for any \( 1 \leq s \leq \left\lfloor \frac{a+1}{2} \right\rfloor \) the sets
\[
\mathcal{U}^{0}_{s} = \left\{ Y^{(a)} \cup \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s-1} \subset \mathcal{U}^{0}_{s-1} \subset \mathcal{U}^{0}_{s-1} \right\}
\]
and consider the corresponding filtration of \( \mathcal{U}^{0}_{s} \)
\[
\mathcal{U}^{3}_{s} = \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s} \subset \mathcal{U}^{0}_{s} \]
Again, we need only to prove (always by induction) that \( \mathcal{U}^{0}_{s} \subset \mathcal{U}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \) for any \( 1 \leq s \leq \left\lfloor \frac{a+1}{2} \right\rfloor \).

For a fixed \( i, 1 \leq i \leq 2j-1 \), let \( Y^{(a)} \cup \mathcal{U}^{0}_{s} \) be of the type
\[
Y^{(a)} \cup \mathcal{U}^{0}_{s} = \left( Y^{(a)}_{2s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \cup \mathcal{U}^{0}_{s} \right) \]
then the element \( Y^{(b)(a)} \cup \mathcal{U}^{0}_{s} \) with \( b(a)^{2s}_{2s-l} = a^{2s}_{2s-l} + 1, 0 \leq l \leq 2s-2i \), \( b(a)^{2s+1}_{2s-l} = a^{2s+1}_{2s-l} - 1 \), \( 0 \leq l \leq 2s-2i \), \( b(a)^{b}_{l} = a^{b}_{l} \) otherwise, belongs to \( \mathcal{U}^{0}_{s-1} \) and we have with the same computations of the previous step and the results of (10):
\[
Y^{(b)(a)}_{2s+1} : Y^{(b)(a)}_{2s+1} \cup \mathcal{U}^{(a)}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \]
which implies that \( Y^{(a)} \cup \mathcal{U}^{0}_{s} \) belongs to \( \mathcal{U}(\mathcal{U}) \cup \mathcal{U}^{(b)(a)}) \). Now suppose by induction Hypothesis that we have already constructed all the elements of \( \mathcal{U}^{0}_{s} \) with \( a^{2s+1}_{2s} = \lambda^{2s+1}_{2s-1} \) of the type
\[
\lambda^{2s+1}_{2s-1}(H_{2s-1}) + a^{2s}_{2s-1} + k.
\]
Then for any element $Y^{(a)}u_1 \in \mathcal{U}_s^0$ of the type
\[
Y^{(a)}u_1 = \left( Y_2^{(a_2^s)} \cdots \right) \left( \cdots \right) \left( Y_1^{(a_1^s)} \right) \left( \sum_{l=0}^{2s-1} Y^{(a_{2s-1}^l)} \right) \cdots \left( \sum_{l=0}^{2s+1} Y^{(a_{2s+1}^l)} \right) \cdots u_1
\]
the element $Y^{(b)}u_1$ with $b(a)^{2s}_{2s-l} = a_{2s-l}^2$, $0 \leq l \leq 2s - 2i$, $b(a)^{2s+1}_{2s+1-l} = a_{2s+1-l}^2 - 1$, $0 \leq l \leq 2s - 2i + 1$, $b(a)_{i}^h = a_{i}^h$ otherwise, belongs to $\mathcal{U}_s^0$ and we have
\[
Y_{2s+1} \cdot Y^{(b)}u_1 = \sum_{k=1}^{2s+2i+1} \left( \Pi_{i=0}^{2s-i} Y^{(a_{2s-i}^l)} \right) \left( \Pi_{i=k+1}^{2s+1-l} Y^{(a_{2s+1-i}^l-1)} \right) \left( \sum_{k=2i}^{2s+1} Y^{(a_{2s+1-i}^l)} \right) \left( \Pi_{i=0}^{2s-1-l} Y^{(a_{2s-i}^l)} \right) u_1
\]
which once again implies that $Y^{(b)}u_1 \in \mathcal{U}(a_n)(\mathcal{U}_s^0)$. This closes the proof of the Theorem.

In the first non trivial case beyond that treated by Douglas and Premat [5] namely the restriction of irreducible finite dimensional $\mathfrak{sl}(4, \mathbb{C})$ module $V(\lambda)$, $\lambda = n\Lambda_1 + m\Lambda_2 + p\Lambda_3$ to the abelian three dimensional Lie algebra spanned by the element $Y_1, Y_2, Y_3$ the set of generators $\mathfrak{g}_3$ is:
\[
\mathfrak{g}_3 = \left\{ Y_2^{m-j+i+h} Y_1^i Y_3^j Y_2 u_\lambda \right\} 0 \leq j \leq m \quad 0 \leq i \leq h \leq p \quad 0 \leq i \leq j + n \quad j \neq 0 \Rightarrow i \neq 0
\]
if $\lambda = n\Lambda_1 + m\Lambda_2 + p\Lambda_3$ with $n > 0$ and:
\[
\mathfrak{g}_3 = \left\{ Y_2^{m+h} Y_1 Y_3^{n+h} Y_2 u_\lambda \right\} 0 \leq h \leq p
\]
if $\lambda = m\Lambda_2 + p\Lambda_3$.

Although we do not need this fact in order to prove that the $a_n$–module $V(\lambda)$ is indecomposable, let us first show that the set of generators $\mathfrak{g}_3$ is a minimal set of generators. We begin with

**Lemma 3.3** No proper subset $\mathfrak{g}_3'$ of $\mathfrak{g}_3$ ($\mathfrak{g}_3' \subsetneq \mathfrak{g}_3$) generates $\mathfrak{g}_3$.

**Proof** It suffices to show that any expression of the form
\[
\sum_{g \in \mathfrak{g}_3} P_g(X_{2j}, Y_{2j+1}) Y^{(ag)}u_1 = \sum_{g \in \mathfrak{g}_3} P_g(X_{2j}, Y_{2j+1}) Y^{(ag)}u_1 \neq 0 \quad \forall g \in \mathfrak{g}_3
\]
where $P_g(X_{2j}, Y_{2j+1})$ are non trivial polynomials in the operators $X_{2j}, j = 1, \cdots, \left[ \frac{n}{2} \right]$, $Y_{2j+1}$, $i = 1, \cdots, \left[ \frac{n-1}{2} \right]$, does not belong to the linear span $\langle \mathfrak{g}_3 \rangle$ of $\mathfrak{g}_3$.

Let us denote by $V(\lambda)^-$ the linear span of all element of the Littelmann basis with $(a_k)^{2j}_{2j} < \lambda^{2j}_{2j-1}(H_{2j})$, since for every element of $\mathfrak{g}_3$ yields $a^{2j}_{2j} = \lambda^{2j}_{2j-1}(H_{2j})$, we have $V(\lambda)^- \cap \langle \mathfrak{g}_3 \rangle = \{0\}$. 

\[10\]
Now from the proof of Theorem 3.2 point 1, for any element $Y^{(a)}u_1$ in $S$, and any operator $X_{2j}$, we have $X_{2j}Y^{(a)}u_1 = \sum c_i Y^{(a_i)}u_1 \in V(\lambda^-)$. Therefore it remains only to consider those combinations of the type (3.1) where there exists at least a monomial which contains only operators of odd index. For any such monomial $P$ if $V(\lambda)^+$ is a subspace of $V(\lambda)$ such that $V(\lambda) = V(\lambda)^+ \oplus (V(\lambda)^- \oplus (S_3))$, then from the proof of Theorem 3.2 points 3. and 4, it follows that for any $g \in S_3, P_g = v^p_g + w^p_g$ with $v^p_g \in V(\lambda)^+$, $w^p_g \in (V(\lambda)^- \oplus (S_3))$ and $v^p_g \neq 0$, moreover if $g' \neq g, g' \in S_3$ or $P \neq Q$ then $v^p_g$ is linear independent from $v^Q_g$. But then for any expression of type (3.1) where there exist at least a monomial which is a product of only the operators $Y_{2j+1} (i = 1, \ldots, \left[\frac{n+1}{2}\right])$ we have
\[
\sum_{g \in G} P^g(X_{2j}, Y_{2j+1}) Y^{(a)}u_1 \notin (V(\lambda)^- \oplus (S_3))
\]

\[\square\]

**Theorem 3.4** The set $S_3$ is a minimal set of generators.

**Proof** Let $\{w_1, \ldots, w_k\}$ be another set of generators, then for all $1 \leq l \leq k$, choosing any ordering $G_3 = \{1, \ldots, \#(G_3)\}$ (where $\#(S)$ denotes the number of elements in the set $S$) of the set $G_3$, we have:
\[
w_l = \sum_{g \in G_3} a_{lg} Y^{(a)}u_1 + \sum_{g \in G_3} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(a)}u_1
\]
where $a_{lg} \in \mathbb{C}$ and $P_{lg}(X_{2j}, Y_{2j+1})$ are polynomials in the operators $X_{2j}, j = 1, \ldots, \left[\frac{n}{2}\right]$, $Y_{2j+1}, i = 1, \ldots, \left[\frac{n+1}{2}\right]$ without constant term. Since the set $\{w_1, \ldots, w_k\}$ generates $V(\Lambda)$ we may obtain acting on it the elements of $S_3$. Let $T = \{w_r \mid a_{rg} \neq 0 \text{ for some } g \in G_3\}$ and $T' = \{j \mid 1 \leq j \leq k \mid w_r \in S\}$. Let $g \in G_3$. Then
\[
Y^{(a)}u_1 = \sum_{i \in T} b_{gi} \left(\sum_{g' \in G_3} a_{lg'} Y^{(a)}u_1\right) + \sum_{g' \in G_3} P'_{lg'}(X_{2j}, Y_{2j+1}) Y^{(a)}u_1
\]
where $P'_{lg'}$ are polynomials in the variables $X_{2j}, j = 1, \ldots, \left[\frac{n}{2}\right], Y_{2j+1}, i = 1, \ldots, \left[\frac{n+1}{2}\right]$ without constant term. From Proof of Lemma 3.3 it follows that $\sum_{g' \in G_3} P'_{lg'}(X_{2j}, Y_{2j+1}) Y^{(a)}u_1$ can not be equal to any combination of elements of $S_3$. Hence
\[
Y^{(a)}u_1 = \sum_{i \in T} b_{gi} \left(\sum_{g' \in G_3} a_{lg'} Y^{(a)}u_1\right).
\]
This implies that if we put $B = (b_{gi})_{g \in G_3}$ and $A = (a_{lg})_{i \in T}$, then $BA$ is the identity matrix. Hence $k \geq \#(T) \geq \text{rank}(B) \geq \#(G_3)$, so $S_3$ is a minimal set of generators. The argument of this proof is due to Premat [12].

\[\square\]
Corollary 3.5 Let \( \mathcal{W} = \{w_1, \ldots, w_k\} \) be a set (non necessarily minimal) of generators, then there exist a injective map \( \phi_{\mathcal{W}} : \mathfrak{G}_A \rightarrow \mathcal{W} \), such that for every \( Y(a_k)u_\lambda \in \mathcal{W} \):

\[
Y(a_k)u_\lambda \mapsto w_{Y(a_k)u_\lambda} = \phi_{\mathcal{W}}(Y(a_k)u_\lambda) = a_g Y(a_k)u_\lambda + \sum_{g' \in \mathcal{G}_A} P_{gg'}(X_{2j}, Y_{2j+1}) Y(a_k')u_\lambda
\]  

(3.2)

for some \( a_g \in \mathbb{C}, a_g \neq 0 \), where \( P_{gg'} \) are polynomials in the variables \( X_{2j}, j = 1, \ldots, \frac{g-1}{2} \), \( Y_{2i+1}, i = 1, \ldots, \left[ \frac{g}{2} \right] \), and the polynomial \( P_{gg} \) has no constant term.

Proof In the proof of Theorem 3.4 we have shown that the elements of \( \mathcal{W} \) can be written in the form

\[
w_i = \sum_{g \in \mathcal{G}_A} a_g Y(a_k)u_\lambda + \sum_{g' \in \mathcal{G}_A} P_{lg}(X_{2j}, Y_{2j+1}) Y(a_k')u_\lambda
\]

where \( A = (a_{lt})_{t=1,\ldots,\#(\mathcal{G}_A)} \) is a matrix of rank at least \( \#(\mathcal{G}_A) \) (recall that \( \#(\mathcal{W}) \leq \#(\mathcal{G}_A) \)). This implies that for any \( g \in \{1, \ldots, \#(\mathcal{G}_A)\} \) we can construct a map \( \phi : \{1, \ldots, \#(\mathcal{G}_A)\} \rightarrow \{1, \ldots, \#(\mathcal{G}_A)\} \) such that for any \( g \in \{1, \ldots, \#(\mathcal{G}_A)\} \), \( a_{\phi(g),g} \) is different from zero and \( g \neq g' \) implies \( \phi(g) \neq \phi(g') \). Then the map

\[
\mathfrak{G}_A \rightarrow \mathcal{W}
\]

\[
Y(a_k)u_\lambda \mapsto \phi_{\mathcal{W}}(Y(a_k)u_\lambda) = a_{\phi(g),g} Y(a_k)u_\lambda + \sum_{g' \in \mathcal{G}_A} a_{\phi(g),g'} Y(a_k')u_\lambda + \sum_{g' \in \mathcal{G}_A} P_{\phi(g),g'}(X_{2j}, Y_{2j+1}) Y(a_k')u_\lambda
\]

is the searched map.

\( \square \)

Proposition 3.6 Any set \( \mathcal{W} = \{w_1, \ldots, w_k\} \) of homogeneous generators contains an element \( w_{\overline{g}} \) such that:

\[
w_{\overline{g}} = a_{\overline{g}g} \quad a_{\overline{g}} \neq 0 \in \mathbb{C}
\]

where \( \overline{g} \) is the element of the set of generators \( \mathfrak{G}_A \) given by

\[
\overline{g} = Y_{2j}^{(i_1)} \cdots Y_{2j}^{(i_{l_1})}(H_{2j}) \cdots Y_{2j}^{(i_2)}(H_{2j}) \cdots Y_{2j}^{(i_1)}(H_{2j}) u_\lambda.
\]

Proof The \( \mathfrak{sl}(n+1, \mathbb{C}) \)–weight \( \mu_{\overline{g}} \) of the element \( \overline{g} \) is \( \mu_{\overline{g}} = \lambda - \sum_{i=1}^{\frac{g}{2}} \alpha_{2i}^{(H_{2i})} \) since

\[
\mu_{\overline{g}} = s_{\alpha_{2j}^{(H_{2j})}} \cdots s_{\alpha_{2j}^{(H_{2j})}}(\lambda)
\]

we have \( \dim(V_{\overline{g}}) = 1 \), but then the claim follows from Corollary 3.5 and the first part of this proposition.
Observe that we do not need the fact that $\mathfrak{g}_{\lambda}$ is a minimal set of generators in order to prove Proposition 3.6. The simple fact that $\mathfrak{g}$ is a set of generators implies

**Lemma 3.7** Let $\mathfrak{B} = \{w_1, \ldots, w_k\}$ be a set (non necessarily minimal) of generators then there exists an $w_\mathfrak{g}$ in $\mathfrak{B}$ such that

$$w_\mathfrak{g} = \overline{a} \overline{g} + \sum_{g \in \mathfrak{g}, g \neq \overline{g}} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(a_k)u_k}$$

with $\overline{a}$ a complex number different from zero.

**Proof** Since:

$$\overline{g} \not\in X_{2j}(V(\lambda)) \quad j = 1, \cdots, \left[\frac{n}{2}\right] \quad \overline{g} \not\in Y_{2j+1}(V(\lambda)) \quad i = 1, \ldots, \left[\frac{n-1}{2}\right],$$

the set $\mathfrak{B}$ is a set of generators of $V(\lambda)$ only if it contains an element $w$ of the form

$$w = \overline{a} \overline{g} + \sum_{g \in \mathfrak{g}, g \neq \overline{g}} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(a_k)u_k}$$

with $\overline{a}$ a complex number different from zero.

Using Lemma 3.7 is obviously possible to prove directly Proposition 3.6.

Let $\mathfrak{s}_n = \mathfrak{h} \rtimes \mathfrak{a}_n$ be the subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ given by the semidirect product between the Cartan subalgebra $\mathfrak{h}$ and the subalgebra $\mathfrak{a}_n$. The $\mathfrak{sl}(n + 1, \mathbb{C})$–module ($\mathfrak{a}_n$–module) $V(\lambda)$ is also a $\mathfrak{s}_n$–module, on which the subalgebra $\mathfrak{h}$ acts diagonally. Obviously any set of generators of the $\mathfrak{a}_n$–module $V(\lambda)$ is also a set of generators of the $\mathfrak{s}_n$–module $V(\lambda)$. Moreover for what said above any $\mathfrak{s}_n$–submodule of $V(\lambda)$ is a $\mathfrak{sl}(n + 1, \mathbb{C})$–weight module, i.e., it can decomposed as a direct sum of $\mathfrak{sl}(n + 1, \mathbb{C})$–weight spaces. From these facts it follows the

**Proposition 3.8** If the $\mathfrak{s}_n$–module $V(\lambda)$ decomposes in a direct sum of two subspaces: $V(\lambda) = U \oplus T$, then $\overline{g}$ belongs either to $U$ or to $T$.

**Proof** Let $\mathfrak{B}_U = \{w_1, \ldots, w_j\}$ and $\mathfrak{B}_T = \{w_{j+1}, \ldots, w_k\}$ be respectively a set of generators of $U$ and of $W$. Since $U$ and $T$ are $\mathfrak{sl}(n + 1, \mathbb{C})$–weight modules we may suppose that both $\mathfrak{B}_U$ and $\mathfrak{B}_T$ are made by homogeneous elements, and therefore $\mathfrak{B} = \mathfrak{B}_U \cup \mathfrak{B}_T = \{w_1, \ldots, w_j, w_{j+1}, \ldots, w_k\}$ is a set of homogeneous generators of $V(\lambda)$. Then from proposition 3.7 it follows that there exists an index $\overline{g}$, $1 \leq \overline{g} \leq h$ such that $\overline{g} = c_\overline{g}w_\mathfrak{g}$. Hence $\overline{g}$ belongs either to $U$ or to $T$. 

□

13
Theorem 3.9  The $\mathfrak{a}_n$–module $V(\lambda)$ is indecomposable.

Proof Let us first show that the $\mathfrak{s}_n$–module $V(\lambda)$ is indecomposable. Let us suppose that $V(\lambda)$ is the direct sum $V(\lambda) = U \oplus T$ of two $\mathfrak{s}_n$–modules $U$ and $T$ and let $\mathfrak{B}_U = \{ w_1, \ldots, w_l \}$ (res. $\mathfrak{B}_T = \{ w_{l+1}, \ldots, w_{l+k} \}$) be a set of homogeneous generators of $U$ (res. of $T$). We know from Proposition 3.8 that either $\overline{g}$ belongs to $U$ or to $T$. Say $\overline{g} \in U$, then we shall show that $V(\lambda) = U$.

We say that an element $Y(a^u)u_\lambda$ of the Littelmann basis is of level $l$ if $l$ is the minimal nonnegative integer such that $Y(a^u)u_\lambda = P_1 \cdots P_{l}u_\lambda$ and any monomial $P_j 1 \leq j \leq l$ is a product of elements $Y_i$ of index either odd or even.

It is immediate to see that all the elements of the Littelmann basis of length 1 and 0 are in $\mathcal{U}(a^{[\rho]}g)$ and therefore in $U$. Let now us suppose by induction that any element in $\mathfrak{g}_1$ of level less or equal $l$ is in $U$. We need to show that any element in $\mathfrak{g}_1$ of level $l + 1$ also belongs to $U$. First, since any element $Y(a^u)u_\lambda$ in $\mathfrak{g}_1$ is of the type $Y(a^u)u_\lambda = Y^{u_{2h}}_{2h} \cdots u_\lambda$ with $a_{2h}^u \neq 0$, $0 < h \leq \left\lfloor \frac{l}{2} \right\rfloor$, $\mathfrak{g}_1$ decomposes as

$$
\mathfrak{g}_1 = \bigcup_{1 \leq j_1 < \cdots < j_s \leq \left\lfloor \frac{l}{2} \right\rfloor} \mathfrak{g}_{j_1, j_2, \ldots, j_s}
$$

where

$$
\mathfrak{g}_{j_1, j_2, \ldots, j_s} = \left\{ g \in \mathfrak{g}_1 \mid g = Y(a^u)u_\lambda = Y^{u_{2j_1}}_{2j_1} \cdots Y^{u_{2j_s}}_{2j_s} Y^{u_{2k+1}}_{2k+1} u_\lambda \right\}
$$

with $a_{2j_i}^u > 0$ for $i = 1, \ldots, s$ and $a_0^u = 0$, $k < j$, $r > 2j_r$. Therefore it is enough to show that for any fixed set $\{ j_1, \ldots, j_s \}$ ($1 \leq j_1 = \cdots = j_s \leq \left\lfloor \frac{l}{2} \right\rfloor$) the elements of length $l + 1$ in $\mathfrak{g}_{j_1, j_2, \ldots, j_s}$ belong to $U$. We shall do it by induction over the orderings $\{ j_1, \ldots, j_s \}$ defined in the proof of Theorem 3.2. If $g \in \mathfrak{g}_1$ is minimal with respect all the ordering $\{ j_1, \ldots, j_s \}$ then

$$
X^{u_{2j_1}}_{2j_1} \cdots X^{u_{2j_s}}_{2j_s} g = c^{2j_1, \ldots, 2j_s}_{2j_1, \ldots, 2j_s} Y(a^u)u_\lambda = c^{2j_1, \ldots, 2j_s}_{2j_1, \ldots, 2j_s} Y^{u_{2k+1}}_{2k+1} \cdots u_\lambda
$$

with $c^{2j_1, \ldots, 2j_s}_{2j_1, \ldots, 2j_s} \neq 0$, $a_0^{2k+1} = 0$ and $Y(a^u)u_\lambda = Y^{u_{2k+1}}_{2k+1} \cdots u_\lambda$ element of $\mathcal{U}$ of level $l$. Since $X^{u_{2j_1}}_{2j_1} \cdots X^{u_{2j_s}}_{2j_s} g$ has been obtained from an element of the set $\mathfrak{g}_1$ of level $l + 1$ by erasing the operators $Y_{2h}$, $1 \leq h \leq s$, and it is of the type $Y^{u_{2k+1}}_{2k+1} \cdots u_\lambda$ with $a_0^{2k+1} > 0$, $0 \leq k \leq \left\lfloor \frac{l}{2} \right\rfloor$, by the very definition of the set $\mathfrak{g}_1$ it can be generated by an element of $\mathfrak{g}_1$ of level $l - 1$ and it is therefore by induction hypothesis a non trivial element in $U$. We can now decompose $g$ as $g = g_U + g_T$ with $g_U = \sum_{k=1}^l P_k (X_{2k}, Y_{2j_k}) w_k \in U$ and $g_T = \sum_{k=l+1}^{h} P_k (X_{2k}, Y_{2j_k}) w_k \in T$ and since all the elements $w_k$ are homogeneous, $g_U$ and $g_T$ are of the same $\mathfrak{sl}(n + 1, \mathbb{C})$–weight of $g$. Now

$$
X^{u_{2j_1}}_{2j_1} \cdots X^{u_{2j_s}}_{2j_s} g_T = X^{u_{2j_1}}_{2j_1} \cdots X^{u_{2j_s}}_{2j_s} g - X^{u_{2j_1}}_{2j_1} \cdots X^{u_{2j_s}}_{2j_s} g_U \in U \implies X^{u_{2j_1}}_{2j_1} \cdots X^{u_{2j_s}}_{2j_s} g_T = 0.
$$
But the fact that $g_U$ and $g_T$ has the same weight of $g$ implies that they have also the same weight of $g$ with respect any subalgebra $g_{2j_i}$ spanned by the vector $H_{2j_i}, X_{2j_i}, Y_{2j_i}$, $1 \leq r \leq s$ and equivalent to the complex simple Lie algebra $sl(2, \mathbb{C})$. Since $H_{2j_i}, g = -a_{2j_i}^2 g$ with $a_{2j_i}^2 > 0$ for $1 \leq r \leq s$ the theory of the $sl(2, \mathbb{C})$–finite dimensional modules implies that for $1 \leq r \leq s$, $X_{2j_i} g_T = 0$ if and only if $g_T = 0$. Hence $g = g_U \in U$. Now, since for any element $g$ in $\mathfrak{g}_{1,j_1,...,j_s}$ which is not a minimal element for at least one of the ordering $\leq j_s, (1 \leq r \leq s)$ we have
\begin{equation}
X_{2j_1}^{a_{2j_1}} \cdots X_{2j_r}^{a_{2j_r}} g = c_{2j_1,...,2j_r}^{j_1,...,j_r} Y_{2j_{k+1}} d_{j_{k+1}} \cdots u_A + \sum_{(b) < (a)} c_{(b)} Y^{(b)} u_A
\end{equation}
by induction over the orderings $\leq j_s, (1 \leq r \leq s)$ we have that $X_{2j_1}^{a_{2j_1}} \cdots X_{2j_r}^{a_{2j_r}} g \in U$. Then from the same argument used above $g \in U$. We have therefore proved that any element of $\mathfrak{g}_A$ of length $l + 1$ belong to $U$, if any element of $\mathfrak{g}_A$ of length $l$ does. Therefore by induction the set of generators $\mathfrak{g}_A$ belongs to $U$. Since $\mathfrak{g}_A$ generates $V(\lambda)$ under the action of $a_n$, we have $V(\lambda) = U$ also as $a_n$–module. Hence the $a_n$–module $V(\lambda)$ is indecomposable for any integer dominant weight $\lambda$.

References

[1] T. Bliem Weight Multiplicities for so5($\mathbb{C}$) Proceedings of the 2008 International Conference on Information Theory and Statistical Learning, CSREA Press, 2008 pp. 80–86, \texttt{arXiv:0902.1744v1}
[2] P. Casati, S. Minniti, V. Solari Indecomposable Representations of the Diamond Lie Algebra to appear on Journal of Mathematical Physics.
[3] P. Casati, G. Ortenzi New Integrable Hierarchies from Vertex Operator Representations of Polynomial Lie Algebras Journal of Geometry and Physics 56 (2006) 418–449
[4] L. Cagliero P. Tirao A closed formula for weight multiplicities of representations of $Sp_{2}(\mathbb{C})$ manuscripta mathematica 115, 417-426 (2004)
[5] A. Douglas, A. Premat A class of nonunitary, Finite Dimensional Representations of the Euclidian Lie Algebra $\mathfrak{o}(2)$ Communication in Algebra 35 (2007) 1433–1448
[6] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer Verlag, Berlin, Heidelberg, New York NY, 1972.
[7] A. Knapp Lie Groups Beyond an Introduction Progress in Mathematics, vol. 140, Birkhäuser, Boston, 2002. ISBN: 0-8176-4259-5. 812+xviii pages.
[8] P. Littelmann, Cones, crystals, and patterns. Transform. Groups 3 (1998), no. 2, 145–179.
[9] P. Littelmann, *Paths and Root Operators in Representation Theory* The Annals of Mathematics, Second Series, Vol. 142, No. 3 (Nov., 1995), pp. 499-525

[10] P. Littelmann, *An algorithm to compute bases and representation matrices for \( \mathfrak{S} \mathfrak{L}_{n+1} \)–representations* Journal of Pure and Applied Algebra 117 & 118 (1997) 447-468.

[11] A. Medina, P. Revoy, *Algèbres de Lie et produit scalaire invariant.* Ann. Sci. École Norm. Sup. (4) 18 (1985), no. 3, 553–561.

[12] A. Premat *Indecomposable Representations of the Euclidean Algebra* Preprint University of York Toronto 2007