A Second Quantized Approach to the Rabi Problem

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In the present work the Rabi Problem, involving the response of a spin \(\frac{1}{2}\) particle subjected to a magnetic field, is considered in a second quantized approach. In this concrete physical scenario, we show that the second quantization procedure can be applied directly in a non-covariant theory. The proposed development explicits not only the relation between the full quantum treatment of the problem and the semiclassical Rabi model, but also the connection of these approaches with the Jaynes-Cummings model. The consistency of the method is checked in the semiclassical limit. The treatment is then extended to the matter component of the Rabi problem so that the Schrödinger equation is directly quantized. Considering the spinorial field, the appearance of a negative energy sector implies a specific identification between Schrödinger’s and Maxwell’s theories. The generalized theory is consistent, strictly quantum and non-relativistic.

Keywords: Rabi problem; second quantization; Jaynes-Cummings model; semiclassical limit

I. INTRODUCTION

Two level systems are paramount in Quantum Mechanics. From the purely theoretical point of view, they are the simplest and best understood quantum system. From the phenomenological side, they adequately describe important physical scenarios, that include lasers \(^1\), optical resonance \(^2\), resonance absorption \(^3\), nuclear induction experiments \(^4\). Recently two state quantum systems modeling atoms and molecules in cavities subjected to electromagnetic fields \(^5\) have been receiving attention due to their relation with possible implementations of quantum computers (see for example \(^6\), \(^7\)).

In the semiclassical approach, that is when external fields are considered classically, to treat a two level problem is to solve a spin equation with the form \(^8\)

\[
\frac{d\psi}{dt} = \hat{H}\psi, \quad \hat{H} = (F\sigma),
\]

where \(F = (F_1(t), F_2(t), F_3(t))\) is an arbitrary vector field which depends on time, and \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) denotes the Pauli matrices.

An important particular case of a two state quantum system is the Rabi Problem, a spin \(\frac{1}{2}\) particle subjected to a constant magnetic field orthogonal to a second rotating field \(^4\). The Rabi problem can be dealt with Eq. \(^1\) when the external field has the form \(F = (B\cos\omega t, B\sin\omega t, B_3)\), where \(B\), \(\omega\) and \(B_3\) are real constants. Another important problem, related to the Rabi case, is obtained with a linearly polarized field, \(F = (2B\cos\omega t, 0, B_3)\). This latter system does not have exact solution, but can be adequately described by a decomposition in the associated Hamiltonian in the form

\[
\hat{H} = \hat{H}_+ + \hat{H}_-, \quad \hat{H}_\pm = (B_\pm\sigma), \quad B_\pm = (B\cos\omega t, \pm B\sin\omega t, B_3).
\]

The main point in this decomposition is that, near the resonance, for \(\omega B_3 > 0\), the term connected to the component \(B_-\) does not contribute to the transitions. In this way the linearly polarized field problem is reduced to the Rabi problem. The quantity \(B_-\) is the so-called counter-rotating term, and the discarding of this term is usually referred as the rotating wave approximation \(^8\).

The interaction of a two level system with an quantized electromagnetic field was presented in the work of Jaynes-Cummings \(^10\). For the case of a lossless cavity with a single mode and a uniform field, the interaction Hamiltonian of the Jaynes-Cummings model is given by

\[
\hat{H}_{\text{int}} = \Omega \hat{B}\hat{S}, \quad \hat{B} = \hat{b} + \hat{b}^+, \quad \hat{S} = \sigma_1.
\]
In this expression \( \hat{B} \) is the field operator, written in terms of the creation and annihilation operators \((\hat{b}^+ \text{ and } \hat{b})\) of the electromagnetic field. The operator \( \hat{S} \) is the spin operator, that is, the polarization operator of the two level system.

The original Rabi Hamiltonian \([4]\) presents a semiclassical approach involving a circularly polarized field. The Jaynes-Cummings model involves a second quantized treatment of a linearly polarized field and cannot be obtained from the Rabi setup without approximations \([11]\). Although the connection between Rabi and Jaynes-Cummings models were already investigated in \([10]\), the theme is far from exhausted. For instance, in a recent development \([11]\), a new approximation (the so-called “intermediate rotating wave approximation”) is introduced. Based on this approach, the authors relate the Jaynes-Cummings and Rabi models, even for large values of detuning and coupling. However, the quantum “Rabi’s Hamiltonian” treated in \([11]\) is not the Hamiltonian that is obtained from the (second) quantized version of the problem introduced in Rabi’s original developments \([4, 12]\). As one of the contributions of present work, we will give a new view of this relation.

The usual methods for second quantization \([13]\) can only be directly applicable to relativistic and covariant (or non-interacting) theories \([14-17]\). However, as it will be demonstrated here, in some cases this restriction can be relaxed if proper caution is exercised. Besides, non-equivalence between representations turns the definition of a semi-classical limit for a second-quantized theory into a highly non-trivial problem \([18]\). Therefore, the consistency of the semi-classical limit must be tested case-by-case. We will see in the specific Rabi’s scenario that the quantization with Schrödinger and Heisenberg picture generate distinct and non-equivalent theories.

Considering the issues discussed, we will present a consistent method for the second quantization of the (non-covariant and interacting) Rabi problem. This quantization is made without the rotating wave approximation, and therefore our development is not equivalent to the Jaynes-Cummings approach. This will be the main contribution of the present work. In addition, we will show how the method can be extended to the treatment of a linear field, analyzing the connection between the Rabi model and the semiclassical limit of the Jaynes-Cummings approach.

The structure of this work is commented in the following. In section II the semiclassical Rabi problem is reviewed. In section III we present our main results, introducing a second quantized approach for the Rabi setup. Also in this section the semiclassical limit of the model is discussed, confirming the consistency of the quantization method. The connection between our development and the original Jaynes-Cummings is commented. In section IV we extend our treatment to include the spinorial field that describes the spin \( \frac{1}{2} \) particle of the Rabi setup. Final considerations are presented in section V.

## II. THE SEMICLASSICAL RABI PROBLEM

For a spin \( \frac{1}{2} \) particle with mass \( m \), \( g \)-factor, electric charge \( q \) and magnetic moment \( \mu \), confined in a region with a magnetic field \( \mathbf{B} \), the semiclassical description is given by a Hamiltonian with the Stern-Gerlach term,

\[
\hat{H} = -\mu (\mathbf{B} \sigma) , \quad \mu = g \frac{q}{2m} .
\]

In Eq. (4) the field \( \mathbf{B} \) is determined by functions that represent the classical field being considered. We will set \( \mu = -1 \) to simplify notation. We are interested in Pauli’s theory for a circular field, i.e., a field in the form

\[
\mathbf{B} = ( B \cos \omega t, B \sin \omega t, B_3 ) .
\]

For this configuration, we have the Rabi’s Hamiltonian \([12]\)

\[
\hat{H}_r = \sigma_3 B_3 + B (\sigma_1 \cos \omega t + \sigma_2 \sin \omega t) .
\]

The problem can be elegantly dealt with the use of a rotating frame \([4, 5]\). This non-inertial reference system rotates with the field and hence the field is constant according to rotating observers. The passage to the rotating frame is done with the unitary transformation

\[
\hat{R}_z (\omega t) = \exp \left( -i \frac{\omega}{2} \sigma_3 t \right) .
\]

The Hamiltonian \( \hat{H}' \), which describes the problem in the rotating coordinate system, has the form

\[
\hat{H}' = \hat{R}_z^+ \hat{H}_r \hat{R}_z - i \hat{R}_z^+ \frac{\partial \hat{R}_z}{\partial t} = \frac{\delta}{2} \sigma_3 + \sigma_1 B ,
\]

\( ^1 \) In many references, the “Rabi model” is understood as a two level system coupled with a quantum harmonic oscillator. That approach does not constitute a second quantization of the original Rabi’s setup \([4, 12]\) that describes the circularly polarized field. For example, the Hamiltonian presented in Eq. (1) of \([11]\) actually represents the interaction of a spin \( \frac{1}{2} \) particle with a quantized harmonic oscillator.
where \( \delta = 2B_3 - \omega \) is the so-called detuning parameter. The semiclassical problem has exact solution, and the spin transition is given by \[19\]

\[
|(+|U(t)|-)|^2 = \frac{B^2}{\Omega^2} \sin^2 \left( \tilde{\Omega} t \right), \quad \sigma_3 |\pm\rangle = \pm |\pm\rangle ,
\]

with \( \tilde{\Omega}^2 = (\delta/2)^2 + B^2 \). The quantity \( \tilde{\Omega} \) is denoted as Rabi’s frequency. The transition probability in Eq. (9) has a maximum for \( B_3 = \omega/2 \), and the resonant frequency of the system is \( \omega_R = 2B_3 \).

### III. QUANTIZATION OF THE RABI PROBLEM

#### A. A second quantized approach

Since we are interested in the problem of a particle with spin, fixed in space and interacting with a magnetic field, we can ignore the electric field and the spatial dependence of the field, and hence its associated Hamiltonian density is given by

\[
H_f = \frac{1}{2} |B|^2 = \frac{1}{2} \left( |B_1|^2 + |B_2|^2 + |B_3|^2 \right).
\]

We will consider \( B_3 = 0 \). Introducing the quantity

\[
b = \sqrt{\frac{1}{2\alpha} (B_1 - iB_2)} ,
\]

where \( \alpha \) is a positive real constant, the Hamiltonian assumes the form

\[
H_f = \alpha b^* b .
\]

The quantization is implemented promoting \( b \) to an operator \( \hat{b} \), and imposing the commutation rule

\[
[\hat{b}, \hat{b}^+] = 1 \Rightarrow [\hat{H}_f, \hat{b}] = -\alpha \hat{b} .
\]

The constant \( \alpha \) is fixed if we impose that the dynamics is determined by the Heisenberg equation

\[
\frac{db}{dt} = i \left[ \hat{H}_f, \hat{b} \right] + U^+ \frac{\partial \hat{b}_S}{\partial t} U = \left( \frac{db}{dt} + \frac{\partial b}{\partial t} \right) \bigg|_{b \rightarrow \hat{b}} ,
\]

where \( \hat{b}_S \) is the \( \hat{b} \) operator in the Schrödinger representation. In the last equality in Eq. (14), the notation used means that we first calculate the classical quantities into the parenthesis, afterwards promoting them to operators. The previous requirement, summarized in Eq. (14), can be interpreted as a correspondence principle for the second quantization approach, ensuring that the quantization has the correct semiclassical limit.

We are interested in the circular magnetic field presented in Eq. (5), whose intensity is conserved. For this field we have

\[
b(t) = \frac{B}{\sqrt{2\alpha}} \exp (-i\omega t) ,
\]

implying that

\[
\dot{b} = -i\omega b .
\]

In Eq. (15) it was explicitly considered that \( \omega > 0 \). For \( \omega < 0 \) we should redefine \( b \) with the transformation \( B_2 \rightarrow -B_2 \). As we will see, this latter prescription is necessary in order that \( \alpha > 0 \). In the second quantization scheme, the field \( \mathbf{B} \) is decomposed in a basis of orthogonal functions \( \phi_i \),

\[
\mathbf{B} = \sum_i c_i(t) \phi_i(x) , \quad c_i = c_i \hat{e}_i , \quad \langle \phi_i | \phi_j \rangle = \delta_{ij} .
\]
The particular basis of functions to be chosen is determined by boundary conditions of the classical problem and symmetries of the system. Besides, since $B$ obeys the wave equation (vacuum Maxwell’s equations), the expansion coefficients $\{c_i\}$ obey the classical relation

$$
\dot{c}_i = -\omega^2 c_i \Rightarrow \dot{c}_i = \pm i \omega c_i , \quad \omega \in \mathbb{R} .
$$

In Eq. (18) we used the positivity of the Hermitian operator $(\mathbf{i} \nabla)^2$, with any appropriate boundary condition that makes it a self-adjoint operator.

In the general case, the coefficients $c_i$ that appear in Eq. (18) are linear combinations of the coefficients $b_j$ present in the Hamiltonian (12). But in the special case of a circular fields, expressions (18) and (15) show that we can identify $c = b$. We stress that this result is a consequence of the fact that, for the circular field, the terms $b^*$ and $b$ that appear in the Hamiltonian in Eq. (12) are automatically the field expansion coefficients. That implies the choice of periodic boundary conditions, and also establishes the physical interpretation of the creation and annihilation operators $\hat{b}^+$ and $\hat{b}$. As we will see, $c = b$ sets that $\hat{b}^*$ creates (annihilates) photons with circular polarization.

We now need to determine the temporal dependence of $\hat{b}_S$. In analogy to the semiclassical treatment, we have the option of considering the field temporal dependence both the Heisenberg and Schrödinger picture. However, the physical meaning of the field is different when comparing with the simpler approach in section II. This happens because the classical fields to be quantized must be described in an inertial frame. In addition, once the dynamic is given by Eq. (14), the Heisenberg representation must be associated with this frame. It turns out that we have no freedom in the association of the representation and the frame that describe the semi-classical theory. In fact, we could try to implement the temporal dependence of the circular field in the Schrödinger picture, i.e., make $\partial \hat{b}_S / \partial t = -i \omega \hat{b}_S$. But using Eqs. (15) and (13), we have from Eq. (14) that

$$
\frac{\partial \hat{b}}{\partial t} = -i \alpha \hat{b} + U + \frac{\partial \hat{b}_S}{\partial t} U = -i \omega \hat{b} .
$$

From Eq. (19), insisting with the Schrödinger picture, we would obtain

$$
- i \alpha \hat{b} = - i \omega (\hat{b} - \hat{b}) = 0 \Rightarrow \alpha = 0 ,
$$

and we could not proceed with the quantization process.

Therefore, the choice of the original frame (classically an inertial frame) is crucial to the second quantization of a non-covariant theory. Physically this is reminiscent of the fact that a static field, i.e., a field which does not depend on time in an inertial frame, does not produce radiation. That is why we do not try to include a non-null $B_3$ in the above procedure.

As a consequence, the temporal dependence must be implemented with the Heisenberg representation and the field operators do not depend on time in the Schrödinger representation, that is $\partial \hat{b}_S / \partial t = 0$. Therefore Eq. (19) implies

$$
i \alpha \hat{b} = i \omega \hat{b} \Rightarrow \alpha = \omega \Rightarrow H_f = \omega \hat{N} , \quad \hat{N} = \hat{b}^+ \hat{b} .
$$

Besides, since in all quantization schemes (in flat spacetime) the time enters as a parameter [21], we rewrite Eq. (16) as

$$
\hat{b}(t) = \hat{b} \exp (-i \omega t) , \quad \hat{b} = \hat{b}(t = 0) .
$$

Since in $t = 0$ we have a magnetic field pointing in the $\hat{x}$ direction, the operator $\hat{b}$ (which do not depend on time) also describes our system in a reference frame that rotates with the field.

It should be observed that $\hat{N}$ does not depend on time in both representations. The fact that it is possible to define a single number operator which is time independent is a consequence of the fact that the intensity of the classical field $B$ is constant. Moreover, both $\hat{b}(t)$ and $\hat{b}$ respect the same algebra [18]. This implies that the quantum theories constructed with these operators, although different, are unitary equivalent.

To complete the description of the physical system discussed here, we introduce the circular polarization unit vector $\epsilon$ and, inverting (11), we write

$$
B = B_1 \hat{e}_1 + B_2 \hat{e}_2 = \sqrt{\omega} \left[ b \epsilon + (b \epsilon)^* \right] , \quad \epsilon = \frac{1}{\sqrt{2}} (\hat{e}_1 - i \hat{e}_2) .
$$

That is, $b$ and $b^*$ are the coefficients of the field $B$ in the circular polarization basis. It follows, from Eqs. (14) and (23), that $\hat{b}^+$ and $\hat{b}$ create and annihilate photons with polarization $\epsilon^*$ and energy $\omega$. For $\omega < 0$ the factors in Eq. (23) interchange.
With a full description for the free field sector of the model, we can reintroduce the spin and return to Rabi’s Hamiltonian. In Dirac notation, using Eq. (22), Eq. (6) can be written as

\[
\hat{H}_r = \Omega \left( e^{-i\omega t} \hat{b} |+\rangle \langle -| + e^{i\omega t} \hat{b}^+ |+\rangle \langle +| \right) + B_3 \sigma_3 ,
\]

where \( \Omega (m, q, \omega) \equiv -\mu \sqrt{2} \omega \). The total Hamiltonian \( \hat{H} = \hat{H}_f + \hat{H}_r \), obtained after the second quantization process, is then

\[
\hat{H} = \hat{H}_0 + \hat{H}_I , \quad \hat{H}_0 = \omega \hat{N} + B_3 \sigma_3 , \quad \hat{H}_I = \Omega e^{-i\omega t} \hat{b} |+\rangle \langle -| + \Omega e^{i\omega t} \hat{b}^+ |+\rangle \langle +|. \]

Notice that Eq. (25) is a result of the formalism, obtained from the quantization of the Rabi’s Hamiltonian in Eq. (6). No ad hoc terms were added to Eq. (6) for the derivation of Eq. (25).

It should be observed that for any self-state \( |n\rangle \) of the operator \( \hat{N} \) we have

\[
|\Omega \hat{b}|^2 |n\rangle = |\Omega|^2 \hat{b}^+ \hat{b} |n\rangle = \Omega^2 n |n\rangle = |\hat{B}|^2 |n\rangle ,
\]

where the normal ordering is chosen for \( |\hat{b}|^2 \). Hence, the classical field intensity is proportional to the number of photons and their coupling to the particle,

\[
n\Omega^2 = |\hat{B}|^2 .
\]

Let us now consider the Schrödinger representation, where

\[
\hat{H}_S = \hat{H} (t = 0) = \hat{H}_0 + \hat{H}_SI \, , \\
\hat{H}_SI = \Omega \left( \hat{b} |+\rangle \langle -| + \hat{b}^+ |+\rangle \langle +| \right) .
\]

We observe that (28) can also be obtained if we apply a rotation (7) in relation (25). This shows that both Schrödinger and Heisenberg representations are connected by a rotation. But in the second quantization approach used in this work, the rotation operation no longer has its original meaning. Although the observer associated with the rotating frame detects a constant field, this same observer also detects a non-zero number of photons. This remark would not be true if this observer tries to quantize this constant field. In conclusion, both frames are not physically equivalent with the second quantization process.

B. Semiclassical limit of the electromagnetic field

A well-defined semiclassical limit is important to guarantee the consistency of the theory. We will investigate this limit presently. In special, we will see that in our development the semiclassical limit is not associated neither to the Schrödinger nor to the Heisenberg representation. The interaction representation is the relevant one.

Considering the Hamiltonian in Eq. (28), we can write

\[
\hat{H}_S = \hat{H}_0' + \hat{H}_1 ,
\]

with

\[
\hat{H}_0' = \omega \left( \hat{N} + \frac{\sigma_3}{2} \right) , \quad \hat{H}_1 = \frac{\delta}{2} \sigma_3 + \hat{H}_SI .
\]

The important point concerning Eq. (28) is that

\[
[\hat{H}_0', \hat{H}_1] = 0 ,
\]

and therefore we can use an interaction picture where the Hamiltonian \( \hat{H}_0' \) can be eliminated and the dynamics is given by the evolution operator \( U_1 \), with

\[
U_1 = e^{i\hat{H}_0't} e^{-i\hat{H}_S t} = \exp \left( -i\hat{H}_1 t \right) .
\]
From relation [31] we have
\[ \exp \left( i \hat{H}_0' t \right) \hat{H}_1 \exp \left( -i \hat{H}_0' t \right) = \hat{H}_1, \] (33)

implying that
\[ e^{i\omega \tilde{N} t} b e^{-i\omega \tilde{N} t} = e^{-i\omega t} b. \] (34)

That is, \( \exp(i\omega \tilde{N} t) \) is equivalent to a rotation \( \tilde{R}_z(\omega t) \). Consequently, the transformation that links the interaction and Schrödinger representations,
\[ \exp \left( i \hat{H}_0' t \right) = e^{i\omega \tilde{N} t} \tilde{R}_z^+(\omega t), \] (35)
do not alter the frame (the rotating frame). This happens because \( e^{i\omega \tilde{N} t} \) and \( \tilde{R}_z^+(\omega t) \) represent rotations in opposite directions. But the transformation (35) discounts the energy associated to the Hamiltonian \( \hat{H}_0' \). In this way, the vacuum energy in the interaction picture is redefined so that the energy of the field is not considered.

As will be seen in the following, the interaction frame gives the semiclassical description. First, by noting that \( |n, -\rangle \) is an eigenstate of \( \hat{H}_1^2 \), with eigenvalues \( \tilde{\omega}^2 = (\delta/2)^2 + n\Omega^2 \), one can determine the transition probability
\[ \left| \langle +, n-1 \mid e^{-i\hat{H}_1 t} |n, -\rangle \right|^2 = \frac{n\Omega^2}{\tilde{\omega}^2} \sin^2 (\tilde{\omega} t), \] (36)

which, from Eq. [27], we can identify with [9]. Another consequence of decomposition in Eq. (29) is that, for any value of \( n \), the Hamiltonian \( \hat{H}_0' \) is degenerate for the states \( \{ |n, \pm\rangle, |n \pm 1, \mp\rangle \} \). Moreover, the Hamiltonian \( \hat{H}_1 \) has non-null matrix elements only for transitions between the same states, or for transitions in the form \( |n, \pm\rangle \rightarrow |n \pm 1, \mp\rangle \). In this way, if the base elements \( \{|n, \pm\rangle\} \) can be reorganized as \( \{|n, +\rangle, |n + 1, -\rangle\} \), the Hamiltonian (29) assumes a block diagonal form, where each block in position \( n \) is a matrix \( 2 \times 2 \), which we will denote as \( \hat{H}^{(n)} \)
\[ \hat{H}^{(n)} = \hat{H}_0^{(n)} + \hat{H}_1^{(n)}, \] (37)

with
\[ \hat{H}_0^{(n)} = \omega \left( n + \frac{1}{2} \right) \mathbb{I}, \quad \hat{H}_1^{(n)} = \frac{\delta}{2} \sigma_3 + \Omega \sqrt{n + 1} \sigma_1. \] (38)

The energy associated to the Hamiltonian \( \hat{H}_0^{(n)} \), which is discounted in the interaction representation, do not correspond to the magnetic field energy only. The term \( \omega/2 \) of the constant multiplying the unit matrix \( \mathbb{I} \) in \( \hat{H}_0^{(n)} \) is originated from the operator \( \sigma_3/2 \), and it is associated to a photon that the particle can emit to the cavity. This energy will also be discounted in the semiclassical Hamiltonian that refers to the spin only. And since [B] is connected with the intensity of the field [27], for large \( n \) we write\(^2\)
\[ \hat{H}_1^{(n)} = \frac{\delta}{2} \sigma_3 + B \sigma_1, \] (39)

which is the classical Hamiltonian [8] in the rotating picture.

It should be pointed out that, although the decomposition in Eq. (29) is well known, our treatment clarifies the relation between the results in Schrödinger picture with the semiclassical limit of the quantum theory. For this purpose, the interpretation of the operator \( \exp(i\omega \tilde{N} t) \) as a rotation is essential. Besides, the semiclassical limit must be taken in the Schrödinger picture because, since the Hamiltonian is time independent in this representation, only in the rotating frame the (semiclassical) Schrödinger and Heisenberg Hamiltonians coincide.

\(^2\) The explicit form of relation (24) depends on the ordering definition for \( |\tilde{B}|^2 \). Semiclassical and fully quantized quantities can only be compared in a limit where ordering is not relevant.
C. Relation with the Jaynes-Cummings model

In the present section, we will establish connections between the Jaynes-Cumming model, that involves a second quantized treatment of a linearly polarized field, and the original Rabi setup that presents a semiclassical approach involving a circularly polarized field.

Let us consider a linear field,
\[
\tilde{\mathbf{B}} = (2B \cos \omega t, 0, 0)
\]
In this case, since the field intensity is not constant, we can not discard the electric field in the Hamiltonian. Still, this problem can be fixed if we define
\[
c = \sqrt{\frac{1}{2\omega}} (\tilde{B}_1 - iE_2),
\]
where we use that, from the Maxwell’s equations in vacuum, \(\mathbf{E}\) and \(\tilde{\mathbf{B}}\) are orthogonal, with a phase difference of \(\pi/2\). It follows that
\[
\hat{H}_f' = \frac{1}{2} (B^2 + E^2) = \omega c^* c.
\]
But, since the spin interacts only with the magnetic field, we obtain that
\[
\hat{H}_I = \tilde{\mathbf{B}}_1 \sigma_1 = \sqrt{\frac{\omega}{2}} (c + c^*) \sigma_1 \rightarrow \sqrt{\frac{\omega}{2}} \hat{B} \hat{S}, \quad \hat{B} = \hat{c} + \hat{c}^+,
\]
where \(\hat{S}\) is the polarization operator introduced in Eq. (3).

The problem can be simplified writing the interaction Hamiltonian in the form (2), that is
\[
\hat{H}_I = \hat{H}_+ + \hat{H}_-.
\]
In Dirac notation,
\[
\hat{H}_\pm = (\sigma \mathbf{B}_\pm) = \mathbf{B} \left[ e^{\mp i\omega t} |+\rangle \langle -| + e^{\pm i\omega t} |−\rangle \langle +| \right] .
\]
Previous expression shows that it is possible to apply the second quantization scheme to theories equipped with the Hamiltonians \(\hat{H}_\pm\), using the quantity \(\sqrt{\omega/2h} = (\tilde{B}_1 - iB_2) = B e^{-i\omega t}\) defined in Eq. (12) for the circular field \(\mathbf{B}\). It should be observed that
\[
\tilde{B}_1 = \sqrt{\frac{\omega}{2}} (b + b^*) = \sqrt{\frac{\omega}{2}} (c + c^*) ,
\]
and so the field operator \(\hat{B}\) (43) has the same form when written in terms of \(\hat{b}\) or \(\hat{c}\), that is, \(\hat{B} = \hat{c} + \hat{c}^+ = \hat{b} + \hat{b}^+\). It follows that, even working with \(b\) in the interaction Hamiltonian, it is still possible to use the same free field Hamiltonian (42). The Jaynes-Cummings model represents the second quantization of this hybrid description.

The second quantization of the Hamiltonians in Eq. (44) gives
\[
\hat{H}_\pm = \sqrt{\frac{\omega}{2}} \left( \hat{b} (t) |\pm\rangle \langle \mp| + \hat{b}^+ (t) |\mp\rangle \langle \pm| \right) .
\]

The interaction Hamiltonian in this system is
\[
\hat{H}_I = \hat{H}_+ + \hat{H}_- = \sqrt{\frac{\omega}{2}} \hat{B} \hat{S}, \quad \hat{S} = |+\rangle \langle +| - |−\rangle \langle −| ,
\]
which (for \(\mu = -1\)) is the same original Hamiltonian (43). This was the interaction Hamiltonian suggested by Jaynes-Cummings (10). The fact that
\[
\hat{H}_f' = \omega \hat{c}^+ \hat{c} = \omega \hat{b} \hat{b}^+
\]
is the cause of the apparent instantaneous non conservation of energy in the Jaynes-Cummings model due to the counter-rotating terms\(^3\). In other words, \(\hat{b}^+ \hat{b}\) is not the number operator of the problem.

As it is well known, focusing on the interaction Hamiltonian only, the Jaynes-Cummings model complemented with the rotating wave approximation is equivalent to the Rabi model with circular field. However, it should be observed that, in this approximation, \(\hat{H}_I \approx \hat{H}_+\) and \(\hat{c} + \hat{c}^+ \neq \hat{b} + \hat{b}^+\). Therefore, the free field Hamiltonian is not given by (12). This discrepancy does not change the transition probability between states near the resonance frequency. But must be taken into account if, for example, the interaction picture is used in models far from resonance point.

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\(^3\) This term has the form \(\hat{b}^+ |+\rangle \langle −|\), representing the spin excitation and the simultaneous photon emission. In our description, they promote transitions between sectors described by \(\hat{H}^{(n)}\) with different \(n\).
IV. QUANTIZATION OF THE SPINORIAL FIELD

The second quantization procedure can be extended to all fields in the Rabi problem, including the spinorial field. Here, we will consider the Pauli spinorial field and a non-covariant formulation. In this development, some attention should be given to possible failures of a classical symmetry in its quantized version. As it is well known, anomalies are a common feature of second quantized models. In particular, the relation between the spin and the Bloch sphere (i.e., the $SU(2)$ symmetry) is not necessarily maintained in a second quantized approach.

Following the work in the previous section, we will construct a Hamiltonian for the free spinorial field. In this case, the presence of the $B_3$ term is associated to the energy content of the system, and therefore we consider a free field when the field component in the plane $x \times y$ is null. The free spin Hamiltonian is

$$\hat{H}_{sf} = B_3 \sigma_3.$$  \hspace{1cm} (49)

We can associate the measured energy of the system with the average value of the energy. Hence the energy of the free spinorial field is

$$H_{sf} = \langle \psi | \hat{H}_{sf} | \psi \rangle = B_3 \left( |\psi_1|^2 - |\psi_2|^2 \right),$$  \hspace{1cm} (50)

where $\{\psi_i\}$ are the components of the Pauli spinor $|\psi\rangle$. This Hamiltonian can be written in the form

$$H = \sum_i \alpha_i b_i^\dagger b_i, \quad \alpha_i \in \mathbb{R},$$  \hspace{1cm} (51)

with the identification $b_i = \psi_i$. Since we are treating fermions, the spinors obey the anti-commutation rules

$$\left\{ \psi_i, \psi_j^\dagger \right\} = \delta_{ij}, \quad \left\{ \psi_i, \psi_j \right\} = \left\{ \psi_i^\dagger, \psi_j^\dagger \right\} = 0.$$  \hspace{1cm} (52)

Eq. (50) is now written as

$$\hat{H}_{sf} = B_3 \left( \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \right).$$  \hspace{1cm} (53)

In the present description, for $\omega B_3 > 0$, the operator $\hat{\psi}_1$ acts as an annihilation operator of the positive energy states and as an identity operator in the space of negative energy states. An analogous procedure is applied to the operator $\psi_2$. After the quantization of the magnetic and Pauli spinorial field, we have

$$\hat{H}_{sf} = \Omega \left( \hat{\psi}_1^\dagger \hat{\psi}_2 + \hat{\psi}_1 \hat{\psi}_2^\dagger \right).$$  \hspace{1cm} (54)

We stress that in our approach, unlike Pauli’s theory, the definition of which state $\{|+\rangle, |-\rangle\}$ has positive energy is not determined only by the sign of $B_3$, but also by $\omega$. This happens because the quantization of the Maxwell field was developed in such a way that the photons always have positive energy (given by $|\omega|$). It follows that, for $\omega < 0$, we should exchange $b \leftrightarrow b^\dagger$ in Eq. (54). For $B_3 > 0$, $\psi_1^\dagger$ ($\psi_2^\dagger$) creates particles in the positive sector with $\omega > 0$ ($\omega < 0$).

From this latter observation, we obtain an interpretation for the fact that only photons associated to the rotating term are capable to promote the excitation of the spinning particle near the resonance point. Notice that this happens despite the fact that photons associated to both rotating and counter-rotating terms have the same energy. The point is that, while in the semiclassical limit this is a consequence of the conservation of angular momentum, in the full quantum theory the counter-rotating photons are not in the adequate energy sector.

We now consider the semiclassical limit of the spinorial field. This limit is given by the matrix elements of the Hamiltonian in a suitable basis, the physical states of the system $\{0, 1\}$, $\{1, 0\}$. This basis is chosen since its elements are the only states coupled by the interaction Hamiltonian (54). Let us introduce the following operators

$$\hat{J}_+ = \hat{\psi}_1^\dagger \hat{\psi}_2, \quad \hat{J}_- = \hat{\psi}_2^\dagger \hat{\psi}_1, \quad \hat{J}_3 = \frac{1}{2} \left( \hat{\psi}_1^\dagger \hat{\psi}_1 - \hat{\psi}_2^\dagger \hat{\psi}_2 \right).$$  \hspace{1cm} (55)

With the elements presented in Eq. (55), the interaction Hamiltonian is written as

$$\hat{H}_{sf} = \Omega \left( \hat{J}_+ b + \hat{J}_- b^\dagger \right).$$  \hspace{1cm} (56)

\footnote{The approach here differs from many usual physics treatments of spinor quantization, where the starting point is a relativistic wave equation.}
The important aspect in the definitions of $\hat{J}_\pm$ and $\hat{J}_3$ is that, using the anti-commuting relation rule in Eq. (52), we observe that the operators $\hat{J}_\pm$ and $\hat{J}_3$ satisfy the relations

$$[\hat{J}_3, \hat{J}_\pm] = \pm \hat{J}_\pm,$$
$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_3,$$  \hspace{1cm} (57)

which is the algebra of the angular momentum operators,

$$\hat{J}_+ = \hat{J}_1 + i\hat{J}_2 = \hat{J}^+,$$
$$[\hat{J}_i, \hat{J}_k] = i\varepsilon_{ijk}\hat{J}_k.$$  \hspace{1cm} (58)

That is, the operator algebra generated by $\{\hat{J}_i\}$ can be satisfied setting $\hat{J}_i = \sigma_i/2$. The same conclusion can be obtained by the direct calculation of the matrix elements of $\hat{J}_\pm$ in Eq. (55) with the basis of physical states $\{|-\rangle \equiv |0,1\rangle, |+\rangle \equiv |1,0\rangle\}$. In this basis

$$\hat{J}_+ = |+\rangle \langle -| = \frac{1}{2}(\sigma_1 + i\sigma_2), \hat{J}_- = (\hat{J}_+)^+ = |-\rangle \langle +|.$$  \hspace{1cm} (59)

Using this representation, we can write Eqs. (50) and (56) as

$$\hat{H}_{s\ell} = B_3\sigma_3, \hat{H}_{sI} = \Omega\hat{b}\langle +| -\rangle \langle +|.$$  \hspace{1cm} (60)

The operators $\hat{H}_{s\ell}$ and $\hat{H}_{sI}$ in Eq. (60) are the components associated to the spin and to the interaction in the Hamiltonian (28). Therefore, the Hamiltonian obtained with the second quantization approach in this work has the correct semiclassical limit.

V. FINAL COMMENTS

In the present work, a second quantization approach is developed for the Rabi Problem. In this setting, we explicitly show that a non-covariant theory with interaction can be directly quantized. In our development, the quantization is made without the rotating wave approximation, and therefore our results are not equivalent to the Jaynes-Cumming approach. We also show how the Pauli spinorial field can be directly quantized in an important (although particular) setup. Moreover, in our work a “complete field” approach is developed, where both electromagnetic and matter sectors are quantized. The appearance of a negative energy sector for the spinorial field implies a specific identification between semi-classical and purely quantum physical quantities.

Unlike the usual development for the electromagnetic field, the procedure presented here gives a simple and direct method to quantize the magnetic field, in a concrete scenario. We stress that, besides being simpler, the direct quantization of the electromagnetic field (instead of the usual quantization of the electromagnetic potential) can avoid problems. For instance, as already observed by Lamb [23], a perturbative treatment of a dipole transition based on the electromagnetic potential (and not on the electromagnetic field) can result in spurious terms and a significant distortion in the resonant curve of the system.

One important point addressed is the fact that the classical quantities in the right side of the Heisenberg equation (14) should be associated to an inertial frame, when non-covariant theories are considered. Only then the semi-classical limit of the quantum theory furnishes the original classical model. Particularly for the Rabi problem, despite the quantization obtained in both Schrödinger and Heisenberg pictures are unitarily equivalent (as expected), the procedure is only consistent if the rotating frame is associated to the Schrödinger picture. As done in the usual quantization of covariant field theories, where vacuum is chosen using Lorentz symmetry as criteria, in our non-covariant approach the correct quantization is chosen using as criteria the presence of classical symmetries in the final quantum theory. This point can be interpreted as a correspondence principle in the second quantization approach. In fact, the development here would not be possible without the explicit use of the mentioned correspondence principle. This remark is very important, but it appears not to have been fully considered in the pertinent literature.

The fact that it is possible to define in the Rabi problem a number operator which is time independent is a consequence of the constancy of the magnetic field magnitude. This is the reason why we could disregard the electric field and still obtain a Hamiltonian where the energy is conserved. This would not be true if a linearly polarized field were considered. But even in this latter case the development presented here could be applied. The work developed here explicits the similarities (and differences) between the Rabi problem and the analogous scenario with a linearly polarized field.
The present work also clarifies the difference between the Rabi problem and the Jaynes-Cummings model with the rotating wave approximation. That is, although the interaction Hamiltonians in both cases coincide, the free field Hamiltonians remain distinct. Still, it is possible to define a consistent number operator for the Jaynes-Cummings model, which can be used in the construction of the interaction picture and in the treatment of transitions with high detuning.

Finally, we expect that the main features of the approach might be relevant even when more general non-covariant theories are considered. For example, in the treatment of semiclassical models that use exact solutions of the spin equation, such as the adiabatic magnetic pulse. This setup is potentially important in the manipulation of quantum dots, and should be considered in a future development of the present work.

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