The coefficients of the HOMFLYPT and the Kauffman polynomials are pointwise limits of Vassiliev invariants.

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Abstract
The Vassiliev conjecture states that the Vassiliev invariants are dense in the space of all numerical link invariants in the sense that any link invariant is a pointwise limit of Vassiliev invariants.

In this article, we prove that the Vassiliev conjecture holds in the case of the coefficients of the HOMFLY and the Kauffman polynomials.

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A well-known conjecture in the theory of Vassiliev invariants, called Vassiliev conjecture, is that these invariants are dense in the space of all numerical link invariants. This was posed as a problem in [BL93] as follows: Given any numerical link invariant $f: \mathcal{L} \to \mathbb{Q}$, does there exist a sequence of Vassiliev invariants $\{v_n^f: \mathcal{L} \to \mathbb{Q}\}_{n \in \mathbb{N}}$ such that, for any fixed $L$, $\lim_{n \to \infty} v_n^f(L) = f(L)$?

In other words, is any link invariant a pointwise limit of Vassiliev invariants?

Vassiliev conjecture arouses interest because, if it holds, it would imply that the Vassiliev invariants can distinguish any two links, as can be seen using the
approximation by Vassiliev invariants of the link invariant $\delta_{L_1}$ defined by $\delta_{L_1}(L) = 1$ if $L = L_1$, $\delta_{L_1}(L) = 0$ otherwise.

It has been proven in [KR00] that the coefficients of the Jones polynomial of a knot are pointwise limit of Vassiliev invariants. As noted in the article, the result carries over with slight modifications to links.

But when one tries to extend these results to the coefficients of the HOMFLYPT and the Kauffman 2-variable polynomial, the method used to achieve the previous result fails. Indeed, the idea was to produce some Vassiliev invariants that are linear combinations of the coefficients of the Jones polynomial, through a change of variable in the polynomial. Then one would solve back using a matrix.

In the case of 2-variable polynomials, one has a double sum so one cannot solve back the same way and the standard change of variable gives a pretty complicated formula for the Vassiliev invariants. The approach of this paper differs from the previous one in three ways: First, we will use a non standard change of variable to produce Vassiliev invariants, which leads to easier formulas. This is explained in subsection 2.2. Second, we will introduce intermediate variables in order to get sums on one index only. These intermediate variables may not be Vassiliev invariants, but their restrictions to links with a fixed number of components are, as explained in subsection 2.3. So, and this is the third difference, we need to prove an equivalent but seemingly weaker formulation of the Vassiliev conjecture which says that working with links with a fixed number of components is enough. This is explained in section 1.

All these steps together reduce the problem to a point where the method of [KR00] applies. We will supply an alternative proof for this last step, which uses complex analysis. These two proofs can be found in subsection 2.4.

In section 3, we explain how a very similar argument can be used in the case of the Kauffman 2-variable polynomial.

All this will allow us to show that Vassiliev conjecture also holds for the coefficients of the HOMFLYPT and the Kauffman 2-variable polynomial.

1 Reducing the problem: the weak Vassiliev conjecture

In order to prove this result, we will use a reformulation of Vassiliev conjecture, which we call the weak Vassiliev conjecture, that seems to be weaker but which we will prove to be equivalent to Vassiliev conjecture.

Let $\mathcal{L}$ be the set of all oriented links and let $\mathcal{L}^{(\mu)}$ be the set of oriented links with $\mu$ components. We can define what it means for a link invariant defined on $\mathcal{L}^{(\mu)}$ only to be a Vassiliev invariant the following way: A link with self-intersections is said to be a $\mu$ component link if by replacing all the singular crossings by positive or negative crossing we get a $\mu$ component link. This definition makes sense because switching crossings does not affect the number of com-
ponents. We denote $X_i^{(\mu)}$ the set of $\mu$ component links with $i$ self-intersections. Let $v^\mu : \mathcal{L}^{(\mu)} \to \mathbb{Q}$ be a link invariant defined on the set of oriented links with $\mu$ components. We can extend it by induction to all the $X_i^{(\mu)}$’s using Vassiliev relation $v^\mu(L_x) = v^\mu(L_+) - v^\mu(L_-)$ where the diagrams of $L_x, L_+$ and $L_-$ are the same except in a neighborhood of a self intersection, denoted by $\times$, which is replaced by a positive crossing in $L_+$ and by a negative crossing in $L_-$. If $v^\mu$ vanishes on all link with $n + 1$ singular crossings, $v^\mu$ is said to be a Vassiliev invariant of order at most $n$.

We are now ready to state the weak Vassiliev conjecture: For any link invariant $f : \mathcal{L} \to \mathbb{Q}$, for any $\mu \in \mathbb{N}^* (= \mathbb{N} \setminus \{0\})$, there exists a sequence of Vassiliev invariants $\{v^\mu_n : \mathcal{L}^{(\mu)} \to \mathbb{Q}\}_{n \in \mathbb{N}}$ such that, for any fixed $L \in \mathcal{L}^{(\mu)}$, $\lim_{n \to \infty} v^\mu_n(L) = f(L)$.

**Proposition 1.** If all the restrictions of a link invariant to links with a fixed number of component are pointwise limits of Vassiliev invariants then the link invariant itself is a pointwise limit of Vassiliev invariants. In other words, the weak Vassiliev conjecture and Vassiliev conjecture are equivalent.

**Proof.** Let $f : \mathcal{L} \to \mathbb{Q}$ be a link invariant such that for any $\mu \in \mathbb{N}^*$, there exists a sequence of Vassiliev invariants $\{v^\mu_n : \mathcal{L}^{(\mu)} \to \mathbb{Q}\}_{n \in \mathbb{N}}$ satisfying the condition that, for any fixed $L \in \mathcal{L}^{(\mu)}$, $\lim_{n \to \infty} v^\mu_n(L) = f(L)$. Let $\{w^\mu_n : \mathcal{L}^{(\mu)} \to \mathbb{Q}\}_{n \in \mathbb{N}}$ be defined by $w^\mu_n = v^\mu_n$ if $n \geq \mu$ and 0 otherwise. Each $w^\mu_n$ is still a Vassiliev invariant. For any given value of $\mu$, $\lim_{n \to \infty} w^\mu_n(L) = f(L)$ because perturbing a finite number of terms in a sequence doesn’t change the limit.

Let $w_n : \mathcal{L} \to \mathbb{Q}$ be the link invariant defined by $w_n(L) = w^\mu_n(L)$ if $L \in \mathcal{L}^{(\mu)}$, $w_n$ is a Vassiliev invariant with order $= \sup_{\mu \in \mathbb{N}^*} \{\text{order} (w^\mu_n)\} = \max_{\mu \leq n} \{\text{order} (w^\mu_n)\}$ which is well defined because the second set is finite.

And obviously, $\lim_{n \to \infty} w_n(L) = f(L)$.

## 2 Approximation of the coefficients of the HOMFLYPT polynomial by Vassiliev invariants

### 2.1 Notations

Let $P_L(v, z)$ denote the HOMFLYPT polynomial of a link $L$. The version of the HOMFLYPT polynomial we are working with is the one defined by:

1. $v^{-1}P_{L+}(v, z) - vP_{L-}(v, z) = zP_{L_0}(v, z)$.
2. $P_T(v, z) = 1$ where $T$ is the unknot of one component.

Suppose the HOMFLYPT polynomial of a link $L$ is

$$P_L(v, k) = \sum_{-k_1 \leq i \leq k_2} \sum_{-j_1 \leq j \leq j_2} a_{k_1, j_2} v^i z^j$$

where $k_1, k_2, j_1, j_2 \in \mathbb{N}$

We call the degree of such a Laurent polynomial $d = \max\{k_1, k_2, j_1, j_2\}$. 

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Lemma 2. Let $\mu$ be the number of components of a link $L$. Its HOMFLYPT polynomial is a polynomial (not a Laurent polynomial) in the variables $v, v^{-1}, z$ and $\frac{v-v^{-1}}{2}$, the maximum possible power of $\frac{v-v^{-1}}{2}$ being $\mu - 1$.

Proof. The unknotting number of a link diagram $D$ is the minimum number of crossings one needs to switch to get the trivial link. Note that it is not necessarily equal to unknotting number of the link, which takes into account all the diagrams.

The lemma can be proven by induction on $(c(D), u(D))$ where $D$ is a diagram of the link $L$, $c(D)$ is the number of crossing of $D$, $u(D)$ is the unknotting number of $D$ and the order on $\mathbb{N} \times \mathbb{N}$ is the lexicographic order.

2.2 Step 1 : Get Vassiliev invariant through a power series expansion

It is proved in [BL93] that the change of variables $v = e^{-\frac{N}{2}x}$ and $z = e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}$ in $P_L(v, z)$ gives a power series expression for $P_L(v, z)$ in variable $x$ whose coefficients are Vassiliev invariants.

This change of variables has become the standard one and is the one that was used in [KR00]. But in the case of the HOMFLYPT polynomial, it leads to very complicated formulas when it comes to expressing the Vassiliev invariants in terms of the coefficients of the polynomial.

It happens that we get much easier computations using the change of variables explained in the following lemma.

Lemma 3. Let $W_{N,x}(L)$ be the polynomial obtained from the HOMFLYPT polynomial of $L$ by setting $v := e^{Nx}$ and $z := x$. After power series expansion, each $w_{Nq}(L)$ in $W_{N,x}(L) = \sum_{q=0}^{\infty} w_{Nq}(L)x^q$ is a Vassiliev invariant of order (at most) $q$, for all $N \in \mathbb{Z}$.

Notice that lemma (2) ensures that they are no negative powers of $x$ after power series expansion because each of the variables $v, v^{-1}, z$ and $\frac{v-v^{-1}}{2}$ has a power series expansion.

Proof. The proof is similar to the one used in [BL93] to show that if we let $v = e^{-\frac{N}{2}x}$ and $z = e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}$ in $P_L(v, z)$ we get, after power series expansion, $\tilde{W}_K(N, x) = \sum_{q=0}^{\infty} \tilde{w}_{Nq}(K)x^q$ where $\tilde{w}_{Nq}(K)$ is a Vassiliev invariant of order (at most) $q$, for all $N$.

By lemma (2), the HOMFLYPT polynomial of a link $L$ with $\mu$ components has no power of $z$ with exponent less than $-\mu + 1$, hence we can rewrite it $P_L(v, k) = \sum_{-d \leq k \leq d} \sum_{-\mu+1 \leq j \leq d} a_{k,j}(L)v^k z^j$ where $d$ is the degree of the HOMFLYPT polynomial of $L$, with the convention $a_{k,j} = 0$ if $|k| > d$ or $|j| > d$. 

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Let $W_{N,x}(L)$ be the polynomial obtained from the HOMFLYPT polynomial of $L$ by setting $v := e^{N x}$ and $z := x$. Let us compute the power series expansion:

$$W_{N,x}(L) = \sum_{-d \leq k \leq -\mu +1} \sum_{d \leq j \leq d} a_{kj}(L) \sum_{s=0}^{\infty} \frac{N^s k^s x^s}{s!} x^j.$$ 

Let $q := s + j$, we get

$$W_{N,x}(L) = \sum_{q \geq -\mu + 1} \sum_{d \leq j \leq d} a_{kj}(L) \sum_{s=0}^{\infty} \frac{N^s k^s j^s}{(q-j)!} x^q.$$ 

Exchanging the sums over $k$ and $q$ yields

$$W_{N,x}(L) = \sum_{q \geq -\mu + 1} \sum_{d \leq j \leq d} a_{kj}(L) \sum_{s=0}^{\infty} \frac{N^s k^s j^s}{(q-j)!} x^q.$$ 

After the change of variable $p = q - j$, we get:

$$w_{Nq}(L) = \sum_{0 \leq p \leq q+\mu -1} \sum_{d \leq j \leq d} a_{kj}(L) \frac{k^p}{p!}$$

for all $q, N \in \mathbb{Z}$ with $q \geq -\mu + 1$.

By Lemma 3, we know that the $w_{Nq}(L)$ are Vassiliev invariants. The above formula shows that they are related to the initial coefficients of the HOMFLYPT polynomial $a_{kj}(L)$ by

$$w_{Nq}(L) = \sum_{-d \leq k \leq -\mu + 1} \sum_{d \leq j \leq d} a_{kj}(L) \frac{k^q j^q}{(q-j)!}$$

for all $q, N \in \mathbb{Z}$ with $q \geq -\mu + 1$.

Our goal is now to show that the coefficients $a_{kj}$ of the HOMFLYPT polynomial satisfy the weak Vassiliev conjecture.

Note that because of the double sum, the method used in [KR00] to prove that the coefficients of the Jones polynomial of a link are limits of Vassiliev invariants cannot be applied to prove that the coefficients of the HOMFLYPT polynomial of a link are limits of Vassiliev invariants. Therefore, we will introduce new variables:

### 2.3 Step 2: The Intermediate variables $B^\mu_{mj}$ are Vassiliev invariants.

**Definition 4.** Let $B_{mj}(L) = \sum_{k=-d}^{d} a_{kj}(L) k^m$ for all $m \in \mathbb{N}, j \in \mathbb{Z}$.

**Definition 5.** Let $B_{mj}^\mu$ (resp. $a_{kj}^\mu$) be the restriction of $B_{mj}$ (resp. $a_{kj}$) to the links of $\mu$ components.
Therefore, it is a Vassiliev invariant. 

**Proof.** Let \( \mu \in \mathbb{N}^* \), \( m \in \mathbb{N} \), \( j \in \mathbb{Z} \).

\( \blacktriangleleft \) **Case 1**: Assume \( j < -\mu + 1 \).

We already noticed that \( P_L(v, k) \) can be written \( P_L(v, k) = \sum_{-d \leq k \leq d} \sum_{-\mu+1 \leq j \leq d} a_{k_j}(L)v^k z^j \), so \( a_{k_j}(L) = 0 \) for all \( k \) when \( j < -\mu + 1 \), hence \( B_{m\mu}^\mu(L) = 0 \) for all \( L \in \mathcal{L}(\mu) \). Therefore, it is a Vassiliev invariant.

\( \blacktriangleleft \) **Case 2**: Assume \( j \geq -\mu + 1 \).

Let \( q := m + j \). \( m \in \mathbb{N} \) so \( q = m + j \geq -\mu + 1 \). By formula (11), the \( w_{Nq}^\mu \) can be expressed in terms of the \( B_{m\mu}^\mu \)'s by

\[
    w_{Nq}^\mu(L) = \sum_{0 \leq \rho \leq q + \mu - 1} N^\rho B_{p,q-\rho}^\mu(L) \quad \text{for all } N \in \mathbb{Z}, \text{ for all } L \in \mathcal{L}(\mu) \quad (2)
\]

By letting \( N = 1, 2, ..., q + \mu \) , we get a set of equalities that can be summarized in a matrix equality,

\[
    \begin{pmatrix}
        1 & 1 & \cdots & 1 \\
        1 & (2)^1 & \cdots & (2)^{q+\mu-1} \\
        1 & (3)^1 & \cdots & (3)^{q+\mu-1} \\
        \vdots & \vdots & \ddots & \vdots \\
        1 & (q+\mu)^1 & \cdots & (q+\mu)^{q+\mu-1}
    \end{pmatrix}
    \begin{pmatrix}
        B_{0,q}^\mu(L) \\
        B_{1,q-1}^\mu(L) \\
        \vdots \\
        B_{q+\mu-1,-\mu+1}^\mu(L)
    \end{pmatrix}
    =
    \begin{pmatrix}
        w_{1,q}^\mu(L) \\
        w_{2,q}^\mu(L) \\
        \vdots \\
        w_{q+\mu,q}^\mu(L)
    \end{pmatrix}
\]

Let \( n \geq q + \mu \). Let \( A_n = \begin{pmatrix}
        1 & 1 & \cdots & 1 \\
        1 & (2)^1 & \cdots & (2)^{n-1} \\
        1 & (3)^1 & \cdots & (3)^{n-1} \\
        \vdots & \vdots & \ddots & \vdots \\
        1 & (n)^1 & \cdots & (n)^{n-1}
    \end{pmatrix} \). As seen in case 1,

\[
    B_{m\mu}^\mu(L) = 0 \quad \text{whenever } j \leq -\mu, \text{ so }
\]

\[
    \begin{pmatrix}
        B_{0,q}^\mu(L) \\
        B_{1,q-1}^\mu(L) \\
        \vdots \\
        B_{n-1,q-n+1}^\mu(L)
    \end{pmatrix}
    =
    \begin{pmatrix}
        B_{0,q}^\mu(L) \\
        B_{1,q-1}^\mu(L) \\
        \vdots \\
        0
    \end{pmatrix}
\]

Hence, for all \( n \geq q + \mu \), \( A_n \cdot \begin{pmatrix}
        B_{0,q}^\mu(L) \\
        B_{1,q-1}^\mu(L) \\
        \vdots \\
        B_{n-1,q-n+1}^\mu(L)
    \end{pmatrix}
    =
    \begin{pmatrix}
        w_{1,q}^\mu(L) \\
        w_{2,q}^\mu(L) \\
        \vdots \\
        w_{n,q}^\mu(L)
    \end{pmatrix}
\]
A\textsubscript{n} is a \( n \times n \) Vandermonde matrix with distinct parameters, thus \( A\textsubscript{n} \) is invertible.

Hence, for all \( n \geq q + \mu \),

\[
\begin{pmatrix}
B^\mu_{0, q}(L) \\
B^\mu_{1, q-1}(L) \\
\vdots \\
B^\mu_{n-1, q-n+1}(L)
\end{pmatrix} = (A\textsubscript{n})^{-1} \cdot 
\begin{pmatrix}
w^\mu_{0, q}(L) \\
w^\mu_{2, q}(L) \\
\vdots \\
w^\mu_{n-q}(L)
\end{pmatrix}
\]

Since the coefficients and the size of \( (A\textsubscript{n})^{-1} \) do not depend on \( L \) (remember that \( \mu \) is fixed), each \( B^\mu_{i, q-i} \), \( 0 \leq i \leq n-1 \) is a linear combination of the Vassiliev invariants \( w^\mu_{i, q} \)'s, so each is a Vassiliev invariant. Letting \( n := \max(m+1, q+\mu) \), we get that \( B^\mu_{m,j} \) is a Vassiliev invariant. \( \square \)

### 2.4 Last step : The initial coefficient are limits of Vassiliev invariants.

We are now ready to prove our main result, namely that the \( a_{k,j}(L) \)'s, the initial coefficient of the HOMFLYPT polynomial are limits of Vassiliev invariants. We know that it suffices to show that they satisfy the weak Vassiliev conjecture.

At this point, given that the \( B^\mu_{m,j} \)'s are Vassiliev invariant, the method exposed by Y. Rong and I. Kofman in [KR00] can be implemented. We will therefore give a first proof based on this method.

The idea that this result could probably also be proven using complex analysis techniques was suggested by João Faria Martins and lead to a second proof, exposed thereafter.

**Proposition 7.** The coefficients \( a_{k,j}(L) \) of the HOMFLYPT polynomial satisfy the weak Vassiliev conjecture i.e., their restrictions to links with a fixed number of components is a pointwise limit of Vassiliev invariants.

**Proof.** 1, Using Linear Algebra

\[ \blacktriangle \] Let \( j \in \mathbb{Z} \). By letting \( m = 0, 1, 2, \ldots \) in \( B_{m,j}(L) = \sum_{k=-d}^{d} a_{k,j}(L) \frac{h^{m}}{m!} \), we get a system of equations that can be expressed with an infinite matrix.

\[
\begin{pmatrix}
\ldots & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & -2 & -1 & 0 & 1 & 2 & \ldots \\
\ldots & (-2)^2 & (-1)^2 & (0)^2 & (1)^2 & (2)^2 & \ldots \\
\ldots & (-2)^3 & (-1)^3 & (0)^3 & (1)^3 & (2)^3 & \ldots \\
\ldots & (-2)^4 & (-1)^4 & (0)^4 & (1)^4 & (2)^4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
a_{-2,j}(L) \\
a_{-1,j}(L) \\
a_{0,j}(L) \\
a_{1,j}(L) \\
a_{2,j}(L) \\
\vdots
\end{pmatrix}
= 
\begin{pmatrix}
0!B_{0,j}(L) \\
1!B_{1,j}(L) \\
2!B_{2,j}(L) \\
3!B_{3,j}(L) \\
\vdots
\end{pmatrix}
\]

with the convention \( a_{k,j} = 0 \) if \( |k| > d \) or \( |j| > d \)

(3)
We use an infinite matrix because we do not want its size to depend on the knot and the degree \( d \) does.

Let \( M_n \) be the finite \((2n+1) \times (2n+1)\) matrix extracted from the above infinite matrix and defined by

\[
M_n = \begin{pmatrix}
1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
-n & \ldots & -1 & 0 & 1 & \ldots & n \\
(-n)^2 & \ldots & (-1)^2 & 0^2 & (1)^2 & \ldots & (n)^2 \\
(-n)^3 & \ldots & (-1)^3 & 0^3 & (1)^3 & \ldots & (n)^3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(-n)^{2n} & \ldots & (-1)^{2n} & 0^{2n} & 1^{2n} & \ldots & (n)^{2n}
\end{pmatrix}
\]

The matrix \( M_n \) is a Vandermonde matrix with distinct parameters, and thus is invertible.

For all \( n \in \mathbb{N} \), the linear equation \( M_n \cdot X_{2n+1} = \begin{pmatrix} 0!B_{0j}(L) \\ 1!B_{1j}(L) \\ 2!B_{2j}(L) \\ \vdots \\ (2n)!B_{2n,j}(L) \end{pmatrix} \) has a unique solution \( s^{n,j}(L) = \begin{pmatrix} s^{n,j}_n(L) \\ s^{n,j}_{n+1}(L) \\ \vdots \\ s^{n,j}_{2n}(L) \end{pmatrix} = (M_n)^{-1} \cdot \begin{pmatrix} 0!B_{0j}(L) \\ 1!B_{1j}(L) \\ 2!B_{2j}(L) \\ \vdots \\ (2n)!B_{2n,j}(L) \end{pmatrix} \).

Let \( \mu \in \mathbb{N^*} \). We now restrict all the link invariants to links of \( \mu \) components, and we get:

\[
s^{n,j,\mu}(L) = \begin{pmatrix} s^{n,j,\mu}_n(L) \\ s^{n,j,\mu}_{n+1}(L) \\ \vdots \\ s^{n,j,\mu}_{2n}(L) \end{pmatrix} = (M_n)^{-1} \cdot \begin{pmatrix} 0!B^\mu_{0j}(L) \\ 1!B^\mu_{1j}(L) \\ 2!B^\mu_{2j}(L) \\ \vdots \\ (2n)!B^\mu_{2n,j}(L) \end{pmatrix}.
\]

Since the coefficients and the size of \((M_n)^{-1}\) do not depend on \( L \), each \( s^{n,j,\mu}_k(L) \) is a linear combination of the Vassiliev invariants \( B^\mu_{mj}(L) \)'s, so each \( s^{n,j,\mu}_k(L) \) is itself a Vassiliev invariant for all \( k \) such that \(-n \leq k \leq n\).

\( \blacktriangleleft \) We are now ready to prove that for any fixed \( \mu, j, k \) and any fixed \( L \in \mathcal{L}(\mu) \),
\[
\lim_{n \to \infty} s^{n,j,\mu}_k(L) = a_{kj}(L).
\]

This follows directly from the following more general statement (in which the number of components is no longer fixed): For any fixed \( j, k \) and any fixed \( L \in \mathcal{L} \),
\[
\lim_{n \to \infty} s^{n,j}_k(L) = a_{kj}(L).
\]

Let \( L \) be a link. Let \( k, j \in \mathbb{Z} \). Let \( n \geq d \) where \( d \) is the degree of HOMFLYPT polynomial of \( K \).
Using (4), the coefficients $a_{d,j}(L)$ are announced. Hence the sequence \(\{s_k^{n,j}(L)\}_{n \geq 1}\) becomes stationary so \(\lim_{n \to \infty} s_k^{n,j}(L) = a_k(L)\).

If we now restrict all the link invariants to links of $\mu$ components, we get \(\lim_{n \to \infty} s_k^{n,j-\mu}(L) = a_k(L)\). Since the $s_k^{n,j-\mu}$'s are Vassiliev invariants, the coefficients $a_k(L)$ of the HOMFLYPT polynomial are limits of Vassiliev invariants as announced.

**Proof. 2: Using Complex analysis**

For any link $L$ and any $j \in \mathbb{Z}$, define $f_{L,j} : \mathbb{C}^* \to \mathbb{C}$ by $f_{L,j}(z) = \sum_{d \leq -d \leq d} a_{k,\mu}(L)z^k$ where the $a_{k,\mu}$'s are the coefficients of the HOMFLYPT polynomial written:

\[
P_L(v, z) = \sum_{d \leq -d \leq d} a_{k,\mu}(L)v^kz^j.
\]

\[
\Delta f_{L,j}(e^x) = \sum_{d \leq -d \leq d} a_{k,\mu}(L)e^{dx} = \sum_{d \leq -d \leq d} a_{k,\mu}(L)\sum_{m=0}^{\infty} \frac{x^m}{m!} = \sum_{m=0}^{\infty} \left(\sum_{d \leq -d \leq d} a_{k,\mu}(L)\frac{x^m}{m!}\right) x^m
\]

so \(f_{L,j}(e^x) = \sum_{m=0}^{\infty} B_{m,j}(L)x^m\).

Let $n \in \mathbb{Z}$. From \(f_{L,j}(z) = \sum_{d \leq -d \leq d} a_{k,\mu}(L)z^k\), we get, by a standard result in complex analysis, \(a_{n,j}(L) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma^+} f_{L,j}(z) \frac{dz}{z^{n+1}}\) where $\Gamma^+$ is the unit circle travelled around once in the positive direction.

\[
\Delta \text{Let's compute this integral: } 2\pi \mathrm{i} \cdot a_{n,j}(L) = \int_{\Gamma^+} f_{L,j}(z) \frac{dz}{z^{n+1}} = \int_0^{2\pi} f_{L,j}(e^{\mathrm{i}t}) e^{-\mathrm{i}t} dt
\]

Using (4), \(2\pi \mathrm{i} \cdot a_{n,j}(L) = \int_0^{2\pi} \sum_{m=0}^{\infty} B_{m,j}(L)(it)^m e^{-\mathrm{i}t} dt = i \int_0^{2\pi} \sum_{m=0}^{\infty} B_{m,j}(L)(it)^m e^{-\mathrm{i}t} dt = i \int_0^{2\pi} \sum_{m=0}^{\infty} B_{m,j}(L)(it)^m e^{-\mathrm{i}t} dt \)

\[
2\pi \cdot a_{n,j}(L) = \int_0^{2\pi} \sum_{m=0}^{\infty} B_{m,j}(L)(it)^m e^{-\mathrm{i}t} dt
\]
In order to show that we can exchange \( \int_0^{2\pi} \sum_{m=0}^{\infty} |B_{m\mu}(L)(it)^m e^{-int}| dt \) in the above formula, it suffices to prove that \( \int_0^{2\pi} \sum_{m=0}^{\infty} |B_{m\mu}(L)(it)^m e^{-int}| dt \) is finite.

\[
\int_0^{2\pi} \sum_{m=0}^{\infty} |B_{m\mu}(L)(it)^m e^{-int}| dt = \int_0^{2\pi} \sum_{m=0}^{\infty} |B_{m\mu}(L)| t^m dt \leq \int_0^{2\pi} \sum_{m=0}^{\infty} \sum_{k=-d}^{d} |a_{kj}(L)| \frac{h_m}{m!} t^m dt =
\]

\[
\int_0^{2\pi} \sum_{k=-d}^{d} |a_{kj}(L)| \sum_{m=0}^{\infty} \frac{(kt)^m}{m!} dt = \int_0^{2\pi} \sum_{k=-d}^{d} |a_{kj}(L)| e^{kt} dt = \sum_{k=-d}^{d} |a_{kj}(L)| \int_0^{2\pi} e^{kt} dt
\]

which is finite.

Hence, we can exchange \( \int_0^{2\pi} \sum_{m=0}^{\infty} \) in (6), so \( 2\pi a_{nj}(L) = \sum_{m=0}^{\infty} B_{m\mu}(L) \int_0^{2\pi} (it)^m e^{-int} dt \).

Let \( \lambda_{m,n} := \frac{1}{2\pi} \int_0^{2\pi} (it)^m e^{-int} dt \).

For any link \( L \), for any \( n, j \in \mathbb{Z} \), \( a_{nj}(L) = \sum_{m=0}^{\infty} B_{m\mu}(L) \cdot \lambda_{m,n} \) (6)

Let \( v^n_{nj}(L) = \sum_{m=0}^{N} \lambda_{m,n} \cdot B_{m\mu}(L) \) and let \( \mu \in N^* \). Let \( n, j \in \mathbb{Z} \).

The restriction of \( v^n_{nj} \) to links of \( \mu \) components is \( v^n_{nj}(L) = \sum_{m=0}^{N} \lambda_{m,n} \cdot B_{m\mu}(L) \). It is a linear combination of the Vassiliev invariants \( B_{m\mu} \) so it is a Vassiliev invariant. By (6), for any fixed \( L \in L(\mu) \), \( \lim_{N \to \infty} v^n_{nj}(L) = a_{nj}(L) \).

This proves that the restrictions of the coefficients of the HOMFLYPT polynomial to links with a fixed number of component are pointwise limits of Vassiliev invariants. By Proposition (11), they are pointwise limits of Vassiliev invariants.

**Remark 8.** The approximation we get is totally explicit because we have the following formula to compute the coefficients \( \lambda_{m,n} = \sum_{p=0}^{m-1} \frac{(2\pi i)^{m-p-1}}{n!} \frac{n!}{n! (m-p)!} \).

### 3 Approximation of the coefficients of the Kauffman two-variable polynomial by Vassiliev invariants

We will use the Dubrovnik version of the Kauffman polynomial.

First, recall that \( \Delta \) is defined by the axioms shown in Figure (11). \( \Delta \) is a regular isotopy invariant for unoriented links.

Let \( L \) be an oriented link, let \( \overrightarrow{D} \) be an oriented diagram for \( L \) and let \( D \) be the corresponding unoriented diagram.

The **Dubrovnik polynomial** is the knot invariant defined by \( F_L(a, z) = a^{-w(\overrightarrow{D})} \Delta(D) \) where \( w(\overrightarrow{D}) \) is the Writhe number.
1. $\triangle(\times) - \triangle(\bigcirc) = z\left(\triangle(\times) - \triangle(\bigcirc)\right)$
2. $\triangle(\bigcirc) = 1$
3. $\triangle(\bigtriangleup) = a \triangle(\bigtriangledown)$ and $\triangle(\bigtriangleup) = a^{-1} \triangle(\bigtriangledown)$

Figure 1: Axioms for $\Delta$

The Dubrovnik polynomial is related to the Kauffman polynomial by the following formula:

$F^D_L(a, z) = (-1)^{\text{Comp}(L)} F^K_L(a' = ia, z' = -iz)$

where $F^D_L(a, z)$ is the Dubrovnik polynomial and $F^K_L(a', z')$ is the Kauffman polynomial. Hence, once we prove that the coefficients of the Dubrovnik polynomial are limits of Vassiliev invariants, the same result will hold for the coefficients of the Kauffman polynomial.

The proof for the HOMFLYPT polynomial carries over to the Dubrovnik polynomial with only slight modifications since we can get preliminary results very similar to the ones we had for the HOMFLYPT polynomial, as shown below.

For instance, $F_L(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$, but as in the previous case, we can get a more precise statement.

**Lemma 9.** Let $\mu$ be the number of components of a link $L$. Its Dubrovnik polynomial is a polynomial (not a Laurent polynomial) in the variables $a, a^{-1}, z$ and $\frac{a - a^{-1}}{z}$, the maximum possible power of $\frac{a - a^{-1}}{z}$ being $\mu - 1$.

**Proof.** The proof is similar to the proof of the corresponding result for the HOMFLYPT polynomial.

Also, we can use the same change of variable as in the case of the HOMFLYPT polynomial to produce Vassiliev invariants:

**Lemma 10.** Let $W^D_{N,x}(L)$ be the polynomial obtained from the Dubrovnik polynomial of $L$ by setting $a := e^{Nx}$ and $z := x$. After power series expansion, each $w^D_{Nq}(L)\in W^D_{N,x}(L) = \sum_{q=0}^{\infty} w^D_{Nq}(L)x^q$ is a Vassiliev invariant of order (at most) $q$, for all $N \in \mathbb{Z}$.

Since we used same change of variable as before, the power series expansion will be the same and the proof is the same as the one for the HOMFLYPT polynomial from now on.
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