Strong spatial mixing of \( q \)-colorings on Bethe lattices

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Abstract

We investigate the problem of strong spatial mixing of \( q \)-colorings on Bethe lattices. By analyzing the sum-product algorithm we establish the strong spatial mixing of \( q \)-colorings on \((b+1)\)-regular Bethe lattices, for \( q \geq 1 + \lceil 1.764b \rceil \). We also establish the strong spatial mixing of \( q \)-colorings on binary trees, for \( q = 4 \).

1 Introduction

A \( q \)-coloring of a graph \( G = (V, E) \) is a function \( \sigma : V \to [q] \) such that no edge is monochromatic (that is, for \( \{u, v\} \in E \) we have \( \sigma(u) \neq \sigma(v) \)). A measure \( p \) on the set of \( q \)-colorings of an infinite graph \( G \) is an infinite-volume Gibbs measure if for every finite region \( R \), and for any \( q \)-coloring \( \sigma \) of \( G \), the conditional probability distribution \( p(\cdot \mid \sigma(V \setminus R)) \) is the uniform distribution on \( q \)-colorings of \( R \). It is known that there is at least one infinite-volume Gibbs measure for any graph \( G \). One problem of interest in statistical physics (c.f. [3]) is whether an infinite-volume Gibbs measure has strong spatial mixing.

Given a \( q \)-coloring \( \sigma \) and a set of vertices \( U \subseteq V \), let \( \sigma_U \) be the \( q \)-coloring restricted to \( U \). Given a measure \( p \), a vertex \( v \not\in U \), and a (partial) \( q \)-coloring \( \sigma_U \), let \( p^\sigma_U v \) be the marginal distribution on the colors of \( v \) conditioned on \( \sigma_U \). Let \( \dist(u,v) \) be the distance between \( u, v \) in \( G \), and let \( \dist(v,U) = \min_{u \in U} \dist(v,u) \).

The definition of strong spatial mixing we use is from [5] and [5] (we state the definition only for colorings).

Definition 1. Let \( \delta : \mathbb{N} \to \mathbb{R}^+ \). The infinite-volume Gibbs measure \( p \) on \( q \)-colorings of \( G = (V, E) \) has strong spatial mixing with rate \( \delta(\cdot) \) if and only if for every vertex \( v \), every \( U \subseteq V \), and every pair of \( q \)-colorings \( \sigma_U, \phi_U \),

\[
|p^\sigma_U v - p^\phi_U v| \leq \delta(\dist(v,\Delta)),
\]

where \( \Delta \subseteq U \) is the set of vertices on which \( \sigma_U \) and \( \phi_U \) differ.

Recently, strong spatial mixing received attention because of its connection with efficient approximation algorithms for certain spin systems (c.f. [3, 4]). For colorings of graphs, strong spatial mixing results were established for different lattice graphs [1, 2].

A Cayley tree (also known as Bethe lattice) \( \hat{T}^b \) is an infinite \((b+1)\)-regular tree. In this paper, we prove the strong spatial mixing for \( q \)-colorings on Cayley trees \( \hat{T}^b \).

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Theorem 1. For \( q \geq 1 + \lfloor \sqrt{b} \rfloor \) where \( c \approx 1.764 \) is the root of \( c = \exp(1/c) \), the infinite-volume Gibbs measure \( p \) on \( q \)-colorings of \( \hat{T}^b \) has strong spatial mixing with rate \( \delta(d) = C \exp(-\alpha d) \) for some positive constants \( C \) and \( \alpha \).

We also establish the strong spatial mixing of \( q \)-colorings on binary trees, for \( q = 4 \).

Theorem 2. Let \( q = 4 \). The infinite-volume Gibbs measure \( p \) on \( q \)-colorings of \( \hat{T}^2 \) has strong spatial mixing with rate \( \delta(d) = C \exp(-\alpha d) \) for some positive constants \( C \) and \( \alpha \).

We will prove Theorem 1 and Theorem 2 by analyzing the sum-product algorithm, which we review in the next section.

2 The sum-product algorithm

Let \( T = (V, E) \) be a \( b \)-ary tree, \( U \subseteq V \) be a subset of vertices, and \( \sigma_U : U \rightarrow [q] \) be a \( q \)-coloring on the vertices in \( U \). For every vertex \( v \in V \), a message (according to \( \sigma_U \)) from \( v \) to its parent is a probability distribution \( \alpha \in \mathbb{R}^q \) on \([q]\) where \( \alpha_i \) is proportional to the number of \( q \)-colorings of the subtree rooted at \( v \) such that the color of \( v \) is different from \( i \). The message from \( v \) to its parent can also be defined recursively as follows.

- If \( v \in U \) and \( \sigma_U(v) = k \) for some \( k \in [q] \), then for \( i \in [q] \),
  \[ \alpha_i = \begin{cases} 0, & \text{if } i = k, \\ 1/(q-1), & \text{otherwise}. \end{cases} \]

- If \( v \in V \setminus U \) and \( v \) is a leaf, then for \( i \in [q] \), \( \alpha_i = 1/q \).

- If \( v \in V \setminus U \) and \( v \) is not a leaf, let \( \beta^\ell, \ell \in [b] \), be the message from the \( \ell \)-th child of \( v \) to \( v \). Then \( \alpha = f(\beta^1, \ldots, \beta^b) \), where \( f : \mathbb{R}^b \rightarrow \mathbb{R}^q \) is defined by
  \[ (f(\beta^1, \ldots, \beta^b))_i = \frac{\sum_{j \in [q], j \neq i} \prod_{\ell=1}^b \beta^\ell_j}{(q-1) \sum_{j \in [q]} \prod_{\ell=1}^b \beta^\ell_j}. \tag{1} \]

Note that the right-hand side of (1) is always bounded by \( 1/(q-1) \) and hence all messages are in the set \( S_1 \), where \( S_1 \) is the set of vectors \( \gamma \in \mathbb{R}^q \) satisfying
\[ \sum_{i=1}^q \gamma_i = 1, \quad \text{and} \quad 0 \leq \gamma_i \leq \frac{1}{q-1}, \text{ for all } i \in [q]. \tag{2} \]

The following folklore result gives a connection between strong spatial mixing and the sum-product algorithm.

Lemma 1. Assume that there exists a function \( \delta \) such that for every \( b \)-ary tree \( T = (V, E) \) (with root \( r \)), for any subset of vertices \( U \subseteq V \), and any pair of configurations \( \sigma_U, \phi_U : U \rightarrow [q] \), the message \( \alpha \) from \( u \) (a child of \( r \)) to \( r \) according to \( \sigma_U \) and the message \( \beta \) from \( u \) to \( r \) according to \( \phi_U \) satisfy
\[ \|\alpha - \beta\| \leq \delta(\text{dist}(r, \Delta)), \tag{3} \]
where \( \| \cdot \| \) is some fixed norm and \( \Delta \subseteq U \) is a set where \( \sigma_U \) and \( \phi_U \) differ. Then the infinite-volume Gibbs measure \( p \) on \( q \)-colorings of \( \hat{T}^b \) has strong spatial mixing with rate \( C\delta(d) \) for some positive constant \( C \) (the constant \( C \) depends on \( b \) and the norm used).
We need the following property of the messages in $S_1$.

**Lemma 2.** Let $\alpha^1, \ldots, \alpha^b \in S_1$, then

$$
\sum_{j \in [q]} \prod_{i \in [b]} \alpha^i_j \geq \frac{q-b}{(q-1)^b}.
$$

**Proof.** We prove the statement by induction on $b$. For $b = 1$, the statement is true.

We assume that the statement is true for $b = t \geq 1$. We now prove the statement for $b = t + 1$.

Let $z_j = \prod_{i=1}^t \alpha^i_j$, for $j \in [q]$. We assume, w.l.o.g., that $z_1 \leq \ldots \leq z_q$. We have

$$
\sum_{j \in [q]} \prod_{i \in [t]} \alpha^i_j = \sum_{j \in [q]} z_j \alpha^{t+1}_j \geq z_q \cdot 0 + \sum_{j=1}^{q-1} z_j/(q-1).
$$

Fixing $\alpha^{t+1} = (1/(q-1), \ldots, 1/(q-1), 0)$, we have $\sum_{j \in [q]} \prod_{i \in [t]} \alpha^i_j$ is minimized when $z_q$ is maximized. Note that $z_q = \prod_{i=1}^t \alpha^i_q \leq 1/(q-1)^b$. Hence we have $\sum_{j \in [q]} \prod_{i \in [t]} \alpha^i_j$ is minimized when $\alpha^i_q = 1/(q-1)$, for all $i \in [t]$.

We next bound $\sum_{j=1}^{q-1} z_j$ from below. Let $\beta^i_j = \alpha^i_j(q-1)/(q-2)$, for $i \in [t]$ and $j \in [q-1]$. Note that $\sum_{j \in [q-1]} \beta^i_j = 1$ and $0 \leq \beta^i_j \leq 1/(q-2)$, for $i \in [t]$ and $j \in [q-1]$. By induction hypothesis, we have

$$
\sum_{j \in [q-1]} \prod_{i \in [t]} \beta^i_j \geq \frac{q-1-t}{(q-2)^t}.
$$

Hence we have

$$
\sum_{j=1}^{q-1} z_j = \sum_{j \in [q-1]} \prod_{i \in [t]} \alpha^i_j = \sum_{j \in [q-1]} \prod_{i \in [t]} \frac{\beta^i_j(q-2)}{q-1} = \frac{(q-2)^t}{(q-1)^t} \sum_{j \in [q-1]} \prod_{i \in [t]} \beta^i_j \geq \frac{q-1-t}{(q-1)^t}.
$$

By (4) and (5), we have

$$
\sum_{j \in [q]} \prod_{i \in [b]} \alpha^i_j \geq \frac{q-b}{(q-1)^b}.
$$

\[\square\]

## 3 The messages in the sum-product algorithm contract

### 3.1 Case $q \geq 1 + \lceil cb \rceil$

Theorem 1 will follow from Lemma 1 and the following lemma, which shows that (1) is a contraction in the following sense: if in a node we have a pair of messages from each child then the pair of messages from the node (where the $i$-th component in the pair is obtained by applying (1) to the $i$-th components of pairs from the children) is closer in the $\ell_1$-norm than the $\ell_1$-distance within the pair from at least one child.
Lemma 3. Let $T = (V, E)$ be a $b$-ary tree rooted at $r$. Let $w \neq r$ be a vertex of $T$ and let $u^1, \ldots, u^b$ be the $b$ children of $w$. Let $U \subseteq V$. Let $\sigma_U, \phi_U : U \rightarrow [q]$ be a pair of configurations such that $\text{dist}(w, \Delta) \geq 1$, where $\Delta \subseteq U$ is the set of vertices on which $\sigma_U$ and $\phi_U$ differ. For $\ell \in [b]$, let $\alpha^\ell$ and $\beta^\ell$ be the messages from $u^\ell$ to $w$ according to $\sigma_U$ and $\phi_U$, respectively. Then the messages $\zeta$ and $\eta$ from $w$ according to $\sigma_U$ and $\phi_U$, respectively, satisfy
\[
\|\zeta - \eta\|_1 \leq \frac{b}{q} \left(1 - \frac{1}{q - b}\right)^{-b + b^2/q} \cdot \max_{\ell \in [b]} \|\alpha^\ell - \beta^\ell\|_1.
\]

Remark 1. In the previous version of the paper, we stated an incorrect version of Lemma 3 using $\ell_{\infty}$-norm, thanks to Sidhant Misra and David Gamarnik for pointing out the error.

Proof of Theorem 1. We will claim that $\frac{b}{q} \left(1 - \frac{1}{q - b}\right)^{-b + b^2/q} < 1$ when $q \geq 1 + \lceil cb \rceil$, where $c > 0$ is the root of $\exp(1/c) = c$. Taking the derivative of $\frac{b}{q} \left(1 - \frac{1}{q - b}\right)^{-b + b^2/q}$ w.r.t. $q$, we obtain
\[
\frac{b}{q} \left(q - b + 1\right) \ln \left(\frac{q - b}{q - b + 1}\right) \left(q - b - 1\right) \left(q - b\right)^{b(b - q)/q}.
\]
We will show that $\frac{b}{q} \left(1 - \frac{1}{q - b}\right)^{-b + b^2/q}$ is not positive when $q \geq b + 1$. It is sufficient to prove that $-q^2 + q + b^2(q - b - 1) \ln \left(\frac{q - b}{q - b - 1}\right) \leq 0$. Note that $\ln(1 + x) \leq x$ for all $x \geq 0$. Hence we have
\[
b^2(q - b - 1) \ln \left(\frac{q - b}{q - b - 1}\right) \leq b^2 \leq (q - 1)^2 \leq q^2 - q.
\]
We now prove that $\frac{b}{q} \left(1 - \frac{1}{q - b}\right)^{-b + b^2/q} < 1$ when $q = 1 + cb$ and $b \geq 2$. Let
\[
g(b) = \frac{b}{cb + 1} \left(1 - \frac{1}{(c - 1)b + 1}\right)^{-b + b^2/(cb + 1)}.
\]
Taking the derivative of $g$, we have
\[
\frac{dg}{db} = -\frac{1}{(cb + 1)^3} \left(\frac{c - 1)b}{(c - 1)b + 1}\right)^{b(c - 1)b/(cb + 1)} \left(b + cb^2 - 1 - cb + (b + 2cb^2 + c^2b^3 - 2b^2 - cb^3) \ln \left(\frac{b}{(c - 1)b + 1}\right)\right).
\]
We will show that $\frac{dg}{db} > 0$ for $b \geq 2$. It is sufficient to prove that
\[
b + cb^2 - 1 - cb + (b + 2cb^2 + c^2b^3 - 2b^2 - cb^3) \ln \left(\frac{b}{(c - 1)b + 1}\right) < 0.
\]
Note that $\ln(1 + x) \geq x - x^2/2$ for all $x \geq 0$. We have

\[
-b - cb^2 + 1 + cb + (b + 2cb^2 + c^2b^3 - 2b^2 - cb^3) \ln \left(1 + \frac{1}{(c-1)b}\right)
\geq \frac{(2c^3 - 3c^2 - c + 2)b^2 + (2c^2 - 4c + 2)b - 1}{2(c-1)^2b} > 0,
\]

for all $b \geq 2$. Hence, $\frac{da}{db} > 0$ for $b \geq 2$. Note that $g(b) \to 1$ as $b \to \infty$. We have $g(b) < 1$ for $b \geq 2$.

Hence Theorem 4 follows from Lemma 1 and Lemma 3.

Before proving Lemma 5 we need a more detailed understanding of the messages. Let $S'_1 \subseteq S_1$ be the set of vectors $\gamma \in \mathbb{R}^q$ satisfying the following property:

\[
\text{for every } i \in [q], \text{ we have }\frac{1}{q-1} \left(1 - \frac{1}{q-1}\right) \leq \gamma_i \leq \frac{1}{q-1}.
\]

Let $S_2$ be the set of permutations of $(0, 1/(q-1), \ldots, 1/(q-1))$.

**Claim 1.** Let $\gamma \in S'_1$, then $\gamma$ has at most $b$ entries of value $1/(q-1)$. If $\gamma$ has $b$ entries of value $1/(q-1)$, then all other entries of $\gamma$ have value $(1 - 1/(q-b))/(q-1)$.

**Proof.** Assume that $\gamma$ has $s$ entries of $1/(q-1)$ and $\gamma_1, \ldots, \gamma_s = 1/(q-1)$. Then by (7), we have

\[
1 - \sum_{j \in [q]} \gamma_j = \frac{s}{q-1} + \sum_{j=s+1}^q \gamma_j \geq \frac{s}{q-1} + \frac{q-s}{q-1} \left(1 - \frac{1}{q-b}\right) = \frac{1}{q-1} \left(\frac{q-q-s}{q-b}\right),
\]

which implies $s \leq b$. Note that if $s = b$, we have $\gamma_{b+1} = \ldots = \gamma_q = (1 - 1/(q-b))/(q-1)$.

The following lemma shows that the set $S'_1 \cup S_2$ contains all the possible messages.

**Lemma 4.** For every $\beta^1, \ldots, \beta^b \in S'_1 \cup S_2$, we have $f(\beta^1, \ldots, \beta^b) \in S'_1$.

**Proof.** To establish (7) we use Lemma 2 and the fact $0 \leq \beta^1, \ldots, \beta^b \leq 1/(q-1)$:

\[
(f(\beta^1, \ldots, \beta^b))_i = \frac{1}{q-1} \left(1 - \frac{\prod_{\ell=1}^b \beta^\ell_i}{\sum_{j \in [q]} \prod_{\ell=1}^b \beta^\ell_j}\right) \geq \frac{1}{q-1} \left(1 - \frac{1}{q-b}\right).
\]

Lemma 5 follows from the following two lemmas:

**Lemma 5.** For every $\gamma^1, \ldots, \gamma^{b-1} \in S'_1 \cup S_2$ and every $\alpha, \beta \in S'_1$, we have

\[
\|f(\gamma^1, \ldots, \gamma^{b-1}, \alpha) - f(\gamma^1, \ldots, \gamma^{b-1}, \beta)\|_1 \leq \frac{1}{(q-1)^b A} \|\alpha - \beta\|_1,
\]

where $A = \sum_{j \in [q]} z_j \alpha_j$ and $z_j = \prod_{i \in [b-1]} \gamma^i_j$, for $j \in [q]$. 5
Lemma 6. For every $0 \leq s \leq b < q$, let $\alpha^1, \ldots, \alpha^{b-s} \in S_1^r$ and $\alpha^{b-s+1}, \ldots, \alpha^b \in S_2$, we have
\[
\sum \prod_{j \in [q]} \alpha^j \geq \frac{q-s}{(q-1)^b} \left(1 - \frac{1}{q-b}\right)^{b-(b-s)/q}.
\] (8)

We first prove Lemma 3 and then Lemma 5 and Lemma 6.

Proof of Lemma 3 If $w \in U$ then from the assumption $\text{dist}(w, \Delta) \geq 3$ we have $\sigma_U(w) = \phi_U(w)$ and hence $\zeta = \eta$. From now on we assume that $w \notin U$ and thus $\zeta = f(\alpha^1, \ldots, \alpha^b)$ and $\eta = f(\beta^1, \ldots, \beta^b)$.

Let $s$ be the number of children of $w$ which are in $U$. W.l.o.g., we assume that $u^1, \ldots, u^s$ are in $U$. Note that we have $Q(s) = \sum_{j=1}^s \alpha^j$ as a function of $s$. We first prove Lemma 3, and then Lemma 5 and Lemma 6.

Proof of Lemma 5 We now prove Lemma 5.

Proof of Lemma 6 We will show that for every
\[
\sum_{k \in [q]} s_k \cdot ((f(\gamma^1, \ldots, \gamma^{b-1}, \alpha))_k - (f(\gamma^1, \ldots, \gamma^{b-1}, \beta))_k) \leq \frac{1}{(q-1)^b A} \|\alpha - \beta\|_1.
\]
Let
\[
Q(t, \alpha, \beta) := \|\alpha - ((1-t)\alpha + t\beta)\|_1
\]
\[
- (q-1)^b A \sum_{k \in [q]} s_k \cdot ((f(\gamma^1, \ldots, \gamma^{b-1}, \alpha))_k - (f(\gamma^1, \ldots, \gamma^{b-1}, (1-t)\alpha + t\beta))_k).
\]
Note that we have $Q(0, \alpha, \beta) = 0$ and our goal is to lower bound $Q(1, \alpha, \beta)$. We have
\[
P(t, \alpha, \beta) := \frac{\partial}{\partial t} Q(t, \alpha, \beta)
\]
\[
= \|\alpha - \beta\|_1 - \frac{(q-1)^b A}{B^2} \sum_{k \in [q]} s_k z_k \left( \alpha_k \left( \sum_{j=1}^q z_j \beta_j \right) - \beta_k \left( \sum_{j=1}^q z_j \alpha_j \right) \right),
\]
where $\alpha = (\alpha^1, \ldots, \alpha^b)$, $\beta = (\beta^1, \ldots, \beta^b)$, and $z = (z^1, \ldots, z^b)$. 

\[\|\alpha - \beta\|_1 = \sum_{k \in [q]} \left| \alpha_k - \beta_k \right|.
\]

The last inequality follows from the facts that $(b-s)/(q-s)$ as a function of $s$ is monotonically decreasing for $0 \leq s \leq b < q$ and $(b-s)^2/(q-s) - (b-s)$ as a function of $s$ is monotonically increasing for $0 \leq s \leq b < q$. \]
where
\[ B = \sum_{j=1}^{q} z_j((1-t)\alpha_j + t\beta_j). \]

We are going to lower bound \( P(t, \alpha, \beta) \) for all \( \alpha, \beta \in S'_1 \) and \( t \in [0,1) \). We have
\[ P(t, \alpha, \beta) = \frac{1}{1-t} P(0, (1-t)\alpha + t\beta, \beta), \]
and hence it is enough to consider the case \( t = 0 \) (note that \( S'_1 \) is convex, and hence \((1-t)\alpha + t\beta\) is in \( S'_1 \) if \( \alpha, \beta \) are in \( S'_1 \)). Substituting \( \beta_j = \alpha_j + \varepsilon_j \) into \( P(0, \alpha, \beta) \) we obtain
\[ P(0, \alpha, \beta) = \sum_{j \in [q]} |\varepsilon_j| - \frac{(q-1)^{b-1}}{A} \sum_{k \in [q]} s_k z_k \left( \sum_{j \in [q], j \neq k} \alpha_k z_j \varepsilon_j \right) - \varepsilon_k \left( \sum_{j \in [q], j \neq k} z_j \alpha_j \right) \]
\[ = \sum_{j \in [q]} |\varepsilon_j| - \frac{(q-1)^{b-1}}{A} \sum_{k \in [q]} \varepsilon_k z_k \sum_{j \in [q]} (s_j - s_k) z_j \alpha_j \]
\[ = \sum_{j \in [q]} |\varepsilon_j| - \frac{(q-1)^{b-1}}{A} \sum_{k \in [q]} \varepsilon_k \tau_k, \]
where \( \tau_k := s_k \sum_{j \in [q]} (s_j - s_k) z_j \alpha_j \).

Let \( \varepsilon^+_i = \max \{\varepsilon_i, 0\} \), \( \varepsilon^-_i = \max \{-\varepsilon_i, 0\} \), and \( D = \sum_{i=1}^{q} \varepsilon^+_i = \sum_{i=1}^{q} \varepsilon^-_i = ||\alpha - \beta||_1/2 \). We have
\[ P(0, \alpha, \beta) = 2D - \frac{(q-1)^{b-1}}{A} \sum_{k \in [q]} (\varepsilon^+_k - \varepsilon^-_k) \tau_k \]
\[ \geq 2D - \frac{(q-1)^{b-1}}{A} \left( \sum_{k \in [q]} \varepsilon^+_k \max \tau_k - \sum_{k \in [q]} \varepsilon^-_k \min \tau_k \right) \]
\[ = 2D - \frac{(q-1)^{b-1}}{A} D \left( \max \tau_k - \min \tau_k \right). \]

Claim 2.
\[ \max_{k \in [q]} \tau_k - \min_{k \in [q]} \tau_k \leq \frac{2A}{(q-1)^{b-1}}. \]

**Proof of Claim** We assume, w.l.o.g., that the largest \( \tau_k \) is \( \tau_q \) and the smallest \( \tau_k \) is \( \tau_1 \). We will show
\[ \tau_q - \tau_1 = \sum_{j=1}^{q} (z_q(s_j - s_q) - z_1(s_j - s_1)) z_j \alpha_j \leq \frac{2A}{(q-1)^{b-1}}. \]  \( \text{(10)} \)

Note that \( s_q \) occurs in \( \text{(10)} \) with negative sign and hence we can assume \( s_q = -1 \). Similarly \( s_1 \) occurs in \( \text{(10)} \) with positive sign and hence we can assume \( s_1 = +1 \).

Let \( P \) be the set of \( j \in [q] \) such that \( s_j = +1 \). Let \( \overline{P} = [q] \setminus P \). We have \{1\} \subseteq P and \{q\} \subseteq \overline{P}.

We can rewrite \( \text{(10)} \) as follows
\[ \tau_q - \tau_1 = 2z_q \sum_{j \in P} z_j \alpha_j + 2z_1 \sum_{j \in \overline{P}} z_j \alpha_j. \]  \( \text{(11)} \)
Note that the right-hand side of (11) is symmetric between $z_1$ and $z_q$ and hence we can, w.l.o.g., assume $z_q \geq z_1$. For fixed $\alpha, z$ the right hand-side of (11) is maximized when $P = [q - 1]$. Hence we have

$$\tau_q - \tau_1 \leq 2z_q(A - z_q\alpha_q) + 2z_1z_q\alpha_q = 2z_qA - 2(z_q - z_1)z_q\alpha_q \leq \frac{2A}{(q-1)^{b-1}},$$

where in the last inequality we used $z_1 \leq z_q \leq 1/(q-1)^{b-1}$.

We now continue the proof of Lemma 5. By Claim 2 we have

$$P(0, \alpha, \beta) \geq 0.$$ Hence we have

$$Q(1, \alpha, \beta) = \int_0^1 P(t, \alpha, \beta)dt = \int_0^1 \frac{1}{1-t}P(0, (1-t)\alpha + t\beta, \beta)dt \geq 0. \quad (12)$$

From (9) and (12), we obtain $\|\alpha - \beta\|_1 \geq (q-1)^bA\|f(\gamma^1, \ldots, \gamma^{b-1}, \alpha) - f(\gamma^1, \ldots, \gamma^{b-1}, \beta)\|_1$. \hfill \Box

Before proving Lemma 6 we will show that the inequality of Lemma 2 can be strengthened if we assume that $\alpha^1, \ldots, \alpha^b \in S'_1$.

**Lemma 7.** Let $\alpha^1, \ldots, \alpha^b \in S'_1$, we have

$$\sum_{j \in [q]} \prod_{i \in [b]} \alpha^i_j \geq \frac{q}{(q-1)^b} \left(1 - \frac{1}{q-b}\right)^{b-2b/q}. \quad (13)$$

**Proof.** We first claim that the LHS of (13) is minimized when $\alpha^\ell$ has $b$ entries of value $1/(q-1)$ for all $\ell \in [b]$. Fix $\alpha^1, \ldots, \alpha^{b-1}$. Let $z_j = \prod_{i \in [b-1]} \alpha^i_j$, for $j \in [q]$. W.l.o.g., we assume that $z_1 \leq z_2 \leq \ldots \leq z_q$. Then by Claim 1 the LHS of (13) is minimized when $\alpha^b_1 = \ldots = \alpha^b_q = 1/(q-1)$ and $\alpha^b_{b+1} = \ldots = \alpha^b_q = (1 - 1/(q-1))/q-1$. By the same claim, the LHS of (13) is minimized when $\alpha^\ell$ has $b$ entries of value $1/(q-1)$ for all $\ell \in [b]$. Hence we can assume that $\alpha^\ell$ has $b$ entries of value $1/(q-1)$ for all $\ell \in [b]$.

We next claim that the LHS of (13) is minimized when the number of $1/(q-1)$ in $(\alpha^\ell_j)_{\ell \in [b]}$ is either $\lfloor b^2/q \rfloor$ or $\lfloor b^2/q \rfloor + 1$, for every $j \in [q]$. Let $j_1, j_2 \in [q]$ and $j_1 \neq j_2$. Fix $(\alpha^\ell_j)_{\ell \in [b]}$ for all $j \in [q] \setminus \{j_1, j_2\}$. Let $t_1$ be the number of $1/(q-1)$ in $(\alpha^\ell_{j_1})_{\ell \in [b]}$ and let $t_2$ be the number of $1/(q-1)$ in $(\alpha^\ell_{j_2})_{\ell \in [b]}$. W.l.o.g., we assume that $t_1 \leq t_2$. We claim that the LHS of (13) is minimized when $t_2 - t_1 \leq 1$. We have

$$\prod_{\ell \in [b]} \alpha^\ell_{j_1} + \prod_{\ell \in [b]} \alpha^\ell_{j_2} = \frac{1}{(q-1)^b} \left(1 - \frac{1}{q-b}\right)^{b-t_1} + \frac{1}{(q-1)^b} \left(1 - \frac{1}{q-b}\right)^{b-t_2}$$

$$= \frac{1}{(q-1)^b} \left(1 - \frac{1}{q-b}\right)^{b-t_1-1} + \frac{1}{(q-1)^b} \left(1 - \frac{1}{q-b}\right)^{b-t_2+1} + \frac{1}{(q-1)^b(q-b)} \left(1 - \frac{1}{q-b}\right)^{b-t_1-1} \left(1 - \frac{1}{q-b}\right)^{t_1+1-t_2-1}.$$
If $t_2 - t_1 > 1$, then
\[
\prod_{\ell \in [b]} \alpha_{j_1}^\ell + \prod_{\ell \in [b]} \alpha_{j_2}^\ell > \frac{1}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-t_1} + \frac{1}{(q-1)^q} \left( 1 - \frac{1}{q-b} \right)^{b-t_2+1},
\]
and the LHS of (13) becomes smaller by moving one $1/(q-1)$ from $(\alpha_{j_2}^\ell)_{\ell \in [b]}$ to $(\alpha_{j_1}^\ell)_{\ell \in [b]}$.

The minimum value of the LHS of (13) is:
\[
\frac{1}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-\lfloor b^2/q \rfloor} (q - b^2 + q\lfloor b^2/q \rfloor) + \frac{1}{(q-1)^q} \left( 1 - \frac{1}{q-b} \right)^{b-\lfloor b^2/q \rfloor - 1} (b^2 - q\lfloor b^2/q \rfloor).
\]

Let $\{b^2/q\}$ be the fractional part of $b^2/q$, we can rewrite (14) as
\[
\frac{1}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-b^2/q + (b^2/q)} q(1 - \{b^2/q\}) + \frac{1}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-b^2/q + (b^2/q) - 1} q\{b^2/q\}
= \frac{q}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-b^2/q} \left( (1 - \frac{1}{q-b})^{(b^2/q)} (1 - \{b^2/q\}) + (1 - \frac{1}{q-b})^{(b^2/q) - 1} \{b^2/q\} \right)
\geq \frac{q}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-b^2/q},
\]
where the last inequality follows from the fact that $x^y(1-y) + x^{y-1}y \geq 1$ for $0 < x < 1$ and $0 \leq y < 1$.

We now prove Lemma 6.

**Proof of Lemma 6** We prove the statement by induction on $s$. For $s = 0$, the statement follows from Lemma 7.

We assume that the statement is true for $s = t \geq 0$. We next consider the case when $s = t + 1$. W.l.o.g., we assume $\alpha^b = (1/(q-1), \ldots, 1/(q-1), 0)$. The LHS of (8) is minimized when $\alpha^\ell_q = 1/(q-1)$ for $\ell \in [b-1]$.

We define $\beta^\ell_j = \alpha^\ell_j(q-1)/(q-2)$, for $j \in [q-1]$ and $\ell \in [b-1]$. Note that $\sum_{\ell \in [q]} \beta^\ell_j = 1$, for $\ell \in [b-1]$, and $0 \leq \beta^\ell_j \leq 1/(q-2)$, for $b-s+1 \leq \ell \leq b-1$, and $(1-1/(q-b))/(q-2) \leq \beta^\ell_j \leq 1/(q-2)$, for $\ell \in [b-s]$.

By induction hypothesis, we have
\[
\sum_{j \in [q-1]} \prod_{\ell \in [b-1]} \beta^\ell_j \geq \frac{q-1-t}{(q-2)^{b-1}} \left( 1 - \frac{1}{q-b} \right)^{b-1-t-(b-1-t)^2/(q-1-t)}.
\]
Hence the LHS of (8) is lower bounded by:
\[
\frac{1}{q-1} \sum_{j \in [q-1]} \prod_{\ell \in [b-1]} \alpha^\ell_j = \frac{1}{q-1} \sum_{j \in [q-1]} \prod_{\ell \in [b-1]} (q-2)\beta^\ell_j = \frac{(q-2)^{b-1}}{(q-1)^b} \sum_{j \in [q-1]} \prod_{\ell \in [b-1]} \beta^\ell_j
\geq \frac{q-s}{(q-1)^b} \left( 1 - \frac{1}{q-b} \right)^{b-s-(b-s)^2/(q-s)}.
\]

\]
3.2 Case $q = 4$ and $b = 2$

In this section, we assume that $q = 4$ and $b = 2$. We will prove the following strengthening of Lemma 3 for the special case $q = 4$ and $b = 2$.

**Lemma 8.** Let $T$ be a binary tree rooted at $r$. Let $w \neq r$ be a vertex of $T$ and let $u$ and $u'$ be the two children of $w$. Let $U \subseteq V$ and let $\sigma_U, \phi_U : U \to [4]$ be a pair of configurations such that $\text{dist}(w, \Delta) \geq 3$, where $\Delta \subseteq U$ is the set of vertices on which $\sigma_U$ and $\phi_U$ differ. Let $\alpha, \beta$ be the messages from $u$ to $w$ according to $\sigma_U$ and $\phi_U$, respectively, and let $\alpha'$ and $\beta'$ be the messages from $u'$ to $w$ according to $\sigma_U$ and $\phi_U$, respectively. Then the messages $\zeta$ and $\eta$ from $w$ according to $\sigma_U$ and $\phi_U$, respectively, satisfy

$$\|\zeta - \eta\|_1 \leq \frac{48}{49} \cdot \max\{\|\alpha - \beta\|_1, \|\alpha' - \beta'\|_1\}. \quad (15)$$

Theorem 2 now follows:

**Proof of Theorem 2.** Theorem 2 follows from Lemma 1 and Lemma 8.

Before proving Lemma 8, we need a more detailed understanding of the messages. Let $S_1' \subseteq S_1$ be the set of vectors $\gamma \in \mathbb{R}^4$ satisfying the following three properties:

- for every $i \in [4]$ we have $1/6 \leq \gamma_i \leq 1/3$, (16)
- for every $i \in [4]$ either $\gamma_i = 1/3$ or $\gamma_i \leq 11/36$, (17)
- if $\gamma$ has exactly two entries of value $1/3$, then $\gamma$ is a permutation of $(1/6, 1/6, 1/3, 1/3)$. (18)

Let $S_2$ be the set of permutations of $(0, 1/3, 1/3, 1/3)$.

**Definition 2.** We say that two vectors $\gamma, \xi \in S_1' \cup S_2$ are coupled if for every $i \in [4]$ we have $\gamma_i = 1/3$ if and only if $\xi_i = 1/3$.

**Claim 3.** Let $\gamma, \xi \in S_1$. Then $\sum_{i=1}^{4} \gamma_i \xi_i \leq \frac{1}{3}$.

**Proof.** W.l.o.g., we assume that $\gamma_1 \leq \ldots \leq \gamma_4$. For fixed $\gamma$, the maximum of $\sum_{i=1}^{4} \gamma_i \xi_i$ over $\xi \in S_1$ happens for $\xi = (0, 1/3, 1/3, 1/3)$ and hence

$$\sum_{i=1}^{4} \gamma_i \xi_i \leq \gamma_1 \cdot 0 + \frac{\gamma_2}{3} = (1 - \gamma_1)/3 \leq \frac{1}{3}.$$ 

The following lemma shows that the set $S_1' \cup S_2$ contains all the possible messages.

**Lemma 9.** For every $\gamma, \xi \in S_1' \cup S_2$, we have $f(\gamma, \xi) \in S_1$.

**Proof.** To establish (16) we use Lemma 2 and the fact $0 \leq \gamma_i, \xi_i \leq 1/3$:

$$(f(\gamma, \xi))_i = \frac{1}{3} \left(1 - \gamma_i \xi_i / \left(\sum_{j=1}^{4} \gamma_j \xi_j\right)\right) \geq \frac{1}{3} \left(1 - \frac{1}{2}\right) = 1/6.$$
Note that if \((f(\gamma, \xi))_i \neq 1/3\) then \(\gamma_i \neq 0\) and \(\xi_i \neq 0\). Then (16) implies \(\gamma_i \geq 1/6\) and \(\xi_i \geq 1/6\) which combined with the upper bound of Claim 3 yields (17)

\[
(f(\gamma, \xi))_i \leq \frac{1}{3} \left(1 - \frac{1}{36} \right) = \frac{11}{36}.
\]

Now we show (18). Assume \(f(\gamma, \xi)_i = f(\gamma, \xi)_j = 1/3\) for \(i \neq j\). Then we have \((\gamma_i = 0 \lor \xi_j = 0)\) and \((\gamma_j = 0 \lor \xi_i = 0)\). Note that at most one entry of \(\gamma\) and at most one entry of \(\xi\) can be \(0\) (and then \(\gamma, \xi \in S_2\)). We can, w.l.o.g., assume \(\gamma = (0, 1/3, 1/3, 1/3)\) and \(\xi = (1/3, 0, 1/3, 1/3)\). Hence \(f(\gamma, \xi) = (1/6, 1/6, 1/3, 1/3)\). □

Lemma 8 will follow from the following contraction properties of (1).

**Lemma 10.** Let \(\alpha, \beta \in S'_1\) be coupled, and let \(\gamma \in S'_1 \cup S_2\), we have

\[
\|f(\alpha, \gamma) - f(\beta, \gamma)\| \leq \frac{1}{9} \sum_{i=1}^{4} \alpha_i \beta_i \|\alpha - \beta\|_1.
\]

**Proof.** Note that \(S'_1\) defined by equations (16)–(18) is not a convex set. However, if \(\alpha, \beta \in S'_1\) and \(\alpha, \beta\) are coupled, then \((1 - t)\alpha + t\beta \in S'_1\) and \(\alpha, (1 - t)\alpha + t\beta\) are coupled. The lemma then follows from the same proof of Lemma 5 □

**Lemma 11.** Let \(\alpha, \beta \in S'_1\) be coupled, and let \(\gamma \in S'_1\) be such that \(\gamma\) has at most one entry of value 1/3. Then we have

\[
\frac{49}{24} \|f(\alpha, \gamma) - f(\beta, \gamma)\| \leq \|\alpha - \beta\|_1.
\]

We first show how Lemma 8 follows from Lemma 10 and Lemma 11 and then prove Lemma 11.

**Proof of Lemma 8.** If \(w \in U\) then from the assumption \(\text{dist}(w, \Delta) \geq 3\) we have \(\sigma_U(w) = \phi_U(w)\) and hence \(\xi = \eta\). From now on we assume that \(w \notin U\) and thus \(\xi = f(\alpha, \alpha')\) and \(\eta = f(\beta, \beta')\).

We will now show that \(\alpha \) and \(\beta\) are coupled. If \(u \in U\), we have \(\alpha = \beta\) (this follows from \(\sigma_U(u) = \phi_U(u)\), which is true since \(\text{dist}(u, \Delta) \geq 2\)). Now assume \(u \notin U\). Suppose \(\alpha_i = 1/3\) for \(i \in [4]\). By the definition of \(f\) in (1), we know that at least one child, say \(v\), of \(u\) has color \(i\) in \(\sigma_U\). Note that \(\text{dist}(v, \Delta) \geq 1\) and hence \(\sigma_U(v) = \phi_U(v)\) which implies \(\beta_i = 1/3\). Hence \(\alpha\) and \(\beta\) are coupled. The same argument yields that \(\alpha'\) and \(\beta'\) are coupled.

If \(\alpha' = \beta'\) and \(\alpha \neq \beta\), then \(\alpha, \beta \in S'_1\). Hence (15) follows from Lemma 10 and Lemma 2 as we have

\[
\|f(\alpha, \alpha') - f(\beta, \alpha')\| \leq \frac{1}{2} \|\alpha - \beta\|_1 \leq \frac{48}{49} \|\alpha - \beta\|_1 \leq \frac{48}{49} \max\{\|\alpha - \beta\|_1, \|\alpha' - \beta'\|_1\}.
\]

The same argument applies if \(\alpha = \beta\) and \(\alpha' \neq \beta'\), and hence from now on we assume \(\alpha \neq \beta\) and \(\alpha' \neq \beta'\).

We next claim that if one of \(\alpha, \beta\) has two or more entries of value 1/3, then \(\alpha = \beta\). By the previous paragraph, \(\alpha\) and \(\beta\) have value 1/3 in the same entries (they are coupled). By (18) they are either permutations of \((0, 1/3, 1/3, 1/3)\) or \((1/6, 1/6, 1/3, 1/3)\), and in both cases we have \(\alpha = \beta\) (using the fact that \(\alpha\) and \(\beta\) have value 1/3 in the same entries). The same argument applies to \(\alpha'\) and \(\beta'\).
Now we can assume that each of $\alpha, \beta$ has most one entry of $1/3$ (otherwise, by the previous paragraph, $\alpha = \beta$, a case that we already covered). Similarly each of $\alpha', \beta'$ has most one entry of $1/3$. Using triangle inequality and Lemma 11 we obtain
\[
\|f(\alpha, \alpha') - f(\beta, \beta')\|_1 \leq \|f(\alpha, \alpha') - f(\alpha, \beta')\|_1 + \|f(\beta, \beta') - f(\alpha, \beta')\|_1 \leq \frac{24}{49}\|\alpha - \beta\|_1 + \frac{24}{49}\|\alpha' - \beta'\|_1 \leq \frac{48}{49}\max\{\|\alpha - \beta\|_1, \|\alpha' - \beta'\|_1\}.
\]

Before we prove Lemma 11 we need the following strengthening of Lemma 2.

Lemma 12. Let $\gamma, \xi \in S'_1 \cup S_2$. Then either
\[
\sum_{i=1}^{4} \gamma_i \xi_i = 2/9, \quad (19)
\]
or $\sum_{i=1}^{4} \gamma_i \xi_i \geq 49/216 > 2/9$, where (19) is attained only when
\begin{itemize}
  \item $\gamma = (0, 1/3, 1/3, 1/3)^\pi$ and $\xi = (1/3, \xi_2, \xi_3, \xi_4)^\pi$, or
  \item $\xi = (0, 1/3, 1/3, 1/3)^\pi$ and $\gamma = (1/3, \gamma_2, \gamma_3, \gamma_4)^\pi$, or
  \item $\gamma = (1/6, 1/6, 1/3, 1/3)^\pi$ and $\xi = (1/3, 1/3, 1/6, 1/6)^\pi$,
\end{itemize}
where $\pi$ is a permutation of [4].

Proof. There are three cases depending on the numbers of $1/3$ in $\gamma$ and $\xi$.
\begin{itemize}
  \item Case: $\gamma \in S_2$ or $\xi \in S_2$. We assume, w.l.o.g., $\gamma = (0, 1/3, 1/3, 1/3)$. We have
    \[
    \sum_{i=1}^{4} \gamma_i \xi_i = (1 - \xi_1)/3 \geq 2/9,
    \]
    where the last inequality is attained only when $\xi_1 = 1/3$. If $\sum_{i=1}^{4} \gamma_i \xi_i \neq 2/9$, we have $\xi_1 \neq 1/3$, and by (17) we have $\xi_1 \leq 11/36$. Hence
    \[
    \sum_{i=1}^{4} \gamma_i \xi_i = (1 - \xi_1)/3 \geq 25/108 > 49/216. \quad (20)
    \]
  \item Case: $\gamma = (1/6, 1/6, 1/3, 1/3)^\pi$ and $\xi \in S'_1$, or $\xi = (1/6, 1/6, 1/3, 1/3)^\pi$ and $\gamma \in S'_1$, where $\pi$ is a permutation of [4]. We assume, w.l.o.g., $\gamma = (1/6, 1/6, 1/3, 1/3)$ and $\xi \in S'_1$. We have
    \[
    \sum_{i=1}^{4} \gamma_i \xi_i = (\xi_1 + \xi_2)/6 + (\xi_3 + \xi_4)/3 = 1/3 - (\xi_1 + \xi_2)/6 \geq 2/9,
    \]
    where the last inequality is attained only when $\xi_1 = \xi_2 = 1/3$. If $\sum_{i=1}^{4} \gamma_i \xi_i \neq 2/9$, we have $\xi_1 \neq 1/3$ or $\xi_2 \neq 1/3$. By the definition of $S'_1$, we have
    \[
    \sum_{i=1}^{4} \gamma_i \xi_i = 1/3 - (\xi_1 + \xi_2)/6 \geq \frac{1}{3} - \frac{1}{6} \cdot \left(\frac{11}{36} + \frac{1}{3}\right) = \frac{49}{216}. \quad (21)
    \]
\end{itemize}
Case: both $\gamma$ and $\xi$ have at most one entry of $1/3$. We assume, w.l.o.g., $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4$. Then the minimum of $\sum_{i=1}^{4} \gamma_i \xi_i$ over $\xi \in S'_{i}$ is achieved for $\xi = (1/3, 11/36, 7/36, 1/6)$, where the first entry is made as big as possible, the second entry is made as big as possible (subject to (17)), and the last entry is made as small as possible (subject to (16)). We have

$$\sum_{i=1}^{4} \gamma_i \xi_i \geq \frac{\gamma_1}{3} + \frac{11\gamma_2}{36} + \frac{7\gamma_3}{36} + \frac{\gamma_4}{6} \geq \frac{1}{3} \cdot \frac{1}{6} + \frac{11}{36} \cdot \frac{7}{36} + \frac{7}{36} \cdot \frac{11}{36} + \frac{1}{6} \cdot \frac{1}{3} = \frac{149}{648} > \frac{49}{216}. \quad (22)$$

The claim then follows from (20), (21) and (22).

We next prove Lemma 11.

**Proof of Lemma 11.** Lemma 11 follows from Lemma 10 and Lemma 12.

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