Fractional fermion charges induced by vector-axial and vector
gauge potentials in planar graphene-like structures

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Abstract

We show that fermion charge fractionalization can take place in a recently proposed chiral gauge model for graphene even in the absence of Kekulé distortion of the graphene honeycomb lattice. We extend the model by adding the coupling of fermions to an external magnetic field and show that the fermion charge can be fractionalized by means of only gauge potentials. It is shown that the chiral fermion charge can also have fractional value. We also relate the fractionalization of the fermion charge to the parity anomaly in an extended Quantum Electrodynamics which involves vector and vector-axial gauge fields.

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I. INTRODUCTION

It is known that the single-particle spectrum for the charge carriers on a honeycomb lattice described by means of a tight-binding Hamiltonian contains two zero-energy Dirac points. The tight-binding Hamiltonian, linearized around the Dirac points, reveals that the kinetics of the charge carriers is governed by a single free Dirac equation for a two-flavor spinor whose components designate massless fermions on the two triangular sublattices of the honeycomb lattice. A convenient coupling of the Fermi surface charge carriers to distortions of the lattice, which preserves the invariance of the system under time and space reversion symmetries, opens a mass gap in the single-particle energy dispersion relation. Recently, Hou, Chamon and Mudry [1] have shown that when such lattice distortions exhibits a vortex-like profile, there are zero modes excitations in the single-particle energy spectrum. The vorticity of the distortion, either \( n \geq 1 \) or \( n \leq -1 \), determines which one of the sublattices supports \( |n| \) zero modes. From the existence of fermion zero modes and from the sublattice symmetry, they show that the fermion quantum charge is fractionalized, similarly to what was shown to occur in one dimensional systems [3], such as polyacetylene [4].

Jackiw and Pi [5] have proposed a chiral gauge theory for graphene. They show that the system considered in [1] is invariant under global chiral gauge transformation and the invariance of the system under local chiral gauge transformation (\( U_A(1) \)) is preserved if the space component of a vector-axial gauge potential coupled in a chiral manner to the fermions is added to the system. Such a naturally incorporated gauge potential is the vector field which enters the phenomenological Nielsen-Olesen-Landau-Ginzburg-Abrikosov model of a complex scalar field minimally coupled to the \( U(1) \) gauge field potential and whose classical minimum energy solutions exhibit vortices profiles as those binding zero-energy fermions in the scenario for fermion charge fractionalization in [1]. In this way, Jackiw and Pi have specified the dynamics for the complex scalar field, which plays the role of the distortion of the graphene lattice. Moreover, they show that sublattice symmetry is preserved and that the fermion zero modes, as well as the fermion charge fractionalization are persistent even when the self-consistent vortex solution for the gauge field is incorporated to the system.

Approximately two decades ago, Semenoff presented a physical realization of the parity anomaly in 2+1 space-time dimensions also in a honeycomb lattice [6]. This time the lattice contains two species of atoms and could describe for example a monolayer of boron nitride.
compound. The difference in energy of electrons on the atoms leads to a mass term for the fermions in the effective Dirac Hamiltonian in the continuum limit. The addition of an external gauge field coupled minimally to the fermion current leads to an abnormal induced current for each one of the fermion species, since the fermion mass breaks the parity symmetry. Such abnormal current is proportional to the sign of the mass of the fermion, and it is persistent when the mass is taken to zero. Since the sign of the mass of one species of fermion is contrary to the other one, the total fermion current is null, but an odd combination of the different currents leads to a nonzero abnormal current.

For each one of the fermion species in the system considered in [6], the induced charge in the presence of an external magnetic field is due to the localized fermion zero modes. The induced charge may be fractional and proportional to the magnetic flux on the plane [7]. In the case of a vortex magnetic field one has a finite magnetic flux and the number of fermion zero modes is proportional to the magnetic flux [8].

We show that a vortex magnetic field, as in the system considered in [5], binds, by itself, zero-energy fermion states whose spinor structure is equal to that found in the fermion zero modes in [1] and [5], that is, the zero-energy states are supported on only one of the triangular sublattices. The breaking of the sublattice symmetry by the fermion zero modes implies into an induced fractional fermion charge which is proportional to the magnetic flux. In fact, one can see from the spinor structure of zero modes that they are eigenstates of a Dirac matrix which anticommutes with the Dirac Hamiltonian and that, the introduction of a staggered chemical potential term proportional to that Dirac matrix reveals the origin of the sign of the fractional charge. Whether the chemical potential is positive or negative implies into the contribution of positive or negative energy states. In this way the realization of induced fractional fermion charge in graphene is provided in a manner very similar to that one proposed originally in [6], that is, by means of only vector-axial gauge field.

Furthermore, by imposing the invariance of the Dirac Hamiltonian under the usual $U(1)$ local gauge transformation, we extend the system by minimally coupling the fermions to a vector gauge potential. We show that a fractional fermion charge may also be induced by a vortex-like magnetic field associated to the added vector potential with and without the magnetic field associated to the original vector-axial gauge potential. We analyze the discrete symmetries of the isolated fermion zero modes and find that they are supported on both triangular sublattices. Based on that symmetry, we show that there is room to the
fractionalization of the fermion (chiral) charge as well.

In the next section we consider the Dirac Hamiltonian resulting from the linearization, around the Dirac points, of the tight-binding Hamiltonian for the fermions on the graphene honeycomb structure. In this case we start with the extended Dirac Hamiltonian proposed in \[5\] and in the absence of the complex scalar field and show the isolated fermion modes as well as the induced fermion charge in the presence of a vortex magnetic field associated to the vector-axial gauge potential alone. Next, we extend the system by coupling the fermions to the usual vector gauge potential and show the isolated fermion modes. By analyzing the discrete symmetries of the problem we calculate the induced fermion charges in the presence of the two kinds of magnetic vortices (or solenoids). The third section is dedicated to a detailed analysis of the fermion states when the magnetic fields are homogeneous and, from the behavior of the lowest Landau level (LLL), we illustrate the origin of the induced fermion charges. In the fifth section we obtain the effective Chern-Simons (C-S) action in an extended Quantum Electrodynamics (QED) in 2+1 dimensions with vector and vector-axial gauge fields and from that we calculate the associated induced fermion currents which recovers the fermion charges obtained in the previous sections. The sixth section is left to the conclusions.

II. FERMION ZERO MODES ON VORTEX-LIKE MAGNETIC FIELDS

The field theory Hamiltonian describing the dynamics of the electrons on the graphene honeycomb structure in the presence of a complex scalar field, which play the role of the lattice distortions, together with a vector gauge potential coupled in a chiral way to the electrons is written in a simplified version as

\[\mathcal{H} = \int d^2 \vec{r} \Psi^\dagger(\vec{r}) \left[ -i \vec{\alpha} \cdot \vec{\nabla} - \gamma_5 \vec{\alpha} \cdot \vec{A}_5 + g \beta (\varphi^r - i \gamma_5 \varphi^l) \right] \Psi(\vec{r}),\]  

where \(\Psi(\vec{r})\) is the four component spinor

\[\Psi(\vec{r}) = \begin{pmatrix} \psi^b_+ (\vec{r}) \\ \psi^a_+ (\vec{r}) \\ \psi^a_-(\vec{r}) \\ \psi^b_- (\vec{r}) \end{pmatrix},\]
and $\vec{\alpha}$, $\beta$ and $\gamma_5$ are Dirac matrices in the representation explicit below. The superscripts in the spinor components designate the triangular sublattice, $A$ or $B$, the electrons are supported on, while the subscripts stand for each one of the Dirac points (the single-particle energy spectrum obtained from the original tight binding Hamiltonian exhibits two Dirac points). The Kekulé distortion is represented by the complex field $\varphi(\vec{r}) = |\varphi(\vec{r})|e^{i\chi(\vec{r})} = \varphi^r(\vec{r}) + i\varphi^i(\vec{r})$ and $\vec{A}_5(\vec{r})$ is the vector-axial gauge potential added to make the Hamiltonian invariant under local chiral gauge transformation $\varphi \to e^{i\gamma_5\omega(\vec{r})}\varphi$, $\Psi \to e^{i\gamma_5\omega(\vec{r})}\Psi$. The matrix structure in (1) is revealed by means of the following gamma matrices

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \vec{\alpha} = \beta\gamma = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \quad \alpha^3 = \beta\gamma^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma_5 = -i\alpha^1\alpha^2\alpha^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

(3)

where $I$ is the $2 \times 2$ identity matrix and $\vec{\sigma} = (\sigma_1, \sigma_2)$ and $\sigma_3$ are the Pauli matrices.

Now, we consider the Hamiltonian (1) by dropping out the interaction term of the electrons with the Kekulé distortion and show that if the vector-axial gauge potential has a convenient asymptotic behavior, there will be isolated fermion zero modes in the single-particle energy spectrum. For the sake of simplicity we consider

$$A_5^i = \varepsilon^{ij}\partial_j A_5(\vec{r}) \Rightarrow B_5 = -\vec{\nabla}^2 A_5(\vec{r}),$$

(4)

where $A_5(\vec{r})$ is a scalar function of the radial coordinate only. The time-independent Dirac equation

$$\vec{\alpha}.(-i\vec{\nabla} - q\gamma_5\vec{A}_5)\Psi(\vec{r}) = E\Psi(\vec{r}),$$

(5)

can be rewritten as four equations for the spinor components (the coupling constant $q > 0$ is introduced here for future convenience). Two of them, in a simple form, are

$$-2i[\partial_z - q\partial_z A_5] \psi^b_+ = E\psi^a_+ \quad \text{and} \quad -2i[\partial_z + q\partial_z A_5] \psi^a_+ = E\psi^b_+, \quad (6)$$

whilst the equations for $\psi^b_-$ and $\psi^a_-$ are obtained by taking the complex conjugate of the above equations and by making the identifications $\psi^b_+ = (\psi^b_+)^*$ and $\psi^a_+ = (\psi^a_+)^*$. In the above equations we are using the notation $z = x + iy$ and $\partial_z = (\partial_x - i\partial_y)/2$. In such a simple notation, one can easily check that the zero-energy non-normalized eigenstates would
be given by

$$
\psi^b_+ = f(z)e^{\frac{qA_5}{r}}, \quad \psi^a_+ = h(z)e^{\frac{-qA_5}{r}}, \quad \psi^b_- = \overline{f(z)}e^{-\frac{qA_5}{r}}, \quad \psi^a_- = \overline{h(z)}e^{-\frac{-qA_5}{r}},
$$

(7)

where $f(z)$ and $h(z)$ are holomorphic functions. The normalization of the eigenfunctions depends on the asymptotic behavior of $A_5(r)$ and, from the functional dependence of the spinor components, either $\psi^b_\pm(r)$ or $\psi^a_\pm(r)$ is normalizable, thereby one obtains one of the following zero-energy eigenstates

$$
\Psi_0(r) = N \begin{pmatrix}
0 \\
e^{+il\theta}
\end{pmatrix} r^l e^{\frac{qA_5}{r}} \quad \text{or} \quad \Psi_0(r) = N \begin{pmatrix}
e^-il\theta \\
e^{+il\theta}
\end{pmatrix} r^l e^{-\frac{qA_5}{r}},
$$

(8)

where $l$ is taken to be an integer number and $\theta$ is the angular variable in cylindrical coordinates. The degeneracy of the zero-energy level depends on the number of values $l$ can assume and it is linked to the magnetic flux according to the theorems proved in [8]. In this section we are concerned with finite magnetic flux configurations only.

To show that a fractional fermion charge can be induced with such magnetic field configurations, we note that the zero modes are eigenfunctions of $\alpha^3$ and that the Hamiltonian operator $\overline{\alpha}((\vec{p} - \gamma_5\vec{A}^\alpha_5))$ anticommutes with the matrix $\alpha^3$, then one can show that each eigenfunction of negative energy is obtained from an eigenfunction of positive energy by $\Psi_{-|E|}(\vec{r}) = \alpha^3 \Psi_{|E|}(\vec{r})$. In other words, the energy spectrum is symmetric around the zero-energy level whose corresponding eigenfunctions are self-conjugate. From this symmetry and from the fact that the zero modes contain representatives of only one of the triangular sublattices one obtains, by following the procedures in [7], [2] and [10] that the induced fermion number is given by

$$
N = \int d^2\vec{r} \frac{1}{2} \langle 0 | [\Psi^\dagger(\vec{r}), \Psi(\vec{r})] | 0 \rangle = \pm \frac{q\Phi_5}{4\pi},
$$

(9)

where $\Phi_5 = \int d^2\vec{r}B_5$ is the magnetic flux and the upper (lower) sign on the right-hand side of (9) comes out when the zero modes are assigned to fermions (antifermions).

The induced fermion charge is finite provided the magnetic flux is finite and that can be achieved by means of for example a solenoid or vortex magnetic field, such that $A_5(r \to \infty) \sim -(\Phi_5/2\pi) \ln r$. For $\Phi_5 > 0$ one has $\Psi^\dagger_0(r \to \infty) \sim r^{l - \frac{q\Phi_5}{2\pi}} (e^{+il\theta} 0 0 e^{-il\theta})$ and
for $\Phi_5 < 0 \Psi_0^T (r \to \infty) \sim r^{\frac{\Phi_5}{2\pi}} (0 \ e^{-il\theta} \ e^{+il\theta} \ 0)$, where we are assuming that $l < \lfloor \frac{\Phi_5}{2\pi} \rfloor - 1$, with $[\nu]$ denoting the largest integer less than $\nu$. The charge is fractional if the magnetic flux is an integer odd number ($n_o$) of quantum of flux $h v_F / q$, where $h$ is the Planck constant, $v_F$ is the Fermi velocity and $q$ is the coupling constant associated to the chiral coupling, that is,

$$Q = e \int d^2r \frac{1}{2} \langle 0 | [\Psi^\dagger(\vec{r}), \Psi(\vec{r})] | 0 \rangle = \mp n_o \frac{e}{2},$$

where $e$ is the unit of electric charge.

A. Minimal coupling to a vector gauge potential, zero modes and fermion charges fractionalization:

The Hamiltonian (11) is also invariant under the usual global gauge transformation $\Psi \to e^{i\xi} \Psi$, with $\xi$ uniform and constant. By promoting such a transformation to a local one, a vector gauge potential $\vec{A}$ must be added to the kinetic term in such a way that $\vec{A} \to \vec{A} + \vec{\nabla} \xi$.

We are now concerned with the appearance of zero-energy bound states of the system when such a vector gauge potential is also taken into account, namely we are going to consider the Dirac equation

$$\vec{\alpha}.(-i\vec{\nabla} - q\gamma_5 \vec{A} - e\vec{A})\Psi(\vec{r}) = E\Psi(\vec{r}),$$

particularly when both gauge potentials engender finite magnetic flux. Here we have also explicit the coupling constant $e > 0$.

An interesting aspect of the Hamiltonian operator in (11) is that it anticommutes with $\alpha^3$ and with $\alpha^3 \gamma_5$. In the previous case the possible zero-energy solutions are eingenstates of $\alpha^3$, whilst in the case $\vec{A} = 0$ and $\vec{A}$ given by

$$A^i = \epsilon^{ij} \partial_j A(r) \Rightarrow B = -\vec{\nabla}^2 A(r),$$

the zero-energy solutions are eingenstates of $\alpha^3 \gamma_5$. In this case the spinor components satisfy the equations

$$-2i [\partial_z + e\partial_z A] \psi^a_+ = E \psi^b_+, \quad -2i [\partial_z - e\partial_z A] \psi^b_+ = E \psi^a_+,$n
$$2i [\partial_z + e\partial_z A] \psi^a_- = E \psi^b_-, \quad 2i [\partial_z - e\partial_z A] \psi^b_- = E \psi^a_-,$$

$$2i [\partial_z + e\partial_z A] \psi^a = E \psi^b, \quad 2i [\partial_z - e\partial_z A] \psi^b = E \psi^a.$$
whose eigenfunctions for $E = 0$ are given by

$$
\Psi_0(\vec{r}) = \mathcal{N} \begin{pmatrix} e^{+i\theta} \\ 0 \\ e^{+i\theta} \\ 0 \end{pmatrix} r^l e^{\epsilon A(r)} \quad \text{or} \quad \Psi_0(\vec{r}) = \mathcal{N} \begin{pmatrix} 0 \\ e^{-i\theta} \\ 0 \\ e^{-i\theta} \end{pmatrix} r^l e^{-\epsilon A(r)},
$$

(14)

We notice that fermion zero modes are supported on both sublattices and are eigenstates of $\alpha^3 \gamma_5$. The energy spectrum is symmetric around $E = 0$, the negative energy eigenstates are obtained from the positive energy ones by means of the norm preserving operation

$$
\Psi_{-|E|}(\vec{r}) = \alpha^3 \gamma_5 \Psi_{|E|}(\vec{r}).
$$

Thus, the induced fermion number is given as in (9) with $q\Phi_5$ replaced by $e\Phi$, with $\Phi = \int d^2\vec{r}B$. By considering $A(r \to \infty) \sim -(\Phi/2\pi) \ln r$ we have

$$
\Psi_0^T(r \to \infty) \sim r^{l-\frac{q\Phi}{2\pi}}(e^{+i\theta} 0 e^{+i\theta} 0) \quad \text{for} \quad \Phi > 0 \quad \text{and} \quad \Psi_0^T(r \to \infty) \sim r^{l+\frac{q\Phi}{2\pi}}(0 e^{-i\theta} 0 e^{-i\theta}) \quad \text{for} \quad \Phi < 0.
$$

An induced fermion charge is obtained similarly to the previous case and if the magnetic flux is quantized as an integer odd number of quantum of flux $h v_F/e$ the fermion charge is fractionalized and given by (11).

The fractionalization of the fermion number (charge) is also possible in very special circumstances when both quantized magnetic fluxes acts simultaneously. Moreover, another charge - we call it chiral charge - can also be fractionalized. That is what we are going to show next.

The spinor components satisfy the following equations

$$
-2i [\partial_+ + \partial_z A_+] \psi^a_+ = E \psi^a_+, \quad -2i [\partial_- - \partial_z A_+] \psi^a_- = E \psi^a_-,
$$

$$
2i [\partial_+ + \partial_z A_-] \psi^a_- = E \psi^a_+, \quad 2i [\partial_- - \partial_z A_-] \psi^a_+ = E \psi^a_+,
$$

(15)

In the equations above we have set $\hbar = v_F = 1$, $A_\pm = eA \pm qA_5$ and, for sake of simplicity, we consider the following asymptotic behaviors $A_5(r \to \infty) \sim -\frac{2\pi}{q} \ln r$ and $A(r \to \infty) \sim -\frac{2\pi}{e} \ln r$, with $n_5$ and $n$ positive integer numbers.

Normalized zero-energy states are obtained in some particular cases, namely

$$
\Psi_{0,l_+,l_-}^{n_n_5}(\vec{r}) = \begin{pmatrix} 1 \\ \sqrt{4\pi \int_0^\infty r^{d+1} e^{2A_+(r')} dr'} r^l e^{il_+ \theta} e^{A_+(r)} \end{pmatrix}, \quad n > n_5, \quad l_\pm = 0, 1, \ldots, \lfloor n \pm n_5 - 1 \rfloor,
$$

(16)
Note that hand, when the magnetic fluxes are both negative, the zero modes are numbers, which depend on the magnetic fluxes as \( \Psi \) and \( \Psi^* \) for \( |n| < n_5 \). Again, the upper (lower) sign on the right-hand side of (18) and (19) depends on the zero-Hamiltonian.

When a staggered chemical potential and a parity-breaking mass term are added to the electrons in the honeycomb structure and in the presence of homogeneous magnetic fields chemical potential. In the next section we analyze in detail the single-particle states for a mass term, in the representation we have been working here, is equivalent to a staggered species of electrons which belong to the two different atoms implies into a mass term. Such fermion charge found previously in the Boron-Nitride honeycomb structure \([6]\) where two kinds of zero modes given by expressions (16) and (17) lead to induced fermion numbers, which depend on the magnetic fluxes as

\[
N = \int d^2r \frac{1}{2} \langle 0 | [\Psi^d(\vec{r}), \Psi(\vec{r})] | 0 \rangle = \pm \frac{n}{2}, \text{ for } |n| > |n_5|, \tag{18}
\]
and

\[
N = \int d^2r \frac{1}{2} \langle 0 | [\Psi^d(\vec{r}), \Psi(\vec{r})] | 0 \rangle = \pm \frac{n_5}{2}, \text{ for } |n| < |n_5|, \tag{19}
\]

Again, the upper (lower) sign on the right-hand sign depends on the zero-energy states are taken as fermions (antifermions).

We have also found that a fractional (chiral) fermion number can also be induced, namely

\[
N_5 = \int d^2r \frac{1}{2} \langle 0 | [\Psi^d(\vec{r}), \gamma_5 \Psi(\vec{r})] | 0 \rangle = \pm \frac{n_5}{2}, \text{ for } |n| > |n_5|, \tag{20}
\]
and

\[
N_5 = \int d^2r \frac{1}{2} \langle 0 | [\Psi^d(\vec{r}), \gamma_5 \Psi(\vec{r})] | 0 \rangle = \pm \frac{n}{2}, \text{ for } |n| < |n_5|, \tag{21}
\]
if \( |n| \) and \( |n_5| \) are odd numbers.

This (chiral) fermion number is proportional to the non-vanishing fractional (anomalous) fermion charge found previously in the Boron-Nitride honeycomb structure \([6]\) where two species of electrons which belong to the two different atoms implies into a mass term. Such a mass term, in the representation we have been working here, is equivalent to a staggered chemical potential. In the next section we analyze in detail the single-particle states for electrons in the honeycomb structure and in the presence of homogeneous magnetic fields when a staggered chemical potential and a parity-breaking mass term are added to the Hamiltonian.
III. LANDAU LEVELS AND INDUCED CHARGES ON UNIFORM MAGNETIC FIELDS

In this section we obtain the fermion bound states when the electrons on the honeycomb structure are under the action of vector gauge potentials for homogeneous magnetic fields. In this analysis we introduce terms in the Hamiltonian density which provide gaps in the single-particle energy spectrum. Such terms are a staggered chemical potential $\mu \alpha^3$ and a parity breaking “mass term” $m_\tau \alpha^3 \gamma_5 = m_\tau \beta \tau$. The staggered chemical potential was already considered in [10], [11] to show the realization of irrational fermion charge induced by scalar fields and still maintaining the time-reversal symmetry. The parity-breaking mass matrix $m_\tau \beta \tau$ in this version of the graphene honeycomb structure is the continuous limit of the Haldane mass (energy) [12] linearized around the Dirac points [13]. We consider it as constant as in [14], but the same term was taken to be position-dependent (a domain wall) in [15] to investigate how the electronic properties of graphene are modified under such a domain wall.

When such terms are added to the Hamiltonian, the corresponding Dirac operator anticommutes neither with $\alpha^3$ nor with $\alpha^3 \gamma_5$ and the Hamiltonian does not admit any norm-preserving (conjugation) symmetry, that is, normalizable negative-energy eigenstates are no longer obtained from the normalizable positive-energy eigenstates and the lowest energy states are not necessarily self-conjugate under the operations of either $\alpha^3$ or $\alpha^3 \gamma_5$. Nevertheless, it may be found a non-unitary operator which conjugates negative to positive-energy eigenstates and vice-versa, but, due to the lack of unitarity, the density of positive-energy states is different from the density of negative-energy states belonging to the continuous energy spectrum and, as a consequence, an irrational fermion number may be induced. Since we do not know exactly the fermion single-particle energy states in a general magnetic field then we are going to consider the case of homogeneous magnetic fields whose single-particle energy levels are the familiar Landau levels. Once the energy spectrum is discrete, there is no way to realize irrational fermion number. Here the fermion number density is proportional to the surface density of states of the LLL whose associated eigenstates are eigenfunctions of either $\alpha^3$ or $\alpha^3 \gamma_5$, depending on the magnitude of the applied magnetic fields.

Now, the Dirac equation is

$$[\vec{\alpha}.(-i\vec{\nabla} - q\gamma_5 \vec{A}_5 - e\vec{A}) + \mu \alpha_3 + m_\tau \beta \tau] \Psi(\vec{r}) = E \Psi(\vec{r}),$$

(22)
where the vector potentials are written as in (4) and (12) with $A(r)$ and $A_5(r)$ given by

$$A(r) = -\frac{Br^2}{4} \quad \text{and} \quad A_5(r) = -\frac{B_5r^2}{4} \quad \text{with} \quad B, B_5 > 0. \quad (23)$$

The set of equations for the spinor components can be written as

$$-i \left[ 2 \partial_z - \frac{\omega_+}{2} z \right] \psi^a_+ = (E - m_+) \psi^b_+, \quad -i \left[ 2 \partial_z + \frac{\omega_+}{2} z \right] \psi^b_+ = (E + m_+) \psi^a_+$$
$$i \left[ 2 \partial_z - \frac{\omega_-}{2} z \right] \psi^a_- = (E + m_-) \psi^b_-, \quad i \left[ 2 \partial_z + \frac{\omega_-}{2} z \right] \psi^b_- = (E - m_-) \psi^a_- \quad (24)$$

where $\omega_\pm = eB \pm qB_5$ and $m_\pm = \mu \pm m_\tau$. This problem is exactly solvable, but the spinorial structure of the eigenstates of the Hamiltonian operator depends on whether $\mu > m_\tau$ or $\mu < m_\tau$ as well as on whether $eB > qB_5$ or $eB < qB_5$. Hereafter, we take $\mu, m_\tau \geq 0$ and $eB, qB_5 > 0$, although the cases when $eB, qB_5 < 0$ and/or $\mu, m_\tau \leq 0$ can also be analyzed straightforwardly. Even in the cases we are concerned here, there are four different situations, which can be better appreciated if examined separately.

**Case 1: $qB_5 > eB$.** The energy spectrum is $|E_\pm| = \sqrt{2n |\omega_\pm| + m_\pm^2}$, with $n \in \mathbb{N}$ and the eigenstates are

$$\Psi^E_{+>0} = e^{-i|E_+|t} \left( \begin{array}{c} \sqrt{1 + \frac{m_+}{|E_+|}} R_{n,l,\omega_+}(r, \theta) \\ i \sqrt{1 - \frac{m_+}{|E_+|}} R_{n-1,l+1,\omega_+}(r, \theta) \\ 0 \\ 0 \end{array} \right),$$

$$\Psi^E_{->0} = e^{-i|E_-|t} \left( \begin{array}{c} 0 \\ 0 \\ -i \sqrt{1 - \frac{m_-}{|E_-|}} R^*_n(r, \theta) \\ \sqrt{1 + \frac{m_-}{|E_-|}} R^*_{n,l,\omega_-}(r, \theta) \end{array} \right), \quad (25)$$

and
\[
\begin{align*}
\Psi_{E_+<0}^E &= \frac{e^{iE_+|t|}}{\sqrt{2}} 
\begin{pmatrix}
\sqrt{1 - \frac{m_+}{\mu_+}} R_{n,l,\omega_+} (r, \theta) \\
-i \sqrt{1 + \frac{m_+}{\mu_+}} R_{n-1,l+1,\omega_+} (r, \theta) \\
0 \\
0
\end{pmatrix} \\
\Psi_{E_-<0}^E &= \frac{e^{iE_-|t|}}{\sqrt{2}} 
\begin{pmatrix}
0 \\
0 \\
i \sqrt{1 + \frac{m_-}{|E_-|}} R_{n-1,l+1,\omega_-}^* (r, \theta) \\
\sqrt{1 - \frac{m_-}{|E_-|}} R_{n,l,\omega_-}^* (r, \theta)
\end{pmatrix},
\end{align*}
\]

where

\[
R_{n,l,\omega_\pm} (r, \theta) = \sqrt{\frac{(|\omega_\pm|)^{l+1}}{2}} \frac{n!}{\pi(n+l)!} e^{i\theta} e^{-\frac{|\omega_\pm|^2}{4} r^2} L_n^l (|\omega_\pm| r^2/2) \cos \theta \in \mathbb{N},
\]

are the Gauss-Laguerre modes in cylindrical coordinates and expressed in terms of the associated Laguerre polynomials \(L_n^l (|\omega_\pm| r^2/2)\). Here \(R^*\) stands for the conjugate to \(R\) whose normalization is \(\int |R_{n,l,\omega_\pm} (r, \theta)|^2 r dr d\theta = 1\), and we are assuming that \(R_{-1,l,\omega_\pm}^* (r, \theta) = 0\).

For \(n = 0\) and for the case \(\mu > m_\tau (m_\pm > 0)\) the lowest energy states are

\[
\Psi_{E_+=m_+}^E = e^{-im_+ t} \begin{pmatrix} R_{0,l,\omega_+} (r, \theta) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_{E_-=m_-}^E = e^{-im_- t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ R_{0,l,\omega_-}^* (r, \theta) \end{pmatrix}.
\]

We notice that when \(m_\tau \to 0\), the energy states above are degenerate such that one has only one eigenspinor with the same structure found in \((17)\), which is normalizable even when \(\mu \to 0\), that is

\[
\Psi_{E_0=\mu} = \begin{pmatrix} \psi_+^b (\vec{r}, t) \\ 0 \\ 0 \\ \psi_-^b (\vec{r}, t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} R_{0,l,\omega_+} (r, \theta) \\ 0 \\ 0 \\ R_{0,l,\omega_-}^* (r, \theta) \end{pmatrix},
\]

Had we started with \(m_\tau = 0\) and \(\mu < 0\) the LLL would have energy \(E_0 = \mu < 0\) and the
corresponding eigenspinor would be

$$\Psi_{E=\pm|\mu|} = \begin{pmatrix}
\psi^b_+ (\vec{r}, t) \\
0 \\
0 \\
\psi^b_- (\vec{r}, t)
\end{pmatrix} = e^{i|\mu| t} \sqrt{2} \begin{pmatrix}
R_{0, t, \omega_+} (r, \theta) \\
0 \\
0 \\
R_{0, t, \omega_-}^* (r, \theta)
\end{pmatrix}. \quad (30)$$

Then the induced fermion density numbers, for $\mu \to 0$, are given by:

$$\rho = \frac{1}{2} \langle 0 | [\psi^\dagger, \psi] | 0 \rangle = -\text{sgn}(\mu) \frac{qB_5}{4\pi}, \quad (31)$$

and

$$\rho_5 = \frac{1}{2} \langle 0 | [\psi^\dagger, \gamma_5\psi] | 0 \rangle = -\text{sgn}(\mu) \frac{eB}{4\pi}. \quad (32)$$

The above results are compatible with those found in the previous section, (19) and (21), which reveals that the fermion numbers are proportional to the flux of the magnetic field. The dependence on the $\text{sgn}(\mu)$ is explained by the fact that when $\mu$ is taken initially as positive, the LLL is occupied by electrons on the conduction band ($E = \mu > 0$), whereas when $\mu < 0$ the LLL is occupied by electrons on the valence band. The above result is only valid for $\mu \to 0$, otherwise there would be a contribution (parity invariant contribution) coming from the higher Landau levels.

Although there is no norm-preserving operator which conjugates negative to positive-energy eigenstates, when $\mu \neq 0$, the energy spectrum is discrete and the eigenstates can be normalized, such that the density of states from the positive and negative energy spectrum (except for the isolated mode $E = \mu$) is equal to each other. Then their contribution vanishes when one calculates the fermion numbers themselves and it is possible to show that the fermion numbers are proportional to the magnetic flux. In fact, the results in (31) and (32) express that the fermion density numbers are equal to the surface density of states of each Landau level (or the surface density of zero modes) and, as a consequence, the degeneracy is proportional to the magnetic flux.

We emphasize that in this case, as was already revealed in equations (19) and (21), the fermion charge is proportional to the flux of the (axial) magnetic field, whilst the fermion chiral charge is proportional to the flux of the magnetic field. This will be explored in the next section to analyse the effective action for the gauge fields in the relativistic context, where we will show that the derivative expansion approximation to the effective action leads to a crossed Chern-Simons term.
The next case we treat is well known and has been recently analyzed in [14] by resorting to another representation for the reducible gamma matrices in 2+1 (space-time) dimensions. Although they do not consider the vector-axial gauge potential, their results are more general than ours in the sense that they take inhomogeneous magnetic fields and analyze also the behavior of the chiral condensates.

**Case 2:** $qB < eB$. The energy spectrum is identical to that in the previous case and the eingestates are expressed as

\[
\psi_{E>0}^{E+} = \frac{e^{-iE+|t}}{\sqrt{2}} \begin{pmatrix}
\sqrt{1 + \frac{m_+}{|E_+|}} R_{n,l,\omega_+} (r, \theta) \\
\frac{i}{\sqrt{2}} \left( \frac{1 - m_+}{|E_+|} \right) R_{n-1,l+1,\omega_+} (r, \theta) \\
0 \\
0 
\end{pmatrix}
\]

\[
\psi_{E>0}^{E-} = \frac{e^{-iE-|t}}{\sqrt{2}} \begin{pmatrix}
0 \\
0 \\
\frac{i}{\sqrt{2}} \left( \frac{1 - m_-}{|E_-|} \right) R_{n-1,l+1,\omega_-} (r, \theta) \\
\sqrt{1 - m_- \frac{1}{|E_-|}} R_{n,l,\omega_-} (r, \theta) 
\end{pmatrix}
\]

(33)

\[
\psi_{E<0}^{E+} = \frac{e^{iE+|t}}{\sqrt{2}} \begin{pmatrix}
\sqrt{1 - \frac{m_+}{|E_+|}} R_{n,l,\omega_+} (r, \theta) \\
- \frac{i}{\sqrt{2}} \left( \frac{1 + m_+}{|E_+|} \right) R_{n-1,l+1,\omega_+} (r, \theta) \\
0 \\
0 
\end{pmatrix}
\]

\[
\psi_{E<0}^{E-} = \frac{e^{iE-|t}}{\sqrt{2}} \begin{pmatrix}
0 \\
0 \\
\frac{i}{\sqrt{2}} \left( \frac{1 - m_-}{|E_-|} \right) R_{n-1,l+1,\omega_-} (r, \theta) \\
\frac{1 - m_-}{|E_-|} R_{n,l,\omega_-} (r, \theta) 
\end{pmatrix}
\]

(34)

For $\mu = 0$ and $m_+ > 0$ the eigenstates of the LLL is

\[
\psi_{E=m_+} = \begin{pmatrix}
\psi^b_+ (\vec{r}, t) \\
0 \\
\psi^a_- (\vec{r}, t) \\
0
\end{pmatrix}
= \frac{e^{-im_+t}}{\sqrt{2}} \begin{pmatrix}
R_{0,l,\omega_+} (r, \theta) \\
0 \\
R_{0,l,\omega_-} (r, \theta) \\
0
\end{pmatrix}
\]

(35)
and exhibits the same spinor structure as the one in [15], whilst for \( \mu = 0 \) and \( m_\tau < 0 \) we have

\[
\Psi^{E=-|m_\tau|} = \begin{pmatrix}
\psi^b(\vec{r}, t) \\
0 \\
\psi^a(\vec{r}, t) \\
0
\end{pmatrix} = \frac{e^{im_\tau t}}{\sqrt{2}} \begin{pmatrix}
R_{0,\omega+}(r, \theta) \\
0 \\
R_{0,\omega-}(r, \theta) \\
0
\end{pmatrix}.
\]

(36)

In obtaining the induced fermion charges it is convenient to separate the contribution of the LLL from the higher LL, which also contribute in this case as well, but the last contributions vanishes when \( m_\tau \to 0 \) and one finds:

\[
\rho = \frac{1}{2} \langle 0 | [\psi^\dagger, \psi] | 0 \rangle = -\text{sgn}(m_\tau) \frac{eB}{4\pi},
\]

(37)

\[
\rho_5 = \frac{1}{2} \langle 0 | [\psi^\dagger, \gamma_5 \psi] | 0 \rangle = -\text{sgn}(m_\tau) \frac{qB_5}{4\pi},
\]

(38)

which are consistent with [15] and [20].

IV. INDUCED FERMION CHARGES FROM FIELD THEORETICAL CALCULATIONS

As one knows QED in 2+1 space-time dimensions can be enriched by adding a topological mass to the gauge field [16] and that such a C-S term can be generated by means of perturbation calculation [17]. The C-S term comes from the first-order in external momentum contribution of the vacuum polarization diagram.

Here, we show that C-S terms can also be induced in the extended QED, whose Lagrangian density is

\[
\mathcal{L} = \bar{\Psi} \left[ \gamma^\mu (\partial_\mu + eA_\mu + gA_\mu^{(5)}) - \mu \gamma^3 - m_\tau \gamma^\tau \right] \Psi.
\]

(39)

We also compute the fermion induced currents from the C-S effective action.

The vacuum polarization diagrams we have to consider are those shown in FIG. 1 which correspond to the vacuum polarization operator

\[
\Pi^{\mu\nu}(k) = -i \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \frac{1}{(\hat{p} + \hat{k}) - \mu \gamma^3 - m_\tau \gamma^\tau} \gamma^\mu (e + q\gamma^5) \frac{1}{\hat{p} - \mu \gamma^3 - m_\tau \gamma^\tau} \gamma^\nu (e + q\gamma^5) \right],
\]

(40)

where \( S_F(p) = (\hat{p} - \mu \gamma_3 - m_\tau \gamma^\tau)^{-1} \) is the fermion propagator given in terms of momenta and under a staggered chemical potential and parity-breaking mass.
FIG. 1:

Here, as in the previous section, we are going to consider the contributions of the staggered chemical potential and of the parity-breaking mass separately. This is not only because we want to see the dependence of the fermion currents on the signs of $\mu$ and $m_\tau$ as in the previous section, but also because the fermion propagator can be easily handled when the approximations

$$S_F(p) \approx S_{F,\mu}(p) = \frac{1}{p - \mu \gamma^3}$$ \hspace{0.5cm} (41)

and

$$S_F(p) \approx S_{F,m_\tau}(p) = \frac{1}{p - m_\tau \gamma^\tau}$$ \hspace{0.5cm} (42)

are employed.

In both cases we look for the term which is first-order in the external momentum, namely $\Pi^{\mu\nu}_1 \sim \epsilon^{\mu\nu\alpha\beta} k_\alpha$. By taking the first approximation for the fermion propagator, Eq. (41), we note that the third diagram in FIG. 1 is the only one which gives a contribution to the C-S terms. This can be appreciated through the following expression for $\Pi^{\mu\nu}$

$$\Pi^{\mu\nu}(k) = -i \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \frac{(\not{p} + \not{k}) + \mu \gamma^3 \gamma^\nu}{(p + k)^2 - \mu^2} \frac{\not{p} + \mu \gamma^3}{p^2 - \mu^2} \gamma^\mu (e + q \gamma^5) \frac{\not{p} + \mu}{p^2 - \mu^2} \gamma^\nu (e + q \gamma^5) \right],$$ \hspace{0.5cm} (43)

and by noting that the Levi-Civita tensor comes from $\text{tr} (\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\tau) = -4i \epsilon^{\mu\nu\alpha\beta}$. By isolating the relevant terms to the C-S contribution one finds

$$\Pi^{\mu\nu}_1 = -\frac{\mu q}{2\pi^3} \epsilon^{\mu\nu\alpha\beta} k_\alpha \lim_{k \to 0} \int d^3p \frac{1}{(p^2 - \mu^2)} \frac{1}{(p + k)^2 - \mu^2} =$$

$$= -\frac{\mu q}{2\pi^3} \epsilon^{\mu\nu\alpha\beta} k_\alpha \lim_{k \to 0} \int \frac{dz}{z} \int d^3p \frac{1}{[p^2 - \mu^2 + k^2 z(1 - z)]^2} =$$

$$= i \frac{\mu q}{2\pi} \epsilon^{\mu\nu\alpha\beta} k_\alpha \lim_{k \to 0} \int \frac{dz}{[\mu^2 - k^2 z^2 + k^2 z - \mu^2]^{1/2}} = i \frac{\mu q}{2\pi} \epsilon^{\mu\nu\alpha\beta} k_\alpha \frac{\mu}{|\mu|}. \hspace{0.5cm} (44)$$

From the effective action up to second-order on the gauge fields corresponding to the third diagram

$$S^{(2)}_{eff} = \frac{1}{2} \int d^3x \left\{ A_\mu(x) \Pi^{\mu\nu}(k_\alpha = i \partial_\alpha) A^{(5)}_\nu(x) + A^{(5)}_\mu(x) \Pi^{\mu\nu}(k_\alpha = i \partial_\alpha) A_\nu(x) \right\}, \hspace{0.5cm} (45)$$
one obtains a crossed C-S term, namely

\[ S_{\text{eff}}^{(2,1)} = -\frac{e q \mu}{4\pi |\mu|} \varepsilon^{\mu\nu\alpha} \int d^3 x \{ A_\mu(x) \partial_\alpha A_\nu^{(5)}(x) + A_\mu^{(5)}(x) \partial_\alpha A_\nu(x) \} . \]  

(46)

The induced fermion current density is obtained from this contribution to the effective action through

\[ \langle j^\mu \rangle = \frac{\delta S_{\text{eff}}^{(2,1)}}{\delta A_\mu} = -\frac{eq \mu}{4\pi |\mu|} \varepsilon^{\mu\nu\alpha} \partial_\alpha A_\nu^{(5)} \]

\[ = -\frac{eq \mu}{4\pi |\mu|} F_\mu^{(5)} , \]  

(47)

where \( F_\mu^{(5)} = (-1/2) \varepsilon^{\mu\nu\alpha} F_{\nu\alpha}^{(5)} \) is the dual of the (axial) field strength, whereas the (axial) fermion current density can be obtained as

\[ \langle j_5^\mu \rangle = \frac{\delta S_{\text{eff}}^{(2,1)}}{\delta A_\mu^{(5)}} = -\frac{eq \mu}{4\pi |\mu|} \varepsilon^{\mu\nu\alpha} \partial_\alpha A_\nu^{(5)} \]

\[ = -\frac{eq \mu}{4\pi |\mu|} F_\mu^{(5)} . \]  

(48)

By taking the time-component of (47) and (48) one recovers the results for the fermion charge densities, equations (31) and (32), respectively.

Now, we consider the effective C-S action coming from the contribution of the first two diagrams. We notice that such contributions are obtained by taking the second approximate fermion propagator given in (42). Here we calculate both contributions together, that is

\[ \tilde{\Pi}^{\mu\nu}(k) = -i \int \frac{d^3 p}{(2\pi)^3} \text{tr} \left[ \frac{(\hat{k} + \hat{p})}{(p + k)^2 - m_\tau^2} e^{i p_\tau} e^{i k_\tau} \gamma^\mu (e + q \gamma_5) \gamma^\nu (e + q \gamma_5) \right] . \]  

(49)

This time there will be no crossed C-S terms because \( \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\tau) = -4i\varepsilon^{\mu\nu\alpha} \) is obtained from the contributions proportional to \( m_\tau \) and to \( e^2 \) and \( q^2 \), thus one has

\[ \tilde{\Pi}_{1,1st}^{\mu\nu} = i \frac{m_\tau}{|m_\tau|} \frac{e^2}{2\pi} \varepsilon^{\mu\nu\alpha} k_\alpha \]  

and \( \tilde{\Pi}_{1,2nd}^{\mu\nu} = i \frac{m_\tau}{|m_\tau|} \frac{q^2}{2\pi} \varepsilon^{\mu\nu\alpha} k_\alpha \),

(50)

for the first and second diagram, respectively, and the corresponding C-S effective action is

\[ \tilde{S}_{\text{eff}}^{(2,1)} = -\frac{1}{4\pi |m_\tau|} \int d^3 x \left\{ e^2 A_\mu(x) \partial_\alpha A_\nu(x) + q^2 A_\mu^{(5)}(x) \partial_\alpha A_\nu^{(5)}(x) \right\} . \]  

(51)

From (51) one obtains the induced fermion currents

\[ \langle j^\mu \rangle = \frac{\delta \tilde{S}_{\text{eff}}^{(2,1)}}{\delta A_\mu} = -\frac{e^2}{4\pi |m_\tau|} F_\mu \]  

and \( \langle j_5^\mu \rangle = \frac{\delta \tilde{S}_{\text{eff}}^{(2,1)}}{\delta A_\mu^{(5)}} = -\frac{q^2}{4\pi |m_\tau|} F_\mu^{(5)} \),

whose time-components recovers the results (37) and (38) respectively.
V. CONCLUSIONS

We have obtained induced fermion charges in a continuum chiral theory for massless planar electrons in a honeycomb structure by means of only vector gauge potentials. For some configurations of the vector gauge potentials, particularly quantized magnetic vortices, we show that the induced charges can have fractional values. Since isolated and localized zero-energy states together with symmetry of sublattices are crucial for the appearance of fractional charges, we have analyzed the spinorial structure of the zero-modes as well as the spectral-symmetry conjugation of the first-quantized Hamiltonian. Moreover, in order to understand and highlight the origin of zero-energy states, that is, if they come from electrons on either valence or conduction bands we have studied in detail the energy eigenstates of fermions in homogeneous magnetic fields by taking also into account a staggered chemical potential ($\mu$) and a parity-breaking mass term ($m_\tau$), which open a mass gap in the energy spectrum (Landau levels). By taking $\mu, m_\tau \to 0$, we calculate the induced fermion charges and the results are in consonance with the results found when the electrons are under the influence of finite magnetic fluxes.

In pursuit of the realization of quantum anomalies in condensed matter systems, we also discuss, through field-theoretical calculation, the relation of induced fermion currents with the parity-anomaly in an extended Quantum Electrodynamics which involves a vector and a vector-axial gauge fields. Unfortunately, this relation is not established beyond doubt, since the parity-anomaly is only realized if parity symmetry-breaking terms, such as the staggered chemical potential and the Haldane mass, are present in the physical system, and that is not the case in graphene.

VI. ACKNOWLEDGMENTS

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