Convex pentagons and convex hexagons that can form rotationally symmetric tilings

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Abstract

In this study, the properties of convex hexagons that can form rotationally symmetric edge-to-edge tilings are discussed. Because the convex hexagons are equilateral convex parallelohexagons, convex pentagons generated by bisecting the hexagons can form rotationally symmetric non-edge-to-edge tilings. In addition, under certain circumstances, tiling-like patterns with an equilateral convex polygonal hole at the center can be formed using these convex hexagons or pentagons.

Keywords: pentagon, hexagon, tiling, rotationally symmetry, monohedral

1 Introduction

In \cite{1}, Klaassen has demonstrated that there exist countless rotationally symmetric non-edge-to-edge tilings with convex pentagonal tiles\textsuperscript{1}. The convex pentagonal tiles are considered to be equivalent to bisecting equilateral convex parallelohexagons, which are hexagons where the opposite edges are parallel and equal in length. Thus, there exist countless rotationally symmetric edge-to-edge tilings with convex hexagonal tiles.

Figure 1 shows a five-fold rotationally symmetric edge-to-edge tiling by a convex hexagonal tile (equilateral convex parallelohexagon) that satisfies the conditions, \(A = D = 72^\circ, B = C = E = F = 144^\circ, a = b = c = d = e = f\). Figure 2 shows examples of five-fold rotationally symmetric non-edge-to-edge tilings with convex pentagonal tiles generated by bisecting the equilateral convex parallelohexagon in Figure 1. Because the equilateral convex parallelohexagons have two-fold rotational symmetry, the number of ways to form convex pentagons generated by bisecting the convex hexagons (the dividing line needs to be a straight line that passes through the rotational center of the convex hexagon and intersects the opposite edge) is countless. The bisecting method can be divided into three cases: Case (i) the dividing line intersects edges \(c\) and \(f\), as shown in Figures 2(a) and 2(b); Case (ii) the dividing line intersects edges \(a\) and \(d\), as shown in Figure 2(c); Case (iii) the dividing line intersects the

\textsuperscript{1}A tiling (or tessellation) of the plane is a collection of sets that are called tiles, which covers a plane without gaps and overlaps, except for the boundaries of the tiles. The term “tile” refers to a topological disk, whose boundary is a simple closed curve. If all the tiles in a tiling are of the same size and shape, then the tiling is monohedral \cite{1,6}. In this study, a polygon that admits a monohedral tiling is called a polygonal tile \cite{4}. Note that, in monohedral tiling, it admits the use of reflected tiles.

\textsuperscript{2}A tiling by convex polygons is edge-to-edge if any two convex polygons in a tiling are either disjoint or share one vertex or an entire edge in common. Then other case is non-edge-to-edge \cite{1,3}.
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Figure 1: Rotationally symmetric tiling with $D_5$ symmetry that formed by a convex hexagonal tile with $D_2$ symmetry (Note that the gray area in the figure is used to clearly depict the structure)

edges $b$ and $e$, as shown in Figure 2(d). If the dividing line is selected such that the opposite vertices of the equilateral convex parallelohexagons are connected, the parallelohexagons contain two congruent convex quadrangles. In such cases, as shown in Figure 3, there exist five-fold rotationally symmetric edge-to-edge tilings with convex quadrangles.

The equilateral convex parallelohexagons that satisfy “$A = D \neq B = C = E = F$, $a = b = c = d = e = f$,” as demonstrated in Figure 1, have two-fold rotational symmetry and two axes of reflection symmetry passing through the center of the rotational symmetry (hereafter, this property is described as $D_2$ symmetry$^3$). Therefore, the parallelohexagon and reflected parallelohexagon have identical outlines. (In the parallelohexagon with $D_2$ symmetry, Cases (ii) and (iii) can be regarded as having a reversible relationship.) If two convex pentagons are generated in the equilateral convex parallelohexagon with $D_2$ symmetry, which is bisected by a dividing line that does not overlap with the axis of reflection symmetry, as shown in Figures 2(b), 2(c), and 2(d), the reflected parallelohexagon has the same outline. However, the arrangement of the inner convex pentagons is different. By using this property, the reflected convex pentagons can be freely incorporated into the tiling. Figure 4 shows a random tiling based on the five-fold rotationally symmetric tiling structure of a convex pentagonal tile

$^3$“$D_2$” is based on the Schoenflies notation for symmetry in a two-dimensional point group [7, 8]. “$D_n$” represents an $n$-fold rotation axis with $n$ reflection symmetry axes. The notation for symmetry is based on that presented in [3].
In this study, the properties of convex hexagons and pentagons that can form rotationally symmetrical tilings presented by Klaassen are explored. The convex hexagons and pentagons can form rotationally symmetric tilings and rotationally symmetric tiling-like patterns with an equilateral convex polygonal hole at the center. Note that the tiling-like patterns are not considered tilings due to the presence of a gap, but are simply called tilings in this study. Herein, the various types of convex hexagons and pentagons are introduced and explored.

2 Rotationally symmetric tilings

In [3], the theorem and proof that countless non-edge-to-edge tilings can be formed with convex pentagonal tiles are presented, and the figures of five-fold and seven-fold rotationally symmetric tilings are shown. The conditions of the convex polygonal tiles that can be formed with an $n$-fold rotationally symmetric edge-to-edge tiling are expressed in [1], while
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Figure 3: Five-fold rotationally symmetric edge-to-edge tilings with convex quadrangles

<Case of n = 5>
A = D = 72°,
B = C = E = F = 144°,
a = b = c = d = e = f.

Figure 4: Random tiling based on the five-fold rotationally symmetric tiling structure of a convex pentagonal tile
considering the pentagonal tilings as convex hexagonal tilings. Note that the vertices (interior angles) and edges of the convex hexagon will be referred to using the nomenclature shown in Figure 1.

\[
\begin{align*}
A &= D = \frac{360^\circ}{n}, \\
B &= E, \\
C &= F, \\
A + B + C &= 360^\circ, \\
a &= b = c = d = e = f,
\end{align*}
\]

where \( n \) is an integer greater than or equal to three, because \( 0^\circ < A < 180^\circ \). In (1), the \( A \) and \( D \) pair are selected as the vertices for \( \frac{360^\circ}{n} \). However, due to the symmetry of the equilateral convex parallelohexagon, the \( B \) and \( E \) pair, as well as the \( C \) and \( F \) pair, are also possible vertex symbol starting points.

When convex hexagonal tiles satisfy (1), where \( B = C = E = F \), they become equilateral convex parallelohexagons with \( D_2 \) symmetry, as shown in Figure 1. The conditions of the convex hexagonal tiles, in this case, are expressed in (2).

\[
\begin{align*}
A &= D = \frac{360^\circ}{n}, \\
B &= C = E = F = 180^\circ - A/2 = 180^\circ - \frac{180^\circ}{n}, \\
a &= b = c = d = e = f.
\end{align*}
\]

The convex hexagons that satisfy (1) belong to the Type 1 family. In contrast, the convex hexagons that satisfy (2) belong to the Type 1 and Type 2 families. When \( n = 3 \) in (2), a regular hexagon has \( A = B = C = D = E = F = 120^\circ \), which belongs to the Type 1, Type 2, and Type 3 families. The convex pentagons generated by bisecting the convex hexagons belong to the Type 1 family.

Table 1 presents some of the relationships between the interior angles of convex hexagons satisfying (2) that can form the \( n \)-fold rotationally symmetric edge-to-edge tilings. (For \( n = 3 \sim 8 \), tilings with convex hexagonal tiles and convex pentagonal tiles generated by bisecting them are drawn. For further details, Figures 1 [2] [15, 19].) As mentioned above, when \( n = 3 \) in (2), the convex hexagonal tile becomes a regular hexagon. Owing to the \( D_6 \) symmetry of the regular hexagon, the pentagonal pair corresponding to the bisected regular hexagon can be arranged by freely combining operations of \( 120^\circ \) or \( 240^\circ \) turnings and their reflections. (In [2], there are figures that depict a few tilings with such operations.)

For \( n = 3 \), the convex hexagonal tile that satisfies (2) is a regular hexagon, and thus, its tiling has \( D_6 \) symmetry. For \( n \geq 4 \) each \( n \)-fold, rotationally symmetric edge-to-edge tiling by a convex hexagonal tile that satisfies (2) has \( D_n \) symmetry. Therefore, the tiling in

\[\text{\(^{4}\)It is known that convex hexagonal tiles belong to at least one of the three families referred to as a “Type.”} \]
\[\text{\(^{5}\)To date, fifteen families of convex pentagonal tiles, each of them referred to as a “Type,” are known.} \]
\[\text{\(^{6}\)For example, if the sum of three consecutive angles in a convex pentagonal tile is 360^\circ, the pentagonal tile belongs to the Type 1 family.} \]
\[\text{\(^{7}\)Known convex pentagonal tiles can form periodic tiling. In May 2017, Micha?el Rao declared that the complete list of Types of convex pentagonal tiles had been obtained (i.e., they have only the known 15 families), but it does not seem to be fixed as of March 2020.} \]
Table 1: Interior angles of convex hexagons satisfying (2) that can form the n-fold rotationally symmetric edge-to-edge tilings

| n  | Value of interior angle (degree) | Figure number |
|----|----------------------------------|---------------|
|    | A  | B  | C  | D  | E  | F  |              |
| 3  | 120 | 120 | 120 | 120 | 120 | 120 | 15            |
| 4  | 90  | 135 | 135 | 90  | 135 | 135 | 16            |
| 5  | 72  | 144 | 144 | 72  | 144 | 144 | 12            |
| 6  | 60  | 150 | 150 | 60  | 150 | 150 | 17            |
| 7  | 51.43 | 154.29 | 154.29 | 51.43 | 154.29 | 154.29 | 18            |
| 8  | 45  | 157.5 | 157.5 | 45  | 157.5 | 157.5 | 19            |
| 9  | 40  | 160 | 160 | 40  | 160 | 160 |              |
| 10 | 36  | 162 | 162 | 36  | 162 | 162 |              |
| 11 | 32.73 | 163.64 | 163.64 | 32.73 | 163.64 | 163.64 |              |
| 12 | 30  | 165 | 165 | 30  | 165 | 165 |              |
| 13 | 27.69 | 166.15 | 166.15 | 27.69 | 166.15 | 166.15 |              |
| 14 | 25.71 | 167.14 | 167.14 | 25.71 | 167.14 | 167.14 |              |
| 15 | 24  | 168 | 168 | 24  | 168 | 168 |              |
| 16 | 22.5 | 168.75 | 168.75 | 22.5 | 168.75 | 168.75 |              |
| 17 | 21.18 | 169.41 | 169.41 | 21.18 | 169.41 | 169.41 |              |
| 18 | 20  | 170 | 170 | 20  | 170 | 170 |              |
| ...| ... | ... | ... | ... | ... | ... | ...            |

Figure 1 has $D_5$ symmetry. The n-fold rotationally symmetric tilings with convex pentagons or convex quadrangles generated by bisecting the convex hexagons that satisfy (2) along the axis of reflection symmetry have $D_n$ symmetry. Therefore, the tilings of Figures 2(a) and 3(a) have $D_5$ symmetry.

When the convex hexagonal tile that satisfies (1) has $B = E \neq C = F$, the equilateral convex parallelohexagon has two-fold rotational symmetry but no axis of reflection symmetry (hereafter, this property is described as $C_2$ symmetry). The n-fold rotationally symmetric edge-to-edge tilings by equilateral convex parallelohexagons with $C_2$ symmetry have $C_n$ symmetry because they have rotational symmetry but no axis of reflection symmetry. For example, the five-fold rotationally symmetric edge-to-edge tiling by the convex hexagonal tile, where “$A = D = 72^\circ$, $B = E = 134^\circ$, $C = F = 154^\circ$, a = b = c = d = e = f” shown in Figure 5(a), has $C_5$ symmetry. Additionally, the five-fold rotationally symmetric non-edge-to-edge tiling of the convex pentagonal tile generated by bisecting the convex hexagon, shown in Figure 5(b), has $C_5$ symmetry. As described above, for $n > 4$ each n-fold, rotationally symmetric edge-to-edge tiling by a convex hexagonal tile that satisfies (2) has $D_n$ symmetry. Conversely, the n-fold rotationally symmetric tilings by a convex pentagonal tile or a convex quadrangle generated from the convex hexagonal tile, which satisfies (2) and is bisected by a dividing line that does not overlap with the axis of reflection symmetry, have $C_n$ symmetry. Therefore, the tilings of Figures 2(b), 2(c), 2(d), and 3(b) have $C_5$ symmetry.

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7, 8: “$C_2$” is based on the Schoenflies notation for symmetry in a two-dimensional point group.
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Figure. 5: Rotationally symmetric edge-to-edge tiling with $C_5$ symmetry by a convex hexagonal tile that satisfies “$A = D = 72^\circ$, $B = E = 134^\circ$, $C = F = 154^\circ$, $a = b = c = d = e = f$,” and the five-fold rotationally symmetric non-edge-to-edge tiling by a convex pentagonal tile generated by bisecting the convex hexagon

Here, the formation of rotationally symmetric edge-to-edge tiling with convex hexagonal tiles is briefly explained. First, as shown in STEP 1 in Figure 6, create a unit connecting the convex hexagonal tiles so that they form $n$-fold rotationally symmetric edge-to-edge tiling in one direction so that $B + D + F = 360^\circ$ and $A + C + E = 360^\circ$. The tiles can then be assembled in such a way as to increase the number of pieces from one to two to three, and so on, in order. Then create a similar unit with the reflected hexagons. Next, connect the two units created in STEP 1, so that $A + E + F = 360^\circ$ as in STEP 2 in Figure 6. Subsequently, take the unit from STEP 2 and rotate it by the value of the interior angle of vertex $A$. When the original unit and the rotated unit are arranged, so that $A + B + F = 360^\circ$ and $A + B + C = 360^\circ$ as shown in STEP 3 in Figure 6, $\frac{2}{n}$ parts in the $n$-fold rotationally symmetric tiling can be formed. Then, by repeating this process as many times as necessary, an $n$-fold rotationally symmetric edge-to-edge tiling with convex hexagonal tiles can be formed. When the convex hexagons are bisected as depicted in Figure 2 it will result in an $n$-fold rotationally symmetric non-edge-to-edge tiling with convex pentagonal tiles. Note that when the hexagons with the $D_2$ symmetry are used, as presented in Table 1 a unit of reflected hexagons is not required. In this case, the units are arranged so that $A + B + E = 360^\circ$ and $A + C + F = 360^\circ$ as in STEP 2, and $A + B + F = 360^\circ$, $A + C + F = 360^\circ$, and $A + B + E = 360^\circ$ as in STEP 3.

3 Rotationally symmetric tilings (tiling-like patterns) with a regular polygonal hole at the center

In [2], there are figures of rotationally symmetric tilings with a hole in the center of a regular hexagon, octagon, or 12-gon formed by using convex pentagons (or elements that can be regarded as convex hexagons). The regular hexagonal hole can be filled with convex pentagons.
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Figure 6: Formation method of rotationally symmetric edge-to-edge tiling with convex hexagonal tiles

However, because the other holes cannot be filled with convex pentagons, they are not exactly tilings. The convex pentagons that can form rotationally symmetric tilings with a regular \( m \)-gonal hole at the center can be generated using the convex hexagons that satisfy (1). The conditions of the convex hexagonal tiles that can be formed with a rotationally symmetric tiling with a regular \( m \)-gonal hole are expressed in (3).

\[
\begin{align*}
A &= D = \frac{720^\circ}{m}, \\
B &= C = E = F = 180^\circ - \frac{A}{2} = 180^\circ - \frac{360^\circ}{m}, \\
a &= b = c = d = e = f,
\end{align*}
\]

where \( m \) is an integer greater than or equal to five, because \( 0^\circ < A < 180^\circ \). The convex hexagons that satisfy (3) are equilateral convex parallelohexagons with \( D_2 \) symmetry. Note that, because “\( 180^\circ - \frac{360^\circ}{m} \)” corresponds to one inner angle (interior angle) of a regular \( m \)-gon, the value of “\( A + F \)” in (3) is equal to the outer angle \( (180^\circ + \frac{360^\circ}{m}) \) of one vertex of a regular \( m \)-gon.

Table 2 presents some of the relationships between the interior angles of convex hexagons satisfying (3) that can form the rotationally symmetric tilings with a regular \( m \)-gonal hole at the center. (For \( m = 5-10, 12, 14, 16 \), tilings with a regular \( m \)-gonal hole at the center by convex hexagons and convex pentagons generated by bisecting them are drawn. For
Table 2: Interior angles of convex hexagons satisfying (3) that can form the rotationally symmetric tilings with a regular \( m \)-gonal hole at the center

| \( m \) | Value of interior angle (degree) | \( n \) of Table [1] | Figure number |
|---|---|---|---|
| 5 | 144 108 108 144 108 108 | | 20 |
| 6 | 120 120 120 120 120 120 | 3 | 21 |
| 7 | 102.86 128.57 128.57 102.86 128.57 128.57 | | 22 |
| 8 | 90 135 135 90 135 135 | 4 | 23 |
| 9 | 80 140 140 80 140 140 | | 24 |
| 10 | 72 144 144 72 144 144 | 5 | 25 |
| 11 | 65.45 147.27 147.27 65.45 147.27 147.27 | | |
| 12 | 60 150 150 60 150 150 | 6 | 26 |
| 13 | 55.38 152.31 152.31 55.38 152.31 152.31 | | |
| 14 | 51.43 154.29 154.29 51.43 154.29 154.29 | 7 | 27 |
| 15 | 48 156 156 48 156 156 | | |
| 16 | 45 157.5 157.5 45 157.5 157.5 | 8 | 28 |
| 17 | 42.35 158.82 158.82 42.35 158.82 158.82 | | |
| 18 | 40 160 160 40 160 160 | 9 | |
| 19 | 37.89 161.05 161.05 37.89 161.05 161.05 | | |
| 20 | 36 162 162 36 162 162 | 10 | |
| 21 | 34.29 162.86 162.86 34.29 162.86 162.86 | | |
| 22 | 32.73 163.64 163.64 32.73 163.64 163.64 | 11 | |
| 23 | 31.30 164.35 164.35 31.30 164.35 164.35 | | |
| 24 | 30 165 165 30 165 165 | 12 | |
| 25 | 28.8 165.6 165.6 28.8 165.6 165.6 | | |
| ... | ... ... ... ... ... | ... | ... |
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Figure 7: Formation method of rotationally symmetric tiling with a regular $m$-gonal hole at the center by convex hexagons in Table 2

Figure 7 create a unit connecting the convex hexagons satisfying in one direction so that $B + D + F = 360^\circ$ and $A + C + E = 360^\circ$. The hexagons can then be assembled in such a way as to increase the number of pieces from one to two to three, and so on, in order. Then, copy the unit in STEP 1, and rotate it by the half value of the interior angle of vertex $A$. Subsequently, connect the two units of STEP 1, so that $A + B + E = 360^\circ$ and $A + C + F = 360^\circ$ as in STEP 2 in Figure 7. The series of two edges, $AF$, of the unit in STEP 2 are edges of the contour of a regular $m$-gon. Then, by repeating this process as many times as necessary, a rotationally symmetric tiling with a regular $m$-gonal hole at the center with convex hexagons can be formed. STEP 3 in Figure 7 is $\frac{4}{10}$ parts of a regular 10-gon (The completed state is shown in Figure 25). As shown in Table 2 and Figure 21, the hexagon, $m = 6$, is a regular hexagon, so it is possible to fit the regular hexagon into the center hole.

4 Rotationally symmetric tilings by an equilateral convex parallelohexagon with $C_2$ symmetry

The conditions of the equilateral convex parallelohexagons with $C_2$ symmetry are expressed in (4).

$$\begin{cases} A = D \neq B, \\ B = E \neq C, \\ C = F \neq A, \\ A + B + C = 360^\circ, \\ a = b = c = d = e = f. \end{cases}$$ (4)
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Figure 8: Rotationally symmetric tilings with $C_3$ or $C_4$ symmetry by a convex pentagon based on a convex hexagon with $C_2$ symmetry that satisfies “$A = D = 90^\circ$, $B = E = 120^\circ$, $C = F = 150^\circ$, $a = b = c = d = e = f$”.

As mentioned in Section 2, the convex hexagonal tiles that satisfy (4) and have interior angles of $\frac{360^\circ}{n}$, where $n$ is an integer greater than or equal to three, can form rotationally symmetric tilings with $C_n$ symmetry.

In [2], Iliev presents some tilings using a convex pentagon that bisects an equilateral convex parallelohexagon with $C_2$ symmetry that satisfies “$A = D = 90^\circ$, $B = E = 120^\circ$, $C = F = 150^\circ$, $a = b = c = d = e = f$.” They are a rotationally symmetric tiling with $C_4$ symmetry, as shown in Figure 8(a), a rotationally symmetric tiling with $C_4$ symmetry with an equilateral convex octagonal hole with $D_4$ symmetry at the center, as shown in Figure 8(b), and a rotationally symmetric tiling with $C_3$ symmetry with an equilateral convex hexagonal hole with $D_3$ symmetry at the center, as shown in Figure 8(c). Notably, those tilings can be formed with convex octagonal and hexagonal holes that are equilateral but not regular polygons at the center, as shown in Figures 8(b) and 8(c). Thus, the convex hexagonal tiles that satisfy (4) and have interior angles of $\frac{360^\circ}{n}$ can form rotationally symmetric tilings with
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Figure 9: Rotationally symmetric tiling with $C_5$ symmetry with an equilateral convex 10-gonal hole with $D_5$ symmetry at the center using a convex hexagon with $C_2$ symmetry that satisfies “$A = D = 72^\circ$, $B = E = 134^\circ$, $C = F = 154^\circ$, $a = b = c = d = e = f$,” and the version of the tiling with convex pentagons based on the convex hexagons $C_n$ symmetry with an equilateral convex $2n$-gonal hole with $D_n$ symmetry at the center. For example, Figure 9(a) presents a rotationally symmetric tiling with $C_5$ symmetry with an equilateral convex 10-gonal hole with $D_5$ symmetry at the center using an equilateral convex parallelohexagon with $C_2$ symmetry that satisfies “$A = D = \frac{360^\circ}{5} = 72^\circ$, $B = E = 134^\circ$, $C = F = 154^\circ$, $a = b = c = d = e = f$,” and Figure 9(b) is a version of the tiling with convex pentagons bisecting the convex hexagons. Note that the convex hexagon in Figure 9 is the same as the convex hexagon in Figure 5. Then, Figure 10(a) shows a rotationally symmetric tiling with $C_7$ symmetry, and Figure 10(b) shows a rotationally symmetric tiling with $C_7$ symmetry with an equilateral convex 14-gonal hole with $D_7$ symmetry at the center, using an equilateral convex parallelohexagon with $C_2$ symmetry that satisfies “$A = D = \frac{360^\circ}{7} \approx 51.43^\circ$, $B = E \approx 143.57^\circ$, $C = F = 165^\circ$, $a = b = c = d = e = f$.”

Let us supplement the reason why the number of edges of the central polygonal hole, i.e., the equilateral convex polygonal hole at the center that is formed by an equilateral convex parallelohexagon with $C_2$ symmetry as described above, is even. For example, when $m = 5$ in Figure 20 (the five-fold rotationally symmetric tilings with a regular pentagonal hole by a convex hexagonal tile that satisfies \(5\) and $A = D = \frac{360^\circ}{5} = 144^\circ$), it can be observed that the five units created in STEP 1 in Figure 7 are used. Conversely, when forming a rotationally symmetric tiling by an equilateral convex parallelohexagon with the $C_2$ symmetry, as shown in Figures 5, 6, 8, 9 and 10, the units with hexagons and units with reflected hexagons should be connected alternately. Therefore, if the number of edges of the central polygonal hole is odd, the convex hexagons with $C_2$ symmetry cannot form the central polygon shown in Figure 11 (when trying to form a nonagonal hole by an equilateral convex parallelohexagon with $C_2$ symmetry that satisfies $A = D = \frac{720^\circ}{9} = 80^\circ$). Because an equilateral convex $2n$-gon with $D_n$ symmetry has two types of inner angles (interior angles) in vertices, there are two...
Figure 10: Rotationally symmetric tiling with $C_7$ symmetry and rotationally symmetric tiling with $C_7$ symmetry with an equilateral convex 14-gonal hole with $D_7$ symmetry at the center using an equilateral convex parallelohexagon with $C_2$ symmetry that satisfies $A = D = \frac{360}{7} \approx 51.43^\circ$, $B = E \approx 143.57^\circ$, $C = F = 165^\circ$, $a = b = c = d = e = f$

types of outer angles in vertices. If $A = D = \frac{360}{n}$ in (4), values of “$A + B$, $A + F$” are equal to the two types of outer angles (see Figure 10(b)).

Figure 12 shows a rotationally symmetric tiling with $C_3$ symmetry formed by a convex hexagonal tile that satisfies “$A = D = 90^\circ$, $B = E = 120^\circ$, $C = F = 150^\circ$, $a = b = c = d = e = f$” in Figure 8. Then, Figure 13(a) shows a three-fold rotationally symmetric tiling formed by the convex hexagonal tile that satisfies “$A = D = 80^\circ$, $B = E = 120^\circ$, $C = F = 160^\circ$, $a = b = c = d = e = f$” shown in Figure 11. As shown in Figure 13(b), the convex hexagons can also form a rotationally symmetric tiling with $C_3$ symmetry with an equilateral convex hexagonal hole at the center. This arrangement indicates that the convex hexagons that satisfy (4), where one of the vertices $A$, $B$, or $C$ of which is equal to $\frac{360}{n}$, can form an $n$-fold rotationally symmetric tiling, and a tiling with $C_n$ symmetry with an equilateral convex $2n$-gonal hole with $D_n$ symmetry at the center. (It is a matter of starting the vertex symbol from somewhere. As shown in Figure 13 if “$A \rightarrow F''$, $B \rightarrow A'$, . . .” is replaced, then $A' = D' = \frac{360}{3} = 120^\circ$, which corresponds to $A_3$.)

Because the convex hexagonal tile that satisfies “$A = D = 90^\circ$, $B = E = 120^\circ$, $C = F = 150^\circ$, $a = b = c = d = e = f$” shown in Figures 8 and 12 is the equilateral convex parallelohexagon with $\frac{360}{3} = 120^\circ$ and $\frac{360}{4} = 90^\circ$, it can form rotationally symmetric tilings with $C_3$ or $C_4$ symmetry, a rotationally symmetric tiling with $C_3$ symmetry with an equilateral convex hexagonal hole with $D_3$ symmetry, and a rotationally symmetric tiling with $C_4$ symmetry with an equilateral convex octagonal hole with $D_4$ symmetry at the center. As an equilateral convex parallelohexagons with $C_2$ symmetry that has the same properties as above, there is a convex hexagonal tile that satisfies “$A = D = 72^\circ$, $B = E = 120^\circ$, $C = F = 168^\circ$, $a = \ldots$
Figure. 11: State in which a nonagonal hole cannot be formed using convex hexagons that satisfy 
"A = D = \frac{720^\circ}{9} = 80^\circ, B = E \neq C = F, A + B + C = 360^\circ, a = b = c = d = e = f."

Figure. 12: Three-fold rotationally symmetric tiling by a convex pentagon based on a convex hexagon that satisfies “A = D = 90^\circ, B = E = 120^\circ, C = F = 150^\circ, a = b = c = d = e = f.”

b = c = d = e = f.” This convex hexagonal tile can form rotationally symmetric tilings with 
C₅ or C₃ symmetry (see Figures 14(a) and 14(b)), a rotationally symmetric tiling with C₅ 
symmetry with an equilateral convex 10-gonal hole with D₅ symmetry (see Figure 14(c)), and 
a rotationally symmetric tiling with C₃ symmetry with an equilateral convex hexagonal hole 
with D₃ symmetry at the center (see Figure 14(d)). Note that Figure 14 shows the tilings 
with convex pentagons based on the convex hexagons.

The convex hexagonal tiles that satisfy (4) and have particular angles (i.e., angles corresponding to \(\frac{360^\circ}{n}\)) of two or more types can form multiple rotationally symmetric tilings
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Figure. 13: Three-fold rotationally symmetric tiling, and rotationally symmetric tiling with an equilateral convex hexagonal hole at the center by a convex hexagon that satisfies “$A = D = 80^\circ$, $B = E = 120^\circ$, $C = F = 160^\circ$, $a = b = c = d = e = f$” with an equilateral polygonal hole at the center, and multiple rotationally symmetric tilings. Such equilateral convex parallelohexagons with $C_2$ symmetry that can form multiple rotationally symmetric tilings with an equilateral polygonal hole at the center and multiple rotationally symmetric tilings are the above two cases with angles of “$90^\circ$, $120^\circ$, $150^\circ$” and “$72^\circ$, $120^\circ$, $168^\circ$.” It is because that interior angles of the convex hexagons that satisfy (4) can be selected up to two different values corresponding to $\frac{360^\circ}{n}$ when $n$ is an integer greater than or equal to three. Therefore, the two particular angles of this special case can be selected from “$72^\circ$, $90^\circ$, $120^\circ$.”

5 Conclusions

This study summarizes how the convex pentagons and hexagons can form rotationally symmetric tilings based on the information in [2] and [3]. The $n$-fold rotationally symmetric edge-to-edge tilings formed of convex hexagonal tiles that satisfy (1) can be divided into $C_n$ symmetry and $D_n$ symmetry depending on the selection of the interior angles of the vertices in the convex hexagon. The convex pentagonal tiles generated by bisecting the convex hexagonal tiles can form rotationally symmetric non-edge-to-edge tilings with $C_n$ or $D_n$ symmetry depending on the shape or division method of the convex hexagon. In contrast, the $n$-fold rotationally symmetric edge-to-edge tilings with the convex pentagonal tiles shown in [5] can form only $C_n$ symmetry. In addition, this study demonstrated a convex hexagons (convex pentagons) that can form rotationally symmetric tilings with an equilateral convex polygonal hole at the center.

In [2], there are cases where $m = 6, 8, 12$ present in Table 2 and tilings as shown in
Figure 14: Five-fold or three-fold rotationally symmetric tilings, and rotationally symmetric tilings with an equilateral convex 10-gonal or hexagonal hole at the center by a convex pentagon based on a convex hexagon that satisfies “\(A = D = 72^\circ\), \(B = E = 120^\circ\), \(C = F = 168^\circ\), \(a = b = c = d = e = f\)”
Figure 8 are described. However, there is no description of properties (tiles conditions such as (1), (2), and (3)) as discussed in this study. In [3], there is no description of tilings with an equilateral convex polygonal hole at the center, and there is insufficient description regarding the convex hexagonal tile (such as the tile condition).

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Figure 15: Three-fold rotationally symmetric tilings by a convex hexagon and a convex pentagon.
Figure. 16: Four-fold rotationally symmetric tilings by a convex hexagon and a convex pentagon

Figure. 17: Six-fold rotationally symmetric tilings by a convex hexagon and a convex pentagon

Figure. 18: Seven-fold rotationally symmetric tilings by a convex hexagon and a convex pentagon
Figure. 19: Eight-fold rotationally symmetric tilings by a convex hexagon and a convex pentagon

Figure. 20: Rotationally symmetric tiling with $C_5$ symmetry with a regular convex pentagonal hole at the center by a convex pentagon
Convex pentagons and convex hexagons that can form rotationally symmetric tilings (v2)

Figure. 21: Rotationally symmetric tiling with $C_6$ symmetry with a regular convex hexagonal hole at the center by a convex pentagon

$\begin{align*}
&\text{Case of } m = 6 \ (n = 3) \\
A &= B = C = D = E = F = 120^\circ, \\
\alpha &= \beta = \gamma = \delta = \epsilon = \zeta.
\end{align*}$

Figure. 22: Rotationally symmetric tiling with $C_7$ symmetry with a regular convex heptagonal hole at the center by a convex pentagon

$\begin{align*}
&\text{Case of } m = 7 \\
\lambda &= \mu = 102.86^\circ, \\
B &= C = E = F = 128.57^\circ, \\
\alpha &= \beta = \gamma = \delta = \epsilon = \zeta.
\end{align*}$
Convex pentagons and convex hexagons that can form rotationally symmetric tilings (v2)

Figure. 23: Rotationally symmetric tiling with $C_8$ symmetry with a regular convex octagonal hole at the center by a convex pentagon

$<\text{Case of } n=8 \text{ (} a=4 \text{)}>$
$A = D = 90^\circ$,
$B = C = E = F = 135^\circ$,
$a = b = c = d = e = f$.

Figure. 24: Rotationally symmetric tiling with $C_9$ symmetry with a regular convex nonagonal hole at the center by a convex pentagon

$<\text{Case of } n=9>$
$A = D = 80^\circ$,
$B = C = E = F = 140^\circ$,
$a = b = c = d = e = f$. 
Convex pentagons and convex hexagons that can form rotationally symmetric tilings (v2)

Figure 25: Rotationally symmetric tiling with $C_{10}$ symmetry with a regular convex 10-gonal hole at the center by a convex pentagon

$<\text{Case of } m=10 (n=5)>$

$A = D = 72^\circ$,
$B = C = E = F = 144^\circ$,
$a = b = c = d = e = f$.

Figure 26: Rotationally symmetric tiling with $C_{12}$ symmetry with a regular convex 12-gonal hole at the center by a convex pentagon

$<\text{Case of } m=12 (n=6)>$

$A = D = 60^\circ$,
$B = C = E = F = 150^\circ$,
$a = b = c = d = e = f$. 
Convex pentagons and convex hexagons that can form rotationally symmetric tilings (v2)

Figure 27: Rotationally symmetric tiling with $C_{14}$ symmetry with a regular convex 14-gonal hole at the center by a convex pentagon

Figure 28: Rotationally symmetric tiling with $C_{16}$ symmetry with a regular convex 16-gonal hole at the center by a convex pentagon