Multiplicity-free $U_q(sl_N)$ 6-j symbols: relations, asymptotics, symmetries

Victor Alekseev$^{a,b,c}$, Andrey Morozov$^{a,b,c}$, Alexey Sleptsov$^{a,b,c}$

\begin{flushleft}
\textit{a} Institute for Theoretical and Experimental Physics, Moscow 117218, Russia \\
\textit{b} Institute for Information Transmission Problems, Moscow 127994, Russia \\
\textit{c} Moscow Institute of Physics and Technology, Dolgoprudny 141701, Russia
\end{flushleft}

Abstract

A closed form expression for multiplicity-free quantum 6-j symbols (MFS) was proposed in [1] for symmetric representations of $U_q(sl_N)$, which are the simplest class of multiplicity-free representations. In this paper we rewrite this expression in terms of $q$-hypergeometric series $4\Phi_3$. We claim that it is possible to express any MFS through the 6-j symbol for $U_q(sl_2)$ with a certain factor. It gives us a universal tool for the extension of various properties of the quantum 6-j symbols for $U_q(sl_2)$ to the MFS. We demonstrate this idea by deriving the asymptotics of the MFS in terms of associated tetrahedron for classical algebra $U(sl_N)$.

Next we study MFS symmetries using known hypergeometric identities such as argument permutations and Sears’ transformation. As a result we get new symmetries, which are a generalization of the tetrahedral symmetries and the Regge symmetries for $N = 2$.

Contents

1 Introduction
2 Racah coefficients, 6-j symbols and types I, II expression
3 Hypergeometric expression for 6-j symbols
   3.1 $q$-Hypergeometric symmetries
   3.2 6-j symbol as $5\Phi_4$ series
   3.3 Expression of 6-j symbol as $4\Phi_3$ series
4 Relation with $U_q(sl_2)$ 6-j symbols
5 Asymptotics of 6-j symbol
6 Symmetries derivation
   6.1 Hypergeometric symmetries group
   6.2 Type I general symmetries
   6.3 Type II general symmetries
7 Weak symmetries of 6-j symbols
   7.1 Type I
   7.2 Type II

\begin{flushleft}
$^*$alekseev.va@phystech.edu  \\
$^\dagger$Andrey.Morozov@itep.ru  \\
$^\ddagger$sleptsov@itep.ru
\end{flushleft}
1 Introduction

Racah-Wigner coefficients or 6-j symbols play an important role in mathematics and theoretical physics, because they appear in many different problems. From mathematical point of view they describe the associativity data, which are still unknown for $U_q(sl_N)$. The main difficulty is in the appearance of the so-called multiplicities, which happens when the algebra rank $N$ is greater than 2. However, even for multiplicity-free representations analytical formulas for 6j-symbols are known only for a small class of representations, namely, for symmetric representations.

In theoretical physics the algebra $U_q(sl_N)$ is very important especially in quantum physics. Here is an incomplete list of topics, in which 6-j symbols of quantum Lie algebra $U_q(sl_N)$ or its classical version $U(sl_N)$, appear:

- quantum mechanics [2] and quantum computing [3],
- quantum $R$-matrices and integrable systems [4],
- WZW conformal field theory and 3d Chern-Simons theory [5, 6],
- lattice gauge theory [7],
- 3-d quantum gravity [8],
- quantum $sl_N$ invariants of knots [9],
- Turaev-Viro invariants of 3-manifolds and topological field theory [10, 11],
- Drinfeld associator and Kontsevich integral [12, 13],
- orthogonal polynomials [14].

One can see that 6-j symbols are widely used in both classical and modern works. Note that in many situations, e.g. in the quantum gravity or in statistical models, one considers partition functions, which contain a sum over all possible 6-j symbols of the given gauge group. In such problems it would be very useful to use symmetries between different 6-j symbols in order to reduce the sum and simplify the computation.

Quantum 6-j symbols have a lot of symmetries, most of them are still unknown. Nowadays we have different situations for $U_q(sl_2)$ and more general $U_q(sl_N)$ 6-j symbols. All symmetries of $U_q(sl_2)$ 6-j symbols are well known and well studied. In the present paper we are interested in the so-called linear symmetries. Non-linear symmetries (e.g. the pentagon relation), that are more complicated, are out of the scope of this paper. Linear symmetries of $U_q(sl_2)$ Racah coefficients include Regge symmetries, the tetrahedral symmetries and transformation $q \leftrightarrow q^{-1}$ [15]. Known symmetries of $U_q(sl_N)$ include complex conjugation, a $q \leftrightarrow q^{-1}$ and the tetrahedral symmetries [6].

Some symmetries may be obtained with the help of the eigenvalue hypothesis [16, 17] including some generalization for Regge symmetries. It says that the Racah matrices are uniquely defined by the eigenvalues of the $\mathcal{R}$-matrices. All studied examples says that it is true and this hypothesis becomes a useful tool to derive symmetries. Moreover, there is an exact expression for the Racah matrices through the $\mathcal{R}$-matrix eigenvalues for the matrices of the size up to $5 \times 5$ [18] and $6 \times 6$ [19].

The 6-j symbols calculation is a big problem for $U_q(sl_N)$ representations. There are few calculation methods and each of them is extremely tedious. Unlike the $U_q(sl_2)$ case, where the answer is known in a closed form for each representation [20], the analytical expression for arbitrary representations is still unknown. However, for the special case of symmetric and conjugated symmetric $U_q(sl_N)$ representations, the analytical expression was proposed recently [1]. This result gives us plenty of new questions. In particular, which properties of the expression are special for $U_q(sl_2)$ and which can be generalized to the more complex cases. For instance, in this context it was found [21] that 6-j symbols for symmetric representations of $U_q(sl_N)$ can be expressed in terms of orthogonal q-Racah polynomials as well as their counterpart for $U_q(sl_2)$.

In this paper we study the analytical expression from [1] in order to find new symmetries. In
section 2 we start by introducing Racah coefficients and 6-j symbols for $U_q(sl_N)$. In this paper we consider 6-j symbols that have only symmetric and conjugate to symmetric representations. All these 6-j symbols may be transformed via tetrahedral symmetries into either type I and type II [6]. For type I the only conjugate to symmetric representation is the second one, for type II – the third one. Each type can be considered as a natural generalization of $U_q(sl_2)$ 6-j symbols because each tensor product decomposition for this case has no multiplicities and can be enumerated by an integer number rather than a whole Young diagram. We consider the expression for both types as an analytic function and study its special properties to obtain new symmetries.

In section 3 we simplify the expression. Firstly, we prove that the expression may be reduced and the series became much more similar to $U_q(sl_2)$ series. This was done for both types independently and as it appears they can be represented as one universal expression for both types. Then we express it in terms of q-hypergeometric function $_4\Phi_3$ with some factor. Also it is proven that this expression does not have any inequality restrictions on its arguments, as it was proposed in the original article. As a result, the expression becomes more convenient for studying symmetries.

In section 4 we analyze the hypergeometric expression of multiplicity-free 6-j symbol. We find the transformation between the multiplicity-free $U_q(sl_N)$ 6-j symbol and its $U_q(sl_2)$ counterpart. This result creates a lot of possibilities to generalize well-known $U_q(sl_2)$ 6-j symbol properties to the considered case. As an immediate output of such relation in section 5 we derive the classical ($q = 1$) 6-j symbol asymptotics, using known results for $U(sl_2)$. Originally it was written in terms of the associated tetrahedron [22, 23]. The $U(sl_N)$ generalization modifies the expression so that the tetrahedron now depends on $N$ and deforms differently for two types of 6-j symbols.

In section 6 the resulting 6-j symbol expression has been studied for symmetries. Obtained $_4\Phi_3$ series has two known symmetries: permutations of arguments in each row and the Sears’ transformation [24]. The total number of hypergeometric symmetries is 23040 for both types, it was obtained by manual computations on computer. However, only 24 of them may be written as the 6-j symbol symmetries for type I and 12 for type II. Some of them are tetrahedral, others can be described as the Regge symmetry generalization for $N \geq 2$.

We also consider additional symmetries that have fusion conditions on both sides of expressions and therefore have fewer free parameters and a lesser degree of generality. Non-trivial expressions are found for both types and examples are provided. The main results of section 7 are symmetries that generalize permutation in a different from tetrahedral way. They become usual well-known symmetries when $N = 2$, but for $N > 2$ they depend on $N$ explicitly.

## 2 Racah coefficients, 6-j symbols and types I, II expression

To define 6-j symbols we need firstly to remind the Racah matrix definition. Here we work with q-deformed algebra $U_q(sl_N)$. Let us consider 3 irreducible C-modules of representations $R_1, R_2, R_3$ acting in $V_{R_1}$, $V_{R_2}$, $V_{R_3}$. Due to a tensor product associativity, $(V_{R_1} \otimes V_{R_2}) \otimes V_{R_3} = V_{R_1} \otimes (V_{R_2} \otimes V_{R_3})$, hence there is a unitary transformation

$$U : (R_1 \otimes R_2) \otimes R_3 \rightarrow R_1 \otimes (R_2 \otimes R_3).$$

(1)

On the other hand, we can rewrite it in irreducible components, where $M^{R_1,R_2}_X$ is a multiplicity space of all $X$’s in the decomposition $R_1 \otimes R_2$:

$$\begin{align*}
(R_1 \otimes R_2) \otimes R_3 &= \bigoplus_i M^{R_1,R_2}_i \otimes X_i \otimes R_3 = \bigoplus_{i,k} M^{R_1,R_2}_i \otimes M^{X_i,R_3}_k \otimes M^{R_4}_k, \\
R_1 \otimes (R_2 \otimes R_3) &= R_1 \otimes \bigoplus_j M^{R_2,R_3}_j \otimes Y_j = \bigoplus_{j,k} M^{R_1,Y_j}_k \otimes M^{R_2,R_3}_j \otimes M^{R_4}_k. 
\end{align*}

(2)

If we consider some particular $R_4$ in the decomposition, it corresponds to the vector space of representations. A basis constructed from the highest weights’ vectors differs for these two fusions.
Thus, there is a transformation between two vector spaces that is defined by the Racah matrix or Racah-Wigner 6-j symbols.

**Definition 1.** Racah coefficients are elements of Racah matrix that is the map:

$$U \left( \begin{array}{cc} R_1 & R_2 \\ R_3 & R_4 \end{array} \right) : \bigoplus_i M_{R_i}^{X_i} \otimes M_{R_4}^{Y_i} \rightarrow \bigoplus_j M_{R_4}^{X_j} \otimes M_{R_3}^{Y_j}. \quad (3)$$

**Definition 2.** Wigner 6-j symbol is the element of a normalized Racah matrix:

$$\left\{ \begin{array}{ccc} R_1 & R_2 & X_i \\ R_3 & R_4 & Y_j \end{array} \right\} = \frac{1}{\sqrt{\text{dim}_q(X_i) \text{dim}_q(Y_j)}} U_{i,j} \left( \begin{array}{cc} R_1 & R_2 \\ R_3 & R_4 \end{array} \right). \quad (4)$$

Here $\text{dim}_q$ means the quantum deformation of the usual expression for the dimension of the representation [25]. It can be computed for every $U_q(sl_N)$ representation $R$ using the corresponding Young diagram $\lambda$ ($\lambda^T$ is a transposed Young diagram):

$$\text{dim}_q(\lambda) = \prod_{(i,j) \in \lambda} q^{\frac{1}{2}(N+1-i-j)} - q^{-\frac{1}{2}(N+1-i-j)}. \quad (5)$$

In this paper we work with the special class of 6-j symbols, which can be seen as a natural generalization of $U_q(sl_2)$ case for $U_q(sl_N)$ 6-j symbols. The initial representations and the resulting one are either symmetric or conjugated to symmetric for this class. Further we will assume that $R_1, R_2, R_3, R_4$ representations are symmetric. Corresponding Young diagrams are $[r_1], [r_2], [r_3], [r_4]$, here $r_i$ are integers that denote numbers of boxes for $U_q(sl_N)$ symmetric representations. Conjugated Young diagram is written as $\overline{r_n}$ and correspond to $\overline{R_n}$.

**Definition 3.** We shall call two 6-j symbols below type I and type II, $\boxdot$ means $N-1$ vertical boxes.

\begin{align*}
\text{I type:} & \quad \left\{ \begin{array}{c} [r_1] \\ [r_3] \end{array} \right\} \equiv \{ \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \}, \\
\text{II type:} & \quad \left\{ \begin{array}{c} [r_1] \\ [r_3] \end{array} \right\} \equiv \{ \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \boxdot \ldots \boxdot \}. \quad (6, 7)
\end{align*}

Although arguments $R_1, R_2, R_3, R_4$ are very simple and can be parametrized by the width and $N$, the last pair of $X$ and $Y$ Young diagrams has more sophisticated expressions. There are two possible cases of tensor products: $[r_n] \otimes [r_m]$ and $[r_n] \otimes \overline{r_m}$. Each element in the decomposition depends on the initial pair of representations and the ordering number in the sum. From the Littlewood-Richardson rules [26] it is easy to see that the mentioned tensor products are multiplicity-free and all representations in a decomposition have different width. Similarly to $U_q(sl_2)$ case, where it is possible to enumerate diagrams by the only integer parameter $i$, for mentioned $U_q(sl_N)$ decompositions we have the enumerating parameter – the first row length. To shorten the notation we shall write 6-j symbol of type I and type II in a more compact form. Let us denote the type by variable $T \in \{1, 2\}$. Type I 6-j symbol is:

$$\left[ \begin{array}{ccc} r_1 & r_2 & i \\ r_3 & r_4 & j \end{array} \right]_1 \equiv \left\{ \begin{array}{c} [r_1] \\ [r_3] \end{array} \right\} \left\{ \begin{array}{c} i, \frac{r_2 - r_1 + i (N-2)}{2} \\ j, \frac{r_2 - r_3 + j (N-2)}{2} \end{array} \right\}. \quad (8)$$
and type II:

\[
\begin{bmatrix}
  r_1 & r_2 & i \\
  r_3 & r_4 & j
\end{bmatrix}_2 := \begin{bmatrix}
  [r_1] & [r_2] & \left[\frac{r_1 + r_2 + i}{2}, \frac{r_1 + r_2 - i}{2}\right] \\
  [r_3] & [r_4] & \left[\frac{r_2 - r_3 + j}{2}\right]
\end{bmatrix},
\]

where \(i, j\) is defined in such a way in order to have a nice \(N = 2\) limit.

Let us note that the fusion rules restrictions require additional equalities:

\[
\begin{align*}
  r_1 + r_3 &= r_2 + r_4 & \text{for type I}, \\
  r_1 + r_2 &= r_3 + r_4 & \text{for type II}.
\end{align*}
\]

**Definition 4.** The equations (11) between \(U_q(sl_2)\) 6-

j symbols are called Regge symmetries or Regge
transformations [27] (\(\rho = \frac{1 + r_1 + r_2 + r_3 + r_4}{2}, \rho' = \frac{1 + r_1 + r_2 + r_3 + r_4}{2}, \rho'' = \frac{r_1 + r_2 + r_3 + r_4 + r_5 + r_6}{2}\)):

\[
\begin{align*}
  \{r_1 & \ r_2 \ i\} = \{\rho - r_3 & \ \rho - r_4 \ i\} = \{\rho' - r_3 & \ \rho' - r_4 \ \rho' - j\} \\
  \{r_3 & \ r_4 \ j\} = \{\rho - r_1 & \ \rho - r_2 \ j\} = \{\rho'' - r_3 & \ \rho'' - r_4 \ \rho'' - j\} \\
  = \{\rho - r_3 & \ \rho' - r_4 \ \rho'' - j\} \\
  \{\rho - r_1 & \ \rho' - r_2 \ \rho'' - j\}.
\end{align*}
\]

**Definition 5.** The tetrahedral symmetry is a known property of 6-j symbol to be invariant under
transformations [6] (\(\lambda_1, \mu, \nu\) are arbitrary Young diagrams):

\[
\begin{align*}
  \begin{bmatrix}
    \lambda_1 & \lambda_2 & \lambda_3 \\
    \lambda_4 & \lambda_5 & \lambda_6
\end{bmatrix} = \begin{bmatrix}
    \lambda_3 & \lambda_2 & \lambda_1 \\
    \lambda_4 & \lambda_5 & \lambda_6
\end{bmatrix} = \begin{bmatrix}
    \lambda_3 & \lambda_2 & \lambda_1 \\
    \lambda_4 & \lambda_5 & \lambda_6
\end{bmatrix} = \begin{bmatrix}
    \lambda_1 & \lambda_2 & \lambda_3 \\
    \lambda_4 & \lambda_5 & \lambda_6
\end{bmatrix}.
\end{align*}
\]

**Statement 1.** 6-j symbol in \(U_q(sl_N), N > 2\) with symmetric and conjugate to symmetric representations
is either trivial (\(X\) and \(Y\) has the only possible value) or may be equated by tetrahedral symmetry
to either type I or type II.

**Proof.** There are only a few possible variants to write down a 6-j symbol with symmetric and conjugate
to symmetric representations. 6-j symbols are invariant under conjugation of all arguments as it can
be seen from (12), so we can consider only symmetric \(R_4\). Let us now investigate how the first
three arguments may be organized. There are four different cases that correspond to the number of
conjugated representations in the product.

- All three representations are conjugated.

Let us conjugate all terms in the product \([r_1] \otimes [r_2] \otimes [r_3] \supseteq [r_4]\), so we can consider \([r_1] \otimes [r_2] \otimes [r_3] \supseteq [r_4]\) and \(N > 2\). It is obvious from the fusion rules [26] that for \(N > 4\) it is not possible to combine the representations into a conjugated one because there are no more than 3 rows in a resulting
Young diagram, whereas \([r_4]\) has \(N - 1 > 3\) rows.

Now we need to prove that it is not possible even for \(N = 3, 4\). The \(N = 4\) case requires the rows of \(R_4\) to be equal. The Littlewood-Richardson rules [26] say that the resulting diagram is constructed as the first multiplier with the second multiplier’s elements but with some restrictions. For symmetric diagrams they forbid to put the new elements in one column. Hence, if we need to combine diagrams into a rectangular one, the corresponding 6-j symbol is trivial. Indeed, the
only way to combine the diagrams properly is to consider them equal and to put them under
each other.

Here and below we use some non-negative integer parameters \(a, b, c\) that encode a Young diagram,
the aim of these parameters is to specify the shape of a considered diagram.

The \(N = 3\) case has a \([r_4]\) diagram that may be written as \([a, a]\). The \([a, a]\) is trivial, because
there is the only diagram \(X = [r_1 + r_2 - b, b]\) that has width \(a\). Indeed, if the width is smaller,
the third multiplier can not make the second row width equal to \(a\), if it is greater, we can not
make \(R_4\) anymore.

Therefore, all \(N > 2\) 6-j symbols with 3 conjugated representations are trivial.
• All three representations are symmetric.

Obviously, if $R_1, R_2, R_3, R_4$ are symmetric in $U_q(sl_N)$, $N > 3$, then the corresponding 6-j symbol has the only $X = [r_1 + r_2]$, the same for $Y$. If $N = 3$, there is a possibility to make a Young diagram with columns of height $N$. However, the fusion rules restrict $X = [r_1 + r_2 - a, a] = [b + r_4, b]$, hence $X = [r_1 + r_2 + r_4, r_1 + r_2 - r_4]$ and this 6-j symbol is trivial.

• Two representations are conjugated and one is symmetric.

Note, that the multiplicity of $R_4$ in decomposition $R_1 \otimes R_2 \otimes R_3$ does not change under a permutation of multipliers. Hence we may always decompose the product of conjugated representations and then multiply it by the symmetric one. Without loss of generality we consider $[r_1] \otimes [r_2] \otimes [r_3]$. Let us firstly decompose the product of conjugated representations. In general, it has the diagram $[a^{N-2}, b]$, where $b \leq a$. It is obtained from $[(r_1 + r_2)^{N-2}, r_1 + r_2 - c, c]$ by reducing the column of height $N$. If $N > 3$, the product $[a^{N-2}, b] \otimes [r_3]$ may have a symmetric diagram in the decomposition only if $a = b$, but it will be trivial because $X = [(r_3 - r_4)^{N-1}]$. If $N = 3$, $[a, b] \otimes [r_3]$ easily makes symmetric diagram with condition $X = [a, a + r_3 - r_4]$. But we can find $a$ from the $[r_1] \otimes [r_2]$ decomposition and it is unique for fixed $r_1$ and $r_2$.

As a result, there are no non-trivial 6-j symbols with two conjugated symmetric representations and symmetric $R_4$.

• One conjugated representation.

There are three such 6-j symbols:

$$\begin{pmatrix} \{r_1\} & \{r_2\} & X \\ \{r_3\} & \{r_4\} & Y \end{pmatrix}, \begin{pmatrix} \{r_1\} & \{r_2\} & X \\ \{r_3\} & \{r_4\} & Y \end{pmatrix}.$$  

(13)

One can check that they may be nontrivial.

We can apply a tetrahedral symmetry to these 6-j symbols, in particular, row permutation of arguments $(R_1, R_2) \leftrightarrow (R_3, R_4)$. After this transformation the first and the third 6-j symbols are swapped and the second one is invariant. Applying other symmetries, one can check that type I and type II are not equal by tetrahedral symmetries. □

It is worth mentioning that there are tetrahedral symmetries acting within each type. In particular, a type I 6-j symbol is still type I after row permutations and the swap of the first two columns. Type II is conserved only by the row permutation of the first two columns. These are the only tetrahedral symmetries that possible to derive if one consider symmetries of type I or type II. The others either were used earlier to transform 6-j symbol into one of the types, or transform any type into a completely different 6-j symbol, which has non-symmetric representations and much more complicated structure, so they are out of the scope of the present paper.

The expression for 6-j symbol of type I and II was proposed in [1]. It may be written as follows.

$$[r_1 \ r_2 \ \ i \ j \ T] = \theta_N(r_1, r_2, i) \theta_N(r_3, r_4, i) \theta_N(r_1, r_4, j) \theta_N(r_2, r_3, j) [N-1]_q! [N-2]_q! \sum_{z = z_{min}}^{z_{max}} (-1)^z[z + N - 1]_q! \ A_{T, z} \theta_N(a, b, c)$$

$$= \sqrt{[\frac{[a+b-c+1]}{[a+b+c]}] \ \ [\frac{[a+c-b]}{[a+b+c]}] \ \ [\frac{[b+c-a]}{[a+b+c]}] [z - \frac{[a+b+c]}{2} \frac{[a+c-b]}{2} \frac{[b+c-a]}{2}]_q!} \ A_{T, z} = \begin{pmatrix} \frac{[k + z_{min} + N - 2 - z]}{[k + z_{min} + N - 1]}_q! \ [k + z_{min} + N - 2 - z] \theta_N(a, b, c) \ A_{T, z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{[k + z_{min} + N - 1]}{[k + z_{min} + N - 1]}_q! \ [k + z_{min} + N - 2 - z] \theta_N(a, b, c) \ A_{T, z} \end{pmatrix}$$

(14)

(15)

To write the 6-j symbol expression we use quantum numbers notations. It is by the definition $[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. Quantum generalization of factorials for non-negative integers can be written as $[n]_q! = \frac{[n]_q \ [n - 1]_q \ ... \ [1]_q}{\ [1]_q \ [2]_q \ ... \ [n]_q}$.
\[
\prod_{k=1}^{n}[k]_q. \text{ Also } k = \frac{1}{2} \min(i - r_1 + r_2, j - r_3 + r_2) \text{ and } \zeta_{\min}, \zeta_{\max} \text{ are defined as the smallest and the largest integers for which the summand is non-trivial}, \text{i.e. there are no factorials of negative integers.}
\]

The expression differs for two types only in the \(A_{r,z}\) expression. Also the following conditions were imposed in the original paper [1] (as we show below, they are not necessary):

\[
\begin{cases}
0 \leq r_2 \leq r_1 \leq r_3 & \text{for type I}, \\
0 \leq r_1 \leq r_2 & \text{for type II}.
\end{cases}
\] (16)

### 3 Hypergeometric expression for 6-j symbols

In this section we express the 6-j symbol expression in terms of basic q-hypergeometric series \(4\Phi_3\). Firstly, we define the q-hypergeometric functions and remind their symmetric properties. After this we use the inequality properties (16) to simplify the 6-j symbol expression. We prove with the help of tetrahedral symmetries that the 6-j symbol’s domain may be extended beyond the mentioned inequalities. Then we write the obtained series as a \(4\Phi_3\) function. As a result, both types can be written as q-hypergeometric \(4\Phi_3\) series multiplied by some factor.

#### 3.1 q-Hypergeometric symmetries

A q-Pochhammer symbol is defined as

\[
(a, q)_n = \prod_{k=0}^{n-1}(1 - aq^k).
\]

**Definition 6.** The q-hypergeometric series are defined as:

\[
p+1\Phi_p\left(\frac{a_1, \ldots, a_{p+1}}{b_1, \ldots, b_p} ; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1, q)_n \cdots (a_{p+1}, q)_n}{(b_1, q)_n \cdots (b_p, q)_n} z^n.
\] (17)

It can be also rewritten in a form, which is more convenient for us:

\[
p+1\Phi_p\left(\frac{a_1, \ldots, a_p, a_{p+1}}{b_1, \ldots, b_p} ; q, z\right) := p+1\Phi_p\left(\frac{q^{a_1}, \ldots, q^{a_p}, q^{a_{p+1}}}{q^{b_1}, \ldots, q^{b_p}} ; q, z\right).
\] (18)

It is far more convenient because it may be reformulated in terms of q-factorials:

\[
p+1\Phi_p\left(\frac{a_1 + 1, \ldots, a_p + 1, a_{p+1} + 1}{b_1 + 1, \ldots, b_p + 1} ; q, z\right) = \sum_{n=0}^{\infty} \frac{[a_1 + n]_q! \cdots [a_{p+1} + n]_q!}{[a_1]_q! \cdots [a_{p+1}]_q!} \frac{[b_1]_q! \cdots [b_p]_q!}{[b_1 + n]_q! \cdots [b_p + n]_q!} z^n.
\] (19)

And it has the obvious limit \(\lim_{q \to 1}[a]_q! = a!\), where the whole series becomes a usual hypergeometric function.

There are a lot of known symmetries for \(4\Phi_3\) series. Here we consider only permutation symmetry and Sears’ transformation.

**Definition 7.** Permutation symmetry is the obvious property of \(r\Phi_p\) functions to be invariant under permutations \(\omega \in S_r\) and \(u \in S_p\):

\[
r\Phi_p\left(\frac{a_1, \ldots, a_r}{b_1, \ldots, b_p} ; q, z\right) = r\Phi_p\left(\frac{a_{\omega(1)}, \ldots, a_{\omega(r)}}{b_{u(1)}, \ldots, b_{u(p)}} ; q, z\right).
\] (20)

**Definition 8.** Sears’ transformation [24] is the relation between two \(4\Phi_3\) functions:

\[
4\Phi_3\left(\frac{x, y, z, n}{u, v, w} ; q, q\right) = \frac{[v-z-n+1]_q! [u-z-n-1]_q! [v-1]_q! [u-1]_q!}{[v-z-1]_q! [v-n-1]_q! [u-z]_q! [u-n-1]_q!} \cdot \frac{4\Phi_3\left(\frac{w-x, w-y, z, n}{1-u+z+n, 1-v+z+n, w} ; q, q\right)}{4\Phi_3\left(\frac{x, y, z, n}{u, v, w} ; q, q\right)},
\] (21)

where \(x + y + z + n = u + v + w\).
3.2 6-j symbol as $\Phi_4$ series

Let us denote the sum (14) as
\[
\left[ \begin{array}{ccc} r_1 & r_2 & i \\ r_3 & r_4 & j \end{array} \right] \rightarrow T = K' \cdot \sum_m I_m = K' \cdot I,
\]
where $m = \frac{1}{2}(r_1 + r_2 + r_3 + r_4) - z$.

Then it can be easily rewritten as:
\[
I = \sum_{m=m_{\text{min}}}^{m_{\text{max}}} \frac{(-1)^{i+j+r_1+r_2+r_3+r_4-m-N-1}q! \cdot A_{T,m}}{[m]_q ![\frac{r_1+r_2+r_4-i}{2}]_q ![\frac{r_3+r_4-j-i}{2}]_q ![\frac{r_1+r_4-j}{2}]_q ![\frac{r_3+r_4-j}{2}]_q ![\frac{r_1+r_3-i-j}{2}]_q ![\frac{r_2+r_4-i-j}{2}]_q ![\frac{r_2+r_3-j}{2}]_q}.
\]

(22)

\[
K' = \theta_N (r_1, r_2, i) \theta_N (r_3, r_4, i) \theta_N (r_1, r_4, j) \theta_N (r_2, r_3, j) [N-1]_q ![N-2]_q !,
\]

(23)

\[
A_{T,m} = \begin{cases} 
[k - m_{\text{max}} + m]_q ! & \text{for type I}, \\
[k + m_{\text{min}} - m]_q ! & \text{for type II}.
\end{cases}
\]

(24)

The explicit relations for $m_{\text{min}}$ and $m_{\text{max}}$ can be easily found from the denominator factorials, because the summand is zero if and only if there is a negative factorial in the denominator:
\[
m_{\text{max}} = \frac{1}{2} \min \left( \begin{array}{c} r_1 + r_2 - i \\ r_3 + r_4 - i \\ r_1 + r_4 - j \\ r_2 + r_3 - j \end{array} \right), \quad m_{\text{min}} = \frac{1}{2} \max \left( \begin{array}{c} r_1 + r_3 - i - j \\ r_2 + r_4 - i - j \end{array} \right).
\]

(25)

As it can be derived from fusion rules, $k, m_{\text{max}}, m_{\text{min}}$ are always integers when a 6-j symbol exists. Moreover, $k$ has a clear meaning in terms of Young diagrams – it is the minimum width among the conjugated parts of diagrams, corresponding to $X_k$ and $Y_k$.

One can notice, that the considered expression fits the $\Phi_4$ definition (18), if $z = q$. This allows us to claim the following.

Claim 1. Both type I and type II may be written as $\Phi_4$ $q$-hypergeometric series multiplied by simple factors:
\[
\left[ \begin{array}{ccc} r_1 & r_2 & i \\ r_3 & r_4 & j \end{array} \right] \rightarrow T = K'' \cdot \Phi_4 \left( a_1, a_2, a_3, a_4, a_5 ; b_1, b_2, b_3, b_4 ; q, q \right),
\]

(26)

\[
2a_i = \begin{pmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{pmatrix} T = K'' \cdot \Phi_4 \left( a_1, a_2, a_3, a_4, a_5 ; b_1, b_2, b_3, b_4 ; q, q \right),
\]

(27)

\[
K'' = \frac{K' \cdot A_{T,0} \cdot [r_1+r_2+r_4-N+1]_q !}{[r_1+r_2+r_4-N+1]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q ![\frac{r_1+r_2+r_4-N+1}{2}]_q}.
\]

(28)

where $\{e_1, e_2\} T \equiv e_T$ is $e_1$ for type I and $e_2$ for type II.

It can be proven straightforwardly by substitution of q-Pochhammer symbols.

3.3 Expression of 6-j symbol as $\Phi_3$ series

The obtained expression for 6-j symbol is not quite convenient to find its symmetries. Expressions for $k, m_{\text{min}}$ and $m_{\text{max}}$ may be simplified in the following way.

Lemma 1. For all type I 6-j symbols $k = m_{\text{max}} = \frac{i+j-r_1-r_4}{2}$ if the following conditions are satisfied:
\[
\begin{cases} 
2r_2 \leq r_1 \leq r_3, \\
r_1 + r_3 = r_2 + r_4.
\end{cases}
\]

(29)
Proof. Let us consider \( k - m_{\text{max}} = \frac{i + j - r_1 - r_2}{2} \). One can check that there are 2 cases when it is so, hence they may be written as the union of two systems:

\[
\begin{align*}
    r_1 + r_2 - i &\leq r_3 + r_4 - i, & r_2 + r_3 - j &\leq r_3 + r_4 - i, \\
    r_1 + r_2 - i &\leq r_2 + r_3 - j, & r_2 + r_3 - j &\leq r_1 + r_2 - i, \\
    r_1 + r_2 - i &\leq r_1 + r_4 - j, & r_2 + r_3 - j &\leq r_1 + r_4 - j, \\
    j - r_3 &\leq i - r_1; & i - r_1 &\leq j - r_3.
\end{align*}
\]

(30)

If the conditions (29) satisfied, the first three inequalities are true. The union of these two systems may be reduced to the next expression.

\[
\begin{align*}
    j - i &\leq r_4 - r_2, \\
    j - i &\geq r_4 - r_2.
\end{align*}
\]

(31)

Consequently, every 6-j symbol from type I is described by \( k - m_{\text{max}} = \frac{i + j - r_1 - r_2}{2} \).

\[\square\]

Lemma 2. For all type II 6-j symbols \( k + m_{\text{min}} = \frac{r_1 + r_2 - i}{2} \) if the conditions are satisfied:

\[
\begin{align*}
    r_1 &\leq r_2, \\
    r_1 + r_2 &\leq r_3 + r_4.
\end{align*}
\]

(32)

Proof. The proof for type II is analogous to type I.

\[\square\]

Lemma 3. Conditions on arguments of a 6-j symbol (16) are redundant, i.e the expression (14) is valid even if the inequalities are not satisfied.

Proof. We are able to obtain every possible 6-j symbol of types I and II by performing a tetrahedral symmetry (12) that leaves the type invariant:

\[
\left\{ \begin{array}{c}
    r_1 \\
    r_3 \\
\end{array} \right\} \begin{array}{c}
    \overline{r_2} \\
    \overline{r_4} \\
\end{array} \begin{array}{c}
    X \\
    Y \\
\end{array} = \left\{ \begin{array}{c}
    \overline{r_3} \\
    \overline{r_1} \\
\end{array} \right\} \begin{array}{c}
    \overline{r_2} \\
    \overline{r_4} \\
\end{array} \begin{array}{c}
    \overline{X} \\
    \overline{Y} \\
\end{array} = \left\{ \begin{array}{c}
    r_2 \\
    r_4 \\
\end{array} \right\} \begin{array}{c}
    \overline{r_1} \\
    \overline{r_3} \\
\end{array} \begin{array}{c}
    X \\
    Y \\
\end{array}.
\]

(33)

One may immediately notice that these symmetries may transform a 6-j symbol from region \( r_2 \leq r_1 \leq r_3 \) into all possible representations. The problem is that the expression for the transformed 6-j symbols may differ from the initial expression. We can check it by substituting arguments transformed by tetrahedral symmetries. Let us show that in our notations it acts on \( r_1, r_2, r_3, r_4, i, j \) as a permutation.

For \( R_n \), the symmetry obviously acts as a permutation of \( r_n \). There are also representations \( X \) and \( Y \) that is conjugated, we can consider only diagram \( \left[ j, \frac{r_1 + r_2 + j - N^2}{2} \right] \) as an example. Under conjugation it transforms \( \left[ j, \frac{r_1 + r_2 + j - N^2}{2} \right] \) into \( \left[ j, \frac{r_3 + r_2 + j - N^2}{2} \right] \), but the expression depends only on \( j \) that is invariant under conjugation.

Therefore, tetrahedral symmetry acts on the expression as a permutation of arguments. One can check that it is invariant under written tetrahedral symmetry transformation. The same for type II, but we need only one relation (the inequality is \( r_1 \leq r_2 \)):

\[
\left\{ \begin{array}{c}
    r_1 \\
    r_3 \\
\end{array} \right\} \begin{array}{c}
    \overline{r_2} \\
    \overline{r_4} \\
\end{array} \begin{array}{c}
    X \\
    Y \\
\end{array} = \left\{ \begin{array}{c}
    \overline{r_3} \\
    \overline{r_1} \\
\end{array} \right\} \begin{array}{c}
    \overline{r_2} \\
    \overline{r_4} \\
\end{array} \begin{array}{c}
    X \\
    Y \\
\end{array}.
\]

(34)

The symmetry acts non-trivially only on \( r_1, r_2, r_3, r_4 \), we already showed why it is a permutation. It is easy to see that the expression is invariant under such a transformation.

Therefore, the expression does not change when we write a 6-j symbol without additional inequality conditions (16). Then we can get rid of these conditions as even if they are not satisfied the expression is valid.

\[\square\]
We have proven in Lemma 1 that for arguments satisfying the inequality condition (36) there are only one combination of $k - m_{max}$ that is present for type I 6-j symbols. This results into the exact value of $A_{T,m}$ which allow us to reduce the whole series. Then we apply tetrahedral symmetries to prove that the statement is true for all type I 6-j symbols. The same procedure has been done for type II and this allows us to simplify both expressions and write down them as follows.

$$
\begin{align*}
\left[ \begin{array}{ccc}
  r_1 & r_2 & i \\
  r_3 & r_4 & j \\
\end{array} \right]_T &= K' \sum_{m=m_{min}}^{m_{max}} \left( \frac{1}{(1-q)^{m}} \right) \frac{(-1)^{r_1+r_2+r_3+r_4-m}[r_1+r_2+r_3+r_4+N-1-m]_q!}{m_q!} \times \\
&\times \frac{1}{(1-q)^{m}} (N-2)\delta_{T,2} - m_q! \frac{1}{(1-q)^{m}} (N-2)\delta_{T,1} + m_q!
\end{align*}
$$

We can express all factorials as q-Pochhammer symbols. The substitution differs for factorials with $+m$ and $-m$:

$$
[m_0 + m]_q! = [m_0]_q \frac{q^{m+1}}{(1-q)^m} (q, q^{m+1})_m, \quad [m_0 - m]_q! = \frac{(-1)^m [m_0]_q!}{(1-q)^{-m} q^{\frac{m^2}{2}} (q^{-m_0}, q)_m}.
$$

By substituting this to the main expression one can check that among depending on $m$ terms only q-Pochhammer symbol remain. This allows us to write the series as a hypergeometric function:

$$
\begin{align*}
\left[ \begin{array}{ccc}
  r_1 & r_2 & i \\
  r_3 & r_4 & j \\
\end{array} \right]_T &\sim 4\Phi_3 \left( \begin{array}{ccc}
a_1, a_2, a_3, a_4 \\
b_1, b_2, b_3 \\
q, q \\
\end{array} \right).
\end{align*}
$$

The $4\Phi_3$ arguments may be easily obtained using (36). Note, that there is the following relation on the arguments:

$$
a_1 + a_2 + a_3 + a_4 + 1 = b_1 + b_2 + b_3.
$$

And the factorizable part of the expression:

$$
K_T = \frac{\theta_N (r_1, r_2, i) \theta_N (r_3, r_4, i) \theta_N (r_1, r_4, j) \theta_N (r_2, r_3, j) [N-1]_q! N_2 [r_1+r_2+r_3+r_4+N-1]_q!}{[r_1+r_2+r_3+r_4]_q! [r_1+r_2]_q! [r_2+r_3]_q! [r_1+r_3]_q! [r_1+r_4]_q! [r_2+r_4]_q! [r_3+r_4]_q!}
$$

Combing all this into one, we come to the following statement.

**Statement 2.** The considered 6-j symbol expression may me expressed as a $4\Phi_3$ function for both types. The factor $K_T$ is as in (39).

$$
2a_i = \begin{pmatrix}
  -r_1 - r_2 + i - 2(N-2)\delta_{T,2} \\
  -r_3 - r_4 + i \\
  -r_1 - r_4 + j \\
  -r_2 - r_3 + j
\end{pmatrix}, \quad 2b_i = \begin{pmatrix}
  -r_1 - r_2 - r_3 - r_4 - 2(N-1) \\
  i + j - r_2 - r_4 + 2 \\
  i + j - r_1 - r_3 + 2 + 2(N-2)\delta_{T,1}
\end{pmatrix}
$$

This is the most suitable form of 6-j symbol for our aims. As it can be seen, we reduced the $5\Phi_4$ series to the $4\Phi_3$ one. This is a non-obvious result. In order to proceed with this reduction we used tetrahedral symmetry along with the special properties of the considered two types. Due to the fact that $U_q(sl_2)$ 6-j symbols are expressed via $4\Phi_3$ too, we may use the same techniques to obtain new results, also the limit $N = 2$ is very easy to apply. This result gives us an idea of a strong
connection between 6-j symbols and q-hypergeometric series. For example, it is interesting whether all multiplicity-free 6-j symbols can be expressed as \(4\Phi_3\) series.

It is interesting to analyze the number of independent parameters in the obtained expression. Neglecting \(q\), on both sides we have 7 parameters: \(\{r_1, r_2, r_3, r_4, i, j, N\}\) and \(\{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}\). They are not independent, it was mentioned that, on the one hand, each type has restrictions for \(N > 2\) that fix one parameter. On the other hand, obtained \(4\Phi_3\) series satisfies a balance condition \(\sum r_i = \sum b_i\). Thus, for \(N > 2\) there are 6 parameters on both sides. For \(N = 2\), the fusion rules do not fix \(r_n\), so there are 6 parameters on both sides. It is natural to ask whether there is a connection between the fusion rules and the balance condition. It seems like these equalities have different meaning, because the condition on \(\{a_i, b_i\}\) is satisfied even if \(r_1 + r_3 \neq r_2 + r_4\). From this point of view another question arises: what class of 6-j symbols can be described in terms of \(4\Phi_3\) series with such equality? This question is out of our consideration in this paper, but it is still important and interesting to study.

## 4 Relation with \(U_q(sl_2)\) 6-j symbols

In this section we investigate the relation between 6-j symbols in multiplicity-free \(U_q(sl_N)\) and \(U_q(sl_2)\) cases. As we have seen, the core of both expressions are \(4\Phi_3\) hypergeometric series. We have already mentioned the number of independent parameters in the series, but now we analyze it in details. Then we shall see the interesting connection between the usual \(U_q(sl_2)\) 6-j symbol and considered one.

Let us write down the \(4\Phi_3\) arguments as a vector space with the basis \((r_1, r_2, r_3, r_4, i, j, N)\). We put all the additional constants in \(\hat{C}\) since they do not play any role in the next discussion:

\[
\begin{pmatrix}
    r_1 + r_2 + r_3 + r_4 + 2(N - 1) \\
    r_1 + r_2 - i + 2(N - 2)\delta r_{2,2} \\
    r_3 + r_4 - i \\
    r_1 + r_4 - j \\
    r_2 + r_3 - j \\
    -r_2 - r_4 + i + j + 2 \\
    i + j - r_1 - r_3 + 2(N - 1)\delta r_{1,1}
\end{pmatrix}
\begin{pmatrix}
    1 & 1 & 1 & 0 & 0 & 1 \\
    1 & 1 & 0 & 0 & -1 & 0 \\
    0 & 0 & 1 & 1 & -1 & 0 \\
    1 & 0 & 0 & 1 & 0 & -1 \\
    0 & 1 & 1 & 0 & 0 & -1 \\
    0 & 1 & 0 & 1 & -1 & -1 \\
    -1 & 0 & -1 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
    r_1 \\
    r_2 \\
    r_3 \\
    r_4 \\
    j \\
    N
\end{pmatrix} + \hat{C}.
\]  

(42)

The rank of this matrix is 6, so there is a kernel of dimension one. This kernel is described by a zero vector \(\vec{v}\). Note that (38) is a completely different condition that does not depend on the values of parameters. The zero vector can be written as follows

\[
\vec{v} = \begin{cases}
    \{0, 1, 0, 1, 1, 1, -1\}, & \text{Type I,} \\
    \{1, 1, 0, 0, 0, 1, -1\}, & \text{Type II,}
\end{cases}
\]  

(43)

with the corresponding shift in the parameters being

\[
\alpha \vec{v} = \begin{cases}
    \alpha \vec{v} = \alpha(r_2 + r_4 + i + j - N), & \text{Type I,} \\
    \alpha \vec{v} = \alpha(r_1 + r_2 + j - N), & \text{Type II.}
\end{cases}
\]  

(44)

This freedom allows to shift the arguments value without changing the actual value of the hypergeometric series, so it can be considered as a symmetry for 6-j symbol although for hypergeometric series it is tautological equality. If one examines the transformation for type I 6-j symbol, it can be seen that the fusion rules are in conflict with it. Indeed, the non-trivial transformation changes \(r_2 + r_4\), but leaves \(r_1 + r_3\) unchanged, thus (10) forbids such transformation for \(N > 2\), for either type I or type II. However for \(N = 2\) the fusion rules disappear and we can apply it without any problems. So we take \(U_q(sl_N)\) 6-j symbol and make transformation (43) in order to get the expression for \(U_q(sl_2)\) 6-j symbol:

\[
\begin{align*}
4\Phi_3(r_1, r_2, r_3, r_4, i, j, N)_1 &= (-1)^N \cdot 4\Phi_3(r_1, r_2 + N - 2, r_3, r_4 + N - 2, i + N - 2, j + N - 2, 2), \\
4\Phi_3(r_1, r_2, r_3, r_4, i, j, N)_2 &= (-1)^N \cdot 4\Phi_3(r_1 + N - 2, r_2 + N - 2, r_3, r_4, i, j + N - 2, 2).
\end{align*}
\]  

(45)  

(46)
The 6-j symbol asymptotics formula for representations \( \{r_1, r_2, r_3, r_4, i, j, N\} \). It partly replicates the hypergeometric arguments, so only a few terms are left in the relation between of multiplicity free \( U_q(sl_N) \) 6-j symbols and \( U_q(sl_2) \) ones. For the sake of brevity, we will write the hypergeometric function from (40) as \( \Phi_3(r_1, r_2, r_3, r_4, i, j, N) \). The factor \( K' \) changes after transformations, let us write it down explicitly.

\[
K'(N) = \theta_N(r_1, r_2, i) \theta_N(r_3, r_4, i) \theta_N(r_1, r_4, j) \theta_N(r_2, r_3, j) [N - 1]_q! [N - 2]_q! \cdot \Theta_T(N) := \frac{1}{[N-1]_q! [N-2]_q! K'(2)}. \tag{47}
\]

\[
\Theta_1(N) = \left( \prod_{m=1}^{N-2} \left[ \frac{i-r_1+r_2}{2} + m \right]_q \frac{j+r_2-r_3}{2} + m \right] \left[ \frac{j+r_1+r_4}{2} + m \right] \left[ \frac{i-r_3+r_4}{2} + m \right]_q \right]^{-\frac{1}{2}}, \tag{49}
\]

\[
\Theta_2(N) = \left( \prod_{m=1}^{N-2} \left[ \frac{r_1+r_2-i}{2} + m \right]_q \frac{j+r_2-r_3}{2} + m \right] \left[ \frac{j+r_1-r_4}{2} + m \right] \left[ \frac{i+r_3+r_4+1+m}{2} \right]_q \right]^{-\frac{1}{2}}. \tag{50}
\]

The resulting relation between multiplicity-free \( U_q(sl_N) \) and \( U_q(sl_2) \) 6-j symbol is as follows.

\[
\begin{align*}
\begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_1 &= \left\{ \begin{array}{ll} r_1 & r_2 + N - 2 \\ r_3 & r_4 + N - 2 \end{array} \right\} \left( -1 \right)^N [N - 1]_q! [N - 2]_q! \cdot \Theta_1(N), \\
\begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_2 &= \left\{ \begin{array}{ll} r_1 + N - 2 & r_2 + N - 2 \\ r_3 & r_4 \end{array} \right\} \left( -1 \right)^N [N - 1]_q! [N - 2]_q! \cdot \Theta_2(N).
\end{align*}
\tag{51}
\]

It can be easily checked that the remaining fusion rules for \( N = 2 \) (triangle inequality, etc.) are always satisfied and the resulting 6-j symbol is non-trivial. On the other hand, if one tries to transform \( U_q(sl_2) \) 6-j symbol into \( N > 2 \) one, the number of problems arises and it is not possible in general. For example, if \( r_1 + r_3 - r_2 - r_4 > 0 \), there is no corresponding \( N > 2 \) 6-j symbol.

This result is interesting not only because it reveals the hidden relation between two classes of 6-j symbols, but additionally it can be applied to extend a lot of known properties of \( U_q(sl_2) \) to arbitrary \( N \). In the next section we derive the asymptotics formula for the multiplicity-free case. Let us show an example of such a generalization.

## 5 Asymptotics of 6-j symbol

The 6-j symbol asymptotics formula for \( N = 2, q = 1 \) was conjectured by G.Ponzano and T.Regge [22] and later was proven by J. Roberts [23]. It is formulated in terms of tetrahedron that is combined from the edges of length \( J_n := r_n + 1/2, J_5 := i + 1/2, J_6 := j + 1/2 \) and approximates the limit \( \lambda \rightarrow \infty \) for representations \( \{ \lambda r_n, \lambda i, \lambda j \} \):

\[
\left\{ \begin{array}{ll} r_1 & r_2 \\ r_3 & r_4 \end{array} \right\} \sim \frac{1}{\sqrt{24\pi V} \Omega(J_n)} \cos \left( \sum_{n=1}^{\infty} J_n \cdot \Omega(J_n) + \frac{\pi}{4} \right), \tag{52}
\]

where \( V \) is the tetrahedron volume, \( \Omega \) is the external dihedral angle about the edge \( J_i \).

Let us consider 6-j symbols at \( q = 1 \). Using (51) we can find the asymptotics for \( U(sl_N) \) 6-j symbol as an asymptotics for equal \( U(sl_2) \) 6-j symbol. It looks very similar to (52), but with deformed expressions for edges, volume and angles. The tetrahedron is now made of \( J_n \) edges, which can be found from \( U(sl_N) \) \( J_n \):

\[
\begin{align*}
\tilde{J}_m &= J_m, \\
\tilde{J}_m &= J_n + N - 2,
\end{align*}
\tag{53}
\]

where \( m \) and \( n \) are defined differently for two types:

\[
\begin{align*}
m \in \{1, 3\}, \quad n \in \{2, 4, 5, 6\} & \quad \text{Type I}, \\
m \in \{3, 4, 5\}, \quad n \in \{1, 2, 6\} & \quad \text{Type II}.
\end{align*}
\tag{54}
\]
The corresponding volume and angles are denoted by $\tilde{V}$ and $\tilde{\Omega}_n$.

The resulting asymptotics for 6-j symbol corresponding to arbitrary symmetric representations of $U_q(sl_N)$, thus, can be written in terms of the associated tetrahedron, but now the tetrahedron depends on $N$:

$$
\frac{1}{\Theta_T(N)} \begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_T \sim \frac{\theta^N \cdot (N-1)! \cdot (N-2)!}{\sqrt{12\pi \cdot |V(\tilde{J}_n)|}} \cos \left( \sum_{n=1}^{6} \tilde{J}_n \cdot \Omega(\tilde{J}_n) + \frac{\pi}{4} \right).
$$

(55)

Although the factor is quite long for the general case, it becomes much simpler when all $r_n$ coincide, for example, for type I it looks like:

$$
\frac{\left(\frac{r}{2} + N-2\right)!}{\left(\frac{r}{2}\right)!} \left(\frac{\left(\frac{r}{2} + N-2\right)!}{\left(\frac{r}{2}\right)!} \begin{bmatrix} r & r & i \\ r & r & j \end{bmatrix}_{T=1} \sim \frac{(-1)^N(N-1)!(N-2)!}{\sqrt{12\pi|V(\tilde{J}_n)|}} \cos \left( \sum_{i=1}^{6} \tilde{J}_n \cdot \Omega(\tilde{J}_n) + \frac{\pi}{4} \right).
$$

(56)

Let us note, that the generalized formula when all parameters of 6-j symbol are the same does not correspond to the regular tetrahedron if $N > 2$. Due to this fact we can not simplify the relation further. Interestingly, the resulting tetrahedron is deformed for every type differently. In particular, type II corresponds to the trigonal pyramid, whereas type I is a bent tetrahedron, which is combined of 4 equal isosceles triangles.

6 Symmetries derivation

6.1 Hypergeometric symmetries group

In this subsection we do not write any symmetries explicitly. Here we are describing the structure of obtained symmetries. The statements in this subsection are given without analytical proof, but it has been checked manually.

We use both Sears’ transformation and permutation symmetry in order to get all possible 6-j symbol transformations. The arbitrary composition of Sears’ transformations and permutations can be written as:

$$
\begin{aligned}
4\Phi_3 \left(a_1, a_2, a_3, a_4; b_1, b_2, b_3; q, q\right) &= C \cdot 4\Phi_3 \left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3; q, q\right), \\
\end{aligned}
$$

(57)

where variables with $\tilde{}$ denotes the resulting arguments. There is a factor $C$ that appears after Sears’ transformations, but we are not interested in it for now.

To find the symmetries we have to solve the system of equations that equates arguments of equal functions. The rank of the system is 6, because the hypergeometric function has 7 arguments with one condition that defines one variable. Note, that we do not restrict them to the fusion rules when we solve the system. That is done because Sears’ transformation do not respect the fusion rules, but some of its combinations with permutations does, hence we need to obtain all symmetries and then restrict to satisfying fusion rules ones. Also we fix $N = \tilde{N}$ by analogy with $N = 2$ symmetries, although theoretically one can find some relations between 6-j symbols with different $N$’s.

**Statement 3.** The overall set of symmetries $G$ that contains all compositions of permutations and Sears’ transformation is a group and it has 23040 elements in total [28].

This result was obtained via the computer algebra system. Permutations and Sears’ transformations were programmed explicitly and combined multiple times. After reaching the mentioned number of symmetries, it was checked that they are closed under composition. Each symmetry is non-degenerate due to the non-degeneracy of the initial equations, hence all elements are invertible. As a result, 23040 symmetries including identity form a group.

The most of these symmetries are not capable to be used in 6-j symbols because they do not always preserve the positiveness of $r_n,i,j$. Due to this fact we define the following property of a symmetry.
Definition 9. We call a transformation to be positive definite if for arbitrary non-negative \( r_n, i, j \) there is such \( N \geq 2 \) that the resulting \( \Phi_3 \) function arguments \( \tilde{r}_n, \tilde{i}, \tilde{j} \) are non-negative.

Statement 4. The positive definite symmetries form a subgroup \( w \subset G \) and \( |w| = 144 \).

The easiest way to obtain this subset is to substitute \( r_n = 1, i = 1, j = 1 \). The resulting symmetries will depend only on \( N \). For \( N \geq 2 \) only 144 symmetries have positive arguments. For \( N = 2 \) they become the group of permutations and Regge transformations, we will denote it as \( w \subset G \). Due to the obvious positive definiteness of permutations and Regge transformations, \( w \) is indeed the maximal positive definite subgroup of \( G \). Note, that the symmetries from \( w \) are not applicable for arbitrary \( N \geq 2 \), but there is always such \( N_{\text{max}} \) that for \( N = N_{\text{max}} \) one of the Young diagrams is trivial and for lesser \( N \)’s all diagrams are valid. If one considers the positive definiteness for arbitrary \( N \), there are 48 equal elements. We do not separate them from the others in this paper because it is more important to find relations for fixed \( N > 2 \) and this approach will give more symmetries.

Definition 10. For each symmetry we have two conditions from the fusion rules (10): for LHS and RHS. If the conditions coincide for a symmetry, we will call it the general one. All other symmetries we will call the weak ones.

Let us provide this definition with an example of both general and weak symmetries.

The general symmetry:

\[
\begin{bmatrix}
  r_1 & r_2 & i \\
  r_3 & r_4 & j
\end{bmatrix}_1 =
\begin{bmatrix}
  r_2 & r_1 & i \\
  r_4 & r_3 & j
\end{bmatrix}_1.
\]

The fusion rules (10) formally require two equalities, but they coincide. This ensures the applicability of general symmetries for each 6-j symbol.

The weak symmetry:

\[
\begin{bmatrix}
  r_1 & r_2 & i \\
  r_3 & r_4 & j
\end{bmatrix}_1 =
\begin{bmatrix}
  r_1 & i & r_2 \\
  r_3 & j & r_4
\end{bmatrix}_1.
\]

Here we have to restrict representations by two equalities: \( r_1 + r_3 = r_2 + r_4 \) and \( r_1 + i = r_3 + j \). This condition is more strict, so not all 6-j symbols may be transformed by these symmetries. Each weak symmetry induces a subset of 6-j symbols that has such a relation.

Statement 5. The general symmetries of 6-j symbols has the following structure. There are 24 equal elements for type I and 12 elements for type II.

If we leave only general symmetries, we obtain subgroup \( g \subset w \subset G \). It can be checked that \( |g| = 24 \) for type I and \( |g| = 12 \) for type II. By a definition, fusion rules are satisfied for the equated 6-j symbols, so the hypergeometric function can be expressed as a 6-j symbols with some factors. It can be checked that factor \( K_T \) also has these symmetries. This leads to the 24 symmetries for type I and 12 symmetries for type II.

\[
\begin{array}{ccc}
G & \text{pos.def} & w \\
23040 & \text{f.rules} & 144 \\
& & g \\
& 24|12 & \text{N=2} \\
& & H \\
& & 144
\end{array}
\]

The explicit relations are written in the next subsections. The general symmetries from \( g \) may be applied to any 6-j symbol of the corresponding type. In other words, for every \( r_n, i, j \) with the satisfied fusion rules it is possible to write down all symmetries from \( g \).

Weak symmetries originally connect hypergeometric functions, but it can be viewed as a relation between 6-j symbols. There are two problems with this point of view. Firstly, 6-j symbols and \( \Phi_3 \) differs by a factor that is not always invariant under weak symmetries, so we need to normalize the RHS. Secondly, weak symmetries require two independent fusion rules conditions, this results in a
fewer number of free parameters. As a result, it may be applied only to the part of all type I and type II 6-j symbols.

If one considers $U_q(sl_2)$ case, it is possible to obtain all 144 known symmetries [15] that we denote as $H$. For $N = 2$, we do not have such fusion rules restrictions, so $w$ coincides with $g$ and all 144 symmetries are the general ones. Regge symmetry and tetrahedral symmetries may be partly obtained via permutations, but all symmetries require Sears’ transformation.

6.2 Type I general symmetries

In this subsection we write down the general symmetries of type I. These symmetries are very similar to the known ones and can be seen as a natural generalization of the symmetries from $U_q(sl_2)$, although in terms of Young diagrams it’s not obvious. In the shortened notation it is easy to see the correspondence between $U_q(sl_2)$ and $U_q(sl_N)$ symmetries. Although the general symmetries of type I by a definition need $r_1 + r_3 = r_2 + r_4$ to be satisfied, we do not write it explicitly because in every equality either both 6-j symbol exist or both of them do not. The same idea is used for type II general symmetries. To write down the symmetries in a more compact way, we use the following variables:

$$\rho = \frac{r_1 + r_2 + r_3 + r_4}{2}, \quad \rho' = \frac{r_2 + i + r_4 + j}{2} = \rho'' = \frac{r_1 + i + r_3 + j}{2}. \quad (58)$$

All 6-j symbols below are equal and form group $g$. Columns of the equality list correspond to row permutations, rows correspond to Regge symmetries:

$$\begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_1 = \begin{bmatrix} r_1 & r_4 & j \\ r_3 & r_2 & i \end{bmatrix}_1 = \begin{bmatrix} r_3 & r_4 & i \\ r_1 & r_2 & j \end{bmatrix}_1 = \begin{bmatrix} r_3 & r_2 & j \\ r_1 & r_4 & i \end{bmatrix}_1 \quad (59)$$

$$\begin{bmatrix} \rho - r_2 & \rho - r_1 & i \\ \rho - r_4 & \rho - r_3 & j \end{bmatrix}_1 = \begin{bmatrix} \rho - r_2 & \rho - r_3 & j \\ \rho - r_4 & \rho - r_1 & i \end{bmatrix}_1 = \begin{bmatrix} \rho - r_4 & \rho - r_1 & j \\ \rho - r_2 & \rho - r_3 & i \end{bmatrix}_1 = \begin{bmatrix} \rho - r_4 & \rho - r_3 & i \\ \rho - r_2 & \rho - r_1 & j \end{bmatrix}_1$$

$$\begin{bmatrix} \rho'' - i & \rho'' - r_1 \\ \rho'' - j & \rho'' - r_3 \end{bmatrix}_1 = \begin{bmatrix} \rho'' - i & \rho'' - r_3 \\ \rho'' - j & \rho'' - r_1 \end{bmatrix}_1 = \begin{bmatrix} \rho'' - r_2 & \rho'' - r_1 \\ \rho'' - r_3 & \rho'' - i \end{bmatrix}_1 = \begin{bmatrix} \rho'' - r_3 & \rho'' - r_1 \\ \rho'' - r_2 & \rho'' - i \end{bmatrix}_1$$

These 24 symmetries form a representation of group $g$ mentioned above. It has two notable subgroups: row permutations and Regge transformations. Despite the fact that in Young diagram notations tetrahedral symmetries are written differently, it is easy to check that these row permutations are indeed the tetrahedral symmetry. Moreover, the second row reduces to the tetrahedral symmetry due to the fusion rules. Indeed, $\rho = r_1 + r_3 = r_2 + r_4$ and, for example, $\rho - r_2 = r_4$. Let us write down only non-tetrahedral relations (the first column without the second element):

$$\begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_1 = \begin{bmatrix} \rho' - i & \rho' - r_2 \\ \rho' - j & \rho' - r_4 \end{bmatrix}_1 = \begin{bmatrix} \rho' - i & r_2 \\ \rho' - j & r_4 \end{bmatrix}_1. \quad (60)$$

Or in Young diagram notation:

$$\begin{bmatrix} \begin{bmatrix} r_1 \\ r_3 \end{bmatrix} & \begin{bmatrix} i, r_1, r_2 \end{bmatrix} & \begin{bmatrix} r_3, r_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_1 \\ r_3 \end{bmatrix} & \begin{bmatrix} \rho' - i \end{bmatrix} & \begin{bmatrix} \rho' - r_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_1 \\ r_3 \end{bmatrix} & \begin{bmatrix} \rho' - j \end{bmatrix} & \begin{bmatrix} \rho' - r_4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_1 + r_3 - i + j \\ r_1 + r_3 + i + j \end{bmatrix} & \begin{bmatrix} \rho' - i \end{bmatrix} & \begin{bmatrix} \rho' - r_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_1 + r_3 - i + j \\ r_1 + r_3 + i + j \end{bmatrix} & \begin{bmatrix} \rho' - j \end{bmatrix} & \begin{bmatrix} \rho' - r_4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_1 + r_3 + i + j \\ r_1 + r_3 + i + j \end{bmatrix} & \begin{bmatrix} \rho' - j \end{bmatrix} & \begin{bmatrix} \rho' - r_4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_1 + r_3 - i + j \\ r_1 + r_3 + i + j \end{bmatrix} & \begin{bmatrix} \rho' - j \end{bmatrix} & \begin{bmatrix} \rho' - r_4 \end{bmatrix} \end{bmatrix}. \quad (61)$$
Let us give a couple of examples of these symmetries:

- Regge symmetry analogue, type I (1st column is invariant, $N \geq 2$):

\[
\begin{align*}
\{ [8] & \quad [4] \quad [12, 4^{N-2}] \} = \{ [8] & \quad [6] \quad [14, 6^{N-2}] \}, \\
\{ [10] & \quad [14] \quad [6] \} = \{ [10] & \quad [12] \quad [4] \}, \\
\{ [10] & \quad [8] \quad [18, 8^{N-2}] \} = \{ [10] & \quad [5] \quad [15, 5^{N-2}] \}, \\
\{ [12] & \quad [14] \quad [6, 5^{N-2}] \} = \{ [12] & \quad [17] \quad [9, 8^{N-2}] \}, \\
\{ [12] & \quad [6] \quad [16, 5^{N-2}] \} = \{ [12] & \quad [9] \quad [19, 8^{N-2}] \}, \\
\{ [14] & \quad [20] \quad [8] \} = \{ [14] & \quad [17] \quad [5, 5^{N-2}] \}, \\
\{ [12] & \quad [8] \quad [10, 3^{N-2}] \} = \{ [12] & \quad [11] \quad [13, 6^{N-2}] \}, \\
\{ [14] & \quad [18] \quad [6] \} = \{ [14] & \quad [15] \quad [3] \}.
\end{align*}
\]

- Regge symmetry analogue, type I (2nd column is invariant, $N \geq 2$):

\[
\begin{align*}
\{ [4] & \quad [6] \quad [2, 2^{N-2}] \} = \{ [2] & \quad [6] \quad [4, 4^{N-2}] \}, \\
\{ [3] & \quad [1] \quad [5, 4^{N-2}] \} = \{ [5] & \quad [1] \quad [3, 2^{N-2}] \}, \\
\{ [6] & \quad [5] \quad [7, 3^{N-2}] \} = \{ [7] & \quad [5] \quad [6, 2^{N-2}] \}, \\
\{ [3] & \quad [4] \quad [2, 2^{N-2}] \} = \{ [2] & \quad [4] \quad [3, 3^{N-2}] \}, \\
\{ [5] & \quad [6] \quad [7, 4^{N-2}] \} = \{ [4] & \quad [6] \quad [8, 5^{N-2}] \}, \\
\{ [4] & \quad [3] \quad [8, 5^{N-2}] \} = \{ [5] & \quad [3] \quad [7, 4^{N-2}] \}, \\
\{ [4] & \quad [6] \quad [2, 2^{N-2}] \} = \{ [2] & \quad [6] \quad [4, 4^{N-2}] \}, \\
\{ [5] & \quad [3] \quad [7, 4^{N-2}] \} = \{ [7] & \quad [3] \quad [5, 2^{N-2}] \}.
\end{align*}
\]

### 6.3 Type II general symmetries

One can similarly consider type II, there are only 12 symmetries. For brevity we use the following variables:

\[
\rho = \frac{r_1 + r_2 + r_3 + r_4}{2}, \quad \rho' = \frac{r_2 + i + r_4 + j}{2}, \quad \rho'' = \frac{r_1 + i + r_3 + j}{2}.
\]

(62)

All 6-j symbols below are equal. Columns of the table correspond to a column permutation, rows correspond to Regge symmetries.

\[
\begin{align*}
\begin{bmatrix}
  r_1 & r_2 & i \\
  r_3 & r_4 & j
\end{bmatrix}_2 
&= \begin{bmatrix}
  r_2 & r_1 & i \\
  r_4 & r_3 & j
\end{bmatrix}_2, \\
\begin{bmatrix}
  \rho - r_3 & \rho - r_4 & i \\
  \rho - r_1 & \rho - r_2 & j
\end{bmatrix}_2 
&= \begin{bmatrix}
  \rho - r_4 & \rho - r_3 & i \\
  \rho - r_1 & \rho - r_2 & j
\end{bmatrix}_2, \\
\begin{bmatrix}
  r_1 & \rho' - r_4 & \rho' - j \\
  r_3 & \rho' - r_2 & \rho' - i
\end{bmatrix}_2 
&= \begin{bmatrix}
  \rho' - r_4 & r_1 & \rho' - j \\
  \rho' - r_2 & r_3 & \rho' - i
\end{bmatrix}_2, \\
\begin{bmatrix}
  \rho'' - r_3 & r_2 & \rho'' - j \\
  \rho'' - r_1 & r_4 & \rho'' - i
\end{bmatrix}_2 
&= \begin{bmatrix}
  r_2 & \rho'' - r_3 & \rho'' - j \\
  r_4 & \rho'' - r_1 & \rho'' - i
\end{bmatrix}_2, \\
\begin{bmatrix}
  \rho - r_3 & \rho' - r_4 & \rho' - j \\
  \rho - r_1 & \rho' - r_2 & \rho' - i
\end{bmatrix}_2 
&= \begin{bmatrix}
  \rho' - r_4 & \rho - r_3 & \rho' - j \\
  \rho' - r_2 & \rho - r_1 & \rho' - i
\end{bmatrix}_2, \\
\begin{bmatrix}
  \rho'' - r_3 & \rho - r_4 & \rho' - j \\
  \rho'' - r_1 & \rho - r_2 & \rho' - i
\end{bmatrix}_2 
&= \begin{bmatrix}
  \rho - r_4 & \rho' - r_3 & \rho' - j \\
  \rho - r_2 & \rho' - r_1 & \rho' - i
\end{bmatrix}_2.
\end{align*}
\]
The Regge transformation is the only new relation here:

\[
\begin{bmatrix}
 r_1 & r_2 & i \\
 r_3 & r_4 & j
\end{bmatrix}
= \begin{bmatrix}
 r_1' & r_2' & \rho' - i \\
 r_3' & r_4' & \rho' - j
\end{bmatrix}
= \begin{bmatrix}
 r_1'' & r_2'' & \rho'' - i \\
 r_3'' & r_4'' & \rho'' - j
\end{bmatrix}
= \begin{bmatrix}
 r_1'' & r_3' & \rho'' - j \\
 r_2' & r_4' & \rho'' - i
\end{bmatrix},
\]

(64)

Or in Young diagram notation for the first two symmetries (the others can be obtained as a composition):

\[
\begin{bmatrix}
 r_1 & r_2 & \frac{n_1 + r_2 + 1}{2}, \frac{n_1 + r_2 - 1}{2} \\
 r_3 & r_4 & \frac{1}{2}, \frac{N - 2}{2}
\end{bmatrix}
= \begin{bmatrix}
 \frac{r_2 - r_4 + i + j}{2} & \frac{r_1 + r_2 + i, r_1 - r_4 + j}{2} \\
 \frac{r_3 - r_4 + i + j}{2} & \frac{r_1 + r_2 + i, r_2 - r_3 + j}{2}
\end{bmatrix}
= \begin{bmatrix}
 \frac{r_2 + r_4 - i + j}{2} & \frac{r_2 + r_4 - i + j}{2} \\
 \frac{r_1 + r_2 + i, r_2 - r_3 + j}{2} & \frac{r_1 + r_3 - i + j, r_2 + r_1 - i}{2}
\end{bmatrix}.
\]

(65)

Let us give a couple of examples of these symmetries.

- Regge symmetry analogue, type II (1st column is invariant, \(N \geq 2\)):

\[
\begin{bmatrix}
 [5] & [6] & [10, 1] \\
 [3] & [8] & [7, 5^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [5] & [7] & [10, 2] \\
 [3] & [9] & [6, 5^{N-2}]
\end{bmatrix},
\]

\[
\begin{bmatrix}
 [5] & [6] & [11] \\
 [1] & [10] & [7, 6^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [5] & [7] & [11, 1] \\
 [1] & [11] & [6, 6^{N-2}]
\end{bmatrix},
\]

\[
\begin{bmatrix}
 [4] & [6] & [10] \\
 [1] & [9] & [7, 6^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [4] & [7] & [10, 1] \\
 [1] & [10] & [6, 6^{N-2}]
\end{bmatrix},
\]

\[
\begin{bmatrix}
 [3] & [6] & [8, 1] \\
 [4] & [5] & [8, 5^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [3] & [8] & [8, 3] \\
 [4] & [7] & [6, 5^{N-2}]
\end{bmatrix}.
\]

- Regge symmetry analogue, type II (2nd column is invariant, \(N \geq 2\)):

\[
\begin{bmatrix}
 [4] & [2] & [6] \\
 [1] & [5] & [3, 2^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [6] & [2] & [6, 2] \\
 [3] & [5] & [1] \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
 [4] & [3] & [6, 1] \\
 [1] & [6] & [2, 2^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [5] & [3] & [6, 2] \\
 [2] & [6] & [1, 1^{N-2}]
\end{bmatrix},
\]

\[
\begin{bmatrix}
 [5] & [6] & [10, 1] \\
 [4] & [7] & [10, 6^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [10] & [6] & [10, 6] \\
 [9] & [7] & [5, 1^{N-2}]
\end{bmatrix},
\]

\[
\begin{bmatrix}
 [5] & [6] & [9, 2] \\
 [2] & [9] & [4, 4^{N-2}]
\end{bmatrix}
= \begin{bmatrix}
 [7] & [6] & [9, 4] \\
 [4] & [9] & [2, 2^{N-2}]
\end{bmatrix}.
\]

7 Weak symmetries of 6-j symbols

7.1 Type I

Also we may consider weak symmetries of \(q\)-hypergeometric function from the group \(w\). These symmetries have fewer parameters, because the condition \(R_1 + R_3 = R_2 + R_4\) must be satisfied for the 6-j symbol on the right-hand side regardless of the same condition on the 6-j symbol on the left-hand. Therefore it is required to fix one more variable, for example, \(i\). In the previous section, if the condition was satisfied for one 6-j symbol, the other also satisfied this condition. But for further symmetries \(R_1 + R_3 = R_2 + R_4\) is true only for the specific values. Because of this, there are not 5 free parameters, but only 4.
Notation 1. Let us denote by \( \cong \) a weak symmetry between two 6-j symbols with additional fusion rules restrictions \((r_1 + r_3 = r_2 + r_4 \text{ for type I and } r_1 + r_2 = r_3 + r_4 \text{ for type II})\) for both LHS and RHS.

Let us consider weak symmetries of type I. It is convenient to write down not the whole group \( w \), but the maximal subgroup that has no elements of \( g \). Note that in \( U_q(\mathfrak{sl}_2) \) we have the subgroups of Regge transformations, row and column permutations, here we also have similar subgroups. The weak symmetries for type I are analogous to column permutations:

\[
\begin{bmatrix}
    r_1 & r_2 & i \\
    r_3 & r_4 & j \\
\end{bmatrix}_1 \cong \begin{bmatrix}
    i + (N - 2) & r_2 & r_1 - (N - 2) \\
    j + (N - 2) & r_4 & r_2 - (N - 2) \\
\end{bmatrix}_1 \cong \begin{bmatrix}
    i + (N - 2) & r_1 - (N - 2) & r_2 \\
    j + (N - 2) & r_3 - (N - 2) & r_4 \\
\end{bmatrix}_1 \cong C \begin{bmatrix}
    r_2 + (N - 2) & i & r_1 - (N - 2) \\
    r_4 + (N - 2) & j & r_3 - (N - 2) \\
\end{bmatrix}_1 \cong \delta_{N,2} \begin{bmatrix}
    r_2 + (N - 2) & r_1 - (N - 2) & i \\
    r_4 + (N - 2) & r_3 - (N - 2) & j \\
\end{bmatrix}_1 .
\]  

(66)

The factor \( C \) is a function defined as follows. Representations without tilde are arguments of 6-j symbol before applying symmetry and ones with tilde are after application. Some symmetries don’t have this factor because numerator and denominator can be reduced.

\[
C = \frac{K_T(r_1, r_2, r_3, r_4, i, j)}{K_T(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{i}, \hat{j})}.
\]  

(71)

These symmetries are interesting because they cannot be expressed as a combination of any known symmetries. From hypergeometric point of view these symbols have the same value of \( 4\Phi_3 \) but it’s still possible that \( K_T \) is changed by this transformation.

Note, that (70) exist only for \( N = 2 \) as it is impossible to satisfy fusion rules conditions otherwise.

The example of a symmetry with 4 parameters that give us an interesting relation:

\[
\begin{bmatrix}
    r_1 & r_2 & i \\
    r_3 & r_4 & j \\
\end{bmatrix}_1 \cong \begin{bmatrix}
    i + \alpha(N - 2) & r_2 & r_1 - \alpha(N - 2) \\
    j + \alpha(N - 2) & r_4 & r_3 - \alpha(N - 2) \\
\end{bmatrix}_1 \quad \alpha(N - 2) \in \mathbb{Z},
\]  

(72)

where \( r_1 + r_3 = r_2 + r_4, \ r_2 + r_4 = i + j + 2\alpha(N - 2) \).

Let us write down a few examples of these symmetries:

- The first symmetry (66):
  \[
  \left\{ \begin{array}{c}
    6 \\
    3
  \end{array} \right\} \left\{ \begin{array}{c}
    4 \\
    5
  \end{array} \right\} \left\{ \begin{array}{c}
    8, 3^{N-2} \\
    1, 1^{N-2}
  \end{array} \right\} = \left\{ \begin{array}{c}
    6 \\
    3
  \end{array} \right\} \left\{ \begin{array}{c}
    8 \\
    1
  \end{array} \right\} \left\{ \begin{array}{c}
    4, 3^{N-2} \\
    5, 5^{N-2}
  \end{array} \right\} .
  \]

- The second symmetry (67), \( N = 4 \):
  \[
  \left\{ \begin{array}{c}
    9 \\
    7
  \end{array} \right\} \left\{ \begin{array}{c}
    10 \\
    6
  \end{array} \right\} \left\{ \begin{array}{c}
    1, 1^{2} \\
    11, 7^{2}
  \end{array} \right\} = \left\{ \begin{array}{c}
    3 \\
    13
  \end{array} \right\} \left\{ \begin{array}{c}
    10 \\
    6
  \end{array} \right\} \left\{ \begin{array}{c}
    7, 7^{2} \\
    5, 1^{2}
  \end{array} \right\} .
  \]

- The fourth symmetry (69), \( N = 4 \):
  \[
  \left\{ \begin{array}{c}
    9 \\
    6
  \end{array} \right\} \left\{ \begin{array}{c}
    8 \\
    7
  \end{array} \right\} \left\{ \begin{array}{c}
    5, 2^{2} \\
    14, 8^{2}
  \end{array} \right\} = C \left\{ \begin{array}{c}
    10 \\
    9
  \end{array} \right\} \left\{ \begin{array}{c}
    5 \\
    14
  \end{array} \right\} \left\{ \begin{array}{c}
    7, 1^{2} \\
    4, 1^{2}
  \end{array} \right\} ,
  \]

where \( C = 2^{-\frac{1}{4}} \) if \( q = 1 \), else the coefficient is

\[
C = \frac{[2]_q}{[4]_q} \sqrt{\frac{[6]_q}{[10]_q[9]_q}}.
\]
Let us remind that if one consider arbitrary initial 6-j symbol of type I, weak symmetries does not always exist, it can be shown for the symmetry (66):

\[
\begin{bmatrix}
[6] & [4] & [2] \\
[3] & [5] & [1]
\end{bmatrix} \cong \begin{bmatrix}
[6] & [2] \\
[3] & [1]
\end{bmatrix}.
\]

The RHS 6-j symbol does not exist as \(6 + 3 \neq 2 + 1\). To make sure one may check that there are no [1] in the \([6] \otimes [2] \otimes [3]\).

### 7.2 Type II

In a similar way we can consider type II with only 4 free parameters. These symmetries are analogous to a column permutation and row permutations:

\[
\begin{bmatrix}
[r_1 & r_2 & i] \\
r_3 & r_4 & j
\end{bmatrix}_2 \cong \begin{bmatrix}
[r_1 & j & r_4] \\
r_3 & i & r_2
\end{bmatrix}_2 \cong \begin{bmatrix}
[j & r_2 & r_3] \\
i & r_4 & r_1
\end{bmatrix}_2 \cong \begin{bmatrix}
r_1 & r_4 - (N - 2) & j + (N - 2) \\
r_3 & r_2 + (N - 2) & i - (N - 2)
\end{bmatrix}_2 \cong C \begin{bmatrix}
r_1 & r_4 - (N - 2) & j + (N - 2) \\
r_3 & r_2 + (N - 2) & i - (N - 2)
\end{bmatrix}_2 \cong C \begin{bmatrix}
j & r_3 - (N - 2) & r_2 + (N - 2) \\
i & r_1 + (N - 2) & r_4 - (N - 2)
\end{bmatrix}_2 \cong C \begin{bmatrix}
j & r_3 - (N - 2) & r_2 + (N - 2) \\
i & r_1 + (N - 2) & r_4 - (N - 2)
\end{bmatrix}_2 \cong C \begin{bmatrix}
r_3 - (N - 2) & r_4 - (N - 2) & i \\
r_1 + (N - 2) & r_2 + (N - 2) & j
\end{bmatrix}_2 \cong \delta_{N,2}.
\]

7.2.1 Type II weak symmetries for \(N = 4\):

\[
\begin{bmatrix}
[r_1 & r_2 & i] \\
r_3 & r_4 & j
\end{bmatrix}_2 \cong C \begin{bmatrix}
r_1 & r_4 - (N - 2) & j + (N - 2) \\
r_3 & r_2 + (N - 2) & i - (N - 2)
\end{bmatrix}_2 \cong C \begin{bmatrix}
[j & r_3 - (N - 2) & r_2 + (N - 2) \\
i & r_1 + (N - 2) & r_4 - (N - 2)
\end{bmatrix}_2 \cong C \begin{bmatrix}
r_3 - (N - 2) & r_4 - (N - 2) & i \\
r_1 + (N - 2) & r_2 + (N - 2) & j
\end{bmatrix}_2 \cong \delta_{N,2}.
\]

It can be obtained that \(r_4 - r_2 = N - 2\). Hence, the symmetry can be rewritten as:

\[
\begin{bmatrix}
[r_1 & r_2 & i] \\
r_3 & r_4 & j
\end{bmatrix}_2 = C \begin{bmatrix}
r_1 & r_2 & j + r_4 - r_2 \\
r_3 & r_4 & i + r_4 - r_2
\end{bmatrix}_2,
\]

where \(r_1 + r_2 = r_3 + r_4\) and \(r_4 - r_2 = N - 2\). As an example of this symmetry, we write down the following pair of 6-j symbols for \(N = 4\) classical case:

\[
\begin{bmatrix}
[6] & [5] & [7, 4] \\
[4] & [7] & [9, 5^2]
\end{bmatrix} = \sqrt{\frac{65}{225}} \begin{bmatrix}
[6] & [5] & [11] \\
[4] & [7] & [1, 1^2]
\end{bmatrix}.
\]

Let us give an example of the type II weak symmetries for \(N = 4\):

\[
\begin{bmatrix}
[5] & [4] & [9] \\
[3] & [6] & [7, 3^2]
\end{bmatrix} := \begin{bmatrix}
r_1 & r_2 & i \\
r_3 & r_3 & j
\end{bmatrix}_2 \cong \begin{bmatrix}
r_1 & i - (N - 2) & r_2 + (N - 2) \\
r_3 & j + (N - 2) & r_4 - (N - 2)
\end{bmatrix}_2 := \begin{bmatrix}
[5] & [7] & [9, 3] \\
[3] & [9] & [4^3]
\end{bmatrix}.
\]

### 8 Select results

Type I 6-j symbol is written as:

\[
\begin{bmatrix}
[r_1 & r_2 & i] \\
r_3 & r_4 & j
\end{bmatrix}_1 = \begin{bmatrix}
[r_1 & r_2] \\
r_3 & r_4
\end{bmatrix} \begin{bmatrix}
r_2 - r_1 + i^{N-2} \\
r_2 - r_3 + j^{N-2}
\end{bmatrix}_1.
\]

19
where we assume \( r_1 + r_3 = r_2 + r_4 \).

Type II is:

\[
\begin{pmatrix}
[r_1 & r_2 & i \\
r_3 & r_4 & j
\end{pmatrix}_2 = \begin{pmatrix}
[r_1] & [r_2] & \left[ \frac{r_1 + r_2 + i}{2}, \frac{r_1 + r_2 - i}{2} \right] \\
[r_3] & [r_4] & \left[ \frac{r_2 - r_3 + j}{2} \right]^{N-2}
\end{pmatrix} 
\]  
(77)

where \( r_1 + r_2 = r_3 + r_4 \).

- Expression (40) for MFS via q-hypergeometric series:

\[
2a_i = \begin{pmatrix}
r_1 & r_2 & i \\
r_3 & r_4 & j
\end{pmatrix}_T = K_T \cdot 4\Phi_3 \left( \frac{a_1, a_2, a_3, a_4}{b_1, b_2, b_3}; q, q \right),
\]
(78)

\[
2b_i = \begin{pmatrix}
-r_1 - r_2 + i - 2(N - 2)\delta T_{2} \\
-r_3 - r_4 + i \\
-r_1 - r_4 + j \\
-r_2 - r_3 + j
\end{pmatrix},
\]
(79)

Factor \( K_T \) depends on type \( T \) and defined as in (39):

\[
K_T = \frac{\theta_N (r_1, r_2, i) \theta_N (r_3, r_4, i) \theta_N (r_1, r_4, j) \theta_N (r_2, r_3, j) [N - 1]_q [N - 2]_q \left( \frac{r_1 + r_2 + r_3 + r_4}{2} + N - 1 \right)_q}{\left[ \frac{r_1 + r_2}{2} \right]_q \left[ \frac{r_3 + r_4}{2} \right]_q \left[ \frac{r_1 + r_3 - r_4}{2} \right]_q \left[ \frac{r_2 + r_4 - r_3}{2} \right]_q \left[ \frac{r_1 + r_3 + r_4 + i}{4} \right]_q \left[ \frac{r_2 + r_3 + j}{4} \right]_q (N - 2)\delta T_{2}, q \right)_q}.
\]
(80)

- Relation (51) between MFS and \( U_q(\mathfrak{sl}_2) \) 6-j symbols:

\[
\begin{pmatrix}
r_1 & r_2 & i \\
r_3 & r_4 & j
\end{pmatrix}_1 = \begin{pmatrix}
r_1 & r_2 + N - 2 & i + N - 2 \\
r_3 & r_4 + N - 2 & j + N - 2
\end{pmatrix} (-1)^N [N - 1]_q [N - 2]_q \cdot \Theta_1(N),
\]
(81)

\[
\begin{pmatrix}
r_1 & r_2 & i \\
r_3 & r_4 & j
\end{pmatrix}_2 = \begin{pmatrix}
r_1 + N - 2 & r_2 + N - 2 & i \\
r_3 & r_4 & j + N - 2
\end{pmatrix} (-1)^N [N - 1]_q [N - 2]_q \cdot \Theta_2(N),
\]
(82)

with factors \( \Theta_1, \Theta_2 \) defined in (49):

\[
\Theta_1(N) = \left( \prod_{m=1}^{N-2} \left[ \frac{i - r_1 + r_2}{2} + m \right]_q \left[ \frac{j + r_2 - r_3}{2} + m \right]_q \left[ \frac{j - r_1 + r_4}{2} + m \right]_q \left[ \frac{i - r_3 + r_4}{2} + m \right]_q \right)^{- \frac{1}{2}},
\]
(83)

\[
\Theta_2(N) = \left( \prod_{m=1}^{N-2} \left[ \frac{r_1 + r_2 - i}{2} + m \right]_q \left[ \frac{j + r_2 - r_3}{2} + m \right]_q \left[ \frac{j + r_1 - r_4}{2} + m \right]_q \left[ \frac{i + r_3 + r_4 + 1 + m}{2} \right]_q \right)^{- \frac{1}{2}}.
\]
(84)

- The asymptotics (55) of MFS for \( U(\mathfrak{sl}_N) \):

\[
\frac{1}{\Theta_T(N)} \begin{pmatrix}
r_1 & r_2 & i \\
r_3 & r_4 & j
\end{pmatrix}_T \sim \frac{(-1)^N (N - 1)! (N - 2)!}{\sqrt{12\pi \cdot |V(\vec{J}_n)|}} \cos \left( \sum_{n=1}^{6} \vec{J}_n \cdot \Omega(\vec{J}_n) + \frac{\pi}{4} \right),
\]
(85)

where \( \vec{J}_n \) are defined in (54).

8.1 General symmetries

- Regge transformations (61), type I (\( \rho' = \frac{r_1 + r_2 + i + j}{2} \)): 

\[
\begin{pmatrix}
r_1 & r_2 & i \\
r_3 & r_4 & j
\end{pmatrix}_1 = \begin{pmatrix}
r_1 & \rho' - i & \rho' - r_2 \\
r_3 & \rho' - j & \rho' - r_4
\end{pmatrix}_1.
\]
(86)
Or in Young diagram notation:

\[
\left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_2-r_1+i+N-2}{2} \\
 \hline
 j, \frac{r_2-r_1+j+N-2}{2} \\
\end{array} \right] = \left[ \begin{array}{c|c}
 r_1 & \frac{r_2+r_4-i+j}{2} \\
 \hline
 r_3 & \frac{r_2+r_4+i-j}{2} \\
\end{array} \right] \left[ \begin{array}{c|c}
 \frac{-r_2+r_4+i+j}{2}, \frac{r_3-r_2+j+N-2}{2} \\
 \hline
 \frac{r_2-r_4+i+j}{2}, \frac{r_2-r_3+j+N-2}{2} \\
\end{array} \right] = \left[ \begin{array}{c|c}
 \frac{r_1+r_3-i+j}{2} & \frac{-r_1+r_3+i+j}{2} \\
 \hline
 \frac{r_1+r_3+i-j}{2} & \frac{r_1-r_3+j+N-2}{2} \\
\end{array} \right].
\]

(87)

- Regge transformations (65), type II \( (\rho' = \frac{r_1+r_3+i+j}{2}, \rho'' = \frac{r_2+r_4+i+j}{2})\):

\[
\left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_2+r_4+i+j}{2} \\
 \hline
 \rho' - r_4 & \rho' - j \\
\end{array} \right] = \left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 \rho'' - r_3 & \rho'' - r_1 \\
 \hline
 \rho'' - r_1 & \rho'' - i \\
\end{array} \right].
\]

(88)

Or in Young diagram notation:

\[
\left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 \frac{r_2+r_4+i+j}{2} \\
 \hline
 \frac{r_2-r_4-i+j}{2} \\
\end{array} \right] = \left[ \begin{array}{c|c}
 r_1 & \frac{r_2-r_4+i+j}{2} \\
 \hline
 r_3 & \frac{r_2+r_4-i+j}{2} \\
\end{array} \right] \left[ \begin{array}{c|c}
 \frac{r_1+r_2+i-j}{2}, \frac{r_1-r_4+j+N-2}{2} \\
 \hline
 \frac{r_1+r_3+i-j}{2}, \frac{r_1-r_3+j+N-2}{2} \\
\end{array} \right] = \left[ \begin{array}{c|c}
 \frac{r_1-r_3+i+j}{2} & \frac{-r_1+r_3+i+j}{2} \\
 \hline
 \frac{r_3-r_1+i+j}{2} & \frac{r_3+r_1-j+N-2}{2} \\
\end{array} \right].
\]

(89)

8.2 Weak symmetries

- Type I:

\[
\left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_2+r_4+i+j}{2} \\
 \hline
 j, \frac{r_2+r_4+i+j}{2} \\
\end{array} \right] \approx \left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_1+r_3+i+j}{2} \\
 \hline
 j, \frac{r_1+r_3+i+j}{2} \\
\end{array} \right] \approx \left[ \begin{array}{c|c}
 \frac{r_1}{2} - (N-2) & \frac{r_2}{2} \\
 \hline
 \frac{r_3}{2} - (N-2) & \frac{r_4}{2} \\
\end{array} \right] \approx C \left[ \begin{array}{c|c}
 \frac{r_2}{2} + (N-2) & \frac{r_1}{2} - (N-2) \\
 \hline
 \frac{r_3}{2} + (N-2) & \frac{r_4}{2} - (N-2) \\
\end{array} \right].
\]

(90)

(91)

\[
\left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_2+r_4+i+j}{2} \\
 \hline
 j, \frac{r_2+r_4+i+j}{2} \\
\end{array} \right] \approx \left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_1+r_3+i+j}{2} \\
 \hline
 j, \frac{r_1+r_3+i+j}{2} \\
\end{array} \right] \approx \left[ \begin{array}{c|c}
 \frac{r_1}{2} - (N-2) & \frac{r_2}{2} \\
 \hline
 \frac{r_3}{2} - (N-2) & \frac{r_4}{2} \\
\end{array} \right] \approx C \left[ \begin{array}{c|c}
 \frac{r_2}{2} + (N-2) & \frac{r_1}{2} - (N-2) \\
 \hline
 \frac{r_3}{2} + (N-2) & \frac{r_4}{2} - (N-2) \\
\end{array} \right].
\]

(92)

(93)

- Type II:

\[
\left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 i, \frac{r_2+r_4+i+j}{2} \\
 \hline
 j, \frac{r_2+r_4+i+j}{2} \\
\end{array} \right] \approx \left[ \begin{array}{c|c}
 r_1 & r_2 \\
 \hline
 r_3 & r_4 \\
\end{array} \right] \left[ \begin{array}{c|c}
 j, \frac{r_1+r_3+i+j}{2} \\
 \hline
 i, \frac{r_1+r_3+i+j}{2} \\
\end{array} \right] \approx \left[ \begin{array}{c|c}
 \frac{r_1}{2} - (N-2) & \frac{r_2}{2} \\
 \hline
 \frac{r_3}{2} - (N-2) & \frac{r_4}{2} \\
\end{array} \right] \approx C \left[ \begin{array}{c|c}
 \frac{r_2}{2} + (N-2) & \frac{r_1}{2} - (N-2) \\
 \hline
 \frac{r_3}{2} + (N-2) & \frac{r_4}{2} - (N-2) \\
\end{array} \right].
\]

(94)
9 Conclusion

The 6-j symbols beyond $U_q(sl_2)$ are rapidly becoming very complicated to analyze. Even in the case of symmetric and conjugate to symmetric representations where we know the analytic expression, there are many features that hide from our sight. Firstly, 6-j expression in its original form \[1\] is the q-factorial series that can be written as a function $5\Phi_4$, but after some manipulations it became clear that the expression is very similar to $U_q(sl_2)$ one and may be written as \[41\] via $4\Phi_3$.

Secondly, the hypergeometric function has a relation \[38\] that is necessary to use the Sears’ transformation. This allow us to think that there is an important class of 6-j symbols with 6 free parameters that is connected with $4\Phi_3$ series. Considered expression \[41\] is already applicable to $N=2$ case and types I, II. It is an interesting question what else may be expressed via $4\Phi_3$.

The relation \[51\] between multiplicity-free $U_q(sl_N)$ and $U_q(sl_2)$ symbols reveals the nature of multiplicity-free case. In fact, multiplicity-free 6-j symbols tends to be very similar to $U_q(sl_2)$ one. As was found in \[25, 17\], the other class of 6-j symbol with symmetric incoming representations may be expressed via $U_q(sl_2)$ one. The further study of more difficult classes can tell us more about the structure of 6-j symbols, but now we can vividly see that q-hypergeometric series play the main role in this problem.

Obtained symmetries show that there are much more relations for 6-j symbols in $U_q(sl_N)$ than tetrahedral symmetries. As the most bright example of this statement, we show that the Regge symmetry is generalizable to both types as \[61, 65\]. Weak symmetries, on the other hand, are not so convenient to use, but they provide a lot of new relations that depend on $N$ explicitly.

Acknowledgements

We are deeply indebted to Andrei Mironov and Alexei Morozov for numerous stimulating discussions. V.A. is also grateful to Satoshi Nawata for clarifications on fusion rules, to Andrei Zotov and Victor Mishnyakov for useful discussions and comments.

Our work was partly supported by the grant of the Foundation for the Advancement of Theoretical Physics “BASIS” (A.M., A.S. and A.V.), by RFBR grants 19-01-00680 (V.A.), 17-01-00585 (A.M.), 18-31-20046 (A.S.), by joint RFBR grants 19-51-18006 (A.M.), 19-51-50008-Yaf-a (A.M.), 18-51-05015-Arm-a (A.M., A.S.), 18-51-45010-Ind-a (A.M., A.S.), 19-51-53014-GFEN-a (A.M., A.S.), 19-51-18006-Bolg-a (A.M.), by President of Russian Federation grant MK-2038.2019.1 (A.M.).

References

[1] Satoshi Nawata, P. Ramadevi, and Zodinmawia. Multiplicity-free quantum 6j-symbols for $U_q(sl_N)$. 
Lett. Math. Phys., 103:1389–1398, 2013. arXiv:1302.5143, doi:10.1007/s11005-013-0651-4.

[2] L. D. Landau and E. M. Lifshitz. Quantum Mechanics: Non-Relativistic Theory. 
Pergamon Press, 1997.

[3] R. N. Bhatt Adam C. Durst, Genesis Yang-Mejia. Quadrupolar interactions between acceptor pairs in p-doped semiconductors. arXiv:1910.06480.

[4] D. Bernard O. Babelon. A Quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations. Physics Letters B, 375:89–97, 1996. arXiv:q-alg/9511019.

[5] R.K. Kaul P. Rama Devi, T.R. Govindarajan. Three Dimensional Chern-Simons Theory as a Theory of Knots and Links III : Compact Semi-simple Group. Nuclear Physics B, 402:548–566, 1993. arXiv:hep-th/9212110.

[6] Hans Jockers Jie Gu. A note on colored HOMFLY polynomials for hyperbolic knots from WZW models. Communications in Mathematical Physics, 338:393–456, 2015. arXiv:1407.5643.

[7] R. Gambini J. M. Aroca, H. Fort. On the Path Integral Loop Representation of (2+1) Lattice Non-Abelian Theory. Physical Review D, 58:045007, 1998. arXiv:1407.5643.
[8] Etera R. Livine. 3d Quantum Gravity: Coarse-Graining and q-Deformation. *Ann. Henri Poincare*, 18:1465–1491, 2017. [arXiv:1610.02716](https://arxiv.org/abs/1610.02716).

[9] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.*, 127:1–26, 1990. [doi:10.1007/BF02096491](https://doi.org/10.1007/BF02096491).

[10] V. G. Turaev and O. Ya. Viro. S. State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology*, 31:865–902, 1992.

[11] V. G. Turaev. Quantum invariants of knots and 3-manifolds. *de Gruyter Studies in Mathematics*, 18, 1994.

[12] Andrey Smirnov Petr Dunin-Barkowski, Alexey Sleptsov. Explicit computation of Drinfeld associator in the case of the fundamental representation of gl(N). *J. Phys. A*, 45:385204, 2012. [arXiv:1201.0025](https://arxiv.org/abs/1201.0025).

[13] Alexey Sleptsov. Hidden structures of knot invariants. *Int. J. Mod. Phys. A*, 29:1430063, 2014.

[14] Andrei Mironov, Alexei Morozov, Alexey Sleptsov. On 6j-symbols for symmetric representations of $U_q(su_N)$. *JETP Lett.*, 106:630–636, 2017. [arXiv:1709.02290](https://arxiv.org/abs/1709.02290).

[15] A. Klimyk and K. Schmudgen. *Quantum groups and their representations*. Springer, 1997.

[16] Andrey Morozov and Alexey Sleptsov. New symmetries for the $U_q(su_N)$ 6-j symbols from the Eigenvalue conjecture. *JETP Lett.*, 108(10):697–704, 2018. [arXiv:1905.01876](https://arxiv.org/abs/1905.01876), [doi:10.1134/S0021364018220058](https://doi.org/10.1134/S0021364018220058).

[17] Victor Alekseev, Andrey Morozov, and Alexey Sleptsov. Interplay between symmetries of quantum 6-j symbols and the eigenvalue hypothesis. 2019. [arXiv:1909.07601](https://arxiv.org/abs/1909.07601).

[18] H. Itoyama, A. Mironov, A. Morozov, and An. Morozov. Eigenvalue hypothesis for Racah matrices and HOMFLY polynomials for 3-strand knots in any symmetric and antisymmetric representations. *Int. J. Mod. Phys.*, A28:1340009, 2013. [arXiv:1209.6304](https://arxiv.org/abs/1209.6304), [doi:10.1142/S0217751X13400095](https://doi.org/10.1142/S0217751X13400095).

[19] A. Mironov and A. Morozov. Universal Racah matrices and adjoint knot polynomials: Arborescent knots. *Phys. Lett.*, B755:47–57, 2016. [arXiv:1511.09077](https://arxiv.org/abs/1511.09077), [doi:10.1016/j.physletb.2016.01.063](https://doi.org/10.1016/j.physletb.2016.01.063).

[20] A. N. Kirillov and N. Yu. Reshetikhin. Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links. In *New Developments in the Theory of Knots*, pages 202–256. World Scientific, aug 1990. [doi:10.1142/9789812798329_0012](https://doi.org/10.1142/9789812798329_0012).

[21] A. Mironov, A. Morozov, and A. Sleptsov. On 6j-symbols for symmetric representations of $U_q(su_N)$. *JETP Lett.*, 106(10):630–636, 2017. [Pisma Zh. Eksp. Theor. Fiz.106,607(2017)]. [arXiv:1709.02290](https://arxiv.org/abs/1709.02290), [doi:10.1134/S0021364017220040](https://doi.org/10.1134/S0021364017220040).

[22] Giorgio Ponzano and Tullio Eugenio Regge. Semiclassical limit of racah coefficients. In *Spectroscopic and group theoretical methods in physics: Racah memorial volume*, pages 1–58. Amsterdam: North-Holland, 1968.

[23] Justin Roberts. Classical 6j-symbols and the tetrahedron. *Geom. Topol.*, 3:21–66, 1999. [arXiv:math-ph/9812013](https://arxiv.org/abs/math-ph/9812013), [doi:10.2140/gt.1999.3.21](https://doi.org/10.2140/gt.1999.3.21).

[24] G. Gasper and M. Rahman. *Basic hypergeometric series*. Cambridge University Press, 1990.

[25] Saswati Dhara, A. Mironov, A. Morozov, An Morozov, P. Ramadevi, Vivek Kumar Singh, and A. Sleptsov. Multi-Colored Links From 3-strand Braids Carrying Arbitrary Symmetric Representations. *Ann. Henri Poincare*, 20(12):4033 – 4054, 2019. [arXiv:1805.03916](https://arxiv.org/abs/1805.03916), [doi:10.1007/s00023-019-00841-z](https://doi.org/10.1007/s00023-019-00841-z).
[26] William Fulton and Joe Harris. *Representation Theory. A First Course*. Springer, 1991.

[27] Tullio Regge. Symmetry properties of Racah’s coefficients. *Nuovo Cim.*, 11:116–117, 1959. doi:10.1007/BF02724914.

[28] Symmetries evaluation. https://github.com/Victor5597/Hypergeometric_symmetries. Accessed: 15.10.2019.