VIRTUAL BETTI NUMBERS OF COMPACT LOCALLY
SYMMETRIC SPACES

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Abstract. We show that the virtual Betti number of a compact
locally symmetric space with arithmetic fundamental group is ei-
ther 0 or else is infinite.

1. Introduction

Let $G$ be a connected non-compact linear Lie group with finite cen-
tre, such that $G$ is simple modulo its centre. Let $\Gamma$ be a torsion free
cocompact arithmetic (not necessarily congruence ) subgroup in $G$ and
let $i \geq 0$ be an integer. Consider the direct limit cohomology group
$$H^i = \lim H^i(\Delta, \mathbb{C})$$
where the direct limit is over all finite index subgroups $\Delta$ in $\Gamma$; we
emphasize that $\Gamma$ is only assumed to be an arithmetic subgroup of $G$
and is not assumed to be a congruence subgroup of $G$. The dimension
of the direct limit $\mathcal{H}^i$ as a $\mathbb{C}$-vector space is called the virtual $i$-th
Betti number of $\Gamma$.

Theorem 1. If the direct limit $\mathcal{H}^i$ is finite dimensional, then
$\mathcal{H}^i = H^i(G_u/K, \mathbb{C})$ where $G_u/K$ is the compact dual of the symmetric space
$G/K$ of $G$.

As a special case we recover the following result of Cooper, Long and
Reid (see [CLR]).

Corollary 1. If $M$ is a compact arithmetic hyperbolic 3-manifold with
non-vanishing first Betti number, then $M$ has infinite virtual first Betti
number.

Proof. Take $G = SL_2(\mathbb{C})$ in Theorem 1, and observe that the compact
dual $G_u/K = S^3$ has vanishing first cohomology. □
The present note was motivated by the recent preprint [CLR] of Cooper, Long and Reid, where they prove Corollary 1, by using crucially, the fact that $M$ is a hyperbolic 3-manifold. We show that this is true in greater generality. The point of Theorem 1 is that the group $\Gamma$ is not assumed to be a congruence subgroup; if $\Gamma$ is a congruence subgroup, this is a result of A. Borel (see [B]).

2. Proof of Theorem 1

Let $K \subset G$ be a maximal compact subgroup; write $\mathfrak{k}$ and $\mathfrak{g}$ for the complexified Lie algebras of $K$ and $G$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\Gamma$ (and hence the finite index subgroup $\Delta$) is torsion-free and cocompact in $G$. We then get by the Matsushima-Kuga formula (see [BoW]),

\[ H^i(\Delta, \mathbb{C}) = \text{Hom}_K(\wedge^i \mathfrak{p}, C^\infty(\Delta \setminus G)(0)). \]

In this formula, $C^\infty(\Delta \setminus G)(0)$ denotes the space of complex valued smooth functions on the manifold $\Delta \setminus G$ which are annihilated by the Casimir of $\mathfrak{g}$ (the latter space in the Matsushima-Kuga formula may be identified with the space of harmonic differential forms of degree $i$ on $\Delta \setminus G/K$ with respect to the $G$-invariant metric on the symmetric space $G/K$).

Taking direct limits in the Matsushima-Kuga formula yields the equality

\[ H^i = \lim H^i(\Delta, \mathbb{C}) = \text{Hom}_K(\wedge^i \mathfrak{p}, \bigcup_{\Delta \subset \Gamma} C^\infty(\Delta \setminus G)(0)). \]

Here, $\Delta$ runs through finite index subgroups of $\Gamma$. Consider the space

\[ \mathcal{F} = \bigcup_{\Delta \subset \Gamma} C^\infty(\Delta \setminus G)(0). \]

On the space $\mathcal{F}$, $G$ acts on the right (since the Casimir commutes with the $G$-action).

Now, $\Gamma$ is an arithmetic subgroup of $G$. That is, there is a semi-simple (simply connected) algebraic group $G$ defined over $\mathbb{Q}$ and a smooth surjective homomorphism $\pi : G(\mathbb{R}) \to G$ with compact kernel such that $\pi(G(\mathbb{Z}))$ is commensurable to $\Gamma$. We define $G(\mathbb{Q})$ simply to mean the image group $\pi(G(\mathbb{Q}))$. It follows from weak approximation ([PR]) that $G(\mathbb{Q})$ is dense in $G$. 
Now, there is an action on $\mathcal{F}$ by $G(\mathbb{Q})$ on the left (which therefore commutes with the right $G$ action), as follows. Given a function $\phi \in \mathcal{F}$ and given an element $g \in G(\mathbb{Q})$, the function $\phi$ is left $\Delta$-invariant for some finite index subgroup $\Delta$ in $\Gamma$. Consider the function $g(\phi) = x \mapsto \phi(g^{-1}x)$. This function is left-invariant under $g\Delta g^{-1}$ and hence under $\Gamma \cap g\Delta g^{-1}$; since $g \in G(\mathbb{Q})$, it follows that $g$ commensurates $\Gamma$ and hence that the subgroup $\Gamma \cap g\Delta g^{-1}$ is of finite index in $\Gamma$. Therefore, $g(\phi)$ lies in $\mathcal{F}$. This defines an action of $G(\mathbb{Q})$ on the direct limit $\mathcal{H}^i$. Note that under this action, the action of $\Delta$ on the cohomology group $H^i(\Delta, \mathbb{C})$ is trivial.

Suppose that $\mathcal{H}^i$ is finite dimensional. Since $\mathcal{H}^i$ is a direct limit of finite dimensional vector spaces, it follows that it coincides with one of them. Therefore there exists a finite index subgroup $\Delta$ of $\Gamma$ such that

$$\mathcal{H}^i = H^i(\Delta, \mathbb{C}).$$

The last sentence of the foregoing paragraph says that while $G(\mathbb{Q})$ acts on $H^i(\Delta, \mathbb{C})$, the action by $\Delta$ is trivial. Hence the action by the normal subgroup $N$ generated by $\Delta$ in $G(\mathbb{Q})$ is also trivial. The density of $G(\mathbb{Q})$ in $G$ is easily seen to imply the density of the normal subgroup $N$ in $G$. Thus the image of $\wedge^4 p$ under any element of $\mathcal{H}^i$ (viewed via the Matsushima-Kuga formula as a (K-equivariant) homomorphism of $\wedge^4 p$ into $\mathcal{F}$), goes into $G$ invariant functions in $C^\infty(\Delta \backslash G)$, i.e., the constant functions. But $Hom_K(\wedge^4 p, \mathbb{C})$ is the space of harmonic differential forms on the compact dual $G_u/K$, and is therefore isomorphic to $H^4(G_u/K, \mathbb{C})$.

This proves Theorem 1.

Remark. If $\Gamma$ and all the subgroups $\Delta$ are congruence subgroups, then one sees at once from strong approximation, that the above $G(\mathbb{Q})$ action on the direct limit translates into the action of the “Hecke Operators” $G(A_f)$ ($A_f$ are the ring of finite adeles) and amounts to the proof of Borel in [13]. In this sense, the proof of Theorem 1 is an extension of Borel’s proof to the non-congruence case.

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