

W-LIKE MAPS WITH VARIOUS INSTABILITIES OF ACIM’S

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ABSTRACT. This paper generalizes the results of [13] and then provides an interesting example. We construct a family of W-like maps \( \{W_a\} \) with a turning fixed point having slope \( s_1 \) on one side and \(-s_2\) on the other. Each \( W_a \) has an absolutely continuous invariant measure \( \mu_a \). Depending on whether \( \frac{1}{s_1} + \frac{1}{s_2} \) is larger, equal or smaller than 1, we show that the limit of \( \mu_a \) is a singular measure, a combination of singular and absolutely continuous measure or an absolutely continuous measure, respectively. It is known that the invariant density of a single piecewise expanding map has a positive lower bound on its support. In Section 4 we give an example showing that in general, for a family of piecewise expanding maps with slopes larger than 2 in modulus and converging to a piecewise expanding map, their invariant densities do not necessarily have a positive lower bound on the support.

1. Introduction

In practice, due to external noise, or roundoff errors in computation, there is a natural interest in the stability of properties of chaotic dynamical systems under small perturbations. If we consider a family of piecewise expanding maps \( \tau_a : I \to I, \) \( a > 0 \) with absolutely continuous invariant measures (acim’s) \( \mu_a \), converging to a piecewise expanding map \( \tau_0 \) with acim \( \mu_0 \), then under general assumptions \( \mu_a \)'s converge to \( \mu_0 \). One such assumption is that \( \inf |\tau'_a| > 2 \) for all \( a > 0 \) (see [1], [6], [7] or [10]). This is useful in the study of the metastable systems [15], or to approximate the invariant densities [8].

Keller [9] introduced the family of \( \{W_a\} \) maps that are piecewise expanding, ergodic transformations with a “stochastic singularity”, i.e., \( \mu_a \)'s converge to a singular measure. This occurs because of the existence of diminishing invariant neighborhoods of the turning fixed point. The slopes of the Keller’s \( W_a \) maps converge to 2 and -2 on the left and right hand sides of the turning fixed point, respectively.

Given two numbers, \( s_1 \) and \( s_2 \), greater than 1, we consider a W-like map with one turning fixed point having slope \( s_1 \) on one side and \(-s_2\) on the other. In [13], the authors considered the special case where \( s_1 = s_2 = 2 \). Their perturbed maps \( W_a \) are piecewise expanding with slopes strictly greater than 2 in modulus and are exact with their acim’s supported on all of \([0, 1]\). The standard bounded variation method [2] cannot be applied in this setting as the slopes of the maps in that family are not uniformly bounded away from 2. Other methods, for example, those studied in [3], [12] and [14] cannot be applied either. Using the main result of [5], it can
be shown that the $\mu_a$’s converge to $\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{\frac{1}{2}}$, where $\delta_{\frac{1}{2}}$ is the Dirac measure at point $1/2$ and $\mu_0$ is the acim of the $W_0$ map. Thus, the family of measures $\mu_a$ approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map. Similar instability was also shown in [11] for a countable family of transitive Markov maps approaching Keller’s $W_0$ map.

In this paper, we construct a family of maps for which the instability of the acim’s has a global character, not a local one. In the more general case considered in this paper, with $s_1, s_2$ not necessarily equal to 2, we will discuss the limits of the acim’s $\mu_a$ of the $\{W_a\}$ maps. We have three cases:

(I) If $\frac{1}{s_1} + \frac{1}{s_2} > 1$, then $\mu_a$’s converge *-weakly to $\delta_{\frac{1}{2}}$.

(II) If $\frac{1}{s_1} + \frac{1}{s_2} = 1$, then $\mu_a$’s converge *-weakly to

$$\frac{(qs_1 + ps_2 - p - q)(s_2 + 2)\mu_0 + 2rs_1s_2^2}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2} \delta_{\frac{1}{2}},$$

where $p, q$ and $r$ are parameters defining our family of maps.

(III) If $\frac{1}{s_1} + \frac{1}{s_2} < 1$, then $\mu_a$’s converge to $\mu_0$.

Additionally, in Theorem 2, we prove that in case (III) the densities of the $\mu_a$’s are uniformly bounded. The first case of our result contains the example in which Keller [9] obtained the “stochastic singularity.” In the second case, the limit measure is a combination of an absolutely continuous and a singular measure, and this combination is varying according to $p$, $q$ and $r$ for fixed $s_1$ and $s_2$. This is a generalization of the result of [13]. In the third case, we have a map with a stable acim.

At the end of the paper, we use our main results to provide an interesting example. Keller [11] and Kowalski [12] proved that for a piecewise expanding map $\tau: I \to I$ with $|\frac{x}{\tau(x)}|$ being a function of bounded variation, the density of the acim of $\tau$ has a uniform positive lower bound on its support. We construct a family of piecewise expanding, piecewise linear maps $\tau_n$ such that $\tau_n$ converge to $\tau = W_0$ ($s_1 = s_2 = 2$), $|\tau_n'| > 2$ for all $n$ but the densities of the acims $\mu_n$’s do not have a uniform positive lower bound.

In Section 2 we introduce our family of $W_a$ maps and state the main result. In Section 3 we present the proofs. In Section 4 we show the example related to the results of Keller [11] and Kowalski [12].

2. Family of $W_a$ maps and the main result

Let $s_1, s_2 > 1$ and $p, q, r > 0$. We consider the family $\{W_a: 0 \leq a \leq 1\}$ of maps of $[0,1]$ onto itself defined by

$$W_a(x) = \begin{cases} 1 - \frac{2(s_1 + pa)}{s_1 - 1 + pa - 2ra} x, & \text{for } 0 \leq x < \frac{1}{2} - \frac{2ra}{s_1 + pa}, \\ (s_1 + pa)(x - 1/2) + 1/2 + ra, & \text{for } \frac{1}{2} - \frac{2ra}{s_1 + pa} \leq x < 1/2; \\ -(s_2 + qa)(x - 1/2) + 1/2 + ra, & \text{for } 1/2 \leq x < \frac{1}{2} + \frac{2ra}{s_2 + qa}; \\ 1 + \frac{2(s_2 + qa)}{s_2 - 1 + qa - 2ra} (x - 1), & \text{for } \frac{1}{2} + \frac{2ra}{s_2 + qa} \leq x \leq 1. \end{cases}$$

For each choice of $s_1, s_2 > 1$, $p, q, r > 0$, we consider only $a > 0$ such that $0 \leq W_a(x) \leq 1$ for $x \in [0,1]$.

An example of a $W_a$ map is shown in Fig.2. Fig.2(a) is the unperturbed $W_0$ map with turning fixed point at $1/2$ and $s_1 = 3/2$, $s_2 = 3$. Fig.2(b) is the perturbed map $W_a$, with $a = 0.05$, $r = 2$, $p = 3$, $q = 2$. The slope of the second branch is
$s_1 + pa = 1.65$, the slope of the third branch is $s_2 + qa = 3.1$, and $W_{0.05}(1/2) = 1/2 + ra = 0.6$. 

Figure 1. The $W$-like maps with $1/s_1 + 1/s_2 = 1$: (a) $W_0$ with $s_1 = 3/2$ and $s_2 = 3$, (b) $W_a$ with $s_1 = 3/2$, $s_2 = 3$; $a = 0.05$; $r = 2$, $p = 3$, $q = 2$; also several initial points of the trajectory of $1/2$.

Every $W_a$ has a unique absolutely continuous invariant measure $\mu_a$ since all the slopes are greater than 1 in modulus. We will show later that, for $1/s_1 + 1/s_2 \leq 1$, $\mu_a$ is supported on $[0, 1]$ and for $1/s_1 + 1/s_2 > 1$ it is supported on a subinterval around $1/2$. $W_a$ is an exact map with the measure $\mu_a$. Let $h_a$ denote the normalized density of $\mu_a$, $a \geq 0$. Since the $W_0$ map is a Markov one, it is easy to check that

$$ \begin{align*}
(2) \quad h_0 &= \begin{cases} 
\frac{2s_1(s_2+1)}{2s_2(s_1-1)}, & \text{for } 0 \leq x < 1/2; \\
\frac{2s_1s_2^2}{2s_2(s_1-1)}, & \text{for } 1/2 \leq x \leq 1.
\end{cases}
\end{align*} $$

Our main result is the following theorem

**Theorem 1.** As $a \to 0$ the measures $\mu_a$ converge $*$-weakly to the measure

(I) $\delta(1/2)$, if $1/s_1 + 1/s_2 > 1$;

(II) $\mu_0 + \frac{2r s_1 s_2^2}{(q^2 s_1 + ps_2 - p - q)(s_2+2) + 2rs_1 s_2^2} \delta(1/2)$, if $1/s_1 + 1/s_2 = 1$;

(III) $\mu_0$, if $1/s_1 + 1/s_2 < 1$,

where $\delta(1/2)$ is the Dirac measure at point 1/2.

The proof relies on the general formula for invariant densities of piecewise linear maps and direct calculations. Most objects and quantities we use depend on the parameter $a$. We suppress $a$ from the notation to make it simpler.

In case (III), we actually prove a little more:

**Theorem 2.** If $1/s_1 + 1/s_2 < 1$, then the normalized invariant densities $\{h_a\}$ are uniformly bounded for given $p$, $q$ and $r$. Consequently, we obtain Theorem (I)(III).

3. Proofs

This section contains the proofs of Theorems 1 and 2 divided into a number of steps.
3.1. Assume $\frac{1}{s_1} + \frac{1}{s_2} > 1$. Let

$$x_l^* = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)}$$

and

$$x_r^* = \frac{s_2 s_1 - s_2 + (2rs_1 - q + ps_2 + qs_1)a + (2rp + pq)a^2}{2(s_1 - 1 + pa)(s_2 + qa)}.$$ 

$x_l^*$ is the fixed point on the second branch of $W_a$, and $x_r^*$ is the preimage of $x_l^*$ under the third branch of $W_a$. Both $x_r^*$ and $x_l^*$ converge to $\frac{1}{2}$ as $a$ approaches 0. For small $a$, we have

$$W_a(1/2) - x_r^* = \frac{ra[s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}{(s_1 - 1 + pa)(s_2 + qa)} < 0.$$ 

In this case, we have $W_a([x_l^*, x_r^*]) \subseteq [x_l^*, x_r^*]$. $W_a|_{[x_l^*, x_r^*]}$ is a skewed tent map with $W_a(1/2) > 1/2$; it is known that with acim $\mu_a$, it is exact on $[x_l^*, W_a(1/2)]$. Since $\mu_a$ is concentrated on $[x_l^*, x_r^*]$, we conclude that $\mu_a$ converge *-weakly to $\delta_{(1/2)}$. This proves Theorem 1(I).

Fig. 2 shows an example with $a = 0.05, r = 2, p = 3, q = 2; s_1 = 4/3, s_2 = 5/2.$

![Figure 2. The $W_a$ map with $\frac{1}{s_1} + \frac{1}{s_2} > 1$](image)

3.2. Formula for the non-normalized invariant density of $W_a$ if $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$. An example of a map $W_a$ is shown in Fig. 1. We have the following proposition.

**Proposition 1.** For $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$, the map $W_a$ has an absolutely continuous invariant measure $\mu_a$ supported on $[0, 1]$ and the map $W_a$ with respect to $\mu_a$ is exact.

**Proof.** $W_a$ is a piecewise expanding transformation. From the general theory (see for example [2]), it follows that it is enough to show that the images $W_a^n(J)$ grow to cover all $[0, 1]$ as $n \to \infty$, for any interval $J \subset [0, 1]$. Since $W_a$ is expanding, $W_a^n(J)$ grow until some image $W_a^{m_0}(J)$ contains an internal partition point. If this point is not 1/2, then $W_a^{m_0+2}(J)$ contains the repelling fixed point 1. Then its images grow
to cover all of $[0,1]$. If this point is $1/2$, we proceed as follows. First, assume that $\frac{1}{s_1} + \frac{1}{s_2} < 1$. Consider a small neighborhood $J = (z_1, z_2)$ around $1/2$ with length $\ell$, then
\[
\min_{z_2-z_1=\ell} \max \left\{ \frac{1}{2} - z_1 (s_1 + pa), (z_2 - \frac{1}{2})(s_2 + qa) \right\} = \frac{1}{s_1 + pa + s_2 + qa} \ell > \ell.
\]
Thus, the interval $J$ will grow until its image covers two partition points of $W_a$. Then the second iteration afterward will cover $[0,1]$. Therefore, $W_a$ is exact with respect of $\mu_a$.

Assume $\frac{1}{s_1} + \frac{1}{s_2} = 1$. If $a \neq 0$, then $\frac{1}{s_1 + pa + s_2 + qa} > 1$, which implies $W_a$ is exact with respect to $\mu_a$. In the case $a = 0$, we first note that $1/2$ is a turning fixed point. Take again a small interval $J = (z_1, z_2) \supseteq 1/2$. Its image is an interval $(z, 1/2)$. It will grow under iteration and its iterations still contain $1/2$. It will grow until its image covers another partition point of $W_a$. Then, the second iteration afterward will cover all of $[0,1]$. Thus, $W_a$ is again exact with respect to $\mu_a$. □

We adapt the general formulas of [5] to our case and obtain the following lemma:

**Lemma 1.** (I) $N=4$, $K=2$, $L=0$;

(II) $\alpha = (1,1/2 + ra, 1/2 + ra, 1)$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, where $\beta_1 = -\frac{2(s_1 + pa)}{s_1 - 1 + pa - 2ra}$, $\beta_2 = s_1 + pa$, $\beta_3 = -(s_2 + qa)$ and $\beta_4 = \frac{2(s_2 + qa)}{s_2 - 1 + qa - 2ra}$, $\gamma = (0,0,0,0)$;

(III) The digits $A = (a_1, a_2, a_3, a_4)$, where $a_1 = -1$, $a_2 = \frac{s_1 - 1 + pa - 2ra}{2}$, $a_3 = -\frac{s_2 + 1 + qa - 2ra}{2}$, $a_4 = \frac{s_2 + 1 + qa - 2ra}{s_1 - 1 + pa - 2ra}$;

(IV) There are two $c_i$’s, which are $c_1 = (1/2, 2)$ and $c_2 = (1/2, 3)$, and $j(c_1) = 2$, $j(c_2) = 3$. Then, $W_u = \{c_1, c_2\}, W_l = \emptyset$, $U_l = \{c_2\}, U_r = \{c_1\}$;

(V) $\beta(c_1, 1) = s_1 + pa$ since $j(c_1) = 2$, then $\beta(c_1, 2) = -(s_1 + pa)(s_2 + qa)$ and $\beta(c_1, k) = -(s_2 + qa)(s_1 + pa)^{k-1}$ up to some $k$ which is the first moment $j$ when the $W_d^j(1/2)$ is less than $\frac{1}{2} - \frac{1}{2 + ra}{s_1 + pa}$, and is the same one defined in Lemma[4];

(VI) $\beta(c_2, 1) = -(s_2 + qa)$ since $j(c_2) = 3$, then $\beta(c_2, 2) = (s_2 + qa)^2$ and $\beta(c_2, k) = (s_2 + qa)^2(s_1 + pa)^{k-2}$ up to the same $k$ in part (e), $W_d^a(c_1) = W_d^a(c_2)$ for all $n$;

(VII) Based on (VI), we have the following for the matrix $S = (S_{i,j})_{1 \leq i,j \leq 2}$:

For $c_1 \in U_r$,
\[
S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1, n) > 0))\delta(W_d^a(c_1) > 1/2) + \delta(\beta((c_1, n) < 0))\delta(W_d^a(c_1) < 1/2)}{|\beta(c_1, n)|},
\]
\[
S_{1,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1, n) > 0))\delta(W_d^a(c_1) > 1/2) + \delta(\beta((c_1, n) < 0))\delta(W_d^a(c_1) < 1/2)}{|\beta(c_1, n)|},
\]

For $c_2 \in U_l$
\[
S_{2,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_d^a(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_d^a(c_2) < 1/2)}{|\beta(c_2, n)|},
\]
\[
S_{2,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_d^a(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_d^a(c_2) < 1/2)}{|\beta(c_2, n)|}.
\]

**Remark 1.** It follows from (V, VI) of Lemma 1 that
\[
S_{1,1} = S_{1,2}, S_{2,1} = S_{2,2} \text{ and } S_{1,1} = \frac{s_2 + qa}{s_1 + pa} S_{2,2}.
\]
Let \( \text{Id} \) be the \( 2 \times 2 \) identity matrix and let \( V = [1, 1] \). Then, for the solution, \( D = [D_1, D_2] \), of the system:

\[
(-S^T + \text{Id}) D^T = V^T, \tag{1}
\]

we have \( D_1 = D_2 \). Let us denote them by \( \Lambda \).

Let \( I_1, I_2, I_3, I_4 \) be the partition of \( I = [0, 1] \) into maximal intervals of monotonicity of \( W_a \): \( I_1 = [0, \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa)}), I_2 = (\frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa)}, \frac{1}{2}), I_3 = (\frac{1}{2}, \frac{s_2 + 1 + qa + 2ra}{2(s_2 + qa)}) \) and \( I_4 = (\frac{s_2 + 1 + qa + 2ra}{2(s_2 + qa)}, 1] \). We define the following index function:

\[
j(x) = j \text { for } x \in I_j, j = 1, 2, 3, 4,
\]

and

\[
j(c_1) = 2, j(c_2) = 3.
\]

We define the cumulative slopes for iterates of points as follows:

\[
\beta(x, 1) = \beta_j(x), \quad \text{and} \quad \beta(x, n) = \beta(x, n - 1) \cdot \beta_j(W_a^{n-1}(x)), \quad n \geq 2.
\]

In particular, we have

\[
\beta(1/2, n) = (s_1 + pa) \cdot W_a'(W_a(1/2)) \cdot W_a'(W_a^2(1/2)) \cdots W_a'(W_a^{n-1}(1/2)),
\]

which is the cumulative slope along the \( n \) steps of trajectory of \( 1/2 \). Recall that \( k \) is the first moment \( j \) when the \( W_a^j(1/2) \) is less than \( \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa} \). Let \( k_1 = \lfloor \frac{2}{3} k \rfloor \) (the integer part of \( 2k/3 \)). Note that \( k_1 \rightarrow \infty \) as \( a \rightarrow 0 \). Let

\[
\chi^x(t, x) = \begin{cases} 
\chi_{[0, x]} & \text{for } t > 0 \\
\chi_{[x, 1]} & \text{for } t < 0.
\end{cases}
\]

Now, we can obtain the following formula for \( f_a \):

**Lemma 2.** Let

\[
f_a = 1 + (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda \left( \sum_{n=1}^{\infty} \frac{\chi^x(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right).
\]

Then \( f_a \) is \( W_a \) invariant non-normalized density. Furthermore, for small \( a > 0 \), we have:

(I) If \( \frac{1}{s_1} + \frac{1}{s_2} = 1 \), then \( \Lambda < -1 \);

(II) If \( \frac{1}{s_1} + \frac{1}{s_2} < 1 \), the sign of \( \Lambda \) depends on \( s_1 \) and \( s_2 \), can be either positive or negative depending on the sign of \( \vartheta = 1 - \left( \frac{s_1 + qa}{s_1 + s_2} + \frac{s_1 + qa}{s_1 + s_2 - 1} \right) = 1 - \frac{s_1 + qa}{s_1 + s_2} \left( 1 + \frac{s_1}{s_2(s_1 - 1)} \right) \).

The case when \( \vartheta = 0 \) is discussed at the end of Section II.

**Proof.** By the Theorem 2 in [3], it follows from \((IV, V, VI)\) of Lemma 1 that:

\[
f_a = 1 + D_1 \sum_{n=1}^{\infty} \frac{\chi^x(\beta(c_1, n), W_a^n(c_1))}{|\beta(c_1, n)|} + D_2 \sum_{n=1}^{\infty} \frac{\chi^x(-\beta(c_2, n), W_a^n(c_2))}{|\beta(c_2, n)|}
\]

\[
= 1 + \Lambda \sum_{n=1}^{\infty} \frac{\chi^x(\beta(c_1, n), W_a^n(1/2))}{|\beta(c_1, n)|} + \Lambda \sum_{n=1}^{\infty} \frac{\chi^x(-\beta(c_2, n), W_a^n(1/2))}{|\beta(c_2, n)|}
\]

\[
= 1 + (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda \left( \sum_{n=1}^{\infty} \frac{\chi^x(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right).
\]
Since
\[ S_{1,1} \geq \frac{1}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{n=1}^{k_1-1} \frac{1}{(s_1 + pa)^n} = \frac{1}{s_1 + pa} + \frac{1}{s_2 + qa} \frac{1 - \frac{1}{(s_1 + pa)^k}}{s_1 + pa - 1}, \]
and \( \Lambda = \frac{1}{s_2 + qa\Lambda_{1,1} - 1} \), we have
\[ \Lambda \leq \frac{1}{1 - (\kappa + \eta(\kappa + \eta(\kappa + \eta)))} \leq \Lambda \leq \Lambda_h, \]
where \( \kappa = \frac{s_1 + s_2 + pa + qa}{s_1 + s_2}, \eta = \frac{s_1 + s_2 + pa + qa}{s_1 + s_2 - 1} \).

To obtain the upper bound of \( S_{1,1} \), we assume \( s_1 < s_2 \). For \( s_1 > s_2 \) the calculations differ slightly.

(I) Note that for small \( a \) both estimates \( \Lambda_l \) and \( \Lambda_h \) are smaller than \( -1 \) since both \( \kappa \) and \( \eta \) are smaller than \( 1 \) and close to \( 1 \). Furthermore, as \( a \) approaches \( 0 \), both \( \kappa \) and \( \eta \) approach \( 1 \).

(II) As \( a \) approaches \( 0 \), \( \kappa \) and \( \eta \) approach \( \frac{s_1 + s_2}{s_1 s_2} \) and \( \frac{s_1 + s_2}{s_1 - 1} \), respectively. Again, note that for small \( a \), estimates \( \Lambda_l \) and \( \Lambda_h \) can be either positive or negative, and they have the same sign. \( \square \)

For small positive \( a \), the first image of \( 1/2 \) is \( W_a(1/2) = 1/2 + ra \) and the next one falls just below the fixed point \( x^* \) slightly less than \( 1/2 \). The following images form a decreasing sequence until they go below \( \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa} \). Since \( k \) is the first iteration \( j \) when the \( W_a^j(1/2) \) is less than \( \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa} \), the consecutive cumulative slopes of \( 1/2 \) are
\[ (s_1 + pa), -(s_1 + pa)(s_2 + qa), -(s_1 + pa)^2(s_2 + qa), \ldots, -(s_1 + pa)^{k-1}(s_2 + qa), \]
and
\[ f_a = 1 + (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda \left( \frac{\chi[0,W_a(1/2)]}{(s_1 + pa)} + \sum_{j=2}^{k} \frac{\chi[W_a^j(1/2),1]}{(s_1 + pa)^{j-1}(s_2 + qa)} + \ldots \right). \]

3.3. Estimates, normalizations and integrals on \( f_a \) for \( \frac{1}{s_1} + \frac{1}{s_2} \leq 1 \). Remembering that \( k = \min\{j \geq 1 : W_a^j(1/2) \leq \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}\} \) and \( k_1 = \lfloor \frac{2k}{3} \rfloor \) (the integer part of \( 2k/3 \)), we will give the estimates on \( f_a \).

Let us define
\[ g_l = \frac{\chi[0,W_a(1/2)]}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi[W_a^j(1/2),1]}{(s_1 + pa)^{j-1}}, \]
and
\[ g_h = g_l + \frac{1}{s_2 + qa} \sum_{j=0}^{\infty} \frac{1}{(s_1 + pa)^{2+j}} = g_l + \frac{1}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1-1}}. \]

Also, let \( \chi_1 = \chi[0,1/2+ra], \chi_j = \chi[W_a^j(1/2),1/2+ra] \), \( j = 2, 3, \ldots, k_1 \), \( \chi_c = \chi(1/2+ra,1) \).
3.3.1. Estimates on $f_a$ if $\frac{a}{s_1} + \frac{1}{s_2} = 1$. We have the following lemma:

**Lemma 3.** For the family of $W_a$ maps, if $\frac{a}{s_1} + \frac{1}{s_2} = 1$, we have

(I) $W_a(1/2) = 1/2 + ra$, $W_a^2(1/2) = -ra(s_2 + qa) + 1/2 + ra$, and for $3 \leq m \leq k$,
we have $W_a^m(1/2) = -a^2(s_1 + pa)^m - r(s_1 + pa - p - q + rpqa) + \frac{s_1 - 1 + pa - 2ra}{s_1 + pa - 1};$

(II) $\lim_{a \to 0} ak = 0$;

(III) $\lim_{a \to 0} \frac{1}{a(s_1 + pa)^k} = 0$;

(IV) $\lim_{a \to 0} \frac{1}{a(s_1 + pa)} = 0$;

(V) $\lim_{a \to 0} a^2(s_1 + pa)^k = 0$;

(VI) $\lim_{a \to 0} W_a^k(\frac{1}{2}) = \frac{1}{2}$.

**Proof.** Suppose (I) is true. Let us first prove that (II) and (III) are true.

By the definition of $k$, we have:

$$0 \leq -a^2(s_1 + pa)^k - r(qs_1 + ps_2 - p - q + rpqa) + \frac{s_1 - 1 + pa - 2ra}{s_1 + pa} \leq \frac{1}{2} - \frac{1}{4s_1 + pa},$$

The first inequality of (2) implies that $(s_1 + pa)^k - 2a^2(r(qs_1 + ps_2 - p - q + rpqa),$ thus

$$ak \leq a \frac{\ln(s_1 - 1 + pa - 2ra) - \ln 2 - 2 \ln a - \ln(r(qs_1 + ps_2 - p - q + rpqa) + 2a, \ln(s_1 + pa)}{a(s_1 + pa)^k} \leq \frac{2a(r(qs_1 + ps_2 - p - q + rpqa)}{s_1 - 1 + pa - 2ra}.$$

Therefore,

$$\frac{1}{a(s_1 + pa)} \leq \frac{2a(r(qs_1 + ps_2 - p - q + rpqa)}{s_1 - 1 + pa - 2ra}.$$

and as $a \to 0$, we obtain (III).

On the other hand, (2) implies

$$\frac{1}{a(s_1 + pa)^k} \leq \frac{2a(r(qs_1 + ps_2 - p - q + rpqa)}{s_1 - 1 + pa - 2ra}.$$
The fixed point slightly less than 1/2 is 
\[ x_t^* = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)} \]
and
\[ x_t^* - W^2_a(1/2) = \frac{ra^2(q(s_1 - 1) + p(s_2 - 1) + apq)}{s_1 - 1 + pa} > 0, \]
which implies that 
\[ W^m_a(1/2) \]
are all in the domain of the second branch of 
\[ W_a \]
for \( 3 \leq m \leq k \). For a linear map 
\[ T(x) = m_0x + b_0, \]
we have 
\[ T^m(x) = m_0^m x + \frac{m_0^m - 1}{m_0 - 1}b_0. \]
This proves (I).

Using (4) and (3), we see that for the functions 
\[ f_l = 1 + \left(1 + \frac{s_1 + pa}{s_2 + qa}\right)\Lambda_l g_h \]
and
\[ f_h = 1 + \left(1 + \frac{s_1 + pa}{s_2 + qa}\right)\Lambda_h g_1, \]
we have
\[ f_l \leq f_o \leq f_h. \]

Now, we will represent functions 
\[ f_l \]
and 
\[ f_c \]
as combinations of functions 
\[ \chi_j, \]
\( j = 1, \ldots, k \)
and 
\[ \chi_c. \]
After some calculations, we obtain
\[ f_l = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_l \left(\frac{\chi[0,W_a(1/2)]}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi[W_a^2(1/2),1]}{(s_1 + pa)^{j-1}}\right) \]
\[ \quad + \frac{1}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1-1}} \]
\[ \quad \left(\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)}\Lambda_l + 1\right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_l \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}} \]
\[ \quad + \left(\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_l 1 - \frac{1}{s_1 + pa - 1} + 1\right) \chi_c \]
\[ \quad + \frac{s_1 + s_2 + pa + qa}{s_2 + qa} \Lambda_l \]
\[ \quad + \frac{1}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1-1}}. \]

\[ f_h = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_h \left(\frac{\chi[0,W_a(1/2)]}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi[W_a^2(1/2),1]}{(s_1 + pa)^{j-1}}\right) \]
\[ \quad \left(\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)}\Lambda_h + 1\right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}} \]
\[ \quad + \left(\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h 1 - \frac{1}{s_1 + pa - 1} + 1\right) \chi_c. \]

In the case we are considering, (3) implies that both \( \Lambda_l, \Lambda_h \) are smaller than -1. Using this, one can show that all the coefficients in the representation of \( f_l \) and \( f_h \) are negative for sufficiently small \( a \). For example, let us consider the coefficient of \( \chi_1 \) in \( f_h \):
\[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)}\Lambda_h + 1 = \frac{\kappa}{1 - (\kappa + \eta)} + 1 = \frac{1 - \eta}{1 - (\kappa + \eta)} < 0. \]
3.3.2. Normalizations and integrals if \( \frac{s_1}{s_2} + \frac{1}{s_2} = 1 \). Let us define \( J_1 = [0, W^{k_1}(1/2)] \), \( J_2 = [W^{k_1}(1/2), 1/2 + ra] \), \( J_3 = (1/2 + ra, 1] \). We will calculate integrals of \( f_h \) over each of these intervals \( J_1 \), \( J_2 \) and \( J_3 \), and use them to normalize \( f_h \). We have

\[
C_1 = \int_{J_1} f_h \, d\lambda = \int_{J_1} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, d\lambda
\]

\[
= \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] W^{k_1}(1/2) = \left[ \frac{\kappa}{1 - (\kappa + \eta)} + 1 \right] W^{k_1}(1/2)
\]

\[
= \left[ \frac{a(2qs_1s_2 + ps^2 - 2qs_2 - p - q)}{(1 - (\kappa + \eta))(s_2 + qa)^2(s_1 + pa - 1)} \right] W^{k_1}(1/2).
\]

Using Lemma 3, we obtain

\[
\lim_{a \to 0} \frac{C_1}{a} = - \frac{2qs_1s_2 + ps^2 - 2qs_2 - p - q}{2s_2(s_1 - 1)} = - \frac{2qs_1 + ps^2 - p - q}{2s_2s_1}.
\]

In the same way, we can see that for any \( 0 < \theta < 1/2 \), we obtain

\[
\lim_{a \to 0} \frac{1}{a} \int_0^\theta f_h d\lambda = - \frac{2qs_1 + ps^2 - p - q}{s_2s_1} \theta.
\]

On the interval \( J_2 \), the integral of \( f_h \) is:

\[
C_2 = \int_{J_2} f_h \, d\lambda = \int_{J_2} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, d\lambda
\]

\[
+ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \int_{J_2} \frac{\chi_j}{(s_1 + a)^{j-1}} \, d\lambda
\]

\[
= \left[ 1 - \frac{\eta}{1 - (\kappa + \eta)} \right] \left( \frac{1}{2} + ra - W^{k_1}(1/2) \right)
\]

\[
+ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \left[ \frac{ra(s_2 + qa)}{s_1 + pa} + \frac{ra(1 - \frac{1}{(s_1 + pa)^{k_1 - 1}})}{(s_1 + pa - 1)^2} \right]
\]

\[
+ \frac{a^2(k_1 - 2)}{s_1 + pa - 1} r(qs_1 + ps_2 - p - q) + rpqa.
\]

Using Lemma 3, we obtain

\[
\lim_{a \to 0} \frac{C_2}{a} = - \frac{s_1 + s_2}{s_2} \left[ \frac{rs_2}{s_1} + \frac{r}{(s_1 - 1)^2} \right] = -rs_2.
\]

On the interval \( J_3 \), the integral of \( f_h \) is:

\[
C_3 = \int_{J_3} f_h \, d\lambda = \int_{J_3} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \left[ 1 - \frac{1}{(s_1 + pa)^{k_1 - 1}} \right] \chi_c \, d\lambda
\]

\[
= \left[ \left( 1 - \frac{1}{(s_1 + pa)^{k_1 - 1}} \right) \left[ \frac{\eta}{1 - (\kappa + \eta)} + 1 \right] \left( \frac{1}{2} - ra \right) \right]
\]

\[
= \left[ \frac{a(qs_1 + ps_2 - p - q) + qa}{(s_1 + pa)(s_2 + qa)} \right] \left( \frac{1}{1 - (\kappa + \eta)} \left( \frac{1}{2} - ra \right) \right).
\]
Lemma 4. For the family of weakly to the measure

\[ \lim_{a \to 0} \frac{C_3}{a} = \frac{qs_1 + ps_2 - p - q}{2s_1s_2}. \]

In the same way, we can see that for any \( 0 < \theta < 1/2 \), we obtain

\[ \lim_{a \to 0} \frac{1}{a} \int_{1/2+\theta}^1 f_h d\lambda = \frac{qs_1 + ps_2 - p - q}{s_1s_2} \left( \frac{1}{2} - \theta \right). \]

If we define \( B = C_1 + C_2 + C_3 \), then \( \frac{B}{a} \) is a normalized density. We see that

\[ \lim_{a \to 0} \frac{B}{a} = \frac{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2}{2s_1s_2}. \]

Our calculations show that the normalized measures \( \{(f_h/B) \cdot \lambda\} \) converge \(*\)-weakly to the measure

\[ \frac{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2}s_1s_2^2 \delta_{\frac{1}{2}}. \]

Now, we will show the same holds for the normalized measure defined by \( f_1 \). To this end, let us notice that

\[ f_h - f_1 = (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_h g_1 - (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_l g_h \]

\[ = (1 + \frac{s_1 + pa}{s_2 + qa})(\Lambda_h - \Lambda_l)g_1 - \Lambda_l (s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1-1} \]

\[ = (1 + \frac{s_1 + pa}{s_2 + qa})\left[ 1 - (\kappa + \eta)(1 - (\kappa + \eta)(1 - \frac{1}{(s_1 + pa)^{k_1-1}})) \right] g_1 \]

\[ - \Lambda_l (s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1-1}, \]

where \( |g_1| \leq \frac{2}{s_1} \) and \( \lim_{a \to 0} \Lambda_l = -1 \). Using Lemma 3 once again, we can show that for any subinterval \( J \subset [0,1] \), we have

\[ \lim_{a \to 0} \frac{1}{a} \int_J (f_h - f_1) d\lambda = 0. \]

For \( J = [0,1] \) this means that the normalizations of \( f_1 \) and \( f_h \) are asymptotically the same. With this, the limit for a general \( J \) means in particular that the \(*\)-weak limit of normalized measures defined using \( f_1 \) is the same as for those defined using \( f_h \). In view of inequality (7), this proves Theorem (II).

3.3.3. Estimates on \( f_a \) if \( \frac{1}{s_1} + \frac{1}{s_2} < 1 \). We have the following lemma:

**Lemma 4.** For the family of \( W_a \) maps, if \( \frac{1}{s_1} + \frac{1}{s_2} < 1 \), we have

(I) \( W_a(1/2) = 1/2 + ra \), \( W_a^2(1/2) = -ra(s_2 + qa) + 1/2 + ra \), and for \( 3 \leq m \leq k \), we have \( W_a^m(1/2) = -a(s_1 + pa)^{m-2}[s_1s_2 - s_1s_2 + a(qs_1 + ps_2 - p - q)] + a^{m-1} + pa - 2ra \); \( \lambda \)

(II) \( \lim_{a \to 0} a_k = 0 \); \( \lambda \)

(III) \( \lim_{a \to 0} a(s_1 + pa)^{k_1} = 0 \);

(IV) \( \lim_{a \to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{4} \).
Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.
By the definition of \( k \), we have:

\[
0 \leq -a(s_1 + pa)^{k-2} \frac{r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}{s_1 + pa - 1} + \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa - 1)}.
\] (8)

The inequality (8) implies \( a(s_1 + pa)^{k-2} \leq \frac{r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}{2r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}, \) thus

\[
ak \leq a \frac{\ln(s_1 - 1 + pa - 2ra) - \ln 2 + 2 \ln(s_1 + pa) - \ln r - \ln a}{\ln(s_1 + pa)} - a \frac{\ln(2r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)])}{\ln(s_1 + pa)},
\]

\[
a(s_1 + pa)^{k_1} \leq \frac{(s_1 - 1 + pa - 2ra)(s_1 + pa)^2}{2r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)](s_1 + pa)^{k - k_1}},
\]

and since \( \lim_{a \to 0} a \ln a = 0 \), we obtain (II) and (III). (IV) follows from (III).

Now, let us prove (I).

The fixed point slightly less than 1/2 is \( x'_i = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)} \), and

\[
x'_i - W^2_a(1/2) = \frac{ra[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}{s_1 - 1 + pa} > 0,
\]

which implies that \( W^m_a(1/2) \) are all in the domain of the second branch of \( W_a \) for \( 3 \leq m \leq k \). Now, (I) follows by the same reasoning as in Lemma 4. \( \square \)

Lemma 5. If the normalized densities \( \{h_a\}_{a < a_0} \), for some \( a_0 > 0 \), are uniformly bounded, then \( h_a \to h_0 \) in \( L^1 \).

Proof. The uniform boundedness implies \( \{h_a\}_{a < a_0} \) is a weakly precompact set in \( L^1 \). Thus, any limit of \( \{h_a\}_{a < a_0} \) is an invariant density by Proposition 11.3.1 [2]. At the same time, this limit is an \( L^1 \) function, thus defines an absolutely continuous invariant measure. Since the map \( W_0 \) is exact and has only one acim, we conclude that \( h_a \to h_0 \) in \( L^1 \). \( \square \)

Now, we will prove Theorem 2.

The main idea of the proof is the following: since non-normalized densities \( \{f_a\} \) are uniformly bounded (formulas (9), (10), (11)), it is enough to show that \( \{f^i_0 f_a d\lambda\} \) are uniformly separated from zero.

For small \( a \), by Lemma 2 \( \Lambda \) (and then both \( \Lambda_f \) and \( \Lambda_h \)) can be either positive or negative. Thus, we can have the following cases.

Case (i): \( \Lambda_l < 0 \):

Comparing with (1) and (3), we see that for the functions \( \hat{f}_l = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_l g_l \) and \( \hat{f}_h = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_h g_l \), we have

\[
\hat{f}_l \leq f_a \leq \hat{f}_h.
\] (9)

Note that \( \hat{f}_l \) and \( \hat{f}_h \) have the same form as \( f_l \) and \( f_h \) in Section 3.3.1 so their representations as combinations of functions \( \chi_j, j = 1, \ldots, k_1 \) and \( \chi_c \) are similar to
that of \( f_1 \) and \( f_h \). At the same time, now we have \( \frac{1}{s_1} + \frac{1}{s_2} < 1 \), so the representation is as follows:

\[
\hat{f}_1 = \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_1 + 1 \right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_1 \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}}
\]

\[
+ \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_1 \frac{1 - \frac{1}{(s_1 + pa)^{j+1}}}{s_1 + pa - 1} + 1 \right) \chi_c
\]

\[
+ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1-1}}
\]

\[
\hat{f}_h = \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}}
\]

\[
+ \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \frac{1 - \frac{1}{(s_1 + pa)^{j+1}}}{s_1 + pa - 1} + 1 \right) \chi_c.
\]

(3) implies that all the coefficients in the representation of \( \hat{f}_1 \) and \( \hat{f}_h \) are negative for sufficiently small \( a \).

We use the same notations \( J_1, J_2 \) and \( J_3 \) as in Section 3.3.2. First, we do the calculations assuming that \( \vartheta = 1 - \left( \frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)} \right) \neq 0 \).

We will calculate the integrals of \( \hat{f}_h \) over each of \( J_1, J_2 \) and \( J_3 \), and use them to normalize \( \hat{f}_h \). We have

\[
\hat{C}_1 = \int_{J_1} \hat{f}_h \, d\lambda = \int_{J_1} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, d\lambda
\]

\[
= \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] W^{k_1} \left( \frac{1}{2} \right) \frac{\chi}{1 - (\kappa + \eta) + 1} \left[ \frac{k}{1 - (\kappa + \eta) + 1} \right] W^{k_1} \left( \frac{1}{2} \right)
\]

\[
= \left[ \frac{s_1 s_2 - s_1 - s_2 - s_2^2}{1 - (\kappa + \eta)(s_2 + qa)^2(s_1 + pa - 1)} + a(2qs_1s_2 + ps^2_2 - 2qs_2 - p - q) + \frac{2pqs_2 - q^2 + q^2 s_1 + pq^2 a^2}{1 - (\kappa + \eta)(s_2 + qa)^2(s_1 + pa - 1)} \right] W^{k_1} \left( \frac{1}{2} \right)
\]

Using Lemma 3.1 we have

\[
\lim_{a \to 0} \hat{C}_1 = \left[ \frac{1}{2} \left( \frac{s_1 s_2 - s_1 - s_2 - s_2^2}{s_2^2(s_1 - 1)} \right) \right] = \left[ \frac{1}{2} \left( \frac{1 - s_1 + s_2}{s_2^2(s_1 - 1)} \right) \right]
\]

On the interval \( J_2 \), the integral of \( \hat{f}_h \) is:

\[
\hat{C}_2 = \int_{J_2} \hat{f}_h \, d\lambda = \int_{J_2} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, d\lambda
\]

\[
+ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \int_{J_2} \frac{\chi_j}{(s_1 + pa)^{j-1}} \, d\lambda
\]
Using Lemma 4 once again, we have
\[
\frac{1}{1 - (\kappa + \eta)} \left( \frac{1}{2} + ra - W_{\hat{a}}^{k_1} \left( \frac{1}{2} \right) \right)
\]
where
\[
s_1 + s_2 + pa + qa = \Lambda h \left[ ra(s_2 + qa) + \frac{ra(1 - \frac{1}{(s_1 + pa)^{k_1 - 1}})}{s_1 + pa} \right]
\]
and
\[
a(k_1 - 2) r(s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa))
\]
\[
\frac{a(k_1 - 2) r(s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa))}{s_1 + pa - 1}.
\]
Using Lemma 4 we have \( \lim_{a \to 0} \hat{C}_2 = 0. \)

On the interval \( J_3 \), the integral of \( \hat{f}_h \) is:
\[
\hat{C}_3 = \int_{J_3} \hat{f}_h \, d\lambda = \int_{J_3} \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda h \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + 1 \right) \chi_c \, d\lambda
\]
\[
= \left[ \left( 1 - \frac{1}{(s_1 + pa)^{k_1 - 1}} \right) \frac{s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)}{(s_1 + pa)(s_2 + qa)} \right] \frac{\eta}{(s_1 + pa)^{k_1 - 1} - 1} \frac{1}{1 - (\kappa + \eta)} \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} .
\]

Using Lemma 4 once again, we have
\[
\lim_{a \to 0} \hat{C}_3 = \frac{1}{2} \left( 1 - \frac{\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2 s_1}}{(s_1 + pa)^{k_1 - 1}} \right).
\]

Note that if we define \( \hat{B} = \hat{C}_1 + \hat{C}_2 + \hat{C}_3 \), then
\[
\lim_{a \to 0} \hat{B} = \frac{1}{2} \left( 1 - \frac{\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2 s_1}}{(s_1 + pa)^{k_1 - 1}} \right),
\]
which is not 0. Since \( \{ \hat{f}_h \} \) are uniformly bounded, we conclude that the normalized \( \{ \hat{f}_h \} \) are also uniformly bounded.

Now, we will show that the normalized \( \{ \hat{f}_i \} \) are also uniformly bounded. To this end, let us notice that
\[
\hat{f}_h - \hat{f}_i = (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda h g_l - (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda_l g_h
\]
\[
= (1 + \frac{s_1 + pa}{s_2 + qa}) (\Lambda h - \Lambda_l) g_l - \Lambda_l \left( \frac{s_2 + qa}{s_1 + pa} \right) (s_1 + pa - 1)(s_1 + pa)^{k_1 - 1}
\]
\[
= (1 + \frac{s_1 + pa}{s_2 + qa}) \left[ \frac{1}{1 - (\kappa + \eta)} \left[ 1 - \frac{1}{(s_1 + pa)^{k_1 - 1}} \right] \right] g_l
\]
\[
\quad - \Lambda_l \left( \frac{s_2 + qa}{s_1 + pa} \right) (s_1 + pa - 1)(s_1 + pa)^{k_1 - 1},
\]
where \( |g_l| \leq \frac{1}{s_2(s_1 - 1)} \) and \( \lim_{a \to 0} \Lambda_l = \frac{1}{1 - \frac{\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2 s_1}}{(s_1 + pa)^{k_1 - 1}}} \). Thus, \( \lim_{a \to 0} \hat{f}_h - \hat{f}_i = 0 \).

We conclude that the normalized \( \{ \hat{f}_i \} \) are uniformly bounded since the normalized \( \{ \hat{f}_h \} \) are uniformly bounded. Thus, after normalization, \( \{ f_a \} \) are also uniformly bounded.
Case (ii): $\Lambda_l > 0$

This case implies that $f_a$ given by (4) has the following properties:

\[ f_a \geq 1, \]

and all the coefficients of the characteristic functions appearing in (4) are positive. We note that $\Lambda$ is always positive for small $a$. Thus,

\[ f_a \leq 1 + (1 + \frac{s_1 + pa}{s_2 + qa})^2 \Lambda \sum_{n=1}^{\infty} \frac{1}{|\beta(1/2, n)|}, \]

which is finite since our maps $\{W_a\}$ are expanding. In view of (10), we conclude that the normalized $\{f_a\}$ are uniformly bounded.

If $\vartheta = 1 - \frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2(s_1 - 1)} = 0$, then we have $\lim_{a \to 0} \frac{1}{\Lambda_0} = \lim_{a \to 0} \frac{1}{\Lambda_0} = 0$, $\Lambda_l$ and $\Lambda_h$ are still of the same sign. We can renormalize $f_a$. Let us take the $\hat{f}_h$ as an example. Multiplying it by $\frac{1}{\Lambda_h}$, we obtain

\[ \frac{1}{\Lambda_h} \hat{f}_h = \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} + \frac{1}{\Lambda_h} \right) \chi_1 + \frac{1 - \frac{1}{(s_1 + pa)^2}}{s_1 + pa - 1} + \frac{1}{\Lambda_h} \chi_c. \]

Note that the coefficients of $\chi_1$ and $\chi_c$ converge to $\frac{s_1 + s_2}{s_1 s_2}$ and $\frac{s_1 + s_2}{s_2(s_1 - 1)}$, respectively. Thus, $\{\int_0^1 \frac{1}{\Lambda_h} \hat{f}_h \, d\lambda\}$ are separated from 0. This implies $\{\frac{1}{\Lambda_h} \hat{f}_h\}$ are uniformly bounded. A similar procedure can be applied to $\hat{f}_l$. We conclude that $\{\frac{1}{\Lambda} f_a\}$ are uniformly bounded.

4. Example

One of the important properties of a piecewise expanding transformation of an interval is that its invariant density is bounded away from 0 on its support. The following result was proved, by Keller [11] and by Kowalski [12].

Theorem 3. Let a transformation $\tau : I \to I$ be piecewise expanding with $\frac{1}{|\tau'(x)|}$ a function of bounded variation, and let $f$ be a $\tau$-invariant density which can be assumed to be lower semicontinuous. Then there exists a constant $c > 0$ such that $f|_{\text{supp } f} > c$.

We provide an example showing that this result cannot be generalized to a family of expanding maps, even if they all have this property and converge to a limit map also with this property. Let $d(\cdot, \cdot)$ be the metric on the weak topology of measures.

Example 1. Let us fix

\[ s_1 = s_2 = 2, \quad p = q = 1. \]

For small $a > 0$, let $W_{a,r}$ denote the $W_a$ maps with varying parameter $r$, and let $\mu_{a,r}$ denote the absolutely continuous invariant measure of $W_{a,r}$. We know that $\mu_{a,r}$ is supported on $[0, 1]$ and $W_{a,r}$ with $\mu_{a,r}$ is exact. Using Theorem 7, we know that $\{\mu_{a,r}\}$ converge $\ast$-weakly to the measure

\[ \mu_{0,r} = \frac{1}{1 + 2r} \mu_0 + \frac{2r}{1 + 2r} \delta_{\frac{1}{2}}. \]
Let \( r_n = n \), \( n = 1, 2, 3, \cdots \). Also, let \( \{a_n\}_{1}^{\infty} \) satisfy \( r_n a_n < 1/2 \) and be so small that
\[
d(\mu_{a_n,r_n}, \mu_{0,r_n}) < \frac{1}{n}.
\]
Now, for the family of maps \( \tau_n = W_{a_n,r_n}, n = 1, 2, 3, \cdots \), \( \tau_n \) converge to \( W_0 \) with \( |\tau_n'(x)| > 2 \), but the invariant densities \( \mu_{a_n,r_n} \) converge to \( \delta_{\frac{1}{2}} \). This implies that the invariant densities \( \{f_{a_n,r_n}\} \) corresponding to \( \{\mu_{a_n,r_n}\} \) have no uniform positive lower bound.

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