On an integral representation of the normalized trace of the $k$-th symmetric tensor power of matrices and some applications

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ABSTRACT

Let $A$ be an $n \times n$ matrix and let $\bigwedge^k A$ be its $k$-th symmetric tensor power. We express the normalized trace of $\bigwedge^k A$ as an integral of the $k$-th powers of the numerical values of $A$ over the unit sphere $S^n$ of $\mathbb{C}^n$ with respect to the rotation-invariant probability measure. Equivalently, this expression in turn can be interpreted as an integral representation for the (normalized) complete symmetric polynomials over $\mathbb{C}^n$. As applications, we present a new proof for the MacMahon Master Theorem in enumerative combinatorics. Then, our next application deals with a generalization of the work of Cuttler et al. in [Cuttler A, Greene C, Skandera M. Inequalities for symmetric means. Eur J Comb. 2011;32(6):745–761] concerning the monotonicity of products of complete symmetric polynomials. Finally, we give a solution to an open problem that was raised by Roventa and Temereanca in [Roventa I, Temereanca LE. A note on the positivity of the even degree complete homogeneous symmetric polynomials. Mediterr J Math. 2019;16(1):1–16].

ARTICLE HISTORY

Received 8 June 2021
Accepted 15 February 2022

COMMUNICATED BY

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KEYWORDS

Integral formula; trace of symmetric powers; homogeneous polynomials

1. Introduction

Given an $n \times n$ complex matrix $A$ in the space of all complex matrices $M_n(\mathbb{C})$, we denote its trace by $\text{tr}(A)$ and we shall write $\text{Tr}(A) := \frac{\text{tr}(A)}{n}$ for the normalized trace. It is well known [1,2] that $\text{Tr}(A)$ can be obtained via the following functional representation:

$$\int_{S^n} < A\xi, \xi > \, d\sigma(\xi) = \text{Tr}(A),$$

where $S^n$ is the unit sphere of $\mathbb{C}^n$, $\sigma$ is the probability Lebesgue measure on $S^n$, and $< \cdot, \cdot >$ is the standard inner product on $\mathbb{C}^n$. The above formula states that the normalized trace of a matrix is the expectation of the numerical values of the matrix computed with respect to $d\sigma$. Noting the linearity on both sides of Equation (1), one obtains the above formula by checking it on matrix bases. When linearity is lost due to some perturbation of the numerical values, new techniques are needed to obtain a reasonable formula. Our main objective in this paper is to find the expectation of natural powers of the numerical values...
computed with respect to $d\sigma$. More precisely, given $k \in \mathbb{N}$ we aim to find an expression for
\[ \int_{\mathbb{S}^n} (\langle A\xi, \xi \rangle)^k d\sigma(\xi). \] (2)

One of our motivations for studying the above integral is detailed in the following subsection.

1.1. Connection to Szegö projection

The preceding integral is of special interest notably in the analysis of some reproducing kernel Hilbert spaces of analytic functions. First, let us recall that a reproducing kernel space is a Hilbert space of functions in which the corresponding point evaluation map over the functions is a linear continuous functional.

One of the well-known reproducing kernel Hilbert spaces is the Hardy space $H^2(\mathbb{S}^n)$ over the unit ball $B(0,1)$ of $\mathbb{C}^n$. This space is defined to be the closure in $L^2(\mathbb{S}^n)$ of all continuous functions on $\mathbb{S}^n$ having a holomorphic extension to $B(0,1)$. The orthogonal projection $P_\mathcal{S}$ of $L^2(\mathbb{S}^n)$ onto $H^2(\mathbb{S}^n)$ is called the Szegö projection. If $f \in L^2(\mathbb{S}^n)$, then the extension of $P_\mathcal{S}f$ to $B(0,1)$ is given by (cf. Section 1.5 in [3])
\[ P_\mathcal{S}f(z) = \int_{\mathbb{S}^n} \frac{f(\xi)}{(1-\langle v, \xi \rangle)^n} d\sigma(\xi) \quad \text{for } v \in B(0,1). \] (3)

This integral representation for $P_\mathcal{S}$ is due to a classical series representation for the reproducing kernel in terms of an orthonormal basis (cf. [3,4]). Note that for a fixed $v \in B(0,1)$, the above integral is expressed as an infinite series of integral (scalar valued) operators of the form
\[ \epsilon_{n,k} \int_{\mathbb{S}^n} (\langle v, \xi \rangle)^k f(\xi) d\sigma(\xi), \]
where $\epsilon_{n,k} = \binom{n+k-1}{k}$. Thus, it is natural to refer to such operators as the ‘building blocks’ for the Szegö projection. In order to clarify the connection of such analysis to our work, we consider a collection of integral transforms $\{T_{A,k}\}_{A \in M_n(\mathbb{C}), k \in \mathbb{N}}$ on $L^2(\mathbb{S}^n)$ defined by
\[ T_{A,k}f(\mu) = \int_{\mathbb{S}^n} \langle A\mu, \xi \rangle^k f(\xi) d\sigma(\xi) \quad \text{for } f \in L^2(\mathbb{S}^n), \quad \mu \in \mathbb{S}^n. \] (4)

Let $e_1 = (1,0,\ldots,0)^T \in \mathbb{C}^n$, then $\{T_{A,k}(e_1)\}_{A \in M_n(\mathbb{C})}$ provides a generalization for the ‘building blocks’ of $P_\mathcal{S}$. Indeed, given a non-zero $v \in B(0,1)$ consider $A = \|v\|_2 U$ where $\|\cdot\|_2$ is the Euclidean norm and $U$ is a unitary matrix satisfying $Ue_1 = \frac{v}{\|v\|_2}$, then
\[ T_{A,k}(e_1) = \int_{\mathbb{S}^n} (\langle v, \xi \rangle)^k f(\xi) d\sigma(\xi). \]

It is easy to see that $T_{A,k}$ is a finite rank operator and apparently the permutation in the $\mu$-coordinate might not allow us to get a canonical decomposition of $T_{A,k}$ for general $A$. This in turn leads to difficulties in computing the trace of $T_{A,k}$. Alternatively, as the kernel $(\langle A\mu, \xi \rangle)^k$ is continuous in the $\mu$ and $\xi$ variables, then the trace of $T_{A,k}$ is given by
Equation (2). A similar argument holds for the Hilbert–Schmidt norm \( \| \cdot \|_2 \) of \( T_{A,k} \), and we have the following identity:
\[
\| T_{A,k} \|_2^2 = \left( \int_{S^n} \| A \xi \|_2^2 d\sigma(\xi) \right)^{1/2k}.
\] (5)

1.2. Preliminaries

First, we recall the definitions of the symmetric tensor product and elementary symmetric polynomials. For \( k \in \mathbb{N} \) the symmetric tensor product of \( \mathbb{C}^n \), denoted by \( \vee^k \mathbb{C}^n \), is the subspace of \( \otimes^k \mathbb{C}^n \) spanned by all elementary symmetric tensor products \( \vee^k \mathbb{C}^n \) of vectors \( x_i \in \mathbb{C}^n \). Given \( A \in M_n(\mathbb{C}) \), we denote the \( k \)-th symmetric power of \( A \) by \( \vee^k A \), i.e. \( \vee^k A \) can be viewed as a linear operator \( \vee^k A : \vee^k \mathbb{C}^n \rightarrow \vee^k \mathbb{C}^n \) which can be given by
\[
\vee^k A(x_1 \vee x_2 \vee \cdots \vee x_k) := Ax_1 \vee Ax_2 \vee \cdots \vee Ax_k.
\] (6)

The \( k \)-th complete homogeneous symmetric polynomial in \( n \)-variables, \( h_k \), is defined by
\[
h_k(z) = \sum_{|\alpha|=k, \alpha \in \mathbb{N}^n} z^\alpha, \quad z \in \mathbb{C}^n.
\]

If the spectrum \( sp(A) = \{ \lambda_i \mid i = 1, \ldots, n \} \), then we denote the (un-ordered) \( n \)-tuple \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \) by \( \lambda(A) \) or simply by \( \lambda \). Using Equation (6) it is easy to check that \( sp(\vee^k A) = \{ \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n \} \). This leads to the well-known formula for the trace of \( \vee^k A \) in terms of \( \lambda_1, \lambda_2, \ldots, \lambda_n \):
\[
tr(\vee^k A) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} = h_k(\lambda).
\] (7)

The normalized complete homogeneous symmetric polynomial function on \( \mathbb{C}^n \) is denoted by \( H_k \), i.e.
\[
H_k(z) = \frac{h_k(z)}{c_{n,k}}.
\] (8)

Next, we will follow the standard notation used in the study of symmetric polynomials and the theory of majorization. We note that the majorization order which is given below is typically known as the ‘dominance order’ in combinatorics. Given \( m \)-tuples \( \lambda, \mu \in \mathbb{R}_+^m \), we say that \( \lambda \) is majorized by \( \mu \) and we will write \( \lambda \preceq \mu \) if
\[
\sum_{i=1}^j \lambda[i] \leq \sum_{i=1}^j \mu[i], \quad j = 1 \cdots n - 1 \quad \text{and} \quad |\lambda| = |\mu|,
\]
where \( x[i] \) is the \( i \)-th component obtained from \( x = (x_1, \ldots, x_n) \) after arranging its components in decreasing order. In the case where \( \mu \in \mathbb{R}_+^n \) and \( n < m \), then we shall complete the components of \( \mu \) to an \( m \)-tuple by adding zeros so that the preceding notion is still
well defined. If $\lambda$ and $\mu$ are partitions of (possibly different) natural integers, then we shall write

$$\lambda \subseteq \mu \quad \text{if} \quad \frac{\lambda}{|\lambda|} \leq \frac{\mu}{|\mu|}.$$  

Given $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m$, the term-normalized homogeneous complete symmetric function is defined on $\mathbb{C}^n$ by

$$H_\lambda(z) = \prod_{i=1}^{m} H_{\lambda_i}(z). \quad (9)$$

In addition, for $x \in \mathbb{R}_+^n$ we set

$$\bar{H}_\lambda(x) := \sqrt[|\lambda|]{H_\lambda(x)}. \quad (10)$$

Here we will use the majorization order to define Schur-convex functions.

**Definition 1.1:** Let $A \subset \mathbb{R}^n$. A real-valued function $f$ on $A$ is called Schur-convex if $\lambda \preceq \mu$ implies $f(\lambda) \leq f(\mu)$.

### 1.3. Objectives

In this paper, we shall use matrix analysis techniques to find the integral given in (2) and then we apply our results to enumerative combinatorics. More explicitly, in Section 2, our main goal is to prove the following identity (see Theorem 2.4)

$$\int_{\mathbb{S}^n} \left( \langle A\xi, \xi \rangle \right)^k \, d\sigma(\xi) = Tr(\vee^k A),$$

where $Tr(\vee^k A)$ denotes the normalized trace of the $k$-th symmetric power of $A$. Since $Tr(\vee^k A)$ is equal to $H_k(\lambda)$, the preceding integral can be then interpreted as a representation for such polynomials.

In Section 3 we use the above equation to obtain an integral representation for a particular determinant (cf. Corollary 3.1). Then, we provide a new proof for the celebrated Macmahon Master Theorem.

The last section is devoted to present some applications of the above equation in the analysis of complete homogeneous symmetric polynomials.

First, we provide a new proof for the positivity of $H_{2k}$ (cf. Corollary 4.1) on $\mathbb{R}^n$ which has first been proved by Hunter (cf. Theorem 1 in [5]). Main building blocks of Hunter’s proof can be listed as the extrema of $h_{2k}$ on the sphere using Lagrange multipliers, Euler’s theorem for homogeneity as well as a differential operator relating $h_k$ and $h_{k-1}$. Two other proofs for the positivity of $h_{2k}$ are presented by Rovenţa and Temereanca in [6]. The first relies on applying a differential operator on $h_k$ and then expressing it as a finite sum of lower order homogeneous symmetric polynomials (cf. Theorem 2.3 in [6]). The second proof is based on writing $h_k$ as a divided difference of a monomial function and also on a relation between the derivative of $n-1$ order of polynomial function and the positivity of the divided difference of order $n$ (cf. Theorem 3.5 in [6]). However, our proof relies only
on the fact that the integral of a positive function remains positive. Thus, we believe that in
the context of our work, we offer a simpler proof among the previously mentioned work.
Our integral representation for $H_k$ also allows us to extend the definition of $H_\lambda(x)$, whenever $x \in \mathbb{R}_+^n$, from the case where $\lambda \in \mathbb{N}^n$ to the case where $\lambda \in \mathbb{R}^n$ (cf. Definition 4.4). We show that the assignment $\mathbb{R}^n \ni \lambda \mapsto H_\lambda(x)$ is Schur-convex (cf. Corollary 4.6) which is a generalization of the following monotonicity result.

**Theorem 1.2 ([7, Theorem 7.3]):** Given integer partitions $\lambda$ and $\mu$ with $\lambda \subseteq \mu$, then for any $x \in \mathbb{R}_+^n$ we have

$$
\mathcal{S}_\lambda(x) \leq \mathcal{S}_\mu(x).
$$

Our approach uses a result due Tong [8] and is motivated by the work of Sra in [9] for
the monotonicity of the term-normalized Schur-functions $S_\lambda$. More precisely, Sra used the
Harish-Chandra–Itzykson–Zuber integral to represent $S_\lambda$ in order to obtain the sufficiency
(cf. Theorem 2 in [9]) of the following conjecture.

**Conjecture 1.3 ([7, Conjecture 7.4]):** Let $\lambda$ and $\mu$ be integer partitions of the same sizes. Then, $S_\lambda \leq S_\mu$ if and only if $\lambda \leq \mu$.

Finally, we prove that the monotonicity in Theorem 1.2 remains true for even integer
partitions on $\mathbb{R}^n$ and not only on $\mathbb{R}_+^n$ (cf. Corollary 4.7). In particular, we provide a proof
(cf. Inequality (42)) of the following conjecture

**Conjecture 1.4 ([6, Section 4]):** Let $k \leq d$ be even integers. Then, the following sequence of inequalities:

$$
H_1(x) \leq H_{\frac{1}{2}}(x) \leq \cdots \leq H_{\frac{k}{2}}(x) \leq H_{\frac{d}{2}}(x)
$$

holds true for all $x \in \mathbb{R}^n$.

**2. Integral formula for the trace of symmetric powers**

There is a natural way to provide an exact formula for (2) by integrating polynomials over
the unit sphere of arbitrary dimensional complex spaces. For this reason, it is more conven-
tient to use the standard notation from the theory of complex analysis of several variables.
For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ of non-negative integers and for $z \in \mathbb{C}^n$, we shall write

$$
\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)^T, \quad z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n},
\quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \quad \text{and} \quad \alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!.
$$

Given $A \in M_n(\mathbb{C})$, the vectorization of its transpose, $\text{vec}(A^T)$, is denoted by $a := \text{vec}(A^T)$, i.e. $a$ is the vector obtained by stacking the rows of $A$ on top of each other (cf. [10,11]).
Whenever $\alpha \in M_n(\mathbb{N})$, we will not distinguish between its matrix form and its representation $\vec{\alpha}^T \in \mathbb{N}^n$. For example, if $A = (a_{ij})$ and $\alpha = (\alpha_{ij})$, we write

$$|\alpha| = \sum_{ij} \alpha_{ij} \quad \text{and} \quad a^\alpha = \prod_{ij} a_{ij}^{\alpha_{ij}}.$$

Let us first express the integral given in (2) using the above notation.

**Proposition 2.1:** Let $A \in M_n(\mathbb{C})$ and $k \in \mathbb{N}$. Given $z \in \mathbb{C}^n$, the following holds:

$$\left( <Az, z> \right)^k = \sum_{|\alpha| = k \atop \alpha \in M_n(\mathbb{N})} \binom{k}{\alpha} a^\alpha z \left( \sum_i \alpha_{ij} \right) z \left( \sum_j \alpha_{ij} \right). \quad (12)$$

**Proof:** Applying the multinomial theorem to $\left( <Az, z> \right)^k = (\sum_{ij} a_{ij}z_i z_j)^k$ leads to:

$$\left( <Az, z> \right)^k = \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^n} \binom{k}{\alpha} a^\alpha \prod_{ij} z_i^{\alpha_{ij}} z_j^{\alpha_{ij}} = \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^n} \binom{k}{\alpha} a^\alpha \prod_{i,j} \left[ \prod_{j=1}^{n} z_i^{\alpha_{ij}} \right] \prod_{j=1}^{n} z_j^{\alpha_{ij}}$$

$$= \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^n} \binom{k}{\alpha} a^\alpha \prod_{j=1}^{n} z_j^{\sum_i \alpha_{ij}} z \left( \alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj} \right)$$

$$= \sum_{|\alpha| = k \atop \alpha \in M_n(\mathbb{N})} \binom{k}{\alpha} a^\alpha z \left( \sum_i \alpha_{1i}, \sum_i \alpha_{2i}, \ldots, \sum_i \alpha_{ni} \right) z \left( \sum_j \alpha_{1j}, \sum_j \alpha_{2j}, \ldots, \sum_j \alpha_{mj} \right). \blacksquare$$

**Remark 2.1:** Suppose that $A$ is an upper triangular matrix, then only upper triangular $\alpha$’s contribute to the sum on the right-hand side of Equation (12). Indeed, if $\alpha_{i_0 j_0} \neq 0$ for certain $i_0 > j_0$, we get $a^\alpha = 0$.

When integrating Equation (12), the orthogonality of the monomials on $L^2(\mathbb{S}^n)$ implies a condition on the row and column sums of $\alpha \in M_n(\mathbb{N})$. The next lemma examines the type of $\alpha$ under such condition.

**Lemma 2.2:** Let $\alpha \in M_n(\mathbb{N})$ be an upper triangular matrix satisfying the condition:

$$\sum_{i=1}^{n} \alpha_{il} = \sum_{j=1}^{n} \alpha_{lj} \quad \text{for every} \ l = 1, 2, \ldots, n, \quad (13)$$

then $\alpha$ is a diagonal matrix.
Proof: We shall proceed by induction. For \( l = 1 \), we have \( \alpha_{11} = \alpha_{11} + \sum_{j=2}^{n} \alpha_{1j} \) so that \( \alpha_{1j} = 0 \) for all \( j = 2, \ldots, n \). Assume that \( \alpha_{kj} = 0 \) for all \( k \leq l_0 \) and \( k \neq j \). It follows from Equation (13) that the \((l_0 + 1)\) column sum of \( \alpha \) satisfies

\[
\sum_{i=1}^{n} \alpha_{i,l_0+1} = \sum_{j=1}^{n} \alpha_{l_0+1,j}.
\]

Notice that

\[
\sum_{i=1}^{n} \alpha_{i,l_0+1} = \sum_{i \leq l_0}^{n} \alpha_{i,l_0+1} + \alpha_{l_0+1,l_0+1} + \sum_{i > l_0+1}^{n} \alpha_{i,l_0+1},
\]

where the first sum on the right-hand side of the above equation vanishes by making use of the induction hypothesis and the second sum vanishes as \( \alpha \) is upper triangular. Similarly, by using the fact that \( \alpha \) is upper triangular we get

\[
\sum_{j=1}^{n} \alpha_{l_0+1,j} = \sum_{j \leq l_0}^{n} \alpha_{l_0+1,j} + \alpha_{l_0+1,l_0+1} + \sum_{j > l_0+1}^{n} \alpha_{l_0+1,j}.
\]

We conclude that

\[
\alpha_{l_0+1,l_0+1} = \alpha_{l_0+1,l_0+1} + \sum_{j > l_0+1}^{n} \alpha_{l_0+1,j},
\]

which in turn yields that \( \alpha_{l_0+1,j} = 0 \) for all \( j > l_0 + 1 \). Thus, \( \alpha \) is diagonal and the proof is complete. ■

We now give a well-known result on the integration of monomials on spheres. This result has played an important role in real, complex, and harmonic analysis (cf. for example [3,12,13]). In particular, it is essentially used for the study of (commuting) Toeplitz operators on the Segal–Bargmann space [14–16]. A simple proof using complex analysis for the next lemma could be found in [17] and is due to Bargmann and Nelson as indicated by Folland [17] (see also Prop. 1.4.9 in [18]).

Lemma 2.3: Let \( \sigma \) be the rotation-invariant probability measure on the unit sphere \( S^n \) of \( \mathbb{C}^n \). For any \( \alpha, \beta \in \mathbb{N}^n \) we have

\[
\int_{S^n} \xi^\alpha \bar{\xi}^\beta \, d\sigma(\xi) = \begin{cases} 
\frac{(n-1)!\alpha!}{(n-1 + |\alpha|)!}, & \alpha = \beta; \\
0, & \alpha \neq \beta.
\end{cases}
\]

(14)

Now, we are in a position to prove our main result.

Theorem 2.4: Let \( A \in M_n(\mathbb{C}) \) and let \( k \in \mathbb{N} \). The normalized trace of the \( k \)-th symmetric power of \( A \) is denoted by \( \text{Tr}(\vee^k A) \). It follows that

\[
\int_{S^n} \left( \langle A \xi, \xi \rangle \right)^k \, d\sigma(\xi) = \text{Tr}(\vee^k A).
\]

(15)
\textbf{Proof:} Using Schur's decomposition, one can write $A = U^*TU$ with $T$ upper triangular and $U$ is unitary. The change of variable $u = U\xi$ provides

$$\int_{S^n} (\langle A\xi, \xi \rangle)^k \, d\sigma(\xi) = \int_{S^n} (\langle TU\xi, U\xi \rangle)^k \, d\sigma(U^*u) = \int_{S^n} (\langle Tu, u \rangle)^k \, d\sigma(u),$$

where the last equality follows from the invariance of $\sigma$ under unitary transformation. Thus, we may assume that $A$ is upper triangular and $\lambda = (a_{11}, a_{22}, \ldots, a_{nn})$. By Proposition 2.1 and Remark 2.1, we obtain that

$$\int_{S^n} (\langle A\xi, \xi \rangle)^k \, d\sigma(\xi)$$

is equal to

$$\sum_{\substack{|\alpha| = k \\
\alpha \text{ upper triangular}}} \binom{k}{\alpha} a^\alpha \int_{S^n} \xi^{\sum_i a_{i1}, \sum_i a_{i2}, \ldots, \sum_i a_{in}} \frac{\xi^{(\sum_j a_{ij}, \sum_j a_{ij}, \ldots, \sum_j a_{ij})}}{\sigma(\xi)} \, d\sigma(\xi).$$

Following Lemma 2.3, the preceding integral terms vanishes for all $\alpha$ except when

$$\sum_{i=1}^{n} \alpha_{il} = \sum_{j=1}^{n} \alpha_{lj} \quad \text{for every } l = 1, 2, \ldots, n.$$

By Lemma 2.2, $\alpha$ is a diagonal matrix so that

$$a^\alpha = \lambda_1^{\alpha_{11}} \lambda_2^{\alpha_{22}} \cdots \lambda_n^{\alpha_{nn}} = \lambda^\alpha,$$

where in the last equality we considered $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn})$ as an $n$-tuple in $\mathbb{N}^n$. This reduces the preceding integral to the following form:

$$\int_{S^n} (\langle A\xi, \xi \rangle)^k \, d\sigma(u) = \sum_{\substack{|\alpha| = k \\
\alpha \in \mathbb{N}^n}} \binom{k}{\alpha} \lambda^\alpha \int_{S^n} \xi^{(\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn})} \frac{\xi^{(\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn})}}{\sigma(\xi)} \, d\sigma(\xi).$$

Another application of Lemma 2.3 for the norms of monomials yields

$$\int_{S^n} (\langle A\xi, \xi \rangle)^k \, d\sigma(\xi) = \sum_{\substack{|\alpha| = k \\
\alpha \in \mathbb{N}^n}} \binom{k}{\alpha} \lambda^\alpha \frac{(n-1)!}{(n-1+|\alpha|)!} = \sum_{\substack{|\alpha| = k \\
\alpha \in \mathbb{N}^n}} \frac{k! \lambda^\alpha}{\alpha!} \frac{(n-1)!\alpha!}{(n-1+k)!} = \frac{k!(n-1)!}{(n-1+k)!} \sum_{\substack{|\alpha| = k \\
\alpha \in \mathbb{N}^n}} \lambda^\alpha.$$
Equation (15) merits (at least a quick) discussion on its important role in the analysis of function spaces. It is trivial that Equation (15) generalizes Equation (1) from $k = 1$ to arbitrary $k \in \mathbb{N}$. With this in mind, a direct application of Equation (1) to $\vee^k A \in M_{c_n,k}(\mathbb{C})$ together with Equation (15) provides

$$
\int_{\mathbb{S}^n_{c_n,k}} \langle \vee^k A \mu, \mu \rangle \ d\sigma(\mu) = \int_{\mathbb{S}^n} \langle (A \xi, \xi) \rangle^k \ d\sigma(\xi).
$$

The above equation can be viewed as a reduction from integrals over $\mathbb{S}^n_{c_n,k}$ to integrals over $\mathbb{S}^n$. This leads to various applications in the study of the analysis of reproducing kernel Hilbert spaces of the analytic function over specific domains. For example, the left-hand side of Equation (17) is the Szegö projection of

$$
\mathbb{S}^n_{c_n,k} \ni \mu \longrightarrow \langle \vee^k A \mu, \mu \rangle
$$

to $H^2(\mathbb{S}^n_{c_n,k})$ at $z = 0 \in \mathbb{C}^n$, while the right hand side is the Szegö projection of

$$
\mathbb{S}^n \ni \xi \longrightarrow (\langle A \xi, \xi \rangle)^k
$$

to $H^2(\mathbb{S}^n)$ at $z = 0 \in \mathbb{C}^n$. Another application is connected to the study of the composition of Toeplitz operators on the Segal–Bargmann space. We refer the reader to [16,19–21] where integral reduction is naturally used to obtain a composition formula for some classes of Toeplitz operators.

Notice that Equation (15) shows that the integral depends on the spectrum of the matrix. More explicitly, by expressing the complete homogeneous symmetric polynomials as power sums we obtain that the integral depends only on moments of the eigenvalues. For this reason, we present the following well-known result on representing the complete symmetric polynomials in terms of power sums.

**Lemma 2.5 ([22, pp. 24–25]):** Let $k \in \mathbb{N}$ and let $h_k(x) = \sum_{|\alpha| = k} x^\alpha$. For each $r \in \mathbb{N}$ let $p_r(x)$ be the power sum symmetric polynomial $\sum_{i=1}^n x_i^r$. Given a non-zero compactly supported sequence arranged in decreasing order $\beta = (\beta_1, \beta_2, \ldots, \beta_l, 0, 0, 0 \ldots) \in \mathbb{N}^\infty$, define

$$
p_\beta(x) = p_{\beta_1}(x)p_{\beta_2}(x) \cdots p_{\beta_l}(x),
$$

with $\beta_1$ is the leading entry of $\beta$. Then,

$$
h_k(x) = \sum_{|\beta| = k} z_\beta^{-1} p_\beta(x),
$$

where

$$
z_\beta = \prod_{i \geq 1} i^{m_i} m_i!
$$

and $m_i$ is the number of times that $i$ occurs in $\beta$. 

We shall apply the preceding lemma to express the trace of the symmetric tensor product of a matrix \( A \) in terms of product of traces of powers of \( A \). Following the notation in the above lemma, Equation (15) can then be reformulated in terms of the \( r \)th-moments of the eigenvalues \( p_r(\lambda) = \sum_{i=1}^{n} \lambda_i^r \) as follows:

\[
\begin{align*}
  c_{n,k} \int_{S^n} \left( \langle A \xi, \xi \rangle \right)^k \, d\sigma(\xi) &= \sum_{|\beta|=k, \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \geq 0} z_{\beta}^{-1} P_{\beta}(\lambda) = \sum_{|\beta|=k, \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \geq 0} z_{\beta}^{-1} \prod_{i=1}^{l} \text{tr}(A^{\beta_i}). \quad (20)
\end{align*}
\]

Notice that the condition for which \( \beta \in \mathbb{N}^k \) is trivial since \( \beta \) is decreasing with \( |\beta| = k \). Using the basic fact that the product of traces is the trace of tensor products, Equation (20) can be rewritten as follows:

\[
\begin{align*}
  c_{n,k} \int_{S^n} \left( \langle A \xi, \xi \rangle \right)^k \, d\sigma(\xi) &= \sum_{|\beta|=k, \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \geq 0} z_{\beta}^{-1} \text{tr} \left[ \bigotimes_{t=1}^{l} A^{\beta_t} \right]. \quad (21)
\end{align*}
\]

where \( \bigotimes \) denotes the tensor (Kronecker) product of matrices.

With an inspection of Equations (20) and (21), one obtains an ‘exhaustive behaviour’ of the \( l \)-index of \( \beta \). Indeed, in both equations there is no need to worry about \( \beta_i \) whenever \( i > l \), so fixing \( l \) will be helpful in computation. Moreover, regarding Equation (21), one might be interested in summing up tensor powers of matrices (of same dimension). This also amounts to collecting the multi-indices having the same index in the leading term. For this reason, we consider

\[
S^k := \{ \beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{N}^k \mid \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \quad \text{and} \quad |\beta| = k \}
\]

and for each \( l = 1, 2, \ldots, k \), we set

\[
S^k_l = \{ \beta \in S^k \mid \beta_l \neq 0 \text{ and } \beta_i = 0 \text{ for all } i > l \}.
\]

Notice that \( \{S^k_l\}_{l=1,2,...,k} \) forms a partition of \( S^k \). Then, the following corollary follows from Theorem 2.4.

**Corollary 2.6:** Let \( A \in M_n(\mathbb{C}) \) and let \( k \in \mathbb{N} \), then the following holds:

\[
\begin{align*}
  c_{n,k} \int_{S^n} \left( \langle A \xi, \xi \rangle \right)^k \, d\sigma(\xi) &= \sum_{l=1}^{k} \sum_{\beta \in S^k_l} \frac{1}{z_{\beta}} \prod_{i=1}^{l} \text{tr}(A^{\beta_i}) \quad (22) \\
  &= \sum_{l=1}^{k} \text{tr} \left[ \left( \sum_{\beta \in S^k_l} \frac{1}{z_{\beta}} \bigotimes_{t=1}^{l} A^{\beta_t} \right) \right]. \quad (23)
\end{align*}
\]

The next example is devoted to provide the explicit value of the integral formula for the cases \( k = 2, \ldots, 6 \). Since going from Equation (22) to Equation (23) is a routine computation, we shall only apply Equation (23) for the cases \( k = 2 \) and \( k = 3 \).
Example 2.7: For a given \( k, n \in \mathbb{N} \), let \( c_{n,k} = \binom{n+k-1}{k} \) then for any \( A \in M_n(\mathbb{C}) \) the following equalities hold:

\[
c_{n,2} \int_{S^n} (\langle A\xi, \xi \rangle)^2 \, d\sigma(\xi) = \frac{1}{2} tr(A^2) + \frac{1}{2} tr(A \otimes A) = \frac{1}{2} tr(A^2) + \frac{1}{2} (trA)^2. \tag{24}
\]

\[
c_{n,3} \int_{S^n} (\langle A\xi, \xi \rangle)^3 \, d\sigma(\xi) = \frac{1}{3} tr(A^3) + \frac{1}{2} tr(A^2 \otimes A) + \frac{1}{6} tr(A \otimes A \otimes A) = \frac{1}{3} tr(A^3) + \frac{1}{2} [tr(A^2)] tr(A) + \frac{1}{6} [trA]^3. \tag{25}
\]

\[
c_{n,4} \int_{S^n} (\langle A\xi, \xi \rangle)^4 \, d\sigma(\xi) = \frac{1}{4} tr(A^4) + \frac{1}{3} tr(A^3) tr(A) + \frac{1}{8} [tr(A^2)]^2 + \frac{1}{4} tr(A^2)[tr(A)]^2
+ \frac{1}{24} [tr(A)]^4. \tag{26}
\]

\[
c_{n,5} \int_{S^n} (\langle A\xi, \xi \rangle)^5 \, d\sigma(\xi) = \frac{1}{5} tr(A^5) + \frac{1}{4} [tr(A^4)] tr(A) + \frac{1}{6} [tr(A^3)] [tr(A^2)]
+ \frac{1}{6} [tr(A^3)] [tr(A)]^2 + \frac{1}{8} [tr(A^2)]^2 tr(A) + \frac{1}{12} tr(A^2)[tr(A)]^3
+ \frac{1}{120} [tr(A)]^5. \tag{27}
\]

\[
c_{n,6} \int_{S^n} (\langle A\xi, \xi \rangle)^6 \, d\sigma(\xi) = \frac{1}{6} tr(A^6) + \frac{1}{5} [tr(A^5)] tr(A) + \frac{1}{8} [tr(A^4)] [tr(A^2)]
+ \frac{1}{18} [tr(A^3)]^2 + \frac{1}{8} [tr(A^4)] [tr(A)]^2
+ \frac{1}{6} [tr(A^3)] [tr(A^2)] [tr(A)] + \frac{1}{48} [tr(A^2)]^3
+ \frac{1}{18} [tr(A^3)] [tr(A)]^3 + \frac{1}{16} [tr(A^2)]^2 [tr(A)]^2
+ \frac{1}{48} [tr(A^2)] [tr(A)]^4 + \frac{1}{720} [tr(A)]^6. \tag{28}
\]

We conclude this section by noting that Equation (24) can be obtained from Corollary 2.3 and Theorem 2.4 in [1], while to the best of our knowledge, Equations (25)–(28) are new.

3. A new proof of the MacMahon master theorem

This section is devoted to provide a simple proof for the MacMahon Master Theorem. This theory has been central in combinatorics and in the theory of angular momentum of systems of particles as well. The original proof is due to MacMahon [23] while another proof, depending on complex analysis, was provided by Good in [24]. In this work, we followed a different approach to provide a simpler proof for the theorem. Making use of the results given in the previous section, let us first present an integral representation of a particular determinant as follows.
Corollary 3.1: Let \( B \in M_n(\mathbb{C}) \) with \( \|B\|_2 = \sup_{\|x\|=1} \|Bx\|_2 < 1 \), then

\[
\int_{\mathbb{S}^n} \frac{d\sigma(\xi)}{((I_n - B)\xi, \xi)^n} = \det(I_n - B)^{-1}.
\] (29)

Proof: By Theorem 2.4, we know that for each \( k \in \mathbb{N} \) the following identity holds:

\[
\int_{\mathbb{S}^n} \binom{n+k-1}{k} (B\xi, \xi)^k d\sigma(\xi) = \sum_{|\alpha|=k} \lambda^\alpha.
\]

Since \( \|B\|_2 < 1 \) the series \( \sum_{k=0}^{\infty} \binom{n+k-1}{k} (B\xi, \xi)^k \) converges absolutely and is dominated by \( \frac{1}{(1-\|B\|_2)^n} \). Hence, it follows from the Lebesgue-dominated convergence theorem that

\[
\int_{\mathbb{S}^n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} (B\xi, \xi)^k = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \lambda^\alpha,
\]

i.e.

\[
\int_{\mathbb{S}^n} (1 - (B\xi, \xi))^{-n} d\sigma(\xi) = \sum_{\alpha \in \mathbb{N}^n} \lambda^\alpha = \prod_{j=1}^{n} \frac{1}{1 - \lambda_j} = \frac{1}{\det(I_n - B)}.
\] (30)

Remark 3.1: Note that the monotone convergence theorem ensures that Equation (29) remains true for any positive semidefinite matrix \( B \). Also, it is easy to see that for any \( c \in \mathbb{C} \) with \(|c| > 1\) and for any non-zero matrix \( A \in M_n(\mathbb{C}) \), we have

\[
\int_{\mathbb{S}^n} \frac{d\sigma(\xi)}{((c\|A\|_2 I_n - A)\xi, \xi)^n} = \det(c\|A\|_2 I_n - A)^{-1}.
\]

Now, we are ready to present our proof for the MacMahon Master Theorem.

Theorem 3.2 (MacMahon): Let \( A = (a_{ij}) \in M_n(\mathbb{C}), \alpha \in \mathbb{N}^n \) and \( x = (x_1, x_2, \ldots, x_n)^T \) be a formal variable. Then the coefficient of \( x^\alpha \) in the expression of \((Ax)^\alpha\) is equal to the coefficient of \( x^\alpha \) in the expansion of \( \det(I_n - \Delta(x)B)^{-1} \), where \( \Delta(x) \) is the diagonal matrix given by \( \Delta(x) = \text{diag}(x_1, \ldots, x_n) \).

Proof: For \( k \in \mathbb{N} \) and for each \( \beta \in \mathbb{N}^n \) with \(|\beta| = k\), the coefficient of \( x^\beta \) in \((Ax)^\beta\) is denoted by \( c_\beta \). Since \( \lambda(\Delta(x)A) = \lambda(A\Delta(x)) \), then by Equation (16) we have

\[
h_k\left(\lambda(\Delta(x)A)\right) = c_{nk} \int_{\mathbb{S}^n} \left( < A\Delta(x)\xi, \xi > \right)^k d\sigma(\xi)
\]

\[
= c_{nk} \int_{\mathbb{S}^n} \left[ \left< \left( \sum_{j=1}^{n} a_{1j} x_j \xi_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j \xi_j \right)^T, \xi \right> \right]^k d\sigma(\xi).
\]
Applying again the multinomial theorem, we get

\[
h_k(\lambda(\Delta(x)A)) = c_{n,k} \sum_{|\beta|=k, \beta \in \mathbb{N}^n} \binom{k}{\beta} \int_{\mathbb{S}^n} c_\beta x^\beta \xi^\beta \overline{\xi}^\beta \, d\sigma(\xi)
\]

\[
= c_{n,k} \sum_{|\beta|=k, \beta \in \mathbb{N}^n} \binom{k}{\beta} c_\beta x^\beta \frac{(n-1)!\beta!}{(n-1+|\beta|)!}
\]

\[
= \frac{(n+k-1)!}{(n-1)!k!} \sum_{|\beta|=k, \beta \in \mathbb{N}^n} \frac{k!}{\beta!} c_\beta x^\beta \frac{(n-1)!\beta!}{(n-1+k)!} = \sum_{|\beta|=k, \beta \in \mathbb{N}^n} c_\beta x^\beta,
\]

where the second equality in (32) follows from Equation (14). Finally, taking the sum over \( k \in \mathbb{N} \) and using Equation (30), we obtain

\[
\det(I_n - \Delta(x)B)^{-1} = \sum_{\alpha \in \mathbb{N}^n} \lambda^\alpha (\Delta(x)A) = \sum_{k=0}^{\infty} h_k(\lambda(\Delta(x)A)) = \sum_{\beta \in \mathbb{N}^n} c_\beta x^\beta. \]

4. On the monotonicity of complete symmetric polynomials

In this section, we first provide our own proof for the positivity of \( H_{2k} \) on \( \mathbb{R}^n \). Then, we extend the notion of \( k \)-homogeneous complete symmetric polynomials, defined on \( \mathbb{R}^n_+ \), from the case where \( k \in \mathbb{N} \) to arbitrary \( p \in \mathbb{R} \). We exploit the definition to introduce a natural extension for the term-normalized homogeneous polynomial. Using the theory of majorization, we obtain a Schur convexity of this extension. As a consequence, we obtain a generalization of Theorem 1.2 and we answer the Open problem 1.4 as well.

For an arbitrary \( z \in \mathbb{C}^n \), we shall write \( Z := \text{diag}(z_1, \ldots, z_n) \in M_n(\mathbb{C}) \). Thus, by Equation (7) and Equation (15) we have the following representation for the normalized complete symmetric polynomials:

\[
H_k(z) = \int_{\mathbb{S}^n} (\langle Z \xi, \xi \rangle)^k \, d\sigma(\xi). \]

**Corollary 4.1:** The even degree complete symmetric polynomials are positive definite on \( \mathbb{R}^n \).

**Proof:** Using the representation (33), we know that for any \( x \in \mathbb{R}^n \) we have

\[
H_{2k}(x) = \int_{\mathbb{S}^n} (\langle X \xi, \xi \rangle)^{2k} \, d\sigma(\xi),
\]

where \( X = \text{diag}(x_1, \ldots, x_n) \) so that positivity is clear. If \( H_{2k}(x) = 0 \), then the map

\[
\mathbb{S}^n \ni \xi \mapsto (\langle X \xi, \xi \rangle)^2 \in \mathbb{R}_+
\]

is equal to zero almost everywhere on \( \mathbb{S}^n \). By continuity of the preceding map, we obtain \( \langle X \xi, \xi \rangle = 0 \) for all \( \xi \in \mathbb{S}^n \) or equivalently \( x = 0 \).
Motivated by the latter property, we introduce the following definition and we shall use the same notation as in (10).

**Definition 4.2:** Given \( \nu = (\nu_1, \nu_2, \ldots, \nu_m) \in \mathbb{N}^m \), let the map \( H_{2\nu} \) be defined on \( \mathbb{R}^n \) by

\[
H_{2\nu}(x) := 2^{\nu} \sqrt{H_{2\nu}(x)}.
\]

In the following proposition, we examine the possibility of an extension of Equation (34) from the case of integer powers to the case of arbitrary real powers.

**Proposition 4.3:** Let \( X \in M_n(\mathbb{C}) \) be a non-zero positive semidefinite matrix. Then for any \( p \in \mathbb{R} \), the integral

\[
\int_{\mathbb{S}^n} (\langle X\xi, \xi \rangle)^p \, d\sigma(\xi)
\]

is well defined with values in \([0, \infty[\).

**Proof:** Suppose first that \( p \geq 0 \), then clearly the proposition follows from the inequality

\[
\int_{\mathbb{S}^n} (\langle X\xi, \xi \rangle)^p \, d\sigma(\xi) \leq \int_{\mathbb{S}^n} (w(X))^p \, d\sigma(\xi) = (w(X))^p,
\]

where \( w(\cdot) \) denotes the numerical radius norm. On the other hand, if \( p < 0 \) and if \( \nu \) denotes the Lebesgue measure on \( \mathbb{C}^n \), then we consider the following set

\[
F := \left\{ z \in \mathbb{C}^n \mid \langle Xz, z \rangle = 0 \right\}.
\]

As \( X \geq 0 \) and \( X \neq 0 \), then \( F = \ker X \) which is an intersection of hyperplanes in \( \mathbb{C}^n \). So that \( \nu(F) = 0 \) or equivalently \( F\setminus\{0\} \) is measurable and \( \nu(F\setminus\{0\}) = 0 \). Next, we consider the set defined by

\[
E := \left\{ \xi \in \mathbb{S}^n \mid \langle X\xi, \xi \rangle = 0 \right\}.
\]

By the definition of the sigma-algebra on \( \mathbb{S}^n \) (cf. for example Chapter 6, Section 3 in [25]), the set \( E \) is measurable if and only if the set

\[
\tilde{E} := \left\{ z \in B(0, 1) \mid z \neq 0 \text{ and } \frac{z}{|z|} \in E \right\}
\]

is Lebesgue measurable in \( \mathbb{C}^n \). Notice that by representing \( z \) in polar coordinates \( z = r\xi \) with \( r \in [0, \infty[ \) we obtain

\[
\tilde{E} = B(0, 1) \cap F\setminus\{0\}
\]

and hence \( E \) is measurable. Let \( \mu_1 \) be the corresponding radial measure on \( (0, \infty) \), i.e. \( \mu_1(\theta) = \int_0^\theta r^{2n-1} \, dr \) for every Lebesgue measurable set \( \theta \) in \( (0, \infty) \). Since \( F\setminus\{0\} = (0, \infty) \times E = \bigcup_{l \in \mathbb{N}} (0, l) \times E \) then

\[
0 = \nu(F\setminus\{0\}) = \lim_{l \to \infty} \mu_1((0, l)) \sigma(E) = \lim_{l \to \infty} \frac{l^{2n}}{2n} \sigma(E)
\]

which holds only in the case when \( \sigma(E) = 0 \). Let \( f \) be the map on \( \mathbb{S}^n \) defined by \( f(\xi) = \langle X\xi, \xi \rangle^p \) with values in \([0, \infty[\). As \( f \) is real valued continuous function on \( \mathbb{S}^n \setminus\{E\} \),
then $f$ is measurable on $E^c$ and therefore on $\mathbb{S}^n$. Moreover, as $E$ is a null-$\sigma$-set, then

$$\int_{\mathbb{S}^n} \left( \langle X\xi, \xi \rangle \right)^p \, d\sigma(\xi) = \int_{\mathbb{S}^n} \left( \langle X\xi, \xi \rangle \right)^p \chi_{E^c}(\xi) \, d\sigma(\xi). \tag{36}$$

By the unitary invariance of $\sigma$, one can assume that $X$ is diagonal and the preceding equation remains true. Indeed, compared to the work in Section 1 the only factor that needs to be examined is $\chi_{E^c}$. However, writing $X = U^* \Delta U$ with $\Delta$ being diagonal matrix and $U \in U_n$, then by the invariance of the Euclidean norm under unitary transformation the set $E$ is given by

$$\left\{ \xi \in \mathbb{S}^n \mid \langle X\xi, \xi \rangle = 0 \right\} = \left\{ \eta \in \mathbb{S}^n \mid \langle \Delta\eta, \eta \rangle = 0 \right\},$$

where the last equality follows from the fact that $U$ is an isometry. The above equation shows that $\chi_{E^c}(U\xi) = 1$ if and only if $\xi \in \mathbb{S}^n$ and $\langle \Delta U\xi, U\xi \rangle = 0$ which is equivalent to say that $\xi \in E$. Therefore, $\chi_{E^c}(U\xi) = \chi_{E^c}(\xi)$ and we can assume that $X = diag(x_1, \ldots, x_n)$ with $x = \lambda(X)$ being a non-zero vector in $\mathbb{R}_+^n$. It remains to prove that the value of the integral is strictly positive. For this, we let $\hat{x} = \min\{x_i \mid x_i \neq 0\}$ and $\hat{x} = \max\{x_i \mid x_i \neq 0\}$ then clearly $\hat{x} > 0$, $\hat{x} > 0$ and

$$\hat{x}^p \leq \left( \langle X\xi, \xi \rangle \right)^p \leq \hat{x}^p,$$

where the above inequality holds for all $\xi \in E^c$. Therefore, by (36) we get

$$\hat{x}^p \leq \int_{\mathbb{S}^n} \left( \langle X\xi, \xi \rangle \right)^p \, d\sigma(\xi) \leq \hat{x}^p. \tag{37}$$

Motivated by the above result, we introduce the following generalization for normalized (and term normalized) complete homogeneous polynomials on $\mathbb{R}_+^n$.

**Definition 4.4:** Let $p \in \mathbb{R}^*$. The $p$-homogeneous function $H_p$ is the map defined on $\mathbb{R}_+^n$ by

$$H_p(x) = \begin{cases} \int_{\mathbb{S}^n} \left( \langle X\xi, \xi \rangle \right)^p \, d\sigma(\xi) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases} \tag{38}$$

where $X = diag(x_1, \ldots, x_n)$. If $p = 0$, we use the convention $H_0 = 0$. In addition, given $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ we define the term-normalized $\lambda$-function by

$$H_{\lambda}(x) = \prod_{i=1}^{m} H_{\lambda_i}(x).$$

In [8], based on Muirhead’s Theorem, Tong provided an inequality for the expectation of product of independent random variables with a majorization condition. The case where the probability measure is considered on $[0, \infty]$ can be found in [26, p. 107]. We formulate this result in the context of our work and we provide our own proof which is motivated by the work of Sra in [9].
Theorem 4.5: Let \( f : \mathbb{S}^n \rightarrow \mathbb{R}_+ \) be a measurable function. Let \( I \subseteq \mathbb{R} \) be an interval and assume that
\[
\int_{\mathbb{S}^n} f^p(\xi) \, d\sigma(\xi) < \infty
\]
for all \( p \in I \). Then for any \( m \in \mathbb{N} \) the map
\[
F(\lambda) := \prod_{i=1}^m \int_{\mathbb{S}^n} f^{\lambda_i}(\xi) \, d\sigma(\xi)
\]
is Schur convex on \( I^m \). Moreover, if \( \lambda, \mu \in I^m \) are integer partitions with \( \lambda \subseteq \mu \), then
\[
|\lambda| \sqrt{|F(\lambda)|} \leq |\mu| \sqrt{|F(\mu)|}.
\]

Proof: As \( F \) is continuous and symmetric it is sufficient to prove \( F \) is convex (cf. [26, p. 97]). Now, by Fubini’s theorem we write
\[
F(\lambda) = \int_{\mathbb{S}^n \times \mathbb{S}^n \times \cdots \times \mathbb{S}^n} \prod_{i=1}^m f^{\lambda_i}(\xi^i) \, d\xi^i,
\]
where \( \xi^i \) is the coordinates in the \( i \)-th copy of \( \mathbb{S}^n \). Thus, for \( \lambda, \mu \in I^m \), the power mean inequality yields
\[
F\left(\frac{\lambda + \mu}{2}\right) = \int_{\mathbb{S}^n \times \mathbb{S}^n \times \cdots \times \mathbb{S}^n} \sqrt[|\mu|]{\prod_{i=1}^m f^{\lambda_i}} \sqrt[|\mu|]{\prod_{i=1}^m f^{\mu_i}} \leq \int_{\mathbb{S}^n \times \mathbb{S}^n \times \cdots \times \mathbb{S}^n} \frac{\prod_{i=1}^m f^{\lambda_i} + \prod_{i=1}^m f^{\mu_i}}{2} \leq \frac{F(\lambda)}{2} + \frac{F(\mu)}{2}.
\]
In case where \( \lambda, \mu \in I^m \) are integer partitions the condition \( \lambda \subseteq \mu \) is equivalent to \( \lambda^{||\mu||} \leq \mu^{||\lambda||} \) (cf. [7]). Hence applying the Schur convexity of \( F \) on \( I^{m|\mu|} \) we obtain
\[
\prod_{i=1}^m \int_{\mathbb{S}^n} f^{\lambda_i} = \left[ \prod_{i=1}^m \int_{\mathbb{S}^n} f^{\lambda_i} \right]^{|\mu| \over |\mu|} = \left[ \int_{\mathbb{S}^n} f^{\lambda_1} \int_{\mathbb{S}^n} f^{\lambda_1} \cdots \int_{\mathbb{S}^n} f^{\lambda_1} \int_{\mathbb{S}^n} f^{\lambda_2} \int_{\mathbb{S}^n} f^{\lambda_2} \cdots \int_{\mathbb{S}^n} f^{\lambda_2} \cdots \int_{\mathbb{S}^n} f^{\lambda_m} \int_{\mathbb{S}^n} f^{\lambda_m} \cdots \int_{\mathbb{S}^n} f^{\lambda_m} \right]^{1 \over |\mu|}
\]
Applying the preceding theorem to the case $f(ξ) = ⟨Xξ, ξ⟩$, whenever $x ∈ \mathbb{R}^n_+$, we obtain the following generalization of Theorem 1.2.

**Corollary 4.6:** Let $λ ∈ \mathbb{R}^m$ and $μ ∈ \mathbb{R}^l$. If $λ ≤ μ$, then

$$H_λ(x) ≤ H_μ(x), \quad x ∈ \mathbb{R}^n_+.$$

If $λ$, $μ$ are integer partitions with $λ ⊑ μ$, then

$$\mathcal{H}_λ(x) ≤ \mathcal{H}_μ(x), \quad x ∈ \mathbb{R}^n_+.$$

Now let $x ∈ \mathbb{R}^n$ and consider the map $f(ξ) = (⟨Xξ, ξ⟩)^2$, then for any $p ≥ 0$ we have

$$\int_{\mathbb{S}^n} f^p(ξ) \, dσ(ξ) < ∞.$$

Notice that if $λ = (2ν_1, ..., 2ν_m)$ and $μ = (2η_1, 2η_2, ..., 2η_l)$ are two integer partitions, then $λ ⊑ μ$ if and only if $ν ⊑ η$. Applying Theorem 4.5 to $f(ξ) = (⟨Xξ, ξ⟩)^2$ we obtain the following.

**Corollary 4.7:** Let $λ ∈ \mathbb{R}^m_+$ and $μ ∈ \mathbb{R}^l_+$. If $λ ≤ μ$, then

$$\prod_{i=1}^{m} \int_{\mathbb{S}^n} \left( (⟨Xξ, ξ⟩)^2 \right)^{\lambda_i} \, dσ(ξ) ≤ \prod_{i=1}^{l} \int_{\mathbb{S}^n} \left( (⟨Xξ, ξ⟩)^2 \right)^{μ_i} \, dσ(ξ), \quad x ∈ \mathbb{R}^n.$$

If $ν$, $η$ are integer partitions with $ν ⊑ η$, then

$$\mathcal{H}_{2ν}(x) ≤ \mathcal{H}_{2η}(x), \quad x ∈ \mathbb{R}^n. \quad (41)$$
Note that applying the above equation to the cases

\[
\begin{align*}
\nu &= k - 1 \quad \text{and} \quad \eta = k \\
\nu &= (k - 1, k + 1) \quad \text{and} \quad \eta = (k, k)
\end{align*}
\]

provides us with an entire sequence of inequalities similar to Newton’s identities which is valid for all \( x \in \mathbb{R}^n \). More precisely, we have

\[
2^{k-2/\sqrt{H_{2k-2}(x)}} \leq 2^{\sqrt{H_{2k}(x)}}, \quad x \in \mathbb{R}^n
\]  

(42)

and

\[
H_{2k-2}(x)H_{2k+2}(x) \leq H_{2k}^2(x), \quad x \in \mathbb{R}^n.
\]

(43)

Finally, we conclude by noting that In Equation (42) provides a proof for Conjecture 1.4.

**Acknowledgements**

We are grateful to the reviewer for his/her valuable suggestions and comments. We highly appreciate his/her careful corrections on the original version of the paper. The first author would like to thank Prof. Dr W. Bauer for his stress on the importance of Lemma 2.3 during several conversations. The first author would also like to dedicate his work to his family and to Ibn L. Hassan Al-Moaammal.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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