Low-rank tensor completion:  
a Riemannian manifold preconditioning approach

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Abstract

We propose a novel Riemannian manifold preconditioning approach for the tensor completion problem with rank constraint. A novel Riemannian metric or inner product is proposed that exploits the least-squares structure of the cost function and takes into account the structured symmetry that exists in Tucker decomposition. The specific metric allows to use the versatile framework of Riemannian optimization on quotient manifolds to develop preconditioned nonlinear conjugate gradient and stochastic gradient descent algorithms for batch and online setups, respectively. Concrete matrix representations of various optimization-related ingredients are listed. Numerical comparisons suggest that our proposed algorithms robustly outperform state-of-the-art algorithms across different synthetic and real-world datasets.

1 Introduction

This paper addresses the problem of low-rank tensor completion when the rank is a priori known or estimated. We focus on 3-order tensors in the paper, but the developments can be generalized to higher order tensors in a straightforward way. Given a tensor $X^{n_1 \times n_2 \times n_3}$, whose entries $X_{i_1,i_2,i_3}^*$ are only known for some indices $(i_1, i_2, i_3) \in \Omega$, where $\Omega$ is a subset of the complete set of indices $\{(i_1, i_2, i_3) : i_d \in \{1, \ldots, n_d\}, d \in \{1, 2, 3\}\}$, the fixed-rank tensor completion problem is formulated as

$$\min_{X^{n_1 \times n_2 \times n_3}} \frac{1}{|\Omega|} \| P_{\Omega}(X) - P_{\Omega}(X^*) \|_F^2$$

subject to $\text{rank}(X) = r$, (1)

where the operator $P_{\Omega}(X)_{i_1,i_2,i_3} = X_{i_1,i_2,i_3}^*$ if $(i_1, i_2, i_3) \in \Omega$ and $P_{\Omega}(X)_{i_1,i_2,i_3} = 0$ otherwise and (with a slight abuse of notation) $\| : \|_F$ is the Frobenius norm. $|\Omega|$ is the number of known entries. rank$(X)$ (= $r = (r_1, r_2, r_3)$), called the multilinear rank of $X$, is the set of the ranks of for each of mode-$d$ unfolding matrices. $r_d \ll n_d$ enforces a low-rank structure. The mode is a matrix obtained by concatenating the mode-$d$ fibers along columns, and mode-$d$ unfolding of a $D$-order tensor $X$ is $X_d \in \mathbb{R}^{n_d \times n_{d+1} \cdots n_D}$ for $d = \{1, \ldots, D\}$.

Problem (1) has many variants, and one of those is extending the nuclear norm regularization approach from the matrix case [6] to the tensor case. This results in a summation of nuclear norm regularization terms, each one corresponds to each of the unfolding matrices of $X$. While this generalization leads to good results [15, 25, 24], its applicability to large-scale instances is not trivial, especially due to the necessity of high-dimensional singular value decomposition computations. A different approach exploits Tucker decomposition [12, Section 4] of a low-rank tensor $X$ to develop large-scale algorithms for (1), e.g., in [8,13].

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1This paper extends the earlier work [11] to include a stochastic gradient descent algorithm for low-rank tensor completion.
The present paper exploits both the symmetry present in Tucker decomposition and the least-squares structure of the cost function of (1) to develop competitive algorithms. The multilinear rank constraint forms a smooth manifold [13]. To this end, we use the concept of manifold preconditioning. While preconditioning in unconstrained optimization is well studied [20, Chapter 5], preconditioning on constraints with symmetries, owing to non-uniqueness of Tucker decomposition [12], is not straightforward. We build upon the recent work [18] that suggests to use preconditioning with a tailored metric (inner product) in the Riemannian optimization framework on quotient manifolds [1, 7, 18]. The differences with respect to the work of Kressner et al. [13], which also exploits the manifold structure, are twofold. (i) Kressner et al. [13] exploit the search space as an embedded submanifold of the Euclidean space, whereas we view it as a product of simpler search spaces with symmetries. Consequently, certain computations have straightforward interpretation. (ii) Kressner et al. [13] work with the standard Euclidean metric, whereas we use a metric that is tuned to the least-squares cost function, thereby inducing a preconditioning effect. This novel idea of using a tuned metric leads to a superior performance of our algorithms. They also connect to state-of-the-art algorithms proposed in [19, 27, 17, 4].

The paper is organized as follows. Section 2 discusses the two fundamental structures of symmetry and least-squares associated with (1) and proposes a novel metric that captures the relevant second order information of the problem. The optimization-related ingredients on the Tucker manifold are developed in Section 3. The cost function specific ingredients are developed in Section 4. The final formulas in Section 5 are listed in Table 1, which allow to develop preconditioned conjugate gradient descent algorithm in the batch setup and stochastic gradient descent algorithm in the online setup. In Section 5, numerical comparisons with state-of-the-art algorithms on various synthetic and real-world benchmarks suggest a superior performance of our proposed algorithms. Our proposed algorithms are implemented in the Matlab toolbox Manopt [5]. The concrete proofs of propositions, development of optimization-related ingredients, and additional numerical experiments are shown in Sections A and B, respectively, of the supplementary material file. The Matlab codes for first and second order implementations, e.g., gradient descent and trust-region methods, are available at https://bamdevmishra.com/codes/tensorcompletion/.

2 Exploiting the problem structure

Construction of efficient algorithms depends on properly exploiting the problem structure. To this end, we focus on two fundamental structures in (1): symmetry in the constraints and the least-squares structure of the cost function. Finally, a novel metric is proposed.

The symmetry structure in Tucker decomposition. The Tucker decomposition of a tensor \( \mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) of rank \( r = (r_1, r_2, r_3) \) is

\[
\mathcal{X} = \mathcal{G} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \mathbf{U}_3,
\]

where \( \mathbf{U}_d \in \text{St}(r_d, n_d) \) for \( d \in \{1, 2, 3\} \) belongs to the Stiefel manifold of matrices of size \( n_d \times r_d \) with orthogonal columns and \( \mathcal{G} \in \mathbb{R}^{r_1 \times r_2 \times r_3} \) [12]. Here, \( \mathcal{W} \times_d \mathbf{V} \in \mathbb{R}^{n_1 \times \cdots \times n_{d-1} \times \sum_{i \neq d} r_i \times n_{d+1} \times \cdots \times n_D} \) computes the \( d \)-mode product of a tensor \( \mathcal{W} \in \mathbb{R}^{n_1 \times \cdots \times n_D} \) and a matrix \( \mathbf{V} \in \mathbb{R}^{m \times n_d} \). Tucker decomposition (2) is not unique as \( \mathcal{X} \) remains unchanged under the transformation

\[
(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathcal{G}) \mapsto (\mathbf{O}_1 \mathbf{U}_1, \mathbf{O}_2 \mathbf{U}_2, \mathbf{O}_3 \mathbf{U}_3, \mathcal{G} \times_1 \mathbf{O}_1^T \times_2 \mathbf{O}_2^T \times_3 \mathbf{O}_3^T)
\]

for all \( \mathbf{O}_d \in \mathcal{O}(r_d) \), which is the set of orthogonal matrices of size \( r_d \times r_d \). The classical remedy to remove this indeterminacy is to have additional structures on \( \mathcal{G} \) like sparsity or restricted orthogonal rotations [12, Section 4.3]. In contrast, we encode the transformation (3) in an abstract search space of equivalence classes, defined as,

\[
[[(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathcal{G})] := \{(\mathbf{U}_1 \mathbf{O}_1, \mathbf{U}_2 \mathbf{O}_2, \mathbf{U}_3 \mathbf{O}_3, \mathcal{G} \times_1 \mathbf{O}_1^T \times_2 \mathbf{O}_2^T \times_3 \mathbf{O}_3^T): \mathbf{O}_d \in \mathcal{O}(r_d)\}.
\]

The set of equivalence classes is the quotient manifold [14]

\[
\mathcal{M}/\sim := \mathcal{M}/(\mathcal{O}(r_1) \times \mathcal{O}(r_2) \times \mathcal{O}(r_3)),
\]
where $\mathcal{M}$ is called the total space (computational space) that is the product space

$$
\mathcal{M} := St(r_1, n_1) \times St(r_2, n_2) \times St(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3}.
$$

Due to the invariance (7), the local minima of (1) in $\mathcal{M}$ are not isolated, but they become isolated on $\mathcal{M}/\sim$. Consequently, the problem (1) is an optimization problem on a quotient manifold for which systematic procedures are proposed in [1], [7]. A requirement is to endow endow $\mathcal{M}/\sim$ with a Riemannian structure, which conceptually translates (1) into an unconstrained optimization problem over the search space $\mathcal{M}/\sim$. We call $\mathcal{M}/\sim$, defined in (5), the Tucker manifold as it results from Tucker decomposition.

The least-squares structure of the cost function. In unconstrained optimization, the Newton method is interpreted as a scaled steepest descent method, where the search space is endowed with a metric (inner product) induced by the Hessian of the cost function [20]. This induced metric (or its approximation) resolves convergence issues of first order optimization algorithms. Analogously, finding a good inner product for (1) is of profound consequence. Specifically for the case of quadratic optimization with rank constraint (matrix case), Mishra and Sepulchre [18] propose a family of Riemannian metrics from the Hessian of the cost function. Applying this approach directly for the particular cost function of (1) is computationally costly. To circumvent the issue, we consider a simplified cost function by assuming that $\Omega$ contains the full set of indices, i.e., we focus on $||\mathcal{X} - \mathcal{X}^*||_F^2$ to propose a metric candidate. Applying the metric tuning approach of [18] to the simplified cost function leads to a family of Riemannian metrics. A good trade-off between computational cost and simplicity is by considering only the block diagonal elements of the Hessian of $||\mathcal{X} - \mathcal{X}^*||_F^2$. It should be noted that the cost function $||\mathcal{X} - \mathcal{X}^*||_F^2$ is convex and quadratic in $\mathcal{X}$. Consequently, it is also convex and quadratic in the arguments $(U_1, U_2, U_3, \mathcal{G})$ individually. Equivalently, the block diagonal approximation of the Hessian of $||\mathcal{X} - \mathcal{X}^*||_F^2$ in $(U_1, U_2, U_3, \mathcal{G})$ is

$$
((G_1 G_1^T) \otimes I_{n_1}, (G_2 G_2^T) \otimes I_{n_2}, (G_3 G_3^T) \otimes I_{n_3}, I_{r_1 r_2 r_3}),
$$

where $G_d$ is the mode-$d$ unfolding of $\mathcal{G}$ and is assumed to be full rank. $\otimes$ is the Kronecker product. The terms $G_d G_d^T$ for $d \in \{1, 2, 3\}$ are positive definite when $r_1 \leq r_2 r_3$, $r_2 \leq r_1 r_3$, and $r_3 \leq r_1 r_2$, which is a reasonable assumption.

A novel Riemannian metric. An element $x$ in the total space $\mathcal{M}$ has the matrix representation $(U_1, U_2, U_3, \mathcal{G})$. Consequently, the tangent space $T_x \mathcal{M}$ is the Cartesian product of the tangent spaces of the individual manifolds of (6), i.e., $T_x \mathcal{M}$ has the matrix characterization

$$
T_x \mathcal{M} = \{ (Z_{U_1}, Z_{U_2}, Z_{U_3}, Z_{\mathcal{G}}) \in \mathbb{R}^{n_1 \times r_1} \times \mathbb{R}^{n_2 \times r_2} \times \mathbb{R}^{n_3 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} : U_d^T Z_{U_d} + Z_{U_d} U_d = 0, \text{ for } d \in \{1, 2, 3\} \}.
$$

From the earlier discussion on symmetry and least-squares structure, we propose the novel metric or inner product $g_x : T_x \mathcal{M} \times T_x \mathcal{M} \to \mathbb{R}$

$$
g_x(\xi_x, \eta_x) = \langle \xi_{U_1}, \eta_{U_1}, (G_1 G_1^T) \rangle + \langle \xi_{U_2}, \eta_{U_2}, (G_2 G_2^T) \rangle + \langle \xi_{U_3}, \eta_{U_3}, (G_3 G_3^T) \rangle + \langle \xi_{\mathcal{G}}, \eta_{\mathcal{G}} \rangle,
$$

where $\xi_x, \eta_x \in T_x \mathcal{M}$ are tangent vectors with matrix characterizations, shown in (8), $(\xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_{\mathcal{G}})$ and $(\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_{\mathcal{G}})$, respectively and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. It should be emphasized that the proposed metric (9) is induced from (7).

Proposition 1. Let $(\xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_{\mathcal{G}})$ and $(\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_{\mathcal{G}})$ be tangent vectors to the quotient manifold (5) at $(U_1, U_2, U_3, \mathcal{G})$, and $(\xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_{\mathcal{G}})$ and $(\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_{\mathcal{G}})$ be tangent vectors to the quotient manifold (5) at $(U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times O_1^T \times O_2^T \times O_3^T)$. The metric (9) is invariant along the equivalence class (4), i.e.,

$$
g(U_1 U_2 U_3, \mathcal{G})(\langle \xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_{\mathcal{G}} \rangle, (\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_{\mathcal{G}})) = g(U_1 O_1 U_2 O_2 U_3 O_3, \mathcal{G} \times O_1^T \times O_2^T \times O_3^T)(\langle \xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_{\mathcal{G}} \rangle, (\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_{\mathcal{G}})).
$$

3
the horizontal space $\mathcal{H}_x$ into an unconstrained optimization over the Riemannian quotient manifold (5). Below we briefly show how to define algorithms on the quotient manifold. The key is endowing the quotient manifold with a Riemannian structure.

Abstract geometric objects on the quotient manifold $\mathcal{M}/\sim$ call for matrix representatives, shown in solid lines, in the total space $\mathcal{M}$.

### 3 Notions of manifold optimization

Each point on a quotient manifold represents an entire equivalence class of matrices in the total space. Abstract geometric objects on the quotient manifold $\mathcal{M}/\sim$ call for matrix representatives, shown in solid lines, in the total space $\mathcal{M}$.

Similarly, algorithms are run in the total space $\mathcal{M}$, but under appropriate compatibility between the Riemannian structure of $\mathcal{M}$ and the Riemannian structure of the quotient manifold $\mathcal{M}/\sim$, they define algorithms on the quotient manifold. The key is endowing $\mathcal{M}/\sim$ with a Riemannian structure. Once this is the case, a constraint optimization problem, for example (1), is conceptually transformed into an unconstrained optimization over the Riemannian quotient manifold (5). Below we briefly show the development of various geometric objects that are required to optimize a smooth cost function on the quotient manifold (5) with first order methods, e.g., conjugate gradients.

**Quotient manifold representation and horizontal lifts.** Figure 1 illustrates a schematic view of optimization with equivalence classes, where the points $x$ and $y$ in $\mathcal{M}$ belong to the same equivalence class (shown in solid blue color) and they represent a single point $[x] := \{ y \in \mathcal{M} : y \sim x \}$ on the quotient manifold $\mathcal{M}/\sim$. The abstract tangent space $T_{[x]}(\mathcal{M}/\sim)$ at $[x] \in \mathcal{M}/\sim$ has the matrix representation in $T_x\mathcal{M}$, but restricted to the directions that do not induce a displacement along the equivalence class $[x]$. This is realized by decomposing $T_x\mathcal{M}$ into two complementary subspaces, the vertical and horizontal subspaces. The vertical space $\mathcal{V}_x$ is the tangent space of the equivalence class $[x]$. On the other hand, the horizontal space $\mathcal{H}_x$ is the orthogonal subspace to $\mathcal{V}_x$ in the sense of the metric (9). Equivalently, $T_x\mathcal{M} = \mathcal{V}_x \oplus \mathcal{H}_x$. The horizontal subspace $\mathcal{H}_x$ provides a valid matrix representation to the abstract tangent space $T_{[x]}(\mathcal{M}/\sim)$. An abstract tangent vector $\xi_{[x]} \in T_{[x]}(\mathcal{M}/\sim)$ at $[x]$ has a unique element $\xi_x \in \mathcal{H}_x$ that is called its horizontal lift $\hat{\xi}_x$.

A Riemannian metric $g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ at $x \in \mathcal{M}$ defines a Riemannian metric $g_{[x]} : T_{[x]}(\mathcal{M}/\sim) \times T_{[x]}(\mathcal{M}/\sim) \to \mathbb{R}$, i.e., $g_{[x]}(\xi_{[x]}, \eta_{[x]}) := g_x(\xi_x, \eta_x)$ on the quotient manifold $\mathcal{M}/\sim$, if $g_x(\xi_x, \eta_x)$ does not depend on a specific representation along the equivalence class $[x]$. Here, $\xi_{[x]}$ and $\eta_{[x]}$ are tangent vectors in $T_{[x]}(\mathcal{M}/\sim)$, and $\xi_x$ and $\eta_x$ are their horizontal lifts in $\mathcal{H}_x$ at $x$, respectively. Equivalently, the definition of the Riemannian metric is well posed when $g_x(\xi_x, \zeta_x) = g_x(\xi_y, \zeta_y)$ for all $x, y \in [x]$, where $\xi_x, \zeta_x \in \mathcal{H}_x$ and $\xi_y, \zeta_y \in \mathcal{H}_y$ are the horizontal lifts of $\xi_{[x]}, \zeta_{[x]} \in T_{[x]}(\mathcal{M}/\sim)$ along the same equivalence class $[x]$. This holds true for the proposed metric (9) as shown in Proposition 1. From Proposition 1, endowed with the Riemannian metric (9), the quotient manifold $\mathcal{M}/\sim$ is a Riemannian submersion of $\mathcal{M}$. The submersion principle allows to work out concrete matrix representations of abstract objects on $\mathcal{M}/\sim$, e.g., the gradient of a smooth cost function $\Psi_x$.

Starting from an arbitrary matrix (with appropriate dimensions), two linear projections are needed: the first projection $\Psi_x$ is onto the tangent space $T_x\mathcal{M}$, while the second projection $\Pi_x$ is onto the horizontal subspace $\mathcal{H}_x$. The computation cost of these is $O(n_1 r_1^2 + n_2 r_2^2 + n_3 r_3^2)$.

The tangent space $T_x\mathcal{M}$ projection is obtained by extracting the component normal to $T_x\mathcal{M}$ in the ambient space. The normal space $N_x\mathcal{M}$ has the matrix characterization $\{ (U_1^T S U_1 (G_1^T G_1)^{-1}), U_2^T S U_2 (G_2^T G_2)^{-1} \}$. 

![Figure 1](image-url)
The quotient manifold (5) endowed with the metric (9) admits the tangent space projector
Proposition 2.
vertical space. The vertical space
Proposition 3.
The quotient manifold (5) endowed with the metric (9) admits the horizontal projector
defined as

\[ \Psi_x(Y_{U_1}, Y_{U_2}, Y_{U_3}, Y_G) = (Y_{U_1} - U_1 S_{U_1} (G_1 G_1^T)^{-1}, Y_{U_2} - U_2 S_{U_2} (G_2 G_2^T)^{-1}, Y_{U_3} - U_3 S_{U_3} (G_3 G_3^T)^{-1}, Y_G), \]

where \( S_{U_d} \) is the solution to the Lyapunov equation \( S_{U_1} G_d G_d^T + G_d G_d^T S_{U_2} = G_d G_d^T (Y_{U_1} U_d + U_d^T Y_{U_d}) G_d G_d^T \) for \( d \in \{1, 2, 3\} \).

The Lyapunov equations in Proposition 2 are solved efficiently with the Matlab's `lyap` routine.

The horizontal space projection of a tangent vector is obtained by removing the component along the vertical space. The vertical space \( \mathcal{V}_x \) has the matrix characterization \( \{(U_1 \Omega_1, U_2 \Omega_2, U_3 \Omega_3, -(G \times_1 \Omega_1 + G \times_2 \Omega_2 + G \times_3 \Omega_3)) : \Omega_d \in \mathbb{R}^{d \times r_d}, \Omega_d^T = -\Omega_d \text{ for } d \in \{1, 2, 3\} \}. \) Skew symmetric matrices \( \Omega_d \) for all \( d \in \{1, 2, 3\} \) parameterize the vertical space. Finally, the horizontal projection operator \( \Pi_x : T_x \mathcal{M} :\rightarrow \mathcal{H}_x : \eta_x \mapsto \Pi_x(\eta_x) \) is given as follows.

Proposition 3.
The quotient manifold (5) endowed with the metric (9) admits the horizontal projector defined as

\[ \Pi_x(\eta_x) = (\eta_{U_1} - U_1 \Omega_1, \eta_{U_2} - U_2 \Omega_2, \eta_{U_3} - U_3 \Omega_3, \eta_G - (-G \times_1 \Omega_1 + G \times_2 \Omega_2 + G \times_3 \Omega_3)), \]

where \( \eta_x = (\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_G) \in T_x \mathcal{M} \) and \( \Omega_d \) is a skew-symmetric matrix of size \( r_d \times r_d \) that is the solution to the coupled Lyapunov equations

\[
\begin{align*}
G_1 G_1^T \Omega_1 + \Omega_1 G_1 G_1^T - G_1 (I_{r_3} \otimes \Omega_2) G_1^T - G_1 (I_{r_3} \otimes \Omega_2) G_1^T + \text{Skew}(U_1^T \eta_{U_1} G_1 G_1^T) + \text{Skew}(G_1 \eta_{G_1}^T), \\
G_2 G_2^T \Omega_2 + \Omega_2 G_2 G_2^T - G_2 (I_{r_3} \otimes \Omega_1) G_2^T - G_2 (I_{r_3} \otimes \Omega_1) G_2^T + \text{Skew}(U_2^T \eta_{U_2} G_2 G_2^T) + \text{Skew}(G_2 \eta_{G_2}^T), \\
G_3 G_3^T \Omega_3 + \Omega_3 G_3 G_3^T - G_3 (I_{r_2} \otimes \Omega_1) G_3^T - G_3 (I_{r_2} \otimes \Omega_1) G_3^T + \text{Skew}(U_3^T \eta_{U_3} G_3 G_3^T) + \text{Skew}(G_3 \eta_{G_3}^T),
\end{align*}
\]

where Skew(\( \cdot \)) extracts the skew-symmetric part of a square matrix, i.e., \( \text{Skew}(D) = (D - D^T)/2 \).

The coupled Lyapunov equations (11) are solved efficiently with the Matlab's `pcg` routine that is combined with a specific symmetric preconditioner resulting from the Gauss-Seidel approximation of (11). For the variable \( \Omega_1 \), the preconditioner is of the form \( G_1 G_1^T \Omega_1 + \Omega_1 G_1 G_1^T \). Similarly, for the variables \( \Omega_2 \) and \( \Omega_3 \).

Retraction. A retraction is a mapping that maps vectors in the horizontal space to points on the search space \( \mathcal{M} \) and satisfies the local rigidity condition (1). It provides a natural way to move on the manifold along a search direction. Because the total space \( \mathcal{M} \) has the product nature, we can choose a retraction by combining retractions on the individual manifolds, i.e., \( R_x(\xi_x) = (uf(U_1 + \xi_{U_1}), uf(U_2 + \xi_{U_2}), uf(U_3 + \xi_{U_3}), G + \xi_G) \), where \( \xi_x \in \mathcal{H}_x \) and \( uf(\cdot) \) extracts the orthogonal factor of a full column rank matrix, i.e., \( uf(A) = A (A^T A)^{-1}/2 \). The retraction \( R_x \) defines a retraction \( R_{[z]}(\xi_{[z]}) := [R_x(\xi_x)] \) on the quotient manifold \( \mathcal{M}/\sim \), as the equivalence class \( [R_x(\xi_x)] \) does not depend on specific matrix representations of \( [x] \) and \( \xi_{[x]} \), where \( \xi_{[x]} \) is the horizontal lift of the abstract tangent vector \( \xi_{[z]} \in T_{[z]}(\mathcal{M}/\sim) \).

Vector transport. A vector transport on a manifold \( \mathcal{M} \) is a smooth mapping that transports a tangent vector \( \xi_x \in T_x \mathcal{M} \) at \( x \in \mathcal{M} \) to a vector in the tangent space at a point \( R_x(\eta_x) \). It is defined by the symbol \( T_{\eta_x}(\xi_x) \). It generalizes the classical concept of translation of vectors in the Euclidean space to manifolds [Section 8.1.4]. The horizontal lift of the abstract vector transport \( T_{\eta_x}(\xi_{[z]}) \) on \( \mathcal{M}/\sim \) has the matrix characterization \( \Pi_{R_x(\eta_x)}(T_{\eta_x}(\xi_x)) = \Pi_{R_x(\eta_x)}(\Psi_{R_x(\eta_x)}(\xi_x)) \), where \( \xi_x \) and \( \eta_x \) are the horizontal lifts in \( \mathcal{H}_x \) of \( \xi_{[x]} \) and \( \eta_{[x]} \) that belong to \( T_{[x]}(\mathcal{M}/\sim) \). \( \Psi_{x(\cdot)}(\cdot) \) and \( \Pi_{x(\cdot)}(\cdot) \) are projectors defined in Propositions 2 and 3. The computational cost of transporting a vector solely depends on the projection and retraction operations.
Table 1: Tucker manifold related optimization ingredients for [1]

| Matrix representation | $x = (U_1, U_2, U_3, G)$ |
|------------------------|----------------------------|
| Computational space $\mathcal{M}$ | $\text{St}(r_1, n_1) \times \text{St}(r_2, n_2) \times \text{St}(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3}$ |
| Group action | $\{(U_1, O_1, U_2, O_2, U_3, O_3, G, \Omega) : O_d \in O(r_d), \text{ for } d \in \{1, 2, 3\}\}$ |
| Quotient space $\mathcal{M}/\sim$ | $\text{St}(r_1, n_1) \times \text{St}(r_2, n_2) \times \text{St}(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3} / \{O(r_1) \times O(r_2) \times O(r_3)\}$ |
| Ambient space | $\mathbb{R}^{r_1 \times r_2 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3}$ |
| Tangent vectors in $T_x \mathcal{M}$ | $\{(Z_{U_1}, Z_{U_2}, Z_{U_3}, Z_G) \in \mathbb{R}^{r_1 \times r_2 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} : \Omega_d = 0, \text{ for } d \in \{1, 2, 3\}\}$ |
| Metric $g_x(\xi, \eta)$ for any $\xi, \eta \in T_x \mathcal{M}$ | $(\xi_1, \eta_1) \rightarrow \langle \xi_1, \eta_1 \rangle + \langle \xi_2, \eta_2 \rangle (G_2 G_2^T) + \langle \xi_3, \eta_3 (G_3 G_3^T) + \langle \xi_G, \eta_G \rangle)$ |
| Vertical tangent vectors in $\mathcal{V}_x$ | $\{(U_1 \Omega_1, U_2 \Omega_2, U_3 \Omega_3, (G \times_1 \Omega_1 + G \times_2 \Omega_2 + G \times_3 \Omega_3)) : \Omega_d \in \mathbb{R}^{r_1 \times r_2 \times r_3}, \Omega_d^T = -\Omega_d, \text{ for } d \in \{1, 2, 3\}\}$ |
| Horizontal tangent vectors in $\mathcal{H}_x$ | $\{(U_{x1}, U_{x2}, U_{x3}, G) \in T_x \mathcal{M} : (G G_d^T)_{x_d} U_d + (G_d G_d^T)_{x_d} \text{ is symmetric, for } d \in \{1, 2, 3\}\}$ |
| Retraction $R_x(\xi)$ | $(uf(U_1 + \xi_U), uf(U_2 + \xi_U), uf(U_3 + \xi_U), G + \xi_G)$ |
| Horizontal lift of the vector transport $\mathcal{T}_{\eta(x)}(\xi_d)$ | $\Pi_{\eta(x)}(\Psi_{\eta(x)}(\xi_d))$ |

4 Riemannian algorithms for [1]

We propose two Riemannian preconditioned algorithms for the tensor completion problem [1] that are based on the developments in Section 3. The preconditioning effect follows from the specific choice of the metric [7]. In the batch setting, we use the off-the-shelf conjugate gradient implementation of Manopt for any smooth cost function [5]. A complete description of the Riemannian nonlinear conjugate gradient method is in [1] Chapter 8. In the online setting, we use the stochastic gradient descent implementation [2]. For fixed rank, theoretical convergence of the Riemannian algorithms are to a stationary point, and the convergence analysis follows from [2, 21, 2]. However, as simulations show, convergence to global minima is observed in many challenging instances.

In addition to the manifold-related ingredients in Section 3 the ingredients needed are the cost function specific ones. To this end, we show the computation of the Riemannian gradient as well as a way to compute an initial guess for the step-size, which is used in the conjugate gradient method. The concrete formulas are shown in Table 1.

Riemannian gradient computation. Let $f(\mathcal{X}) = \|\mathcal{P}_\Omega(\mathcal{X}) - \mathcal{P}_\Omega(\mathcal{X}^*)\|^2 / |\Omega|$ be the mean square error function of [1], and $\mathcal{S} = 2(\mathcal{P}_\Omega(\mathcal{G} \times_1 U_1 \times_2 U_2 \times_3 U_3) - \mathcal{P}_\Omega(\mathcal{X}^*)) / |\Omega|$ be an auxiliary sparse tensor variable that is interpreted as the Euclidean gradient of $f$ in $\mathbb{R}^{n_1 \times n_2 \times n_3}$. The partial derivatives of $f$ with respect to $(U_1, U_2, U_3, G)$ are computed in terms of the unfolding matrices $S_d$. Due to the specific scaled metric [2], the partial derivatives are further scaled by $(G_1 G_1^T)^{-1}, (G_2 G_2^T)^{-1}, (G_3 G_3^T)^{-1}, \mathcal{I}$, denoted as egrad$_d f$ (after scaling). Finally, from the Riemannian submersions theory [1] Section 3.6.2, the horizontal lift of $\text{grad}_d f$ is equal to $\text{grad}_d f = \Psi(\text{egrad}_d f)$. The total numerical cost of computing the Riemannian gradient depends on computing the partial derivatives, which is $O(|\Omega| r_1 r_2 r_3)$.

Proposition 4. The cost function [7] at $(U_1, U_2, U_3, G)$ under the quotient manifold $\mathcal{M}$ endowed with
the Riemannian metric \( \langle \cdot, \cdot \rangle \) admits the horizontal lift of the Riemannian gradient

\[
\begin{align*}
(S_1(U_3 \otimes U_2)G_1^T G_1 T^{-1} - U_1 B_{U_1} (G_1 T^{-1}), \\
S_2(U_3 \otimes U_1)G_2^T G_2 T^{-1} - U_2 B_{U_2} (G_2 T^{-1}), \\
S_3(U_2 \otimes U_1)G_3^T (G_3 G_3^T)^{-1} - U_3 B_{U_3} (G_3 G_3^T)^{-1},
\end{align*}
\]

\[
\sum_{i=1}^{3} U_i^T U_i = 1, \quad U_i^T U_i \leq 3 U_3^T U_3,
\]

where \( B_{U_i} \) for \( d \in \{1, 2, 3\} \) are the solutions to the Lyapunov equations

\[
\begin{align*}
B_{U_1} G_1^T T + G_1 G_1 T B_{U_1} &= 2 \text{Sym}(G_1 G_1^T T (S_1(U_3 \otimes U_2)G_1^T)), \\
B_{U_2} G_2^T T + G_2 G_2 T B_{U_2} &= 2 \text{Sym}(G_2 G_2^T T (S_2(U_3 \otimes U_1)G_2^T)), \\
B_{U_3} G_3^T T + G_3 G_3 T B_{U_3} &= 2 \text{Sym}(G_3 G_3^T T (S_3(U_2 \otimes U_1)G_3^T)),
\end{align*}
\]

where \( \text{Sym}(\cdot) \) extracts the symmetric part of a matrix.

**Initial guess for the step-size.** Following [17, 26, 13], the least-squares structure of the cost function in (1) is exploited to compute a linearized step-size guess efficiently along a search direction by considering a polynomial approximation of degree 2 over the manifold. Given a search direction \( \xi_x \in \mathcal{H}_x \), the step-size guess is \( \arg\min_{\xi \in \mathbb{R}} \| \mathcal{P}_\Omega (G \times_1 U_1 \times_2 U_2 \times_3 U_3 + sG \times_1 \xi U_1 \times_2 U_2 \times_3 U_3 + sG \times_1 U_1 \times_2 \xi U_2 \times_3 U_3 + sG \times_1 U_1 \times_2 U_2 \times_3 \xi U_3 - \mathcal{P}_\Omega (X^*) \|_{F}^2 \), which has a closed-form expression and the numerical cost of computing it is \( O(|\Omega| r_1 r_2 r_3) \).

**Stochastic gradient descent in online setting.** In the online setting, we update \((U_1, U_2, U_3, G)\) every time a frontal slice, i.e., a matrix \( \in \mathbb{R}^{n_1 \times n_2} \), is randomly sampled from \( X_{s_{1,2,3},i}^* \). Equivalently, we assume that the tensor grows along the third dimension. More concretely, we calculate the rank-one Riemannian gradient (12) for the input slice. \((U_1, U_2, U_3, G)\) are updated by taking a step along the negative Riemannian gradient direction. Subsequently, we retract using \( R_x \). A popular formula for the step-size \( \gamma_k \) at \( k \)-th update is \( \gamma_k = \gamma_0 / (1 + \gamma_0 \lambda k) \), where \( \gamma_0 \) is the initial step-size and \( \lambda \) is a fixed reduction factor. Following [3], we select \( \gamma_0 \) in the pre-training phase using a small sample size of a training set. \( \lambda \) is fixed to \( 10^{-7} \).

**Computational cost.** The total computational cost per iteration of our proposed conjugate gradient implementation is \( O(|\Omega| r_1 r_2 r_3) \), where \( |\Omega| \) is the number of known entries. It should be stressed that the computational cost of our conjugate gradient implementation is equal to that of [13]. In the online setting, each stochastic gradient descent update costs \( O(|\Omega_{slice}| r_1 r_2 + n_1 r_1^2 + n_2 r_2^2 + Tr_3^2 + r_1 r_2 r_3) \), where \( |\Omega_{slice}| \) is the number of known entries of the current frontal slice of the incomplete tensor \( X_{s_{1,2,3},i} \), and \( T \) is the number of slices that we have seen along \( n_3 \) direction.

## 5 Numerical comparisons

In the batch setting, we show a number of numerical comparisons of our proposed conjugate gradient algorithm with state-of-the-art algorithms that include TOpt [8] and geomCG [13], for comparisons with Tucker decomposition based algorithms, and HaLRTC [15], Latent [25], and Hard [24] as nuclear norm minimization algorithms. In the online setting, we compare our proposed stochastic gradient descent algorithm with CANDECOMP/PARAFAC based TeCPSGD [16] and OLSTEC [10]. All simulations are performed in Matlab on a 2.6 GHz Intel Core i7 machine with 16 GB RAM. For specific operations with unfoldings of \( S \), we use the mex interfaces for Matlab that are provided by the authors of geomCG. For large-scale instances, our algorithm is only compared with geomCG as others cannot handle them. Cases S and R are for batch instances, whereas Case O is for online instances.

Since the dimension of the space of a tensor \( \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) of rank \( r = (r_1, r_2, r_3) \) is \( \dim(\mathcal{M}/\sim) = \sum_{d=1}^{3} (n_d r_d - r_d^2) + r_1 r_2 r_3 \), we randomly and uniformly select known entries based on a multiple of the dimension, called the over-sampling (OS) ratio, to create the train set \( \Omega \). Algorithms are initialized randomly, as suggested in [13], and are stopped when either the mean square error (MSE) on the train set \( \Omega \) is below \( 10^{-12} \) or the number of iterations exceeds 250. We also evaluate the mean square error on a test set \( \Gamma \), which is different from \( \Omega \). Five runs are performed in each scenario and the plots show all of them. The time plots are shown with standard deviations. It should be noted that we show most numerical
comparisons on the test set $\Omega$ as it allows to compare with nuclear norm minimization algorithms, which optimize a different (training) cost function. Additional plots are provided as supplementary material.

**Case S1: comparison with the Euclidean metric.** We first show the benefit of the proposed metric (9) over the conventional choice of the Euclidean metric that exploits the product structure of $\mathcal{M}$ and symmetry (3). This is defined by combining the individual natural metrics for $\text{St}(r_d, n_d)$ and $\mathbb{R}^{r_1 \times r_2 \times r_3}$. For simulations, we randomly generate a tensor of size $200 \times 200 \times 200$ and rank $r = (10, 10, 10)$. OS is 10. For simplicity, we compare gradient descent algorithms with Armijo backtracking linesearch for both the metric choices. Figure 2(a) shows that the algorithm with the metric (9) gives a superior performance in $\text{train error}$ than that of the conventional metric choice.

**Case S2: small-scale instances.** Small-scale tensors of size $100 \times 100 \times 100$, $150 \times 150 \times 150$, and $200 \times 200 \times 200$ and rank $r = (10, 10, 10)$ are considered. OS is $\{10, 20, 30\}$. Figure 2(b) shows that our proposed algorithm has faster convergence than others. In Figure 2(c), the lowest test errors are obtained by our proposed algorithm and geomCG.

**Case S3: large-scale instances.** We consider large-scale tensors of size $3000 \times 3000 \times 3000$, $5000 \times 5000 \times 5000$, and $10000 \times 10000 \times 10000$ and ranks $r = (5, 5, 5)$ and $(10, 10, 10)$. OS is 10. Our proposed algorithm outperforms geomCG in Figure 2(d).

**Case S4: influence of low sampling.** We look into problem instances from scarcely sampled data, e.g., OS is 4. The test requires completing a tensor of size $10000 \times 10000 \times 10000$ and rank $r = (5, 5, 5)$.
implementations of TeCPSGD and OLSTEC are computationally more intensive than ours, our plots show lower test errors than geomCG across different ranks.

We consider instances where the dimensions and ranks along certain modes are different than others. Two cases are considered. Case (7.a) considers tensors size 20000 × 7000 × 7000, 30000 × 6000 × 6000, and 40000 × 5000 × 5000 with rank r = (5, 5, 5). Case (7.b) considers a tensor of size 10000 × 10000 × 10000 with ranks (7, 6, 6), (10, 5, 5), and (15, 5, 5). In all the cases, the proposed algorithm converges faster than geomCG as shown in Figure 2(b).

**Case R1: hyperspectral image.** We consider the hyperspectral image “Ribeira” discussed in [23][13]. The tensor size is 1017 × 1340 × 33, where each slice corresponds to a particular image measured at a different wavelength. As suggested in [23][13], we resize it to 203 × 268 × 33. We perform five random samplings of the pixels based on the OS values 11 and 22, corresponding to the rank r=(15, 15, 6) adopted in [13]. This set is further randomly split into 80/10/10–train/validation/test partitions. The algorithms are stopped when the MSE on the validation set starts to increase. While OS = 22 corresponds to the observation ratio of 10% studied in [13]. OS = 11 considers a challenging scenario with the observation ratio of 5%. Figures 2(i) shows the good performance of our algorithm.

**Table 2: Cases R1 and R2:** test MSE on $\Gamma$ and time in seconds

| Ribeira                          | Algorithm | OS = 11 | OS = 22 |
|---------------------------------|-----------|---------|---------|
|                                 | Time      | MSE on $\Gamma$ | Time      | MSE on $\Gamma$ |
| Proposed                        | 33 ± 13   | 8.2095 × 10^{-4} ± 1.7 × 10^{-5} | 67 ± 43   | 6.9516 × 10^{-4} ± 1.1 × 10^{-5} |
| geomCG                          | 25 ± 19   | 8.2322 × 10^{-4} ± 4.2 × 10^{-5} | 150 ± 25  | 6.2990 × 10^{-4} ± 2.5 × 10^{-5} |
| HaLRTC                          | 40 ± 4    | 2.2091 × 10^{-4} ± 3.6 × 10^{-5} | 48 ± 0    | 1.3881 × 10^{-4} ± 3.7 × 10^{-5} |
| Topc                            | 80 ± 32   | 1.7504 × 10^{-3} ± 3.8 × 10^{-4} | 27 ± 21   | 2.1259 × 10^{-3} ± 4.8 × 10^{-4} |
| Latent                          | 60 ± 3    | 2.9236 × 10^{-3} ± 6.4 × 10^{-4} | 58 ± 3    | 1.6339 × 10^{-3} ± 3.3 × 10^{-4} |
| Hard                            | 400 ± 5   | 6.5090 × 10^{-2} ± 6.1 × 10^{-3} | 402 ± 4   | 6.3989 × 10^{-2} ± 9.8 × 10^{-3} |
| MovieLens-10M                   | r         |         |         |         |
|                                 | Time      | MSE on $\Gamma$ | Time      | MSE on $\Gamma$ |
| Proposed                        | 1748 ± 441 | 0.6762 × 1.5 × 10^{-5} | 2981 ± 40 | 0.6956 × 2.8 × 10^{-5} |
| geomCG                          | 5035 ± 22 | 0.6093 ± 3.3 × 10^{-5} | 6053 ± 45 | 0.7308 ± 7.1 × 10^{-5} |
| (10, 10, 10)                    | 11370 ± 103 | 0.7563 ± 7.1 × 10^{-5} | 13853 ± 118 | 0.8085 ± 3.3 × 10^{-5} |

Figure 2(c) shows the superior performance of the proposed algorithm against geomCG. Whereas the test error increases for geomCG, it decreases for the proposed algorithm.

**Case S5: influence of ill-conditioning and low sampling.** We consider the problem instance of Case R4 with OS = 5. Additionally, for generating the instance, we impose a diagonal core $G$ with exponentially decaying positive values of condition numbers (CN) 5, 50, and 100. Figure 2(f) shows that the proposed algorithm outperforms geomCG for all the considered CN values.

**Case S6: influence of noise.** We evaluate the convergence properties of algorithms under the presence of noise by adding scaled Gaussian noise $\mathcal{P}(\mathcal{E})$ to $\mathcal{P}(\mathcal{X}^*)$ as in [13]. The different noise levels are $\epsilon = \{10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}\}$. In order to evaluate for $\epsilon = 10^{-12}$, the stopping threshold on the MSE of the train set is lowered to 10^{-2}. The tensor size and rank are same as in Case S4 and OS is 10. Figure 2(g) shows that the test error for each $\epsilon$ is almost identical to the $\epsilon^2 \|\mathcal{P}(\mathcal{X}^*)\|_F^2$ [13], but our proposed algorithm converges faster than geomCG.

**Case S7: rectangular instances.** We consider instances where the dimensions and ranks along certain modes are different than others. We adopt in [13]. The tensor size is 1017 × 1340 × 33, where each slice corresponds to a particular image measured at a different wavelength. As suggested in [23][13], we resize it to 203 × 268 × 33. We perform five random samplings of the pixels based on the OS values 11 and 22, corresponding to the rank $r=(15, 15, 6)$ adopted in [13]. This set is further randomly split into 80/10/10–train/validation/test partitions. The algorithms are stopped when the MSE on the validation set starts to increase. While OS = 22 corresponds to the observation ratio of 10% studied in [13]. OS = 11 considers a challenging scenario with the observation ratio of 5%. Figures 2(i) shows the good performance of our algorithm.

**Figure 2(c)** shows the superior performance of the proposed algorithm against geomCG. Whereas the test error increases for geomCG, it decreases for the proposed algorithm.

In Table 2, our proposed algorithm consistently outperforms geomCG for all the considered CN values.

**Case R2: MovieLens-10M**[13] This dataset contains 10000054 ratings corresponding to 71567 users and 10681 movies. We split the time into 7-days wide bins results, and finally, get a tensor of size 71567 × 10681 × 731. The fraction of known entries is less than 0.002%. The completion task on this dataset reveals periodicity of the latent genres. We perform five random 80/10/10–train/validation/test partitions. The maximum iteration threshold is set to 500. In Table 2 our proposed algorithm consistently gives lower test errors than geomCG across different ranks.

**Case O: online instances.** We compare the proposed stochastic gradient descent algorithm with its batch counterpart gradient descent algorithm and with TeCPSGD [10] and OLSTEC [10]. As the implementations of TeCPSGD and OLSTEC are computationally more intensive than ours, our plots

[http://grouplens.org/datasets/movielens/](http://grouplens.org/datasets/movielens/)
only show test MSE against the number of outer iterations, i.e., the number of the passes through the data.

Figure 3(a) shows comparisons on a synthetic instance of tensor size $100 \times 100 \times 10000$ with rank $r = (5, 5, 5)$. $\gamma_0$ is selected from the step-size list $\{8, 9, 10, 11, 12\}$ in the pre-training phase. 10% entries are randomly observed. The pre-training uses 10% frontal slices of all the slices. The maximum number of outer loops is set to 100. Figure 3(a) shows five different runs, where the online algorithm has the same asymptotic convergence behavior as the batch counterpart on a test dataset $\Gamma$. Figure 3(b) shows comparisons on the Airport Hall surveillance video sequence dataset of size $176 \times 144$ with 1000 frames. $\gamma_0$ is selected from $\{30, 40, 50, 60, 70\}$ and 10% frontal slices are selected for pre-training. 2% of the entries are observed. In Figure 3(b), both the proposed online and batch algorithms achieve lower test errors than TeCPSGD and OLSTEC.

6 Conclusion

We have proposed preconditioned batch (conjugate gradient) and online (stochastic gradient descent) algorithms for the tensor completion problem. The algorithms stem from the Riemannian preconditioning approach that exploits the fundamental structures of symmetry (due to non-uniqueness of Tucker decomposition) and least-squares of the cost function. A novel Riemannian metric (inner product) is proposed that enables to use the versatile Riemannian optimization framework. Numerical comparisons suggest that our proposed algorithms have a superior performance on different benchmarks.

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http://perception.i2r.a-star.edu.sg/bk_model/bk_index.html
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A Proof and derivation of manifold-related ingredients

The concrete computations of the optimization-related ingredients presented in the paper are discussed below.

The total space is \( \mathcal{M} := \text{St}(r_1, n_1) \times \text{St}(r_2, n_2) \times \text{St}(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3} \). Each element \( x \in \mathcal{M} \) has the matrix representation \((U_1, U_2, U_3, G)\). Invariance of Tucker decomposition under the transformation \((U_1, U_2, U_3, G) \mapsto (U_1 O_1, U_2 O_2, U_3 O_3, G \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)\) for all \( O_d \in \mathcal{O}(r_d) \), the set of orthogonal matrices of size of \( r_d \times r_d \) results in equivalence classes of the form \([x] = [(U_1, U_2, U_3, G)] := \{(U_1 O_1, U_2 O_2, U_3 O_3, G \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T) : O_d \in \mathcal{O}(r_d)\}\).

A.1 Tangent space characterization and the Riemannian metric

The tangent space, \( T_x \mathcal{M} \), at \( x \) given by \((U_1, U_2, U_3, G)\) in the total space \( \mathcal{M} \) is the product space of the tangent spaces of the individual manifolds. From \([1]\), the tangent space has the matrix characterization

\[
T_x \mathcal{M} = \{ (Z_{U_1}, Z_{U_2}, Z_{U_3}, Z_G) \in \mathbb{R}^{n_1 \times r_1} \times \mathbb{R}^{n_2 \times r_2} \times \mathbb{R}^{n_3 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} : U_d^T Z_{U_d} + Z_{U_d}^T U_d = 0, \text{ for } d \in \{1, 2, 3\}\}.
\]

The proposed metric \( g_x : T_x \mathcal{M} \times T_x \mathcal{M} \rightarrow \mathbb{R} \) is

\[
g_x(\xi_x, \eta_x) = \langle \xi_{U_1}, \eta_{U_1} (G_1 G_1^T) \rangle + \langle \xi_{U_2}, \eta_{U_2} (G_2 G_2^T) \rangle + \langle \xi_{U_3}, \eta_{U_3} (G_3 G_3^T) \rangle + \langle \xi_G, \eta_G \rangle,
\]

where \( \xi_x, \eta_x \in T_x \mathcal{M} \) are tangent vectors with matrix characterizations \((\xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_G)\) and \((\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_G)\), respectively and \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product.

A.2 Characterization of the normal space

Given a vector in \( \mathbb{R}^{n_1 \times r_1} \times \mathbb{R}^{n_2 \times r_2} \times \mathbb{R}^{n_3 \times r_3} \times \mathbb{R}^{r_1 \times r_2 \times r_3} \), its projection onto the tangent space \( T_x \mathcal{M} \) is obtained by extracting the component \emph{normal}, in the metric sense, to the tangent space. This section describes the characterization of the \emph{normal space}, \( N_x \mathcal{M} \).

Let \( \zeta_x = (\xi_{U_1}, \xi_{U_2}, \xi_{U_3}, \xi_G) \in N_x \mathcal{M} \) and \( \eta_x = (\eta_{U_1}, \eta_{U_2}, \eta_{U_3}, \eta_G) \in T_x \mathcal{M} \). Since \( \zeta_x \) is orthogonal to \( \eta_x \), i.e., \( g_x(\zeta_x, \eta_x) = 0 \), the conditions

\[
\text{Trace}(G_d G_d^T \xi_{U_d} \eta_{U_d}) = 0, \text{ for } d \in \{1, 2, 3\}
\]

must hold for all \( \eta_x \) in the tangent space. Additionally from \([1]\), \( \eta_{U_d} \) has the characterization

\[
\eta_{U_d} = U_d \Omega + U_d \perp K,
\]

where \( \Omega \) is any skew-symmetric matrix, \( K \) is a any matrix of size \((n_d - r_d) \times r_d\), and \( U_d \perp \) is any \( n_d \times (n_d - r_d) \) that is orthogonal complement of \( U_d \). Let \( \tilde{\zeta}_{U_d} = \xi_{U_d} G_d G_d^T \eta_{U_d} + \xi_G \) and let \( \tilde{\zeta}_{U_d} \) is defined as

\[
\tilde{\zeta}_{U_d} = U_d A + U_d \perp B
\]

without loss of generality, where \( A \in \mathbb{R}^{r_d \times r_d} \) and \( B \in \mathbb{R}^{(n_d - r_d) \times r_d} \) are to be characterized from \([A.3]\) and \([A.4]\). A few standard computations show that \( A \) has to be symmetric and \( B = 0 \). Consequently, \( \tilde{\zeta}_{U_d} = U_d S_{U_d} \), where \( S_{U_d} = S_{U_d}^T \). Equivalently, \( \xi_{U_d} = U_d S_{U_d} (G_d G_d^T)^{-1} \) for a symmetric matrix \( S_{U_d} \). Finally, the normal space \( N_x \mathcal{M} \) has the characterization

\[
N_x \mathcal{M} = \{ (U_1 S_{U_1}, (G_1 G_1^T)^{-1}, U_2 S_{U_2} (G_2 G_2^T)^{-1}, U_3 S_{U_3} (G_3 G_3^T)^{-1}, 0) : S_{U_d} \in \mathbb{R}^{r_d \times r_d}, S_{U_d}^T = S_{U_d} \text{ for } d \in \{1, 2, 3\}\}.
\]
A.3 Characterization of the vertical space

The horizontal space projector of a tangent vector is obtained by removing the component along the vertical direction. This section shows the matrix characterization of the vertical space $V_x$.

$V_x$ is defined as the linearization of the equivalence class $[(U_1, U_2, U_3, \mathcal{G})]$ at $x = [(U_1, U_2, U_3, \mathcal{G})]$. Equivalently, $V_x$ is the linearization of $(U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} O_1 \times O_2 \times O_3)$ along $O_d \in \mathcal{O}(r_d)$ at the identity element for $d \in \{1, 2, 3\}$. From the characterization of linearization of an orthogonal matrix [I], we have the characterization for the vertical space as

$$V_x = \left\{ (U_1 \Omega_1, U_2 \Omega_2, U_3 \Omega_3, - (\mathcal{G} \times_1 \Omega_1 + \mathcal{G} \times_2 \Omega_2 + \mathcal{G} \times_3 \Omega_3)) : \Omega_d \in \mathbb{R}^{r_d \times r_d}, \Omega_d^T = -\Omega_d \mbox{ for } d \in \{1, 2, 3\} \right\}. \quad (A.7)$$

A.4 Characterization of the horizontal space

The characterization of the horizontal space $H_x$ is derived from its orthogonal relationship with the vertical space $V_x$.

Let $\xi = (\xi_1, \xi_2, \xi_3) \in H_x$, and $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_0) \in V_x$. Since $\xi_x$ must be orthogonal to $\zeta_x$, which is equivalent to $g_x(\xi_x, \zeta_x) = 0$ in (A.2), the characterization for $\xi_x$ is derived from (A.2) and (A.7).

$$g_x(\xi_x, \zeta_x) = \langle \xi_1, \xi_1 (G_1 G_d^T) \rangle + \langle \xi_2, \xi_2 (G_2 G_d^T) \rangle + \langle \xi_3, \xi_3 (G_3 G_d^T) \rangle + \langle \zeta_0, \zeta_0 \rangle = \langle \xi_1, (U_1 \Omega_1) (G_1 G_d^T) \rangle + \langle \xi_2, (U_2 \Omega_2) (G_2 G_d^T) \rangle + \langle \xi_3, (U_3 \Omega_3) (G_3 G_d^T) \rangle + \langle \zeta_0, -\mathcal{G} \times_1 \Omega_1 + \mathcal{G} \times_2 \Omega_2 + \mathcal{G} \times_3 \Omega_3 \rangle.$$

(Switch to unfoldings of $\mathcal{G}$.)

$$= \text{Trace}(G_1 G_d^T) \zeta_1 U_1 \Omega_1 + \text{Trace}(G_2 G_d^T) \zeta_2 U_2 \Omega_2 + \text{Trace}(G_3 G_d^T) \zeta_3 U_3 \Omega_3 + \text{Trace}(G_d \zeta_0 \Omega_0).$$

where $G_d$ is the mode-$d$ unfolding of $\mathcal{G}$. Since $g_x(\xi_x, \zeta_x)$ above should be zero for all skew-matrices $\Omega_d$, $\xi_x = (\xi_1, \xi_2, \xi_3, \xi_0) \in H_x$ must satisfy

$$(G_d G_d^T) \zeta_0 U_d + \xi_0 G_d^T G_d = 0 \quad \text{is symmetric for } d \in \{1, 2, 3\}. \quad (A.8)$$

A.5 Proof of Proposition 1

We first introduce the following lemma:

**Lemma 1.** Let $(U_1, U_2, U_3, \mathcal{G}) \in \text{St}(r_1, n_1) \times \text{St}(r_2, n_2) \times \text{St}(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3}$ and $\xi_{[(U_1, U_2, U_3, \mathcal{G})]}$ be a tangent vector to the quotient manifold at $[(U_1, U_2, U_3, \mathcal{G})]$. The horizontal lifts of $\xi_{[(U_1, U_2, U_3, \mathcal{G})]}$ at $(U_1, U_2, U_3, \mathcal{G})$ and $(U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$ are related for $O_d \in \mathcal{O}(r_d)$ as follows,

$$(\xi_1 U_1, \xi_2 U_2, \xi_3 U_3, \xi_0, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T) = (\xi_1 O_1, \xi_2 O_2, \xi_3 O_3, \xi_0, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T). \quad (A.9)$$

**Proof.** Let $f : \text{St}(r_1, n_1) \times \text{St}(r_2, n_2) \times \text{St}(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3} \times \mathcal{O}(r_1) \times \mathcal{O}(r_2) \times \mathcal{O}(r_3) \rightarrow \mathbb{R}$ be an arbitrary smooth function, and define

$$\tilde{f} := f \circ \pi : \text{St}(r_1, n_1) \times \text{St}(r_2, n_2) \times \text{St}(r_3, n_3) \times \mathbb{R}^{r_1 \times r_2 \times r_3} \rightarrow \mathbb{R},$$

where $\pi$ is the mapping $\pi : \mathcal{M} \rightarrow \mathcal{M}/\sim$ defined by $x \mapsto [x]$.

Consider the mapping $h : (U_1, U_2, U_3, \mathcal{G}) \mapsto (U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$,
where $O_d \in \mathcal{O}(r_d)$. Since $\pi(h(U_1, U_2, U_3, \mathcal{G})) = \pi(U_1, U_2, U_3, \mathcal{G})$ for all $(U_1, U_2, U_3, \mathcal{G})$, we have

$$f(h(U_1, U_2, U_3, \mathcal{G})) = f(U_1, U_2, U_3, \mathcal{G}).$$

By taking the differential of both sides,

$$Df(h(U_1, U_2, U_3, \mathcal{G}))(\frac{\partial}{\partial U_1}, \frac{\partial}{\partial U_2}, \frac{\partial}{\partial U_3}, \frac{\partial}{\partial \mathcal{G}}) = D\pi(U_1, U_2, U_3, \mathcal{G})(\mathcal{O}, \xi_1, \xi_2, \xi_3, \xi_\mathcal{G}).$$

By noting the definition of $(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G})$, i.e., $D\pi(U_1, U_2, U_3, \mathcal{G})(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = \xi((\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}))$,

the right side of (A.10) is

$$D\tilde{f}(U_1, U_2, U_3, \mathcal{G})(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = Df(\pi(U_1, U_2, U_3, \mathcal{G}))(\mathcal{O}, \xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = Df(\pi(U_1, U_2, U_3, \mathcal{G}))(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}),$$

where the chain rule is applied to the first equality.

Moreover, from the directional derivatives of the mapping $h$, the bracket of the left side of (A.10) is obtained as

$$Dh(U_1, U_2, U_3, \mathcal{G})(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = (\xi_1, O_1, O_2, O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T).$$

Therefore, (A.10) yields

$$D\tilde{f}(U_1, U_2, U_3, \mathcal{G})(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = Df(\pi(U_1, U_2, U_3, \mathcal{G}))(\mathcal{O}, \xi_1, \xi_2, \xi_3, \xi_\mathcal{G}),$$

where we address the equivalence class $\pi(U_1, U_2, U_3, \mathcal{G}) = (U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$. The left side of (A.11) is further transformed by the chain rule as

$$Df(\pi(U_1, U_2, U_3, \mathcal{G}))(\mathcal{O}, \xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = Df(\pi(U_1, U_2, U_3, \mathcal{G}))(\mathcal{O}, \xi_1, \xi_2, \xi_3, \xi_\mathcal{G})$$

By comparing the right sides of (A.11) and (A.12), since this equality holds for any smooth function $f$, it implies that

$$D\pi(U_1, U_2, U_3, \mathcal{G})(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G}) = \xi((U_1, U_2, U_3, \mathcal{G})).$$

Finally, we check whether $(\xi_1, O_1, O_2, O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$ is an element of

$$\mathcal{H}(U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T).$$

Addressing that the mode-1 unfolding of $\mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T$ is $O_1^T G_1(O_3^T \times O_2^T)^T$, plugging $(\xi_1, O_1, O_2, O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$ into $(G_d G_d^T)\xi_{G_d^T} U_d + \xi_{G_d^T} G_d^T$ in (A.8) yields

$$\xi_{G_1} G_1^T O_1 + O_1^T \xi_{G_1} G_1^T O_1 = \xi_{G_1} (G_1^T)^T \xi_{U_1} U_1 + \xi_{G_1} G_1^T O_1.$$  

(A.14)

Since $(\xi_1, \xi_2, \xi_3, \xi_\mathcal{G})$ is a symmetric matrix, the obtained result is also symmetric. Therefore, $(\xi_1, O_1, O_2, O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$ is a horizontal vector at $(U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$. This implies that $(\xi_1, O_1, O_2, O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$ is the horizontal lift of $\xi$ at $(U_1 O_1, U_2 O_2, U_3 O_3, \mathcal{G} \times_1 O_1^T \times_2 O_2^T \times_3 O_3^T)$, and the proof is completed.

Now, the proof of Proposition 1 is given below using the result in Lemma 1.
Proof. Plugging \( \xi'_{U_1} = \xi_{U_1;_1}, \eta'_{U_1} = \eta_{U_1;_1} \), and \( G'_1 = O^T_3 G_1 (O^T_3 \otimes O^T_2) \) into the first term of (A.2) yields

\[
\langle \xi_{U_1;_1}, \eta_{U_1;_1} (G'_1 G'^T_1) \rangle = \text{Trace}(\xi'_{U_1;_1} \eta_{U_1;_1} (G'_1 G'^T_1)) = \text{Trace}(\langle \xi_{U_1;_1}, \eta_{U_1;_1} (G'_1 G'^T_1) \rangle) = \text{Trace}\left[ (\xi_{U_1;_1} (O^T_1)(O^T_3 \otimes O^T_2) T) (O^T_3 G_1 (O^T_3 \otimes O^T_2) T) \right] = \text{Trace}\left[ (\xi_{U_1;_1} (O^T_1)(O^T_3 \otimes O^T_2) T) (O^T_3 G_1 (O^T_3 \otimes O^T_2) T) G_1 \right] = \text{Trace}\left[ (\xi'_{U_1;_1} \eta_{U_1;_1} (G'_1 G'^T_1) \right] = \langle \xi_{U_1;_1}, \eta_{U_1;_1} (G'_1 G'^T_1) \rangle.
\]

Since the same equalities against the each term in the metric (A.2) corresponding to \( U_2, U_3 \) and \( G \) hold, we finally obtain the invariant property that the proposition claims;

\[
g(U_1, U_2, U_3, G) = \text{Trace}(\xi_{U_1;_1} \eta_{U_1;_1} (G'_1 G'^T_1) \rangle) = \text{Trace}(\langle \xi_{U_1;_1}, \eta_{U_1;_1} (G'_1 G'^T_1) \rangle) = \text{Trace}\left[ (\xi_{U_1;_1} (O^T_1)(O^T_3 \otimes O^T_2) T) (O^T_3 G_1 (O^T_3 \otimes O^T_2) T) \right]\]

A.6 Proof of Proposition 2 (derivation of the tangent space projector)

Proof. The tangent space \( T_x M \) projector is obtained by extracting the component normal to \( T_x M \) in the ambient space. The normal space \( N_x M \) has the matrix characterization shown in (A.6). The operator \( \Psi_x : \mathbb{R}^{n_1 \times r_1} \times \mathbb{R}^{n_2 \times r_2} \times \mathbb{R}^{n_3 \times r_3} \times \mathbb{R}^{r_1 \times r_2} \rightarrow T_x M : (Y_{U_1}, Y_{U_2}, Y_{U_3}, Y_G) \rightarrow \Psi_x (Y_{U_1}, Y_{U_2}, Y_{U_3}, Y_G) \) has the expression

\[
\Psi_x (Y_{U_1}, Y_{U_2}, Y_{U_3}, Y_G) = (Y_{U_1} - U_1 S_{U_1}(G_1 G^T_1)^{-1}, Y_{U_2} - U_2 S_{U_2}(G_2 G^T_2)^{-1}, Y_{U_3} - U_3 S_{U_3}(G_3 G^T_3)^{-1}, Y_G).
\]  

(A.15)

From the definition of the tangent space in (A.1), \( U_d \) should satisfy

\[
\eta_{U_d}^T U_d + U_d^T \eta_{U_d} = (Y_{U_d} - U_d S_{U_d}(G_d G^T_d)^{-1})^T U_d + U_d^T (Y_{U_d} - U_d S_{U_d}(G_d G^T_d)^{-1}) = Y_{U_d}^T U_d - (G_d G^T_d)^{-1} S_{U_d} U_d^T U_d + U_d^T Y_{U_d} - U_d^T U_d S_{U_d}(G_d G^T_d)^{-1} = Y_{U_d}^T U_d - (G_d G^T_d)^{-1} S_{U_d} + U_d^T Y_{U_d} - S_{U_d}(G_d G^T_d)^{-1} = 0.
\]

Multiplying \( (G_d G^T_d) \) from the right and left sides results in

\[
(G_d G^T_d)^{-1} S_{U_d} + S_{U_d}(G_d G^T_d)^{-1} = Y_{U_d}^T U_d + U_d^T Y_{U_d} \]

\[
S_{U_d} G_d G^T_d + G_d G^T_d S_{U_d} = G_d G^T_d (Y_{U_d}^T U_d + U_d^T Y_{U_d}) G_d G^T_d \text{ for } d \in \{1, 2, 3\},
\]

(A.16)

that are solved efficiently with the Matlab’s \texttt{lyap} routine.

\[\Box\]
A.7 Proof of Proposition 3 (derivation of the horizontal space projector)

Proof. We consider the projection of a tangent vector $\eta_x = (\eta_1, \eta_2, \eta_3, \eta_g) \in T_xM$ into a vector $\xi_x = (\xi_1, \xi_2, \xi_3, \xi_g) \in H_x$. This is achieved by subtracting the component in the vertical space $V_x$ in (A.7) as

$$
\begin{align*}
\eta_1 &= \eta_1 - U_1\Omega_1 + U_1\Omega_1, \\
\eta_2 &= \eta_2 - U_2\Omega_2 + U_2\Omega_2, \\
\eta_3 &= \eta_3 - U_3\Omega_3 + U_3\Omega_3, \\
\eta_g &= \eta_g - (G_1\times\Omega_1 + G_2\times\Omega_2 + G_3\times\Omega_3) - (G_1\times\Omega_1 + G_2\times\Omega_2 + G_3\times\Omega_3).
\end{align*}
$$

As a result, the horizontal operator $\Pi_x : T_xM \to H_x : \eta_x \mapsto \Pi_x(\eta_x)$ has the expression

$$\Pi_x(\eta_x) = (\eta_1 - U_1\Omega_1, \eta_2 - U_2\Omega_2, \eta_3 - U_3\Omega_3, \eta_g - (G_1\times\Omega_1 + G_2\times\Omega_2 + G_3\times\Omega_3)), \quad (A.17)$$

where $\eta_x = (\eta_1, \eta_2, \eta_3, \eta_g) \in T_xM$ and $\Omega_d$ is a skew-symmetric matrix of size $r_d \times r_d$. The skew-matrices $\Omega_d$ for $d = \{1, 2, 3\}$ that are identified based on the conditions (A.8).

It should be noted that the tensor $G_1\times\Omega_1 + G_2\times\Omega_2 + G_3\times\Omega_3$ in (A.7) has the following equivalent unfoldings.

$$
\begin{align*}
G_1\times\Omega_1 + G_2\times\Omega_2 + G_3\times\Omega_3 &= \text{mode-1} \left( \Omega_1G_1 + \Omega_1G_1(I_r \otimes \Omega_2)^T + G_1(\Omega_3 \otimes I_r)^T \right) \\
&= \text{mode-2} \left( G_1(I_r \otimes \Omega_1)^T + \Omega_2G_2 + G_2(\Omega_3 \otimes I_r)^T \right) \\
&= \text{mode-3} \left( G_3(I_r \otimes \Omega_1)^T + G_3(\Omega_2 \otimes I_r)^T + \Omega_3G_3 \right).
\end{align*}
$$

Plugging $\xi_1 = \eta_1 - U_1\Omega_1$ and $\xi_1G_1 = \eta_1G_1 + G_1(I_r \otimes \Omega_2)^T + G_1(\Omega_3 \otimes I_r)^T$ into (A.8) and using the relation $G(I \otimes B)^T = A^T \otimes B^T$ results in

$$
(G_1^T \xi_1 U_T + \xi_1 G_1^T = (G_1^T G_1^T)^T)(\eta_1 - U_1\Omega_1)^T U_1 + \{\eta_1G_1 + \Omega_1G_1 + G_1(I_r \otimes \Omega_2)^T + G_1(\Omega_3 \otimes I_r)^T\}) G_1^T
$$

which should be a symmetric matrix due to (A.8), i.e., $(G_1^T G_1^T)(\eta_1 - U_1\Omega_1)G_1 = ((G_1^T G_1^T)^T(\eta_1 - U_1\Omega_1)^T G_1)^T.$

Subsequently,

$$
(G_1^T G_1^T)(\eta_1 - U_1\Omega_1) + G_1G_1^T = (G_1^T G_1^T)(\eta_1 - U_1\Omega_1) + G_1G_1^T = (G_1^T G_1^T)^T(\eta_1 - U_1\Omega_1)^T G_1,
$$

which is equivalent to

$$
G_1^T \Omega_1 + G_1G_1^T = G_1(I_r \otimes \Omega_2)G_1^T - G_1(\Omega_3 \otimes I_r)G_1^T = \text{Skew}(G_1^T G_1^T) + \text{Skew}(G_1^T G_1^T).
$$

Here Skew(·) extracts the skew-symmetric part of a square matrix, i.e., Skew(D) = (D - D^T)/2.

Finally, we obtain the coupled Lyapunov equations

$$
\begin{align*}
G_1^T \Omega_1 + G_1G_1^T &= \text{Skew}(G_1^T G_1^T) + \text{Skew}(G_1^T G_1^T), \\
G_2^T \Omega_2 + G_2G_2^T &= \text{Skew}(G_2^T G_2^T) + \text{Skew}(G_2^T G_2^T), \\
G_3^T \Omega_3 + G_3G_3^T &= \text{Skew}(G_3^T G_3^T) + \text{Skew}(G_3^T G_3^T),
\end{align*}
$$

that are solved efficiently with the Matlab’s pcg routine that is combined with a specific preconditioner resulting from the Gauss-Seidel approximation of (A.18).
A.8 Proof of Proposition 4 (derivation of the Riemannian gradient formula)

Proof. Let \( f(\mathcal{X}) = ||P_\Omega(\mathcal{X}) - P_\Omega(\mathcal{X}^*)||_2^2 / ||\Omega|| \) and \( S = 2(P_\Omega(\mathcal{G} \times_1 U_1 \times_2 U_2 \times_3 U_3) - P_\Omega(\mathcal{X}^*)) / ||\Omega|| \) be an auxiliary sparse tensor variable that is interpreted as the Euclidean gradient of \( f \) in \( \mathbb{R}^{n_1 \times n_2 \times n_3} \).

The partial derivatives of \( f(U_1, U_2, U_3, \mathcal{G}) \) are

\[
\begin{align*}
\frac{\partial f_1(U_1, U_2, U_3, \mathcal{G})}{\partial U_1} & = \frac{2}{||\Omega||}(P_\Omega(U_1 G_1 (U_3 \otimes U_2)^T) - P_\Omega(X_1^*)) U_3 \otimes U_2 G_1^T \\
\frac{\partial f_2(U_1, U_2, U_3, \mathcal{G})}{\partial U_2} & = \frac{2}{||\Omega||}(P_\Omega(U_2 G_2 (U_3 \otimes U_1)^T) - P_\Omega(X_2^*)) U_3 \otimes U_1 G_2^T \\
\frac{\partial f_3(U_1, U_2, U_3, \mathcal{G})}{\partial U_3} & = \frac{2}{||\Omega||}(P_\Omega(U_3 G_3 (U_2 \otimes U_1)^T) - P_\Omega(X_3^*)) U_3 \otimes U_1 G_3^T \\
\frac{\partial f(U_1, U_2, U_3, \mathcal{G})}{\partial \mathcal{G}} & = \frac{2}{||\Omega||}(P_\Omega(\mathcal{G} \times_1 U_1 \times_2 U_2 \times_3 U_3) - P_\Omega(\mathcal{X}^*)) \times_1 U_1^T \times_2 U_2^T \times_3 U_3^T
\end{align*}
\]

where \( X_3^* \) is mode-\( d \) unfolding of \( \mathcal{X}^* \) and

\[
\begin{align*}
S_1 & = \frac{2}{||\Omega||}(P_\Omega(U_1 G_1 (U_3 \otimes U_2)^T) - P_\Omega(X_1^*)) \\
S_2 & = \frac{2}{||\Omega||}(P_\Omega(U_2 G_2 (U_3 \otimes U_1)^T) - P_\Omega(X_2^*)) \\
S_3 & = \frac{2}{||\Omega||}(P_\Omega(U_3 G_3 (U_2 \otimes U_1)^T) - P_\Omega(X_3^*)) \\
S & = \frac{2}{||\Omega||}(P_\Omega(\mathcal{G} \times_1 U_1 \times_2 U_2 \times_3 U_3) - P_\Omega(\mathcal{X}^*))
\end{align*}
\]

Due to the specific scaled metric \( \Omega \), the partial derivatives of \( f \) are further scaled by \( ((G_1 G_1^T)^{-1}, (G_2 G_2^T)^{-1}, (G_3 G_3^T)^{-1}, \mathcal{I}) \), denoted as \( \text{egrad}_x f \) (after scaling), i.e.,

\[
\text{egrad}_x f = (S_1(U_3 \otimes U_2) G_1^T (G_1 G_1^T)^{-1}, S_2(U_3 \otimes U_1) G_2^T (G_2 G_2^T)^{-1}, S_3(U_2 \otimes U_1) G_3^T (G_3 G_3^T)^{-1}, S \times_1 U_1^T \times_2 U_2^T \times_3 U_3^T)
\]

Consequently, from the relationship that horizontal lift of \( \text{grad}_{[x]} f \) is equal to \( \text{grad}_x f = \Psi(\text{egrad}_x f) \), we obtain that, using (A.15),

the horizontal lift of \( \text{grad}_{[x]} f \) = \( (S_1(U_3 \otimes U_2) G_1^T (G_1 G_1^T)^{-1} - U_1 B_{U_1} (G_1 G_1^T)^{-1}, S_2(U_3 \otimes U_1) G_2^T (G_2 G_2^T)^{-1} - U_2 B_{U_2} (G_2 G_2^T)^{-1}, S_3(U_2 \otimes U_1) G_3^T (G_3 G_3^T)^{-1} - U_3 B_{U_3} (G_3 G_3^T)^{-1}, S \times_1 U_1^T \times_2 U_2^T \times_3 U_3^T) \).}

From the requirements in (A.16) for a vector to be in the tangent space, we have the following relationship for mode-1.

\[
B_{U_1} G_1 G_1^T + G_1 G_1^T B_{U_1} = G_1 G_1^T (Y_{U_1} U_1 + U_1^T Y_{U_1}) G_1 G_1^T
\]

where \( Y_{U_1} = (S_1(U_3 \otimes U_2) G_1^T (G_1 G_1^T)^{-1}) \).

Subsequently,

\[
G_1 G_1^T (Y_{U_1} U_1 + U_1^T Y_{U_1}) G_1 G_1^T = G_1 G_1^T \{((S_1(U_3 \otimes U_2) G_1^T (G_1 G_1^T)^{-1}) U_1 + U_1^T (S_1(U_3 \otimes U_2) G_1^T (G_1 G_1^T)^{-1}) G_1 G_1^T) \}
\]

\[
= (S_1(U_3 \otimes U_2) G_1^T)^T U_1 G_1 G_1^T + G_1 G_1^T U_1^T (S_1(U_3 \otimes U_2) G_1^T)
\]

\[
= (G_1 G_1^T U_1^T (S_1(U_3 \otimes U_2) G_1^T) + G_1 G_1^T U_1^T (S_1(U_3 \otimes U_2) G_1^T)
\]

\[
= 2\text{Sym}(G_1 G_1^T U_1^T (S_1(U_3 \otimes U_2) G_1^T)
\]
Finally, $B_{U_d}$ for $d \in \{1, 2, 3\}$ are obtained by solving the Lyapunov equations

$$
\begin{align*}
B_{U_1} G_1 G_1^T + G_1 G_1^T B_{U_1} &= 2 \text{Sym}(G_1 G_1^T U_1^T (S_1 (U_3 \otimes U_2) G_2^T)), \\
B_{U_2} G_2 G_2^T + G_2 G_2^T B_{U_2} &= 2 \text{Sym}(G_2 G_2^T U_2^T (S_2 (U_3 \otimes U_1) G_3^T)), \\
B_{U_3} G_3 G_3^T + G_3 G_3^T B_{U_3} &= 2 \text{Sym}(G_3 G_3^T U_3^T (S_3 (U_2 \otimes U_1) G_1^T)),
\end{align*}
$$

where $\text{Sym}(\cdot)$ extracts the symmetric part of a square matrix, i.e., $\text{Sym}(D) = (D + D^T)/2$. The above Lyapunov equations are solved efficiently with the Matlab’s $\text{lyap}$ routine.

B Additional numerical comparisons

In addition to the representative numerical comparisons in the paper, we show additional numerical experiments spanning synthetic and real-world datasets.

Experiments on synthetic datasets:

Case S1: comparison with the Euclidean metric. We first show the benefit of the proposed metric (A.2) over the conventional choice of the Euclidean metric that exploits the product structure of $M$ and symmetry. We compare steepest descent algorithms with Armijo backtracking line search for both the metric choices. Figure A.1 shows that the algorithm with the metric (A.2) gives a superior performance in test error than that of the conventional metric choice.

Case S2: small-scale instances. We consider tensors of size $100 \times 100 \times 100$, $150 \times 150 \times 150$, and $200 \times 200 \times 200$ and ranks $(5, 5, 5)$, $(10, 10, 10)$, and $(15, 15, 15)$. OS is $\{10, 20, 30\}$. Figures A.2(a)-(c) and Figures A.3(a)-(c) show the convergence behavior of different algorithms on a train set $\Omega$ and on a test set $\Gamma$, where Figures A.2(b) is identical to the figure in the manuscript paper. Figures A.2(d)-(f) and A.3(d)-(f) show the mean square error on $\Omega$ and $\Gamma$ on each algorithm. Furthermore, Figure A.2(g)-(i) and Figure A.3(g)-(i) show the mean square error on $\Omega$ and $\Gamma$ when OS is 10 in all the five runs. From Figures A.2 and Figures A.3, our proposed algorithm is consistently competitive or faster than geomCG, HalRTC, and TOpt. In addition, the mean square errors on a train set $\Omega$ and a test set $\Gamma$ are consistently competitive or lower than those of geomCG and HalRTC, especially for lower sampling ratios, e.g., for OS 10.

Case S3: large-scale instances. We consider large-scale tensors of size $3000 \times 3000 \times 3000$, $5000 \times 5000 \times 5000$, and $10000 \times 10000 \times 10000$ and ranks $\mathbf{r} = (5, 5, 5)$ and $(10, 10, 10)$. OS is 10. We compare our proposed algorithm to geomCG. Figure A.4 and Figure A.5 show the convergence behavior of the algorithms. The proposed algorithm outperforms geomCG in all the cases.

Case S4: influence of low sampling. We look into problem instances which result from scarcely sampled data. The test requires completing a tensor of size $10000 \times 10000 \times 10000$ and rank $\mathbf{r} = (5, 5, 5)$. Figures A.6 and Figure A.7 show the convergence behavior when OS is $\{8, 6, 5\}$. The case of OS = 5 is particularly interesting. In this case, while the mean square errors on $\Omega$ and $\Gamma$ increase for geomCG, the proposed algorithm stably decreases the error in all the five runs.

Case S5: influence of ill-conditioning and low sampling. We consider the problem instance of Case S4 with OS = 5. Additionally, for generating the instance, we impose a diagonal core $\mathcal{G}$ with exponentially decaying positive values of condition numbers (CN) 5, 50, and 100. Figure A.8 shows that the proposed algorithm outperforms geomCG for all the considered CN values on a train set $\Omega$.

Case S6: influence of noise. We evaluate the convergence properties of algorithms under the presence of noise. The tensor size and rank are same as in Case S4 and OS is 10. Figure A.9 shows that the train error on a train set $\Omega$ for each $\epsilon$ is almost identical to the $\epsilon^2 \|P_\Omega(X^*)\|^2_F$, but our proposed algorithm converges faster than geomCG.

Case S7: rectangular instances. We consider instances where dimensions and ranks along certain modes are different than others. Two cases are considered. Case (7.a) considers tensors size $20000 \times 7000 \times 7000$, $30000 \times 6000 \times 6000$, and $40000 \times 5000 \times 50000$ and rank $\mathbf{r} = (5, 5, 5)$. Case (7.b) considers a tensor of size $10000 \times 10000 \times 10000$ with ranks $\mathbf{r} = (7, 6, 6)$, $(10, 5, 5)$, and $(15, 4, 4)$. Figures A.10(a)-(c) and Figures A.11(a)-(c) show that the convergence behavior of our proposed algorithm is superior to
that of geomCG on \(\Omega\) and \(\Gamma\), respectively. Our proposed algorithm also outperforms geomCG for the asymmetric rank cases as shown in Figure A.10(d)-(f) and Figure A.11(d)-(f).

**Case S8: medium-scale instances.** We additionally consider medium-scale tensors of size \(500 \times 500\), \(1000 \times 1000 \times 1000\), and \(1500 \times 1500 \times 1500\) and ranks \(r = (5, 5, 5), (10, 10, 10),\) and \((15, 15, 15)\). OS is \{10, 20, 30, 40\}. Our proposed algorithm and geomCG are only compared as the other algorithms cannot handle these scales efficiently. Figures A.12(a)-(c) and A.13(a)-(c) show the convergence behavior on \(\Omega\) and \(\Gamma\), respectively. Figures A.12(d)-(f) and Figures A.13(d)-(f) also show the mean square error on \(\Omega\) and \(\Gamma\) of rank \(r = (15, 15, 15)\) in all the five runs. The proposed algorithm performs better than geomCG in all the cases.

**Experiments on real-world datasets:**

**Case R1: hyperspectral image.** We also show the performance of our algorithm on the hyperspectral image “Ribeira”. We show the mean square error on \(\Omega\) and \(\Gamma\) when OS is \{11, 22\} in Figure A.14 and Figure A.15 where Figure A.15(a) is identical to the figure in the manuscript paper. Our proposed algorithm gives lower test errors than those obtained by the other algorithms. We also show the image recovery results. Figures A.16 and A.17 show the reconstructed images when OS is \{11, 22\}, respectively. From these figures, we find that the proposed algorithm shows a good performance, especially for the lower sampling ratio.

**Case R2: MovieLens-10M.** Figure A.18 and Figure A.19 show the convergence plots for all the five runs of ranks \(r = (4, 4, 4), (6, 6, 6), (8, 8, 8)\) and \((10, 10, 10)\) on \(\Omega\) and \(\Gamma\), respectively. These figures show the superior performance of our proposed algorithm.

**Experiments for online algorithms:**

**Case O: online instances.** Figure A.20 and A.21 show the convergence plots for all the five runs on tensors of ranks \(100 \times 100 \times 5000\), and \(100 \times 100 \times 10000\) with rank \(r = (5, 5, 5)\) on \(\Omega\) and \(\Gamma\), respectively. These figures show that the proposed stochastic gradient descent algorithm gives similar or faster convergence than the proposed batch gradient descent algorithm.

Figure A.22 and A.23 show the convergence speed comparisons in the train error and the test error of the proposed online and batch algorithms with TeCPSGD and OLSTEC with rank \(r = (5, 5, 5)\) on the real-world video sequence Airport Hall dataset. These figures show that the proposed stochastic gradient descent algorithm gives similar or faster convergence than the proposed batch algorithm. In addition, Table B shows that the final train and test MSEs show the superior performance of the proposed algorithms.

![Figure A.1: Case S1: comparison between metrics (test error)](image-url)
Figure A.2: Case S2: small-scale comparisons on $\Omega$ (train error).
Figure A.3: Case S2: small-scale comparisons on $\Gamma$ (test error).
Figure A.4: **Case S3**: large-scale comparisons on $\Omega$ (train error).

Figure A.5: **Case S3**: large-scale comparisons on $\Gamma$ (test error).
Figure A.6: **Case S4**: low-sampling comparisons on $Ω$ (train error).

Figure A.7: **Case S4**: low-sampling comparisons on $Γ$ (test error).

Figure A.8: **Case S5**: $CN = \{5, 50, 100\}$ on $Ω$ (train error).

Figure A.9: **Case S6**: noisy data on $Ω$ (train error).
Figure A.10: **Case S7**: rectangular comparisons on $\Omega$ (train error).

Figure A.11: **Case S7**: rectangular comparisons on $\Gamma$ (test error).
Figure A.12: Case S8: medium-scale comparisons on $\Omega$ (train error).

Figure A.13: Case S8: medium-scale comparisons on $\Gamma$ (test error).
Figure A.14: **Case R1**: mean square error on $\Omega$ (train error).

Figure A.15: **Case R1**: mean square error on $\Gamma$ (test error).
Figure A.16: **Case R1:** recovery results on the hyperspectral image “Ribeira” (frame = 16, OS = 11).

Figure A.17: **Case R1:** recovery results on the hyperspectral image “Ribeira” (frame = 16, OS = 22).
Figure A.18: **Case R2**: mean square error on $\Omega$ (train error).

(a) $r = (4 \times 4 \times 4)$.
(b) $r = (6 \times 6 \times 6)$.
(c) $r = (8 \times 8 \times 8)$.
(d) $r = (10 \times 10 \times 10)$. 
Figure A.19: Case R2: mean square error on $\Gamma$ (test error).
Figure A.20: **Case O**: mean square error on synthetic instance of size $100 \times 100 \times 5000$.

Figure A.21: **Case O**: mean square error on synthetic instance of size $100 \times 100 \times 10000$.

Table A.1: **Case O**: mean square error (5 runs) on Airport Hall dataset.

| Error type  | Algorithm     | run 1     | run 2     | run 3     | run 4     | run 5     |
|-------------|---------------|-----------|-----------|-----------|-----------|-----------|
| **Training error on Ω** | Proposed (Online) | 7.210000  | 7.211718  | 7.205027  | 7.255203  | 7.230000  |
|             | Proposed (Batch) | 7.215763  | 7.211496  | 7.208463  | 7.282901  | 7.218042  |
|             | TeCPSGD       | 7.335320  | 7.389269  | 7.364065  | 7.393318  | 7.390530  |
|             | OLSTEC        | 7.922385  | 7.653096  | 8.150799  | 8.248936  | 7.753596  |
| **Test error on Γ** | Proposed (Online) | 7.462097  | 7.440332  | 7.452799  | 7.443505  | 7.450065  |
|             | Proposed (Batch) | 7.471942  | 7.440508  | 7.446072  | 7.492786  | 7.218042  |
|             | TeCPSGD       | 7.592109  | 7.601955  | 7.600740  | 7.579759  | 7.600621  |
|             | OLSTEC        | 8.205765  | 7.840107  | 8.599819  | 8.625715  | 7.965405  |
Figure A.22: **Case O**: mean square error on the training set $\Omega$ of the Airport Hall dataset.

Figure A.23: **Case O**: mean square error on $\Gamma$ (test error) for the Airport Hall dataset.