MAJORIZING NORMS ON MAJORIZED SPACES

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ABSTRACT. We introduce a new type of norm for ordered vector spaces majorized by a proper (convex) cone that generalizes the notions of order unit norm and base norm. Then we give sufficient conditions to ensure its completeness. In the case of normed Riesz spaces, we also present necessary completeness conditions and describe the completed norm. Finally, we define a particular family of well-behaving sets. Spaces majorized by the cone generated by these sets display surprisingly interesting structural properties. In particular, we generalize two famous standard results in Banach lattice theory. Namely, every principal ideal in a Banach lattice is an AM-space and every Banach lattice having a generating cone with a compact base admits an equivalent L-norm.

INTRODUCTION

This manuscript investigates ordered vector spaces majorized by a proper (convex) cone. The motivations for conducting a study in this direction are essentially two.

The first one comes from applications in group theory. Spaces of functions majorized by the cone generated by a group orbit have been deeply employed in the study of amenability and supramenability related questions. The first to use such spaces was Rosenblatt in his Ph.D. thesis ([R72]) while studying supramenable actions. After him, majorized spaces were also used to investigate supramenable groups in [K14], and a fixed-point property for group representations on cones in [M17] and in [V21]. The reason for using majorized spaces in these situations is that they are the natural domain of unbounded functionals appearing when handling such groups.

The second motivation is more theoretical. Recall that on every principal ideal in a Banach lattice is possible to define a norm, called order unit norm ([E64, Lemma 1]), that turns the principal ideal into an AM-space with order unit ([E64, Lemma 4 (i)]). Being majorized by a cone is, in some sense, the same as having an order unit. One should expect to generalize the concept of ordered unit norm for majorized spaces as well since principal ideals are in particular majorized spaces. Something was already done for Banach lattices that admit a generating cone with a bounded base. Indeed, on such spaces is possible to define a norm using the Minkowski functional associated with the base of the generating cone ([E64, Lemma 3]). The norm so constructed is called a base norm and it turns the Banach lattice into an AL-space ([E64, Lemma 4 (ii)]). The two norms are equal when considering cones with a one-element base or in the finite-dimensional case. However, they are generally different. Actually, they are mutually dual, i.e., a Banach lattice has an order unit norm (resp. a base norm) if and only if its dual has a base norm (resp. an order unit norm), see [E64, Theorem 5 & Corollary p.737].

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Therefore, we introduced a majorizing norm on majorized space, looking to-
ward the following goals. Generalize the notion of order unit norm and base
norm simultaneously to develop a theory that meets both views, and set the the-
oretical background to employ majorized spaces in different domains, e.g., supra-
menable groups.

After a brief recapitulation about ordered vector spaces, Section 2 introduces
majorizing semi-norms. Here, sufficient conditions to ensure that a majorizing
semi-norm is a norm and about its completeness are presented (Propositions 7
and 10). The following section handles the case of Banach lattices. In this case,
it was possible to completely understand the completeness of majorizing norms
(Theorems 18 and 21). In Section 4, linear functionals on majorized spaces are
studied. In particular, linear functionals on majorized spaces were characterized
using continuity, positivity, and boundedness (Theorem 25). Finally, Section 5
considers ordered vector spaces majorized by particular sets called coherent sets.
Surprisingly, spaces majorized by coherent sets are easier to handle, and the co-
herency of the majorizing set implies interesting phenomena on both the ma-
jorized space and the majorizing norm (Theorem 32).

1. Preliminaries

We consider only real vector spaces, and if $E$ is a topological vector space, then
$E'$ denotes its topological dual, i.e., the set of all continuous real linear functionals
on $E$.

An ordered vector space $(E, \leq)$ is nothing but a vector space equipped with a
partial order $\leq$ such that the relation $x \leq y$ implies $x + z \leq y + z$ and $cx \leq cy$ for
every $x, y, z \in E$ and every positive $c \in \mathbb{R}$. We only write $E$ instead of $(E, \leq)$, since
it will always be clear which order we are considering. The set of positive vectors
$E_+$ of $E$ is the set of all $x \in E$ such that $x \geq 0$. The set of all positive vectors of an
ordered vector space $E$ is also called the positive cone of $E$. A vector $u \in E_+$ is
called an order unit if for every $v \in E$ there is $t \in \mathbb{R}_+$ such that $v \leq tu$. Let $A$ be
a subset of $E$. Then a vector $v \in E$ is said majorized by $A$ if there is $a \in A$ such
that $v \leq a$. The ordered vector space $E$ is said majorized by $A$ if every vector of
$E$ is majorized by $A$. An ordered vector space $E$ is Archimedean if $ny \leq x$ for all
$n \in \mathbb{N}$ implies that $y \leq 0$, where $y \in E$ and $x \in E_+$.

A linear functional $T$ between two ordered vector spaces $E$ and $V$ is said a
positive functional if it sends positive vectors to positive vectors, i.e., if $v \in E_+$,
then $T(v) \in V_+$. The functional $T$ is said strictly positive if it sends non-zero
positive vectors to non-zero positive vectors, i.e., if $v \in E$ such that $v > 0$, then
$T(v) > 0$.

A cone $C$ is a non-empty subset of $E$ which is additive and positive homoge-
neous. The cone $C$ is said proper if $C \cap (-C) = \{0\}$, and generating if $E = C - C$.
If $C$ is a proper cone in a vector space $E$, then the binary relation on $E$ defined by

$$x \leq y \iff y - x \in C$$

turns $E$ into an ordered vector space with positive cone $E_+ = C$. In fact, there is a
one-to-one correspondence between proper cones and ordering on vector spaces
([AT07, §1.1]).
We say that an ordered vector space $E$ is a Riesz space if for every pair of vectors $v, w \in E$ their infimum $v \wedge w$, or equivalently their supremum $v \vee w$, exists in $E$. Consequently, on a Riesz space $E$ makes sense to define a notion of absolute value via the equation $|v| = (v) \vee (v)$ for $v \in E$. A vector subspace $F$ of a Riesz space $E$ is called a Riesz subspace if for every $v, w \in F$ their infimum $v \wedge w$, or equivalently their supremum $v \vee w$, belongs to $F$. From the identity $v \wedge w = \frac{1}{2}(v + w + |v - w|)$, see [AT07, Theorem 1.17(6)] for a proof, a vector subspace $F$ of a Riesz space $E$ is a Riesz subspace if and only if $v \in F$ implies that $|v| \in F$. A vector subspace $S$ of a Riesz space $E$ is called an ideal of $E$ if $|v| \leq |w|$ and $w \in S$ implies that $v \in S$. As an ideal is closed by taking absolute value, then every ideal is automatically a Riesz subspace. Note that the positive cone of a Riesz space $E$ is always generating as it is possible to write every $v \in E$ as the difference of the two positive vector $v \vee 0$ and $(-v) \vee 0$ ([AT07, Theorem 1.17(8)]).

A norm $|| \cdot ||$ on an ordered vector space $E$ is said monotone if it preserves the order, i.e., if $0 \leq v \leq w$ then $||v|| \leq ||w||$ for every $v, w \in E$. An ordered vector space $E$ with a generating positive cone together with a monotone norm is called an ordered normed space. If the topology generated by the norm is complete, then we call $E$ an ordered Banach space. A normed Riesz space is a Riesz space $E$ together with a monotone norm. Finally, a Banach lattice is a complete normed Riesz space. A Banach lattice is called an AM-space if the equality $||v \vee w|| = \max\{|v|, ||w||\}$ holds for every positive $v, w \in E$.

2. Majorized Spaces and Majorizing Norms

Let $(E, \leq)$ be an ordered vector space and let $M$ be a subset of $E$. We say that $M$ is a positive subset of $E$ if it is a non-empty subset of the positive cone of $E$. The proper (convex) cone generated by a positive subset $M$ of $E$ is the cone given by

$$C_M = \{tm : t \geq 0 \text{ and } m \in \text{co}(M) \backslash \{0\}\}$$

where $\text{co}(M)$ is the convex hull of $M$. We have to ask that $0 \notin \text{co}(M)$ to ensure that $C_M$ is a cone, see [AT07, Lemma 3.10].

Note that $C_M = C_{\text{co}(M)}$ and $C_K = K$ for any other proper cone $K$ of $E$. Therefore, every proper cone in $E$ can be viewed as a cone generated by some positive subset of $E$. Moreover, if $M$ is a positive bounded subset of an ordered normed space, then $C_M$ is closed if and only if $\text{co}(M)$ is closed.

**Definition 1.** Given a positive subset $M$ of $E$, we define

$$E_M = \{v \in E : \pm v \leq c \text{ for some } c \in C_M\}.$$

In other words, $E_M$ is the set of all vectors of $E$ majorized by the cone $C_M$.

If $E$ is a Riesz space, then $E_M$ is nothing but the ideal generated by the positive subset $M$. Therefore, if $M = \{u\}$ is composed only by a positive vector, then $E_M = E_u$ is the principal ideal generated by this vector.

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1Be careful that for some authors an ordered normed space is only an ordered vector space equipped with a, possibly non-monotone, norm.
Example 2. Let $C^b(\mathbb{R})$ the vector space of all continuous bounded functions on $\mathbb{R}$ equipped with the pointwise order. Let $f \in C^b(\mathbb{R})$ be a strictly positive function different from the constant function $1_{\mathbb{R}}$. Then $C^b(\mathbb{R}) = C^b(\mathbb{R})f = C^b(\mathbb{R})1_{\mathbb{R}}$ but $C_f \not\subset C_{1_{\mathbb{R}}}$ nor $C_{1_{\mathbb{R}}} \not\subset C_f$.

The following proposition is only a consequence of the fact that $E_M$ is majorized by a convex cone. Thus, we omitted the proof.

Proposition 3. Let $M$ be a positive subset of $E$. Then $E_M$ is a vector subspace of $E$. Moreover, if $E$ is a Riesz space, then $E_M$ is an ideal, hence a Riesz subspace of $E$.

Definition 4. Let $M$ be a positive subset of $E$. Define the possibly infinite value

$$p_M(v) = \inf \left\{ \sum_{j=1}^{n} t_j : \pm v \leq \sum_{j=1}^{n} t_j m_j \text{ for some } m_1, \ldots, m_n \in M \right\}$$

where $v \in E$.

It is easy to see that $p_M(v) < \infty$ if and only if $v \in E_M$. Therefore, it is interesting to study $p_M$ restricted to the vector subspace $E_M$. In this case, $p_M$ is a monotone seminorm for $E_M$.

Example 5. Let $E$ be an Archimedean ordered vector space and let $M = \{m_1, \ldots, m_n\}$ be a positive subset of $E$. Then the (positive) vector $u = \sum_{j=1}^{n} m_j$ is an order unit for the ordered vector space $E_M$. Indeed, $m_j \leq u$ for every $j$. We claim that $(E_M, p_M) = (E,u, \frac{1}{p_M(u)} || \cdot ||_u)$ where $||v||_u = \inf\{t : \pm v \leq tu\}$ is the order unit norm of $E_u$. Indeed, $E_M$ is equal to the principal ideal $E_u$ because $u$ is an order unit for $E_M$ and $u \in E_M$. Now, let $v \in E_M$ and $\epsilon > 0$. Then there are $t, t_1, \ldots, t_n \in \mathbb{R}_+$ and $m_1, \ldots, m_n \in M$ such that

$$\pm v \leq \sum_{j=1}^{n} t_j m_j, \quad \sum_{j=1}^{n} t_j \leq p_M(v) + \epsilon, \quad \pm v \leq tu \quad \text{and} \quad t \leq ||v||_u + \frac{\epsilon}{p_M(u)}.$$

On the one hand, the inequality

$$ \pm v \leq \sum_{j=1}^{n} t_j m_j \leq \sum_{j=1}^{n} t_j u \quad \text{implies that} \quad ||v||_u \leq \sum_{j=1}^{n} t_j ||u||_u \leq p_M(v) + \epsilon. $$

On the other hand, the inequality

$$ \pm v \leq tu \quad \text{implies that} \quad p_M(v) \leq tp_M(u) \leq ||v||_u p_M(u) + \epsilon. $$

We can conclude that $||v||_u \leq p_M(v) \leq p_M(u)||v||_u$ for every $v \in E_M$. But now $p_M(u) \geq 1$ as $u \geq m$ for every $m \in M$. Therefore, $p_M = \frac{1}{p_M(u)} || \cdot ||_u$. Hence, $(E_M, p_M) = (E_u, \frac{1}{p_M(u)} || \cdot ||_u)$ is a Banach space by [AT07, Theorem 2.55 (1)].

We want to understand when $p_M$ is a norm. A necessary condition is that the space $E$ is Archimedean, as shown in the following example.

Example 6. Let $E$ be a non-Archimedean Riesz space, e.g., $\mathbb{R}^2$ with the lexicographic order. We claim that there is at least one positive vector $x \in E$ such that
$p_x$ is not a norm on $E_x$. Indeed by [AT07, Exercice 5 p. 20], there are $x, y \in E_+$ such that $ny \leq x$ for all $n \in \mathbb{N}$ but $y > 0$. Consider the space $E_x$. Then $y \in E_x$ and

$$0 \leq p_x(y) \leq \frac{1}{n}p_x(x) \quad \text{for every } n \in \mathbb{N}.$$ 

This implies that $p_x(y) = 0$. Therefore, the map $p_x$ is not a norm.

A subset $M$ of a normed space $(E, || \cdot ||)$ is said $|| \cdot ||$-uniformly bounded, or only uniformly bounded when the norm is understood, if $\sup \{|||m||| : m \in M\}$ is finite.

**Proposition 7.** Let $(E, || \cdot ||)$ be an ordered normed space and let $M$ be a positive uniformly bounded subset of $E$. Then $(E_M, p_M)$ is an ordered normed space and the equality

$$||v|| \leq C p_M(v) \quad \text{holds for every } v \in E_M.$$ 

Here, $C = \sup \{|||m||| : m \in M\}$.

**Proof.** We start by showing the inequality. Let $\epsilon > 0$ and take $v \in E_M$. Then there are $t_1, ..., t_n \in \mathbb{R}_+$ and $m_1, ..., m_n \in M$ such that $\pm v \leq \sum_{j=1}^n t_j m_j$ and $\sum_{j=1}^n t_j \leq p_M(v) + \epsilon$. Since the norm $|| \cdot ||$ is monotone, the following estimation holds:

$$||v|| \leq \sum_{j=1}^n t_j ||m_j|| \leq C \sum_{j=1}^n t_j \leq C p_M(v) + \epsilon.$$ 

As $\epsilon$ was chosen arbitrarily, we are done. Therefore, we can conclude that $p_M$ is a monotone norm on $E_M$. Indeed, it is clear that $p_M$ is a sub-additive, homogeneous and monotone map. It is left to show that if $p_M(v) = 0$, then $v = 0$. But, $p_M(v) = 0$ implies that $||v|| = 0$. Hence, $v = 0$. □

In particular, if $E$ is a normed Riesz space and $M$ is a positive uniformly bounded subset of $E$, then $(E_M, p_M)$ is a normed Riesz space.

**Corollary 8.** Let $E$ be an Archimedean ordered vector space with order unit $u$. Then for every positive $|| \cdot ||_u$-uniformly bounded subset $M$ of $E$, the pair $(E_M, p_M)$ is an ordered normed vector space.

Note that if $u \in M$, then $(E_M, p_M) = (E_u, \frac{1}{C} || \cdot ||_u)$ where $C = \sup \{|||m||| : m \in M\}$.

**Proof of Corollary 8.** By point 2) of [AT07, Theorem 2.55], $|| \cdot ||_u$ is a norm on $E$. Therefore, we can apply Proposition 7. □

Recall that a functional $\psi$ defined on a vector space $E$ is said uniformly bounded on a subset $A \subset E$ if $\sup\{|\psi(a)| : a \in A\} < \infty$.

**Corollary 9.** Let $E$ be an ordered vector space. Suppose that $E$ admits a strictly positive linear functional $\psi$. Then for every positive subset $M \subset E$ on which $\psi$ is uniformly bounded, the pair $(E_M, p_M)$ is an ordered normed vector space.

**Proof.** Let $\psi$ be a positive functional on $E$, and define the norm $||v|| = |\psi(v)|$ for $v \in E$. Therefore, if $\psi$ is uniformly bounded on a positive subset $M$, then $M$ is $|| \cdot ||$-uniformly bounded. Therefore, we can conclude using Proposition 7. □

We shall conclude that section by giving a slight generalization of Example 5.
Proposition 10. Let $E$ be a Banach lattice and $M$ a positive uniformly bounded subset of $E$. Suppose that $M$ is $p_M$-totally bounded. Then $(E_M, p_M)$ is an AM-bounded with order unit if and only if $(E_M, p_M) = (E_u, \frac{1}{p_M(u)} \cdot ||u||)$ for some positive vector $u \in E_M$.

Proof. Suppose that $(E_M, p_M)$ is an AM-space. Since $M$ is $p_M$-totally bounded, there is a vector $u \in E_M$ such that $u = \text{sup}(M)$ by a result of Krengel [AB85, Theorem 4.30]. We claim that $(E_M, p_M) = (E_u, \frac{1}{p_M(u)} \cdot ||u||)$. Clearly, $E_M = E_u$, since $u = \text{sup}(M)$ and $u \in E_M$. Now, let $v \in E_M$ and $\epsilon > 0$. Then there are $t, t_1, ..., t_n \in \mathbb{R}_+$ and $m_1, ..., m_n \in M$ such that

$$\pm v \leq \sum_{j=1}^{n} t_j m_j, \quad \sum_{j=1}^{n} t_j \leq p_M(v) + \epsilon, \quad \pm v \leq tu \quad \text{and} \quad t \leq ||v||u + \frac{\epsilon}{p_M(u)}.$$

On the one hand, the inequality

$$\pm v \leq \sum_{j=1}^{n} t_j m_j \leq \sum_{j=1}^{n} t_j u \quad \text{implies that} \quad ||v||u \leq \sum_{j=1}^{n} t_j ||u||u \leq p_M(v) + \epsilon.$$

On the other hand, the inequality

$$\pm v \leq tu \quad \text{implies that} \quad p_M(v) \leq tp_M(u) \leq ||v||u p_M(u) + \epsilon.$$

We can conclude that $||v||u \leq p_M(v) \leq p_M(u)||v||u$ for every $v \in E_M$. But now $p_M(u) \geq 1$ as $u \geq m$ for every $m \in M$. Therefore, $p_M = \frac{1}{p_M(u)} ||u||$.

Suppose now that $(E_M, p_M) = (E_{u'}, \frac{1}{p_M(u')} ||u'||)$ for some positive vector $u' \in E_M$. Then $(E_M, p_M)$ is an AM-space. Indeed, $(E_{u'}, ||u'||)$ is an AM-space by [AB99, Theorem 9.28]. Thus,

$$p_M(v \vee w) = \frac{1}{p_M(u)} ||v \vee w||u = \frac{1}{p_M(u)} \max \{ ||v||, ||w|| \}$$

$$= \max \left\{ \frac{1}{p_M(u)} ||v||, \frac{1}{p_M(u)} ||w|| \right\} = \max \{ p_M(v), p_M(w) \}$$

for every positive $v, w \in E$. Note that the third equality is possible thanks to [AT07, Theorem 1.17 (4)].

3. Majorizing at infinity

The previous section presented a simple method to ensure that $(E_M, p_M)$ is an ordered normed space. Now we want to understand when the norm $p_M$ is complete. To this end, we restrict to the case of Banach lattices.

For this section, $E$ will always denote a Banach lattice with norm $|| \cdot ||$.

We recall that a sequence $(t_j)_{j=1}^{\infty} \subset \mathbb{R}$ is called summable if $\sum_{j=1}^{\infty} t_j < \infty$ and absolute summable if $\sum_{j=1}^{\infty} |t_j| < \infty$. Note that given a uniformly bounded set $M \subset E$ and a positive summable sequence $(t_j)_{j=1}^{\infty} \subset \mathbb{R}$, the sequence $(u_n)_{n=1}^{\infty} \subset E$ given by $u_n = \sum_{j=1}^{n} t_j m_j$ converges with respect to the norm $|| \cdot ||$ because the infinite serie $\sum_{j=1}^{\infty} t_j m_j$ converges absolutely in the Banach space $E$. In fact,

$$\sum_{j=1}^{\infty} ||t_j m_j|| = \sum_{j=1}^{\infty} t_j ||m_j|| = \sup_{m \in M} ||m|| \sum_{j=1}^{\infty} t_j,$$
which implies that \( \lim_n u_n \in E \) by [M98, Theorem 1.3.9]. With an abuse of notation, we write \( \sum_{j=1}^{\infty} t_j m_j \) to mean the limit of the partial sum sequence \( u_n = \sum_{j=1}^{n} t_j m_j \) with respect to the \( \| \cdot \| \)-norm.

Given a positive uniformly bounded subset \( M \) of \( E \), define

\[
\overline{M} = \left\{ \sum_{j=1}^{\infty} t_j m_j : (t_j)_{j=1}^{\infty} \subset \mathbb{R}_+ \text{ is a summable sequence and } (m_j)_{j=1}^{\infty} \subset M \right\} \subset E.
\]

Therefore,

\[
E_{\overline{M}} = \left\{ v \in E : |v| \leq \sum_{j=1}^{\infty} t_j m_j \text{ for a summable } (t_j)_{j=1}^{\infty} \subset \mathbb{R}_+ \text{ and } (m_j)_{j=1}^{\infty} \subset M \right\}
\]

and

\[
p_{\overline{M}}(v) = \inf \left\{ \sum_{j=1}^{\infty} t_j : |v| \leq \sum_{j=1}^{\infty} t_j m_j \text{ for a summable } (t_j)_{j=1}^{\infty} \subset \mathbb{R}_+ \text{ and } (m_j)_{j=1}^{\infty} \subset M \right\}
\]

for \( v \in E_{\overline{M}} \).

Clearly, \( E_M \subset E_{\overline{M}} \) and \( p_{\overline{M}} \leq p_M \) on \( E_M \), as \( C_M \subset C_{\overline{M}} \). Moreover, note that \( C_{\overline{M}} = \overline{M} \).

One can think that the choice of the notation \( \overline{M} \) could generate some misunderstanding since it recalls the closure of \( M \) with respect to some topology. However, we will motivate this notation choice at the end of the section.

**Example 11.** Consider \( \ell^\infty(\mathbb{N}) \) the set of all bounded real functions on the integers. Set \( M = \{ \delta_j : j \in \mathbb{N} \} \) where \( \delta_j \) is the function which gives 1 at point \( j \) and 0 otherwise. Then

\[
(\ell^\infty(\mathbb{N}), p_M) = (c_0(\mathbb{N}), \| \cdot \|_1) \quad \text{and} \quad (\ell^\infty(\mathbb{N}), p_{\overline{M}}) = (\ell^1(\mathbb{N}), \| \cdot \|_1).
\]

We introduced the cone \( \overline{M} \) to understand when a general pair \( (E_M, p_M) \) is a Banach space. In fact, this section aims to show that for every positive uniformly bounded subset \( M \) of \( E \), the pair \( (E_{\overline{M}}, p_{\overline{M}}) \) is a Banach lattice. To this end, we need to clarify some technical detail.

**Lemma 12.** Let \( (V, \| \cdot \|_V) \) and \( (E, \| \cdot \|_E) \) be two normed vector spaces such that \( V \subset E \) and \( \| \cdot \|_E \leq \| \cdot \|_V \) on \( V \). Suppose that there is a sequence \( (v_n)_n \) in \( V \) which converges to an element \( v_1 \in V \) in \( \| \cdot \|_V \)-norm and to an element \( v_2 \in E \) in \( \| \cdot \|_E \)-norm. Then \( v_1 = v_2 \).

The proof of this last lemma is only due to the fact that the identity map from \( (V, \| \cdot \|_V) \) to \( (E, \| \cdot \|_E) \) is continuous.

**Lemma 13.** Let \( (E, \| \cdot \|) \) be a Banach lattice and let \( (x_k)_k \) and \( (y_k)_k \) be sequences in \( E \). Suppose that the two sequences \( (x_k)_k \) and \( (y_k)_k \) are absolutely summable, i.e.,

\[
\lim_n \sum_{k=1}^{n} \|x_k\| < +\infty \quad \text{and} \quad \lim_n \sum_{k=1}^{n} \|y_k\| < +\infty,
\]
and that for every $k \in \mathbb{N}$ the inequality $x_k \leq y_k$ holds. Then

$$
\lim_{n} \sum_{k=1}^{n} x_k \leq \lim_{n} \sum_{k=1}^{n} y_k,
$$

where the limit is taken with respect to the $\| \cdot \|$-norm.

**Proof.** Consider the sequence $(z_k)_k$ in $E$ given by $z_k = y_k - x_k$. By hypothesis $(z_k)_k$ is positive, which means that it lives in the positive cone of $E$. Now,

$$
\sum_{k=1}^{n} \|z_k\| \leq \sum_{k=1}^{n} \left(\|y_k\| + \|x_k\|\right) = \sum_{k=1}^{n} \|y_k\| + \sum_{k=1}^{n} \|x_k\| \quad \text{for every } n \in \mathbb{N}.
$$

Taking the limit on both sides, we have that $\lim_{n} \sum_{k=1}^{n} \|z_k\| < \infty$. As $E$ is a Banach space, the limit $\lim_{n} \sum_{k=1}^{n} \|z_k\|$ with respect to the $\| \cdot \|$-norm exists and it is positive, since the positive cone of a Banach lattice is always closed, see [AB99, Theorem 8.43 (1)]. This implies that

$$
\lim_{n} \sum_{k=1}^{n} x_k \leq \lim_{n} \sum_{k=1}^{n} y_k,
$$

where the two limits are taken with respect to the $\| \cdot \|$-norm. \qed

For a normed vector space $(E, \| \cdot \|)$, write $\hat{E}$ for the completion of $E$ with respect to the uniformity given by the norm $\| \cdot \|$ and $|| \cdot ||$ for the completed norm of $\hat{E}$.

Recall that the completion $(\hat{E}, || \cdot ||)$ of a normed Riesz space $(E, \| \cdot \|)$ is a Banach lattice ([AB99, Lemma 9.4]) and that the lattice operation $v \mapsto |v|$ on a normed Riesz space is uniformly continuous, since the topology generated by the norm of a normed Riesz space is locally convex solid ([AB99, Theorem 8.41]).

**Lemma 14.** Let $(E, \| \cdot \|_E)$ be a Banach lattice and $(V, \| \cdot \|_V)$ be a normed Riesz space. Let

$$
\iota : (V, \| \cdot \|_V) \longrightarrow (E, \| \cdot \|_E)
$$

be an injective continuous Riesz homomorphism. Then the (unique) extension

$$
\hat{\iota} : (\hat{V}, || \cdot ||_V) \longrightarrow (E, || \cdot ||_E)
$$

of $\iota$ is an injective continuous Riesz homomorphism.

**Proof.** We start showing that $\hat{\iota}$ is a Riesz homomorphism, i.e., $\hat{\iota}(|\vartheta|) = |\hat{\iota}(\vartheta)|$ for every $\vartheta \in \hat{V}$. Let $\vartheta \in \hat{V}$ and let $(v_n)_n$ be a sequence in $V$ which converges to $\vartheta$ in $|| \cdot ||_V$-norm. Then the sequence $(\hat{\iota}(v_n))_n$ converges to $\hat{\iota}(\vartheta)$ in $|| \cdot ||_E$-norm. Therefore,

$$
\hat{\iota}(|\vartheta|) = \lim_{n} \hat{\iota}(|v_n|) = \lim_{n} \hat{\iota}(|v_n|) = \lim_{n} \hat{\iota}(|v_n|) = \lim_{n} \hat{\iota}(|v_n|) = |\hat{\iota}(\vartheta)|.
$$

It just remains to prove that $\hat{\iota}$ is injective. Suppose it is not the case. Then there is a non-zero vector $\vartheta \in \hat{V}$ such that $\hat{\iota}(\vartheta) = 0$. We can suppose that $\vartheta$ is positive as $\hat{\iota}$ is a Riesz homomorphism. By Theorem 60.4 of [L67], there is a positive increasing sequence $(v_n)_n$ in $V$ which converges to $\vartheta$ in $|| \cdot ||_V$-norm.
Therefore, \((\hat{\iota}(v_n))_n\) converges to \(\hat{\iota}(0) = 0\) in \(\| \cdot \|_E\)-norm. But this means that 
\[
\lim_n \| \hat{\iota}(v_n) \|_E = \lim_n \| \iota(v_n) \|_E = 0.
\]
Moreover, \(\| \iota(v_n) \|_E \leq \| \iota(v_m) \|_E\) for every \(m \geq n\), since \((v_n)_n\) is increasing, \(\iota\) is a positive map and \(\| \cdot \|_E\) is a monotone norm. Passing to a subsequence, we can suppose that \(\| \iota(v_n) \|_E \leq \frac{1}{n}\) for every \(n \in \mathbb{N}\). Fix now \(n_0 \in \mathbb{N}\) such that \(v_{n_0} > 0\). Then
\[
\| \iota(v_{n_0}) \|_E \leq \| \iota(v_n) \|_E \leq \frac{1}{n}
\]
for every \(n \geq n_0\). This implies that \(\| \iota(v_{n_0}) \|_E = 0\). Thus, \(\iota(v_{n_0}) = 0\) and \(v_{n_0} = 0\) as \(\iota\) is injective. But this is a contradiction. \(\square\)

This last lemma is no longer valid if we drop the monotonicity of the norm \(\| \cdot \|_V\).

**Example 15.** Let \((E, \| \cdot \|)\) be an infinite-dimensional Banach space and let \(T\) be a discontinuous linear functional on it. Define the map
\[
\| v \|_T = \| v \| + |T(v)| \quad \text{for } v \in E.
\]
Then \(\| \cdot \|_T\) is a norm on \(E\), which is strictly finer than the norm \(\| \cdot \|\). Therefore, the identity map
\[
\text{Id} : (E, \| \cdot \|') \rightarrow (E, \| \cdot \|)
\]
is an injective continuous linear operator. We claim that the extension
\[
\hat{\text{Id}} : (\hat{E}, \| \cdot \|_T) \rightarrow (E, \| \cdot \|)
\]
is not injective. Indeed, suppose that \(\hat{\text{Id}}\) is injective. Then by the Closed Graph Theorem ([AB99, Theorem 5.20]) the inverse of \(\hat{\text{Id}}\) has to be continuous. But this is not possible.

**Corollary 16.** Let \(M\) be a positive uniformly bounded subset of \(E\). Then the completion \(\hat{E}_M\) of \(E_M\) with respect to the \(p^M\)-norm can be realized as a Riesz subspace of \(E\).

**Proof.** Consider the natural inclusion
\[
\iota : E_M \rightarrow E, \quad v \mapsto \iota(v) = v.
\]
Then the extension
\[
\hat{\iota} : \hat{E}_M \rightarrow E, \quad v \mapsto \hat{\iota}(v)
\]
is an injective Riesz homomorphism by Lemma 14. Therefore, \(\hat{E}_M\) can be realize as a Riesz subspace of \(E\). \(\square\)

**Remark 17.** Let \((E, \| \cdot \|)\) be a Banach lattice and take a positive uniformly bounded subset \(M\) of \(E\). Then, for every absolutely convergent sequence \((t_j)_j \subset R_+\) and every sequence \((m_j)_j \subset M\), the sequence \(u_n = \sum_{j=1}^{n} t_j m_j\) converges for the \(\bar{p}_M\) norm and the \(\| \cdot \|\)-norm to the same limit thanks to Corollary 16.

We are finally ready to show the completeness of the space \((E_M, p^M)\).

**Theorem 18.** The pair \((E_M, p^M)\) is a Banach lattice for every positive uniformly bounded subset \(M\) of \(E\).
Proof. We have to show that, for every sequence \((x_k)_k\) in \(E_M\) such that \(\lim_n \sum^n_k p_M(x_k) < \infty\), the limit \(\lim_n \sum^n_k x_k\) with respect to the \(p_M\) norm exists in \(E_M\).

First of all, note that \(\sum^n_k \hat{p}_M(x_k) = \sum^n_k p_M(x_k)\) for every \(n \in \mathbb{N}\). Thus, \(\lim_n \sum^n_k \hat{p}_M(x_k) = \infty\) which means that the limit \(\ell = \lim_n \sum^n_k x_k\) exists in \(E_M\) for the \(\hat{p}_M\) norm, since \((E_M, \hat{p}_M)\) is a Banach space.

We know that \(\ell \in E\) by Corollary 16. We claim that \(\ell \in E_M\). Therefore, we have to show that there are sequences \((t_j)_j \subset \mathbb{R}_+\) and \((m_j)_j \subset M\) such that

\[
|\ell| \leq \sum_{j=1}^\infty t_j m_j = \lim_n \sum_{j=1}^n t_j m_j
\]

where the limit is taken with respect to the \(\| \cdot \|\)-norm. For every \(k \in \mathbb{N}\), there are \((t_j^{(k)})_j \subset \mathbb{R}_+\) and \((m_j^{(k)})_j \subset M\) such that

\[
|x_k| \leq \sum_{j=1}^\infty t_j^{(k)} m_j^{(k)} \quad \text{and} \quad \sum_{j=1}^\infty t_j^{(k)} \leq p_M(x_k) + 2^{-k}.
\]

Set \(y_k = \sum_{j=1}^\infty t_j^{(k)} m_j^{(k)} x_k\). Then \(|x_k| \leq y_k\) for every \(k \in \mathbb{N}\) and

\[
\sum_{k=1}^N \hat{p}_M(y_k) \leq \sum_{k=1}^N \sum_{j=1}^\infty t_j^{(k)} \leq \sum_{k=1}^N p_M(x_k) + 2^{-k}
\]

for every \(N \in \mathbb{N}\). Taking the limit on both sides of this last inequality, we have that

\[
\lim_{N} \sum_{k=1}^N \hat{p}_M(y_k) < \infty.
\]

Therefore by Lemma 13,

\[
|\ell| \leq \sum_{k=1}^\infty |x_k| \leq \sum_{k=1}^\infty \sum_{j=1}^\infty t_j^{(k)} m_j^{(k)},
\]

where all the limits are taken with respect to the \(\hat{p}_M\)-norm. The last double sum converges also in \(\| \cdot \|\)-norm to the same limit as explained in Remark 17. This implies that \(\ell \in E_M\). Now, it is easy to show that the sequence \((v_n)_n\) given by \(v_n = \sum_{k=1}^n x_k\) converges to \(\ell\) in \(p_M\)-norm. Indeed, for every \(\epsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that \(p_M(\sum_{k=1}^n x_k - \ell) < \epsilon\) for every \(n > n_0\). Thus,

\[
p_M(v_n - \ell) = p_M \left( \sum_{k=1}^n x_k - \ell \right) = \hat{p}_M \left( \sum_{k=1}^n x_k - \ell \right) < \epsilon
\]

for every \(n > n_0\). \(\square\)

The next goal is to understand the relation between the spaces \(E_M\) and \(E_M\).

**Proposition 19.** Let \(M\) be a positive uniformly bounded subset of \(E\). Then \(E_M\) is dense in \(E_M\) with respect to the \(p_M\)-norm.

To prove this last proposition, we need the following lemma, which is only a lattice version of the famous and standard result [AB99, Corollary 5.81]. The latter states that a vector subspace of a locally convex vector space is not dense
exactly when a non-zero continuous linear functional vanishes on it. We require an ideal version of it.

**Lemma 20** (Non-density of ideals). Let $E$ be a Banach lattice and let $V \subset E$ be an ideal. Then $V$ is not dense in $E$ if and only if there is a non-zero positive functional on $E$ which vanishes on $V$.

**Proof.** Suppose a non-zero positive functional $\psi$ vanishing on $V$. In particular, $\psi$ is continuous since positive operators on Banach lattice are automatically continuous ([AB99, Theorem 9.6]). Therefore, we can apply [AB99, Corollary 5.81] to conclude that $V$ is not dense in $E$.

Suppose now that $V$ is not dense in $E$. Then there is a non-zero continuous functional $\psi$ on $E$ which vanishes on $V$ by [AB99, Corollary 5.81]. We claim that $|\psi|$ is the functional we are searching. Clearly, $|\psi|$ is a non-zero positive functional. We have only to show that it is zero on $V$. Let $|\psi| = \psi_+ + \psi_-$, where $\psi_+$ is the positive part of $\psi$ and $\psi_-$ is the negative part of $\psi$. We want to prove that the equality $\psi_+(v) = \psi_-(v) = 0$ holds for every positive vector $v \in V$. Actually, it suffices to prove that $\psi_+(v) = 0$. By [AB99, Theorem 8.24], $\psi_+(v) = \sup \{\psi(w) : w \in E \text{ and } 0 \leq w \leq v\} = \sup \{\psi(w) : w \in V \text{ and } 0 \leq w \leq v\}$, where the second equality is possible thanks to the fact that $V$ is an ideal in $E$. As $\psi_+$ is continuous, we can conclude that $\psi_+$ vanishes on $V$. \hfill \Box

This yields:

**Proof of Proposition 19.** In order to find a contradiction, suppose that $E_M$ is not dense in $E_M$ for the $p_{M^*}$-norm. By Lemma 20, there exists a non-zero positive functional $\psi$ which vanishes on $E_M$. As $\psi$ is non-zero, there is a non-zero positive vector $v \in E_M$ such that $\psi(v) > 0$. Now let $(t_j)_{j=1}^\infty \subset \mathbb{R}_+$ be a summable sequence and $(m_j)_{j=1}^\infty \subset M$ such that $v \leq \sum_{j=1}^\infty t_j m_j$, and compute that

$$0 < \psi(v) \leq \psi \left( \sum_{j=1}^\infty t_j m_j \right) = \sum_{j=1}^\infty t_j \psi(m_j) = 0$$

where the second-to-last equality is possible thanks to [M98, Proposition 1.3.7 (d)]. But this is a contradiction. Thus, $E_M$ is indeed dense in $E_M$ with respect to the $p_{M^*}$-norm. \hfill \Box

The following theorem clarifies once and for all the relationship between $p_M$ and $p_{M^*}$. As before, write $E_M$ for the completion of $E_M$ with respect to the uniformity given by the $p_M$-norm and $\widehat{p}_M$ for the completed norm.

**Theorem 21.** Let $M$ be a positive uniformly bounded subset of $E$. Then

$$(E_M, \widehat{p}_M) = (E_M, p_{M^*}) .$$

**Proof.** Consider the natural inclusion

$$\iota : (E_M, p_M) \longrightarrow (E_M, p_{M^*}) \quad v \longmapsto \iota(v) = v .$$

Note that $\iota$ is uniformly continuous as $p_{M^*} \leq p_M$, and that it is an injective Riesz homomorphism. Therefore by Lemma 14, the extension

$$\widehat{\iota} : (E_M, \widehat{p}_M) \longrightarrow (E_M, p_{M^*}) \quad v \longmapsto \widehat{\iota}(v) = v$$
is an injective Riesz homomorphism.

Now, we want to show that \( \hat{\tau} \) is surjective. To this aim, we start proving that for every element \( v_\infty \in E_M \) of the form \( v_\infty = \sum_{j=1}^{\infty} t_j m_j \), where \( (t_j)_j \subset R_+ \) is an absolutely summable sequence and \( (m_j)_j \subset M \), there is \( \tilde{\nu} \in \hat{E}_M \) such that \( \hat{\tau}(\tilde{\nu}) = v_\infty \). So, let \( v_\infty \) such a vector. Then, for every \( n \in N \), define the vector \( S_n(v_\infty) = \sum_{j=1}^{n} t_j m_j \in E_M \). We claim that the sequence \( (\hat{S}_n(v_\infty))_n \) converges to \( v_\infty \) in \( p_M \)-norm. In fact, for every \( \epsilon > 0 \) there is \( n_0 \in N \) such that \( \sum_{j=n_0}^{\infty} |t_j| < \epsilon \). This implies that

\[
p_M(v_\infty - S_n(v_\infty)) \leq \sum_{j=n}^{\infty} |t_j| < \epsilon,
\]

for every \( n > n_0 \). But at the same time, \( (S_n(v_\infty))_n \) is also a Cauchy sequence with respect to the \( p_M \)-norm. Indeed, let \( \epsilon > 0 \) and \( m \in N \), then there is \( n_0 \in N \) such that \( \sum_{j=n_0}^{\infty} |t_j| < \epsilon \), which implies that

\[
p_M(S_{n+m}(v_\infty) - S_n(v_\infty)) \leq \sum_{j=n_0}^{n+m} |t_j| \leq \sum_{j=n_0}^{\infty} |t_j| < \epsilon,
\]

for every \( n > n_0 \). Consequently, there is \( \tilde{\nu} \in \hat{E}_M \) such that \( \lim_n S_n(v_\infty) = \tilde{\nu} \) in \( \hat{p}_M \)-norm. Hence,

\[
\hat{\tau}(\tilde{\nu}) = \hat{\tau}\left(\lim_n S_n(v_\infty)\right) = \lim_n \hat{\tau}(S_n(v_\infty)) = \lim_n S_n(v_\infty) = v_\infty.
\]

Moreover, \( v_\infty = \tilde{\nu} \) by Lemma 12.

Let’s now take an arbitrary \( v \in E_M \), and we show that it lies in the image of \( \hat{\tau} \). By Proposition 19, there is a sequence \( (v_n)_n \subset E_M \) which converges to \( v \) in \( p_M \)-norm. We claim that \( (v_n)_n \) is a Cauchy sequence for the \( p_M \)-norm. Let \( \epsilon > 0 \) and \( m \in N \), then there is \( n_0 \in N \) such that \( \sum_{j=n_0}^{\infty} |t_j| < \epsilon \) for every \( n > n_0 \). This means that there are \( (t_j)_j \subset R_+ \) a summable sequence and \( (m_j)_j \subset G \) such that \( |v_{n+m} - v_n| \leq \sum_{j=1}^{\infty} t_j m_j \) and \( \sum_{j=1}^{\infty} t_j < \epsilon \). Define \( v_e = \sum_{j=1}^{\infty} t_j m_j \). As seen before, \( v_e \in \hat{E}_M \) and \( \hat{\tau}(v_e) = v_e \). Now, \( \hat{\tau} \) is a Riesz homomorphism which is injective and surjective on its image. By [AB99, Theorem 9.17], \( (\hat{\tau})^{-1} \) is positive. Therefore, \( |v_{n+m} - v_n| \leq v_e \in \hat{E}_M \) and, consequently,

\[
p_M(v_{n+m} - v_n) = \hat{p}_M(v_{n+m} - v_n) \leq \hat{p}_M(v_e) \leq \sum_{j=1}^{\infty} t_j < \epsilon,
\]

for every \( n > n_0 \). This shows that \( (v_n)_n \) is a Cauchy sequence for the \( p_M \)-norm. Thus, there is \( \hat{\omega} \in \hat{E}_M \) such that \( \lim_n v_n = \hat{\omega} \) in \( \hat{p}_M \)-norm. We can finally compute that

\[
\hat{\tau}(\hat{\omega}) = \hat{\tau}\left(\lim_n v_n\right) = \lim_n \hat{\tau}(v_n) = \lim_n v_n = v.
\]

As before, \( v = \hat{\omega} \) by Lemma 12.

Finally, we show that \( \hat{p}_M = p_M \). We already know that \( p_M \leq \hat{p}_M \) because of the fact that \( p_M \leq p_M \). For the inverse inequality, take \( v \in E_M \) and let \( \epsilon > 0 \) arbitrary. Then there are \( (t_j)_j \subset R_+ \) a summable sequence and \( (m_j)_j \subset M \) such
that $|v| \leq \sum_{j=1}^{\infty} t_j m_j$ and $\sum_{j=1}^{\infty} t_j \leq p_M(v) + \epsilon$. Set $v_\infty = \sum_{j=1}^{\infty} t_j m_j$ and compute that
$$\hat{p}_M(v) \leq \hat{p}_M(v_\infty) = \lim_{n} p_M(S_n(v_\infty)) \leq \lim_{n} \sum_{j=1}^{n} t_j = \sum_{j=1}^{\infty} t_j \leq p_M(v) + \epsilon.$$ As $\epsilon$ was chosen arbitrarily, $\hat{p}_M \leq p_M$. \hfill $\square$

As promised, we explain the choice of the notation $\overline{M}$. To this aim, take a Banach lattice $E$, and let $M$ be a positive uniformly bounded subset of $E$. Then a sufficient condition to guarantee that $(E, p_M)$ is a Banach lattice is asking that $C_M = \overline{C_M} = \overline{M}$.

However, note that the cone $C_M$ is not always closed for the $p_M$-norm. But the closures of $C_M$ and $\overline{C_M}$ coincide, as explained in what follows.

For a subset $A$ of a normed vector space $(E, || \cdot ||)$, write $\overline{A}$ for the closure of $A$ in $E$ with respect to the norm $|| \cdot ||$.

**Corollary 22.** Let $M$ be a positive uniformly bounded subset of $E$. Then
$$\overline{C_M}^{p_M} = \overline{C_M}^{p_M} = \overline{C_M}^{p_M}.$$ **Proof.** The first equality is given by Theorem 21. Let’s look at the second equality. On the one hand, $C_M \subset \overline{C_M}$. Therefore, $\overline{C_M}^{p_M} \subset \overline{C_M}^{p_M}$. On the other hand, $\overline{C_M}^{p_M} \subset \overline{C_M}^{p_M}$ as saw in the proof of Theorem 21. In fact, every element $\sum_{j=1}^{\infty} t_j m_j$ of $C_M$ can be approximated by the truncated sequence $u_n = \sum_{j=1}^{n} t_j m_j$ with respect to the $p_M$-norm. Hence, $\overline{C_M}^{p_M} \subset \overline{C_M}^{p_M}$. We can conclude that $\overline{C_M}^{p_M} = \overline{C_M}^{p_M}$. \hfill $\square$

If one is not happy about this justification, we will give a more satisfying one in Section 5.

Finally, it is possible to give a sufficient condition in terms of $M$ to ensure the completeness of $(E, p_M)$.

**Corollary 23.** Suppose that $M$ is a positive uniformly bounded subset of a Banach lattice $E$ such that $\text{co}(M)$ is $p_M$-closed, then $(E, p_M)$ is a Banach lattice.

**Proof.** The fact that $\text{co}(M)$ is $p_M$-closed implies that $C_M$ is also $p_M$-closed. Therefore by Corollary 22,
$$C_M \subset \overline{C_M}^{p_M} = \overline{C_M}^{p_M} = C_M \subset C_M.$$ Thus, $\overline{M} = \overline{C_M}$ and $(E, p_M) = (E, p_M)$ is a Banach lattice. \hfill $\square$

## 4. Positive Functionals on Majorized Spaces

This section aims to understand linear functionals on majorized spaces. First, a condition to ensure continuity of positive functionals w.r.t. a majorizing norm is presented. Then, linear functionals on normed majorized spaces are characterized using positivity, continuity, and boundedness on the majorizing subset.

**Proposition 24.** Let $E$ be an ordered vector space and let $M$ be a positive subset of $E$. Suppose that $\psi$ is a positive functional defined on $E_M$ uniformly bounded on the set $M$. Then $\psi$ is continuous with respect to the semi-norm $p_M$. 

Proof. Let \((v_\alpha)_\alpha\) be a net in \(E_M\) which converges to some \(v \in E_M\) w.r.t. the \(p_M\)-semi-norm. Then there is a positive net \((\varepsilon_\alpha)_\alpha\) in \(\mathbb{R}\) converging to zero and there are \(t^\alpha_1, \ldots, t^\alpha_n \in \mathbb{R}\) and \(m^\alpha_1, \ldots, m^\alpha_n \in M\) such that
\[
±(v_\alpha - v) \leq \sum_{j=1}^{n_\alpha} t^\alpha_j m^\alpha_j \quad \text{and} \quad \sum_{j=1}^{n_\alpha} t^\alpha_j \leq \varepsilon_\alpha \quad \text{for every} \ \alpha.
\]
Set \(C = \sup \{\psi(m) : m \in M\}\). Then
\[
±\psi(v_\alpha - v) \leq \sum_{j=1}^{n_\alpha} t^\alpha_j \psi(m^\alpha_j) \leq M \sum_{j=1}^{n_\alpha} t^\alpha_j \leq Me_\alpha \quad \text{for every} \ \alpha.
\]
Taking the limit of this last inequality, we have that
\[
\lim_\alpha ±\psi(v_\alpha - v) \leq C \lim_\alpha \varepsilon_\alpha = 0.
\]
This implies that the net \((\psi(v_\alpha))_\alpha\) converges to \(\psi(v)\) w.r.t. the \(p_M\)-semi-norm showing the continuity of \(\psi\). \(\Box\)

It is possible to go deeper in the understanding of linear functionals on majorized spaces if we suppose that \(p_M\) is a norm.

**Theorem 25.** Let \(E\) be an ordered normed vector space, and let \(M\) be a positive uniformly bounded subset of \(E\). Suppose that \(\psi\) is a functional on \(E_M\). Then

a) if the functional \(\psi\) is positive and uniformly bounded on the set \(M\), then \(\psi\) is continuous for the \(p_M\)-norm and
\[
\|\psi\|_{op} \leq \sup \{\psi(m) : m \in M\}.
\]
In particular, if \(\psi\) is constant on \(M\), then \(\|\psi\|_{op} = \psi(m)\) for a \(m \in M\);

b) if the functional \(\psi\) is continuous for the \(p_M\)-norm and positively constant on the set \(M\), then \(\psi\) is positive.

Moreover, if \(E\) is a normed Riesz space, then

c) if the functional \(\psi\) is continuous for the \(p_M\)-norm, then
\[
\sup \{\psi(m) : m \in M\} \leq \|\psi\|_{op} \leq \sup \{|\psi|(m) : m \in M\}.
\]
In particular, if \(\psi\) is positive, then \(\|\psi\|_{op} = \sup \{\psi(m) : m \in M\}\);

**Proof.** We start by showing point a). The fact that \(\psi\) is continuous for the \(p_M\)-norm is given by Proposition 24. Let now \(v \in E_M\) and \(C = \sup \{\psi(m) : m \in M\}\). Then for every \(\varepsilon > 0\) there are \(t_1, \ldots, t_n \in \mathbb{R}_+\) and \(m_1, \ldots, m_n \in M\) such that
\[
±v \leq \sum_{j=1}^{n} t_j m_j \quad \text{and} \quad \sum_{j=1}^{n} t_j \leq p_M(v) + \frac{\varepsilon}{C}.
\]
Therefore,
\[
|\psi(v)| \leq \psi \left( \sum_{j=1}^{n} t_j m_j \right) = \sum_{j=1}^{n} t_j \psi(m_j) \leq M \sum_{j=1}^{n} t_j \leq Mp_M(v) + \varepsilon.
\]
As \(\varepsilon\) and \(v\) were chosen arbitrarily, \(\|\psi\|_{op} \leq \sup \{\psi(m) : m \in M\}\). If \(\psi\) is constant on the set \(M\), then \(\psi(m) \leq \|\psi\|_{op}\) as \(p_M(m) = 1\) for every \(m \in M\). This implies that \(\|\psi\|_{op} = \psi(m)\).
We continue by proving point b). In order to find a contradiction, suppose that ψ is not positive. Then there is a non-zero positive v ∈ E such that p_M(v) = 1 and ψ(v) < 0. Consequently, for every ε > 0 there are t_1, ..., t_n ∈ R_+ and m_1, ..., m_n ∈ M such that

\[ v ≤ \sum_{j=1}^n t_j m_j \quad \text{and} \quad 1 ≤ \sum_{j=1}^n t_j ≤ 1 + \frac{ε}{||ψ||_op}. \]

Now, ||ψ||_op = ψ(m) for every m ∈ M by point b). On the one hand,

\[ ψ \left( \sum_{j=1}^n t_j m_j - v \right) = ||ψ||_op \sum_{j=1}^n t_j - ψ(v) > ||ψ||_op. \]

On the other hand,

\[ ψ \left( \sum_{j=1}^n t_j m_j - v \right) ≤ ||ψ||_op p_M \left( \sum_{j=1}^n t_j m_j - v \right) ≤ ||ψ||_op p_M \left( \sum_{j=1}^n t_j m_j \right) ≤ ||ψ||_op + ε. \]

Therefore, ψ (\sum_{j=1}^n t_j m_j - v) ≤ ||ψ||_op, as ε was chosen arbitrarily. But this is the contradiction searched. Hence, ψ is positive.

Before proving point c), recall that if ψ is a continuous functional on a Riesz space, then |ψ| is also a continuous functional with ||ψ||_op = || |ψ| ||_op by [AB99, Theorem 8.48]. Now the estimation

\[ ψ(m) ≤ |ψ(m)| ≤ |ψ|(m) ≤ ||ψ||_op p_d(m) = ||ϕ||_op \]

Hence, sup \{ψ(m) : m ∈ M\} ≤ ||ψ||_op. To prove the second inequality, set C = sup\{|ψ|(m) : m ∈ M\}, and note that this value is finite as |ψ| is also continuous. Let v ∈ E. Then for every ε > 0 there are t_1, ..., t_n ∈ R_+ and m_1, ..., m_n ∈ M such that

\[ |v| ≤ \sum_{j=1}^n t_j m_j \quad \text{and} \quad \sum_{j=1}^n t_j ≤ p_M(v) + \frac{ε}{C}. \]

Thus,

\[ |ψ|(v) ≤ |ψ|(|v|) ≤ |ψ| \left( \sum_{j=1}^n t_j m_j \right) = \sum_{j=1}^n t_j |ψ|(m_j) ≤ M \sum_{j=1}^n t_j ≤ M p_M(v) + ε. \]

As ε and v were chosen arbitrarily, ||ψ||_op ≤ sup \{ |ψ|(m) : m ∈ M\}. □

This yields the following well-known result:

**Corollary 26.** Let E be an ordered vector space with order unit u. Let ψ be a functional on E such that ψ(u) ≥ 0. Then ψ is positive if and only if it is || · ||_u-continuous.

**Proof.** Suppose that ψ is positive. Then ψ is || · ||_u-continuous and ||ψ||_u = ψ(u) by point a) of Theorem 25. On the other hand, if ψ is || · ||_u-continuous, then it is positive by point b) of Theorem 25. □

**Example 27.** Let G be a locally compact group which acts continuously and compactly on a locally compact space X. Then there is K ⊂ X a compact set such that X = \bigcup_{g ∈ G} gK. Using the compactness of K, it is possible to find a relatively compact subset U ⊂ X and an open subset V ⊂ X such that K ⊂ U ⊂ \overline{U} ⊂ V.
Thanks to Uryshon Lemma ([R86, Lemma 2.12]), there is a positive $\psi \in C_{00}(X)$ such that $\psi = 1$ on $\mathcal{U}$ and $\psi = 0$ on $X \setminus V$. Set $M = \{g\psi : g \in G\}$, where $g\psi(x) = \psi(g^{-1}x)$ for every $g \in G$ and $x \in X$. We claim that $C_{00}(X)_M = C_{00}(X)$.

Indeed, let $\phi \in C_{00}(X)$ and let $K' = \text{supp}(\phi)$. As the action of $G$ on $X$ is cocompact, there are $g_1, ..., g_n \in G$ such that $K' \subset \bigcup_{j=1}^{n} g_j K \subset \bigcup_{j=1}^{n} g_j \mathcal{U}$.

This implies that

$$|\phi| \leq ||\phi||_{\infty} 1_{\text{supp}(\phi)} \leq \sum_{j=1}^{n} ||\phi||_{\infty} g_j 1_{\mathcal{U}} = \sum_{j=1}^{n} ||\phi||_{\infty} g_j 1_{\mathcal{U}} \leq \sum_{j=1}^{n} ||\phi||_{\infty} g_j \psi$$

which proves that $C_{00}(X)_M = C_{00}(X)$. By Riesz Representation Theorem ([AB99, Theorem 14.14]), every positive functional on $C_{00}(X)$ is nothing but a Radon measure on $X$. Therefore, every Radon measure $\mu$ on $X$ which is uniformly bounded on $M$ is continuous for the $p_{M}$-norm by point a) of Theorem 25. In particular, every $G$-invariant Radon measure on $X$ is $p_{M}$-continuous.

5. COHERENT SUBSETS

The notion of coherent subset comes originally from paradoxical decomposition phenomena. Actually, the coherency of a particular family of functions associated with a subset prevents any paradoxical decomposition of the set itself, see for example [R72, Chapter 1]. It is not difficult to traduce this notion in an ordered vector space framework.

**Definition 28.** We say that a positive subset $M$ of an ordered vector space $E$ is **coherent** if

$$\sum_{j=1}^{n} t_j m_j \geq 0 \quad \text{implies that} \quad \sum_{j=1}^{n} t_j \geq 0 \quad \text{for every} \quad m_1, ..., m_n \in M.$$  

Clearly, if $M$ consists of one vector, then it is coherent by the definition of ordered vector space. However, for two vectors sets, this is not anymore true already.

**Example 29.** Consider $\mathbb{R}^2$ with the lexicographic order, i.e., the order given by the relation

$$(v_1, v_2) \geq (w_1, w_2) \iff v_1 > w_1 \text{ or } v_1 = w_1 \text{ and } v_2 \geq w_2.$$  

We claim that the set $M = \{(1, 1), (0, 1)\}$ is not coherent. Indeed, fix $\alpha > 1$. Then

$$(1, 1) - \alpha (0, 1) = (1, 1 - \alpha) \geq 0.$$  

But $1 - \alpha < 0$.

The Archimedean property can avoid the phenomena happening in the last example in the following way:

**Proposition 30.** Let $E$ be an ordered vector space. Then $E$ is Archimedean if and only if the set $M = \{v, w\}$ is coherent for every two non-zero positive vectors $v, w \in E$.  

Proof. We start by proving the if part. Let \( x \in E_+ \) and \( y \in E \) such that \( ny \leq x \) for every \( n \in \mathbb{N} \). In order to find a contradiction, suppose that \( y > 0 \). This implies that \( 0 \leq x - ny \). But this is in contradiction with the coherency of the set \( M = \{ x, y \} \). Hence, \( y \leq 0 \). We can conclude that \( E \) is Archimedean.

Let’s turn our attention to the only if direction. Let \( v \) and \( w \) be two non-zero positive vectors of \( E \). Take \( a, b \in \mathbb{R} \) such that \( av + bw \geq 0 \). If \( a = 0 \), then \( bw \geq 0 \). Therefore, \( a + b = b \geq 0 \). Same for the case \( b = 0 \). If both \( a \) and \( b \) are positive, then also \( a + b \) is positive. If both \( a \) and \( b \) are negative, then
\[
0 \leq av + bw = -\left( |a|v + |b|w \right) < 0,
\]
which is a contradiction. Therefore, it is left only the case \( a > 0 \) and \( b < 0 \) (the one \( a < 0 \) and \( b > 0 \) is similar). But one can write \( av + bw = av - |b|w \geq 0 \). Thus, \( av \geq |b|w \). Consider the non-zero positive vector \( u = v + w \). Then \( v, w \in E_u \) and the pair \((E_u, \cdot \| \cdot \|_u)\) is a normed space. We can hence compute that \( a = a \|v\|_u \geq |b| \|w\|_u = |b| \), since \( \|v\|_u \) and \( \|w\|_u \) are both one. It is possible to conclude that \( 0 \leq a - |b| = a + b \). So, we can assert that the set \( M = \{ v, w \} \) is coherent. \( \square \)

This last result doesn’t hold anymore when considering sets of three elements.

Example 31. Let \( F_2 = \langle a, b \rangle \) be the free group on 2 generators and let \( \ell^\infty(F_2) \) be the (Archimedean) Banach lattice of all real bounded functions on \( F_2 \). Let \( A \) be the set of all reduced words of \( F_2 \) starting with the generator \( a \) and let \( M = \{ 1_a^{-1}A, 1_bA, 1_A \} \). We claim that \( M \) is not coherent. Indeed, define the function
\[
f = 1_a^{-1}A - 1_bA - 1_A.
\]
It is possible to observe that if \( w \in F_2 \) is a reduced word starting with \( a \) or \( b \), then \( f(w) = 0 \). On the other hand, if \( w \in F_2 \) is a reduced word starting with \( a^{-1}, b^{-1} \) or the identity element \( e \), then \( f(w) = 1 \). This implies that \( f \geq 0 \) and \( f \neq 0 \). However, the sum of its coefficients is equal to -1.

The following theorem groups together different characterizations of the coherency of a set \( M \). In particular, it shows that the norm \( p_M \) and the cone \( C_M \) encode the data of the coherence property of \( M \) in the space \( E_M \).

Recall that a base for a cone \( C \) in a vector space \( E \) is a non-empty convex set \( B \subset C \) such that every non-zero \( c \in C \) has a unique representation of the form \( c = tb \) for \( b \in B \) and \( t \in \mathbb{R}_+ \).

Theorem 32. Let \( E \) be an ordered vector space and let \( M \) be a positive subset of \( E \). Then the following assertions are equivalent:

a) the subset \( M \) is coherent;

b) there exists a positive functional \( \psi \) on \( E_M \) constant equal 1 on \( M \). Moreover, if \( E \) is an ordered normed space and \( M \) is uniformly bounded, then \( \psi \) is \( p_M \)-continuous with operator norm equal 1;

c) the convex set
\[
B = \left\{ \sum_{j=1}^n t_j m_j : \sum_{j=1}^n t_j = 1, t_j \in \mathbb{R}_+ \text{ and } m_j \in M \text{ for every } j = 1, ..., n \right\}
\]
is a base of \( C_M \). Moreover, if \( E \) is an ordered normed space and \( M \) is uniformly bounded, then \( B \) is \( p_M \)-bounded;
d) the equality
\[ p_M \left( \sum_{j=1}^{n} t_j m_j \right) = \sum_{j=1}^{n} t_j \] holds for every \( t_1, \ldots, t_n \in \mathbb{R}_+ \) and \( m_1, \ldots, m_n \in M \).

In other words, \( p_M \) is additive on \( C_M \).

e) the cones \( C_M \) and \( c_{00}(M)_+ \) are strictly positive isomorphic. Moreover, if \( E \) is an ordered normed space and \( M \) is uniformly bounded, then \( C_M \) and \( c_{00}(M)_+ \) are isometric with respect to the norms \( p_M \) and \( \| \cdot \|_1 \).

The strategy for proving this last theorem is the following. Firstly, we show that points a), b) and c) are equivalent. Secondly, we demonstrate that both d) and e) are equivalent to the coherent property.

**Proof of Theorem 32.** We start by proving that a) implies b). Suppose that \( M \) is coherent and define the linear map
\[
\omega : \text{span}_\mathbb{R} (M) \longrightarrow \mathbb{R}, \quad \sum_{j=1}^{n} t_j m_j \longmapsto \omega \left( \sum_{j=1}^{n} t_j m_j \right) = \sum_{j=1}^{n} t_j.
\]

Note that \( \omega \) is well-defined thanks to the coherent property. Indeed, suppose that \( \sum_{j=1}^{n} t_j m_j = \sum_{j=1}^{n} t_j' m_j \). Then \( \sum_{j=1}^{n} (t_j - t_j') m_j = 0 \). Therefore, \( \sum_{j=1}^{n} (t_j - t_j') \geq 0 \) by the coherency of \( M \). Similarly, \( \sum_{j=1}^{n} (t_j' - t_j) \geq 0 \). We can conclude that \( \sum_{j=1}^{n} (t_j - t_j') = 0 \) and hence \( \sum_{j=1}^{n} t_j = \sum_{j=1}^{n} t_j' \). Moreover, it is positive and \( \omega(m) = 1 \) for every \( m \in M \). As \( \text{span}_\mathbb{R} (M) \) is a majorizing vector subspace of \( E_M \), it is possible to employ Kantorovich Theorem ([AT07, Theorem 1.36]) to extend \( \omega \) in a positive way to all \( E_M \). The extension is a positive functional on \( E_M \) constant equal 1 on \( M \). If \( E \) is an ordered normed space and \( M \) is uniformly bounded, then \( \psi \) is \( p_M \)-continuous and has operator norm equal to 1 thanks to point a) of Theorem 25.

Suppose now that b) is true and let \( \psi \) be the relative positive functional. Then \( \psi(v) > 0 \) for every non-zero vector \( v \in C_M \) and
\[
B = \{ v \in C_M : \psi(v) = 1 \}.
\]

Therefore, \( B \) is a base for \( C_M \) by [AT07, Theorem 1.47], if \( E \) is an ordered normed space and \( M \) is uniformly bounded, then \( B \) is \( p_M \)-bounded.

Finally, suppose that c) holds and we want to show that \( M \) is coherent. Let \( t_1, \ldots, t_n \in \mathbb{R} \) and \( m_1, \ldots, m_n \in M \) such that \( \sum_{j=1}^{n} t_j m_j \geq 0 \). Thanks to [AT07, Theorem 1.47], there is a functional \( \psi \) on \( E_M \) strictly positive on \( C_M \). Since \( M \subset B \) and \( B \) is a base for \( C_M \), we can suppose that \( \psi(m) = 1 \) for every \( m \in M \). Then
\[
0 \leq \psi \left( \sum_{j=1}^{n} t_j m_j \right) = \sum_{j=1}^{n} t_j \psi(m_j) = \sum_{j=1}^{n} t_j.
\]

Therefore, \( \sum_{j=1}^{n} t_j \geq 0 \) as wished.

We show now that point d) is equivalent to the coherent property. Suppose that \( M \) is coherent, and let \( t_1, \ldots, t_n \in \mathbb{R}_+ \) and \( m_1, \ldots, m_n \in M \). Then
\[
p_M \left( \sum_{j=1}^{n} t_j m_j \right) \leq \sum_{j=1}^{n} t_j p_M(m_j) \leq \sum_{j=1}^{n} t_j
\]
only because $p_M$ is monotone. Let now $\psi$ be the positive functional on $E_M$ constant equal 1 on $M$ given by point b). Note that $\psi(v) \leq p_M(v)$ for every $v \in E_M$. Indeed, let $v \in E_M$ and $\epsilon > 0$. Then there are $t_1, ..., t_n \in \mathbb{R}_+$ and $m_1, ..., m_n \in M$ such that $\pm v \leq \sum_{j=1}^n t_j m_j$ and $\sum_{j=1}^n t_j \leq p_M(v) + \epsilon$. Therefore, $\psi(v) \leq \sum_{j=1}^n t_j \leq p_M(v) + \epsilon$. As $\epsilon$ was chosen arbitrarily, $\psi(v) \leq p_M(v)$. This implies that

$$p_M \left( \sum_{j=1}^n t_j m_j \right) \geq \psi \left( \sum_{j=1}^n t_j m_j \right) = \sum_{j=1}^n t_j \psi(m_j) = \sum_{j=1}^n t_j.$$

It is possible to conclude that $p_M \left( \sum_{j=1}^n t_j m_j \right) = \sum_{j=1}^n t_j$ as wished.

Let's look at the reverse implication. In order to find a contradiction, suppose that the subset $M$ is not coherent. Therefore, there are $t_1, ..., t_n \in \mathbb{R}$ and $m_1, ..., m_n \in M$ such that

$$\sum_{j=1}^n t_j m_j \geq 0 \quad \text{but} \quad \sum_{j=1}^n t_j < 0.$$

We can suppose that every $t_j$ is in $\mathbb{Q}$. Indeed, if it is not the case, we can take $\epsilon > 0$ such that $\epsilon < \frac{n}{\sum_{j=1}^n t_j}$, and we can chose $q_j \in \mathbb{Q}$ such that $q_j \geq t_j$ and $q_j - t_j < \epsilon$ for every $j$. Then

$$\sum_{j=1}^n q_j m_j \geq \sum_{j=1}^n t_j m_j \geq 0,$$

and

$$\sum_{j=1}^n q_j = \sum_{j=1}^n (q_j - t_j) + \sum_{j=1}^n t_j \leq n \epsilon + \sum_{j=1}^n t_j < 0.$$

Now, there is $m \in \mathbb{N}$ such that $mq_j = z_j \in \mathbb{Z}$ for every $j \in \{1, ..., n\}$. Then

$$0 \leq \sum_{j=1}^n mq_j m_j = \sum_{j=1}^n z_j m_j = \sum_{z_j \in I_+} z_j m_j - \sum_{z_j \in I_-} |z_j| m_j,$$

where $I_+ = \{z_j : z_j > 0\}$ and $I_- = \{z_j : z_j < 0\}$. Therefore,

$$\sum_{z_j \in I_-} |z_j| = p_M \left( \sum_{z_j \in I_-} |z_j| m_j \right) \leq p_M \left( \sum_{z_j \in I_+} |z_j| m_j \right) \leq \sum_{z_j \in I_+} z_j.$$

On the other side,

$$0 > \sum_{j=1}^n mq_j = \sum_{j=1}^n z_j = \sum_{z_j \in I_+} z_j - \sum_{z_j \in I_-} |z_j|$$

which implies that $\sum_{z_j \in I_-} |z_j| > \sum_{z_j \in I_+} z_j$. But this is a contradiction.
At the end, we show the equivalence between point e) and point a). Therefore, suppose that $M$ as the coherent property, and define the map

$$
\iota : c_00(M)_+ \longrightarrow C_M, \quad \sum_{m \in A} t_m \delta_m \mapsto \iota \left( \sum_{m \in A} t_m \delta_m \right) = \sum_{m \in A} t_m m.
$$

The $\iota$ is well-defined. Indeed,

$$
\sum_{m \in A} t_m \delta_m = \sum_{m \in A} t'_m \delta_m \iff \sum_{m \in A} t_m = \sum_{m \in A} t'_m \iff \sum_{m \in A} t_m m = \sum_{m \in A} t'_m m.
$$

This equivalences are true thanks to the fact that the two sets $M$ and $\{\delta_m : m \in M\}$ are coherent and because the $t_m$'s and the $t'_m$'s are positive. It is clear that $\iota$ is additive. Therefore, we proceed to show that it is a bijection. The surjectivity is clear. Let focus on the injectivity. Suppose that there are

$$
\sum_{a \in A} t_a \delta_a \text{ and } \sum_{b \in B} t_b \delta_b \text{ in } c_00(N) \text{ such that } \iota \left( \sum_{a \in A} t_a \delta_a \right) = \iota \left( \sum_{b \in B} t_b \delta_b \right).
$$

Then

$$
\sum_{a \in A} t_a a = \sum_{b \in B} t_b b.
$$

Take now the positive functional on $\psi$ on $E_M$ constant equal 1 on $M$ given by point b). Then

$$
\sum_{a \in A} t_a \psi(a) = \sum_{a \in A} t_a = \sum_{b \in B} t_b = \sum_{b \in B} t_b \psi(b).
$$

Therefore,

$$
\sum_{a \in A} t_a \delta_a = \sum_{b \in B} t_b \delta_b
$$

as wished. Finally, if $E$ is an ordered normed space and $M$ is uniformly bounded, $\iota$ is an isometry with respect to the norms $p_M$ and $\| \cdot \|_1$ thanks to point d).

The reverse implication is straightforward, since we can identify the subset $M$ to the positive coherent $\| \cdot \|_1$-uniformly bounded subset $\{\delta_m : m \in M\}$ of $c_00(M)$.

\[\square\]

**Remark 33.** Point b) may seem a little strange at first glance but it’s something that happens naturally. Indeed, let $f$ be the function of Example 31. Then

$$
f = 1_{a^{-1}A} - 1_{bA} - 1_A \geq 0 \iff 1_{a^{-1}A} \geq 1_{bA} + 1_A.
$$

Therefore,

$$
p_M(1_{bA} + 1_A) \leq p_M(1_{a^{-1}A}) = 1
$$

where $M = \{1_{a^{-1}A}, 1_{bA}, 1_A\}$. 
Suppose that $C$ is a generating cone in a vector space $E$ with bounded base $B$. Take $M = B$. Then $p_M$ is equal to the base norm associated to the cone $C$. Therefore, Theorem 32 generalizes [E64, Theorem 1]. Hence, majorizing norms generalize base norms. Summarizing:

**Corollary 34.** Let $C$ be a generating cone in a vector space $E$ with bounded base $B$. Then $(E, p_B)$ is an ordered normed space.

**Corollary 35.** Let $E$ be an ordered vector space. Suppose that $E$ has a strictly positive functional $\psi$. Then every positive subset $M$ of $E$ on which $\psi$ is constant is coherent.

**Proof.** Let $M$ be a positive subset of $E$ on which $\psi$ is constant. Then a normalization of the restriction of $\psi$ on $E_M$ is the functional requested by point b) of Theorem 32. Therefore, $M$ is coherent. $\square$

**Corollary 36.** Let $(E, || \cdot ||)$ be an ordered normed space and suppose that $|| \cdot ||$ is additive, i.e., $||v + w|| = ||v|| + ||w||$ for every $v, w \in E_+$. Then every positive $|| \cdot ||$-norm constant subset $M$ of $E$ is coherent.

**Proof.** Let $M$ be a positive $|| \cdot ||$-norm constant subset of $E$. Without loss of generality, we can suppose that $||m|| = 1$ for every $m \in M$. Let $t_1, ..., t_n \in \mathbb{R}_+$ and $m_1, ..., m_n \in G$. Then

$$\sum_{j=1}^{n} t_j \geq p_M \left( \sum_{j=1}^{n} t_j m_j \right) \geq \left\| \sum_{j=1}^{n} t_j m_j \right\| = \sum_{j=1}^{n} t_j ||m_j|| = \sum_{j=1}^{n} t_j.$$ 

Therefore, condition d) of Theorem 32 is satisfied. Thus, $M$ is coherent. $\square$

Before continuing, note that it is possible to extend the isometry $\iota : (c_{00}(M), || \cdot ||_1) \longrightarrow (C_M, p_M)$ of point e) of Theorem 32 to an isometry

$$\hat{\iota} : (\ell^1(M), || \cdot ||_1) \longrightarrow (\overline{C_M}^{p_M}, p_M) = (\overline{C_M}^{p_M}, p_M).$$

Then it is worth recalling that a result of D. and V. Milman stated in [MM64] says that a sufficient and necessary condition for a Banach space $E$ not to be reflexive is that the positive cone of $\ell^1(N)_+$ is not embeddable in $E$. We suggest looking at [M70, Theorem 2.9] for a proof.

This two facts yields:

**Corollary 37.** Let $(E, || \cdot ||)$ be a Banach lattice and let $M$ be a positive coherent uniformly bounded subset of $E$. Then $(E_M, p_M)$ is reflexive if and only if $(E_M, p_M)$ is finite-dimensional.

**Proof.** Suppose that $(E_M, p_M)$ is reflexive. Then we claim that the cone $\ell^1(M)_+$ is finite-dimensional. Indeed, suppose it is not the case. Then we can embed $\ell^1(N)_+$ in $E_M$ in the following way:

$$\ell^1(N)_+ \hookrightarrow \ell^1(M)_+ \hookrightarrow \overline{C_M}^{p_M}.$$ 

But this is a contradiction with Milman's result. Therefore, $\ell^1(M)_+$ is finite-dimensional. This means that there are $m_1, ..., m_n \in M$ such that $M = \{m_1, ..., m_n\}$. Therefore, $(E_M, p_M) = (E_M, p_M) = (E_M, p_M) = (E_M, p_M)$ where $u = \sum_{j=1}^{n} m_j$ as seen in Example 5. Thus, $(E_M, p_M)$ is an AM-space by [AB99, Theorem 9.28] and hence is
finite-dimensional by [AB99, Theorem 9.38]. The reverse implication is straightforward.

Finally, another justification for the notation $\overline{\mathcal{M}}$ is given.

**Corollary 38.** Let $(E, || \cdot ||)$ be a Banach lattice and let $M$ be a positive coherent uniformly bounded subset of $E$. Then the cone $\overline{C_M}$ is $p_M$-closed.

**Proof.** Take a sequence $(v_n)_n \subset C_M$ which converges to some $v \in E_M$ in $p_M$-norm. This implies that the corresponding sequence $(v'_n)_n$ in $\ell^1(M)_+$ converges also in $|| \cdot ||_1$-norm to some $v' \in \ell^1(M)$. Since the positive cone of a Banach lattice is always close ([AB99, Theorem 8.43 (1)]), the vector $v'$ is in $\ell^1(M)_+$. Therefore, we can conclude that $v \in C_M$ showing that $C_M$ is $p_M$-closed. □

This last result together with Corollary 22 gives the following. If $M$ is a positive coherent uniformly bounded subset of a Banach lattice, then $\overline{C_M} = C_M = C_M^{p_M} = \overline{C_M}^{p_M} = \overline{C_M}^{p_M} = \overline{C_M} = \overline{M}$.

Therefore, if $E$ is a Banach lattice and $M$ is a positive coherent uniformly bounded subset of $E$, then $(E_M, p_M)$ is a Banach lattice if and only if $C_M$ is $p_M$-closed if and only if the base $B$ of $C_M$ is $p_M$-closed.

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