Gelfand triples for the Kohn-Nirenberg quantization on homogeneous Lie groups

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Abstract We study the action of the group Fourier transform and of the Kohn-Nirenberg quantization [1] on certain Gelfand triples for homogeneous Lie groups $G$. Even for the Heisenberg group $G = \mathbb{H}$ there seems to be no simple intrinsic characterization for the Fourier image of the Schwartz space of rapidly decreasing smooth functions $\mathcal{S}(G)$, see [2, 3]. But we may derive a simple characterization of the Fourier image for a certain subspace $\mathcal{S}_s(G)$ of $\mathcal{S}(G)$. Motivated by [4, 5, 6], we restrict our considerations to the case, where $G$ admits irreducible unitary representations, that are square integrable modulo the center $Z(G)$ of $G$, and where $\dim Z(G) = 1$. Starting with $\mathcal{S}_s(G)$, we are able to construct distributions and Gelfand triples around $L^2(G, \mu)$ and its Fourier image $L^2(\hat{G}, \hat{\mu})$, such that the Fourier transform becomes a Gelfand triple isomorphism. In this context we show for the Fourier side, that multiplication of distributions with a large class of vector valued smooth functions is possible and well behaved. Furthermore, we rewrite the Kohn-Nirenberg quantization as an isomorphism for our new Gelfand triples and prove a formula for the Kohn-Nirenberg symbol, which is known from the compact group case [7].

1 Introduction

There is an imbalance between the Schwartz space of rapidly decreasing smooth functions $\mathcal{S}(G)$ and its Fourier image $\mathcal{S}(\hat{G})$ for simply connected nilpotent Lie groups $G$. For the space $\mathcal{S}(\hat{G})$ is a space of operator valued functions on the irreducible unitary representations of $G$, that is not easily characterized without relying on the
Fourier transform itself, see [2, 3]. Of course we can see \( S(G) \) as a projective limit of suitable Hilbert spaces, that results in a corresponding representation of \( \hat{S(G)} \) as a projective limit. But this approach is rather cumbersome, if we want to identify multiplication operators on \( \hat{S(G)} \). Though, if \( G \) is also a homogeneous group with one-dimensional center, then the dual \( \hat{G} \) equipped with the Plancherel measure is isomorphic to \( \mathbb{R} \setminus \{0\} = \mathbb{R}^\times \) equipped with a suitable measure modulo null sets. The setting of homogeneous Lie groups with square integrable representations seems to be convenient in general, see e.g. [5, 6]. Now the main idea is to define a subspace \( S_*(G) \) of \( S(G) \), such that its Fourier image can be identified with a tensor product of rapidly decreasing functions on \( \mathbb{R}^\times \) and a well behaved space of operators isomorphic to \( L(S'(\mathbb{R}^n), S(\mathbb{R}^n)) \). Here we choose \( S_*(G) \), such that \( S_*(G) \) is still densely embedded into \( L^2(G, \mu) \) and show that the Fourier transform acts as a Gelfand triple isomorphism. By using the theory of vector valued distributions of L. Schwartz and related results in [8], we may then define multiplication operators on the Fourier side. We will employ the concept of polynomial manifolds, which were used in [9], for the corresponding spaces \( S(\mathbb{R}^\times) \) and \( O_M(\mathbb{R}^\times) \) of smooth functions on \( \mathbb{R}^\times \) with growth conditions.

After our treatise of the Fourier transform on \( S_*(G) \), we will also examine the Kohn-Nirenberg quantization, defined in [1]. We incorporate our new function spaces into Gelfand triples of symbols and operators, on which the Kohn-Nirenberg quantization acts as a Gelfand triple isomorphism. Finally, we will prove a formula for the Kohn-Nirenberg symbol of operators \( A \in L(O_M(G)) \), that is motivated by the corresponding formula \( a(x, \xi) = \xi^\ast \cdot A(\xi) \) for symbols of operators \( A \in L(D(H)) \) for compact Lie groups \( H \) from [7]. Now we start by reminding ourselves of some standard notations and concepts.

### 1.1 Generalities

By \( G \) we will always denote a simply connected nilpotent lie group. Since \( G \) is diffeomorphic to its Lie algebra \( \mathfrak{g} \) via the exponential map, we will just model \( G \) to be \( \mathfrak{g} \) as set. I.e. \( G = \mathfrak{g} \) is a Lie algebra and a group with multiplication \((x, y) \mapsto xy\) given by the Baker-Campbell-Hausdorff formula. We will use the symbol \( G \) or \( \mathfrak{g} \) depending on which property we want to emphasise. The center of \( G = \mathfrak{g} \) will be denoted by \( Z(G) = \mathfrak{z} \). We will denote the space of left invariant differential operators on \( G \) by \( \mathfrak{u}(\mathfrak{g}_L) \). We choose a fixed Haar measure \( \mu \) on the group \( \mathfrak{g} = G \). This measure \( \mu \) is both a Haar measure with respect to the multiplication and addition. Note, that for each \( P \in \mathfrak{u}(\mathfrak{g}_L) \), there is a unique left invariant differential operator \( P^\lambda \), such that

\[
\int_G (P\varphi) \psi \, d\mu = \int_G \varphi P^\lambda \psi \, d\mu, \quad \text{for all } \varphi, \psi \in D(G),
\]

where \( D(M) \) denotes the space of smooth compactly supported functions on a manifold \( M \).
We denote by $\text{Irr}(G)$ the set of strongly continuous, unitary and irreducible representations of $G$. The dual of $G$ is the quotient of $\text{Irr}(G)$ under the equivalence relation of unitary equivalence and is denoted by $\hat{G}$ equipped with the usual topology and measurable structure [10, 11]. The Plancherel measure to $\mu$, will be $\hat{\mu}$. The representation space of some $\pi \in \text{Irr}(G)$ is denoted by $H_{\pi}$ and the corresponding space of smooth vectors will be denoted by $H^\infty_{\pi}$. It is equipped with a Fréchet topology defined by the seminorms

$$v \mapsto \|\pi(P)v\|_{H_{\pi}}, \quad \text{for } P \in \mathfrak{u}(_G), \quad \text{where } \pi(P)v := P_\pi \pi(x)v\bigg|_{x=0}.$$  

Finally we will write $H^\infty_{\pi}^\ast$ for the strong dual space of $H^\infty_{\pi}$, i.e. the dual space of $H^\infty_{\pi}$ equipped with the topology of uniform convergence on bounded sets of $H^\infty_{\pi}$. The group Fourier transform is defined by

$$\mathcal{F}_G \varphi(\pi) := \int_G \varphi(x) \pi(x)^\ast \, d\mu(x), \quad \text{for } \varphi \in S(G), \pi \in \text{Irr}(G),$$

and its inverse reads

$$\varphi(x) := \int_G \text{Tr}[\pi(x) \mathcal{F}_G \varphi(\pi)] \, d\hat{\mu}(\pi), \quad \text{for } \varphi \in S(G), x \in G.$$  

Let $S(\hat{G}) = \mathcal{F}_G S(G)$ with the topology induced by $\mathcal{F}_G$. The space $L^2(\hat{G}, \hat{\mu})$ is defined to be the completion of $S(\hat{G})$ with respect to the inner product

$$(\hat{\varphi}, \hat{\psi})_{L^2(\hat{G}, \hat{\mu})} := \int _{\hat{G}} \text{Tr}[\hat{\varphi}(\pi) \hat{\psi}(\pi)^\ast] \, d\hat{\mu}(\pi), \quad \text{for } \hat{\varphi}, \hat{\psi} \in S(\hat{G}).$$

The Fourier transform is a unitary operator between $L^2(G, \mu)$ and $L^2(\hat{G}, \hat{\mu})$. See e.g. [1] for more details.

Let $E, F$ be locally convex spaces over $\mathbb{C}$. In general, we will denote the strong dual space of $E$, by $E'$. The dual pairing between $E$ and $E'$, will be denoted by $\langle e', e \rangle := e'(e)$ for $\phi \in E'$, $e \in E$. Similarly we will equip the space of continuous operators from $E$ to $F$ with the topology of uniform convergence on bounded sets of $E$ and denote it by $\mathcal{L}(E, F)$, resp. $\mathcal{L}(E)$ for $E = F$. For any subspace $A \subset E$ we denote its annihilator by $A^\circ$. If $E$ and $F$ are Hilbert spaces, the Hilbert-Schmidt operators from $E$ to $F$ will be denoted by $\mathcal{HS}(E, F)$ (again $\mathcal{HS}(E) := \mathcal{HS}(E, E)$). As usual $A^\ast$ is the adjoint of $A \in \mathcal{L}(E)$. The trace of some nuclear operator $A \in \mathcal{L}(E)$, will be denoted by $\text{Tr}[A]$. Furthermore $\mathcal{HS}(E, F)$ is equipped with the inner product $(A, B) \mapsto \text{Tr}[AB^\ast]$.

Often we need to integrate vector valued functions. For this purpose we will use the concept of weak integrals.

**Definition 1.** Suppose $(X, \nu)$ is a measure space and $E$ is a locally convex vector space. We will call a function $f : X \to E$ is integrable, iff there is some $e \in E$, such that for each $e' \in E'$ we have $e' \circ f \in L^1(X, \nu)$ and
\[ e'(e) = \int_X e' \circ f \, dv. \]

The element \( \int_X f \, d\mu := e \) is called integral over \( f \). Usually we will just say, that \( \int_X f \, d\mu \) converges in \( E \).

From this definition automatically follows, that
\[ A \int_X f \, dv = \int_X A \circ f \, dv. \]

for any continuous linear or antilinear operator \( A : E \to F \) into another locally convex space \( F \). Here, the integrability of \( f \) implies the integrability of \( A \circ f \).

### 1.2 Tensor products

By \( E \otimes F \) we denote the algebraic tensor product between \( E \) and \( F \). Their complete injective tensor product will be denoted by \( E \hat{\otimes}_e F \) and their complete projective tensor product by \( E \hat{\otimes}_p F \). If \( E_j \) and \( F_j \), \( j = 1, 2 \), are locally convex spaces and \( A_j \in \mathcal{L}(E_j, F_j) \), then
\[ A_1 \otimes A_2 : E_1 \hat{\otimes}_e E_2 \to : F_1 \hat{\otimes}_e F_2 \]
denotes the tensor product map of \( A_1 \) and \( A_2 \). The linear map \( A_1 \otimes A_2 \) is continuous and is even an isomorphism, if \( A_1 \) and \( A_2 \) are isomorphisms [12] (Proposition 43.7). Notice, \( A_1 \otimes A_2 \) can also be defined, if \( A_1 \) and \( A_2 \) are continuous anti-linear operators. Later, we will need the following Lemma.

**Lemma 1.** Let \( E, F \) and \( G \) be locally convex spaces, then
\[ \mathcal{L}(E, G) \to \mathcal{L}(E \hat{\otimes}_e F, G \hat{\otimes}_e F) : A \mapsto A \otimes 1 \]
is continuous.

**Proof.** The topology on \( \mathcal{L}(E \hat{\otimes}_e F, G \hat{\otimes}_e F) \) is induced by seminorms of the form
\[ A \mapsto \sup_{z \in B} p(Az) \]
where \( B \) is a bounded set in \( E \hat{\otimes}_e F \) and \( p \) is a continuous seminorm on \( G \hat{\otimes}_e F \). For \( p \) it is sufficient to take any semi norm of the form
\[ p(z) := \sup_{\phi \in V} q((1 \otimes \phi)(z)) \]
where \( V \) is an equicontinuous subset of \( F' \) and \( q \) a continuous seminorm on \( G \) [12] (Definition 43.1 and Proposition 36.1). Notice that the set \( B \subset E \hat{\otimes}_e F \) is bounded, iff for all equicontinuous sets \( W \subset F' \) and all continuous seminorms \( r \) on \( E \).
In general, a subset of $E$ is bounded, iff all continuous seminorms $r$ are bounded on the set. Hence the set $B_V := \bigcup_{\phi \in V} (1 \otimes \phi)(B)$ is a bounded subset of $E$. We arrive at

$$
\sup_{z \in B} p((A \otimes 1)z) = \sup_{z \in B} \sup_{\phi \in V} q((A \otimes \phi)(z)) = \sup_{\phi \in V} \sup_{e \in (1 \otimes \phi)B} q(\phi(e)) = \sup_{F} \phi \in V \sup_{e \in (1 \otimes \phi)B} q(\phi(e)),
$$

where the right hand side defines a continuous seminorm on $L(E, G)$.

If either $E$ or $F$ are nuclear, both tensor product topologies result in the same locally convex space and we may just write $E \hat{\otimes} F$ for either of the complete tensor products of $E$ and $F$ [12] (Theorem 50.1). If both $E$ and $F$ are nuclear Fréchet spaces, then so is $E \hat{\otimes} F$ [12] (Proposition 50.1) and [13] (Chapter III corollary to 6.3). One reason for the usage of nuclear spaces is the following abstract kernel theorem.

**Theorem 1.** If $F$ is a complete locally convex space and $E$ is a nuclear Fréchet space or dual to a nuclear Fréchet space, then $F' \hat{\otimes} E' \cong L(E, F)$ via the extension of the canonical map

$$
\sum_j f_j \otimes e_j' \mapsto \left[ e \mapsto \sum_j f_j e_j'(e) \right].
$$

Suppose additionally both $E$ and $F$ are Fréchet spaces, then $F' \hat{\otimes} E' \cong (F \hat{\otimes} E)'$, via the extension of the map

$$
\sum_k f_k' \otimes e_k' \mapsto \left[ \sum_j f_j \otimes e_j \mapsto \sum_{j,k} f_k'(f_j) e_k'(e_j) \right].
$$

**Proof.** This is Proposition 50.5 in combination with Proposition 50.6 for the first statement and Proposition 50.7 for the second statement in [12].

The multiplication between spaces of smooth functions and spaces of distributions is rarely a continuous bilinear map. But often it is hypocontinuous. Here, a bilinear map $u : E \times F \to G$ is defined to be hypocontinuous between the locally convex spaces $E, F$ and $G$, if for all bounded sets $B_E \subset E$ and $B_F \subset F$, the two sets of linear maps

$$
\{ u(e, \cdot) \mid e \in B_E \} \quad \text{and} \quad \{ u(\cdot, f) \mid f \in B_F \}
$$

are equicontinuous.

Linear maps on tensor factors can easily be combined to construct a linear map on the complete tensor product. The situation for bilinear maps is not as simple. However, in the context of nuclear spaces, we may use the following theorem, which is an amalgamation of a proposition of C. Bargetz and N. Ortner and a corollary of L. Schwartz.

**Theorem 2.** Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be complete nuclear locally convex spaces with nuclear strong duals and let $E$, $F$ and $G$ be complete locally convex spaces. Suppose that
are two hypocontinuous bilinear maps. Suppose furthermore, that either one of the
three properties

- \( \mathcal{H} \) and \( E \) are Fréchet spaces
- \( \mathcal{H} \) and \( E \) are strong duals of Fréchet spaces
- the bilinear map \( b \) is continuous

are fulfilled. Then there is a unique hypocontinuous bilinear map

\[
b^u: (\mathcal{H} \hat{\otimes} E) \times (\mathcal{K} \hat{\otimes} F) \to \mathcal{L} \hat{\otimes} G,
\]

that fulfils the consistency property

\[
b^u(S \otimes e, T \otimes f) = u(S, T) \otimes b(e, f).
\]

**Proof.** For the cases, where \( \mathcal{H} \) and \( E \) are both Fréchet or both duals to Fréchet
spaces, this statement can be found in [8] (Proposition 1). For the case, where
\( b \) is continuous, we find the statement in [14] (Corollaire and Remarques on page 38).
However, in the sources the notation \( \mathcal{H}(E) := \mathcal{H}eE \) for the \( e \)-product of Schwartz is
used. The nuclearity and completeness of \( \mathcal{H} \) and the completeness of \( E \) make sure,
that \( \mathcal{H}eE = \mathcal{H} \hat{\otimes} E \) by [15] (Satz 10.17 and Satz 11.18), which fits our notation.

Examples of spaces fulfilling the conditions for \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \), are \( S(\mathbb{R}^n) \)
\( S'(\mathbb{R}^n) \), \( O_M(\mathbb{R}^n) \) [16] (Chapitre II Théorème 16), \( \mathcal{L}(S(\mathbb{R}^n)) \approx \mathcal{L}(S'(\mathbb{R}^n)) \) and
\( O_M(\mathbb{R}^n) \hat{\otimes} \mathcal{L}(S(\mathbb{R}^m)) \) [17].

The Hilbert space tensor product of \( E \) with \( F \) will be denoted by \( E \hat{\otimes}_2 F \). By a
slight abuse of notation, we will also denote

\[
A_1 \hat{\otimes} A_2: E_1 \hat{\otimes}_2 F_2 \to F_1 \hat{\otimes}_2 E_2
\]

to be the tensor product map of continuous linear maps \( A_j \) between the Hilbert
spaces \( E_j \) and \( F_j \). If \( A_1 \) and \( A_2 \) are unitary, then so is \( A_1 \hat{\otimes} A_2 \).

**1.3 Gelfand triples**

We define a Gelfand Triple \( \mathcal{G} := (E, H, E') \), to be a Hilbert space \( H \), a nuclear
Fréchet space \( E \) and its strong dual space \( E' \), such that there is a dense and con-
tinuous embedding \( E \hookrightarrow H \), with corresponding transposed dense and continuous
embedding \( H' \hookrightarrow E' \), and with a structure operator \( C \). This structure operator is
defined on the whole Gelfand triple and fulfils

\[
C: H \to H \text{ is antunitary,}
\]
\[
C: E \to E \text{ is an antilinear isomorphism,}
\]
\[
C^2 = 1 \quad \text{and} \quad (Ce')(e) = e'(Ce).
\]
This automatically means, that $\mathcal{C}$ is an antilinear automorphism of $E'$. Using $\mathcal{C}$ and the antilinear Fréchet-Riesz isomorphism $\mathcal{R}: H' \to H$, we can construct a unitary map $\mathcal{I} = \mathcal{R}^{-1} \mathcal{C}: H \to H'$. We call $\mathcal{C}$ the real structure on $\mathcal{G}$. That means we may regard $E$ and $H$ as subspaces of $E'$, where corresponding dual pairing on $E \times E$ resp. $H \times E$ is defined via $\mathcal{I}$. Classically Gelfand triples resp. Rigged Hilbert spaces are defined without such a map $\mathcal{C}$ and the embedding $H \to H' \hookrightarrow E'$ stays antilinear [18]. Though, because of $\mathcal{C}$, we won’t need to keep jumping between the spaces $E$ and $H$ and their conjugate spaces. Later, when we examine the Fourier image of test function spaces on $G$, this will simplify our notation considerably.

Usually, if $H$ is some $L^2$-space, then $(\mathcal{I} f)(g)$ is just the integral over $f \cdot g$ and $Cf = f$.

If $G_1 := (F, K, F')$ is another Gelfand triple with real structure $C_1$ and $T: E' \to F'$ is linear, such that $TE \subset F$ and $TH \subset K$, we write

$$T: \begin{pmatrix} E \\ H \\ E' \end{pmatrix} \to \begin{pmatrix} F \\ K \\ F' \end{pmatrix}, \quad \text{or} \quad T: \mathcal{G} \to \mathcal{G}_1.$$

If $T$ is an isomorphism between $E$ and $F$, an isomorphism between $E'$ and $F'$ and unitary between $H$ and $K$, then we call $T$ a Gelfand triple isomorphism. Notice, that if $S: H \to K$ is some unitary map, such that $S$ is an isomorphism between $E$ and $F$ and the adjoint map $S'$ restricts to an isomorphism between $F$ and $E$ (with respect to the usual embeddings), then there is a unique Gelfand triple isomorphism $T$ with $T|_H = S$. The sum of the two Gelfand triples $\mathcal{G}$ and $\mathcal{G}_1$ is defined to be

$$\mathcal{G} \oplus \mathcal{G}_1 := \begin{pmatrix} E \oplus F \\ H \oplus K \\ E' \oplus F' \end{pmatrix},$$

where we use the identification $(E \oplus F)' \cong E' \oplus F'$ and real structure $C \oplus C_1$. The tensor product of the two Gelfand triples is defined to be

$$\mathcal{G} \otimes \mathcal{G}_1 := \begin{pmatrix} E \hat{\otimes} F \\ H \hat{\otimes} K \\ E' \hat{\otimes} F' \end{pmatrix},$$

where we use the identification $(E \hat{\otimes} F)' \cong E' \hat{\otimes} F'$ from Theorem 1 and real structure $C \otimes C_1$. Note, that if $\mathcal{G}_2$ and $\mathcal{G}_3$ are two more Gelfand triples and $A: \mathcal{G} \to \mathcal{G}_2$ and $B: \mathcal{G}_1 \to \mathcal{G}_3$ are Gelfand triple isomorphisms, then

$$A \otimes B: \mathcal{G} \otimes \mathcal{G}_1 \to \mathcal{G}_2 \otimes \mathcal{G}_3$$
is a Gelfand triple isomorphism, where $A \otimes B$ is just the tensor product of $A$ and $B$ on each level of the Gelfand triples.

Corresponding to two Gelfand triples we can construct a Gelfand triple of operators

$$L(G, G_1) := \left( \begin{array}{c} L(E', F) \\ HS(H, K) \\ L(E, F') \end{array} \right),$$

where we use the identification $L(E', F') \cong (F \hat{\otimes} E)' \cong F' \hat{\otimes} E'$ and the real structure $A \mapsto C_1 A C_1$. Now, by using the well know characterization of Hilbert–Schmidt operators by tensor products, we can restate part of Theorem 1:

**Lemma 2.** Suppose $G$ and $G_1$ are the Gelfand triples from above, then the canonical

$$K^{-1}: F \otimes E \to L(E', F), \sum_j f_j \otimes e_j \mapsto \left[ e' \mapsto \sum_j f_j e'(e_j) \right].$$

induces a Gelfand triple isomorphism $K: L(G, G_1) \to G_1 \otimes G$.

Since we choose $G$ simply connected and nilpotent, the space $\mathcal{H}_E^{\infty}$ is a nuclear Fréchet space [10] (Corollary 4.1.2). With respect to $\pi$ we may define the Gelfand triples

$$G(\pi) := \begin{pmatrix} H_{\pi}^\infty \\ H_{\pi} \\ H_{-\infty} \end{pmatrix} \text{ and } G_{\text{op}}(\pi) := L(G(\pi), G(\pi)) = \begin{pmatrix} L(H_{\pi}^\infty, H_{\pi}^\infty) \\ HS(H_{\pi}) \\ L(H_{\pi}^\infty, H_{-\infty}) \end{pmatrix},$$

if we associate a real structure $C_{\pi}$ to $G(\pi)$. Of course, this real structure is not unique. Instead, we will define $\text{Irr}^G(G)$ to be pairs consisting of $\pi \in \text{Irr}(G)$ and an associated real structure $C_{\pi}$ on $G(\pi)$. Usually we will just write $\pi \in \text{Irr}^G(G)$ and mean, that we took a choice of $C_{\pi}$ for $\pi$.

## 2 Polynomial manifolds and Gelfand triples for the Fourier transform

Suppose $E$ is a complete locally convex space. In this section we will discuss generalizations of the spaces $S(\mathbb{R}^n; E)$ and $O_M(\mathbb{R}^n; E)$, see [19, 16]. As usual we will just write $S(\mathbb{R}^n)$ and $O_M(\mathbb{R}^n)$ for $E = \mathbb{C}$. Denote by $P(\mathbb{R}^n)$ the vector space of polynomial functions from $\mathbb{R}^n$ to $\mathbb{C}$ and by $\text{Diff}_P(\mathbb{R}^n)$ the set of Differential operators with polynomial coefficients on $\mathbb{R}^n$. The space of $E$-valued Schwartz functions, $S(\mathbb{R}^n; E) = S(\mathbb{R}^n) \hat{\otimes} E$, is the space of smooth functions $\varphi: \mathbb{R}^n \to E$, such that

$$\sup_{x \in \mathbb{R}^n} p(P \varphi(x)) < \infty$$

for each continuous seminorm $p$ on $E$ and each $P \in \text{Diff}_P(\mathbb{R}^n)$. The above expression also defines a set of seminorms which define the topology on $S(\mathbb{R}^n; E)$. The space
of slowly decreasing functions with values in $E$, $O_M(\mathbb{R}^n; E) = O_M(\mathbb{R}^n) \hat{\otimes} E$, is the space of smooth functions $f : \mathbb{R}^n \to E$, such that

$$\varphi \mapsto f \cdot \varphi \in \mathcal{L}(S(\mathbb{R}^n), S(\mathbb{R}^n; E)),$$

equipped with the subspace topology in $\mathcal{L}(S(\mathbb{R}^n), S(\mathbb{R}^n; E))$.

Suppose now $M$ is an $n$-dimensional smooth manifold with finitely many connected components. An atlas $\mathcal{A}$ of $M$ will be called a polynomial atlas, iff each two charts $(\phi, U), (\psi, V) \in \mathcal{A}$ fulfill

- $U, V$ are connected components of $M$ and $\phi(U) = \psi(V) = \mathbb{R}^n$,
- and if $U = V$, then $\phi \circ \psi^{-1}$ is a polynomial function on $\mathbb{R}^n$.

As in the smooth case we define two polynomial atlases $\mathcal{A}, \mathcal{A}'$ to be equivalent, iff $\mathcal{A} \cup \mathcal{A}'$ is a polynomial atlas. A polynomial structure is an equivalence class of polynomial atlases. Together with a polynomial structure $M$ will be called a polynomial manifold. A chart of a polynomial structure of $M$ will be called a polynomial chart on $M$.

The following list provides some basic examples of polynomial manifolds:

- a finite dimensional vector space, with respect to the linear charts
- a finite dimensional affine spaces, with respect to the affine linear charts
- a simply connected nilpotent Lie group, with respect to the exponential map
- a coadjoint orbit to a nilpotent Lie group [9]

Suppose $M, N$ are polynomial manifolds. A function $f : M \to N$ will be called polynomial, iff $\psi \circ f \circ \phi^{-1}$ is a polynomial for any polynomial charts $(\phi, U)$ on $M$ and $(\psi, V)$ on $N$ with $U \subset f^{-1}(V)$. The function $f$ will be called polynomial resp. tempered diffeomorphism, iff $f$ is bijective and both $f$ and $f^{-1}$ are polynomial resp. slowly increasing.

**Definition 2.** Suppose $M$ is a polynomial manifold. Then

$$P(M) := \{ q : M \to \mathbb{C} : q \text{ is a polynomial function} \}$$

$$\text{Diff}_P(M) := \left\{ P \in \mathcal{L}(D(M)) : \varphi \mapsto P(\varphi \circ \phi) \circ \phi^{-1} \in \text{Diff}_P(\mathbb{R}^n) \right\}.$$

Furthermore $S(M)$ resp. $O_M(M)$ are the spaces of functions $f : M \to \mathbb{C}$, such that for any polynomial chart $\phi$ the function $f \circ \phi^{-1}$ is in $S(\mathbb{R}^n)$ resp. $O_M(\mathbb{R}^n)$. We equip $S(M)$ resp. $O_M(M)$ with the corresponding projective topology.

Products and disjoint unions of polynomial manifolds are again polynomial manifolds. On products $M \times N$ we choose the canonical polynomial structure defined by combining charts $\phi$ on $M$ and $\psi$ on $N$ to polynomial charts $(\phi, \psi)$ on $M \times N$. Directly from our definition follows, that

$$S(M \cup N) = S(M) \oplus S(N) \quad \text{and} \quad S(M \times N; E) = S(M) \hat{\otimes} S(N) \hat{\otimes} E,$$
where $E$ is a complete locally convex space. The identities also hold, if we exchange $S$ with $O_M$. Similar identities are true for $P(M)$ and $\text{Diff}_P(M)$.

We will call a Radon measure $\nu$ on $\mathbb{R}^n$ tempered, iff it is equivalent to the Lebesgue measure $dx$ and the Radon-Nikodym derivatives $\frac{d\nu}{dx}$ are slowly increasing almost everywhere. A Radon measure on a polynomial manifold $\mathbb{R}^n$ will be called tempered, if each pushforward by a polynomial chart is tempered.

**Definition 3.** Suppose $M$ is a polynomial manifold and $\nu$ a tempered measure on $M$. Then $\mathcal{G}(M, \nu)$ is defined to be the Gelfand triple

$$S(M) \hookrightarrow L^2(M, \nu) \hookrightarrow S'(M),$$

equipped with the real structure defined by the usual complex conjugation $\varphi \mapsto \overline{\varphi}$.

If $f : M_1 \to M$ is a tempered diffeomorphism, then for each $\phi \in S'(M)$ the pullback $\varphi_f \phi(\varphi) := \phi(\varphi \circ f^{-1})$ is well defined and induces a Gelfand triple isomorphism

$$\mathcal{G}(M, \nu) \to \mathcal{G}(M_1, \nu \circ f^{-1}).$$

Indeed, we defined tempered measures and polynomial manifolds in such a way, that we have a very simple Gelfand-Triple isomorphism

$$\mathcal{G}(M, \nu) \cong k \bigoplus_{j=1}^{k} \mathcal{G}(\mathbb{R}^n, dx),$$
given by pullbacks and multiplications with slowly increasing functions provided that $M$ a $n$-dimensional polynomial manifold with $k$ connected components.

### 2.1 The polynomial manifold $\mathbb{R}^\times$

For us the two most important examples of polynomial manifolds are the half lines $\mathbb{R}^+$ and $\mathbb{R}^-$. Here the polynomial structure is induced by the chart $\sigma : A \mapsto |A| - 1/|A|$. On $\mathbb{R}^+$ the inverse reads $\sigma^{-1}(y) = (y + \sqrt{y^2 + 4})/2$.

**Lemma 3.** If we extend each function in $\mathcal{S}(\mathbb{R}^\times)$ by zero to the whole real line, then

$$\mathcal{S}(\mathbb{R}^\times) = \{ \varphi \in \mathcal{S}(\mathbb{R}) \mid \varphi \equiv 0 \text{ on } \mathbb{R}^\times \}$$

and $\mathcal{S}(\mathbb{R}^\times)$ carries the subspace topology with respect to $\mathcal{S}(\mathbb{R})$.

**Proof.** We will prove the statement for the $\mathbb{R}^+$ case, for $\mathbb{R}^-$ the proof is analogous. In the following, any function $f$ defined on $\mathbb{R}^+$ will be extended to $\mathbb{R}$ by setting $f(x) = 0$ for $x \leq 0$. Let us define $\mathcal{S}_e(\mathbb{R}) := \{ \varphi \in \mathcal{S}(\mathbb{R}) \mid \varphi \equiv 0 \text{ on } \mathbb{R}^- \}$, equipped with the subspace topology with respect to $\mathcal{S}(\mathbb{R})$. We readily see, that $\mathcal{D}(\mathbb{R}^\times)$ is dense in both $\mathcal{S}(\mathbb{R}^\times)$ and $\mathcal{S}_e(\mathbb{R})$, and thus it is enough to show that the topologies of $\mathcal{S}(\mathbb{R}^\times)$
and $S_{\ast}(\mathbb{R})$ coincide on $\mathcal{D}(\mathbb{R}^+)$. The $S(\mathbb{R}^+)$-topology is induced by seminorms of the form

$$\mathcal{D}(\mathbb{R}^+) \to \mathbb{R} : \varphi \mapsto \sup_{x>0} |A^k B^j \varphi(x)|, \quad k, j \in \mathbb{N}_0,$$

where $A \varphi := (\partial(\varphi \circ \sigma^{-1})) \circ \sigma$ and $B \varphi := (m(\varphi \circ \sigma^{-1})) \circ \sigma = \sigma \cdot \varphi$. First of all, we have

$$A \varphi(x) = \frac{x^2}{x^2 + 1} \partial_x \varphi(x) =: \eta(x) \cdot \partial_x \varphi(x), \quad \varphi \in \mathcal{D}(\mathbb{R}^+), x \in \mathbb{R}^+.$$

The rational function $\eta$ and all of its derivatives are bounded. Hence $A$ can be extended to an operator in $\mathcal{L}(S_{\ast}(\mathbb{R}))$.

We show, that $B$ has an extension in $\mathcal{L}(S_{\ast}(\mathbb{R}))$. For this purpose it is enough to prove that $\frac{1}{m} \in \mathcal{L}(S_{\ast}(\mathbb{R}))$, where $\frac{1}{m} \varphi(x) = \varphi(x)/x$. First of all, for each $\varphi \in \mathcal{D}(\mathbb{R}^+)$ and each $x > 1$

$$|x^k \partial_x^n \left(\frac{1}{x} \varphi(x)\right)| \leq \sum_{j=0}^{n} \frac{n!}{(n-j)!} x^{k-j} \varphi^{(n-j)}(x) \leq \sum_{j=0}^{n} \frac{n!}{(n-j)!} \sup_{y} |y^{k} \varphi^{(n-j)}(y)|,$$

for arbitrary $k, n \in \mathbb{N}_0$. Now we only need to bound the left-hand side for $0 < x < 1$. For $k > n$ almost the same inequality as above can be used. We assume now $n \geq k$.

For $0 < x < 1$ and each $m \in \mathbb{N}$

$$|\varphi(x)/x^m| = \left| \frac{1}{x^m} \int_{0}^{x} (x-t)^{m-1} \varphi^{(m)}(t) \, dt \right| \leq \frac{1}{m} \sup_{y} |\varphi^{(m)}(y)|.$$

Hence

$$|x^k \partial_x^n \left(\frac{1}{x} \varphi(x)\right)| \leq \sum_{j=0}^{n} \frac{n!}{(n-j)!} x^{k-j} \varphi^{(n-j)}(x) \leq \sum_{j=0}^{n} \frac{n!}{(n-j)!} x^{-n} \varphi^{(n-j)}(x) \leq \sum_{j=0}^{n} \frac{1}{(n-j)!} \sup_{y} |\varphi^{(2n+1-j)}(y)|,$$

for all $0 < x < 1$, $n \leq k$ and $\varphi \in \mathcal{D}(\mathbb{R}^+)$. In conclusion $\frac{1}{m} \in \mathcal{L}(S_{\ast}(\mathbb{R}))$ and thus also $B \in \mathcal{L}(S_{\ast}(\mathbb{R}))$. Due to the continuity of $A$ and $B$ we arrive at

$$S_{\ast}(\mathbb{R}) \hookrightarrow S(\mathbb{R}^+),$$

i.e. the $S_{\ast}(\mathbb{R})$-topology is finer than the $S(\mathbb{R}^+)$-topology.

For the reverse embedding we will transport our situation to the whole real line.
\[ \varphi \mapsto \varphi \circ \sigma^{-1} \]

is an isomorphism \( D(\mathbb{R}^+) \rightarrow D(\mathbb{R}) \) and \( S(\mathbb{R}^+) \rightarrow S(\mathbb{R}) \). We denote the image of \( S(\sigma)(\mathbb{R}) \) by this isomorphism by \( S(\sigma)(\mathbb{R}) \) and equip it with the transported \( S(\sigma)(\mathbb{R}) \)-topology. Then

\[ D(\mathbb{R}) \hookrightarrow S(\sigma)(\mathbb{R}) \hookrightarrow S(\mathbb{R}), \]

where both embeddings are dense. The topology in \( S(\sigma)(\mathbb{R}) \) is induced by seminorms of the form

\[ S(\sigma)(\mathbb{R}) \rightarrow \mathbb{R}: \varphi \mapsto \sup_{y \in \mathbb{R}} |C^k E^j \varphi(y)|, \quad k, j \in \mathbb{N}_0, \]

where \( C \varphi := (\partial(\varphi \circ \sigma)) \circ \sigma^{-1} \) and \( E \varphi := (m(\varphi \circ \sigma)) \circ \sigma^{-1} = \sigma^{-1} \cdot \varphi \). The operator \( C \) can be rewritten as

\[ C \varphi(y) = \left(1 + \frac{2}{(y + \sqrt{y^2 + 4})^2}\right) \varphi'(y) =: \psi(y) \cdot \varphi'(y), \quad \varphi \in S(\sigma)(\mathbb{R}), y \in \mathbb{R}. \]

Because \( \sigma^{-1}, \psi \in O_M(\mathbb{R}), \) both \( C \) and \( E \) have extensions in \( L(S(\mathbb{R})) \). Thus \( S(\sigma)(\mathbb{R}) = S(\mathbb{R}) \) and finally \( S(\sigma)(\mathbb{R}) = S(\mathbb{R}^+) \).

The most important property of \( S(\mathbb{R}) \) (next to being a closed subspace of \( S(\mathbb{R}) \)) is stated in the following corollary.

**Corollary 1.** The map \( x \mapsto |x|^v \) is in \( O_M(\mathbb{R}^+) \) for each \( v \in \mathbb{R} \).

**Proof.** The continuity \( \frac{1}{m} \) was already shown in the proof to the last lemma with inequalities (1). Of course \( m \varphi(x) := x \varphi(x) \) defines a continuous operator on \( S(\mathbb{R}^+) \), as well. The derivative of \( x \mapsto |x|^v \) can be bounded by terms of the form \( x \mapsto x^k \) for \( k \in \mathbb{Z} \), which concludes the proof.

We now find a characterisation for the functions in \( O_M(\mathbb{R}^+ \times M; E) \). This space will be of importance later on, when we examine the Fourier image of \( S(G) \) in further detail and when we want to discuss the integral formula for the Kohn-Nirenberg quantization.

**Corollary 2.** A smooth function \( f : \mathbb{R}^+ \times M \rightarrow E \) is in \( O_M(\mathbb{R}^+ \times M; E) \), iff for each \( k \in \mathbb{N}_0 \), each \( p \in \text{Diff}_c(M) \) and each continuous seminorm \( p \) on \( E \), there exists an \( l \in \mathbb{N} \) and an \( q \in \mathcal{P}(M) \), such that \( p(\partial^k_\lambda p x f(\lambda, x)) \leq (1 + |\lambda|^l + |\lambda|^{-l}) q(x) \).

**Proof.** We know, that \( O_M(\mathbb{R}^+ \times M; E) \) is the space of all smooth functions \( f \) on \( \mathbb{R}^+ \), such that

\[ [\varphi \mapsto f \cdot \varphi] \in \mathcal{L}(S(\mathbb{R}^+ \times M), S(\mathbb{R}^+ \times M; E)). \]

We prove the statement for \( \mathbb{R}^+ \), then the other statement follows at once, since \( \mathbb{R}^- \) is isomorphic to \( \mathbb{R}^+ \) by \( x \mapsto -x \). Also, it is enough to consider \( M = \mathbb{R}^n \), as the more general case follows by just using polynomial coordinate charts.

Suppose \( f \in O_M(\mathbb{R}^+ \times \mathbb{R}^n; E) \). Because \( f \) induces a continuous multiplication operator and because \( S(\mathbb{R}^+) \) is a subspace of \( S(\mathbb{R}) \), for each \( k \in \mathbb{N}_0 \), \( \alpha \in \mathbb{N}_0^m \) and each continuous seminorm \( p \) on \( E \), there is some \( m \in \mathbb{N} \) and \( C > 0 \) with
Gelfand triples for homogeneous Lie groups

\[
\sup_{\lambda \in \mathbb{R}^+, x \in M} p(\partial_\lambda^k \partial_x^\alpha (f(\lambda, x) \varphi(\lambda, x))) \leq C \max_{[\beta], l \leq m} \sup_{\lambda \in \mathbb{R}^+, x \in M} (1 + |\lambda|^m)(1 + |x|^2)^m |\partial_\lambda^l \partial_x^\beta \varphi(\lambda, x)|,
\]

for all \( \varphi \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^n) \). We choose some \( \varphi \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^n) \), such that \( \varphi \equiv 1 \) on some neighbourhood around \((\lambda, x) = (1, 0)\), and define \( \varphi_{a, y}(x) := \varphi(ax^{-1}, x - y) \) for \( a > 0 \), \( y \in \mathbb{R}^n \). Then

\[
p(\partial^{(k, \alpha)} f(a, y)) = p(\partial_\lambda^k \partial_x^\alpha (f(\lambda, x) \varphi_{a, y}(\lambda, x)))|_{(\lambda, x)=(a,y)} \leq C \max_{[\beta], l \leq m} \sup_{\lambda \in \mathbb{R}^+, x \in \mathbb{R}^n} (1 + |\lambda|^m)(1 + |x|^2)^m |\partial_\lambda^l \partial_x^\beta \varphi_{a, y}(\lambda, x)| \]

\[
= C \max_{[\beta], l \leq m} \sup_{\lambda \in \mathbb{R}^+, x \in \mathbb{R}^n} a^{-l}(1 + |a\lambda|^m)(1 + |x + y|^2)^m |\partial_\lambda^l \partial_x^\beta \varphi(\lambda, x)| \]

\[
\leq C'(1 + a^m + a^{-m})(1 + |y|^2)^m,
\]

where \( k, \alpha, m \) and \( C \) are as above. Of course this implies, that for each \( k \in \mathbb{N}_0 \), \( P \in \text{Diff}_R(\mathbb{R}^n) \) and each continuous seminorm \( p \) on \( E \), there exists an \( l \in \mathbb{N} \) and an \( q \in \mathcal{P}(\mathbb{R}^n) \), such that

\[
p(\partial_\lambda^k P_x f(\lambda, x)) \leq (1 + |\lambda|^l + |\lambda|^{-l})q(x). \tag{1}
\]

Now for the converse implication. Let \( f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{C} \) be any smooth function, such that for \( p, k \) and \( P \) we find \( m \) and \( q \) for the inequality (1). Then for arbitrary \( \varphi \in \mathcal{S}(\mathbb{R}^+ \times M) \),

\[
\sup_{\lambda \in \mathbb{R}^+, x \in \mathbb{R}^n} (1 + |\lambda|^k)(1 + |x|^2)^k p(\partial^\alpha (f \varphi)(\lambda, x)) \leq C \sup_{\lambda \in \mathbb{R}^+, x \in \mathbb{R}^n} (1 + |\lambda|^k)(1 + |x|^2)^k \sum_{\beta \leq \alpha} |\partial^{\alpha-\beta} f(\lambda, x) \partial^\beta \varphi(\lambda, x)| \leq C' \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{k+m}(1 + |x|^{2m} + a^{k-m}) \sum_{\beta \leq \alpha} |\partial^\beta \varphi(x)|.
\]

Since \( \frac{1}{\mathcal{P}} \) is a continuous operator on \( \mathcal{S}(\mathbb{R}^+) \), the last line defines a continuous seminorm on \( \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^n) \). Thus the operator \( \varphi \mapsto f \cdot \varphi \) is continuous.

**Lemma 4.** Suppose \( E \) is a complete locally convex space and \( f \in \mathcal{O}_M(G; E) \) and let \( F(\lambda, x) := f(\delta_\lambda x) \). Then \( F \in \mathcal{O}_M(G \times \mathbb{R}^+; E) \).

**Proof.** It is enough to show, that for each continuous seminorm \( p \) on \( E \), each \( k \in \mathbb{N}_0 \) and each \( P \in \text{Diff}_R(G) \) there is a polynomial \( q \in \mathcal{P}(G) \) and \( l > 0 \), for which

\[
p(\partial_\lambda^k P_x F(\lambda, x)) \leq (1 + |\lambda|^l + |\lambda|^{-l})q(x).
\]

We realize, that there are polynomial differential operators \( P_v \), such that

\[
\partial_\lambda^k P_x F(\lambda, x) = \sum_{v \in \mathbb{R}} \lambda^v (P_v f)(\delta_\lambda x),
\]
as a finite linear combination. Since each $p(P_v f)$ is bounded by a polynomial $q_v$, we may find polynomials $q_v$ such that

$$p(\partial^k \lambda P_v f(\lambda, x)) \leq \sum_{v \in \mathbb{R}} |\lambda|^v q_v(\delta_1 x) = \sum_{v \in \mathbb{R}} |\lambda|^v q_v(x).$$

This concludes the proof.

From the moderate structures on $\mathbb{R}^+$ and $\mathbb{R}^-$, we construct the moderate manifold $\mathbb{R}^\times = \mathbb{R}^+ \cup \mathbb{R}^-$. Its Schwartz space $\mathcal{S}(\mathbb{R}^\times) = \mathcal{S}(\mathbb{R}^+) \oplus \mathcal{S}(\mathbb{R}^-)$ can be seen as the closed subspace of $\mathcal{S}(\mathbb{R})$ of functions $f$, which vanish of arbitrary order in 0, i.e. $\partial^k f(0) = 0$ for all $k \in \mathbb{N}_0$. The dual space and the Fourier image of $\mathcal{S}(\mathbb{R}^\times)$ will play an important role in the coming discussion. The first statement requires no further proof.

**Lemma 5.** The image of $\mathcal{S}(\mathbb{R}^\times)$ under the Fourier transform on $\mathbb{R}$, $\mathcal{F}_R$, is $\mathcal{S}'(\mathbb{R})$, which is defined to be the subspace of Schwartz functions $f$ with vanishing moments of arbitrary order, i.e.

$$\int_{\mathbb{R}} f(x) p(x) \, dx = 0, \quad \text{for all } p \in \mathcal{P}(\mathbb{R}).$$

The next Lemma is less obvious. It is an extension of the well known fact, that $\mathcal{S}'(\mathbb{R})$, as a vector space, can be identified with the quotient $\mathcal{S}'(\mathbb{R})/\mathcal{P}(\mathbb{R})$ e.g. [20] (Proposition 1.1.3).

**Lemma 6.** Let $E$ be a nuclear Fréchet space and $\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with support in $\{0\}$. Then

$$(\mathcal{S}(\mathbb{R}^\times) \hat{\otimes} E)' = (\mathcal{S}'(\mathbb{R}) \hat{\otimes} E')/(\mathcal{E}'(\mathbb{R}) \otimes E'),$$

especially $\mathcal{E}'(\mathbb{R}) \otimes E'$ is a closed subspace of $\mathcal{S}'(\mathbb{R}) \hat{\otimes} E'$.

**Proof.** First we will prove, that $Z := \mathcal{E}'(\mathbb{R}) \otimes E'$ is a closed subspace of $X' = \mathcal{S}'(\mathbb{R}) \hat{\otimes} E$, where $X := \mathcal{S}(\mathbb{R}) \hat{\otimes} E$. The family $(\partial^k \delta_0)_{k \in \mathbb{N}_0}$ is a basis for $\mathcal{E}'(\mathbb{R})$ where $\delta_0$ is the delta distribution. We use Lemma 1 on the sequence $P_N$ of projections onto the subspaces spanned by $\{\delta_0, \ldots, \partial^N \delta_0\}$ and conclude, that $Z$ is sequentially dense in its closure $\overline{Z}$. Furthermore we realize, that for any $\phi \in \overline{Z}$ there is a sequence $(\epsilon_k^\phi) \subset E'$, such that

$$\phi = \lim_{N \to \infty} \phi_N := \lim_{N \to \infty} \sum_{k=0}^N (\partial^k \delta_0) \otimes \epsilon_k^\phi.$$
for all functions $f \in X = S(\mathbb{R}) \otimes E$ and all $N \in \mathbb{N}$.

Now suppose there is one $l > M$, such that $e'_l \neq 0$. Let us define the sequence of
Schwartz functions $f_m(x) := e^{imx}\psi(x)e/m^{l-1}$, where $\psi$ is a Schwartz function equal
to one near zero and $e \in E$ with $e'_l(e) = 1$. We arrive at

$$|\phi_l(f_m)| = \left| \sum_{k=0}^{l} \frac{(im)^k}{m^{l-1}} e'_k(e) \right| \xrightarrow{m \to \infty} \infty.$$  

But also

$$\sup_{m \in \mathbb{N}} \sup_{k \leq M} \sup_{x \in \mathbb{R}} (x^k g(\partial^k f_m(x))) < \infty,$$

which is a contradiction. Hence $\phi \in Z$, i.e. $\phi$ is in the finite span of the $\partial^k \delta$ and $e'_l$.

Now let $Y := Z'$ be the polar of $Z$. Because $X$ is reflexive, we may identify $Y \subset X$.
Since $Z$ is a closed subspace, we also have $Y^\circ = Z^{\circ\circ} = Z$. Since $\partial^k \delta_0 \otimes e \in Z$ for
all $k \in \mathbb{N}_0$, $e' \in E'$ and

$$(\partial^k \delta_0 \otimes e')(\varphi) = e'(\partial^k \varphi(0)), \quad \text{for } \varphi \in X = S(\mathbb{R}; E)$$

it is quite obvious, that $Y = S(\mathbb{R}^\times) \hat{\otimes} E$.

Since $E$ is a nuclear Fréchet space, $X$ is a nuclear Fréchet space. That also means,
that $X$ is an (FS) space. I.e. it is the projective limit

$$X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X$$

of a sequence of Banach spaces $(X_k)_k$ with compact maps $X_k \leftarrow X_{k+1}$ [13] (Chapter
3, Corollary 3 to Theorem 7.3). Notice that the maps $X_k \leftarrow X_{k+1}$ are weakly compact,
too. Now we may conclude the proof, by using Theorem 13 of [21]. The theorem states,
that in our situation $Y$ is closed and $X$ is an (FS) space – we have $Y' \simeq X'/Y^\circ$.

By using the euclidean Fourier transform in combination with the last lemma, we get
the following corollary.

**Corollary 3.** Let $E$ be a nuclear Fréchet space, then

$$ (S_c(\mathbb{R}) \otimes E)' = (S'(\mathbb{R}) \otimes E')/(\mathcal{P}(\mathbb{R}) \otimes E') $$

and $\mathcal{P}(\mathbb{R}) \otimes E'$ is closed in $S'(\mathbb{R}) \hat{\otimes} E'$.

Furthermore, this characterization for the dual spaces of $S(\mathbb{R}^\times) \hat{\otimes} E$ and $S_c(\mathbb{R}) \hat{\otimes} E$
by quotient spaces, enables us to find subspaces of $S'(\mathbb{R}) \hat{\otimes} E'$ which are embedded
into these dual spaces. Suppose $F$ is a Banach space, such that there is a continuous
embedding $E \hookrightarrow F$ with dense range. Then we may see, that the Lebesgue-Bochner
spaces $L^p(\mathbb{R}; F')$ are embedded into $S'(\mathbb{R}) \hat{\otimes} E'$ and into $S(\mathbb{R}^\times) \hat{\otimes} E'$ for $p \in (1, \infty)$.
Here we define the distribution corresponding to $f \in L^p(\mathbb{R}; F')$ by

$$T_f(\varphi) := \int_{\mathbb{R}} \langle f(x), \varphi(x) \rangle \chi, \quad \varphi \in S(\mathbb{R}; E),$$
Proposition 1. The quotient maps

\[ S'(\mathbb{R}) \hat{\otimes} E' \rightarrow S'(\mathbb{R}) \hat{\otimes} E', \]
\[ S'(\mathbb{R}) \hat{\otimes} E' \rightarrow S'(\mathbb{R}) \hat{\otimes} E', \]

restrict to embeddings

\[ \tilde{B}'(\mathbb{R}; E') \hookrightarrow S'(\mathbb{R}) \hat{\otimes} E', \]
\[ B'(\mathbb{R}; E') \hookrightarrow S'(\mathbb{R}) \hat{\otimes} E'. \]

Proof. A short calculation yields

\[ \tilde{B}'(\mathbb{R}; E') \cap E_0' \hat{\otimes} E' = \{0\} = B'(\mathbb{R}; E') \cap P(\mathbb{R}) \hat{\otimes} E'. \]

Together with the above lemma and corollary, this already concludes the proof.

2.2 Flat orbits of Homogeneous Lie groups

Let Ad be the adjoint action of G on g. Denote by Ca\_x \( \xi \) := \( \xi \circ \text{Ad}_x \) the coadjoint action of \( x \in G \) on linear functionals \( \xi \in g' \). A subalgebra \( m \subset g \) is called polarizing to \( \ell \in g' \), iff \( \ell([m, m]) = \{0\} \) and \( m \) is a maximal algebra fulfilling this condition. For any \( \ell \in g' \) we can find at least one polarizing algebra. There is a bijection between the Coadjoint Orbits and the unireps of \( G \). It can be described by \( [\pi] \leftrightarrow \Omega = \text{Ca}_G \xi \), where \( \pi \) is unitarily equivalent to the induced representation of \( \chi(m) = e^{2\pi i \ell(m)} \) for \( m \in m \subset G \) for some maximal subordinate algebra \( m \) of \( \ell \) [10] (Theorems 2.2.1 - 2.2.4). This correspondence only depends on the orbit \( \Omega \) and not on the choice of element \( \xi \) spanning \( \Omega \) or the choice of polarizing algebra \( m \). We will write \( \pi \sim \xi \) or \( \pi \sim \Omega \), if the equivalence class of \( \pi \) corresponds to the orbit \( \Omega = \text{Ca}_G \xi \). The bijection \( [\pi] \leftrightarrow \Omega \) for \( \pi \sim \Omega \) from \( \hat{G} \) to \( g'/G \) is a homeomorphism [11]. For any \( \xi \) the orbit \( \Omega = \text{Ca}_G \xi \) is an even dimensional polynomial manifold [9] (page 521) and [10] (Lemma 1.3.2).
A Jordan-Hölder basis of \( g \), is a basis \((e_j)\), such that the linear hull \( q_k = \text{span}\{e_1, \ldots, e_k\} \), is an ideal in \( g \) for each \( k \leq \dim G \). Let \( q_k \) be the quotient map \( g' \to g'/g_k \). The set of jump indices \( J \) is the set of \( j > 1 \), such that
\[
\dim q_j(\Omega) - \dim q_{j-1}(\Omega) = 1
\]
Let us denote \( g_J := \text{span}\{e_j \mid j \in J\} \). From Corollary 3.1.5 [10] follows, that a polynomial chart of \( \Omega \) is given by
\[
\sigma_\Omega : \Omega \to g_J : \xi \mapsto \xi \vert_{g_J}.
\]
This equivalence between orbits and the corresponding subspaces \( g_J \), leads to the definition of the orbital Fourier transform as the integral
\[
F_\Omega \varphi(x) := \int_\Omega e^{-2\pi i \xi(x)} \varphi(\xi) \, d\theta_\Omega(\xi), \quad x \in g_J, \, \varphi \in S(\Omega),
\]
where \( \theta_\Omega \circ \sigma_\Omega^{-1} \) is a Haar measure on \( g_J' \). The Pedersen quantization [22] is the equivalent of the Weyl quantization for general simply connected nilpotent Lie groups. It is defined by the integral
\[
\text{op}_\pi(\varphi) := \int_{g_J} \pi(x) \int_\Omega e^{-2\pi i \xi(x)} \varphi(\xi) \, d\theta_\Omega(\xi) \, d\nu_\Omega(x),
\]
for some representation \( \pi \sim \Omega \) and a fitting Haar measure \( \nu_\Omega \) on \( g_J \). We can easily see, that the outermost integral converges in \( L(H) \). The following theorem fixes the choice of \( \nu_\Omega \).

**Theorem 3.** The Pedersen quantization to \( \pi \sim O \) extends to a Gelfand triple isomorphism
\[
\text{op}_\pi : G(\Omega, \theta_\Omega) \to G_{\text{op}}(\pi).
\]

**Proof.** This is stated in [22] Theorem 4.1.4 with a few differences. First of all Pedersen uses the convention \( \xi \leftrightarrow \chi(\cdot) = e^{i\xi(\cdot)} \) for bijection between functionals and characters, though the formulas are easily adjusted to our convention by a unitary transformation between functions resp. distributions defined on coadjoint orbits. The concrete choice of \( G \)-invariant measure on the orbits is not important, since this just results in an additional multiplicative constant, which cancels out with an adjusted measure on \( g_J \).

Though in our case, we can simplify this process by a lot, since we are only interested in representations derived from generic resp. flat orbits.

An orbit is called **generic**, if for each \( k \) the dimension of \( q_k(\Omega) \) is maximal compared to all other orbits. Let us denote the set of equivalence classes derived from generic orbits by \( \hat{G}_{\text{gen}} \subset \hat{G} \). Note, that the Plancherel measure \( \hat{\mu} \) is concentrated on \( \hat{G}_{\text{gen}} \).

A representation \( \pi \in \text{Irr}(G) \) is **square integrable**, if \( x \mapsto |(\pi(x)v, w)_{H_v}| \) is square integrable on \( g/J \) with respect to the Haar measure for all \( v, w \in H_v \). Let us denote the set of square integrable representations by \( \text{SI}(G) \subset \text{Irr}(G) \) and pairs of square
integrateable representations together with some matching conjugation by $\text{SL}_2(G)$. Suppose $\pi \sim \Omega = C_a G \xi$, then $\pi \in \text{SI}(G)$, if and only if $\Omega = \xi + z^\circ$ [4]. Furthermore, if $\text{SI}(G) \neq \emptyset$, then the orbits to square integrable representations are exactly those having the maximal possible dimension [10] (Corollary 4.5.6). Also, the jump indices for $\pi \in \text{SI}(G)$ are given by $J = \{k + 1, k + 2, \ldots, \dim G\}$, where $k = \dim z$, and the equivalence class $[\pi] \in \hat{G}$ is uniquely determined by the central character $\pi \Gamma_1 = e^{2\pi i \xi} \text{id}_{H_\pi}$, where $\xi \in \omega^\circ \approx z^\circ'$. For this fact see [10] (Corollaries 4.5.3 and 4.5.4).

For all $\pi \in \text{SI}(G)$ the Pedersen quantization is simpler, for we can just take one Haar measure $\theta$ on $z^\circ$ and translate it to a measure $\theta_\pi$ on $\Omega \sim \pi$ for each $\pi \in \text{SI}(G)$. The subspace $\omega := g_J$ complements $z$ in $g$ and is the same for each representation in SI. We get a Gelfand triple isomorphisms

$$G(z^\circ, \theta) \to G(\Omega, \theta_\pi): \phi \mapsto \phi \circ P_y,$$

where $P_y$ is the projection onto $z^\circ$ along $\omega^\circ$. Using this isomorphism, we adjust the Pedersen quantization.

**Definition 4.** We will use the Pedersen quantization $\text{op}_\pi$ of $G(z^\circ, \theta)$ with respect to $\pi \in \text{SI}(G)$, defined by

$$\text{op}_\pi: G(z^\circ, \theta) \to G_{\text{op}}(\pi), \phi \mapsto \text{op}_\pi(\phi \circ P_y).$$

This version of Pedersen quantization takes on the form

$$\text{op}_\pi(\varphi) = \int_\omega \pi(x) \int_{z^\circ} e^{-2\pi i \xi(x)} \varphi(\xi) \text{d}\theta(\xi) \text{d}v(x),$$

where $v = v_\Omega$ depends on $\theta$. Of course $\text{op}_\pi$ is a Gelfand triple isomorphism, as well.

Now we will discuss the concept of generic orbits and square integrable representation in context with homogeneous groups. The Lie group $G = g$ is called a homogeneous Lie group, if it is equipped with a group of dilations

$$(0, \infty) \to \text{Hom}(G): \lambda \mapsto \delta_\lambda,$$

where $\delta_\lambda x = e^{\log(\lambda) A} x$ is also a Lie algebra isomorphism and $A$ is a diagonalizable map with positive eigenvalues. The number $Q := \text{Tr}[A]$ is the homogeneous dimension of $G$.

We may always decompose $g$ into eigenspaces $E_\kappa$ of $A$ to Eigenvalues $\kappa > 0$, i.e.

$$g = \bigoplus_{\kappa > 0} E_\kappa,$$

where $[E_\kappa, E_{\kappa'}] \subset E_{\kappa + \kappa'}$. For every $\mu > 0$ the space $\bigoplus_{\kappa \geq \mu} E_\kappa$ is an ideal in $g$. We may always choose a Jordan-H"{a}fner basis $(e_j)$ through these ideals [10] (Theorem 1.1.13).

If $\dim z = 1$, then the center fulfills $z = E_{\mu}$ for $\mu = \max\{\kappa > 0 | E_\kappa \neq \{0\}\}$. Hence, one vector of our chosen Jordan-H"{a}fner basis of eigenvectors will always
lie in the center $z$. We also have the unique decomposition
\[ \mathfrak{g} = z \oplus \omega, \quad \omega \text{ is } A\text{-invariant}. \]

Now for $\lambda < 0$ denote $\delta_{\lambda} x := -\delta_{|\lambda|} x$, for $x \in z$, and $\delta_{\lambda} x := \delta_{|\lambda|} x$ for $x \in \omega$. Furthermore, let also $\delta_{\lambda} \xi := \xi \circ \delta_{\lambda}$ for $\lambda \in \mathbb{R}^\times$ and $\xi \in \mathfrak{g}'$.

The question arises whether generic orbits are mapped to generic orbits by $\delta_{\lambda}$. The dilation $\delta_{\lambda}$ on $\mathfrak{g}'/\mathfrak{g}_\omega^\times$ is a well defined vector space isomorphism by $\delta_{\lambda} \circ q_j := q_j \circ \delta_{\lambda}$, since $\mathfrak{g}_\omega$ and thus also $\mathfrak{g}_\omega^\times$ are $\delta_{\lambda}$-invariant. Furthermore
\[ \dim q_j(\delta_{\lambda} \Omega) = \dim \delta_{\lambda} \circ q_j(\Omega) = \dim q_j(\Omega). \] (2)

Thus $\delta_{\lambda} \Omega$ is generic for each $\lambda \in \mathbb{R}^\times$.

Now take any $\pi \in \mathrm{Irr}_\mathbb{Q}(G)$ with real structure $C_{\pi}$ and define $\overline{\pi} := C_{\pi} \pi C_{\pi} \in \mathrm{Irr}_\mathbb{Q}(G)$ equipped with the same real structure. The representation $\overline{\pi}$ is equivalent to the dual representation of $\pi$. Now denote $\pi_{\lambda}(x) := \pi(\delta_{\lambda} x)$ for $\lambda > 0$ and $\pi_{\lambda}(x) := \overline{\pi}(D_{\lambda} x)$ for $\lambda < 0$. All the representations $\pi_{\lambda}$ are irreducible unitary representations acting on $\mathcal{H}_{\pi}$ resp. acting smoothly on $\mathcal{H}_{\pi}^{\omega}$. With these definitions and the discussion above, we get the equivalence of the three statements
\begin{itemize}
  \item $\pi \in \mathrm{Sl}(G)$, if and only if $\pi_{\lambda} \in \mathrm{Sl}(G)$,
  \item $[\pi] \in \hat{G}_{\text{gen}}$, if and only if $[\pi_{\lambda}] \in \hat{G}_{\text{gen}}$,
  \item $\pi \sim \xi$, if and only if $\pi_{\lambda} \sim \delta_{\lambda} \xi$.
\end{itemize}

Suppose, that $\mathrm{Sl}(G) \neq \emptyset$ and $\dim z = 1$. And suppose we chose a Jordan-H\ddot{a}ulder basis of eigenvectors to $A$. Let $\pi \in \mathrm{Sl}(G)$. As every equivalence class of representations in $\hat{G}_{\text{gen}}$ only depends on its central character, we get a bijection between $\mathbb{R}^\times$ and $\hat{G}_{\text{gen}}$. Hence, $\pi \in \mathrm{Sl}(G)$, if and only if $[\pi] \in \hat{G}_{\text{gen}}$.

We can even go one step further. The dilations $\delta_{\lambda}$ help us to understand $\hat{G}$ as measure space. For this purpose we need the Pfaffian $\mathcal{P}(\xi)$ to a coadjoint orbit $\Omega = C_{\mathfrak{g}_\omega} \xi$, which is defined by $\mathcal{P}(\xi) = \det B_{\xi}$ up to a sign. Here $B_{\xi} := (\xi([e_i, e_j]))_{i, j \in J}$ where the $(e_j)_{j \in J}$ span $\omega$.

We define $\kappa > 0$ and $B \in \mathcal{L}(\omega)$, by $\delta_{\lambda} \eta := \operatorname{sgn}(\lambda)|\lambda|^\eta \eta$, for $\eta \in \omega^\circ$ and $A|_\omega = B$.

**Proposition 2.** Suppose $G$ is a homogeneous group, $\pi \in \mathrm{Sl}_{\mathbb{K}}(G)$ and $\dim z = 1$, then
\[ (\mathbb{R}^\times, \kappa |\lambda|^{q-1} \mathcal{P}(\xi) d\lambda) \to (\hat{G}_{\text{gen}}, \overline{\mu}): \lambda \mapsto [\pi_{\lambda}], \]
where $\pi \sim \xi \in \omega^\circ$, is a homeomorphism resp. a strict isomorphism between the Borel measure spaces. Furthermore, if $\Omega$ is a fixed generic orbit, then $\lambda \mapsto \delta_{\lambda} \Omega$ defines a bijection between $\mathbb{R}^\times$ and the generic orbits.

**Proof.** Let $U$ be the Zariski open set of functionals $\xi \in \mathfrak{g}'$, such that $C_{\mathfrak{g}_\omega} \xi$ is a generic orbit with respect to our basis. For $\xi \in U$ we have $\delta_{\lambda} \xi \in U$ for each $\lambda \in \mathbb{R}^\times$ by equation (2). Each orbit meets $U \cap \omega^\circ$ in exactly one point [10] (Theorem 3.1.9 and Theorem 4.5.5). Furthermore, for any $\xi \in \omega^\circ := \omega^\circ \setminus \{0\}$, we have that
\[ \mathbb{R}^\times \to \omega^\circ: \lambda \mapsto \delta_{\lambda} \xi \]
is a homeomorphism. Thus also $U \cap \omega^\circ = \omega^x = \{ \delta \ell \mid \lambda \in \mathbb{R}^x \}$. But $\omega^x$ also induces all maximal flat orbits, so they coincide with the generic orbits. Since the correspondence $g'/G \cong \hat{G}$ is a homeomorphism, we also have $U/G \cong \hat{G}_{\text{gen}}$ with respect to the subspace topologies. Let $q: U \to U/G$ be the quotient map. Now $q|_{\omega^x}$ is a continuous bijection. We show, that it is also open. By [10], Theorem 3.1.9, there is a well define map $\psi: \omega^x \times \mathbb{R}^\circ \to U$, such that

$$\psi(u,v) = w \iff w \in C\alpha_G u \text{ and } P_y w = v,$$

where $P_y$ is the projection onto $\mathbb{R}^\circ$ along $\omega^\circ$. The map $\psi$ is a rational, non singular bijection with rational non singular inverse. Hence $\psi$ is a homeomorphism. If $V \subset \omega^x$ is open in $\omega^x$, then $C\alpha_G V$ is open in $U$, since

$$\psi(V \times \mathbb{R}^\circ) = C\alpha_G V.$$

Now, since $q$ is open and $q(C\alpha_G V) = q(V)$, the restriction $q|_{\omega^x}$ is an open map and thus a homeomorphism. If we now denote

$$\sigma: \mathbb{R}^x \to \hat{G}_{\text{gen}}: \lambda \mapsto [\delta \lambda \pi],$$

then $\sigma$ is a homeomorphism by the discussion above. Let $\varphi: \hat{G} \to [0, \infty)$ be Borel measurable. Then by [10] Theorem 4.3.10 and the subsequent discussion

$$\int_{\hat{G}} \varphi([\pi]) \, d\tilde{\mu}([\pi]) = \int_{U \cap \omega^\circ} \varphi([\pi_\xi]) | Pf(\xi)| \, d\tilde{\mu}(\xi),$$

where $\tilde{\mu}$ is the measure on $U \cap \omega^\circ$, such that $\{ t\ell \mid t \in [0, 1] \}$ has measure equal to one and $\pi_\xi \sim C\alpha_G \xi$. Also, since our chosen Jordan-Hâfelder basis is an eigenbasis to $A$ resp. $\delta \lambda$, we have

$$|Pf(\delta \lambda \ell)| = |\det(\delta \lambda \ell(e_j, e_i))|_{j,i}^{\frac{1}{2}} = |\det(|\lambda|^{m_j-n_j} \ell([e_j,e_i]))|_{j,i}^{\frac{1}{2}} = |\lambda|^{Tr B} |Pf(\ell)|,$$

where $|\lambda|^{m_j}$ is the eigenvalue of $e_i$ to $\delta \lambda$ for $j \in J$. Both $\sigma$ and $\sigma^{-1}$ are measurable and we have $d(\tilde{\mu} \circ \sigma)(\lambda) = |\lambda|^{x-1} \, d\lambda$. Hence

$$\int_{U \cap \omega^\circ} \varphi([\pi_\xi]) |Pf(\xi)| \, d\tilde{\mu}(\xi) = \int_{\mathbb{R}^x} \varphi([\pi_\lambda]) |Pf(\delta \lambda \ell)| \, d(\tilde{\mu} \circ \sigma)(\lambda)
\quad = \int_{\mathbb{R}^x} \varphi([\pi_\lambda]) |\lambda|^{-1+Tr A} |Pf(\ell)| \, d\lambda$$

and $\sigma$ is a strict isomorphism of measure spaces.

We will denote the euclidean Fourier transform on $g$ by

$$\mathcal{F}_g \varphi(\xi) = \int_g e^{2\pi i \xi(x)} \varphi(x) \, d\mu(x), \quad \varphi \in S(g), \: \xi \in g'.$$
Of course, there is exactly one Haar measure $\mu'$ on $g'$, such that the Fourier transform is a Gelfand triple isomorphism $\mathcal{G}(g, \mu) \to \mathcal{G}(g', \mu')$. Suppose $\ell \in \omega^x$. The map

$$\varphi_{\ell} f(\lambda, \xi) := f(\delta_\lambda(\ell + \xi)) \quad \text{for} \quad \xi \in z, \lambda \in \mathbb{R}^x \text{ and } f: g' \to \mathbb{C},$$

together with the euclidean Fourier transform and the Pedersen quantization, will enable us to describe the group Fourier transform on $G$ (see also [5] for a similar statement).

### 2.3 The group Fourier transform on homogeneous groups

Let us from now on always denote by $G$ a homogeneous Lie group with $\dim z = 1$ and $\text{SI}(G) \neq \emptyset$. Trivially, the group Fourier transform is an isomorphism between $S(G)$ and $S(\hat{G})$. Also, the group Fourier transform is a unitary map from $L^2(G, \mu)$ to $L^2(\hat{G}, \hat{\mu})$. Of course, we may define a Gelfand triple

$$\mathcal{G}(\hat{G}, \hat{\mu}) := (S(\hat{G}), L^2(\hat{G}, \hat{\mu}), S'(\hat{G})), $$

such that $\mathcal{F}_G$ becomes a Gelfand triple isomorphism. Now we will use the isomorphism from Proposition 2 in order to find a new representation of the group Fourier transform on $L^2(G)$. This will be the basis for the definition of our new Gelfand triples and a Gelfand triple isomorphism in the form of an equivalent Fourier transform.

**Proposition 3.** Suppose $\varphi \in S(G)$ and $\pi \in \text{SI}_R(G)$ with $\pi \sim \ell \in \omega^x$, then

$$\mathcal{F}_G \varphi(\pi) = \begin{cases} \op_{\pi} \left( \varphi \mathcal{F}_g \varphi(\lambda, \cdot) \right), & \lambda > 0, \\ \op_{\pi} \left( \varphi \mathcal{F}_g \varphi(\lambda, \cdot) \right), & \lambda < 0. \end{cases}$$

**Proof.** First of all, for any $\varphi \in S(G)$, we have

$$\mathcal{F}_G \varphi(\pi) = \int_G \lambda^{-\text{Tr} A} \varphi(\delta^{-1} A x) \mu(x) \, d\mu(x) = \lambda^{-\text{Tr} A} \mathcal{F}_G (\varphi \circ \delta^{-1})(\pi),$$

for $\lambda > 0$. Also

$$\mathcal{F}_g (\varphi \circ \delta^{-1}) = \lambda^{\text{Tr} A} (\mathcal{F}_g \varphi) \circ \delta, $$

for $\lambda > 0$. Let $\mu_i$ resp. $\nu$ be Haar measures on $z$ resp $\omega$, such that $\mu = \mu_i \otimes \nu$, then

$$\mathcal{F}_G \varphi(\pi) = \int_{\omega} \pi(x) \int_z e^{-2\pi i \langle \xi, \cdot \rangle} \varphi(z - x) \, d\mu_i(z) \, d\nu(x) = \int_{\omega} \pi(x) \int_{\nu} e^{-2\pi i \langle \xi, \cdot \rangle} \mathcal{F}_g \varphi(\xi) \, d\theta(\xi) \, d\nu(x),$$

here $\theta$ is the measure associated to $\nu$ as described in Definition 4. This formula indeed holds pointwise. Hence
\[ \mathcal{F}_G \varphi (\pi \lambda) = \lambda^{-\text{Tr}} \circ \text{op}_\pi (\mathcal{F}_\pi (\varphi \circ \delta^{-1})) = \text{op}_\pi ((\mathcal{F}_\pi \varphi) \circ \delta_\lambda) = \text{op}_\pi (\varphi \mathcal{F}_\pi (\lambda \cdot)) \]

for all \( \lambda > 0 \). For \( \lambda < 0 \) we get

\[ \mathcal{F}_G \varphi (\pi \lambda) = \mathcal{F}_G (\overline{\pi} \lambda) = \text{op}_\pi (\varphi \cdot \overline{\mathcal{F}_\pi \varphi (-\lambda \cdot)}), \]

since \( \overline{\pi} \sim -\lambda \). Now we can conclude the proof, by using \( \delta_{-\lambda} (-\ell + \xi) = \delta_\lambda (\ell + \xi) \), for any \( \xi \in \mathfrak{m} \).

The above proposition (c.f. to Theorem 3.3, of [5]), shows that the group Fourier transform splits into operators which are easy to handle in the \( L^2 \)-setting, if \( \dim \mathfrak{m} = 1 \). Namely, if we use the isomorphism \((\hat{G}, \hat{\mu}) \cong (\mathbb{R}^\times, \kappa, \lambda | O^{-1} | | \mathcal{F}(\ell) | d\lambda \cdot) \) then we can see \( \mathcal{F}_G \) as the composition of unitary operators

\[ \mathcal{F}_\pi : L^2(G, \mu) \to L^2(\mathfrak{g}', \mu'), \]
\[ \varphi_\ell : L^2(\mathfrak{g}', \mu') \to L^2(\mathbb{R}^\times \times \mathfrak{z}^\circ; \kappa, | \lambda | O^{-1} | \mathcal{F}(\ell) | d\lambda \cdot \theta(\xi)), \]
\[ \mathfrak{C}_\pi : L^2(\mathbb{R}^\times, \kappa, | \lambda | O^{-1} | \mathcal{F}(\ell) | d\lambda \cdot; L^2(\mathfrak{z}^\circ, \theta)) \to L^2(\mathbb{R}^\times, \kappa, | \lambda | O^{-1} | \mathcal{F}(\ell) | d\lambda \cdot; \text{HS}(H_\mu)), \]

where \( \mathfrak{C}_\pi = P_+ \otimes \text{op}_\pi + P_- \otimes \text{op}_\overline{\pi} \), for the projection \( P_\pm \) of \( L^2(\mathbb{R}^\times) \) onto \( L^2(\mathbb{R}^\pm) \).

It is very convenient, that here the operator component emerges as tensor product factor, which in turn enables us to understand multiplication operators more easily. This motivates us to define the following alternative spaces of test functions.

### 2.4 The Fourier transform on \( S_*(G) \)

In order to know which function space is a good choice, we will first take a look at the pull back \( \varphi_\ell \). Here our earlier discussion of moderate manifolds come into play again. Remember that \( \mathbb{R}^\times \) is equipped with a moderate structure defined by \( \mathbb{R}^\times = \mathbb{R}_+ \cup \mathbb{R}_- \), i.e. defined by the moderate Structures on \( \mathbb{R}^{\pm} \). Similarly we define \( \mathfrak{g}^\pm_\ell \) and \( \mathfrak{g}^\times_\ell \) by

\[ \mathfrak{g}^\pm_\ell := \{ t\ell + \eta \mid t \in \mathbb{R}_\pm, \eta \in \mathfrak{z}^\circ \}, \quad \text{for } \ell \in \omega^\times = \omega^\circ \setminus \{0\} \]

and \( \mathfrak{g}^\times = \mathfrak{g}^+_\ell \cup \mathfrak{g}^-_\ell \) and equip \( \mathfrak{g}^\times_\ell \) with the polynomial structure analogously to the one on \( \mathbb{R}^{\pm} \), i.e. the polynomial structure induced by the map

\[ \mathfrak{g}^\times_\ell \to \mathbb{R} \times \mathfrak{z}^\circ: (t\ell + \eta) \mapsto (t - 1/t, \eta). \]

Then \( \delta_\lambda \) induces a tempered diffeomorphism, as written in the following Lemma. The moderate structure on \( \mathfrak{g}^\times_\ell \) is just the one induced by its connected components. Notice that we just have \( \mathfrak{g}^\times = \mathfrak{g}^\times_\ell \setminus \mathfrak{z}^\circ \) as a sets.

**Lemma 7.** Let \( \ell \in \omega^\times \). The Map \( w_\ell : \mathbb{R}^{\pm} \times \mathfrak{z}^\circ \to \mathfrak{g}^\times_\ell : (\lambda, \xi) \mapsto \delta_\lambda (\ell + \xi) \) is a tempered diffeomorphism.
Proof. We prove, that $\mathbb{R}^+ \times \mathfrak{z} \cong \mathfrak{g}^+$ via $w_\ell$. The proof to the second statement is analogous. Suppose $(\xi_j)$ is the dual basis to our Jordan–Hölder basis $(e_j)$ of eigenvectors. Here $(\xi_j)$ is the basis of $\mathfrak{z}$. Let $\kappa_j$ be the positive number, such that $\delta\lambda\xi_j = \lambda\kappa_j\xi_j$ for $\lambda > 0$.

We use the charts $\sigma$ resp. $\sigma_1$, defined by

$$(\lambda, \sum_{j=1}^{2n} c_j\xi_j) \mapsto (\lambda^{1/\kappa_0}c_1, \ldots, \lambda^{1/\kappa_n}c_{2n}).$$

Then

$$(\sigma_1 \circ w_\ell \circ \sigma_1^{-1})(t, c_1, \ldots, c_{2n}) = \left(\frac{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_0})^{\frac{1}{\kappa_0}}}{2^{\frac{1}{\kappa_0}}} - \frac{2^{\frac{1}{\kappa_0}}}{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_0})^{\frac{1}{\kappa_0}}}, \frac{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_1})^{\frac{1}{\kappa_1}}}{2^{\frac{1}{\kappa_1}}} c_1, \ldots, \frac{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_n})^{\frac{1}{\kappa_n}}}{2^{\frac{1}{\kappa_n}}} c_{2n}\right),$$

which is a slowly increasing function. Similarly

$$(\sigma \circ w_\ell^{-1} \circ \sigma_1^{-1})(t, c_1, \ldots, c_{2n}) = \left(\frac{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_0})^{\frac{1}{\kappa_0}}}{2^{\frac{1}{\kappa_0}}} - \frac{2^{\frac{1}{\kappa_0}}}{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_0})^{\frac{1}{\kappa_0}}}, \frac{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_1})^{\frac{1}{\kappa_1}}}{2^{\frac{1}{\kappa_1}}} c_1, \ldots, \frac{(t + \sqrt{t^2 + 4} \frac{1}{\kappa_n})^{\frac{1}{\kappa_n}}}{2^{\frac{1}{\kappa_n}}} c_{2n}\right)$$

is slowly increasing.

By Lemma 3, we can see $S(\mathfrak{g}^+_\ell)$ as the space

$$S(\mathfrak{g}^+_\ell) = \{\varphi \in S(\mathfrak{g}) | \varphi \equiv 0 \text{ on } \mathfrak{g}^+_\ell\},$$

equipped with the subspace topology in $S(\mathfrak{g})$.

The tempered diffeomorphism from the last lemma induces a Gelfand triple isomorphism.

Lemma 8. The pullback $\varphi_\ell f := f \circ w_\ell$ defines a Gelfand triple isomorphism

$$\varphi_\ell : \mathcal{G}(\mathfrak{g}^+_\ell, \mu') \to \mathcal{G}(\mathbb{R}^+, \kappa|\mathcal{P}(\ell)| |\lambda|Q-1 \text{ d}t) \otimes \mathcal{G}(\mathfrak{z}^+, \theta).$$

Proof. We take an arbitrary $f \in C_c(\mathfrak{g}^+_\ell)$. Define $\omega^* := \mathbb{R}^+ \cdot \ell$, then
\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} f(\delta_1(\ell + \xi)) |\mathcal{P}(\ell)| |\mathcal{A}|^{Q-1} \, \mathrm{d}\mathcal{A} \, \mathrm{d}\theta(\xi) \]

\[ = \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} f((\delta_1 \ell + \xi) \mathcal{P}(\ell)) |\mathcal{A}|^{Q-1} \, \mathrm{d}\mathcal{A} \, \mathrm{d}\theta(\xi) \]

\[ = \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} f(\eta + \xi)) |\mathcal{P}(\ell)| \, d\mu_{\omega^\circ}(\eta) \, \mathrm{d}\theta(\xi) \]

\[ = \int_{\mathbb{R}^2} f(\xi) \, \mathrm{d}\mu'(\xi). \]

Here in the last two lines we used, that the measure \( \mu_{\omega^\circ} \) on \( \omega^\circ \) is defined by the Lebesgue measure and \( \ell \) and that \( \theta \) is defined by \( \mu' = |\mathcal{P}(\ell)| \mu_{\omega^\circ} \otimes \theta \). The rest follows with the fact, that \( \varphi_{\ell} f = f \circ \omega_{\ell} \), where \( \omega_{\ell} \) is the tempered diffeomorphism from Lemma 7.

We also proved, that the restriction of the Haar measure \( \mu' \) to \( \mathcal{A}^\circ \) is actually a tempered measure with respect to our chosen moderate structure.

Now we are ready to define Gelfand triples, with respect to which we get a nice theory for the group Fourier transform.

**Definition 5.** We define the following reduced Schwartz space

\[ S_\pi(G) := \{ \varphi \in S(G) \mid [(\lambda, x) \mapsto \varphi(\lambda z + x)] \in S_\pi(\mathbb{R}) \otimes S(\omega) \} \]

for any choice \( z \in \mathbb{R} \setminus \{0\} \), equipped with the subspace topology in \( S(G) \), and the corresponding Gelfand triple

\[ \mathcal{G}_\pi(G, \mu) := (S_\pi(G), L^2(G, \mu), S'_\pi(G, \mu)), \]

equipped with the real structure given by the pointwise complex conjugation. Furthermore, we define the Gelfand triple

\[ \mathcal{G}(\mathbb{R}^\times; \pi) := \begin{cases} S(\mathbb{R}^\times; \pi) \\ L^2(\mathbb{R}^\times; \pi) \\ S'_\pi(\mathbb{R}^\times; \pi) \end{cases} := \mathcal{G}(\mathbb{R}^\times, \kappa |\mathcal{P}(\ell)| |\mathcal{A}|^{Q-1} \, \mathrm{d}\mathcal{A} \otimes \mathcal{G}_{\text{op}}(\pi). \]

for each \( \text{SI}_\pi(G) \ni \pi \sim \ell \in \omega^\circ \).

That \( \mathcal{G}_\pi(G) \) is indeed a Gelfand triple, this can be seen by using Proposition 1. We use any linear isomorphism \( \mathbb{R} \cong \mathbb{R} \) to define \( \mathcal{B}_\pi(3; S'(\omega)) \), then we see, since \( L^2(G, \mu) \subset \mathcal{B}_\pi(3; S'(\omega)) \), that the space \( L^2(G, \mu) \) is embedded into \( S'(G) = S'_\pi(3; S'(\omega)) \). This embedding is continuous, since the embedding \( L^2(G, \mu) \hookrightarrow S'(G) \) is continuous.

Of course, the canonical map of \( S_\pi(G) \) into \( L^2(G, \mu) \) is a continuous embedding as well. Now the Hahn–Banach theorem implies that both embeddings are also dense, for they are dual to each other.

**Theorem 4.** Let \( \text{SI}_\pi(G) \ni \pi \sim \ell \in \omega^\circ \). Let the Fourier transform in \( \pi \)-picture, \( \mathcal{T}_\pi \)

be defined by
The operators $\mathcal{F}_\pi := \mathcal{C}_\pi \circ \phi_\ell \circ \mathcal{F}_\eta$,

where $\mathcal{C}_\pi = P_+ \otimes \mathcal{C}_\pi + P_- \otimes \mathcal{C}_\pi$ and $P_+ = 1 - P_-$ is the projection of $S(\mathbb{R}^\times)$ onto $S(\mathbb{R}^\times)$. Then $\mathcal{F}_\pi$ is a Gelfand triple isomorphism

$\mathcal{F}_\pi : \mathcal{G}_\pi(G) \rightarrow \mathcal{G}(\mathbb{R}^\times; \pi)$.

**Proof.** The proof essentially writes itself by now and is a summary of previous statements. The euclidean Fourier transform $\mathcal{F}_\pi$ is a Gelfand triple isomorphism between $\mathcal{G}(\mathbb{R}^\times, \mu)$ and $\mathcal{G}(\mathbb{R}^\times, \mu') = \mathcal{G}(\mathbb{R}^\times, |\mathcal{F}(\ell)| |\mu_\omega|) \otimes \mathcal{G}(\mathbb{R}^\times, \theta)$ by Lemma 5, where we choose the Haar measures $\mu_\omega$ and $\theta$, such that $\mu' = |\mathcal{F}(\ell)| \mu_\omega \otimes \theta$ and $\mu_\omega$ is induced by the Lebesgue measure $\text{d}l$ via the map $\mathbb{R} \ni \lambda \rightarrow \lambda \ell \in \omega^\times$.

By Lemma 8, the pull back $\varphi_\ell$ is a Gelfand triple isomorphism between $\mathcal{G}(\mathbb{R}^\times, \mu_\ell)$ and $\mathcal{G}(\mathbb{R}^\times, k |\mathcal{F}(\ell)| |\lambda|^{Q-1} \text{d}l) \otimes \mathcal{G}(\mathbb{R}^\times, \theta)$.

For the last step we just need to use that $\mathcal{C}_\pi = P_+ \otimes \mathcal{C}_\pi + P_- \otimes \mathcal{C}_\pi$ is a Gelfand triple isomorphism between $\mathcal{G}(\mathbb{R}^\times, k_0 |\mathcal{F}(\ell)| \text{d}l) \otimes \mathcal{G}(\mathbb{R}^\times, \mu_\ell)$ and $\mathcal{G}(\mathbb{R}^\times; \pi)$ by Theorem 3 and Definition 4.

Let us now discuss a few properties of $S_\pi(G)$ and $S(\mathbb{R}^\times; \pi)$. Their duals can be identified with quotient spaces, in particular

$S'_\pi(G) \simeq S'_\pi(G)/(\mathcal{P}_\pi \otimes S'((\omega)))$ and

$S'(\mathbb{R}^\times; \pi) \simeq S(\mathbb{R}^\times) \otimes \mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu})/(E'_\pi \otimes \mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu}))$,

by Lemma 6 and Corollary 3. By employing Proposition 1, we can identify a big space of distributions on $G$ resp. $\mathbb{R}$ that are embedded into $S'_\pi(G)$ resp. $S'(\mathbb{R}^\times; \pi)$.

I.e. if we define $\hat{\mathcal{B}}(\mathcal{S'}(\omega))$ by using any isomorphism $\mathbb{R} \approx \mathfrak{g}$, then

$\hat{\mathcal{B}}(\mathcal{S'}(\omega)) \hookrightarrow S'_\pi(G)$ and $\hat{\mathcal{B}}(\mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu})) \hookrightarrow S'(\mathbb{R}^\times; \pi)$.

We may, for example, identify $L^p(G, \mu)$, for $p \in [1, \infty)$, and also $S(G)$ as a subspace of $\hat{\mathcal{B}}(\mathcal{S'}(\omega))$ and the Bochner-Lebesgue spaces $L^p(\mathbb{R}, \text{d}l; \mathcal{L}(H_\eta))$, for $p \in (1, \infty]$, and also $S(\mathbb{R}, \mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu}))$ as subspaces of $\hat{\mathcal{B}}(\mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu}))$.

The definition of $S(\mathbb{R}^\times; \pi)$ and $S'(\mathbb{R}^\times; \pi)$ enables us to define a multiplication with a large class of smooth functions via Theorem 2.

**Proposition 4.** For any $\pi \in \operatorname{SI}(G)$, the multiplications

$O_M(\mathbb{R}^\times; \mathcal{L}(H^\infty_{\pi})) \times S(\mathbb{R}^\times; \pi) : (f, \varphi) \mapsto f \varphi$

$O_M(\mathbb{R}^\times; \mathcal{L}(H^\infty_{\pi})) \times S(\mathbb{R}^\times; \pi) : (f, \varphi) \mapsto \varphi f$,

defined pointwise by composition of operators in $\mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu})$, $\mathcal{L}(H^\infty_{\pi})$ and $\mathcal{L}(H^\infty_{\mu})$, are hypocontinuous bilinear maps.

**Proof.** We just need show that we may apply Theorem 2. The compositions of operators

$\mathcal{L}(H^\infty_{\pi}) \times \mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu}) : (A, B) \mapsto AB$

$\mathcal{L}(H^\infty_{\pi}) \times \mathcal{L}(H^\infty_{\pi} \otimes H^\infty_{\mu}) : (A, B) \mapsto BA$
are hypocontinuous, since separately continuous maps on barrelled spaces are hypocontinuous. Since $H^\omega_\pi = S(\mathbb{R}^\pi)$ and Theorem 2, the above operator spaces are barrelled by [16] (Corollaire 2 on page 128 of chapter 2) and [13] (Corollary to 8.4 in chapter 2). Also, the multiplication of slowly increasing functions and Schwartz functions is hypocontinuous. This follows directly from the definition and comments on pages 243 and 244 of [19]. Now we just need to remind ourselves, that $S(\mathbb{R}^\pi; \pi)$ is a tensor product of nuclear Fréchet spaces.

Now, we will prove the analogous result for the multiplication with the operator valued tempered distributions $S'(\mathbb{R}^\pi; \pi)$. As we used in the proof above, $H^\omega_\pi$ is reflexive for any $\pi \in SI(G)$. Thus, by using the transpose, we get the two isomorphisms of topological vector spaces

$$L(H^\omega_\pi) \ni A \mapsto A^t \in L(H^\omega_\pi') \quad \text{and} \quad L(H^\omega_\pi) \ni B \mapsto B^t \in L(H^\omega_\pi') .$$

Denote for $f$ in $\mathcal{OM}(\mathbb{R}^\pi; L(H^\omega_\pi))$ or in $\mathcal{OM}(\mathbb{R}^\pi; L(H^\omega_\pi))$ the operator valued function $f^t(\lambda) := f(\lambda)^t$. Then we may define multiplications on $S'(\mathbb{R}^\pi; \pi)$ by

$$(f \phi)(\varphi) := \phi(f^t \varphi) \quad \text{and} \quad (\phi g)(\varphi) := \phi(g^t \varphi),$$

for all $\phi \in S'(\mathbb{R}^\pi; \pi)$ and $\varphi \in S(\mathbb{R}^\pi; \pi)$, if we choose $f \in \mathcal{OM}(\mathbb{R}^\pi; L(H^\omega_\pi))$ and $g \in \mathcal{OM}(\mathbb{R}^\pi; L(H^\omega_\pi'))$. We get the following corollary.

**Corollary 4.** For any $\pi \in SI(G)$, the multiplications

$$\mathcal{OM}(\mathbb{R}^\pi; L(H^\omega_\pi)) \times S'(\mathbb{R}^\pi; \pi) : (f, \phi) \mapsto f \phi,$$

$$\mathcal{OM}(\mathbb{R}^\pi; L(H^\omega_\pi)) \times S'(\mathbb{R}^\pi; \pi) : (f, \phi) \mapsto \phi f$$

are hypocontinuous.

**Proof.** This follows directly from the definition of the multiplication and the fact, that the dual pairing is hypocontinuous. Equivalently, we could also employ Theorem 2.

Let us now relate the Fourier transform in $\pi$ picture with the group Fourier transform. Denote by $j_\pi$ the map $j_\pi(\sigma) := [\mathbb{R}^\pi \ni \lambda \mapsto \sigma(\pi_\lambda) \in L(\mathcal{H}^\omega_{\pi}; \mathcal{H}^\omega_{\pi})]$, defined on $S(G)$.

Due to Proposition 2, we know that $j_\pi$ has a unitary extension from $L^2(\hat{G}_{\text{gen}}, \hat{\mu})$ onto $L^2(\mathbb{R}^\pi; \pi)$. Because the Plancherel measure $\hat{\mu}$ is concentrated on $\hat{G}_{\text{gen}}$ [10] (Theorem 4.3.16), we can see $j_\pi$ as a map defined on $L^2(\hat{G}, \hat{\mu})$. Proposition 3 implies the left side of Figure 1.

For the Schwartz spaces and spaces of tempered distributions we get a very similar diagram. Denote also by $S_0(\hat{G})$ the image of $S(\mathbb{R}^\pi; \pi)$ under $j_\pi^{-1}$. The commutative diagram for the $L^2$-spaces implies the commutative diagram for the Schwartz spaces in Figure 1. Then, by duality, we get the commutative diagram for the tempered distributions. However, the group Fourier transformations $\mathcal{F}_G$ are defined by duality on $S'(G)$ resp. $S'_0(G)$ and are not the same map, even though we use the same symbol.
Fig. 1 Commuting diagrams, that show the relation between the group Fourier transform on \( G(\mu) \) and the Fourier transform in \( \pi \)-picture on \( G(\mu) \) for some \( \pi \in S\ell(G) \).

By Corollary 3 the map \( j'_* \) can be seen as the quotient map

\[
S'(G) \to S'(G)/(\mathcal{P}(\mathfrak{g}) \otimes S'(\omega)) \simeq S'_\ell(G),
\]

which is an open map. This also implies, that \( j'_0 \) is surjective and open.

Since \( \psi_\ell : \mathbb{R}^n \times \mathfrak{g}^* \to \mathfrak{g}^* \), \( \psi_\ell(\lambda, \xi) = \delta_\lambda(\ell + \xi) \) is a tempered diffeomorphism, we can also see \( \psi_\ell \) as an isomorphism between \( O_M(\mathfrak{g}^*) \) and \( O_M(\mathbb{R}^n \times \mathfrak{g}^*) \) rep. between \( S(\mathfrak{g}^*) \) and \( S(\mathbb{R}^n \times \mathfrak{g}^*) \). However, in order to examine the Fourier image of \( S(\mathfrak{g}^*) \), it is even better to consider mixed spaces. We equip \( \omega^\times = \mathbb{R}^n \cdot \ell \) with the polynomial structure transported from \( \mathbb{R}^n \).

**Lemma 9.** The Gelfand-Triple isomorphism \( \varphi_\ell \) restricts to an isomorphism

\[
\varphi_\ell : O_M(\omega^\times) \otimes S(\mathfrak{g}^*) \to O_M(\mathbb{R}^n) \otimes S(\mathfrak{g}^*).
\]

**Proof.** We identify \( \omega^\times \simeq \mathbb{R}^n \) and \( \mathfrak{g}^* \simeq \mathbb{R}^{2n} \) and \( \mathfrak{g}^* \simeq \mathbb{R}^{2n} \) via our basis of eigenvectors to the dilations. As usual it is enough to consider the \( \mathbb{R}^+ \)-part, since \( O_M(\mathfrak{g}^*) = O_M(\mathbb{R}^+) \oplus O_M(\mathbb{R}^-) \). With these adjustment, we need to exchange \( \varphi_\ell \) by the map \( \varphi \), where

\[
\varphi \varphi_\ell(\lambda, x) = \varphi(\lambda^{e_0}, (\lambda^j x_j)_{j=1}^{2n}).
\]

First of all, we realize that \( \lambda \mapsto \lambda^{e_0} \) is a tempered diffeomorphism. Hence \( T \in \mathcal{L}(O_M(\mathbb{R}^+)) \), where \( T \psi(\lambda) := \psi(\lambda^{e_0}) \), is an isomorphism.

Now let us define linear isomorphisms \( f_\lambda(x) = (\lambda^{e_j} x_j)_{j=1}^{2n} \) on \( \mathbb{R}^{2n} \). Then it is easy to see, that both \( \lambda \mapsto f_\lambda \) and \( \lambda \mapsto f_\lambda^{-1} \) define functions in \( O_M(\mathbb{R}^+; \mathcal{L}(\mathbb{R}^{2n})) \) with values in \( \text{GL}(\mathbb{R}^{2n}) \). Now we denote by \( F_\lambda \) the corresponding operator \( F_\lambda \varphi := \varphi \circ f_\lambda \) and set \( F : \lambda \mapsto F_\lambda \) resp. \( F^{-1} : \lambda \mapsto F_\lambda^{-1} \). A standard calculation shows, that for any continuous seminorm \( p \) on \( \mathcal{L}(S(\mathbb{R}^{2n})) \) and any \( k \in \mathbb{N}_0 \), there is a polynomial \( q \) on \( \mathcal{L}(\mathbb{R}^{2n}) \) such that

\[
p(\partial^k_{\lambda} H_\lambda) \leq q(h^{-1}_{\lambda}, h_\lambda, \partial_{\lambda} \eta_1, \ldots, \partial^k_{\lambda} h_\lambda).
\]

Of course, an analogous inequality is valid for \( H^{-1} \). Hence, we may conclude

\[
H, H^{-1} \in O_M(\mathbb{R}^+; \mathcal{L}(S(\mathbb{R}^{2n}))).
\]
Here $H^{-1}$ is indeed the inverse of $H$ in the algebra $O_M(\mathbb{R}^+; L(S(\mathbb{R}^{2n}))$. Due to Theorem 2, we know that the multiplication

$$O_M(\mathbb{R}^+; S(\mathbb{R}^{2n})) \ni f \mapsto H f \in O_M(\mathbb{R}^+; S(\mathbb{R}^{2n})), \quad (H f)(\lambda, x) = H_\lambda(f(\lambda, \cdot))(x),$$

is continuous and in fact an isomorphism.

Because $\phi f = (T \otimes 1)(F f)$, we can conclude that $\phi$ is an isomorphism.

Using the above lemma, we may now prove the following continuity property for the Fourier transform in $\pi$-picture on $S(G)$.

**Proposition 5.** Fourier transform in $\pi$-picture restricts to a continuous map

$$\mathcal{F}_\pi : S(G) \to O_M(\mathbb{R}^\times) \hat{\otimes} L(\mathcal{H}_\pi^{-\infty}, \mathcal{H}_\pi^{\infty}).$$

**Proof.** This statement follows from the continuity of the maps

$$S(G) \xrightarrow{\mathcal{F}_\pi} S(\mathbb{G}') \hookrightarrow O_M(\omega^\times) \hat{\otimes} S(\mathcal{Z}) \xrightarrow{\phi^\pi} O_M(\mathbb{R}^\times) \hat{\otimes} S(\mathcal{Z}),$$

where we use the continuous inclusion $S(\omega^\times) \subset O_M(\omega^\times)$, and also from the continuity of

$$\mathcal{C}_\pi = P_+ \otimes \mathcal{F}_\pi + P_- \otimes \mathcal{F}_\pi : O_M(\mathbb{R}^\times) \hat{\otimes} S(\mathcal{Z}) \to O_M(\mathbb{R}^\times) \hat{\otimes} L(\mathcal{H}_\pi^{-\infty}, \mathcal{H}_\pi^{\infty}).$$

### 3 Gelfand triples for the Kohn-Nirenberg quantization

In [1] a pseudo-differential calculus resp. a Kohn-Nirenberg quantization for graded nilpotent Lie groups was developed. We will embed this definition into our context and derive an integral formulation for the Kohn-Nirenberg quantization for a general class of symbols. First, consider the map

$$T : S(G \times G) \to S(G \times G), \quad T f(x, y) := f(x, xy^{-1}).$$

Then, it is easy to see, that $T$ extends to a Gelfand triple isomorphism

$$T : \mathcal{G}(G, \mu) \otimes \mathcal{G}(G, \mu) \to \mathcal{G}(G, \mu) \otimes \mathcal{G}(G, \mu).$$

Denote by $K$ the Schwartz kernel map

$$K : L(\mathcal{G}(G, \mu), \mathcal{G}(G, \mu)) \to \mathcal{G}(G, \mu) \otimes \mathcal{G}(G, \mu)$$

from Lemma 2. We may define the Kohn-Nirenberg quantization as the Gelfand triple isomorphism

$$\text{Op} := K^{-1} T^{-1}(1 \otimes \mathcal{F}_G^{-1}) : \mathcal{G}(G, \mu) \otimes \mathcal{G}(G, \mu) \to L(\mathcal{G}(G, \mu), \mathcal{G}(G, \mu)).$$
That means for \(a \in L^2(G \times G, \mu \otimes \mu)\), we have \(\text{Op}(a) \in \mathcal{HS}(L^2(G))\) and

\[
(\text{Op}(a)f, g)_{L^2(G, \mu)} = \int_G \int_G \text{Tr}[a(x, \pi)((1 \otimes \mathcal{F}_G \text{inv})T g \otimes f)(x, \pi)] \, d\mu(x) \, d\mu([\pi])
\]

for all \(f, g \in L^2(G, \mu)\), where \(\text{inv} f(x) := f(-x)\) and \((\cdot, \cdot)_{L^2(G, \mu)}\) is the inner product in \(L^2(G, \mu)\). Because

\[
\text{Tr}[a(x, \pi)((1 \otimes \mathcal{F}_G \text{inv})T g \otimes f)(x, \pi)] = g(x) \, \text{Tr}[a(x, \pi)\mathcal{F}_G f(\pi) \pi(x)]
\]

for almost all \((x, [\pi]) \in G \times \hat{G}\), we may write the operator \(\text{Op}(a)\) as

\[
\text{Op}(a) \varphi = \int_G \text{Tr}[\pi(\cdot) a(\cdot, \pi)\mathcal{F}_G f(\pi)] \, d\mu([\pi]), \quad \text{for } f \in L^2(G),
\]

where the integral converges in \(L^2(G, \mu)\).

We will now define the Kohn-Nirenberg quantization in the context of the Gelfand triples \(G, (\mu, \pi)\) and \(G(\mathbb{R}^2; \sigma)\) for \(\pi \in \text{SI}_2(G)\). Subsequently, we will discuss an integral formula similar to the \(L^2\)-case above, but for a different class of symbols.

### 3.1 The Kohn-Nirenberg quantization on \(S(G) \hat{\otimes} S_\ast(G)\)

We still take \(G\) to be a homogeneous Lie group with \(\dim Z = 1\) and \(\pi \in \text{SI}_2(G)\). We already saw that \(\mathcal{F}_G\) is a Gelfand triple isomorphism from \(G(G)\) to \(\hat{G}(G)\), by the definition of \(G(\hat{G})\). Now we want to use the corresponding statement for the \(S_\ast(G)\) test functions. Again we need to show, that \(\mathcal{T}\) is a Gelfand triple isomorphism in this context. Although the map \(\mathcal{T}\) is not well defined on \(G_\ast(G) \otimes G_\ast(G)\), it is well defined on \(G(G) \otimes G_\ast(G)\).

**Lemma 10.** The map \(\mathcal{T} \upharpoonright_{S(G \times G)}\) extends to a Gelfand triple from \(G(G) \otimes G_\ast(G)\) onto itself, which we will also call \(\mathcal{T}\) by a slight abuse of notation.

**Proof.** Suppose \(\varphi \in S(G) \hat{\otimes} S_\ast(G)\) and \(q \in \mathcal{P}(G)\), then for all \(x \in G\) and \(y \in \omega\)

\[
\int_{\omega} q(z) \varphi(x, x(-z - y)) \, d\mu_\omega(Z) = \int_{\omega} q((-x)(-z - y)) \varphi(x, z) \, d\mu_\omega(z) = 0
\]

Because \([z \mapsto q((-x)(-z - y))] \in \mathcal{P}(\mathbb{Z})\). Hence \(\mathcal{T} \varphi \in S(G) \hat{\otimes} S_\ast(G)\). Analogously we may prove, that \(\mathcal{T}^{-1}\) maps \(S(G) \hat{\otimes} S_\ast(G)\) onto itself. Because \(S(G) \hat{\otimes} S_\ast(G)\) carries the subspace topology in \(S(G) \hat{\otimes} S(G)\), the continuity of \(\mathcal{T}\) and \(\mathcal{T}^{-1}\) on \(S(G) \hat{\otimes} S_\ast(G)\) is evident. Since also

\[
\int_{G \times G} \psi \mathcal{T} \varphi \, d(\mu \otimes \mu) = \int_{G \times G} \mathcal{T}^{-1} \psi \, d(\mu \otimes \mu),
\]

for all \(\varphi, \psi \in S(G \times G)\), we may extend \(\mathcal{T} \upharpoonright_{S(G \times G)}\) to a Gelfand triple isomorphism.
Now a direct conclusion is the formulation of the Kohn-Nirenberg quantization as a Gelfand triple isomorphism that incorporates the new Gelfand triples \( G_\star(G, \mu) \) and \( G(\mathbb{R}^\times; \pi) \).

**Proposition 6.** The Kohn-Nirenberg quantization in \( \pi \)-picture

\[
\text{Op}_\pi := \mathcal{K}^{-1} \mathcal{T}^{-1} (1 \otimes \mathcal{T}_\pi^{-1}) : G(G) \otimes G(\mathbb{R}^\times; \pi) \to L(G_\star(G, G)),
\]

where \( \mathcal{K} \) is the Schwartz kernel map between \( G(G) \otimes G_\star(G) \) and \( L(G_\star(G, G)) \), is a Gelfand triple isomorphism.

As for the Fourier transformation in \( \pi \)-picture, we may relate \( \text{Op}_\pi \) to the original Kohn-Nirenberg quantization \( \text{Op} \) via Figure 1.

### 3.2 The integral formula

Square integrable representation can also be seen as slowly increasing functions. This is integral to our approach and will be proven in the following proposition.

**Proposition 7.** Suppose \( \pi \in \text{SI}(G) \), then the operator valued function \( (x, \lambda) \mapsto \pi_\lambda(x) \) is both in \( O_M(\mathbb{R}^\times; L(H_\infty^\pi)) \) and in \( O_M(\mathbb{R}^\times; L(H_{-\infty}^\pi)) \).

**Proof.** By Lemma 4 it is enough to show, that \( x \mapsto \pi(x) \) is slowly increasing. For this purpose we choose an equivalent representation, that is more easily understood. There is a representation \( \sigma \sim \pi \) on \( H_\sigma = L^2(\mathbb{R}^n) \), such that \( H_\infty^{\sigma} = S(\mathbb{R}^n) \) and

\[
\sigma(x)f(t) = e^{2\pi i \xi(a(x,t))} f(x^{-1} \cdot t)
\]

where \( \xi \) is a linear functional on a subalgebra \( m \) of \( g \), \( a : G \times \mathbb{R}^n \to m \) is polynomial and \( G \times \mathbb{R}^n \ni (x, t) \mapsto x \cdot t \in \mathbb{R}^n \) is a polynomial action of \( G \) on \( \mathbb{R}^n \) by [9]. Because \( (x, t) \mapsto x^{-1} \cdot t \) is polynomial, we may represent the action of \( G \) on \( \mathbb{R}^n \) by a linear combination

\[
x \cdot t = \sum_{j,k} s_{k,j}(x) u_{k,j}(t) e_j,
\]

where \( (e_j)_j \) is the standard basis on \( \mathbb{R}^n \) and \( s_{k,j}, u_{k,j} \) are polynomials. Thus, we also have

\[
t_j \sigma(x)f(t) = \sum_{j,k} s_{k,j}(x) \sigma(x)(u_{k,j} f)(t).
\]

For the same reason, there are polynomials \( q_{j,k}, \bar{q}_{j,k} \) on \( G \), \( r_{j,k}, \bar{r}_{j,k} \) on \( \mathbb{R}^n \) such that

\[
\partial_{t_j} f(x^{-1} \cdot t) = \sum_k \bar{q}_{j,k}(x) \bar{r}_{j,k}(t) (\partial_k f)(x^{-1} \cdot t)
\]

\[
= \sum_k q_{j,k}(x) r_{j,k}(x^{-1} \cdot t) (\partial_k f)(x^{-1} \cdot t).
\]
Hence, for all $\alpha, \beta \in \mathbb{N}_0$ we find operators $A_k \in \mathcal{L}(S(\mathbb{R}^n))$ and polynomials $v_k \in \mathcal{P}(G)$, such that

$$t^\beta \partial_x^\alpha \sigma(x)f(t) = \sum_k v_k(x) \sigma(x)(A_k f)(t),$$

as a finite linear combination.

The topology on $\mathcal{L}(S(\mathbb{R}^n))$ is induced by the seminorms

$$p: A \mapsto \sup_{f \in B, t \in \mathbb{R}^n} |t^\beta \partial_x^\alpha Af(t)|, \quad B \subset S(\mathbb{R}^n) \text{ bounded}, \, \alpha, \beta \in \mathbb{N}_0^n.$$

Now if $L \in \mathfrak{u}(g)_L$ is any left invariant differential operator on $G$ and $p$ is a seminorm as above, we get

$$p(L \sigma(x)) \leq \sum_k v_k(x) \sup_{f \in B, t \in \mathbb{R}^n} |\sigma(x)(A_k \sigma(L)f)(t)| \leq \sum_k v_k(x) \sup_{f \in B, t \in \mathbb{R}^n} |(A_k \sigma(L)f)(t)|.$$

The right-hand side of the above inequality is a sum of continuous seminorms times polynomials, since $\sigma(L) \in \mathcal{L}(S(\mathbb{R}^n))$. Thus $x \mapsto \sigma(x)$ is slowly increasing. Due to $\pi \sim \sigma$ the map $x \mapsto \pi(x)$ is slowly increasing, too. Now $(x, A) \mapsto \pi_A$ is slowly increasing with values in $\mathcal{L}(H^\infty_n)$ due to Lemma 4. We finish the proof by remarking, that $\mathcal{L}(H^\infty_n)$ and $\mathcal{L}(H^{-\infty}_n)$ are isomorphic by transposition and $\pi_A(x)^i = \pi_A(-x)^i$.

With the help of the above proposition, we want to write the inverse Fourier transform as an integral, which converges in $\mathcal{O}_M(G)$. For this purpose we need to explain a small fact about the dual space $\mathcal{O}'_M(G)$. Denote by $\partial_1, \partial_2, \ldots$ the directional derivative to any basis $v_1, v_2, \ldots$ of $\mathfrak{g}$. Each continuous linear functional on $\mathcal{O}_M(\mathfrak{g})$ can be represented by the set

$$\mathcal{O}'_M(G) = \text{span}_\mathbb{C}\{\partial^\alpha f \mid \alpha \in \mathbb{N}^{\dim(G)}_0 \text{ and } f \in C(G) \text{ is rapidly decreasing}\}, \quad (3)$$

where we used the standard multi-index notation, see [16] (page 130 of chapter 2), if we use the dual pairing

$$\langle \partial^\alpha f, g \rangle := \int_G f(\partial^\alpha) g \, d\mu.$$

Here we say $f: G \to \mathbb{C}$ is rapidly decreasing, iff $q f$ is a bounded function for any $q \in \mathcal{P}(G)$. The differential operators $\partial^\alpha, \alpha \in \mathbb{N}^{\dim(G)}_0$ span the $\mathcal{P}(G)$-module $\text{Diff}_P(G)$. Since the multiplication of Schwartz functions with polynomials is continuous, we may exchange $\partial^\alpha$ with arbitrary $P \in \text{Diff}_P(G)$ in the pairing above. By [10] (Lemma A.2.2) the $\mathcal{P}(G)$-span of the left invariant differential operators $u_\mathfrak{g}_L$ is equal to $\text{Diff}_P(G)$. Now let $w^1, w^2, \ldots$ be the dual basis to $v_1, v_2, \ldots$ and let $X_1, X_2, \ldots$ by the left invariant vector fields associated to $v^1, v^2, \ldots$. A quick calculation shows, that for all $\phi \in S'(G)$ and all $j, k$ there exists a polynomial $q \in \mathcal{P}(G)$ with
Of course, the set of rapidly decreasing continuous functions is invariant under the multiplication with polynomials. In conclusion, we may represent the dual to $\mathcal{O}_M(G)$ by

$$O'_M(G) = \text{span}_C \{ P f \mid P \in w(\mathfrak{gl}) \text{ and } f \in C(G) \text{ is rapidly decreasing} \}.$$  

Lemma 11. If $\varphi \in S(G)$ and $\omega^x \ni \ell \sim \pi \in \text{SI}(G)$, then the integral

$$\varphi = \int_{\mathbb{R}^\times} \text{Tr}[\pi_\lambda \mathcal{T}_\pi(\varphi(\lambda))] d\lambda_\pi$$

converges in $\mathcal{O}_M(G)$, where $d\lambda_\pi := \kappa |f(\ell)| |\lambda|^{2-1} d\lambda$.

Proof. Let $f : G \to \mathbb{C}$ be continuous and rapidly decreasing, let $P \in w(\mathfrak{gl})$ and let $\varphi \in S(G)$. Then $f$ and $P^* \varphi$ are $L^2$ functions and we may apply Plancherel for $\mathcal{T}_\pi$. Hence

$$\langle P^* \mathcal{T}_\pi, \varphi \rangle = \int_{\mathbb{R}^\times} \mathcal{T}_\pi(P\varphi(\lambda)) d\mu = \int_{\mathbb{R}^\times} \text{Tr}[\hat{f}_\lambda \mathcal{T}_\pi(P\varphi(\lambda))] d\lambda_\pi,$$

where we used the shorthand $\hat{g}_\lambda = \mathcal{T}_\pi g(\lambda)$. Because $P\varphi \in S(G)$, we have $\mathcal{T}_\pi(P\varphi)(\lambda) = \pi_\lambda(P) \hat{\varphi}(\lambda) \in \mathcal{L}(H^\omega_{\pi^{-}}, H^\omega_{\pi^+})$, which is a nuclear operator for each $\lambda \in \mathbb{R}^\times$. Using Fubini with respect to the counting measure and $\mu$ results in

$$\text{Tr}[\hat{f}_\lambda^* \mathcal{T}_\pi(P\varphi(\lambda))] = \int_G \int_{\mathbb{R}^\times} \mathcal{T}_\pi(P\varphi(\lambda)) d\mu(x) \text{Tr}[\pi_\lambda(x) \pi_\lambda(P) \hat{\varphi}(\lambda)] d\lambda(x),$$

since $f \in L^1(G, \mu)$. Naturally we have $\pi_\lambda(x) \pi_\lambda(P) = P_x \pi_\lambda(x)$. Because $\text{Tr}$ is a continuous functional an $\mathcal{L}(H^\omega_{\pi^{-}}, H^\omega_{\pi^+})$ and $\pi_\lambda \hat{\varphi}(\lambda)$ is a slowly increasing map from $G$ to $\mathcal{L}(H^\omega_{\pi^{-}}, H^\omega_{\pi^+})$, we get

$$\text{Tr}[\pi_\lambda(x) \pi_\lambda(P) \hat{\varphi}(\lambda)] = P_x \text{Tr}[\pi_\lambda(x) \hat{\varphi}(\lambda)],$$

$$\text{Tr}[\pi_\lambda \hat{\varphi}(\lambda)] \in \mathcal{O}_M(G; \mathcal{L}(H^\omega_{\pi^{-}}, H^\omega_{\pi^+})).$$

In conclusion

$$\langle P^* \mathcal{T}_\pi, \varphi \rangle = \int_{\mathbb{R}^\times} \int_G f(x) P_x \text{Tr}[\pi_\lambda(x) \hat{\varphi}(\lambda)] d\mu(x) d\lambda_\pi$$

$$= \int_{\mathbb{R}^\times} \langle P^* \mathcal{T}_\pi, \text{Tr}[\pi_\lambda(\cdot) \hat{\varphi}(\lambda)] \rangle d\lambda_\pi.$$  

Let us write $\rho(x, \lambda) := \pi_\lambda(x)$ and $\rho^*(\lambda, x) := \pi_\lambda(-x)$ for some $\pi \in \text{SI}(G)$. With Lemma 1 we already proved the continuity of the map

$$\mathcal{L}(\mathcal{O}_M(G)) \to \mathcal{L}(\mathcal{O}_M(G \times \mathbb{R}^\times; \mathcal{L}(H^\omega_{\pi^+}))), \ A \mapsto A \otimes 1.$$  

Of course the evaluation map
\[\mathcal{L}(O_M(G \times \mathbb{R}^N; \mathcal{L}(H^\infty_\pi))) \hookrightarrow O_M(G \times \mathbb{R}^N; \mathcal{L}(H^\infty_\pi)), \quad \Phi \mapsto \Phi(\rho)\]

is continuous, as well. Finally, since the multiplication in \(O_M(G \times \mathbb{R}^N)\) is continuous [19] (page 248) and because of Theorem 2, the map \(S\) defined by

\[S: \mathcal{L}(O_M(G)) \rightarrow O_M(G \times \mathbb{R}^N; \mathcal{L}(H^\infty_\pi)), \quad A \mapsto \rho^\prime \cdot (A \otimes 1)(\rho)\]

is continuous. Now this map looks exactly like the inverse Kohn-Nirenberg quantization on compact Lie groups \(H\) from [7]. Namely, for any \(B \in \mathcal{L}(\mathcal{D}(H))\) the unique Kohn-Nirenberg symbol \(b\) with \(B = \text{Op}(b)\), evaluated at the irreducible unitary representation \(\xi\), is given by \(\xi \cdot (A \otimes 1)(\xi) \in \mathcal{D}(H; \mathcal{L}(H^\infty_{\xi}))\).

**Lemma 12.** The embedding \(S_{\xi}(G) \hookrightarrow O_M(G)\) is continuous and has dense range.

**Proof.** The multiplication on \(S_{\xi}(G)\) is a continuous bilinear map. This implies the continuity of the canonical embedding \(\tau: S_{\xi}(G) \hookrightarrow O_M(G)\), since \(S_{\xi}(G)\) carries the subspace topology in \(S(G)\). Now consider the dual map

\[\tau': O'_M(G) \rightarrow G'(G), \quad \text{where} \quad \langle \tau' \phi, \varphi \rangle = \langle \phi, \varphi \rangle, \quad \text{for all} \ \varphi \in R(G).\]

That this is indeed an embedding, can be seen from Proposition 1 and the representation (3) of the dual space \(O'_M(G)\). By the Hahn-Banach theorem the operator \(\tau\) has dense image.

In Lemma above we saw, that \(S_{\xi}(G) \hookrightarrow O_M(G)\) has dense range. Naturally we also have \(O_M(G) \hookrightarrow S'(G)\) and \(O_M(\mathbb{R}^N) \hookrightarrow S'(\mathbb{R}^N)\), thus we get embeddings

\[\mathcal{L}(O_M(G)) \hookrightarrow \mathcal{L}(S_{\xi}(G), S'(G)), \quad O_M(G \times \mathbb{R}^N; \mathcal{L}(H^\infty_\pi)) \hookrightarrow S'(G; S'(\mathbb{R}^N; \pi)).\]

Notice, that we can exchange \(\mathcal{L}(H^\infty_\pi)\) with \(\mathcal{L}(H^{-\infty}_\pi)\), in the paragraph above. By using the embeddings above, we will see, that the map \(S\) does indeed reproduce the Kohn-Nirenberg symbol. We can even go one step further. Of course for \(A \in \mathcal{L}(O_M(G), S(G))\), we can still define the map \(S\), since \(S(G) \hookrightarrow O_M(G)\). However, we are lacking tools to check, whether \(S(A) \in S(G) \otimes O_M(\mathbb{R}^N; \mathcal{L}(H^\infty_\pi))\) or not, since we cannot apply Theorem 2. We run into the same problem, if we try to define \(S\) for operators \(A \in \mathcal{L}(O_M(G); S'(G))\).

Before we prove, that the definition of \(S\) gives us Kohn-Nirenberg symbols, we need two final lemmata.

**Lemma 13.** Suppose \(a \in S'(G) \otimes S'(\mathbb{R}^N; \pi)\), then

\[\rho \cdot a = (1 \otimes T_{\pi}) \text{inv}(1 \otimes T_{\pi}^{-1}) a,\]

where \(\text{inv} f(x) = f(-x)\), \(f \in S(G)\), continued to distributions.

**Proof.** First we take \(a \in S(G) \otimes S(\mathbb{R}^N; \pi)\). Then we just have

\[(1 \otimes T_{\pi}^{-1})(\rho \cdot a)(x, y) = (1 \otimes T_{\pi}^{-1}) a(x, y) = (1 \otimes \text{inv}) T(1 \otimes T_{\pi}^{-1}) a(x, y),\]
by the integral formula for the inverse Fourier transform from Lemma 11. Now the rest just follows due to the continuity of the involved maps.

**Lemma 14.** Define $\chi_x(\xi) := e^{2\pi i \xi(x)}$ for $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}'$. Then

$$\hat{\mathcal{S}}_{\pi} \varphi(\chi_x) = \pi_{\lambda}(x)$$

for any $\lambda \in \mathbb{R}^\times$ and $\text{SL}_2(G) \ni \pi \sim \ell \in \omega^\times$.

**Proof.** Let $\varphi \in \mathcal{D}((\mathbb{R}^\times)^\alpha)$, such that $\varphi \equiv 1$ on some neighbourhood of zero. If we define $\psi_k(x) := \varphi(x/k)$ for $k \in \mathbb{N}$, then $\psi_k \chi_x \to \chi_x$ for $k \to \infty$ in $S'(\mathfrak{g})$. Due to the continuity of $\hat{\mathcal{S}}_{\pi}$, we may deduce for $\lambda > 0$

$$\hat{\mathcal{S}}_{\pi} \varphi(\chi_x) = \lim_{k \to \infty} e^{2\pi i \xi(x)} \mathcal{S}_{\varphi_{\lambda}}(\psi_k \chi \mid_{\mathfrak{g}'}) = \lim_{k \to \infty} e^{2\pi i \xi(x)} \int_{\omega} \pi(y) \tilde{\psi}_k(y - \delta \lambda \overline{x}) \text{d}\nu(y),$$

where $\tilde{\psi}_k \in S(\omega)$ is the euclidean Fourier transform of $\psi_k$ and $\overline{x}$ is the projection of $x$ onto $\omega$ along $3$. If we consider the functions $\tilde{\psi}_k(\cdot - \delta \lambda \overline{x})$ as distributions

$$\hat{\mathcal{S}}(G) \ni \varphi \mapsto \int_{\omega} \tilde{\psi}_k(y - \delta \lambda \overline{x}) \varphi(y) \text{d}\nu(y)$$

in $S'(G)$, then the sequence of functions $\tilde{\psi}_k(\cdot - \delta \lambda \overline{x})$ converges to the Dirac distribution supported on $\delta \lambda \overline{x}$ in $S'(\omega)$. By Proposition 1 and the continuity of $\mathcal{F}_{\pi}$, we arrive at

$$\hat{\mathcal{S}}_{\pi} \varphi(\chi_x) = e^{2\pi i \xi(x)} \pi(\delta \lambda \overline{x}) = \pi_{\lambda}(x).$$

For $\lambda < 0$ the calculation is analogous, we just need to exchange $\pi$ with $\overline{\pi}$.

**Theorem 5.** For any $A \in \mathcal{L}(\mathcal{O}_M(G); E)$, $E \in \{S(G), O_M(G)\}$, the equality $a := S(A) = \mathcal{O}_M^{-1}(\mathcal{A})$ is valid. Furthermore,

$$A \varphi = \int_{\mathbb{R}^\times} \text{Tr}[\pi_{\lambda} a(\cdot, \lambda) \mathcal{F}_{\pi}(\varphi(\lambda))] \text{d}\lambda, \quad \text{for } \varphi \in S(G),$$

where the integral converges in $E$.

**Proof.** First we will prove the integral formula for $A \in \mathcal{L}(\mathcal{O}_M(G), E)$. From Lemma 11 we know, that for $\varphi \in S(G)$

$$A \varphi = A \int_G \text{Tr}[\pi_{\lambda} \varphi_{\lambda}] \text{d}\lambda = \int_G A(\text{Tr}[\pi_{\lambda} \varphi_{\lambda}]) \text{d}\lambda,$$

where the integral converges in $E$ and where we used the shorthand $\mathcal{F}_{\pi} \varphi(\lambda) = \varphi_{\lambda}$. We may use the tensor product structure of the expression to get

$$A(\text{Tr}[\pi_{\lambda} \varphi_{\lambda}]) = (A \otimes \text{Tr})(\pi_{\lambda} \varphi_{\lambda}) = (1 \otimes \text{Tr})(A \otimes 1)(\pi_{\lambda} \varphi_{\lambda}),$$
for each $\lambda \in \mathbb{R}^\times$. Furthermore,

$$(A \otimes 1)(\rho_\lambda \tilde{\varphi}_\lambda) = \rho_\lambda \cdot \rho_\lambda^* \cdot (A \otimes 1)(\rho_\lambda) \cdot \tilde{\varphi}_\lambda,$$

where the multiplication is defined to be pointwise by the multiplication in $\mathcal{L}(H_\pi^\infty)$. Hence, we can represent $A \varphi$ by the integral

$$A \varphi = \int_{\mathbb{R}^\times} \text{Tr}[\pi_\lambda a \tilde{\varphi}_\lambda] \, d\lambda_\pi,$$

with $a := S(A)$.

Now it is left to check, that indeed $A = \text{Op}_\pi(a)$. First of all, due to Lemma 13

$$T^{-1}(1 \otimes \mathcal{F}_\pi^{-1})a = (1 \otimes \text{inv} \mathcal{F}_\pi^{-1})(\rho \cdot a).$$

We define the function $\chi(x, \xi) := e^{2\pi i \xi(x)}$ for $\xi \in \mathfrak{g}'$, $x \in \mathfrak{g}$, then $\xi \in \mathcal{O}_M(\mathfrak{g} \times \mathfrak{g}^\times)$. Because $(1 \otimes \text{op}_\pi \mathcal{P})\chi(x, \lambda) = \pi_\lambda(x)$, due to Lemma 14, and $\mathcal{F}_\pi = \text{op}_\pi \mathcal{P} \circ \mathcal{F}_0$, we know that

$$(1 \otimes \text{inv} \mathcal{F}_\pi^{-1})(A \otimes 1)(\rho) = (A \otimes \text{inv} \mathcal{F}_\pi^{-1})(\chi) = (A \otimes \mathcal{F}_\pi')(\chi).$$

We choose arbitrary $\varphi \in S(\mathfrak{g})$ and $\psi \in S_0(G)$. The integral

$$\varphi = \int_{\mathfrak{g}'} \chi(\cdot, \xi) \mathcal{F}_0 \varphi(\xi) \, d\mu'(\xi)$$

converges in $\mathcal{O}_M(\mathfrak{g})$. Hence

$$\langle (A \otimes \mathcal{F}_\pi')(\chi, \psi \otimes \phi) \rangle = \langle (A \otimes 1)\chi, \psi \otimes \mathcal{F}_0 \varphi \rangle = \int_{\mathfrak{g}'} \langle A(\chi(\cdot, \xi)), \psi \rangle \mathcal{F}_0 \varphi(\xi) \, d\mu'(\xi) = \langle A \varphi, \psi \rangle.$$

Combining the calculations above implies

$$\mathcal{K}A = T^{-1}(1 \otimes \mathcal{F}_\pi^{-1})a,$$

for the Schwartz kernel map $\mathcal{K}$ for $\mathcal{G}(G) \otimes \mathcal{G}_c(G)$. I.e. $\text{Op}^{-1}(A) = S(A)$.

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