The Das-Mathur-Okubo sum rule for the charged pion polarizability in a chiral model

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Abstract

The Das-Mathur-Okubo (DMO) sum rule for the polarizability of charged pions is evaluated for the Nambu-Jona-Lasinio model Lagrangian in both its minimal and extended forms. A comparison is made with the results obtained using the same sum rule from chiral perturbation theory ($\chi$PT), approximate QCD sum rule calculations, explicit calculations on the lattice by Wilcox, and using the semi-empirical Kapusta-Shuryak spectral densities. The $\chi$PT results from Compton scattering are also given. We point to a delicate cancellation between the intrinsic and recoil contributions to $\alpha_{\pi\pm}$ in the DMO sum rule approach that can lead to calculated polarizabilities of either sign.

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The electric and magnetic polarizability coefficients $\alpha_\pi$ and $\beta_\pi$ of pions are fundamental constants of strong interaction physics and their calculation is an important testing ground for QCD or effective models thereof \[1\]–\[10\], since these coefficients can be measured experimentally. There is however, a reasonable amount of confusion as to their values, and a review of the current status of theory and experiment is given in \[11\]. While $\alpha_\pi$ and $\beta_\pi$ can be identified directly from the soft photon limit of the pion Compton scattering amplitude, such calculations are extremely tedious, even in lowest order \[5\]–\[9\]. To obtain a physical picture and common understanding of the values obtained by several approaches, it is far simpler to use the Das-Mathur-Okubo (DMO) current algebra sum rule \[12\], and we examine this here. We also accept the constraint $\alpha_\pi + \beta_\pi = 0$, valid for chiral pions \[13\], and concentrate on the electrical polarizability of charged pions. For charged pions, Holstein \[13\] has shown that the DMO sum rule offers an alternative route for calculating the intrinsic contribution to $\alpha_\pi^\pm$ by recasting it as

$$
\alpha^{intr} = 2 \sum_{i \neq 0} \frac{|\langle 0 | \vec{d}_i | i \rangle|^2}{E_i - E_0} = -\frac{\alpha}{m_\pi f_\pi^2} \int_0^\infty \frac{ds}{s^2} \left[ \rho_V(s) - \rho_A(s) \right]
$$

(1)

that together with the center of mass recoil contribution \[15\] determines $\alpha_\pi^\pm$:

$$
\alpha_\pi^\pm = \frac{\alpha}{3m_\pi} <r_\pi^2> + \alpha^{intr}.
$$

(2)

In these formulae, $\rho_{V,A}$ are the vector and axial vector spectral densities, $m_\pi$, $<r_\pi^2>$ and $f_\pi$ the pion mass, radius squared and decay constant, and $\alpha \approx 1/137$ is the fine structure constant. The summation in the defining expression for $\alpha^{intr}$ in Eq. (1) runs over all electric dipole excitations $|i\rangle$ of energy $E_i$ that are connected to the pionic ground state by the dipole operator $\vec{d}_i$, and the minus sign on the second term anticipates the fact that $\alpha^{intr} < 0$ for pions. Physically, this comes about \[13\] from an interplay between negative energy intermediate states entering the sum in Eq. (1), and the contributions of disconnected photon and pion decay diagrams that have to be subtracted out \[16\].

The DMO sum rule result for $\alpha^{intr}$ is valid to $\mathcal{O}(p^2)$ in the language of chiral perturbation theory (\(\chi\)PT) \[17\] due to the approximations made in its derivation. This can be seen as follows. By inserting the definitions of the Gasser-Leutwyler coupling constants $\bar{l}_5, \bar{l}_6$ of \(\chi\)PT \[17\],

$$
<r_\pi^2> = \frac{1}{16\pi^2 f_\pi^2} (\bar{l}_6 - 1); \quad \int_0^\infty \frac{ds}{s^2} \left[ \rho_V(s) - \rho_A(s) \right] = \frac{1}{48\pi^2} (\bar{l}_5 - 1)
$$

(3)

into Eqs. (1) and (2), one retrieves the \(\chi\)PT expression at one loop level (i.e. also to $\mathcal{O}(p^2)$) for $\alpha_\pi^\pm$ as has also been extracted from Compton scattering \[3\]–\[13\]

$$
(\alpha_\pi^\pm)_{\chi PT} = \frac{\alpha}{48\pi^2 m_\pi f_\pi^2} (\bar{l}_6 - \bar{l}_5).
$$

(4)

\[1\] The integral on the right carries an extra 1/2 in \[14\] due to the conventional QCD lattice normalization of the spectral densities used there.
This takes the numerical value $2.7 \times 10^{-4} \text{ fm}^3$ using experimentally extracted values of the $\hat{l}_i$, and is a direct prediction of QCD. We may thus only make a comparison with $\chi$PT to this order in using the sum rule. Note that a higher order calculation of $\alpha_{\pi \pm}$ has been made via Compton scattering [3]. This calculation involves the computation of over 100 Feynman diagrams, and lowers the final result from $\alpha_{\pi \pm} = 2.7$ to $2.4 \times 10^{-4} \text{ fm}^3$.

The DMO form for $\alpha_{\text{intr}}^{\text{DMO}}$ in Eq. (I) has the particular advantage that this contribution to $\alpha_{\pi \pm}$ can now be estimated using QCD sum rules as in [1], or calculated directly on the lattice as in [14]. Here, we also implement the DMO calculation of $\alpha_{\pi \pm}$ diagrams, and lowers the final result from $\alpha_{\pi \pm}$ via Compton scattering [6]. This calculation involves the computation of over 100 Feynman diagrams, and lowers the final result from $\alpha_{\pi \pm} = 2.7$ to $2.4 \times 10^{-4} \text{ fm}^3$.

The spectral densities $\rho_J(s) = -(1/4\pi)Im \Pi_J(s)$ at four-momentum transfer squared $s = q^2$ that enter into Eq. (I) are obtained from the two point polarization functions of the vector and axial vector currents $J_\mu^a(x) = V_\mu^a(x)$ or $A_\mu^a(x)$ of isospin index $a$ and have the tensor structure $\Pi_{\mu \nu \alpha \beta}(s) = \Pi_J(s)(g_{\mu \nu} - q_\mu q_\nu / q^2)\delta_{\alpha \beta}$. These amplitudes are purely spin one (“transverse”) in the chiral limit due to current conservation, $\partial^\mu J_\mu^a(x) = 0$. Their difference satisfies the unsubtracted dispersion relation [23,24]

$$\frac{1}{4}[\Pi_A(s) - \Pi_V(s)] = \int_0^\infty \frac{dt}{t - s - i\epsilon}[\rho_V(t) - \rho_A(t)]. \quad (6)$$

We now specialize to the case where the dynamics of the $J_\mu^a(x)$ are assumed to be governed by $\mathcal{L}_{\text{ENJL}}$. Working to leading order in $N_c^{-1}$ [23,24], both the lowest order one-loop expressions $\Pi_{V,A}$, and the resummed expressions for the polarization functions $\Pi_{V,A}$ as given by their Bethe-Salpeter equation, can be explicitly shown to satisfy the unsubtracted dispersion relation, Eq. (3), under Pauli-Villars regularization [24]. Using this dispersion relation, the required integral in Eq. (I) can be rewritten as

$$\int_0^\infty \frac{dt}{t^2}[\rho_V(t) - \rho_A(t)] = \frac{1}{4} \frac{\partial}{\partial s}[\Pi_A(s) - \Pi_V(s)]_{s \to 0}. \quad (7)$$

$^2$In which case the associated spectral density difference in Eq. (3) mimics [27] the QCD asymptotic behavior $\rho_V(t) - \rho_A(t) \sim \mathcal{O}(s^{-2})$ [24,30].
The difference of polarization amplitudes in the ENJL model is found by direct calculation to be given by the relation (Ward identity [28])

\[ \frac{1}{4}[\tilde{\Pi}_V(s) - \tilde{\Pi}_A(s)] = f_\pi^2 F_V(s) F_V(s) F_A(s) \]  

in terms of the pion electromagnetic form factor \( F_\pi(s) \) of the minimal NJL model \((G_2 = 0)\) and the vector and axial vector form factors \( F_V(s) \) and \( F_A(s) \) of the constituent quark currents. Closed expressions for the latter two form factors are obtained by summing the associated Bethe-Salpeter equation to all orders in the coupling \( 2G_2 \). One finds

\[ F_V(s) = [1 + 2G_2 \Pi_V(s)]^{-1}; \quad F_A(s) = [1 + 2G_2 \Pi_A(s)]^{-1} = g_A F_A(s) \]  

in terms of the irreducible one-loop polarization functions \( \Pi_{V,A} \). Here \( g_A = G_A(0) \), or \( 1 - g_A = 8G_2 f_\pi^2 \), is the axial coupling constant [31-33]. The derivative on the right hand side of Eq. (7) that is required to evaluate the integral explicitly is determined by the behavior of the form factors at vanishingly small momentum transfers. This can be found in terms of the \( \rho \) and \( \alpha_1 \) meson masses \( m_{V,A} \) and their common coupling constant \( g_\rho \) to the quarks that can be identified from the poles and residues in the corresponding resummed propagators for the vector and axial vector modes \( -iD_{V,A}(s) = \{-2iG_2 F_V(s), -2iG_2 G_A(s)\} \) after introducing low-energy expansions [33] for the irreducible polarizations that they contain: \( \frac{1}{4}\Pi_V(s) \approx -g_\rho^{-2}s \) and \( \frac{4}{4}\Pi_A(s) \approx -g_\rho^{-2}(s - 6m^2) \). Here \( m \) is the self-consistent quark mass of the ENJL model. One finds \( m_{V,A}^2 = 6m^2 + m_{\rho}^2 = m_{\rho}^2/g_A \) and \( m_{V,A}^2 = g_\rho^2/8G_2 = g_\rho^2 f_\pi^2/(1 - g_A) \). The latter result follows by eliminating \( G_2 \) in terms of its expression for \( g_A \) given above. This last form for \( m_{V,A}^2 \) reduces to the KSRF relation [34] for the choice \( g_A = 1/2 \). All of these low-energy relations also follow immediately by working with the bosonized version of the ENJL Lagrangian from the start [35]. Introducing the same expansions of the irreducible polarizations into the denominators of the form factors defined above in Eq. (8), one obtains \( F_V(q^2) = 1 + g^2/m_{V}^2 + \cdots \) and \( F_A(q^2) = 1 + g^2/m_{A}^2 - g_A(1 - g_A)g^2/(8\pi^2f_\pi^2) + \cdots \) after also recalling that \( F_P(q^2) = 1 + g_Aq^2/(8\pi^2f_\pi^2) + \cdots \). Using this information the value of the integral in Eq. (7) becomes

\[ \int_0^\infty \frac{ds}{s^2} [\rho_V(s) - \rho_A(s)] = \frac{g_A^2}{8\pi^2} + \frac{f_\pi^2}{m_{V}^2} + \frac{f_\pi^2}{m_{A}^2}, \]

The pion radius parameter is found from the expansion of the ENJL pion form factor \( F_\pi(q^2) \neq F_P(q^2) \) to be

\[ \langle r_\pi^2 \rangle = \frac{3g_A^2}{4\pi^2 f_\pi^2} + \frac{3}{m_{V}^2} + \frac{3}{m_{A}^2}, \]

for \( N_c = 3 \). Referring back to Eqs. (1) and (2), one sees that an exact cancellation is going to occur between the meson mass terms in the recoil and intrinsic contributions to the electric polarizability. The final result for the ENJL polarizability as calculated using the DMO sum rule thus reads

\[ \alpha_{\pi \pm} = \frac{\alpha g_A^2}{8\pi^2 m_{\pi}^2 f_\pi^2}. \]
To the order $\mathcal{O}(N_c^0)$ and $\mathcal{O}(p^2)$ we are working, this result depends only on the axial coupling constant of the constituent quarks besides the pion parameters. The lesson from this is that there is a delicate balance between the recoil and intrinsic contributions, so that at a more fundamental level, one should probably compute both terms to the same level of approximation, instead of relying, for example, on the known experimental pion radius squared for the first term. As in [14], this can give negative values for $\alpha_{\pi \pm}$. If, in addition, we insist that the KSRF relation [34] should also hold between the meson parameters generated by the ENJL model in its low energy regime, then we have seen that $g_A = 1/2$, and Eq. (12) becomes a parameter-free result for the charged pion polarizability, of magnitude $\alpha_{\pi \pm} \approx 1.5 \times 10^{-4}$ fm$^3$ for the physical constants. This value is listed in Table I.

The expression for $\alpha_{\pi \pm}$ as obtained from Compton scattering calculations using the NJL model [7,10] also reduce to Eq. (12) (with $g_A=1$) when evaluated to leading order $\mathcal{O}(p^2)$. This confirms that the integrals over the model spectral densities behave correctly and is an important self-consistency check, since these spectral functions give the density of states in the $\bar{q}q$ continuum of the non-confining $L_{ENJL}$, and not that of the physical $\rho \to \pi \pi$ and $a_1 \to \pi \rho$ continua. Notwithstanding this unphysical feature, the sum rules continue to be obeyed. The reason for this is that the model polarization amplitudes satisfy unsubtracted dispersion relations like Eq. (6) and the Ward identity of Eq. (8) under gauge-invariant regularization schemes (such as Pauli-Villars or dimensional regularization). This in turn allows one to re-express the sum rules in terms of the zero momentum transfer values of the model polarization amplitudes and their derivatives that are not sensitive to the non-confining nature of the model. Fig. 1 compares the behavior of the vector minus axial vector spectral density difference for the ENJL model with (a) the semi-empirical fits of Kapusta and Shuryak [18], and (b) recent data from the ALEPH collaboration [19,20]. Both theoretical sets of curves satisfy Weinberg’s first sum rule [23]. The ENJL model density difference accomplishes this by having considerably smaller and wider resonant peaks at the (model) vector masses $m_V = 793$ MeV and $m_A = 1122$ MeV. Note, however, that the value that we obtain for $\alpha_{\pi \pm}$ using the Kapusta-Shuryak densities is negative as is seen in Table I. One sees that the vector peak of the ALEPH data is well-modelled by the phenomenological form of Ref. [18]. At large $s$, the uncertainties in the data are too large to make it possible to evaluate the sum rule precisely, for $s \to \infty$ [20].

We next return to the NJL model and examine the sub-leading in $N_c^{-1}$ contributions to Eqs. (11) and (10) and show that they are not small. We illustrate these corrections in the case of the minimal NJL model ($G_2 = 0$ therefore $g_A \to 1$ and $m_V^2, m_A^2 \to \infty$) that only supports $\pi$ and $\sigma$ meson modes. Such a calculation has already been performed for the pion radius to this order, and one has found [39,40].

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3 Since $G_2 \sim N_c^{-1}$ in order to preserve the correct scaling properties of quark and meson masses, and $f_\pi^2 \sim N_c$, one sees that $1 - g_A = 8G_2f_\pi^2 \sim N_c^0$. The same result follows from $N_c$ counting rules [37].

4 Estimates of $g_A$ using the Adler-Weisberger sum rule [37,38], suggest a much larger ($\sim 0.8 - 0.9$) value for $g_A$. 

5
\[ \langle \rho^2 \rangle = \frac{3}{4\pi^2 f_\pi^2} - \frac{1}{16\pi^2 f_\pi^2} \left( \ln \frac{m_\pi^2}{m^2} + \frac{17}{6} \right) + O(m_\pi^2), \]  

(13)

which is necessary for evaluating the recoil contribution to \( \alpha_{\pi^\pm} \). Note that the \( f_\pi \) which enters this expression (via a Goldberger-Treiman relation that is itself correct \([26]\) to \( O(1/N_c) \)) refers to a pion weak decay constant that is correct to \( O(1/N_c) \). This comes about because \( \langle \rho^2 \rangle \) follows from differentiating an NJL electromagnetic form factor for the pion that includes \( O(1/N_c) \) corrections, and therefore contains a \( \pi qq \) coupling constant \( g_{\pi qq} \) that is also correct to this order. Otherwise the modified form factor does not normalise properly to unity at \( q^2 = 0 \). This is demonstrated explicitly in \([10]\).

A similar analysis can be made for the spectral density function integral. This is briefly sketched here. Corrections arise from two types of diagrams \([26]\): (i) single meson loop dressings of the irreducible one-loop quark polarization diagram, and (ii) meson pairs exchanged between \( \rho \) and \( a_1 \) vertices. While type (i) is essential for maintaining the transverse nature of the vector polarization diagrams to \( O(N_c^0) \), the subset (ii) of irreducible two meson loop diagrams is singled out by the fact that their derivatives at \( q^2 = 0 \) contain chiral logarithms \([11]\) that diverge as \( m_\rho^2 \to 0 \) due to the pion pole in the propagator \( D_\pi \) of the pion. So they dominate over the derivatives\(^5\) of type (i) in the chiral limit. Denoting the meson pair exchange diagrams with the coupling to the external channel factored out by \(-i\Pi_{\mu\nu}^{\rho\pi}(q^2)\) and \(-i\Pi_{\mu\nu}^{a_1\pi\sigma}(q^2)\) respectively, one has

\[
\left( -\frac{i}{2} g_\rho \right)^2 \Pi_{\mu\nu}^{\rho\pi}(q^2) = \int \frac{d^4 l}{(2\pi)^4} \Gamma_{\mu\nu}^{\rho\pi}(l, l + q) D_\pi(l) D_\pi(l^2) \Gamma_{\nu\pi}^{\rho\pi}(l + q, l)
\]

(14)

for the \( \rho \) channel. A similar expression holds for the \( a_1 \) channel. The leading contribution in chiral logarithms to this is found most directly by using the bosonized version \([33,34]\) of the Lagrangian \([3]\) to identify the effective meson vertices \( \Gamma_{\mu\nu}^{\rho\pi}(l) \) and \( \Gamma_{\mu\nu}^{a_1\pi\sigma} \). These vertices are generated by the Lagrangians

\[
\delta L_{\rho\pi} = -g_\rho \epsilon_{abc} \partial_\mu \pi_b \bar{\rho}_c ; \quad \delta L_{a_1\pi\sigma} = g_\rho \sigma(2\bar{a}_1^\mu \cdot \partial_\mu \pi + \bar{\pi} \cdot \partial_\mu \bar{a}_1^\mu)
\]

(15)

where \( (\sigma, \bar{\pi}, \bar{\rho}, \bar{a}_1) \) are isoscalar and isovector meson fields and \( g_\rho \) is the common vector meson coupling constant. Filling in the details, one finds the convergent contribution

\[
\frac{1}{\alpha \partial_s} \left[ \Pi_{\mu\nu}^{a_1\pi\sigma}(s) - \Pi_{\mu\nu}^{\rho\pi}(s) \right] \bigg|_{s \to 0} = \frac{1}{8\pi^2} \int_0^1 d\alpha (1 - \alpha) \ln \left\{ \frac{m_\pi^2(1 - \alpha) + m_\pi^2\alpha}{m_\pi^2} \right\} = -\frac{1}{48\pi^2} \left( \ln \frac{m_\pi^2}{m_\sigma^2} + \frac{5}{6} \right) + O(m_\pi^2),
\]

(16)

to be added to Eq. \([10]\), after projecting out onto the transverse parts of the meson loop polarization diagrams. Here \( m_\pi \) and \( m_\sigma = \sqrt{(m_\pi^2 + 4m_\pi^2)} \) are the \( \pi \) and \( \sigma \) meson masses.

\(^5\)In fact the type (i) diagrams are strictly constants i.e. are independent of the external meson momentum in the bosonized version of the NJL model and thus give no contribution at all to the difference of polarization derivatives in Eq. \([16]\) in that case.
of the bosonized NJL model [33]. The revised value of the integral \( \langle 1 \rangle \) for the NJL case \( (g_A = 1) \) now reads
\[
\int_0^\infty \frac{ds}{s^2} [\rho^V(s) - \rho^A(s)](\text{quark+meson loops}) = \frac{1}{8\pi^2} - \frac{1}{48\pi^2} \left( \ln \frac{m_\pi^2}{m_\sigma^2} + \frac{5}{6} \right) + O(m_\pi^2).
\]
(17)

In Eqs. (17) and (13) we have also retained chiral \( O(N_c^0) \) corrections, which are pure numbers in addition to the chiral logarithms that arise from the meson loop, in order to be completely consistent in \( N_c \) power counting. From Eqs. (3), one obtains values for the two scale-independent Gasser-Leutwyler coupling constants \( \tilde{l}_5 + \ln(m_\pi^2/\mu^2) = 7 - 5/6 = 37/6 \) and \( \tilde{l}_6 + \ln(m_\pi^2/\mu^2) = 13 - 17/6 = 61/6 \), (here evaluated at the natural scale \( \mu = m_\sigma \approx 500-600 \) MeV of the NJL model) that are in the form given by chiral perturbation theory [17]. The chiral logarithms cancel in the revised value of their difference which now becomes \( \tilde{l}_6 - \tilde{l}_5 = 4 \) in Eq. (4), down from \( 13 - 7 = 6 \), due to the remaining \( O(N_c^0) \) mass-independent corrections that the \( \tilde{l}'s \) now contain. This is to be compared with the empirical difference of \( 16.5 - 13.9 = 2.6 \). The \( O(N_c^{-1}) \) meson loop corrections are thus large and negative in this case, leading to a considerable decrease in the predicted NJL value for \( \alpha_{\pi\pm} \), see Table I.

To summarize, the charged pion polarizability is examined in the NJL and ENJL models and compared with the results obtained in \( \chi \text{PT} \), which serves as a benchmark of QCD. For completeness, the Compton scattering result for this quantity at two loop order is also quoted. While this does not suffer from any obvious cancellation problems, it involves the computation of more than 100 graphs. The NJL model can be directly compared with \( \chi \text{PT} \) to one loop order, in that the formal expression (4) holds in the NJL model with low energy constants evaluated within the model. This, in turn, leads to a much higher value of \( \alpha_{\pi\pm} \) than does \( \chi \text{PT} \), unless one includes the next order meson loops in the NJL model \( N_c^{-1} \) expansion. In the ENJL model, to lowest order, the final result is dependent on the value of \( g_A \). In an analysis according to the division into intrinsic and recoil contributions, one notes that the intrinsic contribution to the polarizability changes by about a factor 3 from the minimal NJL calculation \( (G_2 = 0) \) to using the empirical densities of Kapusta and Shuryak. The negative results for \( \alpha_{\pi\pm} \) in the latter case, as well as for the lattice calculations, reflect the delicate balance between the two relatively large recoil and intrinsic contributions to \( \alpha_{\pi\pm} \) that are of opposite sign. This emphasizes the need for calculating both terms in the sum rule approach to the same level of approximation. The ENJL result \( (-11.0) \) for \( \alpha^{\text{intr}} \) is itself nicely bracketed by the QCD sum rule result \( (-9.6) \) and chiral perturbation theory \( (-12.6) \), all in units of \( 10^{-4} \text{ fm}^3 \), having been considerably increased over the minimal NJL case \( (-5.9) \) by the presence of the vector and axial vector modes. We have set \( g_A = 1/2 \) in the ENJL case in order to comply with the KSRF relation. However, even allowing \( g_A \) to fall through its full range \( 1 \geq g_A \geq 0 \) (while scaling \( m \) like \( g_A^{-1/2} \) to keep the same \( f_\pi^2 \)) gives values of the integral that only range between 12.7 and 27.5. In the latter instance the intrinsic and recoil contributions would simply cancel, leaving a polarizability coefficient of zero.

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FIGURES

FIG. 1. The vector minus axial vector spectral densities as calculated from the ENJL model (solid curve), and from the semi-empirical Kapusta-Shuryak expressions of Ref. [18] (broken curve) versus momentum transfer squared. The dotted points with error bars are recent experimental values taken from the ALEPH collaboration. The Pauli-Villars scheme has been used to regulate the ENJL polarization amplitudes. This introduces two further artificial thresholds (only the first one is shown) at equally spaced intervals $4\Lambda^2$ beyond the unphysical $\bar{q}q$ threshold that lies at $4m^2 = 0.28$ GeV$^2$ for the particular ENJL parameter choice $G_1 = 2.47$ GeV$^{-2}$, $G_2 = 3.61$ GeV$^{-2}$, and a regulating cutoff of $\Lambda = 1.06$ GeV, used in this figure.
TABLE I. Breakdown of intrinsic and finite size contributions to the polarizability of charged pions in units of $10^{-4}$ fm$^3$ (1 fm $\approx 197.2$ MeV$^{-1}$) as obtained from various theoretical approaches, together with the available experimental data. The original Serpukov II and Mark II data analyses quote $(\alpha_{\pi^\pm} \pm \beta_{\pi^\pm})$ without assuming the constraint $\alpha_{\pi^\pm} + \beta_{\pi^\pm} = 0$. We have simply averaged the statistical and systematic error bounds when extracting $\alpha_{\pi^\pm}$. Input: $m_{\pi} = 139.6$ MeV, $f_{\pi} = 93$ MeV, and a typical quark mass $m = 324$ MeV from solving the gap equation [21] for the ENJL case, giving $m_V = 793$ MeV and $m_A = 1122$ MeV; empirical values $\bar{l}_5 = 13.9 \pm 1.3$, $\bar{l}_6 = 16.5 \pm 1.1$ for the $\chi$PT case [17], and the measured value $<r_{\pi}^2> = 0.439$ fm$^2$ [42] except where derived from a model.

| Approach                                      | Spectral density integral $a^{intr}$ | $a(r_{\pi}^2)/3m_{\pi}$ | $\alpha_{\pi^\pm}$ |
|-----------------------------------------------|-------------------------------------|--------------------------|-------------------|
| $\chi$PT (Compton scattering to two loops)   | 27.2                                | -12.6                     | 2.4   ± 0.5       |
| $\chi$PT [Eqs. (3, 4)]                       | 12.7                                | -5.9                      | 5.8       |
| NJL [Eqs. (10, 11), $g_A = 1$]               | 15.9                                | -7.4                      | 3.9   |
| NJL+$O(N_c^{-1})$ [Eqs. (17, 13)]            | 21.0                                | -9.6                      | 5.6 ± 0.5  |
| QCD sum rule                                 | 23.8                                | -11.0                     | 1.5       |
| ENJL [Eqs. (10, 11), $g_A = 0.5$ (from KSRF)]| 36.3                                | -17.1                     | -2.0 ± 1.8 ± 1.6 |
| Lattice calc. [14]                           | 39.4                                | -18.2                     | -3.0       |
| Semi-empirical densities [18]                 | -                                   | -                         | 20 ± 12    |
| Lebedev [13]                                 | -                                   | -                         | 6.8 ± 1.4  |
| Serpukov I [14]                              | -                                   | -                         | 8.5 ± 4.8 ± 3.5 |
| Serpukov II [45]                             | -                                   | -                         | 2.5 ± 0.5 ± 0.02 |
| Mark II [16]                                 | -                                   | -                         | 2.2 ± 1.6  |
| $\gamma\gamma \rightarrow \pi\pi$ measurement [17] | -                                   | -                         | -        |
$(\rho_V - \rho_A) (\text{GeV}^2)$ vs $s$ (GeV$^2$)

- $s$ (GeV$^2$)

Legend:
- Solid line
- Dashed line

4$\Lambda^2$