Precise Approximate Solution for the Bohm Sheath Potential

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Abstract. The Poisson equation for the plasma sheath potential near a wall, leads to a non-linear differential equation, whose analytic solution is not known. The usual approximation taking only the first non-null term does not give good accuracy. Other approximations taken two additional terms are better, but they fail to give good accuracy in the intermediate region. Here a new analytic approximated solution is presented with much higher accuracy, and more precise results, not only near and far away of the wall, but also in the transition region. Two figures showing these new analytic solutions as a function of the relevant parameters are presented. The advantages of the present solution compared with those of previous works are shown.

1. Introduction

The plasma potential near a wall has been treated in several ways and the most classic is the Bohm’s treatment [1-3]. The Bohm’s model is given by a non-linear second order differential equation of the form

\[ \frac{\partial^2 \varphi}{\partial x^2} = \frac{ne}{\epsilon_0} \left( \text{exp} \left( \frac{\epsilon \varphi}{k_B T} \right) - \sqrt{\frac{1}{2} \frac{mv^2}{k_B T} - \epsilon \varphi} \right), \]

(1)

where \( v \) is a characteristic ion velocity, \( \varphi \) is the potential, \( n \) is the ion and electron plasma density far away from the wall, \( k_B \) is the Boltzmann constant, \( m \) is the mass, \( T \) is the electron and ion temperature, and \( x \) is the distance to the wall. No analytic solution is known of Eq. (1), furthermore a solution of this equation will slow the position as a function of the potential, but the most usual need is the potential as a function of the distance to the wall [2,3]. In this context, the procedure usually used to obtain \( \varphi \) as a function of \( x \), is to expand all the functions as a power of \( \varphi \) and to keep only the term with the lower power in \( \varphi \). However the accuracy of this approximation is low, and furthermore the results for \( K \leq \frac{1}{2} \), are not suitable. A better approximation will be shown here, which is also simpler than previously presented one [4].

2. Theoretical Model

It is convenient to use dimensionless quantities and variables, so using dimensionless variables:

\[ \bar{y} = \frac{x}{\lambda_D}, \quad K = \frac{1}{2} \frac{mv^2}{k_B T}, \quad \frac{1}{\lambda_D^2} = \frac{2ne^2}{\epsilon_0 k_B T}, \]

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where $\lambda_D$ is the plasma Debye length, $K$ is a dimensionless constant, and $\tilde{y}, \phi$ are dimensionless variables. So, Eq. (1) becomes

$$-2\frac{d^2\phi}{d\tilde{y}^2} = e^{-\phi} - \frac{1}{\sqrt{1 + \frac{\phi}{K}}}. \quad (2)$$

As it is well known, by using the new variable

$$\psi = \frac{d\phi}{d\tilde{y}},$$

can be reduced (2) to the integral equation:

$$\psi^2(\phi) - \psi^2(0) = \int_0^\phi \left( \frac{1}{\sqrt{1 + \frac{\phi}{K}}} - e^{-\phi} \right) d\phi. \quad (3)$$

It is convenient to consider the plasma potential as the reference, that is, $\phi$ is zero for $y = \infty$, and then $\psi(0)$ is also zero. In a more precise way, $\phi$ will be zero at the end of the sheath and beginning of the presheath. In this way, the sheath potential can be calculated by the non-linear first order equation

$$\frac{d\phi}{d\tilde{y}} = \sqrt{F(\phi)},$$

where

$$F(\phi) = 2K \left( \frac{\phi}{K} - 2K + e^{-\phi} - 1 \right). \quad (4)$$

Now, by using a Taylor expansion for $F(\phi)$, it is obtained

$$F(\phi) = \phi^2 \left[ \left( \frac{1}{2} - \frac{1}{4K} \right) + \phi \left( \frac{1}{8K^2} - \frac{1}{6} \right) + \phi^2 \left( \frac{1}{24} - \frac{5}{64K^3} \right) + \phi^3 \left( \frac{7}{128K^4} - \frac{1}{120} \right) + \ldots \right].$$

So, $F(\phi)$ can be written as

$$F(\phi) = \left( \frac{2K - 1}{4K} \right) \phi^2 \left[ 1 + \phi \left( \frac{3 - 4K^2}{12K^2 - 6K} \right) + \phi^2 \left( \frac{18K^3 - 15}{96K^3 - 48K^2} \right) + \ldots \right]. \quad (5)$$

The usual solution is obtained by keeping the first term of the expansion, giving

$$\phi_2(y) = \phi_w exp \left( -\frac{1}{2} \sqrt{\frac{2K - 1}{K}} y \right) = \phi_w exp \left( -\lambda y \right),$$

where $\phi_w$ is the potential at the wall and $\phi_2$ is our notation for this approximation. However the accuracy of this solution is very low, and it is not suitable for $K \leq \frac{1}{2}$. In order to find a simple and better approximation the following procedure will be followed. First, in order to get a simple integration for $\phi$, the function $F(\phi)$ is approximated by a fourth degree polynomial, that is,

$$\frac{d\phi}{d\tilde{y}} = \sqrt{F(\phi)} = -\sqrt{\lambda^2 \phi^2 (1 + \frac{\alpha}{2})^2} = -\lambda \phi (1 + \frac{\alpha}{2} \phi),$$

where the procedure to find $\alpha$ will be shown below.
Integrating now the above equation, it is found

$$-\lambda y = \int_{\phi_w}^{\tilde{\phi}} \frac{d\phi}{\phi (1 + \frac{\alpha}{2} \phi)} = \ln \left( \frac{\tilde{\phi}}{1 + \frac{\alpha}{2} \phi} \right) - \ln \left( \frac{\phi_w}{1 + \frac{\alpha}{2} \phi_w} \right),$$

from which we obtain

$$\tilde{\phi}(y) = e^{\lambda y} \left( 1 + \frac{\phi_w}{2} \right) - \frac{\alpha}{2} \phi_w.$$

The simplest procedure to find $\alpha$ is to compare with the expansion in series (5), giving

$$\alpha_1 = \left( \frac{1}{24K^2} - \frac{1}{6} \right) \frac{\lambda}{\lambda} = \frac{(3 - 4K^2)}{(12K^2 - 6K)}.$$

This value of $\alpha$, denoted here by $\alpha_1$, does not give the best approximation. A second procedure is to choose $\alpha$ in such a way that the slope in $y = 0$ coincides with the slope of the exact function defined in Eq. (5). This procedure leads to

$$\alpha = \frac{2}{\phi_w^2} \left[ \sqrt{\frac{4K \left( 2K \sqrt{1 + \frac{\phi_w}{K}} + e^{-\phi_w} - (2K + 1) \right)}{(2K - 1)}} - \phi_w \right]$$

(6)

3. Results

In Figure 1 and Figure 2, respectively, are shown the potentials for $K = 1$ and $K = 0.6$, obtained by:

1. Numerical calculation, $\phi(y)$;
2. The approximation, $\tilde{\phi}(y)$, calculating $\alpha$ by Eq. (6);
3. The usual approximation, $\phi_2(y)$.

The accuracy of the approximant here found is always higher than previous ones.

![Figure 1. Here, the potential as a function of the distance is shown for $K = 1$.](image)
4. Conclusion

An analytic approximate expression for the plasma potential near the wall has been found. This allows the direct calculation of the sheath potential as a function of the distance to the wall. This potential is given as a combination of exponential and rational functions. The accuracy of this approximant is higher than previous one [4] and enough for most experimental works.

References

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Acknowledgments

Work supported by: 1) Decanato de la Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta, Chile, 2) Grant G-22 (P. Martín), Decanato de investigaciones, Universidad Simón Bolívar, Caracas, Venezuela, and 3) FONDECYT-CONICYT Regular Project # 1121103, Chile (L. Cortés-Vega).

In memoriam, we thank also to Carlos Cereceda, since part of the present work is a continuation of his ideas published together with P. Martin (see here in References).