Abstract. We introduce a new axiomatization of matroid theory that requires the elimination property only among modular pairs of circuits, and we present a cryptomorphic phrasing thereof in terms of Crapo's axioms for flats.

This new point of view leads to a corresponding strengthening of the circuit axioms for oriented matroids.

1. INTRODUCTION

In this paper we take a close look at the definition of a matroid in terms of its set of circuits. We prove that the most usual form of these axioms, requiring the elimination property to hold for each pair of circuits, is redundant. Our result shows that it is enough to require this property to hold for some special pairs of circuits: modular pairs. The main idea is to use the fact that modularity is defined in any lattice, not necessarily geometric.

To make things precise, let us begin by defining the elimination property and stating the circuit axioms for matroids. For an introduction to matroid theory we point to Oxley's book [6].

Definition 1. Let $E$ be a finite set. A collection $C$ of incomparable nonempty subsets of $E$ is the set of circuits of a matroid on the ground set $E$ if $E(C_1,C_2,C)$ holds for all $C_1,C_2 \in C$.

One important feature of matroid theory is the availability of many cryptomorphic axiomatizations, different in spirit but equivalent in substance. For example, notice that the set of all unions of circuits, partially ordered by inclusion, is a geometric lattice (see [6] Chapter 1.7). In fact, the structure of this lattice does encode the full matroid structure.

Given the set $C$ of circuits of a matroid, let us call a pair of circuits $A,B \in C$ a modular pair if $A \lor B$ has rank 2 in the associated geometric lattice. The key observation is then that for this definition to make sense all
we need to know about the family $\mathcal{C}$ is that its members are incomparable (compare Definition 3).

We now consider oriented matroids; a general introductory reference is [2]. We consider not only subsets of $E$, but signed subsets, i.e., functions $X : E \to \{-1, 0, +1\}$ representing the “signature” of the set given by the support of $X$. Notice that the set of signs has a natural $\mathbb{Z}_2$ action (switching sign). The support of a signed set $X$ is $\text{supp}(X) := \{e \in E \mid X(e) \neq 0\}$, and we will call a collection $\mathcal{C} \subseteq \{-1, 0, +1\}$ simple if $\text{supp}(Y) = \text{supp}(X)$ implies $X = \pm Y$ for all $X, Y \in \mathcal{C}$. In order to give a definition that exhibits the similarity with the previous one for matroids, let us define a “oriented elimination” property for any $\mathcal{C} \subseteq \{-1, 0, +1\}^E$ and any $X, Y \in \mathcal{C}$.

$$\mathcal{OE}(X, Y, \mathcal{C}) : \text{for all } e, f \text{ with } X(e) = -Y(e) \neq 0, X(f) \neq Y(f) \text{ there is } Z \in \mathcal{C} \text{ with } Z(e) = 0, Z(f) \neq 0, \text{ and } Z(g) \in \{0, X(g), Y(g)\} \text{ for all } g \in E.$$ Then, one definition of oriented matroids is the following.

**Definition 2** (see [2]). A $\mathbb{Z}_2$-invariant, simple collection $\mathcal{C} \subseteq \{-1, 0, +1\}^E$ is the set of signed circuits of an oriented matroid if

1. the collection $\mathcal{C} := \{\text{supp}(X) \mid X \in \mathcal{C}\}$ is the set of circuits of a matroid on $E$, and
2. $\mathcal{OE}(X, Y, \mathcal{C})$ holds for all $X, Y \in \mathcal{C}$ such that $\text{supp}(X), \text{supp}(Y)$ are a modular pair in $\mathcal{C}$.

The previous definition is usually presented as an interesting “curiosum”, whereas the standard definition requires $\mathcal{OE}(X, Y, \mathcal{C})$ to hold for every pair of elements $X, Y \in \mathcal{C}$. However, recent work on cryptomorphic axiom systems for complex matroids by Laura Anderson and the author [1] strongly hints to the fact that modular pairs of circuits truly encode all of the information. For instance, it turns out that in the setting of complex matroids the only case of circuit elimination where one can in general control the (complex) signs of the involved circuits is elimination among modular pairs - and yet this is enough to make the corresponding axiomatization cryptomorphic to the other ones. In fact, our result is needed in the statement of the circuit elimination axioms for complex matroids in [1].

Our Theorem [1] shows that indeed modular circuits (defined as in Definition 3) suffice to characterize any matroid, whether realizable or not. As a corollary, we can remove condition (1) from Definition [2] With Theorem 2 we then also describe a cryptomorphic weakening of Crapo’s axioms for flats.

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2. Main result

Notation and basics. As a general reference on the combinatorics of posets and lattices we refer to [7] Chapter 3. Here let us only recall that a chain \(J\) in a poset \((P, \leq)\) is any totally ordered subset of \(P\); the length of the chain \(J\) is then \(l(J) := |J| - 1\). Given \(x \in P\) we write \(P_{\geq x} = \{x' \in P \mid x' \geq x\}\) and \(P_{\leq x} = \{x' \in P \mid x' \leq x\}\). The length of the lattice \(P\) is \(l(P) := \max\{l(J) \mid J\ \text{a chain of } P\}\).

Given two elements \(x, y \in P\), we say that \(x\) covers \(y\), written \(x < y\), if \(x \leq y\) and \(|P_{\geq x} \cap P_{\leq y}| = 2\).

If for any \(x, y \in P\) the poset \(P_{\geq x} \cap P_{\geq y}\) has a unique minimal element, this element is denoted \(x \lor y\) and called the meet of \(x\) and \(y\). Analogously we call \(x \land y\), or join of \(x\) and \(y\), the unique maximal element of \(P_{\leq x} \cap P_{\leq y}\), if it exists. The poset \(P\) is called a lattice if meet and join are defined for every pair of elements of \(P\). In particular, every finite lattice has a unique minimal element, called \(\hat{0}\), and a unique maximal element, called \(\hat{1}\). In any poset with a unique minimal element \(\hat{0}\), the elements \(a\) with \(\hat{0} < a\) are called atoms.

Definition 3. Let \(L\) be a lattice. The atoms of \(L\) are the elements that cover \(\hat{0}\) in \(L\). The lattice \(L\) is called atomic if every \(x \in L\) is \(x = \lor A\) for some set \(A\) of atoms of \(L\). We say that two atoms \(a, b\) of \(L\) form a modular pair if \(l(P_{\leq a \lor b}) = 2\).

Given any family \(SS\) of subsets of a set \(E\), consider the set

\[ U(SS) := \{\bigcup T \mid T \subseteq SS\} \]

partially ordered by inclusion - so, for \(A, B \in U(SS)\), \(A \leq B\) if \(A \subseteq B\).

If the members of \(SS\) are incomparable, then \(U(SS)\) is an atomic lattice with meet and join defined by set union and intersection, respectively, and \(\hat{0} = \emptyset\). We will say that two members of \(SS\) are a modular pair if they are a modular pair in \(U(SS)\).

Theorem 1. Let \(C\) be a collection of incomparable finite subsets of a set \(E\). If \(\mathcal{e}(A, B, C)\) for all modular pairs \(A, B \in C\), then \(\mathcal{e}(A, B, C)\) for all pairs \(A, B \in C\).

Proof. Take \(A, B \in C\) with \(A \neq B\), \(e \in A \cup B\) and let \(Z := A \cup B = A \lor B\). We want to show that a \(C \in C\) exists with \(e \notin C \subseteq Z\). The finiteness requirement on the cardinality of the elements of \(C\) ensures that \(l(X) < \infty\) for all \(X \in U(C)\). We will proceed by induction on \(l(Z)\).

If \(l(Z) = 2\) then \(A, B\) is a modular pair and we are done. Suppose now \(l(Z) = n > 2\) and let \(J\) be a chain of maximal cardinality in \(U(C)_{\leq Z}\). The chain \(J\) contains exactly one element \(A' \in C\) and at least an element \(Y\) with \(A' \leq Y \leq Z\). If \(e \notin A'\) we are done with \(C := A'\). Else, since \(U(C)\) is atomic, there is \(B' \in C\) with \(A' \lor B' \leq Y\). Again, if \(e \notin B'\) then \(C := B'\) does it; otherwise \(e \in A' \lor B'\) and we may apply the inductive hypothesis to the pair \(A', B'\) (because \(Y < Z\) implies \(l(Y) < l(Z)\)), obtaining \(C\) as desired.
Restricting $E$ to be a finite set, Theorem 1 gives the desired result.

**Corollary 1.** A collection $\mathcal{C}$ of incomparable subsets of a finite set $E$ is the set of circuits of a matroid on $E$ if $\mathcal{E}(A, B, \mathcal{C})$ for all modular pairs $A, B \in \mathcal{C}$.

As a straightforward consequence we have a corresponding strengthening of the axiomatics of oriented matroids given in Definition 2.

**Corollary 2.** A $\mathbb{Z}_2$-invariant, simple collection $\mathcal{C}$ of elements of $\{-1, 0, +1\}^E$ with incomparable support is the set of signed circuits of an oriented matroid if and only if $\mathcal{E}(X, Y, \mathcal{C})$ holds for all $X, Y \in \mathcal{C}$ such that $\text{supp}(X), \text{supp}(Y)$ are a modular pair in the set of supports of elements of $\mathcal{C}$.

**Proof.** From Corollary 1 we know that, under the hypotheses of the theorem, $\mathcal{C} := \{\text{supp}(X) \mid X \in \mathcal{C}\}$ is the set of circuits of a matroid (indeed, $\mathcal{E}(X, Y, \mathcal{C})$ implies $\mathcal{E}(\text{supp}X, \text{supp}Y, \mathcal{C})$).

In its full generality, Theorem 1 implies the corresponding strengthening of the axioms for finitary matroids (also called independence spaces - see [8, Chapter 20] for an overview, [5] for an extended account).

**Corollary 3.** A family $\mathcal{C}$ of incomparable finite subsets of a (possibly infinite) set $E$ is the family of circuits of a finitary matroid if and only if $\mathcal{E}(A, B, \mathcal{C})$ holds for all modular pairs $A, B \in \mathcal{C}$.

### 3. Flats and geometric lattices

A matroid given by its set of circuits $\mathcal{C}$ as in Definition 1 gives rise to a closure operator on its ground set $E$

$$\text{cl} : \mathcal{P}(E) \to \mathcal{P}(E), \quad A \mapsto \text{cl}(A) := A \cup \{e \in E \mid e \in C \subseteq A \cup \{e\} \text{ for a } C \in \mathcal{C}\}$$

where $\mathcal{P}(E)$ is power set of $E$ (compare [6, Proposition 1.4.10]).

A set $X \subseteq E$ is closed if $\text{cl}(X) = X$. Closed sets of matroids are usually called flats. The collection of flats of a matroid, partially ordered by inclusion, is a lattice. After a preparatory definition we will state a characterization, due to Crapo [3] of the posets that arise as lattices of flats of a matroid.

**Definition 4.** Consider a finite poset $P$ and let $\mathcal{A}$ be its set of atoms. Given $x \in P$, write $A_x := A \cap P_{\leq x}$. We say that $x$ satisfies Crapo’s separation property (essentially axiom $\beta$ in [3]) if the property

$$\mathcal{J}(x, P) := \{A_{x'} \setminus A_x \mid x' \succ x\} \text{ is a partition of } A \setminus A_x$$

is satisfied.

**Lemma 1** (Crapo’s axioms for flats, see [3] or p. 35 of [6]). Let $E$ be a finite set. An intersection-closed family $\mathcal{F}$ of subsets of $E$ with $E \in \mathcal{F}$, partially ordered by inclusion, is the set of flats of a matroid on the ground set $E$ if and only if $\mathcal{J}(X, \mathcal{F})$ holds for all $X \in \mathcal{F}$.

We now come to our strengthening of Crapo’s separation axiom.
Theorem 2. Let $E$ be a finite set. An intersection-closed family $\mathcal{F}$ of subsets of $E$ with $E \in \mathcal{F}$, ordered by inclusion, is the set of flats of a matroid on the ground set $E$ if and only if $\mathcal{I}(X, \mathcal{F})$ holds for all $X \in \mathcal{F}$ with $l(F \geq X) = 2$.

Proof. If $\mathcal{F}$ is the lattice of flats of a matroid then, by Lemma 1, $\mathcal{I}(X, \mathcal{F})$ holds for all $X \in \mathcal{F}$.

To prove the other direction, let $\mathcal{F}$ be an intersection-closed collection of subsets of $E$ such that $E \subseteq \mathcal{F}$. Then, with respect to the partial order given by inclusion, $\mathcal{F}$ is a lattice. Moreover, $\mathcal{C} := \{E \setminus F \mid F \subseteq \mathcal{F}, F \lessdot E\}$ is a collection of incomparable subsets of $E$ such that $U(\mathcal{C}) = F^{\text{op}}$; in particular, $F^{\text{op}}$ is atomic. We collect the following two facts, which are easily checked by complementation.

1. $A, B \in \mathcal{C}$ are a modular pair if and only if $E \setminus A \triangleright F$ and $E \setminus B \triangleright F$ for some $F$ with $l(F \geq F) = 2$.

2. For a modular pair $A, B$ in $\mathcal{C}$, $\mathcal{I}((E \setminus A) \land (E \setminus B), \mathcal{F})$ implies $\mathcal{I}(A, B, C)$.

By (2), $\mathcal{I}(A, B, C)$ holds for a given modular pair $A, B$ of $\mathcal{C}$ if $\mathcal{I}(X, \mathcal{F})$ holds for some $X$ that, by (1), has $l(F \geq F) = 2$. Thus, if $\mathcal{I}(X, \mathcal{F})$ holds for all $X \in \mathcal{F}$ with $l(F \geq X) = 2$, $\mathcal{C}$ satisfies the hypotheses of Corollary 1 and so it is the set of circuits of a matroid. The members of $\mathcal{F}$ are complements of members of $U(\mathcal{C})$ and so, e.g., by [6, p. 78], they are flats of another matroid (dual to the former). In particular, $\mathcal{F}$ is the lattice of flats of some matroid. □

Definition 5. A finite lattice $L$ is geometric if it is the lattice of flats of a matroid.

As the title of the paper [3] itself says, Crapo was interested in what we called property $\mathcal{I}$ as a means for characterizing the structure of geometric lattices. We thus close in the same spirit by stating a corollary of Theorem 2, which strengthens [3, Proposition 4] in the finite case.

Corollary 4. A finite lattice $L$ is geometric if and only if $\mathcal{I}(x, L)$ holds for all $x \in L$ with $l(L \geq x) = 2$.

Proof. Every finite lattice $L$ is order-isomorphic to the poset obtained by ordering the family of sets $\mathcal{F} := \{A_x\}_{x \in L}$ by inclusion. Moreover, $\mathcal{I}(A_x, \mathcal{F})$ if and only if $\mathcal{I}(x, L)$ for all $x \in L$. Now Theorem 2 shows that $\mathcal{F}$ is the lattice of flats of a matroid, hence geometric, and thus so is $L$ as well. □

Question 1. In the previous section we gave a corollary of Theorem 1 that stated our result for infinite matroids in terms of one possible approach via “circuits”. Since duality is precisely one of the features that independence spaces lack [5, Theorem 3.1.13], it is not possible to use the argument of Theorem 2 in the infinite case. However, since infinite geometric lattices are well-studied objects that deserve interest in their own right (see for example
we ask the following question: Is there a lattice-theoretic (“dual”) version of Theorem 1 that would lead to an analogue of Corollary 4 for infinite lattices?

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