A Tutorial on Matrix Perturbation Theory
(using compact matrix notation)

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Abstract
Analytic perturbation theory for matrices and operators is an immensely useful mathematical tech-
nique. Most elementary introductions to this method have their background in the physics literature,
and quantum mechanics in particular. In this note, we give an introduction to this method that is
independent of any physics notions, and relies purely on concepts from linear algebra. An additional
feature of this presentation is that matrix notation and methods are used throughout. In particular, we
formulate the equations for each term in the analytic expansions of eigenvalues and eigenvectors as matrix
equations, namely Sylvester equations in particular. Solvability conditions and explicit expressions for
solutions of such matrix equations are given, and expressions for each term in the analytic expansions
are given in terms of those solutions. This unified treatment simplifies somewhat the complex notation
that is commonly seen in the literature, and in particular, provides relatively compact expressions for
the non-Hermitian and degenerate cases, as well as for higher order terms.

1 Introduction
We want to study the behavior of eigenvalues and eigenvectors of matrices that are a function of a “small”
parameter $\epsilon$ of the form

$$A_\epsilon = A_0 + \epsilon A_1,$$

where $A_0$ and $A_1$ are given matrices. If the eigenvectors and eigenvalues are analytic functions of $\epsilon$ in a
neighborhood of zero, then we can write the eigenvalue/eigenvector relations as power series in $\epsilon$, equate
terms of same powers in $\epsilon$ and derive expressions for each set of terms. These expressions and the related
algebra can get messy. It is shown in this document that by adopting matrix notation, expressions are
simplified and compactified, and additional insight is obtained. It is rather ironic that matrix notation
is not fully utilized in standard treatments of matrix perturbation theory. It can be argued that a better way
to think about finding expansion terms in the eigenvectors is to treat them all together as a matrix, rather
than as individual vectors. Better insight is achieved in this manner, especially for the cases of degenerate
eigenvalues and higher order terms.

Notation and Preliminaries
Before we begin, we set some useful matrix notation for eigenvector/eigenvalue relations, as well as manip-
ulations with diagonal matrices and the Hadamard product.

- A vector $v_i$ ($v_i^*$) is a right (left) eigenvector of a matrix $A$ if
  $$Av_i = \lambda_i v_i, \quad (v_i^* A = \lambda_i v_i^*).$$

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supported by NSF Awards CMMI-1763064 and ECCS-1932777.
Throughout this note, we will assume the semi-simple case, i.e. that $A_\epsilon$ has a full set of eigenvectors (i.e. diagonalizable) for each $\epsilon$ in some neighborhood of zero. In this case, for an $n \times n$ matrix $A$, there are $n$ eigenvalue/vector relations

$$Av_i = \lambda_i v_i \quad (w_i^* A = \lambda_i w_i^*), \quad i = 1, \ldots, n,$$

It is very useful to note that these $n$ relations can be compactly rewritten as a matrix equation

$$
\begin{bmatrix}
Av_1 & \cdots & Av_n
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
v_1 & \cdots & v_n
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & \cdots & \lambda_n
\end{bmatrix}
\Leftrightarrow
AV = VA,
$$

(2)

where $V$ is a matrix whose columns are the eigenvectors of $A$, and $\Lambda$ is the diagonal matrix made up of the eigenvalues of $A$. Similarly, we also have

$$W^* A = \Lambda W^*,
$$

where the rows of $W^*$ are the left eigenvectors. Note that in the special case of Hermitian (or more generally normal) matrices, the left and right eigenvectors coincide, i.e. $W = V$.

- For any square matrix $M$, let $\text{dg}(M)$ be itself a square diagonal matrix made up of only the diagonal elements of $M$. Clearly $\text{dg}(M + N) = \text{dg}(M) + \text{dg}(N)$ for any two matrices $M$ and $N$. If $D$ is a diagonal matrix with compatible dimensions, then it immediately follows that $\text{dg}(MD) = \text{dg}(DM) = \text{dg}(M) \text{dg}(D)$. Thus for any two matrices $M$ and $N$ with compatible dimensions

$$
\text{dg}
\left(
\begin{bmatrix}
M & \text{dg}(N)
\end{bmatrix}
\right)
= \text{dg}(M) \text{dg}(N).
$$

- The Hadamard product $M \circ N$ of two matrices is the element-by-element product. It is distributive over matrix additions, but not over matrix products generally except with diagonal matrices

$$M \circ (N_1 + N_2) = M \circ N_1 + M \circ N_2, \quad M \circ (NA) = (M \circ N)A,$$

when $A$ is diagonal. If $M = uw^*$ is rank 1, then it follows that

$$M \circ N = (uw^*) \circ N = \text{diag}(u) N \text{diag}(w),$$

where $\text{diag}(u)$ is the diagonal matrix formed from the entries of the vector $u$. Thus if $M = \sum_i \lambda_i u_i w_i^*$ is a diagonalizable matrix, then we can obtain an expression for the Hadamard product in terms of a sum of standard matrix products

$$M \circ N = \left( \sum_i \lambda_i u_i w_i^* \right) \circ N = \sum_i \lambda_i \text{diag}(u_i) N \text{diag}(w_i).$$

We will use the notation $M^{k\circ}$ for the Hadamard $k$ power of $M$, i.e. the matrix whose entries are $m_{ij}^{k\circ}$, with $m_{ij}$ being the entries of $M$. Similarly, $M^{-\circ}$ is the matrix with entries $1/m_{ij}$ if all entries are $m_{ij} \neq 0$. The Hadamard pseudo-inverse $M^{\circ \circ}$ is the matrix with entries $1/m_{ij}$ if $m_{ij} \neq 0$, and 0 if $m_{ij} = 0.$
2 Perturbation Expansion

Consider (1) and assume that the eigenvectors and eigenvalues are analytic functions of $\epsilon$ in a neighborhood of 0. The eigenvector/eigenvalue relationship then reads as

$$A_\epsilon V_\epsilon = V_\epsilon \Lambda_\epsilon, \quad W_\epsilon^* A_\epsilon = \Lambda_\epsilon W_\epsilon^*$$

$\Downarrow$

$$(A_0 + \epsilon A_1)(V_0 + \epsilon V_1 + \cdots) = (V_0 + \epsilon V_1 + \cdots)(A_0 + \epsilon A_1 + \cdots)$$

(3)

$$(W_0^* + \epsilon W_1^* + \cdots)(A_0 + \epsilon A_1) = (A_0 + \epsilon A_1 + \cdots)(W_0^* + \epsilon W_1^* + \cdots)$$

(4)

Note that $\Lambda$ and $\Lambda_1$'s are diagonal matrices. These equations describe eigenvectors that each belong to a one dimensional subspace, and thus $V_\epsilon$ and $W_\epsilon$ are not unique unless we impose some normalization constraint. There are many possible such constraints. The first we will use is the reciprocal basis constraint $W^*(\epsilon)V(\epsilon) = I$ which gives

$$W^*(\epsilon)V(\epsilon) = I \quad \Leftrightarrow \quad (W_0^* + \epsilon W_1^* + \cdots)(V_0 + \epsilon V_1 + \cdots) = I. \quad (5)$$

Equating equal powers of $\epsilon$ in (3) gives a sequence of matrix equations

$$A_0 V_0 = V_0 \Lambda_0$$

(6)

$$A_0 V_1 + A_1 V_0 = V_0 \Lambda_1 + V_1 \Lambda_0$$

(7)

$$A_0 V_2 + A_1 V_1 = V_0 \Lambda_2 + V_1 \Lambda_1 + V_2 \Lambda_0$$

(8)

$$\vdots$$

$$A_0 V_k + A_1 V_{k-1} = \sum_{i=0}^{k} V_i \Lambda_{k-i}$$

(9)

$$W_0^* A_0 = \Lambda_0 W_0^*$$

(10)

$$W_0^* V_0 = I, \quad W_0^* V_1 + W_1^* V_0 = 0, \quad \cdots \quad \sum_{i=0}^{k} W_i^* V_{k-i} = 0.$$

In the special case of Hermitian matrices, we have $W_\epsilon = V_\epsilon$, and the eigenvectors are thus normalized to be orthonormal. We also then have the condition $V_0^* V_1 = -V_1^* V_0$, i.e. $V_0^* V_1$ is skew-Hermitian, which implies that its diagonal entries must be imaginary, or zero if the vectors are real. In the real case, the diagonal entries are $v_0^i v_1 i = 0$, i.e. for each $i$, $v_0 i$ and $v_1 i$ are orthogonal. Geometrically, this means that the curve $v_i(\epsilon)$ has a tangent at $\epsilon = 0$ that is orthogonal to the direction of the unperturbed eigenvector $v_0$. More generally, the normalization condition insures that the curves $v_i(\epsilon)$ lie on the sphere in $\mathbb{R}^n$ for any $\epsilon$ (thus the orthogonality of their initial tangents to each $v_0 i$).

3 Calculating First order Terms

We first begin by calculating the first order behavior of the eigenvalues. This is the term $\Lambda_1$, for which we can use equation (7). Left multiplication by $W_0^*$ (to get rid of the $V_0$ factor multiplying $A_1$) gives

$$W_0^* A_0 V_1 + W_0^* A_1 V_0 = W_0^* V_0 \Lambda_1 + W_0^* V_1 \Lambda_0$$

$\Leftrightarrow \quad \Lambda_0 W_0^* V_1 - W_0^* V_1 \Lambda_0 + W_0^* A_1 V_0 = \Lambda_1.$

Now make the following observations

- Since $\Lambda_0$ is diagonal, then $(W_0^* V_1) \Lambda_0$ and $\Lambda_0 (W_0^* V_1)$ have equal diagonals, i.e. $W_0^* V_1 \Lambda_0 - \Lambda_0 W_0^* V_1$ has zeros on the diagonal.
• Since $\Lambda_1$ must be diagonal, then

\[
\Lambda_1 = \text{diag}(W_0^*A_1V_0 - W_0^*V_1A_0 + \Lambda_0W_0^*V_1) = \text{diag}(W_0^*A_1V_0) + 0
\]

\[
\Rightarrow \quad \Lambda_1 = \text{diag}(W_0^*A_1V_0)
\]

(11)

Thus the first order correction to the $i$'th eigenvalue is $\lambda_{1i} = w_{0i}^*A_1v_{0i}$, which is the well known expression\[1\]. To fully appreciate this, expand (11) using partitioned matrix notation

\[
A_1 = \text{diag}\left(\begin{bmatrix} w_{01}^* & \cdots & w_{0n}^* \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_{01} & \cdots & v_{0n} \end{bmatrix}\right) = \begin{bmatrix} (w_{01}^*A_1v_{01}) & \cdots & (w_{0n}^*A_1v_{0n}) \end{bmatrix}.
\]

3.1 Calculating First Order Eigenvector Terms: Distinct Eigenvalues

Equations (7) can be rewritten as matrix equations with $V_1$ and $W_1$ as the unknowns respectively as follows

\[
A_0V_1 + A_1V_0 = V_0\Lambda_1 + V_1\Lambda_0 \quad \Leftrightarrow \quad A_0V_1 - V_1\Lambda_0 = (V_0\Lambda_1 - A_1V_0), \quad (12)
\]

\[
W_0^*A_0 - \Lambda_0W_0^* = (\Lambda_1W_0^* - W_0^*A_1) \quad (13)
\]

The last two equations are Sylvester equations for the matrices $V_1$ and $W_1$ respectively. We first discuss (12). Define the matrix-valued operator $\mathcal{L}(X) := A_0X - X\Lambda_0$, and note that (12) can be rewritten as $\mathcal{L}(V_1) = (V_0\Lambda_1 - A_1V_0).$ The properties of this “Sylvester operator” are discussed in Appendix A.1, and the solvability of equation (12) is determined by those properties. The following matrices appear in the solution: $V_0$, $W_0$, the eigenvectors of $A_0$, $A_0$ respectively, and the matrices whose $ij$'th entries are given by

\[
(\Pi)_{ij} := \lambda_{0i} - \lambda_{0j}, \quad (\Pi^\circ)_{ij} := \frac{1}{\lambda_{0i} - \lambda_{0j}}, \quad i \neq j,
\]

\[
0, \quad i = j.
\]

Entries of $\Pi$ are made up of all possible sums of the eigenvalues of $A_0$ and $-A_0$ (i.e. differences of the eigenvalues of $A_0$), and $\Pi^\circ$ is the Hadamard (element-by-element) pseudo-inverse of $\Pi$. We now apply the machinery in the Appendices to the particular operator $\mathcal{L}$ in equation (12).

• The operator $\mathcal{L}$ has the following spectral decomposition (applying formula (24) with $U = Z = I$)

\[
\mathcal{L}(X) = V_0\left(\Pi \circ (W_0^*X)\right) = \sum_{ij}(\lambda_{0i} - \lambda_{0j}) \left(w_{0i}^*Xe_{0j}\right) v_{0i}e_{0j}^*.
\]

(14)

Since for $i = j$, $\lambda_{0i} - \lambda_{0j} = 0$, $\mathcal{L}$ is rank deficient by at least $n$.

• Even though $\mathcal{L}$ is not of full rank, equation (12) is solvable since its right hand side is in the range of $\mathcal{L}$. This argument is detailed in Appendix A.2.

• The minimum norm solution of (12) is obtained from the formula (eq. (26), where we again use $U = Z = I$) for the pseudo-inverse $\mathcal{L}^+$

\[
V_1 = \mathcal{L}^+(V_0\Lambda_1 - A_1V_0) = V_0\left(\Pi^\circ \circ (W_0^*(V_0\Lambda_1 - A_1V_0))\right) = V_0\left(\Pi^\circ \circ (\Lambda_1 - W_0^*A_1V_0)\right)
\]

\[
\Rightarrow \quad V_1 = -V_0\left(\Pi^\circ \circ (W_0^*A_1V_0)\right), \quad (15)
\]

where we used $W_0^*V_0 = I$, the distributive property of the Hadamard product $\circ$, and the fact that $\Pi^\circ\Lambda_1 = 0$ since $\Lambda_1$ is diagonal and $\Pi^\circ$ has zeros on the diagonal.
• The Sylvester operator in (13) is just $L^*$, and we can similarly derive (see Appendix A.2) the minimum-norm solution

$$W_1^* = \left( \Pi^* \circ (W_0^* A_1 V_0) \right) W_0^*.$$  

To compare the solution formula (15) with standard expressions in the literature, we expand it as

$$V_1 = -V_0 \left( \Pi^* \circ (W_0^* A_1 V_0) \right) = -\sum_{i \neq j} \frac{1}{\lambda_{0i} - \lambda_{0j}} (w_{0i}^* A_1 v_{0j}) v_{0i} e_j^*.$$  

Note that $V_1$ is a matrix whose $k$'th column is $v_{1k}$. The $k$'th column is obtained from $V_1 e_k$

$$v_{1k} = -V_1 e_k = -\sum_{i \neq j} \frac{w_{0i}^* A_1 v_{0j}}{\lambda_{0i} - \lambda_{0j}} v_{0i} e_j^* e_k = -\sum_{i \neq k} \frac{w_{0i}^* A_1 v_{0k}}{\lambda_{0i} - \lambda_{0k}} v_{0i}, \quad \sum_{i \neq k} \frac{w_{0i}^* A_1 v_{0k}}{\lambda_{0k} - \lambda_{0i}} v_{0i}, \quad (16)$$  

which is the standard expression in the literature [1].

**Uniqueness of $V_1$ and $W_1$**

Note that (15) is just one solution to the Sylvester equation. All other solutions can be obtained by adding arbitrary elements of the null space of $L$. In Appendix A.2, it is shown how the null spaces of $L$ and $L^*$ are characterized, and from that we can conclude that all solutions of the Sylvester equation (12) and (13) can be written as

$$V_1 = -V_0 \left( \Pi^* \circ (W_0^* A_1 V_0) \right) + V_0 D_1, \quad W_1^* = \left( \Pi^* \circ (W_0^* A_1 V_0) \right) W_0^* - D_2 W_0^*, \quad (17)$$  

where $D_1$ and $D_2$ are arbitrary diagonal matrices. The normalization conditions (10) now give

$$0 = W_0^* V_1 + W_1^* V_0 = -W_0^* V_0 \left( \Pi^* \circ (W_0^* A_1 V_0) \right) + \left( \Pi^* \circ (W_0^* A_1 V_0) \right) W_0^* V_0 + D_1 - D_2$$  

$$= D_1 - D_2. \quad (18)$$  

We now make several observations

• In the self-adjoint case, we have $W_0 = V_0$ and $W_1 = V_1$, and the normalization conditions (10) give

$$0 = V_0^* V_1 = V_0^* \left( -V_0 \left( \Pi^* \circ (W_0^* A_1 V_0) \right) + V_0 D_1 \right) = -\Pi^* \circ (W_0^* A_1 V_0) + D_1. \quad (19)$$  

Since the first term has zeros on the diagonal ($\Pi^*$ is zero on the diagonal) and $D_1$ is diagonal, we conclude that $D_1 = 0$. Thus, the minimum norm solution (15) to the Sylvester equation does indeed satisfy the normalization conditions (10).

• In the general (non self-adjoint) case, it seems that the following set of solutions

$$V_1 = -V_0 \left( \Pi^* \circ (W_0^* A_1 V_0) \right) + V_0 D, \quad W_1^* = \left( \Pi^* \circ (W_0^* A_1 V_0) \right) W_0^* - D W_0^*$$  

where $D$ is any diagonal matrix will all satisfy the normalization condition (10) up to first order. Since $V_1$ and $W_1$ must be uniqueootnote{The condition $W_1^* V_1 = I$ uniquely determine the functions $V_1$ and $W_1$, and therefore uniquely determines their Taylor series expansion terms $\{V_k\}$ and $\{W_k\}$.}, it seems like consideration of higher order terms is necessary to find the correctly normalized solutions.
In some references [2, Eq. 5.1.31], a non-standard normalization is used. In our notation, this normalization is stated as follows

\begin{equation}
\begin{aligned}
w_0^* v_0 j &= \delta_{i-j}, \\
w_0^* v_{\epsilon i} &= 1, \\
W_0^* V_0 &= I,
\end{aligned}
\end{equation}

where \( \epsilon \geq 1 \) and \( \forall k \geq 1 \), \( \text{dg}(W_0^* V_k) = 0 \).

This normalization will uniquely determine \( v_{\epsilon i} \) unless it becomes orthogonal to \( w_0 i \). For finite matrices, we can guarantee that \( v_{\epsilon i} \) will not be orthogonal to \( w_0 i \) for \( \epsilon \) in some neighborhood of 0.

This non-standard normalization is used because it simplifies the recursive formulae for higher order perturbation terms as we will see later on. For now, observe that if we adopt this normalization and check the general solutions (17)

\[ 0 = \text{dg}(W_0^* V_1) = \text{dg}\left(-W_0^* V_0 \left(\Pi^{i0} \circ (W_0^* A_1 V_0)\right) + W_0^* V_0 D_1\right) \]

\[ = -\text{dg}(\Pi^{i0} \circ (W_0^* A_1 V_0)) + D_1 = 0 + D_1 \]

So we can conclude that the minimum-norm solution (15) is indeed normalized with the non-standard normalization (19).

### 3.2 Calculating First Order Eigenvector Terms \( V_1 \): Repeated (Degenerate) Case

We consider the case when the eigenvalues are repeated, but \( A_0 \) has a full set of eigenvectors. This condition is automatically satisfied in the Hermitian or normal \( A_0 \) case, but may not be always true in the non-normal \( A_0 \) case. We will consider the case of normal \( A_0, A_1 \) here, for which we have \( W_0 = V_0 \), and therefore \( A_0 = V_0 A_0 V_0^* \).

The main difficulty when \( A_0 \) has repeated eigenvalues is that there is not a unique choice of the eigenvectors \( V_0 \) in this case, even after normalization. For any repeated eigenvalue, there corresponds an \( m \)-dimensional invariant subspace, where \( m \) is the geometric multiplicity of the eigenvalue. Any basis of this invariant subspace is composed of eigenvectors. It turns out that there is a special choice of basis which renders the expansion (3) valid. The main task is to find such a basis.

Now assume \( A_0 \) has a repeated eigenvalue \( \lambda \) (multiplicity \( m \)) and define a basis so that \( A_0 \) is block-diagonal with the first \( m \) basis elements a basis for the eigensubspace of \( \lambda \). With this basis, \( A_0, V_0, W_0 \) and \( \Lambda_0 \) have a block partitioning as

\[ A_0 = \begin{bmatrix} (A_0)_{11} & 0 \\ 0 & (A_0)_{22} \end{bmatrix} = \begin{bmatrix} (V_0)_{11} & 0 \\ 0 & (V_0)_{22} \end{bmatrix} \begin{bmatrix} \lambda I & 0 \\ 0 & (\Lambda_0)_{22} \end{bmatrix} \begin{bmatrix} (V_0^*)_{11} & 0 \\ 0 & (V_0^*)_{22} \end{bmatrix} = V_0 \Lambda_0 V_0^*, \]

where \( I \) is the \( m \times m \) identity matrix. Clearly there is not a unique choice for \( (V_0)_{11} \) since replacing it with \( (V_0)_{11} U \), where \( U \) is any unitary matrix will keep the above expression valid.\(^2\)

Now take eq. (7)

\[ \Lambda_0 V_1 + A_1 V_0 = V_0 \Lambda_1 + V_1 \Lambda_0. \]

If we partition all matrices conformably with the above partitions, we get

\[ \begin{bmatrix} \lambda I & 0 \\ 0 & (\Lambda_0)_{22} \end{bmatrix} \begin{bmatrix} (V_1)_{11} & (V_1)_{12} \\ (V_1)_{21} & (V_1)_{22} \end{bmatrix} + \begin{bmatrix} (A_1)_{11} & (A_1)_{12} \\ (A_1)_{21} & (A_1)_{22} \end{bmatrix} \begin{bmatrix} (V_0)_{11} & 0 \\ 0 & (V_0)_{22} \end{bmatrix} = \begin{bmatrix} (V_0)_{11} & 0 \\ 0 & (V_0)_{22} \end{bmatrix} \begin{bmatrix} (\Lambda_1)_{11} & 0 \\ 0 & (\Lambda_1)_{22} \end{bmatrix} + \begin{bmatrix} (V_1)_{11} & (V_1)_{12} \\ (V_1)_{21} & (V_1)_{22} \end{bmatrix} \begin{bmatrix} \lambda I & 0 \\ 0 & (\Lambda_0)_{22} \end{bmatrix}. \]

Taking the 11 block equation we get

\[ \lambda \Lambda_0 (V_1)_{11} + (A_1)_{11} (V_0)_{11} = (V_0)_{11} (\Lambda_1)_{11} + (V_1)_{11} \lambda \Lambda_0, \]

\(^2 U \) unitary means \( UU^* = I \), and therefore \( (V_0)_{11} U \lambda \Lambda U^* (V_0^*)_{11} = (V_0)_{11} \lambda \Lambda (V_0^*)_{11} \).
but $\tilde{I}n$ commutes with any matrix, so we simply get

$$(A_1)_{11}(V_0)_{11} = (V_0)_{11}(A_1)_{11},$$

from which we can obtain $(A_1)_{11}$ by left multiplication by $(V_0^*)_{11}$

$$(A_1)_{11} = (V_0^*)_{11}(A_1)_{11}(V_0)_{11} = (V_0^*A_1V_0)_{11},$$

where the last equality follows from the block-diagonal structure of $V_0$ and $V_0^*$. Now let’s examine the meaning of this. The perturbation expansion (3) assumes that all the $\Lambda_i$’s are diagonal matrices (otherwise (3) is not an eigenvalue/eigenvector relation). This equation states that for $(A_1)_{11}$ to be diagonal as required, we must choose $(V_0)_{11}$ so that its columns are the eigenvectors of $(A_1)_{11}$. Consequently, $(A_1)_{11}$ will simply be the diagonal matrix of eigenvalues of $(A_1)_{11}$.

The meaning of the above is that only this particular choice of $V_0$ will yield the expansion (3). Unlike the non-repeated case, we also have to discover “the special” $V_0$, not just $V_1$ and $\Lambda_1$.

Now to calculate $V_1$ we can apply the pseudo-inverse formula (26) (which produced (15) earlier), but now we pay special attention to the structure of $\Pi^\circ$. In this case, it is

$$((\Pi^\circ))_{ij} := \begin{cases} 
\frac{1}{\lambda_{0i} - \lambda_{0j}}, & \lambda_{0i} \neq \lambda_{0j}, \\
0, & \lambda_{0i} = \lambda_{0j}.
\end{cases}$$

It is instructive to look at the structure of $\Pi^\circ$, it is of the following form

$$\Pi^\circ = \begin{bmatrix} 0 \\
0 \\
\vdots
\end{bmatrix},$$

where the large 0 block is $m \times m$, the diagonal is all zeros, and the remaining terms are $1/(\lambda_{0i} - \lambda_{0j})$. The expression for $V_1$ is now the same as (15), but now we also make sure to use the $V_0$ that made $\Lambda_1$ diagonal above (recall that the derivation of (15) assumed $\Lambda_1$ to be diagonal)

$$V_1 = -V_0 \left( \Pi^\circ \circ (V_0^*A_1V_0) \right).$$

Again, to compare with existing expressions, find $v_{1k}$ using $V_1e_k$

$$v_{1k} = -V_1e_k = -\sum_{\lambda_{0i} \neq \lambda_{0k}} \frac{v^*_0A_1v_{0i}}{\lambda_{0i} - \lambda_{0j}} v_{0i}e^*_j e_k = \sum_{\lambda_{0i} \neq \lambda_{0k}} \frac{v^*_0A_1v_{0k}}{\lambda_{0k} - \lambda_{0i}} v_{0i}, \quad (20)$$

**Example 1.** Consider the simplest case of $2 \times 2$ matrices

$$A_0 + \epsilon A_1 := I + \epsilon M,$$

where $M$ is a normal matrix. $A_0 = I$ has two eigenvalues at 1 with multiplicity 2, and a 2-dimensional eigenspace (all of $R^2$). Let $M$ have eigenvalue/vector pairs $(\alpha, x)$ and $(\beta, y)$. We actually know the eigenvalues of $I + \epsilon M$ explicitly as a function of $\epsilon$ as (since $I + \epsilon M$ has the same eigenvectors as $M$)

$$\lambda_{\epsilon 1} = 1 + \epsilon \alpha, \quad \lambda_{\epsilon 2} = 1 + \epsilon \beta.$$

Let’s see if this can be replicated using the procedure described above. We need to choose $V_0$ so that its columns are eigenvectors of $M$, so the choice is

$$V_0 = \begin{bmatrix} x \\
y
\end{bmatrix},$$

and therefore

$$V_0^*A_1V_0 = \begin{bmatrix} x \\
y
\end{bmatrix}^* M \begin{bmatrix} x \\
y
\end{bmatrix} = \begin{bmatrix} \alpha & 0 \\
0 & \beta
\end{bmatrix}. $$

Note that the off-diagonal elements are zero since the vectors $x$ and $y$ are orthogonal ($M$ is normal).
4 Higher Order Terms

The following normalization will lead to rather compact recursive formulae for terms of all orders

\[ w_{0i}v_{0j} = \delta_{i,j}, \quad w_{0i}v_{ei} = 1, \]

\[ W_0^*V_0 = I, \quad dg(W_0^*V_k) = I \Rightarrow \forall k \geq 1, \quad dg(W_0^*V_k) = 0. \]  \hfill (21)

We find the eigenvalues first. Decompose \( \Lambda = (\Lambda_0 + \epsilon \Lambda_1 + \cdots) =: (\Lambda_0 + \tilde{\Lambda}) \), and rearrange to obtain an equation for \( \tilde{\Lambda} \)

\[
(A_0 + \epsilon A_1) V_{\epsilon} = V_{\epsilon} \left( \Lambda_0 + \tilde{\Lambda} \right) \quad \text{rearrange terms}
\]

\[
V_{\epsilon} \tilde{\Lambda} = -V_{\epsilon} \Lambda_0 + A_0 V_{\epsilon} + \epsilon A_1 V_{\epsilon} \quad \text{left multiply by} \ W_0^* \\
sgn(W_0^*V_{\epsilon} \tilde{\Lambda}) = -sgn(W_0^*V_{\epsilon} \Lambda_0) + sgn(A_0 W_0^*V_{\epsilon}) + sgn(W_0^*A_1 V_{\epsilon}) \quad \text{take diagonals}
\]

\[
sgn(W_0^*V_{\epsilon} \tilde{\Lambda}) = -sgn(W_0^*V_{\epsilon}) \Lambda_0 + sgn(W_0^*A_1 V_{\epsilon}) \quad \text{dg}(\Lambda \Lambda) = dg(M) \Lambda \text{ if } \Lambda \text{ diagonal}
\]

\[
\tilde{\Lambda} = sgn(W_0^*A_1 V_{\epsilon}) \quad \text{dg}(\Lambda) = I
\]

\[
(\epsilon \Lambda_1 + \epsilon^2 \Lambda_2 + \cdots) = sgn(W_0^*A_1 (V_{\epsilon} + \epsilon V_{\epsilon} + \cdots)) \quad \text{expand & equate}
\]

\[
\Lambda_k = sgn(W_0^*A_1 V_{k-1}) \quad \text{distinguish: (23)}
\]

The first of these formulae is the familiar \( \Lambda_1 = dg(W_0^*A_1 V_0) \). Successive \( \Lambda_k \)'s require finding the vectors \( V_{k-1} \) that are normalized according to (21) (and not the standard normalization).

To find \( V_k \)'s, recall and rearrange equation (9) into a Sylvester equation for \( V_k \)

\[
A_0 V_k + A_1 V_{k-1} = V_0 \Lambda_k + V_1 A_{k-1} + \cdots + V_{k-1} \Lambda_1 + V_k \Lambda_0 \\
A_0 V_k - \Lambda_k V_0 = V_0 A_k + V_1 A_{k-1} + \cdots + V_{k-1} A_1 - A_1 V_{k-1}
\]

All solutions to this Sylvester equation are given by the pseudo-inverse formula

\[
V_k = V_0 \left( \Pi^{\top} \circ W_0^* \left( V_0 \Lambda_k + V_1 A_{k-1} + \cdots + V_{k-1} \Lambda_1 - A_1 V_{k-1} \right) \right) + V_0 D_k,
\]

where \( D_k \) is any diagonal matrix. Enforcing the normalization \( W_0^*V_0 = I, \ dg(W_0^*V_k) = 0, \ k \geq 1 \) gives that \( D_0 = I \) and \( D_k = 0 \) for \( k \geq 1 \), and we conclude

\[
V_k = V_0 \left( \Pi^{\top} \circ W_0^* \left( V_0 \Lambda_k + V_1 A_{k-1} + \cdots + V_{k-1} \Lambda_1 - A_1 V_{k-1} \right) \right).
\]

When this formula is combined with (23) for \( \Lambda_k \), we see that we can obtain \( V_k \) using previous calculations for the terms \( V_0, \ldots, V_{k-1} \) and \( \Lambda_0, \ldots, \Lambda_{k-1} \). For computations, the following equivalent form may be easier to implement

\[
V_k = V_0 \left( \sum_{i=0}^{k-1} (\Pi^{\top} \circ W_0^* V_i) \Lambda_{i-1} - \Pi^{\top} \circ (W_0^* A_1 V_{k-1}) \right).
\]

5 Acknowledgement

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A Appendix

A.1 Spectral Decomposition of the Sylvester Operator

Consider the Sylvester matrix equation (in $X$) of the form

$$AX + XB = Q,$$

where $A$, $B$ and $Q$ are given (square) matrices. Solutions and properties of this equation are determined by the Sylvester operator $\mathcal{L}$

$$\mathcal{L}(X) := AX + XB.$$  

We will need to find the “eigen-matrices” of $\mathcal{L}$ and its adjoint $\mathcal{L}^\ast$. First we calculate $\mathcal{L}^\ast$ using the standard inner product on matrices $\langle M, N \rangle = \text{tr}(M^\ast N)$. Starting from the $\equiv$ equality

$$\text{tr}(X(A^\ast Y + YB^\ast)) = \text{tr}(X^\ast(A^\ast Y + B^\ast X^\ast)Y) = \text{tr}((AX + XB)^\ast Y) = \langle \mathcal{L}(X), Y \rangle \equiv \langle X, \mathcal{L}^\ast(Y) \rangle = \text{tr}(X^\ast \mathcal{L}^\ast(Y))$$

we see that

$$\mathcal{L}^\ast(Y) = A^\ast Y + YB^\ast.$$  

Thus in particular, $\mathcal{L}$ is self-adjoint only if $A$ and $B$ are self-adjoint$^3$.

Now we assume that $A$ and $B$ are diagonalizable (i.e. have a full set of linearly independent eigenvectors each). A spectral decomposition of $\mathcal{L}$ can be obtained from the spectral decompositions of $A$ and $B$. This will then allow for applying arbitrary analytic functions on $\mathcal{L}$ including inversion.

Denote the eigenvalues and right/left eigenvectors of $A$ and $B$ as follows

$$Av_i = \lambda_i v_i, \quad A^\ast w_i = \lambda_i^\ast w_i, \quad \text{or in matrix form} \quad AV = VA, \quad A^\ast W = W^\ast A^\ast$$

$$Bu_j = \gamma_j u_j, \quad B^\ast z_j = \gamma_j^\ast z_j, \quad \text{or in matrix form} \quad BU = U\Gamma, \quad B^\ast Z = Z\Gamma^\ast.$$  

Note that $W = V^\ast$ and $Z = U^\ast$. The $n^2$ eigenvalues and “eigen-matrices” of $\mathcal{L}$ and $\mathcal{L}^\ast$ are found as follows

$$\mathcal{L}(v_i z_j^\ast) = A v_i z_j^\ast + v_i z_j^\ast B = \lambda_i v_i z_j^\ast + \gamma_j v_i z_j^\ast = (\lambda_i + \gamma_j) v_i z_j^\ast,$$

$$\mathcal{L}^\ast(w_i u_j^\ast) = A^\ast w_i u_j^\ast + w_i u_j^\ast B^\ast = \lambda_i^\ast w_i u_j^\ast + \gamma_j^\ast w_i u_j^\ast = (\lambda_i^\ast + \gamma_j^\ast) w_i u_j^\ast.$$  

In other words, eigenvalues of $\mathcal{L}$ (resp. $\mathcal{L}^\ast$) are all $n^2$ possible combinations $\lambda_i + \gamma_j$ (resp. $(\lambda_i + \gamma_j)^\ast$) of eigenvalues of $A$ and $B$, and the eigen-matrices are all the corresponding outer products of right/left eigenvectors of $A$ and $B$.

Using the above, a spectral decomposition of $\mathcal{L}$ can now be written as follows

$$\mathcal{L}(X) = \sum_{ij} (\lambda_i + \gamma_j) \langle w_i u_j^\ast, X \rangle \ v_i z_j^\ast.$$  

This can be rewritten in compact notation. First define the matrix $\Pi$ whose $ij$’th entry is

$$(\Pi)_{ij} := \lambda_i + \gamma_j.$$  

Noting that $v_i = V e_i$ (where $e_i$ is the vector of all zeros except 1 in the $i$’th row), and similarly for the other eigenvectors, calculate

$$\mathcal{L}(X) = \sum_{ij} (\lambda_i + \gamma_j) \langle w_i u_j^\ast, X \rangle v_i z_j^\ast = \sum_{ij} (\lambda_i + \gamma_j) \langle w_i^\ast X u_j \rangle V e_i e_j^\ast Z^\ast = V \left( \sum_{ij} (\lambda_i + \gamma_j) \langle w_i^\ast X u_j \rangle e_i e_j^\ast \right) Z^\ast = V \left( \Pi \circ (W^\ast X U) \right) Z^\ast,$$

$$\tag{24}$$

$^3$A simple calculation will show that $\mathcal{L}$ is normal if $A$ and $B$ are normal.
where $\circ$ is the Hadamard (element-by-element) product of matrices. The last equation follows from observing that $W^* X U$ is the matrix whose $ij$'th entry is $(w_i^* X u_j)$. For later reference, we also calculate $L^*$

$$L^*(X) = \sum_{ij} (\lambda_i^* + \gamma_j^*) \langle v_i z_j^*, X \rangle \ w_i u_j^* = \sum_{ij} (\lambda_i^* + \gamma_j^*) \langle v_i^* X z_j \rangle \ W e_i e_j^* U^*$$

$$= W \left( \sum_{ij} (\lambda_i + \gamma_j) \langle v_i^* X z_j \rangle e_i e_j^* \right) U^* = W \left( \bar{\Pi} \circ (V^* X Z) \right) U^*, \tag{25}$$

where $\bar{\Pi}$ is the complex conjugate (without transposing) of $\Pi$.

The inverse $L^{-1}$ (if it exists) and the pseudo-inverse $L^\dagger$ can be calculated from the spectral decomposition

$$L^{-1}(X) = \sum_{ij} \frac{1}{\lambda_i + \gamma_j} \langle w_i u_j^*, X \rangle \ v_i z_j^*, \quad L^\dagger(X) = \sum_{\lambda_i + \gamma_j \neq 0} \frac{1}{\lambda_i + \gamma_j} \langle w_i u_j^*, X \rangle \ v_i z_j^*.$$

We can also give compact formulae if we define $\Pi^{-\circ}$ and $\Pi^{\bar{\circ}}$ using element-by-element operations

$$(\Pi^{-\circ})_{ij} := \frac{1}{\lambda_i + \gamma_j}, \quad (\Pi^{\bar{\circ}})_{ij} := \begin{cases} \frac{1}{\lambda_i + \gamma_j}, & \lambda_i + \gamma_j \neq 0, \\ 0, & \lambda_i + \gamma_j = 0. \end{cases}$$

$L^{-1}$ and $L^\dagger$ can now be rewritten as

$$L^{-1}(X) = V \left( \Pi^{-\circ} \circ (W^* X U) \right) Z^*, \quad L^\dagger(X) = V \left( \Pi^{\bar{\circ}} \circ (W^* X U) \right) Z^*. \tag{26}$$

The above can be generalized using the spectral decomposition to any function $f$ analytic in a neighborhood of the spectrum of $L$ by

$$\left( f(L) \right)(X) = V \left( f^{\circ}(\Pi) \circ (W^* X U) \right) Z^*,$$

where $f^{\circ}(\Pi)$ is the element-by-element application of the function $f$ on each entry of the matrix $\Pi$. For example, given a matrix differential equation

$$\dot{X} = AX + XB = L(X), \quad X(0) = \bar{X}$$

we can write the solution formally as $(e^{tL})(\bar{X})$. The formula above gives

$$X(t) = \left( e^{tL} \right)(\bar{X}) = V \left( e^{t\circ(\Pi)} \circ (W^* X U) \right) Z^*,$$

where $e^{t\circ(\Pi)}$ is the matrix whose $ij$'th entry is $e^{t(\lambda_i + \gamma_j)}$.

### A.2 Solvability of the Sylvester Equations (12) and (13)

We begin with the equation (12) for $V_i$.

- We need to characterize $\mathcal{N}(L)$, the null space of $L$. Using (14) we note that $L(X) = 0$ means

$$\forall i, j, \ (\lambda_{0i} - \lambda_{0j}) \ (w_i^* X e_j) = 0 \iff \forall i \neq j, \ (w_i^* X e_j) = 0 \iff W_0^* X = D,$$

where $D$ is some diagonal matrix (the diagonal entries of $D$ are the numbers $(w_0^* X e_i)$ which cannot be determined from the above condition). Finally we note that $W_0$ and $V_0$ are inverses of each other, so $W_0^* X = D$ is equivalent to $X = V_0 D$ and we conclude

$$X \in \mathcal{N}(L) \iff X = V_0 D, \quad D = \text{any diagonal matrix}.$$
• Similarly for $\mathcal{L}^\ast(X) = A^*_0 X - X A^*_0$. It’s spectral decomposition is 
\[
\sum_{ij} (\lambda^*_0 - \lambda^*_j) \left( v^*_0 X e_j \right) w^*_0 e^*_j,
\]
and again $\mathcal{L}^\ast(X) = 0$ iff $\forall i \neq j, (v^*_i X e_j) = 0$. Repeating the above argument we conclude 
\[
X \in \mathcal{N}(\mathcal{L}^\ast) \iff X = W D, \ D = \text{any diagonal matrix}
\]
• The Sylvester equation (12) is solvable iff $(V_0 \Lambda_1 - A_1 V_0) \in \mathcal{R}(\mathcal{L}) = \mathcal{N}(\mathcal{L}^\ast)$. Thus to verify solvability, we need to show that $V_0 \Lambda_1 - A_1 V_0$ is orthogonal to $\mathcal{N}(\mathcal{L}^\ast)$. Indeed 
\[
X \in \mathcal{N}(\mathcal{L}^\ast) \iff X = W D,
\]
\[
\langle X, V_0 \Lambda_1 - A_1 V_0 \rangle = \langle W D, V_0 \Lambda_1 - A_1 V_0 \rangle = \text{tr}(D^* W^*_0 V_0 \Lambda_1) - \text{tr}(D^* W^*_0 A_1 V_0) = \text{tr}(D^* V_0 \Lambda_1) - \text{tr}(D^* W^*_0 A_1 V_0) = \text{tr}(D^* W^*_0 A_1 V_0) = 0,
\]
where we used $\Lambda_1 = \text{dg}(W^*_0 A_1 V_0)$, and that $D$ is a diagonal matrix.

For equation (13), note that by transposing it we get a Sylvester equation of a similar form to (12)
\[
A^*_0 W_1 - W_1 A^*_0 = W_0 \Lambda^*_1 - A^*_1 W_0.
\]
This equation is $\mathcal{L}^\ast(W_1) = W_0 \Lambda^*_1 - A^*_1 W_0$ where $\mathcal{L}^\ast$ is the adjoint of the Sylvester operator of equation (12), which we have already analyzed. In particular, the spectral decomposition of $\mathcal{L}^\ast$ can be calculated from (25)
\[
\mathcal{L}^\ast(Y) = W_0 \left( \Pi \circ (V^*_0 Y) \right).
\]
Applying the pseudo-inverse to the right hand side
\[
W_1 = \mathcal{L}^\ast(K(W_0 A^*_1 - A^*_1 W_0) = W_0 \left( \Pi^{\ast\circ} \circ (V^*_0 (W_0 A^*_1 - A^*_1 W_0)) \right) = W_0 \left( \Pi^{\ast\circ} \circ (A^*_1 - V^*_0 A^*_1 W_0) \right) = - W_0 \left( \Pi^{\ast\circ} \circ (V^*_0 A^*_1 W_0) \right),
\]
where the last equality follows from $\Pi^{\ast\circ}$ having all zeros on the diagonal. We can rewrite the solution for $W^*_1$ by noting that $\Pi^{\ast\circ} = - \Pi$ and similarly $(\Pi^{\ast\circ})^\ast = - \Pi^{\ast\circ}$
\[
W^*_1 = - \left( W_0 \left( \Pi^{\ast\circ} \circ (V^*_0 A^*_1 W_0) \right) \right)^\ast = \left( \Pi^{\ast\circ} \circ (V^*_0 A^*_1 W_0) \right)^\ast W^*_0 = \left( \Pi^{\ast\circ} \circ (W^*_0 A_1 V_0) \right) W^*_0
\]

### A.3 Background: Spectral Decomposition of a Matrix

Just as matrix partition notation is useful in expressing eigenvalue/eigenvector relations (2), it is also useful to understand diagonalization and spectral decomposition of a matrix. First observe that (2) also gives the left eigenvectors of $A$ as follows
\[
A V = V \Lambda \iff V^{-1} A = \Lambda V^{-1} \iff A^* V^{-*} = V^{-*} \Lambda^*,
\]
thus rows of $V^{-1}$ (columns of $V^{-*}$) are left eigenvectors of $A$ (right eigenvectors of $A^*$). These equations also give the diagonalization
\[
A = V AV^{-1}.
\]
which can also be interpreted as a rank-1 decomposition of $A$ as follows. Let $W^* := V^{-1}$ (or equivalently $W := V^{-*}$), and observe that the diagonalization can be rewritten as

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \cr & \ddots & \cr & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} + \cdots + \lambda_n \begin{bmatrix} v_n \end{bmatrix} \begin{bmatrix} w_n^* \end{bmatrix} = \sum_{i=1}^{n} \lambda_i \, v_i w_i^*. \quad (27)$$

This is a rank-1 (aka dyadic) decomposition of $A$, namely into $n$ rank-1 matrices made up of outer products of the respective columns of $V$ and $W$ scaled by the respective eigenvalue. Note that since $W^*V = I$, there are simple relationships between the columns of $V$ and $W$.

- The sets $\{v_i\}$ and $\{w_i\}$ form a reciprocal basis: Reciprocal bases\(^4\) have the property that $v_i^* w_j = \delta_{i,j}$. This is easily seen to be true from the following partitioning of $W^*V = I$

$$V^{-1} = W^*V = \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} w_1^* v_1 & \cdots & w_n^* v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^* v_1 \\ \vdots \\ w_n^* v_n \end{bmatrix} = I.$$

The reciprocal basis is useful since it allows for writing any vector $x$ in terms of a basis $\{v_i\}$ by observing

$$x = VW^* x = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} x = \sum_{i=1}^{n} v_i \langle w_i, x \rangle. \quad (28)$$

Thus the coefficients of expansion of a vector $x$ in a basis $\{v_i\}$ are the inner products $\langle w_i, x \rangle$ of the vector with the respective elements of the reciprocal basis $\{w_i\}$.

- We can obtain yet another interpretation of the action of a diagonalizable matrix on a vector by acting with (27) on any vector $x$

$$Ax = \sum_{i=1}^{n} \lambda_i \, v_i w_i^* x = \sum_{i=1}^{n} \lambda_i \, v_i \langle w_i, x \rangle = \sum_{i=1}^{n} \lambda_i \, P_i x \quad (29)$$

Each matrix $P_i := v_i w_i^*$ is a (not necessarily orthogonal) projection operator (note: $P^2 = P$) onto an individual eigensubspace of $A$. Note that the inner product is with the corresponding left eigenvector $w_i$. This expression above is known as the spectral decomposition of a linear operator.

- Let $f$ be any function analytic in a neighborhood of the set $\{\lambda_1, \ldots, \lambda_n\}$, then

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) \, P_i$$

Note that when $A$ is Hermitian, columns of $V$ are orthonormal, $W = V$, and the projections above are orthogonal.

### A.4 Repeated Eigenvalues Case for Non-normal $A_0$

Now assume $A_0$ has a repeated eigenvalue $\bar{\lambda}$ (multiplicity $m$) and define a basis so that $A_0$ is block-diagonal with the first $m$ basis elements a basis for the eigensubspace of $\bar{\lambda}$. With this basis, $A_0$, $V_0$, $W_0$ and $\Lambda_0$ have a block partitioning as

$$A_0 = \begin{bmatrix} (A_0)_{11} & 0 \\ 0 & (A_0)_{22} \end{bmatrix}, \quad V_0 = \begin{bmatrix} (V_0)_{11} & 0 \\ 0 & (V_0)_{22} \end{bmatrix}, \quad W_0 = \begin{bmatrix} (W_0)_{11} & 0 \\ 0 & (W_0)_{22} \end{bmatrix}, \quad \Lambda_0 = \begin{bmatrix} \bar{\Lambda} & 0 \\ 0 & (\Lambda_0)_{22} \end{bmatrix}$$

\(^4\)Another common term is dual bases.
where $I$ is the $m \times m$ identity matrix. Now take eq. (7)

$$\Lambda_0 V_1 + A_1 V_0 = V_0 A_1 + V_1 A_0.$$  

If we partition all matrices conformably with the above partitions, we get

$$
\begin{bmatrix}
\lambda I & 0 \\
0 & (\Lambda_0)_{22}
\end{bmatrix}
\begin{bmatrix}
(V_1)_{11} \\
(V_1)_{21}
\end{bmatrix}
+ 
\begin{bmatrix}
(A_1)_{11} & (A_1)_{12} \\
(A_1)_{21} & (A_1)_{22}
\end{bmatrix}
\begin{bmatrix}
(V_0)_{11} \\
(V_0)_{22}
\end{bmatrix}
= 
\begin{bmatrix}
(V_0)_{11} \\
0
\end{bmatrix}
\begin{bmatrix}
\lambda I & 0 \\
0 & (\Lambda_0)_{22}
\end{bmatrix}
+ 
\begin{bmatrix}
(A_1)_{11} & 0 \\
(V_1)_{21} & (V_1)_{22}
\end{bmatrix}
\begin{bmatrix}
(V_1)_{11} \\
(V_1)_{12}
\end{bmatrix}
\begin{bmatrix}
\lambda I & 0 \\
0 & (\Lambda_0)_{22}
\end{bmatrix}.
$$

Taking the $11$ block equation we get

$$\lambda I_n(V_1)_{11} + (A_1)_{11}(V_0)_{11} = (V_0)_{11}(A_1)_{11} + (V_1)_{11} \lambda I_n,$$

but $\lambda I_n$ commutes with any matrix, so we simply get

$$(A_1)_{11}(V_0)_{11} = (V_0)_{11}(A_1)_{11},$$

from which we can obtain $(A_1)_{11}$ by left multiplication by $(W^*_0)_{11}$

$$(A_1)_{11} = (W^*_0)_{11}^*(A_1)_{11}(V_0)_{11} = (W^*_0 A_1 V_0)_{11},$$

where the last equality follows from the block-diagonal structure of $V_0$ and $W_0$. Now let’s examine the meaning of this. The perturbation expansion (3) assumes that all the $\Lambda_i$’s are diagonal matrices (otherwise (3) is not an eigenvalue/eigenvector relation). This equation states that for $(A_1)_{11}$ to be diagonal as required, we must choose $(V_0)_{11}$ so that its columns are the eigenvectors of $(A_1)_{11}$. Consequently, $(A_1)_{11}$ will simply be the diagonal matrix of eigenvalues of $(A_1)_{11}$.

The meaning of the above is that only this particular choice of $V_0$ will yield the expansion (3). Unlike the non-repeated case, we also have to discover $V_0$, not just $V_1$ and $A_1$.

Now to calculate $V_1$ we can apply the pseudo-inverse formula (26) (which produced (15) earlier), but now we pay special attention to the structure of $\Pi^\circ$. In this case, it is

$$(\Pi^\circ)_{ij} := \begin{cases} 
\frac{1}{\lambda_{0i} - \lambda_{0j}}, & \lambda_{0i} \neq \lambda_{0j}, \\
0, & \lambda_{0i} = \lambda_{0j}.
\end{cases}$$

It is instructive to look at the structure of $\Pi^\circ$, it is of the following form

$$\Pi^\circ = \begin{bmatrix}
0 \\
0 \\
\ddots
\end{bmatrix},$$

where the large 0 block is $m \times m$, the diagonal is all zeros, and the remaining terms are $1/(\lambda_{0i} - \lambda_{0j})$. The expression for $V_1$ is now the same as (15), but now we also make sure to use the $V_0$ that made $A_1$ diagonal above (recall that the derivation of (15) assumed $A_1$ to be diagonal)

$$V_1 = - V_0 \left(\Pi^\circ \circ (W^*_0 A_1 V_0)\right).$$

Again, to compare with existing expressions, find $v_{1k}$ using $V_1 e_k$

$$v_{1k} = - V_1 e_k = - \sum_{\lambda_{0i} \neq \lambda_{0j}} \frac{w^*_{0i} A_1 v_{0j}}{\lambda_{0i} - \lambda_{0j}} \lambda_{0i} e_j e_k = \sum_{\lambda_{0i} \neq \lambda_{0k}} \frac{w^*_{0i} A_1 v_{0k}}{\lambda_{0k} - \lambda_{0i}} v_{0i}, \quad (30)$$

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