A covariant action principle for dissipative fluid dynamics: from formalism to fundamental physics

N Andersson¹ and G L Comer²

¹ Mathematical Sciences & STAG Research Centre, University of Southampton, Southampton SO17 1BJ, UK
² Department of Physics, Saint Louis University, St. Louis, MO, 63156-0907, USA

E-mail: comergl@slu.edu and na@maths.soton.ac.uk

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Abstract
We present a new variational framework for dissipative general relativistic fluid dynamics. The model extends the convective variational principle for multi-fluid systems to account for a range of dissipation channels. The key ingredients in the construction are (i) the use of a lower dimensional matter space for each fluid component, and (ii) an extended functional dependence for the associated volume forms. In an effort to make the concepts clear, the formalism is developed step-by-step with model examples considered at each level. Thus we consider a model for heat flow, derive the relativistic Navier–Stokes equations and discuss why the individual dissipative stress tensors need not be spacetime symmetric. We argue that the new formalism, which notably does not involve an expansion away from an assumed equilibrium state, provides a conceptual breakthrough in this area of research. We also provide an ambitious list of directions in which one may want to extend it in the future. This involves an exciting set of problems, relating to both applications and foundational issues.

Keywords: general relativity, dissipative fluids, action principle

(Some figures may appear in colour only in the online journal)

1. Introduction

The marriage between the theory of general relativity and thermodynamics is known to be conceptually challenging. On the one hand, the breakthrough associated with Einstein’s theory was due to an understanding of the covariance of physical laws, leading to the concept
of spacetime and an emphasis on the observer’s role in making measurements. On the other hand, thermodynamics identifies a specific direction of time associated with the second law and the inevitable increase of entropy. Hence, it is not surprising that the development of models for non-equilibrium thermodynamical systems consistent with the tenets of general relativity remains a topical problem [1–3]. A workable model for dissipative fluid dynamics is required for a range of applications, from astrophysics and cosmology (perhaps particularly in the context of numerical simulations [4, 5]) to the description of hot dense plasmas for colliders like RHIC and the LHC [6–8]. We also need to make progress on foundational issues. In particular, one should clarify the link between phenomenological macro-models and the relevant processes on the micro-scale [9–12], and understand the intimate relation between the flow of time and a system’s evolution towards dynamical and thermodynamical equilibrium.

1.1. Motivation and scope

The aim of this paper is to demonstrate, mainly as a proof-of-principle, that the dynamics of a dissipative multifluid system can be obtained from a constrained variational principle. This is an exciting result which promises to lead to significant progress in this problem area. The premise of our discussion may seem at odds with the conventional wisdom, according to which action principles—expressed as an integral of a Lagrangian, whose local extrema satisfy the equations of motion, subject to well-posed boundary constraints—do not exist for dissipative systems. However, given the foundational nature of the problem it is quite natural to consider it and, in fact, a number of more or less successful attempts to make progress can be found in the literature. A common approach has been to combine a variational model for the non-dissipative problem with an argument that constrains the entropy production, often involving Lagrange multipliers (see [13] for a review and [14–22] for samples of the literature). The variational model we will develop is conceptually different. The conservative constraints on the system are built into the variation itself and the model does not involve (at least not in the first instance) an expansion away from equilibrium (in contrast to the celebrated ‘second order’ model of Israel and Stewart from the 1970s [10, 23, 24] or, indeed, any model based on a derivative expansion from the beginning). This means that the new description remains valid (at least formally) also for systems far away from equilibrium, and hence it provides a promising framework for the exploration of nonlinear thermodynamical evolution and associated irreversible phenomena. This is a problem area where a number of challenging issues remain to be resolved, involving for example maximum versus minimum entropy production for non-equilibrium systems [25–28].

Given the slow progress on this problem over the last several decades, why does it make sense to insist that a variational argument for non-equilibrium systems ought to exist? The question is multi-faceted, but as we are working within the context of general relativity let us seek inspiration from that theory. One of the most topical problems in gravitational physics involves two stars (or black holes) in a binary system, that lose orbital energy through the emission of gravitational waves. The data for the celebrated binary pulsar PSR1913+16 (and several similar systems) demonstrate that this phenomenon is described to excellent precision by Einstein’s theory. Gravitational-wave emission is obviously a dissipative mechanism, yet the underlying theory is obtained from an action. This example tells us that you can, indeed, use a variational strategy for dissipative problems (a similar argument was recently made in [29]). The key insight is that all the energy in the system must be accounted for. In many ways this statement is trivial. If you account for all the energy in a given system, including the ‘heat bath’, then there is no dissipation as such. Rather, one would be trying to model the
redistribution of energy within the larger (now closed) system. Obviously, if the proposed binary system is alone in the Universe and the gravitational-wave emission is properly accounted for, then the system is conservative and there is no reason why the dynamics should not derive from an action. This may be an acceptable logical argument, but how do you make it into a practical proposition for a generic dissipative system? This is the question that motivates the present work.

Building on recent efforts on the problem of heat in general relativity [30, 31], where progress was made by treating the system’s entropy as an additional field (at the hydrodynamics level), we aim to establish how the convective variational formulation for relativistic fluids [32–39] can be extended to account for dissipative mechanisms. A central issue in this development concerns the second law of thermodynamics (which singles out the entropy as being ‘special’ and which is intimately linked to any thermodynamical arrow of time argument). In the proposed approach, the functional form for the dissipative equations derives from the choice of action, but (just like in all other proposed formulations) the inequality associated with the second law is imposed by hand. This may seem like a trick—sweeping the problem under the carpet—and we would be the first to agree that the model remains incomplete at the fundamental physics level, but we nevertheless believe that our new approach paves the way for a better understanding of the link between physics on the small and macroscopic (fluid) dynamics.

It is useful to make clear under which conditions the model is intended to apply. Our analysis builds on a long tradition from the study of multi-component mixtures in chemistry [40, 41] and fluid dynamics [42]. That is, the focus is on systems where different constituents retain their identity as the system evolves. It is also worth noting that the final construction shares many features with extended irreversible thermodynamics [43]. As we are working within the framework of fluid dynamics, the construction assumes that a system can be described as a number of distinct, not necessarily co-moving, ‘fluids’. As discussed in [44] this boils down to assuming that each constituent has a short enough internal length scale over which averaging can be carried out (this could be the mean free path associated with scattering of particles of the same species, or the coherence length of a superfluid condensate), while any mechanism that couples the flows acts on a larger scale (or a longer time scale). Archetypal systems of this kind are (i) laboratory superfluids, where an inviscid condensate is weakly coupled to a ‘normal’ fluid consisting of thermal excitations (for descriptions related to the present work, see [45, 46]), and (ii) the coupled neutron superfluid/proton superconductor mixture in the outer core of a mature neutron star [47, 48]. The model we discuss here does not consider systems where one (or more) components are not in the fluid regime. One can think of many such problems of interest, e.g. involving superfluids at low enough temperature that the thermal excitations are in the ballistic regime or systems involving radiation. In principle, the model can be extended to consider such cases but it is not our ambition to do so here.

1.2. A few comments on the state of the art

The model described in this paper provides, in essence, an effective field theory for dissipative fluids in general relativity. Yet, this model is quite distinct from other recent efforts that take quantum field theory as their starting point, e.g. the bulk of the work on holographic fluid dynamics [49]. In order to appreciate the distinction between the different approaches, and understand the actual state-of-the-art in the general area of relativistic fluid dynamics, one has to consider the literature at a level of detail that goes beyond the fluid equations of motion. The key difference stems from the tradition in two different application areas. The classical
theory—the setting for the present work—aims to provide a fluid model suitable for applications in astrophysics and cosmology. A key aspect of this area is the connection with Einstein’s theory of gravity and the role of the dynamical spacetime, e.g. the link between fluid dynamics and gravitational-wave emission. In contrast, holographic fluid dynamics is a high-profile area within high-energy physics, motivated by the celebrated AdS/CFT conjecture. A driving motivation for this programme is the fact that one can use weakly coupled (classical) gravity analogues to probe strongly coupled field theories. This is potentially very important since strongly coupled theories are not within reach of other methods, e.g. perturbation theory. The fact that such problems may be considered using classical gravity analogues (e.g. black hole dynamics, which may be probed relatively straightforwardly) is clearly attractive. In terms of applications, the holography efforts have been (perhaps mainly) driven by particle physics and the need to describe the quark–gluon plasma in colliders like RHIC and the LHC (there has also been a recent drive towards problems in low-temperature physics). This situation is different from that in astrophysics in that the focus of particle physics tends to be on hot low-density matter, rather than the cold high-density matter relevant for neutron stars (say), and the spacetime in the holography approach can be considered flat/fixed without any loss of precision.

One would, of course, expect the different approaches to be compatible at a more fundamental level. The final fluid equations should take the same form and one should be able to identify the ‘same’ dissipative terms [50, 51]. However, the developments have not yet reached the stage where a comparison is non-trivial. Neither theory is complete and there is no easy way to bridge between the two (there is no straightforward link between weakly and strongly coupled theories). The models have also been developed to different degrees of sophistication. The classical gravity approach has been extended to deal with complex systems involving different states of matter, like superfluidity and the entrainment effect (in neutron stars due to the strong interaction or Bragg scattering by the crust lattice) and is being applied in situations with direct relevance for observations. Meanwhile, the bulk of the holography models have been carried out for conformal fluids. This assumption is not relevant for ‘normal matter’, but would apply at extremely high energies. At a sufficiently high energy the thermal energy dominates and you can ignore the scale associated with the mass (= chemical potential) of any particles involved. The problem then has only one scale (the temperature) and simplifies considerably. The assumption would be relevant for strongly coupled QCD (this is manifest in the MIT bag model which accounts for the difference in the quark masses as corrections to a model with conformal symmetry). Moreover, the conformal symmetry leads to the trace of the stress–energy tensor vanishing. This is a significant constraint on the theory. More recently, there have been efforts to move away from these restrictive assumptions, accounting for a finite chemical potential [52, 53] and additional degrees of freedom associated with superfluidity [54, 55] etc. The price you pay for making the hydrodynamics more complicated is increased complexity in the corresponding higher dimensional gravity problem (e.g. the inclusion of conserved charges that require a coupling to a gauge field, like in electromagnetism, or superfluidity which requires the consideration of black branes). This obviously makes the analogue gravity problem more difficult, but it may still be easier to deal with than the strongly coupled field theory on the other side of the correspondence. Finally, it is worth noting that the hydrodynamics obtained from the fluid-gravity correspondence is often in a fixed curved spacetime. In essence, this means that the description includes ‘external forces’. These will have to be removed before these models can be used as inspiration for physical systems like neutron stars and other relevant gravitational-wave sources.
2. Convective variational multi-fluid systems

Building on Carter’s convective variational formulation [34, 39], there has been considerable recent progress on the modelling of multi-fluid systems in general relativity. In addition to the intrinsic elegance of an action principle, an appealing feature of the variational approach is that once an ‘equation of state’ for matter (which here takes on the role of the Lagrangian) is provided the theory provides the relation between the various currents and their conjugate momenta. Another key advantage of the variational derivation is that it is straightforward to incorporate additional fluid components [39]. Hence, the extension to more complicated systems is natural.

The variational discussion takes as its starting point the notion of local fluid elements. These elements must contain enough particles that well-defined averaged state parameters (pressure, temperature, and so on) exist and can be measured reliably (the response of the relevant ‘device’ must be much faster than the local changes in the fluid due to statistical fluctuations). At the same time, the fluid elements must be small enough that their respective number of particles is infinitesimal relative to the entire system. Finally, from the spacetime point-of-view the fluid elements should be particle-like in that they trace out distinct worldlines. In this description, a multi-fluid system is such that several distinct components are able to flow more or less independently [44]. That is, each ‘fluid’ of the system has its own set of worldlines that it follows without losing its ‘chemical identity’. The archetypal multi-fluid system is superfluid helium, which is known to be well described by a two-fluid model [45, 46]. The decoupling of the two components is due to the superfluidity which suppresses particle scattering and friction. Another, less obvious, setting involves the flow of heat. In that case, it has been shown that a model based on treating the entropy component as an additional ‘fluid’ successfully resolves troublesome issues associated with causality and stability and also leads to the emergence of the expected second sound [30, 31, 56, 57].

In the following, we will consider a system with \( N_c \) independent constituents. Not all of these must flow independently. There are situations where it is important to keep track of the chemical composition of the various fluid elements, and a workable model must allow for this. Hence, we allow for the presence of \( N_f (\leq N_c) \) distinct flows, and associated fluxes \( n^a_x \), where the index \( x \) labels the components and \( a \) is the spacetime index. The associated number density (as measured by a comoving observer) is given by \( n^a_x = -g_{ab}n^a_x n^b_x \), where \( g_{ab} \) is the spacetime metric (assumed to have signature +2 in the following), and the ‘fluid particles’ associated with each flux have worldlines that follow from the unit four-velocity \( u^a_{x} = n^a_x / n_x \).

(Throughout the paper we work in geometric units where the speed of light is unity.) When \( N_f = N_c \), each constituent can move independently of the others, but when \( N_f < N_c \), some of the constituents are locked. As an example, this would be the case in a non-zero temperature system with vanishing heat conduction where the heat is advected with the flow (the matter and entropy have the same four-velocity). In general, entropy is accounted for by treating it as a separate component (with zero rest mass).

For an isotropic system the matter Lagrangian, \( \Lambda \), should be a relativistic invariant and hence depend only on covariant combinations of the fluxes. This includes the relative flows between them; one must consider both \( n^a_x \) and \( n^b_y = -g_{ab}n^a_x n^b_y \), with \( y \neq x \). The latter encodes the so-called entrainment effect, which tilts the momenta with respect to the currents when two or more fluids are coupled [39, 58]. In the case of neutron stars, the strong interaction is known to induce entrainment between neutrons and protons in the star’s core [83]. Meanwhile, the entropy-matter entrainment has been shown to be a crucial feature of the multi-fluid approach to heat conduction [30, 31, 57].
An arbitrary variation of $\Lambda$ with respect to the fluxes $n^a_x$ and the metric gives (here and in the following we ignore terms that can be written as total derivatives, that is, we ignore 'surface terms' in the action)

$$\delta ( \sqrt{-g} A ) = \sqrt{-g} \left[ \sum_x \mu^x_a \delta n^x_a + \frac{1}{2} \left( A g^{ab} + \sum_x n^a_x \mu^b_x \right) \delta g_{ab} \right],$$  \hspace{1cm} (1)

where $g$ is the determinant of the metric and $\mu^x_a$ are the individual momenta. These take the form

$$\mu^x_a = g_{ab} \left( B^x n^b_x + \sum_y A^{xy} n^x_y \right),$$  \hspace{1cm} (2)

with

$$B^x = -2 \frac{\partial \Lambda}{\partial n^x_x},$$  \hspace{1cm} (3)

and

$$A^{xy} = A^{yx} = - \frac{\partial \Lambda}{\partial n^{xy}_x}, \ \ \ x \neq y.$$  \hspace{1cm} (4)

Each momentum covector, $\mu^x_a$, is dynamically, and thermodynamically, conjugate to the respective number density current, $n^a_x$, and the magnitude gives the chemical potential. The $A^{xy}$ coefficients quantify the entrainment between the $x$ and $y$ components.

Equation (1) illustrates why a variational derivation of fluid dynamics is nontrivial. As it stands, the variation of $\Lambda$ suggests that the equations of motion would be $\mu^x_a = 0$; in essence, none of the fluids carry energy or momentum. This problem is resolved by imposing constraints on the fluxes. This can be done in different ways, but the route we promote here seems (at least to us) the most natural.

In fluid dynamics, there are two common approaches to monitoring the evolution: Eulerian and Lagrangian. In the former, an army of observers at rest with respect to a generic frame of reference make notes of the evolution as the various fluid elements intersect their worldlines. In the latter, each observer attaches him/herself to a particular fluid element and monitors how that element changes. We take the Lagrangian point-of-view by introducing for each fluid an abstract three-dimensional 'matter' space such that the worldline of a given fluid element is identified with a unique point in this space. The idea, which can be traced back to Taub [32] (see also [59–61]), and which has featured prominently in the development of models for relativistic elasticity [62–73], is illustrated in figure 1. The generalization of the idea to the case were there are as many matter spaces as there are components is illustrated in figure 2. The coordinates of each matter space, $X^A_x$ where $A = \{1, 2, 3\}$, serve as labels that distinguish fluid element worldlines. These labels are assigned at the initial time of the evolution, say $t = 0$. The matter space coordinates can be considered as scalar fields on spacetime, with a unique map (obtained by a pull-back construction) relating them to the spacetime coordinates. We will demonstrate later that the $X^A_x$ do not change along the associated worldlines.
The variational construction involves three key steps. First we note that the conservation of the individual fluxes is ensured provided that the dual three-form \( \epsilon_{\alpha \beta \gamma} = n_{\alpha \beta \gamma} \), \( 1 \leq n \leq 3 \), where \( \epsilon_{\alpha \beta \gamma} \) is the usual volume form associated with the spacetime is closed;

\[ \nabla_a n^a = 0 \]

(the square brackets indicate anti-symmetrization, as usual). In the second step we make use of the matter space to construct three-forms that are automatically closed on spacetime;

\[ n_{\alpha \beta \gamma} = \frac{\partial X^A}{\partial x^\alpha} \frac{\partial X^B}{\partial x^\beta} \frac{\partial X^C}{\partial x^\gamma} n_{ABC}, \]

where the Einstein summation convention applies to repeated matter space indices \( A, B, C \). Here, and in the following, we use notation such that a spacetime object and its matter space image are represented by the same symbol, with only the indices being different (i.e. \( n_{\alpha \beta \gamma} \leftrightarrow n_{ABC} \)). The volume form \( n_{ABC} \), which is assumed to be anti-symmetric, provides matter space with a geometric structure. If integrated over a volume in matter space it provides a measure of the number of particles in that volume.

With the above definition, the three form (7) is closed provided \( n_{ABC} \) is a function of the \( X^A \). In other words, if we take the scalar fields \( X^A \) to be the fundamental variables they yield a representation for each particle number density current that is automatically conserved. Hence, it is natural to express the variations of the spacetime three-form in terms of the \( X^A \).

It is worth pointing out that one can easily construct a variational model where the scalar fields \( X^A \) are the primary variables, satisfying the standard Euler–Lagrange equations (see [74, 75] for early work in this direction). This approach, recently explored in [76–80], is simply a reformulation of Carter’s model which forms the basis of our work [34, 39].
The final step involves introducing Lagrangian displacements $\xi^a_x$ for each fluid. These displacements track the movement of the worldline of a given fluid element. From the standard definition of Lagrangian variations in the relativistic context, see for example [81, 82], we have

$$\Delta \delta \xi^a_x = \xi^a_0, \tag{8}$$

where $\delta \xi^a_x$ is the Eulerian variation and $\xi^a_0$ is the Lie derivative along $\xi^a_x$. This means that convective variations are such that (since $X^A_x$ acts as a scalar field on spacetime)

$$\Delta \delta \xi^a_x = -\partial_\xi^a \xi^a_0. \tag{9}$$

After some algebra, one finds that this leads to

$$\Delta \delta n^{abc}_x = 0, \tag{10}$$

which in turn implies

$$\delta n^{abc}_x = n^b_x \delta \xi^a_x - \xi^b_x \delta n^a_x + n^b_x \left( \xi^c_x \delta g_{bc} + \frac{1}{2} g^{bc} \delta g_{bc} \right). \tag{11}$$

This is the result we require. By expressing the variations of the matter Lagrangian in terms of the displacements $\xi^a_x$, we ensure that the flux conservation is accounted for in the equations of motion. The variation of $\Lambda$ now leads to

$$\delta \left( \sqrt{-g} \Lambda \right) = \sqrt{-g} \left[ \frac{1}{2} \left( \Psi \delta g_{ab} + \sum_x n^a_x \mu^a_{b} \right) \delta g_{ac} - \sum_x J^a_{bc} \delta g_{bc} \right], \tag{12}$$

Figure 2. In the case of systems with several coupled fluids each component can be associated with its own three-dimensional matter space. The coordinates of this space, $X^A_x$, which label the flowlines of the various fluid elements in spacetime, are assigned at the initial time of evolution, say $t = 0$. The illustration relates to a problem with two constituents, labelled $x = r(ed)$ and $x = b(lue)$. The map between each matter space and spacetime plays a key role in establishing the conservation of the matter flows in the variational model.
where we have introduced the fluid force
\[ f^x_b = n^x_a \omega^x_{ab}, \]  
(13)
and the fluid vorticity
\[ \omega^x_{ab} = 2 \mathcal{V}_{(a} \mu^x_{b)}; \]  
(14)

From the constrained variation it thus follows that the equations of motion are simply given by
\[ f^x_a = 0 \quad \rightarrow \quad 2 n^x_a \mathcal{V}_{(a} \mu^x_{b)} = 0. \]  
(15)

We also see that the stress–energy tensor (the variation with respect to the spacetime metric) takes the form
\[ T^a_{\ b} = \Psi \delta^a_{\ b} + \sum_x n^x_a \mu^x_b, \]  
(16)
where
\[ \Psi = \Lambda - \sum_x n^x_a \mu^x_b, \]  
(17)
is the (generalized) pressure. When the set of equation (15) are satisfied then it is automatically true that \( \mathcal{V}^a_b T^a_{\ b} = 0. \)

Over the last decade or so, the variational model has been applied to a range of interesting and relevant problems. This has led to progress in a number of directions. Some of the results have been conceptual while others relate directly to applications. Briefly summarized;

(i) The variational model provides a natural framework to describe superfluid systems, both in the laboratory context and in astrophysics. The associated quantization of vorticity is easily imposed on the canonical momentum, and the implications for the dynamics become quite intuitive. In the case of neutron star modelling, the entrainment plays an important role [83], so the fact that it is naturally included in the model is a great advantage.

(ii) Since the variational construction makes direct use of Lagrangian displacements and the matter space, it is straightforward to include the effects of elasticity in the formalism [64, 65, 70–73]. At the linear level, this simply amounts to keeping track of the deviations away from a relaxed reference configuration for which the strain vanishes. This has allowed realistic modelling of the dynamics of neutron star crusts [84–86].

(iii) The model has allowed us to make progress on the long-standing problem of heat-flux in general relativity [87–90], resolving issues regarding causality and stability [90, 91]. Identifying one of the fluid components as the entropy (appropriate when the ‘phonons’ in the system have a short enough mean-free path) and introducing a phenomenological ‘resistivity’ one readily arrives at a formulation that honours the second law of thermodynamics and exhibits the anticipated second sound for heat in the relevant limit [31]. The entropy entrainment provides a key ingredient in the model, encoding the inertia of heat which is required to ensure causality [30, 57].

(iv) Due to its variational origin, it is relatively easy to extend the model to account for charged components and electromagnetism (via the standard gauge-coupling) [92]. In

\[ \text{At this point we have made a subtle switch: } f^x_a = 0 \text{ is enough if we still have in mind that the fluxes are functions of the } X^x_a, \text{ and those are what we solve for. However, usually we have in mind that we are going to solve for the } n^x_a, \text{ in which case } \mathcal{V}^a_b n^x_b = 0 \text{ also has to be considered as an ‘equation of motion’.} \]
this case, the introduction of a phenomenological resistivity leads to a consistent
derivation of the relativistic Ohm’s law for two-components plasmas [93]. The model can
also be extended to account for finite temperature effects and the route to more complex
models is (at least conceptually) quite clear.

3. A new strategy for dissipative systems

As we have already mentioned, the notion that one cannot use a variational approach to model
dissipative systems seems somewhat at odds with the tenets of general relativity. Einstein’s
field equations can be obtained from a variational principle, and if matter is included in the
model then the stress–energy tensor follows (at least in principle) from a variation with
respect to the metric. There is no reason why this argument should not remain valid also for
dissipative systems. As long as all energy contributions (matter, entropy, etc.) are included the
system is, in fact, ‘closed’ and should lend itself to a variational analysis. Our aim is to
develop this strategy in detail (in a way that differs substantially from Carter’s approach in
[35]). Ultimately, we are hoping to develop a practical model for dissipative multi-fluid
dynamics which can be applied to a wide range of topical problems.

We will now take the first few steps towards this goal by devising a variational argument
that leads to the functional form of the dissipative fluid equations. The relevant dissipation
coefficients are, in principle, calculable within the model although this would require a
specific equation of state (in the form of an energy functional) to be provided. We do not
address that problem here, preferring to focus on the formal construction of the variational
model. In many ways, this is the same attitude as in classical mechanics where the equations
of motion for a system can be written down without actual reference to a particular form for
the energy. The completion of the model is, of course, important but the problem is suffi-
ciently complex that it is sensible to progress in manageable steps. Moreover, we will
demonstrate that we can make progress without considering a specific problem setting.

3.1. Interacting matter spaces

The idea behind the new approach is, conceptually, quite simple. Recalling that the individual
matter spaces (associated with the various fluid components) play a central role in the vari-
ational construction for a conservative system, let us consider the ‘physics’ of a dissipative
system, e.g. with resistivity, shear or bulk viscosity etc. On the micro-scale dissipation arises
due to particle interactions/reactions. On the fluid scale this naturally translates into an
interaction between the matter spaces. As we will demonstrate, this can be accounted for by
letting each matter space be endowed with a volume form which depends on:

(1) the coordinates of all the matter spaces, and

(2) the independent mappings of the spacetime metric into these spaces.

For example, if each \( n_{ABC}^a \) is no longer just a function of its own \( X_A^a \), the closure of \( n_{ABC}^a \)
will be broken. As the fluxes are no longer conserved, the formalism incorporates dissipation.

To see how this could work, let us revisit the conservative problem. Recall that the scalar
fields \( X_A^a \) label the (fluid) particles. If these are conserved, then the \( X_A^a \) must be constant along
the relevant worldlines. That this is, indeed, the case is easy to demonstrate. Letting \( \tau \) be the
proper time of each worldline, we have
Since a fluid element’s matter space coordinates $X^A_x$ are constant along its worldline, it must also be the case that
\[
\frac{dn^{ABC}_{\tau}}{d\tau_x} = 0.
\] (19)

In other words, the volume form $n^{ABC}_{\tau}$ is fixed in the associated matter space. It is clear from the steps required in this demonstration that the key to non-conservation is to allow $n^{ABC}_{\tau}$ to be a function of more than the $X^A_x$. This is quite intuitive. The worldlines of the various fluids will in general cut across each other, leading to interactions/reactions. A more general functional form for the matter space volume forms $n^{ABC}_{\tau}$ may be used to reflect this aspect of the underlying physics. A schematic illustration of how this works is provided in figure 3.

As we will demonstrate in the next few sections, the simple step of enlarging the functional dependence of $n^{ABC}_{\tau}$ does indeed allow us to build a variational model that incorporates the ‘expected’ dissipative terms. However, it also takes us into territory where one has to tread carefully. In particular, one must pay more attention to the various ‘matter space objects’. We are now dealing with geometric objects that actually live in the higher-dimensional combination of all the matter spaces, e.g. we are dealing with an object of the form
\[
n^{ABC}_{\tau}(X^I_x, X^E_y) dX^A_x \wedge dX^B_x \wedge dX^C_x, \quad y \neq x.
\] (20)

That is, a volume form in the $x$-matter space parameterized by points in the $y$-matter spaces. From a presentational point-of-view we can still pretend that the individual matter spaces (related to spacetime via the same maps as in the conserved case) remain somehow ‘distinct’, but in reality this is not the case. This issue requires more detailed analysis in the future.

The new model thus involves a change of emphasis. In the conservative multi-fluid problem one may, once the constrained variation is devised, consider the various fluxes as the main variables and formulate the problem at the spacetime level without bothering too much with the detailed matter space quantities. In the model we advocate here, this is no longer the case. The change is inspired by efforts to model relativistic elasticity [64, 65, 70–73], where the role of the matter space is elevated and the action is constructed at that level. In the case of elasticity, the fact that $n^{ABC}_{\tau}$ is a fixed tensor in matter space allows the introduction of an associated ‘matter space metric’ which can be used to quantify the deviation from the relaxed reference shape and hence account for elastic properties.

When we allow $n^{ABC}_{\tau}$ to be more complex we (inevitably) break some of the attractive features of the conservative model. Obviously, $n^{ABC}_{\tau}$ is no longer a fixed matter space object. This has a number of repercussions, especially for discussions of elastic matter. We will not discuss those here, although it is worth noting that it is a very interesting problem given the obvious connection with visco-elasticity. Instead, we simply note that we can still construct the action from matter space objects. To do this we need the map of the spacetime metric into the relevant matter space:
\[
g^{AB}_{\tau} = \frac{\partial X^A_x}{\partial x^a} \frac{\partial X^B_x}{\partial x^b} g_{ab} = g^{BA}_{\tau}.
\] (21)

Note that $g^{AB}_{\tau}$ is not likely to be a tensor on matter space. In order for that to be the case, the corresponding spacetime tensor must satisfy two conditions: first, it must be flowline
orthogonal (on each index). This holds for the present problem, since the operator which generates projections orthogonal to
\[ \otimes_{x} = + \] (22)
and because of equation (18) we have
\[ \otimes_{x} = \frac{\partial X^{A}}{\partial x^{a}} \frac{\partial X^{B}}{\partial x^{b}} g_{ab} = \frac{\partial X^{A}}{\partial x^{a}} \frac{\partial X^{B}}{\partial x^{b}} \otimes_{x} \] (23)
The second condition that \( \otimes_{x} \) must satisfy so that \( g_{x}^{AB} \) is a matter space tensor is [70]
\[ \mathcal{L}_{u_{x}} \otimes_{x} = 0. \] (24)
This is not the case here; indeed, it is too severe for most relevant applications.
Anyway, it is easy to show that a scalar constructed from the contraction involving \( g^{ab} \) and some tensor \( t_{\alpha}^{\lambda} \) is identical to the analogous contraction of the corresponding matter

Figure 3. An illustration of the notion that a coupling between matter spaces can lead to dissipation. We consider the case of two fluids, labelled r and b (for red and blue). The individual \( X_{x}^{i} \) (assigned at the initial time, \( t = 0 \)) do not vary along their own worldlines, even when the system is dissipative. By adding \( X_{y}^{j} \) (\( y \neq x \)) we get ‘evolution’ since the worldlines cut across each other. Let us choose a particular worldline of the r-fluid, say \( X_{y}^{i} \), meaning that \( X_{y}^{i} \) will take the same value at each spacetime point \( x^{v} \) along the worldline. At an intersection with a worldline of a fluid element of the b-fluid (the point labelled 1 in the figure, say) the other fluid’s worldline will have its own label (in this case \( X_{y}^{i} \)), which is the same at every point on that worldline. At the next intersection (point 2), the worldline we are following has the same value for \( X_{x}^{i} \), but it is intersected by a different worldline from the other fluid (\( X_{y}^{i} \)), meaning that \( X_{y}^{i} \) at each intersection is different. Hence, \( X_{y}^{i} \), when considered as a field in spacetime, must vary along the r-fluid worldlines, and vice versa. This is how the closure of the individual volume three-forms is broken and ultimately why the model is dissipative.
space objects [72]. In particular, the number density follows from

\[ n_x^2 = -g_{ab} n_x^a n_x^b = \frac{1}{3!} g^{ad} g^{be} g^{cf} \mu_{abc} n_x^d n_x^e n_x^f = \frac{1}{3!} \mu_x^{AD} g_x^{BE} g_x^{CF} n_{ABC} n_x^D. \]  

(25)

while the chemical potential

\[ \mu_x = -u_x^a \mu_x^a \]  

(26)

(according to an observer at rest in the respective fluid’s frame) can be obtained from

\[ n_x \mu_x = -n_x^a \mu_x^a = \frac{1}{3!} \mu_x^{abc} n_x^a n_x^b n_x^c = \frac{1}{3!} \mu_x^{ABC} n_{ABC}. \]  

(27)

Here we have introduced the dual to the momentum \( \mu_x^a \):

\[ \mu_x^{abc} = c^{abcd} \mu_x^d, \quad \mu_x^a = \frac{1}{3!} c^{abcd} \mu_x^{bcd}, \]  

(28)

and its matter space image;

\[ \mu_x^{ABC} = \frac{\partial X_x^A}{\partial x^a} \frac{\partial X_x^B}{\partial x^b} \frac{\partial X_x^C}{\partial x^c} \mu_x^{abc}. \]  

(29)

The key take-home message is that we can think of the matter action as being constructed entirely from matter space quantities. In the simplest case of a single component one would have \( \Lambda(n_x) = \Lambda(n_x^x, g^{ab}) \leftrightarrow \Lambda(n_{ABC}^x, g_x^{AB}) \). The specification of such an equation of state, with the functional dependencies discussed later, will eventually be required in order to complete the model we are designing. For the moment, we assume that this problem can be dealt with and move on to the actual variational equations of motion.

3.2. Proof-of-principle: a reactive/resistive system

As a first step towards making the proposal concrete, let us work through the key steps in the variational analysis, this time allowing for general variations of the matter space density. The matter space coordinates still vary according to (9) (this is essentially just the definition of the Lagrangian displacement). Noting that

\[ \Delta_x \left( \frac{\partial X_x^A}{\partial x^a} \right) = \frac{\partial}{\partial x^a} \left( \Delta_x X_x^A \right) = 0, \]  

(30)

we easily arrive at the generic variation

\[ \delta n_{abc} = -\mathcal{L} \mu_x^a n_x^a + \frac{\partial X_x^A}{\partial x^a} \frac{\partial X_x^B}{\partial x^b} \frac{\partial X_x^C}{\partial x^c} \Delta_x n_{ABC}. \]  

(31)

To make contact with (11) we need

\[ \mu_x^a \delta n_x^a = \frac{1}{3!} \mu_x^a \delta \left( c^{abcd} n_x^b n_x^c n_x^d \right) = -\frac{1}{3!} \mu_x^{bcd} \delta n_x^{bcd} + \frac{1}{3!} \mu_x^a n_x^b \delta c^{abcd}, \]  

(32)

and [39]

\[ \delta c^{abcd} = -\frac{1}{2} c^{abcd} g^{ef} \delta g_{ef}. \]  

(33)
Hence, we arrive at

$$\mu_x^a \delta n_x^a = \frac{1}{3!} \mu_x^{abc} \xi_x^c n_{abc} - \frac{1}{2} \mu_x^a n_x^b \delta g_{bc} \varepsilon_x^c n_{abc} - \frac{1}{3!} \mu_x^{ABC} \Delta_x n_{ABC}, \tag{34}$$

and the 'final' expression:

$$\mu_x^a \delta n_x^a = \mu_x^a \left( n_x^b \xi_x^b \varepsilon_x^b - \xi_x^c \varepsilon_x^c n_x^c \varepsilon_x^c n_{abc} - \frac{1}{2} n_x^a \varepsilon_x^a \delta g_{bc} \varepsilon_x^c n_{abc} \right) - \frac{1}{3!} \mu_x^{ABC} \Delta_x n_{ABC}. \tag{35}$$

The terms in the bracket are the same as in the conservative case, see (11). The last term is new.

The functional dependence of the volume form for a given fluid's matter space is the main input for what follows. Obviously, \( n_{ABC}^x \) must depend on \( X^A_x \), the coordinates of the corresponding matter space. This leads to the conservative dynamics. Adding to this, let us include the coordinates \( X^A_y \) from the other, \( y \neq x \), matter spaces. As we have already seen, this breaks the closure of \( n_{ABC}^x \).

The required variation of \( n_{ABC}^x \) is now (in view of (8))

$$\Delta_x n_{ABC}^x = \sum_{y \neq x} \frac{\partial n_{ABC}^x}{\partial X^D_y} \Delta_x X_y^D = \sum_{y \neq x} \frac{\partial n_{ABC}^x}{\partial X^D_y} \left( \varepsilon_x^a - \varepsilon_y^a \right) \partial_a X^D_y. \tag{36}$$

Comparing to (34), we see that it is natural to define

$$R_{xy}^a \equiv \frac{1}{3!} \mu_x^{ABC} \frac{\partial n_{ABC}^x}{\partial X^D_y} \partial_a X^D_y. \tag{37}$$

We then have

$$\mu_x^a \delta n_x^a = \mu_x^a \left( n_x^b \xi_x^b \varepsilon_x^b - \xi_x^c \varepsilon_x^c n_x^c \varepsilon_x^c n_{abc} - \frac{1}{2} n_x^a \varepsilon_x^a \delta g_{bc} \varepsilon_x^c n_{abc} \right) + \sum_{y \neq x} R_{xy}^a \left( \varepsilon_x^a - \varepsilon_y^a \right). \tag{38}$$

The final step of the exercise involves writing down the variation of the matter Lagrangian, \( \Lambda \). Starting from (1), we arrive at

$$\delta \left( \sqrt{-g} \Lambda \right) = -\sqrt{-g} \left\{ \sum_x \left( f_x^a + \mu_x^a \Gamma_x - R_x^a \right) \varepsilon_x^a - \frac{1}{2} \left( \eta_{ab} + \sum_x n_x^a \mu_x^b \right) \delta g_{ab} \right\}, \tag{39}$$

where we have used

$$\sum_x \sum_{y \neq x} R_{xy}^a \varepsilon_y^a = \sum_x \sum_{y \neq x} R_{yx}^a \varepsilon_x^a. \tag{40}$$

We have also defined

$$R_x^a = \sum_{y \neq x} \left( R_{xy}^a - R_{yx}^a \right), \tag{41}$$

and

$$\Gamma_x = \varepsilon_x^a n_x^a. \tag{42}$$

Hence, the individual components are governed by the equations of motion

$$f_x^a + \Gamma_x^a - n_x^a \omega_{ba}^x + \Gamma_x^a \mu_x^b = R_x^a. \tag{43}$$

Since the force term \( f_x^a \) on the left-hand side is orthogonal to \( n_x^a \) (by the anti-symmetry of \( \omega_{ab}^x \)), it is easy to see that this result implies that the particle creation/destruction rates are
given by
\[ \Gamma_\mu = -\frac{1}{\mu^x} u^a_\mu R^x_a. \]  
(44)

Finally, an orthogonal projection of (43) leads to
\[ 2n^a_\mu V^b_{[a} \mu^x_{b]} + \Gamma^x_\mu \rightdownarrow_{ab} \mu^x_0 = \rightdownarrow_{ab} R^x_a. \]  
(45)

These equations provide the dissipative equations of motion for this system.

With equation (39) we have a true action principle—in the sense that the field equations are extrema of the action—for a system of fluids that includes dissipation. In many ways, this demonstration is the key result of this work.

Before we move on, it is worth noting that the stress–energy tensor is still given by (16) and we can show that
\[ V_b T^b_a = \sum_x \left( f^x_a + \mu^x_0 T^x_0 \right) = 0, \]  
(46)

because
\[ \sum_x R^x_a = 0, \]  
(47)

identically. The requirement that the covariant divergence of the stress–energy tensor vanish is automatically guaranteed by the dissipative fluid equations, in keeping with the diffeomorphism invariance of the theory.

### 3.3. The problem of heat revisited

In much of the relevant literature, dissipative terms have been added to the equations of relativistic fluid dynamics in a somewhat ad hoc manner, inspired by some level of intuition of how the system ‘ought to behave’ (for recent examples, see [50, 94–102]). The model developed in the previous section puts us in a rather different position as the dissipative contributions were derived, not postulated. This leads to a number of interesting (and challenging) questions, most of which we are not in a position to answer at this point. It is, however, imperative that we establish that the construction ‘makes sense’. To do this, we need to understand the physics content of the model.

In order to gain insight, let us consider the simplest relevant setting. Assume that we consider a system with two components; matter (labelled n) and heat, represented by the entropy (labelled s). In principle, we need to provide an equation of state (that satisfies relevant physics constraints) in order to complete the model. Once this is provided we can calculate the resistivity coefficients from (37) and then model the system using the momentum equation (43). However, a discussion of suitable equations of state would force our attention away from the main focus here, the variational model itself. Hence, we prefer to consider the problem from a phenomenological point-of-view. This is sufficient if our main aim is to show that the model has the anticipated features. To make the model specific, let us assume that the matter component is conserved, but the entropy does not need to be. This is the problem of relativistic heat flow. This problem was recently considered in [30, 31], and it is useful to compare the present model to the results of that analysis. This problem is simple enough that we should be able to understand what is going on.
First of all, given that we only have two components then
\[ R_a^n = R_a^{sn} - R_a^{ns} = -R_a^s. \] (48)

Secondly, the conservation of the material component implies that
\[ \Gamma_\mu = -u_a^a R_a^u = \frac{1}{\mu} u_a^u R_a^{nu} = 0 \quad \quad \Rightarrow \quad \quad u_a^a R_a^{ns} = 0. \] (49)

The upshot is that \( R_a^{ns} \) must be orthogonal to both \( u_a^a \) and \( u_a^s \). Meanwhile, the entropy change is constrained by the second law. That is
\[ \Gamma_\mu = -u_a^a R_a^u = \frac{1}{\mu} u_a^u R_a^{nu} \geq 0, \] (50)

where we have introduced the temperature \( T = \mu^a \). Note that the constraints affect the two, likely independent, contributions to \( R_a^{ns} \). We cannot infer a link between \( R_a^{ns} \) and \( R_a^{sn} \) at this point.

So far we have not introduced a privileged observer. This is in contrast to most previous work which takes this as starting point for the discussion. This means that a direct comparison with other results, such as those in [30, 94–102], require a bit of effort. In order to facilitate a comparison, let us follow [30] and focus on an observer moving along with the matter flow (in the spirit of Eckart [103]). Thus we have \( u_a^a = u_a^u \) and the relative flow required to express the entropy flux is defined such that
\[ u_s^a = \gamma (u_a^a + w^a), \quad \text{where} \quad u_s^u w_s = 0, \quad \text{and} \quad \gamma = (1 - w^2)^{-1/2}. \] (51)

The relative velocity \( w^a \) is aligned with the heat flux vector (as discussed in [30]).

Given (49) and (50) it makes sense to introduce the decompositions
\[ R_a^{ns} = \epsilon_{abcd} \phi_b^a u_c w_d, \] (52)

and
\[ R_a^{sn} = R_w w_a + \epsilon_{abcd} \phi_b^a u_c w_d, \] (53)

where \( \phi_b^a \) and \( \phi_c^a \) are unspecified vector fields. We then see that (50) leads to
\[ TT_a = \gamma R_a^u w^2 \geq 0 \quad \quad \Rightarrow \quad \quad R_w > 0. \] (54)

Meanwhile, the two components \( \phi_b^a \) and \( \phi_c^a \) are not constrained by the thermodynamics. This leaves a degree of arbitrariness in the model\(^7\). Should we be surprised by this? Not really. A similar issue was, in fact, discussed in [30] where it was demonstrated that the variational model led to the presence of a number of terms in the heat equation that could not be constrained by the second law. It was also pointed out that the difference between the model advocated in [30] and the second-order model of Israel and Stewart appeared at this level [56]. It has not been established whether there are situations where these terms have a notable effect on the dynamics. We leave this as an interesting question for the future.

5 It is worth noting that we are making the standard assumption that the second law must hold locally for each fluid element. It is by no means obvious that this has to be true. The question relates to the size of the system from a statistical perspective. However, without this assumption you will not progress beyond this point.

6 Note that this arbitrariness would be removed if we provided a suitable equation of state, in which case the coefficients could be obtained from the definition (37).
3.4. Adding dissipative stresses

We have demonstrated how dissipation can be included in the variational multi-fluid formalism. This is an important step towards a deeper conceptual understanding of non-equilibrium systems in general relativity. Dissipative contributions that have previously been postulated can now be derived from underlying principles. Moreover, as the comparison with the problem of heat flow demonstrates, the variational model points to new aspects of the problem. However, the example we considered above only accounts for two particular non-equilibrium phenomena, particle non-conservation and resistivity. In order to convincingly argue that our model represents a conceptual breakthrough, we need to demonstrate that the action principle generates terms in the field equations of the tensorial form expected for a more general range of processes. Thus, we turn to the issue of dissipative stresses.

The obvious starting point for an extension of the variational strategy is to ask what other quantities the matter space volume form, \( n^A_{x^0} \), may depend on. The natural object to consider is the mapping of the spacetime metric, \( g_{a\beta} \), into the respective matter spaces. As we will now demonstrate, this leads to a system with dissipative shear stresses.

In mapping the metric into the matter spaces we have in principle three independent possibilities. Let us first consider the most intuitive option, which involves allowing \( n^A_{x^0} \) to depend on \( g^{AB}_{x^0} \), as defined in (21).

Noting that equation (18) implies that the \( X^A_{x^0} \) will still be conserved along the associated flow, the variation of \( n^A_{x^0} \) is now such that

\[
\Delta_x n^A_{x^0} = \frac{\partial n^A_{x^0}}{\partial g^{DE}_{x^0}} \Delta_x g^{DE}_{x^0} + \sum_{y \neq x} \frac{\partial n^A_{x^0}}{\partial X^D_y} \Delta_x X^D_y. \tag{55}
\]

The first term in this expression is new, the second term is the same as in (36). The new term is easily worked out, following the steps from the simpler model. We find that

\[
\Delta_x g^{AB}_{x^0} = \frac{\partial X^A_{x^0}}{\partial x^a} \Delta_x g^{ab}_{x^0} = \frac{\partial X^A_{x^0}}{\partial x^a} \frac{\partial X^B_{x^0}}{\partial x^b} \left[ \delta g^{ab} - 2 V^a \frac{\partial}{\partial x^b} \right]. \tag{56}
\]

where we have used

\[
\Delta_x g^{ab} = \delta g^{ab} - 2 V^a \frac{\partial}{\partial x^b}, \tag{57}
\]

(round brackets indicate symmetrization, as usual.)

As in the previous example, the variation of the matter Lagrangian involves \( \mu^{ABC}_{x^0} \Delta_x n^A_{x^0} \). The new contribution takes the form

\[
\frac{1}{3!} \mu^{ABC}_{x^0} \frac{\partial n^A_{x^0}}{\partial g^{DE}_{x^0}} \Delta_x g^{DE}_{x^0} = \frac{1}{3!} \mu^{ABC}_{x^0} \frac{\partial n^A_{x^0}}{\partial g^{DE}_{x^0}} \frac{\partial X^D_{x^0}}{\partial x^a} \frac{\partial X^E_{x^0}}{\partial x^b} \left[ \delta g^{ab} - 2 V^a \frac{\partial}{\partial x^b} \right] \]

\[
= -\frac{1}{2} S_{ab} \left[ g^{ac} g^{bd} \delta g_{cd} + 2 V^a \frac{\partial}{\partial x^b} \right] = -\frac{1}{2} S_{ab} \delta g_{ab} - S_{ab} V^b \frac{\partial}{\partial x^a}, \tag{58}
\]

where we have defined

\[
S_{ab} = \frac{1}{3!} \mu^{ABC}_{x^0} \frac{\partial n^A_{x^0}}{\partial g^{DE}_{x^0}} \frac{\partial X^D_{x^0}}{\partial x^a} \frac{\partial X^E_{x^0}}{\partial x^b} = S_{ba}, \tag{59}
\]

such that

\[
u^a_{x^0} S_{ba} = 0. \tag{60}
\]
Combining the results, we arrive at
\[ \mu_s \delta n_s^a = u_s^a \left( n_s^b V_b e_s^a - e_b V_b n_s^a - n_s^x V_x e_s^a \right) + S_{ab} V^b e_s^a + \sum_{y \neq x} R_{ay} \left( e_y - e_x \right) + \frac{1}{2} \left[ \mu_s^x n_s^y g_{ab} + S_{ab} \right] \delta_{ab}. \]  
\[ (61) \]

Introducing the total dissipative stresses, in this case trivially setting
\[ D_{ab} = S_{ab}, \]  
\[ (62) \]
we see that equation (39) becomes
\[ \delta \left( \sqrt{-g} \Lambda \right) = -\sqrt{-g} \left\{ \sum_x \left( f_{x}^a + \Gamma_x \mu_x^a + V^b D_{x}^b - R_{a}^x \right) e_x^a \right. \]
\[ \left. - \frac{1}{2} \left[ \Psi g_{ab} + \sum_x \left( n_x^a \mu_x^b + D_{x}^{ab} \right) \right] \delta_{ab} \right\}. \]  
\[ (63) \]
where we have used (40) and (41) for the resistivity currents.

The equations of motion now take the form
\[ f_{x}^a + \Gamma_x \mu_x^a + V^b D_{x}^b = R_{a}^x, \]  
\[ (64) \]
and the stress–energy tensor is
\[ T^{ab} = \Psi g^{ab} + \sum_x \left( n_x^a \mu_x^b + D_{x}^{ab} \right), \]  
\[ (65) \]
where the generalized pressure, \( \Psi \), remains unchanged, see (17). As in the previous problem, it is quite easy to show that
\[ V_b T^b_a = \sum_x \left( f_{x}^a + \Gamma_x \mu_x^a + V^b D_{x}^b \right) = 0, \]  
\[ (66) \]
since (47) still holds.

Finally, we can extract the various creation/destruction rates. We first contract equation (64) with \( u_s^a \), noting that \( u_s^a f_{a}^s = 0 \) and \( u_s^a V^b D_{a}^b = -D_{a}^b V^b u_s^a \), to find
\[ \mu^s \Gamma_s = -R_{a}^s u_s^a - D_{ab} V^b u_s^a. \]  
\[ (67) \]
When \( x = s \) this gives the entropy creation rate which should be constrained by the second law.

3.5. Rediscovering Navier–Stokes

Armed with the more general constraint (67) for the dissipative terms, let us revisit the model problem from section 3.2. In the spirit of that discussion, let us ask what we can learn from the various constraints that follow from the derivation (ignoring the fact that the coefficients involved could, at least in principle, be calculated from (37) and (59) if we provided a suitable equation of state). That is, we consider a two-component system with a material component (n) and entropy/heat (s) with the added physics input that \( \Gamma_s = 0 \). As in the previous discussion of this problem we will use an observer moving along with the matter flow, such that \( u_s^a = u_n^a \) and \( w^a \) represents the relative flow.

Let us first consider the matter component. Since we know that \( R^a_{\text{mat}} \) should be orthogonal to \( u_n^a \) we introduce the decomposition
\[ R_{uu}^{ns} = R_u \left( w^2 u_x + w_a \right) + \epsilon_{abcd} \Phi^b_a u^c w^d. \] (68)

Then (67) implies that
\[ D_{ab}^n \nabla^b u^a = -R_{uu}^{ns} u^a = R_u w^2. \] (69)

Now, there are two cases one may consider. In the general situation, when there is a distinct heat flow, we have \( w^2 > 0 \) which if we take \( R_u > 0 \) implies that the left-hand side of (69) must be positive. To ensure that this is the case, we use the standard decomposition (with the same conventions as in [93])
\[ \nabla_a u_b^x = \sigma_{ab}^x + \sigma_{ab}^x u_b^x + u_a^x u_b^x + \frac{1}{3} \theta^x \delta_{ab}, \] (70)

where
\[ \sigma_{ab}^x = D_{ab} u_b^x, \quad \text{with} \quad D_a u_b^x = \frac{1}{3} \delta_{bc} \theta^x \nabla^c u_b^x. \] (71)

where the angular brackets indicate symmetrization and trace removal,
\[ \sigma_{ab}^x = D_{ab} u_b^x, \quad \text{with} \quad D_a u_b^x = \frac{1}{3} \delta_{bc} \theta^x \nabla^c u_b^x. \] (72)

and
\[ \theta^x = V_a u^a_b. \] (73)

With these definitions, each term in (70) is orthogonal to \( u_b^x \). From the fact that \( S_{ab}^x \) is symmetric and orthogonal to \( u_b^x \) it is easy to see that the condition inferred from (69) is satisfied provided we have
\[ D_{ab}^n = \eta^n \sigma_{ab}^n + \zeta^n \theta^n \delta_{ab}. \] (75)

with \( \eta^n > 0 \) and \( \zeta^n > 0 \). We recognize this as the dissipative (shear- and bulk viscosity) stresses expected in the Navier–Stokes equations. Interestingly, the second law of thermodynamics was not engaged in the derivation of this result.

If, on the other hand, there is no heat flux in the system, then \( w^2 = 0 \) and we must have
\[ \sigma_{ab}^n = 0, \quad \theta^n = 0. \] (76)

These are, of course, the expected conditions for an equilibrium system.

Let us now turn to the entropy condition. Making use of the results from the heat example discussed in section 3.2, noting that we can still use (53) for \( R_u^{ns} \), we see that (67) leads to
\[ T_e = \eta R_u w^2 - D_{ab} \nabla^b u^a \geq 0, \] (77)

as required by the second law. This suggests that, in addition to \( R_u > 0 \) from before, we should have
\[ D_{ab}^n = -\eta^n \sigma_{ab}^n + \zeta^n \theta^n \delta_{ab}. \] (78)

with \( \eta^n > 0 \) and \( \zeta^n > 0 \).

This example is a little bit more ‘confusing’ than the pure heat conduction case. On the one hand, it is impressive that we can arrive at the expected form of the equations from this rather general analysis. On the other hand, it is frustrating that we cannot pin down, for example, the sign of the friction coefficient in (68). To do this, we need to consider a particular physics set-up where (37) can be worked out. We plan to consider this problem in more detail later. A particular system worth considering is superfluid Helium, for which the
presence of several bulk viscosity channels—most likely related to \( \zeta^n \) and \( \zeta^s \) in the present model—are known to exist [104, 105].

### 3.6. Adding dissipative stresses: a more general case

The previous example demonstrates the promise of the new variational model. By allowing each matter space volume form \( n_{ABC}^x \) to depend on the coordinates of the other matter spaces \( X^A_y (y \neq x) \) as well as the mapping of the spacetime metric \( g^{AB}_x \), we arrive at a model that allows for particle non-conservation, resistivity and dissipative stresses. This is a conceptual success, but we now face a new set of questions. For example, it seems legitimate to ask whether the model we developed in section 3.3 is the most general construction. It is relatively easy to see that it is not; we could have considered other mappings of the metric. This problem turns out to be relevant, because it leads to a demonstration that the individual dissipative stress tensors need not be symmetric even though the sum of them is. The relevance of such asymmetries, and their potential role in modelling neutron star superfluids, has already been discussed in [44, 46, 106].

In mapping the metric into the matter spaces in section 3.3 we only considered one of the three independent possibilities. We may also;

1. allow \( n_{ABC}^x \) to depend on \( g^{AB}_y \), the metric mapped into the other matter spaces, or
2. use the mixed mapping\(^7\)

\[
g^{AB}_{xy} = \frac{\partial X^A_x}{\partial x^a} \frac{\partial X^B_y}{\partial x^b} g^{ab}_{xy}, \quad y \neq x. \tag{79}
\]

3. Consider \( n_{ABC}^x \) as a function of \( g^{AB}_{yz} \) where \( y \) and \( z \) are different, but neither is equal to \( x \). (We will not work this case out explicitly; the extension is relatively straightforward given the details below.)

It is worth noting that the only symmetry in exchange of indices for \( g^{AB}_{xy} \) is

\[
g^{AB}_{xy} = g^{BA}_{yx}. \tag{80}
\]

This implies that \( g^{[AB]}_{xy} \) may not vanish, which in turn suggests the presence of the asymmetric terms among the dissipative stresses.

\(^7\) As in the previous model, it is worth noting that \( g^{AB}_x \) and \( g^{AB}_y \) are not tensors in the matter space of the \( x \) component. In this case, the spacetime objects are not even (completely) flowline orthogonal with respect to \( u^a_x \). This is obvious from the fact that

\[
g^{AB}_{xy} = \frac{\partial X^A_x}{\partial x^a} \frac{\partial X^B_y}{\partial x^b} g^{ab}_{xy}. \]
The variation of \( n_{ABC}^x \) is now such that

\[
\Delta_x n_{ABC}^x = \frac{\partial n_{ABC}^x}{\partial g_{DE}^x} \Delta_x g_{DE}^x + \sum_{y \neq x} \left( \frac{\partial n_{ABC}^x}{\partial X^D_y} \Delta_x X^D_y + \frac{\partial n_{ABC}^x}{\partial g_{xy}^D} \Delta_x g_{xy}^D + \frac{\partial n_{ABC}^x}{\partial g_y^D} \Delta_x g_{y}^D \right).
\]

Comparing to (55) we have two new terms;

\[
\Delta_x g_{AB}^{xy} = \frac{\partial X^A_y}{\partial x^a} \frac{\partial X^B_x}{\partial x^b} \left[ \delta^{ab} - 2 V^{(ab)}_{\omega\xi} \right] + g^{ab} \frac{\partial X^A_y}{\partial x^a} \left( L_{\xi_x} - L_{\xi_y} \right) \frac{\partial X^B_x}{\partial x^b},
\]

and

\[
\Delta_x g_{AB}^{xy} = \frac{\partial X^A_x}{\partial x^a} \frac{\partial X^B_y}{\partial x^b} \left[ \delta^{ab} - 2 V^{(ab)}_{\omega\xi} \right] + g^{ab} \left( L_{\xi_x} - L_{\xi_y} \right) \frac{\partial X^A_x}{\partial x^a} \frac{\partial X^B_y}{\partial x^b}.
\]

In order to build the variation of the matter Lagrangian we need

\[
\frac{1}{3!} \mu^{ABC} \frac{\partial n_{ABC}^x}{\partial g_{DE}^x} \Delta_x g_{DE}^x
\]

\[
= \frac{1}{3!} \mu^{ABC} \frac{\partial n_{ABC}^x}{\partial g_{xy}^D} \left[ \frac{\partial X^D_x}{\partial x^a} \frac{\partial X^E_y}{\partial x^b} \left[ \delta^{ab} - 2 V^{(ab)}_{\omega\xi} \right] + g^{ab} \frac{\partial X^D_x}{\partial x^a} \left( L_{\xi_x} - L_{\xi_y} \right) \frac{\partial X^E_y}{\partial x^b} \right]
\]

\[
= \frac{1}{2} S_{ab}^{xy} \delta_{\omegaab} - \frac{1}{2} S_{ab}^{xy} \left( V^{ab}_{\xi_x} + V^{ba}_{\xi_y} \right) + \mathcal{R}_a^{xy} \left( z^a_x - z^a_y \right),
\]

where

\[
S_{ab}^{xy} = \frac{1}{3!} \mu^{ABC} \frac{\partial n_{ABC}^x}{\partial g_{xy}^D} \frac{\partial X^D_x}{\partial x^a} \frac{\partial X^E_y}{\partial x^b},
\]

such that

\[
S_{ab}^{xy} u^a_x = 0, \quad S_{ab}^{xy} u^b_y = 0,
\]

and

\[
\mathcal{R}_a^{xy} = \frac{1}{3!} \mu^{ABC} \frac{\partial n_{ABC}^x}{\partial g_{xy}^D} \left( g^{bc} \frac{\partial X^D_x}{\partial x^b} V_a^c \frac{\partial X^E_y}{\partial x^c} \right).
\]

Which is (notably) not guaranteed to be orthogonal to \( u^a_x \).
Finally, we have
\[
\frac{1}{3!}\mu^{ABC} \frac{\partial n_x^{ABC}}{\partial g^{DE}_{y}} \Delta g^{DE}_{y} \\
= \frac{1}{3!}\mu^{ABC} \frac{\partial n_x^{ABC}}{\partial g^{DE}_{y}} \left[ \frac{\partial X^D}{\partial \xi} \frac{\partial X^E}{\partial \xi} \right] = \frac{1}{2} \sum \delta_{\xi}^{ab} \left( g_{\xi}^{a} - g_{\xi}^{b} \right),
\] 
(88)

where we have used
\[
s_{ab}^{xy} = \frac{1}{3!}\mu^{ABC} \frac{\partial n_x^{ABC}}{\partial g^{DE}_{y}} \frac{\partial X^D}{\partial \xi} \frac{\partial X^E}{\partial \xi} = s_{ab}^{xy},
\] 
(89)

and
\[
r_{a}^{xy} = \frac{1}{3!}\mu^{ABC} \frac{\partial n_x^{ABC}}{\partial g^{DE}_{y}} V_a \left( g_{\xi}^{b} \frac{\partial X^D}{\partial \xi} \frac{\partial X^E}{\partial \xi} \right),
\] 
(90)

In this case it is clear that neither \(s_{ab}^{xy}\) nor \(r_{a}^{xy}\) need to be line orthogonal with respect to \(u^a_x\) (although the former is obviously orthogonal to \(u^a_x\)).

Putting all the results together, we arrive at
\[
\mu^a \delta n^a = \mu^a \left( n^a V_b \xi^a - n^b V_b \xi^a - n^a V_b \xi^b \right) + S_{\xi}^{ab} \nabla \xi^a \nabla \xi^b + \sum_{y \neq x} \left( R_{a}^{xy} + R_{a}^{yx} \right) \xi^a + \frac{1}{2} \sum_{y \neq x} \left( S_{ab}^{xy} + S_{ab}^{yx} \right) \xi^a \\
+ \frac{1}{2} \left[ \mu^c \xi^c \delta_{\xi}^{ab} + S_{\xi}^{ab} + \sum_{y \neq x} \left( S_{\xi}^{ab} + S_{\xi}^{ba} \right) \right] \delta_{\xi}^{ab},
\] 
(91)

Using
\[
\sum_{y \neq x} S_{ab}^{xy} \nabla \xi^a \nabla \xi^b = \sum_{y \neq x} \left( S_{ab}^{xy} + S_{ba}^{yx} \right) \nabla \xi^a \nabla \xi^b,
\] 
(92)

this means that equation (39) becomes
\[
\delta \left( \sqrt{-g} A \right) = -\sqrt{-g} \sum_{x} \left( f^x_a + R_{a}^{x} \mu^x + \nabla b D_{ba} - R_{a}^{x} \right) \xi^a \\
+ \frac{1}{2} \left( \psi_{ab}^{xy} + \sum_{x} n_{x} \mu_{x}^{b} + D_{ab} \right) \delta_{\xi}^{ab},
\] 
(93)

where the dissipation tensor of the x-component is
\[
D_{ba}^{x} = S_{ba}^{x} + \sum_{y \neq x} \left( S_{ba}^{xy} + \frac{1}{2} \left( S_{ab}^{xy} + S_{ba}^{yx} \right) \right),
\] 
(94)
while the dissipative stress entering the stress–energy tensor is given by the sum

\[ D_{ab} = \sum_x D_{(ab)}^x = D_{ba} \]  

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not (the third term). This observation, which tends to be overlooked, is important as it links the entropy creation to the vorticity.

At this point, it would be natural to develop a relativistic version of the Onsager argument used in [44, 46, 106]. This involves identifying thermodynamic forces and fluxes, introducing an expansion with respect to an equilibrium state and making use of the relevant symmetries among the introduced coefficients. We will postpone this step for a future effort. It is natural to do so because, so far we have not actually introduced the notion of an equilibrium state and we have certainly not based our analysis on an expansion away from such a state. In other words, the formalism we have developed is still general and nonlinear.

It is perhaps a tribute to the elegance of the variational argument that we managed to get this far without taking what is often seen as one of the first steps of the analysis. However, the Onsager-type argument requires sacrifices and we will be forced to introduce a formal expansion in order to proceed. This will require some care. Further reason for caution comes from the fact that we are working in spacetime. This means that the expansion of the different dissipative contributions will be more complex than in the three dimensional case. There are additional permissible forces/fluxes and the differential structure is richer. In contrast to the Newtonian case, one must tread carefully as there are causality issues to consider. For all these reasons, it is natural to take a break at this point and return to the problem later.

4. Discussion/speculation

We have presented an action principle for general relativistic multi-fluid systems including dissipation. The usefulness of the new formalism is that it can, at least in principle, circumvent ad hoc arguments often used in the traditional approach to the problem of dissipation in general relativistic fluid systems. Admittedly, this may not be the definitive way to incorporate dissipation, but the new scheme is at least coherent and the line of reasoning is conceptually clear. The extension to more complex systems also seems relatively straightforward. This should be an advantage for astrophysical applications which are involving more detailed physics. For example, the coupling to electromagnetism is unambiguous, involving the usual minimal coupling ansatz [93], and it should also be straightforward to account for issues involving polarisable media (although the details remain to be worked out). When it comes to neutron star models, it may be a matter of ‘turning the crank’ to incorporate the elasticity of the outer crust [84–86]. Issues involving anisotropic lattices ought to be easy to accommodate, but extensions to models including say plastic flow still represent a challenge.

It is, of course, not the case that the variational model is complete at this stage. Eventually we would like to turn the proposal into a plug-and-play scheme for relevant applications, but first we need to carry out a careful comparison between our model and the various alternatives. At the same time, one may speculate about potential extensions. In this final section we consider various issues that one might want to consider in more detail and suggest directions in which the model may be extended in the future.

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8 At first sight, this might seem somewhat peculiar and at odds with the basic principles of thermodynamics. However, it is useful to keep in mind that the (conservative) Euler equations are nonlinear (in the same sense as the present model) and do not have an immediate connection with the notion of an expansion away from an equilibrium. Such a connection may exist, although it would have to be at a deeper level.
4.1. Completing the model

The most pressing issue concerns the relationship between the variational model and its various predecessors. It is natural to ask to what extent the new formulation contains the same information regarding the possible dissipation channels as, for example, the celebrated Israel–Stewart construction [10, 23, 24]. A comparison between the two descriptions may seem straightforward, but is in fact not trivial. That this is the case becomes apparent as soon as we note that the variational derivation did not involve an explicit expansion with respect to thermal equilibrium. In fact, we were never required to consider possible equilibrium states at all. This is in sharp contrast to the usual approach that takes an equilibrium as its starting point and then carries out a formal expansion in terms of deviations from this state. There is, of course, nothing that prevents us from expanding the variational results in a similar fashion. In fact, in most situations of practical relevance this may be precisely what one ought to do.

It would seem natural to base such a construction on the standard Onsager approach [40]. This scheme was developed many decades ago, and provides a systematic formalism for determining the number and structure of dissipation channels of a given system. It also sheds light on how these can be woven together so that the second law is guaranteed to be satisfied.

We have already considered this approach in detail in the Newtonian regime [44, 46, 106], and would expect to draw on those results to guide us in the general relativistic context. Even though it derives from a powerful mathematical framework, we must remember that the variational model is phenomenological. In order to apply it to physical systems, we need to connect the macroscopic model with a microscopic analysis. Such a model is required to provide the various transport coefficients, like the thermal conductivity and the various relaxation times. The standard approach to this problem is to resort to kinetic theory, building on a moment expansion for given velocity distributions together with an evaluation of the relevant collision integrals [99, 107]. More recent developments, which may be particularly relevant in the present context since the underlying Lagrangian for the theory is taken as starting point, derive the fluid dynamics from a field theory point of view [107]. Future work needs to explore the connection between our new formulation and those efforts.

It would seem natural to develop the link between the matter-space view of the present analysis and the coarse graining of phase space in statistical physics (see [108] for a potentially relevant discussion). It then becomes relevant to ask at what level the statistics should be considered. Is it at the spacetime level, or is it in the lower-dimensional configuration space? In principle, both answers seem viable but the latter would be an attractive (possibly quite revolutionary) solution. In analogy with the description of elastic matter [64, 65, 70–73] one may envisage a model based on the notion of an evolving ‘thermal geometry’ (in matter space) directly linked to the entropy change between hypersurfaces in spacetime. The model also requires dynamical map between matter space and spacetime, in order to link the changes in the local geometric structure of the matter configuration (described in terms of normal coordinates, say) to the macroscopic evolution of the system.

4.2. Thermodynamical evolution

Since the variational construction does not rely on an expansion away from thermodynamic equilibrium, it retains nonlinearities that may be relevant for a range of considerations. This may lay a foundation for a deeper understanding of nonlinear non-equilibrium thermodynamics and in the extension lead to a framework to discuss the flow of time. As a starting point one might want to establish to what extent the variational model has an interior sense of time, e.g. associated with the constrained entropy evolution. If time is an emergent
phenomenon, how does it depend on the imposed conditions? This is, obviously, a rather deep question but it is clear that the ‘coordinate-free’ representation of the variational approach provides an interesting starting point for a discussion of such foundational issues.

In the case of a dynamical evolution of a general relativistic system, one must consider the role of different observers. This is a non-trivial issue in thermodynamics, closely related to the nature and interpretation of the entropy. Progress on this problem may require experience with the various formalisms for numerical relativity. In fact, one might think that a variation of the 3+1 formalism would be natural in order to represent the internal clock of a system out of equilibrium. Building on the standard framework, one could consider to what extent the spacetime foliations are constrained by the thermodynamics. Are there a set of preferred observers imposed by (say) the entropy flow?

In addition to exploring issues concerning the foliation of spacetime, it would make sense to consider other physical constraints on the model. It is important to establish to what extent a matter/entropy model is constrained by fundamental principles. Take stability as an example. One would obviously expect any physical equilibrium model to be stable. Yet, at the same time one would want a system to exhibit instabilities in order to develop structures. This is another challenging problem. It is important to understand the difference between unphysical instabilities and ones that are expected in a realistic model. Building on the variational model, it would be interesting to investigate the various instabilities that this system exhibits. In connection with this, it would be natural to consider the role of the various energy conditions of general relativity. One would certainly want to understand how these conditions affect the thermodynamics and whether they constrain the evolution of a system.

### 4.3. Hamiltonian formulation—towards quantum aspects

The consideration of boundary terms (ignored throughout our discussion) in the variational approach suggests that an alternative approach to the problem may prove useful. The boundary forms a spacelike two-surface, which will have associated to it two null directions orthogonal to the surface. By considering the extrinsic curvature of the two-surface in these directions one can invariantly define ingoing and outgoing null vectors which implies that the physics may be naturally represented by a 2+2 foliation of spacetime. This formulation has a clear geometric interpretation and it would make sense to explore analogous ideas for the description of thermodynamics. In doing so, we expect to compare and contrast different approaches to the spacetime evolution problem in order to establish the most natural framework for thermodynamical evolutions.

This research direction is entirely within the realm of classical physics. Yet, one would ultimately need to account for quantum aspects. While we make progress at the classical level, we should prepare the ground for future explorations of the quantum arena. In absence of a theory for quantum gravity this is obviously a huge step, but some basic principles seem clear. For example, if we want to discuss quantization then we need to develop a Hamiltonian description for the relativistic thermo/hydrodynamics. This is known to be a challenging problem, but Dirac’s procedure for developing a Hamiltonian system from a given Lagrangian is (at least in principle) clearly laid out. However, the steps involved are far from straightforward in practice. This is particularly true in the case of a constrained variational model, as in the present case. Nevertheless, one should be able to map out this important problem by considering in detail the involved first- and second class constraints.
4.4. Fundamental physics

That gravity and thermodynamics are intimately linked is clear from the equivalence of energy and mass, which implies that heat must affect the gravitational field. Nevertheless, from a conceptual point of view it is not understood to what extent the gravitational field is ‘hot’. Basically, we do not yet have an operational definition of the entropy associated with an evolving gravitational field [109]. This is a long-standing problem. The variational model for heat accounts for the coupling to (and evolution of) the gravitational field via the Einstein field equations. However, so far the main focus has been on the matter sector of the problem. It would be interesting to broaden the discussion and explore the role of the gravitational field in more detail. The aim would be to establish how the second law of thermodynamics feeds into the evolution of the gravitational field, and (conversely) how variations in the gravitational field affect the entropy and the heat flow. A conceptually interesting issue concerns the link between observers and the increase of gravitational entropy, and the (obvious) link to the microstates associated with a black hole’s event horizon.

The variational approach provides the foundation for the exploration of a range of relevant issues. Let us comment on three, perhaps particularly topical, issues;

1. The 2+2 approach (discussed above) has interesting connections with the Ashtekar formulation of quantum gravity, using the description due to Jacobson and Smolin based on self-dual two-forms [110–112]. The key feature of using a null foliation is that the Hamiltonian constraint is no longer first class and the remaining first class constraints form a Lie algebra [113].

2. The notion of a two-dimensional boundary is obviously very similar to the ideas behind gauge/gravity duality and the holographic principle. In fact, the use of a null foliation plays a key role in Jacobson’s derivation of the Einstein equations from thermodynamic principles [114]. Given this, it would make sense to develop the connection with dissipative holographic fluid dynamics. Many such models consider the fluid limit of conformal field theories, starting from a suitable Lagrangian and generating dissipative terms by a formal derivative expansion [49, 51, 115]. This has led to the identification and exploration of dissipative terms that were not present in the classic Israel–Stewart construction. This is exciting progress, but it is important to keep in mind that much of the holographic-fluid program has so far focussed on rather unphysical models, e.g. systems with conformal symmetry. Nevertheless, the discussion promises a deeper understanding in the future, and it would make sense to investigate the connection between our new dissipative variational approach and the quantum field theory-led holography models in detail. This is particularly interesting since the gauge/gravity approach may provide insight into the microphysics origin of the various dissipation channels.

3. The Hamiltonian formulation of the problem [74, 75, 116–119] would allow us to make direct contact (and build upon) the notion of thermal time [3, 120] (associated with the evolution of pre-symplectic systems). So far, this concept has been developed for systems in thermal equilibrium (for which there is a clear description). It is relevant to ask how the concept is altered by non-equilibrium effects. This is a natural problem to consider given that the thermodynamic arrow of time relies on the second law (irreversibility), and hence ‘applies’ only to non-equilibrium systems.

This list of topics and issues formulates an ambitious research programme based on the new variational model for dissipative systems. Some of the problems are clearly achievable, and one might expect to make swift progress on them. Other problems are more speculative.
and foundational in nature. These targets may be much harder to reach, but at least the new model provides a fresh approach that may lead to the development of interesting perspectives.

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