Time asymmetries in quantum cosmology and the searching for boundary conditions to the Wheeler-DeWitt equation

Mario Castagnino*, Gabriel Catren† and Rafael Ferraro‡

Instituto de Astronomía y Física del Espacio,
Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina

Departamento de Física, Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I,
1428 Buenos Aires, Argentina

Abstract
The paper addresses the quantization of minisuperspace cosmological models by studying a possible solution to the problem of time and time asymmetries in quantum cosmology. Since General Relativity does not have a privileged time variable of the newtonian type, it is necessary, in order to have a dynamical evolution, to select a physical clock. This choice yields, in the proposed approach, to the breaking of the so called clock-reversal invariance of the theory which is clearly distinguished from the well known motion-reversal invariance of both classical and quantum mechanics. In the light of this new perspective, the problem of imposing proper boundary conditions on the space of solutions of the Wheeler-DeWitt equation is reformulated. The symmetry-breaking formalism of previous papers is analyzed and a clarification of it is proposed in order to satisfy the requirements of the new interpretation.

PACS 98.80.Hw, 04.20.Cv

*castagni@iafe.uba.ar
†catren@iafe.uba.ar
‡ferraro@iafe.uba.ar
1 Introduction

The so called ‘problem of time’ in quantum gravity is one of the main conceptual and technical problems of the canonical quantization program (Ref.[1],[2],[3]). This quantization program begins by reformulating General Relativity under a Hamiltonian formulation (ADM formalism, see [4]). Within the framework of this formalism the Lorentzian space-time manifolds $M$ are split in a collection of spacelike hypersurfaces $\sum$ parametrized by a real time parameter $t$ (foliation of the space-time). The role of the canonical variables is played by the Riemannian metrics $g_{ij}$ of these hypersurfaces $\sum$. The corresponding configuration space is the space of all the possible Riemannian metrics $g_{ij}$ which is called superspace. The conjugated momenta $\pi^{ij}$ are related to the extrinsic curvature of the hypersurfaces $\sum$, i.e., to the way in which these hypersurfaces are embedded in the space-time manifold $M$. In this Hamiltonian formulation the general covariance of the theory appears as a set of constraints among the canonical variables (four constraints per each point of space-time). The so-called Hamiltonian constraint assures the invariance of the theory under changes of the foliation of the space-time. The momenta constraints (three per each point of space-time) assure the invariance of the theory under a change of the spatial coordinates used to represent the spatial geometry of each hypersurface $\sum$. The existence of the Hamiltonian constraint means that the theory does not select a privileged time variable. To consider General Relativity as a dynamical system it is thus necessary to choose a physical clock, i.e., a physical degree of freedom with suitable properties, to play the role of time. On the contrary, in quantum mechanics there is a privileged time variable which is an evolution parameter clearly separated from the other degrees of freedom which are associated with quantum operators. This difference between General Relativity and quantum mechanics is the main problem for finding a quantum theory of gravity in the framework of the canonical approach.

In this paper the choice of a physical clock will be treated as a symmetry breaking of a certain kind of time symmetry (clock-reversal transformations) which is induced by the double-sheet structure of the Hamiltonian constraint. It will be addressed the case of a time independent reduced Hamiltonian, leaving for another work the treatment of time dependent Hamiltonians. In section II the parametrized system formalism is reviewed and its conceptual problems are discussed. A reconceptualization of this formalism is proposed in order to solve these problems. In section III an example of a reducible minisuperspace model is presented. In section IV the fundamental distinction between clock-reversal transformations and motion-reversal transformations is proposed for time independent reduced Hamiltonians in order to solve an apparent paradox of the worked approach. Its classical and quantum versions are defined. In section V the consequences of our approach for the problem of the boundary conditions on the space of solutions of the Wheeler-DeWitt equation are studied. In section VI a general formalism to study irreversible processes via a symmetry breaking
framework is reviewed and its adaptation for the case of General Relativity is presented in order to fix the proposed approach in a mathematical framework.

2 Parametrized systems formalism

One of the main properties of the Hamiltonian structure of General Relativity is the presence of the Hamiltonian constraint $H = 0$. As stated above this constraint means that the action does not depend on the foliation chosen to describe the “evolution” in the Hamiltonian formulation, this invariance being a consequence of the covariance of the theory. A well known formalism which has this kind of invariance is the parametrized system formalism (Ref.[1], [5]). In a parametrized system the absolute time $t$ is added to the dynamical variables leaving this increased set of dynamical variables as functions of an ad-hoc introduced irrelevant parameter $\tau$. This kind of systems are frequently used as a paradigm for understanding the Hamiltonian structure of General Relativity. Let us start with an action of the form

$$S[q^\mu, p_\mu] = \int_{t_1}^{t_2} p_\mu dq^\mu - h(q^\mu, p_\mu) dt, \quad \mu = 1, \ldots, n$$

The original set of dynamical variables $\{q_\mu, p_\mu\} (\mu = 1, \ldots, n)$ is extended by identifying $q^0 \equiv t$, $p_0 \equiv -h$. The new set of variables are left as functions of an irrelevant parameter $\tau$. The extended set $\{q^0, q^\mu, p_0, p_\mu\}$ can be varied independently, provided that the definition of $p_0$ is incorporated into the action as a constraint

$$H = p_0 + h = 0 \quad (1)$$

with the corresponding Lagrange multiplier $N$ yielding the following action

$$S[q^i(\tau), p_i(\tau), N(\tau)] = \int_{t_1}^{t_2} \left(p_i \frac{dq^i}{d\tau} - NH\right) d\tau \quad (2)$$

The presence of the Lagrange multiplier $N(\tau)$ means that the dynamics remains ambiguous in the irrelevant parameter $\tau$ (one could say that it has no sense to speak about dynamics until the hidden time is recovered). A constraint like this can be disguised by scaling it with a function $f(q, p)$ of a definite sign on the constraint surface or by performing a canonical transformation

$$\{q^i, p_i\} = \{q^0 = t, p_0 = -h, q^\mu, p_\mu\} \rightarrow \{Q^i, P_i\}$$

where now the time is hidden among the rest of the dynamical variables.

A main argument against this interpretation of the Hamiltonian structure of General Relativity (Ref.[6]) is that its Hamiltonian constraint is quadratic in all its momenta while the Hamiltonian constraint of a parametrized action (1) is linear in the momentum conjugated to time. In fact the parametrized system formalism is not completely fitted to describe General Relativity because
This formalism supposes that an absolute external time is hidden among the dynamical variables. In this approach to reduce the system means to find the hidden time by performing canonical transformations or scaling the Hamiltonian constraint in order to find a constraint linear in the momentum canonically conjugated to time. We think that the existence of this hidden time is an unfounded supposition based on an extrapolation of our experience in other branches of physics in which there is always an external time parameter. The most singular feature of modern Cosmology is that it studies a system which, by definition, does not live in an external scenario. We think that the corresponding physics of this kind of auto-contained systems must depart in many fundamental ways from usual physical theories. In particular, the supposition that there is a privileged time hidden among the canonical variables represents a tentative to reduce General Relativity to the usual pattern of what a physical theory is supposed to be. We consider that one of the most fundamental properties of General Relativity is that its solutions do not represent, in general, a time evolution of certain dynamical variables, but that it is a theory which selects certain relative (not dynamical) configurations of its canonical variables which, under certain conditions, could be considered as dynamical evolutions if proper physical clocks can be selected. It is thus only possible to speak about physical clocks, i.e., degrees of freedom which can play the role of evolution parameters for the others degrees of freedom. The only requirement to be satisfied by a degree of freedom in order to be a proper physical clock is that it should be possible to express any relative configuration of the n canonical variables as n-1 functions of the variable \( q_0 \) chosen as the physical clock. As many authors have pointed out (Ref. [6], [7]), we can never observe the evolution along the newtonian time flow like \( q_1(t) \) and \( q_2(t) \) but rather the evolution of certain variables relative to the change of another variable, i.e., something like \( q_2(q_1) \). In this relational approach we cannot say that reducing the system means to find the hidden time but that reducing the system means to select, among the canonical variables, a proper physical clock. A main consequence of this subtle and fundamental change in the perspective is that, as there is not a privileged time, all the momenta must appear quadratically in the Hamiltonian constraint, i.e., all the momenta must appear on an equal footing (as effectively happens in General Relativity), being this an essential fact of the theory which turns the parametrized system formalism an improper analogy. It is thus necessary to reformulate the model which is intended to mimic General Relativity, in order to properly describe this substantial difference. This reformulation must accomplish the requirement that all the canonical momenta must appear quadratically in the Hamiltonian constraint in order not to privilege a certain clock among others. There is even another and more important reason which, if we follow this new interpretative framework,

\[\text{In 1918 the philosopher Ludwig Wittgenstein wrote in the 6.3611 proposition of his \textit{Tractatus Logico-philosophicus}: \"We can no compare any process with \textquote{the flow of time} -which does not exist-, but with another process (as the motion of a chronometer, for example). Therefore the description of the flow of time is only possible using another process.\}\]
turns essential the fact that the Hamiltonian constraint must be quadratic in all its momenta. If there is not a privileged time the solutions are necessarily statics trajectories, i.e., relative configurations among the different variables, for example $q_2 (q_1)$. If one wants now to select a physical clock, for example $q_1$ (we are supposing that $q_1$ is a monotonic function along the trajectory) there is still an ambiguity, i.e., one still have to choose in which direction the trajectory is being unfold. This means that one can choose $t = q_1$ or $t = -q_1$. The static trajectory does not privilege any direction and so both kind of solutions must appear in the reduced formalism. We will show that, for reducing the system, one has to separate the Hamiltonian constraint in two sheets corresponding each sheet to each choice of the direction in which the trajectory is unfold. In order to make this factorization the Hamiltonian constraint must be quadratic in the momentum conjugated to $q_1$. This is the main difference between our approach and the parametrized system formalism. In this last framework the real time was certainly hidden with its direction of evolution, being thus unnecessary the presence of the other sheet. If, on the contrary, one begins with an static configuration both directions must appear. In this new light the Hamiltonian constraint of General Relativity not only implies that the theory is invariant under a change of the chosen foliation of space-time but also that it is invariant under an inversion in the direction in which the corresponding hypersurfaces of simultaneity are unfold.

We will then suppose that the Hamiltonian of the model under study can be taken (using a suitable canonical transformation and/or scaling the Hamiltonian) to the form

$$H = (p_0 + h) (p_0 - h) = p_0^2 - h^2 (q^\mu, p_\mu)$$

Imposing the constraint $H = 0$ is equivalent to choose a sheet of the constraint surface, i.e., to select a direction for the variable $q_0$ which will play the role of time in the reduced formalism. The action (2) expressed in these new variables is

$$S [q^i (\tau), p_i (\tau), N (\tau)] = \int p_\mu dq^\mu + p_0 dq^0 - N (p_0 + h) (p_0 - h) d\tau$$

The Hamilton equation for $q_0$ is

$$\frac{dq_0}{d\tau} = N \frac{\partial H}{\partial p_0} = 2N p_0$$

As $h > 0$ then $p_0$ never vanishes on the constraint surface. This implies that $q_0$ can be made a monotonous function of $\tau$ along each trajectory by a proper gauge choice. In this way $q_0$ acquires the rank of an internal clock. In fact, we will choose the direction of time $t$ as the increasing direction of the variable
We can do it by means of the gauge fixing condition \( t \equiv q^0 = \tau \), which is equivalent to choose the Lagrange multiplier \( N \):

\[
\frac{dq^0}{d\tau} = \frac{dt}{d\tau} = 2Np_0 = 1
\]

or

\[
N(\tau) = \frac{1}{2p_0(\tau)}
\]

Then the action takes the form

\[
S\left[q^i\left(q^0\right), p_i\left(q^0\right)\right] = \int p_\mu dq^\mu + p_0 dq^0 \frac{1}{2p_0} (p_0 + h) (p_0 - h) dq^0
\]

In order to finish the reduction process one has to deduce in which sheet of the constraint surface one is working: \( p_0 + h = 0 \) or \( p_0 - h = 0 \). The chosen gauge fixing condition \( t \equiv q^0 = \tau \) means that the chosen sheet is \( p_0 + h = 0 \). The other sheet does not yield the action to the standard form of a non parametrized system

\[
S\left[q^\mu\left(q^0\right), p_\mu\left(q^0\right)\right] = \int p_\mu dq^\mu - h(q^\mu, p_\mu) dt
\]

with a positive reduced Hamiltonian \( h(q^\mu, p_\mu) > 0 \).

The constraint \( p_0 + h = 0 \) means that the chosen \( N \) is

\[
N = -\frac{1}{2h}
\]

If one had chosen the decreasing direction of \( q^0 \) as the increasing direction of time, i.e., \( t \equiv -q^0 = \tau \), the gauge fixing condition would have been

\[
\frac{dq^0}{d\tau} = -\frac{dt}{d\tau} = 2Np_0 = -1
\]

The condition \( \frac{dq^0}{d\tau} = -1 \) means now that the Lagrange multiplier is

\[
N = -\frac{1}{2p_0}
\]

In order to take the system to the standard reduced form with a positive reduced Hamiltonian \( h(q^\mu, p_\mu) > 0 \) one has to work on the sheet \( p_0 - h = 0 \). This has as a consequence that \( N \), as a function of the reduced variables \( (q^\mu, p_\mu) \), is still

\[
N = -\frac{1}{2h}
\]

and one reobtains (4).
3 Minisuperspace example

In the literature about minisuperspace models it can be found many examples of reducible models, i.e., cosmological models where a physical clock can be separated from the rest of the dynamical variables (Ref. [3]). These models can be classified in those where time is only a function of the configuration variables (intrinsic time) and those where time is a function of the phase space variables (extrinsic time). The Friedmann-Robertson-Walker universe for \( k = 0, -1 \) with cosmological constant coupled with a massless scalar field and the Kantowski-Sachs model are examples of the first kind (with time dependent reduced Hamiltonians). The Taub model is a particularly interesting case because it does not have an intrinsic time but can be reduced by an extrinsic time with a time independent reduced Hamiltonian. The Taub model represents an homogeneous but anisotropic universe. The corresponding configuration space (minisuperspace) is a two dimensional manifold parametrized by a parameter \( \beta \) measuring the spatial anisotropy and a parameter \( \Omega \) measuring the volume of the Universe. The Hamiltonian constraint for this model is

\[
H = -p_\Omega^2 + p_+^2 + 12\pi^2 e^{-4\Omega}(e^{-8\beta} - 4e^{-2\beta})
\]  

(5)

while the momenta constraint are identically satisfied. This constraint does not have a positive potential an it is thus not possible to appreciate the double sheet Hamiltonian structure of the constraint surface. The reduction of the Taub universe was studied in Ref. [9]. By means of the coordinate transformation

\[
\begin{align*}
\Omega &= v - 2u \\
\beta_+ &= u - 2v 
\end{align*}
\]

the Hamiltonian constraint can be written as

\[
H = \frac{1}{6} (p_v^2 + 36\pi^2 e^{12v}) - \frac{1}{6} (p_u^2 + 144\pi^2 e^{6u})
\]  

(6)

Performing the canonical transformation

\[
\begin{align*}
q &= \text{Arc sinh} \left(-\frac{p_v}{6\pi e^{-6v}}\right) \\
p_q^2 &= \frac{1}{36} (p_v^2 + 36\pi^2 e^{12v})
\end{align*}
\]

whose generating function is

\[
F_1 (v, q) = -\pi e^{6v} \sinh q
\]

it is possible to take the constraint to the form

\[
H = 6p_q^2 - \frac{1}{6} (p_u^2 + 144\pi^2 e^{6u})
\]  

(7)

\[7\]
In this way a physical clock $q$ was separated with a reduced Hamiltonian $h$ which does not depend on time $q$. It is now necessary to choose a direction of $q$ for the increasing direction of time. The last expression can be factorized in the form of (8)

$$H = \left( \sqrt{6}p_q + \frac{1}{\sqrt{6}} \sqrt{p_q^2 + \pi^2 e^{6u}} \right) \left( \sqrt{6}p_q - \frac{1}{\sqrt{6}} \sqrt{p_q^2 + 144\pi^2 e^{6u}} \right)$$

The constraint $H = 0$ is fulfilled if one of the factors vanishes on the constraint surface. To choose which factor is null is equivalent to choose which direction of $q$ is the increasing direction of time. The other factor has, on the constraint surface, a definite sign, so being possible to rescale the Hamiltonian by this factor. In Ref. [9] the increasing direction of $q$ was selected as time, i.e., $q = t$.

4 Clock-reversal and motion-reversal transformations

In some sense one could say that each choice ($t = q$ or $t = -q$) corresponds to a kind of time reversal of the other one. If this were the case the choice of the direction of time would be like a breaking of the time-reversal symmetry of the original theory. But one knows that each sheet of the Hamiltonian constraint contains a classical system with the well known classical and quantum symmetries under time reversals. This point is subtle and deserves special attention in order to circumvent this apparent paradox. Classical mechanics is a theory which is said to be invariant under time reversals. By this one means that, given a classical trajectory $\{q(t), p(t)\}$ which unfolds between $\{q_0, p_0\}$ at time $t_0$ to $\{q_f, p_f\}$ at time $t_f$, there exists another trajectory which seems to be the time reversal of the former, and which is also a solution of the Hamilton equations. This inverted trajectory is

$$q^{mr}\left(q_0^{mr} = q_f, p_0^{mr} = -p_f, t_0, t\right) = q(q_f, -p_f, t_0, t) \quad (9)$$

$$p^{mr}\left(q_0^{mr} = q_f, p_0^{mr} = -p_f, t_0, t\right) = p(q_f, -p_f, t_0, t)$$

and exists provided that the Hamiltonian is quadratic in $p$ and does not depend on $t$ (the meaning of the superindex $\text{mr}$ will be explained below). It is often said that the operation of passing from a certain trajectory $\{q(t), p(t)\}$ to the one defined by (9) is like “playing the film backwards”. Actually this assertion does not do enough justice to the solution (9) because it darkens the role of the clock: if the movie is played backwards one would see also the hands of the clock running backwards. Of course the solution (9) refers to a clock going forward, but with initial conditions which have been inverted with respect to the original trajectory: the new trajectory starts with an inverted velocity from the point where the original one ends, but it starts at the same time.
than the original one and unfolds in the same direction of time. We will call the operation (9) a *motion-reversal transformation* (this is the reason for the superindex \( mr \) in (9)). It is a remarkable fact that the double sheet Hamiltonian constraint surface induces a different kind of time symmetry: passing from one sheet to the other one is equivalent to the change \( t \to -t, \ h \to -h \). We reserve the name of *clock-reversal transformation* for this second kind of time symmetry. The motion-reversal transformation represents a motion with the direction of unfolding of all the canonical variables inverted but the one used as a physical clock, while the clock-reversal transformation represents a motion with the evolution of all the variables inverted including the one representing the physical clock. Summarizing, each solution has its corresponding motion-reversed solution on the same sheet and both of these motions are connected by a clock-reversal transformations with a companion pair on the other sheet. In order to fix ideas let us suppose a dynamical system composed of two variables \( q_1 \) and \( q_2 \) with a Hamiltonian constraint \( H(q_1, q_2, p_1, p_2) = 0 \). Without loss of generality let us suppose that, in a particular solution, the representative configuration point \((q_1, q_2)\) makes a motion passing by \((q_1 = -1, q_2 = A)\) and \((q_1 = 1, q_2 = B)\). As we still did not choose a physical clock this is not really a motion but a static trajectory. Let us suppose that \( q_1 \) behaves as a physical clock, i.e., that there is no two values of \( q_2 \) for the same \( q_1 \). As was said before one has two options: \( t = q_1 \) or \( t = -q_1 \). Let us suppose that the increasing direction of \( q_1 \) is chosen as time, i.e., that \( t = q_1 \). It is only now that one can say that the dynamical variable \( q_2 \) is moving from \( A \) to \( B \) as the time \( t = q_1 \) flows (figure 1(a)). From this “original solution” one can construct three others solutions which corresponds to the motion-reversal of the original one (figure 1(b)), the clock-reversal of the original one (figure 1(c)) and the clock-reversal of the motion-reversal of the original one (figure 1(d)). In fact one could find the so called motion-reversal trajectory of the original solution defined in (9). This trajectory goes from \((q_1 = -1, q_2 = B)\) to \((q_1 = 1, q_2 = A)\). In the configuration space \((q_1, q_2)\) this is another trajectory which solves the Hamilton equations and for which time \( t \) is still increasing in the direction of the increasing \( q_1 \), i.e., the dynamical variable \( q_2 \) moves now from \( B \) to \( A \) as the time \( t = -q_1 \) flows (figure 1(b)). Let us suppose now that we choose the decreasing direction of \( q_1 \) as time, i.e., that \( t = -q_1 \). This choice lead us to the clock-reversals of the former solutions. For example the figure 1(c) represents the clock-reversal of 1(a). The dynamical variable \( q_2 \) moves now from \( B \) to \( A \) as the time \( t = -q_1 \) passes. It is now that the representative point is traveling the original trajectory in the opposed direction. The motion-reversal (9) of this last solution is equivalent to the clock-reversal of the motion-reversal of the original solution (figure 1(d) is the motion-reversal of 1(c) and the clock-reversal of 1(b)). This example should clarify the difference between the motion-reversal operations (9) used in classical and quantum mechanics and the passage from one sheet of the Hamiltonian constraint to the other one (clock-reversal operations). The confusion between this two operations is rooted in the fact that we are used to
think the problem in the reduced configuration space which is like looking at films in which the motions of the hands of the clock have not been recorded. In the reduced configuration space (the axis $q_2$ in the example above) these four related solutions reduces to two and this substantial difference degenerates.

### 4.1 Classical transformations

The Hamilton equations for the reduced variables are

$$
\frac{dq^\mu}{d\tau} = N \frac{\partial H}{\partial p_\mu}, \\
\frac{dp_\mu}{d\tau} = -N \frac{\partial H}{\partial q^\mu}
$$

Choosing $q$ as time, i.e., fixing $t \equiv q = \tau$ the Hamilton equations take the form

$$
\frac{dq^\mu}{dq} = \frac{\partial h}{\partial p_\mu}, \\
\frac{dp_\mu}{dq} = -\frac{\partial h}{\partial q^\mu}
$$

(10)

As it was said before it is known that, given a certain trajectory the motion-reversal trajectory (9) is also a solution of the Hamilton equations of motion. We will now define the clock-reversal solution $\{q^{\mu^c r}, p^{\mu c r}\}$ by noting that it is equal to the motion-reversal one plus an inversion of the physical clock $t \rightarrow t^{cr} = -t$

$$
q^{\mu^c r} \begin{pmatrix} q_0^r \\
p_\mu^c \end{pmatrix} = q_f, p_{\mu_0}^c = -p_f, t_0^c = -t_f, t^c = -t
$$

(11)

$$
p^{\mu c r} \begin{pmatrix} q_0^c \\
p_\mu^c \end{pmatrix} = q_f, p_{\mu_0}^c = -p_f, t_0^c = -t_f, t^c = -t
$$

These functions do not satisfy the Hamilton equations (10). These functions do belong to the space of solutions of the other sheet $p - h = 0$, i.e., they are solutions for the other choice of the direction of time ($t \equiv -q = \tau$). In fact these clock-reversed solutions satisfy the equations

$$
\frac{dq_\mu}{d(-q)} = \frac{\partial h}{\partial p_\mu}, \\
\frac{dp_\mu}{d(-q)} = -\frac{\partial h}{\partial q_\mu}
$$

which are the Hamilton equations corresponding to the choice

$$
q = -t \\
p = h
$$
4.2 Quantum transformations

Given a particular solution $|\Psi (t)\rangle$ to the Schrödinger equation of a quantum system its motion-reversed solution (usually called in the literature “time-reversed” solution for the same reasons mentioned before) is given by

$$|\Psi_{mr} (t)\rangle = T |\Psi (-t)\rangle$$  \hspace{1cm} (12)

where $T$ is an antiunitary operator which, in coordinate representation, is equal to the complex conjugation operator (Ref. [10])

$$T \Psi (q) = \Psi^* (q)$$

The transformation (12) is the quantum version of the classical motion-reversal transformation (9) which means that the transformed solution $|\Psi_{mr} (t)\rangle$ is a solution for the same Schrödinger equation. For example, in the case of a quantum state $\Psi (x, t) = e^{-i(\omega t - kx)}$ corresponding to a free particle, the transformation (12) yields $\Psi_{mr} (x, t) = e^{-i(\omega t + kx)}$ which corresponds to a state with the same energy, unfolding in the same direction of time, but with the linear momentum reversed.

We will now define the quantum version of the classical clock-reversal transformation (11) as

$$|\Psi_{cr} (t)\rangle = T |\Psi (t)\rangle$$

In fact, given a solution $|\Psi (q)\rangle$ of the Schrödinger equation

$$i \frac{\partial}{\partial q} |\Psi (q)\rangle = h |\Psi (q)\rangle$$  \hspace{1cm} (13)

corresponding to the quantization on the sheet $p + h (q, p) = 0$ ($t = q$) with the substitution $p_i \rightarrow -i \frac{\partial}{\partial q_i}$, the time reversed solution $T |\Psi (q)\rangle$ is not a solution of (13), but a solution of the Schrödinger equation in the time $t = -q$:

$$-i \frac{\partial}{\partial q} T |\Psi (q)\rangle = h T |\Psi (q)\rangle$$  \hspace{1cm} (14)

corresponding to the quantization on the sheet $p - h (q, p) = 0$. In fact, let us apply the operator $T$ to both sides of (13)

$$-i \frac{\partial}{\partial q} T |\Psi (q)\rangle = T h T^{-1} T |\Psi (q)\rangle$$  \hspace{1cm} (15)

Assuming that the reduced Hamiltonian $h$ is real (quadratic in $p_\mu$) this equation yields

$$-i \frac{\partial}{\partial q} T |\Psi (q)\rangle = h T |\Psi (q)\rangle$$

which shows that $T |\Psi (q)\rangle$ is a solution of (13).
5 The Wheeler-DeWitt equation

In the framework of the canonical quantization program the physical states of the corresponding quantum theory of gravity are functionals of the spatial metric $g_{ij}$, which satisfy the quantum version of the classical constraints in accordance with the Dirac method for quantifying constrained Hamiltonian systems. The quantization of the momenta constraints implies that the physical states depends on the geometry $g^3$ of the hypersurfaces but not on the particular metric tensor $g_{ij}$ used to represent it. The quantum version of the Hamiltonian constraint is the so called Wheeler-DeWitt equation $\hat{H}\Psi = 0$.

It was pointed many times the analogy between the Wheeler-DeWitt equation and the Klein-Gordon equation: both systems have Hamiltonians which are hyperbolic in the momenta. The space of solutions of the Klein-Gordon equation can be turned into a Hilbert space where a subspace with a positive definite inner product can be defined only if the background is stationary. In this case the Hilbert space of the physical states will be the subspace of positive norm, this being equivalent to consider just one of the sheets of the hyperbolic constraint surface. Beyond that similarity there is an important difference between both equations: the Wheeler-DeWitt equation does not have a physical clock in the configuration space because it does not have a positive definite potential term to play the role of the mass term of the Klein-Gordon equation. The double sheet Hamiltonian structure for the Wheeler-DeWitt equation should therefore be searched in the phase space. In that case a canonical transformation should be implemented in order to translate this double sheet Hamiltonian structure in the phase space of the original canonical variables to the configuration space of the new canonical variables. As shown in Section 3 this has been successfully done for the Taub model (Ref. [9]). Following the analogy between both equations it was supposed in Ref. [9] that each sheet of the Hamiltonian constraint $p + h = 0$ and $p - h = 0$ corresponds to positive and negative energies respectively. In the new approach of this paper each sheet corresponds to each possible choice in the direction in which the static trajectory unfolds, being in both cases positive energy solutions. Actually the stationary solutions to the Schrödinger equation \ref{eq:13} are

$$\Psi_E (t = q, q_\mu) = e^{-iE\varphi} (q_\mu) = e^{-iEt} \varphi (q_\mu)$$

while the stationary solutions to the Schrödinger equation \ref{eq:14} are

$$\Psi E (t = -q, q_\mu) = e^{iE\varphi} (q_\mu) = e^{-iEt} \varphi (q_\mu)$$

which shows that both sets of solutions are positive energy solutions for the two defined times.

It is important to notice that the presence of a square root reduced Hamiltonian due to the factorization of the Hamiltonian constraint leads to a canonical
quantization procedure which is not straightforward. The definition of the operators associated with this kind of reduced Hamiltonians can be done in two steps: it is necessary to define the operator under the square root in order to define, in a second step, the square root itself by means of the spectral theorem (Ref. [1], [11]). This can be done if the operator under the square root is a positive definite self-adjoint operator. While this could be done for the Taub model there is no guarantee that this procedure could be applied to a general case.

5.1 Boundary conditions for the Wheeler-DeWitt equation

It is a fundamental problem in Quantum Gravity to find proper boundary conditions in the space of solutions of the Wheeler-DeWitt equation in order to select the physical solutions. The Schrödinger equation is a parabolic equation while the original Wheeler-DeWitt equation is an hyperbolic one, having thus twice the number of solutions than the former. As the Taub model teaches (Ref. [9]), the connection between the Schrödinger equation and the Wheeler-DeWitt equation is not straightforward, because the non positive definite potential that typically appears in the last (see also Ref. [12]). A canonical transformation is necessary in order to find a Hamiltonian constraint of the form $H = p_0^2 - h^2(q^\mu, p_\mu)$ with a well defined reduced Hamiltonian $h$. The system can thus be quantized by means of the corresponding Schrödinger equation associated with one of the sheets of the constraint surface (which is equivalent to chose the solutions associated with the breaking of the clock reversal symmetry). In Ref. [9] it was shown that if the proper canonical transformation to reduce the system is known, is possible to provide a criterium to select the physical solutions of the Wheeler-DeWitt equation. The proposed formalism chooses the solutions of the Wheeler-DeWitt equation which corresponds to the solutions of the Schrödinger equation for the reduced system. In order to apply this criterium it is necessary to find a correspondence between both spaces of solutions. If this correspondence could be defined it would be possible to transform the solutions of the Schrödinger equation finding in this way the corresponding physical subspace in the space of solutions of the Wheeler-DeWitt equation. This amounts to find a quantum correspondence for relating the wave functions corresponding to a pair of quantum-mechanical systems whose classical Hamiltonians are canonically equivalent. In certain cases this quantum correspondence between both representations can be defined (Ref. [9], [13]) as

$$\Psi (q) = N (E) \int_{-\infty}^{+\infty} dQ e^{iF(q, Q)} \Phi (Q)$$

(16)

where $F(q, Q)$ is the generating function of the corresponding canonical transformation. This is a generalization of the Fourier transformation considered as
the quantum version of the canonical transformation generated by $F(q,Q) = qQ$. Using this quantum correspondence it was possible in Ref. [9] to transform the solutions of the Schrödinger equation, so finding the physical solutions of the Wheeler-DeWitt equation. This procedure amounts to select boundary conditions for the solutions of the Wheeler-DeWitt equation which are associated with the direction of time of the chosen physical clock. In this approach the boundary conditions operates as a symmetry breaking of the original invariance of the theory under a clock-reversal transformation. The relation between proper time $T$ and the time variable $t$ chosen as the physical clock is $dT = Ndt$ where $N$ is the Lagrange multiplier for the Hamiltonian constraint (or lapse function). This is a consequence of the way in which the space-time interval is expressed in the ADM formalism (Ref. [4]). The proposed approach is thus completely different from those in which classical proper time is recovered in a semiclassical regime (Ref. [15],[14]). In these approaches a notion of time is associated with ”... the affine parameter along the histories about which the wave functions is peaked. So time, and indeed spacetime, are only derived concepts appropriate to certain regions of configuration spacetime and contingent upon initial conditions” (Ref. [14]). On the contrary in the proposed approach there is a perfectly defined notion of time at the quantum level which plays the same role as the usual time parameter of the Schrödinger equation in ordinary systems.

The proposed boundary conditions relies on the fact that one knows how to find the reduced variables in the classical level, i.e., how to separate a physical clock from the whole set of variables. It would be a great step if one could apply the underlying physical intuition of this criterium without knowing how to reduce the system, i.e., without having separated a physical clock. In fact a fundamental objection against the reduction formalism applied to canonical quantum gravity is that, if one considers that quantum mechanics is a more fundamental theory than classical mechanics, it is not correct to define the quantum theory using a time variable which was selected by a classical criterium (the physical clock should be a variable which monotonically increases along the classical trajectories). In Ref. [9] it was possible to advance in this direction by proposing a criterium of this kind for time independent reduced Hamiltonian systems. After performing a change of coordinates the Hamiltonian constraint could be taken to the form

$$H = p^2 + V(q) - h_{\mu} (q_{\mu}, p_{\mu})$$

(17)

The basic fact of the proposed approach is that in the region where the potential $V(q)$ goes to zero the variable $q$ is the physical clock, i.e., in that region the Hamiltonian (17) goes to

$$H = p^2 - h_{\mu} (q_{\mu}, p_{\mu})$$

(18)

The boundary conditions to be imposed to the solutions of Wheeler-DeWitt equation corresponding to the quantization of (17) is that its physical solutions
should tend, in the region where the potential \( V(q) \) goes to zero, to the solutions of the Schrödinger equation corresponding to (18), i.e., to functions of the form 
\[
\varphi(q, q_\mu) = \phi(q_\mu) e^{-i\sqrt{\varepsilon}q}.
\]
In this way, just by analyzing the asymptotic form of the solutions of the Wheeler-DeWitt equation, it is possible to select the physical solutions with respect to the chosen physical clock. This kind of boundary conditions for the Wheeler-DeWitt equation is similar to those proposed (although reached by using other methods) in Ref. [16].

In the light of the new interpretative approach, where the choice of the sheet is equivalent to a symmetry breaking of the clock-reversal invariance of the theory, this kind of boundary condition can be reformulated in order to find out the way to generalize it. The physical meaning of the proposed boundary conditions is to separate the wave functions going forward in the time \( t = q \) from those going forward in the time \( t = -t = -q \). These two subspaces of solutions of the Wheeler-DeWitt equation are related to each other by the antiunitary operator \( T \) corresponding, in the coordinate representation, to the complex conjugation operator. The main idea is that the breaking of the clock-reversal invariance by selecting quantum states which belong to just one of these subspaces could be a general criterion for selecting a physical subspace. The real character of the Wheeler-DeWitt operator means that, given a solution \( \Psi(q) \), the function \( \Psi^*(q) \) is also a solution, these solutions being linear independent. This means that the space of solutions of the Wheeler-DeWitt equation \( S \) can be decomposed as \( S = C \oplus C^* \) where \( C \) is a subspace of \( S \). A general solution to the Wheeler-DeWitt equation could be written as a linear combination of functions belonging to the subspaces \( C \) and \( C^* \). It is thus necessary, in order to select the physical subspace, to decompose the general space of solutions in \( C \) and \( C^* \). In the case of the Taub model the general solution of the Wheeler-DeWitt equation can be expressed as a combination of the modified Bessel functions \( K_\nu(z) \) and \( I_\nu(z) \) or as a combination of the modified Bessel functions \( I_\nu(z) \) and \( I_{-\nu}(z) \). The functions \( K_\nu(z) \) and \( I_\nu(z) \) are not related by the clock-reversal operator \( T \) while the functions \( I_\nu(z) \) and \( I_{-\nu}(z) \) are, for a real \( z \) variable, complex conjugated of each other: 
\[
I_{\nu}(z) = T[I_{-\nu}(z)] = [I_{-\nu}(z)]^*.
\]
As a result these two decompositions are not equivalent, being the decompositions in the basis \( \{I_\nu(z), I_{-\nu}(z)\} \) the proper one in order to select the physical subspace. Then the problem of imposing boundary conditions on the space of solutions of the Wheeler-DeWitt equation was reduced to the problem of finding a proper decomposition of the space of solutions \( S \) of the form \( S = C \oplus C^* \). These considerations should be complemented by an analysis of systems with time dependent reduced Hamiltonians. As it was shown in Ref. [17] it is not correct to quantify these kind of systems by using the unfactored form of the Wheeler-DeWitt equation because of the very well known ordering of the quantum operators problem. This case will be address in another work.
6 Symmetry breaking of the clock-reversal invariance and the problem of irreversibility

Traditionally the main approach for trying to understand the way in which the phenomenological irreversibility of the world is compatible with the invariance of the main theories of physics under “time-reversal” (where it is not made the fundamental distinction between motion-reversal and clock-reversal transformations) is to separate the whole set of canonical variables in the relevant and those which are irrelevant for the description of the phenomenological dynamics of the system. Neglecting this set of irrelevant variables, the set of relevant ones is an open system whose evolution can not be described as a Hamiltonian evolution or an unitary evolution in classical or quantum mechanics respectively. In this way it is possible to obtain time-asymmetric evolution equations for these open systems from an initial unstable state condition at time $t = 0$, yielding e. g. the growing of entropy from $t = 0$ towards positive time. But this can only be considered as a complete solution if we ignore negative times. In fact for negative times, entropy decreases in a symmetric way, a fact which reflects nothing but the formation process of the unstable state (see e. g. [23]). Therefore the solution is incomplete and must be complemented with other considerations like those introduced in (Ref. [18],[19],[20]). In these works the emergence of irreversibility depends, in the quantum case, on the existence of instabilities which forces to use generalized spectral decompositions of the Hamiltonian with complex eigenvalues. This generalizations ends in a symmetry breaking of the “time-reversal” invariance of the theory by breaking the evolution group of the theory in two semigroups. As in the previous section the space of solutions is decomposed as $S = C \oplus C^*$ where the subspace $C$ is considered as the space of physical admissible solutions and $C^*$ is the space of the corresponding “time-reversal” solutions (the distinction between motion-reversal and clock-reversal transformations is not made in those works). This formalism depends on a particular choice for the space of states of the admissible wave functions $\varphi(\omega, ...)$ (where $\omega$ is the energy). In non relativistic quantum mechanics, several physical considerations leads to postulate that the space of physical states $C$ is not the usual space of regular states (Schwarz class wave functions $S$) but the space of states belonging to the Hardy class from below or from above $H^2_\pm$ for the variable $\omega$, intersected with $S$ (see Ref.[24]). These spaces are called $\phi_-$ or $\phi_+$ respectively:

$$\phi_\pm = \{ |\psi\rangle / \langle \omega \mp | \psi \rangle \in \theta \left( S \cap H^2_\pm \right) \}$$

(19)

where $S$ denotes the Schwarz class, $H^2_\pm$ the upper (lower) Hardy class, and $\theta$ the Heaviside step function.

A complex wave function $f(x)$ on the real line is a Hardy class function from above (below) if

1) $f(x)$ is the boundary value of a function $f(z)$ of a complex variable $z = x + iy$ that is analytical in the half-plane $y > 0$ ($y < 0$)
2) \[
\int_{-\infty}^{+\infty} |f(x + iy)|^2 \, dx < k < \infty
\]
for all \( y \) such that \( 0 < y < \infty \) (\(-\infty < y < 0\)). The Heaviside step function was introduced in order to have physical states which vanish for negative energies.

This particular choice of the space of physical states have the property that each space is not invariant under the action of the complex conjugation operator (operator \( T \) in our formalism) (see Ref.\([18]\))

\[
T : \phi_\mp \rightarrow \phi_\pm
\]

This means that if one chooses \( \phi_- \) or \( \phi_+ \) as the space of physical states then the complex conjugation operator kicks out any state from the space of admissible functions. Another fundamental property of Hardy functions is stated in the Paley-Wiener theorem (Ref.\([20]\)). This theorem states that if \( f_\pm(q) \in H_\pm \), then the Fourier transformation

\[
g_\pm(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ipq} f_\pm(q) \, dq
\]

has the property

\[
g_+(p) = 0 \text{ for } p < 0
\]
\[
g_-(p) = 0 \text{ for } p > 0
\]

This theorem is used to show that the evolution group of the theory breaks in two semigroups. It can be demonstrated (see Ref.\([18]\)) that if \( |\psi\rangle \in \phi_\pm \) then

\[
e^{-iHt} |\psi\rangle \in \phi_\pm \text{ i.f.t } \leq 0
\]

In other words, if \( |\psi\rangle \in \phi_- \), then the evolution operator \( U(t) = e^{-iHt} \) exists in the physical space \( \phi_- \) but the inverted time operator \( U(t) = e^{iHt} \) does not exist in the space of physical states \( \phi_- \), being this the essence of an irreversible theory.

The underlying intuition of this formalism is similar to the one which is proposed in the present paper in the sense that it uses a symmetry-breaking of a “time-reversal” invariance which, in their case, ends in a time-asymmetric evolution without appealing to any kind of coarse-graining. Nevertheless if one takes into account the fundamental distinction made in Section IV between clock-reversal and motion-reversal it is clear that the formalism proposed in Ref.\([18]\) for non-relativistic quantum mechanics must be adapted to the present relativistic case. The problem is that non-relativistic quantum mechanics is not invariant under \( K \) (clock-reversal transformations) but under \( K \) plus \( t \rightarrow -t \) (motion-reversal transformations), which in the old terminology was known simply as “time-reversal” transformation. In the new terminology classical and
quantum mechanics are invariant under motion-reversal transformations but not under clock-reversal transformations because, in the context of parametrized systems formalism, these theories are defined on one sheet of the Hamiltonian constraint. In the approach of Ref. [18] it is supposed that quantum mechanics is invariant under the operation $K$ and that this invariance is broken by selecting the space of physical functions as $\phi_+$ or $\phi_-$. The new theory defined on these spaces would be no more invariant under $K$ because of [21]. The problem is that, taking into account that quantum mechanics is not invariant under clock-reversal transformations but under motion-reversal transformations, if we used the old formalism we would be trying to break a symmetry which is already broken.

Another conceptual problem of this formalism is that its physical content reduces to the fact that, given a particular unstable state including the formation process and the decaying process, the asymmetry is obtained by cutting this process in two halves corresponding each one to the formation and the decaying process respectively (this is in fact the breaking of the evolution group in two semigroups). The whole process is reversible but each part of it is irreversible. This explanation of irreversibility is equivalent to the one postulated by Boltzmann after his failure of deriving irreversibility from the basic laws of mechanics (H theorem). In a global reversible environment a local unstable situation like a fluctuation would define two local and opposed directions of time. In order to explain the global phenomenological irreversibility it is then necessary to postulate an initial unstable state without a formation process. The problem of irreversibility is then taken back to the problem of postulating certain initial conditions for the universe (see [24]) and it is not really grounded on a symmetry-breaking framework.

So we will now propose a way for complementing this previous approach (Ref. [18]) in order to set a better grounded framework for studying the problem of irreversibility based on a symmetry-breaking of the clock-reversal invariance of General Relativity. By selecting a sheet of the Hamiltonian constraint one is not splitting the time line in two halves but unfolding this whole time line in opposed directions. As it is said in Ref. [21] by choosing a certain direction of a canonical variable as time one forces this variable to be non-inversible. This means that its conjugated momentum $p_t$ can not change its sign. In this way, by choosing a physical clock, one forces its conjugated momentum to have a semi-infinite spectrum. What one wants to split in two halves is the spectrum of the energy, not the spectrum of time. One also wants, following the proposed philosophy, that both sheets of the Hamiltonian constraint correspond to positive energy solutions. In this way, and roughly speaking, our proposal is a kind of “Fourier transformation” of the formalism presented in Ref. [18]. In fact we will impose the requirement that the physical states belong to a Hardy class from below or from above ($\phi_-$ or $\phi_+$ respectively) in time representation:

$$\phi_\pm = \{ |\psi\rangle / \langle t| \psi\rangle \in S \cap H^2_\pm \}$$
Another way of stating this proposal is saying that a Hardy class function is, roughly speaking, a “well behaved” function whose Fourier transformation is null in the negative axis. One wants to have quantum states which, in the energy representation, are null on the negative axis. In order to satisfy this requirement is enough to have quantum states which belong to Hardy class functions in time representation. We will now show that the requirement \( g(E) \neq 0 \) only if \( 0 < E < \infty \) is a direct consequence of this choice. We will call \( \Psi \in H \) the quantum states corresponding to the choice \( t = \mp q \) \( (p_t = \mp p_q = \pm h) \). Then, using the Paley-Wiener theorem (22, 21), one finds

\[
\Phi_{\pm}(p_q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ip_q q} \Psi_{\pm}(q) \, dq = 0 \text{ if } p_q < 0
\]

which yields

\[
\Phi_{\pm}(p_q = -p_t = E) = 0 \text{ if } E < 0
\]

\[
\Phi_{\pm}(p_q = p_t = -E) = 0 \text{ if } E < 0
\]

i.e., for both choices the wave function \( G(E) \) in the energy representation is zero if \( E < 0 \). In this way it was shown that the fact that the physical states vanishes for negative energies is a direct consequence of the chosen space of admissible states (Hardy class functions in time representation). It is thus unnecessary to introduce the Heaviside step function (19), being this prescription a natural consequence of the proposed formalism.

7 Conclusions

The Hamiltonian constraint of General Relativity is quadratic in the canonical momenta, which means that the parametrized systems analogy, with a Hamiltonian constraint linear in the canonical momentum conjugated to the hidden time, is not completely fitted to mimic General Relativity. In fact if there is not a privileged time variable (because of the covariance of the theory under changes of the foliation of the space-time) all the momenta must appear in the constraint on an equal foot. Besides, to reduce the system means to pass from a static trajectory in configuration space to a trajectory in a reduced configuration space where one of the original variables was chosen as time. There is no reason to privilege one or the opposite direction of this variable as time, which forces the Hamiltonian constraint to be quadratic in the momentum conjugated to the variable chosen as time (which could be any monotonically increasing variable along the classical trajectories). It is then completely necessary to have a Hamiltonian constraint quadratic in all its momenta circumventing in this way the objection stated against parametrized systems approach (Ref. [6]) that it cannot explain why the Hamiltonian constraint of General Relativity is not in
fact linear in one of its momenta (the supposed hidden time). Besides this con-
ceptual clarification, the proposed interpretation forces to consider that what
changes from one sheet of the constraint to the other one is not the sign of the
energy (as in the Klein-Gordon analogy) but the direction of time. It was then
clearly shown how both sheets of the constraint corresponds to positive energy
solutions.

General Relativity, differently from ordinary classical or quantum mechanics,
is invariant under the transformation which passes from one sheet of the Hamil-
tonian constraint to the other one (inversion of the direction of time). But it
is well known that ordinary classical or quantum mechanics (theories which, if
parametrized, would be defined on one sheet of the corresponding Hamiltonian
constraint) are also invariant under the so called “time-reversal” transfor-
mations. In order to solve this apparent paradox and to show which are the differ-
ences between General Relativity symmetries and ordinary classical or quantum
mechanics symmetries, it was made a distinction between clock-reversals trans-
formations (a symmetry of General Relativity) and motion-reversal transforma-
tions (a symmetry of General Relativity, classical and quantum mechanics).
This distinction was clearly formulated both in the classical and quantum levels.

In the light of this new perspective, the boundary conditions on the space
of solutions of the Wheeler-DeWitt equation proposed in Ref. [9] were restated.
The main idea is that the boundary conditions to be imposed should act as
a symmetry-breaking of the clock-reversal symmetry of General Relativity. In
the quantum level the Wigner operator $T$ (complex conjugation in coordinate
representation) is the operator which transforms quantum states defined on one
sheet of the constraint to the other one. We can thus say that the problem of
finding proper boundary conditions on the space of solutions $S$ of the Wheeler-
DeWitt equation for time independent systems was reduced to the problem of
finding a realization of that space of the form

$$ S = C \oplus C^* $$

where the passage from one subspace $C$ to the other one $C^*$ is generated by $T$.
If this decomposition can be done (which is always possible for systems with
time independent reduced Hamiltonians), one can break the clock-reversal sym-
metry by defining the space of physical solutions as one of this subspaces. One
obeys a theory which is no more invariant under clock-reversals transforma-
tions but which still has the motion-reversal symmetry (as usual classical or
quantum mechanics). In this way it is possible to define boundary conditions
for Wheeler-DeWitt equation with the definite meaning of selecting one conven-
tional direction of time with respect to a chosen physical clock.

In the context of finding a formalism for understanding irreversible process
in a complementary way with the coarse-graining approaches, it was proposed
in previous papers (Ref. [18]) a similar decomposition of the space of states of the
quantum systems under study of the form $S = C \oplus C^*$, using the Hardy class functions
from above and below. We review this approach, state its main problems and suggest an adaptation of it in order to satisfy the requirements of the proposed interpretation. The main idea is to consider that the subspace of physical states is the subspace of quantum states which are Hardy class functions from above or below in the time representation (and not in the energy representation as in the previous approach, which intended to break the symmetry under motion reversal transformations). We think that, by doing this, we set a much more stronger and clearer ground for studying the emergence of irreversibility taking into account the symmetries of General Relativity under clock-reversal transformations.

8 Appendix: Comment on the time operator.

It could be argued that if one selects a physical clock for measuring time nothing prevents us from considering the chosen dynamical variable as a quantum variable. The rough quantum mechanical distinction between dynamical variables associated with operators and time, which is supposed to be just an evolution parameter, should disappear if one considers that there is not something as **Time** but only dynamical variables playing the role of physical clocks. If one assume that this distinction should not hold any more in the context of an operational definition of time then it should be possible to circumvent the well known impossibility of associating a quantum operator with time, which is said to be a result of the fact that the Hamiltonian is semi-bounded from below. This result was obtained by Pauli in 1933. Briefly the statement that the time operator $T$ does not exist if the spectrum of $H$ is bounded from below (Ref.[10]). If this operator could be defined then a state $|E\rangle$ could be transformed in a state of any energy $E + \alpha$ with arbitrary real $\alpha$ by applying the unitary operator $e^{i\alpha T} |E\rangle$.

In fact the energy of this transformed state is

$$H e^{i\alpha T} |E\rangle = (E + \alpha) e^{i\alpha T} |E\rangle$$  \hspace{1cm} (25)

which is inconsistent with the assumption that the spectrum of $H$ is bounded from below. But, as was stated above, the necessity of defining a time operator is a main problem if one follows the proposed interpretation of Canonical Quantum Gravity. The covariance of General Relativity forbids us to consider a certain variable as a privileged clock and consider it, in the quantum version of the theory, as a c-number. This fact imposes the necessity of facing the problem of defining a time operator. As was said above by choosing a physical clock, one forces its conjugated momentum to have a semi-infinite spectrum. Then, when one breaks the clock-reversal symmetry by choosing a direction for time, the possibility of associating an operator with time is apparently eliminated. But it is known that in fact exist several examples of self-adjoint operators that do not possess spectra spanning the entire real line, e.g., the momentum and position operators of a particle trapped in a box, the angular momentum and the angle
operators, the harmonic oscillator number and phase operators (Ref. [22]). Following this examples it was shown in Ref. [22] that a time operator conjugated to a Hamiltonian with a semi-bounded spectrum can be consistently defined. The fact that the spectrum of the reduced Hamiltonian is bounded from below is not more problematic, for a quantum definition of the corresponding operators, than the case of a particle constrained by certain boundary conditions to move in the positive real semi-axis. The main difference between these cases is that in ordinary quantum mechanics it is enough to consider time as a classical external parameter of evolution, being unnecessary to associate it with an operator. It is only in the context of the quantization of gravity that a physical motivation for defining a time operator appears, because in General Relativity, and according to the proposed formalism, there is not an external classical time, but only dynamical variables which could play the role of physical clocks. In a quantum context it would be inconsistent to treat the physical clock as a classical parameter while the rest of the dynamical variables are associated with quantum operators, because there is not an essential difference between the variable used as a physical clock and the rest of the canonical variables, being this a consequence of the covariance of the theory. If this were the correct physical interpretation, the proposed approach imposes the necessity of using the formal demonstration stated in Ref. [22] for defining a time operator.

The fact that it is forbidden to use transformations like (25) is a common feature of any symmetry breaking formalism. The whole space of solutions of the Wheeler-DeWitt equation has a symmetry generated by the unitary operator $e^{i\alpha T}$. The breaking of this symmetry by imposing the proposed boundary conditions on this space of solutions defines two subspaces which are no more invariant under the group generated by $e^{i\alpha T}$. Certainly this breaking do not yields the non existence of $T$. In fact, if one has quantum states which belong to a Hardy class functions in time representation, the relations (23, 15) change to the assertion that if $|E\rangle_{\pm} \in H_{\pm}$ then

$$e^{-iT\alpha} |E\rangle_{\pm} \in H_{\pm} if \alpha > 0$$

In this way the Pauli theorem can be circumvented by choosing the space of admissible quantum states to be a Hardy class function from above or below, i.e., by an explicit breaking of the clock-reversal symmetry of the space of solutions of the Wheeler-DeWitt equation. After choosing a direction for the evolution of the physical clock, the space of admissible states for $g(E)$ is restricted to those functions $g(E)$ such that $g(E) \neq 0$ if $0 < E < \infty$.

Acknowledgments

This work was supported by Consejo Nacional de Investigaciones Científicas y Técnicas, Universidad de Buenos Aires (Proj. X-143) and Fundación Antorchas.
References

[1] K. V. Kuchar, in *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992).

[2] J. Butterfield and C. J. Isham, in *The Arguments of Time*, edited by J. Butterfield (Oxford University Press, Oxford, 1999).

[3] R. Ferraro, *Gravitation and Cosmology* 5, 195 (1999).

[4] C. Misner, K. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[5] C. Lanczos, *The Variational Principles of Mechanics*, (Dover, New York, 1986).

[6] J. Barbour, Class. Quantum Grav. 11, 2853 (1994).

[7] C. Rovelli, Phys. Rev. D 43, 442 (1991).

[8] C. Simeone, *Deparametrization and path integral quantization of cosmological models*, World Scientific Lectures Notes in Physics 69 (World Scientific 2002).

[9] G. Catren and R. Ferraro, Phys. Rev. D 63, 023502 (2001).

[10] L. E. Ballentine, *Quantum Mechanics* (Prentice Hall, New Jersey, 1990).

[11] K. V. Kuchar, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose and D. W. Sciama (Clarendon Press, Oxford, 1981).

[12] S. C. Beluardi and R. Ferraro, Phys. Rev. D 52, 1963 (1995).

[13] G. I. Ghandour, Phys. Rev. D 35, 1289 (1987).

[14] J. Halliwell, *Int. Lectures to Quantum Cosmology*, Jerusalem Winter School (World Scientific 1991).

[15] H. D. Zeh, *The Physical Basis of the Direction of Time*, (Springer-Verlag 1989).

[16] R. M. Wald, Phys. Rev. D 48, 2377 (1993); A. Higuchi and R. M. Wald, Phys. Rev. Lett. 28, 1082 (1972).

[17] W. F. Blyth and C. J. Isham, Phys. Rev. D 11, 4768 (1975).

[18] M. Castagnino and R. Laura, Phys. Rev. A 56, 108 (1997). M. Castagnino, Phys. Rev. D 57, 750-767, (1998), gr-qc/9604034.
[19] E. C. G. Sudarshan, C. B. Chiu and V. Gorini, Phys. Rev. D 18, 2914 (1978); G. Parravicini, V. Gorini and E. C. G. Sudarshan, J. Math. Phys. (N.Y.) 21, 2208 (1980); E. C. G. Sudarshan and C. B. Chiu, Phys. Rev. D 47, 2602 (1993).

[20] A. Bohm, Quantum Mechanics: Foundations and Applications (Springer-Verlag, Berlin, 1986); A. Bohm, M. Gadella and B. G. Maynland, Am. J. Phys. 57, 1103 (1989); M. Gadella, J. Math. Phys. (N.Y.) 22, 1462 (1981); 24, 2124 (1983); 25, 2461 (1984).x

[21] P. Hájícek, Phys. Rev. D 34, 1040 (1986).

[22] E. A. Galapon, quant-ph/9908033.

[23] L. S. Schulman, Time’s Arrow and Quantum Measurements, Cambridge Univ. Pres., Cambridge (1997)

[24] M. Castagnino, J. Gueron, A. Ordoñez, Time asymmetry as a consequence of a wave packets theorem, J. Math. Phys. 43, 705 (2002).
Figure 1: (a) represents a particular motion in which the variable $q_1$ was chosen as the physical clock $t$ ($t = q_1$), (b) represents the motion reversal of (a) ($q_1$ is still the time $t$), (c) represents the clock-reversal of (a) (the time $t$ is now equal to $-q_1$) and (d) represents the clock-reversal of (b) or the motion reversal of (c).