Enumerating Segmented Patterns in Compositions and Encoding by Restricted Permutations

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Abstract

A composition of a nonnegative integer $n$ is a sequence of positive integers whose sum is $n$. A composition is palindromic if it is unchanged when its terms are read in reverse order. We provide a generating function for the number of occurrences of arbitrary segmented partially ordered patterns among compositions of $n$ with a prescribed number of parts. These patterns generalize the notions of rises, drops, and levels studied in the literature. We also obtain results enumerating parts with given sizes and locations among compositions and palindromic compositions with a given number of parts. Our results are motivated by “encoding by restricted permutations,” a relatively undeveloped method that provides a language for describing many combinatorial objects. We conclude with some examples demonstrating bijections between restricted permutations and other objects.

1 Introduction

A composition of a nonnegative integer $n$ is a sequence $\alpha = \alpha_1 \alpha_2 \cdots \alpha_m$ of positive integers whose sum is $n$. We consider the empty sequence with no terms to be the unique composition of 0. We will sometimes write compositions as sums rather than as words, as in $\alpha_1 + \alpha_2 + \cdots + \alpha_m$, though it must be kept in mind that the order of the terms still matters. It is sometimes helpful to think of a composition of $n$ as a sequence of $n$ stones laid in a row, together with a grouping together of the stones in such a way that every stone belongs to a group, every group contains a stone, no stone belongs to two groups, and two stones belong to the same group only if every stone between them belongs to that group.

Each term $\alpha_i$ in a composition $\alpha$ is called a part of that composition. A part equal to $k$ is called a $k$-part. A split in a composition is an integer that

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can be expressed as the sum of the first \(i\) parts of the composition for some nonnegative integer \(i\). Thus, the composition \(3 + 1 + 1 + 2\) has 5 splits: 0, 3, 4, 5, and 7. Using the imagery of stones, the groups of stones are the parts of \(\alpha\), and the splits of \(\alpha\) correspond to the spaces between groups (including the space to the left of the first group and to the right of the last group). We use \(\langle\alpha\rangle\) to denote the composition comprising the parts of \(\alpha\) written in reverse order. A \textit{palindromic composition}, or a \textit{palindrome}, is a composition for which \(\alpha = \langle\alpha\rangle\).

A \textit{rise} (resp. \textit{drop}) is a part followed by a larger (resp. smaller) part. A \textit{level} is a part followed by a part equal to itself.

Frequencies of occurrences of \(k\)-parts, rises, drops, and levels in (palindromic) compositions, as well as in compositions with additional restrictions, have been studied (e.g., see \cite{4,5} and references therein). Heubach and Mansour \cite{5} give a multivariate generating function for joint distribution of parts, rises, levels, and drops in compositions and palindromes. However, using the results from the literature related to the subject, it does not seem to be possible to answer a question like: how many levels are followed by rises among all compositions of \(n\)? To consider a more general question, we introduce the notion of a \textit{segmented pattern} in a composition. A segmented pattern is a word \(w = w_1w_2\cdots w_k\) in the alphabet of positive integers such that if \(b\) is a letter in \(w\) and \(a < b\), then \(a\) is a letter of \(w\). In other words, the letters in \(w\) constitute an order ideal. For example, 431242 is a segmented pattern, while 41242 is not. We say that \(w\) occurs in a composition \(\alpha = \alpha_1\alpha_2\cdots\alpha_m\) if there is a subword \(\alpha_\ell\alpha_{\ell+1}\cdots\alpha_{\ell+k-1}\) of \(\alpha\) that is order-isomorphic to \(w\). Thus, rises, drops, and levels are occurrences of the patterns 12, 21, and 11, respectively. A level followed by a rise is an occurrence of the pattern 112.

More generally, we study occurrences of so-called \textit{segmented partially ordered patterns} (SPOPs) in compositions. A SPOP \(w\) is a word consisting of letters from a partially ordered alphabet \(A\) such that the letters in \(w\) constitute an order ideal in \(A\). For instance, if we have a poset on three elements labeled by 1, 1', and 2' in which the only relation is \(1' < 2'\), then the sequence 31254 has two occurrences of 11'2', namely 312 and 125. Given a SPOP \(w = w_1w_2\cdots w_m\), we say that a segmented pattern \(v = v_1v_2\cdots v_m\) is a \textit{linear extension} of \(w\) if \(w_i < w_j\) implies that \(v_i < v_j\). Thus the linear extensions of 11'2' are 123, 213, and 312.

This paper is organized as follows. In Section 2 we give our main results. Theorem 2.1 gives a multivariate generating function for the number of occurrences of a given SPOP at a given split among compositions of \(n\) with a given number of parts. By specializing variables, we obtain a generating function for the number of occurrences of a given SPOP among all compositions of \(n\) (Corollary 2.2). In Theorem 2.4 we enumerate the occurrences of \(k\)-parts at a given split in compositions of \(n\) with a given number of parts. This generalizes a result in \cite{3}. Our approach to this problem is to use a method which perhaps can be best described as “encoding by restricted permutations.” The idea here is to encode a set of objects under consideration as a set of permutations satisfying certain restrictions. Under appropriate encodings, this allows us to transfer the interesting statistics from our original set to the set of permutations, where they are easy to handle. In Section 3 we use restricted permutations to enumerate \(k\)-blocks with certain statistics in palindromic compositions, refining results in \cite{4}. In Section 4 we provide short bijective encodings of binary bitonic
sequences, binary strings without singletons, permutations avoiding 1-3-2-4 and having exactly one descent, and lines drawn through the points of intersections of \( n \) straight lines in a plane. Relations of these objects to certain restricted permutations were given in [1], but no bijections were provided. We believe that these examples provide some evidence for the broad applicability of the method of encoding by restricted permutations.

We use the following notations throughout the paper. The set of non-negative integers is denoted by \( \mathbb{N} \), and the set of positive integers is denoted by \( \mathbb{P} \). Given \( m \leq n \in \mathbb{N} \), we write \([m, n] = \{m, m+1, \ldots, n\} \) and \([n] = [1, n] \). The permutations in this paper are written in one-line notation. Given a generating function \( G(t) \), we write \([t^n]G(t)\) to denote the coefficient of \( t^n \) in \( G(t) \). We use \( C(n) \) to denote the set of compositions of \( n \), and we write that \( |\alpha| = n \) if \( \alpha \in C(n) \). Finally, let \( C(n, \ell) \) be the number of compositions of \( n \) with \( \ell \) parts.

It is well known and easy to verify that for a fixed non-negative integer \( \ell \), the generating function for \( C(n, \ell) \) is given by

\[
\sum_{n=0}^{\infty} C(n, \ell) x^n = \frac{x^\ell}{(1-x)^\ell}.
\] (1)

2 Compositions

Given a SPOP \( w = w_1 w_2 \cdots w_m \) with \( m \) parts, let \( c_w(n, \ell, s) \) be the number of occurrences of \( w \) among compositions of \( n \) with \( \ell + m \) parts such that the sum of the parts preceding the occurrence is \( s \). Let \( \Omega_w(x, y, z) \) be the generating function for \( c_w(n, \ell, s) \):

\[
\Omega_w(x, y, z) = \sum_{n, \ell, s \in \mathbb{N}} c_w(n, \ell, s) x^n y^\ell z^s.
\]

Our goal is to derive an explicit rational function for \( \Omega_w(x, y, z) \).

Before proceeding, we define the following notation. Given a segmented pattern \( v \) and \( n \in \mathbb{N} \), let \( P_v(n) \) denote the number of compositions of \( n \) that are order isomorphic to \( v \). The generating function \( P_v(x) \) for \( P_v(n) \) is not difficult to derive. If \( j \) is the largest letter of \( v \), then \( P_v(n) \) is the number of integral solutions \( t_1, \ldots, t_j \) to the system

\[
\mu_1 t_1 + \cdots + \mu_j t_j = n, \quad 0 < t_1 < \cdots < t_j,
\] (2)

where \( \mu_k \) is the number of \( k \)'s in \( v \). By expanding terms into geometric series, one can see that the number of integral solutions to (2) is the coefficient of \( x^n \) in

\[
P_v(x) = \prod_{k=1}^{j} \frac{x^{m_k}}{1-x^{m_k}},
\] (3)

where \( m_k = \mu_{j-k+1} + \cdots + \mu_j \) for \( 1 \leq k \leq j \).

**Theorem 2.1.** Let \( w \) be a SPOP. Then

\[
\Omega_w(x, y, z) = \sum_v \frac{(1-x)(1-xz)P_v(x)}{(1-x-xy)(1-xz-xyz)}
\] (4)

where the sum is over all linear extensions \( v \) of \( w \).
Proof. We begin by computing $\Omega_v(x, y, z)$ when $v$ is a segmented pattern. We think of an occurrence of $v$ as the triple of compositions $(\alpha, \beta, \gamma)$ such that $\alpha$ comprises the parts to the left of the occurrence, $\beta$ comprises the parts in the occurrence, and $\gamma$ comprises the parts to the right of the occurrence. Hence, for given $n, \ell, s \in \mathbb{N}$, $c_v(n, \ell, s)$ is the number of triples $(\alpha, \beta, \gamma)$ such that $|\alpha| + |\beta| + |\gamma| = n$, $|\alpha| = s$, $\beta$ is order isomorphic to $v$, and $\alpha$ and $\gamma$ together have $\ell$ parts. Thus we have that
\[
c_v(n, \ell, s) = \sum_{0 \leq j \leq \ell} C(s, j)P_v(k)C(n - s - k, \ell - j).
\]
Using this equality, together with the generating function (1) for $C(n, \ell)$, we can factor $\Omega_v(x, y, z)$ into a product of $P_v(x)$ and two geometric series:
\[
\Omega_v(x, y, z) = \sum_{n, \ell, s \in \mathbb{N}} c_v(n, \ell, s)x^n y^\ell z^s = P_v(x) \left( \sum_{n, \ell \in \mathbb{N}} C(n, \ell)x^n y^\ell \left( \sum_{s, \ell \in \mathbb{N}} C(s, \ell)(xz)^s x^\ell \right) \right) = P_v(x) \frac{(1 - x)(1 - xz)}{(1 - xy)(1 - xz - xyz)}.
\]
Finally, note that if $w$ is a SPOP, then $c_w(x, y, z) = \sum_v c_v(x, y, z)$, where the sum is over all linear extensions $v$ of $w$. Thus, $\Omega_w(x, y, z) = \sum_v \Omega_v(x, y, z)$, and the theorem follows.

Setting $y = z = 1$ in equation (4) yields the following.

**Corollary 2.2.** Given a segmented pattern $w$, the number of occurrences of $w$ among compositions of $n$ is equal to
\[
[x^n]\Omega_w(x, 1, 1) = [x^n] \sum_v \frac{(1 - x)^2P_w(x)}{(1 - 2x)^2},
\]
where the sum is over all linear extensions $v$ of $w$.

**Example 2.3.** We compute the number of occurrences of $m$ levels immediately followed by a rise. This is an occurrence of the segmented pattern $w = 1\cdots 1$. 2.

The content vector of $w$ is $\mu = (m + 1, 1)$, so we have
\[
P_w(x) = \frac{x^{m+3}}{(1 - x)(1 - x^{m+2})}.
\]
Hence, the number of occurrences of $w$ among all compositions of $n$ is
\[
[x^n]\Omega_w(x, 1, 1) = [x^n] \frac{(1 - x)^2x^{m+3}}{(1 - 2x)^2(1 - x)(1 - x^{m+2})}.
\]
For fixed $m$, it is routine to expand the rational function above into partial fractions to obtain a closed form expression for $[x^n]\Omega_w(x, 1, 1)$. 


We now give an enumerative result that describes the number of $k$-parts located at a given split among compositions of $n$ with a given number of parts. Theorem 2.4 below is our first example of encoding with restricted permutations.\footnote{In fact it is possible to use this result to prove Theorem 2.1, though this approach requires several pages of tedious calculation, and is omitted in favor of the short and self-contained proof given above.} For $n, k, \ell, s \in \mathbb{N}$, define $f(n, k, \ell, s)$ to be the number of $k$-parts occurring among compositions of $n$ with $\ell + 1$ parts such that the sum of the parts preceding the $k$-part is $s$. It immediately follows that $f(n, k, \ell, s) = 0$ if either $n = 0$ or $k = 0$. The case when $n = k > 0$ is also clear: $f(n, n, \ell, s) = 1$ if $\ell = s = 0$, and $f(n, n, \ell, s) = 0$ otherwise. The following theorem gives the value of $f(n, k, \ell, s)$ in all remaining cases.

**Theorem 2.4.** If $n \in \mathbb{P}$ and $k \in [n - 1]$, then

$$f(n, k, \ell, s) = \begin{cases} \binom{n-k-1}{\ell-1}, & \text{if } s \in \{0, n-k\} \text{ and } \ell \in [n-k], \\ \binom{n-k-2}{\ell-2}, & \text{if } s \in [n-k-1] \text{ and } \ell \in [2, n-k], \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

**Proof.** We give a bijection between the $k$-parts that we are enumerating and a particular set of restricted permutations. Let $S$ be the set of permutations of the quotient group $Z = \mathbb{Z}/(n-k+1)\mathbb{Z}$ of the form $w_1 w_2 \cdots w_{n-k}$, where

$$w_1 > w_2 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k},$$

and $s + 1 \in \{w_1, \ldots, w_\ell\}$ (where we’ve identified $s$ with its canonical projection in $Z$). To see that $|S|$ is given by the right hand side of equation (5), observe that an element of $S$ is uniquely specified by choosing which elements of $Z$ will be in $\{w_1, \ldots, w_\ell\}$ other than $s$ (which cannot be in there) and $s + 1$ and $\min(Z\setminus\{s\})$ (which must be in there, but which are equal when $s \in \{0, n-k\}$).

We now show that the elements of $S$ are in bijective correspondence with the $k$-parts that we wish to enumerate. First, we can think of such a $k$-part as an element of

$$T = \{(\alpha, \beta) : \alpha \in \mathcal{C}(s), \beta \in \mathcal{C}(n-k-s), \ \alpha \text{ and } \beta \text{ together have } \ell \text{ parts }\}.$$ 

To be precise, a $k$-part in a composition of $n$ corresponds to the ordered pair $(\alpha, \beta)$ of compositions such that $\alpha$ comprises the parts to the left of the chosen $k$-part and $\beta$ comprises the parts to the right of the chosen $k$-part.

We now give a bijection $T \leftrightarrow S$. For an explicit example of the bijection we are about to describe, see Example 2.5. Given $(\alpha, \beta) \in T$, we produce a permutation in $S$ as follows. Concatenate the compositions $\alpha$ and $\beta$, producing a composition $\gamma \in \mathcal{C}(n-k)$ with $\ell$ parts. Let $\overline{\gamma}_\ell = 0$ and let

$$w_i = \overline{\gamma}_i + 1, \quad \text{if } \overline{\gamma}_i \geq s.$$ 

For $1 \leq i \leq \ell$ let

$$\overline{\gamma}_i = \sum_{j=1}^{i} \overline{\gamma}_j, \quad \text{for } 1 \leq i \leq \ell - 1,$$ 

$$w_i = \overline{\gamma}_i + 1, \quad \text{if } \overline{\gamma}_i \geq s.$$ 

$$w_i = \overline{\gamma}_i, \quad \text{if } \overline{\gamma}_i < s.$$ 

\footnote{In fact it is possible to use this result to prove Theorem 2.1, though this approach requires several pages of tedious calculation, and is omitted in favor of the short and self-contained proof given above.}
Finally, let $w_{k+1}, \ldots, w_{n-k}$ be the elements of $[0, n-k] \setminus \{s, w_1, \ldots, w_{\ell}\}$ written in increasing order (where we’ve identified $[0, n-k]$ with its canonical projection in $\mathbb{Z}$).

It is easy to show that this map yields an element of $S$. We show that the map is a bijection by giving its inverse. Given an element of $S$, one may produce an element of $T$ by letting

$$w_i = \begin{cases} w_i, & \text{if } w_i < s, \\ w_i - 1, & \text{if } w_i > s, \end{cases} \quad \text{for } 1 \leq i \leq \ell,$$

letting $\gamma_i = w_{\ell-i} - w_{\ell-i+1}$ for $1 \leq i \leq \ell$, and letting $\gamma = \gamma_1 \cdots \gamma_\ell$. Because of the requirement that $s + 1 \in \{w_1, \ldots, w_\ell\}$, it follows that for some $i$, $\sum_{j=1}^i \gamma_i = s$. Let $\alpha = \gamma_1 \cdots \gamma_i$ and let $\beta = \gamma_{i+1} \cdots \gamma_\ell$. Then we have that $(\alpha, \beta) \in T$.

**Example 2.5.** We choose as our $k$-part the 6 in the composition $3 1 6 2$. Then we have $n = 12$, $k = 6$, $\ell = 3$, and $s = 4$. The claim is that this corresponds to a permutation of the elements in $\mathbb{Z}/7\mathbb{Z}$.

Applying the maps from the theorem to our chosen $k$-part yields $\alpha = 3 1$, and $\beta = 2$. Thus we have $\gamma = 3 1 2$. Computing the values of $w_i$ yields $w_1 = 0$, $w_2 = 3$, and $w_3 = 0$. Observing that $w_1 \geq s = 4$, while $w_2, w_3 < s$, we compute the $w_i$’s as follows

$$w_1 = w_1 + 1 = 5,$$
$$w_2 = w_2 = 3,$$
$$w_3 = w_3 = 0.$$

Finally, we let $w_4 w_5 w_6$ be the elements of

$$\{0, \ldots, 6\} \setminus \{s, w_1, w_2, w_3\} = \{0, \ldots, 6\} \setminus \{4, 5, 3, 0\} = \{1, 2, 6\}$$

written in increasing order. Therefore, the word corresponding to our original $k$-part is

$$s w_1 w_2 \cdots w_6 = 4 5 3 0 1 2 6.$$

As a corollary to Theorem 2.4, we derive a result that appeared in [3].

**Corollary 2.6.** Given $n \in \mathbb{N}$ and $k \in [n - 1]$, the number of $k$-parts among all compositions of $n$ is $2^{n-k-2}(n-k+3)$.

**Proof.** The result follows from using equation (5) to compute

$$\sum_{\ell \in [n-k]} f(n, k, \ell, s) = 2^{n-k} \sum_{\ell=1}^{n-k} \binom{n-k-1}{\ell-1} + (n-k-1) \sum_{\ell=2}^{n-k} \binom{n-k-2}{\ell-2}$$

$$= 2^{n-k} + 2^{n-k-2}(n-k-1)$$
$$= 2^{n-k-2}(n-k+3).$$

$\Box$
3 Palindromic Compositions

We provide two alternative (nonequivalent) encodings by restricted permutations of \( k \)-parts in palindromes of \( N \), when \( N \) and \( k \) have different parity. We give these encodings explicitly in the case of even palindromes of \( N = 2(n - 1) \) and odd \( k \)-parts. These encodings provide bijective proofs of the known result that the number of \( k \)-parts in palindromic compositions of \( 2(n - 1) \) is \((n - k + 1)2^{n-k-1}\) when \( k \) is odd (see [4]). Such \( k \)-parts will be encoded as permutations \( w_1w_2\cdots w_{n-k+1}\) of \( [n-k+1] \) such that, for some \( \ell \in \{2, \ldots, n\} \), \( w_2 > w_3 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k+1} \). In either encoding, the case of odd palindromes \( N = 2n - 1 \) and even \( k \)-parts can be obtained using similar ideas.

3.1 First encoding

First, observe that a permutation \( w_1w_2\cdots w_{n-k+1} \) of \( [n-k+1] \) with

\[
 w_2 > w_3 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k+1}.
\]

corresponds to an ordered pair \((w_1, \alpha)\) with \( w_1 \in [n-k+1] \) and \( \alpha = \alpha_1\cdots\alpha_{\ell-1} \in C(n-k) \) as follows. If \( \ell = 2 \), let \( \alpha_1 = n-1 \). Otherwise, let

\[
 \alpha_1 = w_{\ell-1},
\]

and put

\[
 \alpha_i = w_{\ell-i} - w_{\ell-i+1}, \quad \text{for } 2 \leq i \leq \ell - 2, \\
 \alpha_{\ell-1} = n-k-w_2.
\]

We now explicitly describe the correspondence between pairs \((w_1, \alpha)\) and odd \( k \)'s in palindromic compositions of \( 2(n - 1) \). It may be helpful to use the imagery of stones discussed in Section 1. In this context, \( w_1 \) can be thought of as distinguishing a gap in the sequence of stones, where the gaps are the spaces between any two adjacent stones (whether they belong to the same group or not), as well the space before the first stone and after the last stone. Hence, a sequence of \( n-k \) stones has \( n-k+1 \) gaps, which are indexed with the set \( [n-k+1] \).

**Case I:** The cases in which \( w_1 \in \{1, n-k+1\} \) correspond to the \( k \)'s that are either the left-most or right-most terms in the compositions containing them. In particular, \((1, \alpha)\) corresponds to the left-most \( k \) in the composition

\[
 k + \sum_{i=1}^{\ell-2} \alpha_i + 2(\alpha_{\ell-1} - 1) + \left( \sum_{i=1}^{\ell-2} \alpha_i \right) + k,
\]

while \((n-k+1, \alpha)\) corresponds to the right-most \( k \).

**Case II:** The cases in which \( 2 \leq w_1 \leq n-k \) and \( w_1 \) is a split in \( \alpha \) correspond to the \( k \)'s which are on the left-hand side of the palindromic compositions containing them, but which are not the left-most terms. In these cases, \((w_1, \alpha)\)
corresponds to the indicated $k$ on the left-hand side of the palindromic composition
\[
\sum_{i=1}^j \alpha_i + k + \sum_{i=j+1}^{\ell-2} \alpha_i + 2(\alpha_{\ell-1} - 1) + \left( \sum_{i=1}^j \alpha_i + k + \sum_{i=j+1}^{\ell-2} \alpha_i \right),
\]
where we have the identity $\sum_{i=1}^j \alpha_i = w_1 - 1$.

**Case III:** The cases in which $2 \leq w_1 \leq n-k$ and $w_1$ is not a split in $\alpha$ correspond to the $k$'s which are on the right-hand side of the palindromic compositions containing them, but which are not the right-most terms. These cases break into two subordinate cases:

**Case IIIA:** Within Case III, those $(w_1, \alpha)$ in which $w_1$ is a gap in the last term of $\alpha$ correspond to the indicated $k$ on the right-hand side of the palindromic composition
\[
\sum_{i=1}^{\ell-2} \alpha_i + \alpha'_{\ell-1} + k + 2(\alpha''_{\ell-1} - 1) + k + \alpha'_{\ell-1} + \left( \sum_{i=1}^{\ell-2} \alpha_i \right),
\]
where we use the identities $\alpha'_{\ell-1} + \alpha''_{\ell-1} = \alpha_{\ell-1}$ and $\sum_{i=1}^{\ell-2} \alpha_i + \alpha'_{\ell-1} = w_1 - 1$.

**Case IIIB:** On the other hand, if $w_1$ is not a gap in the last term of $\alpha$, then $(w_1, \alpha)$ corresponds to the indicated $k$ on the right-hand side of the palindromic composition
\[
\sum_{i=1}^{j-1} \alpha_i + \alpha'_j + k + \alpha''_j + \sum_{i=j+1}^{\ell-2} \alpha_i + 2(\alpha_{\ell-1} - 1)
+ \left( \sum_{i=1}^{j-1} \alpha_i + \alpha'_j + k + \alpha''_j + \sum_{i=j+1}^{\ell-2} \alpha_i \right),
\]
where we use the identities $\alpha'_j + \alpha''_j = \alpha_j$ and $\sum_{i=1}^{j-1} \alpha_i + \alpha'_j = w_1 - 1$. (To deduce the value of $j$ from a given composition $\alpha$, we will also need to use the inequality $0 < \alpha'_j < \alpha_j$.)

### 3.2 Second Encoding

Clearly, a palindrome of $2(n-1)$ has either an odd number of parts with an even part in the center or an even number of parts and no central part. To make all palindromes to be of odd length, we create a central part “0” for palindromes with an even number of parts.

We present an algorithm to produce a permutation given an underlined $k$-part in a palindrome $P$. We only consider the case when the chosen part is to the left of the center in $P$; for a part from the right-hand side, we proceed with the part symmetric to it, and we switch 1 and 2 in the obtained permutation. In the bijection below, a part is to the left of the center if and only if in the
corresponding permutation, 1 precedes 2. In general, we find the permutation corresponding to $k$ by inserting the numbers $n - k + 1$, $n - k$, $n - k - 1$, and so on in decreasing order, into initially empty slots $w_1, w_2, \ldots, w_{n-k+1}$.

Suppose $P = C_k D x D_k C$ where $x = 2t$ for $t \geq 0$.

1. If $D$ is empty and $x = 0$, we set $w_1 = (n - k + 1)$ and proceed with (2) below. Otherwise, we set

$$w_{n-k-t+2} w_{n-k-t+3} \cdots w_{n-k+1} = (n - k - t + 2)(n - k - t + 3) \cdots (n - k + 1)$$

and $w_2 = (n - k - t + 1)$ (in particular, if $t = 0$ we only set $w_2 = (n - k + 1)$). We read the parts in $D$ from right to left and fill in the slots $w_3, w_4, \ldots$ by placing $n - k - t$, then $n - k - t - 1$, and so on: if a current part is $a$, then we place $a - 1$ of the largest unplaced numbers to the right in increasing order, and we place the largest number out of the remaining numbers to the left. We then proceed with the part next to $a$ from the left. The only exception is the part immediately to the right of $k$. In this case, we place $a - 1$ of the largest unplaced numbers to the right in increasing order, and then we set $w_1$ be the largest of yet unplaced numbers. If we get $w_2 = 2$, set $w_1 = 1$ and place 2 in the only one remaining slot. Continue with step (2).

2. If $C$ is empty or if $C = 1$, place the unplaced numbers in increasing order into the empty slots. Otherwise, suppose $C = a_1 a_2 \cdots a_k$. Then we consider the binary vector $0^{a_1 - 1} 1^{a_2 - 1} \cdots 0^{a_k - 1}$ (each block of 0’s but the last one is followed by a 1). We read this binary vector from right to left and whenever we meet a 0, we place the largest unplaced number into the leftmost available slot; otherwise, we place this number into the rightmost available slot. If this procedure can no longer be continued, and 1 or 2 have not yet been placed, place them so that 1 precedes 2.

We provide some examples. Suppose we are interested in 1 in the following palindrome of 16: 212141212. The steps of our recursive bijection are as follows:

$*7** **89 \rightarrow *76** **89 \rightarrow 476** **589 \rightarrow 4763** **589 \rightarrow 476312589.$

As further examples, one can check that the underlined 5’s in 5115, 1551, and 525 correspond to 132, 321, and 123 respectively.

The inverse of this algorithm is easy to find. In particular, if $w_1 = 1$ (resp. $w_1 = 2$) then the corresponding $k$-part is the leftmost (resp. rightmost) one in a composition.

4 Additional Encodings with Restricted Permutations

We now provide some additional examples of encodings of combinatorial objects by restricted permutations to demonstrate various approaches to bijective enumeration. But first, some definitions.

A sequence $a_1, a_2, \ldots, a_n$ is *bitonic* if for some $h$, $1 \leq h \leq n$, we have that $a_1 \leq a_2 \leq \cdots \leq a_h \geq a_{h+1} \geq \cdots \geq a_{n-1} \geq a_n$ or $a_1 \geq a_2 \geq \cdots \geq a_h \leq a_{h+1} \leq \cdots \leq a_n$. 

\[ \cdots \leq a_{n-1} \leq a_n. \] A binary string \( x \) is said to be without singletons if the words 010 and 101 are not factors of \( x \).

Let \( S_1 \) (resp. \( S_2 \)) be the set of \((n+2)\)-permutations \( w_1w_2 \cdots w_{n+2} \) such that, \( w_1w_2 = (n+1)(n+2) \) or \( w_1w_2 = (n+2)(n+1) \), and \( w_3w_4 \cdots w_{n+2} \) avoids simultaneously the patterns 1-2-3 and 2-3-1 (resp. 1-2-3, 1-3-2, and 2-1-3). According to \( \Pi \), \(|S_1| = n^2 - n + 2 \) and \(|S_2| = 2F_n \), where \( F_n \) is the \( n \)-th Fibonacci number with \( F_0 = F_1 = 1 \).

Let \( S_3 \) be the set of \((n+3)\)-permutations \( w_1w_2 \cdots w_{n+3} \) such that, \( w_1 < w_2 < w_3 \) and \( w_4w_5 \cdots w_{n+3} \) is in decreasing order. Clearly, \(|S_3| = \binom{n+3}{3}\).

Let \( S_4 \) be the set of \((n+4)\)-permutations \( w_1w_2 \cdots w_n \) such that, \( w_1 \) is the largest letter among the four leftmost letters, \( w_3 < w_4 \) and \( w_5w_6 \cdots w_n \) is in decreasing order. One can see that \(|S_4| = 3\binom{n}{4}\).

**Bijection 1.** The elements of \( S_1 \) are in one-to-one correspondence with binary bitonic sequences of length \( n-1 \).

In order to avoid the restrictions, \( w_3w_4 \cdots w_{n+2} \) must be either of the form
\[
(i-1) \cdots 1n(i-1) \cdots (i+1) \text{ or the form }
\]
\[
n(i-1) \cdots (i+1)(j+1)j \cdots 1n(i-1) \cdots (j+2)
\]
for some \( i > 0 \) and \( j \geq 0 \).

We describe our bijection in the case \( w_1w_2 = (n+1)(n+2) \). We then use the same bijection for \( w_1w_2 = (n+2)(n+1) \) and replace 0’s by 1’s and 1’s by 0’s in the corresponding sequences.

To the permutation \((n+1)(n+2)i(i-1) \cdots 1n(i-1) \cdots (i+1)\) there corresponds the bitonic sequence \(01^i0^{n-i-2}\) where \( i > 0 \); to the permutation
\[
(n+1)(n+2)n(n-1) \cdots (i+1)(j+1)j \cdots 1n(i-1) \cdots (j+2)
\]
there corresponds the sequence \(00^i1^{n-i-j-2}0^j\) where \( i > 0 \) and \( j \geq 0 \). Clearly, our map involves all the binary bitonic sequences starting from 0 exactly once and the reverse to this map is easy to see. Together with the case \( w_1w_2 = (n+2)(n+1) \) we have a bijection.

**Bijection 2.** The elements of \( S_2 \) are in one-to-one correspondence with binary strings of length \( n+2 \) without singletons.

Clearly, any string under consideration ends with either 00 or with 11. We match the strings ending with 00 with the permutations beginning with \( w_1w_2 = (n+1)(n+2) \). It will suffice to consider this case. The remaining cases are handled by replacing 0’s by 1’s and 1’s by 0’s, proceeding with the first case, and then replacing \((n+1)(n+2)\) with \((n+2)(n+1)\) in the resulting permutation.

We begin with a procedure for construction permutations \( w_3w_4 \cdots w_{n+2} \) that avoid the restricted patterns. Insert the numbers 1, 2, \ldots, \( n \), in that order, into \( n \) slots corresponding to the letters \( w_i \). Any filling of the \( n-i \) empty slots is empty, then we only have two choices: either set \( w_{n-i} = i+1 \) or set \( w_{n-i} = (i+1)(i+2) \). A permutation \( w_3w_4 \cdots w_{n+2} \) that avoids the restricted patterns may be thought of as a tiling of a \( 1 \times n \) board by monominos and dominos.
Now, given such a tiling, we construct a binary string $b_1b_2\cdots b_n00$ corresponding to that tiling. Read the tiling from right to left. If the leftmost tile is a monomino, set $b_n = 0$. Otherwise, set $b_{n-1}b_n = 11$. In general, if the last digit placed in the binary string was $b_i = x \in \{0, 1\}$, and the next unread tile is a monomino, read this tile and set $b_{i-1} = x$. Otherwise, if the next unread tile is a domino, read this domino and set $b_{i-2}b_{i-1} = \bar{x}\bar{x}$, where $\bar{x}$ is the binary complement of $x$. In this way, we avoid the possibility of creating singletons.

This process is reversible: we read a binary word without singletons from right to left while tiling a $1 \times n$ board with monomino and dominoes. Whenever we meet $\bar{x}\bar{x}$ after passing $x$ in the binary string, we place a domino on the board. Otherwise, we place a monomino. The resulting tiling defines the corresponding permutation according to the construction described above. Performing all of these correspondences yields the desired bijection. For example, if $w_1w_2\cdots w_9 = 896753412$, then we produce $b_1b_2\cdots b_9 = 110001100$.

**Bijection 3.** The elements of $S_3$ are in one-to-one correspondence with $(n+2)$-permutations avoiding 1-3-2-4 and having exactly one descent (a descent is an $i$ such that $w_i > w_{i+1}$).

Any $(n+2)$-permutation avoiding 1-3-2-4 and having exactly one descent has the structure $ABCD$, where $A = (i_1 + 1)(i_1 + 2)\cdots i_2$, $B = (i_3 + 1)(i_3 + 2)\cdots (n+2)$, $C = 12\cdots i_1$, $D = (i_2 + 1)(i_2 + 2)\cdots i_3$ (see Figure 1), and one of the following four mutually exclusive possibilities occurs:

1. none of $A$, $B$, $C$, and $D$ is empty: there are $\binom{n+1}{3}$ such permutations, given by the number of ways to choose the least elements in $A$, $B$, and $D$ (we know that 1 belongs to $C$);
2. $C$ is empty: there are $\binom{n+1}{2}$ such permutations, since 1 belongs to $A$ and we choose the least elements in $B$ and $D$;
3. $B$ is empty: there are $\binom{n+1}{2}$ such permutations, since 1 belongs to $C$ and we choose the least elements in $A$ and $D$;
4. $A$ and $C$ are empty: there are $(n+1)$ such permutations, since 1 is in $D$ and we need to choose the length of $D$ ($B$ is not empty).

Note that summing over all the cases gives us exactly $\binom{n+3}{3}$ permutations. Once the permutations in avoiding 1-3-2-4 with exactly one descent have been
partitioned into the four cases above, it is easy to find bijections in each case with permutations in $S_3$ as follows.

1. $abc(n + 3)(n + 2)w_6w_7 \cdot \cdot \cdot w_{n+3}$, where $a < b < c$ and $w_5w_7 \cdot \cdot \cdot w_{n+3}$ is decreasing. Choosing $a$, $b$, and $c$ corresponds to choosing $i_1$, $i_2$, and $i_3$;

2. $ab(n + 3)(n + 2)w_5w_6 \cdot \cdot \cdot w_{n+3}$, where $a < b$ and $w_5w_6 \cdot \cdot \cdot w_{n+3}$ is decreasing. Choosing $a$ and $b$ corresponds to choosing $i_2$ and $i_3$;

3. $ab(n + 2)(n + 3)w_5w_6 \cdot \cdot \cdot w_{n+3}$, where $a < b$ and $w_5w_6 \cdot \cdot \cdot w_{n+3}$ is decreasing. Choosing $a$ and $b$ corresponds to choosing $i_1$ and $i_2$;

4. $a(n+2)(n+3)w_4w_5 \cdot \cdot \cdot w_{n+3}$, where $a < b$ and $w_4w_5 \cdot \cdot \cdot w_{n+3}$ is decreasing. The length of $D$ corresponds to $a$.

Since these cases provide a partition of permutations in $S_3$, the bijection is complete.

**Bijection 4.** The elements of $S_4$ are in one-to-one correspondence with the set of all lines drawn through the points of intersections of $n$ straight lines in a plane, no two of which are parallel, and no three of which are concurrent (we assume here that each of such lines goes through exactly two points of intersections).

If we label the lines by $1, 2, \ldots, n$ then each intersection point can be represented by a pair of numbers $(x, y)$ corresponding to the intersecting lines. Now any line from the set of “new” lines can be described by a pair $((x, y), (z, v))$ where all of $x, y, z,$ and $v$ are different. Assuming that $x < y$, $z < v$, and $y < v$ we construct the corresponding permutation $vzxyw_5w_6 \cdot \cdot \cdot w_n$ where $w_5w_6 \cdot \cdot \cdot w_n$ is decreasing. Clearly this map is a bijection.

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