SUPERSINGULAR SCATTERING

T. Dolinszky

KFKI-RMKI, H-1525 Budapest 114, P.O.B. 49, Hungary

Abstract: In 'supersingular' scattering the potential $g^2U_A(r)$ involves a variable nonlinear parameter $A$ upon the increase of which the potential also increases beyond all limits everywhere off the origin and develops a uniquely high level of singularity in the origin. The problem of singular scattering is shown here to be solvable by iteration in terms of a smooth version of the semiclassical approach to quantum mechanics. Smoothness is achieved by working with a pair of centrifugal strengths within each channel. In both of the exponential and trigonometric regions, integral equations are set up the solutions of which when matched smoothly may recover the exact scattering wave function. The conditions for convergence of the iterations involved are derived for both fixed and increasing parameters. In getting regular scattering solutions, the proposed procedure is, in fact, supplementary to the Born series by widening its scope and extending applicability from nonsingular to singular potentials and from fixed to asymptotically increasing, linear and nonlinear, dynamical parameters.
I. Introduction

We are going to consider scattering by singular potentials $g^2U_A(r)$ at fixed as well as asymptotical values of the linear and nonlinear parameters $g^2$ and $A$, respectively. The pioneering work in treating singular scattering by exact means has been done by Calogero\(^1\) in terms of the phase approach to quantum particle dynamics. In particular, he calculated the high energy limit of scattering by inverse power potentials\(^2\). A further correct approach is due to Froemann and Thylwe\(^3\), who analytically proved the exactness of the first order WKB (Wentzel-Kramers-Brillouin) approximation for the case of pure inverse power potentials in the short wavelength limit. Esposito\(^4\) worked in singular scattering by the wave function polydromy method to get a recursion formula between solutions with potentials of different inverse powers in the radial distance. Notice all the above approaches relied on nonlinear first order or linear second order differential equations solvable analytically, in terms of special functions, for certain simple potential shapes.

Nevertheless, it is the method of integral equations that is the first rank candidate for solving two body scattering problems involving a general shape and stage of the potential singularity. The pertinent Volterra type equations can be, perhaps, best classified in terms of the reference scattering problems implied, as follows.

The most evident reference basis seems to be the unperturbed case $g^2 = 0$ taken at the physical energy $k^2$ and the physical orbital angular momentum $l$. The relevant integral equations set up for the regular wave function $u^+(r)$ work in the case of nonsingular potentials, exclusively (Born series). It is true that for finding irregular solutions $u^-(r)$, there are analogous integral equations available, for cases of nonsingular and singular potentials alike, as shown by Newton\(^5\). However, all the above mentioned procedures can only be employed at fixed sets of the scattering parameters. Namely, the residual potential contained is there invariably given by the expression $\Delta(r) = g^2U_A(r)$, whence the iterated series blows up term-by-
term in the limit $g^2 \to \infty$ (strong coupling). The same is true of the supersingular limit. Namely, the form factor $U_A(r)$, besides being singular for $r \to 0$ at fixed parameters, increases, by definition, beyond all limits at each fixed point off the origin for $A \to \infty$. One has therefore to seek alternative approaches outside of the Born series for solving these asymptotical cases.

An interesting choice of reference basis, as proposed by Newton\textsuperscript{5}, offers itself in the exceptional cases of the solution being known in closed form at zero energy and zero orbital angular momentum. The residual potential is then $k^2 - l(l + 1)/r^2$. It is worthwhile to note that the iteration of the integral equation converges exclusively for singular potentials. Moreover, convergence obviously holds in both limits $g^2 \to \infty$ and $A \to \infty$.

Quite wide is the scope of applicability of the third type of reference problems which we are going to consider in detail. It is, in fact, the semiclassical approximation. The Langer version of this method, see e.g. Newton’s monograph\textsuperscript{5}, is specified by the reference centrifugal strength $(l + \frac{1}{2})^2$. Owing to this very choice, the model wave function reproduces the exact regular solution of quantum mechanics (QM) near the origin in the case of nonsingular potentials. As to singular potentials, the small distance QM wave function and its WKB approximation do coincide, even independently of the selection of the reference centrifugal strength. The problem of a possible convergent iteration and that of asymptotical values taken by the potential parameters had remained thereby still open. Recently, Dolinszky\textsuperscript{6} proposed a smooth version of the semiclassical approach which is free of the highly inconvenient turning point singularity inherent in both the Langer’s and the standard WKB approximation. The new procedure has been applied for developing convergent series expansion of the wave function describing scattering by singular potentials at fixed\textsuperscript{7} as well as increasing\textsuperscript{8} linear dynamical parameters. In the present paper an improved and generalized version of that approach will be proposed, which also includes the asymptotical case of increasing nonlinear parameters present in the Schroedinger equation.
II. Physical and reference problem

The physical problem to be solved first is two-body scattering by a central singular potential say, \( g^2U_A(r) \) at the energy \( k^2 \) in the channel of index \( l \), with \( A \) representing a nonlinear parameter. The potential should fulfill the following requirements:

\[
\begin{align*}
[r^4U_A(r)] &\to \infty, [r \to 0]; \quad U_A(r) > 0, U'_A(r) < 0, [r \geq 0]; \\
[r^3U_A(r)] &\to 0, [r \to \infty]; \quad U_A(r) \to \infty, [r \geq 0, A \to \infty].
\end{align*}
\]  

In order to construct a suitable reference system for treating the problem, we introduce a triad \((\lambda^2_\epsilon, \lambda^2_\tau, \lambda^2)\) of auxiliary centrifugal strengths, subject to the restrictions

\[
\lambda^2(l) = \frac{1}{2} [\lambda^2_\epsilon(l) + \lambda^2_\tau(l)], \quad \lambda^2_\epsilon(l) > \lambda^2_\tau(l). \tag{2.2}
\]

The constants \( \lambda^2_\epsilon \) and \( \lambda^2_\tau \) are for the present freely chosen. The concept ‘matching distance’ is defined as the positive root of the ‘master equation’

\[
k^2R^2 - g^2R^2U_A(R) - \lambda^2 = 0; \quad R = R(k^2, g^2, \lambda^2; A) > 0. \tag{2.3}
\]

The radial distance \( r \) will be in general substituted for by the dimensionless radial coordinate \( t \). We put

\[
t = \frac{r}{R}, \quad [r \geq 0]. \tag{2.4}
\]

Different regions of the space are distinguished as follows:

\[
\begin{align*}
t < 1, \quad & \text{region } \epsilon, \ (\text{exponential region}); \\
t > 1, \quad & \text{region } \tau, \ (\text{trigonometric region}); \\
t = 1, \quad & \text{(matching point).}
\end{align*}
\]  

The scattering process is governed by the radial Schroedinger equation

\[
\left\{ \frac{d^2}{dt^2} + k^2R^2 - g^2R^2U_A(Rt) - \frac{l(l + 1)}{t^2} \right\}u^\pm(t) = 0. \tag{2.6}
\]
A regular-irregular pair $u^\pm(t)$ of its solutions behaves near the origin as

$$u^\pm(t) \to [g^2 R^2 U_A(Rt)]^{-\frac{3}{4}} \exp[\pm gR \int_1^t dt' U_A(Rt')]^\frac{1}{4}, \quad [t \to 0]. \quad (2.7)$$

The reference problem will be a special smooth version of the zero-order semi-classical approximation. The entire argument hinges upon a pair of wavenumber function squares for the exponential and trigonometric regions, respectively, as follows:

$$K_\epsilon^2(t) = K_\tau^2(t) = -k^2 + g^2 U_A(Rt) + \frac{\lambda^2}{R^2 t^2}, \quad [t < 1];$$
$$= K_\tau^2(t) = k^2 - g^2 U_A(Rt) - \frac{\lambda^2}{R^2 t^2}, \quad [t > 1]. \quad (2.8)$$

These definitions imply a number of properties concerning behavior around the matching point. In particular, one extracts from the master equation (2.3) that

$$[K_\epsilon^2(t)]_{t=1} = [K_\tau^2(t)]_{t=1}; \quad [K^2(t)]_{t=1} = \frac{\lambda^2_\epsilon - \lambda^2_\tau}{2R^2}. \quad (2.9)$$

Overall properties also follow from Eq.(2.8). Indeed,

$$K^2(t) > 0, \quad \min\{K^2(t)\} = [K^2(t)]_{t=1}; \quad [0 \leq t];$$
$$K_\epsilon^2(t_1) > K_\tau^2(t_2) \quad \text{if} \quad t_1 < t_2; \quad K_\epsilon^2(t_3) < K_\tau^2(t_4) \quad \text{if} \quad t_3 < t_4. \quad (2.10)$$

These statements issue from the restrictions (2.1) imposed on $U_A(r)$. Small and large distance behavior can also be thence extracted such as

$$K_\epsilon^2(t) \to \infty \quad \text{if} \quad t \to 0; \quad K_\tau^2(t) \to k^2 \quad \text{if} \quad t \to \infty. \quad (2.11)$$

For the derivatives of $K^2(t)$ we shall apply the notation:

$$D_\gamma^{(s)}(t) = \frac{1}{K_\gamma^2(t)} \frac{d^s K_\gamma^2(t)}{dt^s}, \quad [s = 1, 2; \gamma = \epsilon, \tau]. \quad (2.12)$$

Hence one obtains by Eqs.(2.1)-(2.2) and (2.8)

$$[D_\tau^{(1)}(t) - D_\epsilon^{(1)}(t)]_{t=1} = 2^4 \{2\lambda^2 - g^2 R^3 U_A'(R)\} > 0. \quad (2.13)$$
A regular-irregular pair of reference wave functions $w_{\gamma}^\pm(t)$ is defined as

$$w_{\epsilon}^\pm(t) = \eta_{\epsilon}(t) \exp[\pm \omega_{\epsilon}(1, t)], \quad [t < 1];$$  \hspace{1cm} (2.14)

$$w_{\tau}^\pm(t) = \eta_{\tau}(t)[C^\pm \cos \omega_{\tau}(1, t)] + S^\pm \sin \omega_{\tau}(1, t)], \quad [t > 1],$$  \hspace{1cm} (2.15)

where we introduced the ‘amplitude function’ and the ‘phase function’ such as

$$\eta_{\gamma}(t) \equiv \left( \frac{k^2}{K_\gamma^2(t)} \right)^{\frac{1}{2}}, \quad [\gamma = \epsilon, \tau];$$  \hspace{1cm} (2.16)

$$\omega_{\gamma}(t_1, t_2) \equiv R \int_{t_1}^{t_2} dt'[K_{\gamma}(t')], \quad [\gamma = \epsilon, \tau].$$  \hspace{1cm} (2.17)

As to the constants $C^\pm$ and $S^\pm$, these parameters can be, for the moment, freely chosen. Out of them, $C^+$ and $S^+$ will be specified later by smoothness requirements. The choice $C^-$ and $S^-$ will, in turn, remain once for all free but the only restriction

$$C^+ S^- S^+ C^- \neq 0,$$  \hspace{1cm} (2.18)

which warrants independence of the functions $w_{\epsilon}^+(t)$ and $w_{\tau}^-(t)$. Yet, there exists a sophisticated definition of the constants $C^-$ and $S^-$ in terms of $C^+$ and $S^+$, namely the one implied in the identity

$$w_{\epsilon}^-(t) \equiv w_{\gamma}^+(t)\{1 - \int_{1}^{t} \frac{dt'}{w_{\gamma}^+(t')}\}, \quad [\gamma = \epsilon, \tau].$$  \hspace{1cm} (2.19)

This relationship is automatically satisfied in the region $\epsilon$ by the definition (2.14). As regards the point $t = 1 \pm 0$, Eq.(2.19) guarantees there smooth matching of $w_{\tau}^-(t)$ to $w_{\epsilon}^-(t)$ whenever $w_{\tau}^+(t)$ matches there $w_{\epsilon}^+(t)$ smoothly. The particular choice (2.19) is quite irrelevant from the viewpoint of the present argument. Nevertheless, it justifies the use of the same superscripts ($-$) over $w_{\tau}^-(t)$ and $w_{\epsilon}^-(t)$.

The reference wave functions $w_{\gamma}^\pm(t), \quad [\gamma = \epsilon, \tau],$ of Eqs.(2.14)-(2.15) solve the pair ($\epsilon, \tau$) of differential equations

$$\left\{ \frac{d^2}{dt^2} + k^2 R^2 - W_{\gamma}^\pm(t) - \frac{l(l+1)}{t^2} \right\} w_{\gamma}^\pm(t) = 0, \quad [\gamma = \epsilon, \tau].$$  \hspace{1cm} (2.20)
Notice the differential equation (2.20) is common for \( w_+^\gamma(t) \) and \( w_-^\gamma(t) \), in the exponential and trigonometric regions alike. Namely, calculation yields for the reference potential
\[
W_\gamma^\pm(t) \equiv W_\gamma(t), \quad [\gamma = \epsilon, \tau].
\] (2.21)

with the notation
\[
W_\gamma(t) = g^2 R^2 U_A(Rt) + \Delta_\gamma(t), \quad [\gamma = \epsilon, \tau].
\] (2.22)

The expressions of the residual potential \( \Delta_\gamma(t) \) introduced here are extracted from Eqs.(2.14)-(2.15) and (2.20) as
\[
\Delta_\epsilon(t) \equiv -\frac{5}{16} [D_\epsilon^{(1)}(t)]^2 + \frac{1}{4} D_\epsilon^{(2)}(t) - \frac{\lambda_\epsilon^2(t) - l(l+1)}{t^2}, \quad [t < 1],
\] (2.23)

\[
\Delta_\tau(t) \equiv -\frac{5}{16} [D_\tau^{(1)}(t)]^2 + \frac{1}{4} D_\tau^{(2)}(t) - \frac{\lambda_\tau^2(t) - l(l+1)}{t^2}, \quad [t > 1].
\] (2.24)

Recall that \( K^2(t) \) of Eq.(2.8) is by Eqs. (2.9) continuous at \( t = 1 \) but, due to (2.12)-(2.13), not smooth. Therefore, the residual potential develops at the matching point a jump. Indeed,
\[
\Delta_\tau(t + 0) - \Delta_\epsilon(t - 0) \neq 0.
\] (2.25)

For simplicity, see expression (2.24), we shall work hence forward with the centrifugal strengths
\[
\lambda_\epsilon^2 = (l + \frac{1}{2})^2, \quad \lambda_\tau^2 = l(l+1).
\] (2.26)

Observe that this choice is compatible with the inequality contained in the postulates (2.2).

**III. A pair of convergent expansions**

A comparison of the Schroedinger equations (2.6), set up for the exact wave function \( u^+(r) \), and Eq.(2.20), solved by the semiclassical wave function \( w_+(r) \), suggests construction of a pair of integral equations,
\[
v_\epsilon^+(t) = w_\epsilon^+(t) + \int_0^t dt' \Delta_\epsilon(t') G_\epsilon^+(t,t') v_\epsilon^+(t'), \quad [t < 1],
\] (3.1)
\[ v^+_\tau(t) = w^+_\tau(t) + \int_1^t dt' \Delta_\tau(t')G^+_\tau(t, t')v^+_\tau(t'), \quad [t > 1]. \] (3.2)

The solutions \(v^+_\epsilon(t)\) and \(v^+_\tau(t)\), if exist, are solutions of the exact Schroedinger equation within the respective regions. In particular, \(v^+_\epsilon(t)\) is uniquely defined by Eq.(3.1) and furnishes a regular solution of the differential equation (2.6) in the exponential region. The solution \(v^+_\tau(t)\) of Eq.(3.2), in turn, while solving the Schroedinger equation in the trigonometric region, still involves two free constants, \(C^+\) and \(S^+\), as noticed following Eq.(2.15). Smoothness requirement for the overall regular solution \(v^+(t)\) at \(t = 1\) is just sufficient to unequivocally specify these coefficients. As to the notation in (3.1)-(3.2), the residual potentials \(\Delta_\gamma(t)\) have been defined by (2.22)-(2.24). The resolvents involved in the integral equations are formally given as

\[
G^+_\gamma(t, t') = \frac{1}{d^+_\gamma}[w^+_\gamma(t)w^-_\gamma(t') - w^-_\gamma(t)w^+_\gamma(t')], \quad [\gamma = \epsilon, \tau],
\] (3.3)

where the Wronskians contained are by Eq.(2.21) constant and read in general

\[
d^+_\gamma = W_\gamma\{w^+_\gamma(t); w^-_\gamma(t)\} = \text{const.}, \quad [\gamma = \epsilon, \tau]. \tag{3.4a}
\]

In particular, one obtains after some calculations

\[
d^+_\epsilon = -2kR, \quad d^+_\tau = kR(C^+S^- - C^-S^+). \tag{3.4b}
\]

The general expressions (3.3) can be recast in terms of the local wave numbers as

\[
G^+_\epsilon(t, t') = \frac{2 \sinh[\omega_\epsilon(t, t')]}{R[K^2_\epsilon(t)K^2_\epsilon(t')]^{1/4}}, \quad [0 \leq t' \leq t < 1],
\] (3.5)

\[
G^+_\tau(t, t') = \frac{\sin[\omega_\tau(t, t')]}{R[K^2_\tau(t)K^2_\tau(t')]^{1/4}}, \quad [1 \leq t' \leq t]. \tag{3.6}
\]

Irrelevance of any special choice of the basis \(w^+_\tau(t)\) for inclusion in the formula (3.3) manifests itself in the absence of the constants \(C^\pm, S^\pm\) from the last expression.
The solution of the integral equations (3.1)-(3.2) rests upon the recursion schemes

\[
\begin{align*}
  w_{\epsilon n}^+(t) &\equiv \int_0^t dt' \Delta_\epsilon(t')G_\epsilon^+(t, t')w_{\epsilon n-1}^+(t'), \quad [n = 1, 2, 3..]; \quad w_{\epsilon 0}^+(t) \equiv w_{\epsilon}^+(t), \quad (3.7) \\
  w_{\tau m}^+(t) &\equiv \int_1^t dt' \Delta_\tau(t')G_\tau^+(t, t')w_{\tau m-1}^+(t'), \quad [m = 1, 2..]; \quad w_{\tau 0}^+(t) \equiv w_{\tau}^+(t). \quad (3.8)
\end{align*}
\]

A necessary condition for getting the solution sought for by iteration is convergence of each of the infinite series

\[
\begin{align*}
  v_\epsilon^+(t) &= \sum_{n=0}^{\infty} w_{\epsilon n}^+(t), \quad [t < 1]; \quad (3.9a) \\
  v_\tau^+(t) &= \sum_{m=0}^{\infty} w_{\tau m}^+(t), \quad [t > 1]. \quad (3.9b)
\end{align*}
\]

The overall solution \(v^+(t) = \{v_\epsilon^+(t); v_\tau^+(t)\}\) should be throughout smooth. This requirement is realized off the matching point spontaneously. At the matching point, it is the zero order term of the series (3.9b) that exclusively contributes to both the solution \(v_\tau^+(t)\) and its first derivative. If the series (3.9a) is in the region \(\epsilon\) convergent, the smoothness postulate for \(t = 1\) can be recast in a simple form as

\[
[w_\tau^+(t)]_{t=1} = [v_\epsilon^+(t)]_{t=1}; \quad [w_\tau^+(t)']_{t=1} = [v_\epsilon^+(t)']_{t=1}. \quad (3.10)
\]

These conditions simultaneously fix the trigonometric constants \(C^+\) and \(S^+\) thus completing the solution in the region \(\tau\). The convergence proof for the above series becomes more transparent by using the following notation in both regions \(\gamma = \epsilon, \tau\)

\[
\begin{align*}
  q_{\gamma n}^+(t) &\equiv \frac{w_{\gamma n}^+(t)}{w_{\gamma 0}^+(t)}; \quad q_{\gamma 0}^+(t) = 1, \quad (3.11) \\
  p_\gamma(t) &\equiv \frac{\Delta_\gamma(t)}{RK_\gamma(t)}, \quad (3.12) \\
  P_\gamma(t_1, t_2) &\equiv \int_{t_1}^{t_2} dt' |p_\gamma(t')|. \quad (3.13)
\end{align*}
\]
In the exponential region, the formula (3.7) can thus be rewritten as
\[
q_{cn}^+(t) = \int_0^t dt' p_\epsilon(t') \{1 - \exp[2\omega_\epsilon(t, t')]\} q_{cn-1}^+(t'). \tag{3.14}
\]
Hence we get by Eq.(2.17) the inequality
\[
|q_{cn}^+(t)| \leq \int_0^t dt' |p_\epsilon(t')| q_{cn-1}^+(t'), \quad [t < 1]. \tag{3.15}
\]
Iteration yields then by Eq.(3.13)
\[
|q_{cn}(t)| \leq \frac{1}{n!} |P_\epsilon(0, t)|^n, \quad [t < 1], \tag{3.16}
\]
whence one extracts by analysis
\[
\sum_{n=0}^{\infty} |w_{cn}^+(t)| < |w_\epsilon(t)| \exp[P_\epsilon(0, t)], \quad [t < 1]. \tag{3.17}
\]
The series \(v_\epsilon^+(t)\) of Eq.(3.9a) is thus absolutely convergent if and only if the integral \(P_\epsilon(0, t)\) exists and is bounded in \(t = (0, 1)\). As to the trigonometric region, the resolvent formula (3.6) combines with the notation (3.12) to an equivalent form of the recursion relationship (3.8). Accordingly, we get
\[
w_{\tau m}^+(t) = \int_1^t dt' p_\tau(t') \sin \omega_\tau(t, t') \frac{K_{\tau}^{1/2}(t')}{K_{\tau}^{1/2}(t)} w_{\tau m-1}^+(t'), \quad [t > 1]. \tag{3.18}
\]
The monotonicity relationship (2.10) implies then the inequality
\[
|w_{\tau m}^+(t)| < \int_1^t dt' |p_\tau(t')| |w_{\tau m-1}^+(t')|, \quad [t > 1, m = 1, 2, 3...]. \tag{3.19}
\]
Recall now the definition (2.15) and conclude by the zero order, \(m = 0\), identity in the relationships (3.8) that
\[
|w_{\tau 0}^+(t)| < (kR)^{1/2} (|C^+| + |S^+|), \quad [t \geq 1]. \tag{3.20}
\]
\[10\]
Iteration furnishes then by the inequalities (3.19)-(3.20)

\[ |w_{\tau m}^+(t)| < (kR)^{\frac{1}{2}}(|C^+| + |S^+|) \frac{1}{m!} [P_{\tau}(1, t)]^m, \quad [t > 1; m = 1, 2, 3...]. \] (3.21)

Summation over \( m \) yields thus

\[ \sum_{m=0}^{\infty} |w_{\tau m}^+(t)| < (kR)^{\frac{1}{2}}(|C^+| + |S^+|) \exp P_{\tau}(1, t), \quad [t > 1]. \] (3.22)

So the series \( v_{\tau}^+(t) \) of the definition (3.9b) is absolutely convergent whenever the integral \( P_{\tau}(1, t) \) of Eq.(3.13) does exist.

Suppose the regional existence conditions

\[ P_{\varepsilon}(0, t) < \infty, \quad [t < 1], \quad P_{\tau}(1, t) < \infty, \quad [t > 1], \] (3.23)

and, in addition, the smoothness postulate (3.10) are satisfied. If so, then solutions of the integral equations (3.1)-(3.2) together recover the regular solution of the differential equation (2.6). In particular,

\[ u^+(t) = v_{\varepsilon}^+(t), \quad [t < 1]; \quad u^+(t) = v_{\tau}^+(t), \quad [t > 1]. \] (3.24)

Virtually, one always works, instead of the infinite expansions (3.9a)-(3.9b), with cut-off series such as

\[ v_{\varepsilon}^{+(N)}(t) = \sum_{n=0}^{N} w_{\varepsilon n}^+(t), \quad [t < 1]; \] (3.25)

\[ v_{\tau}^{+(NM)}(t) = \sum_{m=0}^{M} w_{\tau m}^{+(N)}(t), \quad [t > 1]. \] (3.26)

Smooth matching of these functions at \( t = 1 \) in terms of the relevant analog of Eqs. (3.10) fixes the trigonometric coefficients \( C^+ \) and \( S^+ \) in terms of the cut-off 'length' \( N \) in the exponential region but independently of its pair \( M \) in the
trigonometric region. These constants therefore should carry the superscripts \((N)\) only. The postulates (3.10), rewritten for the cut-off approach, read

\[
2^{\frac{3}{2}} (kR)^{\frac{3}{2}} C^+(N) = [v_\epsilon^+(N)(t)]_{t=1}, \tag{3.27}
\]

\[
-2^{-\frac{3}{4}} [D^{(1)}_\tau(t)]_{t=1} (kR)^{\frac{3}{4}} C^+(N) + 2^{-\frac{3}{4}} S^+(N) = [v_\epsilon^+(N)(t)']_{t=1}. \tag{3.28}
\]

Notice the cut-off approach must not be used unless the conditions of convergence are fulfilled in both regions \(\epsilon\) and \(\tau\).

**IV. The supersingularity limit**

The term ‘supersingularity’ implies in our terminology a singular potential which is subject to the requirements (2.1). This involves, among others, that it contains a nonlinear parameter \(A\) which may increase beyond all limits. The relevant asymptotical form of the master equation (2.3) reads

\[
k^2 - \frac{g^2}{r_0^2} U_A(R_A) - \frac{\lambda^2}{R_A^2} \rightarrow 0, \quad [A \rightarrow \infty]. \tag{4.1}
\]

Bounded values such as \(R_A \rightarrow 0\) and \(R_A \rightarrow \text{const.}\) are by the properties (2.1) obviously excluded from the large-\(A\) solutions of the Eq.(4.1). One is thus left with the only possibility that \(R_A \rightarrow \infty\) for \(A \rightarrow \infty\). The master equation itself reduces therefore in the supersingularity limit to

\[
U_A(R_A) \rightarrow \frac{k^2 r_0^2}{g^2}, \quad [A \rightarrow \infty]. \tag{4.2}
\]

It is perhaps worth recalling that \(U_A(r) \rightarrow 0\) if \(A = \text{fixed}, r \rightarrow \infty\) while \(U_A(r) \rightarrow \infty\) if \(r = \text{fixed}, A \rightarrow \infty\), as implied in the set of postulates (2.1). In Eq. (4.2), in turn, the singularity parameter \(A\) and the matching radius \(R_A\) vary simultaneously and both increase to infinity so as to keep the left hand side of the equation constant.

Four classes of strongly singular potentials will be introduced with each potential being specified by a variable core parameter \(A\) and a fixed tail parameter
B. The dimensionless form factor is in each case a product of an, for \( r \to 0 \), exponentially or powerlaw increasing core factor and an, for \( r \to \infty \), exponentially or powerlike decreasing tail factor. Concerning the subscripts, we use an obvious notation when writing

\[
U_{\alpha\beta}(r) = \exp\left(\frac{\alpha r_1}{r} - \frac{\beta r}{\rho_2}\right),
\]
\[
U_{a\beta}(r) = (1 + \frac{r_1}{r})^a \exp\left(-\frac{\beta r}{\rho_2}\right),
\]
\[
U_{ab}(r) = \exp\left(\frac{\alpha r_1}{r}(\frac{r_2}{r_2 + r})^b\right),
\]
\[
U_{ab}(r) = (1 + \frac{r_1}{r})^a(\frac{r_2}{r_2 + r})^b,
\]

\[0 \leq r, \quad 0 < \alpha, \beta, r_1, r_2, \rho_1, \rho_2; \quad a > 4; \quad b > 3\].

In the last section, we established general criteria for the convergence of the expansions (3.9a)-(3.9b). In the present section, fulfillment of those requirements will be checked for increasing nonlinear parameters.

In the supersingularity limit \( A \to \infty \), the asymptotical form of the master equation (4.2) is explicitly solvable in cases of exponential tail for both types of core singularity. In fact, one concludes from the definitions (4.3) that

\[
R_{\alpha\beta} \to (\rho_1 \rho_2)^{\frac{1}{2}} \left(\frac{\alpha}{\beta}\right)^{1/2}, \quad [\alpha \to \infty];
\]
\[
R_{a\beta} \to (r_1 \rho_2)^{\frac{1}{2}} \left(\frac{a}{\beta}\right)^{\frac{1}{2}}, \quad [a \to \infty].
\]

In the powerlaw tail cases, in turn, only implicit expressions can be extracted from the formulas (4.2)-(4.3). Indeed, one gets

\[
R_{ab} \to r_2 \exp\left(\frac{\alpha r_1}{b R_{ab}}\right), \quad [\alpha \to \infty],
\]
\[
R_{ab} \to r_2 \exp\left(\frac{a r_1}{b R_{ab}}\right), \quad [a \to \infty].
\]

The matching distance increases in all of our examples slower than the respective variable parameter \( A = \alpha \) or \( a \). Indeed, one finds by the relationships (4.4)-(4.5) that

\[
\frac{R_{AB}}{A} \to 0, \quad [A \to \infty; \quad A = \alpha, a; \quad B = \beta, b].
\]

13
The notation $A \to \infty$ and $R_{AB} \to \infty$ compete in representing the supersingular limit by a single symbol. We prefer the use of the latter alternative where possible. Accordingly, we shall throughout eliminate from the formulas the parameter $A$ in terms of $R_{AB}$.

It is also obvious by analysis that the present scattering formalism should, in the supersingularity limit, become, mutatis mutandis, identical for exponentially and powerlaw increasing cores. The equivalence is realized at the following correspondence of potential parameters:

$$r_1 \alpha \to \rho_1 \alpha,\ [a, \alpha \to \infty, B = \beta, b]. \quad (4.7)$$

In the limit considered, one has thus to treat, out of the four cases in (4.3), only two essentially different ones, namely $(A, \beta)$ and $(A, b)$. We now rewrite the wave number squares of Eqs.(2.8), in terms of the variable $t$ of the definition (2.4), for $R_{AB} \to \infty$. In the cases $(A, \beta)$, this transformation is, owing to the explicit relationships (4.4), straightforward. Indeed, one obtains by the definitions (4.3) in the exponential region

$$K^2_{\epsilon A\beta}(t) \to k^2\left\{\frac{g^2}{k^2 r_0^2} \exp\left[\frac{\beta R_{A\beta}}{\rho_2} (1-t)\right] + \frac{\lambda^2}{k^2 R_{A\beta}^2 t^2} - 1\right\}, \quad [t < 1, R_{A\beta} \to \infty, A = \alpha, a]. \quad (4.8)$$

A little bit more complicated is the incorporation of formula (4.5) into Eq.(2.8) in power tail cases for which one gets still in the region $\epsilon$

$$K^2_{\epsilon A b}(t) \to k^2\left\{\left[\frac{k^2 r_0^2}{g^2} \frac{R_{Ab}}{r_2}\right]^{\frac{1}{b}} - 1\right\} + \frac{\lambda^2}{k^2 R_{Ab}^2 t^2}, \quad [t < 1, R_{Ab} \to \infty, A = \alpha, a]. \quad (4.9)$$

In the trigonometric region, distinction should be made between S-wave and higher partial waves. Indeed, one extracts from the definitions (2.8) and (4.3) for the potential classes $A = \alpha$ and $a$ equally, that

$$K^2_{\tau A\beta}(t) \to k^2\left\{1 - \frac{g^2}{k^2 r_0^2} \exp\left[-\frac{\beta R_{A\beta}}{\rho_2} (t - 1)\right]\right\}, \quad [t > 1, l = 0, R_{A\beta} \to \infty], \quad (4.10)$$

14
\[ K^2_{\tau Ab}(t) \rightarrow k^2 \{ 1 - \left[ \frac{g^2}{k^2 r_0^2} (\frac{r_2}{R_{Ab}})^b \right] \left( 1 - \frac{1}{t^b} \right), \quad [t > 1, l = 0, R_{Ab} \rightarrow \infty] \}, \quad (4.11) \]

\[ K^2_{\tau AB}(t) \rightarrow k^2 \{ 1 - \left[ \frac{\lambda^2}{k^2 R^2_{AB} t^2} \right], \quad [t > 1, l > 0, R_{AB} \rightarrow \infty]. \quad (4.12) \]

Notice for the highest values of the parameter \( A \) the wave number function becomes independent of the potential:

\[ K^2_{\tau AB}(t) \rightarrow k^2, \quad [t > 1, l \geq 0, R_{AB} \rightarrow \infty]. \quad (4.13) \]

The quantities \( D^{(s)}_{\tau}(t) \) of the definition (2.12) will be calculated below from the set of supersingularity expressions (4.8)-(4.12). In the exponential region one finds for both cases \( A = \alpha, a \) in the limit \( R_{AB} \rightarrow \infty \)

\[ D^{(1)}_{\tau A\beta}(t) \rightarrow \frac{\beta R_{A\beta}}{\rho_2} \left( \frac{1}{t^2} + 1 \right), \quad D^{(2)}_{\tau A\beta}(t) \rightarrow \left( \frac{\beta R_{A\beta}}{\rho_2} \right)^2 \left( \frac{1}{t^2} + 1 \right)^2, \quad [t > 1, l \geq 0], \quad (4.14) \]

\[ D^{(1)}_{\tau A}(t) \rightarrow \frac{b}{t^2} \ln(\frac{R_{Ab}}{R_{2}}), \quad D^{(2)}_{\tau A}(t) \rightarrow \frac{b^2}{t^4} \ln(\frac{R_{Ab}}{R_{2}})^2, \quad [t > 1, l \geq 0]. \quad (4.15) \]

As regards the trigonometric region, the formulae are again sensitive to the orbital angular momentum. Indeed, for the potential classes \( A = \alpha, a \) alike, one gets

\[ D^{(1)}_{\tau A\beta}(t) \rightarrow \frac{\beta R_{A\beta}}{\rho_2} \left( \frac{1}{t^2} + 1 \right), \quad D^{(2)}_{\tau A\beta}(t) \rightarrow \left( \frac{\beta R_{A\beta}}{\rho_2} \right)^2 \left( \frac{1}{t^2} + 1 \right)^2, \quad [t > 1, l = 0], \quad (4.16) \]

\[ D^{(1)}_{\tau Ab}(t) \rightarrow \frac{b}{t^2} \ln(\frac{R_{Ab}}{R_{2}})^2, \quad D^{(2)}_{\tau Ab}(t) \rightarrow -\frac{b^2}{t^4} \ln(\frac{R_{Ab}}{R_{2}})^2, \quad [t > 1, l = 0], \quad (4.17) \]

\[ D^{(1)}_{\tau AB}(t) \rightarrow \frac{2\lambda^2}{k^2 R^2_{AB} t^3}, \quad D^{(2)}_{\tau AB}(t) \rightarrow -\frac{6\lambda^2}{R^2_{AB} t^4}, \quad [t > 1, l > 0]. \quad (4.18) \]

The residual potentials (2.23)-(2.24) are quadratic and linear expressions of the quantities (4.14)-(4.18). They read in the exponential region, for exponential and powerlaw cores in like manner,

\[ \Delta_{\tau A\beta}(t) \rightarrow -\frac{1}{16} \left( \frac{\beta R_{A\beta}}{\rho_2} \right)^2 \left( \frac{1}{t^2} + 1 \right)^2, \quad [t < 1, R_{A\beta} \rightarrow \infty], \quad (4.19) \]
\[ \Delta_{\epsilon Ab}(t) \to -\frac{1}{16} \ln(\frac{R_{Ab}}{r_2})^2 \frac{b^2}{t^4}, \quad [t < 1, R_{Ab} \to \infty]. \]  

(4.20)

The analogous formulae of the trigonometric region are sensitive to the orbital angular momentum as

\[ \Delta_{\tau A\beta}(t) \to -\frac{9}{16} \left(\frac{\beta R_{A\beta}}{\rho_2}\right)^2 \left(\frac{1}{t^2} + 1\right)^2, \quad [t > 1, l = 0, R_{A\beta} \to \infty], \]  

(4.21)

\[ \Delta_{\tau Ab}(t) \to -\frac{9}{16} \ln(\frac{R_{Ab}}{r_2})^2 \frac{b^2}{t^4}, \quad [t > 1, l = 0, R_{Ab} \to \infty], \]  

(4.22)

\[ \Delta_{\tau AB}(t) \to \frac{3\lambda^2_\tau}{2k^2 R_{AB}^2 t^4}, \quad [t > 1, l > 0, R_{AB} \to \infty, B = \beta, b]. \]  

(4.23)

The next step towards checking expansions (3.9) for the realization of convergence criteria is calculation of the quantities \( p_{\gamma}(t) \) introduced by Eq.(3.12). The asymptotical expressions (4.8)-(4.13) combine with the ones of (4.19)-(4.23) to yield, first for the exponential region,

\[ p_{\epsilon A\beta}(t) \to -\frac{1}{16} \frac{\beta^2 R_{A\beta} r_0}{g \rho_2^2} \left(\frac{1}{t^2} + 1\right)^2 \exp\left[-\frac{\beta R_{A\beta}}{\rho_2} \left(\frac{1}{t} - t\right)\right], \quad [t < 1, R_{A\beta} \to \infty], \]  

(4.24)

\[ p_{\epsilon Ab}(t) \to -\frac{1}{16} \frac{\ln(\frac{R_{Ab}}{r_2})^2}{k R_{Ab}} \frac{b^2}{t^4} \frac{1}{k^2 r_0^2} \left(\frac{r_2}{R_{Ab}}\right)^b t^{\frac{3}{2} - 2b} \left(\frac{r_2}{R_{Ab}}\right)^b t^{\frac{1}{2} - 1}, \quad [t < 1, R_{Ab} \to \infty], \]  

(4.25)

as well as in the trigonometric region

\[ p_{\tau A\beta}(t) \to -\frac{9}{16} \left(\frac{\beta R_{A\beta}}{\rho_2}\right)^2 \frac{1}{k R_{A\beta}} \left(\frac{1}{t^2} + 1\right)^2, \quad [t > 1, l = 0, R_{A\beta} \to \infty], \]  

(4.26)

\[ p_{\tau Ab}(t) \to -\frac{9}{16} \ln(\frac{R_{Ab}}{r_2})^2 \frac{b^2}{t^4} \frac{1}{k R_{Ab}}, \quad [t > 1, l = 0, R_{Ab} \to \infty], \]  

(4.27)

\[ p_{\tau AB}(t) \to \frac{3\lambda^2_\tau}{2k^2 R_{AB}^2 t^4} \frac{1}{k R_{AB}}, \quad [t > 1, l > 0, R_{AB} \to \infty]. \]  

(4.28)

The supersingularity forms of \( P_{\epsilon}(0, t) \) and \( P_{\tau}(1, t) \) are obtained by first integrating the expressions (4.24)-(4.28) over the relevant intervals and subsequently going over to the limit \( R_{AB} \to \infty \).
Within the exponential region, the limits \( t \to 0 \) and \( R_{AB} \to \infty \) mutually strengthen the rate of vanishing. The factor \((\frac{r_2}{R_{AB}})\) in the expression (4.25) vanishes at \( R_{Ab} > r_2 \) for \( t \to 0 \). The functions \( p_{eAB}(t) \) are integrable near the point \( t = 0 \) even at finite values of \( R_{AB} \). Moreover, its integral vanishes in the limit \( R_{AB} \to \infty \). Thus

\[
P_{eAB}(0, t) \to 0, \quad [t < 1; A = \alpha, a; R_{AB} \to \infty]. \quad (4.29)
\]

By virtue of the inequality (3.17), the condition for the absolute convergence of the series (3.9a) is thus fulfilled in the supersingularity limit along the exponential region for each of the potentials \( U_{AB}(r) \) of the set (4.3).

As regards the trigonometric region, there is a competition between the potential and the centrifugal term and this is the point that governs convergence. In the absence of the latter, one extracts from Eq.(4.26) for both cases \( A = \alpha, a \) that

\[
P_{\tau A\beta}(1, t) \to \infty, \quad [t > 1; l = 0; R_{A\beta} \to \infty]. \quad (4.30)
\]

The convergence of the series (3.9b) is thereby frustrated for the S-wave whenever the physical potential decreases exponentially. Not so for the higher partial waves or the cases of powerlaw tails. Equations (4.27)-(4.28) imply namely that

\[
P_{\nu A\beta}(1, t) \to 0, \quad [t > 1; l = 0; R_{A\beta} \to \infty], \quad (4.31)
\]

\[
P_{\nu AB}(1, t) \to 0, \quad [t > 1; l > 0; R_{AB} \to \infty]. \quad (4.32)
\]

The asymptotical relationships (4.29), (4.31) and (4.32) ensure, by the the inequalities (3.17) and (3.22), for the respective potential classes and partial waves, fulfillment of the convergence conditions (3.23), simultaneously at fixed and increasing values of the nonlinear parameter \( A \), involved in the scattering potential \( U_{AB}(r) \). One can thus write in the limit \( R \to \infty \) that

\[
u^+(t) \to v^+_e(t), \quad [t < 1; B = \beta, b; l \geq 0], \quad (4.33)
\]
\[ u^+(t) \rightarrow v_+^+(t), \quad [t > 1; B = b; l = 0], \tag{4.34} \]
\[ u^+(t) \rightarrow v_+^+(t), \quad [t > 1; B = b, \beta; l > 0]. \tag{4.35} \]

V. Asymptotical exactness

In the last section we studied the question whether a semiclassical expansion that is convergent at a fixed set of dynamical parameters preserves this property invariably in the supersingular limit. A further point we are left with to clear is how the structure of the series changes upon going with the nonlinear parameter to infinity. A characteristic quantity of the argument for finding the answer is the ratio of two neighboring general terms. Recall first the definition (3.11) of \( q_+^e(t) \) along with the relevant recursion relationships (3.14) and (3.18). For simplicity, in the present section we are going to suppress in the formulas the matching distance \( R \). Remember it is a functional of the scattering potential \( g^2U_A(Rt) \) and is, in fact, invisibly present in each quantity and expression below.

In the exponential region, the exact formula (3.14) reduces in the supersingular limit, owing to the definition (2.17), to
\[ q_+^e(t) \rightarrow \int_0^t dt' p_e(t') q_+^e(t'), \quad [t < 1, R \rightarrow \infty]. \tag{5.1} \]
Iteration yields then in the limit considered a familiar expression
\[ q_+^e(t) \rightarrow \frac{1}{n!} [P_e(0, t)]^n, \quad [t < 1, R \rightarrow \infty]. \tag{5.2} \]
The scattering wave function reads thus by the definitions (3.9a) and (3.13) in the limit discussed
\[ v_+^e(t) \rightarrow w_+^e(0) \exp P_e(0, t), \quad [t < 1, R \rightarrow \infty]. \tag{5.3} \]
For treating the trigonometric region, it is useful to introduce higher order trigonometric 'coefficients', which are, as a matter of fact, no more constant. The definition is meant to hold for both fixed or variable dynamical parameters and reads
\[ w_+^e(t) = \eta_e(t)[C_m^e(t) \cos \omega(t) + S_m^e(t) \sin \omega(t)], \quad [t > 1, m = 0, 1..]. \tag{5.4} \]
Recognize the identity (5.4) reproduces at \( m = 0 \) the definition (2.15), on account of which one finds that \( C_0^+ = C^+ \) and \( S_0^+ = S^+ \). Combination of this identity with the recursion formula (3.18) furnishes in our limit, after separating the sine- and cosine- contributions, the following system of equations

\[
C_m^+(t) \to \int_1^t dt' p_\tau(t') \sin[kR(t' - 1)]w_{\tau m-1}(t'), \quad [t > 1, R \to \infty], \quad (5.5)
\]

\[
S_m^+(t) \to \int_1^t dt' p_\tau(t') \cos[kR(t' - 1)]w_{\tau m-1}(t'), \quad [t > 1, R \to \infty]. \quad (5.6)
\]

Insertion of the definition (5.4) into the right hand sides of the last two formulas yields by analysis, owing to the infinitely rapid oscillations in the integrands, a pair of coupled systems of integral equations such as

\[
C_m^+(t) \to \frac{1}{2} \int_1^t dt' p_\tau(t') S_{m-1}^+(t'), \quad [t > 1, R \to \infty], \quad (5.7)
\]

\[
S_m^+(t) \to -\frac{1}{2} \int_1^t dt' p_\tau(t') C_{m-1}^+(t'), \quad [t > 1, R \to \infty]. \quad (5.8)
\]

At the end of the iteration, at \( m = 0 \), one encounters the constant coefficients \( C_0^+, S_0^+ \). The system of integral equations becomes thereby explicitly solvable. The solution reads, in terms of the notation

\[
T_m(t) \equiv \frac{1}{m!} \left[ \frac{1}{2} P_\tau(1, t) \right]^m, \quad [t > 1, m = 0, 1, 2, ..], \quad (5.9)
\]

as follows:

\[
C_{4\mu}^+(t) \to +T_{4\mu}(t)C_0^+, \quad S_{4\mu}^+(t) \to +T_{4\mu}(t)S_0^+, \quad (5.10)
\]

\[
C_{4\mu+1}^+(t) \to +T_{4\mu+1}(t)S_0^+, \quad S_{4\mu+1}^+(t) \to -T_{4\mu+1}(t)C_0^+, \quad (5.11)
\]

\[
C_{4\mu+2}^+(t) \to -T_{4\mu+2}(t)C_0^+, \quad S_{4\mu+2}^+(t) \to -T_{4\mu+2}(t)S_0^+, \quad (5.12)
\]

\[
C_{4\mu+3}^+(t) \to -T_{4\mu+3}(t)S_0^+, \quad S_{4\mu+3}^+(t) \to +T_{4\mu+3}(t)C_0^+, \quad (5.13)
\]
where $\mu = 0, 1, 2, \ldots$. Incorporation of the formulas (5.10)-(5.13) into the definition (3.9b) furnishes

$$v^+_{\tau}(t) \rightarrow \sum_{\mu=0}^{\infty} \left\{ \left[ T_{4\mu}(t) - T_{4\mu+2}(t) \right] w^+_{\tau 0}(t) + \left[ T_{4\mu+1}(t) - T_{4\mu+3}(t) \right] w^-_{\tau 0}(t) \right\}, \quad [t > 1, R \rightarrow \infty],$$

(5.14)

where we used the ad hoc yet not inconsistent notation

$$w^-_{\tau 0}(t) = \eta_{\tau}(t) \left\{ S^+_0 \cos[kR(t-1)] - C^+_0 \sin[kR(t-1)] \right\}, \quad [t > 1],$$

(5.15)

Observe there is only a single quadrature involved in the asymptotical formula (5.14), namely the one implicitly contained in the definition (5.9). Knowledge of the higher order $t$-dependent coefficients thus rests upon the knowledge of the $R \rightarrow \infty$ form of the zero order constants $C^+_0$ and $S^+_0$. These constants are fixed by the claim for the smoothness of the exact but supersingularity solution at $t = 1$ as follows.

At the matching point itself, approaching it from either of the regions $\epsilon$ or $\tau$, one obtains the respective amplitude functions, see Eq.(2.16), along with the relevant derivatives as follows:

$$[\eta_{\gamma}(t)]_{t=1} \rightarrow 2^{\frac{3}{4}}(kR)^{\frac{1}{2}}, \quad [\gamma = \epsilon, \tau; \; R \rightarrow \infty],$$

(5.16)

$$\left[ \frac{d\eta_{\gamma}(t)}{dt} \right]_{t=1} \rightarrow \pm 2^{\frac{7}{4}}(kR)^{\frac{1}{2}} [2\lambda^2 - g^2 R^3 U'(R)], \quad [\gamma = (\epsilon); \; R \rightarrow \infty].$$

(5.17)

The second term in the brackets vanishes in this limit by the restrictions imposed upon the potentials in virtue of the asymptotical relationships (2.1).

As to the exponential region, the supersingularity formula (5.3) can be identically recast by means of the definition (2.14) as

$$v^+_{\epsilon}(t) \rightarrow \eta_{\epsilon}(t) \exp \left[ \omega_{\epsilon}(1, t) + P_{\epsilon}(0, t) \right], \quad [t < 1, R \rightarrow \infty].$$

(5.18)
Approaching the matching point from this region, one concludes by Eqs. (5.16)-(5.18) that
\[
[v_ε^+(t)]_{t=1} \to 2^{\frac{3}{4}} (kR)^{\frac{3}{4}} \exp P_ε(0,1), \quad [R \to \infty],
\]
\[
\left[\frac{dv_ε^+(t)}{dt}\right]_{t=1} \to 2^{\frac{3}{4}} (kR)^{\frac{3}{4}} \exp P_ε(0,1)\{4\lambda^2 + 2^{-\frac{3}{4}} + [p_ε(t)]_{t=1}\}, \quad [R \to \infty].
\]
(5.19)
(5.20)

Recall now the expressions (4.24)-(4.25) and extract from them
\[
[p_ε(t)]_{t=1} = O\left\{\frac{1}{kR}\right\} \to 0, \quad [B = b; R \to \infty],
\]
(5.21)
\[
[p_ε(t)]_{t=1} = O\left\{\frac{R}{\rho_2}\right\} \to \infty, \quad [B = \beta; R \to \infty].
\]
(5.22)

Therefore, we have to restrict further discussion to scattering by potentials of the class \(U_{Ab}(r), [A = \alpha, a]\) of Eqs. (4.3). As regards the trigonometric region, the definition (2.15) yields
\[
[w_τ^+(t)]_{t=1} \to 2^{\frac{3}{4}} (kR)^{\frac{3}{4}} C_0^+, \quad [B = b, R \to \infty],
\]
(5.23)
\[
\left[\frac{dw_τ^+(t)}{dt}\right]_{t=1} \to (kR)^{\frac{3}{4}} [−2^{\frac{15}{4}} \lambda^2 C_0^+ + S_0^+], \quad [B = b, R \to \infty].
\]
(5.24)

Smooth matching is realized by equating Eqs. (5.19) and (5.23) as well as (5.20) and (5.24), respectively. In doing so, one obtains
\[
C_0^+ \to \exp P_ε(0,1), \quad S_0^+ \to \exp P_ε(0,1)[2^{\frac{15}{4}} \lambda^2 + 2^{-\frac{3}{4}}], \quad [B = b, R \to \infty].
\]
(5.25)

As to the region \(\epsilon\), Eqs. (4.29), (4.33) and (5.3), as well as concerning the region \(\tau\), Eqs. (4.31), (4.32), (5.9) and (5.14), combine to the statement that, for potentials \(U_{AB}(r)\) with powerlaw tails, [B=b], the pair of zero order terms in the semiclassical expansions of the respective regions themselves recover in the supersingular limit the exact QM solution. In particular,
\[
u^+(t) \to w_ε^+(t), \quad [t < 1; B = \beta, b; R \to \infty],
\]
(5.26)
\[
u^+(t) \to w_τ^+(t), \quad [t > 1; B = b; l \geq 0]; R \to \infty.
\]
(5.27)
In cases of exponential potential tails, the S-partial waves should be excluded from the present approach.

VI. Discussion

Usefulness of a modified semiclassical approach in treating singular scattering has been checked in three steps. First, the conditions were reconsidered at which a smooth WKB method, proposed recently, produces convergent expansion of the wave function at fixed set of the potential parameters. An inherent new point is that, instead of a single one, a pair of integral equations should be set up, one for each of the exponential and the trigonometric regions. A regular solution working in the exponential region selects by virtue of the smoothness postulate a particular one out of the solutions prepared for the trigonometric region. Another new item is the extension of the argument over potentials involving variable nonlinear parameters through the variation of which one may increase the core singularity to asymptotically high levels. Analyzed are four classes of interactions each of which is a product of a core factor implying exponential or powerlaw singularity and of a tail factor that decays either exponentially or powerlike. Independently of the stage of the singularity, the powerlike decaying potentials invariably develop absolute convergent expansions, along both regions, also in the supersingularity limit. In scattering by exponentially decaying potentials, the criteria of convergence seemingly fail to work within our argument of treating supersingularity limit. A further new feature of the present approach is a discussion of the quality of convergence. The conditions are found which the shape of the potential has to show up so that the infinite series shrink, in the supersingularity limit, to a single term. It is, perhaps, worth mentioning that, by varying the length of the cut-off series in the region beyond the matching point, one can obtain, at the expense of a single quadrature, a solution that becomes correct to any prescribed order in the reciprocal supersingularity parameter at its asymptotically large values.
Recall the Born series furnishes the physical scattering wave function for nonsingular potentials at fixed linear and nonlinear dynamical parameters, exclusively. The proposed smooth WKB approach should work for singular potentials at fixed and asymptotical values of linear and nonlinear parameters involved in the Schrödinger equation.

**Acknowledgement** Many thanks are due to Dr. G. Bencze for useful critical remarks. The author is grateful to Dr. G. Kluge and Dr. I. Racz for very valuable discussions. The work was partly supported by the Hungarian NSF under Grant No. OTKA 00157.

**References**

1. F. Calogero: Variable Phase Approach to Potential Scattering, (Academic Press, New York, 1967)
2. F. Calogero: Phys. Rev. **B 135**, 693 (1964)
3. G. Esposito: J. Phys **A 31**, 9493 (1998)
4. N. Froeman and K.-F. Thylwe: J. Math. Phys **20**, 1716 (1979)
5. R.G. Newton: Scattering Theory of Waves and Particles, (Springer Verlag, 1981)
6. T. Dolinszky: Physics Letters **A 132**, 69 (1988)
7. T. Dolinszky: J. Math. Phys. **36**, 1621 (1995)
8. T. Dolinszky: J. Math. Phys. **38**, 16 (1997).