JULIA’S EQUATION AND DIFFERENTIAL TRANSCENDENCE

MATTHIAS ASCHENBRENNER AND WALTER BERGWEILER

Abstract. We show that the iterative logarithm of each non-linear entire function is differentially transcendental over the ring of entire functions, and we give a sufficient criterion for such an iterative logarithm to be differentially transcendental over the ring of convergent power series.

1. Introduction and Main Results

In 1871, Schröder [33] suggested to study the iteration of a meromorphic function \( f \) by using the functional equation

\[
\phi(\lambda z) = f(\phi(z))
\]

that now bears his name. If \( f \) satisfies this equation, then the compositional iterates \( f^n \) of \( f \) satisfy \( \phi(\lambda^n z) = f^n(\phi(z)) \), so in principle we have an “explicit” expression for the iterates of \( f \) in terms of \( \phi \) and its inverse function. Schröder gave various examples, e.g., \( \phi(z) = \tanh z, \lambda = 2 \) and \( f(z) = 2z/(1 + z^2) \), as well as Jacobian elliptic functions \( \phi \) which satisfy (1.1) for certain rational functions \( f \).

Kœnigs [25] considered the case that \( f \) is holomorphic in a neighborhood of a fixed point \( \xi \) and showed that if the multiplier \( \lambda = f'(\xi) \) of \( f \) at \( \xi \) satisfies \( \lambda \neq 0 \) and \( |\lambda| \neq 1 \), then (1.1) has a unique solution \( \phi \) holomorphic in a neighborhood of 0 such that \( \phi(0) = \xi \) and \( \phi'(0) = 1 \). Poincaré [31, p. 318] observed that if \( |\lambda| > 1 \) and if \( f \) is rational, then \( \phi \) extends to a function meromorphic in the plane, and if \( |\lambda| > 1 \) and \( f \) is entire, then \( \phi \) is entire. Therefore the solution \( \phi \) of (1.1) for \( |\lambda| > 1 \) is also called the Poincaré function of \( f \) at \( \xi \).

Schröder [33, p. 303] expressed the opinion that the functions \( f \) whose iterates can be determined using (1.1) are of rather special type. One may argue that the results of Kœnigs and Poincaré say the opposite, but support for Schröder’s view is given by a result of Ritt [32] which implies that only very few Poincaré functions are elementary functions. In order to state Ritt’s result, we recall that a holomorphic function or, more generally, a formal power series \( g \), is said to be differentially algebraic if it satisfies an algebraic differential equation; that is, an equation of the form

\[
P(z, g(z), g'(z), \ldots, g^{(n)}(z)) = 0
\]

where \( P \) is a non-zero polynomial in \( 2 + n \) indeterminates (for some \( n \)) with constant coefficients; if \( g \) is not differentially algebraic, then \( g \) is called differentially transcendental. Ritt’s result says that a polynomial with a differentially algebraic Poincaré...
function is conjugate to a monomial, a Chebychev polynomial, or the negative of a Chebychev polynomial, the corresponding Poincaré functions being the exponential or trigonometric functions. For rational functions there are additional cases arising from the multiplication theorems of elliptic functions. Poincaré functions of transcendental entire functions are always differentially transcendental [11].

A family of meromorphic functions is called coherent (or uniformly differentially algebraic) if there exists an algebraic differential equation which is satisfied by all functions in the family, and incoherent otherwise. Boshernitzan and Rubel [14, Theorem 6.1] showed that a Poincaré function of a rational or entire function \( f \) is differentially algebraic if and only if the family of iterates of \( f \) is coherent. Thus the above results about differential transcendence of Poincaré functions can be rephrased as results about incoherence of iterates.

We now turn to the case where the multiplier \( \lambda = f'(\xi) \) of \( f \) at its fixed point \( \xi \) does not satisfy the conditions \( \lambda \neq 0 \) and \( |\lambda| \neq 1 \) required for Kœnigs’ theorem. If \( |\lambda| = 1 \), but \( \lambda \) is not a root of unity, Schröder’s equation still has a formal power series solution. The question whether this series converges is rather delicate and forms the subject matter of famous results of Siegel, Brjuno and Yoccoz; see [30, Section 11] for a discussion. However, regardless of whether the series converges or not, it is differentially transcendental whenever \( f \) is a non-linear rational or entire function [8].

If \( \lambda = 0 \), then instead of Schröder’s equation one considers Böttcher’s equation. Again the solutions are differentially transcendental except in special cases [8].

Suppose now that \( \lambda \) is a root of unity. In this case the fixed point \( \xi \) is also called parabolic. Passing to an iterate of \( f \) we may assume that \( \lambda = 1 \). Assuming without loss of generality that \( \xi = 0 \) we then write \( f \) in the form

\[
(1.2) \quad f(z) = z + \sum_{k=p}^{\infty} f_k z^k \quad (p \geq 2, f_k \in \mathbb{C} \text{ for } k \geq 2, f_p \neq 0).
\]

A basic result of complex dynamics, called the Leau-Fatou flower theorem [21, 27], says that there are \( p-1 \) domains \( L_1, \ldots, L_{p-1} \), called petals of \( f \), such that \( f(L_j) \subseteq L_j \) and \( f^n \upharpoonright L_j \to 0 \) as \( n \to \infty \), for \( j = 1, \ldots, p-1 \). (See [30, Section 10].) Moreover, the Abel functional equation \( \phi(z+1) = f(\phi(z)) \) has a holomorphic solution \( \phi_j \) mapping the right half-plane to \( L_j \). The functions \( \phi_j \) are again differentially transcendental [8].

A way to describe the iteration of \( f \) not only in the petals but in a full neighborhood of 0 is based on the functional equation

\[
(1.3) \quad \phi(f(z)) = f'(z)\phi(z)
\]

which is named after Julia (e.g., in [26, Sections 3.5B and 8.5A]) or Jabotinsky (e.g., in [1]). It has a unique formal power series solution

\[
(1.4) \quad \phi(z) = f_p z^p + \sum_{k=p+1}^{\infty} \phi_k z^k \quad (\phi_k \in \mathbb{C} \text{ for } k \geq p + 1),
\]

which is called the iterative logarithm of \( f \) and denoted here by itlog(\( f \)). The name iterative logarithm, introduced by Écalle (see [16, p. 8] or [17]), is explained by the
identity

\[ \text{itlog}(f^n) = n \text{itlog}(f) \quad \text{valid for all } n \in \mathbb{N}. \]

The general solution of (1.3) is given by \( \phi = \alpha \text{itlog}(f) \) where \( \alpha \in \mathbb{C} \).

The series in (1.4) converges only in exceptional cases. For example, a result of Erdős and Jabotinsky [19] in combination with results of Baker [4] and Szekeres [37] shows that the only functions \( f \) meromorphic in \( \mathbb{C} \) and of the form (1.2) for which the series in (1.4) converges in some neighborhood of 0 are the functions \( f(z) = z/(1-cz) \) where \( c \in \mathbb{C} \), with \( \text{itlog}(f)(z) = cz^2 \). (However, Écalle [18] has shown that the iterative logarithm of a function \( f \) holomorphic in a neighborhood of 0 satisfying (1.2) is always Borel summable.)

It follows from the results in [11, 14, 32] that the iterative logarithm \( \text{itlog}(f) \) of a non-linear rational or entire function \( f \) is differentially transcendental; cf. the remarks at the end of section 2.2. This can be viewed as an indication that the coefficient sequence \( (\phi_k)_{k \geq p} \) is very irregular: If a formal power series \( y = \sum_k y_k z^k \in \mathbb{C}[[z]] \) is differentially algebraic, then the coefficient sequence \( (y_k) \) satisfies a certain (in general, non-linear) kind of recurrence relation [29, pp. 186–194]. Of particular importance in combinatorial enumeration is the class of \( D \)-finite (also called holonomic) power series [35, Chapter 6]. These are the formal power series whose coefficient sequence satisfies a homogeneous linear recurrence relation of finite degree with polynomial coefficients; equivalently [35, Proposition 6.4.3] those which satisfy a non-trivial linear differential equation over \( \mathbb{C}[z] \).

In this paper, we show that for entire functions we have an even stronger irregularity result. To formulate this result we need some terminology: Given a subring \( R \) of the ring \( \mathbb{C}[[z]] \) of formal power series over \( \mathbb{C} \) which is closed under differentiation, we say that \( \phi \in \mathbb{C}[[z]] \) is differentially transcendental over \( R \) if \( \phi \) does not satisfy a non-trivial polynomial equation in \( \phi \) and its derivatives with coefficients from \( R \). (Thus “differentially transcendental” is synonymous with “differentially transcendental over \( \mathbb{C}[z] \).”)

**Theorem 1.** Let \( f \) be a non-linear entire function of the form (1.2). Then \( \text{itlog}(f) \) is differentially transcendental over the ring of entire functions.

Under an additional hypothesis we can even show that \( \text{itlog}(f) \) is differentially transcendental over the ring \( \mathbb{C}\{z\} \) of power series with positive radius of convergence. In order to state this hypothesis, for an entire function \( f \) we denote by \( \text{sing}(f^{-1}) \) the set of singularities of the inverse function of \( f \); see [10, Section 4.3] for a discussion of their role in complex dynamics. The set \( \text{sing}(f^{-1}) \) coincides with the set of critical and (finite) asymptotic values of \( f \). If \( f \) is a polynomial, then we only have to consider critical values, since polynomials have no finite asymptotic values. The **Speiser class \( S \)** consists of all non-linear entire functions \( f \) for which \( \text{sing}(f^{-1}) \) is finite. It plays an important role in complex dynamics; cf. [10, 20].

The maximal domains \( U_j \) \((j = 1, \ldots, p - 1)\) containing the petals \( L_j \) such that \( f(U_j) \subseteq U_j \) and \( f^n \upharpoonright U_j \rightarrow 0 \) as \( n \rightarrow \infty \) are called **Leau domains** of \( f \). If \( z \in U_j \), then \( f^n(z) \in L_j \) for large \( n \). A classical result of Fatou (see [10, Theorem 7] or [30, Theorem 10.15]) says that \( U_j \cap \text{sing}(f^{-1}) \neq \emptyset \) for all \( j = 1, \ldots, p - 1 \).
Theorem 2. Let $f \in S$ be of the form (1.2). Denote by $U_1, \ldots, U_{p-1}$ the associated Leau domains and suppose that

$$\text{sing}(f^{-1}) \subseteq \{0\} \cup \bigcup_{j=1}^{p-1} U_j.$$  

Then $\text{itlog}(f)$ is differentially transcendental over $\mathbb{C}(z)$.

Examples to which Theorem 2 applies are $f_1(z) = z + z^2$ and $f_2(z) = e^z - 1$. The function $f_1$ has only one critical point at $-1/2$ and $f(-1/2) = -1/4$ is the corresponding critical value. The function $f_2$ has the only asymptotic value $-1$ and no critical values. Thus $\text{sing}(f_1^{-1}) = \{-1/4\}$ and $\text{sing}(f_2^{-1}) = \{-1\}$. It follows from the result of Fatou mentioned above, or by direct computation, that $f_1$ and $f_2$ satisfy the hypothesis of Theorem 2.

Other examples are $f_3(z) = \sin z$ with $\text{sing}(f_3^{-1}) = \{1, -1\}$ and two Leau domains at 0, one containing 1 and one containing $-1$, and $f_4(z) = ze^z$ with $\text{sing}(f_4^{-1}) = \{0, -1/e\}$.

The results of [12, 15] imply that if $\text{Re} a > 3/4$, then both critical points of $f(z) = z + z^2 + az^3$ are in the Leau domain at 0. Thus $f$ satisfies the hypothesis of Theorem 2 if $\text{Re} a > 3/4$. In fact, this even holds [15, p. 277] if $\text{Re} a \geq 3/4 - 1/(2 \log 3)$.

We prove Theorem 1 in Section 3, after some preliminaries in Section 2, and then use it to deduce Theorem 2 in Section 4.

Theorem 2 suggests the following question, whose answer we do not know:

**Question.** Let $f$ be any transcendental entire function of the form (1.2). Is $\text{itlog}(f)$ differentially transcendental over $\mathbb{C}(z)$?

The iterative logarithm

$$\text{itlog}(e^z - 1) = \frac{1}{2} z^2 - \frac{1}{12} z^3 + \frac{1}{48} z^4 - \frac{1}{180} z^5 + \frac{11}{8640} z^6 - \frac{1}{6720} z^7 + \cdots$$

of $f(z) = e^z - 1$ is of particular interest since it is the exponential generating function (egf) of a sequence

$$0, 0, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{12}, -\frac{3}{4}, -\frac{11}{6}, -\frac{29}{4}, -\frac{493}{12}, -\frac{2711}{6}, -\frac{12406}{15}, -\frac{2636317}{60}, \ldots$$

of rational numbers which recently arose both in a conjecture made by Shadrin and Zvonkine [34] (and proved in [2]) in connection with a generating series for Hurwitz numbers, and also in another context (ongoing joint work of the first-named author with van den Dries and van der Hoeven on asymptotic differential algebra [3]). By Theorem 2, its egf $\text{itlog}(e^z - 1)$ is differentially transcendental over $\mathbb{C}(z)$. We do not know whether the ordinary generating function (ogf) of this sequence is differentially transcendental over $\mathbb{C}(z)$, let alone over $\mathbb{C}(z)$. (See [24] for some differential transcendence results over $\mathbb{C}(z)$ for ogf’s of sequences of combinatorial origin.) We also do not know whether all coefficients $\phi_k$ of the power series $\phi = \text{itlog}(e^z - 1) \in \mathbb{Q}[[z]]$ are non-zero for all $k > 2$. (A computation with MAPLE showed that this is the case for $k = 3, \ldots, 300$.) Some general results about the coefficient sequence $(\phi_k)$ in the case where $\phi = \text{itlog}(f) \in \mathbb{C}[[z]] \setminus \mathbb{C}(z)$ can be found in [23].
Conventions and notations. Throughout the paper, $i$, $j$, $m$, $n$, $p$ range over the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers.

2. Preliminaries

In this section we first introduce some basic terminology concerning differential polynomials used later. We then recall more basic facts on repelling periodic points and Poincaré functions, in addition to the ones already appearing in the introduction. In the last part of this section we state a theorem of Steinmetz which is at the heart of the proof of Theorem 1.

2.1. Algebraic differential equations. Let $R$ be a differential ring, that is, a commutative ring (with 1) equipped with a derivation of $R$, i.e., a map $f \mapsto f'$: $R \to R$ which is additive and satisfies the Leibniz Rule:

$$(f + g)' = f' + g', \quad (f \cdot g)' = f' \cdot g + f \cdot g' \quad \text{for all } f, g \in R.$$  

We let $f \mapsto f^{(n)}$ denote the $n$th compositional iterate of $f \mapsto f'$. A subring $S$ of $R$ which is closed under $f \mapsto f'$ is called a differential subring of $R$, and in this case $R$ is called a differential ring extension of $S$. For any $(r + 1)$-tuple $i = (i_0, \ldots, i_r)$ of natural numbers and an element $y$ in a differential ring extension of $R$, put

$$y^i := y^{i_0}(y')^{i_1} \cdots (y^{(r)})^{i_r}.$$  

Let $Y$ be a differential indeterminate over $R$. Then $R\{Y\}$ (not to be confused with $\mathbb{C}\{z\}$) denotes the ring of differential polynomials in $Y$ over $R$. As ring, $R\{Y\}$ is just the polynomial ring $R[Y, Y', Y'', \ldots]$ in the distinct indeterminates $Y^{(n)}$ over $R$, where as usual we write $Y = Y^{(0)}$, $Y' = Y^{(1)}$, $Y'' = Y^{(2)}$, etc. We consider $R\{Y\}$ as the differential ring whose derivation extends the derivation of $R$ and satisfies $(Y^{(n)})' = Y^{(n+1)}$ for every $n$. For $P(Y) \in R\{Y\}$ and $y$ an element of a differential ring extension of $R$, we let $P(y)$ be the element of that extension obtained by substituting $y, y', \ldots$ for $Y, Y', \ldots$ in $P$, respectively. We say that an element $y$ of a differential ring extension of $R$ is differentially algebraic over $R$ if there is some $P \in R\{Y\}$, $P \neq 0$, such that $P(y) = 0$, and if $y$ is not differentially algebraic over $R$, then $y$ is said to be differentially transcendental over $R$.

For $P \in R\{Y\}$, the smallest $r \in \mathbb{N}$ such that $P \in R[Y, Y', \ldots, Y^{(r)}]$ is called the order of the differential polynomial $P$. Let $P \in R\{Y\}$ have order $r$, and let $i = (i_0, \ldots, i_r)$ range over $\mathbb{N}^{1+r}$. We call $Y^i$ a monomial, and denote by $P_i \in R$ the coefficient of $Y^i$ in $P$. Thus $P$ can be uniquely written as

$$P = \sum_i P_i Y^i,$$

where the support of $P$, defined by

$$\text{supp}\, P := \{i : P_i \neq 0\},$$

is finite. We say that a monomial $Y^i$ occurs in $P$ if $i \in \text{supp}\, P$. We set

$$|i| := i_0 + \cdots + i_r, \quad \|i\| := i_1 + 2i_2 + \cdots + ri_r.$$
For $P \neq 0$ we call
\[
\deg(P) = \max_{i \in \text{supp } P} |i|, \quad \text{wt}(P) = \max_{i \in \text{supp } P} \|i\|
\]
the degree of $P$ respectively the weight of $P$. We say that $P \neq 0$ is homogeneous if $|i| = \deg(P)$ for every $i \in \text{supp } P$ and isobaric if $\|i\| = \text{wt}(P)$ for every $i \in \text{supp } P$.

For $r, s \in \mathbb{N}$ with $r \leq s$ we identify each $i = (i_0, \ldots, i_r, 0, \ldots, 0) \in \mathbb{N}^{1+r}$ with the tuple $(i_0, \ldots, i_r, 0, \ldots, 0) \in \mathbb{N}^{1+s}$ and thus view $\mathbb{N}^{1+r}$ as a subset of $\mathbb{N}^{1+s}$. We set $\mathbb{N}^* = \bigcup_{r \in \mathbb{N}} \mathbb{N}^{1+r}$ and order $\mathbb{N}^*$ anti-lexicographically; that is, for $i = (i_0, i_1, \ldots), j = (j_0, j_1, \ldots) \in \mathbb{N}^*$ we set
\[
i < j \iff \text{there is some } k \in \mathbb{N} \text{ with } i_k < j_k \text{ and } i_l = j_l \text{ for } l \geq k + 1,
\]
and we set $i \leq j :\iff i < j \text{ or } i = j$. It is easy to verify that $\leq$ is a well-ordering of $\mathbb{N}^*$, that is, $\leq$ is a linear ordering of $\mathbb{N}^*$, and every non-empty subset of $\mathbb{N}^*$ has a smallest element with respect to $\leq$. For $P \neq 0$ we let the rank $r = r(P)$ of $P$ be the largest element of $\text{supp } P$ with respect to $\leq$. Below $i, j$ range over $\mathbb{N}^*$. For $i = (i_0, i_1, \ldots)$ and $j = (j_0, j_1, \ldots)$ we put $i + j = (i_0 + j_0, i_1 + j_1, \ldots)$.

We view the ring $\mathbb{C}[[z]]$ of formal power series over $\mathbb{C}$ as a differential ring in the usual way (with derivation $\frac{d}{dz}$), and we work with two differential subrings of $\mathbb{C}[[z]]$: the differential subring $\mathbb{C}\{z\}$ of $\mathbb{C}[[z]]$ consisting of the convergent power series, and the smaller differential subring $\mathbb{C}\{z\}_\infty$ of $\mathbb{C}[[z]]$ consisting of the (Taylor series at 0 of) entire functions. Theorem 1 says that the iterative logarithm of a non-linear entire function is differentially transcendental over $\mathbb{C}\{z\}_\infty$, while Theorem 2 says that—under the hypotheses made on $f$—the iterative logarithm of $f$ is differentially transcendental over $\mathbb{C}\{z\}$.

A differential field is a differential ring whose underlying ring happens to be a field. Sometimes we find it convenient to work in the differential field of meromorphic functions (which may be naturally identified with the fraction field of $\mathbb{C}\{z\}_\infty$, equipped with the unique derivation extending that of $\mathbb{C}\{z\}_\infty$).

### 2.2. Repelling periodic points and Poincaré functions.

Let $f$ be a non-linear entire (or rational) function. A point $\xi \in \mathbb{C}$ is called a periodic point of $f$ if there exists an integer $p \geq 1$ such that $f^p(\xi) = \xi$; the smallest such $p$ is called the period of $\xi$. One calls a periodic point $\xi$ of $f$ with period $p$ repelling if the multiplier $\lambda = (f^p)'(\xi)$ of $f$ at $\xi$ satisfies $|\lambda| > 1$.

The Julia set $J(f)$ of $f$ is the of all points in the plane (or Riemann sphere) where the iterates of $f$ do not form a normal family. A standard result of complex dynamics says that $J(f)$ is the closure of the set of repelling periodic points of $f$. For rational functions this was already proved by Fatou and Julia, by different methods (see [30, Section 14] for an exposition of both proofs), for transcendental entire functions it is due to Baker [5] (see [6] and [13] for simpler proofs). The Julia set of $f$ is always non-empty (in fact, a perfect set).

As mentioned in the introduction, results of Köhig and Poincaré say that if $\xi$ is a repelling periodic point of $f$ with period $p$ and multiplier $\lambda$, then there exists a function $\psi$ holomorphic in a neighborhood of 0 such that $\psi(0) = \xi$, $\psi'(0) = 1$ and $\psi(\lambda z) = f^p(\psi(z))$, called the Poincaré function of $f$ at $\xi$. If $f$ is rational, then $\psi$ is
meromorphic in the plane, and if $f$ is entire, then so is $\psi$. Moreover, $\psi$ is given by (cf. [32, p. 670])

$$\psi(z) = \lim_{n \to \infty} f^{np}(\xi + \lambda^{-n}z).$$

Differentiating (2.1) we also obtain

$$\psi^{(m)}(z) = \lim_{n \to \infty} \lambda^{-mn}(f^{np})^{(m)}(\xi + \lambda^{-n}z) \quad \text{for each } m,$$

hence

$$\psi^i(z) = \lim_{n \to \infty} \lambda^{-\|i\|n}(f^{np})^i(\xi + \lambda^{-n}z) \quad \text{for each } i$$

and thus

$$\psi'(z) = \lim_{n \to \infty} \lambda^{-\|i\|n}(f^{np})'(\xi + \lambda^{-n}z) \quad \text{for each } i.$$

As mentioned in the introduction, Ritt [32] determined all differentially algebraic Poincaré functions of rational functions, while it was shown in [11] that Poincaré functions to transcendental entire functions are differentially transcendental.

It turns out that the rational functions with differentially algebraic Poincaré functions have no parabolic fixed points, so there is no iterative logarithm associated to these functions. Thus in our setting all Poincaré functions are differentially transcendental. Together with the results of Boshernitzan and Rubel [14] quoted earlier this implies the following result already mentioned in the introduction: the iterative logarithm of a non-linear rational or entire function is differentially transcendental.

### 2.3. A result of Steinmetz

The following result is due to Steinmetz [36, Satz 1]. We denote by $T(r, f)$ the Nevanlinna characteristic of a meromorphic function $f$, and as usual in Nevanlinna theory, $S(r, f)$ denotes any term satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside some exceptional set of finite measure. See [22] as a reference for Nevanlinna theory.

**Lemma 1.** Let $F_0, F_1, \ldots, F_m$ and $h_0, h_1, \ldots, h_m$ be not identically vanishing meromorphic functions and let $g$ be a nonconstant entire function such that

$$F_0(g)h_0 + F_1(g)h_1 + \cdots + F_m(g)h_m = 0.$$

Suppose that there exists a positive $K \in \mathbb{R}$ such that

$$\sum_{j=0}^{m} T(r, h_j) \leq KT(r, g) + S(r, g).$$

Then there exist polynomials $P_0, P_1, \ldots, P_m$ with constant coefficients, not all zero, such that

$$P_0(g)h_0 + P_1(g)h_1 + \cdots + P_m(g)h_m = 0.$$
3. Proof of Theorem 1

Let \( f \) be a non-linear entire function as in (1.2), with iterative logarithm \( \phi = \text{itlog}(f) \). Differentiation of (1.3) and induction yield the existence of differential polynomials \( A_{ij} \in \mathbb{Z}\{X\} \) \((i \leq j)\) in a differential indeterminate \( X \), independent of \( f \), such that

\[
\phi^{(j)}(f) \cdot (f')^{2j-1} = \sum_{i=0}^{j} A_{ij}(f') \cdot \phi^{(i)}. 
\]

Each \( A_{ij} \) is homogeneous and isobaric, and if non-zero, of degree \( j \) and weight \( j - i \). (See [2, Section 6.5], where \( H_{ij}(X) = A_{ij}(X') \).) Moreover, \( A_{jj} = X^j \) and the monomial of highest rank occurring in \( A_{0j} \) is \( X^{(j)}X^{j-1} \). For \( j \in \mathbb{N}^* \) this yields

(3.1) \[ \phi^j(f) \cdot (f')^{2\|j\| - |j|} = \sum_{i \leq j, |i| = |j|} B_{ij}(f') \cdot \phi^i \]

with differential polynomials \( B_{ij} \in \mathbb{Z}\{X\} \) \((i \leq j, |i| = |j|)\), independent of \( f \). For \( j = (j_0, \ldots, j_r) \in \mathbb{N}^{1+r} \), we have

\[ B_{jj} = \prod_{k=0}^{r} (A_{kk})^{j_k} = X^{\|j\|} \]

and

\[ B_{(|j|),j} = \prod_{k=0}^{r} (A_{0k})^{j_k}, \]

so \( B_{(|j|),j} \) is homogeneous of degree \( \|j\| \) and isobaric of weight \( \|j\| \), and the monomial of highest rank occurring in \( B_{(|j|),j} \) is \( X^j X^{||j||-|j|} \). Note that for each \( n \), (1.3) also holds with \( f \) replaced by the iterate \( F = f^n \) of \( f \), that is,

(3.2) \[ \phi(F(z)) = F'(z)\phi(z) \quad \text{where} \quad F = f^n, \]

and so (3.1) also holds with \( f \) replaced by \( F \).

Towards a contradiction assume now that \( \phi = \text{itlog}(f) \) is differentially algebraic over \( \mathbb{C}\{z\}_\infty \), that is, \( \phi \) satisfies an equation

(3.3) \[ P(\phi) = \sum_i P_i \phi^i = 0 \]

where \( P = \sum_i P_i Y^i \) is a non-zero differential polynomial with entire coefficients \( P_i = P_i(z) \). We assume that \( P \) is chosen so that its rank \( r = r(P) \) is minimal. Note that \( \|r\| > 0 \), since otherwise (3.3) would show that \( \phi \) is algebraic over \( \mathbb{C}\{z\} \) and hence in \( \mathbb{C}\{z\} \) (since \( \mathbb{C}\{z\} \) is algebraically closed in \( \mathbb{C}\{z\} \) by Puiseux’s Theorem), contrary to the results quoted in the introduction. Allowing the coefficients \( P_i \) to be meromorphic, we may also assume that \( P_r = 1 \).
It follows from (3.3) and (3.1) that
\[ 0 = \sum_{j \leq r} P_j(f) \cdot \phi^j(f) \cdot (f')^{2\|r\| - |r|} \]
\[ = \sum_{j \leq r} P_j(f) \cdot (f')^{2\|r\| - 2\|j\| - |r| + |j|} \sum_{i \leq j, |i| = |j|} B_{ij}(f') \cdot \phi^i \]
so that
\[ \sum_i \left( \sum_{i \leq j \leq r, |i| = |j|} P_j(f) \cdot (f')^{2\|r\| - 2\|j\| - |r| + |j|} \cdot B_{ij}(f') \right) \phi^i = 0. \]

(3.4)

It also follows from (3.3) that
\[ \sum_i P_i \cdot (f')^{\|r\|} \cdot \phi^i = 0. \]

(3.5)

In the last two equations, the coefficient of \( \phi^r \) is \( (f')^{\|r\|} \). By the minimality of \( r \) the two equations are thus equal. (We note that the exponent of \( f' \) might actually be negative for some terms on the right hand side of (3.4), but this does not affect the argument, since we may multiply both equations by a sufficiently high power of \( f' \). Similar adjustments are tacitly made in what follows.) In particular, the coefficients of \( \phi^{\|r\|} \) in (3.4) and (3.5) are equal. With \( a = P_{\{r\}} \) we thus obtain
\[ \sum_{|i| \leq j \leq r, |j| = |r|} P_j(f) \cdot (f')^{2\|r\| - 2\|j\| - |r| + |j|} \cdot B_{(j)\cdot j}(f') = a \cdot (f')^{\|r\|}. \]

(3.6)

The monomial of highest rank occurring in \( B_{(j)\cdot j} \) is \( X^{|j| - |j|} X^j \). Hence among the monomials occurring in the differential polynomials \( X^{2\|r\| - 2\|j\|} B_{(j)\cdot j} \) on the left side of (3.6) there is a unique one of maximal rank given by \( X^{\|r\| - |r|} X^r \). Moreover, since \( B_{(j)\cdot j} \) is homogeneous of degree \( |j| \) and isobaric of weight \( \|j\| \), each \( i \in \text{supp} X^{2\|r\| - 2\|j\|} B_{(j)\cdot j} \) satisfies \(|i| + |i| = 2\|r\| \). Thus, incorporating the terms \( X^{2\|r\| - 2\|j\|} \) into the monomials occurring in \( B_{(j)\cdot j} \), equation (3.6) takes the form
\[ \sum_{i \leq (\|r\| - |r|) + r, |i| + |i| = 2\|r\|} b_i(f) \cdot (f')^i = a \cdot (f')^{\|r\|} \]
with meromorphic functions \( b_i \), and \( b(\|r\| - |r|) + r = 1 \). Let
\[ I = \{ i : i \leq (\|r\| - |r|) + r, |i| + |i| = 2\|r\| \}. \]

By the remarks following (3.1), equation (3.7) also holds for \( f \) replaced by \( F = f^n \), for each \( n \), so
\[ \sum_{i \in I} b_i(F) \cdot (F')^i = a \cdot (F')^{\|r\|} \]
where \( F = f^n \).
As noted in Section 2.2, $f$ has repelling periodic points. (Actually it was shown in [9] that every iterate of $f$ has repelling fixed points.) We may in fact assume that $f$ has a repelling fixed point $\xi$. Moreover, we may assume that $\xi$ is not a pole of $a$. With $\lambda = f'(\xi)$ we define $\psi$ by (2.1). From (2.2) and (2.3) recall that

$$
\left(\psi'\right)^k(z) = \lim_{n \to \infty} \lambda^{-kn} (f^n)'^k (\xi + \lambda^{-n}z),
$$

$$
\left(\psi'\right)^i(z) = \lim_{n \to \infty} \lambda^{-\left(|i|+\|i\|\right)n} (f^n)^i (\xi + \lambda^{-n}z) \quad \text{for each } i.
$$

We substitute $\xi + \lambda^{-n}z$ for $z$ in (3.8), multiply both sides of the equation by $\lambda^{-2\|r\|n}$, take the limit as $n \to \infty$, and using $\|r\| > 0$, obtain

$$
\sum_{i \in I} b_i(\psi) \cdot (\psi')^i = 0.
$$

It is a standard result of Nevanlinna theory [22, p. 56] that

$$
T(r, \psi^{(k)}) \leq T(r, \psi) + S(r, \psi)
$$

for each $k$. This implies that

$$
\sum_{i \in I} T(r, (\psi')^i) \leq KT(r, \psi) + S(r, \psi)
$$

for some constant $K \in \mathbb{R}$. Lemma 1 now implies that there exist polynomials $Q_i$ ($i \in I$) with constant coefficients, not all zero, such that

$$
\sum_{i \in I} Q_i(\psi) \cdot (\psi')^i = 0.
$$

Thus $\psi$ satisfies an algebraic differential equation with constants coefficients, contradicting the results quoted in Section 2.2. \hfill \Box

4. Proof of Theorem 2

Suppose that $\phi = \text{itlog}(f)$ satisfies an equation of the form (3.3) whose coefficients $P_i$ are in $\mathbb{C}\{z\}$. Again we assume the rank $r = r(P)$ of $P$ to be minimal. We choose $\rho > 0$ such that all $P_i$ are holomorphic in $D_\rho = \{z: |z| < \rho\}$. Allowing the coefficients $P_i$ to be meromorphic in $D_\rho$ we may assume that $P_r = 1$. We want to show that the $P_i$ are actually meromorphic in $\mathbb{C}$, thereby obtaining a contradiction to Theorem 1.

Let $L_1, \ldots, L_{p-1}$ be petals of $f$ associated to the fixed point 0 as stipulated in the Leau-Fatou theorem. These petals $L_j$ can be chosen arbitrarily small, and thus we may assume that their closures are contained in $D_\rho$. As $f'(0) = 1$, there exists a branch $\psi$ of the inverse function of $f$ defined in a neighborhood of 0 such that $\psi(0) = 0$ and $\psi'(0) = 1$. The Leau-Fatou theorem may also be applied to $\psi$. We denote by $L'_1, \ldots, L'_{p-1}$ the petals for $\psi$. (These petals are also called repelling petals for $f$.) By (1.5) there exists $n$ such that

$$
f^n(\text{sing}(f^{-1})) \subseteq \{0\} \cup \bigcup_{j=0}^{p-1} L_j \subseteq D_\rho
$$
and thus

\[ f^m(\text{sing}(f^{-1})) \subseteq \{0\} \cup \bigcup_{j=0}^{p-1} L_j \subseteq D_\rho \quad \text{for all } m \geq n. \]  

Again we put \( F = f^n \) and, proceeding as in the proof of Theorem 1, we find that the equations (3.4) and (3.5) are equal. The coefficients of (3.5) are defined in \( D_\rho \) while the coefficients of (3.4), with \( f \) replaced by \( F \), are defined in the component of \( F^{-1}(D_\rho) \) that contains 0. We denote this component by \( V \). So the germs of the coefficients \( P_\ell \) at 0 can be continued meromorphically to both \( D_\rho \) and \( V \).

Actually, by passing to slightly smaller domains \( L_j \) and \( D_\rho \) if necessary, we may assume that these germs can be continued meromorphically to a region containing the closure \( \overline{V} \) of \( V \). Moreover, we may assume that \( f^k(\text{sing}(f^{-1})) \cap \partial D_\rho = \emptyset \) for all \( k \), which implies that \( \partial V \) consists of analytic curves.

By the choice of \( n \) we have \( \text{sing}(f^{-1}) \subseteq V \) and in fact

\[ f^m(\text{sing}(f^{-1})) \subseteq V \quad \text{for all } m. \]

Also, we may choose the petals \( L_j \) and \( L'_j \) so small that \( \overline{L_j} \subseteq V \) and \( \overline{L'_j} \subseteq V \) for \( j = 1, \ldots, p - 1 \).

As mentioned, we want to show that the germs of the coefficients \( P_\ell \) at 0 can be continued to functions meromorphic in \( \mathbb{C} \). By the Monodromy Theorem, it suffices to show that the germs can be continued meromorphically along any curve in \( \mathbb{C} \) starting in 0. We may restrict here to curves which intersect \( \partial V \) only finitely often. For example, this follows since it suffices to consider continuation along polygonal paths and since \( \partial V \) consists of analytic curves.

We now show that it suffices to consider continuation along curves \( \gamma : [0, 1] \to \mathbb{C} \) for which there exists \( t_1 \in (0, 1) \) such that \( \gamma([0, t_1]) \subseteq \overline{V} \) while \( \gamma([t_1, 1]) \subseteq \mathbb{C} \setminus V \). In fact, suppose that continuation along such curves is possible and let \( \sigma : [0, 1] \to \mathbb{C} \) be a curve such that

\[ \sigma([0, s]) \subseteq \mathbb{C} \setminus \overline{V}, \quad \sigma((s_1, s_2)) \subseteq \mathbb{C} \setminus \overline{V}, \quad \sigma([s_2, s_3]) \subseteq \overline{V}, \quad (0 < s_1 < s_2 \leq s_3 \leq 1). \]

Then there exists a curve \( \tau : [s_1, s_2] \to \partial V \) satisfying \( \tau(s_1) = \sigma(s_1) \) and \( \tau(s_2) = \sigma(s_2) \) which is homotopic to \( \sigma \mid [s_1, s_2] \) in \( \mathbb{C} \setminus V \); that is, there exists a continuous function \( \Gamma : [s_1, s_2] \times [0, 1] \to \mathbb{C} \setminus V \) such that \( \Gamma(s, 0) = \sigma(s) \) and \( \Gamma(s, 1) = \tau(s) \) for all \( s \in [s_1, s_2] \) and \( \Gamma(s_1, t) = \sigma(s_1) \) and \( \Gamma(s_2, t) = \sigma(s_2) \) for all \( t \in [0, 1] \). Let \( \sigma^* : [0, 1] \to \mathbb{C} \) be defined by

\[ \sigma^*(t) = \begin{cases} \sigma(t) & \text{for } t \in [0, 1] \setminus [s_1, s_2], \\ \tau(t) & \text{for } t \in [s_1, s_2]. \end{cases} \]

By our assumption, meromorphic continuation is possible along both \( \sigma \mid [0, s_2] \) and \( \sigma^* \mid [0, s_2] \). Moreover, \( \Gamma \) yields a homotopy between \( \sigma \mid [0, s_2] \) and \( \sigma^* \mid [0, s_2] \) with the property that meromorphic continuation is possible along all curves in the homotopy. Thus, by the Monodromy Theorem, meromorphic continuation along \( \sigma \mid [0, s_2] \) and \( \sigma^* \mid [0, s_2] \) leads to the same result. Since \( \sigma \mid [s_2, s_3] = \sigma^* \mid [s_2, s_3] \), meromorphic continuation along \( \sigma \mid [0, s_3] \) and \( \sigma^* \mid [0, s_3] \) also leads to the same result.
Now $\sigma^*$ has the property that $\sigma^*([0, s_3]) \subset \nabla$. Starting with a path $\sigma: [0, 1] \to \mathbb{C}$ which intersects $\partial V$ in finitely many points $\sigma(s_1), \ldots, \sigma(s_n)$, iteration of the above procedure yields a path $\gamma: [0, 1] \to \mathbb{C}$ such that continuation along $\sigma$ and $\gamma$ leads to the same result, and $\gamma$ has the additional property that unless $\gamma([0, 1]) \subseteq \nabla$, there exists $t_1 \in (0, 1)$ such that $\gamma([0, t_1]) \subseteq \nabla$ and $\gamma([t_1, 1]) \subseteq \mathbb{C} \setminus V$. It thus suffices to consider curves $\gamma: [0, 1] \to \mathbb{C}$ with $\gamma(0) = 0$ for which such a $t_1$ exists. We may also assume that $\gamma|[0, t_1]$ is injective.

Let now $\gamma$ be such a curve. We have to show that the germs of the $P_t$ at 0 can be continued meromorphically along $\gamma$. In order to do so, we may deform the part of $\gamma$ which is in $V$, as long as it stays in $V$. Thus all $\gamma(t) \in L'_1$ for $t \in (0, t_0]$ with some $t_0 \in (0, t_1)$, but

\begin{equation}
\gamma([0, 1]) \cap \bigcup_{j=0}^{p-1} L_j = \emptyset.
\end{equation}

Using (4.1), (4.2) and (4.3) and noting that $\text{sing}(f^{-1})$ is finite by hypothesis, we may in fact assume that

\begin{equation}
\gamma((0, 1]) \cap \bigcup_{t=0}^{\infty} f^t(\text{sing}(f^{-1})) = \emptyset.
\end{equation}

This implies that branches of the inverse functions of the iterates of $f$ defined in a neighborhood of 0 can be continued analytically along $\gamma$. In particular, for each $m$ we may continue along $\gamma$ the branch $\psi_{m,0}$ of the inverse function of $f^m$, defined in some neighborhood $U_0$ of 0, which is given by $\psi_{m,0}(0) = 0$. Thus for each $t \in (0, 1]$, there exists a neighborhood $U_t$ of $\gamma(t)$, a holomorphic function $\psi_{m,t}: U_t \to \mathbb{C}$ and $\delta > 0$ such that whenever $|s - t| < \delta$, then $\gamma(s) \in U_t$ and $\psi_{m,s}(z) = \psi_{m,t}(z)$ for all $z$ in some neighborhood of $\gamma(s)$. Moreover, while $U_0$ depends on $m$, it follows from (4.4) that the domains $U_t$ may be chosen independent of $m$ for $t \in (0, 1]$. For example, we may choose $U_t$ as the largest disk around $\gamma(t)$ which does not intersect $\bigcup_{t=0}^{\infty} f^t(\text{sing}(f^{-1}))$.

It also follows from (4.4) that there exists a simply connected domain $\Omega$ containing $\gamma((0, t_0])$ such that

\begin{equation}
\Omega \cap \bigcup_{t=0}^{\infty} f^t(\text{sing}(f^{-1})) = \emptyset.
\end{equation}

Thus all $\psi_{m,0}$ may be continued to functions holomorphic in $\Omega$ and we may in fact assume that $\Omega \subseteq U_0$ for all $m$. By [7, Theorem 9.2.1] the $\psi_{m,0}$ form a normal family in $\Omega$. Since $\psi_{m,0} | (L'_1 \cap \Omega) \to 0$ as $m \to \infty$ we deduce that in fact $\psi_{m,0} | \Omega \to 0$ as $m \to \infty$. This implies that $\psi_{m,t} \to 0$ as $m \to \infty$ for all $t \in (0, 1]$, locally uniformly in the domains $U_t$ where they are defined. Altogether we see that if $m$ is sufficiently large, then $\psi_{m,t}(\gamma(t)) \in D_\rho$ for all $t \in [0, 1]$. For the curve $\sigma: [0, 1] \to \mathbb{C}$ defined by $\sigma(t) = \psi_{m,t}(\gamma(t))$ we thus have $\sigma([0, 1]) \subseteq D_\rho$.

Next we note that (3.2) also holds for negative $n$, with a negative exponent standing for the branch of the inverse function of the appropriate iterate of $f$ which fixes 0.
Thus
\[
\phi(\psi_{m,0}(z)) = \psi'_{m,0}(z)\phi(z)
\]
for \( z \) in a neighborhood of 0. Using this instead of (3.2) we obtain (3.4) and (3.5) with \( f \) replaced by \( \psi_{m,0} \). As before we find that these equations are equal. For the equation corresponding to (3.5) the coefficients are meromorphic in \( D_\rho \). In particular, since \( \sigma([0,1]) \subseteq D_\rho \), the germs of the \( P_i \) at 0 can be continued meromorphically along \( \sigma \). Since the coefficients of the equation corresponding to (3.4) are build from the \( P_i \) and from differential polynomials in \( \psi_{m,0} \), these coefficients can also be continued meromorphically along \( \gamma \). As the equations corresponding to (3.4) and (3.5) are equal, we see that the \( P_i \) can be continued meromorphically along \( \gamma \). □

The basic idea of the above proof appears in a paper of Lewin [28] who proved that \( \log(f) \notin C(z) \) for \( f = e^z - 1 \). Assuming that \( \log(f) \) is holomorphic in \( D_\rho \) but has a singularity \( \zeta \in \partial D_\rho \), it is shown there by elementary estimates that \( w_1 = f(\zeta) \in D_\rho \) or that there exists \( w_2 \in D_\rho \) with \( f(w_2) = \zeta \). These points \( w_j \) are also singularities of \( \log(f) \), leading to a contradiction. Note that \( w_2 = \psi(\zeta) \) for some branch \( \psi \) of the inverse of \( f \). The proof of Theorem 2 also uses the idea that given \( \zeta \in \partial D_\rho \) there exists \( m \) such that \( f^m(\zeta) \in D_\rho \) or \( \psi_m(\zeta) \in D_\rho \) for some branch \( \psi_m \) of the inverse function of \( f^m \). However, in this more general setting we have to be careful about the domain where this branch of the inverse can be defined.

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