A SPLITTER THEOREM ON 3-CONNECTED MATROIDS AND GRAPHS

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Abstract. We prove the following splitter theorem for graphs and its generalization for matroids:
Let $G$ and $H$ be 3-connected simple graphs such that $G$ has an $H$-minor and $k := |V(G)|−|V(H)| \geq 2$. Let $n := \lfloor (k+3)/2 \rfloor$. Then there are sets $X_1, \ldots, X_n \subseteq E(G)$ such that each $G/X_i$ is a 3-connected graph with an $H$-minor, each $X_i$ is a singleton set or the edge set of a triangle of $G$ with 3 degree-3 vertices and $X_1 \cup \cdots \cup X_n$ contains no edge sets of circuits of $G$ other than the $X_i$'s. This implies that $G$ has a forest $F$ with at least $(3/5) \lfloor (k+3)/2 \rfloor$ edges, with the property that $G/e$ is 3-connected with an $H$-minor for each $e \in F$. The main result extends previous ones of Whittle (for $k = 1, 2$) and the Author (for $k = 3$).

Key words: graph, matroid, minor, connectivity, vertical connectivity, splitter theorem.

1. Introduction

We follow the terminology of Oxley [11]. Some notations introduced in the beginning of Section 3 remains fixed for sections 3 and 4. The symbol “◊” is used to indicate the end of a nested proof.

For a 3-connected matroid $M$ with an $N$-minor, we say that a set $X \subseteq E(M)$ is (resp. vertically) $N$-contractible in $M$ if $M/X$ (resp. $\text{si}(M/X)$) is a 3-connected matroid with an $N$-minor. We also define $x \in E(M)$ as $N$-contractible or vertically $N$-contractible in $M$ if $M/x$ or $\text{si}(M/x)$, respectively, is a 3-connected matroid with an $N$-minor. Analogously, we define $N$-deletable and cyclically $N$-deletable sets and elements using deletions instead of contractions and cosimplifications instead of simplifications. When not considering an $N$-minor (equivalently, when $N = U_{0,0}$), we simply say that the element or set is (vertically) contractible or (cyclically) deletable, according to the suitable case.

Deletable and contractible elements are vastly used in matroid and graph theory as inductive tools. For example, Thomassen [16] proved that every 3-connected graph has a vertically contractible edge and used this fact to make simple and elegant proofs for classical theorems about graph planarity. Independently, Cunningham [3] and Seymour [13] proved the more general version of this fact for matroids. Wu [19] proved that, in a 3-connected matroid $M$, the number of vertically contractible elements is at least 3, provided $M$ has at least 3 elements. These results were generalized for vertically $N$-contractible elements by Whittle [18] and the Author [4] (see Theorem 1 of this text). Important studies on (vertically) contractible and (cyclically) deletable elements in matroids and graphs were made earlier by Tutte [17], that proved that wheels and whirls are the unique 3-connected matroids whose all elements are essential (neither contractible nor deletable). Bixby [11] proved that, in a 3-connected matroid, each element is vertically contractible or cyclically deletable. Oxley and Wu, on [13] and [14], studied more structural aspects of essential elements.

When working on a class of matroids or graphs having a common minor $N$, (vertically) $N$-contractible and (cyclically) $N$-deletable elements are more appropriate tools. This happens, for instance, when dealing with excluded minors. Theorems asserting the existence of an intermediate matroid $N < M' < M$ for 3-connected matroids $N < M$ are known as splitter theorems (due to Seymour's Splitter Theorem [15]). A version of Seymour's Splitter Theorem establishes that a 3-connected matroid $M$ with a 3-connected minor $N$ has an $N$-contractible or $N$-deletable element, unless $M$ is a wheel or whirl. Other splitter theorems have been proved by Bixby and Coullard [2], Kingan and Lemos [7] and several others. Splitter theorems have an important role in the study of

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Suppose that $G$ and $H$ are $3$-connected simple graphs, that $G$ has an $H$-minor and that $k := |V(G)| − |V(H)| ≥ 2$. Then, there is a family $\mathcal{F} := \{X_1, \ldots, X_n\}$ of pairwise disjoint subsets of $E(G)$, such that $n ≥ \left\lfloor \frac{k+3}{2} \right\rfloor$, each edge set of a circuit of $G[X_1 ∪ \cdots ∪ X_n]$ is a member of $\mathcal{F}$ and, for each $i = 1, \ldots, n$:

(a) $X_i$ is a singleton set such that $G/X_i$ is $3$-connected with an $H$-minor, or

(b) $X_i$ is the edge set of a triangle of $G$ with three degree-$3$ vertices and $G/X_i$ is $3$-connected and simple with an $H$-minor.

If we combine this last corollary with some basic counting principles we get:

**Theorem 3.** Suppose that $G$ and $H$ are $3$-connected simple graphs, that $G$ has an $H$-minor and that $k := |V(G)| − |V(H)| ≥ 2$. Then there is a forest $I ⊆ E(G)$, with at least $\frac{3}{5} \left\lfloor \frac{k+3}{2} \right\rfloor$ edges, such that, for each $e ∈ I$, $G/e$ is $3$-connected with an $H$-minor.

If, in a graph $G$, $T$ is the edge set of a triangle with three degree-$3$ vertices, then the relation between the paths and circuits of $G$ and $G/T$ is evident. A similar relation also holds in a binary matroid if $T$ is a triangle meeting three different triads. We say that a sequence $a_1, a_2, a_3, b_1, b_2, b_3$ is an $N$-**triweb** of a $3$-connected matroid $M$ if $(b_1, b_2, b_3)$ is an $N$-contractible triangle and, for $(i, j, k) = (1, 2, 3)$, $(a_{ij}, b_j, b_k)$ is a triad of $M$. We say that a family $\{X_1, \ldots, X_n\}$ of subsets of $E(M)$ is a **free** family of $M$ if $M[X_1 ∪ \cdots ∪ X_n] = M[X_1] ⊕ \cdots ⊕ M[X_n]$. Note that, if each $X_i$ is a singleton set, then such family is free if and only if $X_1 ∪ \cdots ∪ X_n$ is independent in $M$. A version of our main result for binary matroids is stated next:

**Corollary 4.** If $M$ is a $3$-connected binary matroid with a $3$-connected simple minor $N$ such that $k := r(M) − r(N) ≥ 2$, then $M$ has a free family with cardinality $\left\lfloor \frac{k+3}{2} \right\rfloor$ and whose members are vertically $N$-contractible singleton sets or triangles of $N$-triwebs of $M$.

In the bond matroid of a simple $3$-connected graph $G$, an $N$-triweb corresponds to a $4$-clique with a degree-$3$ vertex whose deletion preserves the $3$-connectivity of $G$. A dual version of Corollary 4 is given by Corollary 2 applied on $M^*(G)$.

Now we generalize the concept of triweb to non-binary matroids. For a $3$-connected matroid $M$ and $n ≥ 3$, a sequence of elements $K := x_1, \ldots, x_n, y_1, \ldots, y_n$ is said to be an $N$-**carambole** of $M$ if:

- $L := \{y_1, \ldots, y_n\}$ is an $N$-contractible line of $M$ and
- for $i = 1, \ldots, n$, $(L − y_i) ∪ x_i$ is a cocircuit of $M$.

In this case, we say that $L$ is the **filament** and $X := \{x_1, \ldots, x_n\}$ is the **cofilament** of $K$. When $L$ is the filament of some $N$-carambole of $M$, we simply say that $L$ is an $N$-**filament** of $M$. If $n = 3$, the carambole is a triweb. When we use the terms “carambole”, “filament” or “triweb” without
mention to a minor $N$, it is the case that $N = U_{0,0}$. A first property of caramboles is given by the next proposition:

**Proposition 5.** If $X$ is the cofilament of a carambole of a 3-connected matroid $M$ with $r(M) \geq 3$, then $r^*_M(X) = 2$.

Now we state the main theorem:

**Theorem 6.** If $M$ is a 3-connected matroid with a 3-connected simple minor $N$ such that $k := r(M) - r(N) \geq 2$, then $M$ has a free family with cardinality $\left\lceil \frac{k+3}{2} \right\rceil$ and whose members are vertically $N$-contractible singleton sets or $N$-filaments of $M$.

Theorem 6 is proved using an inductive strategy whose initial case is given by Theorem 1. But, in a better initial case, Theorem 6 may be improved:

**Theorem 7.** Suppose that $M$, $N$ and $H$ are 3-connected matroids such that $M \geq H > N$ and $r(M) \geq 6$. Suppose that $H$ has a free family with cardinality $n > 1$, whose members are vertically $N$-contractible singleton sets or $N$-filaments of $M$. Then $M$ has such a free family with $n + \lfloor (1/2)(r(M) - r(H) + 1) \rfloor$ members.

The next proposition presents some properties of caramboles.

**Proposition 8.** Let $M$ be a 3-connected matroid. Suppose that $K := x_1, \ldots, x_n, y_1, \ldots, y_n$ is a carambole of $M$ with filament $L$ and cofilament $X$ and $C \notin L$.

(a) If $C$ is a circuit of $M$ intersecting $K$, then

(a.1) $X \subseteq C$ and $(C - L) \cup A \in \mathcal{E}(M)$ for each 2-subset $A$ of $L$, or

(a.2) For some $1 \leq i \leq n$, $X - C = \{x_i\}$, $(C - L) \cup y_i \in \mathcal{E}(M)$ and, for each 2-subset $A$ of $L - y_i$, $(C - L) \cup A \in \mathcal{E}(M)$.

(b) If $C \in \mathcal{E}(M)$, then $C - L \in \mathcal{E}(M/L)$

Proposition 8 is proved in section 2. The Corollaries with nontrivial proofs are proved in section 5.

The next corollary to Proposition 8 establishes that $M$ can be rebuild in an unique way from $H := M/L$ and shows a connection between $N$-caramboles and generalized Delta-Wye exchanges. Let $X := \{x_1, \ldots, x_n\}$ and $L := \{y_1, \ldots, y_n\}$. Let $\Theta_n$ be the matroid on $X \cup L$ such that $\Theta^*_n$ has $L$ as a line, $X$ as a coline and, for $i = 1, \ldots, n$, $C^*_i := (L - y_i) \cup x_i$ as a cocircuit, as defined by Oxley [Proposition 11.5.1] (see also [12]).

**Corollary 9.** Suppose that $H$ is a cosimple matroid with $X \subseteq E(H)$, $r^*_H(X) = 2$ and $L \cap E(H) = \emptyset$. Suppose also that $M$ is a 3-connected simple matroid having $x_1, \ldots, x_n, y_1, \ldots, y_n$ as carambole. Then $H = M/L$ if and only if $M^* = P_X(H^*, \Theta_n)$.

In the next proposition, we see that the existence of filaments also guarantees the existence of certain independent sets of $N$-contractible elements.

**Proposition 10.** Let $M$ be a 3-connected matroid with an $N$-minor. Suppose that $r(M) \geq 4$. If $X$ is the cofilament of an $N$-carambole with $n$ elements in the filament, then $X$ is an $n$-independent set of $N$-contractible elements.

In our studies, a structure weaker than a triweb raises naturally in the critical cases as an obstruction for some elements to be vertically $N$-contractible. An $N$-biweb is a sequence of elements $a_1, a_2, b_1, b_2, b_3$ such that $\{b_1, b_2, b_3\}$ is a vertically $N$-contractible triangle and $\{a_i, b_{3-i}, b_3\}$ is a triad of $M$ for $i = 1, 2$. The following corollaries strengthen Theorem 5 for $k = 4, 5$.

**Corollary 11.** If, in Theorem 5, $k = 4$, then $M$ has a 4-independent set of vertically $N$-contractible elements or an $N$-biweb.

**Corollary 12.** If, in Theorem 5, $k = 5$ and $r(M) \geq 5$, then $M$ has a 4-independent set of vertically $N$-contractible elements or an $N$-triweb.
2. A LEMMA AND PROOFS FOR ELEMENTARY PROPOSITIONS

Lemma 13. If \( H \) is a connected rank-3 simple matroid and \( a, b \in E(H) \), then there is a 4-circuit of \( H \) containing \( a \) and \( b \) or \( H \) is the parallel connection of two lines with base point \( a \) or \( b \).

Proof. First consider the case that \( a \neq b \). Suppose that the result does not hold in such case. Hence \( H \) has a triangle \( T \) containing \( a \) and \( b \). Since \( H \) has rank 3 and no coloops, there are distinct elements \( c \) and \( d \) in \( E(H) - \text{cl}_H(T) \). The dependent set \( \{a, b, c, d\} \) is not a circuit, so we may assume that \( S := \{b, c, d\} \) is a triangle of \( H \). There is an element \( e \in E(M) - (\text{cl}_H(T) \cup \text{cl}_H(S)) \) as \( H \) is not the parallel connection of \( \text{cl}_H(T) \) and \( \text{cl}_H(S) \). By assumption, the dependent set \( \{a, b, d, e\} \) is not a circuit of \( H \). So, it contains a triangle \( R \). By construction, \( e \notin \text{cl}_H(S) \). Hence \( a \in R \). Also \( d, e \notin \text{cl}_M(\{a, b\}) \) and, therefore, \( R = \{a, d, e\} \). By circuit elimination on \( R, S \) and \( d \), there is a circuit \( C \) of \( H \) contained in \( (R \cup S) - d \). Since \( R - d \) and \( S - d \) are in distinct lines of \( H \), then \( C = (R \cup S) - d \), which is a 4-circuit of \( H \) containing \( a \) and \( b \). A contradiction. Thus the result holds if \( a \neq b \).

Now, for \( a = b \), consider a basis \( \{a, x, y\} \) of \( M \). If \( a \) is in no 4-circuit of \( M \), then applying the previous case to \( a \) and \( x \) and to \( a \) and \( y \), we conclude that \( M \) is the parallel connection of two nontrivial lines with base point \( a \) and the result holds in general. \( \square \)

Next we prove propositions 5 and 8.

Proof of Proposition 5. Let \( x_1, \ldots, x_n, y_1, \ldots, y_n \) be a carambole of \( M \). Define \( X := \{x_1, \ldots, x_n\} \), \( Y := \{y_1, \ldots, y_n\} \) and \( C_i^* := (Y - y_i) \cup x_i \) for each \( i \). Consider a cocircuit \( C^* \subseteq C_i^* \cup C_j^* \) containing \( x_1 \). Note that \( C^* - Y \subseteq \{x_1, x_2\} \), but \( C^* \not\subseteq \{x_1, x_2\} \). Hence \( C^* \) meets \( Y \) and, by orthogonality, \( Y - y_3 \subseteq C^* \). Now consider \( D^* \subseteq (C^* \cup C_3^*) - y_2 \) since \( y_3, y_2 \notin D^* \), then \( D^* \) avoids \( Y \). Therefore \( D^* = \{x_1, x_2, x_3\} \). Thus each three elements of \( X \) are in a triad and the proposition holds. \( \square \)

Proof of Proposition 8. Write \( C_i^* = (L - y_i) \cup x_i \) for \( i = 1, \ldots, n \). Let us prove item (a) first. Choose distinct \( i, j, k \in \{1, \ldots, n\} \) such that \( C \cap L \) is equal to \( \{y_i\} \) or \( \{y_j, y_k\} \). By orthogonality with \( C_j^* \), \( x_j \in C \). By orthogonality with the coline containing \( x_i \), \( |C - X| \leq 1 \) (see Proposition 5). Let \( D \) be a circuit of \( M \) with \( x_j \in D \subseteq (C \cup \{y_i, y_j, y_k\}) - y_i \). By orthogonality with \( C_i^* \), \( y_k \in D \). So let \( D_1 \) be a circuit of \( M \) with \( x_j \in (D_1 \subseteq D \cup \{y_i, y_j, y_k\}) - y_k \). Note that \( D_1 \subseteq C \cup y_j \).

By orthogonality with \( C_j^* \), \( y_j \in D_1 \), since \( D_1 \cap L \subseteq \{y_i, y_j\} \).

Next we analyze two cases:

(i) If \( C \cap L = \{y_i\} \), then by orthogonality with \( C_i^* \), \( C - X = \{x_i\} \). Since \( D_1 \cap L \subseteq \{y_i, y_j\} \) and \( x_i \notin D_1 \), then \( y_j \notin D_1 \) by orthogonality with \( C_i^* \). So \( D_1 = C \) since \( D_1 \subseteq C \cup y_j \) by construction. This implies that \( D = (C - L) \). Since \( x_j, x_k \in D \), by orthogonality with \( C^* - j \) and \( C_k^* \), \( D \cap L = \{y_j, y_k\} \).

(ii) If \( C \cap L = \{y_j, y_k\} \), then by construction, \( D_1 = C \). Again: \( D - L = C - L \). If \( x_k \notin C \), then \( D \cap L = \{y_k\} \) by orthogonality with \( C_k^* \). Otherwise \( D \cap L = \{y_j, y_k\} \) by orthogonality with \( C_j^* \) and \( C_k^* \).

If \( X \subseteq C \) we are in case (ii) with \( x_k \in C \). Thus \( D = (C - x_i) \cup x_k \in \mathcal{E}(M) \). Iterating this argument, we get (a1). Otherwise, since \( |X - C| \leq 1 \), \( X - C = \{x_l\} \) for some \( 1 \leq l \leq n \). In this case, we get (a2) iterating (i) for \( i = l \) and (ii) for \( k = l \).

For item (b), if \( D \) is a circuit of \( M \) such that \( D - L \subseteq C - L \), by item (a), \( (D - L) \cup (C \cap L) \) contains a circuit, but this set is contained in \( C \), so it is \( C \) and \( C - L = D - L \). \( \square \)

3. SOME DEFINITIONS AND PRELIMINARIES

We will use with no mention the fact that \( \text{si}(\text{si}(M/e)/f) = \text{si}(M/e, f) \). For this and the next section, we will have some notations fixed as described next. We always consider \( M \) as a 3-connected matroid with rank at least 3 and a 3-connected simple minor \( N \). When talking about a carambole, by standard, we will denote it by \( K = x_1, \ldots, x_n, y_1, \ldots, y_n \), its filament by \( L \) and \( C_i^* := (L - y_i) \cup x_i \).
Some specific structures will appear recurrently in our proofs. It is convenient to name them, as follows along this section.

A vertically contractible element of \( si(M/x) \) may be not vertically contractible in \( M \). This is the greatest difficulty to apply an inductive strategy in this problem. Whittle [18] characterized the structures that may appear in such situation. We will describe such structures and strengthen their characterizations next.

An \((M, N)\)-vertbarrier is a pair \((C^*, p)\), where \( C^* \) is a rank-3 cocircuit of \( M \), \( p \in cl_M(C^*) - C^* \) and \( si(M/x, p) \) is 3-connected with an \( N \)-minor for some \( x \in C^* \) (and therefore for all \( x \in C^* \) by Lemma [16]. We say that \((C^*, p)\) contains \( x \) if \( x \in C^* \).

**Lemma 14.** Let \( H \) be a vertically connected matroid. Suppose that \( z \) is an element of \( H \) such that \( H/\{z\} \) is vertically 3-connected. If \([A, B]\) is a vertical 2-separation of \( H \) with \( z \in A \), minimizing \(|A|\), then \( A \) is a rank-2 cocircuit of \( H \).

**Proof.** We may write \( H = L \oplus_2 K \) with \( E(L) = A \cup p \), where \( p \) is the base point of the two-sum. Since \( H/\{z\} \) is vertically 3-connected, it follows that \( r(L/z) = 1 \). So \( L \) is a parallel extension of a line and \([z, p]\) is independent in \( L \). By the minimality of \( A \), no element of \( A \) is spanned by \( B \) in \( H \). Hence \( B \) is a flat of \( H \). Considering the ranks, we conclude that \( B \) is a hyperplane of \( H \) and the result holds.

The next Lemma is a slight generalization of Lemma 3.6 of [18].

**Lemma 15.** Suppose that \( x \) and \( p \) are elements of \( M \) such that \([x, p]\) is vertically \( N \)-contractible in \( M \) but \( p \) is not. Then \( r(M) \geq 4 \) and there is an \((M, N)\)-vertbarrier \((C^*, p)\) containing \( x \).

**Proof.** If \(|E(M)| \leq 3\), the result is clear. Assume the contrary. So \( M/p \) is connected, and, therefore, vertically connected. Since \( M/p \) is not vertically 3-connected, then \( r(M/p) \geq 3 \). Therefore, \( r(M) \geq 4 \). Thus \( M/p \) is connected. By Lemma [14] for \((H, z) = (M/p, x)\), \( M/p \) has a rank-2 cocircuit \( C^* \) containing \( x \). Since \( M \) is 3-connected and \( r(M) \geq 4 \), hence \( r_M(C^*) = 3 \). Moreover \( p \in cl_M(C^*) \), because \( r_{M/p}(C^*) = 2 \).

Whittle [18] established the following two lemmas:

**Lemma 16.** Suppose that \( C^* \) is a rank-3 cocircuit of \( M \) and \( p \in cl_M(C^*) - C^* \).

(a) If \( x, y \in C^* \), then \( si(M/p, x) \equiv si(M/p, y) \).

(b) If, for some \( z \in C^* \), \([z, p]\) is vertically \( N \)-contractible, then, for each \( x \in C^* \), \([x, p]\) is vertically \( N \)-contractible.

**Lemma 17.** If \( C^* \) is a rank-3 cocircuit of \( M \) and \( x \in C^* \) has the property that \( cl_M(C^*) - x \) contains a triangle of \( M/x \), then \( si(M/x) \) is 3-connected.

Lemma [17] implies the two next corollaries:

**Corollary 18.** If \( M \) is a 3-connected matroid with a triangle \( T \) intersecting a triad \( T^* \), then, for \( x \in T^* - T \) and \( y \in T - T^* \), \( si(M/x) \) and \( co(M/y) \) are 3-connected.

**Corollary 19.** Suppose that \( C^* \) is a rank-3 cocircuit of \( M \), then each element of \( C^* \) in a 4-circuit of \( M \) is vertically \( N \)-contractible in \( M \).

**Lemma 20.** If \( x_1, x_2, x_3, y_1, y_2, y_3 \) are elements of \( M \) such that \( T := \{y_1, y_2, y_3\} \) is a triangle and, for \( i, j, k = \{1, 2, 3\} \), \( T^*_i := \{x_i, y_j, y_k\} \) is a triad of \( M \), then \( W := x_1, x_2, x_3, y_1, y_2, y_3 \) is a triweb of \( M \). Moreover, if, for some \( i, j \in \{1, 2, 3\} \), \( M/x_i, y_i; M/T \) has an \( N \)-minor, then \( W \) is an \( N \)-triweb of \( M \).

**Proof.** Since \( y_3 \in T - T^*_3 \), then, by Corollary [13], \( co(M/y_3) \) is 3-connected. By orthogonality, \( T^*_1, T^*_2 \) and \( T^*_3 \) are the unique triads of \( M \) meeting \( T \). Hence \( \{x_1, y_2\} \) and \( \{x_2, y_1\} \) are the unique serial pairs of \( M/y_3 \). So, \( M/T \) is 3-connected because \( M/T \equiv co(M/y_3) \). Thus, \( W \) is a triweb of \( M \). Now, say that \( M/x_1, y_1 \) has an \( N \)-minor. By Lemma [16], \( si(M/x_1, y_1) \equiv si(M/y_1, y_2) = M/T \). So, \( M/T \) has an \( N \)-minor and the lemma holds. \( \square \)
Lemma 21. If, for \( n \geq 3 \), \( L = \{y_1, \ldots, y_n\} \) is a line of \( M \) and \( x_1, \ldots, x_n \) are elements of \( M \) such that, for each \( i = 1, \ldots, n \), \( C_i^* := (L - y_i) \cup x_i \) is a cocircuit of \( M \), then \( K := x_1, \ldots, x_n, y_1, \ldots, y_n \) is a carambole of \( M \). Moreover, if, for some distinct \( i \) and \( j \) in \( \{1, \ldots, n\} \), \( M/x_i, y_i \) or \( M/y_j, y_j \) has an \( N \)-minor, then \( K \) is an \( N \)-carambole of \( M \).

Proof. Let \( 1 \leq i < j < k \leq n \) and \( Y := L - \{y_j, y_j, y_k\} \). So, \( M \setminus Y \) is 3-connected. By Lemma 20, \( W := x_i, x_j, x_k, y_1, y_j, y_k \) is a triweb of \( M \setminus Y \). Thus, \( M \setminus Y / \{y_j, y_j, y_k\} = M/L \) is 3-connected. The first part of the lemma is proved. The second part follows from Lemma 16 as in the proof of Lemma 20.

It is a consequence of Proposition 8 that:

Corollary 22. If, in \( M \), \( K \) is a carambole intersecting a triangle \( T \), then \( T \) is contained in the filament of \( K \).

The next lemma has an elementary proof, which is left to the reader.

Lemma 23. If \( H \) is a rank-3 simple matroid with \( |E(H)| \geq 4 \), then one of the following alternatives holds:

(a) \( H \) is the direct sum of a nontrivial line and a coloop.
(b) \( H \) is connected and has a 4-circuit.

Motivated by Lemma 23, for an \((M,N)\)-vertbarrier \((C^*, p)\), if \( H := M/(C^* \cup p) \) is disconnected, we define the coloop and the line of \( H \) respectively as the coloop and line of \((C^*, p)\) and we say that \((C^*, p)\) is disconnected. Otherwise \((C^*, p)\) is said to be connected. From Lemma 17 we have:

Corollary 24. If \( x \) is the coloop of an \((M,N)\)-vertbarrier, then \( x \) is vertically \( N \)-contractible.

From Lemma 21:

Corollary 25. The sequence \( x_1, \ldots, x_n, y_1, \ldots, y_n \) is an \( N \)-carambole of \( M \) if and only if, for each \( i = 1, \ldots, n \) and \( C_i^* := \{y_1, \ldots, y_n\} - y_i \cup x_i \), \((C_i^*, y_i)\) is a disconnected \((M,N)\)-vertbarrier with coloop \( x_i \).

From Corollaries 22, 25 and 24 we get:

Corollary 26. If \( x \) is in the cofilament and \( L \) is the filament of an \( N \)-carambole of \( M \), then \( x \) and \( L \) are \( N \)-contractible.

Lemma 27. Suppose that \((C^*, p)\) is a disconnected \((M,N)\)-vertbarrier with \( L \) and \( y \in L \) is not vertically \( N \)-contractible in \( M \). Then there is an \((M,N)\)-vertbarrier \((D^*, y)\) with \( L - y \subseteq D^* \).

Proof. Since \( \text{si}(M/y) \) is 3-connected but \( \text{si}(M/y) \) is not, then, by Lemma 15 there is an \((M,N)\)-vertbarrier \((D^*, y)\) containing \( p \). By orthogonality between \( L \) and \( D^* \), it follows that \( L - y \subseteq D^* \).

Combining Lemma 13 and Corollary 19 we have:

Corollary 28. Suppose that \((C^*, p)\) is a connected \((M,N)\)-vertbarrier. Then

(a) each element of \( C^* \) is vertically \( N \)-contractible in \( M \) or
(b) \( M/(C^* \cup p) \) is the parallel connection of two nontrivial lines with base point, namely, \( b \) and each element of \( C^* - b \) is vertically \( N \)-contractible in \( M \).

Lemma 29. If \( r(M) \geq 4 \) and \((C^*, p)\) is a disconnected \((M,N)\)-vertbarrier with \( L \), then \( L \) is an \( N \)-filament of \( M \) or \( L \) contains a vertically \( N \)-contractible element of \( M \).

Proof. Suppose that \( L \) contains no vertically \( N \)-contractible elements of \( M \). Let \( C_1^* := C^* \), \( y_1 := p \), \( L := \{y_1, \ldots, y_n\} \) and \( x_1 \in C_1^* - L \). By Lemma 27 for each \( i = 2, \ldots, n \), there is an \((M,N)\)-vertbarrier \((C_i^*, y_i)\) with \( L - y_i \subseteq C_i^* \).

If \((C_i^*, y_i)\) is connected, then, by Corollary 28 on \( C_i^* \), \( L \) contains a vertically \( N \)-contractible element of \( M \). A contradiction. Therefore, each \((C_i^*, y_i)\) is a disconnected \((M,N)\)-vertbarrier with coloop, namely, \( x_i \).

If \( L \) intersects a triangle \( T \not

Hence, there is no such a triangle. Thus, the line of each \((C_i^*, y_i)\) must be \( L \). Now, the result follows from Corollary 25.
We say that a biweb $a_1, a_2, b_1, b_2, b_3$ is **strict** if there is no element $a_3$ of $M$ such that $a_1, a_2, a_3, b_1, b_2, b_3$ is a triweb of $M$.

**Corollary 30.** If $r(M) \geq 4$ and $a_1, a_2, b_1, b_2, b_3$ is a strict biweb of $M$, then $b_3$ is vertically $N$-contractible in $M$.

**Proof.** Since the biweb is strict, then $\{a_2, b_3\}$ is the unique pair of $M \setminus b_1$, thus, by Corollary 18 $\text{co}(M \setminus b_1) \cong M/b_3 \setminus b_1$ is 3-connected. But $b_1$ is in parallel in $M/b_3$. So $\text{si}(M/b_3) = M/b_3 \setminus b_1$ and the result holds. $\square$

**Lemma 31.** If $r(M) \geq 4$ and $C^*$ and $D^*$ are distinct cocircuits of $M$, then $r_M(C^* \cup D^*) \geq 4$.

**Proof.** It is straightforward to check that $C^* \cup D^*$ is a 2-separating set of $M$ otherwise. $\square$

**Lemma 32.** (Cunningham [5] Proposition 3.2) If $x$ is an element of a matroid $H$ other than a coloop and $H' x$ is vertically 3-connected, then so is $H$.

**Corollary 33.** If $M \setminus x$ is 3 connected, then each vertically $N$-contractible element of $M \setminus x$ is vertically $N$-contractible in $M$.

### 4. Lemmas

In this section we prove some more specific lemmas. We will keep the notations set in the beginning of section 3.

**Lemma 34.** Suppose that $r(MN) \geq 4$, $x$ is an $N$-deletable element of $M$ and $L$ is an $N$-filament of $M \setminus x$. Then $\text{cl}_M(L)$ is an $N$-filament of $M$ or $\text{cl}_M(L)$ contains a vertically $N$-contractible element of $M$.

**Proof.** Since $\text{si}(M \setminus x / L) \cong \text{si}(M \setminus x / y_1, y_2)$, then $M \setminus x / y_1, y_2$ is vertically 3-connected. Since $x$ is not a coloop of $M / y_1, y_2$, then by Lemma 32 $M / y_1, y_2$ is a vertically 3-connected matroid. If $\text{si}(M / y_1)$ is 3-connected, then the result holds. Assume the contrary. By Lemma 15 there is an $(M, N)$-vertbarrier $(D^*, y_1)$ containing $y_2$.

Since $y_2 \in D^*$, by orthogonality, $\text{cl}_M(L) - y_1 \subseteq D^*$. If $(D^*, y_1)$ is connected, then Corollary 28 implies that there is a vertically $N$-contractible element of $M$ in $L$ and the Lemma holds. If $(D^*, y_1)$ is disconnected, then $\text{cl}_M(L)$ is the line of $(D^*, y_1)$ because $\text{cl}_M(L) - y_1 \subseteq D^*$. The result follows from Lemma 29 in this case. $\square$

**Lemma 35.** Suppose that $r(M) \geq 5$, that $x$ is an $N$-contractible element of $M$ and that $M$ has no $N$-deletable elements. If $L$ is an $N$-filament of $M \setminus x$, then $L$ is an $N$-filament of $M$ or $L$ contains a pair of vertically $N$-contractible elements of $M$.

**Proof.** Suppose that the result does not hold. Say that $y_1, \ldots, y_{n-1}$ are not vertically $N$-contractible in $M$. For $i < n$, $M / y_i$ and $M / y_j$ have $N$-minors. So, $\text{si}(M / y_i)$ is not 3-connected, and, therefore, $\text{co}(M \setminus y_i)$ is 3-connected. Since $M$ has no $N$-deletable elements, each $y_i$ is in a triad $T_i^*$ of $M$. Next we check:

**35.1. $n = 3$**

Suppose the contrary. Thus there is no triad of $M \setminus x$ meeting $L$. This implies that, for each $i < n$, $T_i^*$ is not a triad of $M \setminus x$, and, hence, $x \in T_i^*$. Also, for $i < j < n$, $y_i$ and $y_j$ are not in a same coline of $M$ with more than 3 elements, and, consequently, $T_i^* \cap T_j^* = \{x\}$. Thus $C^* := T_i^* \Delta T_j^*$ is a cocircuit of $M$ and, therefore, of $M \setminus x$. By orthogonality, $|L \cap C^*| \geq |L| - 1$. Since $n > 3$, there is an an index $k \in \{1, \ldots, n - 1\} - \{i, j\}$, such that $y_k \in C^*$. But this implies that $T_i^* \cap T_k^*$ or $T_j^* \cap T_k^*$ differs from $\{x\}$. A contradiction. $\square$

So, $K$ is a triweb of $M \setminus x$. If $L$ is a triangle of $M$, the result follows from Lemma 20. Let us assume that $L \cup x$ is a circuit of $M$. We prove two assertions next:

**35.2. $M / x_i$ is 3-connected for $i = 1, 2, 3$.**
There is no triangle of \( M \) containing \( x \) because \( M/x \) is 3-connected with rank at least 4. Thus \( x \) is not a loop, not a coloop and neither is in a serial pair in \( M/x_i \). By Corollary 26, \( M/x, M/x, x_i \) is 3-connected. This implies 35.2.

35.3. For \( i = 1, 2, 3 \), \( y_i \) is in no triangle of \( M \) or \( M/x_i \).

Since \( L \) is the unique triangle of \( M/x \) containing \( y_i \) and \( L \cup x \in \mathcal{C}(M) \), then \( y_i \) is in no triangle of \( M \). Now, if \( y_i \) is in a triangle \( S \) of \( M/x_i \), then \( S \cup x_i \) is a circuit of \( M \). By orthogonality with \( C_i^*, y_j \in S \), for some \( i, j, k = \{1, 2, 3\} \). By orthogonality with \( C_i^* \), \( \{y_k, x_j\} \) meets \( S \cup x_i \). So, \( S \cup x_i = \{x_i, x_j, y_i, y_j\} \) or \( L \cup x_i \). But this implies that \( r_M(C_i^* \cup C_j^*) = 3 \). A contradiction to Lemma 31.

35.4. For \( i = 1, 2, 3 \), there is no \((M, N)\)-verbarrier in the form \((D^*, y_i)\).

Suppose the contrary. Let \( i, j, k = \{1, 2, 3\} \). Say that \((D^*, y_j)\) is an \((M, N)\)-verbarrier. By 35.3, there is a 4-circuit \( D \) of \( M \) such that \( y_j \in D \subseteq D^* \cup y_i \). As \( n = 3 \), thus \( C_j^* \cap C_i^* = \{y_i\} \). By orthogonality, there is an independent 3-set \( I \subseteq D \) containing \( y_i \) and meeting both \( C_j^* - C_i^* \) and \( C_k^* - C_j^* \). Then, \( D^* \subseteq cl_M(I) \subseteq cl_M(C_j^* \cup C_k^*) \). If \( D \subseteq C_j^* \cup C_k^* = \{x_i, x_j, y_1, y_2, y_3\} \), then \( r_M(C_i^* \cup C_j^*) = 3 \). A contradiction to Lemma 31. This implies that \( D^* \subseteq C_j^* \cup C_k^* \). Therefore, for \( Z := D^* \cup C_j^* \cup C_k^* \), \( r_M(Z) = 4 \) and \( r_M(Z) = |Z| - 3 \). Since \( Z \) is not a 2-separating set of \( M \), then \( |E(M) - Z| \leq 1 \). This implies that \( r_M = 4 \). A contradiction.

35.5. Let \( 1 \leq i, j \leq 3 \). If \( \text{si}(M/x_i, y_j) \) is 3-connected, then \( \text{si}(M/y_j) \) is 3-connected.

Suppose the contrary. Lemma 15 implies the existence of an \((M, N)\)-verbarrier in the form \((D^*, y_j)\). A contradiction to 35.4.

By 35.3 \( \text{si}(M/x_1, y_1) \) is not 3-connected, because, by assumption, \( y_1 \) is not vertically \( N \)-contractible in \( M \). By 35.2 \( M/x_1 \) is 3-connected. Moreover, since \( x_1, x_2, x_3, y_1, y_2, y_3 \) is a triewe of \( M/x \), then \( \text{si}(M/x_1, y_1) \cong (M/x)/L \) is 3-connected. Therefore, by Lemma 15 on \( M/x_1 \), and \( y_1 \), there is an \((M/x_1, N)\)-verbarrier \((D^*, y_1)\) containing \( x \). By 35.3 \((D^*, y_1)\) is connected and, consequently, \((M/x_1))((D^* \cup y_1)\) has a circuit \( D \) containing \( x \) and \( y_1 \). In \( M/x, x_1 \), \( D - x \) is a triangle containing \( C_2^* \cap C_3^* = \{y_1\} \), thus \( D - x \) meets both \( \{x, y_3\} \) and \( \{x_3, y_2\} \), by orthogonality.

If \( D = L \cup x \), then by Lemma 19 \( y_2 \) and \( y_3 \) are vertically \( N \)-contractible in \( M/x_1 \) and, by 35.5 also in \( M \). But this implies the lemma. So, for some \( j, k = \{2, 3\}, x_k \in C \). By Lemma 24 \( M/x, x_1 \) is 3-connected with rank at least 3. So \( D - x \notin C_i^* = \{y_1, x_k, y_j\} \). This implies that \( D = \{x, y_1, x_2, x_3\} \).

If \( D - x \in \mathcal{C}(M/x) \), then \( r_M(x, x_1) \leq 3 \), contradicting Lemma 31. Thus \( C := D - x \cup x_1 = \{y_1, x_1, x_2, x_3\} \in \mathcal{C}(M/x) \). But this contradicts the orthogonality between \( C \) and \( C_i^* \).

In some cases, it is easier to prove that an element \( p \) is spanned by vertically \( N \)-contractible elements instead of proving that \( p \) is vertically \( N \)-contractible itself. In this case, some exchanges to get a desired vertically \( N \)-contractible element may be applied (we will do it further, in Lemma 48). We say that an element \( p \in E(M) \) is \( N \)-replaceable in \( M \) if \( p \) is spanned by a set of vertically \( N \)-contractible elements of \( M \). For \( S \subseteq E(M) \), we say that \( p \) is \((S, N)\)-replaceable if there is a set \( I \) of vertically \( N \)-contractible elements of \( M \) such that \( p \in cl_M(S \cup I) = cl_M(S) \). From Corollary 28 we have:

**Corollary 36.** If \((C^*, p)\) is a connected \((M, N)\)-verbarrier, then each element of \( C^* \cup p \) is \( N \)-replaceable in \( M \).

**Lemma 37.** Suppose that \( M \) has no \( N \)-deletable elements. If \( \text{si}(M/x_1, p_1) \) is 3-connected with an \( N \)-minor, then \( p_1 \) is \( N \)-contractible in \( M \) or \( M \) has an \( N \)-biweb \( x_1, x_2, p_1, p_2, p_3 \).

**Proof.** Suppose the contrary. By Lemma 15 there is an \((M, N)\)-verbarrier \((C^*, p_1)\) containing \( x_1 \). Moreover, by Corollary 30 \((C^*, p_1)\) is disconnected, with line, say, \( L \). By Lemma 15 \( \text{si}(M/x_1, p_1) \cong \text{si}(M/L) \) is 3-connected with an \( N \)-minor. Then \( M/p_1 \) has an \( N \)-minor. Moreover \( \text{co}(M/p_1) \) is 3-connected. Since \( p_1 \) is not \( N \)-deletable, \( p_1 \) is in a triad \( T^* \) of \( M \). This implies that \( L \) is a triangle. Write \( L := \{p_1, p_2, p_3\}, C^* := \{x_1, x_2, p_3\} \) and \( T^* := \{x_2, p_1, p_3\} \). This proves the Lemma.
**Lemma 31** yields:

**Corollary 38.** If \( r(M) \geq 4 \) and \( W \) is a biweb of \( M \), then \( r_{M}(W) \geq 4 \).

**Lemma 39.** If \( r(M) \geq 5 \), \( M \) has no \( N \)-deletable elements and \( W = a_1, a_2, b_1, b_2, b_3 \) is an \( N \)-biweb of \( M \), then \( \{b_1, b_2, b_3\} \) is the unique triangle of \( M \) intersecting \( W \).

**Proof.** Suppose for a contradiction that \( S \) is a triangle of \( M \) intersecting \( W \) other than \( T \), then, by orthogonality with \( \{a_i, b_3 - i, b_3\} \) for \( i = 1, 2 \) we may assume that \( a_1 \in S \). If \( b_3 \in S \), then \( S \subseteq W \), but this implies that \( r_{M}(W) \leq 3 \). A contradiction to Lemma 38. So \( b_2 \in S \) by orthogonality with \( \{a_1, b_2, b_3\} \). This implies that \( W \) is part of a maximal fan \( F \) containing \( S \). As \( M/M_b_1 \) has an \( N \)-minor and \( b_1 \) is an inner spoke of \( F \), hence each deletion of a spoke of \( F \) in \( M \) has an \( N \)-minor. But \( M \) has no \( N \)-deletable elements, so the extremes of \( F \) are triads. Moreover, if \( T^* \) is a triad in \( F \) intersecting \( a_i \) but different from \( T_i^* \), then \( T^* \) is a rank-2 cocircuit of \( M/T \), which is vertically 3-connected with rank at least 3. A contradiction. So \( F = W \). A contradiction.

**Lemma 40.** Suppose that \( r(M) \geq 5 \), \( W = a_1, a_2, b_1, b_2, b_3 \) is an \( N \)-biweb of \( M \) and \( M \) has no \( N \)-deletable elements. Then \( M/(b_1, b_2, b_3) \) is 3-connected.

**Proof.** Write \( T := \{b_1, b_2, b_3\} \). By the definition of biweb, we just have to prove that \( M/T \) is simple. Suppose the contrary and let \( C \) be a circuit of \( M \) such that \( 1 \leq C - T \leq 2 \). By Lemma 39 \( |C| = 4 \). Therefore \( |C \cap T| = |C - T| = 2 \). By Corollary 38 \( C \not\subseteq W \). So, we may assume that \( a_2 \notin C \). But \( C \) meets \( \{b_1, b_3\} \) because \( |C \cap T| = 2 \). By orthogonality with \( \{a_2, b_1, b_3\} \), \( C \cap T = \{b_1, b_3\} \). So \( b_2 \notin C \). By orthogonality with \( \{a_1, b_2, b_3\} \), \( a_1 \in C \). Let \( D \) be a circuit of \( M \) contained in \( (C \cup T) - b_3 \). By orthogonality with \( \{a_2, b_1, b_3\} \), \( b_1 \notin D \). Therefore, \( D \subseteq (T \cup C) - \{b_1, b_3\} \). This implies that \( |D| \leq 3 \). A contradiction to Lemma 39.

**Lemma 41.** Suppose that \( r(M) \geq 5 \), \( T \) is the triangle of an \( N \)-biweb of \( M \) and \( M \) has no \( N \)-deletable elements. If \( L \) is an \( N \)-filament of \( M/T \), then \( L \) is an \( N \)-filament of \( M \) or \( L \) contains an \( N \)-contractible element of \( M \).

**Proof.** Let \( W = a_1, a_2, b_1, b_2, b_3 \) be an \( N \)-biweb of \( M \) with \( T := \{b_1, b_2, b_3\} \). If \( L \) is not a line of \( M \), then \( \{L, T\} \) is not free in \( M \). So, there is \( C \in \mathcal{E}(M\cup T) \) meeting both \( L \) and \( T \). By orthogonality with \( \{a_1, b_2, b_3\} \) and \( \{a_2, b_1, b_3\} \), \( \{a_1, a_2\} \) meets \( C \) and, therefore, \( L \). By Corollary 18 and Lemma 39 \( a_1 \) and \( a_2 \) are \( N \)-contractible in \( M \). This implies the Lemma in this case.

Now, assume that \( L \) is a line of \( M \). Say that \( L \) is part of a carambole \( K \) of \( M/T \). The cocircuits of \( K \) are also cocircuits of \( M \). Hence Lemma 21 implies that \( L \) is an \( N \)-filament of \( M \).

**Lemma 42.** Suppose that \( H \) is a vertically connected matroid, and \( A \) is a maximal vertical 2-separating set of \( H \). Then \( A \) is a flat of \( H \). Moreover, if \( x \in cl_H^*(A) - A \), then \( x \) is a cocircuit of \( H \setminus A \) and \( A \) is a hyperplane of \( M \) with a rank-2 complementary cocircuit.

**Proof.** Let \( B := E(H) - A \). If \( x \in cl_H(A) - A \) or \( x \in cl_H^*(A) - A \), then \( \lambda_H(A \cup x) \leq \lambda_H(A) \). By the maximality of \( A \), \( r_H(B - x) \leq 1 \) and \( x \) is a cocircuit of \( M/B \). Hence, if \( x \in cl_H(A) - A \), then \( \lambda_H(A \cup x) \leq \lambda_H(A) - 1 \), a contradiction to the vertical connectivity of \( M \). So \( A \) is a flat of \( H \).

Now, if \( x \in cl_H^*(A) - A \), then, as we saw before, \( x \) is a cocircuit of \( H/B \). Since \( H \) is vertically connected, then \( B - x \subseteq cl_H(A \cup x) \). But \( A \) is a flat of \( H \). This implies that \( A \) is an hyperplane of \( H \) and the result holds.

**Lemma 43.** Suppose that \( r(M) \geq 4 \), that \( W = a_1, a_2, b_1, b_2, b_3 \) is an \( N \)-biweb of \( M \) with triangle \( T \) and \( p \in E(M) - cl_M(W) \) is such that \( T \cup p \) is vertically \( N \)-contractible in \( M \) but \( p \) is not. Then \( r(M) = 5 \).

**Proof.** Suppose the contrary. If \( r(M) \leq 3 \), then \( M/p \) is trivially vertically 3-connected. So assume that \( r(M) \geq 4 \). Hence, by Corollary 38 and since \( p \notin cl_M(W) \), it follows that \( r(M) \geq 5 \) and, therefore \( r(M) \geq 6 \). Let \( \{A, A^*\} \) be a 2-separation of \( M/p \) such that \( |A \cap T| \geq 2 \) with \( A \) maximal. By Lemma 42 \( A \) is a flat of \( M/p \) and \( T \subseteq A \).
If \( r_{|M/T∪p|}(A−T) ≤ 1 \), then \( r_M(A∪p) ≤ 4 \). Thus \( \{a_1, a_2\} \) meets \( A^* \). By the second part of Lemma \[42\] \( A^* \) is a cocircuit of \( M/p \) and, therefore, of \( M \). So \( A∪p \) is a hyperplane of \( M \). But \( r_M(A∪p) ≤ 4 \), and, therefore, \( r(M) ≤ 5 \). A contradiction. Thus \( r_{|M/T∪p|}(A−T) ≥ 2 \). Now, note that

\[
\begin{align*}
\frac{r_{|M/T∪p|}(A−T) + r_{|M/T∪p|}(A^*)}{1} &= \frac{r_{M/p}(A) - 2 + r_{M/p}(A^*) + (r_{|M/T∪p|}(A^*) - r_{M/p}(A^*))}{1} \\
&≤ \frac{r(M/p) - 1 + (r_{|M/T∪p|}(A^*) - r_{M/p}(A^*))}{1} \\
&= \frac{r(M/T∪p) + 1 + (r_{|M/T∪p|}(A^*) - r_{M/p}(A^*))}{1}.
\end{align*}
\]

As \( T \) meets two triads, it follows that \( T \) is a flat of \( M/p \) disjoint from \( A^* \). Hence \( r_{M/p}(T∪A^*) ≥ 3 \) and \( r_{|M/T∪p|}(A^*) ≥ 1 \). Since \( M/T∪p \) is vertically connected, the right side of (1) is at least \( r(M/T∪p) + 1 \), so \( r_{|M/T∪p|}(A^*) ≤ r_{M/p}(A^*) + 2 \). Now, (1) contradicts the vertical 3-connectivity of \( M/(T∪p) \).

**Corollary 44.** If \( r(M) ≠ 5 \) and \( T \) is a triangle of an \( N \)-biweb of \( M \), then each vertically \( N \)-contractible element of \( s(M/T) \) is \( (T, N) \)-replaceable in \( M \).

**Proof.** Let \( W \) be a biweb of \( M \) having \( T \) as triangle. The result follows from Lemma \[43\] if \( p ∈ cl_M(W) \). Otherwise, it is straightforward that \( p ∈ cl_M(T) \). By Corollary \[18\] the elements of \( W−A \) are vertically \( N \)-contractible and the result follows.

**Lemma 45.** If \( M \) has no deletable elements, \( r(M) = 5 \) and \( M \) has a biweb, then \( M \) has a triweb or a 4-independent set of vertically contractible elements.

**Proof.** Suppose the contrary. Let \( W := a_1, a_2, b_1, b_2, b_3 \) be a biweb of \( M \), which must be strict by our assumptions. By Lemma \[30\] and Corollary \[18\], \( I := \{a_1, a_2, b_3\} \) is an independent set of vertically contractible elements of \( M \). As \( W \) is a union of two triads and \( r(M) = 5 \), it follows that \( F := E(M)−W \) is a rank-3 flat of \( M \). Since \( r_M(W) ≤ 4 \), \( F \) a cocircuit \( C^* \) complementary to a hyperplane containing \( W \). But \( M \) is 3-connected and \( r_M(F) = 3 \), thus \( F = cl_M(C^*) \). As \( M \) has no 4-independent set of vertically contractible elements, hence \( C^* \) contains no vertically contractible elements of \( M \). If \( M \) has a circuit \( C ⊆ F \), choose \( x ∈ cl_M(C) \), picking \( x ∈ cl_M(C) \) in the case that \( C \) is a triangle. By Lemma \[17\] \( x \) is vertically contractible. A contradiction. This implies that \( F = C^* \) and that \( |C^*| = 3 \). Thus, \( |E(M)| = 8 \) and \( r^*(M) = 3 \). Therefore, \( T = \{b_1, b_2, b_3\} \) is a basis of \( M \). Let \( x ∈ C^* \) and let \( D^* \) be a cocircuit such that \( z ∈ D^* ⊆ T∪z \). Note that \( r_M(D^*) = 3 \) and that \( T \) is a triangle of \( M/z \) contained in \( cl_M(D^*)−z \). By Lemma \[17\] \( z \) is vertically contractible. A contradiction.

The next Lemma has an elementary proof, which will be omitted.

**Lemma 46.** If \( \{A_1, . . . , A_n\} \) is a free family of \( M \) and, for each \( i = 1, . . . , n \), \( B_i ⊆ cl_M(A_i) \), then \( \{B_1, . . . , B_n\} \) is a free family of \( M \).

**Lemma 47.** If \( \{A_1, . . . , A_n\} \) is a free family of \( M/X \) and \( r_{M/X}(A_i) = r_M(A_i) \) for each \( i \), then \( \{X, A_1, . . . , A_n\} \) is a free family of \( M \).

**Proof.** Note that:

\[
\begin{align*}
\frac{r_M(X ∪ A_1 ∪ . . . ∪ A_n)}{1} &≤ \frac{r_M(X) + r_M(A_1) + . . . + r_M(A_n)}{1} \\
&= \frac{r_M(X) + r_{M/X}(A_1) + . . . + r_{M/X}(A_n)}{1} \\
&= \frac{r_M(X) + r_{M/X}(A_1 ∪ . . . ∪ A_n)}{1} \\
&= \frac{r_M(X ∪ A_1 ∪ . . . ∪ A_n)}{1}.
\end{align*}
\]

Thus equality holds above. This implies the lemma.

We say that a singleton subset of \( E(M) \) is \((X, N)\)-replaceable in \( M \) if its element is \((X, N)\)-replaceable in \( M \).

**Lemma 48.** Suppose that \( M/X\setminus Z \) is 3-connected with an \( N \)-minor and \( \{X_1, . . . , X_n\} \) is a free family of \( M/X \setminus Z \). Suppose also that, for each \( X_i \), one of the following alternatives holds:

(a) For some 2-subset \( Y_i ⊆ X_i \), \( cl_M(Y_i) \) contains an \((X, N)\)-replaceable element or an \( N \)-filament of \( M \).
Then $M$ has a free family $\{X, Z_1, \ldots, Z_n\}$ such that, for $k = 1, \ldots, n$, $Z_k$ is an $N$-filament or a vertically $N$-contractible singleton set of $M$.

**Proof.** By Lemma 46 we may define a free family $\{X, Y_1, \ldots, Y_n\}$ of $M/I\backslash Z$, choosing $Y_i$ as a 2-subset of $X_i$ according to (a), provided (a) holds, and choosing $Y_i := X_i$ otherwise. Since $1 \leq r_{M\backslash Z}(Y_i) = |Y_i| = r_{M/I\backslash X_i}(Y_i) \leq 2$ for each $i$, it follows, by Lemma 47, that $\{X, Y_1, \ldots, Y_n\}$ is a free family of $M\backslash Z$, and therefore, of $M$. Next, for $i \geq 1$, define $W_i$ as a subset of $\text{cl}_M(Y_i)$ that is an $N$-filament or an $(X, N)$-replaceable singleton set of $M$. By Lemma 46, $F_0 := \{X, W_1, \ldots, W_n\}$ is free. Now, for $k = 1, \ldots, n$, let us define $Z_k$ inductively in such a way that each $\mathcal{F}_k := \{X, Z_1, \ldots, Z_k, W_{k+1}, \ldots, W_n\}$ is free.

Let $1 \leq k \leq n$. If $W_k$ is an $N$-filament of $M$ or a vertically $N$-contractible singleton set of $M$, simply define $Z_k := W_k$. Otherwise, $W_k$ is an $(X, N)$-replaceable but not vertically $N$-contractible singleton set of $M$. Let $I$ be a set of vertically $N$-contractible elements and $C$ a circuit of $M$ such that $w \in C \subseteq (X \cup I) \cup u$. As $\mathcal{F}_{k-1}$ is free and $w \in W_k$, hence $w \notin F := \text{cl}(X \cup Z_1 \cup \ldots \cup Z_{k-1} \cup W_{k+1} \cup \ldots \cup W_n)$ and, therefore, there is an element $z \in C - (F \cup u) \subseteq I$. Since $z \in I$, then $z$ is vertically $N$-contractible. Define $Z_k = \{z\}$. Since $\mathcal{F}_{k-1} - \{W_k\}$ is free and $z \notin F$, then $\mathcal{F}_k := (\mathcal{F}_{k-1} - \{W_k\}) \cup \{Z_k\}$ is free. The family $\mathcal{F}_n$ satisfies the Lemma.

5. PROOFS FOR THE MAIN RESULTS

**Proof of Theorem 6.** Suppose that $(M, N)$ is a counter-example to the theorem minimizing $|E(M)|$. By Theorem 1 the theorem holds for $k \leq 4$. So $k \geq 5$. It is straightforward to verify that $M$ is not a wheel or whirl. In such case, $N$ would be also a wheel or whirl respectively or $N \cong U_{2,4}$, implying that the elements in the rim of $M$ are vertically $N$-contractible. By Seymour’s Splitter Theorem, exactly one of the following cases occur:

(i) $M$ has an $N$-deletable element.

(ii) $M$ has an $N$-biweb and no $N$-deletable elements.

(iii) $M$ has an $N$-contractible element $x$ and (i) and (ii) do not occur.

Next we complete the proof for each case.

Case (i): Define $X := \emptyset$ and $Z$ as an $N$-deletable singleton set of $M$. Let $\mathcal{F} := \{X_1, \ldots, X_n\}$ be a family satisfying Theorem 6 for $M\backslash Z$. By Corollary 33 and Lemma 34 $\mathcal{F}$ satisfies the hypothesis of Lemma 48. The family $\{Z_1, \ldots, Z_n\}$ obtained from Lemma 48 satisfies the theorem, since $r(M) = r(M\backslash Z)$ in this case.

Case (ii): If $r(M) = 5$, then $r(N) = 0$ and the theorem follows from Lemma 45. Assume that $r(M) \geq 6$ for this case. Let $W$ be an $N$-biweb of $M$ with triangle, namely, $T$. By Lemma 40, $M/T$ is 3-connected with an $N$-minor. Let $\mathcal{F} := \{X_1, \ldots, X_n\}$ be a family satisfying Theorem 6 for $M/T$. Define $X := T$ and $Z := \emptyset$. The hypothesis of Lemma 48 are satisfied because of Corollary 44 and Lemma 41. Let $\{X, Z_1, \ldots, Z_n\}$ be a free family of $M$ as in Lemma 48. If $W$ is part of a triweb, then $X$ is an $N$-filament and the theorem holds. Otherwise, if $W$ is a strict biweb, then, by Corollary 30, $X$ contains a vertically $N$-contractible element $x$ of $M$ and $\{x\}, Z_1, \ldots, Z_n$ satisfies the theorem.

Case (iii): Define $Z := \emptyset$ and $X := \{x\}$. Let $\mathcal{F} := \{X_1, \ldots, X_n\}$ be a family satisfying Theorem 6 for $M/x$. By Lemma 57 all vertically $N$-contractible elements of $M/x$ are $(X, N)$-replaceable in $M$. So, by Lemma 33, the hypothesis of Lemma 48 are once more satisfied. The family $\{X, Z_1, \ldots, Z_n\}$, obtained from Lemma 48 satisfies the theorem, as before. This completes the proof.

**Proof of Theorem 7.** A proof for Theorem 7 is similar to the one of Theorem 6. The unique difference between the proofs is the initial case of the inductive process, which is give by the hypothesis for Theorem 7, while it relies on Theorem 1 for Theorem 6.

**Proof of Theorem 8.** Define $M := M(G)$. Choose a free family $\mathcal{F} := \{X_1, \ldots, X_n\}$ as in Theorem 6 maximizing the number $s$ of singleton sets in $\mathcal{F}$. Let $X := X_1 \cup \cdots \cup X_n$. 

\[ \text{Proof of Theorem 6.} \] \[ \text{Proof of Theorem 7.} \] \[ \text{Proof of Theorem 8.} \]

\[ \text{Proof of Theorem 8.} \]
We shall verify that \( \{x\} \in \mathcal{F} \) if \( X_i \) is a triangle and \( x \in T^* - X_i \) for some triad \( T^* \subseteq X_i \cup x \). Indeed, suppose the contrary. By Corollary 25, \( x \) is \( N \)-contractible in \( M \). Since \( T^* \subseteq X_i \cup x \), hence \( E(M) - (X_i \cup T^*) \) do not span \( x \) in \( M \). So \( \mathcal{F} - \{X_i\} \cup \{\{x\}\} \) contradicts the maximality of \( s \).

Say that for \( j = 1, \ldots, s \), \( \{x_j\} := X_j \) is a singleton set. For \( i = s + 1, \ldots, n \), \( X_i \) is a triangle of a triweb, which intersects 3 triads, each one with an element of \( \{x_1, \ldots, x_s\} \) by what we just proved. Since \( M \) is graphic, each element of \( \{x_1, \ldots, x_s\} \) is in at most two triwebs of \( M \). Thus, \( 2s \geq 3(n - s) \) and, therefore, \( s \geq \frac{3}{5} n \geq \frac{3}{5} \left[ \frac{k+3}{2} \right] \).

**Proof of Proposition 10.** By Lemma 31, \( |X| = n \). Moreover, by orthogonality with the cocircuits of the carambole, \( X \) may not contain circuits. The elements of \( X \) are vertically \( N \)-contractible because of Corollary 20. So the proposition holds.

**Proof of Corollary 11.** The proof is analogous to the proof of Theorem 6. Consider a counterexample minimizing \( |E(M)| \). The proof in case (i) is the same. Case (ii) cannot occur. In case (iii) instead of the minimality of \( M \), we use Theorem 1 for \( k = 3 \) to obtain \( \mathcal{F} \) and, in the same way, finish the proof.

**Proof of Corollary 12.** Consider the family \( \{X_1, \ldots, X_s\} \) given by Theorem 6. We may assume that \( X_1 \) is an \( N \)-filament. Since \( X_1 \) is not the triangle of an \( N \)-triweb, it follows that \( X_1 \) has more than 3 elements. By Proposition 10, \( M \) has a 4-independent set of vertically \( N \)-contractible elements.

With the lemmas we established here, it is possible to give an alternative proof for Theorem 1. This is important the self-sufficiency of this work.

**Proof of Theorem 1.** Suppose that \( M \) and \( N \) contradict the theorem, minimizing \( (k, |E(M)|) \) lexicographically. If \( k = 1 \), the result follows directly from Seymour’s Splitter Theorem and Bixby’s Lemma (this is made with details in [11], Lemma 12.3.11). So \( k \geq 2 \). If \( r(M) \leq 3 \), then \( r(N) \leq 1 \) and the result is trivial. Suppose that \( r(M) \geq 4 \). By Proposition 10 and Lemma 30, \( M \) has neither \( N \)-biwebs nor \( N \)-caramboles. By Lemma 33 and the minimality of \( |E(M)| \), \( M \) has no \( N \)-deletable elements. By Seymour’s Splitter Theorem \( M \) has an \( N \)-contractible element \( x \). By the minimality of \( k \), there is a free family \( \mathcal{F} \) with \( k - 1 \) vertically \( N \)-contractible singleton sets of \( si(M/x) \). The proof is finished as in case (iii) of the proof of Theorem 6.

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