SYMMETRIC 1–DEPENDENT COLORINGS OF THE INTEGERS

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Abstract. In a recent paper by the same authors, we constructed a stationary 1–dependent 4–coloring of the integers that is invariant under permutations of the colors. This was the first stationary k–dependent q–coloring for any k and q. When the analogous construction is carried out for q > 4 colors, the resulting process is not k–dependent for any k. We construct here a process that is symmetric in the colors and 1–dependent for every q ≥ 4. The construction uses a recursion involving Chebyshev polynomials evaluated at √q/2.

1. Introduction

By a (proper) q–coloring of the integers, we mean a sequence (X_i : i ∈ Z) of [q]–valued random variables satisfying X_i ≠ X_{i+1} for all i (where [q] := {1, . . . , q}). The coloring is said to be stationary if the (joint) distribution of (X_i : i ∈ Z) agrees with that of (X_{i+1} : i ∈ Z), and k–dependent if the families (X_i : i ≤ m) and (X_i : i > m + k) are independent of each other for each m. In [2], we gave a construction of a stationary 1–dependent 4–coloring of the integers that is invariant under permutations of the colors. When the same construction is carried out for q > 4 colors, the resulting distribution is not k–dependent for any k. Of course, the 1–dependent 4–coloring is also a 1–dependent q–coloring for every q > 4, and one may obtain other 1–dependent q–colorings by splitting a color into further colors using an independent source of randomness. However, these colorings are not symmetric in the colors. We give here a modification of the process of [2] that is symmetric in the colors and 1–dependent for every q ≥ 4. Here is our main result.

Theorem 1. For each integer q ≥ 4, there exists a stationary 1–dependent q–coloring of the integers that is invariant in law under permutations of the colors and under the reflection (X_i : i ∈ Z) ↦ (X_{-i} : i ∈ Z).

Our construction is given in the next section. Sections 3 and 4 provide some preliminary results and the proof of Theorem 1 respectively.

2. The construction

For x = (x_1, x_2, . . . , x_n) ∈ [q]^n, we will write P(x) = P(X_1 = x_1, . . . , X_n = x_n). To motivate the construction, we begin by noting that the finite-dimensional distributions P of the 4–coloring in [2] are defined recursively by P(∅) = 1 and

\[ P(x) = \frac{1}{2(n + 1)} \sum_{i=1}^{n} P(\tilde{x}_i) \] (1)

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for proper \( x \in [4]^n \), where \( \hat{x}_i \) is obtained from \( x \) by deleting the \( i \)-th entry in \( x \). Of course, even if \( x \) is proper, \( \hat{x}_i \) may not be. So the definition is completed by setting \( P(x) = 0 \) for \( x \)'s that are not proper.

For general \( q \geq 4 \), we will now allow the coefficients in the defining sum to depend on \( i \) as well as \( n \). Considering many special cases, and the constraints imposed by the \( 1 \)-dependence requirement, we were led to define

\[
P(x) = \frac{1}{D(n + 1)} \sum_{i=1}^{n} C(n - 2i + 1) P(\hat{x}_i)
\]

for proper \( x \in [4]^n \), in terms of two sequences \( C \) and \( D \). Again motivated by computations in special cases, we take

\[
C(n) = T_n(\sqrt{q}/2), \quad n \geq 0;
\]
\[
D(n) = \sqrt{q} U_{n-1}(\sqrt{q}/2), \quad n \geq 1,
\]

where \( T_n \) and \( U_n \) are the Chebyshev polynomials of the first and second kind respectively.

There are several standard equivalent definitions of Chebyshev polynomials. One is

\[
T_n(u) = \cosh(nt) \quad \text{and} \quad U_n(u) = \frac{\sinh((n+1)t)}{\sinh(t)}, \quad \text{where} \ u = \cosh(t).
\]

A variant definition using trigonometric functions (e.g. (22:3:3-4) of [3]) is easily seen to be equivalent by taking \( t \) imaginary; the hyperbolic function version is convenient for arguments \( u \geq 1 \). Another definition is

\[
T_n(u) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} u^{n-2k} (u^2 - 1)^k \quad \text{and} \quad U_n(u) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n+1}{2k+1} u^{n-2k} (u^2 - 1)^k.
\]

That this is equivalent to (3) follows from e.g. (22:3:1-2) of [3].

If \( x \) is not a proper coloring, we take \( P(x) = 0 \) as before. We extend both sequences \( C \) and \( D \) to all integer arguments by taking \( C(n) \) and \( D(n) \) to be even and odd functions of \( n \) respectively (in accordance with (3)).

Observe that \( C(n) \) and \( D(n) \) are strictly positive for \( q \geq 4 \) and \( n \geq 1 \), and therefore \( P(x) \) is strictly positive for all proper \( x \). Note also that \( C(n - 2i + 1) / D(n + 1) \) is rational; therefore so is \( P(x) \). (The factors of \( \sqrt{q} \) cancel). When \( q = 4 \) we have \( C(n) = 1 \) and \( D(n) = 2n \), and so (2) reduces to (1) in this case. As we will see, the fact that the coefficients in (2) depend on \( i \) substantially complicates the verifications of the required properties of \( P \).

Here are a few examples of cylinder probabilities generated by (2).

\[
P(1) = \frac{1}{q}, \quad P(12) = \frac{1}{q(q-1)}, \quad P(121) = \frac{1}{q^2(q-1)}, \quad P(123) = \frac{1}{q^2(q-2)},
\]
\[
P(1212) = \frac{q-3}{q^2(q-1)(q^2 - 3q + 1)}, \quad P(1234) = \frac{1}{q^2(q^2 - 3q + 1)}.
\]
3. Preliminary results

Chebyshev polynomials satisfy a number of standard identities. They lead to identities satisfied by the sequences $C$ and $D$. The first three in the proposition below are examples of this. The fourth is a consequence of the third one. Before stating them, we record some values of $C$ and $D$ to facilitate checking computations here and later.

$$C(0) = 1, \quad C(1) = \frac{\sqrt{q}}{2}, \quad C(2) = \frac{q - 2}{2}, \quad C(3) = \frac{\sqrt{q(q - 3)}}{2}, \quad C(4) = \frac{q^2 - 4q + 2}{2}.$$  

$D(0) = 0, \quad D(1) = \sqrt{q}, \quad D(2) = q, \quad D(3) = \sqrt{q(q - 1)}, \quad D(4) = q(q - 2).$

**Proposition 2.** For $j, k, \ell, m, n \in \mathbb{Z}$, the following identities hold.

1. $2C(m)C(n) = C(m + n) + C(n - m)$.
2. $\frac{q - 4}{2q} D(m)D(n) = C(m + n) - C(n - m)$.
3. $2C(m)D(n) = D(m + n) + D(n - m)$.
4. $C(j + k)D(k + \ell) = C(k)D(j + k + \ell) - C(\ell)D(j)$.

**Proof.** The first three parts are immediate consequences of (22.5.7-7) in [3], or 22.7.24-26 in [1], if $m$ and $n$ are nonnegative. None of the identities is changed by changing the sign of either $m$ or $n$. Therefore, they hold for all $m$ and $n$. Alternatively, the identities may be checked directly from (3) using the product formulae for hyperbolic functions. For (4), replace the products of $C$’s and $D$’s by sums of $D$’s using (3), and then use the fact that $D$ is an odd function. \( \square \)

Next we verify some identities that involve both the sequences $C$ and $D$ and the measure $P$ defined by (2). For the statement of the second part of the next result, let

$$Q(x) = \frac{1}{D(n + 1)} \sum_{i=1}^{n} C(2i)P(\hat{x}_i) \quad \text{and} \quad Q^*(x) = \frac{1}{D(n + 1)} \sum_{i=1}^{n} C(2n - 2i + 2)P(\hat{x}_i)$$

for $x \in [q]^n$. The first part of Proposition 3 is needed in proving the second part, which plays a key role in the proof of consistency and 1--dependence of $P$. Note the similarity between the left side of (3) and the right side of (2).

**Proposition 3.** If $n \geq 1$, and $x$ is a proper coloring of length $n$, then

1. $\sum_{i=1}^{n} D(n - 2i + 1)P(\hat{x}_i) = 0$;
2. $Q(x) = Q^*(x) = P(x)C(n + 1)$.

**Proof.** For the first statement, let $R$ be the set of proper colorings, and $\hat{x}_A$ be obtained by deleting the entries $x_i$ for $i \in A$ from $x$. The proof of (3) is by induction on $n$, the length of $x$. The identity is easily seen to be true if $n \leq 2$. Suppose that (3) is true for all $x$ of length $n - 1$, and let $x \in R$ have length $n$. For those $i$ with $\hat{x}_i \in R$, applying (3) gives

$$\sum_{j=1}^{i-1} D(n - 2j)P(\hat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n - 2j + 2)P(\hat{x}_{i,j}) = 0.$$
We must show that (10) and (11) imply that (13) is zero.

The left side of (10) for $x$ can be written, using the definition of $P(\widehat{x}_i)$ and then (6), as

$$\frac{1}{D(n)} \sum_{1 \leq i \leq n; \widehat{x}_i \in R} D(n - 2i + 1) \left[ \sum_{1 \leq j < i} C(n - 2j)P(\widehat{x}_{i,j}) + \sum_{i < j \leq n} C(n - 2j + 2)P(\widehat{x}_{i,j}) \right]$$

and, ignoring the $2D(n)$ in the denominator, gives

$$\sum_{i=1}^{n} \frac{1}{D(n)} \sum_{1 \leq j < i \leq n; \widehat{x}_i \in R} [D(2n - 2i - 2j + 1) + D(2j - 2i + 1)]P(\widehat{x}_{i,j})$$

Rearranging, and ignoring the $2D(n)$ in the denominator, gives

$$\sum_{i=1}^{n} 1[\widehat{x}_i \in R] \left[ \sum_{j=1}^{i-1} [D(2n - 2i - 2j + 1) + D(2j - 2i + 1)]P(\widehat{x}_{i,j}) \right]$$

We must show that (10) and (11) imply that (13) is zero.

We would like to write (13) as a linear combination of expressions that vanish because of (10) and (11) as follows.

$$\sum_{1 \leq i \leq n; \widehat{x}_i \in R} \alpha_i \left[ \sum_{j=1}^{i-1} D(n - 2j)P(\widehat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n - 2j + 2)P(\widehat{x}_{i,j}) \right] + \sum_{1 \leq i \leq n} \beta_{i,j}P(\widehat{x}_{i,j})$$

where $\beta_{i,j} = \beta_{i,i-1} + \beta_{i,i+1} = 0$. If $1 \leq i < j \leq n$, the coefficient of $P(\widehat{x}_{i,j})$ in (13) is

$$1[\widehat{x}_j \in R] \left[ D(2n - 2i - 2j + 1) + D(2i - 2j + 1) \right]$$

We need to choose the $\alpha$'s and $\beta$'s so that (15) and (16) agree. If $\widehat{x}_i, \widehat{x}_j \in R$, this says

$$D(2n - 2i - 2j + 1) + D(2n - 2i - 2j + 3) = \alpha_j D(n - 2i) + \alpha_i D(n - 2j + 2)$$

since $D$ is an odd function. It may sound unreasonable to expect to solve this system, since there are $n$ unknowns and $\binom{n}{2}$ equations. However, $D$ satisfies relations that make this possible. Solving the equations for small $n$ suggests trying $\alpha_i = 2C(n - 2i + 1)$. The fact that this choice solves these equations for all choices of $n, i, j$ then follows from (6)
and the fact that $D$ is odd. If $\hat{x}_i \notin R$ and $\hat{x}_j \notin R$, (15) and (16) agree if $\beta_{i,j} + \beta_{j,i} = 0$. If $\hat{x}_i \in R$ and $\hat{x}_j \notin R$, they agree if

$$D(2n - 2i - 2j + 3) + D(2j - 2i - 1) = \alpha_i D(n - 2j + 2) + \beta_{j,i}.$$ 

Using (5) again gives $\beta_{j,i} = 2D(2j - 2i - 1)$. Similarly, if $\hat{x}_i \notin R$ and $\hat{x}_j \in R$, they agree if $\beta_{i,j} = 2D(2i - 2j + 1)$. With these choices, $\beta$ is anti-symmetric, and $\beta_{k,k-1} = 2D(1)$ and $\beta_{k,k+1} = 2D(-1)$, so $\beta_{k,k-1} + \beta_{k,k+1} = 0$ as required. This completes the induction argument.

For (9), consider the case of $Q$ first. Use the definition of $P$ to write the right side of (9) as

$$\frac{C(n + 1)}{D(n + 1)} \sum_{i=1}^{n} C(n - 2i + 1)P(\hat{x}_i).$$

Using (4), this becomes

$$\frac{1}{2D(n + 1)} \sum_{i=1}^{n} C(2n - 2i + 2)P(\hat{x}_i) + \frac{1}{2}Q(x).$$

Therefore, we need to prove that

$$\sum_{i=1}^{n} [C(2n - 2i + 2) - C(2i)] P(\hat{x}_i) = 0.$$ 

But by (5), this follows from (8). The proof for $Q^*$ is similar. 

4. Proof of the main result

We will often write $x_1x_2 \cdots x_n$ instead of $(x_1, x_2, \ldots, x_n)$ below. If $x \in [q]^m$ and $y \in [q]^n$, let $xy$ denote the word $x_1 \cdots x_my_1 \cdots y_n \in [q]^{m+n}$.

Proof of Theorem 1. We first need to show that the finite dimensional distributions defined in (2) are consistent, i.e., that

$$\sum_{a \in [q]} P(xa) = P(x), \quad x \in [q]^n, \ n \geq 0.$$ 

This is true if $x$ is not proper, since then $xa$ is also not proper, and so both sides vanish. For proper $x$, the proof is by induction on $n$. Note that for $a \in [q]$,

$$P(a) = \frac{C(0)}{D(2)} = \frac{1}{q},$$

so $\sum_{a \in [q]} P(a) = 1$. This gives (17) for $n = 0$. Suppose it holds for all $x \in [q]^{n-1}$ with $n \geq 1$. Then for proper $x \in [q]^n$, using the induction hypothesis in the second equality,

$$\sum_{a \in [q]} P(xa) = \sum_{a \notin x_n} \frac{1}{D(n + 2)} \left[ \sum_{i=1}^{n} C(n - 2i + 2)P(\hat{x}_i a) + C(-n)P(x) \right]$$

$$= \frac{1}{D(n + 2)} \left[ \sum_{i=1}^{n} C(n - 2i + 2)P(\hat{x}_i) - C(-n + 2)P(x) + (q - 1)C(-n)P(x) \right].$$
The middle term in the second line accounts for the missing term \( a = x_n \) when the inductive hypothesis is applied to the case \( i = n \) (since \( \hat{x}_n x_n = x \)). Using \((j, k, \ell) = (1, n - 2i + 1, 2i)\) in \(7\) gives

\[
\frac{C(n - 2i + 2)}{D(n + 2)} = \frac{C(n - 2i + 1)}{D(n + 1)} - \frac{C(2i)D(1)}{D(n + 2)D(n + 1)}.
\]

Therefore

\[
\sum_{a \in [q]} P(xa) = P(x) - \frac{Q(x)}{D(n + 2)} - \frac{C(n - 2)}{D(n + 2)} P(x) + (q - 1) \frac{C(n)}{D(n + 2)} P(x).
\]

This is \( P(x) \), as required, by \(9\) and the fact that

\[(q - 1)C(n) = C(n - 2) + C(n + 1)D(1),\]

which is obtained by taking \((j, k, \ell) = (2, -n, n + 1)\) in \(7\), and then canceling a factor of \( \sqrt{q} \).

Invariance of the measure under permutations of colors and translations is immediate from the definition. Invariance under reflection amounts to checking \( P(x) = P(x_n \cdots x_1) \), which follows from the fact that the coefficients of \( \hat{x}_i \) and \( \hat{x}_{n-i+1} \) in \(2\), which are \( C(n - 2i + 1) \) and \( C(-n + 2i - 1) \) respectively, are equal by the symmetry of \( C \).

For 1-dependence, we need to show that for \( x \in [q]^m \) and \( y \in [q]^n \) with \( m, n \geq 0 \),

\[
P(x \ast y) = P(x)P(y),
\]

where the * means that there is no constraint at the single site between \( x \) and \( y \). This is again true if \( x \) or \( y \) is not proper since then both sides are zero. For proper \( x \) and \( y \), the proof is by induction, but now on \( m + n \). The statement is immediate if \( m = 0 \) or \( n = 0 \). So, we take \( m \geq 1 \) and \( n \geq 1 \).

There are two cases, according to whether or not \( xy \) is a proper coloring, i.e., whether \( x_m \) and \( y_1 \) are equal or different. Assume first that \( x_m = y_1 \). Without loss of generality, take their common value to be 1. Then using the definition of \( P \), including the fact that \( P(xy) = 0 \),

\[
P(x \ast y) = \sum_{a \in [q]} P(xay) = \frac{1}{D(n + m + 2)} \sum_{a \neq 1} \left[ \sum_{i=1}^{m} C(n + m - 2i + 2)P(\hat{x}_i ay) \right. \\
+ C(n - m)P(xy) + \sum_{j=1}^{n} C(n - m - 2j)P(xa \hat{y}_j) \left. \right] \\
= \frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^{m} C(n + m - 2i + 2)P(\hat{x}_i \ast y) + \sum_{j=1}^{n} C(n - m - 2j)P(x \ast \hat{y}_j) \right].
\]

Using the induction hypothesis, this becomes

\[
P(x \ast y) = \frac{1}{D(n + m + 2)} \left[ P(y) \sum_{i=1}^{m} C(n + m - 2i + 2)P(\hat{x}_i) + P(x) \sum_{j=1}^{n} C(n - m - 2j)P(\hat{y}_j) \right].
\]
Similarly, taking \((j, k, l) = (n, m - 2i + 1, i)\) in (7) gives
\[
\frac{C(n + m - 2i + 2)}{D(n + m + 2)} = \frac{C(m - 2i + 1)}{D(m + 1)} - \frac{C(2i)D(n + 1)}{D(m + 1)D(n + m + 2)}.
\]

Similarly,
\[
\frac{C(m + 2j - n)}{D(n + m + 2)} = \frac{C(2j - n - 1)}{D(n + 1)} - \frac{C(2n - 2j + 2)D(m + 1)}{D(n + 1)D(n + m + 2)}.
\]

Therefore, since \(C(\cdot)\) is even,
\[
P(x \ast y) = P(y) \left[ P(x) - \frac{D(n + 1)}{D(n + m + 2)} Q(x) \right] + P(x) \left[ P(y) - \frac{D(m + 1)}{D(n + m + 2)} Q^*(y) \right].
\]

By (9),
\[
P(x \ast y) = P(x)P(y) \left[ 2 - \frac{C(m + 1)D(n + 1) + C(n + 1)D(m + 1)}{D(n + m + 2)} \right].
\]

Assume now that \(x_m \neq y_1\), say \(x_m = 1\) and \(y_1 = 2\). Then
\[
(19) \quad P(x \ast y) = \sum_{a \in [q]} P(xay) = \frac{1}{D(n + m + 2)} \sum_{a \neq 1, 2} \left[ \sum_{i=1}^{m} C(n + m - 2i + 2)P(\hat{x}_iay) + C(n - m)P(xy) + \sum_{j=1}^{n} C(n - m - 2j)P(xay_j) \right]
\]
\[
\frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^{m} C(n + m - 2i + 2)P(\hat{x}_i \ast y) + \sum_{j=1}^{n} C(n - m - 2j)P(x \ast y_j) \right]
\]
as in the previous case. However, in the previous case, the term \(P(xy)\) dropped out because \(xy\) was not a proper coloring. In this case, the term \((q - 2)C(n - m)P(xy)\) is cancelled by the terms \(-P(xy)C(n - m + 2)\) and \(-P(xy)C(n - m - 2)\), which arise from
\[
\sum_{a \neq 1, 2} P(\hat{x}_m ay) = P(\hat{x}_m \ast y) - P(xy) \quad \text{and} \quad \sum_{a \neq 1, 2} P(xa\hat{y}_1) = P(x \ast \hat{y}_1) - P(xy).
\]

The fact that the overall coefficient of \(P(xy)\) vanishes is a consequence (1) with \(m = 2\), since \(2C(2) = q - 2\). The rest of the proof is the same as in the case \(x_m = y_1\) above. □

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