On the Breuil-Schneider conjecture: Generic case

Alexandre Pyrovovar

September 22, 2018

Abstract

We use the Taylor-Wiles-Kisin patching method to prove some new cases of the Breuil-Schneider conjecture.

Contents

1 Introduction 
   1.1 Notation 
   1.2 The Breuil-Schneider conjecture 
   1.3 Typical representations 
   1.4 Main result 
   1.5 Outline of the paper 

2 Locally algebraic vectors. Definition. First properties 

3 Interpolation map
   3.1 Construction in general case 
   3.2 Construction in the Iwahori case 

4 Local deformation rings 

5 Local-Global compatibility 

6 Support of patched modules 

7 Computation of locally algebraic vectors
1 Introduction

The aim of this work is to deduce some new cases of Breuil-Schneider conjecture using the patching construction of [CEG+16]. The conjecture in question predicts the existence of $GL_n(F)$-invariant norms on locally algebraic $GL_n(F)$-representations, where $F$ is a $p$-adic field. This conjecture was first proposed by Breuil and Schneider in [BS07], and it may be seen as first evidence to a $p$-Langlands correspondence. For a brief survey of this conjecture one may consult [Sor15]. This introduction is strongly influenced by this survey. Our main result Theorem 1.6, computes the locally algebraic vectors in unitary Banach space representation obtained from this theory.

1.1 Notation

Let $p$ a prime number such that $p \nmid 2n$. Let $F$ be a finite extension of $\mathbb{Q}_p$ with a finite residue field $k_F$. Let $O_F$ be its complete discrete valuation ring, let $\mathfrak{p}$ be the maximal ideal of $O_F$ with uniformizer $\varpi$, and let $q = |O_F/\varpi O_F|$. Let $G = GL_n(F)$.

Let $E$ be a finite extension of $\mathbb{Q}_p$ (the field of coefficients), $O$ the ring of integers of $E$ and $\mathbb{F}$ the residue field. Fix a residual Galois representation $\pi: G_F \rightarrow GL_n(\mathbb{F})$ of the local Galois group $G_F := \text{Gal}(\overline{F}/F)$. We assume that $E$ is large enough to contain all the embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$.

Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$. In the literature, smooth representations of $GL_n(F)$ are studied on complex vector spaces, hence the coefficients are in $\overline{\mathbb{Q}}_p$. The section 3.13 [CEG+16] provides a framework which shows that those results, which are valid in the classical theory over complex numbers, have an analogue over $E$ by faithfully flat descent $\overline{\mathbb{Q}}_p/E$. For instance if $\pi$ is an irreducible representation of $GL_n(F)$ with coefficients in $E$, after extending scalars to larger extension $E'/E$ we may assume that $\pi$ is absolutely irreducible and hence do the base change to $\overline{\mathbb{Q}}_p$. In this way all the results stated with $E$-coefficients can be proven over $\overline{\mathbb{Q}}_p$ without loss of generality.

With this in mind we will cite results from [Pyv18b] and [Pyv18a], which
were proven for $\mathbb{Q}_p$-coefficients and use them with $E$-coefficients where $E$ is a finite extension of $\mathbb{Q}_p$.

In this paper we will follows the notation and conventions of [CEG+16] (cf. 1.8), unless otherwise is stated.

1.2 The Breuil-Schneider conjecture

Now suppose $r : G_F \to GL_n(E)$ is a potentially semi-stable lift of $\mathbf{r}$, with Hodge-Tate weights $HT_{\kappa} = \{i_{\kappa,1} < \ldots < i_{\kappa,n}\}$, for each embedding $\kappa : F \hookrightarrow E$. By Fontaine’s recipe one associates an $n$-dimensional Weil-Deligne $WD(r)$ representation to $r$ with coefficients in $\mathbb{Q}_p \simeq \mathbb{C}$. Let $\mathrm{rec}$ denote the classical local Langlands correspondence with coefficients in $\mathbb{C}$ normalized in a way such that the central character of an irreducible smooth representation of $GL_n(F)$ corresponds to the determinant of the associated Weil-Deligne representation via the local class field theory. This is compatible with convention in the book [HT01], see Lemma VII.2.6. Let $\mathrm{rec}_p$ denote the local Langlands correspondence over $\mathbb{Q}_p$, defined by $\iota \circ \mathrm{rec}_p = \mathrm{rec} \circ \iota$ and define $r_p(\pi) = \mathrm{rec}_p(\pi \otimes |\det|)$. Let $\pi_{\mathrm{sm}}(r)$ be an irreducible smooth representation of $GL_n(F)$ with coefficients in $E$ defined by $\pi_{\mathrm{sm}}(r) = r_p^{-1}(WD(r)^{F-ss})$, where $F-ss$ denotes the Frobenius semi-simplification of $WD(r)$. Assume that $\pi_{\mathrm{sm}}(r)$ is generic, i.e. admits a Whittaker model. Then there is a model of $\pi_{\mathrm{sm}}(r)$ with coefficients in $E$, denoted again $\pi_{\mathrm{sm}}(r)$ which a smooth irreducible $E$-representation of $GL_n(F)$. We say that $r$ is generic when $\pi_{\mathrm{sm}}(r)$ is generic. In the case when $\pi_{\mathrm{sm}}(r)$ is not generic, we need to do some modifications, see [BS07] for more details. Indeed, by Bernstein-Zelevinsky classification, $\pi_{\mathrm{sm}}(r)$ is a Langlands quotient and there is a unique parabolic induction, denoted $\pi_{\mathrm{gen}}(r)$, such that $\pi_{\mathrm{gen}}(r) \twoheadrightarrow \pi_{\mathrm{sm}}(r)$. This representation has a model over $E$, which we will denote again by $\pi_{\mathrm{gen}}(r)$.

To the multi-set $\{HT_{\kappa}\}_{\kappa : F \hookrightarrow E}$ one can attach an irreducible algebraic representation of $\mathrm{Res}_{\mathbb{F}/\mathbb{Q}_p}(GL_n/F)$, which we evaluate at $E$ to get an algebraic representation $\pi_{\mathrm{alg}}(r)$ of $GL_n(F)$. More precisely, for each $\kappa$, let $\pi_{\mathrm{alg},\kappa}(r)$ be the irreducible algebraic representation of $GL_n(F)$ of highest weight $\{-i_{\kappa,1}, \ldots, -i_{\kappa,n} + n - 1\}$ relative to the upper-triangular Borel. Then define $\pi_{\mathrm{alg}}(r) = \bigotimes_{\kappa} \pi_{\mathrm{alg},\kappa}(r)$, with $GL_n(F)$ acting diagonally. This is the representation $L_\xi \otimes_{\mathbb{C}} E$ with notation of section 1.8 [CEG+16], with $\xi_{\kappa,j} = -i_{\kappa,j} + j - 1$.

Define: $BS(r) := \pi_{\mathrm{gen}}(r) \otimes_{E} \pi_{\mathrm{alg}}(r)$.

3
The conjecture, which we state in the generic case, predicts that irreducible locally algebraic representations of $G$ admit integral structures if and only if they are related to Galois representations. More precisely:

**Conjecture 1.1.** Let $\pi$ be an absolutely irreducible generic representation of $GL_n(F)$ and $\sigma$ an irreducible algebraic representation of algebraic group $Res_{F/Q_p}GL_n/F$, both having coefficients in $E$. Then the following statements are equivalent:

1. $\pi \otimes_E \sigma$ admits a $G$-invariant norm.

2. There is a potentially semi-stable Galois representation $r : G_F \to GL_n(E)$ such that $\pi = \pi_{sm}(r)$ and $\sigma = \pi_{alg}(r)$.

The implication $(1) \Rightarrow (2)$, was proven by Hu in full generality in his paper [Hu09]. The converse is still largely open. In [CEG+16], the authors prove many cases of this conjecture by constructing an admissible unitary $E$-Banach space representation $V(r)$ of $GL_n(F)$, such that the locally algebraic vectors in $V(r)$ are isomorphic to $BS(r)$ as $GL_n(F)$. In [CEG+16] the authors assume that $r$ is potentially crystalline. This corresponds to the case when the monodromy operator $N$ of the Weil-Deligne representation $WD(r)$ is zero.

In this paper we extend the methods of [CEG+16] to handle the case, when $r$ is potentially semi-stable. This corresponds to the case when the monodromy operator $N$ of the Weil-Deligne representation $WD(r)$ is allowed to be arbitrary. We will be mostly concerned with the case when the Galois representation $r$ is generic, in this case $BS(r) = \pi_{sm}(r) \otimes_E \pi_{alg}(r)$ is irreducible. We prove, that the locally algebraic vectors of $V(r)$ are isomorphic to $BS(r)$. This will allow us to deduce new cases for the implication $(2) \Rightarrow (1)$.

It was observed in [Sor15], that norms are related to lattices, up to equivalence. Indeed given a norm $\|\cdot\|$, the lattice $\Lambda$ will be a unit ball for this norm. Conversely, given a lattice $\Lambda$, set $\|x\| = q_E^{-v_\Lambda(x)}$, where $q_E = |E|$ and $v_\Lambda(x)$ is the largest $k$ such that $x \in \omega_E^k \Lambda$ ($\omega_E$ is a uniformizer of $E$). Thus we are looking for integral structures in $BS(r)$.

### 1.3 Typical representations

Recall that the Bernstein decomposition expresses the category of smooth $\Q_p$-valued representations of a $p$-adic reductive group $G$ as the product of cer-
tain indecomposable full subcategories, called Bernstein components. Those components are parametrized by the inertial classes. Let me now recall the definition of an inertial class. Let $M$ be a Levi subgroup of some parabolic subgroup of $G$ and let $\rho$ be an irreducible supercuspidal representation of $M$ and consider a set of pairs $(M, \rho)$ as above. We say that two pairs $(M_1, \rho_1)$ and $(M_2, \rho_2)$ are inertially equivalent if and only if there is $g \in G$ and an unramified character $\chi$ of $M_2$ such that:

$$M_2 = M_1^g \text{ and } \rho_2 \simeq \rho_1^g \otimes \chi$$

where $M_1 := g^{-1}M_1g$ and $\rho_1^g(x) = \rho_1(gxg^{-1}), \forall x \in M_1^g$. An equivalence class of all such pairs will be denoted $[M, \rho]_G$. The set of inertial class equivalences of all such pairs will be denoted by $\mathcal{B}(G)$.

Let $F$ be a finite extension of $\mathbb{Q}_p$ with a finite residue field $k_F$. Let $\mathcal{O}_F$ be its complete discrete valuation ring, let $p$ be the maximal ideal of $\mathcal{O}_F$ with uniformizer $\varpi$, and let $q = |\mathcal{O}_F/\varpi\mathcal{O}_F|$. In this paper we only consider the case $G = \text{GL}_n(F)$. Let $E$ be an algebraically closed field of characteristic zero.

Let $\mathcal{R}(G)$ be the category of all smooth $E$-representations of $G$. We denote by $i_P^\text{G} : \mathcal{R}(M) \rightarrow \mathcal{R}(G)$ the normalized parabolic induction functor, where $P = MN$ is a parabolic subgroup of $G$ with Levi subgroup $M$. Let $\overline{P}$ be the opposite parabolic with respect to $M$. We use the notation Ind and $c$-Ind to denote the induction and compact induction respectively.

We are given an inertial class $\Omega := [M, \rho]_G$, where $\rho$ is a supercuspidal representation of $M$ and $D := [M, \rho]_M$. To any inertial class $\Omega$ we may associate a full subcategory $\mathcal{R}^\Omega(G)$ of $\mathcal{R}(G)$, such that $(\pi, V) \in \text{Ob}(\mathcal{R}^\Omega(G))$ if and only if every irreducible $G$-subquotient $\pi_0$ of $\pi$ appear as a composition factor of $i_P^\text{G}(\rho \otimes \omega)$ for $\omega$ some unramified character of $M$ and $P$ some parabolic subgroup of $G$ with Levi factor $M$. The category $\mathcal{R}^\Omega(G)$ is called a Bernstein component of $\mathcal{R}(G)$. We will say that a representation $\pi$ is in $\Omega$ if $\pi$ is an object of $\mathcal{R}^\Omega(G)$. According to [Ber84], we have a decomposition:

$$\mathcal{R}(G) = \prod_{\Omega \in \mathcal{B}(G)} \mathcal{R}^\Omega(G)$$

So in order to understand the category $\mathcal{R}(G)$, it is enough to restrict our attention to the components. We may understand those components via the
theory of types. This is a way to parametrize all the irreducible representations of \( G \) up to inertial equivalence using irreducible representations of compact open subgroups of \( G \).

Let \( J \) be a compact open subgroup of \( G \) and let \( \lambda \) be an irreducible representation of \( J \). We say that \((J, \lambda)\) is an \( \Omega \)-type if and only if for every irreducible representation \( (\pi, V) \in \text{Ob}(\mathcal{R}^\Omega(G)) \), \( V \) is generated by the \( \lambda \)-isotypical component of \( V \) as \( G \)-representation.

Let \( \mathcal{R}_\lambda(G) \) be a full subcategory of \( \mathcal{R}(G) \) such that \((\pi, V) \in \text{Ob}(\mathcal{R}_\lambda(G))\) if and only if \( V \) is generated by \( V^\lambda \) (the \( \lambda \)-isotypical component of \( V \)) as \( G \)-representation.

Let \( K \) be a maximal compact open subgroup of \( G \) containing \( J \). We say that an irreducible representation \( \sigma \) of \( K \) is typical for \( \Omega \) if for any irreducible representation \( \pi \) of \( G \), \( \text{Hom}_K(\sigma, \pi) \neq 0 \) implies that \( \pi \) is an object in \( \mathcal{R}_\lambda^\Omega(G) \).

Define \( \mathcal{H}(G, \lambda) := \text{End}_G(c\text{-Ind}^G_J \lambda) \). Then for any \( \Omega \)-type \((J, \lambda)\), by Theorem 4.2 (ii) [BK98], the functor:

\[
\mathfrak{M}_\lambda : \mathcal{R}_\lambda(G) \to \mathcal{H}(G, \lambda) - \text{Mod} \\
\pi \mapsto \text{Hom}_J(\lambda, \pi) = \text{Hom}_G(c\text{-Ind}^G_J \lambda, \pi)
\]

induces an equivalence of categories. Since \((J, \lambda)\) is an \( \Omega \)-type, we have \( \mathcal{R}_\lambda^\Omega(G) = \mathcal{R}_\lambda(G) \).

Write \( \mathfrak{Z}_\Omega \) for the centre of category \( \mathcal{R}_\lambda^\Omega(G) \) and \( \mathfrak{Z}_D \) for the centre of category \( \mathcal{R}_D^\Omega(M) \), which is defined the same way as \( \mathcal{R}_\lambda^\Omega(G) \). Recall that the centre of a category is the ring of endomorphisms of the identity functor. For example the centre of the category \( \mathcal{H}(G, \lambda) - \text{Mod} \) is \( Z(\mathcal{H}(G, \lambda)) \), where \( Z(\mathcal{H}(G, \lambda)) \) is the centre of the ring \( \mathcal{H}(G, \lambda) \). We will call \( \mathfrak{Z}_\Omega \) a Bernstein centre.

For \( G = \text{GL}_n(F) \), the types can be constructed in an explicit manner (cf. [BK93], [BK98] and [BK99]) for every Bernstein component. Moreover, Bushnell and Kutzko have shown that \( \mathcal{H}(G, \lambda) \) is naturally isomorphic to a tensor product of affine Hecke algebras of type A.

The simplest example of a type is \((I, 1)\), where \( I \) is Iwahori subgroup of \( G \) and 1 is the trivial representation of \( I \). In this case \( \Omega = [T, 1]_G \), where \( T \) is the subgroup of diagonal matrices. We will refer to example as the Iwahori case.

In [SZ99] section 6 (just above proposition 2) the authors define irreducible \( K \)-representations \( \sigma_P(\lambda) \), where \( P \) is partition valued functions with
compact support (cf. section 2 [SZ99]). There is a natural partial ordering, as defined in [SZ99], on the partition valued functions. Let $P_{\text{max}}$ be the maximal partition valued function and let $P_{\text{min}}$ the minimal one. Define $\sigma_{\text{max}}(\lambda) := \sigma_{P_{\text{max}}}(\lambda)$ and $\sigma_{\text{min}}(\lambda) := \sigma_{P_{\text{min}}}(\lambda)$. Both $\sigma_{\text{max}}(\lambda)$ and $\sigma_{\text{min}}(\lambda)$ occur in $\text{Ind}_K^G \lambda$ with multiplicity 1.

In the Iwahori case, $\sigma_{\text{min}}(\lambda)$ is the inflation of Steinberg representation of $GL_n(k_F)$ to $K$ and $\sigma_{\text{max}}(\lambda)$ is the trivial representation.

### 1.4 Main result

Let $v = \{\text{HT}_\kappa\}_{\kappa:F \hookrightarrow E}$ be a multiset of all Hodge-Tate weights and let $\tau : I_F \rightarrow GL_n(E)$ be an inertial type, i.e. $\tau$ is a representation of $I_F$ with open kernel which extends to a representation of the Weil group $W_F$ of $F$, where $I_F$ is the inertia subgroup of $G_{\mathbb{F}}$. We let $R^G_p$ denote the universal $O$-lifting ring of $r$. Then there is a ring $R^G_p(\sigma_{\text{min}}) := R^G_p(\tau, v)$, which is the unique reduced and $p$-torsion free quotient of $R^G_p$ corresponding to potentially semi-stable lifts of weight $\sigma_{\text{alg}}$ (i.e. of weight $v$) and inertial type $\tau$. This ring was constructed in [Kis08]. Moreover there is a "universal admissible filtered $(\varphi, N)$-module" $D^G_p(\tau, v)$ which is a locally free $R^G_p(\tau, v)[1/p] \otimes_{\mathbb{Q}_p} F_0$-module of rank $n$, where $F_0$ is a maximal subfield of $F$ such that $F/F_0$ is totally ramified. The module $D^G_p(\tau, v)$ comes equipped with a universal Frobenius, denoted by $\varphi$.

Let $\sigma_{\text{alg}}$ the restriction to $K$ of $\pi_{\text{alg}}(r)$. Define $\sigma_{\text{min}} := \sigma_{\text{min}}(\lambda) \otimes \sigma_{\text{alg}}$ and $\mathcal{H}(\sigma_{\text{min}}) := \text{End}_G(\text{Ind}_K^G \sigma_{\text{min}})$.

We have fixed a type $\tau$, so there is a finite extension $L$ of $F$ such that the restriction of every Galois representation $r_x$ to $G_L$ is semi-stable. Let $L_0$ its maximal unramified subfield of $L$. We assume $[L_0 : \mathbb{Q}_p] = |\text{Hom}_{\mathbb{Q}_p}(L_0, E)|$ and we let $p^f$ be the cardinality of the residue field of $L_0$. By universal we mean that the specialization $D_x$ of $D^G_p(\tau, v)$ at the closed point $x$ of $R^G_p(\sigma_{\text{min}})[1/p]$ with residue field $E_x$ is an admissible filtered $(\varphi, N, \text{Gal}(L/F))$-module attached to the Galois representation $r_x$ given by the point $x$.

On the other hand one may show using the classical local Langlands correspondence that $\tau$ determines a Bernstein component $\mathcal{R}^G_\varphi(G)$. We prove that there is a map that interpolates the local Langlands correspondence, more precisely:

**Theorem 1.2.** There is an $E$-algebra homomorphism

$$\beta : \mathcal{H}(\sigma_{\text{min}}) \rightarrow R^G_p(\sigma_{\text{min}})[1/p]$$
such that for any closed point $x$ of $R^\infty_p(\sigma_{\text{min}})[1/p]$ with residue field $E_x$, the action of $3\Omega$ on the smooth $G$-representation $\pi_{\text{sm}}(r_x)$ factors as $\beta$ composed with the evaluation map $R^\infty_p(\sigma_{\text{min}})[1/p] \rightarrow E_x$.

This result generalizes Theorem 4.1 [CEG+16] (i.e. if we restrict to the crystalline locus the two maps coincide), however the proof does not follow methods of this paper. Instead we give an explicit construction of this map.

We will sketch the construction of $\beta$ in the Iwahori case, in this case $L = F$ because the lifts we consider are semi-stable. By Satake isomorphism and Corollary 7.2 [Pyv18b], we have $\mathcal{H}(\sigma_{\text{min}}) \simeq E[\theta_1, \ldots, \theta_{n-1}, (\theta_n)^{\pm 1}]$, where $\theta_r$ is a double coset operator $[K(\tilde{\pi}^I_{r0})K]$. All the details of this construction will be discussed in the section 3.2.

If for an embedding $\kappa$ the Hodge-Tate weights are $i_{\kappa,1} < \ldots < i_{\kappa,n}$ define $\varepsilon_{\kappa,j} = -i_{\kappa,j} + (j - 1)$. Then the map $\beta : \mathcal{H}(\sigma_{\text{min}}) \rightarrow R^\infty_p(\tau, v)[1/p]$ is given by the assignment $\theta_r \mapsto -\sum_{i=r}^n \sum_{j=r} \varepsilon_{\kappa,j} q_r \frac{r(1-r)}{2} \text{Tr}(\wedge^r \varphi')$.

For $P$ any partition valued function, define $\sigma_P := \sigma_P(\lambda) \otimes \sigma_{\text{alg}}$, where $\sigma_P(\lambda)$ was defined above and $\sigma_{\text{alg}}$ is the restriction to $K$ of an irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ given by the Hodge-Tate weights. Fix a $K$-stable $O$-lattice $\sigma_P^\circ$ in $\sigma_P$. Set

$$M_\infty(\sigma_P^\circ) := \left(\text{Hom}^\text{cont}_{O[[K]]}(M_\infty, (\sigma_P^\circ)^d)\right)^d$$

where $M_\infty$ is $R_\infty$-module constructed in section 2 [CEG+16] by patching process and $(.)^d = \text{Hom}^\text{cont}(. , O)$ denotes the Shikhof dual. Since $\sigma_P^\circ$ is a free $O$-module of finite rank, it follows from the proof of Theorem 1.2 of [ST02] that Schikhof duality induces an isomorphism

$$\text{Hom}^\text{cont}_{O[[K]]}(M_\infty, (\sigma_P^\circ)^d) \simeq \text{Hom}_K(\sigma_P^\circ, (M_\infty)^d)$$

and Frobenius reciprocity gives

$$\text{Hom}_K(\sigma_P^\circ, (M_\infty)^d) \simeq \text{Hom}_G(\text{c-Ind}_K^G \sigma_P^\circ, (M_\infty)^d).$$

The action of $3\Omega$ on $\text{c-Ind}_K^G \sigma_P$ induces an action on $M_\infty(\sigma_P^\circ)[1/p]$.

To any closed point of $x \in m-\text{Spec} R^\infty_p(\sigma_{\text{min}})[1/p]$ we can attach a partition valued function $P_x$, which encodes information about the shape of
monodromy operator of the admissible filtered $(\varphi, N)$-module $D_x$. We prove that there is a reduced $p$-torsion free quotient $R^\Box_p(\sigma_P)$ of $R^\Box_p(\sigma_{\min})$, such that $x \in m-\text{Spec } R^\Box_p(\sigma_P)[1/p]$ if and only if $P_x \geq P$. When $\sigma_P = \sigma_{\min}$, the ring corresponds to all the potentially semi-stable lift and this is compatible with the notation introduced at the beginning. The other extreme case is $R^\Box_p(\sigma_{\max})$, this ring parametrizes all the potentially crystalline lifts.

As a part of patching construction we know that $R_\infty$ is an $R^\Box_p$-algebra. We define $R_\infty(\sigma_P) \coloneqq R_\infty \otimes_{R^\Box_p} R^\Box_p(\sigma_P)$. Let $R_\infty(\sigma_P)$ be the quotient of $R_\infty$ which acts faithfully on $M_\infty(\sigma_P)$. The usual commutative algebra arguments underlying the Taylor-Wiles-Kisin method, as in [CEG+16], show that $M_\infty(\sigma_P^\circ)$ is a maximal Cohen-Macaulay module over $R_\infty(\sigma_P)$. Moreover we prove an important result about the support of $M_\infty(\sigma_P^\circ)$:

**Proposition 1.3.**

1. The module $M_\infty(\sigma_P^\circ)[1/p]$ is locally free of rank one over the regular locus of $\text{Spec } R_\infty(\sigma_{\min})[1/p]$.

2. $\text{Spec } R_\infty(\sigma_{\min})[1/p]$ is a union of irreducible components of $\text{Spec } R_\infty(\sigma_{\min})'[1/p]$.

The components appearing in the second statement of the Proposition 1.3 are termed *automorphic components*. The proof of the proposition above is similar to Lemma 4.18 [CEG+16]. The action of $3_{\Omega}$ on $M_\infty(\sigma_P^\circ)[1/p]$ induces an $E$-algebra homomorphism:

$$\alpha : 3_{\Omega} \to \text{End}_{R_\infty[1/p]}(M_\infty(\sigma_P^\circ)[1/p])$$

From the Proposition 1.3 we deduce that:

**Theorem 1.4.** We have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec } R_\infty(\sigma_{\min})[1/p]^{\text{reg}} & \xrightarrow{\alpha^\sharp} & \text{Spec } \mathcal{H}(\sigma_{\min}) \\
\downarrow & & \downarrow \\
\text{Spec } R_\infty(\sigma_{\min})[1/p] & \xrightarrow{\text{can}} & \text{Spec } R^\Box_p(\sigma_{\min})[1/p],
\end{array}
$$

where $(\text{Spec } R_\infty(\sigma_{\min})[1/p])^{\text{reg}}$ is the regular locus of $\text{Spec } R_\infty(\sigma_{\min})[1/p]$ and $\alpha^\sharp$ the map induced by $\alpha$. 


Just as in §4.28 [CEG+16], the main technique is to convert information on locally algebraic vectors in the completed cohomology into commutative algebra statements about the module $M_\infty(\sigma_{\text{min}})$ using results on $K$-typical representations that we have explained in the previous section.

Let $x$ be a closed $E$-valued point of $\text{Spec } R_\infty(\sigma_{\text{min}})[1/p]$. The corresponding Galois representation $r_x$ is given by the homomorphism $x : R^1_{\overline{p}} \to \mathcal{O}$, which we extend arbitrarily to homomorphism $x : R_\infty \to \mathcal{O}$. Then

$$V(r_x) := \text{Hom}^{\text{cont}}(M_\infty \otimes_{R_\infty,x} \mathcal{O}, E) \quad (1.5)$$

is an admissible unitary $E$-Banach space representation of $G$. The main result is the following theorem:

**Theorem 1.6.** Let $x$ be a closed $E$-valued point of $\text{Spec } R_\infty(\sigma_{\text{min}})[1/p]$, such that $\pi_{\text{sm}}(r_x)$ is generic and irreducible. Then

$$V(r_x)^{\text{alg}} \simeq \pi_{\text{sm}}(r_x) \otimes \pi_{\text{alg}}(r_x),$$

where $(.)^{\text{alg}}$ denotes the subspace of locally algebraic vectors.

Since the action of $G$ on $V(r_x)$ is unitary, we obtain:

**Corollary 1.7.** Suppose $p \nmid 2n$, and that $r : G_F \longrightarrow GL_n(E)$ is a generic potentially semi-stable Galois representation of regular weight. If $r$ correspond to a closed point $x \in \text{Spec } R_\infty(\sigma_{\text{min}})[1/p]$, then $\pi_{\text{sm}}(r) \otimes \pi_{\text{alg}}(r)$ admits a non-zero unitary admissible Banach completion.

It is conjectured in [CEG+16] that $V(r_x)$ depends only on the Galois representation $r_x$ and that $r_x \mapsto V(r_x)$ realizes the hypothetical $p$-adic local Langlands correspondence. Our Theorem 1.6 provides further evidence of this conjecture.

Now, we will recall some more notation from [CEG+16]. The globalization of $\mathfrak{F}$ constructed in section 2.1 [CEG+16], provides us with a global imaginary CM field $\tilde{F}$ with maximal totally real subfield $\tilde{F}^+$. We refer the reader to the section 2.1 [CEG+16], for more details and precise definitions. Let $S_p$ denote the set of primes of $\tilde{F}^+$ dividing $p$. Fix $p \mid p$. For each $v \in S_p$, let $\tilde{v}$ be a choice of a place in $\tilde{F}$ lying above $v$, as defined in section 2.4 [CEG+16].

The main result of this article is to compute the locally algebraic vectors for $V(r_x)$, a candidate for the $p$-adic local Langlands correspondence, at the smooth points which lie on some automorphic component.
1.5 Outline of the paper

This article is organised as follows: After recalling few definitions in section 2, we will construct the map \( \mathcal{H}(\sigma_{\text{min}}) \rightarrow R^{\text{C}}_p(\sigma_{\text{min}})[1/p] \), which interpolates the local Langlands correspondence in section 3. Then in section 4 we will introduce a stratification of \( R^{\text{C}}_p(\sigma_{\text{min}}) \) with respect to the partition valued function, which will help us to study the support of \( M_\infty(\sigma_{\text{min}}^\circ) \). The goal of the section 5 will be to prove that the action of \( \mathcal{H}(\sigma_{\text{min}}) \) on \( M_\infty(\sigma_{\text{min}}^\circ) \) is compatible with the interpolation map constructed in section 3. In order to deal with the monodromy of potentially semi-stable Galois representations we will use results from [Pyv18a]. This will be stated in more precise manner as Theorem 5.6 and the results about the support of \( M_\infty(\sigma_{\text{min}}^\circ) \) will be given in section 6. In the section 7, we will compute locally algebraic vectors using a global point where we know the result already. The main result of that section is the Theorem 7.7. Finally in the section 8, we will apply this theorem to deduce new cases of the Breuil-Schneider conjecture.

2 Locally algebraic vectors. Definition. First properties

In this section I reproduced some parts of the appendix in [ST01]. Let \( E/\mathbb{Q}_p \) be a finite extension and \( V \) a vector space over \( E \). We begin with a definition, a vector \( v \in V \) is termed locally algebraic if:

The orbit map of the vector \( v \) is locally algebraic, i.e. for \( v \in V \), there is a compact open subgroup \( K_v \) in \( G \), and a finite dimensional subspace \( U \) of \( V \) containing the vector \( v \) such that \( K_v \) leaves \( U \) invariant and operates on \( U \) via the restriction to \( K_v \) of a finite dimensional algebraic representation of the algebraic group scheme Res\( F/\mathbb{Q}_pGL_n \).

Similarly a representation \( \pi \) of \( G = GL_n(F) \) on \( V \) is called locally algebraic if any vector \( v \in V \) is locally algebraic. According to [ST01] Theorem 1] every irreducible locally algebraic representation of \( G \) is of the form \( \pi_{\text{sm}} \otimes \pi_{\text{alg}} \), where \( \pi_{\text{sm}} \) is a smooth irreducible representation of \( G \) and \( \pi_{\text{alg}} \) is an irreducible algebraic representation of \( G \). For any Banach vector space representation \( V \) of \( G \) we have the following functor \( V \mapsto V^{\text{alg}} \), where \( V^{\text{alg}} \) is the subspace of locally algebraic vectors in \( V \).

Notation. Let \( \pi \) be an irreducible representation of \( G \), then we will write
\( \pi^{l, alg} = \pi_{sm} \otimes \pi_{alg} \), where \( \pi_{sm} \) is a smooth irreducible representation of \( G \) and \( \pi_{alg} \) is an irreducible algebraic representation of \( G \).

3 Interpolation map

In this section we will construct an analogous of the map \( \eta \) from Theorem 4.1 [CEG+16] in the potentially semi-stable case. First we extend few results from section 3 of [CEG+16]. Let \( \pi \) be any irreducible representation, then the action of \( Z_{\Omega} \) on \( \pi \) defines an \( E \)-algebra morphism \( \chi_\pi : Z_{\Omega} \to \text{End}_G(\pi) \cong E \).

In the next Lemma we introduce new notation which will be used in this paper.

Lemma 3.1. Let \( \pi \) be an irreducible generic representation of \( G = GL_n(F) \). Then by Bernstein-Zelevinsky classification (cf. [Zel80]) there are pairwise non-isomorphic supercuspidal representations \( \pi_i \) (\( 1 \leq i \leq s \)) and segments \( \Delta_{i,j} = (\pi_i \otimes \chi_{i,j}) \otimes \ldots \otimes (\pi_i \otimes \chi_{i,j} \otimes |\det|^{k_{i,j}-1}) \) (\( 1 \leq j \leq r_i \)), where \( \chi_{i,j} \) are unramified characters and \( k_{i,j} \) are positive integers, such that:

\[
\pi \cong Q(\Delta_{1,1}) \times \ldots \times Q(\Delta_{1,r_1}) \times \ldots \times Q(\Delta_{s,1}) \times \ldots \times Q(\Delta_{s,r_s}),
\]

where \( Q \) denotes the Langlands quotient (cf. [Kud94, Section 1.2]). Notice that all the segments \( \Delta_{i,j} \) and \( \Delta_{i',j'} \) are not linked for \( i \neq i' \), this means that any permutation of blocs \( Q(\Delta_{i,1}) \times \ldots \times Q(\Delta_{i,r_i}) \) gives an isomorphic representation.

Define \( \tilde{\Delta}_{i,j} := (\pi_i \otimes \chi_{i,j} \otimes |\det|^{1-k_{i,j}}) \otimes \ldots \otimes (\pi_i \otimes \chi_{i,j}) \) and consider it as a representation of a corresponding Levi subgroup. Write,

\[
\eta := \tilde{\Delta}_{1,1} \times \ldots \times \tilde{\Delta}_{1,r_1} \times \ldots \times \tilde{\Delta}_{s,1} \times \ldots \times \tilde{\Delta}_{s,r_s}
\]

Notice that any permutation of blocs \( \tilde{\Delta}_{i,1} \times \ldots \times \tilde{\Delta}_{i,r_i} \) gives a representation isomorphic to \( \eta \). Then we have:

\[
c-\text{Ind}^G_K \sigma_{\max}(\lambda) \otimes_{3_{\Omega}, \chi_\pi} E \cong \eta
\]

Moreover the action of \( 3_{\Omega} \) on \( \eta \) is given by the maximal ideal \( \chi_\pi \).

Proof. The result follows by the argument similar to the one given in the proof of [CEG+16, Corollary 3.11]. Let \( \pi' \) be \( G \)-cosocle of \( \eta \). Then by [CEG+16, Proposition 3.10], we have \( c-\text{Ind}^G_K \sigma_{\max}(\lambda) \otimes_{3_{\Omega}, \chi_\pi} E \cong \eta \). However
\( \pi \) occurs as a subquotient of \( \eta \) and the \( G \)-socle of \( \eta \) is irreducible and occurs as a subquotient with multiplicity one, thus \( \pi \) is the \( G \)-socle. Then the action of \( Z \) on \( \eta \) factors through a maximal ideal, which is equal to \( \chi_\pi \). By construction the action of \( Z \) on \( \eta \) is given by \( \chi_\pi' \), thus \( \chi_\pi' = \chi_\pi \).

We are given \( \pi \) an irreducible generic representation as in lemma above. We would like to describe \( \chi_\pi \) in more concrete terms. In facts we would like to have a concrete description of the action of \( Z \) on \( \pi \) in terms of eigenvalues of associated Weil-Deligne representation by local Langlands correspondence.

Let \( W := W(k_F) \) be the ring of Witt vectors of the residue field of \( F \), recall that \( \varpi \) is a uniformizer of \( F \). Let \( F_0 = W(k_F)[1/p] \), then \( F/F_0 \) is totally ramified. We will denote by \( E(u) \in F_0[u] \) the Eisenstein polynomial of \( \varpi \).

Let \( R^{\varphi}(\sigma_{min}) := R^{\varphi}(\tau, \nu) \) the unique reduced and \( p \)-torsion free quotient of \( R^{\varphi} \) corresponding to potentially semi-stable lifts of weight \( \sigma_{alg} \) (i.e. of weight \( \nu \)) and inertial type \( \tau \), which was constructed in [Kis08] and \( \rho_{pst} : G_K \rightarrow GL_n(R^{\varphi}(\tau, \nu)[1/p]) \) the universal lift corresponding to the identity homomorphism \( id : R^{\varphi}(\tau, \nu)[1/p] \rightarrow R^{\varphi}(\tau, \nu)[1/p] \).

It follows from the Theorem 2.5.5 (2) [Kis08] that \( R^{\varphi}(\tau, \nu)[1/p] \) is endowed with a universal \((\varphi, N)\)-module \( D^{\varphi}(\tau, \nu) \), which is a locally free \( R^{\varphi}(\tau, \nu)[1/p] \otimes_{Q_p} F_0 \)-module of rank \( n \).

Let's recall a few facts about \((\varphi, N)\)-modules. We have two finite extensions \( F \) (the base field) and \( E \) (the coefficient field) of \( Q_p \) such that \( [F : Q_p] = |\text{Hom}_{Q_p}(F,E)| \) where \( \text{Hom}_{Q_p}(F,E) \) denotes the set of all \( Q_p \)-linear embeddings of the field \( F \) into the field \( E \). We assume \( F \) is contained in an algebraic closure \( \overline{Q}_p \) of \( Q_p \). We denote by \( q = p^{f_0} \) the cardinality of the residue field of \( F \) and by \( F_0 = \text{Frac}(W(F)) \) its maximal unramified subfield. If \( e := \lfloor L : Q_p \rfloor / f_0 \), we set \( \text{val}_F(x) := e \cdot \text{val}_{Q_p}(x) \) (where \( \text{val}_{Q_p}(p) := 1 \)) and \( |x|_F := q^{-\text{val}_F(x)} \) for any \( x \) in a finite extension of \( Q_p \). We denote by \( W_F = W(\overline{Q}_p/F) \) (resp. \( G_F := \text{Gal}(\overline{Q}_p/F) \)) the Weil (resp. Galois) group of \( F \) and by \( \text{rec}_p : W(\overline{Q}_p/F)^{ab} \rightarrow F^\times \) the reciprocity map sending the geometric Frobenius to the uniformizer.

Let \( L \) be a finite Galois extension of \( L \) and \( L_0 \) its maximal unramified subfield. We assume \( [L_0 : Q_p] = |\text{Hom}_{Q_p}(L_0,E)| \) and we let \( p' \) be the cardinality of the residue field of \( L_0 \) and \( \varphi_0 \) be the Frobenius on \( F \) (raising to the \( p \) each component of the Witt vectors). Consider the following two
categories:

1. the category \( \text{WD}_{L/F} \) of representations \((r, N, V)\) of the Weil-Deligne group of \( F \) on an \( E \)-vector space \( V \) of finite dimension such that \( r \) is unramified when restricted to \( W(\overline{\mathbb{Q}}_p/L) \).

2. the category \( \text{MOD}_{L/F} \) of quadruples \((\varphi, N, \text{Gal}(L/F), D)\) where \( D \) is a free \( L_0 \otimes_{\mathbb{Q}_p} E \)-module of finite rank endowed with a Frobenius \( \varphi : D \to D \), which is \( \phi_0 \)-semi-linear bijective map, an \( L_0 \otimes_{\mathbb{Q}_p} E \)-linear endomorphism \( N : D \to D \) such that \( N\varphi = p\varphi N \) and an action of \( \text{Gal}(L/F) \) commuting with \( \varphi \) and \( N \).

There is a functor (due to Fontaine):

\[
\text{WD} : \text{MOD}_{L/F} \to \text{WD}_{L/F}
\]

The following proposition was proven in [BS07] (Proposition 4.1):

**Proposition 3.2.** The functor \( \text{WD} : \text{MOD}_{L/F} \to \text{WD}_{L/F} \) is an equivalence of categories.

Denote MOD a quasi inverse of the functor WD.

If \( D \) is an object of \( \text{MOD}_{L/F} \), we define:

\[
t_N(D) = \frac{1}{[F : L_0]} \text{val}_F(\det_{L_0}(\varphi^f|D))
\]

For \( \sigma : L \hookrightarrow K \), let \( D_L = D \otimes_{L_0} L \) and :

\[
D_{L,\sigma} = D_L \otimes_{L \otimes_{\mathbb{Q}_p} E} (L \otimes_{F,\sigma} E)
\]

Then one has \( D_L \simeq \prod_{\sigma:F \to E} D_{L,\sigma} \). To give an \( L \otimes_{\mathbb{Q}_p} E \)-submodule \( \text{Fil}^iD_L \) of \( D_L \) preserved by \( \text{Gal}(L/F) \) is the same thing as to give a collection \((\text{Fil}^iD_{L,\sigma})_\sigma\) where \( \text{Fil}^iD_{L,\sigma} \) is a free \( L \otimes_{F,\sigma} E \)-submodule of \( D_{L,\sigma} \) (hence a direct factor as \( L \otimes_{F,\sigma} E \)-module) preserved by the action of \( \text{Gal}(L/F) \). If \((\text{Fil}^iD_{L,\sigma})_{\sigma,i}\) is a decreasing exhaustive separated filtration on \( D_L \) by \( L \otimes_{\mathbb{Q}_p} E \)-submodules indexed by \( i \in \mathbb{Z} \) and preserved by \( \text{Gal}(L/F) \), we define:

\[
t_H(D_L) = \sum_\sigma \sum_{i \in \mathbb{Z}} i \dim_L(\text{Fil}^iD_{L,\sigma}/\text{Fil}^{i+1}D_{L,\sigma})
\]
Recall that such a filtration is called admissible if $t_H(D_L) = t_N(D)$ and if, for any $L_0$-vector subspace $D' \subseteq D$ preserved by $\varphi$ and $N$ with the induced filtration on $D'_L$, one has $t_H(D'_L) \leq t_N(D')$.

Our goal here is to construct a canonical map $\mathcal{H}(\sigma_{min}) \longrightarrow \mathbb{R}(\tau, v)[1/p]$. We proceed in the following steps:

1. Take a smooth closed point $x \in \text{Spec} \mathbb{R}(\tau, v)[1/p]$.

2. The point $x$ corresponds is given by an $E$-algebra homomorphism $x : \mathbb{R}(\tau, v)[1/p] \longrightarrow E$ and it corresponds to $n$-dimensional Galois representation of $G_F$, denoted $V_x$. Let $D_{st,L}(V_x) := (B_{st} \otimes \mathbb{Q}_p V_x)^{GL}$, by construction this is also $D_x$, the specialization of $\mathcal{D}_x(\tau, v)$ at closed point $x$. The admissible filtered $(\varphi, N, \text{Gal}(L/F))$-module $D_x = D_{st,L}(V_x)$ is equipped with Frobenius endomorphism $\phi_x$, which is the specialization of the universal Frobenius $\varphi$ on $\mathcal{D}_x(\tau, v)$ at $x$, i.e. $\varphi \otimes \kappa(x) = \phi_x$.

3. Let $\tilde{\pi}$ be an irreducible representation of $GL_n(F)$ such that

$$\text{rec}_p(\tilde{\pi} \otimes |\det|^{1-n}) = WD(D_x),$$

here $WD(D_x)$ is the Weil-Deligne representation associated to $D_x$ via Fontaine’s recipe. Let $\pi := \text{rec}_p^-1(WD(D_x))$.

4. Theorem 1.2.7 [All16] implies that the representation $\pi$ is generic, because $x$ is a smooth point.

5. Let $\eta := c-\text{Ind}_{K \sigma_{max} \otimes 3_\Omega, \chi^*}^{G} E$, as in Lemma 3.1. By the Lemma 3.1 the action of $3_\Omega$ on $\tilde{\pi} = \pi \otimes |\det|^{1-n}$ is identified with the action of $3_\Omega$ on $\eta \otimes |\det|^{1-n}$. We will try to understand the action of $3_\Omega$ on $\eta \otimes |\det|^{1-n}$.

6. We can interpret the action of $3_\Omega$ on $\eta \otimes |\det|^{1-n}$ in terms of eigenvalues of the linearized canonical map obtained from the specialization of the absolute Frobenius $\varphi$ at point $x$. For this we use the decomposition of spherical Hecke algebra of semi-simple type as a tensor product of Iwahori Hecke algebras and this decomposition restricts to $3_\Omega$. Then we use Satake isomorphism on each factor of $3_\Omega$. In the Iwahori case there is just one factor in that tensor product decomposition.
7. From previous step we can "guess" a ring homomorphism $\beta : \mathfrak{Z}_\Omega \to R^\square_p(\tau, v)[1/p]$. This map is canonical in the sense that if there was another map $\beta'$ it would coincide with $\beta$ on all of the smooth points, and since the smooth points are dense by Theorem 3.3.4 [Kis08], the two maps have to be equal.

8. Finally $\mathcal{H}(\sigma_{\text{min}}(\lambda)) \to \mathcal{H}(\sigma_{\text{min}})$ given by $f \mapsto f.\sigma_{\text{alg}}$ is an isomorphism according to Lemma 1.4 [ST06] and by Corollary 7.2 [Pyv18b] we have a canonical isomorphism $\mathfrak{Z}_\Omega \simeq \mathcal{H}(\sigma_{\text{min}}(\lambda))$. Composing $\beta$ with those isomorphisms gives us the desired map.

Notice that it follows from the step 5 that the map $\beta$ does not depend on monodromy.

The organisation of this section is the following. In the subsection 3.1 we will construct the map $\beta$ in full generality by modifying the argument given in [CEG+16]. Then in section 3.2 we will give an other proof of the same result in the Iwahori case which is based on explicit computation.

### 3.1 Construction in general case

The aim of the next sections is to carry out steps 1.-8. outlined above. Morally the proof is based on the Lemma 4.3 [CEG+16]. We will prove the following Theorem:

**Theorem 3.3.** There is an $E$-algebra homomorphism

$$\beta : \mathcal{H}(\sigma_{\text{min}}) \to R^\square_p(\sigma_{\text{min}})[1/p]$$

such that for any closed point $x$ of $R^\square_p(\sigma_{\text{min}})[1/p]$ with residue field $E_x$, the action of $\mathfrak{Z}_\Omega$ on a smooth $G$-representation $\pi_{\text{am}}(r_x)$ factors as $\beta$ composed with the evaluation map $R^\square_p(\sigma_{\text{min}})[1/p] \to E_x$.

**Proof.** Consider the following map, obtained by specialisation:

$$\gamma_G : \mathfrak{Z}_\Omega \to \prod_{x \in \text{m-Spec } R^\square_p(\tau, v)[1/p]} E'_x$$

where $\gamma_G$ is defined on the factor corresponding to $x$ by evaluating $\mathfrak{Z}_\Omega$ at the closed point in the Bernstein component $\Omega$ determined via local Langlands by $x$, and $E'_x/E_x$ is a sufficiently large finite extension.
Consider as well the following map, also obtained by specialisation:

\[ \gamma_{WD} : R^\square_\tau(\tau, \nu)[1/p] \longrightarrow \prod_{x \in m-\text{Spec } R^\square_\tau(\tau, \nu)[1/p]} E'_x \]

The map \( \gamma_{WD} \) is injective, because the ring \( R^\square_\tau(\tau, \nu)[1/p] \) is reduced and Jacobson.

We have the following diagram:

\[
\begin{array}{ccc}
\mathfrak{Z}_\Omega & \xrightarrow{\gamma} & \prod_{x \in m-\text{Spec } R^\square_\tau(\tau, \nu)[1/p]} E'_x \\
\downarrow T & & \downarrow \gamma_{WD} \\
W_F & \xrightarrow{?} & R^\square_\tau(\tau, \nu)[1/p]
\end{array}
\]

where \( T : W_F \longrightarrow \mathfrak{Z}_\Omega \) be the pseudo-representation constructed in Proposition 3.11 of [Che09] and \( I \) is the map that we want to construct. Observe that the Lemma 3.24 [CEG+16] tells us that the Cheneviers \( E[B] \) is our \( \mathfrak{Z}_\Omega \), so that the definition of the map \( T \) makes sense.

First we will construct a map \( ? \) such that the diagram above commutes. We can apply the Fontaine’s recipe to the absolute Frobenius \( \varphi \) on \( D^\square_\tau(\tau, \nu) \), which is a free \( R^\square_\tau(\tau, \nu)[1/p] \otimes_{\mathbb{Q}_p} F_0 \)-module of rank \( n \). Let’s recall first this construction in the usual setting.

Let \( x \in m-\text{Spec } R^\square_\tau(\tau, \nu)[1/p] \) be a closed \( E \)-valued point. Let \( D_x \) and \( \varphi_x \) be the specializations of \( D^\square_\tau(\tau, \nu) \) and \( \varphi \) at \( x \), respectively. Let \( L \) be a finite extension of \( F \) where all the Galois representation of the given inertial type \( \tau \) are semi-stable and \( L_0 \) a subfield of \( L \) such that \( L/L_0 \) is totally ramified.

Then we deduce from the isomorphism

\[ L_0 \otimes_{\mathbb{Q}_p} E \cong \prod_{\sigma_0 : L_0 \hookrightarrow E} E, \]

the isomorphism

\[ D_x = \prod_{\sigma_0 : L_0 \hookrightarrow E} D_{\sigma_0}, \]

where \( D_{\sigma_0} = (0, \ldots, 0, 1_{\sigma_0}, 0, \ldots, 0) D_x \), is the ”\( \sigma_0 \)-th coordinate of \( D_x \).” Fix now a \( \sigma_0 \). Set \( W_x = D_{\sigma_0} \).
Let \( w \in W_F \), define \( \overline{w} \) to be the image of \( w \) in \( \text{Gal}(L/F) \) and let \( \alpha(w) \in f_0 \mathbb{Z} \) such that the action of \( w \) on \( \overline{F}_p \) is the \( \alpha(w) \)-power of the map \( (x \mapsto x^p) \).

We can define an endomorphism of \( D_x \) by \( r_x(w) := \overline{w} \circ \varphi_x^{-\alpha(w)} \), it can be shown that the restriction of \( r(w) \) to \( W_x \) does not depend on \( \sigma_0 \).

We are interested in the trace of \( r_x(w) \), we have trivially
\[
\text{Tr}(r_x(w) | D_x) = |\text{Hom}_{\mathbb{Q}_p}(L_0, E)| \text{Tr}(r_x(w) | W_x).
\]
However since \( E \) is assumed to be large enough we have \( |\text{Hom}_{\mathbb{Q}_p}(L_0, E)| = [L_0 : \mathbb{Q}_p] \).

Observe that it makes sense to define for each \( w \in W_F \) an endomorphism \( r(w) := \overline{w} \circ \varphi^{-\alpha(w)} \) of \( D_x \) and we can also take its trace.

Define now the following map:
\[
\text{Tr} : W_F \rightarrow R^x(\tau, \nu)[1/p]
\]
\[
w \mapsto \frac{1}{[w:Q_p]} \text{Tr}(r(w) | D_x(\tau, \nu))
\]

Then by the construction of \( T \), we have \( \gamma_G \circ T = \gamma_{WD} \circ \text{Tr} \), i.e. the diagram of sets

\[
\begin{array}{ccc}
\mathfrak{Z}_G & \xrightarrow{\gamma_G} & \prod_{x \in \text{m-Spec} R^x(\tau, \nu)[1/p]} E' \\
T & \xrightarrow{1} & \gamma_{WD} \\
W_F & \xrightarrow{T} & R^x(\tau, \nu)[1/p]
\end{array}
\]

commutes. Now we can define the map \( I \), in order to so, it suffices to show that the image of \( \mathfrak{Z}_G \) under \( \gamma_G \) is contained in the image of \( \gamma_{WD} \). However by Lemma 4.5 \( [\text{CEG+16}] \) the image of \( T \) generates \( \mathfrak{Z}_G \). Then, any element \( a \in \mathfrak{Z}_G \) can be written as \( a = \sum_i \mu_i T(g_i) \). It follows from the commutative diagram above, that the map \( I \) is given by \( \sum_i \mu_i T(g_i) \mapsto \sum_i \mu_i \text{Tr}(g_i) \). Let’s prove that the map \( I \) is a well defined ring homomorphism.

The map \( I \) is well defined. Indeed, if we choose two different presentations of an element \( a \in \mathfrak{Z}_G \), \( a = \sum_i \mu_i T(g_i) = \sum_k \lambda_k T(h_k) \) then the elements \( \sum_i \mu_i \text{Tr}(g_i) \) and \( \sum_k \lambda_k \text{Tr}(h_k) \) should coincide. It is enough to prove that if \( \sum_i \mu_i T(g_i) = 0 \), then \( \sum_i \mu_i \text{Tr}(g_i) = 0 \). Indeed, we have \( 0 = \gamma_G(0) = \gamma_G(\sum_i \mu_i T(g_i)) = \sum_i \mu_i \gamma_G(T(g_i)) = \sum_i \mu_i \gamma_{WD}(\text{Tr}(g_i)) = \gamma_{WD}(\sum_i \mu_i \text{Tr}(g_i)) \), then \( \sum_i \mu_i \text{Tr}(g_i) = 0 \) since \( \gamma_{WD} \) is injective.
Now, we will prove that \( I \) is a ring homomorphism. First notice that \( \gamma_G \) and \( \gamma_{WD} \) are already ring homomorphisms. Let any \( a, b \in \mathcal{O} \), then

\[
\gamma_{WD}(I(a.b) - I(a).I(b)) = \gamma_{WD}(I(a.b)) - \gamma_{WD}(I(a)).\gamma_{WD}(I(b)) = \gamma_{G}(a.b) - \gamma_G(a).\gamma_G(b) = 0
\]

Since \( \gamma_{WD} \) is injective it follows that \( I(a.b) = I(a).I(b) \). Similarly we get \( I(a + b) = I(a) + I(b) \) and \( I(1) = 1 \).

Let \( M \) be the Levi subgroup in the supercuspidal support of any irreducible representation in \( \Omega \), and \( \mathcal{X}(M) \) is the group of unramified characters of \( M \). The group automorphism \( \mathcal{X}(M) \to \mathcal{X}(M) \) given by \( \chi \mapsto \chi | \det | \theta_1^{(1-n)} \) gives rise to an \( E \)-isomorphism \( \text{Spec} \mathcal{O} \to \text{Spec} \mathcal{O} \). The latter map is invariant under the \( W(D) \)-action (the point is that \( | \det | \) is invariant under \( G \)-conjugation) so it descends to an \( E \)-isomorphism \( \text{Spec} \mathcal{O} \to \text{Spec} \mathcal{O} \). Let \( t_W : \mathcal{O} \to \mathcal{O} \) denote the induced isomorphism. Now we construct \( \beta' \) as the following composite map:

\[
\mathcal{O} \xrightarrow{t_W} \mathcal{O} \xrightarrow{I} R_p^\mathcal{O}(\tau, \nu)[1/p]
\]

In order to get the map \( \beta \) as in the statement of the theorem, compose \( \beta' \) with the isomorphisms \( \mathcal{H}(\sigma_{\min}(\lambda)) \to \mathcal{H}(\sigma_{\min}) \) and \( \mathcal{Z} \simeq H(\sigma_{\max}) \). The desired interpolation property of \( \beta' \), follows from the commutative diagram above. This can be easily be checked on points.

3.2 Construction in the Iwahori case

In this subsection we give an explicit construction of the map \( \beta : \mathcal{H}(\sigma_{\min}) \to R_p^\mathcal{O}(\sigma_{\min})[1/p] \) in the semi-stable case.

Assume now, that \( \pi \) has a trivial type \((I, 1)\), i.e. \( \pi^I \neq 0 \) and \( \Omega = [T, 1]_G \). So the inertial type \( \tau \) is also trivial. Let \( \mathcal{H}(\sigma_{\min}) := \text{End}_G(e - \text{Ind}^G_K \sigma_{\min}) \), and by Corollary 7.2 \([\text{Pyv}18b]\) we have a canonical isomorphism \( \mathcal{O} \simeq \mathcal{H}(\sigma_{\min}) \) and also \( \mathcal{O} \simeq \mathcal{H}(\sigma_{\max}) = \mathcal{H}(G, K) \). Moreover the map \( \mathcal{H}(G, K) \to \mathcal{H}(\sigma_{\max}) \) given by \( f \mapsto f.\sigma_{\alg} \) is an isomorphism according to Lemma 1.4 \([\text{ST}06]\). By Satake isomorphism we have \( \mathcal{H}(G, K) \simeq E[\theta_1, \ldots, \theta_{n-1}, (\theta_n)^{\pm 1}] \), where \( \theta_1 \) is a double coset operator \( K \left[ \begin{array}{cc} \varpi I_r & 0 \\ 0 & I_{n-r} \end{array} \right] K \). Putting all these isomorphisms together we have \( \mathcal{H}(\sigma_{\min}) \simeq \mathcal{O} \simeq E[\theta_1, \ldots, \theta_{n-1}, (\theta_n)^{\pm 1}] \).
So in order to describe completely the action of $3_\Omega$ on $\eta \otimes \det |^{\frac{n-1}{2}}$, it would be enough to describe the action of each $\theta_r$.

Let $q$ be the cardinality of residue field $\mathcal{O}_F/p_F$ where $\mathcal{O}_F$ is the ring of integers of $F$ and $p_F$ the maximal ideal. Let $\varpi$ be a uniformizer of $F$.

We describe first the action of $3_\Omega$.

**Lemma 3.4.** Let $\psi := \psi_1 \otimes \ldots \otimes \psi_n$, an unramified character of torus $T$, and $\eta = i_G^B(\psi)$. Then $\theta_r$ acts on $\eta \otimes \det |^{\frac{n-1}{2}}$ by a scalar:

$$q^{r(1-r)} \sum_{\lambda_1 < \ldots < \lambda_r} \psi_{\lambda_1}(\varpi) \ldots \psi_{\lambda_r}(\varpi)$$

where the sum is taken through all the integers $1 \leq \lambda_i \leq n$ such that those inequalities are satisfied.

**Proof.** We follow closely Bump’s lecture notes [Bum] on Hecke algebras, and adapts the argument therein for our needs. One may consult section 9, Proposition 40 in [Bum] for more details. It follows from Iwasawa decomposition that the space of $K$-invariants of $(\eta \otimes \det |^{\frac{n-1}{2}})_K$ is one dimensional and that space generated by the function $f^\circ : bk \mapsto \delta_B^{1/2}(b)\psi'(b)$, with $b \in B$ and $k \in K$ and $\psi'(b) = \psi_1(b_{11})|b_{11}|^{\frac{n-1}{2}} \ldots \psi_n(b_{nn})|b_{nn}|^{\frac{n-1}{2}}$. Hence $\theta_r.f^\circ = c.f^\circ$, then $c = \theta_r.f^\circ(1)$. Using the a double coset decomposition:

$$K \begin{pmatrix} \varpi I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} K = \bigcup_{\beta \in \Lambda} \beta K,$$

where $\Lambda$ is a complete set of representatives, we will compute $\theta_r.f^\circ(1)$. We have a freedom of choice for $\beta$’s, so we can put them in a specific form. More precisely we have

$$K \begin{pmatrix} \varpi I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} K = \bigcup_{S=\{\lambda_1, \ldots, \lambda_r\}} \bigcup_{\beta \in \Lambda_S} \beta K,$$

where $\lambda_1 < \ldots < \lambda_r$ and $\beta \in \Lambda_S$ if and only if the following four conditions are satisfied:

1. $\beta$ is upper triangular;
2. $\beta_{ii} = \varpi$ if $i \in S$ and $\beta_{ii} = 1$ if $i \notin S$;
3. $\beta_{ij}$ is any element of $O_F/p_F$ if $i < j$, $i \in S$ and $j \notin S$

4. all other entries are zero

The number of non-zero entries outside diagonal in a matrix $\beta \in \Lambda_S$ is

$$\sum_{i=1}^{r}(n - r - \lambda_i + i) = r(n - r) + r(r + 1)/2 - \sum_{i=1}^{r} \lambda_i,$$

therefore

$$|\Lambda_S| = q^{r(n-r)+r(r+1)/2 - \sum_{i=1}^{r} \lambda_i}$$

it follows then,

$$\theta_{r}.f^\circ(1) = \sum_{\lambda_1 < \ldots < \lambda_r} \sum_{\beta \in \Lambda_S} f^\circ(\beta) = \sum_{\lambda_1 < \ldots < \lambda_r} |\Lambda_S| f^\circ(\beta)$$

Now let’s compute $f^\circ(\beta) = \delta_B^{1/2}(\beta) \psi'(\beta)$. By definition we have

$$\delta_B^{1/2}(\beta) = \prod_{i=1}^{n} |\beta_{ii}|^{n-2i+1} = q^{-\sum_{i=1}^{r} \frac{n-2\lambda_i+1}{2}} = q^{\frac{r(n+1)}{2} + \sum_{i=1}^{r} \lambda_i},$$

and

$$\psi'(\beta) = q^{\frac{-r(n-1)}{2}} \psi_{\lambda_1}(\varpi) \ldots \psi_{\lambda_r}(\varpi)$$

The total power of $q$ is:

$$r(n - r) + r(r + 1)/2 - \sum_{i=1}^{r} \lambda_i - \frac{r(n + 1)}{2} - \sum_{i=1}^{r} \lambda_i - \frac{r(n - 1)}{2} = -\frac{r(r - 1)}{2}$$

Finally

$$\theta_{r}.f^\circ(1) = q^{\frac{(1-r)}{2}} \sum_{\lambda_1 < \ldots < \lambda_r} \psi_{\lambda_1}(\varpi) \ldots \psi_{\lambda_r}(\varpi)$$

Let $x : \mathbb{R}_{\tau}(\tau, v)[1/p] \rightarrow E$ be an $E$-algebra homomorphism with $V_x$ the corresponding $n$-dimensional Galois representation of $G_F$. Here $V_x$ is already semi-stable, so $L = F$ and $f = f_0$. Let $D_{st}(V_x) := (B_{st} \otimes_{Q_p} V_x)^{G_F}$, by construction this is also $D_x$, the specialization of $D_{\tau}(\tau, v)$ at closed point $x$. Then by Proposition 3.2, $WD(D_{st}(V_x))$ is a Weil-Deligne representation that corresponds to $\pi$ by Local Langlands correspondence, with
normalization as in [HT01]. Assume that $\pi$ is a generic representation. Let $\eta := c \cdot \text{Ind}_K^G \sigma_{\text{max}} \otimes \tau(x,\kappa)$ as in Lemma 3.1. The admissible filtered $(\varphi, N, \text{Gal}(L/F))$-module $D_x = D_{\text{st}}(V_x)$ is equipped with Frobenius endomorphism $\phi_x$, which is the specialization of the universal Frobenius $\varphi$ on $D^\square_{\tau}(\tau, \varpi)$ at $x$, i.e. $\varphi \otimes \kappa(x) = \phi_x$.

**Proposition 3.5.** Let $x : R^\square_{\tau}(\tau, \varpi)[1/p] \to E$, an $E$-algebra homomorphism as above. The double coset operator $\theta_r$ acts on $\eta \otimes |\det|^{\frac{1}{2p}}$ (equivalently on $\pi \otimes |\det|^{\frac{1}{2p}}$) as scalar multiplication by $q^{\frac{r(1-r)}{2}} \text{Tr}(\bigwedge^r(\phi_x)^f)$.

**Proof.** With the notations of Lemma 3.1 we have $s = 1$ and $\pi_1 = 1$. Then there is a partition of $n$, $\sum_{i=1}^t n_i = n$, such that $\pi := Q(\Delta_1) \times \ldots \times Q(\Delta_t)$, with $\Delta_i = \chi_i \otimes \ldots \otimes \chi_i \mid |\cdot|^{-1}$ and $\chi_i \chi_j^{-1} \neq \mid \cdot \mid^{1}$ for all $i \neq j$. Then $\eta = \Delta_1 \times \ldots \times \Delta_t$, where $\Delta_i = \chi_i \mid |\cdot|^{-n_i} \otimes \ldots \otimes \chi_i$. Define $\psi := \psi_1 \otimes \ldots \otimes \psi_n = \Delta_1 \otimes \ldots \otimes \Delta_t$ an unramified character of torus $T$, so that $\eta \simeq i_B^G(\psi)$.

By previous lemma, $\theta_r$ acts on one dimensional space $(\eta \otimes |\det|^{\frac{1}{2p}})^\kappa$ as scalar multiplication by

$$q^{\frac{r(1-r)}{2}} s_r(\chi_1(\varpi)q^{n_1-1}, \ldots, \chi_t(\varpi))$$

where $s_r$ is the $r$th symmetric polynomial in $n$ variables.

The eigenvalues of $\phi_x^f$ are $\chi_1(\varpi)q^{n_1-1}, \ldots, \chi_t(\varpi)q^{n_t-1}$.

Then it follows that

$$s_r(\chi_1(\varpi)q^{n_1-1}, \ldots, \chi_t(\varpi)) = \text{Tr}(\bigwedge^r(\phi_x^f)) = \text{Tr}(\bigwedge^r WD(D_{\text{st}}(V_x))(\text{Frob}_p))$$

where $\text{Frob}_p$ is the geometric Frobenius. Notice that the computations above do not depend on the choice of $\text{Frob}_p$. \hfill $\Box$

If for an embedding $\sigma$ the Hodge-Tate weights are $i_{\kappa,1} < \ldots < i_{\kappa,n}$, define $\xi_{j,\kappa} = -i_{\kappa,j} + (j - 1)$. The highest weight of the algebraic representation $\sigma_{\text{alg}}$ with respect to the upper triangular matrices is given by $\text{diag}(x_1, \ldots, x_n) \mapsto \prod_{j=1}^n \prod_{\kappa} \kappa(x_j^{\xi_{j,\kappa}})$. Then we have to rescale $\theta_r$ by the factor

$$\omega^{-\sum_{\kappa} \frac{r}{2} \sum_{j=1}^n \xi_{j,\kappa}}$$

22
in order to be compatible with isomorphism, \( \mathcal{H}(\sigma_{\text{min}}(\lambda)) \rightarrow \mathcal{H}(\sigma_{\text{min}}) \) given by \( f \mapsto f.\sigma_{\text{alg}} \).

Define \( \tilde{\theta}_r = q^{(r-1)}r - \sum_{i=r}^{n} \sum_{j=1}^{i} \xi_{n,j} \cdot \theta_r \). Then we have a canonical isomorphism \( \mathcal{H}(\sigma_{\text{min}}) \simeq E[\tilde{\theta}_1, \ldots, \tilde{\theta}_{n-1}, (\tilde{\theta}_n)^{\pm 1}] \). We can summarize the results of this section with the following theorem:

**Theorem 3.6.** If \( \tau \) is trivial, then define \( \beta : \mathcal{H}(\sigma_{\text{min}}) \rightarrow R^\square_{\tau}(\tau,v)[1/p] \) by the assignment

\[
\tilde{\theta}_r \mapsto q^{r-1}r - \sum_{i=r}^{n} \sum_{j=1}^{i} \xi_{n,j} \cdot \Theta(\bigwedge^r \phi_f),
\]

where \( \varphi \) is the universal Frobenius on \( D^\square_{\tau}(\tau,v) \). Then the map \( \beta \) is an \( E \)-algebra homomorphism and \( \beta \) interpolates local Langlands correspondence, i.e. such that for any closed point \( x \) of \( R^\square_{\tau}(\sigma_{\text{min}})[1/p] \) with residue field \( E_x \), the action of \( \mathfrak{g}_\Omega \) on a smooth \( G \)-representation \( \pi_{\text{sm}}(r_x) \) factors as \( \beta \) composed with the evaluation map \( R^\square_{\tau}(\sigma_{\text{min}})[1/p] \rightarrow E_x \).

**Proof.** Since \( E[\tilde{\theta}_1, \ldots, \tilde{\theta}_{n-1}, (\tilde{\theta}_n)^{\pm 1}] \) is a polynomial \( E \)-algebra, the previous assignment \( \beta \) a ring homomorphism. Moreover the weak admissibility of \( D_x \)
implies that \( \text{val}_F(q^{r-1}r - \sum_{i=r}^{n} \sum_{j=1}^{i} \xi_{n,j} \cdot \Theta(\bigwedge^r \phi_f)) \geq 0 \), and then \( x(\beta(\theta_r)) \) belongs to the ring of integers, for all \( r \). It follows that the image of the map \( \beta \) is contained in the normalization of \( R^\square_{\tau}(\tau,v)[1/p] \), by Proposition 7.3.6 [dJ95].

As it was observed in the point 7. in the section 3.1 such a map interpolates local Langlands correspondence on all the closed points. \( \Box \)

## 4 Local deformation rings

We begin this section with some elementary linear algebra. Those preparatory results will help us to deal with monodromy of potentially semi-stable Galois representations. Indeed we will introduce locally algebraic representations \( \sigma_P \), where \( P \) is a partition valued function. The properties of smooth part \( \sigma_P(\lambda) \) of \( \sigma_P \) were described in [Pyv18a] and it was explained how the monodromy of an irreducible generic representation can be read of the \( \sigma_P(\lambda) \)'s that it contains. In a similar way, we may study the support of \( M_{\infty}(\sigma_{\text{min}}) \) by introducing a stratification that depends on the \( \sigma_P \)'s. This will be dealt with in the next section. Here we will introduce a stratification of \( R^\square_P(\sigma_{\text{min}}) \) with respect to any partition valued function \( P \), more precisely we will construct
the rings $R_p^\sigma(\sigma\tau)$, which are reduced, $p$-torsion free quotient of $R_p^\sigma(\tau, \nu)$, satisfying the following property: $x \in \text{Spec } R_p^\sigma(\sigma\tau)[1/p]$ if and only if $P_x \geq P$.

Recall a few facts about partitions. Let $(\lambda_1, \ldots, \lambda_l)$ be a partition of $n$, i.e. we have $n = \lambda_1 + \ldots + \lambda_l$ with $\lambda_1 \geq \ldots \geq \lambda_l > 0$. We say that a partition $\lambda^c$ is conjugate of $\lambda = (\lambda_1, \ldots, \lambda_l)$ if it is represented by the reflected diagram of the one associated to $\lambda$ with respect to the line $y = -x$ with the coordinate of the upper left corner is taken to be $(0, 0)$. We have that $\lambda^c_k = |\{k : \lambda_k \geq i\}|$.

Let $M$ be any field, $V$ a $n$-dimensional $M$-vector space and $N : V \to V$ a nilpotent endomorphism. Then the Jordan normal form of $N$ is uniquely determined up conjugacy by a partition $(n_1, \ldots, n_t)$, i.e the blocks are ordered by decreasing size $n_1 \geq \ldots \geq n_t$.

**Lemma 4.1.** Let $M$ be any field, $V$ a $n$-dimensional $M$-vector space, with two nilpotent endomorphisms $N : V \to V$ and $N' : V \to V$. To the endomorphism $N$ (resp. $N'$) corresponds a partition $(n_1, \ldots, n_t)$ (resp. $(n'_1, \ldots, n'_s)$). Then the following statements are equivalent:

1. $\forall i, \dim \ker(N^i) \leq \dim \ker(N'^i)$.
2. $\forall i, \sum_{k=1}^{i} n_k \geq \sum_{k=1}^{i} n'_k$.

**Proof.** The Jordan normal form gives an isomorphism $N \simeq \bigoplus_{k=1}^t N_k$, where $N_k$ is a nilpotent operator of maximal rank on a $n_k$-dimensional vector space. Then:

$$\dim \ker(N^i_k) = \begin{cases} i, & \text{for } i \leq n_k \\ n_k, & \text{for } i > n_k \end{cases}$$

and $\dim \ker(N^i) = \sum_{k=1}^{t} \dim \ker(N^i_k) = \sum_{k=1}^{t} \min(i, n_k)$.

Let $\kappa_j = \dim \ker(N^j) - \dim \ker(N^{j-1})$ for $j \geq 1$ and $\dim \ker(N^0) = 0$. We get $(\kappa_1, \kappa_2, \ldots)$ a partition of $n$ and we will call this partition a kernel partition of the nilpotent operator $N$. By the description of $\dim \ker(N^i)$ in terms of the partition $(n_1, \ldots, n_t)$, we see that $\kappa_i = |\{k : n_k \geq i\}|$. So the partition $(\kappa_1, \kappa_2, \ldots)$ is the dual of the partition $(n_1, \ldots, n_t)$. Let now $(\kappa'_1, \kappa'_2, \ldots)$ the kernel partition of $N'$. Then the inequalities:

$$\dim \ker(N'^i) = \sum_{j=1}^{i} \kappa'_j \leq \dim \ker(N'^i) = \sum_{j=1}^{i} \kappa'_j,$$
∀i, are equivalent to the inequalities from 2. This concludes the proof.

**Lemma 4.2.** Let \( A \) be a commutative ring, \( V \) projective finitely generated \( A \)-module and \( N : V \to V \) a nilpotent \( A \)-linear operator. Then the set

\[
\{ p \in \text{Spec } A | \dim_\kappa(p)(\text{Coker } N) \otimes_A \kappa(p) \geq m \}
\]

is closed for any integer \( m \).

For a point \( x \in \text{Spec } A \), the shape (Jordan normal form) of nilpotent operator \( N \otimes \kappa(x) \) is given by a partition \( P_x \) and this partition determines uniquely, up to conjugacy, a Jordan normal form of a nilpotent operator. Define a partial order \( \leq \) on partitions which is the reverse of so-called natural or dominance partial order ([Knu98] chapter 5 section 5.1.4). Then for all integers \( i \),

\[
\dim_\kappa(x)(\text{Coker } N_i) \otimes_A \kappa(x) \leq \dim_\kappa(y)(\text{Coker } N_i) \otimes_A \kappa(y)
\]

if and only if \( P_x \leq P_y \).

**Proof.** Let’s prove the first assertion. Let \( m_1, \ldots, m_n \), any set of generators of \( C := \text{Coker } N \) over \( A \). It would be enough to prove that the set \( U := \{ p \in \text{Spec } A | \dim_\kappa(p) C \otimes_A \kappa(p) < n \} \) is open. Let \( p \in \text{Spec } A \) and \( \bar{x}_1, \ldots, \bar{x}_k \) be a basis of \( \kappa(p)\)-vector space \( C_p/pC_p \). It follows from Nakayama’s lemma the lifts \( x_1, \ldots, x_k \) to \( C_p \), form a minimal generating set of \( C_p \) over \( A_p \). Write \( m_i/1 = \sum_{j=1}^k (a_{ij}/b_{ij})x_j \) and let \( b = \prod b_{ij} \).

For any \( q \in D(b) \), \( x_1, \ldots, x_k \) is still a generating set of \( C_q \) over \( A_q \). Again, by Nakayama’s lemma it follows that \( \dim_\kappa(p) C \otimes_A \kappa(p) \leq k < n \), so that \( D(b) \subseteq U \). Therefore \( U \) is open.

The second assertion follows from the previous lemma, because

\[
\dim \ker(N^i \otimes \kappa(x)) = \dim \text{Coker}(N^i \otimes \kappa(x))
\]

and we have an isomorphism \( \text{Coker}(N^i \otimes \kappa(x)) \cong (\text{Coker } N^i) \otimes_A \kappa(x) \) since the tensor product is right-exact.

Recall from previous section that we have an endomorphism \( \varphi \) on \( D_\tau(\tau, v) \). Again by Theorem (2.5.5) [Kis08], there is a universal monodromy operator \( N : D_\tau(\tau, v) \to D_\tau(\tau, v) \) which is \( F_0 \otimes_{\mathbb{Q}_p} R_{\tau}(\tau, v)[1/p]\)-linear. Observe, that the monodromy of \( WD(r_x) \) is the specialization of \( N \) at closed point \( x \in \text{Spec } R_{\tau}(\tau, v)[1/p] \).

Let \( \mathcal{P} \) be a partition valued function as in [SZ99]. Apply previous lemma with \( A = F_0 \otimes_{\mathbb{Q}_p} R_{\tau}(\tau, v)[1/p] \) and \( V = D_\tau(\tau, v) \) to get that the set

\[
\{ x \in \text{Spec } R_{\tau}(\tau, v)[1/p] | \mathcal{P}_x \geq \mathcal{P} \}
\]
In this section we will study the support of \( R^\square_{\mathfrak{p}}(\tau, \nu)[1/p] \) for \( \mathfrak{p} \) closed, with \( \lambda, \nu \) stratification that depends on the \( \sigma \) and \( \mathcal{V}(I_{\mathfrak{p}}) \), where \( I_{\mathfrak{p}} \) is an ideal in \( R^\square_{\mathfrak{p}}(\tau, \nu) \), such that the quotient is reduced.

We can now make the following definition:

**Definition 4.3.** Define the ring \( R^\square_{\mathfrak{p}}(\sigma) := R^\square_{\mathfrak{p}}(\tau, \nu)/I_{\mathfrak{p}} \), this ring has the following property: \( x \in \text{Spec} R^\square_{\mathfrak{p}}(\sigma)[1/p] \) if and only if \( \mathcal{P} \mathfrak{p} \geq \mathcal{P} \).

Observe that \( R^\square_{\mathfrak{p}}(\sigma) \) is a reduced, \( p \)-torsion free quotient of \( R^\square_{\mathfrak{p}}(\tau, \nu) \). For \( \mathcal{P} \) maximal partition, which we denote by \( \sigma = \sigma_{\text{max}} \), we get potentially crystalline deformation ring and for \( \mathcal{P} \) minimal, which we denote by \( \sigma = \sigma_{\text{min}} \), we get the potentially semi-stable deformation ring \( R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) := R^\square_{\mathfrak{p}}(\tau, \nu) \).

An easy consequence of the construction above is the following lemma:

**Lemma 4.4.** We have \( \dim R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) = \dim R^\square_{\mathfrak{p}}(\sigma_{\text{max}}) \).

**Proof.** Notice that Theorem (3.3.4) \([\text{Kis08}]\) gives the dimension of \( R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) \) and by Theorem (3.3.8) \([\text{Kis08}]\) we know that \( \dim R^\square_{\mathfrak{p}}(\sigma_{\text{max}}) = \dim R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) \). Since we have, \( R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) \rightarrow R^\square_{\mathfrak{p}}(\sigma) \rightarrow R^\square_{\mathfrak{p}}(\sigma_{\text{max}}) \).

It follows that \( \dim R^\square_{\mathfrak{p}}(\sigma_{\text{max}}) \leq \dim R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) \leq \dim R^\square_{\mathfrak{p}}(\sigma) \), so that \( \dim R^\square_{\mathfrak{p}}(\sigma_{\text{min}}) = \dim R^\square_{\mathfrak{p}}(\sigma) \).

## 5 Local-Global compatibility

In this section we will study the support of \( M_\infty(\sigma_{\text{min}}) \) by introducing a stratification that depends on the \( \sigma_{\mathcal{P}} \)'s. This will allow us to have finer control on the monodromy operator.

The main result of this section is Theorem [5,6]. This result tells us that the action of \( \mathcal{H}(\sigma_{\text{min}}) \) on \( M_\infty(\sigma_{\text{min}}) \) is compatible with the interpolation map \( \mathcal{H}(\sigma_{\text{min}}) \rightarrow R^\square_{\mathfrak{p}}(\sigma_{\text{min}})[1/p] \), constructed previously. Most of the proofs in this section are very similar to the ones given in the section 4 \([\text{CEG}+16]\). Let \( \mathcal{P} \) be a partition valued function. Define \( \sigma_{\mathcal{P}} := \sigma_{\mathcal{P}}(\lambda) \otimes \sigma_{\text{alg}} \), so that \( (\sigma_{\mathcal{P}})_{\text{sm}} = \sigma_{\mathcal{P}}(\lambda) \) and \( (\sigma_{\mathcal{P}})_{\text{alg}} = \sigma_{\text{alg}} \), where \( \sigma_{\mathcal{P}}(\lambda) \) is a smooth type for \( K \) as in \([\text{SZ99}]\) and \( \sigma_{\text{alg}} \) is the restriction to \( K \) of an irreducible algebraic representation of \( \text{Res}_{\mathcal{O}_\mathfrak{p}}GL_n \). Fix a \( K \)-stable \( \mathcal{O} \)-lattice \( \sigma_{\mathcal{P}}^\circ \) in \( \sigma_{\mathcal{P}} \). Set

\[
M_\infty(\sigma_{\mathcal{P}}^\circ) := (\text{Hom}_{\mathcal{O}[K]}(M_\infty, (\sigma_{\mathcal{P}}^\circ)^d))^d
\]

26
where we are considering homomorphisms that are continuous for the profinite topology on $M_\infty$ and the $p$-adic topology on $(\sigma_\mathcal{P})^d$, and where we equip $\text{Hom}_{\mathcal{O}[[K]]}^\text{cont}(M_\infty, (\sigma_\mathcal{P})^d)$ with the $p$-adic topology. Note that $M_\infty(\sigma_\mathcal{P})$ is an $\mathcal{O}$-torsion free, profinite, linear-topological $\mathcal{O}$-module.

Let $R_\infty(\sigma_\mathcal{P})$ be the quotient of $R_\infty$ which acts faithfully on $M_\infty(\sigma_\mathcal{P})$, i.e. $R_\infty(\sigma_\mathcal{P}) = M_\infty(\sigma_\mathcal{P})/\text{ann}(M_\infty(\sigma_\mathcal{P}))$. Set $R_\infty(\sigma_\mathcal{P})' = R_\infty \otimes_{\mathcal{O}} \tilde{R}_\mathfrak{p}(\sigma_\mathcal{P})$.

**Lemma 5.1.** Let $\mathcal{P}$ be a partition valued function, then $R_\infty(\sigma_\mathcal{P})$ is a reduced $\mathcal{O}$-torsion free quotient of $R_\infty(\sigma_\mathcal{P})'$. Moreover the module $M_\infty(\sigma_\mathcal{P})$ is Cohen-Macaulay.

**Proof.** That $R_\infty(\sigma_\mathcal{P})$ is $\mathcal{O}$-torsion free follows immediately from the fact that by definition it acts faithfully on the $\mathcal{O}$-torsion free module $M_\infty(\sigma_\mathcal{P})$.

The fact that it is actually a quotient of $R_\infty(\sigma_\mathcal{P})'$ is a consequence of classical local-global compatibility at $\tilde{\mathfrak{p}}$. The proof of this is identical to the proof of Lemma 4.17(1) in [CEG+16]. Even though that proof is written for $\sigma$ (i.e. $\sigma_{\text{max}}$ with the notation of this paper), all the details remain unchanged if we replace $\sigma$ by $\sigma_\mathcal{P}$, if we observe that by the local-global compatibility (Theorem 1.1 of [Car14]) the restriction to the local factor at $\tilde{\mathfrak{p}}$ of global Galois representation, coming from a closed point of a Hecke algebra, is potentially semi-stable such that the partition valued function associated to monodromy (as in Lemma 4.2) of this local Galois representation bigger then $\mathcal{P}$.

To prove the remaining assertions first notice that the module $M_\infty(\lambda^\circ)$ is a Cohen-Macaulay module, by Lemma 4.30 [CEG+16]. Then $M_\infty(\sigma_\mathcal{P})$ is also a Cohen-Macaulay module because it is a direct summand of $M_\infty(\lambda^\circ)$. Finally, to see that $R_\infty(\sigma_\mathcal{P})$ is reduced, notice that since $R_\infty(\sigma_\mathcal{P})'$ is reduced, any non reduced quotient of the same dimension will have an associated prime, which is not minimal. So $M_\infty(\sigma_\mathcal{P})$ is a faithful Cohen-Macaulay module over $R_\infty(\sigma_\mathcal{P})$, thus this cannot happen, and so $R_\infty(\sigma_\mathcal{P})$ is reduced.

Let $\mathcal{H}(\sigma_{\text{min}}^\circ) := \text{End}_G(c-\text{Ind}_K^G \sigma_{\text{min}}^\circ)$. Note that since $\sigma_\mathcal{P}$ is a free $\mathcal{O}$-module of finite rank, it follows from the proof of Theorem 1.2 of [ST02] that Schikhof duality induces an isomorphism

$$\text{Hom}_{\mathcal{O}[[K]]}^\text{cont}(M_\infty, (\sigma_\mathcal{P})^d) \simeq \text{Hom}_K(\sigma_\mathcal{P}, (M_\infty)^d)$$

and Frobenius reciprocity gives

$$\text{Hom}_K(\sigma_\mathcal{P}, (M_\infty)^d) = \text{Hom}_G(c-\text{Ind}_K^G \sigma_\mathcal{P}, (M_\infty)^d).$$
Thus $M_\infty(\sigma_P^\circ)$ is equipped with an action of $3_\Omega$ which commutes with the action of $R_\infty$.

When $\sigma_P = \sigma_{\text{min}}$, the module $M_\infty(\sigma_{\text{min}}^\circ)$ is equipped with an action of $\mathcal{H}(\sigma_{\text{min}}^\circ)$. Such an action of $\mathcal{H}(\sigma_{\text{min}}^\circ)$ commutes with the action of $R_\infty$. The isomorphism $\mathcal{H}(\sigma_{\text{min}}) \simeq 3_\Omega$ (Corollary 7.2 [Pyv18b]) and the isomorphism of Lemma 1.4 [ST06], $\mathcal{H}(\sigma_{\text{min}}(\lambda)) \to \mathcal{H}(\sigma_{\text{min}})$, allow us to define the action of $\mathcal{H}(\sigma_{\text{min}}^\circ)$ on $M_\infty(\sigma_P^\circ)$.

**Lemma 5.2.** If $z \in \mathcal{H}(\sigma_{\text{min}}^\circ)$ is such that $\beta(z) \in R_\tilde{p}(\sigma_P)$, then the action of $z$ on $M_\infty(\sigma_P^\circ)$ agrees with the action of $\beta(z)$ via the natural map $R_\tilde{p}(\sigma_P) \to R_\infty(\sigma_P)$.

**Proof.** As before, this is a consequence of classical local-global compatibility at $\tilde{p}$. The proof of this is identical to the proof of Lemma 4.17(2) in [CEG+16], where we replace $\sigma$ by $\sigma_P$ and instead of using Lemma 4.17(1) in [CEG+16] we apply Lemma 5.1.

We will now define the space of algebraic automorphic forms. First recall some notation from [CEG+16]. The globalization constructed in section 2.1 provides us with a global imaginary CM field $\tilde{F}$ with maximal totally real subfield $\tilde{F}^+$. We refer the reader to this section for the details of these definitions to section 2.1 [CEG+16].

Recall some notation from section 2.3 [CEG+16]. Let $\tilde{G}/\tilde{F}^+$ a certain definite unitary group as defined in the paper [CEG+16]. Let $U = \prod_v U_v$ any compact open subgroup of $\tilde{G}(\mathbb{A}_{\tilde{F}^+}^\infty)$. Let $S_p$ denote the set of primes of $\tilde{F}^+$ dividing $p$. Fix $p \mid p$. Let $\xi$ the weight as in section 1.2 of this article and $\tau$ the inertial type as in section 1.3. Let $W_{\xi,\tau}$ be the finite free $\mathcal{O}$-module with an action of $\prod_{v \in S_p \setminus \{p\}} U_v$.

For any compact open $U$ and any $\mathcal{O}$-module $V$, let $S_{\xi,\tau}(U, V)$ denote the set of continuous functions

$$f : \tilde{G}(\tilde{F}^+) \setminus \tilde{G}(\mathbb{A}_{\tilde{F}^+}^\infty) \to W_{\xi,\tau} \otimes V$$

such that for $g \in \tilde{G}(\mathbb{A}_{\tilde{F}^+}^\infty)$ we have $f(gu) = u^{-1}f(g)$ for $u \in U$, where $U$ acts on $W_{\xi,\tau} \otimes V$ via the projection to $\prod_{v \in S_p} U_v$. The space $S_{\xi,\tau}(U, V)$ is called the **space of algebraic modular forms**.
Now we can define \( \varpi \)-adically completed cohomology space. For each positive integer \( m \), the compact open subgroups \( U_m \) as defined in the beginning of the section 2.3 \[CEG+16\] have the same level away from \( p \). Let \( U^p \) denote that common level. Define the \( \varpi \)-adically completed cohomology space:

\[
\tilde{S}_{\xi,r}(U^p, \mathcal{O})_m := \lim_{\leftarrow} \left( \lim_{\rightarrow} S_{\xi,r}(U_m, \mathcal{O}/\varpi^s) \right)_{m}
\]

The space is equipped with a natural \( G \)-action, induced from the action of \( G \) on algebraic automorphic forms.

The module \( M_\infty \) comes with an action of \( S_\infty \) (cf. page 27, Section 2.8 \[CEG+16\]). Recall that by Corollary 2.11 \[CEG+16\], we have a \( G \)-equivariant isomorphism

\[
M_\infty /a M_\infty \cong \tilde{S}_{\xi,r}(U^p, \mathcal{O})^d_{m},
\]

where \( a \) is an ideal in \( S_\infty \) generated by some formal variables (cf. page 27, Section 2.8 \[CEG+16\]). Moreover that isomorphism commutes with \( R_\square \)-action on both sides.

Lemma 5.3. Let \( pr : \text{Spec } R_\infty(\sigma_{\text{min}}'[1/p]) \to \text{Spec } R_\infty^\square(\sigma_{\text{min}}'[1/p] \to \text{Spec } R_\square^\square(\sigma_{\text{min}}) \to \text{Spec } R_{\infty}(\sigma_{\text{min}}'[1/p]) \) be the map induced by \( j : R_\square^\square(\sigma_{\text{min}}) \to R_\infty \otimes_{R_\square^\square} R_\square^\square(\sigma_{\text{min}}), x \mapsto 1 \otimes x \). Let \( x \in \text{Spec } R_\square^\square(\sigma_{\text{min}}'[1/p]) \) a closed smooth point. Then any \( y \in pr^{-1}(x) \subset R_{\infty}(\sigma_{\text{min}}'[1/p]) \) is a smooth point of \( \text{Spec } R_{\infty}(\sigma_{\text{min}}) \).

Proof. This is essentially the first part of the proof of Theorem 4.35 \[CEG+16\].

Proposition 5.4. If \( y \in \text{Spec } R_{\infty}(\sigma_{\text{min}}'[1/p]) \cap V(a) \) is a closed point, then \( y \) is a smooth point of \( \text{Spec } R_{\infty}(\sigma_{\text{min}}) \) and \( V(r_y)^{\text{alg}} \cong \pi_{\text{sm}}(r_y) \otimes \pi_{\text{alg}}(r_y) \).

Proof. We follow here quite closely the proof of Theorem 4.35 \[CEG+16\]. By definition

\[
V(r_y) := \text{Hom}_{\mathcal{O}}(M_\infty \otimes_{R_{\infty,y}} \mathcal{O}, E)
\]

Since \( a \subseteq \text{Ker}(y) = m_y \), we have that:

\[
\text{Hom}_{\mathcal{O}}(M_\infty \otimes_{R_{\infty,y}} \mathcal{O}, E) = \text{Hom}_{\mathcal{O}}(M_\infty /a M_\infty \otimes_{R_{\infty,y}} \mathcal{O}, E)
\]

Then by Corollary 2.11 \[CEG+16\], we have

\[
\text{Hom}_{\mathcal{O}}(M_\infty /a M_\infty \otimes_{R_{\infty,y}} \mathcal{O}, E) \cong \text{Hom}_{\mathcal{O}}(\tilde{S}_{\xi,r}(U^p, \mathcal{O})^d \otimes_{R_{\infty,y}} \mathcal{O}, E)
\]

The ideal \( m_y \) is finitely generated, choose a presentation \( m_y = (a_1, \ldots, a_k) \), then we get an exact sequence of \( R_{\infty} \)-modules:

\[
R_{\infty}^k \to R_{\infty} \to \mathcal{O} \to 0
\]
Let $\Pi(\bullet) = \text{Hom}_{\mathcal{O}}^{\text{cont}}(\bullet \otimes_{R, \lambda} \tilde{S}_{\xi, \tau}(U^p, \mathcal{O})^d_m, E)$. Then the functor $\Pi$ is left exact and contravariant, by Lemma 2.20 of [Pas15]. Apply this functor to the exact sequence above to get the following exact sequence:

$$0 \rightarrow V(r_y) \rightarrow \text{Hom}_{\mathcal{O}}^{\text{cont}}(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O})_m^d, E) \xrightarrow{f} \text{Hom}_{\mathcal{O}}^{\text{cont}}(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O})_m^d, E) \oplus_k$$

were $f(l) = (l.a_1, \ldots, l.a_k)$. By the exactness we identify $V(r_y) \simeq \text{Hom}_{\mathcal{O}}^{\text{cont}}(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O})_m^d, E)[m_y]$, but

$$\tilde{S}_{\xi, \tau}(U^p, \mathcal{O})_m \otimes_{\mathcal{O}} E \simeq \text{Hom}_{\mathcal{O}}^{\text{cont}}(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O})_m^d, \mathcal{O}) \otimes_{\mathcal{O}} E \simeq \text{Hom}_{\mathcal{O}}^{\text{cont}}(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O})_m^d, E)$$

Thus

$$V(r_y)^{l,\text{alg}} \simeq (\tilde{S}_{\xi, \tau}(U^p, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l,\text{alg}}[m_y]$$

Proposition 3.2.4 of [Eme06] shows that locally algebraic vectors of a given weight are precisely the algebraic automorphic forms of that weight. Hence:

$$(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l,\text{alg}}[m_y] \simeq \pi_{\text{sm}}(r_y) \otimes \pi_{\text{alg}}(r_y)$$

This isomorphism follows from the classical local-global compatibility (Theorem 1.1 of [Car14]). A priori, $\pi_{\text{sm}}(r_y) \otimes \pi_{\text{alg}}(r_y)$ may appear with some multiplicity. However this multiplicity is seen to be one. Indeed the group $\tilde{G}$ is compact at infinity, so the condition $(\ast)$ from Theorems 5.4 of [Lab11] is automatically satisfied. We may then apply Theorems 5.4 and 5.9 of [Lab11], where $\sigma$, in those Theorems, is our $(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l,\text{alg}}$ and $\pi$ is an automorphic cuspidal representation of $GL_n$, which is a base change of $\sigma$. Then by the choice of $U^p$(section 2.3 [CEG+16] for definition of the $U_m$), the fact that we have fixed the action mod $p$ of the Hecke operators at $\tilde{v}_1$(section 2.3 [CEG+16]) and the irreducibility of the globalization of $\overline{\mathcal{T}}$, we see that the multiplicity of $\pi_{\text{sm}}(r_y) \otimes \pi_{\text{alg}}(r_y)$ is one.

Local factors of $\pi$, as in the paragraph above, are generic according to Corollary of Theorem 5.5 [Sha74]. Then by Theorem 5.9 [Lab11], the local factors of $(\tilde{S}_{\xi, \tau}(U^p, \mathcal{O}) \otimes_{\mathcal{O}} E)^{l,\text{alg}}$ are also generic, since $\tilde{G}$, by construction, is quasi-split at all the finite places. It follows that $\pi_{\text{sm}}(r_y)$ is generic. Moreover, by Theorem 1.2.7[All16], the closed point $pr(y) (pr : \text{Spec } R_{\infty}(\sigma_{\text{min}})|_{[1/p]} \rightarrow \text{Spec } R_{\mathbb{P}}(\sigma_{\text{min}})|_{[1/p]},$ with notation of Lemma 5.3] is smooth if and only if $\pi_{\text{sm}}(r_y)$ is generic. Then by Lemma 5.3 the point $y$ is smooth. \hfill \square
In order to study the support of $M_\infty(\sigma^\circ_{\min})$, we will use the commutative algebra arguments underlying the Taylor-Wiles-Kisin method.

**Proposition 5.5.**

1. The module $M_\infty(\sigma^\circ_{\min})[1/p]$ is locally free of rank one over the regular locus of $R_\infty(\sigma_{\min})[1/p]$.

2. Spec $R_\infty(\sigma_{\min})[1/p]$ is a union of irreducible components of Spec $R_\infty(\sigma_{\min})'[1/p]$.

**Proof.** (1) Let $m$ be a smooth point in the support of $M_\infty(\sigma^\circ_{\min})[1/p]$. Since $M_\infty(\sigma^\circ_{\min})[1/p]$ is a Cohen-Macaulay module we have $\dim M_\infty(\sigma^\circ_{\min})[1/p]_m = \dim M_\infty(\sigma^\circ_{\min})[1/p]_m$. Moreover $\dim M_\infty(\sigma^\circ_{\min})[1/p]_m = \dim R_\infty(\sigma_{\min})[1/p]_m$ since $R_\infty(\sigma_{\min})$ acts faithfully on $M_\infty(\sigma^\circ_{\min})$.

By assumption the ring $R_\infty(\sigma_{\min})[1/p]_m$ is regular, it follows that the module $M_\infty(\sigma^\circ_{\min})[1/p]_m$ has a finite projective dimension over this ring. We also have that $\dim R_\infty(\sigma_{\min})[1/p]_m = \dim R_\infty(\sigma_{\min})[1/p]_m$. Then by Auslander-Buchsbaum formula (Theorem 19.1 [Mat89]), $M_\infty(\sigma^\circ_{\min})[1/p]_m$ is free over $R_\infty(\sigma_{\min})[1/p]_m$. It follows that $M_\infty(\sigma^\circ_{\min})[1/p]$ is locally free (i.e. projective) over regular locus of $R_\infty(\sigma_{\min})[1/p]$.

Let’s check that it is locally free of rank one. Let $x \in \text{Supp} M_\infty(\sigma^\circ_{\min})$ and $y \in \text{Supp}(M_\infty(\sigma^\circ_{\min})) \cap V(\mathfrak{a})$ a smooth closed point that lies on the same irreducible component $V$ as $x$. Such a point $y$ always exist because $V(p) \cap V(\mathfrak{a}) \neq 0$. Since $M_\infty(\sigma^\circ_{\min})[1/p]$ is projective the local rank is constant on irreducible components of the support. It would be enough to compute the local rank at $y$, which is given by

$$\dim_E M_\infty(\sigma^\circ_{\min}) \otimes_{R_\infty} \kappa(y) = \dim_E \text{Hom}_K(\sigma_{\min}, V(r_y)^{\text{alg}}),$$

according to Proposition 2.22 [Pas15]. By Proposition 5.4 we have that $V(r_y)^{\text{alg}} \simeq \pi_{sm}(r_y) \otimes \pi_{\text{alg}}(r_y)$. Moreover, since $\pi_{\text{alg}}$ is an irreducible representation of a Lie algebra of $G$, we have

$$\dim_E \text{Hom}_K(\sigma_{\min}, V(r_y)^{\text{alg}}) = \dim_E \text{Hom}_K(\sigma_{\min}(\lambda), \pi_{sm}(r_y))$$

Then $\dim_E \text{Hom}_K(\sigma_{\min}(\lambda), \pi_{sm}(r_y)) = 1$ by Lemma 3.2 [Pyv18a], because $\pi_{sm}(r_y)$ is generic.

(2). The proof is the same as in Lemma 4.18 [CEG+16].

The action of $3_\Omega$ on $M_\infty(\sigma^\circ_{\min})[1/p]$ induces an $E$-algebra homomorphism:

$$\alpha : 3_\Omega \longrightarrow \text{End}_{R_\infty[1/p]}(M_\infty(\sigma^\circ_{\min})[1/p])$$

From the Proposition 5.3 we deduce that:
**Theorem 5.6.** We have the following commutative diagram:

\[
\begin{array}{ccc}
(S\text{pec } R_\infty(\sigma_{min})[1/p])^{reg} & \xrightarrow{\alpha^*} & \text{Spec } \mathcal{H}(\sigma_{min}) \\
\downarrow & & \downarrow \\
\text{Spec } R_\infty(\sigma_{min})[1/p] & \xrightarrow{pr} & \text{Spec } R_\mathbb{F}_p(\sigma_{min})[1/p],
\end{array}
\]

where (Spec $R_\infty(\sigma_{min})[1/p])^{reg}$ is the regular locus of Spec $R_\infty(\sigma_{min})[1/p]$, $\alpha^*$ the map induced by $\alpha$ and $pr : Spec R_\infty(\sigma_{min})'[1/p] \longrightarrow Spec R_\mathbb{F}_p(\sigma_{min})[1/p]$ the map induced by $j : R_\mathbb{F}_p(\sigma_{min}) \longrightarrow R_\infty \otimes_{R_\mathbb{F}_p} R_\mathbb{F}_p(\sigma_{min})$, $x \mapsto 1 \otimes x$.

**Proof.** We proceed here as in the proof of Theorem 4.19 [CEG+16]. It is enough to check all it on points since all the rings are Jacobson and reduced. Let $x : R_\infty(\sigma_{min})[1/p] \rightarrow E$ a closed point smooth point. Note firstly that if $z \in \mathcal{H}(\sigma_{min}^o)$ is such that $\beta(z) \in R_\mathbb{F}_p(\sigma_{min})$, then $x(\alpha(z)) = x(j(\beta(z)))$ by Lemma 5.2. Since $R_\infty(\sigma_{min})$ is $p$-torsion free by Lemma 5.1, it is therefore enough to show that $\mathcal{H}(\sigma_{min}^o)$ is spanned over $E$ by such elements. But, $\mathcal{H}(\sigma_{min}^o)$ certainly spans $\mathcal{H}(\sigma_{min})$ over $E$, so it is enough to show that for any element $z \in \mathcal{H}(\sigma_{min}^o)$, we have $\beta(p^Cz) \in R_\mathbb{F}_p(\sigma_{min})$ for some $C \geq 0$. The latter condition is obviously true, this concludes the proof. \[
\square
\]

6 Support of patched modules

Let $(J, \lambda)$ be the type, a locally algebraic representation $\lambda \otimes (\sigma_{alg}|J)$ of $J$ will be again denoted by $\lambda$. We have also a patched module $M_\infty(\lambda^o) := (\text{Hom}_{cont}^\sigma[[J]](M_\infty, (\lambda^o)^d))^d$, where $\lambda^o$ is a $J$-stable lattice in $\lambda$. Define also $R_\infty(\lambda) := R_\infty/\text{ann}(M_\infty(\lambda^o))$. We would like to have some statements about a support of patched modules. More precisely, we will prove that $M_\infty(\sigma_{min}^o)$ and $M_\infty(\lambda^o)$ have the same support.

**Proposition 6.1.**

\[\text{Supp}(M_\infty(\sigma_{min}^o)) = \text{Supp}(M_\infty(\lambda^o))\]

**Proof.** It follows from decomposition:

\[\text{Ind}_J^K \lambda = \bigoplus_p \sigma_p^{\oplus m_p}\]
that
\[ M_\infty(\lambda^o) = \bigoplus_p M_\infty(\sigma_p^o)^{\oplus m_p}. \]

Then \( \text{Supp}(M_\infty(\sigma_{\text{min}}^o)) \subseteq \text{Supp}(M_\infty(\lambda^o)) \). By definition, \( \text{Supp}(M_\infty(\sigma_{\text{min}}^o)) = \text{Spec} R_\infty(\sigma_{\text{min}}) \) and also \( \text{Supp}(M_\infty(\lambda^o)) = \text{Spec} R_\infty(\lambda) \). Let \( V(p) \) an irreducible component of the spectrum \( \text{Spec} R_\infty(\lambda) \). It is enough to find a point \( x \in V(p) \) such that \( x \notin V(q) \) for any minimal prime \( q \) of \( R_\infty(\lambda) \) such that \( q \neq p \) and \( x \in \text{Spec} R_\infty(\sigma_{\text{min}}) \). The ideal \( a \) is generated by a regular sequence \( (y_1, \ldots, y_h) \) and \( y_1, \ldots, y_h, \omega \) is a system of parameters for \( \text{Spec} R_\infty(\lambda)/p \). Then by Lemma 3.9 \cite{Pas16}, \( V(p) \) contains a closed point \( x \in \text{Spec} R_\infty(\lambda)/(y_1, \ldots, y_h)[1/p] \). The point \( x \) is smooth by the Lemma \cite{Pas15} hence it does not lies on the intersection of irreducible components.

We have that \( x \in \text{Spec} R_\infty(\lambda)[1/p] \cap V(a) \), so it is a closed point of \( \text{Supp}(M_\infty(\lambda^o)) \), then
\[ M_\infty(\lambda^o) \otimes_{R_\infty} \kappa(x) \neq 0 \]
and by Proposition 2.22 \cite{Pas15}, we have that:
\[ M_\infty(\lambda^o) \otimes_{R_\infty} \kappa(x) = \text{Hom}^\text{cont}_E(\text{Hom}_J(\lambda, V(r_x)^\text{alg}), E) \neq 0 \]

Then by Proposition 5.4 we have that \( V(r_x)^\text{alg} \simeq \pi_{\text{sm}}(r_x) \otimes \pi_{\text{alg}}(r_x) \). Moreover the representation \( \pi_{\text{sm}}(r_x) \) is generic, it follows then from Proposition 3.1 \cite{Pyv18a} that we also have \( \text{Hom}_K(\sigma_{\text{min}}, V(r_x)^\text{alg}) \neq 0 \). This means that \( x \in \text{Spec} R_\infty(\sigma_{\text{min}})[1/p] \cap V(a) \).

\[ \square \]

7 Computation of locally algebraic vectors

By Proposition 6.1 we have \( \text{Supp}(M_\infty(\sigma_{\text{min}}^o)) = \text{Supp}(M_\infty(\lambda^o)) \). In what follows we always identify these two sets, so we have \( \text{Spec} R_\infty(\sigma_{\text{min}}) = \text{Supp}(M_\infty(\sigma_{\text{min}}^o)) = \text{Supp}(M_\infty(\lambda^o)) = \text{Spec} R_\infty(\lambda) \). Let \( x \in \text{m-Spec} R_\infty[1/p] \), such that \( V(r_x) \neq 0 \). Assume moreover that \( x \in \text{Supp}(M_\infty(\lambda^o)) \) and that the representation \( \pi_{\text{sm}}(r_x) := r_p^{-1}(WD(r_x)) \) is generic and irreducible. By definition \( \pi_{\text{sm}}(r_x) \) lies in \( \Omega \).

As always for any partition valued function \( P \), we will write \( \sigma_P := \sigma_P(\lambda) \otimes \sigma_{\text{alg}} \), so that \( (\sigma_P)_{\text{sm}} = \sigma_P(\lambda) \) and \( (\sigma_P)_{\text{alg}} = \sigma_{\text{alg}} \).

By Proposition 4.33 \cite{CEG+16}, we have \( V(r_x)^\text{alg} = \pi_x \otimes \pi_{\text{alg}}(r_x) \), where \( \pi_x \) is an admissible smooth representation which lies in \( \Omega \).
Proof. It follows from Theorem 5.6 and Theorem 3.3 and the isomorphism Lemma 7.1.

It follows that \( \pi \) obtained from Lemma 1.4 \cite{ST06} and Corollary 7.2 \cite{Pyv18b}, we get an \( E \)-algebra morphism \( \chi \) with an isomorphism \( 3_\Omega \cong \mathcal{H}(\sigma_{min}) \), obtained from Lemma 1.4 \cite{ST06} and Corollary 7.2 \cite{Pyv18b}, we get an \( E \)-algebra morphism \( \chi : 3_\Omega \to E \).

Lemma 7.1. The \( E \)-algebra morphisms \( \chi \) and \( \chi_{sm} \) defined above coincide.

Proof. It follows from Theorem 5.6 and Theorem 3.3 and the isomorphism \( 3_\Omega \cong \mathcal{H}(\sigma_{min}) \) as above. \( \square \)

Lemma 7.2. The representation \( \pi_{sm}(r_x) \) is a \( G \)-subquotient of \( \pi_x \).

Proof. Define \( \gamma_x := c\text{-Ind}_K^G(\sigma_{min}(\lambda)) \otimes_{3_\Omega, \lambda} E \). Since \( x \in \text{Supp}(M_\infty(\sigma_{min})) \), we have by definition \( 0 \neq \text{Hom}_K(\sigma_{min}, V(r_x)_{l.alg}) = \text{Hom}_K(\sigma_{min}(\lambda), \pi_x) \). So, there exists a non zero map \( \psi : \gamma_x \to \pi_x \).

Let \( \pi' \) any irreducible quotient of \( \gamma_x \), then \( \text{Hom}_K(\sigma_{min}(\lambda), \pi') \neq 0 \), by Proposition 3.1 \cite{Pyv18a} \( \pi' \) is generic. It follows that by Corollary 3.11 \cite{CEG+16}, the representation \( \pi' \) is the socle of \( c\text{-Ind}_K^G(\sigma_{max}(\lambda)) \otimes_{3_\Omega, \lambda} E \). We write it \( \pi' \simeq \text{soc}_G(c\text{-Ind}_K^G(\sigma_{max}(\lambda)) \otimes_{3_\Omega, \lambda} E) \). Similarly by corollary 3.11 \cite{CEG+16}, \( \pi_{sm}(r_x) \simeq \text{soc}_G(c\text{-Ind}_K^G(\sigma_{max}(\lambda)) \otimes_{3_\Omega, \chi_{sm}} E) \). By Lemma 7.1 \( \chi_{sm} = \chi \), then \( \pi' \simeq \pi_{sm}(r_x) \). So at this stage we proved that the cosocle of \( \gamma_x \), is generic, irreducible and isomorphic to \( \pi_{sm}(r_x) \).

Let \( \kappa = \text{Ker}(\psi) \), then \( \gamma_x / \kappa \hookrightarrow \pi_x \). Let now \( \pi'' \) any irreducible quotient of \( \gamma_x / \kappa \), in particular \( \pi'' \) is a sub-quotient of \( \pi_x \). Moreover \( \gamma_x \to \gamma_x / \kappa \to \pi'' \), so \( \pi'' \) is an irreducible quotient of \( \gamma_x \). By what we have proven above \( \pi' \simeq \pi_{sm}(r_x) \). It follow that \( \pi_{sm}(r_x) \) is a sub-quotient of \( \pi_x \). \( \square \)

Proposition 7.3. Let \( x, y \) be two closed, \( E \)-valued points of \( \text{Spec } R_\infty(\sigma_{min})[1/p] \), lying on the same irreducible component. Let \( \mathcal{P} \) be a partition valued function. If \( x \) is smooth, then

\[
\dim_E \text{Hom}_K(\sigma_{\mathcal{P}}, V(r_x)_{l.alg}) \leq \dim_E \text{Hom}_K(\sigma_{\mathcal{P}}, V(r_y)_{l.alg})
\]

Proof. The proof follows the proof of the Proposition 4.34 \cite{CEG+16} by replacing \( \lambda \) with \( \sigma_{\mathcal{P}} \) everywhere. \( \square \)

34
Lemma 7.4. Let \( x, y \in \text{m-Spec } R_\infty(\sigma_{\text{min}})[1/p] \) smooth points such that the monodromy operators of \( WD(r_x) \) and \( WD(r_y) \) are the same and \( WD(r_x)|_{I_F} \cong WD(r_y)|_{I_F} \). Then \( \pi_{\text{sm}}(r_x)|_K \cong \pi_{\text{sm}}(r_y)|_K \).

Proof. Since \( x \) and \( y \) are both smooth points, the representations \( \pi_{\text{sm}}(r_x) \) and \( \pi_{\text{sm}}(r_y) \) are both irreducible and generic. Moreover, it follows from hypotheses that \( \pi_{\text{sm}}(r_x) \) and \( \pi_{\text{sm}}(r_y) \) have the same inertial support and as well as the same number and the same size of segments for Bernstein-Zelevisky classification. So if \( \pi_{\text{sm}}(r_x) = Q(\Delta_1) \times \ldots \times Q(\Delta_r) \) then there are unramified characters \( \chi_i \) such that \( \pi_{\text{sm}}(r_y) = Q(\Delta_1 \otimes \chi_1) \times \ldots \times Q(\Delta_r \otimes \chi_r) \). Restricting to \( K \), we get \( \pi_{\text{sm}}(r_x)|_K \cong \pi_{\text{sm}}(r_y)|_K \).

Lemma 7.5. Let \( x \in \text{m-Spec } R_\infty(\sigma_{\text{min}})[1/p] \), such that \( \pi_{\text{sm}}(r_x) \) is generic, then \( x \in \text{Supp } M_\infty(\sigma_{\text{P}}) \).

Proof. By Lemma 7.2, \( \pi_{\text{sm}}(r_x) \) is a subquotient of \( \pi_{\text{sm}} \), then for any partition valued function \( \mathcal{P} \), we have:

\[
\dim_E \text{Hom}_K((\sigma_{\mathcal{P}})_{\text{sm}}, \pi_{\text{sm}}(r_x)) \leq \dim_E \text{Hom}_K((\sigma_{\mathcal{P}})_{\text{sm}}, \pi_{\text{sm}})
\]

In particular we have

\[
\dim_E \text{Hom}_K((\sigma_{\mathcal{P}})_{\text{sm}}, \pi_{\text{sm}}(r_x)) \leq \dim_E M_\infty(\sigma_{\mathcal{P}}) \otimes_{R_\infty} \kappa(x)
\]

Since \( \dim_E \text{Hom}_K((\sigma_{\mathcal{P}})_{\text{sm}}, \pi_{\text{sm}}(r_x)) \neq 0 \) then \( \dim_E M_\infty(\sigma_{\mathcal{P}}) \otimes_{R_\infty} \kappa(x) \neq 0 \). This means that \( x \in \text{Supp } M_\infty(\sigma_{\mathcal{P}}) \).

Proposition 7.6. Let \( x \) be any point of \( \text{m-Spec } R_\infty(\sigma_{\text{min}})[1/p] \). Then \( x \in \text{Supp } M_\infty(\sigma_{\mathcal{P}}) \) implies that \( \mathcal{P}_x \geq \mathcal{P} \).

Proof. By Lemma 5.1, the action of \( R_\infty \) on \( M_\infty(\sigma_{\mathcal{P}}) \) is a reduced torsion free quotient of \( R_\infty(\sigma_{\mathcal{P}})' \). So if \( x \in \text{Supp } M_\infty(\sigma_{\mathcal{P}}) \) then \( x \in \text{Spec } R_\infty(\sigma_{\mathcal{P}})' = \text{Spec } R_\infty \otimes_{R_\infty} R_{\mathcal{P}}(\sigma_{\mathcal{P}}) \) then by definition of \( R_{\mathcal{P}}(\sigma_{\mathcal{P}}) \) we have that \( \mathcal{P}_x \geq \mathcal{P} \).

Theorem 7.7. Let \( x \) a closed \( E \)-valued point of \( \text{Spec } R_\infty(\sigma_{\text{min}})[1/p] \), such that \( \pi_{\text{sm}}(r_x) \) is generic and irreducible. Then

\[
V(r_x)^{\text{alg}} \simeq \pi_{\text{sm}}(r_x) \otimes \pi_{\text{alg}}(r_x)
\]
Proof. By Lemma 7.2, \( \pi_{sm}(r_x) \) is a \( G \)-subquotient of \( \pi_x \), and for every partition valued function \( P \)

\[
\dim_E \Hom_K((\sigma P)_{sm}, \pi_{sm}(r_x)) \leq \dim_E \Hom_K((\sigma P)_{sm}, \pi_x)
\]

Let \( y \in \text{Supp}(M_\infty(\lambda)) \cap V(\mathfrak{a}) \) a smooth closed point that lies on the same irreducible component \( V(p) \) as \( x \). Such a point \( y \) always exist because \( V(p) \cap V(\mathfrak{a}) \neq 0 \). Moreover we have that,

\[
V(r_y)^{\text{alg}} \simeq \pi_{sm}(r_y) \otimes \pi_{\text{alg}}(r_y)
\]

by Proposition 5.4. Then it follows from Proposition 7.3, that for every partition valued function \( P \), we have

\[
\dim_E \Hom_K((\sigma P)_{sm}, \pi_x) = \dim_E \Hom_K((\sigma P), V(r_x)^{\text{alg}})
\]

\[
= \dim_E \Hom_K((\sigma P), V(r_y)^{\text{alg}}) = \dim_E \Hom_K((\sigma P)_{sm}, \pi_{sm}(r_y))
\]

because \( x \) and \( y \) are both smooth points, lying on the same component. In particular we have that

\[
\dim_E M_\infty(\sigma P) \otimes_{R_\infty} \kappa(x) = \dim_E M_\infty(\sigma P) \otimes_{R_\infty} \kappa(y)
\]

Then \( y \in \text{Supp} M_\infty(\sigma P) \) if and only if \( x \in \text{Supp} M_\infty(\sigma P) \).

Taking \( P = P_x \), by Lemma 7.5 we have that \( x \in \text{Supp} M_\infty(\sigma P_x) \) hence \( y \in \text{Supp} M_\infty(\sigma P_x) \). Then by Proposition 7.6 \( P_y \geq P_x \). Exchanging the roles of \( x \) and \( y \), we get \( P_y = P_x \). This means that the monodromy operators of \( WD(r_x) \) and \( WD(r_y) \) are the same. All together we have:

\[
\dim_E \Hom_K((\sigma P)_{sm}, \pi_{sm}(r_x)) = \dim_E \Hom_K((\sigma P)_{sm}, \pi_{sm}) = \dim_E \Hom_K((\sigma P)_{sm}, \pi_{sm}(r_y))
\]

Similarly using the Proposition 4.34 [CEG+16],we get

\[
\dim_E \Hom_J(\lambda_{sm}, \pi_{sm}(r_x)) \leq \dim_E \Hom_J(\lambda_{sm}, \pi_x) = \dim_E \Hom_J(\lambda_{sm}, \pi_{sm}(r_y))
\]

We have shown that \( x \) and \( y \) have the same monodromy, then by Lemma 7.4 we get that \( \pi_{sm}(r_x)|K \simeq \pi_{sm}(r_y)|K \), so \( \pi_{sm}(r_x)|J \simeq \pi_{sm}(r_y)|J \). It follows that the inequality above, is an equality.
We know that the functor $\text{Hom}_J(\lambda_{sm}, \cdot)$ is an exact functor. It follows that $\text{Hom}_J(\lambda_{sm}, \pi_{sm}(r_x))$ is a subquotient of $\text{Hom}_J(\lambda_{sm}, \pi_x)$ in the category of $\mathcal{H}(G, \lambda_{sm})$-modules, because by Lemma 7.2 $\pi_{sm}(r_x)$ is a subquotient of $\pi_x$. Since those two $\mathcal{H}(G, \lambda_{sm})$-modules have the same dimension they must be equal. Using the fact that the functor $\text{Hom}_J(\lambda_{sm}, \cdot)$ is an equivalence of categories, we get that $\pi_{sm}(r_x) \simeq \pi_x$.

**Corollary 7.8.** Let $x \in \text{m-Spec } R_\infty(\sigma_{min})[1/p]$ such that $\pi_{sm}(r_x)$ is generic, then $\mathcal{P}_x \geq \mathcal{P}$ implies that $x \in \text{Supp } M_\infty(\sigma_\mathcal{P}_x)$.

**Proof.** From Theorem 3.7 [Sho16], follows that $\mathcal{P}_x$ is the maximal partition $\mathcal{P}$ such that $\text{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \neq 0$. Then by maximality, $\mathcal{P}_x \geq \mathcal{P}$ implies that $\text{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \neq 0$. However by previous proposition combined with Proposition 2.22 of [Pas15] we have that $\text{Hom}_K((\sigma_{\mathcal{P}})_{sm}, \pi_{sm}(r_x)) \neq 0$ if and only if $M_\infty(\sigma_{\mathcal{P}}) \otimes_{R_\infty} \kappa(x) \neq 0$. So $x \in \text{Supp } M_\infty(\sigma_{\mathcal{P}})$. $\square$
8 Points on automorphic components

Recall from section 6, that

$$\text{Spec } R_\infty(\sigma_{\min}) = \text{Supp}(M_\infty(\sigma_{\min}^o)) = \text{Supp}(M_\infty(\lambda^o)) = \text{Spec } R_\infty(\lambda).$$

In what follows we will not differentiate between these four sets.

In this section we will prove that if a Galois representation $r$ is generic and corresponds to a closed point lying on an automorphic component then $BS(r)$ admits a $G$-invariant norm. Let’s say a few words about automorphic components. It follows from Proposition 5.5, that $\text{Spec } R_\infty(\sigma_{\min})$ is a union of irreducible components of $\text{Spec } R_\infty(\sigma_{\min})'$. An irreducible component of $\text{Spec } R_\infty(\sigma_{\min})'$ which is also an irreducible component of $\text{Spec } R_\infty(\sigma_{\min})$ is called an automorphic component.

By Corollary 2.11 [CEG+16], we have $M_\infty/aM_\infty \simeq \tilde{S}_{\xi,\tau}(U^p, \mathcal{O})^d_m$, where the ideal $a = (x_1,\ldots,x_h)$ is generated by a regular sequence $(x_1,\ldots,x_h)$ on $M_\infty(\sigma_{\min}^o)$. We know that $(\omega, x_1,\ldots,x_h)$ is a system of parameters for $M_\infty(\sigma_{\min}^o)$. Then by Lemma 3.9 [PAS16], an irreducible component of $\text{Spec } R_\infty(\sigma_{\min})$ contains a closed point $x \in m-\text{Spec}(R_\infty(\sigma_{\min})/a)[1/p]$. The set $m-\text{Spec}(R_\infty(\sigma_{\min})/a)[1/p]$ is finite, since the ring $(R_\infty(\sigma_{\min})/a)[1/p]$ is zero-dimensional. This point $x \in \text{Supp}(M_\infty(\sigma_{\min}^o)) \cap V(a)$ corresponds to a Galois representation attached to an algebraic automorphic forms (cf. section 2.6 [CEG+16]).

Let’s outline, briefly, how $x$ gives rise to a Galois representation. By Proposition 5.3.2 [EG14] there is a unique lift of a globalization to the Hecke algebra, then by universal property of a global deformation ring we get a surjective map from this global deformation ring to the Hecke algebra. The point $x$ corresponds to a maximal ideal of this Hecke algebra. Thus $x$ corresponds to a maximal ideal of this global deformation ring via the map from global deformation ring to the Hecke algebra. The maximal ideals of global deformation ring correspond to Galois representations of a number field, restricting it to the decomposition group we get a local Galois representation we have been looking for.

The components of $\text{Spec } R_\infty(\sigma_{\min})'$ which contain such a point are precisely the automorphic components. This observation justifies why those components are called automorphic. Now we will deduce some new cases of the Breuil-Schenider conjecture.
Theorem 8.1. Suppose $p \nmid 2n$, and that $r : G_F \rightarrow GL_n(E)$ is a generic potentially semi-stable Galois representation of regular weight. If $r$ correspond to a closed point $x \in \text{Spec} \, R_{\infty}(\sigma_{\text{min}})[1/p]$, then $\pi_{\text{sm}}(r) \otimes \pi_{\text{alg}}(r)$ admits a non-zero unitary admissible Banach completion.

Proof. By Theorem 7.7, we have that $\pi_{\text{sm}}(r) \otimes \pi_{\text{alg}}(r) \simeq V(r)^{t,\text{alg}}$, and by Proposition 2.13 [CEG+16], $V(r)$ is an admissible unitary Banach space representation, hence a $G$-invariant norm on $V(r)$ restricts to a $G$-invariant norm on $\pi_{\text{sm}}(r) \otimes \pi_{\text{alg}}(r)$. \hfill \qed

Remark. It is expected that $\text{Spec} \, R_{\infty}(\sigma_{\text{min}})[1/p] = \text{Spec} \, R_{\infty}(\sigma_{\text{min}})'[1/p]$, i.e. that all the components are automorphic.

Acknowledgments

The results of this paper are the main part of my PhD thesis. I’m tremendously grateful to my advisor Vytautas Paškūnas for sharing his ideas with me and for many helpful discussions. This work was supported by SFB/TR 45 of the DFG.

References

[All16] Patrick B. Allen. Deformations of polarized automorphic Galois representations and adjoint Selmer groups. Duke Math. J., 165(13):2407–2460, 2016.

[BC09] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. Astérisque, (324):xii+314, 2009.

[Ber84] J. N. Bernstein. Le “centre” de Bernstein. In Representations of reductive groups over a local field, Travaux en Cours, pages 1–32. Hermann, Paris, 1984. Edited by P. Deligne.

[BH93] Winfried Bruns and Jürgen Herzog. Cohen-Macaulay rings, volume 39 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993.
[BH06] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for GL(2)*, volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[BK93] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of GL(N) via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[BK98] Colin J. Bushnell and Philip C. Kutzko. Smooth representations of reductive $p$-adic groups: structure theory via types. *Proc. London Math. Soc. (3)*, 77(3):582–634, 1998.

[BK99] Colin J. Bushnell and Philip C. Kutzko. Semisimple types in $GL_n$. *Compositio Math.*, 119(1):53–97, 1999.

[Bor76] Armand Borel. Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Invent. Math.*, 35:233–259, 1976.

[Bou85a] Nicolas Bourbaki. *Éléments de mathématique*. Masson, Paris, 1985. Algèbre commutative. Chapitres 1 à 4. [Commutative algebra. Chapters 1–4], Reprint.

[Bou85b] Nicolas Bourbaki. *Éléments de mathématique*. Masson, Paris, 1985. Algèbre commutative. Chapitres 5 à 7. [Commutative algebra. Chapters 5–7], Reprint.

[Bou03] Nicolas Bourbaki. *Algebra II. Chapters 4–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2003. Translated from the 1981 French edition by P. M. Cohn and J. Howie, Reprint of the 1990 English edition [Springer, Berlin; MR1080964 (91h:00003)].

[Bou06] N. Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitres 8 et 9*. Springer, Berlin, 2006. Reprint of the 1983 original.
[Bou07] N. Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitre 10*. Springer-Verlag, Berlin, 2007. Reprint of the 1998 original.

[Bou12] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Springer, Berlin, 2012. Second revised edition of the 1958 edition [MR0098114].

[BS07] Christophe Breuil and Peter Schneider. First steps towards $p$-adic Langlands functoriality. *J. Reine Angew. Math.*, 610:149–180, 2007.

[Bum] Daniel Bump. Hecke algebras. http://sporadic.stanford.edu/bump/math263/hecke.pdf.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $p$-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[Car14] Ana Caraiani. Monodromy and local-global compatibility for $l = p$. *Algebra Number Theory*, 8(7):1597–1646, 2014.

[Cas] Bill Casselman. Introduction to the Theory of Admissible Representations of $p$-adic reductive groups. http://www.math.ubc.ca/ cass/research/pdf/p-adic-book.pdf.

[CDP15] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paškūnas. Irreducible components of deformation spaces: wild 2-adic exercises. *Int. Math. Res. Not. IMRN*, (14):5333–5356, 2015.

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin. Patching and the $p$-adic local Langlands correspondence. *Camb. J. Math.*, 4(2):197–287, 2016.

[Che04] Gaëtan Chenevier. Familles $p$-adiques de formes automorphes pour $GL_n$. *J. Reine Angew. Math.*, 570:143–217, 2004.

[Che09] Gaëtan Chenevier. Une application de variétés de Hecke des groupes unitaires. 2009. Preprint.
[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations. *Publ. Math. Inst. Hautes Études Sci.*, (108):1–181, 2008. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[dJ95] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, (82):5–96 (1996), 1995.

[EG14] Matthew Emerton and Toby Gee. A geometric perspective on the Breuil-Mézard conjecture. *J. Inst. Math. Jussieu*, 13(1):183–223, 2014.

[Eme06] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.

[Eme07] Matthew Emerton. Locally analytic representation theory of $p$-adic reductive groups: a summary of some recent developments. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 407–437. Cambridge Univ. Press, Cambridge, 2007.

[HH] Urs Hartl and Eugen Hellmann. The universal family of semi-stable $p$-adic Galois representations. Preprint.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[Hu09] Yongquan Hu. Normes invariantes et existence de filtrations admissibles. *J. Reine Angew. Math.*, 634:107–141, 2009.

[Kis06] Mark Kisin. Crystalline representations and $F$-crystals. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 459–496. Birkhäuser Boston, Boston, MA, 2006.

[Kis08] Mark Kisin. Potentially semi-stable deformation rings. *J. Amer. Math. Soc.*, 21(2):513–546, 2008.
[Knu98] Donald E. Knuth. *The art of computer programming. Vol. 3*. Addison-Wesley, Reading, MA, 1998. Sorting and searching, Second edition [of MR0445948].

[Kud94] Stephen S. Kudla. The local Langlands correspondence: the non-Archimedean case. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 365–391. Amer. Math. Soc., Providence, RI, 1994.

[Lab11] J.-P. Labesse. Changement de base CM et séries discrètes. In *On the stabilization of the trace formula*, volume 1 of *Stab. Trace Formula Shimura Var. Arith. Appl.*, pages 429–470. Int. Press, Somerville, MA, 2011.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Paš15] Vytautas Paškūnas. On the Breuil-Mézard conjecture. *Duke Math. J.*, 164(2):297–359, 2015.

[Paš16] Vytautas Paškūnas. On 2-dimensional 2-adic Galois representations of local and global fields. *Algebra Number Theory*, 10(6):1301–1358, 2016.

[Pyv18a] Alexandre Pyrovovarov. Generic smooth representations. *Preprint*, 2018.

[Pyv18b] Alexandre Pyrovovarov. Specialization of a projective generator. *Preprint*, 2018.

[Ren10] David Renard. *Représentations des groupes réductifs p-adiques*, volume 17 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2010.

[Sha74] J. A. Shalika. The multiplicity one theorem for $GL_n$. *Ann. of Math. (2)*, 100:171–193, 1974.

[Sho16] Jack Shotton. Local deformation rings for $GL_2$ and a Breuil-Mézard conjecture when $\ell \neq p$. *Algebra Number Theory*, 10(7):1437–1475, 2016.
[Sor15] Claus M. Sorensen. The Breuil-Schneider conjecture: a survey. In Advances in the theory of numbers, volume 77 of Fields Inst. Commun., pages 219–235. Fields Inst. Res. Math. Sci., Toronto, ON, 2015.

[ST01] P. Schneider and J. Teitelbaum. $U(g)$-finite locally analytic representations. Represent. Theory, 5:111–128, 2001. With an appendix by Dipendra Prasad.

[ST02] P. Schneider and J. Teitelbaum. Banach space representations and Iwasawa theory. Israel J. Math., 127:359–380, 2002.

[ST06] P. Schneider and J. Teitelbaum. Banach-Hecke algebras and $p$-adic Galois representations. Doc. Math., (Extra Vol.):631–684, 2006.

[SZ99] P. Schneider and E.-W. Zink. $K$-types for the tempered components of a $p$-adic general linear group. J. Reine Angew. Math., 517:161–208, 1999. With an appendix by Schneider and U. Stuhler.

[SZ16] Peter Schneider and Ernst-Wilhelm Zink. Tempered representations of $p$-adic groups: special idempotents and topology. Selecta Math. (N.S.), 22(4):2209–2242, 2016.

[Zel80] A. V. Zelevinsky. Induced representations of reductive $p$-adic groups. II. On irreducible representations of $GL(n)$. Ann. Sci. École Norm. Sup. (4), 13(2):165–210, 1980.

Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany.
E-mail address: alexandre.pyvovarov@stud.uni-due.de