Complexity and intelligence

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Abstract

In this paper I will discuss the properties of the Algorithmic Complexity, presenting the most relevant properties. The related concept of logical depth is also introduced. These properties will be used to study the problem of learning from example, paying a special attention to machine learning. We introduce the propensity of a machine to learn a rule and we use it define the intelligence of a machine.

1 Algorithmic Complexity

We have already seen in the chapter […] that Kolmogorov, independently from and more less simultaneously with other peoples, introduced the concept of algorithmic complexity of a string of character (or if we prefer of an integer number) as the length in bits of the shortest message that is needed to identify such a number. This look similar to Shannon’s entropy, but it is deeply different. Shannon’s entropy is the shortest length of the message needed to transmit a generic string of characters belonging to a given ensemble, for example a sentence of given length in Italian. Kolmogorov complexity is is the shortest length of the message needed to transmit a given string of characters.

In a slightly different setting, that is nowadays more familiar to most of us, we can define the Kolmogorov complexity $\Sigma(N)$ of a number $N$ as the length of the shortest computer program that computes such a number (or if we prefer prints that number). This definition of complexity depends on the computer language in which the program is written, so that we should indicate in the definition of $\Sigma(N)$ the language we are using. However if we are interested to large values of complexity the
choice of the language is irrelevant and it contributes in the worst case to constant. It is evident that
\[
\Sigma_{Pascal}(N) < \Sigma_{Pascal}(\text{Fortran compiler written in Pascal}) + \Sigma_{Fortran}(N)
\] (1)
Indeed if we know how to compute something in Fortran and we would like to compute it in Pascal, we can just write a Fortran compiler in Pascal and use it to execute the Fortran code. In the nutshell we can transmit the Fortran compiler written in Pascal and the program that computes the message in Fortran. Inverting the argument we get
\[
|\Sigma_{Fortran}(N) - \Sigma_{Pascal}(N)| < \text{const}
\] (2)
where constant does not depend on \( N \). So for real complex messages the choice of the language is not important.
This is common experience also in human languages. If we are going to have a short discussion on the possible evolution of the snow conditions in the next days in relation to a trip on a dog-trained sled, it is extremely likely that the conversation should be much shorter in some languages and much longer in other ones. On the contrary, if we have to explain to somebody the proof of the last Fermat theorem starting from scratch (i.e. from elementary arithmetics), it would take years and in the process we can introduce (or invent) the appropriate mathematical language, so that the dependence on the natural language we use is rather weak. In other words for messages of small complexity we can profit of the amount of information contained in the language, but this help becomes irrelevant in the case of very complex messages.
It should be clear that in the case of a generic number \( N \) of \( K \) binary digits, e.g. smaller that \( 2^K \) elementary arguments (and also Shannon’s theorem) tell us that the complexity is \( K \). The complexity cannot be larger than \( K \) because we can always write a program that is of length \( K \) plus a small constant:
\[
\text{write } 3178216741756235135123939018297137213617617991101654611661
\]
The length of this program is the same as the length of the output of the program (apart for the characters lost in writing “write”) and it rather likely (I will pay 1000 Euros for a counterexample) that there is no much shorter one \(^1\). Of course the program
\[
\text{write } 11**57
\]
is much shorter that
\[
\text{write } 228761562390246506066453264733492693192365450838991802120171
\]
\(^1\)I beg the reader to forget the inefficiency of writing decimal numbers in ASCII.
although they have the same output.

Let us consider an other example. The program

\[
\begin{align*}
A &= 2^{31}-1 \\
B &= 5^7 \\
I &= 1 \\
\text{for } K = 1 \text{ to } A-1 \\
& \quad I = \text{mod}(I*B, A) \\
\text{endfor}
\end{align*}
\]

generates a sequence of pseudo random numbers that is rather long (about 8 Gbytes) and apart form small variations it is extremely likely that it is the shortest program that generate it. Of course it is much shorter of the program (of 8 Gbytes of length) that explicitly contains the sequence.

\section{2 Some properties of complexity and some apparent paradoxes}

Although most of the numbers of \( K \) bits have complexity \( K \), we have seen that there are notable exceptions. It should be clear that to identify these exceptions and to evaluate of the complexity is not a simple task. A number may have a low complexity because it is a power, or because it is obtained as the iteration of a simple formula, or just because it coincides with the second millions of digits of \( \pi \). A systematic search of these exceptions is not easy.

However one could imagine a simple strategy. We can consider all the programs of length \( K \). Each program will stop and will write a string (maybe empty) or it will never stop: A program that never stops is

\[
\begin{align*}
I &= 1 \\
\text{do forever} \\
& \quad I = I + 1 \\
& \quad \text{if}(I < 0) \text{ then} \\
& \quad \quad \text{write } I \\
& \quad \quad \text{stop} \\
& \quad \text{endif} \\
\text{enddo}
\end{align*}
\]

where \( I \) is an arbitrary length number.

In this case is trivial to show that the program stops. In other cases is much difficult to decide if the programs stops. Let us consider the following example:
\begin{verbatim}
I=1
do forever
  I=I+1
  consider all positive integers a, b, c and n, less than I with n>2.
  if(a**n+b**n=c**n) then
    write a, b, c, n
    stop
  endif
endo
\end{verbatim}

This program stops if and only if Fermat last theorem is false and we know now
that it will never stops. Up to a few years ago it was possible that it would stop
writing a gigantic number. A program that stops only if the Goldbach conjecture is
false quite like will never stop, but we do not know for sure.

There cannot be any computer program that can compute if a computer program
stops or not. Otherwise we would have a contradiction. A computer program would
be able to find out those programs who stops, identifies those program that are the
shortest one and produce the same output: for given length we can sort them in
lexicographic order. If this happens, for example we could identify the first program
of length $K$ of this list. The output of this program has complexity $K$, in the same
way as all the programs of this list, however we would identify the first program of
the list transmitting only the number $K$; this can be done using only $\ln_2(K)$ bits and
therefore the complexity of the output would be $\ln_2(K)$, i.e. a number much smaller
than $K$ and this is a contradiction.

One could try to find out a loophole in the argument. Although it is impossible
to decide if a program does not stop, because we would need to run it for an infinite
amount of time, however we can decide if a program stops in $M$ steps. As far as
there is a finite number of programs of length $K$ for each value of $K$ we can define a
function $f(K)$ that is equal to the largest value of steps where a program of length
$K$ stops. All programs that do not stop in $f(K)$ must runs forever. The previous
construction could be done by checking the output of all the program of length less
than $f(K)$. Therefore if we could compute the function $f(K)$ or an upper bound to
it, we would be able to get the list of all the program that stops.

A simple analysis show that no contradiction is present if we the function $f(K)$
increases faster than any function that is computable with a program of constant
length: e.g. we must have that for large $K$ that
\begin{equation}
  f(K) > e^{e^{e^{e^{e^{e^{e^{e^{e^{e^K}}}}}}}}}
\end{equation}
The complexity of finding an upper bound to $f(K)$ must be $O(K)$.
The existence of a function that grows faster than what can be computed by a program of fixed length seems surprising however it is rather natural. Indeed if we call $M(K)$ the maximum finite number that can be printed by a program of length $K$. Now it is evident that $M(K + 1) > M(K)$, therefore the function $M(K)$ cannot be printed by any program with length less than $K$, so that there is no finite length program that is able to print all the values of $M(K)$. The conclusion is that to find the shortest program (or if we prefer the shorted description) of a number $N$ is something that cannot be done by program: the complexity is a well defined quantity, but it cannot be computed systematically by a program.

Many variations on the same theme can be done: suppose that we add to our computer an oracle: a specialized hardware that that tells the value of $f(K)$ (or directly the complexity of a string). We could now define a complexity $\Sigma_1(N)$ for the system computer + oracle. The same argument as before tell us that the new complexity $\Sigma_1(N)$ cannot be computed for any $N$ by the the system computer + oracle. We could introduce a superoracle and so on. The fact that we are unable, also with the use of any kind of oracles, to find a systems that can compute the computational complexity relative to itself recalls intuitively the Godel theorems: This is not a surprise as far there are deep relations among these arguments and with Godel theorems that have been investigated by Chaitin and presented also in many popular papers.

3 The logical depth

The problem of finding out the minimal complexity of a number (or of a sequence of numbers), intuitively correspond to finding the rule according that some number have been generated. This problem is used in intelligence tests. For example one may be asked to find out the rules that generate the following sequences or equivalently to predict the next numbers of the sequences

1 2 3 4 5 6 7 8 9 10 11 12 13
1 2 4 8 16 32 64 128 256 512 1024 2048
1 1 2 3 5 8 13 21 34 55 89 144
1 1 2 2 4 2 4 6 2 6 4 2 4 6 6 2 6 4 2 6 4 6 8 4 2 4

For the first two lines the rule is clear, the third line is a Fibonacci sequence: the next number is the sum of the of the two previous one, while the last line follow a slightly more complex rule and we leave to the reader the pleasure to find it.

In the case that there are different rules, we consider a natural rule the simplest one: for example we could say that the first line is the sequence of natural integer that cannot be written as $a^2 + b^2 + c^2 + d^2$ with $a$, $b$, $c$ and $d$ are all different (the first integer missing would be 30). However this second rule is much longer and it is
unnatural. The two rules would produce rather different sequences but if we know only the first elements the first choice is much more natural.

A related and somewhat complementary concept is the logical depth of a number [7]: roughly speaking it is the amount of CPU needed to compute it if we use the shortest program that generates it (i.e. the natural definition of the number).

It is evident that if the complexity is given by the upper bound and the program consists of only one write instruction, the execution is very fast. i.e. linear in the number of bits. On the contrary, if a very short program prints a very long sequence, the execution may be still very fast, as in the case of a typical random number generator, or may be very slow, e.g. the computation of the first $10^6$ digits of $\pi$ or to find out the solution of an NP hard problem whose instance has been generated using a simple formula. Only a low complexity sequence may have a large logical depth; moreover the same arguments of the previous section tell us that there must be for any value of the complexity $K$ sequences with extremely large values of the logical depth (i.e. of order $f(K)$).

These results concern us also because science aims to find out the simplest formulation of the law that reproduce the empirical data. The problem of finding the simplest description of a complicated set of data correspond to find the scientific laws of the world. For example both the Newton and the Maxwell laws summarize an enormous quantity of empirical data and are likely the shortest description of them. The ability of finding the shortest description is something that cannot be done in general by a computer and it is often taken a sign of intelligence.

Phenomenological explanations with lot of parameters and no so deep theory are often easy to apply and correspond to rules that have high complexity and low logical depth while simple explanations with few parameter and lot of computation needed have low complexity and rather high logical depths.

For example the computation of chemical properties using the valence theory and the table of electronegativity of the various element belongs to the fists class, while a first principle computation, starting from basic formulae, i.e. quantum mechanics belongs to the second class. A good scientific explanation is a low complexity theory (i.e. the minimal description of the data) that unfortunately may have an extremely large logical depth. Sometimes this leads to the use of approximated theories with higher complexity and smaller logical depth.

4 Learning form examples

Scientific learning is just one case of a general phenomenon: we are able to learn from example and to classify the multitude of external object into different classes. The problem of how it is possible to learn from examples has fascinated thinkers for a
long time. In the nutshell the difficulty is the following: if the rule cannot be logically
derived from the examples, how do we find the rule? The solution put forward by
Plato was that the rule is already contained in the human brain and the examples
have the only effect of selecting the good rule among all the admissible ones.

The opposite point of view (Aristotle) claims that the problem is ill posed and
that human brain is tabula rasa before the experience of the external world.

Plato’s point of view has been often dismissed as idealistic and non-scientific.
Here we want to suggest that this is not the case and that Platonic ideas are correct,
at least in a slightly different contest. Indeed the possibility of having machines which
learn rules from examples has been the subject of intensive investigations in recent
years. A very interesting question is to understand under which conditions the
machine is able to generalize from examples, i.e. to learn the whole rule knowing
only a few applications of the rule. The main conclusion we want to reach is that a
given machine cannot learn an arbitrary rule and the set of rules that can be easily be
learned may be determined by analyzing the architecture of the machine. Learning
by example consists in selecting the correct rule among those contained in this set.
Let us see in the next sections how it may happen and how complexity is related to
intelligence.

5 Learning, generalization and propensities

In order to see how this selectionist principle may work let us start with some def-
inition. In the following I will consider rules which assign to each input vector of
N Boolean variables ($\sigma_i = 0$ or $1$ for $i = 1, N$) an output which consists of a single
Boolean value. In other words a rule is a Boolean valued function defined on a set
of 2^N elements (i.e. the set of all the possible values of the variables $\sigma$; it will be
denoted by $R[\sigma]$).

The rule may be specified either by some analytic formulae or by explicitly stating
the output for all the possible inputs. The number of different rules increases very
fast with $N$: it is given by $2^{2^N} = 2^{N_I}$, where $N_I$ is the number of different possible
input vectors, i.e. $2^N$. In the following we will always consider $N$ to be a large
number: terms proportional to $1/N$ will be neglected.

A learning machine is fully specified if we know its architecture and the learning
algorithm.

Let us firstly define the architecture of the machine. We suppose that the comput-
ations that the machine performs depend on M Boolean variables ($J_k$, $k = 1, M$). In
the nutshell the architecture is a Boolean function $A[\sigma, J]$, which gives the response

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2 Two examples: (a) the horsiness is not the property of any given horse or set of horses (b) there are
many different mathematical rules that generate the same sequence.
of machine to the input $\sigma$‘s for each choice of the control parameters $J$‘s. Typical architectures are the perceptron [?] or a neural network, with discretized synaptic couplings.

For each given rule $R$ and choice of $J$‘s, the machine may make some errors with respect to the rule $R$. The total number of errors ($E[R,J]$) depends on the rule $R$ and on the $J$‘s; it is given by

$$E[R,J] = \sum_{\{\sigma\}} (R[\sigma] - A[\sigma,J])^2.$$  \(4\)

For a given architecture the machine may learn the rule $R$ without errors if and only if there exist a set of $J$‘s such that $E[R,J] = 0$. Simple counting arguments tell us that there are rules which cannot be learned without errors, if $M$ is smaller than $2^N$. In most of the cases $2^N$ is extremely larger than $M$ and therefore the number of admissible rules is an extraordinary tiny fraction of all the possible rules.

In a learning session we give to the machine the information on the values of $R[\sigma]$ for $L$ instances in the $\sigma$‘s ($L$ is generally much smaller than $2^N$). A learning algorithm tries to find the $J$‘s which minimize the error on these $L$ instances. Let us denote by $J^*$ the $J$‘s found by the learning algorithm.

If the error on the other $2^N - L$ instances has decreased as effect of having learned the first $L$ instances, we say that the machine is able to generalize, at least to a certain extent. Perfect generalization is achieved when no error is done on the other $2^N - L$ instances.

For a given machine the propensity to generalize depends on the rule and not all rules will be generalized by the machine. Our aim is to understand how the propensity to learn different rules changes when we change the machine; in this note we are only interested in the effect of changing the architecture.

It was suggested by Carnevali and Patarnello in a remarkable paper [?] that, if we suppose that the learning algorithm is quite efficient, the propensity of the machine to generalize a given rule ($p_R$) depends only on the architecture. The propensity ($p_R$) may be approximated by the number of different control parameters $J$ for which the total number of errors is zero.

In other words we define the propensity as

$$p_R = 2^{-M} \sum_{\{J\}} \delta(E[R,J]),$$ \(5\)

where $E[R,J]$ is given by eq. (1) and obviously depends only on the architecture.

The function $\delta$ is defined in such a way that $\delta(k) = 1$ for $k = 0$, $\delta(k) = 0$ for $k \neq 0$.

\[3\]There are many different learning algorithms and some are faster than others. The choice of the learning algorithm is very important for practical purposes, but we will not investigate this point anymore.
According to Carnevali and Patarnello, rules with very small propensity cannot be generalized, while rules with higher propensity will be easier to generalize. In their approach the propensity of a given architecture in generalizing is summarized by the values of the function \( p_R \) for all the \( 2^N \) arguments (the \( p_R \)’s depend only on the architecture, not by the learning algorithms). Of course the propensity cannot be directly related to the number of examples needed to learn a rule, a more detailed analysis, taking care of the relevance of the presented example, must be done.

6  A statistical approach to propensities

Our aim is to use statistical mechanics techniques [?] to study in details the properties of \( p_R \).

The \( p_R \) are normalized to 1 (\( \sum_R p_R = 1 \)) and it is natural to introduce the entropy of the algorithm \( A \):

\[
S[A] = - \sum_R p_R \ln(p_R).
\] (6)

The entropy \( S[A] \) is a non negative number smaller or equal than \( \ln(2) \min(2^N, M) \).

We could say that if the entropy is finite for large \( N \), the machine is able to represent essentially a finite number of rules, while, if the entropy is too large, too many rules are acceptable.

As an example we study the entropy of the perceptron (without hidden unity). In this case (for \( N = M \) odd) a detailed computation shows that all the \( 2^N \) choices of the \( J \)'s lead to different rules (i.e. two different \( J \)'s produce a different output at least in one case) and therefore \( S[A] = \ln(2)N \).

We note that we could generalize the previous definition of entropy by introducing a partition function \( Z(\beta) \) defined as follows

\[
Z(\beta) = \sum_R \exp(\beta \ln(p_R)) = \sum_R p_R^\beta.
\] (7)

We could introduce the entropy \( S(\beta) \) associated with the partition function [?] (\( S(\beta) \equiv d\ln[Z(\beta)/\beta]/d\beta \)). The previous defined entropy (eq. 6) coincides with \( S(1) \).

The value of the entropy as function of \( \beta \) tells us which is the probability distribution of the \( p_R \)'s. There are many unsolved question whose answer depend on the model: existence of phase transitions, structure of the states at low temperature, breaking of the replica symmetry . . .

Many additional questions may be posed if we consider more than one architecture; in particular we would like to find out properties which distinguish between
architectures which have similar entropies. For example we could consider two different architectures (i.e. a layered perceptron or a symmetric neural network) with $N$ inputs and one output which have the same entropy (this can be achieved for example by adjusting the number of internal layers or hidden neurons). It is natural to ask if these two different architectures are able to generalize the same rules, or if their propensity to generalize is concentrated on rules of quite different nature. Our aim is to define a distance between two architectures, which will help us to compare the different performances.

Let us consider two architectures $A$ and $B$. A first step may consist in defining the entropy of $B$ relative to $A$ as

$$S[B/A] = -\sum_R p_R(A) \ln[p_R(B)/p_R(A)].$$

(8)

It can be shown that $S[B/A]$ is a non-negative quantity that becomes zero if and only if $p_R(A) = p_R(B)$. The relative entropy is not symmetric and we can define the distance (or better the difference) between $A$ and $B$ as

$$d(A, B) = 1/2 S[B/A] + S[A/B].$$

(9)

The introduction of a distance allow us to find out if two different architectures with the same generalization propensity; do they generalize the same rules or the rules they are able to generalize are different? Unfortunately, the explicit computation of the distance among two architectures may be very long and difficult.

A very interesting and less understood question is how many examples are needed to specify the rule for a given architecture. The result obviously depend on the architecture (people with an high value of serendipity guess the write rule after a few examples) and explicit computations are very difficult, if we exclude rather simples cases.

### 7 A possible definition of intelligence

Having in our hands a definition of the distance between two architectures, we can now come to a more speculative question: how to define the intelligence of an architecture. A possibility consists in defining an architecture $I(\sigma, J)$, which is the most intelligent by definition; the intelligence of $A$ can be defined as $-d[A, I]/S[A]$ (the factor $S[A]$has been introduced for normalization purposes). The definition of the intelligent architecture $I$ is the real problem.

We suggest that a sequence of the most intelligent architectures is provided by a Turing machine (roughly speaking a general purpose computer) with infinite amount of time for the computation with a code of length $L$. More precisely the $J$'s are the $L$
bits of a code for the Turing machine (written in a given language) which uses the \( \sigma \) as inputs. The function \( I(\sigma, J) \) is 1, if the program coded by \( J \) stops after some time, and it is 0, if the the program never stops. With this architecture we can compute the function \( s(R) \) (i.e. the simplicity of a rule) \([7]\), defined as \( s(R) \equiv p_R(I) \).

It may be possible to estimate the function \( s(R) \) using the relation \( s(R) \approx 2^{-\Sigma(R)} \), where \( \Sigma(R) \) is the algorithmic complexity (introduced in the first section) i.e. the length of the shortest codes which computes the function \( R[\sigma] \).

If we would like to know the intelligence of an architecture \( A \) with entropy \( S \), we should consider a Turing machine with nearly the same entropy and we should compute the distance between the two architectures.

The previous observations imply that algorithms which compute the rule in a relatively short time are likely unable to implement many rules with low algorithmic complexity and high logical depth. In many of the most common algorithms the number of operations needed to compute the output for a given input is proportional to a power of \( N \). For other algorithms (e.g. an asymmetric neural network, in which we require the computation of a limiting cycle) the computer time needed may be much larger (e.g. proportional to \( 2^L \)). It is natural to suppose that this last class of algorithms will be more intelligent than the previous one and will learn more easily rules with low algorithmic complexity and high logical depth \([7]\).

The puzzled reader may ask: if a general purpose computer is the most intelligent architecture, while people are studying and proposing for practical applications less intelligent architectures likes neural networks? A possible answer is that this definition of intelligence may be useful to disembodied entities that have an unbound amount of time at their disposal. If we have to find a rule and take a decision in real time (one second or one century, it does not matter, what matters is that the time is limited) rules of too low complexity and very large logical depth are completely useless; moreover a computer can be used to define the intelligence but we have seen that a computer is not well suited for finding and executing intelligent rules. The requirement of taking a fast decision may be dramatic in some case, e.g. when you meet a lion on your path; evolution definitely prefers a living donkey to a dead doctor and it quite likely that we have not been selected for learning rules with too large logical depth.

There are also other possible inconveniences with rules with too low complexity, i.e. they may be unstable with the respect of a small variation of the data they have to reproduce. It is quite possible that the shortest program that pronounce words in Italian may have a completely different structure of the shortest program that pronounces words in Spanish, in spite of the similarity of the two languages. We would like to have a program that may be used for many languages, where we can add a new language without changing too much the core of the program; it is quite likely that such a program would be much longer than the shortest one for a given
set of rules. Strong optimization and plasticity are complementary requirements. For example the nervous system of insects is extremely optimized and quite likely it is able to perform the tasks with a minimal number of neurons, but it certain lacks the plasticity of mammalian nervous system.

Architectures that are different from a general purpose computer may realize a compromise producing rules that have low, but not too low complexity and high, but not to high logical depth. Moreover these architectures are specialized: some of them (like neural networks) may be very good in working as associative memories but quite bad in doing arithmetics. The problem of finding the architecture that works in the most efficient way for a given task is difficult and fascinating and in many cases (e.g. reconstructing three dimensional objects from two dimensional images) it has also a very important practical impact.

I hope that this short exposition has convinced the reader of the correctness of the point of view that learning from example can be done only selecting among already existing rules. This is what typically happens in biology in many cases. The genetic information only preselect a large class of behavior and the external stimulus select the behavior among the available ones. This procedure can be seen in action in the immune systems. The number of possible different antibodies is extremely high (O(10^{100})); each given individual at a given moment produces a much smaller number of different antibody (e.g. O(10^8)). The external antigen select the most active antibodies among those present in the actual repertoire and stimulate their production. Eldeman has stressed that it is quite natural that a similar process happens also in the brain as far as learning is concerned.

In conclusion at the present moment we have only started our exploration of the properties of rules that are implicitly defined by an architecture. I am convinced that the future will bring us many interesting results in this direction. It is amazing in how diverse directions Kolmogorov’s ideas are relevant and find an application.