A Numerical Study of Radial Basis Function Based Methods for Options Pricing under the One Dimension Jump-diffusion Model

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Abstract The aim of this paper is to show how option prices in the Jump-diffusion models, mainly on the Merton and Kou models, can be computed using meshless methods based on Radial Basis Function (RBF) interpolation. The RBF technique is demonstrated by solving the partial integro-differential equation (PIDE) in one-dimension for the American vanilla put and the European vanilla call/put options on dividend-paying stocks. The radial basis function we select is the Cubic Spline. We also propose a simple numerical algorithm for finding a finite computational range of an improper integral term in the PIDE so that the accuracy of approximation of the integral can be improved. Moreover, we use a numerical technique called factorization of the Cubic Spline to avoid inverting the ill-conditioned Cubic Spline interpolant. Finally, we will show numerically that in the European case the solution is second order accurate for the spatial and time variables, while in the American case it is second order accurate for spatial variables and first order accurate for time variables.

Key Words: Lévy Processes, the Jump-diffusion model, Partial-Integro Differential Equation, Radial Basis Function, Cubic Spline, European Option, American Option.

1. Introduction

In this paper we show how to compute European and American option prices in the Jump-diffusion model using Radial Basis Function (RBF) interpolation techniques. RBF methods have recently been proposed for numerically solving initial value and free boundary problems for the classical Black and Scholes equation, both in the one and in the multiple asset case (Paussshauer et al. 2004a; Hon and Mao 1999; Larsson et al. 2008). The new feature of the present paper is that in the Jump-diffusion model, as in general Lévy type models, the Black and Scholes PDE is replaced by a Partial Integro-Differential Operator or PIDE, involving a global term in the form of an integral operator. The PIDE has a form:

$$\partial_\tau u(x, \tau) = \frac{1}{2} \sigma^2 \partial_x^2 u + \left( r - q - \frac{1}{2} \sigma^2 - \eta \right) \partial_x u - (r + \lambda) u + \lambda \int_{\mathbb{R}} u(x + y, \tau) f(y) dy$$

(cf. Cont and Tankov 2004a; Schoutens 2003, 2006). Our main contribution is to show how to numerically solve (1) in an efficient way using RBFs, both for initial value and free boundary problems (as for American options). We have chosen the Jump-diffusion model as a typical...
case on which to test the present RBF methodology. Our method extends however without problems to other contexts in which the basic pricing equation is a PIDE, like that of Lévy-type models such as Carr-Geman-Madan-Yor (CGMY) (Carr et al., 2002) or Variance Gamma (VG) (Carr et al., 1998; Madan and Milne, 1991). These will be treated in a future paper.

Currently, PIDEs such as the Merton Model (Merton, 1976) and the Kou Model (Kou, 2002; Kou and Wang, 2001), one have mostly been treated by a traditional Finite Difference Method (FDM) or Finite Elements Method (FEM). In FDM, the idea is to simply fully discretize the PIDE on an equidistant grid, after having (artificially) localized the equations to some bounded interval/domain in \( \mathbb{R} \). The global integral term can be computed by numerical quadrature or by using the Fast Fourier Transform (FFT) (see, Almendral, 2004; Almendral and Oosterlee, 2004; 2005; 2006; 2007; Andersen and Andreasen, 2000; Briani et al., 2007; Cont and Voltchkova, 2005; d’Halluin et al., 2004; 2005; Hirsa and Madan, 2004; Wang et al., 2007). By contrast, FEM is defined as piecewise polynomial functions or wavelet functions on regular triangulations. This technique is used to approximate solutions of the partial differential terms as well as of the integral term (cf. Almendral and Oosterlee, 2005; Matache et al., 2003; 2005).

In general, there is a problem which arises with these current approaches. Some of the literature, e.g. (Andersen and Andreasen, 2000; Briani et al., 2007; Cont and Voltchkova, 2005), plays down the importance of pricing American and European vanilla option values when time to maturity is less than 3 months. The reason is that for short times-to-maturity the numerical methods used to price the option tend to be inaccurate near the strike price where a singularity (kink) exists. A singularity is defined as a point at which the function, or its derivative, is discontinuous. The payoff functions of vanilla call and put options have such a singularity. As a result, standard numerical methods such as FDM with Crank-Nicolson and without any adaptive schemes cannot ensure accuracy of option prices around the strike and a substantial amount of oscillation occurs around the strike when Option Delta \( \Delta \) and Gamma \( \Gamma \) are approximated (Giles and Carter, 2006). Giles and Carter shed light on this kind of problem (Giles and Carter, 2006) by suggesting Rannacher’s time stepping method. This is a mixture of four half-timesteps of backward Euler and Crank-Nicolson methods. Although they solve an one dimensional PDE under the Black-Scholes model and Heston’s volatility model rather than a PIDE under Lévy models, their methods of using backwards Euler time-stepping in one or more initial timesteps have been proved to be achieved second-order convergence in a European case. They also carry out a detail error analysis of their methods by using Fourier analysis and find out four half-timesteps of backward Euler time-marching is the minimum require to recover second-order convergence of solving the PDE. Forysth et al. (d’Halluin et al., 2005) also use the similar idea by suggesting Rannacher’s time stepping method (Rannacher, 1984) to solve a PIDE under the Merton Jump-diffusion model. They demonstrate this technique by approximating an option price whose maturity is a quarter of a year. This method gives second order rates of convergence when pricing European options but not American ones. By using the same idea and combining it with a penalty method and a modified form of a timestep selector suggested in (Johnson, 1987), Forysth et al. (d’Halluin et al., 2004) show how to achieve second order convergence for pricing American options. Unfortunately they do not carry out any stability analysis when they apply Rannacher’s time stepping method to solve the PIDE. Moreover there is no minimum requirement of choosing half-timesteps of backward Euler before Crank-Nicolson methods are applied. All they do is by trial and error.

In most recent research papers in quantitative finance (cf. Faussauer et al., 2004a,b; Hon and Mao, 1999; Larsson et al., 2008) RBF-approximation methods with Multiquadric (MQ) as a basis function have been proposed for numerically solving the classical Black and Scholes PDE, both in the one and in the multiple asset case. In this literature MQ is a more favorite choice than other radial basis functions, such as Thin plate Spline and the like, because of its comparatively higher accuracy. MQ contains a shape parameter which plays an imperative
role in the accuracy of the method (cf. [Wendland 2005]). Most of this recent literature still chooses this parameter by trial and error or some other ad-hoc means. Although there exists a substantial literature on choosing an ”optimal” shape parameter in MQ, e.g. (Fasshauer and Zhang 2007), (Fornberg and Wright 2004) and (Kansa and Carlson 1992), it is still an open question and there is no theoretical proof for selecting an optimal shape parameter (cf. [Wendland 2005]) in MQ. Besides this, the standard approach to the solution of the radial basis function interpolation problem has been recognized as an ill-conditioned problem for many years (cf. Fasshauer 2007, chapter 16). This is especially true when infinitely smooth basic functions such as MQ or Guassian are used with small values of their associated shape parameters. More recently, Fasshauer and Mccourt’s least-squares approximation based on early truncation of the kernel expansion [Fasshauer and Mccourt, 2010] and Fornberg and co-workers’s Contour-Padé integration method (e.g. Driscoll and Fornberg 2002, Fornberg and Wright 2004, Larsson and Fornberg 2005) are successful in solving the ill-conditioning problem of RBF, but the techniques are only restricted to solve the simple interpolation problem rather than to solve PDEs, especially parabolic PDEs. Although Ling and his co-workers (e.g. Ling and Kansa 2004, Brown et al. 2005, Ling and Kansa 2005) address the ill-conditioning problem by using preconditioning methods and extend them to solve PDEs, the methods are not possible to be applied to solving PIDEs.

Our RBF-approximation method with the Cubic Spline as a basis function will circumvent these disadvantages. This paper is divided into five sections, including this introduction. Section 2 is a brief review of both the Merton and Kou Jump-diffusion models. In section 3 we first explain and then define our RBF algorithm for solving PIDEs, which we implement the Jump-diffusion model. Section 4 contains our numerical results for both European and American call and put options, including an analysis of the max error, the root-mean-square error, the rate of convergence and the approximation of $\Delta$ and $\Gamma$ and also a comparison the accuracy of our solution with that of FDM and FEM. Section 5 concludes.

2. PIDE Option Pricing Formula in Jump-diffusion Market

In this short section we will focus on the Merton and the Kou Jump-diffusion Models which are general Lévy processes consisting of Brownian motion and compound Possion jumps. By using these models we can describe the price dynamics of the underlying risky asset, $(S_t)_{t \geq 0}$. The evolution of $(S_t)_{t \geq 0}$ is driven by a diffusion process, punctuated by jumps which describe rare events such as crashes and/or drawdowns at random intervals. As a market model, it is an example of an incomplete market.

The stock price process, $(S_t)_{t \geq 0}$, driven by these models, is given by:

$$S_t = S_0 e^{L_t}$$

where $S_0$ is the stock price at time zero and $L_t$ is defined by:

$$L_t := \gamma_c t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

here, $\gamma_c$ is a drift term, $\sigma$ is a volatility, $W_t$ is a Brownian motion, $N_t$ is a Possion process with intensity $\lambda$, $Y_i$ is an i.i.d. sequence of random variables. Since $\sigma > 0$ in (3), there exists a risk-neutral probability measure $Q$ such that the discounted process $\{e^{-r_q t} S_t\}_{t \geq 0}$ becomes a martingale (cf. [Sato 1999] Theorems 33.1 and 33.2), where $r$ is the interest rate and $q$ is the dividend rate. For a discussion of the issue of choosing $Q$ see, for example, (Cont and Tankov 2004). Then under this new measure $Q$, the risk-neutral Lévy triplet of $L_t$ can be described.
as follows:

\[(\gamma_c, \sigma, \nu)\]

where

\[
\gamma_c = r - q - \frac{1}{2} \sigma^2 - \lambda \eta + \int_{\mathbb{R}} x \nu(dx),
\]

(4)

Here we focus on the case where the Lévy measure is associated to the pure-jump component and hence the Lévy measure \(\nu(dx)\) can be written as \(\lambda f(x)dx\), where the weight function \(f(x)\) can take two forms:

(1) In the classical Merton model, for any \(i \in \{1, 2, \ldots\}\), \(Y_i\) are log-normally distributed variables with \(Y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)\) and as a result,

\[
\eta = e^{\mu_J + \sigma_J^2/2} - 1.
\]

(7)

(2) In the Kou model,

\[
f(x) = p \alpha_1 e^{-\alpha_1 x} \mathbb{1}_{x \geq 0} + (1 - p) \alpha_2 e^{\alpha_2 x} \mathbb{1}_{x \leq 0}.
\]

(6)

**Remark 1:** In the Merton Jump-diffusion model, one should notice that \(Y_i\) is i.i.d so for each \(i \in \{1, 2, 3, \ldots\}\), \(Y_i\) has the same mean and variance. For the sake of simplicity, we use \(\mu_J\) and \(\sigma_J^2\) to represent the mean and variance of each \(Y_i\) respectively.

Also in (4), \(\eta = \int_{\mathbb{R}} (e^x - 1) f(x) dx\) represents the expected relative price change due to a jump. Since we have defined the Lévy density function \(f(x)\) for both Jump-diffusion processes, \(\eta\) can be computed as:

(1) In the Merton model,

\[
\eta = e^{\mu_J + \sigma_J^2/2} - 1.
\]

(7)

(2) In the Kou model,

\[
\eta = \frac{p \alpha_1}{\alpha_1 - 1} + \frac{(1 - p) \alpha_2}{\alpha_2 + 1} - 1.
\]

(8)

This is found by integrating \(e^x\) over the real line by setting \(\alpha_1 > 1\) and \(\alpha_2 > 0\).

For the details of the computation of (7) and (8), we shall refer the reader to (Cont and Tankov, 2004; Boyarchenko and Levendorski, 2002).

The drift-term \(\gamma_c\) in (3) assumes that \(e^{-(r-q)t} S_t\) is a martingale with respect to the natural filtration. We let \(\tau = T - t\), the time-to-maturity, where \(T\) is the maturity of the financial option under consideration and we introduce \(x = \log S_t\), the underlying asset’s log-price. If \(u(x, \tau)\) denotes the values of some (American and European) contingent claim on \(S_t\) when \(\log S_t = x\) and \(\tau = T - t\), then it is well-known, see for example, (Cont and Tankov, 2004) that \(u\) satisfies the following PIDE in the non-exercise region:

\[
\partial_\tau u(x, \tau) = \frac{1}{2} \sigma^2 \partial_x^2 u + \left( r - q - \frac{1}{2} \sigma^2 - \eta \right) \partial_x u - (r + \lambda) u + \lambda \int_{\mathbb{R}} u(x + y, \tau) f(y) dy,
\]

(9)

\(\mathcal{L}[u](x, \tau).\)
with initial value

\[ u(x,0) = g(x) := G(e^x) = \begin{cases} 
\max\{e^x - K, 0\}, & \text{call option} \\
\max\{K - e^x, 0\}, & \text{put option} 
\end{cases} \quad (10) \]

For an American put, we have to take into account the possibility of early exercise (e.g., Cont and Tankov [2004], Schoutens [2003, 2006]). As a result, the highest value of American option can be achieved by maximizing over all allowed exercise strategies:

\[ u(x,\tau) = \operatorname{ess} \sup_{\tau^* \in \Gamma(t,T)} E^Q_t \left[ e^{-r(\tau^* - t)} G(e^{x(\tau^*)}) \right] \quad (11) \]

where \( \Gamma(t,T) \) denotes the set of non-anticipating exercise times \( \tau^* \), satisfying \( t \leq \tau^* \leq T \).

To actually compute the \( u(x,\tau) \) of the American put, one can solve the following linear complementarity problem (Cont and Tankov [2004], Schoutens [2003, 2006]):

\[ \partial_{\tau} u(\tau, x) - Lu(x, \tau) \geq 0, \text{ in } (0,T) \times \mathbb{R} \quad (12) \]
\[ u(x, \tau) - G(e^x) \geq 0, \text{ a.e. in } (0,T) \times \mathbb{R} \quad (13) \]
\[ (u(x, \tau) - G(e^x)) \left( \partial_{\tau} u(\tau, x) - Lu(x, \tau) \right) = 0, \text{ in } (0,T) \times \mathbb{R} \quad (14) \]
\[ u(x, 0) = G(e^x), \quad (15) \]

Since we only deal with a jump-diffusion model with \( \sigma > 0 \) and finite jump intensity in this paper, we know that by Pham (Pham [1997]), the smooth pasting condition,

\[ \frac{\partial u(x_{\tau^*}, \tau^*)}{\partial x} = -1 \]

is valid at time of exercise \( \tau^* \). Therefore the value of an American put option is continuously differentiable with respect to the underlying on \( (0,T) \times \mathbb{R} \); in particular the derivative is continuous across the exercise boundary.

**Remark 2:** One should notice that if we set \( \lambda = 0 \), (9) will become original Black-Scholes PDE.

### 3. Meshfree Numerical Approximation Method

Meshfree radial basis function (RBF) interpolation is a well-known technique for reconstructing an unknown function from scattered data. It has numerous applications in different fields, such as terrain modeling in geology, surface reconstruction in imaging, and the numerical solution of partial differential equations in applied mathematics. In particular, RBFs have recently been used to solve the PDEs of quantitative finance. A number of authors, including Faussauer et al. (Faussauer et al., 2004a; b), Larsson et al. (Larsson et al., 2008), Pettersson et al. (Pettersson et al., 2008) and Hon and Mao (Hon and Mao, 1999), have suggested RBFs as a tool for solving Black-Scholes equations for European as well as American options. This numerical scheme for the estimation of partial derivatives using RBFs was originally proposed by Kansa (Kansa, 1990a), resulting in a new method for solving partial differential equations (Kansa, 1990b). The aim here is to obtain a RBF approximation of the initial value or pay-off of the option. Once we are disposition of such an RBF-interpolant, we implement an RBF-scheme to solve the PIDE with this RBF-interpolant as initial value. The general idea of the proposed numerical scheme is to approximate the unknown function \( u(x, \tau) \) by an RBF-interpolant using the interpolation points found for the initial value using the RBF-scheme, and derive a system of linear constant coefficient ODE by requiring that the PIDE be satisfied in the chosen RBF-interpolation points. After picking interpolation points
x_j \in \mathbb{R}, we approximate, for any fixed time-to-maturity \( \tau \), the solution \( u(x, \tau) \) in (9) by its RBF-interpolant:

\[
  u(x, \tau) \simeq \sum_{j=1}^{N} \rho_j(\tau) \phi(||x - x_j||_2) =: \tilde{u}(x, \tau), \tag{16}
\]

Since the radial basis function does not depend on time, the time derivative of \( \tilde{u}(x, \tau) \) in equation (9) is simply:

\[
  \frac{\partial \tilde{u}(x, \tau)}{\partial \tau} = \sum_{j=1}^{N} \frac{d\rho_j(\tau)}{d\tau} \phi(||x - x_j||), \tag{17}
\]

Moreover, the first and second partial derivatives of \( \tilde{u}(x, \tau) \) with respect to \( x \) are

\[
  \frac{\partial \tilde{u}(x, \tau)}{\partial x} = \sum_{j=1}^{N} \rho_j(\tau) \frac{\partial \phi(||x - x_j||)}{\partial x}, \tag{18}
\]

\[
  \frac{\partial^2 \tilde{u}(x, \tau)}{\partial x^2} = \sum_{j=1}^{N} \rho_j(\tau) \frac{\partial^2 \phi(||x - x_j||)}{\partial x^2}, \tag{19}
\]

where for the particular case when \( \phi \) is the Cubic Spline,

\[
  \frac{\partial \phi(||x - x_j||)}{\partial x} = \begin{cases} 
    3(||x - x_j||^2 & \text{if } x - x_j > 0, \\
    -3(||x - x_j||^2 & \text{if } x - x_j < 0,
  \end{cases} \tag{20}
\]

\[
  \frac{\partial^2 \phi(||x - x_j||)}{\partial x^2} = 6(||x - x_j||). \tag{21}
\]

In this research we choose the Cubic Spline rather than the most popular ones, MQ and IMQ as a basis function because of its simplicity and accuracy and without containing any shape parameters.

### 3.1 Transforming PIDE to A System of ODEs by RBF

Given a set of interpolation points \( x_1, \ldots, x_j, \ldots, x_N \) in \( \mathbb{R} \), and an RBF \( \phi \), we can construct \( N \times N \) matrices \( \mathbf{A}, \mathbf{A}_x \) and \( \mathbf{A}_{xx} \) defined by \( (\phi(||x_i - x_j||))_{1 \leq i,j \leq N} \), \( (\phi'(||x_i - x_j||))_{1 \leq i,j \leq N} \) and \( (\phi''(||x_i - x_j||))_{1 \leq i,j \leq N} \) respectively. Note in case the \( x_j \)'s are chosen according to the Equally Spacing Method, ESM, used in [Fausshauer et al. 2004a,b] [Hon and Mao, 1999]. In brief, Equally Spacing Method is the way to choose equally spaced points in a finite interval. In the ESM, we determine an interval \( [x_{\min}, x_{\max}] \) outside of which we can neglect the contribution of \( u(x, \tau) \) to the global integral term of a PIDE [9], and for given \( N = 0, 1, 2, \ldots \), simply put

\[
  x_j := x_j^{\Delta x} = x_{\min} + j\Delta x, \quad j = 0, 1, 2, \ldots, N - 1 \tag{22}
\]

where \( \Delta x = (x_{\max} - x_{\min})/(N - 1) \). We also define a matrix-valued function \( y \rightarrow \mathbf{A}(y) \) by \( (\phi(||x_i + y - x_j||))_{1 \leq i,j \leq N} \). If we substitute \( \tilde{u}(x, \tau) \) for \( u(x, \tau) \) in (9) and require the PIDE to be satisfied in the interpolation points \( x_j \), we arrive at the following system of ODEs for the
vector $\rho(\tau) := (\rho_1(\tau), \ldots, \rho_N(\tau))$

$$A\rho_\tau = \sigma^2/2 A_{xx}\rho + \left( r - q - \sigma^2/2 - \lambda \eta \right) A_x\rho + (r + \lambda) A\rho +$$

$$\lambda \left( \int_{-\infty}^{\infty} A(y)f(y) \, dy \right) \rho,$$

where $\rho_\tau := \frac{\partial \rho}{\partial \tau}$, and where we recall that $f(y)$ is the probability density of the jump $Y_i \sim N(\mu_J, \sigma^2_J)$ : $f(y) = (\sigma_J \sqrt{2\pi})^{-1} \exp \left( -\frac{(y - \mu_J)^2}{2\sigma^2_J} \right)$ in the Merton model, or $f(y) = p\alpha_1e^{-\alpha_1y}1_{x\geq0} + (1 - p)\alpha_2e^{\alpha_2y}1_{x<0}$ in the Kou model. Before applying a suitable numerical integration algorithm to the integral terms in (23), we truncate the integrals from an infinite computational range to a finite one. Briani et al. (Briani et al., 2007), Cont and Voltchkova (Cont and Voltchkova, 2005), Tankov and Voltchkova (Tankov and Voltchkova, 2009) and d’Halluin et al. (d’Halluin et al., 2004; 2005) have provided different numerical techniques to find out a finite computational range so as to reduce the numerical approximation errors when doing this truncation. In this thesis we shall adopt the Briani et al. numerical technique to truncate the integral domain of our PIDE (cf. [Briani et al., 2007]) in both the Merton and Kou model. See [A] for a proof. Supposed $\epsilon > 0$, a formula of selecting a bounded interval $[y-\epsilon, y_\epsilon]$ for the set of points $y$ in the Merton case is:

$$y_\epsilon = \sqrt{-2\sigma^2_J \log(\epsilon \sigma_J \sqrt{2\pi} / 2) + \mu_J}, \; \forall \; y \geq 0$$

$$y-\epsilon = -y_\epsilon, \; \forall \; y < 0.$$  

(24)  

(25)

In the Kou model we have

$$y_\epsilon = \log \left( \epsilon / p \right) / (1 - \alpha_1), \; \forall \; y \geq 0$$

$$y-\epsilon = -\log \left( \epsilon / (1 - p) \right) / (1 - \alpha_2), \; \forall \; y < 0.$$  

(26)  

(27)

We therefore transform equation (23) into

$$A\rho_\tau = \sigma^2/2 A_{xx}\rho + \left( r - q - \sigma^2/2 - \lambda \eta \right) A_x\rho + (r + \lambda) A\rho +$$

$$\lambda \left( \int_{y-\epsilon}^{y_\epsilon} A(y)f(y) \, dy \right) \rho,$$

(28)

We use matlab’s adaptive Gauss-Kronrod quadrature to evaluate the matrix of the integrals in (28): this amounts to approximating

$$\int_{y-\epsilon}^{y_\epsilon} \phi(|x_i + y - x_j|)f(y) \, dy \approx \sum_{k=1}^{m} w_k \phi(|x_i + y_k - x_j|)f(y_k),$$

(29)

where $w_k$ and $y_k$ are suitable quadrature weights and quadrature points; cf. (Shampine, 2008) for details. To simplify notations, we set

$$F(x_i - x_j) = \sum_{k=1}^{m} w_k \phi(|x_i + y_k - x_j|)f(y_k).$$


Then the integrals in equation (28) will be approximated by
\[
\int_{y_{m-1}}^{y} A(y) f(y) \, dy \approx \begin{bmatrix}
F(x_1 - x_1) & F(x_1 - x_2) & \ldots & F(x_1 - x_N) \\
F(x_2 - x_1) & F(x_2 - x_2) & \ldots & F(x_2 - x_N) \\
\vdots & \vdots & \ddots & \vdots \\
F(x_N - x_1) & F(x_N - x_2) & \ldots & F(x_N - x_N)
\end{bmatrix}
\]
\[
= C(y).
\] (30)

Substituting (30) into equation (28), we arrive at the new approximate equation:
\[
A \rho_r = \frac{\sigma^2}{2} A_{xx} \rho + \left(r - q - \frac{\sigma^2}{2} - \lambda \eta \right) A_{x} \rho + \left(r + \lambda\right) A \rho + \lambda C(y) \rho.
\] (31)

As we have known the Cubic Spline is strictly conditionally positive definite function of order 2, the invertibility of \( A \) is not assumed without adding a real-valued polynomial of degree at most 1 in (16) (cf. Wendland, 2005). Nevertheless, Bos and Salkauskas proved that \( A \) is non-singular in a univariate case (cf. Bos and Salkauskas, 1987, Theorem 5.1). As a result, the invertibility of \( A \) is still guaranteed.

Although the invertibility of \( A \) is able to be shown for all \( \phi \) of the interest, the inverse of \( A \), \( A^{-1} \), may often be very ill-conditioned to solve when its size increases (cf. Fasshauer, 2007, chapter 16). As a result, it may be impossible to solve accurately using standard floating point arithmetic. To address this problem, we factorise \( A \) into the following form (cf. Bos and Salkauskas, 1987, Theorem 3.7):
\[
A = FCF.
\] (32)

Here \( F \) is a \( N \times N \) matrix,
\[
\begin{bmatrix}
|x_1 - x_1| & |x_1 - x_2| & |x_1 - x_3| & \ldots & |x_1 - x_N| \\
|x_2 - x_1| & |x_2 - x_2| & |x_2 - x_3| & \ldots & |x_2 - x_N| \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
|x_N - x_1| & |x_N - x_2| & |x_N - x_3| & \ldots & |x_N - x_N|
\end{bmatrix},
\] (33)

and \( C \) is a near tridiagonal \( N \times N \) matrix,
\[
\begin{bmatrix}
\frac{h - S}{2} & \frac{h}{2} & 0 & \cdots & 0 & \frac{S}{2} \\
\frac{h}{2} & \frac{2h}{2} & \frac{h}{2} & 0 & \cdots & 0 \\
0 & \frac{h}{2} & \frac{2h}{2} & \frac{h}{2} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{S}{2} & 0 & \cdots & \frac{h}{2} & \frac{2h}{2} & \frac{h}{2} \\
0 & \frac{1}{2} & \frac{2h}{2} & \frac{1}{2} & \frac{h}{2} & \frac{h - S}{2}
\end{bmatrix},
\] (34)

where \( h \) is the distance between \( x_{i+1} \) and \( x_i \) for \( 1 \leq i \leq N - 1 \) and \( S = Nh \). We also have an explicit form of \( F^{-1} \) (cf. Bos and Salkauskas, 1987, Lemma 3.6) which is equal to
\[
\begin{bmatrix}
\frac{h - S}{2} & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\
\frac{2h}{2} & \frac{-h}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{-h}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{-h}{2} & \frac{1}{2} & \frac{1}{2} & \frac{h - S}{2} \\
\frac{1}{2} & \frac{2h}{2} & \frac{1}{2} & \frac{1}{2} & \frac{h}{2} & \frac{h - S}{2}
\end{bmatrix},
\] (35)

We perform Gaussian elimination with partial pivoting to calculate \( C^{-1} \). Then, we multiply both sides of (31) by \( C^{-1} \) and \( F^{-1} \) and we finally obtain the following homogeneous system
of ODEs with constant coefficients:

\[
\rho = F^{-1}C^{-1}F^{-1}\left(\frac{\sigma^2}{2}A_{xx} + (r - q - \frac{\sigma^2}{2} - \lambda \eta)A_{x} + (r + \lambda)A + \lambda C(y)\right)\rho
\]

\[\equiv \Theta \rho\]  

(36)

where \( \Theta \) is defined by the left hand side. After some numerical experimentation, we found that the matrix \( \Theta \) is very stiff. To explain why \( \Theta \) is stiff, we shall use the following example to illustrate it. Suppose we select our maximum and minimum logarithm price \( x_{\min}(\log(S_{\min})) \) and \( x_{\max}(\log(S_{\max})) \) in (22) equal to \(-10\) and \(10\) respectively, then we use (22) to generate a list of 100 interpolation points. Based on the procedures and the ideas we have mentioned above we can get a \(100 \times 100\) matrix \( \Theta \) in (36). Then we measure the stiffness ratio of \( \Theta \). The stiffness ratio is the quotient of the largest and the smallest eigenvalues of the Jacobian matrix \( \Theta \). The ratio we have is \(1.2864 \times 10^5\). This implies that (36) is a stiff ODE and therefore we have to solve the ODEs by an implicit method, e.g. backward differentiation formulas (BDFs), a modified Rosenbrock formula of order 2, the trapezoidal rule or TR-BDF2, an implicit Runge-Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two. In this paper we use former one.

4. Numerical Results

4.1 European Vanilla Options

In this section we first present a simple scheme to construct our computational range. We then present the numerical results of our Cubic Spline approximation scheme and compare these with Black-Scholes, Merton and Kou’s analytical option price formula for both puts and calls. Beside this, we also compare the results of our Cubic Spline approximation scheme with those of the Briani et al. finite difference method (FD) with implicit and explicit (IMEX) scheme in Briani et al. (2007) and the Almendral et al. finite element method (FE) with backward differentiation formulas of order two (BDF2) and FD with BDF2 in Almendral and Oosterlee (2005).

We use EMS (22) to choose our interpolation points. Based on this set of interpolation points, we can construct our computational range. We distribute the interpolation point uniformly around the logarithm strike price, \(\log K\), in order to achieve a higher accuracy of pricing European Vanilla Option. Our scheme of distributing the interpolation point is shown in Figure 1. The idea can be explained as follows: We set the range of \([x_{\min}, x_{\max}]\) and then use EMS to create \(N\) interpolation points. We distribute the first \(N/2\) points uniformly in \([x_{\min}, \log(K)]\) and then the rest in \([\log(K), x_{\max}]\).

Figure 1. Uniform distributions of the interpolation points around the strike price by using EMS. The red dots are the interpolation points. The blue cross is the location of the logarithm strike price.

In option trading the region of most interest is when the mean of the stock prices is close to the strike price. Typically, the probability for a stock to default or to be very far from the
strike price is small. Therefore we define the region of interest as follows:

\[ \hat{x}_i \in [\hat{x}_{\min}, \hat{x}_{\max}] := [\log(K/20), \log(2K)]. \] (37)

Based on this region, we can measure the accuracy of our RBF-approximation. We use a set of evaluation points \( \hat{x}_i \), for which we will simply take the grid points

\[ \hat{x}_i := \hat{x}_{\Delta x} = \hat{x}_{\min} + j \Delta \hat{x}, \ j = 0, 1, 2, \ldots, N_{\text{eval}} - 1. \] (38)

Here \( \Delta \hat{x} = (\hat{x}_{\max} - \hat{x}_{\min})/(N_{\text{eval}} - 1) \) with \( x_{\min} \leq \hat{x}_{\min} \leq \hat{x}_{\max} \leq x_{\max} \) and \( N_{\text{eval}} \) is the number of the evaluation points chosen.

It is also of great interest to measure the rate of convergence of our Cubic Spline-approximation scheme. By defining \( \Delta t = 1/M_0 \), where \( M_0 \) is the number of time steps and \( \Delta x = 1/N \), where \( N \) is the number of interpolation points, we assume that

\[ E_{\infty}(\hat{x}_i, T) = C_t(\Delta t)^{R_t} + C_x(\Delta x)^{R_x} \] (39)

for the max error and

\[ E_2(\hat{x}_i, T) = C_t(\Delta t)^{R_t} + C_x(\Delta x)^{R_x} \] (40)

for the root-mean-square (rms) error. Here \( E_{\infty}(\hat{x}_i, T) \) is the max error, \( E_2(\hat{x}_i, T) \) is the rms error, \( \hat{x}_i \) is \( i^{th} \) evaluation point, \( T \) is the maturity time, both \( C_t \) and \( C_x \) are constants, \( R_t \) is the rates of convergence in time and \( R_{\infty} \) and \( R_2 \) are the rates of convergence in space. The formulae of calculating the max error and the rms errors are:

\[ E_{\infty} = \max_{0 \leq i \leq N_{\text{eval}}} |V(e^{\hat{x}_i}, \tau) - \tilde{u}(\hat{x}_i, \tau)|, \] (41)

and

\[ E_2 = \sqrt{\frac{1}{N_{\text{eval}}} \sum_{0 \leq i \leq N_{\text{eval}}} \left| V(e^{\hat{x}_i}, \tau) - \tilde{u}(\hat{x}_i, \tau) \right|^2} \] (42)

respectively. Here \( V(e^{\hat{x}}, \tau) \) and \( \tilde{u}(\hat{x}, \tau) \) are the exact value and approximate value at the point \( (\hat{x}, \tau) \) respectively.

Since we compare the accuracy of our Cubic Spline-approximation scheme with that of FDM and FEM, we use the relative error,

\[ E_{\text{rel}}(\hat{x}, \tau) = \frac{|V(e^{\hat{x}}, \tau) - \tilde{u}(\hat{x}, \tau)|}{V(e^{\hat{x}}, \tau)}, \] (43)

as the measure of the accuracy.

It is known (Merton, 1976) that the analytical price of a European call/put option in the Merton Jump-diffusion model is given by

\[ V_{\text{MJ}}(S_t, \tau, K, r, q, \sigma) = \sum_{k=0}^{\infty} e^{-\lambda(1+\eta)\tau}\frac{((\lambda(1+\eta)\tau)^k}{k!} V_{\text{BS}}(S_t, \tau, K, r_k, \sigma_k, q) \] (44)

where \( \tau = T - t \) is the time to maturity, \( \eta = e^{\mu J} + \frac{\sigma_J^2}{2} - 1 \) represents the expected percentage change in the stock price originating from a jump, \( \sigma_k^2 = \sigma^2 + \frac{k q J}{T} \) the observed volatility, \( r_k = r - \lambda \eta + k \log(1+\eta)/(T-t) \), \( q \) is the dividend and \( V_{\text{BS}} \) the Black-Scholes price of a call.
and put, computed as
\[ V_{BS}(S_t, \tau, K, r_k, \sigma_k, q) = \begin{cases} S_t e^{-q \tau} \Phi(d_{+k}) - Ke^{-r \tau} \Phi(d_{-k}) & \text{call option,} \\ Ke^{-r \tau} \Phi(-d_{-k}) - S_t e^{-q \tau} \Phi(-d_{+k}) & \text{put option,} \end{cases} \]
where \( \Phi(\cdot) \) is the cumulative normal distribution and
\[ d_{+k} = \frac{\log(S_t/K) + (r_k - q + \sigma_k^2/2)\tau}{\sigma_k \sqrt{\tau}}, \quad d_{-k} = d_{+k} - \sigma_k \sqrt{\tau}. \]

For the derivation of \( V_{MJ}(S_t, \tau, K, r, q, \sigma) \), we shall refer to the reader to (Merton 1976; Cont and Tankov 2004).

In general, for models where the characteristic function of the Lévy process is known, an analytical solution of PIDE (9) may be found using Fourier analysis (Carr and Madan 1999; Lewis 2001). For the sake of simplicity and accuracy we propose Jackson et al.’s Fourier Space Time-Stepping method rather than Carr-Madan’s Fast Fourier Transform (FFT) method (Carr and Madan 1999) and Lewis’s FFT method (Lewis 2001). In brief, the idea of this method is based on the Fourier transform of the PIDE. By making use of FFT and inverse Fast Fourier transform (FFT\(^{-1}\)), European Option price can be determined. The pricing formula of evaluating European option can be expressed as follows:
\[ V_{Kou}(S, \tau, K, r, \sigma, q) = FFT^{-1}[FFT[V_{Kou}(S, T) e^{\psi \tau}]], \quad \text{(45)} \]
where \( \psi(z) \) is the characteristic function of the Kou model which can be defined as:
\[ -\frac{\sigma^2 z^2}{2} + iz \gamma_c + \lambda \left( \frac{p \alpha_1}{\alpha_1 - iz} + \frac{(1 - p) \alpha_2}{\alpha_2 + iz} - 1 \right), \]
and \( V_{Kou}(S, T) \) is the payoff function (10). For more details of this method, we shall refer the reader to (Jackson et al. 2008). This method has been reported to have second order convergence in space in European cases.

Our RBF-algorithm for numerically solving (9) with initial condition (10) runs as follows:

1. Find the RBF-approximation to the initial value \( u(x, 0) \) using ESM (see 22). This will provide us with a set of interpolation points \( x_1, \ldots, x_n \), together with an initial vector \( \rho(0) = (\rho_1(0), \ldots, \rho_N(0)) \).
2. Then use \( \rho(0) \) as initial value for the system (36). By using any stiff ODE solver, we find out the \( \rho(T) \) at time \( T \).
3. Finally, substitute \( \rho(T) \) back into \( \sum_{j=1}^{N} \rho_j(T) \phi(|x - x_j|) \) to get an approximate value of \( u(x, T) \).

In our numerical experiment we implement the algorithm in MATLAB R2007b. We select our maximum and minimum logarithm price \( x_{\text{min}} \left( \log(S_{\text{min}}) \right) \) and \( x_{\text{max}} \left( \log(S_{\text{max}}) \right) \), as before, equal to \(-10\) and \(10\) respectively. Because of achieving more accurate approximation of the integral in (28), we also set \( \epsilon \) in both (24) and (26) to be \( 3.72 \times 10^{-40} \) for finding a finite computational interval \([y_{-\epsilon}, y_{+\epsilon}]\). Moreover, we use function \texttt{quadgk} which implements adaptive Gauss-Kronrod quadrature for computing equation (29) as well as function \texttt{ode15s} which implements backward differentiation formulas (BDFs) of order two for the calculation of equation (36). The main reason of choosing it is the following: According to (Iserles 2009) BDFs of orders 1 and 2 are A-stable (the stability region includes the entire left half complex plane). Since (36) is stiff, according to Theorem 4.11 (The Dahlquist second barrier) of (Iserles 2009), the highest order of an A stable multistep method\(^1\), such as BDFs, is only two. We therefore

\(^1\) Multistep methods are used for the numerical solution of ordinary differential equations. Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution.
conclude that our solution is second order convergence in time. By setting $R_t = 2$, (39) and (40) become

$$E_\infty(\hat{x}_i, T) = C_t(\Delta t)^2 + C_x(\Delta x)^R_\infty$$

$$E_2(\hat{x}_i, T) = C_t(\Delta t)^2 + C_x(\Delta x)^R_2$$

for the European option. This conclusion is in line with the finding of (Pettersson et al., 2008). In (Pettersson et al., 2008) Pettersson et al. show that second order in time can be achieved in a European case due to the second order time-stepping scheme, BDFs of order 2. Although they solve Black Schole PDE rather than PIDE in their paper, an similar approach of solving European option like our approximation scheme is applied. The rest of this section, we numerically show that $R_\infty$ and $R_2$ are equal to 2. Besides this, we will numerically approximate $\Delta$ and $\Gamma$ and launch a comparison between our approximation scheme and FDM and FEM.

All the parameters of all the tables except Table 3, 6 and 9 are chosen from different literature. The parameter $\sigma = 1$ in Table 3, 6 and 9 is selected to stress our numerical algorithm. From Table 1 to 9 $E_\infty$ and $E_2$ falls down when the number of the interpolation points N increases. Our Cubic Spline approximation scheme can get second order convergence in space. This is due to the limited smoothness of the Cubic Spline which has second order of convergence (cf. (Wendland, 2005)). In Figure 2, 3 and 4, oscillations do not occur around the strike $K$ for small $T$ when we approximate $\Delta$ and $\Gamma$. In Table 10 we compare the results of the FD used in Briani et al.’s paper (Briani et al., 2007) with those using our Cubic Spline approximation scheme. Our numerical approximation scheme can achieve lower $E_{rel}(\log S, T)$ than ARS-233 scheme and Explicit scheme. Table 10 and 12 are other comparisons of the accuracy between our Cubic Spline approximation scheme and Almendral and Oosterlee’s FD and FE with BDF2. To illustrate a fair comparison, we set our maximum and minimum logarithm price $x_{\min}$ and $x_{\max}$ same as Almendral and Oosterlee proposed in their numerical experiments. Hence we set $[x_{\min}, x_{\max}]$ equal to [-4, 4] and [-6, 6] in the Merton model (Table 11) and the Kou model (Table 12) respectively. Our Cubic Spline approximation scheme can attain lower $E_{rel}(\log S, T)$ than FD and FE with BDF2 in both the Merton and Kou cases.

| N   | $E_\infty(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$ |
|-----|--------------------------|-----------|----------------------|------|
| 100 | 4.207101E-03             | N/A       | 1.864736E-03         | N/A  |
| 600 | 1.195088E-04             | 1.988     | 5.143665E-05         | 2.004|
| 1100| 3.554622E-05             | 2.000     | 1.528321E-05         | 2.002|
| 1600| 1.679290E-05             | 2.001     | 7.219811E-06         | 2.001|
| 2100| 9.745141E-06             | 2.001     | 4.189909E-06         | 2.001|
| 2600| 6.354765E-06             | 2.002     | 2.732818E-06         | 2.001|
| 3100| 4.468110E-06             | 2.003     | 1.921950E-06         | 2.001|
| 3600| 3.311319E-06             | 2.004     | 1.424931E-06         | 2.001|

Table 1. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of a European put under the Black-Scholes model are presented. $N$ is the number of the interpolation points, $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.04$, $q = 0$, $\sigma = 0.29$, $K = 1$ and $T = 1$. The parameters are taken from (Giles and Carter, 2006). The order of convergence is 2 in space.
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Table 2. $E_{\infty}$ and $E_2$ of the Cubic Spline approximation for pricing of a European call under the Black-Scholes model are presented. $N$ is the number of the interpolation points. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0$, $\sigma = 0.2$, $K = 1$ and $T = 2$. The parameters are taken from Shaw (2009). The order of convergence is 2 in space.

| N   | $E_{\infty}(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$  |
|-----|---------------------------|------------|---------------------|--------|
| 100 | 1.924131E-02             | N/A        | 4.690135E-03        | N/A    |
| 600 | 7.143939E-04             | 1.838      | 1.296858E-04        | 2.003  |
| 1100| 2.171519E-04             | 1.965      | 3.870772E-05        | 1.995  |
| 1600| 1.031950E-04             | 1.986      | 1.830673E-05        | 1.998  |
| 2100| 6.002721E-05             | 1.992      | 1.063525E-05        | 1.998  |
| 2600| 3.919766E-05             | 1.995      | 6.934013E-06        | 2.002  |
| 3100| 2.758717E-05             | 1.997      | 4.877540E-06        | 2.000  |
| 3600| 2.406213E-05             | 1.998      | 3.616699E-06        | 2.000  |

Table 3. $E_{\infty}$ and $E_2$ of the Cubic Spline approximation for pricing of a European call under the Black-Scholes model are presented. $N$ is the number of the interpolation points. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.3$, $q = 0.1$, $\sigma = 1$, $K = 1$ and $T = 0.25$. The parameters are taken from Andersen and Andreasen (2000). The order of convergence is 2 in space.

| N   | $E_{\infty}(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$  |
|-----|---------------------------|------------|---------------------|--------|
| 100 | 2.325676E-03             | 0.000      | 1.404611E-03        | N/A    |
| 600 | 6.473617E-05             | 1.999      | 3.856043E-05        | 2.007  |
| 1100| 1.923322E-05             | 2.002      | 1.145625E-05        | 2.002  |
| 1600| 9.079437E-06             | 2.003      | 5.411921E-06        | 2.001  |
| 2100| 5.265272E-06             | 2.004      | 3.140776E-06        | 2.001  |
| 2600| 3.430306E-06             | 2.006      | 2.048580E-06        | 2.001  |
| 3100| 2.406208E-06             | 2.016      | 1.441039E-06        | 2.000  |
| 3600| 1.784242E-06             | 2.007      | 1.068202E-06        | 2.002  |

Table 4. $E_{\infty}$ and $E_2$ of the Cubic Spline approximation for pricing of a European call under the Merton Jump-diffusion model are presented. $N$ is the number of the interpolation points. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0$, $\sigma = 0.2$, $\sigma_J = 0.45$, $\mu_J = -0.9$, $\lambda = 0.1$, $K = 1$ and $T = 0.25$. The parameters are taken from Andersen and Andreasen (2000). The order of convergence is 2 in space.

| N   | $E_{\infty}(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$  |
|-----|---------------------------|------------|---------------------|--------|
| 100 | 1.428497E-02             | N/A        | 3.749983E-03        | N/A    |
| 600 | 4.642130E-04             | 1.912      | 1.011341E-04        | 2.016  |
| 1100| 1.402519E-04             | 1.975      | 3.011378E-05        | 1.999  |
| 1600| 6.640377E-05             | 1.995      | 1.423346E-05        | 2.000  |
| 2100| 3.860331E-05             | 1.995      | 8.262241E-06        | 2.000  |
| 2600| 2.518672E-05             | 1.999      | 5.389115E-06        | 2.001  |
| 3100| 1.772559E-05             | 1.997      | 3.790660E-06        | 2.000  |
| 3600| 1.314288E-05             | 2.000      | 2.810697E-06        | 2.000  |
Table 5. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of a European put under the Merton Jump-diffusion model are presented. $N$ is the number of the interpolation points. $\tilde{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0.02$, $\sigma = 0.15$, $\sigma_J = 0.4$, $\mu_J = -1.08$, $\lambda = 0.1$, $K = 1$ and $T = 0.1$. The parameters are taken from (Andersen and Andreasen, 2000). The order of convergence is 2 in space.

| N   | $E_\infty(\tilde{x}_i, T)$ | $E_2(\tilde{x}_i, T)$ | $R_\infty$ | $R_2$  |
|-----|--------------------------|--------------------------|-------------|--------|
| 100 | 1.956920E-02             | N/A                      | 4.723349E-03 | N/A   |
| 600 | 7.326011E-04             | 1.833                    | 1.305576E-04 | 2.003 |
| 1100| 2.40092E-04              | 1.955                    | 3.898655E-05 | 1.994 |
| 1600| 1.069094E-04             | 1.974                    | 1.844062E-05 | 1.998 |
| 2100| 6.223777E-05             | 1.990                    | 1.071235E-05 | 1.997 |
| 2600| 4.062560E-05             | 1.997                    | 6.985440E-06 | 2.002 |
| 3100| 2.859186E-05             | 1.997                    | 4.913762E-06 | 2.000 |
| 3600| 2.121748E-05             | 1.995                    | 3.643595E-06 | 2.000 |

Table 6. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of a European call under the Merton Jump-diffusion model are presented. $N$ is the number of the interpolation points. $\tilde{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0.01$, $\sigma = 0.2$, $\sigma_J = 0.6$, $\mu_J = -1.08$, $\lambda = 0.1$, $K = 1$ and $T = 1$, whereas the parameter $\sigma = 1$ is selected to stress our numerical algorithm. The order of convergence is 2 in space.

| N   | $E_\infty(\tilde{x}_i, T)$ | $E_2(\tilde{x}_i, T)$ | $R_\infty$ | $R_2$  |
|-----|--------------------------|--------------------------|-------------|--------|
| 100 | 1.026524E-03             | 3.422908E-03             | N/A         | N/A   |
| 600 | 2.819557E-05             | 1.945356E-05             | 2.006       | 2.007 |
| 1100| 8.415823E-06             | 5.762520E-06             | 2.007       | 2.011 |
| 1600| 3.999351E-06             | 2.712396E-06             | 2.011       | 2.059 |
| 2100| 2.463272E-06             | 1.559774E-06             | 2.035       | 2.038 |
| 2600| 1.301472E-06             | 1.004746E-06             | 2.059       | 2.000 |
| 3100| 1.136188E-06             | 7.021072E-07             | 2.038       | 2.000 |
| 3600| 8.358248E-07             | 5.221973E-07             | 1.980       | 2.000 |

Table 7. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of a European put under the Kou Jump-diffusion model are presented. $N$ is the number of the interpolation points. $\tilde{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0$, $q = 0$, $\sigma = 0.2$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\lambda = 0.2$, $p = 0.5$, $K = 1$ and $T = 0.2$. The parameters are taken from (Almendral and Oosterlee, 2005). The order of convergence is 2 in space.

| N   | $E_\infty(\tilde{x}_i, T)$ | $E_2(\tilde{x}_i, T)$ | $R_\infty$ | $R_2$  |
|-----|--------------------------|--------------------------|-------------|--------|
| 100 | 1.239165E-02             | 3.422908E-03             | N/A         | N/A   |
| 600 | 3.932126E-04             | 9.440247E-05             | 2.004       | 2.000 |
| 1100| 1.179555E-04             | 2.808850E-05             | 2.000       | 2.000 |
| 1600| 5.589111E-05             | 1.327392E-05             | 2.000       | 2.000 |
| 2100| 3.246588E-05             | 7.705266E-06             | 2.000       | 2.000 |
| 2600| 2.118103E-05             | 5.025765E-06             | 2.000       | 2.000 |
| 3100| 1.490021E-05             | 3.535171E-06             | 2.000       | 2.000 |
| 3600| 1.105067E-05             | 2.621737E-06             | 2.000       | 2.000 |
Table 8. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of a European call under the Kou Jump-diffusion model are presented. $N$ is the number of the interpolation points. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0$, $\sigma = 0.15$, $\alpha_1 = 3.0465$, $\alpha_2 = 3.0465$, $\lambda = 0.1$, $p = 0.3445$, $K = 1$ and $T = 0.25$. The parameters are taken from (Carr and Mayo, 2007). The order of convergence is 2 in space.

| $N$  | $E_\infty(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$ |
|------|--------------------------|-------------|----------------------|-------|
| 100  | 1.433875E-02             | N/A         | 3.766745E-03         | N/A   |
| 600  | 4.665677E-04             | 1.912       | 1.022079E-04         | 2.013 |
| 1100 | 1.404381E-04             | 1.981       | 3.043034E-05         | 1.999 |
| 1600 | 6.660275E-05             | 1.991       | 1.438190E-05         | 2.000 |
| 2100 | 3.868283E-05             | 1.998       | 8.348098E-06         | 2.000 |
| 2600 | 2.522395E-05             | 2.002       | 5.444331E-06         | 2.001 |
| 3100 | 1.773247E-05             | 2.003       | 3.828943E-06         | 2.001 |
| 3600 | 1.314079E-05             | 2.004       | 2.838628E-06         | 2.001 |

Table 9. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of a European put under the Kou Jump-diffusion model are presented. $N$ is the number of the interpolation points. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.04$, $q = 0.03$, $\sigma = 1$, $\alpha_1 = 4$, $\alpha_2 = 4$, $\lambda = 0.3$, $p = 0.6$ $K = 1$ and $T = 1$, whereas the parameter $\sigma = 1$ is selected to stress our numerical algorithm. The order of convergence is 2 in space.

| $N$  | $E_\infty(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$ |
|------|--------------------------|-------------|----------------------|-------|
| 100  | 1.080306E-03             | N/A         | 7.074108E-04         | N/A   |
| 600  | 2.973137E-05             | 2.005       | 1.940773E-05         | 2.007 |
| 1100 | 8.870629E-06             | 1.995       | 5.757611E-06         | 2.005 |
| 1600 | 4.229400E-06             | 1.977       | 2.712641E-06         | 2.009 |
| 2100 | 2.490583E-06             | 1.947       | 1.567674E-06         | 2.016 |
| 2600 | 1.674611E-06             | 1.859       | 1.014582E-06         | 2.037 |
| 3100 | 1.191565E-06             | 1.935       | 7.096338E-07         | 2.032 |
| 3600 | 9.018770E-07             | 1.863       | 5.232205E-07         | 2.038 |

Figure 2. Put option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Black-Scholes Model. The number of the interpolation points is 3600. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 3.
Figure 3. Call Option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Merton Jump-diffusion Model. The number of the interpolation points is 3600. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 5.

Figure 4. Put Option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Kou Jump-diffusion Model. The number of the interpolation points is 3600. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 8.
Explicit scheme | ARS-233 Scheme
---|---
| \( N \) | \( Value \) | \( E_{rel}(\log S, T) \) | \( Value \) | \( E_{rel}(\log S, T) \) |
| Call | 1024 | 13.286915 | 5.175624E-03 | 13.287427 | 5.214358E-03 |
| Put | 1024 | 8.319940 | 2.57797E-03 | 8.326102 | 1.839249E-03 |

Cubic Spline | N/A
---|---
| \( N \) | \( Value \) | \( E_{rel}(\log S, T) \) | \( Value \) | \( E_{rel}(\log S, T) \) |
| Call | 1024 | 13.219358 | N/A | N/A |
| Put | 1024 | 8.342301 | N/A | N/A |

Table 10. Comparison between Explicit scheme [Briani et al. (2007)], ARS-233 Scheme [Briani et al. (2007)] and Cubic Spline interpolation scheme in evaluating of European call/put under the Merton Jump-diffusion Model. The input parameters are: \( r = 0.05 \), \( q = 0 \), \( \sigma = 0.2 \), \( \sigma_J = 0.8 \), \( \mu = 0 \), \( \lambda = 0.1 \), \( K = 100 \), \( T = 1 \), and \( x = \log 100 \). Reference prices of 13.218501 (call) and 8.341444 (put) and parameters from [Briani et al. (2007)].

FD with BDF2 | FE with BDF2
---|---
| \( N \) | \( Value \) | \( E_{rel}(\log S, T) \) | \( Value \) | \( E_{rel}(\log S, T) \) |
| 1025 | 9.411968E-02 | 1.682457E-04 | 9.412972E-02 | 6.165536E-05 |

Cubic Spline | N/A
---|---
| \( N \) | \( Value \) | \( E_{rel}(\log S, T) \) | \( Value \) | \( E_{rel}(\log S, T) \) |
| 1025 | 9.413023E-02 | N/A | N/A |

Table 11. Comparison of FD with BDF2 [Almendral and Oosterlee (2005)], FE with BDF2 [Almendral and Oosterlee (2005)] and Cubic Spline interpolation scheme in evaluating of a European call (put) under the Merton Jump-diffusion Model. The input parameters are: \( r = 0 \), \( q = 0 \), \( \sigma = 0.2 \), \( \sigma_J = 0.5 \), \( \mu = 0 \), \( \lambda = 0.1 \), \( K = 1 \), \( T = 1 \), and \( S = 1 \). Reference prices of 0.094135525 for both call and put and parameters from [Almendral and Oosterlee (2005)].

FD with BDF2 | FE with BDF2
---|---
| \( N \) | \( Value \) | \( E_{rel}(\log S, T) \) | \( Value \) | \( E_{rel}(\log S, T) \) |
| 513 | 4.240E-02 | 6.346096E-03 | 4.24579E-02 | 5.128562E-03 |

Cubic Spline | N/A
---|---
| \( N \) | \( Value \) | \( E_{rel}(\log S, T) \) | \( Value \) | \( E_{rel}(\log S, T) \) |
| 513 | 4.254583E-02 | N/A | N/A |

Table 12. Comparison of FD with BDF2 [Almendral and Oosterlee (2005)], FE with BDF2 [Almendral and Oosterlee (2005)] and Cubic Spline interpolation scheme in evaluating of a European call (put) under the Kou Jump-diffusion Model. The input parameters are: \( r = 0 \), \( q = 0 \), \( \sigma = 0.2 \), \( \alpha_1 = 3 \), \( \alpha_2 = 2 \), \( \lambda = 0.2 \), \( \rho = 0.5 \), \( K = 1 \), \( T = 0.2 \), and \( S = 1 \). Reference prices of 0.0426761 for both call and put and parameters from [Almendral and Oosterlee (2005)].
4.2 American Vanilla Put Options

In this section we adapt an RBF-algorithm to compute American put-option prices. We then compare the option prices obtained from our RBF-algorithm with the Jackson et al. FST methods of [Jackson et al., 2008]. As mentioned in Section 2, an American put option problem is a free boundary problem because of the possibility of early exercise at any point during its life, leading to the free boundary condition:

\[ u(x, \tau) = \max (K - e^x, u(x, \tau)) \]

Together with the smooth pasting condition mentioned in section 2, this uniquely determines the exercise boundary.

The Jackson et al. FST methods suggest that their solutions can achieve second order in space when they implement their methods to price American put options. They implement their methods in the context of the LCP. As we have seen in Section 2, the value of an American option \( u(\tau, x) \) is always greater than or equal to the payoff function \( G(e^x) \). To numerically keep the condition \( u(\tau, x) - G(e^x) \geq 0 \) to be continuously held (see Section 2), this can be achieved when boundary conditions are applied. The numerical algorithm for this idea can be defined as follows:

\[
V(S, (m + 1)\Delta t, K, r, \sigma, q) = \max\{\text{FFT}^{-1}[\text{FFT} [V(S, m\Delta t, K, r, \sigma, q)] e^{\psi \Delta t}], G(e^x)\}
\]

where time interval \( \Delta t \) is obtained by dividing time-to-maturity \( T \) by the total number \( M \), \( m\Delta \) is the time-step, where \( m \in \{0, 1, 2, \ldots, M - 1\} \), \( \psi(z) \) is the characteristic function of the Merton/Kou models, \( V(S, (m + 1)\Delta t, K, r, \sigma, q) \) is the American put price at time \((m + 1)\Delta t\) and the payoff condition \( G(e^x) \) is equal to \( \max(K - e^x, 0) \). These methods also are required to swap between real and Fourier spaces at each time-step when the American option prices are calculated at each time interval. This is due to no convenient representation of the \( \max(\ldots) \) operator in Fourier space. For the full schematic and numerical description of this method, we refer readers to [Jackson et al., 2008].

As before, we use ESM to approximate \( u(x, 0) = \max(K - e^x, 0) \) and then continue to work with the interpolation points found at \( \tau = 0 \). The algorithm now reads as follows:

1. Divide time-to-maturity \( T \) by total numbers of time-steps \( M \) to obtain time interval \( \Delta t \) and create a list of equally spaced time-points \( m\Delta t, m \in \{0, 1, 2, \ldots, M - 1\} \).
2. Find the RBF-approximation to the initial value \( u(x, 0) \) using ESM. This will provide us with a set of interpolation points \( x_1, \ldots, x_n \), together with an initial vector \( \rho(0) = (\rho_1(0), \ldots, \rho_N(0)) \).
3. Assume we have already determined \( \rho(m\Delta t) \) (if \( m = 0 \), we have \( \rho(0) \)) in equation (36). Solve the system of (stiff) ODEs to find \( \rho((m + 1)\Delta t) \) at the next successive time-step, \((m + 1)\Delta t\).
4. Then at time \((m + 1)\Delta t\), for each interpolation point \( x_i \), define

\[
u(x_i, (m + 1)\Delta t) = \max((K - e^{x_i}), \sum_{j=1}^{N} \rho_j((m + 1)\Delta t) \phi(|x_i - x_j|)).\]

5. Find a new vector \( \rho((m+1)\Delta t) \) such that \( u(x_i, (m+1)\Delta t) = \sum_{j=1}^{N} \rho_j((m+1)\Delta t) \phi(|x_i - x_j|) \) for all \( i \).
6. Repeat Step 3.) to 5.) until \( m = M - 1 \).
7. Finally, substitute \( \rho(T) \) back into \( \sum_{j=1}^{N} \rho_j(T) \phi(|x - x_j|) \) to get an approximate value of \( u(x, T) \).

The settings of our numerical experiment are the same as those in section 4.1. We also
calculate the rate of convergence in time. If we hold $\Delta x$ constant, (39) and (40) become
\[
E_\infty(\hat{x}_i, T) = C_t(\Delta t)^R_t
\]
and
\[
E_2(\hat{x}_i, T) = C_t(\Delta t)^R_t
\]
respectively.

The results from Table 13 to 18 suggest that our Cubic Spline approximation method for pricing of American put options is second order in spatial variables and first order in time variables when the number of interpolation numbers $N$ and the number of time-steps $M_0$ are twofold and fourfold respectively. In table 19 and 20 we implement our own BDF2 with fixed time steps rather than using ode15s with variable time steps. From these two tables, we can achieve first order in time variables when the number of interpolation numbers $N$ is held constant and the number of time-steps $M_0$ is quadrupled. Moreover, from Figure 5 to 8 oscillations do not occur around the strike $K$ for small or big $T$ when we approximate $\Delta$ and $\Gamma$.

| $N$ | $M_0$ | $E_\infty(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$ |
|-----|-------|--------------------------|------------|----------------------|------|
| 225 | 10    | 2.368536E-03             | N/A        | 7.995993E-04        | N/A  |
| 450 | 40    | 7.746936E-04             | 1.612      | 2.740154E-04        | 1.879|
| 900 | 160   | 6.260415E-04             | 1.777      | 6.969946E-05        | 1.975|
| 1800| 640   | 1.639307E-04             | 1.829      | 1.88980E-05         | 1.884|
| 3600| 2560  | 1.613907E-04             | 1.979      | 4.715908E-06        | 2.002|

Table 13. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of an American put under the Merton model are presented. $N$ is the number of the interpolation points. $M_0$ is the number of the time steps. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0$, $\sigma = 0.15$, $\sigma_J = 0.45$, $\mu_J = -0.9$, $\lambda = 0.1$, $K = 1$ and $T = 0.25$. The parameters are taken from (Andersen and Andreasen, 2000). The order of convergence is 2 in space and 1 in time.

| $N$ | $M_0$ | $E_\infty(\hat{x}_i, T)$ | $R_\infty$ | $E_\infty(\hat{x}_i, T)$ | $R_\infty$ |
|-----|-------|--------------------------|------------|--------------------------|------------|
| 225 | 10    | 3.401417E-03             | N/A        | 7.995993E-04             | N/A        |
| 450 | 40    | 1.318325E-03             | 1.367      | 2.451148E-04             | 1.706      |
| 900 | 160   | 3.744579E-04             | 1.816      | 6.873071E-05             | 1.834      |
| 1800| 640   | 1.055849E-04             | 1.826      | 1.927219E-05             | 1.834      |
| 3600| 2560  | 2.823205E-05             | 1.903      | 5.121082E-06             | 1.912      |

Table 14. $E_\infty$ and $E_2$ of the Cubic Spline approximation for pricing of an American put under the Merton model are presented. $N$ is the number of the interpolation points. $M_0$ is the number of the time steps. $\hat{x}_i = \log S_i$ is any evaluation points of a range of $S$ from 0.05 to 2 and the total numbers are 1950. $T$ is the Time-to-maturity. The parameters are: $r = 0.05$, $q = 0.02$, $\sigma = 0.15$, $\sigma_J = 0.4$, $\mu_J = -1.08$, $\lambda = 0.1$, $K = 1$ and $T = 0.1$. The parameters are taken from (Andersen and Andreasen, 2000). The order of convergence is 2 in space and 1 in time.

5. Conclusion

We have implemented an RBF method to solve the PIDE boundary value problem for pricing American put and European call/put options on a dividend-paying stock in the Merton and Kou Jump-diffusion market. By using the numerical scheme of Briani et al., we find out a finite computational range of our global integral. Our results suggest that the Cubic Spline
The order of convergence is 2 in space and 1 in time.

\[ q = 0 \]

points of a range of \( S \) are presented.

\[ N \]

Table 15. \( E_\infty \) and \( E_2 \) of the Cubic Spline approximation for pricing of an American put under the Merton model are presented. \( N \) is the number of the interpolation points. \( M_0 \) is the number of the time steps. \( \hat{x}_i = \log S_i \) is any evaluation point of a range of \( S \) from 0.05 to 2 and the total numbers are 1950. \( T \) is the Time-to-maturity. The parameters are: \( r = 0.05, q = 0.01, \sigma = 1, \sigma_2 = 0.6, \mu = -1.08, \lambda = 0.1, K = 1 \) and \( T = 1 \), whereas the parameter \( \sigma = 1 \) is selected to stress our numerical algorithm. The order of convergence is 2 in space and 1 in time.

| \( N \) | \( M_0 \) | \( E_\infty(\hat{x}_i,T) \) | \( R_\infty \) | \( E_2(\hat{x}_i,T) \) | \( R_2 \) |
|-------|--------|----------------|--------|----------------|--------|
| 225   | 10     | 4.935878E-03  | N/A    | 1.61332E-03   | N/A    |
| 450   | 40     | 1.236617E-03  | 1.997  | 3.725615E-04  | 2.114  |
| 900   | 160    | 3.093198E-04  | 1.999  | 9.101657E-05  | 2.033  |
| 1800  | 640    | 7.734030E-05  | 2.000  | 2.133679E-05  | 2.093  |
| 3600  | 2560   | 1.932168E-05  | 2.001  | 5.074520E-06  | 2.072  |

Table 16. \( E_\infty \) and \( E_2 \) of the Cubic Spline approximation for pricing of an American put under the Kou Jump-diffusion model are presented. \( N \) is the number of the interpolation points. \( M_0 \) is the number of the time steps. \( \hat{x}_i = \log S_i \) is any evaluation point of a range of \( S \) from 0.05 to 2 and the total numbers are 1950. \( T \) is the Time-to-maturity. The parameters are: \( r = 0, q = 0, \sigma = 0.2, \alpha_1 = 3, \alpha_2 = 2, \lambda = 0.2, p = 0.5, K = 1 \) and \( T = 0.2 \). The parameters are taken from Almendral and Oosterlee (2005). The order of convergence is 2 in space and 1 in time.

| \( N \) | \( M_0 \) | \( E_\infty(\hat{x}_i,T) \) | \( R_\infty \) | \( E_2(\hat{x}_i,T) \) | \( R_2 \) |
|-------|--------|----------------|--------|----------------|--------|
| 225   | 10     | 1.508321E-03  | N/A    | 5.589125E-04  | N/A    |
| 450   | 40     | 7.233939E-04  | 1.060  | 1.759571E-04  | 1.667  |
| 900   | 160    | 1.958968E-04  | 1.885  | 4.733738E-05  | 1.894  |
| 1800  | 640    | 5.243753E-05  | 1.901  | 1.271703E-05  | 1.896  |
| 3600  | 2560   | 1.374207E-05  | 1.932  | 3.405083E-06  | 1.901  |

Table 17. \( E_\infty \) and \( E_2 \) of the Cubic Spline approximation for pricing of an American put under the Kou Jump-diffusion model are presented. \( N \) is the number of the interpolation points. \( M_0 \) is the number of the time steps. \( \hat{x}_i = \log S_i \) is any evaluation point of a range of \( S \) from 0.05 to 2 and the total numbers are 1950. \( T \) is the Time-to-maturity. The parameters are: \( r = 0.05, q = 0, \sigma = 0.15, \alpha_1 = 3.0465, \alpha_2 = 3.0465, \lambda = 0.1, p = 0.3445, K = 1 \) and \( T = 0.25 \). The parameters are taken from Carr and Mayo (2007). The order of convergence is 2 in space and 1 in time.

| \( N \) | \( M_0 \) | \( E_\infty(\hat{x}_i,T) \) | \( R_\infty \) | \( E_2(\hat{x}_i,T) \) | \( R_2 \) |
|-------|--------|----------------|--------|----------------|--------|
| 225   | 10     | 1.933354E-03  | N/A    | 8.983577E-04  | N/A    |
| 450   | 40     | 8.487095E-04  | 1.188  | 2.783005E-04  | 1.691  |
| 900   | 160    | 2.497213E-04  | 1.765  | 7.257535E-05  | 1.939  |
| 1800  | 640    | 6.843085E-05  | 1.868  | 1.933309E-05  | 1.908  |
| 3600  | 2560   | 1.827216E-05  | 1.905  | 5.119491E-06  | 1.917  |

Table 18. \( E_\infty \) and \( E_2 \) of the Cubic Spline approximation for pricing of an American put under the Kou Jump-diffusion model are presented. \( N \) is the number of the interpolation points. \( M_0 \) is the number of the time steps. \( \hat{x}_i = \log S_i \) is any evaluation point of a range of \( S \) from 0.05 to 2 and the total numbers are 1950. \( T \) is the Time-to-maturity. The parameters are: \( r = 0.04, q = 0.03, \sigma = 1, \alpha_1 = 4, \alpha_2 = 4, \lambda = 0.3, p = 0.6, K = 1 \) and \( T = 1 \), whereas the parameter \( \sigma = 1 \) is selected to stress our numerical algorithm. The order of convergence is 2 in space and 1 in time.
Table 19. $E_{\infty}$ and $E_2$ of the Cubic Spline approximation for pricing of an American put under the Merton Jump-diffusion model are presented. $N$ is the number of the interpolation points. $M_0$ is the number of the time steps. $\hat{x}_i = \log S_i$ is any evaluation point of a range of $S$ from 0.05 to 2 and the total numbers are 1950. Top: The input parameters are provided in the caption to Table 13. Bottom: The input parameters are provided in the caption to Table 14. The order of convergence is 1 in time.

| N   | $M_0$ | $E_{\infty}(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$ |
|-----|-------|-----------------------------|-------------|----------------------|-------|
| 3600 | 40    | 1.029993E-03               | N/A         | 3.556438E-04         | N/A   |
| 3600 | 160   | 2.974325E-04               | 0.896       | 8.866477E-05         | 1.002 |
| 3600 | 640   | 8.273457E-05               | 0.923       | 2.147058E-05         | 1.023 |
| 3600 | 2560  | 2.123568E-05               | 0.982       | 5.345367E-06         | 1.003 |
| 3600 | 40    | 1.523857E-03               | N/A         | 3.755414E-04         | N/A   |
| 3600 | 160   | 4.561966E-04               | 0.870       | 1.108779E-04         | 0.880 |
| 3600 | 640   | 1.253234E-04               | 0.923       | 2.85790E-05          | 0.978 |
| 3600 | 2560  | 3.230111E-05               | 0.978       | 7.134578E-06         | 1.001 |

Table 20. $E_{\infty}$ and $E_2$ of the Cubic Spline approximation for pricing of an American put under the Kou Jump-diffusion model are presented. $N$ is the number of the interpolation points. $M_0$ is the number of the time steps. $\hat{x}_i = \log S_i$ is any evaluation point of a range of $S$ from 0.05 to 2 and the total numbers are 1950. Top: The input parameters are provided in the caption to Table 16. Bottom: The input parameters are provided in the caption to Table 17. The order of convergence is 1 in time.

| N   | $M_0$ | $E_{\infty}(\hat{x}_i, T)$ | $R_\infty$ | $E_2(\hat{x}_i, T)$ | $R_2$ |
|-----|-------|-----------------------------|-------------|----------------------|-------|
| 3600 | 40    | 1.687267E-03               | N/A         | 1.878158E-04         | N/A   |
| 3600 | 160   | 4.913041E-04               | 0.890       | 5.514562E-05         | 0.884 |
| 3600 | 640   | 1.366624E-04               | 0.923       | 1.590242E-05         | 0.897 |
| 3600 | 2560  | 3.478690E-05               | 0.987       | 3.768285E-06         | 0.923 |
| 3600 | 40    | 8.987023E-04               | N/A         | 2.671173E-04         | N/A   |
| 3600 | 160   | 3.151073E-04               | 0.756       | 7.544376E-05         | 0.912 |
| 3600 | 640   | 9.355218E-05               | 0.876       | 2.184649E-05         | 0.894 |
| 3600 | 2560  | 2.675431E-05               | 0.903       | 6.204563E-06         | 0.908 |

Figure 5. Put option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Merton Jump diffusion Model. The number of the interpolation points $N$ is 1800 and the number of time steps $M_0$ is 640. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 13.
approximation scheme can achieve second-order convergence in both spatial and time variables (due to the second order time-stepping scheme, BDFs of order 2) when it is used to compute European call/put options. Moreover, the results also show that our approximation solution can get second-order convergence in spatial variables and first-order convergence in time variables when the approximation scheme is used to compute American put options. Beside this, we compare our RBF-approximation method against FDM and FEM. Our results suggest that one can achieve a high accuracy by implementing our meshless scheme. Moreover, in terms of meshless interpolation methods, we use cubic spline as a basis function rather than MQ. This basis function can avoid the open question of choosing an optimal shape parameter of MQ. Beside this, by using factorisation of the Cubic Spline, we can avoid inverting an ill-conditioned cubic spline interpolant directly. Finally, throughout the analysis of both $\Delta$ and $\Gamma$, our RBF-approximation method can also avoid the oscillation problem around the strike $K$ in both American and European cases.

**Figure 6.** Put option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Merton Jump diffusion Model. The number of the interpolation points $N$ is 1800 and the number of time steps $M_0$ is 640. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 14.

**Figure 7.** Put Option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Kou Jump-diffusion Model. The number of the interpolation points $N$ is 1800 and the number of time steps $M_0$ is 640. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 17.
Figure 8. Put Option Delta $\Delta$ (Left) and Gamma $\Gamma$ (Right) in the Kou Jump-diffusion Model. The number of the interpolation points $N$ is 1800 and the number of time steps $M_0$ is 640. The number of evaluation points of a range of $S$ from 0.05 to 2 is 1950. The input parameters are provided in the caption to Table 18.

At this stage of development, the Cubic Spline approximation scheme is first order in time for American put options although a second order time-stepping scheme, BDFs of order 2 is implemented. We are investigating various approaches to improve the Cubic Spline approximation for time variables and will treat them in a future paper. Our Method extends in principle to pure jump Lévy type models for the underlying stocks, like the Variance Gamma (VG) model or the CGMY model.
Appendix A. A Finite Computational Range in the Jump-diffusion Model

In the Merton Model suppose in a domain $\Omega \in \mathbb{R}$ European option price $u(x, \tau)$ satisfies Lipchitz inequality such that

$$|u(x_1, \tau) - u(x_2, \tau)| \leq L|\tau_1 - \tau_2|, \forall x_1, x_2 \in \Omega.$$ 

Then we choose a parameter $\epsilon > 0$ and select the bounded intervals $[y_1, y_2]$ as the set of all points $y$ that verify

$$k(y) = \frac{1}{\sqrt{2\pi\sigma_J}} e^{-\frac{(y-\mu_J)^2}{2\sigma_J^2}} \geq \epsilon.$$ 

Because of the symmetry of $k(y)$ we set $y_1 = -\mu_J$. Then the truncation of the integral domain giving an error to approximation of the problem can be estimated by

$$\left| \int_{-\infty}^{\infty} (u(x + y) - u(x))k(y) \, dy - \int_{-\infty}^{y_1} (u(x + y) - u(x))k(y) \, dy \right|$$

$$\leq L \left( \int_{-\infty}^{y_1} y |k(y)| \, dy + \int_{y_1}^{\infty} |y|k(y) \, dy \right) \quad (A1a)$$

$$\leq L \left( \int_{-\infty}^{y_1} y k(y) \, dy + \int_{y_1}^{\infty} y k(y) \, dy \right) \quad (A1b)$$

$$= 2 \int_{y_1}^{\infty} y \frac{1}{\sqrt{2\pi\sigma_J}} \exp\left(-\frac{(y-\mu_J)^2}{2\sigma_J^2}\right) \, dy \quad (A1c)$$

$$= 2 \int_{y_1 - \mu_J}^{\infty} \left(y + \mu_J\right) \frac{1}{\sqrt{2\pi\sigma_J}} \exp\left(-\frac{y^2}{2\sigma_J^2}\right) \, dy \quad (A1d)$$

$$= 2 \int_{y_1 - \mu_J}^{\infty} \left(y + \mu_J\right) \frac{1}{\sqrt{2\pi\sigma_J}} \exp\left(-\frac{y^2}{2\sigma_J^2}\right) \, dy \quad (A1e)$$

$$\leq 2 \int_{y_1 - \mu_J}^{\infty} (y + \mu_J) \frac{1}{\sqrt{2\pi\sigma_J}} \exp\left(-\frac{y^2}{2\sigma_J^2}\right) \, dy \quad (A1f)$$

$$= \frac{4\sigma_J}{\sqrt{2\pi}} \exp\left(-\frac{(y_1 - \mu_J)^2}{2\sigma_J^2}\right) \quad (A1g)$$

$$= 2\sigma_J^2 \epsilon \quad (A1h)$$

Hence by using (A1g) and (A1h),

$$y_1 = \sqrt{-2\sigma_J^2 \log(\epsilon_1 \sqrt{2\pi/2}) + \mu_J} \quad (A2)$$

We use the aforementioned arguments to find the finite computational range $[y_1, y_2]$ in the Kou model. We carry out the reasoning for the positive semi-axis (the reasoning goes similarly for the negative semi-axis) and set $k(y) = p\alpha_1 e^{-\alpha_1 y}$ for $y \geq 0$ ($(1-p)\alpha_2 e^{\alpha_2 y}$ for $y < 0$). Then,
$y_e$ can be found out by the following equations:

\[
\left| \int_0^\infty (u(x+y) - u(x))\lambda f(y) \, dy - \int_0^{y_e} (u(x+y) - u(x))\lambda f(y) \, dy \right|
\leq L \left| \int_0^\infty (x + y - x)\lambda f(y) \, dy - \int_0^{y_e} (x + y - x)\lambda f(y) \, dy \right|
\leq L \int_0^\infty |y| f(y) \, dy
= \int_0^{y_e} |y| p\alpha_1 e^{-\alpha_1 y} \, dy
= p\alpha_1 e^{-y_e} \left( \frac{1}{\alpha_1^2} + \frac{y_e}{\alpha_1} \right)
\]

(Gradshteyn and Ryzhik [1994] equation 3.351)

\[
= \frac{p}{\alpha_1} e^{-y_e} \left( 1 + y_e \alpha_1 \right)
\]

\[
\leq \frac{p}{\alpha_1} e^{-y_e} \alpha_1 \epsilon e^{y_e}
= pe^{y_e} \alpha_1 (1 - \alpha_1)
\]

as a result,

\[
y_e = \log(\epsilon/p)/(1 - \alpha_1).
\]

(A4)

Similar arguments can be applied to $y < 0$, so

\[
y_{-\epsilon} = -\log \left( \frac{\epsilon}{(1 - p)} \right)/(1 - \alpha_2).
\]

(A5)

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