Comparison inequality and two block estimate  
for inhomogeneous Bernoulli measures 

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Abstract. We consider inhomogeneous Bernoulli measures of the form \( \prod_{x \in \Lambda} p_x \) where \( p_x \) are prescribed and uniformly bounded above and below away from 0 and 1. A comparison inequality is proved between the Kawasaki and Bernoulli-Laplace Dirichlet forms. Together with a recent result of Caputo on the gap of the Bernoulli-Laplace model, this proves a spectral gap of the correct order \( L^{-2} \) on cubes of side length \( L \) for the Kawasaki dynamics. The two block estimate of hydrodynamic limits is also obtained.

0. Introduction  

Recently there has been a lot of interest in the transport properties of particle systems in random media [AHL, BE, F, GP, KPW, K, MA, R, Sc]. A simple model for which the hydrodynamic scaling limit can be obtained is the Kawasaki dynamics for random Bernoulli measures studied in [Q], [QY] and recently, [FM]. Such systems have been used to model electron transport in doped crystals. The hydrodynamic limit is a nonlinear diffusion equation with a nontrivial density dependent diffusion coefficient given by a Green-Kubo formula. In fact for such a system even the existence of a diffusive scaling limit is nontrivial. The key input is a so called moving particles lemma which leads to a diffusive spectral gap, as well as a two block estimate [GPV]. The hydrodynamic limit was described in [Q] and an unpublished manuscript [QY] contains most of the details of the proof. Based on these [FM] recently give a complete proof. However they do not use the long jumps method sketched in [Q], [QY], but the more traditional non-gradient method together with subtraction of an appropriate term to compensate for the inhomogeneity in the medium. This restricts one to dimensions \( d \geq 3 \). Another recent article [C] proves the gap of the Bernoulli-Laplace model. Together with a comparison inequality from [QY] he concludes the spectral gap of the Kawasaki dynamics. The purpose of this note is to complete the literature by providing the needed moving particles lemma from the unpublished manuscript [QY]. This work as well as [Q] and [QY] on the hydrodynamic limit, arose out of problems suggested by Herbert Spohn and from joint work with H. T. Yau. Their contribution is gratefully acknowledged.
1. Inhomogeneous Bernoulli measures.

Let $\alpha_x \in [-K, K], x \in \mathbb{Z}^d$ be given and let

$$p_x = \frac{e^{\alpha_x}}{1 + e^{\alpha_x}}, \quad x \in \mathbb{Z}^d.$$ 

For any $\Lambda \subset \mathbb{Z}^d$ the inhomogeneous Bernoulli measure $\mu_\Lambda(\eta)$ on $\{0, 1\}^\Lambda$ is given by

$$\mu_\Lambda(\eta) = \prod_{x \in \Lambda} p_x^{\eta_x} (1 - p_x)^{1 - \eta_x} = Z_{\alpha,\Lambda}^{-1} \exp\{-H_{\alpha,\Lambda}(\eta)\}, \quad H_{\alpha,\Lambda}(\eta) = -\sum_{x \in \Lambda} \alpha_x \eta_x.$$ 

$Z_{\alpha,\Lambda} = \prod_{x \in \Lambda} (1 + e^{\alpha_x})$ is the normalization to make it a probability measure. We can also condition to have a fixed number $N$ of particles in $\Lambda$,

$$\mu_{\alpha,\Lambda,N}(\eta) = \mu_{\alpha,\Lambda}(\eta \mid \sum_x \eta_x = N).$$

For each configuration $\eta$ with exactly $N$ particles, $\mu_{\alpha,\Lambda,N}(\eta) = Z_{\alpha,\Lambda,N}^{-1} \exp\{-H_{\alpha,\Lambda}(\eta)\}$ with

$$Z_{\alpha,\Lambda,N} = \sum_{A \subset \Lambda, |A| = N} \exp\{\sum_{x \in A} \alpha_x\} = \binom{|\Lambda|}{N} \zeta_{\alpha,\Lambda,N}, \quad \zeta_{\alpha,\Lambda,N} = Av_{A \subset \Lambda, |A| = N} \exp\{\sum_{x \in A} \alpha_x\}.$$

Note that

$$\mu_{\alpha,\Lambda}(\eta_x = 1) = \frac{e^{\alpha_x}}{1 + e^{\alpha_x}} = p_x$$

while the corresponding quantity in the canonical ensemble is

$$\mu_{\alpha,\Lambda,N}(\eta_x = 1) = e^{\alpha_x} \frac{Z_{\alpha,\Lambda\setminus\{x\},N-1}}{Z_{\alpha,\Lambda,N}} = \frac{N}{|\Lambda|} \left( \frac{e^{\alpha_x} \zeta_{\alpha,\Lambda\setminus\{x\},N-1}}{\zeta_{\alpha,\Lambda,N}} \right) = p_{\alpha,\Lambda,N,x}.$$

We will use the notation $E_\mu[f; g]$ to denote the covariance $E_\mu[(f - E_\mu[f])(g - E_\mu[g])]$ as well as $\text{Var}_\mu(f)$ for the variance $E_\mu[f; f]$.

2. Dynamics and main results

We define dynamics through Dirichlet forms. There are two basic dynamics: Glauber and Kawasaki.

**Glauber.** Let $\Lambda \subset \mathbb{Z}^d$. The Dirichlet form is given by

$$D_G(\Lambda; f) = E_{\mu_\Lambda}[\sum_{x \in \Lambda} (f(\sigma_x \eta) - f(\eta))^2]$$

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where

$$(\sigma_x \eta)_y = \begin{cases} 
\eta_y & \text{if } y \neq x; \\
1 - \eta_y & \text{if } x = y.
\end{cases}$$

The dynamics corresponding to this Dirichlet form is when each site changes its value from $\eta_x$ to $1 - \eta_x$ at rate $1 + \exp\{\alpha_x (1 - 2\eta_x)\}$.

**Kawasaki.** On a connected set $\Lambda \subset \mathbb{Z}^d$ (in the sense of nearest neighbours, which we write as $x \sim y$), and given a fixed number of particles $0 \leq N \leq |\Lambda|$, the Dirichlet form is

$$D_{\text{Kaw}}(\Lambda, N; f) = E_{\mu_{\Lambda,N}} \left[ \sum_{x \sim y, x,y \in \Lambda} (f(T_{x,y} \eta) - f(\eta))^2 \right]$$

where

$$(T_{x,y} \eta)_z = \begin{cases} 
\eta_y, & \text{if } z = x \\
\eta_x, & \text{if } z = y \\
\eta_z, & \text{otherwise}.
\end{cases}$$

The corresponding dynamics is a system of $N$ particles on $\Lambda = \Lambda_L = \{0, \ldots, L-1\}^d$ moving in the field $\alpha$. At most one particle is allowed at each site. A particle at $x \in \Lambda$ attempts to jump to nearest neighbor site $y \in \Lambda$ at rate

$$1 + e^{\alpha_y - \alpha_x}.$$ 

If there is no particle in the way the particle is allowed to jump. However if there is a particle in the way, the jump is suppressed, and everything starts again. All the particles are doing this independently of each other, and since time is continuous one can ignore the occasion of two particles trying to jump onto each other simultaneously.

**Bernoulli-Laplace.** This is introduced as a tool for proving results about the Kawasaki dynamics. For any $\Lambda \subset \mathbb{Z}^d$, the Dirichlet form is

$$D_{\text{BL}}(\Lambda, N; f) = E_{\mu_{\Lambda,N}} \left[ \sum_{x \sim y, x,y \in \Lambda} (f(T_{x,y} \eta) - f(\eta))^2 \right].$$

The difference between Bernoulli-Laplace and Kawasaki is that in Kawasaki only nearest neighbour jumps are allowed, but in Bernoulli-Laplace we allow jumps to any site.

The following result has been recently been obtained by Caputo [C].

**Theorem 1.** Let $0 \leq K < \infty$. There exists a constant $C = C(K) < \infty$ such that for any field $\alpha$ with $-K \leq \alpha_x \leq K$, any $\Lambda$, and any $0 \leq N \leq |\Lambda|$, for any $f : \{0, 1\}^\Lambda \to \mathbb{R}$,

$$\text{Var}_{\mu_{\Lambda,N}}(f) \leq \frac{C}{|\Lambda|} D_{\text{BL}}(\Lambda, N; f).$$
The main result of this article is

**Lemma 1. (Moving Particles Lemma) [QY]** Suppose \(\mu\) is an inhomogenous Bernoulli measure on \(\mathbb{Z}^1\) with external field \(\alpha\) taking values in \([-K, +K]\), conditioned to have \(N\) particles. Then

\[
E_\mu[(f(T_{1L}\eta) - f(\eta))^2] \leq e^{13KL} \sum_{1 \leq x \leq L-1} E_\mu[(f(T_{x,x+1}\eta) - f(\eta))^2]
\]

The proof is given in section 3.

Once we have the Moving Particles Lemma the spectral gap and two block estimate follow using standard arguments [GPV], [KL], [Q1]. Let

\[m^K_x = Av_y |y - x| \leq K \eta_y\]

be the empirical particle density in a box of radius \(K\) around \(x\). The two block estimate says that, suitably averaged, such a quantity is not substantially different if measured on a large microscale, or a small macroscale. Let \(\Lambda_L\) be a cube of side length \(L\) with periodic boundary conditions and let \(\mathcal{P}_{\Lambda_L,N}\) be the set of probability densities with respect to \(\mu_{\Lambda_L,N}\).

**Theorem 2. (Two block estimate)** Let \(F\) be a continuous function on \([0, 1]\) and \(\varphi\) a smooth function on the \(d\)-dimensional unit torus. If \(K \to \infty\) as \(L \to \infty\) with \(K \leq \delta L\),

\[
\limsup_{\delta \to 0} \limsup_{L \to \infty} \sup_{0 \leq N \leq L^d} \sup_{f \in \mathcal{P}_{\Lambda_L,N}} \left\{ L^{-d} E_{\mu_{\Lambda_L,N}} \left[ \sum_x \varphi(x/N)(F(m^K_x) - F(m^\delta_N))]\right] \right\} \leq 0.
\]

By a box \(\Lambda_L\) of side length \(L\) we mean a set of the form \(\{x \in \mathbb{Z}^d : x_i - y_i \in \{0, 1, \ldots, L-1\}, i = 1, \ldots, d\}\) for some \(y = (y_1, \ldots, y_d) \in \mathbb{Z}^d\). The following theorem gives the spectral gap of the Kawasaki dynamics to correct order. It is stated in [C] based on the Moving Particles Lemma above from [QY].

**Theorem 3.** For each \(K > 0\) there exists a \(C < \infty\) such that for all \(\alpha\) with \(-K \leq \alpha_x \leq K\), all boxes \(\Lambda_L\) of side length \(L\), all \(0 \leq N \leq L^d\), and all \(f : \{0, 1\}^{\Lambda_L} \to \mathbb{R}\),

\[
E_{\mu_{\Lambda_L,N}}[(f - E_{\mu_{\Lambda,L}}[f])^2] \leq CL^2 D_{Kaw}(\Lambda_L, N; f).
\]
Proof. By Theorem 1 we have

\[ E_{\mu_{L,N}}[(f - E_{\mu_{L,N}}[f])^2] \leq \frac{C}{|\Lambda_L|} E_{\mu_{L,N}}[\sum_{x,y \in \Lambda_L} (f(T_{xy}) - f(\eta))^2] \]

For each \( x, y \in \Lambda_L \) choose a canonical path \( x = x_1, x_2, \ldots, x_n = y \) with \( x_i \) and \( x_{i+1} \) by moving first in the first coordinate direction, then in the second coordinate direction, etc. By Lemma 1, we have

\[ E_{\mu_{L,N}}[(f(T_{xy}) - f(\eta))^2] \leq e^{13Kn} \sum_{1 \leq i \leq n-1} E_{\mu}[(f(T_{x_i,x_{i+1}}) - f(\eta))^2]. \]

Summing over \( x \) and \( y \), noting that \( n \leq dL \) and that each nearest neighbour pair is used for the path between \( d(L/2)^{d+1} \) pairs \( x \) and \( y \) we obtain the result. \( \square \)

3. Proof of the main result

Lemma 2. Suppose \( \mu \) is a homogeneous Bernoulli measure (\( \alpha_x \equiv 0 \)) on \( \mathbb{Z}^1 \) conditioned to have \( N \) particles. Let \( k \) be a positive integer and \( \rho_x \) a sequence of positive numbers with \( \sum_{x=1}^{k-1} \rho_x = 1 \). Then

\[ E_\mu[f(T_{1,k\eta}) - f(\eta)]^2 \leq \sum_{x=1}^{k-1} \rho_x^{-1} E_\mu[(f(T_{x,x+1}) - f(\eta))^2]. \]

Proof. By definition, \( T_{1,k\eta} = T_{1,2} \cdots T_{k-2,k-1}T_{k,k-1} \cdots T_{3,2}T_{2,1}\eta \). For notational convenience, let \( T_{k+s-1,k+s} = T_{k-s,k-s-1} \) and \( \eta_{k+s} = \eta_{k-s}, 1 \leq s \leq k-1 \). We have

\[ \eta_1(1-\eta_k)[f(T_{1,k\eta}) - f(\eta)] = \sum_{s=1}^{2k-1} \eta_1(1-\eta_k)q_s(\eta)[f(T_{s+1,s} \cdots T_{3,2}T_{2,1}\eta) - f(T_{s,s-1} \cdots T_{2,1}\eta)] \]

where

\[ q_s(\eta) = \begin{cases} 1 - \eta_{s+1}, & \text{if } s \leq k-1; \\ \eta_{2k-s-2}, & \text{if } s \geq k. \end{cases} \]

Note that the factor \( q_s(\eta) \) can be added free of charge because the summand vanishes exactly when it does. Also, for any \( 1 \leq x \leq k-1 \) fixed, \( q_x + q_{2k-x-2} = 1 \). Hence if one lets \( \rho_{k+s} = \rho_{k-s-2} \) then, by Schwarz’s inequality,

\[ \eta_1(1-\eta_k)[f(\eta^{1k}) - f(\eta)]^2 \]
\[ \leq \eta_1(1-\eta_k) \sum_{s=1}^{2k-1} \rho_{s-1}^{-1}q_s(\eta)[f(T_{s+1}s \cdots T_{21}\eta) - f(T_{ss-1} \cdots T_{21}\eta)]^2 \sum_{s=1}^{2k-2} \rho_{s-1}q_s(\eta) \]
\[ = \sum_{s=1}^{2k-2} \rho_{s-1}^{-1} \eta_1(1-\eta_k)q_s(\eta)[f(T_{s+1,s} \cdots T_{21}\eta) - f(T_{ss-1} \cdots T_{21}\eta)]^2 \]
Taking expectation and change variables $T_{s-1} \cdot \cdots \cdot T_{21} \eta \rightarrow \eta$ and also change the index $s \rightarrow 2k - s - 2$ for $s \geq k$ we find that $E^{\mu}[\eta_1(1 - \eta_k)(f(\eta^k) - f(\eta))^2]$ is bounded by the expectation of

$$\sum_{s=1}^{k-1} \rho_{s-1}^{-1}(1 - \eta_k)(1 - \eta_{s+1}) \left[ f(T_{s+1,s}\eta) - f(\eta) \right]^2 + \sum_{s=1}^{k-2} \rho_{s-1}^{-1} \eta_k(1 - \eta_{s+1}) \left[ f(T_{s+1,s}\eta) - f(\eta) \right]^2$$

$$= \sum_{x=1}^{k-1} \rho_{x-1}^{-1}(1 - \eta_{x+1}) \left[ f(T_{x+1,x}\eta) - f(\eta) \right]^2$$

We have thus proved that

$$E^{\mu}[\eta_1(1 - \eta_k)(f(T_{1,k}\eta) - f(\eta))^2] \leq \sum_{x=1}^{k-1} \rho_{x-1}^{-1} E^{\mu}[(1 - \eta_{x+1})(f(T_{x+1,x}\eta) - f(\eta))^2]$$

Similarly, if one uses the particle-hole duality,

$$E^{\mu}[\eta_k(1 - \eta_1)(f(T_{1,k}\eta) - f(\eta))^2] \leq \sum_{x=1}^{k-1} \rho_{x-1}^{-1} E^{\mu}[(\eta_{x+1}(f(T_{x+1,x}\eta) - f(\eta))^2]$$

The lemma is obtained by adding these two bounds.

**Proof of Lemma 1.** Suppose we change the measure $\mu$ to a new measure $\tilde{\mu}$ by changing each $\alpha_i$ to the nearest value of the form $K j / L$, $j$ an integer. Since there is always such a point with $|\alpha_i - K j / L| \leq K(2L)^{-1}$ the Radon-Nikodym derivative $d\mu / d\tilde{\mu}$ is bounded above and below uniformly by $e^{K/2}$ and $e^{-K/2}$ respectively. Therefore at the cost of a factor of $e^{K}$ we may assume that $\alpha$ takes values in $\{K j / L : j = -L, \ldots, L\}$. By the same reasoning, at the price of a factor $e^{4K}$ we may assume that $\alpha_0 = \alpha_L = K$.

Let $A$ be the set

$$A = \{x_i : \alpha_{x_i} = K, i = 1, \ldots, k.\}$$

By definition,

$$T_{1,L}\eta = T_{x_1,x_2} \cdots T_{x_{k-2},x_{k-1}} T_{x_k,x_{k-1}} \cdots T_{x_3,x_2} T_{x_2,x_1}\eta.$$ 

By Lemma 2 with $\rho_{s}^{-1} = \frac{L}{x_{s+1} - x_s}$,

$$E^{\mu}[(f(T_{1,L}\eta) - f(\eta))^2] \leq e^{4K} \sum_{s=1}^{k-1} \frac{L}{x_{s+1} - x_s} E^{\mu}[(f(T_{x_s,x_{s+1}}\eta) - f(\eta))^2]$$

We have to bound

$$\frac{L}{x_{s+1} - x_s} E^{\mu}[(f(T_{x_s,x_{s+1}}\eta) - f(\eta))^2]$$
We are now in the same situation as before except no $\alpha_x$ can take value $K$ when $x_s < x < x_{s+1}$. Let us change $\alpha_{x_s}$ and $\alpha_{x_s+1}$ to the value $K(L - 1)/L$. The price we pay is a factor $\exp\{2KL^{-1}\}$. Continuing this procedure we have a proof of the lemma.

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