DESINGULARIZATION OF SOME MODULI SCHEME OF STABLE SHEAVES ON A SURFACE

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Abstract. Let $X$ be a nonsingular projective surface over an algebraically closed field with characteristic 0, and $H_-$ and $H_+$ ample line bundles on $X$ separated by only one wall of type $(c_1, c_2)$. Suppose the moduli scheme $M(H_-)$ of rank-two $H_-$-stable sheaves with Chern classes $(c_1, c_2)$ is non-singular. We shall construct a desingularization of $M(H_+)$ by using $M(H_-)$. As an application, we consider whether singularities of $M(H_+)$ are terminal or not when $X$ is ruled or elliptic.

1. Introduction

Let $X$ be a projective non-singular surface over an algebraically closed field with characteristic 0. $H$ an ample line bundle on $X$. Denote by $M(H)$ the coarse moduli scheme of rank-two $H$-stable sheaves with fixed Chern class $(c_1, c_2) \in \text{NS}(X) \times \mathbb{Z}$. In this paper we think about singularity and desingularization of $M(H)$ from the view of wall-crossing problem of $H$ and $M(H)$.

Let $H_-$ and $H_+$ be ample line bundles on $X$ separated by only one wall of type $(c_1, c_2)$. For a parameter $a \in (0, 1)$, one can define the $a$-stability of sheaves on $X$ and have the coarse moduli scheme $M(a)$ of rank-two $a$-stable sheaves with Chern classes $(c_1, c_2)$. Let $a_-$ and $a_+ \in (0, 1)$ be minichambers (see Section 2 for details). Assume $M_- = M(a_-)$ is non-singular; one can find such $a_-$ when $X$ is ruled or elliptic for example. In Section 2 we construct a desingularization $\bar{\pi}_+ : \bar{M} \to M_+$ of $M_+ = M(a_+)$ by using $M_-$ and wall-crossing methods. In Section 3 we calculate $K_{\bar{M}} - \bar{\pi}_+^* K_{M_+}$. In Section 4 we apply it to consider whether singularities of $M_+$ are terminal or not when $X$ is ruled or elliptic.

Here we mention related topics. About singularities of moduli spaces, Vakil [6] shows that every singularity type of finite type over $\mathbb{Z}$ appears on moduli scheme of torsion-free sheaves on $\mathbb{P}^2$, and asks how about moduli scheme of sheaves on surfaces. Thereby one can regard $M(H)$ as a model in which various kinds of singularities can appear. However a little is known about specific way to study singularities of $M(H)$. Methods in this article are suited to study what kind of singularities moduli scheme of sheaves has. Perhaps one can use them to find interesting examples of singularity. Properties of singularities in Section 4 seems to relate with theory of determinantal variety over curve (see Remark 3.3). This topic shall be studied in another article.

Notation. For a $k$-scheme $S$, $X_S$ is $X \times S$ and $\text{Coh}(X_S)$ is the set of coherent sheaves on $X_S$. For $s \in S$ and $E_s \in \text{Coh}(X_S)$, $E_s$ means $E \otimes k(s)$. For $E$ and $F \in \text{Coh}(X)$, $\text{ext}^i(E, F) := \dim \text{Ext}^i_X(E, F)$ and $\text{hom}(E, F) = \dim \text{Hom}_X(E, F)$.

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Ext\(_i^j(X, E)^0\) indicates Ker(tr : Ext\(_i^j(X, E) \to H^i(\mathcal{O}_X))\). For \(\eta \in \text{NS}(X)\), we define \(W^\eta \subset \text{Amp}(X)\) by \(\{H \in \text{Amp}(X) \mid H \cdot \eta = 0\}\).

2. Desingularization of \(M_+\) by using \(M_-\)

We begin with background material. Let \(H_-\) and \(H_+\) be ample divisors lying in neighboring chambers of type \((c_1, c_2) \in \text{NS}(X) \times \mathbb{Z}\), and \(H_0\) an ample divisor in the wall \(W\) of type \((c_1, c_2)\) which lies in the closure of chambers containing \(H_-\) and \(H_+\) respectively. (Refer to [5] about the definition of wall and chamber.) Assume that \(M = H_+ - H_-\) is effective. For a number \(a \in [0, 1]\) one can define the \(a\)-stability of a torsion-free sheaf \(E\) using

\[
P_a(E(n)) = \{(1 - a)\chi(E(H_-(nH_0)) + a\chi(E(H_+(nH_0)))/\text{rk}(E).
\]

There is the coarse moduli scheme \(\mathcal{M}(a)\) of rank-two \(a\)-semistable sheaves on \(X\) with Chern classes \((c_1, c_2)\). Denote by \(\mathcal{M}(a)\) its open subscheme of \(a\)-stable sheaves. When one replace \(H_+\) by \(NH_+\) if necessary, \(\mathcal{M}(0)\) (resp. \(\mathcal{M}(1)\)) equals the moduli scheme of \(H_-\)-semistable (resp. \(H_+\)-semistable) sheaves. There exist finite numbers \(a_1 \ldots a_l \in (0, 1)\) called minichamber such that \(\mathcal{M}(a)\) and \(\mathcal{M}(a)\) changes only when \(a\) passes a miniwall. Refer to [1, Section 3] about these facts. Fix numbers \(a_-\) and \(a_+\) separated by the only one miniwall, and indicate \(\mathcal{M}_- = \mathcal{M}(a_-)\) and \(\mathcal{M}_+ = \mathcal{M}(a_+)\) for short. From [7, Section 2], the subset

\[
\mathcal{M}_- \supset P_- = \{[E] \mid E \text{ is not } a_-\text{-semistable}\}
\]

(resp. \(\mathcal{M}_+ \supset P_+ = \{[E] \mid E \text{ is not } a_-\text{-semistable}\}\)) is contained in \(\mathcal{M}_-\) (resp. \(\mathcal{M}_+)\) and endowed with a natural closed subscheme structure of \(\mathcal{M}_-\) (resp. \(\mathcal{M}_+)\). Let \(\eta\) be a element of

\[
A^+(W) = \{\eta \in \text{NS}(X) \mid \eta \text{ defines } W, 4c_2 - c_1^2 + \eta^2 \geq 0 \text{ and } \eta \cdot H_+ > 0\}\).

After [1, Definition 4.2] we define

\[
T_\eta = M(1, (c_1 + \eta)/2, n) \times M(1, (c_1 - \eta)/2, m),
\]

where \(n\) and \(m\) are numbers defined by

\[
n + m = c_2 - (c_1 - \eta^2)/4 \quad \text{and} \quad n - m = \eta \cdot (c_1 - K_X)/2 + (2a - 1)\eta \cdot M,
\]

and \(M(1, (c_1 + \eta)/2)\) is the moduli scheme of rank-one torsion-free sheaves on \(X\) with Chern classes \(((c_1 + \eta)/2, n)\). If \(F_{T_\eta}\) (resp. \(G_{T_\eta}\)) is the pull-back of a universal sheaf of \(M(1, (c_1 + \eta)/2, n)\) (resp. \(M(1, (c_1 - \eta)/2, m)\)) to \(X_{T_\eta}\), then we have an isomorphism

\[
P_- \simeq \coprod_{\eta \in A^+(W)} P_{T_\eta} \left(\text{Ext}^1_{X_{T_\eta}/T_\eta}(F_{T_\eta}, G_{T_\eta}(K_X))\right)
\]

from [7, Section 5].

**Proposition 2.1** ([7] Proposition 4.9). The blowing-up of \(M_-\) along \(P_-\) agrees with the blowing-up of \(M_+\) along \(P_+\). So we have blowing-ups

\[
M_- \xrightarrow{\pi_-} \tilde{M} \xrightarrow{\pi_+} M_+.
\]
Now we assume that \( P_- \) is nowhere dense in \( M_- \) and that every \( E \in M_- \) satisfies that \( \text{Ext}^2_X(E, E)^0 = 0 \), and explain how to induce a desingularization \( \tilde{M} \to M_+ \) of \( M_+ \) from \( M_- \). The idea is as follows. The problem of comparing \( M_- \) and \( M_+ \) is called wall-crossing problem or polarization-change problem. According to [7], we can endow a natural subset

\[
M_- \supset P = \{ [E] \mid E \text{ is not } H_+\text{-semistable} \}
\]

with a natural closed subscheme structure, and have a morphism from the blowing-up of \( M_- \) along \( P \) to \( M_+ \),

\[
M_- \leftarrow \tilde{M} = B_P(M_-) \to M_+
\]

in such a way that one can compare a universal family of \( M_- \) with that of \( M_+ \), if exists. Since \( M_- \) is non-singular, \( \tilde{M} \) would be a desingularization of \( M_+ \) if the center \( P \) is non-singular. We shall find a natural sequence of blowing-up such that the strict transform of \( P \) becomes smooth, considering a relative property of \( P \) over \( \text{Pic} \times \text{Hilb} \times \text{Hilb} \). As a result we obtain a diagram

\[
\begin{array}{ccc}
\pi_- & \longrightarrow & \pi_+ \\
\downarrow & & \downarrow \\
M_- & \longrightarrow & M_+
\end{array}
\]

where \( \pi_- \) is a sequence of blowing-ups with smooth centers, and so \( \tilde{M} \) is non-singular.

Now suppose that \( A^+(W) = \{ \eta \} \) for simplicity and denote \( T_\eta = T \). For a closed subscheme \( Z \subset V \), let \( B_Z(V) \) mean the blowing-up of \( V \) along \( Z \). First, in case where \( \text{ext}^1_X(F_t, G_t(K_X)) \) is constant for all \( t \in T \), \( P_- \) is non-singular and \( B_{P_-}(M_-) \) is non-singular. Hence from Proposition 2.1 we have a birational morphism \( \pi_+ : B_{P_-}(M_-) \to M_+ \), which we can regard as a desingularization of \( M_+ \).

In general case, set

\[
l_0 = \min \{ \text{ext}^1(F_t, G_t(K_X)) \mid t \in T \} \quad \text{and} \quad l_1 = \max \{ \text{ext}^1(F_t, G_t(K_X)) \mid t \in T \}.
\]

(1)

Since one can readily show \( \text{ext}^2(F_t, G_t(K_X)) = \text{hom}(G_t, F_t) = 0 \) for all \( t \in T \), there is a open covering \( \hat{i} : U \to T \) and a morphism \( F : V^0_U \to V^1_U \) of locally free \( \mathcal{O}_U \)-modules such that \( \text{rk} V^1_U = l_1 \), \( \text{rk} V^1_U - \text{rk} V^0_U = -\chi(F_t, G_t(K_X)) \), and \( P_- \times_T U \cong \text{proj}_U(\text{Cok} F) \).

Denote by \( T_{t_1} \subset T \) the reduced closed subscheme with

\[
T(k) \supset T_{t_1}(k) = \{ t \in T(k) \mid \dim \text{Ext}^1(F_t, G_t(K_X)) = l_1 \}.
\]

Let \( C \subset T_{t_1} \) be any nonsingular subscheme \( C \subset T_{t_1} \) and denote \( P_- \times_T C \) by \( P_C \).

**Lemma 2.2.** We have an open covering \( U' \to B_C(T) \setminus B_C(T_{t_1}) \) and a homomorphism \( F' : V^0_{U'} \to V^1_{U'} \), such that \( \text{Im} F' \supset \text{Im} F \), \( \text{rk}(\text{Cok} F' \otimes k(t)) \leq l_1 - 1 \) for all \( t \in U' \), and

\[
\text{proj}_U(\text{Cok} F') \supset B_{P_C}(P_-) \times_{B_C(T)} U'.
\]

**Proof.** Assume \( U = T \) for simplicity. Let \( (n, A) \) denote the square matrix ring of degree \( n \) with coefficients in a ring \( A \). and \( (n, m, A) \) the set of matrixes of degree \( (n, m) \). Since \( V^1_T \) is locally free, \( F \) locally induces a matrix of degree \( (\text{rk} V^1_T, \text{rk} V^1_T) \). From the definition of \( T_{t_1} \), ideal sheaf \( I_{t_1} \subset \mathcal{O}_T \) equals the radical of \( (D_i) \), where \( D_i \) runs over the set of all minor determinant of degree \( \text{rk} V^1_T - l_1 + 1 \). Hence
Lemma 2.3. The natural immersion $B_{C}(T) \setminus B_{C}(T_{i}) \subset B_{C}(T) = \text{Proj}_{T}(\oplus I_{C}^{d})$. Let $A$ be a square matrix of degree $\text{rk}V_{T}^{1} - l_{1} + 1$ with

$$F = \begin{pmatrix} A & \ast \\ \ast & \ast \end{pmatrix}. $$

Suppose $\det A \neq 0$ in $O_{T}$ and $t \in T$ is the image of a point $s \in D_{+(\det A)}$. In the set of matrixes with coefficients in $O_{T,t}$, $A$ can be transformed into $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ with a square matrix $A'$ with coefficients in $m_{T,t}$.

Since $C$ and $T$ are non-singular, we have a splitting $T \simeq C \times S$ after replacing $T$ by an étale neighborhood of $t$. The point $t \in T$ corresponds to $(t_{1}, t_{2}) \in C \times S$. Let $p_{1} : C \times S \to C$ and $p_{2} : C \times S \to S$ denote projections and $i_{1} : C \simeq C \times \{t_{2}\} \hookrightarrow C \times S$ and $i_{2} : S \simeq \{t_{1}\} \times S \hookrightarrow C \times S$ immersions. For $a \in m_{T,t}$, we have $i_{1*}p_{2*}i_{2*}a = 0$ and so $p_{2*}i_{2*}a \in \text{Ker}i_{1*} = I_{C,t}$. Hence if $A'$ is the cofactor matrix of $A'$, then the matrix $B = p_{2*}i_{2*}(A')/\det A'$ is an element of $M(d, O_{B_{C}(T_{i}), s})$ because of the assumption on $A'$ and the definition of $B_{C}(T)$. Elements of the matrix $A'B - I$ lie in $\text{Ker}i_{2*} = I_{T,t}$, and so $A'$ is invertible in $M(d, O_{B_{C}(T_{i}), s})$. As a result $A$ is invertible in the $M(\text{rk}V_{T}^{1} - l_{1} + 1, O_{B_{C}(T_{i}), s})$ and we obtain the lemma.

\[ \Box \]

**Lemma 2.3.** The natural immersion $B_{P_{C}}(P_{-}) \times B_{C}(T) B_{C}(T_{i}) \subset P_{-} \times T B_{C}(T_{i})$ is isomorphic, and the projection $B_{P_{C}}(P_{-}) \times B_{C}(T) B_{C}(T_{i}) \to B_{C}(T_{i})$ is smooth; that is a $P^{i-1}$-bundle.

\begin{proof}
When we denote by $D \subset B_{C}(T_{i})$ the exceptional divisor, we have

$$P_{T_{i}} \backslash P_{C} \simeq B_{P_{C}}(P_{-}) \times B_{C}(T)[B_{C}(T_{i}) \backslash D] \longrightarrow P_{-} \times T[B_{C}(T_{i}) \backslash D] \longrightarrow B_{P_{C}}(P_{-}) \times B_{C}(T) B_{C}(T_{i}) \longrightarrow P_{-} \times T B_{C}(T_{i}).$$

The first row is isomorphic, and $B_{C}(T_{i}) \backslash D$ is dense in $B_{C}(T_{i})$. The second row is set-theoretically bijective since the dimension of fiber of $B_{C}(T_{i}) \times T P_{-} \to B_{C}(T)$ is upper-semicontinuous. The left-hand side of second row is reduced since $T_{i}$ is reduced so the second row is isomorphic.

There is a sequence of blowing-ups $T_{i}^{N} \to \cdots \to T_{i}^{(1)} \to T_{i} = T_{i}^{(0)}$ with nonsingular center $C^{(i)} \subset T_{i}^{(i)}$ such that $T_{i}^{(N)}$ is nonsingular. They induces a sequence of blowing-ups $T^{(N)} \to \cdots \to T^{(1)} \to T = T^{(0)}$. Put $P^{(0)} = P_{-}$ and $P^{(1)} = B_{P_{C}^{(0)}}(P_{-})$. Then we have a natural morphism $P^{(1)} \to T^{(1)} = B_{C^{(0)}}(T)$ such that the restriction $P^{(1)} \times T^{(1)} C^{(1)} \to C^{(1)}$ is smooth from Lemma 2.3. Set $P^{(2)} = B_{P^{(1)} \times T^{(1)}} C^{(1)}(P^{(1)})$. It is naturally a scheme over $T^{(2)} = B_{C^{(1)}}(T^{(1)})$, and $P^{(2)} \times T^{(2)} C^{(2)} \to C^{(2)}$ is smooth from Lemma 2.3. Denote $M^{(0)} = M_{-}$, $M^{(1)} = B_{P_{C}^{(0)}}(M^{(0)})$ and $M^{(2)} = B_{P^{(1)} \times T^{(1)}} C^{(1)}(M^{(1)})$. $P^{(i)}$ are closed subschemes of $M^{(i)}$ for $i \leq 2$. In the same way we obtain a sequence of blowing-ups $M^{(N)} \to \cdots \to M^{(0)}$ with centers $P^{(i)} \subset M^{(i)}$, which is smooth over $C^{(i)}$ for $0 \leq i \leq N - 1$. The subscheme $P^{(N)} \times T_{i}^{(N)}$ of $P^{(N)}$ is smooth over $T_{i}^{(N)}$ by Lemma 2.3, $T_{i}^{(N)}$ is non-singular by the definition, and hence $P^{(N)} \times T^{(N)} T_{i}^{(N)}$ itself is smooth. Let $M^{(N+1)}$ denote $B_{P^{(N)} \times T^{(N)}} T_{i}^{(N)}(M^{(N)})$. Its closed subscheme $P^{(N+1)} := B_{P^{(N)} \times T^{(N)}} T_{i}^{(N)}(P^{(N)})$
is smooth over a nonsingular scheme $T^{(N+1)} := B_{T_{l_1}}^{(N)}(T^{(N)})$ from Lemma 2.3. Using Lemma 2.2, we can find a homomorphism $F_0 : V^0_{T^{(N+1)}} \rightarrow V^1_{T^{(N+1)}}$ such that $\text{rk Cok } F_0 \otimes k(t) \leq l_1 - 1$ for all $t \in T^{(N+1)}$ and that the $T^{(N+1)}$-scheme $P^{(N+1)}$ is contained in $P_{T^{(N+1)}}((\text{Cok } F_0)$ if we replace $T^{(N+1)}$ with an open covering. Then repeat this process after changing $T$ to $T^{(N+1)}$, $P_-$ to $P^{(N+1)}$, $M_0$ to $M^{(N+1)} := B_{P^{(N)}} \times_{T^{(N)}} T_{l_1}^{(N)}(M^{(N)})$, $F$ to $F_0$ and $l_1$ to $l_1 - 1$. Consequently we obtain a sequence of blowing-ups $T^{(N')} \rightarrow \cdots \rightarrow T^{(0)} = T$ with non-singular center $C^{(i)} \subset T^{(i)}$ and blowing-ups $M^{(N')} \rightarrow \cdots \rightarrow M^{(0)} = M_-$ with center $P^{(i)} \subset M^{(i)}$ as follows. There is a commutative diagram

$$
\begin{array}{ccc}
P^{(i)} & \longrightarrow & T^{(i)} \\
\downarrow & & \downarrow \\
P_- & \longrightarrow & T
\end{array}
$$

and a homomorphism $F^{(m)} : V^0_{T^{(N')}} \rightarrow V^1_{T^{(N')}}$ as follows. It holds that $P_- \subset P_{T^{(N')}}(\text{Cok } F^{(N')})$, $\text{rk Cok } F^{(N')} \otimes k(t) \leq l_0$ for all $t \in t^{(N')}$, and the first row of

$$
\begin{array}{ccc}
P^{(N')} & \longrightarrow & P_{T^{(N')}}(\text{Cok } F^{(N')}) \\
\downarrow & & \downarrow \\
P_- & \longrightarrow & P_{T}(\text{Cok } F)
\end{array}
$$

is isomorphic when it is restricted to the inverse image of $T^{(N')} \setminus T^{(N')} \times_T T_{l_0}$. Thus $P^{(N')} \rightarrow T^{(N')}$ is smooth by the same proof as Lemma 2.3. Set $M^{(N'+1)} = B_{P^{(N')}}(M^{(N')})$. Since $M^{(N'+1)} \rightarrow M_-$ is a composition of blowing-ups of the smooth scheme $M_-$ along nonsingular centers, $M^{(N'+1)}$ itself is nonsingular. One can verify that $M^{(N'+1)} \rightarrow M_-$ splits as $M^{(N'+1)} \rightarrow B_{P_-}(M_-)$, so we obtain a morphism $M^{(N'+1)} \rightarrow B_{P_-}(M_-) = B_{P_+}(M_+) \rightarrow M_+$ from Proposition 2.1, which is a desingularization of $M_+$ since $M^{(N'+1)}$ is nonsingular. Put $\tilde{M} = M^{(N'+1)}$. We have constructed

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\pi_-} & M_- \\
\downarrow \pi & & \downarrow \pi \\
M_+ & \xrightarrow{\pi_+} & M_+
\end{array}
$$

with $\tilde{M}$ nonsingular.

3. Calculation of $K_{\tilde{M}} - \pi_+^* K_{M_+}$

Assume that $M_+ \supset \text{Sing}(M_+) := \{ E \mid \text{ext}^2(E, E)^0 \neq 0 \}$ satisfies $\text{codim}(M_+, \text{Sing}(M_+)) \geq 2$ and that $P_+ \subset M_+$ is nowhere dense, and hence both $M_-$ and $M_+$ are locally complete intersections and so Gorenstein schemes. Let us calculate $K_{\tilde{M}} - \pi_+^* K_{M_+}$. By construction, each step $M^{(i+1)} \rightarrow M^{(i)}$ in $\tilde{M} \rightarrow M_-$ is a $P^{(i-1)}$-bundle over $C^{(i)} \subset T^{(i)} \times_T T_{l(i)} \subset T^{(i)}$, where $l_0 \leq l(i) \leq l_1$ and $T_l \subset T$ is the reduced subscheme
such that $T_i(t) = \{ t \in T \mid \dim \text{Ext}^1(G_t, F_t) \geq l(t) \}$ set-theoretically. If we denote by $D(i) \subset \hat{M}$ the pull-back of the exceptional divisor of $M(i) \to M(i-1)$, then

$$K_M - \pi^*_M \hat{K}_M = \sum_{i=0}^{N-1} \text{codim}(P(i), M(i)) D(i) = \sum_i [\dim M_+ - (l(i) - 1 + \dim C(i))] D(i).$$

(3)

Next consider $\pi^+_M(K_{M+}) - \pi^+_M(K_{M-})$. By the proof of Proposition 2.1, which uses elementary transform, we have the following.

**Proposition 3.1.** Denote the exceptional divisor $\pi^{-1}_-(P_-) = \pi^+_+(P_+)$ by $D$. Suppose we have a universal family $E^+_{M_+} \in \text{Coh}(X_{M_+})$ of $M_+$ and a universal family $E^+_{M_+} \in \text{Coh}(X_{M_+})$ of $M_+$. If $\pi : D \to P_+ \to T$ is a natural map, then there are $T$-flat modules $F_T$ and $G_T$ on $X_T$, line bundles $L_\pm$ on $P_\pm$ and a line bundle $L_0$ on $\hat{M}$ such that we have exact sequences

$$0 \to \pi^+_M E^+_{M_+} \otimes L_0 \to \pi^-_M E^-_{M_+} \to \pi^+_G \otimes \pi^+_L \to 0$$

(4)

in $\text{Coh}(X_{M_+})$ and

$$0 \to \pi^+_M F_T \otimes \pi^-_L \to \pi^-_M (E^-_{M_+})|_{X_D} \to \pi^+_G \otimes \pi^+_L \to 0$$

(5)

in $\text{Coh}(X_{M_+})$.

The exact sequence (5) is the relative $a_+$-Harder Narasimhan filtration of $E^-_{M_+}$. Here we remark that generally a universal family of $M_+$ exists étale-locally, but one can generalize this proposition to general case with straightforward labor. Suppose $L_\pm$ and $L_0$ in this proposition are trivial for simplicity. From (4)

$$\pi^+_M K_{M+} - \pi^+_M K_{M-}$$

$$= \pi^+_M \begin{align*}
\text{det} \text{RHom}_{X_{M_+}/M_+}(E^-_{M_+}, E^-_{M_+}) - \pi^+_M \text{det} \text{RHom}_{X_{M_+}/M_+}(E^+_{M_+}, E^+_{M_+})
\end{align*}

= \text{det} \text{RHom}_{X_{M_+}/M_+}(\pi^+_M E^-_{M_+}, \pi^+_M E^-_{M_+}) - \text{det} \text{RHom}_{X_{M_+}/M_+}(\pi^+_M E^+_{M_+}, \pi^+_M E^+_{M_+})

= \text{det} \text{RHom}_{X_{M_+}/M_+}(E^-_{M_+}, E^+_{M_+}) + \text{det} \text{RHom}_{X_{M_+}/M_+}(E^+_{M_+}, \pi^+_M G_T)

- \text{det} \text{RHom}_{X_{M_+}/M_+}(E^-_{M_+}, E^+_{M_+}) + \text{det} \text{RHom}_{X_{M_+}/M_+}(\pi^+_M G_T, E^+_{M_+})

= \text{det} \text{RHom}_{X_{M_+}/M_+}(E^-_{M_+}, G_D) + \text{det} \text{RHom}_{X_{M_+}/M_+}(G_D, E^+_{M_+}).

$$

If $i : D \to M_+$ is inclusion, then by (5)

$$\text{det} \text{RHom}_{X_{M_+}/M_+}(E^-_{M_+}, G_D) = \text{det} i_* \text{RHom}_{X_{M_+}/D}(E^-_{M_+}|D, G_D) = \text{det} i_* \text{RHom}_{X_{M_+}/D}(F_D, G_D) + \text{det} i_* \text{RHom}_{X_{M_+}/D}(G_D, G_D).$$

(6)

Since $\text{det} \mathcal{O}_D = D$, (6) equals $[\chi(F_t, G_t) + \chi(G_t, G_t)] D$ for any $t \in D$. By the Serre duality

$$\text{det} \text{RHom}_{X_{M_+}/M_+}(G_D, E^+_{M_+}) = \text{det} \text{RHom}_{M_+}(\text{RHom}_{X_{M_+}/M_+}(E^+_{M_+}, G_D(K_X)), \mathcal{O}_{M_+})$$

$$= -\text{det} \text{RHom}_{X_{M_+}/M_+}(E^+_{M_+}, G_D(K_X)) = -\text{det} i_* \text{RHom}_{X_{M_+}/D}(E^+_{M_+}|D, G_D(K_X))$$

$$= -[\chi(F_t, G_t(K_X)) + \chi(G_t, G_t(K_X))] D = -[\chi(G_t, F_t) + \chi(G_t, G_t)] D.$$
Therefore $\tilde{\pi}_+^* K_{M+} - \tilde{\pi}_-^* K_{M-} = [\chi(F_t, G_t) - \chi(G_t, F_t)] D$. On the other hand, we put

$$\tilde{\pi}^* D = \sum_{i=0}^{N'} \lambda_i D^{(i)}. \quad (7)$$

If $\lambda_i$ is determined, then we can calculate $K_M - \tilde{\pi}_+^* K_{M+}$ by (3) and (7). Let $Z_{M+} \subset P_+$ denote the pull-back of $\bigcup_{i=0}^l \text{Sing}(T_i) \subset T$ by $P_+ \to T$, which is a nowhere-dense closed subscheme. Let us consider the induced open subset

$$U_{M+} = M+ \setminus Z_{M+}. \quad (8)$$

One can regard $M_+ \supset P_+ \supset P^{(0)}$ étale-locally as $k[x_1, \ldots, x_m] \supset I_{P^{(0)}} = (x_1, \ldots, x_n) \supset I_{P_+} = (f_1, \ldots, f_m)$. We have

$$B_{P^{(0)}}(M_-)_{(x_n)} = \text{Spec} k[x_1/x_n = x_1', \ldots, x_{n-1}/x_n = x_n', x_n, \ldots, x_m],$$

and if $\pi_0 : M^{(1)} := B_{P^{(0)}}(M_-) \to M_-$ is a natural morphism, then

$$k[x_1, \ldots, x_n, x_n', \ldots, x_m] \supset \pi_0^{-1} I_{P_+} \cdot \mathcal{O}_{P^{(1)}} = (x_1^{N_1} \bar{f}_1(x_1, x_2), \ldots, x_m^{N_m} \bar{f}_m)$$

where $\bar{f}_i$ cannot be divided by $x_n$. $\lambda_0$ equals $\max(N_i)$. It becomes 1 when $\dim M_+ > l_1 - 1 + \dim C^{(0)}$. Indeed for any point $t \in P^{(0)}$, $\text{rk}\Omega_{P^{(0)}} \otimes k(t) \leq l_1 - 1 + \dim C^{(0)}$ since $P^{(0)}$ is a $\mathbf{P}^{l_1-1}$-bundle over $C^{(0)}$. Hence in the exact sequence

$$CN_{P_+/M_-} \otimes k(t) \xrightarrow{\tau} \Omega_{M_-} \otimes k(t) \xrightarrow{\omega} \Omega_{P_-} \otimes k(t) \rightarrow 0,$$

$\tau$ cannot be zero if $\dim M_+ > l_1 - 1 + \dim C^{(0)}$, so $N_j = 1$ for some $j$. Over $U_{M+}$ we can suppose $T_i$ is non-singular, and so $P^{(i)} \times_{M+} U_{M+}$ does not contain any irreducible component of the exceptional divisor of $M^{(i)} \to M$ when $\dim T_i < \dim T_{i-1}$ for all $l_0 < l \leq l_1$. Thereby, similarly to the case where $i = 0$, one can show that $\lambda_i \geq 1$, and that if $\dim M_+ > l^{(i)} - 1 + \dim C^{(i)}$ then $\lambda_i = 1$, since $P^{(i)}$ is a $\mathbf{P}^{l^{(i)}-1}$-bundle over $C^{(i)}$ for $l_0 \leq l^{(i)} \leq l_1$. Thus we have shown the following.

**Proposition 3.2.** In the diagram (2) it holds that

$$K_M - \tilde{\pi}_+^* K_{M+} - \sum_{i=0}^{N'} [\dim M_- - (l^{(i)} - 1 + \dim C^{(i)}) + \lambda_i \{\chi(F_t, G_t) - \chi(G_t, F_t)\}] D^{(i)}. \quad (9)$$

Suppose $\dim T_i < \dim T_{i-1}$ for all $l_0 < l \leq l_1$ and $\dim M_+ > l_1 - 1 + \dim T$. If the image of $T^{(i)} \subset M^{(i)}$ in $T$ agrees with $T_j$ for some $j$, then $\lambda_i = 1$.

Remark that the image of $D^{(i)}$ in $T$ agrees with $T_j$ for some $j$ if $D^{(i)}$ has non-empty intersection with $\tilde{\pi}_+^{-1}(U_{M+})$. Thus one can use this proposition to verify whether singularities in $U_{M+}$ are terminal or not.

**Remark 3.3.** When the image of $D^{(i)}$ in $T$ does not agree with $T_j$ for any $j$, the value $\lambda_i$ seems to relate with determinantal varieties over $C$. 
4. Examples: Ruled or Elliptic Surface

We shall give examples of $M_+$ with $M_-$ non-singular. If a morphism $X \to C$ to a nonsingular curve $C$ exists, then by [2, p.142] we have a $(c_1, c_2)$-suitable polarization, that is, an ample line bundle $H$ such that $H$ does not lie on any wall of type $(c_1, c_2)$, and for any wall $W = W^\nu$ of type $(c_1, c_2)$, we have $\eta \cdot f = 0$ or $\text{Sign}(f \cdot \eta) = \text{Sign}(H \cdot \eta)$. From [2, p.159, p.201], if $X$ is a ruled surface or an elliptic surface, then any rank-two sheaf $E$ of type $(c_1, c_2)$ which is stable respect to $(c_1, c_2)$-suitable polarization is good, i.e., $\text{Ext}^2(E, E)^0 = 0$.

(A) First we suppose that $X$ is a (minimal) ruled surface. When $c_1 \cdot f$ is odd $M(H)$ is empty for $(c_1, c_2)$-suitable polarization. Thus we assume $c_1 = 0$. If a rank-two sheaf $E$ of type $(c_1, c_2)$ is stable with respect to a polarization $H$ such that $H \cdot K_X < 0$, then $E$ is good and so $M(H)$ is nonsingular. Hence we assume that $W^{K_X} \cap \text{Amp}(X) \neq \emptyset$, so $2 \leq g = g(C)$ and $e(X) \leq 2g - 2$ from the description of $\text{Amp}(X)$ [3, Prop. V.2.21]. Since $\dim \text{NS}(X) = 2$, if we move polarization $H$ from a $(c_1, c_2)$-suitable one, then $M(H)$ may begin to admit singularities when $H$ passes the wall $W^{K_X}$. Let $H_-$ and $H_+$ be ample line bundles separated by only one wall $W^{K_X}$. $M(H_-)$ is non-singular, and $E^+ \in P_+$ has a non-trivial exact sequence

$$0 \to G = L \otimes I_{Z_1} \to E^+ 
\to F = L^{-1} \otimes I_{Z_r} \to 0 \quad (10)$$

with $-2L \sim mK_X$. About this filtration we have $\text{Ext}^2(E^+, E^+)^0 = 0$ since $p_g(X) = 0$ (See [4, p. 49] for $\text{Ext}_\pm$), and

$$\text{ext}^2(E^+, E^+) = \text{ext}_+^2(E^+, E^+) = \text{ext}^2(L \otimes I_{Z_1}, L^{-1} \otimes I_{Z_r}) = \text{hom}(I_{Z_r}, \mathcal{O}(K_X + 2L)I_{Z_1}).$$

Since $W^{K_X}$ defines a wall, $H^0(\mathcal{O}(K_X + 2L)) = 0$ unless $2L + K_X = 0$. Hence $\text{ext}^2(E^+, E^+)^0 \neq 0$ if and only if $-2L = K_X$ and $Z_1 \subset Z_r$. As a result when one defines $a$-stability using $H_\pm$,

$$\chi^a(\otimes I_{Z_r}, -K_X \otimes I_{Z_1}) = Aa + B + l(Z_l)$$

for some constant $A$ and $B$, and so the moduli scheme $M(a)$ of $a$-stable sheaves begins to admit singularities just when $a$ passes a miniwall $a_0$ defined by

$$l(Z_l) = \begin{cases} c_2/2 - (g - 1) & \text{if } c_2 \text{ is even} \\ (c_2 - 1)/2 - (g - 1) & \text{if } c_2 \text{ is odd}. \end{cases}$$

Let $a_-$ and $a_+$ be mini-chambers separated by only one miniwall $a_0$. $M(a_+) = M_+$ has singularities along $P_+ \times_T T'$, where

$$T' = \{(L \otimes I_{Z_1}, L^{-1} \otimes I_{Z_r}) \mid -2L = K_X\}_{\text{red}} \subset M(1, K_X/2, l(Z_1)) \times M(1, -K_X/2, l(Z_r)).$$

(B) Suppose that $X$ is an elliptic surface with a section $\sigma$ and $c_1 = \sigma$. In contrast to ruled surfaces, $K_X^2 = 0$ and so $W^{K_X} \cap \text{Amp}(X)$ is always empty, though one can study some singularity appearing in $M(H)$ by Proposition 3.2. Let $\pi : \tilde{X} \to C$ be an elliptic fibration, $f \in \text{NS}(X)$ its fiber class, $d = -\deg R^1\pi_*(\mathcal{O}_X) - \sigma^2 \geq 0$. We have a natural map to a ruled surface $\kappa : X \to \mathbf{P}(\pi_*(\mathcal{O}(2\sigma))) = \mathbf{P}(\mathcal{E}_2)$. Since $\kappa_*(\sigma)$ is a section of $\mathbf{P}(\mathcal{E}_2)$, and since the pull-back of an ample line bundle by a finite map is ample, $L = af$ satisfies $W^{2L-\sigma} \cap \text{Amp}(X) \neq \emptyset$ if $a > 0$ from the description of the ample cone of a ruled surface. Let $c_1$ be $\sigma$ and $c_2 = (c_1 - L) \cdot L = a$. Then any
sheaf $E$ with non-trivial exact sequence
\[ 0 \rightarrow F = L \rightarrow E \rightarrow G = L^{-1} \otimes c_1 \rightarrow 0, \tag{11} \]
whose Chern class equals $(c_1, c_2)$, is stable with respect to a $(c_1, c_2)$-suitable ample line bundle. Indeed, $(2L - c_1) \cdot f < 0$ and so $\pi_*(\mathcal{O}(2L - c_1)) = 0$ and $R^1\pi_*(\mathcal{O}(2L - c_1))$ commutes with base change. Thus the exact sequence
\[ 0 \rightarrow H^1(C, \pi_*(\mathcal{O}(2L - c_1))) \rightarrow H^1(X, \mathcal{O}(2L - c_1)) \rightarrow H^0(E, R^1\pi_*(\mathcal{O}(2L - c_1))) \]
shows that the restriction of the exact sequence (11) to a general fiber is non-trivial, and so a corollary of Artin’s theorem for vector bundles on an elliptic curve [2, p. 144] deduce that $E$ is stable with respect to a suitable polarization. Thereby such $E$ is good. Let $H_-$ and $H_+$ be ample line bundles which lie in no wall of type $(c_1, c_2)$ with $(2L - c_1) \cdot H_- < 0 < (2L - c_1) \cdot H_+$. One can define $\alpha$-stability by them. Let $a_0$ be a miniclaw such that $\chi^{a_0}(\mathcal{O}(L)) = \chi^{a_0}(\mathcal{O}(2L - c_1))$, $a_- < a_0 < a_+$ minichambers, and $M_{\pm} = M(a_{\pm})$. Then some connected components of $P_- \subset M_-$ contains any sheaf $E$ with non-trivial exact sequence (11), and some neighborhood of them in $M_-$ is non-singular.

It induces a desingularization of some open neighborhood of connected components $K_+$ of $P_+$ consisting of sheaves $E^+$ with a non-trivial exact sequence
\[ 0 \rightarrow L^{-1} \otimes c_1 \rightarrow E^+ \rightarrow L \rightarrow 0 \]
as in Section 2.

We have in case of (A) $\text{ext}^1(G, F) \leq 1$, and in case of (B) $\text{ext}^1(G, F) = h^0(c_1 - 2L + K_X) - \chi(c_1 - 2L) \leq 2c_2 + C(X)$ with some constant $C(X)$ independent of $c_2$ because $h^0(c_1 - 2L + K_X) = 0$ if $a = c_2$ is sufficiently large. Thus in both cases one can show that, if $c_2$ is sufficiently large, then all singularities of $M_+^\infty$ along the dense open set $U_{M^+} \cap P_+^{\infty}$ in $P_+^{\infty} \subset \text{Sing}(M_+)$ defined at (8) are terminal. s called by the amsart/book/proc definition of MR.

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