ADIABATIC LIMITS AND SPECTRAL SEQUENCES FOR 
RIEMANNIAN FOLIATIONS

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Abstract. For general Riemannian foliations, spectral asymptotics of the Laplacian is studied when the metric on the ambient manifold is blown up in directions normal to the leaves (adiabatic limit). The number of “small” eigenvalues is given in terms of the differentiable spectral sequence of the foliation. The asymptotics of the corresponding eigenforms also leads to a Hodge theoretic description of this spectral sequence. This is an extension of results of Mazzeo-Melrose and R. Forman.

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1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{F}$ be a $C^\infty$ foliation on a closed Riemannian manifold $(M, g)$, and let $T\mathcal{F} \subset TM$ denote the subbundle of vectors tangent to the leaves. Then the metric $g$ can be written as an orthogonal sum, $g = g_\perp \oplus g_\mathcal{F}$, with respect to the decomposition $TM = T\mathcal{F}^\perp \oplus T\mathcal{F}$; i.e., $g_\perp, g_\mathcal{F}$ are the restrictions of $g$ to $T\mathcal{F}^\perp, T\mathcal{F}$, respectively.

Partially supported by Xunta de Galicia, grant XUGA20701B97.
Partially supported by Dirección General de Enseñanza Superior e Investigación Científica (Spain), sabbatical grant SAB1995-0717.
By introducing a parameter $h > 0$, we can define a family of metrics
\[ g_h = h^{-2} g_\perp \oplus g_F . \] (1.1)

The “limit” of the Riemannian manifolds $(M, g_h)$ as $h \downarrow 0$ is what is known as adiabatic limit. Observe that, in a foliation chart, the plaques get further from each other as $h \downarrow 0$. This form of the adiabatic limit was introduced by E. Witten in [35] for Riemannian bundles over the circle. Witten investigated the limit of the eta invariant of the Dirac operator. This question was also considered in [9], [10] and [12], and extended to general Riemannian bundles in [8] and [14].

New properties of adiabatic limits were discovered by Mazzeo and Melrose for the case of Riemannian bundles, relating them to the Leray spectral sequence [25]. This work was used in [14], and further developed by R. Forman in [16], where the very general setting of any pair of complementary distributions is considered. Nevertheless the most interesting results of [16] are only proved for foliations satisfying very restrictive conditions. The ideas from [25] and [16] were also applied to the Rumin’s complex by Z. Ge [17], [18].

For a general $C^\infty$ foliation $F$ on $M$, the role of (the differentiable version of) the Leray spectral sequence is played by the so called differentiable spectral sequence $(E_k, d_k)$, which converges to the de Rham cohomology of $M$. The definition of $(E_k, d_k)$ is given by filtering the de Rham complex $(\Omega, d)$ of $M$ as in the bundle case: A differential form $\omega$ of degree $r$ is said to be of filtration $\geq k$ if it vanishes whenever $r - k + 1$ of the vectors are tangent to the leaves; that is, roughly speaking, if $\omega$ is of degree $\geq k$ transversely to the leaves. Moreover the $C^\infty$ topology of $\Omega$ induces a topological vector space structure on each term $E_k$ such that $d_k$ is continuous. A subtle problem here is that $E_k$ may not be Hausdorff [20]. So it makes sense to consider the subcomplex given by the closure of the trivial subspace, $\bar{0}_k \subset E_k$, as well as the quotient complex $\hat{E}_k = E_k/\bar{0}_k$, whose differential operator will be also denoted by $d_k$.

The differentiable spectral sequence is known to satisfy certain good properties for the so called Riemannian foliations, which are the foliations with “rigid transverse dynamics”; i.e., foliations with isometric holonomy for some Riemannian metric on smooth transversals. A characteristic property of Riemannian foliations is the existence of a so called bundle-like metric on the ambient manifold, which means that the foliation is locally defined by Riemannian submersions [29], [27], [28]. For such foliations, each term $E_k$ is Hausdorff of finite dimension if $k \geq 2$, and $H(\bar{0}_1) = 0$ [22], [3]. So $E_k \cong \hat{E}_k$ for $k \geq 2$. Moreover it was recently proved by X. Masa and the first author that, for $k \geq 2$, the terms $E_k$ are homotopy invariants of Riemannian foliations [1]—this generalizes previous work showing the topological invariance of the so called basic cohomology [22].

Besides the requirement that $F$ has to be a Riemannian foliation, the mentioned restrictive hypothesis of R. Forman in [16] is that the positive spectrum of the “leafwise Laplacian” on $\Omega$ must be bounded away from zero [4]. Both conditions together are so strong that the only examples we know are Riemannian foliations with compact leaves; i.e., Seifert bundles. The purpose of our paper is to generalize Forman’s work to arbitrary Riemannian foliations. To state our first main result, let $\Delta_{g_h}$ denote the Laplacian defined by $g_h$ on differential forms, and let
\[ 0 \leq \lambda_0^0(h) \leq \lambda_1^1(h) \leq \lambda_2^2(h) \leq \cdots \]

1The leafwise Laplacian is what will be denoted by $\Delta_0$ in this paper.
denote its spectrum on $\Omega^r$, taking multiplicities into account. It is well known that the eigenvalues of the Laplacian on differential forms vary continuously under continuous perturbations of the metric \cite{[14]}, and thus the “branches” of eigenvalues $\lambda^r_i(h)$ depend continuously on $h > 0$. In this paper, we shall only consider the “branches” $\lambda^r_i(h)$ that are convergent to zero as $h \downarrow 0$; roughly speaking, the “small” eigenvalues. The asymptotics as $h \downarrow 0$ of these metric invariants is related to the differential invariant $\hat{E}^r_1$ and the homotopy invariants $E^r_k$, $k \geq 2$, as follows.

**Theorem A.** With the above notation, for Riemannian foliations on closed Riemannian manifolds we have

\begin{align*}
\dim \hat{E}^r_1 &= \# \{ i \mid \lambda^r_i(h) \in O(h^2) \text{ as } h \downarrow 0 \}, \quad (1.2) \\
\dim E^r_k &= \# \{ i \mid \lambda^r_i(h) \in O(h^{2k}) \text{ as } h \downarrow 0 \}, \quad k \geq 2. \quad (1.3)
\end{align*}

As a part of the proof of Theorem A, and also because of its own interest, we shall also study the asymptotics of eigenforms of $\Delta \omega_h$ corresponding to “small” eigenvalues. This study was begun in \cite{[25]} for the case of Riemannian bundles, and continued in \cite{[16]} for general complementary distributions. From both \cite{[25]} and \cite{[16]}, certain rescaling $\Theta_h$ of differential forms, depending on $h > 0$, is crucial to study this asymptotics.

The following well known technicality will be useful to explain $\Theta_h$. The decomposition $TM = TF \perp TF$ induces a bigrading

\begin{equation}
\bigwedge TM^* = \bigoplus_{u,v} \left( \bigwedge^u TF^{\perp*} \otimes \bigwedge^v TF^* \right); \quad (1.4)
\end{equation}

roughly speaking, here $u$ denotes transverse degree and $v$ tangential degree. Then the bigrading of $\Omega$ is defined by considering $C^\infty$ sections of (1.4); i.e., each $\Omega^{u,v}$ is the space of $C^\infty$ sections of $\bigwedge^u TF^{\perp*} \otimes \bigwedge^v TF^*$. Then the de Rham derivative and coderivative decompose as sum of bihomogeneous components,

\begin{equation}
d = d_{0,1} + d_{1,0} + d_{2,-1}, \quad \delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1}; \quad (1.5)
\end{equation}

where the double subindex denotes the corresponding bidegree (see e.g. \cite{[1]}); observe that $d_{i,j}^* = \delta_{-i,-j}$.

Now define $\Theta_h \omega = h^n \omega$ if $\omega \in \bigwedge TM^*$ is of transverse degree $u$. As pointed out in \cite{[25]} and \cite{[16]}, such a $\Theta_h$ is an isometry of Riemannian vector bundles $(\bigwedge TM^*, g_h) \to (\bigwedge TM^*, g)$, where $g, g_h$ also denote the metrics induced by $g, g_h$ on $\bigwedge TM^*$. So we get an isomorphism, also denoted by $\Theta_h$, between the corresponding Hilbert spaces of $L^2$ sections because the volume elements induced by the metrics $g_h$ are multiples of each other. Thus our setting is moved via $\Theta_h$ to the fixed Hilbert space of square integrable differential forms on $M$ with the inner product induced by $g$; this Hilbert space is denoted by $\Omega$ in this paper. Concretely, we have the “rescaled derivative” $d_h = \Theta_h d \Theta_h^{-1}$, whose $g$-adjoint is the “rescaled coderivative” $\delta_h = \Theta_h \delta g_h \Theta_h^{-1}$. It is easy to verify that

\begin{equation}
d_h = d_{0,1} + hd_{1,0} + h^2d_{2,-1} \quad (1.6)
\end{equation}
directly from \( (3.3) \) and the definition of \( \Theta_h \). Thus\(^2 \)

\[
\delta_h = \delta_{0,-1} + h\delta_{-1,0} + h^2\delta_{-2,1}.
\]

(1.7)

The “rescaled Laplacian”

\[
\Delta_h = \Theta_h \Delta_{g_h} \Theta_h^{-1} = d_h \delta_h + \delta_h d_h
\]

is elliptic and essentially self-adjoint in \( \Omega \). Moreover \( \Delta_h \) has the same spectrum as \( \Delta_{g_h} \), and eigenspaces of \( \Delta_{g_h} \) are transformed into eigenspaces of \( \Delta_h \) by \( \Theta_h \). We shall prove that eigenspaces of \( \Delta_h \) corresponding to “small” eigenvalues are convergent as \( h \downarrow 0 \) when the metric \( g \) is bundle-like, and the limit is given by a nested sequence of bigraded subspaces,

\[
\Omega \supset H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_\infty.
\]

The definition of \( H_1, H_2 \) was already given in \( [3] \) as a Hodge theoretic approach to \( (E_1, d_1) \) and \( (E_2, d_2) \), which is based on our study of leafwise heat flow. The other spaces \( H_k \) are defined in this paper as an extension of this Hodge theoretic approach to the whole spectral sequence \( (E_k, d_k) \) (see Sections 2.2 and 5.1 for the precise definition of \( H_k \)). In particular,

\[
H_1 \cong \hat{E}_1, \quad H_k \cong E_k, \quad k = 2, 3, \ldots, \infty,
\]

as bigraded topological vector spaces. Thus this sequence stabilizes\(^3 \) because the differentiable spectral sequence is convergent in a finite number of steps. The convergence of eigenforms corresponding to “small” eigenvalues is precisely stated in the following result, where \( L^2 H_1 \) denotes the closure of \( H_1 \) in \( \Omega \).

**Theorem B.** For any Riemannian foliation on a closed manifold with a bundle-like metric, let \( \omega_i \) be a sequence in \( \Omega^r \) such that \( \| \omega_i \| = 1 \) and

\[
\langle \Delta_h \omega_i, \omega_i \rangle \in o\left(h_i^{2(k-1)}\right)
\]

for some \( k = 1, 2, 3, \ldots \) and some sequence \( h_i \downarrow 0 \). Then some subsequence of the \( \omega_i \) is strongly convergent, and its limit is in \( L^2 H_1 \) for \( k = 1 \), and in \( H_k \) for \( k \geq 2 \).

To simplify notation let \( m_k^r = \dim \hat{E}_k \), and let \( m_k^r = \dim E_k \) for each \( k = 2, 3, \ldots, \infty \). Thus Theorem \( [3] \) establishes \( \lambda_i^r(h) \in O(h^{2k}) \) for \( i \leq m_k^r \), yielding \( \lambda_i^r(h) \equiv 0 \) for \( i \) large enough. For every \( h > 0 \), consider the nested sequence of graded subspaces

\[
\Omega \supset H_1(h) \supset H_2(h) \supset H_3(h) \supset \cdots \supset H_\infty(h),
\]

where \( H_k(h) \) is the space generated by the eigenforms of \( \Delta_h \) corresponding to eigenvalues \( \lambda_i^r(h) \) with \( i \leq m_k^r \); in particular, we have \( H_k(h) = H_\infty(h) = \ker \Delta_h \) for \( k \) large enough. Set also \( H_k(0) = H_k \). We have \( \dim H_k^r(h) = m_k^r \) for all \( h > 0 \), so the following result is a sharpening of Theorem \( [3] \).

**Corollary C.** For any Riemannian foliation on a closed manifold with a bundle-like metric and \( k = 2, 3, \ldots, \infty \), the assignment \( h \mapsto H_k(h) \) defines a continuous map from \( [0, \infty) \) to the space of finite dimensional linear subspaces of \( \Omega^r \) for all \( r \geq 0 \). If \( \dim \hat{E}_1 < \infty \), then this also holds for \( k = 1 \).

\(^2\)Another way to check \( (1.7) \) is by proving directly that

\[
\delta_{g_h} = \delta_{0,-1} + h^2\delta_{-1,0} + h^4\delta_{-2,1}.
\]

\(^3\)We mean \( H_k = H_\infty \) for \( k \) large enough.
In Corollary 4, the continuity of $h \mapsto H_k^r(h)$ for $h > 0$ is a particular case of the general property that eigenspaces of the Laplacian on closed Riemannian manifolds vary continuously as subspaces of $\Omega$ when the metric is perturbed $C^0$-continuously \[1]. \[6]. On the other hand, the continuity of $h \mapsto H_k^r(h)$ at $h = 0$ is a direct consequence of Theorem B.

With an analogous aim, other nested sequences of bigraded subspaces were introduced by Mazzeo-Melrose in \[25\] and by R. Forman in \[16\], which are respectively denoted by

\[\Omega \supset \mathfrak{h}_1 \supset \mathfrak{h}_2 \supset \mathfrak{h}_3 \supset \cdots \supset \mathfrak{h}_\infty, \quad \Omega \supset \mathfrak{f}_1 \supset \mathfrak{f}_2 \supset \mathfrak{f}_3 \supset \cdots \supset \mathfrak{f}_\infty\]

in this paper. These sequences are defined in the following way. According to the expressions \[1.3\] and \[1.7\], we can consider $d_h$ and $\delta_h$ as polynomials on the variable $h$ whose coefficients are the differential operators $d_{i,j}$ and $\delta_{i,j}$. Thus $d_h$ and $\delta_h$ canonically become operators on the polynomial algebra $\Omega[h]$, and $\Delta_h$ as well. Then each $\mathfrak{h}_k$ is the space of differential forms $\omega \in \Omega$ with some extension $\tilde{\omega}(h) \in \Omega[h]$ satisfying

\[\Delta_h \tilde{\omega}(h) \in h^k \Omega[h], \quad (1.10)\]

where extension means $\tilde{\omega}(0) = \omega$. And each $\mathfrak{f}_k$ is the space of differential forms $\omega \in \Omega$ with some extension $\tilde{\omega}(h) \in \Omega[h]$ satisfying

\[d_h \tilde{\omega}(h) \in h^k \Omega[h], \quad \delta_h \tilde{\omega}(h) \in h^k \Omega[h]. \quad (1.11)\]

The sequence $H_k$ also fits in this kind of description as follows (this is a direct consequence of Theorem 5.1): Each $H_k$ is the space of differential forms $\omega \in \Omega$ having sequences of extensions $\tilde{\omega}_1^i(h), \tilde{\omega}_2^i(h) \in \Omega[h]$ satisfying

\[d_h \tilde{\omega}_1^i(h) + h^k \Omega[h] \rightarrow 0, \quad \delta_h \tilde{\omega}_2^i(h) + h^k \Omega[h] \rightarrow 0 \quad (1.12)\]

in $\Omega[h]/h^k \Omega[h]$ as $i \rightarrow \infty$. From \[1.6\], \[1.7\], \[1.11\] and \[1.12\] it easily follows that

\[\mathfrak{f}_k \subset \mathfrak{h}_k \subset \mathfrak{f}_{k/2} \quad (1.13)\]

\[\mathfrak{f}_1 = \mathfrak{h}_1, \quad \mathfrak{f}_k \subset \mathfrak{h}_k, \quad k \geq 2. \quad (1.14)\]

For the case of Riemannian bundles, Mazzeo and Melrose prove in \[25\] that the sequence $\mathfrak{h}_k$ stabilizes, and $\mathfrak{h}_\infty$ is the limit of the spaces $\ker \Delta_h$ as $h \downarrow 0$. And for foliations under the restrictive hypothesis of \[16\], R. Forman proves that the sequence $\mathfrak{f}_k$ is a Hodge theoretic version of the spectral sequence $(E_k, d_k)$, and describes the limit of the eigenspaces of $\Delta_h$ corresponding to “small” eigenvalues. This improves the results of Mazzeo-Melrose by \[1.13\]. But Forman’s sequence $\mathfrak{f}_k$ does not have the same important properties for general Riemannian foliations and bundle-like metrics, as follows from the following result, where the notation $H_k(g)$ and $\mathfrak{f}_k(g)$ is used to emphasize the dependence of $H_k$ and $\mathfrak{f}_k$ on the metric $g$—of course, each $H_k(g)$ is independent of $g$ up to isomorphism by \[1.8\].

**Theorem D.** Let $\mathcal{F}$ be a Riemannian foliation of dimension $p$ on a closed manifold $M$. We have:

(i) There is a bundle-like metric $g$ on $M$ such that $\mathfrak{f}_2^{0,p}(g) = H_2^{0,p}(g)$.

(ii) If $\tilde{\mathfrak{f}}_1^{0,p} \neq 0$, then there is a bundle-like metric $g'$ on $M$ such that $\mathfrak{f}_2^{0,p}(g') = 0$.

The condition $\tilde{\mathfrak{f}}_1^{0,p} \neq 0$ holds for Kronecker’s flows on $T^2$ whose slope is a Liouville’s number \[21\], \[30\]. This was generalized to linear foliations on tori of arbitrary
Moreover $E^{0,p}_2 \cong \mathbb{R}$ in these examples [2],[2]. Therefore Theorem D implies that, in these examples, the dimension of $H^{0,p}_2(g)$ changes when appropriately varying the metric $g$. Thus $H^{0,p}_2(g) \not\cong E^{0,p}_2$ for appropriate choices of $g$; that is, Corollary 4.4 is not completely right with that generality—the possibility that $E_1$ may not be Hausdorff is not considered in that paper. So far it is rather unknown which topological or geometric conditions imply $\bar{0}_1 \neq 0$ for general Riemannian foliations, but the above examples suggest that this may happen “generically”.

A simple argument shows that $H^r_k(h) = H^r_k$ if $h \mapsto H^r_k(h)$ is a $C^\infty$ map: In this case, any $\omega \in H^r_k(h)$ has an extension depending smoothly on $h \geq 0$, whose Taylor polynomial of degree $k$ at zero is easily seen to satisfy (1.11), yielding $\omega \in H^r_k$. Therefore, since both $H^r_k$ and $H^r_k$ obviously stabilize at $k = 2$ for flows on surfaces, Theorem D shows that the map $h \mapsto H^r_k(h)$ is not $C^\infty$ at $h = 0$ for Kronecker’s flows on $T^2$ whose slope is a Liouville’s number and appropriate bundle-like metrics. So [25, Corollary 18] and [14, Corollary 5.22] have no direct generalizations to arbitrary Riemannian foliations and bundle-like metrics.

Nevertheless, the arguments of R. Forman in [16] are right when $\bar{0}_1 = 0$. In particular, Sections 2—4 in [16] show that, in this case, $H^r_k = E_k$ as bigraded vector spaces. Therefore, by (1.8) and (1.14), Forman’s arguments prove the following.

**Theorem E.** Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold $M$. If $\bar{0}_1 = 0$, then $H^r_k(g) = H^r_k(g)$ for every $k \geq 1$ and any bundle-like metric $g$ on $M$.

Theorem D-(ii) is a partial reciprocal of Theorem E, and we could conjecture that its statement holds for any bidegree, but we do not pursue such a result in this paper. A similar question can be raised about Theorem D-(i).

The following are the main ideas of the proofs in this paper. The proof of “$\leq$” in (1.3) (Theorem A) has three main ingredients. The first one is a variational formula for the spectral distribution function of the Laplacian, which is a consequence of the Hodge decomposition, and was used by Gromov and Shubin in another setting [19]. The second ingredient is a direct sum decomposition that holds for general spectral sequences—it is kind of an (only linear) Hodge decomposition. The relation between this decomposition and the formula of Gromov-Shubin can be easily seen, and leads to the proof. But this can not be directly applied to the differentiable spectral sequence $(E_k,d_k)$ because of some technical difficulty (Remark 3). For this reason, we introduce the third ingredient: The $L^2$ spectral sequence $(E^2_k,d^2_k)$, which is another spectral sequence defined in the very same way as $(E_k,d_k)$ but using square integrable differential forms. This change of spectral sequence can be made because we show that $E^2_k \cong E^2_k$ for Riemannian foliations and $k \geq 2$. The proof of this isomorphism heavily depends on the Hodge theoretic approach of the terms $E_1$ and $E_2$ that follows from our work [3] on leafwise heat flow.

The rest of Theorem A is an easy consequence of Theorem E, which in turn is proved by characterizing the terms $H^r_k$ in the appropriate way to apply certain estimation of $\Delta_h$—this estimation is similar to what was done by R. Forman in [16].

Theorem D is an easy consequence of the above theorems and other known results about Riemannian foliations.

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4Indeed [16, Lemma 2.7] is a version of this isomorphism—it must be pointed out that the notation used in [16] is very different from ours—.
Finally, let us mention that a very related study is done in [23], where the second author proves an asymptotical formula for the eigenvalue distribution function of $\Delta_{g}\omega$ in adiabatic limits for Riemannian foliations. That work establishes relationships with the spectral theory of leafwise Laplacian and with the noncommutative spectral geometry of foliations.

2. Differentiable spectral sequence

2.1. General properties. Let $(\mathcal{A}, d)$ be a complex with a finite decreasing filtration

$$A = A_0 \supset A_1 \supset \cdots \supset A_q \supset A_{q+1} = 0$$

by differential subspaces; i.e. $d(A_k) \subset A_{k}$ for all $k$. Recall that the induced spectral sequence $(E_k, d_k)$ is defined in the following standard way [26]:

$$Z_{k}^{u,v} = A_{u}^{v} \cap d^{-1}(A_{u+k}^{v+1})$$

$$B_{k}^{u,v} = A_{u}^{v+1} \cap d(A_{u-k}^{v})$$

$$E_{k}^{u,v} = \frac{Z_{k}^{u,v}}{Z_{k-1}^{u+1,v-1} + B_{k-1}^{u,v}}$$

$$Z_{\infty}^{u,v} = A_{u}^{v+1} \cap \ker d$$

$$B_{\infty}^{u,v} = A_{u}^{v+1} \cap \im d$$

$$E_{\infty}^{u,v} = \frac{Z_{\infty}^{u,v}}{Z_{\infty}^{u+1,v-1} + B_{\infty}^{u,v}}$$

In particular $Z_{0}^{u,v} = Z_{-1}^{u,v} = A_{u}^{v}$. We assume $B_{-1}^{u,v} = 0$, so $E_{0}^{u,v} = A_{u}^{v+1} / A_{u+1}^{v}$. Also, we have $B_{u}^{u,v} = B_{\infty}^{u,v}$ and $Z_{q-u+1}^{u,v} = Z_{\infty}^{u,v}$ since the filtration of $\mathcal{A}$ is of length $q + 1$. Each homomorphism $d_k : E_{k}^{u,v} \rightarrow E_{k+1}^{u,v}$ is canonically induced by $d$.

Now let $\mathcal{F}$ be a $C^{\infty}$ foliation of codimension $q$ on a closed manifold $M$, and $(\Omega, d)$ the de Rham complex of $M$. The differentiable spectral sequence $(E_k, d_k)$ of $\mathcal{F}$ is defined by the decreasing filtration by differential subspaces

$$\Omega = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_q \supset \Omega_{q+1} = 0,$$

where the space of $r$-forms of filtration degree $\geq k$ is given by

$$\Omega_k^{r} = \left\{ \omega \in \Omega^r \mid i_X \omega = 0 \text{ for all } X = X_1 \wedge \cdots \wedge X_{r-k+1}, \text{ where the } X_i \text{ are vector fields tangent to the leaves} \right\}.$$

Moreover, the $C^{\infty}$ topology of $\Omega$ canonically induces a topology on each $E_k^{u,v}$, which becomes a topological vector space. Then each $d_k$ is continuous on $E_k = \bigoplus_{u,v} E_k^{u,v}$ with the product topology. Thus, for each $k$, we have two new bigraded complexes: the closure of the trivial subspace $0 \subset E_k$ and the quotient $\tilde{E}_k = E_k / 0_k$.

Assume $M$ is endowed with a Riemannian metric, and let $\pi_{u,v} : \Omega \rightarrow \Omega^{u,v}$ denote the induced projection defined by the bigrading of $\Omega$. Define the topological vector spaces

$$z_{k}^{u,v} = \pi_{u,v}(Z_{k}^{u,v})$$

$$b_{k}^{u,v} = \pi_{u,v}(B_{k}^{u,v})$$

$$e_{k}^{u,v} = z_{k}^{u,v} / b_{k}^{u,v}$$

$$e_k = \bigoplus_{u,v} e_k^{u,v}.$$
Observe that
\[\Omega_k = \bigoplus_{u \geq k} \Omega^{u,v},\]  
(2.1)
yielding
\[Z_k^{u,v} \cap \ker \pi_{u,v} = Z_{k-1}^{u+1,v-1}.\]

Thus the projection \(\pi_{u,v}\) induces a continuous linear isomorphism \(E_k^{u,v} \xrightarrow{\cong} e_k^{u,v}\). The operator on \(e_k\) that corresponds to \(d_k\) on \(E_k\) by the above linear isomorphisms will be denoted by \(d_k\) as well. We also consider the closure of the trivial subspace, \(\bar{e}_k \subset e_k\), and the quotient \(\bar{e}_k = e_k / \bar{d}_k\). We are going to show that \(d_k\) is continuous on \(e_k\) for \(k = 0, 1\), and thus \(\bar{e}_k\) and \(\bar{d}_k\) become bigraded complexes in a canonical way. But, for \(k \geq 2\), we do not know whether \(d_k\) is continuous on \(e_k\), and whether \(d_k\) induces differentials on \(\bar{e}_k\) and \(\bar{d}_k\). This holds at least for Riemannian foliations as easily follows from Theorem 2.2(vii) in Section 2.2.

By comparing bihomogeneous components in the equality \(d^2 = 0\) we get (see e.g. [1]):
\[
\begin{align*}
d^2_{d,1} &= d_{d,1} = d_{0,1}d_{1,0} + d_{1,0}d_{0,1} = 0, \\
d_{1,0}d_{d,1} + d_{d,1}d_{1,0} &= d_{0,1}^2 + d_{0,1}d_{d,1} + d_{d,1}d_{0,1} = 0. \end{align*}
\]
(2.2)
The term \(d_{d,1}\) is of order zero, and vanishes if and only if \(TF\) is completely integrable. Moreover from (2.1) we get
\[
\begin{align*}
Z_0^{u,v} &= \Omega_u^{v+}, \\
E_0^{u,v} &= d_{0,1} (\Omega^{u,v-1}) \oplus \Omega_{u+1}^{v+}, \\
Z_1^{u,v} &= (\Omega^{u,v} \cap \ker d_{0,1}) \oplus \Omega_{u+1}^{v+} \\
\end{align*}
\]
(2.3)–(2.5)
as topological vector spaces. So
\[
\begin{align*}
z_0^{u,v} &= \Omega^{u,v}, \\
b_0^{u,v} &= d_{0,1} (\Omega^{u,v-1}), \\
\bar{z}_1^{u,v} &= \Omega^{u,v} \cap \ker d_{0,1}, \\
\end{align*}
\]
(2.6)
and the continuous linear isomorphisms \(E_k^{u,v} \xrightarrow{\cong} e_k^{u,v}\), induced by \(\pi_{u,v}\), are homeomorphisms too for \(k = 0, 1\). Thus \(\bar{0}_1 \cong \bar{\bar{0}}_1\) and \(\bar{E}_1 \cong \bar{e}_1\) as topological vector spaces, and \(\bar{\bar{0}}_1\) and \(\bar{e}_1\) become bigraded complexes with the differential induced by \(d_1\). For this reason, using the spaces \(e_1, \bar{\bar{0}}_1, \bar{e}_1\) is rather redundant; we have introduced these spaces to be compared with the corresponding ones for the \(L^2\) spectral sequence (Section 3), where this does not obviously hold. Furthermore (2.3)–(2.5) yield
\[
(e_0, d_0) = (\Omega, d_{0,1}),
\]
(2.7)
and a canonical isomorphism
\[
(e_1, d_1) \cong (H(\Omega, d_{0,1}), d_{1,0*})
\]
(2.8)
of topological complexes. Nevertheless we can not go further keeping full control of the topology. In fact, with this generality, we do not know whether the continuous linear isomorphism \(E_2^{u,v} \xrightarrow{\cong} e_2^{u,v}\), induced by \(\pi_{u,v}\), is a homeomorphism, neither the canonical continuous linear isomorphisms \(E_2 \xrightarrow{\cong} H(E_1, d_1)\) and \(e_2 \xrightarrow{\cong} H(e_1, d_1)\).
2.2. Hodge theory of the terms $E_1$ and $E_2$ for Riemannian foliations. Here, $\mathcal{F}$ is assumed to be a Riemannian foliation and the metric bundle-like.

The de Rham coderivative $\delta$ decomposes as sum of bihomogeneous components $\delta_{i,j} = d^*_{-i,-j}$, and the operators

$$D_0 = d_{0,1} + \delta_{0,-1}, \quad \Delta_0 = D_0^2 = d_{0,1}\delta_{0,-1} + \delta_{0,-1}d_{0,1}$$

are essentially self-adjoint in $\Omega$. But $D_0$ and $\Delta_0$ are not elliptic on $M$—avoiding the trivial case where $q = 0$. The closures of $d, \delta, d_{0,1}, \delta_{0,-1}, D_0$ and $\Delta_0$ in $\Omega$ will be denoted by $\hat{\delta}, \hat{\Delta}$, respectively. Then we have the orthogonal decomposition

$$\Omega = \ker \Delta_0 \oplus \ker(\im d_{0,1}) \oplus \im \delta_{0,-1},$$

(2.9)

where $\im \delta_{0,-1}$ denotes closure in $\Omega$. Moreover

$$\ker \Delta_0 = \ker D_0 = \ker d_{0,1} \cap \ker \delta_{0,-1},$$

(2.10)

Thus let $\Pi, P$ and $Q$ denote the orthogonal projections of $\Omega$ onto $\ker \Delta_0$, $\ker(\im d_{0,1})$ and $\im \delta_{0,-1}$, respectively, and set $\Pi = \im \delta_{0,-1}$, $P = \im \delta_{0,-1}$ and $\im = \im \delta_{0,-1}$. We shall also use the notation $W^k\Omega$ for the $k$th Sobolev space completion of $\Omega$, and let $\im \delta_{0,-1}$ denote closure in $W^k\Omega$. Thus $\Omega = W^0\Omega$.

**Theorem 2.1** (Álvarez-Kordyukov [3]). For each $k \in \mathbb{Z}$, decomposition (2.3) restricts to $W^k\Omega$: i.e.,

$$W^k\Omega = \ker(\Delta_0 \text{ in } W^k\Omega) \oplus \im \delta_{0,-1} + \im \delta_{0,-1} \oplus \im \delta_{0,-1}$$

as topological vector space. Thus (2.3) also restricts to $C^\infty$ differential forms; i.e.,

$$\Omega = \ker \Delta_0 \oplus \im \delta_{0,-1}$$

with respect to the $C^\infty$ topology, where the bar denotes $C^\infty$ closure in $\Omega$. In particular $\Pi, P$ and $Q$ preserve $\Omega$.

From (2.7), (2.8) and Theorem 2.1, we get a canonical isomorphism $\ker \Delta_0 \cong \hat{\delta}_1$ of topological vector spaces, induced by the inclusion

$$\Omega^\ast \cap \ker \Delta_0 \hookrightarrow \Omega^\ast \cap \ker d_{0,1} = z_{1}^\ast.$$

So $\ker \Delta_0 \cong \hat{\delta}_1$ as topological vector spaces. As in [3], let

$$\mathcal{H}_1 = \ker \Delta_0 = \ker D_0 = \ker d_{0,1} \cap \ker \delta_{0,-1},$$

and let $L^2\mathcal{H}_1 = \im \delta_{0,-1}$. From (2.10) and Theorem 2.1 we get

$$\ker \Delta_0 = \ker D_0 = L^2\mathcal{H}_1.$$  (2.11)

Since $\Delta_0$ is bihomogeneous of bidegree $(0,0)$, the bigrading of $\Omega$ restricts to a bigrading of $\mathcal{H}_1$. Moreover, by (2.7), (2.8) and Theorem 2.1 the operator $\delta_1$ on $\hat{\delta}_1$ corresponds to the map $\Pi \delta_1$ on $\mathcal{H}_1$, which will be also denoted by $\delta_1$. Hence $H^\ast(\hat{\delta}_1, d_1) \cong H^\ast(\hat{\delta}_1, d_1) \cong H^\ast(\hat{\delta}_1, d_1)$. Since $\delta_1 = \Pi \delta_1$ is adjoint of $d_1$ in $\mathcal{H}_1$, the operators $D_1 = d_1 + \delta_1$ and $\Delta_1 = D_1^2 = d_1 \delta_1 + \delta_1 d_1$ on $\mathcal{H}_1$ are symmetric. Now, let $\mathcal{H}_2 = \ker \Delta_1$, which inherits the bigrading from $\Omega$ because $\Delta_1$ is bihomogeneous of bidegree $(0,0)$. 

We also define maps \( \tilde{d}_1 \) and \( \tilde{\delta}_1 \) on \( \tilde{\mathcal{H}}_1 \) as follows. First we define the following bigrading on \( \tilde{\mathcal{H}}_1 \):
\[
\tilde{\mathcal{H}}^{u,v}_1 = d_{0,1}(\Omega^{u,v-1}) \oplus \bar{\delta}_{0,-1}(\bar{\Omega}^{u+1,v}).
\]
Let \( \bar{\Pi}_{u,v} \) be the projection of \( \Omega \) onto \( \tilde{\mathcal{H}}^{u,v}_1 \), and set \( \tilde{d}_1 = \bar{\Pi}_{u,v}d \) and \( \tilde{\delta}_1 = \bar{\Pi}_{u,v}\delta \) on \( \tilde{\mathcal{H}}^{u,v}_1 \), which are adjoint of each other. Consider also the symmetric operators \( \bar{D}_1 = \tilde{d}_1 + \tilde{\delta}_1 \) and \( \bar{\Delta}_1 = \bar{D}_1^2 \) on \( \tilde{\mathcal{H}}_1 \).

The closures of \( d_1, \delta_1, D_1 \) and \( \Delta_1 \) in \( L^2\mathcal{H}_1 \), and of \( \tilde{d}_1, \tilde{\delta}_1, \bar{D}_1 \) and \( \bar{\Delta}_1 \) in \( L^2\tilde{\mathcal{H}}_1 \), will be respectively denoted by \( d_1, \delta_1, D_1, \Delta_1, \tilde{d}_1, \tilde{\delta}_1, \bar{D}_1 \) and \( \bar{\Delta}_1 \).

The following theorem collects the main results of [3, Section 7].

**Theorem 2.2** (Álvarez-Kordyukov [3]). We have:

(i) The operators \( D_1 \) and \( \Delta_1 \) are essentially self-adjoint in \( L^2\mathcal{H}_1 \), and the operators \( \bar{D}_1 \) and \( \bar{\Delta}_1 \) are essentially self-adjoint in \( L^2\tilde{\mathcal{H}}_1 \).

(ii) The spectra of \( D_1, \Delta_1, \bar{D}_1 \) and \( \bar{\Delta}_1 \) are discrete subsets of \( \mathbb{R} \) given by eigenvalues of finite multiplicity.

(iii) We have the Hodge type decompositions
\[
L^2\mathcal{H}_1 = \ker \Delta_1 \oplus \operatorname{im} d_1 \oplus \operatorname{im} \delta_1,
\]
\[
L^2\tilde{\mathcal{H}}_1 = \operatorname{im} \tilde{d}_1 \oplus \operatorname{im} \tilde{\delta}_1,
\]
as Hilbert spaces with the \( L^2 \) norm, and moreover
\[
\ker \Delta_1 = \ker D_1 = \ker d_1 \cap \ker \delta_1,
\]
\[
\operatorname{im} \Delta_1 = \operatorname{im} D_1 = \operatorname{im} d_1 \oplus \operatorname{im} \delta_1,
\]
\[
\ker \bar{\Delta}_1 = \ker \bar{D}_1 = 0, \quad \operatorname{im} \bar{\Delta}_1 = \operatorname{im} \bar{D}_1 = L^2\tilde{\mathcal{H}}_1.
\]
Furthermore the operators \( \Delta_1 \) and \( \bar{\Delta}_1 \) satisfy Garding type inequalities.\(^5\)
Thus \( \ker \Delta_1 = \mathcal{H}_2 \), and the above decompositions restrict to \( C^\infty \) differential forms; i.e.,
\[
\mathcal{H}_1 = \ker \Delta_1 \oplus \operatorname{im} d_1 \oplus \operatorname{im} \delta_1,
\]
\[
\tilde{\mathcal{H}}_1 = \operatorname{im} \tilde{d}_1 \oplus \operatorname{im} \tilde{\delta}_1,
\]
as topological vector spaces with the \( C^\infty \) topology, as well as with the restriction of the \( L^2 \) norm topology.

(iv) The space \( \mathcal{H}_2 \) is of finite dimension, and the inclusion \( \mathcal{H}_2 \hookrightarrow \mathcal{H}_1 \) induces isomorphisms
\[
\mathcal{H}_2^{u,v} \cong H^u(\mathcal{H}_1^{u,v}, d_1) \cong H^u(\hat{\mathcal{E}}_1^{u,v}) \cong H^u(\tilde{\mathcal{E}}_1^{u,v}).
\]

(v) We have \( \bar{d}_1^2 = 0 \) and \( H \left( \tilde{\mathcal{H}}_1, \tilde{d}_1 \right) = 0.\)

(vi) Each map \( \tilde{\mathcal{H}}_1^{u,v} \to \tilde{\mathcal{H}}_1^{u,v} = \mathcal{R}_1^{u,v} \cong \mathcal{O}_1^{u,v}, \) defined by the canonical projection
\[
d_{0,1}(\Omega^{u,v-1}) \oplus \bar{\delta}_{0,-1}(\bar{\Omega}^{u+1,v}) \to d_{0,1}(\Omega^{u,v-1})/d_{0,1}(\Omega^{u,v-1}),
\]
induces an isomorphism\(^6\)
\[
0 = H^u(\tilde{\mathcal{H}}_1^{u,v}, d_1) \cong H^u(\hat{\mathcal{O}}_1^{u,v}) \cong H^u(\mathcal{O}_1^{u,v}).
\]

---

\(^5\) Corollary 7.3.

\(^6\) The isomorphism \( H(\mathcal{O}_1) = 0 \) was originally shown by X. Masa [24], as well as property (vii), which is a consequence.
Lemma 2.3. The following properties are satisfied:

(i) We have
\[ d_{1,0} P = P d_{1,0} , \quad d_{1,0} \tilde{Q} = \tilde{Q} d_{1,0} , \quad Q d_{1,0} = Q d_{1,0} Q , \]
\[ \tilde{P} d_{1,0} = \tilde{P} d_{1,0} , \quad \delta_{-1,0} P = Q \delta_{-1,0} Q , \quad \delta_{-1,0} \tilde{P} = \tilde{P} \delta_{-1,0} \tilde{P} , \]
\[ P \delta_{-1,0} = P \delta_{-1,0} P , \quad \tilde{Q} \delta_{-1,0} = \tilde{Q} \delta_{-1,0} \tilde{Q} . \]

(ii) We have
\[ \tilde{P} d_{1,0} P = Q d_{1,0} \tilde{Q} = \tilde{Q} \delta_{-1,0} Q = P \delta_{-1,0} \tilde{P} = 0 . \]

Proof. The equalities involving \( d_{1,0} \) in property (i) follow from (2.2) since
\[ P(\Omega) = d_{0,1}(\Omega) , \quad \tilde{Q}(\Omega) = \ker d_{0,1} . \]

The other equalities in property (i) are obtained by taking adjoints, and property (ii) is a direct consequence of property (i).

Lemma 2.4. The following properties are satisfied:

(i) The following operators on \( \Omega \) define bounded operators on \( \Omega \):
\[ \tilde{P} d_{1,0} , \quad \tilde{P} d_{1,0} , \quad \tilde{P} \delta_{-1,0} , \quad \tilde{P} \delta_{-1,0} . \]
\[ \tilde{Q} d_{1,0} , \quad P d_{1,0} , \quad \tilde{P} \delta_{-1,0} , \quad \tilde{P} \delta_{-1,0} . \]

(ii) The following operators on \( \Omega \) define bounded operators on \( \Omega \) too:
\[ \tilde{P} d_{1,0} , \quad \tilde{P} d_{1,0} , \quad \tilde{P} \delta_{-1,0} , \quad \tilde{P} \delta_{-1,0} . \]
\[ \tilde{P} d_{1,0} , \quad \tilde{P} d_{1,0} , \quad \tilde{P} \delta_{-1,0} , \quad \tilde{P} \delta_{-1,0} . \]

(iii) We have
\[ \text{dom} d_{1} = L^2 \mathcal{H}_1 \cap \text{dom} d , \quad \text{dom} \delta_{1} = L^2 \mathcal{H}_1 \cap \text{dom} \delta , \]
\[ \text{dom} d_{1} = L^2 \tilde{\mathcal{H}}_1 \cap \text{dom} d , \quad \text{dom} \delta_{1} = L^2 \tilde{\mathcal{H}}_1 \cap \text{dom} \delta . \]

Proof. Set \( D_{\perp} = d_{1,0} + \delta_{-1,0} \). Then, by Remark 3.7 and the proof of Lemma 7.2 in [3], the operators
\[ [D_{\perp} , \tilde{P} ] , \quad \tilde{P} v \delta_{-1} D_{\perp} \tilde{P} _{v} , \quad \left( \text{id} - \tilde{P} _{v} \right) D_{\perp} \tilde{P} _{v} \]
on \( \Omega \) define bounded operators on \( \Omega \). This easily yields property (i). Now properties (ii) and (iii) follows from property (i) since \( d_{2,-1} \) and \( \delta_{-2,1} \) are of order zero, and \( d_{0,1} \) and \( \delta_{0,-1} \) vanish on \( \mathcal{H}_1 \) and preserve each \( \tilde{\mathcal{H}}_1^{v} \).
3. L² SPECTRAL SEQUENCE

3.1. General properties. For a $C^\infty$ foliation $\mathcal{F}$ on a closed manifold $M$, what we call the $L^2$ spectral sequence of $\mathcal{F}$ is also a spectral sequence $(E^q, d_k)$ converging to the de Rham cohomology of $M$; in fact, it converges to the $L^2$ cohomology of $M$, but both cohomologies are canonically isomorphic since $M$ is closed. Recall that $d$ denotes the closure of $d$ in $\Omega$. Also, let $\Omega_k$ be the closure of $\Omega_k$ in $\Omega$, and consider the decreasing filtration of the complex $(\operatorname{dom} d, d)$ by the differential subspaces $\Omega_k \cap \operatorname{dom} d$. We define $(E^q, d_k)$ to be the corresponding spectral sequence. Since the inclusion $\Omega \hookrightarrow \operatorname{dom} d$ obviously is a homomorphism of filtered complexes, it induces a canonical homomorphism $(E^q, d_k) \to (E^q, d_k)$ of spectral sequences. We point out that, by the compactness of $M$, the filtered complex $(\operatorname{dom} d, d)$ is well defined independently of any metric, and thus so is the $L^2$ spectral sequence $(E^q, d_k)$.

Each $E^q_{u,v}$ is a topological vector space with the topology induced by the $L^2$ norm of $\Omega$, and consider the product topology on $E^q_{u,v} = E^q_{u,v} \times \cdots \times E^q_{u,v}$.

The notation $Z^u_{k,v}$ and $B^u_{k,v}$ of Section 2.1 will be used for the spaces involved in the definition of the differentiable spectral sequence of $\mathcal{F}$, and the corresponding spaces for the $L^2$ spectral sequence will be denoted by $Z^u_{k,v}$ and $B^u_{k,v}$. We have

$$Z^u_{k,v} = \Omega^u_{u+k} \cap d^{-1}(\Omega^{u+u+1}_{u+k}),$$
$$B^u_{k,v} = \Omega^u_{u+v} \cap d(\Omega^{u+u-1}_{u-k} \cap \operatorname{dom} d),$$
$$Z^u_{\infty} = \Omega^u_{u+v} \cap \ker d,$$
$$B^u_{\infty} = \Omega^u_{u+v} \cap \operatorname{im} d.$$

As in the case of the differentiable spectral sequence, let $\pi_{u,v} : \Omega \to \Omega^u_{u,v}$ be the canonical projection defined by the bigrading of $\Omega$; i.e., $\pi_{u,v} : \Omega \to \Omega^u_{u,v}$ is the continuous extension of $\pi_{u,v} : \Omega \to \Omega^u_{u,v}$. Consider also the topological vector spaces

$$z^u_{k,v} = \pi_{u,v}(Z^u_{k,v}) \quad \text{and} \quad b^u_{k,v} = \pi_{u,v}(B^u_{k,v}) \quad \text{for} \quad k = 0, 1, \ldots, \infty,$$

with the topology induced by the $L^2$ norm of $\Omega$. We clearly have $Z^u_{k,v} \cap \ker \pi_{u,v} = Z^u_{k-1,v}$, and thus each projection $\pi_{u,v}$ induces a continuous linear isomorphism $E^u_{k,v} \cong E^u_{k,v}$. Via these isomorphisms, the differential $d_k$ on $E^u_{k,v}$ induces a differential on $e_k$ that will be denoted by $d_k$ as well. We also have canonical continuous homomorphisms $e^u_{k,v} \to e^u_{k,v}$.

In general, the $L^2$ spectral sequence is more difficult to deal with than the differentiable spectral sequence. For example, we do not know whether the continuous linear isomorphism $E^u_{1,v} \cong e^u_{1,v}$, induced by $\pi_{u,v}$, is a homeomorphism with this generality. Also, the useful expressions (2.3)–(2.8) do not hold for the $L^2$ spectral sequence; indeed, for $r = u + v$, instead of (2.3)–(2.5) we have

$$Z^u_{0,v} = \Omega^r_u \cap \operatorname{dom} d,$$
$$B^u_{0,v} = d(\Omega^{r-1}_u \cap \operatorname{dom} d),$$
$$Z^u_{1,v} = (\Omega^{r}_u \cap \ker d_{0,1}) \cup \Omega^{r+1}_u \cap \operatorname{dom} d.$$

Because of this reason, it will be useful to introduce the spaces

$$D^u_{u,v} = \pi_{u,v}(\Omega^r_u \cap \operatorname{dom} d) \subset \Omega^u_{u,v}, \quad r = u + v,$$
which satisfy
\[(V + \Omega_{u+1}^r) \cap \text{dom } d = ((V \cap D^{u,v}) + \Omega_{u+1}^r) \cap \text{dom } d, \quad r = u + v. \quad (3.4)\]
for any subspace \( V \subset \Omega^{u,v}. \)

Observe that the canonical homomorphism \( E_0^{u,v} \to E_0^{u,v} \) is injective with dense image because it is just the inclusion \( Z_0^{u,v} \to Z_0^{u,v} \), whose image is dense by (2.3) and (3.1). With this generality, at least injectivity holds for \( E_1 \to E_1 \) too, as asserted by the following result.

**Lemma 3.1.** The canonical homomorphism \( E_1 \to E_1 \) is injective.

**Proof.** For \( r = u + v \) we have
\[
Z_0^{u+1,v-1} + B_0^{u,v} = (\Omega_{u+1}^r \cap \text{dom } d) + d(\Omega_{u-1}^r \cap \text{dom } d), \quad \text{by (3.2) and (3.1)},
\]
\[
= (\Omega_{u+1}^r + d(\Omega_{u-1}^r \cap \text{dom } d)) \cap \text{dom } d, \quad \text{since } \text{im } d \subset \text{dom } d,
\]
\[
= (d_0,1D^{u,v-1} + \Omega_{u+1}^r) \cap \text{dom } d. \quad (3.5)
\]
Then
\[
Z_1^{u,v} \cap (Z_0^{u+1,v-1} + B_0^{u,v}) = Z_0^{u+1,v-1} + B_0^{u,v}
\]
by (2.3), (2.4) and (2.5), and the result follows. □

**Lemma 3.2.** We have \( D^{u,v} \subset \text{dom } \partial_{0,1}. \)

**Proof.** Take any \( \alpha \in D^{u,v}. \) For \( r = u + v \), there exists some \( \beta \in \Omega_{r+1}^u \) such that \( \alpha + \beta \in \text{dom } d. \) So \( \pi_{u,v}d(\alpha + \beta) \) is defined in \( \Omega^{u,v}. \) But \( \pi_{u,v}d(\alpha + \beta) = \partial_{0,1}\alpha \) because \( \alpha + \beta \in \Omega_{r}^u. \) □

**Lemma 3.3.** We have
\[
\pi_{u,v}(Z_1^{u,v}) = D^{u,v} \cap \ker \partial_{0,1}, \quad \pi_{u,v}(B_0^{u,v}) = \partial_{0,1}D^{u,v-1},
\]
and thus
\[
e_1^{u,v} = \frac{D^{u,v} \cap \ker \partial_{0,1}}{\partial_{0,1}D^{u,v-1}}.
\]

**Proof.** For \( r = u + v \), we have
\[
\pi_{u,v}(Z_1^{u,v}) = \pi_{u,v}((\Omega^{u,v} \cap \ker \partial_{0,1}) + \Omega_{u+1}^r) \cap \text{dom } d), \quad \text{by (3.3)},
\]
\[
= \pi_{u,v}((D^{u,v} \cap \ker \partial_{0,1}) + \Omega_{u+1}^r) \cap \text{dom } d), \quad \text{by (3.4)},
\]
\[
= D^{u,v} \cap \ker \partial_{0,1},
\]
\[
\pi_{u,v}(B_0^{u,v}) = \pi_{u,v}(d(\Omega_{u-1}^r \cap \text{dom } d) \cap \text{dom } d), \quad \text{by (3.2)},
\]
\[
= \pi_{u,v}(\partial_{0,1}D^{u,v-1} + \Omega_{u+1}^r) \cap \text{dom } d), \quad \text{by (3.4)},
\]
\[
= \partial_{0,1}D^{u,v-1}. \quad \square
\]

As for the differentiable spectral sequence, let \( \bar{0}_1 \subset E_1 \) and \( \bar{e}_1 \subset e_1 \) be the closures of the corresponding trivial subspaces, which are bigraded subspaces with bigraded quotients \( \bar{E}_1 = E_1/\bar{0}_1 \) and \( \bar{e}_1 = e_1/\bar{e}_1 \). Lemma 3.3 has the following direct consequence.
Corollary 3.4. We have

\[
\begin{align*}
\hat{o}_1^{u,v} &= \frac{D^{u,v} \cap \ker d_{0,1} \left( d_{0,1} D^{u,v-1} \right)}{d_{0,1} D^{u,v-1}} = \frac{D^{u,v} \cap \ker d_{0,1} \left( d_{0,1} \Omega^{u,v-1} \right)}{d_{0,1} D^{u,v-1}}, \\
\hat{e}_1^{u,v} &= \frac{D^{u,v} \cap \ker d_{0,1}}{D^{u,v} \cap \ker d_{0,1} \left( d_{0,1} D^{u,v-1} \right)} = \frac{D^{u,v} \cap \ker d_{0,1} \left( d_{0,1} \Omega^{u,v-1} \right)}{D^{u,v} \cap \ker d_{0,1} \left( d_{0,1} \Omega^{u,v-1} \right)}.
\end{align*}
\]

The map \(d_1\), either on \(E_1\) or on \(e_1\), may not be continuous. So \(\hat{o}_1, \hat{e}_1, o_1\) and \(e_1\) may not have canonical structures of bigraded complexes in general. However we shall show that this holds for Riemannian foliations in Section 3.2.

3.2. \(L^2\) spectral sequence of Riemannian foliations.

Theorem 3.5. Let \(F\) be a Riemannian foliation on a closed manifold \(M\). Then the canonical map \(E_k \to E_k\) is injective with dense image for \(k = 0, 1\), and is an isomorphism of topological vector spaces for \(k \geq 2\). In particular \(E_k\) is Hausdorff of finite dimension for \(k \geq 2\).

The goal of this subsection is to prove Theorem 3.5. Thus, from now on, assume \(F\) is a Riemannian foliation. Since its statement is independent of any metric on \(M\), we can take a bundle-like metric on \(M\) to prove it.

In Theorem 3.5, the case \(k = 0\) is obvious, and the case \(k = 1\) follows directly from Lemma 3.1 and the following lemma.

Lemma 3.6. The space \(Z_1^{u,v}\) is dense in \(Z_1^{u,v}\).

Proof. Since the orthogonal projection

\[
\hat{Q} : \Omega^{u,v} \to \Omega^{u,v} \cap \ker d_{0,1}
\]

preserves smoothness on \(M\), the result follows by (3.7) and (3.3). \(\square\)

The proof of Theorem 3.5 for \(k \geq 2\) requires much more work than Lemma 3.6. To establish this, we shall use the Hodge theoretic approach to \(e_1\) and \(e_2\) from Section 2.2, and a similar approach to \(e_1\) and \(e_2\). To begin with, we show that \(d_1\) preserves \(o_1\).

Lemma 3.7. We have \(d_1 (\hat{o}_1) \subset \hat{o}_1\).

Proof. Take any \(\alpha \in D^{u,v} \cap \ker d_{0,1} \left( d_{0,1} \Omega^{u,v-1} \right)\), and fix some \(\beta \in \Omega_{u+1}^{r+1}\) with \(\alpha + \beta \in \text{dom} \, d\), where \(r = u + v\). We know that \(\pi_{u+1,v} d (\alpha + \beta) \in D^{u+1,v}\) by Lemma 3.3. On the other hand, if \(d_{0,1}\) and \(d_{1,0}\) denote the extensions of \(d_{0,1}\) and \(d_{1,0}\) to continuous maps \(\Omega \to W^{-1}\Omega\), we have

\[
\pi_{u+1,v} d (\alpha + \beta) = d_{1,0} \alpha + d_{0,1} \beta_1 \in d_{1,0} \alpha + d_{0,1} \Omega^{u+1,v-1},
\]

where \(\beta_1 = \pi_{u+1,v-1} \beta \in \Omega^{u+1,v-1}\), and

\[
d_{1,0} \alpha \in d_{1,0} \left( \ker \left( d_{0,1} \Omega^{u,v-1} \right) \right) \subset \ker \left( d_{0,1} \Omega^{u+1,v-1} \right).
\]

Hence

\[
\pi_{u+1,v} d (\alpha + \beta) \in D^{u+1,v} \cap \ker \left( d_{0,1} \Omega^{u+1,v-1} \right) = D^{u+1,v} \cap \ker \left( d_{0,1} \Omega^{u+1,v-1} \right)
\]

by Theorem 2.1. Therefore the result follows by Lemma 3.3 and Corollary 3.4. \(\square\)
Now \( \tilde{\phi}_1 \) and \( \hat{\eta}_1 \) canonically are bigraded complexes by Lemma 3.7, and we have the short exact sequence
\[
0 \rightarrow \tilde{\phi}_1 \rightarrow \mathfrak{e}_1 \rightarrow \hat{\eta}_1 \rightarrow 0,
\]
which induces long exact sequences
\[
\cdots \rightarrow H^u(\tilde{\phi}_1) \rightarrow H^u(\mathfrak{e}_1) \rightarrow H^u(\hat{\eta}_1) \rightarrow H^{u+1}(\tilde{\phi}_1) \rightarrow \cdots.
\]

Theorem 2.2-(iii),(iv)

The canonical homomorphism \( e_1 \rightarrow \mathfrak{e}_1 \) is obviously continuous. Hence it induces homomorphisms of complexes \( \tilde{\phi}_1 \rightarrow \tilde{\phi}_1 \) and \( \hat{\eta}_1 \rightarrow \hat{\eta}_1 \), and homomorphisms \( H(\tilde{\phi}_1) \rightarrow H(\hat{\eta}_1) \) and \( H(\tilde{\phi}_1) \rightarrow H(\hat{\eta}_1) \) in cohomology.

Corollary 3.10.

The canonical map \( H(\tilde{\phi}_1) \rightarrow H(\hat{\eta}_1) \) is an isomorphism of topological vector spaces.

Proof. This follows from Theorem 2.2-(iv) and Corollary 3.10.
We also need a Hodge theoretic study of certain complex whose cohomology is isomorphic to $H(\alpha)$. To simplify notation let
\[ Z_v = \bigoplus_u Z^{u,v}_1, \quad B_v = \bigoplus_u \left( Z^{u-1,v+1}_0 + D^{u,v}_0 \right), \]
which are subcomplexes of $(\text{dom } d, d)$. Then
\[ \tilde{\alpha}_1^{u,v} = \text{cl}_0(B_v) / B_v. \] (3.7)
Observe that $Z_{v-1} \subset B_v$.

**Lemma 3.12.** The quotient complex $B_v / Z_{v-1}$ is acyclic. Thus the quotient map $\text{cl}_0(B_v) / Z_{v-1} \to \text{cl}_0(B_v) / B_v = \tilde{\alpha}_1^{u,v}$ induces an isomorphism in cohomology.

**Proof.** The result follows from (3.3) and (3.5) with easy arguments (see Lemma 2.5 in [3] and Lemma 7.4 in [2]). \[ \square \]

Set
\[ \tilde{\Omega}^{u,v} = \Omega^{u,v} + \Omega^{u+1,v-1}, \]
\[ \tilde{\pi}_{u,v} = \pi_{u,v} + \pi_{u+1,v-1} : \Omega \to \tilde{\Omega}^{u,v}, \]
\[ \tilde{D}^{u,v} = \tilde{\pi}_{u,v} (\Omega^{u,v} \cap \text{dom } d), \quad r = u + v. \]

We have
\[ \text{cl}_0(B_v) \cap \ker \tilde{\pi}_{u,v} \subset Z_{v-1} \cap \ker \tilde{\pi}_{u,v}. \]

Hence, for each topological vector space
\[ \tilde{e}_1^{u,v} = \frac{\tilde{\pi}_{u,v}(\text{cl}_0(B_v))}{\tilde{\pi}_{u,v}(Z_{v-1})}, \]
the projection $\tilde{\pi}_{u,v}$ induces a continuous linear isomorphism
\[ \text{cl}_0(B_v) / Z_{v-1} \xrightarrow{\cong} \tilde{e}_1^{u,v}, \quad r = u + v. \] (3.8)

Let $\tilde{d}_1$ be the operator on $\tilde{e}_1 = \bigoplus_{u,v} \tilde{e}_1^{u,v}$ that corresponds to the differential operator on the quotient complex $\text{cl}_0(B_v) / Z_{v-1}$ by the above isomorphisms. Observe that $\tilde{d}_1$ is given as follows: if $\alpha \in \text{cl}_0(B_v)$, and $[\tilde{\pi}_{u,v}\alpha] \in \tilde{e}_1^{u,v}$ denotes the class defined by $\tilde{\pi}_{u,v}\alpha$, then $\tilde{d}_1[\tilde{\pi}_{u,v}\alpha] = [\tilde{\pi}_{u+1,v}d\alpha]$.

The spaces $\tilde{D}^{u,v}$ and $D^{u,v}$ have similar properties. For instance, for any subspace $V \subset \tilde{\Omega}^{u,v}$ we have
\[ (V + \Omega^{r+2}_u) \cap \text{dom } d = \left( (V \cap \tilde{D}^{u,v}) + \Omega^{r+2}_u \right) \cap \text{dom } d, \quad r = u + v. \] (3.9)

**Lemma 3.13.** For $r = u + v$, we have
\[ \tilde{\pi}_{u,v}(\text{cl}_0(B_v)) = \tilde{D}^{u,v} \cap \text{cl}_0(B_v), \]
\[ \tilde{\pi}_{u,v}(Z_{v-1}) = D^{u+1,v-1} \cap Z_{v-1} = D^{u+1,v-1} \cap \ker d_{0,1}, \]
and thus
\[ \tilde{e}_1^{u,v} = \frac{\tilde{D}^{u,v} \cap \text{cl}_0(B_v)}{D^{u+1,v-1} \cap \ker d_{0,1}}. \]

**Proof.** This easily follows from (3.4) and (3.3). \[ \square \]

The following result and Lemma 3.8 are similar, as well as their proofs. \[ ^{7}\text{This notation is used in [1] for the } C^\infty \text{ versions of these complexes.} \]
Lemma 3.14. We have
\[ \tilde{D}^{u,v} \cap cl_0(B_v) = \left( \tilde{D}^{u,v} \cap L^2\tilde{H}_1 \right) \oplus (D^{u+1,v-1} \cap \ker d_{0,1}) \]
as topological vector spaces, and moreover
\[ \tilde{D}^{u,v} \cap L^2\tilde{H}_1 = L^2\tilde{H}_1^{u,v} \cap \text{dom} \tilde{d}_1. \]

Proof. The inclusion \( \supset \) of the first equality is obvious, and the inclusion \( \supset \) of the second equality follows from Lemma 2.4(iii).

To prove the inclusion \( \subset \) of the first equality, by (2.9) it is enough to prove that \( \tilde{\Pi}\alpha \in \tilde{D}^{u,v} \) for all \( \alpha \in \tilde{D}^{u,v} \cap cl_0(B_v) \). This obviously holds if we prove \( \tilde{\Pi}\alpha \in \text{dom} \tilde{d}_1 \) for every such an \( \alpha \) since the inclusion \( \supset \) of the second equality is already proved. This also proves the inclusion \( \subset \) of the second equality by taking \( \alpha \in L^2\tilde{H}_1 \).

Thus take any \( \alpha \in \tilde{D}^{u,v} \cap cl_0(B_v) \). Then there is some \( \beta \in \Omega_{u+2} \) such that \( \alpha + \beta \in \text{dom} d \), where \( r = u + v \). Write \( \alpha = \alpha_1 + \alpha_2 \) with \( \alpha_1 \in \Omega^{u,v} \) and \( \alpha_2 \in \Omega^{u+1,v-1} \). So, since \( \alpha \in cl_0(B_v) \) and \( d_\beta \in \ker \tilde{\Pi}_r \), where \( \tilde{\Pi}_r \) is considered as a projection in \( W^{-1}\Omega \), we get
\[ \tilde{\Omega}^{u+1,v} \ni \tilde{\Pi}_r d(\alpha + \beta) = \tilde{\Pi}_r d\alpha. \]

But
\[ \tilde{\Pi}_r d\alpha = \tilde{\Pi}_r d\tilde{\Pi}_r \alpha + \tilde{\Pi}_r d\Pi_2 \alpha_2 + \tilde{\Pi}_r d\tilde{\Pi}_r \alpha_2 \]
because \( \alpha \in \tilde{D}^{u,v} \cap cl_0(B_v) \), and
\[ \tilde{\Pi}_r d\Pi_2 \alpha_2 + \tilde{\Pi}_r d\tilde{\Pi}_r \alpha_2 \in \Omega \]
by Lemma 2.4(ii). Therefore \( \tilde{\Pi}_r d\tilde{\Pi}_r \alpha_2 \in \tilde{\Omega} \), yielding \( \tilde{\Pi}_r \alpha \in \text{dom} \tilde{d}_1 \) as desired.

Consider the projection
\[ \tilde{D}^{u,v} \cap cl_0(B_v) \to \tilde{D}^{u,v} \cap L^2\tilde{H}_1 = L^2\tilde{H}_1^{u,v} \cap \text{dom} \tilde{d}_1 \]
defined by Lemma 3.14, which is obviously an orthogonal projection.

Corollary 3.15. The inclusions \( L^2\tilde{H}_1^{u,v} \cap \text{dom} \tilde{d}_1 \to \tilde{D}^{u,v} \cap cl_0(B_v) \) induce an isomorphism \( (\text{dom} \tilde{d}_1, \tilde{d}_1) \cong (\tilde{e}_1, \tilde{d}_1) \) of bigraded complexes and topological vector spaces.

Proof. This follows from Lemmas 3.13 and 3.14.

Corollary 3.16. We have \( H(\tilde{0}_1) = 0 \).

Proof. This follows from (3.7), (3.8), Lemma 3.12, Corollary 3.15 and Theorem 2.2(v).

Corollary 3.17. The canonical map \( H(e_1) \to H(\tilde{e}_1) \) is an isomorphism of topological vector spaces. In particular \( H(e_1) \) is Hausdorff of finite dimension.

Proof. The canonical map \( H(e_1) \to H(\tilde{e}_1) \) is a linear isomorphism by Corollary 3.16 and the exactness of (3.6). Moreover it is obviously continuous. Then it is also an homeomorphism because \( H(\tilde{e}_1) \) is a Hausdorff topological vector space of finite dimension.
Corollary 3.18. The canonical map $H(e_1) \to H(e_1)$ is an isomorphism of topological vector spaces.

Proof. By the commutativity of the diagram

$$
\begin{array}{c}
H(e_1) \to H(e_1) \\
\downarrow \quad \quad \downarrow \\
H(\tilde{e}_1) \to H(e_1)
\end{array}
$$

where all maps are canonical, the result follows directly from Theorem 2.2-(vii), Corollaries 3.11 and 3.17.

Corollary 3.19. The canonical map $E_2 \to E_2$ is an isomorphism of topological vector spaces.

Proof. Consider the compositions

$$
E_2 \to H(E_1) \to H(e_1), \quad E_2 \to H(E_1) \to H(e_1),
$$

where the first map of each composition is canonical, and the second one is canonically induced by the projections $\pi_{u,v}$. The first composition is an isomorphism of topological vector spaces by Theorem 2.2-(vii), and we know that the second composition is a continuous linear isomorphism (Section 3.1). Then the second composition is also an homeomorphism because $H(e_1)$ is Hausdorff of finite dimension by Corollary 3.17. So the result follows from Corollary 3.18 and the commutativity of the diagram

$$
\begin{array}{c}
E_2 \to H(e_1) \\
\downarrow \quad \quad \downarrow \\
E_2 \to H(e_1)
\end{array}
$$

where the horizontal arrows denote the above compositions, and the vertical arrows denote canonical maps.

Now Theorem 3.5 for $k \geq 2$ follows from Corollary 3.19 because the canonical map $(E_k, d_k) \to (\hat{E}_k, d_k)$ is a homomorphism of spectral sequences.

4. $L^2$ Spectral Sequence and Small Eigenvalues

4.1. Main results. Let $\mathcal{F}$ be a $C^\infty$ foliation on a closed manifold $M$ with a Riemannian metric $g$, and consider the family of metrics $g_h$, $h > 0$, which were defined in (1.1) and give rise to the adiabatic limit. As in Section 1, let $\Delta_{g_h}$ denote the Laplacian on $\Omega$ defined by $g_h$, and

$$
0 \leq \lambda_0(h) \leq \lambda_1(h) \leq \lambda_2(h) \leq \cdots
$$

its spectrum on $\Omega'$, taking multiplicity into account. The following result suggests that, with this generality, the number of small eigenvalues of $\Delta_h$ may be more related with the $L^2$ spectral sequence than with the differentiable one. Nevertheless, so far we do not know about the relevance its hypothesis for non-Riemannian foliations.
Theorem 4.1. Let $\mathcal{F}$ be a $C^\infty$ foliation on a closed Riemannian manifold. If $Z_{k-1}^{u-1} + Z_{k}^{u,v}$ is closed in $Z_{k}^{u,v}$ for all $u,v$, with $r = u + v$, then
\[
\dim E_k^r \leq \sharp \{ i \mid \lambda'_r(h) \in O(h^{2k}) \text{ as } h \downarrow 0 \}
\]
for all $r$.

The following more understandable result is a direct consequence of Theorem 4.1 because $Z_{u,v}^r \ell$ is a quotient of $E_{u,v}^r$.

Corollary 4.2. Let $\mathcal{F}$ be a $C^\infty$ foliation on a closed Riemannian manifold. If $E_k$ is Hausdorff of finite dimension, then
\[
\dim E_k^r \leq \sharp \{ i \mid \lambda'_r(h) \in O(h^{2k}) \text{ as } h \downarrow 0 \}, \quad \ell \geq k .
\]

Remark 1. Observe that, by Theorem 3.5, Corollary 4.2 holds for Riemannian foliations and $k = 2$, and inequality “≤” of (1.3) in Theorem A follows.

The proof of Theorem 4.1 is given in Section 4.4, and its two main ingredients are described in Sections 4.2 and 4.3: the variational formula of the spectral distribution function used by Gromov-Shubin, and the direct sum decomposition for general spectral sequences.

4.2. Spectral distribution function. For a closed Riemannian manifold $(M,g)$, let $N'(\lambda)$ denote the spectral distribution function of the Laplacian $\Delta$ on $\Omega'$; i.e., $N'(\lambda)$ is the number of eigenvalues of $\Delta$ on $\Omega'$ which are $\leq \lambda$, taking multiplicity into account. Recall that $\Omega$ denotes the Hilbert space of square integrable differential forms with the inner product induced by $g$, and $d$ the closure of the de Rham derivative $d$ in $\Omega$. Let $d : \text{dom } d / \ker d \to \Omega$ denote the map induced by $d$, and consider the quotient Hilbert norm on $\Omega / \ker d$. The following variational expression of $N'(\lambda)$ is a consequence of the Hodge decomposition of $\Omega$.

Proposition 4.3 (Gromov-Shubin [19]). We have
\[
N'(\lambda) = F_{r-1}^r(\lambda) + \beta^r + F_r(\lambda),
\]
where $\beta^r$ is the $r$th Betti number of $M$, and
\[
F_r(\lambda) = \sup_L \dim L ,
\]
with $L$ ranging over the closed subspaces of $\text{dom } d / \ker d$ satisfying
\[
\| d \zeta \| \leq \sqrt{\lambda} \| \zeta \| \text{ for all } \zeta \in L .
\]

Now take again a $C^\infty$ foliation $\mathcal{F}$ on $M$. Then, for each metric $g_h$ of the family (1.1) that gives rise to the adiabatic limit, the spectral distribution function of $\Delta_{g_h}$ will be denoted by $N_h'(\lambda)$, and decomposes as
\[
N_h'(\lambda) = F_{h-1}^r(\lambda) + \beta^r + F_h^r(\lambda) ,
\]
according to Proposition 4.3.

Suppose $\mathcal{F}$ is of codimension $q$, and let $\| \|_h$ be the norm induced by $g_h$ on $\Omega$. The following equality will be also used to prove Theorem 4.1:
\[
\| \omega \|_h = h^{-q/2} h^u \| \omega \| \text{ if } \omega \in \Omega^{u,v} .
\]

(4.1)
This follows from two observations. First, if the metrics induced by \( g \) and \( g_h \) on \( \wedge TM^* \) are also denoted by \( g \) and \( g_h \), then \( g_h = h\mu g \) on forms with transverse degree \( u \). And second, assuming \( M \) is oriented, the volume forms \( \mu \) and \( \mu_h \), induced by \( g \) and \( g_h \), satisfy \( \mu_h = h^{-q}\mu \) since volume forms are of transverse degree \( q \).

By using Proposition 4.3 in the same spirit of [10], we could prove that the asymptotics of the \( \lambda'(h) \), as \( h \downarrow 0 \), are \( C^\infty \) homotopy invariants of \( F \) (with respect to the appropriate definition of homotopy between foliations). However, for our purposes in this paper, it will be enough to prove that the asymptotics of the \( \lambda'(h) \) are independent of the choice of the given metric \( g \) on \( M \). This will not be used to prove Theorem 1 but will play an important role to finish the proof of Theorem 8 in Section 5.2. Such independence of \( g \) is proved in the following way. Let \( g' \) be another metric on \( M \) with corresponding 1-parameter family of metrics \( g'_h \), and let \( \| \| \) and \( \| \|_h \) denote the corresponding norms on \( \Omega \). Compactness of \( M \) implies the existence of some \( C > 0 \) such that

\[
C^{-1} \| \omega \| \leq \| \omega \|' \leq C \| \omega \|
\]

for all \( \omega \in \Omega \), yielding

\[
C^{-1} \| \omega \|_h \leq \| \omega \|'_h \leq C \| \omega \|_h
\]

for all \( \omega \in \Omega \) and \( h > 0 \) by (4.1). Let \( N'_h(\lambda) \) be the spectral distribution function of \( \Delta_{g'_h} \) on \( \Omega' \), and let

\[
N''_h(\lambda) = F''_h(\lambda) + \beta''_h + F''_h(\lambda)
\]

be its decomposition according to Proposition 4.3. Then

\[
F''_h(C^{-4}\lambda) \leq F''_h(\lambda) \leq F''_h(C^4\lambda)
\]

for all \( \lambda \geq 0 \) and \( h > 0 \) by (4.2) and the definition of \( F''_h \) and \( F''_h \). Thus

\[
N''_h(C^{-4}\lambda) \leq F''_h(\lambda) \leq N''_h(C^4\lambda), \tag{4.3}
\]

yielding the metric independence of the asymptotics of the \( \lambda'_h \).

4.3. Direct sum decomposition of spectral sequences. In this subsection we consider the general setting where \( (E_k, d_k) \) is the spectral sequence induced by an arbitrary complex \( (A, d) \) with a finite decreasing filtration

\[
A = A_0 \supset A_1 \supset \cdots \supset A_q \supset A_{q+1} = 0
\]

by differential subspaces.

**Lemma 4.4.** The following properties are satisfied:

(i) There is a (non-canonical) isomorphism

\[
A^\ell \cong E_{\infty}^\ell \oplus \bigoplus_{\ell} \left( (E_{\ell}^u \cap \ker d_{\ell}) \oplus \frac{E_{\ell}^u}{E_{\ell}^u \cap \ker d_{\ell}} \right)
\]

\[
= \bigoplus_{u+v=\ell} \left( E_{\infty}^{u,v} \oplus \bigoplus_{\ell} \left( (E_{\ell}^{u,v} \cap \ker d_{\ell}) \oplus \frac{E_{\ell}^{u,v}}{E_{\ell}^{u,v} \cap \ker d_{\ell}} \right) \right).
\]

(ii) The isomorphism in (i) can be chosen so that \( A^\ell \) corresponds to

\[
\bigoplus_{u \geq k, u+v=\ell} \left( E_{\infty}^{u,v} \oplus \bigoplus_{\ell} \left( (E_{\ell}^{u,v} \cap \ker d_{\ell}) \oplus \frac{E_{\ell}^{u,v}}{E_{\ell}^{u,v} \cap \ker d_{\ell}} \right) \right).
\]
(iii) The isomorphism in (i) can be chosen so that the only possibly non-trivial components of the operator corresponding to \( d \) by (i) are the isomorphisms

\[
\bar{d}_\ell : \frac{E_{\ell}^{u,v}}{E_{\ell}^{u,v} \cap \ker d_\ell} \rightarrow E_{\ell}^{u+\ell,v-\ell+1} \cap \im d_\ell
\]

canonically defined by \( d_\ell \).

Before proving Lemma 4.4, we state three corollaries that will be needed in the proof of Proposition 4.3.

**Corollary 4.5.** There is a (non-canonical) isomorphism

\[
E_k^r \cong E_\infty^r \oplus \bigoplus_{\ell \geq k} \left( (E_\ell^r \cap \im d_\ell) \oplus \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell} \right).
\]

**Proof.** This is a direct consequence of Lemma 4.4. \(\square\)

Let

\[
m_k^r = \dim \bigoplus_{\ell \geq k} \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}.
\]

**Corollary 4.6.** We have

\[
\dim E_k^r = m_k^{r-1} + H^r(\mathcal{A}, d) + m_k^r.
\]

**Proof.** This follows from Corollary 4.5 since each \( d_\ell \) induces isomorphisms

\[
\frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell} \cong E_{\ell+1}^r \cap \im d_\ell. \quad \square
\]

**Corollary 4.7.** For \( r = u + v \), there is a subspace \( L_{k}^{u,v} \subset \mathcal{A}^r/(\mathcal{A}^r \cap \ker d) \) such that:

(i) We have

\[
\frac{Z_k^{u,v} + (\mathcal{A}^r \cap \ker d)}{\mathcal{A}^r \cap \ker d} = L_k^{u,v} \oplus \frac{Z_k^{u+1,v-1} + (\mathcal{A}^r \cap \ker d)}{\mathcal{A}^r \cap \ker d}
\]

as vector spaces. In particular \( \bar{d}(L_k^{u,v}) \subset \mathcal{A}_{u+k}^{r+1} \).

(ii) The direct sum \( L_k^r \) makes sense in \( \mathcal{A}^r/(\mathcal{A}^r \cap \ker d) \), and we have \( \dim L_k^r = m_k^r \).

**Proof.** From Lemma 4.4 we get a (non-canonical) isomorphism

\[
\frac{\mathcal{A}^r}{\mathcal{A}^r \cap \ker d} \cong \bigoplus_{\ell \geq k} \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}, \quad (4.4)
\]

Then let \( L_k^{u,v} \) be the subspace of \( \mathcal{A}^r/(\mathcal{A}^r \cap \ker d) \) that corresponds to

\[
\bigoplus_{\ell \geq k} \frac{E_\ell^{u,v}}{E_\ell^{u,v} \cap \ker d_\ell}
\]

by (4.4). Then property (i) easily follows from Lemma 4.4 and property (ii) is obvious; in fact, \( L_k^r \) corresponds to

\[
\bigoplus_{\ell \geq k} \frac{E_\ell^r}{E_\ell^r \cap \ker d_\ell}
\]

by (4.4). \(\square\)
Remark 2. By Corollary 1.7-(i), the canonical isomorphism

\[
Z_k^{u,v} \to Z_{k-1}^{u+1,v-1} + Z_{k-1}^{u,v} \cong \frac{Z_k^{u,v} + (A^r \cap \ker d)}{Z_{k-1}^{u+1,v-1} + (A^r \cap \ker d)}
\]  

(4.5)
yields

\[
Z_k^{u,v} \to Z_{k-1}^{u+1,v-1} + Z_{k-1}^{u,v} \cong L_k^{u,v}.
\]

When applying Corollary 4.7 to the \(L^2\) spectral sequence of a \(C^\infty\) foliation, the subspaces \(L_i \subset \text{dom } d / \ker d\) of Corollary 4.7 will be the spaces \(L\) needed to apply Proposition 4.3.

The rest of this section will be devoted to prove Lemma 4.4. To begin with, we have \(28\)

\[
E_{\ell}^{u,v} \cap d_{\ell}(E_{\ell}) = \frac{B_{\ell}^{u,v} + Z_{\ell-1}^{u+1,v-1}}{Z_{\ell-1}^{u+1,v-1} + B_{\ell-1}^{u,v}},
\]

(4.6)

\[
E_{\ell}^{u,v} \cap \ker d_{\ell} = \frac{Z_{\ell}^{u,v}}{Z_{\ell-1}^{u+1,v-1} + Z_{\ell+1}^{u,v}}.
\]

(4.7)
canonicall. Here, isomorphism (4.7) is obvious, and (4.6) follows since

\[
B_{\ell-1}^{u,v} \subset B_{\ell}^{u,v}, \quad B_{\ell}^{u,v} \cap Z_{\ell-1}^{u+1,v-1} = B_{\ell+1}^{u+1,v-1}.
\]

Consider the following chain of inclusions for \(0 \leq u \leq q\) and \(r = u + v:

\[
A_{u+1}^r \subset A_{u+1}^r + B_0^{u,v} \subset A_{u+1}^r + B_1^{u,v} \subset \cdots
\]

\[
\cdots \subset A_{u+1}^r + Z_1^{u,v} \subset A_{u+1}^r + Z_2^{u,v} \subset \cdots
\]

\[
\cdots \subset A_{u+1}^r + Z_1^{u,v} \subset A_{u+1}^r + Z_2^{u,v} \subset A_{u+1}^r.
\]

(4.8)
The inclusions in (4.8) have the following quotients:

\[
\frac{A_{u+1}^r + B_{\ell}^{u,v}}{A_{u+1}^r + B_{\ell-1}^{u,v}} \cong \frac{B_{\ell}^{u,v}}{B_{\ell+1}^{u+1,v-1} + B_{\ell-1}^{u,v}},
\]

(4.9)

\[
\frac{A_{u+1}^r + Z_1^{u,v}}{A_{u+1}^r + B_{\ell}^{u,v}} \cong \frac{Z_1^{u,v}}{Z_{\ell+1}^{u+1,v-1} + Z_{\ell}^{u,v}},
\]

(4.10)

\[
\frac{A_{u+1}^r + Z_1^{u,v}}{A_{u+1}^r + B_{\ell}^{u,v}} \cong \frac{Z_1^{u,v}}{Z_{\ell+1}^{u+1,v-1} + Z_{\ell}^{u,v}}.
\]

(4.11)
where these isomorphisms are canonical because

\[
B_{\ell-1}^{u,v} \subset B_{\ell}^{u,v}, \quad B_{\ell}^{u,v} \cap A_{u+1}^r = B_{\ell+1}^{u+1,v-1},
\]

\[
B_0^{u,v} \subset Z_1^{u,v}, \quad Z_1^{u,v} \cap A_{u+1}^r = Z_{\ell+1}^{u+1,v-1},
\]

\[
Z_{\ell+1}^{u,v} \subset A_{u+1}^r, \quad Z_{\ell}^{u,v} \cap A_{u+1}^r = Z_{\ell}^{u+1,v-1}.
\]
The direct sum decomposition in property (i) will depend on the choice of linear complements for the inclusions in (4.8):
\[
A^r_{u+1} + B^u_{\ell,v} = U^u_{\ell,v} \oplus (A^r_{u+1} + B^u_{\ell-1}) ,
\]
\[
A^r_{u+1} + Z^\infty_{\ell,v} = V^u_{\ell,v} \oplus (A^r_{u+1} + B^\infty_{\ell,v}) ,
\]
\[
A^r_{u+1} + Z^\ell_{\ell,v} = W^u_{\ell,v} \oplus (A^r_{u+1} + Z^\ell_{\ell+1}) .
\]

On the one hand, since the chains in (4.8) form a filtration of \( A^r \) when varying \( u \), we have
\[
A^r = \bigoplus_{u+v=r} \left( V^u_{\ell,v} \oplus \bigoplus_{\ell} (U^u_{\ell,v} \oplus W^u_{\ell,v}) \right) \tag{4.12}
\]
as vector space. On the other hand, according to the canonical isomorphisms (4.9), (4.10) and (4.11), the spaces \( U^u_{\ell,v} \), \( V^u_{\ell,v} \) and \( W^u_{\ell,v} \) can be chosen so that
\[
U^u_{\ell,v} \subset B^u_{\ell,v} , \quad V^u_{\ell,v} \subset Z^\infty_{\ell,v} , \quad W^u_{\ell,v} \subset Z^\ell_{\ell,v} ,
\]
yielding direct sum decompositions
\[
B^u_{\ell,v} = U^u_{\ell,v} \oplus \left( B^{u+1,v-1}_{\ell+1} + B^u_{\ell-1} \right) ,
\]
\[
Z^\infty_{\ell,v} = V^u_{\ell,v} \oplus \left( Z^{u+1,v-1}_{\ell+1} + B^\infty_{\ell,v} \right) ,
\]
\[
Z^\ell_{\ell,v} = W^u_{\ell,v} \oplus \left( Z^{u+1,v-1}_{\ell+1} + Z^\ell_{\ell+1} \right) .
\]

Hence
\[
U^u_{\ell,v} \cong E^u_{\ell,v} \cap \text{im} \, d_{\ell} ,
\]
\[
V^u_{\ell,v} \cong E^\infty_{\ell,v} ,
\]
\[
W^u_{\ell,v} \cong E^u_{\ell,v} \cap \ker \, d_{\ell}
\]
by (4.6), (4.7) and (4.14)–(4.16). Therefore property (i) follows from (4.12) and (4.17)–(4.19).

Property (ii) follows from (4.12) because
\[
U^u_{\ell,v} , \quad V^u_{\ell,v} , \quad W^u_{\ell,v} \subset A^r_u .
\]

Now property (iii) is obviously equivalent to the existence of \( U^u_{\ell,v} \), \( V^u_{\ell,v} \) and \( W^u_{\ell,v} \) as above satisfying
\[
d (U^u_{\ell,v}) = d (V^u_{\ell,v}) = 0 , \quad d (W^u_{\ell,v}) = U^u_{\ell,v} - \ell + 1 .
\]
The first equality of (4.20) holds by (4.13). We shall also check that, once the \( W^u_{\ell,v} \) is given satisfying (4.16), the \( U^u_{\ell,v} \) defined by (4.20) satisfies (4.14). This follows because \( d \) canonically induces a map
\[
\hat{d}_{\ell} : \frac{Z^u_{\ell+1,v-1} + Z^\infty_{\ell+1}}{B^{u+1,v-1}_{\ell+1} + B^\infty_{\ell,v-1}} ,
\]
which corresponds to the isomorphism \( \hat{d}_{\ell} \) via (4.7) and (4.8). So \( \hat{d}_{\ell} \) is an isomorphism as well, and thus the above \( U^u_{\ell,v} \) satisfies (4.14) as desired. This finishes the proof of Lemma 4.4.
4.4. **Proof of Theorem 4.1.** Assume $Z_{k-1}^{u+1,v-1} + Z_{\infty}^{u,v}$ is closed in $Z_k^{u,v}$ for all $u, v$. We shall need the following abstract result.

**Lemma 4.8.** Let $L$ be a real complete metrizable topological vector space, and $V \subset L$ a linear subspace. If $V \cap W = 0$, $V$ is closed in $L$, and $W$ is closed in $V + W$, then $V + W = V \oplus W$ as topological vector spaces.

**Proof.** We have $(V + W) \cap \overline{W} = W$ since $W$ is closed in $V + W$, yielding $V \cap \overline{W} = 0$ because $V \cap W = 0$. So $V + \overline{W} = V \oplus \overline{W}$ as topological vector spaces because all spaces involved are closed subspaces of $L$ (see for instance [22, Corollary 3 of Theorem 2.1, Chapter III, page 78]). Now the result follows easily. 

**Lemma 4.9.** For $u + v = r$, the space $(\Omega^r \cap \ker d) + \Omega^{r+1}_u$ is a closed subspace of $Z_k^{u,v} + (\Omega^r \cap \ker d) + \Omega^{r+1}_u$.

**Proof.** The space $(\Omega^r \cap \ker d)$ is closed in $\Omega$ since $d$ is a closed operator, and thus so is its subspace $Z_{\infty}^{u,v} = \Omega_u \cap (\Omega^r \cap \ker d)$. Hence $\Omega^r \cap \ker d = V \oplus Z_{\infty}^{u,v}$ as Hilbert spaces, where $V$ is the orthogonal complement of $Z_{\infty}^{u,v}$ in $\Omega^r \cap \ker d$; in particular $V$ is closed in $\Omega$ too. Obviously,

$$\Omega^r \cap \ker d + \Omega^{r+1}_u = V + Z_{\infty}^{u,v} + \Omega^{r+1}_u,$$

$$Z_k^{u,v} + (\Omega^r \cap \ker d) + \Omega^{r+1}_u = V + Z_k^{u,v} + \Omega^{r+1}_u.$$

On the other hand we clearly have

$$Z_{\infty}^{u,v} + \Omega^{r+1}_u = \Omega_u \cap (\Omega^r \cap \ker d) + \Omega^{r+1}_u,$$

$$Z_k^{u,v} + \Omega^{r+1}_u = \Omega_u \cap (Z_k^{u,v} + (\Omega^r \cap \ker d) + \Omega^{r+1}_u),$$

and thus $Z_{\infty}^{u,v} + \Omega^{r+1}_u$ and $Z_k^{u,v} + \Omega^{r+1}_u$ are respectively closed in $(\Omega^r \cap \ker d) + \Omega^{r+1}_u$ and $Z_k^{u,v} + (\Omega^r \cap \ker d) + \Omega^{r+1}_u$. Therefore Lemma 4.8 yields

$$\Omega^r \cap \ker d + \Omega^{r+1}_u = V \oplus (Z_{\infty}^{u,v} + \Omega^{r+1}_u),$$

$$Z_k^{u,v} + (\Omega^r \cap \ker d) + \Omega^{r+1}_u = V \oplus (Z_k^{u,v} + \Omega^{r+1}_u),$$

as topological vector spaces, and the result follows. 

**Remark 3.** In the proof of Lemma 4.9, the existence of $V$ so that $\Omega^r \cap \ker d = V \oplus Z_{\infty}^{u,v}$ as Hilbert spaces is the technical difficulty we were not able to solve without using square integrable differential forms; that is, we do not know if $\Omega^r \cap \ker d = V \oplus Z_{\infty}^{u,v}$ as topological vector spaces for some subspace $V$. This is the whole reason of introducing the $L^2$ spectral sequence in this paper.

Also, observe that the formula of Gromov-Shubin uses square integrable differential forms. Thus it can be more easily related to the $L^2$ spectral sequence than to the differentiable one. Though this is a minor problem that could be easily solved in the setting of $C^\infty$ differential forms.

We shall use the notation

$$X^r_u = \bigoplus_{a \leq u} \Omega^{a,r-a}, \quad \rho^r_u = \sum_{a \leq u} \pi_{a,r-a} : \Omega^r \longrightarrow X^r_u.$$

With respect to the inner product in $\Omega$ induced by $g$ or any $g_h$, the space $X^r_u$ is the orthogonal complement of $\Omega^r_u$ in $\Omega^r$, and $\rho^r_u$ is an orthogonal projection.

**Corollary 4.10.** For $u + v = r$, the space $\rho^r_u (\Omega^r \cap \ker d)$ is closed in $\rho^r_u (Z_k^{u,v} + (\Omega^r \cap \ker d))$. 

Proof. This follows from Lemma 4.9 since we clearly have
\[ (\Omega^r \cap \ker d) + \Omega^r_{u+1} = \rho^r_u(\Omega^r \cap \ker d) \oplus \Omega^r_{u+1}, \]
\[ Z^u_{k} + (\Omega^r \cap \ker d) + \Omega^r_{u+1} = \rho^r_u(Z^u_{k} + (\Omega^r \cap \ker d)) \oplus \Omega^r_{u+1}, \]
as topological vector spaces.

Recall that \( \tilde{d} : \text{dom } d / \ker d \to \text{im } d \) denotes the map induced by \( d \), and let \( L^u_{k,v} \) and \( L^r_{k} \) be the spaces introduced in Corollary 4.7 in Section 4.3 for the particular case of the \( L^2 \) spectral sequence of \( F \).

Lemma 4.11. We have
\[ \|d\zeta\|_h \leq h^{-q/2}h^{u+k} \|\zeta\| \]
for all \( \zeta \in L^u_{k,v} \) and \( 0 < h \leq 1 \).

Proof. This follows directly from Corollary 1.7 and (4.3).

Let \( \|\cdot\| \) and \( \|\cdot\|_h \) also stand for the quotient Hilbert norms on \( \Omega / \ker \Omega \) induced by the norms \( \|\cdot\| \) and \( \|\cdot\|_h \) on \( \Omega \), respectively. In particular we have the restrictions of \( \|\cdot\| \) and \( \|\cdot\|_h \) to each subspace \( L^u_{k,v} \subset \Omega / \ker \Omega \).

Lemma 4.12. For each subspace \( K \subset L^u_{k,v} \) of finite dimension there is some \( C'_K > 0 \), depending on \( K \), such that
\[ h^{-q/2}h^u \|\zeta\| \leq C'_K \|\zeta\|_h \]
for all \( \zeta \in K \) and \( 0 < h \leq 1 \).

Proof. Let \( u + v = r \). The restriction \( \rho^r_u : Z^u_{k,v} + (\Omega^r \cap \ker d) \to X^r_u \) induces a homomorphism
\[ \tilde{\rho}^r_u : \frac{Z^u_{k,v} + (\Omega^r \cap \ker d)}{\Omega^r \cap \ker d} \to \frac{X^r_u}{\rho^r_u(\Omega^r \cap \ker d)} . \]
We clearly have
\[ \ker \tilde{\rho}^r_u = \frac{\Omega^r_{u+1} + (\Omega^r \cap \ker d)}{\Omega^r \cap \ker d} . \]
So \( \tilde{\rho}^r_u \) induces a continuous linear isomorphism
\[ \frac{Z^u_{k,v} + (\Omega^r \cap \ker d)}{Z^u_{k-1,v+1} + (\Omega^r \cap \ker d)} \approx \text{im } \tilde{\rho}^r_u = \frac{\rho^r_u(Z^u_{k,v} + (\Omega^r \cap \ker d))}{\rho^r_u(\Omega^r \cap \ker d)} . \]
Observe that \( \text{im } \tilde{\rho}^r_u \) is a Hausdorff topological vector space by Corollary 1.10, and thus \( \|\cdot\| \) and \( \|\cdot\|_h \) induce norms on \( \text{im } \tilde{\rho}^r_u \) that will be also denoted by \( \|\cdot\| \) and \( \|\cdot\|_h \), respectively. By (4.21) and Corollary 4.7, the homomorphism \( \tilde{\rho}^r_u \) restricts to an injection \( \rho^r_u : L^u_{k,v} \to \text{im } \tilde{\rho}^r_u \). Since \( \rho^r_u \) is an orthogonal projection for any metric \( g_h \), we easily get
\[ \|\rho^r_u \zeta\|_h \leq \|\zeta\|_h \quad \text{for all } \zeta \in L^u_{k,v} . \]
Here, we use the norm on \( \text{im } \tilde{\rho}^r_u \) in the left hand side of (4.22), and the norm on \( \Omega / \ker \Omega \) in its right hand side. Observe that, by (1.1),
\[ h^{-q/2}h^u \|\omega\| \leq \|\omega\|_h \quad \text{for all } \omega \in X^r_u \quad \text{and } \quad 0 < h \leq 1 , \]
yielding
\[ h^{-q/2}h^u \|\xi\| \leq \|\xi\|_h \quad \text{for all } \xi \in \text{im } \tilde{\rho}^r_u \quad \text{and } \quad 0 < h \leq 1 . \]
Moreover, since $K$ is of finite dimension, $\text{im} \overline{\rho}_u$ is Hausdorff, and the restriction $\overline{\rho}_u^r : L_k^{u,v} \to \text{im} \overline{\rho}_u$ is injective, we get the existence of some $C'_K > 0$ so that

$$\| \zeta \| \leq C'_K \| \overline{\rho}_u \zeta \| \quad \text{for all} \quad \zeta \in K. \quad (4.24)$$

So

$$h^{-q/2} h^u \| \zeta \| \leq C'_K h^{-q/2} h^u \| \overline{\rho}_u \zeta \|, \quad \text{by (4.24)},$$

$$\leq C'_K \| \overline{\rho}_u \zeta \|_h, \quad \text{by (4.23)},$$

$$\leq C'_K \| \zeta \|_h, \quad \text{by (4.22)},$$

for all $\zeta \in K$ and $0 < h \leq 1$ as desired.

**Corollary 4.13.** For each subspace $K \subset L_k^r$ of finite dimension there is some $C_K > 0$, depending on $K$, such that

$$\| \bar{d} \zeta \|_h \leq C_K h^k \| \zeta \|_h$$

for all $\zeta \in K$ and $0 < h \leq 1$.

**Proof.** Since $K$ is of finite dimension, there is some constant $C''_K$, depending on $K$, so that

$$\| \bar{d} \zeta \| \leq C''_K \| \zeta \| \quad \text{for all} \quad \zeta \in K. \quad (4.25)$$

Because $L_k^r = \bigoplus_{u+v=r} L_k^{u,v}$, any finite dimensional subspace $K \subset L_k^r$ is contained in the sum of finite dimensional subspaces $K^{u,v} \subset L_k^{u,v}, u+v=r$. Therefore we can assume $K$ is contained in some $L_k^{u,v}$ with $u+v=r$. Then, for $\zeta \in K$ and $0 < h \leq 1$, we have

$$\| \bar{d} \zeta \|_h \leq h^{-q/2} h^{u+k} \| \bar{d} \zeta \|, \quad \text{by Lemma 4.11},$$

$$\leq C''_K h^{-q/2} h^{u+k} \| \zeta \|, \quad \text{by (4.27)},$$

$$\leq C'_K C''_K h^k \| \zeta \|_h, \quad \text{by Lemma 4.12},$$

and the result follows with $C_K = C'_K C''_K$. \qed

Now the proof of Theorem 4.1 can be finished as follows. If $m^r_k < \infty$, then Corollary 4.13 holds for $K = L_k^r$, and thus

$$F^r_k (C_i h^{2k}) \geq m^r_k.$$

Therefore, in this case, Theorem 4.1 follows from Corollary 4.6 and Proposition 4.3.

If $m^r_k = \infty$, choose any sequence of finite dimensional subspaces $K_i \subset L_k^r$ so that $\dim K_i \uparrow \infty$. Then Corollary 4.13 gives a sequence $C_i > 0$ such that

$$F^r_k (C_i h^{2k}) \geq \dim K_i$$

for $0 < h \leq 1$. Hence Theorem 4.1 also follows in this case by Corollary 4.6 and Proposition 4.3.

5. **Asymptotics of eigenforms**

In the whole of this section, $\mathcal{F}$ is assumed to be a Riemannian foliation and the metric bundle-like.
5.1. The Hodge theoretic nested sequence. So far we have constructed bi-
graded subspaces $\mathcal{H}_1, \mathcal{H}_2 \subset \Omega$, which are respectively isomorphic to $\hat{e}_1, e_2$ as bi-
graded topological vector spaces by Theorem 2.1 and Theorem 2.2-(iv). We con-
continue constructing subspaces $\mathcal{H}_k \subset \Omega$ and isomorphisms $e_k \cong \mathcal{H}_k$ by induction on $k$ as
follows. Suppose we have constructed $\mathcal{H}_k$ and an explicit isomorphism $e_k \cong \mathcal{H}_k$ for some $k \geq 2$. Then the homomorphism $d_k$ corresponds to some homomorphism on $\mathcal{H}_k$ that will be denoted by $d_k$ as well. Thus $\mathcal{H}_k$ becomes a finite dimensional complex. Let $\delta_k$ be the adjoint of $d_k$ on the finite dimensional Hilbert space $\mathcal{H}_k$, and set $\Delta_k = d_k \delta_k + \delta_k d_k$ and $\mathcal{H}_{k+1} = \ker \Delta_k = \ker d_k \cap \ker \delta_k$. We have the orthogonal decomposition

$$\mathcal{H}_k = \mathcal{H}_{k+1} \oplus \text{im } d_k \oplus \text{im } \delta_k,$$

yielding

$$e_{k+1} \cong H(e_k, d_k) \cong H(\mathcal{H}_k, d_k) \cong \mathcal{H}_{k+1},$$

which completes the induction step. So $(\mathcal{H}_k, d_k)$ is, by definition, some kind of a Hodge theoretic version of the sequence $(\hat{e}_1, d_1), (e_2, d_2), (e_3, d_3), \ldots$, and thus of the sequence $(\tilde{E}_1, d_1), (E_2, d_2), (E_3, d_3), \ldots$ as well by Theorem 2.2 and Theorem 2.2-(vii). Furthermore each $\Delta_k$ is bihomogeneous of bidegree $(0,0)$, and thus $\mathcal{H}_k$ inherits the bigrading from $\Omega$, which clearly corresponds to the bigrading of $E_k$ and $e_k$. Observe that the nested sequence

$$\Omega \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \mathcal{H}_3 \supset \mathcal{H}_4 \supset \cdots$$

stabilizes at most at the $(q+1)$th step since so does $E_k$. Then its final term $\mathcal{H}_{q+1} = \mathcal{H}_{q+2} = \cdots$ will be denoted by $\mathcal{H}_\infty$, and we have $E_\infty \cong e_\infty \cong \mathcal{H}_\infty$.

We shall need a better understanding of the new terms $\mathcal{H}_k$ for $k > 2$. Precisely, we shall use the following result.

**Theorem 5.1.** Let $k \geq 3$ and $\omega \in \mathcal{H}_k^{u,v}$. Then $\omega \in \mathcal{H}_k^{u,v}$ if and only if there are sequences $\alpha_i = \sum_{a>0} \alpha_i^a$ and $\beta_i = \sum_{a>0} \beta_i^a$, where $\alpha_i^a \in \Omega_1^{u+a,v-a}$ and $\beta_i^a \in \Omega_1^{u-a,v+a}$, such that

$$\pi_{u+a,v-a+1} d(\omega + \alpha_i) \to 0, \quad \pi_{u-a,v+a-1} \delta(\omega + \beta_i) \to 0$$

strongly in $\Omega$ for $0 < a < k$.

The rest of this section will be devoted to prove Theorem 5.1. To begin with, the nested sequence $\mathcal{H}_k$ is most properly a Hodge theoretic version of another se-
quence of bigraded topological complexes $(\hat{e}_1, d_k)$, which are defined as follows by induction on $k \geq 1$. First, let $\hat{e}_{1,1} = \hat{e}_1$ and $\hat{e}_{1,2} = H(\hat{e}_1)$ with the induced topology in cohomology. We have an explicit isomorphism $e_2 \cong \hat{e}_{1,2}$ of bigraded topological vector spaces given by Theorem 2.2-(vii). Now suppose that, for some fixed $k \geq 2$, we have defined $\hat{e}_{1,k}$ with an explicit isomorphism $e_k \cong \hat{e}_{1,k}$ of bigraded topological vector spaces. Then $\hat{e}_{1,k}$ becomes a topological complex via this isomorphism, and define $\hat{e}_{1,k+1} = H(\hat{e}_{1,k})$. Furthermore the composition $e_{k+1} \cong H(e_k) \cong \hat{e}_{1,k+1}$ is an explicit isomorphism of bigraded topological vector spaces.

**Lemma 5.2.** For $k \geq 1$, we have a canonical isomorphism

$$e_{1,k}^{u,v} \cong \frac{\tilde{e}_{1,k}^{u,v} + b_0^{u,v}}{b_{k-1}^{u,v} + b_0^{u,v}}$$

(5.1)
of topological vector spaces. Moreover, for \( k \geq 2 \), the above isomorphism \( e_k^{u,v} \cong e_{1,k}^{u,v} \) corresponds to the canonical map

\[
\frac{z_k^{u,v}}{b_k^{u,v}} \mapsto \frac{z_k^{u,v} + b_0^{u,v}}{b_k^{u,v} + b_0^{u,v}} \tag{5.2}
\]

when applying (5.1).

**Proof.** The result is proved by induction on \( k \). First, the case \( k = 1 \) is trivial.

Second, the kernel and the image of \( d_1 \) in \( e_1^{u,v} \) respectively are \( z_2^{u,v}/b_0^{u,v} \) and \( b_1^{u,v}/b_0^{u,v} \), whose canonical projections in \( e_1^{u,v} = z_1^{u,v}/b_0^{u,v} \) are

\[
\frac{z_2^{u,v} + b_0^{u,v}}{b_0^{u,v}}, \quad \frac{b_1^{u,v} + b_0^{u,v}}{b_0^{u,v}}, \tag{5.3}
\]

yielding the canonical isomorphism (5.1) for \( k = 2 \). Since the isomorphism \( e_2^{u,v} \cong \hat{e}_1^{u,v} \) is canonically defined, it corresponds to the canonical map (5.2) for \( k = 2 \).

Now assume the result holds for \( k = \ell \geq 2 \) and we prove it for \( k = \ell + 1 \). The kernel and the image of \( d_\ell \) in \( e_\ell \) respectively are \( z_{\ell+1}^{u,v}/b_{\ell-1} \) and \( b_\ell^{u,v}/b_{\ell-1} \), whose images by the canonical isomorphism (5.2) for \( k = \ell \) are

\[
\frac{z_{\ell+1}^{u,v} + b_0^{u,v}}{b_0^{u,v}}, \quad \frac{b_\ell^{u,v} + b_0^{u,v}}{b_0^{u,v}}. \tag{5.4}
\]

These spaces respectively correspond to the kernel and the image of \( d_\ell \) in \( \hat{e}_1^{u,v} \) by (5.1), yielding the canonical isomorphism (5.1) for \( k = \ell + 1 \). Again, because the isomorphism \( e_\ell^{u,v} \cong e_{1,\ell}^{u,v} \) is canonically defined, it is given by the canonical map (5.2) for \( k = \ell + 1 \).

We shall consider each isomorphism (5.1) as an equality from now on.

For \( k \geq 1 \), let \( \Pi_k \) denote the orthogonal projections \( \Omega \to \mathcal{H}_k \); in particular, \( \Pi_1 = \Pi \) with this notation. Let also \( P_0 = P \), \( Q_0 = Q \) and, for \( k \geq 1 \), let \( P_k \) and \( Q_k \) be the orthogonal projections of \( \Omega \) onto \( d_k(\mathcal{H}_k) \) and \( \delta_k(\mathcal{H}_k) \). Finally let \( P_k = \sum_{0 \leq \ell \leq k} P_\ell \) and \( Q_k = \sum_{0 \leq \ell \leq k} Q_\ell \) for \( k \geq 0 \).

**Lemma 5.3.** For \( k \geq 1 \), \( \Pi_k \) induces an isomorphism \( \hat{e}_{1,k}^{u,v} \cong \mathcal{H}_k^{u,v} \), whose composition with the canonical isomorphism \( e_k^{u,v} \cong e_{1,k}^{u,v} \) is the above isomorphism \( e_k^{u,v} \cong \mathcal{H}_k^{u,v} \).

**Proof.** Observe that the first part of the statement means that we have an orthogonal decomposition

\[
z_k^{u,v} + b_0^{u,v} = \mathcal{H}_k^{u,v} \oplus \left( b_{k-1}^{u,v} + b_0^{u,v} \right). \tag{5.5}
\]

Again the result follows by induction on \( k \). We have an orthogonal decomposition

\[
z_1^{u,v} = \mathcal{H}_1^{u,v} \oplus b_0^{u,v} \tag{5.6}
\]

by Theorem 2.1. Thus the isomorphism \( \hat{e}_{1,1} = \hat{e}_1^{u,v} \cong \mathcal{H}_1^{u,v} \) is induced by the orthogonal projection \( \Pi_1 \) onto \( \mathcal{H}_1 \). On the other hand, the kernel and image of \( d_1 \) in \( \mathcal{H}_1^{u,v} \) respectively correspond by this isomorphism to the kernel and image of \( d_1 \) on \( e_1^{u,v} \), which are respectively given by (5.3). So the kernel and image of \( d_1 \) in \( \mathcal{H}_1^{u,v} \) are the orthogonal projections \( \Pi_1 \left( z_2^{u,v} + b_0^{u,v} \right) \) and \( \Pi_1 \left( b_1^{u,v} + b_0^{u,v} \right) \), respectively.
Hence, by definition, $\mathcal{H}_2^{u,v}$ is the orthogonal complement of $\Pi_1 \left( b_1^{u,v} + \bar{b}_0^{u,v} \right)$ in $\Pi_1 \left( z_2^{u,v} + \bar{b}_0^{u,v} \right)$, which is equal to the orthogonal complement of $b_1^{u,v} + \bar{b}_0^{u,v}$ in $z_2^{u,v} + \bar{b}_0^{u,v}$ by (5.6) since 
\[ \bar{b}_0^{u,v} \subset b_1^{u,v} + b_0^{u,v} \subset z_2^{u,v} + \bar{b}_0^{u,v} \subset z_1^{u,v}. \]

Thus the result follows for $k = 2$.

Now suppose the statement holds for $k = \ell \geq 2$. Then, via the isomorphism $\hat{e}_{1,\ell}^{u,v} \cong H_\ell^{u,v}$ induced by $\Pi_\ell$, the kernel and image of $d_\ell$ in $H_\ell^{u,v}$ respectively correspond to the kernel and image of $d_\ell$ in $\hat{e}_{1,\ell}^{u,v}$, which are given in (5.3). So the kernel and image of $d_\ell$ in $H_\ell^{u,v}$ are the orthogonal projections $\Pi_\ell \left( z_{\ell+1}^{u,v} + \bar{b}_0^{u,v} \right)$ and $\Pi_\ell \left( b_\ell^{u,v} + \bar{b}_0^{u,v} \right)$, respectively. Hence, by definition, $\mathcal{H}_\ell^{u,v}$ is the orthogonal complement of $\Pi_\ell \left( b_\ell^{u,v} + \bar{b}_0^{u,v} \right)$ in $\Pi_\ell \left( z_{\ell+1}^{u,v} + \bar{b}_0^{u,v} \right)$, which is equal to the orthogonal complement of $b_\ell^{u,v} + \bar{b}_0^{u,v}$ in $z_{\ell+1}^{u,v} + \bar{b}_0^{u,v}$ by (5.3) for $k = \ell$ since 
\[ b_{\ell-1}^{u,v} + \bar{b}_0^{u,v} \subset b_\ell^{u,v} + \bar{b}_0^{u,v} \subset z_{\ell+1}^{u,v} + \bar{b}_0^{u,v} \subset z_\ell^{u,v} + \bar{b}_0^{u,v}. \]

Thus the result follows for $k = \ell + 1$.

**Remark 4.** The inverse of the isomorphism $\hat{e}_{1,k}^{u,v} \cong H_k^{u,v}$ in Lemma 5.3 is obviously induced by the inclusion $H_k^{u,v} \hookrightarrow z_k^{u,v} + \bar{b}_0^{u,v}$. So we can summarize Lemmas 5.2 and 5.3 by saying that, for $k \geq 2$, the isomorphism $e_k^{u,v} \cong H_k^{u,v}$ is given by the diagram 
\[ e_k^{u,v} = \frac{z_k^{u,v}}{b_{k-1}^{u,v}} \cong e_{1,k}^{u,v} = \frac{z_k^{u,v} + \bar{b}_0^{u,v}}{b_{k-1}^{u,v} + \bar{b}_0^{u,v}} \cong H_k^{u,v}, \] (5.7)

where both isomorphisms are canonically induced by inclusions.

**Remark 5.** In general, we have $z_k^{u,v} \neq H_k^{u,v} \oplus b_k^{u,v}$ because $H_k^{u,v} \nsubseteq z_k^{u,v}$, but the orthogonal decomposition (5.3) always holds. This is the reason the nested sequence $\mathcal{H}_k$ is a Hodge theoretic version of the sequence $(\hat{e}_{1,k}, d_k)$ better than of the sequence $(e_1, d_1), (e_2, d_2), (e_3, d_3), \ldots$.

The following proposition is the key result to prove Theorem 5.1.

**Proposition 5.4.** Let $\omega \in H_k^{u,v}$ and $\gamma \in H_{k+1}^{u,v}$ for $k \geq 2$. If there is a sequence $\alpha_i \in \Omega_{u+1}^{u,v}$ such that
\[ \pi_{u+a,v-a+1}(\omega + \alpha_i) \rightarrow 0, \quad 0 < a < k, \]
\[ \bar{Q}_{k-2} \pi_{u+k,v-k+1}(\omega + \alpha_i) \rightarrow 0, \quad \Pi_k \pi_{u+k,v-k+1}(\omega + \alpha_i) \rightarrow \gamma \]
strongly in $\Omega$, then $d_k \omega = \gamma$. Moreover, in this case the sequence $\alpha_i$ can be chosen so that
\[ \pi_{u+a,v-a+1}(\omega + \alpha_i) \rightarrow 0, \quad 0 < a < k, \]
\[ \pi_{u+k,v-k+1}(\omega + \alpha_i) \rightarrow \gamma \]
with respect to the $C^\infty$ topology in $\Omega$.

The following slightly weaker result will be used as an intermediate step in the proof of Proposition 5.4.
Lemma 5.5. Let $\gamma \in \mathcal{H}_{k}^{u+k, v-k+1}$ for $k \geq 2$. If there is some sequence $\alpha_i \in \Omega_{u+1}^{u+v}$, such that

$$
\pi_{u+a, v-a+1}d\alpha_i \rightarrow 0, \quad 0 < a < k,
\quad \tilde{Q}_{k-2}\pi_{u+k, v-k+1}d\alpha_i \rightarrow 0, \quad \Pi_k\pi_{u+k, v-k+1}d\alpha_i \rightarrow \gamma
$$

strongly in $\Omega$, then $\gamma = 0$.

Both Proposition 5.4 and Lemma 5.5 will be proved simultaneously by induction on $k \geq 2$. For the case $k = 2$ we need the following.

Lemma 5.6. We have $\Pi_2\pi_{u+2, v-1}d\tilde{d}_1\beta = 0$ for any $\beta \in \tilde{\mathcal{H}}_{1}^{u-1, v}$.

Proof. Write $\beta = \beta' + \beta''$ with $\beta' \in P(\Omega^{u-1, v})$ and $\beta'' \in Q(\Omega^{u, v-1})$. Then

$$
\Pi_2\pi_{u+2, v-2}d\tilde{d}_1\beta = \Pi_2(d_{2-1}(d_{1,0}\beta' + d_{0,1}\beta'') + d_{1,0}Q(d_{2-1}\beta' + d_{1,0}\beta''))
$$

$$
= \Pi_2((d_{2-1}d_{1,0} + d_{1,0}d_{2-1})\beta' + (d_{2-1}d_{0,1} + d_{1,0})\beta'')
\quad - \Pi_2d_{1,0}P(d_{2-1}\beta' + d_{1,0}\beta'') - \Pi_2d_{1,0}P(d_{2-1}\beta' + d_{1,0}\beta'')
$$

$$
= -\Pi_2d_{1,0}P(d_{2-1}\beta' + d_{1,0}\beta'') - \Pi_2Pd_{1,0}P(d_{2-1}\beta' + d_{1,0}\beta'')
$$

$$
= 0
$$

by (2.3), Lemma 2.3 and because $\Pi_2d_{0,1} = \Pi_2d_1 = \Pi_2P = 0$.

Lemma 5.7. Let $\alpha_i$ be a sequence in $\tilde{\mathcal{H}}_{1}^{u, v}$ such that $d\tilde{d}_1\alpha_i \rightarrow 0$ strongly in $\Omega$. Then

$$
\Pi_2\pi_{u+2, v-1}d\alpha_i \rightarrow 0
$$

strongly in $\Omega$.

Proof. Since the image of $\tilde{d}_1$ is closed and equal to its kernel, the hypothesis implies the existence of a sequence $\beta_i \in \tilde{\mathcal{H}}_{1}^{u-1, v}$ such that $\alpha_i + \tilde{d}_1\beta_i \rightarrow 0$ strongly in $\Omega$. On the other hand we have

$$
\Pi_2\pi_{u+2, v-1}d = \Pi_2d_{2-1}\pi_{u, v} + \Pi_2d_{1,0}\pi_{u+1, v-1}
= \Pi_2d_{2-1}\pi_{u, v} + \Pi_2d_{1,0}Q\pi_{u+1, v-1}
$$

on $\tilde{\mathcal{H}}_{1}^{u, v}$, and thus the operator $\Pi_2\pi_{u+2, v-1}d : \mathcal{H}_{1}^{u, v} \rightarrow \mathcal{H}_{2}^{u+2, v-1}$ is bounded because $d_{2-1}$ and $\Pi d_{1,0}Q$ are bounded operators in $\Omega$ by Lemma 2.4. Therefore

$$
\Pi_2\pi_{u+2, v-1}d(\alpha_i + \tilde{d}_1\beta_i) \rightarrow 0
$$

strongly in $\Omega$. Then the result follows directly from Lemma 5.6.

Proof of Lemma 5.4 for the case $k = 2$. In this case we have $\gamma \in \mathcal{H}_{2}^{u+2, v-1}$ and $\alpha_i \in \Omega^{u+1, v-1}$, which satisfy

$$
d_{0,1}\alpha_i \rightarrow 0, \quad Qd_{1,0}\alpha_i \rightarrow 0, \quad \Pi_2d_{1,0}\alpha_i \rightarrow \gamma
$$

strongly in $\Omega$. Since

$$
\Pi d_{1,0}\Pi\alpha_i = d_{1,0}\Pi\alpha_i \perp \mathcal{H}_{2}^{u+2, v-1},
$$

we get $\Pi d_{1,0}\Pi\alpha_i \rightarrow \gamma$ strongly in $\Omega$. But $\Pi_2d_{1,0}P\alpha_i = \Pi_2Pd_{1,0}P\alpha_i = 0$ by Lemma 2.3 and thus we get

$$
\Pi_2\pi_{u+2, v-1}dQ\alpha_i = \Pi_2d_{1,0}Q\alpha_i \rightarrow \gamma
$$

(5.8)
strongly in $\Omega$. Now observe that $Q\alpha_i \in \tilde{H}^{u,v}_1$, and
\[ \gamma_1 Q\alpha_i = \tilde{H}^{u,v}_1, \]
and because $d_{0,1}Q = d_{0,1}$ and by Lemma 2.3. Then the result follows by (5.8) and Lemma 5.7.

Now let $\ell \geq 2$ and assume that Lemma 5.5 holds for $2 \leq k \leq \ell$. If $\ell > 2$, assume also that Proposition 5.4 holds for $2 \leq k < \ell$.

Proof of Proposition 5.4 for $k = \ell$. First, we check that the assignment $\omega \mapsto \gamma$, under the conditions in the statement, defines a map $H^u_v \rightarrow H^{u+\ell,v-\ell+1}$—observe that, if such a map is well defined, it is obviously linear. Suppose there is another $\gamma' \in H^{u+\ell,v-\ell+1}$ and another sequence $\alpha'_i \in \Omega^{u,v}_{u+1}$ such that
\[ \pi u+v a\rightarrow \gamma_1 0, \quad 0 < a < \ell, \]
\[ Q_{\ell-2} \pi u+\ell,v-\ell+1 a\rightarrow \gamma_1 \]
strongly in $\Omega$. Then the sequence $\alpha_i - \alpha'_i \in \Omega^{u,v}_{u+1}$ satisfies
\[ \pi u+v a \rightarrow 0, \quad 0 < a < \ell, \]
\[ Q_{\ell-2} \pi u+\ell,v-\ell+1 d(\alpha_i - \alpha'_i) \rightarrow 0, \quad \Pi \pi u+\ell,v-\ell+1 d(\alpha_i - \alpha'_i) \rightarrow \gamma - \gamma' \]
strongly in $\Omega$. Therefore $\gamma = \gamma'$ by Lemma 5.3 for the case $k = \ell$.

Second, we prove that the above map $H^u_v \rightarrow H^{u+\ell,v-\ell+1}$ is $d_{\ell}$; i.e., for each $\omega \in H^u_v$, we prove the existence of a sequence $\alpha_i \in \Omega^{u,v}_{u,v+1}$ such that
\[ \pi u+v a \rightarrow 0, \quad 0 \leq a < \ell, \]
\[ Q_{\ell-2} \pi u+\ell,v-\ell+1 d(\alpha_i + \alpha'_i) \rightarrow 0, \]
\[ \Pi \pi u+\ell,v-\ell+1 d(\alpha_i + \alpha'_i) \rightarrow d\omega \in H^{u+\ell,v-\ell+1} \]
strongly in $\Omega$. According to (5.7), for each $\omega \in H^u_v$ there is a sequence $\omega_i \in z^{u,v}_\ell$ converging to $\omega$ with respect to the $C^\infty$ topology and such that $\omega$ and all the $\omega_i$ define the same class $\tilde{\zeta} \in \xi^{u,v}_\ell$; thus all the $\omega_i$ define the same class $\zeta \in \xi^{u,v}_\ell$. By definition of $z^{u,v}_\ell$, there is another sequence $\alpha_i \in \Omega^{u,v}_{u,v+1}$ such that $\omega_i + \alpha_i \in Z^{u,v}_\ell$. So all the $\omega_i + \alpha_i$ define the same class $\zeta \in F^{u,v}_\ell$, and the class $d\omega \in H^{u+\ell,v-\ell+1}$ is defined by any of the forms $d(\omega_i + \alpha_i) \in Z^{u+\ell,v-\ell+1}_\ell$. Thus
\[ \pi u+v a \rightarrow 0, \quad 0 \leq a < \ell, \]
and any of the forms
\[ \pi u+\ell,v-\ell+1 d(\omega_i + \alpha_i) \in z^{u+\ell,v-\ell+1}_\ell \]
define the class $d\omega \in H^{u+\ell,v-\ell+1}_\ell$ as well as the class $d\zeta \in z^{u+\ell,v-\ell+1}_\ell$, yielding
\[ \pi u+v a \rightarrow 0, \quad 0 \leq a < \ell, \]
\[ Q_{\ell-2} \pi u+\ell,v-\ell+1 d(\omega_i + \alpha_i) = 0, \]
\[ \Pi \pi u+\ell,v-\ell+1 d(\omega_i + \alpha_i) = d\omega \in H^{u+\ell,v-\ell+1}_\ell \]
independently of $i$. Then (5.9)–(5.11) follow by the $C^\infty$ convergence $\omega_i \rightarrow \omega$, as desired.
Finally we prove the last part of the statement. Observe that, in fact, the above arguments yield $C^\infty$ convergence in (5.9)–(5.11), and also the $C^\infty$ convergence
\[ \bar{Q}_{\ell-1}\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i) \rightarrow 0. \]

For each $i$, take $\sigma_i^1 \in Q_i(\Omega^{u+\ell-1,v+\ell+1})$ satisfying
\[ \Pi_1d_1,0\sigma_i^1 = P_i\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i), \]
and take $\sigma_i^0 \in Q_0(\Omega^{u+\ell,v-\ell})$ such that
\[ d_0,1\sigma_i^0 + P_0d_1,0\sigma_i^1 - P_0\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i) \rightarrow 0 \]
with respect to the $C^\infty$ topology. If $\ell > 2$, for each $i$ and $m = 2, \ldots, \ell - 1$ take some $\sigma_i^m \in Q_m(\Omega^{u+\ell-m,v-\ell+m})$ such that
\[ d_m\sigma_i^m = P_m\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i). \]

By Proposition 5.4 for $k < \ell$ there are sequences $\tau_{i,j}^m \in \Omega^{u+\ell-m+1}$ such that
\[ \pi_{u+\ell-m+a,v-\ell+m-a+1}d(\sigma_i^m + \tau_{i,j}^m) \rightarrow 0, \quad 0 < a < m, \]
\[ \pi_{u+\ell,v-\ell+1}d(\sigma_i^m + \tau_{i,j}^m) \rightarrow P_m\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i) \]
with respect to the $C^\infty$ topology in $\Omega$. Then, for each $i, m$ we can clearly choose $j$ depending on $i, m$ so that $\tau_{i,j}^m$ satisfies
\[ \pi_{u+\ell-m+a,v-\ell+m-a+1}d(\sigma_i^m + \tau_{i,j}^m) \rightarrow 0, \quad 0 < a < m, \]
\[ \pi_{u+\ell,v-\ell+1}d(\sigma_i^m + \tau_{i,j}^m) - P_m\pi_{u+\ell,v-\ell+1}d(\omega+\alpha_i) \rightarrow 0 \]
with respect to the $C^\infty$ topology. Let
\[ \beta_i = \alpha_i - \sigma_i^0 - \sigma_i^1 - \sum_{m=2}^{\ell-1}(\sigma_i^m + \tau_{i,j}^m) \in \Omega_{u+1}, \]
where the last term does not show up if $\ell = 2$. From (5.12)–(5.13) we get
\[ \pi_{u+a,v-a+1}d(\omega_i + \beta_i) \rightarrow 0, \quad 0 \leq a < \ell, \]
\[ \pi_{u+\ell,v-\ell+1}d(\omega_i + \beta_i) \rightarrow d_\ell\omega \]
with respect to the $C^\infty$ topology in $\Omega$, and the proof is finished.

We already know that both Proposition 5.4 and Lemma 5.5 hold for $k \leq \ell$, and we have to prove Lemma 5.5 for $k = \ell + 1$. The arguments will be similar to the case $k = 2$, and thus we need an appropriate version of Lemma 5.5. In particular, the generalization of $\tilde{H}_1^{u,v}$ that fits our needs turns out to be the following:
\[ \tilde{H}_\ell^{u,v} = P_0(\Omega^{u,v}) \oplus \bigoplus_{0 < a < \ell} \Omega^{u+a,v-a} \oplus \bar{Q}_{\ell-1}(\Omega^{u+\ell,v-\ell}). \]

Let also $\tilde{H}_\ell^{u,v} = \bigoplus u \tilde{H}_\ell^{u,v}$. We have orthogonal projections $\tilde{\Pi}_{\ell,u,v} : \Omega \rightarrow \tilde{H}_\ell^{u,v}$ and $\tilde{\Pi}_{\ell,v,u} : \Omega \rightarrow \tilde{H}_\ell^{u,v}$ given by
\[ \tilde{\Pi}_{\ell,u,v} = P_0\pi_{u,v} + \sum_{0 < a < \ell} \pi_{u+a,v-a} + \bar{Q}_{\ell-1}\pi_{u+\ell,v-\ell}, \quad \tilde{\Pi}_{\ell,v,u} = \sum_u \tilde{\Pi}_{\ell,u,v}, \]
and let $\tilde{d}_\ell = \tilde{\Pi}_{\ell,v,u}d : \tilde{H}_\ell^{u,v} \rightarrow \tilde{H}_\ell^{u,v}$.

Lemma 5.8. We have $d_\ell^2 = 0$.
Proof. Consider the following subspaces of $\Omega^{u+v}$:

\[
\mathcal{A}_\ell^{u,v} = Z_\ell^{u,v} + B_0^{u,v} + \Omega^{u+v}_{u+1} = (z_\ell^{u,v} + b_0^{u,v}) \oplus \Omega^{u+v}_{u+1},
\]

\[
B^{u,v} = B_0^{u,v} + \Omega^{u+v}_{u+1} = b_0^{u,v} \oplus \Omega^{u+v}_{u+1}.
\]

First, observe that each $\tilde{\mathcal{H}}^{u,v}_\ell$ is the orthogonal complement of $A^{u+\ell,v-\ell}_\ell$ in $\overline{B}^{u,v}$, and thus $\tilde{\mathcal{H}}^{v,u}_\ell$ is the orthogonal complement of $A^{u,v-\ell}_{\ell-1} = \bigoplus_a A^{u,v-\ell}_a$ in $\overline{B}^{v,u} = \bigoplus_a \overline{B}^{u,v}$. So the inclusion $\tilde{\mathcal{H}}^{v,u}_\ell \hookrightarrow \overline{B}^{v,u}/A^{u,v-\ell}_{\ell-1}$

\[(5.16)\]

whose inverse is induced by the orthogonal projection $\bar{\Pi}_{\ell-1,:}\colon \overline{B}^{v,u} \to \tilde{\mathcal{H}}^{v,u}_\ell$. Second, observe that both $\overline{B}^{u,v}$ and $A^{u,v-\ell}_{\ell-1}$ are subcomplexes of $(\Omega, d)$. Moreover $d_\ell$ in $\tilde{\mathcal{H}}^{v,u}_\ell$ clearly corresponds to the differential map in the quotient complex $\overline{B}^{v,u}/A^{u,v-\ell}_{\ell-1}$ via (5.16), and the result follows.

Since $H(\tilde{\mathcal{H}}^{v,u}_1, \tilde{d}_1) = 0$, the following two lemmas generalize Lemma 5.6.

**Lemma 5.9.** For any

\[
\beta \in \bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \tilde{Q}_{\ell-1}(\Omega^{u-1+\ell,v-\ell})
\]

we have

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d\bar{\Pi}_{\ell,:} d\beta = 0.
\]

Proof. By the expression

\[
\bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \tilde{Q}_{\ell-1}(\Omega^{u-1+\ell,v-\ell})
\]

\[
= \bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \tilde{H}^{u+\ell-2,v-\ell+1}_1 + (Q_1 + \cdots + Q_{\ell-1})(\Omega^{u+\ell-1,v-\ell})
\]

it is enough to consider the following three cases. First, assume $\beta \in \bigoplus_{a<\ell} \Omega^{u-1+a,v-a}$ and let $\beta' = \pi_{u+\ell-2,v-\ell+1}\beta$. We clearly have

\[(d - \bar{\Pi}_{\ell,:} d)\beta = (id - \tilde{Q}_{\ell-1})d_{2,-1}\beta' = (Q_\ell + \Pi_{\ell+1} + \tilde{P}_\ell)d_{2,-1}\beta',\]

yielding

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d\bar{\Pi}_{\ell,:} d\beta = -\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d(Q_\ell + \Pi_{\ell+1} + \tilde{P}_\ell)d_{2,-1}\beta'
\]

\[
= -\Pi_{\ell+1} d_{1,0}(Q_\ell + \Pi_{\ell+1} + \tilde{P}_\ell)d_{2,-1}\beta'
\]

\[
= -\Pi_{\ell+1} d_{1,0}(Q_\ell + \Pi_{\ell+1} + P_1 + \cdots + P_\ell)d_{2,-1}\beta'
\]

\[
= -\Pi_{\ell+1} d_{1,0} P_0 d_{2,-1}\beta' = 0
\]

by Lemma 2.3, and because $\Pi_{\ell+1} d_1 = 0$ and $\Pi_{\ell+1} P_0 = 0$.

Second, suppose $\beta \in \tilde{\mathcal{H}}^{u+\ell-2,v-\ell+1}_1$ and write $\beta = \beta' + \beta''$ with

\[
\beta' \in P_0(\Omega^{u+\ell-2,v-\ell+1}), \quad \beta'' \in Q_0(\Omega^{u+\ell-1,v-\ell}).
\]

We clearly have

\[(\bar{\Pi}_{\ell,:} d - \tilde{d}_1)\beta = (\tilde{d}_\ell - \tilde{d}_1)\beta = (Q_1 + \cdots + Q_{\ell-1})(d_{2,-1}\beta' + d_{1,0}\beta'')\]

\[(\tilde{d}_\ell - \tilde{d}_1)\beta = 0,
\]

so

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d\bar{\Pi}_{\ell,:} d\beta = 0.
\]

Finally, assume $\beta \in \Omega^{u-1+\ell,v-\ell}$ and write $\beta = \beta' + \beta''$ with

\[
\beta' \in \bigoplus_{a<\ell} \Omega^{u-1+a,v-a} + \tilde{Q}_{\ell-1}(\Omega^{u-1+\ell,v-\ell})
\]

and

\[
\beta'' \in \tilde{H}^{u+\ell-2,v-\ell+1}_1.
\]

We clearly have

\[(d - \bar{\Pi}_{\ell,:} d)\beta = (id - \tilde{Q}_{\ell-1})d_{2,-1}\beta' = (Q_\ell + \Pi_{\ell+1} + \tilde{P}_\ell)d_{2,-1}\beta',
\]

yielding

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d\bar{\Pi}_{\ell,:} d\beta = -\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d(Q_\ell + \Pi_{\ell+1} + \tilde{P}_\ell)d_{2,-1}\beta'
\]

\[
= -\Pi_{\ell+1} d_{1,0}(Q_\ell + \Pi_{\ell+1} + \tilde{P}_\ell)d_{2,-1}\beta'
\]

\[
= -\Pi_{\ell+1} d_{1,0}(Q_\ell + \Pi_{\ell+1} + P_1 + \cdots + P_\ell)d_{2,-1}\beta'
\]

\[
= -\Pi_{\ell+1} d_{1,0} P_0 d_{2,-1}\beta' = 0
\]
yielding

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d\tilde{\Pi} \pi_{\ell} d\beta = \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d\tilde{d}_1 \beta \\
+ \Pi_{\ell+1} d_1,0 (Q_{1} + \cdots + Q_{\ell-1}) (d_{2,-1} \beta' + d_{1,0} \beta'') \\
= \Pi_{\ell+1} \Pi_2 \pi_{u+\ell+1,v-\ell} d\tilde{d}_1 \beta \\
+ \Pi_{\ell+1} d_1 (Q_{1} + \cdots + Q_{\ell-1}) (d_{2,-1} \beta' + d_{1,0} \beta'') \\
= 0
\]

by Lemma 5.6.

Third, assume \( \beta \in (Q_{1} + \cdots + Q_{\ell-1}) (\Omega^{u+\ell-1,v-\ell}) \), which is contained in \( \mathcal{H}_{1}^{u+\ell-1,v-\ell} \).

Then the result follows because \( \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d \alpha \) on \( \mathcal{H}_{1}^{u,v} \), and

\[
d_1 \mathcal{H}_{1}^{u,v} \subset P_1 (\Omega^{u,v}) \perp \mathcal{H}_{\ell}^{u,v}.
\]

**Lemma 5.10.** For \( \alpha \in \mathcal{H}_{\ell}^{u,v} \), if \( \tilde{d}_1 \beta = 0 \), then \( \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d \alpha = 0 \).

**Proof.** Write \( \alpha = \alpha' + \alpha'' + \alpha''' \) with \( \alpha' \in P_0 (\Omega^{u,v}) \), \( \alpha'' \in \Omega^{u+1,v-1} \) and

\[
\alpha''' = 2 \oplus \Omega^{u+2,v-2} \oplus \overline{Q}_{\ell-1} (\Omega^{u+\ell,v-\ell}).
\]

Observe that \( \alpha' + Q_0 \alpha'' = 0 \). Since \( \tilde{d}_1 \beta = 0 \) on \( \mathcal{H}_{1}^{u,v} \), we have \( \tilde{d}_1 (\alpha' + Q_0 \alpha'') = 0 \). Thus there is some \( \beta \in \mathcal{H}_{1}^{u,v} \) with \( \tilde{d}_1 \beta = \alpha' + Q_0 \alpha'' \) because \( H(\mathcal{H}_{1}^{u,v}, \tilde{d}_1) = 0 \). Then \( \alpha - \tilde{d}_1 \beta \in \mathcal{H}_{1}^{u,v} \) satisfies

\[
\pi_{u,v} (\alpha - \tilde{d}_1 \beta) = Q_0 \pi_{u+1,v-1} (\alpha - \tilde{d}_1 \beta) = 0,
\]

and moreover

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d \alpha = \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d (\alpha - \tilde{d}_1 \beta)
\]

by Lemma 5.9. Therefore we can assume \( \alpha' + Q_0 \alpha'' = 0 \), and thus \( \alpha' = Q_0 \alpha'' = 0 \).

With this assumption, it follows that \( \alpha''' = (P_1 + P_0) \alpha''' \) and

\[
d_1 \Pi_1 \alpha'' = \Pi_1 d_1,0 \Pi_1 + \Pi_1 d_1,0 \alpha'' = \Pi_1 \pi_{u+2,v-1} d \alpha = \Pi_1 \pi_{u+2,v-1} \tilde{d}_1 \alpha = 0
\]

by Lemma 2.1, yielding \( Q_1 \alpha'' = 0 \).

Take a sequence

\[
\phi_i \in Q_0 (\Omega^{u+2,v-2}) \subset \mathcal{H}_{\ell}^{u,v}
\]

such that \( d_{0,1} \phi_i \in C^\infty \) convergent to \( P_0 \alpha'' \). Then the sequence \( \alpha - \tilde{d}_1 \phi_i \in \mathcal{H}_{\ell}^{u,v} \) satisfies

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d \alpha = \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d (\alpha - \tilde{d}_1 \phi_i) \\
\rightarrow \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d (\Pi_1 \alpha'' + \alpha''')
\]

by Lemma 5.9. So

\[
\Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d \alpha = \Pi_{\ell+1} \pi_{u+\ell+1,v-\ell} d (\Pi_1 \alpha'' + \alpha'''), \n\]

and thus we can also assume \( P_0 \alpha''' = 0 \).

For each \( k = 1, \ldots, \ell - 1 \) there is some \( \sigma^k \in Q_k (\Omega^{u-k+1,v+k-2}) \) with \( d_k \sigma^k = P_k \alpha'' \). As above, from the existence of such a \( \sigma^1 \) we can assume \( P_1 \alpha'' = 0 \) by
Lemma 5.9 since $d_1 = \pi_{u,v-1}\Pi_1\tilde{d}$ on $\mathcal{H}_1^{u,v-1}$. If $\ell > 2$, by Proposition 5.4 for $k = 2, \ldots, \ell - 1$ there is a sequence $\tau^k_\ell \in \Omega^{u,v-1}$ such that

$$\pi_{u-k+a+1,v+k-a-1}d(\sigma^k + \tau^k_\ell) \to 0, \quad 0 < a < k,$$

$$\pi_{u+1,v-1}d(\sigma^k + \tau^k_\ell) \to P_\ell \alpha''$$

with respect to the $C^\infty$ topology in $\Omega$. We can thus suppose $P_\ell \alpha'' = 0$ for such a $k$ because

$$\Pi_{\ell+1}\pi_{u+\ell+1,v-\ell}d\Pi_{\ell+1,\ell}d(\sigma^k + \tau^k_\ell) = 0$$

by Lemma 5.9. Therefore

$$\alpha'' \in \mathcal{H}_2^{u,+1,v-1} \otimes \bigoplus_{k=2}^{\ell-1} Q_\ell(\Omega^{u+1,v-1}) \,, \quad (5.17)$$

where the last term does not show up if $\ell = 2$.

Now the condition $d_\ell \alpha = 0$ can be written as

$$\pi_{u+1,a,v-a}d(\alpha'' + \alpha''') = \tilde{Q}_{\ell-1}\pi_{u+\ell+1,v-\ell}d(\alpha'' + \alpha''') = 0, \quad 0 < a < \ell \,. \quad (5.18)$$

Observe that (5.18) summarizes the conditions of the first part of Proposition 5.4 for $k = 2, \ldots, \ell$, with $\omega = \alpha'''$, the constant sequence $\alpha_i = \alpha''$, and $\gamma = 0$ if $2 \leq k < \ell$. Since $\alpha'' \in \mathcal{H}_2^{u,+1,v-1} \otimes \bigoplus_{k=2}^{\ell-1} Q_\ell(\Omega^{u+1,v-1})$ by (5.17), we get inductively on $k = 2, \ldots, \ell - 1$ that $\alpha'' \in \mathcal{H}_2^{u,+1,v-1}$ and $d_k \alpha'' = 0$ by (5.17), (5.18) and Proposition 5.4. Hence $\alpha'' \in \mathcal{H}_2^{u,+1,v-1}$ by (5.17), and thus

$$\Pi_{\ell}\pi_{u+\ell+1,v-\ell}d\alpha = \Pi_{\ell}\pi_{u+\ell+1,v-\ell}d(\alpha'' + \alpha''') = d_\ell \alpha'' \perp \mathcal{H}_{\ell+1}$$

by (5.18) and Proposition 5.4 for $k = \ell$, and the result follows.

We also need the following Hodge theory for the complex $(\tilde{\mathcal{H}}^\ell, \tilde{d}_\ell)$. Let $\tilde{\delta}_\ell = \tilde{\Pi}_{\ell,\cdot,\cdot}d$ on $\tilde{\mathcal{H}}^\ell$, and set $\tilde{D}_\ell = \tilde{d}_\ell + \tilde{\delta}_\ell$ and $\tilde{\Delta}_\ell = \tilde{D}_\ell^2 = \tilde{\delta}_\ell \tilde{d}_\ell + d_\ell \tilde{\delta}_\ell$. Such a $\tilde{\delta}_\ell$ is adjoint of $\tilde{d}_\ell$ in $\tilde{\mathcal{H}}^\ell$ with respect to the $L^2$ inner product, and thus $\tilde{D}_\ell$ and $\tilde{\Delta}_\ell$ are symmetric unbounded operators in the $L^2$ completion $L^2\tilde{\mathcal{H}}^\ell$.

**Lemma 5.11.** The operator $\tilde{D}_\ell$ is essentially self-adjoint in $L^2\tilde{\mathcal{H}}^\ell$.

**Proof.** By Theorem 2.2 in [13], $D = d + \delta$ is essentially self-adjoint in $\Omega$. Then, by using e.g. Lemma XII.1.6-(c) in [13], so is $\pi_{\ell,\cdot,\cdot}D\pi_{\ell,\cdot,\cdot}$ because $\pi_{\ell,\cdot,\cdot}$ is a bounded self-adjoint operator on $\Omega$. But $\pi_{\ell,\cdot,\cdot}D\pi_{\ell,\cdot,\cdot}$ is equal to $\tilde{D}_\ell$ in $L^2\pi_{\ell,\cdot,\cdot}$ and vanishes in its orthogonal complement. Hence $\tilde{D}_\ell$ is essentially self-adjoint.

**Lemma 5.12.** $D\pi_{\ell,\cdot,\cdot} - \tilde{\Pi}_{\ell,\cdot,\cdot}D\pi_{\ell,\cdot,\cdot}$ defines a bounded operator on $\Omega$.

**Proof.** We have

$$D\pi_{\ell,\cdot,\cdot} - \tilde{\Pi}_{\ell,\cdot,\cdot}D\pi_{\ell,\cdot,\cdot} = \tilde{P}_0(\delta_{-1,0} + \delta_{-2,1}) + \delta_{-2,1} + (\Pi_{\ell} + \tilde{\Pi}_{\ell-1})(d_{1,0} + d_{2,-1}) + d_{2,-1} \alpha_{u,v-\ell}.$$ 

But

$$\tilde{P}_0(\delta_{-1,0} + \delta_{-2,1}) = P_0(\delta_{-1,0}) + P_0d_{1,0} + P_0d_{2,-1}$$

on $\tilde{\mathcal{H}}^\ell$. Then the result follows by Lemma 2.4 (i).
For each positive integer \( r \), define the norm \( \| \cdot \|_r' \) on \( \tilde{\mathcal{H}}_\ell^\gamma \) by setting
\[
\| \phi \|_r' = \left\| (id + \tilde{D}_\ell)^r \phi \right\|
\]
and let \( W^k \tilde{\mathcal{H}}_\ell^\gamma \) be the corresponding completion of \( \tilde{\mathcal{H}}_\ell^\gamma \). Then the following result follows directly from Lemma 5.12.

**Corollary 5.13.** The restriction of each \( r \)th Sobolev norm \( \| \cdot \|_r' \) to \( \tilde{\mathcal{H}}_\ell^\gamma \) is equivalent to the norm \( \| \cdot \|_r' \). Thus \( W^k \tilde{\mathcal{H}}_\ell^\gamma \) is the closure of \( \tilde{\mathcal{H}}_\ell^\gamma \) in \( W^k \Omega \).

**Corollary 5.14.** The Hilbert space \( L^2 \tilde{\mathcal{H}}_\ell^\gamma \) has a complete orthonormal system \( \{ \phi_i : i = 1, 2, \ldots \} \subset \tilde{\mathcal{H}}_\ell^\gamma \), consisting of eigenvectors of \( \tilde{\Delta}_\ell \), so that the corresponding eigenvalues satisfy \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \) with \( \lambda_i \uparrow \infty \) if \( \dim \tilde{\mathcal{H}}_\ell^\gamma = \infty \); thus all of these eigenvalues have finite multiplicity. We also have the orthogonal decomposition
\[
\tilde{\mathcal{H}}_\ell^\gamma = (\ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell) \oplus \text{im } \tilde{d}_\ell \oplus \text{im } \tilde{\delta}_\ell,
\]
with
\[
\ker \tilde{\Delta}_\ell = \ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell,
\]
\[
\text{im } \tilde{\Delta}_\ell = \text{im } \tilde{d}_\ell \oplus \text{im } \tilde{\delta}_\ell,
\]
\[
\ker \tilde{d}_\ell = (\ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell) \oplus \text{im } \tilde{d}_\ell,
\]
\[
\ker \tilde{\delta}_\ell = (\ker \tilde{d}_\ell \cap \ker \tilde{\delta}_\ell) \oplus \text{im } \tilde{\delta}_\ell.
\]

**Proof.** Corollary 5.13 implies that each inclusion \( W^{r+1}\tilde{\mathcal{H}}_\ell^\gamma \hookrightarrow W^r \tilde{\mathcal{H}}_\ell^\gamma \) is a compact operator, and \( \bigcap W^r \tilde{\mathcal{H}}_\ell^\gamma = \tilde{\mathcal{H}}_\ell^\gamma \). Then the result follows by Proposition 2.44 in [3] and Lemma 5.11. □

Contrary to the case of \( (\mathcal{H}_\ell^\gamma, d_1) \), it may easily happen that the complex \( (\tilde{\mathcal{H}}_\ell^\gamma, \tilde{d}_1) \) has non-trivial cohomology. But we still can finish the proof of Lemma 5.13.

**Proof of Lemma 5.13 for the case \( k = \ell + 1 \).** Observe that the strong convergence
\[
\pi_{u+a,v-a+1} \delta \alpha_i \to 0, \quad 0 < a \leq \ell,
\]
\[
Q_{\ell-1} \pi_{u+\ell+1,a-\ell} \delta \alpha_i \to 0
\]
in \( \Omega \) just means the strong convergence \( \tilde{d}_\ell \tilde{\Pi}_{\ell+u,v} \alpha_i \to 0 \). Write \( \tilde{\Pi}_{\ell+u,v} \alpha_i = \phi_i + \psi_i \) with \( \phi_i \in \ker \tilde{d}_\ell \) and \( \psi_i \in \text{im } \tilde{\delta}_\ell \), according to Corollary 5.14. Then \( \tilde{d}_\ell \psi_i \to 0 \) strongly in \( \Omega \) by Lemma 5.8, yielding \( \psi_i \to 0 \) strongly in \( \Omega \) by Corollary 5.14. Moreover
\[
\Pi_{\ell+1} \pi_{u+\ell+1,a-\ell} \delta \alpha_i = \Pi_{\ell+1} \pi_{u+\ell+1,a-\ell} \tilde{d} \tilde{\Pi}_{\ell+u,v} \alpha_i = \Pi_{\ell+1} \pi_{u+\ell+1,a-\ell} \tilde{d} \psi_i \to 0
\]
by Lemma 5.10 and because any linear map \( \tilde{\mathcal{H}}_\ell^\gamma \to \mathcal{H}_{\ell+1}^\gamma \) is continuous with respect to the \( L^2 \) norms since \( \mathcal{H}_{\ell+1} \) is of finite dimension. Therefore \( \gamma = 0 \) as desired. □

This finishes the proof of Proposition 5.4, which has the following consequence.
Corollary 5.15. Let $\omega \in \mathcal{H}^{u,v}_k$ and $\gamma \in \mathcal{H}^{v+k-1}_k$ for $k \geq 2$. If there is a sequence $\beta_i \in \bigoplus_{a>0} \Omega^{u-a,v+a}$ such that
\[
\pi_{u-a,v+a-1}\delta(\omega + \beta_i) \rightarrow 0, \quad 0 < a < k,
\]
\[
\hat{P}_{k-2}\pi_{u-k,v+k-1}\delta(\omega + \beta_i) \rightarrow 0, \quad \Pi_k\pi_{u-k,v+k-1}\delta(\omega + \beta_i) \rightarrow \gamma
\]
strongly in $\Omega$, then $\delta_k \omega = \gamma$. Moreover, in this case the sequence $\beta_i$ can be chosen so that
\[
\pi_{u-a,v+a-1}\delta(\omega + \beta_i) \rightarrow 0, \quad 0 < a < k,
\]
\[
\pi_{u-k,v+k-1}\delta(\omega + \beta_i) \rightarrow \gamma
\]
with respect to the $C^\infty$ topology in $\Omega$.

Proof. We can assume that $M$ is oriented by using the two fold covering of orientations with standard arguments. Then it is easy to check that the Hodge star operator, $\star : \Omega \rightarrow \Omega$, satisfies $\star \mathcal{H}_k = \mathcal{H}_k$, and $\star d_k = (-1)^{r+1}\delta_k \star$ on $\mathcal{H}^r_k$ for each integer $r$. Then the result follows from Proposition 5.4. \hfill \Box

Now Theorem 5.1 follows directly from Proposition 5.4 and Corollary 5.15 by induction on $k$.

5.2. Estimates of the rescaled Laplacian. The rescaled Laplacian $\Delta_h$ is the square of the “rescaled Dirac operator” $D_h = d_h + \delta_h$, which will be used here too. The sum of (1.6) and (1.7) gives
\[
D_h = D_0 + hD_\perp + h^2 F,
\]
where
\[
D_0 = d_{0,1} + \delta_{0,-1}, \quad D_\perp = d_{1,0} + \delta_{-1,0}, \quad F = d_{2,-1} + \delta_{-2,1},
\]

Let also $D_\perp = D^2_\perp$.

Lemma 5.16 (Álvarez-Kordyukov [3, Remark 3.5]). There is a zero order differential operator $B$ on $\Omega$ such that
\[
D_\perp D_0 + D_0 D_\perp = BD_0 + D_0 B^*.
\]

Proposition 5.17. There is some $C > 0$ such that\footnote{Recall that, for self-adjoint operators $A, B$ in a Hilbert space $H$, the inequality $A \leq B$ is defined in the sense of quadratic forms: $\langle Au, u \rangle \leq \langle Bu, u \rangle$ for all $u \in H$.}
\[
\Delta_h \geq \frac{1}{2} \Delta_0 + \frac{1}{2} h^2 \Delta_\perp - Ch^2
\]
for $h$ small enough.

Proof. Consider the operators $B, F$ given by Lemma 5.16 and (5.19). Since $B, F$ are of order zero, there is some $C' > 0$ such that $B^* B, F^2 \leq C'$. Because $D_0$ is symmetric, we get
\[
h |\langle (BD_0 + D_0 B^*) \omega, \omega \rangle| \leq 2h |\langle D_0 \omega, B \omega \rangle| \\
\leq 2h \|D_0 \omega\| \|B \omega\| \\
\leq \frac{1}{4} \|D_0 \omega\|^2 + 4h^2 \|B \omega\|^2 \\
= 2 \left( \frac{1}{4} \Delta_0 + 4h^2 B^* B \right) \langle \omega, \omega \rangle
\]
for all $\omega \in \Omega$, yielding
\[ h |BD_0 + D_0B^*| \leq \Delta_0 + h^2 B^* B \leq \Delta_0 + C' h^2. \]

Similarly we get
\[
|FD_0 + D_0F| \leq \Delta_0 + F^2 \leq \Delta_0 + C', \]
\[
|FD_\perp + D_\perp F| \leq \Delta_\perp + F^2 \leq \Delta_0 + C'.
\]

Therefore, from (5.19) and Lemma 5.17 we get
\[
\Delta_h = \Delta_0 + h^2 \Delta_\perp + h^4 F^2 + h(BD_0 + D_0B^*) + h^2(D_0F + FD_0) + h^3(D_\perp F + FD_\perp) \geq \Delta_0 + h^2 \Delta_\perp + h^4 C' - \frac{1}{4}\Delta_0 - C'h^2 \]
\[
- h^2 (\Delta_0 + C') - h^3 (\Delta_\perp + C') \geq \frac{1}{2}\Delta_0 + \frac{1}{2}h^2 \Delta_\perp - Ch^2
\]
for some $C > 0$ and all $h$ small enough.

**Proof of Theorem 7.** In the case $k = 1$, (1.4) just means $\langle \Delta_h, \omega_i, \omega_i \rangle \rightarrow 0$. Therefore
\[
\left\langle \left( \frac{1}{2}\Delta_0 + \frac{1}{2}h^2 \Delta_\perp - Ch^2 \right) \omega_i, \omega_i \right\rangle \rightarrow 0
\]
by Proposition 5.17. Hence
\[
\langle \Delta_0 \omega_i, \omega_i \rangle \rightarrow 0
\]
and $\langle \Delta_\perp \omega_i, \omega_i \rangle$ is uniformly bounded since both $\Delta_0$ and $\Delta_\perp$ are positive operators.

It follows that $\omega_i$ is uniformly bounded in $W^1 \Omega$. Therefore some subsequence of $\omega_i$ is weakly convergent in $W^1 \Omega$ (and thus strongly convergent in $\Omega$) to some $\omega \in W^1 \Omega$.

From (5.20) we also get that $\|D_0 \omega_i\| \rightarrow 0$. So $D_0 \omega_i \rightarrow 0$ strongly in $\Omega$, yielding $\omega \in \ker D_0$ because $D_0$ is a closed operator in $\Omega$. But $\ker D_0 = L^2 \mathcal{H}_1$ by (2.11). Thus the result follows for $k = 1$.

For $k = 2$, it follows from (1.9) that
\[
\|d_{h_i, \omega_i}\| \in o(h_i), \quad \|\delta_{h_i, \omega_i}\| \in o(h_i),
\]
yielding that
\[
\left( \frac{1}{h_i} d_{0,1} + d_{1,0} + h_i d_{2,-1} \right) \omega_i \rightarrow 0, \quad \left( \frac{1}{h_i} \delta_{0,-1} + \delta_{-1,0} + h_i \delta_{-2,1} \right) \omega_i \rightarrow 0,
\]
strongly in $\Omega$ by (1.6) and (1.7). Hence
\[
\Pi (d_{1,0} + h_i d_{2,-1}) \omega_i \rightarrow 0, \quad \Pi (\delta_{-1,0} + h_i \delta_{-2,1}) \omega_i \rightarrow 0
\]
strongly in $\Omega$ as well, and thus so does the sequence $\Pi D_\perp \omega_i$. Then
\[
D_1 \Pi \omega_i = \Pi D_\perp \Pi \omega_i = \Pi D_\perp \omega_i - \Pi D_\perp \Pi \omega_i \rightarrow 0
\]
strongly in $\Omega$ by Lemma 2.4(i). It follows that $\omega \in \ker D_1$ because $D_1$ is a closed operator in $L^2 \mathcal{H}_1$. But $\ker D_1 = \mathcal{H}_2$ by Theorem 2.2(iii), and the result follows for $k = 2$. 


For the case $k > 2$, we can assume $\omega_i \in \Omega^r$ and $\omega \in \mathcal{H}^u_v$ for some integers $u + v = r$. Let $\omega'_i = \pi_{a,r-a} \omega_i$ for each integer $a$, and set

$$\omega'_i = \sum_{a \geq 0} h^{-a}_i \omega_i^{u+a}, \quad \omega''_i = \sum_{a \geq 0} h^{-a}_i \omega_i^{u-a}.$$ 

Now, by Theorem 5.1, the result follows from the following claim.

**Claim 1.** For $0 < a < k$, we have

$$\pi_{u+a,v-a+1} d\omega'_i \longrightarrow 0, \quad \pi_{u-a,v+a-1} \delta \omega''_i \longrightarrow 0,$$

strongly in $\Omega$.

Clearly

$$\pi_{u,v+1} d\omega'_i = d_{0,1} \omega'_1, \quad \pi_{u,v-1} \delta \omega''_i = \delta_{0,-1} \omega''_i.$$ 

Thus both of these components converge strongly to zero because $\omega \in L^2 \mathcal{H}_1$.

To prove Claim 1 for other bihomogeneous components observe that, again from (1.9), both $\|d_h \omega'_i\|$ and $\|d_h \omega''_i\|$ are in $o(h^{k-1}_i)$. Then

$$\|h^2 d_{2,-1} \omega_i^{b-2} + h_i d_{1,0} \omega_i^{b-1} + d_{0,1} \omega_i^{b}\| \in o(h^{k-1}_i), \quad (5.21)$$

$$\|h^2 \delta_{-2,1} \omega_i^{b+2} + h_i \delta_{-1,0} \omega_i^{b+1} + \delta_{0,-1} \omega_i^{b}\| \in o(h^{k-1}_i), \quad (5.22)$$

for every integer $b$, by considering bihomogeneous components of $d_h \omega_i$ and $\delta_h \omega_i$. Now

$$\pi_{u+1,v} d\omega'_i = d_{1,0} \omega'_1 + h^{-1}_i d_{0,1} \omega'_1,$$

$$\pi_{u-1,v} \delta \omega''_i = \delta_{-1,0} \omega''_1 + h^{-1}_i \delta_{0,-1} \omega''_1.$$ 

Both of these components strongly converge to zero in $\Omega$ too by (5.21) and (5.22), since so does $h_i d_{2,-1} \omega_i^{u-1}$ and $h_i \delta_{-2,1} \omega_i^{u+1}$ because $d_{2,-1}$ and $\delta_{-2,1}$ are of order zero and $\|\omega_i\| = 1$.

The other bihomogeneous components of $d\omega'_i$ and $\delta \omega''_i$ are the following ones, where $a \geq 2$,

$$\pi_{u+a,v-a+1} d\omega'_i = h^{-a+2}_i d_{2,-1} \omega_i^{u+a-2} + h^{-a+1}_i d_{1,0} \omega_i^{u+a-1} + h^{-a}_i d_{0,1} \omega_i^{u+a},$$

$$\pi_{u-a,v+a-1} \delta \omega''_i = h^{-a+2}_i \delta_{-2,1} \omega_i^{u-a+2} + h^{-a+1}_i \delta_{-1,0} \omega_i^{u-a+1} + h^{-a}_i \delta_{0,-1} \omega_i^{u-a},$$

which strongly converge to zero in $\Omega$ for $a < k$ by (5.21) and (5.22). This finishes the proof of Claim 1.

**Proof of Theorem 4.** First, we can assume the metric is bundle-like by (4.3). So we can apply the results of this section.

If we had a strict inequality “<” in (1.3) for some $k \geq 2$, by the isomorphism $\mathcal{H}_k \cong E_k^r$ there are sequences $\omega_i \in \Omega^r$ and $h_i \downarrow 0$ such that $\|\omega_i\| = 1$, $\omega_i \perp \mathcal{H}_k$, and

$$\langle \Delta_{h_i, \omega_i, \omega_i} \rangle \in O(h^{2k}_i).$$

But then we get a contradiction by Theorem 3. So inequality “≥” holds in (1.3) for all $k \geq 2$.

The proof of “≥” in (1.2) follows with the same arguments since $E_1^r \cong \mathcal{H}_1^r$, which is of finite dimension if and only if so is $L^2 \mathcal{H}_1^r$.

For $k \geq 2$, inequality “≤” of (1.3) in Theorem 4 follows directly from Corollary 4.2 and Theorem 3.3, as was pointed out in Remark 4.
Now observe that, for each \( h > 0 \) and each \( \omega \in \mathcal{H}_1 \), we have \( D_h \omega = h D_\omega \omega + h^2 F \omega \), according to (5.13). Therefore the inequality “\( \leq \)” in (1.2) follows from the isomorphism \( \mathcal{H}_1^t \cong E_1^t \) by using the well known variational formula \( N_k^t(\lambda) = \sup_\psi \dim V \), where \( V \) runs over the subspaces of \( \Omega^r \) satisfying 
\[
(\Delta_h \omega, \omega) \leq \lambda \| \omega \|^2
\]
for all \( \omega \in V \).

\[\square\]

6. **Forman’s nested sequence**

This section is devoted to the proof of Theorem D. Thus let \( \mathcal{F} \) be a Riemannian foliation of dimension \( p \) on a closed manifold \( M \). We need the following characterization of \( \mathcal{H}_2 \), which is weaker than (1.11) for \( k = 2 \).

**Claim 2.** A differential form \( \omega \in \Omega \) is in \( \mathcal{H}_2 \) if and only if it has extensions \( \tilde{\omega}_1(h), \tilde{\omega}_2(h) \in \Omega[h] \) satisfying
\[
d_h \tilde{\omega}_1(h) = h^2 \Omega[h], \quad \delta_h \tilde{\omega}_2(h) = h^2 \Omega[h].
\]

According to (1.11), it is enough to prove the “if” part of Claim 2. We can assume
\[
\tilde{\omega}_1(h) = \omega + h \omega_1, \quad \tilde{\omega}_2(h) = \omega + h \omega_2
\]
for some \( \omega_1, \omega_2 \in \Omega \) because \( d_h(h^2 \Omega[h]) \) and \( \delta_h(h^2 \Omega[h]) \) are contained in \( h^2 \Omega[h] \). On the other hand, since \( \mathcal{H}_2 \) is a bigraded subspace of \( \Omega \), we can suppose \( \omega \in \Omega^{u,v} \) for some \( u, v \). Then it easily follows from (6.1) that \( \omega_1 \in \Omega^{u+1,v-1} \) and \( \omega_2 \in \Omega^{u,v+1} \). Furthermore we can assume \( \delta_{0,-1} \omega_1 = \delta_{0,1} \omega_2 = 0 \) by Theorem 2.1. Hence the extension
\[
\omega(h) = \omega + h(\omega_1 + \omega_2)
\]
of \( \omega \) is easily seen to satisfy (1.11) for \( k = 2 \), and thus \( \omega \in \mathcal{H}_2 \), finishing the proof of Claim 2.

The statement of Claim 2 seems to hold also for \( \mathcal{H}_k \) with \( k > 2 \), but the proof can not be so easy.

By Theorem A and (1.14), we have \( \mathcal{H}_2^{0,p} = \mathcal{H}_2^{0,p} = 0 \) if \( E_2^{0,p} = 0 \). Therefore we can assume \( E_2^{0,p} \neq 0 \) to prove Theorem D. According to [24] and [2], this assumption implies that \( \mathcal{F} \) is orientable and \( E_2^{0,p} \cong \mathbb{R} \). So \( \mathcal{H}_2^{0,p} \cong \mathbb{R} \) by Theorem A and thus either \( \delta_2^{0,p}(g) = 0 \) or \( \delta_2^{0,p}(g) = \mathcal{H}_2^{0,p}(g) \) by (1.14).

Recall from [31] that the characteristic form, determined by \( \mathcal{F} \) and a metric \( g \) on \( M \), is the unique differential form \( \chi \in \Omega^{0,p} \) whose restriction to the leaves is the leafwise volume form. If \( g \) is a bundle-like metric, then \( \delta_{0,-1} \chi \) corresponds to the leafwise coderivative by restriction to the leaves [2], yielding \( \delta_{0,-1} \chi = 0 \), and thus \( \chi \in \mathcal{H}_2^{0,p} \).

To prove Theorem D(i) just choose the bundle-like metric \( g \) so that \( d_{1,0} \chi = 0 \), which can be done by using Sullivan’s purification [24] (see also [24] and [2]). Hence \( \chi \in \mathcal{H}_2^{0,p} \) by Claim 2, yielding \( \delta_2^{0,p}(g) \neq 0 \).

To prove Theorem D(ii), let us begin with a bundle-like metric \( g \) satisfying Theorem D(i), and the corresponding bigrading of \( \Omega \) and decomposition of \( \delta \) as sum of bihomogeneous components. The hypothesis \( \delta_0^{0,p} \neq 0 \) means that \( \delta_{0,1} \Omega^{0,p-1} \) is not closed in \( \Omega^{0,p} \), and thus we can take some \( \alpha \in \delta_{0,1} \Omega^{0,p-1} \setminus \delta_{0,1} \Omega^{0,p-1} \). Take also some \( \epsilon > 0 \) small enough so that \( \chi + \epsilon \alpha = f \chi \) for some positive function...
Therefore \( \chi' = f\chi \) is the characteristic form of some bundle-like metric \( g' \) on \( M \). Such a \( g' \) can be chosen to define the same bigrading on \( \Omega \) as \( g \), yielding the same decomposition of \( d \) as sum of bihomogeneous components. We have \( \chi' \in \mathcal{H}^{0,p}(g') = \overline{\mathcal{H}}^{0,p}(g') \). Moreover, since \( \alpha \) defines a non-trivial class 
\[
[\alpha] \in \frac{d_{0,1}\Omega_{p-1}}{d_{0,1}\Omega_{0, p-1}} = \overline{\mathcal{H}}_{1}^{0,p} \setminus \mathcal{H}_{1}^{0,p}
\]
and since \( H^{0}(\overline{\mathcal{H}}_{1}^{p}) = H^{0}(\mathcal{H}_{1}^{p}) = 0 \) by Theorem 2.7 (vi), we get
\[
0 \neq d_{1}[\alpha] = [d_{1,0}\alpha] \in \overline{\mathcal{H}}_{1}^{0,1} \setminus \mathcal{H}_{1}^{0,1}.
\]
So
\[
d_{1,0}\chi' = d_{1,0}(\chi + \epsilon\alpha) = \epsilon d_{1,0}\alpha \in \frac{d_{0,1}\Omega_{1,0}}{d_{0,1}\Omega_{0, 1}} \setminus \overline{d_{0,1}\Omega_{1,0}},
\]
yielding \( \chi' \in \mathcal{H}_{2}^{0,p} \setminus \mathcal{H}_{2}^{0,p}(g') \). Therefore \( \mathcal{H}_{2}^{0,p}(g') \neq \mathcal{H}_{2}^{0,p}(g') \), and thus \( \mathcal{H}_{2}^{0,p}(g') = 0 \).

Acknowledgement. The second author gratefully acknowledges the hospitality and support of the University of Santiago de Compostela.

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