On the one fluid limit for vortex sheets

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Abstract
We consider the interface problem between two incompressible and inviscid fluids with constant densities in the presence of surface tension. Following the geometric approach of [14, 15] we show that solutions to this problem converge to solutions of the free–boundary Euler equations in vacuum as one of the densities goes to zero.

1 Introduction

1.1 Description of the problem and main results
We consider the interface problem between two incompressible and inviscid fluids that occupy domains $\Omega^+_{t}$ and $\Omega^-_{t}$ in $\mathbb{R}^n$ ($n \geq 2$) at time $t$. We assume $\Omega^+_0$ is compact and $\mathbb{R}^n = \Omega^+_{t} \cup \Omega^-_{t} \cup S_t$ where $S_t := \partial \Omega^\pm_{t}$. We let $v_\pm, p_\pm$ and $\rho_\pm > 0$ denote respectively the velocity, the pressure and the constant density of the fluid occupying the region $\Omega^\pm_{t}$. We assume the presence of surface tension on the interface which is argued on physical basis to be proportional to the mean curvature $\kappa_+$ of the hypersurface $S_t$. The equations of motion are given by

$$\begin{cases}
\rho (v_t + v \cdot \nabla v) = -\nabla p & x \in \mathbb{R}^n \setminus S_t \\
\nabla \cdot v = 0 & x \in \mathbb{R}^n \setminus S_t \\
v(0, x) = v^0(x) & x \in \mathbb{R}^n \setminus S_0
\end{cases}$$

(E)

with corresponding boundary conditions for the interface evolution and pressure’s jump given by

$$\begin{cases}
\partial_t + v_\pm \cdot \nabla \text{ is tangent to } \bigcup S_t \subset \mathbb{R}^{n+1} \\
p_+(t, x) - p_-(t, x) = \epsilon^2 \kappa_+(t, x), \ x \in S_t.
\end{cases}$$

(BC)

We are interested in analyzing the asymptotic behavior of solutions of the above equations when $\rho_- \to 0$. Our result is convergence to the solution $(v_+, S^\infty_t)$ of the free–boundary problem for Euler equations

$$\begin{cases}
\rho_+ (\partial_t v_+ + v_+ \cdot \nabla v_+) = -\nabla p_+ & x \in \Omega^\infty_t \\
\nabla \cdot v_+ = 0 & x \in \Omega^\infty_t \\
v_+(0, x) = v^0_+(x) & x \in \Omega^0_+
\end{cases}$$

(Eo)

with corresponding boundary conditions

$$\begin{cases}
\partial_t + v_+ \cdot \nabla \text{ is tangent to } \bigcup S^\infty_t \subset \mathbb{R}^{n+1} \\
p_+(t, x) = \epsilon^2 \kappa_\infty(t, x), \ x \in S^\infty_t
\end{cases}$$

(BC0)

where $\kappa_\infty$ denotes the mean curvature of $S^\infty_t := \partial \Omega^\infty_t$. More precisely we will show the following

\footnote{Here we are introducing the notation $f = f_+ \chi_{\Omega^+_t} + f_- \chi_{\Omega^-_t}$ for any $f_\pm$ defined on $\Omega^\pm_t$.}
Theorem 1.1. Let an initial hypersurface $S_0 \in H^{l+1}$ and an initial velocity field $v_0 \in H^l(\mathbb{R}^n \setminus S_0)$ be given for some $l > \frac{n}{2} + 2$. Consider any sequence of local in time solutions of (E)–(BC)

$$S_t^m \in C([0,T],H^{l+1}) \ , \ \ v^m \in C([0,T],H^l(\Omega^m_t))$$

corresponding to values of the density $\rho^m = \rho + \chi_{\Omega^m_1} + \rho^m \chi_{\Omega^m_2}$ with $\rho^m \to 0$ as $m \to \infty$. Then $(v^m, S_t^m)$ converge$^3$ on a small time interval to the solution

$$S_t = \partial \Omega^\infty \in C(H^{l+1}) \ , \ \ v_+ \in C(H^l(\Omega^\infty_t))$$

for any $l' < l$ of (E$_0$)–(BC$_0$). Convergence is in the space $(S_t,v) \in L^\infty(H^{l+\frac{1}{2}}) \times L^\infty(H^l)$ for any $l' < l$.

Free–boundary problems for Euler equations have been extensively studied in recent years following the breakthrough of Wu in [18, 19] where local well–posedness in Sobolev spaces is proved in 2 and 3 dimensions for the irrotational gravity water wave problem. Many works have dealt with the water wave problem also in the general non–zero curl case, see for instance [13, 8, 14, 9].

For the irrotational vortex sheet problem with surface tension Ambrose [1] and more recently Ambrose and Masmoudi [2] proved well–posedness respectively in 2 and 3 dimensions. Cheng, Coutand and Shkoller [6] proved well–posedness in 3–d for the full problem with rotation and well–posedness is also obtained (in any dimension) by Shatah and Zeng [16] for (E)–(BC) and other realted fluid surface problems [16, sec. 6].

In absence of surface tension the vortex sheet problem for the free–boundary motion of two fluids is ill–posed due to the Kelvin–Helmotz instability as shown in [11]. Beale, Hou and Lowengrub [4] showed how the surface tension regularizes the linearized problem. In the next section we will show how the Kelvin–Helmotz instability is very apparent from the infinite–dimensional geometric arguments presented by Shatah and Zeng in [15].

We recall that also the free–boundary problem for Euler equations in vacuum (E$_0$)–(BC$_0$) with $\epsilon = 0$ is known to be ill–posed due to Rayleigh–Taylor instability, see [10], which occurs if one does not assume the sign condition

$$-\nabla \nabla_p(x,t) \geq a > 0.$$  \hspace{1cm} (RT)

In [14] it is shown how also the Rayleigh–Taylor instability is a natural consequence of a geometric calculation and is related to the sign of an operator appearing in the linearization of the Euler flow. Motivated by this we are going to show

**Proposition 1.2.** Let $\Gamma$ be the space of all admissible Lagrangian maps for the interface problem (E)–(BC) defined in (1.7) and let

$$\Gamma^* := \{ \Phi : \Omega^+_0 \to \mathbb{R}^n \ \text{volume–preserving homeomorphisms} \}$$

be the corresponding space for the water wave problem (E$_0$)–(BC$_0$). Consider a point $u \in \Gamma$ and tangent vectors $v_i \in T_u \Gamma$ for $i = 1 \ldots 4$, where $T_u \Gamma$ is endowed with the $L^2(\rho^m dx)$ metric. If we denote $\mathcal{R}$ and $\mathcal{R}^*$ the curvature tensors of $\Gamma$ and $\Gamma^*$ respectively, then

$$(\mathcal{R}(u)(v_1,v_2)v_3,v_4)_{L^2(\rho^m dx)} m \to \infty \to \infty (\mathcal{R}^*(u_+)(v_{1+},v_{2+})v_{3+},v_{4+})_{L^2(\rho_+ dy)}$$ \hspace{1cm} (1.1)

In view of the geometric frame work described below and the linearized equation (1.17), proposition 1.2 can be considered as a first step in showing that solutions of (E)–(BC) converge to solutions of (E$_0$)–(BC$_0$) with $\epsilon = 0$ when $\epsilon, \rho_- \to 0$ at the same time$^5$.

Our paper is organized as follows. The geometry of $\Gamma$ is presented in section 1.2 and an explanation of the geometric intuition behind the Kelvin–Helmotz and Raileigh–Taylor instabilities is given in 1.2.3. Of course we refer to [14, 15] for full details about this general geometric approach. In section 2 we state theorems on energy estimates which are independent of $H^s$–functions.

Theorem 1.2. In section 2 we state theorems on energy estimates which are independent of $H^s$–functions.

1.2 The geometric approach to Euler equations

It is well–known that the interface problem between two fluids has a variational formulation on a subspace of the space of volume–preserving homeomorphisms. For the water wave problem this was observed for the first time by Arnold in his seminal paper [3], where he pointed out that Euler equations for the motion of an inviscid and incompressible fluid can be viewed as the geodesic flow on the infinite–dimensional manifold of volume–preserving diffeomorphisms. This point of view has been adopted by several authors in works such as [17, 5, 12] and more recently by Shatah and Zeng in [14, 15, 16].

$^2$The regularity of hypersurfaces in $\mathbb{R}^n$ is intended in the sense of local coordinates: an hypersurface is $H^s$ for $s > \frac{n}{2}$ if it can be locally represented as the graph of $H^s$–functions.

$^3$Convergence is achieved by reducing the problem to the fixed initial domain $\Omega_m$ using Lagrangian coordinate maps. See section 4 for details.

$^4$Covariant differentiation on $T_u \Gamma$ (and on $T_u \Gamma^*$) is defined in section 1.2.2.

$^5$We believe that some condition of the form $\rho_- = O(\epsilon^\alpha)$ for some $\alpha > 0$ should be needed in this case.
1.2.1 Lagrangian formulation

The surface tension parameter $\epsilon$ will be henceforth set to be one. Multiplying (E) by $v$, integrating over $\mathbb{R}^n \setminus S_t$, using the boundary condition (BC) and the variation of surface area formula, we obtain the conserved energy\(^6\)

$$E = E_0(S_t, v) = \int_{\mathbb{R}^n \setminus S_t} \frac{\rho |v|^2}{2} \, dx + \int_{S_t} dS =: \int_{\mathbb{R}^n \setminus S_t} \frac{\rho |v|^2}{2} \, dx + S(S_t). \quad (1.2)$$

For $y \in \Omega^\pm_0$ we define $u_\pm(t, y)$ to be the Lagrangian coordinate map associated to the velocity field $v_\pm$, i.e. the solution of the ODE

$$\frac{dx}{dt} = v_\pm(t, x), \quad x(0, y) = y \quad \forall \ y \in \Omega^\pm_0; \quad (1.3)$$

for any vector field $w$ on $\mathbb{R}^n \setminus S_t$ we define its material derivative by

$$D_t w := w_t + v \cdot \nabla w = (w \circ u)_t \circ u^{-1}.$$

In [15, sec. 2] the authors derive from (E)–(BC) the following equation for the physical pressure:

$$\begin{align*}
-\Delta p &= \rho \text{tr} \left( Dv^2 \right) \\
p_\pm|_{S_t} &= N^{-1} \left\{ -\frac{1}{|v_+^\pm|^2} \kappa_v^\pm \nabla v_+^\pm \cdot v_+^\pm - \Pi_+(v_+^\pm, v_+^\pm) - \Pi_-(v_+^\pm, v_+^\pm) \\
&- \nabla_{\nu^\pm} \Delta^{-1} \text{tr} \left( Dv^2 \right) - \nabla_{\nu^-} \Delta^{-1} \text{tr} \left( Dv^2 \right) \right\} \quad (1.4)
\end{align*}$$

where $\Pi_\pm$ denotes the second fundamental form of the hypersurface $S_t$ (with respect to the outward unit normal vector $N_\pm$ relative to the domain $\Omega^\pm_t$) and $N$ is given by

$$N = \frac{N_+}{\rho_+} + \frac{N_-}{\rho_-} \quad (1.5)$$

with $N_\pm$ denoting the Dirichlet–to–Neumann operator on the domain $\Omega^\pm_t$.

From (1.3) we see that in Lagrangian coordinates Euler equations assume the form

$$\rho u_{tt} = -\nabla p \circ u \quad u(0) = \text{id}_{\Omega_0} \quad (1.6)$$

with $p$ determined by (1.4).

Since $v$ is divergence free, $u_\pm$ are volume–preserving maps on $\mathbb{R}^n \setminus S_0$. Moreover $u_+(t, S_0) = u_-(t, S_0)$ even if the restriction to $S_0$ of $u_+$ and $u_-$ do not coincide in general. This leads to the definition of the space $\Gamma$ of admissible Lagrangian maps for the interface problem:

$$\Gamma = \left\{ \Phi = \Phi_+ \chi_{\Omega_0^+} + \Phi_- \chi_{\Omega_0^-} : \Phi_\pm : \Omega^\pm_0 \to \Omega^\pm_0 \text{ is volume–preserving homeo. and } \partial \Phi_\pm(\Omega^\pm_0) = \Phi_\pm(\partial \Omega^\pm_0) \right\}.$$

Denoting $S(\Phi) = \int_{\Phi(S_0)} dS$ we can rewrite the energy (1.2) in Lagrangian coordinates as

$$E_0(u, u_t) = \int_{\mathbb{R}^n \setminus S_0} \frac{\bar{\rho}|u|^2}{2} \, dy + S(u)$$

where $\bar{\rho} = \rho \circ u$. The conservation of the above energy suggests that (E)–(BC) has a Lagrangian action

$$I(u) = \int \int_{\mathbb{R}^n \setminus S_0} \frac{\bar{\rho}|u|^2}{2} \, dy \, dt - \int S(u) \, dt. \quad (1.7)$$

\(^6\)Notice that the conserved energy does not control the $L^2$ norm of $v_-$ in the asymptotic regime $\rho_- \to 0$. 

3
1.2.2 The geometry of $\Gamma$

To derive the Euler–Lagrange equations associated to the action $I$ one has to look at the geometry of $\Gamma$ considered as a submanifold of $L^2(\rho dy)$ and identify its tangent and normal spaces. It is easy to see that the tangent space of $\Gamma$ at the point $\Phi$ is given by divergence-free vector fields with matching normal components in Eulerian coordinates$^7$

$$T_\Phi \Gamma = \left\{ \tilde{w} : \mathbb{R}^n \setminus S_0 \to \mathbb{R}^n : \nabla \cdot \tilde{w} = 0 \text{ and } w^+_t + w^-_t|_{\Phi(S_0)} = 0, \text{ where } \tilde{w} = \tilde{w} \circ \Phi^{-1} \right\}.$$ 

while the normal space is

$$(T_\Phi \Gamma)^\perp = \left\{ - \nabla \psi \circ \Phi : \rho_+ \psi_+|_{\Phi(S_0)} = \rho_- \psi_-|_{\Phi(S_0)} = 0 \right\}.$$ 

A critical path $u(t, \cdot)$ of $I$ satisfies

$$(\partial_t u_t + S'(u) = 0)$$

where $S'(u)$ denotes the tangential gradient of $S(u)$ and $\partial_t$ is the covariant derivative on $\Gamma$ (along $u(t)$). In order to verify that the Lagrangian map associated to a solution of (E)–(BC) is indeed a critical path of (1.7) we need to compute $\partial_t$ and $II_{u(t)}$.

Computing $\partial_t$ and $II_{u(t)}$: Given a path $u(t, \cdot) \in \Gamma$ denote $\tilde{v} = u_t$ and $S_t = u(t, S_0)$. For any $\tilde{w}(t, \cdot) \in T_{u(t)} \Gamma$ we must have

$$\tilde{w}_t = \tilde{D}_t \tilde{w} + II_{u(t)}(\tilde{w}, \tilde{v})$$

where $II_{u(t)}(\tilde{w}, \tilde{v}) \in (T_{u(t)} \Gamma)^\perp$ denotes the second fundamental form on $T_{u(t)} \Gamma$. From (1.8) there exists a unique scalar function $p_{v,w}$ defined on $\mathbb{R}^n \setminus S_t$ such that

$$II_{u(t)}(\tilde{w}, \tilde{v}) = -\nabla p_{v,w} \circ u \in (T_{u(t)} \Gamma)^\perp$$

In [15] it is shown that $p_{v,w}$ is given by$^8$

$$\begin{cases}
-\Delta p_{v,w} = \text{tr} (Dv Dw) \\
p^+_v \big|_{S_t} = \frac{1}{\rho_+} N^{-1} \left\{ \nabla v_+ - v_+ \nabla \nabla \nabla - w_+ \nabla \nabla \nabla - \Pi_+ (v_+, w_+) \right\} \\
p^-_v \big|_{S_t} = \frac{1}{\rho_+} \text{tr} (Dv Dw) - \nabla N_+ \nabla \nabla \nabla \text{tr} (Dv Dw) \right\}.
\end{cases}$$

Then in Eulerian coordinates we can write

$$\partial_t w := (\partial_t w) \circ u^{-1} = D_t w + \nabla p_{v,w}.$$ 

Computing $S'(u)$: For any $\tilde{w} \in T_{u(t)} \Gamma$ the formula for the variation of surface area gives

$$\langle S'(u), \tilde{w} \rangle_{L^2(\mathbb{R}^n \setminus S_0, \rho dy)} = \int_{S_t} \kappa_+ w_+^+ dS$$

and it is not hard to verify that the unique representation in Eulerian coordinates of $S'(u)$ as a functional acting on $T_{u(t)} \Gamma$ is

$$S'(u) = \nabla p_k \text{ with } p_k^+ = \frac{1}{\rho_+} \mathcal{H}_k = N^{-1} \nabla N \nabla \kappa_+.$$ 

From (1.4), (1.13) and (1.15) we obtain the identity $p = \rho (p_k + p_{v,w})$. Therefore, taking $\tilde{w} = u_t$, we see from (1.14) and (1.15) that a solution of (1.9) equivalently satisfies

$$D_t v + \nabla p_{v,w} + \nabla p_k = 0$$

which is exactly (1.6) in Eulerian coordinates.

$^7$We follow the convention used in [15] where the Lagrangian description of any vector field $X : \Phi(H) \to \mathbb{R}^n$ is denoted by $X = X \circ \Phi$.

$^8$Let us point out that in the water wave problem with just one fluid in vacuum we have $II_{u(t)}(\tilde{w}, \tilde{v}) = -\nabla p_{v,w} \circ u \in (T_{u(t)} \Gamma)^\perp$ with

$$\begin{cases}
-\Delta p_{v,w} = \text{tr} (Dv Dw) \\
p_{v,w} \big|_{\partial \Gamma} = 0.
\end{cases}$$
1.2.3 Linearized equation and instability

The Lagrangian formulation discussed above provides a convenient setting to study the linearization of the problem. Considering variations around the solution $u_t$ of (1.9) and taking a covariant derivative with respect to the variation parameter, we obtain the following linearization for $\bar{w}(t, \cdot) \in T_{u(t)}\Gamma$:

$$\bar{\partial}_t^2 \bar{w} + \bar{\sigma}(u)(\bar{u}_t, \bar{w}) u_t + \bar{\sigma}^2 S(u) \bar{w} = 0$$  \hspace{1cm} (1.17)

where $\bar{\sigma}$ denotes the curvature tensor of the manifold $\Gamma$ and $\bar{\sigma}^2 S(u)$ is the projection on $T_{u(t)}\Gamma$ of the second variation of the surface area. Both of these linear operators acting on $T_{u(t)}\Gamma$ play a central role in the understanding of the problem and in the definition of high–order energies based on their leading order terms. In [14] an explicit but rather complicated formula is given for $\bar{\sigma}^2 S(u)$; in [14, 15] its leading order term $\bar{\sigma}$ is singled out and turns out to be given$^9$ in Eulerian coordinates by

$$\bar{\sigma}(u)(w) = \nabla f_+ \chi_{\Omega^+} + \nabla f_- \chi_{\Omega^-} \quad \text{with} \quad f_{\pm} = \frac{1}{\rho_{\pm}} \mathcal{H}_{\pm} N^{-1} N^T (-\Delta_{S_t}) w^+_{\pm};$$  \hspace{1cm} (1.18)

it is easy to see that $\bar{\sigma}$ is a third–order$^{10}$ self–adjoint and positive semi–definite operator with $\bar{\sigma}(u)(\bar{w}, \bar{w}) = |\nabla w^\pm|^2_{L^2(S_t)}$. Further computations performed in [15, pp 859 - 860], show that the leading–order term $\bar{\sigma}_0(u)(\bar{v})$ of the unbounded sectional curvature operator $\bar{\sigma}(u)(\bar{v}, \cdot) \bar{v}$ is given in Eulerian coordinates by

$$\bar{\sigma}_0(u)(w) = \nabla f_+ \chi_{\Omega^+} + \nabla f_- \chi_{\Omega^-} \quad \text{with} \quad f_{\pm} = \frac{1}{\rho_{\pm} \rho_{\pm}} \mathcal{H}_{\pm} N^{-1} N^T (\nabla v^\pm_{\Omega^+} - v^\pm_{\Omega^-}) \cdot (w^\pm_{\Omega^+} (v^\pm_{\Omega^+} - v^\pm_{\Omega^-})).$$

Noticing that $\bar{\sigma}_0(u)$ is a second–order negative semidefinite differential operator we immediately see that the linearized Euler equations would be ill–posed if there had been no surface tension generating the operator $\bar{\sigma}$. This is the so–called Kelvin–Helmotz instability for the two fluids interface problem.

We conclude this section recalling that the same geometric setting described above applies to the problem of Euler equations in vacuum. The same Lagrangian approach is of course available and the linearized equation is still given by (1.17). Computations performed in [14, sec 2.2] show how the leading order term of the differential operators involved in the linearization are given by $\bar{\sigma}_0(u)$ and $\bar{\sigma}^* (u)$ satisfying

$$\bar{\sigma}(\bar{v}, \bar{w}) = \bar{\sigma}_0^* (u) + \text{bounded operators} \quad \bar{\sigma}^2 S(u) = \bar{\sigma}^* (u) + \text{second–order differential operators}$$

and

$$\bar{\sigma}_0^* (u) = \int_{S_t} -\nabla N p^*_{\nu, \nu} |\nabla w^\perp|^2 \rho_+ dS, \quad \bar{\sigma}^* (u) = \int_{S_t} |\nabla w^\perp|^2 \rho_+ dS.$$  \hspace{1cm} (1.19)

Since also in this case $\bar{\sigma}^* (u)$ is generated by the presence of surface–tension, we see that (1.17) is ill–posed in absence of surface tension if the Raileigh–Taylor sign condition (RT) is not assumed.

2 Theorems on Energy Estimates

**Definition 2.1.** Let $\Lambda_0 = \Lambda_0(S_0, l - \frac{1}{2}, \delta, L)$ for some $l \geq \frac{n}{2} + 1$, $L > 0$ and $0 < \delta \ll 1$ be the collection of all hypersurfaces $\tilde{S}$ such that a diffeomorphism $F : S_0 \to \tilde{S} \subset \mathbb{R}^n$ exists with

$$|F - \text{id}_{S_0}|_{H^{l-\frac{1}{2}}(S_0)} < \delta$$

and satisfying a uniform bound on the mean curvature $|\kappa|_{H^{l-\frac{1}{2}}(\tilde{S})} < L$.

In [15] the geometric considerations exposed in section 1.2 led the authors to define the following energy for (E)–(BC)

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$^9$Both in the one fluid case and the interface problem the leading order term of $\bar{\sigma}^2 S(u)$ has the same form but its Hilbert space representation does not coincide due to the different orthogonal splitting of $L^2$ in $T_{\bar{q}} \Gamma$ and $(T_{\bar{q}} \Gamma)^\perp$ in the two settings. We refer to [15, pp. 857-858] for the details of the derivation of $\bar{\sigma}$.

$^{10}$Assuming $S_t$ is smooth enough.
\textbf{Definition 2.2.} Consider domains $\Omega^\pm_t$ with $\Omega^+_t$ compact and interface $S_t \in H^{l+1}$. Let $v(t, \cdot) \in H^l(\mathbb{R}^n \setminus S_t)$ be any divergence–free vector field with $v^+_t + v^-_t = 0$, define the energy

$$E(S_t, v(t, \cdot)) = \frac{1}{2} \int_{\mathbb{R}^n \setminus S_t} |\mathbf{R}^\perp v|^2 \rho \, dx + \frac{1}{2} \int_{\mathbb{R}^n \setminus S_t} |\mathbf{R}^\perp \nabla P_\kappa|^2 \rho \, dx + |\omega|^2_{H^{l-1}(\mathbb{R}^n \setminus S_t)}$$

(2.1)

where $\omega$ is the curl of $v$, that is $\omega^i = \partial_j v^j - \partial_j v^i$.

\textbf{Proposition 2.3.} Let $l > \frac{n}{2} + 1$, then for $S_t \in \Lambda_0$ with $S_t \in H^{l+1}$ we have

$$|\kappa|^2_{H^{l-1}(S_t)} \leq C_0(1 + E), \quad |v|^2_{H^l(\mathbb{R}^n \setminus S_t)} \leq C_0(1 + E + E_0)^2$$

where $C_0$ is some positive constant depending only on $\Lambda_0$ and the initial data (in particular it is independent of $\rho_-$).

The above proposition is the equivalent of [15, proposition 4.1]. The proof of bounds which are independent of the density $\rho_-$ just requires some small modification of the argument given in [15]. See section 3.

\textbf{Theorem 2.4 (Energy Estimates for (E) and (BC), [15]).} Let $l > \frac{n}{2} + 1$ and a solution to $(E)$–$(BC)$ be given by $S_t \in H^{l+1}$ and $v \in C^0_t(H^l(\mathbb{R}^n \setminus S_t))$, then there exists $L > 0$ and a positive time $t^*$ independent of $\rho_-$ and depending only on $|v(0, \cdot)|_{H^l(\mathbb{R}^n \setminus S_0)}$, $\Lambda_0$ and $L$, such that $S_t \in \Lambda_0$ and $|\kappa|^2_{H^{l-1}(S_t)} \leq L$ for all $0 \leq t \leq t^*$. Moreover the following energy estimate holds for $0 \leq t \leq t^*$:

$$E(S_t, v(t, \cdot)) \leq 3E(S_0, v(0, \cdot)) + C_1 + \int_0^t P(E_0, E(S_t, v(t, \cdot))) \, dt'$$

(2.2)

where $P$ is a polynomial with positive coefficients determined only by $\Lambda_0$ and the constant $C_1$ depends only on $|v_0|^2_{H^{l-\frac{3}{2}}(\mathbb{R}^n \setminus S_0)}$ and $\Lambda_0$.

The proof of theorem 2.4 is essentially the same as in [15] and is postponed to the appendix.

\textbf{Corollary 2.5.} Consider a sequence of solutions $S_t^m \in C^0_t(H^{l+1})$, $v^m \in C^0_t(H^l(\mathbb{R}^n \setminus S_t^m))$ solving locally in time the Euler system $(E)$–$(BC)$ for values of the density $\rho^m \to 0$. If we denote $E_m(t) := E(S_t^m, v^m(t, \cdot))$ with $E$ given by (2.1), then there exists a positive time $t^*_0$ and a constant $C$ depending only on the set $\Lambda_0$, $|v_0|^2_{H^l(\mathbb{R}^n \setminus S_0)}$ and $|v_0|^2_{H^{l-\frac{3}{2}}(\mathbb{R}^n \setminus S_0)}$ such that

$$\sup_{t \in [0, t_0]} E_m(t) \leq 2E(0) + 2C_1, \quad \forall \ m \in \mathbb{N}.$$  

(2.3)

The above corollary gives as a consequence weak convergence of solutions of the vortex sheet problem to solutions of the one fluid problem in vacuum. Standard compactness arguments are going to give the strong convergence stated in theorem 1.1. See section 4 for details.

For completeness we state here a theorem, proved in [16], based on the above energy estimates and concerning existence of solutions:

\textbf{Theorem 2.6 (Well–posedness for (E)–(BC), [16]).} Given an initial surface $S_0 \in H^{l+1}$ and initial velocity $v_0 \in H^l(\mathbb{R}^n \setminus S_0)$ with $l > \frac{n}{2} + 1$, the free interface problem $(E)$–$(BC)$ has a solution in the space $S_t \in C^0_t(H^{l+1})$, $v \in C^0_t(H^l(\mathbb{R}^n \setminus S_t))$ for $t$ in some small interval $[0, T]$ independent of the density $\rho_-$. Moreover, if $l > 3$ the problem is locally well–posed, i.e. the solution is unique and depends continuously on the initial data.
3 Proof of Proposition 2.3

Using the definition of \( \mathcal{A} \) in (1.18) we can explicitly write the terms appearing in the energy (2.1) as in (B.2), (B.3) and (B.4) with \( k/2 \) replaced by \( l/3 \). From the properties of \( \mathcal{N}_\pm \) and \( \mathcal{N}^{-1} \) in lemma A.2 it follows that there exists a constant \( C \) independent of \( \rho_- \) such that

\[
|\kappa|_{H^{l-1}(S_t)}^2 \leq C(1 + E), \quad |v_\perp|_{H^{l-\frac{1}{2}}(S_t)}^2 \leq C(1 + E).
\]

To estimate \( v \) we proceed in three simple steps:

1) Estimates on the Lagrangian coordinate map: Consider the Lagrangian map \( u_- \) associated to \( v_- \). From lemma A.1 we get

\[
|u_-(t, \cdot) - \text{id}|_{H^l(\Omega^-_t)} \leq C_1 \int_0^t |v_-(s, \cdot)|_{H^l(\Omega^-_t)} |u_-(s, \cdot)|_{H^l(\Omega^-_t)} \, ds
\]

for any \( 0 \leq s \leq l \) where \( C_1 > 0 \) only depends on \( n \) and \( l \). Now, let \( \mu \) be a sufficiently large number to be specified later depending only on the \( H^l \)-norm of the initial velocity, define

\[
t_0 := \sup \left\{ t : |v(t', \cdot)|_{H^l(\mathbb{R}^n \setminus S_{t'})} \leq \mu \quad \forall \ t' \in [0, t] \right\}.
\]

Since \( v \) is assumed to be continuous in time with values in \( H^l \), \( t_0 > 0 \). The previous inequality and an easy bootstrap argument (or Gronwall’s inequality) show that there exists a positive time \( t^- \) and a constant \( C_2 \) depending only on \( l, n, \mu \) and \( \Lambda_0 \) such that

\[
|u_-(t, \cdot) - \text{id}|_{H^l(\Omega^-_t)} \leq C_2 t \leq \frac{1}{2}, \quad \forall \ t \in [0, t^-]
\]

for \( t^- := \min\{t_0, t^-_1, 1/(2C_2)\} \). This shows that \( u_- \) is an \( H^l \)-diffeomorphism so that \( u^{-1}_-(t, \cdot) \) is a well–defined volume preserving map for \( x \in \Omega^-_t \) and for the same range of times we have

\[
|(Du_-)^{-1}|_{H^l(\Omega^-_t)} \leq 2, \quad \forall \ 0 \leq s \leq l - 1.
\]

2) Decomposition of vector fields and control of the lower norm: As it is well-known (and explained in detail in [14, Appendix B]) any divergence–free vector field \( v : \mathbb{R}^n \setminus S_t \rightarrow \mathbb{R}^n \) obeying the condition \( v_\perp^+ + v_\perp^- = 0 \) can be decomposed into two divergence–free components, the rotational part \( v_r \) responsible for the interior motion and an irrotational or gradient component \( v_{ir} = \nabla g \) responsible for the motion of the boundary \( S_t \). More precisely \( g \) is the solution of the elliptic Neumann problem

\[
\begin{aligned}
\Delta g &= 0, & & x \in \mathbb{R}^n \setminus S_t \\
\nabla N \cdot g &= v_\perp, & & x \in S_t
\end{aligned}
\]

and \( v_r := v - v_{ir} \). It is observed in [16] that the invariance of Euler equations under the action of the group of volume preserving diffeomorphisms leads, via Noether’s theorem, to a family of conserved quantities which determine the rotational part of the velocity

\[
v_r(t, \cdot) = P_r \left( S_t, (Du^{-1})^* v(0, u^{-1}(t, \cdot)) \right)
\]

where \( P_r(S_t, X) \) denotes the projection of \( X : \mathbb{R}^n \setminus S_t \rightarrow \mathbb{R}^n \) onto its rotational (gradient–free) part. Therefore we can estimate

\[
|v_r(t, \cdot)|_{L^2(\Omega^-_t)} \leq |v_r|_{L^2(\Omega^-_t)} + |v_{ir}(t, \cdot)|_{L^2(\Omega^-_t)} \leq |(Du^{-1})^* v(0, u^{-1}(t, \cdot))|_{L^2(\Omega^-_t)} + |v_\perp^+ (t, \cdot)|_{H^{l-\frac{1}{2}}(S_t)} + |v_\perp^- (t, \cdot)|_{H^{l-\frac{1}{2}}(S_t)} + CE^{\frac{1}{2}} \leq C(1 + E^{\frac{1}{2}})
\]

with \( C \) depending only on the initial data and \( \Lambda_0 \).

3) Control of \( |v|_{H^l} \): To conclude we use the fact\(^\text{11}\) that any divergence–free vector field can be controlled by its \( \text{curl} \) and normal component:

\[
|v|^2_{H^l(\Omega^-_t)} \leq C(1 + |\kappa_+|_{H^{l-\frac{1}{2}}(S_t)})^2 \left( |\text{curl} v|^2_{H^{l-1}(\Omega^-_t)} + |v_\perp^+|^2_{H^{l-\frac{1}{2}}(S_t)} + |v_\perp^-|^2_{H^{l-\frac{1}{2}}(S_t)} + |v_\perp|^2_{L^2(\Omega^-_t)} \right) \leq C(1 + E + E_0^2)
\]

where the constant \( C \) depends only on the initial data and the set \( \Lambda_0 \).

\(^{11}\)A more general statement is

\[
|v_\perp|_{H^l(\Omega^-_t)} \leq C(1 + |\kappa_+|_{H^{l-\frac{1}{2}}(S_t)}) \left( |\text{div} v_\perp|_{H^{l-1}(\Omega^-_t)} + |\text{curl} v_\perp|_{H^{l-1}(\Omega^-_t)} + |v_\perp|_{H^{l-\frac{1}{2}}(S_t)} + |v_\perp|_{L^2(\Omega^-_t)} \right)
\]

where the constant \( C \) depends only on \( \Lambda_0 \). An essential proof of this can be found in [14, proposition 4.3].
4 Proof Theorem 1.1

In this section we are going to use the uniform bounds provided by corollary 2.5 combined with the non–linear Eulerian frame work introduced in [14] to obtain the strong convergence of solutions stated in theorem 1.1.

4.1 Convergence of Lagrangian maps and velocity fields

As a first step we need to estimate the physical pressure.

**Lemma 4.1.** Let \( v \in H^l \) and \( S_t = \partial \Omega_t \in H^{l+1} \) with \( l > \frac{n}{2} + 2 \) be a given solution of (E)–(BC). Then the pressure \( p \), determined by (1.4), satisfies

\[
|p^+|_{H^\frac{l-3}{2}(\Omega^+_t)} \leq C \left( |v|^2_{H^l(\Omega^+_t)} + \kappa_+|H^{l-1}(S_t) + p^-|_{H^{l-1}(\Omega^-_t)} |N|_{H^{l-1}(S_t)} \right) \tag{4.1}
\]

and for \( p^- \ll 1 \)

\[
|p^-|_{H^\frac{l-3}{2}(\Omega^-_t)} \leq C \rho^- \left( |v|^2_{H^l(\Omega^-_t)} + \kappa_-|H^{l-1}(S_t) + |v|^2_{H^{l-1}(\Omega^-_t)} |N|_{H^{l-1}(S_t)} \right) \tag{4.2}
\]

for some constant \( C \) depending only on the set of hypersurfaces \( \Lambda_0 \).

**Proof** Write \( p_\pm = \Delta^{l-1} \Delta p_\pm + \mathcal{H}_\pm p_\pm^S \) and use lemma A.2 to get

\[
|p_\pm|_{H^\frac{l-3}{2}(\Omega_t)} \leq C \left( \rho_\pm \text{tr} (Dv^2)_{H^\frac{l-3}{2}(\Omega_t)} + |p_\pm^S|_{H^{l-1}(\Omega_t)} \right)
\]

\[
\leq C \rho_\pm |v|^2_{H^l(\Omega_t)} + \frac{\rho^-}{\rho^-} |v|^2_{H^{l-1}(\Omega_t)} + C \rho^- \left( |N|_{H^{l-2}(\Omega_t)} |v|^2_{H^\frac{l-3}{2}(\Omega_t)} \right)
\]

\[
+ |v|_{H^\frac{l-3}{2}(\Omega_t)} |v|^2_{H^\frac{l-3}{2}(\Omega_t)} |N|_{H^{l-1}(\Omega_t)} \]

\[
\square
\]

**Proposition 4.2.** There exists a sequence \( \{m_k\} \), a time \( t^{**} \) depending only on the initial data and an \( H^l \)-diffeomorphism \( u_+ \in C^0_t \left( [0, t^{**}]; H^l(\Omega^+_0) \right) \) with \( \partial_t u_+ \in C^0_t \left( [0, t^{**}]; H^l(\Omega^+_0) \right) \) such that

\[
\lim_{k \to \infty} u^{m_k}_+ = u_+ \text{ in } C^0_t \left( [0, t^{**}]; H^l(\Omega^+_0) \right) \tag{4.3}
\]

\[
\lim_{k \to \infty} \partial_t u^{m_k}_+ = \partial_t u_+ \text{ in } C^0_t \left( [0, t^{**}]; H^l(\Omega^+_0) \right) \tag{4.4}
\]

for any \( l' < l \). Moreover if we define \( \Omega^\infty_t := u_+(t, \Omega_0) \) (4.5) then there exists \( v_+ \in L^\infty \left( H^l(\Omega^\infty_0) \right) \cap L^\infty \left( H^l(\Omega^\infty_0) \right) \) such that

\[
\lim_{k \to \infty} u^{m_k}_+ \circ u^{m_k}_+ = v_+ \circ u_+ \text{ in } C^0_t \left( [0, t^{**}]; H^l(\Omega^+_0) \right) \tag{4.6}
\]

for any \( l' < l \) and \( p_+ \in L^\infty \left( H^l(\Omega^\infty_0) \right) \) such that

\[
\lim_{k \to \infty} p^{m_k}_+ \circ u^{m_k}_+ = p_+ \circ u_+ \text{ weak–star in } L^\infty \left( [0, t^{**}]; H^l(\Omega^+_0) \right) \tag{4.7}
\]

We will still denote these subsequences by the index \( m \).

**Proof** Let us denote \( X(H^l) = X([0, t^{**}]; H^l(\Omega^+_0)) \) for \( X = L^\infty \) or \( C^0_t \) and \( C \) any positive constant depending only the initial data and the set \( \Lambda_0 \). Combining proposition 2.3 and corollary 2.5 we see that

\[
|u^m|_{L^\infty(\Omega^\infty_0)} \leq C_0(1 + E_m) \leq C
\]

for any \( t \leq t^*_0 \). Therefore, arguing as in the proof of proposition 2.3, we can find a positive time \( t^{**} \leq t^*_0 \) depending only on \( \Lambda_0 \) and the initial data, such that for any \( 0 \leq t \leq t^{**} \)

\[
|u^m_+(t, \cdot)|_{H^l(\Omega^+_0)} \leq C t^{**} \leq \frac{1}{2} \tag{4.8}
\]
This show that each map \( u_+^m \) is an \( H^1 \)-diffeomorphism onto its image and is uniformly bounded in \( L^\infty (H^1) \) by a constant depending only on the initial data and the set \( \Lambda_0 \). Then, up to extraction of a subsequence, there exist \( u_+ \in L^\infty (H^1) \) such that \( u_+^m \rightharpoonup u_+ \) weak–star in \( L^\infty (H^1) \). Lemma A.1 and (4.8) imply

\[
|\partial t u_+^m|_{H^1(\Omega_+^m)} \leq |v_+^m|_{H^1(\Omega_+^m)} \leq C.
\]

Again by standard compactness we have, up to extraction, \( \partial_t u_+^m = v_+^m \circ u_+^m \to \partial_t u_+ = : \tilde{v}_+ \) weak–star in \( L^\infty (H^1) \). Since \( u_+^m, u_+ \in W^{1,\infty} (H^1) \), we get \( u_+ \in C^0_\delta (H^1) \) and \( u_+^m \to u_+ \) in \( C^0_\delta (H^1) \).

Passing to the limit in (4.8) we see that \( u_+ \) is also an \( H^1 \)-diffeomorphism. Thus we can define \( v_+ \) by \( v_+ = v_+ \circ u_+ = : \tilde{v}_+ = \partial_t u_+ \). From Euler equations we have \( \partial_t (v_+^m \circ u_+^m) = - \nabla p_+^m \circ u_+^m \) so that lemma 4.1, lemma A.1 and corollary 2.3 together with (A.10) imply

\[
|\partial_t (v_+^m \circ u_+^m)|_{H^{-\frac{1}{2}}(\Omega_+^m)} \leq C |p_+^m|_{H^{\frac{1}{2}}(\Omega_+^m)} \leq C.
\]

In particular this gives continuity of \( v_+^m \circ u_+^m = \partial_t u_+^m \) with values in \( H^{1-1}(\Omega_+^m) \). It also implies the existence of a subsequence (still denoted by the index \( m \)) such that \( \partial_t (v_+^m \circ u_+^m) \to \tilde{V}_+ \) weak–star in \( L^\infty (H^{\frac{1}{2}}) \). Since \( v_+^m \circ u_+^m \to \tilde{v}_+ \) in the sense of distributions, \( \tilde{V}_+ = \partial_t \tilde{v}_+ \). Therefore\(^{13} \) \( \tilde{v}_+ = v_+ \circ u_+ \in C^0_\delta (H^{\frac{1}{2}}) \) and

\[
v_+^m \circ u_+^m \to v_+ \circ u_+ \text{ in } C^0_\delta (\Omega_+^m).
\]

As \( v_+^m \circ u_+^m \) is uniformly bounded in \( L^\infty (H^{1}) \), by interpolating the Sobolev norms we can improve the above convergence obtaining (4.6) and the equivalent (4.4).

Finally, since \( p_+^m \circ u_+^m \) is uniformly bounded in \( L^\infty (H^{1-\frac{1}{2}}) \), up to extraction, we have \( p_+^m \circ u_+^m \to \bar{p}_+ \) weak–star in \( L^\infty (H^{1-\frac{1}{2}}) \) and (4.7) follows just by defining \( p_+ =: \bar{p}_+ \circ u_+^{-1} \).\(\square\)

### 4.2 Verification of (BC0)

Using convergence of the Lagrangian maps \( u_+^m \) associated to \( v_+^m \) established in (4.3), we defined in (4.5) the “limit domain” \( \Omega_+^\infty \) where the evolution of the limit solution is going to take place. From (4.3) and trace estimates we obtain \( u_+^m \big|_{S_0} \to u_+ \big|_{S_0} \in C^0_\delta (H^{1-\frac{1}{2}} (S_0)) \) so that

\[
u_+ (t, S_0) = \partial u_+ (t, \Omega_+^m) = : S_+^\infty \in C^0_\delta (H^{1-\frac{1}{2}} (S_0)) \quad \text{for } t \in [0, \tau^**].
\]

**Proposition 4.3.** The moving boundary condition in (BC0) holds for the set of hypersurfaces \( S_+^\infty \) with \( v_+ \) defined by (4.6).

**Proof** From the definition of Lagrangian maps, (4.4) and (4.6) we have

\[
\partial_t u_+ (t, y) = v_+ (t, u_+ (t, y)) \quad \forall \ (t, y) \in [0, \tau^**] \times \Omega_+^m.
\]

As \( u_+ (t, S_0) = S_+^\infty \) for any \( t \in [0, \tau^**] \), we have that \( (t, u_+ (t, \cdot)) \) is a curve on the space–time boundary \( \partial \Omega_+^\infty \); therefore

\[
\partial_t + \partial_t u_+ \cdot \nu = \partial_t + v_+ \circ u_+ \cdot \nu \text{ is tangent to } \bigcup_t S_+^\infty \subset \mathbb{R}^{n+1}.
\]

The fact that \( u_+ \) is a diffeomorphism from \( S_0 \) to \( S_+^\infty \) for any \( t \in [0, \tau^**] \) gives the claim \(\square\)

**Lemma 4.4.** Let \( N_+^m (t, \cdot) \) be the outward unit normal and \( \kappa_+^m (t, \cdot) \) the mean curvature of \( S_+^m \). Denote by \( N^\infty (t, x) \) and \( \kappa^\infty (t, x) \) respectively the unit normal and the mean curvature of \( S_+^\infty \) at the point \( x \). Then for any \( \ell < l \)

\[
N_+^m \circ u_+^m \to N^\infty \circ u_+ \text{ in } C^0_\delta (H^{l} (S_0)) \quad \text{and} \quad \kappa_+^m \circ u_+^m \to \kappa^\infty \circ u_+ \text{ in } C^0_\delta (H^{l-1} (S_0)).
\]

In particular \( |\kappa^\infty|_{H^{l-1} (S_+^\infty)} \) is uniformly bounded which implies\(^{14} \) \( S_+^\infty \in H^{l+1} \) as stated in theorem 1.1.

\(^{12}\)The standard argument is the following. Consider an arbitrary subsequence of \( \{u_+^m\} \); the boundedness of \( \{\partial_t u_+^m\} \) implies through the Ascoli–Arzelà theorem the existence of a sub-subsequence converging in \( C^0_\delta (H^{l}) \) to a limit which must be \( u_+ \) (the weak * limit of the original sequence \( \{u_+^m\} \)). Therefore \( u_+ \) is the uniform limit of \( \{u_+^m\} \).

\(^{13}\)We use the fact that \( f \in L^p(H^{\frac{1}{2}}) \) and \( f \in L^q(H^{\frac{1}{2}}) \) imply \( f \in C(H^{(\frac{1}{2}+\varepsilon)/2}) \).

\(^{14}\)This can be proved using local coordinates and estimates for quasi–linear elliptic equations. Another proof can be found in [14, proposition A.2].
Proof  Since $\kappa^\infty(t,x)(X,Y) = \text{tr}(Y \cdot \nabla_X N^\infty(t,x))$ for any $X,Y \in T_x S^\infty_t$, it is enough to prove the first statement in (4.9).

We use similar arguments to those in the proof of proposition 4.2. By lemma A.1, (A.10) and (2.3) we obtain uniform bounds on $N^m_+ \circ u^m_+$ in $L^\infty(H^l)$; therefore there exists $N^m_+ \in L^\infty(H^l)$ such that, up to extraction of a subsequence, $N^m_+ \circ u^m_+ \to N_+ =: A^\infty \circ u_+$ weak–star in $L^\infty(H^l)$. Identity (A.6) and estimate (A.10) combined with the uniform energy bounds on $\kappa^m_+$ show that

$$
\left| \frac{d}{dt} (N^m_+ \circ u^m_+) \right|_{H^{l-\frac{1}{2}}(S_0)} \leq C |u^m_+|_{H^l(\Omega^m)} |N^m_+|_{H^{l-\frac{1}{2}}(S_0)} \leq C
$$

with some $C$ uniform in $\Lambda_0$ and $m$. This in particular implies that $N^m_+ \circ u^m_+$ belongs to $C(H^{l-1}(S_0))$ and that, up to further extraction, $\partial_t(N^m_+ \circ u^m_+) \to \partial_t(A^\infty \circ u_+)$ weak–star in $L^\infty(H^{l-1}(S_0))$. As a consequence, $A^\infty \circ u_+ \in C^\infty_l(H^{l-1}(S_0))$ and $N^m_+ \circ u^m_+ \to A^\infty \circ u_+$ in $C^0_l(H^{l-1}(S_0))$ for any $l' < l$.

To show that $A^\infty(t,\cdot)$ is the outward unit normal $N^\infty(t,\cdot)$ to the hypersurface $S^\infty_t$ let $\tau^m \in T_y S^m_t$ be an arbitrary tangent vector.

Since $u^m_+$ is a diffeomorphism from $S_0$ to $S^m_t$, there exists a unique tangent vector $\tau_0 \in T_y S_0$ such that $\tau_m = du^m_+(t,y)\tau_0$, where $du^m_+(t,y)$ denotes the differential of $u^m_+$ as a map from $S_0$ to $S^m_t$ acting on $T_y S_0$ for $y = (u^m_+)^{-1}(t,x)$. Then for any $t \in [0,t^{**}]$

$$
\langle N^m_+ (t, u^m_+(t,y)), du^m_+(t,y) \tau_0 \rangle = 0
$$

Letting $m$ go to infinity using (4.3) we obtain

$$
\langle A^\infty (t, u_+(t,y)), du_+(t,y) \tau_0 \rangle = 0
$$

Since $\tau_m$, and consequently $\tau_0$, was arbitrarily chosen this implies that $A^\infty(t,x) \perp T_x S^\infty_t$ for $x = u_+(t,y)$; by the strong convergence established above $A^\infty$ is unitary and therefore coincides with $N^\infty(t,x)$. \hfill \Box

Proposition 4.5. The boundary condition (BC$\partial_0$) for the pressure is satisfied by the limit solution.

Proof  For the sequence of solutions $(v^m, S^m_t)$ condition (BC) holds for every $m \in \mathbb{N}$. As (E) is also satisfied for every $m$, the boundary condition for the physical pressure $p^m_+$ is the one given in (1.4) (where of course every quantity has to be indexed by $m$). Therefore $(p^m_- - \kappa^m_+) \circ u^m_+ = p^m_- \circ u^m_+$ on $S_0$ and we can use lemma A.1, (4.2) and trace–estimates to obtain

$$
| (p^m_- - \kappa^m_+) \circ u^m_+ |_{H^{l-1}(S_0)} \leq C |p^m_-|_{H^{l-\frac{1}{2}}(\Omega^m_+)} |u^m_+|_{H^{l-\frac{1}{2}}(\Omega^m_+)} \\
\leq C \rho^m_+ \left( |v^m_+|_{H^{l-1}(\Omega^m_+)} + |\kappa^m_+|_{H^{l-1}(\Omega^m_+)}. \right)
$$

Since the expression in parentheses above is uniformly bounded by the energies, letting $m \to \infty$ and using (4.9) we get

$$
p^m_- \circ u^m_+ \to \kappa^\infty \circ u_+ \text{ in } C^0_l(H^{l-1}(S_0))
$$

for any $l' < l$. Using (4.7) we conclude that $p_+(t,x) = \kappa^\infty(t,x)$ for any $t \in [0,t^{**}]$ and $x \in S^\infty_t$. \hfill \Box

4.3 Verification of (E$\partial_0$)

We first need the following estimate:

Lemma 4.6. Let $p_+$ be given by (1.4) then

$$
|D_t p^m_+|_{L^\infty(H^{l-2}(\Omega^m_+))} \leq C.
$$

for some $C$ uniform in $m$.

Proof  In what follows we suppress the use of the index $m$ and let $a \leq b$ denote $a \leq Cb$ for some constant $C$ independent of $\rho_-$. Writing $p_+ = H_+ p_+ + \Delta^{-1} \text{tr}(Dv_+)^2$ we have

$$
D_t p_+ = D_t H_+ p_+ + D_t \Delta^{-1} \text{tr}(Dv_+)^2 = H_+ D_t p_+ + \Delta^{-1} D_t \text{tr}(Dv_+)^2 + R := (I) + (II) + R
$$

where the remainder is given by the sum of the two commutators

$$
R = R^1 + R^2 := [D_t, H_+] p_+ + [D_t, \Delta^{-1}] \text{tr}(Dv_+)^2.
$$

10
We show that every term is bounded in $H^{l-2}$ or better by the quantities $|v|_{H^{l-1}}$, $|p|_{H^{l-\frac{3}{2}}}$, $|\kappa|_{H^{l-1}}$ and $|N|_{H^{l}}$ which are already known to be bounded uniformly in time by the energies independently of $\rho_-$. 

**Estimate of (I):** This is the highest order term in (4.13). Denoting $P := N\rho_+|S_t|$ we have

$$
(I) = H_+D_tN^{-1}P = H_+N^{-1}D_tP + H_+R^3P \quad \text{with} \quad R^3 := [N^{-1}, D_t].
$$

Observe that $R^3 = N^{-1} [N, D_t] N^{-1}$ so that (A.3), (A.5) and (A.13) give

$$
|H_+R^3P|_{H^{l-\frac{3}{2}}(\Omega_t^+)} \lesssim \left| [N^{-1}, D_t] P \right|_{H^{l-1}(S_t)} \lesssim \rho_-|v|_{H^l(\Omega_t)}|P|_{H^{l-2}(S_t)}
$$

Using (A.3) and (A.5) we obtain

$$
|H_+N^{-1}D_tP|_{H^{l-\frac{3}{2}}(\Omega_t^+)} \leq C\rho_-|D_tP|_{H^{l-\frac{3}{2}}(S_t)}.
$$

Now $D_tP$ contains four different terms to be estimated. The term involving the mean curvature is estimated by (A.13) and (A.7):

$$
|D_t \frac{1}{\rho_-}N_\pm \Delta_\pm^{1} tr (D\nu)^2|_{H^{l-2}(S_t)} \lesssim |D_t N_\pm|_{H^{l-2}(S_t)} |H^{l-\frac{3}{2}}(\Omega_t)| + |N_\pm|_{H^{l-2}(S_t)} |D_t \Delta_\pm^{1} tr (D\nu)^2|_{H^{l-\frac{3}{2}}(\Omega_t)}
$$

Notice that the presence of $\rho_-$ in the denominator in this last estimate is compensated by the factor $\rho_-$ in (4.15) so that the bounds remain uniform. For the terms involving $tr(D\nu)^2$ we use (A.6), (A.12) and the identities $D_t\nabla f = \nabla D_t f - (D\nu)^2 \nabla f$ and

$$
D_t tr (D\nu)^2 = -2 tr[(D\nu)^3 - 2\rho_+ D^2 p \cdot D\nu]
$$

to estimate

$$
|D_t \nabla N_\pm \Delta_\pm^{1} tr (D\nu)^2|_{H^{l-2}(S_t)} \lesssim |D_t N_\pm|_{H^{l-2}(S_t)} |H^{l-\frac{3}{2}}(\Omega_t)| + |N_\pm|_{H^{l-2}(S_t)} |D_t \nabla \Delta_\pm^{1} tr (D\nu)^2|_{H^{l-\frac{3}{2}}(\Omega_t)}
$$

Analogously, using (A.8) the terms $D_t \Pi_\pm (v^+ \pm, v^\pm)$ and $D_t (v^\pm \nabla v^\pm)$ can be bounded uniformly in $H^{l-\frac{3}{2}}(S_t)$ and $H^{l-3}(S_t)$ respectively. 

**Estimate of (II):** By the same formula used above to express $D_t tr (D\nu)^2$ we get

$$
|\Delta^{-1} D_t tr (D\nu)^2|_{H^{l-\frac{3}{2}}(\Omega_t^+)} \lesssim |(D\nu)^3|_{H^{l-\frac{3}{2}}(\Omega_t^+)} + \rho_+|D^2 p_+ \cdot D\nu|_{H^{l-\frac{3}{2}}(\Omega_t^+)}
$$

**Estimate of R:** Commutators $R^1$ and $R^2$ are estimated directly by (A.11) and (A.12):

$$
|D_t, H_+|_{H^{l-2}(\Omega_t^+)} \lesssim |v_+|_{H^{l}(\Omega_t^+)}|p_+|_{H^{l-\frac{3}{2}}(\Omega_t^+)} \quad \text{and} \quad |D_t, \Delta^{-1}| tr (D\nu)^2|_{H^l(\Omega_t^+)} \lesssim |v_+|^3_{H^l(\Omega_t^+)}
$$

where as usual the constant $C$ is independent of $\rho_-$.  

\[\text{This identity follows from } D_tDv = D[D_tv - (Dv)^2] \text{ together with Euler equations } \rho D_t v = -\nabla p.\]
Proposition 4.7. Let \( v_+ \) and \( u_+ \) be given as in proposition 4.2 then

\[
\frac{d}{dt} (v_+^m \circ u_+^m) \longrightarrow \frac{d}{dt} (v_+ \circ u_+) \quad \text{in} \quad C^0_t (H^{l+\frac{1}{2}} (\Omega_0^+)) .
\]

and \( v_+ \) satisfies Euler equations (Eo).

**Proof** (4.6) and the uniform bounds on \( p_+^m \) establish the above convergence weak–star in \( L^\infty (H^{l+\frac{1}{2}} (\Omega_0^+)) \). Since \( \partial_t^2 (v_+^m \circ u_+^m) = D_1 \nabla p_+^m \circ u_+^m = \nabla D_1 p_+^m \circ u_+^m - (D_1 \nabla p_+^m) \circ u_+^m \) the bound given in (4.12) implies \( \partial_t^2 (v_+^m \circ u_+^m) \) and the desired strong convergence follows through the usual arguments.

From (4.7) and (4.11) we know that \( p_+^m \circ u_+^m \rightarrow p_+ \circ u_+ \) strongly in \( C^0_t (H^{l+\frac{1}{2}}) \) and therefore \( \nabla p_+^m \circ u_+^m = \nabla (p_+^m \circ u_+^m) (\nabla u_+^m)^{-1} \rightarrow \nabla p_+ \circ u_+ \) in \( C^0_t (H^{l-2} (\Omega_0^+)) \). Since Euler equations in Lagrangian coordinates are \( \partial_t (v_+^m \circ u_+^m) = -\nabla p_+^m \circ u_+^m \) we can take the limit in \( L^\infty (H^{l-2} (\Omega_0^+)) \) obtaining that \( v_+ \) satisfies Euler equations in Lagrangian coordinates too, i.e.

\[
\frac{d}{dt} v_+(t, u_+(t), y) = -\nabla p_+(t, u_+(t), y) \quad \forall \ (t, y) \in [0, t^{**}] \times \Omega_0^+ .
\]

Finally from (4.3) and (4.6) we have \( \nabla (v_+^m \circ u_+^m) \rightarrow \nabla (v_+ \circ u_+) \) in \( C^0_t (H^{l-2} (\Omega_0^+)) \) so that

\[
0 \equiv \nabla \cdot v_+^m \circ u_+^m = \text{tr} (\nabla v_+^m \circ u_+^m) = \text{tr} \left( \frac{\nabla (v_+^m \circ u_+^m)}{(\nabla u_+^m)^{-1}} \right) \rightarrow \nabla \cdot v_+ \circ u_+
\]

which implies \( \nabla \cdot v_+ = 0 \) pointwise in \( \Omega_t^\infty \) for any \( t \in [0, t^{**}] \)

The proof of theorem 1.1 is completed.

### 5 Proof of Proposition 1.2

Let \( \Gamma \) be the infinite–dimensional manifold (1.7) and \( \bar{\mathcal{R}} \) its curvature tensor induced by the covariant differentiation defined in section 1.2. Consider a map \( u(t) : \Omega_0 \rightarrow \Omega_t \) in \( \Gamma \). Let \( \mathcal{R}^m \) denote the sectional curvature of \( \Gamma \) at the point \( u \) as an operator acting on \( T_u \Gamma \) endowed with the \( L^2 (\rho^m dy) \) metric and depending on some \( \bar{v} \in T_u \Gamma \) (and of course on \( u \)). We assume \( v \) and the hypersurfaces \( S_t \) to be sufficiently smooth and single out the leading order term of \( \bar{\mathcal{R}}^m \) analyzing its behavior as \( m \) goes to infinity (or equivalently as the density \( \rho_- \) vanishes). In view of the geometrical frame work discussed in section 1.2, and in particular in 1.2.3, \( \bar{\mathcal{R}}^m \) can be considered as a measurement of the instability occurring in the linearized Euler equations in case surface tension were not present.

Let \( \bar{w} \) be any vector in \( T_u \Gamma \). We assume that \( w \) is uniformly bounded in \( H^l (\mathbb{R}^n \times S_t) \) for some large enough \( l \) and compute the sectional curvature in the direction of \( \bar{v} \) and \( \bar{w} \). Using a well–known formula from Riemannian geometry together with (1.11) we have

\[
\bar{\mathcal{R}}^m = \left\langle \bar{\mathcal{R}} (u) (\bar{v}, \bar{v}) \bar{w}, \bar{w} \right\rangle_{L^2 (\rho^m dx)} = \left\langle II_u (\bar{v}, \bar{v}), II_u (\bar{w}, \bar{w}) \right\rangle_{L^2 (\rho^m dx)} - \left\| II_u (\bar{v}, \bar{w}) \right\|_{L^2 (\rho^m dx)}^2
\]

Again we suppress the use of the index \( m \). Using the divergence theorem the first integral can be written as

\[
\int_{\mathbb{R}^n \times S_t} \nabla p_{v,w} \nabla p_{w,v} \rho \, dx = \int_{S_t} p_{v,w} \nabla N_+^+ \nabla p_{w,v}^+ + \nabla N_- \nabla p_{w,v}^- \right) - \int_{S_t} p_{v,w} \Delta p_{w,v} \rho \, dx
\]

having used \( \nabla p_{v,w}^+ + \nabla p_{w,v}^- = N_+^+ \nabla N_+^+ \Delta p_{w,v}^+ + \nabla N_- \Delta_+^+ \Delta p_{w,v} + \nabla N_- \Delta_-^+ \Delta p_{w,v}^+ \) and (1.13) with \( v = w \). Since \( tr(Dw)^2 = \partial_t w^k \partial_k w^i \) \( \partial_t (w^k \partial_k w^i) \) we can use twice again the divergence theorem obtaining

\[
\int_{\mathbb{R}^n \times S_t} \nabla p_{v,w} \nabla p_{w,v} \rho \, dx = \int_{\mathbb{R}^n \times S_t} D^2 p_{v,w} (w, w) \rho \, dx + \int_{S_t} p_{v,w} \left\{ -2\nabla \bar{w}^+ \cdot \bar{w}^- w_+ + \bar{w}^- (w_+^+, w_+^+) \right\} dS + \int_{S_t} p_{v,w} \bar{w}^+ \Delta p_{w,v} \rho \, dS.
\]

Theorem 1.1 is completed.
To estimate the terms containing \( p_{v,w}^S \), which is the inverse image through \( N \) of a mean zero function on \( S_t \), we use lemma A.2. For any \( f \in L^1(S_t) \), (A.5) yields

\[
\left| \int_{S_t} p_{v,w}^S f \, dS \right| \leq C|p_{v,w}^S|_{H^1(\Omega)} |f|_{L^1(S_t)} \leq C\rho_- |N p_{v,w}^S|_{H^{1-1}(S_t)} |f|_{L^1(S_t)}
\]

whenever \( s > \frac{n-1}{2} \), with \( C \) uniform in \( S_t \in \Lambda_0 \). Since \( |N p_{v,w}^S|_{H^{1-1}(S_t)} \) is uniformly bounded for smooth enough and bounded \( v \) and \( w \), we can easily estimate several terms in (5.1):

\[
\left| \int_{S_t} p_{v,w}^S \nabla w^\pm \cdot \nabla_{in} dS \right|, \quad \left| \int_{S_t} p_{v,w}^S \Pi^\pm (w^\pm, w^\mp) \, dS \right|, \quad \left| \int_{S_t} p_{v,w}^S \nabla w^\pm \cdot N^\pm dS \right| \leq C\rho_- |w|_{H^2(\Omega)}^2.
\]

where \( C \) is some uniform constant depending on \( v \) and the mean curvature of \( S_t \). These bounds imply

\[
\lim_{\rho_- \to 0} \int_{\mathbb{R}^n \setminus S_t} \nabla p_{v,w} \nabla p_{w,v} \rho \, dx = \int_{\Omega^+} D^2 p_{v,w}(w_+, w_+) \rho_+ \, dx - \lim_{\rho_- \to 0} \int_{S_t} \rho_+ w^+_\nabla w_+ p^+_{v,v} + \rho_- w^-_\nabla w_- p^-_{v,v} \, dS.
\]

Next we look at the contribution of \( \|II_u(\bar{v}, \bar{w})\|^2 \), use the decomposition \( f_\pm = \mathcal{H}_\pm f + \Delta^{-1}_\pm \Delta f \) applied to \( p_{v,w} \) and observe that \( \nabla \mathcal{H}_\pm \nabla \Delta^{-1}_\pm \Delta \) to obtain

\[
\int_{\mathbb{R}^n \setminus S_t} |\nabla p_{v,w}|^2 \rho \, dx = \int_{S_t} \nabla \mathcal{N} p_{v,w}^S + \int_{\Omega^+} |\Delta^{-1}_\pm \text{tr} (Dv Dw)|^2 \rho_+ \, dx + \int_{\Omega^-} |\Delta^{-1}_\pm \text{tr} (Dv Dw)|^2 \rho_- \, dx.
\]

In [15] it is shown how the leading order term of the sectional curvature comes from the contribution of the surface integral in the above expression and is a second order negative semi-definite operator. But since \( \nabla \Delta^{-1}_\pm \Delta \) is independent of \( \rho_- \), by the same argument performed above the boundary integral vanishes as \( \rho_- \to 0 \). Therefore

\[
\lim_{m \to \infty} \mathcal{R}^m = \int_{\Omega^+} D^2 p_{v,w}(w_+, w_+) \rho_+ - |\nabla \Delta^{-1}_\pm \text{tr} (Dv Dw)|^2 \rho_+ \, dx
\]

\[
- \lim_{\rho_- \to 0} \int_{S_t} \rho_+ w^+_\nabla w_+ p^+_{v,v} + \rho_- w^-_\nabla w_- p^-_{v,v} \, dS.
\]

By splitting \( w \) into normal and tangential components on the boundary the surface integral in (5.2) is

\[
\int_{S_t} \rho_+ w^+_\nabla w_+ p^+_{v,v} + \rho_- w^-_\nabla w_- p^-_{v,v} \, dS = \int_{S_t} \rho_+ |w^+_\nabla N_+ p^+_{v,v} + \rho_- |w^-_\nabla N_- p^-_{v,v} \, dS + \int_{S_t} \rho_+ w^+_\nabla w_+ p^+_{v,v} + \rho_- w^-_\nabla w_- p^-_{v,v} \, dS.
\]

Writing \( \rho_\pm p_{v,v}^\pm = \mathcal{H}_\pm (p_{v,v}^S) - \rho_\pm \Delta^{-1}_\pm \text{tr} (Dv)^2 \), by the usual estimate for \( p_{v,v}^S \) the right-hand side of (5.3) gives the contribution

\[
\int_{S_t} |w^+_\nabla N_+ p^S_{v,v} + |w^-_\nabla N_- p^S_{v,v} + \rho_+ |w^+_\nabla N_+ p^S_{v,v} + \rho_- |w^-_\nabla N_- p^S_{v,v} \, dS
\]

\[
\lim_{\rho_- \to 0} \int_{S_t} \rho_+ \nabla N_+ \Delta^{-1}_\pm \text{tr} (Dv)^2 |w^+_\nabla p^S_{v,v} \, dS.
\]

Since \( \rho_+ p_{v,v}^+ = \rho_- p_{v,v}^- = p_{v,v}^S \) on \( S_t \) and we are considering only tangential derivatives, the contribution of the term in (5.4) is

\[
\int_{S_t} w^+_\nabla w^+_\nabla p^S_{v,v} \, dS \leq C |w|_{H^1(\mathbb{R}^n \setminus S_t)}^2 p_{v,v}^S \leq C|w|_{H^2(\mathbb{R}^n \setminus S_t)}^2 \rho_- \to 0 \cdot
\]

Gathering (5.2), (5.3), (5.5) and (5.6) we get

\[
\lim_{m \to \infty} \langle \mathcal{R}^m(u)(\bar{v}, \bar{w}), \bar{v} \rangle_{L^2(\rho \, dx)} + \int_{S_t} \rho_+ \nabla N_+ \Delta^{-1}_\pm \text{tr} (Dv)^2 |w^+_\nabla p^S_{v,v} \, dS \leq C|w|_{L^2(\mathbb{R}^n \setminus S_t)}^2.
\]
Lemma A.1. Let
\[ \mathcal{R}_0(v_+) = \left( -\rho_+ \nabla \mathcal{H}_+ \left( \nabla N_+ \Delta^{-1} \text{tr} (Dv_+)^2 \cdot |_{\partial u_+ (\Omega_0)} \right) \right) \circ u_+ \]
and satisfying
\[
\langle \mathcal{R}_0(v_+) \bar{w}_+, \bar{w}_+ \rangle_{L^2 (\Omega, d\rho)} = - \int \nabla N_+ \Delta^{-1} \text{tr} (Dv_+)^2 |w_+|^2 \rho_+ dS.
\]
From (1.12) we see that \( \Delta^{-1} \text{tr} (Dv_+)^2 \) is exactly \( p^*_{+,-} \) for the water wave problem so that (5.8) is equivalent to the first integral in (1.19). Therefore we have shown that as \( \rho_- \to 0 \) the Kelvin–Helmholtz instability for the vortex–sheet problem becomes the Raileigh–Taylor instability, i.e. the leading order term of the sectional curvature of \( \Gamma \) is not definite in general and has a positive sign only provided that the normal gradient of the physical pressure (in absence of surface tension) is negative. We conclude with two observations:

(i) If we do not restrict our attention exclusively to the highest order term of the sectional curvature operator, the above calculations show that

\[
\lim_{m \to \infty} \langle \mathcal{R}^m(u)(\bar{v}, \bar{w}) \rangle_{L^2 (\Omega, d\rho)} = - \int_{S_t} \rho_+ \nabla N_+ \Delta^{-1} \text{tr} (Dv_+)^2 |w_+|^2 dS
- \int_{\mathbb{R}^n \setminus S_t} D^2 \Delta^{-1} \text{tr} (Dv_+)^2 (w_+, w_+) \rho_+ - |\nabla \Delta^{-1} \text{tr} (Dv_+ Dw_+)|^2 \rho_+ dx.
\]

Since the second fundamental form of \( \Gamma^* \) in the water wave problem is given by \( \nabla p^*_{v,w} = \nabla \Delta^{-1} \text{tr} (Dv Dw) \) the above limit is exactly

\[
\int_{\mathbb{R}^n \setminus S_t} \nabla p^*_{v_+, w} \nabla p^*_{w_+, w} \rho_+ dx - \int_{\mathbb{R}^n \setminus S_t} |\nabla p^*_{v_+, w}|^2 \rho_+ dx = \langle \mathcal{R}^* (u_+)(\bar{v}_+, \bar{w}_+) \bar{v}_+, \bar{w}_+ \rangle_{L^2 (\Omega, d\rho)}.
\]

(ii) From a standard argument we conclude that the full curvature tensor of \( \Gamma \) converges to the curvature tensor of \( \Gamma^* \) in the sense stated in (1.1) and this completes the proof of proposition 1.2. ■

### A Supporting material for proofs

In this appendix we gather some technical results needed in the proofs presented and in the proof of theorem 2.4 in appendix B.

**Lemma A.1.** Let \( D_1 \) (resp. \( S_1 \)) be domains (resp. hypersurfaces) in \( \mathbb{R}^n \) for \( i = 0, 1, \) Let \( \eta : D_0 \to D_1 \) (resp. \( \eta : S_0 \to S_1 \)) be an \( H^l \)-diffeomorphism for \( l > \frac{n}{2} + 1 \) (resp. \( l > \frac{n-1}{2} \)) with \( |(\text{det} \ D\eta)|^{-1} \leq a \) (resp. \( |(\text{det} \ D\eta)|^{-1} \leq a \)). Then the operator \( T_\eta : f \to f \circ \eta \) is a bounded operator from \( H^s (D_1) \) to \( H^s (D_0) \) (resp. from \( H^s (S_1) \) to \( H^s (S_0) \)) for any \( s \in [0, l] \) and satisfies

\[
|f \circ \eta|_{H^s (D_0)} \leq C_0 |f|_{H^s (D_1)} |\eta|_{H^l (D_0)}
\]
for some constant \( C_0 \) depending on \( a, s, l \) and the domains \( D_1 \) (resp. the hypersurfaces \( S_1 \)).

**Proof** The case \( s = 0 \) follows immediately from the hypotheses. Assume by induction that (A.1) holds for any integer \( s \) such that \( 0 \leq s \leq k - 1 \leq l - 1 \). We prove the statement for \( s = k \). Write \( D^k (f \circ \eta) = D^{k-1} (Df \circ \eta D\eta) = \sum_{j=0}^{k-1} D^j (Df \circ \eta) D^{k-j} \eta \). Let \( r \geq 2 \) be the integer such that \( \frac{2}{r} - 1 \leq l - r < \frac{n}{2} \); observe that \( D^r \eta \in L^\infty \) for \( i \leq r - 1 \) while it is not uniformly bounded in general for \( i \geq r \) since \( H^{l-1} \) does not embed in \( L^\infty \). According to this we split

\[
\sum_{j=0}^{k-1} D^j (Df \circ \eta) D^{k-j} \eta = \sum_{j=0}^{k-r} D^j (Df \circ \eta) D^{k-j} \eta + \sum_{j=k-r+1}^{k-1} D^j (Df \circ \eta) D^{k-j} \eta =: \Sigma_1 + \Sigma_2.
\]

In \( \Sigma_2 \) all derivatives on \( \eta \) can be taken in \( L^\infty \) and estimated through Sobolev’s embedding:

\[
|\Sigma_2|_{L^2 (D_0)} \leq \sum_{j=k-r+1}^{k-1} |D^j (Df \circ \eta)|_{L^2 (D_0)} |\eta|_{H^k (D_0)} \leq C |Df \circ \eta|_{H^{k-1} (D_0)} |\eta|_{H^l (D_0)}
\]

\[
\leq C |f|_{H^{k-1} (D_1)} |\eta|_{H^l (D_0)}^k |\eta|_{H^l (D_0)} = C |f|_{H^{k} (D_1)} |\eta|_{H^l (D_0)}^k.
\]
The contribution of $\Sigma_1$ is estimated using Hölder's inequality and Sobolev's embeddings. Since $l > \frac{q}{2} + 1$ and $k - j \geq r > 1$, we can choose $2 < p, q < \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \frac{1}{q} > \frac{1}{2} - \frac{l - k + j}{n}, \quad \frac{1}{p} > \frac{1}{2} - \frac{k - 1 - j}{n}.$$  

Using Hölder's inequality and the embeddings $H^{l-k+j} \subset H^{1/2 - 1/2n} \subset L^q, H^{k-1-j} \subset H^{1/2 - 1/2n} \subset L^p$ we get

$$|\Sigma_1|_{L^2(D_0)} \leq \sum_{j=0}^{k-r} |D^j(Df \circ \eta)||_{L^p(D_0)} |D^{k-j}\eta||_{L^q(D_0)} \leq C |D^j(Df \circ \eta)||_{H^{k-1-j}(D_0)} |\eta||_{H^l(D_0)} \leq C |f||_{H^k(D_1)} |\eta||_{H^l(D_0)}$$

with $C$ depending on $a, k$ and the domains $D_0, D_1$. Therefore we proved

$$|D^k(f \circ \eta)||_{L^2(D_0)} \leq C |f||_{H^k(D_1)} |\eta||_{H^l(D_0)}$$

From the inductive hypothesis the same inequality holds for lower order derivatives terms $D^i(f \circ \eta)$, for $0 \leq i \leq k - 1$, replacing $k$ with $i$. Up to further increasing the value of $C$ depending on $a$ and the constants in Sobolev’s embeddings, we can sum this inequalities to obtain (A.1) for $s = k$. The case for non-integer $s$ follows by interpolation.

**Lemma A.2 (About differential operators on $\Lambda_0$ [14]).** Let $\Delta^{-1}$ and $\mathcal{H}$ denote respectively the inverse Laplacian with Dirichlet boundary condition and the Harmonic extension operators on a domain $\Omega$. There exists a uniform constant $C > 0$ such that for every domain $\Omega$ with $\partial\Omega = S \in \Lambda_0$ (see definition 2.1) the following is true:

$$|f||_{H^s(\partial\Omega)} \leq C |f||_{H^{s+1/2}(\Omega)}, \quad \forall s > 0 \quad (A.2)$$

$$|\Delta^{-1}||_{L(H^{s-1}(\Omega),H^{s+1}(\Omega))} + |\mathcal{H}||_{L(H^{s+1/2}(\Omega),H^{s+1}(\Omega))} \leq C, \quad \forall s \in (0, l - 1). \quad (A.3)$$

As a consequence of the Dirichlet–to–Neumann operator satisfies

$$|\mathcal{N}_0||_{L(H^{s+1/2}(\Omega),H^{s+1/2}(\Omega))} + |\mathcal{N}^{-1}_0||_{L(H^{s-1/2}(\Omega),H^{s-1/2}(\Omega))} \leq C, \quad \forall s \in [0, l - 1] \quad (A.4)$$

where $H^s$ denotes zero–mean $H^s$–functions. In particular if $\mathcal{N}$ is the operator defined by (1.5) then $|\mathcal{N}||_{L(H^{s+1/2}(\Omega),H^{s+1/2}(\Omega))} \leq C(\rho_- + \rho_+)/\rho_-$ and

$$|\mathcal{N}^{-1}||_{L(H^{s-1/2}(\Omega),H^{s+1/2}(\Omega))} \leq 2C\rho_- \quad \forall s \in [0, l - 1] \quad (A.5)$$

**Proof** The proof of (A.2), (A.3), (A.4) and more detailed analysis of operators acting on $\partial\Omega$ (and in particular of the Dirichlet–to–Neumann operator) can be found in [14, A.2]. To obtain (A.5) write $\mathcal{N}$ as

$$\mathcal{N} = \left( \frac{N_+}{\rho_+} \rho_- N^{-1}_0 + I \right) \frac{N_-}{\rho_-} =: (B + I) \frac{N_-}{\rho_-}.$$ Estimate (A.4) implies that for $\rho_- \leq \rho_+/(2C^2)$ the linear operator $B$ maps $H^0(\partial\Omega)$ to itself with norm less or equal than $1/2$. Thus $I + B$ is invertible and $N^{-1}_0 = \rho_- N^{-1} \sum_{j=0}^{\infty} (-1)^j B^j$ so that

$$|\mathcal{N}^{-1}||_{L(H^{s-1/2}(\Omega),H^{s+1/2}(\Omega))} \leq C \rho_- \sum_{j=0}^{\infty} |B||^j_{L(H^{s-1/2}(\Omega),H^{s+1/2}(\Omega))} \leq 2C \rho_- \quad \square$$

**Lemma A.3 (Geometric Formulae [14]).** Let $S$ be an hypersurface in $\mathbb{R}^n$ moved by the normal component of a vector field $v$. Let $N, \kappa$ and $\Pi$ denote respectively its unit normal, mean curvature and second fundamental form. Then the following identities hold true:

$$D_N N = -[(Dv)^* \cdot N]^\top \quad (A.6)$$

$$D_N \kappa = -\Delta_S v^\perp + v^\perp |\Pi|^2 + \nabla_{v^\perp} \kappa \quad (A.7)$$

$$D^\top \Pi(\tau) = -D_\tau \left( (Dv)^* N \right)^\top - \Pi \left( (\nabla_{\tau} v)^\top \right) \quad (A.8)$$

$$-\Delta_S \Pi = -D^2 \kappa + (|\Pi|^2 I - \kappa \Pi) \Pi \quad (A.9)$$

---

16 $F = \Delta^{-1} f$ satisfies $\Delta F = f$ in $\Omega$ and $F = 0$ on $\partial\Omega$. $G = \mathcal{H} g$ satisfies $\Delta G = 0$ in $\Omega$ and $G = g$ on $\partial\Omega$.  

17 In view of (A.3) $\mathcal{N}_0$ can be defined for any $f \in H^0(\partial\Omega)$, $s \geq 1/2$ in the weak form $\langle \phi, \mathcal{N}_0(f) \rangle = \int_{\Omega} \nabla \phi \cdot \nabla \mathcal{H} f$. 

15
where $\mathcal{D}$ denotes the covariant derivative on $S$ and $\Delta_S := tr\mathcal{D}^2$. Furthermore there exists a uniform constant $C$ such that for any $S \in \Lambda_0$

$$|\Pi|_{H^l(S)} + |N|_{H^{l+1}(S)} \leq C(1 + |\kappa|_{H^l(S)}) \quad \forall \ l - \frac{5}{2} \leq s \leq l - 1. \tag{A.10}$$

**Lemma A.4 (Commutator Estimates [14]).** There exists a uniform constant $C$ such that for any $\partial \Omega = S \in \Lambda_0$ the following estimates hold:

$$||\mathcal{D}_t, \mathcal{H}||_{L(H^{l-2}(S), H^l(S))} \leq C|\mathcal{H}|_{H^l(S)} \quad \forall \ \frac{1}{2} < s \leq l \tag{A.11}$$

$$||\mathcal{D}_t, \Delta^{-1}||_{L(H^{l-2}(\Omega), H^l(\Omega))} \leq C|\mathcal{H}|_{H^l(\Omega)} \quad \forall \ 2 - l \leq s \leq l \tag{A.12}$$

$$||\mathcal{D}_t, \mathcal{N}_0||_{L(H^l(S), H^{l-1}(S))} \leq C|\mathcal{H}|_{H^l(S)} \quad \forall \ 1 \leq s \leq l - \frac{1}{2} \tag{A.13}$$

$$||\mathcal{D}_t, \Delta \mathcal{S}||_{L(H^l(S), H^{l-2}(S))} \leq C|\mathcal{H}|_{H^l(S)} \quad \forall \ \frac{7}{2} - l \leq s \leq l - \frac{1}{2}. \tag{A.14}$$

**B Proof of Theorem 2.4**

This section is devoted to the proof of Theorem 2.4 and consists essentially of material contained in [15, sec. 4.3, 4.4]. The only difference is that we claim and show independence of the energy estimates on the densities of the two fluids. Therefore, even though the proof is extremely similar to the one performed in [15], we present it here for the reader’s convenience.

**B.1 Estimates on the Lagrangian coordinate map**

We use the same notation in the original proof of theorem 2.4 letting $l := \frac{5}{2} k$. Working on the compact domain $\Omega^+_k$ and arguing as in the proof of proposition 2.3 (see section 3) we obtain the existence of a positive time $t_1$ and a constant $C_1$, only depending on $k, n$ and $\mu$ as in (3.1) such that

$$|u_+(t, \cdot) - \text{id}_{\Omega^+_k}|_{H^{\frac{5}{2}k-\frac{2}{3}}(\Omega^+_k)} \leq C_1 t \quad \forall \ t \in [0, \min\{t_0, t_1\}].$$

This implies the estimate on the mean curvature\(^{18}\)

$$|\kappa_+(t, \cdot)|_{H^{\frac{5}{2}k-\frac{2}{3}}(S_t)} \leq C_2 t + |\kappa_+(0, \cdot)|_{H^{\frac{5}{2}k-\frac{2}{3}}(S_0)} \quad \forall \ t \in [0, \min\{t_0, t_1\}] \tag{B.1}$$

where the constant $C_2$ is only determined by $\mu$ and the set $\Lambda_0$. We conclude that there exists a time $t_2$ again determined only by $\mu$ and the set $\Lambda_0$ such that

$$S_t \in \Lambda_0, \quad \forall \ t \in [0, \min\{t_0, t_2\}] .$$

**B.2 Evolution of the Energy**

The energy defined in (2.1) is made of three terms. The first two involve the operator $\mathfrak{A}$ defined in (1.18) and are used to control the irrotational part of the velocity and the mean curvature (hence the regularity of the evolving domain $S_t$); the third part involves the vorticity $\omega$ and is used to control the rotational part of $v$. More explicitly $E = E_1 + E_2 + |\omega|_{H^{\frac{5}{2}k-1}}^2$ where, using (1.18), the first two terms are given by

$$E_1 := \frac{1}{2} \int_{\mathbb{R}^n \times \Omega_t} |\frac{1}{2} v| \rho \ dx = \frac{1}{2} \int_{S_t} \frac{1}{2} v_+^k(-\Delta \mathcal{N})^{k-1}(-\Delta \mathcal{N}) v_+^k \ dS \tag{B.2}$$

$$E_2 := \frac{1}{2} \int_{\mathbb{R}^n \times \Omega_t} |\frac{1}{2} \mathfrak{A} v| \rho \ dx = \frac{1}{2} \int_{S_t} \kappa_+ \mathcal{N}(-\Delta \mathcal{N})^{k-1} \kappa_+ \ dS \tag{B.3}$$

where

$$\mathcal{N} = \left( \frac{1}{\rho_+} \mathcal{N} \right)^{-1} \left( \frac{1}{\rho_-} \mathcal{N} \right). \tag{B.4}$$

It is clear from lemma A.2 that $\mathcal{N}$ is a first-order self–adjoint operator whose norm and inverse’s norm do not depend on $\rho_-$.\(^{18}\)

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\(^{18}\)This can be checked using the local coordinates constructed in [14, appendix A].
Proposition B.1. There exists a polynomial $Q(t) = \mathcal{Q} \left( \left| v(t, \cdot) \right|_{H^{\frac{1}{2}}(\mathbb{R}^n \times S_t)}, \left| \kappa(t, \cdot) \right|_{H^{\frac{1}{2} - \frac{1}{4}}(S)} \right)$ with positive coefficients depending on the set $\Lambda_0$ and independent of the density $\rho_-$ such that

$$\left| \frac{d}{dt} (E - E_{\text{ex}}) \right| \leq Q \quad (B.5)$$

where the extra energy term $E_{\text{ex}}$, due to the Kelvin–Helmholtz instability, is given by

$$E_{\text{ex}} = \frac{\rho^+}{2(\rho^+ + \rho^-)} \int_{S_t} \nabla v^\top \kappa_+ \cdot \hat{N}(-\Delta_{S_t} \hat{N})^{k - 2} \nabla v^\top \kappa_+ \, dS - \frac{\rho^-}{2(\rho^+ + \rho^-)} \int_{S_t} \nabla v^\top \kappa_+ \cdot \hat{N}(-\Delta_{S_t} \hat{N})^{k - 2} \nabla v^\top \kappa_+ \, dS. \quad (B.6)$$

**Proof** Throughout the proof we denote by $Q$ any generic polynomial satisfying the properties in the statement.

*Evolution of $E_1$: This is the hardest term to deal with and is the one where the extra energy term $E_{\text{ex}}$ appears. We are going to show

$$\left| \frac{d}{dt} (E_1 - E_{\text{ex}}) + \int_{S_t} v^\top_+ (-\Delta_{S_t} \hat{N})^{k - 1} (-\Delta_{S_t} \hat{N}) dS \right| \leq Q. \quad (B.7)$$

From definition (B.4) and (A.13) we have

$$\left| [D_1, \hat{N}] \right|_{L(H^{s}(S_t), H^{s-1}(S_t))} \leq C[v]_{H^{\frac{1}{2} + \frac{1}{4}}(\Omega)} \forall \frac{1}{2} \leq s \leq \frac{3}{2} k - \frac{1}{2}.$$ 

Therefore, using also (A.14), we can commute $D_{t_+}$ with the operators appearing in (B.2) to get

$$\left| \frac{d}{dt} \frac{1}{2} \int_{S_t} v_+^\top (-\Delta_{S_t} \hat{N})^{k-1} (-\Delta_{S_t} \hat{N}) v_+^\top dS - \int_{S_t} v_+^\top (-\Delta_{S_t} \hat{N})^{k-1} (-\Delta_{S_t} \hat{N}) D_{t_+} v_+^\top dS \right| \leq Q.$$ 

Using (A.6) to express $D_{t_+} N_+$, (1.16), (1.13) and (1.15) together with (B.4) we have

$$D_{t_+} v_+^\top = D_{t_+} v \cdot N_+ + v D_{t_+} N_+ = -\nabla N_+ p^S_{v,v} - \nabla N_+ p^\perp_+ - \nabla v^\top v_+ \cdot N_+$$

$$= -\frac{1}{\rho_+} N_+ p^S_{v,v} + \nabla N_+ \Delta^{-1} \text{tr} (Dv)^2 - \hat{N} \kappa_+ - \nabla v^\top v_+^\top + \Pi(v^\top, v^\top)$$

From lemma A.2 and trace–estimates $|\nabla N_+ \Delta^{-1} \text{tr} (Dv)^2|_{H^{\frac{1}{2} - \frac{1}{4}}(S_t)} \leq Q$ so that this term is lower order and

$$\left| \frac{d}{dt} E_1 - \int_{S_t} v_+^\top (-\Delta_{S_t} \hat{N})^{k-1} (-\Delta_{S_t}) \left( -\frac{1}{\rho_+} N_+ p^S_{v,v} - \hat{N} \kappa_+ - \nabla v^\top v_+^\top + \Pi(v^\top, v^\top) \right) dS \right| \leq Q. \quad (B.8)$$

Equation (1.13) for $p^S_{v,v}$ gives

$$-\frac{1}{\rho_+} N_+ p^S_{v,v,v} = -\frac{1}{\rho_+} N_+ \hat{N}^{-1} \left\{ 2 \nabla v^\top - v^\top v_+^\top - \Pi_+(v^\top, v_+^\top) - \Pi_-(v^\top, v_+^\top) - \nabla N_+ \Delta^{-1} \text{tr} (Dv)^2 - \nabla N_+ \Delta^{-1} \text{tr} (Dv)^2 \right\}.$$

Since $N_+ \hat{N}^{-1}$ is an operator of order zero the terms $\nabla N_+ \Delta^{-1} \text{tr} (Dv)^2$ can be treated as before. From Lemma 4.6 in [15] (to which we refer for the proof)

$$\left| (-\Delta_{S_t})^{\frac{1}{2} - S} \right|_{L(H^{s}(S_t))} \leq C \left( 1 + |\kappa(t, \cdot)|_{H^{\frac{1}{2} - \frac{1}{4}}(S_t)} \right) \forall \frac{1}{2} - \frac{3}{2} k \leq s \leq \frac{3}{2} k - \frac{1}{2};$$

this and the definition (1.5) of $\hat{N}$ yield

$$\left| N_+ \hat{N}^{-1} - \frac{\rho^+ \rho^-}{\rho^+ + \rho^-} \right|_{L(H^{\frac{1}{2} - \frac{1}{4}}(S_t), H^{\frac{1}{2} - \frac{1}{4}}(S_t))} \leq Q.$$

Together with (A.10) this gives

$$\left| -\frac{1}{\rho_+} N_+ p^S_{v,v} - \frac{\rho^-}{\rho^+ + \rho^-} \left( 2 \nabla v^\top - v^\top v_+^\top - \Pi_+(v^\top, v_+^\top) - \Pi_-(v^\top, v_+^\top) \right) \right|_{H^{\frac{1}{2} - \frac{1}{4}}(S_t)} \leq Q$$
so that (B.8) becomes
\[
\left| \frac{d}{dt} E_1 \right| = \int_{S_t} v_+^+(\Delta S_t \mathcal{N})^{k-1}(\Delta S_t) \left[ -\frac{\rho_-}{\rho_+ + \rho_-} \Pi_-(v_+^T, v_-^T) + \frac{\rho_+}{\rho_+ + \rho_-} \Pi_+(v_+^T, v_+^T) \right] dS \\
- \nabla v_+^+ \left( \frac{\rho_- - \rho_+}{\rho_+ + \rho_-} \nabla v_+^T - \frac{2\rho_-}{\rho_+ + \rho_-} v_+^T \right) \leq Q.
\]

We now claim that the last two terms in the above integral are lower order. To see this, consider flows $\Phi_\pm(\tau, \cdot)$ on $\Omega_\tau^+$ and apply (A.13) and (A.14) to $D_\tau$ to move outside the tangential derivatives $\nabla v_\perp^+$:
\[
\left| \int_{S_t} v_+^+(\Delta S_t \mathcal{N})^{k-1}(\Delta S_t) \nabla v_+^+ \left( \frac{\rho_- - \rho_+}{\rho_+ + \rho_-} \nabla v_+^T - \frac{2\rho_-}{\rho_+ + \rho_-} v_+^T \right) dS \right| \leq Q;
\]
then integrate by parts in these last two integrals estimating $Dv_\perp^+$ in $L^\infty(S_t)$ and the remaining $\frac{3}{2}k - \frac{1}{2}$ derivatives on $v_+^+$ in $L^2(S_t)$. For the terms involving the second fundamental form, (A.9) gives
\[
\left| \Delta S_t (\Pi_+(v_+^T, v_+^T) - D^2 v_\perp^+(v_+^T, v_+^T)) \right|_{H^{\frac{3}{2}k - \frac{1}{2}}(S_t)} \leq Q.
\]
Since $D^2 \kappa_\pm = \nabla v_\perp^+ \nabla v_\perp^\perp \kappa_\pm - D v_\perp^+ \nabla \kappa_\pm$ and the last term in this sum is lower order, we get
\[
\left| \frac{d}{dt} E_1 \right| = \int_{S_t} v_+^+(\Delta S_t \mathcal{N})^{k-1} \left[ \frac{\rho_-}{\rho_+ + \rho_-} \nabla v_+^+ \nabla \kappa_- - \frac{\rho_+}{\rho_+ + \rho_-} \nabla v_+^+ \nabla \kappa_+ + \Delta S_t \mathcal{N} \kappa_+ \right] dS \leq Q.
\]
Using the same previous argument we can commute one of the factors $\nabla v_\perp^+$ and move it outside to obtain
\[
\left| \frac{d}{dt} E_1 \right| + \int_{S_t} v_+^+(\Delta S_t \mathcal{N})^{k-1} \left[ \frac{\rho_-}{\rho_+ + \rho_-} \nabla v_+^+ \nabla \kappa_- - \frac{\rho_+}{\rho_+ + \rho_-} \nabla v_+^+ \nabla \kappa_+ + \Delta S_t \mathcal{N} \kappa_+ \right] dS \leq Q.
\]

Now, thanks to identity (A.7)
\[
\left| -\Delta S_t v_+^+ - D_{\kappa_+} \kappa_+ \right|_{H^{\frac{3}{2}k - 2}(S_t)} \leq Q
\]
so that we can substitute $D_{\kappa_+} \kappa_+$ to $-\Delta S_t v_+^+$ in (B.9) and (B.10). The usual commutator estimates imply
\[
\left| \frac{d}{dt} E_x \right| + \int_{S_t} v_+^+(\Delta S_t \mathcal{N})^{k-1} \left[ \frac{\rho_-}{\rho_+ + \rho_-} \nabla v_+^+ D_{\kappa_+} \mathcal{N} - \frac{\rho_+}{\rho_+ + \rho_-} \nabla v_+^+ \mathcal{N} D_{\kappa_+} \right] dS \leq Q
\]
and (B.7) follows.

**Evolution of $E_2$:** As before commutator estimates (A.13) and (A.14) give
\[
\left| \frac{d}{dt} E_2 \right| \leq \int_{S_t} \mathcal{N} (\Delta S_t \mathcal{N})^{k-1} D_{\kappa_+} \kappa_+ dS \leq Q.
\]
and in view of (A.7) and (A.10) we obtain
\[ \left| \frac{d}{dt} E_2 - \int_{S_t} \kappa_+ \mathcal{N}(-\Delta S_t, \mathcal{N}) k_{1-1}(-\Delta S_t) v_+^1 dS \right| \leq Q + \int_{S_t} \kappa_+ \mathcal{N}(-\Delta S_t, \mathcal{N}) k_{1-1} \nabla v_+^1 \kappa_+^1 dS \]

The same commutation argument previously adopted shows that
\[ \left| \int_{S_t} \kappa_+ \mathcal{N}(-\Delta S_t, \mathcal{N}) k_{1-1} \nabla v_+^1 \kappa_+^1 dS - \frac{1}{2} \int_{S_t} \nabla v_+^1 \mathcal{N}^2(-\Delta S_t, \mathcal{N}) \kappa_+^1 \kappa_+^1 dS \right| \leq Q. \]

Integrating by parts and estimating \( Dv_+^1 \) in \( L^\infty(S_t) \) and the remaining \( \frac{3}{2} k - 1 \) derivatives on \( \kappa_+ \) in \( L^2 \) shows that this last integral is bounded by \( Q \). Finally use the self–adjointness of \( \mathcal{N} \) and \( \Delta S_t \) to obtain
\[ \left| \frac{d}{dt} E_2 - \int_{S_t} v_+^1(-\Delta S_t, \mathcal{N}) k_{1-1} \kappa_+^1 dS \right| \leq Q. \tag{B.11} \]

**Evolution of the vorticity \( \omega = Dv - (Dv)^* \):** Commuting repeatedly \( D_t \) with \( D \) and using the identity
\[ D_t \omega = DD_t v - (Dv)^2 - (DD_t v)^* + ((Dv)^*)^2 = ((Dv)^*)^2 - (Dv)^2 = -\omega Dv - (Dv)^* \omega \]
we have
\[ \frac{d}{dt} \int_{\mathbb{R}^n \setminus S_t} |D\frac{3}{4}k-1 \omega|^2 dx = \int_{\mathbb{R}^n \setminus S_t} D_t |D\frac{3}{4}k-1 \omega|^2 dx \leq C |v(t, \cdot)|_{H^{2k}(\mathbb{R}^n \setminus S_t)} |\omega(t, \cdot)|_{H^\frac{3}{2}k-1(\mathbb{R}^n \setminus S_t)} \leq Q. \]

Summing up (B.7), (B.11) and (B.12) we get the desired cancellations giving (B.5) □

### B.3 The Energy Inequality

Integrating in time (B.5) gives
\[ E(t) - E(0) - E_{\alpha}(t) + E_{\alpha}(0) \leq \int_0^t Q \left( |v(s, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_t)}, |\kappa(s, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(S_t)} \right) ds \tag{B.12} \]
for any \( 0 \leq t \leq \min\{t_0, t_2\} \). We can estimate the extra energy term (B.6) by
\[ |E_{\alpha}(t)| \leq \frac{1}{2} \int_{S_t} |\mathcal{N}(\Delta S_t, \mathcal{N})^\frac{3}{2} \nabla \kappa_+^1|^2 dS + \frac{1}{2} \int_{S_t} |\mathcal{N}(\Delta S_t, \mathcal{N})^\frac{3}{2} \nabla \kappa_+^1|^2 dS \leq C |v(t, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_t)} |\kappa(t, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(S_t)} \]
where the positive constant \( C \) depends only on the set \( \Lambda_0 \). Interpolating \( v \) between \( H^{\frac{2k}{2}k-\frac{1}{2}} \) and \( H^{\frac{2k}{2}k} \) and \( \kappa \) between \( H^{\frac{2k}{2}k-\frac{1}{2}} \) and \( H^{\frac{2k}{2}k-\frac{1}{2}} \) yields
\[ |E_{\alpha}| \leq \frac{1}{2} E + C_1 \left( 1 + |v|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_t)} \right) \]
for some integer \( m \) where the constant \( C_1 \), which includes \( |\kappa|_{H^\frac{2k}{2}k-\frac{1}{2}} \), depends ultimately only on \( E_0 \) and \( \Lambda_0 \) in view of (B.1). Using Euler equations (1.16) and lemma 4.1 to estimate the pressure, we have
\[ |D_t v|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_t)} = \frac{1}{p_+} |\nabla p_+|_{H^\frac{2k}{2}k-\frac{1}{2}(\Omega_t^\infty)} + \frac{1}{p_-} |\nabla p_-|_{H^\frac{2k}{2}k-\frac{1}{2}(\Omega_t^\infty)} \leq Q. \]

We can then use the Lagrangian coordinate map to estimate
\[ \left| |v(t, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_t)} - |v(0, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_0)} \right| \leq \int_0^t Q(s) ds \]
and obtain
\[ |E_{\alpha}| \leq \frac{1}{2} E + C_1 \left( 1 + |v(0, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_0)} \right) + \int_0^t Q(s) ds \leq \frac{1}{2} E + C_2 + \int_0^t Q(s) ds \]
where \( C_2 \) is determined by \( E_0 \), the set \( \Lambda_0 \) and \( |v(0, \cdot)|_{H^\frac{2k}{2}k-\frac{1}{2}(\mathbb{R}^n \setminus S_0)} \). Inserting this last inequality in (B.12) we finally obtain
\[ E(S_t, v(t, \cdot)) \leq 3E(S_0, v(0, \cdot)) + C_2 + \int_0^t Q(s) ds \]
for some \( C_2 \) as above. Taking \( \mu \) in (3.1) large enough compared to the initial data concludes the proof of theorem 2.4 □
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