Gauge Invariant Matrix Model for the Å-D-È Closed Strings

I. K. Kostov *

Service de Physique Théorique † de Saclay
CE-Saclay, F-91191 Gif-sur-Yvette, France

The models of triangulated random surfaces embedded in (extended) Dynkin diagrams are formulated as a gauge-invariant matrix model of Weingarten type. The double scaling limit of this model is described by a collective field theory with nonpolynomial interaction. The propagator in this field theory is essentially two-loop correlator in the corresponding string theory.

Submitted for publication to: Physics Letters B

SPhT/92-096

8/92

* on leave from the Institute for Nuclear Research and Nuclear Energy, 72 Boulevard Tsari-
gradsko Shosse, 1784 Sofia, Bulgaria

† Laboratoire de la Direction des Sciences de la Matière du Commissariat à l’Energie Atomique
1. Introduction

The string theories with discrete target space are interesting mainly because of their interpretation as theories of two-dimensional quantum gravity with matter fields. The (discrete) degrees of freedom of the matter fields are labeled by the points of the target space.

It appeared that the theories with nontrivial critical behaviour are classified by the two-dimensional integrable statistical systems that can be defined on a generic planar graph. The continuum limit of these models is described by a conformal field theories with central charge \( C \leq 1 \). The explicit construction involving integrable models on planar graphs \([1]\) allowed to apply the formalism called “Coulomb gas picture” and map the string theories with discrete target space onto a loop gas model on a random surface. The latter can be solved order by order in the topology of the random surface by cutting it along the loops into elementary pieces \([2]\).

Although each step in this construction is quite trivial, as a whole it seems heavy to follow, because its constant appeal for geometric imagination. This is why we propose in this letter an alternative, purely algebraic method of solution using a special large \( N \) matrix model. In this way the integrable statistical systems on triangulated random surfaces will be formulated in terms of a collective field theory describing the dynamics of the eigenvalues of the matrix field.

Our motivation is not only the simplicity of presentation. The analysis of the SOS string (target space \( \mathbb{Z} \)) showed that its field content is different than the one of the string with continuous target space \( \mathbb{R} \). In particular, there are no “special states” in the SOS string. This is quite a puzzle because the discrete chain of matrices is known to be equivalent to matrix quantum mechanics after adjusting the parameters \([3]\).

Our conjecture \([2]\) was that the SOS string corresponds to a matrix chain with the critical distance between two subsequent points of the discretized line where a Kosterlitz-Thouless transition occurs. In this case the momenta of the special states are multiples of the period in the momentum space. The equivalence of the SOS string to a matrix field theory allows to prove the above conjecture.

2. The triangulated random surface as a large \( N \) matrix field theory

Our matrix model will be a variant of the Weingarten model \([4]\) which has been recently reconsidered in the interesting paper of Dalley \([5]\). It describes triangulated random surfaces embedded in a discrete target space \( X \). For the moment it is sufficient to assume that \( X \) has a structure of a one-dimensional simplicial complex. In other words, \( X \) is an ensemble of points \( x \) and links \( l \). In particular, this can be the \( D \)-dimensional hypercubic lattice considered as a collection of sites and links. However, a nontrivial continuum limit exists only if \( D = 1 \). More generally, \( X \) can be an (extended) Dynkin diagram of A-D-E type \([1]\).

A link having as extremities the points \( x \) and \( x' \) will be denoted by \( <xx'> \). Below we assume that two points are connected by at most one link only to simplify the notations. The target space \( X \) is defined completely by its connectivity matrix

\[
C_{xx'} = (\text{the number of links connecting } x \text{ and } x') \tag{2.1}
\]

We are going to consider only embeddings compatible with the structure of one-dimensional simplicial complex. This means that the triangulated surface \( S \) is embedded so that the images \( x(\sigma) \) and \( x(\sigma') \) of each pair of nearest neighbour points \( \sigma, \sigma' \in S \) either coincide, or are connected by a link in \( X \). Under the assumption that that the space \( X \) does not contain loops of length 3, each triangle \( \triangle_{\sigma\sigma'\sigma''} \) of \( S \) is mapped onto a single point \( x \) or a link \( <xx'> \) of \( X \).
The entities of the matrix model will be $N \times N$ matrices associated with the points and links of the target space $X$. We introduce a complex matrix field variable $A^{xx'}_{jk} = (A^{xx'}_{kj})$ for each link $< xx' >$ of the target space, and a Hermitean matrix variable $\Phi_x$ for each point $x \in X$. The partition function is defined as

$$Z = \int \prod_{<xx'>} dA^{xx'} dA^{x'x'} \prod_x d\Phi(x) e^{-A[\Phi]}$$

(2.2)

$$A[A, \Phi] = N \text{tr} \left[ \frac{1}{2} \sum_{<xx'>} A^{xx'} A^{x'x'} + \frac{1}{2} \sum_x [\Phi(x)]^2 - \frac{1}{3} \kappa_0 \sum_x [\Phi(x)]^3 \right]$$

$$+ \frac{1}{2} \kappa_1 \sum_{<x,x'>} (A^{xx'} A^{x'x'} \Phi(x) + A^{xx'} \Phi(x') A^{x'x'})$$

(2.3)

The free energy $F = \log Z$ of the model is expressed as a sum over all connected triangulated surfaces $S$ embedded in $X$:

$$F = \sum_S N^x \kappa_0^N \kappa_1^N$$

(2.4)

where $\chi$ is the Euler characteristic of the surface, $N_0$ is the number of its triangles mapped onto a single point, and $N_1$ is the number of triangles mapped onto a link.

Performing the gaussian integration over the link variables $A^{xx'}$ we find

$$Z = \int \prod_x d\Phi(x) e^{N \text{tr}(-\frac{1}{4} [\Phi(x)]^2 + \frac{1}{3} [\Phi(x)]^3)}$$

$$\prod_{<xx'>} \det^{-1}[I \otimes I - \frac{\kappa_1}{2}(I \otimes \Phi(x) - \Phi(x') \otimes I)]$$

(2.5)

where by $I$ we denoted the $N \times N$ unit matrix.

Because of the gauge symmetry $\Phi(x) \rightarrow U_x \Phi(x) U_x^{-1}$ the partition function depends only on the radial degrees of freedom, the eigenvalues of the matrix $\Phi(x)$. It is convenient to perform, together with the diagonalization, a linear change of variables

$$\Phi(x)_{ij} = \frac{1}{\kappa_1} \delta_{ij} + a \sum_{k=1}^N U(x)_{ik} U(x)_{jk} z_k(x)$$

(2.6)

where the small positive constant $a$ plays the role of a cutoff. Introducing the potential

$$V(\phi) = -\frac{N}{2} (\frac{1}{\kappa_1} + az)^2 + \frac{N \kappa_0}{3} (\frac{1}{\kappa_1} + az)^3$$

(2.7)

we can write the matrix integral (2.5) in the form

$$Z = \int \prod_{x \in X} \prod_{j=1}^N dz_j^{(x)} e^{-A[z]}$$

(2.8)
where

\[ A[z] = \sum_{x,j} V(z_j^{(x)}) + \sum_{x,x' \in X} \sum_{j,k=1}^N C_{x,x'} \log |z_j^{(x)} + z_k^{(x')}| \]

\[ - \delta_{x,x'} (1 - \delta_{jk}) \log |z_j^{(x)} - z_k^{(x')}| \]

This integral is divergent but can be given meaning as a formal series in $1/N$. The action (2.9) depends on the two parameters $\kappa_0$ and $\kappa_1$ as well as on the cutoff $a$ through the potential $V(z)$.

The $1/N$ expansion can be performed either as a quasiclassical expansion for the collective field - the eigenvalue density of the matrix $\Phi$. We will explain the method avoiding some technical points arising in passing to the continuum limit which can be reconstructed from the text of ref. [2].

The effective dimension $C$ of the target space (= the central charge of the matter fields) is related to the largest eigenvalue $\beta$ of the connectivity matrix (2.1) by

\[ C = 1 - 6 \frac{(g - 1)^2}{g}, \quad \beta = 2 \cos(\pi g), 0 < g < 2 \]

(2.10)

The continuum limit is achieved along a line in the $(\kappa_0, \kappa_1)$-space where one has to choose the branch $0 < g < 1$. At the endpoint of this line the critical singularity changes and is described by the branch $1 \leq g \leq 2$ [2].

The double scaling limit is achieved if $N$ goes to infinity along the trajectory

\[ a^{1+g} N = \kappa \]

(2.11)

where $\kappa$ is the renormalized string interaction constant. In this limit the model can be studied by means of the collective field method of Das and Jevicki [6].

3. The collective field method

In the large $N$ limit it is convenient to replace the integration in the space of the eigenvalues $z_j^{(x)}$ by a functional integration with respect to the spectral density

\[ \rho_x(z) = \sum_{j=1}^N \delta(z - z_j^{(x)}), \quad x \in X \]

(3.1)

The Jacobian for the change of variables

\[ \int \prod_{j=1}^N dz_j^{(x)} = \int D\rho_x J(x)[\rho(x)] \]

(3.2)

can be represented as a functional integral over a Lagrange multiplier field $\alpha_x(z)$, according to the suggestion of G. Parisi developed by Migdal in [7]

\[ J(x)[\rho_x] = \int D\alpha_x \prod_{j=1}^N dz_j^{(x)} \exp \left( \sum_{x \in X} \int dz \alpha_x(z) [-\rho_x(z) + \sum_{j=1}^N \delta(z - z_j^{(x)})] \right) \]

\[ = \int D\alpha_x \exp \left( \sum_{x \in X} \left[ - \int dz \alpha_x(z) \rho_x(z) + N \log \left( \int dz e^{\alpha_x(z)} \right) \right] \right) \]

(3.3)
Now the partition function (2.9) is given by the functional integral

$$Z = \int \mathcal{D}\rho \mathcal{D}\alpha \ e^{-A_{\text{eff}}[\rho, \alpha]} \quad (3.4)$$

with

$$A_{\text{eff}}[\rho, \alpha] = \sum_{x \in X} \int dz \rho_x(z)[-V(z) + \alpha_x(z)] - N \sum_{x \in X} \log \left[ \int dz \alpha_x(z) \right]$$

$$+ \sum_{x, x' \in X} \int dz \rho_x(z) \int dz' \rho_{x'}(z') [C_{xx'} \log |z + z'| - \delta_{x, x'} \log |z - z'|] \quad (3.5)$$

The $1/N$ expansion is the quasiclassical expansion for the field theory with action (3.5). Let us denote by $\bar{\rho}_x(z), \bar{\alpha}_x(z)$ the classical solution minimizing this action. We parametrize the quantum fluctuations around $\bar{\rho}_x(z)$ and $\bar{\alpha}_x(z)$ as

$$\rho_x(z) = \bar{\rho}_x(z) - \frac{d}{dz} \phi_x(z), \quad \alpha_x(z) = \bar{\alpha}_x(z) + i \epsilon_x(z) \quad (3.6)$$

where the function $\phi(z)$ vanishes for $z \to \pm \infty$. Then the normalization condition

$$\int_{-\infty}^{\infty} dz \rho_x(z) = N \quad (3.7)$$

is automatically satisfied. We are then obliged to introduce a gauge condition suppressing the translational zero mode of $\alpha$

$$\int dz \bar{\rho}_x(z) \epsilon_x(z) = 0 \quad (3.8)$$

Taking the first derivative of the action with respect to the fluctuations $\phi$ and $\epsilon$ we find the classical equations

$$\frac{dV(z)}{dz} + \sum_{x' \in X} \int dz' \bar{\rho}_{x'}(z') [2 \delta_{x, x'} \frac{1}{z' - z} - C_{xx'} \frac{1}{z + z'}] = 0 \quad (3.9)$$

and

$$\bar{\alpha}_x(z) = \log \bar{\rho}_x(z) + \text{constant} \quad (3.10)$$

The constant is fixed to be zero by the gauge condition (3.8). The advantage of this choice is that in the large $N$ limit the logarithm in (3.3) can be replaced by the first term in its expansion

$$N \log \left[ 1 - \frac{1}{N} \int dz \bar{\rho}_x(z) \left[ 1 - e^{i \epsilon_x(z)} \right] \right] \to - \int dz \bar{\rho}_x(z) \left[ 1 - e^{i \epsilon_x(z)} \right] \quad (3.11)$$
The new effective action reads

\[ A_0[\phi, \epsilon, \xi] = \sum_{x, x' \in X} \int dz \int dz' \phi_x(z) \left[ \frac{C_{xx'}}{(z + z')^2} + \frac{\delta_{x,x'}}{(z - z')^2} \right] \phi_{x'}(z') \]

\[ + \sum_{x \in X} \int dz \left[ \log \bar{\rho}_x(z) + i \epsilon(x(z)) \right] \frac{d\phi_x(z)}{dz} + i \sum_{x \in X} \xi_x \int dz \bar{\rho}(z) \epsilon(z) \]

\[ - \int dz \bar{\rho}_x(z) e^{i \epsilon_x(z)} \]

where \( \xi_x \) is a Lagrange multiplier for the gauge condition (3.8). In the continuum limit \( a \to 0 \) the theory depends on the renormalized cosmological constant \( \mu \) and the string coupling \( \kappa \) through the dimensionless combination \( \kappa^2 \mu^{1+g} \); in our normalization \( \kappa = 1 \). The dependence on \( \mu \) is through the classical spectral density \( \bar{\rho}_x(z) \) which will be calculated below.

4. Classical solution

We will confine ourselves to the case of target space \( X = \mathbb{Z} \) of the SOS string where

\[ C_{xx'} = \delta_{x,x'+1} + \delta_{x,x'-1} \]  

(4.1)

In certain sense this target space contains all other one-dimensional target spaces.

We will solve eq. (3.9) assuming that the spectral density is supported by the interval \( -\infty < z < -\sqrt{\mu} \). Then \( \bar{\rho}_x(z) \) can be obtained as the discontinuity of the analytic function

\[ \bar{w}_x(s) = \frac{1}{\pi} \int_{-\infty}^{-\sqrt{\mu}} dz \frac{d\bar{\rho}_x(z')}{z - z'} \]

(4.2)

along the cut.

\[ \bar{\rho}_x(z) = \frac{1}{\pi} \text{Im} \, \bar{w}_x(z) \]

(4.3)

The integral equation (3.9) then implies

\[ \frac{dV(z)}{z} + \text{Re} \, \bar{w}_x(z) + \sum_{x'} C_{x,x'} \bar{w}_{x'}(-z) = 0, \quad -\infty < z < -\sqrt{\mu} \]

\[ \text{Im} \, \bar{w}_x(z) = 0, \quad z > -\sqrt{\mu} \]

(4.4)

The potential term \( dV/dz \) contains no singularities in the limit \( a \to 0 \) and therefore can be eliminated by a shift of the \( \bar{w} \). It only determines the position of the cut.

It is convenient to transform the cut \( z \)-plane into a semi-infinite strip \( \{ \text{Re} \tau \geq 0, |\text{Im} \tau| \leq \pi \} \) by a change of variables \[ z = \sqrt{\mu} \cosh \tau. \]

(4.5)
Then the two sides of the cut are parametrized by the boundaries \( \{ \tau \pm i\pi, \tau > 0 \} \) of the strip. Now eq.(4.4) can be written in the form

\[
\bar{w}_x(\tau + i\pi) + \bar{w}_x(\tau - i\pi) + \bar{w}_{x+1}(\tau) + \bar{w}_{x-1}(\tau) = 0 \tag{4.6}
\]

The general solution of this equation is a superposition of plane waves

\[
\bar{w}(\nu)(\tau) = M^\nu \cosh(\nu \tau) e^{ipx}; \quad -1 < p < 1, \quad \nu = p + 2m + 1, m \in \mathbb{Z} \tag{4.7}
\]

One can show \[2, 9\] that the physical solution is

\[
\bar{w}(\tau) = -\frac{d}{dp} \bar{w}_{(1+p)}(\tau)|_{p=0} = -\sqrt{\mu} \cosh \tau + \text{regular terms} \tag{4.8}
\]

\[
\bar{\rho}(\tau) = \sqrt{\mu} \sinh \tau = \sqrt{z^2 - \mu} \tag{4.9}
\]

In the case of a general target space \( X \) one should diagonalize the connectivity matrix

\[
C_{xx'} = \sum_{p \in P} S_{x(p)}^x 2 \cos(\pi p) S_{x'}^{x'} \tag{4.10}
\]

where \( P \) is the momentum space dual to \( X \). Then the classical solution is given by the plane wave with the minimal momentum \( p_0 = |g - 1| \)

\[
\bar{\rho}_x(z) \sim S_{(p_0)}^x(\sqrt{\mu})^g \sinh(g\tau) \tag{4.11}
\]

All other plane wave solutions are down by positive powers of the cutoff \( a \).

5. The kinetic term

The quadratic kernel in the action (3.12) is essentially the inverse propagator in the string field theory. It is given by

\[
[\hat{G}^{-1} \phi]_x(z) = \frac{\delta A_0}{\delta \phi_x(z)} = -\frac{1}{\pi} \int_{-\infty}^{-\sqrt{\mu}} \frac{dz'}{z - z'} \left[ \frac{2\delta_{xx'}}{z + z'} + \frac{C_{xx'}}{z + z'} \right] \phi_{x'}(z') + \log \bar{\rho}_x(z) \tag{5.1}
\]

If we are interested only in the perturbative expansion in \( \kappa \), we can assume that the quantum field \( \phi \) also vanishes for \( z > -\sqrt{\mu} \). In this case we can introduce the analytic fields

\[
\psi(x, \tau) = \frac{1}{\pi} \int_{-\infty}^{-\sqrt{\mu}} \frac{dz'}{z - z'} \phi_x(z'), \quad z = \sqrt{\mu} \cosh \tau \tag{5.2}
\]

related to the field \( \varphi(x, \tau) \equiv \phi_x(-\sqrt{\mu} \cosh \tau) \) by

\[
\varphi(x, \tau) = \sin(\partial_\tau) \psi(x, \tau) \tag{5.3}
\]
It will be convenient to consider the coordinate $x \in \mathbb{Z}$ as a continuum parameter. The inverse propagator (5.1) then is represented by the differential operator

$$\hat{G}^{-1} = (2 \cos(\pi \partial_\tau) - 2 \cosh \partial_x) \frac{\partial_\tau}{\sin(\pi \partial_\tau)}$$

(5.4)

which is diagonalized by plane waves

$$\langle E, p|\tau, x \rangle = \cos(E\tau)e^{i\pi px}, \quad E \geq 0, -1 \leq p < 1$$

(5.5)

In the case of a general target space the kinetic term reads

$$\langle \phi|\hat{G}^{-1}|\phi \rangle = \sum_{p \in P} \int_0^\infty dE \varphi(E, p) \frac{E}{\sinh(\pi E)}[\cosh(\pi E) - \cos(\pi p)]\varphi(E, p)$$

(5.6)

This quadratic form produces exactly the string propagator which we have found in [2] (eq. (4.52)) by means of the loop diagram technique.

Knowing the kinetic term one can easily calculate the correlation function of the loop operators

$$w_x(\ell) = \int_{-\infty}^{-\sqrt{\mu}} dz e^{\ell z} \phi_x(z) = \int_0^\infty d\tau e^{-\sqrt{\mu} \cosh \tau} \frac{\partial}{\partial \tau} \varphi(x, \tau)$$

(5.7)

It can be represented as an integral over Bessel functions

$$\langle w_x(\ell)w_x'(\ell') \rangle =$$

$$= \sum_{p \in P} \int_0^\infty dE \right S_{(p)}^x K_{iE}(\sqrt{\mu}\ell) \frac{E \sinh(\pi E)}{\sinh(\pi E) - \cos(\pi p)} S_{(p)}^{x'} K_{iE}(\sqrt{\mu}\ell')$$

(5.8)

The two-loop correlator in $C = 0, C = 1/2$ and $C = 1$ strings has been considered from the viewpoint of an effective string field theory in refs. [10], [11], [12]. The expression obtained in the momentum space using the formalism of the large-$N$ matrix quantum mechanics was different than the integrand in (5.8). Now this contradiction is resolved: the discrepancy comes from the fact that the two correlators are considered different momentum spaces, a compact and noncompact one [13]. One can easily check that in the $x$-space the loop-loop correlators coincide, using the identity

$$\langle x_1 | 1 \frac{E}{2 \cosh(\pi E) - 2 \cos \partial_x} \frac{E}{\sinh(\pi E)} | x_2 \rangle$$

$$= \langle x_1 | \frac{1}{\pi^2 E^2 + \partial_x^2} | x_2 \rangle$$

$$= \frac{1}{E} e^{-\pi E|x_1 - x_2|}, \quad x_1 - x_2 \in \mathbb{Z}$$

(5.9)

Amusingly, our string propagator restricted to the $x$-space is just the propagator of a two-dimensional Euclidean massless particle.
6. Discussion

The matrix model considered here gives a unified description of all string theories with rational central charge $C$ of the matter fields. The collective field method should work equally well for $C < 1$ and $C = 1$. Up to two loops it yields the same results as those obtained by the loop gas method \[1\], \[2\].

At the moment we do not know an effective method for performing the integration over the Lagrange multiplier field $\epsilon$. It is however evident that this would lead to a local in the target space interaction. This means that topology changing processes thus occur at a single point of the target space. The effective potential reproducing the loop diagram technique of ref.\[3\] (Eqs. (4.51) - (4.53)) should have the form

$$V[\phi] = \sum_{x \in X} V_x (\partial_\tau \phi(x, \tau)|_{\tau=0}, \partial^3_\tau \phi(x, \tau)|_{\tau=0}, ...)$$  \hspace{1cm} (6.1)

We intend to consider the string interactions in a future publication.

Finally, let us comment briefly the relation between our model and the matrix quantum mechanics which is the most studied model of the $C = 1$ string. If we return to the original definition (2.2)-(2.3) of the matrix model and perform the integration over the $\Phi$-matrices (take the simplest case $\kappa_0 = 0$) we will obtain a particular case of the matrix chains considered in ref.\[4\]. It is defined by the transfer matrix

$$K(y, y') = \exp[(y + y')^2]$$  \hspace{1cm} (6.2)

which can be thought of as a singular limit of the inverse oscillator kernel at time interval which is half of the self-dual radius. This explains why the spectrum of the propagator (5.4) does not contain poles at integer momenta corresponding to the so-called “special states”. The half-wave-lengths of the “special states” are multiples of the distance between two subsequent points of the discrete target space $\mathbb{Z}$. Therefore these states are inobservable.

Acknowledgements

The author thanks Mike Douglas, V. Kazakov, A. Migdal and M. Staudacher for discussions, and V. Pasquier for a critical reading of the manuscript.
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