On the theory of the skewon field: From electrodynamics to gravity

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Abstract

The Maxwell equations expressed in terms of the excitation \( \mathcal{H} = (\mathcal{H}, D) \) and the field strength \( F = (E, B) \) are metric-free and require an additional constitutive law in order to represent a complete set of field equations. In vacuum, we call this law the “spacetime relation”. We assume it to be local and linear. Then \( \mathcal{H} = \mathcal{H}(F) \) encompasses 36 permittivity/permeability functions characterizing the electromagnetic properties of the vacuum. These 36 functions can be grouped into \( 20 + 15 + 1 \) functions. Thereof, 20 functions finally yield the dilaton field and the metric of spacetime, 1 function represents the axion field, and 15 functions the (traceless) skewon field \( S^i_j \) (\( S \) slash), with \( i, j = 0, 1, 2, 3 \). The hypothesis of the existence of \( S^i_j \) was proposed by three of us in 2002. In this paper we discuss some of the properties of the skewon field, like its electromagnetic energy density, its possible coupling to Einstein-Cartan gravity, and its corresponding gravitational energy.

Keywords: Classical electrodynamics, skewon field, general relativity, Einstein-Cartan theory, dilaton field, axion field.

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1 Introduction

At the centennial of the proposition of special relativity theory by Einstein (1905), it is worthwhile to remember that Einstein’s paper was “On the electrodynamics of moving bodies,” see [1]. The task Einstein had taken up was to develop a consistent framework for accommodating Maxwell’s theory of electrodynamics as well as classical mechanics. The tool for achieving this was to study the motion of charged bodies under the action of an electromagnetic field. Maxwell’s theory, suitably interpreted, survived, Newton’s mechanics had to be extended if high relative velocities were involved.

Thus, Maxwell’s theory emerged as a prime example of a special relativistic field theory that is intrinsically related to the Poincaré group (also known as inhomogeneous Lorentz group). Accordingly, electrodynamics is conventionally thought to take place in the flat Minkowski space of special relativity as pointed out by Minkowski in his geometrical formulation of special relativity in 1908, see [1].

The success of special relativity was so striking that the historical fact of the close association of electrodynamics with special relativity stuck in the minds of most physicists and is believed to be a physical fact — even though the development of classical electrodynamics during the last 100 years shows the opposite: The foundations of electrodynamics have nothing to do with special relativity and the Poincaré group, they are rather of a generally covariant (“topological”) nature based on the conservation laws of electric charge and magnetic flux.

This development started with Einstein [2] who shortly after the publication of his general relativity theory observed that the Maxwell equations can be formulated in such a way that neither the metric nor the Christoffel symbols appear in them. In his notation (µ, ν, ... = 0, 1, 2, 3), they read

\[
\frac{\partial F_{\mu \sigma}}{\partial x^\tau} + \frac{\partial F_{\sigma \tau}}{\partial x^\rho} + \frac{\partial F_{\tau \rho}}{\partial x^\sigma} = 0, \quad F^{\mu \nu} = \sqrt{-g} g^{\mu \alpha} g^{\nu \beta} F_{\alpha \beta}, \quad \frac{\partial F^{\mu \nu}}{\partial x^\nu} = J^\mu. \tag{1}
\]

Here we draw on our paper [3]. The Maxwell equations (1) and (3) are apparently metric free. Moreover, since the excitation \( \mathcal{F}^{\mu \nu} \) is considered to be a tensor density of type \([2,0]\) and the field strength \( F_{\mu \sigma} \) a tensor of type \([0,2]\), both equations are — even though only partial derivatives operate in them — covariant under general coordinate transformations (diffeomorphisms). In other words, the system consisting of (1) and (3) doesn’t couple to the gravitational potential as long as (2) is not substituted. A similar presentation of Maxwell’s equations was given by Einstein in his “Meaning of Relativity” [4] in the part on general relativity.

At first sight, this separation of the Maxwell equations into the Maxwell equations proper of (1) and (3) and the spacetime relation (2) may appear to look like a formal trick. However, it is well known that the electric excitation \( D \) and the magnetic excitation \( \mathcal{H} \) are both directly measurable quantities, see, e.g., Raith [5]. Accordingly, the electromagnetic field is represented operationally not only by means of the electric and magnetic field strengths \( E \) and \( B \) — both measured via the Lorentz force — but also by means of the electric and magnetic excitations \( D \) and \( \mathcal{H} \).

On the foundations of general relativity, besides the equivalence principle, there lays the principle of general covariance. And the Maxwell equations (1) and (3) are generally covariant and metric independent. Since in general relativity the metric \( g \) is recognized as gravitational potential, it is quite fitting that the fundamental field equations of electromagnetism do not contain the gravitational potential. Consequently, the Maxwell equations in their premetric form are valid in any 4D differential manifold, provided the latter can be split locally into 1+3. Accordingly, they are not only beyond special relativity, but also beyond general relativity.

\(^1\text{Einstein used subscripts for denoting the coordinates } x, \text{ i.e., } x_\tau \text{ etc. Moreover, we dropped twice the summation symbols } \Sigma.\)
The point of view that the fundamental structure of electrodynamics can be best understood because of the existence of conservation laws that can be formulated generally covariant and metric-free has been mainly developed by Kottler (1922), É.Carlan (1923), and van Dantzig (1934), see [6]. Modern presentations of this “premetric electrodynamics” have been given, e.g., by Truesdell-Toupin [7], Post [8], Kovetz [9], Rubilar [10], Hehl & Obukhov [6], Kiehn [11], Delphenich [12], and Lindell2, see also Itin [14, 15].

2 Premetric electrodynamics

2.1 The Maxwell equations

The conservation of electric charge leads to the inhomogeneous Maxwell equation:

$$dI^H = J \quad (\partial_j \mathcal{H}^{ij} = \mathcal{J}^i).$$

(2)

The first version in exterior calculus is written in terms of the twisted excitation 2-form $I^H = I_{ij} \, dx^i \wedge dx^j / 2$ and the twisted current 3-form $J = J_{ijk} \, dx^i \wedge dx^j \wedge dx^k / 6$. The translation into the corresponding version in components is achieved by $H^{ij} := \varepsilon^{ijkl} I^H_{kl} / 2$ and $J^i := \varepsilon^{ijkl} J_{jkl} / 6$, where $\varepsilon^{ijkl}$ is the totally antisymmetric Levi-Civita symbol with components of value $\pm 1, 0$. Magnetic flux conservation is represented by the homogeneous Maxwell equation

Fig.1. *The tetrahedron of the electromagnetic field.* The excitation $I^H = (\mathcal{H}, \mathcal{D})$ and the field strength $F = (E, B)$ are 4-dimensional quantities of spacetime, namely 2-forms with and without twist, respectively. They describe the electromagnetic field completely. Of electric nature are $\mathcal{D}$ and $E$, of magnetic nature $\mathcal{H}$ and $B$. In 3 dimensions, $\mathcal{H}$ and $E$ are twisted and untwisted 1-forms, respectively; analogously, $\mathcal{D}$ and $B$ are twisted and untwisted 2-forms, respectively. The magnetic and the electric excitations $I^H = (\mathcal{H}, \mathcal{D})$ are extensities, also called quantities (how much?), the electric and the magnetic field strengths $E$ and $B$ are intensities, also called forces (how strong?).

2 Lindell’s presentation of electrodynamics in the framework of exterior differential forms is metric independent. However, in order to make himself understood to his engineering public, he often interpretes the differential-form expressions in terms of metric-dependent Gibbsian vector expressions.
\[ dF = 0 \quad (\partial_i F_{jk} = 0), \] (3)

with the field strength 2-form \( F = F_{ij} \, dx^i \wedge dx^j / 2 \). The decompositions into time and space read

\[ IH = -\mathcal{H} \wedge dt + \mathcal{D}, \quad F = E \wedge dt + B, \] (4)

compare the scheme in Fig.1. Conservation laws can be reduced to counting procedures. No distance concept is required in this context, rather only the ability to circumscribe a definite volume or an area. As a consequence, no metric occurs anywhere in (2), (3), and (4).

### 2.2 Local and linear spacetime relation

Excitation and field strength in vacuum (generalization to media is possible) are assumed to be related by a local and linear relation

\[ IH = \kappa [F] \quad (IH_{\alpha\beta} = \frac{1}{2} \kappa_{\alpha\beta} \gamma^\delta F_{\gamma\delta}). \] (5)

The constitutive tensor \( \kappa_{\alpha\beta} \gamma^\delta = -\kappa_{\beta\alpha} \gamma^\delta = -\kappa_{\alpha\beta} \delta^\gamma \) has 36 independent components. These components can be understood in terms of the tensor-valued 2-form

\[ \mathcal{R}^{\alpha\beta} = \frac{1}{2} \kappa_{\gamma\delta} \alpha^\beta \gamma^\delta. \] (6)

Then (5) can be written explicitly as

\[ IH = \frac{1}{2} \mathcal{R}^{\alpha\beta} e_{\beta} \lfloor e_{\alpha} \rfloor F. \] (7)

The constitutive 2-form \( \mathcal{R}^{\alpha\beta} \) can be split, according to 36=20+15+1, into three irreducible pieces,

\[ \mathcal{R}^{\alpha\beta} = (1) \mathcal{R}^{\alpha\beta}_{\text{principal}} + (2) \mathcal{R}^{\alpha\beta}_{\text{skewon}} + (3) \mathcal{R}^{\alpha\beta}_{\text{axion}} \quad (\kappa_{\alpha\beta} \gamma^\delta = \sum_{A=1}^{3} (A) \kappa_{\alpha\beta} \gamma^\delta). \] (8)

A detailed proof is given in the Appendix. In particular,

\[ (2) \mathcal{R}^{\alpha\beta} = \mathcal{R}^{[\alpha \wedge \gamma] \beta}, \quad (3) \mathcal{R}^{\alpha\beta} = \alpha \vartheta^\alpha \wedge \vartheta^\beta. \] (9)

If the spacetime relation can be derived from a Lagrangian, then the skewon piece \( (2) \mathcal{R}^{\alpha\beta} \) has to vanish. On the other hand, \( (2) \mathcal{R}^{\alpha\beta} \) is a permissible structure provided it is related to dissipative processes. This is indeed the case, see [6]. The hypothesis of the existence of \( (2) \mathcal{R}^{\alpha\beta} \) was proposed by three of us [16]. Its effect on the light propagation has been studied in the meantime [17]. The axion piece \( (3) \mathcal{R}^{\alpha\beta} \) had been proposed much earlier in an elementary particle context, see the axion electrodynamics of Wilczek [18] and the literature given there.

In particular for a comparison with the literature [8] it is convenient to introduce the equivalent constitutive tensor density

\[ \chi^{\alpha\beta\gamma\delta} := \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \kappa_{\mu\nu} \gamma^\delta, \quad \chi^{\alpha\beta\gamma\delta} = \sum_{A=1}^{3} (A) \chi^{\alpha\beta\gamma\delta}, \] (10)
with $36=20+15+1$ independent components. Its skewon and its axion pieces can be mapped to a tensor (15 components) and a pseudo-scalar (1 component), respectively,

$$S_\alpha^\beta := \frac{1}{4!} \hat{\epsilon}^{\alpha\gamma\delta} (2) \chi^\gamma{}_{\delta}{}^{\epsilon}, \quad \alpha := \frac{1}{4!} \hat{\epsilon}_{\alpha\beta\gamma\delta} (3) \chi^{\alpha\beta\gamma\delta}. \quad (11)$$

We have $S_\alpha^\alpha = 0$. For the 1-form $S_\alpha := S_\beta^\alpha \theta^\beta$, with $e_\alpha \mid S_\alpha = 0$, we find by some algebra,

$$S_\alpha = -\frac{1}{2} \hat{R}_\alpha^\alpha, \quad \alpha = \frac{1}{12} \hat{K}. \quad (12)$$

Up to here, our considerations were premetric. If we put the physical dimension of $(1)\hat{R}^{\alpha\beta}$ into a function $\lambda(x)$, the so-called dilaton, then we have 1 dilaton field $\lambda$, 19 remaining components of the principal part $(1)\hat{R}^{\alpha\beta}/\lambda$, 15 skewon components $S_\alpha^\beta$, and 1 axion component $\alpha$. This is as far as we can go with the premetric concept. The study of the properties of light propagation can help to constrain the constitutive tensor of spacetime.

The skewon field, a specific kind of permeability/permittivity of spacetime, will be in the center of our interest. It is non-Lagrangian and dissipative, and it diffracts light. Here we want to address the problem of how it could couple to gravity. However, first we want to turn our attention to its electromagnetic energy-momentum.

### 3 The electromagnetic energy-momentum density of the skewon field

The 3-form of the electromagnetic energy-momentum current can be taken from [6], e.g.:

$$\Sigma_\alpha := \frac{1}{2} [F \wedge (e_\alpha \mid IH) - IH \wedge (e_\alpha \mid F)]. \quad (13)$$

We decompose the 3-form according to $\Sigma_\alpha = \Sigma_{klma} dx^k \wedge dx^l \wedge dx^m /6$. Then the corresponding energy-momentum tensor in tensor calculus can be defined as $T_{ij} := e^{ijkl} \Sigma_{klma} /6$ or

$$T_i^j = \frac{1}{4} \delta_i^j F_{kl} \hat{H}^{kl} - F_{ik} \hat{H}^{jk}. \quad (14)$$

Let us now turn to the skewon part. The excitation can be decomposed in principal, skewon, and axial parts according to $IH = (1)IH + (2)IH + (3)IH$. Since $IH$ enters (13) linearly, we find $\Sigma_\alpha = (1)\Sigma_\alpha + (2)\Sigma_\alpha + (3)\Sigma_\alpha$, with

$$\Sigma_\alpha \mid_{\text{skewon}} \equiv (2)\Sigma_\alpha = \frac{1}{2} \left[ F \wedge (e_\alpha \mid (2)IH) - (2)IH \wedge (e_\alpha \mid F) \right] = \frac{1}{2} e_\alpha \mid (F \wedge (2)IH) - (2)IH \wedge e_\alpha \mid F. \quad (15)$$

The skewon part of the excitation was derived in (75) as

$$(2)IH = \frac{1}{2} \hat{R}^{\alpha} \wedge e_\alpha \mid F. \quad (16)$$

We multiply it by $F$, apply the anti-Leibniz rule for the interior product, and recall that $e_\alpha \mid \hat{R}^{\alpha} = 0$:

$$F \wedge (2)IH = \frac{1}{2} F \wedge \hat{R}^{\alpha} \wedge e_\alpha \mid F = -\frac{1}{2} e_\alpha \mid (F \wedge \hat{R}^{\alpha} \wedge F) + \frac{1}{2} (e_\alpha \mid F) \wedge \hat{R}^{\alpha} \wedge F + \frac{1}{2} F \wedge (e_\alpha \mid \hat{R}^{\alpha}) \wedge F = -\frac{1}{2} F \wedge \hat{R}^{\alpha} \wedge e_\alpha \mid F. \quad (17)$$
Thus, $F \wedge (2) IH = 0$. We substitute this into (15) and find
\begin{equation}
\Sigma_\alpha|_{\text{skewon}} = (e_\alpha \rfloor F) \wedge \mathcal{S}^\beta \wedge (e_\beta \rfloor F) \quad \text{(premetric result)}.
\end{equation}

A similar computation can be performed for the energy-momentum tensor. We have ([6], p.256)
\begin{equation}
(2) IH_{ij} = 2 \mathcal{S}_{[i} \mathcal{F}_{j]k} \quad \text{or} \quad (2) \mathcal{H}^{mn} = \epsilon^{mni} \mathcal{S}^k \mathcal{F}_{jk}.
\end{equation}
On substitution of this into (14), we find the skewon part of the energy-momentum tensor as
\begin{equation}
T_i^j|_{\text{skewon}} = \epsilon^{jklm} \mathcal{F}_{ik} \mathcal{S}^l \mathcal{F}_{nm},
\end{equation}
which is clearly equivalent to (18) and also of premetric nature.

### 3.1 Trace
We transvect (15) with the coframe and recall that $F \wedge (2) IH = 0$:
\begin{equation}
\vartheta^\alpha \wedge (2) \Sigma_\alpha = \vartheta^\alpha \wedge (2) IH \wedge e_\alpha \rfloor F = -2 F \wedge (2) IH = 0.
\end{equation}
Thus, the tracelessness is proved:
\begin{equation}
\vartheta^\alpha \wedge (2) \Sigma_\alpha = 0 \quad \text{(premetric result)}.
\end{equation}
Equivalently, $T_i^i = 0$. Thus, the skewonic part of the energy-momentum is tracefree. Since $\vartheta^\alpha \wedge \Sigma_\alpha = 0$, we find the analogous property for the principal part: $\vartheta^\alpha \wedge (1) \Sigma_\alpha = 0$.

### 3.2 Antisymmetric part
For these considerations we need the existence of a metric. We lower the index of the coframe $\vartheta_\alpha := g_{\alpha \beta} \vartheta^\beta$, multiply the energy-momentum from the left, and antisymmetrize:
\begin{equation}
\vartheta_{[\alpha} \wedge (2) \Sigma_{\beta]} = -(2) IH \wedge \vartheta_{[\alpha} \wedge e_{\beta]} \rfloor F = \vartheta_{[\alpha} \wedge (e_{\beta]} \rfloor F) \wedge \mathcal{S}^\gamma \wedge (e_\gamma \rfloor F).
\end{equation}
This obviously does not vanish. In contrast, in conventional Maxwell-Lorentz vacuum electrodynamics, we have, of course, $\vartheta_{[\alpha} \wedge \Sigma_{\beta]} = 0$, that is, a symmetric energy-momentum, see [6].

Alternatively, one can consider the 2-form $W := e_\alpha \rfloor (2) \Sigma_\alpha$ which is proportional to the left hand side of (23), see Itin [19]. Then,
\begin{equation}
W = -(e_\alpha \rfloor (2) IH) \wedge (e_\alpha \rfloor F).
\end{equation}
Since $IH$ and $F$ are independent, $W$ doesn’t vanish in general. For the Maxwell-Lorentz case, $(2) IH$ is zero and, as a consequence, $W$ vanishes.

Because the electromagnetic skewon energy-momentum has an antisymmetric piece, it would contribute to the first field equation of an Einstein-Cartan-Maxwell (with skewon) system. Thus, we have another (rather indirect) non-Lagrangian type of coupling of the skewon field to gravity.

\footnote{If we translate (20) into the energy-momentum 3-form, then, in components, the corresponding formula reads:
\begin{equation}
\Sigma_{ijk\alpha}|_{\text{skewon}} = \frac{3}{4} \left( \kappa^m \mathcal{F}_{j[m} \mathcal{F}_{k]} + \kappa_{[i} \mathcal{F}_{j]k} \mathcal{F}_{0m} + 2 \mathcal{S}_{[i} \mathcal{F}_{j|m} \mathcal{F}_{\alpha[k]} \right).
\end{equation}
3.3 Energy density

The energy density becomes (spatial indices are \(a, b, c, \ldots = 1, 2, 3\))

\[
T^{0}_{0}|_{\text{skewon}} = \epsilon^{0 k l m} F_{0 k} \mathcal{S}^{n}_{l} F_{n m} = \epsilon^{0 a b c} F_{0 a} \mathcal{S}^{n}_{b} F_{n c}
\]

\[
= \epsilon^{0 a b c} \left( F_{0 a} \mathcal{S}^{0}_{b} F_{0 c} + F_{0 a} \mathcal{S}^{d}_{b} F_{d c} \right) .
\] (25)

The first term in the parenthesis vanishes because of its symmetry in \(a\) and \(c\). Thus,

\[
T^{0}_{0}|_{\text{skewon}} = -\epsilon^{0 a b c} F_{0 a} \mathcal{S}^{d}_{b} F_{d c} .
\] (26)

Now we can substitute the electric and the magnetic field strengths:

\[
T^{0}_{0}|_{\text{skewon}} = -E_{a} \mathcal{S}^{d}_{b} \epsilon^{a b c} \epsilon_{c d e} B^{e} = E_{a} \mathcal{S}^{a}_{b} B^{b} - E_{a} \mathcal{S}^{b}_{b} B^{a} .
\]

We collect the terms and find

\[
T^{0}_{0}|_{\text{skewon}} = \left( \mathcal{S}^{c}_{a} - \delta^{c}_{a} \mathcal{S}^{c}_{c} \right) E_{b} B^{a} .
\] (27)

This is an astonishingly simple premetric result. Note that the second invariant of the electromagnetic field \(I_{2} := F \wedge F = -2 d \sigma \wedge B \wedge E\) (see [6], p.126) enters the energy expression inter alia. Recall also that \(\mathcal{S}^{c}_{c} = -\mathcal{S}^{0}_{0}\).

3.4 Specialization: The spatially isotropic skewon field

The spacetime decomposition of the skewon field reads

\[
\mathcal{S}^{i}_{j} = \begin{pmatrix}
-s^{c}_{c} & m^{a}_{b} \\
0 & s^{a}_{b}
\end{pmatrix} .
\] (28)

Nieves & Pal [20] chose (in nuclear matter) a spatially isotropic skewon field according to

\[
s^{b}_{a} = \frac{s}{2} \delta^{b}_{a} , \quad m^{a}_{a} = 0 , \quad n^{a}_{a} = 0 \quad \text{(Nieves & Pal)} .
\] (29)

In order to be able to substitute this into (27), we compute

\[
s^{b}_{a} - \delta^{b}_{a} s^{c}_{c} = \frac{s}{2} \delta^{b}_{a} - \delta^{b}_{a} \frac{3 s}{2} = -s \delta^{b}_{a} .
\] (30)

We substitute into (27) and find

\[
kT^{0}_{0}|_{\text{skewon N&P}} = -s E_{a} B^{a} .
\] (31)

Hence the energy density here is proportional to the second invariant \(I_{2}\). Since Nieves & Pal didn’t compute the energy of their skewon field, we cannot compare (31) with earlier results.

A direct check of (31) starts from the premetric electromagnetic energy density

\[
u = \frac{1}{2} \left( D^{a} E_{a} + H_{a} B^{a} \right) .
\] (32)

For the Nieves & Pal skewon we have ([6], p.262)

\[
D^{a}|_{\text{skewon N&P}} = -s B^{a} , \quad H_{a}|_{\text{skewon N&P}} = -s E_{a} .
\] (33)

Thus,

\[
u|_{\text{skewon N&P}} = -s E_{a} B^{a} , \quad \text{q.e.d.}
\] (34)
4 Einstein-Cartan theory with skewon, dilaton, and axion interaction

In electrodynamics, one can think of the constitutive 2-form \( K_{\alpha\beta} \) either as a field determined by the electromagnetic properties of some fixed distribution of background matter or as a property of spacetime itself. Whereas the standard matter fields and the electromagnetic potential are dynamical variables, \( K \) is a fixed, non-dynamical (or external) field.

One can try to describe this situation by an “effective” Lagrangian formalism. Consider, for instance, the simple Lagrangian, quadratic in the field strengths, \( L' \sim \chi^{ijkl} F_{ij} F_{kl} \). It is clear that here only the piece of \( \chi^{ijkl} \) symmetric under the exchange \((i,j) \leftrightarrow (k,l)\) survives and thus the related field equations do not contain the skewon piece of \( \chi \). In a way, this result might have been expected: the skewon field is known to cause dissipative effects in electrodynamics \[17\] and, consequently, one does not expect to have a simple local Lagrangian description of the complete dynamics.

In trying to extend our understanding of \( K \) to the gravitational sector, we adopt the interpretation of \( K \) as a property of spacetime, and we will study some of its consequences.

4.1 Specialization: principal part with metric and dilaton

Although it is known that one can construct a gravitational theory without a metric, all such models are limited to the vacuum case, see, e.g., \[21\] \[22\] \[23\] \[24\]. It is unclear whether one can construct a viable gravity theory without a metric in the presence of nontrivial matter sources. Accordingly, we will now specialize to the case when the metric field is available as, e.g., in metric-affine gravity (MAG), see \[25\]. Then, the Maxwell-Lorentz electrodynamics yields the principal part of the form:

\[
\begin{align*}
(1) R_{\alpha\beta} &= \lambda \eta_{\alpha\beta} \\
(1) K_{\gamma\delta} &= \lambda \epsilon_{\gamma\delta\mu\nu} \sqrt{-g} g^{\alpha\mu} g^{\beta\nu} 
\end{align*}
\] (35)

Here \( \eta^{\alpha\beta} \) is defined with the help of the Hodge star * for the spacetime metric \( g \), whereas the scalar field \( \lambda(x) \) is the dilaton field that represents a factor of the principal part of \( K \) absorbing its physical dimension. The dilaton comes as a companion of the skewon and the axion even on the premetric level. When \( \lambda = \text{const} \), and the skewon and axion are absent, we recover from (35) the standard Maxwell-Lorentz electrodynamics.

Now, we recall that the Einstein-Cartan theory (see Blagojević \[26\], Gronwald et al. \[27\], and/or Trautman \[28\]) is determined by the Lagrangian 4-form (in units \( \kappa = 8\pi G = 1 \))

\[
V_{EC} = \frac{1}{2} \eta^{\alpha\beta} \wedge R_{\alpha\beta}.
\] (36)

Noticing that the constitutive 2-form (6) with the principal part (35) provides a natural extension of \( \eta^{\alpha\beta} \) by taking into account the electromagnetic companions of the metric, we can propose a generalization of the Einstein-Cartan theory by means of the Lagrangian

\[
V_{gEC} = \frac{1}{2} R^{\alpha\beta} \wedge R_{\alpha\beta} = \frac{1}{2} (1) R^{\alpha\beta} \wedge (6) R_{\alpha\beta} + \frac{1}{2} (2) R^{\alpha\beta} \wedge (2) R_{\alpha\beta} + \frac{1}{2} (3) R^{\alpha\beta} \wedge (3) R_{\alpha\beta}.
\] (37)

Here we substituted the irreducible decomposition of the curvature into 6 pieces \( R_{\alpha\beta} = \sum_{A=1}^{6} (A) R_{\alpha\beta} \), see \[25\]. Since \( (2) R_{\alpha\beta} \) is the so-called paircommutator and \( (5) R_{\alpha\beta} \) corresponds to the antisymmetric piece
of the Ricci tensor, we recognize that in (37) the skewonic part \((2)K^{\alpha\beta}\) couples only to these specific post-Riemannian pieces of the curvature. More generally, the contributions of the skewon and axion are only nontrivial for a Riemann-Cartan geometry with a nonvanishing torsion 2-form \(T^\alpha\).

The torsion itself can be also irreducibly decomposed according to \(T^\alpha = (1)^\alpha + (2)^\alpha + (3)^\alpha\), with the second and the third irreducible torsion pieces defined as usual by

\[
(2)^\alpha = \frac{1}{3} \vartheta^\alpha \wedge T, \quad (3)^\alpha = -\frac{1}{3} \ast (\vartheta^\alpha \wedge P).
\]

The 1-forms of the trace and the axial trace of torsion are introduced by \(T := e_\alpha T^\alpha\) and \(P := \ast(\vartheta^\alpha \wedge T^\alpha)\), respectively. By making use of the first Bianchi identity \(DT^\alpha = R^\beta_\alpha \wedge \vartheta^\beta\), we can rewrite the above Lagrangian (37) into an equivalent form

\[
V_{gEC} = \frac{1}{2} \lambda \eta^\alpha \wedge (\frac{1}{6} R^\alpha) + \frac{1}{2} \vartheta^\alpha \wedge (1)^\alpha + (2)^\alpha - \frac{1}{2} D(\vartheta^\alpha) \wedge (3)^\alpha + d \Psi.
\]

Here the total derivative term contains the 3-form \(\Psi := -\vartheta^\alpha \wedge ((1)^\alpha + (2)^\alpha) + \frac{1}{2} \vartheta^\alpha \wedge (3)^\alpha\). Obviously, the skewon field \(\vartheta^\alpha\) couples to the tensor and the vector pieces of the torsion, the axion field \(\alpha\), however, to the axial torsion (totally antisymmetric torsion).

### 4.2 Generalized gravitational field equations

The general framework for the derivation of the field equations is provided by the Noether-Lagrange machinery developed in the review paper [25], see its Sec. 5.8.1. The gravitational field equations are given by the system of the so-called first and the second field equations of gravity:

\[
DH_\alpha - E_\alpha = \Sigma_\alpha, \quad (40)
\]

\[
DH_{\alpha\beta} + \vartheta_{[\alpha} \wedge H_{\beta]} = \tau_{\alpha\beta}. \quad (41)
\]

The sources arise as the variational derivatives of the material Lagrangian with respect to the coframe and the connection, and they represent the canonical energy-momentum current \(\Sigma_\alpha\) and the spin current \(\tau_{\alpha\beta}\), respectively.

For the Lagrangian (37) of the generalized Einstein-Cartan theory we find straightforwardly the gravitational gauge field momenta

\[
H_\alpha = -\frac{\partial V_{gEC}}{\partial T^\alpha} = 0, \quad H_{\alpha\beta} = -\frac{\partial V_{gEC}}{\partial R^{\alpha\beta}} = -\frac{1}{2} \vec{K}_{\alpha\beta}, \quad (42)
\]

and the gravitational canonical energy 3-form

\[
E_\alpha = \frac{1}{2} (e_\alpha \wedge \vec{K}) \wedge R^{3\gamma}. \quad (43)
\]

Accordingly, the gravitational field equations read

\[
-\frac{1}{2} (e_\alpha \wedge \vec{K}) \wedge R^{3\gamma} = \Sigma_\alpha, \quad (44)
\]

\[
-\frac{1}{2} D \vec{K}_{\alpha\beta} = \tau_{\alpha\beta}. \quad (45)
\]
Explicitly, the first equation has the form
\[-\frac{1}{2} \lambda \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} - S^\beta \wedge R_{\beta\alpha} + (e_\alpha) S^\beta \, \vartheta^\gamma \wedge R_{\beta\gamma} - \alpha \vartheta^\beta \wedge R_{\alpha\beta} = \Sigma_\alpha. \tag{46}\]

Besides the first Einsteinian term (modified by the scalar dilaton coupling à la Brans-Dicke), we now see that the skewon and the axion fields bring into the first equation new terms which all depend on the Riemann-Cartan curvature.

The second field equation determines the spacetime torsion in terms of the spin current of matter and the additional contributions of skewon, axion, and dilaton. Explicitly, we have:
\[-\frac{\lambda}{2} \eta_{\alpha\beta\gamma} \wedge T^\gamma - \frac{1}{2} \eta_{\alpha\beta} \wedge d\lambda + T_{[\alpha} \wedge S_{\beta]} - \vartheta_{[\alpha} \wedge D S_{\beta]} - \frac{1}{2} \vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge d\alpha - \alpha T_{[\alpha} \wedge \vartheta_{\beta]} = \tau_{\alpha\beta}. \tag{47}\]

In principle, one can resolve this algebraic equation with respect to the components of the torsion. In vacuum, when the matter spin vanishes, we find that the torsion is determined exclusively by the metric companion fields: the dilaton, the skewon, and the axion (and their derivatives).

There is a particular exact solution of the field equations in vacuum, which corresponds to a teleparallel geometry. Indeed, for \(\Sigma = \tau = 0\), we see that \(R_{\alpha\beta} = 0\) solves the first field equation, while (47) defines the intrinsic torsion of spacetime in terms of its \(\mathcal{R}\)-structure (dilaton, skewon, and axion).

### 4.3 Simple vacuum solution

Unfortunately, although the second field equation looks rather simple, it is not easy to find the torsion components from it explicitly. Nevertheless, we can illustrate how the theory works in a simple case when the skewon is absent. Then, the terms with \(S\) disappear from the equation, and we find that the vacuum torsion has the following simple form,
\[T^\alpha = (2)T^\alpha + (3)T^\alpha, \tag{48}\]
that is, the tensor piece \(T^\alpha\) of the torsion vanishes.

From (47), we find:
\[T = \frac{3/2}{\lambda^2 + \alpha^2} (\lambda d\lambda + \alpha d\alpha), \tag{49}\]
\[P = \frac{3}{\lambda^2 + \alpha^2} (\alpha d\lambda - \lambda d\alpha). \tag{50}\]

This can be verified if we notice that the following identities hold true in exterior calculus:
\[\eta_{\alpha\beta\gamma} \wedge (2)T^\gamma = -\frac{2}{3} \eta_{\alpha\beta} \wedge T, \quad \eta_{\alpha\beta\gamma} \wedge (2)T^\gamma = \frac{1}{3} \vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge P, \tag{51}\]
\[(2)T_{[\alpha} \wedge \vartheta_{\beta]} = -\frac{1}{3} \vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge T, \quad (3)T_{[\alpha} \wedge \vartheta_{\beta]} = -\frac{1}{6} \eta_{\alpha\beta} \wedge P. \tag{52}\]

We thus conclude that the trace and the axial trace 1-forms of the torsion are determined, in vacuum, by the dilaton and the axion fields. In particular, when the axion is absent, \(\alpha = 0\), we recover a Brans-Dicke type gravity constructed in the Einstein-Cartan framework in \([29, 30, 31]\). In that case the axial trace vanishes, whereas the torsion trace is proportional to the gradient of the dilaton field. Otherwise, for the case of a constant dilaton, \(\lambda = \lambda_0 = \text{const}\), both torsion 1-forms are nontrivial and depend on the axion only.

Now, when we return to the general case, it is straightforward to verify that the nontrivial skewon induces the trace-free irreducible part of torsion, \((1)T^\alpha\), in addition to the trace and the axial trace 1-forms.
### 5 Gravitational energy in the generalized Einstein-Cartan theory

We now wish to study the energy of the generalized Einstein-Cartan theory (37) in the framework of the canonical formalism [26]. Using simple algebra, we can rewrite the gravitational piece of (37) in the form

\[ V_{gEC} = \mathcal{L}_{gEC} d^4x, \]

with

\[ \mathcal{L}_{gEC} = \frac{1}{4} \chi_{ij} R_{ij} - \frac{1}{2} \Gamma_0 \chi_{ij}^0. \]  

Here,

\[ \chi_{ij} = 2 \lambda \sqrt{-g} e^{ij} + \chi_{ij}, \]
\[ \bar{\chi}_{ij} = e_{k\alpha} e_{l\beta} \left( 2 \epsilon_{klm} \right) + \alpha \chi_{ij} \].

The Latin and Greek indices are raised and lowered with the help of the spacetime metric \( g_{ij} \) and the Lorentz metric \( g_{\alpha\beta} = \text{diag}(+1,-1,-1,-1) \), respectively. Primary constraints are similar to those of the standard Einstein-Cartan theory:

\[ \pi_{\alpha}^0 \approx 0, \quad \pi_{\alpha\beta}^0 \approx 0, \quad \phi_{\alpha} \approx 0, \quad \phi_{\alpha\beta} := \pi_{\alpha\beta} - \chi_{\alpha\beta}^0 \approx 0. \]

Since the Lagrangian is linear in the velocities \( \dot{\Gamma} \), the canonical Hamiltonian is given by

\[ \mathcal{H}_c = -\mathcal{L}_{gEC}(\dot{\Gamma} = 0) = \frac{1}{4} \chi_{ij} R_{ij} - \frac{1}{2} \Gamma_0 \chi_{ij}^0 + \partial_a U^a, \]

where \( U^a = \chi_{\alpha\beta}^0 \Gamma_0^\alpha \). Looking at the form of \( \chi_{\alpha\beta}^{ij} \), we see that the canonical Hamiltonian contains not only the standard Einstein-Cartan piece, modified by the presence of dilaton, but also an additional skewon-axion contribution,

\[ \mathcal{H}_{cSA} = -\frac{1}{4} \bar{\chi}_{ij} R_{ij} - \frac{1}{2} \Gamma_0 \bar{\chi}_{ij}^0 + \partial_a U_{SA}^a, \]

with \( U_{SA}^a = \bar{\chi}_{\alpha\beta}^0 \Gamma_0^\alpha \). The total Hamiltonian has the form

\[ \mathcal{H}_T = \mathcal{H}_c + u_0^\alpha \pi_{\alpha}^0 + \frac{1}{2} u_{\alpha\beta}^0 \pi_{\alpha\beta}^0 + u_{\alpha}^a \pi_{\alpha}^a + \frac{1}{2} u_{\alpha\beta}^a \phi_{\alpha\beta}^a. \]

The simple Hamiltonian structure obtained so far is sufficient to derive the canonical expression for the gravitational energy.

In the Hamiltonian formalism, symmetry properties of a dynamical system are described by the canonical generators \( G[\varphi, \pi] \). Since they act on basic dynamical variables via Poisson brackets, they must be differentiable. A local functional \( F[\varphi, \pi] = \int d^3x f(\varphi, \partial \varphi, \pi, \partial \pi) \) has well defined functional derivatives if its variation can be written in the form \( \delta F[\varphi, \pi] = \int d^3x (A_0 \partial \varphi + B_0 \partial \pi) \), where terms of the form \( \delta(\partial \varphi) \) and \( \delta(\partial \pi) \) are absent. If the generator \( G[\varphi, \pi] \) is not differentiable, its form can be improved by adding a suitable surface term, whereupon it becomes differentiable. On shell, these surface terms represent the values of the related conserved charges.

The canonical generator of time translations is defined by the total Hamiltonian:

\[ P_0 = \int d^3x \mathcal{H}_T, \quad \mathcal{H}_T = \mathcal{H}_c + \partial_a U_a. \]
Looking at the skewon-axion piece of $P_0$, we find that its variation has the form

$$\delta P_0^{\text{SA}} = \int d^3 x \delta \tilde{\Omega}_c^{\text{SA}} + N = -\frac{1}{2} \int d^3 x \partial_a \left( \tilde{\chi}_{\alpha\beta}^{ab} \delta \Gamma_{b}^{\alpha\beta} \right) + N,$$

where $N$ denotes well defined, normal (regular) terms. Thus, $P_0^{\text{SA}}$ can be made differentiable by adding a surface term:

$$P_0^{\text{SA}} \rightarrow \tilde{P}_0^{\text{SA}} = P_0^{\text{SA}} + \mathcal{E}^{\text{SA}},$$

$$\mathcal{E}^{\text{SA}} = \frac{1}{2} \int dS_a \left( \tilde{\chi}_{\alpha\beta}^{ab} \Gamma_{b}^{\alpha\beta} \right).$$

In order to ensure the convergence of the surface integral $\mathcal{E}^{\text{SA}}$, we have to adopt suitable asymptotic conditions. For localized gravitational sources (matter fields decrease sufficiently fast at large distances and give no contribution to surface integrals), we can assume that spacetime is asymptotically flat. The related asymptotic conditions for the gravitational variables, when expressed in the standard spherical coordinates, take the simple form:

$$e_i^\alpha = \delta_i^\alpha + \mathcal{O}(1/r), \quad \Gamma_i^{\alpha\beta} = \mathcal{O}(1/r^2).$$

Hence, $\mathcal{E}^{\text{SA}}$ is convergent if

$$\tilde{\chi}_{\alpha\beta}^{ab} \rightarrow \text{const.} \quad \text{for} \quad r \rightarrow \infty.$$  (62)

The surface term $\mathcal{E}^{\text{SA}}$ represents the value of the skewon-axion contribution to the gravitational energy. It is produced by the interaction between the skewon-axion term $\tilde{\chi}$, and the connection $\Gamma$. One should remember that the complete gravitational energy contains also the standard Einstein-Cartan piece, modified by the presence of the dilaton. The adopted asymptotics, extended naturally to the dilaton field by

$$\lambda(x) \rightarrow \text{const.} \quad \text{for} \quad r \rightarrow \infty,$$

ensures the conservation of the gravitational energy.

### 6 Concluding remarks

(1) According to its definition, the skewon field is some kind of permeability/permittivity of spacetime — and this in a premetric setting when the metric has not yet ”condensed”. In this sense, the skewon field is an elementary electromagnetic property of spacetime. As such, it influences light propagation.

(2) The skewon field contributes non-trivially to the electromagnetic energy. In particular, it induces an asymmetric electromagnetic energy-momentum tensor, which can cause specific gravitational effects as a source term in the Einstein-Cartan-Maxwell system (with skewon).

(3) A smooth deformation of the Einstein-Cartan theory has been introduced and studied as a simple dynamical model incorporating gravitational effects of the skewon field. We found the generalized gravitational field equations and were able to determine the contribution of the skewon field to the gravitational energy.

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Appendix. Decomposition of the local and linear constitutive law

We start with a local (in space and in time) and linear constitutive law

$$IH = \kappa [F] .$$ \hfill (63)

The operator \( \kappa \) acts on the electromagnetic field strength \( F \). Because of its linearity, we have for any 2-forms \( \Psi, \Phi \), if \( a, b \) are two real scalar factors,

$$\kappa [a \Psi + b \Phi] = a \kappa [\Psi] + b \kappa [\Phi] .$$ \hfill (64)

We substitute the decomposition of the field strength \( F \) into (63):

$$IH = \kappa \left[ \frac{1}{2} F_{\alpha \beta} \vartheta^\alpha \wedge \vartheta^\beta \right] = \frac{1}{2} \kappa [\vartheta^\alpha \wedge \vartheta^\beta] F_{\alpha \beta} .$$ \hfill (65)

We introduce the constitutive tensor-valued 2-form

$$\mathcal{R}^{\alpha \beta} := \kappa [\vartheta^\alpha \wedge \vartheta^\beta]$$ \hfill (66)

and recall the decomposition \( F_{\alpha \beta} = e_\beta [e_\alpha F] \). Then the constitutive relation can be brought into the compact form

$$IH = \frac{1}{2} \mathcal{R}^{\alpha \beta} e_\beta [e_\alpha F] , \quad \text{with} \quad \mathcal{R}^{\alpha \beta} = -\mathcal{R}^{\beta \alpha} .$$ \hfill (67)

Here the 2-form \( \mathcal{R}^{\alpha \beta} \) decomposes as

$$\mathcal{R}^{\alpha \beta} = \frac{1}{2} \kappa_{\gamma \delta}^{\alpha \beta} \vartheta^\gamma \wedge \vartheta^\delta .$$ \hfill (68)

Contractions yield a 1-form and a 0-form, respectively:

$$\mathcal{R}^{\beta} := e_\alpha [\mathcal{R}^{\alpha \beta}] = \kappa_{\alpha \delta}^{\beta} \vartheta^\delta := \kappa^{\beta \gamma} \vartheta^\gamma , \quad \mathcal{R} := e_\beta [\mathcal{R}^{\beta}] = \kappa^{\beta \gamma} =: \kappa .$$ \hfill (69)

The tracefree part of the 1-form is

$$\mathcal{R}^{\alpha} := \mathcal{R}^{\alpha} - \frac{1}{4} \mathcal{R} \vartheta^\alpha .$$ \hfill (70)

In this way we can decompose the constitutive antisymmetric tensor valued 2-form into its 3 irreducible pieces,

$$\mathcal{R}^{\alpha \beta} = (1) \mathcal{R}^{\alpha \beta} + (2) \mathcal{R}^{\alpha \beta} + (3) \mathcal{R}^{\alpha \beta} ,$$ \hfill (71)

with the skewon and the axion pieces

$$^{(2)} \mathcal{R}^{\alpha \beta} := \mathcal{R}^{\alpha \beta} \vartheta^\gamma \wedge \vartheta^\delta \quad \text{and} \quad ^{(3)} \mathcal{R}^{\alpha \beta} := \frac{1}{12} \mathcal{R} \vartheta^\alpha \wedge \vartheta^\beta .$$ \hfill (72)

The factors can be determined with some trivial algebra. Note the constraints

$$e_\alpha [^{(1)} \mathcal{R}^{\alpha \beta}] = 0 \quad \text{and} \quad e_\alpha [^{(3)} \mathcal{R}^{\alpha}] = 0 .$$ \hfill (73)

The irreducible pieces in (72) can also be written in components. With the help of the generalized Kronecker delta (see [6]), we find

$$^{(2)} \kappa^{\gamma \delta}_{\alpha \beta} = 2 \delta^{[\alpha}_{\gamma} \delta^{\beta]} , \quad ^{(3)} \kappa^{\gamma \delta}_{\alpha \beta} = \delta^{\alpha \beta}_{\gamma \delta} \alpha .$$ \hfill (74)
If we substitute (71) into (67) and observe \( \vartheta^\alpha \wedge (e_\alpha \lrcorner \omega) = p \omega \), where \( \omega \) is a \( p \)-form, then we finally have

\[
I H = (1) I H + (2) I H + (3) I H \\
= \frac{1}{2} \left( (1) \tilde{R}^{\alpha \beta} e_\beta \lrcorner e_\alpha + \tilde{R}^{\alpha \beta} \lrcorner e_\alpha + \frac{1}{6} \tilde{\varrho} \right) F .
\]

(75)

Thus, the \textit{principal} part of the constitutive 2-form \( \tilde{R}^{\alpha \beta} \) is represented by the \( \begin{bmatrix} 2 & 0 \end{bmatrix} \) antisymmetric tensor-valued 2-form \( (1) \tilde{R}^{\alpha \beta} = - (1) \tilde{R}^{\beta \alpha} \), the \textit{skewon} part by the vector-valued 1-form \( \tilde{\varrho} \), and the \textit{axion} part by the pseudoscalar \( \tilde{\varrho} \). The translation into our usual language is made by \( \tilde{\varrho} = - \frac{1}{2} \tilde{\varrho} \) and \( \alpha = \frac{1}{12} \tilde{\varrho} \).

Incidentally, the IB-medium of Lindell [32] is defined by \( (1) \tilde{R}^{\alpha \beta} = 0 \). If additionally \( (2) \tilde{R}^{\alpha \beta} = 0 \) (vanishing skewon field), then only \( (3) \tilde{R}^{\alpha \beta} = \frac{1}{12} \tilde{\varrho} \vartheta^\alpha \wedge \vartheta^\beta \) is left over, the axion field with 1 component, or, in the language of Lindell & Sihvola [33], the perfect electromagnetic conductor (PEMC).

References

[1] H.A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, \textit{The Principle of Relativity}, a collection of original memoirs on the special and general theory of Relativity, translated from the German (Dover: New York, 1952).

[2] A. Einstein, \textit{Eine neue formale Deutung der Maxwellschen Feldgleichungen der Elektrodynamik} (A new formal interpretation of Maxwell’s field equations of electrodynamics), Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin (1916) pp. 184–188; see also \textit{The collected papers of Albert Einstein}. Vol.6, A.J. Kox et al., eds. (1996) pp. 263–269.

[3] F.W. Hehl and Yu.N. Obukhov, \textit{Dimensions and units in electrodynamics}, Gen. Rel. Grav. \textbf{37} (2005) 733–749; arXiv.org/physics/0407022.

[4] A. Einstein, \textit{The Meaning of Relativity}, 5th ed. (Princeton University Press: Princeton, 1955).

[5] W. Raith, ed., \textit{Bergmann-Schaefer, Lehrbuch der Experimentalphysik, Vol.2, Elektromagnetismus}, new 9th ed. (de Gruyter: Berlin, to be published in autumn 2006).

[6] F.W. Hehl and Yu.N. Obukhov, \textit{Foundations of Classical Electrodynamics — Charge, Flux, and Metric}. Birkhäuser, Boston (2003).

[7] C. Truesdell and R.A. Toupin: \textit{The Classical Field Theories}, in: \textit{Handbuch der Physik}, S. Flügge ed., Vol. III/1 (Springer, Berlin 1960) pp. 226–793.

[8] E.J. Post, \textit{Formal Structure of Electromagnetics – General Covariance and Electromagnetics} (North Holland: Amsterdam, 1962, and Dover: Mineola, New York, 1997).

[9] A. Kovetz, \textit{Electromagnetic Theory} (Oxford University Press: Oxford, 2000).

[10] G.F. Rubilar, \textit{Linear pre-metric electrodynamics and deduction of the lightcone}, Ph.D. Thesis (University of Cologne, June 2002); see \textit{Ann. Phys. (Leipzig)} \textbf{11} (2002) 717–782.

[11] R.M. Kiehn, \textit{Non-equilibrium and irreversible electrodynamics}, November 2003 (76 pages), see \texttt{http://www22.pair.com/csdc/car/carhomep.htm}.

[12] D.H. Delphenich, \textit{On the axioms of topological electromagnetism}, \textit{Ann. Phys. (Leipzig)} \textbf{14} (2005) 347–377; updated version of arXiv.org/hep-th/0311256.

[13] I.V. Lindell, \textit{Differential Forms in Electromagnetics}. IEEE Press, Piscataway, NJ, and Wiley-Interscience (2004).

[14] Y. Itin, \textit{Caroll-Field-Jackiw electrodynamics in the pre-metric framework}, Phys. Rev. \textbf{D70} (2004) 025012 (6 pages); \texttt{arXiv.org/hep-th/0403023}.

[15] Y. Itin and F. W. Hehl, \textit{Is the Lorentz signature of the metric of spacetime electromagnetic in origin?}, \textit{Annals Phys.}(NY) \textbf{312} (2004) 60-83; \texttt{arXiv.org/gr-qc/0401016}. 
[16] F.W. Hehl, Yu.N. Obukhov, G.F. Rubilar, On a possible new type of a T odd skewon field linked to electromagnetism. In: Developments in Mathematical and Experimental Physics, A. Macias, F. Uribe, and E. Diaz, eds. Volume A: Cosmology and Gravitation (Kluwer Academic/Plenum Publishers: New York, 2002) pp.241-256; arXiv.org/gr-qc/0203096.

[17] Yu.N. Obukhov and F.W. Hehl, Possible skewon effects on light propagation, Phys. Rev. D70 (2004) 125015 (14 pages); arXiv.org/physics/0409155.

[18] F. Wilczek, Two applications of axion electrodynamics, Phys. Rev. Lett. 58 (1987) 1799–1802.

[19] Y. Itin, Coframe energy-momentum current. Algebraic properties, Gen. Rel. Grav. 34 (2002) 1819–1837; arXiv.org/gr-qc/0111087.

[20] J.F. Nieves and P.B. Pal, The third electromagnetic constant of an isotropic medium, Am. J. Phys. 62 (1994) 207–216.

[21] A.S. Eddington, The Mathematical Theory of Relativity, 2nd ed. (Cambridge University Press, Cambridge, England, 1924).

[22] E. Schrödinger, Space-Time Structure (Cambridge University Press, Cambridge, England, 1960), reprinted with corrections.

[23] D. Catto, M. Francaviglia, and J. Kijowski, A purely affine framework for unified field theories of gravitation, Bull. Acad. Pol. Sci. (Phys. Astron.) 28 (1980) 179–186.

[24] F. Gronwald, U. Muench, A. Macias, and F.W. Hehl, Volume elements of spacetime and a quartet of scalar fields, Phys. Rev. D58 (1998) 084021 (4 pages); more complete in arXiv.org/gr-qc/9712063.

[25] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne’eman: Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance. Phys. Rep. 258 (1995) 1–171.

[26] M. Blagojević, Gravitation and Gauge Symmetries (IOP Publishing: Bristol, 2002).

[27] F. Gronwald and F.W. Hehl, On the gauge aspects of gravity, in: Proc. Int. School of Cosm. & Gravit. 14th Course: Quantum Gravity, held May 1995 in Erice, Italy. Proceedings. Erice, May 1995. P.G. Bergmann et al. (eds.). World Scientific, Singapore (1996) pp. 148–198; arXiv.org/gr-qc/9602013.

[28] A. Trautman, The Einstein-Cartan theory, in: Encyclopedia of Mathematical Physics, J.P. Françoise et al. (eds.). Elsevier, Oxford, 13 pages, to be published (2005), see http://www.fuw.edu.pl/~amt/ect.pdf.

[29] H.T. Nieh, A spontaneously broken conformal gauge theory of gravitation, Phys. Lett. A88 (1982) 388–390.

[30] T. Dereli and R.W. Tucker, Weyl scaling and spinor matter interactions in scalar-tensor theories of gravitation, Phys. Lett. B110 (1982) 206–210.

[31] Yu.N. Obukhov, Conformal invariance and space-time torsion, Phys. Lett. A90 (1982) 13–16.

[32] I.V. Lindell, The class of IB-media, Helsinki Univ. Tech., Electromagnetics Lab. Report 459, 14 pages (June 2005).

[33] I.V. Lindell, A.H. Sihvola, Perfect electromagnetic conductor, J. Electromag. Waves Appl. 19 (2005) 861–869; arXiv.org/physics/0503232.

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