(BI-)COHEN-MACAULAY SIMPLICIAL COMPLEXES AND THEIR ASSOCIATED COHERENT SHEAVES

GUNNAR FLOYSTAD AND JON EIVIND VATNE

Abstract. Via the BGG correspondence a simplicial complex $\Delta$ on $[n]$ is transformed into a complex of coherent sheaves on $\mathbb{P}^{n-1}$. We show that this complex reduces to a coherent sheaf $\mathcal{F}$ exactly when the Alexander dual $\Delta^*$ is Cohen-Macaulay.

We then determine when both $\Delta$ and $\Delta^*$ are Cohen-Macaulay. This corresponds to $\mathcal{F}$ being a locally Cohen-Macaulay sheaf.

Lastly we conjecture for which range of invariants of such $\Delta$'s it must be a cone, and show the existence of such $\Delta$'s which are not cones outside of this range.

Introduction

To a simplicial complex $\Delta$ on the set $[n] = \{1, \ldots, n\}$ is associated a monomial ideal $I_\Delta$ in the exterior algebra $E$ on a vector space of dimension $n$. Lately there has been a renewed interest in the Bernstein-Gelfand-Gelfand (BGG) correspondence which associates to a graded module $M$ over the exterior algebra $E$ a complex of coherent sheaves on the projective space $\mathbb{P}^{n-1}$ (see [7], [1]). In this paper we study simplicial complexes in light of this correspondence. Thus to each simplicial complex $\Delta$ we get associated a complex of coherent sheaves on $\mathbb{P}^{n-1}$. Our basic result is that this complex reduces to a single coherent sheaf $\mathcal{F}$ if and only if the Alexander dual $\Delta^*$ is a Cohen-Macaulay simplicial complex. So we in yet a new way establish the naturality of the concept of a simplicial complex being Cohen-Macaulay in addition to the well established interpretations via the topological realization, and via commutative algebra and Stanley-Reisner rings.

It also opens up the possibility to study simplicial complexes from the point of view of algebraic geometry. A simple fact is that $\Delta$ is a cone if and only if the support of $\mathcal{F}$ is contained in a hyperplane. Now the nicest coherent sheaves on projective space may be said to be vector bundles, or more generally those sheaves which when projected down as far as possible, to a projective space of dimension equal to the dimension of the support of the sheaf, become vector bundles. This is the class of locally Cohen-Macaulay sheaves (of pure dimension). We show that the coherent sheaf $\mathcal{F}$ is a locally Cohen-Macaulay sheaf iff both $\Delta$ and $\Delta^*$ are Cohen-Macaulay simplicial complexes. We call such $\Delta$ bi-Cohen-Macaulay and try to describe this class as well as possible.
In Section 1 we recall basic facts about the BGG-correspondence. In Section 2 we apply this to simplicial complexes and show the basic theorem, that we get a coherent sheaf $F$ via the BGG-correspondence iff $\Delta^*$ is Cohen-Macaulay. We are also able to give a kind of geometric interpretation of the $h$-vector of $\Delta^*$ in terms of the sheaf $F$.

In Section 3 we consider bi-Cohen-Macaulay simplicial complexes $\Delta$. Then the associated sheaf $F$ on $\mathbf{P}^{n-1}$ when projected down to $\mathbf{P}^{s-1}$, where $s - 1$ is the dimension of the support of $F$, becomes one of the sheaves of differentials $\Omega^c_{\mathbf{P}^{s-1}}$. This gives quite restrictive conditions on the face vector of such $\Delta$. It is parametrized by three parameters, namely $n$, $c$ and $s$.

When $c = 0$, $\Delta$ is just the empty simplex. When $c = 1$ a result of Fröberg [11] enables a combinatorial description of such $\Delta$. If $\Delta$ has dimension $d - 1$ equal to 1 it is a tree and in general $\Delta$ is what is called a $(d - 1)$-tree. When $c \geq 2$ a combinatorial description seems less tractable. There is a classical example of Reisner [13] of a triangulation $\Delta$ of the real projective plane which is bi-Cohen-Macaulay if char $k \neq 2$; but neither $\Delta$ nor $\Delta^*$ are Cohen-Macaulay if char $k = 2$. In particular $\Delta$ is not shellable.

Now suppose $F$ projects down to $\Omega^c_{\mathbf{P}^{s-1}}$. A natural question to ask is whether $F$ is degenerate or not (the support contained in a hyperplane or not). This corresponds to $\Delta$ being a cone or not. In the last section, Section 5, we conjecture that there exists a bi-Cohen-Macaulay $\Delta$ which is not a cone if and only if $s \leq n \leq (c + 1)(s - c)$. We prove this conjecture when $c = 1$ and give examples to show the plausibility of this conjecture for any $c$. Also in the full range we prove the existence of bi-Cohen-Macaulay $\Delta$ which are not cones.

As further motivation for the significance of bi-Cohen-Macaulay simplicial complexes, we refer to the paper [9]. There a natural algebraically defined family of simplicial complexes is defined which depends on the parameters $n, d, c$, and an integer $a \geq 0$. It contains Alexander duals of Steiner systems, when $a = n - d - 1$, cyclic polytopes, when $a = 1$ and $d = 2c$, and bi-Cohen-Macaulay simplicial complexes, when $a = 0$.

This paper started out partly from an observation that the Tate resolution (see Section 1) of the famous Horrocks-Mumford bundle on $\mathbf{P}^4$ contains a $2 \times 5$ matrix of exterior (quadratic) monomials. This paper may be considered as studying (complexes of) coherent sheaves on $\mathbf{P}^{n-1}$ whose Tate resolution involves a $1 \times N$ matrix of exterior monomials (or equivalently a resolution of a monomial ideal in the exterior algebra). In our investigations we have repeatedly had the benefit of computing resolutions over the exterior algebra using Macaulay2 [12], and we express our appreciation of this program.

1. The BGG correspondence

We start by recalling some facts about the BGG correspondence originating from [4]. Our main reference is [7].
Tate resolutions. Let $V$ be a finite dimensional vector space of dimension $n$ over a field $k$. Let $E(V) = \bigoplus \wedge^i V$ be the exterior algebra and for short denote it by $E$. Given a graded (left) $E$-module $M = \bigoplus M_i$ we can take a minimal projective resolution of $M$

$$P : \cdots \to P^{-2} \to P^{-1} \to M$$

where

$$P^{-p} = \bigoplus_{a \in \mathbb{Z}} E(a) \otimes_k \tilde{V}_{-a}^p.$$ 

Now the canonical module $\omega_E$, which is $\text{Hom}_k(E, k)$, is the injective envelope of $k$. Hence we can take a minimal injective resolution

$$I : M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

where

$$I^p = \bigoplus_{a \in \mathbb{Z}} \omega_E(a) \otimes_k V_{-a}^p.$$ 

(For $-p < 0$ we put $V_{-a}^p = \tilde{V}_{-a-n}^p$.) By fixing an isomorphism $k \rightarrow \wedge^n V^*$ where $V^*$ is the dual space of $V$, we get an isomorphism of $E$ and $\omega_E(-n)$ as left $E$-modules, where we have given $V$ degree 1 and $V^*$ degree $-1$.

We can then join together $P$ and $I$ into an unbounded acyclic complex $T(M)$, called the Tate resolution of $M$

$$\cdots \to \bigoplus_a \omega_E(a) \otimes_k \tilde{V}_{-a}^p \xrightarrow{d_p} \bigoplus_a \omega_E(a) \otimes_k \tilde{V}_{-a}^{p+1} \to \cdots$$

such that $M$ is $\text{ker} d^0$ and also $\text{im} d^{-1}$. (One should use $\omega_E$ instead of $E$ in this complex since $\omega_E$ is the natural thing to use in the framework of Koszul duality and hence in the BGG correspondence, see [2].)

BGG correspondence. The terms $T^i$ have natural algebraic geometric interpretations via the BGG correspondence. Let $V$ have a basis $\{e_a\}$ and let $W = V^*$ be the dual space of $V$ with dual basis $\{x_a\}$. Let $S = S(W)$ be the symmetric algebra on $W$. To $M$ we then associate a complex of free $S$-modules

$$L(M) : \cdots \to S(i) \otimes_k M_i \xrightarrow{\delta^i} S(i + 1) \otimes_k M_{i+1} \to \cdots$$

where

$$\delta^i(s \otimes m) = \sum_a sx_a \otimes e_a m.$$ 

If we sheafify $L(M)$ we get a complex of coherent sheaves on the projective space $P(W)$

$$\tilde{L}(M) : \cdots \to \mathcal{O}_{P(W)}(i) \otimes_k M_i \rightarrow \mathcal{O}_{P(W)}(i + 1) \otimes_k M_{i+1} \rightarrow \cdots.$$ 

This, in short, is the BGG correspondence between finitely generated graded (left) $E$-modules and complexes of coherent sheaves on $P(W)$.

Suppose $\tilde{L}(M)$ has only one non-vanishing cohomology group; a coherent sheaf $\mathcal{F}$. Then the terms of the Tate resolution $T(M)$ give the cohomology
groups $H^i(\mathbb{P}(W), \mathcal{F}(a))$ of $\mathcal{F}$ (for short $H^i \mathcal{F}(a)$). More precisely, if $\mathcal{F}$ is $H^0 \tilde{L}(M)$ then

$$T^p(M) = \bigoplus_i \omega_E(p - i) \otimes_k H^i \mathcal{F}(p - i).$$

(1)

Since for a coherent sheaf $\mathcal{F}$ the cohomology $H^i \mathcal{F}(a)$ vanishes for $a \gg 0$ when $i > 0$, we see that for large $p$

$$T^p(M) = \omega_E(p) \otimes_k H^0 \mathcal{F}(p).$$

(2)

Conversely, if $M$ is such that $T^p(M)$ is equal to $\omega_E(p) \otimes_k V^p_p$ for large $p$, then the only non-zero cohomology of $\tilde{L}(M)$ is in degree 0 and so $\tilde{L}(M)$ gives us a coherent sheaf $\mathcal{F}$.

In general all the $\tilde{L}(\ker d^p)[-p]$ for $p$ in $\mathbb{Z}$ have the same cohomology, where $[-p]$ denotes the complex shifted $p$ steps to the left. Hence $\mathcal{F}$ is equal to $H^{-p}(\tilde{L}(\ker d^p))$ for all $p$. Therefore if we find that the only non-vanishing cohomology group of $\tilde{L}(M)$ is $\mathcal{F}$ in degree $-p$, we shall think of $M$ as $\ker d^p$ in $T$. Then (1) and (2) still hold.

**Remark 1.1.** The BGG correspondence induces an equivalence of triangulated categories between the stable module category of finitely generated graded modules over $E$ and the bounded derived category of coherent sheaves on $\mathbb{P}(W)$

$$E - \text{mod} \tilde{L} \simeq D^b(\text{coh/} \mathbb{P}(W)).$$

Due to this remark we may also start with a coherent sheaf $\mathcal{F}$, and there will be a module $M$ over $E$ such that $\tilde{L}(M)$ only has non-zero cohomology in degree 0, equal to $\mathcal{F}$. Forming the Tate resolution $T(M)$ we also denote it by $T(\mathcal{F})$ and say it is the Tate resolution of $\mathcal{F}$.

**Duals.** Consider $\bigwedge^n W$ as a module situated in degree $-n$ and let $M^\vee$ be $\text{Hom}_k(M, \bigwedge^n W)$. Since $\tilde{L}(\bigwedge^n W)$ naturally identifies with the canonical sheaf $\omega_{\mathbb{P}(W)}$ on $\mathbb{P}(W)$ shifted $n$ places to the left, we see that

$$\tilde{L}(M^\vee) = \text{Hom}_k(\tilde{L}(M), \omega_{\mathbb{P}(W)})[n].$$

Hence if $\tilde{L}(M)$ has only one nonvanishing cohomology group $\mathcal{F}$ in cohomological degree $p$, then

$$\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbb{P}(W)}) = H^{i-p-n} \tilde{L}(M^\vee).$$

(3)

Since $\omega_E$ naturally identifies with $\text{Hom}_k(\omega_E, \bigwedge^n W)$ we also get that the Tate resolution of $M^\vee$ is the dual $\text{Hom}_k(T(M), \bigwedge^n W)$ of the Tate resolution of $M$. 
Projections. Given a subspace \( U \subseteq W \) we get a projection \( \pi : \mathbf{P}(W) \rightarrow \mathbf{P}(U) \). If the support of the coherent sheaf \( \mathcal{F} \) does not intersect the center of projection \( \mathbf{P}(W/U) \subseteq \mathbf{P}(W) \) we get a coherent sheaf \( \pi_* \mathcal{F} \) on \( \mathbf{P}(U) \). How is the Tate resolution of \( \pi_* \mathcal{F} \) related to that of \( \mathcal{F} \)? Via the epimorphism \( E \rightarrow E(U^*) \) the latter becomes an \( E \)-module. It then turns out that the Tate resolution \( T(\pi_* \mathcal{F}) = \text{Hom}_E(E(U^*), T(\mathcal{F})) \).

Note that \( \text{Hom}_E(E(U^*), \omega_E) = \omega_{E(U^*)} \).

Hence
\[
T(\pi_* \mathcal{F}) : \cdots \rightarrow \bigoplus_i \omega_{E(U^*)}(p-i) \otimes_k H^i \mathcal{F}(p-i) \rightarrow \cdots .
\]

In particular we see that the cohomology groups \( H^i \pi_* \mathcal{F}(p-i) \) and \( H^i \mathcal{F}(p-i) \) are equal.

Linear subspaces. If \( U \rightarrow W \) is a surjection, we get an inclusion of linear subspaces \( i : \mathbf{P}(W) \hookrightarrow \mathbf{P}(U) \). Then by [14, 1.4 (21)] the Tate resolution of \( i_* \mathcal{F} \) is
\[
\text{Hom}_E(E(U^*), T(\mathcal{F})).
\]

2. Simplicial complexes giving coherent sheaves.

The BGG-correspondence applied to simplicial complexes. Let \( \Delta \) be a simplicial complex on the set \( [n] = \{1, \ldots, n\} \). Then we get a monomial ideal \( I_\Delta \) in \( E \) which is generated by the monomials \( e_{i_1} \cdots e_{i_r} \) such that \( \{i_1, \ldots, i_r\} \) is not in \( \Delta \). Dualizing the inclusion \( I_\Delta \subseteq E(V) \) we get an exact sequence
\[
0 \rightarrow C_\Delta \rightarrow E(W) \rightarrow (I_\Delta)^* \rightarrow 0.
\]

Note that \( E(W) \) is a coalgebra and that \( C_\Delta \) is the subcoalgebra generated by all \( x_{i_1} \cdots x_{i_r} \) such that \( \{i_1, \ldots, i_r\} \) is in \( \Delta \).

Now think of \( \omega_E = E(W) \) as a left \( E(V) \)-module; then \( C_\Delta \) is a submodule of \( \omega_E \). Then we can use the BGG correspondence. A natural question to ask is: When does \( L(C_\Delta) \) have only one non-vanishing cohomology group, a coherent sheaf \( \mathcal{F} \)? It turns out that this happens exactly when the Alexander dual simplicial complex \( \Delta^* \) is Cohen-Macaulay. Let us recall this and some other notions.

A simplicial complex \( \Delta \) is \textit{Cohen-Macaulay} if its Stanley-Reisner ring \( k[\Delta] \) is a Cohen-Macaulay ring. For more on this see Stanley’s book [14].

The \textit{Alexander dual} \( \Delta^* \) of \( \Delta \) consists of subsets \( F \) of \( [n] \) such that \( [n] - F \) is not a face of \( \Delta \). Via the isomorphism \( \omega_E \cong E(n) \), the submodule \( C_{\Delta^*} \) corresponds to the ideal \( I_\Delta \) in \( E \). So we get from [14] an exact sequence
\[
0 \rightarrow C_\Delta \rightarrow \omega_E \rightarrow (C_{\Delta^*})^\vee \rightarrow 0.
\]

Dualizing this we get
\[
0 \rightarrow C_{\Delta^*} \rightarrow \omega_E \rightarrow (C_\Delta)^\vee \rightarrow 0.
\]
Main theorem. A coherent sheaf $\mathcal{F}$ on a projective space is locally Cohen-Macaulay of pure dimension $n$ if for all the localizations $\mathcal{F}_P$ we have depth $\mathcal{F}_P = \dim \mathcal{F}_P = n$. This is equivalent to all intermediate cohomology groups $H^i(\mathcal{F}(p))$ vanishing for $0 < i < n$ when $p$ is large positive or negative. It is also equivalent to $\mathcal{F}$ projecting down to a vector bundle on $\mathbb{P}^n$.

Let $c$ be the largest integer such that all $(c-1)$-simplexes of $[n]$ are contained in $\Delta$.

Theorem 2.1. a) The complex $\bar{L}(C_\Delta)$ has at most one non-vanishing cohomology group, a coherent sheaf $\mathcal{F}$, if and only if $\Delta^*$ is Cohen-Macaulay. In this case $\mathcal{F}$ is $H^{-c} \bar{L}(C_\Delta)$.
b) $\mathcal{F}$ is locally Cohen-Macaulay of pure dimension if and only if both $\Delta$ and $\Delta^*$ are Cohen-Macaulay.
c) The support of $\mathcal{F}$ is contained in a hyperplane if and only if $\Delta$ (or equivalently $\Delta^*$) is a cone.

Proof. By [6] $\Delta^*$ is Cohen-Macaulay if and only if the associated ideal of $\Delta$ in the symmetric algebra has a linear resolution. By [1, Cor.2.2.2] this happens exactly when $I_\Delta$ has a linear resolution over the exterior algebra. Now note that since $I_\Delta$ in $E$ is generated by exterior monomials, in any case a resolution will have terms

$$I_\Delta \leftarrow \oplus_{a \geq c+1} E(-a) \otimes_k \bar{V}_a^1 \leftarrow \oplus_{a \geq c+2} E(-a) \otimes_k \bar{V}_a^2 \leftarrow \cdots$$

with all $\bar{V}_{c+i}^a$ non-zero. But then the injective resolution of the vector space dual $(I_\Delta)^*$ will have "pure" terms $\omega_E(a) \otimes_k \bar{V}_{a-a_0}^{a+a_0}$ for $a \gg 0$, meaning $\bar{L}(C_\Delta)$ is a coherent sheaf, if and only if $I_\Delta$ has a linear resolution from the very start and this then happens exactly when $\Delta^*$ is Cohen-Macaulay.

The fact that $\mathcal{F}$ is locally Cohen-Macaulay means that the terms in the Tate resolution are $\omega_E(a) \otimes_k \bar{V}_{a+a_0}^{e_0+a_0}$ for $a \gg 0$ and similarly for $a \ll 0$.

Now by the dual sequences [5] and [6], the dual of the Tate resolution of $C_\Delta$ is the Tate resolution of $C_{\Delta^*}$. Thus we get that the condition just stated for the Tate resolution of $\mathcal{F}$ must mean that both $\Delta$ and $\Delta^*$ are Cohen-Macaulay.

Suppose now the support of $\mathcal{F}$ is contained in the hyperplane $\mathbb{P}(U) \hookrightarrow \mathbb{P}(W)$ corresponding to a surjection $W \to U$, where the kernel is generated by a form $w$ in $W$ defining the hyperplane. Considering $\mathcal{F}$ as a sheaf on $\mathbb{P}(U)$ denote it by $\mathcal{F}'$. Then the Tate resolutions are related by

$$T(\mathcal{F}) = \text{Hom}_{E(U')} (E, T(\mathcal{F}')).$$

Hence the component of $T(\mathcal{F}')$ in degree $c$ is $\omega_{E(U')}$. Let the image in $T(\mathcal{F}')^c$ of the differential be $C'$. Then $C_\Delta$ is the image of

$$\text{Hom}_{E(U')} (E, C') \hookrightarrow \text{Hom}_{E(U')} (E, \omega_{E(U)}) = E(W)$$

and this is again the sum $C' + wC'$. Since $C_\Delta$ is homogeneous for the multigrading, we see that $C'$ must also be, and then also $w$, so $w = x_i$ for
some $i$. Then we see that $\Delta$ is a cone over the vertex $i$. Since the argument is clearly reversible, we get c). □

Definition 2.2. If $\Delta^*$ is Cohen-Macaulay we denote the corresponding coherent sheaf by $S(\Delta)$.

When $\Delta$ and $\Delta^*$ are both Cohen-Macaulay we say that $\Delta$ is bi-Cohen-Macaulay.

Remark 2.3. The complex $L(C_\Delta)$ is the cellular complex we get from $\Delta$ by attaching the monomial $x_i$ to the vertex $i$. See [3].

Proposition 2.4. When $\Delta^*$ is CM the complex $\tilde{L}((C_\Delta^*)^\vee)[-c-1] : \mathcal{O}_{P(W)}(-c-1)^{\tilde{I}}_\Delta \leftarrow \cdots \leftarrow \mathcal{O}_{P(W)}(-n)$ is a resolution of $S(\Delta)$.

Proof. The exact sequence (5) gives an exact sequence of complexes

$$0 \to \tilde{L}(C_\Delta) \to \tilde{L}(\omega_E) \to \tilde{L}((C_\Delta^*)^\vee) \to 0$$

from which this follows by the long exact cohomology sequence. □

Numerical invariants. For a simplicial complex $\Delta$ on $n$ vertices, let $f_i$ be the number of $i$-dimensional simplices. The $f$-polynomial of $\Delta$ is

$$f_\Delta(t) = 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d$$

where $d - 1$ is the dimension of $\Delta$.

If we form the cone $C_\Delta$ of $\Delta$ over a new vertex, then the $f$-polynomial of $C_\Delta$ is

$$f_{C_\Delta}(t) = (1 + t) f_\Delta(t).$$

The $f$-polynomial of the Alexander dual $\Delta^*$ is related to $f$ by

$$f_i^* + f_{n-i-2} = \binom{n}{i+1}.$$

Note that the invariants $c^*$ and $d^*$ of $\Delta^*$ are related to those of $\Delta$ by

$$c^* + d + 1 = n, \quad c + d^* + 1 = n.$$

Proposition 2.5. Suppose $\Delta^*$ is Cohen-Macaulay. The Hilbert series of $S(\Delta)$ is given by

$$\sum_k h^0(S(\Delta)(k)) t^k = (-1)^{c+1} + (-1)^c f_\Delta(-t)/(1-t)^n.$$

If $f_\Delta$ is $(1 + t)^{n-s} f$ where $f(1)$ is non-zero, then the support of $S(\Delta)$ has dimension $s - 1$. 
Proof. The sheaf $S(\Delta)$ is the cohomology of the complex
$$\mathcal{O}_{P(W)}(-d)^{f_d} \rightarrow \cdots \rightarrow \mathcal{O}_{P(W)}(-c)^{f_c} \rightarrow \cdots \mathcal{O}_{P(W)}$$
at the term $\mathcal{O}_{P(W)}(-c)^{f_c}$. Since the Hilbert series of $\mathcal{O}_{P(W)}(-a)$ is $t^a/(1-t)^n$ we get the proposition by breaking the complex into short exact sequences and running sheaf cohomology on twists of these.

The statement about the dimension of $S(\Delta)$ follows by writing $f$ as a polynomial in $(1+t)$. □

There is also another equivalent set of numerical invariants of $\Delta$. They are related to the $f_i$’s by the following polynomial equation
\begin{equation}
t^d + f_0 t^{d-1} + \cdots + f_{d-1} = (1+t)^d + h_1(1+t)^{d-1} + \cdots + h_d.
\end{equation}
When $\Delta$ is Cohen-Macaulay all the $h_i \geq 0$. [14, II.3].

There is no geometric interpretation of the $h_i$’s in terms of the topological realization of $\Delta$. However the following gives a kind of geometric interpretation of the $h_i$’s for a CM simplicial complex $\Delta^*$ in terms of the sheaf $S(\Delta)$.

**Proposition 2.6.** If $\Delta^*$ is CM then in the Grothendieck group of sheaves on $P(W)$
\begin{equation}
[S(\Delta)(c+1)] = h^*_d [\mathcal{O}_{P_{n-1}}] + h^*_d [\mathcal{O}_{P_{n-2}}] + \cdots + h^*_0 [\mathcal{O}_c]
\end{equation}
More concretely, $S_0 = S(\Delta)(c+1)$ has rank $h^*_d$ and $S_0$ is generated by its sections. Take a general map
$$\mathcal{O}_{P_{n-1}} h^*_d \rightarrow S_0$$
and let $S_1$ be the projection to $P^{n-2}$ of its cokernel. It has rank $h^*_{d-1}$ and is generated by its sections. In this way we continue.

Proof. By Proposition 2.4 there is a resolution
$$S(\Delta) \leftarrow \mathcal{O}_{P(W)}(-c-1)^{f_d} \leftarrow \cdots \leftarrow \mathcal{O}_{P(W)}(-n)$$
so the Hilbert series of $S_0 = S(\Delta)(c+1)$ is
$$\sum_{i=0}^{d^*} (-t)^i f^*_{d^* - i} / (1-t)^n = \sum_{i=0}^{d^*} h^*_{d^* - i} / (1-t)^{n-i}.$$ 
This gives the statement about the class in the Grothendieck group and so the rank of $S_0$ is $h^*_d$. Also note by the Tate resolution of $S(\Delta)$ that $S_0$ is 0-regular as a coherent sheaf. Consider now the sequence
$$\mathcal{O}_{P_{n-1}} h^*_d \rightarrow S_0 \rightarrow \mathcal{T}_1$$
where the first is a general map and $\mathcal{T}_1$ is the cokernel. Since $S_0$ is 0-regular, $\mathcal{T}_1$ will also be. Also the Hilbert series of $\mathcal{T}_1$ is
\begin{equation}
\sum_{i=1}^{d^*} h^*_{d^* - i} / (1-t)^{n-i}.
\end{equation}
Hence letting $S_1$ be the projection of $T_1$ by a general projection to $\mathbb{P}^{n-2}$, then since $S_1$ and $T_1$ have the same cohomology, $S_1$ is 0-regular with Hilbert series (9). In this way we may continue. □

**Remark 2.7.** We thus see that with larger and larger $c$ we are situated in a smaller and smaller part of the Grothendieck group.

### 3. Bi-Cohen-Macaulay simplicial complexes

**Numerical invariants.** The basic types of bi-Cohen-Macaulay simplicial complexes turn out to be the skeletons of simplices of various dimensions. So let

$$f_{s,c}(t) = 1 + st + \binom{s}{2}t^2 + \cdots + \binom{s}{c}t^c$$

be the $f$-polynomial of the $(c-1)$-dimensional skeleton of the $(s-1)$-simplex.

**Proposition 3.1.** If $\Delta$ is bi-CM then

$$f_\Delta(t) = (1 + t)^{n-s}f_{s,c}(t)$$

for some $s$. We then say that $\Delta$ is of type $(n, c, s)$.

**Proof.** By (3) we have that $\text{Ext}^i(S(\Delta), \omega_{\mathbb{P}(W)})$ is $H^{i+c-n}\tilde{L}((C_\Delta)^\vee)$ and by the sequence (5) this identifies with $H^{i+c+1-n}\tilde{L}(C_{\Delta^*})$.

Thus when $\Delta$ is bi-CM and so $S(\Delta)$ is locally Cohen-Macaulay of dimension $s-1$, then

$$\text{Ext}^{n-s}(S(\Delta), \omega_{\mathbb{P}(W)}) = H^{c+1-s}\tilde{L}(C_{\Delta^*})$$

and the other $\text{Ext}$-sheaves vanish. Thus $c+1-s = -c^*$ and since $c^*+d+1 = n$ we get $d = n - s + c$.

Now if for a polynomial $f$ we let $c \geq 1$ be the largest integer for which

$$f(t) = 1 + a + \binom{a}{2}t^2 + \cdots + \binom{a}{c}t^c + \ldots$$

then it is easily seen that $f(t)$ and $(1 + t)f(t)$ have the same invariant $c$. Applying this to $f_\Delta(t) = (1 + t)^{n-s}f(t)$ we see that for the polynomial $f$ the degree must be equal to the invariant $c$ and so $f(t) = f_{s,c}(t)$. □

**Remark 3.2.** This can of course also rather easily be proven in other ways. For instance using the Stanley-Reisner ring $k[\Delta]$. Then $\Delta$ is bi-CM iff $k[\Delta]$ is CM and has a linear resolution by [6]. By Ex.4.1.17 of [5] it is a simple matter to check that the $f$-polynomial has the above form.

It can also be deduced numerically by appealing only to the fact that the $h$-vectors of $\Delta$ and $\Delta^*$ are both non-negative.

**Remark 3.3.** If $\Delta$ is Cohen-Macaulay, the terms of the $h$-vector are all non-negative. If $\Delta$ is bi-CM of type $(n, c, s)$, the terms $h_{c+1} = h_{c+2} = \cdots = 0$. 
So the bi-CM simplicial complexes are in a way numerically extremal in the class of Cohen-Macaulay complexes.

**Algebraic geometric description of bi-CM simplicial complexes.**

Let $\Omega^n_c$ be the sheaf of $c$-differentials on $\mathbb{P}^{n-1}$.

**Proposition 3.4.** a) Let $\Delta$ be the $(c-1)$-skeleton of a simplex on $n$ vertices. Then $\Delta$ is bi-CM with $\mathcal{S}(\Delta) = \Omega^n_c$.

b) When $\Delta$ is bi-CM of type $(n, c, s)$ then $\pi_* \mathcal{S}(\Delta) = \Omega^s_c$ where $\pi$ is a projection $\mathbb{P}^{n-1} \to \mathbb{P}^s$ whose center is disjoint from the support of $\mathcal{S}(\Delta)$.

**Proof.** a) When $\Delta$ is the $(c-1)$-skeleton of a simplex on $n$ elements then $\tilde{L}(C^\Delta) = \Omega^n_c$ is the truncated Koszul complex

$$
\mathcal{O}_{\mathbb{P}^{n-1}}(-c) \otimes_k W \to \cdots \to \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \otimes_k W \to \mathcal{O}_{\mathbb{P}^{n-1}}.
$$

The only cohomology is the kernel of the first map which is $\Omega^c_c$. Since this is a vector bundle, $\Delta$ is bi-CM.

We now prove b). The Tate resolution of $\mathcal{S}(\Delta)$ is

$$
\cdots \to \omega_{E(c+1)} \otimes_k H^0(\mathcal{S}(\Delta)(c+1)) \to \omega_{E(c+2)} \otimes_k H^0(\mathcal{S}(\Delta)(c+2)) \to \cdots.
$$

For a general subspace $U \subseteq W$ of dimension $s$ the projection $\pi_* \mathcal{S}(\Delta)$ on $\mathbb{P}^s = \mathbb{P}(U)$ has (minimal) Tate resolution

$$
T(\mathcal{S}(\Delta)) = \text{Hom}_{E(V)}(E(U^*), T(\mathcal{S}(\Delta))).
$$

Now $\pi_* \mathcal{S}(\Delta)$ and $\mathcal{S}(\Delta)$ have the same Hilbert series and by Proposition 2.5 this is the same as the Hilbert series of $\Omega^s_c$. When we twist the latter with $c+1$ its global sections are $\wedge^{c+1} U$.

Now note that if a map

$$
\omega_{E(U^*)} \to \omega_{E(U^*)}(c+1) \otimes_k \wedge^{c+1} U
$$

is surjective in degree $-c-1$, then it is the map whose graded dual is the unique natural map

$$
\wedge^{c+1} U^* \otimes_k E(U^*)(-c-1) \to E(U^*)
$$
given by $\wedge^{c+1} U^* \otimes_k 1 \to \wedge^{c+1} U^*$.

Hence the maps

$$
\omega_{E(U^*)} \to \omega_{E(U^*)}(c+1) \otimes_k H^0(\mathcal{S}(\Delta)(c+1)).
$$

$$
\omega_{E(U^*)} \to \omega_{E(U^*)}(c+1) \otimes_k H^0(\Omega^n_c)(c+1)
$$

may be identified and so we must have $\pi_* \mathcal{S}(\Delta) = \Omega^s_c$. □

**Remark 3.5.** In the argument above we actually only used the assumption that $\mathcal{S}$ is CM with $f_\Delta$ equal to $(1 + t)^{n-1} f_{s,c}(t)$. Only this thus suffices to conclude that $\Delta$ is bi-CM.
Topological description of bi-CM simplicial complexes. The bi-CM simplicial complexes $\Delta$ correspond by $[6]$ to Stanley-Reisner rings $k[\Delta]$ which are CM and have a linear resolution over the polynomial ring. Since the generators of the ideal of $k[\Delta]$ will have degree $c + 1$, we say the resolution is $(c + 1)$-linear.

In $[11]$ R. Fröberg studies Stanley-Reisner rings $k[\Delta]$ with 2-linear resolution. When $\Delta$ is CM (so $\Delta$ is bi-CM with $c = 1$) he shows that $\Delta$ is what is called a $(d - 1)$-tree. (Strictly speaking he uses this term only for the 1-skeleton of $\Delta$.) They arise as inductively as follows. Start with a $(d - 1)$-simplex, then attach $d - 1$ simplices, one at a time, by identifying one (and only one) $(d - 2)$-face of $\Delta$ with one (and only one) $(d - 2)$-face of the simplex to be attached. This thus describes bi-CM $\Delta$ with $c = 1$. When $c \geq 2$ things appear to be less tractable as the following example shows.

Example 3.6. The following example was first noted in $[13]$. Consider the simplicial complex of dimension 2 with invariants $(n, c, s)$ equal to $(6, 2, 5)$:

![Diagram of a simplicial complex]

This simplicial complex is a triangulation of the real projective plane. It is isomorphic to its Alexander dual. Over any field of characteristic different from two, it is bi-Cohen-Macaulay. However, it has homology in dimension one over $\mathbb{Z}/2\mathbb{Z}$, so it is not Cohen-Macaulay over that field. In particular, it is not shellable.

4. When are CM-simplicial complexes cones?

The following proposition gives rise to the problems and results addressed in this section. In particular we are interested in determining for which range of invariants $(n, c, s)$ a bi-CM simplicial complex necessarily is a cone.
We give a conjecture for this and prove the existence of bi-CM simplicial complexes which are not cones in the whole range of this conjecture.

**Proposition 4.1.** Let \( f \) be a polynomial. Then there exists \( e(f) \) such that for \( e > e(f) \) if \( \Delta \) is a CM simplicial complex with \( f_\Delta = (1 + t)^e f \), then \( \Delta \) is a cone.

**Proof.** The number \( e \) of \( \Delta \) is determined by \( f \) and the \( h^0(S(\Delta)(p)) \) are also determined by \( f \) (Proposition 2.5). Now by the proof of Proposition 2.1, \( h^i(S(\Delta)(c + 1 - i)) \) is zero for \( i > 0 \) so \( S(\Delta) \) is \((c + 1)\)-regular and is generated by its sections when twisted with \( c + 1 \). Letting \( s_1, s_2, \ldots, s_a \) be a basis for these sections, there is a surjection

\[
\bigoplus_{i=1}^a O_{\mathbb{P}^{n-1}} \otimes s_i \rightarrow S(\Delta)(c + 1).
\]

Now let \( b \) be \( h^0(S(\Delta)(c + 2)) \). Then the kernel \( K_i \) of each

\[
H^0(O_{\mathbb{P}^{n-1}}(1) \otimes s_i) \rightarrow H^0(S(\Delta)(c + 2))
\]

is at least \((n - b)\)-dimensional. If \( n > ab \) (which is the case for \( e \) sufficiently large), the intersection of all the \( K_i \) considered as subspaces of \( H^0(O_{\mathbb{P}^{n-1}}(1)) \) is not empty. Thus we get a linear form \( h \) in \( H^0(O_{\mathbb{P}^{n-1}}(1)) \) such that all \( h \otimes s_i \) map to zero. But then \( S(\Delta) \) is contained in the hyperplane \( h = 0 \) in \( \mathbb{P}^{n-1} \) and so \( \Delta \) is a cone by Proposition 2.1. \( \square \)

We now pose the following.

**Problem 4.2.** For each polynomial \( f \) with \( f(-1) \) non-zero, determine the least number, call it \( e(f) \), such that when \( \Delta \) is Cohen-Macaulay with \( f_\Delta = (1 + t)^e f \) and not a cone, then \( e \leq e(f) \).

In the case where \( f \) is \( f_{s,c} \), see [10], we propose the following conjecture for the value of the upper bound of \( e = n - s \) when \( \Delta \) is not a cone.

**Conjecture 1.** Suppose \( \Delta \) is bi-CM of type \((n, c, s)\) and not a cone. Then

\[
n - s \leq c(s - c - 1) \quad \text{(or equivalently \((c + 1)d \leq cn\)).}
\]

**Conjecture 2.** Suppose \( \mathcal{F} \) is a non-degenerate coherent sheaf on \( \mathbb{P}^{n-1} \) which projects down to \( \Omega^c_{\mathbb{P}^{s-1}} \) on \( \mathbb{P}^{s-1} \). Then

\[
n - s \leq c(s - c - 1) \quad \text{(or equivalently \((c + 1)d \leq cn\)).}
\]

Clearly Conjecture 2 implies Conjecture 1 by letting \( \mathcal{F} \) be \( S(\Delta) \). The following shows the existence of non-degenerate coherent sheaves \( \mathcal{F} \) attaining the bound in Conjecture 2 and which cannot be lifted further.

**Proposition 4.3.** The sheaf \( O(-c - 1, 0) \) on the Segre embedding of \( \mathbb{P}^c \times \mathbb{P}^{s-c-1} \) in \( \mathbb{P}^{(c+1)(s-c)-1} \) projects down to \( \Omega^c_{\mathbb{P}^{s-1}} \).

Since the Segre embedding is smooth and projectively normal, this line bundle cannot be lifted further.
Proof. Let us compute the Tate resolution in components $c$ and $c + 1$. For component $c$ we compute

$$h^i(\mathcal{O}(-c - 1, 0)(c - i)) = \begin{cases} 0, & i \neq c \\ 1, & i = c \end{cases}$$

For component $c + 1$ we compute

$$h^i(\mathcal{O}(-c - 1, 0)(c + 1 - i)) = \begin{cases} 0, & i > 0 \\ h^0\mathcal{O}_{\mathbb{P}^n-c-1}(c + 1), & i = 0 \end{cases}$$

Hence components $c$ and $c + 1$ of the Tate resolution are

$$\omega_E \to \omega_{E}(c + 1) \otimes_k H^0\mathcal{O}_{\mathbb{P}^n-c-1}(c + 1).$$

Now note that $h^0\mathcal{O}_{\mathbb{P}^n-c-1}(c + 1)$ is $\binom{n}{c+1}$ which again is $h^0\Omega_{\mathbb{P}^n-1}(c + 1)$. Now the argument proceeds exactly as in the proof of Proposition 3.4 b) and c).

We shall show the existence of bi-CM simplicial complexes which are not cones, in the entire range of Conjecture 1.

Let $p$ and $q$ be positive integers. By thinking of a $p \times q$ matrix we define a vertical path as a non-decreasing function $\alpha : [p] \to [q]$ and a horizontal path as a non-decreasing function $\beta : [q] \to [p]$. By identifying a path with its graph we may consider it as a subset of $[p] \times [q]$. We may note that any horizontal path must intersect any vertical path.

Lemma 4.4. Let $F$ be a subset of $[p] \times [q]$. Then either $F$ contains a horizontal path or the complement $\overline{F}$ contains a vertical path.

Proof. We form a partial horizontal path $\beta : [i] \to [p]$ as follows. Chose $\beta(1)$ minimal such that $(\beta(1), 1)$ is in $F$. Then chose $\beta(2) \geq \beta(1)$ minimal such that $(\beta(2), 2)$ is in $F$. Continuing till the process stops gives a path $\beta : [i] \to [p]$. The block $[\beta(i), p] \times [i + 1, q]$ can then contain no element from $F$. Looking at the block

$$B = [1, \beta(i) - 1] \times [1, i],$$

by construction of the path $\beta$, $F \cap B$ does not contain a horizontal path. By induction $\overline{F} \cap B$ contains a vertical path $\alpha : [1, \beta(i) - 1] \to [i]$ which can then be completed to a path all the way down in $\overline{F}$ by picking elements in the block $[\beta(i), p] \times [i + 1, q]$. \qed

Let $V$ be the vector space with basis $e_{ij}$ where $i = 1, \ldots, p$ and $j = 1, \ldots, q$ and fill the matrix with these elements. Let $Y$ be the simplicial complex defined by the monomial ideal in $E(V)$ generated by horizontal path products

$$e_{\beta(1),1}e_{\beta(2),2} \cdots e_{\beta(q),q}$$

and let $X$ be the simplicial complex defined by the monomial ideal generated by the vertical path products.
Proposition 4.5. The facets of $X$ are the complements of the horizontal paths and the facets of $Y$ are the complements of the vertical paths. In particular $X$ and $Y$ are Alexander dual simplicial complexes.

Proof. The statement about Alexander duals follows from the first statement because the facets of the Alexander dual of $Y$ are the subsets of $[p] \times [q]$ which are the complements of the indexing sets of the monomial generators of $Y$, and so this is $X$.

Now the faces of $X$ are precisely the subsets $F$ of $[p] \times [q]$ which do not contain a vertical path. By the previous lemma $F$ contains a horizontal path and so $F$ is contained in the complement of a horizontal path.

If $F$ is the complement of a horizontal path, then $F$ does not contain a vertical path because any horizontal and vertical paths intersect. Hence $F$ is a face (in fact a facet) of $X$. □

Lemma 4.6. $X$ and $Y$ are shellable simplicial complexes

Proof. The elements of $X$ are complements of horizontal paths $\beta(1) \cdots \beta(q)$ and we represent them as such. We order these lexicographically by letting $1 \succ 2 \succ \cdots \succ p$. This gives a shelling of $X$. Let $\alpha$ and $\beta$ be horizontal paths with $\alpha \succ \beta$ and the cardinality of $\alpha \cap \beta$ less than or equal to $q-2$. If there are at least two values $i$ such that $\alpha(i) < \beta(i)$, let $l$ be maximal among these and let

$$\gamma = \alpha(1) \cdots \alpha(l-1)\beta(l) \cdots \beta(q).$$

If there is only one value $i$ with $\alpha(i) < \beta(i)$ let $i = l$ and

$$\gamma = \alpha(1) \cdots \alpha(l)\beta(l+1) \cdots \beta(q).$$

Then $\gamma \succ \beta$ and the cardinality of $\alpha \cap \gamma$ is greater than that of $\alpha \cap \beta$. The argument for $Y$ is similar. □

Theorem 4.7. Given $s > c$. For all $n$ in the range $s \leq n \leq (c+1)(s-c)$ there exists bi-CM simplicial complexes with invariants $n,c,s$ which are not cones.

There are thus explicit examples in the full range of Conjecture 1.

Proof. Since $X$ and $Y$ are shellable and Alexander duals, the Stanley-Reisner ring $k[X]$ is Cohen-Macaulay of dimension $pq-1$ and has a linear resolution, [6]. Considering the $p \times q$ matrix with entries $x_{ij}$, $k[X]$ is a quotient of $k[x_{ij}]$. The $k$th normal diagonal are the positions $(i,j)$ with $i+j = k-1$. We may make some variables on a normal diagonal equal. Having done so we may form the simplicial complex $X'$ again defined by the ideal of vertical path products. Then $k[X']$ is $k[X]$ divided out by elements $x_{ij} - x_{i'j'}$ each time we make $x_{ij}$ equal to $x_{i'j'}$. Making all the elements on each normal diagonal equal, call the element on the $k$th normal diagonal $x_k$ where $k = 1, \ldots, p+q-1$, we get a simplicial complex $\Delta$ on $[p+q-1]$ whose ideal
is generated by all monomials $x_{i_1} \cdots x_{i_p}$ where $i_1 < \cdots < i_p$. Thus $\Delta$ is the complete $p-2$-dimensional skeleton of the $p+q-2$-simplex and so $k[\Delta]$ is Cohen-Macaulay of dimension $p-1$.

Since we have divided out by $pq-p-q+1$ elements to get from $k[X]$ to $k[\Delta]$, each time we have cut dimension, and so each element must have been regular. Therefore $k[X]$ must be Cohen-Macaulay and have a linear resolution and so is bi-CM. If we divided out by $m$ elements to get from $k[X]$ to $k[X']$ where $m$ is between $0$ and $(p-1)(q-1)$, $X'$ has invariants

\[ n = pq - m, \quad d = pq - q - m, \quad c = p - 1, \quad s = p + q - 1 \]

and so by choosing $p$ and $q$ suitable, we fill up the whole range of the theorem. \hfill \box

Since bi-CM simplicial complexes of type $(n, c, s)$ are Alexander dual to bi-CM simplicial complexes of type $(n, s-c-1, s)$ and $\Delta$ and $\Delta^*$ are cones at the same time, we see that if Conjecture 1 is true for type $(n, c, s)$ it is true for type $(n, s-c-1, s)$. The following easy argument shows the conjecture for $c = 1$ (and thus also for $c = s-2$).

**Proposition 4.8.** If $\Delta$ is bi-CM of type $(n, 1, s)$ and not a cone, then $n-s \leq s-2$.

**Proof.** By Section 3, $\Delta$ is constructed as follows. Start with a $d$-simplex $F_1$. Attach a $d$-simplex $F_2$ on a $(d-1)$-face and continue attaching $F_3, \ldots, F_n$. Now $F_1, \ldots, F_n$ are sets of $d$ elements and for each $j$ there is $i < j$ such that $F_i \cap F_j$ consists of $d-1$ elements. But then $\bigcap^*_1 F_i$ contains at least $d-s+1$ elements and so if $d \geq s$, $\Delta$ must be a cone. Since $d = n-s+1$ this gives the proposition. \hfill \box

**References**

[1] A.Aramova and L.L.Avramov and J.Herzog Resolutions of monomial ideals and cohomology over exterior algebras Trans. AMS 352 (1999) nr.2, pp. 579-594

[2] D. Bayer, H. Charalambous, S. Popescu Extremal Betti numbers and applications to monomial ideals Journal of Algebra 221 (1999) pp.497-512.

[3] D. Bayer, B. Sturmfels Cellular resolutions of monomial modules Journal f"ur die reine und angewandte Mathematik 502 (1998), 123-140.

[4] I.N.Bernstein and I.M.Gel’fand and S.I.Gel’fand Algebraic bundles over $\mathbf{P}^n$ and problems of linear algebra Funct. Anal. and its Appl. 12 (1978) pp.212-214

[5] W.Bruns and J.Herzog Cohen-Macaulay rings Cambridge University Press 1993.

[6] J.A.Eagon and V.Reiner Resolutions of Stanley-Reisner rings and Alexander duality Journal of Pure and Applied Algebra 130 (1998) pp.265-275

[7] D.Eisenbud and G.Fløystad and F.-O. Schreyer Sheaf Cohomology and Free Resolutions over Exterior Algebras Transactions of the AMS, 355 (2003) no.11, pp. 4397-4426.

[8] G. Fløystad Describing coherent sheaves on projective spaces via Koszul duality preprint, math.AG/0012263

[9] G. Fløystad Hierarchies of simplicial complexes via the BGG-correspondence preprint, math.CO/0302313

[10] R. Fröberg Rings with monomial relations having linear resolutions Journal of Pure and Applied Algebra 38 (1985) pp.235-241.
[11] R. Fröberg On Stanley-Reisner rings Banach Center Publications 26, Part 2 (1988), pp. 57-70.
[12] Grayson, Daniel R. and Stillman, Michael E. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/
[13] G.A. Reisner Cohen-Macaulay quotients of polynomial rings Adv. Math. 21 (1975), pp. 30-49.
[14] R. Stanley Combinatorics and Commutative Algebra Second Edition, Birkhäuser 1996.

Matematisk institutt, Johs. Brunsgt. 12, N-5008 Bergen, Norway
E-mail address: gunnar@mi.uib.no and jonev@mi.uib.no