Parafermionic wires at the interface of chiral topological states

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We explore a scenario where local interactions form one-dimensional gapped interfaces between a pair of distinct chiral two-dimensional topological states – referred to as phases 1 and 2 – such that each gapped region terminates at a domain wall separating the chiral gapless edge states of these phases. We show that this type of T-junction supports point-like fractionalized excitations obeying parafermion statistics, thus implying that the one-dimensional gapped interface forms an effective topological parafermionic wire possessing a non-trivial ground state degeneracy. The physical properties of the anyon condensate that gives rise to the gapped interface are investigated. Remarkably, this condensate causes the gapped interface to behave as a type of anyon “Andreev reflector” in the bulk, whereby anyons from one phase, upon hitting the interface, can be transformed into a combination of reflected anyons and outgoing anyons from the other phase. Thus, we conclude that while different topological orders can be connected via gapped interfaces, the interfaces are themselves topological.

Topological phases of matter in 2+1-D are often characterized by a “bulk-boundary” correspondence. Bulk properties such as a topological band structure, quasiparticles exhibiting fractional statistics, or topological ground state degeneracy on manifolds with non-zero genus, go hand in hand with an associated set of boundary/interface states where a topological phase meets a different topological phase such as the vacuum. [1]

Topological phases appear in two general classes: symmetry protected [2–15], or those that have “intrinsic” topological order [16]. There are several important distinctions between these classes, e.g., differing constraints on the ability to open a gap in the edge state spectrum. For the first class, gapped boundaries can exist when the symmetry is broken explicitly or spontaneously. In the latter, interface states with non-vanishing chirality cannot be completely gapped, and, surprisingly, even in the absence of any symmetries, some interface states with vanishing chirality cannot be completely gapped either.[18] This observation may directly impact experiment since such an ungappable edge may exist in the ν = 2/3 fractional quantum Hall effect, or at the interface between two fractional quantum Hall states with, e.g., filling factors ν = 1/3 and ν = 1/5. The latter interface cannot be gapped by any local interaction, essentially due to the completely incompatible bulk properties of the two topological phases.

In this article we focus on the complementary effect that allows disparate topological phases to support gapped interfaces, as they provide a domain for a wide-range of interesting physics. The existence of such an interface requires that a local gapping condition be satisfied [see discussion around Eq. (3)], which, physically amounts to the allowed formation of an “anyon condensate” at the interface. It has been established, for two dimensional Abelian topological phases, that each anyon condensate is in one-to-one correspondence with a mathematical structure called a “Lagrangian subgroup.” [17–19] Each Lagrangian subgroup is a subset M of the set of anyons wherein (i) all quasiparticles have mutual bosonic statistics, and (ii) every quasiparticle not in M has non-trivial statistics with at least one quasiparticle of M. Hence, the simultaneous condensation of the quasiparticles in M is allowed by (i), and will confine all the...
anyons of the theory by (ii). Of great interest are configurations where inequivalent anyon condensates, corresponding to inequivalent choices of \( \mathcal{M} \), are formed in adjoining regions of a topological interface. Indeed, domain walls between these gapped regions have been shown to host non-Abelian defect bound states with parafermionic statistics.\[20,27\] Such bound states could be used as a platform for realizing topological quantum computation.\[28\]

In this letter we characterize a new family of 1D gapped systems that can be formed at the interface between different 2+1D Abelian topological phases. For our examples, we choose single-component chiral phases characterized by the topological invariants (K-matrices) \( k_1 \) and \( k_2 \), respectively. Hereafter we refer to these as phase 1 and phase 2. If these phases arise from quantum Hall states then we have \( k_{1,2} = \nu_{1,2}^{-1} \), where \( \nu \) is the filling fraction that measures the Hall conductance in fundamental units. More generally, for systems without U(1) charge conservation symmetry, \( k_{1,2} \) count the number of distinct bulk quasiparticle types in each phase, and give the topological ground state degeneracy \( g^{k_n} \) of each system defined on a spatial manifold of genus \( g \). For our discussion we will adopt the interface geometry in Fig. 1. The bulk topological phases share a gapped interface with each other, and they have a boundary with the vacuum that contains propagating chiral edge modes. The gapped interface terminates at points separating the gapless edge states of phases 1 and 2.

Our main finding is that such an interface forms a topologically non-trivial, 1D gapped system with a degenerate ground state manifold associated with parafermionic end states. We stress that the parafermions discussed here are not located at the domain walls between distinct gapped interfaces; rather, they are located at domain walls between the gapless edge states of phases 1 and 2, as shown in Fig. 1. Therefore the physical scenario discussed here departs significantly from those of Refs.\[20–24,26,27\], and more closely matches the setup of Ref.\[29\] though here we are focused more on what is happening in the bulk, rather than the edge as in their discussion. Ultimately, our results identify that, while one can find gapped interpolations between 2D phases with different topological order, these are not trivial gapped regions; they are instead topological themselves.

In the following, we will illustrate our result with a bosonization description built from the pair of counter propagating modes stemming from the edge states of phases 1 and 2. We will construct the explicit form of the local, gap-opening interaction, provide a description of the interface anyon condensate, and discuss the onset of the topologically degenerate ground state manifold associated with the expectation value of a non-local operator. Finally, we will discuss the connection between bulk confinement-deconfinement transitions, edge-state transitions, and the bound parafermion modes.

1. Luttinger liquid description of the interface – In Fig. 1[a], we consider an array of 2D topological states in phase 1 (blue) and phase 2 (brown), surrounded by the vacuum. As shown in the Supplementary Material (SM), the most generic gappable interface for one-component states is characterized by \( k_1 = pn^2 \) and \( k_2 = pm^2 \), where \( p, m, \) and \( n \) are positive integers. At each interface along the x-direction, the edge states originating from phases 1 (right-moving) and 2 (left-moving) are propagating in opposite directions. The low energy description of a given interface is given by the Lagrangian density

\[
\mathcal{L}_x = \frac{1}{4\pi} \partial_x \Phi^T K \partial_x \Phi - \frac{1}{4\pi} \partial_x \Phi^T V \partial_x \Phi - \mathcal{H}_{\text{int}}[\Phi],
\]

\[1a\]

\[
K = \begin{pmatrix} pn^2 & 0 \\ 0 & -pm^2 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},
\]

\[1b\]

where \( \phi_{1,2} \) represent the bosonic modes originating from the edge states of phases 1 and 2, \( V \) is a velocity matrix and \( \mathcal{H}_{\text{int}}[\Phi] \) is a local interaction between the edge modes to be discussed below. The symmetric (in this case diagonal) K-matrix gives the structure of the equal-time commutation relations

\[
[\partial_x \phi_i(t, x), \phi_j(t, y)] = -2\pi i K_{ij}^{-1} \delta(x - y).
\]

To simplify our discussion we will choose \( m = 1 \) and provide the details for \( m > 1 \) in the SM. This case is also the most experimentally relevant as it includes interfaces between a \( \nu = 1 \) integer quantum Hall state with, e.g., a \( \nu = 1/9 \) fractional quantum Hall state when \( p = 1, n = 3 \).

Adopting a compactification radius convention with \( \ell = 1 \) for the fields \( \phi_i \), the quasi-particle excitations on the edge are created by the vertex operators \( \exp\left(i \ell^T \Phi \right) \), where \( \ell \) is an integer-valued 2-component vector. In this representation, the exchange statistics associated with taking a quasiparticle \( \ell_a \) adiabatically around another quasiparticle \( \ell_b \) is given by the statistical phase \( S_{ab} = e^{i \theta_{ab}} = e^{i 2\pi \ell_a^T K^{-1} \ell_b} \), and the (topological) spin of each quasiparticle is given by the self-statistics phase \( \theta_a = e^{i \pi \ell_a^T K^{-1} \ell_a} \). Local excitations of the theory are identified with the vertex operator \( \psi = e^{i \Lambda^T K} \Phi \), where \( \Lambda \) is an integer vector.

In Eq. 1[a] \( \mathcal{H}_{\text{int}}[\Phi] \) represents an interaction between the counter-propagating modes that will open a gap at the interface. This interaction, which involves coupling between local degrees of freedom, can be parametrized as \( \mathcal{H}_{\text{int}}[\Phi] = -J \cos \left( \Lambda^T K \Phi \right) \), where \( J \) is a coupling constant, and \( \Lambda^T = (a, b) \) is an integer-valued 2-component vector satisfying the null vector criterion \[32\]

\[
0 = \Lambda^T K \Lambda = p \left(a^2 n^2 - b^2 \right).
\]

The integer vector \( \Lambda = (1, n) \) is a primitive solution\[33\] of (3) that represents a process where a single local operator
\[ ψ_1 = e^{i p n^2 φ_1} \text{ of phase 1 couples with } n \text{ local operators } \psi_2 = e^{-i p φ_2} \text{ of phase 2, giving rise to the interaction } \]
\[ \mathcal{H}_{\text{int}} = -J \cos (n Θ) \propto -J \sum_{n} ψ_1^\dagger ψ_2^\dagger + \text{h.c.}, \]
where \( Θ(x) \equiv p n φ_1(x) - p φ_2(x) \).

This interaction generates an anyon condensate at the interface as we will now describe. In phase 1, there are \( p n^2 \) quasiparticle-types labeled \( ε_1^{p n^2} = 1, \ldots, p n^2 \), and in phase 2, there are \( p \) quasiparticle-types labeled \( ε_2 = 1, \ldots, p \). The set of anyons forms a discrete lattice[26, 34–38], and each anyon is topologically equivalent to anyons that differ only by the attachment of integers among the local quasiparticles \( ψ_i = ε^{k_i}_i, i = 1, 2 \).

In the context of a Laughlin fermionic (bosonic) state, \( ε^{k_1}_1 \) represents the local fermion (boson) of the \( i \)-th phase.

Now we note that in phase 1 the anyon set \( \{ ε_1^{p n^2}_i, x = 1, \ldots, n \} \) contains mutual bosons or fermions with spin \( h(ε_1^{p n^2}_i) = e^{i p x^2}_i \). Moreover we note that the quasiparticle \( χ_1 = ε_1^{p n^2} \) has the same spin as the local excitation \( ψ_2 \) of phase 2, i.e., they are both bosons or fermions depending on the parity of \( p \). Physically this implies that the composite quasiparticle \( σ = ε_1^{p n^2} ψ_2 \) is a boson that can condense, and generate a fully gapped interface between phases 1 and 2. This condensation process, mathematically, is a consequence of the relation \( k_1/k_2 = n^2 \in \mathbb{Z}^2 \), which allows for the existence of a \( p n \)-dimensional Lagrangian subgroup \( \mathcal{M} \) containing \( σ \).

Importantly, the interaction (4) involves \( n \) local particle operators of phase 2 and a single local particle operator of phase 1. This interaction breaks the \( U(1) \times U(1) \) particle conservation symmetries of each phase down to \( Z_1 \times Z_n \), where \( Z_1 \) means no symmetry. Hence the interaction (4) is invariant under \( S_β: ψ_1 \to ψ_1, ψ_2 \to ψ_2 e^{i p x^2}_n, \beta \in \mathbb{Z} \). If the phases began with a \( U(1)_{EM} \) electromagnetic charge conservation symmetry, then this interaction breaks (preserves) the symmetry when the charge vector is \( t^F = (1, 1) \) (\( t^F = (n, 1) \)). This discrete symmetry, it turns out, plays a fundamental role in the identification of the gapped interface as a topological parafermion wire similar to those studied in Refs. [39, 40].

The topological properties of the gapped interface can be more transparently revealed by a description in the zero correlation length limit \( J \to ∞ \), where the interface Hamiltonian density is given solely by Eq. (4), thus leading to a gapped interface as depicted in Fig. 1(c). In this limit the ground state manifold contains \( n \) degenerate minima \( \{ Ψ_q = 2π q/n, q = 1, ..., n \} \) associated with the vacuum expectation value of the composite bosonic operator \( σ(x) = χ_1(x) ψ_2^\dagger(x) = e^{i φ_2(x)} \), which represents a bound state of the quasiparticle \( χ_1 = e^{i p x^2}_n \phi_2 \). We can find a representation of the ground state manifold in terms of the eigenstates of \( σ \) as

\[ ∀x: \ σ(x) | Ψ_q \rangle = ω^q | Ψ_q \rangle, \quad q = 1, ..., n. \]

The structure of the eigenstates (5) is in direct correspondence with symmetry broken states of the ferromagnetic, zero correlation length limit of an \( n \)-state clock model, where \( σ \) naturally acquires the interpretation of a clock operator satisfying \( σ_n = 1 \) and \( σ = σ^{n-1} \).

While it would seem possible to distinguish among the degenerate states by a measurement of \( σ(x) \), \( ⟨Ψ_q | σ(x) | Ψ_q'⟩ = ω^q δ_q q' \) [which is equivalent to adding a perturbation \( δ \mathcal{H} = δ \cos (Θ) \) to the Hamiltonian (4)], the fact that \( σ(x) \) is a non-local operator does not permit such a local distinction, and is a hallmark of the topological nature of the system. With this in mind, the eigenstates (5) indicate a degenerate symmetry breaking manifold associated with the global symmetry

\[ S ≡ S(β = 0) = e^{-i ∫ x L dx φ_2(x)} \frac{d}{dx} \xi(x), \]

whereby the parafermion operator \( α(x) \) is a product of the “order”, \( σ \), and the “disorder”, \( ξ \), operators. The operators (7) satisfy standard parafermionic commutation relations

\[ α(x) α(y) = α(y) α(x) e^{i 2π ξ(y-x)} \].

Importantly, we find that the boundary parafermionic operators \( α(x_L) = σ(x_L) \), \( α(x_R) = σ(x_R) e^{-i ∫ x L dx φ_2(x)} \) commute with the Hamiltonian (4), and the degenerate ground state manifold is given by the eigenstates of the non-local operator \( A = α(x_L) α(x_R) \cdot A | Ω_a⟩ = ω^a | Ω_a⟩, a = 1, ..., n, \) where \( | Ω_a⟩ \) are linear combinations of the \( | Ψ_q⟩ \).

2. Edge transitions – As indicated in Figs. (1)(b) and (1)(c), the formation of the gapped interface prevents the edge modes of phases 1 and 2 from propagating in the \( x \)-direction. Any point \( x \in (x_L, x_R) \) in the interior of the gapped interface establishes a domain wall between distinct gapped bulk topological phases. In turn, the end states of the gapped interface located at \( x = x_{LR} \) correspond to domain walls between topologically distinct gapless edge states. In fact, we shall explicitly demonstrate that, associated to each of the gapless edge transitions there is a parafermion operator with quantum dimension \( √n \) obeying parafermion statistics. Thus, the existence of the parafermions at the end points of the gapped interface, each with quantum dimension \( √n \), is a direct manifestation of the \( n \)-fold degeneracy of the ground state manifold of the gapped interface. Similar physics was first explored in Ref. [29] which focuses on transitions between distinct edge terminations of the
same bulk phase. Here we focus on the interface between different bulk phases, which will have an accompanying transition on the edge.

An important feature of the gappable topological interface is that the bulk phases 1 and 2 can be related to each other by the confinement (or deconfinement) of a 2 + 1-D $\mathbb{Z}_n$ gauge theory. In order to see this, imagine phase 2 is coupled to a $\mathbb{Z}_n$ gauge theory in its deconfined phase. Let the gauge field $\alpha_\mu$ describe the excitations of phase 2, and $(\alpha_\mu, b_\mu)$ the excitations of the $\mathbb{Z}_n$ gauge theory. Hence, the coupled system is described by the Abelian Chern-Simons theory:

$$\mathcal{L}_{2D} = \frac{1}{4\pi} \varepsilon^{\mu\nu\lambda} c_\mu^I K_{IJ}(p, n) \partial_\nu c_\lambda^J,$$  \hspace{1cm} (9a)

where

$$K(p, n) = \begin{pmatrix} p & -1 & 0 \\ -1 & 0 & n \\ 0 & n & 0 \end{pmatrix}, \quad c_\mu = \begin{pmatrix} \alpha_\mu \\ \alpha_\mu' \\ b_\mu \end{pmatrix}. \hspace{1cm} (9b)$$

Einstein summation convention is assumed for repeated indices and Greek indices take values $(0, 1, 2) = (t, x, y)$. In this basis, $e = (0, 1, 0)$ and $m = (0, 0, 1)$ represent the original charge and flux excitations of the $\mathbb{Z}_n$ gauge theory.

It is possible to show that, by a change of basis, K-matrix [96] can be transformed into [49]

$$K_g \equiv W^T K W = p n^2 \oplus \Sigma,$$ \hspace{1cm} (10)

where $W \in \text{GL}(3, \mathbb{Z})$ and $\Sigma$ represents a Pauli matrix, i.e., a trivial sector that can always be gapped out. Thus, Eq. (10) explicitly illustrates that phase 1 can be obtained from phase 2 by a gauging mechanism; conversely, phase 2 descends from phase 1 by confining the $\mathbb{Z}_n$ gauge theory. This kind of gauging mechanism has proven useful in understanding the classification of symmetry enriched topological states [17] and hidden anyonic symmetries [27].

Using this theory we can now explicitly prove the existence of domain-wall parafermions. To do this we analyze the edge states obtained from the bulk theory [9] and (10) in order to study the transitions between the edge phases 1 and 2. For that model we gap the edge states propagating along one of the edges, say $x = x_L$, with the effective theory

$$\mathcal{L}_{x_L, y} = \frac{1}{4\pi} \partial_t \Phi'^T K_g \partial_y \Phi' + \ldots$$

$$+ J_1(y) H_{\text{int}, 1} + J_2(y) H_{\text{int}, 2},$$ \hspace{1cm} (11)

where $\Phi'(t, x_L, y) = (\phi_1(t, x_L, y), \phi_2(t, x_L, y), \phi_3(t, x_L, y))$ are the edge fields. The interactions $H_{\text{int}, 1}$ and $H_{\text{int}, 2}$ will be chosen to stabilize the edge phases 1 and 2, respectively, in different spatial regions. To carry this out we use position-dependent coupling constants $J_1(y)$ and $J_2(y)$ such that, in regions where we want phase 1, $J_1 \to \infty$ and $J_2 = 0$, while for phase 2, $J_1 = 0$ and $J_2 \to \infty$. For the sake of concreteness, we shall take $p = 2q + 1$ and $\Sigma = \sigma_z$ in (10), although similar results can be obtained for the $p = 2q$ case with $\Sigma = \sigma_x$.

The interaction choice

$$H_{\text{int}, 1} = \cos (L_1^T K_g \Phi'), \quad L_1^T = (0, 1, 1)$$ \hspace{1cm} (12)

will gap the trivial modes in $\Sigma$, which then yields the edge states of phase 1. Alternatively, the interaction

$$H_{\text{int}, 2} = \cos (L_2^T K_g \Phi'), \quad L_2^T = (1, q, n, (q + 1)n)$$ \hspace{1cm} (13)

gives rise to the edge state of phase 2, that is, it effectively leads to the confinement of the $\mathbb{Z}_n$ gauge theory. In order to see this, notice that the edge excitations that remain deconfined in the presence of the interaction (13) are described by vertex operators $\exp (i \ell^T \Phi')$, such that $\ell^T \Phi'$ [with $\ell^T = (\ell_1, \ell_2, \ell_3)$] commutes with the argument of the interaction (13). From this condition, which is satisfied when $\ell_1 = -n \ell_2 + (q + 1)\ell_3$, it is easy to see that the deconfined edge excitations are those of the phase 2 described by $k_2 = p = (2q + 1)$. More intuitively, upon rewriting $H_{\text{int}, 2} = \cos (L_2^T K_g \Phi') = \cos (L_2^T K \Phi') = \cos \left( n \frac{\phi_2'}{\phi_2'} \right)$, with $L = WL$ and $\Phi' = W\Phi'$, (13) is seen as the expected “electric”-mass interaction that confines the excitations of the $\mathbb{Z}_n$ gauge theory.

Now let $R_1 = \cup_i (y_{2iL} - \varepsilon, y_{2iL} + \varepsilon)$ and $R_2 = \cup_i (y_{2iL} + \varepsilon, y_{2iL+1} - \varepsilon)$ be the segments in the $x = x_L$ edge corresponding to phases 1 and 2, respectively, where $\varepsilon$ is an infinitesimal positive number that regulates the domain wall transitions (see Fig. (1a)). It then follows that the operators defined as

$$O_i = \exp \left[ \frac{i}{n} \int_{y_{2iL} - \varepsilon}^{y_{2iL} + \varepsilon} dy \partial_y \left( L_2^T K \Phi' \right) \right]$$

$$T_i = \exp \left[ \frac{i}{n} \int_{y_{2iL} + \varepsilon}^{y_{2iL+1} - \varepsilon} dy \partial_y \left( L_2^T K \Phi' \right) \right]$$ \hspace{1cm} (14a)

commute with the edge Hamiltonian and satisfy the non-trivial commutation relations

$$T_i O_i = O_i T_i e^{2\pi i}$$

$$T_{i-1} O_i = O_i T_{i-1} e^{-2\pi i}$$ \hspace{1cm} (14b)

The ground state manifold forms a representation of the algebra (14b), which implies a ground state degeneracy of $n^{k-1}$ in the presence of 2 $k$ domain walls on the boundary, i.e., $k$ gapped interfaces.

Guided by (14), parafermion operators located on the domain walls between phases 1 and 2 along the $x = x_L$
edge can be constructed as:

\[
\alpha_{x_L,2i} = e^{\frac{i}{4} \left[ L^x_1 K_y \Phi'(y_{2i}+\varepsilon)-L^x_1 K_y \Phi'(y_{2i}-\varepsilon) \right]},
\]

\[
\alpha_{x_L,2i+1} = e^{\frac{i}{4} \left[ L^x_2 K_y \Phi'(y_{2i+1}+\varepsilon)-L^x_2 K_y \Phi'(y_{2i+1}-\varepsilon) \right]},
\]

(15a)

which are readily shown to satisfy

\[
\alpha_{x_{L,k}} \alpha_{x_{L,\ell}} \alpha_{x_{L,k}} = e^{\frac{i2\pi}{n} \text{sgn}(k-\ell)(-1)^{k+\ell}}.
\]

(15b)

For a generic gappable interface between one-component states we have the constraint \( k_1 = \frac{n^2}{m} k_2 \) which implies that the phases must be related by the confinement of a \( \mathbb{Z}_m \) gauge theory, and the subsequent gauging and deconfinement of a \( \mathbb{Z}_n \) symmetry. In these cases one would find \( \mathbb{Z}_{mn} \) parafermions (see SM for more detail).

A realization of the algebra [14] has been studied in Ref. [29], for the transitions been chiral bosonic edge states with \( k_1 = 2n^2 \) and \( k_2 = 2 \). While their approach focused solely on the edge transitions of the same bulk phase, our formulation shows that the existence of non-trivial parafermionic modes [15] is a direct consequence of the formation of a gapped interface between different chiral topological states. Hence, we have generalized their result to arbitrary one-component edge transitions, and have shown that such transitions can originate from a bulk phenomenon associated with confinement-deconfinement transitions of discrete gauge theories. Additionally, since these parafermions appear at a “T-junction” between two chiral gapless states and the termination of their gapped interface, they represent a completely new physical phenomenon when compared with the cases studied in Refs. [20–24, 26, 27].

In closing we note that the gapped interface acts like an anyonic Andreev reflector in the bulk. Anyons from, say, phase 1 will hit the interface and be transformed into a combination of outgoing anyons in phase 2 as well as reflected anyons that remain in phase 1. For instance, take \( p = 1, m = 1 \) for simplicity. Then, as, for example, quasiparticle \( \chi_1 = \varepsilon^1_1 \) approaches the interface, a vacuum fluctuation can create a \( (\psi_2, \bar{\psi}_2) \) pair in the region of phase 2 immediately adjacent to the interface; subsequently, the condensation of \( \chi_1 \bar{\psi}_2 \) leaves behind the quasiparticle \( \psi_2 \) in phase 2, as shown in Fig. 1(a). The quasiparticles \( \{ \varepsilon^x_1, x \in \mathbb{Z} \} \) belonging to phase 1 can be absorbed by the gapped interface and fully transmuted into multiples of the local excitation \( \psi_2 \) of phase 2. Other anyons hitting the interface will be partially transmuted and partially reflected by the condensate. For example, if \( \bar{\psi}_1 \) hit the surface it could generate a \( \psi_2 \) in phase 2 as well as a reflected \( \varepsilon^1_{-n+1} \).

In summary, we have shown that a gapped interface between different topologically ordered phases cannot be topologically trivial itself. The interpolation between

the topological orders generates a quasi-1D topological parafermion phase which exhibits characteristic non-Abelian defect modes where the interface intersects the boundary of the system. Although we have only shown this for one-component interfaces, we expect the generalizations to more complicated interfaces to provide a rich set of phenomena. Furthermore, our result may aid in the interpretation of the topological entanglement entropy arising at heterointerfaces of topologically ordered phases as recently calculated in Ref. [48]. We leave this to future work.

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where $\phi$ relations $[a, b]$ are relatively prime.

We immediately see that $p$ plays no role in the determination of this condition, and by construction $\tilde{k}_1$ and $\tilde{k}_2$ are relatively prime.

We can now multiply through to find

\[ a^2 k_1 = b^2 k_2 \]
and this implies, using the relatively prime condition, that $k_1$ divides $b^2$ and that $k_2$ divides $a^2$. As such we have the restrictions
\begin{align}
a^2 &= k_2 \gamma_a \\
b^2 &= k_1 \gamma_b
\end{align}
(S5)
where $\gamma_a$ and $\gamma_b$ are integers. Plugging this into Eq. (S3) we arrive at $\gamma_a = \gamma_b$.

To summarize, we now have the relations
\begin{align}
k_1 &= p k_1, \\
k_2 &= p k_2 \\
a^2 &= k_2 \gamma_a, \\
b^2 &= k_1 \gamma_b
\end{align}
(S6)
Now, in order for $\Lambda$ to be a primitive vector, $a$ and $b$ cannot have common factors, hence $\gamma_a = 1$ if we enforce primitivity. Hence, we arrive at the result that the most generic gappable interface for two, one-component theories is $K = \text{diag}[pm^2, -pn^2]$. Furthermore, a primitive gapping vector that will gap this theory takes the form $\Lambda = (n, m)$.

With this result we can gain a physical understanding. If we take the theory $k_1$ and gauge a $Z_n$ subgroup of the global particle number conservation symmetry and take $k_2$ and gauge a $Z_m$ subgroup of its particle number conservation symmetry then we will arrive at two theories with the same topological order, and which can hence be gapped. In fact, the gapability equation written as $k_1 = \frac{k_2^2}{2} k_2$ essentially encodes that if we take the theory $k_2$, confine a $Z_1$ group and deconfine a $Z_2$ group then we will arrive at the same topological order as $k_1$. From our constraints we see this is exactly true since $a^2$ divides $k_2$. The common factor $g$ is interpreted as a common parent theory from which each topological phase $k_1, k_2$ can be reached by deconfining discrete $Z_m$ and $Z_n$ subgroups of $U(1)$ respectively.

Let us now look at the condensate at the interface. The gapping term can be written as
\begin{align}
H_{\text{int}} &= -J \int_{x_L}^{x_R} dx \cos \left[ mn (pm \phi_1 - pn \phi_2) \right] \\
&= -J \int_{x_L}^{x_R} dx \cos (mn \Theta) + \text{H.c.}
\end{align}
It is important not to factor out $p$ when considering the condensate. To see this let us consider an example for the case when $k_1 = 2$ and $k_2 = 8$ where $p = 2$, $m = 1$ and $n = 2$. In phase 1 the statistics are $\exp(i \pi r^2 / 2)$ where $r \in \mathbb{Z}$. The only fermion or boson in the quasiparticle set is when $r$ is an even integer, i.e., the local boson in this case, which we will call $\psi_1$. For phase 2 the statistics are $\exp(i \pi r^4 / 8)$ and the quasiparticle with $r = 4$ is a (non-local) boson which we call $\chi_2$. For this system we can form $(\psi_1^1, \chi_2^2) = \psi_1^2 \psi_2^1$ to get a local condensate. The interaction that will generate this condensate is $J \cos(4 \phi_1 - 8 \phi_2) = J \cos(2(2 \phi_1 - 4 \phi_2))$. From the form of the condensate we see that the $U(1) \times U(1)$ group structure is broken to $Z_2 \times Z_1$, and hence we expect there to be $Z_2$ parafermions. In the generic case the symmetry is broken to $Z_m \times Z_n$ and hence a $Z_{mn}$ parafermion results. To see this explicitly, note that the second line of (S7) makes explicit that the interaction pairs $n$ operators of phase 1 with $m$ operators of phase 2. The non-local operator $\sigma(x) = e^{i \Theta(x)}$, where $\Theta(x) \equiv p m \phi_1(x) - p n \phi_2(x)$ shall be identified with a $Z_{mn}$ clock variable. From the form of this interaction, we can immediately write down three symmetry operations and their representations in terms of the bosonic fields:
\begin{align}
\tilde{S}_\alpha : \\
\psi_1 &\rightarrow \psi_1 e^{i2\pi n \alpha}, \\
\psi_2 &\rightarrow \psi_2 e^{i2\pi m \alpha}, \\
\alpha &\in \mathbb{R},
\end{align}
(S8)
which is a $U(1)$ symmetry of the interaction Hamiltonian when the topological phases have charge vector $t^T = (m, n)$. Moreover, we have
\begin{align}
S_{1, \beta_1} : \\
\psi_1 &\rightarrow \psi_1 e^{-i2\pi \beta_1 / n}, \\
\psi_2 &\rightarrow \psi_2, \\
\beta_1 &\in \mathbb{Z},
\end{align}
(S9)
which accounts for a discrete $\mathbb{Z}_n$ associated with phase 1 and, equivalently,
\begin{align}
S_{2, \beta_2} : \\
\psi_1 &\rightarrow \psi_1, \\
\psi_2 &\rightarrow \psi_2 e^{-i2\pi \beta_2 / m}, \\
\beta_2 &\in \mathbb{Z},
\end{align}
(S10)
which accounts for a discrete $\mathbb{Z}_m$ associated with phase 2.
Importantly, while $\sigma(x)$ does not transform under the action of $\tilde{S}_\alpha$, it does so under the other two operators as follows:

$$S^\dagger_{1,\alpha} \sigma S_{1,\alpha} = \sigma e^{-i 2\pi \beta_1/mn}, \quad S^\dagger_{2,\beta_2} \sigma S_{2,\beta_2} = \sigma e^{-i 2\pi \beta_2/mn}$$

(S11)

Therefore in the clock representation, the ground state degeneracy of the Hamiltonian (S7) becomes associated with the “symmetry broken” states

$$\forall x : \sigma(x) \mid \Psi_q \rangle = \omega^q \mid \Psi_q \rangle, \quad q = 1, \ldots, mn,$$

(S12)

where $\omega \equiv e^{i \frac{2\pi}{mn}}$. While it would seem that a particular eigenstate could be identified by the local measurement of $\sigma(x)$, $\langle \Psi_q | \sigma(x) | \Psi_{q'} \rangle = \omega^q \delta_{q,q'}$, it is fundamental to recognize that $\sigma(x)$ is not a local operator in terms of the original local degrees of freedom $\psi_1$ and $\psi_2$.

**GAPPED INTERFACE FOR THE $m = 1$ CASE**

We show the existence of an operator that commutes with the interface Hamiltonian

$$H_{int} = \int_{-L}^{L} dx \mathcal{H}_{int} = -J \int_{-L}^{L} dx \cos (n \Theta), \quad \Theta(x) \equiv p n \phi_1(x) - \phi_2(x)$$

(S13)

and yields an $n$-fold degeneracy of the ground state of (S13). For that we seek a unitary transformation

$$\Sigma(a_1, a_2) = e^{i \left[ a_1 \int_{-L}^{L} dx \partial_x \phi_1(x) + a_2 \int_{-L}^{L} dx \partial_x \phi_2(x) \right]}$$

(S14)

parametrized by $a_1, a_2 \in \mathbb{R}$ that commutes with the Hamiltonian (S13). With the equal time commutation relations $[\partial_x \phi_1(x), \phi_1(y)] = -\frac{2\pi i}{n} \delta(x-y)$, $[\partial_x \phi_2(x), \phi_2(y)] = \frac{2\pi i}{p} \delta(x-y)$ and $[\partial_x \phi_1(x), \phi_2(y)] = [\partial_x \phi_2(x), \phi_1(y)] = 0$, we find that $\Sigma^\dagger(a_1, a_2) \phi_1(x) \Sigma(a_1, a_2) = \phi_1(x) - \frac{2\pi i a_1}{p}$ and $\Sigma^\dagger(a_1, a_2) \phi_2(x) \Sigma(a_1, a_2) = \phi_2(x) + \frac{2\pi i a_2}{p}$, where $x_L \leq x \leq x_R$. With that the local operators $\psi_1(x) = e^{i p n \phi_1(x)}$ and $\psi_2(x) = e^{i \phi_2(x)}$ transform as: $\Sigma^\dagger(a_1, a_2) \left( \psi_1(x), \psi_2(x) \right) \Sigma(a_1, a_2) = (\psi_1(x) e^{-i 2\pi a_1}, \psi_2(x) e^{i 2\pi a_2})$. It follows

$$[\Sigma^\dagger(a_1, a_2), H_{int}] = 0 \iff a_1 + na_2 = t \in \mathbb{Z}.$$  

(S15)

Following (S15), we can parametrize the operator $\Sigma$ as

$$\Sigma(\alpha, \beta) = e^{-i\alpha \int_{-L}^{L} dx \partial_x (n\phi_1 - \phi_2)} e^{i(\beta/n) \int_{-L}^{L} dx \partial_x \phi_2} \equiv \tilde{S}_\alpha S_\beta, \quad \alpha \in \mathbb{R}, \beta \in \mathbb{Z}.$$  

(S16)

$\tilde{S}_\alpha$ can be interpreted as a U(1) symmetry generator when the edge modes carry electromagnetic charge with the charge vector $t = (n, 1)$:

$$\tilde{S}_\alpha : \psi_1 \mapsto \psi_1 e^{i 2\pi n \alpha}, \quad \psi_2 \mapsto \psi_2 e^{i 2\pi \alpha}, \quad \alpha \in \mathbb{R}.$$  

(S17a)

The operator $S_\beta$, on the other hand, reflects the fact that the edge Hamiltonian gives rise to an $n$-particle condensate of phase 2; associated to this interaction is the invariance

$$S_\beta : \psi_1 \mapsto \psi_1, \quad \psi_2 \mapsto \psi_2 e^{i 2\pi \beta/n}, \quad \beta \in \mathbb{Z}.$$  

(S17b)

Moreover we find from (S17) that

$$\tilde{S}^\dagger_\alpha \sigma(x) \tilde{S}_\alpha = \sigma(x), \quad S^\dagger_\beta \sigma(x) S_\beta = \omega^{-\beta} \sigma(x).$$  

(S18)

Of particular interest are the operators

$$S = e^{-\frac{i}{n} \int_{-L}^{L} dx \partial_x \phi_2}$$

(S19)

and

$$A = e^{\frac{i}{n} \int_{-L}^{L} dx \partial_x (n\phi_1 - \phi_2)} e^{-\frac{i}{p} \int_{-L}^{L} dx \partial_x \phi_2} = \sigma(x_L) \sigma(x_R) e^{-\frac{i}{p} \int_{-L}^{L} dx \partial_x \phi_2} \equiv \alpha^\dagger(x_L) \alpha(x_R),$$

(S20)

where $\alpha(x_L) = \sigma(x_L)$ and $\alpha(x_R) = \sigma(x_R) e^{-\frac{i}{p} \int_{-L}^{L} dx \partial_x \phi_2}$ are parafermion operators defined at the ends of the interface. The topological property of the gapped interface is manifested by the presence of these zero modes whereby the $n$-fold degeneracy is encoded in the $n$ eigenvalues of the operator $A$, which commutes with $H_{int}$. Furthermore, after mapping parafermion operators into clock operators $(\sigma(x), \tau(x))$, the ground state $n$-fold degeneracy becomes associated with the symmetry breaking of the clock model, where the generator of the global $\mathbb{Z}_n$ symmetry is $S = \prod_x \tau(x)$. In the following we discuss this mapping between clock and parafermion operators in further detail.
Lattice Regularization: Parafermion/Clock Representations

We now discuss a lattice regularization of the interaction Hamiltonian \((S13)\) responsible for gapping the interface between phases 1 and 2. Such regularization serves a useful purpose in allowing a microscopic representation of the parafermion end states, in connection with recent microscopic models of topological parafermion chains \([S2, S8]\).

We address first the lattice regularization in terms of clock variables. As we have seen, the Hamiltonian \((S13)\) commutes with the operator \(S\) defined in Eq. \((S19)\). This operator can be identified with the generator of \(Z_n\) transformations of the clock operator \(\sigma(x)\) via \(S(x)S = \omega(x)S\). In this language, the ground state degeneracy of the Hamiltonian \((S13)\) becomes associated with the “symmetry broken” states

\[
\forall x : \sigma(x) |\Psi_q\rangle = \omega^q |\Psi_q\rangle, \quad q = 1, \ldots, n, \quad (S21)
\]

where \(\omega \equiv e^{i\frac{2\pi}{n}}\). While it would seem that a particular eigenstate could be identified by the local measurement of \(\sigma(x), \langle \Psi_q |\sigma(x) |\Psi_{q'}\rangle = \omega^\delta_{q,q'}\), it is fundamental to recognize that \(\sigma(x)\) is not a local operator in terms of the original local degrees of freedom \(\psi_1\) and \(\psi_2\).

Despite its non-local character, it is still useful to identify \(\sigma(x)\) as a \(Z_n\) clock operator, with \(\sigma_i = \sigma^{(n-1)}(x)\), while acting on the ground state manifold Eq. \((S21)\). Note that, since \(\Theta(x)\) is a bosonic operator, i.e., \([\Theta(x), \Theta(x')] = 0\) for any coordinates \(x\) and \(x'\), so is \(\sigma(x)\sigma(x') = \sigma(x')\sigma(x)\), for \(x \neq x'\). With that we are led to re-express the interaction \((S13)\) using the relation

\[
e^{i\delta_0 \Theta(x)} e^{i(n-1)\Theta(x+\delta)} = \lim_{\delta \to 0} \sigma(x)\sigma(x+\delta) . \quad (S22)
\]

Eq. \((S22)\) motivates introducing a lattice regularization by defining clock operators \(\sigma_i\) at every site \(i \in 1, \ldots, L\) of the open chain and the canonically conjugated operators \(\tau_i\) satisfying \(\tau_i^\dagger \tau_j = \delta_{ij} \sigma_i\). Then a lattice regularization of the interaction \((S13)\) reads \(H_{Lattice} = -J_L \sum_{i=1}^{L} \sigma_i \sigma_{i+1}^\dagger + \text{H.c.}\), whose \(n\) degenerate ground states satisfy \(\sigma_i |\Psi_q\rangle = \omega^q |\Psi_q\rangle\), for \(q = 1, \ldots, n\), in direct correspondence with \((S21)\).

In the lattice representation, introduce of operators \((\alpha_{2i-1}, \alpha_{2i})\) at every site of the open chain defined by \([S1, S3]\)

\[
\alpha_{2j-1} = \sigma_j \prod_{k=1}^{j-1} \tau_k, \quad \alpha_{2j} = \omega^{(N-1)/2} \sigma_j \prod_{k=1}^{j} \tau_k . \quad (S23)
\]

These new operators satisfy parafermion statistics \([S1]\)

\[
\alpha_i \alpha_j = \omega^{\text{sgn}(j-i)} \alpha_j \alpha_i . \quad (S24)
\]

The field theory equivalent of the parafermion operators \([S23]\) reads

\[
\alpha(x) = \sigma(x) e^{-\frac{1}{2} \int_x^y dz \partial_x \varphi(z)} , \quad (S25)
\]

satisfying the commutation relations

\[
\alpha(x) \alpha(y) = \alpha(y) \alpha(x) e^{i\frac{2\pi}{n} \text{sgn}(y-x)} . \quad (S26)
\]

In the parafermion representation the lattice Hamiltonian acquires the form

\[
H_{Lattice} = -J_L \sum_{i=1}^{L} \omega^{(n-1)/2} \alpha_{2i+1}^\dagger \alpha_{2i} + \text{H.c.} . \quad (S27)
\]

Notably, \(\alpha_1\) and \(\alpha_{2L}\) do not appear in the Hamiltonian; for \(n = 2\) these “dangling” parafermions reduce to the Majorana end states in the topological Kitaev chain. \([S2]\) Manifestly, \([H_{Lattice}, A_{lattice}] = 0\), where \(A_{lattice} = \alpha_{2i}^\dagger \alpha_{2L}\) is the non-local operator connecting the parafermion end state through the gapped bulk.

2 + 1D CHIRAL TOPOLOGICAL STATE COUPLED TO \(Z_n\) GAUGE THEORY

\[
\mathcal{L} = \frac{1}{4\pi} e^{\mu\nu\lambda} e^I_{\mu} K_{IJ}(p,n) \partial_{\nu} e^J_{\lambda} . \quad (S28a)
\]
FIG. S1: (color online) Lattice regularization of the Hamiltonian (S13). The non-trivial dimerization pattern encodes the presence of parafermion excitations at the end points of the chain.

where

\[
\tilde{K}_{IJ}(p, n) = \begin{pmatrix}
p & -1 & 0 \\
-1 & 0 & n \\
0 & n & 0
\end{pmatrix}, \quad c = \begin{pmatrix}
\alpha \\
a \\
b
\end{pmatrix}. \tag{S2b}
\]

\(\tilde{K}(2q + 1, n)\) and \(\tilde{K}(2q, n)\) describe, respectively a \(\nu = 1/(2q + 1)\) fermionic and a \(\nu = 1/(2q)\) bosonic Laughlin state coupled to a deconfined \(\mathbb{Z}_n\) gauge theory. The original charge and flux excitations of the \(\mathbb{Z}_n\) gauge theory are described, respectively, by \(e = (0, 1, 0)\) and \(m = (0, 0, 1)\).

For the choice \(p = 2q + 1\), the GL(3, \(\mathbb{Z}\)) transformation

\[
W_F = \begin{pmatrix}
n & 1 & -1 \\
n(2q + 1) & q & -(q + 1) \\
1 & 0 & 0
\end{pmatrix}, \quad \text{Det}[W_F] = -1, \tag{S29}
\]

implements

\[
W_F^T \tilde{K}(2q + 1, n) W_F = (2q + 1) n^2 \oplus \sigma_z \equiv K_g(2q + 1, n). \tag{S30}
\]

For the choice \(p = 2q\), the GL(3, \(\mathbb{Z}\)) transformation

\[
W_B = \begin{pmatrix}
-n & -1 & 0 \\
-2qn & -q & 1 \\
-1 & 0 & 0
\end{pmatrix}, \quad \text{Det}[W_B] = 1, \tag{S31}
\]

implements

\[
W_B^T \tilde{K}(2q, n) W_B = 2q n^2 \oplus \sigma_x \equiv K_g(2q, n). \tag{S32}
\]

The edge states associated with the bulk theory (S28) are described by the 1 + 1D theory

\[
\mathcal{L} = \frac{1}{4\pi} \partial_t \hat{\Phi}^T \tilde{K} \partial_x \hat{\Phi} + \ldots = \frac{1}{4\pi} \partial_t \Phi^T K_g \partial_x \Phi + \ldots, \tag{S33}
\]

where \(\hat{\Phi} = W \Phi\) sets the relation between the edge fields \(\hat{\Phi}^T = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3)\) and \(\Phi^T = (\phi_1, \phi_2, \phi_3)\) of the two representations.

As discussed in the main text, in the representation given by \(K_g\), the interaction for the case \(p = 2q + 1\)

\[
V_1 = \cos \left( L_1^T K_g \Phi \right), \quad L_1^T = (0, 1, 1) \tag{S34}
\]

gaps the “extra” trivial modes, which then yields the edge state of phase 1. Alternatively, the interaction

\[
V_2 = \cos \left( L_2^T K_g \Phi \right), \quad L_2^T = (1, q n, (q + 1) n) \tag{S35}
\]

gives rise to the edge state of phase 2.

We can now establish a relation between null vectors in the two representations by noticing that, if \(L\) is a null vector of \(K_g\), such that, \(L^T K_g L = 0\), then \(L = W L\) is the corresponding null vector of \(\tilde{K}\), since \(L^T \tilde{K} L = L^T W^T K WL = L^T K_g L = 0\). By making use of this relationship, we find

\[
V_1 = \cos \left( \tilde{L}_1^T \tilde{K} \tilde{\Phi} \right) = \cos \left( \tilde{\phi}_1 - n \tilde{\phi}_3 \right), \quad \tilde{L}_1^T = (WL_1)^T = (0, -1, 0) \tag{S36}
\]

and

\[
V_2 = \cos \left( \tilde{L}_2^T \tilde{K} \tilde{\Phi} \right) = \cos \left( n \tilde{\phi}_2 \right), \quad \tilde{L}_2^T = (WL_2)^T = (0, 0, 1). \tag{S37}
\]
It is clear then from Eqs. (S36) and (S37) that the vacuum associated with \( \langle V_2 \rangle \neq 0 \) confines the excitations of the \( \mathbb{Z}_n \) gauge theory yielding the original phase 2. Moreover, in the basis given by \( \vec{K} \), the deconfined excitations in the vacuum where \( \langle V_1 \rangle \neq 0 \) are described by vertex operators \( \exp (\vec{\ell}^T \bar{\Phi}) \), with \( \vec{\ell}^T = (\ell_1, \ell_2, \ell_3) \), such that \( \vec{\ell}^T L_1 = -\ell_1 = 0 \). Then computing the braiding statistics of two excitations \( \bar{\ell} = (\ell_1, 0, \ell_3) \) and \( \bar{m} = (m_1, 0, m_3) \) we find \( \bar{\ell}^T \bar{K}^{-1} \bar{m} = -\ell_1 m_3 = (\ell_1 + n_k m_3) (m_1 + n_m) \), which accounts for the chiral phase with \( k_1 = (2q + 1)n^2 \). This result is consistent with the topological state having \( pn^2 \) classes of quasiparticles \( \{m_1, \ldots, m_{pn^2}\} \), where \( m_1 \) represents the deconfined magnetic flux originating from the \( \mathbb{Z}_n \) gauge theory and \( \psi_1 = m_{pn} \) represents the fermion (boson) of the \( k = p \) theory (for \( p \) even (odd)), which becomes a non-local quasiparticle upon coupling to the deconfined gauge theory.

**GAPPED PARAFAERMION INTERFACE BEHAVES AS AN ANYON ANDREEV REFLECTOR**

Setting \( p = 1 \) for simplicity, suppose a gapped interface between phases 1 and 2 is formed, which physically implies the condensation of the bosonic field \( \Theta(x) = n\phi_1(x) - \phi_2(x) \), so that \( \langle e^{i\Theta} \rangle = \langle e^{i(n\phi_1 - \phi_2)} \rangle \propto \langle \chi_1 \psi_2 \rangle \neq 0 \). (Equivalently, \( e^{-i\Theta} \propto \langle \chi_1 \psi_2 \rangle \neq 0 \).) Hence, the gapped spectrum originates from the condensation of \( \chi_1 \psi_2 \), or, equivalently, of \( \bar{\chi}_1 \bar{\psi}_2 \). This anyon condensate implies that the gapped interface behaves as a anyon “Andreev reflector” in the following sense.

Let an excitation \( \psi_2 \) approach the gapped edge (for \( p = 1 \), this is the only quasiparticle of the phase 2). Moreover, suppose that the pair \( \chi_1 \bar{\chi}_1 \) is produced from vacuum fluctuations on the topological system 1 near the interface. Then due to the nature of the condensation on the interface, the pair \( \bar{\chi}_1 \psi_2 \) can be condensed and absorbed by the interface, while leaving behind the deconfined excitation \( \chi_1 \) on the other side of the interface.

Reversing the logic, we can ask: what happens if a quasiparticle \( \epsilon_1^\ell \) (\( \ell \in \{1, \ldots, n^2\} \)) belonging to phase 1 approaches the interface where it interacts with the condensate? Since the only allowed condensation process involves the quasiparticle \( \chi_1 \), the following non-trivial process that can occur: close to the interface, (1) the incoming quasiparticle decays into \( \epsilon_1^{\ell} = \epsilon_1^{\ell-pn} \chi_1 \) and, (2) the pair \( \psi_2 \bar{\psi}_2 \) is produced from vacuum fluctuations on the topological system 2 near the interface such that, when the condensation \( \langle \chi_1 \psi_2 \rangle \neq 0 \) occurs, it leaves behind the deconfined excitations \( \epsilon_1^{\ell-pn} = \epsilon_1^{\ell-1} \chi_1 \) and \( \psi_2 \) on different sides of the interface. In particular, the case \( \ell = pn \) represents an incoming \( \chi_1 \) quasiparticle, which can be completely absorbed by the interface while the \( \psi_2 \) excitation appears on the other side.

**COMMUTATION RELATION OF VERTEX OPERATORS**

Define the operators:

\[
\Gamma_k = e^{i c_k \int_{y_k}^{z_k} dx \partial_x (L_k^T \cdot \Phi(x))} = e^{i c_k \left[ L_k^T \cdot \Phi(z_k) - L_k^T \cdot \Phi(y_k) \right]},
\]

(S38)

where \( c_k \in \mathbb{R} \) is a real coefficient and \( (y_k, z_k) \) is a finite interval on the line. It follows that

\[
\Gamma_k \Gamma_p = \Gamma_p \Gamma_k e^{i \theta_{kp}},
\]

\[
\Theta_{kp} = -\pi c_k c_p L_k^T \cdot K^{-1} \cdot L_p \left[ \text{sgn}(z_k - z_p) - \text{sgn}(z_k - y_p) - \text{sgn}(y_k - z_p) + \text{sgn}(y_k - y_p) \right].
\]

(S39)

It can be easily seen from the commutation relations of the edge fields that, if two intervals \( (y_k, z_k) \) and \( (y_p, z_p) \) are non overlapping or one of intervals is entirely contained within the other, then \( \Theta_{kp} = 0 \), which implies that \( \Gamma_k, \Gamma_p = 0 \). If, however, the intervals \( (y_k, z_k) \) and \( (y_p, z_p) \) are partially overlapping such that \( y_k < y_p < z_k < z_p \), then it follows that \( \Theta_{kp} = 2\pi c_k c_p L_k^T \cdot K^{-1} \cdot L_p \).

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