Uniqueness of static spherically symmetric vacuum solutions in the IR limit of Hořava-Lifshitz gravity

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(Dated: October 19, 2010)

Abstract

We investigate static spherically symmetric vacuum solutions in the IR limit of projectable nonrelativistic quantum gravity, including the renormalisable quantum gravity recently proposed by Hořava. It is found that the projectability condition plays an important role. Without the cosmological constant, the spacetime is uniquely given by the Schwarzschild solution. With the cosmological constant, the spacetime is uniquely given by the Kottler (Schwarzschild-(anti) de Sitter) solution for the entirely vacuum spacetime. However, in addition to the Kottler solution, the static spherical and hyperbolic universes are uniquely admissible for the locally empty region, for the positive and negative cosmological constants, respectively, if its nonvanishing contribution to the global Hamiltonian constraint can be compensated by that from the nonempty or nonstatic region. This implies that static spherically symmetric entirely vacuum solutions would not admit the freedom to reproduce the observed flat rotation curves of galaxies. On the other hand, the result for locally empty regions implies that the IR limit of nonrelativistic quantum gravity theories does not simply recover general relativity but includes it.

PACS numbers: 04.20.Jb, 04.60.Bc, 04.70.Dy

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I. I. INTRODUCTION

Recently, Hořava [1] proposed a deeply nonrelativistic quantum gravity theory, which is at a Lifshitz point with dynamical critical exponent $z = 3$ in 3+1 dimensions in the UV limit and apparently reduced to general relativity in the IR limit. This theory is called the Hořava-Lifshitz gravity. This is power-counting renormalisable and intended to be ghost-free. The phenomenological aspects of this theory and its variants have been intensively studied, including black holes [2–5], cosmological implications [6–8] and gravitational waves [10, 11] (see e.g. [11] and references therein).

Among the ingredients of Hořava-Lifshitz gravity are the detailed balance condition and the projectability condition. The detailed balance condition strongly restricts the form of the action. Although Hořava originally assumes this condition from a renormalisability point of view, several authors subsequently abandon this condition and extend the theory to more general class of actions (see e.g. [10, 11]).

The projectability condition is very intriguing because it seems to characterise nonrelativistic theories. Under this condition, the Hamiltonian constraint is nonlocal and obtained by the spatial integral. As a result, the Hamiltonian and momentum constraints constitute a closed algebra. This feature appears to be suitable for quantisation compared to the notorious non-closed algebra for general relativity. Although this condition was often neglected in the early stage of phenomenological studies, it was recently shown [8] that this condition in the IR limit results in the emergence of pressureless fluid additional to general relativity and argued that this can play a role of cold dark matter. On the other hand, it has been generally accepted that the rotation curves of galaxies are flatter than expected from the luminous matter distribution [12]. This observational fact suggests that the so-called galactic dark matter is responsible for the considerable fraction of the total mass of galaxies.

In this article, we study static spherically symmetric vacuum solutions of the IR limit of nonrelativistic quantum gravity theories, taking properly into account the projectability condition. We show that as in general relativity such solutions are uniquely given by the Kottler (Schwarzschild-(anti) de Sitter) solutions. This implies that vacuum solutions would not be capable of maintaining the flat rotation curves of galaxies. On the other hand, we also show that static solutions can be locally described by three-sphere or three-hyperboloid if the spacetime is not entirely but only locally empty. This indicates that nonrelativistic
quantum gravity theories would not simply recover general relativity in their IR limit. We retain both the gravitational constant $G$ and the speed of light $c$.

II. FIELD EQUATIONS

The field variables of the Hořava-Lifshitz gravity and other nonrelativistic quantum gravity theories are the spatial metric $g_{ij}$ as well as the lapse $N$ and the shift $N^i$. The line element in the four dimensional spacetime manifold is given in terms of these geometrical quantities as

$$ds^2 = -N^2c^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$  \hspace{1cm} (2.1)

The action of nonrelativistic quantum gravity is constructed so that it is covariant under the foliation-preserving diffeomorphisms $\delta t = f(t), \delta x^i = \zeta^i(t,x^j)$. That is, the theory has no general covariance but the part of it. We focus on the IR limit of the theory, where we can neglect the higher derivative terms in the action. We will use the curvature tensors for the spatial metric $g_{ij}$. We consider the following action \[ S_g = \int dt d^3x \sqrt{gN} \left[ \alpha(K_{ij}K^{ij} - \lambda K^2) + \xi R + \sigma \right], \]  \hspace{1cm} (2.2)

where $\alpha$, $\lambda$, $\xi$ and $\sigma$ are constant parameters. $K_{ij}$ is the extrinsic curvature, defined by

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i),$$  \hspace{1cm} (2.3)

where the dot denotes the partial derivative with respect to $t$. For the Hořava-Lifshitz gravity, the parameters are given in terms of the original parameters by Hořava \[ \] due to the detailed balance condition as

$$\alpha = \frac{2}{\kappa^2}, \quad \xi = \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \Lambda_W, \quad \sigma = \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} (-3\Lambda_W^2).$$  \hspace{1cm} (2.4)

Although this sign of $\sigma$ implies the negative cosmological constant for $\lambda > 1/3$, this can be flipped by making the analytical continuation of Hořava’s original parameters. We can also include the vacuum energy contribution from the matter sector and make the total cosmological constant positive. Moreover, the above IR limit is recovered not only by the Hořava-Lifshitz gravity but also by much wider class of nonrelativistic quantum gravity theories. In the following we mainly choose $\lambda = 1$ to recover the apparent form of general relativity and hence the apparent Lorentz invariance. Then, we can directly compare this
action to that of general relativity. Then, the speed of light $c$, Newton’s constant $G$ and the cosmological constant $\Lambda$ are related to the parameters as follows:

$$\alpha = \frac{1}{16\pi G c}, \quad \xi = c^2 \alpha, \quad \sigma = -2\Lambda c^2 \alpha. \quad (2.5)$$

We can also consider the matter action but here we focus on the vacuum case except for the possible contribution to the cosmological constant.

To obtain the field equations we take the variation of the action with respect to the lapse $N$, the shift $N^i$ and the spatial metric $g_{ij}$. To do this we should recall that Hořava imposes the projectability condition on the lapse function $N$. This demands that the lapse function $N$ be a function only of $t$, i.e., $N = N(t)$, while the shift $N^i$ and the spatial metric $g_{ij}$ are allowed to be functions of both $t$ and $x^i$. The projectability condition is favourable from a quantisation point of view because the Hamiltonian and momentum constraints then constitute a Lie algebra with respect to the Poisson brackets unlike in general relativity.

The variation with respect to $N$ implies

$$\mathcal{H}_0 \equiv \int d^3x \sqrt{g} \left[ -\alpha (K_{ij} K^{ij} - \lambda K^2) + \xi R + \sigma \right] = 0. \quad (2.6)$$

Thus, due to the projectability condition, the Hamiltonian constraint is given in terms of the spatial integral. On the other hand, the variation with respect to $N^i$ implies the momentum constraint

$$H^i \equiv 2\alpha (\nabla_j K^{ji} - \nabla^i K) = 0, \quad (2.7)$$

which is a local equation.

The equations obtained through the variation with respect to $g_{ij}$, which are the evolution equations, are the following:

$$\alpha \left[ \frac{1}{2} K^{lm} K_{lm} g^{ij} - 2 K^{im} K^j_m - \frac{1}{N \sqrt{g}} (\sqrt{g} K^{ij}) \cdot \nabla_p (K^{jp} v^i) - \nabla_p (K^{ij} v^i) + \nabla_p (K^{ij} v^p) \right]$$

$$- \alpha \left[ \frac{1}{2} K^2 g^{ij} - 2 K K^{ij} - \frac{1}{N \sqrt{g}} (\sqrt{g} K g^{ij}) - \nabla_p (K g^{jp} v^i) - \nabla_p (K g^{ij} v^i) + \nabla_p (K v^p g^{ij}) \right]$$

$$+ \xi \left[ \frac{1}{2} g^{ij} R - R^{ij} \right] + \sigma \frac{1}{2} g^{ij} = 0, \quad (2.8)$$

where $v^i \equiv N^i / N$.

We here assume that the spacetime is static, spherically symmetric and vacuum. The line element on the $t =$const spacelike hypersurface in the areal coordinates is given by

$$ds^2 = e^{2\omega(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.9)$$
The shift vector $N^i$ is required to satisfy $N^r = N^r(r)$, $N^\theta = N^\phi = 0$ in these coordinates.

Then, the momentum constraint yields

$$H_r = 2\alpha(\nabla_j K^j_r - \nabla_r K) = -4\alpha \frac{1}{r} \omega' v^r = 0,$$

where the prime denotes the derivative with respect to $r$.

The $(i, j) = (r, r)$ and $(\theta, \theta)$ components of the evolution equations respectively yield

$$\alpha v^r \left[ v^r \left( 2v'' + \frac{v'}{r} + 4\omega' v^r \right) + \xi \frac{1 - e^{-2\omega}}{r^2} + \frac{1}{2} \sigma \right] = 0,$$

$$\alpha \left[ (v'^r + \omega' v^r + \frac{v'}{r})' v^r + (\omega' v^r + \frac{v'}{r}) \left( v'^r + \omega' v^r + \frac{v'}{r} \right) + \left( \frac{v'}{r} \right)^2 \right] + \xi \frac{\omega'}{r} e^{-2\omega} + \frac{1}{2} \sigma = 0$$

The above two equations give all independent components of the evolution equations.

III. III. STATIC SPHERICALLY SYMMETRIC VACUUM SOLUTIONS

We can easily solve the momentum constraint. The result is $\omega' = 0$ or $v^r = 0$. Hereafter, we solve the equations for each case separately.

In the first case, $\omega = \omega_0$, where $\omega_0$ is constant. Then, Eq. (2.11) yields

$$\alpha v^r \left[ v^r \left( 2v'' + \frac{v'}{r} + 4\omega' v^r \right) + \xi \frac{1 - e^{-2\omega}}{r^2} + \frac{1}{2} \sigma \right] = 0.$$

This can be easily integrated. Thus we obtain the following solution:

$$v^r = -\frac{\xi}{\alpha} \frac{(1 - e^{-2\omega_0}) - \frac{1}{2} \sigma}{\omega_0} r^2 + C, \quad \omega = \omega_0$$

where $C$ is a constant of integration. We can easily check the above solution satisfies both Eqs. (2.11) and (2.12). For this solution, we have

$$R^r_r = 0, \quad R^\theta_\theta = R^\phi_\phi = \frac{1 - e^{-2\omega_0}}{r^2},$$

and hence we can see that the IR limit is justified for $r \to \infty$.

Next we will check the global Hamiltonian constraint. For spherically symmetric case, the global Hamiltonian in the IR limit for vacuum is given by

$$H_0 = 4\pi \int dr r^2 e^\omega \left[ -\alpha (K_{ij} K^{ij} - K^2) + \xi R + \sigma \right] = 0.$$
For the static case, we have
\[ K_{ij}K^{ij} - K^2 = \frac{2}{r^2}(rv^2)' . \] (3.5)

When we substitute the solution into the integrand, we can see it identically vanishes. Thus, Eq. (3.2) gives a solution to the Einstein-Hilbert action theory with the projectability condition.

It is interesting to see what this solution describes. For any static metric which is given by the following form
\[ ds^2 = -N^2 c^2 dt^2 + e^{2\omega(r)}(dr + N^r(r)dt)^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \] (3.6)
where \( N \) is constant, we can always transform this into the diagonal form given by
\[ ds^2 = -\left(1 - e^{2\omega \frac{vr^2}{c^2}}\right)c^2dT^2 + \left(1 - e^{2\omega \frac{vr^2}{c^2}}\right)^{-1}e^{2\omega}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \] (3.7)
through the coordinate transformation
\[ Ndt = dT + \frac{e^{2\omega \frac{vr}{c^2}}}{c^2 - e^{2\omega \frac{vr^2}{c^2}}}dr . \] (3.8)

We substitute Eq. (3.2) into Eq. (3.7) and recover \( G, c \) and \( \Lambda \) using Eq. (2.5). Rescaling \( T \) as \( e^\omega T \rightarrow T \) and putting \( C = 2GM \), we can find the Kottler solution in the standard form
\[ ds^2 = -\left[1 - \frac{1}{3}\Lambda r^2 - \frac{2GM}{c^2 r}\right]c^2dT^2 + \left[1 - \frac{1}{3}\Lambda r^2 - \frac{2GM}{c^2 r}\right]^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) . \] (3.9)

Thus, in this case, the Kottler solution is the only solution and there is no degree of freedom to choose a functional form in the distribution of dark matter.

In the second case, \( N = \text{const} \), \( N^r = 0 \) and \( \omega = \omega(r) \) must satisfy the evolution equations. This is called an ultra-static metric in the literature [13]. Eq. (2.11) then yields
\[ \xi \frac{1}{r^2}(1 - e^{-2\omega}) + \frac{1}{2}\sigma = 0 . \] (3.10)

Therefore, we find
\[ e^{2\omega} = \left(1 + \frac{\sigma}{2\xi r^2}\right)^{-1} = (1 - \Lambda r^2)^{-1} . \] (3.11)
It is easily found that this satisfies both Eqs. (2.11) and (2.12). Thus, rescaling $t$, we obtain the following metric

$$ ds^2 = -dt^2 + \frac{1}{1 - \Lambda r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). $$

(3.12)

For $\Lambda > 0$, $0 < r < 1/\sqrt{\Lambda}$ is allowed and the spatial geometry is given by the three-dimensional sphere of radius $1/\sqrt{\Lambda}$. For $\Lambda < 0$, the spatial geometry is given by the three-dimensional hyperboloid. In fact, the above argument would not depend on the symmetry assumption. For the general ultra-static vacuum metric, Eq. (2.8) yields

$$ R^{ij} = 2\Lambda g^{ij}, $$

(3.13)

For the three dimensional space, this directly means that the space is flat, spherical or hyperbolic depending on the sign of $\sigma$. The above equation also implies that the IR limit is justified if $\Lambda$ is sufficiently small.

However, in this case, the global Hamiltonian constraint plays an important role. The integrand of Eq. (3.4) is calculated to give

$$ r^2 e^\omega \left[ -\alpha (K_{ij} K^{ij} - K^2) + \xi R + \sigma \right] = -2\sigma r^2 (1 - \Lambda r^2)^{-1/2}. $$

(3.14)

The integration yields

$$ H_0 = \begin{cases} 
\frac{c}{2G} \left[ \frac{\arcsin(\sqrt{\Lambda} r)}{\sqrt{\Lambda}} - r \sqrt{1 - \Lambda r^2} \right]_r^{r_{\text{max}}} & \text{for } \Lambda > 0 \\
0 & \text{for } \Lambda = 0 \\
-\frac{c}{2G} \left[ r \sqrt{1 - \Lambda r^2} - \frac{\arcsinh(\sqrt{-\Lambda} r)}{\sqrt{-\Lambda}} \right]_r^{r_{\text{max}}} & \text{for } \Lambda < 0,
\end{cases} $$

(3.15)

where the region of the spacetime described by this metric is $r_{\text{min}} < r < r_{\text{max}}$. Therefore, the global Hamiltonian constraint cannot be satisfied with these metrics alone except for $\sigma = 0$, for which the spacetime is Minkowski. On the other hand, since the Hamiltonian constraint is global, it is still possible that the ultra-static metric with $\Lambda \neq 0$ obtained here can describe a vacuum region which is part of the whole spacetime, where the nonvanishing contribution from the region described by the present metric is compensated by the contribution from the region which is not vacuum and/or not static.
We have investigated that static spherically symmetric vacuum solutions to the IR limit of nonrelativistic quantum gravity with the projectability condition. We have shown that such solutions are uniquely given by the Kottler solutions if the spacetime is entirely empty. If the spacetime is instead locally vacuum, we can also have the possibilities that the spacetime is locally given by the Minkowski or ultra-static solutions with sphere or hyperboloid according to the sign of the cosmological constant. In the latter case, the nonvanishing contribution to the global Hamiltonian constraint must be compensated by the contribution from nonempty or nonstatic regions of the spacetime.

Mukohyama [8] showed that the IR limit of nonrelativistic quantum gravity can be regarded as general relativity plus a dust fluid, which emerges not from the matter sector but as a constant of integration due to the nonlocal Hamiltonian constraint and that the four-velocity of this dust is normal to the constant $t$ spacelike hypersurface compatible with the projectability condition. We call this dust ‘dark dust’ to make the discussion clear. It is true that the dark dust plays the same role as dark matter in the homogeneous universe. On the other hand, we have seen that the static spherically symmetric spacetime is uniquely described by the Kottler solution if the spacetime is entirely empty, in spite of the proper implementation of the projectability condition. This means that the theory in spherical symmetry would not have the freedom of choosing the mass function of the dark dust to exhibit the flat rotation curve observed in galaxies. Moreover, since the four-velocity of the dark dust is given by the unit normal of constant $t$ hypersurfaces, i.e., $u_a = cn_a = -cN(t)\nabla_a t$, the dark dust cannot rotate and hence cannot have the stationary configuration with rotating. It would be also impossible that the dark dust obtains some kind of velocity dispersion which makes the static configuration possible. As a result, it would be very unlikely for the dark dust in this category of theories can explain galactic dark matter responsible for flat rotation curves.

The uniqueness of the static spherically symmetric vacuum solution, which holds for locally vacuum regions in general relativity, would not hold in the present theory. The uniqueness in the present case is the following: 1) If the spacetime is static, spherically symmetric and vacuum everywhere, the spacetime is given by the Kottler solution. 2) If the spacetime is static, spherically symmetric and vacuum within a spherical shell of
finite thickness, the spacetime there is given by the Kottler solution or the static three-dimensional sphere or the static three-dimensional hyperboloid, depending upon the sign of the cosmological constant. The projectability condition clearly plays an important role in this conclusion.

It has been believed that the IR limit of Hořava-Lifshitz gravity or other nonrelativistic quantum gravity theories recovers general relativity. On the contrary, the existence of the dark dust clearly questions this belief. Moreover, as we have seen, the uniqueness of the vacuum static spherically symmetric spacetime is quite different from that in general relativity. As a result, we conclude that Hořava-Lifshitz gravity as well as nonrelativistic quantum gravity with the projectability condition would not recover general relativity in their IR limit. It should be however noted that this fact does not immediately mean that the theory is not viable. It is very important to study whether these features which are different from general relativity in the IR limit would leave any difference which is experimentally testable. It can be conjectured that all solutions of general relativity are also solutions of the IR limit of nonrelativistic quantum gravity theories with the projectability condition but not vice versa.

It should be noted that there generally appears a spin-0 scalar mode of gravitational waves in the Minkowski background in this category of theories. This mode is potentially dangerous from a phenomenological point of view $^{10, 11, 14, 15}$.

After this paper was submitted, the authors realised that Tang and Cheng $^{16}$ reported apparently similar results. The present paper focused the uniqueness of the solutions and is complementary to their work.

Acknowledgements:

The authors would like to thank J. Soda, S. Yahikozawa, K. Nakao, T. Kuroki, S. Mukohyama and M. Saijo for valuable comments and discussion. TH was supported by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Culture, Sports, Science and Technology, Japan [Young Scientists (B) 21740190].

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