Abstract

Projective structures have successfully been used for the construction of measures in the framework of loop quantum gravity. In the present paper we establish such a structure for the space $\mathbb{R} \sqcup \mathbb{R}^{Bohr}$ recently constructed in the context of homogeneous isotropic loop quantum cosmology. This space has the advantage to be canonically embedded into the quantum configuration space of the full theory, but, in contrast to the traditional space $\mathbb{R}^{Bohr}$, there exists no Haar measure on $\mathbb{R} \sqcup \mathbb{R}^{Bohr}$. The introduced projective structure, however, allows to construct a family of canonical measures on $\mathbb{R} \sqcup \mathbb{R}^{Bohr}$ whose corresponding Hilbert spaces of square integrable functions we finally investigate.

1 Introduction

In the framework of loop quantum gravity measures usually are constructed by means of projective structures on the quantum configuration spaces of interest. Indeed, the Ashtekar-Lewandowski measure arises in this way [3] and the same is true for the Haar measure on the Bohr compactification of $\mathbb{R}$. [12] This has served as quantum configuration space of isotropic homogeneous loop quantum cosmology so far. [1] Unfortunately, there is no way to embed $\mathbb{R}^{Bohr}$ canonically into the quantum configuration space of the full theory. [5] This arises from the fact that, in contrast to the full theory, for the definition of the cosmological quantum configuration space only linear curves have been taken into account. To solve this problem in [6] the set of embedded analytic curves was used. This leads to the slightly larger cosmological space $\overline{\mathbb{R}} = \mathbb{R} \sqcup \mathbb{R}^{Bohr}$. Unfortunately, $\overline{\mathbb{R}}$ is lacking in continuous group structures so that for this space no Haar measure can exist. [8] In the present paper we will construct reasonable measures by means of projective structures on $\overline{\mathbb{R}}$. Indeed, the introduced

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projective structures allow to derive the normalized Radon measures
\[ \mu_{\rho,t}(A) = t \rho(\lambda)(A \cap \mathbb{R}) + (1-t) \mu_{\text{Bohr}}(A \cap \mathbb{R}_{\text{Bohr}}) \quad \forall A \in \mathcal{B}(\mathbb{R}). \]

Here \(0 \leq t \leq 1\) and \(\rho(\lambda)\) denotes the push forward of the Lebesgue measure \(\lambda\) on \((0,1)\) by the homeomorphism \(\rho: (0,1) \to \mathbb{R}\). We will see that this family of measures gives rise to only two different Hilbert space structures on \(\mathbb{R}\). More precisely, up to \emph{canonical} isomorphisms we have the following three cases
\[ L_2(\mathbb{R}, \lambda), \quad L_2(\mathbb{R}, \lambda) \oplus L_2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}), \quad L_2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}), \]
where \(L_2(\mathbb{R}, \lambda) \oplus L_2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) \cong L_2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}})\) just by dimensional arguments. However, since \(L_2(\mathbb{R}, \lambda)\) is separable and \(L_2(\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}})\) is not, there cannot exist any isometric isomorphism between these spaces.

This paper is organized as follows:

- Section 2 contains notations and the characterization of projective limits most convenient for our purposes. In Section 3 we review the facts on invariant homomorphisms \cite{8} that we need in the main part.
- In the major Section 4 we briefly illustrate the topological and measure theoretical aspects of the cosmological quantum configuration space \(\mathbb{R}\). Then we investigate how to write this space as a projective limit in order to construct reasonable Radon measures thereon. Here we discuss several possibilities leading to the projective structures we present in the third part of this section. Basically, for their definition we use the fact that for each nowhere vanishing \(f \in C_0(\mathbb{R})\) the functions \(\{f\} \cup \{\chi_l\}_{l \in \mathbb{R}}\) generate the \(C^*\)-algebra \(C_0(\mathbb{R}) \oplus C_A(\mathbb{R})\). Then each \(f\) that is in addition injective gives rise to a projective structure similar to that one introduced in \cite{12} for the space \(\mathbb{R}_{\text{Bohr}}\). Finally, we use this structure to construct a family of normalized Radon measures on \(\mathbb{R}\) which we show to give rise to two different non-isomorphic Hilbert spaces of square integrable functions on \(\mathbb{R}\).

\section{Preliminaries}

We start with fixing the the notations. Then we give a short introduction into projective structures and consistent families of normalized Radon measures.

\subsection{Notations}

A curve \(\gamma\) in a manifold \(M\) is a continuous map \(\gamma: I \to M\) where \(I \subseteq \mathbb{R}\) is an interval. Then \(\gamma\) is said to be of class \(C^k\) iff \(M\) is a \(C^k\)-manifold and iff there is a \(C^k\)-curve \(\gamma': (a',b') \to M\) with \(I \subseteq (a',b')\) and \(\gamma'|_I = \gamma\). If \(\omega\) is a smooth connection on a principal fibre bundle \((P, \pi, M, S)\), \(\gamma: [a, b] \to M\)

\footnote{\(\chi_l: x \mapsto e^{ilx}\)

\footnote{This means that \(I\) is of the form \([a, b], [a, b), (a, b]\) or \((a, b)\) for \(a < b\).}

\footnote{We allow \(k \in \{\infty, \omega\}\) where \(\omega\) means analytic.}

\footnote{in the sense of maps between manifolds}

\footnote{\(P\) denotes the total space, \(M\) the base manifold, \(S\) the structure group and \(\pi: P \to M\) the projection map.}
a $C^k$-curve and $p \in \pi^{-1}(\gamma(a))$, then $\gamma_{p}^{\omega} : [a,b] \to P$ denotes the horizontal lift\footnote{This lift always exists and is at least of class $C^1$, cf. \cite{[11]}.} of $\gamma$ w.r.t. $\omega$ in the point $p$. The morphism $\mathcal{P}_{\omega}^{p} : \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(b))$, $p \mapsto \gamma_{p}^{\omega}(b)$ is called parallel transport along $\gamma$ w.r.t. $\omega$. Here morphism means that $\mathcal{P}_{\omega}^{p}(p \cdot s) = \mathcal{P}_{\gamma}^{p}(p) \cdot s$ for all $p \in F_{\pi}(p)$ and all $s \in S$. In the following a path means a curve that is at least $C^\infty$ and defined on a closed interval. $\mathcal{A}$ denotes the set of all smooth connections on $P$. 

For a $C^*$-algebra $\mathfrak{A}$ let $\text{Spec}(\mathfrak{A})$ denote the set of multiplicative, $\mathbb{C}$-valued functionals on $\mathfrak{A}$ equipped with usual Gelfand-topology. The Gelfand transform $\hat{a} \in C_0(\text{Spec}(\mathfrak{A}))$ of $a \in \mathfrak{A}$ is defined by $\hat{a}(\psi) := \psi(a)$. \vspace{10pt}

**Convention 2.1**

Let $X$ be a set

- $B(X)$ denotes the bounded functions on $X$.
- For a $C^*$-algebra $\mathfrak{A} \subseteq B(X)$ let $X_{\mathfrak{A}}$ denote the set of all $x \in X$ for which $\iota_{X} : X \to \text{Hom}(\mathfrak{A}, \mathbb{C})$
  
  $x \mapsto [f \mapsto f(x)]$

  is non-zero, i.e., $X_{\mathfrak{A}} = \{x \in X \mid \exists f \in \mathfrak{A} : f(x) \neq 0\}$. This means that $x \in X_{\mathfrak{A}}$ iff $\iota_{X}(x) \in \text{Spec}(\mathfrak{A})$ and it can be shown that $\iota_{X}(X_{\mathfrak{A}})$ is dense in $\text{Spec}(\mathfrak{A})$. \cite{[10]}, \cite{[6]}, \cite{[8]}.

- Motivated by that the spectrum of $\mathfrak{A}$ is denoted by $\overline{X}$ in the following.
- If $X$ is a locally compact Hausdorff space, then $C_0(X)$ is the set of continuous functions that vanish at infinity.
- By $C_{\text{AP}}(\mathbb{R})$ we denote the almost periodic functions on $\mathbb{R}$. This is the $C^*$-subalgebra of $B(\mathbb{R})$ generated by the set $\mathbb{R}$ of characters $\chi_{t} : \mathbb{R} \to T$, $x \mapsto e^{itx}$ where $T := \{z \in \mathbb{C} \mid |z| = 1\}$.

- We define the Bohr compactification $\mathbb{R}_{\text{Bohr}}$ of $\mathbb{R}$ by $\text{Spec}(C_{\text{AP}}(\mathbb{R}))$. If $D$ denotes the set of all $^*$-homomorphisms $\psi : \mathbb{R} \to T$, then it follows from Subsection 1.8 in \cite{[11]} that the restriction map $\tau : \mathbb{R}_{\text{Bohr}} \to D$, $\psi \mapsto \psi|_{\mathbb{R}}$ is bijective. This means that for the definition of an element in $\mathbb{R}_{\text{Bohr}}$ it suffices to determine its values on $\mathbb{R}$. The measure $\mu_{\text{Bohr}}$ denotes the Haar measure on $\mathbb{R}_{\text{Bohr}}$ that corresponds to the continuous group structure

$\psi_{1} \cdot \psi_{2} := \tau^{-1}(\tau(\psi_{1}) \cdot \tau(\psi_{2}))$ \hspace{10pt} $\psi^{-1} := \tau^{-1}(\tau(\psi))$ \hspace{10pt} $e(f) = 1$

for all $f \in C_{\text{AP}}(\mathbb{R})$ and $\psi, \psi_{1}, \psi_{2} \in \mathbb{R}_{\text{Bohr}}$. Here $(\zeta \cdot \zeta')(x) := \zeta(x)\zeta'(x)$ as well as $\zeta(x) := \zeta(x)$ for $x \in \mathbb{R}$ and $\zeta, \zeta' \in D$. \hfill $\Diamond$

Finally, if $G$ is a group, $X$ a set and $\varphi : G \times X \to X$ a left action, then $\text{Stab}_{\varphi}(x) := \{g \in G \mid \varphi(g,x) = x\}$ as well as $\varphi_{g} : X \to X$, $x \mapsto \varphi(g,x)$. If $G$ is a Lie group and $g \in G$, then $\alpha_{g} : G \to G$, $h \mapsto ghg^{-1}$ denotes the conjugation w.r.t. $g$ and $\text{Ad}_{g} : g \to g$ the differential $d_{e}\alpha_{g}$ at $e \in G$.\vspace{10pt}
2.2 Projective Structures and Radon measures

Definition 2.2
Let \( \{X_\alpha\}_{\alpha \in I} \) be a family of compact Hausdorff spaces where \((I, \leq)\) is a directed set. A compact Hausdorff space \( X \) is called projective limit of \( \{X_\alpha\}_{\alpha \in I} \) iff

i.) For each \( \alpha \in I \) there is a continuous, surjective \( \pi_\alpha : X \to X_\alpha \).

ii.) For \( \alpha_1, \alpha_2 \in I \) with \( \alpha_1 \leq \alpha_2 \) there is a continuous map \( \pi_{\alpha_2}^{\alpha_1} : X_{\alpha_2} \to X_{\alpha_1} \) such that \( \pi_{\alpha_2}^{\alpha_1} \circ \pi_{\alpha_1} = \pi_{\alpha_2} \).

iii.) If \( x, y \in X \) with \( x \neq y \), then there is some \( \alpha \in I \) such that \( \pi_\alpha(x) \neq \pi_\alpha(y) \).

As shown in Lemma B.1 this is equivalent to the usual definition of a projective limit as a subset of the product \( \prod_{\alpha \in I} X_\alpha \). In particular, each two projective limits of the same family of compact Hausdorff spaces are isomorphic. As it provides us with more flexibility in the following we use Definition 2.2 instead of the Cartesian product version.

Definition 2.3
- A Borel measure \( \mu \) on a Hausdorff space \( Y \) is a locally finite measure \( \mu : \mathcal{B}(Y) \to [0, \infty] \) where \( \mathcal{B}(Y) \) denotes the Borel \( \sigma \)-algebra of \( Y \). It is said to be normalized if \( \|\mu\| := \mu(Y) = 1 \).
- A Borel measure \( \mu \) is called inner regular iff for each \( A \in \mathcal{B}(Y) \) we have \( \mu(A) = \sup \{\mu(K) : K \text{ is compact and } K \subseteq A\} \).
- A Radon measure \( \mu \) is an inner regular Borel measure. It is called finite if \( \mu(Y) < \infty \). Remind that each finite Radon measure is outer regular, i.e., for each \( A \in \mathcal{B}(Y) \) we have \( \mu(A) = \inf \{\mu(U) : U \text{ is open and } A \subseteq U\} \).
- Assume that we are in the situation of Definition 2.2 and \( \{\mu_\alpha\}_{\alpha \in I} \) is a family of Radon measures \( \mu_\alpha : \mathcal{B}(X_\alpha) \to [0, \infty] \). Then \( \{\mu_\alpha\}_{\alpha \in I} \) is called consistent iff \( \mu_{\alpha_1} \) equals the push forward measure \( \pi_{\alpha_2}^{\alpha_1}(\mu_2) \) whenever \( \alpha_1 \leq \alpha_2 \) for \( \alpha_1, \alpha_2 \in I \).

Lemma 2.4
Let \( X \) and \( \{X_\alpha\}_{\alpha \in I} \) be as in Definition 2.2. Then the normalized Radon measures on \( X \) are in bijection with the consistent families of normalized Radon measures on \( \{X_\alpha\}_{\alpha \in I} \).

Proof: See, Lemma B.2.

3 Quantum Configuration Spaces in LQG

In this section we give a short introduction into the theory of invariant generalized connections and homomorphisms of paths. For simplicity here we restrict to the case of trivial principal fibre bundles. In the last part we consider the case of homogeneous isotropic loop quantum gravity.

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7This means that \( \leq \) is a reflexive and transitive relation on \( I \) and for each two \( \alpha, \alpha' \in I \) we find some \( \alpha'' \in I \) such that \( \alpha, \alpha' \leq \alpha'' \).

8The general results can be found in [8].
3.1 Generalized Connections and Invariance

Let $P = M \times S$ denote a trivial principal fibre bundle with base manifold $M$ and compact structure group $S$. Moreover, let $\mathcal{P}$ be a fixed set of paths in $M$. For $\gamma \in \mathcal{P}$ with $\dom(\gamma) = [a, b]$ define $h_\gamma : A \to S$, $\omega \mapsto \left( \pr_2 \circ \mathcal{P}_\omega^\gamma \right) (\gamma(a), e)$ and denote by $\mathfrak{S}$ the $\ast$-algebra generated by all functions of the form $f \circ h_\gamma$ with $f \in C(S)$ and $\gamma \in \mathcal{P}$. By compactness of $S$ we have $\mathfrak{S} \subseteq B(A)$ so that we can define the $C^*$-algebra of cylindrical functions $\mathfrak{C}$ to be closure of $\mathfrak{S}$ in $B(A)$. The spectrum of $\mathfrak{C}$ is denoted by $\overline{\mathfrak{A}}$ and its elements are called generalized connections in the following.

Let $(G, \Theta)$ be a Lie group of automorphisms of $P$, i.e., a Lie group $G$ and a smooth left action $\Theta : G \times P \to P$ such that $\Theta(g, p \cdot s) = \Theta(g, p) \cdot s$ holds for all $p \in P$, $g \in G$ and $s \in S$. Then $\Theta$ gives rise to two further left actions:

- $\vartheta : G \times M \to M$, $(g, m) \mapsto \pi(\Theta(g, p_m))$ where $p_m \in F_m$ is arbitrary,
- $\phi : G \times A \to A$, $(g, \omega) \mapsto \Theta^*(g^{-1})\omega$.

The set of invariant connections is defined by $\mathcal{A}_G := \{\omega \in \mathfrak{A} \mid \Stab{\phi}(\omega) = G\}$ and if $\mathcal{P}$ is invariant in the sense that the path $\gamma'(t) := \vartheta(g, \gamma(t))$ is in $\mathcal{P}$ for all $\gamma \in \mathcal{P}$ and all $g \in G$, then $\phi$ can be uniquely extended to $\overline{\mathfrak{A}}$ in the following sense. There exists a unique left action $\Phi : G \times \overline{\mathfrak{A}} \to \overline{\mathfrak{A}}$ such that for each $g \in G$ we have continuity of $\Phi_g$ and commutativity of the diagram

$$
\begin{array}{ccc}
\overline{\mathfrak{A}} & \xrightarrow{\Phi_g} & \overline{\mathfrak{A}} \\
\downarrow{\iota_A} & & \downarrow{\iota_A} \\
A & \xrightarrow{\phi_g} & A,
\end{array}
$$

where $\iota_A$ denotes the map from Convention 2.1. In analogy to $\mathcal{A}_G$ the set of invariant generalized connections is defined by $\mathcal{A}_G := \{\varpi \in \overline{\mathfrak{A}} \mid \Stab{\phi}(\varpi) = G\}$.\footnote{Cf. Corollary 3.8 in \cite{8}.}

The space $\overline{\mathcal{A}}_G$ is compact and for $\mathfrak{R} := \mathfrak{A}_{\mathcal{P}}$ we have

$$
\Spec(\mathfrak{R}) \cong \overline{\mathcal{A}}_G := \{\varpi \in \overline{\mathfrak{A}} \mid \Stab{\phi}(\varpi) = G\}
$$

via $\overline{\iota} : \Spec(\mathfrak{R}) \to \overline{\mathcal{A}}_G$, $\psi \mapsto [f \mapsto \psi(f \circ i)]$ for $i : \mathcal{A}_G \to \mathfrak{A}$ the inclusion map, $\psi \in \Spec(\mathfrak{R})$ and $f \in \mathfrak{C}$.

$$
\begin{array}{ccc}
\Spec(\mathfrak{R}) & \xrightarrow{\overline{\iota}} & \overline{\mathcal{A}}_G \\
\downarrow{\iota_{\mathcal{A}_G}} & & \downarrow{\iota_A} \\
\mathcal{A}_G & \xrightarrow{i} & \mathcal{A}_G
\end{array}
$$

3.2 Homomorphisms of Paths

Let $P = \mathbb{R}^3 \times SU(2)$ and $\mathcal{P}$ the set of the linear or embedded analytic curves in $\mathbb{R}^3$. Recall that two paths $\gamma_1, \gamma_2 \in \mathcal{P}$ are said to be equivalent (write $\gamma_1 \sim_\mathcal{P} \gamma_2$) iff $\mathcal{P}_\gamma = \mathcal{P}_{\gamma'}$ for all $\omega \in \mathfrak{A}$. Let $\gamma \in \mathcal{P}$ with $\dom(\gamma) = [a, b]$.

\footnote{This means that there is an analytic embedding $\gamma' : (a', b') \to \mathbb{R}^3$ such that $[a, b] \subseteq (a', b')$ and $\gamma = \gamma'|_{[a,b]}$. Here an embedding is an immersion which is a homeomorphism onto its image equipped with relative topology.}
• The inverse curve of $\gamma$ is defined by $\gamma^{-1}: [a, b] \ni t \mapsto \gamma(b + a - t)$.
• A decomposition of $\gamma$ is a family of curves $\{\gamma_i\}_{0 \leq i \leq k-1}$ such that $\gamma|_{[\tau_i, \tau_{i+1}]} = \gamma_i$ for $0 \leq i \leq k - 1$ and real numbers $a = \tau_0 < \ldots < \tau_k = b$.

Then $\mathcal{P}$ is stable under inversion and decomposition of its elements and we define the set $\text{Hom}(\mathcal{P}, SU(2))$ of homomorphisms of paths as follows

An element $\epsilon \in \text{Hom}(\mathcal{P}, SU(2))$ is a map $\epsilon: \mathcal{P} \to \mathcal{S}$ such that

• $\epsilon(\gamma^{-1}) = \epsilon(\gamma)^{-1}$ and $\epsilon(\gamma) = \epsilon(\gamma_{k-1}) \cdot \ldots \cdot \epsilon(\gamma_0)$ for each decomposition $\{\gamma_i\}_{0 \leq i \leq k-1}$ of $\gamma \in \mathcal{P}$
• $\epsilon(\gamma_1) = \epsilon(\gamma_2)$ if $\gamma_1, \gamma_2 \in \mathcal{P}$ with $\gamma_1 \sim_\mathcal{A} \gamma_2$

In particular, for each $\omega \in \mathcal{A}$ the map $\gamma \mapsto h_\gamma(\gamma(a), \epsilon)$ is such a homomorphism. Due to denseness of $\iota_\mathcal{A}(\mathcal{A})$ in $\overline{\mathcal{A}}$ for each $\overline{\omega} \in \overline{\mathcal{A}}$ there is a net $\{\omega_\alpha\}_{\alpha \in \mathcal{I}} \subseteq \mathcal{A}$ with $\{\iota_\mathcal{A}(\omega_\alpha)\}_{\alpha \in \mathcal{I}} \to \omega$ and it follows$^{1}$ that

$$
\eta: \overline{\mathcal{A}} \to \text{Hom}(\mathcal{P}, SU(2))
\overline{\omega} \mapsto \left[ \gamma \mapsto \lim_\alpha h_\gamma(\omega_\alpha) \right]
$$

is a well-defined bijection. In particular, we have (see proof of Lemma B.4 in [8])

$$
\eta(\overline{\omega})(\gamma) = \left( \overline{\omega}\left( [h_\gamma]_{ij} \right) \right)_{ij} \quad \forall \overline{\omega} \in \overline{\mathcal{A}}
$$

(2)

where for $1 \leq i, j \leq 2$ by $[s]_{ij}$ we mean the respective matrix entry of $s \in SU(2)$. Finally, if $(G, \Theta)$ is a Lie group of automorphisms of $P$, then we define the corresponding set of invariant homomorphisms by $\text{Hom}_G(\mathcal{P}, SU(2)) := \eta(\overline{\mathcal{A}}_G)$.

### 3.3 Homogeneous Isotropic Loop Quantum Cosmology

Let $P = \mathbb{R}^3 \rtimes SU(2)$, and consider the Lie group $G := \mathbb{R}^3 \rtimes_q SU(2)$ for $q: SU(2) \to SO(3)$ the universal covering map given by $q(\sigma) = \mu^{-1} \circ \text{Ad}_q \circ \mu$ for $\mu \left( \sum_{i=1}^3 v^i e_i \right) := \sum_{i=1}^3 v^i \tau_i$ with $\{e_1, e_2, e_3\}$ the standard basis in $\mathbb{R}^3$ and

$$
\tau_1 := \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_3 := \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.
$$

We define the left action $\Theta: G \times P \to P$, $(g, p) \mapsto g \cdot q p$. This makes sense, since $G$ and $P$ equal as a set. Then the corresponding set of invariant connections is in bijection with $\mathbb{R}$ via the map $\nu: \mathbb{R} \to \mathcal{A}_G$,

$$
\nu(c)_{(x, s)}(\tilde{v}_x, \tilde{\sigma}_x) := c \text{Ad}_{s^{-1}} [\mu(\tilde{v}_x)] + s^{-1} \tilde{\sigma}_s \quad \text{for} \quad (\tilde{v}_x, \tilde{\sigma}_s) \in T_{(x, s)}P.
$$

$^1$For a more general definition cf. Appendix B in [8].

$^{12}$This definition differs from the usual one [2] in the point that we require $\epsilon$ to be compatible w.r.t. decompositions of paths and not w.r.t. their concatenations. This helps to avoid technicalities as it allows to restrict to embedded analytic curves instead of considering all the piecewise ones.

$^{13}$Cf. Appendix B in [8].
Let $\mathcal{C}_l$ and $\mathcal{C}_\omega$ denote the $C^*$-algebras of cylindrical functions that correspond to the set of linear and the set of embedded analytic curves, respectively. Then

\[ \mathfrak{u}^{\ast}\mathcal{R}_l \cong C_{\text{AP}}(\mathbb{R}) \quad \text{and} \quad \mathfrak{u}^{\ast}\mathcal{R}_\omega \cong C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R}) \]

Originally, $\text{Spec}(\mathfrak{u}^{\ast}\mathcal{R}_l) = \mathbb{R}_{\text{Bohr}}$ was used as cosmological quantum configuration space. This space, however, cannot canonically be embedded into the quantum configuration space $\mathcal{A} := \text{Spec}(\mathcal{C}_\omega)$ of the full theory. To fit this problem in the space $\text{Spec}(\mathfrak{u}^{\ast}\mathcal{R}_\omega)$ was introduced. Now for $\epsilon \in \text{Hom}(\mathcal{P}, SU(2))$ we have $\epsilon(v + \sigma(\gamma)) = (\alpha_\sigma \circ \epsilon)(\gamma)$ for all $v, \sigma, \gamma$.

14 Let $\mathfrak{A}$ denote the set of all continuous, bounded function on $\mathbb{R}$ that can be written as a sum $f_0 + f_{\text{AP}}$ for $f_0 \in C_0(\mathbb{R})$ and $f_{\text{AP}} \in C_{\text{AP}}(\mathbb{R})$. Then Corollary B.2 in [6] shows that $\mathfrak{A}$ is a $C^*$-algebra and $\mathfrak{A} = C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R})$. Here $\oplus$ means the direct sum of vector spaces not of $C^*$-algebras.

15 See e.g. Appendix C.

16 See Subsection 3.3.

4 $\mathbb{R}$ as a Projective Limit

In the first part of this section we highlight the crucial properties of the cosmological quantum configuration space $\mathbb{R}$. Then we provide a projective structure for $\mathbb{R}$ that allows to fix a family of canonical Radon measures for this space. Finally, we investigate the corresponding Hilbert spaces of square integrable functions.

In what follows let $\mathcal{P}$ denote the set of embedded analytic curves in $\mathbb{R}^3$. Moreover, let $P = \mathbb{R}^3 \times SU(2)$ and $G = \mathbb{R}^3 \times \text{SO}(2)$ the Lie group with action $\Theta: G \times P \to P$ from Subsection 3.3.

4.1 Topological Aspects

Using the set of embedded analytic curves in order to define the configuration space of homogeneous isotropic loop quantum cosmology leads to the spectrum of the $C^*$-algebra $\mathfrak{A} = C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R})$. If we equip $\mathbb{R} := \mathbb{R} \sqcup \mathbb{R}_{\text{Bohr}}$
with the topology generated by the sets of the following types: [6]

Type 1: \( V \sqcup \emptyset \) with open \( V \subseteq \mathbb{R} \)
Type 2: \( K^c \sqcup \mathbb{R}_{\text{Bohr}} \) with compact \( K \subseteq \mathbb{R} \)
Type 3: \( f^{-1}(U) \sqcup f^{-1}(U) \) with open \( U \subseteq \mathbb{C} \) and \( f \in \mathcal{C}_{\text{AP}}(\mathbb{R}) \),

then Proposition 3.4 in [6] states that \( \text{Spec}(\mathfrak{A}) \cong \mathbb{R} \sqcup \mathbb{R}_{\text{Bohr}} \) for the homeomorphism \( \xi: \mathbb{R} \sqcup \mathbb{R}_{\text{Bohr}} \to \text{Spec}(\mathfrak{A}) \) defined by

\[
\xi(\mathfrak{x}) := \begin{cases} \ f \mapsto f(\mathfrak{x}) & \text{if } \mathfrak{x} \in \mathbb{R} \\ \ f_0 \oplus f_{\text{AP}} \mapsto \psi(f_{\text{AP}}) & \text{if } \mathfrak{x} \in \mathbb{R}_{\text{Bohr}}. \end{cases}
\] (5)

Here \( f_0 \in C_0(\mathbb{R}) \) and \( f_{\text{AP}} \in \mathcal{C}_{\text{AP}}(\mathbb{R}) \). It is straightforward to see that the subspace topologies of \( \mathbb{R} \) and \( \mathbb{R}_{\text{Bohr}} \) w.r.t. the above topology coincide with their usual ones. Then the next step towards physics is to define a reasonable measure on \( \overline{\mathbb{R}} \) that allow to assign Hilbert space structures to this space. Here we have the following canonical approaches:

1) In analogy to \( \mathbb{R}_{\text{Bohr}} \) we can try to find a Haar measure on \( \overline{\mathbb{R}} \).
2) \( \overline{\mathbb{R}} \) is canonically embedded into the quantum configuration space \( \overline{\mathfrak{A}} \), hence measurable w.r.t. the Ashtekar-Lewandowski measure \( \mu_{\text{AL}} \) on \( \mathfrak{A} \). Consequently, we can use the restriction of \( \mu_{\text{AL}} \) to \( \overline{\mathbb{R}} \).
3) We can try to define a projective structure on \( \overline{\mathbb{R}} \) in order to fix reasonable finite Radon measure thereon. This seems to be the most canonical approach since the Ashtekar-Lewandowski measure on \( \mathfrak{A} \) and the Haar measure on \( \mathbb{R}_{\text{Bohr}} \) arise in this way.

It is shown in [8] that there cannot exist any continuous group structure on \( \overline{\mathbb{R}} \) and [11] shows that \( \overline{\mathbb{R}} \) is of measure zero w.r.t. \( \mu_{\text{AL}} \) so that we will follow up the third approach in the present paper. For this recall the following straightforward result from [8] (see Conclusions in [6]) that characterizes the finite Radon measures on \( \overline{\mathbb{R}} \).

**Lemma 4.1**

i.) We have \( \mathfrak{B}(\overline{\mathbb{R}}) = \mathfrak{B}(\mathbb{R}) \sqcup \mathfrak{B}(\mathbb{R}_{\text{Bohr}}) \).

ii.) If \( \mu \) is a finite Radon measure on \( \mathfrak{B}(\overline{\mathbb{R}}) \), then \( \mu|_{\mathfrak{B}(\mathbb{R})} \), \( \mu|_{\mathfrak{B}(\mathbb{R}_{\text{Bohr}})} \) are finite Radon measures, too. Conversely, if \( \mu_{\mathbb{R}}, \mu_{\text{Bohr}} \) are finite Radon measures on \( \mathfrak{B}(\mathbb{R}) \) and \( \mathfrak{B}(\mathbb{R}_{\text{Bohr}}) \), respectively, then

\[
\mu(A) := \mu_{\mathbb{R}}(A \cap \mathfrak{B}(\mathbb{R})) + \mu_{\text{Bohr}}(A \cap \mathfrak{B}(\mathbb{R}_{\text{Bohr}})) \quad \text{for } A \in \mathfrak{B}(\overline{\mathbb{R}}). \] (6)

is a finite Radon measure on \( \mathfrak{B}(\overline{\mathbb{R}}) \).

**PROOF:** Cf. Lemma 4.6 in [8].

In the next subsection we motivate the projective structure on \( \overline{\mathbb{R}} \) that we introduce in the third part.
4.2 Motivation of the Construction

We start with the same maps \( \pi_\alpha \) that we use in Appendix C to define the Ashtekar-Lewandowski measure \( \mu_{\text{AL}} \) on \( \mathcal{A} \). More precisely, this means that we consider the projection maps

\[
\pi_\alpha : \mathbb{R} \rightarrow SU(2)^k
\]

\[
\mathcal{F} \mapsto (\Delta(\mathcal{F})(\gamma_1), \ldots, \Delta(\mathcal{F})(\gamma_k))
\]

for \( \alpha = (\gamma_1, \ldots, \gamma_k) \in \Gamma := \bigsqcup_{l=1}^{\infty} P^l \) and \( \Delta : \mathbb{R} \rightarrow \text{Hom}_G(P, SU(2)) \) defined by

\[
\Delta := \eta \circ i^* \circ v^* \circ \xi.
\]

Here \( v^* (\psi)(h) := \psi(h \circ v) \) for \( h \in \mathcal{C} \) and \( \psi \in \text{Spec}(v^* \mathcal{R}) \).

As in Appendix C we now have to choose a subset \( \Gamma_0 \) of \( \Gamma \) that can be turned into a directed set. Here we have to take some further issues into account:

- \( \Gamma_0 \) has to be large enough to guarantee that an element \( \mathcal{F} \in \mathbb{R} \) is completely determined by all the values \( \pi_\alpha(\mathcal{F}) \) for \( \alpha \in \Gamma_0 \).
- As the elements in \( \mathbb{R} \) correspond to invariant homomorphisms, for each \( \mathcal{F} \in \mathbb{R} \) the values \( \pi_{\gamma_1}(\mathcal{F}) \) and \( \pi_{\gamma_2}(\mathcal{F}) \) are related if the curves \( \gamma_1 \) and \( \gamma_2 \) only differ by an euclidean transformation. This may give some further restriction to the set \( \Gamma_0 \).
- For each \( \alpha \in \Gamma_0 \) we have to find some reasonable measure \( \mu_\alpha \) on \( \text{im}[\pi_\alpha] \subseteq SU(2)^k \). Here it follows from (4) that we cannot stick to the Haar measures on \( SU(2)^k \) if we want to obtain something that is different from the zero measure on \( \mathbb{R} \).

In reference to the first point we recall that the \( C^* \)-algebras \( \mathcal{R}_\omega := \mathcal{C}_\omega|_{A_G} \) and \( \mathcal{R}_{lc} := \mathcal{C}_{lc}|_{A_G} \) coincide \( \mathcal{R} \cong \text{Spec}(\mathcal{R}_\omega) \cong \text{Spec}(\mathcal{R}_{lc}) \). Here \( \mathcal{C}_{lc} \) denotes the \( C^* \)-algebra of cylindrical functions that corresponds to the set of curves \( \mathcal{P}_{lc} := \mathcal{P}_1 \cup \mathcal{P}_c \) for \( \mathcal{P}_1 \) and \( \mathcal{P}_c \) defined as follows:

- \( \mathcal{P}_1 \) denotes the set of linear curves of the form \( x + \gamma_{\bar{v}} t \) for \( x, \bar{v} \in \mathbb{R}^3 \) with \( ||\bar{v}|| = 1 \) and \( \gamma_{\bar{v}} t : [0, l] \rightarrow \mathbb{R}, t \mapsto t \bar{v} \).
- Let \( \mathcal{P}_c \) consist of all circular curves of the form

\[
\gamma^{x,m}_{\bar{n}, \bar{r}} : [0, 2\pi m] \rightarrow \mathbb{R}^3
\]

\[
t \mapsto x + \cos(t) \bar{r} + \sin(t) \bar{n} \times \bar{r}
\]

for \( \bar{n}, \bar{r}, x \in \mathbb{R}^3 \) with \( ||\bar{n}|| = 1 \) as well as \( 0 < m < 1 \).

Consequently, it suffices to consider the curves in \( \mathcal{P}_{lc} \) in order to satisfy condition iii.) from Definition 2.2. Moreover, due to invariance (3) we get by on the set
\( \mathcal{P}_{\text{red}} \) of all linear and circular curves of the form \( \gamma_l := \gamma_{\vec{e}_1,l} \) and \( \gamma_{m,r} := \gamma_{\vec{e}_1,\vec{r} \vec{e}_1} \).

So, in the first instance we end up with

\[ \Gamma_0 := \{ (\delta_1, \ldots, \delta_k) \mid k \in \mathbb{N}_{>0}, \delta_1, \ldots, \delta_k \in \mathcal{P}_{\text{red}} \}. \]

Now for each \( \vec{\tau} \in \overline{\mathbb{R}} \) we have \( \pi_\gamma(\vec{\tau}) \in T_{\vec{e}_1} \), which follows from the last point in Subsection 3.3 or directly from \( \xi(\vec{\tau})(\chi_l) = \chi_l(x_0) \) for some \( x_0 \in \mathbb{R} \), since then

\[
\pi_\gamma(\vec{\tau}) = \xi(\vec{\tau}) \left( \text{Re}(\chi_l) \right) \mathbb{1} - \xi(\vec{\tau}) \left( \text{Im}(\chi_l) \right) \mu(\vec{e}_1)
= \cos(x_0 l) \mathbb{1} - \sin(x_0 l) \mu(\vec{e}_1)
= \exp(x_0 l \mu(\vec{v})).
\]

Here the first equality follows from (1) and (see e.g. Subsection 4.3 in \([8]\))

\[
\text{pr}_2 \circ \mathcal{P}^\alpha(\iota_{x+l \vec{e}_1}, e) = \cos(c \ l \lVert \vec{v} \rVert) \mathbb{1} - \sin(c \ l \lVert \vec{v} \rVert) \mu(\vec{v} / \lVert \vec{v} \rVert).
\]

Then \( \text{im}[\pi_\gamma] = T_{\vec{e}_1} \) is a Lie subgroup of \( SU(2) \) isomorphic to the circle \( T \).

So, it is natural to choose \( \mu_\alpha \) to be the Haar measure on \( H_{\vec{e}_1}^k \cong T^k \) if \( \alpha \) is of the form \( (\gamma_{l_1}, \ldots, \gamma_{l_k}) \). In order to guarantee \( \mu_\alpha(\text{im}[\pi_\gamma]) \neq 0 \) we may only allow indices \( (\gamma_{l_1}, \ldots, \gamma_{l_k}) \) for which \( l_1, \ldots, l_k \) are \( \mathbb{Z} \)-independent. Then it follows from Kronecker’s theorem (cf. Theorem 4.13 in \([4]\)) that \( \pi_\alpha(\mathbb{R}) \) is dense in \( T_{\vec{e}_1}^k \), hence \( T_{\vec{e}_1}^k = \overline{\text{im}[\pi_\alpha]} = \pi_\alpha(\overline{\mathbb{R}}) \) by compactness of \( \overline{\mathbb{R}} \) and denseness of \( \mathbb{R} \) in \( \overline{\mathbb{R}} \)\(^{15} \). On the level of linear curves this is the same as to consider the directed set

\[ I := \{ (l_1, \ldots, l_k) \in \mathbb{R}^k \mid k \in \mathbb{N}_{>0}, l_1, \ldots, l_k \text{ are } \mathbb{Z} \text{-independent} \}, \]

the maps \( \pi_L : \overline{\mathbb{R}} \to T^k, \vec{\tau} \mapsto (\xi(\vec{\tau})(\chi_{l_1}), \ldots, \xi(\vec{\tau})(\chi_{l_k})) \) for \( L = (l_1, \ldots, l_k) \in I \) and to take the Haar measure \( \mu_{|L|} \) on the \( k \)-torus \( T^k \). Moreover, if we restrict to \( \mathbb{R}_{\text{Bohr}} \subseteq \overline{\mathbb{R}} \) and define the transition maps \( \pi_L' : T|L'| \to T|L| \) by

\[
\pi_L'(s_1, \ldots, s_{k'}) := \left( \prod_{i=1}^{k'} s_i^{n_i}, \ldots, \prod_{i=1}^{k'} s_i^{n_i} \right) \text{ if } \quad l_j = \sum_{i=1}^{k'} n_i s_i^{n_i}
\]

with \( n_j \in \mathbb{Z} \) for \( 1 \leq j \leq k \) and \( 1 \leq i \leq k' \), then we obtain a projective structure and a consistent family \( \{ \mu_{|L|} \}_{L \in I} \) of normalized Radon measure that reproduce the Haar measure on \( \mathbb{R}_{\text{Bohr}} \), cf. Section 4 in \([12]\).

However, we also have to take circular curves into account and the first step towards this is to investigate the image of the maps \( \pi_{\gamma_{m,r}} \). For this recall that\(^{19}\)

\[
\text{pr}_2 \circ \mathcal{P}^\alpha(\iota_{\gamma_{m,r}}, \vec{r}, e) = \exp \left( \frac{T}{2} \mu(n) \right) \cdot \alpha_\sigma(A(\tau, c))
\]

\(^{15}\) Alternatively, one could consider Proposition 4.13 (iv).

\(^{16}\) Define \( (l_1, \ldots, l_k) \leq (l_1', \ldots, l_k') \iff l_i \in \text{span} \{ l_1', \ldots, l_k' \} \text{ for all } 1 \leq i \leq k \). Then \( (I, \leq \mathbb{Z}) \) is directed because \( \mathbb{R} \) is a \( \mathbb{Q} \) vector space and \( l_1, \ldots, l_k \in \mathbb{R} \) are \( \mathbb{Z} \)-independent iff they are \( \mathbb{Q} \)-independent.

\(^{19}\) See e.g. Subsection 4.3 in \([8]\).
for \( \sigma \in SU(2) \) with \( \sigma(\vec{e}_3) = \vec{n} \) and
\[
A(\tau, c) := \begin{pmatrix}
\cos(\beta_c \tau) + \frac{i}{2\beta_c} \sin(\beta_c \tau) & \frac{c}{2\beta_c} \sin(\beta_c \tau)
\\
-\frac{c}{2\beta_c} \sin(\beta_c \tau) & \cos(\beta_c \tau) - \frac{i}{2\beta_c} \sin(\beta_c \tau)
\end{pmatrix}.
\]

Here \( \beta_c := \sqrt{c^2 r^2 + \frac{1}{4}} \) and \( \alpha_\sigma \) denotes the conjugation by \( \sigma \) in \( SU(2) \). Observe that \( \exp(\frac{\tau}{\beta_c} \mu(\vec{n})) = \alpha_\sigma(\exp(\frac{\tau}{\beta_c} \mu(\vec{e}_3))) \) and recall that
\[
\exp(\alpha_\mu(\vec{n})) = \cos(\alpha) \mathbb{1} + \sin(\alpha) \mu(\vec{n}) \quad \forall \vec{n} \in \mathbb{R}^3.
\]

The next lemma describes the images of the maps \( \pi_\delta \) for \( \delta \in \mathcal{P}_c \).

**Lemma 4.2**
Let \( \delta := \gamma_{\vec{n},r}^{\mathbb{R}} \in \mathcal{P}_c \) be fixed and \( \sigma \in SU(2) \) with \( \sigma(\vec{e}_3) = \vec{n} \)

i.) There is no proper Lie subgroup \( H \subsetneq SU(2) \) that contains \( \pi_\delta(\mathbb{R}) \).

ii.) \( \pi_\delta(\mathbb{R}) \) is of measure zero w.r.t. the Haar measure on \( SU(2) \).

iii.) The maps \( \pi_\delta, \{\pi_\gamma\}_{\gamma \in \mathbb{R}_{>0}} \) separate the points in \( \mathbb{R} \).

iv.) We have \( \pi_\delta(\mathbb{R}) = \pi_\delta(\mathbb{R}) \cup d \cdot T_{\delta(0)} \) where \( \pi_\delta(\mathbb{R}) \cap d \cdot T_{\delta(0)} = \{d\} \)

for \( d := \exp(\frac{\tau}{\beta_c} \mu(\vec{n})) \).

v.) Define \( a_n := \frac{1}{r} \sqrt{\frac{\pi^2}{r^2} - 1} \) for \( n \in \mathbb{N}_{\geq 1} \). Then we have
\[
\pi_\delta([a_n, a_{n+1}]) \cap \pi_\delta([a_m, a_{m+1}]) = \emptyset \quad \text{for} \quad m, n \in \mathbb{N}_{\geq 0} \quad \text{with} \quad m \neq n
\]
and \( \pi_\delta(a_{2n}) = d, \pi_\delta(a_{2n-1}) = -d \) for all \( n \in \mathbb{N}_{\geq 1} \). Moreover,
\[
\pi_\delta(\mathbb{R}) = \pi_\delta(\mathbb{R}_{\geq 0}) = \pi_\delta([0, a_1]) \cup \bigcup_{n \in \mathbb{N}_{\geq 0}} \pi_\delta([a_n, a_{n+1}])
\]

where for increasing \( n \geq 1 \) the sets \( \pi_\delta([a_{2n}, a_{2(n+1)}]) \) merge to \( d \cdot T_{\delta(0)} \)
in the following sense. For each \( \epsilon > 0 \) we find \( n_\epsilon \in \mathbb{N} \) such that for all \( m \geq n_\epsilon \) we have that
\[
\forall s \in \pi_\delta([a_m, a_{m+1}]) : \exists s' \in d \cdot T_{\delta(0)} : \|s - s'\|_{op} \leq \epsilon.
\]

**Proof:** See Appendix A. \( \square \)

In analogy to the case of \( \mathbb{R}_{Bohr} \) we might try to consider the set \( \Gamma_0 \) of all tuples \( \alpha = (\gamma_{l_1}, \ldots, \gamma_{l_k}, \gamma_{r_1}, \ldots, \gamma_{r_l}) \) such that \((l_1, \ldots, l_k, r_1, \ldots, r_l)\) are \( \mathbb{Z} \)-independent. Then we have to equip \( \Gamma_0 \) with a relation \( \leq_0 \) in such a way that \( (\Gamma_0, \leq_0) \) is a directed set. Let \( \Gamma_0 \ni \alpha = \gamma_{l} \) and \( \Gamma_0 \ni \alpha' = \gamma_{r,l} \) with \( r \tau = l \).

Then the most uncomplicated way to guarantee directedness of \( I' \) would be to define either \( \alpha' \leq_0 \alpha \) or \( \alpha \leq_0 \alpha' \). But then we have to say how to map \( \text{im}[\pi_\alpha] \) onto \( \text{im}[\pi_{\alpha'}] \) or \( \text{im}[\pi_{\alpha'}] \) onto \( \text{im}[\pi_\alpha] \) in a consistent way. This, however, is not possible for the following reasons.
In the case \( \alpha' \leq 0 \) \( \alpha \) there cannot exist any transition map \( \pi_{\alpha'}^\alpha \), since \( \pi_{\alpha}(0) = e = \pi_{\alpha'}(\frac{2\pi}{n}) \) but \( \pi_{\alpha'}(0) = e \neq \pi_{\alpha'}(\frac{2\pi}{n}) \) as \( \pi_{\alpha'}(x) = \pi_{\alpha'}(y) \) for \( x \neq y \) with \( x, y \geq 0 \) enforces \( \pi_{\alpha'}(x) = \pm d \) by Lemma [1,2] iv.

In the other case we have \( \pi_{\alpha'}(\pm a_{2n}) = -d \) for all \( n \in \mathbb{Z} \setminus \{0\} \) so that

\[
\pi_{\alpha'}^\alpha(-d) = \left( \pi_{\alpha'}^\alpha \circ \pi_{\alpha'} \right)(\pm a_{2n}) = \pi_{\alpha}(\pm a_{2n}) \quad \forall n \in \mathbb{Z} \setminus \{0\}.
\]

Now for \( \epsilon > 0 \) we find \( n_\epsilon \in \mathbb{N} \) such that \( l a_{2n} - l a_{2(n+1)} \in B_\epsilon(2\pi) \setminus \{2\pi\} \) for all \( n \geq n_\epsilon \). But \( \pi_{\alpha}(a_{2n}) = \pi_{\alpha}(a_{2(n+1)}) \) implies \( l a_{2(n+1)} - l a_{2n} = k_n2\pi \) for some \( k_n \in \mathbb{Z} \) so that we get a contradiction if we choose \( \epsilon < 2\pi \).

Now the third part of Lemma [12] shows that, in order to fulfil property iii.) from Definition [22] it suffices to take one fixed circular curve \( \gamma_{\tau, r} \) into account. This means that we can stick to the directed set \( I \) from [9] and then we have the following two possibilities.

1.) For \( I \supseteq L = (l_1, \ldots, l_k) \) we could define \( \pi_L : \mathbb{R} \to SU(2)^{k+1} \) by

\[
\pi_L(\vec{x}) := (\pi_{\gamma_{l_1}}(\vec{x}), \ldots, \pi_{\gamma_{l_k}}(\vec{x}), \pi_{\gamma_{\tau, r}}(\vec{x})).
\]

But then \( \text{im}[\pi_L] \) crucially depends on the \( \mathbb{Z} \)-independence of the real numbers \( l_1, \ldots, l_k, r, \tau \) as for instance we have \( \pi_L(\mathbb{R}_\text{Bohr}) = T_{e_1}^k \times \exp(\frac{\pi}{2} \cdot \tau_3) \cdot T_{e_2}^k \) in the \( \mathbb{Z} \)-independent case as well as \( \pi_L(\mathbb{R}_\text{Bohr}) \cong T \) if \( l_1, \ldots, l_k, r, \tau \) are multiples of the same real number. This makes it difficult to find transition maps and suitable consistent families of measures for these spaces.

2.) Basically, Lemma [12] iii.) is due to the fact that for \( \delta \in \mathcal{P}_c \) the \( C_0(\mathbb{R}) \)-part \( [20] \) of the function (cf. [21]) \( a : \mathbb{R} \ni c \mapsto (\pi_\delta(c))_{11} = (h_\gamma \circ v(c))_{11} \) vanishes nowhere. Now we can try to find some analytic curve \( \gamma \) such that for one of the entries \( (\pi_\delta(\cdot))_{ij} \), \( 1 \leq i, j \leq 2 \), the \( C_{AP}(\mathbb{R}) \)-part is zero and the \( C_0(\mathbb{R}) \)-part vanishes nowhere and is injective. Then condition iii.) from Definition [22] would hold for the projection maps \( \pi_L : \mathbb{R} \to \text{im}[\pi_\gamma] \cup T_{e_1}^k \) defined by [21]

\[
\pi_L(\vec{x}) := \begin{cases} 
\pi_\gamma(\vec{x}) & \text{if } \vec{x} \in \mathbb{R} \\
(\pi_{\gamma_{l_1}}, \ldots, \pi_{\gamma_{l_k}}) & \text{if } \vec{x} \in \mathbb{R}_\text{Bohr}
\end{cases}
\] (13)

and we could define the transition maps and measures on \( \text{im}[\pi_\gamma] \) and \( T_{e_1}^k \) separately. However, even if such a curve \( \gamma \) exists it is not to be expected that it is easier to find reasonable measures on \( \pi_\gamma(\mathbb{R}) \) than on \( \pi_\delta(\mathbb{R}) \) for \( \delta \in \mathcal{P}_c \), cf. Lemma [1,2] i., ii.)

In the next subsection we will follow the philosophy of the second approach. Here we use distinguished generators of \( C_0(\mathbb{R}) \oplus C_{AP}(\mathbb{R}) \) in order to define projective structures on \( \mathbb{R} \) in a more straightforward way.

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20If \( \gamma \) is an embedded analytic curve, then each matrix entry of the map \( h_\gamma \circ v \) is an element in \( \mathfrak{a} = C_0(\mathbb{R}) \oplus C_{AP}(\mathbb{R}) \). [8]

21Use [2] and \( \xi(\vec{x}) = \iota_{2\mathbb{K}}(\vec{x}) \) for \( \vec{x} \in \mathbb{R} \subseteq \mathbb{R} \).
4.3 Projective Structures on \( \mathbb{R} \)

In this subsection we will use families of functions \( \{f\} \cup \{\chi_l\}_{l \in \mathbb{R}} \) with suitable \( f \in C_0(\mathbb{R}) \) in order to define projection maps \( \pi_L : \mathbb{R} \to \text{im}[f] \cup T^{|L|} \) analogous to (13). The next lemma shows that for this it suffices to choose \( f \) nowhere vanishing and injective. Here the first part corresponds to the third part of Lemma 4.2 and shows that the elements in \( \mathbb{R} \) are separated by the values \( \xi(\mathbb{R})(f) \) and \( \{\xi(\mathbb{R})(\chi_l)\}_{l \in \mathbb{R}} \) provided that \( f \) vanishes nowhere. Moreover, it motivates to use some injective generator \( f \in C_0(\mathbb{R}) \) in order to split up \( \mathbb{R} = \mathbb{R} \cup \mathbb{R}_{\text{Bohr}} \) into its canonical parts.

**Lemma 4.3**

i.) Let \( f \in C_0(\mathbb{R}) \) vanishes nowhere. Then the functions \( \{f\} \cup \{\chi_l\}_{l \in \mathbb{R}} \) generate a dense \(*\)-subalgebra of \( C_0(\mathbb{R}) \oplus C_{AP}(\mathbb{R}) \). If \( f \) is in addition injective, then \( f \) generates a dense \(*\)-subalgebra of \( C_0(\mathbb{R}) \).

ii.) Each nowhere vanishing, injective \( f \in C_0(\mathbb{R}) \) is a homeomorphism.

**Proof:**

i.) Since \( f(x) \neq 0 \), for all \( x \in \mathbb{R} \) the \(*\)-algebra \( \mathcal{G} \) generated by the functions \( \{f \cdot \chi_l\}_{l \in \mathbb{R}} \subseteq C_0(\mathbb{R}) \) separates the points in \( \mathbb{R} \) and vanishes nowhere. Consequently, \( \mathcal{G} \) is dense in \( C_0(\mathbb{R}) \) by the complex Stone-Weierstrass theorem for locally compact Hausdorff spaces. Moreover, by definition \( C_{AP}(\mathbb{R}) \) is generated by the functions \( \{\chi_l\}_{l \in \mathbb{R}} \) so that the first claim follows. If \( f \) is in addition injective, then the \(*\)-algebra generated by \( f \) is dense in \( C_0(\mathbb{R}) \) because \( f \) separates the points in \( \mathbb{R} \) and vanishes nowhere.

ii.) We have to verify that \( f \) is open as a map from \( \mathbb{R} \) to \( \text{im}[f] \). Due to injectivity of \( f \) it suffices to show this for relatively compact open subsets as those provide a basis for the topology on \( \mathbb{R} \). Let \( V \) be such a subset and assume that \( f(V) \) is not open in \( \text{im}[f] \). Then there is a converging sequence \( \text{im}[f] \setminus f(V) \ni \{c_n\}_{n \in \mathbb{N}} \to c \in f(V) \) and \( n_0 \in \mathbb{N} \) such that \( |c_n| > \frac{|c|}{2} > 0 \) for all \( n \geq n_0 \). Since \( f \) vanishes at infinity, we find a compact subset \( K \subseteq \mathbb{R} \) such that \( c_n \in f(K) \) for all \( n \geq n_0 \). Then \( f(K \setminus V) \) is compact by continuity of \( f \) and \( c_n \in f(K \setminus V) \) for all \( n \geq n_0 \). It follows that \( c \in f(K \setminus V) \) which contradicts that \( c \in V \).

Altogether, have motivated the following

**Definition 4.4**

Assume that \( f \in C_0(\mathbb{R}) \) is injective and \( f(x) \neq 0 \) for all \( x \in \mathbb{R} \).

i.) Let \( I \) consist of all tuples \( (l_1, \ldots, l_k) \in \mathbb{R}^k \) with \( k \in \mathbb{N}_{>0} \) such that \( l_1, \ldots, l_k \) are \( \mathbb{Z} \)-independent.

ii.) For \( L, L' \in I \) define \( L \leq \mathbb{R} L' \) iff \( l_i \in \text{span}_\mathbb{Z}(l_{1}', \ldots, l_{k}') \) for all \( 1 \leq i \leq k \).

iii.) For \( L \in I \) and \( k := |L| \) define \( \pi_L : \mathbb{R} \to \text{im}[f] \cup T^{|L|} =: X_L \) by

\[
\pi_L(\bar{x}) := \begin{cases} 
  f(\bar{x}) & \text{if } \bar{x} \in \mathbb{R} \\
  (\bar{x}(\chi_{l_1}), \ldots, \bar{x}(\chi_{l_k})) & \text{if } \bar{x} \in \mathbb{R}_{\text{Bohr}}
\end{cases}
\]

(14)

and equip the spaces \( X_L \) with the final topology w.r.t. this map.
iv.) For \( L, L' \in I \) with \( L \leq \Xi L' \) define \( \pi_{L'}^L : X_{L'} \to X_L \) by \( \pi_{L'}^L(y) := y \) if \( y \in \text{im}[f] \) as well as

\[
\pi_{L'}^L(s_1, \ldots, s_{k'}) := \left( \prod_{i=1}^{k'} s_i^{n_i}, \ldots, \prod_{i=1}^{k'} s_i^{n_j} \right) \quad \text{if} \quad l_j = \sum_{i=1}^{k'} n_j^i \quad (15)
\]

with \( n_j^i \in \mathbb{Z} \) for \( 1 \leq j \leq k = |L|, 1 \leq i \leq k' = |L'| \) and \( (s_1, \ldots, s_{k'}) \in T^{[L']} \).

For the rest of this subsection we are concerned with showing that \( \overline{\mathbb{R}} \) is indeed a projective limit of \( \{X_L\}_{L \in I} \). Moreover, we determine the Borel \( \sigma \)-algebras of the spaces \( X_L \). This will lead to an analogous decomposition of finite Radon measures as for the space \( \overline{\mathbb{R}} \). Here the crucial point is to show that the subspace topologies of \( \text{im}[f] \) and \( T^{[L]} \) w.r.t. the final topology on \( X_L \) are just the canonical ones. For this we will need the following definitions and facts:

- Let \( \mathcal{T}_f \) and \( \mathcal{T}_L \) denote the standard topologies on \( \text{im}[f] \) and \( T^{[L]} \), respectively. This is subspace topology on \( \text{im}[f] \) inherited from \( \mathbb{R} \) and product topology on \( T^{[L]} \).
- For \( L \in I \) let \( \tilde{\pi}_L : \mathbb{R}_{\text{Bohr}} \to T^{[L]} \) be the restriction of \( \pi_L \) to \( \mathbb{R}_{\text{Bohr}} \).
- Analogously, let \( \tilde{\pi}_L' : T^{[L']} \to T^{[L]} \) denote the restriction of \( \pi_L' \) to \( T^{[L]} \) if \( L, L' \in I \) with \( L \leq \Xi L' \).
- For \( q \in \mathbb{Q} \) define \( \chi_{l,q} := x_{q,l} \).
- Each \( L \in I \) consists of \( \mathbb{Q} \)-independent reals so that we find (and fix) a subset \( L^c \subseteq \mathbb{R} \) for which \( \mathcal{T} := L \cup L^c \) is a \( \mathbb{Q} \)-base of \( \mathbb{R} \). Then \( \{\chi_{l,m}\}_{(l,m) \in \mathcal{T} \times \mathbb{Z} \setminus \{0\}} \) generates a dense *-subalgebra of \( C_{\text{AP}}(\mathbb{R}) \).
- For \( p \in \mathbb{N}_{\geq 1} \) and \( A \subseteq T \) define \( \tilde{p} : T \to T, s \mapsto s^p \) as well as

\[
\tilde{\sqrt[p]{A}} := \{s \in T \mid s^p \in A\} \quad \text{and} \quad A^p := \{s^p \mid s \in A\}.
\]

If \( O \subseteq T \) is open, then \( O^p \) and \( \tilde{\sqrt[p]{O}} = \tilde{p}^{-1}(O) \) are open as well. This is due to the fact that \( \tilde{p} \) is open (inversion theorem) and continuous.
- For \( A \subseteq T \) and \( m \in \mathbb{Z} \setminus \{0\} \) let

\[
A^{\sigma(m)} := \begin{cases} A & \text{if } m > 0 \\ \{\exists \mid z \in A\} & \text{else.} \end{cases}
\]

The next proposition highlights the crucial properties of the maps \( \tilde{\pi}_L \).

**Proposition 4.5**

i.) Let \( L \in I, \psi \in \mathbb{R}_{\text{Bohr}}, q_i \in \mathbb{Q} \) and \( s_i \in T \) for \( 1 \leq i \leq k \). Choose \( x_i \in \mathbb{R} \) such that \( \chi_{i,q_i}(x_i) = s_i \) for \( 1 \leq i \leq k \). For \( (\tau, q) \in \mathcal{T} \times \mathbb{Q} \) define

\[
\zeta(\chi_{\tau,q}) := \begin{cases} 1 & \text{if } l = 0 \\ \chi_{l,q}(x_i) & \text{if } l = l_i \text{ for some } 1 \leq i \leq k \\ \psi(\chi_{\tau,q}) & \text{else.} \end{cases}
\]

Then \( \zeta \) gives rise to an element \( \psi' \in \mathbb{R}_{\text{Bohr}} \) with \( \psi'(\chi_{i,q_i}) = s_i \) for all \( 1 \leq i \leq k \).
ii.) Let \( l \in \mathbb{R}, m_i \in \mathbb{Z} \setminus \{0\} \) and \( O_i \subseteq T \) open for \( 1 \leq i \leq n \). If \( m := |m_1 \cdots m_n| \) and \( p_i := |m_i| \), then
\[
\bigcap_{i=1}^{n} \widehat{\chi}_{l,m_i}(O_i) = \bigcap_{i=1}^{n} \widehat{\chi}_{l,m_i} \left( \left[ \sqrt[p_i]{O_i} \right]^{\sigma(m_i)} \right) = \widehat{\chi}_{l,m}(O). \tag{16}
\]
for the open subset \( O = \left[ \sqrt[p_1]{O_1} \right]^{\sigma(m_1)} \cap \cdots \cap \left[ \sqrt[p_n]{O_n} \right]^{\sigma(m_n)} \subseteq T \).

iii.) Let \( A_1, B_j \subseteq T \) for with \( B_j \neq \emptyset \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq q \). Moreover, let \( h_1, \ldots, h_q \in \mathbb{R} \) such that \( l_1, \ldots, l_k, h_1, \ldots, h_q \) are \( \mathbb{Z} \)-independent. For \( m_1, \ldots, m_k, n_1, \ldots, n_q \in \mathbb{Z} \setminus \{0\} \) define
\[
W := \widehat{\chi}_{l_1, m_1}(A_1) \cap \cdots \cap \widehat{\chi}_{l_k, m_k}(A_k) \cap \widehat{\chi}_{h_1, n_1}(B_1) \cap \cdots \cap \widehat{\chi}_{h_q, n_q}(B_q). \tag{17}
\]
Then \( \hat{\pi}_L(W) = A_1^{m_1} \times \cdots \times A_k^{m_k} \).

iv.) For each \( L \in I \) the map \( \hat{\pi}_L \) is surjective, continuous and open.

proof: i.) If \( l \in \mathbb{R} \), then \( l = \sum_{i=1}^{k} q_i l_i + \sum_{j=1}^{k'} q_j' l_j' \) for unique elements \( l_j' \in L' \) and \( q_i, q_j' \in \mathbb{Q} \) for \( 1 \leq i \leq k, 1 \leq j \leq k' \). We define
\[
\zeta' \left( \chi_l \right) := \prod_{i=1}^{k} \zeta \left( \chi_{l_i, q_i} \right) \cdot \prod_{j=1}^{q} \zeta \left( \chi_{l'_j, q'_j} \right).
\]
Then \( \zeta': \hat{\mathbb{R}} \to T \) is a *-homomorphism, hence \( \widehat{\mathbb{R}} \) defines an element in \( \mathbb{R}_{\text{Bohr}} \) with the claimed properties.

ii.) Obviously, \( O \) is open and since the second equality in (16) is clear, it suffices to show
\[
\widehat{\chi}_{l,m}(A) = \widehat{\chi}_{l,p-m} \left( \left[ \sqrt[p]{A} \right]^{\sigma(p)} \right)
\]
for \( A \subseteq T, l \in \mathbb{R}, p \in \mathbb{Z} \setminus \{0\} \) and \( m \in \mathbb{N}_{\geq 1} \). For the inclusion \( \supseteq \) let \( \psi \in \widehat{\chi}_{l, p-m} \left( \left[ \sqrt[p]{A} \right]^{\sigma(p)} \right) \). Then
\[
\psi(\chi_{l,p-m}) \in \left[ \sqrt[p]{A} \right]^{\sigma(p)} \implies \psi(\chi_{l,m}) = [\psi(\chi_{l,p-m})]^{[p]} \sigma(p) \in A.
\]
For the converse inclusion let \( \psi \in \widehat{\chi}_{l,m}(A) \). Then
\[
[\psi(\chi_{l,p-m})]^{[p]} \sigma(p) = \psi(\chi_{l,m}) \in A \implies \psi(\chi_{l,p-m}) \in \left[ \sqrt[p]{A} \right]^{\sigma(p)}.
\]
iii.) We proceed in two steps:

\[\text{Cf. Convention 2.7}\]
Lemma 4.6

In the next lemma we highlight the crucial properties of the final topology on the spaces $X_L$. In particular, we determine the Borel $\sigma$-algebras of these spaces.

**Lemma 4.6**

1. The subspace topologies of $\text{im}[f]$ and $T^{|L|}$ w.r.t. the final topology $T_F$ on $X_L$ are given by $T_f$ and $T_L$, respectively. For a subset $U \subseteq \text{im}[f]$ we have $U \in T_f$ iff $U$ is open in $X_L$.

2. $X_L$ is a compact Hausdorff space.

---

23This is due to the fact that the Gelfand topology on $\mathbb{R}^\text{Bohr}$ equals the initial topology w.r.t. the Gelfand transforms of the functions $\chi_l$ for $l \in \mathbb{R}$, see e.g. Subsection 2.3 in [6].
iii.) We have \( \mathcal{B}(X_L) = \mathcal{B}(\text{im}[f]) \cup \mathcal{B}([T/L]) \) and

1. If \( \mu \) is a finite Radon measure on \( \mathcal{B}(X_L) \), then \( \mu|_{\mathcal{B}(\text{im}[f])}, \mu|_{\mathcal{B}([T/L])} \) are finite Radon measures, too.

2. If \( \mu_f: \mathcal{B}(\text{im}[f]) \rightarrow [0, \infty) \) and \( \mu_T: \mathcal{B}([T/L]) \rightarrow [0, \infty) \) are finite Radon measures, then

\[
\mu(A) := \mu_f(A \cap \mathcal{B}(\text{im}[f])) + \mu_T(A \cap \mathcal{B}([T/L])) \quad \forall A \in \mathcal{B}(X_L)
\]

is a finite Radon measure on \( \mathcal{B}(X_L) \).

**Proof:**

i.) We first collect the following facts:

(a) The topology on \( \overline{\mathbb{R}} \) induces the standard topologies on \( \mathbb{R} \) and \( \mathbb{R}\text{Bohr} \).

(b) \( U \in \mathcal{T}_f \) iff \( \pi_L^{-1}(U) \) is open in \( \overline{\mathbb{R}} \).

(c) \( W \subseteq \mathbb{R} \) is open in \( \overline{\mathbb{R}} \) iff \( W \) is open in \( \mathbb{R} \).

(d) If \( B \subseteq \mathbb{R}\text{Bohr} \) is open, then there is \( U \subseteq \text{im}[f] \) such that \( f^{-1}(U) \cap B \) is open in \( \overline{\mathbb{R}} \).

(e) \( f: \mathbb{R} \rightarrow \text{im}[f] \) is a homeomorphism.

(f) \( \tilde{\pi}_L: \mathbb{R}\text{Bohr} \rightarrow [T/L] \) is continuous and open.

We show the statements concerning the subspace topologies:

**im[\( f \):** Let \( U \subseteq \text{im}[f] \). Then:

\[
U \text{ is open w.r.t. topology inherited from } X_L \iff 
\exists V \subseteq [T/L] \text{ such that } U \cup V \text{ is open in } X_L
\]

\[
\iff 
\exists V \subseteq [T/L] \text{ such that } \pi_L^{-1}(U \cup V) \text{ is open in } \overline{\mathbb{R}} \quad (b)
\]

\[
\iff 
\exists V \subseteq [T/L] \text{ such that } f^{-1}(U) \cup \tilde{\pi}_L^{-1}(V) \text{ is open in } \overline{\mathbb{R}}
\]

\[
\iff 
U \in \mathcal{T}_f \quad (e)
\]

**[T/L]:** Let \( V \subseteq [T/L] \). Then:

\[
V \text{ is open w.r.t. the topology inherited from } X_L \iff 
\exists U \subseteq \text{im}[f] \text{ such that } U \cup V \text{ is open in } X_L
\]

\[
\iff 
\exists U \subseteq \text{im}[f] \text{ such that } \pi_L^{-1}(U \cup V) \text{ is open in } \overline{\mathbb{R}} \quad (b)
\]

\[
\iff 
\exists U \subseteq \text{im}[f] \text{ such that } f^{-1}(U) \cup \tilde{\pi}_L^{-1}(V) \text{ is open in } \overline{\mathbb{R}}
\]

\[
\iff 
\tilde{\pi}_L^{-1}(V) \text{ is open in } \mathbb{R}\text{Bohr} \quad (a),(d)
\]

\[
\iff 
V \in \mathcal{T}_L \quad (f)
\]

Finally, observe that \( \text{im}[f] \) is open in \( X_L \) because \( \pi_L^{-1}(\text{im}[f]) = \mathbb{R} \) is open in \( \overline{\mathbb{R}} \). Then \( U \subseteq \text{im}[f] \) is open in \( X_L \) iff \( U \) is open in the subspace topology of \( \text{im}[f] \) induced by \( \mathcal{T}_f \). Since this topology equals \( \mathcal{T}_f \), the claim follows.

---

\(^{24}\)By (e) this is equivalent to show that we find an open subset \( W \subseteq \mathbb{R} \) such that \( W \cup B \) is open in \( \overline{\mathbb{R}} \). But this is clear if \( B = \tilde{\chi}_l^{-1}(O) \) for some open subset \( O \subseteq T \) and \( l \in \mathbb{R} \) (see Type 3 sets defined in Subsection [4.1]). As the sets of the form \( \tilde{\chi}_l^{-1}(O) \) provide a subbasis for the topology on \( \mathbb{R}\text{Bohr} \) the claim follows.
Theorem 4.7

Combining Proposition 4.5 and Lemma 4.6 we obtain

We repeat the arguments from the proof of Lemma 4.6 in [8]:

\[
\begin{align*}
ii.) \text{ The spaces } X_L & \text{ are compact by continuity of } \pi_L \text{ and compactness of } \mathbb{R}. \\
\text{For the Hausdorff property first observe that } T_L & \text{ contains all sets of the following types:} \\
\text{Type 1': } & f(V) \cup \emptyset \quad \text{with open } V \subseteq \mathbb{R}, \\
\text{Type 2': } & f(K^c) \cup T^{[L]} \quad \text{with compact } K \subseteq \mathbb{R}, \\
\text{Type 3': } & f(\chi^{-1}_i(O)) \cup \text{pr}^{-1}_i(O) \quad \text{with } O \subseteq T \text{ open, } 1 \leq i \leq k.
\end{align*}
\]

Here pr$_i$: $T^k \to T$, $(s_1, \ldots, s_k) \mapsto s_i$ denotes the canonical projection. For this observe that the preimage of a set of Type $m'$ is a subset of $\mathbb{R}$ of Type $m$, see Subsection 4.1. Then by injectivity of $f$ the elements of im$[f]$ are separated by sets of Type 1'. Moreover, if $x \in$ im$[f]$ and $(s_1, \ldots, s_k) \in T^k$, then we can choose a relatively compact neighbourhood $W$ of $f^{-1}(x)$ in $\mathbb{R}$ in order to define $U := f(W)$ and $V := f(W^c) \cup T^{[L]}$. Finally, if $(s_1, \ldots, s_k), (s'_1, \ldots, s'_k) \in T^k$ are different elements, then $s_i \neq s'_i$ for some $1 \leq i \leq k$. Consequently, for open subsets $O, O' \subseteq T$ with $O \cap O' = \emptyset$ we have $[f(\chi^{-1}_i(O))] \cup \text{pr}^{-1}_i(O) \cap [f(\chi^{-1}_i(O')) \cup \text{pr}^{-1}_i(O')] = \emptyset$.

\[
\begin{align*}
iii.) \text{ We repeat the arguments from the proof of Lemma 4.6 in [8]:} \\
\text{If } U \subseteq X_L \text{ is open, then } U \cap \text{im}[f] \in T_f \text{ and } U \cap T^{[L]} \in T_L \text{ by the first part of this Lemma. This shows } U \in \mathcal{B}(\text{im}[f]) \cup \mathcal{B}(T^{[L]}), \text{ i.e., } \mathcal{B}(X_L) \subseteq \mathcal{B}(\text{im}[f]) \cup \mathcal{B}(T^{[L]}). \text{ For the converse inclusion observe that } U \in T_f \text{ implies that } U \text{ is open in } X_L, \text{ again by the first part, hence } \mathcal{B}(\text{im}[f]) \subseteq \mathcal{B}(X_L). \text{ Finally, if } A \subseteq T^{[L]} \text{ is closed, then } A \text{ is compact w.r.t. } T_L. \text{ This means that } A \text{ is compact w.r.t. subspace topology inherited by } X_L \text{ implying that } A \text{ is compact as a subset of } X_L. \text{ Then } A \text{ is closed since } X_L \text{ is Hausdorff so that } A \in \mathcal{B}(X_L), \text{ hence } \mathcal{B}(T^{[L]}) \subseteq \mathcal{B}(X_L).
\end{align*}
\]

1.) The measures $\mu|_{\mathcal{B}(\text{im}[f])}$ and $\mu|_{\mathcal{B}(T^{[L]})}$ are well-defined and obviously finite. Their inner regularities follow from the fact that subsets of im$[f]$ and $T^{[L]}$ are compact w.r.t. $T_f$ and $T_L$, respectively, iff they are so w.r.t. the topology on $X_L$ just by the first part of this lemma.

2.) For the second statement let $\mu$ be defined as above. Then $\mu$ is a finite Borel measure and its inner regularity follows by a simple $\epsilon/2$ argument from inner regularity of $\mu_f$ and $\mu_T$.

Combining Proposition 4.5 and Lemma 4.6 we obtain

Theorem 4.7

i.) $\mathbb{R}$ is a projective limit of $\{X_L\}_{L \in I}$.

ii.) A family $\{\mu_L\}_{L \in I}$ of measures $\mu_L$ on $X_L$ is a consistent family of normalized Radon measures w.r.t. $\{X_L\}_{L \in I}$ iff the following holds:

(a) There is $t \in [0,1]$ such that for each $L \in I$ and $A \in \mathcal{B}(X_L)$ we have

\[
\mu_L(A) = t \mu_f(A \cap \mathcal{B}(\text{im}[f])) + (1-t) \mu_{T,L}(A \cap \mathcal{B}(T^{[L]}))
\]

for $\mu_f$ a normalized Radon measure on im$[f]$ and $\mu_{T,L}$ a normalized Radon measure on $T^{[L]}$. 

\]
Theorem 4.7 shows that fixing a normalized Radon measure on $X_L$ as follows:

**Proof:**

1. Let $f$ be a family of measures $\{\mu_L\}_{L \in I}$ that are compact and Hausdorff as the second part of Lemma 4.6 shows. Moreover, each $\mu_L$ is surjective by Proposition 4.5. If $L, L' \in I$ with $L \leq L'$, then continuity of the maps $\pi_L^L$ is clear from the multiplicativity of the functions $\chi_L$. Finally, condition iii.) from Definition 2.2 follows from injectivity of $f$ and the fact that the functions $\{\chi_L\}_{L \in I}$ generate $C_0(\mathbb{R})$.

2. Let $\{\mu_L\}_{L \in I}$ be a consistent family of normalized Radon measures w.r.t. $\{X_L\}_{L \in I}$. Then the second part of Lemma 4.6 shows that

\[ \mu_L(A) = \tilde{\mu}_f(A \cap \mathfrak{B}(\text{im}[f])) + \tilde{\mu}_{T,L}(A \cap \mathfrak{B}(T|L|)) \]

for all $A \in \mathfrak{B}(X_L)$. Here $\tilde{\mu}_f, L$ is a finite Radon measure on $\text{im}[f]$ and $\tilde{\mu}_2, L$ a finite Radon measure on $T|L|$. For each $L \in I$. Then consistency forces $\tilde{\mu}_f, L = \tilde{\mu}_{T,L}$ for all $L, L' \in I$. In fact, by Lemma 4.6 there is a normalized Radon measure $\mu$ on $\mathbb{R}$ such that $\mu_L = \pi_L(\mu)$ for all $L \in I$. Consequently, for each $A \in \mathfrak{B}(\text{im}[f])$ and all $L \in I$ we have

\[ \tilde{\mu}_f, L(A) = \mu_L(A) = \pi_L(\mu)(A) = \mu(f^{-1}(A)) =: \tilde{\mu}_f(A). \]

Condition (b) then easily follow from the consistency of the measures $\{\mu_L\}_{L \in I}$. Finally, for $t := \tilde{\mu}_f(\text{im}[f]) \in (0,1)$ condition (a) follows for $\mu_f := \frac{1}{t} \tilde{\mu}_f$ as well as $\mu_T, L := \frac{1}{t} \tilde{\mu}_{2, L}$ for $L \in I$. In the case $t = 0$ we define $\mu_T, L := \tilde{\mu}_{T,L}$ for $L \in I$ and if $t = 1$ we let $\mu_f := \tilde{\mu}_f$.

For the converse implication let $\{\mu_L\}_{L \in I}$ be a family of measures $\mu_L$ on $X_L$ such that (a) and (b) holds. Then the second part of Lemma 4.6 shows that each $\mu_L$ is a finite Radon measure and obviously we have $\mu_L (X_L) = 1$. Finally, from condition (b) for $A \in \mathfrak{B}(X_L)$ we get

\[ \pi_L^L(\mu_L)(A) = \mu_L^L\left((\pi_L^L)^{-1}(A)\right) \]

\[ = t \mu_f\left((\pi_L^L)^{-1}(A) \cap \text{im}[f]\right) + (1 - t) \mu_{T,L}^L\left((\pi_L^L)^{-1}(A \cap T|L|)\right) \]

\[ = t \mu_f\left(A \cap \text{im}[f]\right) + (1 - t) \mu_{T,L}^L\left(A \cap T|L|\right) \]

\[ = t \mu_f\left(A \cap \text{im}[f]\right) + (1 - t) \mu_{T,L}^L\left(A \cap T|L|\right) \]

\[ = \mu_L(A). \]

4.4 Radon Measures on $\overline{\mathbb{R}}$

Theorem 4.7 shows that fixing a normalized Radon measure on $\mathbb{R}$ can be done as follows:

1. Determine a family of normalized Radon measures $\{\mu_{T,L}\}_{L \in I}$ on $T|L|$ that fulfill condition (b).

2. Fix an injective and nowhere vanishing element $f \in C_0(\mathbb{R})$ with suitable image together with a normalized Radon measure $\mu_f$ on $\text{im}[f]$.
3 Adjust \( t \in [0, 1] \).

In the following let \( \lambda \) denote the Lebesgue measure on \( \mathcal{B}(\mathbb{R}) \). Moreover, for \( B \in \mathcal{B}(\mathbb{R}) \) and \( \eta: B \rightarrow \mathbb{R} \) measurable we define \( \eta(\lambda) := \eta(\lambda|_{\mathcal{B}(B)}) \).

**Step 1**

We choose \( \mu_{T,L} \) to be the Haar measure \( \mu|_{L} \) on \( T^{|L|} \) because

- This is canonical from the mathematical point of view.
- As we will see in Lemma 4.8 these measures fulfil the required compatibility conditions.
- This is in analogy to the case \( \mathbb{R}^{\text{Bohr}} \) where this choice results in the usual Haar measure on this space, see Subsection 4.2.
- These measures will imply a canonical choice of \( f \) and \( \mu_f \) in Step 2.

**Lemma 4.8**

Let \( \mu_f: \mathcal{B}(\text{im}[f]) \rightarrow [0, 1] \) be a normalized Radon measure and \( t \in [0, 1] \). For each \( L \in I \) let

\[
\mu_L(A) := t \mu_f(A \cap \text{im}[f]) + (1 - t) \mu|_{L}(A \cap T^{|L|}) \quad \forall \ A \in \mathcal{B}(X_L).
\]

Then \( \{\mu_L\}_{L \in I} \) is a consistent family of normalized Radon measures and the corresponding normalized Radon measure \( \mu \) on \( \mathbb{R}^{\text{Bohr}} \) is given by

\[
\mu(A) = t f^{-1}(\mu_f)(A \cap \mathbb{R}) + (1 - t) \mu^{\text{Bohr}}(A \cap \mathbb{R}^{\text{Bohr}}) \quad \forall \ A \in \mathcal{B}(\mathbb{R}). \tag{18}
\]

**Proof:** Let \( L \in I, \ A \in \mathcal{B}(X_L) \) and \( \mu \) be defined by (18). Then

\[
\pi_L(\mu)(A) = t f^{-1}(\mu_f)(f^{-1}(A \cap \text{im}[f])) + (1 - t) \mu^{\text{Bohr}}(\pi_L^{-1}(A \cap T^{|L|}))
= t \mu_f(A \cap \text{im}[f]) + (1 - t) \pi_L(\mu^{\text{Bohr}})(A \cap T^{|L|}).
\]

So, in order to verify (18) by Lemma 2.4 it suffices to show \( \pi_L(\mu^{\text{Bohr}}) = \mu|_{L} \) for each \( L \in I \) provided that we know that \( \{\mu_L\}_{L \in I} \) is a consistent family of normalized Radon measures w.r.t. \( \{X_L\}_{L \in I} \). This in turn is immediate from Theorem 4.7 if we can verify that for \( L, L' \in I \) with \( L \leq L' \) we have \( \pi_L^{\ast}(\mu|_{L}) = \mu|_{L'} \). But this would follow from \( \pi_L(\mu^{\text{Bohr}}) = \mu|_{L} \) as well, since then

\[
\pi_L^{\ast}(\mu|_{L}) = \left( \hat{\pi}_L' \circ \pi_L \right)(\mu^{\text{Bohr}}) = \hat{\pi}_L(\mu^{\text{Bohr}}) = \mu|_{L}.
\]

Now, to show \( \pi_L(\mu^{\text{Bohr}}) = \mu|_{L} \) it suffices to verify the translation invariance of the normalized Radon measure \( \pi_L(\mu^{\text{Bohr}}) \). To this end let \( \tau \in T^{|L|} \). By surjectivity of \( \pi_L \) we find some \( \psi \in \mathbb{R}^{\text{Bohr}} \) with \( \pi_L(\psi) = \tau \). Since \( \pi_L \) is a homomorphism w.r.t. the group structure \(^{25}\) on \( \mathbb{R}^{\text{Bohr}} \), for \( A \subseteq T^{|L|} \) we have

\[
\pi_L(\psi + \pi_L^{-1}(A)) = \pi_L(\psi) \cdot \pi_L(\pi_L^{-1}(A)) = \tau \cdot A.
\]

\(^{25}\)Cf. Convention 2.1.
Applying $\pi^{-1}_T$ yields $\psi + \pi^{-1}_T(A) \subseteq \pi^{-1}_T(\tau \cdot A)$. For the inverse inclusion let $\psi' \in \pi^{-1}_T(\tau \cdot A)$. Then $\psi' + \psi' \in \pi^{-1}_T(A)$ because $\pi_L(\psi' + \psi') = \tau^{-1} \cdot \pi_L(\psi') \in A$. Consequently, $\psi' \in \psi + \pi^{-1}_L(A)$, hence $\pi^{-1}_L(\tau \cdot A) \subseteq \psi + \pi^{-1}_L(A)$ so that

\[
\pi_L(\mu_{\text{Bohr}})(\tau \cdot A) = \mu_{\text{Bohr}}(\pi^{-1}_L(\tau \cdot A)) = \mu_{\text{Bohr}}(\psi + \pi^{-1}_L(A))
\]

\[
= \mu_{\text{Bohr}}(\pi^{-1}_L(A)) = \pi_L(\mu_{\text{Bohr}})(A)
\]

for all $A \in \mathfrak{B}(T^{|L|})$. This shows that $\pi_L(\mu_{\text{Bohr}})$ is translation invariant. ■

Step 2

If $f, f' \in C_0(\mathbb{R})$ are injective and vanish nowhere, then the respective projective structures from Definition 4.3 are equivalent in the sense that the corresponding spaces $X_L, X'_L$ are homeomorphic via the maps $\Omega_L: X_L \to X'_L$ defined by $\Omega_L|_{T^k} = \text{id}_{T^k}$ and $\Omega_L|_{\text{im}[f]} := f' \circ f^{-1}$. Moreover, if $\mu_f$ is a normalized Radon measure on $\text{im}[f]$, then $\mu_f := (f' \circ f^{-1})(\mu_f)$ is a normalized Radon measure on $\text{im}[f]$ and it is clear from (18) that the corresponding Radon measures $\mu, \mu'$ on $\mathbb{R}$ from Lemma 4.8 coincide. All this makes sense, since in contrast to $C_{AP}(\mathbb{R})$, where we have the canonical generators $\{\chi_l\}_{l \in \mathbb{R}}$, in $C_0(\mathbb{R})$ there is no distinguished, nowhere vanishing, injective generator $f \in C_0(\mathbb{R})$. But this also means that we can restrict to functions with a reasonable image such as the shifted Torus $T_{z} := 1 + T \{ -1 \} \subseteq \mathbb{C}$. In fact, here the analogy to $T^k$ suggests to use the Haar measure $\mu_1$ on $T$. So, in the following we will restrict to the elements in $F := \{ f \in C_0(\mathbb{R}) \mid \text{im}[f] = T_z \}$ where for each $f \in F$ we define $\mu_f := \mu_a: \mathfrak{B}(T_{z}) \to [0,1]$ with $\mu_a(A) := \mu_1(A - 1)$. But then it follows that

\[
\{ f^{-1}(\mu_f) \mid f \in F \} = \{ \rho(\lambda) \mid \rho \in H \}
\]

for $H$ the set of all homeomorphisms $\rho: (0,1) \to \mathbb{R}$.

Proof of (19): We consider the functions $+1: T \{ -1 \} \ni z \mapsto z + 1 \in T_{z}$ and $h: [0,1] \ni t \mapsto e^{i2\pi[t-1/2]} \in T$. Then $\mu_1 = h(\lambda)|_{\mathfrak{B}(T_{z})}$ and $\mu_a = +1(\mu_1)$ and for $f \in F$ we have $f^{-1}(\mu_f) = \rho(\lambda)$ for $H \ni \rho := f^{-1} \circ +1 \circ h|_{[0,1]}$. Conversely, if $\rho \in H$, then $\rho(\lambda) = f^{-1}(\mu_a)$ for $F \ni f := +1 \circ h \circ \rho^{-1}$. ■

So, if we restrict to projective structures arising from elements $f \in F$, then Lemma 4.8 and (19) select the normalized Radon measures of the form

\[
\mu_{\rho(t)}(A) := t \rho(\lambda)(A \cap \mathbb{R}) + (1 - t) \mu_{\text{Bohr}}(A \cap \mathbb{R}_{\text{Bohr}}) \quad \forall A \in \mathfrak{B}(\mathbb{R})
\]

where $\rho: (0,1) \to \mathbb{R}$ is a homeomorphism and $t \in [0,1]$.

Step 3

To adjust the parameters $t \in [0,1]$ we now take a look at the Hilbert spaces $\mathcal{H}_{\rho,t} := L_2(\mathbb{R}, \mu_{\rho,t})$.

Lemma 4.9

For $A \in \mathfrak{B}(\mathbb{R})$ let $\chi_A$ denote the corresponding characteristic function.

i.) If $\rho_1, \rho_2: (0,1) \to \mathbb{R}$ are homeomorphisms and $t_1, t_2 \in (0,1)$, then

\[
\varphi: L_2(\mathbb{R}, \mu_{\rho_1,t_1}) \to L_2(\mathbb{R}, \mu_{\rho_2,t_2})
\]

\[
\psi \mapsto \frac{t_1}{t_2} \left( \chi_\mathbb{R} \cdot \psi \right) \circ (\rho_1 \circ \rho_2^{-1}) + \frac{(1 - t_1)}{(1 - t_2)} \chi_{\mathbb{R}_{\text{Bohr}}} \cdot \psi
\]
is an isometric isomorphism. The same is true for
\[ \varphi : \mathcal{H}_{\rho_1,1} \rightarrow \mathcal{H}_{\rho_2,1}, \quad \psi \mapsto (\chi_\mathbb{R} \cdot \psi) \circ (\rho_1 \circ \rho_2^{-1}) , \]
\[ \varphi : \mathcal{H}_{\rho_1,0} \rightarrow \mathcal{H}_{\rho_2,0}, \quad \psi \mapsto \psi. \]

ii.) If \( t = 1 \), then \( \mathcal{H}_{\rho,1} \cong \mathbb{L}_2 (\mathbb{R}, \rho(\lambda)) \cong \mathbb{L}_2 (\mathbb{R}, \lambda) \) for each \( \rho \in \mathcal{H} \). Here \( \cong \) means canonically isometrically isomorphic.

PROOF:  

i.) This is immediate from the general transformation formula.

ii.) The first isomorphism is due to \( \mu_{\rho,0}(\mathbb{R}_{\text{Bohr}}) = 0 \) and by the first part it suffices to specify the second isomorphism for the case that \( \rho \) is a diffeomorphism. But in this case we have \( \rho(\lambda) = \frac{1}{|\rho|} \lambda \) so that for the isomorphism \( \varphi : \mathbb{L}_2 (\mathbb{R}, \rho(\lambda)) \rightarrow \mathbb{L}_2 (\mathbb{R}, \lambda), \psi \mapsto \frac{1}{\sqrt{|\rho|}} \psi \) we have
\[
\langle \varphi(\psi_1), \varphi(\psi_2) \rangle_\lambda = \int_{\mathbb{R}} \psi_1 \overline{\psi}_2 \frac{1}{|\rho|} d\lambda = \int_{\mathbb{R}} \psi_1 \overline{\psi}_2 \frac{d\rho(\lambda)}{|\rho|} = \langle \psi_1, \psi_2 \rangle_{\rho(\lambda)}. \]

Let \( \rho_0 \in \mathcal{H} \) and \( t_0 \in (0,1) \) be fixed. Then Lemma 4.9 shows that up to canonical isometrically isomorphisms the parameters \( \rho \) and \( t \) give rise to the following three Hilbert space structures:

1) \( \mathbb{L}_2 (\mathbb{R}, \lambda) \cong \mathbb{L}_2 (\mathbb{R}, \rho(\lambda)) \cong \mathcal{H}_{\rho,1} \) for all \( \rho \in \mathcal{H} \) \hspace{1cm} (Lemma 4.9 ii.)

2) \( \mathbb{L}_2 (\mathbb{R}, \mu_{\rho_0, t_0}) \cong \mathcal{H}_{\rho,t} \) for all \( \rho \in \mathcal{H}, t \in (0,1) \) \hspace{1cm} (Lemma 4.9 i.)

3) \( \mathbb{L}_2 (\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) \cong \mathcal{H}_{\rho,0} \) for all \( \rho \in \mathcal{H} \) \hspace{1cm} (\mathbb{R} \text{ is of measure zero})

Here 2) and 3) are isometrically isomorphic because their Hilbert space dimensions coincide. Similarly, the cases 1) and 3) cannot be isometrically isomorphic since \( \mathbb{L}_2 (\mathbb{R}, \lambda) \) is separable and \( \mathbb{L}_2 (\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) \) is not so.

5 Conclusions

We have used distinguished generators of the C*-algebra \( \mathfrak{A} = C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R}) \) in order to establish projective structures on the space \( \mathbb{R} \sqcup \mathbb{R}_{\text{Bohr}} \cong \text{Spec}(\mathfrak{A}) \). These structures then were used to derive the family \( \{ \mu_{\rho,t} \}_{\rho \in \mathcal{H}, t \in [0,1]} \) of normalized Radon measures on this space. Afterwards, we have shown that up to isometric isomorphisms these measures just reproduce the Hilbert spaces \( \mathbb{L}_2 (\mathbb{R}, \lambda) \) and \( \mathbb{L}_2 (\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) \). However, even if \( \mathbb{L}_2 (\mathbb{R}, \mu_{\rho_0, t_0}) \) and \( \mathbb{L}_2 (\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) \) are isometrically isomorphic, on these spaces may exist representations of the algebra of elementary variables that are not unitary equivalent. So, the next step will be to define such representations for \( \mathbb{L}_2 (\mathbb{R}, \mu_{\rho_0, t_0}) \) and then to investigate their relations to the standard representations on \( \mathbb{L}_2 (\mathbb{R}_{\text{Bohr}}, \mu_{\text{Bohr}}) \).

\[ \text{See } \text{[1]}. \]
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Appendix

A Proof of the Lemma in Subsection 4.2

Proof of Lemma 4.2:

\( i. \) It suffices to show the claim for \( \mathbb{R} \subseteq \mathbb{R} \). Now, if \( c \in \mathbb{R} \) and \( g \in \mathbb{C} \), then

\[
(i^*_x \circ \overline{v}^*_x \circ \xi_x)(c)(g) = (i^*_x \circ \overline{v}^*_x \circ \iota_x)(c)(g) = (\iota_x \circ i \circ v_x)(c)(g)
\]

so that \((i^*_x \circ \overline{v}^*_x \circ \xi_x)(c) = (\iota_x \circ i \circ v_x)(c)\) and consequently

\[
\pi_x(c) = h_x \left( \left( i^*_x \circ \overline{v}^*_x \circ \xi_x \right)(c) \right) = h_x \left( (\iota_x \circ i \circ v_x)(c) \right) = \left( \Pr_2 \circ \mathcal{P}_x^\delta(c) \right)(x, e) = \exp \left( \frac{\tau}{2} \right) \cdot \alpha_x(A(\tau, c)).
\]

This shows continuity of \( \pi_x|_\mathbb{R} \) so that the set \( \pi_x(\mathbb{R}) \subseteq SU(2) \) is connected.

Now each proper Lie subalgebra of \( \mathfrak{su}(2) \) is of dimension 1 and since \( SU(2) \) is connected, the same is true for the proper Lie subgroups of \( SU(2) \). Let \( H \subseteq SU(2) \) be such a proper Lie subgroup with \( \pi_x(\mathbb{R}) \subseteq H \). Then \( \mathfrak{h} = \text{span}_\mathbb{R} (s) \) for some \( s \in \mathfrak{su}(2) \) and \( T_x \) is the unique connected Lie subgroup of \( H \) with Lie algebra \( \mathfrak{h} \), i.e., the component of \( e \) in \( H \). But \( e = \pi_x(0_\mathbb{R}) \in \pi_x(\mathbb{R}) \cap H \), hence \( \pi_x(\mathbb{R}) \subseteq T_x \) by connectedness of \( \pi_x(\mathbb{R}) \).

Then each two elements of \( \pi_x(\mathbb{R}) \) have to commute. But this is not true for the parameters \( c_1 = \frac{1}{4} \sqrt{\frac{\tau}{\tau - 3}} - \frac{3}{8} \) and \( c_2 = \frac{1}{3} \sqrt{\frac{\tau}{\tau - 2}} - 1 \) that correspond to the values \( \beta_{c_1} = \frac{\pi}{7} \) and \( \beta_{c_2} = \frac{\pi}{7} \).

\( iv. \) If \( \tau \in \mathbb{R}_{\text{Bohr}} \), then

\[
\pi_x(\tau) = \eta \left( \left( i^*_x \circ \overline{v}^*_x \circ \xi_x \right)(\tau) \right)(\delta) = \left( \left( i^*_x \circ \overline{v}^*_x \circ \xi_x \right)(\tau) \right) \left( [h_x(\cdot)]_{ij} \right)
\]

\[
= \exp \left( \frac{\tau}{2} \right) \cdot \alpha_x \left( \left( \begin{array}{cc} \overline{\mathcal{P}}(\text{Re}(\chi_{rr})) & i \overline{\mathcal{P}}(\text{Im}(\chi_{rr})) \\ i \overline{\mathcal{P}}(\text{Im}(\chi_{rr})) & \overline{\mathcal{P}}(\text{Re}(\chi_{rr})) \end{array} \right) \right)
\]

\[
= d \cdot \alpha_x \left( \left( \begin{array}{cc} \mathcal{P}(c \mapsto \exp(i\tau \mu(\delta_0)))_{ij} \\ i \mathcal{P}(\text{Im}(\chi_{rr})) \end{array} \right) \right)
\]

\[
= d \cdot \left( \mathcal{P}(c \mapsto \exp(i\tau \mu(\delta_0)))_{ij} \right) \in d \cdot T_{\delta(0)}.
\]

Here the third step follows from (10), (11), multiplicativity and linearity of \( \xi(\tau) \) and the fact that the (unique) decompositions of the matrix entries

\[
a: c \mapsto \cos(\beta_\tau) + \frac{i}{\beta_\tau} \sin(\beta_\tau) \quad \text{and} \quad b: c \mapsto \frac{\epsilon}{\beta_\tau} \sin(\beta_\tau)
\]
By invariance (3) of \( \Delta(\pi) \) we have
\[
\pi \in \{ x, y \} \rightarrow \begin{cases} x & \text{if } \sin(\beta x) = 0 \\
y & \text{if } \cos(\beta y) = 1 \end{cases}
\]
for \( s \in \pi_{\delta}(\mathbb{R}) \cap d \cdot T_{\bar{\delta}(0)} \) iff \( \sin(\beta x) = 0 \) and \( \cos(\beta y) = 1 \), hence \( s = d \). But, combining invariance (3) of \( \Delta(\pi) \) with bijectivity of \( \alpha_{\sigma} \) and \( \exp \left( \frac{\tau}{\delta} \mu(\bar{n}) \right) = \alpha(\sigma) \left( \exp \left( \frac{\tau}{\delta} \mu(\bar{e}_1) \right) \right) \) we see that \( \pi_{\delta}(\mathbb{R}) \cap d \cdot T_{\bar{\delta}(0)} = \{ d \} \) iff this is true for \( \bar{n} = \bar{e}_3 \) and \( x = 0 \).

\[
\text{iii.) By iv.) we have } \mu_0(\pi_{\delta}(\mathbb{R})) = \mu_0(\pi_{\delta}(\mathbb{R})) + \mu_0(T_{\bar{\delta}(0)}). \text{ Now the derivative of } \pi_{\delta}(\mathbb{R}) \text{ is given by}
\]
\[
c \mapsto \frac{r^2}{2} \left[ \sin(\beta_c \tau) - \frac{i}{2\beta_c} \cos(\beta_c \tau) + \frac{i}{2\beta_c^2} \sin(\beta_c \tau) \right].
\]

As this vanishes nowhere, \( \pi_{\delta}(\mathbb{R}) \) can be decomposed into countably many 1-dimensional embedded submanifolds each of measure zero w.r.t. the Haar measure on \( SU(2) \). Consequently, \( \mu_0(\pi_{\delta}(\mathbb{R})) = 0 \) and obviously the same is true for \( \mu_0(T_{\bar{\delta}(0)}) \).

\[
\text{iv.) Again it suffices to show the claim for } \delta = \gamma_{\tau, r}. \text{ Now } \pi_{\delta}(\mathbb{R}) = \pi_{\delta}(\mathbb{R}) \text{ is clear from } (20), (11) \text{ and the definition of } \beta_c. \text{ Moreover, a closer look at the entries } \pi_{\delta}(\mathbb{R}) \text{ and } \pi_{\delta}(\mathbb{R}) \text{ shows that for } x \neq y \text{ with } x, y \geq 0 \text{ we have } \pi_{\delta}(x) = \pi_{\delta}(y) \text{ iff either } \tau \beta_x, \tau \beta_y \in \{ 2n\pi \mid n \in \mathbb{N}_{\geq 1} \} \text{ or } \tau \beta_x, \tau \beta_y \in \{ (2n - 1)\pi \mid n \in \mathbb{N}_{\geq 1} \} \text{ which just means } x, y \in \{ a_{2n} \mid n \in \mathbb{N}_{\geq 1} \} \text{ or } x, y \in \{ a_{2n - 1} \mid n \in \mathbb{N}_{\geq 1} \}. \text{ The merging property follows from the formulas (21) and (22).}
\]

\[\text{Cf. the functions } (21) \text{ and (22).}\]
B  Projective Limits

In this section we adapt the standard facts on projective structures to our definitions from Subsection 2.2.

Lemma B.1

Let $X$ be a projective limit of $\{X_\alpha\}_{\alpha \in I}$ w.r.t. the maps $\pi_\alpha : X \to X_\alpha$ for $\alpha \in I$ and $\pi_{\alpha_1}^{\alpha_2} : X_{\alpha_2} \to X_{\alpha_1}$ for $\alpha_1, \alpha_2 \in I$ with $\alpha_2 \geq \alpha_1$. Then $X$ is homeomorphic to

$$\hat{X} := \left\{ \hat{x} \in \prod_{\alpha \in I} X_\alpha \left| \pi_{\alpha_1}^{\alpha_2}(x_{\alpha_2}) = x_{\alpha_1} \quad \forall \alpha_2 \geq \alpha_1 \right. \right\}$$

equipped with subspace topology inherited from Tychonoff topology on $\prod_{\alpha \in I} X_\alpha$.

Proof: The map $\eta : X \to \hat{X}$, $x \mapsto \{\pi_\alpha(x)\}_{\alpha \in I}$ is well-defined by Definition 2.2(ii). Moreover, $\eta$ is continuous as it so as a map from $X$ to $\prod_{\alpha \in I} X_\alpha$. This is because $\text{pr}_\alpha \circ \eta = \pi_\alpha$ is continuous for all projection maps $\text{pr}_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha$ just by Definition 2.2(i). Then $\eta$ is a homeomorphism if it is bijective because $X$ is compact and $\hat{X}$ is Hausdorff. Injectivity of $\eta$ is immediate from Definition 2.2(iii). For surjectivity assume that $\{x_\alpha\}_{\alpha \in I} = \hat{x} \in \hat{X}$ with $\hat{x} \notin \text{im}[\eta]$, i.e., $\bigcap_{\alpha \in I} \pi_\alpha^{-1}(x_\alpha) = \eta^{-1}(\hat{x}) = \emptyset$. By continuity of $\pi_\alpha$ the sets $\pi_\alpha^{-1}(x_\alpha) \subseteq X$ are closed, hence compact by compactness of $X$. Consequently, there are finitely many $\alpha_1, \ldots, \alpha_k \in I$ such that $\pi_{\alpha_1}^{-1}(x_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(x_{\alpha_k}) = \emptyset$. By directedness of $I$ we find some $\alpha \in I$ such that $\alpha_j \leq \alpha$ for all $1 \leq j \leq k$, hence

$$x_{\alpha_j} = (\pi_{\alpha_j} \circ \pi_\alpha)(\pi_\alpha^{-1}(x_\alpha)) = \pi_{\alpha_j}(\pi_\alpha^{-1}(x_\alpha)) \quad \text{for all} \quad 1 \leq j \leq k. \quad (23)$$

due to $\pi_\alpha^{-1}(x_\alpha)$ is non-empty ($\pi_\alpha$ is surjective) and $\pi_{\alpha_j}^{-1}(x_\alpha) = x_{\alpha_j}$ for all $1 \leq j \leq k$. Applying $\pi_\alpha^{-1}$ to both sides of (23) gives $\pi_\alpha^{-1}(x_\alpha) \subseteq \pi_{\alpha_j}^{-1}(x_{\alpha_j})$ for all $1 \leq j \leq k$, which contradicts $\pi_{\alpha_1}^{-1}(x_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(x_{\alpha_k}) = \emptyset$. $\blacksquare$

Lemma B.2

Let $X$ and $\{X_\alpha\}_{\alpha \in I}$ be as in Definition 2.2 Then the normalized Radon measures on $X$ are in one-to-one with the consistent families of normalized Radon measures on $\{X_\alpha\}_{\alpha \in I}$.

Proof: If $\mu$ is a normalized Radon measure on $X$, then it is straightforward to see that $\{\pi_\alpha(\mu)\}_{\alpha \in I}$ is a consistent family of normalized Radon measures on $\{X_\alpha(\mu)\}_{\alpha \in I}$. For the converse statement define $\text{Cyl}(X) := \bigcup_{\alpha \in I} \pi_\alpha(C(X_\alpha)) \subseteq C(X)$. Then $\text{Cyl}(X)$ is closed under involution, separates the points in $X$ and vanishes nowhere. Moreover, $\text{Cyl}(X)$ is closed under addition, since

$$f \circ \pi_{\alpha_1} + g \circ \pi_{\alpha_2} = (f \circ \pi_{\alpha_1}^{\alpha_2}) \circ \pi_{\alpha_3} + (g \circ \pi_{\alpha_2}^{\alpha_3}) \circ \pi_{\alpha_3} \in \text{Cyl}(X), \quad (24)$$

where $\alpha_1, \alpha_2, \alpha_3 \in I$ with $\alpha_1, \alpha_2 \leq \alpha_3$. It follows in the same way that $\text{Cyl}(X)$ is closed under multiplication. By Stone-Weierstrass theorem $\text{Cyl}(X)$ is a dense *-subalgebra of $C(X)$ and the map $I : \text{Cyl}(X) \to \mathbb{C}, f \circ \pi_\alpha \mapsto \int_{X_\alpha} f \, d\mu_\alpha$ is well-defined, linear and continuous w.r.t. supremum norm on $C(X)$. In fact, if
\( f \circ \pi_{\alpha_1} = g \circ \pi_{\alpha_2} \), then 
\( (f \circ \pi_{\alpha_1}^3) \circ \pi_{\alpha_3} = (g \circ \pi_{\alpha_2}^3) \circ \pi_{\alpha_3} \) if \( \alpha_1, \alpha_2 \leq \alpha_3 \). Then \( f \circ \pi_{\alpha_1}^3 = g \circ \pi_{\alpha_2}^3 \) by surjectivity of \( \pi_{\alpha_3} \) and the transformation formula yields

\[
I(f \circ \pi_{\alpha_1}) = \int_{X_{\alpha_1}} f \, d\mu_{\alpha_1} = \int_{X_{\alpha_3}} f \circ \pi_{\alpha_1}^3 \, d\mu_{\alpha_3} = \int_{X_{\alpha_3}} g \circ \pi_{\alpha_2}^3 \, d\mu_{\alpha_3} = \int_{X_{\alpha_2}} g \, d\mu_{\alpha_2} = I(g \circ \pi_{\alpha_2}).
\]

This is well-definedness of \( I \). Linearity follows from \(22\) and for continuity observe that \( |I(f \circ \pi_{\alpha})| \leq \|f\|_{\infty} = \|f \circ \pi_{\alpha}\|_{\infty} \) by surjectivity of \( \pi_{\alpha} \). Since \( I \) is linear and continuous, it extends to a continuous functional on \( C(X) \) which, by Riesz-Markov theorem, defines a finite Radon measure \( \mu \) on \( B(X) \). Then \( \mu(X) = I(1) = 1 \) and for each \( f \in C(X_{\alpha}) \) we have

\[
\int_{X_{\alpha}} f \, d(\pi_{\alpha}^* \mu) = \int_{X} (f \circ \pi_{\alpha}) \, d\mu = I(f \circ \pi_{\alpha}) = \int_{X_{\alpha}} f \, d\mu_{\alpha}
\]

so that \( \pi_{\alpha}^* \mu = \mu_{\alpha} \), again by Riesz-Markov theorem. Finally, if \( \mu' \) is a further finite Radon measure with \( \pi_{\alpha}^* \mu = \mu_{\alpha} \) for all \( \alpha \in I \), then \( I': C(X) \to C \), \( f \mapsto \int_X f \, d\mu' \) is continuous and \( I|_{Cyl(X)} = I'|_{Cyl(X)} \) by transformation formula. Consequently, \( I = I' \) by denseness of \( Cyl(X) \) in \( C(X) \) so that \( \mu = \mu' \).

C The Ashtekar-Lewandowski Measure

In this section we reformulate the results from \(3\) in terms of the Definitions \(2,2\) and \(2,3\). At the same time we introduce the projection maps \( \pi_{\alpha} \) that are used in the motivation subsection \(4,2\). Let \( P = \mathbb{R}^3 \times SU(2) \), \( \mathcal{P} \) be a set of paths in \( \mathbb{R}^3 \) and \( \Gamma := \bigsqcup_{i=1}^{N} \mathcal{P} \). To simplify the notation we will abbreviate \( SU(2) \) by \( S \) in the following.

- A refinement of \((\gamma_1, \ldots, \gamma_l) \in \Gamma \) is an element \((\delta_1, \ldots, \delta_n) \in \Gamma \) such that for every path \( \gamma_j \) we find a decomposition \( \{\gamma_j_i\}_{i \leq j} \) such that each restriction \( \gamma_j_i \) is equivalent to one of the paths \( \delta_r, \delta_r^{-1} \) for \( 1 \leq r \leq n \).
- An element \((\delta_1, \ldots, \delta_n) \in \Gamma \) is said to be independent iff for each collection \( \{s_1, \ldots, s_n\} \subseteq S \) there is some \( \omega \in \mathcal{A} \) such that \( h_{\omega}(\delta_i) = s_i \) for all \( 1 \leq i \leq n \).
- Let \( \Gamma_0 \) be the set of all finite tuples \((\gamma_1, \ldots, \gamma_k) \subseteq \mathcal{P} \) with

\[
\text{im}[\gamma_i] \cap \text{im}[\gamma_j] = \{\gamma_i(a_i), \gamma_i(b_i), \gamma_j(a_j), \gamma_j(b_j)\} \quad \text{for} \quad 1 \leq i \neq j \leq k
\]

where \( \text{dom}[\gamma_i] = [a_i, b_i] \).

- For \((\gamma_1, \ldots, \gamma_k), (\gamma'_1, \ldots, \gamma'_l) \in \Gamma_0 \) write \((\gamma_1, \ldots, \gamma_k) \leq (\gamma'_1, \ldots, \gamma'_l) \) iff each \( \gamma_i \) admits a decomposition \( \{\gamma_i\}_{j \leq \gamma_i} \) such that every restriction \( \gamma_j_i \) is equivalent to one of the paths \( \gamma_1, \ldots, \gamma_k \) or its inverses.

Each \((\gamma_1, \ldots, \gamma_k) \in \Gamma_0 \) is independent\(29\) and \((\Gamma_0, \leq) \) is directed. In fact, if \((\gamma_1, \ldots, \gamma_k), (\gamma'_1, \ldots, \gamma'_l) \in \Gamma_0 \), then the proof of Lemma B.5 in \(3\) shows that for \((\gamma_1, \ldots, \gamma_k, (\gamma'_1, \ldots, \gamma'_l) \subseteq \mathcal{P} \) we find a refinement \( (\delta_1, \ldots, \delta_m) \subseteq \mathcal{P} \) such that \( \text{im}[\delta_i] \cap \text{im}[\delta_j] \) is finite for all \( 1 \leq i \neq j \leq m \). Splitting \( (\delta_1, \ldots, \delta_m) \) at the respective intersection points then gives the desired upper bound in \( \Gamma_0 \).

\[29\text{Cf. Section 3 in} \ 2 \text{ or Proposition A.1 in} \ 4.\]
Definition C.1

i.) For $\alpha = (\gamma_1, \ldots, \gamma_k) \in \Gamma_0$ we define the map $\pi_\alpha : \mathcal{S} \rightarrow X_\alpha := \mathcal{S}^{[\alpha]}$ by

$$\pi_\alpha(\omega) := (\eta(\omega)(\gamma_1), \ldots, \eta(\omega)(\gamma_k)).$$

This map is surjective by independence of $(\gamma_1, \ldots, \gamma_k)$.

ii.) Let $\alpha = (\gamma_1, \ldots, \gamma_k) \leq (\gamma_1', \ldots, \gamma_k') = \alpha'$ and $(\gamma_i)_j \in \{1, -1\}$ be the corresponding decomposition of $\gamma_i$ for $1 \leq i \leq k$. Then $(\gamma_i)_j \sim_\mathcal{A} \gamma_{m_{ij}}$ for $p_{ij} \in \{1, -1\}$ and $1 \leq m_{ij} \leq k'$ uniquely determined by independence of $(\gamma_1', \ldots, \gamma_k')$. Moreover, for all $1 \leq i \leq k$ it follows from injectivity of $\gamma_i$ that $m_{ij} \neq m_{ij'}$ for $1 \leq j \neq j' \leq t_i$. Then we obtain a well-defined and continuous map $\pi^{\alpha'}_\alpha : X_\alpha' \rightarrow X_\alpha$ by

$$\pi^{\alpha'}_\alpha(x_1, \ldots, x_{k'}) := \left( \prod_{j=1}^{t_1} (x_{m_{ij}})^{p_{ij}}, \ldots, \prod_{j=1}^{t_k} (x_{m_{kj}})^{p_{kj}} \right).$$

(25)

Since $\pi^{\alpha'}_\alpha \circ \pi_\alpha = \pi_\alpha$ and due to surjectivity of $\pi_\alpha$ this definition cannot depend on the decompositions $(\gamma_i)_j$.

Lemma C.2

i.) $\mathcal{A}$ is a projective limit of $\{X_\alpha\}_{\alpha \in \Gamma_0}$ w.r.t. the maps $\pi_\alpha$ for $\alpha \in \Gamma_0$ and $\pi^{\alpha'}_\alpha$ for $\alpha, \alpha' \in \Gamma_0$ with $\alpha \leq \alpha'$.

ii.) Let $\mu_k$ be the Haar measure on $\mathcal{S}^k$ for $k \in \mathbb{N}_{\geq 1}$ and $\mu_\alpha := \mu^{[\alpha]}$ for $\alpha \in \Gamma_0$. Then $\{\mu_\alpha\}_{\alpha \in \Gamma_0}$ is a consistent family of normalized Radon measures w.r.t. $\{X_\alpha\}_{\alpha \in \Gamma_0}$.

Proof:

i.) For continuity of the maps $\pi_\alpha$ it suffices to consider the case $\alpha = \gamma \in \mathcal{P}$. Here continuity is clear from (2), since $[h_\gamma]_{ij} \in \mathcal{C}$ and $\omega \in \text{Spec}(\mathcal{C})$. Second, for property iii.) from Definition 2.2 observe that the Gelfand transforms of the functions $[h_\gamma]_{ij}$ separate the points in $\mathcal{A}$ as they generate the continuous functions on $\mathcal{A}$. Then the claim follows from (2).

ii.) We have to show that $\pi^{\alpha'}_\alpha(\mu_\alpha) = \mu_\alpha$ if $\alpha \leq \alpha'$. By Riesz-Markov theorem it suffices to verify $\int_{X_\alpha} f d\mu_\alpha = \int_{X_\alpha} f d\pi^{\alpha'}_\alpha(\mu_\alpha)$ for all $f \in C(X_\alpha)$. Now

$$\mathfrak{B}(X_\alpha') = \bigotimes_{1 \leq i \leq k'} \mathfrak{B}(\mathcal{S})$$

because $\mathcal{S}$ is second countable. Since $\mathcal{S}, \mathcal{S}^{k'-1}$ are $\sigma$-finite, for $g \in C(X_\alpha')$, $x = (x_1, \ldots, x_{k'}) \in \mathcal{S}^{k'}$ and $1 \leq i \leq k'$ we obtain from Fubini’s theorem

$$\int_{X_{\alpha'}} g d\mu_\alpha' = \int_{\mathcal{S}^{k'-1}} \left( \int_{\mathcal{S}} g(x) d\mu_1(x_i) \right) d\mu_{k'-1}(x^i)$$

(26)

where $\mathcal{S}^{k'-1} \ni x^i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k'})$. Hence, for $f \in C(X_\alpha)$

$$\int_{X_\alpha} f d\pi^{\alpha'}_\alpha(\mu_\alpha') = \int_{X_{\alpha'}} \left( f \circ \pi^{\alpha'}_\alpha \right) d\mu_\alpha'$$

$$= \int_{\mathcal{S}^{k'-1}} \left( \int_{\mathcal{S}} \left( f \circ \pi^{\alpha'}_\alpha \right)(x) d\mu_1(x_i) \right) d\mu_{k'-1}(x^i).$$
By definition of $\Gamma_0$ each of the variables $x_1, \ldots, x_{k'}$ occurs in exactly one of the products on the right hand side of (25). Then from left-, right- and inversion invariance of $\mu_1$ we obtain

$$
\int_S \left( f \circ \pi_\alpha' \right)(x) d\mu_1(x_{m_{k_1}}) = \int_S f \left( [\pi_\alpha']_1(x_{m_{k_1}}), \ldots, [\pi_\alpha']_{k-1}(x_{m_{k_1}}), y \right) d\mu_1(y).
$$

Inductively, this gives

$$
\int_{X_\alpha} f d\pi_\alpha'(\mu_\alpha') = \int_{S^{k'}} f(x_1, \ldots, x_k) d\mu_\alpha'(x) = \int_{X_\alpha} f d\mu_\alpha
$$

were the last step follows by induction from (26) and $\mu_1(S) = 1$.

\begin{definition}
C.3
The normalized Radon measure $\mu_{AL}$ on $\mathcal{A}$ that corresponds to the consistent family of normalized Radon measures $\{\mu_\alpha\}_{\alpha \in \Gamma_0}$ from Lemma C.2 ii.) is called Ashtekar-Lewandowski measure on $\mathcal{A}$.
\end{definition}

\begin{thebibliography}{9}

\bibitem{1} Abhay Ashtekar, Martin Bojowald and Jerzy Lewandowski: Mathematical Structure of Loop Quantum Cosmology. Adv. Theor. Math. Phys. \textbf{7} (2003) 233–268. e-print: gr-qc/0304074v4.

\bibitem{2} Abhay Ashtekar and Jerzy Lewandowski: Representation Theory of Analytic Holonomy $C^*$ Algebras. In: Knots and Quantum Gravity (Riverside, CA, 1993), edited by John C. Baez, pp. 21–61, Oxford Lecture Series in Mathematics and its Applications 1 (Oxford University Press, Oxford, 1994). e-print: gr-qc/9311010v2.

\bibitem{3} Abhay Ashtekar, Jerzy Lewandowski: Projective techniques and functional integration. J. Math. Phys. \textbf{36} (1995) no. 5 2170–2191. e-print: gr-qc/9411046v1.

\bibitem{4} Theodor Brücker and Tammo tom Dieck: Representations of Compact Lie Groups. Springer, Berlin, 1985.

\bibitem{5} Johannes Brunnemann and Christian Fleischhack: On the Configuration Spaces of Homogeneous Loop Quantum Cosmology and Loop Quantum Gravity. Math. Phys. Anal. Geom. \textbf{15} (2012) 299–315 e-print: 0709.1621v2 (math-ph).

\bibitem{6} Christian Fleischhack: Loop Quantization and Symmetry: Configuration Spaces. e-print: 1010.0449v1 (math-ph).

\end{thebibliography}

\footnote{Remember $x^{m_{k_1}} = (x_1, \ldots, x_{[m_{k_1}-1]}, x_{[m_{k_1}]+1}, \ldots, x_{k'})$.}
[7] Christian Fleischhack: Parallel Transports in Webs. *Math. Nachr.* **263–264** (2004) 83–102. e-print: 0304001v2 (math-ph).

[8] Maximilian Hanusch: Invariant Connections in Loop Quantum Gravity. e-print: 1307.5303v1 (math-ph).

[9] Shoshichi Kobayashi and Katsumi Nomizu: *Foundations of Differential Geometry*. Wiley Interscience, 1996.

[10] Alan D. Rendall: Comment on a paper of Ashtekar and Isham. *Class. Quant. Grav.* **10** (1993) 605–608.

[11] Walter Rudin: *Representations of Compact Lie Groups*. Wiley Interscience, 1990.

[12] J. M. Velhinho: The Quantum Configuration Space of Loop Quantum Cosmology. *Classical and Quantum Gravity* **24** (2007) 3745–3758. e-print: 0704.2397v2 (gr-qc).