THE GAUSS–MANIN CONNECTION ON THE HODGE STRUCTURES

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The Gauß–Manin connection is an extra structure on the de Rham cohomology of any algebraic variety, \( \nabla : H^*_{dR/k} \rightarrow \Omega^1_k \otimes_k H^*_{dR/k} \) (its definition will appear below). If one believes the Hodge conjecture then for a given pure Hodge structure \( H \) there is at most one connection \( \nabla \) such that \( H \) is a Hodge substructure of a cohomology group of a smooth projective complex variety with \( \nabla \) induced by the Gauß–Manin connection. Independently of the Hodge conjecture, there are at most countably many connections \( \nabla \) on a given pure Hodge structure \( H \) such that \( H \) is a Hodge substructure of a cohomology group of a smooth projective complex variety with \( \nabla \) induced by the Gauß–Manin connection (cf. Corollary 2.4).

The original motivation for this paper were the properties of the forgetful functor

\[
\left\{ \begin{array}{ll}
\text{graded-polarizable Hodge structures equipped with} \\
\text{a connection respecting the weight filtration and} \\
\text{polarizations, and satisfying the Griffiths transversality}
\end{array} \right\} \xrightarrow{\Phi} \{ \text{Hodge structures} \},
\]

meaning the following three questions:

- For a complex algebraic variety \( X \) is the Gauß–Manin connection determined by the Hodge structure \( H^*(X(\mathbb{C})) \)?
- If yes, does there exist a functor \( \Psi \) right inverse to \( \Phi \) such that for each geometric mixed Hodge structure \( H \) the pair \( \Psi(H) \) coincides with \( H \) endowed with the Gauß–Manin connection, and \( \Psi(H \otimes H') \cong \Psi(H) \otimes \Psi(H') \)?
- If the \( \nabla \) is determined by the Hodge structure in a unique way, how to express \( \nabla \) in terms of the Hodge structure? (This assumes that there should be a certain supply of functions on Shimura varieties classifying the Hodge structures.)

In general these questions are very difficult. In this paper we consider some special cases of the problem. In particular, in Proposition 3.2 below we show that such a functor \( \Psi \) exists for the restriction of \( \Phi \) to the subcategory of mixed Tate structures and it is unique. The connection constructed there is non-integrable in general, so, if any Hodge substructure of a geometric Hodge–Tate structure is again geometric, its integrability gives a non-trivial necessary condition for a Hodge–Tate structure to be geometric.

We also compute explicitly the Gauß–Manin connection in terms of the Hodge structure for some geometric Hodge structures of small rank.

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1. The absolute Gauß–Manin connection: definition

Let \( X_\bullet \) be a smooth simplicial quasiprojective variety over a field \( k \) of characteristic zero. For any \( i \geq 0 \) reduction modulo the ideal generated by \( (i+1) \)-forms on the base field \( k \) gives
the following short exact sequences of complexes of sheaves of exterior powers of absolute Kähler differentials on $X_*$

$$0 \to \Omega^{i+1}_k \otimes H^q_{dR/k}(X_*) \to \Omega^i_k \otimes H^q_{dR/k}(X_*) \to 0.$$ \hspace{1cm} (1)

This gives rise to the sequence of homomorphisms with composition $\nabla$ called the Gauß–Manin connection:

$$\begin{array}{ccc}
\Omega^i_k \otimes_k H^q_{dR/k}(X_*) & \xrightarrow{\nabla} & \Omega^{i+1}_k \otimes_k H^q_{dR/k}(X_*) \\
\cong & & \cong \\
\mathbb{H}^{q+i}(\Omega^i_k \otimes_k \Omega^*_{X_*/k}) & \xrightarrow{\text{coboundary}} & \mathbb{H}^{q+i+1}(\Omega^{i+1}_k \otimes_k \Omega^*_{X_*/k})
\end{array} \hspace{1cm} (2)

Let $\overline{X}_*$ be a smooth compactification of $X_*$ by a divisor with normal crossings $D_*$. Then the de Rham cohomology of $\overline{X}_*$ is identified canonically with the hypercohomology of the simplicial logarithmic de Rham complex $\Omega^*_{X_*/k}(\log D_*)$ on $\overline{X}_*$, so one can define the Hodge filtration on the de Rham cohomology groups by

$$F^p H^q_{dR/k}(X_*) = \text{image} \left( \mathbb{H}^q(\overline{X}_*, \Omega^{2p}_{X_*/k}(\log D_*)) \to H^q_{dR/k}(X_*) \right).$$

One has the following short exact sequences of complexes of sheaves of exterior powers of absolute Kähler differentials on $\overline{X}_*$ which is a logarithmic version of the sequence (1):

$$0 \to \Omega^{i+1}_k \otimes \Omega^{2p-1}_{X_*/k}(\log D_*) \to \Omega^i_k \otimes \Omega^{2p}_{X_*/k}(\log D_*) \to 0. \hspace{1cm} (3)$$

Replacing the complex $\Omega^*_{X_*}$ on $X_*$ by the complex $\Omega^{2p}_{X_*/k}(\log D_*)$ on $\overline{X}_*$ in the commutative diagram (2), we get the Griffiths transversality property:

$$\Omega^i_k \otimes_k F^p H^q_{dR/k}(X_*) \xrightarrow{\nabla} \Omega^{i+1}_k \otimes_k F^{p-1} H^q_{dR/k}(X_*). \hspace{1cm} (4)$$

2. Basic properties of the Gauß–Manin connection

Though the most of what follows is presumably valid for arbitrary smooth simplicial schemes, we restrict ourselves to the case of smooth proper varieties.

The following properties of the Gauß–Manin connection are almost immediate.

- As $\nabla$ coincides with the first differential in the Leray spectral sequence $E_1^{s,t} = \Omega^s_k \otimes H^t_{dR/k}(X_*)$ converging to $H^{s+t}_{dR/k}(X_*)$ (associated to the filtration by powers of the ideal $\Omega^{s+1}_k$ in $\Omega^*_k$), one has $\nabla^2 = 0$.
- Since each morphism $f : X_* \to Y_*$ of smooth simplicial schemes induces a morphism of the Leray spectral sequences for $X/k$ and $Y/k$, the Gauß–Manin connection is functorial with respect to pull-backs.
- It is easy to see from the Küneth decomposition $\Omega^*_{X \times Y} = \text{pr}_X^* \Omega^*_{X} \otimes_{\mathcal{O}_{X \times Y}} \text{pr}_Y^* \Omega^*_{Y}$ for a pair of smooth proper schemes $X$, $Y$, that the Gauß–Manin connection on $H^*_{dR/k}(X \times_k Y)$ coincides with the tensor product of the Gauß–Manin connections on $H^*_{dR/k}(X)$ and $H^*_{dR/k}(Y)$.
- Combining the latter with the functoriality with respect to pull-backs applied to the diagonal embedding, we get the Leibniz rule for cup-products.
Proposition 2.1. If the Hodge conjecture for $H^{2q}$ is true then for a given pure effective Hodge structure $H$ of weight $q$ there is at most one connection $\nabla$ such that $H$ is a Hodge substructure of the $q$-th cohomology group of a smooth projective complex variety with $\nabla$ induced by the Gauß–Manin connection.

Proof. Suppose that $H$ as a pure Hodge structure is isomorphic to Hodge substructures of both $H^q(X)$ and of $H^q(Y)$ for some smooth projective complex varieties $X$ and $Y$. By Lefschetz hyperplane section theorem we may suppose that $\dim X = q$. Then there is a morphism of Hodge structures $\alpha : H^q(X) \rightarrow H^q(Y)$ commuting with embeddings $H \hookrightarrow H^q(X)$ and $H \hookrightarrow H^q(Y)$. The class of $\alpha$ is an element in $H^{2q}(X \times Y)(q)$ of Hodge type $(0,0)$, and thus, is presented by an algebraic cycle $\gamma$. By a standard argument $\gamma$ induces a morphism of pairs $(H^q(X), \nabla_X) \rightarrow (H^q(Y), \nabla_Y)$.

The image $\Omega_H$ of the $\mathbb{Q}$-linear map $H \otimes H^\vee \xrightarrow{(\nabla, \cdots)} \Omega^1_\mathbb{C}$, which is a $\mathbb{Q}$-subspace (of dimension $\leq (\text{rk} H)^2$) in $\Omega^1_\mathbb{C}$, is one of basic invariants of the connection $\nabla_H$. Here $H^\vee$ is the Hodge structure dual to $H$. It follows from the compatibility of $\nabla$ with polarizations, duality and tensor products that $\Omega_H = \Omega_{H^\vee}$, $\Omega_H = \Omega_{H \otimes M}$ for any integer $M \geq 1$ and $\Omega_{H_1 \otimes H_2} \subseteq \Omega_{H_1} + \Omega_{H_2}$.

If $X$ is a complex algebraic variety and $H = H^q(X)$ is the group of singular cohomology of $X(\mathbb{C})$ with $\mathbb{Z}$-coefficients for some integer $q \geq 0$, one has the de Rham isomorphism $H \otimes \mathbb{C} \xrightarrow{\sim} H_{dR/\mathbb{C}}$, and polarizations $Q_w : gr^W H \times gr^W H \rightarrow \mathbb{Z}(-w)$, $Q_w(a, b) = (-1)^w Q_w(b, a)$ for each integer $w$. Denote by $\nabla_H$, or simply by $\nabla$, the composition $H \hookrightarrow H \otimes \mathbb{C} \xrightarrow{\sim} H_{dR/\mathbb{C}} \xrightarrow{\nabla} \Omega^1_\mathbb{C} \otimes H_{dR/\mathbb{C}} \xrightarrow{\sim} \Omega^1_\mathbb{C} \otimes H$. After a choice of a basis of $H$ one can view this map as a $(\text{rk} H \times \text{rk} H)$-matrix with entries in $\Omega^1_\mathbb{C}$.

The Gauß–Manin connection respects the polarizations in the sense

$$\nabla_{\mathbb{Z}(-w)} Q_w(a, b) = Q_w(\nabla_{gr^W H}(a, b)) + Q_w(a, \nabla_{gr^W H}(b)).$$
If \( w \) is even one can choose such a basis \( \{e_1, \ldots, e_N\} \) of \( gr^W_\mathcal{V}H \) that the matrix of the polarization is diagonal: \( Q_w(e_i, e_j) = (2\pi \sqrt{-1})^{w} \lambda_i \delta_{ij} \) for some rational \( \lambda_i \)'s. If \( \Omega = (\omega_{ij}) \) is the matrix of \( \nabla_{gr^W_\mathcal{V}H} \) in this basis then \( \omega_{ii} = -\frac{w}{2} \frac{d\pi}{\pi} \) and \( \omega_{ij} = -\frac{\lambda}{2} \omega_{ji} \) for \( i \neq j \).

If \( w \) is odd one can choose a basis \( \{e_1, \ldots, e_N\} \) of \( gr^W_\mathcal{V}H \otimes \mathbb{Q} \) where the matrix of \( Q_w \) is equal to

\[
(2\pi \sqrt{-1})^{-w} \cdot \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

If \( \Omega = (\Omega_{ij})_{1 \leq i, j \leq 2} \) is the matrix of \( \nabla_{gr^W_\mathcal{V}H} \) in this basis then \( \Omega_{12} = \Omega_{12}', \Omega_{21} = \Omega_{21}' \) and

\[
\Omega_{11} + \Omega_{22}' = -w \frac{d\pi}{\pi} \cdot I.
\]

This gives a (very rough) estimate

\[
\dim_\mathbb{Q} \Omega_H \leq \begin{cases} \frac{N(N-1)}{N^2(-1)^w} + 1 & \text{if } H \text{ is a pure structure of weight } 0 \\ 1 & \text{if } H \text{ is a pure structure of weight } w. \end{cases}
\]

We will need the following

**Proposition 2.2** (Katz, [K], [D1]). Let \( S \) be a pathwise connected and locally simply connected topological space and \((H, F, W)\) is a family of mixed Hodge structures on \( S \) such that each of the families \((gr^W_\mathcal{V}H, F, W)\) is polarizable. Suppose that the Hodge filtration \( F \) is locally constant, i.e., comes from a filtration of the complexification \( H_\mathbb{C} \) by local subsystems. Then there exists a finite étale cover \( \pi : S' \longrightarrow S \) such that \( \pi^*(H, F, W) \) is a constant family of mixed Hodge structures on \( S' \).

**Lemma 2.3** (Rigidity). Let \( \overline{f} : \mathcal{X} \longrightarrow T \) be a morphism of complex smooth algebraic varieties. Denote by \( f \) the natural morphism of topological spaces \( \mathcal{X}(\mathbb{C}) \longrightarrow T(\mathbb{C}) \). Suppose that \( R^q f_* \mathbb{Z} \) is a local system of isomorphic Hodge structures.

Then each path \( \gamma : [0, 1] \longrightarrow T(\mathbb{C}) \) gives rise to an isomorphism

\[
((R^q f_* \mathbb{Z})_{\gamma(0)}, \nabla_{\gamma(0)}) \sim (\Omega, \nabla_{\gamma(1)}).
\]

**Proof.** Since the Hodge filtration on the stalks of \( R^q f_* \mathbb{Z} \) defines a filtration of \( R^q f_* \mathbb{C} \) by local subsystems, the Proposition 2.2 implies that the local system \( R^q f_* \mathbb{Z} \) becomes trivial on a finite cover of \( T \). Without a loss of generality we replace \( T \) with such a cover, and moreover, assume \( T \) affine and connected.

Then for any point \( s \) of \( T(\mathbb{C}) \) the restriction \( H_{dR/\mathbb{C}}^q(\mathcal{X}) \longrightarrow H_{dR/\mathbb{C}}^q(\mathcal{X}_s) \) is surjective and its kernel is independent of \( s \). Applying the functoriality of the Gauß–Manin connection to \( r_s \), one gets that \( \nabla(\ker r_s) \subset \Omega^1_\mathbb{C} \otimes \ker r_s \), and thus, \((R^q f_* \mathbb{Q}, \nabla_s) \cong (H^q(\mathcal{X})/\ker r_s, \nabla_\mathcal{X}) \).

**Corollary 2.4.** For a given Hodge structure \( H \) there is at most a countable set of \( \nabla \)'s such that \((H, \nabla)\) is a cohomology group of a smooth projective complex variety with the Gauß–Manin connection.

**Proof.** There exists such a countable set of families of smooth projective complex varieties that each variety is isomorphic to an element of at least one of the families.\footnote{For each pair of integers \( 1 \leq r < N \), a collection of integers \( 2 \leq d_1 \leq d_2 \leq \cdots \leq d_r \), an integer \( s \geq 0 \) and a collection of elements \( P_1, \ldots, P_s \in Sym^* (Sym^{d_1} \mathbb{Q}^N \oplus \cdots \oplus Sym^{d_r} \mathbb{Q}^N) \) one defines a family by homogeneous equations of degrees \( d_1, d_2, \ldots, d_r \) whose coefficients are zeroes of polynomials \( P_1, \ldots, P_s \).} It follows from \([S]\), that the fibers of the period map are algebraic, and thus, we can apply Lemma 2.3 to conclude the proof.

\[\square\]
Lemma 2.3, for any point \( s \) of mixed Hodge structures is a locally constant local system in étale topology. Then, by Proposition 2.5.

Proposition 2.5. Suppose that for a smooth proper algebraic variety \( X \) over \( \mathbb{Q} \), there is a horizontal Hodge structure on the \( j \)-th cohomology group of \( X \). Then, \( H^q(X) \) is the Hodge structure of the \( q \)-th cohomology group of a variety defined over \( \overline{\mathbb{Q}} \).

Proof. Choose a smooth surjective morphism \( \mathcal{X} \to S \) of varieties over \( \overline{\mathbb{Q}} \) and a generic point \( s_0 \in S(\mathbb{C}) \), i.e., an embedding of fields \( \overline{\mathbb{Q}}(S) \hookrightarrow \mathbb{C} \), such that the fiber of \( \mathcal{X} \) over \( s_0 \) is \( X \).

Then, shrinking \( S \), if necessary, we get from Proposition 2.2, that the variation \( R^q\pi_*\mathcal{Z} \) of mixed Hodge structures is a locally constant local system in étale topology. Then, by Lemma 2.3, for any point \( s_1 \in S(\overline{\mathbb{Q}}) \) we have an isomorphism of mixed Hodge structures equipped with connections \( (H^q(\mathcal{X}_{s_0}(\mathbb{C}), \nabla_0) \cong (H^q(\mathcal{X}_{s_1}(\mathbb{C}), \nabla_1)). \)

Example 2.6. For any integer \( m \geq 2 \), those elements of \( \text{Ext}^1_{HS}(\mathbb{Z}, \mathbb{Z}(m)) \) corresponding to cohomology groups of algebraic varieties, correspond, in fact, to cohomology groups of algebraic varieties defined over \( \overline{\mathbb{Q}} \).

Proof. In our case \( F^{1-m} = \cdots = F^0 \), so, by the Griffiths transversality (4), for any integer \( 1 - m \leq p \leq 0 \) one has \( \nabla F^p = \nabla F^0 \subseteq \Omega^1_C \otimes_C F^{-1} = \Omega^1_C \otimes_C F^p. \) This means, that the assumptions of the Proposition 2.3 are the case.

Proposition 2.7. If \( H \) is a geometric pure Hodge structure then \( (H \otimes \mathbb{C})^\vee \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \) coincides with \( H' \otimes \mathbb{C} \) for a Hodge substructure \( H' \) with horizontal Hodge filtration.

Proof. This follows from the exactness of the sequence \( H^*_{dR/\mathbb{Q}}(X) \to H^*_{dR/\mathbb{C}}(X) \to \Omega^1_C \otimes_k H^*_{dR/\mathbb{C}}(X) \) shown in Proposition 4 of [EP].

Conjecture 2.8. For any geometric Hodge structure \( H \) the horizontal subspace \( H^\vee \) is a Hodge substructure isomorphic to a power of \( \mathbb{Q}(0) \).

If \( H^\vee \) is a Hodge substructure of weight \( w \), there is a non-degenerate pairing \( H^\vee \otimes H^\vee \to \mathbb{Q}(w) \) compatible with the connections, so there are no rational horizontal elements in geometric Hodge structures of non-zero weight.

By Propositions 2.5 and 2.7 a horizontal Hodge structure comes from a variety defined over \( \overline{\mathbb{Q}} \), and thus the Conjecture 2.8 is equivalent to transcendence of certain periods of varieties over \( \overline{\mathbb{Q}} \).

3. Hodge–Tate structures

3.1. Calculation of the connection on the logarithmic structures. Consider the first relative cohomology group of \( \mathbb{G}_m \) modulo \( \{1,a\} \) for some \( a \in k \). To calculate this group we present \( \mathbb{G}_m \) as the complement of \( \mathbb{P}^1 \) to the divisor \( (0) + (\infty) \) and then \( H^1_{dR}(\mathbb{G}_m, \{1,a\}) = \mathbb{H}^1(\mathcal{O}(-(1) - (a)) \to \Omega^1_{\mathbb{P}^1}((0) + (\infty))). \)

We can choose a covering \( \mathbb{P}^1 = U_0 \cup U_1 \), say, with \( U_0 = \mathbb{P}^1 \setminus \{a\} \) and \( U_1 = \mathbb{P}^1 \setminus \{1\} \).
1-cocycles in the Čech–de Rham complex are collections \( (f_{ij}, \omega_i) \) with \( i, j \in \{0, 1\} \) and 
\[ df_{ij} = \omega_i - \omega_j, \]
where
\[ f_{ij} \in \mathcal{O}(U_{01}) = \mathbb{C} \left[ \frac{z-1}{z-a}, \frac{z-a}{z-1} \right] \]
and 
\[ \omega_i \in \Omega^1_{\mathbb{P}^1/\mathbb{C}}(0) + (\infty)(U_i). \]

Note, however, that adding a coboundary, we may assume the 1-forms \( \omega_i \) to be regular at the points 1 and \( a \), and therefore the 1-form \( df_{ij} \) to be also regular at the points 1 and \( a \). Since \( df_{ij} \) is regular everywhere on the projective line, \( df_{ij} = 0 \), so \( f_{ij} \) is a constant and \( \omega_0 = \omega_1 \). The latter means that
\[ \omega_0 = \omega_1 \in F^1 H^1_{dR/\mathbb{C}}(\mathbb{G}_m, \{1, a\}) = \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1/\mathbb{C}}((0) + (\infty))) = \left\langle \frac{dz}{z} \right\rangle_\mathbb{C}. \]

Finally,
\[ H^1_{dR/\mathbb{C}}(\mathbb{G}_m, \{1, a\}) = \left\{ \left( \frac{b, c dz}{z} \right) | b, c \in \mathbb{C} \right\}, \]
where \( (1, 0) \) denotes the 1-cocycle presented by the function 1 on \( U_{01} \).

The group \( H^1_{dR/\mathbb{C}}(\mathbb{G}_m, \{1, a\}) \) fits into the exact sequence
\[ 0 \to H^0_{dR/\mathbb{C}}(\mathbb{G}_m) \to H^0_{dR/\mathbb{C}}(\{1, a\}) \to H^1_{dR/\mathbb{C}}(\mathbb{G}_m, \{1, a\}) \to H^1_{dR/\mathbb{C}}(\mathbb{G}_m) \to 0, \]
where \( H^0_{dR/\mathbb{C}}(\{1, a\}) = \mathbb{C} \oplus \mathbb{C} \), the first map is diagonal and the second map is given by \((s, t) \mapsto (s-t, 0)\). In particular, denote by \( e_0 \) the image of \((1, 0)\), equivalently,
\[ e_0 = (1, 0) \in \check{C}^1(\mathcal{O}_{\mathbb{P}^1}(-1) - (a)) \oplus \check{C}^0(\Omega^1_{\mathbb{P}^1/\mathbb{C}}(-1) - (a))). \]

Note, that \( e_0 \) lifts tautologically to a 1-cocycle in the first term
\[ \check{C}^1(\mathcal{O}_{\mathbb{P}^1}(-1) - (a)) \oplus \check{C}^0(\Omega^1_{\mathbb{P}^1/\mathbb{C}}(\log((0) + (a) + (1) + (\infty)))(-1) - (a))) \]
of the absolute Čech–de Rham complex, and therefore, \( \nabla e_0 = 0 \).

To calculate \( \nabla \left( \frac{dz}{z} \right) \) we lift the (relative) form \( \frac{dz}{z} \) to a section \( \eta_j \) of the sheaf of absolute 1-forms
\[ \mathcal{O}_{\mathbb{P}^1}(-(1) - (a)) \otimes \mathcal{O}_{\mathbb{P}^1} \cdot \frac{dz}{z} \cdot \mathcal{O}_U \oplus \frac{dz}{z} \cdot \mathcal{O}_U \to \Omega^1_{\mathbb{P}^1/\mathbb{C}}((0) + (\infty)) \]
over each element \( U_j \) of the covering, say, \( \eta_0 = \frac{dz}{z}, \eta_1 = \frac{dz}{z/(1/a)} \). Then the coboundary of the 1-cochain \( (\eta_j) \), a 2-cocycle in
\[ \check{C}^1(\Omega^1_{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) - (a)) \oplus \check{C}^0(\Omega^1_{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}^1/\mathbb{C}}(-(1) - (a)))) \]
is \( \left( \frac{da}{a}, 0 \right) \).

This gives the formula \( \nabla \left( \frac{dz}{z} \right) = \frac{da}{a} \otimes e_0 \).

\( e_0 \) is an integral generator of \( W_0 H^1(\mathbb{G}_m, \{1, a\}) \). Another generator \( e_2 \) of \( H^1(\mathbb{G}_m, \{1, a\}) \) is 
\[ e_2 = \left[ \frac{dz}{2\pi i z} - \frac{\log a}{2\pi i} e_0 \right] \] for an arbitrary choice of \( \log a \in \mathbb{C} \). Then \( \nabla e_2 = \frac{a^{-1} da - (\log a) \otimes e_0 - \frac{dz}{z} \otimes e_2} \). Finally, for Hodge structure \( H \) of rank 2 and weights 0 and 2, and any element \( \xi \) of \( H \cong H^1(\mathbb{G}_m, \{1, a\}; \mathbb{Q}) \) we get
\[ \nabla \xi = \left( d(\pi z_0) + i e^{-\pi iz_0} d e^{\pi iz_0} \right) \otimes \frac{e_0}{\pi} - \frac{d\pi z}{\pi} \otimes \xi, \]
where \( e_0 \neq 0 \) is an element of \( W_0 \) and \( \xi = z_0 e_0 \in F^1 \). It is easy to see that \( (3) \) is independent of \( e_0 \).
3.2. The Gauß–Manin connection on arbitrary Hodge–Tate structures.

Lemma 3.1. For any Hodge–Tate structure \( H \) inclusion maps induce the decomposition
\[
\bigoplus_k F^k \cap (W_{2k} \otimes \mathbb{C}) \xrightarrow{\sim} H \otimes \mathbb{C}.
\] (6)

Proof. Note that \( F^k \cap (W_{2k-2} \otimes \mathbb{C}) = 0 \), so the subspaces \( F^k \cap (W_{2k} \otimes \mathbb{C}) \) and \( F^l \cap (W_{2l} \otimes \mathbb{C}) \) intersect trivially when \( k \neq l \), and the canonical projection \( \varphi_k : F^k \cap (W_{2k} \otimes \mathbb{C}) \to \text{gr}^{2k} \mathbb{C} \) is an isomorphism. Comparison of dimensions of both sides of (6) ensures it is an isomorphism. □

As a consequence of this Lemma, we get a new \( \mathbb{Q} \)-structure
\[
\bigoplus_k (2\pi i)^k \varphi^{-1}_k(\text{gr}^{2k})
\] for complexification of arbitrary mixed Tate structure.

Let \( \mathcal{H} \) be an abelian category of Hodge–Tate structures containing all logarithmic structures, invariant under Tate twists and containing each Hodge substructure of each its object. Consider the category whose objects are objects of \( \mathcal{H} \) equipped with a connection satisfying the Griffiths transversality and morphisms are morphisms of Hodge structures commuting with connections.

Proposition 3.2.

- There is a unique functor \( H \mapsto (H, \nabla_H) \) right inverse to the forgetful functor
\[
\{ \text{objects of } \mathcal{H} \text{ equipped with a connection satisfying the Griffiths transversality} \} \to \mathcal{H}
\]
such that for a logarithmic structure \( H \) the connection \( \nabla_H \) coincides with the Gauß–Manin connection calculated above, and \( \nabla_H \) induces the same connection on \( H \otimes \mathbb{C} = H(1) \otimes \mathbb{C} \) as \( \nabla_{H(1)} \).

- The above connection on a Hodge–Tate structure \( H \) is integrable if and only if the above connections on \( W_{2k}/W_{2k-6}(k-3) \) are integrable for all integer \( k \).

Proof. It follows from the functoriality, applied to the morphism \( W_k \to H \), that \( \nabla \) respects the weight filtration, and therefore, from the calculation for the logarithmic structures, that the connection on the structure \( \mathbb{Z}(0) \) is zero. This implies that \( \nabla : W_{2k}(k) \to \Omega^1_C \otimes W_{2k-2} \). Combining these with the Griffiths transversality \( \nabla : F^k \to \Omega^1_C \otimes F^{k-1} \), we get
\[
(2\pi i)^k \varphi^{-1}_k(\text{gr}^{2k}) \xrightarrow{\nabla} \Omega_C^1 \otimes_C (F^{k-1} \cap (W_{2k-2})).
\]

It is clear from the decomposition (6) that it suffices to construct the latter maps, and that \( \nabla \) is integrable if and only if the restrictions of \( \nabla^2 \) to \( F^k \cap (W_{2k}) \) vanish for all integer \( k \). As \( F^{k-2} \cap (W_{2k-6}) = 0 \), the vanishing of the latter restrictions is equivalent to the vanishing of the induced maps \( F^k \cap (W_{2k}) \to \Omega^2_C \otimes (W_{2k}/W_{2k-6}) \).

By the functoriality and compatibility with the Tate twists, to construct that map for some \( k \) we may identify the space \( F^k \cap (W_{2k}) \) with the space \( F^1 \cap (W_{2k}) \), and \( F^{k-1} \cap (W_{2k-2}) \) with \( F^0 \cap (W_{2k-4}) \), where \( H' = (W_{2k}/W_{2k-4}(k-1) \). In fact, we may suppose that there is an exact sequence
\[
0 \to \mathbb{Z}(0)^* \to H' \to \mathbb{Z}(-1) \to 0
\]
for some non-negative integer $s$. Then $H'$ can be identified with a Hodge substructure of a direct sum of logarithmic Hodge structures, where we have fixed the connection.

REMARKS. 1. It is easy to see that the connection constructed in Proposition 3.2 on the tensor product of two Hodge–Tate structures coincides with the tensor product of the connections on that Hodge–Tate structures.

2. It follows from Section 3.1 that for a Hodge–Tate structure $H$ of rank 3 with weights 0, 2, 4 the connection $\nabla_H$ is integrable if and only if $e^{\pi iz_0}$ and $e^{\pi iz_2}$ are algebraically dependent, where $z_0$ and $z_2$ are determined by the conditions $e_0 \in W_0 \setminus \{0\}$, $e_2 \in W_2 \setminus W_0$, $e_4 \in W_4 \setminus W_2$, $e_2 - z_0 \cdot e_0 \in F^1$ and $e_4 - z_2 \cdot e_2 \in F^2(W_4/W_0)$. This implies that for each element $a \in \mathbb{C}^\times$ there is a natural embedding $\mathbb{C}^\times/(\mathbb{Q}(a))^{\times} \hookrightarrow \text{Ext}^2(\mathbb{Z}(0), \mathbb{Z}(2))$, where $\text{Ext}^2$ is calculated in the category of flat Hodge–Tate structures.

4. EXAMPLES: SOME HODGE STRUCTURES OF RANK $\leq 3$

Up to the Tate twists and the duality the options for a Hodge structure of rank $\leq 3$ are:

- it is Hodge–Tate;
- it is pure of weight 0, or $-1$;
- it is an extension of $\mathbb{Q}(0)$ by a pure structure of negative weight.

We have calculated the “Gauß–Manin” connection for Hodge–Tate structures in the previous section, so we eliminate them in what follows.

4.1. Pure Hodge structures of rank 2. For any square-free positive integer $D$ denote by $E_D$ an elliptic curve with complex multiplication in $\mathbb{Q}(\sqrt{-D})$. Since $\wedge^2 H^1(E_D) = \mathbb{Q}(-1)$, one has a surjection $\text{End} H^1(E_D) \xrightarrow{\sim} H^1(E_D) \otimes H^1(E_D)(1) \rightarrow \text{Sym}^2 H^1(E_D)(1)$ with the kernel $\mathbb{Q} \cdot \text{id}$. The classes of endomorphisms of the curve generate a Hodge substructure in $\text{End} H^1(E_D)$ isomorphic to $\mathbb{Q}(0) \oplus \mathbb{Q}(0)$, so its image in $\text{Sym}^2 H^1(E_D)(1)$ is isomorphic to $\mathbb{Q}(0)$. Let $\alpha \in \text{Sym}^2 H^1(E_D)(1)$ be the element corresponding to 1 $\in \mathbb{Q}(0)$.

Denote by $M_D^2$ the cokernel of the injection $\text{Sym}^{k-2} H^1(E_D)(-1) \xrightarrow{\sim} \text{Sym}^k H^1(E_D)$. This is a geometric Hodge structure of weight $k$, rank 2 and with Hodge numbers $h^{0,k} = h^{k,0} = 1$.

Alternatively, one can define $M_D^2(k)$ as a Hodge substructure in $H_1(E_D)^{\otimes k}$ as follows. Let $\tau$ be a complex multiplication of $E_D$, $\tau_2$ the induced endomorphism of the first homology group, and $\gamma_1$ a non-trivial element of $H_1(E_D)$. Set $\gamma_2 = \tau_2 \gamma_1$, $\gamma_1^{(k)} = (\gamma_2 - \tau \gamma_1)^{\otimes k} + (\gamma_2 - \tau \gamma_1)^{\otimes k}$ and $\gamma_2^{(k)} = (\tau - \overline{\tau})^{-1} ((\gamma_2 - \tau \gamma_1)^{\otimes k} - (\gamma_2 - \tau \gamma_1)^{\otimes k})$. Then $\gamma_1^{(k)}$ and $\gamma_2^{(k)}$ are integral elements, generating a Hodge substructure $M_D^2(k)$ with induced polarization

$$\langle \gamma_1^{(k)}, \gamma_1^{(k)} \rangle = (1 + (-1)^k) c^k, \quad \langle \gamma_1^{(k)}, \gamma_2^{(k)} \rangle = -\frac{1 - (-1)^k}{(\tau - \overline{\tau})^2} c^k, \quad \langle \gamma_1^{(k)}, \gamma_2^{(k)} \rangle = \frac{1 - (-1)^k}{\tau - \overline{\tau}} c^k,$$

where $c = (\tau - \overline{\tau})\langle \gamma_1, \gamma_2 \rangle$. In particular, when $k$ is even the polarization is of type $x^2 + D y^2$.

Claim 4.1. If $\{\gamma_1^{(k)}, \gamma_2^{(k)}\}$ is a basis of $M_D^2(k)$ with $\gamma_1^{(k)} + (\tau - \overline{\tau})\gamma_2^{(k)} \in F^0$ then

$$\frac{\nabla \gamma_1^{(k)}}{k} = \frac{d \pi}{2\pi} \otimes \gamma_1^{(k)} + (\tau - \overline{\tau}) \left( \frac{d \pi}{4\pi} \sum_{n \in \mathbb{Z}/m} \frac{(-D)}{n} \frac{d \Pi(n/m)}{\Gamma(n/m)} \right) \otimes \gamma_2^{(k)};$$

$$\frac{\nabla \gamma_2^{(k)}}{k} = \frac{1}{\tau - \overline{\tau}} \left( \frac{d \pi}{4\pi} \sum_{n \in \mathbb{Z}/m} \frac{(-D)}{n} \frac{d \Pi(n/m)}{\Gamma(n/m)} \right) \otimes \gamma_1^{(k)} + \frac{d \pi}{2\pi} \otimes \gamma_2^{(k)}.$$
where \( w = 4 \) if \( D = 1 \), \( w = 6 \) if \( D = 3 \), and \( w = 2 \) otherwise; \( h \) is the number of classes of ideals in the ring of integers in \( \mathbb{Q}(\sqrt{-D}) \); and

\[
m = \begin{cases} D, & \text{if } D \equiv 1 \mod 4 \\ 4D, & \text{otherwise.} \end{cases}
\]

**Proof.** It will follow from Example [12] that \( 2\pi i \nabla \gamma_1 = (\omega_2 d\eta_1 - \eta_2 d\omega_1) \otimes \gamma_1 + (\eta_1 d\omega_1 - \omega_1 d\eta_1) \otimes \gamma_2 \) and \( 2\pi i \nabla \gamma_2 = (\omega_2 d\eta_2 - \eta_2 d\omega_2) \otimes \gamma_1 + (\eta_1 d\omega_2 - \omega_1 d\eta_2) \otimes \gamma_2 \). Rewriting the latter using \( \gamma_2 = \tau_\ast \gamma_1 \) as \( 2\pi i \nabla \gamma_2 = 2\pi i(\id \otimes \tau_\ast) \nabla \gamma_1 = \tau \bar{\tau_\ast}(\eta_1 d\omega_1 - \omega_1 d\eta_1) \otimes \gamma_1 + (\ldots) \otimes \gamma_2 \), we get \( \eta_2 d\omega_1 - \omega_2 d\eta_1 = \tau(\eta_1 d\omega_1 - \omega_1 d\eta_1) \), or equivalently, \( d(\eta_2 - \tau_\ast \eta_1) = (\eta_2 - \tau_\ast \eta_1) \frac{d\omega_1}{\omega_1} \). From the Legendre identity \( \omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i \) and \( \omega_2 = \tau \omega_1 \) we get \( d(\eta_2 - \tau_\ast \eta_1) = d\left( \frac{2\pi i}{\omega_1} \right) \). As a corollary of these formulas, \( 2\pi i \nabla (\gamma_2 - \tau_\ast \gamma_1) = \omega_1 d \left( \frac{2\pi i}{\omega_1} \right) \otimes (\gamma_2 - \tau_\ast \gamma_1) \); and \( 2\pi i \nabla (\gamma_2 - \tau_\ast \gamma_1) = 2\pi i \frac{d\omega_1}{\omega_1} \otimes (\gamma_2 - \tau_\ast \gamma_1) \).

Finally,

\[
\nabla \begin{pmatrix} \gamma_1^{(k)} \\ \gamma_2^{(k)} \end{pmatrix} = k \begin{pmatrix} \frac{d\tau}{\tau - \tau'} & \frac{d\omega_1}{\omega_1} - \frac{d\tau}{\tau - \tau'} \\ \frac{d\tau}{\tau - \tau'} & \frac{d\omega_1}{\omega_1} - \frac{d\tau}{\tau - \tau'} \end{pmatrix} \begin{pmatrix} \gamma_1^{(k)} \\ \gamma_2^{(k)} \end{pmatrix}
\]

Due to the Chowla–Selberg formula (cf., e.g., \([6]\)) \( \omega_1 \) is an algebraic multiple of \( \pi^{3/2} \prod_{m=0}^{n-1} \Gamma(n/m)^{w\chi(n)/4h} \), where \( w \) is the number of roots of unity in \( \mathbb{Q}(\sqrt{-D}) \), \( h \) is the number of classes of ideals in the ring of integers in \( \mathbb{Q}(\sqrt{-D}) \), \( m \) is the discriminant of \( \mathbb{Q}(\sqrt{-D}) \) and \( \chi(n) = \left( \frac{-m}{n} \right) \) is the Jacobi symbol.

- Now, a pure polarized Hodge structure of rank 2 of even weight \( w \) is a triplet of a lattice \( H \) of rank 2 with a symmetric definite \( \mathbb{Q}(-w) \)-valued bilinear form on it, and an integer \( k > w/2 \). Then the Hodge filtration is given by \( H_C = F^{w-k} \supset F^k = F^{w-k+1} \supset F^{k+1} = 0 \), where \( F^k \) is the isotropic line, on which the corresponding hermitian form is positive, so for a fixed triplet of integers \( (k > w/2, D > 0) \) there is at most one isogeny class of polarized Hodge structures of rank 2, weight \( w \) with polarization of discriminant \( D \). On the other hand, the Hodge structure \( M_D^{2k}(k - w/2) \) gives the example.

- To determine a polarized Hodge structure of rank 2 and odd weight \( w \) means to fix a lattice \( H \) of rank 2 with an isomorphism \( \wedge^2 H \cong \mathbb{Z}(-w) \), an integer \( k > w/2 \) and a line \( F^k \subset H_C \), on which the corresponding hermitian form is positive. Then the Hodge filtration is given by \( H_C = F^{w-k} \supset F^k = F^{w-k+1} \supset F^{k+1} = 0 \).

If \( k > \frac{w+1}{2} \) then the Griffiths transversality condition and Proposition [23] imply that such geometric Hodge structures correspond to varieties defined over number fields. At least some of the examples of them are constructed in \([72]\). In particular, the Hodge structure \( M_D^{2k-1}(k - w+1/2) \) gives the example.

- Tensoring with \( \mathbb{Z}(\mathbb{Z}(-1)) \), we reduce the case \( k = \frac{w+1}{2} \) to the case of a Hodge structure \( H \) of rank 2 with Hodge numbers \( h^{0,-1} = h^{-1,0} = 1 \). Then we have an embedding \( H \hookrightarrow H_C/F^0 \).

Denote the one-dimensional \( \mathbb{C} \)-space \( H_C/F^0 \) by \( L \) and by \( L^s \) its \( s \)-th tensor power if \( s > 0 \) and the dual of \( L^{-s} \) otherwise. Let \( \zeta = \zeta_H : L - H \to L^{-1} \) be the meromorphic function given by

\[
\zeta_H(z) = \frac{1}{z} + \sum_{\lambda \in H \setminus \{0\}} \left( \frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right) = z^{-1} - \sum_{n \geq 2} z^{2n-1} \left( \sum_{\lambda \in H \setminus \{0\}} \lambda^{-2n} \right).
\]
Set \( A = -15 \sum_{\lambda \in H \setminus \{0\}} \lambda^{-4} \in L^{-4} \) and \( B = -35 \sum_{\lambda \in H \setminus \{0\}} \lambda^{-6} \in L^{-6} \).

The Hodge structure \( H \) is naturally isomorphic to \( H_1(E(\mathbb{C})) \) for the elliptic curve \( E(\mathbb{C}) = L/H \). Our nearest aim is to obtain an algebraic equation (8) of \( E \), and therefore, express a basis (11) of the de Rham cohomology of \( E \) in terms of \( H \). This enables us to calculate the connection in (12) and in Example 4.2. Set

\[
X = \varphi_H(z) = \frac{1}{z^2} + \sum_{\lambda \in H \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda x} \right) = z^{-2} - \frac{1}{5} Az^2 - \frac{1}{7} Bz^4 + O(z^6) \in L^{-2},
\]

\[
Y = -\frac{1}{2} \varphi_H'(z) = \frac{1}{2} \zeta_H''(z) = \frac{1}{z^2} + \sum_{\lambda \in H \setminus \{0\}} \frac{1}{(z - \lambda)^3} = z^{-3} + \frac{1}{5} Az + \frac{2}{7} Bz^3 + O(z^5) \in L^{-3}.
\]

Since any holomorphic elliptic function is constant, we land on the well-known relation

\[
Y^2 = X^3 + AX + B. \tag{8}
\]

Fix a twelve-th root \( \Delta \in L^{-1} \) of \(-4A^3 - 27B^2\) and define the complex numbers \( a = A/\Delta^4 \) and \( b = B/\Delta^6 \), so \( 4a^3 + 27b^2 = -1 \). Set \( x = X/\Delta^2 \) and \( y = Y/\Delta^3 \).

Also, any connection \( \nabla \) on \( L^{-2} \) induces connections on \( L^{-4} \) and \( L^{-6} \). We set

\[
\kappa = \frac{da}{18b} = \frac{db}{4a^2} = \frac{2A\nabla B - 3B\nabla A}{2\Delta^{10}} \in \Omega^1_{\mathbb{C}/\mathbb{Q}}. \tag{9}
\]

Starting with elementary identities in \( \Omega^1_{C(x,y)/\mathbb{C}} \)

\[
-d\left(\frac{1}{y}\right) = \frac{(3x^2 + a)dx}{2y^3}; \quad d\left(\frac{x}{y}\right) = \frac{(2b + ax - x^3)dx}{2y^3}; \quad d\left(\frac{x^2}{y}\right) = \frac{(x^4 + 3ax^2 + 4bx)dx}{2y^4}, \tag{10}
\]

and fixing the following differentials of the second kind on \( E \)

\[
\omega = \frac{dx}{2y} = -dz \quad \text{and} \quad \varphi = \frac{xdx}{2y} = d\zeta(z), \tag{11}
\]

we get then the following congruences modulo exact forms

\[
\omega = \frac{(x^3 + ax + b)dx}{2y^3} \equiv \frac{(2ax + 3b)dx}{2y^3}; \quad \varphi = \frac{(x^4 + ax^2 + bx)dx}{2y^3} \equiv \frac{(2a^2 - 9bx)dx}{6y^3}.
\]

This information is enough to find the Gauss–Manin connection:

\[
\nabla \omega = -\left[ \frac{dy^2 \wedge dx}{4y^3} \right] = -\left[ \frac{(xda + db) \wedge dx}{4y^3} \right] = \kappa \wedge \frac{4a^2 \cdot dx - 18b \cdot xdx}{4y^3}.
\]

So we get

\[
\nabla \omega = 3\kappa \otimes [\varphi]; \quad \text{and similarly}, \quad \nabla \varphi = a\kappa \otimes [\omega]. \tag{12}
\]

Then from the identities

\( \langle \nabla \omega, \gamma \rangle + \langle \omega, \nabla \gamma \rangle = d\langle \omega, \gamma \rangle \quad \text{and} \quad \langle \nabla \varphi, \gamma \rangle + \langle \varphi, \nabla \gamma \rangle = d\langle \varphi, \gamma \rangle, \)

we get

\( \langle \omega, \nabla \gamma \rangle = d\langle \omega, \gamma \rangle - 3\langle \varphi, \gamma \rangle \kappa \quad \text{and} \quad \langle \varphi, \nabla \gamma \rangle = d\langle \varphi, \gamma \rangle - a\langle \omega, \gamma \rangle \kappa. \)

Fix such a basis \( \{\gamma_1, \gamma_2\} \) of the lattice \( H \subset L \) that the imaginary part of \( \gamma_2/\gamma_1 \) is positive. Let \( \omega_1 = \Delta \cdot \gamma_1 \) and \( \omega_2 = \Delta \cdot \gamma_2 \) be the complex numbers corresponding to \( \gamma_1 \) and \( \gamma_2 \), respectively, and \( \eta_1 = \Delta^{-1} \cdot 2\zeta_H(\gamma_j/2). \)

Using the Legendre relation \( \omega_2\eta_1 - \omega_1\eta_2 = 2\pi i \), one easily verifies that the following holds.
Example 4.2. Fix a basis \( \{\gamma_1, \gamma_2\} \) of \( H \) as above. Then, in the above notations,
\[
2\pi i \nabla \gamma_1 = [\omega_2 d\eta_1 - \eta_2 d\omega_1 + (a_1 \omega_2 - 3\eta_1 \eta_2) \kappa] \otimes \gamma_1 \\
+ [\eta_1 d\omega_1 - \omega_1 d\eta_1 + (3\eta_1^2 - a_1 \omega_1^2) \kappa] \otimes \gamma_2;
\]
\[
2\pi i \nabla \gamma_2 = [\omega_2 d\eta_2 - \eta_2 d\omega_2 + (a_2 \omega_2 - 3\eta_2^2) \kappa] \otimes \gamma_1 \\
+ [\eta_1 d\omega_2 - \omega_1 d\eta_2 + (3\eta_1 \eta_2 - a_2 \omega_1) \kappa] \otimes \gamma_2.
\]

4.2. Pure Hodge structures of rank 3. Since any pure Hodge structure of odd rank (3 in our case) is of even weight, we may suppose the weight is 0. To determine such a polarized Hodge structure one has to fix a lattice \( H \) of rank 3 with a non-degenerate symmetric \( \mathbb{Z} \)-valued bilinear form on \( H \), a positive integer \( k \) and an isotropic line \( F^k \subset H_\mathbb{C} \), on which the associated hermitian form is positive. Then the Hodge filtration is given by the classifying space of such structures is

\[
H \text{-symmetric Hodge structures of rank 3 and weights -}k \text{ for } k \geq 2 \text{ and 0.}
\]

Let for some \( t > k/2 \) the Hodge filtration on weight-(\(-k\)) part be in the following range:

\[
F^{-t}(W_{-k})_\mathbb{C} = (W_{-k})_\mathbb{C}
\]

and \( 0 = F^{-k+1}(W_{-k})_\mathbb{C} \subset F^{-k}(W_{-k})_\mathbb{C} = F^{-t}(W_{-k})_\mathbb{C} \neq 0. \)

- If \( t = k + 1 \), or \( k/2 + 1 \leq t < k - 1 \), then the Hodge filtration is obviously horizontal.
- If \( t = k + 1 \) then the filtration looks like \( 0 = F^0 \subset F^1 \subset F^0 = F^{-k} \subset H_\mathbb{C}. \)

Obviously, \( \nabla F^0 \subset \Omega^1 \subset F^0 \). On the other hand, \( F^1 \subset (W_{-k})_\mathbb{C} \), and therefore, one has \( \nabla F^1 \subset \Omega^1 \subset F^1 \cap (W_{-k})_\mathbb{C} = \Omega^1 \subset F^1 \), so Hodge filtration is again horizontal.

- If \( t = k \) then the filtration looks like \( 0 = F^1 \subset F^0 = F^{k-1} \subset F^{-k} = H_\mathbb{C}. \)

Finally, we get that for \( t \geq k \) and \( k/2 + 1 \leq t < k - 1 \) the Hodge structures correspond to varieties over \( \mathbb{Q} \).

- If \( t = k + 1 \) for an odd \( k \geq 5 \), then the connection is trivial on the preimage of \( \mathbb{Z} \) under the map \( F^0 \rightarrow gr^W_0 \otimes \mathbb{C} = \mathbb{C} \), while the rest is a Hodge structure of rank 2, where the connection is already calculated.

- If \( t = k - 1 \) for \( k \geq 3 \) then suppose for simplicity that \( W_{-k} \) is a Hodge substructure of maximal width of \( \hat{H}^{k-2}(E^{k-2})(k-1) \) for an elliptic curve \( E \). Then it is expected that classes of geometric extensions of \( \mathbb{Q}(0) \) by \( H^{k-2}(E^{k-2})(k-1) \) are in one-to-one correspondence with certain classes of motivic cohomology:

\[
Ext^1_{\mathbb{M}, \mathbb{M}}(\mathbb{Q}(0), H^{k-2}(E^{k-2})(k-1)) = Gr^1 \mathcal{H}^{k-2}(E^{k-2}; k-1)_\mathbb{Q}.
\]

As the images of regulators on \( K_{\geq 2} \) are countable (cf. Claim 2.3.4 of [Dr]), such extensions form a countable set.
4.4. Hodge structures of rank 3 and weights $-1$ and 0. These ones are extensions of the Tate structure of weight 0 by Hodge structures of rank 2 and weight $-1$.

- Let for some $t \geq 2$ the Hodge filtration be in the following range: $0 = F^t \subset F^{t-1} = F^1 \subset F^0 \subset F^{1-t} \subset F^{-t} = H_C$.

  If $t \geq 3$ then the Hodge filtration is horizontal for trivial reasons.

  If $t = 2$ then $0 = F^2 \subset F^1 \subset F^0 \subset F^{-1} \subset F^{-2} = H_C$. In particular, $\nabla F^0 \subset \Omega^1_C \otimes \mathbb{C} F^0$.

On the other hand, $F^1 \subset (W_{-1})_C$, and therefore,

$$\nabla F^1 \subset \Omega^1_C \otimes \mathbb{C} \left( F^0 \cap (W_{-1})_C \right) = \Omega^1_C \otimes \mathbb{C} \left( F^1 \cap (W_{-1})_C \right) = \Omega^1_C \otimes \mathbb{C} F^1,$$

so the Hodge filtration is again horizontal.

This means that for any $t \geq 2$ these Hodge structures correspond to varieties over $\overline{\mathbb{Q}}$.

- Now suppose that $t = 1$. We have calculated the Gauß–Manin connection on the weight-$(−1)$ part, the rest is as follows. We keep the notations of §4.1 above with the only exception: we denote by $W_{-1}$ what was $H$ there. Fix an element $\gamma \in H$ projecting to the generator 1 of the Tate structure $\mathbb{Z}(0)$. Subtracting from it an element in $F^0$ projecting to the same element of the Tate structure $\mathbb{Z}(0)$, we get a well-defined element $Z_0$ of the space $L$. Clearly, there is a one-to-one correspondence between elements of the torus $L/W_{-1}$ and extensions of the Tate structure $\mathbb{Z}(0)$ by the Hodge structure $W_{-1}$.

Consider the rational 1-form $\frac{y + \beta}{x - \alpha} \frac{dx}{y} = \frac{\varphi'(z)^{-2}}{\varphi(z)^{-2}} dz = \left(\frac{-2}{z} + O(z)\right) dz$ from the subspace $F^1 H^1_{dR/C}(E \setminus \{0, Q\})$ with logarithmic poles at $Q = (\alpha, \beta)$ and $\infty$, where $\alpha = \varphi(z_0)$ and $\beta = -\frac{1}{2} \varphi'(z_0)$.

$$\nabla \left( \frac{y + \beta}{x - \alpha} \frac{dx}{y} \right) = \nabla \left( \left(1 + \frac{\beta}{y}\right) \frac{d(x - \alpha)}{x - \alpha} \right) = \frac{1}{\beta} \left[ \left( \frac{y^2 d\beta^2}{2y^3} - \frac{\beta^2 dy^2}{2y^3} \right) \wedge \frac{d(x - \alpha)}{x - \alpha} \right].$$

One checks directly, that

$$y^2 d\beta^2 - \beta^2 dy^2 \equiv (x - \alpha)((x^2 + \alpha x + \alpha^2 + a) d\beta^2 - \beta^2 d(3\alpha x + a))$$

modulo the multiples of $d(x - \alpha)$, and therefore,

$$\nabla \left( \frac{y + \beta}{x - \alpha} \frac{dx}{y} \right) = \frac{1}{\beta} \left[ \frac{(x^2 + \alpha x + \alpha^2 + a) d\beta^2 - \beta^2 d(3\alpha x + a)}{2y^3} \wedge d(x - \alpha) \right].$$

Namely, it becomes transparent from the identities

$$y^2 d\beta^2 - \beta^2 dy^2 = (y^2 - 2) d\beta^2 - \beta^2 d(y^2 - 2\beta) = (x - \alpha)((x^2 + \alpha x + \alpha^2 + a) d\beta^2 - \beta^2 d((x - \alpha)^3 + (x - \alpha)(3\alpha x + a))$$.
After some more work\footnote{With a help of the first of the congruences \cite{10} we rewrite} we get

\[ \nabla \left( \frac{y + \beta \, dx}{x - \alpha \, y} \right) = \left[ \frac{d\alpha}{\beta} \wedge \varphi - \frac{a d\alpha}{\beta} \wedge \omega - \frac{2}{\beta} (a \alpha + 3b) \kappa \wedge \omega - \frac{2}{\beta} (2a + 3a^2) \kappa \wedge \varphi \right]. \]

Using an evident expansion \( \zeta(z) - \zeta(z - z_0) = \frac{1}{2} + \zeta(z_0) + O(z) \) of the elliptic function in \( z \) for small \( z \), and notation \( \sigma = \sigma_{W^{-1}}(z) = z \prod_{\lambda \in \mathbb{H} \setminus \{0\}} (1 - \frac{z}{\lambda})e^{z/\lambda + z^2/2\lambda^2} \), we get

\[ \frac{y + \beta \cdot dx}{x - \alpha} = 2(\zeta(z - z_0) - \zeta(z) + \zeta(z_0))dz = 2 \cdot d \log \left( \frac{\sigma(z - z_0)}{\sigma(z)} e^{2\zeta(z_0)z} \right). \]

Obviously, one has \( t \sigma_{W^{-1}}(z) = \sigma_{W^{-1}}(tz) \) for any complex automorphism \( t \) of \( W^{-1} \). In particular, it is an odd function in \( z \). One easily checks that

\[ \sigma(z + \mu) = -e^{(2z + \mu) \zeta(\mu/2)} \sigma(z) \quad \text{for any} \; \mu \in W^{-1} \setminus 2W^{-1}. \]

Namely, second derivatives of logarithm of both sides coincide as they are values of the elliptic function \( -\varphi \) at points \( z + \mu \) and \( z \), respectively. This implies that difference of first derivatives of logarithm of two sides is a constant. Evaluation at \( z = -\mu/2 \) shows that first derivatives of logarithm of both sides coincide. This implies that quotient of two sides is a constant. Evaluation at \( z = -\mu/2 \) shows that the equality holds.

As a corollary, we see that the class \( \gamma \in H^1(E \setminus \{0, (\alpha, \beta)\}; Z(1)) \) is presented by the 1-form

\[ \frac{y + \beta \, dx}{\alpha - x \, 2y} - \zeta(z_0) \cdot \omega - z_0 \cdot \varphi, \quad \text{so finally, we get the following.} \]

**Example 4.3.** For any generator \( \gamma \) as before, the connection is determined by the conditions \( \nabla \gamma \in \Omega^1_{\mathbb{C}} \otimes W^{-1} \) and

\[ \langle \nabla \gamma, \gamma_j \rangle = \omega_j d\zeta(z_0) - \eta_j \frac{d\alpha}{2\beta} - \omega_j \frac{a d\alpha}{2\beta} - \eta_j d\sigma_0 \]

\[ + \left( \alpha_0 \omega_j - 3 \zeta(z_0) \eta_j + \frac{2a + 3a^2}{\beta} \eta_j - \frac{a \alpha + 3b}{\beta} \omega_j \right) \kappa. \, \Box \]

\[ \frac{d\alpha}{\beta} \wedge \varphi - \frac{a d\alpha}{\beta} \wedge \omega + \frac{(2a + 3a^2) (6b \alpha + 4a^2) - (a \alpha + 3b) (4a x + 6b)}{2 \beta} \kappa \wedge \sigma. \]

\[ \frac{d\alpha}{\beta} \wedge \varphi - \frac{a d\alpha}{\beta} \wedge \omega + \frac{(2a + 3a^2) (6b \alpha + 4a^2) - (a \alpha + 3b) (4a x + 6b)}{2 \beta} \kappa \wedge \sigma. \]
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