PERTURBATION THEORY FOR
HOMOGENEOUS EVOLUTION EQUATIONS

DANIEL HAUER

CONTENTS
1. Introduction and main results 1
2. Intermediate results and proofs 4
3. Homogeneous Accretive operators 11
4. Homogeneous completely accretive operators 17
4.1. General framework 18
4.2. Regularizing effect of the semigroup 22
5. Application 25
5.1. Framework 25
5.2. Global regularity estimates of $\frac{du}{dt}$ 27
References 29

ABSTRACT. In this paper, we develop a perturbation theory to show that if a homogeneous operator of order $\alpha \neq 1$ is perturbed by a Lipschitz continuous mapping then every mild solution of the first-order Cauchy problem governed by these operators is strong and the time-derivative satisfies a global regularity estimate. We employ this theory to derive global $L^q-L^\infty$-estimates of the time-derivative of the evolution problem governed by the $p$-Laplace-Beltrami operator and total variational flow operator respectively perturbed by a Lipschitz nonlinearity on a non-compact Riemannian manifold.

1. INTRODUCTION AND MAIN RESULTS

In the pioneering work [7], Bénilan and Crandall showed that for the class of homogeneous operators $A$ of order $\alpha > 0$ with $\alpha \neq 1$ (possibly, multi-valued), defined on a Banach space $(X, \| \cdot \|_X)$, every solution $u$ of the differential inclusion

\begin{equation}
\frac{du}{dt} + A(u(t)) \ni 0
\end{equation}

satisfies

\begin{equation}
\limsup_{h \to 0^+} \frac{\|u(t+h) - u(t)\|_X}{h} \leq 2L \|u(0)\|_X \frac{1}{t^{\alpha - 1}} \quad \text{for every } t > 0.
\end{equation}

Here, an operator $A$ is called homogeneous of order $\alpha \in \mathbb{R}$ (cf., Definition 2.1) if

\begin{equation}
A(\lambda u) = \lambda^\alpha Au \quad \text{for all } \lambda \geq 0 \text{ and } u \in D(A).
\end{equation}
Estimates of type (1.2) are quite important as they describe an instantaneous (for arbitrarily small \( t > 0 \)) and global regularizing effect for solutions \( u \) to the differential inclusion (1.1). In fact, (1.2) implies that the solution \( u \) of (1.1) is locally Lipschitz continuous in \( t \in (0, +\infty) \). If, in addition, the Banach space \( X \) admits the Radon-Nikodým property (see Definition 3.9), then the solution \( u \) of (1.1) is differentiable at almost every (a.e.) \( t > 0 \) or, equivalently (cf., [9]) \( u(t) \in D(A) \) for a.e. \( t > 0 \), and (1.2) becomes

\[
\|A^\alpha u(t)\|_X = \|\frac{du}{dt}(t)\|_X \leq 2 L \frac{\|u(0)\|_X \frac{1}{|\alpha - 1|}}{t} \quad \text{for a.e. } t > 0.
\]

In (1.4),

\[
A^\alpha u := \left\{ v \in X \mid \|v\|_X = \inf_{\hat{v} \in Au} \|\hat{v}\|_X \right\}
\]
denotes the minimal selection of \( Au \) for given \( u \in D(A) \). Moreover, estimate (1.4) also describes a rate of dissipativity related to the differential inclusion (1.1).

Besides the homogeneity property of \( A \), the proof of regularity estimate (1.2) also requires that the Cauchy problem

\[
\begin{align*}
\frac{du}{dt} + A(u(t)) &\ni 0 \quad \text{on } (0, +\infty), \\
u(0) &= 0,
\end{align*}
\]

is well-posed in the mild sense (see Definition 3.2). Thus, there is a strongly continuous semigroup \( \{T_t\}_{t \geq 0} \) of contractions \( T_t : D(A)^\alpha \to D(A)^\alpha \) (see Definition 3.3), which is related to the solutions \( u \) of (1.6) by

\[
T_t u_0 = u(t) \quad \text{for every } t \geq 0, u_0 \in D(A)^\alpha.
\]

Recently, the author and Mazón [17] extended Bénilan and Crandall’s regularity result [7] for the class of homogeneous operators of order \( \alpha = 0 \). This class operators contains, for example, the (negative) total variational flow operator \( Au = -\text{div}(\frac{Du}{|Du|}) \), or the 1-fractional Laplacian \( A = (-\Delta_1)^s \) for \( s \in (0, 1) \), respectively equipped with some boundary conditions.

Our aim in this paper is to establish similar regularity estimates for solutions \( u \) to the Cauchy problem associated with the perturbed operator \( A + F \);

\[
\begin{align*}
\frac{du}{dt} + A(u(t)) + F(u(t)) &\ni 0 \quad \text{on } (0, +\infty), \\
u(0) &= u_0,
\end{align*}
\]

where \( A \) is an homogeneous operator in \( X \) of order \( \alpha \neq 1 \) and \( F : X \to X \) a Lipschitz continuous mapping with constant \( \omega \geq 0 \) satisfying \( F(0) = 0 \). We emphasize that the perturbed operator \( A + F \) is, in general, not homogeneous anymore. Thus, it is not trivial to say that solutions \( u \) to the Cauchy problem (1.8) for the perturbed operator \( A + F \) still admit a regularization effect.

But, our main theorem says the following.

**Theorem 1.1.** Given \( \omega \in \mathbb{R} \), let \( A \) be an \( m \)-accretive operator in \( X \) which is homogeneous of order \( \alpha \neq 1 \) and suppose, the mapping \( F : X \to X \) is Lipschitz continuous.
on $X$ with constant $\omega > 0$, and $F(0) = 0$. For $u_0 \in \overline{D(A)^e}$, suppose that for the corresponding solution $u$ of (1.8),

$$V_\omega(u, t) := \limsup_{h \to 0} \frac{\|u(t + h) - u(t)\|_X}{h}, \ (t > 0),$$

is locally integrable on $[0, T)$, $T > 0$. Then,

$$V_\omega(u, t) \leq \frac{\|u_0\|_X}{1 - \alpha t} \left[ 2e^{L\omega t} + \omega \int_0^t e^{L\omega(t-s)} ds \right].$$

Moreover, if $\frac{du}{dt}$ exists and belongs to $L_{loc}^1([0, T); X)$, then

$$\left\| \frac{du}{dt}(t) \right\|_X \leq \frac{2e^{\omega t}}{t} \left[ (1 + \omega) + \int_0^t (1 + \omega s) \omega e^{\omega(t-s)} ds \right]$$

for a.e. $t \in (0, T)$.

In Theorem 1.1, the hypothesis on $A$ and $F$ guarantee the existence of a semigroup $\{T_t\}_{t \geq 0}$ of $e^{\omega t}$-Lipschitz continuous mappings $T_t$. At the first view, the hypothesis that the function $V_\omega(u, \cdot) \in L_{loc}^1([0, T))$ seems to be quite strong; in fact, this hypothesis, is only given for initial values $u_0 \in D(A)$ if the Banach space $X$ admits the Radon-Nikodým property (cf. [9]). But since the right-hand side in (1.10) and (1.11) only depends on $\|u_0\|_X$, a compactness result can help to remove this condition. We show this in Corollary 3.11 under the assumption that the Banach space $X$ is reflexive. Theorem 1.1 follows from Theorem 2.7.

Even though the statement of [17, Corollary 2.12] is correct, unfortunately, the proof omits some important details. But even worse, the application of Gronwall’s lemma in this paper is not correct. Thus, the proof of Theorem 2.7 intends to fill this gap. We refer to Section 4 for the proof.

In many applications, the Banach space $X$ is given by the classical Lebesgue space $(L^q := L^q(\Sigma, \mu), \|\cdot\|_q)$, $(1 \leq q \leq \infty)$, for a given $\sigma$-finite measure space. If $\{T_t\}_{t \geq 0}$ is a semigroup satisfying an $L^q$-$L^r$ regularity estimate ($q < r < \infty$) of the form (cf., [13])

$$\left\| T_t u_0 \right\|_r \leq C e^{\gamma t} \frac{\|u_0\|_q^\gamma}{t^\delta}$$

for all $t > 0$, and $u_0 \in L^q$, for $\omega \in \mathbb{R}$, $\gamma = \gamma(q, r, d)$, $\delta = \delta(q, r, d) > 0$, and some (or for all) $1 \leq q < r$, then we show in Corollary 2.6 that combining (1.2) with (1.12) yields

$$\limsup_{h \to 0+} \frac{\|u(t + h) - u(t)\|_r}{h} \leq C L 2^{\delta + 2} \frac{\|u_0\|_q^\gamma}{t^\delta \gamma}.$$

Regularity estimates similar to (1.12) have been studied recently by many authors (see, for example, [14, 24, 15] and the references therein for the linear theory, and we refer to [13] and the references therein for the nonlinear one).

In Section 3, we consider the class of quasi accretive operators $A$ (see Definition 3.1) and outline how the property that $A$ is homogeneous of order $\alpha \neq 1$ is passed on to the semigroup $\{T_t\}_{t \geq 0}$ generated by $-A$ (see the paragraph after Definition 3.2). In particular, we discuss when solutions $u$ of (1.1) are differentiable in $X$ at a.e. $t > 0$. 
It is well-known that the Lebesgue space $L^1$ does not admit the Radon-Nikodým property. But on the other hand, from the physical point of view, $L^1$ is for many models not avoidable. In [8], Bénilan and Crandall developed the theory of completely accretive operators $A$ (in $L^1$). For this class of operators, it is known that for each solutions $u$ of (1.1) in $L^1$, the derivative $\frac{du}{dt}$ exists in $L^1$. These results have been extended recently to the notion of quasi completely accretive operators in the monograph [13].

We conclude this paper in Section 5 with an application. We derive global $L^q$-$L^\infty$-regularity estimates of the time-derivative $\frac{du}{dt}$ for solutions $u$ to the perturbed evolution problem (1.8) when $A$ is the negative $p$-Laplacian $-\Delta_p$ in $L^1$ equipped with vanishing condition on a non-compact Riemannian manifold $(M, g)$.

It is worth mentioning that if the operator $A$ in (1.1) is linear (that is, $\alpha = 1$), then inequality (1.14) $\|AT_t u_0\|_X \leq C \frac{\|u_0\|_X}{t}$, holding for all $t \in (0, 1]$ and $u_0 \in D(A)$, yields that $-A$ generates an analytic semigroup $\{T_t\}_{t \geq 0}$ (cf., [2, 19]). Thus, it is interesting to see that a regularity inequality similar to (1.14) also holds for nonlinear operators of the type $A + F$, where $A$ is homogeneous of order $\alpha \neq 1$. Further, if the norm $\|\cdot\|_X$ is induced by an inner product $(\cdot, \cdot)_X$ of a Hilbert space $X$ and $A = \partial \varphi$ is the sub-differential operator $\partial \varphi$ in $X$ of a semi-convex, proper, lower semicontinuous function $\varphi : X \to (-\infty, +\infty]$, then regularity inequality (1.14) is, in particular, satisfied by solutions $u$ of (1.1) (cf., [11, 12]). It is worth mentioning that inequality (1.14) plays an important role in abstract 2nd-order problems of elliptic type for accretive operators $A$ (see, for example, [20, (2.22) on page 525] or, more recently, [16, (1.8) on page 719]).

2. INTERMEDIATE RESULTS AND PROOFS

In this section, we gather some intermediates results to prove the main theorems of this paper.

Suppose $X$ is a linear vector space and $\|\cdot\|_X$ a semi-norm on $X$. Then, the main object of this paper is the following class of operators (cf., [7] and [17]).

**Definition 2.1.** An operator $A$ on $X$ is called **homogeneous of order** $\alpha \in \mathbb{R}$ if for every $u \in D(A)$ and $\lambda \geq 0$, one has that $\lambda u \in D(A)$, $0 \in A0$, and $A$ satisfies (1.3).

For the rest of this section suppose that $A$ denotes a homogeneous operator on $X$ of order $\alpha \neq 1$. We begin by considering the inhomogeneous Cauchy problem

$$\begin{cases}
\frac{du}{dt} + A(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T), \\
u(0) = u_0,
\end{cases}$$

and want to discuss the impact of the homogeneity of $A$ on the solutions $u$ to (2.1). For this, suppose $f \in C([0, T]; X)$, $u_0 \in X$, and $u \in C^1([0, T]; X)$ be a


classical solution of (2.1). Further, for given $\lambda > 0$, set

$$v_{\lambda}(t) = \lambda^{\frac{1}{\tau - 1}} u(\lambda t), \quad (t \in [0, \frac{T}{\lambda}).$$

Then, $v$ satisfies

$$\frac{d v_{\lambda}}{dt}(t) = \lambda^{\frac{1}{\tau - 1} + 1} \frac{d u}{dt}(\lambda t) \in \lambda^{\frac{1}{\tau - 1}} [f(\lambda t) - A(u(\lambda t))]$$

$$= -A(v_{\lambda}(t)) + \lambda^{\frac{1}{\tau - 1}} f(\lambda t)$$

for every $t \in (0, T/\lambda)$ with initial value $v_{\lambda}(0) = \lambda^{\frac{1}{\tau - 1}} u(0) = \lambda^{\frac{1}{\tau - 1}} u_0$.

Now, if we assume that the Cauchy problem (2.1) is well-posed for given $u_0 \in \overline{D(A)^X}$ and $f \in L^1(0, T; X)$, then there is a semigroup $\{T_t\}_{t \in [0, T]}$ of mappings $T_t : \overline{D(A)^X} \times L^1(0, T; X) \rightarrow \overline{D(A)^X}$ given by

$$(2.2) \quad T_t(u_0, f) := u(t) \quad \text{for every } u_0 \in \overline{D(A)^X} \text{ and } f \in L^1(0, T; X),$$

where $u$ is the unique (mild). Then, in terms of this semigroup $\{T_t\}_{t \in [0, T]}$ associated with (2.1), the previous reasoning can be formulated as follows

$$(2.3) \quad \lambda^{\frac{1}{\tau - 1}} T_t(u_0, f) = T_t(\lambda^{\frac{1}{\tau - 1}} u_0, \lambda^{\frac{1}{\tau - 1}} f(\lambda \cdot))$$

holding for every $t \in [0, T/\lambda]$, $\lambda > 0$. Identity (2.3) together with the standard growth estimate

$$(2.4) \quad e^{-\omega t} \|T_t(u_0, f) - T_t(\hat{u}_0, \hat{f})\|_X$$

$$\leq L e^{-\omega s} \|T_s(u_0, f) - T_s(\hat{u}_0, \hat{f})\|_X + L \int_s^t e^{-\omega r} \|f(r) - \hat{f}(r)\|_X \, dr$$

for every $0 \leq s \leq t \leq T$, (for some $\omega \in \mathbb{R}$ and $L > 0$) are the main ingredients to obtain global regularity estimates of the form (1.2). This leads to our first intermediate result. This proposition also generalizes the case of homogeneous operator of order zero (cf., [17, Theorem 2.3]), and the case $\omega = 0$ treated in [7, Theorem 4].

**Proposition 2.2.** Let $\{T_t\}_{t=0}^T$ be a family of mappings $T_t : C \times L^1(0, T; X) \rightarrow C$ defined on a subset $C \subseteq X$, and suppose there are $\omega \in \mathbb{R}$, $L > 0$, and $\alpha \neq 1$ such that $\{T_t\}_{t=0}^T$ satisfies (2.4), (2.3) for every $\lambda > 0$ and $t \in [0, T]$, and $T_t(0, 0) \equiv 0$ for all $t \in [0, T]$. Then for every $u_0 \in C$, $f \in L^1(0, T; X)$,

$$\|T_{t+\alpha}(u_0, f) - T_t(u_0, f)\|_X$$

$$\leq L \left| 1 - (1 + \frac{\alpha}{\lambda T}) \right|^\frac{1}{\alpha} \left[ 2 e^{\omega t} \|u_0\|_X$$

$$+ \int_0^t e^{\omega (t-s)} \|f(s)\|_X + \|f(s + \frac{\alpha}{\lambda T}s)\|_X \, ds \right]$$

$$+ L \left( 1 + \frac{\alpha}{\lambda T} \right)^\frac{1}{\alpha} \int_0^t e^{\omega (t-s)} \left\| f(s + \frac{\alpha}{\lambda T}s) - f(s) \right\|_X \, ds$$
for every $t \in (0, T]$, $h \neq 0$ such that $1 + \frac{h}{T} > 0$ and $t + h \in (0, T]$. In particular, for $V_0(f,t)$ given by (1.9), the family $\{T_t\}_{t \geq 0}$ satisfies

$$
\limsup_{h \to 0^+} \left\| \frac{T_{t+h}(u_0, f) - T_t(u_0, f)}{h} \right\|_X 
\leq \frac{L}{t} \left[ \frac{2 e^{\omega t}}{|1 - \alpha|} \|u_0\|_X + \frac{2}{|1 - \alpha|} \int_0^t e^{\omega(t-s)}\|f(s)\|_X \ ds + V_0(f,t) \right]
$$

(2.6) for every $t > 0$, $u_0 \in C$, and if $f \in W^{1,1}(0, T; X)$, then

$$
\limsup_{h \to 0^+} \left\| \frac{T_{t+h}(u_0, f) - T_t(u_0, f)}{h} \right\|_X
\leq \frac{L}{t} \left[ \frac{2 e^{\omega t}}{|1 - \alpha|} \|u_0\|_X + \frac{2}{|1 - \alpha|} \int_0^t e^{\omega(t-s)}\|f(s)\|_X \ ds + \int_0^t e^{\omega(t-s)}\|f'(s)\|_X s \ ds \right].
$$

(2.7) Moreover, if $\frac{dT_t(u_0, f)}{dt}$ exists (in $X$) at a.e. $t > 0$, then

$$
\left\| \frac{dT_t(u_0, f)}{dt} \right\|_X \leq \frac{L}{t} \left[ \frac{2 e^{\omega t}}{|1 - \alpha|} \|u_0\|_X + \frac{2}{|1 - \alpha|} \int_0^t e^{\omega(t-s)}\|f(s)\|_X \ ds + \int_0^t e^{\omega(t-s)}\|f'(s)\|_X s \ ds \right].
$$

(2.8) Proof. Let $u_0 \in C$, $f \in L^1(0, T; X)$, $t > 0$, and $h \neq 0$ satisfying $1 + \frac{h}{T} > 0$ and $0 < t + h \leq T$. If we choose $\lambda = 1 + \frac{h}{T}$ in (2.3), then

$$
T_{t+h}(u_0, f) - T_t(u_0, f) = T_{\lambda t}(u_0, f) - T_t(u_0, f)
= \lambda^{\frac{1}{\alpha t}} T_t \left[ \lambda^{\frac{1}{\alpha}} u_0, \lambda^{\frac{1}{\alpha}} f(\lambda \cdot) \right] - T_t(u_0, f)
$$

(2.9) and so,

$$
T_{t+h}(u_0, f) - T_t(u_0, f) = \lambda^{\frac{1}{\alpha t}} T_t \left[ \lambda^{\frac{1}{\alpha}} u_0, \lambda^{\frac{1}{\alpha}} f(\lambda \cdot) \right] - T_t(u_0, f)
+ \lambda^{\frac{1}{\alpha t}} \left[ T_t [u_0, f(\lambda \cdot)] - T_t(u_0, f) \right]
+ \left[ \lambda^{\frac{1}{\alpha t}} - 1 \right] T_t(u_0, f).
$$

(2.10) Applying to this (2.4) and by using $T_t(0,0) \equiv 0$, one sees that $\|T_{t+h}(u_0, f) - T_t(u_0, f)\|_X$

$$
\leq \left( 1 + \frac{h}{T} \right)^{\frac{1}{\alpha t}} \|T_t \left[ \lambda^{\frac{1}{\alpha}} u_0, \lambda^{\frac{1}{\alpha}} f(\lambda \cdot) \right] - T_t(u_0, f(\lambda \cdot))\|_X
+ \left( 1 + \frac{h}{T} \right)^{\frac{1}{\alpha t}} \|T_t [u_0, f(\lambda \cdot)] - T_t(u_0, f)\|_X
+ \left( \lambda^{\frac{1}{\alpha t}} - 1 \right) \|T_t(u_0, f)\|_X
$$
Proposition 2.4. \[ \parallel (2.11) \]

\[ \parallel f \parallel \]

V bounded variation

Definition 2.3.

Each \( t \int_0^t e^{\alpha s} \int_0^s e^{-\alpha s} \mid \leq \parallel f \parallel \]

\[ = L e^{\alpha t} \left\{ \begin{array}{l}
(1 + \frac{h}{T}) \frac{1}{\omega} L e^{\alpha t} \\
\parallel f \parallel \]

From this, it is clear that (2.5)-(2.8) follow.

Examples of functions \( f : [0, T] \rightarrow X \) for which \( V_{\omega}(f, t) \) is finite a.e. \( t \) and integrable on \([0, T]\), are functions with bounded variation (cf., [11, Appendix, Section 2.]).

Definition 2.3. For a function \( f : [0, T] \rightarrow X \), one calls

\[ \text{Var}(f; [0, T]) := \sup \left\{ \sum_{i=1}^N \parallel f(t_i) - f(t_{i-1}) \parallel \biggm| 0 = t_0 < \cdots < t_N = T \right\} \]

the total variation of \( f \). Each \( X \)-valued function \( f : [0, T] \rightarrow X \) is said to have bounded variation on \([0, T]\) if \( \text{Var}(f; [0, T]) \) is finite. We denote by \( BV(0, T; X) \) the spaces of all functions \( f : [0, T] \rightarrow X \) of bounded variation and to simplify the notation, we set \( V_f(t) = \text{Var}(f; [0, t]) \) for \( t \in (0, T] \).

Functions of bounded variation have the following properties.

Proposition 2.4. Let \( f \in BV(0, T; X) \). Then the following statements hold.

1. \( f \in L^\infty(0, T; X) \);
2. At every \( t \in [0, T] \), the left-hand side limit \( f(t-) := \lim_{s \rightarrow t-} f(s) \) and right-hand side limit \( f(t+) := \lim_{s \rightarrow t+} f(s) \) exist in \( X \); and the set of discontinuity points in \([0, T]\) is at most countable;
3. The mapping \( t \mapsto V_f(t) \) is monotonically increasing on \([0, T]\), and
4. \[ \parallel f(t) - f(s) \parallel \leq V_f(t) - V_f(s) \] for all \( 0 \leq s \leq t \leq T \);
5. One has that
   \[ \int_0^{t-h} \frac{e^{\alpha (s+h)} - e^{\alpha s}}{h} ds \leq V_f(t) \] for all \( h \in (0, t], 0 < t < T \).
6. One has that \( V_{\omega}(f, \cdot) \in L^\infty(0, T) \) and
   \[ V_{\omega}(f, t) \leq 3t e^{\alpha t} V_f(T) \] for all \( t \in [0, T] \).
We omit the proof of these well-known properties and refer to [11, Section 2., Lemme A.1] or [18, Chapter 2] for further literature to this topic.

In the case $f \equiv 0$, we let $T = \infty$. Then the mapping $T_t$ given by (2.2) only depends on the initial value $u_0$. In other words,

$$T_t u_0 = T_t(u_0, 0) \quad \text{for every } u_0 \in C \text{ and } t \geq 0.$$  

In this case Proposition 2.2 reads as follows (cf., [8]).

**Corollary 2.5.** Let $\{T_t\}_{t \geq 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$, and suppose there are $\omega \in \mathbb{R}$, $L > 0$, and $\alpha \neq 1$ such that $\{T_t\}_{t \geq 0}$ satisfies

$$\|T_t u_0 - T_i u_0\| \leq L e^{\omega t} \|u_0 - u_0\| \quad \text{for all } t \geq 0, u, \hat{u} \in C,$n

$$\lambda^{\frac{h}{\alpha t}} T_{\lambda t} u_0 = T_t [\lambda^{\frac{h}{\alpha t}} u_0] \quad \text{for all } \lambda > 0, t \geq 0 \text{ and } u_0 \in C.$$

Further, suppose $T_0 T_0 \equiv 0 \text{ for all } t \geq 0$. Then, for every $u_0 \in C$,

$$(2.16) \quad \|T_{t+h} u_0 - T_t u_0\| \leq 2 L \left(1 - \left(1 + \frac{h}{\tau} \right)^{\frac{t}{h}}\right) e^{\omega t} \|u_0\|_X.$$

$t > 0$, $h \neq 0$ satisfying $1 + \frac{h}{\tau} > 0$. In particular, the family $\{T_t\}_{t \geq 0}$ satisfies

$$(2.17) \quad \limsup_{h \to 0+} \frac{\|T_{t+h} u_0 - T_t u_0\|}{h} \leq \frac{2 L e^{\omega t}}{1 - \alpha} \|u_0\|_X \quad \text{for every } t > 0, u_0 \in C.$$

Moreover, if for $u_0 \in C$, the right-hand side derivative $\frac{d T_t u_0}{d t}$ exists (in $X$) at $t > 0$, then

$$(2.18) \quad \left\| \frac{d T_t u_0}{d t} \right\|_X \leq \frac{2 L e^{\omega t}}{1 - \alpha} \|u_0\|_X.$$

If the given family $\{T_t\}_{t \geq 0}$ of mappings $T_t$ on $C$ form a semigroup (see Definition 3.3), then one can extrapolate the regularization effect (2.18).

**Corollary 2.6.** Let $\{T_t\}_{t \geq 0}$ be a semigroup of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ and suppose, there is a second vector space $Y$ with semi-norm $\|\cdot\|_Y$ such that $\{T_t\}_{t \geq 0}$ satisfies the following $Y$-regularity estimate

$$(2.19) \quad \|T_t u_0\|_X \leq M e^{\omega t} \|u_0\|_Y^\gamma \quad \text{for every } t > 0 \text{ and } u_0 \in C$$

for some $M, \gamma, \delta > 0$ and $\omega \in \mathbb{R}$. If for some $\alpha \neq 1$ and $L > 0$, $\{T_t\}_{t \geq 0}$ satisfies (2.17) for every $t > 0$, $u_0 \in C$, then

$$\limsup_{h \to 0+} \frac{\|T_{t+h} u_0 - T_t u_0\|}{h} \leq \frac{2^{\delta+2} L M}{1 - \alpha} e^{\frac{t}{2} (\omega + \delta t)} \|u_0\|_Y^\gamma.$$

Moreover, if the right-hand side derivative $\frac{d}{d t} T_t u_0$ exists (in $X$) at $t > 0$, then

$$\left\| \frac{d T_t u_0}{d t} \right\|_X \leq \frac{2^{\delta+2} L M}{1 - \alpha} e^{\frac{t}{2} (\omega + \delta t)} \|u_0\|_Y^\gamma.$$

**Proof.** Let $u_0 \in C$ and $t > 0$. Then by the semigroup property of $\{T_t\}_{t \geq 0}$ and by (2.17) and (2.19), one sees that

$$\limsup_{h \to 0+} \frac{\|T_{t+h} u_0 - T_t u_0\|}{h} = \limsup_{h \to 0+} \frac{\|T_{t+h} (T_t u_0) - T_{t+h} (T_t u_0)\|}{h}.$$
Finally, we turn our attention to the Cauchy problem for the perturbed operator \( A + F \),

\[
\frac{du}{dt} + A(u(t)) + F(u(t)) \geq 0 \quad \text{on } (0, +\infty), \quad u(0) = 0,
\]

involving an \( m \)-accretive and homogenous operator \( A \) in \( X \) of order \( \alpha \neq 1 \) and a Lipschitz continuous perturbation \( F : X \to X \) with Lipschitz constant \( \omega \geq 0 \) satisfying \( F(0) = 0 \). We assume that problem (1.8) is well-posed in \( X \) in the sense that there is a semigroup \( \{ T_t \}_{t \geq 0} \) of Lipschitz continuous mappings \( T_t \) on a set \( C = \overline{D(A)}^\gamma \) with constant \( e^{\omega t} \) and for every given \( u_0 \in \overline{D(A)}^\gamma \), the unique solution \( u \in C([0, \infty); X) \) of (1.8) is given by (1.7).

One important property of the semigroup \( \{ T_t \}_{t \geq 0} \) related to (1.8) is that for given \( u_0 \in \overline{D(A)}^\gamma \), the unique solution \( u \) of Cauchy problem (1.8) is also the unique solution of Cauchy problem (2.1) for \( f : [0, +\infty) \to X \) given by

\[
f(t) := F(u(t)), \quad (t \geq 0).
\]

Therefore, we have that

\[
T_t u_0 = T_t(u_0, f) \quad \text{for every } t \geq 0, u_0 \in \overline{D(A)}^\gamma.
\]

The advantage of this relation is that one can employ inequality (2.4) which is satisfied by the family \( \{ T_t(\cdot, f) \}_{t \geq 0} \).

Now, by Proposition 2.2, the following estimates holds.

**Theorem 2.7.** Let \( F : X \to X \) be a Lipschitz continuous mapping with Lipschitz constant \( \omega > 0 \) satisfying \( F(0) = 0 \). Suppose, there is a subset \( C \subseteq X \), and a family \( \{ T_t \}_{t \geq 0} \) of mappings \( T_t : C \to C \) satisfying

\[
\| T_t u_0 \|_X \leq e^{\omega t} \| u_0 \|_X \quad \text{for all } t \geq 0, u_0 \in C,
\]

and in relation with (2.21), suppose that \( \{ T_t \}_{t \geq 0} \) satisfies (2.3) and (2.4) for \( f \) given by (2.20). Suppose for \( u_0 \in C \), the function \( t \mapsto e^{-\omega t} V_\omega(\{ T_s u_0 \}_{s \geq 0}, t) \) belongs to \( L^1_{loc}(0, T) \), where \( V_\omega \) is defined by (1.9). Then

\[
\limsup_{h \to 0^+} \frac{\| T_{t+h} u_0 - T_t u_0 \|_X}{h} \leq \frac{e^{\omega t} 2 L \| u_0 \|_X}{t |1 - \alpha|} \left[ (1 + \omega t) + \int_0^t (1 + \omega s) L \omega e^{\omega(t-s)} \, ds \right]
\]

for a.e. \( t \in (0, T) \). Moreover, if \( \frac{dT_t u_0}{dt} \) exists at a.e. \( t \in (0, T) \) and belongs to \( L^1(0, T; X) \), then

\[
\left\| \frac{dT_t u_0}{dt} \right\|_X \leq \frac{e^{\omega t} 2 L \| u_0 \|_X}{t |1 - \alpha|} \left[ (1 + \omega t) + \int_0^t (1 + \omega s) L \omega e^{\omega(t-s)} \, ds \right]
\]
for a.e. \( t \in (0, T) \).

For the proof of this theorem, we still need the following version of Gronwall’s lemma.

**Lemma 2.8 ([23, Lemma D.2])**. Suppose \( v \in L^1_{\text{loc}}([0, T]) \) satisfies

\[
(2.25) \quad v(t) \leq a(t) + \int_0^t v(s) b(s) \, ds \quad \text{for a.e. } t \in (0, T),
\]

where \( b \in C([0, T]) \) satisfying \( b(t) \geq 0 \), and \( a \in L^1_{\text{loc}}([0, T]) \). Then,

\[
(2.26) \quad v(t) \leq a(t) + \int_0^t a(s) b(s) e^{\int_s^t b(\tau) \, d\tau} \, ds \quad \text{for a.e. } t \in (0, T).
\]

We now ready to give the proof of Theorem 2.7.

**Proof of Theorem 2.7**. Let \( u_0 \in C, t > 0 \), and \( h \neq 0 \) such that \( |h|/t < 1 \). Then, by the hypotheses of this corollary, we are in the position to apply Proposition 2.2 to \( T_t(u_0, f) \) for \( f \) given by (2.20). Then by (2.5),

\[
\left\| T_{t+h}u_0 - T_tu_0 \right\|_X \\
\leq L \left| 1 - (1 + \frac{h}{T}) \right|^\frac{1}{\alpha} \left[ 2 e^{\omega t} \left\| u_0 \right\|_X \\
+ \int_0^t e^{\omega (t-s)} \left[ \left\| F(T_su_0) \right\|_X + \left\| F(T_{s+h/T}u_0) \right\|_X \right] \, ds \right] \\
+ L \left( 1 + \frac{h}{T} \right)^\frac{1}{2} \int_0^t e^{\omega (t-s)} \left\| F(T_{s+h/T}u_0) - F(T_su_0) \right\|_X \, ds
\]

Since \( F \) is globally Lipschitz continuous with constant \( \omega > 0 \), \( F(0) = 0 \) and by (2.22), it follows that

\[
e^{-\omega t} \left\| T_{t+h}u_0 - T_tu_0 \right\|_X \\
\leq L \left| 1 - (1 + \frac{h}{T}) \right|^\frac{1}{\alpha} \left[ (2 + \omega t) \left\| u_0 \right\|_X + \omega \int_0^t e^{\omega s} \left\| u_0 \right\|_X \, ds \right] \\
+ L \left( 1 + \frac{h}{T} \right)^\frac{1}{2} \omega \int_0^t e^{-\omega s} \left\| T_{s+h/T}u_0 - T_su_0 \right\|_X \, ds
\]

Dividing this inequality by \( |h| > 0 \) and taking the limit superior as \( h \to 0 \) yields

\[
e^{-\omega t} V_\omega(T_tu_0, t) \\
\leq \frac{L_2 \left\| u_0 \right\|_X}{|1 - \alpha|} (1 + \omega t) + L \omega \int_0^t e^{-\omega r} V_\omega(T_tu_0, s) \, dr,
\]

(2.27)

Now, applying Gronwall’s lemma to

\[
v(t) = e^{-\omega t} V_\omega(T_tu_0), \\
a(t) = \frac{L_2 \left\| u_0 \right\|_X}{|1 - \alpha|} (1 + \omega t), \\
b(t) \equiv L \omega,
\]

for a.e. \( t \in (0, T) \).
one gets that \( V_\omega(T_t u_0) \in L^\infty(0, T) \) for every \( T > 0 \) and
\[
e^{-\omega t} V_\omega(T_t u_0) \leq \frac{2 L \| u_0 \|_X}{|1 - \alpha|} \left[ (1 + \omega t) + \int_0^t (1 + \omega s) L \omega e^{L \omega (t-s)} ds \right]
\]
for a.e. \( t \in (0, T) \). From this, one sees that (2.23) holds and (2.24) follows from (2.23). This completes the proof. \( \square \)

3. Homogeneous Accretive Operators

We begin this section with the following definition. Throughout this section, suppose \( X \) is Banach space with norm \( \| \cdot \|_X \).

**Definition 3.1.** An operator \( A \) on \( X \) is called *accretive* in \( X \) if for every \((u, v), (\hat{u}, \hat{v}) \in A \) and every \( \lambda \geq 0 \),
\[
\| u - \hat{u} \|_X \leq \| u - \hat{u} + \lambda (u - \hat{u}) + v - \hat{v} \|_X.
\]

and \( A \) is called *\( m \)-accretive* in \( X \) if \( A \) is accretive and satisfies the range condition (3.1) \( \text{Rg}(I + \lambda A) = X \) for some (or equivalently, for all) \( \lambda > 0, \lambda \omega < 1 \),

More generally, an operator \( A \) on \( X \) is called *quasi-\( (m) \)-accretive operator* in \( X \) if there is an \( \omega \in \mathbb{R} \) such that \( A + \omega I \) is \( (m) \)-accretive in \( X \).

If \( A \) is quasi-\( m \)-accretive in \( X \), then the classical existence theorem [9, Theorem 6.5] (cf [5, Corollary 4.2]), for every \( u_0 \in \overline{D(A)}^X \) and \( f \in L^1(0, T; X) \), there is a unique mild solution \( u \in C([0, T]; X) \) of (2.1).

**Definition 3.2.** For given \( u_0 \in \overline{D(A)}^X \) and \( f \in L^1(0, T; X) \), a function \( u \in C([0, T]; X) \) is called a mild solution of the inhomogeneous differential inclusion (2.1) with initial value \( u_0 \) if \( u(0) = u_0 \) and for every \( \varepsilon > 0 \), there is a partition \( \tau_\varepsilon : 0 = t_0 < t_1 < \cdots < t_N = T \) and a step function
\[
u_{\varepsilon,N}(t) = u_0 \mathbb{I}_{\{t = 0\}}(t) + \sum_{i=1}^{N} u_i \mathbb{I}_{(t_{i-1},t_i]}(t) \quad \text{for every } t \in [0, T]
\]
satisfying
\[
t_i - t_{i-1} < \varepsilon \quad \text{for all } i = 1, \ldots, N,
\]
\[
\sum_{N=1}^N \int_{t_{i-1}}^{t_i} \| f(t) - \overrightarrow{f}_i \| dt < \varepsilon \quad \text{where } \overrightarrow{f}_i := \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(t) dt,
\]
\[
\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + A u_i \geq \overrightarrow{f}_i \quad \text{for all } i = 1, \ldots, N,
\]
and
\[
\sup_{t \in [0,T]} \| u(t) - u_{\varepsilon,N}(t) \|_X < \varepsilon.
\]

Further, if \( A \) is quasi \( m \)-accretive, then the family \( \{T_t\}_{t=0}^T \) of mappings \( T_t : \overline{D(A)}^X \times L^1(0, T; X) \rightarrow \overline{D(A)}^X \) defined by (2.2) through the unique mild solution \( u \) of Cauchy problem (2.1) belongs to the following class.

**Definition 3.3.** Given a subset \( C \) of \( X \), a family \( \{T_t\}_{t=0}^T \) of mapping \( T_t : C \times L^1(0, T; X) \rightarrow C \) is called a *strongly continuous semigroup of quasi-contractive mappings* \( T_t \) if \( \{T_t\}_{t=0}^T \) satisfies the following three properties:


- (semigroup property) for every \((u_0, f) \in \overline{D(A)^x} \times L^1(0, T; X)\),

\[
T_{t+s}(u_0, f) = T_t(T_s(u_0, f), f(s + \cdot))
\]

for every \(t, s \in [0, T]\) with \(t + s \leq T\);

- (strong continuity) for every \((u_0, f) \in \overline{D(A)^x} \times L^1(0, T; X)\),

\[t \mapsto T_t(u_0, f) \text{ belongs to } C([0, T]; X)\];

- (\(\omega\)-quasi contractivity) \(T_t\) satisfies (2.4) with \(L = 1\).

Further, taking \(f \equiv 0\) and only varying \(u_0 \in \overline{D(A)^x}\), defines by

\[
(2.13) \quad T_tu_0 = T_t(u_0, 0) \quad \text{for every } t \geq 0,
\]
a strongly continuous semigroup \(\{T_t\}_{t \geq 0}\) on \(\overline{D(A)^x}\) of \(\omega\)-quasi contractions \(T_t : \overline{D(A)^x} \rightarrow \overline{D(A)^x}\). Given a family \(\{T_t\}_{t \geq 0}\) on \(\overline{D(A)^x}\), the operator

\[
(3.3) \quad A_0 := \left\{ (u_0, v) \in X \times X \mid \lim_{h \downarrow 0} \frac{T_h(u_0, 0) - u_0}{h} = v \text{ in } X \right\}
\]
is an \(\omega\)-quasi accretive well-defined mapping \(A_0 : D(A_0) \rightarrow X\) and called the infinitesimal generator of \(\{T_t\}_{t \geq 0}\). If the Banach space \(X\) and its dual space \(X^*\) are both uniformly convex (see [5, Proposition 4.3]), then one has that

\[-A_0 = A^0,\]

where \(A^0\) is the minimal selection of \(A\) (defined by (1.5)). Even tough by ignoring the additional geometric condition on \(X\), we refer to the two families \(\{T_t\}_{t \geq 0}^N\) defined by (2.2) on \(\overline{D(A)^x} \times L^1(0, T; X)\) and \(\{T_t\}_{t \geq 0}\) defined by (2.13) on \(\overline{D(A)^x}\) as the semigroup generated by \(-A\).

Moreover (cf., [9, Chapter 4.3]), for given \(u_0 \in \overline{D(A)^x}\) and any step function \(f = \sum_{i=1}^N f_i \chi_{(t_{i-1}, t_i]} \in L^1(0, T; X)\), let \(u : [0, T] \rightarrow X\) be given by

\[
(3.4) \quad u(t) = u_0 \chi_{\{t = 0\}}(t) + \sum_{i=1}^N u_i(t) \chi_{(t_{i-1}, t_i]}(t)
\]
is the unique mild solution of (2.1), where \(u_i\) is the unique mild solution of

\[
(3.5) \quad \frac{du_i}{dt} + A(u_i(t)) \supseteq f_i \quad \text{on } (t_{i-1}, t_i), \quad \text{and} \quad u_i(t_{i-1}) = u_{i-1}(t_{i-1}).
\]

Then for every \(i = 1, \ldots, N\), the semigroup \(\{T_t\}_{t \geq 0}^N\) is obtained by the exponential formula

\[
(3.6) \quad T_t(u(t_{i-1}), f_i) = u_i(t) = \lim_{n \rightarrow \infty} \left[ \int_{t_{i-1}}^t f_{\mu - f_i} \right]_t^n u(t_{i-1}) \quad \text{in } C([t_{i-1}, t_i]; X)
\]

for every \(i = 1, \ldots, N\), where for \(\mu > 0\), \(f_{\mu - f_i} = (I + \mu(A - f_i))^{-1}\) is the resolvent operator of \(A - f_i\).

As for classical solutions, the fact that \(A\) is homogeneous of order \(\alpha \neq 1\), is also reflected in the notion of mild solution and, in particular, in the semigroup \(\{T_t\}_{t \geq 0}^T\) as demonstrated in our next proposition.
Proposition 3.4 (Homogeneous accretive operators). Let $A$ be a quasi $m$-accretive operator on $X$ and \{${T_t}$\}$t=0$ the semigroup generated by $-A$ on $D(A) \times L^1(0,T;X)$. If $A$ is homogeneous of order $\alpha \neq 1$, then for every $\lambda > 0$, \{${T_t}$\}$t=0$ satisfies equation

\[\lambda \frac{d\mu}{dt} T_{\lambda t} (u_0, f) = T_t (\lambda \frac{d\mu}{dt} u_0, \lambda \frac{d\mu}{dt} f(\lambda \cdot)) \quad \text{for all } t \in \left[0, \frac{T}{\lambda}\right],\]

for every $(u_0, f) \in D(A) \times L^1(0,T;X)$.

Proof. Let $\lambda > 0$ and $f_i \in X$. Then for every $u, v \in X$ and $\mu > 0$,

\[f_{\mu}^A \frac{d\mu}{dt} f_i \left[ \lambda \frac{d\mu}{dt} v \right] = u \quad \text{if and only if} \quad u + \mu (Au - \frac{\alpha}{\alpha - 1} f_i) \ni \lambda \frac{d\mu}{dt} v.\]

Now, if $A$ is homogeneous of order $\alpha \neq 1$, the right-hand side in the previous characterization is equivalent to

\[\lambda \frac{d\mu}{dt} \mu + \mu (A(\lambda \frac{d\mu}{dt} u) - f_i) \ni v, \quad \text{or} \quad f_{\mu}^A \frac{d\mu}{dt} v = \lambda \frac{d\mu}{dt} u.\]

Therefore, if $A$ is homogeneous of order $\alpha \neq 1$, then

\[\lambda \frac{d\mu}{dt} f_{\mu}^A \frac{d\mu}{dt} f_i \left[ \lambda \frac{d\mu}{dt} v \right] = u \quad \text{for all } \lambda, \mu > 0, \text{and } v \in X.\]

Now, let $u_0 \in D(A) \times \pi_0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$, and $f = \sum_{i=1}^N f_i \Pi_{(t_{i-1}, t_i]} \in L^1(0,T;X)$ a step function. If $u$ denotes the unique mild solution of (2.1) for this step function $f$, then $u$ is given by (3.4), were on each subinterval $(t_{i-1}, t_i]$, $u_i$ is the unique mild solution of (3.5).

Next, let $\lambda > 0$ and set

\[v_{\lambda}(t) := \lambda \frac{1}{\lambda} \mu u(\lambda t) \quad \text{for every } t \in \left[0, \frac{T}{\lambda}\right].\]

Then,

\[v_{\lambda}(t) = \lambda \frac{1}{\lambda} \mu u_0 \Pi_{(t=0)}(t) + \sum_{i=1}^N \lambda \frac{1}{\lambda} \mu u_i(\lambda t) \Pi_{\left(t_{i-1}, \frac{t_i}{\lambda}\right]}(t)\]

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Obviously, $v_{\lambda}(0) = \lambda \frac{1}{\lambda} \mu u_0$. Thus, to complete the proof of this proposition showing that (2.3) holds, it remains to show that $v_{\lambda}$ is a mild solution of

\[\frac{d\mu}{dt} + A(v_{\lambda}(t)) \ni \lambda \frac{1}{\lambda} \mu f(\lambda t) \quad \text{on } \left(0, \frac{T}{\lambda}\right)\]

or, in other words,

\[v_{\lambda}(t) = T_{t}(\lambda \frac{1}{\lambda} \mu u_0, \lambda \frac{1}{\lambda} \mu f(\lambda \cdot))\]

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Let $t \in (0, t_1 / \lambda]$ and $n \in \mathbb{N}$. We apply (3.7) to

\[\mu = \frac{t}{n} \quad \text{and} \quad v = f^A_{\mu} \frac{1}{\lambda} \mu f_i \left[ \lambda \frac{1}{\lambda} \mu u_0 \right].\]

Then, one finds that

\[\left[ f^A_{\mu} \frac{1}{\lambda} \mu f_i \right]^2 \left[ \lambda \frac{1}{\lambda} \mu u_0 \right] = f^A_{\mu} \frac{1}{\lambda} \mu f_i \left[ \lambda \frac{1}{\lambda} \mu f^A_{\mu} \frac{1}{\lambda} \mu f_i u_0 \right] = \lambda \frac{1}{\lambda} \mu \left[ f^A_{\mu} \frac{1}{\lambda} \mu f_i \right]^2 u_0.\]

Applying (3.7) to $\lambda \frac{1}{\lambda} \mu f_i \left[ \lambda \frac{1}{\lambda} \mu u_0 \right]$ iteratively for $i = 2, \ldots, n$ yields

\[\lambda \frac{1}{\lambda} \mu \left[ f^A_{\mu} \frac{1}{\lambda} \mu f_i \right]^n u_0 = \left[ f^A_{\mu} \frac{1}{\lambda} \mu f_i \right]^n \left[ \lambda \frac{1}{\lambda} \mu u_0 \right].\]
By (3.6), sending \( n \to +\infty \) in (3.9) yields on the one side
\[
\lim_{n\to+\infty} \lambda \frac{1}{n} \left( \int_{\frac{n}{2}}^{\frac{n}{2}} f_i \right)^n u_0 = \lambda \frac{1}{n} u_1(\lambda t) = v_\lambda(t),
\]
and on the other side
\[
\lim_{n\to+\infty} \left[ \int_{\frac{n}{2}}^{\frac{n}{2}} f_i \right]^n \left[ \lambda \frac{1}{n} t_i u_0 \right] = T_t(\lambda \frac{1}{n} t_i u_0, \lambda \frac{1}{n} f_1),
\]
showing that (3.8) holds for every \( t \in [0, \frac{T}{T_i}] \). Repeating this argument on each subinterval \( \left[ \frac{i}{T_i}, \frac{i+1}{T_i} \right] \) for \( i = 2, \ldots, N \), where one replaces in (3.9) \( u_0 \) by \( u(t_{i-1}) \), and \( f_i \) by \( f_i \), then shows that \( v_\lambda \) satisfies (3.8) on the whole interval \( [0, \frac{T}{T_i}] \). □

By the above Lemma and Theorem 2.2, we can now state the following.

Corollary 3.5. Let \( A \) be a quasi m-accretive operator on a Banach space \( X \) and \( \{T_i\}_{i=0}^T \)
the semigroup generated by \(-A\) on \( L^1(0, T; X) \times D(A)^\times \). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \((u_0, f) \in D(A)^\times \times L^1(0, T; X) \), \( t \mapsto T_t(\lambda \frac{1}{n} t_i u_0, \lambda \frac{1}{n} f_1) \) satisfies (2.5) and (2.6) on \((0, T]\).
Moreover, for \( f \in W^1(0, T; X) \), \( T_t(u_0, f) \) also satisfies (2.7) (respectively, (2.8)) provided \( \frac{A}{dt} T_t(u_0, f) \) exists in \( X \) at a.e. \( t > 0 \).

To consider the regularizing effect of mild solutions to the Cauchy problem (1.8) for the perturbed operator \( A + F \), we recall the following well-known result form the literature.

Proposition 3.6 ([9, Lemma 7.8]). If \( A \) is quasi accretive in \( X \) and \( f \in BV(0, T; X) \),
then for every \( u_0 \in D(A) := \{ u \in X | Au \neq \emptyset \} \), the mild solution \( u(t) := T_t u_0, \), \( t \geq 0 \), of Cauchy problem (2.1) is Lipschitz continuous and
\[
\limsup_{h \to 0^+} \frac{\|T_{t+h} u_0 - T_t u_0\|_X}{h} \leq e^{\alpha t} \|v\|_X \quad \text{for every } t \in [0, T] \text{ and } v \in A u_0.
\]

To have that the semigroup \( \{T_t\}_{i=0}^T \) generated by \(-A\) satisfies regularity estimate (2.8) (respectively, (2.18)), one requires that each mild solution \( u \) of (2.1) (respectively, of (1.1)) is differentiable at a.e. \( t \in (0, T] \), or in other words, \( u \) is a strong solution of (2.1). The next definition is taken from [9, Definition 1.2] (cf [5, Chapter 4]).

Definition 3.7. A locally absolutely continuous function \( u[0, T] : \to X \) is called a strong solution of differential inclusion
\[
(3.10) \quad \frac{du}{dt} + A(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T),
\]
if \( u \) is differentiable a.e. on \((0, T), \) and for a.e. \( t \in (0, T), \) \( u(t) \in D(A) \) and \( f(t) - \frac{du}{dt}(t) \in A(u(t)) \). Further, for given \( u_0 \in X \) and \( f \in L^1(0, T; X) \),
a function \( u \) is called a strong solution of Cauchy problem (2.1) if \( u \in C([0, T]; X) \), \( u \) is strong solution of (3.10) and \( u(0) = u_0 \).

The next characterization of strong solutions of (3.10) highlights the important point of a.e. differentiability.

Proposition 3.8 ([9, Theorem 7.1]). Let \( X \) be a Banach space, \( f \in L^1(0, T; X) \) and \( A \) be quasi m-accretive in \( X \). Then \( u \) is a strong solution of the differential inclusion (3.10) on \([0, T]\) if and only if \( u \) is a mild solution on \([0, T]\) and \( u \) is “absolutely continuous” on \([0, T]\) and differentiable a.e. on \((0, T).\)
Of course, every strong solution \( u \) of (3.10) is a mild solution of (3.10), and \( u \) is absolutely continuous and differentiable a.e. on \([0,T]\). The differential inclusion (3.10) admits mild and Lipschitz continuous solutions if \( A \) is \( \omega \)-quasi \( m \)-accretive in \( X \) (cf [9, Lemma 7.8]). But, in general, absolutely continuous vector-valued functions \( u : [0,T] \to X \) are not differentiable a.e. on \((0,T)\). Only if one assumes additional geometric properties on \( X \), then the latter implication holds true. Our next definition is taken from [9, Definition 7.6] (cf [3, Chapter 1]).

**Definition 3.9.** A Banach space \( X \) is said to have the Radon-Nikodým property if every absolutely continuous function \( F : [a,b] \to X \), \( a, b \in \mathbb{R}, a < b \), is differentiable almost everywhere on \((a,b)\).

Known examples of Banach spaces \( X \) admitting the Radon-Nikodým property are:

- (Dunford-Pettis) if \( X = Y^* \) is separable, where \( Y^* \) is the dual space of a Banach space \( Y \);
- if \( X \) is reflexive.

We emphasize that \( X_1 = L^1(\Sigma, \mu), X_2 = L^\infty(\Sigma, \mu) \), or \( X_3 = C(\mathcal{M}) \) for a \( \sigma \)-finite measure space \( (\Sigma, \mu) \), or respectively, for a compact metric space \( (\mathcal{M}, d) \) don’t have, in general, the Radon-Nikodým property (cf [3]). Thus, it is quite surprising that there is a class of operators \( A \) (namely, the class of completely accretive operators, see Section 4 below), for which the differential inclusion (2.1) nevertheless admits strong solutions (with values in \( L^1(\Sigma, \mu) \) or \( L^\infty(\Sigma, \mu) \)).

Now, by Corollary 3.5 and Proposition 3.8, we can conclude the following results. We emphasize that one crucial point in the statement of Corollary 3.10 below is that due to the uniform estimate (2.7), one has that for all initial values \( u_0 \in \overline{D(A)}^X \), the unique mild solution \( u \) of (2.1) is strong.

**Corollary 3.10.** Suppose \( A \) is a quasi \( m \)-accretive operator on a Banach space \( X \) admitting the Radon-Nikodým property, and \( \{T_t\}_{t=0}^T \) is the semigroup generated by \(-A\) on \( \overline{D(A)}^X \times L^1(0,T;X) \). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \( u_0 \in \overline{D(A)}^X \) and \( f \in W^{1,1}(0,T;X) \), the unique mild solution \( u(t) := T_t(u_0, f) \) of (2.1) is strong and \( \{T_t\}_{t=0}^T \) satisfies (2.8) for a.e. \( t > 0 \).

Now by Corollary 2.7 and Proposition 3.8, we obtain the following result when \( A \) is perturbed by a Lipschitz mapping \( F \).

**Corollary 3.11.** Let \( A \) be an \( m \)-accretive operator on a reflexive Banach space \( X \) and \( F : X \to X \) a Lipschitz continuous mapping with Lipschitz-constant \( \omega > 0 \) satisfying \( F(0) = 0 \), and \( \{T_t\}_{t \geq 0} \) the semigroup generated by \(-(A + F)\) on \( \overline{D(A)}^X \). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \( u_0 \in \overline{D(A)}^X \), the unique mild solution \( u(t) = T_tu_0, (t \geq 0) \) of Cauchy problem (1.8) is strong, and \( \{T_tu_0\}_{t \geq 0} \) satisfies (2.24) for a.e. \( t > 0 \).

**Proof.** First, let \( u_0 \in D(A) \). Then by Proposition 3.6, the mild solution \( u(t) = T_tu_0 \) is locally Lipschitz continuous on \([0,T)\),

\[
V_\omega(u,t) = \|\frac{du}{dt}(t)\|_X \quad \text{for a.e. } t \in (0,T),
\]

for all \( t \leq T \).
and $V_\omega(u, t)$ belongs to $L^\infty(0, T)$. Thus, by Theorem 2.7, $u$ satisfies (2.24). Now, we square both sides in (2.24) and subsequently integrate the resulting inequality over $(a, b)$ for given $0 < a < b$. Then, one obtains that

$$\int_a^b \left\| \frac{d T_t u_0}{d t} \right\|_X^2 \, dt \leq \frac{2^2 \omega t}{|1 - \alpha|^2} \| u_0 \|_X^2 \left( \frac{2}{a} - \frac{2}{b} \right).$$

Due to this inequality, if $u_0 \in D(A)^\times$ and $(u_{0,n})_{n \geq 1} \subseteq D(A)$ such that $u_{0,n} \to u_0$ in $X$ as $n \to \infty$, then $\left( \frac{d T_t u_{0,n}}{d t} \right)_{n \geq 1}$ is bounded in $L^2(a, b; X)$. Since $X$ is reflexive, also $L^2(a, b; X)$ is reflexive and hence, there is a $v \in L^2(a, b; X)$ and a subsequence of $(u_{0,n})_{n \geq 1}$, which we denote, for simplicity, again by $(u_{0,n})_{n \geq 1}$, such that $\frac{d T_t u_{0,n}}{d t} \to v$ weakly in $L^2(a, b; X)$ as $n \to +\infty$. As $T_t u_{0,n} \to T_t u$ in $C([a, b]; X)$, it follows by a standard argument that $v(t) = \frac{d T_t u_0}{d t}$ in the sense of vector-valued distributions $D'(((a, b); X)$. Since $\frac{d T_t u_0}{d t} \in L^2(a, b; X)$, the mild solution $u(t) = T_t u_0$ is absolutely continuous on $(a, b)$, and since $X$ is reflexive, $u$ is differentiable a.e. on $(a, b)$. Since $0 < a < b < \infty$ were arbitrary, $\frac{d T_t u_0}{d t} \in L^1_{\text{loc}}(0, \infty; X)$. Now, for $\epsilon > 0$, the function $T_{t+\epsilon} u_0$ satisfies the hypotheses of Theorem 2.7 with $\frac{d T_{t+\epsilon} u_0}{d t} \in L^1_{\text{loc}}([0, +\infty); X)$. Thus $T_{t+\epsilon} u_0$ satisfies (2.24) for every $s > 0$. Since $T_s u_0 = T_s (T_t u_0)$ for every $s \geq 0$, choosing $t = s + \epsilon$ in (2.24), we find that

$$\left\| \frac{d T_{t+\epsilon} u_0}{d t} \right\|_X \leq \frac{\omega (t - \epsilon)}{t - \epsilon} 2 L \| T_{t+\epsilon} u_0 \|_X \times \left( 1 + \omega (t - \epsilon) \right) + \int_0^{t-\epsilon} (1 + \omega r) L \omega e^{L \omega (t - \epsilon - r)} \, dr$$

for every $t > 0$ and $\epsilon \in (0, t)$. Sending $\epsilon \to 0^+$ implies that $T_t u_0$ satisfies (2.24). Since $u_0 \in D(A)^\times$ was arbitrary, this completes the proof of this corollary. □

If the Banach space $X$ and its dual space $X^*$ are uniformly convex and $A$ is quasi $m$-accretive in $X$, then (cf., [5, Theorem 4.6]) for every $u_0 \in D(A)$, $f \in W^{1,1}(0, T; X)$, the mild solution $u(t) = T_t u_0(f)$, $(t \in [0, T])$, of (2.1) is a strong one, $u$ is everywhere differentiable from the right, $\frac{du}{dt}$ is right continuous, and

$$\frac{du}{dt}(t) + (A - f(t))^\circ u(t) = 0 \quad \text{for every } t \geq 0,$$

where for every $t \in [0, T]$, $(A - f(t))^\circ$ denotes the minimal selection of $A - f(t)$ defined by (1.5). Thus, under those assumptions on $X$ and by Proposition 3.8, we can state the following three corollaries. We begin by stating the inhomogeneous case.

**Corollary 3.12.** Suppose $X$ and its dual space $X^*$ are uniformly convex, for $\omega \in \mathbb{R}$, $A$ is an $\omega$-quasi $m$-accretive operator on $X$, and $\{T_t\}_{t=0}^T$ is the semigroup on $D(A)^\times \times L^1(0, T; X)$ generated by $-A$. If $A$ is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in D(A)^\times$ and $f \in W^{1,1}(0, T; X)$, the mild solution $u(t) = T_t(u_0, f)$, $(t \in [0, T])$

...
of Cauchy problem (2.1) is strong and
\[
\|(A - f(t)) T_t(u_0, f)\|_X \\
\leq \frac{1}{t} \left[ 2 e^{\omega t} \right] \|u_0\|_X + \frac{2}{|1 - \alpha|} \int_0^t e^{\omega(t-s)} \|f(s)\|_X \, ds \\
+ \int_0^t e^{\omega(t-s)} \|f'(s)\|_X \, s \, ds
\]
for every \( t > 0 \).

Since every uniformly convex Banach space is reflexive, the proof of Corollary 3.12 proceeds in the similar way as the one of Corollary 3.11.

Our next corollary considers the homogeneous case.

**Corollary 3.13.** Suppose \( X \) and its dual space \( X^* \) are uniformly convex, for \( \omega \in \mathbb{R} \), \( A \) is an \( \omega \)-quasi \( m \)-accretive operator on \( X \), and \( \{T_t\}_{t \geq 0} \) is the semigroup on \( \overline{D(A)}^X \), generated by \(-A\). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \( u_0 \in \overline{D(A)}^X \), the mild solution \( u(t) = T_t u_0 , \ (t \geq 0) \) of Cauchy problem (1.1) is a strong solution and
\[
\|A^\alpha T_t u_0\|_X \leq \frac{1}{t} \left[ \frac{2 e^{\omega t}}{|1 - \alpha|} \right] \|u_0\|_X \quad \text{for every} \ t > 0 \ \text{and} \ u_0 \in \overline{D(A)}^X.
\]

The last corollary focuses on the case when the homogeneous operator \( A \) is perturbed by a Lipschitz mapping \( F \). Its statement follows from [5, Theorem 4.6] and Theorem 2.7.

**Corollary 3.14.** Suppose \( X \) and its dual space \( X^* \) are uniformly convex, \( F : X \to X \) be a Lipschitz continuous mapping with Lipschitz-constant \( \omega > 0 \) satisfying \( F(0) = 0 \), \( A \) an \( m \)-accretive operator on \( X \), and \( \{T_t\}_{t \geq 0} \) the semigroup generated by \(-A \oplus F\) on \( \overline{D(A)}^X \). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \( u_0 \in \overline{D(A)}^X \), the mild solution \( u(t) = T_t u_0, \ (t \geq 0) \) of Cauchy problem (1.8) is a strong solution and
\[
\|A^\alpha T_t u_0\|_X \leq \frac{e^{\omega t}}{t} \left[ \frac{2 \|u_0\|_X (1 + \omega t)}{|1 - \alpha|} \right] \left[ (1 + \omega t) + \int_0^t (1 + \omega s) \omega e^{\omega(t-s)} \, ds \right]
\]
for every \( t > 0 \).

### 4. Homogeneous Completely Accretive Operators

In [8], Bénilan and Crandall introduced the class of completely accretive operators \( A \) and showed: even though the underlying Banach spaces does not admit the Radon-Nikodym property, but if \( A \) is completely accretive and homogeneous of order \( \alpha > 0 \) with \( \alpha \neq 1 \), then the mild solutions of differential inclusion (1.1) involving \( A \) are strong. This was extended in [17] to the zero-order case. Here, we provide a generalization to the case of completely accretive operators which are homogeneous of order \( \alpha \neq 1 \) and perturbed by a Lipschitz nonlinearity.
4.1. General framework. In order to keep this paper self-contained, we provide a brief introduction to the class of completely accretive operators, where we mainly follow [8] and the monograph [13].

For the rest of this paper, suppose \((\Sigma, \mathcal{B}, \mu)\) is a \(\sigma\)-finite measure space, and \(M(\Sigma, \mu)\) the space of \(\mu\)-a.e. equivalent classes of measurable functions \(u : \Sigma \to \mathbb{R}\). For \(u \in M(\Sigma, \mu)\), we write \(|u|^+\) to denote \(\max\{u, 0\}\) and \(|u|^− = −\min\{u, 0\}\). We denote by \(L^q(\Sigma, \mu)\), \(1 \leq q \leq \infty\), the corresponding standard Lebesgue space with norm

\[
\|\cdot\|_q = \begin{cases} 
\left(\int_{\Sigma} |u|^q \, d\mu\right)^{1/q} & \text{if } 1 \leq q < \infty, \\
\inf \left\{ k \in [0, +\infty] \left| |u| \leq k \text{ } \mu\text{-a.e. on } \Sigma \right. \right\} & \text{if } q = \infty.
\end{cases}
\]

For \(1 \leq q < \infty\), we identify the dual space \((L^q(\Sigma, \mu))'\) with \(L^{q'}(\Sigma, \mu)\), where \(q'\) is the conjugate exponent of \(q\) given by \(1 = \frac{1}{q} + \frac{1}{q'}\).

Next, we first briefly recall the notion of Orlicz spaces (cf [21, Chapter 3]). A continuous function \(\psi : [0, +\infty) \to [0, +\infty)\) is an \(N\)-function if it is convex, \(\psi(s) = 0\) if and only if \(s = 0\), \(\lim_{s \to 0^+} \psi(s)/s = 0\), and \(\lim_{s \to \infty} \psi(s)/s = \infty\). Given an \(N\)-function \(\psi\), the Orlicz space is defined as follows

\[
L^\psi(\Sigma, \mu) := \left\{ u \in M(\Sigma, \mu) \left| \int_{\Sigma} \psi\left(\frac{|u|}{\alpha}\right) \, d\mu < \infty \text{ for some } \alpha > 0 \right. \right\}
\]

and equipped with the Orlicz-Minkowski norm

\[
\|u\|_\psi := \inf \left\{ \alpha > 0 \left| \int_{\Sigma} \psi\left(\frac{|u|}{\alpha}\right) \, d\mu \leq 1 \right. \right\}.
\]

With these preliminaries in mind, we are now in the position to recall the notion of completely accretive operators introduced in [8] and further developed to the \(\omega\)-quasi case in [13].

Let \(J_0\) be the set given by

\[
J_0 = \left\{ j : \mathbb{R} \to [0, +\infty] \left| j \text{ is convex, lower semicontinuous, } j(0) = 0 \right. \right\}.
\]

Then, for every \(u, v \in M(\Sigma, \mu)\), we write

\[ u \ll v \quad \text{if and only if} \quad \int_{\Sigma} j(u) \, d\mu \leq \int_{\Sigma} j(v) \, d\mu \quad \text{for all } j \in J_0.\]

Remark 4.1. Due to the interpolation result [8, Proposition 1.2], for given \(u, v \in M(\Sigma, \mu)\), the relation \(u \ll v\) is equivalent to the two conditions

\[
\begin{align*}
\int_{\Sigma} |u-k|^+ \, d\mu & \leq \int_{\Sigma} |v-k|^+ \, d\mu \quad \text{for all } k > 0 \quad \text{and} \\
\int_{\Sigma} |u+k|^− \, d\mu & \leq \int_{\Sigma} |v+k|^− \, d\mu \quad \text{for all } k > 0.
\end{align*}
\]

Thus, the relation \(\ll\) is closely related to the theory of rearrangement-invariant function spaces (cf [10]). Another, useful characterization of the relation “\(\ll\)” is the following (cf [8, Remark 1.5]): for \(u, v \in M(\Sigma, \mu)\), \(u \ll v\) if and only if \(u^+ \ll v^+\) and \(u^- \ll v^-\).
Further, the relation \( \ll \) on \( M(\Sigma, \mu) \) has the following properties. We omit the easy proof of this proposition.

**Proposition 4.2.** For every \( u, v, w \in M(\Sigma, \mu) \), one has that

1. \( u^+ \ll u, u^- \ll -u; \)
2. \( u \ll v \) if and only if \( u^+ \ll v^+ \) and \( u^- \ll v^-; \)
3. (positive homogeneity) if \( u \ll v \) then \( \alpha u \ll \alpha v \) for all \( \alpha > 0; \)
4. (transitivity) if \( u \ll v \) and \( v \ll w \) then \( u \ll w; \)
5. (positive homogeneity) if \( u \ll v \) then \( |u| \ll |v|; \)
6. (convexity) for every \( u \in M(\Sigma, \mu) \), the set \( \{ w \mid w \ll u \} \) is convex.

With these preliminaries in mind, we can now state the following definitions.

**Definition 4.3.** A mapping \( S : D(S) \to M(\Sigma, \mu) \) with domain \( D(S) \subseteq M(\Sigma, \mu) \) is called a complete contraction if

\[
Su - S\hat{u} \ll u - \hat{u} \quad \text{for every } u, \hat{u} \in D(S).
\]

More generally, for \( L > 0 \), we call \( S \) to be an \( L \)-complete contraction if

\[
L^{-1}Su - L^{-1}S\hat{u} \ll u - \hat{u} \quad \text{for every } u, \hat{u} \in D(S),
\]

and for some \( \omega \in \mathbb{R} \), \( S \) is called to be \( \omega \)-quasi completely contractive if \( S \) is an \( L \)-complete contraction with \( L = e^{\omega t} \) for some \( t \geq 0 \).

**Remark 4.4.** Note, for every \( 1 \leq q < \infty \), \( j_q(\cdot) = \| |\cdot|^q \| \in \mathcal{J}_0 \), \( j_\infty(\cdot) = \| |\cdot|^+ - k|^{+} \in \mathcal{J}_0 \) for every \( k \geq 0 \) (and for large enough \( k > 0 \) if \( q = \infty \)), and for every \( \mathcal{N} \)-function \( \psi \) and \( \alpha > 0 \), \( j_{\psi, \alpha}(\cdot) = \psi(\| |\cdot|^{\alpha} \|) \in \mathcal{J}_0 \). This shows that for every \( L \)-complete contraction \( S : D(S) \to M(\Sigma, \mu) \) with domain \( D(S) \subseteq M(\Sigma, \mu) \), the mapping \( L^{-1}S \) is order-preserving and contractive respectively for every \( L^q \)-norm \((1 \leq q \leq \infty)\), and every \( L^\infty \)-norm with \( \mathcal{N} \)-function \( \psi \).

Now, we can state the definition of completely accretive operators.

**Definition 4.5.** An operator \( A \) on \( M(\Sigma, \mu) \) is called completely accretive if for every \( \lambda > 0 \), the resolvent operator \( J_\lambda \) of \( A \) is a complete contraction, or equivalently, if for every \( (u_1, v_1), (u_2, v_2) \in A \) and \( \lambda > 0 \), one has that

\[
u_1 - u_2 \ll u_1 - u_2 + \lambda(v_1 - v_2).
\]

If \( X \) is a linear subspace of \( M(\Sigma, \mu) \) and \( A \) an operator on \( X \), then \( A \) is \( m \)-completely accretive on \( X \) if \( A \) is completely accretive and satisfies the range condition (3.1). Further, for \( \omega \in \mathbb{R} \), an operator \( A \) on a linear subspace \( X \subseteq M(\Sigma, \mu) \) is called \( \omega \)-quasi \( (m) \)-completely accretive in \( X \) if \( A + \omega I \) is \( (m) \)-completely accretive in \( X \). Finally, an operator \( A \) on a linear subspace \( X \subseteq M(\Sigma, \mu) \) is called quasi \( (m) \)-completely accretive if there is an \( \omega \in \mathbb{R} \) such that \( A + \omega I \) is \( (m) \)-completely accretive in \( X \).

Before stating a useful characterization of completely accretive operators, we first need to introduce the following function spaces. Let

\[
L^{1+\infty}(\Sigma, \mu) := L^1(\Sigma, \mu) + L^\infty(\Sigma, \mu) \quad \text{and} \quad L^{1\cap \infty}(\Sigma, \mu) := L^1(\Sigma, \mu) \cap L^\infty(\Sigma, \mu)
\]

be the sum and the intersection space of \( L^1(\Sigma, \mu) \) and \( L^\infty(\Sigma, \mu) \), which are equipped, respectively, with the norms

\[
\|u\|_{1+\infty} := \inf \left\{ \|u_1\|_1 + \|u_2\|_\infty \mid u = u_1 + u_2, \ u_1 \in L^1(\Sigma, \mu), u_2 \in L^\infty(\Sigma, \mu) \right\},
\]
\[ \|u\|_{1,\infty} := \max \{ \|u\|_1, \|u\|_\infty \} \]

are Banach spaces. In fact, \( L^{1+\infty}(\Sigma, \mu) \) and and \( L^{1,\infty}(\Sigma, \mu) \) are respectively the largest and the smallest of the rearrangement-invariant Banach function spaces (cf., [10, Chapter 3.1]). If \( \mu(\Sigma) \) is finite, then \( L^{1+\infty}(\Sigma, \mu) = L^{1}(\Sigma, \mu) \) with equivalent norms, but if \( \mu(\Sigma) = \infty \) then \( L^{1+\infty}(\Sigma, \mu) \) contains \( \bigcup_{1 \leq q \leq \infty} L^{q}(\Sigma, \mu) \). Further, we will employ the space

\[ L_0(\Sigma, \mu) := \left\{ u \in M(\Sigma, \mu) \mid \int_{\Sigma} [\|u| - k]^+ \, d\mu < \infty \text{ for all } k > 0 \right\}, \]

which equipped with the \( L^{1+\infty} \)-norm is a closed subspace of \( L^{1+\infty}(\Sigma, \mu) \). In fact, one has (cf., [8]) that \( L_0(\Sigma, \mu) = \overline{L^1(\Sigma, \mu) \cap L^{\infty}(\Sigma, \mu)}^{L^{1+\infty}} \). Since for every \( k \geq 0 \), \( T_k(s) := |||s| - k| \)^+ is a Lipschitz mapping \( T_k : \mathbb{R} \to \mathbb{R} \) and by Chebyshev’s inequality, one sees that \( L^q(\Sigma, \mu) \hookrightarrow L_0(\Sigma, \mu) \) for every \( 1 \leq q < \infty \) (and \( q = \infty \) if \( \mu(\Sigma) < +\infty \)), and \( L^q(\Sigma, \mu) \hookrightarrow L_0(\Sigma, \mu) \) for every \( N \)-function \( \psi \).

**Proposition 4.6 ([13]).** Let \( P_0 \) denote the set of all functions \( T \in C^{\infty}(\mathbb{R}) \) satisfying \( 0 \leq T' \leq 1 \) such that \( T' \) is compactly supported, and \( x = 0 \) is not contained in the support \( \text{supp}(T) \) of \( T \). Then for \( \omega \in \mathbb{R} \), an operator \( A \subseteq L_0(\Sigma, \mu) \times L_0(\Sigma, \mu) \) is \( \omega \)-quasi completely accretive if and only if

\[ \int_{\Sigma} T(u - \tilde{u})(v - \tilde{\omega}) \, d\mu + \omega \int_{\Sigma} T(u - \tilde{u})(u - \tilde{\omega}) \, d\mu \geq 0 \]

for every \( T \in P_0 \) and every \( (u, v), (\tilde{u}, \tilde{\omega}) \in A \).

**Remark 4.7.** For convenience, we denote the unique extension of \( \{T_i\}_{i \geq 0} \) on \( L^q(\Sigma, \mu) \) or \( L^1(\Sigma, \mu) \) again by \( \{T_i\}_{i \geq 0} \).

**Definition 4.8.** A Banach space \( X \subseteq M(\Sigma, \mu) \) with norm \( \| \cdot \|_X \) has the following property:

\[ \left\{ \begin{array}{ll}
& \text{for every } u \in X, v \in M(\Sigma, \mu) \text{ satisfying } v \leq u, \\
& \text{one has that } v \in X \text{ and } \|v\|_X \leq \|u\|_X.
\end{array} \right. \]

Typical examples of normal Banach spaces \( X \subseteq M(\Sigma, \mu) \) are Orlicz-spaces \( L^q(\Sigma, \mu) \) for every \( N \)-function \( \psi \), \( L^q(\Sigma, \mu) \), \( (1 \leq q \leq \infty) \), \( L^{1,\infty}(\Sigma, \mu) \), \( L_0(\Sigma, \mu) \), and \( L^{1+\infty}(\Sigma, \mu) \).

**Remark 4.9.** It is important to point out that if \( X \) is a normal Banach space, then for every \( u \in X \), one always has that \( u^+ \), \( u^- \) and \( |u| \in X \). To see this, recall that by \( (1) \) Proposition 4.2, if \( u \in X \), then \( u^+ \ll u \) and \( u^- \ll -u \). Thus, \( u^+ \) and \( u^- \in X \) and since \( |u| = u^+ + u^- \), one also has that \( |u| \in X \).

The dual space \( (L_0(\Sigma, \mu))^\prime \) of \( L_0(\Sigma, \mu) \) is isometrically isomorphic to the space \( L^{1,\infty}(\Sigma, \mu) \). Thus, a sequence \((u_n)_{n \geq 1} \) in \( L_0(\Sigma, \mu) \) is said to be weakly convergent to \( u \) in \( L_0(\Sigma, \mu) \) if

\[ \langle v, u_n \rangle := \int_{\Sigma} v \, u_n \, d\mu \to \int_{\Sigma} v \, u \, d\mu \quad \text{for every } v \in L^{1,\infty}(\Sigma, \mu). \]

For the rest of this paper, we write \( \sigma(L_0, L^{1,\infty}) \) to denote the weak topology on \( L_0(\Sigma, \mu) \). For this weak topology, we have the following compactness result.

**Proposition 4.10 ([8, Proposition 2.11]).** Let \( u \in L_0(\Sigma, \mu) \). Then, the following statements hold.
(1) The set \( \{ v \in M(\Sigma, \mu) \mid v \ll u \} \) is \( \sigma(L_0, L^{1/\infty}) \)-sequentially compact in \( L_0(\Sigma, \mu) \);

(2) Let \( X \subseteq M(\Sigma, \mu) \) be a normal Banach space satisfying \( X \subseteq L_0(\Sigma, \mu) \) and

\[
(4.1) \quad \begin{align*}
\text{for every } u \in X, (u_n)_{n \geq 1} \subseteq M(\Sigma, \mu) \text{ with } u_n \ll u \text{ for all } n \geq 1 \\
\text{and } \lim_{n \to +\infty} u_n(x) = u(x) \mu\text{-a.e. on } \Sigma, \text{ yields } \lim_{n \to +\infty} u_n = u \text{ in } X.
\end{align*}
\]

Then for every \( u \in X \) and sequence \( (u_n)_{n \geq 1} \subseteq M(\Sigma, \mu) \) satisfying

\[
\begin{align*}
\text{for all } n \geq 1 \quad \text{and } \lim_{n \to +\infty} u_n = u \sigma(L_0, L^{1/\infty})\text{-weakly in } X,
\end{align*}
\]

one has that

\[
\lim_{n \to +\infty} u_n = u \quad \text{in } X.
\]

Note, examples of normal Banach spaces \( X \subseteq L_0(\Sigma, \mu) \) satisfying (4.1) are \( X = L^p(\Sigma, \mu) \) for \( 1 \leq p < \infty \) and \( L_0(\Sigma, \mu) \).

To complete this preliminary section, we state the following Proposition summarizing statements from [13, Proposition 2.9 & Proposition 2.10], which we will need in the sequel (cf., [8] for the case \( \omega = 0 \)).

**Proposition 4.11.** For \( \omega \in \mathbb{R} \), let \( A \) be \( \omega \)-quasi completely accretive in \( L_0(\Sigma, \mu) \).

(1.) If there is a \( \lambda_0 > 0 \) such that \( R_\infty(I + \lambda A) \) is dense in \( L_0(\Sigma, \mu) \), then for the closure \( \overline{A}^0 \) of \( A \) in \( L_0(\Sigma, \mu) \) and every normal Banach space \( X \subseteq L_0(\Sigma, \mu) \), the restriction \( \overline{A}^0_X := \overline{A}^0 \cap (X \times X) \) of \( A \) on \( X \) is the unique \( \omega \)-quasi \( m \)-completely accretive extension of the part \( A_X = A \cap (X \times X) \) of \( A \) in \( X \).

(2.) For a given normal Banach space \( X \subseteq L_0(\Sigma, \mu) \), and \( \omega \in \mathbb{R} \), suppose \( A \) is \( \omega \)-quasi \( m \)-completely accretive in \( X \), and \( \{ T_t \}_{t \geq 0} \) be the semigroup generated by \( -A \) on \( \overline{D(A)}^\omega \). Further, let \( \{ S_t \}_{t \geq 0} \) be the semigroup generated by \( -\overline{A}^0 \), where \( \overline{A}^0 \) denotes the closure of \( A \) in \( \overline{X}^\omega \). Then, the following statements hold.

(a) The semigroup \( \{ S_t \}_{t \geq 0} \) is \( \omega \)-quasi completely contractive on \( \overline{D(A)}^\omega \), \( S_t \) is the restriction of \( S_t \) on \( \overline{D(A)}^\omega \), \( S_t \) is the closure of \( T_t \) on \( L_0(\Sigma, \mu) \), and

\[
S_t u_0 = L_0 - \lim_{n \to +\infty} \left( I + \frac{t}{n} A \right)^{-n} u_0 \quad \text{for all } u_0 \in \overline{D(A)}^\omega \cap X;
\]

(b) If there exists \( u \in L^{1/\infty}(\Sigma, \mu) \) such that the orbit \( \{ T_t u \mid t \geq 0 \} \) is locally bounded on \( \mathbb{R}_+ \) with values in \( L^{1/\infty}(\Sigma, \mu) \), then, for every \( \psi \)-function \( \psi \), the semigroup \( \{ T_t \}_{t \geq 0} \) can be extrapolated to a strongly continuous, order-preserving semigroup of \( \omega \)-quasi contractions on \( \overline{D(A)}^\omega \cap L^{1/\infty}(\Sigma, \mu)^{1/\omega} \) (respectively, on \( \overline{D(A)}^\omega \cap L^{1/\infty}(\Sigma, \mu)^{1/\omega} \)), and to an order-preserving semigroup of \( \omega \)-quasi contractions on \( \overline{D(A)}^\omega \cap L^{1/\infty}(\Sigma, \mu)^{1/\omega} \). We denote each extension of \( T_t \) on those spaces again by \( T_t \).

(c) The restriction \( A_X := \overline{A}^0 \cap (X \times X) \) of \( \overline{A}^0 \) on \( X \) is the unique \( \omega \)-quasi \( m \)-complete extension of \( A \) in \( X \); that is, \( \overline{A} = A_X \).

(d) The operator \( A \) is sequentially closed in \( X \times X \) equipped with the relative \( (L_0(\Sigma, \mu) \times (X, \sigma(L_0, L^{1/\infty}))) \)-topology.
(e) The domain of $A$ is characterized by

$$D(A) = \left\{ u \in D(A)^c \cap X \mid \exists v \in X \text{ such that } e^{-\omega t}S_{t}u \ll v \text{ for all small } t > 0 \right\};$$

(f) For every $u \in D(A)$, one has that

$$\lim_{t \to 0^+} \frac{S_{t}u - u}{t} = -A^c u \quad \text{strongly in } L_0(\Sigma, \mu).$$

4.2. Regularizing effect of the semigroup. As mentioned in Section 3, the Banach space $L^1(\Sigma, \mu)$ does not admit the Radon-Nikodým property and therefore, for a semigroup $\{T_t\}_{t \geq 0}$ on $L^1(\Sigma, \mu)$, the time-derivative $\frac{d}{dt}T_t u_0(t)$ for given $u_0$, does not need to exist in $L^1(\Sigma, \mu)$. In this section, we show that even though the underlying Banach space $X$ is not reflexive, but $A$ is homogeneous of order $\alpha \neq 1$ and quasi-completely accretive, then the time-derivative $\frac{d}{dt}T_t u_0(t)$ exists in $X$. This fact follows from the following compactness result.

Here, the partial ordering “≤” is the standard one defined by $u \leq v$ for $u, v \in M(\Sigma, \mu)$ if $u(x) \leq v(x)$ for $\mu$-a.e. $x \in \Sigma$, and we write $X \hookrightarrow Y$ for indicating that the space $X$ is continuously embedded into the space $Y$.

**Lemma 4.12.** Let $X \subseteq L_0(\Sigma, \mu)$ be a normal Banach space satisfying (4.1). For $\omega \in \mathbb{R}$, let $\{T_t\}_{t \geq 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ of $\omega$-quasi complete contractions satisfying (2.15) and $T_t 0 = 0$ for all $t \geq 0$. Then, for every $u_0 \in C$ and $t > 0$, the set

$$\left\{ \frac{T_{t+h}u_0 - T_t u_0}{h} \bigg| h \neq 0, t + h > 0 \right\}$$

is $\sigma(L_0, L^{1\cap \infty})$-weakly sequentially compact in $L_0(\Sigma, \mu)$.

The proof of this lemma is essentially the same as in the case $\omega = 0$ (cf., [8]). For the convenience of the reader, we include here the proof.

**Proof.** Let $u_0 \in C$, $t > 0$, and $h \neq 0$ such that $t + h > 0$. Then by taking $\lambda = 1 + \frac{h}{t}$ in (2.15), one sees that

$$|T_{t+h}u_0 - T_t u_0| = |\lambda^{\frac{s}{t}} T_t \lambda^{\frac{r}{s}} u_0 - T_t u_0|$$

$$\leq \lambda^{\frac{r}{s}} \left| T_t \left( \lambda^{\frac{r}{s}} u_0 \right) - T_t u_0 \right| + |\lambda^{\frac{r}{s}} - 1| |T_t u_0|.$$ 

Since $T_t$ is an $\omega$-quasi complete contraction and since $T_t 0 = 0, (t \geq 0)$, claim (3) and (5) of Proposition 4.2 imply that

$$\lambda^{\frac{r}{s}} e^{-\omega t} T_t \left( \lambda^{\frac{r}{s}} u_0 \right) - T_t u_0 \ll |1 - \lambda^{\frac{r}{s}}| |u_0|$$

and

$$|\lambda^{\frac{r}{s}} - 1| e^{-\omega t} |T_t u_0| \ll |\lambda^{\frac{r}{s}} - 1| |u_0|.$$ 

Since the set $\{w \mid w \ll |\lambda^{\frac{r}{s}} - 1| |u_0|\}$ is convex (cf., (6) of Proposition 4.2), the previous inequalities imply that

$$\frac{1}{2} e^{-\omega t} |T_{t+h}u_0 - T_t u_0| \ll |\lambda^{\frac{r}{s}} - 1| |u_0|.$$
Using again (3) of Proposition 4.2, gives
\[ |T_{t+h}u_0 - T_tu_0| \leq 2e^{\alpha t} |u_0|. \] (4.5)

Since for every \( u \in M(\Sigma, \mu) \), one always has that \( u^+ \ll |u| \), the transitivity of "\( \ll \)" ((4) of Proposition 4.2) implies that for
\[ f_h := \frac{T_{t+h}u_0 - T_tu_0}{\lambda^{\frac{1}{\mu}} - 1}, \]

one has \( f_h^+ \ll 2e^{\alpha t} |u_0| \). Therefore and since \( |u_0| \in X \), (1) of Proposition 4.10 yields that the two sets \( \{f_h^+ \mid h \neq 0, t + h > 0\} \) and \( \{|f_h^+ \mid h \neq 0, t + h > 0\} \) are \( \sigma(L_0, L^{1, \infty}) \)-weakly sequentially compact in \( L_0(\Sigma, \mu) \). Since \( f_h^+ = f_h^+ - f_h^0 \) and \( f_h = f_h^+ - f_h^- \), and since \( (\lambda^{\frac{1}{\mu}} - 1)/h = ((1 + \frac{\lambda}{\mu})^{\frac{1}{\mu}} - 1)/h \to 1/t(1 - \alpha) \neq 0 \) as \( h \to 0 \), we can conclude that the claim of this lemma holds.

With these preliminaries in mind, we can now state the regularization effect of the semigroup \( \{T_t\}_{t \geq 0} \) generated by a \( \omega \)-quasi \( m \)-completely accretive operator of homogeneous order \( \alpha \neq 1 \).

**Theorem 4.13.** Let \( X \subseteq L^0(\Sigma, \mu) \) be a normal Banach space satisfying (4.1), and \( \|\cdot\| \) denote the norm on \( X \). For \( \omega \in \mathbb{R} \), let \( A \) be \( \omega \)-quasi \( m \)-completely accretive in \( X \), and \( \{T_t\}_{t \geq 0} \) be the semigroup generated by \(-A\) on \( D(A)^\omega \). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \( u_0 \in D(A)^\omega \) and \( t > 0 \), \( \frac{d}{dt} u_0 \) exists in \( X \) and
\[ \|A^\omega T_t u_0\| \leq \frac{2e^{\alpha t}}{|\alpha - 1|} \frac{|u_0|}{t} \quad \mu \text{-a.e. on } \Sigma. \] (4.6)

In particular, for every \( u_0 \in \overline{D(A)^\omega} \),
\[ \left\| \frac{d}{dt}T_t u_0 \right\| \leq \frac{2e^{\alpha t}}{|\alpha - 1|} \frac{|u_0|}{t} \quad \text{for every } t > 0, \] (4.7)

and \( \frac{d}{dt} T_t u_0 \leq \frac{2e^{\alpha t}}{|\alpha - 1|} \frac{|u_0|}{t} \quad \text{for every } t > 0. \) (4.8)

**Proof.** Let \( u_0 \in \overline{D(A)^\omega} \), \( t > 0 \), and \((h_n)_{n \geq 1} \subseteq \mathbb{R} \) be a zero sequence such that \( t + h_n > 0 \) for all \( n \geq 1 \). Due to Lemma 3.4, we can apply Lemma 4.12. Thus, there is a \( z \in L^0(\Sigma, \mu) \) and a subsequence \((h_{k_n})_{n \geq 1}\) of \((h_n)_{n \geq 1}\) such that
\[ \lim_{n \to \infty} \frac{T_{t+h_{k_n}} u_0 - T_t u_0}{h_{k_n}} = z \quad \text{weakly in } L^0(\Sigma, \mu). \] (4.9)

By (2e) of Proposition 4.11, one has that \( (T_t u_0, -z) \in A \). Thus (2f) of Proposition 4.11 yields that \( z = -A^\omega T_t u_0 \) and
\[ \lim_{n \to \infty} \frac{T_{t+h_{k_n}} u_0 - T_t u_0}{h_{k_n}} = -A^\omega T_t u_0 \quad \text{strongly in } L^0(\Sigma, \mu). \] (4.10)

After possibly passing to another subsequence, the limit (4.10) also holds \( \mu \text{-a.e. on } \Sigma \). The argument shows that the limit (4.10) is independent of the choice of the initial zero sequence \((h_n)_{n \geq 1}\). Thus
\[ \lim_{h \to 0} \frac{T_{t+h} u_0 - T_t u_0}{h} = -A^\omega T_t u_0 \quad \text{exists } \mu \text{-a.e. on } \Sigma. \] (4.11)
Theorem 4.14. Suppose \( X \subseteq L_0(\Sigma, \mu) \) be a normal Banach space satisfying (4.1), \( F : X \to X \) be a Lipschitz continuous mapping with Lipschitz-constant \( \omega > 0 \) satisfying \( F(0) = 0 \), \( A \) an \( m \)-completely accretive operator on \( X \), and \( \{ T_t \}_{t \geq 0} \) the semigroup generated by \( -(A + F) \) on \( \overline{D(A)}^\times \). If \( A \) is homogeneous of order \( \alpha \neq 1 \), then for every \( u_0 \in \overline{D(A)}^\times \), the mild solution \( u(t) = T_t u_0 \) \((t \geq 0)\) of Cauchy problem (1.8) is a strong one, and \( \{ T_t u_0 \}_{t \geq 0} \) satisfies (2.24) for \( a.e. \) \( t > 0 \).

Proof of Theorem 4.14. First, let \( u_0 \in D(A) \) and \( t > 0 \). Then by Proposition 3.6, the mild solution \( u(t) := T_t u_0 \) \((t \geq 0)\), is locally Lipschitz continuous on \([0, \infty)\). Thus, by Lemma 4.12, \( u \) is differentiable with values in \( X \) for all \( t > 0 \), \( V_\omega(u, \cdot) \) satisfies (3.11), and \( V_\omega(u, t) \) belongs to \( L^\infty(0, T) \). Therefore we can apply Theorem 2.7 and obtain that \( \{ T_t u_0 \}_{t \geq 0} \) satisfies (2.24) for all \( t > 0 \). Applying (2.24),
we get
\[ \|T_{t+h}u_0 - T_tu_0\| \leq \int_t^{t+h} \| \frac{dT_su_0}{ds} \| \, ds \]
\[ \leq \int_t^{t+h} \frac{\omega_u 2 L\|u_0\|_X}{s} \left( (1 + \omega s) + \int_0^s (1 + \omega r) L \omega e^{L \omega (s-r)} \, dr \right) \, ds \]
\[ (4.14) \]
for every \( h \neq 0, \, t > 0 \) satisfying \( t + h > 0 \).

Next, let \( u_0 \in D(A)^{\mu} \) and \( t > 0 \). Then a standard density argument together with the strong continuity of \( \{T_t\}_{t \geq 0} \) yields that \( \{T_tu_0\}_{t \geq 0} \) satisfies (4.14). Thus, again by Lemma 4.12, the mild solution \( u(t) := T_tu_0, \, (t \geq 0) \), is differentiable with values in \( X \) for all \( t > 0 \) and \( V_\omega(u, \cdot) \) satisfies (3.11). Dividing (4.14) by \( |h| \) and letting \( h \to 0 \) yields that \( \{T_tu_0\}_{t \geq 0} \) satisfies (2.24) for all \( t > 0 \).

5. Application

Our aim in this section is to derive global regularity estimates of the time-derivative \( \frac{du}{dt} \) in the evolution problem (1.8) when for \( 1 \leq p < \infty, \, p \neq 2 \), the operator \( A \) is given by the negative \( p \)-Laplace-Beltrami operator
\[ (5.1) \]
for \( u \in W_0^{1,p}(M) \) (respectively, \( u \in BV_0(M) \) if \( p = 1 \)), on a non-compact Riemannian manifold \( (M, g) \). In the Euclidean space \( M = \mathbb{R}^d \), the total variational flow was studied in [6] and [1].

5.1. Framework. Throughout this section, let \( (M, g) \) denote a non-compact, smooth, \( d \)-dimensional Riemannian manifold with Riemannian metric tensor \( g = \{g(x)\}_{x \in M} \), for \( x \in M \), let \( T_x \) be the tangent space and \( TM \) the tangent bundle of \( M \). We write \( |\xi|_g = \sqrt{\langle \xi, \xi \rangle_g} \) for \( \xi \in T_x \) to denote the induced norm of the inner product \( \langle \cdot, \cdot \rangle_g \) on the tangent space \( T_x \).

For given \( f \in C^\infty(M) \), \( df \) is the differential at \( x \in M \) and \( g = (g_{ij})_{i,j=1}^d \) the matrix of the Riemannian metric \( g \) on \( U \) with inverse \( g^{-1} \), then the corresponding gradient of \( f \) at \( x \) is given by \( \nabla f(x) = g^{-1}(x)df(x) \), and for every \( C^1 \)-vector field \( X = (X^1, \ldots, X^d) \) on \( M \), the divergence
\[ \text{div} (X) := \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left( \sqrt{\det(g)} X^i \right). \]

There exists a unique measure \( \mu \) such that, on any chart \( U \), \( d\mu = \sqrt{\det(g)} \, dx \), where \( dx \) refers to the Lebesgue measure in \( U \). For the measure space \( (M, \mu) \), we denote by \( L^q = L^q(M, \mu) \) (respectively, \( L^q_{\text{loc}}(M, \mu) \)) the classical Lebesgue space of (locally) \( q \)-integrable functions, \( 1 \leq q \leq \infty \), and we denote by \( \|\cdot\|_p \) it standard norm on \( (M, \mu) \). Since a vector field \( v \) on \( M \) is measurable if every component of \( v \) is measurable on all charts \( U \) of \( M \), one can define similarly for \( 1 \leq q \leq \infty \), the space \( \tilde{L}^q = \tilde{L}^q(M, \mu) \) (respectively, \( \tilde{L}^q_{\text{loc}}(M, \mu) \)) of all measurable vector fields \( v \) on \( M \) such that \( |v| \in L^q(M, \mu) \) (respectively, \( |v| \in L^q_{\text{loc}}(M, \mu) \)).
We define the space of test functions $\mathcal{D}(M)$ to be the set $C_0^\infty(M)$ of smooth compactly supported functions equipped with the following convergence: given a sequence $\{\varphi_n\}_{n \geq 1}$ in $C_0^\infty(M)$ and $\varphi \in C_0^\infty(M)$, we say $\varphi_n \rightarrow \varphi$ in $\mathcal{D}(M)$ if all supports $\text{supp}(\varphi_n)$ are contained in the same compact subset $K$ of $M$, and for every chart $U$, and all multi-index $\alpha$, one has $D^\alpha \varphi_n \rightarrow D^\alpha \varphi$ uniformly on $U$. Then the space of distributions $\mathcal{D}'(M)$ is the topological dual space of $\mathcal{D}(M)$. Similarly, one defines the space of test vector fields $\mathcal{D}(M)$ on $M$ and corresponding dual space $\mathcal{D}'(M)$ of distributional vector fields. Given a distribution $T \in \mathcal{D}'(M)$, the distributional gradient $\nabla T \in \mathcal{D}'(M)$ is defined by

$$
\langle \nabla T, \psi \rangle_{\mathcal{D}'(M), \mathcal{D}(M)} = -\langle T, \text{div } \psi \rangle_{\mathcal{D}'(M), \mathcal{D}(M)} \quad \text{for every } \psi \in \mathcal{D}(M).
$$

It is clear that for every $u \in L^1_{\text{loc}}(M, \mu)$, $\langle u, \varphi \rangle_{\mathcal{D}'(M), \mathcal{D}(M)} := \int_M u \varphi \, d\mu$, $\varphi \in \mathcal{D}(M)$, defines a distribution (called regular distribution) on $M$. If for $u \in L^1_{\text{loc}}(M, \mu)$, the distributional gradient $\nabla u \in \mathcal{L}^1_{\text{loc}}(M, \mu)$, then $\nabla u$ is called a weak gradient of $u$. Now, for $1 \leq p < \infty$, the first Sobolev space $W^{1,p} = W^{1,p}(M, \mu)$ is the space of all $u \in L^p(M, \mu)$ with weak gradient $\nabla u \in \mathcal{L}^p(M, \mu)$. Then, $W^{1,p}$ is a Banach space equipped with the norm

$$
\|u\|_{W^{1,p}} := (\|u\|_p + \|\nabla u\|_p)^{1/p}, \quad (u \in W^{1,p}),
$$

and $W^{1,p}$ is reflexive if $1 < p < \infty$. Further, we denote by $W^{1,p}_0(M, \mu)$ the completion of $C_0^\infty(M)$ with respect to $\|\nabla \cdot \|_p$.

Following [?, for a given $u \in L^1_{\text{loc}}(M, \mu)$, the variation of $u$ in the open subset $U$ of $M$ is given by

$$
|\nabla u|(U) := \sup \left\{ \int_M u \text{div } X \, d\mu \bigg| X \in C_0^\infty(M, TM), \|X\|_\infty \leq 1 \right\}
$$

and we defined the space $BV(M, \mu)$ to be the space of all $u \in L^1(M, \mu)$ with finite variation $|\nabla u|(M)$. $BV(M, \mu)$ is a Banach space equipped with norm $\|\cdot\|_1 + |\nabla \cdot|(M)$. Finally, we denote by $BV = BV(M, \mu)$ the completion of $C_0^\infty(M)$ with respect to $|\nabla \cdot|(M)$.

With these preliminaries in mind, for $1 \leq p < \infty$, the negative $p$-Laplace-Beltrami operator $-\Delta_p$ in (5.1) equipped with vanishing conditions on a non-compact manifold can be realized in $L^2(M, \mu)$ as the operator

$$
A = \left\{ (u, h) \in L^2 \times L^2 \bigg| u \in \mathcal{D}_p \text{ and } \forall \psi \in C_0^\infty(M), \int_M |\nabla u|^p \|
abla \psi \|_{\infty}^2 \, d\mu = \int_M h \psi \, d\mu \right\},
$$

where $\mathcal{D}_p := W^{1,p} \text{ if } 1 < p < \infty$, and $\mathcal{D}_1 := BV$. In fact, $A$ is the subdifferential operator $\partial \mathcal{E}$ in $L^2(M, \mu)$ of the convex, proper, lower semicontinuous functional $\mathcal{E} : L^2(M, \mu) \rightarrow [0, +\infty]$ defined by

$$
\mathcal{E}(u) := \begin{cases} 
\frac{1}{p} \int_M |\nabla u|^p \, d\mu & \text{if } u \in \mathcal{D}_p, \\
+\infty & \text{if otherwise,}
\end{cases}
$$

for every $u \in L^2(M, \mu)$.

Thus, by the classical theory (cf., [11]) of maximal monotone operators in Hilbert spaces $A$ is an $m$-accretive operator in $L^2(M, \mu)$. Moreover, since the
effective domain $D(E)$ of $E$ contains $C_c^\infty(M)$ and $C_c^\infty(M)$ is dense in $L^2(M, \mu)$, we have that the domain $D(A)$ is dense in $L^2(M, \mu)$.

Next, suppose $f : M \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous Carathéodory function, that is, $f$ satisfies the following three properties:

\begin{align*}
\text{(5.2)} & \quad f(\cdot, u) : M \to \mathbb{R} \text{ is measurable on } M \text{ for every } u \in \mathbb{R}, \\
\text{(5.3)} & \quad f(x, 0) = 0 \text{ for a.e. } x \in M, \text{ and} \\
& \text{there is a constant } \omega \geq 0 \text{ such that} \\
\text{(5.4)} & \quad |f(x, u) - f(x, \hat{u})| \leq \omega |u - \hat{u}| \quad \text{for all } u, \hat{u} \in \mathbb{R}, \text{ a.e. } x \in M.
\end{align*}

Then, for every $1 \leq q < \infty$, $F : L^q(M, \mu) \to L^q(M, \mu)$ defined by

$$F(u)(x) := f(x, u(x)) \quad \text{for every } u \in L^q(M, \mu)$$

is the associated Nemytskii operator on $L^q(M, \mu)$. Moreover, by (5.4), $F$ is globally Lipschitz continuous on $L^q(M, \mu)$ with constant $\omega > 0$ and $F(0)(x) = 0$ for a.e. $x \in M$.

Under these assumptions, it follows from Proposition 4.6 that the perturbed operator $A + F$ in $L^2(M, \mu)$ is an $\omega$-quasi $m$-completely accretive operator with dense domain $D(A + F) = D(A)$ in $L^2(M, \mu)$ (see [13] for the details in the Euclidean case). Thus, $-(A + F)$ generates a strongly continuous semigroup $\{T_t\}_{t \geq 0}$ of Lipschitz-continuous mappings $T_t$ on $L^2$ with Lipschitz constant $e^{\omega t}$. For every $1 \leq q < \infty$, each $T_t$ admits a unique Lipschitz-continuous extension $T_t^{(q)}$ on $L^q$ with Lipschitz constant $e^{\omega t}$ such that $\{T_t^{(q)}\}_{t \geq 0}$ is a strongly continuous semigroup on $L^q$, and each $T_t^{(q)}$ is Lipschitz-continuous on $L^2 \cap L^{\infty, \infty}$ with respect to the $L^\infty$-norm. By Proposition 4.11, if $\mathcal{A}^{(q)} := \mathcal{A}^{(0)} \cap (L^q \times L^q)$, then the operator $-(\mathcal{A}^{(q)} + F)$ is the infinitesimal generator of $\{T_t^{(q)}\}_{t \geq 0}$ in $L^q$. For simplicity, we denote the extension of $T_t$ in $L^q$ again by $T_t$.

5.2. Global regularity estimates of $\frac{du}{dt}$. Now, fix $1 \leq p < \nu$, and assume that the complete non-compact Riemannian manifold $(M, g)$ satisfies an $(L^p, \nu)$-Sobolev inequality; that is, there is a constant $C(M, p, \mu) > 0$ such that

$$\|u\|_{W_p/(\nu-p)} \leq C(M, p, \mu) \|\nabla u\|_p \quad \text{for all } u \in C_c^\infty(M).$$

Recall, this is the case if the manifold $M$, for example, has non-negative Ricci curvature, or if there exists a $c > 0$ such that $\mu(B(x, 1)) \geq c$ for all $x \in M$ (cf., [4]). For more general cases, we refer to [22].

We choose $q_0 \geq p$ minimal such that $(\frac{\nu}{\nu-p} - 1)q_0 + p - 2 > 0$. Then, it follows from [13, Theorem 1.2 (cf., Theorem 7.4)], the semigroup $\{T_t\}_{t \geq 0}$ generated by $-(A + F)$ satisfies for every $1 \leq q \leq \frac{\nu q_0}{\nu-p}$ satisfying $q > \frac{\nu(2-p)}{p}$, the following $L^q$-Sobolev-regularity estimate

$$\|T_t u\|_\infty \lesssim e^{\omega t} t^{-\frac{\nu}{2}} \|u\|_q^q$$
for every \( t > 0, u \in L^q(\Sigma) \), with exponents
\[
\alpha_q = \frac{\alpha^*}{1 - \gamma^* \left(1 - \frac{q(v-p)}{vq_0}\right)}, \quad \beta_q = \frac{\beta^* + \gamma^* \frac{q(v-p)}{vq_0}}{1 - \gamma^* \left(1 - \frac{q(v-p)}{vq_0}\right)}, \\
\gamma_q = \frac{\gamma^* q(v-p)}{vq_0 \left(1 - \gamma^* \left(1 - \frac{q(v-p)}{vq_0}\right)\right)},
\]
where
\[
\alpha^* = \frac{v - p}{pq_0 + (v - p)(p-2)}, \quad \beta^* = \frac{(\frac{2}{p} - 1)v + p}{pq_0 + (v - p)(p-2)} + 1, \\
\gamma^* = \frac{pq_0}{pq_0 + (v - p)(p-2)}.
\]

Now, the negative \( p \)-Laplacian \( A \) in \( L^2 \) is homogeneous of order \( p - 1 \). Then, for every \( 1 \leq r < \infty \), the restriction \( A^r_{L^p} := A \cap (L^r \times L^r) \) in \( L^r \) is homogeneous of order \( p - 1 \). Thus, by Proposition 4.11 and by Theorem 4.14, the semigroup \( \{T_t\}_{t \geq 0} \) generated by \( -(A + F) \) satisfies
\[
(5.6) \quad \frac{\|dT_tu_0\|}{d} + \left(1 + \omega t\right) + \int_0^t \frac{(1 + \omega s) \omega e^{\omega(1-s)}}{t} ds 
\]
for every \( t > 0, u_0 \in L^r \), and every \( 1 \leq r < \infty \).

Next, for \( 1 \leq q \leq \frac{vq_0}{v-p} \) satisfying \( q > \frac{v(2-p)}{p} \), let \( u_0 \in L^q \cap L^\infty(\Sigma, \mu) \) and \( t > 0 \).
We assume \( \|dT_tu_0\|_\infty > 0 \) (otherwise, there is nothing to show). Then, for every \( s \in (0, \|dT_tu_0\|_{\infty}) \) and every \( q \leq r < \infty \), Chebyshev’s inequality yields
\[
\mu \left( \left\{ \left\| \frac{dT_tu_0}{dt} \right\| + s \right\} \right)^{1/r} \leq \frac{\|dT_tu_0\|_{\infty}}{s}
\]
and so, by (5.6),
\[
s \mu \left( \left\{ \left\| \frac{dT_tu_0}{dt} \right\| + s \right\} \right)^{1/r} \leq \frac{e^{\omega t} 2 \|u_0\|_r}{\left| 1 - \alpha \right|} \left(1 + \omega t\right) + \int_0^t (1 + \omega s) \omega e^{\omega(1-s)} ds.
\]
Thus and since \( \lim_{r \to \infty} \|u_0\|_r = \|u_0\|_{\infty} \), sending \( r \to + \infty \) in the last inequality, yields
\[
s \leq \left[ 2e^{\omega t} \int_0^t e^{\omega s} ds \right] + \omega \int_0^t e^{\omega s} dr ds \frac{e^{\omega t}}{t} \|u_0\|_{\infty}
\]
and since \( s \in (0, \|dT_tu_0\|_{\infty}) \) was arbitrary, we have thereby shown that (5.6) also holds for \( r = \infty \). Applying now, (5.5) to (5.6) and using that \( T_tu_0 = T^2_tu_0(T^2_tu_0) \), one sees that
\[
(5.7) \quad \frac{\|dT_tu_0\|}{d} + \left(1 + \beta_q \right) \|u_0\|_q \left| 1 - \frac{q}{2} \right| \left(1 + \omega \frac{t}{2}\right) + \int_0^{t/2} (1 + \omega s) \omega e^{\omega(1-s)} ds.
\]
A further density argument yields that inequality (5.7) holds, in particular, for all \( u_0 \in L^q \).
REFERENCES

[1] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón, Parabolic quasilinear equations minimizing linear growth functionals, vol. 223 of Progress in Mathematics, Birkhäuser Verlag, Basel, 2004.

[2] W. Arendt, Semigroups and evolution equations: functional calculus, regularity and kernel estimates, in Evolutionary equations. Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 1–85.

[3] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, vol. 96 of Monographs in Mathematics, Birkhäuser/Springer Basel AG, Basel, second ed., 2011.

[4] N. Badr, Gagliardo-Nirenberg inequalities on manifolds, J. Math. Anal. Appl., 349 (2009), pp. 493–502.

[5] V. Barbu, Nonlinear differential equations of monotone types in Banach spaces, Springer Monographs in Mathematics, Springer, New York, 2010.

[6] G. Bellanetti, V. Caselles, and M. Novaga, The total variation flow in $\mathbb{R}^N$, J. Differential Equations, 184 (2002), pp. 475–525.

[7] P. Bénilan and M. G. Crandall, Regularizing effects of homogeneous evolution equations, in Contributions to analysis and geometry (Baltimore, Md., 1980), Johns Hopkins Univ. Press, Baltimore, Md., 1981, pp. 23–39.

[8] P. Bénilan and M. G. Crandall, Completely accretive operators, in Semigroup theory and evolution equations (Delft, 1989), vol. 135 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1991, pp. 41–75.

[9] P. Bénilan, M. G. Crandall, and A. Pazy, Evolution problems governed by accretive operators, book in preparation.

[10] C. Bennett and R. Sharpley, Interpolation of operators, vol. 129 of Pure and Applied Mathematics, Academic Press, Inc., Boston, MA, 1988.

[11] H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Mathematics Studies, No. 5. Notas de Matemática (50), North-Holland Publishing Co., 1973.

[12] R. Chill, D. Hauer, and J. Kennedy, Nonlinear semigroups generated by $j$-elliptic functionals, J. Math. Pures Appl. (9), 105 (2016), pp. 415–450.

[13] T. Coulhon and D. Hauer, Regularisation effects of nonlinear semigroups - Theory and Applications. to appear in BCAM Springer Briefs, 2016.

[14] E. B. Davies, Heat kernels and spectral theory, vol. 92 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1990.

[15] A. Grigor’yan, Heat kernel and analysis on manifolds, vol. 47 of AMS/IP Studies in Advanced Mathematics, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.

[16] D. Hauer, Y. He, and D. Liu, Fractional powers of monotone operators in Hilbert spaces, Adv. Nonlinear Stud., 19 (2019), pp. 717–755.

[17] D. Hauer and J. M. Mazón, Regularizing effects of homogeneous evolution equations: the case of homogeneity order zero, Journal of Evolution Equations, (2019).

[18] G. Leoni, A first course in Sobolev spaces, vol. 105 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2009.

[19] A. Pazy, Semigroups of linear operators and applications to partial differential equations, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[20] E. I. Poppal and S. Reich, An incomplete Cauchy problem, J. Math. Anal. Appl., 113 (1986), pp. 514–543.

[21] M. M. Rao and Z. D. Ren, Theory of Orlicz spaces, vol. 146 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1991.

[22] L. Saloff-Coste, Aspects of Sobolev-type inequalities, vol. 289 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2002.

[23] G. R. Sell and Y. You, Dynamics of evolutionary equations, vol. 143 of Applied Mathematical Sciences, Springer-Verlag, New York, 2002.
[24] N. T. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, vol. 100 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1992.

(Daniel Hauer) School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia
E-mail address: daniel.hauer@sydney.edu.au