SPECTRAL GAPS FOR PERIODIC SCHRÖDINGER OPERATORS WITH HYPERSURFACE MAGNETIC WELLS: ANALYSIS NEAR THE BOTTOM

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Abstract. We consider a periodic magnetic Schrödinger operator $H^h$, depending on the semiclassical parameter $h > 0$, on a noncompact Riemannian manifold $M$ such that $H^1(M, \mathbb{R}) = 0$ endowed with a properly discontinuous cocompact isometric action of a discrete group. We assume that there is no electric field and that the magnetic field has a periodic set of compact magnetic wells. We suppose that the magnetic field vanishes regularly on a hypersurface $S$. First, we prove upper and lower estimates for the bottom $\lambda_0(H^h)$ of the spectrum of the operator $H^h$ in $L^2(M)$. Then, assuming the existence of non-degenerate miniwells for the reduced spectral problem on $S$, we prove the existence of an arbitrary large number of spectral gaps for the operator $H^h$ in the region close to $\lambda_0(H^h)$, as $h \to 0$. In this case, we also obtain upper estimates for the eigenvalues of the one-well problem.

1. Preliminaries and main results

Let $M$ be a noncompact oriented manifold of dimension $n \geq 2$ equipped with a properly discontinuous action of a finitely generated, discrete group $\Gamma$ such that $M/\Gamma$ is compact. Suppose that $H^1(M, \mathbb{R}) = 0$, i.e. any closed 1-form on $M$ is exact. Let $g$ be a $\Gamma$-invariant Riemannian metric and $B$ a real-valued $\Gamma$-invariant closed 2-form on $M$. Assume that $B$ is exact and choose a real-valued 1-form $A$ on $M$ such that $dA = B$.

Thus, one has a natural mapping

$$u \mapsto ih \, du + Au$$

from $C^\infty_c(M)$ to the space $\Omega^1_c(M)$ of smooth, compactly supported one-forms on $M$. The Riemannian metric allows to define scalar products in these spaces and consider the adjoint operator

$$(ih \, d + A)^* : \Omega^1_c(M) \to C^\infty_c(M).$$

A Schrödinger operator with magnetic potential $A$ is defined by the formula

$$H^h = (ih \, d + A)^*(ih \, d + A).$$

Here $h > 0$ is a semiclassical parameter, which is assumed to be small.

Choose local coordinates $X = (X_1, \ldots, X_n)$ on $M$. Write the 1-form $A$ in the local coordinates as

$$A = \sum_{j=1}^n A_j(X) \, dX_j,$$

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the matrix of the Riemannian metric \( g \) as
\[
g(X) = (g_{j\ell}(X))_{1 \leq j, \ell \leq n}
\]
and its inverse as
\[
g(X)^{-1} = (g^{j\ell}(X))_{1 \leq j, \ell \leq n}.
\]
Denote \( |g(X)| = \det(g(X)) \). Then the magnetic field \( B \) is given by the following formula
\[
B = \sum_{j<k} B_{jk} \, dX_j \wedge dX_k, \quad B_{jk} = \frac{\partial A_k}{\partial X_j} - \frac{\partial A_j}{\partial X_k}.
\]
Moreover, the operator \( H^h \) has the form
\[
H^h = \frac{1}{\sqrt{|g(X)|}} \sum_{1 \leq j, \ell \leq n} \left( ih \frac{\partial}{\partial X_j} + A_j(X) \right) \times \left[ \sqrt{|g(X)|} g^{j\ell}(X) \left( ih \frac{\partial}{\partial X_\ell} + A_\ell(X) \right) \right].
\]
For any \( x \in M \), denote by \( B(x) \) the anti-symmetric linear operator on the tangent space \( T_x M \) associated with the 2-form \( B \):
\[
g_x(B(x)u,v) = B_x(u,v), \quad u,v \in T_x M.
\]
Recall that the intensity of the magnetic field is defined as
\[
\text{Tr}^+(B(x)) = \sum_{i: \lambda_i(x) > 0} \lambda_i(x) = \frac{1}{2} \text{Tr}([B^*(x) \cdot B(x)]^{1/2}).
\]
It turns out that in many problems the function \( x \mapsto h \cdot \text{Tr}^+(B(x)) \) can be considered as a magnetic potential, that is, as a magnetic analog of the electric potential \( V \) in a Schrödinger operator \(-h^2 \Delta + V\).

We will also use the trace norm of \( B(x) \):
\[
|B(x)| = \left[ \text{Tr}(B^*(x) \cdot B(x)) \right]^{1/2}.
\]
It coincides with the norm of \( B(x) \) with respect to the Riemannian metric on the space of linear operators on \( T_x M \) induced by the Riemannian metric \( g \) on \( M \).

In this paper we will always assume that the magnetic field has a periodic set of compact potential wells. More precisely, put
\[
b_0 = \min \{ \text{Tr}^+(B(x)) : x \in M \}
\]
and assume that there exist a (connected) fundamental domain \( \mathcal{F} \) and a constant \( \epsilon_0 > 0 \) such that
\[
\text{Tr}^+(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial \mathcal{F}.
\]
For any \( \epsilon_1 \leq \epsilon_0 \), put
\[
U_{\epsilon_1} = \{ x \in \mathcal{F} : \text{Tr}^+(B(x)) < b_0 + \epsilon_1 \}.
\]
Thus \( U_{\epsilon_1} \) is an open subset of \( \mathcal{F} \) such that \( U_{\epsilon_1} \cap \partial \mathcal{F} = \emptyset \) and, for \( \epsilon_1 < \epsilon_0 \), \( U_{\epsilon_1} \) is compact and included in the interior of \( \mathcal{F} \). Any connected component of \( U_{\epsilon_1} \) with \( \epsilon_1 < \epsilon_0 \) and also any its translation under the action of an element of \( \Gamma \) can be understood as a magnetic well. These magnetic wells are separated by potential barriers, which are getting higher and higher when \( h \to 0 \) (in the semiclassical limit).
We will consider the magnetic Schrödinger operator $H^h$ as an unbounded self-adjoint operator in the Hilbert space $L^2(M)$ and study gaps in the spectrum of this operator, which are located below the top of potential barriers, that is, on the interval $[0, h(b_0 + \epsilon_0)]$. Here by a gap in the spectrum $\sigma(T)$ of a self-adjoint operator $T$ in a Hilbert space we will mean any connected component of the complement of $\sigma(T)$ in $\mathbb{R}$, that is, any maximal interval $(a, b)$ such that $(a, b) \cap \sigma(T) = \emptyset$.

The problem of existence of gaps in the spectra of second order periodic differential operators has been extensively studied recently. Some related results on spectral gaps for periodic magnetic Schrödinger operators can be found for example in [2, 7, 8, 15, 18, 19, 20, 21, 22, 23, 24, 25, 27] (see also the references therein).

In our case, the important role is played by the tunneling effect, that is, by the possibility for the quantum particle described by the Hamiltonian $H^h$ with such an energy to pass through a potential barrier. Using the semiclassical analysis of the tunneling effect, we showed in [7] that the spectrum of the magnetic Schrödinger operator $H^h$ on the interval is localized in an exponentially small neighborhood of the spectrum of its Dirichlet realization inside the wells. This result reduces the investigation of some gaps in the spectrum of the operator $H^h$ to the study of the eigenvalue distribution for a “one-well” operator and leads us to suggest a general scheme of a proof of existence of spectral gaps in [8]. We disregard in this paper the analysis of the spectrum in the above mentioned exponentially small neighborhoods.

We consider the case when $b_0 = 0$ and the zero set of the magnetic field has regular hypersurface pieces. More precisely, suppose that there exists $x_0 \in M$ such that $B(x_0) = 0$ and in a neighborhood $U$ of $x_0$ the zero set of $B$ is a smooth oriented hypersurface $S$, and, moreover, there are constants $k \in \mathbb{Z}$, $k > 0$, and $C > 0$ such that, for all $x \in U$, we have:

$$C^{-1}d(x, S)^k \leq |B(x)| \leq Cd(x, S)^k.$$  

(1.2)

On compact manifolds, such a model was introduced for the first time by Montgomery [26] and was further studied in [10, 28, 6, 9].

Denote by $N$ the external unit normal vector to $S$ and by $\tilde{N}$ an arbitrary extension of $N$ to a smooth vector field on $U$. Let $\omega_{0,1}$ be the smooth one form on $S$ defined, for any vector field $V$ on $S$, by the formula

$$\langle V, \omega_{0,1}(y) \rangle = \frac{1}{k!} \tilde{N}^k(B(\tilde{N}, \tilde{V}))(y), \quad y \in S,$$

where $\tilde{V}$ is a $C^\infty$ extension of $V$ to $U$. By (1.2), it is easy to see that $\omega_{0,1}(x) \neq 0$ for any $x \in S$. Denote

$$\omega_{\min}(B) = \inf_{x \in S} |\omega_{0,1}(x)| > 0.$$

For any $\alpha \in \mathbb{R}$, denote by $\lambda_0(\alpha)$ the bottom of the self-adjoint second order differential operator

$$-\frac{d^2}{dt^2} + \left( \frac{t^{k+1}}{k+1} - \alpha \right)^2$$

in $L^2(\mathbb{R})$. Put $\hat{\nu} := \inf_{\alpha \in \mathbb{R}} \lambda_0(\alpha)$. It is clear that $\hat{\nu} \geq 0$. We refer the reader to Section 2 for more properties.

In [9], we have proved the following result.
Theorem 1.1. For any $a$ and $b$ such that

$$\hat{\nu} \omega_{\min}(B) \frac{x^2}{4} < a < b$$

and for any natural $N$, there exists $h_0 > 0$ such that, for any $h \in (0, h_0]$, the spectrum of $H^h$ in the interval

$$[h^{\frac{2k+2}{3}} a, h^{\frac{2k+2}{3}} b]$$

has at least $N$ gaps.

In this paper we will concentrate our analysis on the region close to the bottom of the spectrum. First, we state upper and lower estimates for the bottom $\lambda_0(H^h)$ of the spectrum of the operator $H^h$ in $L^2(M)$

Theorem 1.2. Suppose that the operator $H^h$ satisfies condition \((\ref{condition})\) with some $\epsilon_0 > 0$, and that the zero set of the magnetic field $B$ is a smooth oriented hypersurface $S$. Moreover, assume that there are some $k \in \mathbb{Z}, k > 0$ and $C > 0$ such that, for all $x$ in a neighborhood $U$ of $S$, we have:

$$C^{-1} d(x, S)^k \leq |B(x)| \leq C d(x, S)^k.$$ 

Then there exist $C > 0$ and $h_0 > 0$ such that, for any $h \in (0, h_0]$, we have

$$\hat{\nu} \omega_{\min}(B) \frac{x^2}{4} h \frac{x^2}{4} - Ch \frac{x^2}{4} \frac{x^2}{4} \leq \lambda_0(H^h) \leq \hat{\nu} \omega_{\min}(B) \frac{x^2}{4} h \frac{x^2}{4} + Ch \frac{x^2}{4} \frac{x^2}{4}.$$ 

A similar result was obtained for the bottom of the spectrum of the Neumann realization of the operator $H^h$ in a bounded domain in $\mathbb{R}^2$ by Pan and Kwek \([26]\) in the case $k = 1$ and by Aramaki \([1]\) in the case $k$ arbitrary odd.

As an immediate consequence of Theorems \(1.1\) and \(1.2\) we obtain the following statement.

Corollary 1.1. In addition to the assumptions of Theorem \(1.2\), suppose that $M$ is compact. Denote by $\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots$ the eigenvalues of the operator $H^h$ in $L^2(M)$. Then, for integer $m \geq 0$, we have

$$\lim_{h \to 0} h^{-\frac{4k+2}{3}} \lambda_m(H^h) = \hat{\nu} \omega_{\min}(B) \frac{x^2}{4}.$$ 

In the case when $k = 1$ and $|\omega_{0,1}(x)|$ is constant along $S$, this result was obtained by Montgomery \([20]\). In Theorem 1.4, under some additional assumptions, more generic than in \([20]\), we obtain stronger estimates for the eigenvalues of the one-well problem.

Like in the case of the Schrödinger operator with electric potential (see \([16]\)), one can introduce an internal notion of magnetic well for a fixed hypersurface $S$ in the zero set of the magnetic field $B$. Such magnetic wells can be naturally called magnetic miniwells. They are defined by means of the function $|\omega_{0,1}|$ on $S$. Assuming that there exists a non-degenerate miniwell on $S$, we prove the existence of gaps in the spectrum of $H^h$ on intervals of size $h^{\frac{2k+1}{3}}$, close to the bottom $\lambda_0(H^h)$.

Theorem 1.3. Suppose that the operator $H^h$ satisfies condition \((\ref{condition})\) with some $\epsilon_0 > 0$, and that there exists $x_0 \in M$ such that $B(x_0) = 0$ and in a neighborhood $U$ of $x_0$ the zero set of $B$ is a smooth oriented hypersurface $S$. Suppose moreover that there are constants $k \in \mathbb{Z}, k > 0$ and $C > 0$ such that for all $x \in U$ we have:

$$C^{-1} d(x, S)^k \leq |B(x)| \leq C d(x, S)^k.$$
and the upper bound for the splitting between \( \lambda \in \mathbb{R} \) energy that, using the methods of [4], one can prove the lower bound for the ground state

**Theorem 1.4.** Suppose that \( M \) is a compact manifold of dimension \( n \geq 2 \). Assume that the zero set of \( B \) is a smooth oriented hypersurface \( S \), and there are constants \( k \in \mathbb{Z}, k > 0, \) and \( C > 0 \) such that for all \( x \in S \) we have:

\[
C^{-1}d(x, S)^k \leq |B(x)| \leq Cd(x, S)^k.
\]

Assume that there exists \( x_1, \ldots, x_N \) in some neighborhood of \( x \). Then, for any natural \( N \), there exist \( h_N > 0 \) and \( h_N > 0 \) such that the spectrum of \( H^h \) in the interval

\[
\left[ \tilde{\nu} \omega_{\min}(B)^{\frac{2}{2+k}} h^{\frac{2k+2}{2+k}} \tilde{\nu} \omega_{\min}(B)^{\frac{2}{2+k}} h^{\frac{2k+2}{2+k}} + b_N h^{\frac{2k+3}{2+k}} \right]
\]

has at least \( N \) gaps for any \( h \in (0, h_N) \).

As an immediate consequence of Theorem 1.3, we also obtain upper bounds for the eigenvalues of the one-well problem.

**Theorem 1.4.** Assume finally that there exists \( x_1, \ldots, x_N \) in some neighborhood of \( x \). Assume that there exists \( N \) has at least \( N \) in the interval

\[
\left[ \tilde{\nu} \omega_{\min}(B)^{\frac{2}{2+k}} h^{\frac{2k+2}{2+k}} \tilde{\nu} \omega_{\min}(B)^{\frac{2}{2+k}} h^{\frac{2k+2}{2+k}} + b_N h^{\frac{2k+3}{2+k}} \right]
\]

Under the additional assumption that there exists a unique miniwell, we believe that, using the methods of [4], one can prove the lower bound for the ground state energy \( \lambda_0(H^h) \) of the form

\[
\lambda_0(H^h) \geq \tilde{\nu} \omega_{\min}(B)^{\frac{2}{2+k}} h^{\frac{2k+2}{2+k}} - C h^{\frac{2k+3}{2+k}}
\]

and the upper bound for the splitting between \( \lambda_0(H^h) \) and \( \lambda_1(H^h) \) of the form

\[
\lambda_1(H^h) - \lambda_0(H^h) \leq C h^{\frac{2k+3}{2+k}}.
\]

For this, we have to know a certain property of the ground state energy \( \lambda_0(\alpha) \) of the operator \( [B, \alpha] \), which we state as a conjecture.

**Conjecture 1.1.** Any minimum of \( \lambda_0(\alpha) \) is non-degenerate, that is, for any \( \alpha_{\min} \in \mathbb{R} \) such that \( \lambda_0(\alpha_{\min}) = \tilde{\nu} \) we have

\[
\frac{\partial^2 \lambda_0}{\partial \alpha^2} (\alpha_{\min}) > 0.
\]

Moreover, we believe that one can prove the lower bound for the splitting between \( \lambda_0(H^h) \) and \( \lambda_1(H^h) \) of the form

\[
\lambda_1(H^h) - \lambda_0(H^h) \geq C h^{\frac{2k+3}{2+k}},
\]

if, in addition, the following conjecture is true.

**Conjecture 1.2.** There exists a unique \( \alpha_{\min} \in \mathbb{R} \) such that \( \lambda_0(\alpha_{\min}) = \tilde{\nu} \).

We refer the reader to Section 2 for discussions.
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2. SOME ORDINARY DIFFERENTIAL OPERATORS

For any \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R}, \beta \neq 0 \), consider the self-adjoint second order differential operator in \( L^2(\mathbb{R}) \) given by

\[
Q(\alpha, \beta) = -\frac{d^2}{dt^2} + \left( \frac{1}{k+1} \beta t^{k+1} - \alpha \right)^2.
\]

In the context of magnetic bottles, this family of operators (for \( k = 1 \)) first appears in [26] (see also [10]). Denote by \( \lambda_0(\alpha, \beta) \) the bottom of the spectrum of the operator \( Q(\alpha, \beta) \).

In this section, we recall some properties of \( \lambda_0(\alpha, \beta) \), which were established in [26, 10, 28]. First of all, let us remark that \( \lambda_0(\alpha, \beta) \) is a continuous function of \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \setminus \{0\} \). One can see by scaling that, for \( \beta > 0 \),

\[
\lambda_0(\alpha, \beta) = \beta^{\frac{2}{k+2}} \lambda_0(\beta^{-\frac{1}{k+2}}, 1).
\]

A further discussion depends on the parity of \( k \).

When \( k \) is odd, \( \lambda_0(\alpha, 1) \) tends to \( +\infty \) as \( \alpha \to -\infty \) by monotonicity. For analyzing its behavior as \( \alpha \to +\infty \), it is suitable to do a dilation \( t = \alpha^{\frac{1}{k+1}} s \), which leads to the analysis of

\[
\alpha^2 \left( -h^2 \frac{d^2}{ds^2} + \left( \frac{s^{k+1}}{k+1} - 1 \right)^2 \right)
\]

with \( h = \alpha^{-(k+2)/(k+1)} \) small. One can use the semi-classical analysis (see [3] for the one-dimensional case and [30, 14] for the multidimensional case) to show that

\[
\lambda_0(\alpha, 1) \sim (k+1)^{\frac{2k}{k+2}} \alpha^{\frac{k}{k+1}}, \quad \text{as } \alpha \to +\infty.
\]

In particular, we see that \( \lambda_0(\alpha, 1) \) tends to \( +\infty \).

When \( k \) is even, we have \( \lambda_0(\alpha, 1) = \lambda_0(-\alpha, 1) \), and, therefore, it is sufficient to consider the case \( \alpha \geq 0 \). As \( \alpha \to +\infty \), semi-classical analysis again shows that \( \lambda_0(\alpha, 1) \) tends to \( +\infty \).

So in both cases, it is clear that the continuous function \( \lambda_0(\alpha, 1) \) is nonnegative and that there exists (at least one) \( \alpha_{\min} \in \mathbb{R} \) such that \( \lambda_0(\alpha, 1) \) is minimal:

\[
\lambda_0(\alpha_{\min}, 1) = \tilde{\nu}.
\]

The results of numerical computations performed for us by V. Bonnaillie-Noël for \( \alpha_{\min}, \tilde{\nu} \) and the second eigenvalue \( \lambda_1 \) of the operator \( Q(\alpha_{\min}, 1) \) are given (modulo \( 10^{-2} \)) in Table 1.

| \( k \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|
| \( \alpha_{\min} \) | 0.35  | 0     | 0.16  | 0     | 0.10  | 0     | 0.07  |
| \( \tilde{\nu} \) | 0.57  | 0.66  | 0.68  | 0.76  | 0.81  | 0.87  | 0.92  |
| \( \lambda_1 \) | 1.98  | 2.50  | 2.61  | 2.98  | 3.18  | 3.47  | 3.66  |

Table 1. Numerical results for \( \alpha_{\min}, \tilde{\nu} \) and \( \lambda_1 \)
In Fig. 1 and 2, one can also see the graphs of the function $\lambda = \lambda_0(\alpha, 1)$ and its quadratic approximation at $\alpha = \alpha_{\min}$:

$$\lambda_{\text{quad}}(\alpha) = \lambda_0(\alpha_{\min}, 1) + \frac{1}{2} \frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\min}, 1)(\alpha - \alpha_{\min})^2.$$ 

Numerical computations show that when $k$ is even the minimum is attained at zero: $\alpha_{\min} = 0$. They also suggest that the minimum $\alpha_{\min}$ is non degenerate, supporting Conjecture 1.1 and that the second derivative $\frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\min}, 1)$ tends as $k$ tends to $\infty$ to 2. One more step towards the proof of Conjecture 1.1 can be done by the following considerations.

Let $u_0^\alpha \in L^2(\mathbb{R})$, $\|u_0^\alpha\| = 1$, be the $L^2$ normalized strictly positive eigenvector of the operator $Q(\alpha, 1)$, corresponding to the eigenvalue $\lambda_0(\alpha, 1)$:

$$Q(\alpha, 1)u_0^\alpha = -d^2u_0^\alpha dt^2 + \left(\frac{t^{k+1}}{k + 1} - \alpha\right)^2 u_0^\alpha = \lambda_0(\alpha, 1)u_0^\alpha.$$ 

One can show that $u_0^\alpha$ depends smoothly on $\alpha$. Differentiating (2.2) with respect to $\alpha$, we obtain

$$Q(\alpha, 1)u_0^\alpha = -d^2u_0^\alpha dt^2 + \left(\frac{t^{k+1}}{k + 1} - \alpha\right)^2 u_0^\alpha = \lambda_0(\alpha, 1)u_0^\alpha + \lambda_0(\alpha, 1)\frac{\partial u_0^\alpha}{\partial \alpha}.$$
that implies that
\[
\frac{\partial \lambda_0}{\partial \alpha}(\alpha, 1) = -2 \int \left( \frac{t^{k+1}}{k+1} - \alpha \right) (u_0^\alpha(t))^2 \, dt
\]
and
\[
\frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha, 1) = 2 - 4 \int \left( \frac{t^{k+1}}{k+1} - \alpha \right) \frac{\partial u_0^\alpha}{\partial \alpha} \, dt.
\]

It follows from \((2.4)\) that
\[
\int \left( \frac{t^{k+1}}{k+1} - \alpha_{\text{min}} \right) (u_{\alpha_{\text{min}}}^0(t))^2 \, dt = 0,
\]
and, for \(k\) odd,
\[
\alpha_{\text{min}} = \int \frac{t^{k+1}}{k+1} (u_{\alpha_{\text{min}}}^0(t))^2 \, dt > 0.
\]
It has been claimed that this minimum is unique for \(k = 1\) in \([28]\) and for arbitrary odd \(k\) in \([1]\).

It also follows from \((2.3)\) that
\[
\left( Q(\alpha_{\text{min}}, 1) - \hat{\nu} \right) \frac{\partial u_0^\alpha}{\partial \alpha} = 2 \left( \frac{t^{k+1}}{k+1} - \alpha_{\text{min}} \right) u_{\alpha_{\text{min}}}^0.
\]

Note that the above computations can be made not only for a minimum point \(\alpha_{\text{min}}\), but also for any critical point of \(\lambda_0\).

We will also need the following identity (see \([28]\), Proposition 3.5 and the formula \((3.14)\)):
\[
\left\| \left( \frac{1}{k+1} t^{k+1} - \alpha_{\text{min}} \right) u_{\alpha_{\text{min}}}^0 \right\|^2 = \frac{\hat{\nu}}{k+2}.
\]

Based on numerical computations, one can give a proof of Conjecture \([1]\) for small \(k\) as follows. Since \(\frac{\partial u_0^\alpha}{\partial \alpha}\) is orthogonal in \(L^2(\mathbb{R}, dt)\) to \(u_0^\alpha\), by \((2.7)\) and \((2.8)\), we have
\[
\left\| \frac{\partial u_0^\alpha}{\partial \alpha} \right\|^2 \leq \frac{1}{\lambda_1 - \hat{\nu}} \left( (Q(\alpha_{\text{min}}, 1) - \hat{\nu}) \frac{\partial u_0^\alpha}{\partial \alpha}, \frac{\partial u_0^\alpha}{\partial \alpha} \right) = \frac{2}{\lambda_1 - \hat{\nu}} \int \left( \frac{t^{k+1}}{k+1} - \alpha_{\text{min}} \right) u_{\alpha_{\text{min}}}^0(t) \frac{\partial u_0^\alpha}{\partial \alpha}(t) \, dt \leq \frac{2}{\lambda_1 - \hat{\nu}} \left( \frac{\hat{\nu}}{k+2} \right)^{1/2} \left\| \frac{\partial u_0^\alpha}{\partial \alpha} \right\|.
\]
This implies that
\[
\left\| \frac{\partial u_0^\alpha}{\partial \alpha} \right\| \leq \frac{2}{\lambda_1 - \hat{\nu}} \left( \frac{\hat{\nu}}{k+2} \right)^{1/2}
\]
and
\[
\int \left( \frac{t^{k+1}}{k+1} - \alpha \right) u_{\alpha_{\text{min}}}^0(t) \frac{\partial u_0^\alpha}{\partial \alpha}(t) \, dt \leq \frac{2\hat{\nu}}{(\lambda_1 - \hat{\nu})(k+2)}.
\]
By \((2.6)\), it follows that
\[
\frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\text{min}}, 1) = 2 - \frac{8\hat{\nu}}{(\lambda_1 - \hat{\nu})(k+2)} = \frac{2(k+2)\lambda_1 - (k+6)\hat{\nu}}{(k+2)(\lambda_1 - \hat{\nu})}.
\]
Hence if for some \( k \), we have
\[
(k + 2)\lambda_1 > (k + 6)\hat{\nu} ,
\]
we deduce that the corresponding minimum \( \alpha_{\text{min}} \) is non-degenerate.

Using the data given in Table 1, one easily verifies that the condition (2.9) is satisfied for \( k = 1, \ldots, 7 \). Due to the accuracy of these numerical computations, one can consider that (1.4) is safely controlled for \( k = 1, \ldots, 7 \).

When \( k \) is odd, the potential of the Sturm-Liouville operator \( Q(\alpha, 1) \) is even and, by well-known properties of Sturm-Liouville operators, any eigenfunction associated with the \( m \)-th eigenvalue \( \lambda_m(\alpha) \) of \( Q(\alpha, 1) \) is even if \( m \) is even and odd if \( m \) is odd.

In particular, \( u_0^0 \) is even for any \( \alpha \), and, therefore, \( \partial u_0^0 / \partial \alpha \) is even as well. This shows that one can replace \( \lambda_1 \) by \( \lambda_2 \) in the above arguments, stating that if, for some odd \( k \), we have
\[
(k + 2)\lambda_2 > (k + 6)\hat{\nu} ,
\]
the corresponding minimum \( \alpha_{\text{min}} \) is non-degenerate.

3. Estimates for the bottom

In this section, we will prove Theorem 1.2. So we assume that the zero set of the magnetic field \( B \) is a smooth oriented hypersurface \( S \), and the magnetic field satisfies the estimate (1.2) in a neighborhood \( U \) of \( S \) with some \( k \in \mathbb{Z}, k > 0 \).

Let \( G \) be the Riemannian metric on \( S \) induced by \( g \). Denote by \( dx_G \) the corresponding Riemannian volume form on \( S \). Let \( \omega_0 = i^*_S A \) be the closed one form on \( S \) induced by \( A \), where \( i_S \) is the embedding of \( S \) to \( M \).

For any \( t \in \mathbb{R} \), let \( P^h_S \left( \omega_0, 0 + \frac{1}{k + 1} t^{k+1} \omega_0, 0 \right) \) be a formally self-adjoint operator in \( L^2(S, dx_G) \) defined by
\[
P^h_S \left( \omega_0, 0 + \frac{1}{k + 1} t^{k+1} \omega_0, 0 \right) = \left( ih d + \omega_0, 0 + \frac{1}{k + 1} t^{k+1} \omega_0, 0 \right)^* \times \left( ih d + \omega_0, 0 + \frac{1}{k + 1} t^{k+1} \omega_0, 0 \right) .
\]

Consider the self-adjoint operator \( H^{h,0} \) in \( L^2(\mathbb{R} \times S, dt \, dx_G) \) defined by the formula
\[
H^{h,0} = -h^2 \frac{\partial^2}{\partial t^2} + P^h_S \left( \omega_0, 0 + \frac{1}{k + 1} t^{k+1} \omega_0, 1 \right) .
\]
By Theorem 2.7 of [10], the operator \( H^{h,0} \) has discrete spectrum.

The quadratic form associated with the operator \( H^{h,0} \) is given by
\[
q^h[u] := \int_{-\infty}^{+\infty} \int_S \left[ h^2 \left| \frac{\partial u}{\partial t} \right|^2 + \left| (ih d + \omega_0, 0 + \frac{1}{k + 1} t^{k+1} \omega_0, 1) u \right|_{g_0}^2 \right] \, dt \, dx_G ,
\]
\( u \in C_c^\infty(\mathbb{R} \times S) \).

Without loss of generality, we can assume that \( U \) coincides with an open tubular neighborhood of \( S \) and choose a diffeomorphism
\[
\Theta : I \times S \to U ,
\]
where $I$ is an open interval $(-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$ small enough, such that $\Theta \big|_{\{0\} \times S} = \text{id}$ and
\[
(\Theta g - \tilde{g}_0) \big|_{\{0\} \times S} = 0,
\]
where $\tilde{g}_0$ is a Riemannian metric on $I \times S$ given by
\[
\tilde{g}_0 = dt^2 + G.
\]
By adding to $A$ the exact one form $d\phi$, where $\phi$ is the function satisfying
\[
N(x)\phi(x) = -\langle N, A \rangle(x), \quad x \in U,
\]
\[
\phi(x) = 0, \quad x \in S,
\]
we may assume that
\[
\langle N, A \rangle(x) = 0, \quad x \in U.
\]
Denote by $H^h_D$ the unbounded self-adjoint operator in the Hilbert space $L^2(D)$ defined by the operator $H^h$ in the domain $D = \overline{\Omega}$ with the Dirichlet boundary conditions. The operator $H^h_D$ is generated by the quadratic form
\[
u \mapsto q^h_D[u] := \int_D \big| (i h + A) u \big|^2 dx
\]
with the domain
\[
\text{Dom}(q^h_D) = \{ u \in L^2(D) : (i h + A) u \in L^2 \Omega^1(D), u \big|_{\partial D} = 0 \},
\]
where $L^2 \Omega^1(D)$ denotes the Hilbert space of $L^2$ differential 1-forms on $D$ and $dx$ is the Riemannian volume form on $D$. Denote by $\lambda_0(H^h_D)$ the bottom of the spectrum of the operator $H^h_D$. By Theorem 2.1 in [7], there exist $C, c, h_0 > 0$ such that for any $h \in (0, h_0]$ we have
\[
|\lambda_0(H^h) - \lambda_0(H^h_D)| < Ce^{-c/\sqrt{h}}.
\]
It can be seen from the proof of Theorem 2.7 in [10] that, if there exist $h_0 > 0$, a family of functions $w^h \in C^\infty_c(I \times S)$, $h \in (0, h_0]$, and a function $\lambda^0(h)$ defined on $h \in (0, h_0]$ such that
\[
\lambda^0(h) \leq \tilde{C} h^{(2k+2)/(k+2)}, \quad h \in (0, h_0],
\]
and
\[
\| (H^h, 0 - \lambda^0(h)) w^h \| \leq C_1 h^{(2k+3)/(k+2)} \| w^h \|, \quad h \in (0, h_0],
\]
with some positive constants $\tilde{C}$ and $C_1$, then there exist $h_1 \leq h_0$ and a positive constant $C_2$ such that
\[
\| (H^h_D - \lambda^0(h)) v^h \| \leq C_2 h^{(2k+3)/(k+2)} \| v^h \|, \quad \forall h \in [0, h_1],
\]
where $v^h = (\Theta^{-1})^* w^h \in C_c^\infty(U)$. So, in order to complete the proof, it suffices to prove that there exist $h_0 > 0$ and positive constants $C_1$ and $C_2$ such that, for all $h \in [0, h_0]$,
\[
\hat{\nu} \omega_{\min}(B) h^{\frac{k+2}{k+3}} - C_1 h^{\frac{6k+8}{k+3}} \leq \lambda_0(H^h, 0) \leq \hat{\nu} \omega_{\min}(B) h^{\frac{k+2}{k+3}} + C_2 h^{\frac{6k+8}{k+3}}.
\]
The desired upper bound for $\lambda_0(H^h, 0)$ follows from the construction of approximate eigenfunctions of $H^h, 0$ given in [3]. It remains to prove the lower bound for $\lambda_0(H^h, 0)$.

We will localize the problem in two scales.
First, we choose a covering of $S$ by local coordinate charts, $S = \bigcup_{m=1}^{d} U_m$. Let $\chi_m \in C^\infty(S)$ be a partition of unity subordinate to this covering so that $\text{supp} \chi_m \subset U_m$ for each $m$ and
\[
\sum_{m=1}^{d} \chi_m^2(x) = 1, \quad x \in S.
\]
There exists a $C^\infty$ real valued function $\varphi_m$ such that on $U_m$
\[
\omega_{0,0} = d\varphi_m.
\]
Therefore, for $v \in C^\infty_c(U_m)$, we obtain
\[
q^h[v] = \int_{-\infty}^{+\infty} \int_{U_m} \left[ h^2 \frac{\partial v}{\partial t} \right]^2 dt dx + \left| \left( i h d + \frac{1}{k+1} t^{k+1} \omega_{0,1} \right) \exp \left( \frac{i}{h} \varphi_m \right) v \right|^2_{G} dt dx G.
\]
Secondly, for a fixed $m = 1, 2, \ldots, d$, by scaling a standard partition of unity in $\mathbb{R}^{n-1}$, we can find a partition of unity satisfying
\[
\sum_j |\chi_{m,j}^h(x)|^2 = 1, \quad x \in U_m.
\]
and
\[
\text{supp} \chi_{m,j}^h \subset Q_{m,j} := \{ x \in \mathbb{R}^{n-1} : |x - y_{m,j}| \leq \varepsilon_0 h^\beta \},
\]
where $\beta > 0$ is a parameter which will be determined later.

Choose an arbitrary $z_{m,j} \in Q_{m,j}$. Let $g_{m,j} = G(z_{m,j})$ be a flat Euclidean metric in $\mathbb{R}^{n-1}$. For any $\ell$ and $p$, we have
\[
|G^p(x) - g_{m,j}^p(x)| = O(h^\beta) \text{ as } h \to 0, \quad x \in Q_{m,j}.
\]
We also have a similar estimate for $\omega_{0,1}$:
\[
|\omega_{0,1}(x) - \omega_{0,1}(z_{m,j})| = O(h^\beta) \text{ as } h \to 0, \quad x \in Q_{m,j}.
\]
Consider the self-adjoint operator $H_{m,j}^{h,0}$ in $L^2(\mathbb{R} \times Q_{m,j}, dt dx_{z_{m,j}})$ defined by the formula
\[
H_{m,j}^{h,0} = -h^2 \frac{\partial^2}{\partial t^2} + \left( i h d + \frac{1}{k+1} t^{k+1} \omega_{0,1}(z_{m,j}) \right)^* \left( i h d + \frac{1}{k+1} t^{k+1} \omega_{0,1}(z_{m,j}) \right).
\]
The quadratic form associated with $H_{m,j}^{h,0}$ is given by
\[
q_{m,j}^h[w] := \int_{-\infty}^{+\infty} \int_{Q_{m,j}} \left[ h^2 \frac{\partial w}{\partial t} \right]^2 dt dx + \left| \left( i h d + \frac{1}{k+1} t^{k+1} \omega_{0,1}(z_{m,j}) \right) w \right|^2_{g_{m,j}} |G(z_{m,j})|^{1/2} dt dx,
\]
\[w \in C^\infty_c(\mathbb{R} \times Q_{m,j}).\]
Lemma 3.1. For any \( w \in C_c^\infty(\mathbb{R} \times Q_{m,j}) \), we have
\[
q_{m,j}^h[w] = \tilde{\nu} |\omega_{0,1}(z_{m,j})|^{\frac{2}{k+2}} h^{\frac{2k+2}{k+2}} \|w\|^2.
\]

Proof. Using the simple scaling \( t = h^{1/(k+2)} \tau \), we can write
\[
q_{m,j}^h[w] = h^{\frac{2k+2}{k+2}} \int_{-\infty}^{\infty} \int_{Q_{m,j}} \left( \frac{\partial w}{\partial \tau} \right)^2 \left( (ih \frac{k+2}{k+1} \omega_{0,1}(z_{m,j})) w \right)^2 \frac{d\tau}{|\omega_{0,1}(z_{m,j})|} + h^{\frac{2k+2}{k+2}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \left( \frac{\partial w}{\partial \tau} \right)^2 \left( (ih \frac{k+2}{k+1} \omega_{0,1}(z_{m,j})) w \right)^2 \frac{d\tau d\eta}{|\omega_{0,1}(z_{m,j})|}.
\]

There exists a \( g_{m,j} \)-orthonormal frame \( \{X_1, X_2, \ldots, X_{n-1}\} \) in \( \mathbb{R}^{n-1} \) such that
\[
\omega_{0,1}(z_{m,j})(X_1) = |\omega_{0,1}(z_{m,j})|, \quad \omega_{0,1}(z_{m,j})(X_l) = 0, \quad l \geq 2.
\]

Then, by a linear change of variables in \( \mathbb{R}^{n-1} \), we obtain
\[
q_{m,j}^h[w] = h^{\frac{2k+2}{k+2}} \int_{-\infty}^{\infty} \int_{Q_{m,j}} \left( \frac{\partial w}{\partial \tau} \right)^2 \left( (ih \frac{k+2}{k+1} \omega_{0,1}(z_{m,j})) w \right)^2 \frac{d\tau}{|\omega_{0,1}(z_{m,j})|} + h^{\frac{2k+2}{k+2}} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \left( \frac{\partial w}{\partial \tau} \right)^2 \left( (ih \frac{k+2}{k+1} \omega_{0,1}(z_{m,j})) w \right)^2 \frac{d\tau d\eta}{|\omega_{0,1}(z_{m,j})|}.
\]

Denote by \( \hat{w}(\eta_1, \eta_2, \ldots, \eta_{n-1}) \) the partial Fourier transform of \( w \) in the \( y_1 \)-variable. We have
\[
\int_{\mathbb{R}^n} \left( \frac{\partial w}{\partial \tau} \right)^2 + (\frac{ih}{k+1} \omega_{0,1}(z_{m,j}) - h \frac{k+2}{k+1} \eta_1)^2 \frac{|\hat{w}|^2}{|\omega_{0,1}(z_{m,j})|} d\tau d\eta_1 d\eta_2 \ldots d\eta_{n-1}.
\]

For any fixed \( \eta_1, \eta_2, \ldots, \eta_{n-1} \), the expression
\[
\int_{-\infty}^{\infty} \left( \frac{\partial \hat{w}}{\partial \tau} \right)^2 + \left( \frac{ih}{k+1} \omega_{0,1}(z_{m,j}) - h \frac{k+2}{k+1} \eta_1 \right)^2 \frac{|\hat{w}|^2}{|\omega_{0,1}(z_{m,j})|} d\tau
\]
is the quadratic form of the operator \( Q(\alpha, \beta) \) with
\[
\alpha = h \frac{k+2}{k+1} \eta_1, \quad \beta = |\omega_{0,1}(z_{m,j})|
\]
evaluated on \( \hat{w} \). Therefore, we have
\[
\int_{-\infty}^{\infty} \left( \frac{\partial \hat{w}}{\partial \tau} \right)^2 + \left( \frac{ih}{k+1} \omega_{0,1}(z_{m,j}) - h \frac{k+2}{k+1} \eta_1 \right)^2 \frac{|\hat{w}|^2}{|\omega_{0,1}(z_{m,j})|} d\tau \geq \tilde{\nu} |\omega_{0,1}(z_{m,j})|^{\frac{2}{k+2}} h^{\frac{2k+2}{k+2}} \int_{-\infty}^{\infty} |\hat{w}|^2 d\tau,
\]
that immediately completes the proof. \( \square \)
On the other hand, using the inequality

$$2|ab| \leq \varepsilon^2 a^2 + \varepsilon^{-2} b^2,$$

with $\varepsilon = h^\rho$ (here $\rho > 0$ is a parameter which will be determined later) and estimates \((3.3)\) and \((3.4)\), we get, for any $w \in C_c^\infty(\mathbb{R} \times Q_{m,j})$,

$$q^h[w] = (1 + O(h^3)) \int_{-\infty}^{+\infty} \int_{Q_{m,j}} \left[ h^2 \frac{\partial w}{\partial t} \right]^2 \, dt \, dx$$

$$+ \int_{-\infty}^{+\infty} \int_{Q_{m,j}} \left( i h d + \frac{1}{k + 1} t^{k+1} \omega_{0,j} \right) \left( \exp \left( -\frac{i}{h} \varphi_m \right) w \right)^2 \left| G(z_{m,j}) \right|^{1/2} \, dt \, dx$$

$$\geq (1 - h^{2\rho}) (1 + O(h^3)) q^h_{m,j} \left[ \exp \left( -\frac{i}{h} \varphi_m \right) w \right]$$

$$- h^{-2\rho} (1 + O(h^3)) \int_{-\infty}^{+\infty} \int_{Q_{m,j}} \frac{1}{(k + 1)^2} [t]^{2k+2} \times \left| \omega_{0,1}(x) - \omega_{0,1}(z_{m,j}) \right| w(t,x)^2 \left| G(z_{m,j}) \right|^{1/2} \, dt \, dx$$

$$\geq (1 - h^{2\rho}) (1 + O(h^3)) q^h_{m,j} \left[ \exp \left( -\frac{i}{h} \varphi_m \right) w \right]$$

$$- \frac{1}{(k + 1)^2} h^{2\beta-2\rho} (1 + O(h^3)) \int_{-\infty}^{+\infty} \int_{Q_{m,j}} [t]^{2k+2} w(t,x)^2 \left| G(x) \right|^{1/2} \, dt \, dx.$$

Therefore, using the IMS formula, for any $v \in C_c^\infty(\mathbb{R} \times U_m)$, we obtain

$$q^h[v] = \sum_j q^h_{m,j} v - h^2 \sum_j \| \nabla v \|$$

$$\geq \left[ (1 - h^{2\rho}) (1 + O(h^3)) \hat{\nu} \omega_{\min}(B) \frac{2k+2}{k+1} h^{2\beta-2\rho} - Ch^{2-2\beta} \right] \| v \|^2$$

$$- \frac{1}{(k + 1)^2} h^{2\beta-2\rho} (1 + O(h^3)) \| [t]^{k+1} v \|^2,$$

and, furthermore, for any $u \in C_c^\infty(\mathbb{R} \times S)$, we get

$$q^h[u] = \sum_{m=1}^d q^h_{m} u - h^2 \sum_{m=1}^d \| \nabla u \|^2$$

$$\geq \sum_{m=1}^d q^h_{m} u - Ch^2 \sum_{m=1}^d \| u \|^2$$

$$\geq \left[ (1 - h^{2\rho}) (1 + O(h^3)) \hat{\nu} \omega_{\min}(B) \frac{2k+2}{k+1} h^{2\beta-2\rho} - Ch^{2-2\beta} \right] \| u \|^2$$

$$- \frac{1}{(k + 1)^2} h^{2\beta-2\rho} (1 + O(h^3)) \| [t]^{k+1} u \|^2.$$

It is shown in \([10]\) (see Remark 5.2) that, given some constant $\hat{C}$, then there exists, for any $k \in \mathbb{N}$, a constant $C_k$ such that, for $h \in (0, 1]$ and any eigenvalue $\lambda^0(h)$ of $H^{h,0}$ such that

$$\lambda^0(h) \leq \hat{C} h^{(2k+2)/(k+2)},$$
In particular, we have
\[ \|u^{k+1}u^h\| \leq C_k h^{(k+1)/(k+2)}\|u^h\|. \]
Therefore, we obtain, for some new constants \(C_0, C_1, C_2, C,\)
\[ \lambda_0(h) \geq \left( \hat{\nu} \omega_{\min}(B) \frac{2k}{k+2} - C_0 h^2 \beta - C_1 h^{2\beta} - C_2 h^{2\beta-2} \right) \left( \frac{2k+2}{k+2} \right) - C h^{2-2\beta}. \]
Putting \(\beta = \frac{2}{3(k+2)}, \rho = \frac{1}{3(k+2)},\) this achieves the proof of the lower bound.

4. Existence of gaps

In this section we will prove Theorem 1.3.

4.1. A model operator. First, we will only assume that there exists \(x_0 \in M\) such that \(B(x_0) = 0\) and in a neighborhood \(U\) of \(x_0\) the zero set of \(B\) is a smooth oriented hypersurface \(S,\) and the magnetic field satisfies in \(U\) the estimate \((1.2)\) with some \(k \in \mathbb{Z}, k > 0.\) As above, we will assume that \(U\) coincides with an open tubular neighborhood of \(S\) and choose a diffeomorphism \(\Theta : I \times S \to U,\) where \(I\) is an open interval \((-\epsilon_0, \epsilon_0)\) with \(\epsilon_0 > 0\) small enough, such that \(\Theta |_{\{0\} \times S} = \text{id}.\) We will also assume that the conditions \((1.1)\) and \((1.2)\) hold.

Suppose that \(S\) admits a coordinate system with coordinates \((s_1, \ldots, s_{n-1}).\) Then we have a coordinate system in \(U\) with coordinates \(X = (X_0, X_1, \ldots, X_{n-1})\) with \(X_0 = t \in I\) and \(X_j = s_j, j = 1, \ldots, n - 1.\) Thus, \(S\) is given by the equation \(t = 0.\) We have
\[ H^h = \frac{1}{\sqrt{|g(X)|}} \sum_{0 \leq \alpha, \beta \leq n-1} \nabla^h_{\alpha} \left( \sqrt{|g(X)|} g^{\alpha\beta}(X) \nabla^h_{\beta} \right), \]
where
\[ \nabla^h_{\alpha} = i\hbar \frac{\partial}{\partial X^\alpha} + A_\alpha(X), \quad \alpha = 0, 1, \ldots, n - 1. \]
or, equivalently,
\[ (4.1) \quad H^h = \sum_{0 \leq \alpha, \beta \leq n-1} g^{\alpha\beta}(X) \nabla^h_{\alpha} \nabla^h_{\beta} + i\hbar \sum_{0 \leq \alpha \leq n-1} \Gamma^\alpha \nabla^h_{\alpha}, \]
where
\[ \Gamma^\alpha = \frac{1}{\sqrt{|g(X)|}} \sum_{0 \leq \beta \leq n-1} \frac{\partial}{\partial X^\beta} \left( \sqrt{|g(X)|} g^{\beta\alpha}(X) \right), \quad \alpha = 0, 1, \ldots, n - 1. \]
By \((3.1)\) and \((3.2),\) we have
\[ A_0(t, s) = 0, \]
and
\[ g = dt^2 + G + O(t), \quad t \to 0, \]
where \(G\) is the induced metric on \(S.\) So we can write
\[ g_{00}(t, s) = 1 + g_{00}(s) t + O(t^2), \]
\[ g_{0j}(t, s) = g_{0j}(s) t + O(t^2), \]
\[ g_{j\ell}(t, s) = G_{j\ell}(s) + \hat{g}_{j\ell}(s) t + O(t^2). \]
In particular, we have
\[ |g(t, s)| = |G(s)| + O(t), \quad t \to 0. \]
By assumption, we can write
\[ A_j(t, s) = \omega^{(j)}(s) + \frac{1}{k + 1} \omega^{(j)}_0(s) t^{k+1} \]

\[ + \frac{1}{k + 2} \omega^{(j)}_0(s) t^{k+2} + O(t^{k+3}), \quad t \to 0. \]

Suppose that a family \( u^h \in C^\infty_c(U), h \in (0, h_0), \) satisfies the following assumptions.
For any real \( m > 0, \) there exists \( C_m > 0 \) and \( h_m > 0 \) such that, for any \( h \in (0, h_m), \) we have
\[
\| t^m u^h \| \leq C_m h^{\frac{m}{k+2}} \| u^h \|, \tag{4.2}
\]
\[
\| t^m \nabla^2_j u^h \| + h \| t^m \frac{\partial u^h}{\partial t} \| \leq C_m h^{\frac{m+k+1}{k+2}} \| u^h \|, \tag{4.3}
\]
and
\[
\| t^m \nabla^2_j \nabla^2_k u^h \| + h \left( \| t^m \frac{\partial}{\partial t} \nabla^2_j u^h \| + \| t^m \nabla^2_j \frac{\partial}{\partial t} u^h \| \right) + h^2 \| t^m \nabla^4 u^h \| \leq C_m h^{\frac{m+2k+2}{k+2}} \| u^h \|. \tag{4.4}
\]

As shown in [10], a family \( u^h \in C^\infty_c(U), h \in (0, h_0), \) such that, for some \( C > 0, \) we have
\[ (H^h u^h, u^h) \leq C h^{\frac{2k+2}{k+2}} \| u^h \|^2, \quad h \in (0, h_0), \]
satisfies the conditions (4.2), (4.3) and (4.4). By (4.5), it follows that
\[
H^h u^h = -h^2 g^{(0)}(s) \frac{\partial^2 u^h}{\partial t^2} + ih \sum_{1 \leq j \leq n-1} g^{(j)}(t, s) \left( \frac{\partial}{\partial t} \nabla^2_j + \nabla^2_j \frac{\partial}{\partial t} \right) u^h
\]
\[
+ \sum_{1 \leq j, k \leq n-1} g^{(j)}(t, s) \nabla^2_j \nabla^2_k u^h - h^2 \Gamma_0 \frac{\partial u^h}{\partial t} - h \sum_{1 \leq j \leq n-1} \Gamma^j \nabla^2_j u^h. \]

Observe that the first and the third terms can only contribute to the terms of order \( h^{\frac{2k+2}{k+2}}. \) More precisely, we have
\[
H^h u^h = -h^2 \frac{\partial^2 u^h}{\partial t^2} + \sum_{1 \leq j, k \leq n-1} G^{(j)}(s) \left( ih \frac{\partial}{\partial s_j} + \omega^{(j)}_0(s) + \frac{1}{k + 1} \omega^{(j)}_0(s) t^{k+1} \right)
\]
\[
\times \left( ih \frac{\partial}{\partial s_j} + \omega^{(j)}_0(s) + \frac{1}{k + 1} \omega^{(j)}_0(s) t^{k+1} \right) u^h + O(h^{\frac{2k+3}{k+2}}).
\]

This fact was stated in [10, Theorem 2.7]. For the proof of Theorem [11,3], we have to improve the remainder \( O(h^{\frac{2k+3}{k+2}}) \) in the last formula. For this purpose we will take into account further terms in the expansion of \( H^h u^h \) in powers of \( h^{\frac{2k+2}{k+2}}. \) This leads us to introduce a new model operator \( H^h_0 \) given by
\[
H^h_0 = -h^2 \frac{\partial^2}{\partial t^2} - h^2 g^{(0)}(s) t \frac{\partial^2}{\partial t^2}
\]
\[
+ 2ih \sum_{1 \leq j \leq n-1} g^{(j)}(s) t \left( ih \frac{\partial}{\partial s_j} + \omega^{(j)}_0(s) + \frac{1}{k + 1} \omega^{(j)}_0(s) t^{k+1} \right) \frac{\partial}{\partial t}
\]
\[
+ ih \sum_{1 \leq j \leq n-1} g^{(j)}(s) \omega^{(j)}_0(s) t^{k+1}
\]
+ \sum_{1 \leq j, \ell \leq n-1} G^{j\ell}(s) \left( \frac{ih}{\partial s_j} + \omega^{(j)}_{0,0}(s) + \frac{1}{k+1} \omega^{(j)}_{0,1}(s) t^{k+1} \right)
+ \frac{1}{k+2} \omega^{(j)}_{0,2}(s) t^{k+2} \left( \frac{ih}{\partial s_\ell} + \omega^{(\ell)}_{0,0}(s) + \frac{1}{k+1} \omega^{(\ell)}_{0,1}(s) t^{k+1} \right) 
+ \sum_{1 \leq j, \ell \leq n-1} g^{j\ell}(s) t \left( \frac{ih}{\partial s_j} + \omega^{(j)}_{0,0}(s) + \frac{1}{k+1} \omega^{(j)}_{0,1}(s) t^{k+1} \right) 
\times \left( \frac{ih}{\partial s_\ell} + \omega^{(\ell)}_{0,0}(s) + \frac{1}{k+1} \omega^{(\ell)}_{0,1}(s) t^{k+1} \right) 
- h^2 \Gamma_0^0(s) \frac{\partial}{\partial t} - h \sum_{1 \leq j \leq n-1} \Gamma_j^0(s) \left( \frac{ih}{\partial s_j} + \omega^{(j)}_{0,0}(s) + \frac{1}{k+1} \omega^{(j)}_{0,1}(s) t^{k+1} \right).

Lemma 4.1. Suppose that a family \( u^h \in C_c^\infty(U) \), \( h \in (0, h_0) \), satisfies the conditions (4.2), (4.3) and (4.4). Then, there exists \( C > 0 \) such that, for any \( h \in (0, h_0) \), we have

\[
\| H^h u^h - H_0^h u^h \| \leq C h^2 \| u^h \|.
\]

4.2. Approximate eigenfunctions: main result. Now we additionally assume that there exists \( x_1 \in S \) such that \(|\omega_{0,1}(x_1)| = \omega_{\min}(B)\), a neighborhood \( V \) of \( x_1 \) in \( S \) and a constant \( C_1 > 0 \) such that, for all \( x \in V \),

\[
C_1 d_S(x, x_1)^2 \leq |\omega_{0,1}(x)| - \omega_{\min}(B) \leq C_1 d_S(x, x_1)^2.
\]

Take normal coordinate system \( f: U(x_1) \subset S \to \mathbb{R}^{n-1} \) on \( S \) defined in a neighborhood \( U(x_1) \) of \( x_1 \), where \( f(U(x_1)) = B(0, r) \) is a ball in \( \mathbb{R}^{n-1} \) centered at the origin and \( f(x_1) = 0 \). As above, we will denote local coordinates by \( s = (s_1, s_2, \ldots, s_{n-1}) \). Note that

\[
\omega_{\min}(B) = \left( \sum_{j=1}^{n-1} |\omega^{(j)}_{0,1}(0)|^2 \right)^{1/2}.
\]

Consider the Euclidean space \( \mathbb{R}^{n-1} \) with coordinates \( (\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) \). Take the unit vector

\[
e_\sigma = \frac{1}{\omega_{\min}(B)} \left( \omega^{(1)}_{0,1}(0), \ldots, \omega^{(n-1)}_{0,1}(0) \right) \in \mathbb{R}^{n-1},
\]

and complete it to an orthonormal base \( (e_1 = e_\sigma, e_2, \ldots, e_{n-1}) \) in \( \mathbb{R}^{n-1} \). Denote by \( \hat{e}_\sigma, \hat{e}_2, \ldots, \hat{e}_{n-1} \) the corresponding first order differential operators with constant coefficients in \( \mathbb{R}^{n-1} \). In particular, we have

\[
\hat{e}_\sigma = \frac{1}{\omega_{\min}(B)} \sum_{j=1}^{n-1} \omega^{(j)}_{0,1}(0) \frac{\partial}{\partial \sigma_j}.
\]

The Laplacian \( \Delta = -\sum \frac{\partial^2}{\partial \sigma_j^2} \) in \( \mathbb{R}^{n-1} \) can be written as

\[
\Delta = \Delta_\sigma + \Delta_{\sigma^\perp},
\]

where

\[
\Delta_\sigma = -e_\sigma^2, \quad \Delta_{\sigma^\perp} = -\sum_{j=2}^{n-1} e_j^2.
\]
Consider a second order differential operator \( K \) in \( \mathbb{R}^{n-1} \) given by

\[
K = \frac{1}{2} \frac{\partial^2 \lambda_0}{\partial \alpha^2} (\alpha_{\min}, 1) \Delta_\omega + \Delta_\omega + \sum_{r,m} \Omega_{rm} \sigma_r \sigma_m + A,
\]

where

\[
\Omega_{rm} = \omega_{\min}(B)^{-2} \left[ \frac{\hat{\omega}}{2(k+2)} \frac{\partial^2 |\omega_0,1|^2}{\partial s_r \partial s_m}(0) + \alpha_{\min}^2 \sum_j \frac{\partial \omega_{0,1}^{(j)}(0)}{\partial s_r} \frac{\partial \omega_{0,1}^{(j)}(0)}{\partial s_m} \right]
\]

and

\[
A = -g^{00}(0) \int \tau \frac{\partial^2 u_0^0}{\partial \tau^2} (\tau) u_0^0(\tau) d\tau + i\omega_{\min}(B)^{-1} \sum_{j=1}^{n-1} \frac{\partial \omega_{0,1}^{(j)}(0)}{\partial \sigma_j} \alpha_{\min}
\]
\[
+ 2\omega_{\min}(B)^{-2} \sum_{j=1}^{n-1} \omega_{0,1}^{(j)}(0) \omega_{0,1}^{(j)}(0) \int \frac{\tau^{k+2}}{k+2} \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) (u_0^0(\tau))^2 d\tau
\]
\[
+ \omega_{\min}(B)^{-2} \sum_{1 \leq j, \ell \leq n-1} g^{j\ell}(0) \omega_{0,1}^{(j)}(0) \omega_{0,1}^{(\ell)}(0)
\]
\[
\times \int \tau \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right)^2 (u_0^0(\tau))^2 d\tau.
\]

It should be noted that we have indeed the operator \( K = K_{\alpha_{\min}} \) attached at any minimum \( \alpha_{\min} \) and we can do the same construction at any \( \alpha_{\min} \).

A construction of approximate eigenfunctions of the operator \( H_0^h \) in \( D = U \) is given in the next theorem.

**Theorem 4.1.** For any critical point \( \alpha_{\min} \) and for any \( \lambda \) in the spectrum of \( K = K_{\alpha_{\min}} \), there exist \( C_1 > 0, h_1 > 0 \), and a family \( U^h \in C^\infty_c(D) \), \( h \in (0, h_1) \), such that, for any \( h \in (0, h_1) \), we have

\[
\| (H_0^h - z(h)) U^h \|_{L^2} \leq C_1 h^{\frac{k+7}{k+4}} \| U^h \|_{L^2},
\]

with

\[
z(h) = \hat{\omega}_{\min}(B)^{\frac{k+7}{k+4}} h^{\frac{k+4}{k+7}} + \lambda h^{\frac{k+1}{k+4}}.
\]

**Proof.** The proof of this theorem is long and will be divided into several steps. \( \square \)

### 4.3. Formal expansions near the minimum.

Choose a function \( \phi \in C^\infty(B(0, r)) \) such that \( d\phi = \omega_{0,0} \). For some \( \alpha \in \mathbb{R} \), we make a gauge transformation

\[
u(t,s) = \exp \left( -i \frac{\phi(s)}{h} \right) \exp \left( i \frac{\alpha}{\omega_{\min}(B)} \sum_{j=1}^{n-1} \frac{\omega_{0,1}^{(j)}(0)s_j}{h^{k+7}} \right) v(t,s),
\]

\[
s \in B(0, r), \quad t \in \mathbb{R}.
\]

Then we have

\[
H_0^h u(t, s) = \exp \left( -i \frac{\phi(s)}{h} \right) \exp \left( i \frac{\alpha}{\omega_{\min}(B)} \sum_{j=1}^{n-1} \frac{\omega_{0,1}^{(j)}(0)s_j}{h^{k+7}} \right) P^h v(t, s),
\]

\[
(4.6)
\]
where

$$P^h = P_1^h + P_2^h + P_3^h + P_4^h + P_5^h + P_6^h,$$

and

$$P_1^h = -\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 \hat{g}^{00}(s) t \frac{\partial^2}{\partial t^2},$$

$$P_2^h = 2i\hbar \sum_{1 \leq j \leq n-1} \hat{g}^{ij}(s) t \times \left( \frac{i\hbar}{k} \frac{\partial}{\partial s} + \frac{1}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) \right) \frac{\partial}{\partial t},$$

$$P_3^h = i\hbar \sum_{1 \leq j \leq n-1} \hat{g}^{ij}(s) \omega_{0,1}(s) t^{k+1},$$

$$P_4^h = \sum_{1 \leq j, \ell \leq n-1} G^{ij}(s) \left( \frac{i\hbar}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) - \frac{k+1}{k+\omega_{0,1}(s)} - \frac{1}{k+2} \omega_{0,2}(s) t^{k+2} \right),$$

$$P_5^h = \sum_{1 \leq j, \ell \leq n-1} \hat{g}^{ij}(s) \left( \frac{i\hbar}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) \right) \left( \frac{i\hbar}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) \right),$$

$$P_6^h = -\hbar^2 \Gamma_0^{ij}(s) \frac{\partial}{\partial \ell} - \hbar \sum_{1 \leq j \leq n-1} \Gamma_0^{ij}(s) \left( \frac{i\hbar}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) \right).$$

We now make the change of variables

$$t = \omega_{\min}(B)^{-\frac{1}{k+2}} h^{1/(k+2)} \tau, \quad s = h^{1/2(k+2)} \sigma,$$

and expand the operators in powers of $h$ as $h \to 0$ up to the terms of order $O(h^{(4k+7)/(2(k+2))}).$ For the first three terms, we obtain

$$\tilde{P}_1^h = -\hbar^{\frac{4k+7}{k+2}} \omega_{\min}(B)^{-\frac{1}{k+2}} \frac{\partial^2}{\partial \tau^2} - \hbar^{\frac{4k+7}{k+2}} \omega_{\min}(B)^{-\frac{1}{k+2}} \hat{g}^{00}(s) \tau \frac{\partial^2}{\partial \tau^2} + O(h^{\frac{4k+7}{k+2}}),$$

$$\tilde{P}_2^h = 2i\hbar^{\frac{4k+7}{k+2}} \sum_{1 \leq j \leq n-1} \hat{g}^{ij}(0) \omega_{\min}(B)^{-\frac{1}{k+2}} \omega_{0,1}(0) \tau \left( \frac{\tau^{k+1}}{k+1} - \alpha \right) \frac{\partial}{\partial \tau} + O(h^{\frac{4k+7}{k+2}}),$$

$$\tilde{P}_3^h = i\hbar^{\frac{4k+7}{k+2}} \sum_{1 \leq j \leq n-1} \hat{g}^{ij}(0) \omega_{0,1}(0) \omega_{\min}(B)^{-\frac{1}{k+2}} \tau^{k+1} + O(h^{\frac{4k+7}{k+2}}).$$

For the analysis of the fourth term, let us start with the computation of

$$I = \left( \frac{i\hbar}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) \right) \left( \frac{i\hbar}{k+1} \omega_{0,1}(s) t^{k+1} - \alpha \omega_{\min}(B) \right),$$

$$+ \frac{1}{k+2} \omega_{0,2}(s) t^{k+2}.$$
\[ \times \left( i\hbar \frac{\partial}{\partial s_t} + \frac{1}{k+1} \omega_0^{(t)}(s) k^{k+1} - \alpha \omega_{\min}(B) - \frac{\hbar^2}{2} \omega_0^{(j)}(0) + \frac{1}{k+2} \omega_0^{(j)}(s) k^{k+2} \right). \]

After the change of variables, we obtain

\[
\hat{I} = \omega_{\min}(B) \frac{2 k^{k+2}}{k^{k+2}} \left( \frac{\tau^{k+1}}{k+1} \omega_0^{(j)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) 
\times \left( \frac{\tau^{k+1}}{k+1} \omega_0^{(t)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) 
+ i \omega_{\min}(B) \frac{\tau^{k+1}}{k+1} \omega_0^{(j)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) \frac{\partial}{\partial \sigma_j} (k+1) \omega_0^{(j)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) \frac{\partial}{\partial \sigma_t} 
+ \frac{\hbar^2}{2} \left[ - \frac{\partial^2}{\partial \sigma_j \partial \sigma_t} + \omega_{\min}(B) \frac{2 k^{k+2}}{k+2} \right] 
\times \left( \left( \frac{\tau^{k+1}}{k+1} \omega_0^{(j)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) \omega_{0,1}^{(j)}(h^{1/2(k+2)} \sigma) 
+ \left( \frac{\tau^{k+1}}{k+1} \omega_0^{(t)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) \omega_{0,1}^{(t)}(h^{1/2(k+2)} \sigma) \right). \]

We can write

\[ \omega_{0,1}^{(j)}(h^{1/2(k+2)} \sigma) = \omega_0^{(j)}(0) + \hbar \frac{1}{2\pi^2} \sum \frac{\partial \omega_0^{(j)}(0)}{\partial s_r}(0) \sigma_r + \frac{1}{2} \hbar \frac{1}{2\pi^2} \sum \frac{\partial^2 \omega_0^{(j)}(0)}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m + O(\hbar^2), \quad \hbar \to 0. \]

Since \( s = 0 \) is a minimum of the function

\[ |\omega_{0,1}(s)|^2 = \sum_{j,l} G^{jl}(s) \omega_0^{(j)}(s) \omega_0^{(l)}(s), \]

we have

\[ \left( \frac{\partial}{\partial s_r} |\omega_{0,1}|^2 \right)(0) = 2 \sum_{j} \frac{\partial \omega_0^{(j)}(0)}{\partial s_r}(0) \omega_0^{(j)}(0) = 0. \]

Using (4.8), we obtain that

\[
\left( \frac{\tau^{k+1}}{k+1} \omega_0^{(j)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right) \left( \frac{\tau^{k+1}}{k+1} \omega_0^{(t)}(h^{1/2(k+2)} \sigma) - \alpha \omega_{0,1}(0) \right).
\]
Thus, we get the following expression for \( \hat{I} \):

\[
\hat{I} = \omega_{0,1}(0) \omega_{0,1}(0) \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)^2 + h \frac{1}{2(\pi s)^2} \left( \omega_{0,1}(0) \sum \frac{\partial \omega_{0,1}^{(j)}}{\partial s_r}(0) \sigma_r \right)
\]

\[
+ \omega_{0,1}(0) \sum \frac{\partial \omega_{0,1}^{(j)}}{\partial s_r}(0) \sigma_r \frac{\tau^{k+1}}{k+1} \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)
\]+

\[
+ h \frac{1}{2} \omega_{0,1}(0) \sum \frac{\partial^2 \omega_{0,1}^{(j)}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m
\]

\[
+ \omega_{0,1}(0) \sum \frac{\partial^2 \omega_{0,1}^{(j)}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m \frac{\tau^{k+1}}{k+1} \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)
\]+

\[
+ \sum \frac{\partial \omega_{0,1}^{(j)}}{\partial s_r}(0) \frac{\partial \omega_{0,1}^{(j)}}{\partial s_m}(0) \sigma_r \sigma_m \frac{\tau^{2k+2}}{(k+1)^2} + O(h \frac{1}{2(\pi s)^2}).
\]

Similarly, we have

\[
\frac{\partial}{\partial \sigma_j} \left( \frac{\tau^{k+1}}{k+1} \omega_{0,1}(h^{1/2(k+2)} \sigma) - \omega_{0,1}(0) \right)
\]

\[
+ \left( \frac{\tau^{k+1}}{k+1} \omega_{0,1}(h^{1/2(k+2)} \sigma) - \omega_{0,1}(0) \right) \frac{\partial}{\partial \sigma_j} \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)
\]

\[
= \left[ \omega_{0,1}(0) \frac{\partial}{\partial \sigma_j} + \omega_{0,1}(0) \frac{\partial}{\partial \sigma_j} \left( \frac{\tau^{k+1}}{k+1} - \alpha \right) \right]
\]

\[
+ h \frac{1}{2(k+2)} \left[ \frac{\partial \omega_{0,1}^{(j)}}{\partial \sigma_j}(0) + \sum \frac{\partial \omega_{0,1}^{(j)}}{\partial \sigma_j}(0) \sigma_r \right] \frac{\tau^{k+1}}{k+1} + O(h \frac{1}{2(k+2)}).
\]

Thus, we get the following expression for \( \hat{I} \):

\[
\hat{I} = \omega_{\min}(B) \frac{2k+2}{k+1} h \frac{2k+2}{k+1} \omega_{0,1}(0) \omega_{0,1}(0) \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)^2
\]

\[
+ h \frac{1+k^2}{k+1} \left[ i \omega_{\min}(B) \frac{1}{2} \omega_{0,1}(0) \frac{\partial}{\partial \sigma_j} + \omega_{0,1}(0) \frac{\partial}{\partial \sigma_j} \right] \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)
\]

\[
+ \omega_{0,1}(0) \sum \frac{\partial \omega_{0,1}^{(j)}}{\partial s_r}(0) \sigma_r \frac{\tau^{k+1}}{k+1} \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)
\]+

\[
+ \omega_{0,1}(0) \sum \frac{\partial^2 \omega_{0,1}^{(j)}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m \frac{\tau^{k+1}}{k+1} \left( \frac{\tau^{k+1}}{k+1} - \alpha \right)
\]+

\[
+ \omega_{0,1}(0) \sum \frac{\partial^2 \omega_{0,1}^{(j)}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m \frac{\tau^{2k+2}}{(k+1)^2} + O(h \frac{1}{2(k+2)}).
\]
we obtain that, after the change of variables (4.7), the fourth term takes the form:

$$G^{j\ell}(h^{1/2(k+2)}\sigma) = \delta^{j\ell} + \frac{1}{h} \frac{\partial^2 G^{j\ell}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m + O(h^{2(k+2)}), \quad h \to 0,$$

we obtain that, after the change of variables (4.7), the fourth term takes the form:

$$\hat{P}_4^h = \omega_{\min}(B) \frac{2k+2}{k+1} h^{\frac{2k+2}{k+1}} \left( \frac{x^{k+1}}{k+1} - \alpha \right)^2$$

$$+ \frac{h^{\frac{2k+2}{k+1}}}{\omega_{\min}(B)} \frac{2k+2}{k+1} \left( \frac{x^{k+1}}{k+1} - \alpha \right) \sum_{r,m} \frac{\partial \omega_{0,1}^{(j)}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m \frac{\tau^{k+1}}{k+1} \left( \frac{x^{k+1}}{k+1} - \alpha \right)$$

$$+ \frac{h^{\frac{2k+2}{k+1}}}{\omega_{\min}(B)} \sum_{r,m} \left( \frac{\partial^2 \omega_{0,1}^{(j)}}{\partial s_r \partial s_m}(0) \sigma_r \sigma_m \frac{\tau^{2k+2}}{(k+1)^2} \left( \frac{x^{k+1}}{k+1} - \alpha \right)^2 \right)$$

$$+ O(h^{\frac{2k+2}{k+1}}).$$

Finally, after the change of variables (4.7), the fifth and sixth terms become:

$$\hat{P}_5^h = \omega_{\min}(B) h^{\frac{2k+2}{k+1}} \sum_{1 \leq j, \ell \leq n-1} \frac{\partial G^{j\ell}}{\partial s_r \partial s_m}(0) \omega_{0,1}^{(j)}(0) \omega_{0,1}^{(\ell)}(0) \sigma_r \sigma_m \frac{\tau^{k+1}}{k+1} - \alpha \right)^2$$

$$+ O(h^{\frac{2k+2}{k+1}}),$$

$$\hat{P}_6^h = -\Gamma_0^0(0) \omega_{\min}(B) h^{\frac{2k+2}{k+1}} \frac{\partial}{\partial \tau}$$

$$- \sum_{1 \leq \ell \leq n-1} h^{\frac{2k+2}{k+1}} \frac{\partial G^{j\ell}}{\partial s_r \partial s_m}(0) \omega_{0,1}^{(j)}(0) \left( \frac{x^{k+1}}{k+1} - \alpha \right) + O(h^{\frac{2k+2}{k+1}}).$$

Thus, after the change of variables (4.7), the operator $\hat{P}^h$ has a formal asymptotic expansion

$$\hat{P}^h = \omega_{\min}(B) \frac{2k+2}{k+1} h^{\frac{2k+2}{k+1}} \sum_{\ell=0}^\infty h^{\frac{1}{(k+1)}} \hat{P}_\ell,$$
where

\[
\hat{P}_0 = - \frac{\partial^2}{\partial \tau^2} + \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right)^2 = Q(\alpha, 1),
\]

\[
\hat{P}_1 = 2i \omega_{\min}(B)^{-\frac{k+3}{k+2}} \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right) \sum_j \frac{\omega_{0,1}^{(j)}}{\partial \sigma_j},
\]

and

\[
\hat{P}_2 = - \hat{g}^{00}(0) \omega_{\min}(B)^{-\frac{k+3}{k+2}} \frac{\partial^2}{\partial \tau^2}
\]

\[
+ 2i \sum_{1 \leq j \leq n-1} \hat{g}^{0j}(0) \omega_{\min}(B)^{-\frac{k+3}{k+2}} \omega_{0,1}^{(j)}(0) \tau \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right) \frac{\partial}{\partial \tau}
\]

\[
+ i \sum_{1 \leq j \leq n-1} \hat{g}^{0j}(0) \omega_{0,1}^{(j)}(0) \omega_{\min}(B)^{-\frac{k+3}{k+2}} \tau^{k+1}
\]

\[
- \omega_{\min}(B)^{-\frac{k+3}{k+2}} \sum_j \frac{\partial^2}{\partial \sigma_j^2} + i \omega_{\min}(B)^{-\frac{k+3}{k+2}} \sum_j \frac{\partial \omega_{0,1}^{(j)}}{\partial \sigma_j}(0) \frac{\tau^{k+1}}{k + 1}
\]

\[
+ 2 \omega_{\min}(B)^{-\frac{2k+5}{k+2}} \sum_j \omega_{0,1}^{(j)}(0) \omega_{0,1}^{(j)}(0) \tau^{k+2} \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right)
\]

\[
+ \omega_{\min}(B)^{-2} \left[ \sum_{r,m} \left( \sum_j \frac{\partial \omega_{0,1}^{(j)}}{\partial \sigma_r}(0) \frac{\partial \omega_{0,1}^{(j)}}{\partial \sigma_m}(0) \right) \sigma_r \sigma_m \tau^{k+2} \frac{(k + 1)^2}{(k + 1)^2}
\]

\[
+ \frac{1}{2} \sum_{r,m} \left( \sum_{j,\ell} \frac{\partial^2 G^{\ell}}{\partial \sigma_r \partial \sigma_m}(0) \omega_{0,1}^{(j)}(0) \omega_{0,1}^{(\ell)}(0) \right) \sigma_r \sigma_m \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right)^2 \right]
\]

\[
+ \omega_{\min}(B)^{-\frac{2k+5}{k+2}} \sum_{1 \leq j \leq n-1} \hat{g}^{0j}(0) \omega_{0,1}^{(j)}(0) \omega_{0,1}^{(j)}(0) \tau \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right)^2
\]

\[
- \Gamma_{0}^{0}(0) \omega_{\min}(B)^{-\frac{k+3}{k+2}} \frac{\partial}{\partial \tau}
\]

\[
- \omega_{\min}(B)^{-\frac{k+3}{k+2}} \sum_{1 \leq j \leq n-1} \Gamma_{0}^{0}(0) \omega_{0,1}^{(j)}(0) \left( \frac{\tau^{k+1}}{k + 1} - \alpha \right).
\]

4.4. **Reduction to the zero set.** Now we use the method initiated by Grushin [5] (and references therein) and Sjöstrand [31] in the context of hypoellipticity. We will closely follow the exposition in [41] (see also [4]). We now choose some \( \alpha_{\min} \) and will use the previous construction at

\[
\alpha = \alpha_{\min}.
\]

The starting point is to consider the operator in \( S(\mathbb{R}^n) \times S(\mathbb{R}^{n-1}) \) defined by

\[
P_0 = \begin{pmatrix} P_0 & R_0^- \\ R_0^+ & 0 \end{pmatrix},
\]
where the operator $P_0 : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ is given by
\[
P_0 = -\frac{\partial^2}{\partial \tau^2} + \left(\frac{x^{k+1}}{k+1} - \alpha_{\min}\right)^2 - \tilde{\nu} = Q(\alpha_{\min}, 1) - \tilde{\nu},
\]
the operator $R_0^- : S(\mathbb{R}^{n-1}) \rightarrow S(\mathbb{R}^n)$ is given by
\[
R_0^- \phi(\tau, \sigma) = \phi(\sigma) u^0_{\alpha_{\min}}(\tau), \quad \phi \in S(\mathbb{R}^{n-1}),
\]
and the operator $R_0^+ : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^{n-1})$ is given by
\[
R_0^+ f(\sigma) = \int f(\tau, \sigma) u^0_{\alpha_{\min}}(\tau) d\tau, \quad f \in S(\mathbb{R}^n).
\]
We observe that $\mathcal{P}_0$ considered as an operator in $L^2(\mathbb{R}^n, d\tau d\sigma) \times L^2(\mathbb{R}^{n-1}, d\sigma)$ is formally self-adjoint. In particular $R_0^-$ is the Hilbertian adjoint of $R_0^+$ (considered as an operator from $L^2(\mathbb{R}^n, d\tau d\sigma)$ to $L^2(\mathbb{R}^{n-1}, d\sigma)$).

We also verify that
\[
R_0^+ R_0^- = I_{L^2(\mathbb{R}^{n-1})}, \quad R_0^- R_0^+ = \Pi_0,
\]
where $\Pi_0 : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is the orthogonal projection on the subspace $\{\Re u^0_{\alpha_{\min}}\} \times L^2(\mathbb{R}^{n-1})$ in $L^2(\mathbb{R}^n)$:
\[
\Pi_0 f(\tau, \sigma) = \left(\int f(\tau, \sigma) u^0_{\alpha_{\min}}(\tau) d\tau\right) u^0_{\alpha_{\min}}(\tau), \quad f \in L^2(\mathbb{R}^n).
\]

Define the operator $E_0$ in $S(\mathbb{R}^n)$ by
\[
(4.14) \quad E_0 = (I - \Pi_0) P_0^{-1} (I - \Pi_0),
\]
where, by abuse of notation, we consider $P_0$ as an operator in $L^2(\mathbb{R}^n)$. As shown in Lemma A.5, $E_0$ respects the Schwartz space $S(\mathbb{R}^n)$. Then we have
\[
\mathcal{P}_0 \circ \mathcal{E}_0 = I,
\]
where the operator $\mathcal{E}_0$ in $S(\mathbb{R}^n) \times S(\mathbb{R}^{n-1})$ is given by the matrix
\[
\mathcal{E}_0 = \begin{pmatrix}
E_0 & R_0^- \\
R_0^+ & 0
\end{pmatrix}.
\]

The idea is to consider the more general operator $\mathcal{P}(z)$ in $S(\mathbb{R}^n) \times S(\mathbb{R}^{n-1})$ defined by
\[
(4.14) \quad \mathcal{P}(z) = \begin{pmatrix}
\omega_{\min}(B)^{-\frac{1}{2}} h^{-\frac{2k+2}{2k+2}} (\hat{P} h - z) & R_0^- \\
R_0^+ & 0
\end{pmatrix}.
\]

Note that $\mathcal{P}(z)$ is for $z \in \mathbb{R}$ formally self-adjoint for the original $L^2$-scalar product in $L^2(\mathbb{R}^n)$ but not for the usual $L^2$ associated to the standard Lebesgue measure $d\tau d\sigma$.

We are looking for the right inverse of $\mathcal{P}(z)$ for any small $z \in \mathbb{C}$. One can write
\[
\mathcal{P}(z) = \begin{pmatrix}
P_0 + \delta P - Z & R_0^- \\
R_0^+ & 0
\end{pmatrix},
\]
where
\[
\delta P = \omega_{\min}(B)^{-\frac{1}{2k+2}} \hat{P} h^{-\frac{2k+2}{2k+2}} (\hat{P} h - P_0)
= -\tilde{\nu} + h \frac{2k+2}{2k+2} \hat{P}_1 + h \frac{2k+2}{2k+2} \hat{P}_2 + h \frac{2k+2}{2k+2} Q(h),
\]
\[
Z = \omega_{\min}(B)^{-\frac{1}{2k+2}} h^{-\frac{2k+2}{2k+2}} z - \tilde{\nu},
\]
and \( Q(h) \) admits a complete expansion
\[
Q(h) \sim \sum_{j=0}^{\infty} h^{\frac{j+3}{2(k+2)}} \hat{P}_{j+3}.
\]

We first observe that
\[
P(z)E_0 = I + \mathcal{K},
\]
where
\[
\mathcal{K} = \begin{pmatrix}
(\delta P - Z)E_0 & (\delta P - Z)R_0^+

0 & 0
\end{pmatrix}.
\]

We will assume that \( Z \) is a function of \( h \), \( Z = Z(h) \), which admits a formal asymptotic expansion of the form
\[
(4.17) \quad Z(h) \sim \sum_{\ell \geq 1} Z_\ell h^{\frac{\ell}{2(k+2)}}.
\]

Then we have
\[
(\delta P - Z) \sim \sum_{\ell \geq 1} (\hat{P}_\ell - Z_\ell) h^{\frac{\ell}{2(k+2)}}.
\]

If we define
\[
Q \sim \sum_{j=0}^{+\infty} (-1)^j \mathcal{K}^j,
\]
then the operator is well-defined (after reordering) as a formal expansion in powers of \( h^{\frac{1}{2(k+2)}} \) and
\[
P(z)E_0Q \sim I.
\]

So \( E(z) = E_0Q \) is the right inverse of \( P(z) \). If we write
\[
E(z) = \begin{pmatrix}
E(z) & E^+(z) \\
E^-(z) & E^\pm(z)
\end{pmatrix},
\]
we get, in the sense of formal expansions in powers of \( h^{\frac{1}{2(k+2)}} \),
\[
(4.18) \quad \omega_{\min}(B)^{-\frac{2}{k+2}} h^{-\frac{2k+2}{k+2}} (\hat{P}^h - z)E(z) + R_0^+ E^-(z) \sim I,
\]
\[
(4.19) \quad \omega_{\min}(B)^{-\frac{2}{k+2}} h^{-\frac{2k+2}{k+2}} (\hat{P}^h - z)E^+(z) + R_0^- E^\pm(z) \sim 0.
\]
\[
(4.20) \quad R_0^+ E(z) \sim 0.
\]
\[
(4.21) \quad R_0^- E^\pm(z) \sim I.
\]

Let us introduce a function \( \hat{E}^\pm(Z) \) by
\[
\hat{E}^\pm(Z) = E^\pm(z),
\]
with \( Z \) related to \( z \) by \( 4.16 \). We have
\[
\mathcal{K}^j = \begin{pmatrix}
[\delta P - Z]E_0 & [\delta P - Z]E_0 \cdot [\delta P - Z]R_0^+ & [\delta P - Z]R_0

0 & 0
\end{pmatrix},
\]
and therefore, \( \hat{E}^\pm(Z) \) is the following asymptotic series,
\[
(4.22) \quad \hat{E}^\pm(Z) \sim \sum_{j=1}^{+\infty} (-1)^j R_0^+ [([\delta P - Z]E_0)^j - 1][([\delta P - Z]R_0^-)].
\]
Lemma 4.2. The function $\hat{E}^\pm(Z(h))$ admits the following formal asymptotic expansion in powers of $h^{-1/2(k+2)}$:

$$
\hat{E}^\pm(Z(h)) = \sum_{j=1}^{\infty} E^{\pm}_j h^{j/2(k+2)},
$$

with

\begin{align}
E^{\pm}_1 & = Z_1, \\
E^{\pm}_2 & = Z_2 - \omega_{\text{min}}(B)^{-1/2(k+2)} K.
\end{align}

Proof. It follows from (4.22) that the coefficient of $h^{-1/2(k+2)}$ is given by

$$
E^{\pm}_1 = Z_1 - R_0^+ \hat{P}_1 R_0^-.
$$

Using the definitions of the operators $R_0^+$, $\hat{P}_1$ and $R_0^-$ and (2.6), we get

$$
E^{\pm}_1 = Z_1 - 2i\omega_{\text{min}}(B) \frac{ \tau^{k+1} }{ k+2 } \left( \frac{ \tau^{k+1} }{ k+1 } - \alpha_{\text{min}} \right) \left| u_{0,1}(0) \right|^2 \sum_j \frac{ \partial \omega_j(0) }{ \partial \sigma_j } + R_0^{-} P_0 \hat{P}_1 R_0^-.
$$

By (4.22), the coefficient of $h^{-1/2(k+2)}$ is given by the operator

$$
E^{\pm}_2 = Z_2 - R_0^+ \hat{P}_2 R_0^- + R_0^+ \hat{P}_1 E_0 \hat{P}_1 R_0^-.
$$

By (4.13), it follows that

$$
R_0^+ \hat{P}_2 R_0^- = \omega_{\text{min}}(B)^{-1/2(k+2)} \sum_j \frac{ \partial^2 \omega_j(0) }{ \partial \sigma_j^2 } + \sum_{r,m} b_{rm} \sigma_r \sigma_m + a,
$$

where $b_{rm}$ and $a$ are given by

$$
b_{rm} = -\omega_{\text{min}}(B)^{-2} \left[ R_0^+ \frac{ \tau^{k+1} }{ k+1 } \left( \frac{ \tau^{k+1} }{ k+1 } - \alpha_{\text{min}} \right) \right] \left( \sum_j \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_j } \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_m } \right) + R_0^+ \frac{ \tau^{2k+2} }{ (k+1)^2 } \left( \sum_j \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_m } \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_m } \right).
$$

By (4.13), it follows that

$$
b_{rm} = 2 \left( \frac{ \tau^{k+1} }{ k+1 } - \alpha_{\text{min}} \right) \left[ R_0^+ \frac{ \partial G_{j}(0) }{ \partial \sigma_s } \frac{ \partial G_{j}(0) }{ \partial \sigma_m } \right],
$$

where $b_{rm}$ and $a$ are given by

$$
b_{rm} = -\omega_{\text{min}}(B)^{-2} \left[ R_0^+ \frac{ \tau^{k+1} }{ k+1 } \left( \frac{ \tau^{k+1} }{ k+1 } - \alpha_{\text{min}} \right) \right] \left( \sum_j \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_j } \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_m } \right) + R_0^+ \frac{ \tau^{2k+2} }{ (k+1)^2 } \left( \sum_j \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_m } \frac{ \partial \omega_{0,1}(j)(0) }{ \partial \sigma_m } \right) + \frac{1}{2} R_0^+ \left( \frac{ \tau^{k+1} }{ k+1 } - \alpha_{\text{min}} \right) \left( \sum_j \frac{ \partial^2 G_{j}(0) }{ \partial \sigma_s \partial \sigma_m } \left( \omega_{0,1}(j)(0) \omega_{0,1}(j)(0) \right) \right].
$$
and

\begin{equation}
(4.26)
a = -\dot{g}^{00}(0)\omega_{\min}(B) \int_{\tau} \frac{\partial^2 u_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau \\
+ 2i \sum_{1 \leq j \leq n-1} \dot{g}^{0j}(0) \omega_{0}\omega_{\min}(B) \int_{\tau} \frac{\tau^{k+1}}{k+1} - \alpha_{\min}(\tau) \frac{\partial^2 u_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau \\
+ i \sum_{1 \leq j \leq n-1} g^{0j}(0) \omega_{0,1}(\tau) \omega_{\min}(B) \int_{\tau} \frac{\tau^{k+1}(\omega_{\min}(\tau))^2} d\tau \\
+ \omega_{\min}(B) \int_{\tau} \frac{\partial_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau \\
- \omega_{\min}(B) \int_{\tau} \frac{\partial^2 u_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau.
\end{equation}

By (2.6) and (2.8), it follows that

\begin{align*}
\dot{g}^{00}(0)\omega_{\min}(B) & = \hat{\nu} \int_{\tau} \frac{\tau^{k+1}}{k+1} - \alpha_{\min}(\tau) \frac{\partial^2 u_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau \\
& = \int_{\tau} \frac{\tau^{k+1}}{k+1} - \alpha_{\min}(\tau) \frac{\partial^2 u_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau = \hat{\nu} \frac{\nu}{k+2},
\end{align*}

and

\begin{align*}
R_0^+ \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) \frac{\tau^{k+1}}{k+1} R_0^- \\
& = \int_{\tau} \frac{\tau^{k+1}}{k+1} - \alpha_{\min}(\tau) \frac{\partial^2 u_0^0}{\partial \tau^2}(\tau) u_0^0(\tau) d\tau = \hat{\nu} \frac{\nu}{k+2} + \alpha_{\min}^2.
\end{align*}

Using the fact that $G_{j\ell}(0) = \delta_{j\ell}$ and $\partial G_{j\ell}/\partial s_r(0) = 0$ for any $j$, $\ell$ and $r$, the formula (4.25) implies that, for any $r$ and $m$

\begin{equation}
\frac{\partial^2 \omega_{0,1}^2}{\partial s_r \partial s_m}(0) = \sum_{j,\ell} \frac{\partial^2 G_{j\ell}(0) \omega_{0,1}(0) \omega_{0,1}(0)}{\partial s_r \partial s_m} + 2 \sum_{j} \frac{\partial \omega_{0,1}^0}{\partial s_r}(0) \frac{\partial \omega_{0,1}^0}{\partial s_m}(0) + 2 \sum_{j} \frac{\partial \omega_{0,1}^0}{\partial s_r}(0) \frac{\partial \omega_{0,1}^0}{\partial s_m}(0).
\end{equation}

From the above formulae, it follows that

\begin{equation}
b_{rm} = -\omega_{\min}(B) \frac{\partial^2 \omega_{\min}}{\partial s_r \partial s_m}.\end{equation}

Using integration by parts and (2.6), we get
\[ 2 \int_{\tau} \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) \frac{\partial u^0_0}{\partial \tau}(\tau) u^0_{\alpha_{\min}}(\tau) d\tau + \int_{\tau} \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) \frac{\partial u^0_0}{\partial \tau}(\tau) u^0_{\alpha_{\min}}(\tau) d\tau = 0, \]

that implies that the sum of the second and the third terms in (4.26) equals zero. It is also easy to see that the last two terms in (4.26) equal zero. Thus, we have

(4.27) \[ a = -\omega_{\min}(B)^{-\frac{2}{k+2}} A. \]

We conclude that

(4.28) \[ R_0^+ \hat{P}_2 R_0^- = -\omega_{\min}(B)^{\frac{2}{k+2}} \Delta - \omega_{\min}(B)^{\frac{2}{k+2}} \sum_{r,m} \Omega_{\tau m, \sigma_r, \sigma_m} - \omega_{\min}(B)^{\frac{2}{k+2}} A. \]

Next, by (4.11) and (4.12), we have

\[ E_0 \hat{P}_1 R_0^- = 2i \omega_{\min}(B)^{-\frac{2}{k+2}} (I - \Pi_0) (Q(\alpha_{\min}, 1) - \hat{\nu}^{-1}) (I - \Pi_0) \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) u^0_{\alpha_{\min}}(\tau) \hat{e}_\omega. \]

By (2.7), it follows that

\[ \Pi_0 \left[ \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) u^0_{\alpha_{\min}}(\tau) \right] = \int \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) (u^0_{\alpha_{\min}}(\tau))^2 d\tau = 0. \]

Therefore, by (2.7), we get

\[ (I - \Pi_0) \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) u^0_{\alpha_{\min}}(\tau) = \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) u^0_{\alpha_{\min}}(\tau) = \frac{1}{2} (Q(\alpha_{\min}, 1) - \hat{\nu}) \frac{\partial u^0_0}{\partial \alpha}. \]

We obtain (note that $\Pi_0 \frac{\partial u^0_0}{\partial \alpha} = 0$)

\[ E_0 \hat{P}_1 R_0^- = i \omega_{\min}(B)^{-\frac{2}{k+2}} \frac{\partial u^0_0}{\partial \alpha} \hat{e}_\omega. \]

Next, we have

\[ R_0^+ \hat{P}_1 E_0 \hat{P}_1 R_0^- = 2\omega_{\min}(B)^{-\frac{2}{k+2}} R_0^+ \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) \frac{\partial u^0_0}{\partial \alpha} \Delta_\omega. \]

By (2.7), it follows that

\[ R_0^+ \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) \frac{\partial u^0_0}{\partial \alpha} = \int \left( \frac{\tau^{k+1}}{k+1} - \alpha_{\min} \right) u^0_{\alpha_{\min}}(\tau) \frac{\partial u^0_0}{\partial \alpha}(\tau) d\tau = \frac{1}{4} \left( 2 - \frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\min}, 1) \right). \]

Therefore, we obtain

(4.29) \[ R_0^+ \hat{P}_1 E_0 \hat{P}_1 R_0^- = \frac{1}{2} \omega_{\min}(B)^{-\frac{2}{k+2}} \left( 2 - \frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\min}, 1) \right) \Delta_\omega. \]

From (4.25), (4.28) and (4.29), we get (4.24), that completes the proof of the lemma. □
4.5. **Construction of approximate eigenfunctions.** In this section, we complete the proof of Theorem 4.1. So suppose that \( \lambda \) is in the spectrum of the operator \( K \). Our considerations depend on whether Conjecture 1.1 is true or false. Since this is unknown at the moment, we consider both possible cases.

First, suppose that Conjecture 1.1 is true, that is, the second derivative \( \partial^2 \lambda_n / \partial \alpha^2 (\alpha_{\min}, 1) \) is positive. Then the operator \( K \) has discrete spectrum. Let \( \phi_0 \in \mathcal{S}(\mathbb{R}^{n-1}) \) be an eigenfunction of \( K \) with the corresponding eigenvalue \( \lambda \). Put in (4.17)

\[
Z_1 = 0, \quad Z_2 = \omega_{\min}(B)^{-\frac{3}{4\alpha + 2}}, \quad Z_\ell = 0 \ (\forall \ell \geq 3).
\]

So we have

\[
Z(h) = \omega_{\min}(B)^{-\frac{3}{4\alpha + 2}} \lambda^\frac{1}{4\alpha + 2}, \quad z(h) = \nu \omega_{\min}(B)^{\frac{3}{4\alpha + 2}} h^{\frac{2k+2}{4\alpha + 2} + \lambda h \frac{2k+3}{4\alpha + 2}}.
\]

By Lemma 4.2 we have

\[
\tilde{E}^{\pm}(Z(h)) \phi_0 = O(h^{\frac{3}{4\alpha + 2}}),
\]

and, by (4.19), we obtain

\[
(\tilde{P}^h - z(h))\tilde{E}^{\pm}(Z(h)) \phi_0 = O(h^{\frac{4k+7}{4\alpha + 2}}),
\]

where the function \( \tilde{E}^{\pm}(Z) \) is given by

\[
\tilde{E}^{\pm}(Z) = E^{\pm}(z),
\]

with \( Z \) related to \( z \) by (4.10). From (4.6), it follows that the function

\[
U^h(t, s) = \chi(t, s) h^{-\frac{n+1}{4(\alpha+2)}} \exp \left( -i \frac{\phi(s)}{h} \right) \exp \left( \frac{\alpha_{\min} \sum_{j=1}^{n-1} \omega_{0,j}^{(j)}(0) s_j}{\omega_{\min}(B)^{\frac{3}{4\alpha + 2}} h^{\frac{1}{4\alpha + 2}}} \right)
\]

\[
\times \tilde{E}^{\pm}(Z(h)) \phi_0(\omega_{\min}(B)^{\frac{1}{4\alpha + 2}} h^{-1/(k+2)} t, h^{-1/2(k+2)} s), \quad s \in B(0, r), \quad t \in \mathbb{R},
\]

where \( \chi \in C_c^\infty(U) \) is a cut-off function, satisfies \( ||U_h|| = 1 + o(1) \) and

\[
(H^h_0 - z(h))U^h = O(h^{\frac{4k+7}{4\alpha + 2}}).
\]

From the above, we see that \( \tilde{E}^{\pm}(Z) \) is the following asymptotic series,

\[
\tilde{E}^{\pm}(Z) \sim R_0^+ + \sum_{j=1}^{+\infty} (-1)^j E_0[(\delta P - Z) E_0]^{j-1}[(\delta P - Z) R_0^+],
\]

and, therefore, it admits an asymptotic expansion in powers of \( h^{\frac{1}{4\alpha + 2}} \):

\[
\tilde{E}^{\pm}(Z(h)) \sim \sum_{\ell=1}^{+\infty} E^{\pm}_{\frac{t}{\frac{1}{4\alpha + 2}}} h^{\frac{\ell}{4\alpha + 2}}.
\]

Using this fact, one can easily see that the function \( U^h \) satisfies the conditions (4.12), (4.3) and (4.4). By Lemma 4.1 it follows that

\[
(H^h - z(h))U^h = O(h^{\frac{4k+7}{4\alpha + 2}}),
\]

that completes the proof in this case.
Now consider the case when Conjecture \[ \text{[11]} \] is false. So suppose that the equality \( \frac{\partial^2}{\partial \alpha^2} (a_{\min,1}) = 0 \) is true. Thus, the operator \( K \) has the form
\[
K = \Delta_{\omega} + \sum_{r,m} \Omega_{rm} \sigma_r \sigma_m + A.
\]

**Lemma 4.3.** There exists \( w_0^h \in \mathcal{S}(\mathbb{R}^{n-1}) \), \( \| w_0^h \| = 1 \), which satisfies the following conditions: there exists \( C > 0 \) such that, for any \( h > 0 \), we have
\[
\| (K - \lambda) w_0^h \| \leq C h^{\frac{1}{2k+2}},
\]
and for any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \), there exists a constant \( C_\alpha > 0 \) such that, for any \( h > 0 \), we have
\[
\| \partial_\alpha w_0^h \| \leq C_\alpha h^{\frac{|\alpha|}{2k+2}}.
\]

**Proof.** Take an arbitrary vector \( e_\omega', \in \mathbb{R}^{n-1} \), which is orthogonal to the vectors \( e_2, \ldots, e_{n-1} \) with respect to the positive definite bilinear form \( \Omega \) in \( \mathbb{R}^{n-1} \) given by
\[
\Omega(\sigma, \sigma') = \sum_{r,m} \Omega_{rm} \sigma_r \sigma_m'.
\]
Consider the linear coordinate system in \( \mathbb{R}^{n-1} \) with coordinates \( (\rho_1, \rho_2, \ldots, \rho_{n-1}) \) defined by the base \( (e_\omega', e_2, \ldots, e_{n-1}) \). In these coordinates, the operator \( K \) has the form
\[
K = -\sum_{j=2}^{n-1} \frac{\partial^2}{\partial \rho_j^2} + \Omega_{11}' \rho_1^2 + \sum_{r,m=2}^{n-2} \Omega_{rm} \rho_r \rho_m,
\]
where \( \Omega_{11}' > 0 \) and the quadratic form \( \sum_{r,m=2}^{n-2} \Omega_{rm} \rho_r \rho_m \) is positive definite.

With respect to the decomposition \( L^2(\mathbb{R}^{n-1}) = L^2(\mathbb{R}_{\rho_1}) \otimes L^2(\mathbb{R}_{\rho_2, \ldots, \rho_{n-1}}) \), the operator \( K \) can be written as \( K = V \otimes I + I \otimes K_0 \), where \( V \) is the multiplication operator in \( L^2(\mathbb{R}_{\rho_1}) \) by the function
\[
V(\rho_1) = \Omega_{11}' \rho_1^2, \quad \rho_1 \in \mathbb{R},
\]
and \( K_0 \) is the harmonic oscillator in \( L^2(\mathbb{R}_{\rho_2, \ldots, \rho_{n-1}}) \) given by
\[
K_0 = -\sum_{j=2}^{n-1} \frac{\partial^2}{\partial \rho_j^2} + \sum_{r,m=2}^{n-2} \Omega_{rm} \rho_r \rho_m.
\]
Therefore (see, for instance, \[ \text{[29]} \) Theorem VIII.33)), the spectrum of \( K \) equals
\[
\sigma(K) = \sigma(V) + \sigma(K_0) = [\lambda_0(K_0), +\infty),
\]
where \( \lambda_0(K_0) = \inf \sigma(K_0) \).

Denote by \( u_0 \in \mathcal{S}(\mathbb{R}^{n-2}) \), \( \| u_0 \| = 1 \), the eigenfunction of \( K_0 \) associated with \( \lambda_0(K_0) \). Define the function \( w_0^h \in \mathcal{S}(\mathbb{R}^{n-1}) \) by the formula
\[
(3.31) \quad w_0^h(\rho) = ch^{-1/2(k+2)} e^{-(\rho_1 - a)^2/(2h^{2(k+2)})} u_0(\rho_2, \ldots, \rho_{n-1}), \quad \rho \in \mathbb{R}^{n-1},
\]
where the constant \( c > 0 \) is chosen in such a way that \( \| w_0^h \| = 1 \) and \( a \) satisfies the condition
\[
\lambda = \lambda_0(K_0) + \Omega_{11}' a^2.
\]
Then
\[
K w_0^h(\rho) = \lambda_0(K_0) w_0^h(\rho).
\]
From (4.6), it follows that the function
\[ z \rightarrow \frac{h}{h^{1/2(k+2)}} \int (\beta_0^2 - \alpha^2)^2 e^{-(\beta_0 - \alpha)^2/(2h^{2/(k+2)})} \, d\rho \]
and, therefore, we have
\[ \| Kw_0^h - \omega_0^h \| = c \Omega_{11} h^{-1/2(k+2)} \left( \int (\beta_0^2 - \alpha^2)^2 e^{-(\beta_0 - \alpha)^2/(2h^{2/(k+2)})} \, d\rho \right)^{1/2} \leq \frac{C h^{1/2(k+2)}}{}.
\]
The estimates (4.30) follow immediately from the explicit formula (4.31) for \( w_0^h \).

Take \( w_0^h \) as in Lemma 4.3 and
\[ w_1^h = -(\hat{P}_0 - \hat{\nu})^{-1}(I - \Pi_0)(\hat{P}_2 R_0^- - \hat{P}_1 E_0 \hat{P}_1 R_0^-)w_0^h. \]
The operator \(-(\hat{P}_0 - \hat{\nu})^{-1}(I - \Pi_0)(\hat{P}_2 - \hat{P}_1 E_0 \hat{P}_1)\) has the form of a second order differential operator in \( \sigma \), whose coefficients are bounded operators in \( L^2(\mathbb{R}, dr) \). Using this fact, it can be easily checked that, for any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \), there exists a constant \( C_\alpha > 0 \) such that, for any \( h > 0 \), we have
\[ \| \partial_\sigma^\alpha w_1^h \| \leq C_\alpha h^{-\frac{|\alpha|}{2(k+2)}}. \]

Put
\[ v^h(\tau, \sigma) = (R_0^h - h^{\frac{2(k+3)}{k+2}} E_0 \hat{P}_1 R_0^-)w_0^h(\tau, \sigma) + h^{\frac{3}{k+2}} R_0^- w_1^h(\tau, \sigma) = u_{\omega_{\min}}^0(\tau) \omega_0^h(\sigma) - i h^{\frac{2(k+3)}{k+2}} \omega_{\min}(B) \frac{\partial u_0^h}{\partial \sigma}(\tau) \varphi_0(\omega_0^h(\sigma)) + h^{\frac{3}{k+2}} w_1^h(\tau, \sigma). \]
Then, with \( z(h) = \hat{\nu} \omega_{\min}(B) h^{\frac{2(k+3)}{k+2}} + \lambda h^{\frac{2(k+3)}{k+2}} \), we have
\[ (\hat{P}^h - z(h)) v^h = \omega_{\min}(B) h^{\frac{2(k+3)}{k+2}} \left[ (\hat{P}_0 - \hat{\nu}) R_0^- w_0^h + h^{\frac{3}{k+2}} \left( \hat{P}_1 R_0^- w_0^h - (\hat{P}_0 - \hat{\nu}) E_0 \hat{P}_1 R_0^- w_0^h \right) + h^{\frac{3}{k+2}} \left( (\hat{P}_2 - \omega_{\min}(B))^{-\frac{3}{k+2}} \lambda R_0^- - \hat{P}_1 E_0 \hat{P}_1 R_0^- \right) w_0^h + (\hat{P}_0 - \hat{\nu}) w_1^h \right] + O(h^{\frac{4(k+7)}{2(k+2)}}) = \omega_{\min}(B) h^{\frac{2(k+3)}{k+2}} \left[ \Pi_0 \left( (\hat{P}_2 - \omega_{\min}(B))^{-\frac{3}{k+2}} \lambda R_0^- - \hat{P}_1 E_0 \hat{P}_1 R_0^- \right) w_0^h + O(h^{\frac{4(k+7)}{2(k+2)}}) \right] = h^{\frac{2(k+3)}{k+2}} R_0^- (K - \lambda) w_0^h + O(h^{\frac{4(k+7)}{2(k+2)}}), \]
Therefore, we obtain
\[ \| (\hat{P}^h - z(h)) v^h \| = O(h^{\frac{4(k+7)}{2(k+2)}}). \]
From (4.6), it follows that the function
\[ U^h(t, s) = \chi(t, s) h^{-\frac{3}{k+2}} \exp \left( -i \frac{\phi(s)}{h} \right) \exp \left( \frac{\omega_{\min}^{(n)}}{h^{\frac{k+3}{k+2}}} \right) \]
\[ \times v^h(\omega_{\min}(B) h^{\frac{1}{k+2}} t, h^{-\frac{1}{2(k+2)}} s), \quad s \in B(0, r), \quad t \in \mathbb{R}, \]

where $\chi$ is a cut-off function, satisfies $\|U_h\| = 1 + o(1)$ and
\[
(H^h_z - z(h))U^h = O(h^{\frac{4k+7}{2m+2}}).
\]

Using (4.30) and (4.32), one can easily verify that the function $U^h$ satisfies the conditions (4.2), (4.3) and (4.4). By Lemma 4.1, it follows that
\[
(H^h - z(h))U^h = O(h^{\frac{4k+7}{2m+2}}),
\]
that completes the proof of Theorem 4.1.

If Conjecture 1.1 is true, following the arguments of [4], one can prove the following refined version of Theorem 4.1 (cf. [4, Theorem 3.1]): for any $\lambda$ in the spectrum of $K$, there exist a sequence $\{\zeta_j\}_{j=0}^{\infty} \subset \mathbb{R}$ and a sequence of functions $\{\phi_j\}_{j=0}^{\infty}$ in $C^\infty_c(D)$ such that, for any $N > 0$, there exists $M > 0$ such that, if
\[
z_M(h) = \nu \omega_{\text{min}}(B) \frac{2k}{k+2} h^{\frac{2k+3}{k+2}} + \lambda h^{\frac{2k+3}{k+2}} + h^{\frac{4k+7}{2m+2}} \sum_{j=0}^{M} h^{\frac{4k+j}{2m+j}} \zeta_j,
\]
and
\[
\phi_M^h(x) = \sum_{j=0}^{M} h^{\frac{j}{k+2}} \phi_j(x)
\]
then
\[
\| (H^h - z_M(h)) \phi_M^h \|_{L^2} \leq O(h^N) \| \phi_M^h \|_{L^2}, \quad h \to 0.
\]

Moreover, if, in addition, Conjecture 1.2 is true and a miniwell $x_1 \in S$ is unique, one can show, following the lines of [4], that the $m$-th eigenvalue $\lambda_m(H^h)$ of the operator $H^h$ admits the asymptotic expansion
\[
\lambda_m(H^h) = \nu \omega_{\text{min}}(B) \frac{2k}{k+2} h^{\frac{2k+3}{k+2}} + \lambda_m h^{\frac{2k+3}{k+2}} + h^{\frac{4k+7}{2m+2}} \sum_{j=0}^{\infty} h^{\frac{4k+j}{2m+j}} \zeta_j,
\]
where $\lambda_m$ is $m$-th eigenvalue of the operator $K$.

4.6. **Proof of Theorem 1.3** To complete the proof of Theorem 1.3, we will use a general result on the existence of gaps in the spectrum of the magnetic Schrödinger operator $H^h$ on the interval $[0, h(b_0 + \epsilon_0)]$ obtained in [8, Theorem 2.1]. Fix $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\epsilon_1 < \epsilon_2 < \epsilon_0$, and consider the Dirichlet realization $H^h_D$ on the operator $H^h$ in the domain $D = \overline{U_{\epsilon_2}}$. The operator $H^h_D$ has discrete spectrum.

**Theorem 4.2.** Let $N \geq 1$. Suppose that there is a subset $\mu^0_1 < \mu^1_1 < \ldots < \mu^N_N$ of an interval $I(h) \subset [0, h(b_0 + \epsilon_1))$ such that

1. There exist constants $c > 0$ and $M \geq 1$ such that
   
   \[
   \mu_j^h - \mu_{j-1}^h > ch^M, \quad j = 1, \ldots, N,
   \]
   
   \[
   \text{dist}(\mu_0^h, \partial I(h)) > ch^M, \quad \text{dist}(\mu_N^h, \partial I(h)) > ch^M,
   \]
   
   for any $h > 0$ small enough;

2. Each $\lambda_j, j = 0, 1, \ldots, N$, is an approximate eigenvalue of the operator $H^h_D$:
   
   for some $v_j^h \in C^\infty_c(D)$ we have
   
   \[
   \| H^h_D v_j^h - \lambda_j v_j^h \| = \alpha_j(h) \| v_j^h \|,
   \]
   
   where $\alpha_j(h) = o(h^M)$ as $h \to 0$. 


Then the spectrum of $H^h$ on the interval $I(h)$ has at least $N$ gaps for any sufficiently small $h > 0$.

Fix any $N \geq 1$. If $\frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\text{min}}, 1) > 0$, then $K$ has discrete spectrum $\lambda_0 < \lambda_1 < \ldots, \lambda_j \to \infty$ as $j \to \infty$. Take an arbitrary $b_N > \lambda_N$. By Theorem 1.1 for any $j = 0, 1, \ldots, N$, there exist $C_j > 0, h_{0,j} > 0, \phi_j(h) \in C^\infty_c(D)$ and $z_j(h)$ with

$$z_j(h) = \nu \omega_{\text{min}}(B) \frac{2^j}{2^{j+2}} h^{\frac{2k+2}{k+2}} + \lambda_j h^{\frac{2k+3}{k+2}} + O(h^{\frac{4k+7}{5k+7}})$$

such that, for any $h \in (0, h_{0,j})$, we have

$$\| (H^h - z_j(h)) \phi(h) \| \leq C_j h^{\frac{4k+7}{5k+7}} \| \phi(h) \|.$$  

So we apply Theorem 4.2 with

$$I(h) = \left[ \nu \omega_{\text{min}}(B) \frac{2^j}{2^{j+2}} h^{\frac{2k+2}{k+2}} + \lambda_j h^{\frac{2k+3}{k+2}} + b_N h^{\frac{2k+2}{k+2}} \right]$$

and $\mu^h_j = z_j(h), j = 0, 1, \ldots, N$, that completes the proof in this case.

If $\frac{\partial^2 \lambda_0}{\partial \alpha^2}(\alpha_{\text{min}}, 1) = 0$, then the spectrum of $K$ is a semi-axis $[\lambda_0, \infty)$. Taking an arbitrary $b_N > \lambda_0$ and arbitrary $\lambda_1 < \ldots < \lambda_N$ on the interval $(\lambda_0, b_N)$ and proceeding as above, we complete the proof.

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