Symbolic computation of Schur multipliers with an application to the groups of order dividing $p^6$

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Abstract

We describe an algorithm to compute the Schur multipliers of all nilpotent Lie $p$-rings in the family defined by a symbolic nilpotent Lie $p$-ring. Symbolic nilpotent Lie $p$-rings can be used to describe the isomorphism types of $p$-groups of order $p^n$ for $n \leq 7$ and all primes $p \geq n$. We apply our algorithm to compute the Schur multipliers of all $p$-groups of order dividing $p^6$.

1 Introduction

A Lie ring is an additive abelian group with a multiplication, denoted by $\{,\}$, that is bilinear, alternating and satisfies the Jacobi-identity. A Lie $p$-ring is a Lie ring with $p^n$ elements for some prime power $p^n$. A Lie $p$-ring is nilpotent, if its lower central series terminates at $\{0\}$. A nilpotent Lie $p$-ring $L$ of order $p^n$ can be described by a presentation $P(A)$ on $n$ generators $b_1, \ldots, b_n$ with coefficients $A = (a_{ijk}, a_{ik} \mid 1 \leq i < j < k \leq n)$, where $a_{ijk}$ and $a_{ik}$ in the field $\mathbb{Z}_p = \{0, \ldots, p-1\}$, and relations

$$[b_i, b_j] = \sum_{k=j+1}^{n} a_{ijk}b_k \quad \text{for} \quad 1 \leq i < j \leq n,$$

$$pb_i = \sum_{k=i+1}^{n} a_{ik}b_k \quad \text{for} \quad 1 \leq i \leq n.$$

We generalize this type of presentation: we assume that $p$ is an indeterminate and the elements $a_{ijk}$ and $a_{ik}$ are polynomials in a polynomial ring $\mathbb{Z}[w, x_1, \ldots, x_m]$. Then $P(A)$ is a symbolic nilpotent presentation with respect to a set of primes $\Pi$ if for each $p \in \Pi$ and each $x_1, \ldots, x_m \in \mathbb{Z}_p$ the presentation $P(A)$ evaluated at these values and $w$ being a primitive root mod $p$ defines a nilpotent Lie $p$-ring of order $p^n$. In this case we say that $P(A)$ defines a symbolic nilpotent Lie $p$-ring.

The first aim here is to describe an algorithm to compute the Schur multipliers of the nilpotent Lie $p$-rings described by a symbolic nilpotent Lie $p$-ring. This algorithm translates, modifies and generalizes the method to compute the Schur multiplier of a polycyclic group as introduced by Eick & Nickel [2].

Newman, O’Brien & Vaughan-Lee [6, 7] determined a classification up to isomorphism of the nilpotent Lie $p$-rings of order dividing $p^7$. This is available in the LiePRing package [10] of GAP [8] in the form of finitely many symbolic nilpotent presentations for Lie $p$-rings. We apply our algorithm to this classification and hence obtain a complete list of
the Schur multipliers of the nilpotent Lie $p$-rings of order dividing $p^6$. The following table lists for each $p^n$ with $1 \leq n \leq 6$ and all primes $p \geq 5$ the number of isomorphism types of nilpotent Lie $p$-rings of order $p^n$ with Schur multiplier isomorphic to $M$. A similar result for the primes 2 and 3 can be computed readily and is included in Section 7 below. In the following table we use $A := \gcd(p - 1, 3)$, $B := \gcd(p - 1, 4)$ and $C := \gcd(p - 1, 5)$.

| $M$ | $p^1$ | $p^2$ | $p^3$ | $p^4$ | $p^5$ | $p^6$ |
|-----|-------|-------|-------|-------|-------|-------|
| $\{0\}$ | 1 | 1 | 2 | 2 | $p + 4$ | $p + 3$ |
| $\mathbb{Z}_p$ | 1 | 1 | 5 | $p + 2A + B + 14$ | $6p + 4A + B + 18$ |
| $\mathbb{Z}_p^2$ | 1 | 4 | 9 | $3p^2 + 19p + 14A + 7B + 2C + 42$ |
| $\mathbb{Z}_p^3$ | 1 | 1 | 17 | $9p + 6A + 3B + 106$ |
| $\mathbb{Z}_p^4$ | 1 | 4 |  |  | 4 | 4p + 39 |
| $\mathbb{Z}_p^5$ | 4 |  |  |  |  | 52 |
| $\mathbb{Z}_p^6$ | 1 | 2 |  |  |  | 32 |
| $\mathbb{Z}_p^7$ | 1 |  |  |  | 4 |  |
| $\mathbb{Z}_p^8$ |  |  |  |  | 4 |  |
| $\mathbb{Z}_p^9$ |  |  |  |  | 6 |  |
| $\mathbb{Z}_p^{10}$ |  |  |  | 1 |  | 1 |
| $\mathbb{Z}_p^{11}$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p^{15}$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p^2$ | 1 | 3 |  |  |  | 6 |
| $\mathbb{Z}_p + \mathbb{Z}_p^2$ | 1 |  |  |  | 9 |  |
| $\mathbb{Z}_p^2 + \mathbb{Z}_p^2$ | 1 |  |  |  | 8 |  |
| $\mathbb{Z}_p^3 + \mathbb{Z}_p^2$ |  |  |  |  | 6 |  |
| $\mathbb{Z}_p^4 + \mathbb{Z}_p^2$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p^5 + \mathbb{Z}_p^2$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p^2$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p + \mathbb{Z}_p^2$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p^3 + \mathbb{Z}_p^2$ |  |  |  |  | 1 |  |
| $\mathbb{Z}_p^3$ |  |  |  |  | 1 |  |
is also a table of the Schur multipliers of the $p$-groups of order dividing $p^6$ for all primes $p \geq 5$.
The Schur multipliers of the groups of order dividing $p^5$ have been determined in [3] using
ad-hoc methods and in the unpublished work [4] using an algorithmic approach. It was
one motivation for this work to exhibit a better algorithmic method that would also cover
the groups of order $p^6$.

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2 Preliminaries

First, we recall that the multiplication of a Lie ring is

- **alternating:** $[x, x] = 0$ for $x \in L$,
- **antisymmetric:** $[x, y] = -[y, x]$ for $x, y \in L$,
- **bilinear** $[x + y, z] = [x, z] + [y, z]$ and $[x, y + z] = [x, y] + [x, z]$, for $x, y, z \in L$, and
- **Jacobi-identity** $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for $x, y, z \in L$.

The Schur multiplier of a group $G$ can be defined as the second homology group $H_2(G, \mathbb{Z})$.

For a Lie ring $L$, the Schur multiplier is defined as the second Chevalley-Eilenberg homol-
ogy group of $L$. We recall the following properties of Schur multipliers of nilpotent Lie
$p$-rings from [1].

1 Theorem: Let $L$ be a nilpotent Lie $p$-ring and let $L = F/R$ for a free Lie ring $F$.

- $M(L)$ is a finite abelian $p$-group.
- $M(L) \cong ([F, F] \cap R)/[F, R]$.
- $([F, F] \cap R)/[F, R]$ is the torsion subgroup of $R/[F, R]$.

3 The Schur multiplier of a nilpotent Lie $p$-ring

Let $L$ be a nilpotent Lie $p$-ring defined by a presentation $P(A)$. Let $V_n$ denote the vector
space of dimension $n(n + 1)/2$ over $\mathbb{Z}_p$ with basis

$$B_n = \{t_{ij} \mid 1 \leq i < j \leq n\} \cup \{s_i \mid 1 \leq i \leq n\}.$$ 

Define $t_{ii} = 0$ and $t_{ij} = -t_{ji}$ for $j < i$. Further, we introduce the following elements of $V_n$
for $1 \leq i, j, h \leq n$:

- $u_i = \sum_{k=1}^{n} a_{ik} t_{ki}$,
- $v_{ij} = pt_{ij} + \sum_{k=1}^{n} (a_{ijk} s_k - a_{ik} t_{kj})$ for $i \neq j$, and
- $w_{ijh} = \sum_{k=1}^{n} (a_{ijk} t_{ik} + a_{ijk} t_{hk} + a_{ikj} t_{jk})$ for $i < j < h$.

Let $Mat(A)$ denote the matrix whose rows consist of the coefficient vectors of the elements
$u_i, v_{ij}, w_{ijh}$ with respect to the basis $B_n$.

2 Theorem: Let $P(A)$ define a nilpotent Lie $p$-ring of order $p^n$. Then the abelian
invariants of the Schur multiplier $M(P(A))$ coincide with the the non-zero elementary
divisors of $\text{Mat}(A)$.
Proof: Let $F$ be the free Lie ring on $b_1, \ldots, b_n$ and write $P(A) = F/R$ for $R$ generated by the relations of $P(A)$. Our aim is to show that $R/[R, F] \cong \mathbb{Z}^n/(n+1)/2/\mathrm{Im}(Mat(A))$. This yields the desired claim, as $M(P(A))$ is isomorphic to the torsion subgroup of $R/[R, F]$ by Theorem 1 and the non-zero elementary divisors of $Mat(A)$ describe exactly that, see Sims [8, Sec. 3.8].

Recall that $B_n = \{t_{ij}, s_i \mid 1 \leq i < j \leq n\}$ has $n(n+1)/2$ elements. Let $L(A)$ denote the Lie ring generated by $\{b_1, \ldots, b_n\} \cup B_n$ subject to the relations

\[
[b_i, b_j] = \sum_{k=1}^n a_{ijk} b_k + t_{ij} \text{ for } i \neq j,
\]

\[
 pb_i = \sum_{k=1}^n a_{ik} b_k + s_i,
\]

\[
t_{ij} \text{ and } s_i \text{ central}.
\]

Let $\langle B_n \rangle$ denote the subring generated by $B_n$ in $L(A)$. Then similarly to Lemma 1 of [2] it follows that $L(A) \cong F/[R, F]$ and $\langle B_n \rangle \cong R/[R, F]$.

We now determine a presentation for the abelian subgroup $\langle B_n \rangle$ of $L(A)$. Similar as in [2] this can be obtained by evaluating 'consistency relations' in the generators $b_1, \ldots, b_n$. These consistency relations in $L(A)$ are:

- $[pb_i, b_i] = p[b_i, b_i] = 0$ for all $i$.
- $[pb_i, b_j] = p[b_i, b_j]$ for all $i \neq j$.
- $[b_i, [b_j, b_h]] + [b_h, [b_i, b_j]] + [b_j, [b_h, b_i]] = 0$ for all $i < j < h$.

We note that all these relations hold in $P(A)$, since $P(A)$ defines a nilpotent Lie $p$-ring of order $p^n$ and hence is consistent. We evaluate in $L(A)$ for all $i, j, h$ and $i \neq j$:

\[
p[b_i, b_j] = p \left( \sum_{k=1}^n a_{ijk} b_k + t_{ij} \right)
\]

\[
= \sum_{k=1}^n a_{ijk} pb_k + pt_{ij}
\]

\[
= \sum_{k=1}^n a_{ijk} \left( \sum_{l=1}^n a_{kl} b_l + s_k \right) + pt_{ij}
\]

\[
= \sum_{l=1}^n \left( \sum_{k=1}^n a_{ijk} a_{kl} \right) b_l + \sum_{k=1}^n a_{ijk} s_k + pt_{ij}
\]
\[
[pb_i, b_j] = \left[ \sum_{k=1}^{n} a_{ik}b_k + s_i, b_j \right] \\
= \sum_{k=1}^{n} a_{ik} [b_k, b_j] \\
= \sum_{k=1}^{n} a_{ik} \left( \sum_{l=1}^{n} a_{kj}b_l + t_{kj} \right) \\
= \sum_{l=1}^{n} \left( \sum_{k=1}^{n} a_{ik}a_{kjl} \right) b_l + \sum_{k=1}^{n} a_{ik}t_{kj}
\]

\[
[b_i, [b_j, b_h]] = \left[ b_i, \sum_{k=1}^{n} a_{jkh}b_k + t_{jh} \right] \\
= \sum_{k=1}^{n} a_{jkh} [b_i, b_k] \\
= \sum_{k=1}^{n} a_{jkh} \left( \sum_{l=1}^{n} a_{ikl}b_l + t_{ik} \right) \\
= \sum_{l=1}^{n} \left( \sum_{k=1}^{n} a_{jkh}a_{ikl} \right) b_l + \sum_{k=1}^{n} a_{jkh}t_{ik}
\]

Using that the relations hold in \( P(A) \) and using the definitions for \( u_i, v_{ij} \) and \( w_{ijh} \) it now follows that

- \( u_i = [pb_i, b_i] \),
- \( v_{ij} = p[b_i, b_j] - [pb_i, b_j] \), and
- \( w_{ijh} = [b_j, [b_j, b_h]] + [b_j, [b_h, b_i]] + [b_h, [b_i, b_j]] \).

Hence \( u_i = v_{ij} = w_{ijh} = 0 \) in \( L(A) \) and it follows that \( R/[R, F] \cong \mathbb{Z}^{n(n+1)/2}/\text{Im}(\text{Mat}(A)) \) as desired.

### 4 Schur multipliers for symbolic nilpotent Lie \( p \)-rings

Let \( P(A) \) be a symbolic nilpotent Lie \( p \)-ring; that is, the prime \( p \) of \( P(A) \) is indeterminate and the elements \( a_{ijk} \) and \( a_{ij} \) of \( A \) are integral polynomials in the indeterminates \( w, x_1, \ldots, x_m \).

Then \( P(A) \) defines a family of nilpotent Lie \( p \)-rings of order \( p^n \). Our aims are to determine the Schur multipliers for all members of the family simultaneously. The principal approach towards this aim is very similar to the algorithm described in Section 3.

Given \( P(A) \), we first determine the matrix \( \text{Mat}(A) \). This can be read off readily from \( A \) and it is a matrix with entries in \( \mathbb{Z}[p, w, x_1, \ldots, x_m] \). Our aim now reduces to computing the elementary divisors of \( \text{Mat}(A) \) for all possible evaluations of the indeterminates simultaneously.
We recall the computation of elementary divisors here briefly. Elementary row (column) operations of a matrix are:

- Interchanging two rows (columns),
- Adding a multiple of one row (column) to another,
- Multiplying one row (column) by a unit in the underlying ring.

Let $S = \mathbb{Z}[w, x_1, \ldots, x_m]$ and let $Q = \text{Quot}(S)$ the set of rational functions in $w, x_1, \ldots, x_m$ over $\mathbb{Z}$. An element in $Q[p]$ has the form $s_n p^n + \ldots + s_1 p + s_0$ with $s_n, \ldots, s_0 \in Q$. We call such an element a pseudo-unit if $s_0 \neq 0$. Note that every non-zero element in $Q[p]$ can be written as $p^l u$ for a pseudo-unit $u$.

We now perform a Smith normal form computation on $\text{Mat}(A)$ by allowing elements in $Q[p]$ as entries in the matrix and by using pseudo-units in the third case of elementary operations. Whenever we divide by a pseudo-unit in the algorithm, we store the used pseudo-unit and return the resulting list of pseudo-units together with the elementary divisors of $\text{Mat}(A)$. Note that the elementary divisors are all powers of $p$ as a result.

A pseudo-unit $u$ is a rational function over $S = \mathbb{Z}[w, x_1, \ldots, x_m]$. Let $E_u = \{(x_1, \ldots, x_m) \in Z^m_p \mid u(w, x_1, \ldots, x_m) = 0\}$ the (finite) set of zeros of $u$. Then for all elements outside $E_u$ the pseudo-unit $u$ yields a proper unit in $\mathbb{Z}_p$. Hence for all elements outside

$$E = \bigcup_{u \text{ pseudo-unit}} E_u$$

we can determine the Schur multiplier of $P(A)$ from $\text{Mat}(A)$ via Theorem 2. It remains to consider the finite set $E$. For each tuple in $E$ we evaluate the presentation $P(A)$ in the tuple and then determine the Schur multiplier with the method of the previous section in the evaluated presentation.

## 5 Examples

We exhibit two examples. Both refer to the library of Lie $p$-rings of dimension 6 in the LiePRing package of GAP. The first example gives a first introduction into the algorithm. The second example is the most difficult case for our algorithm among all Lie $p$-rings of dimension 6.

### 5.1 First example

As a first example, we consider the parametrised Lie $p$-ring $L$ number 245 of dimension 6. This defines a family of $p-1$ Lie $p$-rings with parameter $x \in \{1, \ldots, p-1\}$.

```gap
gap> L := LiePRingsByLibrary(6)[245];
<LiePRing of dimension 6 over prime p with parameters [ x ]>
gap> NumberOfLiePRingsInFamily(L);
p-1
gap> LibraryConditions(L);
[ "x ne 0", "" ]
```

We now apply the algorithm of Section 4.
The output asserts that $M(F) = \mathbb{Z}_p^2$ for all Lie $p$-rings $F$ in the family defined by the symbolic nilpotent Lie $p$-ring $L$ with the possible exception of $L$ evaluated at the places where $-x^2 - x = 0$. Clearly, $-x^2 - x = 0$ if and only if $x \in \{0, -1\}$. The case $x = 0$ is excluded in the family defined by $L$. Hence it remains to consider the case $x = -1$. To consider this special case, we evaluate the Lie $p$-ring presentation for $L$ at this place and then apply the algorithm of Section 4 again.

In summary, this parametrised Lie $p$-ring yields $p^2$ Lie $p$-rings with Schur multiplier $\mathbb{Z}_p^2$ and 1 Lie $p$-ring with Schur multiplier $\mathbb{Z}_p^3$.

5.2 Second example
The second example is the Lie $p$-ring $L$ with number 267 of dimension 6. This has four parameters $x, y, z$ and $t$ and it defines a family containing $(2p^2 + p - (p - 1, 4) + 1)/2$ Lie $p$-rings.

In the considered case, the parameters $x, y, z, t$ are elements of $\{0, \ldots, p - 1\}$ so that

$$A = \begin{pmatrix} t & x \\ y & z \end{pmatrix}$$

is non-singular modulo $p$. Two such parameter matrices $A$ and $B$ define isomorphic algebras if and only if

$$B = \frac{1}{\det P}PAP^{-1} \mod p$$

for some matrix $P$ of the form

$$\begin{pmatrix} \alpha & \beta \\ e\omega\beta & e\alpha \end{pmatrix}$$

with $e = \pm 1$, $\omega$ is a primitive element modulo $p$ and $\alpha^2 - \omega\beta^2 \neq 0$ modulo $p$. The set of all matrices $P$ forms a subgroup $U$ of $GL(2, p)$ of size $2(p^2 - 1)$. We apply the algorithm of Section 4 to $L$. 
Hence the algorithm determines three pseudo-units in this case. One of them is $t$. As noted by Vaughan-Lee, every orbit of $U$ on the set of parameter matrices contains a matrix with $t \in \{0,1\}$. Hence it is sufficient to consider the cases $t = 0$ and $t = 1$.

We now analyse the occurring sets of pseudo-units. As $A$ is non-singular, the case $t = 0$ implies $x,y \neq 0$. Thus the pseudo-units reduce to $z$ and it remains to consider $(t,z) = (0,0)$ as special case. If $t = 1$, then the pseudo-units are $\{x, (z+1)(xy-z)\}$. As $A$ is non-singular, it follows that $xy \neq z$. Hence it remains to consider the special cases $(t,x) = (1,0)$ and $(t,z) = (1,-1)$.

If $(t,z) = (0,0)$, then $x \neq 0$, and if $(t,x) = (1,0)$, then $z \neq 0$, as $A$ is invertible. The case $(t,x,z) = (1,0,-1)$ is covered by the third case. Hence the remaining pseudo-units can all be considered as proper units.

Note that there are $p-1$ Lie $p$-rings in the family defined by $L$ with $(t,z) = (0,0)$ and $(p+1)/2$ with $(t,z) = (1,-1)$. In summary, $L$ yields $(2p^2 - 2p - (p - 1,4) + 2)/2$ Lie $p$-rings with Schur multiplier $\mathbb{Z}_p^2$ and $(3p-1)/2$ Lie $p$-rings with Schur multiplier $\mathbb{Z}_p^3$. The following graph visualizes the case-distinctions in this example. Cases that do not play a role due to library conditions are indicated by “N/A”.

If $(t,z) = (0,0)$, then $x \neq 0$, and if $(t,x) = (1,0)$, then $z \neq 0$, as $A$ is invertible. The case $(t,x,z) = (1,0,-1)$ is covered by the third case. Hence the remaining pseudo-units can all be considered as proper units.

Note that there are $p-1$ Lie $p$-rings in the family defined by $L$ with $(t,z) = (0,0)$ and $(p+1)/2$ with $(t,z) = (1,-1)$. In summary, $L$ yields $(2p^2 - 2p - (p - 1,4) + 2)/2$ Lie $p$-rings with Schur multiplier $\mathbb{Z}_p^2$ and $(3p-1)/2$ Lie $p$-rings with Schur multiplier $\mathbb{Z}_p^3$. The following graph visualizes the case-distinctions in this example. Cases that do not play a role due to library conditions are indicated by “N/A”.

6 Limitations

Among the Lie $p$-rings of order $p^7$ in the LiePRing package there is one with 13 parameters and the following presentation:

```
gap> ViewPCPresentation(L);
p*l1 = j*l5 + k*l6 + m*l7
p*l2 = n*l5 + r*l6 + s*l7
p*l3 = t*l5 + u*l6 + v*l7
p*l4 = x*l5 + y*l6 + z*l7
[l2,l1] = l5
[l3,l1] = l6
[l3,l2] = l7
[l4,l2] = w*l6
[l4,l3] = l5
```

This Lie $p$-ring is currently beyond the range of our algorithm and it is a main reason why we limited our application to the groups of order dividing $p^6$.

7 The groups of orders dividing $2^6$ and $3^6$

The following table lists the numbers of groups of order $p^n$ with prescribed Schur multiplier for $n \leq 6$ and $p = 2$ and $p = 3$. We only list those orders for which the result differs from the table in the introduction. The information in this table can be computed readily using GAP [9] and we list it for completeness only.
| $M$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | $3^5$ | $3^6$ |
|-----|-------|-------|-------|-------|-------|-------|
| $\{0\}$ | 2 | 4 | 5 | 9 | 5 | 6 |
| $\mathbb{Z}_p$ | 2 | 3 | 12 | 33 | 19 | 44 |
| $\mathbb{Z}_p^2$ | 1 | 3 | 14 | 56 | 10 | 123 |
| $\mathbb{Z}_p^3$ | 2 | 6 | 41 | 14 | 132 |
| $\mathbb{Z}_p^4$ | 2 | 33 | 4 | 55 |
| $\mathbb{Z}_p^5$ | 4 | 33 | 4 | 44 |
| $\mathbb{Z}_p^6$ | 1 | 2 | 7 | 2 | 32 |
| $\mathbb{Z}_p^7$ | 2 | 1 | 4 |
| $\mathbb{Z}_p^8$ | 4 |
| $\mathbb{Z}_p^9$ | 5 | 6 |
| $\mathbb{Z}_p^{10}$ | 1 | 2 | 1 | 1 |
| $\mathbb{Z}_p^{11}$ | 1 |
| $\mathbb{Z}_p^{15}$ | 1 | 1 |
| $\mathbb{Z}_p^2$ | 1 | 1 | 10 | 4 | 15 |
| $\mathbb{Z}_p + \mathbb{Z}_p^2$ | 2 | 14 | 2 | 12 |
| $\mathbb{Z}_p^2 + \mathbb{Z}_p^2$ | 2 | 9 | 1 | 10 |
| $\mathbb{Z}_p^3 + \mathbb{Z}_p^2$ | 3 | 8 |
| $\mathbb{Z}_p^4 + \mathbb{Z}_p^2$ | 3 | 1 |
| $\mathbb{Z}_p^5 + \mathbb{Z}_p^2$ | 2 | 1 |
| $\mathbb{Z}_p^2$ | 1 |
| $\mathbb{Z}_p + \mathbb{Z}_p^2$ | 1 | 1 |
| $\mathbb{Z}_p^3$ | 1 | 1 |
| $\mathbb{Z}_p^3$ | 1 | 1 |
| $\mathbb{Z}_p^2 + \mathbb{Z}_p^2$ | 1 |

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