Fundraising and vote distribution: a non-equilibrium statistical approach

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The number of votes correlates strongly with the money spent in a campaign, but the relation between the two is not straightforward. Among other factors, the output of a ballot depends on the number of candidates, voters, and available resources. Here, we develop a conceptual framework based on Shannon entropy maximization and Superstatistics to establish a relation between the distributions of money spent by candidates and their votes. By establishing such a relation, we provide a tool to predict the outcome of a ballot and to alert for possible misconduct either in the report of fundraising and spending of campaigns or on vote counting. As an example, we consider real data from a proportional election with 6323 candidates, where a detailed data verification is virtually impossible, and show that the number of potential misconducting candidates to audit can be reduced to only nine.

In an effort towards fair electoral processes, regulations and reforms are constantly on the agenda of many countries around the world [1]. To avoid that the decision-making process is dominated by wealth and influence, the most pertinent processes to legislate are arguably fundraising and spending [2]. Different countries have different rules, but in general, candidates and parties are the ones that report on the financial details of their own campaigns, what raises obvious doubts over the veracity of the reported data. As the number of collected votes correlates with the money spent in the campaign [3], establishing a quantitative relation between the distribution of votes and financial resources among the candidates is instrumental to raise flags about possible misconduct.

Within some regulated boundaries, several individuals or institutions can contribute financially to a campaign. The value of the contribution is very subjective, depending on their interests and on the economic and political conjecture [5–8]. Thus, predicting the distribution of funds raised and money spent in a campaign from “first principles” is likely a hopeless endeavor, challenging the verification of the reported data. In sharp contrast, the distribution of votes among candidates is well studied. It is known to differ for proportional and plural elections, and to depend on the country, number of candidates, and money spent in campaigns [9–14]. Different models were developed to explain this distribution [3, 15–20], as well as methodologies to identify vote-counting irregularities [21–26]. Here we propose an approach based on the Shannon entropy maximization and Superstatistics to disclose a relation between the distribution of financial resources declared by candidates and the distribution of their votes in proportional elections.

Given a certain amount of money $m_i$ spent by a candidate $i$ in the campaign, the conditional probability for $i$ to receive $v$ votes is $p(v|m_i)$. Since the money spent is heterogeneously distributed among candidates, the probability $p(v)$ that a candidate receives $v$ votes is given by,

$$p(v) = \sum_{m=0}^{m_{\text{max}}} p(v|m)p(m),$$

where $p(v|m)p(m)$ is the conditional probability of candidate $i$ spending $m$ money. Figure 1 illustrates this concept.

FIG. 1. By employing the principle of maximum entropy under the constraints of a fixed number of voters and candidates, we derive the conditional probability $p(v|m_i)$, that a candidate $i$ receives $v$ votes, provided that $i$ spends an amount of money $m_i$ in the campaign. Since the amount of money spent usually differ from candidate to candidate, the final distribution of votes should depend on the distribution of money spent. A formalism based on Superstatistics is then used to establish a relation between these two distributions.
where \( p(m) \) is the probability that a candidate spends an amount of money \( m \) in the campaign and \( m_{\text{max}} \) is the maximum amount of money that can be spent (see Fig. 1). For simplicity, we have considered that \( m \) is a discrete variable and a multiple of \( \Delta m \), where \( \Delta m \) is the “price of a vote”. Equation (1) is the basis of Superstatistics for non-equilibrium systems [2], a theoretical framework developed to describe the thermal fluctuations of an ensemble of particles at different effective thermostat temperatures and consequently different weights for each configuration. Analogously, in an election, the amount of money spent differs from candidate to candidate and thus also the probability that they receive a certain number of votes. As a consequence, the variable \( m \) is the analogue for elections of the thermostat temperature in a thermal system.

To calculate \( p(v|m) \), let us consider a proportional election with \( N_c \) candidates and \( N_v \) voters. Based on the principle of maximum entropy [27], \( p(v|m) \) should maximize the Shannon entropy,

\[
S = - \sum_{i=1}^{N_c} \sum_{v=v_0}^{m_i/\Delta m} p(v|m_i) \ln[p(v|m)] ,
\]

where \( v_0 \) and \( m_i/\Delta m \) are the minimum and maximum number of votes that the candidate \( i \) can receive. For simplicity, hereafter we assume that \( v_0 \) is the same for all candidates. At this point, two constraints need to be imposed, as both the number of candidates \( N_c \) and voters \( N_v \) are fixed (see Fig. 1). In this way, the first constraint is then,

\[
N_c \sum_{i=0}^{m_{\text{max}}/\Delta m} \sum_{v=v_0}^{m_i/\Delta m} p(v|m_i) = N_v , \quad (3)
\]

which ensures the normalization of \( p(v|m) \), while the second one is,

\[
N_c \sum_{i=0}^{m_{\text{max}}/\Delta m} \sum_{v=v_0}^{v_0} v p(v|m_i) = N_v . \quad (4)
\]

By maximizing \( S \) subjected to Eqs. (3) and (4), we obtain

\[
p(v|m) = \frac{1}{Z(m)} e^{-\mu v} , \quad (5)
\]

where \( Z(m) \) is a normalization factor that depends on \( m \) and it is the analogue of the partition function in a thermal system, given by,

\[
Z(m) = \frac{e^{\mu(1-v_0)} - e^{-\mu m/\Delta m}}{e^{\mu} - 1} , \quad (6)
\]
where $\mu$ is the Lagrange multiplier related to the second constraint (Eq. (4)). Since the number of votes is limited, $p(v|m)$ decays exponentially for $v \in [v_0, m/\Delta m]$ and it is zero otherwise.

In order to verify if the distribution predicted by Eq. (5) is compatible with real data, we consider the 2014 election for federal deputies in Brazil, using the dataset available in Ref. 28. Each state has its own ballot, with different candidates and voters. Countrywide, this is an election with 6323 candidates, roughly 140 million voters, and more than 280 million dollars invested in campaigns.

We first analyze the results for the top four populated Brazilian states, namely, São Paulo, Rio de Janeiro, Minas Gerais, and Bahia. These states have each more than 10 million voters and between 239 (Bahia) and 1364 (São Paulo) candidates. For each state, we grouped the candidates by the amount of money that they reported to have spent in their campaigns. Figure 2A shows the average number of votes received by a candidate that spent $m$ money in the campaign is,

$$\langle v \rangle = v_0 + \frac{1}{e^\mu - 1} + \frac{1 - v_0 + m/\Delta m}{1 - e^\mu(1-v_0+m/\Delta m)}. \quad (7)$$

The value of $\mu$ is obtained by imposing the second constraint (Eq. (4)) and considering $\Delta m$ as a free parameter. Figure 2A shows the number of votes per candidate against the money spent in the campaign (gray circles) and the average value for candidates in the same money group (orange circles), where the circles in blue correspond to the outliers. The solid line in Fig. 2A is the non-linear fit of Eq. (7) to the numbers of votes of all candidates as a function of their financial resources, which gives $\Delta m = 2.07$. As shown, the excellent agreement be-

![Figure 3](image-url)

**FIG. 3.** Results for the 2014 election for federal deputies in the state of São Paulo, Brazil, with 1364 candidates, more than 32 million voters, and about 58 million dollars of total investment in the campaigns. (A) Number of votes as a function of the money spent in the campaign for all candidates (gray dots) and the average value within each money group (blue-dark dots). The (red-)solid line is obtained by a least-squares fit of Eq. (7) to the average data, taking $\Delta m$ as a free parameter. The black circles are outliers, which we defined as the candidates with a number of votes that deviate more than $6\sigma$ from the average. (B) Distribution of the number of votes per candidate. The (red) dots were obtained from the data and the solid line was obtained by randomly assigning a number of votes $v$ for each candidate from the distribution given by Eq. (5), where $m$ is the amount of money officially declared to have been spent in the campaign. The obtained curve is remarkably consistent with the empirical data over more than five orders of magnitude.
tween this fit and the averaged data points extends over four orders of magnitude, with deviations found only for candidates with very scarce resources, a fact that can be explained as follows. For simplicity, we have considered that the minimum number of votes \( v_0 \) is the same for all candidates, obtained by assuming that \( v_0 \) equals the average number of votes for candidates who spent less than 1200 dollars [3]. In general, however, every candidate has a different \( v_0 \), depending on several factors such as, his/her party, visibility, and social status.

Through Eq. (5), it is also possible to predict, from the reported amount of money spent by each campaign, the distribution of votes for an election. As an example, let us consider again the election in the state of São Paulo. Figure 3B shows the distribution of the number of votes (red dots) obtained by each candidate. To predict the distribution of votes for this election, we assigned randomly a number of votes to each candidate from a distribution given by Eq. (5), with \( m_0 \) equal to the amount of money spent in the campaign, as declared by the candidate. The solid line in Fig. 3B is the predicted outcome, which is in excellent agreement with the empirical data. Once more, in the proposed framework, \( \Delta m \) is a free parameter that relates to the amount of money spent by a candidate campaign and the maximum number of votes that it can receive. Its value was estimated by fitting Eq. (7), as explained before.

Conclusions. We have shown, using the principle of maximum entropy, that the distribution of votes received by a candidate should follow an exponential distribution parameterized by the amount of money that was spent in her/his campaign. This prediction is consistent with real data from a very large proportional election, with 6323 candidates. Furthermore, as the money spent in a campaign is heterogeneously distributed among candidates, we developed a framework based on superstatis-

ics to establish the relation between the distribution of money spent and of votes. Within this framework, it was possible to predict the outcome of a ballot from the distribution of money spent, and identify potential cases of misconduct either in the report of fundraising and spending or on vote counting.

For several proportional elections, the distribution of votes per candidate is fat tailed [29], what has motivated an enthusiastic discussion about the underlying mechanism [11]. As our theoretical approach shows, for an election, if all candidates spent the same amount of money in their campaigns, the expected distribution of votes would be exponential. So, the fat-tailed distribution is a consequence of an heterogeneous distribution of resources. This is consistent with the reported power-law distribution of money spent by candidates in the same elections [3].

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