Computation of q-Binomial Coefficients with the $P(n, m)$ Integer Partition Function

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Abstract

Using $P(n, m)$, the number of integer partitions of $n$ into exactly $m$ parts, which was the subject of an earlier paper, $P(n, m, p)$, the number of integer partitions of $n$ into exactly $m$ parts with each part at most $p$, can be computed in $O(n^2)$, and the q-binomial coefficient can be computed in $O(n^3)$. Using the definition of the q-binomial coefficient, some properties of the q-binomial coefficient and $P(n, m, p)$ are derived. The q-multinomial coefficient can be computed as a product of q-binomial coefficients. A formula for $Q(n, m, p)$, the number of integer partitions of $n$ into exactly $m$ distinct parts with each part at most $p$, is given. Some formulas for the number of integer partitions with each part between a minimum and a maximum are derived. A computer algebra program is listed implementing these algorithms using the computer algebra program of the earlier paper.

Keywords: q-binomial coefficient, integer partition function.

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1 Definitions and Basic Identities

Let the coefficient of a power series be defined as:

$$[q^n] \sum_{k=0}^{\infty} a_k q^k = a_n$$

(1.1)

Let $P(n)$ be the number of integer partitions of $n$, and let $P(n, m)$ be the number of integer partitions of $n$ into exactly $m$ parts. Let $P(n, m, p)$ be the number of integer partitions of $n$ into exactly $m$ parts with each part at most $p$, and let $P^*(n, m, p)$ be the number of integer partitions of $n$ into at most $m$ parts with each part at most $p$, which is the number of Ferrer diagrams that fit in a $m$ by $p$ rectangle:

$$P^*(n, m, p) = \sum_{k=0}^{m} P(n, k, p)$$

(1.2)

Let the following definition of the q-binomial coefficient, also called the Gaussian polynomial, be given.

**Definition 1.1.** The q-binomial coefficient is defined by [2, 4]:

$$\binom{m+p}{m}_q = \prod_{j=1}^{m} \frac{1-q^{p+j}}{1-q^j}$$

(1.3)
The q-binomial coefficient is the generating function of $P^*(n, m, p)$ \([2]\):

$$P^*(n, m, p) = [q^n] \binom{m + p}{m}_q$$ \hspace{1cm} (1.4)

The q-binomial coefficient is a product of cyclotomic polynomials \([6]\).

2 Properties of q-Binomial Coefficients and $P(n, m, p)$

Some identities of the q-binomial coefficient are proved from its definition, and from these some properties of $P^*(n, m, p)$ and $P(n, m, p)$ are derived.

**Theorem 2.1.**

$$\binom{m + p}{m}_q = \binom{m + p - 1}{m - 1}_q + q^m \binom{m + p - 1}{m}_q$$ \hspace{1cm} (2.1)

**Proof.**

\begin{align*}
\prod_{j=1}^{m} \frac{1 - q^{p+j}}{1 - q^j} &= \prod_{j=1}^{m-1} \frac{1 - q^{p+j}}{1 - q^j} + q^m \prod_{j=1}^{m-1} \frac{1 - q^{p+j-1}}{1 - q^j} \hspace{1cm} (2.2) \\
\prod_{j=1}^{m} (1 - q^{p+j}) &= (1 - q^m) \prod_{j=1}^{m-1} (1 - q^{p+j}) + q^m \prod_{j=0}^{m-1} (1 - q^{p+j}) \hspace{1cm} (2.3) \\
1 &= \frac{1 - q^m}{1 - q^{m+p}} + q^m \frac{1 - q^p}{1 - q^{m+p}} \hspace{1cm} (2.4) \\
1 - q^{m+p} &= 1 - q^m + q^m (1 - q^p) \hspace{1cm} (2.5)
\end{align*}

**Theorem 2.2.**

$$P^*(n, m, p) = P^*(n, m - 1, p) + P^*(n - m, m, p - 1)$$ \hspace{1cm} (2.6)

**Proof.** Using the previous theorem:

$$P^*(n, m, p) = [q^n] \binom{m + p}{m}_q = [q^n] \binom{m + p - 1}{m - 1}_q + [q^{n-m}] \binom{m + p - 1}{m}_q$$ \hspace{1cm} (2.7) \\

$$= P^*(n, m - 1, p) + P^*(n - m, m, p - 1)$$

From this theorem and identity \([1, 2]\) follows:

$$P^*(n, m, p) - P^*(n, m - 1, p) = P(n, m, p) = P^*(n - m, m, p - 1)$$ \hspace{1cm} (2.8)

or equivalently:

$$P^*(n, m, p) = P(n + m, m, p + 1)$$ \hspace{1cm} (2.9)

From this theorem and this identity follows:

$$P(n, m, p) = P(n - 1, m - 1, p) + P(n - m, m, p - 1)$$ \hspace{1cm} (2.10)
Theorem 2.3.

\[
\binom{m+p}{m} q = \binom{m+p-1}{m} q + q^p \binom{m+p-1}{m-1} q \quad (2.11)
\]

Proof.

\[
\prod_{j=1}^{m} \frac{1 - q^{p+j}}{1 - q^j} = \prod_{j=1}^{m} \frac{1 - q^{p+j-1}}{1 - q^j} + q^p \prod_{j=1}^{m-1} \frac{1 - q^{p+j}}{1 - q^j} \quad (2.12)
\]

\[
\prod_{j=1}^{m} (1 - q^{p+j}) = \prod_{j=0}^{m-1} (1 - q^{p+j}) + q^p (1 - q^m) \prod_{j=1}^{m-1} (1 - q^{p+j}) \quad (2.13)
\]

\[
1 = \frac{1 - q^p}{1 - q^{m+p}} + q^p \frac{1 - q^m}{1 - q^{m+p}} \quad (2.14)
\]

\[
1 - q^{m+p} = 1 - q^p + q^p (1 - q^m) \quad (2.15)
\]

\[\square\]

Theorem 2.4.

\[P^*(n, m, p) = P^*(n, m, p - 1) + P^*(n - p, m - 1, p) \quad (2.16)\]

Proof. Using the previous theorem:

\[P^*(n, m, p) = [q^n] \binom{m+p}{m} q = [q^n] \binom{m+p-1}{m} q + [q^{n-p}] \binom{m+p-1}{m-1} q \]

\[= P^*(n, m, p - 1) + P^*(n - p, m - 1, p) \quad (2.17)\]

\[\square\]

Using (2.10):

\[P(n, m, p) = P(n, m, p - 1) + P(n - p, m - 1, p) \quad (2.18)\]

The following theorem is a symmetry identity:

Theorem 2.5.

\[
\binom{m+p}{m} q = \binom{m+p}{p} q \quad (2.19)
\]

Proof.

\[
\prod_{j=1}^{m} \frac{1 - q^{p+j}}{1 - q^j} = \prod_{j=1}^{p} \frac{1 - q^{m+j}}{1 - q^j} \quad (2.20)
\]

Using cross multiplication:

\[
\prod_{j=1}^{p} (1 - q^j) \prod_{j=1}^{m} (1 - q^{p+j}) = \prod_{j=1}^{m} (1 - q^j) \prod_{j=1}^{p} (1 - q^{m+j}) = \prod_{j=1}^{m+p} (1 - q^j) \quad (2.21)
\]

\[\square\]
From this theorem follows:
\[ P^*(n, m, p) = P^*(n, p, m) \] (2.22)
and using (2.9):
\[ P(n, m, p) = P(n - m + p - 1, p - 1, m + 1) \] (2.23)
Using (1.2) and (2.8):
\[ P^*(n, m, p) = \sum_{k=0}^{m} P^*(n - k, k, p - 1) \] (2.24)
Combining this identity with (1.4) and using theorem 2.5:
\[ \sum_{k=0}^{m} q^k \left( \frac{p + k - 1}{p - 1} \right) q \] (2.25)
which is identity (3.3.9) in [2].

Taking \( m = n \) in (1.2) and (2.9) and conjugation of Ferrer diagrams:
\[ P(2n, n, p + 1) = P(n, n, p) \] (2.26)
and taking \( p = n \):
\[ P(n) = P(2n, n) = P(2n, n, n + 1) \] (2.27)
The partitions of \( P(n, m, p) - P(n, m, p - 1) \) have at least one part equal to \( p \), and therefore by conjugation of Ferrer diagrams:
\[ P(n, m, p) - P(n, m, p - 1) = P(n, p, m) - P(n, p, m - 1) \] (2.28)

This identity can also be derived from the other identities.

Using (1.2) and (5.7) in [3]:
\[ \sum_{m=0}^{n} P(n, m, p) = P^*(n, n, p) = [q^n] \frac{1}{\prod_{j=1}^{p} (1 - q^j)} \] (2.29)

Theorem 2.6.
\[ \sum_{m=0}^{n} (-1)^m P(n, m, p) = [q^n] \frac{1}{\prod_{j=1}^{p} (1 + q^j)} \] (2.30)

Proof. Using (2.8) and (1.4):
\[ \sum_{m=0}^{n} (-1)^m P(n, m, p) = \sum_{m=0}^{n} (-1)^m P^*(n - m, m, p - 1) \]
\[ = \sum_{m=0}^{n} (-1)^m [q^{n-m}] \left( \frac{m + p - 1}{m} \right) q = [q^n] \sum_{m=0}^{n} (-1)^m q^m \left( \frac{m + p - 1}{m} \right) q \] (2.31)

Using the negative q-binomial theorem [10]:
\[ \sum_{m=0}^{\infty} \left( \frac{m + p - 1}{m} \right) q^m = \frac{1}{\prod_{j=0}^{p-1} (1 - q^j t)} \] (2.32)
Taking \( t = -q \) and the sum up to \( n \), because only the coefficient \([q^n]\) is needed, gives the theorem.
3 The q-Multinomial Coefficient

Let \((m_i)_{i=1}^s\) be a sequence of \(s\) nonnegative integers, and let \(n\) be given by:

\[
    n = \sum_{i=1}^s m_i
\]

The q-multinomial coefficient is a product of q-binomial coefficients \([5, 11]\):

\[
    \binom{n}{m_1 \cdots m_s} q = \prod_{i=1}^s \left( \sum_{j=1}^{m_i} q^{n-j} \right) q = \prod_{i=1}^s \left( \frac{n - \sum_{j=1}^{i-1} m_j}{m_i} \right) q
\]

4 Computation of \(P(n, m, p)\) with \(P(n, m)\)

Let the coefficient \(a_k^{(m,p)}\) be defined as:

\[
    a_k^{(m,p)} = \left[ q^k \right] \prod_{j=1}^m \frac{1}{1 - q^{p+j}}
\]

These coefficients can be computed by multiplying out the product, which up to \(k = n - m\) is \(O(m(n - m)) = O(n^2)\). Using (1.4) and (2.8):

\[
    P(n, m, p) = P^*(n - m, m, p - 1) = [q^{n-m}] \prod_{j=1}^m \frac{1 - q^{p+j-1}}{1 - q^j} = [q^{n-m}] \sum_{k=0}^{n-m} a_k^{(m,p-1)} \prod_{j=1}^m \frac{q^k}{1 - q^j}
\]

\[
    = \sum_{k=0}^{n-m} a_k^{(m,p-1)} [q^{n-k}] \frac{q^m}{\prod_{j=1}^m (1 - q^j)} = \sum_{k=0}^{n-m} a_k^{(m,p-1)} P(n - k, m)
\]

The list of the \(n - m + 1\) values of \(P(m, m)\) to \(P(n, m)\) can be computed using the algorithm in [7] which is also \(O(n^2)\), and therefore this algorithm computes \(P(n, m, p)\) in \(O(n^2)\). For computing \(P^*(n, m, p)\) (2.9) can be used.

5 Computation of q-Binomial Coefficients

From definition (1.3) the q-binomial coefficients are:

\[
    [q^n] \prod_{j=1}^m \frac{1 - q^{p+j}}{1 - q^j} = [q^n] \sum_{k=0}^n a_k^{(m,p)} \prod_{j=1}^m \frac{q^{k}}{1 - q^j} = \sum_{k=0}^n a_k^{(m,p)} [q^{n+m-k}] \prod_{j=1}^m \frac{q^{k}}{1 - q^j}
\]

\[
    = \sum_{k=0}^n a_k^{(m,p)} P(n + m - k, m)
\]

Because \(P^*(n, m, p) = 0\) when \(n > mp\) and (1.4), the coefficients \([q^n]\) are nonzero if and only if \(0 \leq n \leq mp\). The product coefficients \(a_k^{(m,p)}\) can therefore be computed in \(O(m^2p)\), and
the list of \( mp + 1 \) values of \( P(m, m) \) to \( P(mp + m, m) \) can also be computed in \( O(m^2p) \) \[7\]. The sums are convolutions which can be done with \texttt{ListConvolve}, and therefore this algorithm computes the q-binomial coefficients in \( O(m^2p) \). Because of symmetry theorem \[2,5\], \( m \) and \( p \) can be interchanged when \( m > p \), which makes the algorithm \( O(\min(m^2p, p^2m)) \). Using a change of variables:

\[
\binom{n}{m}_q = \binom{m+n-m}{m}_q
\] (5.2)

The algorithm for computing this q-binomial coefficient is \( O(\min(m^2(n-m), (n-m)^2m)) \). From this follows that when \( m \) or \( n-m \) is constant, then the algorithm is \( O(n) \), and when \( m = cn \) for some constant \( c \), then the algorithm is \( O(n^3) \). Because \( P^*(n, m, p) = P^*(mp - n, m, p) \) only \( P^*(n, m, p) \) for \( 0 \leq n \leq [mp/2] \) needs to be computed, which makes the algorithm about two times faster. For comparison with the computer algebra program below an alternative algorithm using cyclotomic polynomials is given.

\begin{verbatim}
QBinomialAlternative[n_, m_] := Block[{result = {1}, temp},
Do[Which[Floor[n/k] - Floor[m/k] - Floor[(n - m)/k] == 1, 
  temp = CoefficientList[Cyclotomic[k, q] q, q];
  result = ListConvolve[result, temp, {1, -1}, 0], {k, n}];
result]
\end{verbatim}

Computations show that this alternative algorithm is \( O(n^4) \).

6 A Formula for \( Q(n, m, p) \)

Let \( Q(n, m, p) \) be the number of integer partitions of \( n \) into exactly \( m \) distinct parts with each part at most \( p \).

**Theorem 6.1.**

\[ Q(n, m, p) = P(n - m(m - 1)/2, m, p - m + 1) \] (6.1)

**Proof.** The proof is with Ferrer diagrams and the "staircase" argument. Let a normal partition be a partition into \( m \) parts, and let a distinct partition be a partition into \( m \) distinct parts. Let the parts of a Ferrer diagram with \( m \) parts be indexed from small to large by \( s = 1 \cdots m \). Each distinct partition of \( n \) contains a "staircase" partition with parts \( s - 1 \) and a total size of \( m(m-1)/2 \), and subtracting this from such a partition gives a normal partition of \( n - m(m-1)/2 \), and the largest part is decreased by \( m - 1 \). Vice versa adding the "staircase" partition to a normal partition of \( n \) gives a distinct partition of \( n + m(m-1)/2 \), and the largest part is increased by \( m - 1 \). When the parts of the distinct partition are at most \( p \), then the parts of the corresponding normal partition are at most \( p - (m - 1) \). Because of this \( 1 - 1 \) correspondence between the Ferrer diagrams of these two types of partitions the identity is valid. \[ \square \]

7 Partitions with Each Part Between \( p_{min} \) and \( p_{max} \)

Let \( P^#(n, p_{min}, p_{max}) \) be the number of partitions of \( n \) with each part between \( p_{min} \) and \( p_{max} \), and let \( Q^#(n, p_{min}, p_{max}) \) be the number of partitions of \( n \) into distinct parts with each part between \( p_{min} \) and \( p_{max} \), and let \( P(n, m, p_{min}, p_{max}) \) be the number of partitions of \( n \) into exactly \( m \) parts...
with each part between \(p_{\text{min}}\) and \(p_{\text{max}}\), and let \(Q(n, m, p_{\text{min}}, p_{\text{max}})\) be the number of partitions of \(n\) into exactly \(m\) distinct parts with each part between \(p_{\text{min}}\) and \(p_{\text{max}}\). These functions are related by:

\[
P^\#(n, p_{\text{min}}, p_{\text{max}}) = \sum_{m=0}^{\lfloor n/p_{\text{min}} \rfloor} P(n, m, p_{\text{min}}, p_{\text{max}}) \\
Q^\#(n, p_{\text{min}}, p_{\text{max}}) = \sum_{m=0}^{\lfloor n/p_{\text{min}} \rfloor} Q(n, m, p_{\text{min}}, p_{\text{max}})
\] (7.1) (7.2)

Because the Ferrer diagrams of the partitions in \(P(n, m, p_{\text{min}}, p_{\text{max}})\) and \(Q(n, m, p_{\text{min}}, p_{\text{max}})\) all have a block of \(m(p_{\text{min}} - 1)\) filled, the following relations are given:

\[
P(n, m, p_{\text{min}}, p_{\text{max}}) = P(n - mp_{\text{min}} + m, m, p_{\text{max}} - p_{\text{min}} + 1) \\
Q(n, m, p_{\text{min}}, p_{\text{max}}) = Q(n - mp_{\text{min}} + m, m, p_{\text{max}} - p_{\text{min}} + 1)
\] (7.3) (7.4)

**Theorem 7.1.**

\[
P(n, m, p_{\text{min}}, p_{\text{max}}) = \sum_{l=0}^{\min(m, p_{\text{min}})} (-1)^l \sum_{m=0}^{n-m+l} P(k - l(l - 1)/2, l, p_{\text{min}} - l) P(n - k, m - l, p_{\text{max}})
\] (7.5)

**Proof.** From (5.11) in [3]:

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p_{\text{min}}, p_{\text{max}}) q^n z^m = \frac{1}{\prod_{j=p_{\text{min}}}^{p_{\text{max}}} (1 - q^j)} = \frac{\prod_{j=1}^{p_{\text{min}} - 1} (1 - q^j)}{\prod_{j=1}^{p_{\text{max}}} (1 - q^j)}
\] (7.6)

From (5.9) and (5.11) in [3]:

\[
\prod_{j=1}^{p_{\text{min}} - 1} (1 - q^j) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p_{\text{min}} - 1)(-1)^m z^m q^n
\] (7.7)

\[
\prod_{j=1}^{p_{\text{max}} - 1} (1 - q^j) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p_{\text{max}}) z^m q^n
\] (7.8)

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p_{\text{min}}, p_{\text{max}}) q^n z^m = \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p_{\text{min}} - 1)(-1)^m q^n z^m \right) \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p_{\text{max}}) q^n z^m \right)
\] (7.9)

\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} Q(n_1, m_1, p_{\text{min}} - 1) P(n_2, m_2, p_{\text{max}})(-1)^{m_1} q^{n_1 + n_2} z^{m_1 + m_2}
\]
The coefficients on both sides must be equal, so \( n_1 + n_2 = n \) and \( m_1 + m_2 = m \), which is equivalent to \( n_2 = n - n_1 \) and \( m_2 = m - m_1 \):

\[
P(n, m, \text{\(p_{\text{min}}\)}, \text{\(p_{\text{max}}\}) = \sum_{l=0}^{m} (-1)^l \sum_{k=l}^{n-m+l} Q(k, l, p_{\text{min}} - 1)P(n - k, m - l, p_{\text{max}}) \quad (7.10)
\]

Application of theorem 6.1 to this identity gives this theorem.

The following is a special case of \( P(n, m, \text{\(p_{\text{min}}\)}, \text{\(p_{\text{max}}\}) \):

\[
P(n, m, p, \text{\(p_{\text{max}}\}) = \begin{cases} 1 & \text{if } mp = n \\ 0 & \text{otherwise} \end{cases} \quad (7.11)
\]

**Theorem 7.2.** Let \( P(n, m, \text{\(p_{\text{min}}\)}, \text{\(p_{\text{max}}\}) \) be the formula of the previous theorem:

\[
Q(n, m, \text{\(p_{\text{min}}\)}, \text{\(p_{\text{max}}\}) = (-1)^m P(n, m, \text{\(p_{\text{max}}\}) + 1, p_{\text{min}} - 1) \quad (7.12)
\]

**Proof.**

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, \text{\(p_{\text{min}}\)}, \text{\(p_{\text{max}}\})q^n z^m = \prod_{j=p_{\text{min}}}^{p_{\text{max}}} (1 + zq^j) = \frac{\prod_{j=1}^{p_{\text{max}}}(1 + zq^j)}{\prod_{j=1}^{p_{\text{min}} - 1}(1 + zq^j)} \quad (7.13)
\]

\[
\prod_{j=1}^{p_{\text{max}}}(1 + zq^j) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, \text{\(p_{\text{max}}\})z^n q^m \quad (7.14)
\]

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, \text{\(p_{\text{min}}\)} - 1)(-1)^m z^n q^m \quad (7.15)
\]

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, \text{\(p_{\text{min}}\)}, \text{\(p_{\text{max}}\})q^n z^m
\]

\[
= \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, \text{\(p_{\text{max}}\})q^n z^m \right) \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, \text{\(p_{\text{min}}\)} - 1)(-1)^m q^n z^m \right) \quad (7.16)
\]

\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} Q(n_1, m_1, \text{\(p_{\text{max}}\})P(n_2, m_2, \text{\(p_{\text{min}}\)} - 1)(-1)^{m_1} q^{n_1 + n_2} z^{m_1 + m_2} \quad (7.17)
\]

The coefficients on both sides must be equal, so \( n_1 + n_2 = n \) and \( m_1 + m_2 = m \), which is equivalent to \( n_2 = n - n_1 \) and \( m_2 = m - m_1 \):

\[
Q(n, m, p, \text{\(p_{\text{max}}\}) = (-1)^m \sum_{l=0}^{m} (-1)^l \sum_{k=l}^{n-m+l} Q(k, l, \text{\(p_{\text{max}}\})P(n - k, m - l, \text{\(p_{\text{min}}\)} - 1) \quad (7.17)
\]

Comparing this with (7.10) in theorem 6.1 gives this theorem.
From the generating functions in [3] follows:

\[ P^\#(n, p_{\min}, p_{\max}) = [q^n] \prod_{j=p_{\min}}^{p_{\max}} \frac{1}{(1 - q^j)} = [q^n] \prod_{j=1}^{p_{\min}-1} \frac{1}{(1 - q^j)} \]

\[ = [q^n] \sum_{k=0}^{n} q^k a_k^{(p_{\min}-1,0)} \prod_{j=1}^{p_{\max}} \frac{1}{(1 - q^j)} = \sum_{k=0}^{n} q^k a_k^{(p_{\min}-1,0)} [q^{p_{\max}+n-k}] \prod_{j=1}^{p_{\max}} \frac{1}{(1 - q^j)} \]

(7.18)

\[ Q^\#(n, p_{\min}, p_{\max}) = [q^n] \prod_{j=p_{\min}}^{p_{\max}} (1 + q^j) = [q^n] \prod_{j=1}^{p_{\max}-p_{\min}+1} (1 + q^{p_{\min}+1+j}) \]

(7.19)

The following are special cases of \( P^\#(n, p_{\min}, p_{\max}) \):

\[ P^\#(n, 1, m) = P(n + m, m) \] (7.20)

\[ P^\#(n, p, p) = \begin{cases} 1 & \text{if } p \text{ divides } n \\ 0 & \text{otherwise} \end{cases} \] (7.21)

The following is lemma (5) in [1]:

**Lemma 7.1.**

\[ \sum_{k=1}^{m} q^k \prod_{j=1}^{k-1} (1 - q^j) = 1 - \prod_{j=1}^{m} (1 - q^j) \] (7.22)

**Proof.** The lemma is true for \( m = 0 \), and using induction on \( m \), when it is true for \( m \), then for \( m + 1 \):

\[ \sum_{k=1}^{m+1} q^k \prod_{j=1}^{k-1} (1 - q^j) = q^{m+1} \prod_{j=1}^{m} (1 - q^j) + \sum_{k=1}^{m} q^k \prod_{j=1}^{k-1} (1 - q^j) \]

\[ = q^{m+1} \prod_{j=1}^{m} (1 - q^j) + 1 - \prod_{j=1}^{m} (1 - q^j) = 1 - (1 - q^{m+1}) \prod_{j=1}^{m} (1 - q^j) = 1 - \prod_{j=1}^{m} (1 - q^j) \]

(7.23)

Dividing this lemma by \( \prod_{j=1}^{m} (1 - q^j) \) and taking the coefficient \( [q^n] \):

\[ [q^n] \sum_{k=1}^{m} q^k \prod_{j=k}^{m} \frac{1}{(1 - q^j)} = \sum_{k=1}^{m} [q^{n-k}] \prod_{j=k}^{m} \frac{1}{(1 - q^j)} = [q^n] \prod_{j=1}^{m} \frac{1}{(1 - q^j)} - [q^n] \]

(7.24)

and using \( P(0, p, p) = 1 \) gives the following identity:

For integer \( m \leq n \):

\[ \sum_{k=1}^{\min(m, \lfloor n/2 \rfloor)} P^\#(n - k, k, m) = P^\#(n, 1, m) - \delta_{n,m} \] (7.25)
Taking $m = n$ and using $P^\#(n, 1, n) = P(n)$:

$$\sum_{k=1}^{\lfloor n/2 \rfloor} P^\#(n - k, k, n) = P(n) - 1 \quad (7.26)$$

The following was proved as lemma 1.1 in [7]:

$$\sum_{k=0}^{m} q^{k} \prod_{j=k+1}^{m} (1 - q^j) = 1 \quad (7.27)$$

Dividing this lemma by $\prod_{j=1}^{m}(1 - q^{j})$ and taking the coefficient $[q^n]$:

For integer $m \leq n$:

$$\sum_{k=1}^{m} P^\#(n - k, 1, k) = P^\#(n, 1, m) \quad (7.28)$$

Taking $m = n$ and using $P^\#(n, 1, n) = P(n)$:

$$\sum_{k=1}^{n} P^\#(n - k, 1, k) = P(n) \quad (7.29)$$

8 The q-Binomial Coefficient for Negative Arguments

From another paper [8] the following was proved:

For integer $n \geq 0$ and integer $k$:

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = 0 \text{ if } k < 0 \text{ or } k > n \quad (8.1)$$

and from that paper theorem 2.4 gives:

For negative integer $n$ and integer $k$:

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \begin{cases} (-1)^k q^{nk - k(k-1)/2} \left(\begin{array}{c} -n + k - 1 \\ k \end{array}\right)_q & \text{if } k \geq 0 \\ (-1)^{n-k} q^{(n-k)(n+k+1)/2} \left(\begin{array}{c} -k - 1 \\ n - k \end{array}\right)_q & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases} \quad (8.2)$$

9 Computer Algebra Program

The following Mathematica® functions are listed in the computer algebra program below.

PartitionsPList[n,pmin,pmax]

Gives a list of the $n$ numbers $P(1, p_{\text{min}}, p_{\text{max}}) \ldots P(n, p_{\text{min}}, p_{\text{max}})$, where $P(n, p_{\text{min}}, p_{\text{max}})$ is the number of partitions of $n$ with each part between $p_{\text{min}}$ and $p_{\text{max}}$. This algorithm is $O(n^2)$.

PartitionsQList[n,pmin,pmax]
Gives a list of the $n$ numbers $Q(1, p_{\text{min}}, p_{\text{max}})\ldots Q(n, p_{\text{min}}, p_{\text{max}})$, where $Q(n, p_{\text{min}}, p_{\text{max}})$ is the number of partitions of $n$ into distinct parts with each part between $p_{\text{min}}$ and $p_{\text{max}}$. This algorithm is $O(n^2)$.

**PartitionsInPartsP[n,m,p]**

Gives the number of partitions of $n$ into exactly $m$ parts with each part at most $p$. This algorithm is $O(n^2)$.

**PartitionsInPartsQ[n,m,p]**

Gives the number of partitions of $n$ into exactly $m$ distinct parts with each part at most $p$, using the formula $Q(n, m, p) = P(n - m(m - 1)/2, m, p - m + 1)$. This algorithm is $O(n^2)$.

**PartitionsInPartsPList[n,m,p]**

Gives a list of $n - m + 1$ numbers of $P(m, m, p)\ldots P(n, m, p)$. This algorithm is $O(n^2)$.

**PartitionsInPartsQList[n,m,p]**

Gives a list of the $n - m(m + 1)/2 + 1$ numbers $Q(m(m + 1)/2, m, p)\ldots Q(n, m, p)$, using the formula $Q(n, m, p) = P(n - m(m - 1)/2, m, p - m + 1)$. This algorithm is $O(n^2)$.

**QBinomialCoefficients[n,m,q]**

Gives the $q$-binomial coefficient $\binom{n}{m}_q$ as a polynomial in $q$, where $n$ and $m$ are integers and $q$ is a symbol. This algorithm is $O(n^3)$.

**QMultinomialCoefficients[mlist,q]**

Gives the $q$-multinomial coefficient $\binom{n}{m_1\ldots m_s}_q$ as a polynomial in $q$, where $s$ is the length of the list `mlist` containing the integers $m_1\ldots m_s$, and where $n = \sum_{i=1}^{s} m_i$, and where $q$ is a symbol.

Below is the listing of a Mathematica® program that can be copied into a notebook, using the package taken from at least version 3 of the earlier paper [7]. The notebook must be saved in the directory of the package file.

```mathematica
SetDirectory[NotebookDirectory[]];
<< "PartitionsInParts.m"
partprod[n_, m_, p_, s_]:=Block[{prod=ConstantArray[0,n+1]},prod[[1]]=1;
  Do[prod[[Range[p+k+1,n+1]]]+=s prod[[Range[1,n-p-k+1]]],{k,Min[m,n-p]}];
  prod]
PartitionsPList[n_Integer?Positive,pmin_Integer?Positive,
  pmax_Integer?Positive]:=If[pmax<pmin,{},
  Block[{prods,parts},prods=partprod[n,pmin-1,0,-1];
  parts=PartitionsInPartsPList[n+pmax,pmax];
  ListConvolve[prods,parts,{1,1},0][[Range[2,n+1]]]]
PartitionsQList[n_Integer?Positive,pmin_Integer?Positive,
  pmax_Integer?Positive]:=If[pmax<pmin,{},
  partprod[n,pmax-pmin+1,pmin-1,1][[Range[2,n+1]]]]
PartitionsInPartsP[n_Integer?NonNegative,m_Integer?NonNegative,
  p_Integer?NonNegative]:=If[n<=m,0,
  Block[{prods,parts,result},prods=partprod[n-m,m,p-1,-1];
  parts=PartitionsInPartsPList[n,m];result=0;
  Do[result+=prods[[k+1]]parts[[n-m-k+1]],{k,0,n-m}];result]]
PartitionsInPartsQ[n_Integer?NonNegative,m_Integer?NonNegative,
  p_Integer?NonNegative]:=If[n-m(m-1)/2<m||p<m,0,
  PartitionsInPartsP[n-m(m-1)/2,m,p-m+1]]
```

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PartitionsInPartsPList[n_Integer?NonNegative, m_Integer?NonNegative, p_Integer?NonNegative] := If[n < m, {}, Block[{prods, parts}, prods = partprod[n - m, m, p - 1, -1]; parts = PartitionsInPartsPList[n, m]; ListConvolve[prods, parts, {1, 1}, 0]]]

PartitionsInPartsQList[n_Integer?NonNegative, m_Integer?NonNegative, p_Integer?NonNegative] := If[n - m(m - 1)/2 < m || p < m, {}, PartitionsInPartsPList[n - m(m - 1)/2, m, p - m + 1]]

QBinomialCompute[N_Integer, M_Integer, q_Symbol, doq_?BooleanQ] := Block[{m = M, p = N - M, result, prods, parts, ceil}, Which[m > p, m = p; p = M; ceil = Ceiling[(m + p + 1)/2]; prods = partprod[ceil - 1, m, p, -1]; parts = PartitionsInPartsPList[ceil + m - 1, m]; result = PadRight[ListConvolve[prods, parts, {1, 1}, 0], m + 1]; result[[Range[ceil + 1, m p + 1]]] = result[[Range[m p - ceil + 1, 1, -1]]]; If[doq, q^Range[0, Length[result] - 1].result, result]]

QBinomialCoefficients[n_Integer, k_Integer, q_Symbol] := If[(n >= 0 && (k < 0 || k > n)) || (n < 0 && n < k < 0), 0, If[n == 0, QBinomialCompute[n, k, q, True], If[k == 0, (-1)^(k - 1) q^((k + 1)/2) QBinomialCompute[-k - 1, n, q, True], (-1)^(n - k) q^((n - k)(n + k + 1)/2) QBinomialCompute[-k - 1, n - k, q, True]]]]

MListQ[alist_List] := (alist!={} && VectorQ[alist, IntegerQ])

QMultinomialCoefficients[mlist_List?MListQ, q_Symbol] := If[!VectorQ[mlist, NonNegative], 0, Block[{length = Length[mlist], bprod = {1}, msum = mlist[[1]], qbin}, Do[msum += mlist[[k]]; qbin = QBinomialCompute[msum, mlist[[k]], q, False]; bprod = ListConvolve[bprod, qbin, {1, -1}, 0], {k, 2, length}]; q^Range[0, Length[bprod] - 1].bprod]]

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