Multialternative Neural Decision Processes

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Abstract

We introduce an algorithmic decision process for multialternative choice that combines binary comparisons and Markovian exploration. We show that a preferential property, transitivity, makes it testable.

1 Introduction

A decision maker aims to find the best alternative within a finite set of $A$ available ones. Had he unconstrained time (or any other relevant resource) and were he able to make exact judgments between alternatives, he could proceed by standard revision. This brute force comparison-and-elimination algorithm sequentially analyzes pairs of alternatives and permanently discards the inferior one. After $|A| - 1$ comparisons, the incumbent solution is an optimal choice.

If time is a scarce resource and comparisons are subject to noise, say because of the stochasticity of the information the decision maker’s neural system is able to gather, a stochastic revision procedure results. In this paper we study a such procedure generated by an algorithmic decision process for multialternative choice that combines binary comparisons and Markovian exploration. We show that a preferential property, transitivity, makes it testable.

2 The mechanics of choice

2.1 Kernels and transitivity

Let $A$ be a menu of alternatives, with typical elements $i$, $j$ and $k$. We consider a decision maker who compares alternatives $i$ and $j$ in $A$ in a pairwise manner through a (binary) stochastic choice kernel $\rho : A^2_{\neq} \to [0, 1]$ where

$$\rho (i \mid j)$$

denotes the probability with which the proposal $i$ is accepted, while $1 - \rho (i \mid j)$ is the probability with which the incumbent (or status quo) $j$ is maintained.

1 The set $A^2_{\neq} = \{(i, j) \in A \times A : i \neq j\}$ consists of all distinct ordered pairs of alternatives in $A$. 
Definition 1 A stochastic choice kernel $\rho$ is:

- (status-quo) unbiased if
  \[ 1 - \rho(i \mid j) = \rho(j \mid i) \]
  for all distinct alternatives $i$ and $j$;

- (strictly) positive if $0 < \rho(i \mid j) < 1$ for all distinct alternatives $i$ and $j$.

These properties have a simple interpretation: a stochastic choice kernel is unbiased when it gives the incumbent alternative no special status, and positive when it selects either alternative with a strictly positive probability.

Definition 2 A stochastic choice kernel $\rho$ is transitive if

\[ \rho(j \mid i) \rho(k \mid j) \rho(i \mid k) = \rho(k \mid i) \rho(j \mid k) \rho(i \mid j) \]

for all distinct alternatives $i$, $j$ and $k$.

In words, a stochastic choice kernel is transitive when violations of transitivity in the choices that it determines are due only to the presence of noise. Indeed, condition (1) amounts to require that the intransitive cycles

\[ i \rightarrow j \rightarrow k \rightarrow i \quad \text{and} \quad i \rightarrow k \rightarrow j \rightarrow i \]

be equally likely.

Intransitive stochastic choice kernels may result in choices between alternatives that feature systematic intransitivities, thus violating a basic rationality tenet. Transitivity ensures that this is not the case.

2.2 Binary choices and neural mechanisms

A behavioral binary choice (BBC) model is a pair of random matrices $(C, RT)$ where:

(i) $C = [C_{i,j}]$ consists of the random choice variables $C_{i,j}$ taking values in $\{i, j\}$ that describe the outcome of the comparison between proposal $i$ and status quo $j$;

(ii) $RT = [RT_{i,j}]$ consists of the random response times $RT_{i,j}$ required to compare proposal $i$ and status quo $j$, which are assumed to have finite mean and variance.

The distributions of $C$ and $RT$ of a BBC model are, in principle, both observable in choice behavior. They induce a stochastic choice kernel $\rho_C$ defined by

\[ \rho_C(i \mid j) = \Pr[C_{i,j} = i] \]

for all proposals $i$ and incumbents $j$. Kernel $\rho_C$ describes the probabilistic choices that a BBC model $(C, RT)$ induces.
A BBC model \((C, RT)\) is *unbiased* if the induced \(\rho_C\) is. This is the case, for instance, when \(C_{i,j} = C_{j,i}\). In a similar vein, we say that a BBC model is *positive* or *transitive* if the induced kernel is. Transitivity is a preferential property of a BBC model that presupposes that a viable choice model should not be prone to systematic errors, whatever the underlying neural mechanism is.

Different neural mechanisms may, indeed, underlie a BBC model. A broad family is given by the *evidence threshold models*. In these models, a stochastic process \(\{X_{i,j}(t)\}_{t \in \mathbb{N}}\) is given for each pair \((i, j) \in A^2_\neq\) of distinct alternatives. The *neural decision variable* \(X_{i,j}(t)\) represents the evidence—accumulated or instantaneous— in favor of \(i\) and against \(j\) that the decision maker takes into account at time \(t\). Given an evidence threshold \(\beta > 0\), a decision is taken when the evidence in favor of either alternative reaches level \(\beta\). This happens at (stochastic) time

\[
RT_{i,j} = \min \{ t \in \mathbb{N} : |X_{i,j}(t)| \geq \beta \}
\]

and the alternative favored by evidence

\[
C_{i,j} = \begin{cases} 
i & \text{if } X_{i,j}(RT_{i,j}) \geq \beta \\
j & \text{if } X_{i,j}(RT_{i,j}) \leq -\beta
\end{cases}
\]

is selected. Evidence threshold models encompass integration models, like the drift-diffusion model and its generalizations, as well as extrema detection models, as recently discussed by Stine, Zylberberg, Ditterich and Shadlen (2020). This is readily seen by considering neural decision variables of the Ornstein-Uhlenbeck form

\[
X_{i,j}(t+1) - X_{i,j}(t) = -\lambda X_{i,j}(t) + (v_i - v_j) \mu(t) + \sigma \varepsilon(t)
\]

where \(\lambda \in (0, 1)\) captures evidence deterioration, \((v_i - v_j) \mu(t)\) is the instantaneous strength of the evidence in favor of \(i\) over \(j\), and \(\sigma\) is the standard deviation of a Gaussian white noise process \(\varepsilon\).

### 3 Neural Metropolis algorithm

We now describe an algorithmic decision process that the neural system of a decision maker might implement when facing a multialternative menu \(A\). The process consists of a sequence of pairwise comparisons conducted via a BBC model, the contestants of which are selected by a Markovian mechanism a la Metropolis et al. (1953).

The algorithm starts according to an initial distribution \(\mu \in \Delta(A)\) that describes the “first fixation” of the decision maker\(^3\) and proceeds through an exploration matrix \(Q\) that describes how his neural system navigates the landscape of alternatives, as suggested by eye-tracking evidence.\(^4\) Pairs of alternatives are compared via a BBC model \((C, RT)\) and the last incumbent is maintained.

The algorithm terminates according to a (stochastic) stopping time \(N\).

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\(^3\)The dependence of \(\mu\) on \(t\) allows to incorporate the presence of urgency signals.

\(^4\)As usual, \(\Delta(A)\) is the set of all probability distributions on \(A\), and \(\Delta_+(A)\) is its subset consisting of all probability distributions with full support.

\(^2\)See Russo and Rosen (1975) and the more recent Krajbich and Rangel (2011) and Reutskaja et al. (2011).
Neural Metropolis Algorithm

Input: Given a stopping time $N > 0$.

Start: Draw $i_0$ from $A$ according to $\mu$ and

- set $t_0 = 0$,
- set $j_0 = i_0$.

Repeat: Draw $i_{n+1}$ from $A$ according to $Q(\cdot \mid j_n)$ and compare it to $j_n$:

- set $t_{n+1} = t_n + RT_{i_{n+1}, j_n}$,
- set $j_{n+1} = C_{i_{n+1}, j_n}$;

until $n + 1 = N$.

Stop: Set $k = j_N$.

Output: Choose $k$ from $A$.

When the underlying BBC Model is the Drift Diffusion Model of Ratcliff (1978) and the stopping rule corresponding to $N$ is given by a fixed deadline $T$, the Neural Metropolis Algorithm coincides with the Metropolis-DDM of Baldassi et al. (2020). Based on the latter paper, and independently of this one, Valkanova (2020) studied a Markovian model of decision-making with a geometric stopping time; but the model only considers “timeless” choice probabilities.

The Neural Metropolis Algorithm generates a Markov chain of incumbents

$$M = \{J_0, J_1, \ldots \}$$

with $Pr [J_0 = j] = \mu(j)$ for all alternatives $j \in A$ and

$$Pr [J_{n+1} = i \mid J_n = j] = \frac{Q(i \mid j) \times \rho_C(i \mid j)}{\text{prob. } i \text{ is proposed} \times \text{prob. that } i \text{ is accepted}} \approx M(i \mid j)$$

for all distinct alternatives $i$ and $j$ in $A$. Thus, $M$ is the transition matrix of this Markov chain.

The first result of this note is that multialternative choice distributions and average decision times of the algorithm can be written in explicit form. To this end, note that, if the incumbent at some iteration is $j$ and the proposal is $i$, the average duration of that iteration is $RT_{i,j}$, but then conditional on $j$ being the incumbent, the average duration of an iteration is

$$\tau_j = \sum_{i \in A} Q(i \mid j) RT_{i,j}$$

Denote by $\tau$ is the vector of those conditional expected durations.
Lemma 1 The Neural Metropolis Algorithm generates the following choice probabilities from $A$

$$p_N = \left( \sum_{m=1}^{\infty} \Pr[N = m] M^{m-1} \right) \mu \quad (2)$$

and the mean decision time is

$$T_N = \tau^t \left( \sum_{m=1}^{\infty} \Pr[N = m] \left( \sum_{n=1}^{m} M^{n-1} \right) \right) \mu \quad (3)$$

$$= \tau^t \left( \sum_{n=1}^{\infty} \Pr[N \geq n] M^{n-1} \right) \mu \quad (4)$$

provided $N$, $C$, and $RT$ are independent.

Note that denoting by $p_{ij}^{m-1} \in [0, 1]$ the generic entry of $M^{m-1}$, then each entry

$$\sum_{m=1}^{\infty} \Pr[N = m] p_{ij}^{m-1}$$

of $\sum_{m=1}^{\infty} \Pr[N = m] M^{m-1}$ is a converging series (the average of the function $m \mapsto p_{ij}^{m-1}$ with respect to the distribution of $N$), so $\sum_{m=1}^{\infty} \Pr[N = m] M^{m-1}$ is a bona fide stochastic matrix and $p_N$ a bona fide probability vector. Moreover, each entry

$$\sum_{n=1}^{\infty} \Pr[N \geq n] p_{ij}^{n-1}$$

of $\sum_{n=1}^{\infty} \Pr[N \geq n] M^{n-1}$ is bounded by $\sum_{n=1}^{\infty} \Pr[N \geq n] = \mathbb{E}[N]$, thus the convergence of $\sum_{n=1}^{\infty} \Pr[N \geq n] M^{n-1}$ is guaranteed if $N$ has finite expectation. When $M$ is reversible (see next section) the previous computation is standard because $M$ can be diagonalized,

$$M = U \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{|A|} \end{bmatrix} U^{-1}$$

where the columns of $U$ form a basis of eigenvectors and hence

$$\sum_{m=1}^{\infty} \Pr[N = m] M^{m-1} = \begin{bmatrix} \sum_{m=1}^{\infty} \Pr[N = m] \lambda_1^{m-1} \\ \ddots \\ \sum_{m=1}^{\infty} \Pr[N = m] \lambda_{|A|}^{m-1} \end{bmatrix} U^{-1}$$

this gives even more explicit expressions for equations (2), (3), and (4).
Example 1 But also in the absence of reversibility, since \( \sum_{m=1}^{\infty} \Pr [N = m] M^{m-1} \) is a power series, some computation is possible. The geometric and Poisson cases are iconic:

- If stopping is geometric with continuation probability \( \zeta \), then
  \[
  \sum_{m=1}^{\infty} \Pr [N = m] M^{m-1} = (1 - \zeta) (I - \zeta M)^{-1}
  \]
  this was first proven by Valkanova (2020).

- Instead, if stopping is Poisson with mean \( \lambda \), then
  \[
  \sum_{m=1}^{\infty} \Pr [N = m] M^{m-1} = e^{-\lambda} e^{\lambda M}
  \]
  this result is new.

4 Transitive algorithms

How can we test whether incumbents are stochastically generated by a neural Metropolis algorithm? The next Proposition 3 shows that transitivity makes it possible to address this key question, but we need a simple lemma before getting to it. Recall that an exploration matrix is nice if and only if it is symmetric and strictly positive off the diagonal.

Lemma 2 If the BBC is positive, and the exploration matrix \( Q \) is nice, then the transition matrix \( M \) has strictly positive entries.

We are now ready to state the second result of this note.

Proposition 3 The following conditions are equivalent for a positive BBC model:

(i) the transition matrix \( M \) is reversible, for every/some nice exploration matrix \( Q \);

(ii) the stochastic choice kernel \( \rho_C \) is transitive;

(iii) there exist a probability \( \pi \in \Delta_+ (A) \) and a symmetric function \( s : A^2 \to (0, \infty) \) such that

\[
\rho_C (i \mid j) = s(i, j) \frac{\pi (i)}{\pi (i) + \pi (j)} \quad \forall i \neq j
\]

In this case, \( \pi \) and \( s \) are unique. Moreover,

1. \( \pi \) is the only element of \( \Delta (A) \) under which \( M \) is reversible and, given any alternative \( i \),

\[
\pi (j) = \frac{\rho (j \mid i)}{\sum_{k \in A} \rho (k \mid i) \rho (i \mid k)} \quad \forall j
\]
2. \( \rho \) is unbiased if and only if \( s \) is constant to 1, that is,

\[
\rho_C (i \mid j) = \frac{\pi(i)}{\pi(i) + \pi(j)} \quad \forall i \neq j
\]

The importance of this proposition, so of transitivity, does not reside in the existence of a unique stationary distribution \( \pi \) for \( M \). Indeed, any neural Metropolis algorithm featuring a positive BBC has an irreducible and aperiodic transition matrix, hence a unique stationary distribution \( \pi \) that may be used to approximate the frequency of incumbents that the algorithm generates (assuming its convergence, i.e., a \( T \) large enough relative to the BBC average response times; see the next section).

Instead, the central feature of our result is that, when the BBC is transitive, the stationary distribution is independent of \( Q \) and can be expressed solely in terms of the BBC kernel \( \rho_C \). Therefore, by knowing the observable elements of a transitive BBC and assuming convergence of the algorithm, we can test the neural Metropolis algorithm.

A functional property like transitivity, which we claim that any viable decision process should feature, thus characterizes a class of testable multialternative neural decision processes.

## 5 Asymptotic heuristics

As observed above we have the following:

**Fact 1** For each positive BBC model, and each nice exploration matrix \( Q \), the transition matrix \( M \) has a unique stationary distribution \( \pi \).

What is the relation between the stationary distribution \( \pi \) and the algorithm’s output, when \( N \) is given by a fixed deadline \( T \)? Heuristically, as \( T \) increases and so does the number \( n \) of iterations performed, the fraction of clock-time in which \( j \) is the incumbent and \( i \) is the proposal is directly proportional to:

- the probability of \( j \) being the incumbent and \( i \) the proposal, which is

\[
(M^n \mu)(j) \times Q(i \mid j) \rightarrow \pi(j) Q(i \mid j)
\]

- the average clock-time it takes to compare \( i \) and \( j \), which is \( \overline{RT}_{i,j} \).

Assuming \( Q \) is null on the diagonal (i.e., that incumbents cannot be re-proposed), a reasonable conjecture is that, as \( T \) increases, the total probability of \( j \) being the incumbent approaches

\[
\pi^* (j) = \frac{\sum_{i \in A \setminus j} \pi(j) Q(i \mid j) \overline{RT}_{i,j}}{\sum_{k \in A} \left( \sum_{i \in A \setminus k} \pi(k) Q(i \mid k) \overline{RT}_{i,k} \right)} \quad \forall j \in A
\]

normalizing constant
Preliminary investigations, and numerical simulations for the very general Ornstein-Uhlenbeck family of BBC models provide preliminary support to the conjecture.

6 Proofs and related material

6.1 Proof of Lemma 1

The algorithm does not stop at iteration 0 and the 0-th iteration is instantaneous. Moreover, if it stops at iteration \( m \) it chooses the incumbent \( j_{m-1} \).

By independence, for each \( i \), the probability of choosing \( i \) conditional on stopping at iteration \( m \) is the \( i \)-th component \( (M^{m-1}\mu)_i \) of the vector \( M^{m-1}\mu \). The probability of stopping at iteration \( m \) is \( \Pr [N = m] \), and so the probability of of choosing \( i \) is

\[
\sum_{m=1}^{\infty} \Pr [N = m] (M^{m-1}\mu)_i
\]

but then the choice probabilities are given by

\[
\sum_{m=1}^{\infty} \Pr [N = m] M^{m-1}\mu = \lim_{l \to \infty} \sum_{m=1}^{l} \Pr [N = m] M^{m-1}\mu = \lim_{l \to \infty} \left( \sum_{m=1}^{l} \Pr [N = m] M^{m-1} \right) \mu
\]

\[
= \left( \sum_{m=1}^{\infty} \Pr [N = m] M^{m-1} \right) \mu
\]

proving equation (2).

By independence, conditional on \( j \) being the incumbent, the average duration of an iteration is

\[
\tau_j = \sum_{i \in A} Q(i \mid j) RT_{i,j}
\]

Now the probability with which \( j \) is an incumbent at iteration \( n \) is \( (M^{n-1}\mu)_j \), then the average duration of iteration \( n \) (if it takes place) is

\[
\sum_{j \in A} \tau_j (M^{n-1}\mu)_j = \tau^T M^{n-1}\mu
\]

By independence, the average processing time, conditional on performing \( m \) iterations is

\[
\sum_{n=1}^{m} \tau^T M^{n-1}\mu = \tau^T \left( \sum_{n=1}^{m} M^{n-1} \right) \mu
\]

But the probability of performing \( m \) iterations is \( \Pr [N = m] \), it follows that

\[
\text{MDT} (\mu) = \sum_{m=1}^{\infty} \Pr [N = m] \tau^T \left( \sum_{n=1}^{m} M^{n-1} \right) \mu = \tau^T \left( \sum_{m=1}^{\infty} \Pr [N = m] \left( \sum_{n=1}^{m} M^{n-1} \right) \right) \mu
\]

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5See, Busemeyer and Townsend (1993), Bogacz et al. (2006), and Stine, Zylberberg, Ditterich and Shadlen (2020).

6Python scripts are available upon request.
This proves (3), as for (4), given any two integers \( n, m \in \mathbb{N} \) set

\[
1(n \leq m) = \begin{cases} 
1 & \text{if } n \leq m \\
0 & \text{if } n > m 
\end{cases}
\]

with this

\[
\sum_{m=1}^{\infty} \Pr [N = m] \left( \sum_{n=1}^{m} M^{n-1} \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{m} \Pr [N = m] M^{n-1} = \sum_{m=1}^{\infty} \sum_{n=1}^{m} 1(n \leq m) \Pr [N = m] M^{n-1} = \sum_{n=1}^{\infty} \Pr [N \geq n] M^{n-1}
\]
as wanted.

\section{Kolmogorov criterion, Luce product rule, and Lemma 2}

Let \( A \) be a finite set, with typical elements \( i, j \) and \( k \). A \textit{(left) stochastic matrix} \( P = [P(i \mid j)]_{i,j \in A} \) is an \( A \times A \) matrix such that \( P(\cdot \mid j) \in A(A) \) for all \( j \in A \). In general, \( P(i \mid j) \) is interpreted as the probability with which a system moves from state \( j \) to state \( i \).

\textbf{Definition 3} A stochastic matrix \( P \) is:

- \textit{reversible} if there exists \( \pi \in A(A) \) such that
  \[
P(i \mid j) \pi(j) = P(j \mid i) \pi(i) \quad \forall i, j \in A
  \]

- \textit{transitive} if
  \[
P(j \mid i) P(k \mid j) P(i \mid k) = P(k \mid i) P(j \mid k) P(i \mid j) \quad \forall i, j, k \in A
  \]

- \textit{full} if \( P(i \mid j) > 0 \) for all \( i, j \in A \), i.e., \( P(\cdot \mid j) \in A(A) \) for all \( j \in A \);

- \textit{an exploration matrix} if
  \[
P(j \mid i) = P(i \mid j) > 0 \quad \forall i \neq j \in A
  \]

A few remarks are in order. First, since (7) is automatically satisfied when \( i = j \), reversibility can be stated as \( P(i \mid j) \pi(j) = P(j \mid i) \pi(i) \) for all \( i \neq j \). Second, transitivity is known as the \textit{Kolmogorov criterion} in the Markov chains literature (see, e.g., Kelly, 1979, p. 24, and Kijima, 1997, p. 60) and as the \textit{product rule} in stochastic choice literature (Luce and Suppes, 1965, p. 341) where it was introduced by Luce (1957).

Third, transitivity is automatically satisfied if at least two of the three states \( i, j \), and \( k \) in \( A \) coincide. In fact,

- if \( i = j \), then
  \[
P(j \mid i) P(k \mid j) P(i \mid k) = P(i \mid i) P(k \mid i) P(i \mid k)
  \]
  \[
P(k \mid i) P(j \mid k) P(i \mid j) = P(k \mid i) P(i \mid k) P(i \mid i)
  \]
• if $i = k$, then

\[
P(j \mid i) P(k \mid j) P(i \mid k) = P(j \mid i) P(i \mid j) P(i \mid i)
\]
\[
P(k \mid i) P(j \mid k) P(i \mid j) = P(i \mid i) P(j \mid i) P(i \mid j)
\]

• if $j = k$, then

\[
P(j \mid i) P(k \mid j) P(i \mid k) = P(j \mid i) P(j \mid j) P(i \mid j)
\]
\[
P(k \mid i) P(j \mid k) P(i \mid j) = P(j \mid i) P(j \mid j) P(i \mid j)
\]

Therefore, transitivity can be restated as

\[
P(j \mid i) P(k \mid j) P(i \mid k) = P(k \mid i) P(j \mid k) P(i \mid j)
\]

for all distinct $i$, $j$ and $k$ in $A$.

**Nota Bene** the argument we just reported applies to any function $P : A \times A \to \mathbb{R}$, and it is independent on the values $P(i \mid i)$ that the function takes on the diagonal.

The next result, which relates reversibility and transitivity, builds upon Kolmogorov (1936) and Luce and Suppes (1965).

**Proposition 4** Let $P$ be a full stochastic matrix. The following conditions are equivalent:

(i) $P$ is reversible, with respect to some $\pi \in \Delta(A)$;

(ii) $P$ is transitive.

In this case, given any $i \in A$, it holds

\[
\pi(j) = \frac{P(j \mid i)}{\sum_{k \in A} P(k \mid i)} P(i \mid j) \quad \forall j \in A
\]

In particular, $\pi$ is unique and has full support.

**Proof** If $P$ is reversible with respect to $\pi$, then

\[
P(i \mid j) \pi(j) = P(j \mid i) \pi(i) \quad \forall i, j \in A
\]  \hspace{1cm} (9)

If $\pi(i^*) = 0$ for some $i^* \in A$, then (since $P$ is full)

\[
\pi(j) = \frac{P(j \mid i^*)}{P(i^* \mid j)} \pi(i^*) = 0 \quad \forall j \in A
\]  \hspace{1cm} (10)
But, this is impossible since \( \sum_{j \in A} \pi(j) = 1 \). Hence, \( \pi \) has full support. Moreover, by (10),

\[
\frac{P(j \mid i^*)}{P(i^* \mid j)} = \frac{P(j \mid i^*)}{P(i^* \mid j)} \pi(i^*) \quad \sum_{k \in A} \frac{P(k \mid i^*)}{P(i^* \mid k)} = \sum_{k \in A} \frac{P(k \mid i^*)}{P(i^* \mid k)} \pi(i^*) = \sum_{k \in A} \pi(k) = \pi(j) \quad \forall j \in A
\]

irrespective of the choice of \( i^* \in A \). Finally, given any \( i, j, k \in A \), by (9) we have:

\[
\frac{\pi(j) \pi(k)}{\pi(i) \pi(j) \pi(k)} = 1 \implies \frac{P(j \mid i) P(k \mid j) P(i \mid k)}{P(i \mid j) P(j \mid k) P(k \mid i)} = 1 \implies \frac{P(j \mid i) P(k \mid j) P(i \mid k)}{P(k \mid i) P(j \mid k) P(i \mid j)} = 1 \implies P(j \mid i) P(k \mid j) P(i \mid k) = P(k \mid i) P(j \mid k) P(i \mid j)
\]

and transitivity holds.

Conversely, if transitivity holds, choose arbitrarily \( i^* \in A \), and set

\[
\pi^*(j) \doteq \frac{P(j \mid i^*)}{\sum_{k \in A} \frac{P(k \mid i^*)}{P(i^* \mid k)} P(i^* \mid j)} \doteq \zeta P(j \mid i^*) P(i^* \mid j) \quad \forall j \in A
\]

where \( 1/\zeta = \sum_{k \in A} P(k \mid i^*)/P(i^* \mid k) > 0 \). With this, for all \( i, j \in A \),

\[
P(i \mid j) \pi^*(j) = P(i \mid j) \zeta P(j \mid i^*) = \frac{P(j \mid i^*)}{P(i^* \mid j)}
\]

Transitivity implies that

\[
P(j \mid i) P(i^* \mid j) P(i \mid i^*) = P(i^* \mid i) P(j \mid i^*) P(i \mid j)
\]

and since \( P \) is full

\[
P(j \mid i) \frac{P(i \mid i^*)}{P(i^* \mid i)} = P(j \mid i) P(i \mid j) \quad \forall j \in A
\]

Thus,

\[
P(i \mid j) \zeta \frac{P(j \mid i^*)}{P(i^* \mid j)} = P(j \mid i) \zeta \frac{P(i \mid i^*)}{P(i^* \mid i)} = P(j \mid i) \pi^*(i)
\]

and reversibility with respect to \( \pi^* \) holds.

**Proof of Lemma 2** We have \( M(\cdot \mid j) \in \Delta_+ (A) \) for all \( j \in A \). Indeed,

\[
M(i \mid j) = \frac{Q(i \mid j)}{\rho(i \mid j) \in (0,1)} \times \rho(i \mid j) \in (0,1) \quad \forall i \neq j
\]

and \( M(j \mid j) = 1 - \sum_{k \neq j} Q(k \mid j) \rho(k \mid j) > 1 - \sum_{k \neq j} Q(k \mid j) = Q(j \mid j) \geq 0 \).
6.3 Proof of Proposition $3$

**Proof** For clarity, we split point (i) in two parts:

(i.a) the transition matrix $M$ is reversible (for every exploration matrix $Q$);

(i.b) the transition matrix $M$ is reversible (for some exploration matrix $Q$);

We show that (i.a) $\Rightarrow$ (i.b) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i.a).

(i.a) $\Rightarrow$ (i.b) Trivial.

(i.b) $\Rightarrow$ (ii) By Proposition $4$ $M$ is transitive. Thus, for all distinct $i$, $j$ and $k$ in $A$,

$$Q(j|i)\rho(j|i)Q(k|j)\rho(k|j)Q(i|k)\rho(i|k) = Q(k|i)\rho(k|i)Q(j|k)\rho(j|k)Q(i|j)\rho(i|j)$$

Since $Q$ is symmetric and strictly positive off the diagonal, then

$$\rho(j|i)\rho(k|j)\rho(i|k) = \rho(k|i)\rho(j|k)\rho(i|j)$$

that is, $\rho$ is transitive.

(ii) $\Rightarrow$ (iii) Arbitrarily extend $\rho$ to $A^2$, say by taking $\rho(i|i) = 1$ for all $i \in A$. Choose $i^* \in A$, and set

$$\pi^*(j) = \frac{\rho(j|i^*)}{\sum_{k \in A} \rho(i^*|k)}\rho(i^*|j)$$

for all $j \in A$.

Note that

$$\pi^*(i^*) = \frac{1}{1 + \sum_{k \neq i^*} \rho(k|i^*)}$$

does not depend on the value of $\rho(i^*|i^*)$. With this

$$\frac{\pi^*(j)}{\pi^*(k)} = \frac{\rho(j|i^*)}{\rho(k|i^*)}$$

for all $j, k \in A$.

The product rule implies, again irrespective of the choice of $\rho(h|h)$ for $h \in A$,

$$\rho(j|i)\rho(k|j)\rho(i|k) = \rho(k|i)\rho(j|k)\rho(i|j)$$

for all $i, j, k \in A$.

and

$$\frac{\rho(j|i)\rho(i|k)}{\rho(j|k)\rho(i|j)} = \frac{\rho(j|k)}{\rho(k|j)}$$

Therefore,

$$\frac{\pi^*(j)}{\pi^*(k)} = \frac{\rho(j|i^*)\rho(i^*|k)}{\rho(i^*|j)\rho(k|i^*)} = \frac{\rho(j|k)}{\rho(k|j)}$$

for all $j, k \in A$. (11)
Then,
\[
\frac{\pi^*(j)}{\pi^*(j) + \pi^*(k)} = \frac{\rho(j | k)}{\rho(j | k) + \rho(k | j)} \quad \forall j \neq k \text{ in } A
\]
and
\[
\rho(j | k) = \frac{\pi^*(j)}{\pi^*(j) + \pi^*(k)} \frac{\rho(j | k) + \rho(k | j)}{\pi(j) + \pi(k)}
\]

This shows that (5) holds. In this case, \(M\) is reversible with respect to \(\pi^*\) since, for all \(k \neq j \in A\),
\[
M(k | j) \pi^*(j) = Q(k | j) \rho(k | j) \pi^*(j) = Q(j | k) \rho(j | k) \pi^*(k) = M(j | k) \pi^*(k)
\]
irrespective of \(Q\). This is the basic idea behind Hastings (1970).

(iii) \(\implies\) (i.a) If there exist \(\pi \in \Delta_+(A)\) and a symmetric \(s : A_\pi^2 \to (0, \infty)\) such that, for all \(k \neq j \in A\),
\[
\rho(k | j) = s(k, j) \frac{\pi(k)}{\pi(k) + \pi(j)}
\]
then for all \(k \neq j \in A\),
\[
M(k | j) \pi(j) = Q(k | j) \rho(k | j) \pi(j) = Q(j | k) s(j, k) \frac{\pi(j)}{\pi(j) + \pi(k)} \pi(k) = Q(j | k) \rho(j | k) \pi(k) = M(j | k) \pi(k)
\]
irrespective of \(Q\). Then, \(M\) is reversible with respect to \(\pi\) for all \(Q\), which proves (11.a).

Moreover, if also \(\tilde{\pi} \in \Delta_+(A)\) and \(\tilde{s} : A_\tilde{\pi}^2 \to (0, \infty)\) are such that (13) holds, the argument we just presented implies that \(M\) is also reversible with respect to \(\tilde{\pi}\) for all \(Q\). Proposition 4 tells us that \(\pi = \tilde{\pi}\), which not only proves uniqueness of \(\pi\), but also, thanks to (12), that \(\pi = \pi^*\) implying (5).

Uniqueness of \(s\) follows by inverting (5). In fact, since \(\pi\) is unique and \(\rho = \{\rho(i | j)\}_{i \neq j}\) is given, then (5) implies that
\[
s(i, j) = \frac{\pi(i) + \pi(j)}{\pi(i)} \rho(i | j) \quad \forall (i, j) \in A_\pi^2
\]
identifying \(s\).

Another application of (5), yields
\[
\frac{\pi(j)}{\pi(k)} = \frac{\pi(j) + \pi(k)}{\pi(k) + \pi(j)} \frac{s(j, k)}{s(k, j)} = \frac{\rho(j | k)}{\rho(k | j)} \quad \forall j, k \in A
\]

Thus, if \(\rho\) is unbiased,
\[
\frac{\pi(j)}{\pi(j) + \pi(k)} = \frac{\rho(j | k)}{\rho(j | k) + \rho(k | j)} = \frac{\rho(j | k)}{1} \quad \forall j, k \in A
\]
and
\[
\frac{\pi(j)}{\pi(i)} = \frac{\pi(i) + \pi(j)}{\pi(i)} \rho(i | j) = 1
\]
and \(s \equiv 1\). Conversely, if \(s \equiv 1\), then (5) implies that \(\rho\) is unbiased. \(\blacksquare\)
References

[1] C. Baldassi, S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and M. Pirazzini, A behavioral characterization of the drift diffusion model and its multialternative extension for choice under time pressure, *Management Science*, 66, 5075-5093, 2020.

[2] A. A. Barker, Monte Carlo calculations of the radial distribution functions for a proton-electron plasma, *Australian Journal of Physics*, 18, 119-134, 1965.

[3] R. Bogacz, E. Brown, J. Moehlis, P. Holmes and J. D. Cohen, The physics of optimal decision making: a formal analysis of models of performance in two-alternative forced-choice tasks. *Psychological Review*, 113, 700-765, 2006.

[4] J. R. Busemeyer and J. T. Townsend, Decision field theory: a dynamic-cognitive approach to decision making in an uncertain environment. *Psychological Review*, 100, 432-459, 1993.

[5] N. J. Higham, *Functions of matrices: theory and computation*, Society for Industrial and Applied Mathematics, 2008.

[6] N. L. Johnson, A. W. Kemp, and S. Kotz, *Univariate discrete distributions*, Wiley, 2005.

[7] F. P. Kelly, *Reversibility and stochastic networks*, Wiley, 1979.

[8] A. N. Kolmogorov, Zur theorie der Markoffschen ketten. *Mathematische Annalen*, 112, 155-160, 1936 (trans. in *Selected Works of A. N. Kolmogorov*, Springer, 1992).

[9] M. Kijima, *Markov processes for stochastic modeling*. Springer, 1997.

[10] I. Krajbich and A. Rangel, Multialternative drift-diffusion model predicts the relationship between visual fixations and choice in value-based decisions, *Proceedings of the National Academy of Sciences*, 108, 13852-13857, 2011

[11] R. D. Luce, A theory of individual choice behavior, mimeo, 1957.

[12] R. D. Luce and P. Suppes, Preference, utility and subjective probability, in *Handbook of Mathematical Psychology* (R. D. Luce, R. R. Bush and E. Galanter, eds.), v. 3, Wiley, 1965.

[13] E. Reutskaja, R. Nagel, C. F. Camerer and A. Rangel, Search dynamics in consumer choice under time pressure: An eye-tracking study, *American Economic Review*, 101, 900-926, 2011.

[14] J. E. Russo and L. D. Rosen, An eye fixation analysis of multialternative choice, *Memory and Cognition*, 3, 267-276, 1975.

[15] G. M. Stine, A. Zylberberg, J. Ditterich, M. N. Shadlen, Differentiating between integration and non-integration strategies in perceptual decision making, *biorXiv*, https://doi.org/10.1101/2020.01.24.918169

[16] K. Valkanova, Markov stochastic choice, mimeo, 2020.

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