THE STANDARD MODEL ANOMALIES IN CURVED SPACE-TIME WITH TORSION

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Abstract

Using the Fujikawa and the heat-kernel methods we make a complete and detailed computation of the global, gauge and gravitational anomalies present in the Standard Model defined on a curved space time with torsion. We find new contributions coming from curvature and torsion terms to the leptonic number anomaly (so that $B - L$ is not conserved any more), to the $U(1)_Y$ gauge and to the mixed $U(1)_Y$-gravitational anomalies, but the gauge anomaly cancellation conditions on the hypercharges remain the same. We also find that the condition, usually related to the cancellation of the mixed $U(1)_Y$-gravitational anomaly, can be reobtained in the context of the Standard Model in flat space-time by requiring the cancellation of the global Lorentz anomaly without any reference to gravitation.

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1 Introduction

In the last two decades an enormous amount of work has been devoted to the formulation of different extensions of the Standard Model (SM). These extensions include Kaluza-Klein models, Grand Unified Theories, Supersymmetry, Supergravity, Superstrings and so on (different reviews on these topics can be found in [1]). The main goal of these theories is to provide an unified framework for all the known interactions including gravitation in many cases. Nevertheless, in spite of the great number of achievements obtained in those fields, no general consensus exists on which should be the most appropriate approach to the description of the known interactions. Even in the case of the heterotic string, which seems to be the most promising theoretical framework in the opinion of many theoreticians, the low-energy predictions rely very much in the particular choice of the details of the compactification of the extra dimensions which are needed for a proper formulation of (critical) string theories.

At this point it seems to be interesting to recapitulate and try to go back to the point where fundamental physics, understood as a positive science, lies today. From this point of view, the amount of knowledge on fundamental interactions confirmed experimentally can be summarized roughly in the SM, considered as a Quantum Field Theory (QFT), and Classical Gravitation (CG) by which we mean General Relativity or other geometrical theories where the gravitational field is described as a space-time curvature.
thus including the Equivalence Principle (EP). By this we mean that any phenomenon ever observed can in principle be accommodated in the SM formulated in a curved space-time background. Of course there are many reasons to think that this is not the final theory (provided such a thing exists at all) but at least it is the minimal one compatible with all the experimental data.

For the above reason we consider an important issue the proper formulation of the SM in curved space-time. The problem of defining a QFT in curved space-time has been considered in detail in the literature some time ago (see [2] for a review). Concerning the particular case of the SM, the most important property is that it is a chiral gauge theory based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. As the matter (quarks and fermions) is described by fermionic fields, one is forced to introduce vierbeins and connections on the space-time manifold. As it is well known, once a metric is given, there is only one connection which is metric compatible and torsion free, namely the Levi-Civita connection which is defined by the Christoffel symbols. In fact this was the connection considered by Einstein in his original formulation of General Relativity. However, one can also consider the vierbein and the connection as independent structures. In this case, if one starts from the standard Einstein-Hilbert action one gets again the Christoffel symbols for the connection as a solution of an equation of motion together with the Einstein field equations for the metric (Palatini formalism). Thus in this case we also obtain the Levi-Civita connection on-shell. However quantum effects or modifications of the action obtained for example by adding higher
derivatives terms to the Einstein-Hilbert action could produce torsion. In addition, fermions, like the ones appearing in the SM, give a non-zero contribution to the torsion (see for example [3]). Finally, most of the extensions of General Relativity introduce the vierbein and the connection as independent entities and this will be our approach in the following. Nevertheless, we will keep the covariant constancy condition for the metric, in order to have a geometrical meaning for the connection. This condition amounts to consider the connection as a $SO(4)$ or $SO(3,1)$ Lie algebra valued one form (for Euclidean or Lorentzian signature respectively).

Thus in the following we will address the problem of defining properly the SM as a QFT in presence of a classical space-time with torsion. As it is well known theories with chiral fermions like the SM are potentially plagued of gauge and gravitational anomalies which can ruin the consistency of the quantum theory even if it is well defined at the classical level. Fortunately, the current assignment of hypercharges for the different fermions appearing in the SM is done in such a subtle way that all those anomalies exactly cancel. In addition we have also anomalies affecting some other global classical symmetries that can give rise to interesting physical effects like the non-conservation of the baryonic or leptonic numbers.

In this work we will compute all those anomalies in the SM defined on a curved space-time with torsion. As we will see, the torsion will give new contributions to most of those gauge, gravitational and global anomalies which are far from trivial. The plan of the paper goes as follows: In section two we develop the formalism for the definition of the SM in a curved background.
space-time with torsion. In section three we discuss the technicalities for the computation of the different anomalies including the appropriate versions of the Fujikawa and heat-kernel methods. In section four we consider the baryon and lepton anomalies in presence of torsion. In section five we compute the anomalies of the gauge $SU(3)_C \times SU(2)_L \times U(1)_Y$ group currents including the new torsion contributions. In section six we study the gravitational anomalies as anomalies in the Lorentz group understood as a gauge (local) group. In section seven we discuss some of the consequences of our previous anomaly computation and in particular those concerning the quantization of the electric charge in the framework of the SM. Finally, in section eight we briefly list the main conclusions of our work.

2 The Standard Model in curved space-time with torsion

The formulation of the SM interacting with classical gravity is based on the Einstein Equivalence Principle (EP) (see for example [4]). The EP makes it possible to obtain the curved space-time Physics (i.e. General Relativity) from that of the flat space-time (i.e. Special Relativity), since it determines the nature of the interaction with gravity of any other field. This principle states that at each point $p$ of space-time it is always possible to find a privileged coordinate system in which physics looks locally like in flat space-time. The procedure one should follow in order to introduce the gravitational interaction in any field theory built in flat space-time goes as follows: take the Lorentz invariant action of the theory and identify the coordinates appea-
ing in it with that of the locally inertial system. Then perform a coordinate change to an arbitrary coordinate system and the gravitational interaction will appear automatically. As in this work we are mainly interested in the effect of gravitation on anomalies, we will start by applying this recipe in detail to work out the gravitational interaction of Dirac spinors. At the end we will obtain also the lagrangians for scalar and gauge fields interacting with gravity.

Let us first introduce some notation. We will use latin indices \( m, n \ldots \) for objects referred to the locally inertial coordinate system and greek indices \( \mu, \nu \ldots \) for any other. If \( \{ \xi^m \} \) are the coordinates in the privileged system and \( \{ x^\mu \} \) the coordinates in any other, then:

\[
g^{\mu\nu}(x) = e^\mu_m(x)e^n_\mu(x)\eta^{mn} \tag{1}
\]

where \( \eta^{mn} = (-, -, - , -) \) is the Euclidean flat metric once the Wick rotation has been done (as usual in functional calculations we will work in Euclidean space): \( x^0 \rightarrow -i\hat{x}^4, x^i \rightarrow \hat{x}^i, \partial_0 \rightarrow i\hat{\partial}_4, \partial_i \rightarrow \hat{\partial}_i \) and the Euclidean gamma matrices: \( \gamma^0 \rightarrow \hat{\gamma}^4, \gamma^i \rightarrow i\hat{\gamma}^i \). We define: \( \hat{\gamma}_5 = -\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3\hat{\gamma}_4 \). The vierbein \( e^\mu_m(x) = \partial x^\mu/\partial \xi^m \) gives at each point the change of coordinates to the privileged system. Analogously it is possible to define an inverse vierbein by \( e^\mu_m e^m_\mu = \delta^\mu_\nu \) and \( e^\mu_m e^m_\nu = \delta^\mu_\nu \). Finally let us introduce the volume form written in terms of vierbeins:

\[
d^4\xi = \sqrt{g}d^4x = (\det e^m_\mu) d^4x \tag{2}
\]

with \( g = | \det g_{\mu\nu} | \).
In flat space-time Dirac spinors change in the following way under Lorentz transformations:

\[ \psi(p) \rightarrow U\psi(p) = e^{\frac{i}{2}\epsilon^{mn}\Sigma_{mn}}\psi(p) \]

\[ \psi^\dagger(p) \rightarrow \psi^\dagger(p)U^\dagger = \psi^\dagger(p)e^{-\frac{i}{2}\epsilon^{mn}\Sigma_{mn}} \] (3)

where \( \Sigma_{mn} = \frac{i}{4}[\gamma_m, \gamma_n] \) are the hermitian generators of the \( SO(4) \) group in the spinor representation.

The Dirac lagrangian in flat space-time

\[ L = \frac{1}{2}(\psi^\dagger \phi \psi - \partial_m \psi^\dagger \gamma^m \psi) \] (4)

is invariant under those global transformations. Notice that we have written the hermitian form of the lagrangian in Euclidean space and with fermions considered as anticommuting variables. In flat space-time it is always possible to integrate by parts and write the lagrangian in the more usual way:

\[ L = \psi^\dagger \phi \psi \] (5)

Now, the EP requires this invariance of the Dirac lagrangian under Lorentz transformations to be not only global but also local when gravitation is included. This fact forces us to introduce a covariant derivative for this gauge transformation in order to make eq.4 invariant. Therefore, let us write the gauged hermitian Dirac lagrangian in the following way:

\[ L = \frac{1}{2}(\psi^\dagger \gamma^m D_m \psi - D_m \psi^\dagger \gamma^m \psi) \] (6)
The EP has allowed us to write the Dirac lagrangian in the privileged system. Now we can write it in any other coordinate system by using the vierbein:

\[ \mathcal{L} = \frac{1}{2}(\psi^\dagger \gamma^\mu D_\mu \psi - D_\mu \psi^\dagger \gamma^\mu \psi) \]  

(7)

where we have introduced the Dirac matrices in curved space-time \( \gamma^\mu = e^\mu_m \gamma^m \). These matrices satisfy a similar algebra in curved space-time: \( \{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \). The covariant derivative is defined as usual by:

\[ D_\mu = (\partial_\mu + \Omega_\mu) \]  

(8)

where \( \Omega_\mu \) is known as the spin connection. The transformations rules of \( \Omega_\mu \) under local Lorentz transformations are those of a gauge connection:

\[ \Omega_\mu \rightarrow \Omega'_\mu = U(x)\Omega_\mu U^{-1}(x) - (\partial_\mu U)U^{-1}(x) \]  

(9)

or infinitesimally:

\[ \Omega_\mu \rightarrow \Omega_\mu + \frac{i}{2}e^{ab}(x)[\Sigma_{ab}, \Omega_\mu] - \frac{i}{2}(\partial_\mu e^{ab}(x))\Sigma_{ab} \]  

(10)

Now, recalling that the components of the connection 1-form in Riemannian geometry have precisely the latter transformation rule \( \bar{\Omega} \), we can identify:
\[ \Omega_\mu = -\frac{i}{2} \hat{\Gamma}_\mu^{ab} \Sigma_{ab} \]  

and define the covariant derivative acting on spinors as:

\[ D_\mu \psi = (\partial_\mu - \frac{i}{2} \hat{\Gamma}_\mu^{ab} \Sigma_{ab}) \psi \]  

Depending on the object this derivative acts on, the generators will appear in the corresponding representation (vector, tensor, etc) of the Lorentz group. It is easy to see that this gauge covariant derivative is nothing but the ordinary geometric covariant derivative but referred to the privileged coordinate system. However, this gauge formulation of the Lorentz group enables to introduce spinors in curved space-time which otherwise would be impossible, since \( GL(4) \) does not posses spinor representations.

Notice that \( \{ \hat{\Gamma}_\mu^{ab} \} \) does not have to be a torsion free Levi-Civita connection, which we will denote \( \{ \Gamma_\mu^{ab} \} \). In general, if we take a connection compatible with the metric, i.e. \( \hat{\nabla}_\nu g_{\alpha\beta} = 0 \), then it can be written as:

\[ \hat{\Gamma}_\mu^{ab} = \Gamma_\mu^{ab} + e^a_\nu e^b_\lambda K^{\nu}_{\mu\lambda} \]  

where \( K^{\nu}_{\mu\lambda} \) is known as the contorsion tensor which in terms of the torsion tensor reads:

\[ K^{\nu\mu\lambda} = \frac{1}{2}(T^{\nu\mu\lambda} + T^{\mu\nu\lambda} + T^{\lambda\nu\mu}) \]
Now, using the decomposition in eq. (13) we can write the Dirac lagrangian with an arbitrary metric connection in terms of the usual Levi-Civita plus an additional term depending on the torsion [6]:

\[
\mathcal{L} = \frac{1}{2}(\gamma^\mu(\partial_\mu - i\hat{\Gamma}_{\mu}^{ab} \Sigma_{ab})\psi - (\partial_\mu \psi^\dagger + \frac{i}{2} \hat{\Gamma}_{\mu}^{ab} \psi^\dagger \Sigma_{ab})\gamma^\mu \psi)
\]

\[
= \psi^\dagger \gamma^\mu(\partial_\mu - i\hat{\Gamma}_{\mu}^{ab} \Sigma_{ab} + \frac{1}{2} T_\mu)\psi = \psi^\dagger \gamma^\mu(\partial_\mu - i\frac{1}{2} \hat{\Gamma}_{\mu}^{ab} \Sigma_{ab} - \frac{1}{8} S_\mu \gamma^5)\psi
\]

where:

\[
S_\alpha = \epsilon_{\mu\nu\lambda\alpha} T^{\mu\nu\lambda}
\]

\[
T_\mu = T^\lambda_{\lambda\mu} = K^\lambda_{\lambda\mu}
\]

Note that with this definition \( S_\mu \) is axial the part of the torsion tensor.

In conclusion, the lagrangian for Dirac fermions in a curved space-time with torsion is that of a fermion in a curved space-time without torsion plus an axial interaction with \( S_\mu \). Nevertheless, there is a difference between the axial coupling of torsion with the usual axial couplings of gauge fields. While the latter breaks the hermiticity of the Dirac operator, the former does not. This similarity will simplify the computation of the anomalies when using the well-known heat kernel expansion in curved space-time.

Notice however that this is the minimal lagrangian. If we had used instead a more general form, some other non-minimal couplings with torsion could also have appeared, as for instance: \( i\psi^\dagger \gamma^\mu T_\mu \psi \) [6]. Such non-minimal coupling can be discarded by anomaly cancellation arguments. In fact, this
term behaves as an hypercharge field which interacts with the left and right components with the same coupling constant (except for the neutrino). We will see that such coupling yields an anomaly in the $SU(2)_L, U(1)_Y$ as well as in the local Lorentz symmetry. As there is no hypercharge assignment that could cancel simultaneously all the anomalous contributions, we will not consider it here.

Now that we know the general expression for the Dirac lagrangian in a curved space-time with torsion, let us apply it to the SM matter sector, which can be written in the following way in the case of massless fermions and without considering the Yukawa couplings to the Higgs field:

\[ \mathcal{L}_m = Q^\dagger \bar{\psi}_Q Q + L^\dagger \bar{\psi}_L L \]  

where the Dirac operators for quarks and leptons are defined as:

\[ i \bar{\psi}_Q = i \gamma^\mu (\partial_\mu + \Omega_\mu + G_\mu + W_\mu P_L + B_\mu + S_\mu \gamma_5) \]

\[ i \bar{\psi}_L = i \gamma^\mu (\partial_\mu + \Omega_\mu + W_\mu P_L + B_\mu + S_\mu \gamma_5) \]  

Here we have organized the matter fields in doublets, so that for the first family we have:

\[ Q = \begin{bmatrix} u \\ d \end{bmatrix}, \quad L = \begin{bmatrix} \nu \\ e \end{bmatrix} \]  

Their left components $Q_L$ and $L_L$ are $SU(2)_L$ doublets, while each component of the right part $Q_R$ and $L_R$ are $SU(2)_L$ singlets. In turn the $u$ and $d$ quarks are $SU(3)_c$ triplets. The gauge fields appearing in the operators are:
• Gluons; which are those corresponding to the $SU(3)_c$ group, that we will denote by

$$G_\mu = -ig_S G^a_\mu \Lambda^a$$  \hspace{1cm} (20)

Here the $\Lambda^a$ are the eight group generators in a product representation: $\Lambda^a = \lambda^a \otimes 1_2$, where $\lambda^a$ are the properly normalized Gell-Mann matrices and $1_2$ is the $2 \times 2$ identity matrix in flavor space.

• $W$-bosons, which are those corresponding to the $SU(2)_L$ symmetry that we will write as

$$W_\mu = -ig W^a T^a$$  \hspace{1cm} (21)

where $T^a$ are the three group generators in the product representation: $T^a = 1_3 \otimes \sigma^a/2$ for quarks and $T^a = \sigma^a/2$ for leptons, with $\sigma^a$ the Pauli matrices.

• Finally there is also the hypercharge boson

$$B_\mu = i g' B_\mu \left( P_L \begin{pmatrix} y^u_L \\ y^d_L \end{pmatrix} + P_R \begin{pmatrix} y^u_R \\ y^d_R \end{pmatrix} \right)$$  \hspace{1cm} (22)

which has been written for the case of quarks. For leptons the expression is obtained using their corresponding hypercharges.

Before commenting on the curvature and torsion terms we should stress that these operators are not hermitian. This is due to the chiral couplings of $SU(2)_L$ and hypercharge fields. Thus the adjoint operators are:
\[(i \bar{\psi}_Q)^\dagger = i\gamma^\mu(\partial_\mu + \Omega_\mu + G_\mu + W_\mu P_R + B_\mu + S_\mu \gamma_5)\]
\[(i \bar{\psi}_L)^\dagger = i\gamma^\mu(\partial_\mu + \bar{\Omega}_\mu + \bar{W}_\mu P_R + \bar{B}_\mu + \bar{S}_\mu \gamma_5)\]

where:
\[
\bar{B}_\mu = i g^' B_\mu \left( P_R \begin{pmatrix} y^a_L & y^d_L \\ y^a_R & y^d_R \end{pmatrix} + P_L \begin{pmatrix} y^b_R & y^d_R \\ y^b_L & y^d_L \end{pmatrix} \right) \]

and analogously for leptons. Notice that, since there is no right neutrino, the spin connection can be written as follows for leptonic operators:

\[
\Omega_\mu = -\frac{i}{2} \Gamma^a_{\mu} \left( P_R \Sigma_{ab} \Sigma_{ab} \right), \quad \bar{\Omega}_\mu = -\frac{i}{2} \Gamma^a_{\mu} \left( P_L \Sigma_{ab} \Sigma_{ab} \right)\]

for the same reason, the torsion terms are:

\[
S_\mu \gamma_5 = -\frac{1}{8} S_\mu \left( P_L \gamma_5 \gamma_5 \right), \quad \bar{S}_\mu \gamma_5 = -\frac{1}{8} \bar{S}_\mu \left( P_R \gamma_5 \gamma_5 \right)\]

Once we have obtained the lagrangian for Dirac spinors in curved space-time with torsion, we will write the corresponding one for scalar fields. The standard lagrangian for a scalar field in flat space-time can be written as follows:

\[
L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi)\]

(we will consider only the case of a real scalar field since the complex case is completely analogous). According to the given prescription to build field
theories interacting with gravity from the EP, we have to identify the coordinates in the lagrangian density with those of the privileged system, and then make local the Lorentz invariance. As the fields are scalars, they do not change under Lorentz transformations and their covariant derivative is just an ordinary derivative. Finally we have to use the vierbein to transform to an arbitrary coordinate system. Then the final expression for the action integral reads

\[
S = \int d^4x \sqrt{g} \left( \frac{1}{2} g^\mu\nu \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m \phi^2 - V(\phi) \right)
\] (28)

which is the minimal coupling to gravity. However the most general lagrangian for a scalar field interacting with gravity, up to a given order in derivatives, could also include terms like \(R \phi^2\), \(T_\mu T^\mu \phi^2\), \(q_{\mu\nu\alpha} q^{\mu\nu\alpha} \phi^2\), \(\nabla_\mu T^\mu \phi^2\), etc, where we have defined:

\[
T_{\alpha\beta\mu} = q_{\alpha\beta\mu} + \frac{1}{3} (T_{\beta g_{\alpha\mu}} - T_{\mu g_{\alpha\beta}}) - \frac{1}{6} S^\nu \epsilon_{\alpha\beta\mu\nu}
\] (29)

and \(q_{\alpha\beta\mu}\) satisfies \(\epsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0\).

Nevertheless, even if we had started with the minimal lagrangian in eq.(28), some of these terms would appear in the renormalization procedure as one-loop counterterms. In the case of scalar fields interacting with some gauge fields, we should turn the derivatives in eq.(28) into gauge covariant derivatives, as it happens for instance for the Higgs fields in the SM.

Once we have worked out the lagrangian for scalars we turn to the gauge fields. The Yang-Mills lagrangian is given by:
We consider the strength tensor $F_{mn}$ as defined in a locally inertial coordinate system. $F_{mn}$ is a Lorentz tensor and $F^a_{mn} F^m_{an}$ is invariant under global and local Lorentz transformations. Therefore we only have to transform it to an arbitrary coordinate system by means of the vierbein:

\[ F^a_{\mu\nu} = e^m_\mu e^n_\nu F^a_{mn} \]  

(31)

and the action integral reads:

\[ S = \int d^4x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \]

(32)

Therefore, the coupling to gravity occurs only through the vierbein and the spin connection does not appear. Then there is no coupling between gauge fields and torsion. As a consequence, the only fields which may potentially couple to torsion are the fermion fields. It is also important to notice that this is the only way to preserve gauge invariance when passing from flat to curved space-time. In fact the other natural possibility, which consists in defining

\[ F_{\mu\nu} = \hat{\nabla}_\mu A_\nu - \hat{\nabla}_\nu A_\mu + [A_\mu, A_\nu] \]

(33)

drives to a Yang-Mills action that is not gauge invariant when torsion is present, since:
Here the first term is gauge covariant, whereas the last term explicitly breaks the gauge invariance in the Yang-Mills action for non-vanishing torsion.

3 The heat-kernel for the Standard Model operators

In this section we will study the possible violation, due to quantum effects, of the SM classical symmetries, which are of two kinds:

- Those which are exact, that can either be gauge symmetries as $SU(3)_c$, $SU(2)_L, U(1)_Y$ and the Lorentz group, or global as those corresponding to the lepton and baryon number conservation.

- There are also those which are approximate which we will not consider here. Some examples are the $U(1)_A$, the $SU(3)_L \times SU(3)_R$ chiral symmetries in low-energy QCD and the $SU(2)_L \times SU(2)_R$ global symmetry of the symmetry breaking sector of the SM which are only exact in certain limits.

The non-conservation of the gauge symmetries due to anomalies leads to the inconsistency of the model. Therefore, it is interesting that the inclusion of the gravitational interaction does not affect the anomaly cancellation in
gauge currents. In addition, the gravitational contribution to the non conservation of lepton and baryon number, could have some relevance concerning to the problem of the baryon number asymmetry of the Universe. Let us then discuss the status of each of these symmetries at the quantum level in a curved space-time with torsion.

There are several techniques proposed for the computation of anomalies in the literature. For our purposes here the most appropriate is to use the functional methods that were first introduced by Fujikawa [7] in a flat space-time and later extended to curved space-time by Yajima [8]. According to these methods we need an hermitian operator in order to regularize the anomalous path integral jacobian of the symmetry transformation. Thus for instance, for the axial anomaly the transformation yields:

\[
[d\psi d\psi^\dagger] \rightarrow [d\psi d\psi^\dagger] \exp(-2 \int d^4 x \sqrt{g} i \alpha(x) A(x))
\] (35)

where the anomaly \( A(x) \) appearing in the (regularized) jacobian reads

\[
A_{\text{reg}} = \lim_{t \to \infty} \sum_n \phi_n^\dagger \gamma_5 e^{-\frac{t^2}{4}} \phi_n = \lim_{t \to \infty} \sum_n \phi_n^\dagger \gamma_5 e^{-\frac{(\partial \phi)^2}{4t^2}} \phi_n = \lim_{t \to \infty} tr \frac{t^2}{(4\pi)^2} \gamma_5 \sum_{n=0}^\infty \frac{a_n(x)}{t^n}
\] (36)

where \( \lambda_n \) are the eigenvalues of the hermitian operator of the theory and in the last step we have performed a heat-kernel expansion. In general, the expression above is divergent, in the \( t \to \infty \) limit, due to the two first terms in the heat-kernel expansion. In such case, certain renormalization prescription will be needed to obtain a finite value for the anomaly. However, it may happen, as it occurs in theories with only vector couplings to gauge fields, that those potentially divergent terms vanish and \( A_{\text{reg}} \) is finite.
One prescription to eliminate the divergent terms in the anomaly consists in removing them directly. This drastic procedure can be justified in some circumstances as follows [9]. Let us define the transformation jacobian as the quotient between the effective action and the transformed effective action, both regularized using \(\zeta\)-function regularization. Thus, for instance, in the case of the axial transformations considered before:

\[
det \mathcal{D} = J \det (e^{\gamma_5 \alpha(x)} \mathcal{D} e^{\gamma_5 \alpha(x)})
\]

where \(J\) is the jacobian of the symmetry transformation. It is then possible to show that, provided the operator is hermitian, the result for the anomaly is the same as the one obtained using the Fujikawa method and removing the divergent terms, i.e: \(A = \frac{1}{(4\pi)^2} tr \gamma_5 a_2(x)\). Therefore, this is the prescription we will use to render the result finite. However, for that purpose, we need an hermitian operator as a regulator but, as we have seen, the Dirac operator in the SM is not hermitian. Several methods have been proposed [10] to avoid this problem, we will mention two of them. In the first one we split the lagrangian and the integration measure in their left and right components:

\[
[d\psi d\psi^\dagger] \rightarrow [d\psi_R^\dagger d\psi_L^\dagger d\psi_R d\psi_L]
\]

\[
\psi^\dagger \mathcal{D} \psi = \psi_R^\dagger \mathcal{D}_L \psi_L + \psi_L^\dagger \mathcal{D}_R \psi_R
\]

In curved space-time without torsion (the torsion term is written between brackets) the Dirac quark operators
\[
\begin{align*}
    i \bar{\psi}_L &= i\gamma^\mu (\partial_\mu + \Omega_\mu + G_\mu + W_\mu + B_\mu^L(-S_\mu)) \\
    i \bar{\psi}_R &= i\gamma^\mu (\partial_\mu + \Omega_\mu + G_\mu + B_\mu^R(+S_\mu))
\end{align*}
\] (39)

are hermitian (the same is true for leptons). Thus they allow the regularization of the corresponding piece of the anomaly. However, the torsion term breaks the hermiticity of these operators and therefore this method does not seem to be suitable in presence of torsion. In spite of this fact, one could rotate \( S_\mu \to iS_\mu \) \[11\]. This makes the operators in the above equations hermitian and then, at the end of the calculation, one can undo the rotation. Such procedure has been proved to be useful in theories with axial gauge couplings and yields the so called consistent anomaly. Notice however that, in presence of torsion, certain inconsistencies appear, since it can be shown that there would not be any appropriate choice of hypercharges in the SM that could cancel the gauge anomalies.

An alternative method \[10\] which does not suffer from this inconsistencies is to regularize separately those pieces in the anomaly coming from the transformation of \( \psi \) and \( \psi^\dagger \). In this case our first step is to build two hermitian operators which preserve all the gauge symmetries in the lagrangian, namely:

\[
\begin{align*}
    H_\psi &= (i \slashed{D})^\dagger (i \slashed{D}) \\
    H_{\psi^\dagger} &= (i \slashed{D}) (i \slashed{D})^\dagger
\end{align*}
\] (40)
Then the hermiticity ensures that their corresponding eigenfunctions form a complete set:

\[ H \psi_n = \lambda_n^2 \psi_n \]
\[ H \psi^\dagger \xi_n = \lambda_n^2 \xi_n \] (41)

Now we expand \( \psi \) and \( \psi^\dagger \) in terms of eigenfunctions of \( H \psi \) and \( H \psi^\dagger \) respectively:

\[ \psi = \sum_n a_n \phi_n \]
\[ \psi^\dagger = \sum_n \bar{b}_n \xi^\dagger_n \] (42)

Under an infinitesimal transformation like (the non-abelian case follows the same steps):

\[ \psi \rightarrow \psi + i\alpha(x)\psi \]
\[ \psi^\dagger \rightarrow \psi^\dagger - i\psi^\dagger \alpha(x) \] (43)

the integration measure changes as:

\[ [d\psi d\psi^\dagger] \propto [da_n d\bar{b}_n] \rightarrow [da'_n d\bar{b}'_n] = [da_n d\bar{b}_n] \exp(- \int d^4 x \sqrt{g} i \alpha(x) A(x)) \] (44)

where, in the present case, \( A(x) \) is the vector abelian anomaly

\[ A(x) = \sum_n \phi_n^\dagger \phi_n - \sum_n \xi_n^\dagger \xi_n \] (45)

As it has already been mentioned, we regularize each piece of the anomaly with the corresponding operator:

\[ A(x) = \lim_{M \rightarrow \infty} \sum_n \phi_n^\dagger e^{-\frac{H \psi}{M^2}} \phi_n - \sum_n \xi_n^\dagger e^{-\frac{H \psi^\dagger}{M^2}} \xi_n \] (46)
In order to obtain a finite result, we have to perform the heat-kernel expansion for the $H_\psi$ and $H_{\psi^\dagger}$ operators and subtract the divergent terms. However, still a new difficulty appears. Although the heat-kernel expansion has been worked out for a wide class of operators even in curved space-time, the coefficients become unmanageable \cite{12} for operators which do not cast the general form:

$$H = D_\mu D^\mu + X \quad (47)$$

where $X$ does not contain derivatives. At first glance, this is not the case of $H_\psi$ and $H_{\psi^\dagger}$. However, with some algebra we can write them in the desired form \cite{13}:

$$H_\psi = (i\not{D})(i\not{D})^\dagger = D_\mu D^\mu + \gamma_5 S_{\mu}^\nu + 2 S_\mu S^\mu - \frac{1}{4} [\gamma^\mu, \gamma^\nu][d_\mu, d_\nu]$$

$$H_{\psi^\dagger} = (i\not{D})(i\not{D}) = \bar{D}_\mu \bar{D}^\mu + \gamma_5 \bar{S}_\mu^\nu + 2 \bar{S}_\mu \bar{S}^\mu - \frac{1}{4} [\gamma^\mu, \gamma^\nu][\bar{d}_\mu, \bar{d}_\nu] \quad (48)$$

where for quarks:

$$d_\mu = \partial_\mu + \Omega_\mu + G_\mu + W_\mu P_L + B_\mu$$

$$\bar{d}_\mu = \partial_\mu + \Omega_\mu + \bar{G}_\mu + \bar{W}_\mu P_R + \bar{B}_\mu \quad (49)$$

and

$$D_\mu = d_\mu - \frac{1}{2} \gamma_5 [\gamma^\mu, \gamma^\nu] S^\nu$$

$$\bar{D}_\mu = \bar{d}_\mu - \frac{1}{2} \gamma_5 [\gamma^\mu, \gamma^\nu] \bar{S}^\nu \quad (50)$$
In the case of leptons we have:

\[
\begin{align*}
d_\mu &= \partial_\mu + \Omega_\mu + W_\mu P_L + B_\mu \\
\bar{d}_\mu &= \partial_\mu + \bar{\Omega}_\mu + W_\mu P_R + \bar{B}_\mu 
\end{align*}
\]

and:

\[
\begin{align*}
D_\mu &= d_\mu - \frac{1}{2}\gamma_5[\gamma^\mu, \gamma^\nu]S^\nu \\
\bar{D}_\mu &= \bar{d}_\mu - \frac{1}{2}\gamma_5[\gamma^\mu, \gamma^\nu]\bar{S}^\nu
\end{align*}
\]

where \(B_\mu, \bar{\Omega}_\mu\) and \(\bar{S}^\nu\) have been defined above. Therefore we have already an appropriate form to use the heat-kernel expansion. Now removing the divergent \(a_1(x)\) coefficient we obtain for the anomaly in the abelian vector currents:

\[
A(x) = \frac{1}{(4\pi)^2}tr(a_2(H_\psi, x) - a_2(H_{\psi^\dagger}, x))
\]

where the second coefficient in the heat-kernel expansion in curved space-time has been worked out in different references [8] [12] [13] using different methods and in our case reads:

\[
a_2(H_\psi, x) = \frac{1}{12}[D_\mu, D_\nu][D^\mu, D^\nu] + \frac{1}{6}[D_\mu, [D^\mu, X]] + \frac{1}{2}X^2 - \frac{1}{6}RX - \frac{1}{30}R_{\mu\nu}^\mu + \frac{1}{72}R^2 + \frac{1}{180}(R_{\mu\nu\rho\sigma}R^\mu_{\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu})
\]
and:

\[
\alpha_2(H_{\Psi^\dagger, x}) = \frac{1}{12} [\bar{D}_\mu, D_\nu] [\bar{D}^\mu, D^n] + \frac{1}{6} [\bar{D}_\mu, [\bar{D}^\mu, \bar{X}]] + \frac{1}{2} \bar{X}^2 - \frac{1}{6} \bar{R} \bar{X} \\
- \frac{1}{30} \bar{R}^\mu + \frac{1}{72} \bar{R}^2 + \frac{1}{180} (\bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma} - \bar{R}_{\mu\nu} \bar{R}^{\mu\nu})
\]  

(55)

where according to eq.48:

\[
X = \gamma_5 S^\mu_{\mu} + 2 S^\mu S^\mu - \frac{1}{4} [\gamma^\mu, \gamma^\nu] [d_\mu, d_\nu]
\]

\[
\bar{X} = \gamma_5 \bar{S}^\mu_{\mu} + 2 \bar{S}^\mu \bar{S}^\mu - \frac{1}{4} [\gamma^\mu, \gamma^\nu] [\bar{d}_\mu, \bar{d}_\nu]
\]  

(56)

Notice that for quarks, the torsion and curvature terms are the same either with or without a bar. The explicit expression for the commutators can be written as follows for quarks:

\[
[D_\mu, D_\nu] = R_{\mu\nu} + G_{\mu\nu} + W_{\mu\nu} P_L + B_{\mu\nu} + [\gamma_5, \gamma_\alpha] \left( \frac{1}{2} \gamma_\alpha S^\alpha_{\mu\nu} - S_\nu S^\alpha \right) - S^\alpha S_\alpha [\gamma_\nu, \gamma_\mu]
\]

(57)

and

\[
[\bar{D}_\mu, \bar{D}_\nu] = \bar{R}_{\mu\nu} + \bar{G}_{\mu\nu} + \bar{W}_{\mu\nu} P_L + \bar{B}_{\mu\nu} + [\gamma_5, \gamma_\alpha] \left( \frac{1}{2} \gamma_\alpha S^\alpha_{\mu\nu} - \bar{S}_\nu S^\alpha \right) - \bar{S}^\alpha S_\alpha [\gamma_\nu, \gamma_\mu]
\]

(58)

For leptons we have:

\[
[D_\mu, D_\nu] = R_{\mu\nu} + W_{\mu\nu} P_L + B_{\mu\nu} + [\gamma_5, \gamma_\alpha] \left( \frac{1}{2} \gamma_\alpha S^\alpha_{\mu\nu} - S_\nu S^\alpha \right)
\]

(59)

\[
- [\gamma_\nu, \gamma_\alpha] \left( \frac{1}{2} \gamma_\alpha S^\alpha_{\mu\nu} - S_\nu S^\alpha \right) - S^\alpha S_\alpha [\gamma_\nu, \gamma_\mu]
\]
and

\[ [\bar{D}_\mu, \bar{D}_\nu] = \bar{R}_{\mu\nu} + W_{\mu\nu}P_R + \bar{B}_{\mu\nu} + [\gamma_\mu, \gamma_\alpha]\left(\frac{1}{2}\gamma_5\bar{S}^\alpha_{\mu} - \bar{S}_\nu\bar{S}^\alpha\right) \]

\[ - \left[\gamma_\nu, \gamma_\alpha\right]\left(\frac{1}{2}\gamma_5\bar{S}^\alpha_{\mu} - \bar{S}_\mu\bar{S}^\alpha\right) - \bar{S}_\alpha\bar{S}_\alpha\left[\gamma_\nu, \gamma_\mu\right] \]

where we have defined for leptons:

\[ R_{\mu\nu} = -\frac{i}{2}R^{ab}_{\mu\nu}\left(PL^{ab}_{\Sigma_{ab}}\right), \quad \bar{R}_{\mu\nu} = -\frac{i}{2}R^{ab}_{\mu\nu}\left(PR^{ab}_{\Sigma_{ab}}\right) \]

and \( G_{\mu\nu}, W_{\mu\nu} \) and \( B_{\mu\nu} \) are the usual strength tensors of the corresponding gauge groups.

Once we have a consistent method for computing anomalies in a curved space-time with torsion, let us apply it to the anomalies present in the SM.

4 Anomalies in the leptonic and baryonic currents

In this section we will make use of the method just presented in the previous section in order to compute the anomalies in two global vector currents \( B \) and \( L \), whose difference \( B - L \) is conserved in flat space-time although separately they are not. However, we will show that in curved space-times the absence of right neutrinos implies that, in some sense, gravity couples chirally, and thus the anomaly in the leptonic current acquires a gravitational contribution. Nevertheless, these gravitational terms are not present in the baryonic sector, thus yielding the above commented \( B - L \) non-conservation.

Let us proceed with the computation. In order to obtain the anomalous Ward identities related to the leptonic and baryonic numbers, we will consider the following local transformations of quarks and leptons:
\[
\psi \rightarrow \psi + i\alpha(x)\psi \\
\psi^\dagger \rightarrow \psi^\dagger - i\psi^\dagger \alpha(x)
\]  

(62)

Note that the classical action would be invariant under these transformations if they were global. In order to calculate how the SM fermionic action changes, we write it in terms of a general connection:

\[
\int d^4x \sqrt{g}L_m = \int d^4x \sqrt{g/2}(\psi^\dagger \gamma^\mu D_\mu \psi - (D_\mu \psi)^\dagger \gamma^\mu \psi)
\]  

(63)

where we denote by \(\psi\) the leptons and quarks and \(D_\mu\) is the gauge and Lorentz covariant derivative with the general connection. Under the above transformations, the classical action changes as follows:

\[
\int d^4x \sqrt{g}L_m \rightarrow \int d^4x \sqrt{g}(L - i\alpha(x) \nabla_\mu (\psi^\dagger \gamma^\mu \psi))
\]  

(64)

where we have integrated by parts with the Levi-Civita covariant derivative \(\nabla_\mu\). On the contrary, the effective action does not change under the transformation since it only affects to fermion fields which are integration variables:

\[
e^{-W[A,\Gamma,e]} = \int [d\psi d\psi^\dagger] e^{-\int d^4x \sqrt{g}(L,\psi^\dagger)} = \int [d\psi' d\psi'^\dagger] e^{-\int d^4x \sqrt{g}L_m(\psi',\psi'^\dagger)}
\]  

(65)

Now using eqs.\#14 and \#64 we obtain for the effective action the following expression:
\[
e^{-W[A,T,e]} = \int [d\psi d\psi^\dagger] e^{-\int d^4x \sqrt{g}\alpha(x) A(x) e^\int d^4x \sqrt{g}\alpha(x) \nabla_\mu j^\mu e^{-\int d^4x \sqrt{g}L_m(\psi,\psi^\dagger)}} \tag{66}
\]

Therefore, identifying the exponents in eqs. 65 and 66 we arrive at:

\[
A(x) = \nabla_\mu j^\mu \tag{67}
\]

As we saw in the previous section the regularized expression for the anomaly is that of eq. 53. In case we applied the transformations in eq. 62 to quarks, we would have obtained the anomaly in the baryonic current which is:

\[
\nabla_\mu j^\mu_B = \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \left( \frac{g^2}{2} W^a_{\mu\nu} W^a_{\alpha\beta} + g'^2 B_{\mu\nu} B_{\alpha\beta} \sum_{u,d} (y^2_L - y^2_R) \right) \tag{68}
\]

where the baryonic current is defined in the usual form:

\[
j_B^\mu = \frac{1}{N_c} Q^\dagger \gamma^\mu Q \tag{69}
\]

We see that the result agrees with the flat space-time case. There is no contribution from the curvature nor the torsion.

Following the same steps with the operators for leptons, we obtain the anomaly in the leptonic current which reads:

\[
\nabla_\mu j^\mu_L = \frac{1}{32\pi^2} \left\{ -\frac{\epsilon^{\alpha\beta\gamma\delta}}{24} R^{\mu\nu}_{\alpha\beta\gamma\delta} R^\mu_{\nu\gamma\delta} + \frac{\epsilon^{\alpha\beta\gamma\delta}}{48} S_{\beta\gamma} S_{\delta\alpha} + \epsilon^{\alpha\beta\gamma\delta} \left( \frac{g^2}{2} W^a_{\gamma\delta} W^a_{\alpha\beta} 
\right. \\
\left. + g'^2 B_{\gamma\delta} B_{\alpha\beta} \sum_{\nu,e} (y^2_L - y^2_R) \right) + \frac{1}{6} \Box S^\alpha_{\alpha \delta} + \frac{1}{96} (S^\alpha S^\nu S^\alpha)_{\nu} - \frac{1}{6} (R^\mu_{\nu\alpha} S_{\alpha} - \frac{1}{2} R S^\nu)_{\nu} \right\} \tag{70}
\]
where we have defined the leptonic current as:

$$j^\mu_L = \bar{L} \gamma^\mu L$$ (71)

We see in this case that, due to the non-existence of right neutrinos, some terms depending on the curvature and torsion (as total divergences) appear in the anomaly. If we had assumed their existence, such terms would have not appeared and $B - L$ would be conserved, as it happens in flat space time, provided the following relation is satisfied:

$$\sum_{u,d} (y_L^2 - y_R^2) = \sum_{\nu,e} (y_L^2 - y_R^2)$$ (72)

which is indeed the case for the usual SM hypercharge assignment.

5 Gauge anomalies

In the previous section we have shown that the formulation of the SM in a curved space-time with torsion may drive to the non-conservation of global currents $B - L$ which however are conserved in flat space-time. In this section we will study whether something similar happens to gauge currents. The non-conservation of gauge currents would destroy the consistency of the model. On the other hand one may wonder, whether the cancellation of new contributions due to curvature and torsion could impose new constraints to hypercharge assignments.

Let us begin by writing the effective action for the SM matter sector:

$$e^{-W[A,\Gamma,e]} = \int [d\psi d\psi^\dagger] e^{-\int d^4x \sqrt{g} \mathcal{L}_m}$$ (73)
The matter lagrangian given in eq. 17 is invariant under the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge transformations, which are given by:

$$Q \rightarrow Q - i\theta^a(x)\Lambda^a Q$$
$$Q^\dagger \rightarrow Q^\dagger + i\theta^a(x)Q^\dagger \Lambda^a$$

(74)

where, following the definitions at the beginning of this chapter, $\Lambda^a$ are the $SU(3)_c$ generators in the appropriate representation. These transformations only affect to quarks since they are the only fields that couple to $SU(3)_c$. We also have:

$$\psi \rightarrow \psi - i\theta^a(x)T^a P_L \psi$$
$$\psi^\dagger \rightarrow \psi^\dagger + i\theta^a(x)\psi^\dagger P_R T^a$$

(75)

Here $T^a$ are the $SU(2)_L$ generators in the appropriate representation and $\psi$ stands for quarks or leptons. Finally, the hypercharge transformation reads:

$$\psi \rightarrow \psi - i\theta(x)(y_L P_L + y_R P_R)\psi$$
$$\psi^\dagger \rightarrow \psi^\dagger + i\theta(x)\psi^\dagger (y_L P_R + y_R P_L)$$

(76)

where $y_L$ and $y_R$ are the diagonal hypercharge matrices in flavor space which appear in eq. 22. As it is well known, in spite of the invariance of the lagrangian under the above gauge transformations, the effective action may have an anomalous variation due to the integration measure. In the following we will obtain the expression for the anomalous variation of the effective
action in the case of SU(3)$_C$ transformations, but the result will be equally valid for the other groups. Let us first introduce the notation $\theta = -i\theta^a \Lambda^a$, $D_\mu \theta = \partial_\mu \theta + [G_\mu, \theta]$. We will use that:

$$\frac{\delta W}{\delta G_\mu^a} = -i g_s \langle Q^\dagger \gamma^\mu \Lambda^a Q \rangle = -i g_s \langle j^{\mu a} \rangle$$

(77)

is the expectation value of the gauge current in presence of the background fields. We will also define:

$$\langle j^\mu \rangle = \langle j^{\mu a} \Lambda^a \rangle$$

(78)

Under the previously mentioned SU(3)$_C$ transformations the gauge fields change as follows:

$$G_\mu \rightarrow G_\mu - D_\mu \theta$$

(79)

or in components:

$$G^c_\mu \rightarrow G^c_\mu - \frac{1}{g_s} \partial_\mu \theta^c + G^b_\mu \theta^a f^{abc}$$

(80)

and the anomalous change in the effective action is given by:

$$W[G - D \theta, \Gamma, e] - W[G, \Gamma, e] = \int d^4x \sqrt{g} \left( \frac{1}{g_s} \partial_\mu \theta^c + G^a_\mu \theta^b f^{abc} \right) \frac{\delta W}{\delta G^c_\mu}$$

$$= \int d^4x \sqrt{g} \theta^b \left( \frac{1}{g_s} \nabla_\mu \frac{\delta W}{\delta G^b_\mu} + G^a_\mu \frac{\delta W}{\delta G^c_\mu} f^{acb} \right)$$

$$= -\int d^4x \sqrt{g} i\theta^b (D_\mu \langle j^{\mu} \rangle)^b$$

(81)
where we have integrated by parts with the Levi-Civita covariant derivative and we have made use of the symmetry properties of the structure constants $f^{abc}$. Notice that we have denoted:

$$D_{\mu} \langle j^{\mu} \rangle = \nabla_{\mu} \langle j^{\mu} \rangle + [G_{\mu}, \langle j^{\mu} \rangle]$$  \hspace{1cm} (82)

and:

$$D_{\mu} \langle j^{\mu} \rangle = (D_{\mu} \langle j^{\mu} \rangle)^a \Lambda^a$$  \hspace{1cm} (83)

The change in the integration measure can be computed in the standard fashion as we did in the abelian case and it yields:

$$[dQ'dQ^\dagger] = [dQdQ^\dagger]e^{i \sum_n \int d^4x \sqrt{g} \phi_n^\dagger \theta^\mu(x) \Lambda^a \phi_n - \xi_n^\dagger \theta^\mu(x) \Lambda^a \xi_n}$$  \hspace{1cm} (84)

where $\phi_n$ and $\xi_n$ are given in eq.41. The anomaly is defined as:

$$A^a(x)_{SU(3)} = \sum_n (\phi_n^\dagger \Lambda^a \phi_n - \xi_n^\dagger \Lambda^a \xi_n}$$  \hspace{1cm} (85)

Therefore we can write the transformed effective action as

$$e^{-W[G',\Gamma,e]} = \int [d\psi d\psi^\dagger]e^{-\int d^4x \sqrt{g} \mathcal{L}_m}e^{-i \int d^4x \sqrt{g} \theta^\mu(x) A^a(x)}$$  \hspace{1cm} (86)

Expanding to first order in $\theta$ and identifying with eq.81 we obtain:

$$(D_{\mu} \langle j^{\mu} \rangle)^a = A^a(x)$$  \hspace{1cm} (87)

This anomalous Ward identity implies that the non-conservation of the gauge current expectation value is given by the anomaly coefficient. Finally the expression for the anomaly in the $SU(2)_L$ and $U(1)_Y$ currents can be computed in the same way and are given by:
\[ A^a(x)_{SU(2)} = \sum_n (\phi_n^\dagger P_L T^a \phi_n - \xi_n^\dagger P_R T^a \xi_n) \]
\[ A(x)_{U(1)} = \sum_n (\phi_n^\dagger (y_L P_L + y_R P_R) \phi_n - \xi_n^\dagger (y_L P_R + y_R P_L) \xi_n) \] (88)

As we have said before these expressions for the anomalies need regularization and, as we did in the abelian case, we will use the operators \( H_\psi \) and \( H_\psi^\dagger \) defined in eq.40 to regularize each piece of the anomaly separately. The results are the following:

**Anomaly in the SU(3)_c gauge current.**

\[ A^a_{SU(3)}(x) = \frac{1}{(4\pi)^2} tr(\Lambda^a(a_2(H_\psi, x) - a_2(H_{\psi^\dagger}, x))) \] (89)

which for the divergence of the current gives:

\[ (D_\mu (j^\mu))^a = -\frac{1}{32\pi^2} g_s g_s' \epsilon^{\mu\nu\alpha\beta} G^a_{\mu\nu} B_{\alpha\beta} \sum_{u,d} (y_L - y_R) \] (90)

This result agrees with that found in flat space-time. There are no new contributions from curvature or torsion. The only term present depends on the strength tensor of the hypercharge fields. The cancellation condition for this anomaly is given by:

\[ \sum_{u,d} (y_L - y_R) = 0 \] (91)
Anomaly in the $SU(2)_L$ gauge current

Following the same steps as before for the $SU(2)_L$ transformations, we find:

$$A_{SU(2)}^{a}(x) = \frac{1}{(4\pi)^2} tr(T^a(a_2(H_\psi, x)P_L - a_2(H_\psi^\dagger, x)P_R)) \quad (92)$$

The expression for the divergence of the gauge current can be obtained after some algebra and it yields:

$$(D_\mu \langle j^\mu \rangle)^a = -\frac{1}{32 \pi^2} gg' e^{\mu\alpha\beta} W^a_{\mu\nu} B_{\alpha\beta} (\sum_{u,d} N_C y_L + \sum_{\nu,e} y_L) \quad (93)$$

We observe that the result is the same as in flat space-time. All the contributions coming from curvature or torsion vanish, and the only term present depends on the strength fields of $SU(2)_L$ and the hypercharge field. The cancellation condition reads in this case:

$$\sum_{u,d} N_C y_L + \sum_{\nu,e} y_L = 0 \quad (94)$$

Anomaly in the $U(1)_Y$ gauge current

Finally, the expression for the anomaly in the $U(1)_Y$ current can be written as:

$$A_{U(1)}(x) = \frac{1}{(4\pi)^2} tr((y_L P_L + y_R P_R) a_2(H_\psi, x) - (y_L P_R + y_R P_L) a_2(H_\psi^\dagger, x)) \quad (95)$$

where $y_L$ and $y_R$ are the hypercharge matrices. The final expression for the divergence of the gauge current is now more involved than the non-abelian...
cases due to the appearance of terms depending on curvature and torsion.

The result goes as follows:

\[
D_\mu \langle j^\mu \rangle = \frac{1}{32\pi^2} \left( \left[ \sum_{u,d} N_C (y_L - y_R) + \sum_{\nu,e} (y_L - y_R) \right] \right) \left( -\frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\alpha\beta} R_\gamma^\nu R_{\delta\gamma}^\mu + \frac{1}{6} \Box S^\mu_{\mu} \right)
+ \frac{1}{96} (S^\alpha S^\nu S_\alpha)_{\nu} + \frac{1}{48} \epsilon^{\alpha\beta\gamma\delta} S_\beta S_\gamma S_\delta \epsilon S_{\alpha} - \frac{1}{6} (R^{\nu\alpha} S_\alpha - \frac{1}{2} R S_{\nu}^{\nu})_{\nu} \right) \nonumber
+ \frac{g^2}{2} \epsilon^{\mu\nu\alpha\beta} C_\mu C_\nu C_\alpha C_\beta \sum_{u,d} (y_L - y_R) + \frac{g^2}{4} \epsilon^{\mu\nu\alpha\beta} W_\mu W_\nu \sum_{u,d} N_C y_L + \sum_{\nu,e} y_L \nonumber
+ \frac{g^2}{2} \epsilon^{\mu\nu\alpha\beta} B_\mu B_\nu \sum_{u,d} N_C (y_L^3 - y_R^3) + \sum_{\nu,e} (y_L^3 - y_R^3) \right) \nonumber
\]

(96)

Notice the appearance of terms depending on curvature and torsion which
did not occur in the case of non-abelian gauge fields although they are also
chiral. The new terms that were not present in flat space-time impose a new
cancellation condition: the vanishing of the sum of all hypercharges:

\[
\sum_{u,d} N_C (y_L - y_R) + \sum_{\nu,e} (y_L - y_R) = 0 \quad \text{(97)}
\]
on the other hand we have that the cancellation of the terms already present
in flat space time gives the conditions:

\[
0 = \sum_{u,d} (y_L - y_R) \quad \text{(98)}
\]

\[
0 = \sum_{u,d} N_C y_L + \sum_{\nu,e} y_L \quad \text{(99)}
\]

\[
0 = \sum_{u,d} N_C (y_L^3 - y_R^3) + \sum_{\nu,e} (y_L^3 - y_R^3) \quad \text{(100)}
\]

The conditions in eqs. (98) and (99) are respectively the same as those in
eqs. (91) and (94), therefore there are four independent conditions and five hy-
percharges, namely: \( y_L^U = y_L^e, y_L^u = y_L^d, y_L^U, y_L^d \).
6 Gravitational anomalies

As we have mentioned above, the EP states that any theory in curved spacetime should be invariant under local Lorentz transformations. In this section, we consider the possible violation of this local symmetry due to quantum effects when chiral fermions are present [14], as indeed happens in the SM. We will conclude that whenever abelian chiral gauge fields are present, as it is the case of the hypercharge field, local Lorentz invariance is violated. However, due to the specific hypercharge assignment in the SM this anomaly is exactly cancelled. The condition for the cancellation of the Lorentz anomaly is the same as that of the cancellation of terms depending on curvature and torsion in the $U(1)_Y$ anomaly eq. 97.

Let us proceed with the explicit computation. Under local Lorentz transformations the spinor, vierbein and connection fields present in the matter lagrangian of the SM, eq.17, change as:

\[
\begin{align*}
\psi(p) & \rightarrow e^{\frac{\epsilon_{mn}(x)}{2} \Sigma_{mn}} \psi(p) \\
\psi^\dagger(p) & \rightarrow \psi^\dagger(p) e^{-\frac{\epsilon_{mn}(x)}{2} \Sigma_{mn}} \\
e^a_{\mu} & \rightarrow e^a_{\mu} - \epsilon^a_{\mu}(x) e^b_{\mu} \\
\Gamma^a_{\mu} & \rightarrow \Gamma^a_{\mu} + \epsilon^a_c(x) \Gamma^c_{\mu} - \epsilon^b_c(x) \Gamma^b_{\mu} - \partial_{\mu} e^{ab}(x)
\end{align*}
\]  

(101)

Under this set of transformations the matter lagrangian is invariant. However, it may happen that the effective action in eq.73 suffers an anomalous change as it occurs for gauge fields. This change is given by:
\[ W[A, \Gamma - D\epsilon, e - \epsilon e] = \\
W[A, \Gamma, e] - \int d^4x \sqrt{g} \left( -\epsilon^a_c(x) \Gamma^b_{\mu} + \epsilon^b_c(x) \Gamma^c_{\mu} + \partial_\mu \epsilon^{ab}(x) \right) \frac{\delta W}{\delta \Gamma^a_{\mu}} + \epsilon^a_b(x) \epsilon^b_{\mu} \frac{\delta W}{\delta \epsilon^a_{\mu}} \]

(102)

Here we have denoted by \( A \) all the gauge fields in the theory. Now, if we integrate by parts and use the antisymmetry of the connection components \( \Gamma^a_{\mu} \) in \( a \) and \( b \), we can rewrite this expression as

\[ W[A, \Gamma - D\epsilon, e - \epsilon e] = \\
W[A, \Gamma, e] + \int d^4x \sqrt{g} \epsilon^a_{\mu} \left( \nabla_\mu \Gamma^a_{\mu} + \Gamma^c_{\mu} \frac{\delta W}{\delta e^{ab}_{\mu}} - \Gamma^c_{\mu} \frac{\delta W}{\delta \Gamma^a_{\mu}} - T_{ab} \right) \]

(103)

where \( T_{ab} = \epsilon^b_{\mu} \frac{\delta W}{\delta \epsilon^a_{\mu}} \) is the expectation value of the energy-momentum tensor in presence of the background fields. We can write this result more conveniently using the following definitions:

\[
\frac{\delta W}{\delta \Gamma^a_{\mu}} = -\frac{i}{4} \langle \bar{\psi} \gamma^\mu (\Sigma^{ab} + \Sigma^{ab} \gamma^\mu) \psi \rangle = -\frac{i}{2} \langle j^{ab}_\mu \rangle \\
\langle j^a_\mu \rangle = \langle j^{ab}_\mu \Sigma_{ab} \rangle \\
D_\mu (j^a_\mu) = \nabla_\mu (j^a_\mu) + [\Gamma^a_{\mu}, \langle j^a_\mu \rangle] \\
\Gamma^a_{\mu} = \Gamma^a_{\mu} \Sigma_{ab} \\

(104)

Therefore we can rewrite:

\[ W[A, \Gamma - D\epsilon, e - \epsilon e] = W[A, \Gamma, e] + \int d^4x \sqrt{g} \epsilon^{ab}(x) \left( -\frac{i}{2} (D_\mu (j^a_\mu))^{ab} - T^{ab} \right) \]

(105)

In addition, we can calculate the change in the effective action due to the change in the integration measure as we did for the gauge anomaly and
obtain:

$$e^{-W[A',x']} = \int [d\psi d\psi^\dagger] e^{-\int d^4x \sqrt{g} \mathcal{L}_m} e^{-\frac{i}{2} \int d^4x \sqrt{g} (\epsilon_{ab}(x) A^{ab}(x))}$$  \hspace{1cm} (106)

where:

$$A^{ab}(x) = \sum_n \int d^4x \sqrt{g} (\phi_n^\dagger \Sigma^{ab} \phi_n - \xi_n^\dagger \Sigma^{ab} \xi_n)$$  \hspace{1cm} (107)

Finally, expanding eq.106 to first order in $\epsilon$ and identifying the terms in eq.105, we find the anomalous identity:

$$A^{ab}(x) = -(D_\mu \langle j^\mu \rangle)^{ab} + i(T^{ab} - T^{ba})$$  \hspace{1cm} (108)

The expression for the anomaly in eq.107 needs regularization. As we did in all the previous cases we use the operators $H_\psi$ and $H_\psi^\dagger$ to regularize the first and second terms respectively. The result can be expressed as follows:

$$A_{SO(4)}^{mn}(x) = \frac{1}{(4\pi)^2} tr(\Sigma^{mn} (a_2(H_\psi, x) - a_2(H_\psi^\dagger, x)))$$  \hspace{1cm} (109)

After a lengthy calculation we arrive to the final expression for the Lorentz anomaly:

$$A^{mn}(x) = \frac{g'}{32\pi^2} \left(\frac{1}{6} \epsilon^{mnab} R_{\mu\nu ab} B^{\mu\nu} - \frac{1}{6} (B^\nu_{\mu} S^{m;\nu} - B^m_{\nu} S^{m;\nu}) - \frac{1}{24} \epsilon^{mnab} (B_{ac} S^c S_b + B_{ab} S^2) - \frac{1}{6} \epsilon^{mnab} B_{ab} R - \frac{1}{2} S^\mu_{\nu} B^{mn} - \frac{1}{3} \epsilon^{mnab} \Box B_{ab} (\sum_{u,d} N_c (y_L - y_R) + \sum_{\nu,e} (y_L - y_R)) \right)$$  \hspace{1cm} (110)
Notice that pure gravity terms do not occur. Indeed it has been shown that there are no pure gravitational anomalies in four dimensions \[15\]. Observe also that all the terms depend on the \(B_{ab}\) field, which is abelian, whereas there is no contribution from non-abelian gauge fields. Finally, the cancellation condition agrees with that of eq.\[97\] which ensures the vanishing of the gravity terms in the \(U(1)_Y\) anomaly and, as we have already commented, is satisfied in the SM. It is also interesting to realize that eq.\[110\] without curvature and torsion terms reduces to:

\[
A^{mn}(x) = \frac{g'}{32\pi^2} \left( -\frac{1}{3} \epsilon^{mnab} \Box B_{ab} \right) \left( \sum_{u,d} N_c (y_L - y_R) + \sum_{\nu,e} (y_L - y_R) \right)
\]

(111)

This is the expression for the anomaly in the global Lorentz current in flat space-time, which classically is a basic symmetry in any relativistic quantum field theory. Notice again that the specific hypercharge assignment in the SM allows its cancellation. The last remark is important since, the cancellation condition in eq.\[97\] or eq.\[110\] were obtained in curved space-time and are also referred as mixed gauge-gravitational anomalies. However eq.\[111\] is obtained in flat space-time. Therefore, in flat space-time without any reference to gravitation, it is also possible to obtain the four anomaly cancellation conditions mentioned before.

## 7 Charge quantization in the Standard Model

In this section we will discuss the consequences of the requirement of the cancellation of the above computed gauge and gravitational anomalies. The set of eqs.\[91, 94, 97\] for the vanishing of gauge anomalies and eq.\[111\] for the
absence of global Lorentz anomalies in each family provide four equations for
the five unknowns \( y^\nu_L = y^\nu_R, \ y^u_L = y^u_R, \ y^e_L = y^e_R, \ y^u_R, \ y^d_R \). Let us try to solve the sys-

tem explicitly and accordingly to check whether they fix all the hypercharges
up to a normalization factor [16]. First, we note that the four equations can
be reduced to just one equation for two unknowns, namely:

\[
21y^u_R y^d_R + 21y^u_R y^d_R + 6y^u_R + 6y^d_R = 0 \quad (112)
\]

This equation, in turn, can be expressed in terms of one variable for \( y^d_R \neq 0 \):

\[
1 + \left( \frac{y^u_R}{y^d_R} \right)^3 + \frac{21}{6} \left( \frac{y^u_R}{y^d_R} \right)^2 + \frac{21}{6} \frac{y^u_R}{y^d_R} = 0 \quad (113)
\]

Now it is not difficult to see that there are three real solutions for this
equation:

\[
\frac{y^u_R}{y^d_R} = -1, \ -2, \ -\frac{1}{2} \quad (114)
\]

The quotient determines the rest of hypercharges as follows:

\[
y^u_L = y^d_L = \frac{1}{2}(y^u_R + y^d_R), \\
y^e_L = y^e_R = -\frac{3}{2}(y^u_R + y^d_R) \\
y^e_R = -3(y^u_R + y^d_R)
\]

Therefore, there are three possible sets of hypercharges (up to a normal-
ization factor) which explicitly read:
\[ y^u_R = -y^d_R \]  
\[ y^u_L = y^d_L = y^e_L = y^e_R = 0, \]  
\[ y^u_R = -2y^d_R \]  
\[ y^u_L = y^d_L = -\frac{1}{2}y^d_R \]  
\[ y^e_L = y^e_R = \frac{3}{2}y^d_R \]  
\[ y^e_L = y^e_R = 3y^d_R, \]

and

\[ y^d_R = -2y^u_R \]  
\[ y^u_L = y^d_L = -\frac{1}{2}y^u_R \]  
\[ y^e_L = y^e_R = \frac{3}{2}y^u_R \]  
\[ y^e_L = y^e_R = 3y^u_R. \]

The second set provides the usual assignment in the SM. The first one is the "bizarre" hypercharge assignment obtained in [17], and finally the third one can be obtained from the usual one by exchanging the hypercharges of \( u \) and \( d \) quarks. With the standard weak isospin assignment, the last set leads to different electric charges for the left and right components of the quark fields and therefore to chiral electromagnetism. To summarize, anomaly cancellation arguments alone do not fix the hypercharges in the SM; further physical constraints as the vector character of the electromagnetism and the existence of charged electrons are needed for that purpose.
8 Conclusions

In this work we have carefully computed the different anomalies that appear in the Standard Model (SM) defined in a classical background space-time with torsion.

The Equivalence Principle can be used to completely define the nature of the coupling of the Standard Model fields to the vierbein and the metric connection. In particular only fermions need to be minimally coupled to the torsion. The addition of other non minimal couplings give rise in some cases to gauge anomalies that cannot be cancelled by any hypercharge assignment.

Concerning the anomalies affecting global currents we arrive to the following results. The baryonic current anomaly is not modified by any curvature or torsion term and then it is the same as in flat space-time. However, due to the absence of the right-handed neutrinos, the leptonic current anomaly gets new contributions coming from curvature and torsion terms. Therefore, the conservation of the total baryonic minus leptonic number \( (B - L) \), which is known to apply for the SM in flat space-time, is violated when curvature and torsion are present. This fact could be relevant in connection with the problem of the baryonic asymmetry of the Universe.

The gauge anomalies corresponding to the groups \( SU(3)_C \) and \( SU(2)_L \) do not get new contributions and then we find the standard conditions for their cancellation in terms of the fermion hypercharges. For the \( U(1)_Y \) anomaly we obtain contributions from all the SM gauge fields and also new curvature and torsion terms. The cancellation of these gauge and gravitational terms gives rise to two new conditions on the hypercharge assignments in addition
to the other two mentioned above.

The gravitational anomaly has been computed as a gauge anomaly corresponding to the local (Euclidean) Lorentz group $SO(4)$. This anomaly has contributions from terms which are products of the hypercharge gauge field and curvature (mixed gauge-gravitational anomalies), hypercharge and torsion and hypercharge alone. This is consistent with the well known fact of the absence of pure gravitational anomalies in four dimensions. On the other hand, the only condition on the hypercharges found to cancel this terms is exactly the same found to cancel the curvature and the torsion terms appearing in the $U(1)_Y$ gauge anomaly. At this point we would like to stress that, even when the curvature and the torsion vanish, we find a term contributing to this anomaly which depends only on the $U(1)_Y$ field. Therefore, the corresponding anomaly equation is just the same that one find when computing the anomaly of the global current corresponding to the Lorentz group for the SM in flat space-time. Therefore, the condition on the hypercharge assignment that is usually referred as coming from the cancellation of the mixed gauge-gravitational anomaly in the SM, can be obtained without any reference to gravitation just by demanding the cancellation of the global Lorentz anomaly in the SM. In spite of the fact that this condition does not correspond to the cancellation of a gauge but a global anomaly (which in principle does not destroy the consistency of the theory), it is quite natural to be required since it amounts to the preservation of the Special Relativity Principle at the quantum level.

Finally we have deal with the problem of the hypercharge assignments
(family by family) from the cancellation of the gauge and mixed gauge-gravitational anomalies (or global Lorentz anomalies according to the discussion above) in the SM. In principle we have four equations and five unknowns. If the solution to this equations were unique one could fix the hypercharges modulo the global normalization that could be determined for example from the electron charge (provided it were different from zero). However, as a result of our analysis we find three independent kind of solutions. One is the .bizarre. solution found in [17]. The second is the one that contains the standard hypercharge assignment of the SM and then leads to the (fractional) quantization of the electric charge. The third one is analogous to the second one but exchanging the \(u\) quark hypercharge by the \(d\) quark one. From the phenomenological point of view only the second solution is acceptable since the first one produce a chargeless electron and the third one chiral electromagnetism. Those are the facts and we consider a matter of personal taste to decide if the cancellation of the anomalies determines or almost determines the SM hypercharges and the (fractional) quantization of the electric charges in the SM. In any case we would like to remark again that, as discussed above, the four hypercharge conditions can be obtained entirely in the context of the SM in flat space without the introduction of gravitation at all.

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