Tubular free by cyclic groups and the strongest Tits alternative

J. O. Button

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Abstract

We show, using Wise’s equitable sets criterion, that every tubular free by cyclic group acts freely on a CAT(0) cube complex. We also show that these groups have a finite index subgroup satisfying the strongest Tits alternative, which means that every subgroup either surjects a non abelian free group or is torsion free abelian. In particular the Gersten group is the first known group virtually having this property but which is not virtually special nor virtually residually free.

1 Introduction

When dealing with finitely presented groups, it may well be the case that knowledge of how they act geometrically allows us to prove purely group theoretic facts. An extreme example of this occurs with a word hyperbolic group, because if it acts properly and cocompactly on a CAT(0) cube complex (which will therefore be finite dimensional) we can conclude by the Agol - Wise results that it has a finite index subgroup which embeds in a right angled Artin group (a RAAG). In this case we say our group is virtually special and then any property that holds for all subgroups of RAAGs will hold virtually for our hyperbolic group. Two useful examples of this are linearity and (if non abelian) largeness. In fact very strong versions of both hold in this case; first we obtain linearity not just over $\mathbb{C}$ but over $\mathbb{Z}$. As for largeness, it was shown in [H] that RAAGs have the property that any subgroup (even if not finitely generated) either surjects the free group $F_2$
or is torsion free abelian, which they named the strongest Tits alternative. Thus our hyperbolic group possessing such an action will have a finite index subgroup satisfying the strongest Tits alternative. However hyperbolicity is certainly needed here to go from this geometric action to these group theoretic properties. If we remove this condition then the Burger - Mozes groups act properly and cocompactly on a 2 dimensional CAT(0) cube complex, but they can be simple groups which will fail nearly all of the usual group theoretic properties of interest.

In this paper we will consider two families of groups which all contain \( \mathbb{Z}^2 \) and therefore cannot be hyperbolic but which are generally considered to be well behaved. The first is free by cyclic groups, or more accurately (finite rank free) by \( \mathbb{Z} \) groups. It was proved in [11] that if a free by cyclic group is word hyperbolic then it does act properly and cocompactly on a CAT(0) cube complex and so has the above strong group theoretic properties. Now it is known that if a free by cyclic group is not word hyperbolic then it contains \( \mathbb{Z}^2 \), so we can ask: does such a group always have a “nice” geometric action on a CAT(0) cube complex? Of course it depends on what is meant by nice but in [10] Gersten displayed a free by cyclic group that cannot act properly and cocompactly on any CAT(0) space. Moreover this group is not virtually special but we can still ask whether the strong properties of virtually special groups also hold for the Gersten group, even though we will not be able to establish them by finding such geometric actions. Here we have nothing to say about linearity but we do show that the Gersten group virtually satisfies the strongest Tits alternative (though virtually is needed here).

The other class of groups we consider in this paper are what have been called the tubular groups: namely the fundamental group of a finite graph of groups with all vertex groups isomorphic to \( \mathbb{Z}^2 \) and all edge groups isomorphic to \( \mathbb{Z} \). These have been considered from both a geometric and group theoretic point of view. In [2] they were shown to have interesting Dehn functions and their quasi-isometric classes were considered in [8]. Now Wise showed in [22] that they can fail to be Hopfian groups, so certainly they can be badly behaved group theoretically as they need not be linear or even residually finite. As for the existence of nice geometric actions, the Gersten group is in fact also a tubular group, so proper and cocompact actions on a CAT(0) cube complex will not always exist. However in [23] Wise looked at the question of when a tubular group has a free action on a CAT(0) cube complex. As all our groups are torsion free and because proper here means topologically proper (a compact set can only have finitely many elements
which translate it to an image that intersects this set), acting properly and acting freely mean the same thing in this context. In [23] Corollary 5.10 the tubular groups acting freely and cocompactly on a CAT(0) cube complex are classified, though these are quite restrictive. However, on removing the compactness hypothesis, the main result in Theorem 1.1 of that paper gives a condition (in terms of what are called equitable subsets of the vertex groups) that determines exactly when a tubular group acts freely on a CAT(0) cube complex (though this complex might not be finite dimensional, nor locally finite). From this he was able to show that a wider range of tubular groups have a free action on a CAT(0) cube complex, in particular his non Hopfian example and the Gersten group, but he also used this condition to give examples of tubular groups with no such actions. Afterwards in [24] a criterion for when a tubular group acts freely on a finite dimensional CAT(0) space was developed, and in the forthcoming paper [25] it is shown that this is equivalent to the group being virtually special.

Here our purpose is to examine the intersection of these two classes of groups, namely the tubular groups $G(\Gamma)$ which are also free by cyclic. We first identify in Theorem 2.1 exactly which tubular groups are free by cyclic, which are those groups having a homomorphism to $\mathbb{Z}$ that is non zero on all edge groups. This always holds if the underlying graph $\Gamma$ is a tree. In [23], Conjecture 1.8 states that every free by cyclic group acts freely on a CAT(0) cube complex. Using our homomorphism we can show in Theorem 2.4 that any tubular free by cyclic group satisfies Wise’s equitable subsets condition and therefore does have a free action on a CAT(0) cube complex.

However, as mentioned above, this result will not allow us to establish group theoretic properties for tubular free by cyclic groups such as the Gersten group. Thus in Section 3 we look at whether tubular free by cyclic groups satisfy the strongest Tits alternative. Although they need not in general, they are not far from doing so in that they all have a finite index subgroup that does. In fact we consider tubular groups where all edge groups embed as maximal cyclic subgroups of the vertex groups, as it is straightforward to establish in Proposition 3.1 that any free by cyclic tubular group has a finite cover of this form. We then show in Theorem 3.7 that all tubular groups with maximal edge inclusions do virtually satisfy the strongest Tits alternative, though virtually is needed here. The method of proof is to look purely at the graph of groups and the associated action on the Bass - Serre tree, resulting in a parity argument that is established at the end of the proof, once the appropriate finite index subgroup is revealed.
In Section 4 we take a quick look at some other group theoretic properties which are implied if the group is virtually special, but which also hold for all or many tubular groups. Largeness is quickly established for all tubular groups, regardless of the edge embeddings, in Proposition 4.1 and virtual biorderability for all free by cyclic tubular groups in Theorem 4.2, using the Perron - Rolfsen criterion in [16]. The other property considered here is that of being residually free, whereupon it is pointed out at the end of this Section that very few tubular groups are residually free or even virtually residually free. The reason for mentioning this negative result is that a group which is virtually residually free will of course virtually satisfy the strongest Tits alternative. In particular we have shown that the Gersten group is the first group known virtually to satisfy the strongest Tits alternative which is not either virtually special or virtually residually free.

We finish in Section 5 with a range of examples of tubular groups that have already appeared in the literature, with comments on which group theoretic properties are known to hold for them. The first three are all free by cyclic groups and it will be seen that a whole host of strong group theoretic properties hold for all tubular free by cyclic groups (though not subgroup separability, whereas linearity is still in question). Meanwhile the other three examples seem to behave in a less predictable way. A straightforward method to create tubular groups with bad properties is to ensure the existence of non Euclidean Baumslag - Solitar subgroups (namely groups with presentation \(\langle a, t | ta^mt^{-1} = a^n \rangle \) where \(|m| \neq |n|\)). However these last three examples do not contain such subgroups, and in particular it is not the case that every tubular group either contains a non Euclidean Baumslag - Solitar group ("badly behaved") or has a finite index subgroup which is free by cyclic ("well behaved"), but there are also "strangely behaved" tubular groups too.

## 2 Tubular free by cyclic groups acting freely

Given a finite graph of groups \(G(\Gamma)\) with all vertex groups isomorphic to \(\mathbb{Z}^2\) and all edge groups isomorphic to \(\mathbb{Z}\), we can produce a presentation for the resulting fundamental group \(G\), which will be referred to as a tubular group, in the usual way. We first pick a maximal tree in \(\Gamma\) and contract each edge by forming an amalgamated free product. As \(\mathbb{Z}^2\) has an obvious 2-generator 1-relator presentation and we need to add 1 relator each time when performing the amalgamation, this process creates a presentation with
2v generators and 2v − 1 relations if there are v vertices. Having done this, we then introduce a stable letter for each of the b edges left (b being the first Betti number of the graph Γ) and form HNN extensions identifying the remaining cyclic subgroups, thus resulting in a presentation for G which has 2v + b generators and 2v + b − 1 relators. In particular G has a presentation of deficiency 1, that is where the number of generators is 1 more than the number of relators.

Moreover any finite presentation for G has deficiency at most 1, which can be seen because the standard presentation 2-complex forms an aspherical graph of spaces as in [19] Section 3. Thus G is also of cohomological and geometric dimension 2 and its Euler characteristic will equal 0, where 1 minus the Euler characteristic is an upper bound for the deficiency of G. Also G is well known to be coherent, namely every finitely generated subgroup of G is finitely presented.

Another well behaved class of groups sharing these nice properties are the free by \( \mathbb{Z} \) groups, which here refer to finitely presented groups of the form \( F \rtimes \alpha \mathbb{Z} \) where \( F \) is free of finite rank. Therefore it is of interest to determine when a tubular group is actually free by \( \mathbb{Z} \), with the next theorem giving us a complete answer.

**Theorem 2.1** A tubular group G is isomorphic to a free by \( \mathbb{Z} \) group if and only if there exists a homomorphism from G to \( \mathbb{Z} \) which is non zero on every edge group.

**Proof.** We use the following two results:

**Proposition 2.2** (i) ([13] Corollary 1) Let G be a group with an amalgamated product decomposition \( G = A \ast_C B \), and let \( \phi : G \to \mathbb{Z} \) be a homomorphism such that \( \ker(\phi|_C) \) is finitely generated and not equal to C. Then \( \ker(\phi) \) is finitely generated if and only if \( \ker(\phi|_A) \) and \( \ker(\phi|_B) \) are finitely generated.

(ii) ([4] Proposition 2.2) If G splits over the edge group C and N is finitely generated and normal in G then either N is in C or else CN has finite index in G.

First suppose that we have such a homomorphism \( \chi \). We begin with the case where our graph Γ is a tree and work by induction on the number of vertices by taking an amalgamated free product at each stage. Thus let us say that we have the subgroup \( A \) of G formed by amalgamation of the vertices in a subtree
of $\Gamma$. Our inductive hypothesis is that $\ker(\chi|_A)$ is finitely generated and free, whereupon on making the next amalgamation we are forming $H = A \ast_C B$ with $B \cong \mathbb{Z}^2$ and $C \cong \mathbb{Z}$. Now $\chi$ is not zero on the edge group $C$, so that $\ker(\chi|_C)$ is certainly finitely generated and not equal to $C$ as it is trivial. Moreover $\ker(\chi|_A)$ and $\ker(\chi|_B)$ are finitely generated, so Proposition 2.2 (i) tells us that $\ker(\chi|_H)$ is too. To see that it is free, we have by [20] Section I.5.5 Theorem 14 that a subgroup $S$ of $H = A \ast_C B$ that misses all conjugates of $C$ is a free product of a free group with factors of the form $S \cap hAh^{-1}, S \cap hBh^{-1}$ for various elements $h \in H$. But on setting $S = \ker(\chi|_H)$, we have that $S \cap hAh^{-1}$ is equal to $h\ker(\chi|_A)h^{-1}$, which is free by hypothesis. As for $S \cap hBh^{-1} \cong \ker(\chi|_B) \leq B \cong \mathbb{Z}^2$, this is trivial or isomorphic to $\mathbb{Z}$ because $C$ is not in the kernel.

If now $\Gamma$ is not a tree then we can still perform the above process on a spanning tree. On contracting this tree we are left with a single vertex where we have a subgroup $H$ of $G$ which is free by $\mathbb{Z}$ because $\ker(\chi|_H)$ is a free group of finite rank $r$ say, along with some self loops at this vertex corresponding to HNN extensions. Let $t$ be an element of $H$ generating the image of $\chi$ so that we can express $H$ as

$$K \rtimes_{\alpha} \langle t \rangle = \langle k_1, \ldots, k_r, t | tk_1t^{-1} = \alpha(k_1), \ldots, tk_rt^{-1} = \alpha(k_r) \rangle$$

for the appropriate automorphism $\alpha$ of $\ker(\chi|_H)$ which we now call $K$. Now when we take a loop and form the HNN extension $E$ of $H$ with stable letter $s$ conjugating $x \in H$ to $y \in H$, we can write $x = ut^m$ and $y = vt^n$ for $u, v \in K$. But $sx^{-1}s^{-1} = y$ implies that $m = n$ on application of $\chi$, which is defined on $E$ too. Moreover $n \neq 0$ as $x$ generates an edge group. Thus we have

$$E = \langle s, k_1, \ldots, k_r, t | tk_1t^{-1} = \alpha(k_1), \ldots, tk_rt^{-1} = \alpha(k_r), sut^n s^{-1} = vt^n \rangle$$

which, on rearranging the last relation and introducing new generators $s_1 = s$, as well as $s_i = ts_{i-1}t^{-1}$ for $2 \leq i \leq n$ when $n \geq 2$, admits the following presentation:

$$E = \langle k_1, \ldots, k_r, s_1, \ldots, s_n, t | tk_1t^{-1} = \alpha(k_1), \ldots, tk_rt^{-1} = \alpha(k_r), \quad \text{1) } ts_1t^{-1} = s_2, \ldots, ts_nt^{-1} = v^{-1}s_1u \rangle$$

whereupon we see that $E$ is also free by $\mathbb{Z}$. We can now remove the self loops one at a time to conclude that $G$ is also free by $\mathbb{Z}$.

Finally suppose that we have any homomorphism $\chi$ from a given tubular group $G(\Gamma)$ to $\mathbb{Z}$ with kernel $K$, along with an edge group $E$ with $\chi(E) = 0$. 

We can form $G$ by leaving this edge until last, whereupon either $G = A *_E B$ or $G = H *_E$. Now Proposition 2.2 (ii) implies that if $K$ is finitely generated then, as $E \subseteq K$ so $EK$ has infinite index in $G$, we have $E = K \cong \mathbb{Z}$ which implies that $G = \mathbb{Z}^2$ and $\Gamma$ is a single vertex. We now let $\chi$ vary over all homomorphisms from $G$ to $\mathbb{Z}$.

Note: more recently it was shown in [9] how to calculate the BNS invariant of a graph of groups, thus telling us which homomorphisms have finitely generated kernel.

**Corollary 2.3** If $\Gamma$ is a tree then $G(\Gamma)$ is free by $\mathbb{Z}$.

**Proof.** We pick a vertex $v_0$ with which to start defining $\chi$, which for now will be a homomorphism from $G$ to $\mathbb{Q}$, and then inductively extend to the vertices at distance $n$ from $v_0$. Say this has been done in such a way that $\chi$ is non zero on every edge group and take a vertex $v$ at distance $n + 1$. We can let this vertex group equal $\langle x, y \rangle$ where the edge group into $v$ from distance $n$ is generated by a power of $x$ and thus we have $\chi(x) \neq 0$. We then have other generators for the edge groups out of $v$ which will all be of the form $x^i y^j$. As this is a finite list, we can choose a value of $\chi(y) \in \mathbb{Q}$ such that no edge group in $v$ is sent to zero by $\chi$. We then extend $\chi$ to all of the vertex groups, hence to $G$, and this will be well defined because $\Gamma$ is a tree.

We end up with a homomorphism $\chi$ from a finitely generated group to $\mathbb{Q}$, thus the image is cyclic and we can multiply $\chi$ by a suitable integer so that this image now lies in $\mathbb{Z}$.

Next we examine the necessary and sufficient condition stated in [23] Theorem 1.1 for a tubular group to act freely on a CAT(0) cube complex (which in general here need not be finite dimensional nor locally finite). This condition is that there exists an equitable set, which is a choice of a finite family of elements from each vertex group such that

(i) each family generates a finite index subgroup of the respective vertex group and

(ii) the compatibility condition is satisfied on each edge $e$, namely on taking a generator of this edge group which embeds as $x$ in one vertex group and $y$ in the other, we have

$$\sum_{i=1}^{m} \# [x, s_i] = \sum_{j=1}^{n} \# [y, t_j].$$
Here \( \{s_1, \ldots, s_m\} \) is the chosen family for the first mentioned vertex group, \( \{t_1, \ldots, t_n\} \) is for the second and \( \#[x, s] \) is the intersection number. If we have \( x = (x_1, x_2) \) and \( s = (s_1, s_2) \) for some choice of basis of the first vertex group then \( \#[x, s] \) is equal to the modulus of the determinant \( x_1 s_2 - x_2 s_1 \) of the matrix \( \begin{pmatrix} x_1 & s_1 \\ x_2 & s_2 \end{pmatrix} \), so that we will write \( |(x s)| \) for \( \#[x, s] \).

It was shown in [23] Example 1.4 that the Gersten free by cyclic tubular group (with graph a single vertex and a bouquet of two circles) satisfies the above condition, thus although it does not act properly and cocompactly on any CAT(0) metric space by [10], or even properly and semisimply by [3], it does act freely on a CAT(0) cube complex. Conjecture 1.8 of this paper says that every free by \( \mathbb{Z} \) group possesses such an action. Here we are able to confirm this conjecture for the tubular free by \( \mathbb{Z} \) groups.

**Theorem 2.4** If \( G(\Gamma) \) is a tubular free by \( \mathbb{Z} \) group then there exists an equitable set for \( G(\Gamma) \) and so \( G \) acts freely on a CAT(0) cube complex.

**Proof.** We use Theorem 2.1 so that we have \( \chi : G \to \mathbb{Z} \) which is non zero on all the edge groups.

Our first case is when \( \Gamma \) is a bouquet of circles, so that we have only one vertex \( v \) but many self loops. We pick a basis for our vertex group \( v \) and therefore represent the elements as ordered pairs of integers. Therefore suppose that, having directed the \( k \) edges arbitrarily, we have the following generators when the edge groups are injected via the negative and the positive ends respectively:

\[
\begin{align*}
a_1 &= (a_1, b_1), \ldots, a_k = (a_k, b_k) \\
c_1 &= (c_1, d_1), \ldots, c_k = (c_k, d_k)
\end{align*}
\]

which are all elements of \( \mathbb{Z}^2 \). We now require a choice of \( N \) elements \( x_1, \ldots, x_N \in \mathbb{Z}^2 \) such that for for each \( j \) with \( 1 \leq j \leq k \) the following equation holds:

\[
\sum_{i=1}^{N} |(x_i a_j)| = \sum_{i=1}^{N} |(x_i c_j)|. \tag{2}
\]

We will be able to do this with \( N = 2 \) by proceeding as follows. First note that we can work over \( \mathbb{Q}^2 \) instead of \( \mathbb{Z}^2 \) throughout, as on finding \( x_1, \ldots, x_N \in \mathbb{Q}^2 \) satisfying (2), we can clear out denominators by multiplying both sides of (2) by an appropriate integer.
Suppose that the generators of the vertex group \((1,0)\) and \((0,1)\) are mapped to \(m\) and \(n\) respectively under \(\chi\), with the basis chosen such that \(m \neq 0\). Set \(l_j\) to be \(\chi(a_j) = ma_j + nb_j\) so that also \(l_j = \chi(c_j) = mc_j + nd_j\), with no \(l_j\) being equal to 0 by the properties of \(\chi\). On setting \(l = l_1\), the idea is to consider as candidates for the \(x_i\) points of the form \((\frac{l-ny_i}{m}, y_i)\) for \(y_i \in \mathbb{Q}\) to be determined. Note that as all of these points lie on the line \(mx + ny = l \neq 0\), these will span \(\mathbb{Q}^2\) provided only that we take more than one distinct point of this form. We find for each \(j\) between 1 and \(k\) that

\[
\sum_{i=1}^{N} |(x_i\ a_j)| = \sum_{i=1}^{N} \left| \left( \frac{l-ny_i}{m} \ m \ l_j-nb_j \ b_j \right) \right| = \sum_{i=1}^{N} \left| \frac{lb_i - l_jy_i}{m} \right| = \frac{|l_j|}{m} \sum_{i=1}^{N} |y_i - l_j b_j| \]

and so we also have

\[
\sum_{i=1}^{N} |(x_i\ c_j)| = \frac{|l_j|}{m} \sum_{i=1}^{N} |y_i - l_j d_j|. \]

Now if given two \(k\)-tuples of rationals \((q_1, \ldots, q_k)\) and \((r_1, \ldots, r_k)\), we can find \(N\) and \(y_1, \ldots, y_N \in \mathbb{Q}\) such that

\[
\sum_{i=1}^{N} |y_i - q_j| = \sum_{i=1}^{N} |y_i - r_j| \]

holds for each \(j\) between 1 and \(k\) by setting \(N = 2\), then letting \(y_1\) be the minimum \(m\) of our \(2k\) rational entries and \(y_2\) be the maximum \(M\) whereupon both sums become \(M - m\) regardless of \(j\). (In order to ensure that \(y_1 \neq y_2\) for spanning, we can increase \(M\) and decrease \(m\) if needed.)

For a general graph \(\Gamma\), we will reduce to the above case by contracting edges one by one in a maximal tree. However this will not be the usual process used when calculating the fundamental group of a graph of groups. Rather, given an oriented edge \(e\) running from the vertex \(v \in V(\Gamma)\) to the vertex \(w \neq v\), we will replace the graph of groups \(G(\Gamma)\) by another graph of groups \(G_1(\Gamma_1)\) where although \(\Gamma_1\) is the result of contracting the edge \(e\) in \(\Gamma\) onto the vertex \(w\), the fundamental groups \(G\) and \(G_1\) will in general be different. The procedure is not to amalgamate the vertex groups \(G_v\) and \(G_w\) but rather to replace \(G_v\) with \(G_w\) using a suitable isomorphism between them that respects the homomorphism \(\chi\). We then reinterpret the inclusions into \(G_v\) of the edge groups for the other edges ending at \(v\) as inclusions into \(G_w\) using this isomorphism.
We start by choosing a basis for each vertex group so that all the elements in a particular vertex group can be thought of as lying in $\mathbb{Z}^2$. However when finding our equitable vectors, we again work over $\mathbb{Q}^2$ and convert these into elements of $\mathbb{Z}^2$ by clearing out denominators at the end.

Given the first edge $e_1$ to be contracted, with vertex groups $V_1^+$ and $V_1^-$ at the vertices $v_1^+$, $v_1^-$ of $e_1$ which are now both isomorphic to $\mathbb{Q}^2$, let $g_1^+$ be the injection into $V_1^+$ of the generator of the edge group at $e_1$, and let $k_1^+$ be a non zero element of $V_1^+$ such that $\chi(k_1^+) = 0$. Note that $\chi(g_1^+) \neq 0$ by hypothesis, so $g_1^+$ and $k_1^+$ span $\mathbb{Q}^2$. We also do the same at the other end of $e_1$ to obtain two elements $g_1^-$ and $k_1^-$ of $V_1^-$ and we then define the isomorphism $\theta_1 : V_1^- \to V_1^+$ by sending $g_1^-$ and $k_1^-$ to $g_1^+$ and $k_1^+$ respectively. However, as the elements $k_1^\pm$ are only defined up to multiplication by a non zero scalar, we choose them such that the matrix $P_1$, which represents $\theta_1$ with respect to the original bases chosen for the vertex groups, has determinant 1 in modulus. (This will mean that $P_1$ has entries in $\mathbb{Q}$, though not necessarily in $\mathbb{Z}$.) Moreover as $\chi(g_1^+) = \chi(g_1^-)$ and $\chi(k_1^+) = \chi(k_1^-)$ we have that $\chi(\theta_1(\lambda g_1^- + \mu k_1^-)) = \chi(\lambda g_1^+ + \mu k_1^+)$ on temporarily writing these groups additively.

We now contract the edge $e_1$ from the vertex $v_1^-$ to $v_1^+$. This means that all other edges (if any) meeting $v_1^-$ get moved to meet $v_1^+$ and we replace each generator of the image of these edge groups in $v_1^-$ by applying $\theta_1$ to this element. This results in the graph $\Gamma_1$ which is the result of contracting the edge $e_1$ in $\Gamma$, along with cyclic edge groups and embeddings of these into the neighbouring vertex groups except that we can have fractional powers of elements. Note that as $\chi$ respects this substitution, we still have a homomorphism from the new fundamental group $G_1$ to $\mathbb{Z}$ which is non zero on all edge groups and we call this $\chi$ as well.

We now continue this process, producing further isomorphisms $\theta_2, \theta_3, \ldots, \theta_{l-1}$ until $\Gamma$ has been contracted to the graph $\Gamma_l$ with just one vertex $v_l$, though there could be many self loops at this vertex. By the previous argument, we can find an equitable set $\{x_1, x_2\}$ consisting of a pair of elements that are both in $\mathbb{Q}^2$ for the labelled graph $\Gamma_l$. This means that for each of the $k$ edges and pair of elements $a_j, c_j \in \mathbb{Q}^2$ at either end of the $j$th edge, we have

$$|\langle x_1, a_j \rangle| + |\langle x_2, a_j \rangle| = |\langle x_1, c_j \rangle| + |\langle x_2, c_j \rangle|.$$  

(3)

(If $\Gamma$ were a tree then we would have no edges left so in this case we would not contract the final edge. It is then straightforward to find an equitable set consisting of a pair of elements for each of the two remaining vertices.)
Now we reverse our contracting process and our isomorphisms. This means that if immediately prior to the application of the final isomorphism $\theta_{l-1}$, we have that there is some $j$ where $a_j$ is based at $v_l$ but $c_j$ is based at the other vertex $v_{l-1}$, we replace $c_j$ by the element $\theta_{l-1}^{-1}(c_j)$ so that this element now lies back in the vertex group $V_{l-1}$. However we also pick the elements $\theta_{l-1}^{-1}(x_1)$ and $\theta_{l-1}^{-1}(x_2)$ for the part of our equitable set based at $v_{l-1}$.

To see this does make an equitable set, note that we already had equation (3) holding and we are now replacing (for $i = 1, 2$) $x_i$ by $P_{l-1}^{-1}(x_i)$ and $c_j$ by $P_{l-1}^{-1}(c_j)$, because $P_{l-1}$ is the matrix representing $\theta_{l-1}$. But for any 2 by 2 matrix $M$ and $x, y \in \mathbb{Q}^2$ we have

$$|(Mx, My)| = |M(x, y)| = |M||(x, y)|$$

and as here $M = P_{l-1}^{-1}$ has determinant $\pm 1$, we see that the right hand side of (3) is unchanged when we insert the new elements. Similarly both sides of (3) are unchanged if for another value of $j$ both $a_j$ and $c_j$ get moved back to the vertex $v_{j-1}$, and if they both stay at $v_j$ then nothing at all in (3) is changed.

We can now reverse this process, introducing equitable sets at each new vertex until we return to $\Gamma$, whereupon we have our original tubular group $G(\Gamma)$ but now with a pair of equitable elements at every vertex.

Combining Theorems 2.1 and 2.4, we immediately obtain:

**Corollary 2.5** Any tubular group which is also free by $\mathbb{Z}$ acts freely on a CAT(0) cube complex.

### 3 The strongest Tits alternative

A group that has a homomorphism onto the free group $F_2$ must necessarily have a subgroup isomorphic to $F_2$ by the universal property of free groups. In [1] a group $G$ is said to satisfy the strongest Tits alternative if every subgroup $H$ of $G$ (not necessarily finitely generated) either has a homomorphism onto $F_2$ or is a torsion free abelian group. The authors showed in this paper that right angled Artin groups (and hence their subgroups) satisfy the strongest Tits alternative. Thus any group that is known to be virtually special, that is having a finite index subgroup which embeds in a right angled Artin group,
will virtually satisfy the strongest Tits alternative. Here we are interested in establishing this property for various tubular groups. However a tubular group need not have a finite index subgroup which embeds in a right angled Artin group, for instance we will see this in Example 5.2 for the Gersten group, thus we will need an alternative argument.

Our first problem is that given a graph of groups $G(\Gamma)$ with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups, it might well be that the inclusion of an edge group $\langle x \rangle$ in one of its vertex groups is not maximal, meaning that there is $g \in \mathbb{Z}^2 \setminus \{id\}$ and $n > 1$ such that $x = g^n$ (which is of course not the same as saying that $\langle x \rangle$ is a maximal subgroup of $\mathbb{Z}^2$). Certainly these can give rise to subgroups of the fundamental group $G$ which are not abelian but which do not surject to $F_2$, even if $\Gamma$ is a tree. For instance

$$\langle a, b, x, y, c, d | [a, b], [x, y], [c, d], b^2 = x, c^3 = y \rangle$$

has the subgroup $\langle b, x, y, c | [x, y], b^2 = x, c^3 = y \rangle$ which is non abelian (as it surjects to the modular group $C_2 * C_3$) but any surjection to a free group would have to send $\langle x, y \rangle$ to a cyclic subgroup so the image of this surjection would have first Betti number 1. However we will see that actually this group has a finite index subgroup obeying the strongest Tits alternative.

A more extreme situation occurs when the fundamental group $G$ contains a non Euclidean Baumslag-Solitar subgroup, namely a subgroup isomorphic to $\langle a, t | ta^nt^{-1} = a^n \rangle$ for $m, n \neq 0$ and where $|m| \neq |n|$. These groups do not surject $F_2$ and moreover, by replacing $a$ with $b = a^d$ where $d$ is the highest common factor of $m$ and $n$, we have a further subgroup isomorphic to $B = \langle b, t | tb^m/dt^{-1} = b^{n/d} \rangle$ where $m/d$ and $n/d$ are coprime. If $G$ contains such a subgroup $B$ then no finite index subgroup $H$ of $G$ can satisfy the strongest Tits alternative. This is because $B$ is not large, so the finite index subgroup $H \cap B$ of $B$ cannot surject $F_2$ but it is never abelian either.

However both these examples use non maximal inclusions of edge groups. In [7] we gave an exact criterion for a finite graph of groups with $\mathbb{Z}$ edge groups and a wide variety of possible vertex groups, certainly including $\mathbb{Z}^n$, to contain a non Euclidean Baumslag-Solitar group and it is immediate from this that it never happens if all edge inclusions are maximal. Moreover it also cannot occur if the fundamental group is free by $\mathbb{Z}$ (which covers the case when $\Gamma$ is a tree by Corollary 2.3), but there are examples where all edge inclusions are maximal and the fundamental group is free by $\mathbb{Z}$ but does not surject $F_2$. For instance the Burns group $\langle x, y, t | [x, y], txt^{-1} = y \rangle$ in
Example 5.1 where the graph is a single self loop does not surject $F_2$ as it is $2$-generated but not free and $F_2$ is Hopfian. Therefore we can at least hope that the fundamental group virtually satisfies the strongest Tits alternative in the two cases where it is free by $\mathbb{Z}$ and also where all edge inclusions are maximal. The rest of this section will provide a proof of this, starting by reducing the first case to the second.

**Proposition 3.1** Let $G(\Gamma)$ be a finite graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups such that $G$ is free by $\mathbb{Z}$. Then there exists a finite index subgroup $H$ of $G$ which can be written as a finite graph of groups $H(\Delta)$ with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups where all inclusions of edge groups into vertex groups are maximal.

**Proof.** As we have a homomorphism $\chi : G(\Gamma) \to \mathbb{Z}$ which is non zero on every edge group by Theorem 2.1, for a generator $x_i$ of each edge group $\langle x_i \rangle$ we can set $\chi(x_i) = m_i \neq 0$ and let $M$ be the lowest common multiple over all of the finitely many $m_i$. We take for $H$ the kernel of the map $\chi$ modulo $M$ which is a surjective homomorphism from $G$ to the cyclic group $C_M$. Consider the action of $G(\Gamma)$ on its Bass - Serre tree $T$, where of course we have $\mathbb{Z}^2$ vertex stabilisers and $\mathbb{Z}$ edge stabilisers and such that there are only finitely many orbits of vertices and edges under $G(\Gamma)$. Now $H$ also acts on this tree with vertex stabilisers $H \cap \mathbb{Z}^2$ and edge stabilisers $H \cap \mathbb{Z}$, which are isomorphic to $\mathbb{Z}^2$ or $\mathbb{Z}$ respectively because $H$ has finite index in $G$. Thus by taking the quotient $H \backslash T$ we see that $H$ is also a graph of groups $H(\Delta)$ with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups, and $\Delta$ is a finite graph because $H$ having finite index in $G$ also means there are finitely many vertices and edges in $\Delta$.

Finally we consider an edge subgroup $\langle x_i \rangle$ in $G$ with $\chi(x_i) = m_i$ and we set $d_i = M/m_i$. This tells us that $\langle x_i \rangle \cap H$ is $\langle x_i^{d_i} \rangle$ and so in the graph of groups $H(\Delta)$ there will be two inclusions of $x_i^{d_i}$, each of which is into a vertex subgroup isomorphic to $\mathbb{Z}^2$. But if $x_i^{d_i} = h^n$ for $h \in \mathbb{Z}^2 \subseteq H$ then $M = |n| \cdot |\chi(h)|$, but $\chi(h)$ must be zero modulo $M$ so $|n| = 1$ and $x_i^{d_i}$ is a maximal element of this vertex subgroup of $H(\Delta)$.

$\square$

In looking for homomorphisms from the fundamental group of a graph of groups onto $F_2$, we begin with the following straightforward but useful lemma, giving us homomorphisms onto $\mathbb{Z}$. 
Lemma 3.2 Suppose that $G(T)$ is a (possibly infinite) graph of groups with $T$ a tree where all vertex groups are free abelian and all edge groups embed as direct summands of the vertex groups. Then any non-trivial homomorphism $h$ from a vertex group to $\mathbb{Z}$ can be extended to a homomorphism $h : G \to \mathbb{Z}$.

Proof. Let the root vertex of the tree $T$ be the vertex where $h$ is defined. This then determines an integer valued homomorphism on all the edge groups out of the root vertex. Thus for any vertex joined to the root, we can extend $h$ from the inclusion of the edge subgroup to the whole vertex group, because it is a direct summand. We can now continue this process over all vertices to get the surjective homomorphism $h : G(T) \to \mathbb{Z}$, and then take a direct limit if $T$ is infinite.

We can now use this for the following theorem.

Theorem 3.3 Let $G(\Gamma)$ be a finite graph of groups having $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups and such that all inclusions of edge groups in vertex groups are maximal. Let $S$ be any subgroup of $G$, so that by restricting the action of $G$ on its Bass-Serre tree $T$ we have that $S$ is also a graph of groups $S(\Sigma)$ with vertex stabilisers equal to $S \cap \mathbb{Z}^2$ and edge stabilisers equal to $S \cap \mathbb{Z}$. Then either $S$ is free abelian (thus isomorphic to $\{id\}, \mathbb{Z}$ or $\mathbb{Z}^2$), or it surjects the free group $F_2$, or the graph $\Sigma$ contains only one loop.

Proof. As $S$ is any subgroup of $G$ it might have infinite index in $G$ or even be infinitely generated, so we cannot assume $\Sigma$ is a finite graph. However in any event it is well known that there is a surjection from $S$ to the fundamental group $\pi_1(\Sigma)$ of the underlying graph, so that if $\Sigma$ contains at least two loops then $S$ surjects to $F_2$ in this case.

Thus we can now suppose that $\Sigma$ is a tree where the vertex subgroups of $S(\Sigma)$ are all isomorphic to $\{id\}, \mathbb{Z}$ or $\mathbb{Z}^2$ and the edge subgroups $S \cap \mathbb{Z}$ either are trivial or are copies of $\mathbb{Z}$. Moreover if here the vertex subgroup $S \cap \mathbb{Z}^2$ is itself isomorphic to $\mathbb{Z}^2$ then we would have that $S \cap \mathbb{Z}^2$ has finite index in the original $\mathbb{Z}^2$ vertex subgroup of $G(\Gamma)$, so therefore any edge subgroup $E$ of $G(\Gamma)$ that has an inclusion into this vertex subgroup $\mathbb{Z}^2$ will have $S \cap E$ also isomorphic to $\mathbb{Z}$. Furthermore if $\{id\}$ is now a vertex subgroup in $S(\Sigma)$ then any edge with endpoint this vertex will also have trivial stabiliser. However in the case when the new vertex subgroup $S \cap \mathbb{Z}^2$ is isomorphic to $\mathbb{Z}$, we
have that the edge subgroups including in this vertex subgroup can either be trivial or themselves isomorphic to \(\mathbb{Z}\).

In summary we have a (possibly infinite) graph of groups \(S(\Sigma)\) where the underlying graph \(\Sigma\) is actually a tree and where each vertex group is \(\{id\}, \mathbb{Z}\) or \(\mathbb{Z}^2\), with edge groups \(\{id\}\) joining what we will call \(\{id\}\) type vertices, edge groups \(\{id\}\) or \(\mathbb{Z}\) joining \(\mathbb{Z}\) type vertices and edge groups \(\mathbb{Z}\) joining \(\mathbb{Z}^2\) type vertices. Moreover we know that in \(G(\Gamma)\) all inclusions of edge groups into vertex groups are maximal and so the same is true here because we are just intersecting each edge and vertex group with the same subgroup \(S\).

We begin by assuming here that \(\Sigma\) is a finite tree. If every vertex group of \(\Sigma\) is \(\mathbb{Z}^2\) then every edge group must be \(\mathbb{Z}\). In this case we can take any edge in \(\Sigma\) and by choosing the relevant bases of the two neighbouring vertex groups, we can suppose that these two vertex groups are \(\langle x, y \rangle \cong \mathbb{Z}^2\) and \(\langle a, b \rangle \cong \mathbb{Z}^2\), where \(x\) is amalgamated with \(a\). Hence on removing this edge to obtain two disjoint trees \(T_1\) and \(T_2\), we have the graph of groups \(G_1(T_1)\) containing the \(\langle x, y \rangle\) vertex and \(G_2(T_2)\) containing the \(\langle a, b \rangle\) vertex. Thus we can apply Lemma 3.2 to get a homomorphism \(h\) from \(G_1(T_1)\) onto \(\mathbb{Z}\) which sends \(x\) to zero and \(y\) to 1. We now do the same with \(G_2(T_2)\) and as \(x\) and \(a\) are sent to zero by these separate homomorphisms to \(\mathbb{Z}\), we have that \(S = G_1 \ast_{x=a} G_2\) surjects to \(F_2\).

Indeed if there exists anywhere in \(\Sigma\) two \(\mathbb{Z}^2\) type vertices which are joined by an edge (so that this edge group must be \(\mathbb{Z}\)) then the argument just given works here too. Another case that is easily dealt with is when there are two \(\mathbb{Z}\) type vertices which are joined by an edge having trivial stabiliser. On removal of this edge to form trees \(T_1\) and \(T_2\) with notation as before, we now have that \(S\) is the free product \(G_1 \ast G_2\) and so we can apply Lemma 3.2 to each of these trees to conclude that \(G_1, G_2\) both surject \(\mathbb{Z}\) and thus \(S\) surjects \(F_2\).

However there are other cases to consider here and we need to do this in a way which will extend to infinite trees. First suppose that we have two distinct vertices \(v, w\) of type \(\mathbb{Z}^2\) and consider the unique path in \(\Sigma\) running between them. If a vertex \(u\) in this path has type \(\{id\}\) then we can ignore it, because on contracting one of the two edges that lie in the path and pass through \(u\) onto its other endpoint, we are forming a free product where one of the factors is trivial. As the stabiliser of both these edges must be trivial, the process of contracting edges in this path can only end when we obtain an edge with trivial stabiliser which joins vertices that are both of \(\mathbb{Z}\) type and this has been dealt with above. If however there are no \(\{id\}\) type
vertices then either we use the above argument on a \( \{id\} \) type edge which will join two \( \mathbb{Z} \) type vertices as before, or a very similar argument works by removing all \( \mathbb{Z} \) type vertices in this path. This is because the edge groups must now be \( \mathbb{Z} \) and the edge inclusions are maximal, thus the inclusion of any edge subgroup is also equal to any vertex group \( V_1 \) which is of \( \mathbb{Z} \) type. Thus the amalgamation formed by contracting an edge joining this vertex to a neighbouring vertex with group \( V_2 \) results in the trivial amalgamation \( V_2 \ast V_1 \). 

This covers the case when \( \Sigma \) has two \( \mathbb{Z}^2 \) type vertices but in fact essentially the same argument applies whenever we have a path between vertices \( v \) and \( w \) with vertex groups \( V, W \) respectively such that both the inclusion into \( V \) of the first edge group and the inclusion into \( W \) of the last edge group are proper. Thus now suppose this never occurs in \( S(\Sigma) \). If there is exactly one \( \mathbb{Z}^2 \) type vertex then the whole tree can be contracted back to this vertex without changing the fundamental group, so that \( S = \mathbb{Z}^2 \). This also works if there is a \( \mathbb{Z} \) type vertex joined to an edge with trivial stabiliser, leaving only the case where all vertex and edge groups are isomorphic to \( \mathbb{Z} \), in which case \( S \) is \( \mathbb{Z} \) too as all edge inclusions are maximal, and the trivial case where all stabilisers equal \( \{id\} \).

As for the case where \( \Sigma \) is an infinite tree so that \( S \) is defined as a direct limit of the fundamental group of finite subtrees \( \Sigma_n \), we see that once we have taken a finite subtree \( \Sigma_N \) that includes a path with proper edge inclusions at each end, our use of Lemma 3.2 twice to obtain a surjection from the fundamental group of \( \Sigma_n \) to \( F_2 \) will be consistently defined for \( n \geq N \), because each homomorphism to \( \mathbb{Z} \) is defined from the starting vertex outwards, so that this surjection extends to \( S \) as well. If however this never occurs in \( \Sigma \) then the fundamental groups stabilise as \( \mathbb{Z}^2, \mathbb{Z} \) or \( \{id\} \).

\[ \square \]

Note: the first paragraph of the proof, along with Lemma 3.2, allows us to conclude that if \( G(\Gamma) \) is a graph of groups with free abelian vertex groups and all edge subgroups embed as direct summands of the vertex groups then every non trivial subgroup of \( G \) has a surjective homomorphism to \( \mathbb{Z} \), thus \( G \) is locally indicable, left orderable, has the unique product property, has no zero divisors in its group algebra and so on. Moreover the same techniques show that if \( G(\Gamma) \) is a finite graph of groups with finitely generated free abelian vertex groups then, regardless of the edge embeddings, \( G \) is locally indicable and so has the other properties. This is because any edge group
will embed as a finite index subgroup of a direct factor, so in the case where the subgroup $S$ is such that $S \backslash T$ is a tree (where $T$ is the Bass-Serre tree for $G$), application of Lemma 3.2 will obtain a non trivial homomorphism $h$ from $S$ to $Q$. But if $S$ is finitely generated then there is a compact subgraph of $S \backslash T$ such that the inclusion of the corresponding graph of groups induces an isomorphism of fundamental groups. Consequently we can assume that this tree is finite and so $h$ has discrete image.

We now have to go back to finding surjective homomorphisms to $F_2$, where $\Sigma$ having a single loop is the case that remains, though we have seen in the example of the Burns group that the conclusion is false if this loop is a self loop. Thus we next look at the case where all vertex groups are $\mathbb{Z}^2$, all inclusions of edge groups are maximal, and our graph has a single loop which is not a self loop.

**Proposition 3.4** Suppose that $G(\Gamma)$ is a (possibly infinite) graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups and such that all inclusions of edge groups in vertex groups are maximal. If the graph has a single loop and this is not a self loop then the fundamental group $G$ of the graph of groups $G(\Gamma)$ surjects to $F_2$.

**Proof.** We can form a graph of groups using just the loop $\Lambda$ in $\Gamma$, which we will call $L(\Lambda)$, and we will initially define our homomorphism on the group $L$.

First suppose that there is a vertex $v$ lying in the loop $\Lambda$ with vertex group $\langle x, y \rangle$ say, having the following property: on taking the two edges joining $v$ that also lie in $\Lambda$, the inclusions of these edge groups into $\langle x, y \rangle$ are the same subgroup, which we will take to be $\langle x \rangle$. Then we can define a homomorphism from $L(\Lambda)$ to $F_2 = \langle u, v \rangle$ as follows: we send $y$ to $u$ and every other vertex subgroup in $\Lambda$ to the identity, along with the element $x$. But $L(\Lambda)$ will have another generator, namely the stable letter $t$ coming from the loop and we send this to $v$ which will provide a well defined surjection from $L$ to $F_2$.

If there is no vertex in $\Lambda$ with this property then we will take any edge in the loop and this will span distinct vertex groups $\langle x, y \rangle$ and $\langle a, b \rangle$, with the edge group inclusions assumed to be $\langle x \rangle$ and $\langle a \rangle$ respectively. When forming $L(\Lambda)$ we will remove this edge to form a maximal tree, so we can assume that in $L$ we have $txt^{-1} = a$, where $t$ is the stable letter corresponding to the loop $\Lambda$. We further assume that the edge group of the other edge lying in $\Lambda$ and joined to the vertex with vertex group $\langle x, y \rangle$ is generated by $x^iy^j$ for coprime $i$ and $j$, and similarly we have the edge group generated by $a^kb^l$ which is
joined to the \(\langle a, b \rangle\) vertex. We suppose that \(F_2\) has free basis \(u, v\) and we start by sending \(t\) to \(v\), \(x\) to \(u^j\) and \(y\) to \(u^{-il}\) so that the edge group \(\langle x^i y^j \rangle\) is sent to the identity. Similarly we send \(a\) to \(vu^jv^{-1}\) and \(b\) to \(vu^{-jk}v^{-1}\). We can now send all other vertex groups in the loop \(\Lambda\) to the identity.

Having defined in either case our homomorphism on \(L(\Lambda)\), back in \(\Gamma\) we may also have trees emanating from the vertices in the loop, so we extend the definition by sending all vertex groups in a tree to zero if its root vertex group was sent there. This now leaves the two trees \(T_x\) with root \(\langle x, y \rangle\) and \(T_a\) with \(\langle a, b \rangle\) (or just the tree \(T_x\) in the first case). For each of these we can define homomorphisms \(h_x, h_a\) to \(\mathbb{Z}\) with \(h_x(x) = h_a(a) = jl, h_x(y) = -il, h_a(b) = -jk\) using Lemma 3.2 (or \(h_x(x) = 0, h_x(y) = 1\) in the first case).

We then compose so that an element \(g_x\) in the fundamental group formed from \(T_x\) is sent to \(u^{h_x(g_x)} \in F_2\) and likewise any \(g_a\) in the fundamental group formed from \(T_a\) goes to \(vu^{h_a(g_a)}v^{-1}\). This now allows us to extend our homomorphism to the whole of \(G(\Gamma)\), even when \(\Gamma\) is infinite, and we have that the image is a non abelian subgroup of \(F_2\), so certainly \(G\) surjects to \(F_2\).

However this result above requires the vertex groups to equal \(\mathbb{Z}^2\) and the edge groups to equal \(\mathbb{Z}\), not subgroups of these which will be encountered when dealing with an arbitrary subgroup \(S\) of \(G(\Gamma)\). We now deal with this case, though we will need to avoid some subgroups of \(G\) which, as well as the situation when \(\Gamma\) contains self loops, will require dropping to a finite index subgroup \(H\) of \(G\) before establishing that the strongest Tits alternative holds for \(H\).

**Proposition 3.5** Suppose that \(G(\Gamma)\) is a finite graph of groups having \(\mathbb{Z}^2\) vertex groups and \(\mathbb{Z}\) edge groups with \(\Gamma\) containing no self loops, and such that all inclusions of edge groups in vertex groups are maximal. Let \(S\) be any subgroup of \(G\). Then either \(S\) surjects to \(F_2\) or is free abelian (therefore isomorphic to \(\{id\}, \mathbb{Z}\) or \(\mathbb{Z}^2\)) or the graph of groups \(S(\Sigma)\) obtained by setting \(\Sigma = S\setminus T\) for \(T\) the Bass-Serre tree of \(G\) results in \(\Sigma\) being a graph with a single loop \(\Lambda\) which is not a self loop, and such that all edges in \(\Lambda\) are of \(\mathbb{Z}\) type, whereas at most one vertex in \(\Lambda\) is of \(\mathbb{Z}^2\) type, with all other vertices in \(\Lambda\) of \(\mathbb{Z}\) type.

**Proof.** Let \(S\) be any subgroup of \(G\), so that as mentioned it is also a graph of groups \(S(\Sigma)\) with vertex stabilisers equal to \(S \cap \mathbb{Z}^2\) and edge stabilisers equal to \(S \cap \mathbb{Z}\). As before we are done unless it turns out that \(\Sigma\) contains
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a single loop $\Lambda$. Moreover as $\Gamma$ does not contain any self loops, we conclude that $\Sigma$ does not either. This is because if we consider the action of $G$ on its Bass-Serre tree $T$ with quotient graph $\Gamma = G\backslash T$, an element in $S$ which identifies the endpoints of an edge in $T$ will also lie in $G$, and $G$ acts without edge inversions. Now $S(\Sigma)$ may have vertex groups which are not $\mathbb{Z}^2$ and edge groups which are not $\mathbb{Z}$, but we still have maximal inclusions of edge groups into vertex groups as before.

First suppose that there is an $\{id\}$ type edge lying in $\Lambda$. Then removal of this edge from $\Sigma$ gives us a graph of groups where the graph is now a tree, and this group will surject to $\mathbb{Z}$ by Lemma 3.2 (unless every vertex group is trivial, in which case we have the trivial group). Now on putting back the missing edge, we find that $S$ is isomorphic to the free product of this group with $\mathbb{Z}$ because here the edge group is trivial. So in this case either $S$ surjects to $F_2$ or it is equal to $\{id\} \ast \mathbb{Z}$.

Now we are left with the case where the loop $\Lambda$ has no $\{id\}$ type edges (thus no $\{id\}$ type vertices), though there could well be $\mathbb{Z}$ type vertices. However whenever this occurs then at such a vertex all edge inclusions will equal the vertex group by maximality. Moreover we are assuming that there are at least 2 vertices of $\mathbb{Z}^2$ type in $\Lambda$. We now tidy up the loop $\Lambda$ in a similar fashion to that in the proof of Theorem 3.3, so that by contracting edges we remove all $\mathbb{Z}$ type vertices from $\Lambda$ to form $\Lambda'$, which will not change the subgroup $S$. This leaves us with two graphs of groups $L(\Lambda')$ and $S(\Sigma')$ for new graphs $\Lambda', \Sigma'$ but the same underlying groups $L, S$. Now in $\Lambda'$ all vertices are of $\mathbb{Z}$ type and there is a single loop that is not a self loop (because the two vertices of $\mathbb{Z}^2$ type in $\Lambda$ remain as distinct vertices in the new loop). This certainly allows us to apply Proposition 3.4 to $L(\Lambda')$, but also to $S(\Sigma')$ because it is easily seen that the proof of Proposition 3.4 does not require any restriction on the types of vertices or edges that lie outside the loop $\Lambda$.

Thus we are left with $S(\Sigma)$ having a single loop $\Lambda$ with at most one $\mathbb{Z}^2$ type vertex in $\Lambda$ and all others of $\mathbb{Z}$ type. Although this is now our only remaining case, the problem is that in general these will not satisfy the conclusions of our proposition, even if $\Lambda$ is not a self loop. For instance let us suppose that the graph of groups $S(\Sigma)$ is such that $\Sigma$ consists only of the loop $\Lambda$, with the vertices labelled $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ in order, and such that all of $v_1, \ldots, v_{n-1}$ are type $\mathbb{Z}$ vertices. Thus on forming the underlying group $S'$,
we can contract Λ to a self loop at \( v_0 \), so if this is a \( \mathbb{Z}^2 \) type vertex with vertex group \( \langle x, y \rangle \) and neighbouring edge groups \( \langle x^k y^l \rangle \) towards \( v_{n-1} \) and \( \langle x \rangle \) towards \( v_1 \) say, we see that \( S \) is of the form \( \langle x, y, t | [x, y], tx^{\pm 1}t^{-1} = x^k y^l \rangle \) which will not surject to \( F_2 \) unless \( l = 0 \). On the other hand if \( v_0 \) is of type \( \mathbb{Z} \) with vertex group \( \langle x \rangle \) then we would have \( S = \langle x, t | txt^{-1} = x^{\pm 1} \rangle \) which can be the Klein bottle group. Although these examples could certainly occur as subgroups of our original group \( G \), the remainder of this section involves showing that they do not appear in the finite index subgroup of \( G \) that we will now take.

**Proposition 3.6** Suppose that \( G(\Gamma) \) is a graph of groups with \( \mathbb{Z}^2 \) vertex groups and \( \mathbb{Z} \) edge groups, with \( G \) acting on the Bass - Serre tree \( T \) to form the finite graph \( \Gamma = G\setminus T \) with natural projection \( \pi_G : T \to \Gamma \). Then if \( b \) is the first Betti number of \( \Gamma \), there exists a subgroup \( H \) of index \( 2^b \) in \( G \), thus which is also a graph of groups \( H(\Delta) \) with \( \mathbb{Z}^2 \) vertex groups and \( \mathbb{Z} \) edge groups for \( \Delta = H\setminus T \) and natural projection \( \pi_H : T \to \Delta \), that gives rise to a finite cover \( q : \Delta \to \Gamma \) of these graphs where \( \pi_G = q\pi_H \) and with the following property:

Suppose that we have a closed path \( p \) in \( \Delta \) and consider its image \( q(p) \) in \( \Gamma \) which is also closed. Then in tracing out the path \( q(p) \) we pass through every (non oriented) edge an even number of times. In particular there are no self loops in \( \Delta \).

**Proof.** We have the natural homomorphism \( h \) from \( G \) to the fundamental group \( \pi_1(\Gamma) \) of the underlying graph, which is free of rank \( b \), formed by quotienting out by all vertex and edge stabilisers. Thus there is also a natural homomorphism factoring through this from \( G \) to \( (C_2)^b \) and the kernel \( H \) of this is an index \( 2^b \) subgroup of \( G \) containing all vertex and edge stabilisers of \( \Gamma \). Consequently \( H \) can also be considered as the fundamental group of a graph of groups \( H(\Delta) \) for \( \Delta = H\setminus T \). Now as \( H \) is normal in \( G \) we have that the action of \( G \) on \( T \) with quotient \( \Gamma \) factors through \( \Delta \) to give us a map \( q : \Delta \to \Gamma \) such that \( \pi_G = q\pi_H \). Moreover the action of the group \( G/H \) on \( \Delta \) with quotient \( \Gamma \) is without fixed points because \( H \) contains all stabilisers, thus \( q \) is a regular covering map. We then have, say by [12] Proposition 1.40 (c), that \( G/H \) is isomorphic to \( \pi_1(\Gamma)/q(\pi_1(\Delta)) \), therefore the subgroups \( h(H) \) and \( q(\pi_1(\Delta)) \) are both normal in \( \pi_1(\Gamma) \) with quotient \( (C_2)^b \) so must be equal.

Given a finite graph \( \Gamma \), the cycle space of \( \Gamma \) is the vector subspace of the edge space (functions from the edges of \( \Gamma \) to \( C_2 \)) which is the span of the
closed reduced paths in $\Gamma$. Now an element $\gamma \in \pi_1(\Gamma)$ can be thought of as first a concatenation of closed paths corresponding to the generators of $\pi_1(\Gamma)$ which is then reduced. But this reduction does not change the parity of the number of times an edge is passed through, thus we obtain a function from $\pi_1(\Gamma)$ to the edge space with image equal to the cycle space, which is isomorphic to $(C_2)^b$. Moreover this is a homomorphism from $\pi_1(\Gamma)$ onto $(C_2)^b$ so its kernel must equal the characteristic subgroup $h(H)$. Thus although our path $p$ might not be reduced, it can still be thought of as an element of $\pi_1(\Delta)$ where $q(p)$ lies in this kernel $h(H)$, without changing the parity of the number of edge visits. Hence we have that the edge set of $q(p)$ induces the zero map from the edges of $\Gamma$ to $C_2$, so every edge in $\Gamma$ has been passed through an even number of times.

\[\blacksquare\]

We now have our finite index subgroup of $G$ in place, allowing us to present our main result of this section. The proof is all about ensuring there is a finite index subgroup which avoids the examples given after Proposition 3.5.

**Theorem 3.7** Suppose that $G(\Gamma)$ is a finite graph of groups having $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups and such that all inclusions of edge groups in vertex groups are maximal. Then there is a finite index subgroup $H$ of $G$ such that for every subgroup $S$ of $H$ we have that either $S$ surjects to $F_2$ or $S$ is free abelian (therefore isomorphic to $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$).

**Proof.** We will assume that we have obtained $H(\Delta)$ from $G(\Gamma)$ as in the statement of Proposition 3.6. Now suppose that $S$ is any subgroup of $H$, which itself has finite index in $G$. We have that the action of $G$ on its Bass-Serre tree $T$ with projection $\pi_G : T \to \Gamma = G\backslash T$ gives rise to the graph of groups $G(\Gamma)$, but by restricting this action on $T$ to the subgroup $S$ we obtain the graph of groups $S(\Sigma)$ for $\Sigma = S\backslash T$. Now by earlier results in this section, we can assume that the quotient graph $\Sigma$ (which could be infinite) has a single loop $\Lambda$. Moreover this is not a self loop because there are none in $\Delta$, so by Proposition 3.5 we are done unless the vertices of $\Lambda$ are all of $\mathbb{Z}$ type apart from possibly one of $\mathbb{Z}^2$ type, and such that the edges of $\Lambda$ are all of $\mathbb{Z}$ type, with maximal inclusions into the vertex subgroups of $\Lambda$. Our method of proof is to lift $\Lambda$ to a path $p$ in $T$ and then give a parity argument utilising the fact that $S$ is a subgroup of $H$ in order to get a contradiction.
We will proceed by labelling the vertices in Λ consecutively as \( v_0, v_1, \ldots, v_n = v_0 \) with \( v_0, v_1 \) joined by the edge \( e_0 \), and so on for edges \( e_1 \) up to \( e_{n-1} \) which joins \( v_{n-1} \) to \( v_n = v_0 \). We assume that if there is one vertex of \( \mathbb{Z}^2 \) type then it is \( v_0 \), with all others of \( \mathbb{Z} \) type. Now as \( \Sigma = S \setminus T \), we can take a vertex \( t_0 \in T \) which lies above \( v_0 \) and which will have stabiliser in \( S \) equal to the group label on the vertex \( v_0 \) in the graph of groups \( S(\Sigma) \) which we have obtained from the action of \( S \) on \( T \). There will next be an edge \( d_0 \in T \) joining \( t_0 \) that lies above \( e_0 \) and also with stabiliser in \( S \) equal to the label on its image in \( \Sigma \), given as a subgroup of the stabiliser in \( S \) of the vertex \( v_0 \).

We continue this process until we have found vertices \( t_0, \ldots, t_{n-1}, t_n \) above \( v_0, \ldots, v_{n-1}, v_n = v_0 \) and edges \( d_0, \ldots, d_{n-1} \) in \( T \) above \( e_0, \ldots, e_{n-1} \) so that \( d_i \) joins the vertices \( t_i \) and \( t_{i+1} \). Note that although \( t_n \) also lies above \( v_0 \) it is not equal to \( t_0 \) because \( T \) is a tree, though there will be an element \( s \in S \) with \( s(t_0) = t_n \) and so the stabiliser (in \( S \) or in \( G \)) of \( t_n \in T \) will be the conjugate by \( s \) of the stabiliser (in \( S \) or in \( G \)) of \( t_0 \). We put an orientation on the straight line path from \( t_0 \) to \( t_n \), which we will call \( p \), so that the edge \( d_i \) points from \( t_i \) to \( t_{i+1} \) and this induces an orientation on the loop \( \Lambda \). We also take another lift of the edge \( e_{n-1} \) which we call \( d_{-1} \), this time joining \( s^{-1}(t_{n-1}) \) and \( t_0 \) and such that its stabiliser in \( S \) is the subgroup of the vertex stabiliser in \( S \) of \( t_0 \) which is the edge label in \( S(\Sigma) \) on \( e_{n-1} \) lying nearest to \( v_0 \).

We next consider the image under \( \pi_G \) of our path \( p \). As the element \( s \in S \) sending \( t_0 \) to \( t_n \) is also in \( G \), we know \( \pi_G(p) \) is clearly a closed path in \( \Gamma \) but it need not be reduced. For instance there can be backtracks in \( \pi_G(p) \) when say \( t_{i-1} \) and \( t_{i+1} \) are identified in \( \Gamma \) by an element of the vertex stabiliser of \( t_i \) in \( G \) which does not lie in \( S \). But as our subgroup \( S \) lies in the finite index subgroup \( H \) of \( G \), we have that this path is the image under the covering map \( q \) of the closed path \( \pi_H(p) \) in \( \Delta \). Thus by Proposition 3.6 every (non oriented) edge is passed through an even number of times when tracing out \( \pi_G(p) \) in \( \Gamma \).

Our purpose now is to note the relationship between the vertex and edge stabilisers of \( G \) acting on \( T \) and the vertex and edge groups in the graph of groups \( G(\Gamma) \) which we are thinking of as labels appearing on the vertices and edges of \( \Gamma \). We only need this information for the path \( p \), so to this end we define \( V_i \) (where \( 0 \leq i \leq n - 1 \)) to be the group \( V_{\pi_G(t_i)} \) at the vertex \( \pi_G(t_i) \in \Gamma \). Similarly given the edge \( d_j \) for \( 0 \leq j \leq n - 1 \) with orientation as above, we set \( E_j \) to be the edge subgroup \( E_{\pi_G(d_j)} \) of \( V_{\pi_G(t_j)} \) which is the outgoing edge label of \( \pi_G(d_j) \) in \( G(\Gamma) \) and \( F_{j+1} \) equal to the edge subgroup
Let $F_{\pi_G(d_i)}$ of $V_{\pi_G(t_{j+1})}$ by reading the incoming edge label. In particular both $E_i$ and $F_i$ are subgroups of $V_i$ for $0 < i < n$ and $E_0, F_n$ are subgroups of $V_0$.

In order to form the fundamental group $G$ from the graph of groups $G(\Gamma)$, we first pick a maximal tree in $\Gamma$ and then associate the remaining edges (once given an arbitrary orientation) to a free basis of the fundamental group $\pi_1(\Gamma)$. Earlier we took a lift such that the initial vertex $t_0 \in T$ actually has stabiliser equal to the label $V_0$ at the vertex $\pi_G(t_0) \in \Gamma$. Then we have that the vertex stabiliser of the point $t_i$, the edge stabiliser of the edge $d_{i-1}$ coming into $t_i$, and the edge stabiliser of the edge $d_i$ leaving $t_i$ are all simultaneously conjugate in $G$ to the labels $V_i, F_i, E_i$ respectively of their images under $\pi_G$ in the graph of groups $G(\Gamma)$. In particular we have that $E_i$ and $F_i$ are equal subgroups of $V_i$ if and only if the stabilisers $\text{Stab}_G(d_{i-1})$ and $\text{Stab}_G(d_i)$ are equal subgroups of $\text{Stab}_G(t_i)$.

We now look more closely at the stabilisers of $G$ acting on $T$. Thus let us set the vertex stabiliser in $G$ of the point $t_i \in T$ (where $0 \leq i \leq n$) to be $\langle x_i, y_i \rangle$ and let the (infinite cyclic) edge stabiliser in $G$ of the edge $d_j$ for $0 \leq j \leq n - 1$ be as follows: by changing bases if necessary we can assume by maximality that it is equal to the cyclic subgroup $\langle x_j \rangle$ of $\text{Stab}_G(t_j)$, although this edge stabiliser is also a maximal cyclic subgroup of the vertex stabiliser $\langle x_{j+1}, y_{j+1} \rangle$ of $t_{j+1}$.

Next we perform a similar comparison between the vertex and edge labels on $S(\Sigma)$ and the stabilisers of $S$ acting on $T$, at least for the loop $\Lambda$ in $\Sigma$. Of course the latter are found by simply intersecting $S$ with the relevant vertex or edge stabiliser of $G$ acting on $T$. Thus the stabiliser in $S$ of $t_i$ for $0 \leq i \leq n - 1$ will equal $S \cap \langle x_i, y_i \rangle$, and we know these groups are all infinite cyclic, except possibly for $t_0$ which could have $\mathbb{Z}^2$ type. Also the edge stabilisers $S \cap \langle x_j \rangle$ in $S$ of $d_j$ for $0 \leq j \leq n - 1$ are all of $\mathbb{Z}$ type, thus we will have $S \cap \langle x_j \rangle = \langle x_j^{a_j} \rangle$ for some $a_j \neq 0$. However the stabiliser in $S$ of the edge $d_{j-1}$, which is also of $\mathbb{Z}$ type, must therefore be an infinite cyclic subgroup of the vertex stabiliser $S \cap \langle x_j, y_j \rangle$ and this latter subgroup is equal to $\langle x_j^{a_j} \rangle$ when $1 \leq j \leq n - 1$ because these vertices are also of $\mathbb{Z}$ type. This implies by maximality in $G(\Gamma)$ that $\langle x_j \rangle$ is equal to the incoming stabiliser in $G$ of $d_{j-1}$. But as the incoming and outgoing stabiliser of the edge $d_j$ in $T$ are the same group, we get

$$\langle x_0 \rangle = \langle x_1 \rangle = \langle x_2 \rangle = \ldots = \langle x_{n-1} \rangle,$$

thus by replacing elements with their inverse if necessary we have $x_0 = x_1 = \ldots = x_{n-1} = x$ say and $a_0 = a_1 = \ldots = a_{n-1} = a$ say. In particular the
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element \( x \) stabilises the geodesic line between \( t_0 \) and \( t_n \) as it lies in all of the relevant edge and vertex stabilisers. However if the vertex \( v_0 \) is of \( \mathbb{Z}^2 \) type then we need not have the incoming edge stabiliser of \( d_{-1} \) in \( S \) equal to the outcoming edge stabiliser \( S \cap \langle x \rangle = \langle x^a \rangle \) of \( d_0 \) at the vertex \( t_0 \in T \) but instead the incoming edge stabiliser of \( d_{-1} \) would equal \( \langle x_0^{k} y_0^{l} \rangle \) in \( G \) say, and \( \langle x_0^{m}, y_0^{n} \rangle \) say in \( S \).

At last we are ready to compare our path \( p \) from \( t_0 \) to \( t_n \) in \( T \) with its image \( \pi_G(p) \) in \( \Gamma \). We noted earlier that as we walk along the path \( p \) or \( \pi_G(p) \), a vertex having the property that the inclusions of the two edge groups are equal in this vertex group will display this behaviour both in \( T \) and in \( G(\Gamma) \).

Let us first consider the case where the vertex \( v_0 \) has \( \mathbb{Z}^2 \) type. Here we know that the final incoming edge group \( F_n \) in \( G(\Gamma) \) and the initial outgoing edge group \( E_0 \) are both subgroups of the vertex group \( V_0 \) but that they need not be equal. However we have that the subgroups \( F_i \) and \( E_i \) are equal in \( V_i \) for all other vertices because of the correspondence with the stabilisers. If in fact we do have \( E_0 = F_n \) then as the vertex \( v_0 \) has \( \mathbb{Z}^2 \) type, the first case of the proof of Proposition 3.4 immediately applies here for \( S(\Sigma) \), telling us that \( S \) surjects \( F_2 \). Otherwise we now define an equivalence relation on the set \( X \) of edges \( d_i \) \((0 \leq i \leq n - 1)\) in our path \( p \) for which the image \( \pi_G(d_i) \) (here regarded as a non oriented edge) has an endpoint equal to \( \pi_G(t_0) \) as follows: first suppose that there are no self loops in \( \pi_G(p) \). Then for each edge \( d_i \) in \( X \) we let the subgroup \( G_i \) of \( V_0 \) be whichever one of the edge subgroups \( E_i \) or \( F_{i+1} \) of \( \pi_G(d_i) \) lies nearest to \( \pi_G(t_0) \) (so if the pair of edges \( \pi_G(d_{i-1}) \) and \( \pi_G(d_{i}) \) both pass through \( \pi_G(t_0) \) then we will have \( G_{i-1} = F_i \) and \( G_i = E_i \) which are both subgroups of \( V_i = V_0 \), whereas if it is the pair \( \pi_G(d_i) \) and \( \pi_G(d_{i+1}) \) then now \( G_i = F_{i+1} \) and \( G_{i+1} = E_{i+1} \) which both lie in \( V_{i+1} = V_0 \). Then if \( d_i \) and \( d_j \) are two edges in \( X \), we say that they are equivalent exactly when \( G_i \) and \( G_j \) are equal subgroups of \( V_0 \). Now we know that \( G_0 = E_0 \) and \( G_{n-1} = F_n \) are not equal, so \( d_0 \) and \( d_{n-1} \) must lie in different equivalence classes. But every edge in \( \Gamma \) is passed through an even number of times by Proposition 3.6 and as \( \pi_G(d_i) = \pi_G(d_j) \) clearly implies that \( G_i = G_j \) (even if the images of these edges inherit opposite orientations from \( p \)) because we are reading the same edge label from \( G(\Gamma) \) in both cases, we must have that each equivalence class has even size.

However we will now look at the equivalence class \( C \) of \( d_0 \) in a different way by gradually building it up. Let us start by setting \( C \) equal to \( \{d_0\} \) so that we have a set of odd size. Now the image of \( d_0 \) will appear again as every edge in \( \pi_G(p) \) is visited an even number of times. Suppose then that
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we find $0 < i \leq n - 1$ with $\pi_G(d_i) = \pi_G(d_0)$. If $\pi_G(d_i)$ is travelling away from
the vertex $\pi_G(t_0)$ then certainly $i \neq n - 1$ because $\pi_G(d_{n-1})$ ends at $\pi_G(t_0)$,
and similarly $i \neq 1$ by considering $\pi_G(d_{i-1})$. In this case we have $d_{i-1}, d_i \in X$
with $G_{i-1} = F_i$ and $G_i = E_i$ and we know that $E_i = F_i$ here. However we
also have $G_i = G_0$ because $\pi_G(d_i) = \pi_G(d_0)$.

If however $\pi_G(d_i)$ travels towards $\pi_G(t_0)$ then we now have $G_i = F_{i+1}$
but again $G_i = G_0 = E_0$, therefore we cannot have $i = n - 1$ because
$G_{n-1} = F_n \neq E_0$. Thus $0 < i < n - 1$ and we also have $d_{i+1} \in X$ with
$G_{i+1} = E_{i+1} = F_{i+1} = G_i = G_0$. Thus in either case we see that two more
edges $d_{i-1}, d_i$ or $d_i, d_{i+1}$ are equivalent to $d_0$. Consequently we now add them
to $C$ but $|C|$ is still odd. Thus some edge in $C$ must have its projection
visited again so this argument can be repeated, to find a new edge with its projection
having one end at $\pi_G(t_0)$. Moreover the edge immediately before
or after this edge that also projects to one with an endpoint at $\pi_G(t_0)$ will not
yet have appeared in $C$ because at each stage we add to $C$ a pair of edges
with image passing through $\pi_G(t_0)$. Thus two more elements can now be
added to $C$ and the argument can again be repeated until $C$ is of arbitrarily
large but odd size, which is a contradiction.

If there are self loops then the previous argument can now be made to
work as follows. If we obtain a self loop from $X$, namely the edge $d_i$ is such
that the initial and terminal endpoint of $\pi_G(d_i)$ are both equal to $\pi_G(t_0)$,
then we split the edge $d_i$ into two new edges $d_i, d_{i+1}$ (with the later edges
renumbered accordingly). We think of these as an outgoing edge $d_i$ where
we set $G_i$ (in the new numbering) equal to $E_i$ (in the old numbering) and
$G_{i+1}$ (in the new numbering) equal to $F_{i+1}$ (in the old numbering). Our
parity argument will now work as before.

The remaining case is where all vertices and edges have $\mathbb{Z}$ type, so that
the fundamental group of the graph of groups obtained from the loop $\Lambda$ in
$\Sigma$ will be isomorphic either to $\mathbb{Z}^2$ or to the Klein bottle group and we will
now eliminate the latter under the assumption that this group lies in $H$.
Consequently we have $E_i = F_i$ for $1 \leq i \leq n - 1$ as before, and now also
$F_n = E_0$. The image in $\Gamma$ under $\pi_G$ of the path $p$ can be thought of as a
connected subgraph $\Pi$ of $\Gamma$. We have that all edge groups in $\Pi$ are infinite
cyclic and for the edge $d_i$ we know that the outgoing edge stabiliser label
$E_i$ in $G(\Gamma)$ is a subgroup of $V_i$ and the incoming label $F_{i+1}$ is a subgroup of
$V_{i+1}$ (with $F_n \leq V_0$). Thus for each vertex $w \in \Pi$ and then for each edge
e in $\Pi$ that leaves $w$, we have that the outward pointing edge stabiliser $E_e$
is an infinite cyclic subgroup and we choose a preferred generator $g_e$ of $E_e$. 
However in our graph of groups $G(\Gamma)$ we do not just have group labels $E_i$ and $F_{i+1}$ at each end of the edge $\pi_G(d_i)$, which we have used up to now, but also a given isomorphism of the edge stabiliser to each of the infinite cyclic subgroups $E_i$ and $F_{i+1}$, thus providing a given isomorphism between them and hence between $E_i$ and $E_{i+1} = F_{i+1}$. We now label the edge $d_i$ with $+$ or $-$ as follows: by looking at $\pi_G(d_i)$ in $\Pi$, we will find that the given isomorphism from $E_i$ to $E_{i+1}$ either sends the preferred generator of $E_i$ to that of $E_{i+1}$, in which case the edge $d_i$ is labelled $+$, or to its inverse in which case $d_i$ is labelled $-$. In particular if $\pi_G(d_i) = \pi_G(d_j)$ then (regardless of orientation) $d_i$ and $d_j$ have the same label, because these are chosen after applying $\pi_G$.

Thus on forming the subgroup $S$ from $S(\Sigma)$, we can first consider the loop $\Lambda$. We have that $S$ conjugates the stabiliser $\langle x^a \rangle$ in $S$ of the vertex $t_0$ to that of $t_n$ which is also equal to $\langle x^a \rangle$, but in order to work out whether we have $sx^a s^{-1} = x^a$ or $sx^a s^{-1} = x^{-a}$, we follow the image path $\pi_G(p)$ in $\Pi$, whereupon we need to keep track of how the element $x$, which lies in all of $\text{Stab}_G(t_0), \ldots, \text{Stab}_G(t_{n-1}), \text{Stab}_G(t_n)$, relates to these generators $g_i$. Now by the description earlier of the relationship between stabilisers in $G$ and labels in $G(\Gamma)$, we can take $x = g_0$ where we are setting $g_i$ equal to the preferred generator of $E_i$. Then as we walk along $p$ we have at each vertex $t_i$ that $x$ is conjugate in $G$ to one of $g_i^{\pm 1}$, and we can keep track of the sign when moving to $t_{i+1}$ by changing it if $\pi_G(d_i)$ is labelled with $-$, so that $g_i^{\pm 1}$ is replaced by $g_{i+1}^{\mp 1}$ respectively, but keeping the sign if $\pi_G(d_i)$ has a $+$ so that $g_i^{\pm 1}$ becomes $g_{i+1}^{\mp 1}$. When we arrive at $t_n$ our element will be $g_n$ or $g_n^{-1}$ according to the parity of the number of negative edges we walked over in $\Pi$ and this element $g_n^{\pm 1}$ is a conjugate (by $s$) of $x = g_0$. But $g_n = g_0$ because they are both the same preferred generator as $\pi_G(t_n) = \pi_G(t_0)$. Thus we have $x^{\pm 1} = sx s^{-1}$ but as every edge is walked over an even number of times, we conclude that $x^{\pm 1}$ is actually equal to $x$.

Consequently in the graph of groups $S(\Sigma)$ we find that the loop $\Lambda$ gives rise to the subgroup $\langle x, s | sx^a s^{-1} = x^a \rangle \cong \mathbb{Z}^2$ of $S$. As for $S$ itself, we are now left with trees emanating from here, so we will find that either $S$ surjects to $F_2$ or is just $\mathbb{Z}^2$ by the proof of Theorem 3.3 as if $\Sigma$ were a tree.
4 Other properties of tubular groups

A group is said to be large if it has a finite index subgroup that surjects $F_2$. Thus any group which virtually satisfies the strongest Tits alternative and which is not virtually abelian, such as virtually special groups or the tubular groups in the last section with maximal edge inclusions, is automatically large. However it is not hard to see that actually all tubular groups are large, including those containing non Euclidean or even non residually finite Baumslag - Solitar groups. Here we give a proof of this fact using results from the last section which shows that tubular groups come very close to always surjecting $F_2$.

Proposition 4.1 If $G(\Gamma)$ is a tubular group and $\Gamma$ is not just a single vertex (whereupon $G \cong \mathbb{Z}^2$) or a single vertex with a single self loop then $G$ surjects $F_2$.

If $\Gamma$ is a single vertex with a single self loop, so that without loss of generality we have

$$G = \langle x, y, t | [x, y], tx^it^{-1} = x^jy^k \rangle$$

for $i \neq 0$ and $j, k$ not both 0 then $G$ surjects $F_2$ if and only if $k = 0$. If not then $G$ has a subgroup of index 2 surjecting $F_2$.

Proof. We know by the previous section that we are done if $\Gamma$ has at least two loops, or if $\Gamma$ is a tree by the first part of the proof of Theorem 3.3 which does not require maximal inclusions of edge groups because $\Gamma$ is a finite tree. The same point holds using the proof of Proposition 3.4 if $\Gamma$ has a single loop which is not a self loop, on replacing any edge inclusion with the maximal cyclic subgroup of the relevant vertex group which contains it. Thus suppose that

$$G = \langle x, y, t | [x, y], tx^it^{-1} = x^jy^k \rangle$$

so that if $k = 0$ then we can remove $x$ to surject $F_2$. But in any homomorphism of $G$ onto $F_2$, we know that $x$ and $y$ have to map into the same cyclic subgroup of $F_2$, so that this map would factor through

$$\langle x, y, t, z | [x, y], tx^it^{-1} = x^jy^k, x = z^l, y = z^m \rangle$$

for some $l, m$ not both 0

$$\cong \langle t, z | tz^it^{-1} = z^{j+lkm} \rangle$$

which is 2-generated but not isomorphic to $F_2$ unless $l = 0$ and $k = 0$. 
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However as $\Gamma$ is a self loop then, even though $G$ does not surject $F_2$ in this case, we can make an immediate application of Proposition 3.6 with $b = 1$ to obtain the graph of groups $H(\Delta)$ where the index 2 subgroup $H$ of $G$ surjects $F_2$ by the start of this proof, because $\Delta$ is not just a single self loop. 

As for group theoretic properties which are held by all free by $\mathbb{Z}$ tubular groups, as opposed to all tubular groups, we also have residual finiteness (which in fact holds for all free by $\mathbb{Z}$ groups). The other property, also shared by virtually special groups, which we will show here is that of being virtually biorderable. This will be a consequence of [16] Corollary 2.2 and Theorem 2.6, which together imply that if the free by $\mathbb{Z}$ group $G = F_n \rtimes_\alpha \mathbb{Z}$ is such that all eigenvalues of the matrix given by the action of $\alpha$ on the abelianisation $\mathbb{Z}^n$ are real and positive then $G$ is biorderable.

**Theorem 4.2** If $G(\Gamma)$ is a tubular free by $\mathbb{Z}$ group then $G$ is virtually biorderable.

**Proof.** We assume that we have a homomorphism $\chi : G \to \mathbb{Z}$ as described in Theorem 2.1 which is non zero on all edge groups. This gives rise to a particular decomposition of $G$ as $F_n \rtimes_\alpha \mathbb{Z}$ for $F_n$ the kernel of $\chi$ and the characteristic polynomial of the abelianised matrix is also equal to the Alexander polynomial $\Delta_{G,\chi}(t)$. This can be evaluated using the Fox calculus as described in many places: see [6] for a treatment of groups with a deficiency 1 presentation, as we have here.

We first suppose that $\Gamma$ is a tree of $v$ vertices, giving rise to the standard $2v$ generator, $2v - 1$ relator presentation of $G$ which we read off from the graph of groups. We next obtain, using the Fox calculus, the $2v - 1$ by $2v$ Alexander matrix $A$ with entries in the ring of Laurent polynomials $\mathbb{Z}[t^{\pm 1}]$, where $t$ generates the image of $\chi$. We then delete each column in turn and take determinants to obtain the $2v$ minors, with the Alexander polynomial (defined up to units) being the highest common factor of these minors. We will show by induction on $v$ that if the first column is deleted then the corresponding minor is (up to units) a product of cyclotomic polynomials, thus so is $\Delta_{G,\chi}$. For $v = 2$ let us take a basis $\langle a_1, b_1 \rangle$ for the first vertex, where the inclusion into $\langle a_1, b_1 \rangle$ of the single edge group lies in $\langle a_1 \rangle$, so that $\chi(a_1) = p \neq 0$ and we can choose $b_1$ so that $\chi(b_1) = 0$. Similarly for the second vertex, giving us a basis $\langle a_2, b_2 \rangle$ where $\chi(b_2) = 0$ and $\chi(a_2) = q \neq 0$. Thus our three relators here will be $[a_1, b_1], a_1^k a_2^{-l}$ (where $k, l \neq 0$ and $kp = \ldots$
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\[ a, b \], \quad \left[ a_2, b_2 \right]. \]

On ordering the generators \( a_1, b_1, a_2, b_2 \) and the relators as above, we obtain the Alexander matrix

\[
\begin{pmatrix}
0 & t^p - 1 & 0 & 0 \\
\frac{t^{pq} - 1}{t^p - 1} & 0 & \frac{1 - t^q}{t^r - 1} & 0 \\
0 & 0 & 0 & t^q - 1
\end{pmatrix}
\]

thus our result holds so far. Now suppose this is true for \( 2v \) vertices and we add a leaf to \( \Gamma \), thus introducing two new generators and columns \( a_{v+1}, b_{v+1} \), where again we assume that the inclusion into \( \langle a_{v+1}, b_{v+1} \rangle \) of the new edge subgroup lies in \( \langle a_{v+1} \rangle \) and \( \chi(a_{v+1}) = r \neq 0, \chi(b_{v+1}) = 0 \). If this new vertex joins the \( d \)th vertex then we will have the two new relators (and consequent rows) \( a_d^{-1}b_d^{-1}a_{v+1}^{-s} \) and \( [a_{v+1}, b_{v+1}] \). But as our new generators \( a_{v+1}, b_{v+1} \) did not appear in any of the previous relations, this means we are adding two new columns to our Alexander matrix, with the first that corresponds to \( a_{v+1} \) having zeros in all but the penultimate row where we have \( (1 - t^{rs})/(t^r - 1) \), whereas the second new column corresponding to \( b_{v+1} \) has all zeros except \( t^r - 1 \) at the very bottom. Thus, regardless of what else lies in our new rows, on crossing off the first column and taking the determinant we obtain \( 1 - t^{rs} \) times the previous determinant so our induction is complete.

If \( \Gamma \) is not a tree then we now add extra edges to a maximal tree \( T \) with graph of groups \( H(T) \) for which the above applies. But by the description in the proof of Theorem 2.1 we can replace \( \chi : G \rightarrow \mathbb{Z} \) by the related homomorphism \( \chi_1 : G \rightarrow \mathbb{Z} \) where \( \chi_1 \) sends the stable letters to zero but agrees with \( \chi \) on the vertex groups. Then on adding the first edge we see from the presentation (1) in Theorem 2.1 and the description of the Alexander polynomial in terms of the characteristic polynomial of the abelianised matrix that \( \Delta_{E, \chi_1} \) is the product of the characteristic polynomial \( \Delta_{H, \chi} \) as found above and the determinant of the matrix with 1s along the lower diagonal and in the top right hand corner, which is \( \pm(t^n - 1) \).

We then repeat this for all remaining stable letters, resulting in \( G \) being expressed as \( F_N \rtimes_{\alpha} \mathbb{Z} \) where the natural homomorphism to \( \mathbb{Z} \) has Alexander polynomial whose zeros are all roots of unity. But on taking the finite cyclic cover \( H = F_N \rtimes_{\alpha} \mathbb{Z} \) of index \( i \) in \( G \), we have that here the abelianised matrix is the \( i \)th power of that for \( \alpha \), so for suitable \( i \) the eigenvalues are all 1. Thus \( H \) is biorderable by the results quoted just before this theorem.

\[ \square \]
We finish this section with some properties that tubular groups do not possess. They are obviously not word hyperbolic but also are never relatively hyperbolic by [7] Corollary 5.5. However Theorem 4.2 in that paper shows they are nearly always acylindrically hyperbolic: a tubular group is not acylindrically hyperbolic exactly when for every vertex we have that all edge inclusions into that vertex group lie in a single cyclic subgroup. (Acylindrical hyperbolicity implies the property of being SQ universal but this is weaker than largeness which we already have for tubular groups.)

The final property we consider is that of being residually free. As a subgroup of a residually free group is also clearly residually free, we have that any group known to be residually free will also immediately satisfy the strongest Tits alternative. Therefore we should confirm that tubular groups cannot be residually free (or virtually residually free) in order to ensure that the results in the last section do not follow from this property. However there is one type of tubular group that is obviously residually free, namely anything isomorphic to $F_n \times \mathbb{Z}$ and there are various graphs of groups with this as the fundamental group.

**Proposition 4.3** If $G(\Gamma)$ is a tubular group then $G$ is residually free if and only if $G$ is isomorphic to $F_n \times \mathbb{Z}$ for some $n \in \mathbb{N}$. This occurs if and only if

(i) For every vertex we have that all edge inclusions are equal to a single maximal cyclic subgroup of this $\mathbb{Z}^2$ vertex group and

(ii) We can assign an orientation of each of these maximal cyclic subgroups, namely a choice of one of the two possible generators, such that on following this generator around the edge subgroups of any closed loop in the graph $\Gamma$, we return to the original orientation.

**Proof.** Here we only give a sketch. It can be shown without much problem, by mimicking the approach of [21] Theorem 2.9 for surface by $\mathbb{Z}$ groups, that a free by $\mathbb{Z}$ group is residually free if and only if it is isomorphic to $F_n \times \mathbb{Z}$ for some $n \in \mathbb{N} \cup \{0\}$. Thus first suppose that we have a tubular group $G(\Gamma)$ where $G$ is residually free and $\Gamma$ is a tree, so that $G$ is free by $\mathbb{Z}$ from Corollary 2.3 and hence isomorphic to $F_n \times \mathbb{Z}$. Moreover the same holds for any subtree of $\Gamma$, from which we build up $G$ using amalgamated free products. By [14] Corollary 4.5 we have that the centre $Z(G)$ of $G = A \ast_C B$ is equal to $C \cap Z(A) \cap Z(B)$, so that by induction on the number of edges $G$ has a non trivial centre if and only if for every vertex we have that all edge inclusions into that vertex group lie in a single cyclic subgroup. However in $G \cong F_n \times \mathbb{Z}$ the centre is a maximal cyclic subgroup, thus requiring that at
every vertex all edge inclusions equal the same maximal cyclic subgroup, as otherwise we would have an element of a vertex group not lying in the centre of \( G \) but a power of it would.

Next we assume that \( \Gamma \) is a general finite graph but without self loops. Then we can take different maximal trees of \( \Gamma \), where every edge will lie in one of these trees, and use the conclusion above in combination with these various trees which will still force each edge group at a vertex to equal the same maximal cyclic subgroup. We then go back to our original maximal tree and add the remaining edges one by one, whereupon we require the orientation condition in following the edge subgroups around a loop, so that the relevant stable letter commutes with a generator of this edge group rather than conjugating it to its inverse. Finally we can add on any self loops at a single vertex, whereupon we still require the orientation condition.

By [15] Lemma 3.9 we have that being acylindrically hyperbolic is a commensurability invariant. Therefore suppose \( G(\Gamma) \) is a tubular group such that at some vertex in the graph the edge groups do not all lie in a single cyclic subgroup of this vertex group. Then as mentioned \( G \) is not acylindrically hyperbolic, so it cannot be virtually residually free because if so it is virtually \( F_N \times \mathbb{Z} \) by the above and this is certainly not acylindrically hyperbolic. Therefore the tubular groups which virtually satisfy the strongest Tits alternative by consequence of being virtually residually free are a very small collection amongst all tubular groups.

5 Examples of tubular groups

In this final section we look at examples that have already appeared in the literature and discuss their properties. Interestingly all graphs here will consist of a single vertex with one or two self loops.

Example 5.1: The Burns, Karass, Solitar group

This group \( \langle a, b, t | [a, b], tat^{-1} = b \rangle \) was introduced in [3] as the first 3-manifold group which was not subgroup separable (but then neither are many RAAGs).

However it is coherent and of cohomological and geometric dimension 2, it does not contain non Euclidean Baumslag - Solitar subgroups, it is free by \( \mathbb{Z} \), it is acylindrically but not relatively hyperbolic, every non trivial subgroup has a surjection to \( \mathbb{Z} \), it is biorderable, has the unique product property, and
has no zero divisors in its group algebra. Further it acts freely on a CAT(0) cube complex; indeed freely and cocompactly by [23] Corollary 5.10. Moreover by [13] or [17] it is in fact virtually special and so has a finite index subgroup satisfying the strongest Tits alternative (of which 2 is the smallest such index by Theorem 3.7 and Proposition 4.1). Being virtually special also implies that it is linear, even over $\mathbb{Z}$.

**Example 5.2**: The Gersten group $G = \langle a, b, s, t | [a, b], sas^{-1} = a^{-1}b^2, tat^{-1} = b \rangle$ with two self loops contains the previous example and so is not subgroup separable. However, in common with the Burns, Karass, Solitar group above, it is coherent and of cohomological and geometric dimension 2, it does not contain non Euclidean Baumslag - Solitar subgroups, it is free by $\mathbb{Z}$, it is acylindrically but not relatively hyperbolic, every non trivial subgroup has a surjection to $\mathbb{Z}$, it is biorderable, has the unique product property, and has no zero divisors in its group algebra. However it was introduced in [10] as an example of a group which does not act properly and cocompactly on a CAT(0) space. Also it is not virtually special: we thank Mark Hagen for explaining this. If there was a finite index subgroup $H$ of $G$ which was a subgroup of a RAAG (without loss of generality finitely generated because $H$ is) then this RAAG acts properly and cocompactly on the Salvetti complex, so $H$ acts properly on a finite dimensional cube complex. Then we can use quasiconvex walls obtained from $H$ to induce an action of $G$ on a CAT(0) cube complex, of higher but still finite dimension. This will also be a proper action and finite dimensionality tells us that $G$ acts properly and semisimply on this cube complex. However [3] points out that Gersten’s result extends to proper and semisimple actions on a CAT(0) space, thus giving us a contradiction. The Gersten group does act freely and hence properly on a CAT(0) cube complex by [23] but this will necessarily be infinite dimensional.

Now by Theorem 3.7 $G$ has a finite index subgroup (of index 4) satisfying the strongest Tits alternative, yet we see that this property cannot be established for $G$ by a cubulation type argument. We believe that $G$ is the best behaved example known, in terms of its abstract group theoretic properties, that does not have a correspondingly well behaved geometric action. However we do not know about its linearity, over $\mathbb{C}$ or over $\mathbb{Z}$.

**Example 5.3**: The Woodhouse group

In [24] a criterion was given for when a tubular group acts freely on a finite dimensional CAT(0) cube complex. We now look at Example 5.1 of that
paper, which again comes from two self loops:

\[ \langle a, b, s, t | [a, b], sabs^{-1} = a^2, tabt^{-1} = b^2 \rangle \]

and which was shown there, despite being free by \( \mathbb{Z} \) and so having a free action on a CAT(0) cube complex by Theorem 2.4, to have no such action on one which is finite dimensional.

As for its group theoretic properties, we again see that it is coherent and of cohomological and geometric dimension 2, it does not contain non Euclidean Baumslag - Solitar subgroups, it is free by \( \mathbb{Z} \), it is acylindrically but not relatively hyperbolic, every non trivial subgroup has a surjection to \( \mathbb{Z} \), it is left orderable, has the unique product property, and has no zero divisors in its group algebra. It does not have maximal edge inclusions but has a cover of index 2 that does by Proposition 3.1. It is also virtually biordeable and has a subgroup of index 8 satisfying the strongest Tits alternative by Theorem 3.7. In the follow up paper [25] it is shown that for tubular groups, being virtually special is equivalent to acting freely on a finite dimensional CAT(0) cube complex, so like the Gersten group this group does not virtually embed in a RAAG. Again we do not know about its linearity.

We now consider some rather stranger examples.

**Example 5.4:** The Wise simple curve examples

Example 1.7 in [23] introduced the following tubular groups:

\[ G_n = \langle a, b, s, t | [a, b], sa^nbs^{-1} = ab, tab^n t^{-1} = ab \rangle \]

for \( n \geq 2 \) where all edge inclusions are maximal. Even so it was shown there that they do not have equitable sets and so \( G_n \) does not act freely on any CAT(0) cube complex, thus it is not virtually special. As in common with all tubular groups it is coherent and of cohomological and geometric dimension 2, is not relatively hyperbolic, it is locally indicable and hence left orderable, has the unique product property, and has no zero divisors in its group algebra. Moreover it does not contain non Euclidean Baumslag - Solitar subgroups, and it is acylindrically hyperbolic. However it is not free by \( \mathbb{Z} \). Nevertheless, because all edge inclusions are maximal, we can conclude that every non trivial subgroup has a surjection to \( \mathbb{Z} \) and that (by Theorem 3.7) there is a subgroup of index 4 that satisfies the strongest Tits alternative. We do not know if \( G_n \) is linear or virtually biordeable.
Example 5.5: The Wise non simple curve examples
Example 1.6 of [23] was
\[ G_n = \langle a, b, s, t | sa^n s^{-1} = ab, tb^n t^{-1} = ab \rangle \]
so that \( n = 2 \) is actually the Woodhouse group in Example 5.3. It was shown in [23] that \( G_n \) does not act freely on a CAT(0) cube complex if \( n \geq 3 \) and so we know it is not virtually special. Again we conclude it is coherent and of cohomological and geometric dimension 2, it does not contain non Euclidean Baumslag - Solitar subgroups, it is acylindrically but not relatively hyperbolic, it is locally indicable and hence left orderable, has the unique product property, and has no zero divisors in its group algebra. But as \( G_n \) is not free by \( \mathbb{Z} \) and the edge inclusions are not all maximal, we do not know if it virtually satisfies the strongest Tits alternative.

Example 5.6: Wise’s non Hopfian tubular group
The group
\[ a, b, s, t | [a, b], sas^{-1} = a^2b^2, tbt^{-1} = a^2b^2 \]
already appears in [22] where it was shown to be non Hopfian, so it is certainly not virtually special. Of course it is coherent and of cohomological and geometric dimension 2, it does not contain non Euclidean Baumslag - Solitar subgroups, it is acylindrically but not relatively hyperbolic, it is locally indicable and hence left orderable, has the unique product property, and has no zero divisors in its group algebra. Moreover [23] shows that it acts freely on a CAT(0) cube complex, though this cannot be finite dimensional by [25]. However the edge inclusions are not maximal and it is not free by \( \mathbb{Z} \). Indeed as it is not residually finite, it is not even virtually free by \( \mathbb{Z} \).

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Selwyn College, University of Cambridge, Cambridge CB3 9DQ, UK

E-mail address: j.o.button@dpmms.cam.ac.uk