STABILISATION OF THE LHS SPECTRAL SEQUENCE FOR ALGEBRAIC GROUPS

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ABSTRACT. In this note, we consider the Lyndon–Hochschild–Serre spectral sequence corresponding to the first Frobenius kernel of an algebraic group $G$ and computing the extensions between simple $G$-modules. We state and discuss a conjecture that $E_2 = E_\infty$ and provide general conditions for low-dimensional terms on the $E_2$-page to be the same as the corresponding terms on the $E_\infty$-page, i.e. its abutment.

1. Introduction

Let $G$ be a reductive algebraic group over an algebraically closed field $k$ of characteristic $p > 0$. Let $\lambda$ and $\mu$ be two dominant weights for $G$. This paper concerns the representation theory of $G$ and its first Frobenius kernel $G_1$; we refer to [Jan03] for notation. It is the purpose of this short note to state and provide some evidence towards the following conjecture.

Conjecture. Suppose all $G_1$-injective hulls have the structure of $G$-modules, for instance if $p \geq 2h - 2$. Then the Lyndon–Hochschild–Serre spectral sequence

(*) \[ E_2^{ij} = \text{Ext}_{G/G_1}^i(k, \text{Ext}_{G_1}^j(L(\lambda), L(\mu))) \Rightarrow \text{Ext}_{G}^{i+j}(L(\lambda), L(\mu)) \]

stabilises (i.e. reaches its abutment) at the $E_2$-page. That is, $E_2^{ij} \cong E_\infty^{ij}$ for all $i, j$.

Hence

\[ \text{Ext}_G^n(L(\lambda), L(\mu)) \cong \bigoplus_{i+j=n} \text{Ext}_{G/G_1}^i(k, \text{Ext}_{G_1}^j(L(\lambda), L(\mu))). \]

Note that it is an open conjecture of Humphreys and Verma that all $G_1$-injective hulls do indeed have the structure of $G$-modules, possibly making the first hypothesis trivially satisfied.

Let us underline the fact that we are unaware of any occasion where any differential in the spectral sequence (*) is known to be non-zero—even after replacing $G$ with an arbitrary connected algebraic group and replacing $L(\lambda)$ and $L(\mu)$ by arbitrary $G$-modules. Showing that certain differentials in the spectral sequence are zero has some history; we pick out a few cases. For a large class of naturally occurring modules $V$ and $W$, it was shown in [Par07] that when $G = \text{SL}_2$ the spectral sequence does stabilise at the $E_2$-page. In particular the conjecture is confirmed for the case $G = \text{SL}_2$, with no condition on $p$. It was shown by Donkin in [Don82] that the differentials $d_{m,1} : E_2^{m,1} \rightarrow E_2^{m+2,0}$ are zero, also with no condition on $p$. Some other special cases involving maps needed to compute second cohomology were considered in work of McNinch [McN02], the second author [Ste10,Ste12], and Ibraev [Ibr11,Ibr12].

Another case in which the conjecture is true is if $\lambda$ and $\mu$ are $p$-regular restricted weights, $p \geq 2h - 2$ and $p$ is large enough that the Lusztig Character Formula holds. Then [PS13, Theorem 5.3] shows that the $G$-module $\text{Ext}_{G_1}^i(L(\lambda), L(\mu))^{[[-1]}$ has a good filtration for each $n$. Under these
circumstances the spectral sequence moreover degenerates to a line; in particular the conjecture is true.

Note that the conjecture is not true if $G$ is replaced by an arbitrary group. See [BF94, §6], [Lea93] and [Sie00] for examples of non-zero differentials.

The main theorem of this paper is a confirmation of the conjecture in a generic sense. Here, the vanishing of differentials of degree much lower than $p$ is guaranteed.

**Theorem.** Suppose $p \geq (r + 1)(h - 1)$. Then the differentials $d^{ij}_n$ in the spectral sequence (*) satisfying $i \leq r - 1$ and $n \geq 2$ or $j = 0$ and $n \geq 2$ or $j = 1$ and $n \geq 2$ are all zero.

In particular,

$$\text{Ext}^i_G(L(\mu), L(\lambda)) \cong \bigoplus_{j=0}^{i} E^{i-j,j}_2$$

for $i \leq r + 1$.

We prove the above theorem by applying techniques from [Par07]. First, we show, in a proposition, that part of a minimal $G_1$-injective resolution has a compatible $G$-structure. We then reconstruct the spectral sequence (*) in such a way that the bottom-most complex in the double complex giving the $E_0$-page contains this part of a minimal $G_1$-injective resolution. It follows that many maps in the $E_0$-page are zero. Then some derived couple arguments prove the theorem.

2. **Proposition and proof of the theorem**

In the proposition below, note that the case $r = 0$ would be a special case of the Humphreys–Verma conjecture. (It is not known if the bound $p \geq 2h - 2$ could be reduced to $p \geq h - 1$ for $G_1$-injective hulls to lift to $G$-modules.)

**Proposition.** Let $r \geq 1$ and let $\mu \in X_i$. Provided $p \geq (r + 1)(h - 1)$, there is a minimal $G_1$-resolution

$$0 \rightarrow L(\mu) \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_r \rightarrow \cdots$$

such that the sequence up to term $I_r$ has a $G$-structure.

**Proof.** We prove a fortiori that there is such a sequence of $G$-modules with $I_r$ having weights $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_i$, which satisfy $(\lambda_1, \alpha^G_0) \leq (r + 1)(h - 1)$.

First, let us treat the case $r = 1$. Set $I_0 = Q_1(\mu)$. The hypotheses imply that $p \geq 2h - 2$; thus we know that $Q_1(\mu)$ has the structure of a $G$-module. The injection $L(\mu) \rightarrow Q_1(\mu)$ is then a map of $G$-modules.

Let $M := Q_1(\mu)/L(\mu)$. We may write $\text{Soc}_{G_1} M = \bigoplus_{\nu} L(\nu_0) \otimes M_\nu^F$ where $\nu_0 \in X_i$ and $M_\nu$ is some $G$-module. Set $I_1 = \bigoplus_{\nu} Q_1(\nu_0) \otimes M_\nu^F$. So $\text{Soc}_{G_1} I_1 = \text{Soc}_{G_1} Q_1(\mu)/L(\mu)$. (It is worth noting that the condition on the weights here is enough to ensure that $\text{Soc}_{G_1} M = \text{Soc}_G M$ but we do not need this fact explicitly.) Thus $I_1$ is the $G_1$-injective hull of $M$, hence if there is a $G$-map $I_0 \rightarrow I_1$, this will be part of a minimal resolution. It remains to show that there is indeed a map $I_0 \rightarrow I_1$ of $G$-modules whose kernel is $L(\mu)$, i.e. a map $I_0/L(\mu) \rightarrow I_1$. Note that we do have a map $\text{Soc}_{G_1} M \rightarrow I_1$ by construction, so consider the exact sequence

\begin{equation}
\text{Hom}_G(M, I_1) \rightarrow \text{Hom}_G(\text{Soc}_{G_1} M, I_1) \rightarrow \text{Ext}^1_G(M/\text{Soc}_{G_1} M, I_1).
\end{equation}
If we could show that the third term in this sequence is zero then we would have that the first map were surjective, hence the $G$-map $\text{Soc}_{G_1} M \rightarrow I_1$ would lift to a map $M = I_0/L(\mu) \rightarrow I_1$ and we would be done.

Now $\text{Ext}^3_{G_2}(M/\text{Soc}_{G_1} M, I_1)$ has a filtration by spaces $E = \text{Ext}^3_{G_2}(M/\text{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^F)$ over certain weights $\nu = \nu_0 + p

Thus the composition factors of $I_0$ (hence of $M$) are of the form $L(\xi_0) \otimes L(\xi_1)^F$. In particular, we have that the weights $\nu_1$ satisfy $(\nu_1, \alpha_0^\vee) \leq h-1$. Thus $(\xi_1 + p, \alpha_0^\vee), (\nu_1 + p, \alpha_0^\vee) \leq 2h-2$ and our condition on $p$ implies that they are both in the closure of the lowest alcove, $\mathcal{C}_Z$. So let $L(\xi_0) \otimes L(\xi_1)^F$ be a composition factor of $M/\text{Soc}_{G_1} M$. We compute:

$$
\text{Ext}^1_{G/G_1}(k, \text{Hom}_{G_1}(M/\text{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^F)) \rightarrow E
$$

Now the third term here is zero, as $Q_1(\nu_0)$ is injective for $G_1$, hence, to show $E = 0$, it suffices to show that the first term is zero.

Now $I_0 = Q_1(\mu)$, being a direct $G_1$-summand of $St_1 \otimes L((p-1)\rho - \mu)$, has weights satisfying $\xi = \xi_0 + p\xi_1$ with $(\xi_1, \alpha_0^\vee) \leq h-1$. Thus the composition factors of $I_0$ (hence of $M$) are of the form $L(\xi_0) \otimes L(\xi_1)^F$. In particular, we have that the weights $\nu_1$ satisfy $(\nu_1, \alpha_0^\vee) \leq h-1$. Thus $(\xi_1 + p, \alpha_0^\vee), (\nu_1 + p, \alpha_0^\vee) \leq 2h-2$ and our condition on $p$ implies that they are both in the closure of the lowest alcove, $\mathcal{C}_Z$. So let $L(\xi_0) \otimes L(\xi_1)^F$ be a composition factor of $M/\text{Soc}_{G_1} M$. We compute:

$$
\text{Ext}^1_{G/G_1}(k, \text{Hom}_{G_1}(L(\xi_0) \otimes L(\xi_1)^F, Q_1(\nu_0) \otimes L(\nu_1)^F)) \cong \text{Ext}^1_{G}(L(\xi_1), \text{Hom}_{G_1}(L(\xi_0), Q_1(\nu_0))[-1] \otimes L(\nu_1))
$$

Now $\text{Hom}_{G_1}(L(\xi_0), Q_1(\nu_0))$ is non-zero, hence equal to $k$, if an only $\xi_0 = \nu_0$; in that case, the term on the right becomes $\text{Ext}^1_{G}(L(\xi_1), L(\nu_1))$, and since $\xi_1, \nu_1 \in \mathcal{C}_Z$, this vanishes by the linkage principle. This concludes the proof in case $r = 1$.

Now by induction we may assume that we have a sequence of $G$-modules

$$
0 \rightarrow I_0 \rightarrow \cdots \rightarrow I_{r-1} \xrightarrow{\pi} I_r,
$$

which is minimal as an injective $G_1$-resolution, such that the composition factors of $I_{r-1}$ have high weights $\lambda$ satisfying $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1$ and $(\lambda_1, \alpha_0^\vee) \leq r(h-1)$. We construct $I_r$ in a similar way to before: set $I_r = \bigoplus_{\nu_0} Q_1(\nu_0) \otimes M_\nu$, where the sum is over the $G$-composition factors of $\text{Soc}_{G_1} I_{r-1}/\pi I_{r-2}$, where $\nu_0 \in X_1$ and a weight $\nu_0$ of $M_\nu$ satisfies $(\nu_0, \alpha_0^\vee) \leq r(h-1)$. Thus a weight $\xi$ of $I_r$, say $\xi_0 + p\xi_1$ with $\xi_0 \in X_1$ satisfies $(\xi_1, \alpha_0^\vee) \leq (\nu_0, \alpha_0^\vee) + (\rho, \alpha_0^\vee) = (r+1)(h-1)$ as required. Note that $I_r$ is again a $G_1$-injective hull of $I_{r-1}/\text{im} \pi$ so if we can show there is a $G$-module map $I_{r-1} \rightarrow I_r$ with kernel im $\pi$, we will be done.

Of course, it is equivalent to produce a map from $M := I_{r-1}/\text{im} \pi$ to $I_r$. By construction we do have a map from $\text{Soc}_{G_1} M \rightarrow I_r$. Now the same argument as before shows that the third term in the sequence (*) (with $I_r$ replacing $I_1$) is zero. This completes the proof.

\[\square\]

**Proof of the theorem.** We write $L(\mu) = L(\mu_0) \otimes L(\mu_1)^F$ using Steinberg’s tensor product theorem where $\mu_0 \in X_1$ and $\mu_1 \in X^+$. Using the proposition we have a $G$-resolution which is also a $G_1$-injective resolution:

$$
0 \rightarrow L(\mu_0) \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_r \rightarrow \cdots,
$$

where, up to $I_r$, the resolution is minimal for $G_1$. 

We denote the differentials by $\delta_i : I_i \to I_{i+1}$ and the kernels by $K_i := \ker \delta_i$. Dimension shifting gives us $\text{Ext}^i_{G_1}(L(\lambda_0), L(\mu_0)) \cong \text{Ext}^i_{G_1}(L(\lambda_0), K_{i-1})$. Minimality gives us for $\mu_1 \in X_1$ that $\text{Ext}^i_{G_1}(L(\lambda_0), L(\mu_0)) \cong \text{Hom}_{G_1}(L(\lambda_0), K_i) \cong \text{Hom}_{G_1}(L(\lambda_0), I_i)$ for $i \leq r$.

We now have a $G$-resolution:
\[
0 \to L(\mu) \to I_0 \otimes L(\mu_1)^F \xrightarrow{\delta_0} I_1 \otimes L(\mu_1)^F \xrightarrow{\delta_1} \cdots
\]
where $\delta_i = \delta_i \otimes \text{id}$, as tensoring is exact. Also note that such a resolution stays injective as a $G_1$-resolution as $L(\mu_1)^F$ is trivial as a $G_1$-module.

Now consider the $E_0$-page of the LHS spectral sequence that converges to $\text{Ext}^*_G(L(\lambda), L(\mu))$ as constructed in [Par07, §2]
\[
E_0^{mn} = \text{Hom}_{G/G_1}(k, \text{Hom}_{G_1}(L(\lambda), I_n \otimes L(\mu_1)^F) \otimes J_m^F)
\]
where we have a $G$-injective resolution of the trivial module:
\[
0 \to k \to J_0 \to J_1 \to \cdots
\]
and this spectral sequence has $E_1$ and $E_2$ page
\[
E_1^{mn} = \text{Hom}_{G/G_1}(k, \text{Ext}^n_{G_1}(L(\lambda), L(\mu)) \otimes J_m^F)
\]
\[
E_2^{mn} = \text{Hom}(G/G_1, \text{Ext}^n_{G_1}(L(\lambda), L(\mu))).
\]

Consider the induced maps $\partial^*_m$ in the following complex, which has homology $\text{Ext}^*_G(L(\lambda), L(\mu))$:
\[
\text{Hom}_{G_1}(L(\lambda), I_0 \otimes L(\lambda_1)^F) \xrightarrow{\partial^*_0} \text{Hom}_{G_1}(L(\lambda), I_1 \otimes L(\mu_1)^F) \xrightarrow{\partial^*_1} \cdots.
\]

Now
\[
\text{Ext}^m_{G_1}(L(\lambda), L(\mu)) \cong \text{Ext}^m_{G_1}(L(\lambda_0), L(\mu_0)) \otimes L(\mu_1)^F \otimes L(\lambda_1)^F
\]
\[
\cong \text{Hom}_{G_1}(L(\lambda_0), I_m) \otimes L(\mu_1)^F \otimes L(\lambda_1)^F \cong \text{Hom}_{G_1}(L(\lambda), I_m \otimes L(\mu_1)^F)
\]
for $m \leq r$. Thus all the differentials $\partial^*_m$ for $m \leq r$ must be zero.

Now by [Ben98, §3.2, §3.4] we know that the spectral sequence can be constructed using derived couples. We have
\[
D_0^{mn} = \bigoplus_{m+n=e+f, e \geq m} E_0^{ef}
\]
\[
E_1^{mn} = H(E_0^{mn}, d_0)
\]
\[
D_1^{mn} = H(E_0^{mn} \oplus E_0^{m+1,n-1} \oplus \cdots, d_0 + d_1)
\]

We define the higher derived couples by taking the derived couple of the previous one. We have an exact diagram of doubly graded $k$-modules
\[
\begin{array}{ccc}
D_0 & \xrightarrow{i} & D_1 \\
\downarrow{k} & & \downarrow{j} \\
E_l & \xrightarrow{f} & E_l
\end{array}
\]
The derived couple (for \( l \geq 1 \)) is defined by

\[
D_{l+1}^{mn} = \text{im} \left. i_l^{m+1,n-1} \right|_{D_l}\quad E_{l+1}^{mn} = H(E_l^{mn}, d_l)
\]

\[
\begin{align*}
\pi_l^{mn} & = \pi_l^{mn} \quad j_l^{mn}(j_l^{m+1,n-1}(x)) = j_l^{m+1,n-1}(x) + \text{im}(d_l) \\
D_{l+1}^{mn} & = H(E_{l+1}^{mn}, d_{l+1}) \\
E_{l+1}^{mn} & = \pi_l^{mn} (z + \text{im}(d_l)) = k_l^{mn}(z) \\
\end{align*}
\]

And the degrees of the maps \( k, j \) and \( d \) are:

\[
\deg(i_n) = (-1, 1), \quad \deg(j_n) = (n - 1, n + 1), \quad \deg(k_n) = (1, 0).
\]

Now using [Par07, Lemma 2.1] we have that \( d_0^{mn} = 0 \) implies that \( k_1^{m-1,n+1} = 0 \). Thus since \( d_l^{mn} = 0 \) for \( m \leq r \) we have \( k_l^{mn} = 0 \) for \( m \leq r - 1 \). Thus all \( k_l^{mn} = 0 \) for all \( l \geq 2 \) and \( m \leq r - 1 \).

As \( d_l^{mn} = j_l^{m+1,n} \circ k_l^{mn} \) we also get \( d_l^{mn} = 0 \) for all \( l \geq 2 \) and \( m \leq r - 1 \).

In other words, as all these differentials are zero on the \( E_2 \) page and remain zero, the terms \( E_2^{mn} \) with \( m \leq r - 1 \) must already be the stable value. That is, \( E_\infty^{mn} = E_2^{mn} \) for \( m \leq r - 1 \).

This easily gives us that

\[
\text{Ext}_G^i(L(\lambda), L(\mu)) = \bigoplus_{j=0}^i E_2^{i-j,j}
\]

for \( i < r \). To get the result for \( i = r \), we note that all the terms in the sum

\[
\bigoplus_{j=0}^r E_\infty^{r-j,j}
\]

stabilise at the \( E_2 \) page by the above, except, possibly the term \( E_\infty^{r,0} \). But here clearly \( E_\infty^{r,0} = E_2^{r,0} \) as all incoming differentials are zero by the above, and the leaving differential \( d_1^{r,0} \) is always zero as our spectral sequence is first quadrant.

We may similarly argue for \( r + 1 \). We consider

\[
\bigoplus_{j=0}^{r+1} E_\infty^{r+1-j,j}
\]

As before all terms except possibly \( E_\infty^{r,1} \) and \( E_\infty^{r+1,0} \) stabilise at the \( E_2 \) page. The same argument as in the previous case gives \( E_\infty^{r+1,0} = E_2^{r+1,0} \).

Now note that all incoming differentials to \( E_1^{r,1} \) are zero for \( l \geq 2 \) by the above. We also have that \( d_l^{r,1} = 0 \) for \( l \geq 3 \), again since the spectral sequence is first quadrant. So we need only check that \( d_1^{r,1} = 0 \), but this is true using [Don82, Main Theorem]. Thus we also get the result for \( r + 1 \). \( \square \)

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