STABILITY EQUIVALENCE AMONG STOCHASTIC DIFFERENTIAL EQUATIONS AND STOCHASTIC DIFFERENTIAL EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS AND CORRESPONDING EULER-MARUYAMA METHODS

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Abstract. In this paper, we consider the equivalence of the $p$th moment exponential stability for stochastic differential equations (SDEs), stochastic differential equations with piecewise continuous arguments (SDEPCAs) and the corresponding Euler-Maruyama methods EMSDEs and EMSDEPCAs. We show that if one of the SDEPCAs, SDEs, EMSDEs and EMSDEPCAs is $p$th moment exponentially stable, then any of them is $p$th moment exponentially stable for a sufficiently small step size $h$ and $\tau$ under the global Lipschitz assumption on the drift and diffusion coefficients.

Key words. Exponential stability, Stochastic differential equations, Numerical solutions, Piecewise continuous arguments

AMS subject classifications. 60H10, 65C20, 65L20, 60H35

1. Introduction. Stochastic differential equations (SDEs) have been widely used in many branches of science and industry [1, 4, 8, 9, 28, 34]. There is an extensive literature in stochastic stability (e.g. the moment exponential stability or almost sure exponential stability) [1, 5, 9, 18, 25, 36, 37]. One of the powerful techniques in the study of stochastic stability is the method of Lyapunov functions. In the absence of an appropriate Lyapunov function, we may carry out careful numerical simulations using a numerical method, say the Euler-Maruyama (EM) method [see e.g. 2, 12, 16, 17, 19, 26, 33, 39] with a small step size. Does the main question arise whether the numerical solutions can reproduce and predict the stability of the underlying solutions?

The case that stochastic stability of the general nonlinear equation and that of the numerical method are equivalent for a sufficiently small step size can be founded in [6, 13, 15, 22, 27, 30], while for the linear equation in [3, 11, 35]. Higham et al. in [14] showed that when the SDE obeys a linear growth condition, the EM method recovers almost surely exponential stability.

In this paper, we consider the following stochastic differential equation with piecewise continuous argument (SDEPCA)

\[ dx(t) = (f(x(t)) + u_1(x([t/\tau]\tau)))\,dt + (g(x(t)) + u_2(x([t/\tau]\tau)))\,dw(t) \]

and the stochastic differential equation (SDE)

\[ dy(t) = (f(y(t)) + u_1(y(t)))\,dt + (g(y(t)) + u_2(y(t)))\,dw(t). \]

We also consider the applications of EM method to SDEPCA (1.1) and SDE (1.2), respectively

\[ X_{n+1} = X_n + (f(X_n) + u_1(X_{[n/m]m}))h + (g(X_n) + u_2(X_{[n/m]m}))\Delta w_n, \]

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\[ Y_{n+1} = Y_n + (f(Y_n) + u_1(Y_n))h + (g(Y_n) + u_2(Y_n))\Delta w_n, \]

where \( h = \frac{\tau}{m} \), \( m \in \mathbb{N}^+ \). We refer to (1.3) and (1.4) by the terms EMSDEPCA (1.3) and EMSDE (1.4), respectively. The main purpose of the present paper is to show that if one of the SDEPCAs (1.1), SDEs (1.2), EMSDEPCA (1.3) and EMSDE (1.4) is \( p \)th moment exponentially stable, then so are the others for a sufficiently small step size \( h \) and \( \tau \) under a global Lipschitz assumption on the drift and diffusion coefficients.

In order to do this, we shall concentrate on the following questions:

(Q1) If for a sufficiently small \( \tau \), the SDEPCA (1.1) is \( p \)th moment exponentially stable, can we confidently infer that the SDE (1.2) is \( p \)th moment exponentially stable?

(Q2) For a sufficiently small step size \( h \), does the EMSDE (1.4) reproduce the \( p \)th moment exponential stability of the underlying SDE (1.2)?

(Q3) For a sufficiently small \( \tau \), the EMSDEPCA (1.3) can preserve the \( p \)th moment exponential stability of EMSDE (1.4)?

(Q4) If the EMSDEPCA (1.3) is \( p \)th moment exponentially stable, will the SDEPCA (1.1) be the \( p \)th moment exponentially stable for a sufficiently small step size \( h \)?

It is known that the positive answer to (Q2) for SDE in case \( p = 2 \) can be founded in [13]. The stochastic differential equation with piecewise continuous arguments (SDEPCA) has been studied extensively [see e.g. 7, 21, 29, 31, 32, 40], and in the case of \( \tau = 1 \), we refer to [23, 24]. Mao in [29] is the first paper that investigated the th moment exponential stability of EMSDE (1.4)?

In this paper, we will give the positive answer for (Q1), (Q2), (Q3), (Q4). In section 2, we describe the SDEPCA and EM methods along with the definitions of th moment exponential stability for SDE, SDEPCA, EMSDE, EMSDEPCA. Section 3, section 4, section 5, section 6 answer the questions (Q1), (Q3), (Q4), (Q2) respectively, the final conclusions are stated in the last section.

2. Perilimaries. Throughout this paper, unless otherwise specified, we will use the following notations. If \( A \) is a vector or matrix, its transpose is denoted by \( A^T \). If \( x \in \mathbb{R}^n \), then \( |x| \) is the Euclidean norm. If \( A \) is a matrix, we let \( |A| = \sqrt{\text{trace}(A^T A)} \) be its trace norm. If \( D \) is a set, its indicator function is denoted by \( 1_D \). Moreover, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (that is, it is right continuous and increasing while \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets), and let \( \mathbb{E} \) denote the expectation corresponding to \( \mathbb{P} \). Let \( B(t) \) be a \( m \)-dimensional Brownian motion defined on the space. Throughout this paper, we set \( p \geq 2 \).

In this paper, we deal with the following \( d \)-dimensional nonlinear stochastic differential equations with piecewise continuous arguments (SDEPCAs)

\begin{equation}
\begin{aligned}
&dx(t) = [f(x(t)) + u_1(x([t/\tau])\tau)]\,dt + [g(x(t)) + u_2(x([t/\tau])\tau)]\,dw(t) \\
x(0) = x_0 \in \mathbb{R}^d
\end{aligned}
\end{equation}

on \( t \geq 0 \), where \( w(t) \) is an \( m \)-dimensional Brownian motion, \( f: \mathbb{R}^d \to \mathbb{R}^d \), \( g: \mathbb{R}^d \to \mathbb{R}^{d \times m} \), \( u_1: \mathbb{R}^d \to \mathbb{R}^d \) and \( u_2: \mathbb{R}^d \to \mathbb{R}^{d \times m} \). \( \tau \) is a positive constant, \( [t/\tau] \) is the integer part of \( t/\tau \). We denote \( x(t) \) the solution of (2.1) with initial data \( x(0) = x_0 \) and \( y(t) \) the solution of the following SDEs

\begin{equation}
\begin{aligned}
dy(t) = [f(y(t)) + u_1(y(t))]\,dt + [g(y(t)) + u_2(y(t))]\,dw(t)
\end{aligned}
\end{equation}
on \( t \geq 0 \) with initial data \( y(0) = x_0 \).

In the present paper, we also deal with the application of EM method to SDEPCA (2.1) and SDE (2.2). We note that \( [t/\tau]\tau = nt \) for \( t \in [nt, (n + 1)\tau), n = 0, 1, 2, \cdots \), a natural choice for \( h \) is \( h = \frac{\tau}{m}, m \in \mathbb{N}^+ \). Hence, we have

\[
X_{n+1} = X_n + (f(X_n) + u_1(X_{n/m}))h + (g(X_n) + u_2(X_{n/m}))\Delta w_n,
\]

\[
Y_{n+1} = Y_n + (f(Y_n) + u_1(Y_n))h + (g(Y_n) + u_2(Y_n))\Delta w_n,
\]

where \( X_n \) and \( Y_n \) are the approximations of \( x(t) \) and \( y(t) \) at grid points \( t = nh, n = 0, 1, 2, \cdots \), respectively, \( \Delta w_n = w(t_{n+1}) - w(t_n) \). Let \( n = km + l, k \in \mathbb{N}^+, l = 0, 1, \cdots, m - 1 \). Then (2.3) and (2.4) would reduce to

\[
X_{km+l+1} = X_{km+l} + (f(X_{km+l}) + u_1(X_{km+l}))h + (g(X_{km+l}) + u_2(X_{km+l}))\Delta w_{km+l},
\]

\[
Y_{km+l+1} = Y_{km+l} + (f(Y_{km+l}) + u_1(Y_{km+l}))h + (g(Y_{km+l}) + u_2(Y_{km+l}))\Delta w_{km+l}.
\]

Remark 2.1. If we choose \( h = \tau \), then (2.3) and (2.4) are the same and (2.5) and (2.6) are the same.

In spite of the simplicity of the EM method, explicit EM method is the most popular for approximating the solution of the SDE under global Lipschitz condition [see 12, 19, 33] and has often been used successfully in actual calculations. For further analysis it is more convenient to use continuous-time approximations,

\[
x_{\Delta}(t) = x_0 + \int_0^t f(\bar{x}_{\Delta}(s)) + u_1(\bar{x}_{\Delta}(s/\tau))ds + \int_0^t g(\bar{x}_{\Delta}(s)) + u_2(\bar{x}_{\Delta}(s/\tau))dw(s),
\]

\[
y_{\Delta}(t) = x_0 + \int_0^t f(\bar{y}_{\Delta}(s)) + u_1(\bar{y}_{\Delta}(s))ds + \int_0^t g(\bar{y}_{\Delta}(s)) + u_2(\bar{y}_{\Delta}(s))dw(s),
\]

where

\[
\bar{x}_{\Delta}(t) = \sum_{n=0}^\infty X_n 1_{[t_n, t_{n+1})}(t), \quad \bar{y}_{\Delta}(t) = \sum_{n=0}^\infty Y_n 1_{[t_n, t_{n+1})}(t), \quad \forall t \geq 0.
\]

We observe that \( x_{\Delta}(t_n) = \bar{x}_{\Delta}(t_n) = X_n \) and \( y_{\Delta}(t_n) = \bar{y}_{\Delta}(t_n) = Y_n \). Consequently,

\[
x_{\Delta}([t/\tau]\tau) - \bar{x}_{\Delta}([t/\tau]\tau) = 0, \quad y_{\Delta}([t/\tau]\tau) - \bar{y}_{\Delta}([t/\tau]\tau) = 0.
\]

In this paper, we impose the following standing hypothesis.

Assumption 2.2. Assume that there exists a positive constant \( K \) such that

\[
|f(x) - f(y)| \vee |g(x) - g(y)| \vee |u_1(x) - u_1(y)| \vee |u_2(x) - u_2(y)| \leq K|x - y|,
\]

for all \( x, y \in \mathbb{R}^d \). Assume also that \( f(0) = 0, g(0) = 0, u_1(0) = 0 \) and \( u_2(0) = 0 \).

Assumption 2.2 implies that

\[
|f(x)| \vee |g(x)| \vee |u_1(x)| \vee |u_2(x)| \leq K|x|
\]

for all \( x \in \mathbb{R}^d \).

We now give our basic definitions, which is cited from [28].
DEFINITION 2.3. The equations SDEPCA (2.1) and SDE (2.2) are said to be \( p \)th moment exponentially stable if there exist positive constants \( M_1, \gamma_1, M_2, \) and \( \gamma_2 \) such that

\[
\mathbb{E}|x(t)|^p \leq M_1|x_0|^p e^{-\gamma_1 t}, \quad \forall t \geq 0,
\]

and

\[
\mathbb{E}|y(t)|^p \leq M_2|x_0|^p e^{-\gamma_2 t}, \quad \forall t \geq 0,
\]

for any \( x_0 \in \mathbb{R}^d \).

DEFINITION 2.4. For any given step size \( h > 0 \), the Euler-Maruyama numerical methods EMSDEPCA (2.3) and EMSDE (2.4) are said to be \( p \)th moment exponentially stable, if there exist positive constants \( \lambda_1, L_1, \lambda_2, L_2 \) such that

\[
\mathbb{E}|X_n|^p \leq L_1|x_0|^p e^{-\lambda_1 nh},
\]

\[
\mathbb{E}|Y_n|^p \leq L_2|x_0|^p e^{-\lambda_2 nh},
\]

for any \( x_0 \in \mathbb{R}^d, n \in \mathbb{N} \).

It is known that under Assumption 2.2, for any initial value \( x_0 \) given at time \( t = 0 \), the SDEPCA (2.1) and SDE (2.2) have a unique continuous solutions on \( t \geq 0 \) (see [28]). To emphasize the role of the initial value, we denote the solution \( x(t) \) and \( y(t) \) by \( x(t;0,x_0) \) and \( y(t;0,x_0) \), respectively. Of course, we may consider a more general case, for example, where the SDEs and the SDEPCAs have a random initial data \( x(0) = \xi \) which is an \( F_0 \)-measurable \( \mathbb{R}^d \)-valued random variable such that \( \mathbb{E}(|\xi|^p) < \infty, \forall \, p \geq 0 \). In this case, by the Markov property of the solution, we can easily see that the solution satisfies

\[
\mathbb{E}|x(t)|^p = \mathbb{E}(\mathbb{E}(|x(t)|^p|\mathcal{F}_0)) \leq \mathbb{E}(M_1|\xi|^p e^{-\gamma_1 t}) = M_1\mathbb{E}(|\xi|^p e^{-\gamma_1 t}).
\]

It is therefore clear why it is enough to consider only the deterministic initial value \( x(0) = x_0 \).

Let \( y(t;s,y(s)) \) be the solution of SDE (2.2) for \( t > s \) with initial value \( y(s) \). It is also known that the solutions to SDE (2.2) have the following flow property,

\[
y(t;0,x_0) = y(t;s,y(s)), \quad \forall \, t \geq s > 0.
\]

Moreover, the solutions of SDE (2.2) also have the time-homegeneous Markov property. Hence (2.10) implies

\[
\mathbb{E}|y(t;s,\xi)|^p \leq M_2\mathbb{E}|\xi|^p e^{-\gamma_2 (t-s)}, \quad \forall \, t \geq s.
\]

Given \( y_k \) for some \( k \in \mathbb{N}^+ \), the process \( \{y_n\}_{n \geq k} \) can be regard as the process which is produced by EM method applied to the SDE (2.2) on \( t \geq kh \) with the initial value \( y(kh) = y_k \). In other words, the process \( \{y_n\}_{n \geq k} \) is time-homegeneous Markov process. Hence, (2.12) is equivalent to the following more general form.

\[
\mathbb{E}|y_n|^p \leq L_2\mathbb{E}|y_k|^p e^{-\lambda_2 (n-k)h}.
\]
Due to the special feature of the SDEPCA (2.1), the solution \( x(t) \) has flow property and the Markov property at the discrete time \( t = k\tau \ (k \in \mathbb{N}^+) \). Hence

\[
x(t;0,x_0) = x(t;k\tau,x(k\tau))
\]

and (2.9) implies

\[
(2.14) \quad \mathbb{E}|x(t)|^p \leq M_1\mathbb{E}|x(k\tau)|^p e^{-\gamma_1(t-k\tau)}, \quad t \geq k\tau.
\]

Given \( x_{km} \) for some \( k \in \mathbb{N}^+ \), the process \( \{x_n\}_{n \geq km} \) can be regard as the process which is produced by EM method applied to the SDEPCA (2.1) on \( t \geq k\tau \) with the initial value \( x(k\tau) = x_{km} \). The process \( \{x_n\}_{n \geq km} \) is time-homogeneous Markov process. Hence, (2.11) is equivalent to the following more general form.

\[
(2.15) \quad \mathbb{E}|x_n|^p \leq L_1\mathbb{E}|x_{km}|^p e^{-\lambda_1(n-km)\tau}.
\]

### 3. SDE (2.2) shares the stability with SDEPCA (2.1).

In this section, we shall investigate that if the SDEPCA (2.1) is \( p \)th moment exponentially stable with a sufficiently small \( \tau \), then the SDE (2.2) is also \( p \)th moment exponentially stable, i.e. give the positive answer to (Q1). To show this, we need several lemmas. The last lemma estimates the difference in the \( p \)th moment between the solution of the SDE (2.2) and that of the SDEPCA (2.1).

**Lemma 3.1.** Assume that Assumption 2.2 holds. Then for any given constant \( T \geq 0 \), we have

\[
(3.1) \quad \sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^p \leq H_1(T,p,K)|x_0|^p,
\]

where \( H_1(T,p,K) = e^{2pK[1+(p-1)K]T} \).

**Proof.** In view of Itô formula and Assumption 2.2, we obtain

\[
\mathbb{E}|x(v)|^p \leq |x_0|^p + \mathbb{E} \int_0^v p|x(s)|^{p-1}|f(x(s)) + u_1(x([s/\tau]\tau))| \, ds + \frac{p(p-1)}{2} |x(s)|^{p-2}|g(x(s)) + u_2(x([s/\tau]\tau))|^2 \, ds
\]

\[
\leq |x_0|^p + \mathbb{E} \int_0^v pK|x(s)|^{p-1}(|x(s)| + |x([s/\tau]\tau)|) \, ds + p(p-1)K^2|x(s)|^{p-2}(|x(s)|^2 + |x([s/\tau]\tau)|^2) \, ds
\]

\[
\leq |x_0|^p + 2pK[1+(p-1)K] \int_0^v \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p \, ds.
\]

Taking the supremum value of both sides over \( v \in [0,t] \), we have

\[
\sup_{0 \leq v \leq t} \mathbb{E}|x(v)|^p \leq |x_0|^p + 2pK[1+(p-1)K] \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p \, ds.
\]

The desired result (3.1) follows from the well-known Gronwall inequality. \( \square \)

**Lemma 3.2.** Assume that Assumption 2.2 holds. Then for any \( t \geq 0 \),

\[
\mathbb{E}|x(t) - x([t/\tau]\tau)|^p \leq C_1(K,p,\tau)\tau^{\frac{p}{2}} e^{2pK[1+(p-1)K]t}|x_0|^p.
\]

where \( C_1(K,p,\tau) = 2^{2p-1}K^p \left[ \tau^{rac{p}{2}} + (p(p-1)/2)\frac{\tau^{p}}{2} \right] \).
Proof. By basic inequality, Hölder inequality, moment inequality and Assumption 2.2, we obtain

\[
\mathbb{E}[x(t) - x([t/\tau])]^p \leq 2^{p-1} \tau^{p-1} \mathbb{E} \int_{[t/\tau]}^t |f(x(s)) + u_1(x([s/\tau])|^p ds
\]
\[
+ 2^{p-1} \left( \frac{p(p-1)}{2} \right) \tau^{\frac{p}{2}} \mathbb{E} \int_{[t/\tau]}^t |g(x(s)) + u_2(x([s/\tau])|^p ds
\]
\[
\leq C_1(K,p,\tau)^{\frac{p}{2}-1} \int_{[t/\tau]}^t \left( \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p \right) ds
\]

It comes from (3.1) that

\[
\mathbb{E}[x(t) - x([t/\tau])]^p \leq C_1(K,p,\tau)^{\frac{p}{2}-1} \int_{[t/\tau]}^t e^{2pK[1+(p-1)K]s|x_0|^p ds
\]
\[
\leq C_1(K,p,\tau)^{\frac{p}{2}} e^{2pK[1+(p-1)K]t|x_0|^p}
\]

The lemma is proved.

The following lemma estimates the difference in the $p$th moment between $x(t)$ and $y(t)$.

**Lemma 3.3.** Let Assumption 2.2 hold. Then

\[
\mathbb{E}[x(t) - y(t)]^p \leq C_2(K,p,\tau)^{\frac{p}{2}} |x_0|^p \left( e^{C_3(p,K)t} - 1 \right),
\]

for all $x_0 \in \mathbb{R}^d$ and $t \geq 0$, where $C_2$ and $C_3$ are defined as (3.4) and (3.5), respectively.

**Proof.** Using Itô formula and Assumption 2.2, we have

\[
\mathbb{E}|x(t) - y(t)|^p
\]
\[
\leq \mathbb{E} \int_0^t pK|x(s) - y(s)|^{p-1}(|x(s) - y(s)| + |x([s/\tau]) - y(s)|)
\]
\[
+ (p(p-1)K^2|x(s) - y(s)|^{p-2}(|x(s) - y(s)|^2 + |x([s/\tau]) - y(s)|^2) ds
\]
\[
= (pK + p(p-1)K^2) \int_0^t \mathbb{E}|x(s) - y(s)|^p ds
\]
\[
+ pKE \int_0^t |x(s) - y(s)|^{p-1}|x([s/\tau]) - y(s)| ds
\]
\[
+ p(p-1)K^2 \int_0^t |x(s) - y(s)|^{p-2}|x([s/\tau]) - y(s)|^2 ds
\]

By Young inequality, we have

\[
\mathbb{E}|x(t) - y(t)|^p
\]
\[
\leq \left( (2p - 1 + 2^{p-1})K + 2(p-1)(p-1 + 2^{p-1})K^2 \right) \int_0^t \mathbb{E}|x(s) - y(s)|^p ds
\]
\[
+ 2^{p-1}(K + 2(p-1)K^2) \int_0^t \mathbb{E}|x([s/\tau]) - x(s)|^p ds
\]

(3.2)
In view of Lemma 3.2, we have
\[
2^{p-1}(K + 2(p - 1)K^2) \int_0^t \mathbb{E}|x([s/\tau]\tau) - x(s)|^p ds \\
\leq 2^{p-1}(K + 2(p - 1)K^2) \int_0^t C_1(K, p, \tau)\tau^\frac{p}{2} e^{2p[1+(p-1)K]}|x_0|^p ds \\
= \frac{2^{p-1}(1+2(p-1)K}\tau^\frac{p}{2} C_1(K, p, \tau)|x_0|^p}{p(2+2(p-1)K)} \left( e^{2p[1+(p-1)K]\tau} - 1 \right) \\
\leq \frac{2^{p-1}\tau^\frac{p}{2} C_1(K, p, \tau)|x_0|^p}{p} \left( e^{2p[1+(p-1)K]\tau} - 1 \right)
\]
(3.3)

Substituting (3.3) into (3.2) and using Gronwall inequality, we show that
\[
\mathbb{E}|x(t) - y(t)|^p \leq C_2(K, p, \tau)\tau^\frac{p}{2} |x_0|^p \left( e^{C_3(p, K)\tau} - 1 \right),
\]
where
\[
C_2(K, p, \tau) = \frac{2^{p-1}C_1(K, p, \tau)}{p},
\]
(3.4)
\[
C_3(p, K) = [4p - 1 + 2^{p-1} + 2(p - 1)(2p - 1) + 2^{p-1}K]K.
\]
(3.5)

This completes the proof of Lemma 3.3.

Our positive answer to (Q1) is stated in the following theorem.

**Theorem 3.4.** Let Assumption 2.2 hold and the SDEPCA (2.1) is pth moment exponentially stable, i.e., \( \mathbb{E}|x(t)|^p \leq M_1 e^{-\gamma_1 t}|x_0|^p \). Choose \( \delta \in (0, 1) \), if \( \tau \) satisfies
\[
R(\tau) = \delta + 2^{p-1}C_2(K, p, \tau)\tau^\frac{p}{2} e^{\frac{\ln \left( \frac{2^{p-1}M_1}{\gamma_1^2} \right)}{\gamma_1^2} + \tau} - 1 < 1,
\]
then the SDE (2.2) is also pth moment exponentially stable, where \( C_2(K, p, \tau) \) and \( C_3(p, K) \) are defined in Lemma 3.3.

**Proof.** Step 1. Let us choose a positive integer \( \hat{n} \) such that
\[
\frac{\ln \left( \frac{2^{p-1}M_1}{\delta} \right)}{\gamma_1^2} \leq \hat{n} < \frac{\ln \left( \frac{2^{p-1}M_1}{\delta} \right)}{\gamma_1^2} + 1.
\]
So \( 2^{p-1}M_1 e^{-\gamma_1 \hat{n} \tau} \leq \delta \). Hence,
\[
2^{p-1}\mathbb{E}|x(\hat{n}\tau)|^p \leq 2^{p-1}M_1 e^{-\gamma_1 \hat{n} \tau}|x_0|^p \leq \delta|x_0|^p.
\]
(3.7)

By virtue of Lemma 3.3, we obtain
\[
\mathbb{E}|x(\hat{n}\tau) - y(\hat{n}\tau)|^p \leq C_2(K, p, \tau)\tau^\frac{p}{2} |x_0|^p \left( e^{C_3(p, K)\hat{n} \tau} - 1 \right),
\]
which together with (3.7), we arrive at
\[
\mathbb{E}|y(\hat{n}\tau)|^p \leq \left[ \delta + 2^{p-1}C_2(K, p, \tau)\tau^\frac{p}{2} \left( e^{C_3(p, K)\hat{n} \tau} - 1 \right) \right]|x_0|^p \leq R(\tau)|x_0|^p
\]
In view of (3.6), there is a positive constant $\gamma_2$ such that $R(\tau) = e^{-\gamma_2 \bar{\tau}}$. Consequently,

$$\mathbb{E}|y(\hat{n}\tau)|^p \leq e^{-\gamma_2 \bar{\tau}}|x_0|^p.$$ 

Step 2. For any given $k \in \mathbb{N}^+$, let $\bar{x}(t)$ be the solution to the SDEPCA (2.1) for $t \geq k\hat{n}\tau$ with the initial value $\bar{x}(k\hat{n}\tau) = y(k\hat{n}\tau)$. We have from (2.14) that

$$(3.8) \quad \mathbb{E}|\bar{x}((k+1)\hat{n}\tau)|^p \leq M_1 \mathbb{E}|y(k\hat{n}\tau)|^p e^{-\gamma_2 \bar{\tau}}.$$ 

In view of Lemma 3.3, we arrive at

$$(3.9) \quad \mathbb{E}|\bar{x}((k+1)\hat{n}\tau) - y((k+1)\hat{n}\tau)|^p \leq C_2(K, p, \tau)\tau^\delta \mathbb{E}|y(k\hat{n}\tau)|^p \left(e^{C_3(p,K)\hat{n}\tau} - 1\right).$$

Using (3.8) and (3.9), we can show, in the same way as we did in Step 1, that

$$\mathbb{E}|y((k+1)\hat{n}\tau)|^p \leq \mathbb{E}|y(k\hat{n}\tau)|^p e^{-\gamma_2 \bar{\tau}}.$$ 

Consequently,

$$(3.10) \quad \mathbb{E}|y(k\hat{n}\tau)|^p \leq e^{-\gamma_2 \bar{\tau}} \mathbb{E}|y((k-1)\hat{n}\tau)|^p \leq \cdots \leq e^{-k\gamma_2 \bar{\tau}}|x_0|^p.$$ 

Now, for any $t > 0$, there is a unique $k$ such that $k\hat{n}\tau \leq t < (k+1)\hat{n}\tau$. In view of Itô formula and Assumption 2.2, similarly as the proof of Lemma 3.1, we arrive at

$$\mathbb{E}|y(t)|^p \leq \mathbb{E}|y(k\hat{n}\tau)|^p + 2pK(1 + (p - 1)K) \int_{k\hat{n}\tau}^t \mathbb{E}|y(s)|^p ds.$$ 

By the Gronwall inequality and (3.10), we can derive

$$\mathbb{E}|y(t)|^p \leq \mathbb{E}|y(k\hat{n}\tau)|^p e^{2pK(1 + (p - 1)K)(t - k\hat{n}\tau)} \leq \mathbb{E}|y(k\hat{n}\tau)|^p e^{2pK(1 + (p - 1)K)\hat{n}\tau} \leq e^{-k\gamma_2 \bar{\tau}}|x_0|^p e^{2pK(1 + (p - 1)K)\hat{n}\tau} \leq M_2|x_0|^p e^{-\gamma_2 t},$$

where $M_2 = e^{\gamma_2 + 2pK(1 + (p - 1)K)\hat{n}\tau}$. The proof is hence complete. \qed

4. EMSDEPCA (2.3) shares the stability with EMSDE (2.4). In this section, we shall show that if the EMSDE (2.4) is $p$th moment exponentially stable, then the EMSDEPCAs (2.3) is also $p$th moment exponentially stable, i.e. give the positive answer to (Q3). It is known from Remark 2.1 that if $h = \tau$, then EMSDE (2.4) and EMSDEPCA (2.3) are the same, and the answer for (Q3) is obviously positive. So in this section, we assume $h \neq \tau$.

**Theorem 4.1.** Assume that Assumption 2.2 holds. For a step size $h = \tau/m$, the EMSDE (2.4) is $p$th moment exponentially stable , i.e. $\mathbb{E}|Y_n|^p \leq L_2 e^{-\lambda_2 nh}|x_0|^p$. Choose $\delta \in (0, 1)$, if $\tau$ satisfies

$$(4.1) \quad 2^{p-1}H_4 \left(2 \left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right) \tau^\delta + \delta < 1$$

where $H_4(T, K, \tau, p)$ is defined in Lemma 4.3, then the EMSDEPCA (2.3) is also $p$th moment exponentially stable.
The above theorem will be proved below by making use of the following lemmas.

**Lemma 4.2.** Assume that Assumption 2.2 holds. Then for any given $T > 0$ such that

$$\sup_{0 \leq t_n \leq T} E|X_n|^p \leq H_3(T, p, K)|x_0|^p,$$

where $H_3(T, p, K) = e^{2pK(1+(p-1)K)T}$.

**Proof.** The proof follows from Lemma 3.1. But to highlight the importance of numerical solutions, it is given here. In view of Itô formula and Assumption 2.2, we have

$$E|x_\Delta(u)|^p = |x_0|^p + E \int_0^t p|x_\Delta(s)|^{p-2}x_\Delta(s)^T(f(x_\Delta(s)) + u_1(x_\Delta([s/\tau]\tau)))$$

$$+ \frac{p(p-1)}{2} |x_\Delta(s)|^{p-2}g(x_\Delta(s)) + u_2(x_\Delta([s/\tau]\tau)))^2 ds$$

$$\leq |x_0|^p + E \int_0^t pK|x_\Delta(s)|^{p-1}(|x_\Delta(s)| + |x_\Delta([s/\tau]\tau)|)ds$$

$$+ p(p-1)K^2 E \int_0^t |x_\Delta(s)|^{p-2} (|\bar{x}(s)|^2 + |\bar{x}([s/\tau]\tau)|^2) ds$$

$$\leq |x_0|^p + 2pK (1 + (p-1)K) \int_0^t \sup_{0 \leq u \leq s} E|x_\Delta(u)|^p ds$$

According to the Gronwall inequality, we obtain

$$\sup_{0 \leq t \leq T} E|x_\Delta(t)|^p \leq |x_0|^p e^{2pK(1+(p-1)K)T}. \tag{4.2}$$

The proof is completed by noting that $x_\Delta(t_n) = X_n$, i.e.

$$\sup_{0 \leq t_n \leq T} E|X_n|^p \leq |x_0|^p e^{2pK(1+(p-1)K)T}. \tag{4.3}$$

The following lemma estimates the difference in the $p$th moment between approximation of EMSDE (2.4) and that of EMSDEPCA (2.3).

**Lemma 4.3.** Let Assumption 2.2 hold. Then for any given positive constant $T > 0$,

$$\sup_{0 \leq t_n \leq T} E|X_n - Y_n|^p \leq H_4(T, K, \tau, p) \tau^\frac{p}{2} |x_0|^p,$$

where $H_4(T, K, \tau, p) = C_2(K, p, \tau) (e^{C_3(p,K)T} - 1)$, $C_2$ and $C_3$ are defined in Lemma 3.3.
Proof. According to (2.7), (2.8), Itô formula and Assumption 2.2, we have
\begin{align*}
\mathbb{E}|x_{\Delta}(v) - y_{\Delta}(v)|^p
\leq & \mathbb{E}\int_0^v p|x_{\Delta}(s) - y_{\Delta}(s)|^{p-1}|f(\bar{x}_{\Delta}(s)) - f(\bar{y}_{\Delta}(s)) + u_1(\bar{x}_{\Delta}([s/\tau]) - u_1(\bar{y}_{\Delta}(s))| \\
& + \frac{p(p-1)}{2}|x_{\Delta}(s) - y_{\Delta}(s)|^{p-2}|g(\bar{x}_{\Delta}(s)) - g(\bar{y}_{\Delta}(s)) + u_2(\bar{x}_{\Delta}([s/\tau]) - u_2(\bar{y}_{\Delta}(s))|^2ds \\
\leq & \mathbb{E}\int_0^v pK|x_{\Delta}(s) - y_{\Delta}(s)|^{p-1}(|\bar{x}_{\Delta}(s) - \bar{y}_{\Delta}(s)| + |\bar{x}_{\Delta}([s/\tau]) - \bar{y}_{\Delta}(s)|) \\
& + p(p-1)K^2|x_{\Delta}(s) - y_{\Delta}(s)|^{p-2}(|\bar{x}_{\Delta}(s) - \bar{y}_{\Delta}(s)|^2 + |\bar{x}_{\Delta}([s/\tau]) - \bar{y}_{\Delta}(s)|^2)ds \\
\leq & (pK + p(p-1)K^2)\int_0^v \sup_{0\leq u \leq s} \mathbb{E}|x_{\Delta}(u) - y_{\Delta}(u)|^pds \\
& + \mathbb{E}\int_0^v pK|x_{\Delta}(s) - y_{\Delta}(s)|^{p-1}|\bar{x}_{\Delta}([s/\tau]) - \bar{y}_{\Delta}(s)|ds \\
& + \mathbb{E}\int_0^v p(p-1)K^2|x_{\Delta}(s) - y_{\Delta}(s)|^{p-2}|\bar{x}_{\Delta}([s/\tau]) - \bar{y}_{\Delta}(s)|^2ds
\end{align*}
(4.3)

Similarly as in the proof of Lemma 3.2, we obtain
\begin{align*}
\mathbb{E}|\bar{x}_{\Delta}(t) - \bar{x}_{\Delta}([t/\tau])|^p = \mathbb{E}|X_{km+t} - X_{km}|^p \\
\leq C_1(K, p, \tau)\tau^\frac{p}{2}|x_0|p2^pK(1+(p-1)K)t
\end{align*}
(4.4)

Substituting (4.4) into (4.3), we have
\begin{align*}
\mathbb{E}|x_{\Delta}(v) - y_{\Delta}(v)|^2
\leq & \left[(2p - 1 + 2^{p-1}) + 2(p - 1)(p - 1 + 2^{p-1})K\right]K\int_0^v \sup_{0\leq u \leq s} \mathbb{E}|x_{\Delta}(u) - y_{\Delta}(u)|^pds \\
& + 2^{p-1} (1 + 2(p-1)K)K\int_0^v C_1(K, p, \tau)\tau^\frac{p}{2}|x_0|p2^pK(1+(p-1)K)t ds
\end{align*}

Applying the Gronwall inequality, we have
\begin{equation*}
\sup_{0\leq u \leq T} \mathbb{E}|x_{\Delta}(v) - y_{\Delta}(v)|^p \leq C_2(K, p, \tau)\tau^\frac{p}{2}|x_0|p \left(e^{C_3(p, K)T} - 1\right),
\end{equation*}

where $C_2(K, p, \tau)$ and $C_3(p, K)$ are defined in Lemma 3.3. For ease of notations, set $H_4(T, K, p, \tau) = C_2(K, p, \tau)\left(e^{C_3(p, K)T} - 1\right)$. The proof is completed by noting that $x_{\Delta}(t_n) = X_n$ and $y_{\Delta}(t_n) = Y_n$, i.e.
\begin{equation*}
\sup_{0\leq t_n \leq T} \mathbb{E}|X_n - Y_n|^p \leq H_4(T, K, p, \tau)\tau^\frac{p}{2}|x_0|p.
\end{equation*}

The proof of Theorem 4.1. Let
\begin{equation*}
\hat{n} = \left[ \frac{\ln\left(\frac{2^{p-1}L_2}{\lambda^2}\right)}{\lambda^2} \right] + 1,
\end{equation*}
which implies that
\[
2^{p-1}L_2 e^{-\lambda_2 \bar{n}\tau} \leq \delta, \quad \text{and} \quad \bar{n}\tau \leq \frac{\ln\left(2^{p-1}L_2/\delta\right)}{\lambda_2} + \tau.
\]
By \(|a + b|^p \leq 2^{p-1}|a|^p + 2^{p-1}|b|^p\), we have
\[
\mathbb{E}|X_n|^p \leq 2^{p-1}\mathbb{E}|X_n - Y_n|^p + 2^{p-1}\mathbb{E}|Y_n|^p.
\]
According to the \(p\)th moment exponentially stability of EMSDE (2.4) and Lemma 4.3, we have
\[
\sup_{\bar{n}\tau \leq t_n \leq 2\bar{n}\tau} \mathbb{E}|X_n|^p \leq 2^{p-1} \sup_{0 \leq t_n \leq 2\bar{n}\tau} \mathbb{E}|X_n - Y_n|^p + 2^{p-1} \sup_{\bar{n}\tau \leq t_n \leq 2\bar{n}\tau} \mathbb{E}|Y_n|^p
\]
\[
\leq \left(2^{p-1}H_4(2\bar{n}\tau, K, \tau, p)^{\frac{p}{p+1}} + 2^{p-1}L_2 e^{-\lambda_2 \bar{n}\tau}\right) |x_0|^p
\]
\[
\leq \left(2^{p-1}H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right)^{\frac{p}{p+1}} + \delta\right) |x_0|^p
\]
Let \(R(\tau) = 2^{p-1}H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right)^{\frac{p}{p+1}} + \delta\). It is known from (4.1) that \(R(\tau) < 1\). Therefore, we can find a positive constant \(\lambda_1\) such that
\[
R(\tau) < e^{-\lambda_1 \bar{n}\tau},
\]
and
\[
\sup_{\bar{n}\tau \leq t_n \leq 2\bar{n}\tau} \mathbb{E}|X_n|^p \leq e^{-\lambda_1 \bar{n}\tau} |x_0|^p.
\]
Let \(\{\bar{Y}_n\}_{t_n \geq \bar{n}\tau}\) be the solution of EMSDE (2.4) with initial data \(\bar{Y}_{\bar{n}m} = X_{\bar{n}m}\) at initial time \(t = \bar{n}\tau\). According to Lemma 4.3, we have
\[
\sup_{\bar{n}\tau \leq t_n \leq 3\bar{n}\tau} \mathbb{E}|X_n - Y_n|^p \leq H_4\left(2\left(\frac{\ln(2^{p-1}L_2/\delta)}{\lambda_2} + \tau\right), K, \tau, p\right)^{\frac{p}{p+1}} \mathbb{E}|X_{\bar{n}m}|^p.
\]
It comes from (2.13) that
\[
\mathbb{E}|\bar{Y}_n|^p \leq L_2 e^{-\lambda_2 (\bar{n}h - \bar{n}\tau h)} \mathbb{E}|X_{\bar{n}m}|^p.
\]
Using similar arguments that produced (4.6), we obtain
\[
\sup_{2\bar{n}\tau \leq t_n \leq 3\bar{n}\tau} \mathbb{E}|X_n|^p \leq R(\tau) \mathbb{E}|X_{\bar{n}m}|^p \leq e^{-\lambda_1 \bar{n}\tau} \mathbb{E}|X_{\bar{n}m}|^p \leq e^{-\lambda_1 \bar{n}\tau} \sup_{\bar{n}\tau \leq t_n \leq 2\bar{n}\tau} \mathbb{E}|X_n|^p
\]
By (4.6), we obtain
\[
\sup_{2\bar{n}\tau \leq t_n \leq 3\bar{n}\tau} \mathbb{E}|X_n|^p \leq e^{-2\lambda_1 \bar{n}\tau} |x_0|^p
\]
Continuing this approach and using (4.5), we have, for any \(i = 1, 2, \ldots\),
\[
\sup_{i\bar{n}\tau \leq t_n \leq (i+1)i\bar{n}\tau} \mathbb{E}|X_n|^p \leq e^{-\lambda_1 \bar{n}\tau} |x_0|^p \leq L_1 e^{-\lambda_1 nh} |x_0|^p
\]
where $L_1 = e^{\lambda_1 \bar{n}t}$. For $i = 0$, by using Lemma 4.2, we get
\[ \sup_{0 \leq t_n \leq \bar{n}T} \mathbb{E}|x_n|^p \leq H_3(\bar{n}T, p, K)|x_0|^p \leq L_1|x_0|^p e^{-\lambda_1 nh}, \]
where $L_1 = H_3(\bar{n}T, p, K)e^{\lambda_1 \bar{n}T} > e^{\lambda_1 \bar{n}T} = \bar{L}_1$. This, together with (4.7), we arrive at for all $n \in \mathbb{N}$
\[ \mathbb{E}|x_n|^p \leq L_1|x_0|^p e^{-\lambda_1 nh}. \]
\[ \square \]

### 5. SDEPCA (2.1) shares the stability with EMSDEPCA (2.3)
In this section, we shall show that for a given step size $h$, if the EMSDEPCA (2.3) is $p$th moment exponentially stable, then the SDEPCA (2.1) is also $p$th moment exponentially stable with some restriction with $h$, i.e. give the positive answer to (Q4). The first lemma shows that the EMSDEPCA (2.3) is convergent in the $p$th moment to SDEPCA (2.1).

**Lemma 5.1.** Assume that Assumption 2.2 holds. For $T > 0$,
\[ \sup_{0 \leq t_n \leq T} \mathbb{E}|x(t_n) - X_n|^p \leq H_6(T, K, p)h^\frac{p}{2}|x_0|^p, \]
where $H_6(T, K, p)$ is defined as (5.3).

**Proof.** For any $t \geq 0$, by Itô formula, Assumption 2.2 and Young inequality, we obtain
\[ \mathbb{E}|x(t) - x_\Delta(t)|^p \leq \mathbb{E} \int_0^t pK|x(s) - x_\Delta(s)|^{p-1} |(x(s) - x_\Delta(s)) + |x([s/\tau]T) - x_\Delta([s/\tau]T)| |x_\Delta([s/\tau]T) - x_\Delta([s/\tau]T)| ds \]
\[ + p(p-1)K^2|x(s) - x_\Delta(s)|^{p-2}(|x(s) - x_\Delta(s)|^2 + |x([s/\tau]T) - x_\Delta([s/\tau]T)|^2) ds \]
\[ \leq 2pK(1 + 2(p-1)K) \int_0^t \mathbb{E} \sup_{0 \leq u \leq s} |x(s) - x_\Delta(s)|^p ds \]
\[ + pK \mathbb{E} \int_0^t |x(s) - x_\Delta(s)|^{p-1}|x_\Delta(s) - \bar{x}_\Delta(s)| ds \]
\[ + pK \mathbb{E} \int_0^t |x(s) - x_\Delta(s)|^{p-1}|x_\Delta([s/\tau]T) - \bar{x}_\Delta([s/\tau]T)| ds \]
\[ + 2p(p-1)K^2 \mathbb{E} \int_0^t |x(s) - x_\Delta(s)|^{p-2}|x_\Delta(s) - \bar{x}_\Delta(s)|^2 ds \]
\[ + 2p(p-1)K^2 \mathbb{E} \int_0^t |x(s) - x_\Delta(s)|^{p-2}|x_\Delta([s/\tau]T) - \bar{x}_\Delta([s/\tau]T)|^2 ds \]
By noting $x_\Delta([s/\tau]T) = \bar{x}_\Delta([s/\tau]T) = 0$, we have
\[ \mathbb{E}|x(t) - x_\Delta(t)|^p \leq K[3p - 1 + 2(p-1)(3p-2)K] \int_0^t \mathbb{E} \sup_{0 \leq u \leq s} |x(u) - x_\Delta(u)|^p ds \]
\[ + K[1 + 4(p-1)K] \int_0^t \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^p ds \]
\[ \tag{5.1} \]
Now, we shall give the estimation of the second term of the right hand. For any $t > 0$, there exists $k$ and $l$ such that $tk_{m+l} \leq t < tk_{m+l+1}$. Then $\bar{x}_\Delta(t) = X_{km+l} =$
By noting (5.3) we have
\[ E[x(t) - \bar{x}(t)]^p \leq 2^{2p-1}h^{\frac{p}{2}}K^p |x_0|^p e^{2pK(1+(p-1)K)t} \]

Applying (4.2), we obtain
\[ E[x(t) - \bar{x}(t)]^p = E[(t - t_{k+1}) (f(X_{k+1}) + u_1(X_k)) + (g(X_{k+1}) + u_2(X_k)) (w(t) - w(t_{k+1}))]^p \leq 2^{2p-1}h^{\frac{p}{2}}K^p (E|X_{k+1}|^p + E|X_k|^p) \]

By Gronwall inequality, we have
\[ \sup_{0 \leq t \leq T} E|X(t) - \bar{X}(t)|^p \leq H_6(T, p, K)h^{\frac{p}{2}} |x_0|^p, \]
where
\[ H_6(T, p, K) = [1 + 4(p-1)K]^{2pK+1}e^{KT[5p-1+4(p-1)(2p-1)K]}T. \]

By noting \( x(t) = X(t) \), we get for \( t = t_n \)
\[ \sup_{0 \leq t_n \leq T} E|X(t_n) - X_n|^p \leq H_6(T, p, K)h^{\frac{p}{2}} |x_0|^p \]

The proof is completed.

**Lemma 5.2.** Assume that Assumption 2.2 holds. Then for any \( 0 \leq t_n \leq t \leq t_{n+1} \leq T \),
\[ \sup_{0 \leq t \leq T} E|x(t) - x(t_n)|^p \leq H_7(T, p, K)h^{\frac{p}{2}} |x_0|^p, \]
where
\[ H_7(T, p, K) = 2^{2p-1}K^p \left[ T^{\frac{p}{2}} + \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} \right] e^{2pK[1+(p-1)K]T}. \]

**Proof.** For any \( 0 \leq t_n \leq t \leq t_{n+1} \leq T \), we have
\[ x(t) - x(t_n) = \int_{t_n}^t f(x(s)) + u_1(x([s/\tau]))ds + \int_{t_n}^t g(x(s)) + u_2(x([s/\tau]))dw(s). \]

In view of Hölder inequality, Assumption 2.2 as well as moment inequality, we have
\[ E|x(t) - x(t_n)|^p \leq 2^{2p-2} (t - t_n)^{\frac{p}{2} - 1}K^p \left[ (t - t_n)^{\frac{p}{2}} + \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} \right] \int_{t_n}^t E|x(s)|^p + E|x([s/\tau])|^p ds \]
\[ \leq 2^{2p-1} (t - t_n)^{\frac{p}{2} - 1}K^p \left[ (t - t_n)^{\frac{p}{2}} + \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}} \right] \int_{t_n}^t \sup_{0 \leq u \leq s} E|x(u)|^p ds \]
It follows from (3.1) that

\[ \mathbb{E}|x(t) - x(t_n)|^p \leq 2^{p-1}(t - t_n)^2K^p \left( t - t_n \right)^2 + \left( \frac{p(p-1)}{2} \right)^\frac{p}{2} e^{2pK|1+(p-1)K|t} |x_0|^p \]

\[ \leq 2^{p-1}h^2K^p \left[ t^2 + \left( \frac{p(p-1)}{2} \right)^\frac{p}{2} e^{2pK|1+(p-1)K|t} |x_0|^p \right] \]

Hence,

\[ \sup_{0 \leq t \leq T} \mathbb{E}|x(t) - x(t_n)|^p \leq 2^{p-1}K^p \left[ T^2 + \left( \frac{p(p-1)}{2} \right)^\frac{p}{2} e^{2pK|1+(p-1)K|T} h^2 |x_0|^p \right]. \]

The proof is complete.

**Theorem 5.3.** Assume that Assumption 2.2 holds. For a step size \( h = \frac{T}{n_0} \), the EMSDEPCA (2.3) is \( p \)th moment exponentially stable, i.e., \( \mathbb{E}|X_n|^p \leq L_1 e^{-\lambda_1 n h} |x_0|^p \).

If the step size \( h \) satisfies

\[ 3^{p-1} H_6(2\hat{n}\tau, K, p) h^2 + e^{-\frac{3}{4}\lambda_1 \hat{n}\tau} \leq e^{-\frac{3}{4}\lambda_1 \hat{n}\tau}, \]

where \( \hat{n} = \left[ \frac{4\ln(3^{p-1}L_1)}{\lambda_1 \tau} \right] + 1 \) and \( H_6(2\hat{n}\tau, K, p) = H_7(2\hat{n}\tau, K, p) + H_6(2\hat{n}\tau, K, p) \), then the SDEPCA (2.1) is also \( p \)th moment exponentially stable, where \( H_6(2\hat{n}\tau, K, p) \) is defined in Lemma 5.1 and \( H_7(2\hat{n}\tau, K, p) \) in Lemma 5.2.

**Proof.** For any \( t \geq 0 \), there exist \( n \in \mathbb{N} \) such that \( t_n \leq t < t_{n+1} \),

\[ \mathbb{E}|x(t)|^p \leq 2^{p-1} \mathbb{E}|x(t) - x(t_n)|^p + 3^{p-1} \mathbb{E}|x(t_n) - X_n|^p + 3^{p-1} \mathbb{E}|X_n|^p \]

According to Lemma 5.1, we have

\[ \sup_{0 \leq t_n \leq 2\hat{n}\tau} \mathbb{E}|x(t_n) - X_n|^p \leq H_6(2\hat{n}\tau, K, p) h^2 |x_0|^p. \]

By Lemma 5.2, we have

\[ \sup_{0 \leq t \leq 2\hat{n}\tau} \mathbb{E}|x(t) - x(t_n)|^p \leq H_7(2\hat{n}\tau, K, p) h^2 |x_0|^p. \]

Since \( \hat{n} = \left[ \frac{4\ln(3^{p-1}L_1)}{\lambda_1 \tau} \right] + 1 \), we have \( 3^{p-1} L_1 e^{-\lambda_1 \hat{n}\tau} \leq e^{-\frac{3}{4}\lambda_1 \hat{n}\tau} \), Therefore,

\[ \sup_{\hat{n}\tau \leq t \leq 2\hat{n}\tau} \mathbb{E}|x(t)|^p \leq 3^{p-1} \sup_{0 \leq t \leq 2\hat{n}\tau} \mathbb{E}|x(t) - x(t_n)|^p + 3^{p-1} \sup_{0 \leq t_n \leq 2\hat{n}\tau} \mathbb{E}|x(t_n) - X_n|^p + 3^{p-1} \sup_{\hat{n}\tau \leq t_n \leq 2\hat{n}\tau} \mathbb{E}|X_n|^p \]

\[ \leq \left( 3^{p-1} H_7(2\hat{n}\tau, K, p) h^2 + 3^{p-1} H_6(2\hat{n}\tau, K, p) h^2 + 3^{p-1} L_1 e^{-\lambda_1 \hat{n}\tau} \right) |x_0|^p \]

\[ \leq \left( 3^{p-1} H_7(2\hat{n}\tau, K, p) h^2 + e^{-\frac{3}{4}\lambda_1 \hat{n}\tau} \right) |x_0|^p \]

where \( H_6(2\hat{n}\tau, K, p) = H_7(2\hat{n}\tau, K, p) + H_6(2\hat{n}\tau, K, p) \). Recalling (5.4), we have

\[ \sup_{\hat{n}\tau \leq t \leq 2\hat{n}\tau} \mathbb{E}|x(t)|^p \leq e^{-\frac{3}{4}\lambda_1 \hat{n}\tau} |x_0|^p. \]
Denote by \( \{\tilde{X}_n\}_{n \geq n_0} \) the numerical solution of (2.3) with initial data \( \tilde{X}_{n_0} = x(\hat{n}\tau) \) at \( t = \hat{n}\tau \). Then from (2.15), we have
\[
\mathbb{E}|\tilde{X}_n|^p \leq L_1 e^{-\lambda_1(n-n_0)\hat{n}\tau} \mathbb{E}|x(\hat{n}\tau)|^p.
\]
Using Lemma 5.1 and Lemma 5.2, we get
\[
\sup_{\tilde{n}\tau \leq t \leq 3\hat{n}\tau} \mathbb{E}|x(t) - \tilde{X}_n|^p \leq H_6(2\hat{n}\tau, K, p) h^{\frac{p}{2}} \mathbb{E}|x(\hat{n}\tau)|^p.
\]
Therefore,
\[
\sup_{2\hat{n}\tau \leq t \leq 3\hat{n}\tau} \mathbb{E}|x(t)|^p \leq \left(3^{p-1} H_7(2\hat{n}\tau, K, p) h^{\frac{p}{2}} + 3^{p-1} H_6(2\hat{n}\tau, K, p) h^{\frac{p}{2}} + L_1 e^{-\lambda_1\hat{n}\tau}\right) \mathbb{E}|x(\hat{n}\tau)|^p
\]
\[
\leq \left(3^{p-1} H_6(2\hat{n}\tau, K, p) h^{\frac{p}{2}} + e^{-\frac{1}{4}\lambda_1\hat{n}\tau}\right) \mathbb{E}|x(\hat{n}\tau)|^p
\]
\[
\leq e^{-\frac{1}{4}\lambda_1\hat{n}\tau} \sup_{\tilde{n}\tau \leq t \leq 2\hat{n}\tau} \mathbb{E}|x(t)|^p
\]
By (5.5), we obtain
\[
\sup_{2\hat{n}\tau \leq t \leq 3\hat{n}\tau} \mathbb{E}|x(t)|^p \leq e^{-\frac{1}{4}\lambda_12\hat{n}\tau}|x_0|^p.
\]
Repeating this procedure, we find for \( i = 1, 2, \cdots \),
\[
(6.6) \quad \sup_{i\hat{n}\tau \leq t \leq (i+1)\hat{n}\tau} \mathbb{E}|x(t)|^p \leq e^{-\frac{1}{4}\hat{n}\tau}|x_0|^p \leq \bar{M}_1 e^{-\frac{1}{4}t}|x_0|^p,
\]
where \( \bar{M}_1 = e^{\frac{1}{4}\hat{n}\tau} \). On the other hand, by means of Lemma 3.1, we can show that
\[
\sup_{0 \leq t \leq \hat{n}\tau} \mathbb{E}|x(t)|^p \leq H_1(\hat{n}\tau, p, K)|x_0|^p \leq M_1|x_0|^p e^{-\frac{1}{2}t},
\]
where \( M_1 = H_1(\hat{n}\tau, p, K)e^{\frac{1}{2}\hat{n}\tau} > e^{\frac{1}{2}\hat{n}\tau} = \bar{M}_1 \), this, together with (6.6), we arrive at for any \( t \geq 0 \),
\[
\mathbb{E}|x(t)|^p \leq M_1|x_0|^p e^{-\frac{1}{2}\lambda_1t}.
\]
This completes the proof. \( \Box \)

6. EMSDE (2.4) shares the stability with SDE (2.2). [13] gives the positive answer to (Q2) only for the case \( p = 2 \). In this section, we shall show that for \( p > 2 \), if the SDE (2.2) is \( p \)th moment exponentially stable, then the EMSDE (2.4) is also \( p \)th moment exponentially stable with some restriction on \( h \), i.e. give the positive answer to (Q2). The first lemma shows that the EMSDE (2.4) is convergent in the \( p \)th moment to SDE (2.2).

**Lemma 6.1.** Assume that Assumption 2.2 holds. For any \( T > 0 \),
\[
\sup_{0 \leq t_n \leq T} \mathbb{E}|y(t_n) - Y_n|^p \leq H_9(T, K, p) h^{\frac{p}{2}} |x_0|^p,
\]
where \( H_9(T, K, p) \) is defined as (6.4).
Applying (6.2), we obtain
\[
\begin{align*}
\mathbb{E}|y(t) - y_\Delta(t)|^p \\
\leq \mathbb{E} \int_0^t p|y(s) - y_\Delta(s)|^{p-1} |f(y(s)) - f(y_\Delta(s)) + u_1(y(s)) - u_1(y_\Delta(s))| \\
+ \frac{p(p-1)}{2} |y(s) - y_\Delta(s)|^{p-2} |g(y(s)) - g(y_\Delta(s)) + u_2(y(s)) - u_2(y_\Delta(s))|^2 ds \\
\leq \mathbb{E} \int_0^t 2pK|y(s) - y_\Delta(s)|^{p-1}|y(s) - y_\Delta(s)| \\
+ 2p(p-1)K^2|y(s) - y_\Delta(s)|^{p-2}|y(s) - y_\Delta(s)|^2 ds \\
\leq (2K(2p-1) + 8(p-1)^2K^2) \int_0^t \mathbb{E}|y(s) - y_\Delta(s)|^p ds \\
(6.1) + (2K + 8(p-1)K^2) \int_0^t \mathbb{E}|y(s) - y_\Delta(s)|^p ds
\end{align*}
\]

Now, we shall give the estimation of the second term of the right hand. For any \( t > 0 \), there exists \( n \) such that \( t_n \leq t < t_{n+1} \), and \( y_\Delta(t) = Y_n = y_\Delta(t_n) \). Hence from (2.4) we have
\[
\mathbb{E}|y_\Delta(t) - y_\Delta(t)|^p \\
= \mathbb{E}|(t-t_n)(f(Y_n) + u_1(Y_n)) + g(Y_n) + u_2(Y_n)\tilde{g}(t-t_n)|^p \\
\leq 2^{p-1} \mathbb{E}|(t-t_n)(f(Y_n) + u_1(Y_n))|^p + \mathbb{E}|(g(Y_n) + u_2(Y_n))\tilde{g}(t-t_n)|^p \\
\leq 2^{p^2} K^p \mathbb{E}|Y_n|^p h_\Delta^p
\]

Similarly as the proof of Lemma 3.1, we have
\[
\sup_{0 \leq s \leq t} \mathbb{E}|y_\Delta(s)|^p \leq |x_0|^p e^{2pK(1+(p-1)K)t}
\]

Applying (6.2), we obtain
\[
\mathbb{E}|y_\Delta(t) - y_\Delta(t)|^p \leq 2^{2p} K^p e^{2pK(1+(p-1)K)t} h_\Delta^p |x_0|^p
\]

Substituting (6.3) into (6.1), we obtain
\[
\mathbb{E}|y(t) - y_\Delta(t)|^p \leq (2K(2p-1) + 8(p-1)^2K^2) \int_0^t \mathbb{E}|y(s) - y_\Delta(s)|^p ds \\
+ (2K + 8(p-1)K^2) \int_0^t 2^{2p^2} K^p e^{2pK(1+(p-1)K)t} h_\Delta^p |x_0|^p ds
\]

By Gronwall inequality, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E}|y(t) - y_\Delta(t)|^p \leq H_\theta(T, p, K) h_\Delta^p |x_0|^p,
\]

where
\[
H_\theta(T, p, K) = [1 + 4(p-1)K]2^{2p+1}K^{p+1} e^{2pK(3p-1+(p-1)(5p-4)K)} T.
\]

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By noting $y_\Delta(t_n) = Y_n$, we get for $t = t_n$

$$
\sup_{0 \leq t_n \leq T} \mathbb{E}|y(t_n) - Y_n|^p \leq H_9(T, p, K)h^{\frac{p}{2}}|x_0|^p
$$

The proof is completed.

**Theorem 6.2.** Let Assumption 2.2 hold. Assume that the SDE (2.2) is $p$th moment exponentially stable and satisfies (2.10). Let $T = 1 + 4\ln(2^{p-1}M_2)/\gamma_2$. If $h$ satisfies

\begin{equation}
2^{p-1}H_9(2T, p, K)h^\frac{p}{2} + e^{-\frac{3}{2}\gamma_2 T} \leq e^{-\frac{3}{2}\gamma_2 T}.
\end{equation}

Then the EMSDE (2.4) is $p$th moment exponentially stable.

**Proof.** Since $T = 1 + 4\ln(2^{p-1}M_2)/\gamma_2$, we have

$$2^{p-1}M_2 e^{-\gamma_2 T} < e^{-\frac{3}{2}\gamma_2 T}.$$

Now, for any given $i \in \mathbb{N}$, let $\{\hat{y}(t)\}_{t \geq iT}$ be the solution to the SDE (2.2) for $t \in [iT, \infty)$, with the initial condition $y_\Delta(iT)$. Then using basic inequality, Lemma 6.1, (2.10) and (6.5), we have

$$
\sup_{iT \leq t \leq (i+1)T} \mathbb{E}|y_\Delta(t)|^p \leq 2^{p-1} \sup_{iT \leq t \leq (i+1)T} \mathbb{E}|y_\Delta(t) - \hat{y}(t)|^p + 2^{p-1} \sup_{(i+1)T \leq t \leq (i+2)T} \mathbb{E}|\hat{y}(t)|^p
$$

$$\leq \left(2^{p-1}H_9(2T, p, K)h^\frac{p}{2} + 2^{p-1}M_2 e^{-\gamma_2 T}\right) \mathbb{E}|y_\Delta(iT)|^p
$$

$$\leq \left(2^{p-1}H_9(2T, p, K)h^\frac{p}{2} + e^{-\frac{3}{2}\gamma_2 T}\right) \mathbb{E}|y_\Delta(iT)|^p
$$

\begin{equation}
\leq e^{-\frac{3}{2}\gamma_2 T} \sup_{iT \leq t \leq (i+1)T} \mathbb{E}|y_\Delta(t)|^p.
\end{equation}

According to (6.2),

\begin{equation}
\sup_{0 \leq t \leq T} \mathbb{E}|y_\Delta(t)|^p \leq |x_0|^p e^{2pK(1+(p-1)K)T} \leq L_2 e^{-\frac{3}{2}\gamma_2 T}|x_0|^p,
\end{equation}

where $L_2 = e^{\frac{3}{2}\gamma_2 T+2pK(1+(p-1)K)T}$. Combining (6.7) and (6.6), we obtain that

$$
\sup_{(i+1)T \leq t \leq (i+2)T} \mathbb{E}|y_\Delta(t)|^p \leq e^{-\frac{3}{2}(i+1)\gamma_2 T} \sup_{0 \leq t \leq T} \mathbb{E}|y_\Delta(t)|^p
$$

$$\leq e^{-\frac{3}{2}(i+1)\gamma_2 T}|x_0|^p e^{2pK(1+(p-1)K)T}
$$

\begin{equation}
\leq L_2 e^{-\frac{3}{2}\gamma_2 T}|x_0|^p.
\end{equation}

Due to (6.8) and (6.7), the proof is completed by using $t = t_n$.

**7. Conclusion.** In this paper, we have shown from Theorem 3.4, Theorem 4.1, Theorem 5.3 and Theorem 6.2 that, under the standing Assumption 2.2,

$\text{SDE}(2.2) \xRightarrow{Q^2} \text{EMSDE}(2.4) \xRightarrow{Q^3} \text{EMSDEPCA}(2.3) \xRightarrow{Q^4} \text{SDEPCA}(2.1) \xRightarrow{Q^1} \text{SDE}(2.2).$

Hence we have the following theorem.

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Theorem 7.1. Under Assumption 2.2, if one of SDEPCA (2.1), SDE (2.2), EMSDEPCA (2.3) and EMSDE (2.4) is \( p \)th moment exponentially stable, then the other three are also \( p \)th moment exponentially stable for sufficiently small step size \( h \) and \( \tau \).

By examining the proof of the Theorem 3.4, Theorem 4.1, Theorem 5.3 and Theorem 6.2, we see that the \( p \)th moment exponential stability of SDEPCA (2.1), SDE (2.2), EMSDEPCA (2.3) and EMSDE (2.4) are equivalent as long as their solutions are \( p \)th moment bounded and arbitrarily close for sufficiently small \( \tau \) and \( h \).

For \( V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+) \), we define an operator \( \mathcal{L}V \) by

\[
\mathcal{L}V(y,t) = V_t(y,t) + V_y(y,t)F(y(t)) + \frac{1}{2} \text{trace} \left[ G^T(y) V_{yy}(y,t) G(y) \right].
\]

The sufficient criterion for \( p \)th moment exponential stability via a Lyapunov function is given by Theorem 4.4 in [28, P130]. Now we quote it here.

Theorem 7.2. Assume that there is a function \( V(y,t) \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+) \), and positive constants \( c_1, c_2, c_3 \) such that

\[
c_1|y|^p \leq V(y,t) \leq c_2|y|^p \quad \text{and} \quad \mathcal{L}V(y,t) \leq -c_3V(y,t)
\]

for all \((y,t) \in \mathbb{R}^d \times \mathbb{R}_+\). Then for the SDE (2.2), we have

\[
\mathbb{E}|y(t)|^p \leq \frac{c_2}{c_1}|x_0|^p e^{-c_3t},
\]

for all \( x_0 \in \mathbb{R}^d \). In other words, the SDE (2.2) is \( p \)th moment exponentially stable.

For convenience, we impose the following hypothesis.

Assumption 7.3. There exists a pair of positive constants \( p \) and \( \lambda \) such that

\[
|y|^2 (2y^T F(y) + |G(y)|^2) - (2 - p)|y^T G(y)|^2 \leq -\lambda|y|^4, \quad \forall \ y \in \mathbb{R}^d.
\]

Applying the Theorem 7.2 with \( V(y,t) = |y|^p \), we easily obtain the following theorem [see 20].

Theorem 7.4. Under Assumption 7.3, the SDE (2.2) is \( p \)th moment exponentially stable, i.e.

\[
\mathbb{E}|y(t)|^p \leq |x_0|^p e^{-\frac{\lambda}{2}t}, \quad \forall \ t > 0,
\]

where \( p \) and \( \lambda \) are given in Assumption 7.3.

In combination with Theorem 7.1, the following theorem provides an interesting result.

Theorem 7.5. Assume that Assumption 2.2 and Assumption 7.3 hold, then SDE (2.2) is \( p \)th moment exponentially stable and SDEPCA (2.1), EMSDEPCA (2.3), EMSDE (2.4) are also \( p \)th moment exponentially stable as long as step size \( h \) and \( \tau \) are sufficiently small.

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