MULTIFRACTAL ANALYSIS AND LOCALIZED ASYMPTOTIC BEHAVIOR FOR ALMOST ADDITIVE POTENTIALS

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Abstract. We conduct the multifractal analysis of the level sets of the asymptotic behavior of almost-additive continuous potentials \((\phi_n)_{n=1}^\infty\) on a topologically mixing subshift of finite type \(X\) endowed itself with a metric associated with such a potential. We work without bounded distortion property assumption. We express the whole Hausdorff spectrum in terms of a conditional variational principle, as well as a new large deviations principle. Our approach provides a new description of the structure of the spectrum in terms of weak concavity. Another new point is that we consider sets of points at which the asymptotic behavior of \(\phi_n(x)\) is localized, i.e. depends on the point \(x\) rather than being equal to a constant. Specifically, we compute the Hausdorff dimension of sets of the form \(\{x \in X : \lim_{n \to \infty} \phi_n(x)/n = \xi(x)\}\), where \(\xi\) is a given continuous function. This is naturally related to Birkhoff’s ergodic theorem and has interesting geometric applications to fixed points in the asymptotic average for dynamical systems in \(\mathbb{R}^d\), as well as the fine local behavior of the harmonic measure on conformal planar Cantor sets.

1. Introduction

We say that \((X,T)\) is a topological dynamical system (TDS) if \(X\) is a compact metric space and \(T\) is a continuous mapping from \(X\) to itself. We denote by \(\mathcal{M}(X,T)\) the set of invariant probability measures on \((X,T)\).

We say that \(\Phi = (\phi_n)_{n=1}^\infty\) is almost additive if \(\phi_n\) is continuous on \(X\) and there is a positive constant \(C(\Phi) > 0\) such that

\[
-C(\Phi) + \phi_n + \phi_p \circ T_n \leq \phi_{n+p} \leq C(\Phi) + \phi_n + \phi_p \circ T_n, \quad \forall n, p \in \mathbb{N}.
\]

By subadditivity, for every \(\mu \in \mathcal{M}(X,T)\), \(\Phi_*(\mu) := \lim_{n \to \infty} \frac{1}{n} \int_X \phi_n \, d\mu\) exists, and we define the compact convex set \(L_\Phi = \{\Phi_*(\mu) : \mu \in \mathcal{M}(X,T)\}\). We denote by \(C_{aa}(X,T)\) the collection of almost-additive potentials on \(X\).

The ergodic theorem naturally raises the following question. Given \(\Phi\) an almost additive potential taking values in \(\mathbb{R}^d\) (this means that \(\Phi = (\Phi^1, \ldots, \Phi^d)\) with each \(\Phi^i \in C_{aa}(X,T)\)) and \(\xi : X \to \mathbb{R}^d\) a continuous function, what is the Hausdorff dimension of the set

\[
E_\Phi(\xi) := \left\{x \in X : \lim_{n \to \infty} \frac{\phi_n(x)}{n} = \xi(x)\right\}?
\]

When \(\xi(x) \equiv \alpha\) is constant, this question has been solved for some \(C^{1+\varepsilon}\) conformal dynamical systems, sometimes assuming restrictions on the regularity of \(\Phi\), and this problem

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is known as the multifractal analysis of Birkhoff averages, and more generally almost additive potentials \[12, 32, 31, 30, 29, 26, 14, 5, 15, 6, 20, 19, 16, 27, 4\]. Moreover, the optimal results are expressed in terms of a variational principle of the following form: \( E_\Phi(\alpha) \neq \emptyset \) if and only if \( \alpha \in L_\Phi \) and in this case

\[
(1.1) \quad \dim_H E_\Phi(\alpha) = \max \left\{ \frac{h_\mu(T)}{\int_X \log \| DT \| \, d\mu} : \mu \in \mathcal{M}(X, T), \, \Phi_*(\mu) = \alpha \right\},
\]

the supremum being attained by a unique Gibbs measure if \( \Phi \) is the sequence of Birkhoff sums of a Hölder potential, and \( \alpha \) is in the interior of \( L_\Phi \). To our best knowledge no result is known for \( \dim_H E_\Phi(\xi) \) for non constant \( \xi \). We are going to give an answer to this question when \( (X, T) \) is a topologically mixing subshift of finite type endowed with a metric associated with a negative almost additive potential, and consider geometric realizations on Moran sets like those studied in \[2\], the main examples being \( C^1 \) conformal repellers and \( C^1 \) conformal iterated function systems (see section \[3\] for precise definitions and statements). In the setting outlined above, if \( d = 1 \) and \( \xi \) takes its values in \( L_\Phi \), we find the natural variational formula

\[
\dim_H E_\Phi(\xi) = \max \left\{ \frac{h_\mu(T)}{\int_X \log \| DT \| \, d\mu} : \mu \in \mathcal{M}(X, T), \, \Phi_*(\mu) \in \xi(X) \right\}.
\]

As an application of this kind of results, we obtain unexpected results like the following one: Let \( d \in \mathbb{N}_+ \) and \( (m_1, \ldots, m_d) \) be \( d \) integers \( \geq 2 \). Let \( T : [0, 1]^d \to [0, 1]^d \) be the mapping \( (x_1, \ldots, x_d) \mapsto (m_1 x_1 \,(\text{mod } 1), \ldots, m_d x_d \,(\text{mod } 1)) \). Consider

\[
\mathcal{F} = \left\{ x \in [0, 1]^d : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k x = x \right\},
\]

the set of those points \( x \) which are fixed by \( T \) in the asymptotic average. Then \( \mathcal{F} \) is dense and of full Hausdorff dimension in \([0, 1]^d\).

Another application concerns harmonic measure. Let us consider here the special case of the set \( J = C^2 \subset \mathbb{R}^2 \), where \( C \) is the middle third Cantor set. The harmonic measure on \( J \) is the probability measure \( \omega \) such that for each \( x \in J \) and \( r > 0 \), \( \omega(B(x, r)) \) is the probability that a planar Brownian motion started at \( \infty \) attains \( J \) for the first time at a point of \( B(x, r) \) (see Section \[3.4\] for more general examples and a reference). For \( x \in J \), one defines the local dimension of \( \omega \) at \( x \) as \( d_\omega(x) = \lim_{r \to 0^+} \log \omega(B(x, r))/\log(r) \) whenever this limit exists. Let \( I \) stand for the set of all possible local dimensions for \( \omega \). By using the fact that \( \omega \) is a Gibbs measure, we prove that if \( \xi : J \to \mathbb{R}_+ \) is continuous and \( \xi(J) \subset I \), then the set \( E_\omega(\xi) = \{ x \in J : d_\omega(x) = \xi(x) \} \) is dense in \( J \) and the following variational formula holds:

\[
\dim_H E_\omega(\xi) = \sup \{ \dim_H E_\omega(\alpha) : \alpha \in \xi(J) \}, \text{ where } E_\omega(\alpha) = \{ x \in J : d_\omega(x) = \alpha \}.
\]

Our approach necessitates to revisit the case where \( \xi \) is constant. At this occasion, we complete the work achieved in \[14, 15, 20\] by identifying, in our general framework, the Hausdorff dimensions of the sets \( E_\Phi(\alpha) \) with a large deviation spectrum which is equal to the Legendre transform of a kind of "metric" pressure; this is a new kind of large deviation principle in this context. Moreover, our approach brings out an interesting new property
for the structure of the Hausdorff spectrum $\alpha \mapsto \dim_H E_\Phi(\alpha)$. We call this property *weak concavity*; it is between concavity and quasi-concavity. This structure turns out to be crucial both in establishing the large deviation principle and our results on fixed points in the asymptotic average.

The paper is organized as follows. In Section 2 we give basic definitions and state our main results on subshift of finite type. In Section 3 we give the geometric realizations. The other sections provide the proofs of our results.

2. Definitions and main results

2.1. Definitions. Recalls on thermodynamic formalism.

2.1.1. Thermodynamic formalism for almost additive potentials. Given $\Phi \in C_{\text{aa}}(X,T)$, define $\Phi_\text{max} := \max(\phi_1) + C(\Phi)$ and $\Phi_\text{min} := \min(\phi_1) - C(\Phi)$. Define $\|\Phi\| := |\Phi_\text{max}| \lor |\Phi_\text{min}|$. By the almost additivity property we easily get

$$n\Phi_\text{min} \leq \phi_n(x) \leq n\Phi_\text{max}, \quad \forall \ n \in \mathbb{N}.$$ 

Consequently we have $\|\phi_n\|_\infty \leq n\|\Phi\|$.

Define two collections of special almost additive potentials on $X$ as

$C_{\text{aa}}^+(X,T) := \{\Phi \in C_{\text{aa}}(X,T) : \Phi_\text{min} > 0\}$ and $C_{\text{aa}}^-(X,T) := \{\Phi \in C_{\text{aa}}(X,T) : \Phi_\text{max} < 0\}$.

For $\Phi \in C_{\text{aa}}(X,T)$ we get $\phi_{n+1}(x) \leq \phi_n(x) + \phi_1(T^nx) + C(\Phi) \leq \phi_n(x) + \Phi_\text{max} \leq \phi_n(x)$, so $\{\phi_n : n \in \mathbb{N}\}$ is a strictly decreasing sequence of functions.

If $\Phi = (\Phi_1, \ldots, \Phi^d)$ is such that each $\Phi^j \in C_{\text{aa}}(X,T)$, then we call $\Phi$ a *vector-valued almost additive potential* and write $\Phi \in C_{\text{aa}}(X,T,d)$. In this case $\Phi = (\phi_n)_{n=1}^\infty$ with $\phi_n = (\phi_1^n, \ldots, \phi^n_d)$. We set $\Phi_\text{max} := (\Phi_\text{max}^1, \ldots, \Phi_\text{max}^d)$ and $\Phi_\text{min} := (\Phi_\text{min}^1, \ldots, \Phi_\text{min}^d)$. Define $\|\Phi\| := (\sum_{j=1}^d |\Phi^j|^2)^{1/2}$ and $\|\Phi\|_\text{lim} := \limsup_{n \to \infty} \|\phi_n\|_\infty / n$. We have $\|\phi_n\|_\infty \leq n\|\Phi\|$.

Given $u, v \in \mathbb{R}^d$, we write $[u, v] := \{tu + (1-t)v : 0 \leq t \leq 1\}$ to denote the closed interval connecting $u$ and $v$. If $u_i \leq v_i$ for $i = 1, \ldots, d$, then we write $u \leq v$. For $\Phi \in C_{\text{aa}}(X,T,d)$ define $C(\Phi) := (C(\Phi^1), \ldots, C(\Phi^d))$, then we also have the following vector version formula:

$$-C(\Phi) + \phi_n + \phi_p \circ T^n \leq \phi_{n+p} \leq C(\Phi) + \phi_n + \phi_p \circ T^n, \quad \forall \ n, p \in \mathbb{N}.$$ 

For $\mu \in \mathcal{M}(X,T)$, define $\Phi_\mu(\mu) := (\Phi_1(\mu), \ldots, \Phi^d(\mu))$. Define $L_\Phi := \{\Phi_\mu(\mu) : \mu \in \mathcal{M}(X,T)\}$. Given $\Phi, \Psi \in C_{\text{aa}}(X,T,d)$, define $\Phi + \Psi := (\phi_n + \psi_n)_{n=1}^\infty$. We have $\Phi + \Psi \in C_{\text{aa}}(X,T,d)$ with $C(\Phi + \Psi) = C(\Phi) + C(\Psi)$.

The simplest almost additive potentials are the additive ones. Given $\phi : X \to \mathbb{R}^d$ continuous, define $\phi_n = S_n \phi := \sum_{j=0}^{n-1} \phi \circ T^j$ and define $\Phi = (\phi_n)_{n=1}^\infty$. In this case $\phi_{n+p} = \phi_n + \phi_p \circ T^n$, thus $\Phi \in C_{\text{aa}}(X,T,d)$. Such a $\Phi$ is called an *additive potential*. In fact $\phi_n$ is the $n$-th Birkhoff sum of $\phi$. Given an additive potential $\Phi = (S_n \phi)_{n=1}^\infty$, if $\phi$ is Hölder continuous, we say that $\Phi$ is *Hölder continuous*. The simplest Hölder continuous potentials are the constant potentials $(n\alpha)_{n=1}^\infty$, $\alpha \in \mathbb{R}^d$, that we also denote as $\alpha$. 

We collect some useful facts here, see [18] for proofs.

**Proposition 2.2.** Assume $(X,T)$ is a TDS with specification. Let $\Phi = (\Phi_1,\cdots,\Phi^d) \in \mathcal{C}_{aa}(X,T,d)$. Then $L_{\Phi}$ is of dimension $d$ if and only if $\Phi^1,\cdots,\Phi^d \in C_{aa}(X,T)/\sim$ are linearly independent.

The thermodynamic formalism for almost additive potentials has been studied in several works [13, 2, 19, 17, 3, 25, 4, 11]. For our purpose, we only need to consider the subshift of finite type case. Let $(\Sigma_A,T)$ be a subshift of finite type. Given $\Phi \in \mathcal{C}_{aa}(\Sigma_A,T)$, the topological pressure can be defined as

$$P(T,\Phi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \Sigma_A,n} \exp(\sup_{x \in \{w\}} \phi_n(x)).$$

Usually we write $P(\Phi)$ for $P(T,\Phi)$ when there is no confusion. The following extension of the classical variational principle (see [21]) holds:

**Theorem 2.1.** [3, 4, 11] Let $(\Sigma_A,T)$ be a subshift of finite type. For any $\Phi \in \mathcal{C}_{aa}(\Sigma_A,T)$, we have $P(T,\Phi) = \sup\{h_\mu(T) + \Phi_*(\mu) : \mu \in \mathcal{M}(\Sigma_A,T)\}$.

2.1.2. Weak Gibbs metric on subshift of finite type. Let $(\Sigma_A,T)$ be a topologically mixing subshift of finite type with alphabet $\{1,\cdots,m\}$, where $A$ is a $m \times m$ matrix with entries 0 and 1 such that $A^{p_0} > 0$ for some $p_0 \in \mathbb{N}$ and $T$ is the shift map. We endow $\Sigma_A$ with a metric naturally associated with a potential $\Psi \in \mathcal{C}_{aa}^- (\Sigma_A,T)$. This kind of metrics have been considered in [21] and [23] associated with additive potentials.

Note that by endowing $\Sigma_A$ with the standard metric $d_1$ defined as $d_1(x,y) = m^{-|x \land y|}$ (where $|x \land y|$ is the length of the common prefix of $x$ and $y$), $(\Sigma_A,d_1)$ is a compact metric space and $(\Sigma_A,T)$ is a TDS satisfying the specification property. Let $\Sigma_{A,n}$ be the set of the admissible words of length $n$ and let $\Sigma_{A,*} := \bigcup_{n \geq 0} \Sigma_{A,n}$. For $w \in \Sigma_{A,*}$ and $w = w_1 \cdots w_n$, we denote the length of $w$ by $|w| = n$. Given $w \in \Sigma_{A,*} \cup \Sigma_A$ with $|w| \geq n$, we denote $w_1 \cdots w_n$ by $|w|_n$. Given $u \in \Sigma_{A,*} \cup \Sigma_A$ and $v \in \Sigma_{A,*} \cup \Sigma_A$, if $u_j = v_j$ for $j = 1,\cdots,|u|$, then we say $u$ is a prefix of $v$ and write $u \prec v$. For $u = u_1 \cdots u_n \in \Sigma_{A,n}$, $u^*$ stands for $u|_{n-1}$. For $x,y \in \Sigma_{A,*} \cup \Sigma_A$ such that $x \neq y$, $x \land y$ stands for the common prefix of $x$ and $y$ of maximal length.
Recall that $A^{p_0}(i,j) > 0$ for all $1 \leq i, j \leq m$, consequently $A^{p_0+2}(i,j) > 0$. For each $i, j$ we fix $w(i,j) \in \Sigma_{A, p_0}$ such that $iw(i,j)$ is admissible. Define $\Xi := \{w(i,j) : 1 \leq i, j \leq m\}$.

Given a continuous function $\phi : \Sigma_A \to \mathbb{R}^d$, we define
\begin{equation}
\|\phi\|_n := \sup_{x_n = y_n} |\phi(x) - \phi(y)|,
\end{equation}
and for $\Phi \in C_{aa}(\Sigma_A, T, d)$ we write $\|\Phi\|_n := \|\phi_n\|_n$. Writing $\Phi = (\Phi^1, \ldots, \Phi^d)$, we have
\begin{equation}
\sum_{j=1}^d \|\Phi_j\|_n^2^{1/2} \leq \sqrt{d} \|\Phi\|_n.
\end{equation}

For $\Phi \in C_{aa}(\Sigma_A, T)$ and $w \in \Sigma_{A,n}$ we define
\[\Phi[w] := \sup\{\exp(\phi_n(x)) : x \in [w]\}.
\]
Now we fix a $\Psi \in C_{aa}(\Sigma_A, T)$. For $x, y \in \Sigma_A$ define
\[d_\Psi(x, y) := \begin{cases} \Psi[x \wedge y], & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}
\]

**Proposition 2.3.** $d_\Psi$ is an ultra-metric on $\Sigma_A$. If $x \in \Sigma_A$ and $r > 0$, the closed ball $B(x, r)$ is the cylinder $[x|_n]$, where $n$ is the unique integer such that $\Psi[x|_{n-1}] > r$ and $\Psi[x|_n] \leq r$. Each cylinder $[w]$ is a ball with $\text{diam}([w]) = \Psi[w]$.

The proof is elementary and we omit it. For the metric space $(\Sigma_A, d_\Psi)$ we define
\[B_n(\Psi) = \{w \in \Sigma_{A,s} : [w] \text{ is a closed ball of } \Sigma_A \text{ with radius } e^{-n}\} \quad (n \geq 0).
\]
We note that $\Sigma_A = \bigcup_{w \in B_n(\Psi)} [w]$ for each $n \geq 0$.

2.1.3. **Three dimension functions.** We introduce three functions which will turn out to take the same values on $L_\Psi$ and provide the Hausdorff and packing dimensions of the sets $E_\Phi(\alpha)$. They correspond to different point of views to estimate these dimensions, namely box-counting of balls intersecting $E_\Phi(\alpha)$, variational principle for entropy like (1.1) and Legendre transform of a kind of metric pressure. The proofs of the propositions stated in this section are given in Section 4.

(1) **Box-counting type function, the large deviation spectrum:** fix $\Psi \in C_{aa}^- (\Sigma_A, T)$ and $\Phi \in C_{aa}(\Sigma_A, T, d)$. Define $d_\Psi$ and $B_n(\Psi)$ as above. Given $\alpha \in L_\Psi$, $n \geq 1$ and $\epsilon > 0$, define
\[F(\alpha, n, \epsilon, \Phi, \Psi) := \left\{ u \in B_n(\Psi) : \text{ there exists } x \in [u] \text{ such that } \left| \frac{\Phi[u](x)}{|u|} - \alpha \right| < \epsilon \right\}.
\]
Let $f(\alpha, n, \epsilon, \Phi, \Psi)$ be the cardinality of $F(\alpha, n, \epsilon, \Phi, \Psi)$.

**Proposition 2.4.** For any $\Psi \in C_{aa}(\Sigma_A, T)$, the limit
\begin{equation}
D(\Psi) := \lim_{n \to \infty} \frac{\log \#B_n(\Psi)}{n}
\end{equation}
exists. Moreover there exist constants $C_2(\Psi) > C_1(\Psi) > 0$ such that
\begin{equation}
C_1(\Psi) \log m \leq D(\Psi) \leq C_2(\Psi) \log m.
\end{equation}
For any \( \Phi \in C_{aa}(\Sigma_A, T, d) \) and any \( \alpha \in L_\Phi \), we have
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} =: \Lambda_\Phi^\Psi(\alpha).
\]
The function \( \Lambda_\Phi^\Psi : L_\Phi \to \mathbb{R} \) is upper semi-continuous.

We will prove that \( \Lambda_\Phi^\Psi(\alpha) \) is the Hausdorff dimension of \( E_\Phi(\alpha) \) for all \( \alpha \in L_\Phi \). The function \( \Lambda_\Phi^\Psi \) has more regularity than upper semi-continuity. To state it we need several standard notations from convex analysis. Recall that a subset \( M \) of \( \mathbb{R}^d \) is an affine subspace if \( \lambda x + (1 - \lambda)y \in M \) for every \( x, y \in M \) and \( \lambda \in \mathbb{R} \). Given \( A \subset \mathbb{R}^d \), the affine hull of \( A \) is the smallest affine subspace \( M \) of \( \mathbb{R}^d \) such that \( M \supset A \) and is denoted by \( \text{aff}(A) \). For a convex set \( A \), we define \( \text{ri}(A) \), the relative interior of \( A \) as \( \text{ri}(A) := \{ x \in \text{aff}(A) : \exists \epsilon > 0, (x + \epsilon B) \cap \text{aff}(A) \subset A \} \), where \( B = B(0,1) \subset \mathbb{R}^d \) is the unit ball. Let \( A \subset \mathbb{R}^d \) be a convex set and \( h : A \to \mathbb{R} \) be a function. If there exists \( c \geq 1 \) such that for any \( \alpha, \beta \in A \), we can find \( \gamma_1 = \gamma_1(\alpha, \beta), \gamma_2 = \gamma_2(\alpha, \beta) \in [c^{-1}, c] \) such that for any \( \lambda \in [0,1] \)
\[
(2.7) \quad \lambda h(\alpha) + (1 - \lambda)h(\beta) \leq h\left( \frac{\lambda \gamma_1 \alpha + (1 - \lambda) \gamma_2 \beta}{\lambda \gamma_1 + (1 - \lambda) \gamma_2} \right),
\]
then we call \( h \) a weakly concave function on \( A \). Note that if \( c = 1 \), we go back to the usual concept of concave function. Also, \( h(\gamma) \geq \min(h(\alpha), h(\beta)) \) if \( \gamma \in [\alpha, \beta] \subset A \), thus \( h \) is quasi-concave.

**Proposition 2.5.** The function \( \Lambda_\Phi^\Psi : L_\Phi \to \mathbb{R} \) is bounded, positive and weakly concave. It is continuous on any closed interval \( I \subset L_\Phi \) and on \( \text{ri}(A) \), where \( A \subset L_\Phi \) is any convex set. Consequently it is continuous on \( \text{ri}(L_\Phi) \). If moreover \( L_\Phi \) is a convex polyhedron, then \( \Lambda_\Phi^\Psi \) is continuous on \( L_\Phi \). Assume \( I = [\alpha_0, \alpha_1] \subset L_\Phi \) and \( \alpha_{\text{max}} \in I \) such that \( \Lambda_\Phi^\Psi(\alpha_{\text{max}}) = \max\{\Lambda_\Phi^\Psi(\alpha) : \alpha \in I\} \), then \( \Lambda_\Phi^\Psi \) is decreasing from \( \alpha_{\text{max}} \) to \( \alpha_j \), \( j = 0, 1 \).

**Remark 2.1.** Large deviations spectra for the Hausdorff dimension estimation of sets like \( E_\Phi(\alpha) \) have been considered since the first studies of multifractal properties of Gibbs or weak Gibbs measures and then extended to the study of Birkhoff averages \[2\] \[32\] \[10\] \[31\] \[30\] \[29\] \[26\] \[14\] \[15\] \[6\] \[20\]. Until now, in the situations where such a spectrum may be non-concave \[6\] \[4\] \[20\], no description of its regularity like that of Proposition 2.5 had been given. Moreover, the methods used in the papers mentioned above seem not adapted to provide this information.

(2) **Function associated with a conditional variational principle:** For \( \alpha \in L_\Phi \), let
\[
E_\Phi^\Psi(\alpha) := \sup \left\{ \frac{h_\nu(T)}{-\Psi_s(\mu)} : \mu \in \mathcal{M}(\Sigma_A, T) \text{ such that } \Phi_s(\mu) = \alpha \right\}.
\]

(3) **Pressure type function and its Legendre transform-like associated function:** at first we define a kind of pressure function.

**Proposition 2.6.** Fix \( \Phi \in C_{aa}(\Sigma_A, T, d) \) and \( \Psi \in C_{aa}(\Sigma_A, T) \). Let \( z, \alpha \in \mathbb{R}^d \). Then the equation
\[
(2.8) \quad P(\langle z, \Phi - \alpha \rangle + \tau_\Phi^\Psi(z, \alpha) \Psi) = 0
\]
has a unique solution \( \tau_\Psi(z, \alpha) \). Moreover the following variational principle holds

\[
\tau_\Psi(z, \alpha) = \sup \left\{ \frac{h_\mu(T) + \langle z, \Phi_\alpha(\mu) - \alpha \rangle}{-\Psi_\alpha(\mu)} : \mu \in \mathcal{M}(\Sigma_A, T) \right\},
\]

and one also has

\[
\tau_\Psi(z, \alpha) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \mathcal{B}_n(\Psi)} \exp \left( \sup_{x \in [w]} \langle z, (\phi_{|w|}(x) - |w|\alpha) \rangle \right).
\]

We will identify the function

\[
(2.11) \quad \tau_\Psi^*(\alpha) := \inf \{ \tau_\Psi(z, \alpha) : z \in \mathbb{R}^d \}
\]

with \( \Lambda_\Phi \) on \( L_\Phi \). This is the large deviations principle announced in the introduction.

**Remark 2.2.** (1) The function \( \tau_\Psi(z, \alpha) \) defined as in (2.8) was first introduced in \([6]\) when \( \Phi \) and \( \Psi \) are Hölder potentials, and also in \([1]\) where Hölder condition on \( \Phi \) is replaced by the bounded distortion property for almost additive potentials.

(2) \( \tau_\Psi^* \) is a generalization of Legendre transform as noted in \([6]\). For the special constant potential \( \Psi = (-n)_{n=1}^\infty \), \( \tau_\Psi(z, \alpha) = P(\langle z, \Phi \rangle) - \langle z, \alpha \rangle \), thus \( \tau_\Psi^* \) is the classical Legendre transform of the pressure function \( P(\langle z, \Phi \rangle) \).

2.2. Main results on topologically mixing subshift of finite type. Throughout this subsection we fix \( \Phi \in \mathcal{C}_{aa}(\Sigma_A, T, d) \) and \( \Psi \in \mathcal{C}_{a-}(\Sigma_A, T) \). We work on the metric space \( (\Sigma_A, d_\Psi) \). We write \( \dim_H E, \dim_P E, \dim_B E \) for the Hausdorff, packing and box dimensions of \( E \subset \Sigma_A \). For convenience we write \( D^\Psi_\Phi(\alpha) := \dim_H E_\Phi(\alpha) \).

**Theorem 2.2 (Multifractal analysis of the level sets \( E_\Phi(\alpha) \)).**

1. \( E_\Phi(\alpha) \neq \emptyset \) if and only if \( \alpha \in L_\Phi \). For \( \alpha \in L_\Phi \) we have

\[
D^\Psi_\Phi(\alpha) = \Lambda_\Phi^\Psi(\alpha) = \mathcal{E}_\Phi^\Psi(\alpha) = \tau_\Psi^*(\alpha),
\]

and the function \( D^\Psi_\Phi \) is weakly concave.

2. \( \dim_H \Sigma_A = \dim_B \Sigma_A = D(\Psi) = \max \{ \Lambda_\Phi^\Psi (\alpha) : \alpha \in L_\Phi \} \).

**Theorem 2.3 (Localized asymptotic behavior).** Assume \( \xi : \Sigma_A \to \mathbb{R}^d \) is continuous and \( \xi(\Sigma_A) \subset \text{aff}(L_\Phi) \).

1. \( \dim_H E_\Phi(\xi) \geq \sup \{ D^\Psi_\Phi(\alpha) : \alpha \in \xi(\Sigma_A) \cap \text{ri}(L_\Phi) \} \).
2. If \( \xi(\Sigma_A) \subset L_\Phi \) then \( E_\Phi(\xi) \) is dense in \( \Sigma_A \).
3. If \( \sup \{ D^\Psi_\Phi(\alpha) : \alpha \in \xi(\Sigma_A) \cap \text{ri}(L_\Phi) \} = \sup \{ D^\Psi_\Phi(\alpha) : \alpha \in \xi(\Sigma_A) \cap L_\Phi \} \), then \( \dim_H E_\Phi(\xi) = \dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup \{ D^\Psi_\Phi(\alpha) : \alpha \in \xi(\Sigma_A) \cap L_\Phi \} \).
4. If \( d = 1 \) and \( \xi(\Sigma_A) \subset L_\Phi \), then \( E_\Phi(\xi) \) is dense and \( \dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup \{ D^\Psi_\Phi(\alpha) : \alpha \in \xi(\Sigma_A) \} \).

**Remark 2.3.** (1) Recall Remark 2.2. In \([14, 15, 16]\), where the metric is the standard one, the equality \( \Lambda_\Phi^\Psi(\alpha) = \inf_{z \in \mathbb{R}^d} P(\langle z, \Phi \rangle) - \langle z, \alpha \rangle \) is established, and both functions are concave. In our work, the weak concavity of \( \Lambda_\Phi^\Psi \) turns out to be crucial in proving the equality \( \Lambda_\Phi^\Psi(\alpha) = \tau_\Psi^*(\alpha) \) in full generality. This equality can be read as a new large deviation principle thanks to the expression (2.10) giving \( \tau_\Psi^* \) as a kind of metric pressure.
In [6, 4], assuming more regularity than continuity for \( \Phi \) and \( \Psi \), namely bounded distorsion property or Hölder continuity in the case where the potential is additive, the equality 
\[
D\Psi\Phi(\alpha) = E\Psi\Phi(\alpha) = \tau\Psi\circ\Phi(\alpha)
\]
is shown only for \( \alpha \in \text{int}(L\Phi) \), where \( \text{int}(A) \) denotes the interior of \( A \subset \mathbb{R}^d \). The argument is strongly based on the differentiability of the pressure function in these cases.

In [20], the authors consider the case of additive potentials \( \Phi \) and \( \Psi \), and work under the assumption that \( \Psi \) corresponds to a Hölder potential. They show 
\[
D\Psi\Phi(\alpha) = E\Psi\Phi(\alpha)
\]
for all \( \alpha \in L\Phi \). Here we work under weaker regularity assumptions on \( \Psi \), and both \( \Phi \) and \( \Psi \) are almost additive. Also, we use a different method to compute the function \( D\Psi\Phi(\alpha) \), namely concatenation of Gibbs measures. Such a method has been used successfully in [23] to deal with the special sets \( E\Psi\Psi(\alpha) \) when \( \Psi \) is additive as well as in [11] to deal with the asymptotic behavior of almost additive potentials in the different context of full-shifts endowed with self-affine metrics (the spectrum is always concave in this case).

Here, we need to refine such approach in order to remove some delicate points in our geometric application to attractors of \( C^1 \) conformal iterated function systems. We also mention that in the case where the metric is the standard one on a full-shift, the equalities 
\[
D\Psi\Phi(\alpha) = E\Psi\Phi(\alpha) = \tau\Psi\circ\Phi(\alpha)
\]
(there the spectrum is concave) have been obtained in [16] when \( \Phi \) is built from Birkhoff products of continuous positive matrices. There, the computation of Hausdorff dimension uses concatenation of words, like in [14, 15, 20].

Remark 2.4. (1) The proof Theorem 2.3 uses the weak concavity of the spectrum \( D\Phi \).
It also requires to concatenate Gibbs measures in a more elaborated way than to determine \( D\Psi \).

(2) In fact we shall prove a slightly more general result than Theorem 2.3(1): (1') Suppose that \( \xi \) is bounded and continuous outside a subset \( E \) of \( \Sigma_A \), and \( \xi(\Sigma_A \setminus E) \subset \text{aff}(L\Phi) \). If 
\[
\dim_H E < \sup\{D\Phi(\alpha) : \alpha \in \xi(\Sigma_A \setminus E) \cap \text{ri}(L\Phi)\}
\]
then 
\[
\dim_H E(\xi) \geq \sup\{D\Phi(\alpha) : \alpha \in \xi(\Sigma_A \setminus E) \cap \text{ri}(L\Phi)\}.
\]

(3) An extension of Theorem 2.3(4) is given in the final remark of Section 3.4.

3. Geometric results

In this section we show how the main results of the previous section can be applied to multifractal analysis on conformal repellers and on attractors of conformal IFS satisfying the strong open set condition. Such sets fall in the Moran-like geometric constructions considered in [2, 29]. At first we describe this kind of construction (Section 3.1). Then we state the geometric results deduced from Theorems 2.2 and 2.3 (Section 3.2). We give our application to fixed points in the asymptotic average for dynamical systems in \( \mathbb{R}^d \) in Section 3.3. Finally, we give an application to the local scaling properties of weak Gibbs measures in Section 3.4, special example of which is the harmonic measure on planar conformal Cantor sets.

3.1. General setting of geometric realization. Let \( (\Sigma_A, T) \) be a topologically mixing subshift of finite type with alphabet \( \{1, \cdots, m\} \) and \( \Psi \in C^{aa}_0(\Sigma_A, T) \). Let \( X \) be \( \mathbb{R}^d \) or be a connected, \( d' \)-dimensional \( C^1 \) Riemannian manifold. Consider a family of sets
\{R_w : w \in \Sigma_{A,s}\}$, where each $R_w \subset X$ is a compact set with nonempty interior. We assume that this family of compact sets satisfies the following conditions:

1. $R_w \subset R_{w'}$ whenever $w' < w$.
2. For any integer $n > 0$, the interiors of distinct $R_w, w \in \Sigma_{A,n}$ are disjoint.
3. Each $R_w$ contains a ball of radius $\mathcal{U}_w$ and is contained in a ball of radius $\mathcal{T}_w$.
4. There exists a constant $K > 1$ and a negative sequence $\eta_n = o(n)$ such that for every $w \in \Sigma_{A,s}$,
   \begin{equation}
   K^{-1} \exp(\eta_{|w|})\Psi[w] \leq \mathcal{T}_w \leq \mathcal{U}_w \leq K \Psi[w].
   \end{equation}

Let $J = \bigcap_{n \geq 0} \bigcup_{w \in \Sigma_{A,n}} R_w$. We call $J$ the limit set of the family \{R_w : w \in \Sigma_{A,s}\}. We can define the coding map $\chi : \Sigma_A \to J$ as $\chi(x) = \bigcap_{n \geq 1} R_{x|n}, \forall x \in \Sigma_A$. It is clear that $\chi$ is continuous and surjective.

We say that $J$ is a Moran type geometric realization of $\Sigma_A$ with potential $\Psi$.

For this kind of construction we have the following useful observation:

**Proposition 3.1.** Let $J$ be a Moran type geometric realization of $\Sigma_A$ with almost additive potential $\Psi$, then for any $E \subset J$ we have $\dim_H E = \dim_{\Psi}^\Psi(\chi^{-1}(E))$.

In this paper we consider two classes of Moran type geometric realizations of $\Sigma_A$.

1. **Topologically mixing** $C^1$ conformal repeller $(J, g)$. We refer the book [29] for the definitions and the basic properties related to conformal repellers. It is well known that in this case $(J, g)$ has a Markov partition \{R_1, \ldots, R_m\}. For each $w = w_1 \cdots w_n$, define $R_w := R_{w_1} \cap g^{-1}(R_{w_2}) \cap \cdots \cap g^{-n+1}(R_{w_n})$. Define $\psi(x) = -\log|g'(\chi(x))|$ and $\Psi = (S_n \psi)\Sigma_A$. By the definition of $R_w$ and the property of Markov partition, the condition (1) and (2) are checked directly. (3) and (4) are stated in [29] (Proposition 20.2), except that for (4) we have an additional term $\exp(\eta_{|w|}) = \exp(-||\Psi||_{|w|})$. This is because we only assume $\psi$ to be continuous rather than Hölder continuous. Thus $J$ is a Moran type geometric realization of $\Sigma_A$ for some primitive matrix $A$ and the potential $\Psi$. Moreover in this case we have $\chi \circ T = g \circ \chi$.

2. **Attractors of** $C^1$ conformal IFS satisfying the strong open set condition. For completeness we recall the related definitions. Let $U \subset \mathbb{R}^d$ be a non-empty open set. A map $f : U \to U$ is contracting if there exists $0 < \gamma < 1$ such that $|f(x) - f(y)| \leq \gamma|x - y|$ for all $x, y \in U$. Let $\{f_1, \ldots, f_m\}$ be a collection of contracting maps from $U$ to $U$ and suppose that for some closed set $X \subset U$ we have $f_j(X) \subset X$ for each $j$. Then, it is well known that there is a unique non-empty compact set $J \subset X$ such that $J = \bigcup_{j=1}^m f_j(J)$. Such a family is called an Iterated Function System (IFS), of which $J$ is the attractor. This IFS is said to satisfy the open set condition (OSC) if there is a non-empty open set $V \subset U$ such that $f_i(V) \subset V$ for each $j$ and $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$. The strong open set condition (SOSC) holds if moreover this open set $V$ can be chosen with $V \cap K \neq \emptyset$. 
A $C^1$-map $f : U \to \mathbb{R}^d$ is conformal if the differential $f'(x) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|f'(x)y| = |f'(x)||y| \neq 0$ for all $x \in U$ and $y \in \mathbb{R}^d$, $y \neq 0$. We say that an IFS $\{f_1, \cdots, f_m\}$ is a $C^1$ conformal IFS if each $f_j$ is an injective conformal map. We refer to \cite{28} for more details.

Assume $\{f_1, \cdots, f_m\}$ is a $C^1$ conformal IFS satisfying the SOSC. Let $J$ be its attractor. Define $\psi(x) = \log|J_{x1}(\chi(Tx))|$ and $\Psi = (S_n\psi)_{n=1}^\infty$. Let $V$ be an open set such that the SOSC holds. For $w = w_1 \cdots w_n$, define $R_w = f_w(\bar{V})$, where $f_w := f_{w_1} \circ \cdots \circ f_{w_n}$. Due to the SOSC, (1) and (2) hold for $\{R_w : w \in \Sigma_{A,s}\}$. Moreover, arguments similar to those used to prove Proposition 20.2 in \cite{29} show that (3) and (4) also hold. Thus, $\{R_w : w \in \Sigma_{A,s}\}$ is a Moran type geometric realization of $\Sigma_A$ with potential $\Psi$. Notice that here $\Sigma_A$ is the full shift $\Sigma_n$. By the uniqueness of the attractor it is easy to verify that the attractor $J$ is the limit set of the family $\{R_w : w \in \Sigma_{A,s}\}$.

### 3.2. Multifractal analysis on Moran type geometric realizations.

We are going to conduct multifractal analysis on Moran type geometric realizations, thus we need a dynamics $g$ on $J$ so that $(J,g)$ be a factor of some $(\Sigma_A,T)$. For $C^1$ conformal repellers, there is such a natural dynamics. For the attractor of a $C^1$ conformal IFS, there is no such one in general, the difficulty coming from those points having several codings. However, under the SOSC, we can naturally define such a $g$ by removing a "negligible" part of $J$:

Let $\{f_1, \cdots, f_m\}$ be a $C^1$ conformal IFS satisfying the SOSC. Let $V$ be an open set such that the SOSC holds. By \cite{28}, such an open set always exists as soon as the mappings $f_i$ are $C^{1+\epsilon}$ and the OSC holds. Define $\tilde{Z}_\infty := \bigcup_{w \in \Sigma_{A,s}} f_w(\partial V)$ and $Z_\infty := \chi^{-1}(\tilde{Z}_\infty)$. We have the following lemma (proved in Section \cite{7}):

**Lemma 3.1.** The set $\Sigma_A \setminus Z_\infty$ is not empty and $\chi : \Sigma_A \setminus Z_\infty \to J \setminus \tilde{Z}_\infty$ is a bijection. Moreover $T(\Sigma_A \setminus Z_\infty) \subset \Sigma_A \setminus Z_\infty$, $T(Z_\infty) \subset Z_\infty$ and for any Gibbs measure $\mu$ on $\Sigma_A$ we have $\mu(Z_\infty) = 0$.

By the previous lemma we can define the mapping $\tilde{g} : J \setminus \tilde{Z}_\infty \to J \setminus \tilde{Z}_\infty$ as $\tilde{g}(x) = \chi \circ T \circ \chi^{-1}$. By construction we have $\chi \circ T = \tilde{g} \circ \chi$ over $\Sigma_A \setminus Z_\infty$.

Let $J$ be a Moran type geometric realization of $(\Sigma_A,T)$. We set $\tilde{J} = J$ when $J$ is a $C^1$ conformal repeller and $\tilde{J} = J \setminus \tilde{Z}_\infty$ when $J$ is the attractor of a $C^1$ conformal IFS satisfying the SOSC.

Given a sequence of functions $\Phi = (\phi_n)_{n=1}^\infty$ from $\tilde{J}$ to $\mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$, we set $E_\Phi(\alpha) = \{ x \in \tilde{J} : \lim_{n \to \infty} \phi_n(x)/n = \alpha \}$. We also use the notation $D_\Phi(\alpha) = \dim_H E_\Phi(\alpha)$ and we define $L_\Phi = \{ \alpha \in \mathbb{R}^d : E_\Phi(\alpha) \neq \emptyset \}$.

When $J$ is a conformal repeller the system $(J,g)$ is naturally a TDS. For $\Phi \in C_{aa}(J,g,d)$, if we define $\tilde{\Phi} := (\phi_n \circ \chi)_{n=1}^\infty$, since $g \circ \chi = \chi \circ T$, we have $\tilde{\Phi} \in C_{aa}(\Sigma_A,T,d)$ with $C(\tilde{\Phi}) = C(\Phi)$. And for $\alpha \in \mathbb{R}^d$ we have $E_\Phi(\alpha) = \chi(E_{\tilde{\Phi}}(\alpha))$.

When $J$ is the attractor of a $C^1$ conformal IFS satisfying the SOSC, if $\varphi$ is a continuous function from $J$ to $\mathbb{R}^d$, it generates the additive potential $\tilde{\Phi} = (S_n\varphi)_{n=1}^\infty$ on $(\Sigma_A,T)$, where $\phi = \varphi \circ \chi$, and it also defines $\tilde{\Phi} = (S_n\varphi)_{n=1}^\infty$ on $(\tilde{J},\tilde{g})$. Then for $\alpha \in \mathbb{R}^d$ we have $E_\Phi(\alpha) = \chi(E_{\tilde{\Phi}}(\alpha) \setminus Z_\infty)$.
Theorem 3.1. Let $J$ be a Moran type geometric realization of $(\Sigma_A, T)$. If $J$ is a $C^1$ conformal repeller, let $\Phi \in C_{aa}(J, g, d)$ and define $\tilde{\Phi}$ as above. If $J$ is the attractor of a $C^1$ conformal IFS satisfying the SOSC, let $\varphi$ be a continuous map from $J$ to $\mathbb{R}^d$, and define the additive potential $\tilde{\Phi} = (S_n\varphi)_{n=1}^\infty$ on $(\Sigma_A, T)$ with $\phi = \varphi \circ \chi$ and $\Phi = (S_n\varphi)_{n=1}^\infty$ on $(\tilde{J}, \tilde{g})$. Then

1. $L_\Phi = L_{\tilde{\Phi}}$; for $\alpha \in L_\Phi$ we have $\dim_H E_\Phi(\alpha) = \dim_P E_\Phi(\alpha)$ and

$$D_\Phi(\alpha) = D_{\tilde{\Phi}}(\alpha) = \Lambda_f^\Phi(\alpha) = \mathcal{E}_f^\Phi(\alpha) = \tau_f^\Phi(\alpha).$$

2. $\dim_H J = \dim_H \tilde{J} = D(\Psi) = \max\{D(\Phi(\alpha) : \alpha \in L_\Phi)\}$.

Remark 3.1. For the case of conformal repellers, the connection between Theorem 3.1 and the other works [6, 21, 4] is similar to that done in Remark 2.3(2) and (3).

For the set $E_\Phi(\xi)$ we have the following result:

Theorem 3.2. Let $J$ be a Moran type geometric realization of $(\Sigma_A, T)$, which is either a $C^1$ conformal repeller or the attractor of a $C^1$ conformal IFS satisfying the SOSC. Let $\Phi$ and $\tilde{\Phi}$ be the same as in Theorem 3.1. Let $\xi : J \to \mathbb{R}^d$ be continuous and $E_\Phi(\xi) = \{x \in J : \lim_{n \to \infty} \phi_n(x)/n = \xi(x)\}$. If $\xi(J) \subset \text{aff}(L_\Phi)$, then

1. $\dim_H E_\Phi(\xi) \supset \max\{D(\Phi(\alpha) : \alpha \in \xi(J) \cap \text{ri}(L_\Phi))\}$, and $E_\Phi(\xi)$ is dense if $\xi(J) \subset L_\Phi$.
2. If $\sup\{D(\Phi(\alpha) : \alpha \in \xi(J) \cap \text{ri}(L_\Phi))\} = \sup\{D(\Phi(\alpha) : \alpha \in \xi(J) \cap L_\Phi)\}$, then $\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup\{D(\Phi(\alpha) : \alpha \in \xi(J) \cap L_\Phi)\}$.
3. If $d = 1$ and $\xi(J) \subset L_\Phi$, then $\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup\{D(\Phi(\alpha) : \alpha \in \xi(J))\}$ and $E_\Phi(\xi)$ is dense.

3.3. Application to fixed points in the asymptotic average for dynamical systems in $\mathbb{R}^d$. Suppose that $(J, g)$ is a dynamical system with $J \subset \mathbb{R}^d$. We say that $x \in J$ is a fixed point of $g$ in the asymptotic average if $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g^k x = x$. We are interested in the Hausdorff dimension of the set of all such points:

$$\mathcal{F}(J, g) = \{x \in J : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g^k x = x\}.$$ 

If $\xi$ stands for the identity map on $J$ and $\Phi$ stands for the additive potential associated with the potential $\xi$, in our setting we have $\mathcal{F}(J, g) = E_\Phi(\xi)$.

The set $L_\Phi$ is contained in the convex hull of $J$, and it contains the set of the fixed points of $g$. An example of trivial situation is provided by the unit circle endowed with dynamic $g(z) = z^2$ in $\mathbb{C}$. There, $\mathcal{F}(J, g) = \{1\}$. How about general conformal repellers and attractors of conformal IFS? This question is non trivial in general. We are going to describe a class of conformal IFS, namely self-similar generalized Sierpinski carpets, for which the situation is non trivial and we have a complete answer.

We consider a special self-similar IFS $\{f_1, \cdots, f_m\}$ on $\mathbb{R}^d$: $f_j(x) = \rho_j x + c_j$, $0 < \rho_j < 1$, $(1 \leq j \leq m)$. We assume further the SOSC fulfills. Let $x_j$ stand for the unique fixed
point of $f_j$ and let $J$ be the attractor of this IFS. Notice that the mappings $f_j$ have no rotation part, thus the convex hull of $J$ satisfies $\text{Co}(J) = \text{Co}\{x_1, \ldots, x_m\} =: \Delta$, and is a polyhedron. We further assume that $\text{Co}(J)$ has dimension $d$ (otherwise we can define this IFS in a smaller affine subspace).

Let $W$ stand for the open set such that the SOSC holds. It is ready to see that $V := W \cap \Delta$ is also an open set such that SOSC holds. We can define the dynamics $\tilde{g}$ on $\tilde{J} = J \setminus \bar{Z}_\infty$, where $\bar{Z}_\infty$ is defined as in the previous subsection.

Now we have the following result whose proof is given in Section 7.

**Theorem 3.3.** Let $\Phi = \text{id}_J$. Then $\mathcal{F}(\tilde{J}, \tilde{g})$ is dense and $\dim_H \mathcal{F}(\tilde{J}, \tilde{g}) = \sup \{\mathcal{D}_\Phi(\alpha) : \alpha \in J\}$. Moreover if the point at which $\mathcal{D}_\Phi$ attains its maximum belongs to $J$, then $\mathcal{F}(\tilde{J}, \tilde{g})$ is of full Hausdorff dimension.

We have the following corollary, in which the lower bound for the Hausdorff dimension follows directly from Theorem 3.3 and the upper bound follows from standard estimates based on the bounds provided in Section 5.1.

**Corollary 3.1.** Let $N \in \mathbb{N}_+$ and let $d_1, \ldots, d_N$ be $N$ positive integers. Consider $N$ self-similar IFS without rotations components $\{f^{(j)}_1, \ldots, f^{(j)}_{d_j}\}_{1 \leq j \leq N}$ living respectively in $\mathbb{R}^{d_j}$.

Denote by $J_j$, $1 \leq j \leq N$, their respective attractors as well as the corresponding dynamical systems $(\tilde{J}_j, \tilde{g}_j)$. Let $\tilde{J} = \prod_{j=1}^N \tilde{J}_j \subseteq \mathbb{R}^{\sum_{j=1}^N d_j}$ be endowed with the dynamics $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_N)$. We have $\dim_H \mathcal{F}(\tilde{J}, \tilde{g}) = \sum_{j=1}^N \dim_H \mathcal{F}(\tilde{J}_j, \tilde{g}_j) = \sum_{j=1}^N \mathcal{D}_\Phi(\alpha) : \alpha \in J_j$, where $\Phi_j = \text{id}_{\mathbb{R}^{d_j}}$.

Both the previous results yield the result presented in the introduction of the paper:

**Theorem 3.4.** Let $d \in \mathbb{N}_+$ and $(m_1, \ldots, m_d)$ be $d$ integers $\geq 2$. Set $J = [0,1]^d$ and let $g : J \to J$ be the mapping $(x_1, \ldots, x_d) \mapsto (m_1x_1 \text{ (mod 1)}, \ldots, m_dx_d \text{ (mod 1)})$. Then $\mathcal{F}(J, g)$ is dense and of full Hausdorff dimension in $[0,1]^d$.

To see this, for a fixed integer $m \geq 2$ let $g_m : [0,1] \to [0,1]$ be the mapping $x \mapsto mx \text{ (mod 1)}$. Let $(\Sigma_m, T)$ be the full shift over alphabet $\{0, \ldots, m-1\}$, where $\Sigma_m$ is endowed with the usual metric $d_1(x,y) = m^{-\lfloor x/y \rfloor}$. If we take $\psi = -\log m$ and $\Psi = (n\psi)_{n=1}^\infty$, then $d_1 = d_\Psi$. Define a map $\chi : \Sigma_m \to [0,1]$ as $\chi(x) = \sum_{n=1}^{\infty} x_n/m^n$. Then $\chi$ is continuous and surjective. Consider the IFS $\{f_j : j = 0, \ldots, m-1\}$ defined as $f_j(x) = (x+j)/m$. It is seen that the SOSC holds with $V = (0,1)$. Let $\bar{Z}_\infty := \left\{ \sum_{j=1}^n x_j m^{-j} : n \in \mathbb{N}; x_j = 0, \ldots, m-1 \right\} \cup \{1\}$. Define the dynamics $\tilde{g}$ on $\tilde{J}_m = [0,1] \setminus \bar{Z}_\infty$ as in the previous section. Then it is easy to check that $\tilde{g} = g_m|_{\tilde{J}_m}$. Let $\Phi = \text{id}_{[0,1]}$. By theorem 3.3 we get $\dim_H \mathcal{F}(\tilde{J}_m, g_m) = \sup \{\mathcal{D}_\Phi(\alpha) : \alpha \in [0,1]\}$. By the law of large number applied to the measure of maximal entropy we get $\mathcal{D}(1/2) = 1$. We conclude by noticing that $\mathcal{F}(J, g) = \prod_{i=1}^d \mathcal{F}([0,1], g_{m_i})$.

Next we consider concrete examples of carpets in the unit square.

**Heterogeneous carpets in the unit square.** In order to fully illustrate our purpose, we consider an IFS $S_0 = \{f_1, \ldots, f_N\}$ in $\mathbb{R}^2$ made of contractive similitudes without
rotations such that the squares $f_i([0,1]^2)$ form a tiling of $[0,1]^2$. All these situations have been determined in [9]. In this way, $[0,1]^2$ can be chosen as the open set such that the SOSC holds, and the boundaries of the sets $f_i([0,1]^2)$ have big intersections. The picture in the left of Figure 1 give an example of this kind of IFS. This IFS contains 15 dilation maps, and the dynamics on this attractor is highly non-trivial.

Let $\Phi$ denote $\text{Id}_{\mathbb{R}^2}$. For each $\emptyset \neq S \subset S_0$, we denote by $J_S$ the attractor of the IFS $S$. The dynamics $\tilde{g}_S$ defined on $\tilde{J}_S$ is the restriction of $\tilde{g}_{S_0}$ to $\tilde{J}_S$. The set $\mathcal{F}(\tilde{J}_{S_0}, \tilde{g}_{S_0})$ is of full Hausdorff dimension, since $J_{S_0} = [0,1]^2$. If $S \neq S_0$, we have the variational formula $\dim H \mathcal{F}(\tilde{J}_S, \tilde{g}_S) = \sup \{ D_\phi (\alpha) : \alpha \in J_S \}$, and in general it is hard to know whether $\mathcal{F}(\tilde{J}_S, \tilde{g}_S)$ is of full dimension or not in $J_S$. However, here are two simple examples illustrating both possibilities.

We consider the case of the regular tiling associated with the IFS $S_0 = \{ f_{i,j} : x \mapsto \frac{x}{3} + \frac{(i,j)}{3} : 0 \leq i, j \leq 2 \}$. Then, let $S_1 = \{ f_{0,0}, f_{0,2}, f_{2,0}, f_{2,2} \}$ and $S_2 = S_1 \cup \{ f_{1,1} \}$. We claim that $\mathcal{F}(\tilde{J}_{S_1}, \tilde{g}_{S_1})$ is not of full Hausdorff dimension, while $\mathcal{F}(\tilde{J}_{S_2}, \tilde{g}_{S_2})$ is.

The simpler situation is that of $S_2$. In this case, $G = (1/2,1/2)$, the center of symmetry of $J_{S_2}$ is the fixed point of $f_{1,1}$ and it belongs to $L_\Phi$. Moreover, it is obvious that the uniform measure (or Parry measure) on $J_{S_2}$ is carried by the set $E_\Phi(G)$. This yields the result by Theorem 3.3 and $\dim H \mathcal{F}(\tilde{J}_{S_2}, \tilde{g}_{S_2}) = \log(5)/\log(3)$.

In the case of $S_1$, the point $G$ is still the center of symmetry of $J_{S_1}$, so $\mathcal{D}_\Phi$ reaches its maximum at $G$. However, $G$ does not belong to $J_{S_1}$. Since $\Phi$ is H"older continuous and the tiling is regular, we know that $\mathcal{D}_\Phi$ is strictly concave. By using the symmetry, one deduces that the restriction of $\mathcal{D}_\Phi$ to $J_{S_1}$ reaches its maximum at any of the four points $(1/3,1/3)$, $(1/3,2/3)$, $(2/3,1/3)$ and $(2/3,2/3)$. This yields $\dim H \mathcal{F}(\tilde{J}_{S_1}, \tilde{g}_{S_1}) = \mathcal{D}_\Phi((1/3,1/3)) < \log(4)/\log(3) = \dim H J_{S_1}$.

![Figure 1](image)

### 3.4. Localized results for Gibbs measures

Let $\{ f_1, \cdots, f_m \}$ be a homogenous self-similar IFS in $\mathbb{C}$ satisfying the strong separation condition, that is, each function $f_j$ has the form $f_j(z) = a_jz + b_j$ where $0 < \rho = |a_j| < 1$, and there exists a topological closed disk $D$ such that $f_j(D) \subset D$ and the $f_j(D)$ are pairwise disjoint. There is a natural coding map $\chi : \Sigma_m \to J$. Moreover if we define $\psi(x) \equiv \log \rho$ for $x \in \Sigma_m$, and $\Psi = (S_n \psi)_{n=1}^\infty$, then $\chi : (\Sigma_m, d_\Psi) \to (J, |\cdot|)$ is a bi-Lipschitz homeomorphism.
Let \( \phi : J \to \mathbb{R} \) be continuous and define \( \tilde{\phi} = \phi \circ \chi \). By subtracting a constant potential if necessary, we can assume \( P(T, \tilde{\phi}) = 0 \). There exists a weak Gibbs measure \( \tilde{\mu} \) on \( \Sigma_m \) (see [22]) such that
\[
    d_{\tilde{\mu}}(x) := \lim_{n \to \infty} \log \frac{\mu(B(x, r))}{\log r} = \lim_{n \to \infty} \frac{S_n \tilde{\phi}(x)}{n \log \rho}
\]
in the sense that either both the limits do not exist, either they exist and are equal. Define \( \mu := \chi_* (\tilde{\mu}) \). By the bi-Lipschitz property of \( \chi \) and the strong separate condition, we can easily conclude that \( d_{\mu}(y) = \lim_{n \to \infty} S_n \phi(y) / n \log \rho \) for any \( y \in J \). Let \( \Phi = (S_n \phi)^{\infty}_{n=1} \). If we define \( E_\mu(\alpha) = \{ y \in J : d_{\mu}(y) = \alpha \} \), then we get \( E_\Phi(\alpha) = E_\mu(\alpha / \log \rho) \) for any \( \alpha \in L_\Phi \).

By applying Theorem [3.2] for \( d = 1 \), we have the following property regarding the local dimension for weak Gibbs measure:

**Corollary 3.2.** Let \( \mu \) be the weak Gibbs measure associated with \( \phi \). Then the set of all possible local dimension for \( \mu \) is the interval \( L_\Phi / \log \rho \). Assume \( \xi : J \to \mathbb{R} \) is continuous and \( \xi(J) \subset L_\Phi / \log \rho \), then
\[
\dim_H \{ x \in J : d_{\mu}(x) = \xi(x) \} = \sup \{ \dim_H E_\mu(\alpha) : \alpha \in \xi(J) \}.
\]

Now let \( \omega \) stand for the harmonic measure on \( J \). It is well known that (see for example the survey paper [24]) there exists a Hölder continuous function \( \phi : J \to \mathbb{R} \) such that \( w \asymp \mu \), where \( \mu \) is the equilibrium state of \( \phi \). By a direct application of the above corollary we have the following property:

**Corollary 3.3.** Assume \( \omega \) is the harmonic measure on \( J \) and \( I \) is the set of all possible local dimension for \( \omega \). Assume \( \xi : J \to \mathbb{R} \) is continuous and \( \xi(J) \subset I \). Then
\[
\dim_H \{ x \in J : d_{\omega}(x) = \xi(x) \} = \sup \{ \dim_H E_\omega(\alpha) : \alpha \in \xi(J) \}.
\]

**Final remark.** At least when \( d = 1 \), it is not difficult to extend the results obtained in this paper by considering \( \Upsilon = (\gamma_n)_{n \geq 1} \in C^1_{ac}(\Sigma_A, T) \) and the more general level sets \( E^w_{\Phi/T}(\xi) = \{ x \in \Sigma_A : \lim_{n \to \infty} \phi_n(x)/\gamma_n(x) = \xi(x) \} \); when \( \xi \) is constant, such sets have been considered in the contexts examined in [6, 4]. The formula is that if the continuous function \( \xi \) takes values in the set \( L_{\Phi/T} = \{ \Phi_s(\nu)/\Upsilon_s(\nu) : \nu \in \mathcal{M}(\Sigma_A, T) \} \), then
\[
\dim_H (E^w_{\Phi/T}(\xi)) = \sup \{-h_\nu(T)/\Phi_s(\nu) : \nu \in \mathcal{M}(\Sigma_A, T), \Phi_s(\nu)/\Upsilon_s(\nu) \in \xi(\Sigma_A) \}.
\]
When \( \Upsilon = -\Psi \), this can be applied to the local dimension of Gibbs measures associated with Hölder potentials \( \varphi \) on any \( C^1 \) conformal repeller of a map \( f \), since in this case we know from [29] that such a measure is doubling so that the local dimension is directly related to the asymptotic behavior of \( S_n \varphi / S_n (-\log \|DF\|) \). Consequently, Corollary 3.3 can be extended to harmonic measure on more general conformal repellers (see [24]).

### 4. Proofs of Propositions 2.2, 2.4, 2.5 and 2.6

#### 4.1. Proof of proposition 2.2
Suppose that \( \Psi, \Psi' \in C_{ac}(X, T) \) and \( \| \Psi - \Psi' \|_\text{lim} = 0 \). Then \( \mu_s(\Psi) = \mu_s(\Psi') \) for every \( \mu \in \mathcal{M}(X, T) \). From this and the definition of \( L_\Phi \), we can easily show that if \( L_\Phi \) is of dimension \( d \) then \( \Phi^1, \ldots, \Phi^d \) are linearly independent.
To show the other direction, at first we assume \( d = 1 \). By [[18] lemma 3.5, we have \( L_\Phi = [\beta_1, \beta_2] \), where \( \beta_1 = \lim_{n \to \infty} \inf_{x \in X} \phi_n(x)/n \) and \( \beta_2 = \lim_{n \to \infty} \sup_{x \in X} \phi_n(x)/n \). We need to show that if \( \Phi \not\sim 0 \), then \( L_\Phi \) is a non degenerate interval in \( \mathbb{R} \). Otherwise, we get \( \beta_1 = \beta_2 \), then we get \( \|\Phi - \beta_1\|_{\infty} = 0 \), thus \( \Phi \sim 0 \), which is a contradiction.

Now assume \( d > 1 \) and \( \Phi^1, \ldots, \Phi^d \) are linearly independent. If \( L_\Phi \) has dimension strictly less than \( d \), then there exists a non-zero vector \( p \in \mathbb{R}^d \) such that \( p \cdot L_\Phi \) is a singleton, that is \( L_{p \cdot \Phi} = p \cdot L_\Phi \) is a singleton. By what has been shown, we conclude that \( p \cdot \Phi \sim 0 \), that is \( p_1 \Phi^1 + \cdots + p_d \Phi^d = 0 \), which is a contradiction.

4.2. Proof of Proposition 2.4. We need some preliminary results.

**Lemma 4.1.** Let \( \Phi \in \mathcal{C}_{aa}(\Sigma_A, T) \). Let \( C = C(\Phi) \).

1. \( \lim_{n \to \infty} \|\Phi\|_n/n = 0 \).
2. For any \( u, v \in \Sigma_A \) such that \( uv \in \Sigma_A \), we have
   \[
   \exp(-C - \|\Phi\|_n)\Phi[u] \Phi[v] \leq \Phi[uv] \leq \exp(C)\Phi[u] \Phi[v].
   \]
3. For \( w = u_1 w_1 \cdots u_n w_n u_{n+1} \in \Sigma_A \), let \( k = \sum_{j=1}^{n+1} |u_j| \). We have
   \[
   \exp(-2nC + k\Phi_{\min}) \prod_{j=1}^{n} \Phi[w_j] \exp(-\|\Phi\|_{w_j}) \leq \Phi[w] \leq \exp(2nC + k\Phi_{\max}) \prod_{j=1}^{n} \Phi[w_j].
   \]
4. If \( \Phi \in \mathcal{C}_{aa}^\infty(\Sigma_A, T) \), then \( \Phi[v] \leq \Phi[u] \) for \( u < v \).

**Proof.**

1. Fix \( k \in \mathbb{N} \), let \( n = kp + l \) with \( 0 \leq l < k \). Then by almost additivity we get
   \[
   \sum_{j=0}^{p-1} \phi_k(T^{kj}x) + \phi_l(T^{pk}x) - pC \leq \phi_n(x) \leq \sum_{j=0}^{p-1} \phi_k(T^{kj}x) + \phi_l(T^{pk}x) + pC.
   \]
   This yields \( \|\Phi\|_n \leq \sum_{j=1}^{p} \|\phi_k\|_{kj} + 2k\|\Phi\| + 2pC \) and
   \[
   \frac{\|\Phi\|_n}{n} \leq \frac{\sum_{j=1}^{p} \|\phi_k\|_{kj}}{kp} + 2\|\Phi\| + \frac{2C}{k}.
   \]
   When \( k \) is fixed, since \( \phi_k \) is continuous, we know that \( \sum_{j=1}^{p} \|\phi_k\|_{kj}/p \to 0 \) as \( p \to \infty \). Then the result follows easily.

2. Let \( |u| = n \) and \( |v| = k \). Given \( x \in [uv] \) we have \( x \in [u] \) and \( T^px \in [v] \) and \( \phi_{n+k}(x) \leq C + \phi_n(x) + \phi_k(T^nx) \). Thus \( \sup_{x \in [uv]} \phi_{n+k}(x) \leq C + \sup_{x \in [u]} \phi_n(x) + \sup_{y \in [v]} \phi_k(y) \). Consequently, we have \( \Phi[uv] \leq \exp(C)\Phi[u] \Phi[v] \).

On the other hand take \( x_0 \in [v] \) and \( y_0 \in [u] \) such that \( \phi_k(x_0) = \sup_{x \in [v]} \phi_k(x) \) and \( \phi_n(y_0) = \sup_{y \in [u]} \phi_n(y) \). Let \( \bar{x} = u x_0 \), then we have
\[
\sup_{x \in [uv]} \phi_{n+k}(x) \geq \phi_{n+k}(\bar{x}) \geq \phi_n(\bar{x}) + \phi_k(x_0) - C \geq \phi_n(y_0) + \phi_k(x_0) - C - \|\Phi\|_n.
\]
Consequently, \( \exp(-C - \|\Phi\|_n)\Phi[u] \Phi[v] \leq \Phi[uv] \).

3. It is similar to the proof of (2).

4. If \( \Phi \in \mathcal{C}_{aa}^\infty(\Sigma_A, T) \), then \( \phi_n(x) \leq \phi_m(x) \) for any \( m \leq n \). Since \( [v] \subset [u] \) for \( u < v \), by definition we get \( \Phi[v] \leq \Phi[u] \). \( \square \)
Remark 4.1. By repeating the proof of (1), one can show that we still have $\|\Phi\|_n/n \to 0$ as $n \to \infty$ for $\Phi \in C_{aa}(\Sigma_A, T, d)$.

Lemma 4.2. Let $\Psi \in C_{aa}(\Sigma_A, T)$.

1. Let $C_1(\Psi) = 1/|\Psi_{\text{min}}|$ and $C_2(\Psi) = 1 + 1/|\Psi_{\text{max}}|$. For any $w \in B_n(\Psi)$ we have

$$C_1(\Psi)n \leq |w| \leq C_2(\Psi)n.$$  \hspace{1cm} (4.2)

2. For any $w \in B_n(\Psi)$ we have

$$\exp(-C(\Psi) - \|\Psi\|_{|w|} + \Psi_{\text{min}})e^{-n} \leq \Psi[w] \leq e^{-n}.$$  \hspace{1cm} (4.3)

3. The balls in $\{[w] : w \in B_n(\Psi)\}$ are pairwise disjoint.

4. If $u < v$ are such that $u \in B_{n_1}(\Psi)$ and $v \in B_{n_2}(\Psi)$, then

$$|v| - |u| \leq \frac{\Psi_{\text{min}} - \|\Psi\|_{|v|} - (n_2 - n_1) - 2C}{\Psi_{\text{max}}}.$$  \hspace{1cm} (4.4)

Proof. (1) By (2.1) we have $e^{\Psi[w]} \leq e^{\Psi_{\text{min}}}$ for any $w \in \Sigma_{A,*}$. If $w \in B_n(\Psi)$, we have $e^{\Psi[w]} \leq \Psi[w] \leq e^{-n} < \Psi[w^*] \leq e^{\Psi_{\text{max}}(w^*) - 1}$. Thus $C_1(\Psi)n = n\Psi_{\text{min}} \leq |w| \leq n(1 - 1/\Psi_{\text{max}}) = C_2(\Psi)n$.

(2) By definition $\Psi[w] \leq e^{-n}$. Assume $|w| = k$. By Lemma 4.1(2)

$$\Psi[w] = \Psi[w^*w_k] \geq \exp(-C - \|\Psi\|_{k-1})\Psi[w^*]\Psi[w_k] \geq \exp(-C - \|\Psi\|_k - n + \Psi_{\text{min}}).$$

(3) $d_\Psi$ is ultra-metric.

(4) Write $v = uw$. Then $|w| = |v| - |u|$ and

$$e^{-C - \|\Psi\|_{|v|} - n_2 + \Psi_{\text{min}}} \leq \Psi[v] = \Psi[uv] \leq e^C\Psi[u]\Psi[w] \leq e^{C - n_1 + |w|\Psi_{\text{max}}}.$$  \hspace{1cm} □

Given $\Phi \in C_{aa}(\Sigma_A, T, d)$ and two constants $C_2 \geq C_1 > 0$. For each $n \in \mathbb{N}$ we define

$$\|\Phi\|_n^* := \max\{|\Phi|_l : C_1n \leq l \leq C_2n\}.$$  \hspace{1cm} (4.4)

By remark 4.1 we have $\|\Phi\|_n^*/n \to 0$ when $n \to \infty$.

Proof of Proposition 2.3. We fix $\Phi$ and $\Psi$ and write $F(\alpha, n, \epsilon), f(\alpha, n, \epsilon)$ and $\Lambda$ for $F(\alpha, n, \epsilon, \Phi, \Psi), f(\alpha, n, \epsilon, \Phi, \Psi)$ and $\Lambda^\Psi$ to simplify the notation.

At first we show that log $f(\alpha, n, \epsilon)$, as a sequence of $n$, has a kind of subadditivity property. More precisely, for any $\epsilon > 0$, there exist an $N \in \mathbb{N}$ and $\beta_n > 0$ such that log $\beta_n = o(n)$ and $f(\alpha, n, \epsilon)p \leq \beta_n f(\alpha, (n + \tilde{c})p, 2\epsilon)$ for any $n \geq N$, and any $p \geq 1$, where $\tilde{c} = \lceil -p_0\Psi_{\text{max}} - 2C(\Psi) \rceil$. Recall that $p_0$ is a fixed positive integer such that $A^{p_0} > 0$.

In fact for $w_1, \ldots, w_p \in F(\alpha, n, \epsilon)$, let $w = w_1 \cdots w_p$, where $w_j = w_ju_j$ with $u_j \in \Xi$ such that $u_ju_jw_j+1$ is admissible. Recall (see (2.3)) that for any cylinder $[u]$ and any $x, \bar{x} \in [u]$, we have $|\psi_{[u]}(x) - \psi_{[u]}(\bar{x})| \leq \|\Psi\|_{|u|}$. Now for any $x \in [w]$, let $s_0 = 0, s_k =$
\[
\sum_{j=1}^{k} (|w_j| + p_0) \ (1 \leq k \leq p) \text{ and define } x^k = T^{s_{k-1}} x. \text{ We have } |w| = s_p \text{ and } x^k \in \lbrack w_k \rbrack \text{ for } k = 1, \cdots, p. \text{ Take } y^k \in \lbrack w_k \rbrack \text{ such that } \Psi[w_k] = \exp(\psi_{|w|}(y^k)). \text{ Then }
\]

\[
\psi_{|w|}(x) \geq \sum_{k=1}^{p} \psi_{|w_k|}(x^k) + p_0 p \Psi_{\min} - (2p - 1)C(\Psi)
\]

\[
\geq \sum_{k=1}^{p} \psi_{|w_k|}(y^k) - \sum_{k=1}^{p} \|\Psi\|_{|w_k|} + p(p_0 \Psi_{\min} - 2C(\Psi)).
\]

Thus by Lemma 4.2(2)

\[
\Psi[w] \geq \exp(\psi_{|w|}(x))
\]

\[
\geq \exp \left( \sum_{k=1}^{p} \psi_{|w_k|}(y^k) - \sum_{k=1}^{p} \|\Psi\|_{|w_k|} + p(p_0 \Psi_{\min} - 2C(\Psi)) \right)
\]

\[
= \left( \prod_{k=1}^{p} \Psi[w_k] \right) \exp \left( - \sum_{k=1}^{p} \|\Psi\|_{|w_k|} + p(p_0 \Psi_{\min} - 2C(\Psi)) \right)
\]

\[
\geq \exp(-pm) \exp(-2 \sum_{k=1}^{p} \|\Psi\|_{|w_k|} + p((p_0 + 1)\Psi_{\min} - 3C(\Psi)))
\]

\[
\geq \exp(-pm) \exp \left( p((p_0 + 1)\Psi_{\min} - 3C(\Psi)) - 2p\|\Psi\|_{n}^{*} \right) \geq \exp(-p(n + c_1(n)),
\]

where \(c_1(n) = -(p_0 + 1)\Psi_{\min} + 3C(\Psi) + 2\|\Psi\|_{n}^{*} > 0 \) and \(\|\Psi\|_{n}^{*} \) is defined as in (4.3) with constants \(C_2(\Psi) \geq C_1(\Psi) > 0\). Lemma 4.1(1) yields \(c_1(n)/n \rightarrow 0\).

By Lemma 4.1(3) we also have

\[
\Psi[w] \leq \exp(2pC(\Psi) + p_0 p \Psi_{\max}) \left( \prod_{k=1}^{p} \Psi[w_k] \right) \leq \exp(-p(n - p_0 \Psi_{\max} - 2C(\Psi))).
\]

By definition of \(\overline{c}\), there exists \(u \in B_{p(n + \overline{c})}(\Psi)\) such that \(u\) is the prefix of \(w\). Write \(w = uw'\).

**Claim:** \(|w'| \leq p(ac_1(n) + b) \text{ for some constant } a, b > 0\).

Indeed, we have \(e^{-p(n + c_1(n))} \leq \Psi[w] \leq e^{C \Psi[u] \Psi[w']} \leq e^{C \Psi(u) \Psi[w']}. \text{ Thus } |w'| \leq p(c_1(n) + p_0 \Psi_{\max} + 3C)/(-\Psi_{\max}).

Now since \(w_k \in F(\alpha, n, \epsilon)\) we can find \(x_k \in \lbrack w_k \rbrack\) such that \(|\phi_{|w_k|}(x_k)| |w_k| - \alpha| < \epsilon\). Take \(x \in \lbrack w \rbrack\); in particular, \(x \in \lbrack u \rbrack\). Define \(s_k\) and \(x^k\) as above. We have \(|w| = s_p\) and \(x^k \in \lbrack w_k \rbrack\) for \(k = 1, \cdots, p\). By almost additivity, we get

\[
\phi_{|u|}(x) + \phi_{|w'|}(T^{|u|} x) - C(\Phi) \leq \phi_{|w|}(x) \leq \phi_{|u|}(x) + \phi_{|w'|}(T^{|u|} x) + C(\Phi)
\]

(this is a vector inequality). Notice that if \(\beta_1, \beta_2 \in \mathbb{R}^d\) are such that \(\beta_1 > 0\) and \(-\beta_1 \leq \beta_2 \leq \beta_1\), then \(|\beta_2| \leq |\beta_1|\). Thus we have

\[
\phi_{|w|}(x) = \phi_{|w|}(x) + \eta_0 = \sum_{k=1}^{p} \phi_{|w_k u_k|}(x^k) + \eta_0 = \sum_{k=1}^{p} \phi_{|w_k|}(x^k) + \eta_2 + \eta_1 + \eta_0
\]
\[
\sum_{k=1}^{p} \phi|w_k|(x_k) + \eta_3 + \eta_2 + \eta_1 + \eta_0 = \left( \sum_{k=1}^{p} |w_k| \right) \alpha + \eta_4 + \eta_3 + \eta_2 + \eta_1 + \eta_0,
\]

where \(|\eta_0| \leq |w'| ||\Phi|| + |C(\Phi)| + \sum_{k=1}^{p} |w_k| \alpha \leq p(\alpha c_1(n) + b) ||\Phi|| + |C(\Phi)|; |\eta_1| \leq (p - 1)|C(\Phi)|; |\eta_2| \leq p(\alpha c_1(n) + b) ||\Phi|| + |C(\Phi)|); |\eta_3| \leq \sum_{k=1}^{p} |w_k| \alpha \leq p|\Phi|; |\eta_4| \leq (\sum_{k=1}^{p} |w_k|) \epsilon.\n
Since \(s_p = \sum_{k=1}^{p} |w_k| + p\alpha p\) and \(|w_k| \geq C_1 n\), we have \(s_p \geq C_1 np\), and
\[
\left| \frac{\phi|w|(x')}{|u|} - \alpha \right| \leq \frac{\left( \sum_{k=1}^{p} |w_k| \right) \epsilon + |\eta_4| + |\eta_3| + |\eta_2| + |\eta_1| + |\eta_0|}{|u|}.
\]

Moreover, \(|u| = s_p - |w'| \geq pC_1 n - p(\alpha c_1(n) + b)\) and \(c_1(n)/n, ||\Phi||^*/n \to 0\), so that we can choose \(N(\epsilon)\) big enough such that \(|\phi|w|(x)/|u| - \alpha| \leq 2 \epsilon\) when \(n \geq N(\epsilon)\). Consequently \(u \in F(\alpha, p(n + \bar{c}), 2\epsilon)\). Thus, \(f(\alpha, p(n + \bar{c}), 2\epsilon) \geq f(\alpha, n, \epsilon^p)/m^{p(\alpha c_1(n) + b)}\). If we take \(\beta_n = m^{\alpha c_1(n) + b}\), then we get the desired subadditivity.

Next we show that
\[
\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log f(\alpha, n, \epsilon)}{n} = \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon)}{n}.
\]

Note that both limit exist since \(f(\alpha, n, \epsilon)\) is an increasing function in the variable \(\epsilon\). Denote by \(\beta\) the left-hand side limit. Then for any \(\delta > 0\), there exists \(\epsilon_0 > 0\) such that \(\liminf_{n \to \infty} \log f(\alpha, n, \epsilon_0)/n < \beta + \delta\). Fix \(\delta > 0\) and \(\epsilon_0 > 0\) as above. To show the equality we only need to show that \(\limsup_{n \to \infty} \log f(\alpha, n, \epsilon_0)/n \leq \beta + \delta\). Fix \(n \in \mathbb{N}\). Take a sequence of integers \(n_k \to \infty\) such that \(f(\alpha, n_k, \epsilon_0) < e^{n_k(\beta + \delta)}\) for any \(n_k\) in \(\mathbb{N}\). For each \(k\), write \(n_k = (n + \bar{c})p_k - l_k\) with \(0 \leq l_k < n + \bar{c}\). By the subadditivity property, we have \(f(\alpha, n, \epsilon_0/4)p_k \leq \beta_n p_k f(\alpha, n + \bar{c})p_k, \epsilon_0/2)\). If \(w = w_1w_2\) is such that \(w_1 \in B_l(\Psi)\) and \(w \in B_{l+s}(\Psi)\) with \(1 \leq s \leq (n + \bar{c})\), then by Proposition \ref{prop:subadditivity}(4) we have
\[
|w_2| \leq \frac{\Psi_{\min} - ||\Psi|||w| - (n + \bar{c}) - 2C(\Psi)}{\Psi_{\max}}.
\]

Thus \(|w_2|/|w| \to 0 \) when \(|w| \to \infty\). Choose \(l_0\) large enough so that when \(l \geq l_0\) we have
\[
\frac{|w|}{|w_1|} \leq 3, \quad \frac{|C(\Phi)|}{|w_1|} \leq \frac{\epsilon_0}{8} \quad \text{and} \quad \frac{|w_2|}{|w_1|} \leq \frac{\epsilon_0}{8(||\Phi|| + |\alpha|)}.
\]

Let \(k_0\) such that \((n + \bar{c})p_{k_0} \geq l_0 + (n + \bar{c})\). Fix \(k \geq k_0\). Fix \(w \in F(\alpha, (n + \bar{c})p_k, \epsilon_0/2)\). There exists \(x \in \{w\}\) such that \(|\phi|w|(x) - |w|\alpha| \leq |w|\epsilon_0/2\). Let \(w_1 < w\) such that \(|\Psi|w_1| \leq e^{-n_k}\) and \(\Psi[w_1] > e^{-n_k}\). Thus \(w_1 \in B_{n_k}(\Psi)\). Write \(w = w_1 w_2\). By (4.6) we have
\[
|\phi|w_1|(x) - |w_1|\alpha| \leq |\phi|w_1|(x) - |w_1|\alpha| + |\phi|w_1|(x) - |w|\alpha| + |w_2|\alpha| \leq |\phi|w_1|(x) - |w|\alpha| + |w_2|(|\Phi| + |\alpha|) + |C(\Phi)| \leq \frac{|w|\epsilon_0}{2} + |w_2|(|\Phi| + |\alpha|) + |C(\Phi)| \leq \frac{3|w_1|\epsilon_0}{4} + \frac{|w_1|\epsilon_0}{8} + \frac{|w_1|\epsilon_0}{8} = |w_1|\epsilon_0,
\]

which means that \(w_1 \in F(\alpha, n_k, \epsilon_0)\). Moreover by (4.5), we have \(|w_2|/|w| \to 0 \) when \(|w| \to \infty\). Thus we can find a sequence \(\gamma_k\) such that \(|w_2| \leq \gamma_k = o(|w|) = o(n_k) = o(p_k)\).
We can conclude that
\[ f(\alpha, (n+\bar{c})p_k, \epsilon_0/2) \leq m^{\gamma_k} f(\alpha, n_k, \epsilon_0). \]
This yields
\[ f(\alpha, n, \epsilon_0/4) \leq \beta_n m^{\gamma_k/p_k} f(\alpha, n_k, \epsilon_0)^{1/p_k} \leq \beta_n m^{\gamma_k/p_k} e^{n_k(\beta+\delta)/p_k}. \]

Letting \( k \to \infty \) we get \( f(\alpha, n, \epsilon_0/4) \leq \beta_n e^{(n+\bar{c})(\beta+\delta)}. \) Then, letting \( n \to \infty \) we have
\[ \limsup_{n \to \infty} \log(f(\alpha, n, \epsilon_0/4))/n \leq \beta + \delta. \]

Next we show the upper semi-continuity of \( \Lambda(\alpha) \). Let \( \alpha \in L_\Phi \). For any \( \eta > 0 \) there is \( \epsilon > 0 \) such that \( \liminf_{n \to \infty} f(\alpha, n, \epsilon) < \Lambda(\alpha) + \eta \). Let \( \beta \in L_\Phi \) with \( |\beta - \alpha| < \epsilon/3 \).

Given \( w \in F(\beta, n, \epsilon/3) \), there exists \( x \in [w] \) such that \( |\phi|_w(x)/|w| - \beta| \leq \epsilon/3 \). Hence \( |\phi|_w(x)/|w| - \alpha| \leq |\phi|_w(x)/|w| - \beta| + |\beta - \alpha| < \epsilon \), which means \( w \in F(\alpha, n, \epsilon) \). This proves that \( F(\beta, n, \epsilon/3) \subset F(\alpha, n, \epsilon) \). It follows that \( f(\beta, n, \epsilon/3) \leq f(\alpha, n, \epsilon) \), therefore
\[ \Lambda(\beta) \leq \liminf_{n \to \infty} \frac{f(\beta, n, \epsilon/3)}{n} \leq \limsup_{n \to \infty} \frac{f(\alpha, n, \epsilon)}{n} < \Lambda(\alpha) + \eta. \]
This establishes the upper semi-continuity of \( \Lambda \) at \( \alpha \).

By essentially repeating the proof above (in fact it is much easier), we can show
\[ \liminf_{n \to \infty} \frac{\log \#B_n(\Psi)}{n} = \limsup_{n \to \infty} \frac{\log \#B_n(\Psi)}{n}. \]

We denote the limit by \( D(\Psi) \). By (4.2) there exist constants \( 0 < C_1 = C_1(\Psi) < C_2 = C_2(\Psi) \) such that for any \( w \in B_n(\Psi), C_1 n \leq |w| \leq C_2 n \). This yields \( \#\Sigma_{A_1|C_1n]} \leq \#B_n(\Psi) \leq \#\Sigma_{A_2[C_2n]} \) and \( C_1 \log m \leq D(\Psi) \leq C_2 \log m. \)

Now we come to the weak concavity of the function \( \Lambda_\Psi^\gamma \).

4.3. Proof of Proposition 2.5 Let \( A \subset \mathbb{R}^d \). We say that \( x \in A \) is a local cone point, or an \( \epsilon \)-cone point, if there exists \( \epsilon > 0 \) such that for any \( y \in A \cap B(x, \epsilon) \), the interval \([x, y_\epsilon] \subset A \), where \( y_\epsilon := x + \epsilon(y - x)/|y - x| \).

Lemma 4.3. Let \( A \subset \mathbb{R}^d \) be a convex set and \( h : A \to \mathbb{R} \) be a bounded weakly concave function. Then \( h \) is lower semi-continuous at each local cone point of \( A \). Especially \( h \) is lower semi-continuous on \( \text{ri}(A) \) and on any closed interval \( I \subset A \). It is lower semi-continuous on \( A \) if \( A \subset \mathbb{R}^d \) is a convex closed polyhedron.

Proof. Let \( \beta \in A \) be a \( \epsilon \)-cone point of \( A \) for some \( \epsilon > 0 \). Suppose that \( h \) is not lower semi-continuous at \( \beta \). Thus we can find \( \eta > 0 \) and \( \alpha_n \in A \cap B(\beta, \epsilon) \) such that \( \alpha_n \to \beta \) and \( h(\alpha_n) \leq h(\beta) - \eta \). Define \( \alpha_n' = \beta + \epsilon(\alpha_n - \beta)/|\alpha_n - \beta| \), then \( \alpha_n' \in A \) since \( \beta \) is a \( \epsilon \)-cone point. Let \( \lambda_n \in [0, 1] \) such that
\[ \alpha_n = \frac{\lambda_n \gamma_1(\alpha_n', \beta) \alpha_n' + (1 - \lambda_n) \gamma_2(\alpha_n', \beta) \beta}{\lambda_n \gamma_1(\alpha_n', \beta) + (1 - \lambda_n) \gamma_2(\alpha_n', \beta)}. \]

Since \( \gamma_1, \gamma_2 \in [c^{-1}, c] \) and \( \alpha_n \to \beta \) we conclude that \( \lambda_n \to 0 \). Since \( h \) is bounded, by (2.7) we get \( h(\alpha_n) \geq \lambda_n h(\alpha_n') + (1 - \lambda_n) h(\beta) \to h(\beta) \) (as \( n \to \infty \)), which is in contradiction with the choice of \( \alpha_n \). So \( h \) is lower semi-continuous at \( \beta \).

Since each \( x \in E \) is a local cone point of \( E \) when \( E \) is \( \text{ri}(A) \), or \( E \) is a closed interval in \( A \), or \( E \) is \( A \) itself and \( A \) is a convex closed polyhedron, the other results follow. \( \square \)
Proof of Proposition 2.5. At first we show that $\Lambda^\Psi_f$ is bounded and positive. Fix $\alpha \in L_\Psi$. By definition $\Lambda^\Psi_f(\alpha) \leq D(\Psi)$. On the other hand since $\alpha \in L_\Psi$, for any $\epsilon > 0$, when $n$ large enough, $F(\alpha, n, \epsilon) \neq \emptyset$. Consequently $\Lambda^\Psi_f(\alpha) \geq 0$. Thus $\Lambda^\Psi_f(L_\Psi) \subset [0, D(\Psi)]$.

Next we show that $\Lambda^\Psi_f$ is weakly concave. Let $\alpha, \beta \in L_\Psi$. For any $w_1, \cdots, w_p \in F(\alpha, n, \epsilon)$ and any $w_{p+1}, \cdots, w_{p+q} \in F(\beta, n, \epsilon)$, let $w = \overline{w}_1 \cdots \overline{w}_{p+q}$ where $\overline{w}_j = w_j u_j$ with $u_j \in \Xi$ such that $w$ is admissible. By the same argument as for Proposition 2.4 we can show that $\exp(-(p+q)(n+c_1(n))) \leq \Psi[w] \leq \exp(-(p+q)(n+c))$ with the same $c_1(n)$ and $c$ as in Proposition 2.4 which means that there exists $u < w$ such that $u \in B_{(p+q)(n+c)}(\Psi)$. Write $w = uw'$. We also have $|w'| \leq (p+q)(ac_1(n) + b)$ with the same $(a, b)$ as in that proposition.

Now define $F_k(\alpha, n, \epsilon) = \Sigma_{A,k} \cap F(\alpha, n, \epsilon)$. We have $F(\alpha, n, \epsilon) = \bigcup_{C_1 \leq k \leq C_2} F_k(\alpha, n, \epsilon)$, where $C_i = C_i(\Psi)$ for $i = 1, 2$. Define $f_k(\alpha, n, \epsilon) = \#F_k(\alpha, n, \epsilon)$. Choose $k_0$ such that $f_{k_0}(\alpha, n, \epsilon) = \max_{C_1 \leq k \leq C_2} F_k(\alpha, n, \epsilon)$. Then $f_{k_0}(\alpha, n, \epsilon) \geq f(\alpha, n, \epsilon)/(C_2 - C_1)n$. Write $k_0 = \gamma_n(\alpha)$, thus $\gamma_n(\alpha) \in [C_1, C_2]$. Likewise we can find $\gamma_n(\beta) \in [C_1, C_2]$ such that $f_{\gamma_n(\beta)n}(\beta, n, \epsilon) \geq f(\beta, n, \epsilon)/(C_2 - C_1)n$.

Fix a subsequence $n_k \uparrow \infty$ such that $\gamma_n(\alpha) \rightarrow \gamma(\alpha)$ and $\gamma_n(\beta) \rightarrow \gamma(\beta)$ as $k \rightarrow \infty$. Take $w_1, \cdots, w_p \in F_{\gamma_n(\alpha)n}(\alpha, n, \epsilon)$ and $w_{p+1}, \cdots, w_{p+q} \in F_{\gamma_n(\beta)n}(\beta, n, \epsilon)$. Choose $x_j \in [w_j]$ such that

\[
\begin{align*}
|\phi_{[w_j]}(x_j) - |w_j|\alpha| & \leq |w_j|\epsilon, \quad \text{if } 1 \leq j \leq p, \\
|\phi_{[w_j]}(x_j) - |w_j|\beta| & \leq |w_j|\epsilon, \quad \text{if } p+1 \leq j \leq p+q.
\end{align*}
\]

Let $w = \overline{w}_1 \cdots \overline{w}_{p+q}$ and write $w = uw'$ such that $u \in B_{(p+q)(n_k+c)}(\Psi)$. Then we know that $|w| = p(\gamma_n(\alpha)n_k + p_0) + q(\gamma_n(\beta)n_k + p_0)$ and $|u| = |w| - |w'|$. Now for any $x \in [w]$, define $x^1 = x$ and $x^j = T\Sigma_{i=1}^{j-1} |w_i|/p_0 x$ for $j \geq 2$. Then we have

\[
\phi_{[w]}(x) = \phi_{[w]}(x) + \eta_0 = \sum_{j=1}^{p+q} \phi_{[w_j]}(x^j) + \eta_1 + \eta_0 = \sum_{j=1}^{p+q} \phi_{[w_j]}(x_j) + \eta_2 + \eta_1 + \eta_0
\]

\[
= p\gamma_n(\alpha)n_k\alpha + q\gamma_n(\beta)n_k\beta + \eta_3 + \eta_2 + \eta_1 + \eta_0
\]

\[
= p\gamma(\alpha)n_k\alpha + q\gamma(\beta)n_k\beta + \eta_4 + \eta_3 + \eta_2 + \eta_1 + \eta_0,
\]

where $|\eta_0| \leq |w'|\|\Phi\| + |C(\Phi)| \leq (p+q)(ac_1(n_k) + b)|\Phi| + |C(\Phi)|, |\eta_1| \leq (p+q)(p_0)|\Phi| + 2C(\Phi)|; |\eta_2| \leq (p+q)|\Phi||n_k|; |\eta_3| \leq n_k(p\gamma_n(\alpha) + q\gamma_n(\beta))\epsilon$, and $|\eta_4| \leq pm n_k|\alpha|\gamma_n(\alpha) - \gamma(\alpha)| + qn_k|\beta|\gamma_n(\beta) - \gamma(\beta)|$.

This yields that for $k$ large enough, $u \in F((p\gamma(\alpha) + q\gamma(\beta)\beta)/(p\gamma(\alpha) + q\gamma(\beta)), (n_k + c)(p + q), 2\epsilon)$. Thus we conclude that

\[
f\left(\frac{p\gamma(\alpha) + q\gamma(\beta)\beta}{p\gamma(\alpha) + q\gamma(\beta)}\right)(n_k + c)(p + q), 2\epsilon
\]

\[
\geq f_{\gamma_n(\alpha)n_k}\alpha, n_k, \epsilon \cdot f_{\gamma_n(\beta)n_k}(\beta, n_k, \epsilon) p^{(p+q)(ac_1(n_k)+b)}
\]

\[
\geq f(\alpha, n_k, \epsilon) p f(\beta, n_k, \epsilon) q[(C_2 - C_1)n_k]^{p-q} m^{-(p+q)(ac_1(n_k)+b)}.
\]
Combining this with Proposition 2.4, we get
\[
\lambda \Lambda_\phi^\Psi(\alpha) + (1 - \lambda) \Lambda_\phi^\Psi(\beta) \leq \Lambda_\phi^\Psi \left( \frac{\lambda \gamma(\alpha) + (1 - \lambda) \gamma(\beta) \beta}{\lambda \gamma(\alpha) + (1 - \lambda) \gamma(\beta)} \right)
\]
for any \( \lambda = \frac{p}{m+q} \in [0, 1] \cap \mathbb{Q} \). Since \( \Lambda_\phi^\Psi \) is upper semi-continuous, we conclude that this formula holds for any \( \lambda \in [0, 1] \). Thus \( \Lambda_\phi^\Psi \) is weakly concave.

Assume \( A \subset L_\phi \) is a convex set, and \( I \subset L_\phi \) is a closed interval. By Lemma 4.3, \( \Lambda_\phi^\Psi \) is lower semi-continuous on \( \mathrm{ri}(A) \) and \( I \). Combining this with the upper semi-continuity yields the continuity on \( \mathrm{ri}(A) \) and \( I \). Taking \( A = L_\phi \) we get the continuity on \( \mathrm{ri}(L_\phi) \).

Now assume \( L_\phi \) is a polyhedron. By Lemma 4.3 \( \Lambda_\phi^\Psi \) is lower semi-continuous on \( L_\phi \), this, together with the upper semi-continuity yields the continuity on \( L_\phi \).

Let \( I = [\alpha_1, \alpha_2] \subset L_\phi \) and \( \alpha_{\text{max}} \in I \) as defined in the proposition. Assume \( \Lambda_\phi^\Psi \) is not decreasing from \( \alpha_{\text{max}} \) to \( \alpha_1 \). Since \( \Lambda_\phi^\Psi \) is continuous on \( I \), we can find \( \beta_1, \beta_2, \beta_3 \in [\alpha_1, \alpha_{\text{max}}] \) such that \( \beta_2 \in [\beta_1, \beta_3] \) and \( \Lambda_\phi^\Psi(\beta_1) = \Lambda_\phi^\Psi(\beta_3) > \Lambda_\phi^\Psi(\beta_2) \), which is in contradiction with the fact that \( \Lambda_\phi^\Psi \) is quasi-concave, since it is weakly concave. Thus \( \Lambda_\phi^\Psi \) is decreasing from \( \alpha_{\text{max}} \) to \( \alpha_1 \). The same proof shows that \( \Lambda_\phi^\Psi \) is decreasing from \( \alpha_{\text{max}} \) to \( \alpha_2 \). \( \square \)

4.4. Proof of Proposition 2.6. We begin with a lemma about the Lipschitz continuity of the pressure functions.

**Lemma 4.4.** Let \( \Phi, \Psi \in C_{\alpha_0}(\Sigma_A, T) \) and define \( f(\lambda) = P(\Phi + \lambda \Psi) \). If \( \lambda_1 < \lambda_2 \) we have \( \Psi_{\text{min}}(\lambda_2 - \lambda_1) \leq f(\lambda_2) - f(\lambda_1) \leq \Psi_{\text{max}}(\lambda_2 - \lambda_1) \).

**Proof.** Let \( w \in \Sigma_{A,n} \), and pick up \( x_1, x_2 \in [w] \) such that \( (\phi_n + \lambda_j \psi_n)(x_j) = \sup_{x \in [w]} (\phi_n + \lambda_j \psi_n)(x) \). Then we have
\[
(\phi_n + \lambda_2 \psi_n)(x_2) \leq (\phi_n + \lambda_2 \psi_n)(x_1) + \|\Phi + \lambda_2 \Psi\|_n
= (\phi_n + \lambda_1 \psi_n)(x_1) + (\lambda_2 - \lambda_1) \psi_n(x_1) + \|\Phi + \lambda_2 \Psi\|_n
\leq (\phi_n + \lambda_1 \psi_n)(x_1) + (\lambda_2 - \lambda_1) \Psi_{\text{max}} + \|\Phi + \lambda_2 \Psi\|_n;
\]
since \( \|\Phi + \lambda_2 \Psi\|_n / n \to 0 \), this yields \( f(\lambda_2) - f(\lambda_1) \leq \Psi_{\text{max}}(\lambda_2 - \lambda_1) \). The other inequality can be proved similarly. \( \square \)

**Proof of Proposition 2.6.** Define \( f(\lambda) = P(\langle z, \Phi - \alpha \rangle + \lambda \Psi) \). By Lemma 4.4 for \( \lambda_1 < \lambda_2 \) we have \( \Psi_{\text{min}}(\lambda_2 - \lambda_1) \leq f(\lambda_2) - f(\lambda_1) \leq \Psi_{\text{max}}(\lambda_2 - \lambda_1) \), where \( \Psi_{\text{min}} \leq \Psi_{\text{max}} < 0 \). Thus \( f(\lambda) = 0 \) has a unique solution, which is \( \tau_\phi^\Psi(z, \alpha) \). By Theorem 2.1 we have
\[
0 = P(\langle z, \Phi - \alpha \rangle + \tau_\phi^\Psi(z, \alpha) \Psi) = \sup \{ h_\mu + (\langle z, \Phi - \alpha \rangle + \tau_\phi^\Psi(z, \alpha) \Psi)_\mu(\mu) : \mu \in \mathcal{M}(\Sigma_A, T) \}.
\]
From this (2.9) follows easily.

Now we show (2.10). By considering the potential \( \Phi' = \Phi - \alpha \), we can restrict ourselves to the case \( \alpha = 0 \). At first we assume \( \Phi = (S_n \phi)_{n=1}^\infty \) and \( \Psi = (S_n \psi)_{n=1}^\infty \) with \( \phi \) and \( \psi \) Hölder continuous. Since \( \psi \) is Hölder continuous, for any \( w \in B_n(\Psi) \) and any \( x \in [w] \) we have \( S_{|w|} \psi(x) \approx -n \). Let \( \mu \) be the unique equilibrium state of \( \langle z, \phi \rangle + \tau(z, 0) \psi \). For any \( w \in B_n(\Psi) \), by the Gibbs property we have
\[
\mu([w]) \approx \exp \left( \langle z, S_{|w|} \phi(x) \rangle + \tau(z, 0) S_{|w|} \psi(x) \right) \quad (\forall x \in [w])
\]
\[ \approx \exp \left( \langle z, S_{|w|}\phi(x) \rangle - \tau(z, 0)n \right) \quad (\forall x \in [w]) \]
\[ \approx \exp \left( \sup_{x \in [w]} \langle z, S_{|w|}\phi(x) \rangle - \tau(z, 0)n \right). \]

From this and \( \sum_{w \in B_n(z)} \mu([w]) = 1 \), (2.10) follows. The general case requires an approximation argument. We postpone its proof to the end of Section 8. \( \square \)

5. Proof of Theorem 2.2

Our plan is the following: at first we show that \( D_\Phi^\Psi(\alpha) \leq \Lambda_\Phi^\Psi(\alpha) \leq E_\Phi^\Psi(\alpha) \leq D_\Phi^\Psi(\alpha) \), then we show \( \Lambda_\Phi^\Psi(\alpha) = \tau_\Phi^\Psi(\alpha) \). We divide this into four steps:

5.1. \( D_\Phi^\Psi(\alpha) \leq \Lambda_\Phi^\Psi(\alpha) \). We prove a slightly more general result for the upper bound. Given \( \Phi \in C_{aa}(\Sigma_A, T, d) \) and \( \Omega \subset L_\Phi \), define \( E_\Phi(\Omega) := \bigcup_{\alpha \in \Omega} E_\Phi(\alpha) \).

**Proposition 5.1.** For any compact set \( \Omega \subset L_\Phi \) we have \( \dim_\Psi^\Psi E_\Phi(\Omega) \leq \sup \{ \Lambda_\Phi^\Psi(\alpha) : \alpha \in \Omega \} \). In particular, if \( \alpha \in L_\Phi \) we have \( D_\Phi^\Psi(\alpha) \leq \dim_\Psi^\Psi E_\Phi(\alpha) \leq \Lambda_\Phi^\Psi(\alpha) \).

**Proof.** Let \( \Lambda_\Phi^\Psi(\alpha, \epsilon) := \lim_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} \), then \( \Lambda_\Phi^\Psi(\alpha, \epsilon) < \Lambda_\Phi^\Psi(\alpha) \) when \( \epsilon < 0 \).

Fix \( \eta > 0 \), for each \( \alpha \in \Omega \), there exists \( \epsilon_\alpha > 0 \) such that for any \( 0 < \epsilon \leq \epsilon_\alpha \) we have \( \Lambda_\Phi^\Psi(\alpha, \epsilon) < \Lambda_\Phi^\Psi(\alpha) + \eta \).

Since \( \{B(\alpha, \epsilon_\alpha) : \alpha \in \Omega \} \) is an open covering of \( \Omega \), we can find a finite covering \( \{B(\alpha_1, \epsilon_1), \cdots, B(\alpha_s, \epsilon_s)\} \), where \( \epsilon_j = \epsilon_{\alpha_j} \). For each \( n \in \mathbb{N} \) define
\[
H(n, \eta) := \bigcup_{j=1}^{s} \bigcup_{w \in F(\alpha_j, n, \epsilon)} [w] \quad \text{and} \quad G(k, \eta) := \bigcap_{n \geq k} H(n, \eta)
\]

**Claim:** \( E_\Phi(\Omega) \subset \bigcup_{k \in \mathbb{N}} G(k, \eta) \).

Indeed, for any \( x \in E_\Phi(\Omega) \), there exists \( \alpha \in \Omega \) such that \( \phi_n(x)/n \to \alpha \). There exists \( j \in \{1, \cdots, s\} \) such that \( \alpha \in B(\alpha_j, \epsilon_j) \). Take \( N \) large enough so that \(|\phi_n(x)/n - \alpha| < \epsilon_j - |\alpha - \alpha_j|\) for any \( n \geq N \). For such an \( n \) we have \(|\phi_n(x)/n - \alpha| < |\phi_n(x)/n - \alpha| + |\alpha - \alpha_j| < \epsilon_j|\), hence \( x \in H(n, \eta) \) for all \( n \) large enough.

By the previous claim we have
\[
(5.1) \quad \dim_\Psi^\Psi E_\Phi(\Omega) \leq \sup_{k \in \mathbb{N}} \dim_\Psi^\Psi G(k, \eta).
\]

Now we find the desired upper bound for the packing dimension of \( G(k, \eta) \). By definition it is covered by \( \{[w] : w \in F(\alpha_j, n, \epsilon) ; j = 1, \cdots, s\} \) for any \( n \geq k \). Since each element in \( \{[w] : w \in F(\alpha_j, n, \epsilon)\} \) is a ball with radius \( \epsilon^{-n} \), we conclude that
\[
\dim_\Psi^\Psi G(k, \eta) \leq \dim_\Psi^\Psi B(k, \eta) \leq \limsup_{n \to \infty} \frac{\log \sum_{j=1}^{s} f(\alpha_j, n, \epsilon_j)}{n}
\]
\[
\leq \sup_{j=1, \cdots, s} \limsup_{n \to \infty} \frac{\log f(\alpha_j, n, \epsilon_j)}{n} = \sup_{j=1, \cdots, s} \Lambda_\Phi^\Psi(\alpha_j, \epsilon_j)
\]
Thus \[ \log \Phi_k(\alpha_j) + \eta \leq \sup \{ \Lambda^\Psi_k(\alpha) : \alpha \in \Omega \} + \eta. \]

Combining this with (5.1) we get \( \dim^\Psi \text{E}_\Phi(\Omega) \leq \sup \{ \Lambda^\Psi_k(\alpha) : \alpha \in \Omega \} + \eta. \) Since \( \eta \) is arbitrary, we get the result. \( \square \)

5.2. \( \Lambda^\Psi_k(\alpha) \leq E^\Psi_k(\alpha). \) Our approach is inspired by that of [14], which deals with the case that \( \Psi \) is additive and built from a constant negative potential.

To show this inequality we need to approximate the almost additive potentials \( \Phi \) and \( \Psi \) by two sequences of Hölder potentials. We describe this procedure as follows.

Given \( \Phi \in C_\sigma(\Sigma_A, T, d) \), for each \( k \in \mathbb{N} \) we define \( \Phi^k \in C_\sigma(\Sigma_A, T, d) \) as follows. For each \( w \in \Sigma_A \) choose \( x_w \in [w] \). For any \( x \in [w] \) define \( \phi_k(x) := \phi_k(x_w)/k \). Define \( \phi_n^k := S_n \phi_k \) and \( \Phi^k := \phi_k^\infty \).

Thus \( \Phi^k \) is additive and \( \phi_k \) depends only on the first \( k \) coordinates of \( x \in \Sigma_A \). Consequently \( \Phi^k \) is Hölder continuous.

**Lemma 5.1.** We have \( \Phi_{\min} \leq \phi_n^k \leq \Phi_{\max} \leq \Phi_{\max}. \) Moreover \( \| \phi_n - \phi_n^k \| \leq 2 \cdot |\Phi| + 4k \| \Phi \| + \sqrt{\alpha} \| | \Phi \| / k \). Consequently \( \| \Phi - \Phi^k \|_{\text{lim}} \rightarrow 0 \) when \( k \rightarrow \infty \).

This lemma will be proved at the end of this subsection.

**Proof of** \( \Lambda^\Psi_k(\alpha) \leq E^\Psi_k(\alpha). \) Now, for \( \Phi \in C_\sigma(\Sigma_A, T, d) \) and \( \Psi \in C_\sigma^{-}(\Sigma_A, T) \) define \( \Phi^k \) and \( \Psi^k \) according to (5.2). Fix \( \epsilon > 0 \). By Lemma 5.1 we can find \( K(\epsilon) \) such that for each \( k \geq K(\epsilon) \) and sufficiently large \( n \) (related to \( k \)) we have \( \| \phi_n - \phi_n^k \| \leq n\epsilon/2 \) and \( \| \psi_n - \psi_n^k \| \leq ne/2 \). Then \( F(\alpha, n, \epsilon/2, \Phi, \Psi) \subset F(\alpha, n, \epsilon, \Phi^k, \Psi) \), and consequently \( f(\alpha, n, \epsilon/2, \Phi, \Psi) \leq f(\alpha, n, \epsilon, \Phi^k, \Psi). \)

For any word \( w \) such that \( |w| \geq k \), we define the integer valued function \( \theta_w : \Sigma_{A,k} \rightarrow \mathbb{N} \) as \( \theta_w(u) = \#\{j : w_j \cdots w_{j+k-1} = u\} \). It is clear that \( (5.3) \)

\[ \sum u \theta_w(u) = |w| - k + 1. \]

Let \( P_\nu^{(n)} = \{ u : w = w_1w_2, \text{admissible}, w_1 \in F(\alpha, n, \epsilon, \Phi^k, \Psi), |w_2| = k - 1 \} \). Since \( w_1 \in B_\epsilon(\Psi) \) we have \( |w_1| \leq C_2(\Psi)n \), thus \( |P_\nu^{(n)}| \leq (C_2n)^{m_k} \). For each \( \theta \in P_\nu^{(n)} \), let \( T(\theta) \) be the collection of all \( w_1w_2 \) such that \( w_1 \in F(\alpha, n, \epsilon, \Phi^k, \Psi), |w_2| = k - 1 \) and \( \theta_{w_1w_2} = \theta \). Then we have \( f(\alpha, n, \epsilon, \Phi^k, \Psi) \leq \sum \#T(\theta) \leq (C_2n)^{m_k} \max \#T(\theta) \).

Thus \( \log f(\alpha, n, \epsilon/2, \Phi, \Psi) / n \leq \log f(\alpha, n, \epsilon, \Phi^k, \Psi) / n \leq \max_{\theta \in P_\nu^{(n)}} \log \#T(\theta) / n + m_k O(\log n / n). \)

Following [14] we define \( \Delta^+_k \), the set of all positive functions \( p \) on \( \Sigma_{A,k} \) satisfying the following two relations:

\[ \sum_{w \in \Sigma_{A,k}} p(w) = 1; \quad \sum_{w} p(w_1w_2 \cdots w_{k-1}) = \sum_{w} p(w_1w_2 \cdots w_{k-1}). \]
It is known (see [14]) that for any \( \eta > 0 \), there is a positive integer \( N = N(\eta) \) such that for any \( w \in \Sigma_{A,l+k-1} \) with \( l > N \), there exists a probability vector \( p \in \Delta_k^+ \) such that

\[
\left| \frac{\theta_w(u)}{l} - p(u) \right| < \eta, \quad p(u) > \frac{\eta}{m^{k+1}}.
\]

We discard the trivial case where \( \Phi \equiv 0 \) and fix \( \eta > 0 \) such that \( \eta < \epsilon/(m^k \| \Phi \|) \).

Now we fix \( \theta \in \mathcal{P}_k^{(n)} \) with \( n \) large enough, and write \( \theta = \theta_{ww'} \) with \( w \in F(\alpha, n, \epsilon, \Phi^k, \Psi) \) and \( |w'| = k - 1 \). Notice that any word \( v \in \mathcal{T}(\theta) \) can be written as \( v_1 v_2 \) with \( |v_2| = k - 1 \) and \( |v_1| \) also equal to a constant (this is due to (5.3)) that we denote by \( l_\theta \). Fix a \( p \in \Delta_k^+ \) as described above. Consider the Markov measure \( \nu_p \) corresponding to \( p \) (see [14] for the definition and related properties). For any word \( v = v_1 v_2 \in \mathcal{T}(\theta) \) with \( |v_1| = l_\theta \) (and \( v_1 \in F(\alpha, n, \epsilon, \Phi^k, \Psi) \) and \( |v_2| = k - 1 \), we have

\[
\nu_p([v_1 v_2]) = p(v_1 v_2|k) \prod_{|u| = k} t(u)^{\theta(u)} \geq \frac{\eta}{m^{k+1}} \prod_{|u| = k} t(u)^{\theta(u)} := \rho,
\]

where \( t(a_1 \cdots a_k) = \frac{p(a_1 \cdots a_k)}{\sum p(a_1 \cdots a_{k-1} \epsilon)} \). Also \( \rho \# \mathcal{T}(\theta) \leq \nu_p(\bigcup_{v \in \mathcal{T}(\theta)} [v]) \leq 1 \). Thus,

\[
\# \mathcal{T}(\theta) \leq \frac{1}{\rho} = \frac{m^{k+1}}{\eta} \prod_{|u| = k} t(u)^{-\theta(u)}.
\]

Since \( C_1(\Psi)n \leq l_\theta \leq C_2(\Psi)n \) and \( \eta/m^{k+1} \leq t(u) \leq 1 \), we have

\[
\frac{\log \# \mathcal{T}(\theta)}{l_\theta} \leq O\left(\frac{k}{n}\right) + O\left(\frac{\log \eta}{n}\right) - \sum_{|u| = k} \frac{\theta(u)}{l_\theta} \log t(u)
\]

\[
\leq O\left(\frac{k}{n}\right) + O\left(\frac{\log \eta}{n}\right) - \sum_{|u| = k} p(u) \log t(u) + m^k \eta (| \log \eta | + (k + 1) \log m)
\]

\[
= h(\nu_p) + O\left(\frac{k}{n}\right) + O\left(\frac{\log \eta}{n}\right) + m^k \eta (| \log \eta | + (k + 1) \log m).
\]

Next we estimate \( n/l_\theta \). Let \( x_0 \in [w w'] \). By [13] we have

\[-n - C(\Psi) - 2 \| \Psi \|_\theta + \Psi_{\min} \leq \psi_{\theta}(x_0) \leq \sup_{x \in [w]} \psi_{\theta}(x) \leq -n,
\]

and by Lemma 5.1 when \( k \) and \( n \) are large enough we have \( \| \psi_{\theta} - \psi_{\theta}^k \| \leq l_\theta \epsilon \), thus

\[
- n - C(\Psi) - 2 \| \Psi \|_n^* + \Psi_{\min} - C_2 n \epsilon \leq \psi_{\theta}^k(x_0) \leq -n + C_2 n \epsilon.
\]

Also,

\[
\frac{\psi_{\theta}^k(x_0)}{l_\theta} = \sum_{|u| = k} \frac{\theta(u)}{l_\theta} \tilde{\psi}_k(x_u) = \sum_{|u| = k} p(u) \tilde{\psi}_k(x_u) + m^k O(\eta)
\]

\[
= \int \tilde{\psi}_k d\nu_p + m^k O(\eta) = \Psi_k(\nu_p) + m^k O(\eta) = \Psi(\nu_p) + O(\epsilon) + m^k O(\eta).
\]

Combining this with (5.4) and the fact that \( \| \Psi \|_n^*/n = o(1) \) we get

\[
\frac{n}{l_\theta} = - \Psi(\nu_p) + O(\epsilon) + m^k O(\eta) + o(1).
\]
As a result we get
\[
\frac{\log \# T(\theta)}{n} = \log \# T(\theta) \cdot \frac{l_\theta}{n} \leq \frac{h(\nu_p) + O(\frac{1}{n}) + O(\frac{\log n!}{n}) + m^k \eta (| \log \eta | + (k + 1) \log m)}{-\Psi_*(\nu_p) + O(\epsilon) + m^k O(\eta) + o(1)}.
\]

Since \( w \in F(\alpha, n, \epsilon, \Phi^k, \Psi) \), there exists \( y_0 \in [w] \) such that \( |\phi^k_{\nu_p}(y_0)/l_\theta - \alpha| \leq \epsilon \). We have
\[
|\Phi_*(\nu_p) - \alpha| \leq |\Phi^k_*(\nu_p) - \alpha| \leq |\int \tilde{\phi}_k d\nu_p - \alpha| + \epsilon = \sum_{|u|=k} p(u) \tilde{\phi}_k(x_u) - \alpha| + \epsilon
\]
\[
\leq |\phi^k_{\nu_p}(x)/\theta - \alpha| + m^k \eta \|\Phi\| + \epsilon \quad \text{(for any } x \in [w^\prime])
\]
\[
\leq |\phi^k_{\nu_p}(x)/\theta - \phi^k_{\nu_p}(y_0)/\theta| + |\phi^k_{\nu_p}(y_0)/\theta - \alpha| + m^k \eta \|\Phi\| + \epsilon \leq \frac{k \|\Phi\|}{C_1 n} + m^k \eta \|\Phi\| + 2 \epsilon.
\]

By our choice of \( \eta \) we have \( m^k \eta \|\Phi\| < \epsilon \). Moreover, when \( n \geq k \|\Phi\|/(C_1 \epsilon) \) we have \( k \|\Phi\|/(C_1 n) \leq \epsilon \). Letting \( n \to \infty \) and then \( \eta \to 0 \) we conclude that
\[
\limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon/2)}{n} \leq \sup_{|\Phi_*(\nu) - \alpha| \leq 4 \epsilon} \frac{h(\nu)}{-\Psi_*(\nu) + O(\epsilon)}.
\]

Notice that the set of invariant measures \( \nu \) such that \( |\Phi_*(\nu) - \alpha| \leq 4 \epsilon \) is compact, so by using the upper semi-continuity of \( h(\nu) \) and letting \( \epsilon \) tend to 0 we can find an invariant measure \( \nu_0 \) such that \( \Phi_*(\nu_0) = \alpha \) and
\[
A^\Psi_\Phi(\alpha) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon/2)}{n} \leq \frac{h(\nu_0)}{-\Psi_*(\nu_0)} \leq c^\Psi_\Phi(\alpha).
\]

**Proof of Lemma 5.1** At first we assume \( \Phi \in C_{aa}(\Sigma_A, T) \). By (21) we get \( \Phi_{\min} \leq \tilde{\phi}_k \leq \Phi_{\max} \). Since \( \Phi^k \) is additive, we have \( \Phi_{\min} \leq \tilde{\phi}_{k_{\min}} = \Phi_{k_{\min}} \leq \Phi_{k_{\max}} = \Phi_{k_{\max}} \leq \Phi_{\max} \).

For \( n \in \mathbb{N} \), write \( n = pk + l \) with \( 0 \leq l < k \). For \( 1 \leq j \leq k \) we have
\[
\phi_n(x) \leq \phi_j(x) + \sum_{l=0}^{p-2} \phi_k(T^{j+l}x) + pC(\Phi) + \phi_{k+l-j}(T^{j+(p-1)k})
\]
\[
\leq \sum_{l=0}^{p-2} \phi_k(T^{j+l}x) + pC(\Phi) + 2k \|\Phi\|.
\]

So we get
\[
\phi_n(x) \leq \sum_{j=1}^{k} \sum_{l=0}^{p-2} \frac{\phi_k(T^{j+l}x)}{k} + pC(\Phi) + 2k \|\Phi\| = \sum_{j=1}^{n-1} \phi_k(T^jx)/k + pC(\Phi) + 2k \|\Phi\| \leq \sum_{j=0}^{n-1} \phi_k(T^jx)/k + pC(\Phi) + 4k \|\Phi\|.
\]

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Similarly we have \( \phi_n(x) \geq \phi_n^k(x) - pC(\Phi) - 4k\|\Phi\| - \frac{\|\Phi\|}{k}n \), hence \( \|\phi_n - \phi_n^k\|_{\infty} \leq pC(\Phi) + 4k\|\Phi\| + \frac{\|\Phi\|}{k}n \), and \( \|\Phi - \Phi^k\|_{\text{lim}} \leq \frac{C(\Phi)}{k} + \frac{\|\Phi\|}{k} \to 0 \) (as \( k \to \infty \)).

If \( \Phi = (\Phi(1), \ldots, \Phi(d)) \in C_{aa}(\Sigma_A, T, d) \), applying the result just proven to each component of \( \Phi \) and using (2.4) we get the result. \( \square \)

5.3. \( \mathcal{E}_\Phi^\Psi(\alpha) \leq D_\Phi^\Psi(\alpha) \). It is the content of Proposition 6.2.

Until now we have shown that \( D_\Phi(\alpha) = \mathcal{E}_\Phi(\alpha) = \Lambda_\Phi^\Psi(\alpha) \).

5.4. The large deviation principle \( \Lambda_\Phi^\Psi(\alpha) = \tau_\Phi^\Psi(\alpha) \), and \( \dim_H(\Sigma_A) = \dim_B(\Sigma_A) = \text{dim}(\Psi) = \max\{\Lambda_\Phi^\Psi(\alpha) : \alpha \in L_\Phi\} \).

At first we show the following simple fact:

**Lemma 5.2.** For any \( \alpha \in L_\Phi \) we have \( \Lambda_\Phi^\Psi(\alpha) \leq \tau_\Phi^\Psi(\alpha) \).

**Proof.** Fix \( \alpha \in L_\Phi \). By the variational principle, for any \( \mu \in \mathcal{M}(\Sigma_A, T) \)

\[
0 = P((z, \Phi - \alpha) + \tau_\Phi^\Psi(z, \alpha)\Psi) \geq h_\mu(T) + \langle z, \Phi_\alpha(\mu) - \alpha \rangle + \tau_\Phi^\Psi(z, \alpha)\Psi(\mu).
\]

Thus if \( \Phi_\alpha(\mu) = \alpha \), we get \( \tau_\Phi^\Psi(z, \alpha) \geq -h_\mu(T)/\Psi_\alpha(\mu) \). This implies that \( \tau_\Phi^\Psi(\alpha) = \inf\{\tau_\Phi^\Psi(z, \alpha) : z \in \mathbb{R}^d\} \geq \mathcal{E}_\Phi^\Psi(\alpha) \). Since \( \mathcal{E}_\Phi^\Psi(\alpha) = \Lambda_\Phi^\Psi(\alpha) \), the result follows. \( \square \)

Next we show \( \Lambda_\Phi^\Psi(\alpha) = \tau_\Phi^\Psi(\alpha) \). We do this at first for H"older potentials, then we deal with the general case by using an approximation procedure.

5.4.1. \( \Lambda_\Phi^\Psi = \tau_\Phi^\Psi \) when \( \Phi \) and \( \Psi \) are H"older potentials and \( L_\Phi \) has dimension \( d \).

**Lemma 5.3.** Assume \( \Phi \) and \( \Psi \) are H"older continuous potentials and \( L_\Phi \) has dimension \( d \). Then

1. \( \tau_\Phi^\Psi(0, \alpha) = D(\Psi) \), consequently \( \tau_\Phi^\Psi(\alpha) \leq D(\Psi) \) for any \( \alpha \in \mathbb{R}^d \). Moreover \( D(\Psi) \)

2. Let \( (z, \alpha, \alpha') \in (\mathbb{R}^d)^3 \). If \( (z, \alpha' - \alpha) \geq 0 \), then

\[
C_1(\Psi)(z, \alpha' - \alpha) \leq \tau_\Phi^\Psi(z, \alpha) - \tau_\Phi^\Psi(z, \alpha') \leq C_2(\Psi)(z, \alpha' - \alpha).
\]

3. Let \( \alpha \in \mathbb{R}^d \). Then \( \tau_\Phi^\Psi(\cdot, \alpha) \) is convex.

4. If \( \alpha \in \text{int}(L_\Phi) \) and \( \delta_0 > 0 \) is such that \( B(\alpha, \delta_0) \subset L_\Phi \), then for any \( z \in \mathbb{R}^d \),

\[
\tau_\Phi^\Psi(z, \alpha) \geq \delta_0 C_1|z|/2, \text{ where } C_1 = C_1(\Psi) = 1/|\Psi_{\text{min}}|.
\]
Proof. (1) By (2.5) and (2.10) we get \( \tau^\Psi_\Phi(0, \alpha) = D(\Psi) \). By definition \( D(\Psi) = \tau^\Psi_\Phi(0, \alpha) \) is the unique root of \( P(\lambda \Psi) = 0 \).

(2) By Proposition 2.6 we have

\[
\tau^\Psi_\Phi(z, \alpha) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \mathcal{B}_n(\Psi)} \exp(\sup_{x \in [w]} \langle z, \phi_{|w|}(x) - |w|\alpha \rangle).
\]

For a fixed \( w \in \mathcal{B}_n(\Psi) \) and any \( x, y \in [w] \), since \( \Phi \) is H"older continuous and \( \langle z, \alpha' - \alpha \rangle \geq 0 \), we get

\[
\langle z, \phi_{|w|}(x) - |w|\alpha \rangle = \langle z, \phi_{|w|}(x) - |w|\alpha' + |w|\alpha \rangle = \langle z, \phi_{|w|}(y) - |w|\alpha' \rangle + C_2(\Psi)n(z, \alpha' - \alpha) + C(\Phi, z).
\]

Thus \( \max_{x \in [w]} \langle z, \phi_{|w|}(x) - |w|\alpha \rangle \leq \max_{y \in [w]} \langle z, \phi_{|w|}(y) - |w|\alpha' \rangle + C_2(\Psi)n(z, \alpha' - \alpha) + C(\Phi, z) \). From this we get \( \tau^\Psi_\Phi(z, \alpha) - \tau^\Psi_\Phi(z, \alpha') \leq C_2(\Psi)\langle z, \alpha' - \alpha \rangle \). The other inequality follows similarly.

(3) It is a classical result, see for example Lemma 6 in [30].

(4) Assume \( \alpha \in \text{int}(L_\Phi) \) and \( \delta_0 > 0 \) such that \( B(\alpha, \delta_0) \subset L_\Phi \). For any \( z \in \mathbb{R}^d \), let \( \alpha' = \alpha + \delta_0 z/|z| \). We have

\[
\sum_{w \in \mathcal{B}_n(\Psi)} \exp(\sup_{x \in [w]} \langle z, (\phi - \alpha)|w|(x) \rangle) \geq \sum_{w \in \mathcal{B}_n(\alpha', \delta_0/2, \Phi, \Psi)} \exp(\sup_{x \in [w]} \langle z, (\phi - \alpha)|w|(x) \rangle) \\
\geq \sum_{w \in \mathcal{B}_n(\alpha', \delta_0/2, \Phi, \Psi)} C \exp(\langle |w||z|\delta_0/2 \rangle) \geq C \exp(C_1 n |z|\delta_0/2) f(\alpha', n, \delta_0/2, \Phi, \Psi).
\]

This yields \( \tau^\Psi_\Phi(z, \alpha) \geq \delta_0 C_1 |z|/2 + \limsup_{n \to \infty} \frac{\log f(\alpha', n, \delta_0/2, \Phi, \Psi)}{n} \geq \delta_0 C_1 |z|/2. \)

\[ \square \]

Lemma 5.4. If \( \Phi \) and \( \Psi \) are H"older continuous and \( L_\Phi \) has dimension \( d \), then \( \Lambda^\Psi_\Phi(\alpha) = \tau^\Psi_\Phi^*(\alpha) \) over \( L_\Phi \).

Proof. When \( \alpha \in \text{int}(L_\Phi) \), the result has been shown in [6]. Specifically, there exists \( z \in \mathbb{R}^d \) such that \( \tau^\Psi_\Phi^*(\alpha) = \tau^\Psi_\Phi(z, \alpha). \)

Now we assume \( \alpha \) is in the boundary of \( L_\Phi \). When \( \Phi \) and \( \Psi \) are H"older continuous, it is known that \( \dim(\Lambda^\Psi_\Phi(\alpha)) = \dim^B_A = \dim^\Psi_A + \dim^H_A \dim^\Psi_A \) is the unique root of \( P(\lambda \Psi) = 0 \) [20]. Thus by Lemma 5.3 (1), \( \dim^\Psi_A + \dim^H_A \dim^\Psi_A = D(\Psi) \). If \( \Lambda^\Psi_\Phi(\alpha) = D(\Psi) \), then by Lemma 5.3 (1) we have \( \tau^\Psi_\Phi^*(\alpha) \leq \Lambda^\Psi_\Phi(\alpha) \). So in the following we assume \( \Lambda^\Psi_\Phi(\alpha) < D(\Psi) \). By the regularity of \( \Lambda^\Psi_\Phi \) we can find \( \alpha_* \in \text{int}(L_\Phi) \) such that \( \Lambda^\Psi_\Phi(\alpha_*) > \Lambda^\Psi_\Phi(\alpha) \). The line passing through \( \alpha \) and \( \alpha_* \) intersects the boundary of \( L_\Phi \) at another point \( \alpha_1 \). Let \( \alpha_{\text{max}} \in [\alpha, \alpha_1] \) such that \( \Lambda^\Psi_\Phi(\alpha_{\text{max}}) = \max\{\Lambda^\Psi_\Phi(\beta) : \beta \in [\alpha, \alpha_1]\} \). By Proposition 2.5 \( \Lambda^\Psi_\Phi \) is non-increasing from \( \alpha_{\text{max}} \) to \( \alpha \). Let \( \beta_0 \) be a point in the open interval \( (\alpha_{\text{max}}, \alpha) \) such that \( \Lambda^\Psi_\Phi(\beta_0) > \Lambda^\Psi_\Phi(\alpha) \). We have \( \beta_0 \in \text{int}(L_\Phi) \). Let \( \alpha_t = t\beta_0 + (1-t)\alpha \) \((0 < t < 1)\). Then \( \alpha_t \in \text{int}(L_\Phi) \) and \( \alpha_t \to \alpha \) as \( t \to 0 \). Let \( z_t \in \mathbb{R}^d \) such that \( \Lambda^\Psi_\Phi(\alpha_t) = \tau^\Psi_\Phi^*(\alpha_t) = \tau^\Psi_\Phi(z_t, \alpha_t) \).

We claim that \( \langle z, \beta_0 - \alpha_t \rangle \leq 0 \). Otherwise \( \langle z, \beta_0 - \alpha_t \rangle > 0 \), and by [6, 5] we have \( \tau^\Psi_\Phi(z_t, \beta_0) \leq \tau^\Psi_\Phi(z_t, \alpha_t) = C_1(\Psi) \langle z, \beta_0 - \alpha_t \rangle < \tau^\Psi_\Phi(z_t, \alpha_t) = \Lambda^\Psi_\Phi(\alpha_t) \). On the other hand we should have \( \tau^\Psi_\Phi(z_t, \beta_0) \geq \tau^\Psi_\Phi^*(\beta_0) = \Lambda^\Psi_\Phi(\beta_0) \geq \Lambda^\Psi_\Phi(\alpha_t) \), which is a contradiction.
Consequently, due to the definition of $\beta_0$ we have $\langle z_t, \alpha - \alpha_t \rangle \geq 0$. Still by (5.5) we get
\[
\tau_\Phi^\Psi(z_t, \alpha) \leq \tau_\Phi^\Psi(z_t, \alpha_t) - C_1(\Psi)(z_t, \alpha - \alpha_t) \leq \Lambda_0^\Psi(\alpha_t).
\]
Thus $\tau_\Phi^\Psi(\alpha) \leq \Lambda_0^\Psi(\alpha_t)$. Letting $t$ tend to $0$, by the continuity of $\Lambda_0^\Psi$ on the closed interval $[\alpha, \alpha_1]$, we get $\tau_\Phi^\Psi(\alpha) \leq \Lambda_0^\Psi(\alpha)$. Combining this with Lemma 5.2 we get the equality. \[\square\]

5.4.2. $\dim_H^\Psi(\Sigma_A) = \dim_B^\Psi(\Sigma_A) = D(\Psi) = \max \{ \Lambda_0^\Psi(\alpha) : \alpha \in L_\Phi \}$ in the general case.

We need to describe the $\Psi$- and $\Phi$- dependence of the function $\Lambda_0^\Psi$. Recall that in Lemma 5.1 we have defined $\Lambda_0^\Psi(\alpha, \epsilon) = \limsup_{n \to \infty} \log f(\alpha, n, \epsilon, \Phi, \Psi)) / n$ and we know that $\Lambda_0^\Psi(\alpha, \epsilon) \supset \Lambda_0^\Psi(\alpha)$ as $\epsilon \to 0$. The following lemma will be proved in Section 5.

**Lemma 5.5.**

1. Assume $\Psi, \Upsilon \in C_0^\infty(\Sigma_A, T)$, then we have
\[
D(\Psi) - D(\Upsilon) \leq 2 \log m \cdot \left(1 + \frac{1}{\max |\Psi|}\right) \left(1 + \frac{1}{\max |\Upsilon|}\right) \|\Psi - \Upsilon\|_{\text{lim}}.
\]

2. Let $\Phi, \Theta \in C_0^\infty(\Sigma_A, T, d)$. Let $\beta \in L_\Phi$. For any $\alpha \in B(\beta, \eta) \cap L_\Phi$ and $\delta_0 < 1/2C_2(\Psi)$ we have
\[
\Lambda_0^\Psi(\alpha) \leq \frac{2C_2(\Psi) \log m}{\max |\Upsilon|} \delta_0 + (1 - C_2(\Psi)\delta_0)\Lambda_0^\Psi(\beta, \alpha + \kappa\delta_0 + 2\eta),
\]
where $\delta_0 = \|\Psi - \Upsilon\|_{\text{lim}}$, $a_0 = \|\Phi - \Theta\|_{\text{lim}}$, $C_2(\Psi) = 1 + 1/|\max |\Psi||$ and $\kappa = \kappa(\Psi, \Upsilon, \Phi) = 14\|\Phi\|C(\Psi)\|\Upsilon\|_{\text{lim}} / \max |\Upsilon|$. \[\square\]

**Proof of the fact** that $\dim_H^\Psi(\Sigma_A) = \dim_B^\Psi(\Sigma_A) = D(\Psi) = \max \{ \Lambda_0^\Psi(\alpha) : \alpha \in L_\Phi \}$. At first assume that $\Psi$ is Hölder continuous. By Lemma 5.3 (1), $P(D(\Psi) = 0$. Let $\mu$ be the unique equilibrium state of $D(\Psi)\Psi$. It is well known (see [7]) that $\mu$ is ergodic, and $D(\Psi) = \dim_H^\mu(\Sigma_A) = \dim_B^\mu(\Sigma_A) = \dim_B^\Psi(\Sigma_A)$ (5). Let $\alpha = \Phi(\mu)$. By the sub-additive ergodic theorem we have $\mu(E_\Phi(\alpha)) = 1$, consequently $\Lambda_0^\Psi(\alpha) = D(\Psi)$. Thus, when $\Psi$ is a Hölder potential the result holds.

Next we assume $\Psi \in C_0^\infty(\Sigma_A, T)$. Define $\Psi^n$ according to (5.2), then $\lim_{n \to \infty} \|\Psi - \Psi^n\|_{\text{lim}} = 0$ and $|\max |\Psi^n| | \leq |\max |\Psi| |$. By (5.6) we have $\lim_{n \to \infty} D(\Psi^n) = D(\Psi)$. Let $\mu_n$ be the unique equilibrium state of $D(\Psi^n)\Psi^n$ and define $\alpha_n = \Phi(\mu_n)$. Then $\alpha_n \in L_\Phi$ and $\Lambda_0^\Psi(\alpha_n) = D(\Psi^n)$. Let $\alpha$ be a limit point of the sequence $\{\alpha_n : n \in \mathbb{N}\}$. Without loss of generality we assume $\alpha_n \to \alpha$. By (5.7) we have
\[
\Lambda_0^\Psi(\alpha_n) \leq \frac{2C_2(\Psi) \log m}{\max |\Psi|} \delta_n + (1 - C_2(\Psi)\delta_n)\Lambda_0^\Psi(\alpha, \kappa\delta_n + 2\eta),
\]
where $\delta_n := \|\Psi - \Psi^n\|_{\text{lim}}$, $C_2(\Psi) = 1 + 1/|\max |\Psi| |$, $\kappa_n = 14\|\Phi\|C(\Psi)\|\Psi^n\|_{\text{lim}} / \max |\Psi^n|$, $\eta_n = |\alpha - \alpha_n|$. By Lemma 5.1 we have $C_2(\Psi) \leq 1 + 1/|\max |\Psi| |$, thus we can rewrite (5.8) as $D(\Psi^n) \leq d_1\delta_n + \Lambda_0^\Psi(\alpha, d_2\delta_n + 2\eta_n)$. Letting $n$ tend to $\infty$ we get $D(\Psi) \leq \Lambda_0^\Psi(\alpha)$. By the definition of box dimension we have $\dim_B^\Psi(\Sigma_A) \leq D(\Psi)$. Thus we have $D(\Psi) = \Lambda_0^\Psi(\alpha) = \dim_B^\Psi(\Sigma_A)$. \[\square\]

As a consequence of the previous lemma we have the following corollary:

**Corollary 5.1.** Given $\Phi \in C_0^\infty(\Sigma_A, T)$ and $\Psi \in C_0^\infty(\Sigma_A, T)$. Define $\Psi^n$ as in (5.2). Assume $\Phi_n \in C_0^\infty(\Sigma_A, T, d)$ is such that $\|\Phi - \Phi^n\|_{\text{lim}} \to 0$ when $n \to \infty$. Assume $\alpha_n \in L_\Phi$ and $\lim_{n \to \infty} \alpha_n = \alpha$. Then $\Lambda_0^\Psi(\alpha) \geq \limsup_{n \to \infty} \Lambda_0^\Psi(\alpha_n)$. 

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Proof. By (5.7), \( \Lambda^\Psi_{\Phi^0}(\alpha_n) \leq \frac{2C_2(\Psi^n)\log n}{|\Psi_{\text{max}}|} \delta_n + (1-C_2(\Psi^n)\delta_n)\Lambda^\Psi_{\Phi^0}(\alpha, \|\Phi^n - \Phi\|_{\text{lim}} + \kappa_n \delta_n + 2\eta_n) \), where \( \delta_n, \kappa_n \) and \( \eta_n \) are the same as in the proof above. Letting \( n \to \infty \) we get the result.

5.4.3. \( \Lambda^\Psi_{\Phi^0}(\alpha) = \tau_{\Phi^0}(\alpha) \) when \( \Phi \) and \( \Psi \) are general and \( L_\Phi \) has dimension \( d \). We need a last intermediate result regarding the \( \Phi^- \) and \( \Psi^- \) dependence of the function \( \tau_{\Phi^0}^\Psi \).

Given \( \Phi \in C_0(\Sigma_A, T, d) \) and \( \Psi \in C^-_0(\Sigma_A, T) \), let \( \Psi^n \) as in (5.2). Assume \( \Phi^n \in C_0(\Sigma_A, T, d) \) are Hölder and \( \lim_{n \to \infty} \|\Phi - \Phi^n\|_{\text{lim}} = 0 \).

For \( (z, \alpha) \in (\mathbb{R}^d)^2 \) let \( \tau_{\Phi^0}^\Psi(z, \alpha, n) \) be the solution of \( P((z, \Phi^n - \alpha) + \tau_{\Phi^0}^\Psi(z, \alpha, n)\Psi^n) = 0 \) and let \( \tau_{\Phi^0}^\Psi(\alpha, n) := \inf \{ \tau_{\Phi^0}^\Psi(z, \alpha, n) : z \in \mathbb{R}^d \} \).

**Lemma 5.6.** Assume \( L_\Phi \) has dimension \( d \). Then

1. Let \( (z, \alpha) \in (\mathbb{R}^d)^2 \). We have \( \lim_{n \to \infty} \tau_{\Phi^0}^\Psi(z, \alpha, n) = \tau_{\Phi^0}^\Psi(z, \alpha) \). In particular \( \tau_{\Phi^0}^\Psi(0, \alpha) = D(\Psi^0) \), and consequently \( \tau_{\Phi^0}^\Psi(\alpha) = D(\Psi^0) \). If \( \alpha \in \text{int}(L_\Phi) \), then \( \lim_{n \to \infty} \tau_{\Phi^0}^\Psi(\alpha, n) = \tau_{\Phi^0}^\Psi(\alpha) \).

2. Let \( (z, \alpha, \alpha') \in (\mathbb{R}^d)^3 \). If \( \langle z, \alpha' - \alpha \rangle \geq 0 \), then

\[
C_1 |z, \alpha' - \alpha| \leq \tau_{\Phi^0}^\Psi(z, \alpha) - \tau_{\Phi^0}^\Psi(z, \alpha') \leq C_2 |z, \alpha' - \alpha|.
\]

where \( C_1 = 1/|\Psi_{\text{min}}| \) and \( C_2 = 1 + 1/|\Psi_{\text{max}}| \).

The proof will be given in Section 8.

**Proof of \( \Lambda^\Psi_{\Phi^0}(\alpha) = \tau_{\Phi^0}^\Psi(\alpha) \) when \( L_\Phi \) has dimension \( d \).** At first assume \( \alpha \in \text{int}(L_\Phi) \).

Since \( \Lambda^\Psi_{\Phi^0}(\alpha) = \mathcal{E}_\Phi^\Psi(\alpha) \), there exists an invariant measure \( \mu \) such that \( \Phi_*(\mu) = \alpha \) and \( \Lambda^\Psi_{\Phi^0}(\alpha) = -h_\mu(T)/\mathcal{E}_\Phi^\Psi(\mu) \). Choose a sequence of ergodic Markov measures \( \mu_n \) which converges to \( \mu \) in the weak-star topology and such that \( h_{\mu_n}(T) \) also converges to \( h_\mu(T) \) as \( n \) tends to \( \infty \). We claim that we can choose a sequence \( \{\Phi^n : n \geq 1\} \) of Hölder continuous potentials such that

\[
\forall \ n \geq 1, \ \Phi^n_*(\mu_n) = \alpha \ \text{and} \ \lim_{n \to \infty} \|\Phi - \Phi^n\|_{\text{lim}} = 0.
\]

Indeed at first, let \( \tilde{\Phi}^n \) be a sequence associated with \( \Phi \) as in (5.2). Then we know that \( \lim_{n \to \infty} \|\Phi - \tilde{\Phi}^n\|_{\text{lim}} = 0 \) and each \( \tilde{\Phi}^n \) is a Hölder potential. Let \( \delta_n = \Phi_*(\mu) - \tilde{\Phi}^n_*(\mu_n) \). We have \( |\delta_n| \leq |\Phi_*(\mu) - \Phi_*(\mu_n)| + |\Phi_*(\mu_n) - \tilde{\Phi}^n_*(\mu_n)| \leq |\Phi_*(\mu) - \Phi_*(\mu_n)| + |\Phi - \tilde{\Phi}^n|_{\text{lim}} \). Since \( \Phi_* \) is continuous, we have \( \lim_{n \to \infty} \delta_n = 0 \). Define \( \Phi^n := \Phi^n + \delta_n \). Thus \( \Phi^n \) satisfies (5.10).

Since \( \|\Phi^n - \Phi\|_{\text{lim}} \to 0 \), it is ready to show that \( d_H(L_{\Phi^n}, L_\Phi) \to 0 \). Since \( L_{\Phi^n} \) and \( L_\Phi \) are all compact convex sets and \( L_\Phi \) has dimension \( d \), \( L_{\Phi^n} \) has nonempty interior for \( n \) large enough. Consequently \( L_{\Phi^n} \) has dimension \( d \) for \( n \) large enough.

Now, let \( \Psi^n \) be associated with \( \Psi \) as in (5.2). By Corollary 5.1, we have \( \Lambda^\Psi_{\Phi^0}(\alpha) \geq \limsup_{n \to \infty} \Lambda^\Psi_{\Phi^0}(\alpha) \). On the other hand since \( \Phi^n_*(\mu_n) = \alpha \), we have \( \Lambda^\Psi_{\Phi^0}(\alpha) \geq -h_{\mu_n}(T)/\Psi^n_*(\mu_n) \). So we get \( \lim\inf \Lambda^\Psi_{\Phi^0}(\alpha) \geq \lim\inf_{n \to \infty} \frac{h_{\mu_n}(T)}{-\Psi^n_*(\mu_n)} = \Lambda^\Psi_{\Phi^0}(\alpha) \). Thus \( \Lambda^\Psi_{\Phi^0}(\alpha) = \lim_{n \to \infty} \Lambda^\Psi_{\Phi^0}(\alpha) \). For large \( n \), since \( \Phi^n \) and \( \Psi^n \) are Hölder continuous and \( L_{\Phi^n} \) has dimension \( d \), by Lemma 5.4, we have \( \tau_{\Phi^0}^\Psi(\alpha, n) = \Lambda^\Psi_{\Phi^0}(\alpha) \). Now by Lemma 5.6(1) since \( \alpha \in \text{int}(L_\Phi) \) we have \( \tau_{\Phi^0}^\Psi(\alpha, n) = \lim_{n \to \infty} \tau_{\Phi^0}^\Psi(\alpha, n) \). Then the result follows.
Next we assume $\alpha$ is in the boundary of $L_\Phi$. We have shown that $D(\Psi) = \max\{\Lambda^\Psi_\Phi(\alpha) : \alpha \in L_\Phi\}$. Moreover, by Lemma 5.6, $\tau^\Psi_\Phi(\alpha) \leq D(\Psi)$. Thus, since relation (5.9) holds, the same proof as in the Hölder case shows that $\tau^\Psi_\Phi(\alpha) = \Lambda^\Psi_\Phi(\alpha)$.

5.4.4. $\tau^\Psi_\Phi(\alpha) = \Lambda^\Psi_\Phi(\alpha)$ in the general case.

We only need to show the equality when $\dim L_\Phi = s < d$. Write $\Phi = (\Phi^1, \ldots, \Phi^d)$, $\hat{\Phi}^l := (\Phi^1, \ldots, \Phi^s, \bar{\Phi}^l)$ for $l = s + 1, \ldots, d$ and $\hat{\Phi} := (\Phi^1, \ldots, \Phi^s)$. The set $L_{\hat{\Phi}}$ is a projection of $L_{\Phi}$ and $L_{\hat{\Phi}^l}$ is a projection of $L_{\Phi}$ for each $l = s + 1, \ldots, d$. Thus $\dim L_{\hat{\Phi}^l} \leq \dim L_{\hat{\Phi}} \leq s$ for any $l = s + 1, \ldots, d$. It is clear that $\langle \hat{\Phi}^1, \ldots, \hat{\Phi}^d \rangle$ has dimension at least $s$, otherwise, the dimension of $L_{\Phi}$ will be strictly less than $s$. By relabeling we can assume $\hat{\Phi}^1, \ldots, \hat{\Phi}^s$ are linearly independent. By Proposition 2.2, we have $\dim L_{\hat{\Phi}} = s$. Thus we have $\dim L_{\hat{\Phi}^l} = s$, for any $l = s + 1, \ldots, d$. Again by Proposition 2.2 we conclude that for each $l = s + 1, \ldots, d$, $\hat{\Phi}^l$ is a linear combination of $\hat{\Phi}^1, \ldots, \hat{\Phi}^s$. Thus there exists a unique $d \times s$-matrix $A$ of rank $s$ and vector $b \in \mathbb{R}^d$ such that $\| A\hat{\Phi} + b - \hat{\Phi} \|_{\lim} = 0$. Consequently $L_{\Phi} = AL_{\hat{\Phi}} + b$.

For $\Phi \in C_{ud}(\Sigma_A, T, s)$, since $\dim L_{\hat{\Phi}} = s$, by the result proven in section 5.4.3 for any $\beta \in L_{\hat{\Phi}}$, we have $\Lambda^\Psi_{\hat{\Phi}}(\beta) = \tau^\Psi_{\hat{\Phi}}(\beta)$. Fix $\alpha \in L_{\Phi}$ and let $\beta \in L_{\hat{\Phi}}$ be the unique vector such that $\alpha = A\beta + b$. Then $E_{\Phi}(\alpha) = E_{\hat{\Phi}}(\beta)$, so $\Lambda^\Psi_{\Phi}(\alpha) = \tau^\Psi_{\Phi}(\alpha)$.

On the other hand, $\tau^\Psi_{\Phi}(\alpha) = \inf_{\xi \in \mathbb{R}^d} \tau^\Psi_{\Phi}(\xi, \alpha) \leq \inf_{\xi \in \mathbb{R}^d} \tau^\Psi_{\Phi}(A\xi, \alpha)$, where $\tau^\Psi_{\Phi}(\xi, \alpha)$ satisfies $P(\langle \xi, \Phi - \alpha \rangle + \tau^\Psi_{\Phi}(\xi, \alpha) \Psi) = 0$. Since $\| A\hat{\Phi} + b - \hat{\Phi} \|_{\lim} = 0$, for any $\lambda \in \mathbb{R}$ we have $P(\langle A\xi, \Phi - \alpha \rangle + \lambda \Psi) = P(\langle A\xi, A(\hat{\Phi} - \beta) \rangle + \lambda \Psi) = P(\langle A^*A\xi, A^*A(\hat{\Phi} - \beta) \rangle + \lambda \Psi)$. Thus we get $\tau^\Psi_{\Phi}(A\xi, \alpha) = \tau^\Psi_{\Phi}(A^*A\xi, \beta)$. Since $A$ has rank $s$, the square matrix $A^*A$ also has rank $s$. This yields $\tau^\Psi_{\Phi}(\alpha) \leq \inf_{\xi \in \mathbb{R}^d} \tau^\Psi_{\Phi}(A\xi, \alpha) = \inf_{\xi \in \mathbb{R}^d} \tau^\Psi_{\Phi}(A^*A\xi, \beta) = \inf_{\xi \in \mathbb{R}^d} \tau^\Psi_{\Phi}(\xi, \beta) = \tau^\Psi_{\Phi}(\beta) = \Lambda^\Psi_{\Phi}(\beta)$. Combining this with the inverse inequality, we get the result.

6. Proof of Theorem 2.3

We prove the slightly more general result mentioned in Remark 2.4(2). Suppose that $\xi$ is bounded and continuous outside a subset $E$ of $\Sigma_A$, and $\xi(\Sigma_A \setminus E) \subset \text{aff}(L_\Phi)$. Also, suppose that $\dim_H^\Psi E < \lambda := \sup\{D^\Psi_\Phi(\alpha) : \alpha \in \xi(\Sigma_A \setminus E) \cap \text{ri}(L_\Phi)\}$.

In order to prepare the proof of our geometric result, we prove a slightly more general result than necessary.

Proposition 6.1. Assume that $Z \subset \Sigma_A$ is a closed set such that $\mu(Z) = 0$ for any Gibbs measure $\mu$ supported on $\Sigma_A$. For any $\delta > 0$ such that $\lambda - \delta > \dim_H^\Psi(E)$, we can construct a Moran subset $\Theta \subset \Sigma_A$ such that $\Theta \setminus E \subset E_\Phi(\xi)$, $\dim_H^\Psi \Theta \geq \lambda - \delta$ and there exists an increasing sequence of integers $(g_j)_{j \geq 1}$ such that $T^g x \notin Z$ for any $x \in \Theta$ and any $j \geq 1$.

Proof. Fix $\delta > 0$ such that $\lambda - \delta > \dim_H^\Psi(E)$. Choose $\alpha_0 \in \xi(\Sigma_A \setminus E) \cap \text{ri}(L_\Phi)$ such that $\Lambda^\Psi_\Phi(\alpha_0) > \lambda - \delta/2$. Assume $L_\Phi$ has dimension $d_0 \leq d$ and $\text{aff}(L_\Phi) = \alpha_0 + U(\mathbb{R}^{d_0} \times \{0\}^{d-d_0})$, where $U$ is an orthogonal matrix.
Since $D_0^\Psi$ is continuous in $\text{ri}(L_\Phi)$, we can find $\eta > 0$ such that $B(\alpha_0, \eta) \cap \text{aff}(L_\Phi) \subset \text{ri}(L_\Phi)$ and for any $\alpha \in B(\alpha_0, \eta) \cap \text{aff}(L_\Phi)$ we have $|D_0^\Psi(\alpha) - D_0^\Psi(\alpha_0)| < \delta/2$. Consequently $D_0^\Psi(\alpha) > \lambda - \delta$ for all $\alpha \in B(\alpha_0, \eta) \cap \text{aff}(L_\Phi)$. Let $n_0 \in \mathbb{N}$ such that $2^{-n_0} \sqrt{d_0} < \eta$ and define a sequence of sets as follows:

$$\Delta_n := B(\alpha_0, \eta) \cap \text{aff}(L_\Phi) \cap (\alpha_0 + 2^{-n-n_0}U(\mathbb{Z}^{d_0} \times \{0\}^{d-d_0})), \ n \geq 0.$$ 

Then $\Delta_0 \neq \emptyset$ and $\Delta_n \subset \Delta_{n+1}$ for any $n \geq 0$ and each $\Delta_n$ is a finite set. For each $\alpha \in \bigcup_{n \geq 0} \Delta_n$, we can find a measure $\mu_\alpha$ such that $\Phi_*(\mu_\alpha) = \alpha$ and $D_0^\Psi(\alpha) = E_0^\Psi(\alpha) = h_{\mu_\alpha}(T) / \gamma_\alpha$, where $\gamma_\alpha = -\Psi_*(\mu_\alpha)$.

Let $(\varepsilon_j)_{j \geq 1}$ be a positive sequence such that $\sum_j \varepsilon_j < \infty$. For each $j \geq 1$ and each $\alpha \in \Delta_j$, we can find a Markov (hence Gibbs) measure $\mu_{\alpha,j}$ such that

$$\max(|h_{\mu_{\alpha,j}}(T) - h_{\mu_\alpha}(T)|, |\beta_{\alpha,j} - \alpha|, |\gamma_{\alpha,j} - \gamma_\alpha| < \varepsilon_j < 1,$$

where $\beta_{\alpha,j} = \Phi_*(\mu_{\alpha,j})$ and $\gamma_{\alpha,j} = -\Psi_*(\mu_{\alpha,j})$. Let $(\varphi^j)_{j \geq 1}$ and $(\psi^j)_{j \geq 1}$ be two sequences of Hölder potentials defined on $\Sigma_A$ such that $\|\Phi^j - \Phi\|_{\text{lim}} < \varepsilon_j$ and $\|\Psi^j - \Psi\|_{\text{lim}} < \varepsilon_j$, where $\Phi^j = (S_0^j \varphi^j)_{n=1}^\infty$ and $\Psi^j = (S_0^j \psi^j)_{n=1}^\infty$. For each $j \geq 1, \alpha \in \Delta_j$ and $s \in \{1, \ldots, m\}$ we denote by $\nu_{s,j}^j$ the restriction of $\mu_{s,j}$ to $[s]$ and $\nu_{\alpha,j}^j$ the probability measure $\nu_{\alpha,j}^j / \nu_{\alpha,j}([s])$.

For $N \geq 1$ let

$$E_N^j(\alpha) = \bigcap_{n \geq N} \left\{ x \in \Sigma_A : \left| \frac{\phi^j_n(Tx)}{n} - \alpha \right|, \left| \frac{\log \nu_{\alpha,j}^j([x|n])}{n} - h_{\mu_\alpha}(T) \right|, \left| \frac{n^j(Tx)}{n} - \gamma_\alpha \right| \leq 2\varepsilon_j \right\}.$$ 

Notice that each $\Delta_j$ is a finite set, thus the ergodicity of each $\mu_{\alpha,j}$ imply that we can fix an integer $N_j$ such that

$$\forall \ N \geq N_j, \ \forall \alpha \in \Delta_j, \ \forall s \in \{1, \ldots, m\} \ \nu_{s,j}^j(E_N^j(\alpha)) \geq 1 - \varepsilon_j/2.$$ 

Define $V_N := \{ v \in \Sigma_{A,N} \cap Z = \emptyset \}$. There exists $\hat{N}_j \geq 1$ such that for each $N \geq \hat{N}_j$,

$$\nu_{s,j}^j \left( \bigcup_{v \in V_N} \{ v \} \right) \geq 1 - \varepsilon_j/2, \ \forall \alpha \in \Delta_j.$$ 

Define $V_N^j(\alpha) = \{ v \in V_N, \ [v] \cap E_N^j(\alpha) \neq \emptyset \}$. Thus, if $N \geq \max(N_j, \hat{N}_j)$ we have

$$\nu_{s,j}^j \left( \bigcup_{v \in V_N^j(\alpha)} \{ v \} \right) \geq 1 - \varepsilon_j, \ \forall \alpha \in \Delta_j.$$ 

Now we can build a measure on $\Sigma_A$ as follows. At first we define $\vartheta \in \Sigma_{A,*}$ and inductively a sequence of integers $\{g_j : j \geq 0\}$ and a sequence of measures $\{\rho_j : j \geq 0\}$ such that $\rho_j$ is a measure on $(\{\vartheta\}, \sigma([u] : \vartheta < u \in \Sigma_{A,g_j}))$ for each $j \geq 0$, and the measures $\rho_j$ are consistent: for each $j \geq 0$ the restriction of $\rho_{j+1}$ to $\sigma([u] : \vartheta < u \in \Sigma_{A,g_j})$ is equal to $\rho_j$.

Fix $x^0 \in \Sigma_A \setminus E$ such that $\xi(x^0) = 0$ and write $x^0 = x_1^0 x_2^0 \cdots$. Choose $g_0 \in \mathbb{N}$ such that $O(\xi([x^0|g_0])) \leq 2^{-g_0}$, where $O(\xi, V)$ stands for the oscillation of $\xi$ over $V$. Write $\vartheta := x^0|g_0$. Define the probability measure $\rho_0$ to be the trivial probability measure on $(\{\vartheta\}, \{\emptyset, \{\vartheta\}\})$. Suppose we have defined $(g_k, \rho_k)_{0 \leq k \leq j}$ for $j \geq 0$ as desired. To obtain $(g_{j+1}, \rho_{j+1})$ from $(g_j, \rho_j)$, choose any $L_{j+1} \geq \max\{N_{j+1}, \hat{N}_{j+1}\}$, define $g_{j+1} = g_j + L_{j+1}$. 

For every $w \in \Sigma_{A, g_j}$ with $\vartheta < w$, choose $x_w \in [w]$. Since $x_w \in [w] \subset [\vartheta]$ we have $|\xi(x_w) - \alpha_0| = |\xi(x_j) - \xi(x^0)| \leq 2^{-n_0} \leq \eta$. Notice that by our assumption $\xi(\Sigma_A) \subset \text{aff}(L_0)$, thus $\xi(x_w) \in B(\alpha_0, \eta) \cap \text{aff}(L_0)$. Take $\alpha_w \in \Delta_{j+1}$ such that $|\xi(x_w) - \alpha_w| \leq 2^{-j-1-n_0} \sqrt{d_0}$. For each $v \in \Sigma_{A, L_{j+1}}$ such that $wv$ is admissible, let $(\tilde{w})$ stands for the last letter of $w$.

$$
\rho_{j+1}([wv]) := \rho_j([w]) \rho_{\alpha_w, j+1}([\tilde{w}v]).
$$

By construction the family $\{\rho_j : j \geq 0\}$ is consistent. Denote by $\rho$ the Kolmogorov extension of the sequence $(\rho_j)_{j \geq 0}$ to $[\vartheta], \sigma([u] : \vartheta < u \in \Sigma_{A,*})$.

If $\vartheta < u$ and $u \in \Sigma_{A,n}$ with $g_j \leq n < g_{j+1}$, writing $u = \vartheta w^1 \cdots w^j \cdot v$ with $|w^k| = L_k$ and $|v| = n - g_i$, and denoting $\vartheta w^1 \cdots w^k$ by $\tilde{w}_k$, we have the useful formula

$$
(6.1)
\rho([u]) = \left( \bigcup_{k=1}^j \omega_{\alpha_{\tilde{w}_k-1}, \tilde{w}_k} \right) \rho_{\alpha_{\tilde{w}_j}, j+1}([\tilde{w}^j v]).
$$

Let $\Theta = \{ x \in [\vartheta] : \forall j \geq 1, T^{g_j-1} x |_{L_{j+1}} \in V_{L_{j+1}}^{j+1}(\alpha_{x|_{g_j}}) \}$. By construction, $T^{g_j-1} x \notin Z$ for any $x \in \Theta$ and any $j \geq 1$.

Write $\alpha_k := \alpha_{x|_{g_{k-1}}}$, By construction of $\rho$, for each $j \geq 1$, by using (6.1) we can get

$$
\rho(\{ x \in [\vartheta] : [x|_{g_j}] \cap \Theta \neq \emptyset \})
\leq \left( \sum_{\vartheta w^1 \cdots w^j \text{ admissible}, \ \forall 1 \leq k \leq j, w^k-1 w^k \in V_{L_k}^{k}(\alpha_k)} \rho_j(\vartheta w^1 \cdots w^j) \right)
\leq \left( \sum_{\vartheta w^1 \cdots w^{j-1} \text{ admissible}, \ \forall 1 \leq k \leq j-1, w^k-1 w^k \in V_{L_k}^{k}(\alpha_k)} \rho_{\alpha_{\tilde{w}_j, j}}([\tilde{w}^j-1, w^j]) \right)
\geq \left( \sum_{\vartheta w^1 \cdots w^{j-1} \text{ admissible}, \ \forall 1 \leq k \leq j-1, w^k-1 w^k \in V_{L_k}^{k}(\alpha_k)} \rho_{\alpha_{\tilde{w}_j, j}}(1 - \varepsilon_j) \right) \geq \prod_{k=1}^j (1 - \varepsilon_j).
$$

Consequently, $\rho(\Theta) \geq \prod_{j \geq 1} (1 - \varepsilon_j) > 0$ since we assumed that $\varepsilon_j < 1$ and $\sum_{j \geq 1} \varepsilon_j < \infty$.

For $\eta \in \{\varphi, \psi\}$ and $j \geq 1$, let

$$
c(\eta^j) = \sup_{n \geq 1} \max_{v \in \Sigma_{A,n}} \max_{x, y \in [v]} |S_n \eta^j(x) - S_n \eta^j(y)|.
$$

This number is finite since each $\eta^j$ is Hölder continuous. Let $M_j \not\rightarrow \infty$ such that

$$
\forall n \geq M_j, \max(\|\phi_n^j - \phi_n\|_\infty, \|\psi_n^j - \psi_n\|_\infty) \leq 2\varepsilon_j n.
$$

The sequence $(L_j)_{j \geq 1}$ can be specified to satisfy the additional properties

$$
L_j \geq M_{j+1} \quad \text{and} \quad \max(K_1(j), K_2(j), K_3(j)) \leq \varepsilon_j g_j,
$$
(recall that \( g_j = g_0 + \sum_{k=1}^{j} L_k \), where
\[
\begin{align*}
K_1(j) &= \sum_{k=1}^{j+1} (c(\varphi^k) + c(\psi^k)) \\
K_2(j) &= \max_{\alpha \in \Delta_{j+1}} \max_{0 \leq s \leq m-1} \| n|\alpha| \cdot \| \phi_n^{j+1} \|_{\infty} \| \log \nu_{\alpha,j+1}(\lfloor n \rfloor) \|_{\infty}, \| \psi_n^{j+1} \|_{\infty}) \\
K_3(j) &= (j+1) \max_{1 \leq n \leq M_{j+1}} \max(\| \phi_n^{j+1} - \phi_n \|_{\infty}, \| \psi_n^{j+1} - \psi_n \|_{\infty})
\end{align*}
\]

Let us check that \( \Theta \setminus E \subset E_4(\xi) \). Let \( x \in \Theta \setminus E \), \( n \geq g_1 \) and \( j \geq 1 \) such that \( g_j \leq n < g_{j+1} \). Since \( g_j > L_j \geq M_j \), we have
\[
|\phi_n(x) - n\xi(x)| \leq \| \phi_n^{j+1} - \phi_n \|_{\infty} + |\phi_n^{j+1}(x) - n\xi(x)| \leq 2\varepsilon_{j+1} n + |\phi_n^{j+1}(x) - n\xi(x)|.
\]

We have (with \( g_{-1} = 0 \), \( \alpha_k = \alpha |_{g_{k-1}} \) and \( L_0 = g_0 \))
\[
|\phi_n^{j+1}(x) - n\xi(x)| \\
\leq |\phi_n^{j+1}(g_j x) - g_j \xi(x)| + |\phi_n^{j+1}(T^{g_j} x) - (n - g_j) \xi(x)| \\
= |\phi_n^{j+1}(x) - \sum_{k=0}^{j} L_k \alpha_k | + \sum_{k=0}^{j} L_k |\alpha_k - \xi(x)| \\
\leq \sum_{k=0}^{j} |\phi_n^{j+1}(T^{g_k-1} x) - L_k \alpha_k | + \sum_{k=0}^{j} L_k |\alpha_k - \xi(x)| (=: (I) + (II)) + |\phi_n^{j+1}(T^{g_j} x) - (n - g_j) \alpha_{j+1}| + |(n - g_j) |\alpha_{j+1} - \xi(x)| (=: (III) + (IV))
\]

At first we have
\[
(I) + (III) \leq \sum_{k=0}^{j} \| \phi_n^{j+1} - \phi_L \|_{\infty} + \sum_{k=0}^{j} \| \phi_L - \phi_L \|_{\infty} + \left( \sum_{k=0}^{j} |\phi_n^{j+1}(T^{g_k-1} x) - L_k \alpha_k | + |\phi_n^{j+1}(T^{g_j} x) - (n - g_j) \alpha_{j+1}| \right)
\]

If \( L_k \leq M_{j+1} \), then \( \| \phi_n^{j+1} - \phi_L \|_{\infty} \leq K_3(j) /(j+1) \); if \( L_k > M_{j+1} \), then \( \| \phi_n^{j+1} - \phi_L \|_{\infty} \leq 2\varepsilon_k L_k \). Thus we have \( \sum_{k=0}^{j} \| \phi_n^{j+1} - \phi_L \|_{\infty} \leq K_3(j) + 2 \sum_{k=0}^{j} \varepsilon_k L_k \). Since \( L_k \geq M_{k+1} \geq M_k \) we also have \( \sum_{k=0}^{j} \| \phi_n^{j+1} - \phi_L \|_{\infty} \leq 2 \sum_{k=0}^{j} \varepsilon_k L_k \). Thus both terms are \( o(g_j) \) as \( n \to \infty \). Consequently both terms are \( o(n) \).

For \( k = 0, \ldots, j \), by the construction of \( \Theta \), we have \( T^{g_k-1} x \mid_{L_k+1} = x_{g_{k-1}} \cdot (T^{g_k-1} x \mid_{L_k}) \in V_{L_k} \), so \( x_{g_{k-1}} \cdot (T^{g_k-1} x \mid_{L_k}) \setminus E_{N_k} (\alpha_k) \neq \emptyset \). Since \( L_k \geq N_k \), there exists \( y \in [T^{g_k-1} x \mid L_k] \) such that \( \| \phi_{L_k}(y) - L_k \alpha_k \| \leq 2\varepsilon_k L_k \), hence \( \| \phi_{L_k}(T^{g_k-1} x) - L_k \alpha_k \| \leq 2\varepsilon_k L_k + c(\varphi^k) \). Similarly, we have \( |\phi_{n-g_j}(T^{g_j} x) - (n - g_j) \alpha_{j+1}| \leq 2\varepsilon_{j+1} (n - g_j) + c(\varphi^{j+1}) \) if \( n - g(j) \geq N_{j+1} \), and we trivially have \( |\phi_{n-g_j}(T^{g_j} x) - (n - g_j) \alpha_{j+1}| \leq 2K_2(j) \) otherwise. This yields
\[
\left( \sum_{k=0}^{j} \| \phi_{L_k}(T^{g_k-1} x) - L_k \alpha_k | \right) + |\phi_{n-g_j}(T^{g_j} x) - (n - g_j) \alpha_{j+1}|
\]
That is the local lower dimension of Proposition 6.2.

and there exists an integer sequence αj := αx|gk−1

Together we get (I) + (III) = o(n). On the other hand, by construction,

and limk→∞ O(ξ, |x|gk)) + 2−k−no√d0 = 0 since ξ is continuous at x. Thus we conclude that (II) + (IV) = o(n). Finally, |φn(x) − nξ(x)| = o(n), and Θ \ E ⊂ EΦ(ξ).

Similarly, for any x ∈ Θ we can prove that (αk := αx|gk−1)

By construction liminfj→∞ hμαj(T)/γαj ≥ λ − δ. For any y ∈ [x|n] we have |ψn(y) − ψn(x)| = o(n), thus we get diam([x|n]) = Ψ|x|n| = exp(ψn(x) + o(n)). Combining the above two relations we conclude that liminfj→∞ log ρ([x|n]) = O(ξ, |x|gk)) + 2−k−no√d0.

That is the local lower dimension of ρ at each x ∈ Θ is ≥ λ − δ, hence dim_H(Θ) ≥ λ − δ by the mass distribution principle (see [29] for instance).

By essentially repeating the same proof as above (in fact, it is easier), we can get the following property:

Proposition 6.2. Assume Z ⊂ ΣA is a closed set such that μ(Z) = 0 for any Gibbs measure μ supported on ΣA. For any α ∈ LΦ, we can construct a subset Θ ⊂ EΦ(α) such that dim_H(Θ) ≥ E^Φ(α) and there exists an integer sequence gj ↗ ∞ such that Tgjx ⊄ Z for any x ∈ Θ and any j ≥ 1. In particular, E^Φ(α) ≤ D^Φ(α).

Proof of Theorem 2.3 (1) Since dim_H E < λ − δ, by the proposition above we have dim^Φ_H(Θ \ E) = dim^Φ_H(Θ) ≥ λ − δ. Consequently dim^Φ_H EΦ(ξ) ≥ dim^Φ_H(Θ \ E) ≥ λ − δ. Since δ > 0 is arbitrary, we get dim^Φ_H EΦ(ξ) ≥ λ.

(2) If ξ(ΣA) ⊂ LΦ, the construction of a Moran subset of EΦ(ξ) can be done around any point of ΣA, like in the proof of Proposition 6.1. The only difference is that in this case the dimension of this set is of no importance. Hence, EΦ(ξ) is dense.

(3) Now we assume ξ is continuous everywhere. If moreover

sup{D^Φ_H(α) : α ∈ ξ(ΣA) ∩ ri(LΦ)} = sup{D^Φ_H(α) : α ∈ ΣA ∩ LΦ} =: θ,

then at first we have dim^Φ_H EΦ(ξ) ≥ θ. On the other hand by definition we have EΦ(ξ) ⊂ EΦ(ξ(ΣA)∩LΦ). Thus by Proposition 5.1 we have dim^Φ_H EΦ(ξ) ≤ θ. So we get dim^Φ_H EΦ(ξ) = dim^Φ_H EΦ(ξ) = θ.
(4) Assume $d = 1$, $\xi$ is continuous everywhere and $\xi(\Sigma_A) \subset L_{\Phi}$. Notice that in this case $L_{\Phi} = [\alpha_1, \alpha_2]$ is an interval. Assume $\alpha_0 \in \xi(\Sigma_A)$ such that $D_\Phi^\Psi(\alpha_0) = \sup\{D_\Phi^\Psi(\alpha) : \alpha \in \xi(\Sigma_A)\}$. If $\alpha_0 \in (\alpha_1, \alpha_2)$, by (2) we conclude. Now assume $\alpha_0 = \alpha_1$. If $\alpha_1$ is not isolated in $\xi(\Sigma_A)$, still by (2) and the continuity of $D_\Phi^\Psi$, we get the result. If $\alpha_1$ is isolated in $\xi(\Sigma_A)$, then by the continuity of $\xi$, we can find a cylinder $[w] \subset \Sigma_A$ such that $\xi([w]) = \alpha_1$. From this we get $E_{\Phi}(\xi) \supset E_{\Phi}(\alpha_1) \cap [w]$. Thus $\dim_H E_{\Phi}(\xi) \geq D_\Phi^\Psi(\alpha_1)$ and the result holds. If $\alpha_0 = \alpha_2$, the proof is the same.

\hspace{1cm} \square

7. Proofs of results in Section 3

We will use the following lemma, which is standard and essentially the same as Lemma 5.1 in [21] (the proof is elementary).

Lemma 7.1. Let $X$ and $Y$ be metric spaces and $\chi : X \to Y$ a surjective mapping with the following property: there exists a function $N : (0, \infty) \to \mathbb{N}$ with $\log N(r)/\log r \to 0$ when $r \to 0$ such that for any $r > 0$, the pre-image $\chi^{-1}(B)$ of any $r$-ball in $Y$ can be covered by at most $N(r)$ sets in $X$ of diameter less than $r$. Then for any set $E \subset Y$ we have $\dim_H E \geq \dim_H \chi^{-1}(E)$.

Proof of Proposition 3.1. Condition (4) implies that $\chi : (\Sigma_A, d_{\Psi}) \to (J, d)$ is Lipschitz continuous, thus we have $\dim_H E \leq \dim_H \chi^{-1}(E)$.

For the converse inequality, let us check the condition of the above lemma. Let $B \subset J$ be a ball of radius $r$, let $n \in \mathbb{N}$ such that $e^{-n} \leq r < e^{1-n}$. define

$$G^r_B = \{w \in B_n(\Psi) : R_w \cap B \neq \emptyset\}.$$ 

One checks that $\{[w] : w \in G^r_B\}$ is an $r$-covering of $\chi^{-1}(B)$. Define $N(r) := \#G^r_B$. Let us estimate the number $\#G^r_B$. Clearly, $\#G^r_B \geq 1$. By condition (4), for each $w \in G^r_B$, $R_w$ is contained in a ball of radius $K\Psi[w] \leq K e^{-n}$, thus $\bigcup_{w \in G^r_B} R_w \subset B(y, r + 2K e^{-n}) \subset B(y, (e + 2K)e^{-n})$, where $y$ is the center of $B$. On the other hand, by Lemma 4.2(1) there exists $C > 0$ such that $|w| \leq Cn$ for any $w \in B_n(\Psi)$, thus $\eta[w] = o(|w|) = o(n)$ for any $w \in B_n(\Psi)$. By construction, the interiors of the sets $R_w$, $w \in G^r_B$, are disjoint and each $R_w$ contains a ball of radius $K^{-1} \exp(\eta[w]) \Psi[w] = K^{-1}e^{o(n)}\Psi[w] = K^{-1}e^{-n+o(n)}$ by Lemma 4.2(2). Thus $\#G^r_B \leq K^d (e + 2K)^d e^{o(n)}$. So we conclude that $\log N(r)/\log r = \log \#G^r_B/\log r \to 0$ as $r \to 0$. Thus by Lemma 7.1 we can conclude that $\dim_H E \geq \dim_H \chi^{-1}(E)$.

Proof of Lemma 3.1. At first we show that $J \cap V = J \setminus \tilde{Z}_\infty$, consequently by the SOSC, $J \setminus \tilde{Z}_\infty \neq \emptyset$ and we get $\emptyset \neq \chi^{-1}(J \setminus \tilde{Z}_\infty) = \Sigma_A \setminus Z_\infty$. In fact

$$y \in J \setminus \tilde{Z}_\infty \iff y \in J \text{ and } \forall n \geq 1 \exists x \in \Sigma_A \text{ s.t. } y \in \text{int}(R_{x|n}) = f_{x|n}(V)$$

$$\iff y \in J \text{ and } \forall n \geq 1 \exists x \in \Sigma_A \text{ s.t. } y \in \text{int}(R_{x|n}) = f_{x|n}(V) \iff y \in J \cap V.$$

By construction, $\chi : \Sigma_A \setminus Z_\infty \to J \setminus \tilde{Z}_\infty$ is surjective. Since $J \setminus \tilde{Z}_\infty = J \cap V$, it is ready to show that $\chi$ is also injective.
Next we show that $T(\Sigma_A \setminus Z_\infty) \subset \Sigma_A \setminus Z_\infty$. Take $x \in \Sigma_A \setminus Z_\infty$. If $Tx \in Z_\infty$, then we can find $n_0 \in \mathbb{N}$ such that $\chi(Tx) \in f_{Tx|_{n_0}}(\partial V)$. Consequently $\chi(x) = f_{x_1}(\chi(Tx)) \in f_{x_1}(f_{Tx|_{n_0}}(\partial V)) = f_{x_1} \circ f_{Tx|_{n_0}}(\partial V) = f_{x_1|_{n_0+1}}(\partial V)$, which is a contradiction. At last we show that for any Gibbs measure $\mu$ we have $\mu(Z_\infty) = 0$. Define $\tilde{Z}_n := \bigcup_{w \in \Sigma_A} \chi^{-1}(\tilde{Z}_n)$ and $Z_n = \chi^{-1}(\tilde{Z}_n)$. The sequence $(Z_n)_{n \geq 1}$ is nondecreasing and $Z_\infty = \bigcup_{n \geq 1} Z_n$. Since the IFS is conformal we can easily get $T(Z_n) \subset Z_n$ for $n \geq 1$ and $T(Z_0) \subset Z_0$. Consequently $T(Z_n) \subset Z_n$. By the ergodicity we have $\mu(Z_n) = 0$ or 1. By the SOSC, $\Sigma_A \setminus Z_n$ is nonempty and open, thus by the Gibbs property of $\mu$ we get $\mu(\Sigma_A \setminus Z_n) > 0$, hence $\mu(Z_n) = 0$. Consequently $\mu(Z_\infty) = 0$.

From $T(Z_n) \subset Z_n$ we easily get $T(Z_\infty) \subset Z_\infty$. □

**Proof of Theorem 3.1.** (1) At first we notice that by the property (4) assumed in the construction of $J$ the mapping $\chi$ is Lipschitz. This is enough to get the desired upper bounds from Theorem 2.2(1).

Now we deal with the lower bound for dimensions and the equality $L_\Phi = L_{\tilde{\Phi}}$. We notice that the inclusion $L_\Phi \subset L_{\tilde{\Phi}}$ holds by construction.

**Suppose $J$ is a conformal repeller.** Since we have $\chi \circ T = g \circ \chi$ on $\Sigma_A$ and $\chi$ is surjective, it is seen that $\chi^{-1}(E_\Phi(\alpha)) = E_{\tilde{\Phi}}(\alpha)$ for any $\alpha \in L_{\tilde{\Phi}}$. Thus $L_\Phi = L_{\tilde{\Phi}}$ and by Proposition 3.1 we have $D_\Phi(\alpha) = D_{\tilde{\Phi}}(\alpha)$.

**Suppose $J$ is the attractor of a conformal IFS with SOSC.** Let $\alpha \in L_{\tilde{\Phi}}$. Let $Z = \chi^{-1}(\partial V)$. The set $Z$ is closed and by Lemma 3.1 $\mu(Z) = 0$ for any Gibbs measure $\mu$. By Proposition 6.2 we can construct a Moran set $\Theta \subset E_{\tilde{\Phi}}(\alpha)$ such that $\dim_H(\Theta) \geq E_{\tilde{\Phi}}(\alpha) = D_{\tilde{\Phi}}(\alpha)$ and there exists a sequence $g_j \nearrow \infty$ such that $T^n x \notin Z$ for any $x \in \Theta$ and any $j \geq 1$. The last property means that $\Theta \subset \Sigma_A \setminus Z_\infty$. Since $\chi$ is a bijection between $\Sigma_A \setminus Z_\infty$ and $J \setminus Z_\infty$, we conclude that $\chi^{-1} \circ \chi(\Theta) = \Theta$, thus by Proposition 3.1 $\dim_H(\chi(\Theta)) = \dim_H(\Theta) \geq D_{\tilde{\Phi}}(\alpha)$. Since we also have $\chi \circ T = g \circ \chi$ on $\Sigma_A \setminus Z_\infty$, we get that $\chi(\Theta) \subset E_\Phi(\alpha)$. Thus $\alpha \in L_\Phi$ and $D_\Phi(\alpha) \geq D_{\tilde{\Phi}}(\alpha)$.

(2) Take $E = J$ in Proposition 3.1 then use (1) and Theorem 2.2(2).

**Proof of Theorem 3.2.** Define $\tilde{\xi} := \xi \circ \chi$.

Case 1: $J$ is a conformal repeller. One checks easily that $\chi^{-1}(E_\Phi(\xi)) = E_{\tilde{\Phi}}(\tilde{\xi})$. Then the result is a consequence of Theorem 3.1 and Theorem 2.3.

Case 2: $J$ is the attractor of a conformal IFS with SOSC. By using Proposition 6.1 Theorem 2.3 and the same argument as in the proof of Theorem 3.1 we get the result. □

**Proof of Theorem 3.3.** Let $\xi = \Phi$, then $\mathcal{F}(\tilde{J}, \tilde{g}) = E_\Phi(\xi)$. To show the result we need only to check the condition of Theorem 3.2 and the only condition we need to check is that

$$\sup\{D_\Phi(\alpha) : \alpha \in \xi(J) \cap \text{ri}(L_\Phi)\} = \sup\{D_\Phi(\alpha) : \alpha \in \xi(J) \cap L_\Phi\}. \quad(*)$$

Notice that in this special case we have $\xi(J) = J$ and $L_\Phi = \text{Co}(J)$, thus $\xi(J) \cap L_\Phi = J$. Recall that in this case $L_\Phi$ is a convex polyhedron, thus by Proposition 2.5 and Theorem 3.1 $D_\Phi$ is continuous on $L_\Phi$. Thus the supremum in the right hand side of $(*)$ can be
By using (2.6) we get

Combining (8.1), (8.2) and (2.5) we get

The same proof as that of the claim in Proposition 2.4 yields

\[ (8.2) \]

\[ B \]

\[ C \]

(8.1)

\[ \#B_{[n(1-C_2(\Psi))\delta]}(Y) \leq \#B_n(\Psi). \]

Let \( c_1(n) = -\Psi_{\text{min}} + C(\Psi) + \|\Psi\|_n, \) then \( c_1(n) > 0 \) and \( c_1(n) = o(n). \) Write \( w = uw'. \)

The same proof as that of the claim in Proposition 2.4 yields \( w' \leq (c_1(n) + 2nC_2(\Psi)\delta + C(Y))/|\gamma_{\text{max}}|. \) Thus we can conclude that

\[ (8.2) \]

\[ \#B_{[n(1-C_2(\Psi))\delta]}(Y) \geq \#B_n(\Psi)m^{-(c_1(n)+2nC_2(\Psi)\delta+C(Y))/|\gamma_{\text{max}}|}. \]

Combining (8.1), (8.2) and (2.5) we get

\[ (1 - C_2(\Psi)\delta)D(Y) \leq D(\Psi) \leq (1 - C_2(\Psi)\delta)D(Y) - 2C_2(\Psi)\delta \log m / |\gamma_{\text{max}}|. \]

By using (2.6) we get \( |D(\Psi) - D(Y)| \leq a(m, \Psi, Y)\delta, \) where

\[ a(m, \Psi, Y) = 2C_2(\Psi)C_2(\log m = 2\left( 1 + \frac{1}{\Psi_{\text{max}}}, \frac{1}{\gamma_{\text{max}}} \right) \log m. \]

Since \( \delta > \|\Psi - Y\|_{\text{lim}} \) is arbitrary, we get \( |D(\Psi) - D(Y)| \leq a(m, \Psi, Y)\|\Psi - Y\|_{\text{lim}}. \)

(2) Now given \( \Phi, \Theta \in C_{aa}(\Sigma_A, T, d). \) Assume \( 0 < \epsilon < \|\Phi\| \) and \( \beta \in L_\Theta. \) Fix \( \alpha \in B(\beta, \eta) \cap L_\Phi. \) Fix \( \delta > \|\Psi - Y\|_{\text{lim}}. \)

For any \( \epsilon > 0, \) pick up \( w \in F(\alpha, n, \epsilon, \Phi, \Psi). \) Then \( w \in B_n(\Psi) \) and there exists \( x \in [w] \) such that \( |\phi|_{[w]}(x) - |w|\alpha| \leq \|w|\epsilon. \) We have seen in proving (1) that \( w = uw', \) where \( u \in B_{[n(1-C_2(\Psi)\delta)]}(Y) \) and \( |w'| \leq (c_1(n) + 2nC_2(\Psi)\delta + C(Y))/|\gamma_{\text{max}}|. \) Notice that \( \text{diam}(L_\Phi) \leq \|\Phi\|, \) thus \( |\alpha| \leq \|\Phi\|. \) So we have

\[ |\phi|_{[w]}(x) - |w|\alpha| \leq |\phi|_{[w]}(x) - |w|\alpha| + |w'|(||\Phi| + |\alpha| + |C(\Phi)| \]

\[ \leq |w| \epsilon + 2|w'||\Phi| + |C(\Phi)| \]

\[ \leq |u| \left( \epsilon + \frac{3||\Phi|||w'| + |C(\Phi)|}{|u|} \right). \]

Since \( 0 < c_1(n) = o(n), \) for large \( n \) we have

\[ 3||\Phi||C_1(n) + C(Y) + C(\Phi)|\gamma_{\text{max}}| \leq n||\Phi||C_2(\Psi)\delta. \]
Combining this with \( |u| \geq C_1(\Psi_\min)(1 - C_2(\Psi)\delta) \) we get that for \( \delta < 1/(2C_2(\Psi)) \),

\[
(3\|\Phi\||u'\| + |C(\Phi)|/|u| \leq 14\|\Phi\||C_2(\Psi)\| \frac{M_{\min}}{M_{\max}} \delta =: \kappa(\Psi, \Theta, \Phi)\delta = \kappa \delta.
\]

Fix any \( a > \|\Theta - \Phi\|_{\text{lim}} \). For \( n \) large enough we have

\[
|\theta_{u_1}(x) - |u|\beta| = |\theta_{u_1}(x) - \phi_{u_1}(x)| + |\phi_{u_1}(x) - |u|\alpha + |u|\alpha - \beta| \leq a|u| + (\epsilon + \kappa \delta)|u| + \|u\|\eta/\|u\|.
\]

As a result \( u \in F(\beta, n(1 - C_2(\Psi)\delta), a + \epsilon + \kappa \delta + \eta, \Theta, \Phi) \). Thanks to our control of \(|u'|\), we have

\[
f(\beta, n(1 - C_2(\Psi)\delta), a + \epsilon + \kappa \delta + \eta, \Theta, \Phi) \geq f(\alpha, n, \epsilon, \Phi, \Psi)m^{-\Gamma(n) + 2nC_2(\Psi)\delta + \Gamma(\Theta))/\|u'\|_{\text{lim}}.}
\]

This yields \( \Lambda^\Psi(\alpha, \epsilon) \leq \frac{2C_2(\Psi)\log m}{\|u'\|_{\text{lim}}} \delta + (1 - C_2(\Psi)\delta)\Lambda^\Psi(\beta, a + \epsilon + \kappa \delta + \eta) \). Letting \( \epsilon \downarrow 0 \) and then \( a \downarrow a_0 \) and \( \delta \downarrow \delta_0 \) we get

\[
\Lambda^\Psi(\alpha) \leq 2C_2(\Psi)\log m \delta_0 + (1 - C_2(\Psi)\delta_0)\Lambda^\Psi(\beta, (a_0 + \kappa \delta_0 + \eta) + 2\eta).
\]

**Proof of Lemma 5.6** (1) For \( \lambda \in \mathbb{R} \) define \( f_n(\lambda) := P(\langle z, \Phi^n - \lambda \rangle + \lambda \Psi^n) \) and \( f(\lambda) := P(\langle z, \Phi - \lambda \rangle + \lambda \Psi) \). Since \( \langle z, \phi_k^n(x) - k\alpha \rangle + \lambda \psi_k^n(x) - \langle z, \phi_k(x) - k\alpha \rangle + \lambda \psi_k(x) \rangle \leq |z||\phi_k^n - \phi_k|| + |\lambda|\psi_k^n - \psi_k| \), we have \( |f_n(\lambda) - f(\lambda)| \geq |z|||\Phi^n - \Phi||_{\text{lim}} + |\lambda|\||\Psi^n - \Psi||_{\text{lim}}.\n\]

Thus \( f_n \) converges uniformly to \( f \) over any bounded interval \( I \). By Lemma 4.3, \( f_n(\lambda) = 0 \) and \( f(\lambda) = 0 \) have unique solutions. Assume \( f(\lambda_0) = 0 \) and \( f_n(\lambda_n) = 0 \). Then we have \( \lambda_n \to \lambda_0, \) i.e. \( \tau^\Psi(z, \alpha, n) \to \tau^\Psi(z, \alpha) \).

Since \( \tau^\Psi(0, \alpha, n) = D(\Psi^n) \) and we have shown that \( D(\Psi^n) \to D(\Psi) \), we get \( \tau^\Psi(0, \alpha, n) \to \lim_{n \to \infty} \tau^\Psi(0, \alpha, n) = D(\Psi) \), and then \( \tau^\Psi(0, \alpha, n) \to \tau^\Psi(0, \alpha, n) = D(\Psi) \).

Now assume \( \alpha \in \text{int}(L_\Phi) \). Since \( \|\Phi^n - \Phi\|_{\text{lim}} \to 0 \), it is easy to show that \( d_H(L_{\Phi^n}, L_{\Phi}) \to 0 \). Since \( L_{\Phi^n} \) and \( L_{\Phi} \) are all compact convex sets and \( L_{\Phi} \) has dimension \( d \), it is seen that \( L_{\Phi^n} \) has nonempty interior for large \( n \). Moreover we can find \( N \in \mathbb{N} \) and \( \delta_0 > 0 \), such that \( B(\alpha, \delta_0) \subset L_{\Phi^n} \) for any \( n \geq N \). By Lemma 5.3(4) for any \( z \in \mathbb{R}^d \) and any \( n \geq N \)

\[\tau^\Psi(z, \alpha, n) \leq \delta_0 c_1(\Psi^n)\|z\|/2 = \delta_0 c_1(\Psi^n) \geq \delta_0 \|z\|/2 \|\Psi^n\|_{\text{min}} \geq \delta_0 \|z\|/2 \|\Psi\|_{\text{min}}.\]

Letting \( n \) tend to \( \infty \) and then \( |z| \to \infty \) we get \( \lim_{|z| \to \infty} \tau^\Psi(0, \alpha, n) = +\infty. \) Thus we can find a \( z_0 \in \mathbb{R}^d \) such that \( \tau^\Psi(z_0, \alpha) \geq \tau^\Psi(z, \alpha) \). By Lemma 5.3(3) \( \tau^\Psi(z, \alpha, n) \) is convex. By a well known theorem in convergence analysis [33] (p. 90), \( \tau^\Psi(\cdot, \alpha, n) \) is convex. Moreover the convergence is uniform on any compact domain. Now by the uniform convergence of \( \tau^\Psi(z, \alpha, n) \) to \( \tau^\Psi(z, \alpha) \) over the closed ball \( B(z_0, R) \) with \( R > 0 \) large enough, we can easily show that \( \tau^\Psi(\alpha, n) \to \tau^\Psi(\alpha) \) as \( n \to \infty \).

(2) Since \( \Phi^n \) and \( \Psi^n \) are Hölder continuous, by Lemma 5.3(2), if \( \langle z, \alpha' - \alpha \rangle \geq 0 \), then

\[
C_1^n(\langle z, \alpha' - \alpha \rangle) \leq \tau^\Psi(z, \alpha, n) - \tau^\Psi(z, \alpha', n) \leq C^n(\langle z, \alpha' - \alpha \rangle).
\]

where \( C^n_1 = 1/\|\Psi_{\text{min}}\| \) and \( C^n_2 = 1 + 1/\|\Psi_{\text{max}}\| \). By Lemma 5.1, we know that \( |\Psi_{\text{min}}| \geq |\Psi_{\text{min}}^n| \) and \( |\Psi_{\text{max}}| \leq |\Psi_{\text{max}}| \). Since \( \tau^\Psi(z, \cdot, n) \to \tau^\Psi(z, \cdot) \), letting \( n \to \infty \) we get the result. \[ \square \]

Now we complete the proof of the Proposition 2.6.

**Proof of Proposition 2.6 (Continued)** We prove it in two steps:
(1) For $\Phi$ Hölder and $\Psi \in C_{\text{aa}}^-(\Sigma_A, T)$ the result holds: Let 
\[ c(\Psi, \Phi, n) := \sum_{w \in B_n(\Psi)} \exp(\sup_{x \in [w]} \langle z, \phi_{|w|}(x) \rangle). \]
Fix $\Upsilon$ a Hölder potential and $\delta > \|\Psi - \Upsilon\|_{\text{lim}}$. For any $w \in B_n(\Psi)$ the proof of Lemma 5.5 yields $u \prec w$ such that $u \in B_{n(1 - C_2 \delta)}(\Upsilon)$ and $|w| - |u| \leq C_2(o(n) + n \delta)$. Thus we get 
\[ c(\Psi, \Phi, n) = \sum_{u \in B_{n(1 - C_2 \delta)}(\Upsilon)} \sum_{v \in B_n(\Psi)} \exp(\sup_{x \in [w]} \langle z, \phi_{|u|}(x) \rangle) \leq C_3^{o(n) + n \delta} c(\Upsilon, \Phi, [n(1 - C_2 \delta)]), \]
where $C(\Phi, z)$ is a constant depending on $\Phi$ and $z$ only. Similarly we can get $c(\Psi, \Phi, n) \geq C_4^{o(n) + n \delta} c(\Upsilon, \Phi, [n(1 - C_2 \delta)])$. Since $\|\Psi - \Upsilon\|_{\text{lim}}$ and hence $\delta$ can be taken arbitrarily small, this yields $\tau_{\Psi}^{\Phi}(z, 0) = \lim_{n \to \infty} \ln c(\Psi, \Phi, n)/n$.

(2) For general $\Phi$ and general $\Psi$ the result holds: Indeed, once the previous step is established, by taking a sequence of Hölder potentials $\Phi^j$ such that $\|\Phi^j - \Phi\|_{\text{lim}} \to 0$ one can easily conclude. □

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