From 1-matrix model to Kontsevich model

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Abstract

Loop equations of matrix models express the invariance of the models under field redefinitions. We use loop equations to prove that it is possible to define continuum times for the generic hermitian 1-matrix model such that all correlation functions in the double scaling limit agree with the corresponding correlation functions of the Kontsevich model expressed in terms of kdV times. In addition the double scaling limit of the partition function of the hermitian matrix model agree with the $\tau$-function of the kdV hierarchy corresponding to the Kontsevich model (and not the square of the $\tau$-function) except for some complications at genus zero.
1 Introduction

Since the discovery of the Kontsevich model as a matrix model realization of a τ-function of the kdV hierarchy [1] there has been attempts to relate the partition function of this model to the continuum partition function of various 1-matrix models. In reference [2] a limiting procedure which allows one, on path integral level, to pass from the partition function of the reduced hermitian matrix model to the square of the partition function of the Kontsevich model was presented. However this procedure relied very heavily on the assumption of a symmetrical potential for the 1-matrix model. In reference [3] a different limiting procedure provided a way of passing from the partition function of the generic hermitian 1-matrix model to the non squared partition function of the Kontsevich model. However the latter limiting procedure was rather unconventional, involving an analytic continuation of the size of the matrix of the original 1-matrix model from $N$ to $-\xi N$.

Here we will show using the conventional double scaling limit how one can define continuum times for the generic hermitian matrix model so that its continuum partition function apart from some complications at genus zero turn directly into the non squared partition function of the Kontsevich model. Our proof is based on the loop equations of the two models and is facilitated by the use of the so called moment description of the 1-matrix model introduced in reference [3]. (See also [4] and [5] for earlier versions.)

The content of section 2 is a short review of the moment description away from and in the double scaling limit. In section 3 we have collected some useful formula for the Kontsevich model. Section 4 contains a motivation for our choice of continuum time variables, a derivation of an appropriate version of the loop equations for the hermitian matrix model as well as for the Kontsevich model and finally the proof of our statement. In section 5 we discuss how our procedure can be applied to the complex matrix model and the symmetrical hermitian one. In particular we show that there is no contradiction between the results of this paper and those of references [2], [6] and [7].

2 The hermitian 1-matrix model

2.1 Discrete description

The hermitian 1-matrix model is defined by the partition function

$$Z[\{g_{ij}\}, N] = e^{N^2F} = \int_{N \times N} d\phi \exp(-N \text{ Tr } V(\phi))$$

(2.1)
where the integration is over hermitian $N \times N$ matrices and

$$V(\phi) = \sum_{j=1}^{\infty} \frac{g_j}{j} \phi^j$$  \hspace{1cm} (2.2)

If $g_{2k+1} = 0$ for all values of $k$ we will denote the model as symmetrical. Otherwise we will denote it as generic. The observables of the model are the $s$-loop correlators

$$W(p_1, \ldots, p_s) = N^{s-2} \left\langle \frac{1}{p_1 - \phi} \cdots \frac{1}{p_s - \phi} \right\rangle_{\text{conn.}}$$  \hspace{1cm} (2.3)

where the expectation value is defined in the conventional way and where $\text{conn}$ refers to the connected part. The multi-loop correlators can be found from the free energy by application of the loop insertion operator, $\frac{d}{d\phi(p)}$:

$$W(p_1, \ldots, p_s) = \frac{d}{dV(p_s)} \frac{d}{dV(p_{s-1})} \cdots \frac{d}{dV(p_1)} F$$  \hspace{1cm} (2.4)

where

$$\frac{d}{dV(p)} \equiv -\sum_{j=1}^{\infty} \frac{j}{p^{j+1}} \frac{d}{dg_j}$$  \hspace{1cm} (2.5)

With the normalization chosen above, the genus expansion of the correlators reads

$$W(p_1, \ldots, p_s) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_g(p_1, \ldots, p_s) \quad (s \geq 1).$$  \hspace{1cm} (2.6)

Similarly we have

$$F = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} F_g$$  \hspace{1cm} (2.7)

The genus $g$ contribution to the 1-loop correlator can be found by solving by iteration the loop equations of the model. These equations express the invariance of (2.1) under field redefinitions of the form

$$\phi \rightarrow \phi + \epsilon \sum_{n \geq 0} \frac{\phi^n}{p^{n+1}} = \phi + \epsilon \frac{1}{p - \phi}$$  \hspace{1cm} (2.8)

Under a transformation of this type the measure changes as

$$d\phi \rightarrow d\phi \left( 1 + \epsilon \left[ \text{Tr} \left( \frac{1}{p - \phi} \right) \right]^2 \right)$$  \hspace{1cm} (2.9)

and we get to first order in $\epsilon$

$$\int d\phi \left\{ \left( \text{Tr} \left( \frac{1}{p - \phi} \right) \right)^2 - N \text{Tr} \left( V'(\phi) \frac{1}{p - \phi} \right) \right\} e^{-N \text{Tr} V(\phi)} = 0$$  \hspace{1cm} (2.10)
or introducing (2.3)

\[ \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p - \omega} W(\omega) = (W(p))^2 + \frac{1}{N^2} W(p, p) \]

(2.11)

where \( V(\omega) = \sum_j g_j^{\omega_j} / j \) and \( C \) is a curve which encloses all possible eigenvalues of \( \phi \) i.e. all singularities of \( W(\omega) \). Here we will assume that the density of eigenvalues of \( \phi \) has support only on the interval \([y, x]\) and is normalized to 1. Then the singularities of \( W(p) \) consist of only one square root branch cut on the real axis, \([y, x]\) and \( W(p) \) behaves as \( 1/p \) as \( p \to \infty \). With this assumption the genus-0 contribution to the one loop correlator can be written as

\[ W_0(p) = \frac{1}{2} \oint_C \frac{d\omega}{2\pi i} V'(\omega) \left\{ \frac{(p - x)(p - y)}{(\omega - x)(\omega - y)} \right\}^{1/2} \]

(2.12)

where \( x \) and \( y \) are determined by the following boundary equations

\[ B_1(x, y) = \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} = 0, \]

(2.13)

\[ B_2(x, y) = \oint_C \frac{d\omega}{2\pi i} \frac{\omega V'(\omega)}{\sqrt{(\omega - x)(\omega - y)}} = 2. \]

(2.14)

From the 1-loop correlator all multi-loop correlators can be found by application of the loop insertion operator and the free energy by application of the inverted loop insertion operator. The higher genera contributions are most easily expressed by introducing instead of the couplings, \( \{g_i\} \), some moments, \( \{M_i, J_i\} \). One of the advantages of this change of variables is that the genus \( g \) contribution to the \( s \)-loop correlator, \( W_g(p_1, \ldots, p_s) \), depends only on a finite number of moments namely at most \( 2 \times (3g - 2 + s) \) for the generic hermitian matrix model. As opposed to this \( W_g(p_1, \ldots, p_s) \) depends on the entire set of coupling constants, \( \{g_i\} \). Furthermore the description in terms of moments allows one to access very easily the continuum limit. For the hermitian matrix model the moments are defined by

\[ M_k(x, y, \{g_i\}) = \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - x)^{k+1/2} (\omega - y)^{1/2}} \]

(2.15)

\[ J_k(x, y, \{g_i\}) = \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - x)^{1/2} (\omega - y)^{k+1/2}} \]

(2.16)

Normally we would think of \( x \) and \( y \) as being fixed by the boundary conditions (2.13) and (2.14) for given values of the coupling constants. However it will prove convenient for the following to consider \( M_k \) and \( J_k \) as functions of \( x \) and \( y \) as well as of the coupling constants. Furthermore we shall make use of the following rewriting of \( W_0(p) \)

\[ W_0(p) = \frac{1}{2} V'(p) - \frac{1}{4} (p - x)^{1/2} (p - y)^{1/2} \sum_{q=1}^{\infty} \{ (p - x)^{q-1} M_q + (p - y)^{q-1} J_q \} \]

(2.17)
It is derived by deforming the contour of integration in (2.12) into two, one which encloses the point \( \omega = p \) and one which encircles infinity. To find the contribution from the latter we write \((p - \omega)^{-1}\) as

\[
\frac{1}{p - \omega} = \frac{1}{2} \left\{ \frac{1}{(p - x) - (\omega - x)} + \frac{1}{(p - y) - (\omega - y)} \right\} \tag{2.18}
\]

and expand i powers of \( \frac{p-x}{\omega-x} \) and \( \frac{p-y}{\omega-y} \) respectively. The expansion procedure is justified by the fact that the contour of integration encircles infinity which allows us to assume that \( \omega >> p \). We note that the two bracketed terms in (2.17) are actually identical. This is an example of another appealing feature of the moment description. All expressions are invariant under the interchanges \( x \leftrightarrow y, J \leftrightarrow M \) which of course just reflects the fact that a priori there is nothing which allows us to distinguish between the two endpoints of the cut. (In the continuum limit, which we are going to consider, this will not be true.) Furthermore it is worth noting that the expression (2.17) allows us to determine very easily the inverse transformations \( g_i = g_i(x, y, \{M_i\}, \{J_j\}) \) since we know that \( W_0(p) \) only contains negative powers of \( p \).

In reference [3] explicit expressions for \( W_g(p) \) and \( F_g \) in terms of the moments were presented for the lowest genera. Furthermore their general form were conjectured and proven by induction. Here we will only be interested in terms which are relevant for the double scaling limit.

### 2.2 The continuum limit

Certain points in the (infinite dimensional) coupling constant space are of particular interest, namely the Kazakov multi-critical points [10]. The \( m \)'th multi-critical points are in the case of a generic hermitian matrix model characterized by the fact that the eigenvalue distribution which under normal circumstances vanishes as a square root at both endpoints of its support acquires \((m - 1)\) additional zeros at one end point, say \( x \). The condition for being at a \( m \)'th multi-critical point is in this case

\[
M_1 = M_2 = \ldots = M_{m-1} = 0, \quad M_m \neq 0, \quad J_1 \neq 0 \tag{2.19}
\]

as is seen from (2.17). In other words the \( m \)'th multi-critical points constitute a subspace of coupling constant space characterized by the constraints (2.19). Let \( \{g_i, c\} \) denote a particular multi-critical point in this subspace for which the eigenvalues of the matrix model are confined to the interval \([y_c, x_c]\). If we change the coupling constants we will move away from the subspace, the \( M_k \)'s, \( k \in [1, m - 1] \) will no longer be zero and the support of the eigenvalue distribution will move to \([y, x]\). For a variation of the coupling constants of the type

\[
g_i = g_i, + \delta g_i, \quad \delta g_i \sim o(a^m) \tag{2.20}
\]
we will have

\[ M_k \sim a^{m-k}, \quad k \in [0, m] \quad (2.21) \]

while the J-moments do not scale. This means that \( x \) and \( y \) must scale in the following way

\[ x - x_c \sim a, \quad y - y_c \sim a^m. \quad (2.22) \]

The continuum limit is defined as the limit where we send \( a \) to zero keeping however the string coupling constant \( a^{-2m-1}N^{-2} \) fixed. The definite scaling properties of the moments make these parameters well suited for studying the continuum limit of the 1-matrix model. Making use of the scaling properties of the moments an iterative procedure which allows one to calculate directly the double scaling relevant versions of \( F_g \) and \( W_g(p_1, \ldots, p_s) \) was developed [3]. The iterative procedure provided a proof that the continuum relevant version of \( F_g \) for a generic model, in the following denoted as \( F_g^{(NS)} \), takes the following form

\[ F_g^{(NS)} = \sum_{1<\alpha_j \leq m} \langle \alpha_1 \ldots \alpha_s | \alpha \rangle_g \frac{M_{\alpha_1} \ldots M_{\alpha_s}}{M_1^\alpha d_c^{-1}} \quad g \geq 1 \quad d.s.l. \quad (2.23) \]

where the sum is over all sets of indices obeying the following restrictions

\[ \sum_{j=1}^{s} \alpha_j = 3g - 3 + s \quad \alpha = 2g - 2 + s \quad (2.24) \]

The parameter \( d_c \) is the distance between the endpoints of the support of the eigenvalue distribution, \( d_c = x_c - y_c \) and the quantity in brackets is a real number. All terms in (2.23) are of the same order, namely \( a^{-(2m+1)(1-g)} \). Bearing in mind the relation (2.21) we see that the moments \( M_k \) appear as bare coupling constants or mass parameters of the theory and that continuum coupling constants, \( \mu_k \), can be introduced by

\[ M_k = a^{m-k} \mu_k, \quad k \in [0, m] \quad (2.25) \]

As in ordinary statistical mechanics the physical coupling constants are defined not at the critical point but by the approach to the critical point. In the neighbourhood of an \( m \)'th multi-critical point we can consequently write

\[ F_g^{(NS)} = \sum_{1<\alpha_j \leq m} \langle \alpha_1 \ldots \alpha_s | \alpha \rangle_g \frac{\mu_{\alpha_1} \ldots \mu_{\alpha_s}}{\mu_1^\alpha} \frac{1}{[a^{2m+1}d_c]^{g-1}}, \quad g \geq 1 \quad d.s.l. \quad (2.26) \]

In the following we will consider a specific type of deformation \( \delta g_i = o(a^m) \) away from the \( m \)'th multi-critical point \( \{g_{i,c}\} \), namely one for which \( y \) is kept fixed at \( y_c \). This choice is not essential but will simplify our derivations.
3 The Kontsevich model

The Kontsevich model is defined by the partition function

\[ Z_{\text{Kont}}[N, M] = e^{N^2 F_{\text{Kont}}} = \frac{\int dX \exp \left\{ -N \text{Tr} \left( \frac{MX^2}{2} + iX^3 \right) \right\}}{\int dX \exp \left\{ -N \text{Tr} \left( \frac{MX^2}{2} \right) \right\}} \]  (3.1)

where the integration is over \( N \times N \) hermitian matrices. This partition function only depends on the parameters \( t_k \)

\[ t_k = \frac{1}{N} \text{Tr} \ M^{-(2k+1)} \]  (3.2)

and expressed in terms of these it is a \( \tau \)-function of the kdv hierarchy. As is the case for 1-matrix models the free energy \( F_{\text{Kont}} \) has a genus expansion (Cf. to equation (2.7)).

The genus-0 contribution to \( F_{\text{Kont}} \) reads

\[ F_{0,\text{Kont}} = \frac{1}{3} \sum_{i=1}^{N} m_i^3 - \frac{1}{3} \sum_{i=1}^{N} (m_i^2 - 2u_0)^{3/2} - u_0 \sum_{i=1}^{N} (m_i^2 - 2u_0)^{1/2} \]

\[ + \frac{u_0^3}{6} - \frac{1}{2} \sum_{i,j=1}^{N} \ln \left( \frac{(m_i^2 - 2u_0)^{1/2} + (m_j^2 - 2u_0)^{1/2}}{m_i + m_j} \right) \]  (3.3)

where the \( m_i \)'s are the eigenvalues of the matrix \( M \) and the parameter \( u_0 \) is given by the boundary condition

\[ u_0 + \frac{1}{N} \sum_i \frac{1}{\sqrt{m_i^2 - 2u_0}} = 0 \]  (3.4)

It can be derived by means of the Dyson Schwinger equations of the model as done in references \[ \text{[11, 12]} \]. Alternatively it can be found by exploiting the fact that \( Z_{\text{Kont}}[N, \{t_k\}] \) is a \( \tau \)-function of the kdv hierarchy. The latter approach was followed in reference \[ \text{[13]} \] where it was also shown that the higher genera contributions can be written in the following form

\[ F_{g,\text{Kont}} = \sum_{\alpha_j>1} \langle \alpha_1 \ldots \alpha_s | \alpha \rangle_{g,\text{Kont}} \frac{I_{\alpha_1} \ldots I_{\alpha_s}}{(1 - T_1)^a} \quad g \geq 1 \]  (3.5)

where the sum is over sets of indices obeying the following restrictions

\[ \sum_{j=1}^{s} \alpha_j = 3g - 3 + s \quad a = 2g - 2 + s \]  (3.6)

and where the moments \( I_k \) are defined by

\[ I_k = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{(m_j^2 - 2u_0)^{k+1/2}} \quad k \geq 0 \]  (3.7)
The quantities in brackets can be given an interpretation in terms of intersection indices on moduli space \([1, 14]\). The formula (3.5) encodes all informations about the \(k\)dV hierarchy and hence all information about the string equations corresponding to the different multi-critical points mentioned in the previous section. As shown in reference \([13]\) to obtain the string equation of the \(m\)th multi-critical model from the free energy of the Kontsevich model one should neglect \(I_k\) with \(k > m\), keep \(I_m\) constant, send \(I_1 - 1, I_2, \ldots, I_{m-1}\) to zero and introduce a scaling parameter \(s\) such that

\[
v_q = \frac{I_q(1 - I_1)^{q-2}}{I_2^{q-1}} \quad 3 \leq q \leq m
\]

remains finite while

\[
s = \frac{1 - I_1}{I_2^{3/5}}
\]

tends to zero. In this limit we hence have

\[
F_g^{(NS)} = \sum_{1<\alpha_1 \leq m} \langle \alpha_1 \ldots \alpha_s | \alpha \rangle_g v_{\alpha_1} \ldots v_{\alpha_s} \frac{1}{s^{3(g-1)}} \quad g \geq 1
\]

We see that this prescription for fine tuning the \(I\)'s is exactly the same fine tuning as the one for the \(M_k\)'s dictated by the double scaling limit, \(s^5\) playing a role similar to that of \(a^{(2m+1)}\). Furthermore it was shown in reference \([3]\) that

\[
\langle \alpha_1 \ldots \alpha_s | \alpha \rangle_{g}^{Kont} = \langle \alpha_1 \ldots \alpha_s | \alpha \rangle_g
\]

Hence it is obvious that we have a 1-1 mapping between \(F_g^{(NS)}\) and \(F_g^{Kont}\) for \(g \geq 1\). However the proof of reference \([3]\) was based on a somewhat unconventional limiting procedure as already noted. It also left unanswered the question what parameters of the original 1-matrix model play the role of continuum time variables. Here we will show using the usual double scaling prescription that it is possible to define continuum time variables for the generic hermitian matrix model such that the double scaling limit of its partition function apart from some complications for genus zero turns into the partition function of the Kontsevich model.

4 From 1-matrix model to Kontsevich model

The connecting link between the hermitian 1-matrix model and the Kontsevich model is the boundary equations of the two models. Below we first show how these equations immediately tell us how to define continuum time variables for the hermitian 1-matrix model. The rest of the section is devoted to proving the following conjecture. With the definition of continuum times given in equation (4.6) the continuum partition function
of the generic hermitian 1-matrix model deviates from the one of the Kontsevich model only at genus zero. The proof is based on the loop equations of the two models and is yet another example of the strength of these equations.

4.1 Definition of continuum time variables.

To motivate our choice of continuum time variables $T_k$ for the non symmetrical hermitian 1-matrix model let us write the boundary equation for the Kontsevich model as

$$\sum_{k=0}^{\infty} c_k \left( t_k + \delta_{k,1} \right) (2u_0)^k = 0, \quad c_k = \frac{(2k - 1)!!}{k! 2^k}$$

and let us remind the reader that to study the $m$’th kdV flow we should keep in this equation only $t_0$ and $t_m$.

The idea is now to define $T_k$ in such a way that by taking the double scaling limit of the boundary equations (2.13) and (2.14) we reproduce equation (4.1) with the $T_k$’s replacing the $t_k$’s. Here and in the following we will consider a 1-matrix model for which the eigenvalues, at the critical point, are confined to the interval $[y_c, x_c]$. As mentioned earlier, when we move away from a given $m$’th multi-critical point by a change of coupling constants $\delta g_i \sim o(a^m)$ in general both $x$ and $y$ will change. For the simplicity of the presentation we restrict ourselves to the subclass of deformations for which $y$ is kept fixed at $y_c$. Expanding equation (2.13) in powers of $(x - x_c)$ we find

$$\sum_{p=0}^{\infty} c_p (x - x_c)^p M_p^c \left( \{ g_i \} \right) = 0$$

where

$$M_p^c \left( \{ g_i \} \right) = M_p \left( x_c, y_c, \{ g_i \} \right),$$

Note that the coupling constants entering $M_p^c$ are completely arbitrary. Rewriting the boundary equation (2.14) we find

$$\sum_{p=0}^{\infty} c_p (x - x_c)^p \left\{ x_c M_p^c \left( \{ g_i \} \right) + M_{p-1}^c \left( \{ g_i \} \right) \right\} = 2$$

Since $M_p^c \sim a^{m-p}$, for $0 \leq k \leq m$ the term $M_{p-1}^c$ is subleading when compared to $M_p$ except for $M_{-1}$ which is equal to 2 up to subleading terms of $o(a^{m+1})$. If we set

$$x - x_c = a (2u_0)$$

and define our continuum time variables by

$$T_k + \delta_{k,1} = a^{k+1/2} d_c^{1/2} M_k^c \left( \{ g_i \} \right) \quad k \geq 0$$

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both (4.2) and (4.4) turn into the boundary equation of the Kontsevich model (4.1).
(The reason why we include an additional factor $\sqrt{a} d^{1/2}$ will become clear later.)

4.2 Continuum loop equations for the 1-matrix model

In view of the scaling behaviour (4.5) and the general form of $W_g(p)$ it is natural to introduce continuum momentum variables $\pi_i$ by

$$p_i = x_c + a\pi_i$$  (4.7)

We define continuum correlators $W^{cont}(\pi_1, \ldots, \pi_s)$ by

$$W(p) - \frac{1}{2} V'(p) \xrightarrow{d.s.l.} \frac{1}{a} \left( W^{cont}(\pi) - \frac{1}{2} \sum_{k=0}^{\infty} (T_k + \delta_{k,1})\pi^{k-1/2} \right)$$  (4.8)

$$W(p_1, p_2) + \frac{1}{2} \left( p_1 - p_2 \right)^2 \xrightarrow{d.s.l.} \frac{1}{a^2} \left( W^{cont}(\pi_1, \pi_2) + \frac{\frac{1}{2}(\pi_1 + \pi_2)}{2(\pi_1 - \pi_2)^2}\sqrt{\pi_1 \pi_2} \right)$$  (4.9)

$$W(p_1, \ldots, p_s) \xrightarrow{d.s.l.} \frac{1}{a^s} W^{cont}(\pi_1, \ldots, \pi_s)$$  (4.10)

The subtraction needed to make contact with continuum physics only concerns the genus zero contribution to the 1- and 2-loop correlators. This is a well known feature of the double scaling limit. In particular we have

$$W(p, p) \xrightarrow{d.s.l.} \frac{1}{a^2} \left( W^{cont}(\pi, \pi) + \frac{1}{16\pi^2} \right)$$  (4.11)

We note that the prescription (4.7) for taking the double scaling limit implies that in this limit the cut $[y, x]$ on the real axis for $W(p)$ is replaced by a cut of the type $[-\infty, 2u_0]$. This is of course a pleasing result since this is exactly the analyticity structure of the 1-loop correlator of the Kontsevich model [11]. Let us comment on the $a$ factors extracted on the right hand side. As shown in reference [3] in the double scaling limit the loop insertion operator reduces to

$$\frac{d}{dV(p)} = \sum_{n=1}^{\infty} \frac{dM_n}{dV(p)} \frac{\partial}{\partial M_n} + \frac{dx}{dV(p)} \frac{\partial}{\partial x} \xrightarrow{d.s.l.}$$  (4.12)

where

$$\frac{dM_n}{dV(p)} = -(n + 1/2) \left\{ d_{e^{-1/2}}(p - x)^{-n-3/2} - M_{n+1} \frac{dx}{dV(p)} \right\} \xrightarrow{d.s.l.}$$  (4.13)

$$\frac{dx}{dV(p)} = \frac{1}{M_1} d_{e^{-1/2}}(p - x)^{-3/2} \xrightarrow{d.s.l.}$$  (4.14)

1 We note that the subclass of deformations considered above does not include the deformations that are associated with $m$'th multi-critical behaviour in the usual 1-matrix model sense. These deformations correspond to the situation where only $T_m$ and $T_0$ are different from zero. Such a situation can only be obtained if we scale the coupling constants as $g_i = g \cdot g_{i,e}$ which requires that both $x$ and $y$ must scale. However the arguments of this and all following sections can be repeated for the case where $y$ is not kept fixed. The expressions just become more involved.
Now expanding the moment $M_p$ in powers of $x - x_c$ gives
\[
M_p = \sum_{l=p}^{\infty} a^{-p-1/2} (T_l + \delta_{1,l}) (2u_0)^{l-p} \frac{\Gamma(l+1/2)}{\Gamma(p+1/2) (l-p)!}, \quad p \geq 1
\] (4.15)

(From this expression one can read off the continuum scaling behaviour of the moments for a given $m$'th multi-critical model and the relation (2.21) is easily reproduced.) Using equation (4.15) the scaling relation (4.3) for $x$ and the boundary equation expressed in terms of $T_k$'s one can by means of the chain rule show that the following relation holds
\[
\frac{d}{dV(p)} \xrightarrow{d.s.l.} \frac{1}{a} \frac{d}{dV_{cont}(\pi)} = \frac{1}{a} \left\{ -\sum_{k=0}^{\infty} (k + 1/2) \frac{1}{\pi^{k+3/2}} \frac{d}{dT_k} \right\}
\] (4.16)

Bearing in mind the relation (2.4) it appears natural to extract one power of $a^{-1}$ for each loop in a given correlator. The relation (4.16) will be essential for the proof of our conjecture. Due to the peculiarities of the genus zero contributions to the 1- and 2-loop correlators it is convenient to write the loop equations in a genus expanded version. Inserting the genus expansion (2.6) into (2.11) it is seen that $W_g(p), \ g \geq 1$ obeys the following equation.
\[
\left\{ \hat{K} - 2W_0(p) \right\} W_g(p) = \sum_{g'=1}^{g-1} W_{g'}(p) W_{g-g'}(p) + W_{g-1}(p,p), \quad g \geq 1
\] (4.17)

where
\[
\hat{K} f(p) = \oint_C \frac{d\omega}{2\pi i} \frac{V''(\omega)}{p - \omega} f(\omega)
\] (4.18)

Let us now introduce in this equation the continuum correlators. First we note that since $W_0(p)$ itself does not scale we have in the double scaling limit
\[
\left\{ \hat{K} - 2W_0(p) \right\} W_g(p) \xrightarrow{d.s.l.} \frac{1}{a^2} \oint_C \frac{d\omega}{2\pi i} \frac{V''(\omega) - 2W_0(\omega)}{p - \omega} W_{g\text{ cont}}(\omega)
\] (4.19)

Using the definition (4.8) the right hand side (times $a^2$) can be written as
\[
\text{rhs} = -\left\{ 2W_0^{\text{cont}}(\pi) - \sum_k (T_k + \delta_{k,1}) \pi^{k-1/2} \right\} W_g^{\text{cont}}(\pi)
\]
\[-\oint_{\infty} \frac{d\omega}{2\pi i} \left\{ 2W_0^{\text{cont}}(\omega) - \sum_k (T_k + \delta_{k,1}) \omega^{k-1/2} \right\} W_g^{\text{cont}}(\omega)
\] (4.20)

where $\oint_{\infty}$ denotes a contour integral where the contour encircles infinity. To proceed let us write $W_g^{\text{cont}}(\omega)$ as
\[
W_g^{\text{cont}}(\omega) = \sum_{q=0}^{\infty} \omega^{-q-3/2} W_g^{\text{cont},q}
\] (4.21)
That $W_g^{\text{cont}}$ allows an expansion of this type is obvious for $g \geq 1$ since for $g \geq 1$ we have $W_g^{\text{cont}}(\omega) = \frac{d}{dV^{\text{cont}}(\omega)} F_g$, where $\frac{d}{dV^{\text{cont}}(\omega)}$ is given by equation (4.16). (It is also evident from the explicit expression for $W_0(p)$ given in reference [3].) That the same is true for $g = 0$ will become clear in section (4.4). Performing the contour integral in (4.20) and making use of the definition (4.9) one obtains the following continuum loop equation

$$
\left[ 2W_0^{\text{cont}}(\pi) - \sum_k (T_k + \delta_{k,1}) \pi^{-k+1/2} \right] W_g^{\text{cont}}(\pi) - \sum_{q,a} (T_{2+a+q} + \delta_{2+a+1}) W_g^{\text{cont},q} \pi^a
$$

$$
= g - 1 \sum_{g'=1}^{g-1} W_g^{\text{cont}}(\pi) W_{g'-1}(\pi, \pi) + \delta_{g,1} \cdot \frac{1}{16\pi^2} \tag{4.22}
$$

4.3 Loop equations for the Kontsevich model

Inspired by the equation (4.16) let us introduce a loop insertion operator for the Kontsevich model by

$$
\frac{d}{dV^{\text{Kont}}(\pi)} \equiv - \sum_k (k + 1/2) \frac{1}{\pi^{k+3/2}} \frac{d}{dt_k} \tag{4.23}
$$

and multi-loop correlators by

$$
W^{\text{Kont}}(\pi_1, \ldots, \pi_s) = \frac{d}{dV^{\text{Kont}}(\pi_s)} \cdots \frac{d}{dV^{\text{Kont}}(\pi_1)} F^{\text{Kont}}, \quad s \geq 1 \tag{4.24}
$$

Using the relation (3.2) it is easy to show by means of the chain rule that

$$
\frac{d}{dV^{\text{Kont}}(\pi)} = N \left. \frac{\partial}{\partial m_i^2} \right|_{m_i^2 = \pi} \tag{4.25}
$$

In this form the loop insertion operator can readily be applied to (3.3) to yield

$$
W_0^{\text{Kont}}(\pi) = \frac{1}{2} \left[ -\sqrt{\pi - 2u_0} + \sqrt{\pi} + \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\pi - m_j^2} \left( \frac{\sqrt{\pi - 2u_0}}{\sqrt{m_j^2 - 2u_0}} - m_j \right) \right] \tag{4.26}
$$

and

$$
W_0^{\text{Kont}}(\pi_1, \pi_2) = \frac{1}{4(\pi_1 - \pi_2)^2} \left\{ \frac{\pi_1 + \pi_2 - 4u_0}{\sqrt{\pi_1 - 2u_0} \sqrt{\pi_2 - 2u_0}} - \frac{\pi_1 + \pi_2}{\sqrt{\pi_1} \sqrt{\pi_2}} \right\} \tag{4.27}
$$

With the definitions given above the master equation of the Kontsevich model can be written as

$$
\frac{1}{N^2} \left\{ W^{\text{Kont}}(\pi, \pi) + \frac{1}{16\pi^2} \right\} + (W^{\text{Kont}}(\pi))^2 + \frac{1}{N} \sum_{j=1}^{N} \frac{\tilde{W}^{\text{Kont}}(m_j^2)}{m_j^2 - \pi} = \frac{\pi}{4} \tag{4.28}
$$
where
\[ \tilde{W}^{Kont}(\pi) = W^{Kont}(\pi) - \frac{1}{2} \sum_k (t_k + \delta_{k,1})\pi^{k-1/2} \] (4.29)

To make contact with the previous section we will rewrite this equation in a genus expanded version. To deal with the sum appearing on the left hand side above let us note that the genus \( g \) contribution to the 1-loop correlator can be expanded in the following way
\[ W_g^{Kont}(\pi) = \sum_{q=0}^{\infty} \pi^{-q-3/2} W_{g,q}^{Kont} \] (4.30)

Then making use of the definition of the \( t_k \)'s, (3.2) we arrive at the following form of the loop equation.
\[ -\left[ 2W_0^{Kont}(\pi) - \sum_k (t_k + \delta_{k,1})\pi^{k-1/2} \right] W_g^{Kont}(\pi) - \sum_{q,a} (t_{2+a+q} + \delta_{2+q+a,1}) W_{g,q}^{Kont} \pi^a \]
\[ = \sum_{g'=1}^{g-1} W_{g'}^{Kont}(\pi) W_{g-g'}^{Kont}(\pi) + W_{g-1}^{Kont}(\pi, \pi) + \delta_{g,1} \cdot \frac{1}{16\pi^2} \] (4.31)

4.4 Proof of our conjecture

The task of proving our conjecture amounts to proving the following two identities
\[ W_0^{cont}(\pi) = W_0^{Kont}(\pi) \] (4.32)
\[ W_0^{cont}(\pi_1, \pi_2) = W_0^{Kont}(\pi_1, \pi_2) \] (4.33)

where it is understood that the quantities on the left hand side should be expressed in terms of \( T_k \)'s whereas those on the right hand side should be expressed in terms of \( t_k \)'s. By comparing equation (4.22) and (4.31) it is easily seen that once this task has been fulfilled it follows by induction that
\[ W_g^{cont}(\pi, \{T_k\}) = W_g^{Kont}(\pi, \{t_k\}), \quad g \geq 1 \] (4.34)

since now the two sets of loop equations and corresponding boundary equations only differ by \( \{T_k\} \) appearing in one case and \( \{t_k\} \) in the other. Furthermore by taking a glance at equation (4.14) and (4.23) bearing in mind the relations (2.4) and (4.8) – (4.10) one easily convinces oneself that
\[ W_g^{cont}(\pi_1, \ldots, \pi_s, \{T_k\}) = W_g^{Kont}(\pi_1, \ldots, \pi_s, \{t_k\}), \quad g, s \geq 1 \] (4.35)

From equation (4.8) and (4.34) it follows that
\[ W_g(p) \overset{d.s.t.}{\longrightarrow} \frac{1}{\alpha} W_g^{Kont}(\pi), \quad g \geq 1 \] (4.36)
since both the second term on the left hand side and the second term on the right hand side of (4.32) are of zeroth order in genus. Now due to the similarity between \( \frac{d}{dV_{\text{cont}}(\pi)} \) and \( \frac{d}{dV_{\text{cont}}(\pi)} \) we immediately find

\[
F_g \xrightarrow{\text{d.s.l.}} F_{g}^{\text{Kont}}, \quad g \geq 1
\]

(4.37)

To address the \( g = 0 \) case we note that equation (4.9) and (4.33) imply that the double scaling limit of \( W_0(p_1, p_2) \) differs from \( W_0^{\text{Kont}}(\pi_1, \pi_2) \) only by a term which does not depend on any couplings. Therefore we have

\[
W_0^{\text{cont}}(\pi_1, \ldots, \pi_s) = W_0^{\text{Kont}}(\pi_1, \ldots, \pi_s), \quad s \geq 3
\]

(4.38)

However, equation (4.32) does not allow us to conclude anything about the relation between the genus zero contributions to the partition functions of the two models because of the subtractions appearing in equation (4.8).

Let us now turn to the proof of the relations (4.32) and (4.33). The proof of the latter is by far the most straightforward since \( W_0(p_1, p_2) \) is universal, i.e. it does not contain any explicit reference to the coupling constants. By acting with \( \frac{d}{dV(p_2)} \) on \( W_0(p_1) \) one gets

\[
W_0(p_1, p_2) = \frac{1}{2(p_1 - p_2)^2} \left\{ \frac{p_1 p_2 - \frac{1}{2}(p_1 + p_2)(x + y) + x y}{\sqrt{(p_1 - x)(p_1 - y)(p_2 - x)(p_2 - y)}} - 1 \right\}
\]

(4.39)

and taking the double scaling limit following the prescriptions (4.5) and (4.7) one easily reproduces (4.9) with \( W_0^{\text{Kont}}(\pi_1, \pi_2) \) replacing \( W_0^{\text{cont}}(\pi_1, \pi_2) \). To prove the relation (4.32) we will prove that

\[
W_0(p) = \frac{1}{2} V'(p) + \frac{1}{2} \sum_{k=0}^{\infty} (T_k + \delta_{k,1}) \pi^{k-1/2} \xrightarrow{\text{d.s.l.}} \frac{1}{a} W_0^{\text{Kont}}(\pi)
\]

(4.40)

First we note that due to equation (2.12) we can write the two first terms as

\[
W_0(p) - \frac{1}{2} V'(p) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \left\{ \frac{(p - x)(p - y)}{(\omega - x)(\omega - y)} \right\}^{1/2}
\]

(4.41)

Furthermore inserting our definition of continuum times (4.6) into the remaining term on the left hand side of (4.40) we get

\[
\frac{1}{a} \sum_{k=0}^{\infty} (T_k + \delta_{k,1}) \pi^{k-1/2} = \frac{1}{2} \sum_{k=0}^{\infty} (a \pi)^{k-1/2} d_c^{1/2} M_k^c (\{ g_i \})
\]

\[
= - \left\{ W_0(p) - \frac{1}{2} V'(p) \right\} \bigg|_{x = -x_c, y = y_c} + \frac{1}{2} (p - x_c)^{-1/2} (p - y_c)^{1/2} M_0^c (\{ g_i \})
\]

\[
= - \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} V'(\omega) \left\{ \frac{(p - y_c)(\omega - x_c)}{(p - x_c)(\omega - y_c)} \right\}^{1/2}
\]

(4.42)
To obtain the second equality sign we have made use of the rewriting of $W_0(p)$ given in (2.17) and the scaling relation for $p$, (4.7). (In particular we have used that $(p - y)$ in the continuum limit can be replaced by $d_c$.) We note that it was in order to be able to carry out this step that we had to multiply our boundary equation (4.2) by $d_c^2 \sqrt{a}$. To obtain the third equality sign we have made use of the relation (4.41). So our statement is now the following

$$W^{\text{Kont}}_0 = \lim_{d.s.l.} a \cdot \left[ \frac{1}{2} \int_\infty^{2\pi i} \frac{d\omega}{p - \omega} \left\{ \frac{\sqrt{(p - x)(p - y_c)}}{\sqrt{(\omega - x)(\omega - y_c)}} - \frac{\sqrt{(\omega - x_c)(p - y_c)}}{\sqrt{(p - x_c)(\omega - y_c)}} \right\} \right]$$ (4.43)

The similarity with equation (4.26) is striking and the equality is straightforward to prove. To do so one expands $(p - \omega)^{-1}$ in powers of $\frac{x - x_c}{\omega - x_c}$ and the quantities $(p - x)$ and $(\omega - x)$ in powers of $\frac{\omega - x_c}{\omega - x_c}$ and $\frac{\omega - x_c}{\omega - x_c}$ respectively. The factor $(p - y_c)$ can simply be replaced by $d_c$. The $\frac{1}{\sqrt{\pi}}$ term of (4.43) vanishes as it should. This is actually ensured by the boundary equation. In the process of expanding the integrand it proves convenient to pull out a factor $(p - x)^{1/2}$. The result of the expansion procedure is the following expression for the right hand side of (4.43)

$$\text{rhs} = \lim_{d.s.l.} \left\{ \frac{1}{2} \frac{1}{2} (p - x)^{1/2} (p - y_c)^{1/2} \sum_{b=1}^{\infty} \sum_{m=1}^{c_b} \frac{(x - x_c)^b}{(p - x_c)^{b-m+1} M_c^m} \right\}$$ (4.44)

$$= \frac{1}{2} (\pi - 2u_0)^{1/2} \sum_{b=1}^{\infty} \sum_{m=1}^{c_b} \frac{(2u_0)^b}{\pi^{b-m+1}} (T_m + \delta_{m,1})$$ (4.45)

By rewriting (4.26) using the definition (3.2) of times it is easy to show that the expression (4.45) is exactly $W^{\text{Kont}}_0(\pi)$.

5 Discussion

We have in the present paper considered the generic hermitian matrix model. However the same strategy can be applied to the complex matrix model. The complex matrix model is in all respects very similar to the hermitian matrix model. It has a set of loop equations which can be written in the same form as that of equation (2.11). The appropriate requirement concerning the analyticity structure of its 1-loop correlator is that it has only one square root branch cut $[\sqrt{z}, \sqrt{z}]$ on the real axis. With this requirement one can solve the loop equations genus by genus. The solution of course depends on the parameter $z$ which is determined by a boundary condition similar to (2.14). As before the multi-loop correlators can be found by applying a loop insertion operator to the 1-loop correlator and as before expressing the higher genera contributions to the correlators is facilitated by introducing a moment description. To
relate the double scaling of the partition function of the complex matrix model to the one of the Kontsevich model one takes the same line of action as for the hermitian matrix model. Appropriate continuum time variables are defined by the requirement that the boundary equation of the complex matrix model reproduces the boundary equation of the Kontsevich model when the double scaling limit is taken. The resulting variables turn out to be related to the moments of the complex matrix model by an equation similar to (4.6). Furthermore for the loop insertion operator one has again a relation like (4.16). However, a closer analysis of the loop equations shows that in stead of (4.37) we have

$$F_g^{\text{C}} \xrightarrow{\text{d.s.l.}} \frac{1}{4g-1} \left(2F_g^{\text{Kont}}\right), \quad g \geq 1$$

(5.1)

and that in the double scaling limit the partition function of the complex matrix model involving matrices of size $N \times N$ turn into the square of the partition function of the Kontsevich model involving matrices of size $2N \times 2N$ except for some complications at genus zero. The same is true for the continuum partition function of the reduced hermitian matrix model since in reference [3] it was shown that

$$F_g^{\text{C}} = \frac{1}{4g-1} F_g^{(S)} \quad \text{d.s.l.}$$

(5.2)

where $F_g^{(S)}$ is the genus $g$ contribution to the free energy of the reduced model. (We refer to [3] for details.) Let us mention that the factor two in difference between the double scaling limit of the free energy of the symmetrical and the generic hermitian matrix model is easy to understand in the eigenvalue picture. It arises because the double scaling relevant part of $F_g$ in the case of a symmetrical model gets contributions from both ends of the support of the eigenvalue distribution while in the generic case there is only critical behaviour associated with one end of the eigenvalue distribution. And let us stress that it is not possible to pass from the generic to the symmetrical case after the continuum limit has been taken since the possibility of having different behaviour at the two endpoints exists only in the generic case.

Acknowledgements We thank L. Chekhov, Yu. Makeenko, A. Marshakov and A. Mironov for interesting and valuable discussions.

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