Stability solutions of a dumbbell-like system in an elliptical orbit

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Abstract. A dumbbell-like system is analyzed, considering two mass points connected with a massless and rigid tether with length variations, and the center of mass described by Keplerian orbits. This kind of system, in a certain type of configuration, is a simple conceptualization of the space elevator. The system motion is obtained with the Lagrangian Formulation in a Central Gravitational Field, and the perturbations of motion are neglected. Laws of control are considered for the angle of systems rotation around the center of mass. Those include uniform rotations or no rotation at all. Stability conditions were obtained for the first case, analyzing its neighborhood and using Floquet Theory. The results shown there are regions of eccentricities were stability is found. Lastly, a dynamic numerical simulator was created, where the implementation of the results could be tested. The dynamic behavior of the system showed regular and chaotic properties.

1. Introduction
Researchers have been studying, since 1960th, the possibilities to control attitude motion of space systems with variable mass distributions. Perturbations of motion can occur for several such atmospheric drag, solar pressure, system deployment, n-body perturbations else which will not be discussed here. It will be analyzed stability conditions inside solutions in Keplerian orbits for a model for two-body tether systems in a Central Gravitational Field (Newtonian). The motions equation will be written by Lagrangian formulation, explaining the Kinetic and Potential Energy, considering the possibility to control the tether length or the angle of rotation around the center of mass. Several authors studied attitude dynamics of TSS of different configurations. Two-body and three-body systems have been mostly studied, but even tethered satellite constellations were analyzed. In [1] the dynamics of three-body TSS were examined, with the cm (center of mass) following a circular orbit. A. Burov, I. Kosenko and A. Guerman, [2] – [5], studied the dynamics of a moon-anchored tether with a material point at its end, for variable tether length. The bi-dimensional problem can be addressed as a simple model for the lunar elevator. In this problem, some particular solutions of the motion equation are derived, choosing a specific control law for the tether length and finding radial and oblique configurations for the system geometry, while defining regions of stability and instability for each case.
2. Mathematical model of a two point tethered satellite system (TSS)

The main objective of this paper is to analyze equilibrium conditions for a two masses point TSS in an elliptical orbit, where the local vertical of the system must align with an inert axis fixed on Earth. The model is a dumbbell-like system, with the two body connected with a rigid and massless tether. The system's center of mass is described by a Keplerian orbit fixed on an inert axis in the primary body's center [6].

![Model of the satellite system connected with tethers.](image1)

**Figure 1.** Model of the satellite system connected with tethers.

**Figure 2.** Dumbbell-like system composed by a massless tether and two mass points.

The simplifying scheme can be seen in Figure 2. Although the simplification is rough, since the moment of inertia of a 3D system may not be aligned with the local vertical, it allows a fundamental mission conceptualization. In this model, the position of the two masses will be described in order to the position of the center of mass (C), and the system's motion in an elliptic orbit. The angles \( \nu \) and \( \varphi \) represent the true anomaly and the angle between the tether and \( \rho \), respectively. \( \rho \) is the distance from \( E \) to \( C \), \( l_1 \) and \( l_2 \) are the tether lengths from the mass points \( m_1 \) and \( m_2 \) to \( C \), respectively.

2.1. Positions of the system

This allows the model to be described by planar Keplerian motion:

\[
\rho = \frac{p}{1 + e \cos(\nu)} \tag{1}
\]

where \( p \) is the focal parameter, \( e \) the eccentricity and \( \nu \) the true anomaly. The system coordinates are:

\[
\begin{align*}
    x_0 &= \rho \cos(\nu) \\
    y_0 &= \rho \sin(\nu) \\
    x_1 &= x_0 + l_1 \cos(\nu + \varphi) \\
    y_1 &= y_0 + l_1 \sin(\nu + \varphi) \\
    x_2 &= x_0 - l_2 \cos(\nu + \varphi) \\
    y_2 &= y_0 - l_2 \sin(\nu + \varphi)
\end{align*}
\]
where, \( x_0 \) and \( y_0 \) being the coordinates of the position of the center of mass, \((\vec{r}_0)\); \( x_i \) and \( y_i \) being the coordinates of the position of the point mass \( i \), \((\vec{r}_i)\).

\[
\begin{align*}
\vec{r}_0 &= (x_0, y_0) \\
\vec{r}_1 &= (x_1, y_1) \\
\vec{r}_2 &= (x_2, y_2)
\end{align*}
\]  

(3)

2.2. Lagrange Equations of Motion

Lagrange equations of motion are known as differential equations of system motion in generalized coordinates. In fact, these Lagrange equations provide a simple way to obtain the system positions in function of time. It was decided to choose the Lagrangian formulation because it allows us to define the global positions of the system through Potential and Kinetic Energy and in function of generalized coordinates, unlike the needs to define all the forces involved, as in Newtonian formulation. The general formula of the Lagrange Equation is:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{dL}{dq_i} = Q_i
\]  

(4)

where \( L = T - V \), \( q_i \) is the generalized coordinate and \( Q_i \) is the generalized force actuating in the system. For the following analysis, the generalized coordinates are \( \varphi \) and \( l \) and the system is only under the gravity-gradient. Defining \( \varphi \) or \( l \) allows to control the system, as can be seen in [2]. It has been chosen to define \( \varphi \) and consecutively obtain the \( l \) behavior.

\[
l \left( 3\mu_0 \sin(2\varphi)(1 + e\cos(\nu))^3 + 2p^3(\dot{\nu} + \dot{\varphi}) \right) + 4p^3l(\dot{\nu} + \dot{\varphi}) = 0
\]  

(5)

All these variables are time-related. For a more advantageous formulation, consider a timeless relation, based only on general positions and given by the new independent variable true anomaly \( \nu \) [6]. Substituting all these variables it can be rearranged in:

\[
(1 + e\cos(\nu))\varphi'' + 2 \left( \frac{\nu'}{l}(1 + e\cos(\nu)) - e\sin(\nu) \right) (\varphi' + 1) + 3\cos(\varphi)\sin(\varphi) = 0
\]  

(6)

a well-known equation ([2] – [6]).

3. \( \varphi \) Control laws

At this stage, the Lagrange Equations of Motion were derived and control laws can now be considered for \( \varphi \) or \( l \). Opting for one is sufficient and convenient to know the system motion. The chose to define over one or other parameter is arbitrary and, therefore, were chosen \( \varphi \) control laws. The tether length \( (l) \) is small compared to the orbit parameter \( (p) \). Two control laws for the behavior of \( \varphi \) are presented: fixed angle and uniform rotations. Each case will be analyzed independently.

3.1. Fixed Angle

The fixed angle \( (\varphi = \varphi_0) \) for non-rotational motion about the center of mass, allows the satellite to point its face (or antenna) through Earth, not rotating locally. This has practical and real-life meaning, as GPS satellites, for instance, need to be pointing every time through Earth. This control law (Eq. 6)can assume the following expression:

\[
\frac{l'(\nu)}{l(\nu)} = \frac{e\sin(\nu)}{1 + e\cos(\nu)} - \frac{3\sin(2\varphi_0)}{4(1 + e\cos(\nu))}
\]  

(7)

Chosen different \( \varphi_0 \), the logarithmic behaviours of the tether are: (Figures 3 - 10).
There are uniform solutions to values $\varphi_0 = \frac{k\pi}{2}$ (k is integer). For an analytical solution, an approximation of 4th order using Taylor Series was also done allowing the integration of Eq. 7 showed in Figure 5, analyzed for various cases with different eccentricities.

3.2. Uniform Rotation
For uniform rotations, the control law can assume the following equation ($\varphi = \omega \nu + \varphi_0$). This relation means that the actual angle of rotation depends on the initial condition $\varphi_0$, the variation of $\nu$ and a constant $\omega$. Equation 6 is rewritten

$$\frac{l'(\nu)}{l(\nu)} = \frac{e \sin(\nu)}{1 + e \cos(\nu)} - \frac{3 \sin(2(\omega \nu + \varphi_0))}{4(\omega + 1)(e \cos(\nu) + 1)}$$

(8)

The Eq. 8 allows to know the tether length for each $\nu$. A numerical integration was done, where $\omega$ and $\varphi_0$ were substituted previously. Also, for $\omega = -1$ nothing can be analyzed since the Eq. 8 has singularities around that point.

The monodromy matrix is used to analyze stability conditions around small oscillations ($\delta\varphi$) of $\varphi$. Replacing ($\varphi = \omega \nu + \delta\varphi$) in Eq. (6) allows one obtain the nonlinear equation of perturbed motion, and linearized equation constricts this analysis to small variations of $\varphi$.

$$\left(1 + e \cos(\nu)\right)\delta\varphi'' + 3 \left(2 \cos(2\omega \nu)\delta\varphi - \frac{\sin(2\omega \nu)\delta\varphi'}{\omega + 1}\right) = 0$$

(9)

The Eq. 9 is the equation for the variations. It is now possible to analyze the stability using the Floquet theory, the linearized equation constricts this analysis to small variations of $\varphi$. Applying the Floquet theory, find the monodromy matrix for this system.
Figure 7. The control law for the tether length for $\varphi_0 = 0$ and $\omega = 1$.

Figure 8. The control law for the tether length for $\varphi_0 = 0$ and $\omega = 2$.

Figure 9. The control law for the tether length for $\varphi_0 = 0$ and $\omega = \frac{3}{4}$.

Figure 10. The control law for the tether length for $\varphi_0 = 0$ and $\omega = \frac{5}{3}$.

Figure 11. The control law for the tether length for $\varphi_0 = 0$ and $\omega = -\frac{3}{4}$. 
4. Conclusion

Laws of control are considered for the angle of systems rotation around the center of mass. The necessary conditions of stability for uniform rotations were analyzed using the Floquet Theory analyzing the parameters $\omega$ and the eccentricity of the orbit, generating the control laws for the tether length. Stability conditions give the values of $e$ and $\omega$ where the system is stable. As reported before, this means that small perturbations wont change the behavior of the system. Stability is found for several values of eccentricity, as shown in stability curves for the parameter $\omega$. The uniform rotations of $\varphi$ proved to have smooth tether variations, for every simulation done. For the fixed angle analysis, there was only one value where $\varphi$ behave properly, which is $\varphi_0 = 0$. At last, the simulations should always be considered in an engineering study. They allow to fully understand the dynamics of the system, and serve as a guide to future work.
results obtained in the simulator were always expected.

Acknowledgments
This work was accomplished with the support of CAPES – Brazil, under Contract BEX 10689-13-3, INPE – National Institute for Space Research – Brazil and Centre for Mechanical and Aerospace Science and Technologies (C-MAST) – UBI Portugal.

References
[1] Misra, A. K., Z. Amier, and V. J. Modi. "Attitude dynamics of three-body tethered systems." Acta Astronautica (1988):1059-1068.
[2] A.A. Burov and H. Troger, The relative equilibria of an orbital pendulum suspended on a tether, Journal of Applied Mathematics and Mechanics, Vol. 64, Issue 5, 2000b, pp. 723-728.
[3] A. Burov, O. I. Kononov, and A. D. Guerman, Relative equilibria of a Moon - tethered spacecraft, Advances in the Astronautical Sciences, v. 136, 2011a, pp. 2553-2562.
[4] A. Burov and I.I. Kosenko, Plane oscillations of a body with variable mass distribution in an elliptic orbit, Proc. of ENOC 2011, July 24-29, 2011b, Rome, Italy.
[5] A.A. Burov, I.I. Kosenko, and A. D. Guerman, Dynamics of a moon-anchored tether with variable length. Advances in the Astronautical Sciences, 2012, Vol. 142, pp.3495-3507.
[6] Simão Antônio da Rocha e Brito de Aguiã Morant. Space Tether Systems – Stability Solutions of a Dumbbell-like System, Dissertation Master Engineering Aeronautical - UBI - Portugal 2013.
[7] Belestsky, Vladimir V.; Levin, Evgenii M., 1993. Dynamics of Space Tethers Systems. San Diego - California: Advances in the Astronautical Sciences, Vol. 83 - American Astronautical Society ISSN 0-87703-370-6.
[8] M.P. Cartmell, D.J. McKenzie, A review of space tether research, Progress in Aerospace Sciences, Volume 44, Issue 1, January 2008, Pages 1-21, ISSN 0376-0421, http://dx.doi.org/10.1016/j.paerosci.2007.08.002.