Newtonian Versus Relativistic Cosmology

Master Thesis

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1 Introduction

Cosmology is the study of the origin and structure of the universe. We have good observational evidence that the universe was isotropic and homogeneous at early times, the best evidence being the small fluctuations (about $\sim 10^{-5}$) of the cosmic microwave background (CMB) radiation [1]. However, the structure of the universe that we observe today is very inhomogeneous on scales below $\sim 100$ Mpc\(^1\) there are galaxies (on scales of $\sim 1$ Mpc) and clusters of galaxies (on scales of $\sim 10$ Mpc). On scales of $\sim 100$ Mpc there are superclusters of galaxies, which are separated by large voids [2], see figure 1. The theory of structure formation connects the early homogeneous universe with the inhomogeneous universe today: It is assumed that there exist initially small density perturbations, which grow due to gravitational attraction. The mathematical tool to describe this scenario accurately is relativistic cosmological perturbation theory, where one considers small perturbations on the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric. However, in this theory there are complications arising due to the freedom of gauge, i.e. choosing the correspondence between perturbed and background quantities.

Another approach is Newtonian cosmological perturbation theory, which results in perturbed Newtonian equations (continuity equation, Euler equation and Poisson equation) on a FRW background.

\footnote{\textsuperscript{1}Distances in cosmology are usually measured in units of Millions of parsec (Megaparsec, Mpc), where one parsec (pc) is roughly 3.26 lightyears.}
One has to keep in mind that Newtonian theory is wrong in that it assumes instantaneous gravitational interaction and infinite speed of light. However, it turns out that Newtonian and relativistic cosmological perturbation theory coincide for scales much smaller than the Hubble horizon, which is basically the size of the observable universe.

Cosmological $N$-body simulations use Newtonian cosmology to move particles and simulate the density growth in the universe. These simulations can be used to test our understanding of structure formation. A famous simulation is the Millennium run (see figure 2), which simulated the behaviour of $\sim 10^{10}$ "particles" (each particle representing a dark matter clump with a mass of $10^9$ suns) in a box of $\sim 1\text{Gpc}$ side length \cite{3}. But the question is, how reliable are these simulations, since they use Newtonian rather than relativistic equations. In this work, we want to study the relativistic corrections to the Newtonian equations. Cosmological perturbation theory is a major tool to do this.

The outline is as follows: In part 2, we will summarize our current understanding of the universe and give the most important equations describing it. In part 3, we are going to consider Newtonian perturbation theory on an expanding FRW background. We will derive and solve the perturbed Newtonian equations up to second order. In part 4, we will introduce relativistic cosmological perturbation theory and derive and solve the relativistic evolution equations for the perturbation variables. In part 5, we will quantify the relativistic corrections to cosmological simulations and the observed matter power spectrum.
2 The homogeneous and isotropic universe

2.1 Basic equations

This section resumes the basic equations in cosmology. For a reference, see e.g. [4, 5]. We set \( c = 1 \).

The *Friedmann equation* relates the Hubble parameter to the content of the universe:

\[
H^2 = \frac{8\pi G}{3} \rho(t) - \frac{k}{a^2},
\]

(2.1)

where \( a \) is the scale factor, \( H \equiv \frac{1}{a} \frac{\partial a}{\partial t} = \frac{\dot{a}}{a} \) is the Hubble parameter, \( k \) is the curvature of the universe and \( \rho(t) \) is its energy density. This can consist of matter density, radiation density, or something else. Define the critical energy density

\[
\rho_c \equiv \frac{3H^2}{8\pi G},
\]

(2.2)

and the relative energy density

\[
\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)}.
\]

(2.3)

Then we can rewrite the Friedmann equation as

\[
1 - \Omega(t) = -\frac{k}{a^2H^2}.
\]

(2.4)

From this equation it can be seen that the universe has critical energy density \( (\Omega = 1) \) if and only if the curvature \( k \) vanishes. Furthermore, an overcritical universe \( (\Omega > 1) \) has positive curvature and an undercritical universe \( (\Omega < 1) \) has negative curvature.

The *equation of state* connects the energy density of a given source of energy and the pressure \( P \) that is created by it,

\[
P = w\rho,
\]

(2.5)

where \( w \) is the equation of state parameter.

Another important equation is the *continuity equation*, which tells how much the energy density will change in time,

\[
\dot{\rho} + 3H(\rho + P) = 0.
\]

(2.6)

Inserting the equation of state, we find

\[
\dot{\rho} + 3H\rho(1 + w) = 0,
\]

(2.7)

which has the solution

\[
\rho = \rho_0 a^{-3(1+w)},
\]

(2.8)

where \( \rho_0 = \rho(t_0) \) is the energy density today (we scale \( a \) such that \( a(t_0) = 1 \)). For a spatially flat single fluid universe, the Friedmann equation then becomes

\[
\dot{a}^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+3w)}.
\]

(2.9)
Given an equation of state parameter $w$ we can solve this differential equation for the scale factor $a(t)$.

Finally, we introduce the Raychaudhuri equation, which can be derived from the Friedmann equation and the continuity equation. It gives the acceleration of the scale factor in terms of energy density and pressure,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P).$$  \hspace{1cm} (2.10)

### 2.2 Content of the universe

Now we discuss the different components of our universe. The universe contains:

- **Matter.** This is to a small part ($\Omega_{b,0} \approx 0.05$) baryonic matter, which is mostly hydrogen. Everything that we see, stars, galaxies and clusters of galaxies, but also interstellar dust is contained in this number. However, various observations suggest that there is another type of matter that we cannot see directly, but its gravitational effects. We call this type of matter dark matter. Matter density $\rho_m$, defined as mass per volume, scales as $\rho_m(t) \sim a^{-3}$, because volume scales as $a^3$. Thus, the equation of state parameter is $w_m = 0$. In other words, matter does not create any pressure. Today, baryonic and dark matter together make up a relative energy density of $\Omega_{m,0} \approx 0.28$, but the biggest part of it comes from dark matter, $\Omega_{dm,0} \approx 0.23$.

- **Radiation.** Radiation density scales as $\rho_\gamma \sim a^{-4}$, because the number density of photons scales as $n_\gamma \sim a^{-3}$ and the energy of each photon scales as $E_\gamma \sim \frac{1}{\lambda} \sim a^{-1}$. Hence, the equation of state parameter is $w_\gamma = \frac{1}{3}$. Measurements show that today $\Omega_{\gamma,0} \approx 10^{-4}$, which comes mainly from CMB (cosmic microwave background) photons.

- **Curvature.** It is not known for sure if the universe is spatially flat. If curvature exists, then it acts effectively as an energy density, as can be seen from the Friedmann equation, with $\rho_k \sim a^{-2}$, so that $w_k = -\frac{1}{3}$. However, measurements show that the effect of curvature cannot be large today, $|\Omega_{k,0}| \lesssim 10^{-2}$.  

Figure 3: Content of the universe today. [source: map.gsfc.nasa.gov/universe/uni_matter.html]
Figure 4: Overview over the different epochs of our universe (in a logarithmic scaling). Because radiation, matter and dark energy density scale differently with time, there exist eras where one of them is dominating.

- $\Lambda$. Dark energy (also called vacuum energy or cosmological constant $\Lambda$) is a component of the universe whose energy density does not change with time. Consequently, the equation of state parameter is $w_\Lambda = -1$, whence it creates negative pressure. Measurements of Type Ia supernovae show that the universe today is expanding with an accelerating scale factor. These measurements can be explained by the existence of a cosmological constant with a relative energy density of $\Omega_{\Lambda,0} \approx 0.72$. Thus, the biggest contribution to the energy density of the universe today comes from dark energy $[7]$.

Now that we know how the different components of the universe scale, we can give an approximation when which component was dominant. Current cosmological models say that after the big bang, there was a short time when the universe was dark energy dominated, so that it expanded exponentially. This period is called inflation. It broke down when the slow-roll conditions were violated, to be explained below. After this, the universe got radiation-dominated ($a \sim t^{1/2}$) until the time of radiation-matter equality at $a_{eq} = \frac{\Omega_{\gamma,0}}{\Omega_{m,0}} \approx 3 \cdot 10^{-4}$. Then it became matter-dominated ($a \sim t^{2/3}$) until the time of matter-$\Lambda$ equality at $a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \approx 0.7$. Thus, today ($a_0 = 1$) we live at a time just after the cosmological constant overtook matter and started to accelerate the expansion of the universe, see figure 4 for an illustration.

### 2.3 Inflation

Inflation is a phase of accelerated expansion of the universe which is assumed to have happened for a very short time after the Big Bang. The precise definition of inflation is an epoch with accelerating scale factor,

$$\ddot{a} > 0.$$ (2.11)
In order to have inflation, one typically introduces a scalar field with negative pressure, that drives the inflation. We call this scalar field the \textit{inflaton} $\varphi$. Let its Lagrangian be

$$\mathcal{L} = -\frac{1}{2} \varphi, \mu \varphi^{\mu} - V(\varphi)$$

(2.12)

with some potential $V(\varphi)$. Then the inflaton field has the energy density

$$\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi)$$

(2.13)

and the pressure

$$P_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi).$$

(2.14)

Note that for slowly varying $\varphi$, we have $P_\varphi \simeq -\rho_\varphi$, which gives $w_\varphi \simeq -1$, whence the inflaton field has negative pressure. Now we substitute the above equations into the Friedmann and Raychaudhuri equations, which gives:

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right),$$

(2.15)

$$\ddot{\varphi} + 3H \varphi = -V,_{\varphi},$$

(2.16)

where $V,_{\varphi} \equiv \frac{dV}{d\varphi}$ and $M_{\text{Pl}}$ is the \textit{reduced Planck mass},

$$M_{\text{Pl}} \equiv \frac{1}{\sqrt{8\pi G}}.$$  

(2.17)

One can show that these relations reduce to

$$H^2 \simeq \frac{1}{3M_{\text{Pl}}^2} V(\varphi),$$

(2.18)

$$3H \varphi \simeq -V,_{\varphi},$$

(2.19)

if the \textit{slow-roll conditions},

$$\epsilon \ll 1,$$

(2.20)

$$|\eta| \ll 1,$$

(2.21)

are satisfied, where $\epsilon$ and $\eta$ are the \textit{slow-roll parameters}, defined as:

$$\epsilon \equiv \frac{M_{\text{Pl}}^2}{2} \left( \frac{V,_{\varphi}}{V} \right)^2,$$

(2.22)

$$\eta \equiv M_{\text{Pl}}^2 \frac{V,_{\varphi}^2}{V}.$$  

(2.23)
Table 1: Evolution of the scale factor, the Hubble parameter and the Hubble horizon during dark energy, radiation and matter domination.

To see the connection between slow-roll conditions and inflation, rewrite the condition for inflation as

$$\ddot{a} / a = \dot{H} + H^2 > 0,$$

or, equivalently,

$$-\frac{\dot{H}}{H^2} < 1.$$  \hspace{1cm} (2.25)

Substituting the slow-roll equations gives

$$-\frac{\dot{H}}{H^2} \simeq \frac{M_{Pl}^2}{2} \left( \frac{V_\varphi}{V} \right)^2 = \epsilon.$$  \hspace{1cm} (2.26)

Hence, if the slow-roll conditions are satisfied ($\epsilon \ll 1$), then inflation takes place ($\ddot{a} > 0$). The duration of inflation is controlled by the magnitude of $\eta$. The potential has to be flat enough (which corresponds to a small $\eta$) so that $\epsilon$ stays small.

As an example, consider the inflaton potential

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2.$$  \hspace{1cm} (2.27)

The slow-roll parameters are

$$\epsilon = \eta = \frac{2M_{Pl}^2}{\varphi^2}.$$  \hspace{1cm} (2.28)

Hence, inflation occurs as long as $\varphi^2 \gg 2M_{Pl}^2$, and it breaks down near the minimum of the potential, when $\varphi^2 \sim 2M_{Pl}^2$ [8].

2.4 Evolution of scales

Later we will discuss the evolution of density perturbations in Fourier space, that is, the perturbations on a given comoving scale $k^{-1}$. An important question is if the considered scale is larger or smaller than the Hubble horizon at the time. The (comoving) Hubble horizon is given by the inverse of the comoving Hubble parameter, $H^{-1} = (aH)^{-1}$. Let us evaluate how this horizon develops in time during different periods of the universe. In order to do this, we use eq. (2.9) in order to find the scale factor $a(t)$. The results are listed in table 1. Note that during inflation ($\Lambda$-domination) the comoving Hubble scale decreases, or in other words, the horizon shrinks. During this period, all scales leave the horizon. Later, when inflation ends, the scales start to re-enter the horizon during the radiation- and matter-dominated era, small scales before large scales (see figure 5). We normalize the horizon so that today
Figure 5: A given comoving scale $k^{-1}$ leaves the horizon during inflation and re-enters later, during radiation or matter domination. (not to scale!)

Table 2: Different scales with their scale factor $a_H$ and redshift $z_H = \frac{1}{a_H} - 1$ of horizon entry. The scale factor at radiation-matter equality is given for a $\Lambda$CDM model and the Hubble horizon today is given for $h = 0.7$.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Scale} & k^{-1} [\text{Mpc}] & k [\text{Mpc}^{-1}] & a_H & z_H \\
\hline
4286 (\text{horizon today}) & 2.33 \cdot 10^{-4} & 1 & 0 \\
1000 & 10^{-3} & 5.44 \cdot 10^{-4} & 17.4 \\
100 & 0.01 & 5.44 \cdot 10^{-4} & 1837 \\
\hline
74.23 (\text{equality scale}) & 0.0135 & 3 \cdot 10^{-4} & 3332 \\
10 & 0.1 & 4.04 \cdot 10^{-6} & 24.7 \cdot 10^4 \\
1 & 1 & 4.04 \cdot 10^{-6} & 24.7 \cdot 10^4 \\
0.1 & 10 & 4.04 \cdot 10^{-7} & 24.7 \cdot 10^4 \\
\hline
\end{array}
\]

(at $a = 1$) it is the radius of the whole visible universe, $R_H \simeq 3000 h^{-1} \text{Mpc}$. Thus,

\[
H^{-1} = R_H a^{1/2} \text{ for } a \geq a_{\text{eq}}. \tag{2.29}
\]

We then normalize the horizon before $a_{\text{eq}}$ to match the above relation:

\[
H^{-1} = \frac{R_H}{a_{\text{eq}}^{1/2}} a \text{ for } a < a_{\text{eq}}. \tag{2.30}
\]

Table 2 shows some scales with their scale factor of horizon entry $a_H$.

\footnote{$h \approx 0.7$ is the reduced Hubble constant, defined by $H_0 = 100h \text{ km s}^{-1} \text{Mpc}^{-1}$.}
3 The perturbed universe: Newtonian treatment

3.1 Definitions

In cosmology, there are two different types of coordinates used, *proper coordinates* \( r \) and *comoving coordinates* \( x \). These are related to each other by the scale factor \( a \),

\[
  r = ax. \tag{3.1}
\]

Consequently, we need to define the two different Nabla operators, \( \nabla \equiv \frac{\partial}{\partial x} \) and \( \nabla_r \equiv \frac{\partial}{\partial r} \), which are related to each other by

\[
  \nabla_r = \frac{1}{a} \nabla. \tag{3.2}
\]

In the same way, we can define two different times, related to each other by the scale factor:

\[
  dt = ad\tau. \tag{3.3}
\]

We call \( t \) the *proper time* and \( \tau \) the *conformal time*. In our notation, a dot denotes a partial derivative with respect to proper time \( t \), a prime with respect to conformal time \( \tau \). We define the Hubble parameter \( H \equiv \frac{a'}{a} \) and the conformal Hubble parameter \( \mathcal{H} \equiv \frac{a'}{a} \).

The absolute velocity \( u \) is defined as

\[
  u \equiv \frac{dr}{dt} = \frac{d(ax)}{ad\tau} = \frac{dx}{a} = Hx + v = Hr + v = v_H + v, \tag{3.4}
\]

where \( v_H \equiv Hr = \mathcal{H}x \) is the *Hubble velocity* or *Hubble flow*, while the *comoving (peculiar) velocity* \( v \equiv \frac{dx}{d\tau} \) measures the velocity relative to the Hubble flow.

The *convective derivative* in the \((t, r)\)-system is given by

\[
  \frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla_r, \tag{3.5}
\]

which follows simply from the chain rule. In the \((\tau, x)\)-system the convective derivative is given by

\[
  \frac{d}{d\tau} = \frac{\partial}{\partial \tau} + v \cdot \nabla. \tag{3.6}
\]

Because proper time and conformal time are related by \( dt = ad\tau \), the convective derivative transforms as

\[
  \frac{d}{dt} = \frac{1}{a} \frac{d}{d\tau}. \tag{3.7}
\]

3.2 Perturbed fields

Now we are going to consider perturbations up to the second order. We start from the *continuity equation*, *Poisson equation* and *Euler equation* on an expanding, spatially flat FRW background. We will consider a universe filled only with pressureless matter and no cosmological constant. This model
is called the *Einstein-de Sitter model*. The relevant fields are the matter density $\rho$, the gravitational potential $\phi$ and the absolute velocity $u$.

Now we split the fields into a *background part* (denoted with a superscript $(0)$) and a *perturbed part*. The perturbed part itself can be further split into first order perturbations, second order perturbations (superscripts $(1)$, $(2)$) and so on:

$$
\rho = \rho^{(0)} + \delta\rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)} + ... = \rho^{(0)}(1 + \delta^{(1)} + \delta^{(2)} + ...),
$$

(3.8)

$$
\phi = \phi^{(0)} + \delta\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + ...,
$$

(3.9)

$$
u = \nu_H + \nu = \nu_H + \nu^{(1)} + \nu^{(2)} + ..., \quad (3.10)
$$

where we have defined the $n$-th order *density contrast* $\delta^{(n)}$,

$$
\delta^{(n)} \equiv \frac{\rho^{(n)}}{\rho^{(0)}}. \quad (3.11)
$$

Here, $\nu_H = Hr = H(x)$ is considered as the background velocity, and the peculiar velocity $\nu = \frac{dx}{dt}$ is assumed to be small, $|\nu| \ll |\nu_H|$. Furthermore, according to the *Helmholtz theorem*\(^3\) $\nu$ can be separated into a part with zero curl (*longitudinal part* $\nu_\parallel$) and a part with zero divergence (*transverse part* $\nu_\perp$):

$$
\nu = \nu_\parallel + \nu_\perp, \text{ with } \nabla \times \nu_\parallel = 0 \text{ and } \nabla \cdot \nu_\perp = 0. \quad (3.12)
$$

Define a scalar potential $v$ and a vector potential $\Omega$ such that:

$$
v_\parallel \equiv \nabla v \text{ and } v_\perp \equiv \nabla \times \Omega. \quad (3.13)
$$

### 3.3 Perturbed Newtonian equations

#### 3.3.1 Continuity equation

The continuity equation in physical coordinates and proper time is

$$
\frac{\partial}{\partial t} \rho + \nabla r (\rho u) = 0. \quad (3.14)
$$

Using the Leibniz rule we obtain

$$
\frac{\partial}{\partial t} \rho + u \cdot \nabla r \rho + \rho \nabla r \cdot u = 0. \quad (3.15)
$$

The first two terms of this equation together form the convective derivative of $\rho$,

$$
\frac{d}{dt} \rho + \rho \nabla r \cdot u = 0. \quad (3.16)
$$

\(^3\)The Helmholtz decomposition exists and is unique under the assumption that $|\nu| \to 0$ as $|x| \to \infty$, see appendix A for a proof.
Now we go from the \((t, r)\)-system to the \((\tau, x)\)-system,
\[
\frac{1}{a} \frac{d}{d\tau} \rho + \frac{1}{a} \rho \nabla \cdot \mathbf{u} = 0. \tag{3.17}
\]
Multiplying this equation by \(a\) and using the definitions of the conformal convective derivative \(\frac{d}{d\tau}\) and the absolute velocity \(\mathbf{u}\), we find:
\[
\rho' + 3\mathcal{H} \rho + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.18}
\]
To the background order, this is
\[
\rho^{(0)'} + 3\mathcal{H} \rho^{(0)} = 0, \tag{3.19}
\]
which is the well known mass continuity equation for \(p = 0\) and has the solution \(\rho^{(0)} \sim a^{-3}\).

To the linear order we find the relation
\[
\rho^{(1)'} + 3\mathcal{H} \rho^{(1)} + \rho^{(0)} \nabla \cdot \mathbf{v}^{(1)} = 0. \tag{3.20}
\]
Now we use
\[
\rho^{(n)'} = (\rho^{(0)} \delta^{(n)})' = \rho^{(0)} \delta^{(n)'} - 3\mathcal{H} \rho^{(0)} \delta^{(n)} \tag{3.21}
\]
with \(n = 1\) to find:
\[
\delta^{(1)'} + \nabla \cdot \mathbf{v}^{(1)} = 0. \tag{3.22}
\]
Note that only the scalar part of \(\mathbf{v}\) survives, because we take its divergence. Hence, we can also write
\[
\delta^{(1)'} + \Delta \mathbf{v}^{(1)} = 0, \tag{3.23}
\]
where \(\Delta \equiv \nabla^2\) is the Laplace operator in comoving coordinates.

To the second order we find:
\[
\rho^{(2)'} + 3\mathcal{H} \rho^{(2)} + \nabla \cdot (\rho^{(1)} \mathbf{v}^{(1)}) + \rho^{(0)} \nabla \cdot \mathbf{v}^{(2)} = 0. \tag{3.24}
\]
Again, we make use of eq. \((3.21)\) with \(n = 2\) to find:
\[
\delta^{(2)'} + \nabla \cdot (\delta^{(1)} \mathbf{v}^{(1)}) + \nabla \cdot \mathbf{v}^{(2)} = 0. \tag{3.25}
\]

### 3.3.2 Poisson equation

The Poisson equation connects the gravitational potential with the matter density. In proper coordinates it is
\[
\Delta_r \phi = 4\pi G \rho.
\]
In comoving coordinates it becomes
\[
\Delta \phi = 4\pi G a^2 \rho. \tag{3.26}
\]
To the **background order**, we have

\[ \Delta \phi^{(0)} = 4\pi G a^2 \rho^{(0)}, \]  
(3.27)

from which we find the background value for \( \phi \),

\[ \phi^{(0)} = \frac{2}{3} \pi G a^2 \rho^{(0)} x^2 + C(t), \]  
(3.28)

where \( C(t) \) is an arbitrary function of time.

To the **linear order**, we have

\[ \Delta \phi^{(1)} = 4\pi G a^2 \rho^{(0)} \delta^{(1)}, \]  
(3.29)

and to the **second order**, we find

\[ \Delta \phi^{(2)} = 4\pi G a^2 \rho^{(0)} \delta^{(2)}. \]  
(3.30)

### 3.3.3 Euler equation

The Euler equation tells us how the velocity field changes in time given a gravitational potential. In proper coordinates it is

\[ \frac{\partial u}{\partial t} + u \cdot \nabla x u = -\nabla \phi. \]  
(3.31)

Using the convective derivative we can express it in the shorter form

\[ \frac{du}{dt} = -\nabla \phi. \]  
(3.32)

In the \((\tau, x)\)-system the equation becomes after multiplying with the scale factor

\[ \frac{du}{d\tau} = -\nabla \phi. \]  
(3.33)

Using the definitions of the total velocity and the convective derivative we find

\[ \mathcal{H}' x + v' + \mathcal{H} v + v \cdot \nabla v = -\nabla \phi. \]  
(3.34)

To the **background order**, we have, using the solution for \( \phi^{(0)} \),

\[ \mathcal{H}' = \frac{4}{3} \pi G a^2 \rho^{(0)}, \]  
(3.35)

which is the Raychaudhuri equation for \( p = 0 \).

The perturbed order is

\[ v' + \mathcal{H} v + v \cdot \nabla v = -\nabla \delta \phi. \]  
(3.36)

Note that

\[ v \cdot \nabla v = \frac{1}{2} \nabla v^2 - v \times \omega, \]  
(3.37)
where we have defined the vorticity
\[ \omega \equiv \nabla \times \mathbf{v}, \quad (3.38) \]
so that eq. (3.36) reads
\[ \mathbf{v}' + H \mathbf{v} + \frac{1}{2} \nabla \mathbf{v}^2 - \mathbf{v} \times \omega = -\nabla \delta \phi. \quad (3.39) \]
Taking the curl of this equation, we find the vorticity equation:
\[ \omega' + H \omega = \nabla \times (\mathbf{v} \times \omega). \quad (3.40) \]
This differential equation for the vorticity \( \omega \) has an interesting solution: If \( \omega = 0 \) everywhere initially, then the vorticity will stay zero at all times. However, the situation changes if we include a pressure gradient in the perturbed part of the Euler equation,
\[ \mathbf{v}' + H \mathbf{v} + \frac{1}{2} \nabla \mathbf{v}^2 - \mathbf{v} \times \omega = -\nabla \delta \phi - \frac{1}{\rho} \nabla \rho. \quad (3.41) \]
Then the vorticity equation becomes
\[ \omega' + H \omega = \nabla \times (\mathbf{v} \times \omega) + \frac{1}{\rho^2} \nabla \rho \times \nabla \rho. \quad (3.42) \]
The second term on the r.h.s. is called the baroclinic contribution. It is zero in the case of a vanishing entropy gradient, \( \nabla S = 0 \) \[9\]. Hence, if we consider a model with irrotational, isentropic initial conditions and adiabatic evolution, it follows that \( \omega = 0 \) always. In the Einstein-de Sitter model, which we consider here, there is no entropy at all. In order to define entropy, one needs at least two different types of fluids. Hence, it is reasonable to assume
\[ \omega = 0, \quad (3.43) \]
which is equivalent to
\[ \mathbf{v}_\perp = 0. \quad (3.44) \]
Hence, in Newtonian theory, the velocity perturbation is a pure potential flow\[4\] \( \mathbf{v} = \mathbf{v}_\parallel = \nabla \phi \). Furthermore, it follows that the vector potential \( \Omega \) is a solution of the Laplace equation, since
\[ 0 = \omega = \nabla \times (\nabla \times \Omega) = \nabla (\nabla \cdot \Omega) - \Delta \Omega = -\Delta \Omega, \]
where we set \( \nabla \cdot \Omega = 0 \) without any loss of generality.

To the linear order, the Euler equation reads
\[ \mathbf{v}^{(1)'} + H \mathbf{v}^{(1)} = -\nabla \phi^{(1)}, \quad (3.45) \]

\[4\]Indeed, a constant non-zero \( \mathbf{v}_\perp \) would also give \( \omega = 0 \), but this would be in contradiction to the isotropy of the universe. Also, the Helmholtz decomposition is not unique if we allow constant parts in the vector.
and to the second order, we find

\[ \mathbf{v}^{(2)'} + \mathcal{H}\mathbf{v}^{(2)} + \mathbf{v}^{(1)} \cdot \nabla \mathbf{v}^{(1)} = -\nabla \phi^{(2)}. \]  

(3.46)

### 3.3.4 Summary

The important results of this section are the perturbed parts of the continuity, Poisson and Euler equation. In comoving coordinates and conformal time these are:

|       | 0. order | 1. order | 2. order |
|-------|----------|----------|----------|
| C     | \( \rho^{(0)''} + 3\mathcal{H}\rho^{(0)} = 0 \) | \( \delta^{(1)'} + \nabla \cdot \mathbf{v}^{(1)} = 0 \) | \( \delta^{(2)'} + \nabla \cdot (\delta^{(1)}\mathbf{v}^{(1)}) + \nabla \cdot \mathbf{v}^{(2)} = 0 \) |
| P     | \( \Delta\phi^{(0)} = 4\pi Ga^2\rho^{(0)} \) | \( \Delta\phi^{(1)} = 4\pi Ga^2\rho^{(0)}\delta^{(1)} \) | \( \Delta\phi^{(2)} = 4\pi Ga^2\rho^{(0)}\delta^{(2)} \) |
| E     | \( \mathcal{H}' = -\frac{4}{7}\pi Ga^2\rho^{(0)} \) | \( \mathbf{v}^{(1)'} + \mathcal{H}\mathbf{v}^{(1)} = -\nabla \phi^{(1)} \) | \( \mathbf{v}^{(2)'} + \mathcal{H}\mathbf{v}^{(2)} + \mathbf{v}^{(1)} \cdot \nabla \mathbf{v}^{(1)} = -\nabla \phi^{(2)} \) |

Table 3: Continuity equation (C), Poisson equation (P) and Euler equation (E) in background, linear and quadratic order perturbation theory.

### 3.4 Solutions

#### 3.4.1 First order solutions

In linear order the equations decouple into scalar and vector parts. The decoupled equations are:

|       | scalar part | vector part |
|-------|-------------|-------------|
| continuity equation | \( \delta^{(1)'} + \nabla \cdot \mathbf{v}^{(1)} = 0 \) | - |
| Poisson equation | \( \Delta\phi^{(1)} = 4\pi Ga^2\rho^{(0)}\delta^{(1)} \) | - |
| Euler equation | \( \mathbf{v}^{(1)'} + \mathcal{H}\mathbf{v}^{(1)} = -\nabla \phi^{(1)} \) | \( \mathbf{v}^{(1)'} + \mathcal{H}\mathbf{v}^{(1)} = 0 \) |

Table 5: Decoupled first order Newtonian equations.

The vector contribution to \( \mathbf{v}^{(1)} \) can be neglected because it decays \( \sim a^{-1} \) (this is a special case of the above statement that the vorticity vanishes at all orders in Newtonian theory), so we can set \( \mathbf{v}^{(1)} = \mathbf{v}^{(1)}_\parallel \). We do a spatial Fourier transformation of the remaining equations. In momentum space these equations become:

\[ \delta^{(1)'}_k + i\mathbf{k} \cdot \mathbf{v}^{(1)}_k = 0, \]  

(3.47)

\[ -k^2\phi^{(1)}_k = \frac{6}{\tau^2}\delta^{(1)}_k, \]  

(3.48)

\[ \mathbf{v}^{(1)'}_k + \frac{2}{\tau}\mathbf{v}^{(1)}_k = -i\mathbf{k}\phi^{(1)}_k, \]  

(3.49)

where we have used the Friedmann equation to replace the expression \( 4\pi Ga^2\rho^{(0)} \):

\[ \frac{4}{\tau^2} = \mathcal{H}^2 = \frac{8\pi G}{3}\rho^{(0)}a^2 \Rightarrow 4\pi G\rho^{(0)}a^2 = \frac{6}{\tau^2}. \]
The solutions can be obtained as follows: First, solve the Poisson equation for $\delta_k^{(1)}$:

$$\delta_k^{(1)} = -\frac{1}{6} \tau^2 k^2 \phi_k^{(1)}. \quad (3.50)$$

The Euler equation can be written as

$$\frac{1}{a} (a v_k^{(1)})' = -i k \phi_k^{(1)}, \quad (3.51)$$

whence

$$v_k^{(1)} = -i k \frac{1}{a} \int_0^\tau d\tau' a \phi_k^{(1)}. \quad (3.52)$$

Now we insert the solutions for $\delta_k^{(1)}$ and $v_k^{(1)}$ into the time derivative of the continuity equation, $\delta_k^{(1)''} + i k \cdot v_k^{(1)'} = 0$ and find the following equation for $\phi_k^{(1)}$:

$$\phi_k^{(1)''} + \frac{\tau}{6} \phi_k^{(1)'} = 0. \quad (3.53)$$

This is a second order differential equation for $\phi_k^{(1)}$ and has two solutions. Whenever this is the case, we call the dominating solution the growing mode and the subdominant solution the decaying mode. Here, the decaying solution is $\phi_k^{(1)} \sim \tau^{-5}$ and the growing solution is a constant,

$$\phi_k^{(1)} = \phi_k^{(1)}(k). \quad (3.54)$$

In summary, the growing first order solutions are:

$$\phi_k^{(1)} = \phi_k^{(1)}(k), \quad (3.55)$$

$$\delta_k^{(1)} = -\frac{1}{6} \tau^2 k^2 \phi_k^{(1)} = -\frac{2}{3} a k^2 R_H^2 \phi_k^{(1)}(k), \quad (3.56)$$

$$v_k^{(1)} = -\frac{i}{3} k \tau \phi_k^{(1)} = -\frac{2}{3} i k a^{1/2} R_H \phi_k^{(1)}(k). \quad (3.57)$$

Note that, in order to convert $\tau$ to $a$, we use the definition of the Hubble parameter:

$$\frac{2}{\tau} = \dot{H} = a H = a \frac{2}{3t} = a \frac{2}{3t_0} \frac{t_0}{t} = a H_0 a^{-3/2} = H_0 a^{-1/2},$$

whence

$$\tau = 2 a^{1/2} H_0^{-1} = 2 a^{1/2} \frac{c}{H_0} c^{-1} = 2 a^{1/2} R_H c^{-1},$$

where $\frac{c}{H_0} = R_H \simeq 4300 \text{Mpc}$ is the Hubble radius. Here, we use $c = 1$-units, so that the connection between $\tau$ and $a$ is simply

$$\tau = 2 a^{1/2} R_H. \quad (3.58)$$
The decaying solutions for \( \delta_k^{(1)} \) and \( v_k^{(1)} \) can be obtained using the decaying solution for \( \phi_k^{(1)} \),

\[
\begin{align*}
\dot{\phi}_k^{(1)} &\sim \tau^{-5}, \\
\delta_k^{(1)} &\sim \tau^{-3}, \\
v_k^{(1)} &\sim \tau^{-4}.
\end{align*}
\]

### 3.4.2 Initial conditions

Now we want to answer the question how big \( \phi_k \) is, since all the other quantities depend on it (we drop the superscripts \(^{(1)}\) for the moment). For this we have to go deeper into the theory of the primordial power spectrum. Note that this is not part of Newtonian theory.

Inflation creates curvature perturbations. Denote the curvature perturbation field by \( \zeta(x, \tau) \), and its Fourier transform by \( \zeta_k \). The precise definition will be given in subsection 4.2.4. An important physical quantity is its power spectrum. It is defined by

\[
P_\zeta(k) \equiv \frac{k^3}{2\pi^2} |\zeta_k|^2.
\]

Measurements of WMAP give a nearly scale-invariant \((Harrison-Zel’dovich)\) power spectrum for the primordial curvature perturbation \[10\],

\[
P_\zeta(k) = A^2, \quad A \simeq 5 \cdot 10^{-5}.
\]

Bardeen showed in \[11\] that in the matter dominated era of the universe the Bardeen potential \( \Phi_k \) (we will give the definition later) is related to \( \zeta_k \) via

\[
\Phi_k = -\frac{3}{5} \zeta_k.
\]

As we will show later, the Bardeen potential \( \Phi_k \) can be identified with the gravitational potential perturbation \( \phi_k \). Hence, \( \phi_k \) has the following power spectrum:

\[
P_\phi(k) = \frac{k^3}{2\pi^2} |\phi_k|^2 = \frac{9}{25} \frac{k^3}{2\pi^2} |\zeta_k|^2 = \frac{9}{25} P_\zeta(k) = \frac{9}{25} A^2.
\]

Thus, also the spectrum for \( \phi_k \) is scale-invariant. However, for the spectrum of \( \delta_k \) we find a scale-dependence:

\[
P_\delta(k) = \frac{k^3}{2\pi^2} |\delta_k|^2 = \frac{k^3}{2\pi^2} \frac{4}{9} a^2 k^4 R_H^4 |\phi_k|^2 = \frac{4}{9} a^2 k^4 R_H^4 P_\phi(k) = \frac{4}{25} a^2 k^4 R_H^4 A^2 \sim k^4,
\]

or equivalently,

\[
P_\delta(k) = \frac{8}{25} \pi^2 R_H^4 A^2 a^2 k \sim k.
\]

This means that there is more power (and therefore more structure) on smaller scales. Measurements show that the \( k \)-dependence of \( P_\delta(k) \) is correct on large scales, but at a scale of about \( k_{\text{eq}} \simeq 0.01 \text{ Mpc}^{-1} \),
The matter power spectrum today, measured using various techniques [12]. The bend at \( k = k_{\text{eq}} \approx 0.01 \text{ Mpc}^{-1} \) is a result of the transition between radiation domination and matter domination.

which is the scale that enters the horizon at the time of radiation-matter equality, the power spectrum has a bend and decreases roughly as \( k^{-3} \) for higher \( k \) (see figure 6). This bend comes from the fact that the universe was not always matter-dominated, but there was a radiation-dominated era (which is not included so far in our model). In order to explain the bend, we have to find out how density perturbations grow during radiation domination.

### 3.4.3 Transfer function

As we discussed above, it is important that in the history of the universe there was a radiation-dominated era, where the pressure was not zero. Therefore we consider now matter density perturbations during radiation domination. However, we are going to neglect perturbations in the radiation density, which turns out to be a good approximation. If we consider a universe that contains both matter and radiation, we have to modify the Euler equation, containing both a gravitational potential gradient and a pressure gradient. Then the first order Newtonian equations are:

\[
\begin{align*}
\delta^{(1)}' + \nabla \cdot \mathbf{v}^{(1)} &= 0, \\
\Delta \phi^{(1)} &= 4\pi G a^2 \bar{\rho}_m \delta^{(1)}, \\
\mathbf{v}^{(1)}' + \mathcal{H} \mathbf{v}^{(1)} &= -\nabla (\phi^{(1)} + \frac{p^{(1)}}{\bar{\rho}}),
\end{align*}
\]
where \( \bar{\rho} = \bar{\rho}_m + \bar{\rho}_\gamma \) is the total background energy density. Now we take the time derivative of eq. (3.68) and subtract the divergence of eq. (3.70), using eq. (3.69) to replace \( \Delta \phi^{(1)} \). This gives the following relation:

\[
\delta^{(1)'} + \mathcal{H} \delta^{(1)} = 4 \pi G \bar{\rho}_m a^2 \delta^{(1)} + \Delta c_s^2 \delta^{(1)},
\]

(3.71)

where \( c_s^2 = w = \rho^{(1)} / \bar{\rho} \) is the square of the speed of sound. In Fourier space this equation reads:

\[
\delta_k^{(1)''} + \mathcal{H} \delta_k^{(1)'} = 4 \pi G \bar{\rho}_m a^2 \delta_k^{(1)} - k^2 c_s^2 \delta_k^{(1)}.
\]

(3.72)

This equation is known as the growth equation or Jeans equation for \( \delta_k^{(1)} \) \[13\]. The l.h.s. represents the time change of \( \delta_k^{(1)} \), including a friction term \( \mathcal{H} \delta_k^{(1)'} \), which is also called "Hubble damping" or "Hubble friction" (the expansion of the universe slows down the growth of density perturbations). On the r.h.s. there are two competitive sources: gravity, which supports the growth of density perturbations, and pressure, which prevents it. It is convenient to introduce the comoving Jeans wavenumber,

\[
k_J \equiv \left( \frac{4 \pi G \bar{\rho}_m a^2}{c_s^2} \right)^{1/2},
\]

(3.73)

and the corresponding comoving Jeans length or Jeans scale,

\[
\lambda_J \equiv k_J^{-1} = \left( \frac{4 \pi G \bar{\rho}_m a^2}{c_s^2} \right)^{-1/2}.
\]

(3.74)

Then the Jeans equation can be written as

\[
\delta_k^{(1)''} + \mathcal{H} \delta_k^{(1)'} = (k_J^2 - k^2) c_s^2 \delta_k^{(1)}.
\]

(3.75)

Now we have two cases:

1. For scales much smaller than the Jeans length, \( k \gg k_J \), we can approximate the Jeans equation by

\[
\delta_k^{(1)''} + \mathcal{H} \delta_k^{(1)'} = -k^2 c_s^2 \delta_k^{(1)}.
\]

(3.76)

The solutions are oscillating with the frequency \( k c_s \), and the oscillations are damped by the Hubble friction term, whence the amplitude of the oscillations decays with time. There is no structure growth on sub-Jeans scales.

2. For scales much larger than the Jeans length, \( k \ll k_J \), we can approximate the Jeans equation by

\[
\delta_k^{(1)''} + \mathcal{H} \delta_k^{(1)'} = 4 \pi G \bar{\rho}_m a^2 \delta_k^{(1)}.
\]

(3.77)

During matter domination, this gives the growing solution \( \delta_k^{(1)} \sim a \), which we already know. During radiation domination, we have

\[
\bar{\rho}_m \ll \bar{\rho} \simeq \bar{\rho}_\gamma
\]

(3.78)
and hence
\[ 4\pi G a^2 \bar{\rho}_m = \frac{3}{2} H^2 \bar{\rho}_m \ll 1, \tag{3.79} \]
so that we can neglect the source term on the right hand side. Then the Jeans equation becomes
\[ \delta_k^{(1)''} + H \delta_k^{(1)'} = 0, \tag{3.80} \]
which has the general solution
\[ \delta_k^{(1)} = C_1 + C_2 \ln a. \tag{3.81} \]

Hence, during radiation domination density perturbations grow at most logarithmically on super-Jeans scales.

Now we estimate the Jeans scale. During matter domination we have \( \lambda_J = 0 \) since \( c_s = 0 \), so that perturbations grow on all scales. During radiation domination the speed of sound is \( c_s = \sqrt{\frac{1}{3}} \) and therefore \( k_J = \sqrt{3} \sqrt{4\pi G \bar{\rho}_m a^2} \). As we approach the time of radiation-matter equality, we have \( \bar{\rho}_m \approx \bar{\rho}_\gamma \approx \frac{\bar{\rho}}{2} \), whence \( k_J \approx H \). Thus, the Jeans scale grows to the size of the horizon at radiation-matter equality and reduces to zero when matter dominates. Consequently, it makes a difference if a scale enters the horizon during radiation domination or during matter domination because perturbations on scales that enter during radiation domination experience a suppression in growth. Now we want to quantify this effect. First consider a density perturbation on a scale \( k^{-1} > k_{\text{eq}}^{-1} \), which enters the horizon during matter domination. At a later time \( a_f \) during matter domination it has grown by a factor:
\[ \frac{\delta_k^{(1)}(a_f)}{\delta_k^{(1)}(a_{\text{enter}})} = \frac{H_{\text{enter}}^2}{H_f^2} = \frac{k^2}{H_f^2}. \tag{3.82} \]
Now consider a perturbation on a scale \( k^{-1} < k_{\text{eq}}^{-1} \). This scale enters the horizon during radiation domination. But then, between the time it enters and the time matter takes over nothing happens to the perturbation. Thus,
\[ \frac{\delta_k^{(1)}(a_f)}{\delta_k^{(1)}(a_{\text{enter}})} = \frac{\delta_k^{(1)}(a_f)}{\delta_k^{(1)}(a_{\text{eq}})} = \frac{H_{\text{eq}}^2}{H_f^2} = \frac{k_{\text{eq}}^2}{H_f^2} \]
Hence, the growth on this scale is suppressed by a factor
\[ T(k) \equiv \frac{k_{\text{eq}}^2}{k^2} \tag{3.84} \]
compared to a scale that enters during matter domination. In the last equation we introduced the transfer function \( T(k) \), which gives the deficit in growth that perturbations on scales smaller than \( k_{\text{eq}}^{-1} \) experience. It should have the following properties:
\[ T^2(k) = 1 \text{ for } k \ll k_{\text{eq}}, \tag{3.85} \]
\[ T^2(k) \sim \frac{k_{\text{eq}}^4}{k^4} \text{ for } k \gg k_{\text{eq}}. \tag{3.86} \]
Figure 7: Two different transfer functions. The simple one, \( T^2(k) = \left(1 + \frac{k^4}{k_{eq}^4}\right)^{-1} \), cuts away too much power on small scales, because it neglects the growth of the CDM density contrast during radiation domination on these scales.

A simple choice is

\[
T^2(k) = \frac{1}{1 + \frac{k^4}{k_{eq}^4}}.
\]

(3.87)

If we want to make the transfer function more accurate, we have to account for cold dark matter (CDM) perturbations. Since CDM does not interact with radiation, perturbations do not feel the radiation pressure and grow independently of the Jeans scale (see appendix D). Hence, our simplified transfer function suppresses too much power on small scales. The correction is roughly logarithmic, so that the transfer function should go as:

\[
T^2(k) = 1 \quad \text{for} \quad k \ll k_{eq},
\]

(3.88)

\[
T^2(k) \sim \frac{k_{eq}^4}{k^4} \ln^2\left(\frac{k}{k_{eq}}\right) \quad \text{for} \quad k \gg k_{eq}.
\]

(3.89)

The most prominent transfer function that includes this correction and gives power spectra in good agreement with measurements is the \textit{BBKS transfer function}, introduced in 1986 by Bardeen, Bond, Kaiser and Szalay [14],

\[
T_{BBKS}(k) = \frac{\ln(1 + 2.34q)}{2.34q} [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4},
\]

(3.90)

23
where \( q \equiv \frac{k}{\Omega h \text{Mpc}} \) and the shape parameter \( \Gamma \) is given by

\[
\Gamma \equiv \Omega_0 h \exp(-\Omega_b - \sqrt{2} h \Omega_b \Omega_0),
\]

where \( \Omega_0 \) is the relative energy density of the universe today and \( \Omega_b \) is the relative baryon density today. Henceforth, we will use the BBKS transfer function with the parameters \( \Omega_0 = 1, \Omega_b = 0.05 \) and \( h = 0.7 \). We show a plot of the two different transfer functions in figure [7].

The transfer function connects the "real" power spectrum and the power spectrum from the Einstein-de Sitter model. Hence, we make the following transformation:

\[
P(k) \rightarrow T^2(k) P(k),
\]

The transfer function explains the bend in matter power spectrum (see figure [11]). Furthermore, now we can answer the question what the initial value for \( \phi^{(1)}_k \) is. Including the transfer function, eq. (3.65) becomes

\[
P(\phi) = \frac{k^3}{2\pi^2} |\phi^{(1)}_k|^2 = \frac{9}{25} A^2 T^2(k).
\]

From this we find the following initial value for \( \phi^{(1)}_k \):

\[
|\phi^{(1)}_k| = \frac{3}{5} A(2\pi^2)^{1/2} k^{-3/2} T(k).
\]

### 3.4.4 Second order solutions

The second order equations do not decouple a priori; we have to neglect the vector parts of the first order quantities, which can be done since \( v^{(1)}_\perp \) decays. The decoupled second order equations are:

| scalar part | vector part |
|-------------|-------------|
| continuity equation | \( \delta^{(2)\prime} + \nabla \cdot v^{(2)}_\parallel = -\nabla \cdot (\delta^{(1)} v^{(1)}_\parallel) \) | - |
| Poisson equation | \( \Delta \delta^{(2)} - 4\pi G a^2 \rho^{(0)} \delta^{(2)} = 0 \) | - |
| Euler equation | \( v^{(2)\prime}_\parallel + H v^{(2)}_\parallel + \nabla \phi^{(2)} = -v^{(1)}_\parallel \cdot \nabla v^{(1)}_\parallel \) | \( v^{(2)\prime}_\perp + \mathcal{H} v^{(2)}_\perp = 0 \) |

Table 6: Decoupled second order equations.

The vector part of the Euler equation gives \( v^{(2)}_\perp \sim a^{-1} \) (again, this is a special case of the general result that the vorticity vanishes at all orders in Newtonian theory, as we showed earlier), so we can neglect the vector contribution to \( v^{(2)} \) and set \( v^{(2)} = v^{(2)}_\parallel \). Now consider the scalar equations: The sources for the second order perturbations are products of first order perturbations, which we have written on the r.h.s. of each equation. There are hence two solutions. The homogeneous solutions are obtained by ignoring the source terms, which gives the same time dependence as in first order perturbation theory. However, the specific solutions grow faster due to the source terms, so that we can neglect the homogeneous solutions. Note that the term \( v^{(1)}_\parallel \cdot \nabla v^{(1)}_\parallel \) can also be written as \( \nabla \frac{v^{(1)}_\parallel^2}{2} \).

Now we need to be careful when transforming to momentum space, because products in \( x \)-space become convolutions in \( k \)-space.
• In momentum space the expression $v^{(1)}$ becomes:

$$F[v^{(1)} \cdot v^{(1)}](k) = \frac{1}{(2\pi)^3} (v \ast v)(k)$$

$$= \frac{1}{(2\pi)^3} \int d^3k' v(k') \cdot v(k - k')$$

$$= -\frac{1}{(2\pi)^3} \frac{1}{9} f \int d^3k' \phi(k') \cdot (k - k') \phi(|k - k'|)$$

$$= -\frac{1}{(2\pi)^2} \frac{1}{9} f \int \frac{\pi}{2} \frac{d\theta'}{2} \frac{d\phi'}{2} \sin \theta' (kk' \cos \theta' - k'^2) \phi(k') \phi(\sqrt{k^2 + k'^2 - 2kk' \cos \theta'}).$$

• The expression $\nabla \cdot (\delta^{(1)}v^{(1)})$ becomes:

$$F[\nabla \cdot (\delta^{(1)}v^{(1)})](k) = ik \cdot \frac{1}{(2\pi)^3} (\delta^{(1)} \ast v^{(1)})(k)$$

$$= ik \cdot \frac{1}{(2\pi)^3} \int d^3k' \delta^{(1)}(k') v(k - k')$$

$$= ik \cdot \frac{1}{(2\pi)^3} \frac{i}{18} f \int d^3k' k'^2 \frac{\phi(k')}{|k - k'|} (k' \cdot (k - k'))$$

$$= -\frac{1}{(2\pi)^3} \frac{1}{18} f \int d^3k' k'^2 \phi(k') (|k - k'|) k \cdot (k - k')$$

$$= -\frac{1}{(2\pi)^3} \frac{1}{18} f \int \frac{\pi}{2} \frac{d\theta'}{2} \frac{d\phi'}{2} \sin \theta' (kk' \cos \theta' - k'^2) \phi(k') \phi(\sqrt{k^2 + k'^2 - 2kk' \cos \theta'}).$$

where we set $k = k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $k' = k' \begin{pmatrix} \sin \theta' \cos \phi' \\ \sin \theta' \sin \phi' \\ \cos \theta' \end{pmatrix}$, so that $k \cdot k' = kk' \cos \phi'.

The second order evolution equations in momentum space are:

$$\delta_k^{(2)} = -F[\nabla \cdot (\delta^{(1)}v^{(1)})], \quad k \neq 0$$

$$k^2 \phi_k^{(2)} + \frac{6}{\tau^2} \delta_k^{(2)} = 0,$$  \hspace{1cm} (3.95)

$$v_k^{(2)} = -i k \cdot v^{(2)} - \frac{1}{2} F[v^{(1)} \cdot v^{(1)}].$$  \hspace{1cm} (3.97)

Note that $\delta^{(1)}v^{(1)} \sim \tau^3$ and $v^{(1)} \cdot v^{(1)} \sim \tau^2$ in leading order. There are other solutions as well, which can be obtained by combining decaying and growing modes of $\delta^{(1)}$ and $v^{(1)}$. However, these solutions are all decaying and can be neglected. Hence, it is reasonable to make the following ansatz by adjusting
the time dependence of the second order perturbations to the time dependence of the source terms:

\[
\begin{align*}
\delta^{(2)}_k &= C_k \tau^4, \\
v^{(2)}_k &= D_k \frac{k}{\tau^3}, \\
\phi^{(2)}_k &= E_k \tau^2.
\end{align*}
\] (3.98, 3.99, 3.100)

With this ansatz, we find the following system of linear equations for the coefficients \(C_k, D_k, E_k\):

\[
\begin{align*}
4C_k - D_k k &= -\mathcal{F}[(\nabla \cdot \delta^{(1)} v^{(1)})(k)] \tau^{-3}, \\
E_k k^2 + 6C_k &= 0, \\
5D_k + E_k k &= -\frac{1}{2} k \mathcal{F}[(v^{(1)} \cdot v^{(1)})(k)] \tau^{-2}.
\end{align*}
\] (3.101, 3.102, 3.103)

The solutions are obtained by first solving the convolution integrals and then solving the system of linear equations for the coefficients. This is done numerically for three fixed scales that we are interested in: the galaxy scale \((k^{-1} = 1 \text{ Mpc})\), the cluster scale \((k^{-1} = 10 \text{ Mpc})\), and the supercluster scale \((k^{-1} = 100 \text{ Mpc})\). Figures 8, 9 and 10 show the plots of the first and second order perturbations on these scales. Note that we plot the dimensionless quantities \(\Delta^{(1)}_\delta = \sqrt{P_\delta} = \frac{k^{3/2}}{2\pi} |\delta^{(1)}_k| \) etc.

![Figure 8: Evolution of first and second order perturbations for the scale \(k^{-1} = 1 \text{ Mpc}\).](image)
Figure 9: Evolution of first and second order perturbations for the scale $k^{-1} = 10$ Mpc.

Figure 10: Evolution of first and second order perturbations for the scale $k^{-1} = 100$ Mpc.
For early times and large scales, linear perturbation theory is valid, since all second order perturbations are substantially smaller than all first order perturbations. Now a difficulty arises to define the time when a scale "goes non-linear". Strictly speaking, linear perturbation theory becomes invalid when the largest second order contribution becomes as important as the smallest first order contribution. Here this is the crossing of $\delta^{(2)}_k$ and $\phi^{(1)}_k$. We call the scale factor at this crossing $a_{NL1}$.

$$\delta^{(2)}_k(a_{NL1}) \simeq \phi^{(1)}_k(a_{NL1}).$$

(3.104)

However, the perturbations series $\delta_k = \delta^{(1)}_k + \delta^{(2)}_k$ still make sense, since $\delta^{(2)}_k$ is much smaller than $\delta^{(1)}_k$ at this time. According to the spherical collapse model, collapse happens much later, roughly when the density contrast reaches unity \[8, 13\], which is a commonly accepted indicator for the beginning of nonlinearity. A density contrast of 1 means that the density perturbations become as large as the background density. In other words, there can be regions which become twice as dense as the background. We call the scale factor when this happens $a_{NL2}$.

$$\Delta \delta^{(1)}_k(a_{NL2}) \simeq 1.$$ 

(3.105)

The transition between the linear system and the nonlinear system happens certainly between $a_{NL1}$ and $a_{NL2}$ (see Table 7). Shortly after $a_{NL2}$, the system has virialized into a gravitational bound object, where the angular momentum prevents further collapse \[8\]. This has already happened for galaxies and clusters of galaxies, while perturbations on scales $\gtrsim 10$ Mpc still behave linear today. The fact that small scales collapse before large scales leads to the bottom-up picture of structure formation: small scale structures form before large scale structures.

Figures 11 shows the evaluated matter power spectrum $P_\delta(k) = |\delta^{(1)}_k|^2$ today. It increases $\sim k$ for $k < k_{eq}$ and decreases $\sim k^{-3} \ln^2(k)$ for $k > k_{eq}$ due to the implemented transfer function and shows good correspondence with measurements, as shown in Figure 6. However, nonlinear effects become important as $P_\delta(k)$ reaches unity, which happens at about $k = 0.078$ Mpc$^{-1}$ today. Figure 12 shows the power spectrum for the absolute value of the peculiar velocity $P_v(k) = |v^{(1)}_k|^2$ today. It decreases $\sim k^{-1}$ for $k < k_{eq}$ and $\sim k^{-5} \ln^2(k)$ for $k > k_{eq}$.

### Table 7: $a_{NL1}$ and $a_{NL2}$ for different scales.

| Scale | 1 Mpc | 10 Mpc | 100 Mpc |
|-------|-------|--------|---------|
| $a_{NL1}$ | 0.005 | 0.092 | 9 |
| $a_{NL2}$ | 0.12 | 0.75 | 30 |
Figure 11: The matter power spectrum today. The top figure shows $P_\delta(k)$ and the bottom figure shows $P_\delta(k)$. Linear theory fails when $P_\delta(k) \gtrsim 1$ (vertical line).
Figure 12: The power spectrum of $|\mathbf{v}|$ today. The top figure shows $\mathcal{P}_v(k)$ and the bottom figure shows $P_v(k)$. 
3.4.6 Solutions during $\Lambda$-domination

As we approach $a = 1$, the cosmological constant becomes more and more important, as matter scales $\sim a^{-3}$. During $\Lambda$-domination the Friedmann and Raychaudhuri equations become

$$
\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho_\Lambda, \quad (3.106)
$$

$$
\mathcal{H}' = \frac{8\pi G}{3} a^2 \rho_\Lambda. \quad (3.107)
$$

However, if we neglect dark energy perturbations, there are no variations in the first order equations. Hence, the form of the Jeans equation remains unchanged,

$$
\delta^{(1)''}_k + \mathcal{H}\delta^{(1)'}_k = 4\pi G \bar{\rho}_m a^2 \delta^{(1)}_k. \quad (3.108)
$$

Note that

$$
4\pi G \bar{\rho}_m a^2 = \frac{3}{2} \mathcal{H}^2 \bar{\rho}_m \ll 1, \quad (3.109)
$$

so that the source term of the Jeans equation becomes negligible. Then we have two solutions, the growing solution

$$
\delta^{(1)}_k = \text{const.} \quad (3.110)
$$

and the decaying solution

$$
\delta^{(1)}_k \sim a^{-2}. \quad (3.111)
$$

Hence, as we approach $\Lambda$-domination, the density perturbations freeze out. A more general solution in the intermediate regime, when dark energy and matter density are equally important, will be discussed later, in section 5.3.

3.5 Newtonian cosmological simulations

Cosmological $N$-body simulations use Newton’s equation of motion to simulate the behaviour of a fixed number of particles under the influence of their reciprocal gravitational interaction. These simulations are usually run in a box with periodic boundary conditions. In the $(t,r)$-system the equation of motion can be written as

$$
\frac{d^2 r_i(t)}{dt^2} = -\nabla r \phi_{\text{sim}}, \quad (3.112)
$$

where $r_i(t)$ is the trajectory of the $i$-th particle. Transforming to the $(\tau,x)$-system, we find

$$
\frac{d}{dr} \frac{d}{d\tau} (ax_i(\tau)) = \frac{1}{a} \nabla \phi_{\text{sim}}. \quad (3.113)
$$

The l.h.s. is

$$
\frac{d}{d\tau} \frac{d}{dr} (ax_i(\tau)) = \frac{d}{d\tau} \left( \mathcal{H} x_i(\tau) + \frac{dx_i(\tau)}{d\tau} \right) = \frac{1}{a} \left( \mathcal{H}' x_i(\tau) + \mathcal{H} \frac{d^2 x_i(\tau)}{d\tau^2} + \frac{dx_i(\tau)}{d\tau} \right),
$$
and on the r.h.s. the potential can be split into a background part and a perturbed part,

\[ \frac{1}{a} \nabla \phi_{\text{sim}} = \frac{1}{a} \nabla (\bar{\phi}_{\text{sim}} + \delta \phi_{\text{sim}}). \]

Subtracting the background (this is the Raychaudhuri equation), we find

\[ \frac{d^2 x_i(\tau)}{d\tau^2} + \mathcal{H} \frac{dx_i(\tau)}{d\tau} = -\nabla \delta \phi_{\text{sim}}. \] (3.114)

From this equation it can be seen that all freely falling observers (for which \( \delta \phi_{\text{sim}} = 0 \)) will become resting observers after sufficiently long time, since \( \frac{dx_i(\tau)}{d\tau} \sim \frac{1}{a} \). The gravitational potential perturbation is obtained using the Poisson equation,

\[ \Delta \delta \phi_{\text{sim}} = 4\pi G a^2 \delta \rho_{\text{sim}}, \] (3.115)

which is solved in Fourier space, where the matter density perturbation is calculated by counting particles in cells,

\[ \delta \rho_{\text{sim}}(\mathbf{x}, \tau) = a^{-3} \sum_i m_i \delta \rho_D(\mathbf{x} - \mathbf{x}_i(\tau)). \] (3.116)

4 The perturbed universe: relativistic treatment

4.1 The perturbed metric

In relativistic cosmological perturbation theory we consider small perturbations on the homogenous and isotropic FRW metric tensor and on the energy-momentum tensor and calculate how these perturbations develop in time, using the energy-momentum conservation equation and the Einstein field equations. For a detailed reference, see [15]. Our conventions are the following: Greek indices will range from 0 to 3 and Latin indices from 1 to 3. \( \nabla_\mu \) as well as an index \( \mu \) denote a covariant derivative with respect to the perturbed metric. We are going to drop the superscripts \(...^{(1)}\) here, since we only consider linear perturbations. Background quantities are denoted with a bar.

The FRW metric in a flat universe, using comoving coordinates and conformal time, is

\[ \bar{g}_{\mu\nu} \equiv a^2 \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix}. \] (4.1)

Now consider a small perturbation of the background metric: \( g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \). We define the metric perturbation

\[ \delta g_{\mu\nu} \equiv a^2 \begin{pmatrix} -2\phi & w_i \\ w_i & -2c\delta_{ij} + 2h_{ij} \end{pmatrix}. \] (4.2)

\( w_i \) and \( h_{ij} \) can be further decomposed into scalar, vector and tensor parts,

\[ w_i = w_i^s + w_i^v + \] (4.3)
### Table 8: Degrees of freedom in linear perturbation theory.

| type               | fields       | constraints          | degrees of freedom |
|--------------------|--------------|----------------------|--------------------|
| scalar perturbations | \( \phi, \psi, w, h \) | -                    | 4                  |
| vector perturbations | \( w_i^\perp, h_i \) | \( \nabla^i w_i^\perp = \nabla^i h_i = 0 \) | 4                  |
| tensor perturbations | \( h_{ij}^T \) | \( \nabla^i h_{ij}^T = (h^T)^i_j = 0, h_{ij}^T = h_{ji}^T \) | 2                  |

Table 8: Degrees of freedom in linear perturbation theory.

and

\[
h_{ij} = D_{ij} h + h_{(ij)} + h_{ij}^T,
\]

where \( D_{ij} \) is the symmetric traceless double-gradient operator,

\[
D_{ij} = \nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \Delta.
\]

Note that \( w_i^\perp \) and \( h_i \) are pure vector parts with zero divergence \((\nabla^i w_i^\perp = \nabla^i h_i = 0)\) and \( h_{ij}^T \) is transverse \((\nabla^i h_{ij}^T = 0)\), traceless \(((h^T)^i_j = 0)\) and symmetric \((h^T = h^T_{ji})\). Thus, in first order the metric perturbations decouple:

\[
\delta g_{\mu\nu} = \delta g_{\mu\nu}^S + \delta g_{\mu\nu}^V + \delta g_{\mu\nu}^T
\]

\[
\begin{align*}
\delta g_{\mu\nu} &= a^2 \left( \begin{array}{cc}
-2\phi & w_i^\perp \\

w_i^\perp & -2\psi \delta_{ij} + 2h_{ij}
\end{array} \right) + a^2 \left( \begin{array}{cc}
0 & w_i^\perp \\
w_i^\perp & 2h_{(ij)}
\end{array} \right) + a^2 \left( \begin{array}{cc}
0 & 0 \\
0 & 2h_{ij}^T
\end{array} \right),
\end{align*}
\]

where we have introduced a new variable

\[
\psi = \psi + \frac{1}{3} \Delta h.
\]

Now that we have decomposed the metric perturbation, it is instructive to count the degrees of freedom, see table 8. The result is that we have altogether 10 degrees of freedom in the metric perturbation. However, four of them correspond to the freedom of coordinate choice, so that we can put further restrictions on the metric, as will be discussed later. The decomposition into scalar, vector and tensor parts is important for the physical interpretation. The scalar perturbations are the most important, because they couple to density perturbations and give rise to structure formation. The vector perturbations are not very interesting in linear order, because they couple to rotations only, which are decaying in an expanding universe due to angular momentum conservation. Tensor perturbations give rise to gravitational waves.

### 4.2 Gauge transformations

There is no unique mapping between points in the background spacetime and points in the perturbed spacetime. Indeed, there are infinitely many mappings, all close to each other. Consequently, for

\footnote{Some authors introduce this quantity as the curvature perturbation. However, we will later introduce the curvature perturbation according to the definition of Bardeen.}
a given coordinate system of the background, there exist infinitely many coordinate systems of the perturbed spacetime. A coordinate transformation between these coordinate systems is called a gauge transformation. Consider a point \( \bar{P} \) in the background spacetime, whose coordinates are \( \{ x^\alpha \} \). Now consider two different coordinate systems in the perturbed spacetime, and denote them by \( \{ \tilde{x}^\alpha \} \) and \( \{ \hat{x}^\alpha \} \). In the \( \tilde{x}^\alpha \)-coordinates, the background point \( \bar{P} \) corresponds to a point \( \tilde{P}(\tilde{x}) \). In the \( \hat{x}^\alpha \)-coordinates, this point corresponds to another point \( \hat{P}(\hat{x}) \), see figure 13 for an illustration. We have by construction

\[ x^\alpha(\bar{P}) = \tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}), \]

(4.9)
because all coordinates refer to the same point in the manifold. Now one can ask the question: what are the \( \hat{x}^\alpha \)-coordinates of the point \( \tilde{P} \)? These clearly differ from the \( \tilde{x}^\alpha \)-coordinates and we denote this difference by \( \xi^\alpha \), which is first-order small due to the fact that all coordinate systems are close to each other:

\[ \tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) + \xi^\alpha, \]

(4.10)
\[ \tilde{x}^\alpha(\hat{P}) = \hat{x}^\alpha(\hat{P}) + \xi^\alpha. \]

(4.11)

Note that it does not matter, in which coordinate system we give \( \xi^\alpha \), because the difference is second order. The above equations represent a coordinate transformation, which in cosmological perturbation theory is called gauge transformation. They are equivalent to:

\[ \tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) - \xi^\alpha, \]

(4.12)
\[ \tilde{x}^\alpha(\hat{P}) = \hat{x}^\alpha(\hat{P}) - \xi^\alpha. \]

(4.13)
4.2.1 Gauge transformations of scalars, vectors and tensors

Using these transformations, we can determine how various geometric objects (scalars, vectors and tensors) transform under a gauge transformation.

• First, let us determine the transformation of a scalar. Using Taylor expansion, we find

\[ s(\tilde{P}) = s(\hat{P}) + \frac{\partial s}{\partial \hat{x}^\alpha} (\hat{x}^\alpha(\hat{P}) - \hat{x}^\alpha(\tilde{P})) = s(\hat{P}) - \frac{\partial \tilde{s}}{\partial \hat{x}^\alpha} \xi^\alpha = s(\hat{P}) - \tilde{s}' \xi^0. \]  

(4.14)

Here we have used that (1) the difference between \( \frac{\partial s}{\partial \hat{x}^\alpha} \) and \( \frac{\partial \tilde{s}}{\partial \hat{x}^\alpha} \) is first order small and can be neglected when multiplied by \( \xi^\alpha \) and (2) the background value \( \tilde{s} \) only depends on conformal time \( \tau \), because the background spacetime is isotropic. The scalar perturbation in a given gauge is defined as \( \delta \tilde{s} \equiv s(\tilde{P}) - \tilde{s}(\hat{P}) \) (or \( \delta s \equiv s(\tilde{P}) - s(\hat{P}) \)). The transformation law is

\[ \delta \tilde{s} = s(\tilde{P}) - \tilde{s}(\hat{P}) = s(\hat{P}) - \tilde{s}(P) - \tilde{s}' \xi^0 = \delta \tilde{s} - \tilde{s}' \xi^0. \]  

(4.15)

Hence, any scalar that is constant in the background \( (\tilde{s}' = 0) \) is gauge-invariant.

• For a vector \( w_\hat{\mu}(\hat{P}) \) we have \( (\hat{\mu} \) refers to the coordinates \( \hat{x}^\mu \):

\[ w_\hat{\mu}(\tilde{P}) = w_\hat{\mu}(\hat{P}) - \frac{\partial \tilde{w}_\mu}{\partial x^\alpha} \xi^\alpha \]  

(4.16)

and therefore

\[ w_\tilde{\mu}(\tilde{P}) = \frac{\partial \hat{x}_\sigma}{\partial \hat{x}^\mu} w_\hat{\sigma}(\hat{P}) = \frac{\partial (\hat{x}_\sigma - \xi_\sigma)}{\partial \hat{x}^\mu} (w_\hat{\sigma}(\hat{P}) - \frac{\partial \tilde{w}_\sigma}{\partial x^\alpha} \xi^\alpha) \]

\[ = (\delta_\sigma^\mu - \xi_\mu^\sigma)(w_\hat{\sigma}(\hat{P}) - \frac{\partial \tilde{w}_\sigma}{\partial x^\alpha} \xi^\alpha) \]

\[ = w_\tilde{\mu}(\tilde{P}) - \tilde{w}_\mu,\alpha \xi^\alpha - \tilde{w}_\sigma \xi^\sigma_{,\mu}. \]  

(4.17)

For a vector perturbation we find the transformation

\[ \delta \tilde{w}_\mu = w_\tilde{\mu}(\tilde{P}) - \tilde{w}_\mu(\hat{P}) \]

\[ = \delta \hat{w}_\mu - \hat{w}_\mu,\alpha \xi^\alpha - \hat{w}_\sigma \xi^\sigma_{,\mu}. \]  

(4.18)

Obviously, any vector that vanishes in the background is gauge-invariant.

• For a \((0,2)\)-tensor \( B_\hat{\mu}\hat{\nu}(\hat{P}) \) we have:

\[ B_{\tilde{\mu}\tilde{\nu}}(\tilde{P}) = B_{\hat{\mu}\hat{\nu}}(\hat{P}) - B_{\hat{\mu}\hat{\nu},\alpha} \xi^\alpha = B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \tilde{B}_{\mu\nu,\alpha} \xi^\alpha \]  

(4.19)
and therefore

\[ B_{\hat{\mu}\hat{\nu}}(\hat{P}) = \frac{\partial \hat{x}^\rho}{\partial \hat{x}^\mu} \frac{\partial \hat{x}^\sigma}{\partial \hat{x}^\nu} B_{\rho\sigma}(\hat{P}) \]

\[ = \frac{\partial (\hat{x}^\rho - \xi^\rho)}{\partial \hat{x}^\mu} \frac{\partial (\hat{x}^\sigma - \xi^\sigma)}{\partial \hat{x}^\nu} (B_{\rho\sigma}(\hat{P}) - \bar{B}_{\rho\sigma,\alpha} \xi^\alpha) \]

\[ = (\delta^\rho_\mu - \xi^\rho_\mu)(\delta^\sigma_\nu - \xi^\sigma_\nu)(B_{\rho\sigma}(\hat{P}) - \bar{B}_{\rho\sigma,\alpha} \xi^\alpha) \]

\[ = B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \bar{B}_{\mu\nu,\alpha} \xi^\alpha - \xi^\rho_\mu \bar{B}_{\rho\nu} - \xi^\sigma_\nu \bar{B}_{\mu\sigma}. \]  

(4.20)

For a tensor perturbation we find

\[ \delta \bar{B}_{\mu\nu} = B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \bar{B}_{\mu\nu}(\bar{P}) \]

\[ = \delta \bar{B}_{\mu\nu} - \bar{B}_{\mu\nu,\alpha} \xi^\alpha - \xi^\rho_\mu \bar{B}_{\rho\nu} - \xi^\sigma_\nu \bar{B}_{\mu\sigma}. \]  

(4.21)

Again, any tensor that vanishes on the background is gauge-invariant.

### 4.2.2 Gauge transformation of metric perturbations

Now we can evaluate how the metric perturbation transforms. Using that the background metric is symmetric and homogeneous, eq. (4.21) gives:

\[ \delta g_{\mu\nu} = \bar{g}_{\mu\nu}' - 2 \xi_{(\mu,\nu)}. \]  

(4.22)

- The \((0,0)\)-component of eq. (4.22) reads, using the metric tensor from eq. (4.1) and the perturbed part of the metric from eq. (4.2):

\[ -2a^2 \hat{\phi} = -2a^2 \hat{\phi} + 2aa' \xi^0 + 2a^2 \xi^{0\prime} \]

\[ \Rightarrow \hat{\phi} = \hat{\phi} - \mathcal{H} \xi^0 - \xi^{0\prime}. \]  

(4.23)

- From the \((0,i)\)-component of eq. (4.22) we find the transformation of \(w_i\):

\[ -a^2 \hat{w}_i = -a^2 \hat{w}_i - 2 \xi_{(0,i)} \]

\[ = -a^2 \hat{w}_i - a^2 \xi^0_i + a^2 \xi^{i\prime}_i \]

\[ \Rightarrow \hat{w}_i = \hat{w}_i + \xi^0_i - \xi^{i\prime}_i. \]  

(4.24)

This can be further decomposed using \(\xi_i = \xi^0_i + \xi^i_\perp\),

\[ \hat{\xi} = \hat{\xi} + \xi^0_i - \xi^{i\prime}_i, \]  

(4.25)

\[ \hat{\xi}_\perp = \hat{\xi}_\perp - \xi^{i\prime}_i. \]  

(4.26)
• The \((i,j)\)-component of eq. (4.22) is:

\[
-2a^2\delta_{ij} + 2a^2\tilde{h}_{ij} = -2a^2\delta_{ij} + 2a^2\hat{h}_{ij} - 2aa'\delta_{ij}\xi^0 - 2\xi_{(i,j)}. \tag{4.27}
\]

Note that \(\xi_{(i,j)}\) can be split into a traceless part and a diagonal part:

\[
\xi_{(i,j)} = \frac{1}{3}\xi_{k,k}\delta_{ij} + \xi_{(i,j)} - \frac{1}{3}\xi_{k,k}\delta_{ij}. \tag{4.28}
\]

The trace of eq. (4.27) gives the transformation of \(e\):

\[
\tilde{e} = \hat{e} + H\xi^0 + \frac{1}{3}\xi^k. \tag{4.29}
\]

If we take the traceless part of eq. (4.27), we find the transformation of \(h_{ij}\):

\[
\tilde{h}_{ij} = \hat{h}_{ij} - \xi_{(i,j)} + \frac{1}{3}\delta_{ij}\xi^k. \tag{4.30}
\]

This can be further decomposed,

\[
\begin{align*}
\tilde{h} &= \hat{h} - \xi, \tag{4.31} \\
\tilde{h}_i &= \hat{h}_i - \xi^\perp_i, \tag{4.32} \\
\tilde{h}_{ij} &= \hat{h}_{ij}. \tag{4.33}
\end{align*}
\]

• From eq. (4.31) and eq. (4.29) we find the transformation law for \(\psi\),

\[
\tilde{\psi} = \hat{\psi} + H\xi^0. \tag{4.34}
\]

4.2.3 Gauge transformations of velocity and density perturbations

Next, consider the transformation of the peculiar velocity \(v\). We have:

\[
\begin{align*}
\tilde{v}^i &= \frac{d\tilde{x}^i}{d\tilde{\tau}} = \frac{d(\hat{x}^i + \xi^i)}{d(\hat{\tau} + \xi^0)} = \frac{d(\hat{x}^i + \xi^0)}{d\hat{\tau}} \left(\frac{d(\hat{\tau} + \xi^0)}{d\hat{\tau}}\right)^{-1} \\
&= (\tilde{v}^i + \xi''^0)(1 + \xi''^0)^{-1} = (\tilde{v}^i + \xi''^0)(1 - \xi''^0) = \tilde{v}^i + \xi'',
\end{align*}
\]

so that the velocity potential transforms as

\[
\tilde{\psi} = \hat{\psi} + \xi^0, \tag{4.35}
\]

and the vector part transforms as

\[
\tilde{v}_i^\perp = \hat{v}_i^\perp + \xi^\perp_i. \tag{4.36}
\]

37
For the density contrast \( \delta \) we find the transformation:

\[
\tilde{\delta} = \frac{\delta \rho}{\bar{\rho}} = \delta - \bar{\rho}^\prime \xi^0 = \bar{\rho} \xi^0 = 3 \mathcal{H} \xi^0.
\] (4.37)

This means that \( v \) and \( \delta \) depend on the gauge choice; they are different in each gauge. Hence, they cannot be unique physical observables. However, it is possible to construct gauge-invariant quantities, which do not depend on the gauge choice, as we will show in the next subsection.

### 4.2.4 Gauge-invariant quantities

As mentioned before, unique physical quantities need to be gauge-invariant. A good example for this is the theory of electromagnetism. Consider the free Lagrangian,

\[
\mathcal{L}_{\text{EM}} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu},
\] (4.38)

where \( F^{\mu \nu} \equiv 2 A(\mu, \nu) \) is the electromagnetic tensor and \( A_\mu \) is the electromagnetic 4-potential. Note that the Lagrangian is invariant under the gauge transformation

\[
A_\mu \rightarrow A_\mu + \chi_\mu.
\] (4.39)

However, we can construct two gauge-invariant quantities out of \( A_\mu \), the electric field

\[
E \equiv \dot{\mathbf{A}} + \nabla A^0
\] (4.40)

and the magnetic field

\[
B \equiv \nabla \times \mathbf{A}.
\] (4.41)

We can learn from this example that the question of gauge only arises in the theory. Once we evaluate measurable quantities, the gauge should become irrelevant. However, here we ignored the role of the observer. For example, if we consider a charge at a fixed point, a resting observer would measure no magnetic field, while a moving observer would. This shows that also in electrodynamics the role of the observer plays a crucial role.

Now let us come back to cosmological perturbation theory. We can construct gauge-invariant combinations out of the perturbed quantities. The most famous ones are the Bardeen potentials [11],

\[
\Phi \equiv \phi + \frac{1}{a} [(w - h') a]',
\] (4.42)

\[
\Psi \equiv \psi - \mathcal{H} (w - h').
\] (4.43)

Another important gauge-invariant quantity is the curvature perturbation, which we define according to Bardeen [11],

\[
\zeta \equiv \frac{1}{3} \delta + \psi.
\] (4.44)

We introduced this quantity already in subsection 3.4.2 where we used its power spectrum to find the
Table 9: Gauges-overview. One can choose two different gauge conditions. A slash denotes combinations that are not possible. For some important gauges we introduce names. Stars denote residual gauge freedom (see text for explanation).

|   | \(\phi = 0\) | \(\psi = 0\) | \(\delta = 0\) | \(w = 0\) | \(h = 0\) | \(v = 0\) |
|---|---|---|---|---|---|---|
| \(\phi = 0\) | - | / | / | S * | ** | ** |
| \(\psi = 0\) | / | - | / | UC * | SE | * |
| \(\delta = 0\) | / | / | - | UD | * |
| \(w = 0\) | S * | UC * | UD | - | N | C * |
| \(h = 0\) | ** | SE | UD | N | - | / |
| \(v = 0\) | * | * | C * | / | - |

4.2.5 Gauges

We can fix the gauge by putting constrains on the metric perturbation fields or the fluid perturbation fields. Henceforth, we will only consider scalar perturbations and neglect all vector and tensor perturbations. This can be done since in first order scalar, vector and tensor modes decouple. Then the following transformations are relevant:

\[
\begin{align*}
\tilde{\phi} &= \dot{\phi} - H\xi^0 + \xi^0', \\
\tilde{\psi} &= \dot{\psi} + H\xi^0, \\
\tilde{\omega} &= \dot{\omega} + \xi^0 - \xi', \\
\tilde{h} &= \dot{h} - \xi, \\
\tilde{v} &= \dot{v} + \xi', \\
\tilde{\delta} &= \ddot{\delta} + 3H\xi^0.
\end{align*}
\]

Now we can choose two of these fields to be zero, which corresponds to choosing \(\xi^0\) and \(\xi\). Note that not all combinations are possible. One can only choose \(h\) or \(v\) to be zero (which fixes \(\xi\)), as well as one can only choose \(\phi\) or \(\psi\) or \(\delta\) to be zero (which fixes \(\xi^0\)). However, one can always choose \(w\) to be zero, because this only fixes the difference between \(\xi^0\) and \(\xi\). In table 9 we show an overview over the possible gauges one can construct by combining constrains on these perturbation fields. A slash indicates a gauge that does not exist. For example, it is not possible to construct a gauge in which both \(\psi\) and \(\phi\) are zero, because then eq. (4.45) and eq. (4.46) would give contradicting expressions for \(\xi^0\). The stars indicate the following residual gauge freedoms:

- *: There is a residual gauge freedom \(\xi \to \xi + C(x)\), where \(C(x)\) is an arbitrary function.
There is a residual gauge freedom \( \xi^0 \to \xi^0 + \frac{1}{a} D(x) \), where \( D(x) \) is an arbitrary function.

Hence, gauges with one and/or two stars are not unique, and we have to fix the residual gauge freedom (e.g. by setting \( C(x) = D(x) = 0 \)). Furthermore, we have given some important gauges names in order to refer to them:

- **UC**: uniform curvature gauge. In this gauge we set \( \psi = w = 0 \). One can show that in a spatially flat universe the 3-Ricci-scalar is given by \( (3)^R = 4\Delta \psi \) (see [11]). Hence, in the uniform curvature gauge, as well as in any other gauge with \( \psi = 0 \) there is no intrinsic curvature. These gauges are hence a good choice for the comparison to Newtonian cosmology, where curvature does not exist.

- **SE**: spatially Euclidean gauge. In this gauge we set \( \psi = h = 0 \), so that again the intrinsic curvature vanishes. Furthermore, in this gauge the spatial part of the metric has no perturbation, i.e. looks Euclidean. All perturbations are in the \((0,0)\)-part and the \((0,i)\)-part of the metric.

- **UD**: uniform density gauge. In this gauge there are no density perturbations, since we set \( \delta = 0 \).

- **N**: conformal Newtonian gauge. In this gauge we set \( w = h = 0 \). It is equivalent to the zero shear gauge, \( \sigma = w = 0 \), where \( \sigma = h' + w \) generates the traceless part of the the extrinsic curvature tensor and hence can be interpreted as the shear in the normal worldlines [11]. Another common name for it is longitudinal gauge. Note that the metric perturbations \( \phi_N \) and \( \psi_N \) coincide with the Bardeen potentials in this gauge, \( \phi_N = \Phi \) and \( \psi_N = \Psi \). Furthermore, it can be shown that \( \Phi \) corresponds to the gravitational potential introduced in Newtonian cosmology. For this, consider a static universe\(^6\). The line element in conformal Newtonian gauge has the form

\[
\text{ds}^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1 + 2\Phi)d\tau^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j.
\]

The proper time between two events along a worldline is given by

\[
\int \sqrt{-ds^2} = \int d\tau \sqrt{1 + 2\Phi - (1 - 2\Psi)\delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}}.
\]

Now we expand the integrand and keep only terms linear in \( \Phi, \Psi \) and quadratic in \( v \):

\[
\int \sqrt{-ds^2} = \int dt (1 + \Phi - \frac{v^2}{2}).
\]

The proper time is extremized if and only if the Lagrangian \( L = 1 + \Phi - \frac{v^2}{2} \) satisfies the Euler-Lagrange equation

\[
\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial v},
\]

which yields Newton’s law for the motion of particles,

\[
\dot{v} = -\nabla \Phi.
\]

\(^6\) A static universe has the nice simplifications \( a = 1, \ dt = d\tau, \ H = 0, \ u = v, \ x = r \).
Hence, the Bardeen potential $\Phi$ can be identified with the gravitational potential.

- **C - comoving gauge.** In this gauge we set $w = v = 0$. To explain what this means more vividly, we need to introduce some new vocabulary. A gauge can be interpreted as a way of "cutting" the 4-dimensional spacetime into space and time. Hypersurfaces of constant $\tau$ give 3-dimensional slices and worldlines of constant $x_i$ give 1-dimensional threads. The shift vector $w$ tells how much the coordinates of a point are shifted from one slice to the next slice. The shift vector vanishes if and only if slicing and threading are orthogonal (see figure 14). The slicing is said to be comoving, if the time slices are always orthogonal to the fluid 4-velocity, which can be achieved by choosing $w = v$. The threading is said to be comoving, if the threads are worldlines of comoving observers, i.e. $v = 0$. The comoving gauge is defined by requiring both comoving slicing and comoving threading [16, 5], which gives $v = w = 0$, or in the case of scalar perturbations, $v = w = \phi = 0$. Hence, in the comoving gauge slicing and threading are orthogonal and the threads are worldlines of comoving observers.

- **S - synchronous gauge.** In this gauge we set $w = 0$, so that comoving observers do not change their coordinates from one time-slice to the next one, and $\phi = 0$, so that observers at different places have synchronous clocks (note that $\phi$ only affects the time-time component of the metric tensor).
4.3 Relativistic equations in the conformal Newtonian gauge

In the conformal Newtonian gauge, the metric tensor is
\[ g_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2\Phi & 0 \\ 0 & (1 - 2\Psi)\delta_{ij} \end{pmatrix} \]  
(4.56)

and the inverse metric is
\[ g^{\mu\nu} = a^{-2} \begin{pmatrix} -1 + 2\Phi & 0 \\ 0 & (1 + 2\Psi)\delta_{ij} \end{pmatrix}. \]  
(4.57)

Using the metric tensor, we can construct the connection coefficients,
\[ \Gamma^{\mu}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\lambda} (g_{\lambda\beta,\mu} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda}). \]  
(4.58)

A calculation gives:
\[ \Gamma^0_{00} = \mathcal{H} + \Phi', \quad \Gamma^0_{0k} = \Phi_k, \quad \Gamma^0_{ij} = (\mathcal{H} - 2\mathcal{H}(\Phi + \Psi))\delta_{ij}, \]
\[ \Gamma^i_{00} = \Phi_{,i}, \quad \Gamma^i_{0j} = (\mathcal{H} - \Psi')\delta_{ij}, \quad \Gamma^i_{kl} = -(\Phi_{,l}\delta^i_k + \Psi_{,k}\delta^i_l) + \Psi_{,ij}. \]

The Ricci tensor is defined as
\[ R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\alpha\nu,\mu} + \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu} \Gamma^\beta_{\alpha\nu}. \]  
(4.59)

The components are:
\[ R_{00} = -3\mathcal{H}' + 3\psi'' + \Delta\Phi + 3\mathcal{H}(\Phi' + \Psi'), \]
\[ R_{0i} = 2(\Psi' + \mathcal{H}\Phi),_i, \]
\[ R_{ij} = (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} + [\Psi'' + \Delta\Psi - \mathcal{H}(\Phi' + 5\Psi) - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)]\delta_{ij} + (\Psi - \Phi),_{ij}. \]

Raising an index we find
\[ R^0_0 = 3a^{-2}\mathcal{H}' + a^{-2}[\mathcal{H}' + 3\psi'' - \Delta\Phi + 3\mathcal{H}(\Phi' + \Psi') - 6\mathcal{H}'\Phi], \]
\[ R^0_i = -2a^{-2}(\Psi' + \mathcal{H}\Phi),_i, \]
\[ R^i_0 = 2a^{-2}(\Psi' + \mathcal{H}\Phi),_i, \]
\[ R^i_j = a^{-2}(\mathcal{H}' + 2\mathcal{H}^2)\delta^i_j + a^{-2}[\Psi'' + \Delta\Psi - \mathcal{H}(\Phi' + 5\Psi) - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)]\delta_{ij} + a^{-2}(\Psi - \Phi),_{ij}. \]
The Ricci scalar is

\[ R \equiv R^0_0 + R^i_i = 6a^{-2}(H' + H^2) + a^{-2}(-6\Psi'' + 2\Delta(2\Psi - \Phi) - 6\mathcal{H}(\Phi' + 3\Psi') - 12(H' + H^2)\Phi). \]

Finally, we can construct the Einstein tensor \( G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu, \)

\[
\begin{align*}
G^0_0 &= -3a^{-2}H^2 + a^{-2}[2\Delta\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi], \\
G^i_0 &= R^i_0 = -2a^{-2}(\Psi' + \mathcal{H}\Phi)_i, \\
G^0_i &= R^i_0 = 2a^{-2}(\Psi' + \mathcal{H}\Phi)_i, \\
G^i_j &= R^i_j - \frac{1}{2}R\delta^i_j \\
&= a^{-2}(-2H' - H^2)\delta^i_j \\
&\quad + a^{-2}[2\Psi'' + \Delta(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4H' + 2H^2)\Phi]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi),ij.
\end{align*}
\]

The energy-momentum tensor for a perfect fluid with zero pressure is

\[ T^\mu_\nu \equiv \rho u^\mu u_\nu = \bar{\rho}(1 + \delta)u^\mu u_\nu. \] (4.60)

For the 4-velocity \( u^\mu \) we find, using the normalization condition \( u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu = -1, \)

\[
\begin{align*}
u^\mu &= \frac{1}{a} \begin{pmatrix} 1 - \Phi \\ v_{N,i} \end{pmatrix}, \\
u^\mu &= a \begin{pmatrix} -1 - \Phi \\ v_{N,i} \end{pmatrix}.
\end{align*}
\] (4.61) (4.62)

Note that \( v_N = \nabla v_N \), since we only consider scalar perturbations. Thus, the energy-momentum tensor up to linear order is:

\[ T^\mu_\nu = \begin{pmatrix} -\bar{\rho}(1 + \delta_N) & \bar{\rho}v_{N,i} \\ -\bar{\rho}v_{N,i} & 0 \end{pmatrix}. \] (4.63)

Now we consider the Bianchi identity or relativistic energy-momentum conservation equation,

\[ \nabla_\mu T^\mu_\nu = 0. \] (4.64)

First consider the \((\nu = 0)\)-component of this equation. At background order, this gives the continuity equation that we know already from Newtonian physics,

\[ \bar{\rho}' + 3\mathcal{H}\bar{\rho} = 0, \] (4.65)
while the perturbed part gives the relativistic version of the perturbed continuity equation,

$$\delta'_{N} + \Delta v_{N} - 3\Psi' = 0. \quad (4.66)$$

Note the extra term $-3\Psi'$, which does not appear in Newtonian cosmology. Now consider the $(\nu = i)$-component of eq. (4.64). There is no background order of this equation because $\bar{T}^{\nu}_{\mu} = 0$. However, the perturbed part of the $(\nu = i)$-component gives the relativistic version of the perturbed Euler equation,

$$\nabla v'_{N} + \mathcal{H} \nabla v_{N} = -\nabla \Phi, \quad (4.67)$$

which coincides exactly with the Newtonian equation.

Now consider the Einstein field equations,

$$G^\mu_{\nu} = 8\pi G T^\mu_{\nu}. \quad (4.68)$$

To the background order, both sides are diagonal, and we have two equations: The $(0,0)$-component gives the Friedmann equation,

$$\mathcal{H}^2 = \frac{8\pi G}{3} \bar{\rho} a^2, \quad (4.69)$$

and the trace of the $(i,j)$-components gives the Raychaudhuri equation,

$$\mathcal{H}' = -\frac{4\pi G}{3} \bar{\rho} a^2. \quad (4.70)$$

At linear order however, there are 4 equations, the $(0,0)$-component, the $(0,i)$-component, the trace of the $(i,j)$-components, and the traceless part of the $(i,j)$-components. These are in conformal Newtonian gauge:

$$3\mathcal{H}^2 \Phi + 3\mathcal{H} \Psi' - \Delta \Psi = -4\pi G a^2 \bar{\rho} \delta_{N}, \quad (4.71)$$

$$\Psi' + \mathcal{H} \Phi = -4\pi G a^2 \bar{\rho} v_{N}, \quad (4.72)$$

$$3\Psi'' + 3\mathcal{H}(\Phi' + 2\Psi') + \Delta(\Psi - \Phi) + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 0, \quad (4.73)$$

$$(\Psi - \Phi)_{,ij} = 0. \quad (4.74)$$

Note that the $(0,0)$-component is the relativistic version of the Poisson equation, but the other equations do not have any counterpart in Newtonian physics. Hence, in relativistic cosmological perturbation theory we have twice as much equations as in Newtonian cosmology.

### 4.4 Solutions

#### 4.4.1 Solutions in the conformal Newtonian gauge

From eq. (4.74) we see that $\Phi$ and $\Psi$ can only vary by some function of time, $\Phi - \Psi = f(\tau)$. However, since the mean value of a Gaussian perturbation vanishes (see appendix B), we must have $f(\tau) = 0$. 

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Thus,

$$\Phi = \Psi.$$  \hfill (4.75)

Using this and the Raychaudhuri equation, eq. (4.73) becomes

$$\Phi'' + 3H\Phi' = 0.$$  \hfill (4.76)

The decaying solution is $\Phi \sim \tau^{-5} \sim a^{-3/2}$ and the growing one is

$$\Phi = \Phi(x).$$  \hfill (4.77)

Both the growing and the decaying mode coincide with the solution for the gravitational perturbation in the Newtonian case. From eq. (4.67) we find the velocity potential,

$$v_N = -\frac{2\Phi}{3H}.$$  \hfill (4.78)

Again, this coincides with the velocity potential in the Newtonian case. Now consider the relativistic Poisson equation, eq. (4.71), in Fourier space. Using the Friedmann equation $4\pi G a^2 \bar{\rho} = \frac{3}{2} H^2$, $\Phi = \Psi$ and $\Psi' = 0$, we obtain:

$$\delta_N = -2\Phi - \frac{2k^2}{3H^2} \Phi.$$  \hfill (4.79)

For scales that are small compared to the horizon, $\frac{k}{H} \gg 1$, this corresponds to the Newtonian result. However, on scales larger than the horizon, $\frac{k}{H} \ll 1$, the density contrast stays constant.

### 4.4.2 Solutions in other gauges

We can construct solutions in other gauges using gauge transformations. For example, consider a gauge transformation from the conformal Newtonian gauge ($w_N = h_N = 0$) to the uniform curvature gauge ($\psi_{UC} = w_{UC} = 0$). Equations (4.46) and (4.47) give

$$\xi^0 = \xi' = \frac{\psi_N}{H} = \frac{\Phi}{H}.$$  \hfill (4.80)

Then the density contrast in uniform curvature gauge is, according to eq. (4.50),

$$\delta_{UC} = \delta_N - 3H\xi^0 = \delta_N - 3\Phi = -5\Phi - \frac{2k^2}{3H^2} \Phi.$$  \hfill (4.81)

Note that we express the density contrast in terms of the Bardeen potential $\Phi$, so that the Newtonian correspondence on small scales is satisfied. Hence, also in the uniform curvature gauge the density contrast stays constant outside the horizon, however the value of this constant differs by a factor of $\frac{5}{2}$.

The density contrast in other gauges can be constructed in the same way. An overview can be found in table 10, where we set all residual gauge modes to zero. Note that the density contrast in the comoving gauge and in the synchronous gauge have the same form as in Newtonian cosmology. However, in other gauges the density contrast stays constant outside the horizon. Plots of $\Delta_\delta = \frac{k^{3/2}}{\sqrt{2\pi}} |\delta|$
in different gauges can be found in figures 15 and 16. A discussion will follow in part 5.

### 4.4.3 Gauge-invariant solutions

In order to avoid the problem of gauge-dependence, one can consider only gauge-invariant perturbations. However, then the difficulty arises in interpreting these quantities, which is only possible if one evaluates the gauge-invariant quantity in a specific gauge. As an example, consider the following three different gauge-invariant combinations containing the density contrast, all introduced by Bardeen [17]:

- The quantity $\delta_{\text{GI, UC}} = \delta - 3\psi = 3\zeta$ reduces to $\delta$ in gauges where $\psi = 0$. Hence, it measures the density contrast on hypersurfaces with zero curvature.

- The quantity $\delta_{\text{GI, N}} = \delta - 3H(w - h')$ reduces to $\delta$ in a gauge where $w = h = 0$. Hence, it measures the density contrast on longitudinal hypersurfaces, i.e. hypersurfaces whose normal unit vectors have zero shear.

- The quantity $\delta_{\text{GI, C}} = \delta - 3H(v + w)$ reduces to $\delta$ in a gauge where $v = w = 0$. Hence, it measures the density contrast on comoving hypersurfaces, i.e. “from the point of view of the matter” [17].

Expressed in terms of the Bardeen potential, these quantities are

\[
\begin{align*}
\delta_{\text{GI, UC}} = & -5\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi, \\
\delta_{\text{GI, N}} = & -2\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi, \\
\delta_{\text{GI, C}} = & -\frac{2}{3} \frac{k^2}{H^2} \Phi.
\end{align*}
\]

All three quantities coincide on scales well within the horizon, $k \gg H$, but they differ significantly outside the horizon. Which of these gauge-invariant quantities can be considered as the physically most appropriate is, as Bardeen formulated it, “a matter of taste” [17]. Thus, the introduction of gauge-invariant variables does not solve the gauge problem. Instead of choosing a gauge, one has to choose which gauge-invariant variable to take.

---

**Table 10: Overview - perturbations in conformal Newtonian (N), uniform curvature (UC), spatially Euclidean (UC), synchronous (S), comoving (C) and uniform density (UD) gauge. The quantity $3\zeta = \delta - 3\psi$ is gauge-invariant. (\* This is a result of the calculations and not the gauge condition.)**

| gauge | N | UC | SE | S | C | UD |
|-------|---|----|----|---|---|----|
| $\phi$ | $\Phi$ | $\frac{5}{2} \Phi$ | $\frac{5}{2} \Phi$ | 0 | 0* | $\frac{5}{3} \frac{k^2}{H^2} \Phi$ |
| $\psi$ | $\Phi$ | 0 | $\frac{5}{3} \Phi$ | $\frac{2}{3} \Phi$ | $\frac{5}{3} \Phi + \frac{2}{3} \frac{k^2}{H^2} \Phi$ |
| $w$ | 0 | 0 | $-\frac{2}{3} \Phi$ | 0 | 0 | 0 |
| $h$ | 0 | $\frac{\Phi}{H^2}$ | 0 | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi - \frac{1}{3} \frac{k^2}{H^2} \Phi$ |
| $v$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{5}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | 0* | 0 | $\frac{2}{3} \frac{k^2}{H^2} \Phi$ |
| $\delta$ | $-2\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-5\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-5\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | 0 |
| $3\zeta$ | $-5\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-5\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-5\Phi - \frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | $-\frac{2}{3} \frac{k^2}{H^2} \Phi$ | 0 |
Figure 15: Evolution of the density contrast for $k = 0.01 \text{Mpc}^{-1}$ in different gauges.

Figure 16: Evolution of the density contrast for $k = 0.001 \text{Mpc}^{-1}$ in different gauges.
4.5 Vector and tensor perturbations

So far we have ignored vector and tensor perturbations and considered only scalar perturbations. This is possible, because in first order they decouple from each other. For the same reason we can choose the gauge condition for the scalar perturbations (this corresponds to choosing $\xi^0$ and $\xi$) and the vector perturbations (this corresponds to choosing $\xi^{\perp}$) independently. Note that pure tensor perturbations are per se gauge-invariant, so there is no gauge-choice for them.

As an example, we can choose a gauge with the scalar gauge condition $w = h = 0$ and the additional vector gauge condition $h = 0$. We call this gauge the Poisson gauge (P), in agreement with [9]. The gauge conditions can be reformulated in the form

\begin{align}
\nabla_i w_P &= 0, \quad (4.85) \\
\nabla_i h_P^{ij} &= 0. \quad (4.86)
\end{align}

Note that the scalar perturbations in the Poisson gauge are the same as the scalar perturbations in the conformal Newtonian gauge. However, in the Poisson gauge, we have some additional vector perturbation in the metric, $w_P = w_P^P \neq 0$. Hence, in this gauge we find the same equations for scalar perturbations as in the conformal Newtonian gauge, but we find some additional equations for the vector perturbations, as shown in [9]. These are the vector parts of the Euler equation, the $(0, i)$-component and the $(i, j)$-component of the Einstein equations (we drop the index $P$ now - all equations are given in Poisson gauge):

\begin{align}
(v_i^+ + w_i^+)' + 2H(v_i^+ + w_i^+) &= 0, \quad (4.87) \\
\frac{1}{2} \Delta w_i^+ &= 8\pi G a^2 \bar{\rho}(v_i^+ + w_i^+), \quad (4.88) \\
\left(w_{(i,j)}^+\right)' + 2Hw_{(i,j)} &= 0. \quad (4.89)
\end{align}

From eq. (4.87) we find a decaying solution for the vector perturbations,

\begin{equation}
(v_i^+ + w_i^+) \sim a^{-1}. \quad (4.90)
\end{equation}

If there would have been some vector perturbations initially, they would have decayed, so our initial approach to neglect all vector perturbations from the beginning is justified. However, note that this argument only holds in first order perturbation theory. At second order, there are couplings between scalar and vector perturbations [18].

The only pure tensor perturbation that occurs is the transverse-traceless part of $h_{ij}$. In Poisson gauge, we have $h_{ij} = h_{ij}^T$, since $h = 0$ and $h = 0$. The only equation with a pure tensor part is the transverse-traceless part of the $(i, j)$-component of the Einstein field equations,

\begin{equation}
h_{ij}'' + 2H h_{ij}' - \Delta h_{ij} = 0. \quad (4.91)
\end{equation}
Neglecting the Hubble friction, the solutions are gravitational waves,

\[ h_{ij} \sim e^{i(\omega t + k \cdot x)}, \]  

with the dispersion relation

\[ \frac{\omega}{k} = \frac{\partial \omega}{\partial k} = 1, \]  

which means that gravitational waves propagate with the speed of light. The Hubble drag term causes the amplitude of these gravitational waves to decrease with time due to the expansion of the universe.

5 Quantification of relativistic corrections

In this part we quantify the relativistic corrections in cosmological \( N \)-body simulations and the observed matter power spectrum. Furthermore, we discuss briefly how the situation changes if one considers a \( \Lambda \)CDM model instead of an Einstein-de Sitter model.

5.1 Relativistic corrections to Newtonian simulations

In the conformal Newtonian gauge and in the uniform curvature gauge the relativistic corrections for the density perturbations on large scales and for early times become significantly large. This can have an impact on cosmological \( N \)-body simulations, which use Newtonian instead of relativistic equations. These simulations typically start at \( z = 100 \) and the largest scales they simulate are about \( k^{-1} \simeq 1 \) Gpc.

We define the relative error of the Newtonian density contrast compared to the relativistic one,

\[ D(a, k) \equiv \frac{\delta_{\text{GR}} - \delta_{\text{Newton}}}{\delta_{\text{Newton}}}, \]  

where we fix the initial value for all solutions according to eq. (3.94). Plots of \( D(a, k) \) can be found in figure 17. The relative error increases dramatically for large scales and early times: It is of the order \( \sim 10^{3\%} \) \((\sim 4000\% \) in the uniform curvature gauge and \( \sim 2000\% \) in the conformal Newtonian gauge) on the 1 Gpc-scale at \( z = 100 \). For scales below 10 Mpc (galaxy and cluster scales) and redshifts below \( z = 100 \), the error does not exceed 1\%. Hence, Newtonian simulations are more reliable on these scales.

Some authors [19, 20, 21] argue that Newtonian simulations are exact at linear order, if one uses the following "dictionary" (one-to-one correspondence) between the simulated quantities and gauge-invariant relativistic quantities:

\[ \delta_{\text{sim}} \leftrightarrow \Phi, \]  

\[ v_{\text{sim}} \leftrightarrow v_{\text{GLN}}, \]  

\[ \delta_{\text{sim}} \leftrightarrow \delta_{\text{GLC}}, \]  

where \( v_{\text{GLN}} = v + h' \), so that it reduces the velocity potential on zero shear hypersurfaces. However, there is a problem arising with this dictionary: While there is one set of equations connecting the quantities on the left hand side (namely the Newtonian equations), there is not one set of equations
Figure 17: The relative error $D(a, k)$ in the uniform curvature gauge (top) and the conformal Newtonian gauge (bottom).
connecting the quantities on the right hand side. Indeed, this dictionary identifies the simulated gravitational and velocity potentials with the relativistic counterparts on zero shear hypersurfaces, but the simulated density contrast with the relativistic one on comoving hypersurfaces. This means that Newtonian simulations are practically a mixture of relativistic simulations on two different hypersurfaces. Furthermore, note that we cannot observe $\delta_{GLC}$ since we are not observers on a comoving hypersurface. A consistent dictionary would be the following,

$$\delta \phi_{\text{sim}} \leftrightarrow \Phi,$$  \hspace{1cm} (5.5)  
$$v_{\text{sim}} \leftrightarrow v_{\text{GLN}},$$  \hspace{1cm} (5.6)  
$$\delta \sim \leftrightarrow \delta_{\text{GLN}} + 2\Phi,$$  \hspace{1cm} (5.7)  

where we identify all simulated quantities with gauge-invariant quantities that have a physical interpretation on zero shear hypersurfaces.

### 5.2 Relativistic corrections to the observed power spectrum

Another interesting result is the rise of the matter power spectrum at very large scales, see figure 18. At scales near the horizon today the relativistic power spectrum in the uniform curvature gauge is about two orders of magnitude higher than in the the comoving/synchronous gauge. This rise of the
Table 11: The minimal observable wavenumber for various redshifts.

| $z$  | $k_{\text{min}}$ [Mpc$^{-1}$] |
|------|-------------------------------|
| 1    | $1.25 \cdot 10^{-3}$          |
| 0.6  | $1.75 \cdot 10^{-3}$          |
| 0.3  | $2.98 \cdot 10^{-3}$          |

power spectrum at large scales can be misinterpreted as primordial non-Gaussianity in the gravitational potential perturbation, as pointed out by Yoo in [22] and by Zhang in [23]. As a next step we want to find out if this is an observable effect in the measured power spectrum at a given redshift. For this, we introduce the distance-redshift relation in an Einstein-de Sitter universe [4],

$$d(z) = \frac{2}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right). \quad (5.8)$$

The minimal wavenumber one can observe at a given redshift is given by the inverse of $d(z)$,

$$k_{\text{min}} = \frac{H_0}{2} \frac{\pi}{1 - \frac{1}{\sqrt{1+z}}}, \quad (5.9)$$

where the extra factor $\pi$ comes from the fact that we want to measure at least half a wavelength (note that the relation between wavenumber and wavelength is $k = \frac{2\pi}{\lambda}$). Table 11 shows $k_{\text{min}}$ for the redshifts $z = 0.3$, $z = 0.6$ and $z = 1$. In figure 19 we show the matter power spectra for these redshifts, where the vertical bar denotes $k_{\text{min}}$. The difference due to the gauge choice can become significantly large: At a redshift of $z = 1$, the power spectrum at the minimal observable wavenumber is about twice as high in the uniform curvature gauge as it is in the comoving gauge.

For completeness we also show the velocity power spectrum $P_v(k) = |\mathbf{v}_k|^2$ today in different gauges in figure 20. In the conformal Newtonian and the spatially Euclidean gauge it corresponds to the Newtonian power spectrum, see figure 12. However, in the uniform curvature gauge the spectrum is higher by a factor of $\frac{25}{4}$. In the uniform density gauge the spectrum has a totally different shape. Thus, the property that all gauges coincide on sub-horizon scales, which we found in the case of density perturbations, is not true for velocity perturbations. Note that in the comoving gauge and in the synchronous gauge the spectrum is zero because the peculiar velocity vanishes in these gauges, see table 10.

### 5.3 Relativistic corrections in $\Lambda$CDM models

As we approach the present time, the cosmological constant gets more and more dominant. Today we have $\Omega_\Lambda \simeq 0.7$, so that we should not neglect the effect of $\Lambda$. We can include dark energy into our equations by setting $\rho = \rho_m + \rho_\Lambda$ and $p = p_m + p_\Lambda$, where $\rho_\Lambda = \frac{\Lambda}{8\pi G} = \text{const.}$ and $p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}$. Then the modified energy-momentum tensor is

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu. \quad (5.10)$$


Figure 19: The matter power spectrum between $k = 0.001 \, \text{Mpc}^{-1}$ and $k = 0.01 \, \text{Mpc}^{-1}$ in different gauges for the redshifts $z = 1$, $z = 0.6$ and $z = 0.3$. The vertical line denotes $k_{\text{min}}$, the minimal observable wavenumber at the corresponding redshift.
Figure 20: The velocity power spectrum today in different gauges.

Note that \( \rho + p = \rho_m + \rho + p_m + p_{\Lambda} = \rho_m \) and \( p = p_m + p_{\Lambda} = -\frac{\Lambda}{8\pi G} \), so that we have

\[
T_{\nu}^{\mu} = \rho_m u^\mu u_\nu - \rho_{\Lambda} \delta^\mu_\nu.  \tag{5.11}
\]

Hence, the only change is in the background,

\[
\bar{T}^\mu_\nu = \begin{pmatrix}
-\bar{\rho}_m - \rho_{\Lambda} & 0 \\
0 & -\rho_{\Lambda} \delta_{ij}
\end{pmatrix},  \tag{5.12}
\]

while the perturbations remain unchanged,

\[
\delta T^\mu_\nu = \begin{pmatrix}
-\bar{\rho}_m \delta_N & \bar{\rho}_m v_N,i \\
-\bar{\rho}_m v_N,i & 0
\end{pmatrix}.  \tag{5.13}
\]

Now we calculate the corrections that appear in the background equations. First, consider the Bianchi identity \( \nabla_\mu T^\mu_\nu = 0 \). For \( \nu = 0 \) we found the mass continuity equation,

\[
\bar{\rho}_m' + 3\mathcal{H} \bar{\rho}_m = 0,  \tag{5.14}
\]

which remains unchanged, since \( \Lambda \) is constant in time. The \( (\nu = i) \)-component of the Bianchi identity still has no background part, which comes from the fact that \( \Lambda \) is constant in space. Now consider the Einstein field equations, \( G^\mu_\nu = 8\pi G T^\mu_\nu \). In the background, the \((0,0)\)-component gives the Friedmann
equation, which becomes now
\[ \mathcal{H}^2 = \frac{8\pi G}{3} a^2 (\bar{\rho}_m + \rho_\Lambda). \] (5.15)

The Raychaudhuri equation (trace of the \((i,j)\)-components) becomes
\[ 2\mathcal{H}' + \mathcal{H}^2 = 8\pi G a^2 \rho_\Lambda, \] (5.16)
or, using the Friedmann equation,
\[ \mathcal{H}' = -\frac{4\pi G}{3} a^2 (\bar{\rho}_m - 2\rho_\Lambda). \] (5.17)

In terms of proper time the background equations are
\[ \dot{\bar{\rho}}_m + 3H \bar{\rho}_m = 0, \] (5.18)
\[ H^2 = \frac{8\pi G}{3} (\bar{\rho}_m + \rho_\Lambda), \] (5.19)
\[ \frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\bar{\rho}_m - 2\rho_\Lambda). \] (5.20)

One can find solutions to these background equations in terms of hyperbolic functions \[24]\,
\[ a(t) = \left[ \frac{\Omega_m}{1 - \Omega_m} \right]^{1/3} \sinh^{2/3} \left( \frac{t}{t_\Lambda} \right), \] (5.21)
\[ H(t) = \frac{2}{3t_\Lambda} \coth \left( \frac{t}{t_\Lambda} \right), \] (5.22)
\[ \bar{\rho}_m(t) = \rho_\Lambda \sinh^{-2} \left( \frac{t}{t_\Lambda} \right), \] (5.23)

where \( t_\Lambda = \frac{2}{\sqrt{8\pi G \rho_\Lambda}} \). Now consider the perturbed equations in conformal Newtonian gauge:
\[ \delta' - k^2 v - 3\Phi' = 0, \] (5.24)
\[ v' + H v = -\Phi, \] (5.25)
\[ 3H^2 \Phi + 3H \Phi' + k^2 \Phi = -4\pi G a^2 \bar{\rho}_m \delta, \] (5.26)
\[ \Phi' + H \Phi = -4\pi G a^2 \bar{\rho}_m v, \] (5.27)
\[ \Phi'' + 3H \Phi' = -8\pi G a^2 \rho_\Lambda \Phi. \] (5.28)

These are the perturbed parts of the 0-component and the \(i\)-component of the energy-momentum conservation equation, the \((0,0)\)-component, \((0,i)\)-component and \((i,i)\)-component of the Einstein field equations, respectively, where we already used \(\Phi = \Psi\), which follows from the traceless part of the \((i,j)\)-components, which remains unchanged. Note that the only change occurs in the \((i,i)\)-component, eq. [5.28], because the term \(2\mathcal{H}' + \mathcal{H}^2\) is not zero any more as we leave matter domination. An analytic solution of these equations is very difficult due to the non-triviality of the background solutions. However, one can do a numerical integration of the above equations to find the evolution of
Figure 21: Relative error $D(z,k)$ (here denoted as $\Delta$) between the relativistic density contrast in conformal Newtonian gauge and the Newtonian one. From top to bottom, the curves are for $k = 0.001\,\text{Mpc}^{-1}$, $k = 0.01\,\text{Mpc}^{-1}$, $k = 0.1\,\text{Mpc}^{-1}$ and $k = 1\,\text{Mpc}^{-1}$, respectively [25].

The relative errors become even larger ($\sim 10^4\%$ today on 1 Gpc-scales) in $\Lambda$CDM models than in the Einstein-de Sitter model (where we found at most $\sim 10^3\%$ on these scales). Note that the relative error remains constant in time due to the fact that the density perturbations freeze out as dark energy takes over.
6 Summary and outlook

In this thesis we studied the validity of Newtonian equations, as used in cosmological $N$-body simulations at large cosmological scales. These simulations are used to test our understanding of the universe, in particular the formation of galaxies and clusters of galaxies. We linearized these equations in order to compare them to linear relativistic cosmological perturbation theory. The linear equations need to be corrected on small scales due to nonlinear effects and on large scales due to relativistic effects. The important results are the following:

- For typical density fluctuations ($\lesssim 1\sigma$) linear theory is valid until the density contrast reaches unity. This happened at a scale factor of $a_{NL2} = 0.12$ (corresponding to a redshift of $z_{NL2} \simeq 7$) on galaxy scales and $a_{NL2} = 0.75$ ($z_{NL2} \simeq 0.3$) on cluster scales. Perturbations on supercluster scales still behave linear today (all second order quantities are smaller than all first order quantities).

- The behaviour of density perturbations on scales larger than the Hubble horizon is strongly gauge-dependent. It turns out that in the synchronous and the comoving gauge the density contrast behaves like in Newtonian theory. However, in the conformal Newtonian gauge, the spatially Euclidean gauge and the uniform curvature gauge the density contrast is constant outside the Hubble horizon.

- The relativistic corrections to the simulated density perturbations in Newtonian $N$-body simulations can become significantly large. Assuming that the accurate description of the density contrast on superhorizon scales is given in the uniform curvature gauge or the conformal Newtonian gauge, the error of the density contrast in Newtonian simulations reaches up to $\sim 10^3\%$ on the 1 Gpc-scale at a redshift of $z = 100$. This error increases to $\sim 10^4\%$ if we consider a $\Lambda$CDM model instead of an Einstein-de Sitter model. For scales below 10 Mpc and redshifts below $z = 100$, the error does not exceed 1\%, whence Newtonian simulations on these scales are more reliable.

- The observed matter power spectrum differs on large scales in different gauges. For example, at a redshift of $z = 1$, the power spectrum at the minimal observable wavenumber is about twice as large in the uniform curvature gauge as it is in the comoving gauge. It remains an open question which power spectrum we expect to measure from a theoretical point of view. Maybe the gauge-dependence of the power spectrum is a cue that this is not at all the quantity we should consider.

- We should not ask which gauge is the "correct" one. Indeed, perturbations in all gauges are correct solutions of Einstein’s equations - this is the basic principle of general relativity. Instead, we should ask the following questions: Is there a chosen gauge which gives predictions that are closest to our measurements? If yes, which gauge and why? Another possibility is that there is no such chosen gauge and indeed all quantities that we measure are truly gauge-invariant quantities, which we do not yet understand theoretically. We have to answer these questions first before we really know how reliable Newtonian $N$-body simulations at large cosmological scales are.
Appendix A
Helmholtz decomposition theorem

Bounded domain

Let \( \mathbf{v}(\mathbf{r}) \) be a smooth vector function on a bounded domain \( D \subset \mathbb{R}^3, \mathbf{v} : D \to \mathbb{R}^3 \). Then there exists a scalar potential \( v \) and a vector potential \( \mathbf{\Omega} \) such that:

\[
\mathbf{v} = v_\parallel + v_\perp, \\
v_\parallel = \nabla v + \nabla \times \mathbf{\Omega}, \\
v_\perp = \nabla \times \mathbf{\Omega},
\]

with \( v_\parallel = \nabla v \) and \( v_\perp = \nabla \times \mathbf{\Omega}, \ \forall \mathbf{r} \in D \). It follows that \( \nabla \cdot v_\perp = 0 \) and \( \nabla \times v_\parallel = 0 \).

The proof is done by construction. One can solve

\[
\nabla \cdot v_\parallel = \nabla \cdot \mathbf{v}, \\
\nabla \times v_\perp = \nabla \times \mathbf{v},
\]

using the relation

\[
\Delta \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}').
\]

The solutions are given by

\[
v = -\frac{1}{4\pi} \int_D d^3\mathbf{r}' \frac{\nabla' \cdot \mathbf{v}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi} \int_{\partial D} d\mathbf{S}' \frac{\mathbf{v}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \\
\mathbf{\Omega} = \frac{1}{4\pi} \int_D d^3\mathbf{r}' \frac{\nabla' \times \mathbf{v}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi} \int_{\partial D} d\mathbf{S}' \frac{\mathbf{v}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.
\]

Unbounded domain

If the domain \( D \) is unbounded, i.e. \( D = \mathbb{R}^3 \), we have to make the restriction

\[
\frac{|\mathbf{v}(\mathbf{r})|}{|\mathbf{r}|} \to 0 \text{ as } |\mathbf{r}| \to \infty,
\]

so that the second terms in \( v \) and \( \mathbf{\Omega} \) vanish. Then the solutions are given by

\[
v = -\frac{1}{4\pi} \int_D d^3\mathbf{r}' \frac{\nabla' \cdot \mathbf{v}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \\
\mathbf{\Omega} = \frac{1}{4\pi} \int_D d^3\mathbf{r}' \frac{\nabla' \times \mathbf{v}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.
\]

Hypertorus

Now let the domain \( D \) be a Hypertorus, \( D = \mathbb{T}^3 \), as used in numerical simulations, where one considers a box with periodic boundary conditions. Since we have \( \mathbf{v}(\mathbf{r}_a) = \mathbf{v}(\mathbf{r}_b) \) and \( d\mathbf{S}(\mathbf{r}_a) = -d\mathbf{S}(\mathbf{r}_b) \) for two
opposite sites $a$ and $b$ of the hypertorus, the boundary terms in $v$ and $\Omega$ vanish as in the case of the unbounded domain.

**Uniqueness**

The Helmholtz decomposition is not unique. If the vector contains a constant part, which is both divergence- and curl-free, it can be included in the longitudinal part or in the transverse part. Note that in cosmology a constant vector stands in contradiction to the isotropy of the universe. In order to have a unique decomposition, one has to assume natural boundary conditions, i.e. $v = 0$ at the boundary of the domain. In particular, in the case of an unbounded domain, the boundary condition is

$$v(|x| \to \infty) \to 0.$$  \hfill (A.11)

For consistency, we show that the peculiar velocity we found satisfies this condition in all gauges considered. We found $|v_k| \sim k \Phi \sim k^{-1/2}T(k)$ in the conformal Newtonian gauge, the uniform curvature gauge and the spatially Euclidean gauge and $|v_k| \sim k^3 \Phi \sim k^{1/2}T(k)$ in the uniform density gauge (see table 10). Now we will consider the general case $|v_k| \sim k^n T(k)$ and show that its Fourier transform vanishes for $|x| \to \infty$ for all $n \in \mathbb{R}$. We have

$$|v(x)| \sim \int d^3k |v_k| e^{ik \cdot x} \sim \int d^3k k^n T(k) e^{ik \cdot x} \leq \int d^3k k^n \sim \int_{k_{IR}}^{k_{UV}} dk \int_0^\pi d\theta \sin \theta k^{n+2} e^{i k x \cos \theta} \sim \int_{k_{IR}}^{k_{UV}} dk k^{n+2} \sin(k x) / k x,$$

where we introduced the ultraviolet cutoff wavenumber $k_{UV}$ and the infrared cutoff wavenumber $k_{IR}$ (this is a reasonable assumption for a Harrison-Zel’dovich spectrum, see appendix B). Now we make the substitution $p \equiv k x$, so that

$$|v(x)| \sim \frac{1}{x^{n+3}} \int_{k_{IR}}^{k_{UV}} \frac{dpp^{n+2} \sin(p)}{p} \leq \frac{1}{x^{n+3}} \int_{k_{IR}}^{k_{UV}} pp^{n+1} = \frac{1}{x^{n+3}} \frac{1}{n+2} \int_{k_{IR}}^{k_{UV}} (k_{UV}^{n+2} - k_{IR}^{n+2}) \sim \frac{1}{x}.$$
for $n \neq -2$. For $n = -2$ we have

$$|v(x)| \leq \frac{1}{x} \int_{x}^{k_{UV}} dpp^{-1} \int_{x}^{k_{IR}} \ln \left( \frac{k_{UV}}{k_{IR}} \right) \sim \frac{1}{x}.$$ 

Thus, the boundary condition $v(|x| \to \infty) \to 0$ is satisfied in all gauges we considered in this thesis.

## Appendix B

### Statistical properties of Gaussian perturbations

A Gaussian perturbation

$$g(x) = \int d^3k g(k) e^{ik \cdot x}$$  \hspace{1cm} (B.1)

is defined as a perturbation whose Fourier components have a Gaussian probability distribution,

$$\text{Prob}[g(k)] = \frac{1}{2\pi s_k^2} \exp \left( -\frac{|g(k)|^2}{2s_k^2} \right),$$  \hspace{1cm} (B.2)

where $s_k^2$ is the variance. From this we see directly that the mean value of each Fourier component is zero,

$$\langle g(k) \rangle = 0.$$  \hspace{1cm} (B.3)

Furthermore, if we assume homogeneity and isotropy of the perturbations, it follows that different Fourier components satisfy the following normalization condition:

$$\langle g(k) g(k') \rangle = \delta(k + k') P(k),$$  \hspace{1cm} (B.4)

where

$$P(k) \equiv |g(k)|^2$$  \hspace{1cm} (B.5)

is the power spectrum.\(^7\)

Now we go back to coordinate space and find the following relation:

$$\langle g(x) \rangle = \langle \int d^3k g(k) e^{ik \cdot x} \rangle = \int d^3k \langle g(k) \rangle e^{ik \cdot x} = 0.$$

\(^7\)This commonly used definition of the power spectrum is formally wrong. Indeed, one should divide by a unit volume, so that the power spectrum has the correct dimension, namely $\text{Mpc}^3$. However, this is only a formal issue, so this unit volume is usually dropped.
Hence, the mean value of a Gaussian perturbation in coordinate space vanishes. This comes from the fact that positive and negative deviations from the background value are equally probable. In other words, the mean value of a perturbed quantity is its background quantity. Next, consider

\[
\langle g(x) g(y) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \int \frac{d^3k'}{(2\pi)^3} e^{ik' \cdot y} g(k) g(k')
\]

\[
= \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x-y)} P(k)
\]

\[
= \int_0^\infty \frac{k^2 dk \sin(kr)}{2\pi^2 kr} P(k)
\]

\[
= \int_0^\infty \frac{dk \sin(kr)}{k} \frac{k^3}{kr} \frac{2\pi^2}{P(k)} \equiv P(k)
\]

where \( r \equiv |x - y| \) is the distance and

\[
P(k) = \frac{k^3}{2\pi^2} P(k) = \frac{k^3}{2\pi^2} |g(k)|^2
\]

is the dimensionless (band) power spectrum. For \( r = 0 \) we have

\[
\langle g(x)^2 \rangle = \int_{-\infty}^{\infty} d\ln k P(k).
\]

This means that the power spectrum \( P(k) \) of \( g \) gives the contribution of a logarithmic scale interval to the variance of \( g(x) \). A power-law spectrum is a power spectrum of the form

\[
P(k) = A^2 \left( \frac{k}{k_{\text{pivot}}} \right)^{n-1},
\]

where \( k_{\text{pivot}} \) is a chosen reference scale, the "pivot scale", \( A \) is the amplitude at this scale, and \( n \) is the spectral index. For \( n = 1 \), the spectrum is scale-invariant,

\[
P(k) = A^2.
\]

This spectrum is also called the Harrison-Zel’dovich spectrum. Note that \( \langle g(x)^2 \rangle \) is logarithmically divergent for a Harrison-Zel’dovich spectrum. Therefore one usually defines a cutoff-scale at small scales (ultraviolet-cutoff) and at large scales (infrared-cutoff) in order to make the integral finite.
Appendix C

Perturbations in the inflaton field

For completeness, we consider here perturbations in the inflaton field, that lead to the important curvature perturbations after inflation. We start with the Lagrangian for the inflaton field:

\[ \mathcal{L} = -\frac{1}{2} \phi^{\mu} \phi_{,\mu} - V(\phi). \]  

(C.1)

Applying the Euler-Lagrange equations, we find the field equation for the inflaton,

\[ \ddot{\phi} + 3H \dot{\phi} + \nabla^2 \phi + V,_{\phi} = 0. \]  

(C.2)

Now we split the inflaton field into a background part and a perturbed part,

\[ \varphi(r,t) = \bar{\varphi}(t) + \delta \varphi(r,t). \]  

(C.3)

Then, in zeroth order, the field equation becomes

\[ \ddot{\bar{\varphi}} + 3H \dot{\bar{\varphi}} + V,_{\varphi} = 0, \]  

(C.4)

which we already found in section 2.3. In linear order the field equation reads

\[ \ddot{\delta \varphi} + 3H \dot{\delta \varphi} - \nabla^2 \delta \varphi + \delta \varphi V,_{\varphi\varphi} = 0. \]  

(C.5)

In slow-roll approximation, i.e. \( V,_{\varphi\varphi} \ll \frac{V}{M^2} \simeq H^2 \), we can neglect the last term on the left hand side. The resulting equation becomes in Fourier space:

\[ \ddot{\delta \varphi_k} + 3H \dot{\delta \varphi_k} + \frac{k^2}{a^2} \delta \varphi_k = 0, \]  

(C.6)

where \( k \) is the comoving wavenumber of the considered scale. Now we can distinguish two cases:

- For early times, when \( 3H \ll \frac{k^2}{a^2} \), that means before horizon exit, the inflaton perturbation is oscillating with a decaying amplitude, \( |\delta \varphi_k| \sim \frac{1}{n} \).

- For late times, when \( 3H \gg \frac{k^2}{a^2} \), that means after horizon exit, the solution is \( \delta \varphi_k = \text{const.} \), or in other words, \( \delta \varphi_k \) freezes out.

Note that, because \( H \approx \text{const.} \) during inflation, all scales have similar behaviour. This leads to a nearly scale-invariant power spectrum of inflaton-, and hence, curvature perturbations. Indeed, one can even show that the spectral index is related to the slow roll parameters, as defined in section 2.3,

\[ n - 1 = -6\epsilon + 2\eta, \]  

(C.7)

\(^8\)Bardeen showed in [11] that curvature perturbations are related to inflaton perturbations by \( \zeta = -H \frac{\dot{\delta \varphi}}{\dot{\varphi}} \). Hence, their spectra differ only by a factor \( \frac{\dot{\varphi}}{\dot{\varphi}} \), which is constant in the slow-roll approximation.
Appendix D
Meszaros equation

The Meszaros equation gives the growth of cold dark matter density perturbations in the intermediate regime between radiation domination and matter domination. Assume that the universe is filled with cold dark matter and radiation to a comparable amount. Then the Friedmann and Raychaudhuri equations are:

\[ H^2 = \frac{8\pi G}{3}(\bar{\rho}_m + \bar{\rho}_\gamma), \quad (D.1) \]
\[ \frac{\dot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho}_m + 2\bar{\rho}_\gamma), \quad (D.2) \]

where \( \bar{\rho}_m \) is the (dark) matter density and \( \bar{\rho}_\gamma \) is the radiation density. It is helpful to introduce a new time variable

\[ y \equiv \frac{a}{a_{eq}} = \frac{\bar{\rho}_m}{\bar{\rho}_\gamma}, \quad (D.3) \]

so that

\[ y \ll 1 \iff \text{radiation domination}, \]
\[ y \gg 1 \iff \text{matter domination}. \]

For cold dark matter perturbations the Jeans equation gives

\[ \ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}_m\delta = 0. \quad (D.4) \]

Note that there is no pressure source term because we consider only cold dark matter, which does not interact with radiation. Now we want to write this equation in terms of the new time variable \( y \). To do this, we use

\[ 4\pi G\bar{\rho}_m = \frac{3}{2}H^2 \frac{y}{1+y}, \quad (D.5) \]

and the transformation of the time derivatives,

\[ \frac{d}{dt} = \frac{dy}{dt} \frac{d}{dy} = H y \frac{d}{dy}. \quad (D.6) \]

After some calculation one finds the Jeans equation in terms of the new time variable,

\[ \frac{d^2\delta}{dy^2} + \frac{2 + 3y}{2y(y+1)} \frac{d\delta}{dy} - \frac{3}{2y(y+1)}\delta = 0. \quad (D.7) \]
This equation was first derived by Meszaros [26] and hence carries his name, the Meszaros equation. The general solution is given by
\[ \delta(y) = C_1 \delta_1(y) + C_2 \delta_2(y), \] (D.8)
where \( C_1 \) and \( C_2 \) are constants and
\[ \delta_1(y) = 1 + \frac{3}{2} y, \] (D.9)
\[ \delta_2(y) = 3 \sqrt{1 + y} - \left(1 + \frac{3}{2} y\right) \ln \frac{\sqrt{1 + y} + 1}{\sqrt{1 + y} - 1}. \] (D.10)

We can reconstruct the solutions during radiation domination and matter domination by considering the limits \( y \ll 1 \) and \( y \gg 1 \), respectively:

- During radiation domination (\( y \ll 1 \)) the leading order in \( \delta_2 \) is \( \delta_2 \sim \ln y \sim \ln a \), and the leading order in \( \delta_1 \) is a constant. Hence, \( \delta_2 \) is the growing mode and \( \delta_1 \) is the decaying mode.

- During matter domination (\( y \gg 1 \)) the leading order in \( \delta_1 \) is \( \delta_1 \sim y \sim a \), and the leading order in \( \delta_2 \) is \( \delta_2 \sim y^{-3/2} \sim a^{-3/2} \). Thus, \( \delta_1 \) is the growing mode and \( \delta_2 \) is the decaying mode.

**Appendix E**

**Identity of synchronous and comoving gauge**

We claim that, neglecting residual gauge modes, the synchronous gauge, where \( \phi_S = w_S = 0 \), and the comoving gauge, where \( v_C = w_C = 0 \), are identical gauges during matter domination in that all perturbation variables are the same in both gauges. To see that, first consider a gauge transformation from a generic gauge \( G \) to the conformal Newtonian gauge \( N \), where \( w_N = h_N = 0 \), which is given by
\[ \xi^0 = h_G - w_G, \] (E.1)
\[ \xi = h_G. \] (E.2)

This transformation is unique. Now consider a gauge transformation from the conformal Newtonian gauge to another gauge where two of the triple \((\phi, w, v)\) are zero. The transformation rules are:
\[ \phi = \Phi - \mathcal{H} \xi^0 - \xi^\nu, \] (E.3)
\[ w = \xi^0 - \xi^\nu, \] (E.4)
\[ v = v_N + \xi^\nu. \] (E.5)

\[^9\) It is also possible to construct the Meszaros equation for the intermediate time interval between matter domination and dark energy domination. However, this equation is difficult to solve analytically. Böhmer et al. constructed a \( w \)-Meszaros equation for the time between matter domination and domination of a component with arbitrary \( w \), and they were only able to find two additional analytic solutions in the parameter space \( w \in [-1, 1] \), one for \( w = -\frac{1}{3} \) and one for \( w = \frac{1}{3} \) [27].\]
Note that the transformation law for $\phi$ implies $\xi^0 = -v_N \Leftrightarrow \phi = 0$, which follows from the Euler equation during matter domination. Now there are three possibilities:

- Setting $\phi = w = 0$ (synchronous gauge) implies $\xi^0 = \xi' = -v_N + \frac{C}{a}$, whence $v = \frac{C}{a}$. We can set the constant to zero, so that $v = 0$.

- Setting $v = w = 0$ (comoving gauge) implies $\xi^0 = \xi' = -v_N$, whence also $\phi = 0$.

- Setting $\phi = v = 0$ implies $\xi^0 = -v_N + \frac{C}{a}$ and $\xi' = -v_N$, whence $w = \frac{C}{a}$. Again, we can set the constant to zero, so that $w = 0$.

Since all three gauges, in which two of the triple $(\phi, w, v)$ are zero, are achieved by choosing $\xi^0 = \xi' = -v_N$, they are identical. In particular, the comoving and the synchronous gauge are identical during matter domination. Note that this only holds if we set all residual gauge modes to zero.
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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe. Bielefeld, im Januar 2012.

(Samuel Flender)