RELATIVISTIC EFFECTS IN S-WAVE QUARKONIUM DECAYS

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Abstract

The decay widths of S-wave quarkonia ($\eta_c, \eta_b \rightarrow \gamma\gamma$ and $J/\psi, \Upsilon \rightarrow e^+e^-$) are calculated on the basis of a quasipotential approach. The nontrivial dependence on relative quark motion of decay amplitude is taken into consideration via quarkonium wave function. It is shown that relativistic corrections may be large (10-50%) and comparable with QCD corrections.

1 Introduction

The constituent quark models [1] are being not well grounded from the point of view of quantum field theory, as the QCD sum rules method [2] or lattice QCD approach [3], but they give us an opportunity to describe most of all existing experimental data. The great success in description of heavy meson static characteristics and decay widths has been obtained on the basis of a relativistic quark models [4, 5]. As it was shown, relativistic corrections are not small for mesons containing comparably light charm quark [6]. In this paper we present the results of calculation of the relativistic corrections in S-wave quarkonium decays ($\eta_c, \eta_b \rightarrow \gamma\gamma$ and $J/\psi, \Upsilon \rightarrow e^+e^-$) using the approach based on a Logunov-Tavkhelidse local quasipotential equation [7] and perturbative QCD methods.

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2 Decay amplitude in quasipotential approach

The meson decay amplitude may be presented as the product of an parton decay amplitude $\mathcal{M}_H(q_1, q_2)$ with amputate quark lines times a Bethe-Salpeter wave function $\Psi(q_1, q_2)$:

$$\mathcal{M} = \int \frac{d^4q_1}{(2\pi)^4} Sp[\Psi(q_1, q_2)\mathcal{M}_H(q_1, q_2)],$$  \hspace{1cm} (1)$$

where $\Psi(q_1, q_2)$ describes two-quark bound state, $\mathcal{M}_H(q_1, q_2)$ describes transition between initial two-quark state and final state of free particles, $Sp$ means the trace over spin and colour indexes. The wave function $\Psi(q_1, q_2)$ satisfies two-particle Bethe-Salpeter equation, which has not physically interesting exact solutions. That is why the redefinition of the wave function for calculation of decay amplitude $\mathcal{M}$ is needed. We must to introduce new wave function, which has clear physical interpretation and it is determined by more simple bound state equation.

Let us define vertex function $\Gamma(q_1, q_2)$ as follows:

$$\Psi(q_1, q_2) = \frac{\hat{q}_1 + m}{q_1^2 - m^2 + io} \Gamma(q_1, q_2) \frac{\hat{q}_2 - m}{q_2^2 - m^2 + io}. \hspace{1cm} (2)$$

Transforming (1) to the center-of-mass reference frame and using projection operator decomposition for quark propagator on positive and negative energy states

$$\frac{\hat{p} + m}{p^2 - m^2 + io} = \frac{1}{2\varepsilon(\hat{p})} \left[ \frac{U^\alpha(\hat{p})\bar{U}^\alpha(\hat{p})}{p^\alpha - \varepsilon(\hat{p}) + io} + \frac{V^\alpha(-\hat{p})\bar{V}^\alpha(-\hat{p})}{p^\alpha + \varepsilon(\hat{p}) - io} \right], \hspace{1cm} (3)$$

we can write amplitude $\mathcal{M}$ in following form:

$$\mathcal{M} = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2\varepsilon(\hat{q})} \int dq^\alpha \left\langle \frac{(M - 2\varepsilon(\hat{q}))}{2\pi (q^\alpha - \varepsilon(\hat{q}) + io)(q^\alpha - M + \varepsilon(\hat{q}) - io)} \right\rangle \times \hspace{1cm} (4)$$

$$Sp \left[ \Psi_M(\hat{q}) \hat{R}_s(\hat{q}) \mathcal{M}_H(q_1, q_2) \right],$$

where $\varepsilon(\hat{p}) = \sqrt{\hat{p}^2 + m^2}$, $q_1 = (q^\alpha, \hat{q})$, $q_2 = (M - q^\alpha, -\hat{q})$, $M$ is the mass of bound state, $\Psi_M(\hat{q})$ is single time bound state wave function which was obtained via projection the Bethe-Salpeter amplitude on positive energy state [3]:

$$\hat{U}(\hat{q}) \Gamma(q_1, q_2) \hat{V}(\hat{q}) = 2\varepsilon(\hat{q})(M - 2\varepsilon(\hat{q})) \Psi_M(\hat{q}). \hspace{1cm} (5)$$

Note, that for S-wave bound states which are discussed here, $\Psi_M(\hat{q}) = \Psi_M(|\hat{q}|)$. 

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The spin property of the two-quark bound state is described by relativistic projection operator, in which the exact dependence on vector momentum $\vec{q}$ must be considered. We write it as in [9]:

$$\hat{R}_s(\vec{q}) = \frac{(\hat{q} + m)(1 + \gamma_0)\hat{s} - m - \hat{q}'}{2\sqrt{2}(\varepsilon(\vec{q}) + m)},$$ (6)

where $\hat{s} = \hat{e}_\mu\gamma^\mu$ for vector $^3S_1$ state, $e_\mu$ is polarization vector, and $\hat{s} = \gamma_5$ for pseudoscalar $^1S_0$ state, $q = (\varepsilon(\vec{q}), \vec{q})$, $q' = (\varepsilon(\vec{q}), -\vec{q})$.

The relative energy $q^0$ integral in (4) is defined by poles of integrated function. The pole $q^0 = \varepsilon(\vec{q}) - i0$ in lower halfplane of complex variable $q^0$ gives main contribution. It is obviously that nontrivial contribution of other poles in parton amplitude $M_{H(q_1, q_2)}$ is suppressed by factor $\Delta \approx (M - 2\varepsilon(\vec{q}))/m$, which however is not very small in the case of charmonium state decays and it will be discussed separately.

The nonrelativistic approximation corresponds to limit $\vec{q} \to 0$. In this case the meson decay amplitude is equal to

$$M_0 = \frac{\Psi_M(0)}{\sqrt{2M}}Sp\left[\hat{R}_{s0}M_H(0, 0)\right],$$ (7)

where

$$\Psi_M(0) = \int \frac{d^3q}{(2\pi)^3}\Psi_M(\vec{q}),$$

and $\hat{R}_{s0} = m(1 + \gamma_0)\hat{s}/\sqrt{2}$.

The expression (7) is agree with usual used formula in potential quark models [1].

The wave function $\Psi_M(\vec{p}, \vec{q})$ of two-particle bound system with mass $M$ satisfies to quasipotential equation in the center-of-mass reference frame:

$$(M - 2\varepsilon(\vec{p}))\Psi_M(\vec{p}) = \int \frac{d^3q}{(2\pi)^3}V(\vec{p}, \vec{q}, M)\Psi_M(\vec{q}),$$ (8)

where $V(\vec{p}, \vec{q}, M)$ is so-called quasipotential, which is calculated using two-particle scattering off-shell amplitude, $\vec{p}$ is a relative momentum of particles: $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$. For more comfortable investigation bound state wave function and mass spectrum, we transform eq. (8) to a local form [10]:

$$\left(\frac{b^2(M)}{2\mu_R} - \frac{(\vec{p}^2)}{2\mu_R}\right)\Psi_M(\vec{p}) = \int \frac{d^3q}{(2\pi)^3}V(\vec{p}, \vec{q}, M)\Psi_M(\vec{q}),$$ (9)
where \( b^2(M) = (M^2 - 4m^2)/4 \) is the square of relative momentum of particles on energy shell \((M = 2\varepsilon(\vec{p}))\), \(\mu_R = M/4\) is relativistic reduced mass. Then we obtain from eq.(9) the bound state equation in coordinate space:

\[
\left( -\frac{\Delta^2}{2\mu_R} + U(\vec{r}) \right) \Psi_M(\vec{r}) = E_R \Psi_M(\vec{r}),
\]

(10)

where

\[
E_R = \frac{b^2_R}{2\mu_R} = \frac{M^2 - 4m^2}{2M}.
\]

In the case of spherical symmetry potential \( U = U(r) \) we have for radial part of wave function:

\[
\frac{d^2\chi}{dr^2} - 2\mu_R (U(r) - E_R) \chi = 0,
\]

(11)

where

\[
\chi(r) = \sqrt{4\pi r}\Psi_M(r).
\]

Taking into account the accuracy of our calculations, we shall use non-relativistic approximation of quark-antiquark interaction potential. As usual we present it using a composition of Coulomb and linear components:

\[
U(r) = -\frac{b}{r} + ar + c.
\]

(12)

In our calculation we use following numerical value for parameters of interactive potential: \( b = 0.25, \ a = 0.27 \text{ GeV}^2, \ c = -0.76 \text{ GeV} \) for \( c \)-quarks; \( b = 0.4, \ a = 0.25 \text{ GeV}^2, \ c = -0.3 \text{ GeV} \) for \( b \)-quarks. Because we don’t take into consideration spin dependence of quark-antiquark potential, it is necessary to use quark masses as free parameters for right description of mass spectra of quarkonium states. We obtained: \( m_c = 1.62 \text{ GeV} \) and \( m_b = 4.87 \text{ GeV} \) for \(^3S_1\) state, \( m_c = 1.55 \text{ GeV} \) \( m_b = 4.84 \text{ GeV} \) for \(^1S_0\) state. In this case we have: \( M_{J/\psi} = 3.10 \text{ GeV}, M_{\eta_c} = 2.98 \text{ GeV}, M_\Upsilon = 9.45 \text{ GeV}, M_{\eta_b} = 9.4 \text{ GeV}. \)

So, solving numerically bound state equation (11) with potential (12), we obtain quarkonium wave function in coordinate space. After numerical Fourier transformation we obtained the wave function in momentum space, which was used for estimation of relativistic effects in S-wave quarkonium decays.
3 Two-lepton decays $J/\psi, \Upsilon \rightarrow e^+e^-$

Two-lepton decay of $^3S_1$ state of heavy quark-antiquark bound system (for example $J/\psi$) is very interesting both for theory and experiment in particle physics. First, two-lepton mode of $J/\psi$ decay is good trigger in experiment, where $J/\psi$ production is studied. Second, as it is usually expected, we can used in first approximation simple non-relativistic description for bound state of heavy quarks and study relativistic effects as small corrections. From the point of view perturbative QCD and motivation of quark-antiquark potential, the small value of running constant $\alpha_s(m_c^2) = 0.3$ is also very important. That is why calculation $J/\psi$ or $\Upsilon$ decay widths is a good test of our knowledge about quark-antiquark interaction at large distance as well as about role of relativism in description of heavy quark bound states.

The amplitude of decays $J/\psi, \Upsilon \rightarrow e^+e^-$ has the form:

$$\mathcal{M}(J/\psi \rightarrow e^+e^-) = \sqrt{2M} \int \frac{d^3q}{2\varepsilon(q)(2\pi)^3} \int \frac{dq^o}{2\pi} \mathcal{S}_p \left[ R_J(q)M_H(q_1, q_2) \right] F_c \frac{(M - 2\varepsilon(q))\Psi_M(q)}{(q^o - \varepsilon(q) + io)(q^o - M + \varepsilon(q) - io)},$$

(13)

where $q_1 = (q^o, \vec{q}), q_2 = (q^o - M, -\vec{q}), F_c = \sqrt{3}$ is colour factor. The relativistic normalization for quasipotential wave function gives the multiplier $\sqrt{2M}$ in (16). The "hard" part of amplitude can be written as:

$$\mathcal{M}_H(q_1, q_2) = \frac{e^2 e_c}{M^2} \gamma^\mu \bar{U}_e(k_1)\gamma_\mu V_e(k_2).$$

In the case of $J/\psi, \Upsilon \rightarrow e^+e^-$ decays, amplitude $\mathcal{M}_H$ doesn't depend on relative quark energy and the $q^o$ integration is carried out simply. After sum over lepton polarizations, the squared modulus of amplitude takes the form

$$|\mathcal{M}(J/\psi \rightarrow e^+e^-)|^2 = \frac{e^4 e_c^2}{M^3} \int \frac{d^3q}{(2\pi)^3} \frac{\Psi_M(q)}{\varepsilon(q)(\varepsilon(q) + m)} \int \frac{d^3p}{(2\pi)^3} \frac{\Psi_M(p)}{\varepsilon(p)(\varepsilon(p) + m)} L_{\mu\nu}(k_1, k_2) \mathcal{H}^\mu(q) \mathcal{H}^\nu(p),$$

(14)

where

$$L_{\mu\nu}(k_1, k_2) = 4 \left[ k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - (k_1k_2)g_{\mu\nu} \right],$$

$$\mathcal{H}^\mu(q) = \mathcal{S}_p \left[ (\hat{q}_1 + m)(1 + \gamma_\nu)\hat{e}_J(\hat{q}_2 - m)\gamma^\mu \right].$$

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Using the condition for $J/\psi$ polarization vector
\[ \sum s J \varepsilon_{\mu} \varepsilon_{\nu} = -g_{\mu \nu} + P_{\mu} P_{\nu}/M^2, \]
we average (14) over $J/\psi$ polarizations and than carry out the angle integration. Finally we obtain:
\[ \frac{1}{3} \sum |M|^2 = \frac{16 \alpha^2 e^2}{9 \pi^3 M} (m^2 I_0 + 3m^3 I_1 + 2I_2)^2, \] (15)
where
\[ I_0 = \int \frac{q \chi(q) dq \varepsilon(q) \varepsilon(q) dq}{\varepsilon(q) (\varepsilon(q) + m)}, I_1 = \int \frac{q \chi(q) dq \varepsilon(q) \varepsilon(q) dq}{(\varepsilon(q) + m)}, I_2 = \int \frac{\varepsilon(q) q \chi(q) dq \varepsilon(q) \varepsilon(q) dq}{(\varepsilon(q) + m)}. \] (16)
The width of $J/\psi \rightarrow e^+ e^-$ decay and amplitude are connected as follows:
\[ \Gamma(J/\psi \rightarrow e^+ e^-) = \frac{1}{16 \pi M} \frac{1}{3} \sum |M|^2. \] (17)
In non-relativistic approach it reads:
\[ \Gamma_0(J/\psi \rightarrow e^+ e^-) = 16 \pi \varepsilon^2 \alpha^2 |\Psi_M(0)|^2 / M^2. \] (18)
Our numerical calculation gives for $J/\psi$ particle $\Gamma/\Gamma_0 = 0.89$ and for $\Upsilon$ particle - $\Gamma/\Gamma_0 = 0.97$. So, the values of relativistic corrections in two-lepton decays of $J/\psi, \Upsilon$ particles are not small, but they are less than QCD corrections in next order of $\alpha_s$ [11]:
\[ \Gamma_{QCD} = \Gamma_0 \left(1 - \frac{16 \alpha_s}{3 \pi} \right). \] (19)
Putting up $\alpha_s(m^2_c) = 0.3$ and $\alpha_s(m^2_b) = 0.2$, we obtain $\Gamma_{QCD}/\Gamma_0(J/\psi) \approx 0.5$ and $\Gamma_{QCD}/\Gamma_0(\Upsilon) \approx 0.7$. Note, that our calculations give more large relativistic corrections than estimation in binding energy $\varepsilon$ approximation [5]:
\[ \Gamma = \Gamma_0 \left(1 - \frac{1}{3 \varepsilon/m} \right), \] (20)
This equation gives $\Gamma/\Gamma_0 = 0.96$ for $J/\psi$ and $\Gamma/\Gamma_0 = 0.98$ for $\Upsilon$ particles. The difference may be explained using following fact. We exactly take into consideration relative quark motion in amplitude and in wave function, but in $\tilde{p}^2/m^2$ approximation in amplitude is used and wave function is presented only in origin.
4 Two-photon decays $\eta_c, \eta_b \rightarrow \gamma\gamma$

The two-photon decays of $\eta_c$ and $\eta_b$ particles are not studied sufficiently in experiment opposite to lepton decays of $J/\psi$ and $\Upsilon$ particles. The experimental uncertainty of decay width is equal to 50% for $\eta_c \rightarrow \gamma\gamma$ decay and data is absent for $\eta_b \rightarrow \gamma\gamma$ decay [12].

Contradict to this fact, these decays are very interesting for discrimination of many models which describe heavy quark bound state and QCD properties at large distance [13].

The decay amplitude $\eta_c \rightarrow \gamma\gamma$ ($\eta_b \rightarrow \gamma\gamma$) may be presented in the form:

$$|M|^2 = 32F_c^2\epsilon_c^4(4\pi\alpha)^2\frac{MJ^2}{4m^2},$$

(24)

where

$$J = \frac{m^2}{(2\pi)^3} \int \frac{d^3q}{2\varepsilon(q)} \frac{\Psi_M(q)}{\varepsilon(q)} \ln \frac{\varepsilon(q) + q}{\varepsilon(q) - q}$$

$$\Psi_M(q) = \int d^3q' \frac{R_M(q')}{2\varepsilon(q')} F_c(q' - \varepsilon(q') + io)(q' - M + \varepsilon(q') - io).$$

(22)

The integral function in (32) has two poles in plane of complex variable $q^o$: $q^o_1 = \varepsilon(q) - io$ or $q^o_1 = M - \varepsilon(q) + io$ as well as $q^o_1 = k_1^2 \pm \sqrt{k_1^2 + m^2} + io$ and $q^o_2 = k_2^2 \pm \sqrt{k_2^2 + m^2} + io$.

The contribution of other poles from "hard" part of amplitude is suppressed comparably the contribution of pole $q^o_1 = \varepsilon(q) - io$ by factor $\Delta \approx 0.1$ for $\eta_c$ and $\Delta \approx 0.03$ for $\eta_b$ particles. We will discuss it below.

After integration equation (32) over $q^o_1$, the contribution of the pole $q^o_1 = \varepsilon(q) - io$ reads:

$$\mathcal{M}(\eta_c \rightarrow \gamma\gamma) = \sqrt{2M} \int \frac{d^3q}{2\varepsilon(q)(2\pi)^3} \int \frac{dq''}{2\pi} Sp[R_M(q''),\mathcal{M}_H(q_1, q_2)] F_c(q'' - \varepsilon(q'') + io)(q'' - M + \varepsilon(q'') - io).$$

(21)

The "hard" part of amplitude $\mathcal{M}(\eta_c \rightarrow \gamma\gamma)$ may be written as follows:

$$\mathcal{M}_H(q_1, q_2) = e^2\epsilon_\mu^\nu(k_1)e^\nu(k_2)[\gamma^\mu(\hat{q}_1 - \hat{k}_1 + m)/(q_1 - k_1)^2 - m^2\gamma^\nu + \gamma^\mu(\hat{q}_2 - \hat{k}_2 + m)/(q_2 - k_2)^2 - m^2\gamma^\nu].$$

(22)
In non-relativistic approach we obtain well known result from (32):

\[ |M_0|^2 = 32F^2 e^4 (4\pi\alpha)^2 \left| \frac{\Psi_M(0)}{M} \right|^2. \] (25)

After numerical calculation of (35) we obtained \( \frac{\Gamma}{\Gamma_0} = 0.53 \) for \( \eta_c \to \gamma \gamma \) decay and \( \frac{\Gamma}{\Gamma_0} = 0.71 \) for \( \eta_b \to \gamma \gamma \) decay. The increase of the relativistic corrections in two-photon decays as compare with two-lepton decay is explained by the nontrivial dependence of the two-photon decay amplitude from relative quark momentum \( \vec{q} \). Obtained value of relativistic correction for two-photon decay width is approximately equal to QCD correction [11]:

\[ \Gamma_{QCD} = \Gamma_0 \left( 1 - \frac{\alpha_s}{\pi} \left( \frac{20}{3} - \frac{\pi^2}{3} \right) \right). \]

For estimation of contributions from the other poles in "hard" part of decay amplitude, we keep in mind that we can to ignore the dependence of amplitude on the vector momentum \( \vec{q} \). Thus the given contribution elementary related with non-relativistic approximation:

\[ M^2 = M_0^2 (1 + \delta), \] (26)

where

\[ \delta = \frac{M - 2m}{\sqrt{8m}}. \]

For \( \eta_c \to \gamma \gamma \) decay we obtained: \( \delta = -0.027 \). Note, that the sign of this correction is strongly dependent on choice of quark-antiquark potential.

5 Conclusions

Using relativistic quasipotential approach, we have calculated decay widths for S-wave quarkonia: \( \eta_c, \eta_b \to \gamma \gamma, \ J/\psi, \ U \to e^+e^- \). Nontrivial dependence on relative quark motion of decay amplitude is taken into account both in parton amplitude as in quarkonium wave function.

Our conclusions are:

1. Relativistic corrections for S-wave quarkonium decay are not small (10-50%) and comparable with QCD corrections.
2. Quasipotential approach gives more large value for relativistic corrections than binding energy approximation.

3. Additional poles in a relative-energy integral determine non vanishing contribution in decay width of quarkonia.

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