Element matrix formulation for bi-quadratic infinite element

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Abstract. In reality, no physical domain extends to infinity but for the convenience of calculation in certain instances like unbounded domains it is better to develop mathematical models with the assumption that it extends to infinity. Unbounded domains are present in a wide variety of practical engineering problems. Specific examples can be found in fields like solid mechanics, fluid flow, acoustics, heat and mass transfer. The common engineering approach when it comes to defining an unbounded domain is to limit it to a very large finite area. This method of using finite element analysis over a very large domain is called truncation. The determination of the finite boundary requires a lot of experience and intuition. This method results in an approximated approach, which takes up a significant amount of computational effort and time due to the large number of elements required to mesh the region. In this research paper, the use of a bi-quadratic infinite element to solve infinite domain structural problems is studied. The effect of truncation to represent an infinite domain element is examined. The infinite solution is found using infinite element method.

1. Introduction

It is impossible for any physical domain to extend to infinity. However, for the convenience of calculation, in certain instances mathematical models are developed with the assumption that the computational domain extends to infinity. This approach is especially preferred in cases of unbounded domains. Examples can be found in several fields of engineering and science. Various problems with domains that are assumed to extend to infinity, such as soil-structure interaction (Sáez, Esteban, Fernando Lopez-Caballero, and Arezou Modaressi-Farahmand-Razavi, 2013 [1]), underground excavation (Kumar, P, 2000 [2]), fluid flow (Peter Bettess, 1977 [3]), wave propagation (Jacqueline A. Bettes and Peter Bettes, 1998 [4]), and heat transfer (Demidem, M, 2005 [5]) can be found in literature. For such problems, analysis extends to huge distances in one or more directions to represent the far field domain or unbounded domain. The key struggle encountered during the process of obtaining the numerical solution to these problems is the discretization of the unbounded domain. A popular engineering approach which is been used to solve unbounded regions is simple truncation, in which the infinite region is limited to an artificial boundary at a position where the influence of the load applied is negligible and the infinite boundary conditions are tentatively applied to the artificial boundary. However, the placement of the finite boundary is critical to obtain an accurate solution and the choice
of the minimum distance from the region of interest is often dependent on experience and ingenuity. When the standard finite element method (FEM) is utilized to discretize the system, an extensive number of elements are required, as the domain tends to be huge. This will ostensibly take a considerable amount of computational effort and memory. Plus, in case of dynamic analysis, an artificial boundary can steer significant error due to reflection of waves being propagated in the system. In case of static problems, the artificial boundary location is not accurate, and computational concession is required (i.e. more distant the artificial boundary is positioned from the area of interest; the more accurate is the solution that will be obtained but at the expense of more computational power).

To evade the use of numerous elements and to more accurately represent the infinite domain, infinite elements are utilized. Zienkiewicz, Emson and Bettess and Morques and Owenzv are few of the authors who originated the development of these elements. The infinite element method is an extension of the finite element method and works alongside it as the edge compatibility between finite and infinite elements based on the continuity of deformations of adjoining nodes being satisfied in a similar manner to that between finite elements. Thus, the inherent procedure used to solve finite element problems will be unchanged. The key strides of the infinite element is that the mapping function of the geometry extend to infinity in the desired direction.

In his formulation, Bettes (1977) [3] utilized a Lagrangian polynomial multiplied by an exponential decay term with a negative power exp(-r). Later, utilizing similar concepts serendipity infinite elements were developed by Beer and Meek (1981) [7], in which the geometry of the infinite element is represented by the parametric mapping of the element such that it extends to infinity in one natural coordinate direction using mapping functions which have a singularity at $\xi=1$ and is interpolated along the finite boundary of the element using the usual serendipity-type functions. RAGHAVARAO and SANYASIRAJU (1994) [8] presents an easier method to develop the infinite elements of any order. The elements are obtained by the mixture of Lagrangian and Serendipity types of functions employing a methodology which was formerly used by Zienkiewicz for developing rectangular serendipity finite elements, and is similar to the elements derived by Beer and Meek (1981) [7]. The Serendipity family of elements were initially derived by examination, the development of higher-order elements is rather difficult and also necessitate considerable ingenuity.

1.1. Bi-quadratic infinite element

The infinite domain is represented on a finite region using geometric interpolation functions called mapping functions. These functions are used to represent the infinite region by a square element of edge dimension 2 units in a natural coordinate system ($\xi \eta$ coordinate system).

![Figure 1. Bi-quadratic infinite element](image)
The element is derived with the geometry tending to infinity at $\eta=1$. The element can be divided into two regions, the region from $\eta=-1$ to 0 as the finite region and the region from $\eta=0$ to 1 as the region which tends to infinity. The nodes in the finite region with the use of the corresponding mapping function is used to represent the geometry of the element. The mapping functions are derived such that it is equal to one at the corresponding nodes, zero at the remaining nodes and tend to infinity at $\eta=1$. The mapping functions are derived and multiplied with $\frac{1}{1-\eta}$ to make it tend to infinity at $\eta=1$.

A comprehensive set of geometry and field variable mapping functions are derived for two-dimensional bi-quadratic infinite element, to find the solution for the combination of infinite and finite element model. The mapping functions are derived as

\begin{align}
M1 &= (1-\xi) * (-\xi-\eta-1)/(1-\eta) \\
M2 &= (1 + \xi) * (\xi-\eta-1)/(1-\eta) \\
M3 &= (1 + \eta) * (1 + \xi)/(2*(1-\eta)) \\
M4 &= (1-\xi) * (1 + \eta)/(2*(1-\eta)) \\
M5 &= 2 * (1 + \xi) * (1-\xi)/(1-\eta)
\end{align}

The usual shape functions for 8 noded element are used to represent the field variable within the element. All the eight nodes are used to represent the field variable within the element.

The shape function is taken as

\begin{align}
N1 &= -1/4 * (1-\xi) * (1-\eta) * (1 + \xi + \eta) \\
N2 &= -1/4 * (1 + \xi) * (1-\eta) * (1-\xi + \eta) \\
N3 &= 1/2 * (1 + \xi) * (1-\eta^2) \\
N4 &= 1/2 * (1-\xi) * (1-\eta^2) \\
N5 &= 1/2 * (1-\xi^2) * (1-\eta) \\
N6 &= -1/4 * (1 + \xi) * (1 + \eta) * (1-\xi-\eta) \\
N7 &= 1/2 * (1-\xi^2) * (1 + \eta) \\
N8 &= -1/4 * (1-\xi) * (1 + \eta) * (1 + \xi-\eta)
\end{align}

These mapping and shape functions are used to derive the stiffness matrix of the problem.

2. Validation of results obtained by infinite element with analytical solution

The results obtained by using infinite element need to be validated with an analytical solution to show that they are accurate. Therefore, a simple problem for which an analytical solution is present has been selected and the results compared.

2.1. Pressurized hole in infinite medium

The infinite element technique is applied to a simple problem of a hole in an infinite medium under uniform internal pressure and compared with the analytical solution. The inner diameter of the hole is taken as 20 units. The young’s modulus is being taken as 100 and poisons ratio 0.3. Here plane stress condition is taken for the analysis.
Taking advantage of symmetry of the problem only the top half is used for the analysis. The problem is meshed with 5 eight noded finite quadrilateral element in the radial direction with a step size of 2 meter, and a step of 10 degree in the tangential direction. The infinite elements are then placed around the circumference of the mesh. Since the disc deforms radially the constrains are given as shown in Figure.2 to provide a radial deformation. The pressure intensity is applied to all the nodes in the inner surface of the disc. The loads are calculated by multiplying the pressure with the inner perimeter and then dividing it with the number of nodes on the surface. Since only one half of the disc is used for the analysis the load on the edge nodes should be halved. The loads are then split into its components in x and y axis and applied.

The problem for a pressurized hole in an infinite medium is a special case of thick walled cylinder under uniform boundary pressure on both sides for which an analytical solution is present. Here the outer diameter is taken as infinity, the outer pressure is zero and the equation is re-arranged to get the analytical solution can be give as

$$ U_r = \frac{1 + \nu}{E} \frac{p \times r_i^2}{r} $$

Table 1: Analytical solution and Infinite element solution

| P   | r_i | R  | Analytical Deflection | Deflection with infinite element |
|-----|-----|----|------------------------|----------------------------------|
| 11.45915 | 10  | 10 | 1.48968                | 1.50427                         |
| 11.45915 | 10  | 11 | 1.35425                | 1.34078                         |
| 11.45915 | 10  | 12 | 1.24140                | 1.20185                         |
| 11.45915 | 10  | 13 | 1.14591                | 1.11157                         |
| 11.45915 | 10  | 14 | 1.06406                | 1.03558                         |
| 11.45915 | 10  | 15 | 0.99312                | 0.96087                         |
| 11.45915 | 10  | 16 | 0.93105                | 0.89533                         |
| 11.45915 | 10  | 17 | 0.87628                | 0.84331                         |
| 11.45915 | 10  | 18 | 0.82760                | 0.79887                         |
| 11.45915 | 10  | 19 | 0.78404                | 0.75973                         |
| 11.45915 | 10  | 20 | 0.74484                | 0.72369                         |
The radial deflection at different radial position is found using the infinite element and the obtained analytical solution are tabulated in Table.1. The data is plotted in Figure.3, which shows that the combination of finite and infinite elements gives a solution that is close enough to the one given by the analytical equation.

3. Comparison of truncation method and use of infinite element
   The common engineering approach when it comes to defining an unbounded domain is to limit it to a very large finite area. This method of using finite element analysis over a very large domain is called truncation. The determination of the finite boundary requires a lot of experience and intuition. This method results in an approximated approach, which takes up a lot of computational effort and time due to the large number of elements required to mesh the region. In the first problem, the effects of truncation are studied to represent an infinite domain. A simple problem is selected and solved for different finite boundary positions and compared with the solution found by using the infinite element. The example is used to show that truncation is not suitable for most problems and the correctness of the solution is difficult to determine.

3.1. Static analysis of plate with hole
   A problem of a thin circular disc of inner diameter 10mm is selected. The top part of the inner surface is constrained, and force is applied at the bottom as shown in Figure.5. An initial diameter of 50mm is selected and divided into six elements in the radial direction. The area is divided into 24 elements in the tangential direction. The infinite elements are placed along the outer surface of this area and solved for the maximum displacement in the vertical direction. To compare the results with truncated finite element method, outer diameters of 200mm, 300mm, 400mm are selected and the problem solved by dividing the additional area into 5,7,10,13 elements in the radial direction respectively. To reduce discretization error 8 noded quadrilateral element is used to represent the circular area. Young’s modulus is taken as 100 and Poisons ratio as 0.3. Here plane stress condition is taken for the analysis.
Table 2. Geometric parameters and displacement

| Geometrical parameters | Deflection |
|------------------------|------------|
| Inner diameter 10      | Outer diameter 50 | -4.342660604 |
| 10                     | 200        | -3.852763766 |
| 10                     | 300        | -3.835748791 |
| 10                     | 400        | -3.829784338 |
| 10                     | 500        | -3.827066925 |
| 10                     | Infinity   | -3.481864484 |

Figure 4. Mesh for static analysis of plate with hole

Figure 5. Boundary conditions for static analysis of plate with hole

Figure 6. Deflection with respect to boundary diameter
The results are given in Table 2 and Figure 6. It evident that an even larger area is required to get an acceptable solution. Also, it is difficult to find the location where the artificial finite boundary need to be fixed as no mathematical model can be generated for arbitrary problems to calculate the deviation from the exact solution. Thus, it is evident that the use of infinite element method is superior to the truncation method.

4. Conclusion
The mapping function which parametrically represent the unbounded region by a finite region has been derived. The element stiffness matrix for structural element extending to infinity has been computed using 2-point Gauss quadrature (G4P) based on the defined mapping function and standard shape function. Two numerical examples have been taken to study the behavior. The first numerical problem confirms the results obtained by using a combination of finite and infinite elements to get the analytical solution for the simple problem of a pressurized hole in an infinite medium. The second numerical example shows the trend of the solution as different large finite regions are used for truncated finite analysis and is compared with the infinite region solution. It is evident that an even larger area should be considered to get an acceptable solution. The use of the infinite element is found to generate better results than truncation and also provides a significant reduction in computation time.

5. References
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