Control by quantum dynamics on graphs

Chris Godsil
Department of Combinatorics & Optimization, University of Waterloo, N2L 3G1 Waterloo, ON, Canada

Simone Severini
Department of Physics and Astronomy, University College London, WC1E 6BT London, United Kingdom

We address the study of controllability of a closed quantum system whose dynamical Lie algebra is generated by adjacency matrices of graphs. We characterize a large family of graphs that renders a system controllable. The key property is a novel graph-theoretic feature consisting of a particularly disordered cycle structure. Disregarding efficiency of control functions, but choosing subfamilies of sparse graphs, the results translate into continuous-time quantum walks for universal computation.

I. INTRODUCTION

The study of classical control theory is ubiquitous across engineering disciplines. The concept of controllability at the quantum level is a fundamental notion which expresses the ability of implementing any dynamics in a given quantum mechanical set-up (see [2] for an introductory monograph). From the practical point of view, the successful realization of quantum devices for a variety of information processing tasks strongly depends on the ability of manipulating systems with sufficient freedom. The design of protocols to control closed quantum systems mainly deals with schemes for efficient controllability by acting on subspaces [3].

A large number of systems have been shown to exhibit characteristics that allow controllability. For example, almost any pair of Hamiltonians that can be coherently applied to a finite-dimensional quantum system renders it controllable, and almost any quantum logic gate is universal [4]. However, controllability of large systems does not directly give an efficient implementation of control functions, and therefore processes like universal quantum computation. Moreover, control criteria are generally not computable for large systems and are not immediately scalable with the system size [5].

Here we consider controllability in relation to a class of dynamics that can be interpreted as Schrödinger evolutions of a particle hopping between the vertices of a graph. The corresponding Hamiltonians are matrices with nonzero entries only where transitions are permitted by the graph structure. This class naturally embraces continuous-time quantum walks (for short, QWs) [7], and the evolution of a single (or multiple [8]) excitation subspace for systems of spin-half particles with various kind of interaction (e.g., XY, XYZ, etc.) [17]. QWs have been shown to build gates by scattering off a set of small graphs attached to wires representing basis states [9]. QWs give examples of matrices for sufficient control, when we can arbitrarily modify edge-weights (or, equivalently, strength of couplings) [10].

During our discussion, an n-level system is controllable by a given set of Hamiltonians (possibly acting on specific subsystems only) if every element of the unitary group U(n) can be approximated by the matrices of the subgroup obtained via the dynamical Lie algebra of the set. This general definition is useful to isolate the main difference between the notions of controllability and universality. For instance, global evolution of a spin system may require complex protocols to implement 2-qubit gates on distant sites, even if it permits complete controllability.

The specific problem addressed here is the following one: we study controllability by the alternate application of two Hamiltonians. One of the Hamiltonian describes a nearest-neighbour interaction defined by some graph. The other Hamiltonian is a projector given by the characteristic vector of a subset of vertices. This describes an interaction between every two spins associated to the elements of the subset.

Our main result is to characterize a large family of graphs that give a pair of Hamiltonians implementing any quantum dynamics, thereby rendering a system controllable (Section II). The result can be seen as an analogue of the Burgarth-Giovannetti infectivity criterion [11] in this setting involving two Hamiltonians. A comparison with the infectivity criterion, possibly with the use of the notion of zero-forcing [12], remains an open question. It is also an open problem to determine the necessity of our condition.

The proofs follow easily from facts of Lie theory and algebraic properties of graphs. The members of our family present a particularly disordered cycle structure [13]. Specifically we require that the number of cycles of a certain length, starting from different vertices, cannot be written as a sum of the numbers of smaller cycles. We will show that this is a property responsible for controllability, when the Hamiltonian is the adjacency matrix of the graph. Indeed, we will use powers of the adjacency matrix. These encode the cycles of a graph (see the definition of a walk matrix below). In analogy with a known result in quantum control theory of spin systems, the path graph turns out to be the arguably most simple example [2, 5]. The setting is directly equivalent to a single excitation evolving on an XY spin chain with constant couplings (here XY means XX + YY, à la Bose [6]). The path is a connected graph with minimal number of edges, therefore it corresponds to a very sparse Hamiltonian. This is a fact to take into account, since sparse Hamiltonians can be simulated efficiently in a quantum
computer \[14\].

In general, focusing only on the dynamics restricted to the \(n\)-dimensional subspace, the physical device for implementation consists of any machine realizing QWs (e.g., an optical waveguide lattice \[13\]). While we do not modify the circuitry for different tasks, an external clock is necessary, since we need to know the time of application of each Hamiltonian, even if the resulting operation is just a phase factor. We will give evidence that our property is almost sure (Section II). This is parallel to the fact that almost every (generic) Hamiltonian gives sufficient control. For many types of graphs, the construction of an infinite family with a typical property may not be straightforward (e.g., expanders, small diameter graphs, etc.). We will present a method to construct infinite families of graphs (without special constraints) that satisfy our required property (Section III).

The results of the paper may translate into valuable information in the perspective of designing schemes for scalable quantum computation via local control (e.g., help in the selection of the systems, the engineering of control functions, etc.). (See \[2, 4\] for extended treatments of this topic.) From a wider angle, the results consist of a step towards a better and more general understanding quantum evolution on networks. Additionally, we introduce concepts that propose an interface between control theory and graph theory.

The paper has four further sections: Section II contains the general result; Examples are in Section III; we draw some brief conclusions and state open problems in Section IV.

## II. CONTROLLABILITY

Let \(X = (V(X), E(X))\) be a (simple) graph with a set of \(n\) vertices \(V(X)\) and a set of edges \(E(X) \subseteq V(X) \times V(X) - \{(i, i) : i \in V(X)\}\). The adjacency matrix of \(X\), denoted by \(A(X)\), is an \(n \times n\) matrix with \(A(X)_{i,j} = 1\) if \((i, j) \in E(X)\) and \(A(X)_{i,j} = 0\), otherwise. The walk matrix of a graph contains information about its cycle structure \[16\]. Let \(X\) be a graph on \(n\) vertices and let \(z \in \mathbb{R}^n\). We define and denote by \(W_z(X) = (z \ A(X)z \ldots A(X)^{n-1}z)\) an \(n \times n\) matrix with entries in \(\mathbb{Z}^{\geq 0}\) associated to \(X\). When \(z\) is the characteristic vector of some set \(S \subseteq V(X)\), the matrix \(W_z(X)\) is called a walk matrix of \(X\) with respect to \(S\). In this case, we may write \(W_S(X)\) instead of \(W_z(X)\). Let \(X\) be a graph on \(n\) vertices and let \(z \in \mathbb{R}^n\). The pair \((X, z)\) is said to be controllable if the matrix \(W_z(X)\) is invertible (i.e., \(\det(W_z(X)) \neq 0\)). When \(z\) is the characteristic vector of some set \(S \subseteq V(X)\), we may write \((X, S)\) instead of \((X, z)\). The definition of a controllable graph arises as a special case of a controllable pair. Let \(X\) be a graph and let \(1\) be the all-ones vector. This is also the characteristic vector of \(V(X)\). The graph \(X\) is said be controllable if \((X, 1)\) (or, equivalently, \((X, V(X))\)) is controllable.

Let us recall that a walk of length \(l\) in a graph \(X\) is a sequence of vertices \(1, 2, \ldots, l, l + 1\), such that \(\{i, i + 1\} \in E(X)\), for every \(1 \leq i \leq l\). The \(ij\)-th entry of the walk matrix, \(W_{l}(X)_{i,j} = \sum_{t=1}^{n} A_{i,j}^{t-1}(X)\), counts the number of all walks of length \(l - 1\) from vertex \(i\). Let \(d(i) := \{|j : (i, j) \in E(X)\}|\) be the degree of a vertex \(i\). A graph \(X\) is regular if \(d(i)\) is constant over \(V(X)\). Notice that a controllable graph can not be regular. In fact, the walk matrix of a regular graphs has rank 1, because \(1\) is one of its eigenvectors. One can verify by exhaustive search that there are no controllable graphs on \(n \leq 5\) vertices. Fig. 1 below contains drawings of all connected (non-isomorphic) controllable graphs on six vertices. Numerics show that the ratio \([\text{number of graphs}] / [\text{number of controllable graphs}]\) decreases with \(n\) (see caption of Fig. 1 for small examples). We expect that asymptotically almost surely every graph is controllable, also considering that the automorphisms fixing the vertices of a controllable graphs are trivial.

![FIG. 1: Drawings of all connected non-isomorphic controllable graphs on six vertices.](image)

A (continuous-time) quantum walk on a graph \(X\), starting from a state \(|\psi_0\rangle \in \mathbb{C}^n\), is the process induced by the rule \(U_{M(X)}(t)|\psi_0\rangle \rightarrow |\psi_t\rangle\), where \(U_{M(X)}(t) := e^{-iM(X)t}\) \((t \in \mathbb{R}^+)\) and \(M(X)\) is a symmetric matrix with nonzero entries corresponding to the edges of \(X\) (e.g., adjacency matrix, combinatorial Laplacian, etc.). A probability distribution supported by \(V(X)\) is obtained by performing a projective measurement on the state \(|\psi_t\rangle\). The matrix \(M(X)\) can also be seen as governing the dynamics of a system of spin-half particles restricted to a single excitation subspace. The dimension of such a subspace is in fact \(n\). Here we work with adjacency matrices only, but the results described are valid for any symmetric matrix. Studies of perfect state transfer and entanglement transfer in spin systems are often carried on with respect to this restriction \[12\]. QWs and their discrete analogues (e.g., coined quantum walks, scalar quantum walks, etc.) have found a number of algorithmic applications. The views \[2\] give a detailed perspective on this and related topics.
When choosing Hamiltonians of the form of \( M(X) \), the question to ask about controllability is the following one: can we obtain any quantum dynamics on an \( n \)-level system by performing repeated applications of QWs? Moreover, how much can we limit our resources (e.g., number of non-null interactions, number of different Hamiltonians, etc.)? In particular, can we use just a single QW (i.e., a fixed graph) plus an extra operation acting on a subspace of a relatively small dimension? The latter one is linked to core questions in quantum control theory, where we are interested in driving global dynamics by directly modifying only a limited portion of the system under a parsimony criterion. In the quantum mechanical set-up, controllability occurs together with the ability of constructing with reasonable accuracy any unitary matrix of the appropriate dimension (see Chapter 3 of [2]).

The corresponding property is expressed if we guarantee density in \( U(n) \) of the group of unitaries realized as sequences of QWs. The next technical lemma describes a relation between controllable pairs and Lie algebras.

**Lemma 1** Let \( X \) be a graph and let \( z \) be the characteristic vector of a set \( S \subseteq V(X) \). Let us define the symmetric \((0,1)\)-matrix \( L = zz^T \). If \((X,S)\) is a controllable pair then the real Lie algebra generated by the matrices \( A(X) \) and \( L \) is \( \text{Mat}_{n \times n}(\mathbb{R}) \), the algebra of all \( n \times n \) real matrices. The real Lie algebra generated by \( iA(X) \) and \( iL \) is the vector space of all skew-Hermitian matrices.

**Proof.** We prove by induction on \( k \) that the Lie algebra generated by \( A = A(X) \) and \( L \) contains the matrices \( A^{k-i}LA^i \), with \( i = 0, \ldots, k \). The first claim in the lemma will follow at once from this. We note that there are integers \( c_i \) such that \( LA^iL = c_i L \). If our Lie algebra contains the matrices \( A^{k-i}LA^i \), then it contains the Lie products \( A^{k+1-i}LA^i - A^{k-i}LA^i+1 \), with \( i = 0, \ldots, k \), and the partial sums \( A^{k+1-i}LA^i - A^{k+1}LA^i+1 \), for all \( i \). In particular, it contains \( A^{k+1}L - LA^{k+1} \) and therefore also

\[
LA^{k+1} - L^2A^{k+1} - A^{k+1}L^2 + LA^{k+1}L = 2c_{k+1}L - c_0(A^{k+1}L + LA^{k+1}).
\]

for all \( i \). From this, it follows that it contains \( LA^{k+1} \), and therefore all the monomials \( A^{k+1-i}LA^i \). Let us now consider the second claim of the lemma. We say a matrix is a *commutator of degree* \( r \) if, it can be written as \( AX-XA \) or \( LX-XL \) for some commutator of degree \( r \), where the commutators of degree zero are the matrices in the span of \( A \) and \( L \). Since \( A \) and \( L \) are symmetric, we see that all commutators of even weight are symmetric and all commutators of odd weight are skew-symmetric. The intersection of the space of symmetric matrices with the space of skew-symmetric matrices is the zero subspace, from which we deduce that the even-weight commutators span the space of real symmetric matrices and the odd-weight commutators span the space of skew-symmetric matrices. This implies that the even-weight commutators in \( iA \) and \( iL \) span the space of skew symmetric matrices, with dimension \((n^2-n)/2\), while the odd-weight commutators span a complementary space of dimension \((n^2+n)/2\). This proves our second claim.

We remark that it is not hard to show that the matrices

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]

generate \( \text{Mat}_{2 \times 2}(\mathbb{R}) \), but the real Lie algebra they generate is \( \mathfrak{sl}(2, \mathbb{R}) \) rather than \( \mathfrak{gl}(2, \mathbb{R}) \). Lemma 1 is reminiscent of the Lie Algebra Rank Condition in quantum control theory. The following result gives a sufficient condition to render a system controllable by a QW:

**Theorem 2** Let \( X \) be a graph and let \( z \) be the characteristic vector of a set \( S \subseteq V(X) \). If \((X,S)\) is a controllable pair then the unitary matrices \( U_{A(X)}(s) = e^{-iA(X)s} \) and \( U_L(t) = e^{-itL}, s, t \in \mathbb{R}^+, \) generate a dense subgroup of the unitary group \( U(n) \), \( n \geq 2 \).

**Proof.** Let \( G \) be the closed subgroup generated by the given elements. Then it is a Lie subgroup of \( U(n) \), and its tangent space is the Lie algebra generated by \( iA(X) \) and \( iL \). Since \( X \) is controllable, by Lemma 1 this Lie algebra is the space of all skew-Hermitian matrices, which is the tangent space to the unitary group \( U(n) \). It follows that \( G = U(n) \).

If \( S = V(X) \) then \( z = 1 \) and the Hamiltonian \( L = J \), the all-ones matrix. For its unitary, we have \( U_j(t)_{k,l} = \frac{1}{n} \left( n + e^{-int} - 1 \right) \) if \( k = l \) and \( U_j(t)_{k,l} = \frac{1}{n} \left( e^{-nit} - 1 \right) \) otherwise. In fact \( U_M(t) \) is a polynomial in \( M \) with degree at most the degree of the minimal polynomial of \( M \). Among adjacency matrices, this is the *full* possible Hamiltonian. Its implementation has been discussed in several works [18]. The matrix \( U_f(t) \) is essentially the same as the Grover operator used in quantum search algorithms [19]. Turning our attention to different characteristic vectors, we can prove a similar result concerning the path graph. We denote by \( P_n \) the path of length \( n - 1 \), i.e., the graph on \( n \) vertices \( \{ 1, 2, ..., n \} \) and edges \( \{1, 2\}, \{2, 3\}, ..., \{n-1, n\} \) (1 and \( n \) are called end-vertices). Weighted paths are often used to model 1D spin chains. A control criterion concerning the global space of 1D spin chains have been isolated in [8]. Controllability of these systems and scalable quantum computation has also been discussed by in [2].

**Corollary 3** Let \( P_n \) be the path on \( n \) vertices. The unitary matrix \( U_{A(P_n)}(s) = e^{-iA(P_n)s} \) together with the diagonal unitary matrix \( U_{e_1e_1^T}(t) = e^{-it_{e_1}e_1^T} \) \([e^{-it}, 1, ..., 1] \), \( s, t \in \mathbb{R}^+ \), where \( e_1 = (1, 0, ..., 0)^T \), generate a dense subgroup of the unitary group \( U(n) \), with \( n \geq 2 \).

**Proof.** We need to prove that \( W(P_n) = (e_1 A(P_n) e_1 \ldots A(P_n)^{n-1} e_1) \) is invertible. Observe that the first entry of the vector \( A(P_n)^{l} e_1 \) is a Catalan number \( C_{l/2} = \frac{2l}{l+1} {l\choose l/2} \) if \( l \) is even and zero, otherwise; the second entry behaves similarly, with \( C_{l+1/2} = \)
Given $S \subseteq V(X)$, the cone of $X$ relative to $S$ is the graph $\hat{X}_S$ such that $V(\hat{X}_S) = V(X) \cup \{0\}$, for a new vertex 0, and $E(\hat{X}_S) = E(X) \cup \{0, i \mid i \in S\}$. We denote by $X \setminus i$ the graph obtained from $X$ by deleting the vertex $i$ and all its incident edges.

**Theorem 4** Given a graph $X$ and a vertex 1 $\in V(X)$, the pair $(\hat{X}_1, \{0\})$ is controllable if $(X, \{1\})$ is.

**Proof.** We will show that if $u \in V(Z)$ for some graph $Z$, then $(Z, \{u\})$ is controllable if and only if the characteristic polynomials (of the adjacency matrices) $\phi(Z, t)$ and $\phi(Z, u)$ are coprime. From the properties of $\hat{X}_1$, one can prove that $\phi(\hat{X}_1, t) = t\phi(X, t) - \phi(X \setminus 1, t)$. From this, we deduce that if $\phi(X, t)$ and $\phi(X \setminus 1, t)$ are coprime then so are $\phi(\hat{X}_1, t)$ and $\phi(X, t)$. Now we derive our characterization of controllability. Assume $n = |V(X)|$. Let $e_1$ be the first vector of the standard basis, and let $E_{\theta}$ denote the idempotent in the spectral decomposition of $A = A(X)$ that corresponds to $\theta$ (see [21], pp. 186–187) it follows that

$$\frac{\phi(X \setminus 1, t)}{\phi(X, t)} = [(tI - A)^{-1}]_{1,1} = \sum_{\theta} (t - \theta)^{-1} e_1^T E_{\theta} e_1.$$

We observe that the number of poles in the rational function here is equal to the number of eigenvalues $\theta$ such that $e_1^T E_{\theta} e_1 \neq 0$; in other words, it is equal to the number of eigenvalues $\theta$ such that the projection $E_{\theta} e_1 \neq 0$. Note also that this number is $n$ if and only if $\phi(X, t)$ and $\phi(X \setminus 1, t)$ are coprime. To complete the argument, consider the walk matrix $W_{e_1}(X)$. By spectral decomposition (again) $W_{e_1} = \sum_{\theta} E_{\theta} e_1$, from which it follows that the column space of $W_{e_1}(X)$ lies in the span of the nonzero vectors $E_{\theta} e_1$. Since each projection $E_{\theta}$ is a polynomial in $A$, we conclude that $\text{rk}(W)$ is equal to the number of eigenvalues $\theta$ of $X$ such that $E_{\theta} e_1 \neq 0$. This proves our characterization.}

A method based on the theorem can be used to construct infinite families of controllable graphs:

**Corollary 5** Let $(X, S)$ be a controllable pair. If $Y$ is the graph obtained by joining one end-vertex of the path to each vertex in $S$, then $Y$ is controllable.

**IV. CONCLUSIONS**

We have considered controllability and QWs. As a technical tool, we have introduced the combinatorial notion of a controllable pair. A graph and a subset of its vertices form a controllable pair, when the structure of the graph exhibit a certain type of disorder. The disorder is expressed by the cycle structure of the graph, encoded in the entries of powers of the adjacency matrix. We have proved that a QW involving such a pair renders a system controllable. By this result, we can in principle perform universal quantum computation as an alternating sequence of QWs on two graphs, or on the same graph, but interspersed with phase factors. Including fault tolerance in this picture would encounter hard obstacles because of the sensitivity to phenomena linked to decoherence and Anderson localization [22]. An issue related to the more abstract aspects is the lack of transparency when trying to design algorithms with a logic that requires operations on specific subsystems. We conclude by stating four problems: **Problem 1:** Let $G$ be a subgroup of $U(n)$ which fixes $|\psi\rangle \in \mathbb{C}^n$ $(n \geq 3)$ and let $V \subseteq U(n) - G$. Then, the group $(G, V)$ (i.e., the subgroup of $U(n)$ generated by $G$ and $V$) is dense in $U(n)$ (see, e.g., Lemma 20 in [1]). Is there an analogue to this statement for dynamical Lie algebras generated by adjacency matrices? In particular, let $z$ be the characteristic vector of $S \subseteq V(X)$ and let $P$ be a permutation matrix corresponding to an automorphism of $X$. When $P z = z$, it follows that $P W_S(X) = W_S(X)$ and then $P = I$ because $\det(W_S(X)) \neq 0$. This means that the automorphisms of $X$ fixing $S$ are trivial if $(X, S)$ is controllable. Is this the most general condition for controllability? **Problem 2:** Can we lift the combinatorial criterion for controllability introduced in this paper to general criteria for controllability of spin systems? **Problem 3:** Study controllability by adjacency matrices, when the time of application of each Hamiltonian is constrained. Determining relations between quantum control by nearest-neighbour interaction on graphs and classical simulatability of the associated dynamics is an open problem. **Problem 4:** What can be said about controllability by acting only on a connected induced subgraph? If the number of vertices is constrained, the optimum may be difficult to compute. This would be parallel to the Burgarth-Giovannetti criterion whose optimum is difficult to approximate [23].

**Acknowledgments.** The authors would like to thank Daniel Burgarth and Alastair Kay for very valuable conversation about quantum control theory and for reading earlier drafts; Domenico D’Alessandro for several important remarks; Alessandro Cosentino for drawing the graphs in Fig. 1; the anonymous referees for suggestions that helped to improve the paper. SS is supported by a Newton International Fellowship.
[1] D. Aharonov, *Noisy quantum computation*, Ph.D. Thesis, Hebrew University, 1998.
[2] D. D’Alessandro, *Introduction to Quantum Control and Dynamics* (Taylor and Francis, Boca Raton, 2008).
[3] S. Lloyd, A. J. Landahl, and J.-J. E. Slotine, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[4] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[5] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[6] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[7] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[8] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[9] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[10] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[11] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[12] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[13] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[14] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);
[15] D. Deutsch, A. Barenco, and A. Ekert, Proc. Roy. Soc. London A 449, 669 (1995); S. Lloyd, Phys. Rev. A 69, 012305 (2004); A. Kay, Phys. Rev. Lett. 98, 010501 (2007); S. Montangero, T. Calarco, and R. Fazio, Phys. Rev. Lett. 99, 170501 (2007); H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett., 70, 548 (1993);