Conductance fluctuations in disordered superconductors with broken time-reversal symmetry near two dimensions

S. Ryu, A. Furusaki, A. W. W. Ludwig, and C. Mudry

1Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
2Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama 351-0198, Japan
3Department of Physics, University of California, Santa Barbara, CA 93106 USA
4Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland

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We extend the analysis of the conductance fluctuations in disordered metals by Altshuler, Kravtsov, and Lerner (AKL) to disordered superconductors with broken time-reversal symmetry in $d = (2 + \epsilon)$ dimensions (symmetry classes C and D of Altland and Zirnbauer). Using a perturbative renormalization group analysis of the corresponding non-linear sigma model (NL$\sigma$M) we compute the anomalous scaling dimensions of the dominant scalar operators with $2s$ gradients to one-loop order. We show that, in analogy with the result of AKL for ordinary, metallic systems (Wigner-Dyson classes), an infinite number of high-gradient operators would become relevant (in the renormalization group sense) near two dimensions if contributions beyond one-loop order are ignored. We explore the possibility to compare, in symmetry class D, the $\epsilon = (2 - d)$ expansion in $d < 2$ with exact results in one dimension. The method we use to perform the one-loop renormalization analysis is valid for general symmetric spaces of Kähler type, and suggests that this is a generic property of the perturbative treatment of NL$\sigma$Ms defined on Riemannian symmetric target spaces.

I. INTRODUCTION

A. Conductance fluctuations and the non-linear sigma model

The non-linear sigma model (NL$\sigma$M) provides, as first proposed by F. Wegner, an efficient framework within which to formulate (Anderson-type) localization transitions of noninteracting quantum mechanical particles subject to (static) random potentials. This provides some of the simplest models of electrons in disordered solids. In a seminal piece of work, Altshuler, Kravtsov and Lerner (AKL) showed in the 1980’s how to incorporate the important phenomenon of conductance fluctuations into this framework. Within a perturbative renormalization group (RG) analysis of the weakly coupled NL$\sigma$M in $d = 2$ (or $d = 2 + \epsilon$) spatial dimensions, i.e., in the regime of large (dimensionless) conductance $g$, these authors stressed the importance of terms in the low-energy effective action of the NL$\sigma$M which possess an arbitrary large number ($= 2s$) of spatial derivatives but do not violate any symmetries. These terms are called “high-gradient operators” and are generalizations of the kinetic term in the NL$\sigma$M which has two derivatives ($2s = 2$). High-gradient operators are highly irrelevant in the RG sense by “power counting” (i.e., without incorporation of fluctuation corrections). Specifically, AKL and Yudson studied the anomalous (RG) dimensions of these operators to one-loop order in the fluctuation corrections in $d = 2$ and in the $\epsilon = (d - 2)$-expansion for $d > 2$. It was found that (i) these operators would become strongly relevant (in the RG sense) for a large enough number $2s$ of derivatives, in spite of their strong irrelevance by power counting, if contributions beyond one-loop order are ignored, and that (ii) they would (consequently) dominate the cumulants of the conductance of sufficiently large order $2s$, in a sample of (large) linear size $L$. Based on these observations AKL were led to argue that for large mean (dimensionless) conductance $\bar{g}$, the main weight of the probability distribution of the conductance arises from a narrow peak at $g = \bar{g}$, while there exist anomalously long tails (close to being log-normal). The latter arise from the high moments of the conductance, which are dominated by the above-mentioned high-gradient operators appearing in the low-energy effective action. Upon approaching the regime of stronger disorder (smaller mean conductance $\bar{g}$), and in particular upon approaching a metal-insulator transition, the probability distribution becomes increasingly broad. AKL find similar behavior for the probability distributions of other observables, including, e.g., the local density of states (LDOS), current densities, and long-time tails of relaxation currents. Very recently the basic observations of AKL for the LDOS were confirmed in a chiral symmetry class (nomenclature of Refs. 11 and 12), and were analyzed more completely in this case.

Ten years ago, in another seminal piece of work, Zirnbauer and Altland and Zirnbauer provided a general classification scheme for the behavior of non-interacting quantum mechanical particles subject to random potentials, on length scales much larger than the mean free path in terms of a total of ten symmetry classes. In every realization of disorder, the Hamiltonian has the same symmetries as a corresponding ensemble of random matrices. This encompasses the three previously known so-called “Wigner-Dyson symmetry classes” relevant for the physics of Anderson localization of electrons in disordered solids (corresponding to “orthogonal, unitary, and symplectic” random matrix ensembles). Altland and Zirnbauer’s exten-
sion includes four novel symmetry classes describing the (Anderson-like) localization physics of non-interacting Bogoliubov-De Gennes (BdG) quasiparticles within a mean-field treatment of pairing in disordered superconductors. (Four symmetry classes arise since SU(2) spin-rotational, or time-reversal invariance may be present or absent.) As parameters are varied, the wave functions of the (BdG) quasiparticles may be extended or localized due to disorder, and (Anderson localization type) transitions between such phases may occur. Since BdG quasiparticles do not carry a conserved electric charge, the difference between phases of localized and extended quasiparticle wave functions manifests itself not in the electrical conductivity, but instead in the thermal conductivity $\kappa$, and if spin is conserved, also in the spin conductivity $\sigma_{\text{spin}}$. In complete analogy with the fluctuations of the electrical conductance in a finite sample of disordered metals (described by the Wigner-Dyson symmetry classes), the system of BdG quasiparticles also exhibits fluctuations of the appropriate conductance in a finite-size sample. These are the fluctuations of the thermal conductance (divided by temperature $T$), or of the spin conductance. Henceforth, for notational convenience, we will refer to these quantities simply as “the conductance”, and denote the corresponding “dimensionless conductance” again by the symbol $g$.

Just as for disordered metals (i.e., for the Wigner-Dyson symmetry classes), the NL$\sigma$M also provides an efficient framework to formulate (Anderson-type) localization physics of BdG quasiparticles in superconductors. In this article we focus on systems where time-reversal symmetry is broken in every realization of disorder: the corresponding AZ symmetry classes are conventionally called “class D” if SU(2) spin-rotational invariance is broken and “class C” if it is conserved, in every disorder realization. In parallel, and for comparison, we will also discuss here localization physics of ordinary electrons in the absence of time-reversal symmetry, which is described by the Wigner-Dyson symmetry class conventionally called “unitary class” or “class A”. (See also Table I) When disorder averaging is implemented using fermionic replicas, the target manifolds of the resulting NL$\sigma$Ms are the symmetric spaces $O(2N)/U(N)$ for symmetry class D, $Sp(N)/U(N)$ for symmetry class C. We remind the reader that for the familiar unitary symmetry class A, the corresponding target space is $U(2N)/[U(N) \times U(N)]$. (In all three cases the number $N$ of replicas is taken to zero at the end of calculations.)

The aim of this paper is to investigate perturbatively, in the spirit of the original work of AKL for ordinary metals, the conductance fluctuations for the novel AZ symmetry classes D and C, i.e., for BdG quasiparticles in superconductors lacking time-reversal symmetry, in the absence (class D) or presence (class C) of spin-rotational symmetry. Our main results are briefly summarized in Sec. 11 below. We also point out that it might be possible in the case of symmetry class D to make contact with exact nonperturbative results obtained in $d = 1$ dimensions from the so-called Dorokhov-Mello-Pereyra-Kumar (Fokker-Planck) analysis of the probability distribution of the conductance.

We end this section by recalling, as an aside, that the NL$\sigma$Ms in all three symmetry classes, D, C, and A, share certain essential features associated with time-reversal symmetry breaking. First, in $d = 2$ dimensions, all three target manifolds allow for a topological term (a term that does not modify the equations of motion), the so-called “theta- or Pruisken term”. Its presence is responsible for the appearance of thermal, spin, or charge Hall insulating phases, and corresponding plateau transitions in $d = 2$ dimensions. Second, the target manifold for symmetry class D stands out in that it is not simply connected. Consequently, there are certain discrete $\mathbb{Z}_q$ domain wall excitations that may give rise to a rich phase diagram for certain types of disorders.

In this paper we shall ignore the physics associated with such $\mathbb{Z}_q$ domain walls, which requires an extension of the NL$\sigma$M description in symmetry class D. These features are not present in certain realizations of this symmetry class, on which we focus in this paper. Finally, all three target spaces have in common that they are Kähler manifolds from a geometrical point of view. This means that each target space defines a complex manifold endowed with a Hermitian metric derived from a Kähler potential. This property will allow us to perform the perturbative RG analysis in a unified way for symmetry classes D, C, and A. (See Sec. 11 for a brief summary.)

### B. Summary of main results

We begin with the summary of our main results by quoting the standard one-loop result for the (RG) beta function:

$$\beta(t) = -et + (N - \vartheta)t^2 + O(t^3)$$

In $d = 2 + \epsilon$ spatial dimensions, obeyed by the dimensionless coupling constant $t$ of the relevant NL$\sigma$M [defined in (2.11) below] whose inverse is proportional to the mean dimensionless conductance $\bar{g}$, once the replica

| RMT | $T$ | SU(2) | NL$\sigma$M target space | $\vartheta$ | $N$ |
|-----|-----|-------|-----------------|-----|-----|
| D   | no  | no    | $O(2N)/U(N)$   | +1  | $N$ |
| C   | no  | yes   | $Sp(N)/U(N)$   | $-1$ | $N$ |
| A   | no  | —     | $U(p + q)/[U(p) \times U(q)]$ | 0   | $p + q$ |

### TABLE I: Fermionic replica target spaces of the NL$\sigma$M describing weakly disordered superconductors and metals when time-reversal symmetry ($T$) is broken. Symmetry classes D, C, and A refer to the symmetric spaces defined by the RMT theory. The number of replicas $N$ is $N$ for the two BdG symmetry classes whereas it is $p = q = N$ for the unitary symmetry class. The index $\vartheta = 0, \pm 1$ distinguishes the behavior of the beta function for the NL$\sigma$M in symmetry classes D, C, and A.
limit \( N \rightarrow 0 \) has been taken. To treat the symmetry classes D, C, and A on equal footing, we have introduced the replica index \( N \) that takes the value \( N \) for symmetry classes D and C and \( p + q \) for symmetry class A. The index \( \vartheta \) is also needed to distinguish the one-loop beta function for the NLO\( p, q \)M corresponding to symmetry classes D \((\vartheta = +1)\), symmetry classes C \((\vartheta = -1)\), and symmetry classes A \((\vartheta = 0)\). The correspondence between the pair \((\vartheta, N)\) and the symmetry classes D, C, and A is summarized in Table II. The beta function in Eq. (1.1) possesses a trivial fixed point,

\[
t = 0,
\]

which is infrared (IR) stable for \( d = 2 + \epsilon > 2 \) (i.e., \( \epsilon > 0 \)), while it is IR unstable when \( d = 2 + \epsilon < 2 \) (i.e., \( \epsilon < 0 \)). Upon taking the replica limit \( N \rightarrow 0 \) this describes a diffusive metallic phase. Whenever \( 0 < \epsilon/(N - \vartheta) \) is small, At the corresponding fixed point \( t^* \) of the beta function \( (1.2a) \),

\[
t^* := \frac{\epsilon}{N - \vartheta} + O(\epsilon^2).
\]

For symmetry classes C and A this describes the replica limit and in \( d = 2 + \epsilon > 2 \) dimensions, a metallic-insulator transition separating a metallic from an insulating phase. For symmetry class D, on the other hand, this describes (in the replica limit) a stable fixed point (“critical phase”) in \( d = 2 - |\epsilon| < 2 \) dimensions. The IR flows for symmetry classes D and C are summarized in Fig. 1. A possible evolution of the stable fixed point in symmetry class D for dimensionality \( d = (2 - |\epsilon|) \) between one and two dimensions, to be discussed in the following subsection, is sketched in Fig. 2.

In this paper we find the dominant anomalous scaling dimension \( x^{(s)} \) among all high-gradient operators containing 2s gradients, to one-loop order. These are

\[
x^{(s)} = 2s - 2(s^2 - s)\epsilon + O(\epsilon^2)
\]

for \( U(p + q)/[U(p) \times U(q)] \),

\[
x^{(s)} = 2s - 2(s^2 - s)\epsilon + O(\epsilon^2)
\]

for \( \text{Sp}(N)/U(N) \),

\[
x^{(s)} = 2s - (s^2 - s)\epsilon + O(\epsilon^2)
\]

for \( \text{SO}(2N)/U(N) \),

when evaluated in the vicinity of the zero-coupling fixed point \( (1.2a) \) where \( t \) is small. At the corresponding fixed points in \( d = 2 + \epsilon \) dimensions, Eq. (1.2b), these anomalous scaling dimensions become, consequently [for generic numbers of replicas, \((p, q)\) or \( N\)]

\[
x^{(s)} = 2s - 2(s^2 - s)\frac{\epsilon}{p + q} + O(\epsilon^2)
\]

for \( U(p + q)/[U(p) \times U(q)] \),

\[
x^{(s)} = 2s - 2(s^2 - s)\frac{\epsilon}{N + 1} + O(\epsilon^2)
\]

for \( \text{Sp}(N)/U(N) \),

\[
x^{(s)} = 2s - (s^2 - s)\frac{\epsilon}{N - 1} + O(\epsilon^2)
\]

for \( \text{SO}(2N)/U(N) \),

Equations (1.3a) and (1.4a) were derived a long time ago by Lerner and Wegner\(^{38,39}\). Equations (1.3) and (1.4) are, otherwise, new and are the main results of this paper.

Taking the replica limit of Eq. (1.3) yields the dominant anomalous scaling dimensions at the nontrivial fixed points of the localization problems in \( d = 2 + \epsilon \) dimensions. In particular, in symmetry class C one obtains, in the replica limit, the one-loop anomalous scaling dimensions

\[
x^{(s)} = 2s - 2(s^2 - s)\epsilon + O(\epsilon^2)
\]
of dominant 2s-gradient operators at the metal-insulator transition in \( d = 2 + \epsilon > 2 \) dimensions (\( \epsilon > 0 \)). On the other hand, in symmetry class D in \( d = 2 - |\epsilon| < 2 \) dimensions, the dominant 2s-gradient operators have the anomalous scaling dimensions

\[
x^{(s)} = 2s - (s^2 - s)|\epsilon| + \mathcal{O}(\epsilon^2) \quad (1.5b)
\]

at the stable, non-trivial fixed point (in the replica limit).

### C. Discussion

When taken at face value, the one-loop expressions in Eqs. (1.3), (1.4), and (1.5) would appear to imply that the anomalous scaling dimension \( x^{(s)} \) always turns negative for sufficiently large \( s \).

If this behavior was to persist when all higher orders in \( \epsilon \) are included, an infinite number of high-gradient operators would be relevant at the nontrivial fixed points of these NL\( \sigma \)Ms, i.e., infinitely many coupling constants would have to be fine-tuned to reach these nontrivial fixed points. The perturbative treatment of these NL\( \sigma \)Ms would then appear to contradict the belief that the underlying microscopic models undergo continuous phase transitions whose properties are accessible perturbatively when all higher orders in \( \epsilon \) are included. An infinite number of high-gradient operators would be rele-

vant at the nontrivial fixed points of these NL\( \sigma \)Ms. Whether the apparent breakdown of one-parameter scaling due to the relevance of high-gradient operators at one loop is an artifact of the 2 + \( \epsilon \) expansion or has a deeper meaning remains an outstanding problem for the description of Anderson localization using the NL\( \sigma \)M approach.

Symmetry classes D and DIII, on the other hand, may perhaps offer a somewhat different opportunity for the following reason. On the one hand, there exists a critical phase described by a stable fixed point in \( d = 2 - |\epsilon| < 2 \) dimensions whose properties are accessible perturbatively within the \( \epsilon \) expansion. These include the properties of the conductance fluctuations and of the high-gradient operators discussed above. On the other hand, a critical phase of these models is also known to exist in \( d = 1 \) dimensions. Therefore, one might expect that the results obtained from the \( \epsilon \) expansion in \( d = 2 - |\epsilon| < 2 \) dimensions would eventually, when \( |\epsilon| = 1 \), turn into the results obtained from the exact solutions of the models in \( d = 1 \) dimensions, as is schematically depicted in Fig. 2. If this was the case, then one could “test” the properties of the high-gradient operators and the nature of the conductance fluctuations obtained from them following the work of AKL, against exact results in \( d = 1 \) dimensions. Of course, as with all \( \epsilon \) expansions, one would not be able to predict exactly the actual numerical values of critical exponents directly at \(|\epsilon| = 1\). However, based on experience with the \( \epsilon \) expansion in, say, ordinary Landau-Ginzburg theory, one might expect that the structure of the entire probability distribution of the conductance would evolve, in some sense, “smoothly” with \( \epsilon \), possibly all the way down to \( d = 1 \) dimensions (\(|\epsilon| = 1\)). Let us now pursue this possibility.

Using perturbation theory of the NL\( \sigma \)M, AKL demonstrate that in general (i.e., for any NL\( \sigma \)M) there are two contributions to the \( s \)th cumulant of the (dimensionless) conductance.

(Below, we write the corresponding expressions appearing directly at the stable fixed point in \( d = 2 - |\epsilon| \) dimensions.) The first, “standard” contribution reads:

\[
\langle \langle g^s \rangle \rangle_{\text{std}} \propto (\bar{g})^{2-s} \quad (1.6)
\]

where \( \bar{g} \equiv \langle g \rangle \) is the mean conductance. However, when a high-gradient operator with 2s derivatives is added to the action of the NL\( \sigma \)M with coupling constant \( z_s \), then there is an additional contribution

\[
\langle \langle g^s \rangle \rangle_{\text{add}} \propto z_s \left( \frac{L}{\ell} \right)^{d-x^{(s)}} \quad (1.7)
\]

to this cumulant. Here, \( L \) is the linear system size and \( \ell \) the characteristic microscopic length scale (the “mean free path”) for the disorder. We see that it is when \( d - x^{(s)} > 0 \) that the additional contribution in Eq. (1.7) dominates over the standard contribution in Eq. (1.6). This hence happens precisely when the high-gradient operator with 2s derivatives becomes a relevant operator in the action of the NL\( \sigma \)M.

Now, AKL argue that, if one was to take the one-loop expressions for the anomalous scaling dimensions of the high-gradient operators at face value, which for symmetry class D would amount to the result in Eq. (1.5b), derived by us in this paper, then for cumulants of sufficiently higher order \( s > \frac{2}{|\epsilon|} [1 + \mathcal{O}(\epsilon)] \) the additional contribution would dominate over the “standard” one, yielding

\[
\langle \langle g^s \rangle \rangle \sim A_s \left( \frac{L}{\ell} \right)^{d-x^{(s)}}, \quad s > \frac{2}{|\epsilon|} [1 + \mathcal{O}(\epsilon)] \quad (1.8)
\]
as \( L/\ell \to \infty \), where \( A_s \) are certain amplitudes. Following AKL, the lower cumulants of order \( s < \frac{2}{|\epsilon|} [1 + \mathcal{O}(\epsilon)] \), on the other hand, would be determined by Eq. (1.6). In summary, the reasoning of AKL implies, when specialized to the stable fixed point in symmetry class D in \( d = 1 \) dimensions, as is schematically depicted in Fig. 2.
$2 - |\epsilon| < 2$ spatial dimensions, the following form of the conductance cumulants

$$\langle g^s \rangle \sim \begin{cases} (g)^{2-s}, & \text{if } s < s_0, \\ A_s (L/\ell)^{d-x(s)}, & \text{if } s > s_0, \end{cases} \quad (1.9a)$$

where

$$s_0 \approx \frac{2}{|\epsilon|} \left[ 1 + O(\epsilon) \right] \quad (1.9b)$$

and

$$d - x(s) = |\epsilon| s^2 - (2 + |\epsilon|) s + (2 - |\epsilon|) + O(\epsilon^2) \quad (1.9c)$$

Let us now discuss the known exact results for the $d = 1$ dimensional case. Specifically, the leading dependence of all cumulants of the conductance of a very long but thick quasi one-dimensional wire of length $L$, believed to be described by a NL$\sigma$M in $d = 1$ dimensions, has been extracted exactly from the DMPK approach for symmetry classes D and DIII.\textsuperscript{34,54,55} For both symmetry classes, these cumulants decay algebraically with system size,

$$\langle g^s \rangle \propto (L/\ell)^{-1/2}, \quad s = 1, 2, \ldots, \quad (1.10)$$

with the same exponent for all cumulants ("$\ell$" is again the mean free path). Moreover, the logarithm of the conductance exhibits the behavior

$$\langle \ln g \rangle \propto -(L/\ell)^{1/2}, \quad \text{var} \ln g \sim L/\ell. \quad (1.11)$$

Thus, the typical conductance exhibits stretched exponential behavior, in contrast with the mean conductance. This indicates a broad conductance distribution.

For symmetry class DIII the result\textsuperscript{10} for the mean conductance has been reproduced by Lamacraft, Simons, and Zirnbauer\textsuperscript{56} using the one-dimensional NL$\sigma$M (which is a principal chiral model for symmetry class DIII) and instanton methods.

If one assumes that the exact results in $d = 1$ describe the end point (at $\epsilon = 1$) of the line of critical points in $2 - |\epsilon|$ dimensions (see Fig.2), there are then two possibilities. First, the one-loop relevance of high-gradient operators is an artifact of the $\epsilon$-expansion and these operators are in fact irrelevant once all orders in the $\epsilon$-expansion are taken into account (see Refs.\textsuperscript{44,45}). Second, this is not the case and high-gradient operators are truly relevant. In this case, however, deducing the scaling of the cumulants based on the one-loop anomalous scaling dimensions does not appear to be sufficient to determine their scaling behavior. Indeed, in this case the running coupling constants $z_\sigma$ of all the high-gradient operators in the effective action of the NL$\sigma$M grow rapidly, more so the higher the number $2s$ of gradients. Under these conditions, then, there is no rationale to neglect the nonlinear coupling between the running coupling constants $z_\sigma$ of the high-gradient operators as was done to reach the conclusion\textsuperscript{19}, when extracting the scaling of the cumulants of the conductance. In effect, one needs to implement a functional RG analysis which can treat an infinite set of running (and increasing) coupling constants on equal footing. We have performed such an analysis (on which we will report elsewhere) for symmetry class A, but have so-far been unable to solve the resulting RG equations. Since in the present case the coupling constants $z_\sigma$ of the high-gradient operators all become large, it is clear (as already mentioned) that they will not flow independently of each other in the IR limit, but will strongly mix with each other under the RG flow. In this situation, their flow does not need to result in the breakdown of single parameter scaling. Indeed, a precisely analogous situation is encountered when computing the local DOS of the two-dimensional random hopping problem\textsuperscript{51} in which case one-parameter scaling for the local DOS is recovered using a functional RG analysis.

**D. Summary of the method of calculation**

The RG scheme that we adopt in this paper is the covariant background field method.\textsuperscript{52} In Refs.\textsuperscript{44,45} this method was used to compute, up to two loop order, the RG flows of the high-gradient operators for the $O(N)/O(N-1)$ NL$\sigma$M, relevant to statistical models such as the classical Heisenberg magnet. The covariant background field method consists of two key steps. First, the fields of the NL$\sigma$M are separated into short wavelength ("fast") and long wavelength ("slow") degrees of freedom where the slow degrees of freedom are defined as the solutions to the classical equations of motion. Second, the fast fields are integrated out. When using this method, no redundant operator is generated in the second step of the covariant background field method, a great advantage when one needs to consider mixing of many scaling operators by the RG transformation (as we do here). Another advantage is that explicit invariance under a coordinate reparametrization of the group manifold is maintained, allowing for a geometrical formulation at all stages of the calculation.

In this paper, we shall also extend the covariant background field method to the cases when the target spaces of the NL$\sigma$M are Riemannian or Kähler supermanifolds.\textsuperscript{53} We derive a master formula, Eq. (1.13), in the Kähler case for the RG flow of high-gradient operators, all of which possess a vanishing covariant derivative. [See Eq. (C19) for the super Riemannian cases and Eq. (C40) for the super Kähler cases, respectively. To obtain the corresponding formula for the non-supersymmetric cases (for fermionic replicated NL$\sigma$M, i.e., for compact NL$\sigma$M) we set all factors $(-1)^p$ to $+1$, where $p = 0, 1$ is here the grade in the supersymmetric expression].\textsuperscript{52} This master formula is expressed solely in terms of the geometrical data of the target manifold and should apply to all symmetry classes of $\mathbb{Z}_{11,12}$ but is not limited to them. The generality of the master formula suggests that high-
gradient operators are generically relevant to 1-loop order at the nontrivial fixed point of a NLσM defined on a Riemannian target space.

E. Geometric interpretation of high-gradient operators

All high-gradient operators that enter the computation of the cumulants for the conductance can be expressed in terms of tensor fields on the target manifold. We will show that high-gradient operators together with the metric and curvature tensors on the target manifolds of the NLσM form a family of tensors with vanishing covariant derivative.

F. Outline of the paper

This paper is organized as follows. Section II introduces the NLσM on Kähler manifolds for disordered superconductors with broken time-reversal symmetry as well as for a disordered metal in the standard unitary symmetry class. Section III describes how to use the covariant background field method to deduce perturbative RG flows. The dominant anomalous scaling dimensions for high-gradient operators are computed in Sec. IV to one-loop order. We relegate to appendices some key intermediary steps in the derivation of Eq. (1.3). The master formula, Eq. (1.3), which expresses the RG transformation law obeyed by high-gradient operators solely in terms of the geometrical data of the target manifold is derived in appendix C for any Riemannian or Kählerian supermanifold on which the high-gradient operators have vanishing covariant derivatives.

II. NLσM FOR WEAKLY DISORDERED SUPERCONDUCTORS

A. NLσM on Kähler manifolds

We start from the NLσM description for the symmetry classes D, C and A of Anderson localization in terms of fermionic replicas. Alternatively, one can use either bosonic replica or supersymmetric methods to perform the disorder average. These three options are equivalent to non-interacting quasi-particles were identified in the seminal work of Zirnbauer, and Altland and Zirnbauer. For the cases of interest in this paper, i.e., disordered superconductors with broken time-reversal symmetry, the target spaces of the relevant NLσM turn out to be Kähler manifolds which are the Hermitian symmetric spaces SO(2N)/U(N) for symmetry class D and Sp(N)/U(N) for symmetry class C. The target space of the NLσM that describes a disordered metal when time-reversal symmetry is broken is the Hermitian symmetric space U(2N)/[U(N) × U(N)], which is isomorphic to the complex Grassmannian G_{2N,N}(C).

Within the mesoscopic community, the usual representation of the NLσM in symmetry classes D, C, and A is in terms of matrices belonging to the relevant symmetric spaces, “the Q-matrix representation” after the seminal work by Efetov et al. in Ref. 19. This representation emphasizes the pattern of symmetry breaking but quickly becomes cumbersome when performing an RG analysis of the flow of composite operators. We opt instead to emphasize more directly the geometrical properties of the target spaces, namely the fact that they are examples of Kähler manifolds for symmetry classes D, C, and A. We refer the reader to appendix A for a relation between the representation of the NLσM that we shall introduce below and the more common Q-matrix representation thereof.

We now present the definition of a NLσM on a Kähler manifold starting from a NLσM on a 2M-dimensional Riemannian manifold \( (\mathcal{M}, g) \),

\[
Z := \int \mathcal{D}[\sigma] e^{-S[\sigma]}, \quad (2.1a)
\]

\[
S[\sigma] := \frac{1}{4\pi^2} \int_r g_{AB} (\phi) \partial_\mu \phi^B \partial_\mu \phi^A. \quad (2.1b)
\]

Our conventions are here the following. We reserve the greek alphabet to label the \( d \) coordinates \( r_\mu \) where \( \mu = 1, \ldots, d \) of \( r \in \mathbb{R}^d \), and use the short-hand notation \( \int_r \equiv \int_{|r|>a} d^d r/a^{d-2} \) for the integration over \( \mathbb{R}^d \) with the short-distance cut-off \( a \). The dimensionless coupling constant of the NLσM is here denoted by \( \tau \). The dimensionless conductance shares with \( \tau \) the same engineering dimension. To each point \( r \) from the base space we have assigned a point \( p \) on the manifold \( \mathfrak{M} \) with the coordinates \( (\phi^A) \in \mathbb{R}^{2M} \). In the definition of the NLσM on a Riemannian manifold, Eq. (2.1), the \( 2M \times 2M \) components of the metric tensor field \( g \) of rank (0, 2) are represented by \( (g_{AB}) \). We reserve the capital latin alphabet to label the \( 2M \) coordinates \( \phi^A=1, \ldots, 2M \) of \( p \in \mathfrak{M} \).

The components of the metric tensor field \( g_{AB} \) on the Riemannian manifold \( (\mathfrak{M}, g) \) are real and symmetric under the exchange of the indices

\[
g_{AB} = g_{BA}, \quad A, B = 1, \ldots, 2M. \quad (2.1c)
\]

There is a distinction between upper/lower capital latin indices. The components of \( g^{-1} \) are denoted by \( g^{AB} \) and satisfy

\[
g_{AC} g^{CB} = \delta^B_\delta^B, \quad g^{AC} g_{CB} = A^\delta_\delta^B, \quad (2.2)
\]

and so on for \( A, B, C = 1, \ldots, 2M \). On the other hand, there is no distinction between upper and lower greek indices and we will always choose them to be lower indices.
A Hermitian manifold $(\mathcal{M}, g)$ is a $2M$-dimensional Riemannian manifold endowed with a rank $(1,1)$ tensor field $J$ that is defined globally on $\mathcal{M}$ and is called a complex structure. On a Hermitian manifold, we can always make a coordinate transformation from $(\phi^A)$ to $(x^a, y^a)$ that satisfies, for each point $p \in \mathcal{M}$,

$$3_p \left( \frac{\partial}{\partial x^a} \right) = \frac{\partial}{\partial y^a}, \quad 3_p \left( \frac{\partial}{\partial y^a} \right) = -\frac{\partial}{\partial x^a}, \quad (2.3)$$

where $a = 1, \ldots, M$. It is then natural to introduce complex coordinates $(z^{*a}, z^a)$ through

$$z^{*a} := x^a - iy^a, \quad z^a := x^a + iy^a, \quad a = 1, \ldots, M. \quad (2.4)$$

A NL$\sigma$M on a Hermitian manifold $(\mathcal{M}, g)$ is thus defined by the partition function

$$Z := \int D[z^*, z] e^{-S[z^*, z]}, \quad (2.5a)$$

$$S[z^*, z] := \frac{1}{2\pi} \int_r g_{a^*b^*}(z^*, z) \partial_{\mu} z^b \partial_{\mu} z^{*a}, \quad (2.5b)$$

where we have assigned to each point $r$ from the base space a point $p$ on the manifold $\mathcal{M}$ with the coordinates $(z^{*a}, z^a) \in \mathbb{C}^M$. Components of tensor fields on $\mathcal{M}$ have now two types of indices, holomorphic ($a, b, \ldots$) and antiholomorphic ones ($a^*, b^*, \ldots$). Holomorphic indices can be contracted with holomorphic components of the coordinates $z^a$ whereas antiholomorphic ones with $z^{*a}$, the complex conjugate of $z^a$. In the definition of the NL$\sigma$M on a Hermitian manifold, Eq. (2.5), the $M \times M$ components of the Hermitian metric tensor field $g$ of rank $(0,2)$ are represented by $(g_{a^*b^*})$. We reserve the latin alphabet to label the $M$ holomorphic and antiholomorphic coordinates $z^{*a}$ and $z^a$ where $a^*, a = 1, \ldots, M$, respectively, of $p \in \mathcal{M}$. The use of a capital latin letter on a Hermitian manifold, say $\mathcal{A}$, as an index denotes either an holomorphic ($\mathcal{A} = a$) or an antiholomorphic ($\mathcal{A} = a^*$) index.

The components of the metric tensor field $g_{AB}$ on the Hermitian manifold $(\mathcal{M}, g)$ are symmetric under the exchange of the indices and Hermitian

$$g_{AB} = g_{BA}, \quad (g_{AB})^* = g_{A^*B^*}. \quad (2.5c)$$

with $A = a$ or $a^*, B = b$ or $b^*$, $A^* = a^*$ or $a$, and $B^* = b^*$ or $b$ running from 1 to $M$. The components of $g^{-1}$ are denoted by $g^{AB}$ and satisfy

$$g_{AC}g_{CB} = A^\delta_B, \quad g^{AC}g_{CB} = A_\delta_B. \quad (2.6)$$

and so on for $A, B, C^* = 1, \ldots, M$.

A Kähler manifold $(\mathcal{M}, g)$ is a Hermitian manifold whose Hermitian metric can be derived from a Kähler potential $K(z^*, z)$,

$$g_{ab}(z^*, z) = \partial_{\rho} \partial_{\sigma} K(z^*, z), \quad (2.7)$$

with $a, b^* = 1, \ldots, M$. Not all Hermitian manifolds are Kählerian.

The Kähler manifolds corresponding to the Hermitian symmetric spaces $U(p + q)/[U(p) \times U(q)]$ for symmetry class A, $Sp(N)/U(N)$ for symmetry class C, and $SO(2N)/U(N)$ for symmetry class D are specified once the corresponding Kähler potentials have been constructed. To this end, we begin by parametrizing the complex Grassmannian $U(p + q)/[U(p) \times U(q)]$ for symmetry class A by the stereographic coordinates defined by the independent set

$$\{\varphi^a_A, \varphi_A^a, \ A = 1, \ldots, p, \ a = 1, \ldots, q \} \quad (2.8a)$$

of $p \times q$ complex-valued matrices $62, 63, 64$. With this coordinate system, the Kähler potential on $U(p + q)/[U(p) \times U(q)]$ is given by

$$K(\varphi, \varphi) = c \ln \det (I_q + \varphi^\dagger \varphi) \quad (2.8b)$$

where $c$ is an arbitrary constant (the scale of the metric) which is chosen to be unity in the following. (Another choice for $c$ just amounts to a rescaling of $t$.) The Kähler metric induced by the Kähler potential (2.8) is

$$g_{(AA)(Bb)}(\varphi^* \varphi) = Z_{ab} Y_{AB} \quad (2.9a)$$

where we have introduced $q \times q$ and $p \times p$ Hermitian matrices,

$$Z_{ab} := (I_q + \varphi^\dagger \varphi)^{-1}, \quad a, b = 1, \ldots, q, \quad (2.9b)$$

$$Y_{AB} := (I_p + \varphi^\dagger \varphi)^{-1}, \quad A, B = 1, \ldots, p. \quad (2.9c)$$

The cases of $Sp(N)/U(N)$ for symmetry class C and $SO(2N)/U(N)$ for symmetry class D are obtained by setting $p = q = N$ and by imposing the constraint

$$\varphi + \partial_\varphi T = 0 \quad (2.10)$$

on the complex Grassmannian $U(p + q)/[U(p) \times U(q)]$ where $\partial = -1$ for $Sp(N)$ (symmetry class C) and $\partial = +1$ for $SO(2N)$ (symmetry class D). Due to the constraint (2.10), independent entries of $\{\varphi^a_A, \varphi_A^a, \ A = 1, \ldots, N \}$ can be chosen to be those with the indices $A \leq a$ for $Sp(N)$ (symmetry class C) and $A < a$ for $SO(2N)$ (symmetry class D). In turn, the metric for $Sp(N)/U(N)$ (symmetry class C) and the metric for $SO(2N)/U(N)$ (symmetry class D) are given by

$$g_{(AA)(Bb)}(\varphi^* \varphi) = \alpha_{AA} \alpha_{Bb} S^\theta_{(AA)(Bb)} Z_{ab} Z_{AB}, \quad (2.11a)$$

where

$$\alpha_{AA} := 1 - \delta_{AA}/2, \quad (2.11b)$$

$$S^\theta_{(AA)(Bb)} := S^\theta_{(AA)} S^\theta_{(Bb)} \cdots, \quad (2.11c)$$

$$S^\theta_{(AA)} T_{\ldots A \ldots a} := T_{\ldots A \ldots a} - \partial T_{\ldots A \ldots a}, \quad (2.11d)$$

and

$$Z_{ab} := (I_N + \varphi^\dagger \varphi)^{-1}, \quad a, b = 1, \ldots, N. \quad (2.11e)$$

Observe that $Z$ in Eq. (2.9e) and $Y$ in Eq. (2.9c) are not independent anymore because $Y = Z^T$ when $p = q = N$ and the constraint (2.10) holds. By a slight abuse of notation we will sometimes refer to the complex Grassmannian case by $\partial = 0$. 


B. High-gradient operators

The seminal contribution by Altshuler, Krawtsov, and Lerner was to observe that the perturbative RG flow obeyed by the cumulants of the conductance necessarily involves an infinite number of composite operators and to identify a criterion for the selection of this infinite family of composite operators. The selection criterion is that the flow must be closed within a family of composite operators characterized by the most dominant scaling dimensions. Applied to the three cases at hand, we thus seek a family of composite operators that transform as scalars under reparametrization of the target manifold and contain as many derivatives as possible subject to the condition that their anomalous scaling dimension be dominant. We are led to the family of high-gradient operators

\[
T_{IJK\ldots(z^*, z)} \frac{\partial z^I_{\mu} \partial z^J_{\nu} \partial z^K_{\rho}}{2s} \ldots
\]  

(2.12)

where \( T_{IJK\ldots(z^*, z)} \) is a tensor on the target manifold with \( s \) holomorphic and antiholomorphic indices; we do not yet specify how the greek indices must be contracted. Although composite operators containing derivatives of the form \( \partial^nz \) (\( n \geq 2 \)) are generated by the RG flow for the cumulants of the conductance they are neither expected to contribute to the dominant RG anomalous dimensions. Applied to the three cases at hand, we thus contend with two independent tensors of rank \( (0,2s) \) is simpler for the Grassmannian since we only need to know that the high-gradient operators \( (2.12) \) as we shall see below.

As the simplest example of a high-gradient operator belonging to the family \( (2.12) \) is the Lagrangian \( g_{IJ} \partial^I z^J \) of the NLsM action. This is in fact the only scalar under both reparametrization of the target manifold and global rotations in the base space with \( s = 1 \) that belongs to the family \( (2.12) \). To see this, observe that the number of independent tensors with two indices on the target manifold is related to that of independent bilinear forms in the group (coset). For simple Lie groups (and their analogues for cosets), there is only one such bilinear form (the Casimir), and hence \( g_{IJ} \) is uniquely fixed. The scaling flow obeyed by \( g_{IJ} \partial^I z^J \) controls the scaling flow obeyed by the mean conductance, i.e., the first cumulant of the conductance once the replica limit has been taken.

Life already becomes more complicated if we seek all the high-gradient operators with \( s = 2 \) in the family \( (2.12) \). On a generic complex manifold more than \( 2 \) independent tensors of rank \( (0,4) \) are available,

\[
g_{IJKL}, \quad R_{IJKL}, \quad R_{IJMN} R^{MN}_{KL} \]  

(2.13)

and so on, where \( R_{IJKL} \) is the curvature (Riemann) tensor on the target manifold. Fortunately, the situation is simpler for the Grassmannian since we only need to contend with two independent tensors of rank \( (0,4) \),

\[
S^{(2)}_{(Aa)(bb)(Cc)(Dd)} = Z_{ab} Y_{BC} Z_{cd} Y_{DA}, \]  

(2.14a)

\[
S^{(1)}_{(Aa)(bb)(Cc)(Dd)} = Z_{ab} Y_{BA} Z_{cd} Y_{DC}. \]  

(2.14b)

[See Eq. (1.9).]

Observe that \( S^{(1)}_{AB} \) is nothing but the metric tensor \( g_{AB} \).

For symmetry class A and for given \( s \in \mathbb{N} \), all the independent tensors \( T_{IJK\ldots} \) of rank \( (0,2s) \) of the complex Grassmannian can be expressed by taking the products of the building blocks,

\[
S^{(l)}_{(Aa)(bb)(Cc)(Dd)} \ldots := Z_{ab} Y_{BC} Z_{cd} Y_{DE} \ldots Y_{ZA} \]  

(2.15a)

made of \( l \) pairs of \( Z \) and \( Y \), according to the decomposition

\[
T_{I_1J_1K_1\ldots} = \prod_i S^{(l_i)}_{I_iJ_iK_i\ldots} \quad \sum_i l_i = s. \]  

(2.15b)

For the cases of \( \text{Sp}(N)/U(N) \) (symmetry class C) and \( \text{SO}(2N)/U(N) \) (symmetry class D) we need to replace Eq. (2.15a) by

\[
S^{(l)}_{(Aa)(bb)(Cc)(Dd)} \ldots := \cdots \alpha_{CC} S^B_{Cc} Y_{BC} Z_{cd} \ldots . \]  

(2.15c)

Another simplification attached to the Kähler manifolds for symmetry classes A, C, and D is that contraction between \( S^{(l_i)} \) and the \( \partial^I \) 's reduces to a trace,

\[
S^{(l)}_{(Aa)(bb)(Cc)(Dd)} \ldots \partial_{\mu_1} Y_{Aa} \partial_{\mu_2} Y_{bb} \partial_{\mu_3} Y_{Cc} \ldots = \text{tr} \left( A^+_{\mu_1} A^-_{\mu_2} A^+_{\mu_3} \ldots \right), \]  

(2.16a)

where we have introduced

\[
A^+ = Y \partial_{\mu} \varphi, \quad A^- = Z \partial_{\mu} \varphi, \]  

(2.16b)

for the complex Grassmannian (symmetry class A). For \( \text{Sp}(N)/U(N) \) (symmetry class C) and for \( \text{SO}(2N)/U(N) \) (symmetry class D) one must impose the additional constraint

\[
(A^+)^T = -\partial A^+, \quad (A^-)^T = -\partial A^- . \]  

(2.16c)

Any member \( T \) of the family of tensor fields constructed in Eq. (2.15) has the remarkable property that it vanishes under the action of the covariant derivative on any one of its indices

\[
\nabla_T T_{IJKL\ldots} = \partial_T T_{IJKL\ldots} - \Gamma^Q_{PT} T_{QJKL\ldots} - \Gamma^Q_{PJT} T_{IQKL\ldots} - \cdots = 0. \]  

(2.17)

Here, \( \Gamma_{BC}^{\alpha} \) are the components of the Levi-Civita connection whose non-vanishing entries are given by

\[
\Gamma_{\alpha bc} = g^{ap\alpha} \partial_c g_{pb}, \quad \Gamma_{\alpha bc} = g^{ap\alpha} \partial_c g_{pb}. \]  

(2.18)

This is also true for the NLsM on the real Grassmannian \( O(p+q)/[O(p) \times O(q)] \) which is related to the symplectic symmetry class of Anderson localization, and for the cases of simpler target manifolds such as \( O(N)/O(N-1) \).
or \( \mathbb{C}P^{n+m-1|m} \) for which the tensor fields \( T \) are just products of the metric tensor. In all these cases, the fact that all high-gradient operators have vanishing covariant derivatives greatly simplifies the computation of their one-loop RG flows.

For a given number of derivatives, 2s, there are many degenerate operators of the type (2.12) with \( T_{ijk...} \) given by Eq. (2.15) that share the same anomalous scaling dimensions 2s at the metallic fixed point. They are specified by the choice of \( \{l_i\} \) in Eq. (2.15) as well as that of the greek indices in the \( \partial^\mu z^i \)'s. This degeneracy is, however, lifted in perturbation theory as we depart from the metallic fixed point.

We close this section by noting that it is more convenient to introduce the conformal indices + and −

\[ \partial_{\pm} = \partial_x \pm i \partial_y, \]

instead of the Euclidean indices \( \mu = x, y \) in 2d 2.11.

III. THE COVARIANT BACKGROUND FIELD METHOD

A. Kähler normal coordinates for NLσM

To perform the RG program, we divide the fields \((z^a, z)\) in the NLσM into slow \((\psi^*, \psi)\), and fast \((\pi^*, \pi)\), modes. To this end, we write

\[ z^{*a} = \psi^a + \pi^{*a}, \quad z^a = \psi^a + \pi^a, \]

where we require that \((\psi^*, \psi)\) satisfy the classical equations of motion. The fast modes \((\pi^*, \pi)\) represent the quantum fluctuations around the classical background \((\psi^*, \psi)\). They are integrated out, thereby generating an effective action for the slow modes \((\psi^*, \psi)\). In practice, this effective action is computed by order by order within an expansion, the so-called loop expansion, which is nothing but the cumulant expansion. One drawback with the separation (3.1) into fast and slow modes is that the perturbative expansion of the action in powers of the fast modes \((\pi^*, \pi)\) with respect to the action evaluated at \((\psi^*, \psi)\) need not transform covariantly under reparametrization of the manifold. This danger can be avoided by choosing Kähler normal coordinates (KNC) in terms of which we denote the fast modes by \((\omega^*, \omega)\). The KNC coordinates are defined by demanding the manifest covariance of the Kähler potential when expanded in terms of \((\omega^*, \omega)\) 35,66. The perturbative expansion of the action in powers of the KNC about the background field is then manifestly covariant, i.e., only covariant objects such as the metric, the Riemann tensor, the covariant derivative on the manifold, and so on appear in the expansion, as we shall shortly see. One advantage of this RG scheme, the covariant background field method, is that no redundant operator is induced upon renormalization. This is a very useful property when dealing with the mixing of the composite operators induced by an RG transformation. A related advantage is that there is no need to break the symmetry of the NLσM to tame IR divergences.

The relation between the generic parametrization \((\pi^*, \pi)\) and the KNC parametrization \((\omega^*, \omega)\) of the fast modes is non-linear. For examples,

\[ \partial_{\mu} (\psi^a + \pi^a) = \partial_{\mu} \psi^a + D_{\mu} w^a \]

\[ + \frac{1}{2} R^a_{bc} e^c \partial^a \psi^b \psi^c w^d \partial^a w^e + \cdots, \]

\[ g_{i,j}(\psi^* + \pi^*, \psi + \pi) = g_{i,j}(\psi^*, \psi) - R_{i,j,k,l} \partial^a w^k \partial^a w^l + \cdots. \]

As promised, the right-hand side is covariant under reparametrization as the pair \((D^a, D^a)\) denotes the holomorphic and antiholomorphic covariant derivatives

\[ D^a w^j = \nabla_{a w^j} \partial^a \psi^a \]

\[ = \partial^a w^j + \Gamma^a_{bj} w^b \partial^a \psi^a, \]

\[ D^a w^j = (D^a w^j)^*, \]

\[ = \nabla_{a w^j} \partial^a \psi^a \]

\[ = \partial^a w^j + \Gamma^a_{bj} w^b \partial^a \psi^a, \]

respectively, with the components

\[ R^a_{b c d} = -\partial^a \Gamma^a_{b c d}, \quad R^a_{b c d} = -\partial^a \Gamma^a_{b c d}, \]

\[ R_{E D A B} = g_{E C} R^C_{D A B} \] and so on of the curvature tensor. The expansion of the action in powers of the KNC is

\[ S[\psi^* + \pi^*, \psi + \pi] - S[\psi^*, \psi] = \]

\[ + \frac{1}{2\pi i} \int_{r} g_{i,j}(\psi^*, \psi) D_{\mu} w^j D^a w^i \]

\[ - \frac{1}{2\pi i} \int_{r} R_{i,j,k,l}(\psi^*, \psi) w^i \partial^a \psi^b \partial^a \psi^c \]

\[ - \frac{1}{2} w^i \partial^a \psi^b \partial^a \psi^c w^i - \frac{1}{2} \partial^a \psi^b \partial^a \psi^c w^i + \cdots. \]

B. Canonical form of the metric tensor and vielbeins

The covariant Gaussian expansion 35,66 about an extremum \((\psi^*, \psi)\) of the action can still be simplified further by performing a local \(GL(M, \mathbb{C})\) transformation of the tangent space at \((\psi^*, \psi)\) that brings the metric tensor at \((\psi^*, \psi)\) to the canonical form. To this end, introduce the fast degree of freedom \((\zeta^*, \zeta)\) by the linear transformation 57,68

\[ \zeta^{*a} := w^{*a} e^{*a}, \quad w^{*a} = \zeta^{*a} _{\hat{a}} \hat{a}^{*a}, \]

\[ \zeta_{\hat{a}} := w^a e^a, \quad w^a = \zeta^a _{\hat{a}} \hat{a}^a, \]

by demanding that

\[ g_{a,b}(\psi^*, \psi) w^a w^b = \eta_{a,b} \zeta_{ab}. \]
where the transformed metric is given by

\[ \eta_{\hat{a}\hat{b}} = \text{diag}(\mathbb{I}_{M_1}, -\mathbb{I}_{M_2}), \quad M_1 + M_2 = M. \] (3.6d)

The two background dependent matrices \((a\hat{a})\) and \((\hat{a}\epsilon^a)\) from \(\text{GL}(M, \mathbb{C})\) that implement this transformation in the antiholomorphic sector are inverse to each other. The same is true of the matrices \((a\hat{a})\) and \((\hat{a}\epsilon^a)\) in the holomorphic sector. These four matrices are called the vielbeins and must obey

\[ g_{\hat{a}\hat{b}} = \epsilon^a \, \eta_{\hat{a}\hat{b}} \epsilon^a = g_{\hat{a}\hat{b}}, \quad \eta_{\hat{a}\hat{b}} = \hat{g}_{\hat{a}\hat{b}}. \] (3.7)

by condition (3.6a). From now on, latin letters with a hat refer to the coordinates of the Kähler manifold in the vielbein basis (3.6). Under transformation (3.6) the covariant expansion (3.5) becomes

\[
S[\psi^* + \pi^* , \psi + \pi] - S[\psi^*, \psi] = + \frac{1}{2\pi i} \int r \eta_{\hat{a}\hat{b}} \hat{D} \mu \zeta^{\hat{a}} \hat{D} \nu^* \zeta^{\hat{a}} - \frac{1}{2\pi i} \int r R_{\nu^{(ijk)}} \left[ (\zeta^i \epsilon^j \epsilon^k)(\zeta^i \epsilon^j \epsilon^k) \partial_\mu \psi^j \partial_\mu \psi^j \right] - \frac{1}{2} \left( \partial_\mu \psi^k \partial_\mu \psi^k (\zeta^i \epsilon^j \epsilon^k) \partial_\mu \psi^j \right) + \cdots
\]

where yet another pair of covariant derivatives

\[
\hat{D}_\mu \zeta^{\hat{a}} = \partial_\mu \zeta^{\hat{a}} - \zeta^{\hat{a}} \epsilon^a (A_\mu)^a, \quad (3.9a)
\]

\[
\hat{D}_\mu \zeta^{\hat{a}} = \partial_\mu \zeta^{\hat{a}} - \hat{\epsilon}^b \epsilon^a (A_\mu)^a, \quad (3.9b)
\]

have been introduced. The field \(A_\mu\) is the \(U(M)\) gauge field defined through the \(U(M)\) spin connection \(\omega_{\mu c}^b\), \(\epsilon^a\) by

\[
\omega_{\mu c}^b := \epsilon^a \left( \partial_{\mu a} e^b - \Gamma_{a c}^r e^b \right), \quad (3.10a)
\]

\[
\omega_{\mu c}^b := \epsilon^a \left( \partial_{\mu a} e^b - \Gamma_{a c}^r e^b \right), \quad (3.10b)
\]

\[
\hat{a} (A_\mu)^b := \epsilon^a \omega_{\mu c}^b \partial_\mu \psi^c + \hat{a} \omega_{\mu c}^b \partial_\mu \psi^c. \quad (3.10c)
\]

C. One-loop beta functions

We are in position to integrate out the fast degrees of freedom for the complex Grassmannian, \(\text{SO}(2N)/U(N)\), and \(\text{Sp}(2N)/U(N)\). The partition function for slow and fast modes is

\[
Z := \int \mathcal{D}[\psi^*, \psi] \mathcal{D}[\zeta^*, \zeta] e^{-S[\psi^*, \psi; \zeta^*, \zeta]}. \quad (3.11a)
\]

The action \(S[\psi^*, \psi; \zeta^*, \zeta] := S[\psi^* + \pi^*, \psi + \pi]\) is separated into three contributions

\[
S[\psi^*, \psi; \zeta^*, \zeta] := S[\psi^*, \psi] + S_0[\zeta^*, \zeta] + S_1[\psi^*, \psi; \zeta^*, \zeta], \quad (3.11b)
\]

where \(S_0[\zeta^*, \zeta]\) is the free Gaussian part for the fast modes, and \(S_1[\psi^*, \psi; \zeta^*, \zeta]\) represents the interactions between the slow and fast modes. Integration over the fast mode \((\zeta^*, \zeta)\) is performed with the help of the cumulant expansion

\[
Z = \int \mathcal{D}[\psi^*, \psi] e^{-S[\psi^*, \psi] + \delta S[\psi^*, \psi]}, \quad (3.12a)
\]

\[
\delta S[\psi^*, \psi] := \langle S_I \rangle c = \frac{1}{2} \left[ \left( \langle S_I \rangle_c^2 \right) - \left( \langle S_I \rangle_c \right)^2 \right] + \cdots, \quad (3.12b)
\]

where

\[
\langle (\cdots) \rangle_0 := \frac{\int \mathcal{D}[\zeta^*, \zeta] e^{-S_0[\zeta^*, \zeta] (\cdots)}}{\int \mathcal{D}[\zeta^*, \zeta] e^{-S_0[\zeta^*, \zeta]}}.
\]

To evaluate the cumulant expansion, we introduce the Green function

\[
\langle \zeta^a(x) \zeta^b(y) \rangle_c = \eta^{ab} 2\pi t \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{e^{-ik(x-y)}}{k^2} = \eta^{ab} 2\pi t G_0(x-y), \quad (3.13)
\]

where \(\Lambda = a^{-1}\) is the UV cutoff. The leading contribution to the increment of the action of the slow mode \((\psi^*, \psi)\) resulting from integrating out the fast mode \((\zeta^*, \zeta)\) within the second-order cumulant expansion is

\[
\delta S[\psi^*, \psi] = -G_0(0) \int r R_{\alpha b} \psi^b \partial_\alpha \psi^a, \quad (3.14)
\]

with

\[
R_{\alpha b} := R_{\alpha c}^{\ast} \epsilon^a = R_{\alpha c}^{\ast} \epsilon^a b, \quad (3.15)
\]

the Ricci tensor.

For symmetry classes C and D, the Ricci tensor is proportional to the metric (the target space is an Einstein manifold) and given by

\[
R_{(C_c)(Ee)} = (N - \tau) g_{(C_c)(Ee)}, \quad (3.16)
\]
Thus, from Eq. (3.14), the one-loop beta functions in (2 + \epsilon)d for the NL\sigmaM coupling \beta(t) = dt/dl is

\[ \beta(t) = -\epsilon t + (N - \vartheta)t^2 + O(t^3). \] (3.17)

Here, dl is the infinitesimal rescaling of the ultraviolet cutoff, a \rightarrow ae^{-\epsilon dl} for which \Gamma(0) = dl/2\pi. This one-loop result agrees with the known three-, four-, and five-loop results.\textsuperscript{35,36,37} Another check comes from the group isomorphisms\textsuperscript{36,61}.

\[ U(2)/[U(1) \times U(1)] \simeq \text{SO}(4)/U(2) \simeq \text{Sp}(1)/U(1), \] (3.18a)

\[ U(4)/[U(1) \times U(3)] \simeq \text{SO}(6)/U(3), \] (3.18b)

\[ \text{Sp}(N)/U(N) \simeq \text{SO}(-2N)/U(-N). \] (3.18c)

One verifies that the beta functions match if the couplings are appropriately rescaled for isomorphisms (3.18a) and (3.18b) and with the additional substitution \( t \rightarrow -t \) for isomorphisms (3.18c). The nontrivial zeros of the beta function in symmetry classes A, C, and D are

\[ 0 < t^* = \frac{\epsilon}{N - \vartheta} + O(\epsilon^2). \] (3.19)

IV. ONE-LOOP RG FOR HIGH-GRADIENT OPERATORS

A. Strategy

We now turn to the computation of the dominant anomalous scaling dimension \( x_i^{(s)} \) associated to any high-gradient operator \( O_i^{(s)} \) of type \textsuperscript{(2,12)}, where the collective index \( i \) runs over any allowed choice for \( \{ t_i \} \) in Eq. (2.15), and over the greek indices in \( \partial_{\mu}z^I \partial_{\nu}z^J \partial_{\rho}z^K \cdots \). The one-loop anomalous scaling dimension \( x_i^{(s)} \) of \( O_i^{(s)} \) is extracted from the one-loop RG transformation law

\[ \left\langle \left[ O_i^{(s)}(z^*, z) \right]_{\zeta^2} \right\rangle = tdl\hat{\Delta}_{ij}^{(s)}(\psi^*, \psi). \] (4.1a)

Here, \( \left[ O_i^{(s)}(z^*, z) \right]_{\zeta^k} \) denotes all the terms that are of order \( k \) in \( \zeta^4 \) in the KNC expansion of \( O_i^{(s)}(z^*, z) \), and

\[ \langle \cdots \rangle = \left\langle e^{-S_i(\cdots)} \right\rangle / \langle e^{-S_i} \rangle \] (4.1b)

is evaluated up to first order in \( tdl \) [recall Eq. (3.12c)].

Once the mixing matrix

\[ \hat{\Delta}^{(s)} = \left( \hat{\Delta}_{ij}^{(s)} \right) \] (4.2a)

is known, its eigenvalues \( \{ \alpha_i^{(s)} \} \) yield the spectrum of anomalous scaling dimensions

\[ x_i^{(s)} = 2s - \alpha_i^{(s)}t \] (4.2b)

and the spectrum of anomalous scaling dimensions

\[ y_i^{(s)} := d - x_i^{(s)} \] (4.2c)

associated to the high-gradient operator \( O_i^{(s)} \). Any eigenoperator of Eq. (4.1) is relevant (marginal, irrelevant) when its eigenvalue \( y_i^{(s)} \) is positive (zero, negative). We are going to construct the eigenoperator of Eq. (4.1) with the largest eigenvalue of the mixing matrix \( \hat{\Delta}^{(s)} \) up to one-loop accuracy close to the metallic fixed point.

To this end, we need to expand \( O_i^{(s)}(z^*, z) \) in the KNC. By taking advantage of the fact that all the covariant derivatives of the tensor \( T_{IJ...}(z^*, z) \) vanish and that \( \langle [\partial z^I]_{\zeta^2} \rangle = 0 \), we infer that

\[ \left\langle [T_{IJ...}(z^*, z)\partial_{\mu}z^I\partial_{\nu}z^J \cdots]_{\zeta^2} \right\rangle = \left\langle [T_{IJ...}]_{\zeta^2} \right\rangle \partial_{\mu}\psi^{I}\partial_{\nu}\psi^{J} \cdots + T_{IJ...}(\psi, \psi^{*}) \left\langle [\partial_{\mu}z^I]_{\zeta^1} [\partial_{\nu}z^J]_{\zeta^1} \cdots \right\rangle. \] (4.3a)

Thus, all we need are

\[ \left\langle [\partial_{\mu}z^I]_{\zeta^1} [\partial_{\nu}z^J]_{\zeta^1} \right\rangle = +tdl\delta_{\mu\nu}\partial_{\rho}\psi^{k}\partial_{\tau}\psi^{l}R^{k}_{\rho^{l}j} \tau^{j}, \] (4.3b)

\[ \left\langle [\partial_{\mu}z^I]_{\zeta^1} [\partial_{\nu}z^J]_{\zeta^1} \right\rangle = -tdl\delta_{\mu\nu}\partial_{\rho}\psi^{k}\partial_{\tau}\psi^{l}R^{k}_{\rho^{l}j}, \] (4.3c)

\[ \left\langle [\partial_{\mu}z^I]_{\zeta^1} [\partial_{\nu}z^J]_{\zeta^1} \right\rangle = +tdl\delta_{\mu\nu}\partial_{\rho}\psi^{k}\partial_{\tau}\psi^{l}R^{l}_{k^{j}j}. \] (4.3d)
on the one hand and
\[
\left\langle \left[ T_{b_1 \cdots b_\ell} b_\ell^* \cdots b_2^* \right] \zeta^2 \right\rangle (z^*, z) = t dl^\dagger \sum_{i=1}^\ell R_i^\dagger a_i^* b_i^* T_{b_1 \cdots b_\ell} b_\ell^* \cdots b_2^* \cdots b_i^* b_i^* \cdots b_2^* (\psi^*, \psi)
\]
\[
= t dl^\dagger \sum_{i=1}^\ell R_k^\dagger a_i^* b_i^* T_{b_1 \cdots b_\ell -1 a b_\ell} b_\ell^* \cdots b_2^* (\psi^*, \psi),
\]
(4.3c)

on the other hand. Here, we are using the conformal indices defined in Eq. (2.19). Note also that since holomorphic \( \nabla_i \) and antiholomorphic \( \nabla_j \) covariant derivatives do not commute in general, we have two distinct representations for \( \left\langle [T_{IJK \cdots}] \zeta^2 \right\rangle \).

**B. RG transformation**

The RG transformation law (4.3) obeyed by any high-gradient operator for a NL\( \sigma \)M on the Hermitian symmetric spaces \( U(p+q)/[U(p) \times U(q)] \), \( SO(2N)/U(N) \), and \( Sp(2N)/U(N) \) becomes explicit with the help of the

(a) intra-trace formulæ
\[
\left\langle \text{tr} \left[ [(A_\mu^0)_{\zeta^1}]_\zeta \mathcal{M} [A_\nu^0]_{\zeta^1} \mathcal{N} \right] \right\rangle = -\frac{2}{2} t dl^\dagger \mu_{-\nu} \left[ \text{tr} (A_\mu^0 N A_\nu^0 \mathcal{M}) \right] + \text{tr} (A_\mu^0 \mathcal{M}) \text{tr} (A_\nu^0 N) + \text{tr} (A_\mu^0 \mathcal{M} A_\nu^0 N) \right],
\]
(4.4a)
\[
\left\langle \text{tr} \left[ [(A_\mu^0)_{\zeta^1}]_\zeta \mathcal{M} [A_\nu^-]_{\zeta^1} \mathcal{N} \right] \right\rangle = \frac{2}{2} t dl^\dagger \mu_{-\nu} \left[ \text{tr} (A_\mu^0 A_\nu^- N) \right] - \text{tr} (A_\mu^0 A_\nu^- \mathcal{M} \mathcal{N}) \right],
\]
(4.4b)

(b) inter-trace formulæ
\[
\left\langle \text{tr} \left[ [(A_\mu^0)_{\zeta^1}]_\zeta \mathcal{M} \right] \text{tr} \left[ [(A_\nu^0)_{\zeta^1}]_\zeta \mathcal{N} \right] \right\rangle = \frac{2}{2} t dl^\dagger \mu_{-\nu} \left[ \text{tr} (A_\mu^0 N A_\nu^0 \mathcal{M}) \right] - \text{tr} (A_\mu^0 N A_\nu^0 \mathcal{M}) \right],
\]
(4.5a)
\[
\left\langle \text{tr} \left[ [(A_\mu^0)_{\zeta^1}]_\zeta \mathcal{M} \right] \text{tr} \left[ [(A_\nu^-)_{\zeta^1}]_\zeta \mathcal{N} \right] \right\rangle = \frac{2}{2} t dl^\dagger \mu_{-\nu} \left[ \text{tr} (A_\mu^0 A_\nu^- \mathcal{N}) \right] - \text{tr} (A_\mu^0 A_\nu^- \mathcal{M} \mathcal{N}) \right],
\]
(4.5b)

and (c) diagonal contributions
\[
\left\langle \left[ T_{a_1 \cdots a \delta_1 \cdots \delta_1^\ast \delta_2^*} \right] \zeta^2 \right\rangle = -t dl s (N - \vartheta) T_{a_1 \cdots a \delta_1 \cdots \delta_1^\ast \delta_2^*} (\psi^*, \psi),
\]
(4.6)

where \( \mathcal{M} \) and \( \mathcal{N} \) represent arbitrary rectangular matrices of the proper size and the matrix-valued \( A_{\mu}^0 \) were defined in Eq. (2.19). This result, that follows from substituting the components of the curvature tensor (2.8) when expressed in the stereographic coordinates (2.8) into the master formula (4.3), agrees with the one-loop RG result for symmetry class A (\( \vartheta = 0 \)) in Ref. [38].

**C. Diagonalization of the RG equation within the subspace of maximal number of switches**

As is shown in Appendix B, the RG transformation law (4.3) has an upper (or lower) triangular structure when appropriate quantum numbers (“number of switches”) are defined. Since the number of switches never increases under an RG transformation, we can solve the eigenvalue and eigenvector problem (4.3) for each subspace with a fixed number of switches. We will limit ourselves to operators with the maximal number of switches, as the eigenscaling operator with the most dominant RG anomalous scaling dimension is known to be among them for symmetry class A [39]. In this subspace, any operator can be expressed as a polynomial of the ob-
\begin{equation}
\left\{ \text{tr} \left[ (A_{\uparrow \uparrow}^s A_{\downarrow \downarrow}^s)^{-1} \right], \text{tr} \left[ (A_{\uparrow \downarrow}^s A_{\downarrow \uparrow}^s)^{-1} \right] \right\}_{l=1,2,\ldots} \tag{4.7}
\end{equation}

i.e., conformal indices in a single trace are completely alternating. It can also be shown that the spaces spanned by \{ \text{tr} \left[ (A_{\uparrow \uparrow}^s A_{\downarrow \downarrow}^s)^{-1} \right] \}_{l=1,2,\ldots} \text{ and } \{ \text{tr} \left[ (A_{\uparrow \downarrow}^s A_{\downarrow \uparrow}^s)^{-1} \right] \}_{l=1,2,\ldots} \text{ are not mixed by the RG transformation. We can thus consider these two subspaces separately as long as we do not require composite operators to be Hermitian.}

We now compute the action of the RG transformation \((4.3)\) on a composite operator of the form

\begin{equation}
\mathcal{O}_{(r')}^{(s)}(z^*, \bar{z}) = \Omega_1^{(s)} \Omega_2^{(s)} \cdots \Omega_L^{(s)} \sum_{l=1}^{L} \Im r_l = s \tag{4.8}
\end{equation}

where all \(\Omega_l\) on the right-hand side are a short-hand notation for objects of type \(\text{tr} \left[ (A_{\uparrow \downarrow}^s A_{\downarrow \uparrow}^s)^{-1} \right] \) without loss of generality. Since the diagonal contributions \((c)\) to the RG eigenvalues are trivial, we focus on the intra-trace contributions \((a)\) and inter-trace contributions \((b)\) for the moment. If we define the off-diagonal mixing matrix \(\hat{R}^{(s)} = \left( \hat{R}^{(s)}_{(r')}(r') \right) \) by

\begin{equation}
\left\langle \mathcal{O}_{(r')}^{(s)}(z^*, \bar{z}) \right\rangle_{(a),(b)} = \text{td} \hat{R}^{(s)}_{(r')(r')} \mathcal{O}_{(r')}^{(s)}(\psi^*, \psi), \tag{4.9}
\end{equation}

i.e., we have only collected the off-diagonal contributions \((4.4)\) and \((4.5)\) to the RG transformation law obeyed by \(\mathcal{O}_{(r')}^{(s)}(z^*, \bar{z})\), the action of \(\hat{R}^{(s)}\) can be cast into the form

\begin{equation}
\hat{R}^{(s)} = -\partial \sum_{k} k^2 \Omega_k \frac{\partial}{\partial \Omega_k} + N \sum_{k} k \Omega_k \frac{\partial}{\partial \Omega_k} + \sum_{l,n} \left[ 2 l n \Omega_{l+n} \frac{\partial}{\partial \Omega_l} \frac{\partial}{\partial \Omega_n} \right. \right.
+ (2 - \partial^2)(l + n) \Omega_{l+n} \frac{\partial}{\partial \Omega_{l+n}} \tag{4.10}
\end{equation}

The eigenvalue problem \((4.10)\) can be solved by brute force for small \(s\) in which cases we have verified that the sign of the eigenvalues of the off-diagonal mixing matrix \(\hat{R}^{(s)}\) for \(\text{SO}(2N)/U(N)\) are opposite to those for \(\text{Sp}(N)/U(N)\). In particular, the largest eigenvalue of \(\hat{R}^{(s)}\) for \(\text{SO}(2N)/U(N)\) is given by minus the smallest eigenvalue for \(\text{Sp}(N)/U(N)\) and vice versa. This fact is consistent with the group isomorphism \((3.15b)\), and the fact that the beta functions for the corresponding \(\text{NL}\sigma\text{M}\) match upon the replacement \(t \rightarrow -t\).

More generally, one verifies that

\begin{equation}
\lambda^{(s)}(\theta = 0) = 2(s^2 - s) + s(p + q), \tag{4.11a}
\end{equation}

\begin{equation}
\lambda^{(s)}(\theta = +1) = s^2 - 2s + sN, \tag{4.11b}
\end{equation}

\begin{equation}
\lambda^{(s)}(\theta = -1) = 2s^2 - s + sN, \tag{4.11c}
\end{equation}

is an eigenvalue of the off-diagonal mixing operator \(\hat{R}^{(s)}\) for any given \(s\) with the eigenvector

\begin{equation}
u^{(s)} = \sum_{l} \prod_{j} \frac{1}{r_j l_{j}} (\Omega_j)^{r_j}, \quad \theta = 0, \tag{4.12a}
\end{equation}

\begin{equation}
u^{(s)} = \sum_{l} \prod_{j} \frac{1}{r_j l_{j}} (\Omega_j)^{r_j}, \quad \theta = +1, \tag{4.12b}
\end{equation}

\begin{equation}
u^{(s)} = \sum_{l} \prod_{j} \frac{2j-1}{r_j l_{j}} (\Omega_j)^{r_j}, \quad \theta = -1. \tag{4.12c}
\end{equation}

As it is known that \(\lambda^{(s)}\) is the largest eigenvalue of \(\hat{R}^{(s)}\) in symmetry class A, we believe this to be also true for symmetry classes C and D. If so, the largest RG eigenvalues for a given number \(s\) of gradients is given by, after adding the diagonal contributions \((c)\),

\begin{equation}
\alpha^{(s)} = \lambda^{(s)} - s(p + q) = 2(s^2 - s), \quad \theta = 0, \tag{4.13a}
\end{equation}

\begin{equation}
\alpha^{(s)} = \lambda^{(s)} - s(N - 1) = s^2 - s, \quad \theta = +1, \tag{4.13b}
\end{equation}

\begin{equation}
\alpha^{(s)} = \lambda^{(s)} - s(N + 1) = 2s^2 - s, \quad \theta = -1. \tag{4.13c}
\end{equation}

Note that these results are valid up to one loop before taking the replica limit. Equation \((1.3)\) then follows by combining Eq. \((4.13)\) with Eq. \((4.2)\).

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APPENDIX A: GROUP-THEORY VERSUS GEOMETRIC REPRESENTATIONS OF THE NL\sigma\text{M}

The purpose of this appendix is to bridge the geometrical representation of the \(\text{NL}\sigma\text{M}\) used in this paper and the one in terms of the so-called “Q-matrix”.

First, recall that an element \(Q \in U(p+q)/[U(p) \times U(q)]\) can be written as\(^{19}\)

\[ Q = U^{-1} A U, \quad A = \text{diag}(i_p, -i_q), \quad U \in U(p + q). \tag{A1} \]

As noticed by Lerner and Wegner,\(^{22}\) high-gradient operators in the Q-matrix representation as well as the action of the \(\text{NL}\sigma\text{M}\) can be conveniently expressed in terms of the objects

\[ A_{\mu}^\tau := \frac{1}{2} P^{\tau \tau} \left( \partial_\mu Q \right) P^{-\tau}, \quad (A2a) \]
where $\tau = \pm$ and the projectors $P^\pm$

$$P^\pm := \frac{1}{2} (1 \pm Q), \quad (P^\pm)^2 = P^\pm \quad (A2b)$$

were introduced. A generic high-gradient operator is then built by taking a product of

$$\text{tr} \left( A^+_\mu A^-_\nu A^+_{\beta\rho} \cdots \right). \quad (A3)$$

For example, the action of the NLσM is

$$S = \frac{1}{16\pi t} \int \text{tr} \left( \left( \partial_\mu Q \right) \left( \partial_\nu Q \right) \right) = \frac{1}{4\pi t} \int \text{tr} \left( A^+_\mu A^-_\mu \right). \quad (A4)$$

In order to establish explicitly the connection between the $Q$-matrix representation (A2)–(A4) and the geometrical representation in the stereographic coordinates (2.8), we parametrize $P^+$ and $P^- = P^+ - Q$ in Eq. (A2b) through a $(p + q) \times q$ matrix $\Phi_{\alpha i}$ according to

$$P^+_{\alpha\beta} = (\Phi \Phi^\dagger)_{\alpha\beta} = \sum_{i=1}^{q} \Phi_{\alpha i} (\Phi^\dagger)_{i\beta} = \sum_{i=1}^{q} \Phi_{\alpha i} \Phi^*_{\beta i} \quad (A5a)$$

where $\Phi$ satisfies the constraint

$$\Phi^\dagger \Phi = I_q. \quad (A5b)$$

The matrix $\Phi$ is thus a collection of $q$ orthonormal vectors $(\Phi_\alpha \in \mathbb{C}^{p+q})_{\alpha = 1, \ldots, q}$. We can use this constraint to fix the upper $q \times q$ block of $\Phi$. We write

$$\Phi = \left( \begin{array}{c|c} \varphi X & \d_1 \\ \hline \varphi X^T & 1_p \end{array} \right) \quad (A6a)$$

where $\Phi_{\alpha i}, \varphi_{ij}, X_{ij}$, and $\varphi_{ij}, X_{jk}$, are $(p + q) \times q$, $p \times q$, $q \times q$, and $p \times p$, matrices, respectively, and the indices $i, j, I$, and $\alpha$ run over the sets $i = 1, \ldots, q$, $I = 1, \ldots, p$, $\alpha = 1, \ldots, p + q$, respectively. Next, we express $X$ in terms of $\varphi$ by making use of the constraint (A5a),

$$X_{ni} Z_{mi} X_{mk} = \delta_{ik} \left( X^\dagger Z^{-1} X = I_q \right). \quad (A6b)$$

Here the Hermitian $q \times q$ matrix $Z$ was introduced in Eqs. (2.9b) and (2.9c). Constraint (A6b) is satisfied if $X$ is chosen to be

$$X^\dagger = X = Z^{1/2}. \quad (A6c)$$

A generic high-gradient operators (A3) in the $Q$-matrix representation is thus expressed in terms of the stereographic coordinate $(\varphi^* \varphi)$ from Eq. (2.8). For example, one verifies that the action of the NLσM (A4) reduces to Eq. (2.5) with the metric (2.9a) derived from the Kähler potential (2.8). Once the parametrization (A5) with Eq. (A6) is used. Similarly,

$$\text{tr} \left( A^+_\mu A^-_\nu A^+_{\beta\rho} \cdots \right) = \text{tr} \left[ \left( \partial_\mu \varphi^* \right) Y \left( \partial_\nu \varphi \right) Z \left( \partial_\rho \varphi^* \right) Y \left( \partial_\sigma \varphi \right) \right]. \quad (A7)$$

Observe that the alternating structure in $A^+$ and $A^-$ in the high-gradient operators has a one-to-one correspondence to that in $Z$ and $Y$.

### APPENDIX B: UPPER TRIANGULAR STRUCTURE OF THE RG TRANSFORMATION

We prove in this appendix that the RG transformation law obeyed by high-gradient operators has an upper (or lower) triangular structure when appropriate “quantum” numbers (“number of switches”) are defined. A trace made of an even product of $A$’s,

$$\text{tr} \left( A^+_{\sigma_1} A^-_{\sigma_2} A^+_{\sigma_3} A^-_{\sigma_4} \cdots \right), \quad (B1)$$

is said to have $n^+\tau$ number of switches from $A^+_{\sigma_1}$ to $A^-_{\sigma_2}$ when, dividing the sequence into pairs according to

$$\text{tr} \left[ \left( A^+_{\sigma_1} A^-_{\sigma_2} \right) \left( A^+_{\sigma_3} A^-_{\sigma_4} \right) \cdots \right], \quad (B2)$$

there are $n^+\tau$ pairs $(A^+_{\sigma_1} A^-_{\sigma_2})$ in the sequence. The number $n^+\tau$ does not depend on the cyclic permutations of the pairs $(A^+_{\sigma_1} A^-_{\sigma_2})$. For a given trace of the form (B2) we can define

$$n_{tr} := \sum_{\tau, \sigma = \pm} n^+\tau. \quad (B3)$$

We are going to show that, for the symmetry classes A, C, and D, the total number

$$n := \sum_{tr} n_{tr} \quad (B4)$$

of switches in a composite operator of type (2.12) never increases under the RG transformation law (4.3). Here, the summation extends over the product over all traces of the form (B2) making up the high-gradient operator. In other words, the RG transformation law (4.3) is triangular with respect to the quantum number $\vartheta$.

#### 1. Symmetry class A

We begin with symmetry class A. This case ($\vartheta = 0$) was already treated in Refs. 38, 39, 43 and is needed for symmetry classes C ($\vartheta = -1$) and D ($\vartheta = +1$). Consider any string $L_i A^+_{\sigma_1} A^-_{\sigma_1} R_1$ and any string $L_2 A^+_{\sigma_2} A^-_{\sigma_2} R_2$ of 4 matrices each that enter a high-gradient operator.

First, we assume that each string belongs to two distinct traces. We seek the one-loop RG transformation law obeyed by the number of switches

$$\delta_{\sigma_1, -\sigma_1'} + \delta_{\sigma_2, -\sigma_2'} \quad (B5a)$$

contained in

$$\text{tr} \left( L_1 A^+_{\sigma_1} A^-_{\sigma_1} R_1 \right) \text{tr} \left( L_2 A^+_{\sigma_2} A^-_{\sigma_2} R_2 \right). \quad (B5b)$$

By making use of Eq. (4.5) with $\vartheta = 0$ we deduce that the
one-loop RG transformation law obeyed by Eq. (B5b) is
\[
\left( \text{tr} \left( \left[ A_{\sigma_1}^+ A_{\sigma_1}^- \right] \right) \left| R_1 L_1 \right) \right) \left( \text{tr} \left( \left[ A_{\sigma_2}^+ A_{\sigma_2}^- \right] \right) \left| R_2 L_2 \right) \right) =
\text{tdl} \left( -\delta_{\sigma_1,-\sigma_2} + \delta_{\sigma_1,-\sigma_1} + \delta_{\sigma_1,-\sigma_2} - \delta_{\sigma_1,-\sigma_1} \right) 
\times \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_1}^- R_2 L_2 A_{\sigma_2}^+ A_{\sigma_2}^- R_1 L_1 \right)
\] (B6a)
\[-\text{tdl} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( R_2 L_2 A_{\sigma_1}^+ A_{\sigma_2}^- A_{\sigma_1}^- R_1 L_1 \right)
\] (B6b)
\[+\text{tdl} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_2}^- R_2 L_2 A_{\sigma_2}^+ A_{\sigma_1}^- R_1 L_1 \right)
\] (B6c)
\[-\text{tdl} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( A_{\sigma_2}^+ A_{\sigma_1}^- R_2 L_2 A_{\sigma_2}^+ A_{\sigma_2}^- R_1 L_1 \right)
\] (B6d)
\[-\text{tdl} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( A_{\sigma_2}^+ A_{\sigma_1}^- R_2 L_2 A_{\sigma_2}^+ A_{\sigma_2}^- R_1 L_1 \right)
\] (B6e)

Line (B6a) contains \( \delta_{\sigma_1,-\sigma_2} + \delta_{\sigma_2,-\sigma_1} \) switches. Lines (B6b) to (B6e) contain \( \delta_{\sigma_1,-\sigma_2} + \delta_{\sigma_2,-\sigma_1} \) switches. Hence, only line (B6a) has the potential to increase the initial number (B5a) of switches. However, an increase over the initial number of switches is not possible in view of the identity 
\[-\delta_{\sigma_1,-\sigma_2} + \delta_{\sigma_1,-\sigma_1} + \delta_{\sigma_2,-\sigma_2} - \delta_{\sigma_1,-\sigma_1} = 2\sigma_1 \sigma_2 \delta_{\sigma_1,-\sigma_2} \delta_{\sigma_2,-\sigma_1} \] Line (B6a) only contributes if \( \sigma_1' = -\sigma_1 \) and \( \sigma_2' = -\sigma_2 \) both hold, i.e., when the initial number of switches (B5a) is maximum (2). Consequently, the number of switches \( 2\delta_{\sigma_1,\sigma_2} \) induced by line (B6a) equals the initial number of switches 2 when \( \sigma_1 = \sigma_2 \) while it decreases otherwise.

Second, we assume that each string belongs to the same trace. We seek the one-loop RG transformation law obeyed by the number of switches 
\[\delta_{\sigma_1,-\sigma_2} + \delta_{\sigma_2,-\sigma_1} \] contained in
\[
\text{tr} \left( L_1 A_{\sigma_1}^+ A_{\sigma_1}^- L_2 R_1 L_1 A_{\sigma_2}^+ A_{\sigma_2}^- R_2 L_2 \right).
\] (B7b)

By making use of Eq. (4.3) with \( \vartheta = 0 \) one verifies again that the number of switches \( n_\tau^\varphi \) cannot increase after integrating over the fast modes up to one loop.

We have proven that the RG flows of high-gradient operators made of any given number of switches decouple from the RG flows of high-gradient operators made of a lesser number of switches in symmetry class A. As a corollary, the mixing matrix (4.3) is triangular in symmetry class A.

2. Symmetry classes C and D

We turn our attention to symmetry classes \( |\vartheta| = 1 \), i.e., SO(2N)/U(N) and Sp(N)/U(N). We first consider the inter-trace formula of the RG transformation (4.5) applied to
\[\cdots \times \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_1}^- | M \right) \times \text{tr} \left( A_{\sigma_2}^+ A_{\sigma_2}^- | N \right) \times \cdots \) (B8)

where \( M := R_1 L_1 \) and \( N := R_2 L_1 \). It is sufficient to consider the contributions proportional to \( \vartheta \) in Eq. (4.5) as the remaining terms can be treated along the lines of symmetry class A. There are four such contributions (a) – (d):
\[\delta_{\sigma_2,-\sigma_2} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( A_{\sigma_2}^+ A_{\sigma_2}^- A_{\sigma_1}^+ A_{\sigma_1}^- M \right), \] (B9a)
\[\delta_{\sigma_1,-\sigma_1} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_1}^- A_{\sigma_2}^+ A_{\sigma_2}^- M \right), \] (B9b)
\[\delta_{\sigma_1,-\sigma_1} \delta_{\sigma_2,-\sigma_2} \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_2}^- A_{\sigma_2}^+ A_{\sigma_1}^- M \right), \] (B9c)
\[\delta_{\sigma_2,-\sigma_2} \delta_{\sigma_1,-\sigma_2} \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_2}^- A_{\sigma_1}^+ A_{\sigma_2}^- M \right), \] (B9d)

We assign to
\[\mathcal{A} := \text{tr} \left( A_{\sigma_1}^+ A_{\sigma_1}^- | M \right) \times \text{tr} \left( A_{\sigma_2}^+ A_{\sigma_2}^- | N \right) \] (B10)
the quartet of numbers
\[\left( n_+^+, n_+^-, n_+^-, n_- \right) \] (B11)
where \( n_\tau^\varphi \) counts the total number of switches of type \( A_{\sigma_1}^+ A_{\sigma_2}^- \) in \( \mathcal{A} \). In symmetry classes D and C, we can combine \( \text{tr} \mathcal{O} = \text{tr} \mathcal{O}^T \) and the symmetry \( (A_{\sigma_1}^\mu)^T = -\vartheta A_{\sigma_1}^\mu \), to infer that
\[\left( n_+^+, n_+^-, n_+^-, n_- \right) = \left( n_-^-, n_-^+, n_-^+, n_- \right). \] (B12)

As opposed to symmetry class A, individual switches \( n_\tau^\varphi \) are not separately conserved but the total number of switches
\[n = \sum_{\sigma, \tau} n_\tau^\varphi \] (B13)
will be shown to be conserved.

Without loss of generality we choose \( \tau = + \in A \). The total number of switches \( n_+^\varphi \) and \( n_+^\varphi \) are
\[n_+^+ = n_+^+ \left[ A_{\sigma_1}^+ A_{\sigma_1}^- + n_+^+ | M \right] + n_+^+ \left[ A_{\sigma_2}^+ A_{\sigma_2}^- + n_+^+ | N \right], \] (B14a)
\[n_-^- = n_-^- \left[ A_{\sigma_1}^+ A_{\sigma_1}^- + n_-^- | M \right] + n_-^- \left[ A_{\sigma_2}^+ A_{\sigma_2}^- + n_-^- | N \right], \] (B14b)
respectively. The total number of switches \( n_+^- \) and \( n_-^- \) follow from using \( \mathcal{A} = \text{tr} \left( A_{\sigma_1}^+ \mathcal{M} A_{\sigma_1}^+ \right) \times \text{tr} \left( A_{\sigma_2}^\varphi \mathcal{N} A_{\sigma_2}^\varphi \right) \) and are given by
\[n_+^- = n_-^- \left[ A_{\sigma_1}^+ \mathcal{M} A_{\sigma_1}^+ + n_-^- | M \right] \mathcal{M} A_{\sigma_1}^+ \right) \times \left[ A_{\sigma_2}^+ \mathcal{N} A_{\sigma_2}^+ \right] \] (B14c)
\[n_-^- = n_-^- \left[ A_{\sigma_1}^+ \mathcal{M} A_{\sigma_1}^+ + n_-^- | M \right] \mathcal{N} A_{\sigma_1}^+ \right) \times \left[ A_{\sigma_2}^+ \mathcal{N} A_{\sigma_2}^+ \right] \] (B14d)
respectively.
The number of switches \((B14)\) in \(\mathcal{A}\) changes under the RG transformation \((B9)\) as follows. The total number of switches \(n^+_\pm\) and \(n^-_\pm\) from line \((B9a)\) is given by

\[
\begin{align*}
n^+_\pm[(a)] &= n^+_\pm[A^\pm_{\sigma_2}N^T A^\pm_{\sigma_2}] + n^+_\pm[A^\pm_{\sigma_1} A^\pm_{\sigma_1}] + n^+_\pm[\mathcal{M}] \\
n^-\pm[(a)] &= n^-\pm[A^\pm_{\sigma_2} N A^\pm_{\sigma_2}] + n^+_\pm[A^\pm_{\sigma_1} A^\pm_{\sigma_1}] + n^+_\pm[\mathcal{M}],
\end{align*}
\]

respectively, where we again made use of the symmetry condition \((A^\pm_{\mu})^T = -\sigma^\pm_{\mu}\), and the fact that the number of \(A^\pm_{\mu}\) in a trace is always even. Similarly, the total number of switches \(n^+_\pm\) and \(n^-_\pm\) from line \((B9a)\) is given by

\[
\begin{align*}
n^+_\pm[(a)] &= n^+_\pm[A^\pm_{\sigma_2}N^T A^\pm_{\sigma_2}] + n^+_\pm[A^\pm_{\sigma_1} A^\pm_{\sigma_1}] + n^+_\pm[\mathcal{M}] \\
n^-\pm[(a)] &= (\text{as above with } \sigma \to -\sigma),
\end{align*}
\]

respectively. Thus, the total number of switches changes from

\[
n = \sum_{\sigma} \left\{ n^+_{\sigma}[A^+_{\sigma_1} A^-_{\sigma_1}] + n^+_{\sigma}[\mathcal{M}] + n^+_{\sigma}[A^+_{\sigma_2} A^-_{\sigma_2}] \right\}
\]

\[
+ n^+_\sigma[N] + n^+_\sigma[A^-_{\sigma_1} A^+_{\sigma_1}] + n^+_\sigma[A^-_{\sigma_2} N A^+_{\sigma_2}],
\]

\[
(B15c)
\]

to

\[
n[(a)] = \sum_{\sigma} \left\{ n^-_{\sigma}[A^-_{\sigma_1} N A^+_{\sigma_2}] + n^+_{\sigma}[A^+_{\sigma_1} A^-_{\sigma_1}] + n^+_{\sigma}[\mathcal{M}] \right\}
\]

\[
+ n^-_\sigma[N] + n^+_\sigma[A^-_{\sigma_2} A^+_{\sigma_2}] + n^+_\sigma[A^-_{\sigma_1} A^+_{\sigma_1}],
\]

\[
(B17)
\]

for the contribution from line \((B9a)\). The net change \(\Delta n_{(a)} = n - n_{[(a)]}\) is

\[
\Delta n_{(a)} = \sum_{\sigma} \left\{ n^+_{\sigma}[A^+_{\sigma_2} A^-_{\sigma_1}] - n^-_{\sigma}[A^-_{\sigma_2} A^+_{\sigma_1}] \right\}
\]

\[
+ n^+_\sigma[A^-_{\sigma_1} M A^+_{\sigma_2}] - n^+_\sigma[A^-_{\sigma_2} M A^+_{\sigma_1}],
\]

\[
(B18)
\]

When \(\delta_{\sigma_2, -\sigma_2} \delta_{\sigma_1, \sigma_2} = 1\), we find

\[
\Delta n_{(a)} = \sum_{\sigma} \left\{ n^+_{\sigma}[A^+_{\sigma_2} A^-_{\sigma_1}] - n^-_{\sigma}[A^-_{\sigma_2} A^+_{\sigma_1}] \right\}
\]

\[
+ n^+_\sigma[A^-_{\sigma_1} M A^+_{\sigma_2}] - n^+_\sigma[A^-_{\sigma_2} M A^+_{\sigma_1}],
\]

\[
= 0
\]

\[
(B19)
\]

with the help of the prefactor \(\delta_{\sigma_2, -\sigma_2} (\delta_{\sigma_1, \sigma_2} - \delta_{\sigma_1, -\sigma_2})\). When \(\delta_{\sigma_2, -\sigma_2} \delta_{\sigma_1, \sigma_2} = 1\), we find

\[
\Delta n_{(a)} = \sum_{\sigma} \left\{ n^+_{\sigma}[A^+_{\sigma_2} A^-_{\sigma_1}] - n^-_{\sigma}[A^-_{\sigma_2} A^+_{\sigma_1}] \right\}
\]

\[
+ n^+_\sigma[A^-_{\sigma_1} M A^+_{\sigma_2}] - n^+_\sigma[A^-_{\sigma_2} M A^+_{\sigma_1}] \}
\]

\[
\geq 0.
\]

\[
(B20)
\]

We conclude that the total number of switches does not increase under the RG transformation \((B9a)\).

Repeating the same analysis for the contributions \((b) - (d)\) in Eq. \((B9)\) yields that the change in the total number of switches is \(\geq 0\).

We leave it to the reader to verify that the same conclusion holds for the intra-trace formula \((B4)\).

This completes the proof for the upper triangular structure of the RG equation with respect to the total number of switches in symmetry classes D and C.

APPENDIX C: RG FLOWS OF HIGH-GRADIENT OPERATORS FOR NL\(\sigma\)M ON SUPERMANIFOLDS

We derive in this appendix two “master formulae” needed to perform the one-loop RG analysis of high-gradient operators for NL\(\sigma\)M defined on an arbitrary Riemannian or Kählerian supermanifold. Master formulae \((C19)\) and \((C40)\) are supersymmetric versions of Eq. \((4.3)\) and can be used for any target supermanifold once the geometrical data of the manifold are known. An expansion in terms of normal coordinate of tensor fields plays again an essential role. For the non-supersymmetric Riemannian case, an expansion in terms of Riemann normal coordinate (RNC) based on the notion of geodesics was used for the computation of the beta function in all generality \(22,25\) and the RG analysis of high-gradient operators for the special case of the \(O(N)/O(N - 1)\) NL\(\sigma\)M \(24,45\). The RNC expansion can also be applied to Hermitian manifolds. For the (non-supersymmetric) Kählerian case, however, one can also use the Kähler potential to find a normal coordinate system, the Kähler normal coordinates (KNC) introduced in Refs. \(62\) and \(63\). In this Appendix, we will generalize these normal coordinate expansions to the case of supersymmetric target manifolds. All the results that follow reduce to the non-supersymmetric ones if we ignore the commuting-anticommuting nature of the objects and fermionic sign factors.

1. NL\(\sigma\)M on Riemannian supermanifolds

a. Notations and conventions for Riemannian supermanifolds  We begin with a quick review of concepts and
notation. We refer the reader to Ref. 59 for a textbook on the geometry of supermanifolds.

- A tensor field of rank \((r, s)\) on a supermanifold has \(r\) contravariant (upper) and \(s\) covariant (lower) indices, \(T^a\). On a supermanifold, components of tensor fields can be either commuting or anti-commuting supernumbers.

- In order to define their mutual statistics, the grade \(\epsilon(\mathcal{O}) = 0, 1\) is assigned to each object \(\mathcal{O}\) on a supermanifold, where \(\mathcal{O}\) can be a supernumber, a super tensor field, a component of a supertensor field, etc. When the ordering of two objects \(\mathcal{O}_1\) and \(\mathcal{O}_2\) is exchanged, a factor \((-1)^{\epsilon(\mathcal{O}_1)\epsilon(\mathcal{O}_2)}\) arises, \(\mathcal{O}_1\mathcal{O}_2 = (-1)^{\epsilon(\mathcal{O}_1)\epsilon(\mathcal{O}_2)}\mathcal{O}_2\mathcal{O}_1\). In order to define the grade for the component of a tensor field of arbitrary rank, we first assign a grade to each of its indices. If index \(a\) corresponds to a commuting component of the coordinates of a point on the supermanifold \(\epsilon(a) = 0\), while \(\epsilon(a) = 1\) otherwise. For simplicity, a statistical factor \((-1)^{\epsilon(a)\epsilon(b)}\) is often abbreviated by \((-1)^{ab}\), e.g., \(X^aY^b = (-1)^{ab}Y^bX^a\). The grade of the component \(T^a\) of the tensor field \(T\) is then \(\epsilon(T^{a...}) = \epsilon(a)+\epsilon(b)+\epsilon(c)+\epsilon(d)+\ldots\).

- It is convenient to introduce a shifting rule for the left most index of tensors,

\[
aT^{b...} := T^{ab...}, \quad aT^{b...} := (-1)^{\alpha(a)+\beta} a \overline{T}^b...a,
\]

(C1a)

(C1b)

- As with all other objects on a supermanifold, derivatives also carry a grade. Thus, derivatives from the left \(\overline{T}_a\) and the right \(T_a\) need to be distinguished. They are related by

\[
\overline{T}_a f = (-1)^{\alpha(f)+\beta} f \overline{T}_a,
\]

(C2)

where \(f\) is a function over the manifold. In this appendix we will mainly use derivatives \(\overline{T}_a\) and covariant derivatives \(\nabla_a\) that act from the right.

- Super transposition of matrices, denoted by \(\sim\), is defined by

\[
\begin{align*}
\sim K_{ij} & = (-1)^{j(i+j)} K_{ij}, \\
\sim M_{ij} & = (-1)^{j(i+j+l)} M_{ij}, \\
\sim N_{ij} & = (-1)^{j(i+j)} N_{ij}.
\end{align*}
\]

(C3a)

(C3b)

(C3c)

For illustration, we shall apply these rules to the components \(g_{ab}\) of the metric tensor field, and components \(g^{ab}\) of the inverse of the metric tensor field. From the shifting rule

\[
\begin{align*}
ag_b = g^{ab}, \quad ag_b = (-1)^{a} g_{ab}, \\
ag_c g^b = ab, \quad ag_c g^b = a \delta_b.
\end{align*}
\]

(C4a)

(C4b)

The supersymmetry of the metric tensor field implies

\[
a g_b = (-1)^a g_{ab}, \quad a g_b = (-1)^{a+b} g_{ab}, \quad a g_{cb} = (-1)^{a+b+c} g_{cb}, \quad a g = g^b = (-1)^{ab} g_{ab}^b = (-1)^{ab} g^b a.
\]

(C5a)

(C5b)

An index of a component \(X^a\) of a supervector field can be raised, lowered, and shifted according to the rules

\[
\begin{align*}
a X & = X^a, \quad a X = (-1)^a X_a, \\
ag_b X & = X^b g_{ab}, \quad X_a = X^b g_b a, \quad a X = g_b^a X, \quad X^a = X^b g_b a.
\end{align*}
\]

(C6a)

(C6b)

(C6c)

b. The NL\(\sigma\)M on a Riemannian supermanifold The NL\(\sigma\)M on a Riemannian supermanifold is defined by the partition function

\[
Z := \int D[\phi] e^{-S[\phi]}, \quad S[\phi] := \frac{1}{4\pi t} \int R g^{a\mu b} \phi_a \partial_\mu \phi b,
\]

(C7a)

(C7b)

where \(\phi^a(r) \in \mathbb{R}^M_c \times \mathbb{R}^N_a\) represents the \(M+N\) components of the coordinates of point \(r\) on the target manifold, with \(M\) referring to the number of commuting coordinates whereas \(N\) that of the number of anti-commuting ones and \(t\) is the NL\(\sigma\)M coupling constant.

c. The covariant background field method and the super Riemann normal coordinate expansion In the background field method, the covariant vector field \(\phi^a(r)\) is separated into two parts,

\[
\phi^a(r) = \psi^a(r) + \pi^a(r),
\]

(C8)

whereby \(\psi^a(r)\) is assumed to be a slowly varying solution to the classical equations of motion which transforms like a contravariant vector and \(\pi^a(r)\) represents fluctuations around the slow degrees of freedom \(\psi^a(r)\).

Two steps are needed to perform the one-loop RG program in a covariant fashion. First, we need to trade the expansion in terms of \(\pi^a\), which is not covariant, for that of fields \(\xi^a\) that transform like contravariant vectors. Second, we need to rotate the RNC by a vielbein to evaluate the functional integral over the fast modes.

The RNC is based on the notion of geodesics on Riemannian manifolds. Since almost all the notions on a Riemannian manifold such as forms, connection, the Riemann tensor, geodesics, etc., have their counterpart on a Riemannian supermanifold, it is straightforward to develop the super extension of the RNC expansion. The expansion of a covariant tensor field \(T\) in terms of the super RNC \(\xi^a\) is given by
The RNC expansion for $\partial_{\mu}(\psi^a + \pi^a)$ is given by

$$
\partial_{\mu}(\psi^a + \pi^a) = \partial_{\mu}\psi^a + \xi^a \tilde{D}_{\mu} + \frac{1}{3} R^a_{\ e \ c \ d} \partial_{\mu} \psi^c \xi^e \xi^d + \cdots,
$$

(C10)

where $\tilde{D}_{\mu}$ is the right covariant derivative for $\xi^a$

$$
\xi^a \tilde{D}_{\mu} = \partial_{\mu}\xi^a + \Gamma^a_{\ bc} \partial_{\mu}\psi^c \xi^b,
$$

(C11)

and $\Gamma^a_{\ bc}$ is a component of the connection. For a Riemannian supermanifold the connection can be derived from the metric. With the covariant expansion Eqs. (C9) and (C10) the action is expanded in terms of $\xi^a$ according to

$$
S[\psi, \pi] - S[\psi] = \frac{1}{4\pi t} \int_\mathcal{R} \left[ \xi^a \tilde{D}_{\mu} a \theta_b(\psi) \xi^b \tilde{D}_{\mu} + R_{abcd} \partial_{\mu} \psi^d \xi^c \xi^b \partial_{\mu} \psi^a \right] + \cdots,
$$

(C12)

d. Vielbeins on a Riemannian supermanifold  To integrate over the fast modes in the expansion (C12), we rotate $\xi^a$ to $\xi^a$ with the linear transformation

$$
\xi^a := \xi^a \hat{e}_a(\psi), \quad \xi^a = \xi^a \hat{e}_a(\psi), \quad (\xi^a \hat{e}_a(\psi) \xi^b = \xi^b \hat{e}_b(\psi) \xi^a),
$$

(C13a)

\[
\hat{a} \xi = \hat{a} \hat{e}_a(\psi) \xi, \quad b \xi = b \hat{e}_b(\psi) \xi
\]

(C13b)

where

\[
a \hat{e}_a \hat{e}_b = a \delta^b_1, \quad \hat{a} \hat{e}_a \hat{e}_b = \hat{a} \delta^b_1, \quad (C14a)
\]

\[
\hat{a} \hat{e}_a \hat{e}_b = \hat{a} \delta^b_1, \quad \hat{a} \hat{e}_a \hat{e}_b = \hat{a} \delta^b_1, \quad (C14b)
\]

\[
(-1)^{\hat{a}(\hat{a} + a)} \hat{e}_a = a \hat{e}_a, \quad (-1)^{\hat{a}(\hat{a} + a)} \hat{e}_a = a \hat{e}_a.
\]

(C14c)

Here, the vielbeins (and their inverse) $\hat{e}$ are defined by

$$
\hat{a} \hat{e}_a a \theta_b \hat{e}_b = \hat{a} \eta_b, \quad a \hat{e}_a \hat{e}_b \hat{b} = a \theta_b
$$

(C15a)

with

$$
\hat{a} \eta_b = \begin{pmatrix} D^M_0 & 0 \\ 0 & D^N \end{pmatrix},
$$

(C15b)

\[
D^M := \text{diag} \left( -1, \ldots, -1, +1, \ldots, +1 \right), \quad (C15c)
\]

\[
D^N := \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}, \quad (C15d)
\]

Observe that we are allowing the bosonic part of $\eta$ to be pseudo Riemannian, i.e., $\eta$ is allowed to have $-1$ as well as $+1$ as its diagonal elements. From now on, latin letters with a hat refer to the coordinates of the Riemannian manifold in the vielbein basis.

The covariant expansion of the action in terms of $\zeta^a$ is given by

$$
S[\psi, \pi] - S[\psi] = \frac{1}{4\pi t} \int_\mathcal{R} \left[ \hat{a} \hat{D}_{\mu} \hat{a} \eta_b \hat{a} \hat{e}_b \hat{a} \hat{D}_{\mu} + R_{\hat{a} \hat{b} \hat{c} \hat{d}} \partial_{\mu} \psi^d \hat{a} \hat{e}_d \hat{a} \hat{e}_b \hat{a} \hat{e}_c \partial_{\mu} \psi^b \right] + \cdots,
$$

(C16a)

where yet another covariant derivative from the right

$$
\hat{a} \hat{D}_{\mu} := \partial_{\mu} \hat{a} \zeta - \hat{a} \hat{D}_{\mu} (A_{\mu}(\psi)) \hat{b} \zeta, \quad (C16b)
\]

has been introduced together with the $\text{OSp}(M|N)$ spin connection [if the bosonic part of $\eta$ is pseudo Riemannian, $\text{OSp}(M|N)$ should be replaced by $\text{OSp}(M_1, M_2|N)$ where $M_1, M_2$ is the number of $\pm 1$ in the diagonal part of $\eta$ with $M_1 + M_2 = M$],

$$
\hat{a} (\omega_c) := \hat{a} \hat{e}_a \left[ (-1)^{\hat{b}} \left( \hat{a} \hat{e}_b \hat{D}_c + \Gamma^a_{\ bc} \hat{b} \hat{e}_b \right) \right],
$$

(C16d)

the $\text{OSp}(M|N)$ gauge field

$$
\hat{a} (A_{\mu})_b := (-1)^{\hat{b}} \hat{a} (\omega_c)_b \partial_{\mu} \psi^c,
$$

(C16e)

\[
\hat{a} (A_{\mu})_b = -(\hat{a} \hat{D}_{\mu} (A_{\mu}))_b \hat{a},
\]

(C16f)

and the $\text{OSp}(M|N)$ field strength tensor

\[
\hat{a} (F_{\mu \nu})_b = \partial_{\mu} \hat{a} (A_{\nu})_b - \partial_{\nu} \hat{a} (A_{\mu})_b + \hat{a} (A_{\mu})_c \hat{e}^c (A_{\nu})_b - \hat{a} (A_{\nu})_c \hat{e}^c (A_{\mu})_b = - \hat{a} \hat{e}_a R^a_{\ bcd} \partial_{\mu} \psi^d \partial_{\nu} \psi^c \hat{b} \hat{e}_b.
\]

(C16g)

The components of the Riemann tensor are here $R^a_{\ bcd}$. 

e. One-loop beta function  The integration over the fast modes \( \zeta \) is performed with the help of the cumulant expansion. The effective action for the slow modes \( \psi \) is, up to one-loop,

\[
Z := \int \mathcal{D}[\psi] e^{-S[\psi]} e^{i\delta S[\psi]}, \tag{C17a}
\]

\[
\delta S[\psi] = \frac{1}{2} G_0(0) \int \partial_{\mu} \psi^i (-1)^i R_{ij} \partial_{\mu} \psi^j, \tag{C17b}
\]

where

\[
R_{ij} := (-1)^{k(i+1)+l} g^{kl} R_{iklj} \tag{C17c}
\]

\[
\left\langle \left[ T_{ij...} (\phi) \partial_{\mu} \phi^i \partial_{\nu} \phi^j \cdots \right] \right\rangle = \left\langle \left[ T_{ij...} \right] \right\rangle \partial_{\mu} \psi^i \partial_{\nu} \psi^j \cdots
\]

\[
+ T_{ij...} (\psi) \left\langle \left[ \partial_{\mu} \phi^i \right] \right\rangle \partial_{\nu} \psi^j \cdots + T_{ij...} (\psi) \partial_{\nu} \psi^i \left\langle \left[ \partial_{\mu} \phi^j \right] \right\rangle \cdots
\]

\[
+ T_{ij...} (\psi) \left\langle \left[ \partial_{\mu} \phi^i \right] \right\rangle \left\langle \left[ \partial_{\mu} \phi^j \right] \right\rangle \cdots + \cdots.
\]

Thus, all we need are

\[
\left\langle \left[ \partial_{\mu} \phi^i \right] \right\rangle \left\langle \left[ \partial_{\mu} \phi^j \right] \right\rangle = \frac{-\delta_{l,2}^b}{2} R^i_{bcd} \partial_{\mu} \psi^d \partial_{\nu} \psi^e g^{b} - \frac{\delta_{l,2}^b}{3} \delta_{\mu,-\nu} (-1)^j R^i_{tkl} \left[ t^{l} g^{i} \left[ \partial_{\nu} \psi^k \partial_{\mu} \psi^l \right] \right], \tag{C19b}
\]

\[
\left\langle \left[ \partial_{\mu} \phi^i \right] \right\rangle = \frac{\delta_{l,2}^b}{3} R^i_{bcd} \partial_{\mu} \psi^d \partial_{\nu} \psi^e g^{b}, \tag{C19c}
\]

on the one hand and

\[
\left\langle \left[ T_{b_1...b_l} (\phi) \right] \right\rangle = -\frac{\delta_{l,2}^b}{2} I_{i} b_{1}...b_{l-1} b_{l+1}...b_{l} (-1)^{b_{1}+...+b_{l}} \left( R^b_{c_1 c_2} + (-1)^b c_1 R^b_{c_1 c_2} \right) c_2 g^{c_1}, \tag{C19d}
\]

on the other hand. Here, we are using the conformal indices defined in Eq. \( \text{(2.19)} \).

2. NL\( \sigma \)M on Kähler supermanifolds

\[
g. \text{Notations and conventions for Hermitian supermanifolds} \]  Some additional notation is needed to deal with Hermitian and Kählerian manifolds.

- With the convention that complex conjugation of a composite objects reverts the ordering, \((O_1 O_2 O_3 \cdots)^* = \cdots (O_3)^* (O_2)^* (O_1)^*\), it is useful to introduce sign factors associated to the operation of complex conjugation. For covariant and contravariant holomorphic and anti-holomorphic vectors, the action of the complex conjugation is defined by

\[
(V^a)^* = V^* a, \quad (V_a)^* = (-1)^a V^* a. \tag{C20a}
\]

For tensor fields of interest in this paper, the complex conjugation of a tensor field is given by

\[
\left[ T^{A_1...A_{r+s}} \right] = (-1)^{\delta_{r+s}(A)} T^{A_1...A_{r+s}}, \tag{C20b}
\]

with \( \delta_{r+s}(A) = \sum_{t_i u=1, t < u} A_i A_u \).

- The grade for an antiholomorphic index is equal to that of its holomorphic counterpart, \( \epsilon(a^*) = \epsilon(a) \).

For illustration, we shall apply these rules to the components \( _a g_{a^*} \) of the metric tensor field. Combining the rule \( \text{(C20b)} \) with the supersymmetry of the metric \( \text{(C20)} \),

\[
_a g_{a^*} = (1)^{a+b+ab} (a^* g_b)^* \quad \text{as well and can be used to compute their anomalous scaling dimensions. We shall assume that all the covariant derivatives of the tensor field } T_{ij...}(\phi) \text{ vanish. If so we infer the RG transformation law}
\]

\[
T_{ij...}(\phi) \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \ldots
\]

\[
\text{is the Ricci tensor.}
\]

f. RG flows of high-gradient operators  The covariant expansion in terms of the super RNC applies to the high-gradient operators

\[
T_{ij...}(\phi) \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \ldots \tag{C18}
\]

we infer the metric tensor field is superhermitian

\[
g_{b^* a} = (g_{a^* b})^*. \tag{C22}
\]
h. **The NLσM on a Hermitian supermanifold**  
The NLσM on a Hermitian supermanifold is defined by the partition function

\[ Z := \int \mathcal{D}[z^*, z] e^{-S[z^*, z]}, \quad (C23a) \]

\[ S[z^*, z] := \frac{1}{2\pi i} \int_r \partial_{\mu} z^* \partial_{\mu} z, \quad (C23b) \]

where \((z^*, z)\) represents coordinates on the complex supermanifold. For a Kählerian supermanifold, the metric is derived from the Kähler potential \(K(z^*, z)\),

\[ g_{ab} = K(z^*, z) \partial_a \hat{\partial}_b, \quad (C24) \]

[Here, observe that the shifting rule for indices \(\text{(C1b)}\) and the relation between the left- and right-derivatives \(\text{(C2)}\) are compatible.]

i. **The covariant background field method and the super Kähler normal coordinate expansion**  
In the background field method, the covariant vector field \((z^a, z^a)(r)\) is separated into two parts,

\[ z^a = \psi^a + \pi^a, \quad z^a = \psi^a + \pi^a, \quad (C25) \]

whereby \((\psi^a, \pi^a)(r)\) is assumed to be a slowly varying solution to the classical equations of motion which transforms like a contravariant vector and \((\pi^a, \pi^a)(r)\) represents fluctuations around the slow degrees of freedom \((\psi^a, \psi^a)(r)\).

The expansion in terms of \((\pi^a, \pi)\) is traded off for one in terms of normal coordinates \((w^a, \omega)\), the super version of the KNC. The super KNC expansion for a tensor field of rank \((0, r+s)\) with \(r\) holomorphic and \(s\) antiholomorphic indices \(T_{b_1\ldots b_r c_1\ldots c_s}(z^*, z)\) is

\[ T_{b_1\ldots b_r c_1\ldots c_s}(z^*, z) = T_{b_1\ldots b_r c_1\ldots c_s}(\psi^*, \psi) + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} w^{k_1} + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{l_1} w^{l_1} + \frac{1}{2} T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} \nabla_{l_1} w^{k_1} w^{l_1} + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} w^{l_1} w^{k_1} + \ldots \]

\[ = T_{b_1\ldots b_r c_1\ldots c_s}(\psi^*, \psi) + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} w^{k_1} + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{l_1} w^{l_1} + \frac{1}{2} T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} \nabla_{l_1} w^{k_1} w^{l_1} + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} w^{l_1} w^{k_1} + \ldots \]

\[ = T_{b_1\ldots b_r c_1\ldots c_s}(\psi^*, \psi) + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} w^{k_1} + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{l_1} w^{l_1} + \frac{1}{2} T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} \nabla_{l_1} w^{k_1} w^{l_1} + T_{b_1\ldots b_r c_1\ldots c_s} \nabla_{k_1} w^{l_1} w^{k_1} + \ldots \]

\[ = \ldots \]

\[ (C26) \]

The super KNC expansion for \(\partial_{\mu}(\psi^A + w^A)\) is given by

\[ \partial_{\mu}(\psi^A + w^A) = \partial_{\mu}\psi^A + w^A \nabla_{\mu} + \frac{1}{2} R_{c_1 c_2 c_3}(\psi, \psi^*) \partial_{(\mu} \psi^{c_3} w^{c_2} w^{c_1} + \ldots \]

\[ \partial_{(\mu} \psi^{c_3} \partial_{c_2} \psi^{c_1} + \frac{1}{2} R_{c_1 c_2 c_3}(\psi, \psi^*) \partial_{(\mu} \psi^{c_3} w^{c_2} w^{c_1} + \ldots \]

\[ (C27) \]

where \(\nabla_{\mu} \) is the right covariant derivative for \((w^A, \psi^A)\)

\[ w^j \nabla_{\mu} = w^j \partial_{\mu} \psi^A = \partial_{\mu} w^j + \Gamma_{ab}^{c} w^b \partial_{\mu} \psi^a, \quad (C28a) \]

\[ j w^{*j} \nabla_{\mu} = j w^{*j} \partial_{\mu} \psi^A = \partial_{\mu} j w^{*j} + \Gamma_{aj}^{b} j w^{*j} \partial_{\mu} \psi^a + \Gamma_{aj}^{j} j w^{*j} \partial_{\mu} \psi^a \]

\[ (C28b) \]

with \(\Gamma_{ab}^{c}\) the coefficient of the connection that satisfies \((\Gamma_{bc}^{a})^* = (-1)^{c+b+a(k+r)+bc} \Gamma_{bc}^{a} \) [see \(\text{(C20b)}\)]. With the covariant expansion Eqs. \(\text{(C20)}\) and \(\text{(C27)}\), the action is expanded in terms of \((w^a, w^a)\) according to

\[ S[\psi^* + \pi^*, \psi + \pi] - S[\psi^*, \psi] = \]

\[ \frac{1}{2\pi i} \int_r (\nabla_{\mu} w^a)^* \partial_a g_b (w^b \nabla_{\mu}) - \frac{1}{2\pi i} \int_r R_{\mu \nu}j^{(k+l)} (w^k \nabla_{\mu} \partial_{\nu} \psi^l \partial_{\nu} \psi^l - \frac{1}{2} \partial_{\nu} \psi^l w^k \nabla_{\mu} \partial_{\nu} \psi^l) + \ldots \]

\[ (C29) \]

j. **Vielbeins on a Hermitian supermanifold**  
Introduce the fast degrees of freedom \((\zeta^*, \zeta)\) by the linear
The gauge field is skew superhermitian,
\[ b(A_\mu)_{c^r} = - (c(A_\mu)_{b^r})^* \]  
where indices of \( A_\mu \) are raised and lowered by \( \eta \). The explicit form of the gauge field in terms of the vielbeins is given by the U(\( M_1, M_2|N \)) spin connection
\[ \omega^b_{c^r} = - a_{c^r} \left( e_{b^r} \frac{\partial}{\partial e^c} \right), \]
and
\[ \omega^b_{c^r} = a_{c^r} \left( e_{b^r} \frac{\partial}{\partial e^c} \right) = - e_{a} a_{c^r} \left( e_{b^r} \frac{\partial}{\partial e^c} \right) - (1)^{a + \eta + (r + b)} \Gamma^{\rho}_{ac^r} c^\rho_{b^r}, \]
whereby
\[ a_{c^r} = e_{c^r} e_{b^r} \partial_{\mu} \psi^{b^r} + e_{b^r} e_{c^r} \partial_{\mu} \psi^{b^r}. \]

The U(\( M_1, M_2|N \)) field strength tensor is given by
\[ F_{\mu \nu} = \partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu} + \partial_{\mu} \partial_{\nu} a_{\alpha} \Gamma^\alpha_{\mu \nu}, \]
where
\[ a_{c^r} = a_{c^r} a_{b^r} \partial_{\mu} \psi^{b^r} + a_{b^r} a_{c^r} \partial_{\mu} \psi^{b^r}. \]

The Ricci tensor,
\[ R_{i^u j} := -(1)^{k+l+i} g_{i^u j} \partial_{i^u j}, \]
is introduced together with the U(\( M_1, M_2|N \)) gauge field \( A_\mu \). The gauge field with different index-structures are related by
\[ \hat{a}_{\mu} \hat{A}_{\mu} = \hat{a}_{\mu} \hat{A}_{\mu} \Rightarrow \hat{a}_{\mu} \hat{A}_{\mu} = -(1)^{\mu + b} \hat{a}_{\mu} \hat{A}_{\mu} \hat{b}, \]
\[ (\hat{a}_{\mu} \hat{A}_{\mu})^* = \hat{A}_{\mu}^* \hat{a}_{\mu}^* \Rightarrow - (\hat{A}_{\mu})^* = \left[ \hat{a}_{\mu} \hat{A}_{\mu} \right]^*. \]
\[
\left\langle T_{IJ\ldots}(z^*, z) \partial_{\mu \nu} z^I \partial_{\nu} z^J \ldots \right\rangle_{c^2} = \left\langle \left[ T_{IJ\ldots}(z^*, z) \right]_{c^2} \right\rangle \partial_{\mu} z^I \partial_{\nu} z^J \ldots + T_{IJ\ldots}(z^*, z) \left\langle \left[ \partial_{\mu} z^I \right]_{c^2} \left[ \partial_{\nu} z^J \right]_{c^2} \right\rangle \ldots \ldots \ldots .
\]
(C40a)

Thus, all we need are
\[
\left\langle \left[ \partial_{\mu} z^{si} \right]_{c^1} \left[ \partial_{\nu} z^{sj} \right]_{c^1} \right\rangle = -t dl \delta_{\mu,-\nu} (-1)^{j(k+l)} R_{i'k'1 \ldots l} R_{j'k'1 \ldots l} g^{i'k'} \partial_{\mu} z^k \partial_{\nu} z^l,
\]
(C40b)
\[
\left\langle \left[ \partial_{\mu} z^{si} \right]_{c^1} \left[ \partial_{\nu} z^j \right]_{c^1} \right\rangle = -t dl \delta_{\mu,-\nu} R_{bc} \partial_{\mu} z^b \partial_{\nu} z^c g^{ij},
\]
(C40c)
\[
\left\langle \left[ \partial_{\mu} z^i \right]_{c^1} \left[ \partial_{\nu} z^j \right]_{c^1} \right\rangle = -t dl \delta_{\mu,-\nu} R_{ikl}^{a} l_1^{i} \partial_{\mu} z^k \partial_{\nu} z^l,
\]
(C40d)
on the one hand and
\[
\left\langle \left[ T_{b_1 \ldots b_r c^1 \ldots c^s} (z^*, z) \right]_{c^2} \right\rangle = -t dl \sum_{i=1}^{r} (-1)^{(a+b_i)(b_{i+1}+\ldots+c_s)} T_{b_1 \ldots b_r c^1 \ldots c^s} R_{b_1 k_1 l_1}^{a} g^{k_1 \ldots l_1},
\]
(C40f)
on the other hand. Here, we are using the conformal indices defined in Eq. (2.19).

Since a Kählerian supermanifold is a Riemannian supermanifold, one can make use of either the RNC or KNC expansion. Indeed, all the results in this appendix derived from the KNC expansion can actually be obtained from the RNC expansion as well. A subtle thing here is that the RNC and KNC expansion for a tensor field \( T_{IJK\ldots}(z^*, z) \) and \( \partial_{\mu} z^A \) are different. However, the RNC and KNC expansions are identical when applied to the high-gradient operators (C39) and thus give us the same results.

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In symmetry class A, in the replica limit
the beta function of $t$ encodes the change to $t$ induced by the infinitesimal rescaling of the short distance cutoff, $a \to a e^{-\delta t^d}$.

In symmetry class A, in the replica limit $p + q \to 0$ (see Table 1), the coefficient of the $t^2$ term in the beta function happens to vanish, and one needs to compute the beta function up to order $t^3$ in order to find a metal-insulator fixed point in symmetry class A.\textsuperscript{35,36,47}

The antisymmetry of the Riemann tensor $R^{A}_{BC\cdot D}$ has to be used.

As we will not include domain walls in symmetry class D, we replace $O(N)$ by its connected counterpart $SO(N)$ in the following.

Here, we are considering $c$-type (commuting) tensor fields only. If the tensor field itself is $a$-type (anticommuting), components of the tensor field pick up an extra statistical factor.\textsuperscript{55}

In this paper, we are always dealing with real $c$-type vector and tensor fields and their components in the so-called standard basis.\textsuperscript{52}