GENERALIZED NOWICKI CONJECTURE
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Abstract. Let B be an integral domain over a field K of characteristic 0. The derivation δ of B[Yd] = B[y1, . . . , Yd] is elementary if δ(B) = 0 and δ(yi) ∈ B, i = 1, . . . , d. Then the elements ui,j = δ(yi)yi,j − δ(yj)yi, 1 ≤ i < j ≤ d, belong to the algebra B[Yd]δ of constants of δ and it is a natural question whether the B-algebra B[Yd]δ is generated by all ui,j. In this paper we consider the special case of B = K[Xd] = K[x1, . . . , xd]. If δ(yi) = xi, i = 1, . . . , d, this is the Nowicki conjecture from 1994 which was confirmed in several papers based on different methods. The case δ(yi) = xni, ni > 0, i = 1, . . . , d, was handled by Khoury in the first proof of the Nowicki conjecture given by him in 2004. As a consequence of the proof of Kuroda in 2009 if δ(yi) = f(xi), for any nonconstant polynomials f1(xi), i = 1, . . . , d, then B[Yd]δ = K[Xd, Yd]δ is generated by Xd and Ud = {ui,j = f1(xi)yi,j − yi,fj(xi) | 1 ≤ i < j ≤ d}. In the present paper we have found a presentation of the algebra

K[Xd, Yd]δ = K[Xd, Ud, | R = S = 0],
R = \{r(i, j, k, l) | 1 ≤ i < j < k < l ≤ d\},
S = \{s(i, j, k) | 1 ≤ i < j < k ≤ d\},
and a basis of K[Xd, Yd]δ as a vector space. As a corollary we have shown that the defining relations R ∪ S form the reduced Gröbner basis of the ideal which they generate with respect to a specific admissible order. This is an analogue of the result of Makar-Limanov and the author in their proof of the Nowicki conjecture in 2009.

1. Introduction

In the present paper we consider only commutative algebras over a field K of characteristic 0. A linear operator δ of an algebra A is a derivation if it satisfies the Leibniz rule

δ(a1a2) = δ(a1)a2 + a1δ(a2), \quad a1, a2 ∈ A.

The kernel Aδ of δ is the algebra of constants of δ. Let B be an integral domain over K and let A = B[Yd] = B[y1, . . . , Yd]. The derivation δ of A is elementary if δ(B) = 0 and δ(yi) ∈ B, i = 1, . . . , d. Then the determinants

ui,j = \begin{vmatrix} \delta(yi) & \delta(yj) \\ yi & yj \end{vmatrix} = \delta(yi)yj − δ(yj)yi, \quad 1 ≤ i < j ≤ d,

belong to B[Yd]δ and it is a natural question whether the B-algebra B[Yd]δ is generated by the elements \(\mathbb{1}\). In the sequel we assume that B = K[Xd] = K[x1, . . . , xd] (and δ(B) = 0). In the special case

δ(yi) = xi, \quad i = 1, . . . , d,

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the finite generation of $A^\delta = B[Y_d]^\delta = K[X_d,Y_d]^\delta$ follows from a more general result of Weitzenböck [12] in 1932. In 1994 Nowicki [10] conjectured that for $\delta$ from [2] the algebra $K[X_d,Y_d]^\delta$ is generated by $X_d$ and

$$u_{ij} = x_i y_j - x_j y_i, \quad 1 \leq i < j \leq d.$$ 

This was confirmed in several papers based on different methods, see, e.g., [9] and [2] for details.

In the first proof of the Nowicki conjecture given in his Ph.D. thesis Khoury [6] [8] made one more step and established a result which gives an answer to a generalization of the Nowicki conjecture. If $u_i, i = 1, \ldots, d,$ are positive integers, and the derivation $\delta$ of $B[Y_d] = K[X_d,Y_d]$ is defined by

$$\delta(y_i) = x_i^{u_i}, \quad i = 1, \ldots, d,$$

then the algebra $K[X_d,Y_d]^\delta$ of constants of $\delta$ is generated again by $X_d$ and

$$u_{ij} = x_i^{u_i} y_j - x_j^{u_j} y_i, \quad 1 \leq i < j \leq d.$$ 

The most general result in this direction belongs to Kuroda [9].

**Theorem 1.1.** Let $B$ be an integral domain over $K$ and let $\delta$ be an elementary derivation of $B[Y_d]$ such that $f_i = \delta(y_i), i = 1, \ldots, d,$ are algebraically independent over $K$. If $B$ is flat over $K[f_d] = K[f_1, \ldots, f_d]$, then the $B$-algebra $B[Y_d]^\delta$ is generated by $u_{ij} = f_i y_j - f_j y_i, 1 \leq i < j \leq d$.

It seems that it is difficult to find further generalizations. Khoury [7] showed that the algebra of constants of the elementary derivation $\delta$ of $B[Y_4], B = K[X_3]$, for $F_4 = \{x_1^2, x_2^2, x_3^2, x_1 x_2 x_3\}$ is finitely generated but cannot be generated by expressions which are linear in $Y_4$. Also, many of the modern counterexamples to the Fourteenth Hilbert problem are constructed in terms of elementary derivations, see, e.g., the surveys by Freudenburg [5] and Nowicki [11].

It is easy to see that Theorem 1.1 holds for the elementary derivation $\Delta$ of $B[Y_d], B = K[X_d], and

$$\Delta(y_i) = f_i(x_i), \quad i = 1, \ldots, d,$$

where $f_i(x_i)$ is a nonconstant polynomial in the variable $x_i, i = 1, \ldots, d$. But there is an essential difference between the Nowicki conjecture and this result. The Nowicki conjecture is equivalent to a statement of classical invariant theory. When the polynomials $f_i(x_i)$ are not linear, we cannot see how to restate the result in the language of invariant theory. In the present paper we apply the methods developed in our proof with Makar-Limanov [3] of the Nowicki conjecture and find a presentation of the algebra $K[X_d,Y_d]^\Delta$ and a basis of $K[X_d,Y_d]^\Delta$ as a vector space. As a consequence we show that our defining relations form the reduced Gröbner basis of the ideal which they generate with respect to a specific admissible order.

## 2. Generators of the Algebra of Constants

We shall need the following easy lemma. We include the proof for self-containedness of the exposition.

**Lemma 2.1.** Let $f_i = f_i(x_i), i = 1, \ldots, d,$ be nonconstant polynomials. Then $K[X_d]$ is a free $K[f_1, \ldots, f_d]$-module.
Proof. Let \( m_i = \deg(f_i) \) be the degree of \( f_i \). Every monomial \( x_i^m \) can be written as a linear combination of polynomials \( q_n(x_i)f_i^n(x_i) \), \( \deg(q_n) < m_i \), \( n = 0, 1, 2, \ldots \), and hence as a \( K[f_1, \ldots, f_d] \)-module \( K[X_d] \) is generated by the monomials
\[
x_i^{r_1} \cdots x_d^{r_d}, \quad 0 \leq r_i < m_i, \quad i = 1, \ldots, d.
\]
The leading terms of the products
\[
x_1^{r_1} f_1^{r_1}(x_1) \cdots x_d^{r_d} f_d^{r_d}(x_d)
\]
with respect to the lexicographic order are equal to \( x_1^{n_1}m_i^{r_1} \cdots x_d^{n_d}m_d^{r_d} \) and are pairwise different. Hence the polynomials \( 3 \) are linearly independent which implies that \( K[X_d] \) is a free \( K[f_1, \ldots, f_d] \)-module. \( \Box \)

**Theorem 2.2.** Let \( f_i = f_i(x_i), \ i = 1, \ldots, d, \) be nonconstant polynomials in one variable and let \( \Delta \) be the derivation of \( K[X_d, Y_d] \) defined by
\[
\Delta(y_i) = f_i(x_i), \quad \Delta(x_i) = 0, \quad i = 1, \ldots, d.
\]
Then the algebra \( K[X_d, Y_d]_\Delta \) of constants of \( \Delta \) is generated by \( X_d \) and the determinants from \( 1 \)
\[
u_{ij} = f_i(x_i)y_j - f_j(x_j)y_i, \quad 1 \leq i < j \leq d.
\]

**Proof.** By \( \cite{4} \) Corollary 6.6, p. 165 if a \( B \)-module \( M \) is finitely generated then it is flat if and only if it is a summand of a free \( B \)-module. Hence by Lemma \( \cite{2} \) \( K[X_d] \) is a flat \( K[f_1, \ldots, f_d] \)-module. Obviously \( f_i(x_i), \ i = 1, \ldots, d, \) are algebraically independent over \( K \) and Theorem \( \cite{1} \) immediately gives that the algebra \( K[X_d, Y_d]_\delta \) is generated by \( X_d \) and the polynomials \( 4 \). \( \Box \)

**Remark 2.3.** If some of the polynomials \( f_i(x_i) \) in Theorem \( \ref{2.2} \) is a constant, then the description of \( K[X_d, Y_d]_\Delta \) is trivial. Let, for example, \( f_1(x_1) = \alpha \in K \setminus \{0\} \). We replace the variables \( Y_d \) by \( Z_d \), where
\[
z_1 = y_1, \quad z_i = \alpha y_i - f_i(x_i)y_1, \quad i = 2, \ldots, d.
\]
Then the definition of \( \Delta \) becomes
\[
\Delta(z_1) = \alpha, \quad \Delta(z_i) = 0, \quad i = 2, \ldots, d, \quad \Delta(X_d) = 0.
\]
Since \( \alpha \neq 0 \), we obtain that \( K[X_d, Z_d]_\Delta = K[X_d, z_2, \ldots, z_d] \).

3. The main result

In this section we follow our paper with Makar-Limanov \( \cite{3} \) and use the methods developed there. Since we shall work with \( \text{Gröbner} \) bases, we refer, e.g., to the book by Adams and Loustaunau \( \cite{1} \) for a background on the topic. We fix the degrees \( m_1, \ldots, m_d \) of the polynomials \( f_1(x_1), \ldots, f_d(x_d) \) and the set
\[
U_d = \{ u_{ij} \mid 1 \leq i < j \leq d \},
\]
where the elements \( u_{ij} \) are defined in \( \cite{2} \).

**Lemma 3.1.** The subsets \( X_d \) and \( U_d \) of \( K[X_d, Y_d] \) satisfy the relations \( R = S = 0 \), where
\[
R = \{ r(i, j, k, l) = u_{ij}u_{kl} - u_{ik}u_{jl} + u_{il}u_{jk} \mid 1 \leq i < j < k < l \leq d \},
\]
\[
S = \{ s(i, j, k) = f_i(x_i)u_{jk} - f_j(x_j)u_{ik} + f_k(x_k)u_{ij} \mid 1 \leq i < j < k \leq d \}.
\]
Proof. The annihilating of the relations (5) and (6) in \(K[X_d, U_d]\) can be verified directly. Instead, the expansions of the determinants

\[
\begin{vmatrix}
 f_i(x_i) & f_j(x_j) & f_k(x_k) & f_l(x_l) \\
 y_i & y_j & y_k & y_l \\
f_i(x_i) & f_j(x_j) & f_k(x_k) & f_l(x_l) \\
y_i & y_j & y_k & y_l
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
 f_i(x_i) & f_j(x_j) & f_k(x_k) & f_l(x_l) \\
 y_i & y_j & y_k & y_l \\
f_i(x_i) & f_j(x_j) & f_k(x_k) & f_l(x_l) \\
y_i & y_j & y_k & y_l
\end{vmatrix}
\]

relative to the first two rows and to the first row give, respectively, (5) and (6). \(\square\)

Now we shall work in the polynomial algebra \(K[X_d, U_d]\). By Theorem 2.2 there is a canonical epimorphism

\[(7) \quad \pi : K[X_d, U_d] \to K[X_d, U_d]^\Delta.\]

Since there will be no misunderstanding, we shall use the same symbols \(x_i \in X_d\) and \(u_{ijk} \in U_d\) for the generators of the polynomial algebra \(K[X_d, U_d]\) and their images under \(\pi\) which generate the algebra of constants \(K[X_d, U_d]^\Delta\).

Our first goal is to show that the kernel of \(\pi\) is generated by the relations \(r(i, j, k, l)\) and \(s(i, j, k)\) from (5) and (6). We associate with every \(u_{ijk} \in U_d\) the open interval \((i, j, k)\). As in (3) we define an ordering of \(K[X_d, U_d]\) called degree–interval length–lexicographic order (DILL order). We order the monomials of \(K[X_d, U_d]\) first by the degree in \(X_d\) and in \(U_d\), then by the total length of the intervals associated with the participating variables \(u_{ij}\) and finally lexicographically. If

\[v = x_{i_1} \cdots x_{i_p} u_{j_1k_1} \cdots u_{j_qk_q}\]

and

\[v' = x_{i'_1} \cdots x_{i'_p} u_{j'_1k'_1} \cdots u_{j'_qk'_q},\]

where \(1 \leq i_1 \leq \cdots \leq i_p \leq d, 1 \leq j_1 \leq \cdots \leq j_q \leq d\) and \(1 \leq k_1 \leq k_2 \leq \cdots \leq k_a \leq d\) if \(j_a = j_{a+1}\), with similar restrictions on \(i'_1, j'_1, k'_1\), we define \(v \succ v'\) if

(i) \(p > p'\) (we compare the degrees of \(v\) and \(v'\) in \(X_d\));

(ii) \(p = p'\) and \(q > q'\) (we compare the degrees of \(v\) and \(v'\) in \(U_d\));

(iii) \(p = p'\), \(q = q'\) and

\[\sum_{b=1}^{q} (k_b - j_b) > \sum_{b=1}^{q} (k'_b - j'_b)\]

for all \(\omega > \omega'\) for the \((p + 2q)\)-tuples

\[\omega = (i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_q)\]

and

\[\omega' = (i'_1, \ldots, i'_p, j'_1, \ldots, j'_q, k'_1, \ldots, k'_q)\]

where \((a_1, \ldots, a_n) > (b_1, \ldots, b_n)\) if \(a_1 = b_1, \ldots, a_e = b_e, a_{e+1} > b_{e+1}\) for some \(e\) (we compare lexicographically \(v\) and \(v'\)).

Obviously the DILL-order is admissible, i.e., it is linear, satisfies the descending chain condition, and if \(v \succ v'\) for two monomials \(v\) and \(v'\), then \(vw \succ wv'w\) for all monomials \(w\). For \(0 \neq f \in K[X_d, U_d]\) we denote by \(\bar{f}\) the leading monomial of \(f\).
Lemma 3.2. The set of normal words in $K[X_d, U_d]$ with respect to DILL order and modulo the relations $R \cup S$ consists of the monomials

\[ v = x_{d_1}^{a_1} \cdots x_{d_d}^{a_d} u_{j_1, k_1} \cdots u_{j_q, k_q}, \quad 1 \leq j_b < k_b \leq d, \]

such that

(i) If $(j_b, k_b) \cap (j_c, k_c) = \emptyset$ for two different $u_{j_b, k_b}$ and $u_{j_c, k_c}$ in (8), then one of the intervals $(j_b, k_b)$ and $(j_c, k_c)$ is contained in the other;

(ii) If $i \in (j_b, k_b)$ for some $u_{j_b, k_b}$ in (8), then $a_i < m_i$.

As a vector space the algebra $K[X_d, Y_d]^\Delta$ is spanned by the images under $\pi$ of the products (8).

Proof. The leading monomials with respect to the DILL order of $r(i, j, k, l)$ and $s(i, j, k)$ from (7) and (10) are, respectively,

\[ u_{b,j,l} k_i, \quad 1 \leq i < j < k < l \leq d, \quad \text{and} \quad x_j^{m_j} u_{1,k}, \quad 1 \leq i < j < k \leq d. \]

(i) Let $u_{j_b, k_b} u_{j_c, k_c}$ divide the monomial $v$, and let $j_b \leq j_c$ and if $j_b = j_c$, then $k_b \leq k_c$. If the intervals $(j_b, k_b)$ and $(j_c, k_c)$ have a nontrivial intersection and are not contained in each other, then $j_b < j_c < k_b < k_c$. Hence the monomial $u_{j_b, k_b} u_{j_c, k_c}$ is the leading term of $r(j_b, j_c, k_b, k_c)$. In this way $v$ is not a normal word and this proves (i).

(ii) If the monomial $v$ is a normal word and $x_j^{m_j} u_{j_b, k_b}$ divides $v$ with $i \in (j_b, k_b)$, then $x_j^{m_j} u_{j_b, k_b}$ is not divisible by a monomial from (10) and hence $a_i < m_i$. This proves (ii).

Let us take some set of polynomials $W = \{w_1, \ldots, w_n\} \subset K[X_d, U_d]$ which are in the kernel of $\pi$ from (7). The images in $K[X_d, Y_d]^\Delta$ of the normal words in $K[X_d, U_d]$ with respect to $W$ span the vector space $K[X_d, Y_d]^\Delta$. By Lemma 3.1, the images in $K[X_d, Y_d]^\Delta$ from (7) and (9) belong to the kernel of $\pi$. Hence $K[X_d, Y_d]^\Delta$ is spanned by the images of the normal words from (8).

Lemma 3.3. The images under the epimorphism $\pi$ from (7) of the normal words (8) from Lemma 3.2 form a basis of the vector space $K[X_d, Y_d]^\Delta$.

Proof. Let $v = x_{d_1}^{a_1} \cdots x_{d_d}^{a_d} u_{j_1, k_1} \cdots u_{j_q, k_q} \in K[X_d, U_d]$ be a normal word. Then

\[ \pi(v) = x_{d_1}^{a_1} \cdots x_{d_d}^{a_d} \prod_{b=1}^{q} (f_{j_b, k_b} (x_{j_b}, y_{k_b}) - f_{k_b, j_b} (x_{j_b}, y_{k_b})). \]

We shall follow the proof of [3] Step 3 in the proof of Theorem 5]. We compare the monomials in $K[X_d, Y_d] = K[x_1, y_1, \ldots, x_d, y_d]$ lexicographically:

\[ x_{d_1}^{a_1} y_{b_1}^{b_1} \cdots x_{d_d}^{a_d} y_{d_d}^{b_d} > x_{d_1}^{a_1} y_{b_1}^{b_1} \cdots x_{d_d}^{a_d} y_{d_d}^{b_d}, \]

if $(a_1, b_1, \ldots, a_d, b_d) > (a'_1, b'_1, \ldots, a'_d, b'_d)$ if the usual lexicographic order. If $j_b < k_b$, then the leading monomials of $f_{j_b, k_b} (x_{j_b}, y_{k_b})$ and $f_{k_b, j_b} (x_{j_b}, y_{k_b})$ are $x_{j_b}^{m_{j_b}} y_{k_b}$ and $y_{j_b} x_{k_b}^{m_{k_b}}$, respectively. Hence $x_{j_b}^{m_{j_b}} y_{k_b} > y_{j_b} x_{k_b}^{m_{k_b}}$ and the leading monomial of (10) is

\[ \text{lead}(\pi(v)) = x_{d_1}^{a_1} \cdots x_{d_d}^{a_d} x_{j_1}^{m_{j_1}} y_{k_1} \cdots x_{j_1}^{m_{j_1}} y_{k_1}. \]

We shall show that we can recover the normal word $v$ from the leading monomial $\text{lead}(\pi(v))$. Hence the monomials $\text{lead}(\pi(v))$ are pairwise different and the polynomials $\pi(v)$ are linearly independent. By Lemma 3.2 this implies that $\pi(v)$ form a basis of the vector space $K[X_d, Y_d]^\Delta$. 

It is sufficient to compare those $\pi(v)$ which are of the same degree in $Y_d$. As in [3] we use induction on the degree with respect to $Y_d$. Since $\deg_{Y_d}(v) = \deg_{Y_d}(\lead(\pi(v)))$, if $\deg_{Y_d}(\lead(\pi(v))) = 0$, then $\deg_{Y_d}(v) = 0$ and $v$ coincides with $\lead(\pi(v))$. Now let $\deg_{Y_d}(\lead(\pi(v))) > 0$. We rewrite (11) in the form

$$\lead(\pi(v)) = x_1^{p_1} \cdots x_k^{p_k} y_k y_k^{q_k} \cdots x_d^{p_d} y_d^{q_d}, \quad q_k > 0.$$  

The variable $y_k$ participates in $\lead(\pi(v))$ because $u_{ijk}$ participates in $v$ from (8) for some $j < k$ and $p_j \geq m_j$. We choose the maximal $i$ such that $i < k$ and $p_i \geq m_i$. Lemma 3.2 implies that $i$ does not belong to the ideal $(j, k)$. Hence $i = j$ and the factor $u_{ijk} = u_{ijk}$ of $v$ is uniquely determined by the form (12) of $\lead(\pi(v))$. In this way $v = u_{ijk}v_1$ for some normal word $v_1 \in K[X_d, U_d]$ and

$$\lead(\pi(v)) = x_i^{m_i} y_k \lead(\pi(v_1)).$$  

By inductive arguments $\lead(\pi(v_1))$ determines $v_1$ and this completes the proof. \hfill $\Box$

The following theorem is the main result of the paper.

**Theorem 3.4.** Let $\Delta$ be the derivation of $K[X_d, Y_d]$ defined by

$$\Delta = \sum_{i=1}^d f_i(x_i) \frac{\partial}{\partial y_i},$$

where $f_i(x_i)$ are polynomials of positive degree. Then

(i) The algebra of constants $K[X_d, Y_d]^\Delta$ has the presentation

$$K[X_d, Y_d]^\Delta \cong K[X_d, U_d]/(R, S),$$

where $U_d = \{u_{ij} \mid 1 \leq i < j \leq d\}$ and the ideal $(R, S)$ is generated by $R$ and $S$ from (5) and (6).

(ii) The set $R \cup S$ is a reduced Gröbner basis of the ideal $(R, S)$ with respect to the DILL order of $K[X_d, U_d]$.

(iii) The basis of the vector space $K[X_d, Y_d]^\Delta$ consists of the images in $K[X_d, Y_d]^\Delta$ of the normal words $v$ from Lemma 3.2.

**Proof.** By Lemma 3.4 $R \cup S$ belongs to the kernel of the epimorphism $\pi$ from (7). A subset of the ideal $\ker(\pi)$ is its Gröbner basis if and only if the set of normal words forms a basis of the factor algebra $K[X_d, U_d]/\ker(\pi) \cong K[X_d, Y_d]^\Delta$. Hence the statements (i), (ii), and (iii) follow immediately from Lemma 3.3. Since the leading terms of the relations $r(i, j, k, l) \in R$ and $s(i, j, k) \in S$ with respect to the DILL order do not divide the monomials participating in the other relations in $R$ and $S$, we conclude that the Gröbner basis $R \cup S$ is reduced. \hfill $\Box$

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