Tilting mutation for \( m \)-replicated algebras

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Abstract

Let \( A \) be a finite dimensional hereditary algebra over an algebraically closed field \( k \), \( A^{(m)} \) be the \( m \)-replicated algebra of \( A \) and \( \mathcal{C}_m(A) \) be the \( m \)-cluster category of \( A \). We investigate properties of complements to a faithful almost complete tilting \( A^{(m)} \)-module and prove that the \( m \)-cluster mutation in \( \mathcal{C}_m(A) \) can be realized in \( \text{mod } A^{(m)} \), which generalizes corresponding results on duplicated algebras established in [Z1].

Keywords: \( m \)-replicated algebras, \( m \)-cluster category, tilting mutation

1 Introduction

Cluster categories were introduced in [BMRRT] and for type \( A_n \), also in [CCS], as a categorical model for better understanding of cluster algebras of Fomin and Zelevinsky in [FZ1, FZ2]. Now, cluster categories have become a successful model for acyclic cluster algebras, see the surveys [BM], [Re] for backgrounds and recent

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developments of cluster tilting theory. Later, $m$-cluster categories were introduced in [Th] as a generalization of cluster categories. Another good interpretation of $m$-cluster category and its tilting objects is the $m$-replicated algebras, see [ABST2] and also see [ABST1] for the case of $m = 1$.

Throughout this paper, we always assume that $A$ is a finite dimensional hereditary algebra over an algebraically closed field $k$. Furthermore, we assume that $A$ has $n$ simple modules and $n \geq 3$ provided $A$ is representation finite. Let $A^{(m)}$ be the $m$-replicated algebra of $A$. Then $\text{gl.dim} A^{(m)} = 2m + 1$. $A^{(1)}$ is called duplicated algebra.

Cluster category $\mathcal{C}(A)$ is the orbit category $D^b(A)/\langle \tau^{-1}[1] \rangle$ of a bounded derived category $D^b(A)$ of $A$ which is a triangulated category by [K], $m$-cluster category $\mathcal{C}_m(A)$ is the orbit category $D^b(A)/\langle \tau^{-1}[m] \rangle$ which also is a triangulated category by [K].

It is well known that there is a one-to-one correspondence between basic tilting $A^{(m)}$-modules with projective dimension at most $m$ and basic tilting objects in $m$-cluster category $\mathcal{C}_m(A)$, see [ABST2] and see [ABST1] for $m = 1$. This motivates further investigates in this kind of algebras. Some interesting results were proved in [LLZ], [Z1] and [Z2], for example, cluster mutation can be realized in duplicated algebra (see [Z1]). A faithful almost complete tilting $A^{(m)}$-module with projective dimension at most $m$ has exactly $m+1$ non-isomorphic complements with projective dimension at most $m$ (see [LLZ]). Furthermore, any partial tilting $A^{(m)}$-module admits a complement and partial tilting $A^{(m)}$-module is tilting if and only if the number of its non-isomorphic indecomposable summands equals to the rank of Grothendieck group of $A^{(m)}$ (see [Z2]).

The aim of this paper is to investigate further properties of complements to a faithful almost complete tilting $A^{(m)}$-module and to prove that $m$-cluster mutation in $\mathcal{C}_m(A)$ can be realized in mod $A^{(m)}$. This paper is arranged as the following. In section 2, we collect necessary definitions and basic facts needed for our research. In section 3, we prove a structure theorem for complements to a faithful almost complete tilting $A^{(m)}$-module (see Theorem 3.4), and also show that $m$-cluster
mutation in $C_m(A)$ can be realized as tilting mutation in mod $A^{(m)}$ (see Theorem 3.9). In section 4, we prove that complements to a faithful almost tilting $A^{(m)}$-module with projective dimension at most $m$ induce an AR-$(m+3)$-angle in $C_m(A)$ in the sense of [IY] (see Theorem 4.2).

2 Preliminaries

Let $\Lambda$ be an Artin algebra. We denote by mod $\Lambda$ the category of all finitely generated right $\Lambda$-modules. The derived category of bounded complexes of mod $\Lambda$ is denoted by $D^b(\Lambda)$ and the shift functor by $[1]$. For a $\Lambda$-module $M$, we denote by add $M$ the subcategory of mod $\Lambda$ whose objects are the direct summands of finite direct sums of copies of $M$ and by $\Omega^1_{\Lambda} M$ the first cosyzygy which is the cokernel of an injective envelope $M \hookrightarrow I$. The projective dimension of $M$ is denoted by pd $M$, the global dimension of $\Lambda$ by gl.dim $\Lambda$ and the Auslander-Reiten translation of $\Lambda$ by $\tau_{\Lambda}$.

Let $C$ be a full subcategory of mod $\Lambda$, $C_M \in C$ and $\varphi : C_M \longrightarrow M$ with $M \in$ mod $\Lambda$. The morphism $\varphi$ is a right $C$-approximation of $M$ if the induced morphism $\text{Hom}(C, C_M) \longrightarrow \text{Hom}(C, M)$ is surjective for any $C \in C$. A minimal right $C$-approximation of $M$ is a right $C$-approximation which is also a right minimal morphism, i.e., its restriction to any nonzero summand is nonzero. The subcategory $C$ is called contravariantly finite if any module $M \in$ mod $\Lambda$ admits a (minimal) right $C$-approximation. The notions of (minimal) left $C$-approximation and of covariantly finite subcategory are dually defined. It is well known that add $M$ is both a contravariantly finite subcategory and a covariantly finite subcategory. We call a morphism $\psi : X \longrightarrow Y$ in $C$ is a sink map of $Y$ if $\psi$ is right minimal and $\text{Hom}(C, X) \longrightarrow \text{Rad}(C, Y) \longrightarrow 0$ is exact. A source map can be defined dually.

Let $T$ be a $\Lambda$-module. $T$ is said to be exceptional if $\text{Ext}^i_{\Lambda}(T, T) = 0$ for all $i \geq 1$. An exceptional module $T$ is called a partial tilting module provided pd $T < \infty$. A partial tilting module $T$ is called a tilting module if there exists an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_d \longrightarrow 0$$
with each $T_i \in \text{add } T$. A partial tilting module $T$ is called an almost complete tilting module if there exists an indecomposable $\Lambda$-module $N$ such that $T \oplus N$ is a tilting module.

From now on, let $A$ be a finite dimensional hereditary algebra over an algebraically closed field $k$. The repetitive algebra $\hat{A}$ of $A$ is the infinite matrix algebra

$$\hat{A} = \begin{pmatrix} \ddots & 0 & & & & & & \\ & A_{i-1} & & & & & & \\ & Q_i & A_i & & & & & \\ & Q_{i+1} & A_{i+1} & & & & & \\ & & & & \ddots & & & \\ & 0 & & & & & & \end{pmatrix}$$

where matrices have only finitely many non-zero coefficients, $A_i = A$ and $Q_i = DA$ for all $i \in \mathbb{Z}$, where $D = \text{Hom}_k(-,k)$ is the dual functor, all the remaining coefficients are zero and multiplication is induced from the canonical isomorphisms $A \otimes_A DA \simeq A DA_A \simeq DA \otimes_A A$ and the zero morphism $DA \otimes_A DA \to 0$ (see [HW] and [H]).

**Lemma 2.1.** ([H]) The derived category $\mathcal{D}^b(A)$ is equivalent, as a triangulated category, to the stable module category $\text{mod } \hat{A}$.

**Lemma 2.2.** Let $M$ be an indecomposable $\hat{A}$-module which is not projective-injective. Then there exists an indecomposable $A$-module $N$ such that $M \simeq \Omega^{-l}_A N$ for some $l \in \mathbb{Z}$. We denote by $l$ the degree of $M$, that is, $\deg M = l$.

The $m$-replicated algebra $A^{(m)}$ of $A$ is defined as the quotient of the repetitive algebra $\hat{A}$, that is,

$$A^{(m)} = \begin{pmatrix} A_0 & & & & & & \\ Q_1 & A_1 & & & & & \\ & Q_2 & A_2 & & & & \\ & & \ddots & \ddots & & & \\ & & & & \ddots & & \\ & & & & & Q_m & A_m \\ 0 & & & & & & \end{pmatrix}.$$

**Remark.** $A^{(1)}$ is the duplicated algebra of $A$ (see [ABST1]).


Let $\mathcal{C}_m(A)$ be the $m$-cluster category of $A$. An object $X$ in $\mathcal{C}_m(A)$ is said to be exceptional if $\text{Ext}^i_{\mathcal{C}_m(A)}(X, X) = 0$ for all $i$ with $1 \leq i \leq m$ and is called an $m$-cluster tilting object if it is exceptional and maximal respect to this property. The object $X$ is said to be almost complete tilting if there is an indecomposable object $Y$ such that $X \oplus Y$ is an $m$-cluster tilting object and $Y$ is called a complement to $X$. It follows from [ZZ] that, for an almost complete tilting object $T$ in $\mathcal{C}_m(A)$, it has exactly $m + 1$ indecomposable non-isomorphic complements $X_0, X_1, \ldots, X_m$ and there are $m + 1$ connecting triangles:

$$X_i \xrightarrow{f_i} T_i \xrightarrow{g_i} X_{i+1} \rightarrow X_i[1],$$

where $f_i$ is the minimal left add $T$-approximation of $X_i$ and $g_i$ the minimal right add $T$-approximation of $X_{i+1}$. We call $\mu_{X_i}(T \oplus X_i) = T \oplus X_{i+1}$ an $m$-cluster mutation in direction $X_i$.

The following definition is taken from [IY].

**Definition 2.3.** Let $M$ be an $m$-cluster tilting object in $\mathcal{C}_m(A)$. The $(m+3)$-angle

$$X_0 \xrightarrow{a_0} M_0 \rightarrow \cdots \rightarrow M_m \xrightarrow{b_{m+1}} X_0$$

induced by $(m+1)$ triangles

$$X_i \xrightarrow{a_i} M_i \xrightarrow{b_{i+1}} X_{i+1} \rightarrow X_i[1], \quad i = 0, 1, \ldots, m$$

is called an AR $(m+3)$-angle if the following conditions are satisfied:

1. $X_0$ and $M_i$ ($0 \leq i \leq m$) all belong to add $M$;
2. $a_0$ is a source map of $X_0$ in add $M$ and $b_{m+1}$ is a sink map of $X_{m+1} = X_0$ in add $M$;
3. $a_i$ is a minimal left add $M$-approximation of $X_i$ for $1 \leq i \leq m$;
4. $b_i$ is a minimal right add $M$-approximation of $X_i$ for $1 \leq i \leq m$.

We denote by $\pi$ the following composition functor,

$$\pi : \text{mod } A^{(m)} \hookrightarrow \text{mod } \hat{A} \rightarrow \text{mod } \hat{A} \cong D^b(A) \rightarrow \mathcal{C}_m(A).$$
By abuse of notation, we often denote objects and modules by the same letter even when they are considered as objects in different categories.

We follow the standard terminology and notation used in the representation theory of algebras, see [ARS],[H] and [Ri].

3 Tilting mutation in $\text{mod } A^{(m)}$

The following lemmas are useful and can be easily proved.

**Lemma 3.1.** For $\hat{A}$-modules $X$ and $Y$,

$$\text{Ext}^s_{\hat{A}}(X,Y) \simeq \text{Hom}_{\hat{A}}(X,\Omega^{-s}Y).$$

**Lemma 3.2.** Let $M$ be an indecomposable exceptional $\hat{A}$-module, which is not projective-injective. Then $\text{End}_{\hat{A}}(M) = k$.

Let $T$ be a faithful almost complete tilting $A^{(m)}$-module. According to [LLZ,Z2], we know that $T$ has $t+1$ non-isomorphic indecomposable complements $X_0, \cdots, X_t$ with $2m \leq t \leq 2m+1$ which are connected by $t$ connecting sequences:

$$0 \rightarrow X_i \rightarrow T_i \rightarrow X_{i+1} \rightarrow 0, \ 0 \leq i \leq t-1.$$  

It is easy to see that $\text{Hom}_{A^{(m)}}(X_j, X_i) = 0$ and $\text{Ext}^1_{A^{(m)}}(X_i, X_j) = 0$ provided $j > i$, and that $X_0$ is the Bongartz-complement to $T$, which means that $X_0$ can not be generated by any tilting modules $T \oplus X_i$ for $1 \leq i \leq t-1$. For convenience, we also call $X_t$ the sink complement to $T$, that is, $X_t$ can not be cogenerated by any tilting modules $T \oplus X_i$ for $1 \leq i \leq t-1$.

**Lemma 3.3.** $\text{End}_{A^{(m)}}(X_i) = k$ for $0 \leq i \leq t$.

**Theorem 3.4.** Taking the notation as above. We have that

$$\dim_k \text{Ext}^s_{A^{(m)}}(X_j, X_i) = \begin{cases} 1, & \text{if } i + s = j \text{ and } s \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
Furthermore, the $i$-th connecting sequence $0 \to X_i \to T_i \to X_{i+1} \to 0$ is a $k$-basis of $\text{Ext}^1_{A(m)}(X_{i+1}, X_i)$. Moreover, for any $0 \leq i \leq t-1$ and $0 \leq i + s \leq t$, the exact sequence $0 \to X_i \to T_i \to T_{i+1} \to \cdots \to T_{i+s-1} \to X_{i+s} \to 0$ is a $k$-basis of $\text{Ext}^s_{A(m)}(X_{i+s}, X_i)$.

**Proof.** By applying $\text{Hom}_{A(m)}(-, X_i)$ to the $j$-th connecting sequence

$$0 \to X_j \to T_j \to X_{j+1} \to 0,$$

we get $\text{Ext}^{j+1-i}_{A(m)}(X_{j+1}, X_i) \cong \text{Ext}^{j-i}_{A(m)}(X_j, X_i)$ for $0 \leq i < j \leq t-1$.

In particular, we also have an exact sequence

$$\text{Hom}_{A(m)}(X_i, X_i) \to \text{Ext}^1_{A(m)}(X_{i+1}, X_i) \to \text{Ext}^1_{A(m)}(T_i, X_i) = 0.$$

Note that $\text{Ext}^1_{A(m)}(X_{i+1}, X_i) \neq 0$ and $\text{End}_{A(m)}(X_i) = k$ by Lemma 3.3, it follows that $\text{Ext}^1_{A(m)}(X_{i+1}, X_i) \cong \text{End}_{A(m)}(X_i) = k$, and that $\text{Ext}^{j-i}_{A(m)}(X_j, X_i) = k$ for $0 \leq i < j \leq t$. We can take the $i$-th connecting sequence

$$0 \to X_i \to T_i \to X_{i+1} \to 0$$

as a $k$-basis of $\text{Ext}^1_{A(m)}(X_{i+1}, X_i)$. It is easy to see that

$$0 \to X_i \to T_i \to T_{i+1} \to \cdots \to T_{i+s-1} \to X_{i+s} \to 0$$

is non-zero in $\text{Ext}^s_{A(m)}(X_{i+s}, X_i)$, which is a $k$-basis of $\text{Ext}^s_{A(m)}(X_{i+s}, X_i)$.

Now we shall show that $\text{Ext}^s_{A(m)}(X_j, X_i) = 0$ for $s \neq j-i$.

If $s > j - i$, $\text{Ext}^s_{A(m)}(X_j, X_i) \cong \text{Ext}^s_{A(m)}(X_i, X_i) = 0$ since $X_i$ is exceptional.

Now, we claim that $\text{Hom}_{A(m)}(X_j, X_i) = 0$ provided $j > i$.

In fact, if $j > i + 1$, $\text{Hom}_{A(m)}(X_j, X_i) = 0$ since $\deg X_j > \deg X_i$.

We only need to prove that $\text{Hom}_{A(m)}(X_{i+1}, X_i) = 0$. On the contrary we assume that $\text{Hom}_{A(m)}(X_{i+1}, X_i) \neq 0$.

Applying $\text{Hom}_{A(m)}(X_{i+1}, -)$ to the $i$-th connecting sequence

$$0 \to X_i \to T_i \to X_{i+1} \to 0,$$

we have an exact sequence

$$0 \to \text{Hom}_{A(m)}(X_{i+1}, X_i) \to \text{Hom}_{A(m)}(X_{i+1}, T_i) \to \text{Hom}_{A(m)}(X_{i+1}, X_{i+1}) \to \text{Ext}^1_{A(m)}(X_{i+1}, X_i).$$
It follows that $\text{Hom}_{A(m)}(X_{i+1}, T_i) \simeq \text{Hom}_{A(m)}(X_{i+1}, X_i) \neq 0$ since $\text{Hom}_{A(m)}(X_{i+1}, X_{i+1}) \simeq \text{Ext}^1_{A(m)}(X_{i+1}, X_i) \simeq k$.

In particular, the quiver of algebra $\text{End}_{A(m)}(T \oplus X_{i+1})$ will have an oriented cycle, which contradicts with that $T \oplus X_{i+1}$ being a tilting $A(m)$-module. Our claim is proved.

If $s < j - i$, according to our claim,

$$\text{Ext}^s_{A(m)}(X_j, X_i) \simeq \text{Ext}^1_{A(m)}(X_{j-(s-1)}, X_i) \simeq \text{Hom}_{A(m)}(X_{j-s}, X_i) = 0.$$ 

This completes the proof. 

Now we are going to show that the converse of Theorem 3.4 is partly true.

**Lemma 3.5.** Let $M$ be an indecomposable non-injective-projective $\hat{A}$-module which satisfies that $\text{Ext}^s_{\hat{A}}(M, M) = 0$ for $s \geq 1$. Then $\text{Ext}^s_{\hat{A}}(\Omega^i_{\hat{A}}M, \Omega^i_{\hat{A}}M) = 0$, for any $i \geq 0$.

**Proof.** By Lemma 3.1, we have that

$$\text{Ext}^s_{\hat{A}}(\Omega^i_{\hat{A}}M, \Omega^i_{\hat{A}}M) \simeq \text{Hom}_{\hat{A}}(\Omega^i_{\hat{A}}M, \Omega^{i-s}_{\hat{A}}M)$$

$$\simeq \text{Hom}_{D^b(A)}(M[-i], M[s-i])$$

$$\simeq \text{Hom}_{D^b(A)}(M, M[s])$$

$$\simeq \text{Hom}_{\hat{A}}(M, \Omega^{-s}M)$$

$$\simeq \text{Ext}^s_{\hat{A}}(M, M)$$

$$= 0.$$ 

Definition 3.6. A set of indecomposable non-projective-injective $A^{(m)}$-modules $\{X_0, X_1, \cdots, X_t\}$ is called a mutation team in $\text{mod } A^{(m)}$ if it satisfies Theorem 3.4, i.e.,

$$\dim_k \text{Ext}^s_{A^{(m)}}(X_j, X_i) = \begin{cases} 
1, & \text{if } i + s = j \text{ and } s \geq 0, \\
0, & \text{otherwise}.
\end{cases}$$

and is maximal with respect to this property.

**Remark.** Every $X_i$ in a mutation team $\{X_0, X_1, \cdots, X_t\}$ is exceptional with $\text{Hom}_{A^{(m)}}(X_j, X_i) = 0$ and $\text{deg} X_j \geq \text{deg} X_i$ for $j > i \geq 0$. 


Lemma 3.7. Let \( \{X_0, X_1, \cdots, X_t\} \) be a mutation team in \( \text{mod } A^{(m)} \). Then

1. \( \deg X_0 = 0 \).
2. For any \( 0 \leq i \leq t - 1 \), \( 0 \leq \deg X_{i+1} - \deg X_i \leq 1 \).
3. There are at most two elements in \( \{X_0, X_1, \cdots, X_t\} \) with same degree.
4. \( 2m \leq t \leq 2m + 1 \).

Proof. (1) Suppose that \( \deg X_0 = r \geq 1 \). Without loss of generality, suppose that \( \deg X_0 = 1 \). Then there is a non-split exact sequence

\[
0 \rightarrow \Omega_{A^{(m)}} X_0 \rightarrow I \rightarrow X_0 \rightarrow 0,
\]

where \( I \rightarrow X_0 \rightarrow 0 \) is a projective cover and \( I \) is projective-injective. Clearly, \( \Omega_{A^{(m)}} X_0 = \Omega_X X_0 \). Since \( X_0 \) is exceptional, it follows from Lemma 3.5 that \( \Omega_{A^{(m)}} X_0 \) is also exceptional. Applying \( \text{Hom}_{A^{(m)}}(\cdot, \Omega_{A^{(m)}} X_0) \) to the sequence above, we have

\[
\cdots \rightarrow \text{Hom}_{A^{(m)}}(I, \Omega_{A^{(m)}} X_0) \rightarrow \text{Hom}_{A^{(m)}}(\Omega_{A^{(m)}} X_0, \Omega_{A^{(m)}} X_0) \rightarrow \cdots.
\]

It is easy to see that \( \text{Hom}_{A^{(m)}}(I, \Omega_{A^{(m)}} X_0) = 0 \) and that \( \text{Ext}^1_{A^{(m)}}(I, \Omega_{A^{(m)}} X_0) = 0 \). By Lemma 3.2, \( \text{Hom}_{A^{(m)}}(\Omega_{A^{(m)}} X_0, \Omega_{A^{(m)}} X_0) \cong \text{Hom}_A(\Omega_{A^{(m)}} X_0, \Omega_{A^{(m)}} X_0) \cong k \), which implies that \( \text{Ext}^1_{A^{(m)}}(X_0, \Omega_{A^{(m)}} X_0) \cong k \).

Denote \( \Omega_{A^{(m)}} X_0 \) by \( X_{-1} \). Then we have that

\[
\dim_k \text{Ext}^{s+1}_{A^{(m)}}(X_j, X_{-1}) = \begin{cases} 
1 & \text{if } s = j \text{ and } s \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

It follows that \( \{X_{-1}, X_0, \cdots, X_t\} \) form a mutation team in \( \text{mod } A^{(m)} \), which is a contradiction. This completes the proof of (1).

2. Suppose that \( \deg X_i = r \) and \( \deg X_{i+1} = r + p \), that is, there are two indecomposable \( A \)-modules \( M \) and \( N \) such that \( X_i \cong \Omega^{-r}_{A^{(m)}} M \) and \( X_{i+1} \cong \Omega^{-(r+p)}_{A^{(m)}} N \). Then we see that

\[
k \cong \text{Ext}^1_{A^{(m)}}(X_{i+1}, X_i) \\
\cong \text{Hom}_{A^{(m)}}(X_{i+1}, \Omega^{-1}_{A^{(m)}} X_i) \\
\cong \text{Hom}_{D^b(A)}(X_{i+1}, X_i[1]) \\
\cong \text{Hom}_{D^b(A)}(N, M[1 - p])
\]
It follows that $p = 0$ or $p = 1$. This finishes the proof of (2).

(3) For $p \geq 2$ and $0 \leq i \leq t$, we claim that $\deg X_i \neq \deg X_{i+p}$.

Otherwise, $\deg X_i = \deg X_{i+p}$ and $p \geq 2$ imply that $\text{Ext}^p_{D^b(A)}(X_{i+p}, X_i) = 0$.

On the other hand, we have that

$$k \cong \text{Ext}^p_{A^m}(X_{i+p}, X_i)$$

$$\cong \text{Hom}_{A^m}(X_{i+p}, \Omega A^m X_i)$$

$$\cong \text{Hom}_{D^b(A)}(X_{i+p}, X_i[p])$$

$$\cong \text{Ext}^p_{D^b(A)}(X_{i+p}, X_i),$$

which is a contradiction.

Now we suppose that $X_i$, $X_{i+1}$ and $X_j$, $X_{j+1}$ are the first four elements in

\{ $X_0, X_1, \cdots, X_t$ \} such that $\deg X_i = \deg X_{i+1}$ and $\deg X_j = \deg X_{j+1}$. It is easy to see that $j > i+1$. According to (1) and (2), we have that $\deg X_i = \deg X_{i+1} = i$ and $\deg X_{j+1} = \deg X_j = j - 1$ and $\deg X_{j+1} = \deg \Omega X_0$.

On the other hand,

$$k = \text{Ext}^{j+1}_{A^m}(X_{j+1}, X_0)$$

$$\cong \text{Hom}_{A^m}(X_{j+1}, \Omega^{j+1} X_0)$$

$$\cong \text{Hom}_{D^b(A)}(X_{j+1}, X_0[j + 1])$$

$$\cong \text{Ext}^2_{D^b(A)}(X_{j+1}, X_0[j - 1]),$$

which is a contradiction. This completes the proof of (3).

(4) Since $\text{gl.dim} A^m = 2m + 1$, the consequence follows from (1), (2) and (3).

Recall from [ABST2], the $m$-left part $L_m(A^m)$ of mod $A^m$ consists of the indecomposable $A^m$-modules all of whose predecessors have projective dimension at most $m$.

**Corollary 3.8.** Let \{ $X_0, X_1, \cdots, X_t$ \} be a mutation team in mod $A^m$, and \{ $X_0, X_1, \cdots, X_t$ \} = \{ $X_0, X_1, \cdots, X_t$ \} $\cap L_m(A^m)$ be the partial mutation team in the $m$-left part of mod $A^m$. Then $m - 1 \leq l \leq m$.

**Theorem 3.9.** Let $N$ be a partial mutation team in the $m$-left part of mod $A^m$.

Assume that $N$ has exactly $m + 1$ elements \{ $X_0, \cdots, X_m$ \}. Then there exists a
faithful almost complete tilting $A^{(m)}$-module $T$ such that $T \oplus X_i, 0 \leq i \leq t$ are all tilting $A^{(m)}$-modules.

**Proof.** The case of $m = 1$ has been proved in [Z1] and then we assume that $m \geq 2$. We only need to prove that

$$\text{Ext}^l_{A^{(m)}}(X_j, X_i) = \text{Ext}^l_{C^{(m)}}(\pi(X_j), \pi(X_i))$$

for $1 \leq l \leq m$ and $0 \leq i \leq j \leq m$, that is, to show that $\pi(X_0), \pi(X_1), \ldots, \pi(X_m)$ form an exchange team in $C^{(m)}$ in sense of [ZZ]. Then according to Theorem 5.8 in [ZZ], there exists an almost complete tilting object $\pi(T')$, where $T'$ is a non-projective-injective exceptional $A^{(m)}$-module, such that $\pi(T') \oplus X_i, 0 \leq i \leq m$, are all $m$-cluster tilting objects. By Theorem 29 in [ABST2], $T'$ has projective dimension at most $m$ and $T' \oplus P$ is a faithful almost complete tilting $A^{(m)}$-module, where $P$ is the direct sum of all indecomposable projective-injective $A^{(m)}$-modules. Let $T = T' \oplus P$. Then $T$ is just what we want.

Firstly, we assume that $i + l = j$. Then $\dim_k \text{Ext}^l_{A^{(m)}}(X_{i+l}, X_i) = 1$. Let

$$0 \rightarrow X_i \rightarrow T_i \rightarrow T_{i+1} \rightarrow \cdots \rightarrow T_{i+l-1} \rightarrow X_{i+l} \rightarrow 0$$

be a $k$-basis of $\text{Ext}^l_{A^{(m)}}(X_{i+l}, X_i)$ given by a chain of non-split short exact sequences:

$$0 \rightarrow X_{i+s} \rightarrow T_{i+s} \rightarrow X_{i+s+1} \rightarrow 0, \ 0 \leq s \leq l - 1.$$

By [H], each

$$0 \rightarrow X_{i+s} \rightarrow T_{i+s} \rightarrow X_{i+s+1} \rightarrow 0$$

gives rise to a triangle

$$X_{i+s} \rightarrow \bar{T}_{i+s} \rightarrow X_{i+s+1} \rightarrow X_{i+s}[1]$$

in $D^b(A)$, which is non-zero in

$$\text{Hom}_{D^b(A)}(X_{i+s+1}, X_{i+s}[1]) = \text{Ext}^1_{D^b(A)}(X_{i+s+1}, X_{i+s}).$$

Then the induced map $X_{i+l} \rightarrow X_{i+l-1}[1] \rightarrow \cdots \rightarrow X_i[l]$ is non-zero in $\text{Hom}_{D^b(A)}(X_{i+l}, X_i[l])$ and thus

$$\text{Hom}_{D^b(A)}(X_{i+l}, \Omega^{-l}X_i) \simeq \text{Hom}_{D^b(A)}(X_{i+l}, X_i[l]) \neq 0.$$
By the assumption $\text{Ext}^l_A(X_{i+l}, X_i) \simeq \text{Ext}^l_{A(m)}(X_{i+l}, X_i) \simeq k$ and Lemma 3.1, we get that

$$\text{Hom}_{D^b(A)}(X_{i+l}, X_i[l]) \simeq \text{Hom}_{A}(X, \Omega^{-l}X) \simeq \text{Ext}^l_A(X_{i+l}, X_i) \simeq k.$$ 

Since $m \geq 2$, we have that $\text{Hom}_{D^b(A)}(X_{i+l}, \tau X_i[l - m]) = 0$ and that $\text{Hom}_{D^b(A)}(X_{i+l}, \tau^{-1}X_i[l + m]) = 0$. Therefore

$$\text{Ext}^l_{A(m)}(\pi(X_{i+l}), \pi(X_i)) \simeq \text{Hom}_{A}(\pi(X_{i+l}), \pi(X_i)[l]) \simeq \text{Hom}_{D^b(A)}(\pi(X_{i+l}), \pi(X_i)[l]) \oplus \text{Hom}_{D^b(A)}(\pi(X_{i+l}), \tau\pi(X_i)[l - m]) \oplus \text{Hom}_{D^b(A)}(\pi(X_{i+l}), \tau^{-1}\pi(X_i)[l + m]) \simeq \text{Hom}_{D^b(A)}(X_{i+l}, X_i[l]) \oplus \text{Hom}_{D^b(A)}(X_{i+l}, \tau X_i[l - m]) \oplus \text{Hom}_{D^b(A)}(X_{i+l}, \tau^{-1}X_i[l + m]) \simeq \text{Hom}_{D^b(A)}(X_{i+l}, X_i[l]) \simeq k.$$ 

This finishes the proof for the case of $i + l = j$.

Now we assume that $i + l \neq j$. Without loss of generality, we assume that $i + l < j$. Then it is easy to see that $\deg X_j - \deg X_i \geq 1$. Since

$$\text{Ext}^l_{A(m)}(X_j, X_i) \simeq \text{Ext}^l_{A}(X_j, X_i) \simeq \text{Hom}_{A}(X_j, \Omega^{-l}X_i) = 0,$$

we have that

$$\text{Hom}_{D^b(A)}(\pi(X_j), \pi(X_i)[l]) \simeq \text{Hom}_{D^b(A)}(X_j, X_i[l]) \simeq \text{Hom}_{A}(X_j, \Omega^{-l}X_i) = 0.$$ 

Since $m \geq 2$, it is easy to see that

$$\text{Hom}_{D^b(A)}(\pi(X_j), \tau\pi(X_i)[l - m]) \simeq \text{Hom}_{D^b(A)}(X_j, \tau X_i[l - m]) = 0.$$ 

We now claim that $\text{Hom}_{D^b(A)}(\pi(X_j), \tau^{-1}\pi(X_i)[l + m]) = 0$. Note that $l + i + m - j \geq 1$ since $l \geq 1$ and $m \geq j$. Then if $l + i + m - j \geq 2$, our claim is true since $\deg \pi(X_j) \leq \deg \tau^{-1}\pi(X_i)[j - i] \leq \deg \tau^{-1}\pi(X_i)[l + m] - 2$. For the case of $l + i + m - j = 1$, that is, $l = 1$, $i = 0$ and $j = m$, we will show that

$$\text{Hom}_{D^b(A)}(\pi(X_m), \tau^{-1}\pi(X_0)[m][1]) \simeq \text{Hom}_{D^b(A)}(X_m, \tau^{-1}X_0[m][1]) = 0.$$
By Lemma 3.4 in [LLZ], $\text{pd}_{A^{(m)}} X_m = m$ and thus $\deg X_m$ is either $m - 1$ or $m$.

If $\deg X_m = m - 1$, our claim holds because $\deg \tau^{-1} X_0[m + 1] \geq m + 1$.

If $\deg X_m = m$, there exists an indecomposable projective $A$-module $P$ such that $X_m = P[m]$. Then

$$\text{Hom}_{\mathcal{D}^b(A)}(X_m, \tau^{-1} X_0[m][1]) \simeq \text{Hom}_{\mathcal{D}^b(A)}(P, \tau^{-1} X_0[1]) = 0.$$  

By the arguments above, we get that, for the case of $l + i < j$,

$$\text{Ext}_{\mathbb{D}^b_m(A)}^{l}(\pi(X_{i+l}), \pi(X_i)) \simeq \text{Hom}_{\mathbb{D}^b_m(A)}(\pi(X_{i+l}), \pi(X_i)[l])$$

$$\simeq \text{Hom}_{\mathcal{D}^b(A)}(\pi(X_{i+l}), \pi(X_i)[l])$$

$$\oplus \text{Hom}_{\mathcal{D}^b(A)}(\pi(X_{i+l}), \tau \pi(X_i)[l - m])$$

$$\oplus \text{Hom}_{\mathcal{D}^b(A)}(\pi(X_{i+l}), \tau^{-1} \pi(X_i)[l + m])$$

$$\simeq \text{Hom}_{\mathcal{D}^b(A)}(X_{i+l}, X_i[l])$$

$$\oplus \text{Hom}_{\mathcal{D}^b(A)}(X_{i+l}, \tau X_i[l - m])$$

$$\oplus \text{Hom}_{\mathcal{D}^b(A)}(X_{i+l}, \tau^{-1} X_i[l + m])$$

$$= 0.$$  

This finishes the proof of the theorem. \qed

4 Relation with AR $(m + 3)$-angle in $\mathcal{C}_m(A)$

In this section, we shall give a further explanation about the relationship between the tilting mutation in $\text{mod } A^{(m)}$ and the $m$-cluster mutation in $\mathcal{C}_m(A)$.

Let $T$ be a faithful almost complete tilting $A^{(m)}$-module with $\text{pd}_{A^{(m)}} T \leq m$. By [LLZ], $T$ has exactly $m + 1$ indecomposable non-isomorphic complements $X_0, \ldots, X_m$ with projective dimensions at most $m$, which are connected by the long exact sequence:

$$(*): 0 \rightarrow X_0 \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{m-1} \rightarrow X_m \rightarrow 0,$$

where $T_i \in \text{add } T$ for all $0 \leq i \leq m - 1$, $X_i = \text{Coker } g_{i-1}$ for $1 \leq i \leq m$ and each of the induced monomorphisms $X_i \hookrightarrow T_i$ is a minimal left add $T$-approximation.

It follows from Theorem 29 in [ABST2] that $\pi(T)$ is an almost complete $m$-cluster
tilting object in $\mathcal{C}_m(A)$, and that $\pi(X_0), \ldots, \pi(X_m)$ are its $m+1$ indecomposable non-isomorphic complements, which are connected by the connecting triangles:

$$
\pi(X_i) \xrightarrow{\overline{T}_i} \pi(T_i) \xrightarrow{\overline{g}_i} \pi(X_{i+1}) \longrightarrow \pi(X_i)[1],
$$

where $\overline{T}_i$ is the minimal left add $\pi(T)$-approximation of $\pi(X_i)$ and $\overline{g}_i$ is the minimal right add $\pi(T)$-approximation of $\pi(X_{i+1})$. Then we have a long angle:

$$\pi(X_0) \longrightarrow \pi(T_0) \longrightarrow \pi(T_1) \longrightarrow \cdots \longrightarrow \pi(T_m) \longrightarrow \pi(X_0).$$

It is easy to see that (**) is an AR $(m + 3)$-angle for $m \geq 2$ by Corollary 4.4 in [ZZ] and for $m = 1$ by Lemma 6.13 in [BMRRT]. Now, we want to show that (**) is induced by (*).

**Lemma 4.1** For all $0 \leq i \leq m - 1$, the $i$-th connecting sequence

$$0 \longrightarrow X_i \xrightarrow{f_i} T_i \xrightarrow{g_i} X_{i+1} \longrightarrow 0$$

in mod $A^{(m)}$ induces a triangle in $\mathcal{C}_m(A)$:

$$\pi(X_i) \xrightarrow{T_i} \pi(T_i) \xrightarrow{g_i} \pi(X_{i+1}) \longrightarrow \pi(X_i)[1],$$

where $\overline{T}_i$ is the minimal left add $\pi(T)$-approximation of $\pi(X_i)$ and $\overline{g}_i$ is the minimal right add $\pi(T)$-approximation of $\pi(X_{i+1})$.

**Proof.** By [H], the short exact sequence in mod $A^{(m)}$ (also in mod $\hat{A}$)

$$0 \longrightarrow X_i \xrightarrow{f_i} T_i \xrightarrow{g_i} X_{i+1} \longrightarrow 0$$

gives rise to a triangle $X_i \longrightarrow T_i \longrightarrow X_{i+1} \longrightarrow X_i[1]$ in $\mathcal{D}^b(A)$ and hence a triangle in $\mathcal{C}_m(A)$:

$$\pi(X_i) \xrightarrow{T_i} \pi(T_i) \xrightarrow{\overline{g}_i} \pi(X_{i+1}) \longrightarrow \pi(X_i)[1].$$

Note that $\pi$ is an exact functor and deg$X_i \leq m$ for all $0 \leq i \leq m$. By Lemma 2.1 and Lemma 3.1, non-split sequence $0 \longrightarrow X_i \xrightarrow{f_i} T_i \xrightarrow{g_i} X_{i+1} \longrightarrow 0$ induces a non-split triangle $X_i \longrightarrow T_i \longrightarrow X_{i+1} \longrightarrow X_i[1]$ in $\mathcal{D}^b(A)$. Moreover, $\pi(X_i) \xrightarrow{T_i} \pi(T_i) \xrightarrow{\overline{g}_i} \pi(X_{i+1}) \longrightarrow \pi(X_i)[1]$ is not-split in $\mathcal{C}_m(A)$.

It follows from [ZZ] that $\dim_k \text{Ext}_{\mathcal{C}_m(A)}^1(\pi(X_{i+1}), \pi(X_i)) = 1$. 

14
Let $\pi(X_i) \xrightarrow{\alpha} E_i \xrightarrow{\beta} \pi(X_{i+1}) \xrightarrow{} \pi(X_i)[1]$ be a basis of $\text{Ext}^1_{C_m(A)}(\pi(X_{i+1}), \pi(X_i))$, where $\alpha$ (resp. $\beta$) is the minimal left (resp. right) add $\pi(T)$-approximation of $\pi(X_i)$ (resp. $\pi(X_{i+1})$). Then $\pi(X_i) \xrightarrow{T_i} \pi(T_i) \xrightarrow{\overline{T}_i} \pi(X_{i+1}) \xrightarrow{} \pi(X_i)[1]$ is isomorphic to this basis and hence $\overline{T}_i$ is the minimal left add $\pi(T)$-approximation of $\pi(X_i)$ and $\overline{T}_i$ is the minimal right add $\pi(T)$-approximation of $\pi(X_{i+1})$.

**Theorem 4.2.** Let $T$ be a faithful almost complete tilting $A^{(m)}$-module with projective dimension at most $m$ and $X_0, \cdots, X_m$ be its indecomposable non-isomorphic complements with projective dimension at most $m$. Then the induced $(m+3)$-angle in $C_m(A)$ by the $m+1$ connecting sequences is just the AR $(m+3)$-angle in the sense of [IY].

**Proof.** By Lemma 4.1, the long exact sequence in mod $A^{(m)}$

\[(*) \quad 0 \rightarrow X_0 \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{m-1} \rightarrow X_m \rightarrow 0\]

induces a long angle in $C_m(A)$

\[(1*) \quad \pi(X_0) \rightarrow \pi(T_0) \rightarrow \pi(T_1) \rightarrow \cdots \rightarrow \pi(T_{m-1}) \rightarrow \pi(X_m).\]

Let $\pi(X_m) \rightarrow \pi(T_m)$ be the minimal left add $\pi(T)$-approximation, which induces a triangle in $C_m(A)$

\[(2*) \quad \pi(X_m) \rightarrow \pi(T_m) \rightarrow \pi(X_0) \rightarrow \pi(X_m)[1],\]

where $\pi(T_m) \rightarrow \pi(X_0)$ is the minimal right add $\pi(T)$-approximation by [BMRRT]. Now we get the long angle $(**)$ by connecting $(1*)$ and $(2*)$. By Corollary 4.4 in [ZZ], the proof is finished.

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