QUANTISATION OF THE SU(N) WZW MODEL AT LEVEL $k$

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ABSTRACT

The quantisation of the Wess-Zumino-Witten model on a circle is discussed in the case of $SU(N)$ at level $k$. The quantum commutation of the chiral vertex operators is described by an exchange relation with a braiding matrix, $Q$. Using quantum consistency conditions, the braiding matrix is found explicitly in the fundamental representation. This matrix is shown to be related to the Racah matrix for $U_t(SL(N))$. From calculating the four-point functions with the Knizhnik-Zamolodchikov equations, the deformation parameter $t$ is shown to be $t = \exp(i\pi/(k + N))$ when the level $k \geq 2$. For $k = 1$, there are two possible types of braiding, $t = \exp(i\pi/(1 + N))$ or $t = \exp(i\pi)$. In the latter case, the chiral vertex operators are constructed explicitly by extending the free field realisation a la Frenkel-Kac and Segal. This construction gives an explicit description of how to chirally factorise the $SU(N)_{k=1}$ WZW model.

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Introduction

Rational conformal field theory\cite{1} has been very useful and instructive for many aspects of two-dimensional physics. This is largely due to the nice properties that follows from its symmetry algebras\cite{1,2}. The generally accepted definition of rational conformal field theory\cite{3} is that the symmetry algebras are formed by two commuting chiral algebras, $\mathcal{A} \times \bar{\mathcal{A}}$, and each algebra contains a copy of Virasoro algebra which ensures the conformal symmetry. In general, it also contains other chiral algebras such as the Kac-Moody algebras in the case of Wess-Zumino-Witten models. Consequently, the Hilbert space can be decomposed into a \textit{finite} number of highest weight representations of $\mathcal{A} \times \bar{\mathcal{A}}$.

$$\mathcal{H} = \bigoplus_{\lambda} H_{\lambda} \otimes \mathcal{H}_\lambda$$  \hspace{1cm} (1)

In the Euclidean formulation, there is a state/vertex operator correspondence which says that the highest weight state in $H_{\lambda} \otimes \mathcal{H}_\lambda$ can be obtained by applying the corresponding primary field at the origin to the vacuum, $\Phi_{\lambda,\bar{\lambda}}(0,0)|0\rangle$. Under conformal transformations, the behaviour of each primary field is determined by its conformal weight $(\Delta_{\lambda}, \Delta_{\bar{\lambda}})$. One important consequence of these properties is that one can calculate all the correlation functions in the theory from those of the primary fields using the Ward identities derived from the two conserved chiral currents. Since the two chiral algebras commute, these correlation functions are decomposed into a sum of products of holomorphic and antiholomorphic functions, which are called the \textit{conformal blocks}\cite{1}. This suggests that the primary field $\Phi_{\lambda,\bar{\lambda}}(z, \bar{z})$ is also decomposed into a sum of products of holomorphic primary fields, $U_{a,\lambda}(z)$, and antiholomorphic primary fields, $V_{\bar{a},\bar{\lambda}}(\bar{z})$,

$$\Phi_{\lambda,\bar{\lambda}}(z, \bar{z}) = \sum_{a,\bar{a}} C_{a,\bar{a},\lambda,\bar{\lambda}} U_{a,\lambda}(z) V_{\bar{a},\bar{\lambda}}(\bar{z}).$$  \hspace{1cm} (2)

From this point of view, conformal blocks are formed by the correlation functions of the chiral primary fields and one should be able to derive the braiding and fusion properties\cite{3} of the conformal blocks from the local properties of these operators. In particular, they are nonlocal operators and obey braiding relations of the following type,

\begin{align*}
U(z) \otimes U(w) &= (1 \otimes U(w)) (U(z) \otimes 1) Q(z, w), \\
V(\bar{z}) \otimes V(\bar{w}) &= \overline{Q}(\bar{z}, \bar{w}) (1 \otimes V(\bar{w})) (V(\bar{z}) \otimes 1) .
\end{align*}  \hspace{1cm} (3)

They generate the entire field content of the chiral models and the chiral Hilbert space can be determined completely from them. The chiral primary fields satisfying these descriptions are called the \textit{chiral vertex operators}\cite{3-8}. (In particular, we are refering to the \textit{vertex-chiral vertex operators} discussed in \cite{4}.)

It is therefore natural to ask how one can formulate the chiral conformal field theory from these chiral vertex operators such that their correlation functions reproduce the conformal blocks. Such a formulation is especially desirable from a phenomenological point
of view when one discusses the compactification of string theories. Whilst there has been a number of studies\cite{3-11} indicating that a quantum group related symmetry acts in the chiral theory, a description of precisely how the factorisation occurs and how the physical states in the chiral Hilbert space transform under the quantum group is lacking.

This paper is a continuation of the investigation of this problem using the approach of canonical quantisation begun in\cite{11} and briefly reviewed below. The aim is to quantise the chiral primary fields in the Wess-Zumino-Witten (WZW) models associated with a group $G$ and compactified on a circle. As explained in\cite{10-12}, these chiral primary fields obey a quantum exchange relation given by a braiding matrix, $Q$. An explicit solution for $Q$ was obtained for $SU(2)$. In this paper, we solve for the braiding matrix in the fundamental representation for the model associated with $SU(N)$ at level $k$. Because the braiding matrix can be related to the Racah matrix for the quantum group $U_t(SL(N))$, this implies indirectly that $U_t(SL(N))$ is acting on the chiral Hilbert space. Instead of using the conformal scaling argument in\cite{11}, we show that the quantum value of the deformation parameter $t$ can be determined using the Knizhnik-Zamolodchikov (KZ) equations. For $k \geq 2$, our results agree with the usual observation that $t = \exp(i\pi/(k + N))$. In the level-one case, we find that $U_t(SL(N))$ with $t = e^{i\pi}$ also describes the additional symmetry in the quantised chiral theory. Notice that when the deformation parameter is a root of unity, the tensor products of the irreducible representations of $U_t(SL(N))$ are truncated to contain only the integrable representations. These tensor products can not be made co-associative for all irreducible representations\cite{13,14}. Consequently, the corresponding symmetry algebra should be a quasi quantum group, i.e. a quasi-Hopf algebra\cite{15}.

For the simplest models with the level $k = 1$, the chiral symmetry algebras, i.e. level-one Kac-Moody algebras, has an unitary realisation in terms of free scalar fields a la Frenkel-Kac and Segal\cite{16}. In this paper, we extend this construction to give the chiral vertex operators such that their local properties can be examined directly. This provides an explicit description of the chiral Hilbert space acted by two symmetry algebras, $SU(N)_{k=1}$ Kac-Moody algebra and the quantum group $U_{-1}(SU(N))$ (co-associative when $t = -1$). This construction not only exemplifies the results obtained from the canonical quantisation, but it also provides a useful insight for formulating the corresponding chiral WZW models.

**Canonical Structures of the WZW Models**

Let us first briefly review the Poisson structures of the WZW models derived in\cite{11}. Consider the defining field $g(\tau, x)$ which takes values in a compact simple Lie group $G$ of rank $r$ and $x$ is the spatial coordinate for a circle of length $2\pi$ such that $g(\tau, x + 2\pi) = g(\tau, x)$. Let $\psi$ be the longest root and $\{t^a\} \equiv \{H^i, E^a\}$ denote a basis for the generators of the Lie algebra, $\mathcal{G}$. We shall normalise these generators so that $\text{Tr}(t^a t^b) = N \delta^{ab}$. The action of the WZW model has an infra-red fixed point and it can be written as

$$A[g] = -\frac{k \psi^2}{16\pi N} \left\{ \int_{\mathcal{M}} \text{tr} \left( g^{-1} \partial_\mu gg^{-1} \partial^\mu g \right) d^2x - \frac{2}{3} \int_{\mathcal{B}} \epsilon^{\lambda\mu\nu} \text{tr} \left( \hat{g}^{-1} \partial_\lambda \hat{g} \hat{g}^{-1} \partial_\mu \hat{g} \hat{g}^{-1} \partial_\nu \hat{g} \right) d^3x \right\},$$

(4)

where $\mathcal{B}$ is a three-dimensional manifold whose boundary is the cylinder $\mathcal{M}$ and $\hat{g}(\tau, x, y) \in G$ is defined on $\mathcal{B}$ such that it maps to $g(\tau, x)$ at $y = 0$. In order for the action to be well
defined, \( k \) must be an integer so that \( e^A \) is single valued.

The equations of motion, i.e. the Euler-Lagrange equations obtained from (4), can be written in terms of two chirally conserved currents:

\[
\partial_+(g^{-1}\partial_-g) = 0, \quad \partial_-(\partial_+gg^{-1}) = 0, \quad x^\pm \equiv \tau \pm x.
\]  

(5)

In the gauge fixed approach of \([11]\), a general solution of the equations of motion can be written in terms of the periodic coset variables \( \tilde{u} \in LG/H \) and \( \tilde{v} \in H\backslash LG \):

\[
g(x, t) = \tilde{u}(x^+)e^{iq\cdot \nu H}e^{i2(x^+ + x^-)\nu \cdot H} \tilde{v}(x^-).
\]  

(6)

Substituting the parameterisation of (6) into the symplectic two form, we can invert it and obtain the Poisson brackets. In terms of the new chiral variables,

\[
u(x^+) \equiv \tilde{u}(x^+)e^{iq\cdot H}e^{i2x^+\cdot H} \tilde{v}(x^-),
\]  

(7)

the Poisson brackets are listed as follows.

\[
\{q^i, \nu^j\} = \frac{1}{2} \delta^{ij}, \quad \{\tilde{u}(x^+), \tilde{v}(x^-)\} = 0, \quad \{\tilde{u}(x^+), \nu^j\} = 0, \quad \{\tilde{v}(x^-), \nu^j\} = 0,
\]

\[
\{u_1(x^+), v_2(y^-)\} = (x^+ - y^-)\beta u_1(x^+)\overline{v}_2(y^-) + \{\tilde{u}_1(x^+), iq_\nu H_2\}e^{i2x^+\cdot H_1}e^{iq_\nu H_1}v_2(y^-) + u_1(x^+)e^{iq_\nu H_2}e^{i2y^-\nu H_2}\{iq_\nu H_1, \overline{v}_2(y^-)\},
\]

\[
\{u_1(x^+), u_2(y^+)\} = u_1(x^+)u_2(y^+) \quad r(x^+ - y^+),
\]

\[
\{v_1(x^-), v_2(y^-)\} = \overline{r}(x^- - y^-) \quad v_1(x^-)v_2(y^-),
\]  

(8)

where \( \overline{H} \equiv H_1 \cdot H_2 = \sum_{j=1}^{r} H^j \otimes H^j \) and \( \beta = 4\pi/\psi^2 k \). The classical r-matrices in the last two brackets of (8) are

\[
r(x^+) = \frac{\beta}{2} \eta(x^+)|H| + i\frac{\beta}{2} \sum_{\alpha} \frac{1}{\sin(\alpha \cdot \nu)} e^{-i\alpha \cdot \nu \eta(x^+)} E^\alpha \otimes E^{-\alpha},
\]

(9)

\[
\overline{r}(x^-) = \frac{\beta}{2} \eta(x^-)|H| - i\frac{\beta}{2} \sum_{\alpha} \frac{1}{\sin(\alpha \cdot \nu)} e^{i\alpha \cdot \nu \eta(x^-)} E^\alpha \otimes E^{-\alpha},
\]

with \( \eta(x) = 2[x/2\pi] + 1 \) and \([y]\) denotes the maximal integer less than \( y \). Notice that the Poisson bracket, \( \{u_1, v_2\} \) in (8), is different from the one given in Eq. (36) of [11] because \( \{\tilde{u}, q_\nu\} \) and \( \{\tilde{v}, q_\nu\} \) do not vanish in general. They can be made to vanish in a natural way for \( \tilde{u} \) and \( \tilde{v} \) at the identity but will be present away from it. Their precise forms will depend on the way the coset representations \( \tilde{u} \) and \( \tilde{v} \) are chosen. We have not specified them explicitly and they are not needed for the discussion in the rest of this paper because each chiral sector can be quantised separately. The fact that the Poisson bracket in (36) of [11] was not consistent was drawn to our attention by G. Papadopoulos. One can easily verify that the Poisson brackets in (8) satisfy the Jacobi’s identities.
Classically, we can use these Poisson brackets to verify that \( u (v) \) transforms as a primary field of the right-moving (left-moving) Kac-Moody algebra, i.e.

\[
\{ J^a_+ (x^+) , u(y^+) \} = it^a u(y^+) \delta (x^+ - y^+) , \quad \{ J^a_- (x^-) , v(y^-) \} = iv(y^-) t^a \delta (x^- - y^-) .
\]

(10)

The Kac-Moody currents can be written in terms of the chiral group elements \( u(x^+) \) and \( v(x^-) \) as

\[
J_+ (x^+) = (ik/4\pi) \partial_+ uu^{-1} , \quad J_- (x^-) = (ik/4\pi) v^{-1} \partial_- v .
\]

(11)

**Quantisation and the Braiding Matrix**

Let us proceed to quantise these Poisson brackets. The first four brackets in (8) can simply be replaced by Dirac commutators,

\[
[q_i^j , \nu^j] = i \frac{\beta \hbar}{2} \delta^{ij} , \quad [\tilde{u}_1 (x^+) , \tilde{v}_2 (y^-)] = 0 , \quad [\tilde{u} (x^+) , \nu^j] = 0 , \quad [\tilde{v} (x^-) , \nu^j] = 0 .
\]

(12)

The quantum correction to the constant \( \beta \hbar \) will be determined later from consistency conditions. Following [11], we quantise the last two brackets in (8) by the following exchange relation,

\[
u_1 (x^+) u_2 (y^+) = u_2 (y^+) u_1 (x^+) Q (x^+ - y^+) ,
\]

(13a)

\[
u_1 (x^-) v_2 (y^-) = Q^{-1} (x^- - y^-) v_2 (y^-) v_1 (x^-) .
\]

(13b)

where \( Q(x) \) is the braiding matrix which can be determined from the consistency of quantisation as follows.

**Unitarity,**

\[
Q^\dagger (x) = Q^{*T} (x) = Q^{-1} (x) .
\]

(14a)

**Antisymmetry property,**

\[
Q (-x) = \mathbb{P} Q(x) \mathbb{P}^{-1} .
\]

(14b)

**Monodromy property,**

\[
Q (x + 2\pi) = (M(\nu)^{-1} \otimes \mathbb{I}) Q(x) (M(\nu_2) \otimes \mathbb{I})
\]

\[
= (\mathbb{I} \otimes M(\nu_1)) Q(x) (\mathbb{I} \otimes M(\nu)^{-1})
\]

(14c)

**Classical limit,**

\[
Q (x) = \mathbb{I} \otimes \mathbb{I} + i \hbar r (x) + O (\hbar^2)
\]

(14d)

**Locality condition,**

\[
[Q(x) , e^{iq(\nu_1 + \nu_2)} H] = 0
\]

(14e)

**Jacobi identities,** with \( x_{ij} \equiv x_i - x_j \),

\[
Q_{23} (\nu_1 , x_{23}) Q_{13} (\nu , x_{13}) Q_{12} (\nu_3 , x_{12}) = Q_{12} (\nu , x_{12}) Q_{13} (\nu_2 , x_{13}) Q_{23} (\nu , x_{23}) ,
\]

(14f)

The subscript of \( \nu \) denotes a shift, e.g. \( \nu_2 = \nu + \frac{1}{2} \hbar \beta H_2 \), due to the commutation relation,

\[
[u(x) , \nu] = -\frac{1}{2} \beta \hbar u(x) \cdot H .
\]

(15)

These six requirements are fairly obvious except the locality condition, (14f), which is added to ensure that two \( g \)'s at a space-like separation commute with each other.
In the following, we solve for the braiding matrix in the \( N \)-dimensional fundamental representation for \( SU(N) \) at level \( k \). Let \( \Phi \) denotes the space of roots and \( \vec{\lambda}_a \) for \( a = 1, 2, \ldots N \) be the weights in the fundamental representation whose highest weight is \( \lambda_1 \). The normalisation of these vectors is chosen to be \( \vec{\lambda}_a \cdot \vec{\lambda}_b = \delta_{ab} - 1/N \). Let \( e_{ab} \) denote a \( N \times N \) matrix which has only one non-zero element, 1, located at the \( a\)th row and the \( b\)th column. Then, the Cartan-Weyl basis is given by \( \vec{H} = \sum_{a=1}^{N} \vec{\lambda}_a e_{aa} \) and the step-operator \( E_{\alpha ab} = e_{ab} \) for \( \vec{\alpha} = \vec{\lambda}_a - \vec{\lambda}_b \in \Phi \). With this normalisation, \( N = 1 \) and the length of the longest root is \( \psi^2 = 2 \).

In order to solve for \( Q \) in (14), we exponentiate the classical \( r \)-matrix in (9) and replace its coefficients by their quantum values: namely, \( \chi \), \( \theta \), and \( \gamma \).

\[
Q(x) = \exp \left\{ i\chi \eta(x)\vec{H} - \sum_{\alpha \in \Phi} \theta(\alpha \cdot \nu)e^{-i\gamma_\alpha \eta(x)}E_\alpha \otimes E_{-\alpha} \right\},
\]

\[
= t^{\frac{N-1}{N}} \eta(x) \left( 1 \otimes 1 - \sum_{\alpha \in \Phi} E_\alpha E_{-\alpha} \otimes E_{-\alpha} E_\alpha \right) + t^{-\frac{1}{N}} \eta(x) \sum_{\alpha \in \Phi} \cos \theta(\alpha \cdot \nu)E_\alpha E_{-\alpha} \otimes E_{-\alpha} E_\alpha - t^{-\frac{1}{N}} \eta(x) \sum_{\alpha \in \Phi} e^{-i\gamma_\alpha \eta(x)} \sin \theta(\alpha \cdot \nu)E_\alpha \otimes E_{-\alpha}.
\]

Without specifying the values of these coefficients, (16) already satisfies most of the requirements in (14) except the monodromy property and the Jacobi’s identities. These two properties impose constraints on the coefficients in \( Q \) and they can be solved as follows.

\[
\gamma_\alpha = \alpha \cdot \nu, \quad \sin \theta(\alpha \cdot \nu) = \frac{\sin(\chi)}{\sin(\alpha \cdot \nu)}, \quad \chi = \frac{1}{2} \beta \hbar \longrightarrow \frac{\pi}{k} + O(\hbar).
\]

Therefore, the braiding matrix is determined completely up to the braiding parameter \( t \). The quantum value of this parameter will be determined in the next section when we study the monodromy properties of the four-point functions by solving the Knizhnik-Zamolodchikov equations.

Notice that this braiding matrix can be related to the Racah matrix of \( U_t(SL(N)) \) and consequently indicates that \( U_t(SL(N)) \) is an additional symmetry of the chiral theory. This identification comes from the following observation. From the conformal scaling argument given in the appendix, we know that \( \alpha \cdot \nu/\chi \in \mathbb{Z} \) because the monodromy takes values of \( \nu/\chi = \Lambda + \delta \) for some highest weight \( \Lambda \) and \( \delta \) denotes one half of the sum of all positive roots. Subsequently, when \( t \neq \pm 1 \), the braiding matrix \( Q \) in (16) is a \( N^2 \times N^2 \) matrix related to the deformed Racah matrix[17–19] for \( U_t(SL(N)) \) up to some normalisation:

\[
Q_{mn,m'n'}(\nu) \propto \delta(\lambda_m + \lambda_n - \lambda_{m'} - \lambda_{n'}) \left\{ \begin{array}{ccc} \lambda_1 & \Lambda & \Lambda - \lambda_{n'} \\ \Lambda - \lambda_m & \lambda - \lambda_n & \lambda_1 \end{array} \right\}^{RW}_{q=t^2}.
\]
This identification is valid only when all the entries in the bracket are highest weights. In order to avoid confusion, we have denoted the quantum group by $t = q^\frac{1}{r}$ rather than $q$ such as in [17,18]. In the limit when $t^2 = 1$, $Q$ can be related to the undeformed Racah matrix of $SU(N)$.

For example, when $N = 2$ and $t \neq \pm 1$, we can substitute $\sqrt{2}\nu/\chi = (2j + 1)$ for $j \geq 1$ and obtain the following $4 \times 4$ matrix in terms of the deformed-Racah matrix of $U_t(SL(2))$,

$$Q_{m,n,m',n'}(\nu) = \delta_{m+n,m'+n'}(-1)^{\ell+L-J} \left\{ \begin{array}{c} j \ell - \frac{m+n}{2} \\ j - \frac{n'}{2} \end{array} \right\}_{q=t^2}^{\mathrm{RW}} R W_{j,j},$$

with $C_j = j(j+1)$, $\ell = j - \frac{n'}{2}$, $L = j - \frac{m}{2}$, $J = j - \frac{m+n}{2}; \forall m, n, n' = \pm 1.$

In the limit when $t = e^{i\pi}$, this gives the Racah($\{6j\}$)-matrix for $SU(2)$ according to

$$Q_{m,n,m',n'}(\nu) = \delta_{m+n,m'+n'}(-1)^{\frac{m+n}{2}} \left\{ \begin{array}{c} j \frac{m+n}{2} \\ j - \frac{n'}{2} \end{array} \right\}_{q=t^2}^{\mathrm{RW}}.$$

As explained in the next section, the quantum value of $\chi$ is typically given by $\pi$ divided by a positive integer. This raises a concern that (16) and (17) become singular when $\alpha \cdot \nu / \chi \in \mathbb{Z}$. However, for the WZW models which we are studying, only the primary fields in the integrable representations[20] contribute to correlation functions, i.e. their highest weights satisfy $0 < \lambda^0 \cdot \psi < k + 1$. Restricted to these representations only[4], the monodromy momentum takes the value of $\nu / \chi = \lambda^0 + \delta$ and it satisfies $0 < \alpha \cdot \nu / \pi < 1$ when $\chi = \pi / (k + N)$. When the level $k = 1$, the coefficient $\sin(\alpha \cdot \nu)$ in (17) becomes a finite integer in the limit of $\chi = \pi$ for $\alpha \cdot \nu / \pi \in \mathbb{Z}$. Therefore, the braiding matrix in (16) is regular for the physical representations considered in the chiral $SU(N)_k$ WZW models.

**Braiding Matrix and Monodromy Matrix**

In order to determine the quantum value of the braiding parameter, $t = e^{i\chi}$, one can follow the conformal scaling argument given in [11]. We give the result of that argument for $SU(N)$ in the appendix. The drawback of this argument is that it does not apply to the case when $k = 1$. In this section, we give a new derivation of the braiding parameter based on quantum consistency, i.e. the Knizhnik-Zamolodchikov (KZ) equations. Since $Q$ in (16) describes the braiding relation of two chiral primary fields in the fundamental representation, we shall consider the four-point functions in the representations: $\Lambda_1, \Lambda, \Lambda, \Lambda_4$; namely, the two vertex operators in the middle are in the fundamental representation, $\Lambda$. Let us denote the weights in these representations by $a_j, b_j$ and define the following four-point functions,

$$F_{a_1a_2a_3a_4}^{b_1b_2b_3b_4} (z_1, z_2, z_3, z_4) \equiv \langle 0 | U_{a_1b_1}^\Lambda (z_1) U_{a_2b_2}^\Lambda (z_2) U_{a_3b_3}^\Lambda (z_3) U_{a_4b_4}^\Lambda (z_4) | 0 \rangle. \quad (21)$$

By exchanging the vertex operators at $z_2$ and $z_3$ according to (13a), we find the following braiding relation between the four-point functions,

$$F_{a_1a_2a_3a_4}^{b_1b_2b_3b_4} (z_1, z_2, z_3, z_4) = \sum_{r_2,r_3} F_{a_1a_3a_2a_4}^{b_1r_3r_2b_4} (z_1, z_3, z_2, z_4) Q_{r_2r_3,b_2b_3} (\frac{z_2}{z_3}, t). \quad (22)$$
The main idea is to solve for the four-point functions using the Ward-identities and then determine the braiding parameter $t$ from (22). As shown in [2], these four-point functions satisfy the following set of differential equations,

$$
\sum_{j=1}^4 z_j^n (z_j \partial z_j + (n+1)\Delta_j) F^{(b)} = 0; \quad n = -1, 0, 1 \quad (23a)
$$

$$
\left(\sum_{j=1}^4 t_j^a\right) F^{(b)} = 0, \quad (23b)
$$

$$
(k + N) \partial_z F^{(b)} = \sum_{j=1, j \neq i}^4 \frac{t_i \cdot t_j}{z_i - z_j} F^{(b)}, \quad (23c)
$$

where $\{t_j\}$ denote the $SU(N)$ generators in $\Lambda_j$ representation and $\Delta_j$ denotes the conformal weight of the $j$-th vertex. These weights are given by their highest weight vectors according to $\Delta_j = (\Lambda_j, \Lambda_j + 2\delta)/2(k + N)$. For example, $\Delta(\Lambda) = (N^2 - 1)/2N(k + N)$ for the fundamental representation.

Before solving the KZ equations, let us first recall that, following from (23a), the correlation functions depend only on the coordinate differences, $z_{ij} \equiv z_i - z_j$. Once a pre-factor independent of the indices $\{a\}$ and $\{b\}$ has been extracted, the rest will depend only on the cross-ratio, $y \equiv (z_{12} z_{34})/(z_{13} z_{24})$,

$$
F^{(b)}_{\{a\}}(z_1, z_2, z_3, z_4) = \left(\prod_{i<j} z_{ij}^{-\gamma_{ij}}\right) f^{(b)}_{\{a\}}(y), \quad (24)
$$

where

$$
\begin{align*}
\gamma_{12} &= \gamma_{13} = 0, \\
\gamma_{14} &= 2\Delta_1, \\
\gamma_{23} &= \Delta_1 + \Delta_2 + \Delta_3 - \Delta_4, \\
\gamma_{24} &= -\Delta_1 + \Delta_2 - \Delta_3 + \Delta_4, \\
\gamma_{34} &= -\Delta_1 - \Delta_2 + \Delta_3 + \Delta_4.
\end{align*}
$$

Thus, omitting the $\{b\}$ indices for the moment, we can rewrite the KZ equation as follows,

$$
(k + N) \partial_y f(y) = \left(\frac{t_1 \cdot t_2}{y} + \frac{t_1 \cdot (t_2 + t_3)}{1 - y}\right) f(y). \quad (25)
$$

For example, let us first consider the case when $\Lambda_1 = \Lambda_4 = \overline{\Lambda}$ conjugate to the fundamental representation $\Lambda$. The weights in $\Lambda$ are denoted by $m \equiv \lambda_m$ for $m = 1, 2, \ldots, N$ and those in $\overline{\Lambda}$ by $\overline{m} \equiv -\lambda_m$. Eq. (23b) implies that $f^{(b)}_{\{a\}}$ is an isotropic tensor on the lower indices $\{a\}$ and so can be written in terms of a basis of such tensors; there are two independent ones in this case,

$$
f_{\overline{m}, n, l, \overline{n}}(y) = \delta_{m, n} \delta_{l, k} f_1(y) + \delta_{m, l} \delta_{n, k} f_2(y). \quad (26)
$$

In this case, the KZ equation in (25) becomes a set of differential equations in terms of the two scalars,

$$
(k + N) \partial_y \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(y) = \begin{pmatrix} 1/y & (1 - N^2)/N \\ -1/N & 1 - y \end{pmatrix} \begin{pmatrix} 1/N^2 - 2 \\ 1/N \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(y). \quad (27)
$$
This equation can be rewritten as second-order differential equations whose solutions are hypergeometric functions. In order to relate to the braiding relation in (22), it is sufficient to consider the solutions in the limit of \( z_{12} \to 0 \) or \( y \to 0 \). There are two independent sets of solutions:

\[
\begin{align*}
    f_1^{-}(y) &= y^{-2\Delta}(1-y)^{\frac{N^2+N+2}{N(k+N)}} F\left(1, \frac{k+1}{k+N}, \frac{k}{k+N}, y \right), \\
    f_2^{-}(y) &= -\frac{1}{k} y^{-1-2\Delta}(1-y)^{\frac{N^2+N+2}{N(k+N)}} F\left(1+\frac{1}{k+N}, \frac{k+1}{k+N}, 2-\frac{N}{k+N}, y \right),
\end{align*}
\]

and

\[
\begin{align*}
    f_1^{+}(y) &= y^{\frac{1}{N(k+N)}}(1-y)^{\frac{N^2+N+2}{N(k+N)}} F\left(1+\frac{1}{k+N}, \frac{N+1}{k+N}, 1+\frac{N}{k+N}, y \right), \\
    f_2^{+}(y) &= -Ny^{\frac{1}{N(k+N)}}(1-y)^{\frac{N^2+N+2}{N(k+N)}} F\left(1, \frac{N+1}{k+N}, \frac{N}{k+N}, y \right).
\end{align*}
\]

It is easy to check that these solutions are the same as those in Eq. (4.10) of [2] where the KZ equation was solved in terms of the cross ratio \( x \equiv y/(y-1) = (z_{12}z_{34})/(z_{14}z_{32}) \). The reason why we have used \( y \) in this paper is because the monodromy properties of these solutions are easier to study on the \( y \)-plane. We shall explain the details in the next paragraph. Meanwhile, one can understand why there are these two sets of solutions as follows. According to the operator product expansions (OPE) of the chiral primary fields,  

\[
U^{\Delta_1}(z_1)U^{\Delta_2}(z_2) \sim \sum_{\Delta_3} (z_1 - z_2)^{\Delta_3 - \Delta_1 - \Delta_2} U^{\Delta_3}(z_2) \left(1 + O(z_1 - z_2)\right),
\]

we know that \( U^{\Delta_3} \) corresponds to the intermediate states contributing to the four-point functions. Thus, the singular behaviours of the solutions in (28) near \( z_{12} \to 0 \), together with the prefactor \( \prod_{i<j} z_{ij}^{-\gamma(ij)} \), imply that the first set of solutions, \( f^{-}(y), \) is contributed from the intermediate states in the singlet representation of \( SU(N)_k \) and the second set of solutions, \( f^{+}(y), \) is contributed from those in the adjoint representation. These are the two intermediate representations expected from the fusion rule\(^{[3]}\).

We can now determine the braiding parameter in (22) from the monodromy properties of the four-point functions. In general, the \( SU(N) \)-invariant correlation function is a linear combination of these two solutions:

\[
F^{(b)} = \left( \prod_{i<j} z_{ij}^{-\gamma(ij)} \right) \sum_{\rho=\pm} Y^{(b)}_{\rho} f^{\rho}(y).
\]

We can denote the solutions of (28) graphically and write the braiding relation in (22) as

\[
\sum_{\rho=\pm} 2,m_{\rho} \longrightarrow 3,n_{\rho} Y^{(b)}_{\rho} = \sum_{\rho'=\pm} 3,n_{\rho'} \longrightarrow 2,m_{\rho'} Y^{b_1 r_3 b_2 b_3}_{\rho'} Q_{r_2 r_3 b_2 b_3}.
\]
When we exchange $z_2$ with $z_3$, the cross-ratio $y$ becomes $1/y$. Thus, we need to relate the hypergeometric functions of $y$ to those of $1/y$. Since $0 < y < 1$ for $0 < z_3 < z_2 < \infty$, we can choose the cut of these functions to be the negative real axis. Using the properties of the hypergeometric functions, we find that $f_1(y)$ and $f_2(1/y)$ are linearly related by a connection matrix $\mathcal{K}$ as follows.

$$f_1^\rho(y) = e^{i\pi \frac{N^2-1}{N(k+N)}} \sum_{\mu=\pm} f_2^\mu \left( \frac{1}{y} \right) \mathcal{K}^\rho_\mu,$$  \hspace{1cm} (32a)

$$\mathcal{K} = e^{-i \frac{\pi}{N(k+N)}} \begin{pmatrix}
   e^{i \frac{N\pi}{Nk+N}} \sin(\frac{\pi y}{N+1}) & k(\frac{1-N^2}{N}) \frac{\Gamma^2(1+\frac{N}{N+N+1})}{\Gamma(1+\frac{N}{N+1})} \\
   -\frac{1}{N} \frac{\Gamma^2(1-\frac{N}{N+1})}{\Gamma(1-\frac{N}{N+1})} & e^{-i \frac{N\pi}{Nk+N}} \sin(\frac{\pi y}{N+1})
\end{pmatrix}. \hspace{1cm} (32b)$$

In deriving for $\mathcal{K}$, we have implicitly chosen a contour of analytic continuation by specifying $-y = e^{i\pi\eta}y$ with $\eta \equiv \eta(\arg(z_2/z_3)) = 1$. One can check that the following results apply to the other range of $z_2/z_3$. Notice that the normalisation of $f_1$ determines the normalisation of $f_2$ and the definition of the connection matrix depends on this normalisation. However, the eigenvalues of the connection matrix are not affected. From (28), (30) and (32), we find that the braiding relation in (22) will hold provided

$$\sum_{\mu=\pm} \mathcal{K}^\rho_\mu Y^{b_1b_2b_3b_4}_{\emptyset} = \sum_{r_2,r_3=1}^N Y^{b_1r_3r_2b_4}_{\rho} Q_{r_2r_3,b_2b_3} (\nu_4^0,t). \hspace{1cm} (33)$$

Here, $\nu_4^0$ denotes the value of the momentum $\nu$ acting on the vacuum with a shift due to the vertex operator at $z_4$, see (15). This condition requires that $\mathbb{P}Q$ and $\mathcal{K}$ have at least one common eigenvalue. $\mathbb{P}$ is the permutation matrix which interchanges the left two indices of $Q$. From (16), $\mathbb{P}Q$ has two distinct eigenvalues, $t^{\frac{N+1}{k+N}}$ of multiplicities $\frac{1}{2}N(N+1)$ and $-t^{\frac{N-1}{k+N}}$ of multiplicities $\frac{1}{2}N(N-1)$. While, the connection matrix $\mathcal{K}$ has two eigenvalues: $e^{i\pi(N-1)/N(k+N)}$ and $-e^{-i\pi(1+N)/N(k+N)}$. This would give totally four possible values for $t$. In general, we do not expect $Y^{(b)}$ to vanish when $b_2 = b_3$. Then, only two of the four possibilities for $t$ should be considered.

$$t = e^{i\pi/(k+N)}, \hspace{1cm} (34a)$$

or

$$t^{\frac{N-1}{k+N}} = (1) e^{-i\pi \frac{N+1}{N(k+N)}}. \hspace{1cm} (34b)$$

Further consideration is needed in order to determine which of these two gives the correct braiding parameter. In the following, we will discuss the case for $k = 1$ and $k \geq 2$ separately.

When the level $k = 1$, it is known that only the chiral primary fields in the minimal representations can contribute to the correlation functions. Therefore, the adjoint solution $f^+$ in (28b) is not allowed, i.e. $Y^{(b)}_{+} = 0$ in (30). Substituting the singlet solution $f^-$ in (28a) into the braiding relation in (31), we find that

$$e^{i2\pi \Delta_2} Y^{b_1b_2b_3b_4}_{-} = \sum_{r_2,r_3=1}^N Y^{b_1r_3r_2b_4}_{-} Q_{r_2r_3,b_2b_3} (\nu_4^0,t). \hspace{1cm} (35)$$
If the diagonal coefficients \( Y^{b_1 r r b_4} \neq 0 \), then the braiding parameter must be \( t = e^{i \pi} \) as in (34b). However, if one can impose an extra symmetry in the chiral theory\(^{[22]}\) so that \( Y^{b_1 r r b_4} = 0 \), then (34a) can also be a solution of (35). To summarise, for the \( SU(N)_{k=1} \) WZW model,

\[
\begin{align*}
    t &= e^{i \frac{\pi}{N}} \quad \text{only if} \quad Y^{b_1 r r b_4} = 0, \\
    t &= e^{i \pi} \quad \text{if} \quad Y^{b_1 r r b_4} \neq 0.
\end{align*}
\]

(36a) \hspace{1cm} (36b)

This seems to indicate that there may be two different ways of factorising the level-one \( SU(N) \) WZW model by implementing the two different internal symmetries. Previous treatments\(^{[3-7,14]}\) have proposed the quasi-quantum group \( U_t(SL(N)) \) with \( t = e^{i \frac{\pi}{1+N}} \) in (36a) to be the internal symmetry although explicit implementations have not been given. On the other hand, the possibility of \( U_{-1}(SL(N)) \) symmetry has not been considered, and we shall show that it is realised explicitly by the free field construction of the chiral vertex operators in the next section.

Let us now consider what happens when \( k \geq 2 \). Since both solutions in the singlet and adjoint representations are present, we have to study other types of four-point functions in order to determine the quantum value for \( t \). In the following, we shall consider the four-point functions in representations: \( \Lambda_1, \Lambda, \Lambda, \Lambda \), with \( \Lambda_1 \) conjugates to the totally symmetric representation of \( N(N+1)(N+2)/6 \) dimensions. The conformal weight for \( \Lambda_1 \) is given by

\[
\Delta_1 = \frac{3(N^2 + 3N + 2)}{2N(k + N)}. 
\]

(37)

Using the fact that \( \Lambda_2 = \Lambda_3 = \Lambda_4 = \Lambda \) and \( C_{a_1,a_2,a_3,a_4} \) is totally symmetric in the last three indices, we have

\[
(t_1 \cdot t_2) C = - \left( \frac{N^2 - 1}{N} + 2 \frac{N - 1}{N} \right) C. 
\]

(38)

Similarly for \( (t_1 \cdot t_3) \) on \( C \), we can solve for the scalar \( S^{(b)} \) from (25):

\[
S^{(b)} = Y^{(b)} \left( z_{24}z_{34} \right)^{\Delta_1 - \Delta_N} z_{14}^{-2\Delta_1} z_{23}^{-\Delta_1 - \Delta_N} \left( y(1-y)^2 \right)^{-\frac{(N-1)(N+3)}{N(k+N)}}, 
\]

(39)

where \( Y^{(b)} \) is some normalisation constant. Substituting (37) and (39) into the braiding relation in (22), we find that the braiding parameter is given by (34a),

\[
t = \exp(i \frac{\pi}{k + N}) \quad \text{for} \quad k \geq 2.
\]

(40)

Therefore, when \( k \geq 2 \), this result agrees with the previous observations in [3-11] that the additional symmetry acting on the chiral sectors of \( SU(N)_k \) WZW model is the quasi-quantum group \( U_t(SL(N)) \) with the parameter \( t \) given in (40).
For $N = 2$, our results on the monodromy properties of the four-point functions agree with those of Tsuchiya and Kanie\cite{8} for both $k = 1$ and $k \geq 2$, but it should be noted that we have used a different basis to solve the KZ equations in order to be able to relate the connection matrix $K$ with the braiding matrix $Q$ as in (33). This distinction is significant when we are considering the construction of the chiral parts $U(x^+, x^-)$ of the group element $g(x^+, x^-)$, rather than just chiral primary fields. A relevant observation is that the (quasi-)quantum group should act on $U(x^+)$ from the right-hand side while the Kac-Moody current acts from the left-hand side. In this way, the (quasi-)quantum group appears only in the chiral theory and it plays the role of an internal symmetry of the original WZW model.

In this section, we have completed solving for the braiding matrix specifying the exchange relation of two chiral vertex operators in the fundamental representation. In principle, this allows us to determine also the braiding relations involving chiral vertex operators in other representations. This is because the OPE’s of the chiral vertex operators in the fundamental representation generate the rest of the chiral vertex operators. Indirectly, this result also reveals the symmetry structures of the Hilbert space of the chiral model. In order to make this more precise, we discuss in the next section the explicit construction of the chiral vertex operators in terms of free fields for the level-one $SU(N)$ WZW model.

**Free Field Construction of the Chiral Vertex Operators at Level One**

There are two stages in our construction: the first stage is to obtain the chiral primary fields by extending the construction of Frenkel-Kac and Segal\cite{16} for the affine algebras. It turns out that these chiral primary fields obey a diagonal braiding relation. The second stage is to determine the zero-modes in order to complete the construction of the chiral vertex operators defined in (13a) and (16).

Let us briefly review the free field realisation of the level-one Kac-Moody algebras in \cite{16}. Further details can be obtained in \cite{20}. The main advantage of this construction is that unitarity is manifest. In the Cartan-Weyl basis, this affine algebra can be written as the following with the structure constants $\epsilon(\alpha, \beta)$ normalised to be $\pm 1$.

\[
[J_{i}^{j}, J_{n}^{j}] = m\delta_{i}^{j}\delta_{m,-n} \quad \text{if } i, j = 1, 2, \ldots N - 1.
\]

\[
[J_{i}^{j}, J_{m}^{n\alpha}] = \alpha^{i}J_{m+n}^{n\alpha} \quad \text{if } \alpha + \beta \text{ is a root}
\]

\[
[J_{i}^{j}, J_{m}^{n\beta}] = \begin{cases} \epsilon(\alpha, \beta)J_{n}^{\alpha+\beta} & \text{if } \alpha + \beta = 0 \\ \sum_{j=1}^{r} \alpha^{r}J_{m+n}^{j} + m\delta_{m,-n} & \text{otherwise.} \end{cases}
\]

(41)

Let us introduce $N - 1$ right-moving scalar fields $\phi^{j}(z)$ which is expanded on a complex plane of $z = e^{ix^+}$,

\[
\phi^{j}(z) = q_{\phi}^{j} - ip_{\phi}^{j} \ln z + i \sum_{n\neq0, \in \mathbb{Z}} \frac{1}{n} \phi_{n}^{j} z^{-n},
\]

(42a)

where

\[
\phi_{n}^{j\dagger} = \phi_{-n}^{j}, \quad [\phi_{m}^{j}, \phi_{n}^{j}] = m\delta_{m,-n}\delta^{ij}, \quad j = 1, 2, \ldots N - 1;
\]

\[
p_{\phi}^{j\dagger} = p_{\phi}^{j}, \quad [q_{\phi}^{j}, p_{\phi}^{j}] = i\hbar\delta^{ij}.
\]

(42b)
For each root $\beta$, we define a normal ordered vertex operator,

$$
O^{\beta}(z) \equiv z^{\beta^2/2} \exp(i\beta \cdot \phi(z)) = \sum_{n \in \mathbb{Z}} z^{-n} O^{\beta}_n
$$

$$
\equiv z^{\beta^2/2} \exp(-\beta \cdot \sum_{n<0} \frac{1}{n} \phi_n z^{-n}) \exp(\beta \cdot \sum_{n>0} \frac{1}{n} \phi_n z^{-n}).
$$

(43)

The product of two such operators has a short distance expansion obtained by normal ordering,

$$
O^{\alpha}(z) O^{\beta}(w) = (z - w)^{\alpha \cdot \beta} \times O^{\alpha}(z) O^{\beta}(w) \times \quad \text{for } |z| > |w|.
$$

(44)

This allows us to evaluate the following contour integral by integrating $z$ over the contour $C_w$ which winds around $w$,

$$
\oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} (z - w)^{\alpha \cdot \beta} z^m w^{n-1} \times O^{\alpha}(z) O^{\beta}(w) \times = O^{\alpha}_m O^{\beta}_n - (-1)^{\alpha \cdot \beta} O^{\beta}_n O^{\alpha}_m.
$$

(45)

The right-hand side of (45) almost gives the commutator of $O^{\alpha}$ and $O^{\beta}$ except for the extra phase $(-1)^{\alpha \cdot \beta}$ in front of the second term. In order to remove this phase, we introduce an operator $\hat{C}_{\alpha}$ defined on the root lattice $\Lambda_R$ such that they obey the following cocycle condition,

$$
\hat{C}_{\alpha} \hat{C}_{\beta} = S(\alpha, \beta) \hat{C}_{\beta} \hat{C}_{\alpha} = \epsilon(\alpha, \beta) \hat{C}_{\alpha + \beta},
$$

with

$$
S(\alpha, \beta) = e^{i\pi \alpha \cdot \beta}, \quad \forall \alpha, \beta \in \Lambda_R.
$$

(46)

The structure constant $\epsilon(\alpha, \beta)$ and the symmetry factor $S(\alpha, \beta)$ have to satisfy the following consistency requirements,

$$
\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma),
$$

(47a)

$$
\epsilon(\alpha, \beta) = S(\alpha, \beta)\epsilon(\beta, \alpha),
$$

(47b)

$$
S(\alpha, \beta)S(\gamma, \beta) = S(\alpha + \gamma, \beta),
$$

(47c)

$$
S(\alpha, \beta) = S^{-1}(\beta, \alpha).
$$

(47d)

These conditions determine $\hat{C}$ up to some gauge transformation.[20,23] We will discuss this gauge freedom in more details later on. In general, without introducing new degrees of freedom, one can construct $\hat{C}$ as functions of the zero modes $p, q$ of the free scalar fields. From (42)-(47), the current generators can be realised as follows.[16]

$$
J^{\alpha}(z) = O^{\alpha}(z) e^{-i\alpha \cdot q \phi} \hat{C}_{\alpha}, \quad \forall \alpha \text{ is a root},
$$

$$
J^j(z) = iz \partial_z \phi^j(z), \quad j = 1, 2, ..N - 1.
$$

(48)
Conformal symmetry is generated by the Virasoro generators which can be obtained from the affine currents according to the Sugawara construction,

\[
L(z) = \frac{1}{2(1 + N)} \times \sum_{a=1}^{N^2-1} J^a(z)J^a(z) = \frac{1}{2} \times (iz \partial_z \phi)^2 \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times 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On this enlarged lattice, the cocycles for the Kac-Moody generators correspond to \( \hat{C}_{(\alpha,0)} \) and \( \hat{C}_{(0,\alpha)} \) respectively. Then, the primary fields will have the correct commutators with the current generators provided that the symmetry factors are given by

\[
S(\vec{\mu}, \vec{\nu}) = e^{i\pi(\mu \cdot \nu - \vec{p} \cdot \vec{q})}, \quad \forall \ \vec{\mu}, \vec{\nu} \in \Lambda_W \times \Lambda_W.
\]  

One can easily check that (55) satisfy the consistency conditions in (47c) and (47d) on the new lattice \( \Lambda_W \times \Lambda_W \).

Once the symmetry factor is given, we can determined \( \epsilon(\vec{\mu}, \vec{\nu}) \) using (47b) up to the following gauge transformation with \( t(\vec{\mu}) = t(\mu)\vec{t}(\vec{\nu}) \),

\[
\hat{C}_{\vec{\mu}} \rightarrow t(\vec{\mu})\hat{C}_{\vec{\mu}}; \quad \epsilon(\vec{\mu}, \vec{\nu}) \rightarrow \frac{t(\vec{\mu})t(\vec{\nu})}{t(\vec{\mu} + \vec{\nu})}\epsilon(\vec{\mu}, \vec{\nu}).
\]  

Subsequently, the cocycles can be constructed as functions of the zero-modes, \( \vec{p} \equiv (p_\phi, \vec{p}_\vec{\phi}) \).

\[
\hat{C}_{\vec{\chi}} \equiv e^{i\vec{\chi} \cdot \vec{q}}\epsilon(\vec{\chi}, \vec{p}).
\]  

The gauge freedom in (56) allows us to normalise the symmetry factor \( S(\vec{\mu}, \vec{\mu}) = 1 \). This implies that the left and the right weights, \( \mu \) and \( \vec{\mu} \) are actually in the “diagonal” subspace,

\[
\Lambda_D^\pm \equiv \bigcup_{\lambda=0,\text{minimal}} \Lambda_{\lambda} \times \Lambda_{\pm\lambda}.
\]  

In the following discussion, we shall mainly consider the cocycles in \( \Lambda_D^+ \) since it is compatible with the original Hilbert space of the WZW models.

Thus, we have completed the construction of the primary field \( g(z, \vec{z}) \) in terms of free scalar fields. This allows us to verify directly that \( g \) is local in the sense that

\[
g_{rs}(z, \vec{z})g_{r's'}(w, \vec{w}) = g_{r's'}(w, \vec{w})g_{rs}(z, \vec{z}), \quad \text{for} \quad |z| > |w|.
\]  

Since we consider the WZW models compactified on a circle, \( g(z, \vec{z}) \) is periodic and this amounts to impose an extra constraint on the left-moving and the right-moving momenta,

\[
\lambda_r \cdot p_\phi - \lambda_s \cdot \vec{p}_\vec{\phi} \in \mathbb{Z}, \quad \forall \lambda_r, \lambda_s \in [\lambda].
\]  

This is manifestly satisfied when \( (p_\phi, \vec{p}_\vec{\phi}) \) takes values in \( \Lambda_D^+ \). This can be understood as the gauge-fixed approach discussed in [11].

Now, it is fairly clear that the left-moving and the right-moving degrees of freedom in the primary field of (53) are nicely separated except for the cocycles which depend on the zero-modes of both the left-moving and the right-moving scalar fields. One way to separate these degrees of freedom is to enlarge the phase space by introducing, in each chiral sector, \((N-1)\) new pairs of harmonic operators, \([q^l_\pm, p^j_\pm] = i\delta^{lj}\). These new degrees of freedom are
used to replace the zero-modes of the opposite sector in the cocycles. To be more explicit, we write down the right-moving primary field in the fundamental representation as

$$\hat{U}_{rs}(z) = \times e^{i\lambda_r \cdot \phi(z)} \times e^{i\lambda_s \cdot q_- C(\lambda_r, \lambda_s)}(p, p_-).$$

(61)

The left-moving primary field $V(z)$ can be constructed in a similar fashion using $q_+, p_+$. The advantage of having such an explicit construction is that we can calculate the correlation functions and verify the KZ equation (23c) directly using free scalar fields and cocycles. Also, the local properties of the chiral primary fields can be determined directly. In fact, we find the following braiding,

$$\hat{U}_1(z)\hat{U}_2(w) = \hat{U}_2(w)\hat{U}_1(z)R(\arg(z/w)), \quad \text{for } |z| > |w|,$$

where

$$R_{rs,r's'}(x) = \delta_{rr'}\delta_{ss'}\exp\{i\pi \lambda_s \cdot \lambda_{s'}\eta(x)\}. \quad (62)$$

When $\eta(x) = 1$, $R(x)$ can be written as

$$R(t) = t^{-\frac{1}{N}} \left\{ \sum_{r=1}^{N} e_{rr} \otimes e_{rr} + \sum_{r \neq s=1}^{N} e_{rs} \otimes e_{rs} + (t - t^{-1}) \sum_{r > s=1}^{N} e_{rs} \otimes e_{sr} \right\}, \quad t = e^{i\pi}.$$  

(63)

Thus, we can identify $e^{i\frac{\pi}{N}\eta(x)}R(x)$ as the $R$-matrix in the fundamental representation of $U_t(SU(N))$ with $t = e^{i\pi}$. Note that we have replaced $U_t(SL(N))$ by $U_t(SU(N))$ since $t = -1 \in \mathbb{R}$. This braiding parameter agrees with the one in (36b) obtained from solving the KZ equations.

However, the braiding relation in (62) is given by the R-matrix rather than the braiding matrix $Q$ in (16). In order not to alter the defining properties of chiral primary fields, we can multiply the free-field vertex $\hat{U}(z)$ by a $N$-dimensional matrix, $A$, which commutes with the Kac-Moody currents. In particular, let the matrix $A$ depend on the momentum, $p_-$ and denote $\hat{A} \equiv e^{iHq_-}A(p_-)$ such that

$$U(z) \equiv \times e^{i\phi(z)H} \times (e^{-iq_-H}) \hat{C}(e^{-iq_-H}) \hat{A}. \quad (64)$$

Then, the braiding of $U$ will be given by $Q$ provided that $\hat{A}$ satisfies the following equation,

$$R(t)\hat{A}_1\hat{A}_2 = \hat{A}_2\hat{A}_1Q(\nu, t). \quad (65)$$

It is sufficient to show that (65) holds for $\eta(x) = 1$ because the same result will follow automatically for other $x$. In fact, this requirement gives the operator identity for the IRF-Vertex transformation in $U_t(SL(N))$ when $\hat{A}$ is identified as the Wigner-matrix which relate states in different irreducible representations of $U_t(SL(N))$.

For $U_t(SL(2))$, this operator identity has been realised explicitly in [24] in terms of two pairs of harmonic oscillators. In particular, we take their result in the limit of $[Y]_t = Yt^{Y-1}$ for $Y \in \mathbb{Z}$ when $t = e^{i\pi}$ and obtain

$$\hat{A} \equiv e^{i\frac{\sigma}{\sqrt{2}} q_-} \left( e^{i\frac{\nu + p_-}{\sqrt{2}} \sin \theta} \cos \theta \quad -e^{-i\frac{\nu + p_-}{\sqrt{2}} \sin \theta} \sin \theta \right) e^{i\frac{\sigma}{\sqrt{2}} q_+}, \quad \sin \theta \equiv \sqrt{\frac{\nu - p_-}{2\nu}}. \quad (66)$$
Notice that \( \hat{A} \) depends on two pairs of harmonic oscillators, \((q_-, p_-)\) and an additional pair \((q_\nu, \nu)\) where \(\nu\) is the monodromy momentum in the braiding matrix \(Q\). These zero modes are independent degrees of freedom in the sense that they commute with the Kac-Moody current. We will come back to the discussion of the monodromy in a short while.

Thus, we have completed the construction of the chiral vertex operator, \( U(z) \) in (64). Its unitarity is manifest because both \( \hat{C} \) and \( \hat{A} \) are unitary. According to [24], \( \hat{A} \) transforms covariantly under \( U_{-1}(SU(2)) \), i.e.

\[
\varrho (A_{+b}, A_{-b}) = (A_{+b}, A_{-b}) (\Pi^{\frac{1}{2}} \otimes id) \Delta(\varrho), \quad \forall \ \varrho \in U_{-1}(SU(2)),
\]

(67)

where \( \Pi^{\frac{1}{2}} \) denotes the spin \( \frac{1}{2} \) representation and the coproduct \( \Delta(\varrho) \) can be obtained from those of the quantum group generators,

\[
\Delta(t^{S_3}) = t^{S_3} \otimes t^{S_3}, \quad \Delta(S_\pm) = t^{\pm S_3} \otimes S_\pm + S_\pm \otimes t^{\pm S_3}.
\]

(68)

Since \( t^2 = 1 \), the symmetry algebra \( U_{-1}(SU(2)) \) is isomorphic to \( SU(2) \). Its generators can be realised in terms of the harmonic oscillators as

\[
S^3 \equiv \frac{1}{\sqrt{2}} p_-, \quad S^\pm \equiv e^{\pm i\sqrt{2} q_-} (t^{\mp S_3}) \sqrt{\left(\frac{\nu}{\sqrt{2}}\right)^2 - \left(\frac{p_-}{\sqrt{2}} \pm \frac{1}{2}\right)^2}.
\]

(69)

The transformation laws of (67) can be verified directly with (66) and (69). Moreover, the quadratic Casimir operator of the algebra is given by \((\nu/\sqrt{2})^2 - 1/4\). Thus, its irreducible representations, \( \mathcal{V}_j \), are labelled by the eigenvalues of \((\nu/\sqrt{2})^2 - 1/4\) for \(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\).

The Wigner operator \( \hat{A} \), consequently the chiral vertex operator \( U \), takes a state in \( \mathcal{V}_j \) to another state in \( \mathcal{V}_{j+\frac{1}{2}} \) of \( U_{-1}(SU(2)) \). Because this algebra commutes with the chiral affine algebras, it plays the role of an internal symmetry in the original WZW model.

Let us now discuss the monodromy momentum, \( \nu \). From its definition given in (7), the chiral vertex operator has the following monodromy,

\[
U_{rs}(ze^{i2\pi}) = U_{rs}(ze^{i2\pi\nu \cdot \lambda_r}).
\]

(70)

Comparing with the explicit construction of \( U \) in (64), we need to impose a constraint between \( \nu \) and the free-field momentum \( p_\phi \) such that

\[
e^{i2\pi p_\phi \cdot \lambda_r} = e^{i2\pi \nu \cdot \lambda_r}, \quad \forall \lambda_r, \lambda_s = \pm \frac{1}{\sqrt{2}}.
\]

(71)

This condition is manifestly satisfied when the spins of \( U_{-1}(SU(2)) \) and the spin of the Kac-Moody representation differ by an integer. Applying the chiral vertex operator and the Kac-Moody current successively on the vacuum, we obtain all the states in the chiral Hilbert space. It can be decomposed as follows.

\[
\mathcal{H}_R = \left( H_0 \otimes (\oplus_{j=0,1,2,3,\ldots} \mathcal{V}_j) \right) \bigoplus \left( H_{\frac{1}{2}} \otimes (\oplus_{j=\frac{1}{2},\frac{3}{2},\ldots} \mathcal{V}_j) \right).
\]

(72)
Hilbert Space for the Chiral $SU(N)_{k=1}$ WZW Model

The results obtained in the last section for the chiral vertex operators in the fundamental representation of $SU(2)$ can be generalised to other basic representations of $SU(N)$. The chiral Hilbert space is decomposed as

\[ H_R = \bigoplus_{\lambda=0, \text{minimal}} H_\lambda \otimes \left( \bigoplus_{\xi \in \tilde{\Lambda}_\lambda} V_\xi \right), \tag{73} \]

where $H_\lambda$’s are the minimal representations of the right-moving $SU(N)_{k=1}$ Kac-Moody algebra and $\tilde{\Lambda}_\lambda$ denotes the space of all the highest weights of $U_{-1}(SU(N))$ contained in the weight lattice, $\Lambda_\lambda$. Then, the monodromy constraint in (71) will hold for any two weights $\lambda_r$ and $\lambda_s$ in the same representation.

The Hilbert space of the left-moving sector can be constructed in a similar way using the left-moving free scalar fields and $(N-1)$ additional pairs of harmonic oscillators.

\[ H_L = \bigoplus_{\lambda=0, \text{minimal}} \left( \bigoplus_{\xi \in \tilde{\Lambda}_\lambda} V_\xi \right) \otimes \overline{\Pi}_\lambda. \tag{74} \]

These new degrees of freedom do not appear in the original WZW models and they should be gauged away when we “join” the left-moving and the right-moving sectors together. In particular, we should recover the original defining field $g(z, \bar{z})$ in (53) and also the original Hilbert space

\[ H_{WZW} = \bigoplus_{\lambda=0, \text{minimal}} H_\lambda \otimes \overline{\Pi}_\lambda. \tag{75} \]

One has to make this gauging process more precise in order to obtain the Lagrangian formulation of the chiral WZW model. We hope to investigate this question in the near future.

From the free field construction of the chiral vertex operators, we obtain an explicit description of how to factorise the $SU(N)_{k=1}$ WZW model into the right-moving and the left-moving sectors. In contrast to the previous treatments\cite{3-11,14}, the chiral Hilbert space acquires an additional symmetry given by $U_{-1}(SU(N))$ instead of $U_1(SL(N))$ with $t = e^{i\pi/(1+N)}$. From the study of the KZ equations, see (36), we learn that these correspond to the only two possible braiding which describe the local properties of the chiral vertex operators in the level-one chiral models. However, an explicit realisation of the quasi-quantum group symmetry $U_t(SL(N))$ with $t = e^{i\pi/(1+N)}$ is still lacking. When $k \geq 2$, we know from the braiding matrix that there is a unique chiral factorisation with the additional symmetry given by the quasi-quantum group $U_t(SL(N))$ with $t = \exp(i\pi/(k+N))$. We are currently investigating its implementation in the chiral WZW models of higher levels. At the moment, the different nature of the chiral factorisation in the level-one WZW models remains somewhat mysterious.
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Appendix: Conformal Scaling and the Braiding Parameter

In this appendix, we use the scaling argument to determine the braiding parameter \( t \) for the \( SU(N)_k \) WZW model when \( k \geq 2 \). Let us denote \( \zeta = e^{i2\pi x/l} \). For some constant \( \mu \), \( \zeta^\mu u(\zeta) \) transforms as a primary field under the Sugawara stress tensor,\)
\[
L = \frac{1}{2(k + N)} \sum_{a=1}^{N} J^a J^a \times
\]
The conformal weight of \( u(\zeta) \) in the \([\Lambda]\) representation is given by \( h = \frac{<\Delta \Lambda + 2\delta >}{2(k+N)} \). Under a scaling, \( \zeta \mapsto s\zeta \), with \( s = e^{2\pi i} \), \( u \) transforms according to
\[
s^{L_0} u^\Lambda(\zeta) s^{-L_0} = s^h - \mu u^\Lambda(\zeta) e^{2i\nu \cdot H}.
\]
The left-hand side comes from the monodromy of the \( u \)-vertex. Consider the matrix elements of (A2) between two states \( \langle \Lambda_L, \rho_L | \) and \( | \Lambda_R, \rho_R \rangle \) in the highest weight representations \( \Lambda_L \) and \( \Lambda_R \). We find that the non-zero elements of this matrix for \( u_{rs} \) satisfy
\[
\rho_L = \rho_R + \lambda_r, \quad \Lambda_L = \Lambda_R + \lambda_s.
\]
For \( k > 1 \), we can consider \( \Lambda_{L,R} \) in the range \( 0 < \Lambda_L \cdot \psi < k \) and \( 0 < \Lambda_R \cdot \psi < k \). Then, with \( \nu|\Lambda_R, \rho_R \rangle = \nu_R|\Lambda_R, \rho_R \rangle \) and similarly for \( \nu_L \), (A2) gives the following constraints for all \( \lambda_s \) in the fundamental representation,
\[
\frac{1}{\pi} \nu_R \cdot \lambda_s - \frac{(\Lambda_L - \Lambda_R, \Lambda_R + \delta)}{k + N} + \frac{1}{2}(\Lambda_L - \Lambda_R)^2 + h - \mu \in \mathbb{Z};
\]
\[
(\nu_L - \nu_R) \cdot \lambda_s - \frac{1}{2} \hbar \beta \lambda_s \in \mathbb{Z}.
\]
By choosing \( \nu_R \) in the positive Weyl chamber and excluding the extreme representations such as \( \Lambda_L \cdot \psi = 0, k \) and \( \Lambda_R \cdot \psi = 0, k \), we find
\[
\begin{cases}
\nu_L = (\Lambda_L + \delta) \left( \frac{\pi}{k+N} \right), \\
\nu_R = (\Lambda_R + \delta) \left( \frac{\pi}{k+N} \right),
\end{cases}
\]
\[
\nu_L - \nu_R = \frac{1}{2} \hbar \beta (\Lambda_L - \Lambda_R).
\]
Subsequently, we are lead to the following result:
\[
\chi = \frac{1}{2} \hbar \beta = \frac{\pi}{k+N}, \quad \text{i.e.} \quad t \equiv e^{i\chi} = e^{i\frac{\pi}{k+N}}, \quad \text{for} \quad k \geq 2.
\]
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