Toric Varieties and Codes, Error-correcting Codes, Quantum Codes, Secret Sharing and Decoding

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Abstract

We present toric varieties and associated toric codes and their decoding. Toric codes are applied to construct Linear Secret Sharing Schemes (LSSS) with strong multiplication by the Massey construction. Asymmetric Quantum Codes are obtained from toric codes by the A.R. Calderbank P.W. Shor and A.M. Steane construction of stabilizer codes (CSS) from linear codes containing their dual codes.

Keywords: Toric Varieties, Toric Codes, Quantum Codes, Stabilizer Code, Multiplicative Structure, Linear Secret Sharing Schemes (LSSS), Strong Multiplication, Decoding.

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1 Introduction

1.1 Notation

$q = p^n$, where $p$ is a prime number.

$\mathbb{F}_q$ – the finite field with $q$ elements of characteristic $p$.

$\mathbb{F}_q^*$ – the invertible elements in $\mathbb{F}_q$.

$k = \overline{\mathbb{F}_q}$ – an algebraic closure of $\mathbb{F}_q$.

$M \cong \mathbb{Z}^r$ – a free $\mathbb{Z}$-module of rank $r$.

$H = \{0, 1, \ldots, q - 2\} \times \cdots \times \{0, 1, \ldots, q - 2\} \subseteq M$.

$\square \subseteq M_R = M \otimes_{\mathbb{Z}} \mathbb{R}$ – an integral convex polytope.

$X = X_\square$ – the toric variety associated to the polytope $\square$.

$T = T_N = U_0 \subseteq X$ – the torus.

1.2 Error-correcting codes

Codes are used in communication and storage of information.

The message is divided into blocks and extra information is appended before transmission allowing the receiver to correct a few errors.

Let $\mathbb{F}_q$ be the field $q$ elements. A word of length $n$ in the alphabet $\mathbb{F}_q$ is a vector $c = (c_1, c_2, \ldots, c_n) \in \mathbb{F}_q^n$.

The Hamming weight $w(c)$ is the number of non-zero coordinates. The Hamming distance $d(c_1, c_2)$ between two words is the Hamming weight $w(c_1 - c_2)$.

A linear code is a linear subspace $C \subseteq \mathbb{F}_q^n$. The minimum distance $d(C)$ is the minimal Hamming distance $d(c_1, c_2) = w(c_1 - c_2)$ between two distinct code words $c_1, c_2 \in C$ and $c_1 \neq c_2$.

A linear code $C$ can correct $t$ errors if and only if $t < d(C)/2$.

Example 1.1 (Reed–Solomon code). Let $x_1, x_2, \ldots, x_n \in \mathbb{F}_q$ be $n$ distinct elements and let $0 < k \leq n$.

To the word $(a_0, a_1, \ldots, a_{k-1}) \in \mathbb{F}_q^k$ of length $k$ we associate the polynomial

$f(X) = a_0 + a_1 X + \cdots + a_{k-1} X^{k-1} \in \mathbb{F}_q[X]$

and upon evaluation the Reed–Solomon code word

$\left(f(x_1), f(x_2), \ldots, f(x_n)\right) \in \mathbb{F}_q^n$.

The Reed–Solomon code $C_{n,k} \subseteq \mathbb{F}_q^n$ is the subspace of all Reed–Solomon code words $f(X) \in \mathbb{F}_q[X]$ with $\deg f(X) < k \leq n$.

The Reed–Solomon code $C_{n,k} \subseteq \mathbb{F}_q^n$ has dimension $k$, minimum distance $d(C_{n,k}) = n - k + 1$ and correct $t < d(C_{n,k})/2$ errors.
For a general presentation of the theory, see Justesen and Høholdt (2017).

2 Toric varieties and surfaces

The toric codes are obtained from evaluating certain rational functions in rational points on toric varieties. For the general theory of toric varieties we refer to Cox et al. (2011), Fulton (1993) and Oda (1988).

Here we will be using toric surfaces and we recollect some of the theory and the construction of toric codes.

2.1 Polytopes, normal fans and support functions

Let $M \cong \mathbb{Z}^r$ be a free $\mathbb{Z}$-module of rank $r$ over the integers $\mathbb{Z}$. Let $\square$ be an integral convex polytope in $M_\mathbb{R} = M \otimes \mathbb{R}$, i.e. a compact convex polyhedron such that the vertices belong to $M$.

Let $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ be the dual lattice with canonical $\mathbb{Z}$-bilinear pairing $\langle -, - \rangle : M \times N \to \mathbb{Z}.$

Let $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N_\mathbb{R} = N \otimes \mathbb{R}$ with canonical $\mathbb{R}$-bilinear pairing $\langle -, - \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}.$

The $r$-dimensional algebraic torus $T_N = (\mathbb{F}_q^*)^r$ is defined by $T_N := \text{Hom}_\mathbb{Z}(M, \mathbb{F}_q^*)$.

The multiplicative character $e(m), m \in M$ is the homomorphism $e(m) : T \to \mathbb{F}_q^*$ defined by $e(m)(t) = t(m)$ for $t \in T_N$. Specifically, if $\{n_1, \ldots, n_r\}$ and $\{m_1, \ldots, m_r\}$ are dual $\mathbb{Z}$-bases of $N$ and $M$ and we denote $u_j := e(m_j), j = 1, \ldots, r$, then we have an isomorphism $T_N = (\mathbb{F}_q^*)^r$ sending $t$ to $(u_1(t), \ldots, u_r(t))$. For $m = \lambda_1 m_1 + \cdots + \lambda_r m_r$ we have

$$e(m)(t) = u_1(t)^{\lambda_1} \cdots u_r(t)^{\lambda_r}.$$  

Given an $r$-dimensional integral convex polytope $\square$ in $M_\mathbb{R}$. The support function $h_\square : N_\mathbb{R} \to \mathbb{R}$ is defined as $h_\square(n) := \inf \{\langle m, n \rangle \mid m \in \square\}$ and $\square$ can be reconstructed from the support function

$$\square h = \{m \in M \mid \langle m, n \rangle \geq h(n) \quad \forall n \in N\}. \quad (2.1)$$

The support function $h_\square$ is piecewise linear in the sense that $N_\mathbb{R}$ is the union of a non-empty finite collection of strongly convex polyhedral cones in $N_\mathbb{R}$ such that $h_\square$ is linear on each cone.

A fan is a collection $\Delta$ of strongly convex polyhedral cones in $N_\mathbb{R}$ such that every face of $\sigma \in \Delta$ is contained in $\Delta$ and $\sigma \cap \sigma' \in \Delta$ for all $\sigma, \sigma' \in \Delta$.

The normal fan $\Delta$ is the coarsest fan such that $h_\square$ is linear on each $\sigma \in \Delta$, i.e. for all $\sigma \in \Delta$ there exists $l_\sigma \in M$ such that

$$h_\square(n) = \langle l_\sigma, n \rangle \quad \forall n \in \sigma. \quad (2.2)$$
Let \( n \in \mathbb{N} \cap \rho \) such that \( \rho = \mathbb{R}_{\geq 0} n(\rho) \).

Upon refinement of the normal fan, we can assume that two successive pairs of \( n(\rho) \)'s generate the lattice and we obtain the refined normal fan.

Figure 1 and Figure 2 present examples of polytopes with Figure 3 and Figure 4 showing their corresponding refined normal fans.

### 2.1.1 Examples

Let \( q = p^n \), where \( p \) is a prime.

**Example 2.1.** Let \( d \) be a positive integer and let \( \square \) be the polytope in \( M_\mathbb{R} \) with vertices \( (0,0), (d,0), (0,d), (2d,0) \), see Figure 1. Assume that \( 2d < q - 1 \). We have that \( n(\rho_1) = \left( \frac{1}{d} \right), n(\rho_2) = \left( \frac{n}{d} \right), n(\rho_3) = \left( \frac{1}{d} \right) \) and \( n(\rho_4) = \left( \frac{1}{d} \right) \).

Let \( \sigma_1 \) be the cone generated by \( n(\rho_1) \) and \( n(\rho_2) \), \( \sigma_2 \) be the cone generated by \( n(\rho_2) \) and \( n(\rho_3) \), \( \sigma_3 \) the cone generated by \( n(\rho_3) \) and \( n(\rho_4) \) and \( \sigma_4 \) the cone generated by \( n(\rho_4) \) and \( n(\rho_1) \).

The support function is:

\[
\begin{aligned}
    h_{\sigma_1}(n_1, n_2) &= \begin{cases}
        \left( \frac{1}{d} \right) \left( \frac{n_1}{n_2} \right) & \text{if } (n_1, n_2) \in \sigma_1, \\
        \left( \frac{d}{n_1} \right) \left( \frac{n_2}{n_1} \right) & \text{if } (n_1, n_2) \in \sigma_2, \\
        \left( \frac{n_1}{n_2} \right) & \text{if } (n_1, n_2) \in \sigma_3, \\
        \left( \frac{2}{n_1} \right) \left( \frac{n_2}{n_1} \right) & \text{if } (n_1, n_2) \in \sigma_4.
    \end{cases}
\end{aligned}
\]

**Example 2.2.** Let \( d \) be a positive integer and let \( \square \) be the polytope in \( M_\mathbb{R} \) with vertices \( (0,0), (d,0), (0,d), (d,d) \), see Figure 1. Assume that \( d < q - 1 \). We have that \( n(\rho_1) = \left( \frac{1}{d} \right), n(\rho_2) = \left( \frac{1}{d} \right), n(\rho_3) = \left( \frac{1}{d} \right) \).

Let \( \sigma_1 \) be the cone generated by \( n(\rho_1) \) and \( n(\rho_2) \), \( \sigma_2 \) be the cone generated by \( n(\rho_2) \) and \( n(\rho_3) \) and \( \sigma_3 \) the cone generated by \( n(\rho_3) \) and \( n(\rho_4) \).

The support function is:

\[
\begin{aligned}
    h_{\sigma_1}(n_1, n_2) &= \begin{cases}
        \left( \frac{1}{d} \right) \left( \frac{n_1}{n_2} \right) & \text{if } (n_1, n_2) \in \sigma_1, \\
        \left( \frac{d}{n_1} \right) & \text{if } (n_1, n_2) \in \sigma_2, \\
        \left( \frac{n_1}{n_2} \right) & \text{if } (n_1, n_2) \in \sigma_3.
    \end{cases}
\end{aligned}
\]

**Example 2.3.** Let \( d, e \) be positive integers and let \( \square \) be the polytope in \( M_\mathbb{R} \) with vertices \( (0,0), (d,0), (d,e), (0,e) \), see Figure 1. Assume that \( d < q - 1 \) and that \( e < q - 1 \). We have that \( n(\rho_1) = \left( \frac{1}{d} \right), n(\rho_2) = \left( \frac{1}{d} \right), n(\rho_3) = \left( \frac{1}{d} \right) \) and \( n(\rho_4) = \left( \frac{1}{d} \right) \).

Let \( \sigma_1 \) be the cone generated by \( n(\rho_1) \) and \( n(\rho_2) \), \( \sigma_2 \) be the cone generated by \( n(\rho_2) \) and \( n(\rho_3) \), \( \sigma_3 \) the cone generated by \( n(\rho_3) \) and \( n(\rho_4) \) and \( \sigma_4 \) the cone generated by \( n(\rho_4) \) and \( n(\rho_1) \).

The support function is:

\[
\begin{aligned}
    h_{\sigma_1}(n_1, n_2) &= \begin{cases}
        \left( \frac{1}{d} \right) \left( \frac{n_1}{n_2} \right) & \text{if } (n_1, n_2) \in \sigma_1, \\
        \left( \frac{d}{n_1} \right) \left( \frac{n_2}{n_1} \right) & \text{if } (n_1, n_2) \in \sigma_2, \\
        \left( \frac{n_1}{n_2} \right) & \text{if } (n_1, n_2) \in \sigma_3, \\
        \left( \frac{2}{n_1} \right) \left( \frac{n_2}{n_1} \right) & \text{if } (n_1, n_2) \in \sigma_4.
    \end{cases}
\end{aligned}
\]
**Example 2.4.** Let $d, e, r$ be positive integers and let $\square$ be the polytope in $M_\mathbb{R}$ with vertices $(0,0), (d,0), (d,e+rd), (0,e)$, see Figure 2. Assume that $d < q - 1$, that $e < q - 1$ and that $e + rd < q - 1$. We have that $n(\rho_1) = \left( \frac{1}{d} \right)$, $n(\rho_2) = \left( \frac{0}{d} \right)$, $n(\rho_3) = \left( \frac{0}{e} \right)$ and $n(\rho_4) = \left( \frac{1}{e} \right)$. Let $\sigma_1$ be the cone generated by $n(\rho_1)$ and $n(\rho_2)$, $\sigma_2$ be the cone generated by $n(\rho_2)$ and $n(\rho_3)$, $\sigma_3$ the cone generated by $n(\rho_3)$ and $n(\rho_4)$ and $\sigma_4$ the cone generated by $n(\rho_4)$ and $n(\rho_1)$. The support function is:

$$h_{\square} \left( \frac{n_1}{n_2} \right) = \begin{cases} 
\left( \frac{0}{d} \right) \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_1, \\
\left( \frac{0}{d} \right) \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_2, \\
\left( \frac{d}{e+rd} \right) \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_3, \\
\left( \frac{0}{e} \right) \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_4.
\end{cases}$$

Figure 1: The polytope of Example 2.1 is the left triangle with vertices $(0,0), (d,d), (0,2d)$. The polytope of Example 2.2 is the right triangle with vertices $(0,0), (d,0), (0,d)$. The polytope of Example 2.3 is the square with vertices $(0,0), (d,0), (d,e), (0,e)$.

### 2.2 The toric variety and the Cartier divisor associated to a fan

Notation as in Section 1.1 and Section 2.1.

The **toric surface** $X_\square$ associated to the refined normal fan $\Delta$ of $\square$ is

$$X_\square = \bigcup_{\sigma \in \Delta} U_\sigma$$
Figure 2: The polytope of Example 2.4 is the polytope with vertices \((0,0), (d, 0), (d, e + rd), (0, e)\).

Figure 3: The refined normal fans of the polytopes in Figure 1.

Figure 4: The normal fan of the polytope in Figure 2.
where $U_\sigma$ is the $\mathbb{F}_q$-valued points of the affine scheme $\text{Spec}(\mathbb{F}_q[S_\sigma])$, i.e.

$$U_\sigma = \{ u : S_\sigma \to \mathbb{F}_q \mid u(0) = 1, u(m + m') = u(m)u(m') \quad \forall m, m' \in S_\sigma \},$$

where $S_\sigma$ is the additive subsemigroup of $M$

$$S_\sigma = \{ m \in M \mid \langle m, y \rangle \geq 0 \quad \forall y \in \sigma \}.$$

The toric surface $X_\sigma$ is irreducible, non-singular and complete, see Oda (1988, Chapter 1). If $\sigma, \tau \in \Delta$ and $\tau$ is a face of $\sigma$, then $U_\tau$ is an open subset of $U_\sigma$. Obviously, $S_0 = M$ and $U_0 = T_N$ such that the algebraic torus $T_N$ is an open subset of $X_\sigma$.

$T_N$ acts algebraically on $X_\sigma$. On $u \in U_\sigma$ the action of $t \in T_N$ is obtained as

$$(tu)(m) := t(m)u(m) \quad m \in S_\sigma$$

such that $tu \in U_\sigma$ and $U_\sigma$ is $T_N$-stable. The orbits of this action is in one-to-one correspondance with $\Delta$. For each $\sigma \in \Delta$ let

$$\text{orb}(\sigma) := \{ u : M \cap \sigma \to \mathbb{F}_q^* \mid u \text{ is a group homomorphism} \}.$$ 

Then $\text{orb}(\sigma)$ is a $T_N$ orbit in $X_\sigma$. Define $V(\sigma)$ to be the closure of $\text{orb}(\sigma)$ in $X_\sigma$.

A $\Delta$-linear support function $h$ gives rise to the Cartier divisor $D_h$. Let $\Delta(1)$ be the 1-dimensional cones in $\Delta$, then

$$D_h := - \sum_{\rho \in \Delta(1)} h(n(\rho)) V(\rho).$$

In particular

$$D_m = \text{div}(e(-m)) \quad m \in M.$$

Following Oda (1988, Lemma 2.3), we have the lemma.

**Lemma 2.5.** Let $h$ be a $\Delta$-linear support function with associated Cartier divisor $D_h$ and convex polytope $\Box_h$ defined in (2.1). The vector space $H^0(X, O_X(D_h))$ of global sections of $O_X(D_h)$, i.e. rational functions $f$ on $X_\sigma$ such that $\text{div}(f) + D_h \geq 0$ has dimension $\#(M \cap \Box_h)$ and has $\{ e(m) \mid m \in M \cap \Box_h \}$ as a basis.

**2.2.1 Examples**

In Example 2.1

$$D_h := - \sum_{\rho \in \Delta(1)} h(n(\rho)) V(\rho) = d V(\rho_3) + 2 d V(\rho_4)$$

and

$$\dim H^0(X, O_X(D_h)) = (d + 1)^2.$$ 

In Example 2.2

$$D_h := - \sum_{\rho \in \Delta(1)} h(n(\rho)) V(\rho) = d V(\rho_3)$$

6
and
\[ \dim H^0(X, O_X(D_h)) = \frac{(d+1)(d+2)}{2}. \]

In Example 2.3
\[ D_h := - \sum_{\rho \in \Delta(1)} h(n(\rho)) V(\rho) = d V(\rho_3) + e V(\rho_4) \]
and
\[ \dim H^0(X, O_X(D_h)) = (d+1)(e+1). \]

In Example 2.4
\[ D_h := - \sum_{\rho \in \Delta(1)} h(n(\rho)) V(\rho) = d V(\rho_3) + e V(\rho_4) \]
and
\[ \dim H^0(X, O_X(D_h)) = (d+1)(e+1) + r \frac{d(d+1)}{2}. \]

### 2.3 Polytopes, Cartier divisors and Intersection theory

For a fixed linebundle \( L \) on \( X \), given an effective divisor \( D \) such that \( L = O_X(D) \), the fundamental question to answer is: How many points from a fixed set \( \mathcal{P} \) of rational points are in the support of \( D \). This question is treated in general in Hansen (2001b) using intersection theory, see Fulton (1998). Here we will apply the same methods when \( X \) is a toric surface.

For a \( \Delta \)-linear support function \( h \) and a 1-dimensional cone \( \rho \in \Delta(1) \), we will determine the intersection number \((D_h; V(\rho))\) between the Cartier divisor \( D_h \) and \( V(\rho) = \mathbb{P}^1 \). This number is obtained in Oda (1988, Lemma 2.11). The cone \( \rho \) is the common face of two 2-dimensional cones \( \sigma', \sigma'' \in \Delta(2) \). Choose primitive elements \( n', n'' \in \mathbb{N} \) such that
\[
\begin{align*}
n' + n'' &\in \mathbb{R} \rho \\
\sigma' + \mathbb{R} \rho &= \mathbb{R}_{\geq 0} n' + \mathbb{R} \rho \\
\sigma'' + \mathbb{R} \rho &= \mathbb{R}_{\geq 0} n'' + \mathbb{R} \rho
\end{align*}
\]

**Lemma 2.6.** For any \( l_\rho \in M \), such that \( h \) coincides with \( l_\rho \) on \( \rho \), let \( \overline{h} = h - l_\rho \). Then
\[
(D_h; V(\rho)) = -\langle \overline{h}(n') + \overline{h}(n'') \rangle.
\]

In the 2-dimensional non-singular case, let \( n(\rho) \) be a primitive generator for the 1-dimensional cone \( \rho \). There exists an integer \( a \) such that
\[
n' + n'' + an(\rho) = 0,
\]
\( V(\rho) \) is itself a Cartier divisor and the above gives the self-intersection number
\[
(V(\rho); V(\rho)) = a.
\]

More generally, the self-intersection number of a Cartier divisor \( D_h \) is obtained in Oda (1988, Prop. 2.10).
Lemma 2.7. Let \( D_h \) be a Cartier divisor and let \( \square_h \) be the polytope associated to \( h \), see (2.1). Then

\[
(D_h; D_h) = 2 \text{vol}_2(\square_h),
\]

where \( \text{vol}_2 \) is the normalized Lesbesgue-measure.

In case of Example 2.4 the intersection table becomes

\[
\begin{array}{cccc}
V(\rho_1) & V(\rho_2) & V(\rho_3) & V(\rho_4) \\
V(\rho_1) & -r & 1 & 0 & 1 \\
V(\rho_2) & 1 & 0 & 1 & 0 \\
V(\rho_3) & 0 & 1 & r & 1 \\
V(\rho_4) & 1 & 0 & 1 & 0
\end{array}
\]

3 Toric Codes

In Hansen (1998), Hansen (2000) and Hansen (2002) we introduced toric codes and presented a general method to obtain the dimension and a lower bound for the minimal distance of a toric code.

3.1 The construction

Let \( M \cong \mathbb{Z}^r \) be a free \( \mathbb{Z} \)-module of rank \( r \) over the integers \( \mathbb{Z} \).

Definition 3.1. For any subset \( U \subseteq M \), let \( \mathbb{F}_q[U] \) be the linear span in \( \mathbb{F}_q[X_1^{\pm 1}, \ldots, X_r^{\pm 1}] \) of the monomials

\[
\{X^u = X_1^{u_1} \cdots X_r^{u_r} | u = (u_1, \ldots, u_r) \in U\}.
\]

This is an \( \mathbb{F}_q \)-vector space of dimension equal to the number of elements in \( U \).

Let \( T(\mathbb{F}_q) = \mathbb{F}_q^* \) be the \( \mathbb{F}_q \)-rational points on the torus and let \( S \subseteq T(\mathbb{F}_q) \) be any subset. The linear map that evaluates elements in \( \mathbb{F}_q[U] \) at all the points in \( S \) is denoted by \( \pi_S \):

\[
\pi_S : \mathbb{F}_q[U] \to \mathbb{F}_q^{|S|},
\]

\[
f \mapsto (f(P))_{P \in S}.
\]

In this notation \( \pi_{\{P\}}(f) = f(P) \).

The toric code is the image \( C_U = \pi_S(\mathbb{F}_q[U]) \).

Remark 3.2 \((r = 1, \text{Reed–Solomon codes})\). Consider the special case, where \( M \cong \mathbb{Z}, U = \square = [0, k-1] \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \) and \( S = T(\mathbb{F}_q) = \mathbb{F}_q^* \).

The toric code \( C_\square \) associated to \( \square \) is the linear code of length \( n = (q - 1) \) presented in Definition 1.1 with \( S = \{x_1, \ldots, x_n\} \).

Remark 3.3 \((r = 2)\). Consider the special case, where \( M \cong \mathbb{Z}^2, U = \square \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \) is an integral convex polytope and \( S = T(\mathbb{F}_q) = \mathbb{F}_q^* \times \mathbb{F}_q^* \).

Let \( \xi \in \mathbb{F}_q \) be a primitive element. For any \( i \) such that \( 0 \leq i \leq q-1 \) and any \( j \) such that \( 0 \leq j \leq q-1 \), we let \( P_{ij} = (\xi^i, \xi^j) \in S = \mathbb{F}_q^* \times \mathbb{F}_q^* \). Let \( m_1, m_2 \) be a \( \mathbb{Z} \)-basis for \( M \). For any \( m = \lambda_1 m_1 + \lambda_2 m_2 \in M \cap \square \), we let \( e(m)(P_{ij}) := (\xi^i)^{\lambda_1} (\xi^j)^{\lambda_2} \).
The toric code $C_\square$ associated to $\square$ is the linear code of length $n = (q - 1)^2$ generated by the vectors $$\{(e(m)(P_j))_{i=0,\ldots,q-1; j=0,\ldots,q-1} \mid m \in M \cap \square\}.$$  

**Remark 3.4.** The toric codes can as evaluation codes be presented in the context of toric varieties in the notation of Section 2.2.

For each $t \in T(\mathbb{F}_q) = (\mathbb{F}_q^\times)^r$ we evaluate the rational functions in $H^0(X, O_X(D_h))$ and minimal distance is equal to $(q - 1)^2 - 2d(q - 1)$.

Let $H^0(X, O_X(D_h))_{\text{Frob}}$ denote the rational functions in $H^0(X, O_X(D_h))$ that are invariant under the action of the Frobenius. Evaluating in all points in $T(\mathbb{F}_q)$, we obtain the code $C_\square$:

$$H^0(X, O_X(D_h))_{\text{Frob}} \to C_\square \subset (\mathbb{F}_q^\times)^\#T(\mathbb{F}_q)$$

$$f \mapsto (f(t))_{t \in T(\mathbb{F}_q)}.$$ 

as in **Definition 3.1**.

We obtained the following results.

### 3.2 Results and examples

**Theorem 3.5.** Let $d$ be a positive integer and let $\square$ be the polytope in $M_\mathbb{R}$ with vertices $(0,0), (d, 0), (0, 2d)$, see Figure 1. Assume that $2d < q - 1$. The toric code $C_\square$ has length equal to $(q - 1)^2$, dimension equal to $\#(M \cap \square) = (d + 1)^2$ (the number of lattice points in $\square$) and minimal distance is equal to $(q - 1)^2 - 2d(q - 1)$.

**Theorem 3.6.** Let $d$ be a positive integer and let $\square$ be the polytope in $M_\mathbb{R}$ with vertices $(0,0), (d, 0), (0, d)$, see Figure 1. Assume that $d < q - 1$. The toric code $C_\square$ has length equal to $(q - 1)^2$, dimension equal to $\#(M \cap \square) = (d + 1)(d + 2)/2$ (the number of lattice points in $\square$) and minimal distance is equal to $(q - 1)^2 - d(q - 1)$.

**Theorem 3.7.** Let $d, e$ be positive integers and let $\square$ be the polytope in $M_\mathbb{R}$ with vertices $(0,0), (d, 0), (d, e), (0, e)$, see Figure 1. Assume that $d < q - 1$ and that $e < q - 1$. The toric code $C_\square$ has length equal to $(q - 1)^2$, dimension equal to $\#(M \cap \square) = (d + 1)(e + 1)$ (the number of lattice points in $\square$) and minimal distance is equal to $(q - 1)^2 - (d(q - 1) + (q - 1 - d)e) = (q - 1 - d)(q - 1 - e)$.

**Theorem 3.8.** Let $d, e, r$ be positive integers and let $\square$ be the polytope in $M_\mathbb{R}$ with vertices $(0,0), (d, 0), (d, e + rd), (0, e)$, see Figure 2. Assume that $d < q - 1$, that $e < q - 1$ and that $e + rd < q - 1$. The toric code $C_\square$ has length equal to $(q - 1)^2$, dimension equal to $\#(M \cap \square) = (d + 1)(e + 1) + rd(d + 1)/2$ (the number of lattice points in $\square$) and minimal distance is equal to

$$\text{Min}[(q - 1 - d)(q - 1 - e), (q - 1)(q - 1 - e - rd)].$$

In Figure 5, we have plotted for $q = 16$ and $q = 32$ the $xy$-diagrams for the codes obtained, where $x$ for a given code is the rate of the code, that is the fraction $\frac{\text{dimension}}{\text{length}}$, and $y$ is the relative minimal distance $\frac{\text{minimal distance}}{\text{length}}$. 

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Figure 5: For all possible codes obtained by Theorem 3.8 a point is marked in the usual $xy$-diagram, where $x$ for a given code is the rate of the code, i.e., the fraction $\frac{\text{dimension}}{\text{length}}$, and $y$ is the relative minimal distance $\frac{\text{minimal distance}}{\text{length}}$. The left diagram is for the case $q = 16$ and the right is for the case $q = 32$.

In Section 3.2.1 we present the method using toric varieties, their cohomology and intersection theory to obtain bounds for the number of rational zeroes of a rational function, which can be used to prove the theorems on dimension and minimal distance of the codes $C$ presented above.

### 3.2.1 Determination of parameters using intersection theory

Let $m_1 = (1, 0)$. The $\mathbb{F}_q^*\times \mathbb{F}_q^*$-rational points of $T = \mathbb{F}_q^* \times \mathbb{F}_q^*$ belong to the $q - 1$ lines on $X$ given by $\prod_{\eta \in \mathbb{F}_q^*} (e(m_1) - \eta) = 0$. Let $0 \neq f \in H^0(X, O_X(D_h))$ and assume that $f$ is zero along precisely $a$ of these lines. As $e(m_1) - \eta$ and $e(m_1)$ have the same divisors of poles, they have equivalent divisors of zeroes, so

$$(\text{div}(e(m_1) - \eta))_0 \sim (\text{div}(e(m_1)))_0.$$ 

Therefore,

$$\text{div}(f) + D_h - a(\text{div}(e(m_1)))_0 \geq 0$$

or equivalently,

$$f \in H^0(X, O_X(D_h - a(\text{div}(e(m_1)))_0)).$$

In the cases of all the theorems this implies that $a \leq d$ according to Lemma 2.5. On any of the other $q - 1 - a$ lines, the number of zeroes of $f$ is according to Hansen (2001b) at most the intersection number:

$$(D_h - a(\text{div}(e(m_1))))_0; (\text{div}(e(m_1)))_0). \quad (3.1)$$

This number can be calculated using Lemma 2.6 and Lemma 2.7. In the situation of Theorem 3.5, the number is $2d - a \cdot 2 \cdot (\frac{1}{2} \cdot 1 \cdot 2) = 2d - 2a$ and in the situation of Theorem 3.6, it is $d - a \cdot 2 \cdot (\frac{1}{2} \cdot 1 \cdot 1) = d - a$ (in both cases the volume-element is shown as gray in Figure 1). In the situation of Theorem 3.7, the volume-element is the line segment shown in bold in Figure 1 and the number is $e$. As $0 \leq a \leq d$ the total number of zeroes for $f$ in the three cases is at
most:
\[ \begin{align*}
a(q - 1) + (q - 1 - a)(2d - 2a) & \leq (q - 1)2d \\
a(q - 1) + (q - 1 - a)(d - a) & \leq d(q - 1) \\
a(q - 1) + (q - 1 - a)e & \leq d(q - 1) + (q - 1 - d)e
\end{align*} \]

In case of Theorem 3.8 the intersection number (3.1) is easily calculated using the intersection table above and that \((\text{div}(e(m_1)))_0 = V(\rho_1) + rV(\rho_4)\). We get
\[
(D_h - a(\text{div}(e(m_1))))_0; (\text{div}(e(m_1)))_0) = e + (d - a)r.
\]

As \(0 \leq a \leq d\), the total number of zeroes for \(f\) is at most
\[
a(q - 1) + (q - 1 - a)(e + (d - a)r) \\
\leq \max\{d(q - 1) + (q - 1 - d)e, (q - 1)(e + dr)\}.
\]

This implies in all cases that the evaluation maps
\[
H^0(X, O_X(D_h))^\text{Frob} \to C_{\square} \subset (\mathbb{F}_q^*)^{\text{dim}(\mathbb{F}_q)}
\]

\[ f \mapsto (f(t))_{t \in T(\mathbb{F}_q)} \]

are injective and that the dimensions and the lower bounds for the minimal distances of the toric codes are as claimed.

To see that the lower bounds for the minimal distances are in fact the true minimal distances, we exhibit codewords of minimal weight.

In the case of Theorem 3.5, we let \(b_1, \ldots, b_{2d} \in \mathbb{F}_q^*\) be pairwise different elements. Then the function
\[
(y - b_1) \cdots (y - b_{2d}) \in H^0(X, O_X(D_h))^\text{Frob}
\]
evaluates to zero in the \((q - 1)(2d)\) points
\[(x, b_j), \quad x \in \mathbb{F}_q^*, j = 1, \ldots, 2d\]
and gives a codeword of weight \((q - 1)^2 - 2d(q - 1)\).

In the case of Theorem 3.6, we let \(b_1, \ldots, b_d \in \mathbb{F}_q^*\) be pairwise different elements. Then the function
\[
(y - b_1) \cdots (y - b_d) \in H^0(X, O_X(D_h))^\text{Frob}
\]
evaluates to zero in the \((q - 1)d\) points
\[(x, b_j), \quad x \in \mathbb{F}_q^*, j = 1, \ldots, d\]
and gives a codeword of weight \((q - 1)^2 - 2d(q - 1)\).

In the case of Theorem 3.7 and Theorem 3.8, we let \(b_1, \ldots, b_{e + rd} \in \mathbb{F}_q^*\) be pairwise different elements. Then the function
\[
x^d(y - b_1) \cdots (y - b_{e + rd}) \in H^0(X, O_X(D_h))^\text{Frob}
\]
evaluates to zero in the \((q - 1)(e + rd)\) points
\[(x, b_j), x \in \mathbb{F}_q^*, \quad j = 1, \ldots, e + rd\]
and gives a codeword of weight \((q - 1)^2 - (q - 1)(e + rd) = (q - 1)(q - 1 - (e + rd))\).

On the other hand, we let \(a_1, \ldots, a_d \in \mathbb{F}_q^*\) be pairwise different elements and let \(b_1, \ldots, b_e \in \mathbb{F}_q^*\) be pairwise different elements. Then the function

\[
(x - a_1) \cdots (x - a_d)(y - b_1) \cdots (y - b_e) \in H^0(X, O_X(D))^{\text{Frob}}
\]
evaluates to zero in the \(d(q - 1) + (q - 1)e - de\) points

\((a_i, y), (x, b_j), \quad x, y \in \mathbb{F}_q^*, i = 1, \ldots, d, j = 1, \ldots, e\)

and gives a codeword of weight \((q - 1 - d)(q - 1 - e)\).

Remark 3.9. Our method to estimate the minimum distance of toric codes has subsequently been supplemented, e.g., Little and Schenck (2006), Soprunov and Soprunova (2008/09), Little and Schwarz (2007), Ruano (2007), Beelen and Ruano (2009), Little (2013) Soprunov (2015), and Little (2015).

3.3 Translation

Let \(U \subseteq M\) be a subset, let \(v \in M\) and consider translation \(v + U := \{v + u \mid u \in U\} \subseteq M\).

Lemma 3.10. Translation induces an isomorphism of vector spaces

\[
\mathbb{F}_q[U] \rightarrow \mathbb{F}_q[v + U]
\]

\[
f \mapsto f^v := X^v \cdot f.
\]

We have that

(i) The evaluations of \(\pi_{T(\mathbb{F}_q)}(f)\) and \(\pi_{T(\mathbb{F}_q)}(f^v)\) have the same number of zeroes on \(T(\mathbb{F}_q)\).

(ii) The minimal number of zeros on \(T(\mathbb{F}_q)\) of evaluations of elements in \(\mathbb{F}_q[U]\) and \(\mathbb{F}_q[v + U]\) are the same.

(iii) For \(v = (v_1, \ldots, v_r)\) with \(v_i\) divisible by \(q - 1\), the evaluations \(\pi_S(f)\) and \(\pi_S(f^v)\) are the same for any subset \(S \subseteq T(\mathbb{F}_q)\).

The lemma and generalizations have been used in several articles classifying toric codes, e.g., Little and Schenck (2006).

An immediate consequence of (iii) above is the following corollary, which also can be found in Ruano (2007, Theorem 3.3).

Corollary 3.11. Let \(U \subseteq M\) be a subset and let

\[
\bar{U} := \{(\bar{u}_1, \ldots, \bar{u}_r) \mid \bar{u}_i \in \{0, \ldots, q - 2\} \text{ and } \bar{u}_i \equiv u_i \mod q - 1\}
\]

be its reduction modulo \(q - 1\). Then \(\pi_S(\mathbb{F}_q[U]) = \pi_S(\mathbb{F}_q[\bar{U}])\) for any subset \(S \subseteq T(\mathbb{F}_q)\).
3.4 Dual toric code

Proposition 3.13 exhibits the dual code of the toric code \( C = \pi_T(\mathbb{F}_q[U]) \) defined in Definition 3.1.

Let \( U \subseteq M \) be a subset, define its opposite as \( -U := \{-u \mid u \in U\} \subseteq M \). The opposite maps the monomial \( X^u \) to \( X^{-u} \) and induces by linearity an isomorphism of vector spaces

\[
\mathbb{F}_q[U] \rightarrow \mathbb{F}_q[-U]
\]

\( X^u \mapsto X^{-u} \)

\[ f \mapsto \hat{f}. \]

On \( \mathbb{F}_q^{|T(\mathbb{F}_q)|} \), we have the inner product

\[
(a_0, \ldots, a_n) \star (b_0, \ldots, b_n) = \sum_{i=0}^{n} a_i b_i \in \mathbb{F}_q,
\]

with \( n = |T(\mathbb{F}_q)| - 1 \).

Lemma 3.12. Let \( f, g \in \mathbb{F}_q[M] \) and assume \( f \neq \hat{g} \), then

\[
\pi_T(\mathbb{F}_q)(f) \star \pi_T(\mathbb{F}_q)(g) = 0.
\]

Let

\[ H = \{0, 1, \ldots, q - 2\} \times \cdots \times \{0, 1, \ldots, q - 2\} \subseteq M. \]

With this inner product we obtain the following proposition, e.g. Bras-Amorós and O’Sullivan (2008, Proposition 3.5) and Ruano (2009, Theorem 6).

Proposition 3.13. Let \( U \subseteq H \) be a subset. Then we have

(i) For \( f \in \mathbb{F}_q[U] \) and \( g \in \mathbb{F}_q[-H \setminus U] \), we have that \( \pi_T(\mathbb{F}_q)(f) \star \pi_T(\mathbb{F}_q)(g) = 0. \)

(ii) The orthogonal complement to \( \pi_T(\mathbb{F}_q)(\mathbb{F}_q[U]) \) in \( \mathbb{F}_q^{|T(\mathbb{F}_q)|} \) is

\[
\pi_T(\mathbb{F}_q)(\mathbb{F}_q[-H \setminus U]),
\]

i.e., the dual code of \( C = \pi_T(\mathbb{F}_q[U]) \) is \( \pi_T(\mathbb{F}_q)[\mathbb{F}_q[-H \setminus U]) \).

An example is shown in Figure 6.

4 Secret Sharing Schemes from toric varieties and codes

Massey’s method for constructing linear secret sharing schemes from error-correcting codes is applied to codes obtained using toric varieties. The schemes obtained are ideal and the number of players is \((q - 1)r - 1\) for any positive integer \( r \). Examples of schemes which are quasi-threshold and have strong multiplication with respect to certain adversary structures are also presented. In particular, for any pair of integers \( a, b \) with \( 0 \leq b \leq a \leq q - 2 \), using toric surfaces, schemes with \((q - 1)a - 1\) players are given whose reconstruction threshold (i.e., the smallest integer \( r \) such that any set of at least \( r \) of the shares
Figure 6: Hirzebruch surfaces. The convex polytope $H$ with vertices $(0,0)$, $(q-2,0)$, $(q-2,q-2)$, $(0,q-2)$, the convex polytope $\square$ with vertices $(0,0)$, $(d,0)$, $(d,e+rd)$, $(0,e)$ and their opposite convex polytopes $-H$ and $-\square$. Also the (non-convex) polytope $-H \setminus -\square$ is depicted.

determines the secret) is at most $1 + (q - 2) - (q - 1 - a)$ and whose privacy threshold (i.e., the largest integer $t$ such that no set of $t$ or fewer shares determines the secret) is at least $b - 1$. The schemes have $t$-strong multiplication (i.e., privacy threshold $t$, and the product of any subset of $n - t$ shares (where $n$ is the number of players) obtained by removing any $t$ shares determines the product of the secrets) with respect to the threshold adversary structure if $t \leq \min\{b - 1, (q - 2 - 2a) - 1\}$.

For publication, see Hansen (2017b).

4.1 Secret sharing

Secret sharing schemes were introduced in Blakley (1979) and Shamir (1979) and provide a method to split a secret into several pieces of information (shares) such that any large enough subset of the shares determines the secret, while any small subset of shares provides no information on the secret.
Example 4.1 (Shamir Secret Sharing). Let \( x_1, \ldots, x_n \in \mathbb{F}_q \) be distinct elements. Let \( s_0 \in \mathbb{F}_q \) be the secret to be shared. Choose \( a_1, \ldots, a_d \in \mathbb{F}_q \) at random and let 

\[
 f(X) = s_0 + a_1 X + \ldots + a_d X^d \in \mathbb{F}_q[X].
\]

The \( n \) shares are the values \( f(x_i), i = 1, \ldots, n \). Knowing at least \( d + 1 \) shares, we can reconstruct \( f(X) \) by interpolation, and determine the secret \( s_0 \), whereas knowing \( d \) or fewer shares gives no information on the secret.

Secret sharing schemes have found applications in cryptography, when the schemes have certain algebraic properties. Linear secret sharing schemes (LSSS) are schemes where the secrets \( s \) and their associated shares \( (a_1, \ldots, a_n) \) are elements in a vector space over some finite ground field \( \mathbb{F}_q \). The schemes are called ideal if the secret \( s \) and the shares \( a_i \) are elements in that ground field \( \mathbb{F}_q \). Specifically, if \( s, \tilde{s} \in \mathbb{F}_q \) are two secrets with share vectors \( (a_1, \ldots, a_n), (\tilde{a}_1, \ldots, \tilde{a}_n) \in \mathbb{F}_q^n \), then the share vector of the secret \( s + \lambda \tilde{s} \in \mathbb{F}_q \) is \( (a_1 + \lambda \tilde{a}_1, \ldots, a_n + \lambda \tilde{a}_n) \in \mathbb{F}_q^n \) for any \( \lambda \in \mathbb{F}_q \).

The reconstruction threshold of the linear secret sharing scheme is the smallest integer \( r \) such that any set of at least \( r \) of the shares \( a_1, \ldots, a_n \) determines the secret \( s \). The privacy threshold is the largest integer \( t \) such that no set of \( t \) (or fewer) elements of the shares \( a_1, \ldots, a_n \) determines the secret \( s \). The scheme is said to have \( t \)-privacy.

An ideal linear secret sharing scheme is said to have multiplication if the product of the shares determines the product of the secrets. It has \( t \)-strong multiplication if it has \( t \)-privacy and has multiplication for any subset of \( n - t \) shares obtained by removing any \( t \) shares.

The properties of multiplication was introduced in Cramer et al. (2000). Such schemes with multiplication can be utilized in the domain of multiparty computation (MPC), see Chaum et al. (1988), Ben-Or et al. (1988), Cramer et al. (2015) and Cascudo (2010).

4.1.1 Basic definitions and concepts – Linear Secret Sharing Schemes (LSSS)

This section presents basic definitions and concepts pertaining to linear secret sharing schemes as introduced in Massey (2001), Cramer et al. (2000), Chen and Cramer (2006) and Chen et al. (2007).

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements.

An ideal linear secret sharing scheme \( \mathcal{M} \) over a finite field \( \mathbb{F}_q \) on a set \( \mathcal{P} \) of \( n \) players is given by a positive integer \( e \), a sequence \( V_1, \ldots, V_n \) of \( 1 \)-dimensional linear subspaces \( V_i \subset \mathbb{F}_q^e \) and a non-zero vector \( u \in \mathbb{F}_q^e \).

An adversary structure \( \mathcal{A} \), for a secret sharing scheme \( \mathcal{M} \) on the set of players \( \mathcal{P} \), is a collection of subsets of \( \mathcal{P} \), with the property that subsets of sets in \( \mathcal{A} \) are also sets in \( \mathcal{A} \). In particular, the adversary structure \( \mathcal{A}_{t,n} \) consists of all the subsets of size at most \( t \) of the set \( \mathcal{P} \) of \( n \) players, and the access structure \( \Gamma_{r,n} \) consists of all the subsets of size at least \( r \) of the set \( \mathcal{P} \) of \( n \) players.

For any subset \( A \) of players, let \( V_A = \sum_{i \in A} V_i \) be the \( \mathbb{F}_q \)-subspace spanned by all the \( V_i \) for \( i \in A \).

The access structure \( \Gamma(\mathcal{M}) \) of \( \mathcal{M} \) consists of all the subsets \( B \) of players with \( u \in V_B \), and \( \mathcal{A}(\mathcal{M}) \) consists of all the other subsets \( A \) of players, that is \( A \notin \Gamma(\mathcal{M}) \).
A linear secret sharing scheme \( M \) is said to reject a given adversary structure \( A \), if \( A \subseteq A(M) \). Therefore \( A \in A(M) \) if and only if there is a linear map from \( \mathbb{F}_q^e \) to \( \mathbb{F}_q \) vanishing on \( V_A \), while non-zero on \( u \).

The scheme \( M \) works as follows. For \( i = 1, \ldots, n \), let \( v_i \in V_i \) be bases for the 1-dimensional vector spaces. Let \( s \in \mathbb{F}_q \) be a secret. Choose at random a linear morphism \( \phi : \mathbb{F}_q^e \to \mathbb{F}_q \), subject to the condition \( \phi(u) = s \), and let \( a_i = \phi(v_i) \) for \( i = 1, \ldots, n \) be the shares.

\[
\phi : \mathbb{F}_q^e \to \mathbb{F}_q \\
u \mapsto s \\
v_i \mapsto a_i \quad \text{for } i = 1, \ldots, n.
\]

Then

- the shares \( \{a_i = \phi(v_i)\}_{i \in A} \) determine the secret \( s = \phi(u) \) uniquely if and only if \( A \in \Gamma(M) \),
- the shares \( \{a_i = \phi(v_i)\}_{i \in A} \) reveal no information on the secret \( s = \phi(u) \), i.e., when \( A \in \mathcal{A}(M) \).

**Definition 4.2.** Let \( M \) be a linear secret sharing scheme.

The reconstruction threshold of \( M \) is the smallest integer \( r \) so that any set of at least \( r \) of the shares \( a_1, \ldots, a_n \) determines the secret \( s = \phi(u) \), i.e., \( \mathcal{V}_r \subseteq \Gamma(M) \).

The privacy threshold is the largest integer \( t \) so that no set of \( t \) (or less) elements of the shares \( a_1, \ldots, a_n \) determine the secret \( s = \phi(u) \), i.e., \( \mathcal{A}_t \subseteq \mathcal{A}(M) \). The scheme \( M \) is said to have \( t \)-privacy.

**Definition 4.3.** An ideal linear secret sharing scheme \( M \) has the strong multiplication property with respect to an adversary structure \( A \) if the following holds.

1. \( M \) rejects the adversary structure \( A \).
2. Given two secrets \( s \) and \( \tilde{s} \). For each \( A \in \mathcal{A} \), the products \( a_i \cdot \tilde{a}_i \) of all the shares of the players \( i \in A \) determine the product \( s \cdot \tilde{s} \) of the two secrets.

### 4.2 Secret sharing from toric codes – the Massey construction

Linear secret sharing schemes obtained from linear codes were introduced by James L. Massey in [Massey (2001)](#) and were generalized in Chen et al. (2007, Section 4.1). A scheme with \( n \) players is obtained from a linear \( C \) code of length \( n + 1 \) and dimension \( k \) with privacy threshold \( t = d' - 2 \) and reconstruction threshold \( r = n - d + 2 \), where \( d \) is the minimum distance of the code and \( d' \) the minimum distance of the dual code.

We utilize the Massey construction to obtain linear secret sharing schemes from toric codes.

Under certain conditions the linear secret sharing schemes from toric codes have the strong multiplication property.

This method of toric varieties also applies to construct algebraic geometric ideal secret sharing schemes (LSSS) defined over a finite ground field \( \mathbb{F}_q \) with \( q \).
elements. In a certain sense our construction resembles that of Chen and Cramer (2006), where LSSS schemes were constructed from algebraic curves. However, the methods of obtaining the parameters are completely different.

The linear secret sharing schemes we obtain are ideal and the number of players can be of the magnitude $q^r$ for any positive integer $r$. They are obtained by evaluating certain rational functions in $\mathbb{F}_q$-rational points on toric varieties.

The thresholds and conditions for strong multiplication are derived from estimates on the maximal number of zeroes of rational functions obtained via the cohomology and intersection theory on the underlying toric variety. In particular, we focus on toric surfaces.

We present examples of linear secret sharing schemes which are quasi-threshold and have strong multiplication Cramer et al. (2000) with respect to certain adversary structures.

Specifically, for any pair of integers $a,b$ with $0 \leq b \leq a \leq q-2$, we produce linear secret sharing schemes with $(q-1)^2 - 1$ players which are quasi-threshold, i.e., the reconstruction threshold is at most $1 + (q-1)^2 - (q-1-a)$ and the privacy threshold is at least $b-1$. The schemes have $t$-strong multiplication with respect to the threshold adversary structure if $t \leq \min\{b-1, (q-2-2a) - 1\}$.

### 4.2.1 The construction of Linear Secret Sharing Schemes (LSSS)

With notation as in Definition 3.1.

**Definition 4.4.** Let $S \subseteq T(\mathbb{F}_q)$ be any subset so that $P_0 \in S$. The linear secret sharing schemes (LSSS) $M(U)$ with support $S$ and $n = |S| - 1$ players is obtained as follows:

- Let $s_0 \in \mathbb{F}_q$ be a secret value. Select $f \in \mathbb{F}_q[U]$ at random, such that $\pi_{\{P_0\}}(f) = f(P_0) = s_0$.

- Define the $n$ shares as
  \[
  \pi_{S \setminus \{P_0\}}(f) = (f(P))_{P \in S \setminus \{P_0\}} \in \mathbb{F}_q^{n-1} = \mathbb{F}_q^n.
  \]

The main objectives are to study privacy, reconstruction of the secret from the shares and the property strong multiplication of the scheme as introduced in Definition 4.2 and Definition 4.3.

**Theorem 4.5.** Let $M(U)$ be the linear secret sharing schemes of Definition 4.4 with $(q-1)^r - 1$ players.

Let $r(U)$ and $t(U)$ be the reconstruction and privacy thresholds of $M(U)$ as defined in Definition 4.2.

Then

\[
\begin{align*}
  r(U) & \geq \text{(the maximum number of zeros of } \pi_{T(\mathbb{F}_q)}(f)) + 2 \\
  t(U) & \leq (q-1)^r - \text{(the maximum number of zeros of } \pi_{T(\mathbb{F}_q)}(g)) - 2
\end{align*}
\]
for some \( f \in \mathbb{F}_q[U] \) and for some \( g \in \mathbb{F}_q[-H \setminus U] \), where

\[
\pi_{T(\mathbb{F}_q)} : \mathbb{F}_q[U] \to \mathbb{F}_q^{[T(\mathbb{F}_q)]}, \\
f \mapsto \pi_{T(\mathbb{F}_q)}(f) = (f(P))_{P \in T(\mathbb{F}_q)}, \\
\pi_{T(\mathbb{F}_q)} : \mathbb{F}_q[-H \setminus U] \to \mathbb{F}_q^{[T(\mathbb{F}_q)]}, \\
g \mapsto \pi_{T(\mathbb{F}_q)}(g) = (g(P))_{P \in T(\mathbb{F}_q)}.
\]

**Proof.** The minimal distance of an evaluation code and the maximum number of zeros of a function add to the length of the code.

The bound for \( r(U) \) is based on the minimum distance \( d \) of the code \( C = \pi_{T(\mathbb{F}_q)}(\mathbb{F}_q[U]) \subseteq \mathbb{F}_q^{[T(\mathbb{F}_q)]} \), the bound for \( t(U) \) is based on the minimum distance \( d' \) of the dual code \( C' = \pi_{T(\mathbb{F}_q)}(\mathbb{F}_q[-H \setminus U]) \subseteq \mathbb{F}_q^{[T(\mathbb{F}_q)]} \), using Proposition 3.13 to represent the dual code as an evaluation code.

The codes have length \( |T(\mathbb{F}_q)| \), hence,

\[
2r(U) \geq |T(\mathbb{F}_q)| - d + 2
\]

(\text{the maximum number of zeros of } \pi_{T(\mathbb{F}_q)}(f)) + 2

\[
t(U) \leq d' - 2
\]

(\text{the maximum number of zeros of } \pi_{T(\mathbb{F}_q)}(g)) - 2.

The results follow from the construction in Massey (2001, Section 4.1). \( \square \)

**Theorem 4.6.** Let \( U \subseteq H \subseteq M \) and let \( U + U = \{u_1 + u_2 \mid u_1, u_2 \in U\} \) be the Minkowski sum. Let

\[
\pi_{T(\mathbb{F}_q)} : \mathbb{F}_q[U + U] \to \mathbb{F}_q^{[T(\mathbb{F}_q)]}
\]

\[
h \mapsto \pi_{T(\mathbb{F}_q)}(h) = (h(P))_{P \in T(\mathbb{F}_q)}.
\]

The linear secret sharing schemes \( \mathcal{M}(U) \) of Definition 4.4 with \( n = (q-1)^r - 1 \) players, has strong multiplication with respect to \( A_{t,n} \) for \( t \leq t(U) \), where \( t(U) \) is the adversary threshold of \( \mathcal{M}(U) \), if

\[
t \leq n - 1 - (\text{the maximal number of zeros of } \pi_{T(\mathbb{F}_q)}(h))
\]

for all \( h \in \mathbb{F}_q[U + U] \).

**Proof.** For \( A \in A_{t,n} \), let \( B := T(\mathbb{F}_q) \setminus (\{P_0\} \cup A) \) with \( |B| = n - t \) elements. For \( f, g \in \mathbb{F}_q[U] \), we have that \( f \cdot g \in \mathbb{F}_q[U + U] \). Consider the linear morphism

\[
\pi_B : \mathbb{F}_q[U + U] \to \mathbb{F}_q^{|B|} \quad (4.1)
\]

\[
h \mapsto (h(P))_{P \in B} \quad (4.2)
\]

evaluating at the points in \( B \).

By assumption \( h \in \mathbb{F}_q[U + U] \) can have at most \( n - t - 1 < n - t = |B| \) zeros, therefore \( h \) cannot vanish identically on \( B \), and we conclude that \( \pi_B \) is injective. Consequently, the products \( f(P) \cdot g(P) \) of the shares \( P \in B \) determine the product of the secrets \( f(P_0) \cdot g(P_0) \), and the scheme has strong multiplication by definition. \( \square \)

To determine the product of the secrets from the product of the shares amounts to decoding the linear code obtained as the image in (4.1).
4.2.1.1 Hirzebruch surfaces and their associated Linear Secret Sharing Schemes (LSSS).

Let \( d, e, r \) be positive integers and let \( □ \) be the polytope in \( M_\mathbb{R} \) with vertices \((0,0),(d,0),(d,e+rd),(0,e)\) rendered in Figure 2 and with refined normal fan depicted in Figure 4. We obtain the following result as a consequence of Theorem 4.5 and the bounds obtained in Theorem 3.8 on the number of zeros of functions on such surfaces.

**Theorem 4.7.** Let \( □ \) be the polytope in \( M_\mathbb{R} \) with vertices \((0,0),(d,0),(d,e+rd),(0,e)\) Assume that \( d \leq q-2, e \leq q-2 \) and that \( e+rd \leq q-2 \). Let \( U = M \cap □ \) be the lattice points in \( □ \).

Let \( M(U) \) be the linear secret sharing schemes of Definition 4.4 with support \( T(F_q) \) and \((q-1)^2 - 1\) players.

Then the number of lattice points in \( □ \) is

\[
|U| = |M \cap □| = (d+1)(e+1) + \frac{d(d+1)}{2}.
\]

The maximal number of zeros of a function \( f \in F_q[U] \) on \( T(F_q) \) is

\[
\max\{d(q-1) + (q-1-d)e, (q-1)(e+dr)\}
\]

and the reconstruction threshold as defined in Definition 4.2 of \( M(U) \) is

\[
r(U) = 1 + \max\{d(q-1) + (q-1-d)e, (q-1)(e+dr)\}.
\]

**Remark 4.8.** The polytope \(-H \setminus -□\) is not convex, so our method using intersection theory does not determine the privacy threshold \( t(U) \). It would be interesting to examine the methods and results of Little and Schenck (2006), Soprunov and Soprunova (2008/09), Little and Schwarz (2007), Ruano (2007), Beelen and Ruano (2009), Little (2013) Soprunov (2015), and Little (2015) for toric codes in this context.

**Toric surfaces and their associated Linear Secret Sharing Scheme (LSSS) with strong multiplication.** Let \( a,b \) be positive integers \( 0 \leq b \leq a \leq q-2 \), and let \( □ \) be the polytope in \( M_\mathbb{R} \) with vertices \((0,0),(a,0),(b,q-2),(0,q-2)\) rendered in Figure 7 and with normal fan depicted in Figure 8.

Under these assumptions the polytopes \( □, -H \setminus -□ \) and \( □ + □ \) are convex and we can use intersection theory on the associated toric surface to bound the number of zeros of functions and thresholds.

The primitive generators of the 1-dimensional cones are

\[
n(\rho_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n(\rho_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad n(\rho_3) = \begin{pmatrix} -a-b \\ d(a-bq-2) \end{pmatrix}, \quad n(\rho_4) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},
\]

For \( i = 1, ..., 4 \), the 2-dimensional cones \( \sigma_i \) are shown in Figure 8. The faces of \( \sigma_1 \) are \( \{\rho_1, \rho_2\} \), the faces of \( \sigma_2 \) are \( \{\rho_2, \rho_3\} \), the faces of \( \sigma_3 \) are \( \{\rho_3, \rho_4\} \) and the faces of \( \sigma_4 \) are \( \{\rho_4, \rho_1\} \).
The support function of □ is:

\[
h_\square \left( \frac{n_1}{n_2} \right) = \begin{cases} 
\left( \frac{a}{b} \right) \cdot \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_1, \\
\left( \frac{b}{a} \right) \cdot \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_2, \\
\left( \frac{b}{a} \right) \cdot \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_3, \\
\left( \frac{a}{b} \right) \cdot \left( \frac{n_1}{n_2} \right) & \text{if } \frac{n_1}{n_2} \in \sigma_4.
\end{cases}
\tag{4.3}
\]

The related toric surface is in general singular as \([n(p_2), n(p_3)]\) and \([n(p_3), n(p_4)]\) are not bases for the lattice \(M\). We can desingularize by subdividing the cones \(\sigma_2\) and \(\sigma_3\), however, our calculations will only involve the cones \(\sigma_1\) and \(\sigma_2\), so we refrain from that.

For all pairs of 1-dimensional cones \(\rho_i, \rho_j \in \Delta(1), i = 1, \ldots, 4\), the intersection numbers \((V(\rho_i); V(\rho_j))\) are determined by the methods above, however, we only need the self-intersection number \((V(\rho_1); V(\rho_1))\), and as

\[
n(p_2) + n(p_4) + 0 \cdot n(p_1) = 0,
\]
we have that

\[
(V(\rho_1); V(\rho_1)) = 0 \tag{4.4}
\]

by the remark following Lemma 2.6.

**Theorem 4.9.** Assume \(a, b\) are integers with \(0 \leq b \leq a \leq q - 2\).

Let \(\square\) be the polytope in \(M_\mathbb{R}\) with vertices \((0, 0), (a, 0), (b, q - 2), (0, q - 2)\) rendered in Figure 7, and let \(U = M \cap \square\) be the lattice points in \(\square\).

Let \(\mathcal{M}(U)\) be the linear secret sharing schemes of Definition 4.4 with support \(\mathcal{T}(\mathbb{F}_q)\) and \(n = (q - 1)^2 - 1\) players.

(i) The maximal number of zeros of \(\pi_{\mathcal{T}(\mathbb{F}_q)}(f)\) for \(f \in \mathbb{F}_q[U]\) is less than or equal to

\[
(q - 1)^2 - (q - 1 - a).
\]

(ii) The reconstruction threshold as defined in Definition 4.2 satisfies

\[
r(U) \leq 1 + (q - 1)^2 - (q - 1 - a).
\]

(iii) The privacy threshold as defined in Definition 4.2 satisfies

\[
t(U) \geq b - 1.
\]

(iv) Assume \(2a \leq q - 2\). The secret sharing scheme has \(t\)-strong multiplication for

\[
t \leq \min\{b - 1, (q - 2 - 2a) - 1\}.
\]

**Proof.** Let \(m_1 = (1, 0)\). The \(\mathbb{F}_q\)-rational points of \(T \simeq \mathbb{F}_q^* \times \mathbb{F}_q^*\) belong to the \(q - 1\) lines on \(X\) given by

\[
\prod_{\eta \in \mathbb{F}_q^*} (e(m_1) - \eta) = 0.
\]

Let \(0 \neq f \in H^0(X, O_X(D_h))\). Assume that \(f\) is zero along precisely \(c\) of these lines.

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As \( e(m_1) - \eta \) and \( e(m_1) \) have the same divisors of poles, they have equivalent divisors of zeroes, so
\[
(e(m_1) - \eta) \sim (e(m_1))_0.
\]
Therefore
\[
\text{div}(f) + D_h - c(e(m_1))_0 \geq 0
\]
or equivalently
\[
f \in H^0(X, \mathcal{O}_X(D_h - c(e(m_1))_0).
\]
This implies that \( c \leq a \) according to Lemma 2.5.

On any of the other \( q - 1 - c \) lines the number of zeroes of \( f \) is at most the intersection number
\[
(D_h - c(e(m_1))_0; (e(m_1))_0).
\]
This number can be calculated using Lemma 2.6 using the observation that \( (e(m_1))_0 = V(p_1) \).

We get from (4.3) and (4.4) that
\[
(D_h - c(e(m_1))_0; (e(m_1))_0) = (D_h; (e(m_1))_0) - c(e(m_1))_0; (e(m_1))_0
\]
\[
= -h_F(\frac{q}{1}) - h_F(\frac{q}{1}) = q - 2,
\]
as \( l_{p_1} = (\frac{q}{1}, \frac{q}{1}) \in M. \)

As \( 0 \leq c \leq a \), we conclude the total number of zeroes for \( f \) is at most
\[
c(q - 1) + (q - 1 - c)(q - 2) \leq a(q - 1) + (q - 1 - a)(q - 2) = (q - 1)^2 - (q - 1 - a)
\]
proving (i).

According to Theorem 4.5, we have the inequality of (ii)
\[
r(U) \leq 1 + (q - 1)^2 - (q - 1 - a).
\]

We obtain (iii) by using the result in (i) on the polytope \( (q - 2, q - 2) + (-H \setminus -\square) \) with vertices \( (0, 0), (q - 2 - b, 0), (q - 2 - a, q - 2) \) and \( (q - 2, q - 2) \). The maximum number of zeros of \( \pi_{T(x)}(g) \) for \( g \in \Phi_q[-H \setminus -U] \) is by Lemma 3.10 and the result in (i) less than or equal to \( (q - 1)^2 - (q - 1 - (q - 2 - b)) = (q - 1)^2 - 1 - b \) and (iii) follows from Theorem 4.5.

To prove (iv) assume \( t \leq (q - 2 - 2a) - 1 \) and \( t \leq b - 2 \). We will use Theorem 4.6.

Consider the Minkowski sum \( U + U \) and let \( V = U + U \) be its reduction modulo \( q - 1 \) as in Corollary 3.11. Under the assumption \( 2a \leq q - 2 \), we have that \( V = U + U \) is the lattice points of the integral convex polytope with vertices \( (0, 0), (2a, 0), (2b, q - 2) \) and \( (0, q - 2) \).

By the result in (i) the maximum number of zeros of \( \pi_{T(x)}(h) \) for \( h \in \Phi_q[V] \) is less than or equal to \( (q - 2)^2 - (q - 1 - 2a) \). As the number of players is \( n = (q - 1)^2 - 1 \), the right hand side of the condition (4.6) of Theorem 4.6 is at least \( (q - 2 - 2a) - 1 \), which by assumption is at least \( t \).

By assumption \( t \leq b - 1 \) and from (iii) we have that \( b - 1 \leq t(U) \). We conclude that \( t \leq t(U) \). \( \square \)
Figure 7: The convex polytope $H$ with vertices $(0, 0), (q-2, 0), (q-2, q-2), (0, q-2)$ and the convex polytope $\square$ with vertices $(0, 0), (a, 0), (b, q-2), (0, q-2)$ are shown. Also their opposite convex polytopes $-H$ and $-\square$, the complement $-H \setminus -\square$ and its translate $(q-2, q-2) + (-H \setminus -\square)$ are depicted. Finally the convex hull of the reduction modulo $q-1$ of the Minkowski sum $U + U$ of the lattice points $U = \square \cap M$ in $\square$, is rendered. It has vertices $(0, 0), (2a, 0), (2b, q-2)$ and $(0, q-2)$.

5 Asymmetric Quantum Codes on Toric Surfaces

5.1 Introduction

In Hansen (2013) we applied our construction of Section 3 to obtain toric codes suitable for constructing quantum codes by the Calderbank-Shor-Steane method. Our constructions extended similar results obtained by A. Ashikhmin, S. Litsyn and M.A. Tsfasman in Ashikhmin et al. (2001) from Goppa codes on algebraic curves.

Works of Shor (1995) and Steane (1996c), Steane (1996a) initiated the study and construction of quantum error-correcting codes. Calderbank and Shor (1996) Shor (1996) and Steane (1999b) produced stabilizer codes (CSS) from linear codes containing their dual codes. For details see for example Ashikhmin and Knill (2001), Calderbank et al. (1998) and Steane (1998).

Asymmetric quantum error-correcting codes are quantum codes defined over biased quantum channels: qubit-flip and phase-shifter errors may have equal or different probabilities. The code construction is the CSS construction.
\[ n(\rho_3) = \left\{ \frac{1}{q} (q - 2), \frac{a - b}{\gcd(a, q - 2)}, \frac{a - b}{\gcd(a, q - 2)} \right\}. \]

Figure 8: The normal fan and its 1-dimensional cones \( \rho_i \), with primitive generators \( n(\rho_i) \), and 2-dimensional cones \( \sigma_i \) for \( i = 1, \ldots, 4 \) of the polytope \( \square \) in Figure 7.

Based on two linear codes. The construction appeared originally in Evans et al. (2007), Ioffe and Mézard (2007) and Stephens et al. (2008). We present new families of toric surfaces, toric codes and associated asymmetric quantum error-correcting codes.

5.2 The toric surfaces \( X_b \) and their intersection theory

Let \( \mathbb{F}_q \) be the field with \( q \) elements and let \( r \) be an integer dividing \( q \). Let \( b \in \mathbb{Z} \) such that \( 0 \leq b \leq q - 2 \) with \( a := b + \frac{q - 2}{r} \leq q - 2 \).

Let \( M \) be an integer lattice \( M \cong \mathbb{Z}^2 \). Let \( N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \) be the dual lattice with canonical \( \mathbb{Z} \)-bilinear pairing \( \langle -,- \rangle : M \times N \to \mathbb{Z} \). Let \( M_\mathbb{R} = M \otimes \mathbb{R} \) and \( N_\mathbb{R} = N \otimes \mathbb{R} \) with canonical \( \mathbb{R} \)-bilinear pairing \( \langle -,- \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R} \).

Let \( \square_b \) in \( M_\mathbb{R} \) be the 2-dimensional integral convex polytope in \( M_\mathbb{R} \) with vertices \((0,0), (a,0), (b,q-2) \) and \((0,q-2)\) properly contained in the square \([0,q-2] \times [0,q-2]\), see Figure 9. It is the Minkowski sum of the line segment from \((0,0)\) to \((b,0)\) and the polytope \( \square_0 \), see Figure 10.

The support function \( h_b : N_\mathbb{R} \to \mathbb{R} \) for \( \square_b \) is defined as \( h_b(n) := \inf \{ \langle m, n \rangle \mid m \in \square_b \} \) and the polytope \( \square_b \) can be reconstructed from the support function

\[ \square_b = \{ m \in M \mid \langle m, n \rangle \geq h(n) \quad \forall n \in N \}. \quad (5.1) \]

The normal fan \( \Delta_b \) is the coarsest fan such that \( h_b \) is linear on each \( \sigma \in \Delta_b \),
Figure 9: The polytope $\square_b$ is the polytope with vertices $(0,0)$, $(a = b + \frac{q-2}{r}, 0)$, $(b, q-2)$, $(0, q-2)$.

Figure 10: The polytope $\square_0$ is the polytope with vertices $(0,0)$, $(a = \frac{q-2}{r}, 0)$, $(0, q-2)$. 
i.e. for all $\sigma \in \Delta_b$ there exists $l_{\sigma} \in M$ such that
\[ h_b(n) = \langle l_{\sigma}, n \rangle \quad \forall n \in \sigma. \] (5.2)

Upon refinement of the normal fan, we can assume that the generators of any two successive pairs of 1-dimensional cones generate the lattice and we obtain the refined normal fan.

The 1-dimensional cones in the refined normal fan $\Delta_0$ of the polytope $\Box_0$ are generated by unique primitive elements $n(\rho)$ such that $\rho = R_{\geq 0}n(\rho)$, specifically
\[ n_{\rho_1} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad n_{\rho_2} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad n_{\rho_3} = \left( \begin{array}{c} -1 \\ 0 \end{array} \right), \quad n_{\rho_4} = \left( \begin{array}{c} -r-1 \\ 1 \end{array} \right), \] (5.3)
see Figure 11.

There are four 2-dimensional cones $\sigma_i$ in the refined normal fan $\Delta_0$ with corresponding $l_{\sigma_i}$ as in (5.2):

- $\sigma_1$ with faces $\rho_1$, $\rho_2$ and $l_{\sigma_1} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$
- $\sigma_2$ with faces $\rho_2$, $\rho_3$ and $l_{\sigma_1} = \left( \begin{array}{c} 2 \\
-2 \\ 0 \end{array} \right)$
- $\sigma_3$ with faces $\rho_3$, $\rho_4$ and $l_{\sigma_1} = \left( \begin{array}{c} 2 \\ 0 \\ 0 \end{array} \right)$
- $\sigma_4$ with faces $\rho_4$, $\rho_1$ and $l_{\sigma_1} = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)$.

The resulting toric surface $X_b$ is irreducible, complete and non-singular under the assumption that we are working with the refined normal fan.

Let $h_0$ be the support function of the refined normal fan $\Delta_0$. Let $D_{h_0}$ be the associated Cartier divisor of, see Section 2.2.

By the methods of Section 2.3, we obtain:
\[ (D_{h_0}; V(\rho_1)) = -(h_0(n_{\rho_1})) = q - 2, \] (5.4)
and
\[ (V(\rho_1); V(\rho_1)) = r. \] (5.5)
Theorem 5.1. Let $\mathbb{F}_q$ be the field with $q$ elements and let $r$ be an integer dividing $q$. Let $b \in \mathbb{Z}$ such that $0 \leq b \leq q - 2$ with $a := b + \frac{q-2}{r} \leq q - 2$.

Let $\square_b$ in $M_{\mathbb{R}}$ be the 2-dimensional integral convex polytope in $M_{\mathbb{R}}$ with vertices $(0,0), (a,0), (b,q-2)$ and $(0,q-2)$ contained in the square $[0,q-2] \times [0,q-2]$, see Figure 9.

Let $C_b$ be the corresponding toric code as defined in Section 3.

Then
(i) \( n = \text{length } C_b = (q - 1)^2 \).

(ii) \( k = \dim C_b = \frac{1}{2} \left( \frac{q^2}{r} + 1 \right) q + b(q - 1) \).

(iii) \( d(C_b) = (q - 1 - a)(q - 1) \) (the minimum distance).

Proof. As we evaluate in \((q - 1)^2\) points on \(X_b\), the length is as claimed. The dimension \( \dim C_b \) equals the number of integral points in \( \Delta_b \), which is \( \frac{1}{2} \left( \frac{q^2}{r} + 1 \right) q + b(q - 1) \).

**Minimum distance of the toric code in the special case \( b = 0 \).** See figures 10 and 11. We bound the number of points in the support \( S = \mathbb{F}_q^* \times \mathbb{F}_q^* \subseteq X_b \), where the rational functions in \( H^0(X_0, O_X(D_{b_0}))^{\text{Frob}} \) evaluates to zero.

The support \( S \) is stratified by the intersections with the zeros of \( e(m_1) - \psi \), where \( \psi \in \mathbb{F}_q^* \). A rational function \( f \) can either vanish identically on a stratum or have a finite number of zeroes along the stratum.

**Identically vanishing on strata:** Assume that \( f \) is identically zero along precisely \( A \) of these strata. As \( e(m_1) - \psi \) and \( e(m_1) \) have the same divisors of poles, they have equivalent divisors of zeroes, so

\[
(e(m_1) - \psi)_0 \sim (e(m_1))_0.
\]

Therefore

\[
\text{div}(f) + D_{b_0} - A(e(m_1))_0 \geq 0
\]

or equivalently

\[
f \in H^0(X_0, O_X(D_{b_0} - A(e(m_1))_0).
\]

Therefore \( A \leq a \) by Lemma 2.5.

**Vanishing in a finite number of points on a stratum:** On any of the \( q - 1 - A \) other strata, the number of zeroes of \( f \) is at most the intersection number

\[
(D_{b_0} - A(e(m_1))_0; (e(m_1))_0) = (q - 2) - Ar
\]

following (5.4) and (5.5), see Hansen (2001a).

Consequently, the number of zeroes is at most \( A(q - 1) + (q - 1 - A)(q - 2 - Ar) \leq (q - 1)^2 - (q - 1 - a)(q - 1) \) as \( A \leq a \) and therefore \( d(C_0) \geq (q - 1 - a)(q - 1) \).

**Minimum distance of the toric code in the general case \( b > 0 \).** See Figure 9. The polytope \( \Gamma_a \) with vertices \((0, 0), (a, 0), (b, q - 2) \) and \((0, q - 2) \) is the Minkowski sum of the line segment from \((0, 0)\) to \((b, 0)\) and the polytope \( \Delta_a \), see Figure 10. Applying Little and Schenck (2006, Proposition 2.3) and the special case \( b = 0 \), we have the inequality \( d(C_a) \geq (q - 1 - a)(q - 1) \) also in the general case.

For pairwise different \( x_1, \ldots, x_a \in \mathbb{F}_q^* \) the function \( (e(m_1) - x_1)(e(m_1) - x_2) \ldots (e(m_1) - x_a) \in H^0(X_0, O_X(D_{b_0})) \) vanishes in the \( a(q - 1) \) points \((x_i, y), i = 1, \ldots, a \) and \( y \in \mathbb{F}_q^* \). In conclusion, we have the equality \( d(C_b) = (q - 1 - a)(q - 1) \).
5.3 Asymmetric Quantum Codes

5.3.1 Notation

Let $H$ be the Hilbert space $H = \mathbb{C}^n = \mathbb{C}^q \otimes \mathbb{C}^q \otimes \cdots \otimes \mathbb{C}^q$. Let $|x\rangle, x \in \mathbb{F}_q$ be an orthonormal basis for $\mathbb{C}^q$. For $a, b \in \mathbb{F}_q$, the unitary operators $X(a)$ and $Z(b)$ in $\mathbb{C}^q$ are

$$X(a)|x\rangle = |x + a\rangle, \quad Z(b)|x\rangle = \omega^{\text{tr}(bx)}|x\rangle,$$

where $\omega = \exp(2\pi i/p)$ is a primitive $p$th root of unity and $\text{tr}$ is the trace operation from $\mathbb{F}_q$ to $\mathbb{F}_p$.

For $a = (a_1,\ldots,a_n) \in \mathbb{F}_q^n$ and $b = (b_1,\ldots,b_n) \in \mathbb{F}_q^n$

$$X(a) = X(a_1) \otimes \cdots \otimes X(a_n)$$

$$Z(b) = Z(b_1) \otimes \cdots \otimes Z(b_n)$$

are the tensor products of $n$ error operators.

With

$$E_x = \left\{ X(a) = \bigotimes_{i=1}^n X(a_i) \left| a \in \mathbb{F}_q^n, a_i \in \mathbb{F}_q \right. \right\},$$

$$E_z = \left\{ Z(b) = \bigotimes_{i=1}^n Z(b_i) \left| b \in \mathbb{F}_q^n, b_i \in \mathbb{F}_q \right. \right\}$$

the error groups $G_x$ and $G_z$ are

$$G_x = \{ \omega^c E_x \omega^c X(a) \left| a \in \mathbb{F}_q^n, c \in \mathbb{F}_p \right. \},$$

$$G_z = \{ \omega^c E_z \omega^c Z(b) \left| b \in \mathbb{F}_q^n, c \in \mathbb{F}_p \right. \}.$$

It is assumed that the groups $G_x$ and $G_z$ represent the qubit-flip and phase-shift errors.

**Definition 5.2** (Asymmetric quantum code). A $q$-ary asymmetric quantum code $Q$, denoted by $[[n,k,d_x/d_z]]_q$, is a $q^k$-dimensional subspace of the Hilbert space $\mathbb{C}^q^n$ and can control all bit-flip errors up to $d_x$ and all phase-flip errors up to $d_z$. The code $Q$ detects $(d_x - 1)$ qubit-flip errors as well as detect $(d_z - 1)$ phase-shift errors.

Let $C_1$ and $C_2$ be two linear error-correcting codes over the finite field $\mathbb{F}_q$, and let $[n,k_1,d_1]_q$ and $[n,k_2,d_2]_q$ be their parameters. For the dual codes $C_i^\perp$, we have $\dim C_i^\perp = n - k_i$ and if $C_i^\perp \subseteq C_2$ then $C_i^\perp \subseteq C_1$.

**Lemma 5.3.** Let $C_i$ for $i = 1, 2$ be linear error-correcting codes with parameters $[n,k_i,d_i]_q$ such that $C_1^\perp \subseteq C_2$ and $C_2^\perp \subseteq C_1$. Let $d_x = \min\{\text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp)\}$, and $d_z = \max\{\text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp)\}$. Then there is an asymmetric quantum code with parameters $[[n,k_1 + k_2 - n, d_x, d_z]]_q$. The quantum code is pure to its minimum distance, meaning that if $\text{wt}(C_1) = \text{wt}(C_1 \setminus C_2^\perp)$, then the code is pure to $d_x$, also if $\text{wt}(C_2) = \text{wt}(C_2 \setminus C_1^\perp)$, then the code is pure to $d_z$.

This construction is well-known, see for example Ashikhmin and Knill (2001), Calderbank et al. (1998), Shor (1995), Steane (1996c), Steane (1996b), Steane (1999a) Aly and Ashikhmin (2010). The error groups $G_x$ and $G_z$ can be mapped to the linear codes $C_1$ and $C_2$.
5.4 New Asymmetric Quantum Codes from Toric Codes

Let \( \mathbb{F}_q \) be the field with \( q \) elements and let \( r \) be an integer dividing \( q \). Let \( b \in \mathbb{Z} \) such that \( 0 \leq b \leq (r-1)(q-2)/r \). Then the polytope \( \square_b \) with vertices \((0,0), (a = b + \frac{r-2}{r}, 0), (0, q-2) \) is contained in \([0, q-2] \times [0, q-2] \). Consider the associated toric code \( C_b \).

From the results in Section 3.4 we conclude that the dual code \( C_b^\perp \) is the toric code associated to the polytope \( \square_{b,\perp} \) with vertices \((0, q-2), (a^\perp = b^\perp + \frac{q-2}{r}, 0), (0, q-2) \) where \( b^\perp = \frac{(r-1)(q-2)}{r} - b \) such that \( a^\perp = q-2-b \).

For \( i = 1,2 \) let \( b_i \in \mathbb{Z} \) with \( 0 \leq b_i \leq (r-1)(q-2)/r \) and \( b_1 + b_2 \geq (r-1)(q-2)/r \).

We have the inclusions of polytopes \( \square_{b_i} \subseteq \square_{b_1} \) and \( \square_{b_i} \subseteq \square_{b_2} \), see Figure 12, and corresponding inclusions of the associated toric codes.

\[
C_{b_1}^\perp = C_{b_1} \subseteq C_{b_1} \cap C_{b_2} \subseteq C_{b_2}. 
\]

The nested codes gives by the construction of Lemma 5.3 and the discussion above rise to an asymmetric quantum code \( Q_{b_1, b_2} \).

**Theorem 5.4** (Asymmetric quantum codes \( Q_{b_1, b_2} \)). Let \( \mathbb{F}_q \) be the field with \( q \) elements and let \( r \) be an integer dividing \( q \). For \( i = 1, 2 \) let \( b_i, a_i = b_i + \frac{q-2}{r} \in \mathbb{Z} \).

Then there is an asymmetric quantum code \( Q_{b_1, b_2} \) with parameters \( \left(\frac{(q-1)^2}{4\left(\frac{r-2}{r} + 1\right)q + (b_1 + b_2)(q-1), d_x/d_z}\right)_q \), where

\[
d_x = (q - 1 - \min\{b_1, b_2\})(q - 1) 
\]

\[
d_z = (q - 1 - \max\{b_1, b_2\})(q - 1) 
\]

If \( b_1 + b_2 \neq (r-1)(q-2)/r \) the quantum code is pure to \( d_x \) and \( d_z \).

**Proof.** The parameters and claims follow directly from Lemma 5.3 and Theorem 5.1.

\( \square \)

6 Toric Codes, Multiplicative Structure and Decoding

The main theme is the inherent multiplicative structure on toric codes. The multiplicative structure allows for decoding, resembling the decoding of Reed-Solomon codes and aligns with decoding by error correcting pairs.

Toric codes have an inherent multiplicative structure.

In Hansen (2017a) we utilized the multiplicative structure to decode toric codes, resembling the decoding of Reed-Solomon codes and decoding by error correcting pairs, see

Pellikaan (1992), Kötter (1992) and Márquez-Corbella and Pellikaan (2016).

6.1 Multiplicative structure

In the notation of Section 3 let \( \square \) and \( \square \) be polyhedra in \( \mathbb{R}^r \) and let \( \square + \square \) denote their Minkowski sum. Let \( U = \square \cap \mathbb{Z}^r \) and \( \bar{U} = \square \cap \mathbb{Z}^r \). The map

\[
\mathbb{F}_q[U] \otimes \mathbb{F}_q[\bar{U}] \rightarrow \mathbb{F}_q[U + \bar{U}]
\]

\((f, g) \mapsto f \cdot g, \)

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induces a multiplication on the associated toric codes

\[ C_{\square} \oplus C_{\tilde{\square}} \rightarrow C_{\square + \tilde{\square}} \]

with coordinatewise multiplication of the codewords – the \textit{Schur} product.

Our goal is to use the multiplicative structure to correct \( t \) errors on the toric code \( C_{\square} \). This is achieved choosing another toric code \( C_{\tilde{\square}} \) that helps to reduce error-correcting to a linear problem.

Let \( \square \) and \( \tilde{\square} \) be polyhedra as above in \( \mathbb{R}^2 \), let \( \square + \tilde{\square} \) denote their Minkowski sum. Assume from now on:

1. \( |\tilde{\mathcal{U}}| > t \), where \( \tilde{\mathcal{U}} = \tilde{\square} \cap \mathbb{Z}^2 \).
2. \( d(C_{\square + \tilde{\square}}) > t \), where \( d(C_{\square + \tilde{\square}}) \) is the minimum distance of \( C_{\square + \tilde{\square}} \).
3. \( d(C_{\tilde{\square}}) > n - d(C_{\square}) \), where \( d(C_{\square}) \) and \( d(C_{\tilde{\square}}) \) are the minimum distances of \( C_{\square} \) and \( C_{\tilde{\square}} \).

### 6.1.1 Error-locating

Let the received word be \( y(P) = f(P) + e(P) \) for \( P \in T(\mathbb{F}_q) \), with \( f \in \mathbb{F}_q[U] \) and error \( e \) of Hamming-weight at most \( t \) with support \( \mathcal{T} \subseteq T(\mathbb{F}_q) \), such that \( |\mathcal{T}| \leq t \).

From (1), it follows that there is a \( g \in \mathbb{F}_q[\tilde{\mathcal{U}}] \) such that \( g|_\mathcal{T} = 0 \) – an \textit{error-locator}. To find \( g \), consider the linear map:

\[
\mathbb{F}_q[\tilde{\mathcal{U}}] \oplus \mathbb{F}_q[U + \tilde{\mathcal{U}}] \rightarrow \mathbb{F}_q^n,
\]

\[
(g, h) \mapsto \left( g(P)y(P) - h(P) \right)_{P \in T(\mathbb{F}_q)}.
\]
As $y(P) - f(P) = 0$ for $P \not\in T$ (recall that the support of the error $e$ is $T$), we have that $g(P)y(P) - (g \cdot f)(P) = 0$ for all $P \in T(\mathbb{F}_q)$. That is $(g, h = g \cdot f)$ is in the kernel of $(6.1)$.

**Lemma 6.1.** Let $(g, h)$ be in the kernel of $(6.1)$. Then $g|_T = 0$ and $h = g \cdot f$.

**Proof.**

\begin{equation}
\begin{align*}
e(P) &= y(P) - f(P) & \text{for } P \in T(\mathbb{F}_q).
\end{align*}
\end{equation}

Coordinate wise multiplication yields by $(6.1)$

\begin{equation}
\begin{align*}
g(P)e(P) &= g(P)y(P) - g(P)f(P) \\
&= h(P) - g(P)f(P)
\end{align*}
\end{equation}

for $P \in T(\mathbb{F}_q)$. The left hand side has Hamming weight at most $t$, the right hand side is a code word in $C_{\ominus \oplus C}$ with minimal distance strictly larger than $t$ by assumption $(2)$. Therefore both sides equal $0$. \hfill \square

### 6.1.2 Error-correcting

**Lemma 6.2.** Let $(g, h)$ be in the kernel of $(6.1)$ with $g|_T = 0$ and $g \neq 0$. There is a unique $f$ such that $h = g \cdot f$.

**Proof.** As in the above proof, we have

\begin{equation}
\begin{align*}
g(P)y(P) - g(P)f(P) &= 0 & \text{for } P \in T(\mathbb{F}_q).
\end{align*}
\end{equation}

Let $Z(g)$ be the zero-set of $g$ with $T \subseteq Z(g)$. For $P \not\in Z(g)$, we have $y(P) = f(P)$ and there are at least $d(C_{\ominus}) > n - d(C_{\ominus})$ such points by $(3)$. This determines $f$ uniquely as it is determined by the values in $n - d(C_{\ominus})$ points. \hfill \square

**Example 6.3.** Let $\Box$ be the convex polytope with vertices $(0, 0)$, $(a, 0)$ and $(0, a)$. Let $\bigcirc$ be the convex polytope with vertices $(0, 0)$, $(b, 0)$ and $(0, b)$. Their Minkowski sum $\Box + \bigcirc$ is the convex polytope with vertices $(0, 0)$, $(a + b, 0)$ and $(0, a + b)$, see Figure 13.

From Hansen (2002, Theorem 1.3), we have that $n = (q-1)^2$, $|\Box| = (b+1)(b+2)/2$, $d(C_{\bigcirc}) = (q-1)(q-1-a)$, $d(C_{\bigcirc} + \bigcirc) = (q-1)(q-1-b)$ and $d(C_{\bigcirc} + \bigcirc) = (q-1)(q-1-(a+b))$ for the associated codes over $\mathbb{F}_q$.

Let $q = 16$, $a = 4$ and $b = 8$. Then $n = 225$, $|\Box| = 45$, $d(C_{\bigcirc}) = 165$, $d(C_{\bigcirc} + \bigcirc) = 105$ and $d(C_{\bigcirc} + \bigcirc) = 45$.

As $d(C_{\bigcirc} + \bigcirc) = 105 > 60 = n - d(C_{\bigcirc})$, the procedure corrects $t$ errors with $t < \min(d(C_{\bigcirc} + \bigcirc), |\Box|) = 45$.

**Remark 6.4** (Error correcting pairs). Pellikaan (1992) and Kötter (1992) introduced the concept of error-correcting pairs for a linear code, see also Márquez-Corbella and Pellikaan (2016). Specifically for a linear code $C \subseteq \mathbb{F}_q^n$, a $t$-error correcting pair consists of two linear codes $A, B \subseteq \mathbb{F}_q^n$, such that

\begin{equation}
A \perp B, \quad \dim_{\mathbb{F}_q} A > t, \quad A \perp B^+, \quad d(A) + d(C) > n.
\end{equation}

Here $A \perp B = \{a \cdot b \mid a \in A, b \in B\}$ and $\perp$ denotes orthogonality with respect to the usual inner product. They described the known decoding algorithms for decoding $t$ or fewer errors in this framework.

Also the decoding in the present paper can be described in this framework, taking $C = C_{\bigcirc}, A = C_{\bigcirc}$ and $B = (C \star A)^\perp$ using Proposition 3.13.
Figure 13: The convex polytope $\Box$ with vertices $(0,0),(a,0)$ and $(0,a)$. The convex polytope $\tilde{\Box}$ with vertices $(0,0),(b,0)$ and $(0,b))$. Their Minkowski sum $\Box + \tilde{\Box}$ having vertices $(0,0),(a+b,0)$ and $(0,a+b)$.

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