THE $RO(G)$-GRADED COEFFICIENTS OF $(\mathbb{Z}/2)^n$-EQUIVARIANT $K$-THEORY

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1. Introduction

In physics, in type IIA and IIB string theory, D-brane charges are calculated by $K$-theory (see [8] or [9] for a survey). Kriz, Pando and Quiroz [7], following up on previous investigation of other authors (e.g. [4, 5]) investigated the basic case of D-brane charges of orbifolds obtained by linear operation of finite groups on flat spacetime. In the case of the finite group $(\mathbb{Z}/2)^n$, the groups relevant can be expressed as the $RO(G)$-graded coefficients of $(\mathbb{Z}/2)^n$-equivariant $K$-theory. In case of what is known in physics as discrete torsion (cf. [5]), we get twisted $(\mathbb{Z}/2)^n$-equivariant $K$-groups with compact supports of representations. A physical phenomenon called $T$-duality predicts further certain relations between these groups.

This is an amusing problem, and we found a direct elementary solution, which is the subject of the present note. We show that $K((\mathbb{Z}/2)^n)(S^V)$ for any finite representation $V$ is always concentrated in dimensions of one parity (even or odd) essentially $RO((\mathbb{Z}/2)^m)$ for some $m \leq n$ (see Theorem 1). The exponents are calculated by an explicit combinatorial algorithm. The twisted groups are the same with a certain shift, which corresponds to the physical $T$-duality prediction.

Since originally posting this note, Max Karoubi pointed out to us that Theorem 1 can in fact be deduced as an easy consequence of his much more general result, namely Theorem 1.8 of [6]. We thank him for bringing this to our attention. We also thank John Greenlees for previous discussions.

2. The Main Theorem

For a group $G$, we consider the reduced $G$-equivariant $K$-theory $\tilde{K}_G$. The goal of this note is to prove the following theorem:

**Theorem 1.** For any finite dimensional representation $V$ of $G = (\mathbb{Z}/2)^n$, there exists an $\epsilon \in \{0, 1\}$ and $m \in \{0, 1, \ldots, n\}$, such that

$$\tilde{K}^*_G(S^V) = \mathbb{Z}^{2m} \text{ where } k \equiv \epsilon \mod 2$$

(1)

Here, the number $m$ can be calculated by induction.

**Proof:** The basic observation is that by equivariant Bott periodicity [1], the $K$-theory group [1] does not change when we add a complex representation $W$ to $V$. 

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Also, the result is obviously true when \( V \) is a sum of \( \ell \) irreducible real representations \( \alpha_1, \ldots, \alpha_\ell \), which are \( \mathbb{Z}/2 \)-linearly independent in the character group. In that case, we have \( m = n - \ell \) and \( \epsilon \equiv 0 \) by smashing the usual cofiber sequence

\[
G/Ker(\alpha_i) \to S^0 \to S^\alpha_i
\]

for \( i = 1, \ldots, \ell \).

Next, we recall that we also have Bott periodicity for equivariant Spin\(^c\)-representations \((\Pi)\). If \( \alpha, \beta, \gamma \) are real irreducible, \n
\[
U = 1 + \alpha + \beta + \gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\beta\gamma
\]

is spin. To this end, note that if we set \( H^*((\mathbb{Z}/2)^3, \mathbb{Z}/2) = \mathbb{Z}[x, y, z] \) then the total Stiefel-Whitney class of \( U \) is

\[
(1 + x)(1 + y)(1 + z)(1 + x + y)(1 + x + z)(1 + y + z)(1 + x + y + z)
\]

which has even coefficients in degree < 3. Consequently, \( w_1(U) = w_2(U) = 0 \). So we have proved

\[
\tilde{K}^k_G(S^U) \cong \tilde{K}^k_G(S^0).
\]

It is now time for us to set up our induction process. We will choose irreducible real representations \( \alpha_1, \ldots, \alpha_p \), independent in the class group, such that \( V \) is a sum of tensor products of subsets of \( \alpha_1, \ldots, \alpha_p \). In fact, we shall represent \( V \) as a “hypergraph”, i.e. the system \( \Gamma \) of subsets of \( \{\alpha_1, \ldots, \alpha_p\} \) whose sum is \( V \) (assuming it contains no complex sub-representations). By \((\Pi)\), we may assume \( \Gamma \) is in fact just a “graph”, i.e. it does not contain any sets of cardinality > 2 (because otherwise we can add representations of the form \((3)\) and subtract complex representations to reduce the number of sets of highest cardinality). Note that we do not, however, assume \( \Gamma \) contains all its “vertices”, i.e. 1-element sets.

Let us now look at a vertex \( v \). We will do induction with respect to the number of edges adjacent to \( v \), or alternately the number of vertices. If \( v \) is attached to at least two edges \( a, b \) in \( \Gamma \), which in turn attached to vertices \( u, w \) (whether \( v, w, u \) are included in \( \Gamma \) or not), apply \((3)\) with \( \alpha = v, \beta = w, \gamma = u \). Then the graph \( \Gamma \) will turn into a hypergraph again, which will contain the set \( \{u, v, w\} \) not contain either \( a \) or \( b \), and may contain some of the sets \( \{u\}, \{v\}, \{w\}, \{u, w\} \) (in fact precisely those which were not contained in \( \Gamma \)). Now apply a base change which uses all the original basis elements (=vertices) with the exception of \( w \), which will be replaced by \( w' = w + u \). Then representing the representation again as a hypergraph with respect to the new basis, the set \( \{u, v, w\} \) will turn into \( \{v, w'\} \). We see that there are now now more sets of cardinality > 2 attached to \( v \), and the number of edges attached to \( v \) has decreased by 1. Any sets of cardinality > 2 in the new graph not containing \( v \) can be eliminated again as before by (possibly repeated) application of \((3)\).

By this induction, it suffices to consider the case where \( v \) is attached by a single edge \( a \) in \( \Gamma \) (whether \( v \) is included in \( \Gamma \) or not). But the case when \( v \) is not included in \( \Gamma \) in fact equivalent to the graph obtained by erasing \( v \). Suppose therefore that \( v \) is included in \( \Gamma \). Then let \( v \) be attached by an edge to a vertex \( w \) (which may or may not be included in \( \Gamma \)). Now the point is that the entire induction described above
Hypergraph will contain the sets \( \{v, w\} \), possibly (but not necessarily) the set \( \{w\} \), and not any other sets containing \( v \) or \( w \). Let us recapitulate then: we know that \( \Gamma \) contains the sets \( \{u, v, w\} \), \( \{u, v\} \), \( \{u, v\} \), possibly (but not necessarily) the set \( \{w\} \) (in fact, precisely when \( \Gamma \) didn’t contain it), but no other sets containing \( v \) or \( w \). Then, change basis by including all the vertices of \( \Gamma \) with the exception of one single edge, ending in a vertex \( v \) or \( w \). Then apply \( \mathfrak{sl}_2 \) with \( \alpha = v, \beta = w, \gamma = u \). Then the new hypergraph will contain the sets \( \{u, v, w\} \), \( \{u, v\} \), \( \{u, v\} \), possibly (but not necessarily) the set \( \{w\} \). We see then that the hypergraph is a disjoint union of two hypergraphs, which reduces this case to lower rank cases by the Künneth theorem.

Perhaps surprisingly, there is still a case which remains to be treated, namely the case of \( n = 2 \), \( V = \alpha + \beta + \alpha \beta \) where \( \alpha, \beta \) are the generators of the character group. Let \( A, B, C \cong \mathbb{Z}/2 \) be the subgroups of \((\mathbb{Z}/2)^2\) on which the representations \( \alpha, \beta, \alpha \beta \) vanish. Then consider the cofiber sequence

\[
(\mathbb{Z}/2)^2/C_+ \wedge S^{\alpha + \beta} \to S^{\alpha + \beta} \to S^{\alpha + \beta + \alpha \beta}.
\]

The \( K_{(\mathbb{Z}/2)^2} \)-groups of the first two terms vanish, theis \( K_{(\mathbb{Z}/2)^2}^0 \)-groups are \( \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \) respectively, so we need to identify the map

\[
Z \oplus Z \leftarrow Z
\]

induced on \( K_{(\mathbb{Z}/2)^2}^0 \) by the first map \( \mathfrak{sl}_2 \). Note that the interesting information is just the image of this map, which can be calculated in \( C \cong \mathbb{Z}/2 \)-equivariant \( K \)-theory. When restricted to \( C \), \( \alpha \cong \beta \) are isomorphic to the sign representation of \( C \). Now in \( K_C^0(S^\alpha) \), we have an element \( c \) which, under the inclusion

\[
S^0 \subset S^\alpha
\]

restricts to

\[
1 - \alpha \in R(C) = \tilde{K}_C^0(S^0).
\]

Then the restriction of the generator of \( \tilde{K}_{(\mathbb{Z}/2)^2}^0(S^{\alpha + \beta}) \) to \( \tilde{K}_C^0(S^{2\alpha}) \) is \( c^2 \). Our question is thus equivalent to finding the image of \( c^2 \) in \( \tilde{K}_C^0(S^0) \) under Bott periodicity.

Now the Bott element \( u \in \tilde{K}_C^0(S^{2\alpha}) \) maps to \( c \) (by the construction of the Bott element as \( 1 - H \) where \( H \) is the tautological line bundle on \( \mathbb{C}P^1 \) where \( \mathbb{Z}/2 \) acts as \(-1\) on \( \mathbb{C} \subset \mathbb{C}P^1 \)). So, we need to find the image of \( u^2 \in \tilde{K}_C^0(S^{4\alpha}) \) under the composition \( Bf^* \) where \( f : S^{2\alpha} \to S^{4\alpha} \) is the inclusion (all such inclusions are equivariantly homotopic) and \( B \) is Bott periodicity. But we already know

\[
f^*(u^2) = u(1 - \alpha)
\]

so

\[
Bf^*(u^2) = 1 - \alpha.
\]

This generates a direct summand in the left hand side of \( \mathfrak{sl}_2 \). Hence, we conclude

\[
\tilde{K}_{(\mathbb{Z}/2)^2}^1(S^V) = \mathbb{Z},
\]

\[
\tilde{K}_{(\mathbb{Z}/2)^2}^0(S^V) = 0.
\]
3. An example: $n = 3$.

To demonstrate the algorithm described in the proof of Theorem 1 in action, let us consider $n = 3$, with the character group generated by representations $\alpha, \beta, \gamma$. Then the cases which do not immediately reduce to $n = 2$ are when the representation $V$ is related by an automorphism of $(\mathbb{Z}/2)^3$ by one of the following:

(8) \[ \alpha + \beta + \gamma + \alpha \beta \gamma \]

(9) \[ \alpha + \beta + \gamma + \alpha \beta + \beta \gamma \]

(10) \[ \alpha + \beta + \gamma + \alpha \beta + \gamma \alpha + \beta \gamma \]

(11) \[ \alpha + \beta + \gamma + \alpha \beta + \beta \gamma + \alpha \beta \gamma \]

(12) \[ \alpha + \beta + \gamma + \alpha \beta + \beta \gamma + \alpha \beta \gamma \]

(13) \[ \alpha + \beta + \gamma + \alpha \beta + \beta \gamma + \alpha \gamma + \alpha \beta \gamma. \]

First note that in the case of (13), as noted above, $1 + V$ is spin, so in this case,

\[ m = 3, \epsilon = 1. \]

In the case (8), the algorithm will first convert the hypergraph into a graph using (3), but then immediately add the same representation (3) again, to change to the basis $\alpha, \beta' = \beta \gamma, \gamma$. In this basis $V = \alpha + \alpha \beta' + \beta' \gamma + \gamma$. The algorithm then adds (3) again to give $\alpha \gamma + \beta' + \alpha \beta' \gamma$, and changes base again to $\alpha, \beta', \gamma' = \gamma \alpha$, which gives $\beta' + \gamma' + \beta' \gamma'$. So, keeping track of dimensions, we get

\[ m = 2, \epsilon = 0. \]

In the case (9), the algorithm adds (3) to get $\alpha \gamma + \alpha \beta \gamma$, which under the base change $\alpha, \beta, \gamma' = \alpha \gamma$ becomes $\gamma' + \beta \gamma'$. So the answer is

\[ m = 1, \epsilon = 1. \]

For (10), the algorithm applies (3) to reduce to the graph $\alpha \beta + \beta \gamma$ which are independent, so

\[ m = 1, \epsilon = 1. \]

For (11), the algorithm applies (3) to get $\alpha \beta \gamma$, and then base change to $\alpha, \beta, \gamma' = \beta \gamma$, to give $\alpha \gamma'$, so the answer is

\[ m = 2, \epsilon = 1. \]

For (12), again the algorithm reduces to a graph by applying (3), giving $\alpha \gamma$. So again,

\[ m = 2, \epsilon = 1. \]
4. The twisted case

It is interesting to note that Theorem 1 also gives a calculation of all twisted reduced \((\mathbb{Z}/2)^n\)-equivariant \(K\)-groups of representations (see [2] for the definition of twisted \(K\)-theory). To see this, first note that since \(S^V = V^c\) (the 1-point compactification of \(V\)), reduced \(K\)-theory of \(S^V\) is \(K\)-theory with compact support of \(V\). But \(K\)-theory with compact support on a space \(X\) is, by definition, twisted by the same group as \(K\)-theory for \(X\) (since it is a direct limit of related \(K\)-theories of pairs \((X, U)\) where \(U\) is compact). But \(V\) is contractible, so the ordinary “lower” twistings (note that there are no higher twistings over a point) are classified by

\[H^3((\mathbb{Z}/2)^n, \mathbb{Z}/2) = \mathbb{Z}/2[x_1, \ldots, x_n].\]

Indeed, the Bockstein \(\beta : H^2((\mathbb{Z}/2)^n, \mathbb{Z}/2) \to H^3((\mathbb{Z}/2)^n, \mathbb{Z})\) is onto (the group is annihilated by 2), and \(\beta(x_i^2) = 0\), which by (15) gives the isomorphism (16).

Now let \(\alpha_1, \ldots, \alpha_n\) be the generators of the character group of \((\mathbb{Z}/2)^n\), so that

\[w_1(\alpha_i) = x_i.\]

Then we note that

\[w(1 + \alpha_i + \alpha_j + \alpha_i\alpha_j) = (1 + x_i)(1 + x_j)(1 + x_i + x_j)\]
\[= (1 + x_i + x_j)^2 + x_ix_j\]
\[= 1 + x_i^2 + x_j^2 + x_ix_j,\]

so

\[\beta w_2(1 + \alpha_i + \alpha_j + \alpha_i\alpha_j) = \beta(x_i x_j),\]

which are the generators of (16).

Now, however, the following fact holds in twisted \(K\)-theory. Suppose \(X\) is a \(G\)-space, and \(\eta\) is a \(G\)-equivariant even-dimensional orientable vector bundle over \(X\) with total space \(V\). Then \(\eta\) induces a non-equivariant vector bundle on the Borel construction \(EG \times_G X\), so we have equivariant Stiefel-Whitney classes in Borel cohomology

\[w_i(\eta) \in H^i_{Borel}(X, \mathbb{Z}/2).\]

The class

\[w_3(\eta) = \beta w_2(\eta)\]

is the obstruction to the bundle \(\eta\) being a \(G\)-equivariant \(Spin^c\)-bundle. Now there is a Bott periodicity isomorphism on twisted \(K\)-theory with compact support:

\[K^i,c_G(V) \cong K^{i,c}_{G,\tau + w_3(\eta)}(X).\]

(The proof is just a straightforward modification of Atiyah-Bott’s index argument to the twisted case; there is also a formal argument using parametrized equivariant spectra. A very similar argument is made in [3].) But by (18), we see that the
$w_3$-classes of the representations $1 + \alpha_i + \alpha_j + \alpha_i\alpha_j$ generate the twisting group. When a twisting has
\[
\tau = \sum \epsilon_{ij} \beta(x_ix_j),
\]
then by (19), therefore
\[
\tilde{K}_i(S^V) = \tilde{K}_i(S^V + \sum \epsilon_{ij}(1 + \alpha_i + \alpha_j + \alpha_i\alpha_j))
\]
which reduces the twisted case to the untwisted.

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