Convexity of a Small Ball Under Quadratic Map

Anatoly Dymarsky $^a$

$^a$Center for Theoretical Physics, MIT, Cambridge, MA, USA, 02139

$^1$Skolkovo Institute of Science and Technology, Novaya St. 100, Skolkovo, Moscow Region, Russia, 143025

Abstract

We derive an upper bound on the size of a ball such that the image of the ball under quadratic map is strongly convex and smooth. Our result is the best possible improvement of the analogous result by Polyak [1] in the case of quadratic map. We also generalize the notion of the joint numerical range of $m$-tuple of matrices by adding vector-dependent inhomogeneous terms and provide a sufficient condition for its convexity.

Keywords: convexity, quadratic transformation (map), joint numerical range

1 Introduction and Main Result

1.1 Polyak Convexity Principle

Convexity is a highly appreciated feature which can drastically simplify analysis of various optimization and control problems. In most cases, however, the problem in question is not convex. In [1] Polyak proposed the following approach which proved to be useful in many applications [2]: to restrict the optimization or control problem to a small convex subset of the original set. More concretely, for a map $y_i = f_i(x)$

$^1$Permanent address.
from $\mathbb{R}^m$ to $\mathbb{R}^n$, instead of the full image $\mathcal{F}(f) \equiv f(\mathbb{R}^m) = \{f(x) : x \in \mathbb{R}^n\}$, which is not necessarily convex, let us consider an image of a small ball $B_{\varepsilon}(x_0) = \{x : |x - x_0|^2 \leq \varepsilon^2\}$. For a regular point $x_0$ of $f_i(x)$ there is always small $\varepsilon$ such that the image $f(B_{\varepsilon}(x_0))$ is convex. The underlying idea here is very simple: for any $x$ from a small vicinity of a regular point $x_0$, where rank ($\partial f(x_0)/\partial x$) = $m$, the map $f(x)$ can be approximated by a linear map

$$y_i(x) - y_i(x_0) \simeq \frac{\partial f_i}{\partial x^a}|_{x_0} (x - x_0)^a.$$  

(1.1)

Since the linear map preserves strong convexity, so far the nonlinearities of $f(x)$ are small and can be neglected, the image of a small ball around $x_0$ will be convex. Reference [1] computes a conservative upper bound on $\varepsilon \leq \varepsilon_P$ in terms of the smallest singular value $\nu$ of the Jacobian $J(x_0) \equiv \frac{\partial f}{\partial x}|_{x_0}$ and the Lipschitz constant $L$ of the Jacobian $\partial f(x)/\partial x$ inside $B_{\varepsilon}(x_0)$,

$$\varepsilon^2_P = \frac{\nu^2}{4L^2}.$$  

(1.2)

The resulting image of $B_{\varepsilon}(x_0)$ satisfies the following two properties.

1. The image $f(B_{\varepsilon}(x_0))$ is strictly convex.

2. The pre-image of the boundary $\partial f(B_{\varepsilon}(x_0))$ belongs to the boundary $\partial B_{\varepsilon}(x_0) = \{x : |x - x_0|^2 = \varepsilon^2\}$. The interior points of $B_{\varepsilon}(x_0)$ are mapped into the interior points of $f(B_{\varepsilon}(x_0))$.

### 1.2 Local Convexity of Quadratic Maps

In this paper we consider quadratic maps from $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}^m$ of general form

$$f_i(x) = x^* A_i x - v_i^* x - x^* v_i,$$  

(1.3)

defined through an $m$–tuple of symmetric (hermitian) $n \times n$ matrices $A_i$ and an $m$–tuple of vectors $v_i \in \mathbb{R}^n$ (or $v_i \in \mathbb{C}^n$). Most of the results are equally applicable to both real $x \in \mathbb{R}^n$ and complex $x \in \mathbb{C}^n$ cases. The symbol * denotes transpose or hermitian conjugate correspondingly. Occasionally we will also use $^T$ to denote transpose for the explicitly real-valued quantities.
Applying the general theory of [1] toward (1.3) one obtains (1.2), where \( \nu^2 \) is the smallest eigenvalues of the symmetric \( m \times m \) matrix \( \text{Re}(v_i^* v_j) \) and the Lipschitz constant \( L \) for (1.3) can be defined through

\[
L = \max_{|x_1|^2 = |x_2|^2 = 1} \sqrt{\sum_{i=1}^{n} \text{Re}(x_1^* A_i x_2)^2} . \tag{1.4}
\]

We see from (1.2) that \( \varepsilon_P \) is non-zero only if the point \( x_0 = 0 \) is regular and the \( m \times 2n \) matrix \( \text{Re}(v_i) \oplus \text{Im}(v_i) \) has rank \( m \).

Since any linear transformations of \( x \) respects the form (1.3), generalizations to different central points \( x_0 \neq 0 \) or non-degenerate ellipsoids \( (x - x_0)^* G (x - x_0) \leq \varepsilon^2 \), with some positive-definite \( G \) instead of \( |x - x_0|^2 \leq \varepsilon^2 \) is trivial.

The bound (1.2) is usually very conservative, and one can normally find a much larger ball \( B_{\varepsilon'}(x_0) \) with \( \varepsilon' > \varepsilon_P \) such that the properties 1, 2 from section 1.1 are still satisfied. The main result of this paper is the new improved bound \( \varepsilon^2_{\max} \geq \varepsilon^2_P \), where

\[
\varepsilon^2_{\max} \equiv \lim_{\epsilon \to 0^+} \min_{|c| = 1} \left| (c \cdot A - \lambda_{\text{min}}(c \cdot A) + \epsilon)^{-1} c \cdot v \right|^2 , \tag{1.5}
\]

\[
\lambda_{\text{min}}(A) = \min \{ \lambda_{\text{min}}(A), 0 \} . \tag{1.6}
\]

Here the minimum is over the unit sphere from the dual space \( c \in \mathbb{R}^m, c \cdot y \equiv c^t y_i \), and \( \lambda_{\text{min}}(A) \) denotes the smallest eigenvalue of a symmetric (hermitian) matrix \( A \). In (1.5) and in what follows the sum of a matrix and a number always understood in a sense that the number is multiplied by \( I \), the \( n \times n \) identity matrix.

**Proposition 1.** For any ball \( B_{\varepsilon}(x_0 = 0) \), \( \varepsilon^2 < \varepsilon^2_{\max} \) the image \( f(B_{\varepsilon}(0)) \) is strongly (strictly for \( \varepsilon^2 = \varepsilon^2_{\max} \)) convex and smooth and the pre-image of the boundary \( \partial f(B_{\varepsilon}(0)) \) belongs to the boundary of the sphere \( \partial B_{\varepsilon}(0) \) (properties 1, 2 from section 1.1). The value of (1.5) is maximally possible such that \( f(B_{\varepsilon}(0)) \) is stably convex (remains convex under infinitesimally small variation of \( A_i, v_i \)). In this sense (1.5) is the best possible improvement of the Polyak’s bound (1.2).

Sometimes propery 2 is not important and can be relaxed. We would still want the image of \( \partial B_{\varepsilon}(0) \) to be convex, but it is no longer important that the pre-image of the boundary \( \partial f(B_{\varepsilon}(0)) \) belongs solely to the boundary \( \partial B_{\varepsilon}(0) \). In such a case the bound (1.5) can be improved

\[
\varepsilon^2_{\max} \equiv \lim_{\epsilon \to 0^+} \min_{c \in \mathcal{C}} \left| (c \cdot A - \lambda_{\text{min}}(c \cdot A) + \epsilon)^{-1} c \cdot v \right|^2 , \tag{1.7}
\]

\[
\mathcal{C} = \{ c : c \in \mathbb{R}^m, |c|^2 = 1, \lambda_{\text{min}}(c \cdot A) \leq 0 \} . \tag{1.8}
\]
Proposition 2. For any ball $B_\varepsilon(x_0 = 0)$, $\varepsilon^2 \leq \varepsilon_{\text{max}}^2$ the image $f(B_\varepsilon(0))$ is strictly convex (property 1 from section 1.1).

To solve the minimization problems (1.5) and calculate the exact value of $\varepsilon_{\text{max}}$ is a nontrivial task. In section 4 we introduce simplifications which lead to a number of easy-to-calculate conservative estimates of $\varepsilon_{\text{max}}$. A reader primarily interested in practical applications of our results can look directly there. By further simplifying the bound (1.5) we recover (1.2). This is a first proof of the main result of [1] (in a particular case of quadratic maps) which is not based on the Newton’s method.

2 Inhomogeneous Joint Numerical Range

Before discussing convexity of the a ball $B_\varepsilon(x_0)$ under quadratic map, it would be convenient first to understand the geometry of the image of a unit sphere $|x|^2 = 1$,

$$\mathcal{F}(A, v) = \{ y_i : \exists \ x, \ y_i = f_i(x), \ |x|^2 = 1 \} \ .$$

(2.1)

Functions $f_i(x)$ are defined in (1.3). We introduced a new notation $\mathcal{F}(A, v)$ instead of the colloquial $f(|x|^2 = 1)$ to stress that (2.1) is an interesting object in its own right. We propose to call (2.1) inhomogeneous joint numerical range because of its resemblance to the original definition: $\mathcal{F}(A, 0)$ is the joint numerical range of the $m-$tuple of matrices $A_i$. Below we formulate a sufficient condition for $\mathcal{F}(A, v)$ to be convex.

Proposition 3. Inhomogeneous joint numerical range $\mathcal{F}(A, v)$ defined in (2.1) is strongly convex and smooth if

$$\lim_{\varepsilon \to 0^+} \min_{|c|^2 = 1} \left| (c \cdot A - \lambda_{\text{min}}(c \cdot A) + \varepsilon)^{-1} c \cdot v \right| > 1 \ ,$$

(2.2)

and $n > m$ ($2n > m$ in the complex case). When $n = 2$ ($2n = m$) generalized joint numerical range $\mathcal{F}(A, v)$ is an “empty shell” $\mathcal{F}(A, v) = \partial \text{Conv}[\mathcal{F}(A, v)]$ where Conv denotes the convex hull.

Comment. The inequality (2.2) is a sufficient but not a necessary condition. Say, when all $v_i = 0$, inequality (2.2) is not satisfied, but the joint numerical range $\mathcal{F}(A, 0)$ can nevertheless be (strongly) convex. This happens, for example, if the rank of the smallest eigenvalue of $c \cdot A$ is the same for all non-zero $c_i$ [3] (see also [4]).

4
Proof of Proposition 3. We will prove strong convexity of $F(A, v)$ using the supporting hyperplanes technique. Our logic closely follows the proof of convexity of the joint numerical range [3], [4]. Provided that (2.2) is satisfied we will show that for any non-zero covector $c_i \in \mathbb{R}^m$ the corresponding supporting hyperplane touches $F(A, v)$ at exactly one point. This will establish strict convexity of $\text{Conv}[F(A, v)]$. By calculating the Hessian at the boundary and showing it is strictly positive we establish that $\text{Conv}[F(A, v)]$ is strongly convex. Last, we provide a topological argument that for $n > m$ ($2n > m$ in the complex case) $f(x)$ is surjective inside $\partial F(A, v)$.

Strict convexity of $\text{Conv}[F(A, v)]$. Let us consider a non-zero covector $c_i \in \mathbb{R}^m$ and corresponding family of hyperplanes $c \cdot y = \text{const}$. First, we would like to find a minimum $F_c = c \cdot y$ among all $y$ from $F(A, v)$, which is the same as to minimize $c \cdot f(x)$ with the constraint $x^*x = 1$. After introducing a Lagrange multiplier $\lambda$ to enforce the constraint, the equation determining $x$ takes the form

$$(c \cdot A - \lambda)x = c \cdot v .$$

(2.3)

Once $c_i$ is fixed it is convenient to diagonalize $c \cdot A$ and rewrite $c \cdot v$ using the eigenbasis

$$(c \cdot A)x_k = \lambda_k x_k, \quad x_k^*x_l = \delta_{kl} ,$$

(2.4)

$$c \cdot v = \sum_{k=1}^n \alpha_k x_k .$$

(2.5)

Let us assume for now that all $\alpha_k \neq 0$. The minimum of $F_c(x)$ is given by a minimum of

$$F_c(\lambda) = \lambda - \sum_k |\alpha_k|^2 / \lambda_k - \lambda ,$$

(2.6)

subject to constraint

$$\frac{dF_c}{d\lambda} = 1 - \sum_k |\alpha_k|^2 / (\lambda_k - \lambda)^2 = 0 .$$

(2.7)

In other words we need to find a local extremum of $F_c(\lambda)$ where $F_c(\lambda)$ is minimal. This is not the same as the global minimum of $F_c(\lambda)$ because this function is not bounded from below and approached minus infinity when $\lambda \to -\infty$. In general the constraint $dF_c/d\lambda = 0$ has many solutions (from 2 to $2n$). Smallest $\lambda$ solving...
\[ \frac{dF_c}{d\lambda} = 0 \] corresponds to the smallest \( F_c(\lambda) \). Indeed, let \( \tilde{\lambda}_1 > \tilde{\lambda}_2 \) be two solution of \( \frac{dF_c}{d\lambda} = 0 \). Then
\[
F_c(\tilde{\lambda}_1) - F_c(\tilde{\lambda}_2) = \sum_k \frac{|\alpha_k|^2(\tilde{\lambda}_1 - \tilde{\lambda}_2)^3}{2(\lambda_k - \lambda_1)^2(\lambda_k - \lambda_2)^2} > 0.
\] (2.8)
The minimal \( \lambda \) solving (2.7) is smaller than all \( \lambda_k \)'s. On the interval from \(-\infty\) to \( \lambda_{\min} \) the derivative \( \frac{dF_c}{d\lambda} \) monotonically decreases from 1 to \(-\infty\). Therefore it vanishes at exactly one point. At that point matrix \((c \cdot A - \lambda)\) is positive-definite and therefore \( x \) minimizing \( c \cdot y(x) \) is unique.

Now we have to consider an important case when \( \alpha_k = 0 \) for some \( k = \tilde{k} \). Of course when \( \lambda \) is generic \( \lambda \neq \tilde{\lambda}_k \), function \( F_c(\lambda) \) and the minimization problem remains the same. But in the special case \( \lambda = \lambda_{\tilde{k}} \) matrix \( c \cdot A - \lambda \) develops a zero mode \( x_{\tilde{k}} \) and the constraint \( x^*x = 1 \) is no longer given by \( \frac{dF_c}{d\lambda} = 0 \), but
\[
1 \geq \sum_{k \neq \tilde{k}} \frac{|\alpha_k|^2}{(\lambda_k - \lambda_{\tilde{k}})^2} = 1 - |x_{\tilde{k}}^*x|^2 \geq 0.
\] (2.9)
Comparing \( F_c(\lambda) \) at two different solutions of (2.8) or (2.7) we find that \( F_c \) is minimal at minimal \( \lambda \). Hence if \( c \cdot A \) has an eigenvalue \( \lambda_{\tilde{k}} \) such that \( x_{\tilde{k}}^*v = 0 \), \( \lambda_{\tilde{k}} \) satisfies (2.9), and it is smaller than the smallest solution of (2.7) (which would imply \( \lambda_{\tilde{k}} < \lambda_k \) for all \( k \) with \( \alpha_k \neq 0 \)) the resulting \( x \) minimizing \( c \cdot y(x) \) is not unique. This is because only the absolute value of the component of \( x \) along \( x_{\tilde{k}} \) is fixed by (2.9), but not its sign (or the complex phase).

Let us repeat what we understood so far. For a given \( c_i \), in case the projection of \( c \cdot v \) on the eigenspace of the smallest eigenvalue of \( c \cdot A \) is non-trivial, \( (c \cdot v)^*x_{\min} \neq 0 \), the supporting hyperplane orthogonal to \( c_i \) always touches \( F(A,v) \) “from below” at one point. In case \( (c \cdot v)^*x_{\min} = 0 \) for all \( x_{\min} \) corresponding to \( \lambda_{\min} \), vector \( x = (c \cdot A - \lambda_{\min}(c \cdot A) + \epsilon)^{-1}(c \cdot v) \) is well defined when \( \epsilon \to 0^+ \) and there are two options. If \( |x| \geq 1 \), the supporting hyperplane still touches \( F(A,v) \) at one point, but if \( |x| < 1 \) there are two (or continuum) points \( x \) minimizing \( F_c = c \cdot y(x) \) for \( |x|^2 = 1 \), although these point may still correspond to the same \( y_i(x) \).

Above we explained that (2.2) is a sufficient condition for \( \text{Conv}[F(A,v)] \) to be strictly convex.

\textit{The convex hull \( \text{Conv}[F(A,v)] \) is strongly convex and smooth. The boundary \( \partial \text{Conv}[F(A,v)] \) is an embedding of \( S^{m-1} \) in \( \mathbb{R}^m \).} For \( c_i \neq 0 \) we define a map
$y_i(c) = y_i(x(c))$ where $x(c)$ minimizes $c \cdot y(x)$ for $|x|^2 = 1$. As was demonstrated above such $x(c)$ is unique for each $c$ and hence $y(c)$ is well-defined. Since $y(c) = y(\mu c)$ for any positive $\mu$ function $y(c)$ is defined on the sphere $S^{m-1} \in \mathbb{R}^m$. First we will show that $y : S^{m-1} \rightarrow \mathbb{R}^m$ is an immersion by proving that the rank of $\frac{\partial y}{\partial c}$ is $m-1$ for all $c \neq 0$. Let us introduce a “time” parameter $\tau$, such that $c_i(\tau = 0) = c_i$ and $\dot{c} \equiv \frac{dc}{d\tau}|_{\tau=0}$ is a given vector from $TS^{m-1}$ which can be identified with the orthogonal complement of $c$ inside $\mathbb{R}^m$. Given (2.2) is satisfied there is a unique $x(\tau)$ that minimizes $c(\tau) \cdot y(x)$ over $|x|^2 = 1$. At the point $\tau = 0$ we have

$$(c \cdot A - \lambda)x = c \cdot v,$$  \hspace{1cm} (2.10)

$$(c \cdot A - \lambda)x = -(\dot{c} \cdot A - \dot{\lambda})x + \dot{c} \cdot v.$$  \hspace{1cm} (2.11)

Coefficient $\lambda(\tau)$ must be chose such such that $|x(\tau)|^2 = 1$, i.e. $x$ is orthogonal to $\dot{x}$. This is always possible because (2.11) becomes a non-degenerate linear equation on $\dot{\lambda}$ after multiplication by $x^*(c \cdot A - \lambda)^{-1}$ from the left (as was discussed above matrix $(c \cdot A - \lambda)$ is positive-definite and hence non-degenerate).

It follows from (2.11) that $\dot{x} = 0$ if and only if

$$(\dot{c} \cdot A - \dot{\lambda})x = \dot{c} \cdot v.$$  \hspace{1cm} (2.12)

We will show momentarily that (2.10) combined with (2.12) would contradict the main assumption (2.2).

**Lemma 1.** If there is a vector $x$ of unit length, $|x|^2 = 1$, which solves

$$(c_i \cdot A - \lambda_i)x = c_i \cdot v \quad \text{for} \quad i = 1, 2,$$ \hspace{1cm} (2.13)

for two non-collinear $c_1, c_2$, $(c_1 - c_2) \cdot v \neq 0$ and two numbers $\lambda_1, \lambda_2$, and the matrix $c_1 \cdot A - \lambda_1$ is positive-definite, then there exist $c \neq 0$, $\lambda$ such that $(c \cdot A - \lambda)$ is semi-positive definite, has a zero eigenvalue, and solves

$$(c \cdot A - \lambda)x = c \cdot v.$$ \hspace{1cm} (2.14)

**Proof of Lemma 1.** Let us consider a one-dimensional family of vectors $c(\mu) = c_1(1 + \mu) - c_2 \mu$ and function $\lambda(\mu) = \lambda_1(1 + \mu) - \lambda_2 \mu$. Because $c_1, c_2$ are non-collinear vector $c(\mu) \neq 0$ for any $\mu$. We know that

$$(c(\mu) \cdot A - \lambda(\mu))x = c(\mu) \cdot v, \quad |x|^2 = 1,$$ \hspace{1cm} (2.15)
for any $\mu$ and that for $\mu = 0$ matrix $(c(\mu) \cdot A - \lambda(\mu))$ is positive-definite. When $\mu \to \infty$, or $\mu \to -\infty$, or both, matrix $(c(\mu) \cdot A - \lambda(\mu))$ will develop negative eigenvalues (unless $(c_1 - c_2) \cdot A = \lambda_1 - \lambda_2$; but then (2.15) would imply $(c_1 - c_2) \cdot v = 0$). Then by continuity there will be a value of $\mu$ when matrix $(c(\mu) \cdot A - \lambda(\mu))$ is semi-positive definite with a zero eigenvalue.

Now, using Lemma 1 with $c_1 = c$ and $c_2 = \dot{c}$ we find a contradiction with (2.2), which finishes our proof that $\dot{x} \neq 0$. Hence $x(c)$ is an immersion of $S^{m-1}$ into $S^{n-1}$ (or $S^{2n-1}$).

Finally we want to show that $\dot{y} \neq 0$ for any $\dot{c}$ and hence rank$(\partial y / \partial c) = m - 1$. It is enough to calculate

$$\dot{c} \cdot \dot{y} = \dot{x}^* (c \cdot A - \lambda(c)) \dot{x} > 0,$$

where we used $x^* \dot{x} = 0$ and positive-definiteness of $(c \cdot A - \lambda(c))$. Hence, $y(c)$ is an immersion of $S^{m-1}$ into $\mathbb{R}^m$ and $\partial \text{Conv}[\mathcal{F}(A, v)]$ is smooth.

As a side note we observe that the second derivative of $c \cdot y(\tau)$ is strictly positive as well

$$c \cdot \ddot{y} = \dot{x}^* (c \cdot A - \lambda(c)) \dot{x} .$$

Hence Conv[$\mathcal{F}(A, v)$] is strongly convex.

To prove, that $y(c)$ is an embedding we have to show that it is injective, i.e. $y(c_1) = y(c_2)$ implies $c_1 = c_2$. Clearly this would imply $x = x(c_1) = x(c_2)$ as was discussed above. The vector $x$ would satisfy

$$(c_i \cdot A - \lambda_i) x = c_i \cdot v \quad \text{for} \quad i = 1, 2,$$

such that both matrices $(c_i \cdot A - \lambda_i)$ are positive-definite and therefore according to Lemma 1 this is inconsistent with (2.2) unless $c_1 = c_2$.

**Topological argument proving surjectivity of** $y_i = f_i(x)$ **on Conv[$\mathcal{F}(A, v)$]**. Our last step is to show that $\mathcal{F}(A, v)$ coincides with its own convexification, i.e. $\mathcal{F}(A, v)$ includes all points contained inside $\partial \text{Conv}[\mathcal{F}(A, v)]$. Let us assume this is not the case and there is a point $y_0$ in the interior of Conv[$\mathcal{F}(A, v)$] which does not belong to $\mathcal{F}(A, v)$. Then we can define a continuous retraction $\varphi$ of $\mathcal{F}(A, v)$ on $\partial \text{Conv}[\mathcal{F}(A, v)] = S^{m-1}$. For any $y \in \mathcal{F}(A, v)$, we define $\varphi(y)$ as the intersection point of the ray from $y_0$ to $y$ and the boundary $\partial \text{Conv}[\mathcal{F}(A, v)]$. Because the set confined by the boundary
∂Conv[\mathcal{F}(A, v)] is convex \varphi(y) is well-defined. Next, the embedding \( y(x(c)) \) from \( S^{m-1} \) to \( \partial\text{Conv}[\mathcal{F}(A, v)] \) can be inverted \( y^{-1} : \partial\text{Conv}[\mathcal{F}(A, v)] \to S^{m-1} \subset S^{n-1} \) (\( S^{m-1} \subset S^{2n-1} \) in the complex case). The combination \( \phi = y^{-1} \circ \varphi \circ f \) defines a map \( \phi : S^{n-1} \to S^{m-1} \subset S^{n-1} \) (or \( \phi : S^{2n-1} \to S^{m-1} \subset S^{2n-1} \) in the complex case) from the sphere \( |x|^2 = 1 \) to the preimage of the boundary \( \partial\text{Conv}[\mathcal{F}(A, v)] \) inside \( S^{n-1} \). Because \( S^{m-1} \) is mapped by \( \phi \) into itself it must be homologically non-trivial inside \( S^{n-1} \) (or \( S^{2n-1} \)). This is possible only if \( m = n \) \((m = 2n)\). In this case \( \mathcal{F}(A, v) \) is an “empty shell”, \( \mathcal{F}(A, v) = \partial\text{Conv}[\mathcal{F}(A, v)] \). Otherwise, when \( n > m \) \((2n > m)\), all \( y_0 \) confined by \( \partial\text{Conv}[\mathcal{F}(A, v)] \) belong to \( \mathcal{F}(A, v) \) which coincides with its convex hull. Because of \( (\ref{eq:2.2}) \) matrix \( v_i^a \) (or \( \text{Re}(v_i^a) \otimes \text{Im}(v_i^a) \)) must have rank \( m \) which excludes \( n < m \) \((2n < m)\).

**Corollary 2.** Sufficient condition \( (\ref{eq:2.2}) \) can be relaxed to include equality. In such a case the set \( \mathcal{F}(A, v) \) remains to be strictly convex when \( n > m \) \((2n > m)\).

If \( n = m \) \((2n = m)\), \( \text{Conv}[\mathcal{F}(A, v)] \) remains to be strictly convex. At the same time \( (\ref{eq:2.2}) \) can not be made much stronger. In case \( (\ref{eq:2.2}) \) is larger than 1, for some \( c_i \) the minimum \( F_c = c \cdot y(x) \) would be achieved at more than one \( x \). Although the corresponding \( \mathcal{F}(A, v) \) may still be convex, even strictly convex if all such \( x \)'s are mapped into the same point \( y = f(x) \), convexity would be lost upon an infinitesimal variation \( v_i \to v_i + \epsilon c_i^\perp x_{\min} \) (here \( c_i^\perp \) is any vector orthogonal to \( c_i \) and \( \epsilon \) is an infinitesimal parameter).

### 3 Convexity of Image of a Small Ball

#### 3.1 Mapping Interior Into Interior

In this section we consider the main question, convexity of the image of a ball \( B_\varepsilon(0) = \{ x : |x|^2 \leq \varepsilon^2 \} \) under the map \( (\ref{eq:1.3}) \) while the interior points of \( B_\varepsilon(0) \) are mapped strictly into interior point of \( f(B_\varepsilon(0)) \). More precisely, we want to find a bound on \( \varepsilon \) such that the image \( f(B_\varepsilon(0)) \) satisfies the properties 1, 2 from section \( (\ref{eq:1.1}) \).

It is convenient to think of the ball \( B_\varepsilon(0) \) as a collection of spheres \( |x|^2 = z \), \( \varepsilon^2 \geq z \geq 0 \). Using results from the previous section we can easily find a point where each such sphere touches a supporting hyperplane defined by a vector \( c \neq 0 \). The
minimal value of $c \cdot y(x)$ over $|x|^2 = z$ is given by

$$F_c(z) = z\lambda - \sum_k \frac{|\alpha_k|^2}{\lambda_k - \lambda} ,$$

(3.1)

where $\lambda(z)$ is the smallest solution of (we assume there is no $\lambda_k < \lambda(z)$, $\alpha_k = 0$)

$$z = \sum_k \frac{|\alpha_k|^2}{(\lambda_k - \lambda)^2} .$$

(3.2)

From $\frac{dF_c}{dz} = \lambda$ we conclude that the “outer layer” of $f(B_\varepsilon(0))$ in the direction $c_i$ corresponds to $|x|^2 = z$ with the maximal value of $z = \varepsilon^2$ if $\lambda(\varepsilon^2) < 0$, or to $|x|^2 = z < \varepsilon^2$ such that $\lambda(z) = 0$ otherwise. In the latter case property 2 of section 1.1 will not be satisfied: pre-image of $\partial f(B_\varepsilon(0))$ does not lie within the boundary $\partial B_\varepsilon(0)$. Combining the constraint that $f(|x|^2 = \varepsilon^2)$ is the “outer layer” for all $c_i$ with the sufficient condition from section 2 ensuring its convexity we obtain our main result $\varepsilon^2 \leq \tilde{\varepsilon}_\text{max}^2$, where $\tilde{\varepsilon}_\text{max}^2$ is defined in (1.5).

For $n > m$ ($2n > m$) the topological argument from section 2 ensures that $f(|x|^2 = \varepsilon^2)$ is a convex set and therefore $f(B_\varepsilon(0)) = f(\partial B_\varepsilon(0))$. Hence Proposition 1 is proved in this case. For $n = m$ ($2n = m$) the image $f(|x|^2 = \varepsilon^2)$ is an “empty shell” $f(\partial B_\varepsilon(0)) = \partial f(B_\varepsilon(0))$ and we yet have to show that $f(B_\varepsilon(0))$ is convex by demonstrating that all points inside $\partial f(B_\varepsilon(0))$ belong to $f(B_\varepsilon(0))$. To that end we slightly modify the topological argument from section 2. Let us assume there is $y_0$ inside $\partial f(B_\varepsilon(0))$ which does not belong to $f(B_\varepsilon(0))$. Then one can define a retraction $\varphi$ from $f(B_\varepsilon(0))$ to $\partial f(B_\varepsilon(0))$. Hence we obtain the map $\phi = y^{-1} \circ \varphi \circ f : B_\varepsilon(0) \to \partial f(B_\varepsilon(0)) = S^{n-1}$, where possibility to invert $y$ on $\partial f(B_\varepsilon(0))$ was proved in section 2. The boundary $S^{n-1}$ is mapped into itself by $\phi$ and therefore it is homologically non-trivial inside $B_\varepsilon(0)$, which is a contradiction. Hence such $y_0$ can not exist which finishes the proof of Proposition 1.

### 3.2 Mapping Interior Into Anything

What if we relax property 2 of section 1.1 and will no longer require the pre-image of $\partial f(B_\varepsilon(0))$ to belong solely to the boundary $\partial B_\varepsilon(0)$? We still would want to preserve strict convexity of $f(B_\varepsilon(0))$ (property 1). Recycling the results of section 3.1 we conclude that for any $c_i$ the corresponding supporting hyperplane touches $f(B_\varepsilon(0))$ at exactly one point, provided $\varepsilon^2 \leq \tilde{\varepsilon}_\text{max}^2$, where $\tilde{\varepsilon}_\text{max}^2$ is given by (1.7). Hence the
Conv\([f(B_\varepsilon(0))]\) is strictly convex and \(\partial \text{Conv}[f(B_\varepsilon(0))]\) is homeomorphic to a sphere \(S^{m-1}\).

Now the points from \(\partial f(B_\varepsilon(0))\) may correspond not only to \(\partial B_\varepsilon(0)\) but also to the interior of \(B_\varepsilon(0)\). The remaining challenge is to modify the topological argument from the previous section to prove that all points confined by \(\partial \text{Conv}[f(B_\varepsilon(0))]\) belong to \(f(B_\varepsilon(0))\).

Let \(x = x(c)\) minimize \(F_c \equiv c \cdot y(x)\) over \(f(B_\varepsilon(0))\) for some \(c_i \neq 0\). So far \(\varepsilon \leq \tilde{\varepsilon}_{\text{max}}\) such \(x\) is unique for each \(y \in \partial \text{Conv}[f(B_\varepsilon(0))]\). Hence we can define the continuous map \(y^{-1}\) from \(S^{m-1} = \partial \text{Conv}[f(B_\varepsilon(0))]\) to \(S^{m-1} \subset S^{n-1}\) (or \(S^{m-1} \subset S^{2n-1}\)). This is absolutely analogous to the cases considered previously, although now the map \(y^{-1}\) is merely a continuous homeomorphism, not an embedding as before.

The rest is straightforward. Assuming there is \(y_0\) inside \(\partial \text{Conv}[f(B_\varepsilon(0))]\) which does not belong to \(f(B_\varepsilon(0))\) we first form a retraction \(\varphi : f(B_\varepsilon(0)) \to \partial \text{Conv}[f(B_\varepsilon(0))]\) and then the continuous map \(\phi = y^{-1} \circ \varphi \circ f : B_\varepsilon(0) \to S^{m-1} \subset B_\varepsilon(0)\). Since \(S^{m-1}\) is mapped into itself it must be homologically non-trivial inside \(B_\varepsilon(0)\) which is a contradiction. This finishes the proof of Proposition 2.

4 Conservative Estimate of \(\varepsilon_{\text{max}}^2\)

Calculating \(\varepsilon_{\text{max}}^2\) from (1.5) explicitly could be a difficult task. Therefore it would be useful to derive a conservative estimate \(\varepsilon_{\text{est}}^2 \leq \varepsilon_{\text{max}}^2\) which would be easy to calculate in practice. For any given \(c_i \neq 0\) the length of the vector \((c \cdot A - \lambda_{\text{m}}(c \cdot A) + \varepsilon)^{-1}c \cdot v\) can be bound from below as follows

\[
\lim_{\varepsilon \to 0^+} \left| (c \cdot A - \lambda_{\text{m}}(c \cdot A) + \varepsilon)^{-1}c \cdot v \right|^2 \geq \frac{|c \cdot v|^2}{||c \cdot A - \lambda_{\text{m}}(c \cdot A)||^2}. \tag{4.1}
\]

Here the matrix norm \(||A||\) is defined as

\[
||A|| \equiv \max_{||x||^2 = 1} |Ax| = \lambda_{\text{m}}^{1/2}(A^*A) = \max\{\lambda_{\text{max}}(A), \lambda_{\text{max}}(-A)\}, \tag{4.2}
\]

where the last identity holds for a symmetric (hermitian) \(A\). In most cases the estimate (4.1) is very conservative. For example, if the projection of \(c \cdot v\) on the zero eigenvector of \(c \cdot A - \lambda_{\text{min}}(c \cdot A)\) is non-vanishing, the LHS of (4.1) will be infinite while the RHS will stay finite. Nevertheless, the advantage of (4.1) is that it is much easier to deal with than the original expression.
The norm $||c \cdot A - \lambda_m(c \cdot A)||$ can be estimated from above by $2||c \cdot A||$. This estimate is tight if $\lambda_{\text{max}}(c \cdot A) = -\lambda_{\text{min}}(c \cdot A)$ and is conservative otherwise. Hence we arrive at the following easy-to-calculate estimate

$$\varepsilon_{\text{est}}^2 = \min_{|c|^2=1} \frac{|c \cdot v|^2}{4||c \cdot A||^2}. \quad (4.3)$$

This expression still can be simplified. To proceed further we would need the following lemma (below $||c_i||$ stands for the conventional definition of the vector norm $||c_i|| \equiv \sqrt{\sum_{i=1}^n |c_i|^2}$).

**Lemma 2.** For a $m$-tuple of symmetric (hermitian) matrices $A_i$

$$L(A) \equiv \max_{|x_1|^2=|x_2|^2=1} ||\text{Re}(x_1^* A_i x_2)|| = \max_{|x|^2=1} ||x^* A_i x|| = \max_{|c|^2=1} \lambda_{\text{max}}(c \cdot A). \quad (4.4)$$

**Proof of Lemma 2.** For any symmetric (hermitian) matrix $A$, $\lambda_{\text{max}}(A) = \max_{|x|^2=1} (x^* A x)$. Besides, for any real-valued vectors $a_i, c_i$, $\max_{|c|^2=1} (c \cdot a) = ||a||$. Therefore

$$\max_{|c|^2=1} \lambda_{\text{max}}(c \cdot A) = \max_{|c|^2=1} \max_{|x|^2=1} ||x^* A_i x|| = \max_{|c|^2=1} \lambda_{\text{max}}(c \cdot A), \quad (4.5)$$

which proves the second equality of (4.4).

It is obvious that $L(A) \equiv \max_{|x_1|^2=|x_2|^2=1} ||\text{Re}(x_1^* A_i x_2)|| \geq \max_{|x|^2=1} ||x^* A_i x||$. Let $x_1^m, x_2^m$ be the vectors of unit length maximizing $||\text{Re}(x_1^* A_i x_2)||$. Let us also define a real-valued vector of unit length $c_i^m = \text{Re}((x_1^m)^* A_i x_2^m)/||\text{Re}((x_1^m)^* A_i x_2^m)||$. Then

$$L(A) = \text{Re}((x_1^m)^* (c^m \cdot A) x_2^m) \leq ||c^m \cdot A|| \leq \max_{|c|^2=1} \lambda_{\text{max}}(c \cdot A), \quad (4.6)$$

which finished the proof.

Let us now return back to (4.3). Because of $c_i$-dependence in both numerator and denominator this quantity may look difficult to calculate. Let us make one last simplification and minimize/maximize numerator and denominator separately

$$\varepsilon_P^2 = \frac{\min_{|c|^2=1} |c \cdot v|^2}{4 \max_{|c|^2=1} \lambda_{\text{max}}^2(c \cdot A)} \leq \varepsilon_{\text{est}}^2. \quad (4.7)$$

The obtained estimate $\varepsilon_P$ is nothing but the Polyak’s bound (1.2, 1.4). Hence we rederived Polyak’s result in case of a quadratic map without using Newton’s method, something which has not been done before [1].
In fact we can do systematically better than (4.7). Expression (4.3) is homogeneous of zero degree in \(c_i\) and therefore minimizing over \(|c| = 1\) or any other non-degenerate ellipse \(c^*gc = 1\) (where \(g\) is a positive-definite \(m \times m\) symmetric matrix) would yield the same result. This observation allows us to effectively get rid of the numerator of (4.3) by means of preconditioning. We introduce matrix \(g_{ij} = (v^*_iv_j + v^*_jv_i)/2\) and \(\Lambda_j\) which transforms it into identity matrix, \(\Lambda^Tg\Lambda = \mathbb{I}_{m \times m}\). Since \(|c \cdot v|^2 = c^*gc\) we obtain for (4.3)

\[
\varepsilon^2_{est} = \frac{1}{4L^2(A)} , \quad \hat{A}_i = \Lambda_j^4A_j .
\]  

(4.8)

4.1 Approximate Estimate of \(L\)

The problem of calculating or effectively estimating \(L(A)\) is interesting in its own right. Originally Polyak provided an estimate

\[
L_P(A) = ||\lambda_{\text{max}}(A_i)|| = \sqrt{\sum_{i=1}^{n} ||A_i||^2} \geq L(A) ,
\]  

(4.9)

which is very conservative. Recently reference [5] put forward a number of improvements, including a convex semidefinite programming algorithm which they claim accurately estimates \(L(A)\) from above. We believe the representation

\[
L(A)^2 = \max_{|x|^2 = 1} \sum_{i=1}^{n} (x^*A_ix)^2 ,
\]  

(4.10)

established in Lemma 2, would allow to reduce the problem of calculating \(L(A)\) to one of the known problems of convex optimization or provide the best possible effective algorithm to accurately estimate (4.10). Thus, by introducing the matrix \(X = x \otimes x^*\) and relaxing the condition \(\text{rank}(X) = 1\), (4.10) can be recast as a minimization of a quadratic function over a convex space of positive-definite matrices. Furthermore, by treating \(z = x \otimes x\) as a vector in the \(n^2\)-dimensional space and introducing \(Z = z \otimes z^*\) after relaxing \(\text{rank}(Z) = 1\) condition the problem reduces to a standard question of semi-definite programming. This trick was used in the algorithm of [5], which they conjectured to be the tightest estimate of \(L(A)\) to date. We believe our method will be more precise in estimating \(L(A)\) because it is based on (4.10), which is an exact expression for \(L(A)\), while the algorithm of [5] relied on an approximate expression (also quartic in \(x\)) which bounds the true value of \(L(A)\) from above.
Besides the sophisticated algorithms to estimate $L(A)$ it would be of practical value to outline more elementary yet less precise ways to bound $\varepsilon_{\text{est}}^2$. The original estimate by Polyak (4.10) is very easy to calculate but it is too conservative. Reference [5] suggests a systematic improvement over that result:

$$L_P(A) \geq L_{\text{new}}(A) \equiv \lambda_{\text{max}}^{1/2} \left( \sum_{i=1}^{n} A_i^2 \right) \geq L(A). \quad (4.11)$$

Using last representation from Lemma 2 we can provide another estimate based on the identity $\lambda_{\text{max}}^2 (c \cdot A) \leq \text{Tr} ((c \cdot A)^2)$,

$$L_n(A) \equiv \lambda_{\text{max}}^{1/2} (\text{Tr}(A_i A_j)) \geq L(A). \quad (4.12)$$

This bound can be further improved using identity (12) of [5] (see there for original reference)

$$L_{\text{nov}} = \max_{|c|^2=1} (c \cdot a + \sqrt{c^T M c}), \quad (4.13)$$

$$a_i = \frac{\text{Tr}(A_i)}{m}, \quad M_{ij} = \text{Tr}(A_i A_j) - m a_i a_j. \quad (4.14)$$

Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $M$ and $a_k$ – the projections of $a_i$ on the $k$-th eigenvector of $M$. Then calculating $L_{\text{nov}}$ is reduced to finding roots of an algebraic equation

$$L_{\text{nov}} = \max_{\lambda \in \mathcal{L}} \sqrt{F(\lambda)} , \quad F(\lambda) = \lambda \left( 1 + \sum_k \frac{a_k^2}{\lambda - \lambda_k} \right) , \quad (4.15)$$

$$\mathcal{L} = \left\{ \lambda : \frac{dF(\lambda)}{d\lambda} = 0 \right\} \cup \left\{ \lambda = \lambda_k : a_k = 0 \text{ and } \frac{dF(\lambda_k)}{d\lambda} > 0 \right\}. \quad (4.16)$$

Neither $L_{\text{nov}}$ nor $L_{\text{new}}$ is systematically better.

\footnote{The original paper [5] provides a slightly different formula $L_{\text{new}}(A) \equiv \lambda_{\text{max}}^{1/2} \left( \sum_{i=1}^{n} A_i^* A_i \right)$. This is of course the same for symmetric (hermitian) matrices $A_i$. A few examples considered in [5] include non-symmetric real matrices $A_i$. While one can define quadratic map (1.3) with any real-valued $n \times n$ matrices $A_i$ (when $x \in \mathbb{R}^n$), only their symmetrization $(A_i + A_i^T)/2$ truly matters. Similarly the symmetrized matrices should be used to calculate $L_{\text{new}}(A)$. Using non-symmetric $A_i$ would unnecessarily increase the estimate $L_{\text{new}}$.}
Acknowledgments

I would like to thank Tudor Dimofte, Konstantin Turitsyn and Jamin Sheriff for useful discussions and gratefully acknowledge support from the grant RFBR 12-01-00482.

References

[1] B. T. Polyak, “Local Programming,” Comp. Math. and Math. Phys., 41(9), 1259-1266 (2001).

[2] B. T. Polyak, “Convexity of Nonlinear Image of a Small Ball with Applications to Optimization,” Set-Valued Analysis, 9(1/2), 159-168 (2001).
B. T. Polyak, “The convexity principle and its applications,” Bull.Braz.Math.Soc. (N.S.), 34(1), 59-75 (2003).

[3] E. Gutkin, E. A. Jonckheere, M. Karow, “Convexity of the joint numerical range: topological and differential geometric viewpoints,” Linear Algebra Appl. 376 (2004), 143171.

[4] J. Sheriff, “The Convexity of Quadratic Maps and the Controllability of Coupled Systems,” Ph.D. thesis, Harvard University, 2013, UMI publication number 3567063.

[5] Y. Xia, “On Local Convexity of Quadratic Transformations,” [arXiv:1405.6042 [math.OC]].