Analysis of a Galerkin scheme to avoid the locking phenomenon: solid-fluid interaction problem

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Abstract. Determining the response to the forces applied to an elastic solid containing an ideal fluid with constant density is essential in the engineering and biomedical fields. Therefore this paper aims to present and analyze a mixed finite element method for an interaction problem solid-fluid that contributes to understanding these areas. It is assumed transmission conditions are maintained at the fluid boundary and are given by the balance of forces and the equality of normal displacements. The mixed variational formulation that avoids the locking phenomenon, for the coupled problem is in terms of displacement, stress tensor, and rotation in the solid and by pressure and scalar potential in the fluid, the main contribution of this work. The first transmission condition is imposed in the definition of the space and the rest of the conditions appear naturally, which means Lagrange multipliers are not needed at the coupling border. The unknowns for the fluid and the solid are approximated by finite element subspaces of Lagrange and Arnold-Falk-Winther of order 1, which lead to a Galerkin scheme for the continuous problem. Also, the resulting Galerkin scheme is convergent and derives optimal convergence rates. Finally, the model is illustrated using a numerical example.

1. Introduction
The coupling between solid and fluid mechanics allows studying the interaction between rigid or deformable structures with one or more fluids. The problems of fluid-structure interaction are common in the engineering and biomedical fields. Particularly in [1, 2], the authors make biomechanical simulations of brain deformations, seen as a coupled source problem involving an elastic material containing an almost incompressible fluid. This type of phenomenon covers a significant amount of the issues of diverse complexity. In general, these systems do not have an analytical solution that approximates the real behavior of these systems; this is because the velocities and pressures in the flow cause large displacements in the solid that, in turn, generate changes in the field of velocities and pressures of the fluid.

Finite elements are the most widely used tool to solve problems of fluid-structure interaction; in [3], the authors study a primal-mixed formulation where the formulation suffers the locking phenomenon. In [4–6], the authors introduce varied mixed formulations to avoid the locking phenomenon; in [4], the researchers derive a mixed formulation in terms of the tension tensor, the rotation, the pressure gradient, the traces corresponding to the displacement of the solid and the pressure; the finite element scheme is given in terms of the plaint elasticity elements with reduced symmetry (PEERS) for the unknowns of the solid, Raviart-Thomas elements of
obtained by applying Babuska-Brezzi’s classical theory [9,10].

The objective of this paper was to propose a mixed variational formulation, in which the unknowns present in the solid and the fluid. It provides direct approximations of the stress tensor and the rotation, which appears as an additional unknown by imposing the symmetry of the stress tensor. A Galerkin scheme is defined in terms of Arnold-Falk-Winther and Lagrange finite elements to approximate the unknowns present in the solid and the fluid.

2. Mathematical model

Let Ω_S and Ω_F be the domain occupied by the fluid and the solid. The outer boundary to Ω_S is the union of Γ_N and Γ_D, the structure is fixed on Γ_D and affected by the surface force g on Γ_N. Γ_I denotes the interface between the solid and the fluid, while ν is the normal vector outside Ω_S and n is the normal vector outside Ω_F. The pressure p and the scalar potential ϕ satisfy [3]: \nabla p = f_F in Ω_F and \frac{1}{\rho_F} p + ∆ϕ = 0 in Ω_F, where \rho_F is the density of the fluid, c is the speed of sound of the fluid and f_F is a term source. For its part, the stress tensor σ and the displacement u in the solid are linked by the elasticity equation linear: \sigma = Cε(u) in Ω_S and −divσ = f_S in Ω_S, where ε(u) = \frac{1}{2}(∇u + (∇u)^t) is the strain tensor, f_S is a source term and C is the elasticity operator determined by Hooke’s Law: Cζ := λ tr(ζ)I + 2μζ, ζ ∈ [L^2(Ω_S)]^{2×2}, I is the identity matrix, tr is the trace of a matrix, λ and μ are the Lamé constants defined by the Equation (1).

\[
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad μ = \frac{E}{2(1+\nu)}, \quad (1)
\]

where E is Young’s modulus and ν is the Poisson radius [10]; furthermore, the transmission conditions satisfy [3,8,11]: \sigma n = −p n on Γ_I and \frac{\partial u}{\partial n} = u \cdot n on Γ_I and the Dirichlet-Neumann conditions [4–6]: u = 0 on Γ_D and σv = g on Γ_N. Then, to approach a model through a mixed formulation that describes the pressure, scalar potential, stress tensor, and displacement of the solid, the linear elasticity equation σ = Cε(u) is included to the coupled problem introduced in [3] as shown in Equation (2).

\[
\begin{align*}
\nabla p &= f_F in Ω_F, \quad \frac{1}{\rho_F} p + ∆ϕ = 0 in Ω_F, \quad σ = Cε(u) in Ω_S, \quad −divσ = f_S in Ω_S, \\
\frac{\partial ϕ}{\partial n} &= u \cdot n on Γ_I, \quad σ n = −p n on Γ_I, \quad σv = g on Γ_N, \quad u = 0 on Γ_D.
\end{align*}
\]

2.1. Variational formulation

To derive the mixed variational formulation of the coupled problem, the rotation γ := as(∇u) = \frac{1}{2}(∇u − (∇u)^t), belonging to the space of asymmetric tensioners |L^2(Ω_S)|^{2×2}_ASIM, is introduced as an additional unknown in the solid. Besides, the \mathcal{H} and \mathcal{Q} spaces defined in Equation (3) are obtained by applying Babuska-Brezzi’s classical theory [9,10].

\[
\mathcal{H} = \{(\tau, q) ∈ \mathcal{H}_Γ(\divv, Ω_S) × H^1(Ω_F) : τ n + q n = 0 on Γ_1\} \quad \text{and}
\]

\[
\mathcal{Q} = L^2(Ω_S) × H^1(Ω_F) × [L^2(Ω_S)]^{2×2}_ASIM,
\]

where \mathcal{H}_Γ(\divv, Ω_S) = \{τ ∈ \mathcal{H}(\divv, Ω_S) : τv = 0 on Γ_N\} is a Hilbert space with the norm \|τ\|_{\mathcal{H}_Γ(\divv, Ω_S)}^2 = \|τ\|^2_{0, Ω_S} + \|\divvτ\|^2_{0, Ω_S} and \hat{H}^1(Ω_F) = \{ψ ∈ H^1(Ω_F) : ∫_{Ω_F} ψ = 0\} is a
subspace of $H^1(\Omega_F)$ with trace over $\Gamma_f$. The spaces defined in the Equation (3) have as norm $\| (\tau, q) \|_{\Omega} = |\tau|_{H(\text{div}, \Omega)} + \|q\|_{\Omega_F}$ and $\| (\mathbf{v}, \psi, \eta) \|_{\Omega} = \|\mathbf{v}\|_{H_0^1(\Omega)} + \|\psi\|_{\Gamma_1} + \|\eta\|_{\Gamma_0(\Omega)}$; then, multiplying by appropriate test functions and integrating by parts, it gets the mixed variational formulation defined in the Equation (4); find $(\sigma, p, (\mathbf{u}, \varphi, \gamma)) \in \mathbb{H} \times \mathbb{Q}$.

\[
A(\sigma, p, (\tau, q)) + B((\tau, q), (\mathbf{u}, \varphi, \gamma)) = 0 \quad \forall (\tau, q) \in \mathbb{H},
\]

\[
B((\sigma, p), (\mathbf{v}, \psi, \eta)) = F(\mathbf{v}, \psi, \eta) \quad \forall (\mathbf{v}, \psi, \eta) \in \mathbb{Q},
\]

where the linear functional $F : \mathbb{H} \to \mathbb{R}$ and the bilinear forms $A : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, $B : \mathbb{H} \times \mathbb{Q} \to \mathbb{R}$ are defined by Equation (5).

\[
F(\mathbf{v}, \psi, \eta) = -\int_{\Omega} f_S \cdot \mathbf{v} - \int_{\Omega} f_F \cdot \nabla \psi, \quad A(\sigma, p, (\tau, q)) = \int_{\Omega_S} \sigma : \tau + \int_{\Omega_F} \frac{1}{\rho_F} pq
\]

and $B((\tau, q), (\mathbf{v}, \psi, \eta)) = \int_{\Omega_S} \mathbf{v} \cdot \nabla \tau + \int_{\Omega_S} \nabla \varphi \cdot \nabla \psi$.

3. Approximation of the solution

Let $\{T_h(\Omega_S)\}_{h>0}$ and $\{T_h(\Omega_F)\}_{h>0}$ be two families of partitions on a regular basis of the polygonal regions $\Omega_S$ and $\Omega_F$, by triangles $T$ in $\mathbb{R}^2$ or tetrahedra $T$ in $\mathbb{R}^3$ of diameter $h_T$, with mesh size $h = \max\{h_T : T \in T_h(\Omega_S) \cup T_h(\Omega_F)\}$. The vertices of the families coincide on the $\Gamma_1$ interface; furthermore, given an integer $k \geq 1$ and a subset $S$ of $\mathbb{R}^n$, $P_k(S)$ denotes the space of polynomial functions defined in $S$ of degree at most $k$. Now, the global discrete subspaces for $\mathbb{H}$ and $\mathbb{Q}$ are introduced in Equation (6).

\[
\mathbb{H}_h = \{(\tau_h, q_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} : \tau_h \mathbf{n} + q_h \mathbf{n} = 0 \text{ on } \Gamma_1\} = (\mathbb{H}_{1,h} \times \mathbb{H}_{2,h}) \cap \mathbb{H},
\]

\[
\mathbb{Q}_h = Q_{1,h} \times Q_{3,h} \times Q_{2,h},
\]

where $\mathbb{H}_{1,h} = \{\tau_h \in \mathbb{H}_{1}^S(\text{div}, \Omega_S) : \tau_h|_T \in P_1(T) \forall T \in T_h(\Omega_S)\}$, $Q_{1,h} = \{\mathbf{v}_h \in L^2(\Omega_S) : \mathbf{v}_h|_T \in P_0(T) \forall T \in T_h(\Omega_S)\}$ and $Q_{2,h} = \{\eta_h \in [L^2(\Omega_S)]^{2 \times 2^2 : \eta_h|_T \in P_0(T) \forall T \in T_h(\Omega_S)\}$ are discrete subspaces of finite dimension for the solid, while the fluid is discretized by the subspaces $\mathbb{H}_{2,h} = \{q_h \in C(\overline{\Omega_F}) : q_h|_T \in P_1(T) \forall T \in T_h(\Omega_F)\}$ and $Q_{3,h} = \{\psi_h \in C(\overline{\Omega_F}) : \psi_h|_T \in P_1(T) \forall T \in T_h(\Omega_F)\}$ of diameter $3$ for $\mathbb{H}_{2,h}$ and $\mathbb{H}_{2,h}$ of $\mathbb{Q}_{3,h}$.

Now, the finite element scheme for the mixed formulation defined in Equation (5) consists of finding $(\sigma_h, p_h, (\mathbf{u}_h, \varphi_h, \gamma_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ satisfying Equation (8).

\[
A(\sigma_h, p_h, (\tau_h, q_h)) + B((\tau_h, q_h), (\mathbf{u}_h, \varphi_h, \gamma_h)) = 0 \quad \forall (\tau_h, q_h) \in \mathbb{H}_h,
\]

\[
B((\sigma_h, p_h), (\mathbf{v}_h, \psi_h, \eta_h)) = F(\mathbf{v}_h, \psi_h, \eta_h) \quad \forall (\mathbf{v}_h, \psi_h, \eta_h) \in \mathbb{Q}_h,
\]

where the linear functional $F : \mathbb{H}_h \to \mathbb{R}$ and the bilinear forms $A : \mathbb{H}_h \times \mathbb{H}_h \to \mathbb{R}$, $B : \mathbb{H}_h \times \mathbb{Q}_h \to \mathbb{R}$ are defined analogously to the linear operators defined in the Equation (5).
3.1. Analysis of discrete variational formulation

Our next objective is to prove the existence and uniqueness of the mixed formulation defined in Equation (8). For this end, the null discrete space of the bilinear form \( B(\cdot, \cdot) \) is characterized by: \( \text{ker}_h(B) = \{(\tau_h, q_h) \in H^h : \text{div} \tau_h = 0 \text{ in } \Omega_h, \int_{\Omega_h} \tau_h : \eta_h = 0 \forall \eta_h \in Q_{2,h}, \int_{\Omega_h} \nabla q_h \cdot \nabla \psi_h = 0 \forall \psi_h \in Q_{3,h}\} \). The bilinear form \( A(\cdot, \cdot) \) is elliptical in \( \text{ker}_h(B) \), that is, there is a constant \( \tilde{C} > 0 \) independent of \( h \) and \( \lambda \) satisfying \( A((\tau_h, q_h), (\tau_h, q_h)) \geq \tilde{C} \|(\tau_h, q_h)\|_{H^h}^2, \forall (\tau_h, q_h) \in \text{ker}_h(B) \). Then, show that \( B(\cdot, \cdot) \) satisfies the discrete inf-sup condition; Lemma 1, Lemma 2, and Lemma 3 described below are consequences of Babuška-Brezzi theory for mixed variational formulations [9, 10].

Lemma 1. There exists \( C_1 > 0 \) independent of \( h \) and \( \lambda \), such that for each \( (v_h, \psi_h, \eta_h) \in Q_h \), satisfies Equation (9).

\[
\sup_{(\tau_h, q_h) \in \text{ker}_h(B)} \frac{B((\tau_h, q_h), (v_h, \psi_h, \eta_h))}{\|(\tau_h, q_h)\|_H} \geq C_1(\|v_h\|_{0, \Omega_S} + \|\eta_h\|_{0, \Omega_S}). \tag{9}
\]

**Proof.** From [10, 14, 15] an auxiliary problem defined in Equation (10) is constructed: given \( (v_h, \psi_h, \eta_h) \in Q_h \), we can find a tensor \( \tilde{\tau}_h = \alpha \tau_h + \beta \tau_h^\perp \in H^h \), with \( \alpha, \beta > 0 \) such that \( \text{div} \tilde{\tau}_h = v_h \) and \( \text{div} \tau_h^\perp = 0 \). Indeed, let \( (\tilde{z}, \tilde{\tau}) \in L^2(\Omega_S) \times H^1(\text{div}, \Omega_S) \) be the unique solution of the linear elasticity problem with Neumann boundary condition.

\[
\begin{align*}
-\text{div} \tilde{\tau} &= v_h \text{ in } \Omega_S, & \tilde{\tau} &= \varepsilon(\tilde{z}) \text{ in } \Omega_S, \\
\tilde{\tau} n &= 0 \text{ on } \Gamma, & \tilde{z} &= 0 \text{ on } \Gamma_D, & \tilde{\tau} \nu &= 0 \text{ on } \Gamma_N, \tag{10}
\end{align*}
\]

then it exists \( s > 0 \) such that \( \tilde{z} \in H^{1+s}(\Omega_S), \tilde{\tau} \in H^s(\Omega_S) \) and \( \|\tilde{\tau}\|_{s, \Omega_S} + \|\tilde{z}\|_{1+s, \Omega_S} \leq \tilde{C}_s \|v_h\|_{0, \Omega_S} \) [5, 6, 11]. Given that \( \tilde{\tau} \in H^s(\Omega_S) \cap H^s(\text{div}, \Omega_S) \), then of Equation (7) it follows that \( \tau_h^\perp := \Pi_h(\tilde{\tau}) \). By the continuous dependence of linear elasticity problems and by the approximation property of the BDM interpolation operator \( \Pi_h \) [10, 14, 15], we have Equation (11).

\[
\text{div}(\tau_h^\perp) = \text{div}(\Pi_h(\tilde{\tau})) = L_h(\text{div}\tilde{\tau}) = v_h, \\
\|\tau_h^\perp\|_{H(\text{div}, \Omega_S)} = \|\Pi_h \tilde{\tau}\|_{H(\text{div}, \Omega_S)} \leq C \|	ilde{\tau}\|_{H(\text{div}, \Omega_S)} \leq \tilde{C} \|v_h\|_{0, \Omega_S}. \tag{11}
\]

We introduce the finite element spaces \( E_h := \{v_h \in [C(\Omega_S)]^n : \varphi_h|_T \in P_2(T) \forall T \in \mathcal{T}_h^S \} \cap H^1(\Omega_S) \) and \( W_h := \{q_h \in L^2(\Omega_S) : q_h|_T \in P_0(T) \forall T \in \mathcal{T}_h^S \} \cap L^2(\Omega_S) \) to construct the tensor \( \tau_h^\perp \). The pair \((E_h, W_h)\) is stable for Stokes’ problem. Given \( \eta_h \in Q_{2,h} \), there exists \( v_h \in W_h \) such that \( \eta_h = S^2(v_h) \), in particular \( v_h = \eta_{12h} - \eta_{21h} \), \( S^2(v_h) \) is the antisymmetric tensor. Now, in the Equation (12) includes the discrete Stokes problem [16]; given \( \eta_h = S^2(v_h) \), find \((\psi_h, p_h) \in (E_h, W_h)\).

\[
\begin{align*}
\int_{\Omega_S} \nabla \psi_h : \nabla z_h - \int_{\Omega_S} p_h \text{div} z_h &= 0 \forall z_h \in E_h, \\
-\int_{\Omega_S} q_h \text{div} \psi_h &= -\int_{\Omega_S} v_h q_h \forall q_h \in W_h. \tag{12}
\end{align*}
\]

When \( q_h = v_h \), the Equation (13) is obtained due to the uniqueness of the above-mixed formulation.

\[
\|\psi_h\|_{1, \Omega_S} \leq c \|v_h\|_{0, \Omega_S} = \frac{\sqrt{2}}{2} \|\eta_h\|_{0, \Omega_S} \quad \text{and} \quad \int_{\Omega_S} \text{div} \psi_h v_h = \frac{1}{2} \|\eta_h\|_{0, \Omega_S}^2. \tag{13}
\]
The second step is to correct $\hat{\tau}_h$ for a free divergence tensor $\tau_h^2$. The tensor $\tau_h^2$ is defined using the curl operator restricted to each component of the vector $\psi_h = (\psi_1^h, \psi_2^h)$, that is, $\tau_h^2 = (\tau_{1h}, \tau_{2h})$, for $\tau_{1h} = (-\partial_2 \psi_1^h, -\partial_2 \psi_2^h)^T$ and $\tau_{2h} = (\partial_1 \psi_1^h, \partial_1 \psi_2^h)^T$. Then $\tau_h^2 \in P_1(T)$, $\text{div}\tau_h^2 = 0$ in $\Omega_S$ and $\psi_h = 0$ on $\Gamma$, also $\tau_h^2 \nu|_{\Gamma} = \nabla \psi_h t = 0$ for $t = (n_2, n_1)$. From the Equation (5), Equation (11), Equation (13) and the fact that $(\tau_h^1, 0)$, $(\tau_h^2, 0) \in H_h$, the Equation (14) is obtained.

$$B((\hat{\tau}_h, 0), (v_h, \psi_h, \eta_h)) \geq \tilde{C}_1 \|v_h\|^2_{0, \Omega_S} + \tilde{C}_2 \|\eta_h\|^2_{0, \Omega_S}. \quad (14)$$

Applying the triangular inequality, Equation (11), Equation (13) and the inequality $\|\tau_h^2\|^2_{0, \Omega_S} \leq 4\|\psi\|^2_{1, \Omega_S}$ the Equation (15) is concluded.

$$\|\hat{\tau}_h\|^2_{\text{div}, \Omega_S} \leq \tilde{c}^2 (\|v_h\|^2_{0, \Omega_S} + \|\eta_h\|^2_{0, \Omega_S}). \quad (15)$$

Equation (14) and Equation (15) conclude the proof.

**Lemma 2.** There exists $C_2 > 0$ independent of $h$ and $\lambda$, such that for each $(v_h, \psi_h, \xi_h) \in Q_h$, the inequality of Equation (16) is satisfied.

$$\sup_{(\tau_h, q_h) \in H_h \setminus 0} \frac{B((\tau_h, q_h), (v_h, \psi_h, \xi_h))}{\|((\tau_h, q_h))\|_{H}} \geq C_2 |\psi_h|_{1, \Omega_F}. \quad (16)$$

**Proof.** The demonstration is similar to Lemma 1. The discrete inf-sup condition for bilinear form $B(\cdot, \cdot)$ can be established as a direct consequence of Lemma 1 and Lemma 2 [9, 10, 16].

**Lemma 3.** There exists a constant $C > 0$ independent of $h$ and $\lambda$, such that it satisfies inf-sup condition defined in the Equation (17).

$$\sup_{(\tau_h, q_h) \in H_h \setminus 0} \frac{B((\tau_h, q_h), (v_h, \psi_h, \eta_h))}{\|((\tau_h, q_h))\|_{H}} \geq C \|v_h, \psi_h, \eta_h\|_Q \quad \forall (v_h, \psi_h, \eta_h) \in Q_h. \quad (17)$$

The unique solution, stability and convergence of the mixed formulation defined in Equation (8) is guaranteed by Theorem 1.

**Theorem 1.** There exists a single solution $((\sigma_h, p_h), (u_h, \varphi_h, \gamma_h)) \in H_h \times Q_h$ of the mixed formulation defined in the Equation (8) and there exist $c_1, c_2 > 0$ independent of $h$ and $\lambda$, such that (Equation (18))

$$\|((\sigma_h, p_h))\|_{H} + \|(u_h, \varphi_h, \gamma_h)\|_Q \leq c_1 (\|f_S\|_{0, \Omega_S} + \|f_F\|_{0, \Omega_F}),$$

$$\|((\sigma, p)) - ((\sigma_h, p_h))\|_{H} + \|(u, \varphi, \gamma) - (u_h, \varphi_h, \gamma_h)\|_Q \leq c_2 \left(\inf_{(\tau_h, q_h) \in H_h} \|((\tau_h, q_h))\|_{H} + \inf_{(v_h, \psi_h, \eta_h) \in Q_h} \|(u, \varphi, \gamma) - (v_h, \psi_h, \eta_h)\|_Q\right),$$

where $((\sigma, p), (u, \varphi, \gamma)) \in H \times Q$ is a single solution of the mixed formulation defined in Equation (5).

**Proof.** It is a direct result of Lemma 3 applying the theory of Babuška-Brezzi [9, 10, 16].
3.2. Convergence order

An interpolant is constructed in space $H$ with the interface condition $[6,11]$. Given $q \in H^1(\Omega_F)$, let $(\hat{u},\hat{\sigma}) \in L^2(\Omega_S) \times \mathbb{H}^1_N(\text{div},\Omega_S)$ be the solution of a problem of linear elasticity with mixed condition, form regularity results, there exists $s \in (0,1]$ such that $\hat{\sigma} \in \mathbb{H}^s(\Omega_S)$ satisfying $\|\hat{u}\|_{1+s,\Omega_S} + \|\hat{\sigma}\|_{s,\Omega_S} \leq C\|q\|_{1,\Omega_F}$. Then, it is possible to define the operator $E : H^1(\Omega_F) \to \mathbb{H}^1_N(\text{div},\Omega_S)$. $E_q = -\hat{\sigma}$ linear and bounded of null divergence in the solid. It is observed that $E_q \in \mathbb{H}^s(\Omega_S)$ with $s \in (0,1]$ and $\|E_q\|_{s,\Omega_S} \leq C\|q\|_{1,\Omega_F}$. Furthermore, it is possible to define the pair $(E_h,q) = (E(q),q)$ belonging to space $H$. Now $E_hq = \Pi_h E(\pi_hq) \in \mathbb{H}_{1,h}$ and $\hat{E}_hq = (E_hq,\pi_hq)$, $\forall q \in H^1(\Omega_F)$ are the discrete counterparts of $E$ and $\hat{E}$, where $\pi_h : H^1(\Omega_F) \to \mathbb{H}_{2,h}$ is the orthogonal projection of $H^1(\Omega_F)$ to $\mathbb{H}_{2,h}$. Therefore, any $(\tau,q) \in H$ is well approximated in $\mathbb{H}_h$, taking $q_h = \pi_hq$, obtaining $\Pi_h\tau$ in $\Omega_S$ to a tensor $\tau_h$ satisfying $\tau_hu + q_hu = 0$ on $\Gamma_1$.

Theorem 2 establishes the good position and convergence of the discrete scheme defined in Equation (8) $[10,11]$.

**Theorem 2.** There exists $c > 0$ and $s \in (0,1]$ independent of $h$ and $\lambda$, such that (Equation 19).

$$
\|((\sigma,p) - (\sigma_h,p_h))\|_2 + \|((u,\varphi,\gamma) - (u_h,\varphi_h,\gamma_h))\|_Q \leq c h^s \left(\|\sigma\|_{\mathbb{H}^s(\text{div},\Omega_S)} + ||p||_{1+s,\Omega_F} + \|u\|_{s,\Omega_S} + |\varphi|_{1+s,\Omega_F} + \|\gamma\|_{s,\Omega_S}\right),
$$

where $((\sigma,p),(u,\varphi,\gamma)) \in \mathbb{H} \times Q$ and $((\sigma_h,p_h),(u_h,\varphi_h,\gamma_h)) \in \mathbb{H}_h \times Q_h$ are unique solutions for the mixed formulations defined in Equation (4) and Equation (8), respectively.

**Proof.** The result is a consequence of $[11]$ and the Theorem 1.

4. Numerical results

In this section we report some experiments that evidence the theoretical results. The numerical method has been implemented in a MATLAB code using a fine mesh as reference. The equations for errors and convergence rates were taken from $[4,15]$. Taking into account the Loking phenomenon, the Lamé parameters are chosen: Young’s modulus $E = 1$ and Poisson’s radius $\nu = 0.49$ along with parameters $\rho_F = 1$ and $c = 1$. Then, taking the domain $\Omega = [0,1]^2$ and the source term $f_F = (\cos(x^2 + y^2)) + x^2e^{\sin(x+y^2)}, x(x-1)y(y-1)^t$ for the fluid; the domain $\Omega_S = [-0.5,1.5]^2 \backslash \Omega_F$, the source term $f_S = (\sin(2\pi x)cos(x)e^{x^2+y^2},e^{4x}\sin(4y) + x^2 + y^2)^t$ and the surface force $g = (0,0)^t$ for the solid.

Table 1 shows the parameters associated with the error $e(\sigma)$, $e(u)$, $e(\gamma)$, $e(p)$, $e(\varphi)$ and the convergence rates $r(\sigma)$, $r(u)$, $r(\gamma)$, $r(p)$, $r(\varphi)$ corresponding to the solid and the fluid when the mesh size $h$ is varied. As the mesh size decreases the errors tend to 0 and the convergence rates stabilize around 1, that is, a good approximation was obtained since the aim is to decrease the error and reach an order of convergence $s = 1$ by the Theorem 2. Furthermore, the process is not affected by choice of Lamé parameters, confirming the correct functioning of the method.

| $h$     | $e(\sigma)$ | $r(\sigma)$ | $e(u)$ | $r(u)$ | $e(\gamma)$ | $r(\gamma)$ | $e(p)$ | $r(p)$ | $e(\varphi)$ | $r(\varphi)$ |
|---------|-------------|--------------|--------|--------|-------------|-------------|--------|--------|-------------|-------------|
| 0.1768  | 8.0410      | -            | 1.1412 | -      | 2.8587      | -           | 0.2043 | -      | 1.7865      | -           |
| 0.0884  | 4.1292      | 0.96         | 0.4653 | 1.29   | 1.4712      | 0.96        | 0.1054 | 0.95   | 0.8171      | 1.13         |
| 0.0442  | 2.0936      | 0.98         | 0.1940 | 1.26   | 0.7917      | 0.89        | 0.0520 | 1.02   | 0.3822      | 1.10         |
| 0.0221  | 0.9817      | 1.09         | 0.0807 | 1.27   | 0.4049      | 0.97        | 0.0233 | 1.16   | 0.1684      | 1.18         |
5. Conclusion
The stability of the problem was studied at a continuous and discrete level, taking into account the phenomenon of locking that arises when the Poisson radius tends to 0.5. The results show that to be can obtain optimal convergence rates using the mixed formulation proposed in this paper. Furthermore, the research carried out allowed the estimation of the stress tensor and the rotation in the solid, unknowns that contribute to a better understanding of phenomena associated with linear elasticity problems.

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