SO(4)-symmetry of mechanical systems with 3 degrees of freedom

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Abstract

We answered the old question: does there exist a mechanical system with 3 degrees of freedom, except for the Coulomb system, which has 6 first integrals generating the Lie algebra \( \mathfrak{o}(4) \) by means of the Poisson brackets? We presented a system which is not centrally symmetric, but has such 6 first integrals. We showed also that not every mechanical system with 3 degrees of freedom possesses such Lie algebra \( \mathfrak{o}(4) \).

1 Introduction

It is well-known (see, e.g., [7]) that in the Coulomb field, i.e., in the mechanical system with 3 degrees of freedom (3d mechanical system) with the Hamiltonian

\[
H = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \quad \text{where} \quad \mathbf{p}^2 := \sum_{i=1,2,3} p_i^2, \quad r := \left( \sum_{i=1,2,3} q_i^2 \right)^{1/2}
\]

(1)

the symmetry group of canonical transformations has a subgroup isomorphic to SO(4) acting in the domain \( H < 0 \). This fact, found by V. Fock [5], helps to explain the structure of the spectrum of the hydrogen atom. Sometimes this symmetry is called hidden.

An important property of this SO(4) is that the Casimirs of its Lie algebra \( \mathfrak{o}(4) \) restore the Hamiltonian. The Hamiltonian in Eq. (1) describes, for example, the motion of two particles interacting via gravity, and the motion of two charged particles with the charges of opposite sign. The number of works investigating this Hamiltonian is huge It is therefore astonishing that the literature does not give (at least, we could not find it) the definite answer to a natural question: “does there exist a mechanical system with 3 degrees of freedom, except for the Coulomb system, which has 6 first integrals generating the Lie algebra \( \mathfrak{o}(4) \)?” posed, e.g., in [9, 11]. Two different answers to the question were given fifty years ago:

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Recall, that in Classical Hamiltonian Mechanics, the transformations of the phase space that preserve the Hamiltonian form of the Hamilton equations, whatever the Hamiltonian function is, are said to be canonical.

See, for example, [3, 10] and references therein.
1) Mukunda\[9\] claimed that every mechanical system with \( n \) degrees of freedom has a subgroup of canonical transformations locally isomorphic to \( \text{O}(n + 1) \).

2) Szymacha and Werle\[11\] claimed that there are no other mechanical system with the same property, assuming that \( \mathfrak{o}(4) \) contains the Lie algebra of spatial rotations of \( \mathbb{R}^3 \).

To prove that not any system with 3 degrees of freedom has \( \text{SO}(4) \) symmetry or at least \( \mathfrak{o}(4) \) Lie algebra of the first integrals, we offer a simple necessary condition for existence of \( \mathfrak{o}(4) \) symmetry, see Section 4, and in Section 5 we give an example for which this condition is violated.

In Section 6 we consider the Hamiltonian of a charged particle in an homogeneous electric field. For this Hamiltonian, there exists a family of sextuples of first integrals such that every sextuple generates (by means of the Poisson bracket) the Lie algebra \( \mathfrak{o}(4) \).

To avoid misunderstanding, we should note that we consider the symmetry algebra (consisting of some set of the first integrals) of the systems, not the Lie algebra of dynamical symmetry group introduced in\[4\], which is also called noninvariance group, see\[8\].

2 Generalities (following\[1\])

Recall the definition of the symmetry group of canonical transformations and its Lie algebra.

Let \( H(q_i, p_i) \), where \( i = 1, 2, 3 \), be a Hamiltonian of some mechanical system. We will also denote the whole set of the \( q_i \) and \( p_i \) for \( i = 1, 2, 3 \) by \( z_\alpha \), where \( \alpha = 1, \ldots, 6 \).

Let the first integral \( F \) of this system be a real function on some domain \( U_F \subset \mathbb{R}^6 \). Let \((q, p) \in U_F \); the case \( F = H \) is not excluded. Then \( F \) generates a 1-dimensional Lie group \( \mathcal{L}_F \) of canonical transformations \((q, p) \mapsto (q^F(\tau|q, p), p^F(\tau|q, p))\) leaving the Hamiltonian \( H \) and the domain \( U_F \) invariant if

\[
(q^F(\tau|q, p), p^F(\tau|q, p)) \in U_F \quad \text{for any } \tau \in \mathbb{R}.
\]

The transformations are defined by the relations

\[
\frac{dq_i^F}{d\tau} = \{q_i^F, F\} = \frac{\partial F(q^F, p^F)}{\partial p_i^F},
\]

\[
\frac{dp_i^F}{d\tau} = \{p_i^F, F\} = -\frac{\partial F(q^F, p^F)}{\partial q_i^F},
\]

\[
q_i^F(0|q, p) = q_i, \quad p_i^F(0|q, p) = p_i,
\]

where \( \{\cdot, \cdot\} \) is the Poisson bracket\[3\] in \( \mathbb{R}^6 \):

\[
\{F, G\} := \sum_{i=1,2,3} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) = \sum_{\alpha, \beta = 1, \ldots, 6} \frac{\partial F}{\partial z_\alpha} \omega_{\alpha\beta} \frac{\partial G}{\partial z_\beta}.
\]

Here the symplectic form \( \omega \) is of shape \( \omega = \begin{pmatrix} 0_3 & 1_3 \\ -1_3 & 0_3 \end{pmatrix} \), where \( 1_3 \) and \( 0_3 \) are \( 3 \times 3 \) matrices.

\[3\]The definition\[3\] has the opposite sign as compared with the one given in\[7\], but coincides with the definition of the Poisson bracket given in\[2\] \[12\] \[6\] \[1\].
We call the transformations Eq. (2) – (4) the Hamiltonian flow, generated by the Hamiltonian \( F \), and denote it \( L_F \).

If a certain finite set of first integrals \( \mathcal{F} = \{ F_\alpha \mid \alpha = 1, 2, \ldots \} \) has the same domain \( U \) invariant under the action of all Hamiltonian flows \( L_{F_\alpha} \), then these flows generate a Lie group, the space of its Lie algebra being generated by the set \( \mathcal{F} \) by means of the bracket (5).

3 The case of the Coulomb field (following [7])

Here we briefly consider the mechanical system (1) with the Hamiltonian

\[
H = \frac{p^2}{2} - \frac{1}{r}, \quad \text{where} \quad p^2 := \sum_{i=1,2,3} p_i^2, \quad r := \left( \sum_{i=1,2,3} q_i \right)^{1/2}.
\]

This Hamiltonian has two well-known triples of first integrals: one consists of the coordinates \( L_i \) of the angular momentum vector, the other one consists of the coordinates of the Runge-Lenz vector \( R_i \), defined in the domain

\[
U = \{ z \in \mathbb{R}^6 \mid H(z) < 0 \},
\]
or in any of the domains \( E_{\min} < H < E_{\max} < 0 \), by the formulas

\[
L_i := \sum_{j,k=1,2,3} \varepsilon_{ijk} q_j p_k, \quad R_i := (-2H)^{-1/2} \left( \sum_{j,k=1,2,3} \varepsilon_{ijk} L_j p_k + \frac{q_i}{r} \right),
\]

where \( \varepsilon_{ijk} \) is an anti-symmetric tensor such that \( \varepsilon_{123} = 1 \).

These first integrals satisfy the following commutation relations:

\[
\{ H, L_i \} = 0,
\]

\[
\{ H, R_i \} = 0,
\]

and

\[
\{ L_i, L_j \} = \sum_{k=1,2,3} \varepsilon_{ijk} L_k,
\]

\[
\{ R_i, R_j \} = \sum_{k=1,2,3} \varepsilon_{ijk} L_k,
\]

\[
\{ L_i, R_j \} = \sum_{k=1,2,3} \varepsilon_{ijk} R_k.
\]

Due to relations (9) and by definition of the domain \( U \), the later is invariant under the action of Hamiltonian flows generated by the first integrals \( L_i \) and \( R_i \).

The relations (10) show that these first integrals generate the Lie algebra \( \mathfrak{o}(4) \).

Since \( \mathfrak{o}(4) \simeq \mathfrak{o}(3) \oplus \mathfrak{o}(3) \), we can introduce two commuting triples of first integrals

\[
G_i := \frac{1}{2} (L_i + R_i), \quad \text{where} \quad i = 1, 2, 3,
\]

\[
G_{3+i} := \frac{1}{2} (L_i - R_i), \quad \text{where} \quad i = 1, 2, 3,
\]

satisfying the commutation relations

\[
\{ G_i, G_j \} = \sum_{k=1,2,3} \varepsilon_{ijk} G_k, \quad \text{where} \quad i, j = 1, 2, 3,
\]

\[
\{ G_{3+i}, G_{3+j} \} = \sum_{k=1,2,3} \varepsilon_{ijk} G_{3+k}, \quad \text{where} \quad i, j = 1, 2, 3,
\]

\[
\{ G_i, G_{3+j} \} = 0, \quad \text{where} \quad i, j = 1, 2, 3.
\]
4 Restrictions on the rank

Let some 3d mechanical system have the Hamiltonian \( H \) and 6 first integrals \( G_\alpha \) satisfying the commutation relations Eq. (12).

Consider two \( 6 \times 6 \) matrices: the Jacobi matrix \( J \) with elements
\[
J^\beta_\alpha := \frac{\partial G_\alpha}{\partial z_\beta}, \quad \text{where } \alpha, \beta = 1, \ldots, 6, \tag{13}
\]
and the matrix \( P \) with elements
\[
P_{\alpha\beta} := \{G_\alpha, G_\beta\}, \quad \text{where } \alpha, \beta = 1, \ldots, 6. \tag{14}
\]
Then definitions (13) of Jacobi matrix and (5) of brackets imply that
\[
P_{\alpha\beta} = \sum_{\gamma,\delta=1,\ldots,6} J^\gamma_\alpha \omega_{\gamma\delta} J^\delta_\beta. \tag{15}
\]

Suppose that \( G_1^2 + G_2^2 + G_3^2 \neq 0 \) and \( G_4^2 + G_5^2 + G_6^2 \neq 0 \). Then the matrix \( P \) has two independent null-vectors
\[
(G_1, G_2, G_3, 0, 0, 0) \quad \text{and} \quad (0, 0, 0, G_4, G_5, G_6) \tag{16}
\]
due to relations (12), and so \( \text{rank}(P) = 4 \).

Since the symplectic form \( \omega \) is non-degenerate, the relation Eq. (15) and degeneracy of the matrix \( P \) imply that
\[
\text{rank}(P) \leq \text{rank}(J) < 6. \tag{17}
\]
So either \( \text{rank}(J) = 4 \), or \( \text{rank}(J) = 5 \). Both these cases can be realized: \( \text{rank}(J) = 5 \) for the Coulomb system while \( \text{rank}(J) = 4 \) for some of the systems described in Section 6.

5 Not all 3d systems have \( o(4) \) symmetry

To give an example of a 3d mechanical system without \( o(4) \) symmetry, consider the Hamiltonian
\[
H = H_1 + H_2 + H_3, \quad \text{where } H_i = \frac{1}{2}p_i^2 + \frac{\omega_i^2}{2}q_i^2 \tag{18}
\]
and where the \( \omega_i \) for \( i = 1, 2, 3 \) are incommensurable.

Evidently, each of the functions \( H_i \) is a first integral.

Let us show that each first integral of this system is a function of the \( H_i \), where \( i = 1, 2, 3 \). Indeed, let \( F \) be a first integral. So, \( F \) is constant on every trajectory defined for the system under consideration by relations
\[
q_i = \frac{\sqrt{2}}{\omega_i}r_i \sin(\omega_i t + \varphi_i), \quad p_i = \sqrt{2}r_i \cos(\omega_i t + \varphi_i) \quad \text{for } i = 1, 2, 3, \tag{19}
\]
where the \( r_i \) and \( \varphi_i \) are constants specifying the trajectory. Since every trajectory given by Eq. (19) is everywhere dense on the torus
\[
T(r_1, r_2, r_3) := \left\{ z \in \mathbb{R}^6 \mid \frac{1}{2}p_i^2 + \frac{\omega_i^2}{2}q_i^2 = r_i^2 \quad \text{for } i = 1, 2, 3 \right\}, \tag{20}
\]
it follows that \( F \) is constant on every torus \( T(r_1, r_2, r_3) \), and hence \( F \) is a function of the \( r_i \). This implies \( F = F(H_1, H_2, H_3) \).

Now suppose that the system has 6 first integrals \( G_\alpha \) satisfying commutation relations (12) of the Lie algebra \( \mathfrak{o}(4) \). Then, since \( G_\alpha = G_\alpha(H_1, H_2, H_3) \), it follows that the Jacobi matrix \( J \) in Eq. (13) is of rank \( \leq 3 \), and so due to Eq. (15) the matrix \( P \), see Eq. (14), is of rank \( \leq 3 \). But this fact contradicts the easy to verify fact that if \( G_1^2 + G_2^2 + G_3^2 \neq 0 \) and \( G_4^2 + G_5^2 + G_6^2 \neq 0 \), then \( \text{rank}(P) = 4 \).

So, the system under consideration has no \( \mathfrak{o}(4) \) symmetry in any domain invariant with respect to Hamiltonian flow generated by \( H \).

6 An example of non-Coulomb 3d mechanical system with \( \mathfrak{o}(4) \) Lie algebra of the first integrals

Consider a particle in an homogeneous field with potential \(-q_3\). This is a system with 3 degrees of freedom with Hamiltonian

\[
H = \frac{p^2}{2} - q_3. \tag{21}
\]

Let \( U := \{ z \in \mathbb{R}^6 \mid p_1^2 < a_1^2, \ p_2^2 < a_2^2 \} \),

where each \( a_s \) is any smooth function of Hamiltonian \( H \). We denote the boundary of \( U \) by \( \partial U \) and its closure by \( \bar{U} \).

Then the real functions

\[
\begin{align*}
G_1 &= p_1, \\
G_2 &= \sqrt{a_1^2 - p_1^2} \cos(q_1 - p_1p_3), \\
G_3 &= \sqrt{a_2^2 - p_1^2} \sin(q_1 - p_1p_3), \\
G_4 &= p_2, \\
G_5 &= \sqrt{a_2^2 - p_2^2} \cos(q_2 - p_2p_3), \\
G_6 &= \sqrt{a_2^2 - p_2^2} \sin(q_2 - p_2p_3),
\end{align*}
\]

are the first integrals defined in \( \bar{U} \) and smooth in \( U \). Let \( \mathcal{A} \) be the space generated by \( G_\alpha \).

The space \( \mathcal{A} \), with Poisson brackets as an additional operation, is the Lie algebra isomorphic to \( \mathfrak{o}(4) \). It is subject to a direct verification that the integrals (23) indeed satisfy the relations (12) for \( \mathfrak{o}(4) \)-generators.

The Casimirs, defined by the formulas

\[
K_1 := \sum_{i=1,2,3} G_i^2, \quad K_2 := \sum_{i=1,2,3} G_{3+i}^2
\]

are equal to

\[
K_1 = a_1^2, \quad K_2 = a_2^2
\]

and do not define the Hamiltonian only if the \( a_s \) are constant. In the case where the \( a_s \) are constant, the Jacobi matrix for the functions (23) has rank 4 at the generic point. Otherwise, \( \text{rank}(J) = 5 \) at the generic point.
6.1 Non-Invariance of the domain $U$ under the flows $L_G$.

For $\lambda_2$ and $\lambda_3$ real, such that $\sqrt{\lambda_2^2 + \lambda_3^2} = 1$, $\lambda_2 = \cos \varphi$, $\lambda_3 = -\sin \varphi$, we see that $G := \lambda G_1 + \lambda_2 G_2 + \lambda_3 G_3$ is of the shape

$$G = \lambda p_1 + Q \cos(q_1 - p_1 p_3 + \varphi),$$

where $Q := \sqrt{a_1^2 - p_1^2}$.

Set

$$Q_H := \frac{dQ}{dH} = \frac{a_1 da_1}{Q dH}$$

so that

$$\{z_\alpha, Q\} = Q_H\{z_\alpha, H\} - \frac{p_1}{Q}\{z_\alpha, p_1\},$$

$$\{z_\alpha, H\} = \sum_i \{z_\alpha, p_i\} p_i - \{z_\alpha, q_3\}.$$  

Introduce a new variable $u$ instead of $q_1$:

$$u := q_1 - p_1 p_3 + \varphi. \tag{25}$$

Let $z(\tau_0) \in U$. The equations of the Hamiltonian flow $L_G$ are then of the form

$$\frac{d}{d\tau} z_\alpha = \{z_\alpha, G\}, \ i.e.,$$

$$\frac{d}{d\tau} p_3 = Q_H \cos(u),$$

$$\frac{d}{d\tau} q_3 = Q p_1 \sin(u) + Q_H p_3 \cos(u),$$

$$\frac{d}{d\tau} p_2 = 0, \ \frac{d}{d\tau} q_2 = Q_H p_2 \cos(u) \tag{26}$$

$$\frac{d}{d\tau} p_1 = Q \sin(u),$$

$$\frac{d}{d\tau} q_1 = \lambda + Q p_3 \sin(u) - \frac{p_1}{Q} \cos(u) + Q_H p_1 \cos(u).$$

Since $\{G, H\} = 0$, it is clear, that $dH/d\tau = 0$ and $da_s/d\tau = 0$ along the trajectories $z(\tau)$ defined by Eqs (26).

**Proposition 1.** For any $z(\tau_0) \in U$, there exists a first integral $G_z \in \mathcal{A}$ such that the Hamiltonian flow $L_{G_z}$ leads the point $z(\tau_0)$ to the boundary of $U$ for a finite time.

**Proof.** We have

$$\frac{d}{d\tau} u = \lambda - \frac{p_1}{Q} \cos(u), \tag{27}$$

$$\frac{d}{d\tau} p_1 = Q \sin(u), \tag{28}$$

$$\frac{d}{d\tau} Q = -p_1 \sin(u). \tag{29}$$
and hence
\[ \frac{d^2}{d\tau^2}p_1 = -p_1 \sin^2(u) + Q \cos(u)(\lambda - \frac{p_1}{Q} \cos(u)) = -p_1 + \lambda Q \cos(u) \]

Further on we consider only the case \( \lambda = 0 \). In this case
\[ \frac{d^2}{d\tau^2}p_1 = -p_1. \] (30)
and
\[ p_1 = p_1^{\text{max}} \sin(\tau + \psi), \] (31)
where \( p_1^{\text{max}} \geq 0 \) and \( \psi \) are constant on the trajectories.

We have
\[ (p_1^{\text{max}})^2 = p_1^2 + \left( \frac{d}{d\tau}p_1 \right)^2 = p_1^2 + (a_1^2 - p_1^2) \sin^2(u) \]
\[ = a_1^2 \sin^2(u) + p_1^2 \cos^2(u) = a_1^2 - (a_1^2 - p_1^2) \cos^2(u) \]
and
\[ (p_1^{\text{max}})^2 = a_1^2 - (a_1^2 - p_1^2(\tau)) \cos^2(u(\tau)) \] (32)
for any \( \tau \) since \( p_1^{\text{max}} \) is constant on each trajectory.

If \( \cos(u(\tau_0)) \neq 0 \) and \( p_1^2(\tau_0) < a_1^2(\tau_0) \), then
\[ (p_1^{\text{max}})^2 = a_1^2 - (a_1^2 - p_1^2(\tau_0)) \cos^2(u(\tau_0)) < a_1^2. \] (33)
Eqs (33) and (31) imply that
\[ p_1^2(\tau) < a_1^2(\tau) \quad \text{for any } \tau \]
i.e., \( z(\tau) \in U \) for any \( \tau \in \mathbb{R} \). Besides, conditions (32) and (33) imply that
\[ \cos(u(\tau)) \neq 0 \quad \text{for any } \tau. \]

Now, observe that for every \( z(\tau_0) \) it is possible to choose \( \lambda_2 \) and \( \lambda_3 \) (i.e., \( \varphi \)) such that \( \cos(u(\tau_0)) = 0 \). Then, for this \( \varphi \), we have \( (p_1^{\text{max}})^2 = a_1^2 \) and \( Q(\pi/2 - \psi) = 0 \), i.e., \( z(\pi/2 - \psi) \in \partial U \).

Comment 1. The proof of Proposition 1 shows also that for each fixed \( \varphi \), the domain
\[ U_{\varphi} := \{ z \in U \mid \cos(q_1 - p_1p_3 + \varphi) \neq 0 \} \] (35)
is invariant under the action of Hamiltonian flow \( \mathcal{L}_Q \cos(q_1 - p_1p_3 + \varphi) \) acting on \( U_{\varphi} \) as 1-dimensional Lie group.

Comment 2. There is no domain \( U_{\text{common}} \subset U \) invariant under Hamiltonian flows \( \mathcal{L}_Q \cos(q_1 - p_1p_3 + \varphi) \) for all \( \varphi \in [0, 2\pi) \).

Indeed, \( U_{\text{common}} \subset \bigcap_{\varphi} U_{\varphi} \), and \( \bigcap_{\varphi} U_{\varphi} = \emptyset \) since for any \( z \in U \) there exists \( \varphi \in [0, 2\pi) \) such that \( \cos(q_1 - p_1p_3 + \varphi) = 0 \).
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