Double-port measurements for robust quantum optical metrology

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It has been proposed and demonstrated that taking path-entangled Fock states (PEFSs) can give better phase sensitivities than using NOON states in lossy optical interferometry [Huver et al., Phys. Rev. A 78, 063828 (2008)]. However, the demonstration was based on a measurement scheme which was yet to be implemented in experiments. In this work, we quantitatively illustrate the advantage of PEFSs over NOON states in the presence of photon losses by analytically calculating the quantum Fisher information. To realize such advantage in practice, we then investigate the achievable sensitivities with Bayesian estimation method by employing two types of feasible detection: parity and photon number resolving. We here apply a double-port measurement scheme where the photons at the two output ports of the interferometer are simultaneously detected with the aforementioned types of detection schemes.

I. INTRODUCTION

An essential task in quantum optical metrology is to reach the ultimate sensitivity limit to the phase measurement imposed by quantum theory [1–7]. By now, enormous strategies have been proposed to improve the sensitivities of phase measurements. Among these, one celebrated strategy is the use of NOON states [1]

\[ |N::0⟩ = \frac{1}{\sqrt{2}} (|N, 0⟩ + |0, N⟩). \tag{1} \]

It has been experimentally demonstrated that this strategy can acquire a Heisenberg-scaling sensitivity in the lossless optical interferometry [8–12], which is a \( \sqrt{N} \) factor improvement over the shot-noise limit (SNL) [1–6]. However, such a quantum improvement will be completely gone even if an individual photon loss occurs [13–15].

To mitigate this problem, two main kinds of protocols have been raised: active and passive. The former one tries to actively reduce the effects of losses by managing experimental processes, e.g., applying quantum error correction [16, 17]. The latter one tries to passively retain sub-SNL sensitivities by using certain probe states which are more robust against losses but at expense of a bit of sensitivity [13, 18–23]. Among these passive protocols, one of the most representative proposed by Huver et al is that by taking the PEFS [18]

\[ |m::n⟩ = \frac{1}{\sqrt{2}} (|m, n⟩ + |n, m⟩), \quad (m > n) \tag{2} \]

as the probe state. The authors in [18] then demonstrated that the PEFSs could outperform NOON states in the presence of photon losses under the constraint of \( \Delta \equiv m - n = N \). However, the demonstration was based on a measurement scheme which was yet to be implemented. Later, Jiang et al. proposed a single parity (SP) measurement scheme for showing such advantage [23]. Unfortunately, they found that no explicit benefit of PEFSs over NOON states can be exploited in the lossy interferometry.

In this manuscript, we address this issue by finding effective measurements to present the robustness of the PEFS scheme in practical metrological experiments. To quantitatively demonstrate the advantage of the PEFS scheme, we first derive the lower sensitivity bounds in the presence of photon losses by invoking the quantum Fisher information (QFI). We then discuss the achievable sensitivities with two different types of detection: parity and photon number resolving. Here, we employ a double-port measurement scheme where the photons at the two output ports of the interferometer are simultaneously detected with the aforementioned two types of detection schemes. The reason why we consider the double-port measurement scheme is because we expect to extract more information w.r.t. the single-port measurement scheme. The efficiency of the double measurement comparing to the single one has been evaluated w.r.t. the photon number resolving detection [24, 25]. In our work, we find that the double parity (DP) measurement scheme brings an additional enhancement in phase sensitivity over the SP measurement scheme. However, it still fails to show the advantage of PEFSs over NOON states in the lossy interferometry. Thereafter, we consider the double photon number resolving (DPNR) measurement [24, 25]. Interestingly, we find that such a measurement scheme allows the PEFSs to outperform the NOON states in the lossy interferometry. We also find that this measurement can always saturate the sensitivity bounds for lossy NOON states.

This paper is organized as follows. In Sec. II, we revisit the robust optical metrology with PEFSs proposed by Huver et al. [18] and derive the lower sensitivity bounds based on quantum estimation theory. In Sec. III, we discuss the achievable sensitivities with two different types of detection schemes.
of measurement. Finally, our conclusions are given in Sec. IV.

II. ROBUST QUANTUM-OPTICAL INTERFEROMETRY WITH PEFSS

A general optical two-mode interferometer consists of three optical elements: two beam splitters (BSs) denoted by $\hat{B}$ and a phase shifting denoted by $\hat{U}$ (see Fig. 1). This device serves to illustrate the principle of phase estimation. The input light is divided into two beams through the first BS. Then they accumulate an unknown phase $\phi$ of interest under the phase shifting. The value of the phase parameter is finally read out by detecting the photons in the beams out from the second BS.

From quantum estimation theory [26–28], the sensitivity of estimating the phase $\phi$ is statistically measured by the units-corrected mean-square deviation of the estimator $\phi_{\text{est}}$ from the true value $\phi$,

$$(\delta \phi_{\text{est}})^2 = \left( \frac{\phi_{\text{est}}(\phi_{\text{est}})_{\text{av}} - \phi}{\langle \phi_{\text{est}} \rangle_{\text{av}}} \right)^2,$$

where the brackets $(\bullet)_{\text{av}}$ denotes the statistical average and the derivative $\partial_{\phi}(\phi_{\text{est}})_{\text{av}}$ removes the local difference in the ‘units’ of $\phi_{\text{est}}$ and $\phi$. Whichever measurement scheme is employed, the ultimate limit to the sensitivity of the unbiased estimator is given by the quantum Cramér-Rao bound

$$(\delta \phi_{\text{est}})^2 \geq \frac{1}{v F_Q},$$

where $v$ is the repetitions of the experiment and $F_Q$ is the so-called QFI [26–28]. This bound is asymptotically achieved for large $v$ under optimal measurements, followed by the maximum likelihood estimator [26–28]. Although optimal measurements for saturating this bound have been formally demonstrated in Refs. [28, 29], they may be not feasible in realistic experiments.

Assume that $a$ and $b$ denote the annihilation operators of upper and lower modes, respectively. We then define the Schwinger representation [30] as follows

$$\hat{J}_x = \frac{1}{2} \left( a^\dagger b + ab \right), \quad \hat{J}_y = \frac{1}{2i} \left( a^\dagger b - ab \right),$$

$$\hat{J}_z = \frac{1}{2} \left( a^\dagger a - b^\dagger b \right),$$

which satisfy the commutation relations for Lie algebra of $su(2)$

$$[\hat{J}_x, \hat{J}_y] = i \hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i \hat{J}_x, \quad [\hat{J}_x, \hat{J}_z] = i \hat{J}_y.$$  (7) 

and commute with the total photon number operator $\hat{N} = a^\dagger a + b^\dagger b$. For simplicity, we denote $\hat{J}_0 = \hat{N}/2$ below. Within this representation, the operations of the BS and phase shifting are represented by

$$\hat{B} = \exp \left( -i \frac{\pi}{2} \hat{J}_y \right), \quad \hat{U} = \exp (i \hat{a}^\dagger \hat{a} \cdot) \hat{a},$$

where $\hat{B}$ refers to a balanced BS and $\hat{U}$ accounts for the phase imprinting solely on mode $a$. An interferometry with asymmetric BSs has been discussed for various states [31, 32]. As for phase shifting $\hat{U}$, another both-arm configuration is represented by $\exp (i \phi \hat{J}_x)$, which has been widely considered in previous studies. Although these two types of phase shifting have subtle difference in phase estimation [33], they have the same effect in our cases. That is because that the states of consideration here are definite photon numbers, thus the phase shifts accumulating by single- and both-arm phase shifting are up to an irrelevant global phase factor, i.e., $\exp (i \phi N/2)$.

In our work, as shown in Fig. 1, the probe state refers to the state after the first BS, i.e., $|\psi\rangle = \hat{B} |\psi_{\text{in}}\rangle$, where $|\psi_{\text{in}}\rangle$ is the input state powering at the input port of the interferometer. After the phase shifting $\hat{U}$, it becomes $|\psi(\phi)\rangle = \hat{U} |\psi\rangle$ acquiring an unknown phase to be estimated. Taking the NOON state of Eq. (1) as the probe state, one can acquire the Heisenberg-limit sensitivity associating with $F_Q = N^2$. However, in practical situation, the interferometry is often subjected to photon losses. This causes that the dedicated NOON state may rapidly lose its quantum advantage for phase resolution. To circumvent this problem, Huyer et al. proposed a robust metrological scheme by applying the PEFS as the probe state [18]. In the lossless case, the QFI for the PEFS of Eq. (2) reads $F_Q = \Delta^2$.

Below, we revisit this robust metrological scheme and derive the sensitivity bounds in the presence of photon losses by using the QFI. Here, photon losses are modeled by inserting fictitious BSs with transmissivities $\eta_x, (x = a, b)$ into two arms of the interferometer [13, 18, 21]. Their effects can be formally represented in a Kraus form as

$$\rho = \sum_{i_a, i_b=0}^{\infty} \hat{K}_{a,i_a} \hat{K}_{b,i_b} \rho_{\text{in}} \hat{K}_{b,i_b}^\dagger \hat{K}_{a,i_a}^\dagger,$$

with Kraus operators

$$\hat{K}_{x,t} = (1 - \eta_x)^{t/2} \left( \sqrt{\eta_x} \right)^{\dagger x} x^t / \sqrt{t!}.$$  (10)
Thanks to the commutation relationship between phase shifting and photon losses [13, 21], one can freely exchange the order of operations between the phase shift and the photon loss. For simplicity, we assume that photon losses act after the phase shifting, as shown in Fig. 1.

Successively going through the phase shifting and the photon losses, the probe state of Eq. (2) of $\Xi \equiv m + n$ evolves into a parametric mixed state as [18, 23]

$$\rho(\phi) = \frac{1}{2} \sum_{l_a=0}^{\Xi} \sum_{l_b=0}^{\Xi-l_a} \left[ B^m_{l_a l_b} C^m_{l_a l_b} |m-l_a, n-l_b\rangle \langle m-l_a, n-l_b| + B^n_{l_a l_b} C^n_{l_a l_b} |n-l_a, m-l_b\rangle \langle n-l_a, m-l_b| \right]$$

$$+ \sqrt{B^m_{l_a l_b}} B^n_{l_a l_b} C^m_{l_a l_b} C^n_{l_a l_b} e^{i\Delta \varphi} |m-l_a, n-l_b\rangle \langle n-l_a, m-l_b| + h.c. \right],$$

where $\binom{k}{l} \equiv \binom{k}{l} \left( \Xi - k \right) \binom{\Xi - k - 1}{l_a} \binom{\Xi - k - 1}{l_b} \eta_a^{-1} \eta_b^{-1}$, (12)

$$C^k_{l_a l_b} \equiv H[k - l_a] - H[k - \Xi + l_b - 1],$$

and $\Delta \varphi$ denotes the binomial coefficient and $H[n]$ is the Heaviside step function of a discrete form. For NOON states, i.e., $m = N$ and $n = 0$, Eq. (11) can be simplified as a direct sum form of

$$\rho(\phi) = |\xi(\phi)\rangle \langle \xi(\phi)| \oplus \rho_D,$$ (14)

where the $\phi$-dependent state $|\xi(\phi)\rangle$ is given by

$$|\xi(\phi)\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{\eta_a N} e^{iN \varphi} |N, 0\rangle + \sqrt{\eta_b N} |0, N\rangle \right),$$ (15)

up to a normalization constant and another part is a $\phi$-independent diagonal matrix of dimension $2N$ as

$$\rho_D = \frac{1}{2} \sum_{l=1}^{N} \left( B^0_{l l} |N-l, 0\rangle \langle N-l, 0| + B^0_{0 l} |0, N-l\rangle \langle 0, N-l| \right).$$ (16)

To calculate the QFI, one should determine the eigenvalues and eigenstates of $\rho(\phi)$ of Eq. (11). However, an analytical diagonalization of $\rho(\phi)$ may be not easily obtained in the case with photon losses in both beams. Following Ref. [13], we obtain the upper bound of the QFI for PEFSs, when losses occur in two arms, as follows

$$F_Q \leq 2 \left( m^2 + n^2 \right) - 2 \sum_{l_a=0}^{\Xi} \sum_{l_b=0}^{\Xi-l_a} \frac{\left( mB^m_{l_a l_b} C^m_{l_a l_b} + nB^n_{l_a l_b} C^n_{l_a l_b} \right)^2}{B^m_{l_a l_b} C^m_{l_a l_b} + B^n_{l_a l_b} C^n_{l_a l_b}}.$$ (17)

The above inequality is saturated when losses occur only in one arm, i.e., $\eta_b = 1$, in which case, Eq. (17) reduces to [13]

$$F_Q = 2 \left( m^2 + n^2 \right) - 2 \sum_{l_a=0}^{\Xi} \frac{\left( mB^m_{l_a l_a} C^m_{l_a l_a} + nB^n_{l_a l_a} C^n_{l_a l_a} \right)^2}{B^m_{l_a l_a} C^m_{l_a l_a} + B^n_{l_a l_a} C^n_{l_a l_a}}.$$ (18)

Besides, it is also saturated for lossy NOON states of Eq. (14) as

$$F_Q^{\text{NOON}} = 2N^2 \left( \eta_a N \eta_b N \right),$$ (19)

This can be easily obtained by employing the property that the QFI for a density matrix of direct sum form $\rho(\phi) = \bigoplus_{l=1}^N \rho_l(\phi)$ is given by the sum over all amounts of the QFI in terms of each sub-matrices, i.e., $F_Q(\rho(\phi)) = \sum_{l=1}^N F_Q[\rho_l(\phi)]$. For lossy NOON states, the $\rho_D$ of Eq. (16) does not contribute to the amount of the QFI due to the fact that $\rho_D$ is independent of $\phi$. Hence the QFI for lossy NOON states is given by $F_Q = F_Q[|\xi(\phi)\rangle \rangle$ and then derived as Eq. (19) with Eq. (15).

In order to compare the performance of the two strategies in lossy optical interferometry, we plot in Fig. 2 the sensitivity bounds for $|10 : 4\rangle$ and $|6 : 0\rangle$ according to Eqs. (17), (18) and (19). It explicitly shows that $|10 : 4\rangle$ is indeed superior to $|6 : 0\rangle$ in the presence of losses. Although similar conclusion has been obtained by Huver et al. [18], based on a measurement scheme which was yet to be implemented. In what follows, we address how to realize this superiority in practice with feasible measurements.

III. PHASE SENSITIVITIES WITH TWO FEASIBLE MEASUREMENTS

In quantum theory, a generic measurement can be described by a positive-operator-valued measurement $\hat{M} \equiv \{M_{\chi}\}$ with $\chi$ being the results of measurement. Given an operator $\hat{M}$, the accessible phase sensitivity is limited by a classical analog of inequality (4) $(\delta \phi)^2 \geq (\nu F_C)^{-1}$, where $F_C$ is the classical Fisher information (CFI) defined as

$$F_C = \sum_{\chi} \frac{1}{p(\chi|\phi)} \left( \frac{dp(\chi|\phi)}{d \phi} \right)^2.$$ (20)

Here, $p(\chi|\phi) \equiv \text{Tr}[\hat{M}_\chi \rho(\phi)]$ is the probability of the outcome $\chi$ with the specific value of $\phi$. This bound is
achievable for the maximum likelihood estimator with Bayesian estimation method when \( v \) is sufficiently large [26, 27, 34, 35]. If the equality of \( F_C = F_Q \) holds, the \( M \) is the optimal measurement. In the lossless interferometry, it has been demonstrated that parity measurement [36] and photon number resolving measurement [25, 37, 38] are optimal for all path-symmetric pure states.

Below, we apply these two types of measurements to the above phase resolution problem and investigate the accessible sensitivities with these two measurements.

### A. Parity measurement

Parity measurement was originally proposed to probe atomic frequency in trapped ions by Bollinger \textit{et al.} [39] and later employed for optical interferometry by Gerry [40]. It accounts for distinguishing the states with even and odd numbers of photons. Specifically, the parity is assigned as the value of +1 when the photon number of a state is even, and the value of −1 if odd. A parity operator acting on the output mode \( c \) can be described by

\[
\hat{\Pi}_c = (-1)^{N_c} \exp\left[i\pi \left(J_0 + \hat{J}_z\right)\right].
\]  

(21)

Analogously, the one acting on the output mode \( d \) is denoted by

\[
\hat{\Pi}_d = (-1)^{N_d} \exp\left[i\pi \left(J_0 - \hat{J}_z\right)\right].
\]  

(22)

Obviously, they satisfy \( \hat{\Pi}_i^2 = \mathbb{1}, (i = c, d) \), with \( \mathbb{1} \) being the identity matrix. To facilitate our calculation below, we here consider the second BS operation as a part of measurement. The parity measurement through the BS is transformed into [23, 41]

\[
\hat{\pi}_c = \hat{B} \hat{\Pi}_c \hat{B}^\dagger = \sum_{N=0}^{\infty} \sum_{k=0}^{N} \left| k, N - k \right\rangle \langle N - k, k |,
\]  

(23)

\[
\hat{\pi}_d = \hat{B} \hat{\Pi}_d \hat{B}^\dagger = \sum_{N=0}^{\infty} \sum_{k=0}^{N} (-1)^{2j} \left| k, N - k \right\rangle \langle N - k, k |.
\]  

(24)

Then, we have

\[
\hat{\pi}_c \hat{\pi}_d = \hat{\Pi}_c \hat{\Pi}_d = \sum_{N=0}^{\infty} \sum_{k=0}^{N} (-1)^{2j} \left| k, N - k \right\rangle \langle k, N - k |.
\]  

(25)
as a result of the commutation of $[\hat{J}_0, \hat{B}] = 0$.

The single parity measurement scheme has been employed to probe the phase shift of the optical interferometer [23, 36, 40–49]. In contrast, we here apply a DP measurement scheme, in which one simultaneously performs parity detection on each output port of the interferometer. To our knowledge, it has never been discussed in previous studies. Formally, it can be expressed as a projection operator $\hat{M}_p = \langle p_c, p_d | p_c, p_d \rangle \delta_{p_c, p_d = \pm 1}$, with $(p_c, p_d)$ the outcome results of the parities at two output ports. Assume the second BS differs from the first one up to a $\pi$ phase factor. The conditional probability w.r.t. $(p_c, p_d)$ is defined by

$$p(p_c, p_d | \phi) = \langle p_c, p_d | \hat{B} \rho(\phi) \hat{B}^\dagger | p_c, p_d \rangle,$$  \hspace{0.5cm} (26)

where we have employed the cyclic property of the trace operation. Using Eq. (20) and Eq. (26) yields the CFI w.r.t. the DP measurement.

To calculate exactly the CFI, we further express the $\hat{F}_c^D$ into an alternative form of Eq. (A6), as shown in Appendix A, which depends on three expectation values $\langle \hat{\Pi}_a \rangle$, $\langle \hat{\Pi}_b \rangle$ and $\langle \hat{\Pi}_a \hat{\Pi}_b \rangle$. With Eqs. (11), (23), (24) and (25), we obtain

$$\langle \hat{\Pi}_a \rangle = D + E \cos \Delta \phi,$$ \hspace{0.5cm} (27)

$$\langle \hat{\Pi}_b \rangle = D + E (-1)^\Delta \cos \Delta \phi,$$ \hspace{0.5cm} (28)

$$\langle \hat{\Pi}_a \hat{\Pi}_b \rangle = \frac{1}{2} \left[ (1 - 2 \eta_a)^m (1 - 2 \eta_b)^n + (1 - 2 \eta_b)^m (1 - 2 \eta_a)^n \right],$$ \hspace{0.5cm} (29)

where

$$D = \frac{1}{2} F_1 \left( -m, -n; 1; \frac{\eta_a \eta_b}{\gamma_a \gamma_b} \right) \left( \gamma_a^m \gamma_b^n + \gamma_a^n \gamma_b^m \right),$$ \hspace{0.5cm} (30)

$$E = \left( \frac{\Delta}{\eta_a \eta_b} \right) F_1 \left( -m, -n; 1 + \Delta; \frac{\eta_a \eta_b}{\gamma_a \gamma_b} \right) \left( \gamma_a^\Delta \gamma_b^m + \gamma_a^m \gamma_b^\Delta \right),$$ \hspace{0.5cm} (31)

with $\gamma_i \equiv 1 - \eta_i$, $(i = a, b)$ and $F_1(a, b; c; z)$ the ordinary hyper-geometric function [23]. If photon losses only occur on mode $a$, the above terms $D$ and $E$ can be simplified as

$$D = \frac{1}{2} \left( \frac{m}{n} \right) \eta_a^{\Delta} \gamma_a^m, \hspace{0.5cm} E = \eta_a^{\Delta/2 + n},$$ \hspace{0.5cm} (32)

and $\langle \hat{\Pi}_a \hat{\Pi}_b \rangle$ becomes

$$\langle \hat{\Pi}_a \hat{\Pi}_b \rangle = \frac{(-1)^\Delta}{2} \left[ (2 \eta_a - 1)^m + (2 \eta_a - 1)^n \right].$$ \hspace{0.5cm} (33)

We plot in Fig. 3 the CFIs for the single and double parity measurements as a function of $\phi$ for $[6 : 0]$ and $[10 : 4]$. In the ideal case, i.e., $\eta = 1$, the CFIs w.r.t. two measurements merge and they are equal to the QFIs over the whole phase interval. Thus, the two parity measurements are equivalent (see Appendix A for details) and optimal globally in the absence of losses [36]. Nevertheless, in the lossy cases, the CFIs critically depends on both $\eta$ and $\phi$. An oscillation with a period of $\pi/\Delta$ takes place for both single and double parity measurements. It shows that the CFIs based on the two parity measurements are extremely sensitive to photon losses. Similar results also take account when photon losses occur on one arm.

From Figs. 3(a) and (b) we see that the decreasing rate of the CFI for $[10 : 4]$ is faster than that for $[6 : 0]$. This can be seen more clearly on Fig. 2, in which we plot the phase sensitivities with the DP measurement as a function of $\eta$ by setting $\phi = \pi/2$. Such a faster deterioration accounts for the disadvantage of PEFSs over NOON states in the presence of photon losses, as also shown in Ref. [23]. Hence, although an additional enhancement of sensitivity may be achievable by the DP measurement comparing with the single one, such an enhancement does not suffice to reflect the advantage of the PEFS strategy, as expected by Huver et al. [18].

**B. Photon number resolving measurement**

A DPNR measurement can be represented as the projection operator $\hat{M}_N = \{ | n_c, n_d \rangle \langle n_c, n_d | \}_{n_c, n_d = 0}^{n_c, n_d = \infty}$ [25], where the pairs of outcomes $(n_c, n_d)$ are the photon numbers detected at the $c$ and $d$ output ports of the interferometer. To evaluate the CFI, we should determine the conditional probability in terms of $(n_c, n_d)$ as

$$p(n_c, n_d | \phi) = \langle n_c, n_d | B \rho(\phi) B^\dagger | n_c, n_d \rangle.$$ \hspace{0.5cm} (34)

The analytical evaluation of the CFI for PEFSs is computationally involved (except for NOON states). We thus employ numerical calculations for the CFI.

As mentioned in Sec. II, the lossy NOON state is given by Eq. (14). By identifying $2j = n_c + n_d$ and $2\mu = n_c - n_d$, we have $p(n_c, n_d | \phi) = p(j, \mu | \phi)$, and then obtain

$$p(j, \mu | \phi) = \sqrt{\gamma_a^N \gamma_b^N} \left( \frac{\pi}{2} \right)^2 (-1)^{j+\mu} \cos (N \phi) \delta_{j, \frac{N}{2}} + \frac{1}{2} \sum_{k=0}^{N} \left[ d_{j, \mu}^k \left( \frac{\pi}{2} \right)^2 \right] \left( \begin{array}{c} N \\ k \end{array} \right) \left( \eta_a^{N-k} \gamma_a^k + \eta_b^{N-k} \gamma_b^k \right) \delta_{j, \frac{N-k}{2}},$$ \hspace{0.5cm} (35)

function and $\delta_{i,j}$ denotes the Kronecker delta function.
Replacing the result of Eq. (35) into Eq. (20) and then doing the cancellation of terms in terms of \( j \neq N/2 \) as a consequence of the property of \( \delta_{i,j} \), finally yields

\[
F_C^{\text{NOON}} = g(\eta_a, \eta_b) F_Q^{\text{NOON}},
\]

(36)

with

\[
g(\eta_a, \eta_b) = \frac{(\eta_a^N + \eta_b^N) \sin^2(N\phi)}{(\eta_a^N + \eta_b^N) - 4N^2\eta_a^N\eta_b^N \cos^2(N\phi)}.\]

(37)

In the above derivation, we used the following identity [25]

\[
\sum_{k=\text{seven}} \left[ \frac{d_{k-j}}{k-j} \left( \frac{\pi}{2} \right) \right]^2 = \sum_{k=\text{odd}} \left[ \frac{d_{k-j}}{k-j} \left( \frac{\pi}{2} \right) \right]^2 = \frac{1}{2}.
\]

(38)

Equation (36) suggests that the DPNR measurement serves as the optimal measurement when \( g(\eta_a, \eta_b) = 1 \). This equality holds, according to Eq. (37), when either of the following two conditions is satisfied: (a) \( \eta_a = \eta_b \); or (b) \( \phi \to \pi/2N \). Specially, for the condition of \( \eta_a = \eta_b \), the equality of \( F_C^{\text{NOON}} = F_Q^{\text{NOON}} \) always holds independently of \( \phi \). It means that the DPNR measurement is globally optimal over the whole range of values of phase parameter for equal loss rates of two arms. Additionally, as for PEFSs, we numerically evaluate the maximum sensitivity over \( \phi \) with the DPNR measurement in the presence of photon losses.

As shown in Fig. (2), we numerically plot the phase sensitivities achievable with the DPNR measurement for \( |6 : 0 \rangle \) and \( |10 : 4 \rangle \), respectively. Similar to the parity measurements, the photon number resolving measurement also saturates the sensitivity bounds for the two probe states in the lossless case, i.e., \( \eta = 1 \). This is an expected result, as demonstrated in previous works [25, 37, 38]. Our numerical evidence shows that the sensitivity bounds for \( |6 : 0 \rangle \) is indeed reachable with the DPNR measurement for both single- and two-arm losses, as analytically predicted above. From Fig. (2), although the DPNR fails to be optimal for \( |10 : 4 \rangle \), it is more effective than the parity measurement, especially, in the case of losses on one arm, it is nearly optimal for \( |10 : 4 \rangle \). More importantly, the advantage of robust metrological protocol with PEFSs can be fully reflected with the DPNR measurement.

**IV. CONCLUSION**

We investigated in this paper the phase sensitivities of robust quantum optical interferometry with PEFSs by invoking the quantum estimation theory. We analytically derived the QFIs for PEFSs and NOON states in the presence of photon losses and quantitatively demonstrated that the PEFS strategy outperforms the NOON state strategy in the lossy interferometry. We then addressed how to implement this advantage with two different feasible measurement schemes by invoking the Bayesian estimation method.

Unlike the SP measurement scheme employed in Ref. [23], we alternatively applied a DP measurement scheme. We found that it can provide an additional enhancement in phase sensitivities in comparison to the SP measurement, but it still fails to observe the robustness of the PEFS strategy. Finally, we considered the DPNR measurement scheme. Our finding indicated that the DPNR measurement scheme allows the PEFSs to outperform the NOON states in the lossy interferometry. Moreover, this type of measurement is always optimal for lossy NOON states. Our results are readily applicable to other robust quantum metrological schemes [13, 19, 20, 22].

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**APPENDIX A: CFI WITH DOUBLE PARITY MEASUREMENT**

In this appendix, we present an alternative expression of the CFI w.r.t. the DP measurement. As mentioned in the main text, the SP detection has two outcome results: +1 and −1. For simplicity, we here denote them as + and −, respectively. Thus, the DP detection comes out four outcome results: ++, +−, −+, and −−. According to Eq. (20), the CFI with the DP measurement can be expanded into four terms as follows

\[
F_C = \frac{[\partial_\phi P(++)|^2]}{P(++)} + \frac{[\partial_\phi P(+-)|^2]}{P(+-)} + \frac{[\partial_\phi P(+-)|^2]}{P(+-)} + \frac{[\partial_\phi P(--)|^2]}{P(--)}. \tag{A1}
\]

We find that the conditional probabilities in terms of the four outcomes satisfy the following equations
\[ P(\pm |\phi\rangle + P(-|\phi\rangle + P(-|\phi\rangle = 1, \quad (A2) \]

\[ P(\pm |\phi\rangle + P(-|\phi\rangle - P(-|\phi\rangle = (\Pi_{c}), \quad (A3) \]

\[ P(\pm |\phi\rangle + P(-|\phi\rangle - P(-|\phi\rangle = (\Pi_{d}), \quad (A4) \]

\[ P(\pm |\phi\rangle + P(-|\phi\rangle - P(-|\phi\rangle = (\Pi_{c}\Pi_{d}). \quad (A5) \]

One can reverse these equations and solve for the conditional probabilities in terms of the expectation values \((\Pi_{c}), (\Pi_{d}),\) and \((\Pi_{c}\Pi_{d})\). Finally, replacing them into Eq. (A1) yields

\[ F_{C} = \frac{1}{2} \frac{1}{\langle \Pi_{c} \rangle} \frac{\partial_{\phi} \langle \Pi_{c} \rangle}{\langle \Pi_{c} \rangle - \langle \Pi_{d} \rangle} \left[ \partial_{\phi} \langle \Pi_{c} \rangle - \langle \Pi_{d} \rangle \right]^{2} + \frac{1}{2} \frac{1}{\langle \Pi_{c} \Pi_{d} \rangle} \frac{\partial_{\phi} \langle \Pi_{c} \Pi_{d} \rangle}{\langle \Pi_{c} \Pi_{d} \rangle - \langle \Pi_{d} \Pi_{d} \rangle} \left[ \partial_{\phi} \langle \Pi_{c} \Pi_{d} \rangle - \langle \Pi_{d} \Pi_{d} \rangle \right]^{2}. \quad (A6) \]

This is the main result of the appendix. If defining operators \(\Pi_{c} = \Pi_{c} \pm \Pi_{d},\) and associating with \(\Pi_{c}^{2} = 2(\pm \Pi_{c} \Pi_{d})\), one can rewrite Eq. (A6) in a concise form as

\[ F_{C} = \sum_{i=1}^{2} \frac{\langle \Pi_{c} \rangle \left( \partial_{\phi} \langle \Pi_{c} \rangle \right)^{2}}{\langle \Pi_{c} \rangle - \langle \Pi_{d} \rangle}, \quad (A7) \]

Let us take more insight on Eq. (A6). Suppose that the total photon number of the system is definite, which is denoted as \(N\). For even \(N\), we have \(\langle \Pi_{c} \rangle = \langle \Pi_{d} \rangle\) and \(\langle \Pi_{c} \Pi_{d} \rangle = 1\). Similarly, for odd \(N\), we have \(\langle \Pi_{c} \rangle = -\langle \Pi_{d} \rangle\) and \(\langle \Pi_{c} \Pi_{d} \rangle = -1\). With these results, Eq. (A6) can be simplified as

\[ F_{C} = \frac{\left( \partial_{\phi} \langle \Pi_{c} \rangle \right)^{2}}{1 - \langle \Pi_{c} \rangle}, \quad (A8) \]

which corresponds to the CFI w.r.t. the SP measurement [36]. It means that the DP measurement is metrologically equivalent to the SP measurement when the number of total photons is fixed.

This equivalence can be well understood. For probe states with fixed photon number, the outcome of the parity on one output port of the interferometer is determined by the outcome of the parity on another port. As shown in the main text, the probe states of consideration have definite photon numbers. This seems that the DP measurement here does not allow any advantage to be gained in comparison of the SP measurement. It is true for the lossless case, but the situation is different when photon losses occur, in which case, the initial probe state with definite number of photons evolves to a mixed state with fluctuating photon number. The metrological equivalence between the SP and DP may not persist in such circumstance. One can identify the relationship between the CFIs for the SP and DP as \(F_{DP} \geq F_{SP}\). This inequality can be obtained by replacing \(p(\rho_{c}|\phi) = \sum_{n_{d}} p(\rho_{c}, \rho_{d}|\phi)\) into Eq. (20) and invoking the Cauchy-Schwarz inequality, as a similar derivation w.r.t. the DPNR measurement [25].

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