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A frozen glass phase in the multi-index matching problem

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The multi-index matching is an NP-hard combinatorial optimization problem; for two indices it reduces to the well understood bipartite matching problem that belongs to the polynomial complexity class. We use the cavity method to solve the thermodynamics of the multi-index system with random costs. The phase diagram is much richer than for the case of the bipartite matching problem: it shows a finite temperature phase transition to a completely frozen glass phase, similar to what happens in the random energy model. We derive the critical temperature, the ground state energy density, and properties of the energy landscape, and compare the results to numerical studies based on exact analysis of small systems.

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It has been recognized early on that one important motivation of the research in spin glass theory is the ubiquity of systems with frustration and disorder. In particular, recent statistical physics studies have brought interesting new results in some important computer science problems. Notable examples are found in optimization (e.g., K-satisfiability (K-SAT)), graph coloring, or vertex cover) and information theory (e.g., error correcting codes). So far, the most interesting applications of spin glass theory have been obtained in this emerging field, which witnesses an upsurge of interdisciplinary studies involving physicists, computer scientists, and probabilists.

One of the first optimization problems studied analytically by physics methods was the random Bipartite Matching Problem (BMP). It is also a simple problem: from the computer science point of view, it belongs to the class P of polynomial complexity; from the physics point of view, it has no phase transition at finite temperature, and its solution with the replica method shows a simple replica symmetric behavior. Interestingly, the validity of this solution has been recently confirmed by a rigorous mathematical study.

In this work we study the Multi-Index Matching Problem (MIMP), a natural extension of the BMP. This is a more complicated problem: it belongs to the NP-hard class, and as we will see it also exhibits a finite temperature phase transition, with a low temperature glassy phase. Using the cavity method, we find that this phase consists of isolated configurations, and we conjecture that our method yields an exact solution to this problem. Because of its structural resemblance to the BMP, one may hope that the MIMP will also be amenable to rigorous study, generalizing the construction of to a problem with a glass phase.

The random MIMP — In the BMP one is given two sets of M points, S1 and S2. Each point of S1 must be "matched" or assigned to one point of S2. This matching must be one-to-one, and it can be represented by the "occupation" of the edges between the points of the two sets; we define n1, i2 = 1 if the points (1, 2) ∈ S1 × S2 are matched, while n1, i2 = 0 otherwise. To each matching we associate a cost or energy, which is the sum of the costs of each occupied edge.

The MIMP is a straightforward generalization of this problem to more than two indices. Given d sets S1, ..., Sd, each of M sites, a hyperedge is a d-uplet where exactly one site from each set appears. For each hyperedge we introduce a cost ℓ1, ..., ℓd, and the total cost of a (multi-index) matching is given, in terms of the occupation numbers of hyperedges, by:

\[ H(\{n_{1,1,1,\ldots,1,2}\}) = \sum_{1,1,\ldots,1,2} \ell_{1,1,\ldots,1,2} n_{1,1,\ldots,1,2} \]  

(1)

The occupation numbers of hyperedges, n1, ..., nd ∈ {0, 1} must be such that each site appears exactly once:

\[ \forall r \in [1,d], \quad \forall i_r, \quad \sum_{1,1,\ldots,1,1,2,1,\ldots,1,2} n_{1,1,\ldots,1,2} = 1. \]  

(2)

The optimization problem consists in finding the minimum cost matching. What makes this problem difficult is the constraint of having each site appear exactly in just one hyperedge; for d ≥ 3 the MIMP is NP-hard. MIMP arise for instance when assigning tasks (jobs) to people in particular time slots or in different places. An application also arises in the context of track reconstruction: given the positions of M unlabeled particles at d different times, one is to determine the tracks or trajectories of each. This kind of formulation is in fact used in track reconstruction in high energy physics.

We shall study the random MIMP where the individual costs ℓ1, ..., ℓd are independent identically distributed random variables. For definiteness we shall take ℓ1, ..., ℓd to have uniform distribution in [0, 1], although other distributions can be studied similarly.

For a given sample ℓ, characterized by the values of ℓ1, ..., ℓd, the partition function at inverse temperature β is
The inverse temperature and the sum is over all possible matchings. In the thermodynamic limit ($M \to \infty$), only the behavior at the lowest values of $\ell_{i_1 \ldots i_d}$ matters. Indeed, if we consider a given site in any of the $d$ sets, it is to be assigned to a low cost hyperedge; generally it is possible to assign it to one of the first shortest such hyperedges. This means that at large $M$, the typical cost of an occupied hyperedge in the low temperature regime should scale as $1/M^{d-1}$. It is thus convenient to work with rescaled quantities that are extensive (i.e. proportional to $M$):

$$E = M^{d-1} H.$$  

This amounts to considering thermodynamic quantities and having $\beta = \tilde{\beta}/M^{d-1}$ as the control parameter: one should keep $\beta$ fixed when taking the large $M$ limit.

Given these considerations, we conjecture that the free energy density is self-averaging as in most disordered systems, and in particular as rigorously proved for $d=2$.

Numerical study of the ground state — For a given sample of the quenched disorder, we want to determine the ground state energy $E_0$ which is the minimum of all $E\{\{n_{i_1}, \ldots, i_d\}\}$. An exhaustive search over all matchings works only for very small $M$ (typically $M \leq 6$ when $d \geq 3$) because of the rapid growth of the number of legal matchings, in $(M!)^{d-1}$. We have followed instead a branch and bound (B&B) approach which allows us to study intermediate $M$. From such an algorithm, we can test numerically whether $E_0$ is self-averaging and study its large $M$ limit.

The determination of the best matching uses a search tree. All the nodes at level $p$ of this tree correspond to having chosen hyperedges for the first $p$ points of the set $S_1$ (ordered arbitrarily). To go from level $p$ to level $(p+1)$, we branch on all possible $M^{d-1}$ choices for the next hyperedge. Then a path from the tree’s root (level 0) to a leaf (level $M$) is a choice of $M$ hyperedges which may or not correspond to a legal matching. The cost of a node in the tree is defined as: the sum of the costs of its associated hyperedges when they don’t overlap, or $\infty$ if the hyperedges overlap (i.e. they use a point of the $d$ sets $S_i$, more than once).

The B&B algorithm searches the tree and implements pruning. For this, it needs an upper bound $E_{ub}$ on $E_0$; we initialize this quantity before performing the search via the cost of a legal matching obtained by a greedy assignment of the hyperedges. Then the algorithm starts at the root of the tree (level 0) and searches it recursively. At each level, one branches on the $M^{d-1}$ choices of hyperedges that take one to the next level. Every time the current node has a cost greater than $E_{ub}$, all of its descendent nodes can be ignored as they cannot contain the ground state. If we reach level $M$, we have a legal matching which we keep if its cost is less than $E_{ub}$ (and we update $E_{ub}$ accordingly). Upon termination, we have the ground state and $E_0 = E_{ub}$.

We have implemented this algorithm along with a number of optimizations. For our computer program, one $d=3$ sample at $M = 20$ takes typically 5 seconds on a 2 GHz PC, and the CPU time grows by a factor around 2.2 every time $M$ is increased by 1. We have performed runs for $M \leq 22$ with 20000 samples at each $M$. From these data, we have extracted $E_0$, the disorder average of $E_0$; the mean cost per node is shown in Fig. 1. The data for $M \geq 10$ are well fitted by a quadratic curve in $1/M$, giving $E_0/M \to 3.06 \pm 0.03$; a power law fit of the same quality gives $E_0/M \to 3.09 \pm 0.03$.

In the inset of the figure, we show that the standard deviation $\sigma(E_0/M)$ scales as $1/\sqrt{M}$, which is evidence for self-averaging and also suggests a central limit theorem behavior when $M \to \infty$.

Finally, we have also investigated a bit the case of $d = 4$; however, we were limited to $M \leq 15$ and used only 5000 samples. (The CPU time grows by about the same factor when $M$ is increased by 1 as when $d = 3$.) Our best fit in this case leads to $E_0/M \to 7.2(3)$. 

Thermodynamics and the cavity approach — The recent formulation of the cavity method [3] for diluted systems offers a choice tool to study the thermodynamics of the MIMP analytically. Building on the idea that the optimal matching selects preferentially the hyperedges with the lowest costs, we dilute the initially complete hypergraph by suppressing hyperedges with $\ell_{i_1 \ldots i_d} > CM^{1-d}$ [4]. In the resulting graph, the degree of each site is a Poisson distributed variable of mean $C$. When increasing $M$ to $M + 1$, a new serie of $d$ sites is added. Each of them is connected to a finite number of neighbours. The partition function of one new site is easily computed.
in terms of the probability, \( \exp[\beta(x_i - C/d)] \), of unoccupation of each of its neighbors (say neighbor \( i \)) in the \( Md \) sites problem. Assuming a replica symmetric (RS) structure, the order parameter is the probability \( \mathcal{P}(x) \) that a randomly chosen site \( i \) has \( x_i = x \), which satisfies the self-consistent equation:

\[
\mathcal{P}(x) = \sum_{k=0}^{\infty} \frac{C^k e^{-C}}{k!} \int_0^C \int_{a=1}^{k} \frac{d\xi_a}{C} \int_{j=1}^{d} dx_j \mathcal{P}(x_j) \delta \left[ x + \frac{1}{\beta} \ln \left( e^{-\beta C/d} + \sum_{a=1}^{k} e^{-\beta(\xi_a - \sum_{j=1}^{d-1} x_j)} \right) \right].
\]

The free energy \( f_{rs}(\beta) \) can be obtained from \( \mathcal{P}(x) \) as:

\[
f_{rs}(\beta) = -\frac{d}{\beta} \left\{ \ln \left( e^{-\beta C/d} + \sum_{a=1}^{k} e^{-\beta(\xi_a - \sum_{j=1}^{d-1} x_j)} \right) \right\} + \frac{(d-1)C}{\beta} \left\{ \ln \left( 1 + e^{-\beta(\sum_{j=1}^{d-1} x_j)} \right) \right\}.
\]

where \( \langle \rangle \) stands for the averages of the cavity fields \( x \) with the distribution \( \mathcal{P} \), of the connectivities \( k \) with the Poissonian distribution of mean \( C \) and of the truncated costs \( \xi \) with the uniform distribution in \([0, C] \), as in [5].

In the zero temperature limit, \( \beta \to \infty \), we obtain formulae for the ground state energy that directly generalize the ones of the BMP case [1, 3].

However, while correctly describing the \( d = 2 \) problem, these RS equations are inconsistent when \( d > 2 \). We shall discuss specifically the \( d = 3 \) case. First, the entropy becomes negative for \( \beta > \beta_s = 0.412 \pm 0.001 \), as shown on Fig. 3. Second, we have found the RS solution to be unstable for \( \beta > \beta_s \), with \( \beta_s \simeq 0.6 \) [10]. These two facts show that a discontinuous phase transition takes place at some inverse temperature \( \beta_s < \beta \). Such transitions are also present in other NP-hard combinatorial optimization problems like \( K \)-SAT, and are usually overcome by passing to a one-step replica symmetry broken (1RSB) formalism [11]. Here however, the direct application of the 1RSB cavity method at zero temperature turns out to be inadapted.

The originality of the MIMP comes from the peculiar nature of the low temperature phase. This phase is dominated by isolated configurations, instead of clusters of configurations that generally arise in 1RSB systems: the 1RSB clusters have no internal entropy here, a situation which is also found in some other disordered systems, the REM (random energy model) [12], the directed polymer on disordered tree and the binary perceptron [13]. Upon cooling, these systems freeze when reaching the temperature \( 1/\beta_s \) where the entropy becomes zero. As a result, the thermodynamical properties can be derived from the knowledge of the RS solution only [14]. The free energy is given by:

\[
f(\beta) = \begin{cases} 
  f_{rs}(\beta) & \text{if } \beta \leq \beta_s, \\
  f_{rs}(\beta_s) & \text{if } \beta \geq \beta_s.
\end{cases}
\]

Necessary conditions for this frozen 1RSB Ansatz to hold include the existence of a finite \( \beta_s \) where the RS entropy becomes negative, the stability of the RS solution up to (at least) \( \beta_s \) and the absence of any discontinuous 1RSB transition before \( \beta_s \) (as we have checked from a finite \( \beta \) 1RSB population dynamics investigation).

On top of these properties, a crucial necessary condition for the frozen 1RSB Ansatz to hold is that the system must be subject to a restricted class of constraints, that we call hard constraints [13]. For matching problems, hard constraints reflect the requirement to realize perfect matchings and basically mean that the occupancy of a hyperedge is uniquely determined by that of its neighbors; this is to be contrasted with the situation in coloring for instance, where the color of a site is not necessarily uniquely prescribed by the colors of its neighbors. Notice that the \( d = 2 \) case satisfies all these requirements, except for the fact that \( \beta_s = \infty \).

The prediction (7) yields a ground state energy density \( f_0/M = f_{rs}(\beta_s) = 3.126 \pm 0.002 \) (see Fig. 3): our B&B numerical estimate is compatible with this value considering the systematic effects arising from the small \( M \) used there. When \( d = 4 \), we find similarly a ground state energy density \( f_0/M = 7.703 \pm 0.002 \) (with \( \beta_s = 0.135 \pm 0.002 \)); here again the B&B estimate we obtained is close to this value.

Overlaps — The cavity method gives access to the typical overlap \( q \) between equilibrium configurations, de-
FIG. 3: Distribution of overlaps between the ground state and the first excited state for \( d = 3 \) MIMP for \( M = 8 \) to \( M = 22 \) (from bottom to top at \( q = .32 \)).

We have derived the full phase diagram; we conjecture these results to be exact, and the numerical checks which we have performed on relatively small systems, through an efficient B&B algorithm, are consistent with the predictions. It will be extremely interesting to generalize to this problem the rigorous mathematical methods developed for the BMP.

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[1] M. Mézard, G. Parisi, and M. A. Virasoro, Spin-Glass Theory and Beyond, vol. 9 of Lecture Notes in Physics (World Scientific, Singapore, 1987).
[2] M. Mézard, G. Parisi, and R. Zecchina, Science 297, 812 (2002).
[3] R. Mulet, A. Pagnani, M. Weigt, and R. Zecchina, Phys. Rev. Lett. 89, 268701 (2002).
[4] M. Weigt and A. K. Hartmann, J. Phys. A 43, 11069 (2003).
[5] A. Montanari, Eur. Phys. J. B. 23, 121 (2001).
[6] M. Mézard and G. Parisi, J. Physique 46, L771 (1985).
[7] D. J. Aldous, Rand. Struct. Algo. 18, 381 (2001).
[8] W. P. Pierskalla, Operations Research 16, 422 (1968).
[9] R. Karp, in Complexity of Computer Computations, edited by R. Miller and J. Thatcher (Plenum Press, 1972), pp. 85–103.
[10] A. B. Poore, Comput. Opt. Appl. 3, 27 (1994).
[11] J. Pusztaszeri, P. E. Rensing, and T. M. Liebling, Journal of Global Optimization 16, 422 (1995).
[12] D. J. Aldous, Rand. Struct. Algo. 1, 383 (1990).
[13] M. Mézard and G. Parisi, Eur. Phys. J. B 20, 217 (2001).
[14] We must then allow for unoccupied sites: we introduce a chemical potential \( \mu = C/d \), conjugate to the number of occupied sites, which is sent to \( \infty \) with \( C \) at the end of the computation.
[15] W. Krauth and M. Mézard, Europhys. Lett. 3, 213 (1989).
[16] O. C. Martin, M. Mézard, and O. Rivoire, in preparation.
[17] B. Derrida, Phys. Rev. Lett 45, 79 (1980).
[18] B. Derrida and H. Spohn, J. Stat. Phys. 51, 817 (1988).
[19] W. Krauth and M. Mézard, J. Phys. France 50, 3057 (1989).
[20] J. Houdayer and O. C. Martin, Phys. Rev. Lett. 81, 2554 (1998).
[21] D. J. Aldous and A. G. Percus, PNAS 100, 11211 (2003).