Steiner Decomposition Number of Complete $n$ –Sun graph

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Abstract: If each subgraph of the decomposition $\pi$ of the graph $G$ has the Steiner number same as $G$ then $\pi$ is said to be a Steiner decomposition of $G$. The maximum cardinality among the Steiner decomposition $\pi$ of $G$ is the Steiner decomposition number of $G$ and is denoted by $\pi_{st}(G)$. In this paper, we present the Steiner decomposition number for Complete $n$ –Sun graph.

Keywords: Steiner number, Steiner decomposition number, Complete $n$ –Sun graph

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1. Introduction

Let $G = (V,E)$ be a simple, undirected, connected graph. Charland & Zhang introduced the concept of Steiner number of a graph.

Definition 1.1[1] Let $G$ be a connected graph. For a set $W \subseteq V(G)$, a tree contained in $G$ is a Steiner tree with respect to $W$ if $T$ is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all vertices in $G$ that lie on some Steiner tree with respect to $W$. The set $W$ is a Steiner set for $G$ if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of $G$ is the Steiner number $s(G)$.

Definition 1.2[4] If $G_1, G_2, ..., G_n$ are connected edge-disjoint subgraphs of $G$ with $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$, then $\pi = \{G_1, G_2, ..., G_n\}$ is said to be a decomposition of $G$.

In [6] we introduced the concept Steiner decomposition number of graphs.

Definition 1.3[6] Let $\pi = \{G_1, G_2, ..., G_n\}$ be a decomposition of a graph $G$. If $s(G) = s(G_i), (1 \leq i \leq n)$ then $\pi$ is said to be a Steiner decomposition of $G$. The maximum cardinality of Steiner decomposition $\pi$ of $G$ is called the Steiner decomposition number of $G$ and is denoted as $\pi_{st}(G)$. This paper investigates and presents the Steiner decomposition number of Complete $n$ –Sun graph.

2. Steiner decomposition number of Complete $n$ –Sun graph

Definition 2.1 Complete $n$ –Sun graph is a graph on $2n$ vertices consisting of a central complete graph $K_n$ with an outer ring of $n$ vertices, each of which is joined to both endpoints of the closest outer edge of $K_n$. Let us denote the Complete $n$ –Sun graph as $CS_n$.

Definition 2.2 A vertex of degree one is called a pendant vertex and its incident edge is said to be a pendant edge. The vertex adjacent to the pendant vertex is called as support vertex.

Definition 2.3 $B_{m,n} (m, n \geq 2)$ denotes the graph bistar which is obtained by joining the central vertices of stars $K_{1,m}$ and $K_{1,n}$ with an edge.
Definition 2.4 \(U_{3,k}\) denote a unicyclic graph created from the cycle \(C_3\) by attaching \(k\) pendant edges to a vertex of \(C_3\).

Remark 2.5 Let \(G\) be a graph with \(s(G) \geq 3\). Each subgraph in the Steiner decomposition \(\pi\) of \(G\) has at least \(s(G)\) number of edges.

Theorem 2.6 Steiner decomposition number of \(CS_n\) \((n \geq 3)\) is

\[
\pi_{st}(CS_n) = \begin{cases} 
3 & \text{if } n = 3 \\
\frac{n+1}{2} & \text{if } n \text{ odd and } n > 3 \\
\frac{n}{2} + 1 & \text{if } n \text{ even}
\end{cases}
\]

Proof:

Let \(V(CS_n) = \{v_r/1 \leq r \leq n\} \cup \{u_r/1 \leq r \leq n\}\) where \(u_r, 1 \leq r \leq n\) the vertices on the outer ring of \(CS_n\). Let \(E(CS_n) = \{v_r v_s/r \neq s\} \cup \{u_r v_r/1 \leq r \leq n\} \cup \{u_r v_{r+1}/1 \leq r \leq n-1\} \cup \{u_nv_1\}\). The minimum Steiner set of \(CS_n = \{u_r/1 \leq r \leq n\}\) and so \(s(CS_n) = n\).

Case 1: \(n = 3\)

\(CS_3\) can be decomposed into either three copies of \(K_3\) or three copies of \(K_{1,3}\). Since \(s(K_3) = s(K_{1,3}) = 3 = s(CS_3)\), both decompositions are Steiner decompositions and these are the Steiner decompositions of maximum cardinality. Hence \(\pi_{st}(CS_3) = 3\).

Case 2: \(n\) odd and \(n > 3\)

Subcase 1: \(n = 5\)

A decomposition \(\pi\) of \(CS_5\) is given in figure 1.

Since \(s(G_1) = s(G_2) = s(G_3) = 5 = s(CS_5)\), \(\pi = \{G_1, G_2, G_3\}\) is a Steiner decomposition.

Moreover, it is easily verified that \(\pi\) is the Steiner decomposition of maximum cardinality and so \(\pi_{st}(CS_5) = 3\).
Subcase 2: \( n > 5 \)

Decompose \( CS_n \) as follows:

Step 1:

Construct a subgraph \( K_{1,n} \) of \( CS_n \) by considering \( v_n \) as central vertex and \( v_2, v_3, ..., v_{n-2}, v_{n-1}, u_{n-1}, u_n \) as pendant vertices. The Steiner number of \( K_{1,n} \) is \( n \).

Step 2:

Construct a subgraph \( U_{3,n-2} \) from \( CS_n \) by taking vertices \( v_1, v_{n-2}, v_{n-1} \) as vertices of the cycle and \( v_2, v_3, ..., v_{n-3}, u_{n-2}, u_{n-1} \) as \((n-2)\) pendant vertices attached to the vertex \( v_{n-1} \). The minimum Steiner set of \( U_{3,n-2} = \{ v_r / 1 \leq r \leq n-2 \} \cup \{ u_{n-2}, u_{n-1} \} \) and hence \( s(U_{3,n-2}) = n \).

Step 3:

Construct \( \binom{n-5}{2} \) copies of bistar \( B_{\frac{n+1}{2}, \frac{n-1}{2}} \) as given below:

Let \( S_i = \{ v_i, v_{n-1+i} \} \), \( 1 \leq i \leq \frac{n-5}{2} \). Each \( S_i \) denote the set which consists of the central vertices of \( i^{th} \)

While constructing \( B_{\frac{n+1}{2}, \frac{n-1}{2}} \),

- \( v_1 \) is the support vertex of \( v_{n-1}, v_{n-2}, ..., v_{n-3}, v_n, u_n, u_1 \).
- \( v_i, 2 \leq i \leq \frac{n-5}{2} \), is the support vertex of \( v_{n-1+i}, v_{n-2+i+2}, ..., v_{n-2}, v_{n-1+i}, v_1, v_2, ..., v_{i-1}, u_{i-1}, u_i \).
- \( v_{n-1+i}, 1 \leq i \leq \frac{n-5}{2} \), is the support vertex of \( v_{i+1}, v_{i+2}, ..., v_{n-1+i-2}, u_{n-1+i-2}, u_{n-1+i-1} \).

The Steiner number of \( B_{\frac{n+1}{2}, \frac{n-1}{2}} = n \).

Step 4:

Construct the edge induced subgraph from the remaining \( \frac{3(n+1)}{2} \) edges. This results in the following graph \( G^* \).

![Figure 2: G*](image)
Minimum Steiner set of $G^* = \{v_1, v_2, \ldots, v_{n-5}, u_{n-5}, u_{n-3}, u_{n-2}, u_{n-3}, u_{n-4}, v_2, \ldots, v_{n-3}, v_2, \ldots\}$ and hence $s(G^*) = n$. Thus $CS_n$ gets decomposed into a star graph $K_{1,n}$, a copy of $U_{3,n-2}$, $(\frac{n-5}{2})$ copies of bistar $B_{\frac{n+1}{2}, \frac{n-1}{2}}$ and a copy of graph $G^*$. Let this decomposition of $CS_n$ be denoted as $\pi$. Since $s(K_{1,n}) = s(U_{3,n-2}) = s(B_{\frac{n+1}{2}, \frac{n-1}{2}}) = s(G^*) = n = s(CS_n)$, $\pi$ is a Steiner decomposition. Also the cardinality of the Steiner decomposition $\pi$ is $\frac{n+1}{2}$.

Now to prove $\pi$ is the Steiner decomposition of maximum cardinality. If not $\pi_{st}(CS_n) = \frac{n+3}{2}$. This implies that each subgraph in the decomposition is a star graph $K_{1,n}$. But this contradicts to the fact that atmost three disjoint copies of $K_{1,n}$ only can be obtained in any decomposition of $CS_n$. Therefore $\pi_{st}(CS_n) \neq \frac{n+3}{2}$. Hence $\pi$ is the Steiner decomposition of maximum cardinality and so $\pi_{st}(CS_n) = \frac{n+1}{2}$.

**Case 3: $n$ even**

**Subcase 1: $n = 4$**

A decomposition $\pi$ of $CS_4$ is given in figure 3.

![Diagram of CS4](image)

**Figure 3: Steiner decomposition of CS4**

Since $s(G_1) = s(G_2) = s(G_3) = 4 = s(CS_4)$, $\pi = \{G_1, G_2, G_3\}$ is a Steiner decomposition. Also it is the Steiner decomposition of maximum cardinality and so $\pi_{st}(CS_4) = 3$.

**Subcase 2: $n > 4$**

Construct a decomposition $\pi$ of $CS_n$ by following the steps given.

**Step 1:**

Similar to step 1 of subcase 2 in case 2.

**Step 2:**

Similar to step 2 of subcase 2 in case 2.
Step 3:

Obtain \( \left( \frac{n}{2} - 1 \right) \) copies of \( B_{\frac{n}{2}} \) from \( CS_n \) as given below:

Let \( S_i = \{ v_i, v_{\frac{n}{2}+i-1} \}, \; 1 \leq i \leq \frac{n}{2} - 1 \). Each \( S_i \) denote the set which consists of the central vertices of \( i^{th} \)
copy of \( B_{\frac{n}{2}} \).

While constructing \( B_{\frac{n}{2}} \),

- \( v_1 \) is the support vertex of \( v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, ..., v_{n-3}, v_n, u_n, u_1 \).
- \( v_{\frac{n}{2}+2}, ..., v_{\frac{n}{2}-2}, u_{\frac{n}{2}-2}, u_{\frac{n}{2}-1} \) is the support vertex of \( v_1, v_2, ..., v_{\frac{n}{2}-2}, u_{\frac{n}{2}-2}, u_{\frac{n}{2}-1} \).
- \( v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, ..., v_{\frac{n}{2}+i-2}, u_{\frac{n}{2}+i-2}, u_{\frac{n}{2}+i-1} \) is the support vertex of \( v_{i+1}, v_{i+2}, ..., v_{\frac{n}{2}+i-2}, u_{\frac{n}{2}+i-2}, u_{\frac{n}{2}+i-1} \).

The Steiner number of \( B_{\frac{n}{2}} \) is \( n \).

The decomposition \( \pi \) of \( CS_n \) consists of a star graph \( K_{1,n} \), a copy of \( U_{3,n-2} \) and \( \left( \frac{n}{2} - 1 \right) \) copies of bistar \( B_{\frac{n}{2}} \). Also \( s(K_{1,n}) = s(U_{3,n-2}) = s \left( B_{\frac{n}{2}} \right) = n = s(CS_n) \). Hence \( \pi \) is a Steiner decomposition and the cardinality of \( \pi \) is \( \left( \frac{n}{2} \right) + 1 \).

Now to prove \( \pi_{st}(CS_n) = \frac{n}{2} + 1 \). If \( \pi_{st}(CS_n) = \left( \frac{n}{2} \right) + m \) where \( m \geq 2 \) then the requirement of edges in \( CS_n \geq \frac{n(n+2m)}{2} \). This is not possible since the required edges exceeds the number of edges of \( CS_n \). Hence \( \pi \) is a Steiner decomposition of maximum cardinality and so \( \pi_{st}(CS_n) = \frac{n}{2} + 1 \).

Thus Steiner decomposition number of \( CS_n \) \( (n \geq 3) \) is

\[
\pi_{st}(CS_n) = \begin{cases} 
3 & \text{if } n = 3 \\
\frac{n+1}{2} & \text{if } n \text{ odd and } n \geq 3 \\
\frac{n}{2} + 1 & \text{if } n \text{ even}
\end{cases}
\]

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