Barycentric Bak-Sneppen model

Philip Kennerberg* and Stanislav Volkov*

June 17, 2019

Abstract

We study the behaviour of the interacting particle system, arising from the Bak-Sneppen model and Jante’s law process. Let \( N \) vertices be placed on a circle, such that each vertex has exactly two neighbours. To each vertex assign a real number, called fitness. Now find the vertex which fitness deviates most from the average of the fitnesses of its two immediate neighbours (in case of a tie, draw uniformly among such vertices), and replace it by a random value drawn independently according to some distribution \( \zeta \). We show that in case where \( \zeta \) is a uniform or a discrete uniform distribution, all the fitnesses except one converge to the same value.

Keywords: Bak-Sneppen model, Jante’s law process, interacting particle systems.
Subject classification: 60J05, 60K35, 91D10.

1 Introduction

The model we study in the current paper is a “marriage” between Jante’s law process and the Bak-Sneppen model.

Jante’s law process refers to the interacting particle model studied in [4] under the name “Keynesian beauty contest process”, and generalized in [6]. This model runs as follows. Fix an integer \( N \geq 3, \ d \geq 1 \), and some \( d \)-dimensional random variable \( \zeta \). Let the initial configuration consist of \( N \) arbitrary points in \( \mathbb{R}^d \). The process runs in discrete time according to the following algorithm: first, compute the centre of mass \( \mu \) of the given configuration of \( N \); then replace the point which is the most distant from \( \mu \) by a new \( \zeta \)-distributed point drawn independently each time. It was shown in [4] that if \( \zeta \) has a uniform distribution on the unit cube, then

*Centre for Mathematical Sciences, Lund University, Box 118 SE-22100, Lund, Sweden
all but one points converge to some random point in $\mathbb{R}^d$. This result was further generalized in [6], by allowing $\zeta$ to have an arbitrary distribution, and additionally removing not just 1, but $K \geq 1$ points chosen to minimize a certain functional. The term “Jante’s law process” was also coined in [6], to reflect that this process is reminiscent of the “Law of Jante” principle, which a pattern of group behaviour towards individuals within Scandinavian countries that criticises individual success and achievement as unworthy and inappropriate; in other words, it is better to be “like everyone else”. The origin of this “law” dates back to Aksel Sandemose [9]. Another modification of this model in one dimension, called the $p$-contest, was introduced in [4, 5] and later studied e.g. in [7]. This model runs as follows: fix some constant $p \in (0, 1) \cup (1, \infty)$, and replace the point which is the farthest from $p\mu$ (rather than $\mu$).

Pieter Trapman (2018, personal communications) suggested to study Jante’s law model with local interactions, thus making it very similar to the famous Bak-Sneppen (BK) model see e.g. [1]. In the BK model, $N$ species are located around a circle, and each of them is associated with a so-called “fitness”. The algorithm consists in choosing the least fit individual, and then replacing it and both of its two closest neighbours by a new species, with a new random and independent fitness. After a long time, there will be a minimum fitness, below which species do not survive. The model proceeds through certain events, called “avalanches”, until it reaches a state of relative stability where all fitnesses are above a certain threshold level. There is a version of the model where fitnesses take only values 0 and 1, but even this simplified version turns out to be notoriously difficult to analyse, see e.g. [8]. Some more recent results can be found in [2, 10].

The barycentric Bak-Sneppen model, or, equivalently, Jante’s law process with local interactions, is defined as follows.
Fix an integer $N \geq 3$, and let $S = \{1, 2, \ldots, N\}$ be the set of nodes uniformly spaced on a circle. At time $t$, each node $i \in S$ has a certain “fitness” $X_i(t) \in \mathbb{R}$; let $X(t) = (X_1(t), \ldots, X_N(t))$.

Next, for the vector $x = (x_1, \ldots, x_N)$, define

$$d_i(x) = \left| x_i - \frac{x_{i+1} + x_{i-1}}{2} \right|,$$

as the measure of “non-conformism” of the fitness at node $i$ (here and further we will use the convention that $N + 1 \equiv 1$, $N + 2 \equiv 2$, and $1 - 1 \equiv N$ for indices on $x$). Let also $d(x) = \max_{i \in S} d_i(x)$.

The process runs as follows. Let $\zeta$ be some fixed one-dimensional random variable. At time $t$, $t = 0, 1, 2, \ldots$, we chose the “least conformist node” $i$, i.e. the one maximizing $d_i(X(t))$ (in case of a tie, choose it randomly, with equal probabilities), and replace it by a $\zeta$-distributed random variable. Let $j(x)$ denote the index of such a node in the configuration $x = (x_1, \ldots, x_N)$, that is

$$d_{j(x)}(x) = d(x)$$

(see Figure 1). Also assume that all the coordinates of the initial configuration $X(0)$ lie in the support of $\zeta$. We are interested in the long-term dynamics of this process.

We start with a somewhat easier case, where $\zeta$ takes finitely many distinct values (Section 2), and then extend this result to the case where $\zeta \sim U[0, 1]$ (Section 3). We will show that in both cases all the fitnesses (except the one which has just been updated) converge to the same (random) value.

**Remark 1.** One can naturally extend this model to any connected non-oriented graph $G$ with vertex set $V$, as follows. For any two vertices $v, u \in V$ that are connected by an edge we write $u \sim v$. To each vertex $v$ assign a fitness $x_v \in \mathbb{R}$, and define the measure of non-conformity of this vertex as

$$d_v(x) = \left| x_c - \frac{\sum_{u : u \sim v} x_u}{N_v} \right|,$$

where $N_v = |u \in V : u \sim v|$ denotes the number of neighbours of $v$, and the replacement algorithm runs exactly as it is described earlier.

In particular, if $G$ is a cycle graph, we obtain the model studied in the current paper. On the other hand, if $G$ is a complete graph, we obtain the model equivalent to that studied in [4, 6].

**Remark 2.** Unfortunately, our results cannot be extended to a general model, described in Remark 1. Indeed, assume that $\text{supp} \, \zeta = \{0, 1\}$. It is not hard to show that if for some $v$ we have $N_v = 1$, then the statement of Theorem 1 does not have to hold.

Moreover, it turns out that even when all the vertices have at least two neighbours (i.e., $N_v \geq 2$ for all $v \in V$), then there are still counterexamples: please see Figure 3.
Figure 2: On this graph with $N = 6$ vertices, only values $x$ and $y \in \{0, 1\}$ are updated all the time; infinitely often half of the fitnesses equal 0, while the other half equals 1.

Figure 3: Transition probabilities within the recurrent class $D_L$.

2 Discrete case

In this Section we study the case when fitnesses take finitely many values, equally spaced between each other. Due to the shift- and scale-invariance of the model, without loss of generality we may assume that $\text{supp } \zeta = \{1, 2, \ldots, M\} =: \mathcal{M}$, and that $p = \min_{j \in \mathcal{S}} \mathbb{P}(\zeta = j) > 0$. In this case $X(t)$ becomes a finite state-space Markov chain on $\mathcal{M}^N$.

Note that if $N - 1$ fitnesses coincide and are equal to some $L \in \mathcal{M}$, then it is the fitness that differs from $L$ that will keep being replaced, until it finally coincides with the others. When this happens, we will have to choose randomly among all the vertices, and replace its fitness. The replaced fitness may or may not differ from $L$, and then this procedure will repeat over and over again.

Formally, let

$$D^1_L = \{x \in \mathcal{M}^N : x_1 = x_2 = \cdots = x_{i-1} = x_{i+1} = x_N = L, \ x_i \neq L\}$$
$$D^*_L = \{x \in \mathcal{M}^N : x_1 = x_2 = \cdots = x_{i-1} = x_i = x_{i+1} = x_N = L\}$$

Then

$$D_L = D^*_L \cup \bigcup_{i \in \mathcal{S}} D^i_L$$
forms a recurrent class, with the transitions depicted in Figure 3. We will show that the $D_Ls$ ($L = 1, 2, \ldots, M$) are the only recurrent classes of the chain.

Remark 3. The fact that the values of $\zeta$ are equally spaced is, surprisingly, crucial. Let supp$\zeta = \{0, 1, 5, 6\} =: \mathcal{M}$ and $N = 8$. Then the set of configurations

$$[0, 1, x, 5, 6, 5, y, 1], \quad x, y \in \mathcal{M}$$

is stable; the maximum distance from the average of the fitnesses of the neighbours is always at nodes 3 or 7, and it equals 2 or 3, while the other distances are at most 1.5 or 2 respectively.

Theorem 1. Let $\hat{D} = \bigcup_{L \in \mathcal{M}} D_L$ and $\tau = \inf\{t \geq 0 : X(t) \in \hat{D}\}$. Then

$$\mathbb{P}(\tau < \infty | X(0) = x) = 1$$

regardless of the starting configuration $x \in \mathcal{M}^N$. Moreover, $X(t) \in \hat{D}$ for all $t \geq \tau$.

We say that the process has converged by time $t$, if $X(t) \in \hat{D}$. Then the Theorem will immediately follow from the next statement and from the fact that $\hat{D}$ is a union of the recurrent classes.

Lemma 1. The process converges within at most $T = (N-1)^2M^2$ steps, with a positive probability depending on $p, n, M$ only. More precisely,

$$\mathbb{P}(X(T) \in \hat{D} | X(0) = x) \geq \left(\frac{p}{N}\right)^T$$

for any $x \in \mathcal{M}^N$.

Proof. Recall that all vertex indices are to be understood modulo $N$, e.g. $X_{N+1}(t) = X_1(t)$. Let us define $f(t) = \sum_{i=1}^{N}(X_i(t) - X_{i+1}(t))^2$. Then $f(t) = 0$ if and only if all the fitnesses coincide; in this case $X(t) \in \bigcup_{L \in \mathcal{M}} D_L^* \subseteq \hat{D}$.

Claim 1. $f(t) = 0$ if and only if $d(t) = 0$.

Proof. Let $X(t) = x = (x_1, \ldots, x_N)$. If $f(t) = 0$, then $x_i \equiv x_1$ for all $i \in S$ and hence $d_i(x) = 0$ for all $i \in S \iff d(t) = 0$.

On the other hand, suppose that $d_i(x) = 0$ for all $i$. If not all $x_i$’s are equal, there must be an index $j$ such for which $x_j = \max_{i \in S} x_i$ and either $x_{j-1} \neq x_j$ or $x_{j+1} \neq x_j$. This, in turn, implies that

$$2d_j(x) = |(x_j - x_{j-1}) + (x_j - x_{j+1})| = (x_j - x_{j-1}) + (x_j - x_{j+1}) > 0$$

yielding a contradiction. □
If $X_i(t)$ is replaced by some $X_i(t+1) = a$ then

$$f(t + 1) - f(t) = (X_{i-1}(t) - a)^2 + (a - X_{i+1}(t))^2 - (X_{i-1}(t) - X_i(t))^2 - (X_i(t) - X_{i+1}(t))^2. \tag{2.1}$$

Claim 2. Suppose that $X(t) = x = (x_1, \ldots, x_N)$ and

$$X(t+1) = x' = (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_N) \text{ where } a = \left\lfloor \frac{x_{i-1} + x_{i+1}}{2} \right\rfloor.$$ 

Then $f(t+1) \leq f(t)$. Moreover if $d_i(x) \geq 1$ then $f(t+1) \leq f(t) - 1$.

Proof. It follows from (2.1) that

$$\frac{f(t + 1) - f(t)}{2} = (a - x_i)(a + x_i - x_{i-1} - x_{i+1}) = \left(a - \frac{x_{i-1} + x_{i+1}}{2}\right)^2 - \left(x_i - \frac{x_{i-1} + x_{i+1}}{2}\right)^2 = d_i(x')^2 - d_i(x)^2.$$

Since

$$d_i(x) = \left|x_i - \frac{x_{i-1} + x_{i+1}}{2}\right| \geq \left|a - \frac{x_{i-1} + x_{i+1}}{2}\right| = \begin{cases} 0, & \text{if } x_{i-1} + x_{i+1} \text{ is even}, \\ 1/2, & \text{if } x_{i-1} + x_{i+1} \text{ is odd}, \end{cases}$$

and $f$ takes only integer values, we get the required result.

Let $j_t \in S$ be the index such that

(a) $d_{j_t}(X(t)) = d(X(t))$, i.e., the fitness at $j_t$ is a possible candidate for replacement;

(b) $X_{j_t}(t)$ is (one of) the largest among such fitnesses.

If there is more than one index satisfying (a) and (b), choose one of them arbitrarily. Let

$$A_t = \left\{ X_{j_t}(t) \text{ is replaced by } X_{j_t}(t+1) = \left\lfloor \frac{X_{j_t-1}(t) + X_{j_t+1}(t)}{2} \right\rfloor \right\},$$

and let $B_s = \bigcap_{t=0}^{N-2} A_{s+t}$. Observe that

$$\mathbb{P}(B_s \mid F_s) \geq \left(\frac{p}{N}\right)^{N-1}. \tag{2.2}$$

Claim 3. If $f(s) > 0$ then $f(s + N - 2) \leq f(s) - 1$ on $B_s$. 
Proof. Note that $d(x)$ can take only values $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$. W.l.o.g. we assume that $s = 0$. There are three possibilities if $B_s$ occurs.

(a) If $d(X(t)) \geq 1$ for some $0 \leq t < N - 2$, then $f(t + 1) \leq f(t) - 1$ by Claim 2 and by monotonicity $f(N - 2) \leq f(t + 1) \leq f(t) - 1 \leq f(0) - 1$.

(b) If $d(X(t)) = 0$ for some $0 \leq t < N - 2$, then $f(t) = 0$ by Claim 1 and again by monotonicity $f(N - 2) \leq 0 = f(t) \leq f(0) - 1$ since $f(0) \geq 1$.

(c) If $d(X(s + t)) = \frac{1}{2}$ for all $0 \leq t < N - 2$, then the set $M(t) = \{j \in S : X_j(t) = \max_{i \in S} X_i(t)\}$ must have between 2 and $N - 2$ elements (a single maximum would imply $d(t) \geq 1$, the same holds if there are $N - 1$ coinciding maxima; finally, $\text{card}(M(t)) = N$ would imply that $d(t) = 0$). However, on $A_t$ we have $M(t + 1) \subseteq M(t)$ and $\text{card}(M(t + 1)) = \text{card}(M(t)) - 1$, and $M(0) \leq N - 2$, hence this case is impossible.

Let $L = M^2(N - 1)$ and

$$C = \bigcap_{t=0}^{L-1} B_t(N-1) = \bigcap_{t=0}^{L(N-1)-1} A_t = \bigcap_{t=0}^{T-1} A_t.$$

Since trivially $f(0) \leq L$, by Claim 3 on $C$ we see that $f(L) = 0$; hence

$$C \subseteq \{X((N - 1)^2M^2) \in \bar{D}\} \equiv \{X(T) \in \bar{D}\}.$$

Moreover,

$$\mathbb{P}(C \mid X(0) = x) = \mathbb{P}(B_{(L-1)(N-1)} \mid B_{(L-2)(N-1)}, \ldots, B_0, X(0) = x) \times \mathbb{P}(B_{(L-2)(N-1)} \mid B_{(L-3)(N-1)}, \ldots, B_0, X(0) = x) \times \ldots \times \mathbb{P}(B_{N-1} \mid B_0, X(0) = x) \times \mathbb{P}(B_0 \mid X(0) = x) \geq \left(\left(\frac{p}{N}\right)^{(N-1)^2M^2}\right)^L = \left(\frac{p}{N}\right)^{(N-1)^2M^2}$$

by (2.2). As a result,

$$\mathbb{P}\left(X(T) \in \bar{D} \mid X(0) = x\right) \geq \mathbb{P}(C \mid X(0) = x) \geq \left(\frac{p}{N}\right)^{(N-1)^2M^2}.$$
3 Continuous case

Throughout this section, we assume that $\zeta \sim U[0,1]$, and $X_i(t) \in [0,1]$ for all $i \in S$ and $t = 0, 1, 2, \ldots$.

**Theorem 2.** There exists a.s. a random variable $\bar{X} \in [0,1]$ such that as $t \to \infty$

$$(X_1(t), X_2(t), \ldots, X_{j(X(t)) - 1}(t), X_{j(X(t)) + 1}(t), \ldots, X_N(t)) \to (\bar{X}, \bar{X}, \ldots, \bar{X}) \in [0,1]^{N-1} \text{ a.s.}$$

A weaker statement, which will follow from Lemma 8, is

**Proposition 1.** As $t \to \infty$, $|X_i(t) - X_{i+1}(t)| \cdot 1_{j(X(t)) \notin \{i, i+1\}} \to 0$ a.s.

It turns out that it is much easier to work with the embedded process, for which either

the non-conformity of the node at which the value is replaced, is smaller than the initial non-conformity, or at least

the location of the “worst” node (i.e. the one where $d_i$ is the largest) has changed, whichever comes first. Formally, let $\nu_0 = 0$ and recursively define for $k = 0, 1, 2, \ldots$

$$\nu_{k+1} = \inf \{t > \nu_k : j(X(t)) \neq j(X(\nu_k)) \text{ or } d(X(t)) < d(X(\nu_k))\}$$

Note that all $\nu_k$ are finite a.s.

**Examples:**

(a) $x = (\ldots 0.5, 0.6, 0.5, 0.3, \ldots)$. The “worst” node is the second one (with the fitness of 0.6) and $d = d_2(x) = 0.1$; it is replaced, say, by 0.32. Now the configuration becomes

$$x' = (\ldots, 0.5, 0.32, 0.5, 0.3, \ldots)$$

and the worst node is the third one with $d(x') = d_3(x') = 0.19 > 0.1 = d(x)$;

(b) $x$ is the same as in (a), but $x_2$ is replaced by 0.58. Now the configuration becomes

$$x = (\ldots, 0.5, 0.58, 0.5, 0.3, \ldots)$$

and the worst node is still the second one with $d(x') = d_2(x') = 0.08 < 0.1 = d(x)$.

Now let $\tilde{X}(s) = X(\nu_s)$ and $\mathcal{F}_s = \sigma(\tilde{X}(1), \ldots, \tilde{X}(s))$ be the filtrations associated with this embedded process. Since for $t \in [\nu_k, \nu_{k+1})$ we have

$$X_i(t) = X_i(\nu_k) \text{ for all } i \neq j(X(t))$$

one can see that Theorem 2 follows immediately from
Theorem 3. There exists a.s. a random variable \( \bar{X} \in [0, 1] \) such that as \( s \to \infty \)

\[
(X_1(s), X_2(s), \ldots, X_N(s)) \to (\bar{X}, \bar{X}, \ldots, \bar{X}) \in [0, 1]^N \quad \text{a.s.}
\]

We will use the Lyapunov functions method, with a clever choice of the function. For \( x = (x_1, x_2, \ldots, x_N) \) define

\[
h(x) = 2 \cdot \sum_{i \in S} (x_i - x_{i+1})^2 + \sum_{i \in S} (x_i - x_{i+2})^2 = 2 \sum_{i \in S} (3x_i^2 - 2x_ix_{i+1} - x_ix_{i+2}).
\]

Lemma 2. \( \xi(s) = h \left( \bar{X}(s) \right) \) is a non-negative supermartingale.

Proof. The non-negativity of \( \xi(s) \) is obvious. To show that it is a supermartingale, assume w.l.o.g. that \( j(X(s)) = 3 \), and let \( \bar{X}(s) = (x_1, x_2, x_3, x_4, x_5, \ldots) \). Assuming that the allowed range (i.e., for which either \( d \) decreases or the location of the minimum changes) for the newly sampled point is \([a, b] \subseteq [0, 1] \), we have

\[
\Delta := (b - a) \mathbb{E}(\xi(s + 1) - \xi(s)|\mathcal{F}_s) = \int_a^b \{2(x_2 - u)^2 + 2(u - x_4)^2 + (x_1 - u)^2 + (u - x_5)^2
\]

\[\quad - \left[2(x_2 - x_3)^2 + 2(x_3 - x_4)^2 + (x_1 - x_3)^2 + (x_3 - x_5)^2\right]\} \, du \quad (3.3)
\]

\[= 2(a^2 + b^2 + ab) + (2x_3 - a - b)(x_1 + 2x_2 + 2x_4 + x_5) - 6x_3^2.
\]

Now we need to compute the appropriate \( a \) and \( b \), and then show that \( \Delta \leq 0 \).

W.l.o.g. we can assume that \( x_3 > \frac{x_2 + x_4}{2} \), the other case can be studied identically. When the fitness at node 3 is replaced by some value \( u \), let the new value of the non-conformity at node 3 be \( d_3' = d_3(x_1, x_2, u, x_4, x_5, \ldots) \).

- If \( x_3 \) is replaced by \( u > x_3 \), then this value will be “rejected”, in the sense that \( d \) has only increased while the \( \text{arg max}_{i \in S} d_i \) is still at the same node (i.e., 3). Indeed, when \( x_3 \) increases by some \( \delta > 0 \), so does \( d_3 \), while \( d_2 \) and \( d_4 \) can potentially increase only by \( \delta/2 \) and thus cannot overtake \( d_3 \).

- When \( u \in \left(\frac{x_2 + x_4}{2}, x_3\right) \), \( d_3' \) is definitely smaller than the original \( d_3 \).

Assume from now on that \( u \in \left(0, \frac{x_2 + x_4}{2}\right) \). When \( x_3 \) is replaced by \( u \), it might happen that while the new \( d_3 \) is larger than the original one, the value of \( d_2 \) or \( d_4 \) overtakes \( d_3 \).

- When \( u \in \left(0, \frac{x_2 + x_4}{2}\right) \) the condition that \( d_3' < d_3 \) is equivalent to

\[
\frac{x_2 + x_4}{2} - u < x_3 - \frac{x_2 + x_4}{2} \iff u > x_2 + x_4 - x_3 =: Q_0.
\]
• For $d_2$ to overtake $d_3$, we need

$$\left| x_2 - \frac{x_1 + u}{2} \right| > \frac{x_2 + x_4}{2} - u \iff \begin{cases} u > x_1 - x_2 + x_4 =: Q_1 \\ \text{or} \\ u > \frac{-x_1 + 3x_2 + x_4}{3} =: Q_2 \end{cases}$$

• For $d_4$ to overtake $d_3$, we need

$$\left| x_4 - \frac{u + x_5}{2} \right| > \frac{x_2 + x_4}{2} - u \iff \begin{cases} u > x_2 - x_4 + x_5 =: Q_3 \\ \text{or} \\ u > \frac{x_2 + 3x_4 - x_5}{3} =: Q_4 \end{cases}$$

As a result, the condition for $d_3$ to be overtaken by some other node, or $d'_3 < d_3$ is

$$u > \min_{j=0,1,2,3,4} Q_j.$$

Consequently, we must set

$$a = \max \left\{ 0, \min \{Q_0, Q_1, Q_2, Q_3, Q_4 \} \right\} = \max \left\{ 0, \min \left\{ x_2 + x_4 - x_3, x_1 - x_2 + x_4, \frac{-x_1 + 3x_2 + x_4}{3}, x_2 - x_4 + x_5, \frac{x_2 + 3x_4 - x_5}{3} \right\} \right\},$$

$$b = x_3.$$

Note that we are guaranteed that $a \leq b$. This is trivial when $a = 0$; on the other hand, when $a > 0$ we have

$$a \leq x_2 + x_4 - x_3 = \frac{x_2 + x_4}{2} - \left[ x_3 - \frac{x_2 + x_4}{2} \right] < \frac{x_2 + x_4}{2} < x_3 = b$$

since $x_3 > \frac{x_2 + x_4}{2}$.

By substituting $b = x_3$ into the expression for the drift (3.3), we get

$$\Delta = (x_3 - a)(x_1 + 2x_2 - 4x_3 + 2x_4 + x_5 - 2a)$$

and to establish $\Delta \leq 0$ it suffices to show

$$x_1 + 2x_2 - 4x_3 + 2x_4 + x_5 \leq 2a = 2 \max \left\{ 0, \min \{Q_0, Q_1, Q_2, Q_3, Q_4 \} \right\}$$

under the assumption that

$$x_3 - \frac{x_2 + x_4}{2} > \max \left\{ \left| x_2 - \frac{x_1 + x_3}{2} \right|, \left| x_4 - \frac{x_3 + x_5}{2} \right| \right\}.$$
that is, equivalently,
\[ x_3 > \max\{Q_1, Q_2, Q_3, Q_4\}. \] (3.5)

In order to show (3.4) we consider a number of cases. First, assume that \( x_2 + x_4 < x_3 \). Then \( Q_0 < 0 \) and \( a = 0 \). From (3.5) we get that \( 2x_3 > Q_1 + Q_3 = x_1 + x_5 \), thus
\[ x_1 + 2x_2 - 4x_3 + 2x_4 + x_5 = (x_1 + x_5 - 2x_3) + 2(x_2 + x_4 - x_3) < 0 = a \]
and (3.4) is fulfilled.

The next case is when \( \frac{x_2 + x_4}{2} < x_3 < x_2 + x_4 \). We need to verify if all of the following holds:
\[ x_1 + 2x_2 - 4x_3 + 2x_4 + x_5 \leq 0 \quad \text{subject to} \]
\[ Q_0 \geq 0, \ x_3 \geq Q_1 \geq 0, \ x_3 \geq Q_2 \geq 0, \ x_3 \geq Q_3 \geq 0, \ x_3 \geq Q_4 \geq 0 \]
and
\[ x_1 + 2x_2 - 4x_3 + 2x_4 + x_5 \leq 0 \quad \text{subject to} \]
\[ Q_j \leq 0, \ x_3 \geq Q_1, \ x_3 \geq Q_2, \ x_3 \geq Q_3, \ x_3 \geq Q_4 \]
for \( j = 0, 1, 2, 3, 4 \). This can be done using Linear Programming method. Thus \( \Delta \leq 0 \).

**Lemma 3.** Let \( x = (x_1, \ldots, x_N) \) and \( \Delta_i(x) := x_i - x_{i-1}, \ i \in S \). Then
\[ d(x) \leq \max_{i \in S} |\Delta_i| \leq N d(x), \]
\[ 2d(x)^2 \leq h(x) \leq 6N^3 d(x)^2. \]

**Proof.** Note that \( \Delta_1 + \cdots + \Delta_N = 0 \) and
\[ h(x) = \sum_{i \in S} \left[ 2\Delta_i^2 + (\Delta_i + \Delta_{i+1})^2 \right], \]
\[ d(x) = \frac{1}{2} \max_{i \in S} |\Delta_{i+1} - \Delta_i|. \]

Let \( j \) be such that \( d_j(x) = d(x) \), then by the triangle inequality
\[ |\Delta_{j+1}| + |\Delta_j| \geq |\Delta_{j+1} - \Delta_j| = 2d(x) \]
so at least one of the two terms on the LHS \( \geq d(x) \), hence \( \max_{i \in S} |\Delta_i| \geq d(x) \).

Now we will show that \( \max_{i \in S} |\Delta_i| \leq N d(x) \). Indeed, suppose that this is not the case, and w.l.o.g. \( \Delta_1 > N d(x) \). For all \( i \) we have \( |\Delta_{i+1} - \Delta_i| \leq 2d(x) \), hence by induction and the triangle
inequality we get
\[
\begin{align*}
\Delta_2 &> (N - 2) d(x), \\
\Delta_3 &> (N - 4) d(x), \\
\vdots \\
\Delta_{N-1} &> (N - 2(N - 2)) d(x), \\
\Delta_N &> (N - 2(N - 1)) d(x).
\end{align*}
\]
As a result, \( \Delta_1 + \Delta_2 + \cdots + \Delta_N > [N^2 - 2(1 + 2 + \cdots + (N - 1))] d(x) = Nd(x) \geq 0 \), which yields a contradiction, since the LHS is identically equal to 0.

Thus \( |\Delta_i| \leq Nd(x) \), and so \( |\Delta_i + \Delta_{i+1}| \leq 2Nd(x) \) for all \( i \in S \). Consequently, \( h(x) \leq 2N(Nd(x))^2 + N(2Nd(x))^2 = 6N^3d(x)^2 \). On the other hand, \( h(x) \geq \max_{i \in S} 2\Delta_i^2 \geq 2d(x)^2. \)

**Lemma 4.** Suppose that \( X(t) = x = (x_1, x_2, x_3, x_4, \ldots) \), and \( d_3(x) > \max \{d_2(x), d_4(x)\} \).

Let \( \mu = \frac{x_3 + x_4}{2} \) and \( \delta = |x_3 - \mu| = d_3(x) \). If \( x_3 \) is replaced by some \( u \in [\mu - \delta/6, \mu + \delta/6] \) then \( \Delta_h := h(X(t+1)) - h(X(t)) \leq -\frac{5}{6}\delta^2. \) (Note that the Lebesgue measure of \([\mu - \delta/6, \mu + \delta/6] \cap [0,1]\) is always at least \( \delta/6 \); also after this replacement \( d_3 \) must decrease.)

**Proof.** Note that the change in \( h \) equals
\[
\Delta_h = -2(x_3 - u)(3u + A), \quad \text{where } A = 3x_3 - x_1 - 2x_2 - 2x_4 - x_5.
\]
W.l.o.g. assume \( x_3 > \mu \). Then
\[
x_3 - u \geq \mu + \delta - \left( \mu + \frac{\delta}{6} \right) = \frac{5}{6}\delta.
\]
At the same time
\[
\min_{x_1, \ldots, x_5 \geq 0} A \quad \text{subject to} \quad x_3 - \mu > \max \left\{ \left| \frac{x_2 - \frac{1}{2}(x_1 + x_3)}{2} \right|, \left| x_4 - \frac{x_3 + x_5}{2} \right| \right\}
\]
equals \(-3\mu + \delta\). Hence
\[
3u + A \geq 3 \left( \mu + \frac{\delta}{6} \right) - 3\mu + \delta = \frac{\delta}{2}
\]
and thus \( \Delta_h \leq -\frac{5}{6}\delta^2 \cdot \frac{\delta}{2}. \)

**Lemma 5.** Suppose that \( X(t) = x = (x_1, x_2, x_3, x_4, \ldots) \), and \( d_3(x) = d(x) \). Let \( \mu = \frac{x_3 + x_4}{2} \) and \( \delta = |x_3 - \mu| = d_3(x) \). Given that \( x_3 > \mu \), if \( x_3 \) is replaced by some \( u \notin [\mu - 3\delta, x_3] \) then \( d_3\left( x' \right) > d_3(x) \) and \( d_3\left( x' \right) \) is still the largest of \( d_i(x') \), where \( x' = (x_1, x_2, u, x_4, x_5, \ldots) \). The same conclusion holds if \( x_3 < \mu \) and \( x_3 \) is replaced by some \( u \notin [x_3, \mu + 3\delta] \).
Before presenting the proof of Lemma 5, we state the obvious

**Corollary 1.** Let $\delta = d(\tilde{X}(s))$. If $i = j(\tilde{X}(s))$ then

$$\tilde{X}_i(s + 1) \in [\tilde{X}_i(s) - 4\delta, \tilde{X}_i(s) + 4\delta]$$

(and if $i \neq j(\tilde{X}(s))$ then trivially $X_i(s + 1) = X_i(s)$). Hence we always have

$$\max_{i \in S} |\tilde{X}_i(s + 1) - \tilde{X}_i(s)| \leq 4\delta.$$

The next implication of Lemma 5 requires a bit of work.

**Corollary 2.** Let $\rho = 1 - \frac{5}{36N^2} < 1$. Then

$$\mathbb{P}(\xi(s + 1) < \rho \xi(s) \mid \mathcal{F}_s) \geq \frac{1}{48}.$$

**Proof of Corollary 2.** From Corollary 1 we know that given $x = \tilde{X}(s)$, the allowed range for the newly sampled point to be in $\tilde{X}(s + 1)$ is at most $8\delta$ where $\delta = d(x)$. At the same time if the newly sampled point falls into the interval $[\mu - \delta/6, \mu + \delta/6]$ (see Lemma 5), at least half of which lies in $[0, 1]$, then $\xi(s + 1) - \xi(s) \leq -\frac{5}{6}\delta^2$; the probability of this event is no less than $\frac{\delta/6}{8\delta} = \frac{1}{48}$. Since $\xi(s) = h(x)$ and by Lemma 3 we have $d(x)^2 \geq \frac{h(x)}{6N}$, the inequality $\xi(s + 1) - \xi(s) \leq -\frac{5}{6}\delta^2$ implies $\xi(s + 1) - \xi(s) \leq -\frac{5}{36N^2}\xi(s)$.

**Proof of Lemma 5.** By symmetry, it suffices to show just the first part of the statement. First, observe that

$$d_j(x') = d_j(x) \leq d_3(x) \text{ for } j \in S \setminus \{2, 3, 4\};$$

$$d_2(x') = \left| \left( \frac{x_1 + x_3}{2} - x_2 \right) + \frac{u - x_3}{2} \right| \leq d_2(x) + \left| \frac{u - x_3}{2} \right| \leq d_3(x) + \left| \frac{u - x_3}{2} \right|. \quad (3.6)$$

If $u > x_3 > \mu$, then from (3.6)

$$d_3(x') = u - \frac{x_2 + x_4}{2} > x_3 - \frac{x_2 + x_4}{2} = d_3(x);$$

$$d_2(x') \leq d_3(x) + \left| \frac{u - x_3}{2} \right| = d_3(x') - (u - x_3) + \left| \frac{u - x_3}{2} \right| = d_3(x') - \left| \frac{u - x_3}{2} \right| < d_3(x');$$

$$d_4(x') < d_3(x') \quad \text{(by the same argument as } d_2)$$

so indeed $d_3(x) < d_3(x') = \max_{i \in S} d_i(x')$. 

13
On the other hand, if $u < \mu - 3\delta < x_3 = \mu + \delta$, then $d_j$ for $j \in S \setminus \{2, 3, 4\}$ still remain unchanged, but

\[
d_3(x') = \mu - u > 3\delta > d_3(x);
\]

\[
d_2(x') \leq d_3(x) + \left| \frac{u - x_3}{2} \right| = \delta + \frac{x_3 - u}{2} = \delta + \frac{x_3 - \mu}{2} + \frac{\mu - u}{2} = \frac{3\delta}{2} + \frac{\mu - u}{2};
\]

\[
d_4(x') < d_3(x') \quad \text{(by the same argument as } d_2)\]

hence $d_3(x) < d_3(x') = \max_{i \in S} d_i(x')$ in this case as well.

\[\square\]

**Lemma 6.** $\xi(s + 1)/\xi(s) \leq r$ for some $r > 0$.

**Proof.** By Corollary 1 it follows that the worst outlier (w.l.o.g. $x_3$) can be replaced only by a point at most at the distance $4\delta$ from $x_3$ at time $\nu_{s+1}$. Let the new value of the fitness at node 3 be $x_3 + v$, $|v| \leq 4\delta$. The change in the Lyapunov function is given by

\[
\xi(s + 1) - \xi(s) = [2((x_3 + v) - x_2)^2 + 2((x_3 + v) - x_4)^2 + ((x_3 + v) - x_1)^2 + ((x_3 + v) - x_5)^2] \\
- [2(x_3 - x_2)^2 + 2(x_3 - x_4)^2 + (x_3 - x_1)^2 + (x_3 - x_5)^2] \\
= (12x_3 - 2x_2 - 2x_4 - 4x_1 - 4x_5)v + 6v^2 \\
= (3.7)
\]

Since

\[
|12x_3 - 2x_2 - 2x_4 - 4x_1 - 4x_5| = \left| 8\left( x_2 - \frac{x_1 + x_3}{2} \right) + 8\left( x_4 - \frac{x_5 + x_3}{2} \right) + 20\left( x_3 - \frac{x_2 + x_4}{2} \right) \right| \\
\leq 8\delta + 8\delta + 20\delta = 36\delta
\]

from (3.7) and the fact that $\delta = d(\tilde{X}(s)) \leq \sqrt{\frac{\xi(s)}{2}}$ by Lemma 3

\[
|\xi(s + 1) - \xi(s)| \leq 36\delta \times 4\delta + 6(4\delta)^2 = 240\delta^2 \leq 120\xi(s),
\]

so we can take $r = 121$.

\[\square\]

**Lemma 7.** Fix a $k > 1$. Let $\tau_1 = \inf \{ s > 0 : \xi(t + s) < \xi(t)/k \}$ and $\tau_2 = \inf \{ s > 0 : \xi(t + s) > k\xi(t) \}$. Then $\tau = \min(\tau_1, \tau_2)$, given $\mathcal{F}_t$, is stochastically smaller than some random variable with a finite mean, the distribution of which does not depend on anything except $N$ and $k$.

**Proof.** Fix a positive integer $L$. Define

\[
B_t = \left\{ \xi(t + L) \leq \frac{\xi(t)}{k^2} \right\}.
\]

14
It suffices to show that \( P(B_t \mid \mathcal{F}_t) \geq p \) for some \( p > 0 \) uniformly in \( t \), since obviously
\[
B_t \cap \{ \tau_2 > L \} \subseteq \{ \tau_1 \leq L \}
\]
implying \( B_t \subseteq \{ \tau_2 \leq L \} \cup \{ \tau_1 \leq L \} = \{ \tau \leq L \} \). As a result, \( \tau \) is stochastically smaller than \( L \) multiplied by a geometric random variable with parameter \( p = p(N, k) \).

To show that \( P(B_t \mid \mathcal{F}_t) \geq p \), note that by Corollary 2,
\[
P(B^*_m \mid \mathcal{F}_{m-1}) \geq \frac{1}{48}, \quad \text{where } B^*_m = \{ \xi(m) < \rho \xi(m - 1) \}, \quad \rho = 1 - \frac{5}{36N^3}.
\]

Let \( L \) be so large that \( \rho^L < \frac{1}{k^2} \). Then, on one hand,
\[
\bigcap_{m=1}^{L} B^*_{t+m} \subseteq B_t \quad \text{whence} \quad P(B_t \mid \mathcal{F}_t) \geq P \left( \bigcap_{m=1}^{L} B^*_{t+m} \mid \mathcal{F}_t \right),
\]
while on the other hand
\[
P \left( \bigcap_{m=1}^{L} B^*_{t+m} \mid \mathcal{F}_t \right) \geq \frac{1}{48^L} =: p
\]
which depends on \( N \) and \( k \) only. \( \square \)

The proof of the next statement requires a bit more work than that of Lemma 2.4 in [4].

**Lemma 8.** \( \xi(s) \to 0 \) a.s. as \( s \to \infty \) (and as a result \( \Delta_i(\hat{X}(s)) \to 0 \) a.s. and \( d(\hat{X}(s)) \to 0 \) a.s. as \( s \to \infty \)).

**Proof.** From Lemma 2 it follows that \( \xi(s) \) converges a.s. to a non-negative limit, say \( \xi_\infty \). Let us show that \( \xi_\infty = 0 \). From Corollary 2 we have
\[
P(\xi(s + 1) \leq \rho \xi(s) \mid \mathcal{F}_s) \geq \frac{1}{48}. \quad (3.8)
\]

Fix an \( \varepsilon > 0 \) and a \( T \in \mathbb{N} \). Let \( \sigma_{\varepsilon,T} = \inf \{ s \geq T : \xi(s) \leq \varepsilon \} \). Then \( (3.8) \) implies
\[
P(A_{s+1} \mid \mathcal{F}_s) \geq \frac{1_{s < \sigma_{\varepsilon,T}}}{48}, \quad \text{where } A_{s+1} = \{ \xi(s + 1) \leq \xi(s) - (1 - \rho)\varepsilon \}
\]
(Compare this with the inequality (2.18) in [4]). From the non-negativity of \( \xi(s) \), we know that only finitely many of \( A_s \) can occur. By the Levy’s extension to the Borel-Cantelli lemma, we get that \( \sum_{s=T}^{\infty} P(A_{s+1} \mid \mathcal{F}_s) < \infty \) a.s., and hence \( \sum_{s=T}^{\infty} 1_{s < \sigma_{\varepsilon,T}} < \infty \). This, in turn, implies that \( \sigma_{\varepsilon,T} < \infty \) a.s. Consequently, since \( T \) is arbitrary,
\[
\lim \inf_{s \to \infty} \xi(s) \leq \varepsilon \quad \text{a.s.}
\]
Since \( \varepsilon > 0 \) is also arbitrary and \( \xi(s) \) converges, \( \lim_{s \to \infty} \xi(s) = \lim \inf_{s \to \infty} \xi(s) = 0 \) a.s. \( \square \)
The next general statement may be known, but since we could not find it in the literature, we present its fairly short proof.

**Proposition 2.** Suppose that \( \xi(s) \) is a positive bounded supermartingale with respect to a filtration \( F_s \). Suppose there is a constant \( r > 1 \) such that \( \xi(s+1) \leq r \xi(s) \) a.s. and that for all \( k \) large enough the stopping times

\[
\tau_s = \inf\{t > s : \xi(t) > k\xi(s) \text{ or } \xi(t) < k^{-1}\xi(s)\}
\]

are stochastically bounded above by some finite–mean random variable \( \bar{\tau} > 0 \), which depends on \( k \) only (and, in particular, independent of \( F_s \)). Then

\[
\limsup_{s \to \infty} \frac{\ln \xi(s)}{s} < 0 \quad \text{a.s.}
\]

i.e., \( \xi(s) \to 0 \) exponentially fast.

**Proof.** First, observe that by the Optional Stopping Theorem

\[
\mathbb{E}(\xi(\tau_s) | F_s) \leq \xi(s), \quad (3.9)
\]

while, on the other hand,

\[
\mathbb{E}(\xi(\tau_s) | F_s) = \mathbb{E}(\xi(\tau_s), \xi(\tau_s) > k\xi(s) | F_s) + \mathbb{E}(\xi(\tau_s), \xi(\tau_s) < k^{-1}\xi(s) | F_s)
\geq \mathbb{E}(\xi(\tau_s), \xi(\tau_s) > k\xi(s) | F_s) \geq k\xi(s) \cdot \mathbb{P}(\xi(\tau_s) > k\xi(s) | F_s). \quad (3.10)
\]

From (3.9) and (3.10) we conclude

\[
p := \mathbb{P}(\xi(\tau_s) > k\xi(s) | F_s) < \frac{1}{k}. \quad (3.11)
\]

Now let us define a sequence of stopping times as follows: \( \eta_0 = 0 \) and for \( n = 1, 2, \ldots, \)

\[
\eta_n = \inf\{s > \eta_{n-1} : \xi(s) > k\xi(\eta_{n-1}) \text{ or } \xi(s) < k^{-1}\xi(\eta_{n-1})\}
\]

and let

\[
N_s = \{n : \eta_n \leq s < \eta_{n+1}\} = \max\{n : \eta_n \leq s\}.
\]

From the definition of the stopping times \( \eta \), it follows

\[
\xi(s) \leq k\xi(\eta_s), \quad \xi(\eta_{n+1}) \leq rk\xi(\eta_n). \quad (3.12)
\]

Consider now the sequence of random variables \( \xi(\eta_n) \). From (3.11) and (3.12) we obtain that

\[
\log_k \frac{\xi(\eta_n)}{\xi(\eta_{n-1})}
\]

is stochastically bounded above by a random variable \( X_n \in \{-1, 1 + \log_k r\} \) such that

\[
1 - \mathbb{P}(X_n = -1) = \mathbb{P}(X_n = 1 + \log_k r) = \frac{1}{k}
\]

16
yielding
\[ \mathbb{E}X_n = \frac{2 + \ln r}{k} - 1 =: g(r, k); \]
we can also assume that \( X_n \) are i.i.d. One can choose \( k > 1 \) so large\(^1\) that \( g(r, k) < -\frac{1}{2} \). Then, by the Strong Law applied to \( \sum_{i=1}^{n} X_i \), we get
\[
\limsup_{n \to \infty} \frac{\log_k \xi(\eta_n)}{n} \leq \limsup_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} < -\frac{1}{2} \quad \text{a.s.}
\]

From the condition of the proposition we know that the differences \( \eta_n - \eta_{n-1}, n = 1, 2, \ldots, \) are stochastically bounded by independent random variables with the distribution of \( \bar{\tau} \) with \( \mathbb{E}\bar{\tau} =: \mu < \infty \). Then by the Strong Law for renewal processes (see e.g. [3], Theorem I.7.3) applied to the sum of independent copies of \( \bar{\tau} \), we get
\[
\liminf_{s \to \infty} \frac{N_s}{s} \geq \frac{1}{\mu} \quad \text{a.s.} \implies s \leq 2\mu N_s \text{ for all large enough } s. \tag{3.13}
\]

Combining (3.12) and (3.13), we get
\[
\limsup_{s \to \infty} \frac{\log_k \xi(s)}{s} \leq \limsup_{s \to \infty} \frac{\log_k (k\xi(\eta_{N_s}))}{s} = \limsup_{s \to \infty} \frac{\log_k \xi(\eta_{N_s})}{s} \\
\leq \limsup_{s \to \infty} \frac{\log_k \xi(\eta_{N_s})}{2\mu N_s} = \frac{1}{2\mu} \limsup_{n \to \infty} \frac{\log_k \xi(\eta_n)}{n} < -\frac{1}{4\mu} \quad \text{a.s.}
\]

since the numerators above are negative, and \( N_s \to \infty \) as \( s \to \infty \) a.s. \( \square \)

The next statement strengthens Lemma 8.

**Corollary 3.** \( \xi(s) \to 0 \) exponentially fast.

**Proof.** The statement follows immediately from Proposition that we have by Lemma 8 the other condition follows from Lemma 7. \( \square \)

**Proof of Theorem 8** According to Corollary 8 there exist \( a, b > 0 \) which are a.s. finite and such that \( \xi(t) \leq ae^{-bt} \). If we take \( s_0 \) such that \( ae^{-bs} \leq \epsilon \) for all \( s \geq s_0 \) then if \( s_0 \leq s < t \),
\[
|\tilde{X}(t) - \tilde{X}(s)| \leq \sum_{k=s+1}^{t} 4d(\tilde{X}(k)) \leq \sum_{k=s+1}^{t} \sqrt{8\xi(k)} \\
\leq \sqrt{8\epsilon} \sum_{k=s+1}^{t} e^{-bk/2} \leq \frac{\sqrt{8\epsilon}}{1 - e^{-b/2}}.
\]

\(^1\)if \( r > 4.1 \), then \( k = \ln(r) \) will be sufficient.
where we used Corollary 1 in the first inequality and Lemma 3 in the second inequality. We can thus conclude that \( \{ \bar{X}_i(t) \} \) is a Cauchy sequence in the a.s. sense; therefore the limit \( \bar{X}_i(\infty) = \lim_{t \to \infty} \bar{X}_i(t) \) exists a.s.

Furthermore, assuming w.l.o.g. that \( i < j \),

\[
|\bar{X}_i(\infty) - \bar{X}_j(\infty)| = \lim_{t \to \infty} |\bar{X}_i(t) - \bar{X}_j(t)| \leq \lim_{t \to \infty} \sum_{k=i+1}^{j} |\Delta_k(\bar{X}(t))| = 0
\]

by Lemma 3 which completes the proof.

\[ \square \]

4 Discussion and open problems

One may be interested in the speed of convergence, established in Theorem 3. In Lemma 6 we can take \( r = 121 \) and from the proof of Proposition 2, \( k = \ln r = \ln(121) = 2\ln(11) \) will be sufficient. Then, for Lemma 7, find \( L \) such that

\[
\left( 1 - \frac{5}{36N^3} \right)^L < \frac{1}{23} < \frac{1}{k^2}
\]

We can take, e.g.,

\[
L \approx 7.2N^3 \cdot \ln(23) \approx 22.6N^3
\]

This, in turn, will provide a bound on \( \mu = \mathbb{E} \bar{\tau} \leq \frac{L}{\rho} = L \cdot 48L \) for Proposition 2 and hence the speed of the convergence for large \( s \):

\[
2 [d(\bar{X}(s))]^2 \leq h(\bar{X}(s)) = \xi(s) \leq k^{-\frac{s}{48}} \leq \exp \left\{ -\frac{s}{8L48L \ln(11)} \right\} \approx \exp \left\{ -\frac{s}{433 \cdot 10^{38N^3}} \right\}
\]

This bound is, however, far from the optimal one. The simulations seem to indicate that, depending on \( N \),

\[
\xi(s) \sim e^{-\rho_N s},
\]

where e.g. \( \rho_5 \in (0.47, 0.77) \), \( \rho_{10} \in (0.14, 0.23) \), \( \rho_{20} \in (0.02, 0.03) \), \( \rho_{40} \in (0.003, 0.006) \), suggesting that (a) \( \rho_N \) can be, in fact, random, and (b) the average value of \( \rho_N \) decays roughly like \( 5/N^2 \).

We leave the study of the properties of \( \rho_N \) for further research.

We believe that the convergence, described by Theorems 2 and 3 holds for a much more general class of replacement distributions \( \zeta \), not just uniform; for example, for the continuous distributions with the property that their density is uniformly bounded away from zero. Unfortunately, our proof is based on the construction of the Lyapunov function which cannot be easily transferred to other cases (obviously, it will work for any \( \zeta \sim U[a, b] \), where \( a < b \).
One can also attempt to generalize the theorems for more general graphs as described in Remark 1; this should be done, however, with care, as it will not work for all the distributions (see Remark 2).

References

[1] Bak, P. and Sneppen, K. (1993). Punctuated equilibrium and criticality in a simple model of evolution. Physical Review Letters 71, 4083–4086.

[2] Ben-Ari, I., Silva, R. (2018). On a local version of the Bak-Sneppen model. J. Stat. Phys. 173, 362–380.

[3] Durrett, R. (2010). Probability: Theory and Examples. Cambridge Series in Statistical and Probabilistic Mathematics.

[4] Grinfeld, M., Volkov, S., Wade, A. R. (2015). Convergence in a multidimensional randomized Keynesian beauty contest. Adv. in Appl. Probab. 47, 57–82.

[5] Teck-Hua Ho, Colin Camerer and Keith Weigelt. (1998). Iterated Dominance and Iterated Best Response in Experimental “p-Beauty Contests” The American Economic Review 88, 947–969.

[6] Kennerberg, P., and Volkov, S. (2018). Jante’s law process. Adv. in Appl. Probab. 50, 414–439.

[7] Kennerberg, P. and Volkov, S. (2018). The p-contest with \( p \neq 1 \). https://arxiv.org/abs/1812.00629, preprint.

[8] Meester, R., and Znamenski, D. (2003). Limit behavior of the Bak-Sneppen evolution model. Ann. Probab. 31, 1986–2002.

[9] Sandemose, A. (1936). A fugitive crosses his tracks. translated by Eugene Gay-Tifft. New York: A. A. Knopf.

[10] Veeerman, J., Prieto, F. (2014). On rank driven dynamical systems. J. Stat. Phys. 156 455–472.