VOLUME GROWTH OF COMPLETE SUBMANIFOLDS IN GRADIENT RICCI SOLITONS WITH BOUNDED WEIGHTED MEAN CURVATURE

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Abstract. In this article, we study properly immersed complete noncompact submanifolds in a complete shrinking gradient Ricci soliton with weighted mean curvature vector bounded in norm. We prove that such a submanifold must have polynomial volume growth under some mild assumption on the potential function. On the other hand, if the ambient manifold is of bounded geometry, we prove that such a submanifold must have at least linear volume growth. In particular, we show that a properly immersed complete noncompact hypersurface in the Euclidean space with bounded Gaussian weighted mean curvature must have polynomial volume growth and at least linear volume growth.

1. INTRODUCTION

In recent years, motivated by the research of the mean curvature flows in the Euclidean space $\mathbb{R}^m$, self-shrinkers, self-expanders and translators have been studied extensively as they are the singularity models for the flows. It is known that they are the critical points of the corresponding weighted volume functionals for all compactly supported variations, which indicates that it is important to study submanifolds under weighted volume measures.

Recently, McGonagle-Ross [24] began to study a special class of hypersurfaces in $\mathbb{R}^m$: the solutions to the Gaussian isoperimetric problem, which satisfy the equation

$$H - \frac{\langle x, n \rangle}{2} = \lambda,$$

where $\lambda \in \mathbb{R}$ is constant. These hypersurfaces are not only constant weighted mean curvature (CWMC) hypersurfaces in $\mathbb{R}^m$, but also the critical points of the Gaussian weighted area functional for compactly supported variations preserving enclosed Gaussian weighted volume. A trivial example is any hyperplane as $H = 0$ and $\langle x, n \rangle$ is constant. In [24], they proved that hyperplanes are the only stable smooth, complete, properly immersed solutions to the Gaussian isoperimetric problem, and that there are no hypersurfaces of index one. Later, in [6], Q.M.Cheng-G.Wei called hypersurfaces satisfying \((1.1)\) $\lambda$-hypersurfaces. Also, they extended the concept of $F$-functional of
self-shrinkers introduced by Colding-Minicozzi in [14] to \( \lambda \)-hypersurfaces and studied the related \( F \)-stability. There are some other works, for instance, by Q.M.Cheng-Ogata-G.Wei [7], Guang [17], and etc. It is worth noting that if \( \lambda = 0 \) in (1.1), the hypersurface is just a self-shrinker.

The Gaussian isoperimetric problem in \( \mathbb{R}^m \) can be generalized to a general ambient manifold. In general, a constant weighted mean curvature (CWMC) hypersurface (see its definition in Section 2) is a critical point of the weighted area functional for compactly supported variations preserving enclosed weighted volume. In this paper, the ambient smooth metric measure space \((M, g, f)\) we consider is a shrinking gradient Ricci soliton, that is, the triple \((M, g, f)\) satisfies that

\[
\overline{\text{Ric}} + \nabla^2 f = \frac{1}{2} g,
\]

where, for convenience, we choose the constant \( \frac{1}{2} \) and the potential function \( f \) to be normalized (see the meaning of the normalization of \( f \) in Section 4). It is known that shrinking gradient Ricci solitons are very important in research of the Ricci flow since they are singularity models of type I of the Ricci flow. Our consideration on shrinking Ricci soliton ambient manifolds not only includes the \( \lambda \)-hypersurfaces in Gaussian isoperimetric problem in \( \mathbb{R}^m \), but also arises from the study of mean curvature flows of hypersurfaces in an ambient manifold evolving by Ricci flow. In this aspect, Lott [22] and Magni-Mantegazza-Tsatis [23] showed that Huisken’s monotonicity formula holds when the ambient is a gradient Ricci soliton solution to the Ricci flow. Later, Yamamoto [29] studied the asymptotic behavior of a Ricci-mean curvature flow moving along a gradient shrinking Ricci soliton when it develops a singularity of type I. Moreover, Lott [22] introduced the concept of mean curvature soliton for the mean curvature flow evolving in a gradient Ricci soliton solution. Its definition implies that a mean curvature soliton is just an \( f \)-minimal hypersurface in a gradient Ricci soliton, which is the critical point of weighted area functional with the weight \( e^{-f} \). Here \( f \) is the potential function of the ambient gradient Ricci soliton. There are some studies on the properties of geometry and topology of \( f \)-minimal hypersurfaces in a gradient shrinking soliton, or more general, an ambient manifold with Bakry-Émery Ricci curvature bounded below by a positive constant. In [8] and [9], Mejia, the first and third authors of the present paper discussed the stability of \( f \)-minimal surfaces and proved some compactness theorems for \( f \)-minimal surfaces. There are also other works, for instance, see [10], [12], [19], [21], [27] and etc.

The class of \( f \)-minimal hypersurfaces is the particular case of CWMC hypersurfaces with weighted mean curvature zero. In this article, motivated by studying the volume growth of CWMC hypersurfaces in shrinking gradient Ricci solitons, we deal with properly immersed submanifolds in shrinking gradient Ricci solitons with weighted mean curvature vector bounded in norm. Indeed, the shrinking gradient Ricci soliton ambience leads to strong
restrictions on the volume growth of such submanifolds. It is known that the volume of a properly immersed minimal hypersurface in $\mathbb{R}^m$ has at least Euclidean growth and may grow exponentially. But, Ding-Xin [15] proved that a complete properly immersed self-shrinker hypersurface in $\mathbb{R}^m$ has polynomial volume growth. Further, in [11], the first and third authors of the present paper, proved that for a complete self-shrinker in $\mathbb{R}^m$, properness of immersion, polynomial volume growth and finiteness of weighted volume are equivalent to each other. Recently, in the arxiv version of [6], Q.M.Cheng-G.Wei used Theorem 1.1 in [11] to the $\lambda$-hypersurfaces in $\mathbb{R}^m$ satisfying the equation $H - \langle x, n \rangle = \lambda$ and proved that such a $\lambda$-hypersurface also has a polynomial volume growth. On the other hand, in [9], Mejia, the first and third authors of the present paper studied $f$-minimal submanifolds in a shrinking gradient Ricci soliton and showed the equivalence of the properness of immersion, polynomial volume growth and finiteness of weighted volume of a $f$-minimal submanifold under the assumption of the convexity of $f$. In this paper, we prove that for a complete properly immersed submanifold in a shrinking gradient Ricci soliton, if its weighted mean curvature vector is bounded in norm, then it has polynomial volume growth (see Theorem 1.2). Unfortunately, Theorem 1.1 in [11] cannot be applied directly to prove this result when the ambient manifold is a shrinking gradient Ricci soliton satisfying (1.2), even including the case of CWMC hypersurfaces in $\mathbb{R}^m$ satisfying (1.1). In order to prove it, we need the following Theorem 1.1 which is the key in the proof of Theorem 1.2 and of independent interest.

**Theorem 1.1.** Let $(X, g)$ be a complete noncompact Riemannian manifold. Let $f$ be a proper nonnegative $C^2$ function. Assume that there exist constants $\alpha > 0$, $\beta > 0$, and $a$ so that $f$ satisfies

\begin{equation}
\Delta f - \alpha |\nabla f|^2 + \beta f \leq a
\end{equation}

and

\begin{equation}
\Delta f \leq a.
\end{equation}

Then

(i) The volume of the set $D_r = \{ x \in X; 2\sqrt{f} \leq \sqrt{2\beta r} \}$, denoted by $\text{Vol}(D_r)$, satisfies

\begin{equation}
\text{Vol}(D_r) \leq e^{\frac{\alpha \beta}{\alpha + \beta}} r^{\frac{2\alpha}{\beta}} \int_{D_r} e^{-\alpha f} dv, \quad \text{for} \quad r \geq 1.
\end{equation}

(ii) The integral

\begin{equation}
\int_X e^{-\alpha f} dv < \infty.
\end{equation}

(iii) The volume of the set $D_r$ has polynomial growth. More precisely, for $r \geq 1$,

\begin{equation}
\text{Vol}(D_r) \leq C r^{\frac{2\alpha}{\beta}}, \quad \text{where} \quad C = e^{\frac{\alpha \beta}{\alpha + \beta}} \int_X e^{-\alpha f} dv.
\end{equation}
Remark 1.1. Theorem 1.1 generalizes Theorem 1.1 in [11] proved by the first and third authors of the present paper.

Next, based on Theorem 1.1 and the identities of submanifolds in a shrinking gradient Ricci soliton, we show

**Theorem 1.2.** Let \((M, g, f)\) be an \(m\)-dimensional complete shrinking gradient Ricci soliton satisfying (1.2). Let \(\Sigma\) be an \(n\)-dimensional, \(n < m\), properly immersed complete submanifold in \(M\) with the trace of Hessian of \(f\) restricted on the normal bundle of \(\Sigma\) satisfying \(\text{tr}_{\Sigma} \nabla^2 f \geq \frac{k}{2}\) for some constant \(k\). If \(\Sigma\) has a weighted mean curvature vector \(H_f\) bounded in norm, then

(i) The volume of the set \(D_r := \{ x \in \Sigma ; 2\sqrt{f} \leq r \}\) satisfies

\[
\text{Vol}(D_r) \leq C r^l, \quad \text{for} \quad r \geq 1,
\]

where \(l = m - k - \inf_{\Sigma} (R + |H|^2) - \inf_{\Sigma} (R + |(\nabla f)^{1/2}|^2) + \sup_{\Sigma} |H_f|^2\) is a nonnegative constant, and \(C = e^{\frac{1}{2} + \frac{1}{2} \inf_{\Sigma} (R + |(\nabla f)^{1/2}|^2)} \int_{\Sigma} e^{-\frac{1}{2} f}.\)

(ii) \(\Sigma\) has finite weighted volume with respect to the weighted volume element \(e^{-f} dv_{\Sigma} :\)

\[
\int_{\Sigma} e^{-f} < \infty.
\]

(iii) \(\Sigma\) must have polynomial volume growth. More precisely, for a fixed point \(p \in M\), there exist constants \(C\) and \(r_0\) so that for all \(r \geq r_0\),

\[
\text{Vol}(B^M_r (p) \cap \Sigma) \leq C r^l,
\]

where \(l\) is the same constant as in (i), \(B^M_r (p) = \{ x \in M ; d_M(p, x) \leq r \}\) denotes the geodesic ball in \(M\) of radius \(r\) centered at \(p\), and \(\text{Vol}(B^M_r (p) \cap \Sigma)\) denotes the volume of \(B^M_r (p) \cap \Sigma\).

It is worth noting that the polynomial volume growth rate \(r^l\) in Theorem 1.2 is optimal. Observe that the cylinder shrinkers \(S^d(\sqrt{2} d) \times \mathbb{R}^{n-d}\) in \(\mathbb{R}^{n+1}\) and cylinder Gaussian CWMC hypersurfaces \(S^d(R) \times \mathbb{R}^{n-d}\) in \(\mathbb{R}^{n+1}\), \(0 \leq d < n\), have the volume growth rate \(r^{n-d}\) and in both cases, \(l = n - d\). (see Remark 4.1 for details).

In [1], Alencar-Rocha applied Theorem 1.1 in [11] and proved the following result ([1], Proposition 1.2): Let \((M, g, f)\) be an \(m\)-dimensional complete shrinking gradient Ricci soliton satisfying (1.2) with convex potential function \(f\), i.e., \(\nabla^2 f \geq 0\). If \(\Sigma\) is a properly immersed complete noncompact submanifold in \(M\) satisfying \(b = \text{sup}_{\Sigma} \langle H_f, \nabla f \rangle < \infty\), then \(\Sigma\) has finite weighted volume and its volume has polynomial growth with the rate \(r^{m+2b}\).

We mention that for a CWMC hypersurface in \(\mathbb{R}^n\) with nonzero constant \(H_f\), it is not clear whether the assumption \(\text{sup}_{\Sigma} \langle H_f, \nabla f \rangle < \infty\) is satisfied in general.

Using Proposition 1.1 in [1], Proposition 5 in [9] and Theorem 1.2 in this paper, we prove the equivalence of properness of immersion, polynomial
volume growth and finite weighted volume for submanifolds in a shrinking gradient soliton with weighted mean curvature bounded in norm. More precisely,

**Theorem 1.3.** Let \((M, \mathcal{g}, f)\) be an \(m\)-dimensional complete shrinking gradient Ricci soliton. Let \(\Sigma\) be an \(n\)-dimensional, \(n < m\), complete submanifold immersed in \(M\) with the trace of Hessian of \(f\) restricted on the normal bundle of \(\Sigma\) satisfying \(\text{tr}_{\Sigma_{-}} \nabla^{2} f \geq k^{2}\) for some constant \(k\). If the weighted mean curvature vector \(H_{f}\) of \(\Sigma\) is bounded in norm, then the following items are equivalent to each other:

(i) \(\Sigma\) is properly immersed.
(ii) \(\Sigma\) has polynomial volume growth.
(iii) \(\Sigma\) has finite weighted volume:
\[
\int_{\Sigma} e^{-f} < \infty.
\]

**Remark 1.2.** Theorem 1.3 generalizes Theorem 1.3 in [11] for the case of self-shrinkers and Corollary 1 in [9] for the case of \(f\)-minimal submanifolds.

As a special case, we get

**Theorem 1.4.** Let \(f(x) = \frac{|x|^{2}}{4}\), \(x \in \mathbb{R}^{n+1}\) be the Gaussian potential function. Let \(\Sigma\) be a complete hypersurface immersed in \(\mathbb{R}^{n+1}\) with bounded Gaussian weighted mean curvature \(H_{f} = H - \langle x, n \rangle^{2}\). Then the following items are equivalent to each other:

(i) \(\Sigma\) is properly immersed.
(ii) \(\Sigma\) has at most the following polynomial volume growth rate: For a fixed point \(p \in M\), there are some constants \(C > 0\) and \(r_{0}\) so that for all \(r \geq r_{0}\),
\[
\text{Vol}(B_{r}(p) \cap \Sigma) \leq Cr^{l},
\]
where \(l = n - 1 - \inf_{\Sigma} |H|^{2} - \frac{1}{4} \inf_{\Sigma} |\langle x, n \rangle|^{2} + \sup_{\Sigma} |H_{f}|^{2}\) and \(B_{r}(p)\) denotes the geodesic ball in \(\mathbb{R}^{n+1}\) of radius \(r\) centered at \(p\).
(iii) \(\Sigma\) has polynomial volume growth.
(iv) \(\Sigma\) has finite weighted volume:
\[
\int_{\Sigma} e^{-\frac{|x|^{2}}{4}} < \infty.
\]

In Section 5 of this paper, we study the lower bound of volume growth of a complete noncompact submanifold properly immersed in a shrinking gradient Ricci soliton with weighted mean curvature vector bounded in norm and prove that it has at least linear growth. Here we give a brief related history. Calabi and Yau proved independently that a complete noncompact Riemannian manifold with nonnegative Ricci curvature must have at least linear volume growth. Recently, Munteanu-Wang showed that this lower bound estimate also holds for a shrinking gradient Ricci soliton in [25] and later generalized to the complete smooth metric measure space \((\Sigma, g, e^{-f} dv)\)
with $Ric_f \geq \frac{1}{2}g$ and $|\nabla f|^2 \leq f$ in [26]. On the other hand, for submanifolds, Cheung-Leung [13] proved that the volume of any complete noncompact submanifold in $\mathbb{R}^n$, $\mathbb{H}^n$ or more generally, in an ambient manifold of bounded geometry, must have at least linear growth. Recently, Li-Y. Wei [20] proved that a properly immersed complete non-compact self-shrinker in $\mathbb{R}^{n+1}$ also has at least linear volume growth. Later, the result of Li-Wei was extended to the $\lambda$-hypersurfaces in $\mathbb{R}^{n+1}$ by Q.M.Cheng-G.Wei in [6], and to the case of $f$-minimal submanifolds in a shrinking gradient Ricci soliton with convex potential function $f$ by Y. Wei in [28].

Recall that a Riemannian manifold is said to have bounded geometry if the sectional curvature is bounded above and the injectivity radius is bounded below by a positive constant. In this paper, we prove

**Theorem 1.5.** Let $(M, g, f)$ be an $m$-dimensional complete shrinking gradient Ricci soliton satisfying (1.2). Assume that $M$ has bounded geometry. Let $\Sigma$ be an $n$-dimensional complete properly immersed submanifold in $M$ with weighted mean curvature vector bounded in norm. Then $\Sigma$ must have at least linear volume growth, that is, for a fixed point $p \in M$, there are some constants $C$ and $r_0$ so that

$$\text{Vol}(B_r^M(p) \cap \Sigma) \geq Cr, \quad \text{for all} \quad r \geq r_0,$$

where $B_r^M(p) = \{x \in M; d_M(p, x) \leq r\}$ denotes the geodesic ball in $M$ of radius $r$ centered at $p$.

It is easy to show that the volume growth rates of submanifolds we consider are independent of the choice of the point $p \in M$. Theorem 1.5 has the following special case:

**Theorem 1.6.** Let $f = \frac{|x|^2}{4}$, $x \in \mathbb{R}^{n+1}$ be the Gaussian potential function. Let $\Sigma$ be a complete hypersurface properly immersed in $\mathbb{R}^{n+1}$ with bounded Gaussian weighted mean curvature $H_f = H - \frac{\langle x, n \rangle}{2}$. Then $\Sigma$ must have at least linear volume growth.

The rest of the paper is organized as follows: In Section 2 we give some notation and facts as preliminaries. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2. In Section 5 we prove Theorem 1.5.

2. Preliminaries

In this section, we give some notation and conventions.

Let $(X^d, g)$ denote a $d$-dimensional smooth Riemannian manifold. A smooth metric measure space, denoted by $(X^d, g, e^{-f}dv)$, is $(X^d, g)$ together with a weighted volume form $e^{-f}dv$ on $X$, where $f$ is a smooth function on $X$ and $dv$ is the volume element induced by the metric $g$. For $(X^d, g, e^{-f}dv)$, the drifted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is a densely defined self-adjoint second order elliptic operator in $L^2(X, e^{-f}dv)$. For $u$ and $v$ in $C^0_0(X)$, it
holds that
\begin{equation}
(2.1) \int_X (\Delta f) e^{-f} dv = - \int_X \langle \nabla u, \nabla v \rangle e^{-f} dv.
\end{equation}

In this article, we will study submanifolds in a Riemannian manifold. Let \((M, \mathcal{g})\) be a smooth \(m\)-dimensional Riemannian manifold and \(f\) a smooth function on \(M\). Let \(i : (\Sigma^n, \mathcal{g}) \to (M^m, \mathcal{g})\), \(n < m\), denote the smooth isometric immersion of an \(n\)-dimensional submanifold \(\Sigma\) into \(M\). Clearly the function \(f\) restricted on \(\Sigma\), still denoted by \(f\), induces a weighted volume element \(e^{-f} dv \Sigma\) on \(\Sigma\) and thus a smooth metric measure space \((\Sigma, \mathcal{g}, e^{-f} dv \Sigma)\), where \(dv \Sigma\) denote the volume element of \((\Sigma, \mathcal{g})\). When we deal with submanifolds, unless otherwise specified, the notations with a bar denote the quantities corresponding to the metric \(\mathcal{g}\). On the other hand, the notations without a bar denote the quantities corresponding to the intrinsic metric \(\mathcal{g}\) on \(\Sigma\).

The submanifold \(\Sigma\) is said to be properly immersed if, for any compact subset \(\Omega\) in \(M\), the pre-image \(i^{-1}(\Omega)\) is compact in \(\Sigma\).

For \(\Sigma\), its mean curvature vector \(H\) is
\begin{equation}
H(p) := \sum_{i=1}^{n} (\nabla_{e_i} e_i) \perp = \sum_{i=1}^{n} A(e_i, e_i), \quad p \in \Sigma,
\end{equation}
where \(\{e_1, e_2, \cdots, e_n\}\) is a local orthonormal frame of \(\Sigma\) at \(p\), \(\perp\) denotes the projection onto the normal bundle of \(\Sigma\), and \(A\) denotes the second fundamental form of \(\Sigma\).

The weighted mean curvature vector \(H_f\) of \(\Sigma\) is defined by
\begin{equation}
(2.2) \quad H_f := H + \langle \nabla f \rangle \perp.
\end{equation}

A submanifold \(\Sigma\) is called \(f\)-minimal if its weighted mean curvature \(H_f\) vanishes identically, or equivalently \(H = -\langle \nabla f \rangle \perp\).

The weighted volume of a measurable subset \(S \subset \Sigma\) is defined by
\begin{equation}
V_f(S) = \int_S e^{-f} dv \Sigma.
\end{equation}

It is known that an \(f\)-minimal submanifold is a critical point of the weighted volume functional. On the other hand, it is also a minimal submanifold under the conformal metric \(\hat{g} = e^{-\frac{f}{2}} g\) on \(M\) (see, e.g. [9], [8]).

In the case of hypersurfaces, the mean curvature of \(\Sigma\) is defined by
\begin{equation}
(2.3) \quad H = -Hn,
\end{equation}
where \(n\) is the unit normal field on \(\Sigma\).

The weighted mean curvature is defined as
\begin{equation}
(2.4) \quad H_f = -H_f n, \quad \text{or equivalently} \quad H_f = H - \langle \nabla f, n \rangle
\end{equation}
A hypersurface \(\Sigma\) is said to be a constant weighted mean curvature (denoted simply by CWMC) hypersurface if it satisfies
\begin{equation}
(2.5) \quad H - \langle \nabla f, n \rangle = C.
\end{equation}
So an \( f \)-minimal hypersurface \( \Sigma \) is the case of \( C = 0 \), that is, \( H = \langle \nabla f, n \rangle \). When \((M, \mathcal{g})\) is the Euclidean space \((\mathbb{R}^m, g_0)\) with the Gaussian measure \( e^{-\frac{|x|^2}{4}} dv_{g_0} \), a CWMC hypersurface is just the solution to the Gaussian isoperimetric problem in [24] or the \( \lambda \)-hypersurface in [6] (observe that in [6], the weighted measure is taken to be \( e^{-\frac{|x|^2}{2}} dv_{g_0} \), which has no essential difference).

In [2] and [3], Barbosa-do Carmo-Eschenburg showed that a constant mean curvature (CMC) hypersurface in a Riemannian manifold is the critical point of its area functional for compactly supported variations which preserve enclosed volume. McGonagle-Ross [24] proved the same property for CWMC hypersurfaces in \( \mathbb{R}^m \) under the Gaussian weighted area and Gaussian weighted volume, respectively. Using the similar argument to the one in [3] and [24], one can show that a CWMC hypersurface in a Riemannian manifold is still the critical point of its weighted area functional for compactly supported variations which preserve enclosed weighted volume, where the weighted volume element in \( M \) is \( e^{-f} dv_M \) and \( dv_M \) denotes the volume element of \( M \).

In this paper, we estimate the volume growth of submanifolds. Recall that \( \Sigma \) is said to have polynomial volume growth if, for a fixed point \( p \in M \), there are constants \( C \), \( s \) and \( r_0 \) so that for all \( r \geq r_0 \),

\[
(2.6) \quad \text{Vol}(B^M_r(p) \cap \Sigma) \leq C r^s,
\]

where \( B^M_r(p) \) is the geodesic ball in \( M \) of radius \( r \) centered at \( p \), \( \text{Vol}(B^M_r(p)) \) denotes the volume of \( B^M_r(p) \cap \Sigma \). When \( s = n \) in (2.6), \( \Sigma \) is said to be of Euclidean volume growth.

In this article, the ambient manifold \( M \) is assumed to be a shrinking gradient Ricci soliton. The triple \((M, \mathcal{g}, f)\) is called a shrinking gradient Ricci soliton if it satisfies that

\[
(2.7) \quad \overline{\text{Ric}} + \nabla^2 f = \rho \mathcal{g},
\]

where \( \rho > 0 \) is a constant. Gaussian shrinking soliton \((\mathbb{R}^{n+1}, g_{\text{can}}, |x|^2)\) and Cylinder shrinking solitons are examples. If \( f \) is constant, a shrinking gradient Ricci soliton is just an Einstein manifold.

3. Proof of Theorem 1.1

Recall that a function on a Riemannian manifold is called proper if for any compact set in \( \mathbb{R} \), its pre-image is compact. Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** Since \( f \) is proper and nonnegative, the minimum of \( f \) can be achieved in some point of the manifold \( X \). Thus (1.3) implies that the constant \( a \geq 0 \). Denote \( \gamma = \frac{a}{\alpha} \). For \( t \geq 1 \), define the function

\[
\phi_t = a \log t + \frac{\alpha}{t^\gamma} f
\]
and consider the drifted Laplacian \( \Delta_{\phi_t} = \Delta - \langle \nabla \phi_t, \nabla \cdot \rangle \).

Using Assumptions (1.3) and (1.4), we have

\[
(\Delta_{\phi_t} f) e^{-\phi_t} + t \frac{d}{dt} e^{-\phi_t} = \left( \Delta f - \alpha \frac{1}{t^\gamma} |\nabla f|^2 - a + \frac{\beta}{t^\gamma} f \right) e^{-\phi_t}
\]

\[
= \left[ \frac{1}{t^\gamma} (\Delta f - \alpha |\nabla f|^2 + \beta f - a) \right] e^{-\phi_t}
\]

\[
+ \left[ (1 - \frac{1}{t^\gamma})(\Delta f - a) \right] e^{-\phi_t}
\]

\[
\leq 0.
\]

(3.1)

Then, (3.1) implies that, at a regular value of \( f \) and for \( t \geq 1 \),

\[
\frac{d}{dt} \int_{D_r} e^{-\phi_t} = \int_{D_r} \frac{d}{dt} e^{-\phi_t}
\]

\[
\leq - \frac{1}{t} \int_{D_r} (\Delta_{\phi_t} f) e^{-\phi_t} = - \frac{1}{t} \int_{\partial D_r} \langle \nabla f, \nabla \phi_t \rangle e^{-\phi_t}
\]

\[
= - \frac{1}{t} \int_{\partial D_r} |\nabla f| e^{-\phi_t} \leq 0,
\]

(3.2)

where we used the Stokes’ theorem in the second equality. Integrating (3.2) from \( t = 1 \) to \( t = T \geq 1 \) gives

\[
\int_{D_r} e^{-\phi_T} \leq \int_{D_r} e^{-\phi_1}.
\]

Taking \( T = r^{2 \gamma}, r \geq 1 \) and noting \( \phi_1 = \alpha f \), we have

\[
r^{-\frac{\alpha}{\gamma}} \int_{D_r} e^{-\frac{\alpha}{r^{2 \gamma}} f} \leq \int_{D_r} e^{-\alpha f}.
\]

(3.3)

Since the integrals in (3.3) is right continuous in \( r \), (3.3) holds for all \( r \geq 1 \).

By \( D_r = \{ x \in \mathcal{X}; 2\sqrt{f} \leq \sqrt{2\beta r} \} \) and \( \gamma = \frac{\beta}{\alpha} \), (3.3) implies

\[
\text{Vol}(D_r) \leq e^{\frac{\alpha \beta}{2} \int_{D_r} e^{-\alpha f}}, \quad \text{for} \quad r \geq 1,
\]

which is (1.5). Now we prove (ii). We have that, for \( r \geq 1 \),

\[
\int_{D_r} e^{-\alpha f} - \int_{D_{r-1}} e^{-\alpha f} = \int_{D_r \setminus D_{r-1}} e^{-\alpha f} \leq e^{-\frac{\alpha \beta (r-1)^2}{2}} \int_{D_r \setminus D_{r-1}} 1
\]

\[
\leq e^{-\frac{\alpha \beta (r-1)^2}{2}} \text{Vol}(D_r)
\]

\[
\leq e^{\frac{\alpha \beta}{2} \int_{D_r} e^{-\alpha f}} \cdot e^{-\frac{\alpha \beta (r-1)^2}{2}} \int_{D_r} e^{-\alpha f}.
\]

(3.5)

In (3.5), the last inequality used Inequality (3.4). Noting that there is a large number \( r_0 \geq 1 \) such that for \( r \geq r_0 \), \( e^{\frac{\alpha \beta}{2} \int_{D_r} e^{-\alpha f}} \leq e^{-r} \), we have,
by \(3.5\),
\[
\int_{D_r} e^{-\alpha f} \leq \frac{1}{1 - e^{-r}} \int_{D_{r-1}} e^{-\alpha f}.
\]
Thus, given any positive integer \(N\) and \(r \geq r_0\),
\[
\int_{D_{r+N}} e^{-\alpha f} \leq \left( \prod_{i=0}^{N} \frac{1}{1 - e^{-(r+i)}} \right) \int_{D_{r-1}} e^{-\alpha f}.
\]
(3.6)
Noting that the infinite product \(\prod_{i=0}^{\infty} \left(1 - e^{-(r+i)}\right)\) converges to a positive number and letting \(N\) tend to infinity in (3.6), we get
\[
\int_{X} e^{-\alpha f} < \infty,
\]
which is (ii). Finally (3.7) and (3.6) give (iii):
\[
\text{Vol} \left( D_r \right) \leq C r^{\frac{a}{\alpha}} = C r^{\frac{2a}{\beta}}, \quad \text{for} \; r \geq 1,
\]
where \(C = e^{\frac{a}{2}} \int_{X} e^{-\alpha f}.\)
\(\square\)

**Remark 3.1.** Theorem 1.1 generalizes Theorem 1.1 in [11] by the following fact: since \(\Delta f - |\nabla f|^2 + f \leq a\) and \(|\nabla f|^2 \leq f\) imply the inequality \(\Delta f \leq a\), we can take \(\alpha = 1, \beta = 1\) in Theorem 1.1.

### 4. Upper estimate of volume growth

In what follows, we assume that the ambient space \((M^m, \bar{g}, f)\) is an \(m\)-dimensional smooth shrinking gradient Ricci soliton. Without loss of generality, we take \(\rho = \frac{1}{2}\) in (2.7), that is, \((M, \bar{g}, f)\) satisfies
\[
\overline{\text{Ric}} + \nabla^2 f = \frac{1}{2} \bar{g}.
\]
(4.1)
Equation (4.1) implies that
\[
\overline{R} + \Delta f = \frac{m}{2}.
\]
(4.2)
Here \(\overline{R}\) denotes the scalar curvature of \(M\). It is well known that the potential function \(f\) satisfying (4.1) can be normalized by adding a suitable constant to it such that the following identity holds:
\[
\overline{R} + |\nabla f|^2 = f.
\]
(4.3)
Then (4.2) and (4.3) give that
\[
\overline{\Delta} f - |\nabla f|^2 + f = \frac{m}{2}.
\]
(4.4)
It was proved by B. Chen [5] that the scalar curvature of $M$ satisfies $\overline{R} \geq 0$. Hence
\begin{equation}
|\nabla f|^2 \leq f. \tag{4.5}
\end{equation}

By applying Theorem 1.1, we may prove Theorem 1.2.

**Proof of Theorem 1.2** In [1], Cao and the third author proved the following result: For a fixed point $p \in M$, let $\overline{r}(x) \coloneqq d_M(p, x)$, $x \in M$, denote the distance between $x$ and $p$ in $M$. Then there are positive constants $c$ and $r_0$ depending only of $f(p)$ and dimension $m$ such that for any $x \in M$ and $\overline{r}(x) \geq r_0$,
\begin{equation}
\frac{1}{4}(\overline{r}(x) - c)^2 \leq f(x) \leq \frac{1}{4}(\overline{r}(x) + c)^2. \tag{4.6}
\end{equation}

Inequality (4.6) implies that $f$ is proper on $M$. This property of $f$ and the proper immersion of $\Sigma$ in $M$ imply that the restriction $f|_\Sigma$ of $f$ on $\Sigma$ is also proper. Identity (4.3) implies that
\begin{equation}
|\nabla f|^2 = |\nabla f|^2 - |(\nabla f)\perp|^2 = f - \overline{R} - |(\nabla f)\perp|^2. \tag{4.7}
\end{equation}

Using (4.2) and $H_f = H + (\nabla f)\perp$, we have
\begin{align*}
\Delta f &= tr_{\Sigma}(\nabla^2 f + \langle \nabla f, H \rangle) \\
&= \Delta f - tr_{\Sigma}(\nabla^2 f + \langle (\nabla f)\perp, H_f \rangle) - |(\nabla f)\perp|^2 \\
&= \frac{1}{2}m - \overline{R} - tr_{\Sigma}(\nabla^2 f - \frac{1}{2}|H|^2 + \frac{1}{2}|H_f|^2 - \frac{1}{2}|(\nabla f)\perp|^2), \tag{4.8}
\end{align*}

where we used the basic identity: $\langle u, v \rangle = \frac{1}{2}(|u|^2 + |v|^2 - |u - v|^2)$ in the last equality of (4.8). Thus, by $\overline{R} \geq 0$, the boundedness of $H_f$, and the assumption $tr_{\Sigma}(\nabla^2 f) \geq \frac{l}{2}$, we have
\begin{equation}
\Delta f \leq \frac{l}{2}, \tag{4.9}
\end{equation}

where $l = m - k - \inf_{\Sigma}(\overline{R} + |H|^2) - \inf_{\Sigma}(\overline{R} + |(\nabla f)\perp|^2) + \sup_{\Sigma}|H_f|^2$. Since $f = f|_\Sigma$ is a proper nonnegative function on $\Sigma$, the minimum of $f$ is achieved on $\Sigma$. So $\Delta f \geq 0$ at the minimal point and hence $l \geq 0$. On the other hand, using (4.7) and (4.8), we have
\begin{align*}
\Delta f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}f &= \frac{1}{2}m - \overline{R} - tr_{\Sigma}(\nabla^2 f - \frac{1}{2}|H|^2 + \frac{1}{2}|H_f|^2 - \frac{1}{2}|(\nabla f)\perp|^2 \\
&\quad - \frac{1}{2}(f - \overline{R} - |(\nabla f)\perp|^2) + \frac{1}{2}f \\
&= \frac{1}{2}m - \frac{1}{2}\overline{R} - tr_{\Sigma}(\nabla^2 f - \frac{1}{2}|H|^2 + \frac{1}{2}|H_f|^2). \\
\text{Letting} \quad \tilde{f} = f - \inf_{\Sigma}(\overline{R} + |(\nabla f)\perp|^2), \quad \text{by (4.7), we have} \quad \tilde{f} \geq 0. \quad \text{By (4.9),}
\end{align*}
\begin{equation}
\Delta \tilde{f} \leq \frac{l}{2}. \tag{4.10}
\end{equation}
Also, (4.10) implies
\[
\Delta \tilde{f} - \frac{1}{2} |\nabla \tilde{f}|^2 + \frac{1}{2} \tilde{f} \leq \frac{l}{2}.
\]

Now in Theorem 1.1 we let \( X = \Sigma, f = \tilde{f}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}, \) and \( a = \frac{1}{2} \).
By the conclusion (ii) in Theorem 1.1, we have that
\[
e^{-\frac{1}{2}\inf_{\Sigma}(\nabla f)^2} \int_{\Sigma} e^{-\frac{1}{2} f} = \int_{\Sigma} e^{-\frac{1}{2} f} < \infty.
\]
The conclusion (iii) in Theorem 1.1 implies that the volume of the set \( \{ x \in \Sigma; 2\sqrt{f} \leq r \} \) satisfies
\[
\text{Vol}(\{ x \in \Sigma; 2\sqrt{f} \leq r \}) \leq Cr^l, \quad \text{for} \quad r \geq 1,
\]
where the constant \( C = e^{\frac{1}{8} + \frac{1}{2}\inf_{\Sigma}(\nabla f)^2} \int_{\Sigma} e^{-\frac{1}{2} f} \). Since the level set \( D_r = \{ x \in \Sigma; 2\sqrt{\tilde{f}} \leq r \} \subseteq \{ x \in \Sigma; 2\sqrt{\tilde{f}} \leq r \} \), its volume satisfies that
\[
\text{Vol}(D_r) \leq Cr^l, \quad \text{for} \quad r \geq 1,
\]
where the constant \( C \) is the same as in (4.13). Thus we proved (i) of Theorem 1.2. Noting that \( \int_{\Sigma} e^{-f} \leq \int_{\Sigma} e^{-\frac{1}{2} f} \), we get (ii) of Theorem 1.2. Finally, we prove (iii). Inequality (4.9) implies that for \( r \geq \max\{r_0, 1\} \),
\[
B_r^M(p) \cap \Sigma \subseteq \{ x \in \Sigma : f \leq \frac{1}{4}(r + c)^2 \} = D_{r+c}.
\]
So, for \( r \geq \max\{r_0, 1, c\} \),
\[
\text{Vol}(B_r^M(p) \cap \Sigma) \leq \text{Vol}(D_{r+c}) \leq C(r + c)^l \leq C_1 r^l,
\]
where the constant \( C_1 = 2^l C = 2^l e^{\frac{1}{8} + \frac{1}{2}\inf_{\Sigma}(\nabla f)^2} \int_{\Sigma} e^{-\frac{1}{2} f} \). This completes the proof of Theorem 1.2.

\[\square\]

Remark 4.1. Theorem 1.2 extends the known results in [9] and [15]. Moreover, the order \( l \) of the polynomial volume growth estimate is optimal. In \( f \)-minimal case, i.e., \( H_f = 0 \), we may take the cylinder self-shrinkers \( S^d(\sqrt{2}d) \times \mathbb{R}^{n-d} \) in \( \mathbb{R}^n \), \( 0 \leq d < n \). Then \( |H| = \frac{\sqrt{d}}{\sqrt{2}} \) and \( l = n - d \). Theorem 1.2 implies that \( \text{Vol}(B_r(0) \cap \Sigma) \leq C r^{n-d} \). Obviously the order \( n - d \) is achieved. In the case of \( \mathbf{H}_f \neq 0 \), we may take any cylinder CWMC hypersurface \( S^d(R) \times \mathbb{R}^{n-d} \) in \( \mathbb{R}^{n+1} \), where \( 0 \leq d < n \). Then \( H = \frac{d}{R}, \langle x, n \rangle = R \), and \( H_f = H - \frac{1}{2} \langle x, n \rangle = \frac{d}{R} - \frac{R}{R} \). Thus \( l = n - d \), which is also sharp.

Remark 4.2. Theorem 1.1 can be applied for all the values of the constant \( p \) when \( M \) satisfies the general shrinking gradient Ricci soliton equation (2.7): \( \overline{Ric} + \nabla^2 f = pf \) with the constant \( p > 0 \) and the normalized \( f \), that is, \( \overline{R} + |\nabla f|^2 = 2pf \). In this case, we assume the condition \( \text{tr}_{\Sigma} \nabla^2 f \geq \rho k \) for some constant \( k \) instead in Theorem 1.2. Using the same argument
as in the proof of Theorem 1.2 by choosing $\alpha = \frac{1}{2}, \beta = \rho, a = \rho(m - k) + -\inf_{\Sigma}(\overline{\text{Ric}} + |H|^2) - \inf_{\Sigma}(\overline{\text{Ric}} + |\nabla f|^2) + \sup_{\Sigma}|H| f$, and $\tilde{f} = f - \frac{\inf_{\Sigma}(\overline{\text{Ric}} + |\nabla f|^2)}{2\rho}$, we get that $\Sigma$ has polynomial volume growth with the rate $r^l$, where the constant $l = (m - k) + -\inf_{\Sigma}(\overline{\text{Ric}} + |H|^2) - \inf_{\Sigma}(\overline{\text{Ric}} + |\nabla f|^2) + \sup_{\Sigma}|H| f$.

**Remark 4.3.** If $M$ is a complete shrinking gradient soliton with its Ricci curvature tensor $\overline{\text{Ric}}$ bounded above, $tr_{\Sigma} \nabla^2 f$ must be bounded below by some constant.

In [9], it was proved that if a complete properly immersed submanifold $\Sigma$ in a complete Riemannian manifold $M$ with $\overline{\text{Ric}} + \nabla^2 f \geq \frac{1}{2}g$ and $|\nabla f|^2 \leq f$ has polynomial volume growth, it must have finite weighted volume (Proposition 5 in [9]). On the other hand, in [1], Alencar-Rocha proved that if $\Sigma$ is a complete submanifold immersed in a complete Riemannian manifold with weighted mean curvature vector bounded in norm, then the finite weighted volume of $\Sigma$ implies properness of immersion (Proposition 1.1 in [1]). Using these two properties and Theorem 1.2, we get Theorem 1.3.

### 5. Lower estimate of volume growth

In this section, we estimate the lower bound of volume growth of the submanifolds with weighted mean curvature vector bounded in norm. We assume that $(M, \overline{g}, f)$ is a complete $m$-dimensional shrinking gradient Ricci soliton:

$$\overline{\text{Ric}} + \nabla^2 f = \frac{1}{2}g$$

with the normalized potential function $f$. We have that $f$ satisfies (4.2), (4.3), (4.4), and (4.5). Moreover, the scalar curvature of $M$ satisfies $R \geq 0$.

Let $n < m$ and $\Sigma$ be a complete $n$-dimensional properly immersed submanifold in $M$ with bounded $|H| f$. We first prove that the volume of $\Sigma$ is infinite. Before doing this, we need some facts.

Observing that the conclusion of Lemma 4.1 in [28] is still true when the assumption of boundedness of the sectional curvature of the ambient manifold is changed to that the sectional curvature is only bounded above by a constant, we get the following lemma.

**Lemma 5.1.** Let $(M^m, \overline{g})$ be an $m$-dimensional Riemannian manifold whose sectional curvature is bounded above by $K_0 > 0$ and injectivity radius is bounded below by $i_0 > 0$. Let $\Sigma$ be an $n$-dimensional complete properly immersed submanifold in $(M^m, \overline{g})$. For any $p \in \Sigma$ and $r < \min\{1, i_0, 1/\sqrt{K_0}\}$, if $|H| \leq C/r$ in $B^M_r(p) \cap \Sigma$ for some positive constant $C > 0$, then the volume of $B^M_r(p) \cap \Sigma$ satisfies

$$\text{Vol}(B^M_r(p) \cap \Sigma) \geq \tau r^n,$$

where $\tau = \omega_n e^{-2(n\sqrt{K_0} + C)}$. Here $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. 

Proof. Since $M$ has bounded geometry, by the Hessian comparison theorem and direct computation, the following inequality holds:

$$\nabla^2 r(x)(V, V) - \frac{1}{r(x)} |V - \langle V, \nabla r(x) \rangle \nabla r(x)|^2 \geq -\sqrt{K_0},$$

where $x \in M$, $V$ denotes any unit tangential vector in $T_xM$, and $r(x) = d_M(x, p)$ is the distance between $p$ and $x$ in $M$.

Following the same argument as in the proof of Lemma 4.1 in [28], one can prove (5.2).

To continue our proof, we need a logarithmic Sobolev inequality on submanifolds. In [16], Ecker proved the logarithmic Sobolev inequality for submanifolds in $\mathbb{R}^m$ and pointed out that by applying the Sobolev inequality for submanifolds of Riemannian manifolds (see [18]), his theorem extends to this more general context restricted only by the additional assumptions on the support of the admissible functions imposed there and with constants depending apart from the dimension of the submanifold also on the sectional curvatures of the ambient manifold. For the sake of completeness, we state it here:

**Proposition 5.1.** (see Proposition 3.1, [28]) Let $\Sigma$ be an $n$-dimensional submanifold immersed in an $m$-dimensional Riemannian manifold $(M^m, \mathbf{g})$.

Assume that the sectional curvature of $M$ satisfies $K_M \leq b^2$ and the injectivity radius of $M$ restricted to $\Sigma$ is bounded below by a constant $i_0 > 0$, where $b$ denotes a positive real number or a pure imaginary one. Let $\mu$ be a smooth positive function on $M$ and $\lambda > 0$ be a positive constant. Then the following inequality

$$\int_\Sigma h^2(\ln h)\mu d\sigma - \frac{1}{2} \left( \int_\Sigma h^2\mu d\sigma \right) \ln \left( \int_\Sigma h^2\mu d\sigma \right)
\leq \lambda \int_\Sigma |\nabla h|^2\mu d\sigma + \frac{\lambda}{4} \int_\Sigma |H - (\nabla \ln \mu)^\perp|^2 h^2\mu d\sigma
+ \int_\Sigma h^2\mu \left( c(n, \alpha, \lambda^{-1}) - \frac{\lambda}{4} |\nabla \ln \mu|^2 - \frac{\lambda}{2} \text{div}(\nabla \ln \mu) - \frac{1}{2} \ln \mu \right) d\sigma$$

(5.3)

holds for any nonnegative $C^1$ function $h$ on $\Sigma$ vanishing on $\partial \Sigma$ provided that the volume of the support of $h$ (denoted by $|\text{supp}(h)|$) satisfies the following restriction

$$b^2(1 - \alpha)^{-2} \left( \omega_n^{-1} |\text{supp}(h)| \right)^{2/n} \leq 1 \quad \text{and} \quad 2\rho_0 \leq i_0,$$

where

$$\rho_0 = \left\{ \begin{array}{ll}
b^{-1} \arcsin \left( b(1 - \alpha)^{-1} \left( \omega_n^{-1} |\text{supp}(h)| \right)^{2/n} \right) & \text{if } b \text{ is real}, \\
(1 - \alpha)^{-1} \left( \omega_n^{-1} |\text{supp}(h)| \right)^{2/n} & \text{if } b \text{ is imaginary}. \end{array} \right.$$

Here $\alpha$ is a free parameter satisfying $0 < \alpha < 1$ and $\text{div}(\nabla \ln \mu)$ denotes the divergence of $\nabla \ln \mu$ with respect to $\Sigma$. 

□

Now we go back to the gradient shrinking Ricci soliton \((M, g, f)\). Take \(\mu = e^{-f}\). Then
\[
\nabla \ln \mu = -\nabla f,
\]
\[
\mathbf{H} - (\nabla \ln \mu)^\perp = \mathbf{H} + (\nabla f)^\perp = \mathbf{H}_f,
\]
\[
div(\nabla \ln \mu) = -\nabla^2 f = -\Delta f + \left\langle \mathbf{H}_f, (\nabla f)^\perp \right\rangle
\]
\[
= -\Delta f - |(\nabla f)^\perp|^2 + \left\langle \mathbf{H}_f, (\nabla f)^\perp \right\rangle.
\]

Let \(\mu = e^{-f}, \alpha = \frac{n}{n+1}\), and \(\lambda = 2\) in Proposition 5.1. Using the second equality in (4.8), (4.2) and (4.3), we have
\[
c(n, \alpha, \lambda^{-1}) - \frac{\lambda}{4} |\nabla \ln \mu|^2 - \frac{\lambda}{2} \nabla \ln \mu - \frac{1}{2} \mu
\]
\[
= c(n) - \frac{1}{2} |\nabla f|^2 + \Delta f + |(\nabla f)^\perp|^2 - \left\langle \mathbf{H}_f, (\nabla f)^\perp \right\rangle + \frac{f}{2}
\]
\[
= c(n) - \frac{1}{2} |\nabla f|^2 + \Delta f - tr_{\Sigma^\perp} \nabla^2 f + \frac{f}{2}
\]
\[
= c(n) + \frac{1}{2} m - R - \frac{1}{2} |\nabla f|^2 + \frac{f}{2} - tr_{\Sigma^\perp} \nabla^2 f
\]
\[
= c(n) + \frac{1}{2} m - \frac{1}{2} R - tr_{\Sigma^\perp} \nabla^2 f.
\]
(5.4)

Therefore Proposition 5.1 implies that

Lemma 5.2. Let \(M\) and \(\Sigma\) be the same as in Proposition 5.1. Assume that \((M^m, g, f)\) is a gradient shrinking soliton. Then
\[
\int_{\Sigma} h^2 (\ln h) e^{-f} d\sigma - \frac{1}{2} \int_{\Sigma} h^2 e^{-f} d\sigma \ln \left( \int_{\Sigma} h^2 e^{-f} d\sigma \right)
\]
\[
\leq 2 \int_{\Sigma} |\nabla h|^2 e^{-f} d\sigma + \frac{1}{2} \int_{\Sigma} |\mathbf{H}_f|^2 h^2 e^{-f} d\sigma
\]
\[
+ \int_{\Sigma} h^2 \left( c(n) + \frac{1}{2} m - \frac{1}{2} R - tr_{\Sigma^\perp} \nabla^2 f \right) e^{-f} d\sigma,
\]
(5.5)

holds for any nonnegative \(C^1\) function \(h\) on \(\Sigma\) vanishing on \(\partial \Sigma\) provided that the volume of the support of \(h\) (denoted by \(|\text{supp}(h)|\)) satisfies the following restriction
\[
b^2 \left( (n+1) \omega_n^{-1} |\text{supp}(h)| \right)^\frac{n}{2} \leq 1 \quad \text{and} \quad 2\rho_0 \leq i_0,
\]
(5.6)

where
\[
\rho_0 = \begin{cases} 
\frac{b^{-1} \arcsin \left[ b \left( (n+1) \omega_n^{-1} |\text{supp}(h)| \right)^\frac{1}{2} \right]}{\left( (n+1) \omega_n^{-1} |\text{supp}(h)| \right)^\frac{1}{n}} & \text{if } b \text{ is real}, \\
\left( (n+1) \omega_n^{-1} |\text{supp}(h)| \right)^\frac{1}{n} & \text{if } b \text{ is imaginary}.
\end{cases}
\]
(5.7)

Now we are ready to prove the following result.
Theorem 5.1. Let \((M, \nabla, f)\) be an \(m\)-dimensional complete shrinking gradient Ricci soliton whose sectional curvature is bounded above by a constant \(K_0 > 0\) and injectivity radius is bounded below by a constant \(i_0 > 0\). Let \(\Sigma\) be an \(n\)-dimensional complete properly immersed submanifold in \(M\). If \(\Sigma\) has weighted mean curvature vector bounded in norm, then its volume is infinite.

Proof. Suppose, to the contrary, that \(\Sigma\) has finite volume. Let \(\rho(x) = 2\sqrt{f(x)}\), \(x \in M\). Fix a point \(p \in \Sigma\) and denote \(\tau(x) = d_M(p, x)\). Since \(\Sigma\) is non-compact and properly immersed, the image of \(\tau\) on \(\Sigma\) is \([0, \infty)\). Note (4.6) says that there are constants \(r_0\) and \(c\) so that \(|\rho(x) - \tau(x)| \leq c\) for \(\tau(x) \geq r_0, x \in M\). Thus the image of \(\rho(x)\) on \(\Sigma \setminus B_{r_0}(p) \cap \Sigma\) contains \((r_0 + c, \infty)\). For \(0 < k_1 < k_2\) very large, denote
\[
A(2^{k_1}, 2^{k_2}) = \{ x \in \Sigma; 2^{k_1} \leq \rho(x) \leq 2^{k_2} \},
\]
\[
V(2^{k_1}, 2^{k_2}) = \text{Vol}(A(2^{k_1}, 2^{k_2})).
\]
By \(\text{Vol}(\Sigma) < \infty\), it holds that for every \(\epsilon > 0\), there exists \(k_0 > 0\) so that for \(k_2 > k_1 \geq k_0\),
\[
V(2^{k_1}, 2^{k_2}) \leq \epsilon. \tag{5.8}
\]

Claim 1. We may choose some \(k_1\) and \(k_2\) in (5.8) so that \(k_1\) and \(k_2\) also satisfy the following inequality:
\[
V(2^{k_1}, 2^{k_2}) \leq 2^{4n} V(2^{k_1+2}, 2^{k_2-2}). \tag{5.9}
\]

Now we show the claim. For a very large \(k\), the set
\[
\{ x \in \Sigma; 2^k + c \leq \tau(x) \leq 2^{k+1} - c \}
\]
contains at least \(2^{2k-1}\) disjoint balls in \(M\) of radius \(r = \frac{2^k - 2c}{2^k}\) centered at \(x_i \in \Sigma\) and restricted on \(\Sigma\):
\[
B_r(x_i) = (B^M_r(x_i)) \cap \Sigma = \{ x \in \Sigma; \tau(x) = d_M(x_i, x) < r \}.
\]
Since \(\{ x \in \Sigma; 2^k + c \leq \tau(x) \leq 2^{k+1} - c \} \subseteq A(2^k, 2^{k+1})\), the set \(A(2^k, 2^{k+1})\) also contains all the balls \(B_r(x_i), i = 1, \ldots, 2^{2k-1}\).
For \(x \in A(2^k, 2^{k+1})\),
\[
|H| \leq |H| + |(\nabla f)^{-1}| \leq \sup_{\Sigma} |H| + \sqrt{f} \leq \sup_{\Sigma} |H| + 2^k \leq 2^{k+1} \leq \frac{2}{r}.
\]
In the above inequality, we used Inequality (1.13) and \(\sup_{\Sigma} |H| \leq 2^k\) for \(k\) very large. Furthermore, let \(k\) be sufficiently large so that
\[
2^{-(k+1)} \leq r = 2^{-k} - 2^{-2k+1} c \leq \min\{1, i_0, 1/\sqrt{K_0}\}.
\]
Applying Lemma 5.1 to the balls \(B_r(x_i)\), we have
\[
\text{Vol}(B_r(x_i)) \geq \tau r^n \geq \tau 2^{-(k+1)n},
\]
where \(\tau = \omega_n e^{-2(n\sqrt{K_0}+2)}\). Hence
\[
V(2^k, 2^{k+1}) \geq \tau 2^{2k-1} 2^{-(k+1)n} = \tau 2^{-kn+2k-n-1}. \tag{5.10}
\]
Let $K$ be a very large integer satisfying $\tau 2^{8K-n-1} > \text{Vol}(\Sigma)$ and the conditions of $k$. Take $k_1 = 2K, k_2 = 6K + 1$. If (5.9) in Claim 1 fails, then
\[
V(2^{k_1}, 2^{k_2}) > 2^{4n} V(2^{k_1+2}, 2^{k_2-2}).
\]
If
\[
V(2^{k_1+2}, 2^{k_2-2}) \leq 2^{4n} V(2^{k_1+4}, 2^{k_2-4}),
\]
we are done, otherwise we repeat the process. After $j$ steps we have
\[
(5.11) \quad V(2^{k_1}, 2^{k_2}) > 2^{4nj} V(2^{k_1+2j}, 2^{k_2-2j}).
\]
Take $j = K$. Then $k_1 + 2j = 4K, k_2 - 2j = 4K + 1$. Thus (5.11) and (5.10) imply that
\[
(5.12) \quad \text{Vol}(\Sigma) \geq V(2^{k_1}, 2^{k_2}) \geq 2^{4nK} V(2^{4K}, 2^{4K+1}) \geq \tau 2^{8K-n-1}.
\]
This contradicts the assumption of $K$. Hence after finitely many steps, (5.9) must hold. Thus we proved Claim 1.

Now for each $\epsilon > 0$, let $k_1$ and $k_2$ satisfy (5.8) and (5.9). Take a smooth cut-off function $\varphi(t)$ with $0 \leq \varphi(t) \leq 1, |\varphi'(t)| \leq 1$ and
\[
(5.13) \quad \varphi(t) = \begin{cases} 
0, & t \leq 2^{k_1} \\
0 \leq \varphi'(t) \leq \frac{c_1}{2^{k_1}}, & 2^{k_1} \leq t \leq 2^{k_1+2} \\
n \frac{c_2}{2^{k_2}} \leq \varphi'(t) \leq 0, & 2^{k_2-2} \leq t \leq 2^{k_2} \\
0, & t \geq 2^{k_2}
\end{cases}
\]
where $c_1$ and $c_2$ are some positive constants. Define
\[
(5.14) \quad h(x) = e^{L + \frac{f(x)}{n}} \varphi(\rho(x)), \quad x \in \Sigma,
\]
where $L$ is a constant and $h(x)$ satisfies
\[
(5.15) \quad 1 = \int_{\Sigma} h^2(x) e^{-f} = e^{2L} \int_{A(2^{k_1}, 2^{k_2})} \varphi^2(\rho(x)).
\]
Choose a very small $\epsilon_0 > 0$ satisfying
\[
(5.16) \quad \sqrt{K_0} \left((n + 1) \omega_n^{-1} \epsilon_0\right)^{\frac{1}{n}} \leq 1,
\]
\[
(5.17) \quad \text{arcsin} \left[\sqrt{K_0} \left((n + 1) \omega_n^{-1} \epsilon_0\right)^{\frac{1}{n}}\right] \leq i_0.
\]
Note that for each $\epsilon < \epsilon_0$, by the definition of $h$ and (5.8), it holds that
\[
|\text{supp}(h)| \leq V(2^{k_1}, 2^{k_2}) \leq \epsilon \leq \epsilon_0.
\]
Hence (5.16) and (5.17) imply that $h(x)$ satisfies the restriction of $|\text{supp}(h)|$ provided in Lemma 5.2. Substituting $h(x)$ in (5.14) into the logarithmic
In the following, we estimate the integrals in (5.19). Thus (5.19) implies that

\[
\int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi^2(\rho(x)) \left( L + \frac{f}{2} + \ln(\varphi(\rho(x))) \right)
\leq C + 2 \int_{A(2^{k_1}, 2^{k_2})} e^{2L} |\varphi'(\rho(x))\nabla \rho + \frac{1}{2} \varphi(\rho(x)) \nabla f|^2,
\]

(5.18)

where \( C = \frac{1}{2} \sup_{\Sigma} |H_f|^2 + c(n) + \frac{m}{2} - \inf_{\Sigma} R - \inf_{\Sigma} \text{tr}_\Sigma \nabla^2 f \). Here we used the fact that \( \text{tr}_\Sigma \nabla^2 f \) must be bounded below by some constant. The reason is the following: Since \( \overline{M} \) has the sectional curvature bounded above by a positive constant, its Ricci curvature tensor \( \overline{Ric} \) is bounded above. By the equation \( \overline{Ric} + \nabla^2 f = \frac{1}{2} g \), we have that \( \nabla^2 f \) is bounded below by some constant 2-tensor. Now (5.18) implies that

\[
C \geq L + \int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi^2(\rho(x)) \left( f - \frac{\nabla f^2}{2} \right)
+ \int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi^2(\rho(x)) \ln(\varphi(\rho(x)))
- 2 \int_{A(2^{k_1}, 2^{k_2})} e^{2L} |\varphi'(\rho(x))\nabla \rho|^2
- 2 \int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi'(\rho(x)) \varphi(\rho(x)) \langle \nabla \rho, \nabla f \rangle.
\]

(5.19)

In the following, we estimate the integrals in (5.19).

By \( \nabla f^2 \leq f \), the first integral is non-negative.

Since the inequality \( t \ln t \geq -\frac{1}{e} \) holds for any \( 0 \leq t \leq 1 \),

\[
\int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi^2(\rho(x)) \ln(\varphi(\rho(x))) \geq -\frac{1}{2e} e^{2L} V(2^{k_1}, 2^{k_2}).
\]

By \( |\varphi'(t)| \leq 1 \) and \( |\nabla \rho| \leq 1 \),

\[
-2 \int_{A(2^{k_1}, 2^{k_2})} e^{2L} |\varphi'(\rho(x))\nabla \rho|^2 \geq -2e^{2L} V(2^{k_1}, 2^{k_2}).
\]

Note that \( 0 \leq \langle \nabla \rho, \nabla f \rangle = \frac{\nabla f^2}{\sqrt{f}} \leq \sqrt{f} = \frac{f}{2} \) and the property that \( 0 \leq \varphi' \leq \frac{a}{\sqrt{f}} \) in \( [2^{k_1}, 2^{k_1+2}] \), \( \varphi' \leq 0 \) in \( [2^{k_1+2}, 2^{k_2}] \). Then

\[
-2 \int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi'(\rho(x)) \varphi(\rho(x)) \langle \nabla \rho, \nabla f \rangle \geq -2e^{2L} \int_{A(2^{k_1}, 2^{k_1+2})} \frac{c_1}{2^{k_1+2}} \frac{2^{k_1+2}}{2^{k_1}} e^{2L} V(2^{k_1}, 2^{k_2}).
\]

Thus (5.19) implies that

\[
C \geq L - \left( \frac{1}{2e} + 2 + 4c_1 \right) e^{2L} V(2^{k_1}, 2^{k_2}).
\]

(5.20)
Using (5.9), we have
\[
C \geq L - \left( \frac{1}{2e} + 2 + 4c_1 \right) e^{2L} 2^{4n} V(2^{k_1}, 2^{k_2})
\]
\[
\geq L - \left( \frac{1}{2e} + 2 + 4c_1 \right) 2^{4n} \int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi^2(\rho(x))
\]
\[
\geq L - \left( \frac{1}{2e} + 2 + 4c_1 \right) 2^{4n}.
\]
(5.21)

So \( L \) is bounded above by a universal constant. On the other hand, using (5.8), we have
\[
1 = \int_{A(2^{k_1}, 2^{k_2})} e^{2L} \varphi(\rho(x)) \leq e^{2L} V(2^{k_1}, 2^{k_2}) \leq e^{2L} \epsilon.
\]
(5.22)

Since \( \epsilon \) can be arbitrarily small, \( L \) cannot be bounded above by a universal constant, which is a contradiction. This contradiction implies that the volume of \( \Sigma \) is infinite. So we finished the proof.

Before proving Theorem 1.5, we show some inequalities about the volume of \( B^M_r(p) \cap \Sigma \), where \( p \in \Sigma \). Suppose that \( \text{tr}_\Sigma \nabla^2 f \geq \frac{k}{2} \) for some constant \( k \). So (4.8) implies
\[
\Delta f + \frac{1}{2} R + \frac{1}{2} |(\nabla f)^\perp|^2 \leq \frac{1}{2} s,
\]
(5.23)

where \( s = m - k - \inf_{\Sigma} (R + |H|^2) + \sup_{\Sigma} |H_f|^2 \) is a nonnegative constant. Denote the set \( D_r = \{ x \in \Sigma : \rho(x) \leq r \} \), for \( r \geq r_0 \), where \( r_0 \) is chosen to let \( D_r \) have the positive measure. We define
\[
V(r) = \text{Vol}(D_r) = \int_{D_r} dv\Sigma,
\]
\[
\chi(r) = \int_{D_r} R dv\Sigma, \quad \text{and} \quad \eta(r) = \int_{D_r} |(\nabla f)^\perp|^2 dv\Sigma.
\]

By the co-area formula, it holds that
\[
V'(r) = \int_{\partial D_r} \frac{1}{|\nabla \rho|} dA,
\]
\[
\chi'(r) = \int_{\partial D_r} \frac{R}{|\nabla \rho|} dA, \quad \text{and} \quad \eta'(r) = \int_{\partial D_r} \frac{|(\nabla f)^\perp|^2}{|\nabla \rho|} dA.
\]

We will prove the following inequality:

**Proposition 5.2.** Under the above notation and assumption, for \( r \geq r_0 \), it holds that
\[
rV'(r) - sV(r) \leq \frac{4}{r} \eta'(r) - \eta(r) + \frac{4}{r} \chi'(r) - \chi(r).
\]
(5.24)
Proof. By Inequality \[ \text{(5.23)} \], the Stokes’ formula, and \[ \text{(4.3)} \], we have
\[
\frac{s}{2}V(r) - \frac{1}{2}r \eta(r) - \frac{1}{2}r \chi(r) \geq \int_{D_r} \Delta f dv_{\Sigma} = \int_{\partial D_r} \left( \nabla f, \frac{\nabla \rho}{|\nabla \rho|} \right)
\]
\[
= \int_{\partial D_r} \left( \nabla f, \frac{\nabla f}{|\nabla \rho| \sqrt{f}} \right) = \int_{\partial D_r} \frac{|\nabla f|^2 - |(\nabla f)^{\perp}|^2}{|\nabla \rho| \sqrt{f}}
\]
\[
= \frac{2}{r} \int_{\partial D_r} \frac{f - R - |(\nabla f)^{\perp}|^2}{|\nabla \rho|}
\]
\[
= \frac{r}{2} V'(r) - \frac{2}{r} \eta'(r) - \frac{2}{r} \chi'(r).
\]
\[
\square
\]

Remark 5.1. From the above proof, we know that for all \( r \geq r_0 \),
\[
\eta(r) + \chi(r) \leq sV(r).
\]

Proposition \[ \text{(5.2)} \] implies that

Lemma 5.3. Under the same notation and assumption as in Proposition \[ \text{(5.4)} \], for any \( r_2 \geq r_1 \geq \max\{r_0, 2\sqrt{s + 2}\} \),
\[
r_2^{-s}V(r_2) - r_1^{-s}V(r_1) \leq 4sr_2^{-s-2}V(r_2).
\]

Proof. Using \[ \text{(5.24)} \], we have
\[
(r^{-s}V(r))' = r^{-s-1} \left[ rV'(r) - sV(r) \right]
\]
\[
\leq r^{-s-2} \left[ 4\eta'(r) - r\eta(r) + 4\chi'(r) - r\chi(r) \right].
\]

Integrate (5.27) from \( r_1 \) to \( r_2 \). Then
\[
r_2^{-s}V(r_2) - r_1^{-s}V(r_1)
\]
\[
\leq \int_{r_1}^{r_2} 4r^{-s-2} \left[ \eta'(r) + \chi'(r) \right] dr - \int_{r_1}^{r_2} r^{-s-1} [\eta(r) + \chi(r)] dr
\]
\[
= 4r_2^{-s-2} [\eta(r_2) + \chi(r_2)] - 4r_1^{-s-2} [\eta(r_1) + \chi(r_1)]
\]
\[
+ \int_{r_1}^{r_2} [4(s + 2)r^{-s-3} - r^{-s-1}] [\eta(r) + \chi(r)] dr.
\]

Since \( r_2 \geq r_1 \geq 2\sqrt{s + 2} \),
\[
r_2^{-s}V(r_2) - r_1^{-s}V(r_1) \leq 4r_2^{-s-2} [\eta(r_2) + \chi(r_2)] - 4r_1^{-s-2} [\eta(r_1) + \chi(r_1)]
\]
\[
+ [\eta(r_1) + \chi(r_1)] \int_{r_1}^{r_2} [4(s + 2) - r^2] r^{-s-3} dr
\]
\[
\leq 4r_2^{-s-2} [\eta(r_2) + \chi(r_2)] - 4r_1^{-s-2} [\eta(r_1) + \chi(r_1)]
\]
\[
+ [\eta(r_1) + \chi(r_1)] [-4(r_2^{-s-2} - r_1^{-s-2})]
\]
\[
= 4r_2^{-s-2} [\eta(r_2) + \chi(r_2) - \eta(r_1) - \chi(r_1)]
\]
\[
(5.28) \leq 4sr_2^{-s-2}V(r_2).
\]
In the last inequality in (5.28), we used (5.25).

Lemma 5.3 also gives an alternative proof of polynomial growth of a properly immersed submanifold in a shrinking gradient Ricci soliton with bounded \(|H_f|\). More precisely,

**Corollary 5.1.** Let \(\Sigma\) be a properly immersed complete \(n\)-dimensional submanifold in an \(m\)-dimensional complete shrinking gradient Ricci soliton \((M^m, \mathbf{g}, f)\) with \(\text{tr}_{\Sigma} \nabla^2 f \geq \frac{k^2}{2}\) for some constant \(k\). Then, for \(p \in M\) fixed, there are some constants \(C > 0\) and \(r_0\) so that

\[
\text{Vol}(B^M_r(p) \cap \Sigma) \leq Cr^s \quad \text{for} \quad r \geq r_0,
\]

where \(s = m - k - \inf_{\Sigma} (\overline{R} + |H|^2) + \sup_{\Sigma} |H_f|^2\).

**Proof.** In Proposition 5.3 letting \(r_1\) fixed and taking \(r_2 = r\), we have

\[
r^{-s}V(r) - r_1^{-s}V(r_1) \leq 4sr^{-s-2}V(r).
\]

Then

\[
V(r) \leq \frac{1}{1 - 4sr^{-2}} \frac{V(r_1)}{r_1^s} \leq \frac{V(r_1)}{(1 - 4sr^{-2})r_1^s} \leq Cr^s.
\]

Now using the same argument as in the proof of Theorem 1.2 (iii), we get (5.29).

Remark 5.2. It is worth noting that the polynomial volume growth estimate in Theorem 1.2 is better than the one in Corollary 5.1 in the sense that \(l \leq s\).

To prove Theorem 1.5, we also need the following logarithmic Sobolev inequality by taking \(\mu = 1\), \(\alpha = \frac{n}{n+1}\) and \(\lambda = 1\) in Proposition 5.1.

**Lemma 5.4.** Let \(M\) and \(\Sigma\) be the same as in Proposition 5.1. Then

\[
\int_{\Sigma} h^2(\ln h) d\sigma - \frac{1}{2} \left( \int_{\Sigma} h^2 d\sigma \right) \ln \left( \int_{\Sigma} h^2 d\sigma \right) \leq \int_{\Sigma} |\nabla h|^2 d\sigma + \frac{1}{4} \int_{\Sigma} |H|^2 h^2 d\sigma + c(n) \int_{\Sigma} h^2 d\sigma,
\]

where the function \(h\) is the same as in Proposition 5.1 and \(c(n)\) is the constant \(c(n, \frac{n}{n+1}, 1)\) in Proposition 5.1.

Now we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** First, note that the property of bounded geometry of \(M\) and the soliton equation (5.1) of \(M\) imply that \(\text{tr}_{\Sigma} \nabla^2 f \geq \frac{k}{2}\) for some constant \(k\). Now we give the following
Claim 2. There are constants $C > 0$ and $\bar{r} > 0$ so that
\begin{equation}
V(r) \geq Cr \quad \text{for all } \ r \geq \bar{r},
\end{equation}
where $V(r) = \text{Vol}(D_r)$.

We will prove the claim by contradiction. Assume that for any $\epsilon > 0$ and any $\tilde{r} > 0$, there exists a number $r \geq \tilde{r}$ depending on $\epsilon$ and $\tilde{r}$ so that
\begin{equation}
V(r) \leq \epsilon r.
\end{equation}

For $t \geq \max\{r_0, 2\sqrt{s} + 2\}$, Lemma 5.3 implies
\begin{equation}
V(t + 1) \leq V(t) \frac{(t + 1)^s}{t^s} - \frac{1}{(t + 1)^2}.
\end{equation}

Using the basic inequality $\frac{1}{1-a} \leq 1 + 2a$ for $0 < a \leq \frac{1}{2}$, we have
\begin{equation}
V(t + 1) \leq V(t) \left(1 + \frac{1}{t}\right)^s \left(1 + \frac{8s}{(t + 1)^2}\right).
\end{equation}

Using $(1 + \frac{1}{t})^s \leq 2^s$, we get
\begin{equation}
V(t + 1) - V(t) \leq V(t) \left(1 + \frac{1}{t}\right)^{s+1} + \frac{C_1(s)}{t^2} - 1,
\end{equation}
where $C_1(s)$ is a constant depending on $s$. So
\begin{equation}
V(t + 1) - V(t) \leq V(t) \left(1 + \frac{1}{t}\right)^{[s]+1} + \frac{C_1(s)}{t^2} - 1
\end{equation}
\begin{equation}
\leq V(t) \frac{C_2(s)}{t},
\end{equation}
where $C_2(s)$ is a constant depending on $s$. The last inequality in (5.33) used Bernoulli inequality: $(1 + z)^\alpha \leq 1 + (2^\alpha - 1)z$ for $z \in [0, 1]$ and $\alpha \geq 1$. By (5.33), there exists a $\bar{r} \geq \max\{r_0, 2\sqrt{s} + 2\}$ depending on $s$, such that, for $t \geq \bar{r}$,
\begin{equation}
V(t + 1) \leq 2V(t).
\end{equation}

Hence, for $t \geq \bar{r} + 1$, (5.33) and (5.34) imply that
\begin{equation}
V(t + 2) - V(t - 1) \leq C_2(s) \left(\frac{V(t + 1)}{t + 1} + \frac{V(t)}{t} + \frac{V(t - 1)}{t - 1}\right)
\leq C_2(s) V(t) \left(\frac{2}{t + 1} + \frac{1}{t} + \frac{1}{t - 1}\right)
\end{equation}
\begin{equation}
\leq C(s) \frac{V(t)}{t},
\end{equation}
where $C(s) = 5C_2(s)$. In the last inequality of (5.35), we used $\frac{1}{t-1} \leq \frac{1}{t}$ for $t \geq 2$. 


We may further assume that for an \( \epsilon > 0 \) to be determined later, there is an integer \( r \geq \overline{r} + 1 \) so that
\[
V(r) \leq \epsilon r. \tag{5.36}
\]
Otherwise, (5.32) fails and hence Claim 2 is proved. For \( \epsilon \) and \( r \) in (5.36), define the set \( S \):
\[
S := \{ k \in \mathbb{N} : V(t) \leq 2\epsilon r \quad \text{for all integers } t \in [r, k]\}.
\]
Clearly \( r \in S \) and \( S \neq \emptyset \). By (5.35) and the definition of \( S \), for \( k \in S \) and \( t \in [r, k] \),
\[
V(t + 2) - V(t - 1) \leq C(s)\frac{2\epsilon r}{t} \leq 2\epsilon C(s). \tag{5.37}
\]

**Claim 3.** Any integer \( k \geq r \) is in \( S \).

We will prove Claim 3 by induction. Assume \( k \in S \). Let \( h(x) \) be the Lipschitz function on \( \Sigma \) with compact support defined by
\[
(5.38) \quad h(x) = \begin{cases} 
0, & \text{in } D(t - 1) \\
\rho(x) - (t - 1), & \text{in } D(t) \setminus D(t - 1) \\
1, & \text{in } D(t + 1) \setminus D(t) \\
t + 2 - \rho(x), & \text{in } D(t + 2) \setminus D(t + 1) \\
0, & \text{in } \Sigma \setminus D(t + 2).
\end{cases}
\]
Since \( \text{supp}(h) \subseteq D(t + 2) \setminus D(t - 1) \), (5.37) implies \( |\text{supp}(h)| < 2\epsilon C(s) \). We may choose \( \epsilon \) small enough such that \( 2\epsilon C(s) < \epsilon_0 \), where \( \epsilon_0 \) is the constant chosen by (5.16) and (5.17). So \( |\text{supp}(h)| < \epsilon_0 \). Substitute \( h(x) \) into the Log-Sobolev inequality in Lemma 5.4. Then
\[
\int_{D(t+2) \setminus D(t-1)} h^2(\ln h) d\sigma - \frac{1}{2} \left( \int_\Sigma h^2 d\sigma \right) \ln (V(t + 2) - V(t - 1)) \\
\leq \int_{D(t+2) \setminus D(t-1)} |\nabla h|^2 d\sigma + c(n)[V(t + 2) - V(t - 1)] + \frac{1}{4} \int_\Sigma |\nabla h|^2 h^2 d\sigma.
\]
Noting \( |\nabla h| \leq 1 \), the basic inequality \( s \ln s \geq -\frac{1}{e} \), for \( 0 \leq s \leq 1 \), the inequality \( |\nabla h|^2 \leq 2(|\nabla h| + |\nabla - \nabla h|^2) \), and the boundedness of \( |\nabla h|^2 \), we have
\[
- \frac{1}{2} \left( \int_\Sigma h^2 d\sigma \right) \ln (V(t + 2) - V(t - 1)) \\
\leq C[V(t + 2) - V(t - 1)] + \frac{1}{2} \int_\Sigma |\nabla - \nabla h|^2 h^2 \\
\leq C[V(t + 2) - V(t - 1)] + \frac{1}{2}[\eta(t + 2) - \eta(t - 1)], \tag{5.39}
\]
where \( C = c(n) + 1 + \frac{1}{2\epsilon} + \frac{1}{2} \sup_\Sigma |\nabla h|^2 \). We may let \( \epsilon \) be very small so that \( 2\epsilon C(s) \leq 1 \). By (5.37),
\[
- \ln (V(t + 2) - V(t - 1)) \geq -\ln(2\epsilon C(s)) \geq 0.
\]
Noting \( \int_{\Sigma} h^2 \geq V(t+1) - V(t) \) and using the above inequality, (5.39) implies

\[
[V(t+1) - V(t)] \ln(2eC(s))^{-1} \leq 2C[V(t+2) - V(t-1)] + \eta(t+2) - \eta(t-1).
\]

Iterating (5.40) from \( t = r \) to \( t = k \) and summing up give that

\[
[V(k+1) - V(r)] \ln(2eC(s))^{-1} \leq 2C[V(k+2) + V(k+1) + V(k) - V(r+1) - V(r) - V(r-1)]
+ \eta(k+2) + \eta(k+1) + \eta(k) - \eta(r+1) - \eta(r) - \eta(r-1)
\leq 6C[V(k+2) + 3\eta(k+2)]
\leq (6C + 3s) V(k+2)
\leq (12C + 6s) V(k+1).
\]

Here we used (5.25) and (5.34) in the third and last inequalities in (5.41) respectively. Then (5.41) implies that

\[
V(k+1) \leq V(r) \frac{\ln(2eC(s))^{-1}}{\ln(2eC(s))^{-1} - 12C - 6s}.
\]

Further choose \( \epsilon \) very small so that

\[
\frac{\ln(2eC(s))^{-1}}{\ln(2eC(s))^{-1} - 12C - 6s} \leq 2.
\]

Using the assumption (5.36), (5.42) reduces to

\[
V(k+1) \leq 2V(r) \leq 2\epsilon r.
\]

So \( k+1 \in S \). By induction, Claim 3 holds. Now, for any integer \( k \geq r \),

\[
V(k) \leq 2\epsilon r.
\]

Hence \( \Sigma \) must have finite volume, which contradicts Theorem 1.5. By this contradiction, we have proved Claim 2.

Now given a fixed point \( p \in M \), by (4.6), there are constants \( c \) and \( r_1 \) so that \( |\rho(x) - \Phi(x)| < c \) for \( \Phi(x) \geq r_1 \), where \( x \in \Sigma \). Then for \( r \geq r_1 \),

\[
D_r \subseteq B_{r+c}(p) \cap \Sigma.
\]

By (5.41) and Claim 2 for \( r \geq \max\{\bar{r} + c, r_1 + c, 2c\} \),

\[
\text{Vol}(B_r^M(p) \cap \Sigma) \geq \text{Vol}(D_{r-c}) \geq C(r - c) \geq \frac{C}{2} \bar{r}.
\]

Theorem 1.5 is proved. \( \Box \)
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