Quantum Gravity and the Algebra of Tangles

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Abstract

In Rovelli and Smolin’s loop representation of nonperturbative quantum gravity in 4 dimensions, there is a space of solutions to the Hamiltonian constraint having as a basis isotopy classes of links in $\mathbb{R}^3$. The physically correct inner product on this space of states is not yet known, or in other words, the $\ast$-algebra structure of the algebra of observables has not been determined. In order to approach this problem, we consider a larger space $\mathcal{H}$ of solutions of the Hamiltonian constraint, which has as a basis isotopy classes of tangles. A certain algebra $\mathcal{T}$, the “tangle algebra,” acts as operators on $\mathcal{H}$. The “empty state” $\psi_0$, corresponding to the class of the empty tangle, is conjectured to be a cyclic vector for $\mathcal{T}$. We construct simpler representations of $\mathcal{T}$ as quotients of $\mathcal{H}$ by the skein relations for the HOMFLY polynomial, and calculate a $\ast$-algebra structure for $\mathcal{T}$ using these representations. We use this to determine the inner product of certain states of quantum gravity associated to the Jones polynomial (or more precisely, Kauffman bracket).

1 Introduction

In the Rovelli-Smolin approach to nonperturbative quantum gravity in 4 dimensions \cite{1,2}, the “kinematical state space” has as a basis generalized links in $\mathbb{R}^3$. One obtains the physical state space by Dirac’s procedure for quantizing systems with constraints. Thus, one first takes a quotient of the kinematical state space by the action of diffeomorphisms of $\mathbb{R}^3$ (more precisely, those connected to the identity). Then, inside this quotient space one seeks solutions to the Hamiltonian constraint; these form the physical state space. A large set of solutions is known, corresponding simply to isotopy classes of unoriented links in $\mathbb{R}^3$. While there is much to be done towards finding the complete solution space of the Hamiltonian constraint, and clarifying technical issues related to regularization and operator ordering, the main “practical” problem is that the inner product on the physical state space is not known; it is presently merely a vector space, not a Hilbert space. (Indeed, it is not even known whether isotopy classes of unoriented links are truly normalizable elements of the physical
This is important because an inner product is essential in the probabilistic interpretation of a quantum theory. Alternatively, one could say that the problem is the lack of an adjoint on the algebra of linear operators on the physical state space.

Recall that an algebra $A$ is a $\ast$-algebra if there is a map $\ast : A \to A$ such that

$$(a^\ast)^\ast = a, \quad (\lambda a)^\ast = \overline{\lambda} a^\ast, \quad (a + b)^\ast = a^\ast + b^\ast, \quad (ab)^\ast = b^\ast a^\ast.$$ 

Until an operator algebra has been given the structure of a $\ast$-algebra, one can identify neither the observables (self-adjoint elements) nor the physical states (functionals $\psi : A \to \mathbb{C}$ such that $\psi(1) = 1$ and $\psi(a^\ast a) \geq 0$ for all $a \in A$). Thus the algebra of operators for a physical system should be a $\ast$-algebra.

For reasons of mathematical convenience and physical principle, it is usually assumed that the algebra of observables is a C*-algebra. Recently, beginning with the work of Jones [3], a profound connection has been found between C*-algebras, especially type II$_1$ factors, and link invariants. In this paper we attempt to exploit this connection to shed some light on the $\ast$-algebraic aspects of quantum gravity.

We begin by considering the structure of quantum gravity at spacelike infinity. Our treatment will be brief, as it is a natural generalization of the original work by Rovelli and Smolin, and it will only serve as a heuristic preparation for the material in the next section. We fix an asymptotically flat structure on $\mathbb{R}^3$, and instead of working only with links as states, we consider a generalization of links in which certain strands extend to infinity in an asymptotically geodesic fashion in $\mathbb{R}^3$. These will be regarded as embedded in the closed unit ball, $D^3$, and are a slight variation on what are commonly known in knot theory as tangles. In an obvious generalization of their original construction, there is a representation of the Rovelli-Smolin quantized loop observables on the vector space built up from generalized tangles having certain types of self-intersection. This is the kinematical state space. The key point is that the loop observables are “local,” that is, they do not affect the structure of the tangles at spacelike infinity. Thus the geometry of a tangle at spatial infinity, which in the next section we call “boundary data,” defines superselection sectors of the kinematical state space.

Then, rather than taking the quotient by all diffeomorphisms of $\mathbb{R}^3$, we do so only by those that extend to diffeomorphisms of $D^3$ equal to the identity on the boundary, $\partial D^3 = S^2$. This yields the diffeomorphism-invariant state space, which has as its basis isotopy classes of tangles with intersections, where isotopies are required to leave $S^2$ fixed. Finally, vectors in the diffeomorphism-invariant state space that are annihilated by the Hamiltonian constraint span the physical state space. More precisely, we only require that

$$\left( \int_{\mathbb{R}^3} H(x)f(x)d^3x \right)\psi = 0$$

for compactly supported $f$, so that the analysis of Rovelli and Smolin extends to the case of framed tangles. In particular, isotopy classes of honest framed tangles (with
no intersections) span a subspace of the space of physical states. For the most part we only consider such states. Thus we work with a theory whose state space $\tilde{H}$ has as a basis isotopy classes of tangles.

While somewhat technical, it is important here to mention the issue of equipping tangles with framings and orientations. Rovelli and Smolin’s original construction worked with unoriented links in $\mathbb{R}^3$. More recent work of Brügmann, Gambini and Pullin suggests that framed links are required to deal with regularization issues. Their work is closely related to the need for framings in Chern-Simons theory. In work whose relation to the above is not yet quite clear, Ashtekhar and Isham have introduced framed links, or more precisely “strips,” in a rigorous treatment of some aspects of the loop representation. Certainly, to make contact with the Reshetikhin-Turaev theory of link invariants, framings should be taken into account. In addition to framings, orientations may be required in theories of gravity coupled to matter. We thus take as a basis for $\tilde{H}$ isotopy classes of framed oriented tangles. This is probably the most economical approach in the long run, as the modifications necessary to treat the unframed or unoriented cases are easy, and tangles of all types are a promising approach to a central problem of quantum gravity: describing local excitations in a manifestly diffeomorphism-invariant manner.

In Section 2 we develop a framework for handling symmetries and other natural operators on the space $\tilde{H}$. The group of orientation-preserving diffeomorphisms of $S^2$ acts on $\tilde{H}$, and one may mod out by almost all of this group to obtain a reduced state space $H$. The remaining symmetries are described by a discrete group, the “tangle group.” The action of this group together with certain annihilation and creation operators generates an algebra we call the “tangle algebra.” It seems that the whole space $H$ may be built up from a certain vector $\psi_0$, somewhat analogous to the vacuum in ordinary quantum field theory, by applying operators in the tangle algebra.

The next step is to determine the physically correct $\ast$-structure of the tangle algebra. Since the elements of the tangle group represent symmetries, it is natural to assume that they are unitary. The difficulty is to find the adjoints of the annihilation and creation operators. To gain insight into this issue, in Section 3 we consider simpler representations of the tangle algebra, or “tangle field theories,” obtained as quotients of $H$ by certain relations, essentially the skein relations for the HOMFLY polynomial. The HOMFLY polynomial is closely related to the representation theory of the quantum groups $SL_q(n)$, and for unitarity of the tangle group action $q$ must be a root of unity. In fact, unitarity imposes enough constraints on these tangle field theories to calculate the adjoint of the annihilation operator in terms of the creation operator. As a special case of these tangle field theories, we obtain an unoriented tangle field theory corresponding to the Kauffman bracket (a normalized form of the Jones polynomial). This theory may be regarded as a reduction of quantum gravity, and states of this theory correspond to the states of quantum gravity obtained via Chern-Simons theory by Brügmann, Gambini and Pullin. Our calculations essentially determine the inner product of such states from the unitarity of the tangle
group action. In Section 4 we sketch a general method of obtaining tangle field theories from representations of quantum groups. A systematic investigation of these tangle field theories should shed more light on the $*$-algebraic aspects of quantum gravity.

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2 The Tangle Algebra

We define a tangle in a manifold with boundary $M$ to be an oriented smooth 1-dimensional submanifold $X$ of $M$, possibly with boundary, such that $\partial X \subset \partial M$ and such that $X$ meets $\partial M$ transversally. A tangle $X$ is thus a disjoint union of connected components, which are oriented circles contained in the interior of $M$ or oriented paths connecting two points on $\partial M$. The connected components of a tangle will be called strands, and call the points in $\partial X$ boundary points. Using the orientation on $X$, we may describe these boundary points as either incoming or outgoing points. If there are $n$ incoming boundary points, there are $n$ outgoing boundary points, and we call $n$ the boundary number of the tangle.

Now suppose that $M$ is a 3-manifold. A framing of a tangle $X \subset M$ is a smooth section $v$ of the tangent bundle $TM$ over $X$ such that for all $x \in X$, $v_x \not\in T_x X$; for boundary points $x \in X$ we require that $v_x$ is tangent to $\partial M$. A framed tangle may be visualized as a disjoint union of ribbons. Diffeomorphisms of $M$ act on framed tangles in $M$ in a natural way. We say two framed tangles $X$ and $X'$ are isotopic if there is a continuous one-parameter family $f_t$ of diffeomorphisms of $M$ such that $f_0$ is the identity, $X' = f_1(X)$, and $f_t$ is the identity on $\partial M$ for all $t$.

Usually knot theory considers tangles in $[0,1] \times \mathbb{R}^2$. For us, tangles will be taken in the ball $D^3$ unless otherwise specified. Let $\tilde{H}$ denote the vector space having as its basis isotopy classes of framed tangles. Note that

$$\tilde{H} = \bigoplus_{n=0}^{\infty} \tilde{H}_n$$

where $\tilde{H}_n$ is spanned by isotopy classes of framed tangles with boundary number $n$. We may further decompose the spaces $\tilde{H}_n$ as follows. For any tangle with boundary number $n$, we write the incoming points as $x^- = (x_1^-, \ldots, x_n^-)$ and the outgoing points as $x^+ = (x_1^+, \ldots, x_n^+)$. We have $x = (x^-, x^+) \in (S^2)^{2n} - \Delta$, where $\Delta$ is the set of even $2n$-tuples of points in $S^2$ at least two of which are equal. To each framed tangle we may associate the pair $(x, v)$, where $x \in (S^2)^{2n} - \Delta$ is as above, and

$$v = (v_{x_1^-}, \ldots, v_{x_n^-}, v_{x_1^+}, \ldots, v_{x_n^+}).$$

However, this association is not canonical, since we may choose any ordering of the incoming/outgoing boundary points. Let $C(n)$ denote the space of pairs $(x, v)$ such
that \( x \in (S^2)^{2n} - \Delta \) and \( v = (v_1^-, \ldots, v_n^-, v_1^+, \ldots, v_n^+) \), where \( v_i^\pm \) is a nonzero vector in \( T_{x_i}^\pm S^2 \). Let \( S_n \) denote the symmetric group. A pair \((\sigma, \tau) \in S_n \times S_n\) acts on \((x, v) \in C(n)\) by permuting the incoming boundary points \( x_i^-\) and tangent vectors \( v_i^-\) with \( \sigma\), and the outgoing boundary points \( x_i^+\) and tangent vectors \( v_i^+\) with \( \tau \). Thus to each framed tangle we may canonically associate boundary data \([x, v] \in B(n)\), where \( B(n)\) is the quotient space \( C(n)/(S_n \times S_n)\). Each space \( \mathcal{H}_n \) is thus a direct sum

\[
\mathcal{H}_n = \bigoplus_{[x,v] \in B(n)} \mathcal{H}_n[x, v],
\]

where \( \mathcal{H}_n[x, v] \) denotes the space spanned by isotopy classes of framed tangles with boundary data \([x, v] \in B(n)\). The reader may find this uncountable direct sum surprising. It is, of course, required by the fact that \( \mathcal{H} \) has an uncountable basis, while each \( \mathcal{H}_n[x, v] \) has a countable basis. It seems likely that in a more analytically sophisticated approach this direct sum would be replaced by a direct integral. Here, however, we take advantage of the fact that the spaces \( \mathcal{H}_n[x, v] \) are the fibers of a flat vector bundle over \( B(n) \).

To see this, first note that each diffeomorphism of \( S^2 \) extends to a diffeomorphism of \( D^3 \), unique up to isotopy. This intuitive but highly nontrivial result is due to Cerf, Munkres, and Smale \([3, 4, 5]\). It follows that \( \text{Diff}^+(S^2)\), the group of orientation-preserving diffeomorphisms of \( S^2 \), acts on isotopy classes of framed tangles. There is thus a representation \( \rho \) of \( \text{Diff}^+(S^2)\) on \( \mathcal{H} \). Note that if \( g \in \text{Diff}^+(S^2) \),

\[
\rho(g): \mathcal{H}_n[x, v] \to \mathcal{H}_n(g[x, v])
\]

where we define

\[
g[x, v] = [g(x_1^-), \ldots, g(x_n^+), dg(v_1^-), \ldots, dg(v_n^+)].
\]

It follows that for any \([x, v], [x', v'] \in B(n)\), we may identify the spaces \( \mathcal{H}_n[x, v] \) and \( \mathcal{H}_n[x', v'] \). This identification is not unique, since there are many \( g \in \text{Diff}^+(S^2) \) with \( g[x, v] = [x', v'] \). However, if \([x, v]\) and \([x', v']\) are close, we may take any \( g \in \text{Diff}^+(S^2) \) sufficiently close to the identity with \( g[x, v] = [x', v'] \), and identify \( \mathcal{H}_n[x, v] \) with \( \mathcal{H}_n[x', v'] \) via \( \rho(g) \). The map \( \rho(g) \) does not depend on the choice of such \( g \), so the spaces \( \mathcal{H}_n[x, v] \) are the fibers of a flat vector bundle over \( B(n) \).

This device reduces the study of the state space \( \mathcal{H} \) to the study of the spaces \( \mathcal{H}_n[x, v] \), where for each \( n \geq 0 \), \([x, v]\) is a single arbitrary element of \( B(n) \). Thus we fix once and for all distinct points \( x_i^\pm \) on \( S^2 \) and nonzero tangent vectors \( v_i^\pm \in T_{x_i}^\pm S^2 \) for \( i = 1, 2, 3, \ldots \), and let

\[
\mathcal{H}_n = \mathcal{H}_n[x, v]
\]

where \( x = (x_1^-, \ldots, x_n^-, x_1^+, \ldots, x_n^+) \) and \( v = (v_1^-, \ldots, v_n^-, v_1^+, \ldots, v_n^+) \). We may picture the points \( x_1^- , x_2^- , \ldots \) as lined up from left to right near the north pole of \( S^2 \), and \( x_1^+ , x_2^+ , \ldots \) as lined up from left to right near the south pole, with the tangent vectors
of $v_i^\pm$ pointing to the right. Of course, the actual geometry of the situation is irrelevant, as any choice of points may be brought into this position by a diffeomorphism. We let

$$H = \bigoplus_{n=0}^{\infty} H_n.$$ 

By forming the space $H_n$, we have almost reduced the space $\tilde{H}_n$ by the action of all orientation-preserving diffeomorphisms of $D^3$. However, diffeomorphisms which fix the equivalence class $[x,v]$ still act as symmetries of $H_n$. Indeed, even a diffeomorphism which fixes $(x,v)$ can act nontrivially on $H_n$. Since the space $H_n[x,v]$ is the fiber of a flat vector bundle over $B(n)$, the tangle group

$$T_n = \pi_1(B(n))$$

acts on $H_n$ by holonomy. This discrete symmetry group action on $H_n$ captures the diffeomorphism-invariance of the space $\tilde{H}$. We denote the representation of $T_n$ on $H_n$ by $\rho_n$, and we will also regard the operators $\rho_n(g)$ as operators on $H$ by extending them to be zero on all $H_m$ with $m \neq n$.

An element of $T_n$ may be represented as a loop in the space $B(n)$, or as a path in $C(n)$. It may thus be represented as a framed tangle in $[0,1] \times S^2$ with boundary number $2n$, having the points $(1,x_i^-)$ and $(0,x_i^+)$ as incoming and $(0,x_i^-)$ and $(1,x_i^+)$ as outgoing boundary points. Each strand must go either from a point of the form $(1,x_i^-)$ to one of the form $(0,x_j^-)$, or from one of the form $(0,x_i^+)$ to one of the form $(1,x_j^+)$. The framing at the boundary points must match up with the standard vectors $v_i^\pm$ in the obvious way. Moreover, the tangle cannot contain any embedded circles, and all the embedded line segments must be, up to orientation, of the form

$$t \mapsto (t,f(t))$$

for some $f:[0,1] \to S^2$. This sort of framed tangle is, in fact, a particular sort of framed braid on $S^2$. The action of an element of $T_n$ on an element of $H_n$ is illustrated in Figure 1. In the case of a theory of unoriented framed tangles we would enlarge the symmetry group from $T_n$ to the whole framed braid group $FB_{2n}(S^2)$, since there would no longer be a fundamental distinction between incoming and outgoing strands. In the case of a theory of tangles with neither orientation nor framing the symmetry group would just be the group $B_{2n}(S^2)$ of braids on $S^2$. (See [12] for the definition of these groups.)

Let the empty state, $\psi_0 \in H_0$, be the basis vector corresponding to empty set, which is vacuously a 1-dimensional submanifold of $D^3$. The state $\psi_0$ is not a non-degenerate ground state of a Hamiltonian, since all states in $H$ are annihilated by the Hamiltonian constraint. However, $\psi_0$ is analogous to the vacuum, in that many, perhaps all, vectors in $H$ may be obtained from $\psi_0$ by applying certain operators and taking linear combinations, as we now describe.
The creation operator

\[ c: H_n \to H_{n+1} \]

simply adds an extra strand to any tangle with boundary number \( n \); the new strand is required to be unknotted, untwisted, and to remain to the right of the existing strands. Figure 2 shows a tangle representing \( \psi \in H_n \) and the tangle representing \( c\psi \). For \( n \geq 1 \), the annihilation operator

\[ a: H_n \to H_{n-1} \]

moves the rightmost boundary points, \( x_n^- \) and \( x_n^+ \), slightly into the interior of \( D^3 \), and connects them with a smooth arc that is unknotted, untwisted, and remains to the right of the existing strands. We define \( a \) to be zero on \( H_0 \). Figure 3 shows a tangle representing \( \psi \in H_n \) and the tangle representing \( a\psi \). It is essential, but easy, to check that the annihilation and creation operators are well-defined.

We call the algebra of operators on \( H \) generated by the annihilation and creation operators together with the operators representing the tangle groups the tangle algebra, \( T \). It is a consequence of Alexander’s theorem \[13\] that all isotopy classes of framed oriented links may be written as \( a^n \rho_n(g) c^n \psi_0 \) for some sufficiently large \( n \), where \( g \in T_n \). It follows that every vector in \( H_0 \) is of the form \( A\psi_0 \) for some \( A \in T \). We conjecture that in fact every vector in \( H \) is of this form, i.e., that the empty state is a cyclic vector for the tangle algebra.

Simple physical considerations give some information about the Hilbert space structure of \( \tilde{H} \) and the reduced space \( H \). First, it is natural to assume that the representation \( \rho \) of \( \text{Diff}^+(S^2) \) on \( \tilde{H} \) is unitary, as it acts as symmetries. Since the spaces \( \tilde{H}_n \) are superselection sectors it is also natural to assume that they are orthogonal. At the level of reduced state spaces, we may assume that each representation \( \rho_n \) is unitary and the direct sum decomposition \( H = \bigoplus H_n \) is orthogonal. Let \( p_n: H \to H_n \) be the orthogonal projection, so that

\[ p_n = p_n^*, \quad p_np_m = \delta_{nm}p_n. \]

Then for all \( g \in T_n \) we have

\[ \rho_n(g)^* = \rho_n(g^{-1}), \]

\[ \rho_n(g)\rho_n(g)^* = \rho_n(g)^*\rho_n(g) = p_n. \]

It follows that to determine a *-algebra structure of \( T \), it suffices to describe the adjoints \( a^* \) and \( c^* \). Of course, it is possible that these do not lie in \( T \), in which case \( T \) must be extended by the operators \( a^*, c^* \) to become a *-algebra. In the next section we present a simplified model in which \( a^* \) is a constant times \( c \). This, of course, further justifies the analogy with annihilation and creation operators.
3 Tangle Field Theories

It is plausible that in some sense, quantum gravity should be 4-dimensional topological quantum field theory. A definition of topological quantum field theories has been given by Atiyah [14], and examples have been constructed in 3 dimensions [15, 16]. However, one expects quantum gravity to differ considerably from 3-dimensional topological quantum field theories, since in gravity there are local excitations. This is reflected in the fact that the reduced state space $\mathcal{H}$ is infinite-dimensional, with isotopy classes of tangles representing local excitations. Additionally, we have seen that $\mathcal{H}$ is a tangle field theory, that is, a representation of the tangle algebra $T$. It seems that some insight into quantum gravity may be gained by considering simpler tangle field theories formed as quotients of $\mathcal{H}$. As an example, we construct a family of tangle field theories related to the HOMFLY polynomial invariant of links. As shown by Reshetikhin and Turaev [7, 8], this invariant is associated to the quantum groups $SL_q(n)$. We obtain tangle field theories that are unitary representations of the tangle group only when the parameter $q$ is a root of unity.

Let $t, x, y$ be arbitrary nonzero complex numbers. For $n \geq 1$, let $I_n$ be the subspace of $\mathcal{H}_n$ spanned by the elements

$$t^{-1}y^{-1}\phi_+ - ty\phi_ - x\phi_0,$$

(1)

where $\phi_+, \phi_-, \phi_0 \in \mathcal{H}_n$ are the isotopy classes of three framed tangles with identical pictures except within a small disk, where they appear as in Figure 4, and

$$y\psi - \psi',$$

(2)

where $\psi, \psi' \in \mathcal{H}_n$ are identical except within a small disk, where they appear as in Figure 5. Let $I_0$ be the subspace of $\mathcal{H}_0$ be the space spanned by the elements (1) and (2), together with

$$\psi_{unknot} - t^{-1} - t x \psi_0,$$

(3)

where $\psi_{unknot} = ac\psi_0 \in \mathcal{H}_0$ is the isotopy class of an unknotted circle in $D^3$. Let $K_n = \mathcal{H}_n/I_n$. It is easy to check that the representation of $T$ on $\mathcal{H}$ gives rise to a representation of $T$ on the space

$$K = \bigoplus_{n=0}^{\infty} K_n.$$

Thus $K$ is a tangle field theory formed as quotient of $\mathcal{H}$. For any $\psi \in \mathcal{H}$, we write $[\psi]$ for the corresponding vector in $K$.

It is easy to calculate in the tangle field theory $K$, because the relations (1) and (2) are the skein relations for the HOMFLY polynomial invariant of links [17] (normalized as in the paper by Jones [3]) multiplied by $y^w$, where $w$ is the writhe of the link in
question. For any nonempty framed link $\psi \in H_0$, repeated use of these relations, together with (3), allows one to express $[\psi] \in K_0$ as a multiple of $[\psi_0]$.

Next, note that each space $H_n$ is an algebra, with the product $\psi \psi'$ of two framed tangles $\psi$ and $\psi'$ given by attaching the $i$th outgoing boundary point of $\psi'$ to the $i$th incoming boundary point of $\psi$. (This remarkable fact, that a space of states should have an algebra structure, also holds for the state space of $S^n$ in any topological quantum field theory satisfying Atiyah’s axioms.) The identity of $H_n$ is the vector $\psi_n = c^n \psi_0$.

Moreover, the subspace $I_n$ is an ideal of $H_n$, so the quotient $K_n$ inherits an algebra structure from $H_n$. We can use this to calculate $K_n$ for $n \geq 0$, as follows.

The tangle group $T_n$ has the braid group $B_n$ as a subgroup. Recall that $B_n$ is generated by elements $s_i$, $1 \leq i < n$, with relations

\[
s_i s_j = s_j s_i \quad \text{if} \quad |i - j| > 1,
\]

\[
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.
\]

We may regard $B_n$ as a subgroup of $T_n$ by letting $s_i$ switch the $i$th and $(i + 1)$st outgoing strands as in Figure 6. The braid group action on $H_n$ is related to its algebra structure as follows:

\[
\rho_n(gh)\psi_n = (\rho_n(g)\psi_n)(\rho_n(h)\psi_n)
\]

for any $g, h \in B_n$. We may use the skein relations (3) to show that elements of the form $[\rho_n(g)\psi_n]$ span $K_n$, where $g \in B_n$. Moreover, the skein relations imply that

\[
[\rho_n(s_i^2)\psi_n] = tyx[\rho_n(s_i)\psi_n] + (ty)^2[\psi_n].
\]

Let $H(q, n)$ denote the Hecke algebra with generators $g_i$, $1 \leq i < n$, and relations

\[
g_i g_j = g_j g_i \quad \text{if} \quad |i - j| > 1,
\]

\[
g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},
\]

\[
g_i^2 = (q - 1)g_i + q.
\]

We define $H(q, 1)$ to be $\mathbb{C}$. Suppose that

\[
ty = \lambda^{1/2} q^{1/2}, \quad x = q^{1/2} - q^{-1/2},
\]

where we choose, once and for all, square roots of $\lambda$ and $q$. Then for there is clearly a homomorphism

\[
H(q, n) \to K_{n-1}
\]

given by

\[
g_i \mapsto \lambda^{-1/2}[\rho_n(s_i)\psi_n].
\]
In fact, it follows from results of Turaev [18] that this is an isomorphism. As we shall show, in certain cases this isomorphism gives rise to a natural inner product on a quotient

\[ L = \bigoplus_{n=0}^{\infty} L_n \]

of \( K \). The summands \( L_n \) are orthogonal with respect to this inner product, and the tangle group \( T_n \) acts as unitary operators on \( L_n \). These are precisely the properties we argued for on physical grounds in the previous section, and should probably be added to the definition of a tangle field theory. Moreover, relative to this inner product, \( a^* \) is just a scalar multiple of \( c \).

As shown by Ocneanu [17] and Wenzl [19], for any \( z \in \mathbb{C} \) there exists a unique trace \( \tau \) on the inductive limit of the Hecke algebras such that \( \tau(1) = 1 \) and the Markov property \( \tau(g_n x) = z \tau(x) \) holds for all \( x \in H(q, n) \). Using the isomorphism above we obtain traces \( \tau_n \) on the algebras \( K_n \) determined by the properties

\[
\begin{align*}
\tau_n([\psi]) &= 1 \\
\tau_{n+1}([c\psi]) &= \tau_n([\psi]) \\
\tau_{n+1}([\rho_{n+1}(s_n) c\psi]) &= \lambda^{1/2} z \tau_n([\psi])
\end{align*}
\]

for all \( n \) and \( \psi \in H_n \). Note that

\[
\tau_{n+1}([\rho_{n+1}(s_n^{-1}) c\psi]) = \lambda^{-1/2}(q^{-1}z + q^{-1} - 1) \tau_n([\psi]).
\]

If

\[
y = (\lambda z/(q^{-1}z + q^{-1} - 1))^{1/2}
\]

and we define

\[
N = (z(q^{-1}z + q^{-1} - 1))^{1/2}
\]

with a consistent choice of square roots, we thus have

\[
\begin{align*}
\tau_{n+1}([\rho_{n+1}(s_n) c\psi]) &= y N \tau_n([\psi]), \\
\tau_{n+1}([\rho_{n+1}(s_n^{-1}) c\psi]) &= y^{-1} N \tau_n([\psi]).
\end{align*}
\]

These equations describe how the Markov moves affect the trace.

Henceforth we assume that equation (4) holds. Note that it implies \( N^{-1} = (t^{-1} - t)/x \), so that

\[
\psi_{unknot} = N^{-1} \psi_0.
\]

By the theory relating link and tangle invariants to Markov traces [3, 8, 18], it follows that

\[
[a^n \psi]/[\psi_0] = N^{-n} \tau_n([\psi])
\]

for all \( \psi \in H_n \). Moreover, if \( \psi \) is the isotopy class of a framed tangle, the quantity \( [a^n \psi]/[\psi_0] \) depends only on the isotopy class of the framed link \( a^n \psi \), the closure of \( \psi \).
When \(|q| = 1\) there is a unique \(*\)-structure on \(H(q, n)\) such that \(g_i^* = g_i^{-1}\). We transfer this to \(K_n\), making it into a \(*\)-algebra. Note that if \(|\lambda| = 1\),
\[
[\rho_n(g)\psi_n]^* = [\rho_n(g^{-1})\psi_n].
\]
If in addition \(|y| = 1\), the relation (2) implies that for \(\psi \in H_n\) the isotopy class of a tangle,
\[
[\psi]^* = [\psi^*],
\]
where \(\psi^* \in H_n\) is the isotopy class of the tangle with the opposite orientation, reflected about the \(xy\)-plane as in Figure 7. Here we assume, without loss of generality, that reflection of the points \(x_i^+\) and vectors \(v_i^+\) about the \(xy\)-plane yields \(x_i^-\) and \(v_i^-\), respectively. This operation on tangles extends uniquely to a \(*\)-structure on \(H_n\).

As shown by Ocneanu and Wenzl, the traces \(\tau_n\) are positive, that is, \(\tau_n(\psi^*\psi) \geq 0\) for all \(\psi \in K_n\), if and only if \(q\) and \(z\) satisfy
\[
qu = e^{\pm 2\pi i/\ell}, \quad z = \frac{q - 1}{1 - q^k},
\]
where \(k, \ell\) are integers with \(0 < k < \ell\). From now on we assume these conditions on \(q\) and \(z\), and assume \(|\lambda| = 1\). It follows from equations (4) and (5) that
\[
N = \frac{q^{-k/2} - q^{k/2}}{q^{1/2} - q^{-1/2}}, \quad y = \lambda^{1/2}q^{(1-k)/2}.
\]

We give the space \(K_n\) an “inner product” by setting
\[
\langle \psi, \phi \rangle = \begin{cases} \tau_n(\psi^*\phi) & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}
\]
for \(\psi \in K_n\) and \(\phi \in K_m\). This has all the properties of a true inner product except that it is not definite, that is, there are typically nonzero states \(\psi \in K_n\) with \(\langle \psi, \psi \rangle = 0\). We deal with these states of norm zero later; first we show that the tangle group action preserves this “inner product” on \(K_n\) and compute \(a^*\).

Let \(\psi, \phi \in H_n\) be isotopy classes of tangles. To show that the tangle group \(T_n\) preserves the “inner product” on \(K_n\) it suffices to note that
\[
\tau_n([\rho_n(g)\psi]^*(\rho_n(g)\phi)]) = \tau_n([\psi^*\phi])
\]
for any \(g \in T_n\). In fact, the isotopy classes \((\rho_n(g)\psi)^*(\rho_n(g)\phi)\) and \(\psi^*\phi\) are equal. This is easily seen using pictures. We also claim that
\[
\langle \psi, c\phi \rangle = N\langle a\psi, \phi \rangle
\]
for all \( \psi \in K_n \) and \( \phi \in K_{n-1} \). It suffices to consider the case where
\[
\psi = [\rho_n(g)\psi_n], \quad \phi = [\rho_{n-1}(h)\psi_{n-1}],
\]
for \( g \in B_n \) and \( h \in B_{n-1} \). We have
\[
\langle \psi, c\phi \rangle = N^n[a^n((\rho_n(g)\psi_n)^*(c\rho_{n-1}(h)\psi_{n-1}))]/[\psi_0].
\]
From Figure 8, it is easy to see that
\[
a^n((\rho_n(g)\psi_n)^*(c\rho_{n-1}(h)\psi_{n-1})) = a^{n-1}((a\rho_n(g)\psi_n)^*(\rho_{n-1}(h)\psi_{n-1))).
\]
It follows that
\[
\langle \psi, c\phi \rangle = N^n[a^{n-1}(a(\rho_n(g)\psi_n)^*(\rho_{n-1}(h)\psi_{n-1}))]/[\psi_0] = N\langle a\psi, \phi \rangle.
\]
We thus have \( a^* = N^{-1}c \).

Finally, to obtain a tangle field theory with a positive definite inner product, we simply define \( L_n \) to be the quotient of \( K_n \) by the subspace of \( \psi \in K_n \) such that
\[
\langle \psi, \phi \rangle = 0 \quad \forall \phi \in K_n.
\]
The space \( L_n \) inherits an inner product from \( K_n \), and since \( L_n \) is finite-dimensional it is a Hilbert space. Let \( L \) denote the Hilbert space direct sum of the spaces \( L_n \). Since the tangle group \( T_n \) preserves the “inner product” on \( K_n \), it has a unitary representation on \( L_n \). Note also that if \( \psi \in K \) has the property that \( \langle \psi, \phi \rangle = 0 \) for all \( \phi \in K \), the vectors \( a\psi \) and \( c\psi \) share this property, since
\[
\langle \psi, c\phi \rangle = N\langle a\psi, \phi \rangle.
\]
It follows that the operators \( a, c \) on \( K \) define operators on the quotient space \( L \), and \( L \) becomes a representation of the whole tangle algebra. In addition to the relation \( a^* = N^{-1}c \), it is worth noting that the operator \( ac \), which adds an extra unknotted circle to any tangle, satisfies
\[
ad = N^{-1}.
\]
To show this holds when applied to \( \psi_0 \) requires equation (8).

If we choose \( k = 2 \) and \( \lambda^{1/2} = -q^{1/4} \), the link invariant \( [a^n\psi]/[\psi_{\text{unknot}}] \) is equal to the the Kauffman bracket [20], which is just the Jones polynomial times \( (-q)^{-3w/4} \), where \( w \) is the writhe. The Kauffman bracket is an invariant of unoriented framed links. Thus in this case we obtain an unoriented tangle field theory, hence a reduction of quantum gravity. Moreover, in this case \( L_n \) may be identified with the Temperley-Lieb algebra [3]. It should be noted that the Kauffman bracket is implicit in the work of Brügmann, Gambini and Pullin [4], who obtain the Jones polynomial times a function of the writhe when constructing states of quantum gravity from Chern-Simons theory. The above results effectively determine the inner product on “Chern-Simons states” of quantum gravity on \( D^3 \) from the unitarity of the tangle group action.
4 Conclusions

While the above argument obtains an inner product on $L$, hence a $*$-structure for the tangle algebra as represented on $L$, it would be preferable to have physical grounds for choosing an inner product on $H$, or its unoriented analog. The problem, of course, is that we have little idea what the physical observables are in a manifestly diffeomorphism-invariant formulation of quantum gravity. Thus it seems worthwhile to examine a variety of other tangle field theories formed as reductions of $H$. For example, there should be an unoriented tangle field theory based on skein relations for the Kauffman polynomial, in which the Birman-Wenzl algebra takes the place of the Temperley-Lieb algebra [21].

More generally, given a representation $V$ of a quantum group (or more precisely, ribbon Hopf algebra), the work of Reshetikhin and Turaev [4, 8, 18] shows how to associate to any element of $H_n$ a linear transformation of $V^\otimes n$. If we let $K_n$ denote the range of $H_n$ in the space of linear transformations $\text{Hom}(V^\otimes n)$, then $K = \bigoplus K_n$ becomes a tangle field theory by the methods of the previous section. In particular, the HOMFLY tangle field theories are associated to the quantum groups $SL_q(n)$, while the Kauffman tangle field theories are associated to $SO_q(n)$ and $Sp_q(n)$. To further understand the $*$-algebraic aspects of the tangle algebra, it will be useful to determine which tangle field theories arising from quantum group representations admit inner products for which the tangle group action is unitary, and to calculate $a^*$ in these theories. These reductions of $H$ are especially interesting because they have many tantalizing connections with conformal field theory and 3-dimensional topological quantum field theories [22, 23].

As a further generalization, it would be interesting to consider theories based on framed tangles admitting some sort of self-intersections. This is desirable because only states built from self-intersecting links are not annihilated by the determinant of the metric. Indeed, whether states built from non-self-intersecting links are “physical” is a matter of dispute [2, 24], and there has been considerable work on finding solutions to the Hamiltonian constraint built from links with intersections [4, 23, 20, 27, 28]. While this topic is still not well understood, relevant mathematical techniques for dealing with links admitting self-intersections have recently been developed by Vassiliev [29] and, in subsequent work, Bar-Natan [30], Birman and Lin [31].

Finally, it is worth noting that tangle field theories are closely related to what knot theorists call “skein modules,” and that skein modules suggest generalizations of tangle field theories to 3-manifolds with boundary other than $S^2$. There is a review article on skein modules by Hoste and Przytycki [32].

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