Shadows of the Planck Scale:
The Changing Face of Compactification Geometry

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By studying the effects of the shape moduli associated with toroidal compactifications, we demonstrate that Planck-sized extra dimensions can cast significant “shadows” over low-energy physics. These shadows can greatly distort our perceptions of the compactification geometry associated with large extra dimensions, and place a fundamental limit on our ability to probe the geometry of compactification simply by measuring Kaluza-Klein states. We also discuss the interpretation of compactification radii and hierarchies in the context of geometries with non-trivial shape moduli. One of the main results of this paper is that compactification geometry is effectively renormalized as a function of energy scale, with “renormalization group equations” describing the “flow” of geometric parameters such as compactification radii and shape angles as functions of energy.

I. INTRODUCTION

In a recent paper [1], it was shown that the shape moduli associated with toroidal compactifications can have a number of important effects on the corresponding Kaluza-Klein spectrum: they induce level-crossing, they modify the mass gap, and in certain cases they permit extra dimensions to grow infinitely large without violating experimental constraints. These results suggest that shape moduli have the potential to drastically change our naïve expectations based on studying simple compactifications in which shape moduli are ignored or held fixed.

The purpose of this paper is to demonstrate several further surprising consequences of shape moduli. One of the crucial differences between volume moduli and shape moduli is that volume moduli are dimensionful and necessarily have energy scales associated with them; by contrast, shape moduli are dimensionless. Thus, even when the radii and volume of certain extra dimensions are taken to zero, the corresponding shape moduli do not vanish and can still play a significant role in affecting our understanding of low-energy phenomenology. Indeed, much like the smile of the Cheshire cat, the shape moduli can survive and ultimately distort our perceptions of purely low-energy physics — even when the extra dimensions to which they correspond are no larger than the Planck length!

As we shall see, this observation implies that it is impossible to verify the “true” compactification geometry experimentally — indeed, the whole notion of a fixed compactification geometry becomes experimentally meaningless. By contrast, we shall show that compactification geometry, much like other “constants” of nature, is effectively renormalized as a function of energy scale, with quantities such as compactification radii changing their apparent values as functions of the energy with which the compactification manifold is probed.

II. CORRESPONDENCE RELATIONS AND SHADOWING

In order to illustrate these ideas concretely, let us consider compactifications on general one-, two-, and three-dimensional tori. These tori are illustrated in Fig. 1, where opposite edges in all diagrams are periodically identified. The one-dimensional torus, of course, is nothing but a circle and has no corresponding shape moduli. By contrast, the two- and three-dimensional tori are described not only by radii but also by the shift angles θ and α_{ij}, which mix the periodicities associated with translations along the corresponding directions. In such cases, the shape moduli are the shift angles as well as the ratios of the radii. Note that tori with different shift angles are topologically distinct (up to modular transformations which will be discussed below). However, despite the appearance of such shape moduli, in all cases the compactification manifolds are flat.

Our goal is to study the extent to which various low-energy observers can determine the shapes of these tori by studying their associated Kaluza-Klein spectra. Towards this end, let us assume that the “true” compactification geometry is given by the three-torus shown in Fig. 1(c). Furthermore, let us assume that there is a hierarchy of length scales such that R_3 ≪ R_2 ≪ R_1. For example, R_3 might be near the Planck scale, while R_1 might be at the inverse TeV scale and R_2 might be at some intermediate scale. Of course, to a high-energy observer with access to energies E_{max} ≫ O(R_3^{-1}), the Kaluza-Klein spectrum will reveal the presence of all three dimensions of the torus. Such an observer can then determine all three radii R_i and shape angles α_{ij} through a detailed spectral analysis of the Kaluza-Klein states. However, for an observer with access to only intermediate energies O(R_3^{-1}) ≪ E_{max} ≪ O(R_1^{-1}), the third dimension will be inaccessible; the compactification man-
The main point of this paper is to demonstrate that the correspondence relations in Eqs. (1) and (2) are incorrect, even in the presence of a large hierarchy $R_3 \ll R_2 \ll R_1$, and must be replaced by relations which are far more non-trivial. Indeed, as we shall see, the relations in Eqs. (1) and (2) hold only when the shape moduli are ignored (i.e., when all shape angles are taken to be $\pi/2$). In the presence of non-rectangular shape angles, by contrast, we shall see that these relations completely fail to describe the process by which small extra dimensions can be “integrated out” when passing to larger and larger length scales. We stress that this failure occurs no matter how small the smallest radii become. Thus, this failure can have dramatic phenomenological consequences at low energies.

It is straightforward to determine the correct correspondence relations by comparing the Kaluza-Klein spectra in each case. In the case of the circle in Fig. 1(a), the Kaluza-Klein spectrum is given by

$$\mathcal{M}^2 = \frac{n_1^2}{\rho^2} ,$$  

(3)

where $n_1 \in \mathbb{Z}$. By contrast, for the general two-torus shown in Fig. 1(b), the Kaluza-Klein spectrum is instead given by

$$\mathcal{M}^2 = \frac{1}{\sin^2 \theta} \left[ \sum_{i=1}^{2} \frac{n_i^2}{\rho_i^2} - \sum_{i \neq j} n_i n_j \cos \theta \right]$$  

(4)

where $n_i \in \mathbb{Z}$. An explicit derivation of this result can be found, e.g., in Ref. [1]; note that the periodicities of the two-torus allow us to restrict our attention to the range $0 \leq \theta \leq \pi/2$ without loss of generality. Finally, for the general three-torus shown in Fig. 1(c), the Kaluza-Klein spectrum is given by [2]

$$\mathcal{M}^2 = \frac{1}{K} \left[ \sum_{i,j=1}^{3} \frac{n_i^2}{R_i^2} s_{jk} - \sum_{i \neq j} \frac{n_i n_j}{R_i R_j} (c_{ij} - c_{ik} c_{jk}) \right]$$  

(5)

where $k \neq i, j$, where $c_{ij} \equiv \cos \alpha_{ij}$ and $s_{ij} \equiv \sin \alpha_{ij}$, and where $K$ (the dimensionless squared volume of the parallelepiped in Fig. 1(c)) is given by

$$K \equiv 1 - \sum_{i < j} c_{ij}^2 + 2 \prod_{i < j} c_{ij} = \sum_{i < j} s_{ij}^2 + 2 \left( \prod_{i < j} c_{ij} - 1 \right) .$$  

(6)

Note that such a torus is physical only if $\alpha_{ij} + \alpha_{jk} > \alpha_{ik}$ for all combinations of unequal $(i, j, k)$; this bound is saturated in the degenerate limit when one of the torus periodicities lies in the plane of the other two.
Given these results, we can now determine the appropriate correspondence relations. In deriving these relations, we shall assume a hierarchy \( R_3 \ll R_2 \ll R_1 \) so that we can successively integrate out small extra dimensions when passing to larger length scales. Our procedure will be to disregard all Kaluza-Klein states whose masses exceed the appropriate reference energy (either high, intermediate, or low) and are therefore inaccessible to the corresponding observer.

The observer at highest energy clearly sees three dimensions worth of Kaluza-Klein states, and deduces the “true” geometry of the compactified space by comparing the measured Kaluza-Klein masses with Eq. (5). However, the observer at intermediate energy cannot perceive excitations in the \( R_3 \) direction, since functionally \( R_3 \to 0 \) for this observer. His attention is therefore restricted to states with \( n_3 = 0 \), and he attempts a spectral analysis of the remaining states via comparison with Eq. (6). This leads to the identifications

\[
\frac{1}{\sin^2 \theta} \frac{1}{r_i^2} = \frac{s_{2k}}{KR_i^3} \quad (i = 1, 2)
\]

\[
\frac{\cos \theta}{\sin^2 \theta} \frac{1}{r_{12}} = \frac{c_{12} - c_{13}c_{23}}{KR_1R_2}.
\]

This observer therefore deduces that the compactified space is a two-torus parametrized by \((r_1, r_2, \theta)\) given by

\[
\begin{cases}
  r_i &= s_{3i}R_i \\
  \cos \theta &= (c_{12} - c_{13}c_{23})/s_{13}s_{23}.
\end{cases}
\]

Note that both the radii \( r_i \) and the shape angle \( \theta \) are affected, leading to apparent values for \((r_1, r_2, \theta)\) which are not present in the original three-torus.

The lowest-energy observer, by contrast, misses the \( n_2 \) excitations as well. Upon comparing with Eq. (6), he therefore concludes that the compactified space is a circle of radius

\[
\rho = (\sin \theta)r_1 = \frac{\sqrt{K}}{s_{23}}R_1.
\]

Once again, this radius does not correspond to any periodicity in the original three-torus.

Mathematically, these results reflect the geometric “shadows” that successive smaller extra dimensions cast onto the larger extra dimensions when they are integrated out. As such, they indicate that the low-energy observer can see only those “projections” of the compactification space which are perpendicular to the extinguished dimensions. But given the assumed large hierarchy of length scales, the physical implications of this shadowing effect are rather striking. A small extra spacetime dimension — even one no larger than the Planck length! — is able to cast a huge shadow over all other length scales and their associated dimensions, completely distorting our low-energy perception and interpretation of the compactification geometry. Indeed, the physics which we would normally associate with the Planck scale (such as the angles that parametrize the shape of the Planck-sized extra dimensions relative to the larger extra dimensions) fail to decouple at low energies.

Let us consider an extreme example to illustrate this point. If \( \sqrt{K}/s_{23} \ll 1 \) in Eq. (6), then \( \rho \) can appear to be very small at low energies even though \( R_1 \) itself might be huge. Thus, the original three-torus would have a huge dimension \( R_1 \), yet this dimension would be completely invisible at low energies because of the “shadow” cast by the additional dimensions associated with \( R_2 \) and \( R_3 \). This distortion occurs even though these additional dimensions might be at the Planck scale! While this is similar to the invisibility mechanism discussed in Ref. [4], our point here is that the shadowing phenomenon is completely general and holds even when \( R_2 \) and \( R_3 \) are vanishingly small. In other words, our normal expectations concerning decoupling do not apply when non-trivial shape moduli are involved. Other examples and situations will be discussed below.

Of course, no observer at any energy scale can use this shadowing phenomenon in order to deduce the existence of an extra spacetime dimension beyond his own energy scale. Nevertheless, the observer’s interpretation of that portion of the compactification geometry accessible to him is completely distorted, leading him to deduce geometric radii and shape angles that have no basis in reality. Since the existence of an even smaller extra dimension beyond those already perceived can never be ruled out, this shadowing effect implies that one can never know the “true” compactification geometry. Even when light Kaluza-Klein states are detected and successful fits to the Kaluza-Klein mass formulae are obtained via spectral analyses, the presence of further additional dimensions with appropriate shape moduli can always reveal the previous successes to have been illusory.

We are not claiming that no “true” compactification geometry can ever exist. Indeed, if one takes the predictions of string theory seriously, then there is ultimately a true, maximum number of compactified dimensions, with associated radii and shape moduli. However, as an experimental question, one can never be satisfied concerning the true number of extra dimensions. Thus, our result implies that one can correspondingly never be certain of the nature of whatever compactification geometry is ultimately discovered. In this sense, the concept of a “true” compactification geometry does not exist.

* This restriction to states with \( n_3 = 0 \) is applicable unless there exist special cancellations in the Kaluza-Klein mass formula. We will discuss such special cases below, but they do not affect our results.
III. MODULAR TRANSFORMATIONS AND COMPACTIFICATION RADI I

It is important to stress that the effects embodied in the relations (3) and (4) cannot generally be undone either by changes of coordinate basis or by modular transformations of the higher- or lower-dimensional tori. Since the case of modular transformations is particularly important, it merits some discussion.

Because of the modular symmetries of the torus, it is possible to describe the topology of a given torus using a multitude of different values for the compactification radii and shape angles; only the corresponding Kaluza-Klein spectrum is physical and invariant under modular transformations. Thus, modular transformations are analogous to gauge transformations, providing redundant descriptions of the same physics.

Given this, the question then emerges as to whether the shadowing effect can be undone via such modular transformations. Might there exist an alternative, modular-equivalent description of the compactification radii and twist angles of either the original torus or the effective low-energy torus (or both) such that the relations in Eqs. (1) and (2) can be restored? It is relatively straightforward to show that the answer to this question is ‘no’, and we shall see an explicit example of this below. Thus, the effects of shadowing cannot be eliminated by exploiting modular symmetries; they persist no matter which modular-equivalent descriptions are used to describe the higher- or lower-dimensional tori. In other words, they are not “pure-gauge”.

On the other hand, the existence of modular symmetries indicates that we must more carefully refine our concept of “compactification radii” when discussing compactification manifolds with non-trivial shape moduli. In the case of toroidal compactifications, only the one-dimensional periodicity radius \( \rho \) has an absolute meaning (since circles have no associated modular symmetries); the periodicity radii of all higher-dimensional tori are not invariant under modular transformations, and can be adjusted. For example, a two-torus with parameters \((r_1, r_2, \theta)\) is topologically the same (and thus has the same Kaluza-Klein spectrum) as the infinite set of two-tori with parameters \((r_1', r_2', \theta')\) given by

\[
\begin{align*}
r_1' &= r_2 \sqrt{(c \cos \theta + d r_1/r_2)^2 + c^2 \sin^2 \theta} \\
r_2' &= r_2 \sqrt{(a \cos \theta + b r_1/r_2)^2 + a^2 \sin^2 \theta} \\
\sin \theta' &= \sin \theta (r_1 r_2/r_1' r_2')
\end{align*}
\]

or \( r_2' = r \sin \theta \) for general \( r_1, r_2, \) and \( \theta \). This explicitly demonstrates that shadowing is a physical effect rather than a modular (gauge) artifact.

However, note that the ratio \( r_2/r_1 \) is not modular invariant; this ratio can be adjusted even though the corresponding Kaluza-Klein spectrum is unaltered. For example, if \( r_2/r_1 = 1 \) with \( \theta = \pi/2 \), we can set \( r_2'/r_1' = \sqrt{1 + b^2} \) for arbitrarily large \( b \in \mathbb{Z} \) simply by taking \( a = d = 1 \) and \( c = 0 \). Such a hierarchy is clearly unphysical. How then can we properly define the notion of “hierarchy” that we have exploited in this paper?

The key to a proper definition of “hierarchy” is to focus purely on the (modular-invariant) Kaluza-Klein spectrum. Let us consider the case of a three-torus for simplicity; a sketch with two physical hierarchies is given in Fig. 2. If the low-energy Kaluza-Klein spectrum resembles that of a circle, with a single tower of equally spaced states with masses \( k/\rho \), and if this pattern exists up to some mass scale \( M' \) before additional unexpected states appear, then we may say that a hierarchy exists of magnitude \( \rho M' \gg 1 \). A similar procedure can be used to define the hierarchy \( M'' \gg M' \) for a third extra dimension, and so forth.

Hierarchies in Kaluza–Klein spectrum

\[
\begin{array}{cccc}
| \mathbf{M'} | & | \mathbf{M''} | & | \mathbf{M'''} | & | \mathbf{M''''} | \\
\hline
(n_1,0,0) & (n_1,n_2,0) & (n_1,n_2,n_3) \\
\hline
\end{array}
\]

FIG. 2. Physical hierarchies in the Kaluza-Klein spectrum. For this sketch, we have taken \( R_1 = 10R_2 = 20R_3 \) on a rectangular three-torus with \( \alpha_{12} = \alpha_{13} = \alpha_{23} = \pi/2 \). The sizes of the modular-invariant hierarchies are given by \( M' \) and \( M'' \).

Note that this also resolves the issue raised above concerning whether it is legitimate to restrict our attention to states with vanishing \( n_3 \) and \( n_2 \) when passing to lower energy scales. Anomalously light states with non-vanishing \( n_3 \) or \( n_2 \) exist only when we have chosen a poor modular “slice” (analogous to a gauge slice) on which to describe the physics. In other words, the lightest states may no longer be those states for which \( n_2 \)
or $n_3$ vanish. In all cases, however, the true modular-invariant size of the hierarchy can always be determined as discussed above, namely by taking $M'$ to be the mass scale at which a new tower of states appears. In a rough sense, this is equivalent to defining the hierarchy as the ratio of radii on the modular slice for which the sines of all shape angles are maximized (thereby minimizing the non-diagonal terms in the Kaluza-Klein mass formulas). We shall discuss these issues more fully in Ref. [2].

A similar remark holds for the correspondence relations in Eqs. (3) and (4). Strictly speaking, these relations hold as written only when the light states are those for which $n_3$ and $n_2$ respectively vanish. As discussed above, this occurs when the sines of the shape angles are large or when the hierarchies in the radii are sufficiently large. For example, in the case of a two-torus, Eq. (3) holds as written only when $r_1 \sin \theta \gg r_2$. On other modular slices, these relations must be modified according to Eq. (10).

Finally, we emphasize that in this paper we are considering the compactification geometry as deduced through the Kaluza-Klein spectrum. If string theory is the fundamental theory, then winding modes also exist; whether or not such modes affect the light spectrum depends on the relation between the compactification radii and the string scale. If the string scale exceeds $R_3^{-1}$ where $R_3$ is the smallest radius, then the winding modes will all be heavy and play no role in this analysis. Other configurations may be more complicated, and will be discussed in Ref. [2]. Similar remarks also apply for tori with background antisymmetric tensor fields [2].

IV. EXAMPLES OF SHADOWING

Let us now give some examples of the shadowing effects embodied in Eqs. (3) and (4). Our purpose is not to propose a particular set of numerical parameters for specific phenomenological purposes, but merely to illustrate the different phenomenological possibilities that shadowing provides. In each case, we shall assume that the “true” compactification manifold is a three-torus and determine how this three-torus is perceived at various energies.

Let us begin by assuming $R_1^{-1} = 10^3$ GeV, $R_2^{-1} = 10^{11}$ GeV, and $R_3^{-1} = 10^{19}$ GeV. We shall also take $\alpha_{12} = \pi/2$, and $\alpha_{13} = \alpha_{23} = \pi/3$. The most straightforward way to analyze the Kaluza-Klein spectrum is to write the Kaluza-Klein masses in Eq. (3) in the “diagonal” form

$$M^2 = \sum_{i=1}^{3} \tilde{n}_i^2 R_i^2$$

where the eigenvalues are given by

$$\tilde{R}_1^{-1} \approx 10^3 \text{ GeV} ,$$

$$\tilde{R}_2^{-1} \approx 10^{11} \text{ GeV} ,$$

$$\tilde{R}_3^{-1} \approx \sqrt{2} \times 10^{19} \text{ GeV}$$

and where the eigenvectors are given by

$$\tilde{n}_1 \approx n_1 - (2.5 \times 10^{-25}) n_2 - (5 \times 10^{-17}) n_3$$

$$\tilde{n}_2 \approx (2.5 \times 10^{-41}) n_1 + n_2 - (5 \times 10^{-9}) n_3$$

$$\tilde{n}_3 \approx (5 \times 10^{-17}) n_1 + (5 \times 10^{-9}) n_2 + n_3 .$$

Given this form, we immediately see that the lowest-lying Kaluza-Klein states are those with only $n_1$ non-zero. For example, if $n_2$ is non-zero, then we immediately obtain contributions to $M$ in the neighborhood of $10^{11}$ GeV. Note that it is indeed possible to cancel $n_2$ even if $n_2 = 1$ by taking $n_1 \approx -4 \times 10^{40}$ or $n_3 \approx 2 \times 10^8$. However, these values make $\tilde{n}_1$ or $\tilde{n}_3$ extremely large, again inducing contributions to $M$ of size exceeding $10^{11}$ GeV. A similar argument applies for contributions with non-zero $n_3$. From this, we conclude that the Kaluza-Klein spectrum in this example exhibits two physical hierarchies: one between $10^3$ GeV and $10^{11}$ GeV, and one between $10^{11}$ GeV and $10^{19}$ GeV. The Kaluza-Klein spectrum in this example therefore resembles that shown in Fig. 2.

The observer with energies below $10^{11}$ GeV can detect only the $n_1$ excitations. Adding together the contributions from $\tilde{n}_1$, $\tilde{n}_2$, and $\tilde{n}_3$, we see that the masses of these states are given by

$$M^2 \approx n_1^2 \left( 1.5 \times 10^6 \text{ GeV}^2 \right) .$$

Thus, this observer concludes that the compactification space is a circle of radius $\rho^{-1} \approx \sqrt{3/2} \times 10^3$ GeV, in accordance with Eq. (3). Note that while the above results are only approximate, the result in Eq. (4) is indeed exact. Likewise, the observer with energies below $10^{19}$ GeV sees a two-torus with radii $r_1^{-1} = (2/\sqrt{3}) \times 10^3$ GeV, $r_2^{-1} = (2/\sqrt{3}) \times 10^{11}$ GeV, and twist angle $\theta = 71^\circ$. This occurs even though the true “base” of the original three-torus is completely rectangular!

In the above example, the numerical distortions of the low-energy parameters relative to the parameters of the original three-torus are not large. However, these distortions are significant, they persist over the whole hierarchy, and they do not disappear even as the smallest dimension(s) are taken to zero size. Indeed, the lowest-energy observer sees the regularly spaced Kaluza-Klein states in Eq. (4) stretching over eight orders in magnitude in energy, yet no corresponding radius of this size actually exists in the “true” compactification geometry. It is only the presence of two further extra dimensions, many orders of magnitude smaller, that causes this distortion!

These shadowing effects become even more dramatic in cases where we approach a limit $\alpha_{13} + \alpha_{23} \to \alpha_{12}$ in the original three-torus. In such cases, the orientation of the Planck-sized extra dimension associated with $R_3$ is highly “squashed” relative to the two larger dimensions associated with $R_1$ and $R_2$. For example, let us assume
\(\alpha_{12} = \pi/2\), as before, but let us now take \(\alpha_{13} = \pi/3 + t\) and \(\alpha_{23} = \pi/6 + t\) where \(t \ll 1\). Even though the “base” of this three-torus is actually rectangular with \(\alpha_{12} = \pi/2\), this base will appear to a low-energy observer as if it has a nearly vanishing shape angle \(\theta \sim \sqrt{7}\) after the Planck-sized extra dimension is integrated out. In other words, the squashing of the original Planck-sized extra dimension relative to the large dimensions is perceived by a low-energy observer as a squashing of the two large dimensions with respect to each other! We stress that this illusion is wholly due to the existence of the third Planck-sized extra dimension. Note that even though this example involves “squashed” extra dimensions, the use of Eqs. (8) and (9) is justified provided the hierarchy of radii is sufficiently large compared to the degree of squashing.

Further developing this example, let us also assume that the base radii in the original three-torus have equal lengths, \(R_1 = R_2\). We then find effective radii \(r_1/r_2 = \sqrt{3}\) for the resulting two-torus. Remarkably, this result (a nearly squashed two-torus whose two radii have an algebraic irrational ratio) are exactly the preconditions needed for the invisibility mechanism presented in Ref. [1]. Indeed, we now see that the large values for the parameter \(\tau\) discussed in Ref. [1] can be realized purely as the result of shadowing from a three-torus in which \(s_{23}/s_{13} \gg 1\).

Other interesting phenomenologies are also possible. For example, even a rectangular two-torus can sometimes be nothing but a low-energy illusion; one configuration that accomplishes this is to take \(\alpha_{12} = \pi/3\), while \(\alpha_{13} = \alpha_{23} = \pi/4\). Thus, even if the perceived shape moduli appear to be trivial to a low-energy observer, non-trivial shadowing may still be at work in producing this effect.

Non-trivial shape moduli can also be used to generate physical hierarchies when extra dimensions are integrated out. For example, even if \(R_1\) and \(R_2\) are of equal magnitude with \(\alpha_{12} = \pi/2\), the third dimension may have \(\alpha_{13} \to 0\). Even though this third dimension is Planck-sized, its severe orientation induces a physical hierarchy between the two large dimensions. In other words, a low-energy observer will observe \(r_2/r_1 \gg 1\) even though \(R_1 = R_2\).

Further examples and their phenomenological implications will be discussed in Ref. [3].

V. CONCLUSIONS: SHADOWING, GEOMETRY, AND RENORMALIZATION

The main result of this paper is that our perception of the compactification geometry associated with already-discovered large extra dimensions can be significantly distorted by the presence of additional, as-yet-undiscovered smaller dimensions. As we go to higher and higher energies and discover these additional dimensions, our description of the compactification manifold changes — not merely in its dimensionality but also in the radii and shape angles that parametrize all length scales of this geometry. Indeed, our perception of very simple geometric quantities such as the radii and shape moduli associated with the largest (and experimentally accessible) extra dimensions continually evolves as a function of the energy with which we probe this manifold — even though the largest extra dimensions are already detected and their geometric properties are already presumed known.

Of course, this is not a new concept in physics: this is nothing but renormalization. Thus, in this sense, we see that the apparent compactification geometry is not fixed at all, but rather undergoes renormalization much like other “constants” of nature. Indeed, the correspondence relations in Eqs. (8) and (9) serve as “renormalization-group equations” which describe the flow of the perceived geometric parameters associated with the largest extra dimensions as we pass through the thresholds associated with additional, smaller extra dimensions. Moreover, as we discussed, this renormalization-group evolution cannot be undone through modular transformations. This evolution is therefore a truly physical effect, one which corresponds to perceived changes in topology as well as geometry.

It is important to understand the precise sense in which shadowing can be considered as renormalization. Clearly, in different energy ranges, we are employing a series of different effective field theories in order to describe the Kaluza-Klein spectrum: the effective field theory at the lowest energy scales has a single parameter \(\rho\); the effective field theory at intermediate energies has three parameters \((r_1, r_2, \theta)\); and the effective field theory at still higher energies has six parameters \(\{R_i, \alpha_{ij}\}\). The correspondence relations in Eqs. (8) and (9) are thus properly viewed as matching conditions (or threshold relations) between different effective field theory descriptions of the same physics at different energy scales. Indeed, these matching conditions reflect nothing more than our requirement that the physical Kaluza-Klein spectrum remain invariant as we change our description of the physics by changing the cutoffs inherent in our sequence of effective field theories.

However, the cumulative effect of such threshold corrections as we pass between different effective theories is precisely what is usually meant by renormalization. Indeed, if we imagine extrapolating our calculations to incorporate a continuing series of hierarchies corresponding to a continuing series of extra dimensions, then the corresponding series of matching conditions constitutes a renormalization group “flow”. Under this flow, the values of parameters such as the radius of the largest extra dimension, be it \(\rho\) or \(r_1\) or \(R_1\), evolve in a non-trivial way due to the presence of non-trivial shape moduli. In other words, such parameters are renormalized.

We stress that this “renormalization” is a purely classical effect, one which arises for purely geometric rea-
sons. As such, it does not incorporate further quantum-mechanical effects which may arise due to the quantum field-theoretic renormalization of the Kaluza-Klein masses. Indeed, implicit in our previous discussions has been the assumption that the Kaluza-Klein mass spectrum is itself a physical observable, one which can be measured independently of other parameters in the theory. Of course, in a more general interacting theory, the renormalization flow of the compactification geometry may receive further quantum-mechanical contributions.

We also stress that in this paper we have considered only the simplest case of flat, toroidal compactifications without torsion. In principle, one can also study more complicated manifolds with more complicated Kaluza-Klein spectra.

In all cases, however, our main observation stands: quantities such as compactification radii — quantities which one might have naively assumed to be fixed once the corresponding extra dimensions are discovered — are not fixed at all. Instead, they are effectively renormalized as we pass to higher and higher energies and as additional extra dimensions become apparent. Since one can never be satisfied experimentally that one has discovered the totality of possible extra dimensions, this process need not terminate. It then becomes experimentally meaningless to speak of a “true” compactification geometry, in exactly the same way as it is meaningless to speak of the “true” electron charge.

There have been several recent discussions relating compactification geometry and renormalization. These include the “deconstruction” idea, as well as the AdS/CFT correspondence in the context of higher-dimensional models with localized gravity. Yet each of these cases is quite different from the shadowing effect we are discussing here. In the localized gravity/AdS case, the apparent compactification geometry is fixed; what changes is the renormalization scale on a particular brane as it moves through the AdS geometry. Likewise, in the deconstruction case, extra dimensions are generated as the result of certain fields condensing; this change has nothing to do with the geometry of the space itself. By contrast, our results hinge purely on the geometric properties of the compactification space and its manifestations at different energy scales.

The implications of the shadowing effect are likely to be profound. Rather than think of compactification geometry as fixed and immutable, we instead must think of it as something renormalizable. This clearly raises a number of provocative questions. What are the properties of the renormalization flow of compactification geometry as we approach the fundamental scale of quantum gravity where the whole notion of a continuous spacetime might break down? If string theory is the correct underlying theory, how can we incorporate winding modes (and ultimately T-duality) into this picture? Conversely, might our own four-dimensional spacetime only appear to be large and flat as a consequence of shadowing from additional spacetime dimensions? Might this provide a new approach to the cosmological constant problem? Indeed, in what sense is spacetime geometry knowable at all — are there analogues of renormalization-group invariants?

There is an old question in mathematical physics: Can one hear the shape of a drum? Clearly, our answer is that drums have no absolute shape. Instead, the shape of the drum depends on how well one listens.

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[1] K.R. Dienes, hep-ph/0108115 (Phys. Rev. Lett., in press).
[2] K.R. Dienes and A. Mafi, Compactification on Manifolds with Non-Trivial Shape Moduli, to appear.
[3] See, e.g., N. Kaloper et al., Phys. Rev. Lett. 85 (2000) 928 [hep-ph/0002001].
[4] N. Arkani-Hamed, A.G. Cohen and H. Georgi, Phys. Rev. Lett. 86 (2001) 4757 [hep-ph/0104005]; C.T. Hill, S. Pokorski and J. Wang, hep-th/0104035.
[5] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231 [hep-th/9711200].
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [hep-ph/9905221]; Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].