On the Twisted $K$-Homology of Simple Lie Groups

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Abstract

We prove that the twisted $K$-homology of a simply connected simple Lie group $G$ of rank $n$ is an exterior algebra on $n - 1$ generators tensor a cyclic group. We give a detailed description of the order of this cyclic group in terms of the dimensions of irreducible representations of $G$ and show that the congruences determining this cyclic order lift along the twisted index map to relations in the twisted $Spin^c$ bordism group of $G$. 

Contents

1 Introduction

1.1 Six Interpretations of Twisted $K$-Theory

1.2 Results

1.3 Techniques and Overview

2 Twisted $K$-Theory and the Rothenberg-Steenrod Spectral Sequence

2.1 Twisted Homology Theories

2.2 The Twisted Rothenberg-Steenrod Spectral Sequence

3 Tate Resolutions and Tor$_{K^*G}(\mathbb{Z}, \mathbb{Z})$

3.1 Tor for $SU(n + 1)$ and $Sp(n)$

3.2 Tor for the Exceptional Groups

3.3 Proof of Theorem 1.1

4 Generating Varieties, the Cyclic Order of $K^*G$, and Tor$_{K^*G}(\mathbb{Z}, \mathbb{Z})$

4.1 Generating Varieties and Holomorphic Induction

4.2 Subvarieties of $\Omega SU(n + 1)$ and $\Omega G_2$

4.3 Generating Varieties for $\Omega Sp(n)$

4.4 The Tor Calculation for $Spin(n)$

4.5 Poincare-Dual Bases and the Cyclic Order of $K^*G$

5 Twisted $Spin^c$ Bordism and the Twisted Index

5.1 A Cocycle Model for Twisted $Spin^c$ Bordism

5.2 Twisted Nullbordism and the Geometry of the Cyclic Order

5.3 Representing the Exterior Generators of Twisted $K$-Homology

References

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1 Introduction

By way of motivation we present six interpretations of twisted $K$-theory. These interpretations inform the methods and perspectives adopted in the paper but are otherwise unnecessary for what follows. We then summarize our results on the twisted $K$-homology of simple Lie groups and overview our main techniques, namely the twisted Rothenberg-Steenrod spectral sequence, Tate resolutions, Bott generating varieties, and twisted $Spin^c$ bordism.

1.1 Six Interpretations of Twisted $K$-Theory

1.1.1 1-Dimensional Elements in Elliptic Cohomology

A twisting on a space $X$ of a cohomology theory represented by a spectrum $R$ is a bundle of spectra on $X$ with fibre $R$ and the associated twisted cohomology of $X$ is given by the homotopy classes of sections of this bundle. Such twistings are classified by maps from $X$ to the classifying space $BAut R$ of homotopy automorphisms of the spectrum $R$. If $R$ is an $A_\infty$ ring spectrum, the classifying space $BGL_1 R$ of homotopy units in $R$ maps to $BAut R$ and thereby classifies a subset of the twistings—we refer to these twistings as elementary.

The classifying space $BGL_1 HC$ for elementary twistings of ordinary cohomology with complex coefficients is $B\mathbb{C}^*$; here $HC$ denotes the Eilenberg-MacLane spectrum for $\mathbb{C}$, and both $\mathbb{C}$ and $\mathbb{C}^*$ have the discrete topology. There is a map $B\mathbb{C}^* \to \mathbb{Z} \times BU$ of this classifying space into the representing space for $K$-theory; any twisting $X \to BGL_1 HC$ for the ordinary cohomology of $X$ therefore determines a $K$-theory class on $X$. Of course, there is a natural geometric interpretation of the $K$-theory classes arising in this way, namely as the classes represented by flat line bundles on $X$. The twisted cohomology of $X$ is simply the cohomology of $X$ with coefficients in the line bundle, reinterpreted as the homotopy classes of sections of an associated $HC$ bundle.

The classifying space $BGL_1 K$ for elementary twistings of complex $K$-theory splits, as an infinite loop space, as $T \times S$. The factor $T$ is a $K(\mathbb{Z},3)$ bundle over $K(\mathbb{Z}/2,1)$ which splits as a space but has nontrivial infinite loop structure classified by $\beta Sq^2 \in H^3(H(\mathbb{Z}/2);\mathbb{Z})$. There is a natural infinite loop map $T \to TMF$ from $T$ to the representing space for topological modular forms, and so by projecting through $T$ a map $BGL_1 K \to TMF$. In particular an elementary twisting of $K$-theory for $X$ determines a TMF-class on $X$. (Notice that TMF is the analog of real $K$-theory, that is of $KO$, and so the map $BGL_1 K \to TMF$ corresponds to the composite $BGL_1 HC \to \mathbb{Z} \times BU \to \mathbb{Z} \times BO$; it is not known whether there exists an appropriate factorization $BGL_1 K \to E \to TMF$ for every elliptic spectrum $E$.) The geometric interpretation of these TMF classes is simplified if we restrict our attention to those classes coming from twistings involving only the $K(\mathbb{Z},3)$ factor of $T$. Such a twisting is determined by a map $X \to K(\mathbb{Z},3)$ or equivalently by a $BS^1$ bundle on $X$. We think of this bundle as a stack locally isomorphic to the sheaf of line bundles on $X$ and as such a 1-dimensional 2-vector bundle on $X$. In this sense we imagine the TMF classes coming from $K$-theory twistings as 1-dimensional elliptic elements and twisted $K$-theory as $K$-theory with coefficients in this "elliptic line bundle".
1.1.2 Projective Hilbert Space Bundles

There is a very simple and well known reformulation of twistings of K-theory as projective Hilbert space bundles and of the corresponding twisted K-theory groups as families of Fredholm operators on these bundles. Indeed, the space of unitary operators on Hilbert space is contractible, so the group of projective unitary operators has the homotopy type of $BS^1$. As such a twisting $\alpha : X \to K(\mathbb{Z}, 3)$ of K-theory determines a projective bundle $\mathcal{H}(\alpha)$ of Hilbert spaces on $X$. The space of Fredholm operators on a Hilbert space has the homotopy type of $\mathbb{Z} \times BU$ and depends only on the projectivization of the Hilbert space. Sections of the $\mathbb{Z} \times BU$ bundle associated to the twisting $\alpha$ can therefore be thought of as Fredholm operators on the projective bundle $\mathcal{H}(\alpha)$. It remains to develop a general index theory for elliptic operators on these projective bundles, but substantial progress has been made by Mathai, Melrose, and Singer [20], who prove an index theorem in the case that the twisting $\alpha$ is a torsion class in $H^3(X; \mathbb{Z})$.

1.1.3 $K(\mathbb{Z}, 2)$-Equivariant K-Theory

We would like to discuss an algebro-geometric model for twisted K-theory, and the proper formulation is suggested by reinterpreting twisted K-theory as a $K(\mathbb{Z}, 2)$-equivariant theory; this formulation will also hint at connections with the representation theory of loop groups. As before, a twisting is a map $\alpha : X \to BK(\mathbb{Z}, 2)$ defining a principal $K(\mathbb{Z}, 2)$-bundle $P(\alpha)$ on $X$. The set of sections of the associated bundle $P(\alpha) \times_{K(\mathbb{Z}, 2)} (\mathbb{Z} \times BU)$ is the same as the set of $K(\mathbb{Z}, 2)$-equivariant maps from $P(\alpha)$ to $\mathbb{Z} \times BU$; that is, the twisted K-theory of $X$ is the “$K(\mathbb{Z}, 2)$-equivariant” $K$-theory of $P(\alpha)$. In particular, elements of the twisted K-theory of $X$ are represented by virtual vector bundles on the total space $P(\alpha)$ of the $K(\mathbb{Z}, 2)$-principal bundle associated to the twisting; these vector bundles $V$ are required to be $K(\mathbb{Z}, 2)$-equivariant in the sense that for a line $L \in K(\mathbb{Z}, 2)$, the virtual vector space $V_{L \cdot x}$ at the point $L \cdot x \in P(\alpha)$ is equal to $L \otimes V_x$, for all points $x \in P(\alpha)$.

1.1.4 Perfect Complexes of $\alpha$-Twisted Sheaves

Our ‘space’ $X$ will now be a scheme, and a twisting of K-theory is a $\mathbb{G}_m$-gerbe on $X$. These gerbes are classified by $H^2(X; \mathbb{G}_m)$ and can be thought of as stacks locally isomorphic to the category of invertible sheaves. Elements of the twisted K-theory of $X$ for a twisting gerbe $\alpha$ should be virtual sheaves of locally free sheaves on $\alpha$ that are $BG_m$-equivariant in an appropriate sense. More precisely an element of the twisted K-theory of $X$ is a perfect complex of $\alpha$-twisted sheaves on the gerbe $\alpha$, that is a complex of $\alpha$-twisted sheaves locally quasiisomorphic to a finite length complex of free finite rank sheaves on $\alpha$-modules. In the topological situation the analogue of the perfect complex on $\alpha$ is a two term complex of bundles on $P(\alpha)$, each of countably infinite rank, with a differential that is locally an isomorphism off of a finite rank subbundle. We would like to emphasize that this notion of $\alpha$-twisted K-theory elements on the scheme $X$ does not depend on the class $\alpha \in H^2(X; \mathbb{G}_m)$ being torsion.

1.1.5 Central Extensions of Loop Groups

We now specialize to the case (which indeed will be our primary focus in this paper) that our space is a connected simply connected compact Lie group $G$. A twisting map $\alpha : G \to K(\mathbb{Z}, 3)$ gives a map from the free loop space $LG$ to the classifying space $BS^1$ by
the composition \( LG \to LK(\mathbb{Z}, 3) \to \Omega K(\mathbb{Z}, 3) \simeq BS^1 \), and thereby gives a principal \( S^1 \)-bundle on \( LG \). The total space \( \tilde{LG} \) of this principal bundle can be given a group structure as an \( S^1 \)-central extension of \( LG \). The classifying space \( B\tilde{LG} \) of the based loop central extension \( \tilde{LG} \subset \tilde{LG} \) is precisely the total space \( P(\alpha) \) of the principal \( K(\mathbb{Z}, 2) \) bundle over \( G \). Moreover, to an irreducible highest-weight representation of \( \tilde{LG} \) one can associate an equivariant map from \( P(\alpha) \) to \( \mathbb{Z} \times BU \) and thereby an element of the twisted \( K \)-theory of \( G \) [22]. The precise relation between the representation theory of loop groups and twisted \( K \)-theory is described by Freed, Hopkins, and Teleman [14]—they prove that the group of positive energy unitary representations of \( \tilde{LG} \) is the twisted \( G \)-equivariant \( K \)-theory of \( G \).

1.1.6 B-Fields and D-Branes

A great deal of the limelight focused on twisted \( K \)-theory has come from the widespread realization that certain boundary conditions in string theory naturally represent elements in the twisted \( K \)-theory of spacetime. In this context the twistings are represented by nontrivial Neveu-Schwarz B-fields; the elements of twisted \( K \)-theory are D-branes, submanifolds of spacetime with a twisted \( Spin^c \) structure on their normal bundles. More generally, such a submanifold \( M \) may be equipped with a vector bundle \( V \) and the class represented by the pair \( (M, V) \) is the pushforward of \( V \) to the twisted \( K \)-theory of the ambient spacetime \( X \). When the space \( X \) is a Lie group, as in this paper, the twisted \( K \)-theory can be thought of as a topological model for the space of D-branes in a Wess-Zumino-Witten model for conformal field theory. Frequently the spacetime \( X \) is itself a \( Spin^c \) manifold; the D-branes are then twisted \( Spin^c \) submanifolds and represent elements in the twisted \( K \)-homology of \( X \). In this case, a D-brane \( M \) naturally represents a class in a more refined group, the twisted \( Spin^c \) bordism of \( X \), and there is a twisted index map that recovers the twisted \( K \)-homology class of \( M \). This perspective guides the discussion of the twisted \( Spin^c \) bordism of Lie groups in the last section of this paper.

1.2 Results

We prove that the twisted \( K \)-homology ring of a simple Lie group is an exterior algebra tensored a cyclic group, we give a detailed description of the orders of these cyclic groups in terms of the dimensions of irreducible representations of related groups, and we show that these orders originate, via a twisted index map, from relations in the twisted \( Spin^c \) bordism group.

**Theorem 1.1.** Let \( G \) be a compact, connected, simply connected, simple Lie group of rank \( n \). The twisted \( K \)-homology ring of \( G \) with nonzero twisting class \( k \in H^3(G; \mathbb{Z}) \simeq \mathbb{Z} \) is an exterior algebra of rank \( n - 1 \) tensored a cyclic group:

\[
K^{\tau(k)}(G) \cong \Lambda[x_1, \ldots, x_{n-1}] \otimes \mathbb{Z}/c(G, k).
\]

Here \( c(G, k) \) is an integer depending on the group and the twisting.

This fact was first noticed in the case of \( SU(n) \) by Hopkins. The proof is in section 3 for groups other than \( Spin(n) \), and in section 4.4 for \( Spin(n) \).
Theorem 1.2. For the classical groups, the cyclic orders $c(G, k)$, $k > 0$, of the twisted $K$-homology groups of $G$ are:

$$c(SU(n + 1), k) = \gcd \left\{ \binom{k+i}{i} - 1 : 1 \leq i \leq n \right\}$$
$$c(Sp(n), k) = \gcd \left\{ \sum_{-k \leq j \leq -1} \binom{2j + 2(i-1)}{2(i-1)} : 1 \leq i \leq n \right\}$$
$$c(\text{Spin}(4n - 1), k) = \gcd \left\{ \binom{k}{i} : 1 \leq i \leq 2n - 2 \right\}$$
$$\quad \cup \left\{ \binom{k}{2i+1} : n \leq i \leq 2n - 2 \right\}$$
$$c(\text{Spin}(4n + 1), k) = \gcd \left\{ \binom{k}{i} : 1 \leq i \leq 2n - 1 \right\} \cup \left\{ \binom{k}{2i+1} : n \leq i \leq 2n - 1 \right\}$$
$$c(\text{Spin}(4n + 2), k) = \gcd \left\{ \binom{k}{i} : 1 \leq i \leq 2n \right\}$$
$$\quad \cup \left\{ \binom{k}{2i+1} : n + 1 \leq i \leq 2n - 1 \right\}$$
$$c(\text{Spin}(4n), k) = \gcd \left\{ \binom{k}{i} : 1 \leq i \leq 2n - 1 \right\} \cup \left\{ \binom{k}{2i+1} : n \leq i \leq 2n - 2 \right\}.$$

(Note that $c(G, -k) = c(G, k)$. The formulas for $c(\text{Spin}(4n - 1), k)$ and $c(\text{Spin}(4n), k)$ exclude the degenerate case $n = 1$.) The proofs for $SU(n)$, $Sp(n)$, and $\text{Spin}(n)$ occur respectively in sections 4.2, 4.3, and 4.4. A general method for computation, applicable to the exceptional groups, is discussed in section 4.5, and the cyclic order for $G_2$ is given in section 4.2.

The referee has drawn our attention to an intriguing conjecture by Volker Braun concerning these cyclic orders [8], namely that $c(G, k)$ is the greatest common divisor of the dimensions of the representations generating the Verlinde ideal. (The Verlinde ideal is the ideal $I$ in the representation ring $R[G]$ such that the quotient $R[G]/I$ is the Verlinde algebra; by Freed, Hopkins, and Teleman [14] the Verlinde algebra is the twisted $G$-equivariant $K$-theory of $G$.) Using a Kunneth spectral sequence in equivariant $K$-theory due to Hopkins, Braun is able to show that the conjecture holds for any group for which the Verlinde algebra is a complete intersection—for such groups $G$ Braun’s argument also shows that the twisted $K$-theory of $G$ is a group isomorphic to the exterior algebra described in Theorem 1.1 above; Braun does not address the question of which groups satisfy the complete intersection condition. Using a twisted Bousfield-Kan spectral sequence one can see that for the groups $SU(n)$ and $Sp(n)$ the Verlinde algebra is a complete intersection [11]. Computational evidence strongly suggests that Braun’s conjecture is true for all groups, but at the moment it remains open for the spin and for the exceptional groups.

Proposition 1.3. Let $G$ be as in Theorem 1.1. Suppose $M_i$ is a collection of $\text{Spin}^c$ manifolds over $\Omega G$ whose fundamental classes generate $K.\Omega G$ as an algebra. Then there are twisted $\text{Spin}^c$ structures on the bordisms $W_i = M_i \times I$ such that the cyclic order of the twisted $K$-homology of $G$ is $\gcd(\text{ind}(\partial W_1), \ldots, \text{ind}(\partial W_n))$, where $\text{ind} : M\text{Spin}^c \to K.\ast$ is the index map from $\text{Spin}^c$ bordism to $K$-homology.

The proof of this proposition is the focus of section 5.2.
1.3 Techniques and Overview

The primary tool for calculating twisted $K$-homology rings is the twisted Rothenberg-Steenrod spectral sequence; this is the original method used by Hopkins in the case $G = SU(n)$. The spectral sequence is:

$$E^2 = \text{Tor}^{K.\Omega G}(\mathbb{Z}, \mathbb{Z}_{\tau(k)}) \Rightarrow K_{\tau}^*(G),$$

where $\mathbb{Z}_{\tau(k)}$ is the integers with a twisted $K.\Omega G$-module structure depending on $k$. In section 2.1 we present various generalities about twisted homology theories; then in section 2.2 we use a method of Segal [26] to construct this Rothenberg-Steenrod spectral sequence in twisted $K$-homology.

As the $K$-homology rings of loop spaces of simple Lie groups are known, our primary task is computing the Tor groups over these rings. Remarkably, for $G \neq \text{Spin}(n)$ this can be done without identifying the twisted $K.\Omega G$-module structure on $\mathbb{Z}$. These Tor groups are calculated in section 3 by an iterated series of filtration spectral sequences applied to a judiciously chosen Tate resolution. The spectral sequences are seen to collapse and to be extension-free, completing the proof of Theorem 1.1 for $G \neq \text{Spin}(n)$.

The Tor computation for $\text{Spin}(n)$ requires a detailed knowledge of the twisted module structure on $\mathbb{Z}$; this module structure is also precisely what is needed to identify the cyclic orders of the twisted $K$-homology groups. The best way to identify this module structure is via generating varieties for the loop space of the group, and this is the subject of section 4. Sections 4.2, 4.3, and 4.5 describe generating varieties for various groups, compute the cyclic orders in the corresponding cases, and discuss a general method for determining the cyclic order. Section 4.4 describes the twisted module structure for $\text{Spin}(n)$ and presents the belated Tor calculation for this group.

The computation in section 4 of the cyclic order in terms of the dimensions of irreducible representations does not give much geometric insight into these torsion groups. We give, in section 5, an interpretation of these orders in terms of relations in the twisted $\text{Spin}^c$ bordism group of $G$. The main tool, presented in section 5.1, is a cocycle model for twisted $\text{Spin}^c$ bordism. This model allows explicit descriptions of nullbordisms of particular $\text{Spin}^c$ manifolds over $G$ corresponding to relations in the twisted $K$-homology of $G$—see section 5.2. We conclude in section 5.3 by discussing potential representatives in $\text{MSpin}^{\ast\tau}(G)$ for the exterior generators of $K_{\tau}(G)$.

2 Twisted $K$-Theory and the Rothenberg-Steenrod Spectral Sequence

2.1 Twisted Homology Theories

We review the definitions and basic properties of twisted homology and cohomology theories. There are by now various models for these theories, but the following perspective owes as much to Goodwillie as to folklore.

For a spectrum $F$, the cohomology of a space $X$ with coefficients in $F$ can be defined as

$$F^n(X) := \text{colim} \Gamma_h(X, X \times \Omega^i F_{i+n});$$
here $\Gamma_h(X, E)$ refers to homotopy classes of sections of the (here trivial) bundle $E$ on $X$. The maps in the colimit are induced by applying the usual structure maps $\Omega^i F_{i+n} \to \Omega^{i+1} \Sigma F_{i+n} \to \Omega^{i+1} F_{i+1+n}$ fibrewise to the bundle $X \times \Omega^i F_{i+n} \to X$. Now let $E$ be a bundle of based spectra over $X$, with fibre spectrum $F$; this means in particular that for each $i$ we have a fibration $E_i \to X$, a section $X \to E_i$, and a fibrewise structure map $\Sigma X E_i \to E_{i+1}$. (Note that $\Sigma X$ denotes fibrewise suspension and $\Omega X$ will denote the fibrewise loops.) The cohomology of $X$ with coefficients in $E$ is defined to be

$$E^n(X) := \text{colim} \Gamma_h(X, \Omega^i X E_{i+n})$$

where the colimit maps are, as expected, induced by $\Omega^i X E_{i+n} \to \Omega^{i+1} \Sigma X E_{i+n} \to \Omega^{i+1} X E_{i+1+n}$.

The parallel in homology is similar. The homology of $X$ with coefficients in $F$ is

$$F_n(X) := \text{colim} [S^{i+n}, (X \times F_i)/X],$$

with maps induced by $\Sigma((X \times F_i)/X) = (X \times \Sigma F_i)/X \to (X \times F_{i+1})/X$. As above, when $E$ is a bundle of based spectra, we have a ‘base point’ section $X \to E_i$ for all $i$. The homology of $X$ with coefficients in $E$ is

$$E_n(X) := \text{colim} [S^{i+n}, E_i/X];$$

the colimit maps are induced by $\Sigma(E_i/X) = (\Sigma X E_i)/X \to E_{i+1}/X$.

For completeness we also mention the reduced analogs of homology and cohomology with coefficients in a bundle of spectra. The reduced cohomology with coefficients in a trivial $F$ bundle can be given as

$$\tilde{E}^n(X) := \text{colim} \Gamma_h^b(X, X \times \Omega^i X E_{i+n}),$$

that is as the colimit of homotopy classes of sections taking the base point of $X$ to the basepoint of $\Omega^i X E_{i+n}$. The reduced cohomology with coefficients in $E$ is then

$$\tilde{E}^n(X) := \text{colim} \Gamma_h^b(X, \Omega^i X E_{i+n});$$

the maps are induced as before. Similarly, the reduced homology with coefficients in a trivial bundle is

$$\tilde{F}_n(X) := \text{colim} [S^{i+n}, (X \times F_i)/(X \vee F_i)].$$

The twisted reduced homology is finally

$$\tilde{E}_n(X) := \text{colim} [S^{i+n}, E_i/(X \vee F_i)];$$

the maps are induced by $\Sigma(E_i/(X \vee F_i)) = (\Sigma X E_i)/(X \vee \Sigma F_i) \to E_{i+1}/(X \vee F_{i+1})$. Of course, these reduced groups are special cases of the relative groups:

$$E^n(X, A) := \text{colim} \Gamma_h(X, A; \Omega^i X E_{i+n}, s(A)),$$
groups are then, of course, the stable homotopy groups \( \pi_{K} B\text{Spin} \) smashing over particular universal \( K \) class. For any basepoint-preserving action of \( K \) on \( X \), classified up to isomorphism by the homotopy class of the map. For any basepoint-preserving action of \( K \) on a spectrum \( F \), we can form the associated bundle to \( P(\alpha) \). The resulting bundle \( P(\alpha) \times_{K(\mathbb{Z},2)} F \) is a bundle of based spectra on \( X \), as above. Note that on the level of spaces, the action of \( K \) on \( X \) is given by maps \( K(\mathbb{Z},2)_+ \wedge F_i = (K(\mathbb{Z},2) \times F_i)/(K(\mathbb{Z},2) \times *) \rightarrow F_i \), and we often denote the spectrum action simply by a map \( K(\mathbb{Z},2)_+ \wedge F \rightarrow F \).

Our primary examples are twisted \( \text{Spin}^c \)-bordism and twisted \( K \)-theory. The \( K(\mathbb{Z},2) \) bundle

\[
K(\mathbb{Z},2) = BU(1) \rightarrow B\text{Spin}^c \rightarrow BSO
\]

is principal, with classifying map \( BSO \xrightarrow{\beta_{w_2}} BBU(1) = K(\mathbb{Z},3) \) classifying the integral Bockstein of the second Stiefel-Whitney class. In particular we have an action \( K(\mathbb{Z},2) \times B\text{Spin}^c \rightarrow B\text{Spin}^c \); on Thom spaces this action is \( K(\mathbb{Z},2)_+ \wedge M\text{Spin}^c \rightarrow M\text{Spin}^c \), that is, a based action of \( K(\mathbb{Z},2) \) on the \( \text{Spin}^c \) Thom spectrum. The \( \alpha \)-twisted \( \text{Spin}^c \)-bordism groups are then, of course, the stable homotopy groups \( \pi_i((P(\alpha) \times K(\mathbb{Z},2) M\text{Spin}^c)/X) \).

The \( K \)-theory spectrum \( K \) is a module over \( \text{Spin}^c \)-bordism by the usual index map \( M\text{Spin}^c \xrightarrow{\text{ind}} K \). Taking the above based action \( K(\mathbb{Z},2)_+ \wedge M\text{Spin}^c \xrightarrow{\phi} M\text{Spin}^c \) and smashing over \( M\text{Spin}^c \) with \( K \), we have a compatible based action on \( K \)-theory:

\[
\begin{array}{ccc}
K(\mathbb{Z},2)_+ \wedge M\text{Spin}^c & \xrightarrow{\phi} & M\text{Spin}^c \\
\text{ind} & & \\
\text{id} \wedge \text{ind} & & \\
K(\mathbb{Z},2)_+ \wedge K & \xrightarrow{\phi \wedge \text{ind}(\text{id})} & K
\end{array}
\]

The corresponding map on associated principal bundles \( P(\alpha) \times_{K(\mathbb{Z},2)} M\text{Spin}^c \rightarrow P(\alpha) \times_{K(\mathbb{Z},2)} K \) induces a map from twisted \( \text{Spin}^c \)-bordism to twisted \( K \)-theory which we call the twisted index map. This map will be important in section 5.

Twisted \( K \)-theory can be defined more directly by choosing an explicit model for \( \mathbb{Z} \times BU \) (typically the space of Fredholm operators on a fixed Hilbert space \( \mathcal{H} \)) that admits an explicit action by some model for \( BU(1) \) (typically the space of projective unitary
operators on $\mathcal{H}$); see, for example, Atiyah [3]. (The referee has pointed out that the recent exposition by Atiyah and Segal [5] is another source for operator-theoretic definitions of twisted $K$-theory.) Whatever the formal definition, the geometric action being modeled is the following: a complex line $L$ (representing a point in $BU(1)$) acts on a virtual-dimension-zero (or stable) vector space $V$ (representing a point in $BU$) by tensor product, that is, $V \mapsto L \otimes V$.

It is worth noting, though, that this heuristic action of tensoring a vector bundle with a line can be misleading if we pay insufficient attention to the virtual dimension zero condition. It is tempting to think of elements of $\alpha$-twisted $K$-cohomology as sections of an $\alpha$-twisted gerbe of rank $n$, for some sufficiently large $n$; (such a section is locally a rank-$n$ vector bundle, twisted globally by $\alpha$). However, in this paper we are dealing with non-torsion twistings, and therefore no nontrivial element of twisted $K$-cohomology is representable by a section of any finite rank gerbe. We are inescapably in either a virtual-dimension-zero or an infinite-dimensional situation—which would seem to be a matter of personal penchant.

2.2 The Twisted Rothenberg-Steenrod Spectral Sequence

The “twisted” Rothenberg-Steenrod spectral sequence computing the twisted $K$-homology of a space is in fact the ordinary Rothenberg-Steenrod (a.k.a. homology Eilenberg-Moore) spectral sequence in an appropriate category, and as such requires little comment. We briefly recall the spectral sequence in generality, then describe its application to the geometric bar complex on the loop space of a simple Lie group.

We work in the category $\mathcal{K}$ of pairs $(X; E)$, where $X$ is a space and $E$ is a bundle of based spectra on $X$ with fibre the $K$-theory spectrum; the morphisms are those bundle maps that are homotopy equivalences on each fibre. Similarly, we have a category of triples $(X, A; E)$ where $A$ is a closed subspace of $X$ and $E$ is again a bundle on $X$. As mentioned in the last section, the functors

$$(X, A; E) \mapsto E_n(X, A) = \text{colim}[S^{i+n}, (E_i/X)/(E_i|_A)/A]$$

form a homology theory in the classical sense. In particular, for any simplicial object $S$ in $\mathcal{K}$, there is a spectral sequence a la Segal [26] with $E^2$ term $H_p(E_q(S))$ converging to the homology of the realization $E_{p+q}(|S|)$.

Let $G$ be a simple, simply connected Lie group and $k \in H^2(\Omega G; \mathbb{Z}) = \mathbb{Z}$ an integer describing a line bundle $L^{-k}$ on the loop space $\Omega G$. On the one hand, there is the trivial projection map in $\mathcal{K}$ from $(\Omega G; \Omega G \times K)$ to $(*; K)$. On the other hand, there is a twisted map $\tau(k) : (\Omega G; \Omega G \times K) \to (\ast; K)$ given by $\Omega G \times K \xrightarrow{k \times 1} K(\mathbb{Z}, 2) \times K \to K$, where the last map is the $K(\mathbb{Z}, 2)$ action on the spectrum $K$ described in section 2.1. The geometric bar construction $B_\tau \Omega G = B.(\ast, \Omega G, \ast_\tau)$ is a simplicial object in $\mathcal{K}$. To describe the corresponding spectral sequence we need only compute the effect of $\tau(k)$ in homology and identify the realization $|B_\tau \Omega G|$.

Given a class $\phi$ in the $K$-homology of $\Omega G$ the image of $\phi$ under $\tau(k)$ is evidently equal to the evaluation $\langle \tau(k)^*(1), \phi \rangle$, where $\langle -, - \rangle$ denotes the Kronecker pairing. The
pullback $\tau(k)^*(1)$ is $L^k$, and the resulting map $K.\Omega G \xrightarrow{(L^k, -)} K.$ defines a module structure on $K.$ which we denote $(K.)_\tau$. The $E^2$ term of our spectral sequence is therefore $\text{Tor}^{K.\Omega G}(K.,(K.)_\tau)$.

As a space the realization of $B_3\Omega G$ is evidently $B\Omega G \simeq G$; we identify the $K$-bundle. The $K$-bundle on the realization is defined by a 1-cocycle $\tau(k)$ with values in $K(\mathbb{Z}, 2)$ and as such is classified by the image of $\tau(k)$ in $H^3(B\Omega G; \mathbb{Z})$. We have $H^3(B\Omega G) \cong H^3(\Sigma \Omega G)$ and it is enough to identify the restriction of $\tau(k)$ to the 1-skeleton $\Sigma \Omega G$ of $B\Omega G$. It is, however, immediate that this cocycle on the 1-skeleton of the geometric bar construction $B_3\Omega G$ has homology invariant $k \in H^3(\Sigma \Omega G)$. In summary:

**Proposition 2.1.** There is a spectral sequence of algebras with $E^2$ term

$$E^2_{pq} = \text{Tor}^{K.\Omega G}_{p,q}(K.,(K.)_\tau)$$

converging as an algebra to the twisted $K$-homology $K^\tau_{p+q}(G)$.

The twisted $K$-homology of $G$ is by definition the homotopy of the spectrum $E/G$, where $E$ is the bundle of spectra over $G$ determined by the twisting class $\tau$. There is a pairing of spectra $E/G \wedge E/G \to E/G$ induced by the multiplication on $G$, and this pairing gives the algebra structure on $K^\tau(G)$. Note that the existence of this pairing depends essentially on the fact that the twisting class $\tau$ is primitive as an element of $H^3(G)$; because $G$ is simple and simply connected, all such elements are indeed primitive.

The multiplicative structure of the above spectral sequence can be seen as follows. The filtration of $B\Omega G \simeq G$ by the standard skeleta $B_3\Omega G$ induces a filtration of $E/G$ by a tower $T$ of spectra $T_i = (E_i[B_3\Omega G])/B_3\Omega G$. The spectral sequence in question is the homotopy spectral sequence associated to this tower. The multiplication on $G$ corresponds to a filtration-preserving multiplication on $B\Omega G$. The pairing on $E/G$ therefore induces a pairing of towers $T \wedge T \to T$ and this pairing of towers descends to the algebra structure on the homotopy spectral sequence—see for example the careful exposition of homotopy spectral sequence pairings by Dugger [13]. That the resulting algebra structure on the $E^2$ term of the spectral sequence agrees with the usual pairing on $\text{Tor}$ follows by comparing the algebraic and geometric bar constructions for $K.\Omega G$ and $\Omega G$ respectively, as in for instance [25,21,24].

## 3 Tate Resolutions and $\text{Tor}^{K.\Omega G}(\mathbb{Z}, \mathbb{Z}_\tau)$ for $G \neq \text{Spin}(n)$

For each group $G$, we describe the $K$-homology of the loop space of $G$, give an appropriate Tate resolution of $K.\star = \mathbb{Z}$ over $K.\Omega G$, and compute the torsion group using a series of filtration spectral sequences.

We recall Tate’s main result on algebra resolutions over a commutative Noetherian ring $R$. An ideal $I \subset R$ is said to be generated by the regular sequence $a_1, \ldots, a_r \in R$ if $I = (a_1, \ldots, a_r)$ and $a_i$ is not a zero-divisor in $R/(a_1, \ldots, a_{i-1})$ for all $i$.

**Theorem 3.1** (Tate [28]). Let $A \subset B$ be ideals of $R$ generated respectively by the regular sequences $(s_1, \ldots, s_m)$ and $(t_1, \ldots, t_n)$. For any choice of constants $c_{ji} \in R$ such that
is a resolution of $R/B$ as an $R/A$-module. Here the $T_i$ are strictly skew commutative generators of degree 1, and the $S_j$ are divided power algebra generators of degree 2.

In particular, $\text{Tor}_{R/A}(R/B, Q)$ will be given as the homology $H(D \otimes_{R/A} Q)$. In our applications, $R$ will be a polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, the ideal $A$ will depend on the group, the ideal $B$ will be $(x_1, \ldots, x_n)$, and $Q$ will be an $R/A$-module $\mathbb{Z}_r$ on which $x_i$ acts by an integer $c_i$ depending on the group and the twisting class.

3.1 Tor for $SU(n + 1)$ and $Sp(n)$

Elementary calculation shows that the integral cohomology rings of $SU(n+1)$ and $Sp(n)$ are exterior algebras on $n$ generators. Application of the spectral sequence $\text{Ext}^{H^i(G;k)}(k, k) \Rightarrow H.(\Omega G; k)$, $k$ a field, then implies that the integral Pontryagin rings $H.(\Omega SU(n+1))$ and $H.(\Omega Sp(n))$ are both polynomial on $n$ generators, all in even degree. In each case the Atiyah-Hirzebruch spectral sequence for $K$-theory then collapses, and the $K$-theory Pontryagin ring is again polynomial.

The Tate resolution in this case is especially simple, as the ideal $A$ is trivial. Let $G$ denote either $SU(n+1)$ or $Sp(n)$ and $k \in \mathbb{Z} \approx H^3(G; \mathbb{Z})$ the twisting class. Choose reduced generators $x_i$ of $K.\Omega G$, so that $K.\Omega G \cong \mathbb{Z}[x_1, \ldots, x_n]$. (Note that, unless otherwise noted, we treat $K$-theory as $\mathbb{Z}/2$-graded.) The $K.\Omega G$ module structure on $\mathbb{Z}_r$ is given, as in section 2.2, by the map $K.\Omega G \to K.*$ sending a class $x$ to $\langle L^k, x \rangle$, where $L$ is a generating line bundle. We defer the explicit evaluation of these maps to section 4. For now, we denote by $c_i$ the image of $x_i$ in $\mathbb{Z}_r$; of course this constant depends on both the group and the twisting, but we tend to omit both dependencies from the notation. By Tate’s theorem,

$$\text{Tor}^{K.\Omega G}(\mathbb{Z}, \mathbb{Z}_r) = H(\mathbb{Z}[x_1, \ldots, x_n]/\langle T_1, \ldots, T_n \rangle \otimes_{\mathbb{Z}[x_1, \ldots, x_n]} \mathbb{Z}_r; d) = H(\mathbb{Z}(T_1, \ldots, T_n); dT_i = c_i).$$

To evaluate this homology group we employ the following general procedure. Suppose we know the homology of the subalgebra generated by $T_1, \ldots, T_i$. We filter the subalgebra generated by $T_1, \ldots, T_{i+1}$ by powers of $T_{i+1}$ and look at the associated spectral sequence. The only differential is $d^1$, which is given by multiplication by $c_{i+1}$, and by induction we can thus compute the homology of the original algebra.

We assume for now that $c_1$ is not zero; this is indeed the case (see sections 4.2 and 4.3). The homology of $(\mathbb{Z}(T_1), d)$ is $\mathbb{Z}/c_1$. The, quite degenerate, spectral sequence of the filtration of $(\mathbb{Z}(T_1, T_2), d)$ by $T_2$ is therefore

$$\mathbb{Z}/c_1 \xrightarrow{\xi^2} \mathbb{Z}/c_1.$$
The homology is \( \mathbb{Z}/g_{12}(y_2) \), where \( g_{12} = \gcd\{c_1, c_2\} \) and \( y_2 \) is an exterior class. More generally we will denote by \( g_{1,i} \) the greatest common divisor \( \gcd\{c_1, c_2, \ldots, c_i\} \). The induction step is, as expected, the homology of
\[
\mathbb{Z}/g_{1,i}(y_2, \ldots, y_i) \xleftarrow{c_{i+1}} \mathbb{Z}/g_{1,i}(y_2, \ldots, y_i),
\]
and the Tor groups are given by
\[
\text{Tor}^{K,\Omega SU(n+1)}(\mathbb{Z}, \mathbb{Z}_r) = \mathbb{Z}/(g_{1,n}(SU(n+1), k))(y_2, \ldots, y_{n-1})
\]
\[
\text{Tor}^{K,\Omega Sp(n)}(\mathbb{Z}, \mathbb{Z}_r) = \mathbb{Z}/(g_{1,n}(Sp(n), k))(y_2, \ldots, y_{n-1}).
\]

We belabor this calculation only because, when we come to more complicated examples, especially \( Spin(n) \), it will help to have a clear model.

3.2 Tor for the Exceptional Groups

The exceptional Lie groups are nature’s best attempts to make a finite dimensional Lie group out of \( K(\mathbb{Z}, 3) \). In particular they are homotopy equivalent to \( K(\mathbb{Z}, 3) \) through a range of dimensions, and so their loop spaces are homotopy equivalent to \( K(\mathbb{Z}, 2) \) through a similar range. The \( K \)-homology of \( K(\mathbb{Z}, 2) \) is the subalgebra of \( \mathbb{Q}[a] \) generated by \( \{a, \left(\begin{smallmatrix}a \\ 2\end{smallmatrix}\right), \left(\begin{smallmatrix}a \\ 3\end{smallmatrix}\right), \ldots\} \); see [1]. Extensive computations by Duckworth [12] show that for \( G \) exceptional, the \( K \)-homology \( KG \) differs from a polynomial ring only in the aforementioned low-dimensional flirtation with \( K(\mathbb{Z}, 2) \). For example, Duckworth proves that \( K\Omega E_8 \) is a polynomial ring on seven generators tensor the subalgebra of \( \mathbb{Q}[a] \) generated by the elements \( \{a, \left(\begin{smallmatrix}a \\ 2\end{smallmatrix}\right), \left(\begin{smallmatrix}a \\ 3\end{smallmatrix}\right), \left(\begin{smallmatrix}a \\ 4\end{smallmatrix}\right), \left(\begin{smallmatrix}a \\ 5\end{smallmatrix}\right)\} \). In order to use Tate resolutions, we must give explicit algebra presentations of these \( K \)-homology rings:

**Proposition 3.2.** The \( K \)-homology rings of the loop spaces of the exceptional Lie groups are given by
\[
K\Omega G_2 = \frac{\mathbb{Z}[a, b, x_3]}{(a(a - 1) - 2b)}
\]
\[
K\Omega F_4 = \frac{\mathbb{Z}[a, b, c, x_4, x_5, x_6]}{(a(a - 1) - 2b, b(a - 2) - 3c)}
\]
\[
K\Omega E_6 = \frac{\mathbb{Z}[a, b, c, x_4, x_5, x_6, x_7, x_8]}{(a(a - 1) - 2b, b(a - 2) - 3c, c(a - b) - c - 1)}
\]
\[
K\Omega E_7 = \frac{\mathbb{Z}[a, b, c, d, x_5, x_6, x_7, x_8, x_9, x_{10}]}{(a(a - 1) - 2b, b(a - 2) - 3c, b(b + 1) - a(b + c) - 2d)}
\]
\[
K\Omega E_8 = \frac{\mathbb{Z}[a, b, c, d, e, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]}{(a(a - 1) - 2b, b(a - 2) - 3c, b(b + 1) - a(b + c) - 2d, d(a - 4) - 5e)}
\]

Note that the unsightly third relation in the rings for \( E_7 \) and \( E_8 \) is essential and cannot be replaced by the more sensible relation \( c(a - 3) - 4d \). We remark that, because the ‘lettered’ generators in these \( K \)-homology rings come from corresponding generators in \( K(K(\mathbb{Z}, 2)) \), the twisted pushforwards of these elements are easily computed. In particular, the twisted
pushforward of $a$, denoted again by $c_1$, is just $k$, the twisted pushforward of $b$ is $c_2 = \binom{k}{2}$, of $c$ is $c_3 = \binom{k}{3}$, and so on, with each generator mapping to its respective binomial coefficient.

As always, our starting point is the Tate resolution:

$$\text{Tor}^{K,ΩG_2}(\mathbb{Z}, \mathbb{Z}_p) = H(\mathbb{Z}\langle T_1, T_2, T_3 \rangle \{S_1 \}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2).$$

Consider the subalgebra generated by $T_1$, $T_2$, and $S_1$. If $k$ is even, we can rewrite this DGA as

$$(\mathbb{Z}\langle T'_1, T'_2 \rangle \{S_1 \};dT'_1 = 0, dT'_2 = k, dS_1 = 2T'_1),$$

where $T'_1 = (k-1)T_1 - 2T_2$ and $T'_2 = \frac{k}{2}T_1 - T_2$. The Kunneth theorem immediately shows that the homology of this DGA is $\mathbb{Z}/(k)$. If $k$ is odd, we instead change the basis to $T'_1 = \frac{k+1}{2}T_1 - T_2$ and $T'_2 = kT_1 - 2T_2$. The algebra then takes the form

$$(\mathbb{Z}\langle T'_1, T'_2 \rangle \{S_1 \};dT'_1 = 0, dT'_2 = k, dS_1 = 2T'_1),$$

and by the Kunneth theorem its homology is $\mathbb{Z}/k$. In other words, the homology of the subalgebra in question is, in any case, $\mathbb{Z}/g_{12}$, where as before $g_{12} = \gcd\{c_1, c_2\}$. Filtering as in section 3.1 we see that the full Tor group is $\mathbb{Z}/g_{123}\langle y_3 \rangle$.

The Tate resolution for $F_4$ gives

$$\text{Tor}^{K,ΩF_4}(\mathbb{Z}, \mathbb{Z}_p) = H(\mathbb{Z}\langle T_1, T_2, T_3, T_4, T_5, T_6 \rangle \{S_1, S_2 \};
\quad dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2, dS_2 = (c_1 - 2)T_2 - 3T_3),$$

We focus on the subalgebra generated by $\{T_1, T_2, T_3, S_1, S_2\}$. The method used for $G_2$, of changing basis to split the algebra into simpler pieces, works here as well; the basis change now depends on $k$ modulo 6. We spell out only the case $k = 1$ (mod 6). As basis change for the $T_i$’s we take

$$\begin{pmatrix}
\frac{k-1}{2} & -1 & 0 \\
-k-1 & \frac{k-1}{3} & -1 \\
3k-1 & -1 & 3
\end{pmatrix}.$$

The algebra then has the form

$$(\mathbb{Z}\langle T'_1, T'_2, T'_3 \rangle \{S_1, S_2 \};dT_1 = dT_2 = 0, dT_3 = k, dS_1 = 2T'_1, dS_2 = 3T'_2 + T'_1).$$
The spectral sequence associated to the filtration of the \( \{T'_1, T'_2, S_1, S_2\} \) subalgebra by powers of \( S_2 \) is

\[
\begin{array}{cccc}
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \cdots \\
(s^1_1 T'_1 r'_2) & (s^1_2 s^3_1 T'_1) & (s^1_2 s^3_1 T'_1) & \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \cdots \\
(s^1_1 T'_1) & (s^3_2 s^1_1 T'_2) & (s^3_2 s^1_1 T'_2) & \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \cdots \\
(s_1 T'_2) & (s^3_2 s^1_1 T'_1) & (s^3_2 s^1_1 T'_1) & \\
\mathbb{Z}/2 & \mathbb{Z}/2 & (1,1) & \mathbb{Z}/2 + \mathbb{Z} & \cdots \\
(s_1 T'_2) & (s^3_2 T'_2) & (1,3) & \mathbb{Z} & (s^3_2) \\
\mathbb{Z}/2 + \mathbb{Z} & (1,3) & \mathbb{Z} & (s^3_2) \\
(r'_2, T'_2) & (s^3_2 T'_2, s^3_2 T'_2) & (s^3_2 T'_2, s^3_2 T'_2) & \\
\mathbb{Z} & (1) & & \\
\end{array}
\]

There are, of course, no differentials beyond \( d^1 \) and the homology of the \( \{T'_1, T'_2, S_1, S_2\} \) subalgebra is therefore \( \mathbb{Z}/6 \) in odd degree, 0 in positive even degree, and \( \mathbb{Z} \) in degree zero; consequently the homology of the \( S_2 \) subalgebra is therefore \( \mathbb{Z}/6 \) in degree zero. In general, ie for \( k \) not necessarily congruent to 1 modulo 6, this \( \mathbb{Z}/k \) is replaced by \( \mathbb{Z}/g_{123} \) and the full Tor group for \( F_1 \) is \( \mathbb{Z}/g_{12}(y_4, y_5, y_6) \). The computation for \( E_6 \) is identical, but for two additional exterior generators in the final Tor group.

This basis change approach quickly becomes impractical: for \( E_8 \) the congruence of \( k \) modulo 60 determines the structure of the basis change and of the subsequent homology computation. If we are willing to give up our ability to write down explicit generators modulo 60 determines the structure of the basis change and of the subsequent homology computation. If we are willing to give up our ability to write down explicit generators for the Tor groups, we can do the computation without such a case by case analysis. We briefly reconsider the groups \( G_2 \) and \( F_4 \). For \( G_2 \) the main step was computing the homology of the DGA

\[
D = (\mathbb{Z} \langle T_1, T_2 \rangle \{S_1\}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2);
\]

recall that \( c_1 = k \) and \( c_2 = \binom{k}{2} \). The homology of the \( \{T_1, T_2\} \) subalgebra is \( \mathbb{Z}/g_{12}(y_2) \), where the generator \( y_2 \) can be taken to be \(-(c_1/g_{12})T_2\) modulo terms involving \( T_1 \). (We will refer to terms with lower indices, sensibly enough, as ‘lower terms’ and so say, for example, that “\( y_2 \) is \(-(c_1/g_{12})T_2\) modulo lower terms”). Thus, when we filter \( D \) by powers of \( S_1 \), the homology of \( D \) becomes the homology of

\[
(\mathbb{Z}/g_{12}(y_2) \{S_1\}; dS_1 = (2g_{12}/c_1)y_2).
\]

Note that \( 2g_{12}/c_1 \) is an integer, so this expression makes sense. We observe that \( 2g_{12}/c_1 \) is actually a unit in \( \mathbb{Z}/g_{12} \); indeed \( g_{12} = c_1/g_{\text{gcd}(2, c_1)} \) so

\[
\text{gcd}(2g_{12}/c_1, g_{12}) = \text{gcd}(2/\text{gcd}(2, c_1), c_1/\text{gcd}(2, c_1)) = 1.
\]
The homology of $D$ is therefore simply $\mathbb{Z}/g_{12}$, as previously noted, and thus the full Tor group is again an exterior algebra tensor a cyclic group.

The case of $F_4$ (and therefore of $E_6$) is again similar. The main step is the computation of the homology of the DGA

$$D = (\mathbb{Z}(T_1, T_2, T_3)\{S_1, S_2\}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2, dS_2 = (c_1 - 2)T_2 - 3T_3).$$

(Here again $c_i = (i)$. Using the $G_2$ result we see that the homology of the $\{T_1, T_2, T_3, S_1\}$ subalgebra is $\mathbb{Z}/g_{123}\langle y_3 \rangle$ where $y_3$ is $-(g_{12}/g_{123})T_3$ modulo lower terms. As above the homology of $D$ is thereby reduced to the homology of

$$(\mathbb{Z}/g_{123}\langle y_3 \rangle\{S_2\}; dS_2 = (3g_{123}/g_{12})y_3).$$

Again, this differential is an isomorphism, ie $3g_{123}/g_{12}$ is a unit in $g_{123}$. The trick is the same: observe that $g_{123} = g_{12}/\gcd(3, g_{12}) = c_1/(\gcd(3, c_1) \gcd(2, c_1))$; from this we have

$$\gcd(3g_{123}/g_{12}, g_{123}) = \gcd(3/\gcd(3, c_1), c_1/(\gcd(3, c_1) \gcd(2, c_1)))$$

$$= \gcd(3/\gcd(3, c_1), c_1/\gcd(3, c_1)) = 1.$$

The homology of $D$ is thus, again, $\mathbb{Z}/g_{123}$. The full Tor group follows.

Despite the increased complexity of the $K$-homology rings of $\Omega E_7$ and $\Omega E_8$, the Tor calculations in these cases are no more elaborate than for the other exceptional groups. The presentation in Proposition 3.2 suggests an appropriate Tate resolution and the Tor computation for $\Omega E_7$ is given by the homology of the DGA

$$(\mathbb{Z}(T_1, \ldots, T_{10})\{S_1, S_2, S_3\}; dT_i = c_i, dS_1 = (c_1 - 1)T_1 - 2T_2,$$

$$dS_2 = (c_1 - 2)T_2 - 3T_3, dS_3 = (c_2 + 1)T_2 - (c_2 + 3)T_1 - 2T_4).$$

Using the $F_4$ computation, we see that the homology of the $\{T_1, T_2, T_3, T_4, S_1, S_2 \}$ subalgebra is $\mathbb{Z}/g_{1234}\langle y_4 \rangle$, where $y_4$ is $-(g_{123}/g_{1234})T_4$ modulo lower terms. The homology of the $\{T_1, T_2, T_3, T_4, S_1, S_2, S_3 \}$ subalgebra is therefore the homology of

$$(\mathbb{Z}/g_{1234}\langle y_4 \rangle\{S_3\}; dS_3 = (2g_{1234}/g_{123})y_4).$$

We observe that $2g_{1234}/g_{123}$ is a unit in $\mathbb{Z}/g_{1234}$ and so the homology of this subalgebra is $\mathbb{Z}/g_{1234}$ concentrated in degree zero. The full Tor group is finally $\mathbb{Z}/g_{1\ldots10}\langle y_5, y_6, \ldots, y_{10} \rangle$. In this calculation it is critical that the third relation in the presentation of $K.\Omega E_7$ is $b(b + 1) - a(b + c) - 2d$ and not the expected $c(a - 3) - 4d$. The latter relation would produce a differential $dS_3 = (4g_{1234}/g_{123})y_4$ and thereby (because $4g_{1234}/g_{123}$ is not always a unit in $\mathbb{Z}/g_{1234}$) a plethora of nontrivial higher torsion.

The Tor computation for $E_8$ is entirely analogous. The Tate resolution is dictated by the presentation in Proposition 3.2 and the necessary combinatorial fact is that $5g_{12345}/g_{1234}$ is a unit in $\mathbb{Z}/g_{12345}$. 

15
3.3 Proof of Theorem 1.1

We can now establish the bulk of our main theorem. We assume the computation of the torsion group for $\text{Spin}(n)$, which is carried out in section 4.4:

$$\text{Tor}^{K,\Omega\text{Spin}(n)}(K, (K, *)) = \Lambda[x_1, \ldots, x_{n-1}] \otimes \mathbb{Z}/c(\text{Spin}(n), k).$$

Though we have treated $K$-homology as $\mathbb{Z}/2$-graded in our Tor computations, properly it is $\mathbb{Z}$-graded, and the $E^2$ term of the Rothenberg-Steenrod spectral sequence has the appearance:

$$
\begin{array}{cccc}
\text{Tor}_0^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \text{Tor}_1^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \text{Tor}_2^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \cdots \\
0 & 0 & 0 & \\
\text{Tor}_0^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \text{Tor}_1^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \text{Tor}_2^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \cdots \\
0 & 0 & 0 & \\
\text{Tor}_0^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \text{Tor}_1^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \text{Tor}_2^{K_0\Omega G}(\mathbb{Z}, \mathbb{Z}) & \cdots \\
\end{array}
$$

In our cases these torsion groups are generated in Tor-degree 1; the (homological) differentials vanish on the generators and thus the spectral sequence collapses at the $E^2$ term.

We show that there are no additive extensions. We have established that the $E^\infty$ term of the spectral sequence is a cyclic group, say $\mathbb{Z}/c$, tensor an exterior algebra. The filtration is homological, so the subgroup $(\mathbb{Z}/c)\{1\} \subset E^\infty$ generated by the identity element of the torsion group $\text{Tor}^{K_0\Omega G} = E^2 = E^\infty$ is actually a subgroup of the $K$-homology $K^\tau(G)$. The construction of the spectral sequence shows that this identity element in the torsion group corresponds to the identity element in the $K$-homology. The identity element $1 \in K^\tau(G)$ is therefore killed by multiplication by $c$, and so the entire $K$-homology ring is $c$-torsion, as desired.

For degree reasons, the only possible multiplicative extension is $y_i^2 = d \in K^\tau(G)$; that is the square of the $K$-homology class represented by an exterior generator $y_i \in \text{Tor}_1$ could be a constant integer, an element of $\text{Tor}_0$. However, by construction the exterior classes $y_i$ are represented by reduced classes in $K^\tau(G)$ and so their squares are also certainly reduced, eliminating the possibility of multiplicative extensions. □

4 Generating Varieties, the Cyclic Order of $K^\tau G$, and $\text{Tor}^{K,\Omega\text{Spin}(n)}(\mathbb{Z}, \mathbb{Z})$

The twisted $K$-homology of a simple Lie group is an exterior algebra tensor a cyclic group. The order of this cyclic group depends on the twisting class and is, as yet, determined by a mysterious set of constants. We will see that this cyclic order of the twisted $K$-homology $K^\tau G$ is the greatest common divisor of the dimensions of a particular set of representations of $G$. The main ingredient in computing the cyclic order is a detailed understanding of the twisted module structure of $\mathbb{Z}_\tau$, that is of the twisting map $K.\Omega G \xrightarrow{\tau(k)} K.*$. Bott’s theory of generating varieties allows us to produce explicit
representatives of classes in $K.\Omega G$, as fundamental classes of complex algebraic varieties, and thereby to describe the twisting map.

4.1 Generating Varieties and Holomorphic Induction

4.1.1 Bott Generating Varieties

A generating variety for $\Omega G$ is, for us, a space $V$ and a map $i: V \to \Omega G$ such that the images $i_* (H.V)$ and $i_* (K.V)$ of the homology and $K$-theory of $V$ generate $H.\Omega G$ and $K.\Omega G$, respectively, as algebras, where $\Omega G$ is the identity component of $\Omega G$. In [6], Bott produced a beautiful, systematic family of generating varieties of the form $G/H$, as we now describe; these particular homogeneous spaces are better known as coadjoint orbits and as such are smooth complex algebraic varieties with an even-dimensional cell decomposition.

We briefly review Bott’s construction. Let $G$ be a compact and connected but not necessarily simply connected Lie group. Denote by $\Gamma_G = \ker(\exp: t \to T)$ the coweight lattice of $G$; we do not distinguish between a coweight and the corresponding circle in $G$. A coweight $\ell \in \Gamma_G$ is called generating if for every root $r \in t^*$ of $G$, there is an element $w$ of the Weyl group such that $r(w \cdot \ell) = 1$. Note that the coweight lattice $\Gamma_G$ of the group is contained in the coweight lattice $\Gamma_W$ of the Lie algebra, which is also the coweight lattice of the adjoint form of $G$. Though a group may not have a generating coweight, its adjoint form always will. The simple rank 2 groups with generating coweights, namely $PSU(3)$, $PSp(2)$, and $G_2$, are illustrated in Figure 1.

![Fig. 1. Generating Coweights for Rank 2 Lie Groups](image)

Suppose $\ell \in \Gamma_G$ is a generating coweight for $G$, and let $C(\ell) \subset G$ denote the pointwise centralizer of the corresponding circle; note that $C(\ell)$ can also be described as the image under the exponential map of the subalgebra of $\mathfrak{g}$ generated by the root spaces associated to roots $r$ perpendicular to $\ell$, that is to roots where $r(\ell) = 0$. The map

$$G \to \Omega G$$

$$g \mapsto g\ell g^{-1}\ell^{-1}$$

descends to a map on cosets $G/C(\ell) \to \Omega G$. The main theorem, which is due to Bott in homology and to Clarke [9] in $K$-theory, is that $G/C(\ell)$ is a generating variety for $\Omega G$. 

17
Suppose $G$ is simply connected and $\ell$ is a generating circle for $PG$, the adjoint form of $G$. Then $PG/C_{PG}(\ell) = G/C_G(\tilde{\ell})$ where $\tilde{\ell}$ denotes a loop in $G$ covering $\ell$. The composite

$$G/C_G(\tilde{\ell}) = PG/C_{PG}(\ell) \to \Omega'PG = \Omega G$$

is therefore a generating variety for $\Omega G$. For example, the generating varieties corresponding to the marked coweights in Figure 1 are $SU(3)/U(2)$, $Sp(2)/U(2)$, and $G_2/U(2)$ respectively. In general there may be more than one Bott generating variety for a group; we list a Bott generating variety for each of the classical groups in the following table:

| Group          | Generating Variety                      |
|----------------|----------------------------------------|
| $SU(n + 1)$    | $SU(n + 1)/U(n)$                       |
| $Spin(2n + 1)$ | $Spin(2n + 1)/(Spin(2n - 1) \times \mathbb{Z}/2 \ Spin(2))$ |
| $Sp(n)$        | $Sp(n)/U(n)$                           |
| $Spin(2n)$     | $Spin(2n)/(Spin(2n - 2) \times \mathbb{Z}/2 \ Spin(2))$ |

Here the $\mathbb{Z}/2$ action on $Spin(n)$ is the one whose quotient is $SO(n)$.

~~~~~~~~~

We need to compute the twisted map $K.\Omega G \xrightarrow{\tau(k)} K.\ast$. To this end we want to represent the algebra generators of $K.\Omega G$ in a way that allows us to compute their twisted images. In our computations we utilize generating varieties to represent algebra generators in three independent ways; we refer to these briefly as representing them via subvarieties, via an evaluation dual basis, and via a Poincare dual basis.

In some cases we have a sufficiently explicit handle on the generating variety $V$ for $\Omega G$ that we can describe a collection of maps $W_i : V \to V \to \Omega G$ such that $W_i$ is a $K$-oriented manifold and the images in $K.\Omega G$ of the $K$-homology fundamental classes of the $W_i$ are the desired algebra generators; frequently, though not always, the $W_i$ are subvarieties of the generating variety $V$. Variants of this ‘subvariety’ representation are used for $SU(n)$, $G_2$, and $Sp(n)$ in sections 4.2 and 4.3.

The $K$-cohomology of the Bott generating variety $V$ is easily determined from the representation theory of $G$. Specifically, if the Bott generating variety $V$ is the quotient $G/H$ with $G$ simply connected, then $K^*V = R[H]/i^*(I[G])$, where $i : H \to G$ is the inclusion and $I[G]$ is the augmentation ideal of the representation ring $R[G]$. If there is a minor miracle and we can write down a clean basis for this ring, then we can take an evaluation dual basis for the $K$-homology $K.V$; the image of this basis in $K.\Omega G$ will generate as an algebra and the twisting map will be easily computable. This is the approach taken for $Spin(n)$ in section 4.4.

More commonly, any apparent basis for the $K$-cohomology of the generating variety is quite haphazard. In this case we consider the Poincare dual basis (to some chosen basis) for the $K$-homology $K.V$. Again the images of these classes in $K.\Omega G$ will generate, but computing their twisted images requires a bit more work. Specifically, we use holomorphic induction to write these images in terms of the dimensions of irreducible representations.
of $G$, as described in detail in the next section. This Poincare dual approach is the one that provides a general procedure and is the subject of section 4.5.

### 4.1.2 The Twisting Map via Holomorphic Induction

In section 2.2 we described the $K.\Omega G$-module structure on $\mathbb{Z}_r$ by the twisting map

$$K.\Omega G \xrightarrow{\tau(k)} K.*$$

$$x \mapsto \langle L^k, x \rangle,$$

where $L \in K'.\Omega G$ is a generating line bundle. The purpose of this section is to outline the computation of the twisted image $\langle L^k, x \rangle$ when $x$ is represented as the image of the Poincare dual of a bundle on an appropriate $K$-oriented manifold.

Let $i : W \to \Omega G$ be a map from a $K$-oriented manifold $W$ to $\Omega G$ and let $\eta \in K'.W$ be a bundle on $W$ such that $i_*(D\eta) = x \in K.\Omega G$; here, $D$ denotes the Poincare duality map. We first translate the evaluation $\langle L^k, x \rangle$ into a pushforward on $W$:

$$\langle L^k, i_*(D\eta) \rangle = \langle i^*(L^k), D\eta \rangle = \langle i^*(L^k), \eta \cap [W] \rangle = \langle i^*(L^k) \cup \eta, [W] \rangle.$$

The third equality is simply

$$\langle i^*(L^k), \eta \cap [W] \rangle = \pi_*(i^*(L^k) \cap (\eta \cap [W])) = \pi_*(i^*(L^k) \cup \eta) \cap [W] = \langle i^*(L^k) \cup \eta, [W] \rangle,$$

where $\pi : W \to *$ denotes projection. We are thereby reduced to computing $K$-theory pushforwards $\langle \mu, [W] \rangle = \pi_!(\mu)$.

Suppose that, as in our computations it always is, $W$ is a homogeneous space $G/H$ which is a Kähler manifold, and $\mu \in K'(G/H)$ is the bundle associated to an irreducible representation of $H$. (If our original bundle $\mu$ is not irreducible, we decompose it into irreducible components and work with each component separately.) In this case $\mu$ lifts to a $G$-equivariant bundle, also denoted $\mu$, and the pushforward $\pi_!(\mu)$ is the dimension of the equivariant pushforward $\pi^G_!(\mu)$. This equivariant pushforward $\pi^G_! : R[H] \cong K^*_G(G/H) \to K^*_G(*) \cong R[G]$ is usually referred to as holomorphic induction. Thanks to the Atiyah-Segal fixed point theory [4], pushforwards in equivariant $K$-theory, particularly those involving homogeneous spaces, are easily computable. To avoid reviewing the whole of Atiyah-Segal’s theory, we describe only the form it takes in the context of holomorphic induction.

Let $G$ be a compact connected simply connected Lie group and $H$ a centralizer of a circle in $G$; in particular $H$ and $G$ share a maximal torus and their weight lattices coincide. In this situation, as remarked earlier, the $K$-theory of the quotient $G/H$ is simply the quotient of representation rings: $K^*(G/H) = R[H]/i^*(I[G])$. We may further assume that we have chosen an order on the roots of $G$ such that $H$ is generated by a subset of the simple roots of $G$; in particular this determines standard Weyl chambers for $G$ and $H$. (That there is such a choice of order is the content of Wang’s theorem and depends on $H$ being the centralizer of a torus in $G$; see Bott [7].) Let $\mu$ denote simultaneously a weight in the Weyl chamber of $H$, the corresponding irreducible representation of $H$, and the
associated bundle on $G/H$. Let $\rho$ denote half the sum of the positive roots of $G$, and let $S$ denote the union of the hyperplanes perpendicular to the roots of $G$. Further, for a weight $\omega$ of $G$, let $T(\omega)$ denote the unique weight in the Weyl chamber of $G$ that is the image of $\omega$ under an element of the Weyl group. The index $\text{ind}(\omega)$ of a weight $\omega$ not in $S$ is the number of hyperplanes of $S$ intersecting a straight line connecting $\omega$ and $T(\omega)$. Holomorphic induction on the representation $\mu$ is described by the following well known theorem:

**Theorem 4.1.** In the above situation,

1. The character of the representation $\pi^G_1(\mu) \in R[G]$ is $$\sum_{w \in W} \det(w)(\mu - \rho)^w.$$ Here $\mu \cdot \rho$ is the one-dimensional representation with weight $\mu + \rho$, and $W$ is the Weyl group of $G$.

2. If $\mu + \rho \notin S$ then $\pi^G_1(\mu) \in R[G]$ is $(-1)^{\text{ind}(\mu + \rho)}$ times the irreducible representation of $G$ with highest weight $T(\mu + \rho) - \rho$.

The first part of the theorem is a consequence of the Atiyah-Segal fixed point formula [4], and the second part follows from the first using the Weyl character formula. Strictly speaking, we only need the first part, but it will be convenient, using the second part of the theorem, to refer to appropriate pushforwards as irreducible representations.

The nonequivariant pushforwards follow immediately. When $\mu + \rho \in S$ we say that $\mu$ is singular. Thus, when $\mu$ is singular, $\pi_1(\mu) = 0$, and otherwise

$$\pi_1(\mu) = (-1)^{\text{ind}(\mu + \rho)} \dim([T(\mu + \rho) - \rho]_G),$$

where $[-]_G$ denotes the irreducible representation of $G$ with the specified highest weight. The character of that representation is given in 4.1 above, and in any particular case its dimension is easily computed using the Weyl dimension formula. This method provides a systematic approach to computing the twisting map on a class represented by the image of the Poincare dual to a $K$-cohomology class of an appropriate homogeneous space. We proceed to specific examples.

### 4.2 Subvarieties of $\Omega SU(n+1)$ and $\Omega G_2$

A generating variety for $\Omega SU(n+1)$ is $SU(n+1)/U(n) = \mathbb{CP}^n \rightarrow \Omega SU(n+1)$, and the induced map in homology is

$$\overline{H}.\mathbb{CP}^n = \mathbb{Z}\{z_1, \ldots, z_n\} \rightarrow \mathbb{Z}[w_1, \ldots, w_n] = H.\Omega SU(n+1)$$

$$z_i \mapsto w_i.$$ 

Here the classes $z_i$ are represented by the fundamental homology classes of the subvarieties $\mathbb{CP}^i \subset \mathbb{CP}^n$. The Atiyah-Hirzebruch spectral sequence collapses for both $\Omega SU(n+1)$ and $\mathbb{CP}^n$ and there are no extensions. In particular $K.\Omega SU(n+1)$ is polynomial on $n$ generators, as previously noted, and $\overline{K}.\mathbb{CP}^n$ is free abelian of rank $n$.

**Lemma 4.2.** The set $\{[\mathbb{CP}^i]\}_{i=1}^n$ of fundamental $K$-homology classes of the subvarieties $\mathbb{CP}^i \subset \mathbb{CP}^n$ forms a basis for $\overline{K}.\mathbb{CP}^n$.
Proof. By induction it is enough to show that under the projection $K \cdot \mathbb{C}^i \rightarrow K.(\mathbb{C}^i, \mathbb{C}^{i-1}) = \mathbb{Z}$ the fundamental class of $\mathbb{C}^i$ maps to a generator. This follows immediately from the naturality of Poincare duality,

$$
\begin{array}{c}
K \cdot \mathbb{C}^i \longrightarrow K.(\mathbb{C}^i, \mathbb{C}^{i-1}) \\
K^* \mathbb{C}^i \longrightarrow K^*(\mathbb{C}^i - \mathbb{C}^{i-1}),
\end{array}
$$

because the unit in $K^* \mathbb{C}^i$ certainly maps to a generator of $K^*(\mathbb{C}^i - \mathbb{C}^{i-1})$. $\square$

The images $i_*(\mathbb{C}^i)$ generate $K.\Omega SU(n+1)$ as an algebra and we may therefore take $\{x_i = i_*(\mathbb{C}^i) - 1\}$ to be the reduced polynomial generators.

We now have to evaluate the pushforward $\langle L^k, \mathbb{C}^i \rangle$, where we use $L$ to denote the generating line bundle on $\mathbb{C}^i = SU(i+1)/U(i)$; this $L$ is of course the pullback of the generating line bundle on $\Omega SU(n+1)$. The bundle $L$ corresponds to an irreducible representation of $U(i)$, thus to a weight of $U(i)$ and so a weight, also denoted $L$, of $SU(i+1)$; this weight $L$ is in the Weyl chamber of $SU(i+1)$. The irreducible representation of $SU(i+1)$ corresponding to $L$ is the dual of the standard representation. (That it is the dual of the standard representation and not the standard representation is the effect of a sign choice—see the remark at the end of this section). It happens that the $k$-fold symmetric power of this representation is irreducible, and so the dimension of the irreducible representation corresponding to $L^k$ is $\binom{k+i}{i}$. The image of $x_i = i_*(\mathbb{C}^i) - 1 \in K.\Omega SU(n+1)$ in $\mathbb{Z}$ is therefore $c_i = \binom{k+i}{i} - 1$, and the cyclic order of $K^* G_2$ is

$$
c(SU(n+1), k) = \gcd \left\{ \binom{k+1}{1} - 1, \binom{k+2}{2} - 1, \ldots, \binom{k+n}{n} - 1 \right\}.
$$

The procedure for calculating the cyclic order of $K^* G_2$ is similar: we find fundamental class representatives for algebra generators of the homology of $\Omega G_2$ and then show that the corresponding $K$-homology fundamental classes also generate. The map $\Omega G_2 \rightarrow \mathbb{C}P^\infty$ classifying the generating line bundle is a homology equivalence through degree 4. Using this, the Serre spectral sequence for $\Omega SU(3) \rightarrow \Omega G_2 \rightarrow \Omega S^6$ shows that

$$
H.\Omega G_2 \cong \mathbb{Z}[a_2, a_4, a_{10}]/a_2^2 = 2a_4.
$$

The composition

$$
\mathbb{C}P^2 \rightarrow \Omega SU(3) \rightarrow \Omega G_2 \rightarrow \mathbb{C}P^\infty
$$

is simply the inclusion and as such, $a_2$ and $a_4$ in $H.\Omega G_2$ are represented respectively by the fundamental classes $[\mathbb{C}P^1]$ and $[\mathbb{C}P^2]$. The Bott generating variety for $\Omega G_2$ is $G_2/U(2)$, where the $U(2)$ in question is included in $G_2$ along a pair of complex-conjugate short roots. The manifold $G_2/U(2)$ has dimension 10 and the image of its homology generates $H.\Omega G_2$; we may therefore choose $a_{10}$ to be the image of the fundamental homology class $[G_2/U(2)]$. 

21
The Atiyah-Hirzebruch spectral sequence for $\Omega G_2$ collapses, and the low-degree equivalence between $\Omega G_2$ and $\mathbb{C}P^\infty$ resolves the extension. The $K$-homology $K\Omega G_2$ is thereby isomorphic to $\mathbb{Z}[a, b, x_3]/(a^2 + 3a = 2b)$.

**Lemma 4.3.** The reduced algebra generators of $K\Omega G_2 \cong \mathbb{Z}[a, b, x_3]/(a^2 + 3a = 2b)$ may be taken to be the reduced fundamental $K$-homology classes $[\mathbb{C}P^1] - 1$, $[\mathbb{C}P^2] - 1$, and $[G_2/U(2)] - 1$ respectively.

**Proof.** Let $(G_2/U(2))^8$ denote the 8-skeleton of the generating variety, that is everything except the top cell. As in Lemma 4.2, the fundamental $K$-homology class of $G_2/U(2)$ maps to a generator of $K(G_2/U(2), (G_2/U(2))^8)$. Comparing the Atiyah-Hirzebruch spectral sequences for $G_2/U(2)$ and $\Omega G_2$ we see that $[G_2/U(2)]$ lives in filtration 10 in $K\Omega G_2$ and projects to the generator $a_{10}$ in $H_{10}\Omega G_2$. The fundamental $K$-homology classes $[\mathbb{C}P^1]$ and $[\mathbb{C}P^2]$ certainly project to the generators $a_2$ and $a_4$ respectively in the appropriate filtration quotients, and this completes the proof.

We remark that these algebra generators differ by a change of basis from those implicitly chosen in section 3.2 and this explains the difference in the relation; the Tor computation and the cyclic order are not affected by the change.

We need only compute the pushforward $\langle L^k, [G_2/U(2)] \rangle$. The bundle $L$ corresponds to the shortest weight $\mu$ perpendicular to the roots of $U(2)$; as a weight of $G_2$, $\mu$ is the long root of $G_2$ in the Weyl chamber. The pushforward is therefore the dimension of the irreducible representation of $G_2$ with highest weight $k\mu$. By the Weyl dimension formula (see for example [15]) this is

$$\dim([k\mu]_{G_2}) = \frac{(k + 1)(k + 2)(2k + 3)(3k + 4)(3k + 5)}{120}.$$ 

The cyclic order of $K^*G_2$ is finally

$$c(G_2, k) = \gcd \left\{ \left( \begin{array}{c} k + 1 \\ 1 \end{array} \right) - 1, \left( \begin{array}{c} k + 2 \\ 2 \end{array} \right) - 1, \frac{(k + 1)(k + 2)(2k + 3)(3k + 4)(3k + 5)}{120} - 1 \right\}.$$ 

A remark on signs is in order. If we have chosen a generating line bundle $L$ on $\Omega G_2$ a priori, the weight corresponding to $L$ may be $-\mu$ instead of $\mu$ as claimed above. The dimension resulting from holomorphic induction on the weight $k(-\mu)$ is wildly different from the dimension associated to $k(\mu)$, and this might be cause for worry. However, the greatest common divisor is in all cases unaffected by the change. The easiest way around this ambiguity is to chose $L$ such that the corresponding weight is $\mu$ and not $-\mu$; we must then pick the generating variety $\mathbb{C}P^2$ for $\Omega SU(3)$ in such a way that the given $L$ corresponds there to the dual of the standard representation (and not to the standard representation) as described in the discussion of $SU(n + 1)$ above—this is easily accomplished. Similar remarks apply to all our computations and we make convenient sign choices without comment.
4.3 Generating Varieties for $\Omega Sp(n)$

The homology and $K$-homology of $\Omega Sp(n)$ are polynomial in $n$ generators. The natural Bott generating variety for $\Omega Sp(n)$ is $Sp(n)/U(n)$, which has homology and $K$-homology of rank $n^2+n$. Identifying the $n$ elements which generate therefore requires more doing—we return to this question later. Luckily, $\Omega Sp(n)$ has smaller generating varieties—see [16,23]; in particular we work with $(\mathbb{CP}^{2n-2})_2$, the Thom complex of the square of the tautological bundle.

Let $P_i(V)$ or $P(V)$ denote the projectivization of the bundle $V$ on $\mathbb{CP}^i$; note that we can rewrite our generating variety $V(n) = (\mathbb{CP}^{2n-2})_2$ as $P_{2n-2}(L^2 + 1)/P_{2n-2}(L^2)$. We think of the quotient map $P(L^2 + 1) \to P(L^2 + 1)/P(L^2)$ as a resolution of our (quite singular) generating variety, and we represent homology and $K$-homology classes in $V(n)$ (and thus in $\Omega Sp(n)$) as the images of fundamental classes of subvarieties of $P(L^2 + 1)$. The reduced homology of $V(n)$ is free of rank one in each even degree between 2 and $4n - 2$, and the degree $2i$ group is generated by the image of the fundamental class $[P_{i-1}(L^2 + 1)]$. In particular, the algebra generators $\{a_{4i-2}\}$ of $H\Omega Sp(n) = \mathbb{Z}[a_2, a_6, a_{10}, \ldots, a_{4n-2}]$ are represented by the fundamental classes $[P_{2(i-1)}(L^2 + 1)]$, for $1 \leq i \leq n$.

The $K$-homology situation is the same.

**Lemma 4.4.** The reduced polynomial generators of the $K$-homology $K\Omega Sp(n) \cong \mathbb{Z}[x_1, \ldots, x_n]$ can be taken to be the reduced $K$-homology fundamental classes $f_*[P_{2(i-1)}(L^2 + 1)] - 1$, $1 \leq i \leq n$; here $f$ is the composite

$$P_{2(i-1)}(L^2 + 1) \to P_{2(n-2)}(L^2 + 1) \to P_{2(n-2)}(L^2 + 1)/P_{2(n-2)}(L^2) \to \Omega Sp(n).$$

The $K$-homology fundamental classes map, in the appropriate filtration quotients, to the homology fundamental classes; the proof is the same as for $SU(n+1)$ and $G_2$.

To evaluate the twisting map, specifically to calculate $\langle L^k, f_*[P_{2(i-1)}(L^2 + 1)] \rangle$, we need to identify the bundle $f^*(L^k)$. We do this by writing down a bundle on $P(L^2 + 1) = P_{2(i-1)}(L^2 + 1)$ that is trivial on $P(L^2) = P_{2(i-1)}(L^2)$, and show that the corresponding bundle on the quotient $V(i)$ is the pullback $f^*(L)$ where $f'$ is the inclusion $V(i) \to V(n) \to \Omega Sp(n)$. Let $\gamma$ be the tautological bundle on the total space $P(L^2 + 1)$ and let $\pi : P(L^2 + 1) \to \mathbb{CP}^{2(i-1)}$ be the bundle projection. The subspace $P(L^2)$ is of course just the base $\mathbb{CP}^{2(i-1)}$ and so $\gamma$ restricts to $\pi^*(L^2)|_{P(L^2)}$ on $P(L^2)$. In particular then, the bundle $\gamma \otimes \pi^*(L^2)$ is trivial on the subspace $P(L^2)$ and so pulls back from a bundle $\phi$ on $V(i)$. To see that $\phi$ is equal to $f^*(L)$ (up to our usual sign ambiguity), and therefore that $\gamma \otimes \pi^*(L^2) = f^*(L)$, it is enough to check that the first Chern class of $\gamma \otimes \pi^*(L^2)$ is a module generator of $H^2(P(L^2 + 1)) = \mathbb{Z}\{c_1(\gamma), \pi^*(c_1(L))\}$; this much is clear.

We now compute the pushforward

$$\langle L^k, f_*[P_{2(i-1)}(L^2 + 1)] \rangle = \langle (\gamma \otimes \pi^*(L^2))^k, [P_{2(i-1)}(L^2 + 1)] \rangle.$$

First pushforward along the fibres:

$$\langle \gamma^k \otimes \pi^*(L^{-2k}), [P_{2(i-1)}(L^2 + 1)] \rangle = \langle \text{Sym}^k(L^2 + 1) \otimes L^{-2k}, [\mathbb{CP}^{2(i-1)}] \rangle.$$

23
This is a parameterized version of the pushforward
\[
\langle \gamma^k_{\text{taut}}, [\mathbb{C}P^n] \rangle = \langle \gamma^k_{\text{taut}}, [P(\mathbb{C}^{n+1})] \rangle = \dim(\text{Sym}^k(\mathbb{C}^{n+1})) = \binom{k+n}{k}
\]
used in the preceding section. Next
\[
\text{Sym}^k(L^2 + 1) \otimes L^{-2k} = L^{-2k} + L^{-2k+2} + \ldots + 1
\]
and so
\[
\langle \text{Sym}^k(L^2 + 1) \otimes L^{-2k}, [\mathbb{C}P^{2(i-1)}] \rangle = \left( \frac{-2k + 2(i - 1)}{2(i - 1)} \right) + \left( \frac{-2k + 2 + 2(i - 1)}{2(i - 1)} \right) + \ldots + 1.
\]
Here we use \( \binom{a+b}{b} \) to denote \( \frac{(a+b)(a+b-1)\ldots(a)}{(b)(b-1)\ldots(1)} \) even when \( a + b \) is negative; we implicitly observe that this expression does give the correct pushforward even when the bundle, as in the case of \( L^{-2k+2l} \), corresponds to a weight that is not in the Weyl chamber of \( SU(2(i - 1) + 1) \). This finishes our calculation of the cyclic order of \( K^r Sp(n) \):
\[
c(Sp(n), k) = \gcd \left\{ \sum_{-k \leq j \leq -1} \left( \frac{2j + 2(i - 1)}{2(i - 1)} \right) : 1 \leq i \leq n \right\}.
\]

It would be more natural to express the cyclic order of \( K^r Sp(n) \) in terms of the dimensions of irreducible representations of symplectic groups. This is possible if we work with subvarieties of the Bott generating variety \( V = Sp(n)/U(n) \). There is a natural collection of \( n \) subvarieties of \( V \), namely \( \{Sp(i)/U(i)\} \). It is not the case that the fundamental homology classes of these subvarieties represent algebra generators for \( H.\Omega Sp(n) \); indeed, the algebra generators are in dimensions \( \{4i - 2\} \), while these subvarieties have dimensions \( \{i^2 + 2\} \). It is therefore remarkable that the \( K \)-homology fundamental classes of these subvarieties do appear to generate the \( K \)-homology of \( \Omega Sp(n) \).

Conjecture. The \( K \)-homology ring \( K.\Omega Sp(n) \) is polynomial on the classes represented by the reduced \( K \)-homology fundamental classes \( [Sp(i)/U(i)] - 1, \) for \( 1 \leq i \leq n \).

Using the Weyl character formula, this immediately gives a description of the cyclic order:
\[
c(Sp(n), k) = \gcd \left\{ \prod_{1 \leq j \leq i} \frac{(l-j)(2k+2i+2-j+l)}{(2i-1)!(2i-3)!\ldots3!1!} : 1 \leq i \leq n \right\}.
\]

These gcd’s agree with those determined using the generating variety \( (\mathbb{C}P^{2n-2})^2 \).
4.4 The Tor Calculation for Spin$(n)$

We now pay our debt to the proof of Theorem 1.1 by calculating Tor$^{K, \Omega\text{Spin}(n)}(\mathbb{Z}, \mathbb{Z}_r)$; in the process we determine the cyclic order of $K^*\text{Spin}(n)$. For the other simple groups, we were able to calculate the Tor group without knowing the map $K.\Omega G \to (K.*)_r$ and we determined, after the fact, the structure of this twisting map. The ring $K.\Omega\text{Spin}(n)$ is too complicated to permit this a-priori Tor calculation; we must first identify algebra generators of $K.\Omega\text{Spin}(n)$ and compute the twisting map. It happens that the reduced $K$-cohomology of the Bott generating variety for $\text{Spin}(n)$ admits a particularly simple representation-theoretic basis, and an evaluation dual basis maps to a set of algebra generators in $K.\Omega\text{Spin}(n)$. Once we know the twisted pushforwards of these algebra generators, the Tor computation becomes tractable.

We concentrate on the odd Spin groups; at the end of the section we delineate the corresponding steps for the even Spin groups. The structure of the $K$-homology ring of $\Omega\text{Spin}(2n+1)$ was described by Clarke [9]:

$$K.\Omega\text{Spin}(2n+1) = \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, 2\sigma_n, 2\sigma_{n+1} + \sigma_n, \ldots, 2\sigma_{2n-1} + \sigma_{2n-2}],$$

$$\rho_k = \sigma_k^2 + \sum_{i=0}^{k-1} (-1)^{k-i} \sigma_i \sum_{j=k}^{2k-i-1} \left(\frac{k-i-1}{j-k}\right) (2\sigma_{j+1} + \sigma_j).$$

One can see why the a-priori Tor calculation is unlikely to be fruitful. The $K$-cohomology of the Bott generating variety $V = \text{Spin}(2n+1)/(\text{Spin}(2n-1) \times \mathbb{Z}/2 \text{Spin}(2))$ is simply the quotient of the representation ring of $\text{Spin}(2n-1) \times \mathbb{Z}/2 \text{Spin}(2)$ by the image of the augmentation ideal of the representation ring of $\text{Spin}(2n+1)$. Clarke writes this quotient in a convenient form:

$$K^*V = \mathbb{Z}[\mu, \gamma]/(\mu^n - 2\gamma - \mu\gamma, \gamma^2);$$

here $\mu = L - 1$ where $L$ is the generating line bundle whose $k$-th power determines the twisting. Note that $\mu^{2n} = 0$ in this ring, and so $(\mu, \mu^2, \ldots, \mu^{2n-1})$ is a basis for $K^*(V) \otimes \mathbb{Q}$. Letting $(\sigma'_1, \ldots, \sigma'_{2n-1})$ be the evaluation dual basis of $K_*(V) \otimes \mathbb{Q}$, we see that

$$(\sigma'_1, \sigma'_2, \ldots, \sigma'_{n-1}, 2\sigma'_n, 2\sigma'_{n+1} + \sigma'_n, \ldots, 2\sigma'_{2n-1} + \sigma'_{2n-2})$$

is a basis for $\widetilde{K}_*V$; these elements map, respectively, to the given generators of $K.\Omega\text{Spin}(2n+1)$. The twisting map $K.\Omega\text{Spin}(2n+1) \to (K.*)_r$ takes a generator $g$ to $\langle L^k, g \rangle \in \mathbb{Z}$. Because $\mu^{2n} = 0$, we have

$$\langle L^k, \sigma'_i \rangle = \langle (\mu + 1)^k, \sigma'_i \rangle = \binom{k}{i},$$

and the images of our integral generators are respectively

$$\left(\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{n-1}, 2\binom{k}{n}, 2\binom{k}{n+1} + \binom{k}{n}, \ldots, 2\binom{k}{2n-1} + \binom{k}{2n-2}\right).$$

We can now prove that Tor$^{K.\Omega\text{Spin}(2n+1)}(\mathbb{Z}, \mathbb{Z}_r)$ is an exterior algebra on $n-1$ generators
tensor a cyclic group. We first rewrite the above presentation of $K.\Omega Spin(2n + 1)$ in a way that suggests a propitious choice of Tate resolution. Let $(a_1, \ldots, a_{2n-1})$ denote the given generators of $K.\Omega Spin(2n + 1)$. For $i$ sufficiently large, the relation $\rho_i$ expresses the generator $a_{2i}$ in lower terms; in particular
\[
\begin{align*}
K.\Omega Spin(4n - 1) &= \mathbb{Z}[a_1, a_2, \ldots, a_{2n-2}, a_{2n-1}, a_{2n+1}, a_{2n+3}, \ldots, a_{4n-5}, a_{4n-3}] / (\rho_1, \rho_2, \ldots, \rho_{n-1}) \\
K.\Omega Spin(4n + 1) &= \mathbb{Z}[a_1, a_2, \ldots, a_{2n-2}, a_{2n-1}, a_{2n+1}, a_{2n+3}, \ldots, a_{4n-3}, a_{4n-1}] / (\rho_1, \rho_2, \ldots, \rho_{n-1})
\end{align*}
\]
The remaining relations can be written
\[
\rho_i = 2a_{2i} + r_i a_{2i-1} + \ldots,
\]
with $r_i$ odd and all unspecified monomials containing some $a_j$ with $j < 2i - 1$, except for $\rho_1$ which is $2a_2 + a_1 - a_2^2$. If we can show that Tor over the subring $R_n = \mathbb{Z}[a_1, \ldots, a_{2n-2}] / (\rho_1, \ldots, \rho_{n-1})$ is exterior on $n - 2$ generators, the desired result follows. Rather than presenting the general induction immediately, we discuss the first few cases explicitly.

The case $n = 1$ corresponding to $Spin(3)$ requires no comment. The ring $K.\Omega Spin(7)$ is $\mathbb{Z}[a_1, a_2, a_3, a_4] / (2a_2 + a_1 - a_2^2)$. This is reminiscent of $K.\Omega G_2$ and indeed the Atiyah-Hirzebruch spectral sequence for the fibration $\Omega G_2 \to \Omega Spin(7) \to \Omega S^7$ collapses; there are no possible multiplicative extensions and so this confirms that $K.\Omega Spin(7)$ is $K.\Omega G_2$ adjoin a generator in degree 6. As in section 3.2, the Tor group in question is
\[
\text{Tor}^{R_2}(\mathbb{Z}, \mathbb{Z}_r) = \text{Tor}^{\mathbb{Z}[a_1, a_2] / (2a_2 + a_1 - a_2^2)}(\mathbb{Z}, \mathbb{Z}_r) = \mathbb{Z}/g_{12}.
\]
(Note that the generator $a_i$ of the subring $R_n$ has image under the twisting map $c_i = \binom{k}{i}$ and as before we abbreviate $\gcd\{c_1, c_2, \ldots, c_i\}$ by $g_{1..i}$.)

The relevant subring of $K.\Omega Spin(11)$ is
\[
R_3 = \mathbb{Z}[a_1, a_2, a_3, a_4] / (\rho_1, 2a_4 + 3a_3 + (a_2 + 1)a_2 + (-2a_3 - a_2)a_1).
\]
This presentation suggests the Tate resolution
\[
\text{Tor}^{R_3} = H(\mathbb{Z}[T_1, T_2, T_3, T_4] / \{S_1, S_2\});
\]
d$T_1 = c_1$, $dS_1 = 2T_2 + (1 - c_1)T_1$, $dS_2 = 2T_4 + 3T_3 + (c_2 + 1)T_2 + (-2c_3 - c_2)T_1$.

The $E_1$ term of the spectral sequence associated to the filtration of this complex by $S_2$ is
\[
\begin{array}{cccc}
\mathbb{Z}/g & \mathbb{Z}/g & \mathbb{Z}/g & \mathbb{Z}/g \\
\mathbb{Z}/g & \mathbb{Z}/g \oplus \mathbb{Z}/g & \mathbb{Z}/g \\
\mathbb{Z}/g \oplus \mathbb{Z}/g & \mathbb{Z}/g \\
\mathbb{Z}/g &
\end{array}
\]
26
where \( g = g_{1234} \) and the generator in degree \((1,1)\) is \( S_2 \). At first blush the generators in degree \((0,1)\) have the form \( t_3 = (g_{12}/g_{123})T_3 + O(2) \) and \( t_4 = (g_{123}/g_{1234})T_4 + O(3) \), where the omitted terms contain only terms involving \((T_2 \text{ and } T_1)\) and \((T_3, T_2, \text{ and } T_1)\) respectively. In order to determine the differential on \( S_2 \) we need control over the \( T_3 \) term in the generator \( t_4 \). The basic observation is that if there exists a cocycle \( t'_4 \) of the form \((g_{123}/g_{1234})T_4 + O(2)\), then some linear combination \( t_4 + ct_3 \) is cohomologous to \( t'_4 \) and so we may take the generators in degree \((0,1)\) to be \( t'_4 \) and \( t_3 \). The existence of this cocycle is ensured by the fact that \((g_{123}/g_{1234})g_4\) is divisible by \( g_{12} \), as is easily checked. The differential on \( S_2 \) is therefore \((2g_{1234}/g_{123})t'_4 + (3g_{123}/g_{12})t_3\). Because the greatest common divisor of \( 2g_{1234}/g_{123} \) and \( 3g_{123}/g_{12} \) is always 1, the torsion group is finally

\[ \text{Tor}^{R_3} = \mathbb{Z}/g_{1234}(x_4); \]

here we can choose the generator \( x_4 \) to be \((g_{123}/g_{1234})T_4 + O(3)\).

The case of \( \text{Spin}(15) \) proceeds similarly. The relevant subring of \( K.\Omega \text{Spin}(15) \) is

\[ R_4 = \mathbb{Z}[a_1, a_2, a_3, a_4, a_5, a_6]/(\rho_1, \rho_2, 2a_6 + 5a_5 + 4a_4 + O(3)), \]

and we take the corresponding Tate resolution. Filtering by \( S_3 \) we have the spectral sequence (here condensed)

\[ \mathbb{Z}/g\langle x_4, x_5, x_6 \rangle \quad \mathbb{Z}/g\langle x_4, x_5, x_6 \rangle \quad \mathbb{Z}/g\langle x_4, x_5, x_6 \rangle \quad \ldots \]

The torsion \( g \) is \( g_{1.6} \) and the generators in degree \((0,1)\) are

\[ x_4 = (g_{123}/g_{1234})T_4 + O(3) \]
\[ x_5 = (g_{1234}/g_{1.5})T_5 + O(4) \]
\[ x_6 = (g_{1.5}/g_{1.6})T_6 + O(5). \]

It happens that \((g_{1234}/g_{1.5})g_5 \) and \((g_{1.5}/g_{1.6})g_6 \) are both divisible by \( g_{123} \); we can therefore adjust our generators so that they are

\[ x_4 = (g_{123}/g_{1234})T_4 + O(3) \]
\[ x_5 = (g_{1234}/g_{1.5})T_5 + O(3) \]
\[ x_6 = (g_{1.5}/g_{1.6})T_6 + O(3). \]

The differential on \( S_3 \) is thus \((2g_{1.6}/g_{1.5})x_6 + (5g_{1.5}/g_{1234})x_5 + (4g_{1234}/g_{123})x_4 \). Because \( 2g_{1.6}/g_{1.5} \) and \( 5g_{1.5}/g_{1234} \) are relatively prime, there exist constants \( z_1 \) and \( z_2 \) so that if we set

\[ y_6 = x_6 + z_1x_4 = g_{1.5}/g_{1.6}T_6 + O(4) \]
\[ y_5 = x_5 + z_2x_4 = g_{1234}/g_{1.5}T_5 + O(4), \]

then \( \{dS_3, y_6, y_5\} \) forms a basis for the degree \((0,1)\) group. Finally, then, the Tor group is

\[ \text{Tor}^{R_4} = \mathbb{Z}/g_{1.6}(y_5, y_6) \]

as desired.
The general case is now clear. Suppose we know that

$$\text{Tor}^{R_n} = \mathbb{Z}/g_{1..(2n-2)}\langle x_{n+1}, \ldots, x_{2n-2} \rangle,$$

where \(x_i = (g_{1..(i-1)}/g_{1..i})T_i + O(i - 1)\). The ring \(R_{n+1}\) has two additional generators \(a_{2n-1}\) and \(a_{2n}\) and one additional relation \(\rho_n\). Filter the appropriate Tate resolution by powers of \(S_n\), then adjust the generators of the degree \((0, 1)\) group in the spectral sequence so that the single generator \(x_{2n}\) involving \(T_{2n}\) does not contain any terms involving \(T_{2n-1}\).

This is possible because \(g_{1..(2n-2)}\) divides \((g_{1..(2n-1)}/g_{1..(2n)})g_{2n}\). The differential of \(S_n\) then has the form

$$dS_n = (2g_{1..(2n)}/g_{1..(2n-1)})x_{2n} + (rg_{1..(2n-1)}/g_{1..(2n-2)})x_{2n-1} + \ldots.$$

As those two leading terms are relatively prime, this ensures that \(\text{Tor}^{R_{n+1}}\) again has the desired form. Note that in theory there could be multiplicative extensions in the filtration spectral sequence calculating the \(\text{Tor}\) group, but the above procedure gives a sufficiently explicit handle on the generating classes as to eliminate this possibility.

This completes the proof of Theorem 1.1 for the odd \(\text{Spin}\) groups and also establishes the odd \(\text{Spin}\) cyclic orders given in Theorem 1.2. The calculation for the even \(\text{Spin}\) groups is analogous and proceeds as follows. The relevant \(K\)-homology ring, initially described by Clarke [9], is

$$K.\Omega\text{Spin}(2n+2) = \frac{\mathbb{Z}\langle \sigma_1, \ldots, \sigma_{n-1}, \sigma_n + \epsilon, -2\epsilon, 2\sigma_{n+1} - \epsilon, 2\sigma_{n+2} + \sigma_{n+1}, \ldots, 2\sigma_{2n} + \sigma_{2n-1} \rangle}{\langle \rho_1, \ldots, \rho_{n-1}, \rho_n - \epsilon^2 \rangle},$$

where the polynomial expressions \(\rho_k\) are as in the odd orthogonal case. The \(K\)-cohomology of the corresponding Bott generating variety \(V_{2n+2} = \text{Spin}(2n+2)/\text{Spin}(2n) \times \mathbb{Z}/2 \text{Spin}(2)\) is

$$K^*V_{2n+2} = \begin{cases} \mathbb{Z}[\mu, \gamma]/(\mu^{n+1} - 2\mu\gamma - \mu^2\gamma, \gamma^2 - \mu^n\gamma, \mu^{n+1}\gamma) & \text{if } n \text{ is even}, \\ \mathbb{Z}[\mu, \gamma]/(\mu^{n+1} - 2\mu\gamma - \mu^2\gamma, \gamma^2) & \text{if } n \text{ is odd}. \end{cases}$$

This presentation [10] is a slight correction of the one given in Clarke’s paper. Note that \(\mu^{2n+1} = 0\) in either ring, and thus \((1, \mu, \ldots, \mu^{n-1}, \mu^n, \beta, \mu^{n+1}, \mu^{n+2}, \ldots, \mu^{2n})\) is a basis for \(K^*(V) \otimes \mathbb{Q}\), where \(\beta = \mu^n - 2\gamma - \mu\gamma\). Let \((1, \sigma'_1, \ldots, \sigma'_n, \epsilon, \sigma'_{n+1}, \ldots, \sigma'_{2n})\) be the evaluation dual basis for \(K(V) \otimes \mathbb{Q}\) and observe that

$$(1, \sigma'_1, \ldots, \sigma'_{n-1}, \sigma_n' + \epsilon', -2\epsilon', 2\sigma'_{n+1} - \epsilon', 2\sigma'_{n+2} + \sigma'_{n+1}, \ldots, 2\sigma'_{2n} + \sigma'_{2n-1})$$

is a basis for \(K.V\). These basis elements (excepting the initial 1) map to the generators of \(K.\Omega\text{Spin}(2n+2)\) listed above. The twisting map \(K.\Omega\text{Spin}(2n+2) \to (K.\star)_r\) takes these generators in turn to the integers

$$\left(\begin{array}{c} k \\ 1 \end{array}\right), \ldots, \left(\begin{array}{c} k \\ n-1 \end{array}\right), \left(\begin{array}{c} k \\ n \end{array}\right), 0, 2\left(\begin{array}{c} k \\ n+1 \end{array}\right), 2\left(\begin{array}{c} k \\ n+2 \end{array}\right) + \left(\begin{array}{c} k \\ n+1 \end{array}\right), \ldots, 2\left(\begin{array}{c} k \\ 2n \end{array}\right) + \left(\begin{array}{c} k \\ 2n-1 \end{array}\right).$$

As we did for the odd \(\text{Spin}\) groups, we can simplify the presentation of the \(K\)-homology of the loop spaces of these even \(\text{Spin}\) groups before embarking on the Tor computation.
Abbreviate the generators of $K \Omega Spin(2n + 2)$ as $(a_1, \ldots, a_{n-1}, \hat{a}_n, b, \check{a}_{n+1}, a_{n+2}, \ldots, a_{2n})$ respectively. In $K \Omega Spin(4n + 2)$, the relation $\rho_n$ eliminates the generator $b$, while $\rho_i$ eliminates $a_{2i}$ for $n < i < 2n$, and the relation $\rho_{2n} - \epsilon^2$ eliminates $a_{4n}$. This leaves

$$K \Omega Spin(4n + 2) = \frac{\mathbb{Z}[a_1, \ldots, a_{2n-1}, \hat{a}_{2n}, \check{a}_{2n+1}, a_{2n+3}, \ldots, a_{4n-1}]}{(\rho_1, \ldots, \rho_{n-1})}.$$ 

Similarly in $K \Omega Spin(4n)$, the relation $\rho_n$ eliminates the generator $\hat{a}_{2n}$, while $\rho_i$ eliminates $a_{2i}$ for $n < i < 2n - 1$, and $\rho_{2n-1} - \epsilon^2$ eliminates $a_{4n-2}$, leading to the presentation

$$K \Omega Spin(4n) = \frac{\mathbb{Z}[a_1, \ldots, a_{2n-2}, \hat{a}_{2n-1}, b, a_{2n+1}, a_{2n+3}, \ldots, a_{4n-3}]}{(\rho_1, \ldots, \rho_{n-1})}.$$ 

The crucial subring $R_n = \mathbb{Z}[a_1, \ldots, a_{2n-2}]/(\rho_1, \ldots, \rho_{n-1})$ that we considered for the odd Spin groups is precisely the relation subring of both $K \Omega Spin(4n + 2)$ and $K \Omega Spin(4n)$. As such our previous Tor calculation carries over without modification. This completes the proof of Theorem 1.1 for the even Spin groups and the resulting even Spin cyclic orders are recorded in Theorem 1.2.

4.5 Poincare-Dual Bases and the Cyclic Order of $K^* G$

We describe a general procedure for computing the cyclic order of $K^* G$ for any simple $G$ and illustrate the method with the group $G_2$. The referee has pointed out that Braun [8] conjectures that this cyclic order is the greatest common divisor of the representations generating the Verlindde ideal. This conjecture is almost certainly correct and therefore provides an efficient (as compared with the method below) means of computing these cyclic orders.

Let $V = G/H$ denote the Bott generating variety for $\Omega G$; recall that the $K$-cohomology of $V$ is $R[H]/i^* I[G]$ where $i : H \to G$ is the inclusion. Pick a module basis $\{w_i\}$ for this ring and consider the Poincare-dual basis $\{Dw_i\}$ of $KV$. The image of this basis in $K \Omega G$, which we also denote by $\{Dw_i\}$, is a set of algebra generators for $K \Omega G$. Note that for any set $\{y_i\}$ of algebra generators for $K \Omega G$, the cyclic order of $K^* G$ is given by $\gcd\{\tau_k(y_i) - \tau_0(y_i)\}$, where $\tau_k$ and $\tau_0$ are respectively the twisted and untwisted maps from $K \Omega G$ to $K_*$. In section 4.1.2 we saw that $\tau_k(Dw_i) = \langle L^k \cup v_i, [V] \rangle$ where $L$ denotes the generating line bundle on $V$. Decompose $w_i$ into a sum of irreducible representations $\Sigma v_{ij}$, and let $h(v_{ij})$ denote the highest weight corresponding to $v_{ij}$. The product $L^k \cup v_{ij}$ is again irreducible, with highest weight $kL + h(v_{ij})$, and theorem 4.1 therefore applies: the pushforward $\langle L^k \cup v_{ij}, [V] \rangle$ is either 0 or is (plus or minus) the dimension of the irreducible representation of $G$ with highest weight $T(kL + h(v_{ij}) + \rho) - \rho$, where $T$ reflects a weight into the fundamental Weyl chamber. This procedure expresses the cyclic order of $K^* G$ as the greatest common divisor of a finite set of differences of dimensions of irreducible representations of $G$.

Recall that the Bott generating variety for $G_2$ is $G_2/U(2)$ for the short-root inclusion of $U(2)$. Let $a$ and $b$ denote the fundamental weights of $G_2$ corresponding to the 7 and 14 dimensional representations; in particular $R[G_2] = \mathbb{Z}[a, b]$. Similarly $R[U(2)] = \mathbb{Z}[f, t, t^{-1}]$, 

29
where $f$ and $t$ are respectively the standard representation and the determinant representation. The restriction map is

\[ i^*(a) = f + f^2t^{-1} - 1 + ft^{-1} \]
\[ i^*(b) = t + f^3t^{-1} - 2f + f^2t^{-1} + f^3t^{-2} - 2ft^{-1} + t^{-1} \]

Let $s = t^{-1}$; the $K$-cohomology of the generating variety is then

\[ K^*V = \mathbb{Z}[f, s]/(f + f^2s - 1 + fs, 1 + f^3s^2 - 2fs + f^2s^2 + f^3s^3 - 2fs^2 + s^2). \]

(Note that the description of $K^*V$ in Clarke [9] omits certain relations, as the ring given there is not finitely generated.) An integral basis for $K^*V$ is then \{1, $s$, $s^2$, $f$, $fs$, $f^2$\}. These representations of $U(2)$ are irreducible except for $f^2$ which splits as $(f^2 - t) + t$.

Consider the diagram of weights in Figure 2. The solid lines are the Weyl walls, the dotted lines describe the set of singular weights, and the seven highest weights $h_i$ under consideration, namely \{0, $s$, 2$s$, $f$, $f + s$, 2$f$, $t$\}, are circled. Note that $L = t$ and as such, for $k > 0$, the weight $kL + h_i$ is either singular or is already in the fundamental Weyl chamber. The basis for $K.V$ is of course \{D1, D$s$, D$(s^2)$, D$f$, D$(fs)$, D$(f^2)$\} and we are interested in the differences $\tau_k(Dw) - \tau_0(Dw)$. Letting $\Gamma_{(n,m)}$ denote the dimension of the irreducible representation of $G_2$ with highest weight $na + mb$, the six differences are respectively

\[ \Gamma_{(0,k)} - \Gamma_{(0,0)} \]
\[ \Gamma_{(0,k-1)} - 0 \]
\[ \Gamma_{(0,k-2)} - 0 \]
\[ \Gamma_{(1,k)} - \Gamma_{(1,0)} \]
\[ \Gamma_{(1,k-1)} - 0 \]
\[ \Gamma_{(2,k)} + \Gamma_{(0,k+1)} - \Gamma_{(2,0)} - \Gamma_{(0,1)}. \]
Applying the Weyl dimension formula, we arrive at the cyclic order

\[ c(G_2, k) = \gcd\{k(422 + 585k + 400k^2 + 135k^3 + 18k^4)/120, \]
\[ k(2 + 15k + 40k^2 + 45k^3 + 18k^4)/120, \]
\[ k(2 - 15k + 40k^2 - 45k^3 + 18k^4)/120, \]
\[ k(601 + 660k + 350k^2 + 90k^3 + 9k^4)/30, \]
\[ k(16 + 60k + 80k^2 + 45k^3 + 9k^4)/30, \]
\[ k(2867 + 2550k + 1090k^2 + 225k^3 + 18k^4)/30\}. \]

Indeed, this agrees with the result from section 4.2.

5 Twisted Spin\(^c\) Bordism and the Twisted Index

The ordinary \(K\)-homology of a space \(X\) is entirely determined by the Spin\(^c\) bordism of \(X\); see [17]. This suggests that much of the structure in twisted \(K\)-homology ought to be visible in twisted Spin\(^c\) bordism. In section 3 we saw that the cyclic order of the twisted \(K\)-homology of a group \(G\) is determined by a collection of relations of the form \(\tau_k(x) - \tau_0(x) = 0\), where \(\tau_j\) is the \(j\)-twisted map from \(K.\Omega G\) to \(K.\ast\). When the class \(x \in K.\Omega G\) is represented as the image of the fundamental class of a Spin\(^c\) manifold \(M\), there is a natural Spin\(^c\) manifold \(M(j)\) such that the fundamental class \([M(j)] \in M\text{Spin}^c;\ast\) maps via the index to the element \(\tau_j(x) \in K.\ast\). Moreover, there is an explicitly identifiable twisted Spin\(^c\) nullbordism over \(G\) of \(M(k) - M(0)\). In short, the relations determining the cyclic order of twisted \(K\)-homology have realizations in twisted Spin\(^c\) bordism. The construction of these nullbordisms is the focus of sections 5.1 and 5.2. Section 5.3 discusses the possibility of representing the exterior generators of the twisted \(K\)-homology of \(G\) by twisted Spin\(^c\) manifolds.

5.1 A Cocycle Model for Twisted Spin\(^c\) Bordism

In order to describe twisted Spin\(^c\) manifolds explicitly, we need a more geometric, less homotopy-theoretic description of twisted Spin\(^c\) structures; in particular we present a cocycle model for twisted Spin\(^c\) bordism. This model is presumably well known and in any case takes cues from the Hopkins-Singer philosophy of differential functions [18].

Recall that Spin\(^c\) is the total space of a \(U(1)\)-principal bundle over \(SO\). Correspondingly there is a principal bundle \(BU(1) \to B\text{Spin}^c \to BSO\) which is classified by \(\beta w_2 : BSO \to BBU(1)\), the integral Bockstein of the second Stiefel-Whitney class. A Spin\(^c\) structure on an oriented manifold \(M\) is a lift to \(B\text{Spin}^c\) of the classifying map \(\nu : M \to BSO\) of the (stable) normal bundle of \(M\). Such a lift is determined by a nullhomotopy of the composite \(\beta w_2(\nu) : M \to BSO \to BBU(1)\). Specifying such a nullhomotopy is equivalent to choosing a 2-cochain \(c\) on \(M\) such that the coboundary of \(c\) is \(\beta w_2(\nu(M))\). (Note that we have chosen once and for all a 3-cocycle \(g\) representing the generator of \(H^3(BBU(1); \mathbb{Z})\), and the condition on the cochain \(c\) is that \(\delta c = \nu^*((\beta w_2)^*(g))\)). Ordinary Spin\(^c\) bordism of \(X\) is therefore equivalent to bordism of oriented manifolds \(M\) over \(X\) equipped with a
2-cochain $c$ on $M$ such that 

$$\delta c = \beta w_2(\nu(M)).$$

The model for twisted $Spin^c$ bordism is similar. We first recall the homotopy-theoretic definition of twisted $Spin^c$ bordism from section 2.1. Given a twisting map $\tau : X \to K(\mathbb{Z}, 3)$, we have a $K(\mathbb{Z}, 2)$-principal bundle $P$ on $X$ and so an associated $BSpin^c$ bundle $Q = P \times_{K(\mathbb{Z}, 2)} BSpin^c$. More particularly we have a series of bundles

$$Q_n = P \times_{K(\mathbb{Z}, 2)} BSpin^c(n)$$

and universal vector bundles

$$UQ_n = (P \times_{K(\mathbb{Z}, 2)} ESpin^c(n)) \times_{Spin^c(n)} \mathbb{R}^n.$$ 

The corresponding Thom spectrum

$$\text{Th}(UQ) = P_+ \wedge_{K(\mathbb{Z}, 2)_+} MSpin^c$$

has as its homotopy groups the twisted $Spin^c$ bordism groups of $X$. The twisted index map to twisted $K$-homology is induced by the map $id \wedge \text{ind} : P_+ \wedge_{K(\mathbb{Z}, 2)_+} MSpin^c \to P_+ \wedge_{K(\mathbb{Z}, 2)_+} K$.

The principal bundle $P$ and the associated $BSpin^c$ bundle $Q$ are defined by the pullbacks

$$
\begin{array}{ccc}
P & \xrightarrow{i} & EK(\mathbb{Z}, 2) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tau} & K(\mathbb{Z}, 3) \\
\end{array} \quad \begin{array}{ccc}
Q & \xrightarrow{\beta w_2} & BSO \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tau} & K(\mathbb{Z}, 3) \\
\end{array}
$$

On the other hand $BSO$ is precisely the quotient $\ast \times_{K(\mathbb{Z}, 2)} BSpin^c$, and the diagram

is therefore a homotopy pullback. Twisted $Spin^c$ bordism is the homotopy of $\text{Th}(UQ)$; a map from a sphere into $\text{Th}(UQ)$ transverse to the zero section $Q$ determines a manifold $M$ equipped with a map $M \to Q$. This map $M \to Q$ specifies maps $i : M \to X$ and $\nu : M \to BSO$ (classifying the normal bundle of $M$) and a chosen homotopy between $\tau i$ and $\beta w_2 \nu$. The choice of this homotopy is equivalent to the choice of a 2-cochain $c$ with coboundary equal to the difference $\nu^*((\beta w_2)^* g) - i^*(\tau^* g)$, where $g$ is as before a 3-cocycle representing the generator of the third cohomology of $K(\mathbb{Z}, 3)$. In summary, the $\tau$-twisted $Spin^c$ bordism of $X$ is bordism of oriented manifolds $M$ equipped with a map $i : M \to X$ and a 2-cochain $c$ such that

$$\delta c = \beta w_2(\nu(M)) - i^*(\tau),$$

where $\nu(M)$ is the stable normal bundle of $M$. 

32
5.2 Twisted Nullbordism and the Geometry of the Cyclic Order

In section 3 we saw that the cyclic order of the twisted $K$-homology of $G$ is the greatest common divisor of the collection of differences $\{\tau_k(x_i) - \tau_0(x_i)\}$, where $\{x_i\}$ is a set of algebra generators for $K\Omega G$ and $\tau_j$ denotes the $j$-twisted map from $K\Omega G$ to $K$. Frequently, these generators $\{x_i\}$ can be described as the images of the fundamental classes of $Spin^c$ manifolds $M_i$. (For example, we gave such a description for $SU(n+1)$, $Sp(n)$, and $G_2$ in sections 4.2 and 4.3). In this case the manifolds $M_i$ admit modified $Spin^c$ structures $M_i(j)$ and the index of $M_i(j)$ is $\tau_j(x_i) \in K$. Moreover, there is a twisted $Spin^c$ structure (over $G$) on $M_i \times I$ cobounding the difference $M_i(k) - M_i(0)$; the relations $\tau_k(x_i) - \tau_0(x_i) = 0$ determining the cyclic order of twisted $K$-homology therefore have realizations in twisted $Spin^c$ bordism.

Before constructing these twisted $Spin^c$ bordisms, we recall that a $Spin^c$ structure can be altered by a line bundle and we discuss how this alteration affects the pushforward of the fundamental class. A twisted $Spin^c$ manifold is, as before, an oriented manifold $M$ together with a 2-cochain $c$ such that $\delta c = \beta w_2(\nu(M)) - i^*(\tau)$. In the examples we consider, the underlying manifold $M$ is almost complex and so has a canonical ordinary $Spin^c$ structure; in particular $M$ comes equipped with a 2-cochain $b$ such that $\delta b = \beta w_2(\nu(M))$. A twisted structure on $M$ is then given by a choice of 2-cochain $d$ such that $\delta d = -i^*(\tau)$. If the twisting class $\tau$ is zero on $M$, then the ‘twisted’ $Spin^c$ structure corresponding to the cochain $b + d$ is of course ordinary, but it nevertheless differs from the $Spin^c$ structure determined by the original cochain $b$. We denote by $M(d)$ this modification of the canonical $Spin^c$ structure on $M$ by the 2-cocycle $d$; we also refer to the alteration as a modification by the corresponding line bundle $L(d)$. Let $\pi : M \to \ast$ be the projection to a point; the pushforward in $K$-theory depends on the $Spin^c$ structure on $M$ as follows:

$$\pi^!(M(d))(1) = \pi^!(M)(L(d)).$$

This relation follows more or less immediately from the fact that the Thom class defined by the $Spin^c$ structure on $M(d)$ is $L(d)$ tensor the Thom class defined by the structure on $M$; see [19]. In terms of our Rothenberg-Steenrod spectral sequence approach to twisted $K$-homology, this tells us that the twisted image $\tau_k([M]) = \langle L(k), [M] \rangle$ of the fundamental class $[M]$ is reinterpretable as the ordinary image $\tau_0([M(k)]) = \langle 1, [M(k)] \rangle$ of the fundamental class $[M(k)]$.

We present twisted $Spin^c$ bordisms realizing the relations in the twisted $K$-homology of $SU(n+1)$. In section 4.2 we saw that the fundamental classes $\{[\mathbb{CP}^i]\}$ are algebra generators for $K\Omega SU(n+1)$, and the relations determining the cyclic order of $K^* SU(n+1)$ are

$$0 = \langle L^k, [\mathbb{CP}^i] \rangle - 1 = \langle L^k, [\mathbb{CP}^i] \rangle - \langle 1, [\mathbb{CP}^i] \rangle.$$

(These classes take values in the twisted $K$-theory of $SU(n+1)$ via the inclusion $\mathbb{CP}^i \to \Sigma \mathbb{CP}^i \to \Sigma \mathbb{CP}^n \to SU(n+1)$, which is of course nullhomotopic.) By the above remarks
we can rewrite the relation as

\[ 0 = \langle 1, [\mathbb{CP}^i(k)] \rangle - \langle 1, [\mathbb{CP}^i] \rangle = \langle 1, [\mathbb{CP}^i(k) - \mathbb{CP}^i] \rangle. \]

If we can produce a nullbordism of \( \mathbb{CP}^i(k) - \mathbb{CP}^i \), we will have pulled the given relation back to twisted \( \text{Spin}^c \) bordism.

Write \( \Sigma \mathbb{CP}^i = ([−2, 2] \times \mathbb{CP}^i)/(\{−2, 2\} \times \mathbb{CP}^i \cup [−2, 2] \times * ) \) and consider the inclusion \( i : \Sigma \mathbb{CP}^i \rightarrow \Sigma \mathbb{CP}^n \rightarrow SU(n + 1) \). Choose a 3-cocycle representing the twisting \( \tau \) on \( SU(n + 1) \) such that \( i^*(\tau) \) is \( k \) times the cocycle locally Poincare dual to the submanifold \( \mathbb{CP}^{i−1} \times \{0\} \) in \( \Sigma \mathbb{CP}^i \). The product \( \mathbb{CP}^i \times [−1, 1] \) has a canonical \( \text{Spin}^c \) structure coming from the complex structure of \( \mathbb{CP}^i \). There is a twisted structure on \( \mathbb{CP}^i \times [−1, 1] \) defined by the 2-cochain \( d \) that is \( k \) times the cocycle locally Poincare dual to the submanifold \( \mathbb{CP}^{i−1} \times [−1, 0] \); denote this twisted structure by \( (\mathbb{CP}^i \times [−1, 1])(k|0) \).

The coboundary of \( d \) is precisely \( −i^*(\tau) \). Moreover, the cochain \( d \) restricts to \( k \) times the generator of \( H^2(\mathbb{CP}^i \times \{−1\}) \) and to zero on \( \mathbb{CP}^i \times \{1\} \). The difference \( \mathbb{CP}^i(k) - \mathbb{CP}^i \) is therefore zero in \( M\text{Spin}^{c,\tau}SU(n + 1) \), as desired. Notice that the same argument shows that \( \mathbb{CP}^i(l + k) - \mathbb{CP}^i(l) \) is null for all \( l \), which implies that \( \left( \binom{l+k+i}{l} \right) - \left( \binom{l+i}{l} \right) \) is zero in \( K^{\tau}SU(n + 1) \). In fact, for any sequence of integers \( \{l_i\}, \ 1 \leq i \leq n \), the gcd of the set \( \{\binom{l_i+k+i}{l_i} - \binom{l_i+i}{l_i}\} \) is again the cyclic order of \( K^{\tau}SU(n + 1) \).

Whenever algebra generators of \( K.\Omega G \) are represented as the fundamental classes of \( \text{Spin}^c \) manifolds, the same argument produces nullbordisms in \( M\text{Spin}^{c,\tau}G \) realizing the cyclic order of \( K^{\tau}G \); we forgo details. Note though that in general the twisting cochain \( d \) will no longer be locally Poincare dual to a submanifold but merely to an appropriate singular chain.

### 5.3 Representing the Exterior Generators of Twisted \( K \)-Homology

We would like to represent the algebra generators of \( K^{\tau}G \) as the fundamental classes of twisted \( \text{Spin}^c \) manifolds over \( G \). Here we merely suggest an approach for further investigation, taking cues from the structure of the Rothenberg-Steenrod spectral sequence; in the process we produce a candidate representative for the generator of \( K^{\tau}SU(3) \). Finding representatives in general will require a more thorough investigation of \( M\text{Spin}^{c,\tau}G \) and of the associated map to \( K^{\tau}G \).

The structure of the ordinary \( \text{Spin}^c \) bordism group is governed by \( \text{Spin}^c \) characteristic numbers; we briefly recall how to compute these invariants. In section 5.1 we considered the principal bundle \( BU(1) \rightarrow B\text{Spin}^c \rightarrow BSO \) classified by \( \beta w_2 : BSO \rightarrow BBU(1) \). There is another principal bundle \( B\text{Z}/2 \rightarrow B\text{Spin}^c \rightarrow BSO \times BU(1) \) classified by \( (w_2 \times r) : BSO \times BU(1) \rightarrow B\text{Z}/2 \), where \( r \) is the nontrivial map \( BU(1) \rightarrow B\text{Z}/2 \). This latter bundle is usually more convenient for computations of \( \text{Spin}^c \) characteristic classes. The relationship between the two bundles is encoded in the matrix

\[ \text{matrix} \]

34
Indeed this diagram shows that the total spaces of the two fibrations are the same.

Following Anderson, Brown, and Peterson \cite{AndersonBrownPeterson}, Stong \cite{Stong} showed that a Spin\(^c\) manifold \(M\) is zero in Spin\(^c\) bordism if and only if all of its rational and mod 2 characteristic numbers vanish. The map 

\[
(\pi \times \lambda)^* : H^*(BSO \times BU(1); \mathbb{Q}) \to H^*(BSpin^c; \mathbb{Q})
\]

is an isomorphism and 

\[
(\pi \times \lambda)^* : H^*(BSO \times BU(1); \mathbb{Z}/2) \to H^*(BSpin^c; \mathbb{Z}/2)
\]

is an epimorphism. In particular a 2\(n\)-dimensional Spin\(^c\) manifold \(M\) is nullbordant if all the characteristic classes of the underlying oriented manifold vanish and the single Spin\(^c\) characteristic number \(\langle \lambda(M)^n, [M]_H \rangle\) is zero. The characteristic class \(\lambda\) depends on the Spin\(^c\) structure on \(M\) as follows. Let \(M(d)\) denote as in the last section the modification of the Spin\(^c\) structure on \(M\) by the line bundle or 2-cocycle \(d\). The class \(\lambda(M(d))\) is then \(\lambda(M) + 2d\), as is easily checked by noting that the composite \(BU(1) \to BSpin^c \xrightarrow{\lambda} BU(1)\) is multiplication by 2.

\[\sim\sim\sim\sim\]

We produce a candidate twisted Spin\(^c\) representative for the exterior generator of \(K^7 SU(3)\) by investigating the corresponding class in the \(E^2\) term of the Rothenberg-Steenrod spectral sequence. For simplicity we assume the twisting class \(k\) is odd; the even case is entirely analogous.

In section 3.1 we saw that the generator of \(K^7 SU(3)\) is represented at the \(E^2\) term of the Rothenberg-Steenrod spectral sequence by \(x_2 - \frac{k+3}{2}x_1\); here \(x_2\) and \(x_1\) are elements of \(\text{Tor}_1^{K,\Omega SU(3)}(\mathbb{Z}, \mathbb{Z})\), therefore of the \(E^1\) term of the spectral sequence, and their differentials are given by

\[
d^1 x_2 = \langle 1, [\mathbb{CP}^2(k)] \rangle - 1 = \langle 1, [\mathbb{CP}^2(k) - \mathbb{CP}^2(0)] \rangle
\]

\[
d^1 x_1 = \langle 1, [\mathbb{CP}^1(k)] \rangle - 1 = \langle 1, [\mathbb{CP}^1(k) - \mathbb{CP}^1(0)] \rangle
\]

In section 5.2 we found a twisted Spin\(^c\) bordism \(X_2 = (\mathbb{CP}^2 \times I)(k|0)\) whose boundary has index

\[\text{ind}(\partial X_2) = d^1 x_2.\]
Because of this index property, we consider \( X_2 \) a geometric representative of the algebraic class \( x_2 \). Note that the bordism \( X_2 \) is over \( \Sigma \mathbb{CP}^2 \) and therefore over \( SU(3) \).

Similarly, we have a bordism \( \widetilde{X}_1 = (\mathbb{CP}^1 \times I)/(k|0) \) whose boundary has index \( \text{ind}(\partial \widetilde{X}_1) = d^1x_1 \). Given our selection of \( X_2 \), the manifold \( \widetilde{X}_1 \) is a poor choice for a geometric representative of \( x_1 \); we would like to have a five-dimensional bordism \( X_1 \), still living over \( \Sigma \mathbb{CP}^2 \), with the same index property as \( \widetilde{X}_1 \). A natural choice for the underlying oriented bordism is \( P(\nu + 1) \times I \), where \( P(\nu + 1) \) is the projectivization of the sum of a trivial bundle and the normal bundle of \( \mathbb{CP}^1 \) in \( \mathbb{CP}^2 \); this projectivization is a resolution of the Thom space of the normal bundle and as such the bordism maps to \( \mathbb{CP}^2 \times I \subset \Sigma \mathbb{CP}^2 \). There is moreover a twisted \( \text{Spin}^c \) structure on this bordism, denoted \( X_1 = (P(\nu + 1) \times I)(k|0) \) and produced as in section 5.2, such that

\[
\text{ind}(\partial X_1) = d^1x_1.
\]

The linear combination \( C = X_2 - \frac{k+3}{2}X_1 \) wants to be an element of \( MS\text{Spin}^{c_7} \Sigma \mathbb{CP}^2 \) mapping to the exterior generator of \( K^7SU(3) \). The trouble of course is that \( C \) is not a closed manifold and so does not properly represent an element of \( MS\text{Spin}^{c_7} \Sigma \mathbb{CP}^2 \). Note that the map \( \partial C \rightarrow \Sigma \mathbb{CP}^2 \) is nullhomotopic by a nullhomotopy on which the twisting class is zero. Suppose there is a nullbordism \( W \) of \( \partial C \) in \( MS\text{Spin}^{c_7} \); then the union \( W \cup_{\partial C} C \) is a closed twisted \( \text{Spin}^c \) manifold over \( \Sigma \mathbb{CP}^2 \), as desired.

The boundary of \( C \) is

\[
\partial C = (\mathbb{CP}^2(k) - \mathbb{CP}^2) - \frac{k+3}{2}(P(\nu + 1)(k) - P(\nu + 1)).
\]

All the \( SO \)-characteristic numbers of \( \partial C \) certainly vanish. The cohomology ring of \( P(\nu + 1) \) is \( H^*(P(\nu + 1)) = \mathbb{Z}[y,x]/(y^2, x^2 + yx) \), where \( y \) is the first Chern class of the tautological bundle on the base \( \mathbb{CP}^1 \) and \( x \) is the first Chern class of the fibrewise tautological bundle on the total space. The tangential \( \text{Spin}^c \) characteristic class of \( P(\nu + 1)(k) \) is

\[
\lambda(T(P(\nu + 1)(k))) = \lambda(T_{\text{horiz}}) + \lambda(T_{\text{vert}}) = -(2 + 2k)y - 2x,
\]

and the associated characteristic number is

\[
\langle \lambda(T(P(\nu + 1)(k)))^2, [P(\nu + 1)]_H \rangle = 8k + 4.
\]

Similarly the characteristic number for \( \mathbb{CP}^2(k) \) is

\[
\langle \lambda(T(\mathbb{CP}^2(k)))^2, [\mathbb{CP}^2]_H \rangle = 4k^2 + 12k + 9.
\]

The vanishing of the \( \text{Spin}^c \) characteristic number for \( \partial C \) follows:

\[
\langle \lambda(T(\partial C))^2, [\partial C]_H \rangle = 4k^2 + 12k + 9 - \frac{k+3}{2}(8k + 4 - 4) = 0.
\]

Picking any \( \text{Spin}^c \) nullbordism \( W \) of \( \partial C \), the five-dimensional twisted \( \text{Spin}^c \) manifold \( W \cup_{\partial C} C \) should represent the generator of \( K^7SU(3) \).

36
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