Expansion Theorem for Sturm-Liouville problems with transmission conditions

O. Sh. Mukhtarov† and K. Aydemir†

†Department of Mathematics, Faculty of Science, Gaziosmanpaşa University, 60250 Tokat, Turkey
e-mail: omukhtarov@yahoo.com, kadriye.aydemir@gop.edu.tr

Abstract: The purpose of this paper is to extend some spectral properties of regular Sturm-Liouville problems to the special type discontinuous boundary-value problem, which consists of a Sturm-Liouville equation together with eigenparameter-dependent boundary conditions and two supplementary transmission conditions. We construct the resolvent operator and Green’s function and prove theorems about expansions in terms of eigenfunctions in modified Hilbert space $L_2[a,b]$.

Keywords: Boundary-value problems, transmission conditions, Resolvent operator, expansion theorem.

AMS subject classifications: 34L10, 34L15

1 Introduction

With historical roots in the application of Fourier series to heat flow, the Sturmian theory is one of the most extensively developing fields in pure and applied mathematics. As is well-known the eigenvalue parameter takes part linearly only in the differential equation in the classical Sturm-Liouville problems. However, in mathematical physics are encountered such problems, where eigenvalue parameter appear in both differential equation and boundary conditions. The first, we cite the works of Walter [18] and Fulton [7] both of which have extensive bibliographies, in the case of [7], a discussion of physical applications. Afterwards, we mention the results [3, 4, 11, 14] and corresponding references cited therein. In recent years there has been increasing interest of some Sturm-Liouville type problems which may have discontinuities in the solution or its derivative at interior point (see [1, 2, 5, 6, 8, 9, 10, 17, 19]).

In this paper we shall investigate some spectral properties of one discontinuous Sturm-Liouville problem for which the eigenvalue parameter takes part in both
differential equation and boundary conditions. Moreover, two supplementary
transmission conditions at one interior point are added to boundary conditions.
Namely, we consider the Sturm-Liouville equation,
\[ \tau u := -u''(x) + q(x)u(x) = \lambda u(x) \] (1.1)
to hold in finite interval \((a,b)\) except at one inner point \(c \in (a,b)\), where discon-
tinuity in \(u\) and \(u'\) are prescribed by transmission conditions
\[ \gamma_1 u(c - 0) - \delta_1 u(c + 0) = 0, \] (1.2)
\[ \gamma_2 u'(c - 0) - \delta_2 u'(c + 0) = 0, \] (1.3)
together with the eigenparameter- dependent boundary conditions
\[ \alpha_1 u(a) + \alpha_2 u'(a) = 0, \] (1.4)
\[ (\beta'_1 \lambda + \beta_1)u(b) - (\beta'_2 \lambda + \beta_2)u'(b) = 0, \] (1.5)
where the potential \(q(x)\) is real-valued, continuous in each interval \([a,c)\) and
\((c,b]\) and has a finite limits \(q(c \mp 0)\); \(\alpha_i, \beta_i, \beta'_i, \delta_i, \gamma_i \) \((i = 1,2)\) are real
numbers; \(\lambda\) is a complex eigenparameter. Naturally we exclude each of the trivial
conditions \(\gamma_1 = \delta_1 = 0, \gamma_2 = \delta_2 = 0, \alpha_1 = \alpha_2 = 0, \beta'_1 = \beta_1 = \beta'_2 = \beta_2 = 0\). In
contrast to previous works, eigenfunctions of this problem may have discontinuity
at the one inner point of the considered interval.

This kind of problems are connected with discontinuous material properties,
such as heat and mass transfer, varied assortment of physical transfer problems,
vibrating string problems when the string loaded additionally with point masses,
and diffraction problems [10, 16]. The study of the structure of the solution
of the matching region leads to the consideration of an eigenvalue problem for
a second order differential operator with piecewise continuous coefficients and
transmission conditions at interior points. A. Boumenir [5] use sampling tech-
niques to reconstruct the characteristic function associated with the eigenvalues
of two linked Sturm-Liouville operators by a transmission condition. In [19],
Wang et al. studied a class of Sturm-Liouville problems with eigenparameter-
dependent boundary conditions and transmission conditions at an interior point.
B. Chanane [6] computed the eigenvalues of Sturm-Liouville problems with several
discontinuity conditions inside a finite interval and parameter dependent boundary
conditions using the regularized sampling method. In [2] E. Bairamov and E.
Uğurlu examined the determinants of dissipative Sturm-Liouville operators with
transmission conditions. J. Ao et al. [1] have considered the finite spectrum of
Sturm-Liouville problems with transmission conditions. Such properties as iso-
morphism, coerciveness with respect to the spectral parameter, completeness of
root functions, distributions of eigenvalues of some discontinuous boundary value problems with transmission conditions and its applications to the corresponding initial-boundary value problems for parabolic equations have been investigated in [8, 9, 10] and [16].

2 The fundamental solutions and Green’ s function

By following the procedure of [9] we can define four solutions \( \phi_1(x, \lambda) \), \( \phi_2(x, \lambda) \), \( \chi_1(x, \lambda) \) and \( \chi_2(x, \lambda) \) of the equation (1.1) under the initial conditions

\[
\begin{align*}
u(a) &= \alpha_2, \quad u'(a) = -\alpha_1, \quad (2.1) \\
u(c + 0) &= \frac{\gamma_1}{\delta_1} \phi_1(c - 0, \lambda), \quad u'(c + 0) = \frac{\gamma_2}{\delta_2} \frac{\partial \phi_1(c - 0, \lambda)}{\partial x} \quad (2.2) \\
u(b) &= \beta_2 \lambda + \beta_2, \quad u'(b) = \beta_1 \lambda + \beta_1, \quad (2.3)
\end{align*}
\]

and

\[
\begin{align*}
u(c - 0) &= \frac{\delta_1}{\gamma_1} \chi_2(c + 0, \lambda), \quad u'(c - 0) = \frac{\delta_2}{\gamma_2} \frac{\partial \chi_2(c + 0, \lambda)}{\partial x} \quad (2.4)
\end{align*}
\]

respectively. Consequently, each of the functions

\[
\phi(x, \lambda) = \begin{cases} 
\phi_1(x, \lambda) & \text{for } x \in [a, c) \\
\phi_2(x, \lambda) & \text{for } x \in (c, b]
\end{cases} \quad \chi(x, \lambda) = \begin{cases} 
\chi_1(x, \lambda) & \text{for } x \in [a, c) \\
\chi_2(x, \lambda) & \text{for } x \in (c, b]
\end{cases}
\]

satisfies the equation (1.1) and the both transmission conditions (1.2) and (1.3). Moreover, the solution \( \phi(x, \lambda) \) satisfies the first of boundary condition (1.4), but \( \chi(x, \lambda) \) satisfies the other boundary condition (1.5). By applying the same method as in [17] we can prove that the solutions \( \phi(x, \lambda) \) and \( \chi(x, \lambda) \) are entire functions of complex parameter \( \lambda \) for each fixed \( x \in [a, c) \cup (c, b] \).

It is known from ordinary linear differential equation theory that each of the Wronskians \( w_1(\lambda) = W(\phi_1(x, \lambda), \chi_1(x, \lambda)) \) and \( w_2(\lambda) = W(\phi_2(x, \lambda), \chi_2(x, \lambda)) \) are independent of \( x \) in \([a, c) \) and \((c, b] \) respectively. By using (2.2) and (2.3) we have

\[
\begin{align*}
w_1(\lambda) &= \phi_1(c - 0, \lambda) \frac{\partial \chi_1(c - 0, \lambda)}{\partial x} - \chi_1(c - 0, \lambda) \frac{\partial \phi_1(c - 0, \lambda)}{\partial x} \\
&= \frac{\delta_1 \delta_2}{\gamma_1 \gamma_2} \left( \phi_2(c + 0, \lambda) \frac{\partial \chi_2(c + 0, \lambda)}{\partial x} - \chi_2(c + 0, \lambda) \frac{\partial \phi_2(c + 0, \lambda)}{\partial x} \right) \\
&= \frac{\delta_1 \delta_2}{\gamma_1 \gamma_2} w_2(\lambda) \quad (2.5)
\end{align*}
\]
Denote
\[ w(\lambda) := \gamma_1\gamma_2 w_1(\lambda) = \delta_1\delta_2 w_2(\lambda). \tag{2.6} \]

Again, similarly to [9] it can be proven that, there are infinitely many eigenvalues \( \lambda_n, \ n = 1, 2, ... \) of the BVTP (1.1) – (1.5) which are coincide with the zeros of characteristic function \( w(\lambda) \).

Let us consider the nonhomogeneous differential equation
\[ u'' + (\lambda - q(x))u = F_1(x), \tag{2.7} \]
on \([a, c) \cup (c, b]\) subject to nonhomogeneous boundary conditions
\[ \alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{2.8} \]
\[ (\beta_1 u(b) - \beta_2 u'(b)) + \lambda(\beta_1' u(b) - \beta_2' u'(b)) = F_2 \tag{2.9} \]
and homogeneous transmission conditions
\[ \gamma_1 u(c - 0) - \delta_1 u(c + 0) = 0, \tag{2.10} \]
\[ \gamma_2 u'(c - 0) - \delta_2 u'(c + 0) = 0. \tag{2.11} \]

and let \( \lambda \) is not eigenvalue. Making use of the definitions of the functions \( \phi_i, \chi_i \) \( (i = 1, 2) \) we find that the general solution of the differential equation (2.7) can be written in the form
\[
\begin{align*}
    u(x, \lambda) = & \begin{cases}
        \frac{\chi_1(x, \lambda)}{\omega_1(\lambda)} \int_a^x \phi_1(y, \lambda) F_1(y) dy + \frac{\phi_1(x, \lambda)}{\omega_1(\lambda)} \int_x^c \chi_1(y, \lambda) F_1(y) dy \\
        + c_{11} \phi_1(x, \lambda) + c_{12} \chi_1(x, \lambda), & \text{for } x \in (a, c) \\
        \frac{\chi_2(x, \lambda)}{\omega_2(\lambda)} \int_c^x \phi_2(y, \lambda) F_1(y) dy + \frac{\phi_2(x, \lambda)}{\omega_2(\lambda)} \int_x^b \chi_2(y, \lambda) F_1(y) dy \\
        + c_{21} \phi_2(x, \lambda) + c_{22} \chi_2(x, \lambda), & \text{for } x \in (c, b)
    \end{cases}
\end{align*}
\tag{2.12}
\]

where \( C_{ij} \ (i, j = 1, 2) \) are arbitrary constants. By substitution into the boundary conditions (2.8) and (2.9) we see at once that
\[ c_{12} = 0, \ C_{21} = \frac{F_2}{\omega_2(\lambda)}. \tag{2.13} \]

Further, substitution (2.12) into transmission conditions (2.10) and (2.11) we have the inhomogeneous linear system of equations for \( c_{11} \) and \( c_{22} \), the determinant of which is equal to \(-\omega(\lambda)\) therefore is not vanish by assumption. Solving that system we find
\[ c_{11} = \frac{1}{\omega_2(\lambda)} \int_c^b \chi_2(y, \lambda) F_1(y) dy + \frac{F_2}{\omega_2(\lambda)}, \tag{2.14} \]
\[ c_{22} = \frac{1}{\omega_1(\lambda)} \int_a^c \phi_1(y, \lambda) F_1(y) dy. \] (2.15)

Putting (2.13), (2.14) and (2.15) in (2.12) we deduce that problem (2.7)-(2.11) has an unique solution,

\[
u(x, \lambda) = \begin{cases} 
\chi_1(x, \lambda) \frac{\gamma_1}{\omega_1(\lambda)} \int_a^x \phi_1(y, \lambda) F_1(y) dy + \frac{\delta_1 \delta_2}{\gamma_1 \gamma_2} \int_c^b \chi_2(y, \lambda) F_1(y) dy 
+ \delta_1 \delta_2 \gamma_1 \gamma_2 F_2 
& \text{for } x \in (a, c) \\
\chi_2(x, \lambda) \frac{\gamma_2}{\omega_2(\lambda)} \int_a^c \phi_2(y, \lambda) F_1(y) dy + \delta_1 \delta_2 \gamma_1 \gamma_2 F_2 
& \text{for } x \in (c, b) 
\end{cases}
\] (2.16)

By defining the Green’s function as

\[
G_1(x, y; \lambda) = \begin{cases} 
\frac{\phi(x, \lambda) \chi(x, \lambda)}{\omega(\lambda)} & \text{for } a \leq y \leq x \leq b, \ x, y \neq c \\
\frac{\phi(y, \lambda) \chi(x, \lambda)}{\omega(\lambda)} & \text{for } a \leq x \leq y \leq b, \ x, y \neq c 
\end{cases}
\] (2.17)

the formula (2.16) can be rewritten in the following form

\[
\begin{aligned}
u(x, \lambda) & = \gamma_1 \gamma_2 \int_a^c G_1(x, y; \lambda) F_1(y) dy + \delta_1 \delta_2 \int_c^b G_1(x, y; \lambda) F_1(y) dy \\
& + \delta_1 \delta_2 \gamma_1 \gamma_2 \frac{\phi(x, \lambda)}{\omega(\lambda)} 
\end{aligned}
\] (2.18)

3 Operator formulation in modified Hilbert space

In this section we shall introduce a special equivalent inner product in the Hilbert space \( L_2[a, b] \oplus \mathbb{C} \) and define a symmetric operator \( A \) in this space such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. For this we assume that, \( \rho := \beta_1 \beta_2 - \beta_1 \beta_2^* > 0 \) and for the sake of shorting we restrict ourselves to the investigation only the case \( \gamma_i \neq 0, \delta_i \neq 0 (i = 1, 2) \).

In the Hilbert Space \( H = L_2[a, b] \oplus \mathbb{C} \) of two-component vectors we define an equivalent inner product by

\[
< F, G > := |\gamma_1 \gamma_2| \int_a^c F_1(x) \overline{G_1(x)} dx + |\delta_1 \delta_2| \int_c^b F_1(x) \overline{G_1(x)} dx + \frac{|\delta_1 \delta_2| }{\rho} F_2 \overline{G_2}
\]
for $F = (F_1(x), F_2(x))$, $G = (G_1(x), G_2(x)) \in H$ and apply operator theory in the modified Hilbert space $H_1 = (L_2[a,b] \oplus \mathbb{C}, <.,.>)$. Below we shall use the following notations:

$$(u)_\beta := \beta_1 u(b) - \beta_2 u'(b), \quad (u)'_\beta := \beta_1' u(b) - \beta_2' u'(b).$$

Let us define a linear operator $H : A \rightarrow A$ with the domain

$$D(A) := \left\{ F = (F_1(x), (F_1)_\beta^') : F_1(x) and F_1^'(x) are absolutely continuous in each interval \([a,c) and (c,b]\), and has a finite limits \(F_1(c \mp 0) and F_1^'(c \mp 0), \tau F_1 \in L_2[a,b], a_1 u(a) + a_2 u'(a) = 0, \gamma_1 F_1(c - 0) = \delta_1 F_1(c + 0), \tau_1 F_1^'(c - 0) = \delta_2 F_1^'(c + 0)\right\}$$

and action low

$$A(F_1(x), (F_1)_\beta^') = (\tau F_1, (-F_1)_\beta^').$$

Consequently the problem (1.1) - (1.5) can be written in the operator form as

$$AU = \lambda U, \quad U := (u(x), (u)_\beta^') \in D(A)$$

in the Hilbert space $H_1$.

**Lemma 3.1.** The domain $D(A)$ is dense in $H_1$.

**Proof.**

**Theorem 3.2.** If $\gamma_1 \gamma_2 \delta_1 \delta_2 > 0$ then the linear operator $A$ is symmetric.

**Proof.**

**Remark 3.3.** Having in view this property of the problem (1.1) - (1.5), we shall assume $\gamma_1 \gamma_2 \delta_1 \delta_2 > 0$ everywhere in below. Also without loss of generality we shall let $\gamma_1 \gamma_2 > 0 and \delta_1 \delta_2 > 0$.

**Remark 3.4.** By Lemma 3.2 all eigenvalues of the problem (1.1) - (1.5) are real. Consequently, we can now assume that all eigenfunctions are real-valued.

**Corollary 3.5.** Let $u(x)$ and $v(x)$ be eigenfunctions corresponding to distinct eigenvalues. Then

$$\gamma_1 \gamma_2 \int_a^c u(x)v(x)dx + \delta_1 \delta_2 \int_c^b u(x)v(x)dx + \frac{\delta_1 \delta_2}{\rho} (u)_\beta'(v)_\beta' = 0. \quad (3.1)$$
Proof. The proof is immediate from the fact that, the eigenelements 
\((u(x), (u)')_\beta\) and \((v(x), (v)')_\beta\) of the symmetric linear operator \(A\) is ortogonal in 
the Hilbert space \(H_1\). \(\square\)

4 The Resolvent operator and Self-adjointness of the problem

In this section we shall construct the Resolvent operator and prove self-adjointness of the problem.

Lemma 4.1. Let \(\lambda_0\) be zero of \(w(\lambda)\). Then the solutions \(\phi(x, \lambda_0)\) and \(\chi(x, \lambda_0)\) are linearly dependent.

Proof. \(\square\)

Theorem 4.2. Each eigenvalue of the problem (1.1)-(1.5) is the simple zero of \(w(\lambda)\).

Proof. \(\square\)

Let \(A\) be defined as above and let \(\lambda\) not be an eigevalue of this operator. For construction the resolvent operator \(R(\lambda, A) := (\lambda - A)^{-1}\) we shall solve the 
operator equation 
\[(\lambda - A)U = F\] (4.1)
for \(F = (F_1(x), F_2) \in H_1\). This operator equation equivalent to the problem 
(2.7)-(2.11).

Using the equalities we see that 
\[(G_1(x, ,; \lambda))' = \frac{\phi(x, \lambda)}{\omega(\lambda)} (\chi(x, \lambda))' = \frac{\rho(\chi(x, \lambda))}{\omega(\lambda)}.\] (4.2)

Hence, the solution \(u(x, \lambda)\) may be written as 
\[u(x, \lambda) = \gamma_1 \gamma_2 \int_a^c G_1(x, y; \lambda) F_1(y) dy + \delta_1 \delta_2 \int_c^b G_1(x, y; \lambda) F_1(y) dy + \frac{\delta_1 \delta_2}{\rho} (G_1(x, ,; \lambda))' F_2\] (4.3)

Consequently, for the solution 
\[U(F, \lambda) := (u(x, \lambda), (u(., \lambda))')\] (4.4)
of the nonhomogeneous operator equation (4.1) we obtain the following formula
\[ U(F, \lambda) := (G_{x,\lambda}^{1}(x,;\lambda), (G_{x,\lambda}^{1}(x,;\lambda))'_{\beta}) \] (4.5)
where
\[ G_{x,\lambda} := (G_{1}^{1}(x,.;\lambda), (G_{1}^{1}(x,.;\lambda))'_{\beta}) \] (4.6)
Now, making use (2.17), (4.3), (4.4), (4.5) and (4.6) we see that if \( \lambda \) not an eigenvalue of \( A \) then
\[ U(F, \lambda) \in D(A) \text{ for } F \in H_{1}, \] (4.7)
\[ U((\lambda - A)F, \lambda) = F, \text{ for } F \in D(A) \] (4.8)
and
\[ \|U(F, \lambda)\| \leq |\text{Im}\lambda|^{-1}\|F\| \text{ for } F \in H_{1}, \text{ Im}\lambda \neq 0. \] (4.9)
Hence, each nonreal \( \lambda \in \mathbb{C} \) is a regular point of an operator \( A \) and
\[ R(\lambda, A)F = (G_{x,\lambda}^{1}, (G_{x,\lambda}^{1})'_{\beta}) \text{ for } F \in H_{1} \] (4.10)
Because of (4.7) and (4.10)
\[ (\lambda - A)D(A) = R(\lambda, A)D(A) = H_{1} \text{ for } \text{Im}\lambda \neq 0. \] (4.11)
**Theorem 4.3.** The linear operator \( A \) is self-adjoint.

**Proof.** From the equality (4.11) and the fact that \( A \) is symmetric it follows by the standard theorems for symmetric operators in Hilbert space s that \( A \) is self-adjoint in \( H_{1} \) (see, for example, [12], Theorem 2.2.p. 198).

5 Expansion is series of eigenfunctions

Let \( \lambda_{n}, n = 1, 2, \ldots \) be eigenvalues of the operator \( A \) and let \( \phi_{n}(x) := \phi(x, \lambda_{n}), n = 1, 2, \ldots \) be defined as in section 2. By virtue of Lemma 4.1 the two-component vectors
\[ \Phi_{n} := (\phi(x, \lambda_{n}), (\phi(., \lambda_{n}))'_{\beta}), \] (5.1)
are the eigenelements of \( A \). Moreover,
\[ <\Phi_{n}, \Phi_{m}>_{1} = 0 \text{ for } n \neq m \] (5.2)
since $A$ is self-adjoint in $H_1$. Denote the normalized eigenelements by

$$\Psi_n := (\psi_n(x), (\psi_n(x))'),$$

(5.3)

where

$$\psi_n(x) := \frac{\phi(x, \lambda_n)}{\|\Phi_n\|_1}.$$  

(5.4)

Let $k_n \neq 0$ denote the real constant for which

$$\chi(x, \lambda_n) = k_n \phi(x, \lambda_n), \quad n = 0, 1, 2, \ldots, x \in (a, c) \cup (c, b).$$

(5.5)

Then

$$\frac{\phi_n(x)}{\beta} = \frac{\rho}{k_n}.$$  

(5.6)

Writing for $\lambda_n$ instead of $\lambda_0$ we obtain

$$\|\phi_n\|_1^2 = \frac{\omega'(\lambda_n)}{k_n}.$$  

(5.7)

Now, making use the representation (4.3) of the solution $u(x, \lambda)$, the equalities (2.17), (5.3)- (5.5) and the fact that each eigenvalue $\lambda_n$ is simple zero of $\omega(\lambda)$ we derive that

$$\text{Res}_{\lambda = \lambda_n} u(x, \lambda) = <F, \Psi_n >_1 \psi_n(x).$$

(5.8)

Consequently,

$$\text{Res}_{\lambda = \lambda_n} R(\lambda, A)F := <F, \Psi_n >_1 \Psi_n = C_n(F) \Psi,$$

(5.9)

where

$$C_n(F) := <F, \Psi_n >_1$$

(5.10)

are Fourier coefficients.

**Theorem 5.1.** (i) The modified Parseval equality

$$\gamma_1 \gamma_2 \int_a^c f^2(x)dx + \delta_1 \delta_2 \int_c^b f^2(x)dx = \sum_{n=0}^{\infty} |\gamma_1 \gamma_2 \int_c^c f(x) \psi_n(x)dx|$$

$$+ \delta_1 \delta_2 \int_c^b f(x) \psi_n(x)dx |^2$$

(5.11)

is hold for each $f \in L_2[a, c] \oplus L_2[c, b]$. 

9
Proof.

**Theorem 5.2.** Let \((f(x), (f)'_\beta) \in D(A)\). Then

\[
(i) \quad f(x) = \sum_{n=0}^{\infty} (\gamma_1 \gamma_2 \int_a^c f(x) \psi_n(x)dx + \delta_1 \delta_2 \int_c^b f(x) \psi_n(x)dx + \frac{\delta_1 \delta_2}{\rho} (f)'_\beta(\psi_n)'_\beta \psi_n(x) \tag{5.12}
\]

where, the series converges absolutely and uniformly in whole \([a, c) \cup (c, b]\). (ii) The series \((5.12)\) may also be differentiated, the differentiated series also being absolutely and uniformly convergent in whole \([a, c) \cup (c, b]\).

Proof.

6 **Counterexample**

Recall that we had derived all results in this study under condition \(\gamma_1 \gamma_2 \delta_1 \delta_2 > 0\). Let us show that this simple condition on the sign of the coefficients of the transmission conditions can not be omitted without putting any other condition on this coefficients. For this, consider the following special case of the problem (1.1)–(1.5) for which \(\gamma_1 \gamma_2 \delta_1 \delta_2 < 0\):

\[
- u'' = \lambda u, \quad x \in [-1, 0) \cup (0, 1] \tag{6.1}
\]

\[
u(-1) = 0, \quad \lambda u(1) = u'(1) \tag{6.2}
\]

\[
u(0 -) = u(0 + 0), \quad u'(0 -) = -u'(0 + 0) \tag{6.3}
\]

It is easy to verify that this problem has only the trivial solution \(u = 0\) for any \(\lambda \in \mathbb{C}\). Thus, if \(\gamma_1 \gamma_2 \delta_1 \delta_2 < 0\) then the spectrum of the problem (1.1)–(1.5) may be empty.

References

[1] J. Ao, J. Sun and M. Zhang *Matrix representations of Sturm-Liouville problems with transmission conditions*, Comput. Appl. Math., 63 (2012) 1335-1348.

[2] E. Bairamov and E. Uğurlu, *The determinants of dissipative Sturm-Liouville operators with transmission conditions*, Math. Comput. Modelling, Vol. 53, Nr. 5-6 (2011), 805-813.
[3] P. A. Binding, P. J. Browne and B. A. Watson, *Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. I.* Proceedings of the Edinburgh mathematical Society, Series II, 45(3) (2002), 631-645.

[4] P. A. Binding, P. J. Browne and B. A. Watson, *Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter II*, J. Comput. Appl. Math. 148(2002), 147-169.

[5] A. Boumenir, *Sampling the miss-distance and transmission function*, J. Math. Anal. Appl. 310(2005), 197-208.

[6] B. Chanane, *Sturm-Liouville problems with impulse effects*, Appl. Math. Comput., 190/1(2007) pp. 610-626.

[7] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edin. 77A(1977), 293-308.

[8] O. Sh. Mukhtarov and H. Demir, *Coersiveness of the discontinuous initial-boundary value problem for parabolic equations*, Israel J. Math., Vol. 114(1999), 239-252.

[9] O. Sh. Mukhtarov, M. Kadakal and F.S. Muhtarov On discontinuous Sturm-Liouville Problems with transmission conditions. J. Math. Kyoto Univ., 2004, Vol. 44, No. 4, 779-798.

[10] O. Sh. Mukhtarov and S. Yakubov, *Problems for ordinary differential equations with transmission conditions*, Appl. Anal., Vol 81(2002), 1033-1064.

[11] A. Schneider, *A note on eigenvalue problems with eigenvalue parameter in the boundary condition* Math. Z. 136, 163-167, 1974.

[12] S. Lang, *Real Analysis (Second edition)* Addison-Wesley, Reading, Mass. 1983.

[13] A. V. Likov and Y. A. Mikhailov, *The theory of Heat and Mass Transfer*, Qosenergaizdat,1963 (In Russian).

[14] A. A. Shkalikov, *Boundary value problems for ordinary differential equations with a parameter in boundary condition*, Trudy Sem. Imeny I.G. Petrowsgo, 9, 190-229., 1983.

[15] E. C. Titchmarsh, *Eigenfunctions Expansion Associated with Second Order Differential Equations I*, second edn. Oxford Univ. Press, London, 1962.
[16] I. Titeux, Ya. Yakubov, *Completeness of root functions for thermal condition in a strip with piecewise continuous coefficients*, Math. Models Methods Appl. Sci. 7 (1997) 1035-1050.

[17] E. Tunç and O.Sh. Muhtarov, *Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions*, Appl. Math. Comput., 157(2004), 347-355.

[18] J. Walter, *Regular eigenvalue problems with eigenvalue parameter in the boundary conditions*, Math. Z., 133(1973), 301-312.

[19] A. Wang, J. Sun, X. Hao and S. Yao, *Completeness of Eigenfunctions of Sturm-Liouville Problems with Transmission Conditions*, Methods Appl. Anal. Vol. 3(2009), 299-312.