CLASSIFICATION OF LINKS UP TO SELF #-MOVE

TETSUO SHIBUYA
Department of Mathematics, Osaka Institute of Technology
Omiya 5-16-1, Asahi, Osaka 535-8585, Japan
e-mail: shibuya@ge.oit.ac.jp

AKIRA YASUHARA
Department of Mathematics, Tokyo Gakugei University
Nukuikita 4-1-1, Koganei, Tokyo 184-8501, Japan
Current address:
Department of Mathematics, The George Washington University
Washington, DC 20052, USA
e-mail: yasuhara@u-gakugei.ac.jp

Dedicated to Professor Shin’ich Suzuki for his 60th birthday

Abstract
A pass-move and a # move are local moves on oriented links defined by L.H. Kauffman and H. Murakami respectively. Two links are self pass-equivalent (resp. self # equivalent) if one can be deformed into the other by pass-moves (resp. # moves), where non of them can occur between distinct components of the link. These relations are equivalence relations on ordered oriented links and stronger than link-homotopy defined by J. Milnor. We give two complete classifications of links with arbitrarily many components up to self pass-equivalence and up to self # equivalence respectively. So our classifications give subdivisions of link-homotopy classes.

1. Introduction

We shall work in piecewise linear category. All links will be assumed to be ordered and oriented.

A pass-move (resp. # move) is a local move on oriented links as illustrated in Figure 1.1(a) (resp. 1.1(b)). If the four strands in Figure 1.1(a) (resp. 1.1(b)) belong to the same component of a link, we call it a self pass-move (resp. self # move) (12). We note that the first author called pass-move and # move # (II) move and # (I) move respectively in his prior papers (12), (13), (14), etc. Two links are self pass-equivalent (resp. self #-equivalent) if one can be deformed into the other by a finite sequence of self pass-moves (resp. self # moves). Two links are link-homotopic if one can be deformed into the other by finite sequence of self crossing changes. Since both self pass-move and self # move are realized by self crossing changes, self pass-equivalence and self # equivalence are

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stronger than link-homotopy. Link-homotopy classification is already done by N. Habegger and X.S. Lin [3]. In this paper we give two complete classifications of links with arbitrarily many components up to self pass-equivalence and up to self #-equivalence respectively. So our classifications give subdivisions of link-homotopy classes.

![Diagram](image.png)

Figure 1.1

An n-component link \( l = k_1 \cup \cdots \cup k_n \) is proper if the linking number \( \text{lk}(l - k_i, k_i) \) is even for any \( i = 1, \ldots, n \). We define that a knot is a proper link. For a proper link \( l = k_1 \cup \cdots \cup k_n \), we call \( \text{Arf}(l) - \sum_{i=1}^{n} \text{Arf}(k_i) \pmod{2} \) the reduced Arf invariant [12] and denote it by \( \overline{\text{Arf}}(l) \), where Arf is the Arf invariant [10].

**Theorem 1.1.** Let \( l = k_1 \cup \cdots \cup k_n \) and \( l' = k'_1 \cup \cdots \cup k'_n \) be n-component links. Then the following (i) and (ii) hold.

(i) \( l \) and \( l' \) are self pass-equivalent if and only if they are link-homotopic and \( \overline{\text{Arf}}(k_{i_1} \cup \cdots \cup k_{i_p}) = \overline{\text{Arf}}(k'_{i_1} \cup \cdots \cup k'_{i_p}) \) for any proper links \( k_{i_1} \cup \cdots \cup k_{i_p} \subseteq l \) and \( k'_{i_1} \cup \cdots \cup k'_{i_p} \subseteq l' \).

(ii) \( l \) and \( l' \) are self #-equivalent if and only if they are link-homotopic and \( \overline{\text{Arf}}(k_{i_1} \cup \cdots \cup k_{i_p}) = \overline{\text{Arf}}(k'_{i_1} \cup \cdots \cup k'_{i_p}) \) for any proper links \( k_{i_1} \cup \cdots \cup k_{i_p} \subseteq l \) and \( k'_{i_1} \cup \cdots \cup k'_{i_p} \subseteq l' \).

For two-component links, both self pass-equivalence classification and self #-equivalence classification are done by the first author [13]. His proof can be applied to only two-component links. So we need different approach to proving Theorem 1.1.

A link \( l = k_1 \cup \cdots \cup k_n \) is \( \mathbb{Z}_2 \)-algebraically split if \( \text{lk}(k_i, k_j) \) is even for any \( i, j \) \( (1 \leq i < j \leq n) \). We note that if \( l = k_1 \cup \cdots \cup k_n \) is \( \mathbb{Z}_2 \)-algebraically split link, then \( l \) and \( k_i \cup k_j \) \( (1 \leq i < j \leq n) \) are proper.

**Theorem 1.2.** Let \( l = k_1 \cup \cdots \cup k_n \) and \( l' = k'_1 \cup \cdots \cup k'_n \) be n-component \( \mathbb{Z}_2 \)-algebraically split links. If \( l \) and \( l' \) are link-homotopic, then

\[
\overline{\text{Arf}}(l) + \sum_{1 \leq i < j \leq n} \overline{\text{Arf}}(k_i \cup k_j) \equiv \overline{\text{Arf}}(l') + \sum_{1 \leq i < j \leq n} \overline{\text{Arf}}(k'_i \cup k'_j) \pmod{2}.
\]
2. Preliminaries

In this section, we collect several results in order to prove Theorems 1.1 and 1.2.

Let \( l = k_1 \cup \cdots \cup k_n \) and \( l' = k'_1 \cup \cdots \cup k'_n \) be \( n \)-component links. Suppose that there is a disjoint union \( A = A_1 \cup \cdots \cup A_n \) of \( n \) annuli in \( S^3 \times [0, 1] \) with \( (\partial (S^3 \times [0, 1]), \partial A_i) = (S^3 \times \{0\}, k_i) \cup (-S^3 \times \{1\}, -k'_i) \) \((i = 1, \ldots, n)\) such that

(i) \( A \) is locally flat except for finite points \( p_1, \ldots, p_m \) in the interior of \( A \), and

(ii) for each \( p_j \) \((j = 1, 2, \ldots, m)\), there is a small neighborhood \( N(p_j) \) of \( p_j \) in \( S^3 \times [0, 1] \) such that \( (\partial N(p_j), \partial (N(p_j) \cap A)) \) is a link as illustrated in Figure 2.1,

where \(-X\) denotes \( X \) with the opposite orientation. Then \( A \) is called a pass-annuli between \( l \) and \( l' \).

Figure 2.1

The following is proved by the first author in [12].

**Lemma 2.1.** Two links \( l \) and \( l' \) are self pass-equivalent if and only if there is a pass-annuli between them. \( \Box \)

It is known that a pass-move is realized by a finite sequence of \#-moves [8]. Thus we have the following.

**Lemma 2.2.** If two links \( l \) and \( l' \) are self pass-equivalent, then they are self \#-equivalent. \( \Box \)

A \( \Gamma \)-move [4] is a local move on oriented links as illustrated in Figure 2.2.

Figure 2.2
The following is known \cite{4}.

**Lemma 2.3.** A $\Gamma$-move is realized by a single pass-move. \qed

Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k'_1 \cup \cdots \cup k'_n$ be $n$-component links such that there is a 3-ball $B^3$ in $S^3$ with $B^3 \cap (l \cup l') = l$. Let $b_1, \ldots, b_n$ be mutually disjoint disks in $S^3$ such that $b_i \cap l = \partial b_i \cap k_i$ and $b_i \cap l' = \partial b_i \cap k'_i$ are arcs for each $i$. Then the link $l \cup l' \cup (\bigcup_{i=1}^n \partial b_i) - (\bigcup \text{int}(b_i \cap (l \cup l'))) \text{ is called a band sum (or a product fusion \cite{11}) of } l \text{ and } l' \text{ and denoted by } (k_1 b_1 k'_1) \cup \cdots \cup (k_n b_n k'_n). \text{ Note that a band sum of } l \text{ and } l' \text{ is } \mathbb{Z}_2\text{-algebraically split if } lk(k_i, k_j) \equiv lk(k'_i, k'_j) \pmod{2} \text{ (}1 \leq i < j \leq n\text{).}

By the definition of the Arf invariant via 4-dimensional topology \cite{10}, we have the following.

**Lemma 2.4.** Two links $l$ and $l'$ are link-homotopic if and only if there is a band sum of $l$ and $-l'$ that is link-homotopic to a trivial link, where $(S^3, -l) \cong (-S^3, -l')$. \qed

By the definition of the Arf invariant via 4-dimensional topology \cite{10}, we have the following.

**Lemma 2.5.** Let $l$ and $l'$ be proper links and $L$ a band sum of $l$ and $-l'$. Then $L$ is proper and $\operatorname{Arf}(L) \equiv \operatorname{Arf}(l) + \operatorname{Arf}(l') \pmod{2}$. \qed

The following lemma forms an interesting contrast to the lemma above.

**Lemma 2.6.** Let $l = k_1 \cup k_2$ and $l' = k'_1 \cup k'_2$ be 2-component links with $\text{lk}(k_1, k_2)$ and $\text{lk}(k'_1, k'_2)$ odd. Let $L = (k_1 b_1 (-k'_1)) \cup (k_2 b_2 (-k'_2))$ be a band sum and $L'$ a band sum obtained from $L$ by adding a single full-twist to $b_2$; see Figure 2.3. Then $L$ and $L'$ are proper and link-homotopic, and $\operatorname{Arf}(L) \neq \operatorname{Arf}(L')$.

![Figure 2.3](imageURL)

**Proof.** Clearly $L$ and $L'$ are proper and link-homotopic. So we shall show $\operatorname{Arf}(L) \neq \operatorname{Arf}(L')$.

Let $a_i$ be the $i$th coefficient of the Conway polynomial. Then we have

$$a_3(L) - a_3(L') = a_2((k_1 b_1 (-k'_1)) \cup k_2 \cup (-k'_2)).$$

It is known that the third coefficient of the Conway polynomial of two-component proper link is mod 2 congruent to the sum of the Arf invariants of the link and the components
This and Lemma 2.5 imply \( \text{Arf}(L) - \text{Arf}(L') \equiv a_3(L) - a_3(L') \) (mod 2). By 8,

\[
\begin{align*}
a_2((k_1 \# b_1(-k'_1)) \cup k_2 \cup (-k'_2)) &= \text{lk}(k_1 \# b_1(-k'_1), k_2)\text{lk}(k_2, -k'_2) + \text{lk}(k_2, -k'_2)\text{lk}(-k'_2, k_1 \# b_1(-k'_1)) \\
&\quad + \text{lk}(-k'_2, k_1 \# b_1(-k'_1))\text{lk}(k_1 \# b_1(-k'_1), k_2).
\end{align*}
\]

Thus we have \( \text{Arf}(L) - \text{Arf}(L') \equiv 1 \) (mod 2). \( \square \)

A \( \Delta \)-move is a local move on links as illustrated in Figure 2.4. If at least two of the three strands in Figure 2.4 belong to the same component of a link, we call it a quasi self \( \Delta \)-move. Two links are quasi self \( \Delta \)-equivalent if one can be deformed into the other by a finite sequence of quasi self \( \Delta \)-moves.

![Figure 2.4](image)

The following is proved by Y. Nakanishi and the first author in 9.

**Lemma 2.7.** Two links are link-homotopic if and only if they are quasi self \( \Delta \)-equivalent. \( \square \)

### 3. Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.2.** Since \( l \) is link-homotopic to \( l' \), by Lemma 2.7, \( l \) is quasi self \( \Delta \)-equivalent to \( l' \). It is sufficient to consider the case that \( l' \) is obtained from \( l \) by a single quasi self \( \Delta \)-move.

Suppose that the three strands of the \( \Delta \)-move that is applied to the deformation from \( l \) into \( l' \) belong to one component of \( l \). Without loss of generality we may assume that the component is \( k_1 \). Note that \( k_i \) and \( k'_i \) are ambient isotopic for any \( i \neq 1 \), and that \( k_i \cup k_j \) and \( k'_i \cup k'_j \) are ambient isotopic for any \( i < j \) (\( i \neq 1 \)). Since a \( \Delta \)-move changes the value of Arf invariant, we have \( \text{Arf}(l) \neq \text{Arf}(l') \), \( \text{Arf}(k_1) \neq \text{Arf}(k'_1) \) and \( \text{Arf}(k_1 \cup k_j) \neq \text{Arf}(k'_1 \cup k'_j) \). Thus we have \( \text{Arf}(l) = \text{Arf}(l') \) and \( \text{Arf}(k_1 \cup k_j) = \text{Arf}(k'_1 \cup k'_j) \). So we have the conclusion.

We consider the other case, i.e., the three strands of the \( \Delta \)-move belong to exactly two components of \( l \). Without loss of generality we may assume that the two components
are $k_1$ and $k_2$. Note that $k_i$ and $k'_i$ are ambient isotopic for any $i$, and that $k_i \cup k_j$ and $k'_i \cup k'_j$ are ambient isotopic for any $i < j$ ($(i, j) \neq (1, 2)$). Since $\text{Arf}(l) \neq \text{Arf}(l')$ and $\text{Arf}(k_1 \cup k_2) \neq \text{Arf}(k'_1 \cup k'_2)$, $\text{Arf}(l) + \text{Arf}(k_1 \cup k_2) \equiv \text{Arf}(l') + \text{Arf}(k'_1 \cup k'_2) \pmod{2}$. This completes the proof. □

Lemma 3.1. Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k'_1 \cup \cdots \cup k'_n$ be $n$-component $\mathbb{Z}_2$-algebraically split links. If $l$ and $l'$ are link-homotopic, $\text{Arf}(k_i) = \text{Arf}(k'_i)$ ($i = 1, \ldots, n$) and $\text{Arf}(k_i \cup k_j) = \text{Arf}(k'_i \cup k'_j)$ ($1 \leq i < j \leq n$), then $l$ and $l'$ are self pass-equivalent.

Proof. Since $l$ is link-homotopic to $l'$, by Lemma 2.7, $l$ is quasi self $\Delta$-equivalent to $l'$. Let $u$ be the minimum number of quasi self $\Delta$-moves which are needed to deform $l$ into $l'$. By Theorem 1.2, $\text{Arf}(l) = \text{Arf}(l')$. Since a $\Delta$-move changes the value of the Arf invariant, $u$ is even. It is sufficient to consider the case $u = 2$. Therefore there is a union $A = A_1 \cup \cdots \cup A_n$ of level-preserving $n$ annuli in $S^3 \times [0, 1]$ with $(\partial(S^3 \times [0, 1]), \partial A_i) = (S^3 \times \{0\}, k_i) \cup (-S^3 \times \{1\}, -k'_i)$ ($i = 1, \ldots, n$) such that

(i) $A$ is locally flat except for exactly two points $p_1, p_2$ in the interior of $A$, and

(ii) for each $p_t$ ($t = 1, 2$) there is a small neighborhood $N(p_t)$ of $p_t$ in $S^3 \times [0, 1]$ such that $(\partial N(p_t), \partial(N(p_t) \cap A))$ is the Borromean ring $R_t$, at least two components of which belong to some $A_i$.

A singular points $p_t$ is called type (i) if the three components of $R_t$ belong to $A_i$ and type (i, j) ($i < j$) if one or two componets are in $A_i$ and the others in $A_j$. For each $i$ (resp. $i, j$), let $u_t$ (resp. $u_{i,j}$) be the number of the singular points of type (i) (resp. type (i, j)). We note that a number of $\Delta$-moves which are needed to deform $k_i$ into $k'_i$ (resp. $k_i \cup k_j$ into $k'_i \cup k'_j$) is equal to $u_i$ (resp. $u_{i,j} + u_i + u_j$). By the hypothesis of this lemma, we have $u_i$ and $u_{i,j} + u_i + u_j$ are even. Hence $u_i$ and $u_{i,j}$ are even. This implies that both $p_1$ and $p_2$ are the same type.

Suppose that $p_1$ and $p_2$ are type (i, j). Without loss of generality we may assume that $(i, j) = (1, 2)$ and two components of the Borromean ring $R_1$ belong to $A_2$. Let $\alpha$ be an arc in the interior of $A_1$ that connects two singular points $p_1$ and $p_2$ of type $(1, 2)$, and let $(S^3, L) = (\partial N(\alpha), \partial(N(\alpha) \cap (A_1 \cup A_2)))$. Then $L$ is a 5-component link as illustrated in either Figure 3.1(a) or (b). In the case that $L$ is as Figure 3.1(a), we can deform $L$ into a trivial link by applying $\Gamma$-moves to the sublink $L \cap A_2$; see Figure 3.2. In the case that $L$ is as Figure 3.1(b), we can deform $L$ into the link as in Figure 3.2(a) by two $\Gamma$-moves, one is applied to $L \cap A_1$ and the other to $L \cap A_2$; see Figure 3.3. It follows from this and Figure
3.2 that $L$ can be deformed into a trivial link by $\Gamma$-moves, one is applied to $L \cap A_1$ and the others to $L \cap A_2$.

Suppose that $p_1$ and $p_2$ are type $(i)$. Let $\alpha$ be an arc in the interior of $A_i$ that connects two singular points $p_1$ and $p_2$ of type $(i)$, and let $(S^3, L) = (\partial N(\alpha), \partial (N(\alpha) \cap A_i))$. By the argument similar to that in the above, $L$ can be deformed into a trivial link by applying $\Gamma$-moves to $L \cap A_i$.

Therefore, by Lemma 2.3, we can construct pass-annuli in $S^3 \times [0, 1]$ between $l$ and $l'$. Lemma 2.1 completes the proof. $\square$
Lemma 2.5, we have $\text{Arf}(C)$

For a link $l = k_1 \cup \cdots \cup k_n$, let $G_l^0$ (resp. $G_l^c$) be a graph with the vertex set $\{k_1, ..., k_n\}$ and the edge set $\{k_i k_j | \text{lk}(k_i, k_j) \text{ is odd} \}$ (resp. $\{k_i k_j | \text{lk}(k_i, k_j) \text{ is even} \}$). Note that $G_l^0 \cup G_l^c$ is the complete graph with $n$ vertices. For a band sum $L = K_1 \cup \cdots \cup K_n (= (k_1 \# b_1 (-\overrightarrow{K_1}))) \cup \cdots \cup (k_n \# b_n (-\overrightarrow{K_n}))$ of $l$ and $-\overrightarrow{G}$, let $A_L$ be a graph with the vertex set $\{K_1, ..., K_n\}$ and the edge set $\{K_i K_j | \text{Arf}(K_i \cup K_j) = 0 \}$. (Note that $L$ is a $\mathbb{Z}_2$-algebraically split link since $l$ and $l'$ are link-homotopic.)

Claim. There is a band sum $L$ of $l$ and $-\overrightarrow{G}$ such that $L$ is link-homotopic to a trivial link and $A_L$ is the complete graph with $n$ vertices.

Proof. Let $T$ be a maximal subgraph of $G_l^0$ that does not contain a cycle. Since $T$ does not contain a cycle, by Lemmas 2.4 and 2.6, there is a band sum $L$ of $l$ and $l'$ such that $L$ is link-homotopic to a trivial link and $T \subset h(A_L)$, where $h : A_L \rightarrow G_l^0 \cup G_l^c$ is the natural map defined by $h(K_i) = k_i$ and $h(K_i K_j) = k_i k_j$. By Lemma 2.5, we have $G_l^c \subset h(A_L)$. Since $h$ is injective and $G_l^0 \cup G_l^c$ is the complete graph, it is sufficient to prove that $h$ is surjective. Let $E$ be the set of edges which are not contained in $h(A_L)$, and $H^o = h(A_L) \cap G_l^0$. Suppose $E \neq \emptyset$. Then there is an edge $e \in E$ such that there is a cycle $C$ in $H^o \cup e$ containing $e$ whose any ‘diagonals’ are not contained in $G_l^c$. (In fact, for each $e_i \in E$, consider the minimum length $l_i$ of cycles in $H^o \cup e_i$ containing $e_i$ and choose an edge $e$ and a cycle $C$ in $H^o \cup e$ containing $e$ so that length $C$ is equal to $\min\{l_i | e_i \in E\}$.) Without loss of generality we may assume that $C = k_1 k_2 \cdots k_c k_1$ and $e = k_1 k_2$. Set $l_c = k_1 \cup \cdots \cup k_c$ and $L_c = K_1 \cup \cdots \cup K_c$. Since $C$ has no diagonals in $G_l^0$, all diagonals are in $G_l^c$. Thus we have $k_i k_j \subset H^c \cup G_l^c (= h(A_L))$ for any $i, j$ ($1 \leq i < j \leq c$) except for $(i, j) = (1, 2)$. This implies $\text{Arf}(K_i \cup K_j) = 0$ for any $i, j$ ($1 \leq i < j \leq c$, $(i, j) \neq (1, 2)$). The fact that $C$ has no diagonals in $G_l^c$ implies $l_c$ is a proper link. By the hypothesis about the Arf invariants and Lemma 2.5, we have $\text{Arf}(L_c) \equiv 2\text{Arf}(l_c) \equiv 0 \pmod{2}$ and $\text{Arf}(K_i) \equiv 2\text{Arf}(k_i) \equiv 0 \pmod{2}$.

Figure 3.3
Since $L_c$ is link-homotopic to a trivial link, by Theorem 1.2, $\text{Arf}(K_1 \cup K_2) = 0$. This contradicts $e = k_1k_2 \in E$. □

By Claim, there is a band sum $L = K_1 \cup \cdots \cup K_n$ of $l$ and $-\overline{l}$ such that $L$ is link-homotopic to a trivial link, $\text{Arf}(K_i) = 0$ ($i = 1, \ldots, n$) and $\text{Arf}(K_i \cup K_j) = 0$ ($1 \leq i < j \leq n$). By Lemma 3.1, $L$ is self pass-equivalent to a trivial link. Since $L$ is a band sum of $l$ and $-\overline{l}$, we can construct a pass-annuli between $l$ and $l'$. Lemma 2.1 completes the proof.

(ii) Since a $\#$-move changes the value of the Arf invariant, by applying self $\#$-moves, we may assume that $\text{Arf}(k_i) = \text{Arf}(k'_i)$ for any $i$. Theorem 1.1(i) and Lemma 2.2 complete the proof. □

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