THE COBURN-SIMONENKO THEOREM FOR TOEPLITZ OPERATORS ACTING BETWEEN HARDY TYPE SUBSPACES OF DIFFERENT BANACH FUNCTION SPACES

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Abstract. Let Γ be a rectifiable Jordan curve, let X and Y be two reflexive Banach function spaces over Γ such that the Cauchy singular integral operator S is bounded on each of them, and let M(X, Y) denote the space of pointwise multipliers from X to Y. Consider the Riesz projection \( P = (I + S) / 2 \), the corresponding Hardy type subspaces \( P_X \) and \( P_Y \), and the Toeplitz operator \( T(a) : P_X \to P_Y \) defined by \( T(a)f = P(af) \) for a symbol \( a \in M(X, Y) \). We show that if \( X \hookrightarrow Y \) and \( a \in M(X, Y) \setminus \{0\} \), then \( T(a) \in L(P_X, P_Y) \) has a trivial kernel in \( P_X \) or a dense image in \( P_Y \). In particular, if \( 1 < q \leq p < \infty \), \( 1/r = 1/q - 1/p \), and \( a \in L^r \equiv M(L^p, L^q) \) is a nonzero function, then the Toeplitz operator \( T(a) \), acting from the Hardy space \( H^p \) to the Hardy space \( H^q \), has a trivial kernel in \( H^p \) or a dense image in \( H^q \).

1. Introduction

Let Γ be a Jordan curve, that is, a curve that homeomorphic to a circle. We suppose that Γ is rectifiable and equip it with the Lebesgue length measure \( |d\tau| \) and the counter-clockwise orientation. The Cauchy singular integral of a measurable function \( f : \Gamma \to \mathbb{C} \) is defined by

\[
(Sf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t, \varepsilon)} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,
\]

where the “portion” \( \Gamma(t, \varepsilon) \) is

\[
\Gamma(t, \varepsilon) := \{ \tau \in \Gamma : |\tau - t| < \varepsilon \}, \quad \varepsilon > 0.
\]

It is well known that \( (Sf)(t) \) exists a.e. on Γ whenever \( f \) is integrable (see [10, Theorem 2.22]).

For two normed spaces \( X \) and \( Y \), we will write \( X \hookrightarrow Y \) if there is a constant \( c \in (0, \infty) \) such that \( \|f\|_Y \leq c\|f\|_X \) for all \( f \in X \), \( X = Y \) if \( X \) and \( Y \) coincide as sets and there are constants \( c_1, c_2 \in (0, \infty) \) such that \( c_1\|f\|_X \leq \|f\|_Y \leq c_2\|f\|_X \) for all \( f \in X \), and \( X \equiv Y \) if \( X \) and \( Y \) coincide as sets and \( \|f\|_X = \|f\|_Y \) for all \( f \in X \). As usual, the space of all bounded linear operators from \( X \) to \( Y \) is denoted by \( \mathcal{L}(X, Y) \). We adopt the standard abbreviation \( \mathcal{L}(X) \) for \( \mathcal{L}(X, X) \).

Let \( \gamma \) be a measurable subset of \( \Gamma \) of positive measure. The set of all measurable complex-valued functions on \( \gamma \) is denoted by \( M(\gamma) \). Let \( M^+(\gamma) \) be the subset

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of functions in $\mathcal{M}(\gamma)$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \gamma$ is denoted by $\chi_E$.

Following [1] Chap. 1, Definition 1.1], a mapping $\rho_\gamma : \mathcal{M}^+(\gamma) \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in \mathcal{M}^+(\gamma)$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets $E$ of $\gamma$, the following properties hold:

(A1) $\rho_\gamma(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho_\gamma(a f) = a \rho_\gamma(f)$, $\rho_\gamma(f + g) \leq \rho_\gamma(f) + \rho_\gamma(g)$,

(A2) $0 \leq g \leq f$ a.e. $\Rightarrow \rho_\gamma(g) \leq \rho_\gamma(f)$ (the lattice property),

(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \rho_\gamma(f_n) \uparrow \rho_\gamma(f)$ (the Fatou property),

(A4) $\rho_\gamma(\chi_E) < \infty$,

(A5) $\int_E f(\tau)|d\tau| \leq C_E \rho_\gamma(f)$

with the constant $C_E \in (0, \infty)$ that may depend on $E$ and $\rho_\gamma$, but is independent of $f$. When functions differing only on a set of measure zero are identified, the set $X(\gamma)$ of all functions $f \in \mathcal{M}(\gamma)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\gamma)$, the norm of $f$ is defined by

$$
\|f\|_{X(\gamma)} := \rho(|f|).
$$

The set $X(\gamma)$ under the natural linear space operations and under this norm becomes a Banach space (see [1] Chap. 1, Theorems 1.4 and 1.6) and

$$
L^\infty(\gamma) \hookrightarrow X(\gamma) \hookrightarrow L^1(\gamma).
$$

If $\rho_\gamma$ is a Banach function norm, its associate norm $\rho_\gamma'$ is defined on $\mathcal{M}^+(\gamma)$ by

$$
\rho_\gamma'(g) := \sup \left\{ \int_\gamma f(\tau)g(\tau)|d\tau| : f \in \mathcal{M}^+(\gamma), \rho_\gamma(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+(\gamma).
$$

It is a Banach function norm itself [1] Chap. 1, Theorem 2.2]. The Banach function space $X'(\gamma)$ determined by the Banach function norm $\rho_\gamma'$ is called the associate space (Kôthe dual) of $X(\gamma)$. The associate space $X'(\gamma)$ can be viewed a subspace of the dual space $X^*(\gamma)$.

Recall that, since the Lebesgue length measure $|d\tau|$ is separable (see, e.g., [12] Section 6.10), a Banach function space $X(\gamma)$ over $\gamma$ is separable if and only if its Kôthe dual space $X'(\gamma)$ is isometrically isomorphic to the Banach dual space $X^*(\gamma)$ (see, e.g., [1] Chap. 1, Corollaries 4.3, 4.4]). A Banach function space $X(\gamma)$ reflexive if and only if $X(\gamma)$ and $X'(\gamma)$ are separable (see, e.g., [1] Chap. 1, Corollary 5.6]).

For Banach function spaces $X(\gamma)$ and $Y(\gamma)$, let $M(X(\gamma), Y(\gamma))$ denote the space of pointwise multipliers from $X(\gamma)$ to $Y(\gamma)$ defined by

$$
M(X(\gamma), Y(\gamma)) := \{ f \in \mathcal{M}(\gamma) : fg \in Y(\gamma) \text{ for all } g \in X(\gamma) \}.
$$

It is a Banach function space with respect to the operator norm

$$
\|f\|_{M(X(\gamma), Y(\gamma))} = \sup \{ \|fg\|_{Y(\gamma)} : g \in X(\gamma), \|g\|_{X(\gamma)} \leq 1 \}.
$$

In particular, $M(X(\gamma), X(\gamma)) \equiv L^\infty(\gamma)$. Note that it may happen that the space $M(X(\gamma), Y(\gamma))$ contains only the zero function. For instance, if $1 \leq p < q < \infty$, then $M(L^p(\gamma), L^q(\gamma)) = \{0\}$. The continuous embedding $L^\infty(\gamma) \hookrightarrow M(X(\gamma), Y(\gamma))$ holds if and only if $X(\gamma) \hookrightarrow Y(\gamma)$. For example, if $1 \leq q \leq p \leq \infty$, then $L^p(\gamma) \hookrightarrow L^q(\gamma)$ and $M(L^p(\gamma), L^q(\gamma)) \equiv L^r(\gamma)$, where $1/r = 1/q - 1/p$. For these and many
other properties and examples, we refer to [16, 18, 20, 21, 22] (see also references therein).

For the brevity, we will write \( X := X(\Gamma) \) if \( \Gamma \) is a rectifiable Jordan curve. If \( X \) is a reflexive Banach function space over a rectifiable Jordan curve \( \Gamma \) and the Cauchy singular integral operator defined by (1) is bounded on \( X \), then in view of [13, Theorem 6.1] and the Hölder inequality for Banach function spaces (see, e.g., [1, Chap. 1, Theorem 2.4]), the curve \( \Gamma \) is a Carleson curve (or Ahlfors-David regular curve), that is,

\[
\sup_{t \in \Gamma} \sup_{\varepsilon > 0} \frac{|\Gamma(t, \varepsilon)|}{\varepsilon} < \infty.
\]

Moreover, by [13, Lemma 6.4], the operators

\[
P := (I + S)/2, \quad Q := (I - S)/2
\]

are bounded projections both on \( X \) and on \( X' \), the latter means that \( P^2 = P \) and \( Q^2 = Q \). Then we can define Hardy type subspaces \( PX, QX \) of \( X \) and \( PX', QX' \) of \( X' \).

In what follows we will always assume that \( X \) and \( Y \) are reflexive Banach function spaces and \( S \) is bounded on both \( X \) and \( Y \). For \( a \in M(X, Y) \), define the Toeplitz operator \( T(a) : PX \to PY \) with symbol \( a \) by

\[
T(a)f = Pf, \quad f \in PX.
\]

It is clear that \( T(a) \in \mathcal{L}(PX, PY) \) and

\[
\|T(a)\|_{\mathcal{L}(PX, PY)} \leq \|P\|_{\mathcal{L}(Y)} \|a\|_{M(X,Y)}.
\]

We note that there is a huge literature dedicated to Toeplitz operator acting between the same Hardy spaces \( H^p = PL^p, 1 < p < \infty \), see, e.g., the monographs by Douglas [7], Böttcher and Silbermann [3], Gohberg, Goldberg, Kaashoek [11], Nikolski [23] for Toeplitz operators on Hardy spaces over the unit circle and the monograph by Böttcher and Karlovich [2] for Toeplitz operators on weighted Hardy spaces over Carleson curves.

Surprisingly enough, we could find only one paper by Tolokonnikov [27] dedicated to Toeplitz operators acting between different Hardy spaces \( H^p \) and \( H^q \) over the unit circle. In particular, he described in [27, Theorem 4] all symbols generating bounded Toeplitz operators from \( H^p \) to \( H^q \) for \( 0 < p, q \leq \infty \). Very recently, Lešnik [17] proposed to study Toeplitz and Hankel operators between abstract Hardy spaces \( H[X] \) and \( H[Y] \) built upon different separable rearrangement-invariant Banach function spaces \( X \) and \( Y \) over the unit circle such that \( X \hookrightarrow Y \) and the space \( Y \) has nontrivial Boyd indices. Notice that the latter condition is equivalent to the boundedness of the operator \( S \) on the space \( Y \), whence \( H[Y] = PY \). Lešnik obtained analogues of the Brown-Halmos and Nehari theorems (see [17, Theorem 4.2] and [17, Theorem 5.5], respectively), extending results of the author [14] for the case of a reflexive rearrangement-invariant Banach function space \( X \) (that is, \( X = Y \) with nontrivial Boyd indices. He also proved [17, Theorem 6.1] that a Toeplitz operator \( T(a) : H[X] \to H[Y] \) is compact if and only if \( a = 0 \).

Inspired by the work of Lešnik [17], we prove the following analogue of the Coburn-Simonenko theorem for Toeplitz operators \( T(a) : PX \to PY \) in the case when \( X \) and \( Y \) are different Banach function spaces. Notice that we do not assume that the spaces \( X \) and \( Y \) are rearrangement-invariant.
Theorem 1.1. Let $X$ and $Y$ be reflexive Banach function spaces over a rectifiable Jordan curve $\Gamma$. Suppose $X \rightarrow Y$ and the Cauchy singular integral operator $S$ given by (1) is bounded on $X$ and on $Y$. If $a \in M(X,Y) \setminus \{0\}$, then $T(a) \in \mathcal{L}(PX, PY)$ has a trivial kernel in $PX$ or a dense image in $PY$.

The above result was proved by Coburn [4] for the case of $X = Y = L^2$ over the unit circle and by Simonenko [26] in a more general setting of $X = Y = L^p$, $1 < p < \infty$, over so-called Lyapunov curves. We also refer to [2] Theorem 6.17, where the above theorem is proved in the case $X = Y = L^p(w)$, where $L^p(w)$, $1 < p < \infty$, is a Lebesgue space with a Muckenhoupt weight over a Carleson Jordan curve.

The statement of Theorem 1.1 has a more precise form for concrete Banach function spaces $X, Y$ when $M(X,Y)$ can be calculated and conditions for the boundedness of $S$ are known. Here we mention only the case of Toeplitz operators acting from the Hardy space $H^p = PL^p$ to the Hardy space $H^q = PL^q$ as the simplest example.

Corollary 1.2. Let $1 < q \leq p < \infty$ and $1/r = 1/q - 1/p$. Suppose $\Gamma$ is a Carleson Jordan curve. If $a \in L^r \setminus \{0\}$, then the Toeplitz operator $T(a) \in \mathcal{L}(H^p, H^q)$ has a trivial kernel in $H^p$ or a dense image in $H^q$.

It seems that the above corollary is new even in the case of the unit circle.

The paper is organized as follows. In Section 2 we collect properties of Banach function spaces and their Hardy type subspaces proved elsewhere. In Section 3 we first relate the triviality of the kernel (resp. the density of the image) of a Toeplitz operator $T(a) \in \mathcal{L}(PX, PY)$ with the density of the range (resp. triviality of the kernel) of its companion operator $\bar{T}(a) : \mathcal{L}(QY', QX')$ defined by $\bar{T}(a)f = Q(af)$. Then show that one of the operators $T(a)$ or $\bar{T}(a)$ is injective with the aid of the Lusin-Privalov theorem and other results stated in Section 2. In Section 4 we recall the definition of variable Lebesgue spaces $L^{p(\cdot)}$, which give a non-trivial example of Banach function spaces. Further, we describe the space $M(L^{p(\cdot)}, L^{r(\cdot)})$ and formulate conditions for the boundedness of the operator Cauchy singular operator $S$ on $L^{p(\cdot)}$. These results allow us to reformulate Theorem 1.1 for Toeplitz operators between $PL^{p(\cdot)}$ and $PL^{q(\cdot)}$ in terms of variable exponents $p, q : \Gamma \rightarrow (1, \infty)$. In particular, we immediately get Corollary 1.2 taking all exponents constant.

2. Preliminaries

2.1. The Lusin-Privalov theorem. Let $\Gamma$ be a rectifiable Jordan curve. It divides the plane into a bounded connected component $D^+$ and an unbounded connected component $D^-$. We provide $\Gamma$ with the counter-clockwise orientation, that is, we demand that $D^+$ stays on the left of $\Gamma$ when the curve is traced out in the positive direction. Without loss of generality we suppose that $0 \in D^+$. Put

$$L^1_+ := \left\{ f \in L^1 : \int_{\Gamma} f(\tau) \tau^n d\tau = 0 \quad \text{for} \quad n \geq 0 \right\},$$

$$\langle L^1 \rangle^0_+ := \left\{ f \in L^1 : \int_{\Gamma} f(\tau) \tau^n d\tau = 0 \quad \text{for} \quad n < 0 \right\},$$

$$L^1_- := \langle L^1 \rangle^0_+ \oplus \mathbb{C}.$$

From [24] pp. 202–206 one can extract the following result.
Lemma 2.1. We have \( L^1_+ \cap (L^1)_0 = \{0\} \) and \( L^1_+ \cap L^1_- = \mathbb{C} \).

The proof of the following important theorem is contained in [24, p. 292] or [9, Theorem 10.3].

**Theorem 2.2** (Lusin-Privalov). Let \( \Gamma \) be a rectifiable Jordan curve. If \( f \in L^1_\pm \), then \( f \) vanishes either almost everywhere on \( \Gamma \) or almost nowhere on \( \Gamma \).

2.2. Properties of Banach function spaces and pointwise multipliers. In this subsection we collect some well known properties of Banach function spaces and pointwise multipliers between them.

**Lemma 2.3** ([11, Chap. 1, Proposition 2.10]). Let \( X, Y \) be Banach function spaces over a rectifiable Jordan curve \( \Gamma \) and let \( X', Y' \) be their associate spaces, respectively. If \( \Gamma \rightharpoonup Y \), then \( \Gamma \rightharpoonup X' \).

**Lemma 2.4** ([16, Section 2, property (vii)]). Let \( X, Y \) be Banach function spaces over a rectifiable Jordan curve \( \Gamma \) and let \( X', Y' \) be their associate spaces, respectively. Then \( M(X,Y) \equiv M(Y',X') \).

**Lemma 2.5**. Let \( X, Y \) be separable Banach function spaces over a rectifiable Jordan curve \( \Gamma \) and \( a \in M(X,Y) \). Then the adjoint of the operator \( aI \in \mathcal{L}(X,Y) \) of multiplication by the function \( a \) is the operator \( (aI)^* = \overline{a}I \in \mathcal{L}(Y',X') \).

**Proof.** Since \( X \) (resp., \( Y \)) is separable, its Banach dual space \( X^* \) (resp., \( Y^* \)) is isometrically isomorphic to the the associate (Köthe dual) space \( X' \) (resp., \( Y' \)) and

\[
G(f) = \int_\Gamma f(\tau)\overline{g(\tau)}d\tau
\]

gives the general form of a linear functional on \( X \) (resp., \( Y \)) and \( \|G\|_{X^*} = \|g\|_{X'} \) (resp., \( \|G\|_{Y^*} = \|g\|_{Y'} \)), see, e.g. [11, Chap. 1, Corollary 4.3]. The desired statement follows immediately from the above observation and Lemma 2.4.

2.3. Hardy type subspaces of a Banach function space. Suppose \( X \) is a reflexive Banach function space in which the Cauchy singular integral operator \( S \) is bounded. Put

\[
X_+ := PX, \quad X^0 := QX, \quad X_- := X^0_0 \oplus \mathbb{C}.
\]

The corresponding subspaces \( X^+_0, (X')^0_0, X'_- \) are defined analogously.

For \( f \in X \subset L^1 \), consider the Cauchy type integrals

\[
(C_\pm f)(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - z}d\tau, \quad z \in D^\pm.
\]

It is well known [24, p. 189] that the functions \((C_\pm f)(z)\) are analytic in \( D^\pm \), they have nontangential boundary values \((C_\pm f)(t)\) as \( z \to t \) almost everywhere on \( \Gamma \). These boundary values can be found by the Sokhotsky-Plemelj formulas

\[
(C_\pm f)(t) = \frac{1}{2} f(t) \pm \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t}d\tau,
\]

that is,

\[
(C_0 f)(t) = (Pf)(t), \quad (C^- f)(t) = (Qf)(t).
\]

Since the function \( f \in X_+ \) (respectively, \( f \in X^0_0 \)) coincides on \( \Gamma \) with the boundary value of the function \( C_0 f \) (respectively, \( C^- f \)) defined in \( D^+ \) (respectively, \( D^- \)), we will think of functions from \( X_+ \) (respectively, \( X^0_0 \)) as of functions defined in \( D^+ \) (respectively, in \( D^- \)) by \( f(z) := (C_0 f)(z) \) (respectively, by \( f(z) := (C^- f)(z) \)).
Lemma 2.6 ([13] Lemma 6.9). Let \( \Gamma \) be a rectifiable Jordan curve and \( X \) be a reflexive Banach function space in which the Cauchy singular integral operator \( S \) is bounded.

(a) If \( f \in X_\pm \) and \( g \in X'_\pm \), then \( fg \in L^1_\pm \). If, in addition, \( f \in X^0_\pm \) or \( g \in (X')^0_\pm \), then \( fg \in (L^1)^0_\pm \).

(b) We have
\[
X_+ = L^1_+ \cap X, \quad X^0_0 = (L^1)^0_0 \cap X, \quad X_- = L^1_- \cap X.
\]

2.4. Adjoint operators of the projections \( P \) and \( Q \). On a rectifiable Jordan oriented curve \( \Gamma \), we have
\[
d\tau = e^{i\theta_\Gamma(\tau)}|d\tau|,
\]
where \( \theta_\Gamma(\tau) \) is the angle made by the positively oriented real axis and the naturally oriented tangent of \( \Gamma \) at \( \tau \) (which exists almost everywhere). Let \( X \) be a Banach function space over \( \Gamma \). Define the operator \( H_\Gamma : X \to X \) by
\[
(H_\Gamma f)(\tau) := e^{-i\theta_\Gamma(\tau)}\overline{f(\tau)}.
\]
Note that the operator \( H_\Gamma \) is additive but
\[
H_\Gamma(\alpha f) = \overline{\alpha} \cdot H_\Gamma f \quad \text{for} \quad \alpha \in \mathbb{C} \quad \text{and} \quad f \in X.
\]
It is clear that \( H_\Gamma \) is bounded on \( X \) and \( H^2_\Gamma = I \).

Lemma 2.7 ([13] Lemma 6.6). Let \( \Gamma \) be a rectifiable Jordan curve and \( X \) be a reflexive Banach function space in which the Cauchy singular integral operator \( S \) is bounded. Then the adjoint of \( S \in \mathcal{L}(X) \) is the operator \( S^* = -H_\Gamma SH_\Gamma \in \mathcal{L}(X') \) and consequently,
\[
P^* = H_\Gamma QH_\Gamma, \quad Q^* = H_\Gamma PH_\Gamma.
\]

3. Proof of the main results

3.1. Companion operator of a Toeplitz operator. Let \( X \) and \( Y \) be reflexive Banach function spaces over a rectifiable Jordan curve \( \Gamma \). Suppose \( a \in M(X,Y) \equiv M(Y',X') \) and the operator \( S \) is bounded on \( X \) and on \( Y \). In view of Lemma 2.4 the operator \( S \) is also bounded on \( Y' \) and on \( X' \). Then, along with the Toeplitz operator
\[
T(a) : X_+ \to Y_+,
\]
we consider its companion operator \( \tilde{T}(a) : (Y')^0_- \to (X')^0_- \) defined by
\[
\tilde{T}(a)f = Q(af), \quad f \in (Y')^0_-
\]
It is obvious that \( \tilde{T}(a) \in \mathcal{L}((Y')^0_-, (X')^0_-) \) and
\[
\|\tilde{T}(a)\|_{\mathcal{L}((Y')^0_-, (X')^0_-)} \leq \|Q\|_{\mathcal{L}(X')}\|a\|_{M(X,Y)}.
\]

Lemma 3.1. Let \( X \) and \( Y \) be reflexive Banach function spaces over a rectifiable Jordan curve. Suppose \( X \to Y \) and the Cauchy singular integral operator \( S \) given by \([11]\) is bounded on \( X \) and on \( Y \). If \( a \in M(X,Y) \), then the Toeplitz operator \( T(a) : X_+ \to Y_+ \) has a trivial kernel in \( X_+ \) (resp., a dense image in \( Y_+ \)) if and only if its companion operator \( \tilde{T}(a) : (Y')^0_- \to (X')^0_- \) has a dense image in \( (X')^0_- \) (resp., a trivial kernel in \( (Y')^0_- \)).

Proof. Let \( \text{Im} A \) and \( \ker A \) denote the image and the kernel, respectively, of a bounded linear operator \( A \) acting between Banach spaces.
Since \( X \hookrightarrow Y \), we have \( Q \in \mathcal{L}(X,Y) \) and \( PaP + Q \in \mathcal{L}(X,Y) \). The spaces \( X \) and \( Y \) decompose into the direct sums \( X = X_+ \oplus X^0 \) and \( Y = Y_+ \oplus Y^0 \). Accordingly, the operator \( PaP + Q \) may be written as an operator matrix

\[
\begin{pmatrix}
T(a) & 0 \\
0 & I
\end{pmatrix} : \begin{pmatrix} X_+ \\ X^0 \end{pmatrix} \to \begin{pmatrix} Y_+ \\ Y^0 \end{pmatrix}.
\]

Hence

\[
\text{Im}(PaP + Q) = \text{Im} T(a) \oplus Y^0, \quad \text{Ker}(PaP + Q) = \text{Ker} T(a). \tag{2}
\]

On the other hand, \( Y' \hookrightarrow X' \) by Lemma 2.3 and \( a \in M(Y', X') \) by Lemma 2.4. Then \( P \in \mathcal{L}(Y', X') \) and \( P + QAQ \in \mathcal{L}(Y', X') \). Since the spaces \( Y' \) and \( X' \) decompose into the direct sums \( Y' = (Y')_+ \oplus (Y')^0 \) and \( X' = (X')_+ \oplus (X')^0 \), the operator \( P + QAQ \) may be written as an operator matrix

\[
\begin{pmatrix} I & 0 \\
0 & \tilde{T}(a) \end{pmatrix} : \begin{pmatrix} (Y')_+ \\ (Y')^0 \end{pmatrix} \to \begin{pmatrix} (X')_+ \\ (X')^0 \end{pmatrix}.
\]

Therefore

\[
\text{Im}(P + QAQ) = (X')_+ \oplus \text{Im} \tilde{T}(a), \quad \text{Ker}(P + QAQ) = \text{Ker} \tilde{T}(a). \tag{3}
\]

Lemmas 2.3 and 2.4 yield

\[
(PaP + Q)^* = P^* \pi P^* + Q^* = (H_T Q H_T)(H_T a H_T)(H_T Q H_T) + H_T P H_T = H_T (P + QAQ) H_T. \tag{4}
\]

From the second identity in (2) it follows that \( T(a) \in \mathcal{L}(X_+, Y_+) \) has a trivial kernel in \( X_+ \) if and only if \( PaP + Q \in \mathcal{L}(X, Y) \) has a trivial kernel in \( X \). On the other hand, from (1) and \( H_2^T = I \) we deduce that the latter fact is equivalent to the fact that \( P + QAQ \in \mathcal{L}(Y', X') \) has a dense image in \( X' \) (see, e.g., [25, Section 4.12]). In turn, in view of the first identity in (3), the operator \( P + QAQ \) has a dense image in \( X' \) if and only if the operator \( \tilde{T}(a) \in \mathcal{L}((Y')^0_-, (X')^0_+) \) has a dense image in \( (X')^0_+ \).

The proof of the equivalence of the density of the image of \( T(a) \) in \( Y_+ \) and the triviality of the kernel of \( \tilde{T}(a) \) in \( (Y')^0_+ \) is analogous. \( \square \)

3.2. Proof of Theorem 1.1. In view of Lemma 3.1 it is sufficient to show that \( T(a) : X_+ \to Y_+ \) is injective on \( X_+ \) or \( \tilde{T}(a) : (Y^0)_- \to (X^0)_+ \) is injective on \( (Y^0)_+ \).

Assume the contrary, that is, that there exist \( f_+ \in X_+ \) and \( g_- \in (Y^0)_+ \) such that \( f_+ \not= 0, g_- \not= 0 \), and

\[
Pa f_+ = 0, \quad Qa g_- = 0. \tag{5}
\]

By Lemma 2.6(b), \( f_+ \in X_+ \subset L^1_+ \) and \( g_- \in (Y^0)^- \subset L^1_- \). Since \( f_+ \not= 0 \) and \( g_- \not= 0 \), from the Lusin-Privalov Theorem 2.2 it follows that \( f_+ \not= 0 \) a.e. on \( \Gamma \) and \( g_- \not= 0 \) a.e. on \( \Gamma \).

Put \( f_- := a f_+ \) and \( g_+ := a g_- \). Then from [15] it follows that \( P a f_+ = P f_- = 0 \) and \( Q a g_- = Q g_+ = 0 \). Therefore,

\[
f_- = a f_+ = P a f_+ + Q a f_+ = Q a f_+ \in Y^0_+.
g_+ = a g_- = P a g_- + Q a g_- = P a g_- \in (X^0)_+.
\]

Then

\[
f_+ g_+ = f_+(a g_-) = (f_+ a) g_- = f_- g_- \tag{6}
\]
From Lemma 2.6(a) we deduce that \( f_+ g_+ \in L^1_+ \) and \( f_- g_- \in (L^1)^0 \). Lemma 2.7 and identity (3) imply that \( f_+ g_+ = f_- g_- = f_+ g_- = 0 \). Since \( f_+ \neq 0 \) a.e. on \( \Gamma \) and \( g_- \neq 0 \) a.e. on \( \Gamma \), we conclude that \( a = 0 \) a.e. on \( \Gamma \), but this contradicts our hypothesis and, thus, completes the proof. \( \Box \)

4. Toeplitz operators between Hardy type subspaces of variable Lebesgue spaces

4.1. Variable Lebesgue spaces. Given a rectifiable Jordan curve \( \Gamma \), let \( P(\Gamma) \) be the set of all measurable functions \( p : \Gamma \to [1, \infty] \). For \( p \in P(\Gamma) \) and a measurable subset \( \gamma \subset \Gamma \), put

\[
\gamma_p^{(\cdot)} := \{ t \in \gamma : p(t) = \infty \}.
\]

For a measurable function \( f : \gamma \to \mathbb{C} \), consider

\[
\varrho_{p(; \gamma)}(f) := \int_{\gamma_p^{(\cdot)}} |f(t)|^p dt + \|f\|_{L_\infty(\gamma_p^{(\cdot)})}.
\]

According to [5, Definition 2.9], the variable Lebesgue space \( L^p(\cdot ; \gamma) \) is defined as the set of all measurable functions \( f : \gamma \to \mathbb{C} \) such that \( \varrho_{p(; \gamma)}(f/\lambda) < \infty \) for some \( \lambda > 0 \). This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

\[
\|f\|_{L^p(\cdot ; \gamma)} := \inf\{ \lambda > 0 : \varrho_{p(; \gamma)}(f/\lambda) \leq 1 \}
\]

(see, e.g., [5, Theorems 2.17, 2.71 and Section 2.10.3]). If \( p \in P(\Gamma) \) is constant, then \( L^p(\cdot ; \gamma) \) is nothing but the standard Lebesgue space \( L^p(\gamma) \). Variable Lebesgue spaces are often called Nakano spaces. We refer to Maligranda’s paper [19] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces.

The following property of the unit ball of variable Lebesgue spaces is well known (see, e.g., [5, Corollary 2.22]).

Lemma 4.1. Let \( \gamma \) be a measurable subset of a rectifiable Jordan curve \( \Gamma \). If \( p \in P(\Gamma) \) and \( f \) is a measurable function on \( \gamma \), then the inequalities \( \varrho_{p(; \gamma)}(f) \leq 1 \) and \( \|f\|_{L^p(\cdot ; \gamma)} \leq 1 \) are equivalent.

For the brevity, we will simply write \( L^p(\cdot) \) for \( L^p(\cdot ; \gamma) \). For \( p \in P(\Gamma) \), put

\[
p_- := \text{ess inf}_{t \in \Gamma} p(t), \quad p_+ := \text{ess sup}_{t \in \Gamma} p(t).
\]

Lemma 4.2 ([5, Corollary 2.81]). Let \( \Gamma \) be a rectifiable Jordan curve and \( p \in P(\Gamma) \). Then \( L^p(\cdot) \) is reflexive if and only if \( 1 < p_- \leq p_+ < \infty \).

Embeddings of variable Lebesgue spaces are characterized as follows.

Lemma 4.3 ([5, Corollary 2.48]). Let \( \Gamma \) be a rectifiable Jordan curve. Suppose \( p, q \in P(\Gamma) \). Then \( L^p(\cdot) \hookrightarrow L^q(\cdot) \) if and only if \( q(t) \leq p(t) \) for almost all \( t \in \Gamma \).

4.2. Pointwise multipliers between variable Lebesgue spaces. In this subsection we will describe the space of pointwise multipliers between variable Lebesgue spaces. The next lemma follows from [21] Section 2, Property (f) and Theorem 1 and the fact that variable Lebesgue spaces are Banach function spaces [5, Section 2.10.3].
Lemma 4.4. Let $\gamma$ be a measurable subset of a rectifiable Jordan curve $\Gamma$ and $p \in \mathcal{P}(\Gamma)$. Then

$$M(L^\infty(\gamma), L^{p(\cdot)}(\gamma)) \equiv L^{p(\cdot)}(\gamma), \quad M(L^{p(\cdot)}(\gamma), L^{q(\cdot)}(\gamma)) \equiv L^\infty(\gamma).$$

Now we state the following two simple statements.

Lemma 4.5. Let $\Gamma$ be a rectifiable Jordan curve and $\gamma_1, \ldots, \gamma_k$ be measurable subsets of $\Gamma$ such that

$$\gamma_i \cap \gamma_j = \emptyset \quad \text{for} \quad i, j \in \{1, \ldots, k\}, \quad \gamma_1 \cup \cdots \cup \gamma_k = \Gamma.$$ (7)

If $p \in \mathcal{P}(\Gamma)$, then

$$L^{p(\cdot)} = L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k),$$

where the norm in the direct sum $L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k)$ is defined by

$$\|f\|_{L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k)} = \|f\chi_{\gamma_1}\|_{L^{p(\cdot)}(\gamma_1)} + \cdots + \|f\chi_{\gamma_k}\|_{L^{p(\cdot)}(\gamma_k)}.$$ (8)

Lemma 4.6. Let $\Gamma$ be a rectifiable Jordan curve and $\gamma_1, \ldots, \gamma_k$ be measurable subsets of $\Gamma$ satisfying (7). If $p, q \in \mathcal{P}(\Gamma)$ and $q(t) \leq p(t)$ for almost all $t \in \Gamma$, then

$$M(L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k), L^{q(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{q(\cdot)}(\gamma_k)) \equiv M(L^{p(\cdot)}(\gamma_1), L^{q(\cdot)}(\gamma_1)) \oplus \cdots \oplus M(L^{p(\cdot)}(\gamma_k), L^{q(\cdot)}(\gamma_k)).$$

The proofs of the above two lemmas are straightforward and they are omitted. We will need the following generalized Hölder inequality.

Lemma 4.7 ([5 Corollary 2.28]). Let $\Gamma$ be a rectifiable Jordan curve. Suppose $p, q, r \in \mathcal{P}(\Gamma)$ are related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, \quad t \in \Gamma.$$ (9)

Then there exists a constant $C > 0$ such that for all $f \in L^{p(\cdot)}$ and $g \in L^{r(\cdot)}$, one has $fg \in L^{q(\cdot)}$ and

$$\|fg\|_{L^{q(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{r(\cdot)}}.$$
Similarly, from (8) we also obtain $\Gamma_{p'(\ell)}^{(1)} \subset \Gamma_{p'\ell}^{(p)} \cap \Gamma_{\infty}^{(p)}$, whence $\gamma_2 = \Gamma_{p'(\ell)}^{(1)} \setminus \Gamma_{\infty}^{(p)}$. Therefore, $p(t) = q(t) < \infty$ and $r(t) = \infty$ for $t \in \gamma_2$. Then, from Lemma 4.7 we get

$$M(L^{p'(\ell)}(\gamma_2), L^{p(\ell)}(\gamma_2)) \equiv M(L^{p'(\ell)}(\gamma_2), L^{p(\ell)}(\gamma_2)) \equiv L^{\infty}(\gamma_2) \equiv L^{r(\ell)}(\gamma_2). \quad (10)$$

The rest of the proof is developed by analogy with the proof of [29, Theorem 4]. Let $f \in M(L^{p'(\ell)}(\gamma_3), L^{p(\ell)}(\gamma_3))$. The multiplication operator $Tg = fg$ maps $L^{p'(\ell)}(\gamma_3)$ into $L^{q(\ell)}(\gamma_3)$ and has a closed graph. Hence there exists a constant $c \in (0, \infty)$ such that

$$\|fg\|_{L^{q(\ell)}(\gamma_3)} \leq c\|g\|_{L^{p'(\ell)}(\gamma_3)} \quad \text{for all } \ g \in L^{p'(\ell)}(\gamma_3). \quad (11)$$

For $\varepsilon > 0$, put

$$f_\varepsilon(t) = \begin{cases} \frac{c + \varepsilon}{f(t)} \left( \frac{|f(t)|}{c + \varepsilon} \right)^{r(t)/q(t)} & \text{if } f(t) \neq 0, \\ 0, & \text{if } f(t) = 0. \end{cases} \quad (12)$$

Let us show that

$$\varphi_{p(\cdot), \gamma_3}(f_\varepsilon) \leq 1. \quad (13)$$

Assume the contrary, that is, $\varphi_{p(\cdot), \gamma_3}(f_\varepsilon) > 1$. Then from [8, Propositions A.1 and A.8] it follows that there exists a measurable set $\gamma \subset \gamma_3$ such that

$$\varphi_{p(\cdot), \gamma_3}(\chi_{\gamma} f_\varepsilon) = 1. \quad (14)$$

From (8) and (12) we get

$$|f_\varepsilon(t)| = \left( \frac{|f(t)|}{c + \varepsilon} \right)^{r(t)/q(t) - 1} = \left( \frac{|f(t)|}{c + \varepsilon} \right)^{r(t)/p(t)}, \quad t \in \gamma. \quad (15)$$

Equality (14) and Lemma 4.1 imply that $\|\chi_{\gamma} f_\varepsilon\|_{L^{p'(\ell)}(\gamma_3)} \leq 1$. Applying (11) with $g = \chi_{\gamma} f_\varepsilon$, we obtain

$$\left\| \frac{\chi_{\gamma} f_\varepsilon f}{c} \right\|_{L^{p'(\ell)}(\gamma_3)} \leq \|\chi_{\gamma} f\|_{L^{p'(\ell)}(\gamma_3)} \leq 1.$$  

Then, in view of Lemma 4.1 we get

$$\varphi_{q(\cdot), \gamma_3} \left( \frac{\chi_{\gamma} f_\varepsilon f}{c} \right) \leq 1. \quad (16)$$

Combining (14), (12), (8), and (16), we arrive at

$$1 = \varphi_{p(\cdot), \gamma_3}(\chi_{\gamma} f_\varepsilon) = \varphi_{r(\cdot), \gamma_3} \left( \frac{\chi_{\gamma} f_\varepsilon}{c + \varepsilon} \right) = \varphi_{q(\cdot), \gamma_3} \left( \frac{\chi_{\gamma} f_\varepsilon f}{c + \varepsilon} \right) \leq \frac{c}{c + \varepsilon} \varphi_{q(\cdot), \gamma_3} \left( \frac{\chi_{\gamma} f_\varepsilon f}{c} \right) \leq \frac{c}{c + \varepsilon} < 1,$$

and we get a contradiction. Hence (13) is fulfilled. Applying Lemma 4.1 to (13), we deduce that $\|f_\varepsilon\|_{L^{p'(\ell)}(\gamma_3)} \leq 1$. Then, in view of (11), we obtain

$$\|f_\varepsilon f\|_{L^{q(\ell)}(\gamma_3)} \leq \|f_\varepsilon\|_{L^{p'(\ell)}(\gamma_3)} \leq c.$$ 

Taking into account the above inequality, equality (12) and Lemma 4.1 we see that

$$\varphi_{r(\cdot), \gamma_3} \left( \frac{f_\varepsilon}{c + \varepsilon} \right) = \varphi_{q(\cdot), \gamma_3} \left( \frac{f_\varepsilon f}{c + \varepsilon} \right) \leq \varphi_{q(\cdot), \gamma_3} \left( \frac{f_\varepsilon f}{c} \right) \leq 1,$$
The following class of nice variable exponent.

Let \( \Gamma \) be a rectifiable Jordan curve. We say that an exponent \( p \) is locally \( \log\)-Hölder continuous (cf. [5, Definition 2.2]) if \( 1 < p < \infty \) and there exists a constant \( C_{p,\Gamma} \in (0, \infty) \) such that

\[
|p(t) - p(\tau)| \leq \frac{C_{p,\Gamma}}{-\log |t - \tau|} \quad \text{for all} \quad t, \tau \in \Gamma \quad \text{satisfying} \quad |t - \tau| < 1/2.
\]

The class of all locally \( \log\)-Hölder continuous exponent will be denoted by \( LH(\Gamma) \). Notice that some authors also denote this class by \( \mathcal{P}^{\log}(\Gamma) \), see, e.g., [15] Section 1.1.4.

**Theorem 4.9** ([15] Theorems 2.45 and 2.49). Let \( \Gamma \) be a rectifiable Jordan curve and \( p \in LH(\Gamma) \). Then the Cauchy singular integral operator \( S \) is bounded on \( L^p(\Gamma) \) if and only if \( \Gamma \) is a Carleson curve.

### 4.4. The Coburn-Simonenko theorem for Toeplitz operators acting between Hardy type subspaces of variable Lebesgue spaces

Now we are in a position to give a more precise formulation of Theorem [15] in the case of Toeplitz operators acting between Hardy type subspaces \( PL^p(\Gamma) \) and \( PL^q(\Gamma) \) of variable Lebesgue spaces \( L^p(\Gamma) \) and \( L^q(\Gamma) \), respectively.
Theorem 4.10. Let $\Gamma$ be a Carleson Jordan curve. Suppose variable exponents $p, q \in LH(\Gamma)$ and $r \in \mathcal{P}(\Gamma)$ are related by (8). If $a \in L^r(\Gamma) \setminus \{0\}$, then the Toeplitz operator $T(a) \in \mathcal{L}(PL^p(\Gamma), PL^q(\Gamma))$ has a trivial kernel in $PL^p(\Gamma)$ or a dense image in $PL^q(\Gamma)$.

Proof. We know from Lemma 4.2 that the spaces $L^p(\cdot)$ and $L^q(\cdot)$ are reflexive because $1 < p_-, q_- < 1$ and $p_+, q_+ < \infty$ (in view of $p, q \in LH(\Gamma)$). Since $r \in \mathcal{P}(\Gamma)$, we have $1 \leq r(t) \leq \infty$ for almost all $t \in \Gamma$. Then we deduce from (8) that $q(t) \leq p(t)$ for almost all $t \in \Gamma$. Therefore, by Lemma 4.3, $L^p(\cdot) \hookrightarrow L^q(\cdot)$. It follows from Theorem 4.9 that the Cauchy singular integral operator $S$ is bounded on $L^p(\cdot)$ and $L^q(\cdot)$. Now we observe that $L^r(\cdot) = M(L^p(\cdot), L^q(\cdot))$ in view of Theorem 4.8. It remains to apply Theorem 4.1.

Corollary 1.2 follows immediately from Theorem 4.10 if we take all exponents $p, q$, and $r$ constant.

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