Smoothness of the Gap Function in the BCS-Bogoliubov Theory of Superconductivity

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Abstract

We deal with the gap equation in the BCS-Bogoliubov theory of superconductivity, where the gap function is a function of the temperature $T$ only. We show that the squared gap function is of class $C^2$ on the closed interval $[0, T_c]$. Here, $T_c$ stands for the transition temperature. Furthermore, we show that the gap function is monotonically decreasing on $[0, T_c]$ and obtain the behavior of the gap function at $T = T_c$. We mathematically point out some more properties of the gap function.

I. INTRODUCTION

Since the surprising discovery by Onnes that the electrical resistivity of mercury drops to zero below the temperature 4.2 K in 1911, the zero electrical resistivity is observed in many metals and alloys. Such a phenomenon is called superconductivity. In 1957 Bardeen, Cooper and Schrieffer \[1\] proposed the highly successful quantum theory of superconductivity, called the BCS theory. In 1958 Bogoliubov \[3\] obtained the results similar to those in the BCS theory using the canonical transformation called the Bogoliubov transformation. The theory by Bardeen, Cooper, Schrieffer and Bogoliubov is called the BCS-Bogoliubov theory.

As an experimental fact, it is observed that it takes a finite energy to excite a quasi particle from the superconducting ground state to an upper energy state. This energy gap is described in terms of the gap function and results from the existence of the electron pairs called the Cooper pairs. Let $k_B > 0$ and $\omega_D > 0$ stand for the Boltzmann constant and the Debye frequency, respectively. We denote Planck’s constant by $\hbar$ ($> 0$) and set $\hbar = h/(2\pi)$. Let the temperature $T$ satisfy $0 \leq T \leq T_c$, where $T_c > 0$ is called the transition temperature (the critical temperature). Let $m > 0$ and $\mu > 0$ stand for the electron mass and the chemical potential, respectively. Let $k \in \mathbb{R}^3$ denote wave vector and set $\xi_k = \hbar^2 |k|^2/(2m) - \mu$. The gap function, denoted by $\Delta_k(T) \geq 0$, is a function
both of the temperature $T$ and of wave vector $k \in \mathbb{R}^3$. In the BCS-Bogoliubov theory, the gap function satisfies the following nonlinear equation called the gap equation:

$$\Delta_k(T) = -\frac{1}{2} \sum_{k'} \frac{U_{k,k'} \Delta_{k'}(T)}{\sqrt{\xi_{k'}^2 + \Delta_{k'}(T)^2}} \tanh \frac{\sqrt{\xi_{k'}^2 + \Delta_{k'}(T)^2}}{2k_B T}$$

for $0 \leq T \leq T_c$. Here, $k' \in \mathbb{R}^3$ denotes wave vector and the potential $U_{k,k'}$ is a function of $k$ and $k'$ satisfying $U_{k,k'} \leq 0$. In this connection, see [8] for a new gap equation of superconductivity.

The sum in (1) is often replaced by an integral, and accordingly the gap equation is often regarded as a nonlinear integral equation. In such a situation, Odeh [6] and Billard and Fano [2] established the existence and uniqueness of the positive solution to the gap equation in the case $T = 0$. In the case $T \geq 0$, Vansevenant [7] and Yang [9] determined the transition temperature and showed that there is a unique positive solution to the gap equation. Recently Hainzl, Hamza, Seiringer and Solovej [4], and Hainzl and Seiringer [5] proved that the existence of a positive solution to the gap equation is equivalent to the existence of a negative eigenvalue of a certain linear operator to show the existence of a transition temperature.

Suppose that $U_{k,k'}$ is given by (see [1])

$$U_{k,k'} = \begin{cases} -U_0 & (|\xi_k| \leq \hbar \omega_D \text{ and } |\xi_{k'}| \leq \hbar \omega_D), \\ 0 & \text{(otherwise)}, \end{cases}$$

where $U_0 > 0$ is a constant. Then $\Delta_k(T)$ depends only on the temperature $T$ when $|\xi_k| \leq \hbar \omega_D$, whereas $\Delta_k(T) = 0$ when $|\xi_k| > \hbar \omega_D$. Let $|\xi_k| \leq \hbar \omega_D$. Then (1) leads to

$$1 = \frac{U_0}{2} \sum_{k' \text{ (} |\xi_{k'}| \leq \hbar \omega_D\text{)}} \frac{1}{\sqrt{\xi_{k'}^2 + \Delta(T)^2}} \tanh \frac{\sqrt{\xi_{k'}^2 + \Delta(T)^2}}{2k_B T}.$$

Here the symbol $k'$ ($|\xi_{k'}| \leq \hbar \omega_D$) stands for $k'$ satisfying $|\xi_{k'}| \leq \hbar \omega_D$, and the gap function $\Delta_k(T)$ is denoted by $\Delta(T)$ simply because it does not depend on $k$ when $k$ satisfies $|\xi_k| \leq \hbar \omega_D$. Accordingly, in this case, the gap function $\Delta(T)$ becomes a function of the temperature $T$ only.

We now replace the sum in (3) by the following integral (see [1]):

$$1 = \frac{U_0N_0}{2} \int_{-\hbar \omega_D}^{\hbar \omega_D} \frac{1}{\sqrt{\xi^2 + \Delta(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta(T)^2}}{2k_B T} d\xi,$$

where $0 \leq T \leq T_c$, and $N_0 > 0$ stands for the density of states per unit energy at the Fermi surface.

The gap equation of the form (1) as well as the hypothesis (2) is accepted widely in condensed matter physics (see e.g. [1] and [10], (11.45), p.392). In this paper we deal with the gap equation (4) to discuss smoothness of the squared gap function $\Delta(T)^2$ as well as its properties. We show that the squared gap function is of class $C^2$ on the closed interval $[0, T_c]$. Furthermore, we show that the gap function is monotonically decreasing.
on $[0, T_c]$ and obtain the behavior of the gap function at $T = T_c$. We mathematically point out some more properties of the gap function.

It is well known that superconductivity occurs at temperatures below the temperature $T_c > 0$ called the transition temperature. Let us now define it.

**Definition 1** ([1]). The transition temperature is the temperature $T_c > 0$ satisfying

$$\frac{1}{U_0 N_0} = \int_0^{\hbar \omega_D/(2 k_B T_c)} \frac{\tanh \eta}{\eta} d\eta.$$  

**Remark 2.** The equality in the definition above is rewritten as

$$1 = \frac{U_0 N_0}{2} \int_{-\hbar \omega_D}^{\hbar \omega_D} \frac{1}{\sqrt{\xi^2}} \tanh \frac{\sqrt{\xi^2}}{2 k_B T_c} d\xi,$$

which is obtained by setting $\Delta(T) = 0$ and $T = T_c$ in (4).

The paper proceeds as follows. In section 2 we state our main results without proof. In sections 3 and 4 we study some properties of the function $F$ defined by (5) below. In section 5, on the basis of this study, we prove our main results in a sequence of lemmas.

**II. MAIN RESULTS**

Let

$$h(T, Y, \xi) = \begin{cases} \frac{1}{\sqrt{\xi^2 + Y}} \tanh \frac{\sqrt{\xi^2 + Y}}{2 k_B T} & (0 < T \leq T_c, \ Y \geq 0), \\ \frac{1}{\sqrt{\xi^2 + Y}} & (T = 0, \ Y > 0) \end{cases}$$

and set

$$F(T, Y) = \int_0^{\hbar \omega_D} h(T, Y, \xi) d\xi - \frac{1}{U_0 N_0}. \quad (5)$$

We consider the function $F$ on the following domain $W \subset \mathbb{R}^2$:

$$W = W_1 \cup W_2 \cup W_3 \cup W_4,$$

where

$$W_1 = \{(T, Y) \in \mathbb{R}^2 : 0 < T < T_c, \ 0 < Y < 2 \Delta_0^2\},$$
$$W_2 = \{(0, Y) \in \mathbb{R}^2 : 0 < Y < 2 \Delta_0^2\},$$
$$W_3 = \{(T, 0) \in \mathbb{R}^2 : 0 < T \leq T_c\},$$
$$W_4 = \{(T_c, Y) \in \mathbb{R}^2 : 0 < Y < 2 \Delta_0^2\}.$$

Here,

$$\Delta_0 = \frac{\hbar \omega_D}{\sinh \frac{1}{U_0 N_0}}. \quad (6)$$
Remark 3. The gap equation (4) is rewritten as $F(T, Y) = 0$, where $Y$ corresponds to $\Delta(T)^2$.

The following are our main results.

**Theorem 4.** Let $F$ be as in (5) and $\Delta_0$ as in (6). Then there is a unique solution: $T \mapsto Y = f(T)$ of class $C^2$ on the closed interval $[0, T_c]$ to the gap equation $F(T, Y) = 0$ such that the function $f$ is monotonically decreasing on $[0, T_c]$:

$$f(0) = \Delta_0^2 > f(T_1) > f(T_2) > f(T_c) = 0, \quad 0 < T_1 < T_2 < T_c.$$

Let $g$ be given by

$$g(\eta) = \begin{cases} \frac{1}{\eta^2} \left( \frac{1}{\cosh^2 \eta} - \frac{\tanh \eta}{\eta} \right) & (\eta > 0), \\ -\frac{2}{3} & (\eta = 0). \end{cases}$$

(7)

Note that $g(\eta) < 0$, as is pointed out by Lemma 9 below. Let $G$ be given by

$$G(\eta) = \begin{cases} \frac{1}{\eta^2} \left\{ 3g(\eta) + 2 \frac{\tanh \eta}{\eta \cosh^2 \eta} \right\} & (\eta > 0), \\ -\frac{16}{15} & (\eta = 0). \end{cases}$$

(8)

See Lemma 16 below for some properties of $G$.

**Proposition 5.** Let $f$ be as in Theorem 4. Then the values of the derivative $f'$ at $T = 0$ and at $T = T_c$ are given as follows:

$$f'(0) = 0, \quad f'(T_c) = 8k_B^2T_c \int_0^{\hbar\omega_D/(2k_BT_c)} \frac{d\eta}{\cosh^2 \eta} \frac{d\eta}{g(\eta)} < 0.$$

Consequently, the behavior of $f$ at $T = T_c$ is given by

$$f(T) \approx -f'(T_c)(T_c - T) = -8k_B^2 \int_0^{\hbar\omega_D/(2k_BT_c)} \frac{d\eta}{\cosh^2 \eta} \frac{d\eta}{g(\eta)} T_c(T_c - T).$$

(9)

Remark 6. The behavior of $f$ similar to (9) was already obtained by a different method in the context of theoretical, condensed matter physics. However, Proposition 5 gives the new form (9) explicitly in the context of mathematics.

Let $\phi$ be a function of $\eta$ and let $\eta$ be a function of $\xi$. Set

$$I[\phi(\eta)] = \int_0^{\hbar\omega_D} \phi(\eta) d\xi.$$
Proposition 7. Let $f$ be as in Theorem 4 and $I[\cdot]$ as in (10). Then the values of the second derivative $f''$ at $T = 0$ and at $T = T_c$ are given as follows:

\[ f''(0) = 0, \]

\[
\begin{align*}
f''(T_c) &= 16 k_B^2 \frac{I \left[ \frac{\eta_0 \tanh \eta_0 - 1}{\cosh \eta_0} \right]}{I \left[ g(\eta_0) \right]} - 32 k_B^2 \frac{I \left[ \frac{\tanh \eta_0}{\eta_0 \cosh^2 \eta_0} \right]}{\{ I \left[ g(\eta_0) \right] \}^2} \\
&
+ 8 k_B^2 \frac{\left\{ I \left[ \frac{1}{\cosh \eta_0} \right] \right\}^2 I \left[ G(\eta_0) \right]}{\{ I \left[ g(\eta_0) \right] \}^2}, \quad \eta_0 = \frac{\xi}{2k_BT_c}. \end{align*}
\]

Combining Theorem 4 with Propositions 5 and 7 immediately implies the following.

Corollary 8. There is a unique gap function: $T \mapsto \Delta(T) = \sqrt{f(T)}$ on the closed interval $[0, T_c]$ such that it is of class $C^2$ on the interval $[0, T_c)$, and is monotonically decreasing on $[0, T_c)$:

\[ \Delta(0) = \Delta_0 > \Delta(T_1) > \Delta(T_2) > \Delta(T_c) = 0, \quad 0 < T_1 < T_2 < T_c. \]

Furthermore, $\Delta'(0) = \Delta''(0) = 0$ and $\lim_{T \uparrow T_c} \Delta'(T) = -\infty$.

III. THE FIRST-ORDER PARTIAL DERIVATIVES OF $F$

In this section we deal with the first-order partial derivatives of the function $F$ and show that $F$ is of class $C^1$ on $W$.

A straightforward calculation yields the following.

Lemma 9. Let $g$ be as in (7). Then the function $g$ is of class $C^1$ on $[0, \infty)$ and satisfies

\[ g(\eta) < 0, \quad g'(0) = 0, \quad \lim_{\eta \to \infty} g(\eta) = \lim_{\eta \to \infty} g'(\eta) = 0. \]

Lemma 10. The values of the partial derivatives $\frac{\partial F}{\partial T}$ and $\frac{\partial F}{\partial Y}$ exist at each point in $W_1$.

Proof. Let $(T, Y) \in W_1$. Then there is a $\theta$ ($0 < \theta < 1$) satisfying $\theta T_c < T < T_c$. Therefore,

\[ \left| \frac{\partial h}{\partial T} (T, Y, \xi) \right| \leq \frac{1}{2k_B \theta^2 T_c^2}, \]

where the right side is integrable on $[0, \hbar \omega_D]$. Hence the value of $\frac{\partial F}{\partial T}$ exists at each point in $W_1$. Here,

\[ \frac{\partial F}{\partial T}(T, Y) = -\frac{1}{2k_B T^2} \int_0^{\hbar \omega_D} \frac{d\xi}{\cosh^2 \eta}, \quad \eta = \frac{\sqrt{\xi^2 + Y}}{2k_B T}. \] (11)
On the other hand,
\[
\frac{\partial h}{\partial Y} (T, Y, \xi) = \frac{g(\eta)}{2(2k_BT)^3}, \quad \eta = \frac{\sqrt{\xi^2 + Y}}{2k_BT}.
\]
Hence, by Lemma 9,
\[
\left| \frac{\partial h}{\partial Y} (T, Y, \xi) \right| \leq \max_{\eta \geq 0} |g(\eta)| \frac{2}{(2k_BT)^3},
\]
where the right side is also integrable on \([0, \hbar \omega_D]\). Hence the value of \(\frac{\partial F}{\partial Y}\) exists at each point in \(W_1\). Here,
\[
\frac{\partial F}{\partial Y} (T, Y) = \frac{1}{2(2k_BT)^3} \int_0^{\hbar \omega_D} g(\eta) \, d\xi, \quad \eta = \frac{\sqrt{\xi^2 + Y}}{2k_BT}.
\]
(12)

\[\boxed{\text{Lemma 11. The values of the partial derivatives } \frac{\partial F}{\partial T} \text{ and } \frac{\partial F}{\partial Y} \text{ exist at each point in } W.}\]

\[\text{Proof. We show that the values of } (\partial F/\partial T) \text{ and } (\partial F/\partial Y) \text{ exist at each point in } W_2. \text{ Let } (0, Y_0) \in W_2. \text{ Then}
\]
\[
\left| \frac{F(T, Y_0) - F(0, Y_0)}{T} \right| \leq \int_0^{\hbar \omega_D} \frac{1}{T \sqrt{\xi^2 + Y_0}} \left( 1 - \tanh \frac{\sqrt{\xi^2 + Y_0}}{2k_BT} \right) \, d\xi
\]
\[
\leq \frac{4k^2_BT}{Y_0} \int_0^{\hbar \omega_D} \frac{d\xi}{\sqrt{\xi^2 + Y_0}},
\]
and hence
\[
\frac{\partial F}{\partial T}(0, Y_0) = 0.
\]
On the other hand, for \(Y > Y_0/2\),
\[
\frac{F(0, Y) - F(0, Y_0)}{Y - Y_0} = -\int_0^{\hbar \omega_D} \frac{d\xi}{\sqrt{\xi^2 + Y} \sqrt{\xi^2 + Y_0} \left( \sqrt{\xi^2 + Y} + \sqrt{\xi^2 + Y_0} \right)}.
\]
Note that
\[
\frac{1}{\sqrt{\xi^2 + Y} \sqrt{\xi^2 + Y_0} \left( \sqrt{\xi^2 + Y} + \sqrt{\xi^2 + Y_0} \right)} \leq \frac{2(\sqrt{2} - 1)}{Y_0^{3/2}},
\]
where the right side is integrable on \([0, \hbar \omega_D]\). Therefore,
\[
\frac{\partial F}{\partial Y}(0, Y_0) = -\frac{1}{2} \int_0^{\hbar \omega_D} \frac{d\xi}{(\sqrt{\xi^2 + Y_0})^3} = -\frac{\hbar \omega_D}{2Y_0 \sqrt{\hbar^2 \omega_D^2 + Y_0}}.
\]
(13)
Similarly we can show that those exist at each point in \( W_3 \), and in \( W_4 \). Their values are given as follows: For \((T_0, 0) \in W_3\),

\[
\frac{\partial F}{\partial T}(T_0, 0) = -\frac{1}{2k_B T_0^2} \int_0^{h_0 D} \frac{d\xi}{\cosh^2 \frac{\xi}{2k_B T_0}},
\]

\[
\frac{\partial F}{\partial Y}(T_0, 0) = \frac{1}{2(2k_B T_0)^3} \int_0^{h_0 D} g \left( \frac{\xi}{2k_B T_0} \right) d\xi,
\]

and for \((T_c, Y_0) \in W_4\),

\[
\frac{\partial F}{\partial T}(T_c, Y_0) = -\frac{1}{2k_B T_c^2} \int_0^{h_0 D} \frac{d\xi}{\cosh^2 \sqrt{\frac{\xi^2 + Y_0}{2k_B T_c}}},
\]

\[
\frac{\partial F}{\partial Y}(T_c, Y_0) = \frac{1}{2(2k_B T_c)^3} \int_0^{h_0 D} g \left( \sqrt{\frac{\xi^2 + Y_0}{2k_B T_c}} \right) d\xi.
\]

The result follows.

Lemmas 9, 10 and 11 immediately give the following.

**Lemma 12.** At each \((T, Y) \in W \setminus W_2\),

\[
\frac{\partial F}{\partial T}(T, Y) < 0, \quad \frac{\partial F}{\partial Y}(T, Y) < 0.
\]

We now study the continuity of the functions \( F \), \( \partial F/\partial T \) and \( \partial F/\partial Y \) on \( W \).

**Lemma 13.** The partial derivatives \( \frac{\partial F}{\partial T} \) and \( \frac{\partial F}{\partial Y} \) are continuous on \( W_1 \). Consequently, the function \( F \) is of class \( C^1 \) on \( W_1 \).

**Proof.** It is enough to show that the functions: \((T, Y) \mapsto I_1(T, Y)\) and \((T, Y) \mapsto I_2(T, Y)\) (see (11) and (12)) are continuous at \((T_0, Y_0) \in W_1\). Here,

\[
I_1(T, Y) = \int_0^{h_0 D} \frac{d\xi}{\cosh^2 \eta}, \quad I_2(T, Y) = \int_0^{h_0 D} g(\eta) d\xi, \quad \eta = \sqrt{\frac{\xi^2 + Y_0}{2k_BT}}. \quad (14)
\]

Set \( \eta_0 = \frac{\sqrt{\xi^2 + Y_0}}{2k_BT_0} \). Since \((T, Y) \in W_1\) is close to \((T_0, Y_0) \in W_1\), it follows that \( T > T_0/2 \). Then

\[
|I_1(T, Y) - I_1(T_0, Y_0)| \\
\leq \int_0^{h_0 D} \left| \frac{1}{\cosh \eta} + \frac{1}{\cosh \eta_0} \right| \frac{\cosh \eta - \cosh \eta_0}{\cosh \eta \cosh \eta_0} d\xi \\
\leq 2h_0 D \sinh \frac{\sqrt{h^2 \omega^2_D + 2 \Delta^2_0}}{k_B T_0} \left( \frac{\sqrt{h^2 \omega^2_D + 2 \Delta^2_0}}{k_B T_0} |T - T_0| + \frac{|Y - Y_0|}{k_B T_0 \sqrt{Y_0}} \right),
\]

\[
|I_2(T, Y) - I_2(T_0, Y_0)| \\
\leq \int_0^{h_0 D} |g(\eta) - g(\eta_0)| d\xi \\
\leq h_0 D \max_{\eta \geq 0} |g'(\eta)| \left( \frac{\sqrt{h^2 \omega^2_D + 2 \Delta^2_0}}{k_B T_0^2} |T - T_0| + \frac{|Y - Y_0|}{k_B T_0 \sqrt{Y_0}} \right).
\]
Thus the functions: \((T, Y) \mapsto I_1(T, Y)\) and \((T, Y) \mapsto I_2(T, Y)\), and hence \((\partial F/\partial T)\) and \((\partial F/\partial Y)\) are continuous at \((T_0, Y_0) \in W_1\).

\[\square\]

**Lemma 14.** The function \(F\) is continuous on \(W\).

**Proof.** Note that \(F\) is continuous on \(W_1\) by Lemma 13. We then show that \(F\) is continuous on \(W_2\).

Let \((0, Y_0) \in W_2\) and let \((T, Y) \in W_1 \cup W_2\). Since \((T, Y)\) is close to \((0, Y_0)\), it follows that \(Y > Y_0/2\). Then, by (5),

\[
|F(T, Y) - F(0, Y_0)| \\
\leq \int_0^{h\omega_D} \left\{ \frac{1 - \tanh \sqrt{\xi^2 + Y}}{\sqrt{\xi^2 + Y_0}} + \frac{1}{\sqrt{\xi^2 + Y} - \sqrt{\xi^2 + Y_0}} \right\} d\xi \\
\leq h\omega_D \left\{ \frac{1}{\sqrt{Y_0}} \left( 1 - \tanh \frac{\sqrt{Y_0/2}}{2k_B T} \right) + \frac{2|Y - Y_0|}{(\sqrt{2} + 1)Y_0^{3/2}} \right\}.
\]

Thus \(F\) is continuous on \(W_2\). Similarly we can show the continuity of \(F\) on \(W_3\), and on \(W_4\).

\[\square\]

**Lemma 15.** The partial derivatives \(\frac{\partial F}{\partial T}\) and \(\frac{\partial F}{\partial Y}\) are continuous on \(W\). Consequently, the function \(F\) is of class \(C^1\) on \(W\).

**Proof.** Note that \((\partial F/\partial T)\) and \((\partial F/\partial Y)\) are continuous on \(W_1\) by Lemma 13. We then show that \((\partial F/\partial T)\) and \((\partial F/\partial Y)\) are continuous at \((T_c, 0) \in W_3\). We can show their continuity at other points in \(W\) similarly.

**Step 1.** Let \((T, Y) \in W_1\). We show

\[
\frac{\partial F}{\partial T}(T, Y) \rightarrow \frac{\partial F}{\partial T}(T_c, 0), \quad \frac{\partial F}{\partial Y}(T, Y) \rightarrow \frac{\partial F}{\partial Y}(T_c, 0) \quad \text{as} \quad (T, Y) \rightarrow (T_c, 0).
\]

Since \((T, Y)\) is close to \((T_c, 0)\), it then follows that \(T_c/2 < T < T_c\). Set \(\eta_0 = \frac{\sqrt{\hbar^2 \omega_0^2 + 2 \Delta_0^2}}{k_B T_c}\).

Then

\[
\left| \frac{1}{T^2 \cosh^2 \frac{\sqrt{Y_0/2}}{2k_B T}} - \frac{1}{T_c^2 \cosh^2 \frac{\xi}{2k_B T_c}} \right| \\
\leq \frac{8 \cosh \eta_0}{T_c^3} \left\{ |T - T_c| (\cosh \eta_0 + \eta_0 \sinh \eta_0) + \frac{\sqrt{Y}}{4k_B} \sinh \eta_0 \right\},
\]

and hence \((\partial F/\partial T)(T, Y) - (\partial F/\partial T)(T_c, 0) \rightarrow 0\) as \((T, Y) \rightarrow (T_c, 0)\).

Since

\[
\left| g \left( \frac{\sqrt{\xi^2 + Y}}{2k_B T} \right) - g \left( \frac{\xi}{2k_B T_c} \right) \right| \leq \max_{\eta \geq 0} |g'(\eta)| \left( \frac{\hbar \omega_D |T - T_c|}{k_B T_c^2} + \frac{\sqrt{Y}}{k_B T_c} \right),
\]

Thus

\[
\frac{\partial F}{\partial Y}(T, Y) \rightarrow \frac{\partial F}{\partial Y}(T_c, 0) \quad \text{as} \quad (T, Y) \rightarrow (T_c, 0).
\]

\[\square\]
it follows that
\[
\int_{0}^{\hbar \omega D} \left\{ g \left( \frac{\sqrt{\xi^2 + Y}}{2k_B T} \right) - g \left( \frac{\xi}{2k_B T_c} \right) \right\} d\xi \to 0 \quad \text{as} \quad (T, Y) \to (T_c, 0),
\]
and hence \((\partial F/\partial Y)(T, Y) - (\partial F/\partial Y)(T_c, 0) \to 0 \quad \text{as} \quad (T, Y) \to (T_c, 0).

**Step 2.** When \((T, Y) = (T, 0) \in W_3 \) and \((T, Y) = (T_c, Y) \in W_4\), an argument similar to that in Step 1 gives
\[
\frac{\partial F}{\partial T}(T, 0) \to \frac{\partial F}{\partial T}(T_c, 0), \quad \frac{\partial F}{\partial Y}(T, 0) \to \frac{\partial F}{\partial Y}(T_c, 0) \quad \text{as} \quad (T, 0) \to (T_c, 0)
\]
and
\[
\frac{\partial F}{\partial T}(T_c, Y) \to \frac{\partial F}{\partial T}(T_c, 0), \quad \frac{\partial F}{\partial Y}(T_c, Y) \to \frac{\partial F}{\partial Y}(T_c, 0)
\]
as \((T_c, Y) \to (T_c, 0)\). The result follows. \(\Box\)

**IV. THE SECOND-ORDER PARTIAL DERIVATIVES OF \(F\)**

In this section we deal with the second-order partial derivatives of the function \(F\) and show that \(F\) is of class \(C^2\) on \(W_1\).

A straightforward calculation yields the following.

**Lemma 16.** Let \(G\) be as in (8) and \(g\) as in (7). Then the function \(G\) is of class \(C^1\) on \([0, \infty)\) and satisfies
\[
g'(\eta) = -\eta G(\eta), \quad G'(0) = 0, \quad \lim_{\eta \to \infty} G(\eta) = \lim_{\eta \to \infty} G'(\eta) = 0.
\]

**Lemma 17.** The values of the partial derivatives \(\frac{\partial^2 F}{\partial T^2}, \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial Y} \right)\), \(\frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right)\), and \(\frac{\partial^2 F}{\partial Y^2}\) exist at each point in \(W_1\). Furthermore,
\[
\frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right) = \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial Y} \right) \quad \text{on} \quad W_1.
\]

**Proof.** Let \((T, Y) \in W_1\). Then there is a \(\theta\) \((0 < \theta < 1)\) satisfying \(\theta T_c < T < T_c\). Set \(\eta = \frac{\sqrt{\xi^2 + Y}}{2k_B T}\). Then
\[
\left| \frac{\partial}{\partial T} \left( \frac{1}{\cosh^2 \eta} \right) \right| \leq \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_B \theta^2 T_c^2},
\]
where the right side is integrable on \([0, \hbar \omega D]\). So, the function: \((T, Y) \mapsto I_1(T, Y)\) (see (11)), and hence \((\partial F/\partial T)\) (see (11)) is differentiable with respect to \(T\) on \(W_1\), and the second-order partial derivative is given by
\[
\frac{\partial^2 F}{\partial T^2}(T, Y) = \frac{1}{k_B T^3} \left\{ I_1(T, Y) - \int_{0}^{\hbar \omega_D} \eta \tanh \eta \frac{d\xi}{\cosh^2 \eta} \right\}, \quad \eta = \frac{\sqrt{\xi^2 + Y}}{2k_B T}.
\]
Similarly we can show that \( \frac{\partial F}{\partial T} \) is differentiable with respect to \( Y \) on \( W_1 \), that \( \frac{\partial F}{\partial Y} \) is differentiable with respect to \( T \) on \( W_1 \), and that \( \frac{\partial F}{\partial Y} \) is differentiable with respect to \( Y \) on \( W_1 \). The corresponding second-order partial derivatives are given as follows:

\[
\frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right)(T, Y) = \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial Y} \right)(T, Y) = \frac{1}{(2k_BT)^3} \int_0^{\hbar\omega_D} \frac{\tanh\eta}{\eta \cosh^2\eta} \, d\xi,
\]
\[
\frac{\partial^2 F}{\partial Y^2}(T, Y) = \frac{1}{4(2k_BT)^5} \int_0^{\hbar\omega_D} G(\eta) \, d\xi, \quad \eta = \frac{\sqrt{\xi^2 + Y^2}}{2k_BT}.\]

Here, \( G \) is that in Lemma 16 (see also (8)).

**Lemma 18.** The partial derivatives \( \frac{\partial^2 F}{\partial T^2} \), \( \frac{\partial}{\partial Y} \left( \frac{\partial F}{\partial T} \right) \) and \( \frac{\partial^2 F}{\partial Y^2} \) are continuous on \( W_1 \).

**Proof.** We show that \( \frac{\partial^2 F}{\partial Y^2} \) is continuous on \( W_1 \). Similarly we can show the continuity of other second-order partial derivatives.

By the form of \( \frac{\partial^2 F}{\partial Y^2} \) given in the proof of Lemma 17 it suffices to show that the function: \((T, Y) \mapsto I_3(T, Y)\) is continuous at \((T_0, Y_0) \in W_1\). Here,

\[
I_3(T, Y) = \int_0^{\hbar\omega_D} G(\eta) \, d\xi, \quad \eta = \frac{\sqrt{\xi^2 + Y^2}}{2k_BT}.\]

Since \((T, Y)\) is close to \((T_0, Y_0)\), it then follows that \( T_0/2 < T \). A straightforward calculation then gives

\[
|I_3(T, Y) - I_3(T_0, Y_0)| \leq \hbar\omega_D \max_{\eta \geq 0} |G'(\eta)| \left( \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_BT_0^2} |T - T_0| + \frac{|Y - Y_0|}{k_BT_0 \sqrt{Y_0}} \right).
\]

Hence the function: \((T, Y) \mapsto I_3(T, Y)\) is continuous at \((T_0, Y_0) \in W_1\).

**V. PROOFS OF OUR MAIN RESULTS**

In this section we prove Theorem 4, Propositions 5 and 7 in a sequence of lemmas.

**Remark 19.** One may prove the theorem and the propositions above on the basis of the implicit function theorem. In this case, an interior point \((T_0, Y_0)\) of the domain \( W \) satisfying \( F(T_0, Y_0) = 0 \) need to exist. But there are the two points \((0, \Delta_0^2)\) and \((T_c, 0)\) in the boundary of \( W \) satisfying

\[
F(0, \Delta_0^2) = F(T_c, 0) = 0. \tag{15}
\]

So one can not apply the implicit function theorem in its present form.

**Lemma 20.** There is a unique solution: \( T \mapsto Y = f(T) \) to the gap equation \( F(T, Y) = 0 \) such that the function \( f \) is continuous on the closed interval \([0, T_c]\) and satisfies \( f(0) = \Delta_0^2 \) and \( f(T_c) = 0 \).
Proof. By Lemmas 12, 15 and 15, the function: \( Y \mapsto F(T_c, Y) \) is monotonically decreasing and there is a \( Y_1 \quad (0 < Y_1 < 2\Delta_0^2) \) satisfying \( F(T_c, Y_1) < 0 \). Note that \( Y_1 \) is arbitrary as long as \( 0 < Y_1 < 2\Delta_0^2 \). Hence, by Lemma 15, there is a \( T_1 \quad (0 < T_1 < T_c) \) satisfying \( F(T_1, Y_1) < 0 \). Hence, \( F(T, Y_1) < 0 \) for \( T_1 \leq T \leq T_c \). On the other hand, by Lemmas 12, 13 and 15, the function: \( T \mapsto F(T, 0) \) is monotonically decreasing and there is a \( T_2 \quad (0 < T_2 < T_c) \) satisfying \( F(T_2, 0) > 0 \). Note that \( T_2 \) is arbitrary as long as \( 0 < T_2 < T_c \). Hence, \( F(T, 0) > 0 \) for \( T_2 \leq T < T_c \).

Let \( \max(T_1, T_2) \leq T < T_c \) and fix \( T \). It then follows from Lemmas 12 and 15 that the function: \( Y \mapsto F(T, Y) \) with \( T \) fixed is monotonically decreasing on \( [0, Y_1] \). Since \( F(T, 0) > 0 \) and \( F(T, Y_1) < 0 \), there is a unique \( Y \) \( (0 < Y < Y_1) \) satisfying \( F(T, Y) = 0 \). When \( T = T_c \), there is a unique value \( Y = 0 \) satisfying \( F(T_c, Y) = 0 \) (see (13)).

Since \( F \) is continuous on \( W \) by Lemma 15 there is a unique solution: \( T \mapsto Y = f(T) \) to the gap equation \( F(T, Y) = 0 \) such that \( f \) is continuous on \( [\max(T_1, T_2), T_c] \) and \( f(T_c) = 0 \).

Since \( (\partial F/\partial Y)(0, Y) < 0 \quad (0 < Y < 2\Delta_0^2) \) by (13), there is a unique value \( Y = \Delta_0^2 \) satisfying \( F(0, Y) = 0 \). Combining Lemma 15 with Lemma 12 therefore implies that the function \( f \) is continuous on \( [0, T_c] \) and that \( f(0) = \Delta_0^2 \) and \( f(T_c) = 0 \).

Lemma 21. The function \( f \) given by Lemma 20 is of class \( C^1 \) on \( [0, T_c] \), and the derivative \( f' \) satisfies

\[
f'(0) = 0, \quad f'(T_c) = 8k_B^2T_c\int_{0}^{\frac{\hbar\omega_D/(2k_BT_c)}{\cosh^2 \xi}} \frac{g(\xi)}{\cosh^2 \xi} \, d\xi.
\]

Proof. Lemmas 10, 11, 12 and 15 immediately imply that the function \( f \) is of class \( C^1 \) on the interval \([0, T_c]\) and that its derivative is given by

\[
f'(T) = -\frac{F_T(T, f(T))}{F_Y(T, f(T))}.
\]

The values of \( f'(0) \) and \( f'(T_c) \) are derived from (16). \( \square \)

Combining (16) with Lemma 12 immediately yields the following.

Lemma 22. The function \( f \) given by Lemma 20 is monotonically decreasing on \( [0, T_c] \):

\[
f(0) = \Delta_0^2 > f(T_1) > f(T_2) > f(T_c) = 0, \quad 0 < T_1 < T_2 < T_c.
\]

Lemma 23. Let \( f \) be as in Lemma 20 and \( I[\cdot] \) as in (10). Then the function \( f \) is of class \( C^2 \) on \( [0, T_c] \), and the second derivative \( f'' \) satisfies

\[
f''(0) = 0 \quad \text{and} \quad f''(T_c) = 16k_B^2 \left[ I \left[ \frac{\eta_0 \tanh \eta_0 - 1}{\cosh^2 \eta_0} \right] \right] - 32k_B^2 \left[ I \left[ \frac{1}{\cosh^2 \eta_0} \right] I \left[ \frac{\tanh \eta_0}{\eta_0 \cosh^2 \eta_0} \right] \right]
\]

\[
+ 8k_B^2 \left[ I \left[ \frac{1}{\cosh^2 \eta_0} \right] \right]^2 I \left[ G(\eta_0) \right] \left[ I \left[ \frac{\eta_0 \tanh \eta_0}{\cosh^2 \eta_0} \right] \right] \left[ I \left[ \frac{1}{\cosh^2 \eta_0} \right] \right]^3, \quad \eta_0 = \frac{\xi}{2k_BT_c}.
\]
Proof. Lemma 18 implies that \( f \) is of class \( C^2 \) on the open interval \((0, T_c)\) and that

\[
f''(T) = \frac{-F_{TT} F_Y^2 + 2 F_{TY} F_T F_Y - F_{YY} F_T^2}{F_Y^3}, \quad 0 < T < T_c. \tag{17}
\]

So we have only to deal with \( f \) and its derivatives at \( T = 0 \) and at \( T = T_c \).

**Step 1.** We show that \( f' \) is differentiable at \( T = 0 \) and that \( f'' \) is continuous at \( T = 0 \).

Note that \( f'(0) = 0 \) by Lemma 21. Since \( T \) is close to \( T = 0 \), the inequality \( f(T) > \Delta_0^2/2 \) holds. It then follows from (16), (11) and (12) that

\[
\left| \frac{f'(T) - f'(0)}{T} \right| \leq 4 \left\{ \sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2} \exp \left( -\frac{\sqrt{\Delta_0^2/2}}{k_B T} \right) \right\} \left( \tanh \eta_1 - \frac{\eta_1}{\cosh^2 \eta_1} \right) \to 0 \quad (T \downarrow 0).
\]

Here, \( \eta_1 = \frac{\sqrt{\xi_1^2 + f(T)}}{2k_B T} \to \infty \) as \( T \downarrow 0 \) \((0 < \xi_1 < \hbar \omega_D)\). Hence \( f' \) is differentiable at \( T = 0 \) and \( f''(0) = 0 \).

By (17), a similar argument gives \( \lim_{T \downarrow 0} f''(T) = 0 \). Hence \( f'' \) is continuous at \( T = 0 \).

**Step 2.** We show that \( f' \) is differentiable at \( T = T_c \) and that \( f'' \) is continuous at \( T = T_c \).

Note that

\[
f'(T_c) = 8 k_B^2 T_c \left\{ \frac{1}{\cosh^2 \eta_0} \right\} \tag{I}, \quad \eta_0 = \frac{\xi}{2k_B T_c}
\]

by Lemma 21. It follows from (16), (11) and (12) that

\[
f'(T) = 8 k_B^2 T \left\{ \frac{1}{\cosh^2 \eta} \right\}, \quad \eta = \frac{\sqrt{\xi^2 + f(T)}}{2k_B T}.
\]

Hence

\[
\frac{f'(T_c) - f'(T)}{T_c - T} = 8 k_B^2 \left\{ \frac{1}{\cosh^2 \eta_0} \right\} \left\{ \frac{1}{\cosh^2 \eta} \right\} \frac{I [g(\eta)] - I [g(\eta_0)]}{I [g(\eta)] I [g(\eta_0)]} \left\{ I \left[ \frac{1}{\cosh^2 \eta} \right] - I \left[ \frac{1}{\cosh^2 \eta_0} \right] \right\}.
\]

Note that \( g(\eta) - g(\eta_0) = (\eta - \eta_0) g'(\eta) \) and \( \cosh \eta - \cosh \eta_0 = (\eta - \eta_0) \sinh \eta_2 \). Here,

\[
\eta_0 = \frac{\xi}{2k_B T_c} < \eta_i < \eta = \frac{\sqrt{\xi^2 + f(T)}}{2k_B T}, \quad i = 1, 2
\]
\[ \eta - \eta_0 = \frac{1}{2k_B T} \left\{ \frac{f(T)}{\sqrt{\xi^2 + f(T)} + \xi} + \xi \frac{T_c - T}{T_c} \right\}. \]

Since \( T \) is close to \( T_c \), the inequality \( T > T_c / 2 \) holds. Therefore, by Lemma 16,
\[ \left| \frac{g'(\eta_1)}{\sqrt{\xi^2 + f(T)} + \xi} \right| \leq \frac{1}{k_B T_c} \max_{\eta \geq 0} |G(\eta)| \]
and
\[ \left| \frac{\sinh \eta_2}{\sqrt{\xi^2 + f(T)} + \xi} \right| \leq \frac{1}{k_B T_c} \max_{0 \leq \eta \leq M} \left| \frac{\sinh \eta}{\eta} \right|, \quad M = \frac{\sqrt{\hbar^2 \omega_D^2 + 2 \Delta_0^2}}{k_B T_c}. \]

So \( f' \) is differentiable at \( T = T_c \), and it is easy to see that the form of \( f''(T_c) \) is exactly the same as that mentioned just above.

Furthermore, it follows from (17) that \( f'' \) is continuous at \( T = T_c \).

\[ \square \]

References

[1] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Theory of superconductivity, Phys. Rev. 108, 1175–1204 (1957).

[2] P. Billard and G. Fano, An existence proof for the gap equation in the superconductivity theory, Commun. Math. Phys. 10, 274–279 (1968).

[3] N. N. Bogoliubov, A new method in the theory of superconductivity I, Soviet Phys. JETP 34, 41–46 (1958).

[4] C. Hainzl, E. Hamza, R. Seiringer and J. P. Solovej, The BCS functional for general pair interactions, Commun. Math. Phys., in press. arXiv: 0703086.

[5] C. Hainzl and R. Seiringer, Spectral properties of the BCS gap equation of superfluidity, arXiv: 0802.0446.

[6] F. Odeh, An existence theorem for the BCS integral equation, IBM J. Res. Develop. 8, 187–188 (1964).

[7] A. Vansevenant, The gap equation in the superconductivity theory, Physica 17D, 339–344 (1985).

[8] S. Watanabe, Superconductivity and the BCS-Bogoliubov theory, JP Journal Algebra, Number Theory and Appl. 11, 137–158 (2008).

[9] Y. Yang, On the Bardeen-Cooper-Schrieffer integral equation in the theory of superconductivity, Lett. Math. Phys. 22, 27–37 (1991).

[10] J. M. Ziman, Principles of the Theory of Solids, Cambridge University Press, Cambridge, 1972.