Uncertainty relations for time-delayed Langevin systems

Tan Van Vu* and Yoshihiko Hasegawa†
Department of Information and Communication Engineering,
Graduate School of Information Science and Technology,
The University of Tokyo, Tokyo 113-8656, Japan
(Dated: February 20, 2019)

The thermodynamic uncertainty relation, which establishes a universal trade-off between the nonequilibrium current fluctuations and the dissipation, has been found for various Markovian systems. However, this relation has not been revealed for non-Markovian systems. Thus, we investigate the thermodynamic uncertainty relation for time-delayed Langevin systems. We prove that the fluctuation of arbitrary dynamical observables at a steady state is constrained by the Kullback-Leibler divergence between the distributions of the forward path and its reversed counterpart. Specifically, for observables that are antisymmetric under time reversal, the fluctuation is bounded from below by a function of a quantity, which can be considered a generalization of the total entropy production in Markovian systems. We also provide a lower bound for arbitrary observables that are odd under position reversal. Our results hold for finite observation time and a large class of time-delayed systems since the detailed underlying dynamics is not required in the derivation. We numerically verify the derived uncertainty relation with two linear systems.

I. INTRODUCTION

In the last two decades, substantial progress has been made in stochastic thermodynamics (ST) relative to describing small systems that fluctuate and are far from thermal equilibrium [1–8]. The first and second laws of ST have been generalized for individual trajectory levels, and fluctuation theorems [4, 9, 10] that express the universal properties of the probability distribution of thermodynamic quantities, such as work, heat, and entropy production, have been derived. ST has been used to investigate various systems, such as optical and colloidal particle systems and biochemical reaction networks [4].

In recent years, the thermodynamic uncertainty relation (TUR) stating that smaller current fluctuation cannot be attained without higher thermodynamic cost, has been discovered in various Markovian dynamical processes [11–16]. The TUR was first proved for large time limit by using the large deviation theory [12], and after that was shown to be valid even for finite observation time [15]. The general form of the TUR is expressed in the inequality

\[
\text{Var}[\sigma] \geq 2k_B \langle \sigma \rangle \text{Var}[\langle \sigma \rangle],
\]

where \(k_B\) is Boltzmann’s constant, \(\langle \sigma \rangle\) and \(\text{Var}[\langle \sigma \rangle]\) are the mean and variance of the current, respectively, and \(\Sigma\) is the average of the total entropy production. Analogous precision-cost trade-off relations have been reported in the literature [17, 18]. Various forms of the TUR have been proposed and studied intensively in many other contexts [19–34].

To date, the TUR has only been investigated in Markovian systems. However, the time delay that causes non-Markovian dynamical behavior inevitably exists in many real-world stochastic processes, such as gene regulatory [35, 36] and biochemical reaction networks [37], as well as control systems involving a feedback protocol [38–40]. It is well known that time delay can completely alter system dynamics, e.g., delay-induced oscillations [35]. In addition, Ref. [41] has recently shown that even a small delay time also leads to finite heat flow in the system. Despite the importance of the delay in many classical and quantum systems, thermodynamic analysis of such systems remains challenging [42, 43].

In this paper, we study the TUR for general dynamical observables that are antisymmetric under conjugate operations, which can be time reversal or position reversal. First, we define a trajectory-dependent quantity \(\sigma\) [cf. Eq (6)], whose average is the Kullback-Leibler (KL) divergence between the distributions of the forward path and its conjugate counterpart. In the absence of time delay and under time reversal, \(\sigma\) is identified as the trajectory-dependent total entropy production in Markovian systems. Starting from the point that the joint probability distribution of \(\sigma\) and the observable obeys the strong detailed fluctuation theorem (DFT) [4], we prove that the relative fluctuation of the observable is lower bounded by \(2/(e^{(\langle \sigma \rangle)} - 1)\). This implies that the time irreversibility in the system constrains the fluctuation of observables that are odd under time reversal. For observables that are antisymmetric under position reversal, the bound on the fluctuation reflects the position-symmetry breaking in the system. The derived TUR holds for arbitrary observation time and a large class of time-delayed systems, for instance, for the systems with multiple delays and distributed delays. We numerically verify the validity of the derived inequality in two linear systems, wherein \(\langle \sigma \rangle\) can be obtained analytically.

* tan@biom.t.u-tokyo.ac.jp
† hasegawa@biom.t.u-tokyo.ac.jp
II. MODEL

For the sake of clear illustration of the results, we consider here a single time-delayed system with dynamical variables \( \mathbf{x}(t) = [x_1(t), \ldots, x_N(t)]^\top \) described by the following set of coupled Langevin equations:

\[
\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{x}_r) + \xi,
\]

where \( \mathbf{x}_r = \mathbf{x}(t - \tau) \), \( \mathbf{F}(\mathbf{x}, \mathbf{x}_r) \in \mathbb{R}^N \) is a drift force, \( \xi(t) = [\xi_1(t), \ldots, \xi_N(t)]^\top \) is zero-mean white Gaussian noise with covariance \( \langle \xi_i(t)\xi_j(t') \rangle = 2D_i\delta_{ij}\delta(t - t') \), and \( \tau \geq 0 \) denotes the delay time in the system. Here, \( D_i \) denote the noise intensity. Equation (2) is interpreted as Ito stochastic integration. Throughout the paper, Boltzmann’s constant is set to \( k_B = 1 \). Let \( P(\mathbf{x}, t) \) be the probability distribution function for the system to be in state \( \mathbf{x} \) at time \( t \). The corresponding Fokker-Planck equation (FPE) is expressed as [44, 45]

\[
\partial_t P(\mathbf{x}, t) = -\sum_{i=1}^{N} \partial_{x_i} J_i(\mathbf{x}, t),
\]

where

\[
J_i(\mathbf{x}, t) = \int d\mathbf{y} F_i(\mathbf{x}, \mathbf{y}) P(\mathbf{y}, t - \tau; \mathbf{x}, t) - D_i \partial_{x_i} P(\mathbf{x}, t)
= \mathcal{F}_i(\mathbf{x}) P(\mathbf{x}, t) - D_i \partial_{x_i} P(\mathbf{x}, t)
\]

is the probability current. Here,

\[
\mathcal{F}_i(\mathbf{x}) = \int d\mathbf{y} F_i(\mathbf{x}, \mathbf{y}) P(\mathbf{y}, t - \tau | \mathbf{x}, t)
\]

is an effective force obtained by taking the delay-averaged integration on the variable \( \mathbf{y} \), and \( P(\mathbf{y}, t - \tau; \mathbf{x}, t) \) is a joint probability density that the system takes value \( \mathbf{y} \) at time \( t \) and \( \mathbf{y} \) at time \( t - \tau \). Generally, solving \( P(\mathbf{y}, t - \tau; \mathbf{x}, t) \) results in an infinite hierarchy of equations, where \( n \)-time probability distribution depends on the \( (n + 1) \)-time one. Therefore, it is difficult to analytically obtain the effective force \( \mathcal{F}_i(\mathbf{x}) \), except linear systems. Hereafter, the system is assumed to be in a steady state, where the probability distribution and probability current are \( P^\text{ss}(\mathbf{x}) \) and \( J^\text{ss}(\mathbf{x}) \), respectively.

We define \( \mathbf{X}_{[s,e]} = \{ \mathbf{x}(t) | t \in [s, e] \} \) as a trajectory that begins from \( t = s \) and ends at \( t = e \). Let \( \mathcal{P}(\mathbf{X}_{[s,e]}) \) be the probability of observing the trajectory \( \mathbf{X}_{[s,e]} \). For each trajectory \( \mathbf{X}_{[s,e]} \), we consider a conjugate trajectory \( \mathbf{X}^\dagger_{[s,e]} \) defined by \( \mathbf{X}^\dagger_{[s,e]} = \{ \mathbf{x}^\dagger(t) | t \in [s, e] \} \). Assuming that we observe the system during a time interval \( [0, T] \), we then define a trajectory-dependent quantity \( \sigma(\mathbf{X}_{[0,T]}) \), which is the ratio of the probabilities of observing the forward path and its conjugate counterpart, as follows:

\[
\sigma(\mathbf{X}_{[0,T]}) = \ln \frac{P(\mathbf{X}_{[0,T]} | \mathbf{X}^\dagger_{[0,T]})}{P(\mathbf{X}^\dagger_{[0,T]} | \mathbf{X}_{[0,T]})),}
\]

is the ratio of the probabilities of observing the forward path and its conjugate counterpart, as follows:

\[
\sigma = -\ln \frac{P(\mathbf{x}(T))}{P(\mathbf{x}(0))} + \ln \frac{P(\mathbf{X}(\mathbf{x}(0))}{P(\mathbf{X}^\dagger(\mathbf{x}^\dagger(0))) = \Delta s + \Delta s_m.
\]
which was identified as the total entropy production along a trajectory in Markovian systems [4]. Here, \( \Delta s \) and \( \Delta s_m \) are the changes in the system entropy and the medium entropy, respectively. Under time reversal, \( \langle \sigma \rangle \) can be considered a generalization of total entropy production in time-delayed systems. Another possible conjugate operation is position reversal, i.e., \( x^\dagger(t) = \kappa - x(t) \). Here, \( \kappa \in \mathbb{R}^N \) is a constant and basically can take an arbitrary value, except the system involving the odd variables, wherein \( \kappa \) must be carefully chosen to satisfy the requirements. Specifically, \( \kappa \) is set to \( \kappa_i = 0 \) for all odd variables \( x_i \). Under this conjugate operation, \( \langle \sigma \rangle \) reflects the position-symmetry breaking with respect to the position \( \kappa / 2 \) of the system. In the remaining part of the paper, we consider the \( \kappa = 0 \) case. To distinguish when which operation is employed, we use subscripts \( t \) and \( p \) to refer time reversal and position reversal, respectively.

Now, we investigate a more detailed form of \( \langle \sigma \rangle \) with respect to conjugate operations for the system defined in Eq. (2). For \( T > \tau \), the path probability can be rewritten

\[
\mathcal{P}(X_{[\tau,T]}|X_{[0,\tau]}) = \mathcal{P}(X_{[\tau,T]}|X_{[0,\tau]}),
\]

\[
\mathcal{P}(X_{[0,T]}|X_{[0,\tau]}) = \mathcal{P}(X_{[\tau,T]}|X_{[0,\tau]}),
\]

where \( \mathcal{P}(X_{[\tau,T]}|X_{[0,\tau]}) \) is the probability of observing \( X_{[\tau,T]} \), conditioned on \( X_{[0,\tau]} \). We note that under time reversal, \( X^\dagger_{[\tau,T]} = \{ \epsilon x(T - t) | t \in [\tau,T] \} \). The conditional probability can be calculated via the path integral as

\[
\mathcal{P}(X_{[\tau,T]}|X_{[0,\tau]}) = \mathcal{N} \exp \left( - \sum_{i=1}^{N} \mathcal{S}_i(X_{[0,T]}) \right),
\]

where \( \mathcal{S}_i(X_{[0,T]}) \) is the stochastic action given by

\[
\mathcal{S}_i(X_{[0,T]}) = \int_{\tau}^{T} dt \left[ \frac{(\dot{x}_i - F_i(x, \bar{x}_r))^2}{4D_i} + \frac{\partial x_i F_i(x, \bar{x}_r)}{2} \right],
\]

and \( \mathcal{N} \) is a positive term independent of the trajectory.

In the long-time limit, i.e., \( T \to \infty \), the first term in Eq. (14) becomes the dominant term as the second term is only a boundary term. By plugging Eq. (12) into Eq. (14), \( \langle \sigma_t \rangle \) and \( \langle \sigma_p \rangle \) can be expressed further as

\[
\langle \sigma_t \rangle = \sum_{i=1}^{N} \left( -\mathcal{S}_i(X_{[0,T]}) + \int_{\tau}^{T} dt \left[ \frac{(\dot{x}_i + F_i(x, \bar{x}_r))^2}{4D_i} \right] + \frac{\partial x_i F_i(x, \bar{x}_r)}{2} \right) + \left( \ln \frac{\mathcal{P}(X_{[0,T]})}{\mathcal{P}(X^\dagger_{[0,T]})} \right),
\]

\[
\langle \sigma_p \rangle = \sum_{i=1}^{N} \left( \int_{\tau}^{T} dt \left[ \frac{(\dot{x}_i - F_i(x, \bar{x}_r))}{D_i} \right] \left( F_i(x, \bar{x}_r) + F_i(-x, -\bar{x}_r) \right) \right) + \left( \ln \frac{\mathcal{P}(X_{[0,T]})}{\mathcal{P}(X^\dagger_{[0,T]})} \right).
\]

For general systems, it is difficult to obtain more detailed forms of \( \langle \sigma_t \rangle \) and \( \langle \sigma_p \rangle \) than those in Eq. (15). Since \( \langle \sigma_t \rangle \) characterizes the time reversibility of the system, \( \langle \sigma_t \rangle \) becomes zero when the system is in equilibrium state. On the other hand, \( \langle \sigma_p \rangle \) can be positive even in the equilibrium system, as far as the symmetry with respect to position reversal is broken.

### III. DERIVATION OF UNCERTAINTY RELATION

In this section, we derive the TUR for an arbitrary dynamical observable \( j(X) \) that is antisymmetric under the conjugate operation, i.e., \( j(X^\dagger) = -j(X) \). This antisymmetric property can be satisfied, for instance, for the generalized currents of the form \( j(X) = \int_{0}^{T} dt \Lambda(x)^\dagger \circ \dot{x} \) under time reversal, or for the observable \( j(X) = \int_{0}^{T} dt \Gamma(x) \) under position reversal. Here, \( \Gamma(x) \) is an arbitrary odd function.

In Ref. [46], we have derived a generalized TUR from the fluctuation theorem for Markovian processes. Regardless of the underlying dynamics, the generalized TUR holds as far as the fluctuation theorem is valid. Here we apply the same technique and derive the TUR for time-delayed systems. First, we show that the joint probability distribution of \( \sigma \) and \( \gamma \), \( \mathcal{P}(\sigma, \gamma) \), obeys the strong DFT. This can be proved analogously as in the following:

\[
\mathcal{P}(\sigma, j) = \int \mathcal{D}X \delta(\sigma - \sigma(X)) \delta(j - j(X)) \mathcal{P}(X)
\]

\[
= \int \mathcal{D}X \delta(\sigma - \sigma(X)) \delta(j - j(X)) e^{\sigma(X)} \mathcal{P}(X^\dagger)
\]

\[
= e^{\sigma} \int \mathcal{D}X \delta(\sigma - \sigma(X)) \delta(j - j(X)) \mathcal{P}(X^\dagger)
\]

\[
= e^{\sigma} \int \mathcal{D}X^\dagger \delta(\sigma + \sigma(X^\dagger)) \delta(j + j(X^\dagger)) \mathcal{P}(X^\dagger)
\]
\[ = e^\sigma P(-\sigma, -\sigma). \] (16)

Inspired by Ref. [47], where statistical properties of entropy production were obtained from the strong DFT, we derive the TUR solely from Eq. (16). By observing that

\[ 1 = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\eta P(\sigma, \eta) \]
\[ = \int_{0}^{\infty} d\sigma \int_{-\infty}^{\infty} d\eta (1 + e^{-\sigma}) P(\sigma, \eta), \] (17)

we introduce a probability distribution \( Q(\sigma, \eta) \equiv (1 + e^{-\sigma}) P(\sigma, \eta) \), defined over \([0, \infty) \times (-\infty, \infty)\). Using the distribution \( Q(\sigma, \eta) \), the moments of \( \sigma \) and \( \eta \) can be expressed in an alternative way as follows:

\[ \langle \sigma^{2k} \rangle_Q = \langle \sigma^{2k} \rangle_Q, \]
\[ \langle \eta^{2k} \rangle_Q = \langle \eta^{2k} \rangle_Q, \]
\[ \langle \sigma^{2k+1} \rangle = \langle \sigma^{2k+1} \tanh(\frac{\sigma}{2}) \rangle_Q, \]
\[ \langle \eta^{2k+1} \rangle = \langle \eta^{2k+1} \tanh(\frac{\sigma}{2}) \rangle_Q, \] (18)

where \( \langle \ldots \rangle_Q \) denotes the expectation with respect to \( Q(\sigma, \eta) \). Applying the Cauchy-Schwartz inequality to \( \langle \eta \rangle \), we obtain

\[ \langle \eta \rangle^2 = \langle \eta \tanh(\frac{\sigma}{2}) \rangle_Q^2 \leq \langle \eta^2 \rangle_Q \langle \tanh(\frac{\sigma}{2}) \rangle_Q \] (19)

The last term in the right-hand side of Eq. (19) can be further upper bounded. We find that

\[ \langle \tanh(\frac{\sigma}{2}) \rangle_Q \leq \tanh(\frac{\langle \sigma \rangle}{2}). \] (20)

Equation (20) is obtained by first noticing that \( \tanh(\frac{\sigma}{2}) \leq \tanh(\frac{\sigma}{2}) \) for all \( \sigma \geq 0 \). After that, by applying Jensen’s inequality to the concave function \( \tanh(x) \), we obtain

\[ \langle \tanh(\frac{\sigma}{2}) \rangle_Q \leq \tanh(\langle \frac{\sigma}{2} \rangle) \]
\[ = \tanh(\frac{\langle \sigma \rangle}{2}). \] (21)

From Eqs. (19) and (20), we have

\[ \langle \eta \rangle^2 \leq \langle \eta^2 \rangle \tanh(\frac{\langle \sigma \rangle}{2}). \] (22)

By transforming Eq. (22), we obtain the following TUR for the observable \( \eta \):

\[ \text{Var}[\eta] = \frac{\langle \eta^2 \rangle - \langle \eta \rangle^2}{\langle \eta \rangle^2} \geq \frac{2}{e^{\langle \sigma \rangle} - 1}. \] (23)

The inequality in Eq. (23) is the main result of the paper. For observables that are antisymmetric under time (or position) reversal, the term \( \langle \sigma \rangle \) in the bound of Eq. (23) should be replaced by \( \langle \sigma \rangle \) (or \( \langle \sigma \rangle \)).

In the limit \( \tau \to 0 \), the system becomes a continuous-time Markovian process, where the conventional TUR provides a lower bound on the current fluctuations as in Eq. (1). Since \( e^{\langle \sigma \rangle} - 1 \geq \langle \sigma \rangle \), the derived TUR has a looser bound than the conventional TUR. Regarding this difference, there are two possible reasons for explaining. Firstly, that is because there is no requirement on the details of the underlying dynamics of the system in the derivation. It was proven that the conventional TUR does not hold for discrete-time Markovian processes [26, 48]. In contrast, the derived TUR holds not only for continuous-time but also for discrete-time delay systems [49]. The lower bound in Eq. (23) is the same as that in Ref. [26], which derived the TUR in the long-time limit for discrete-time Markovian processes. Secondly, the derived TUR also holds for non-current observables, which is different from the conventional TUR that holds only for current-type observables defined by \( \eta(X) = \int_0^T dt \Lambda(x)^T \circ \dot{x} \).

IV. EXAMPLES

A. One-dimensional system

We study a one-dimensional linear system, whose drift term is given by

\[ F(x, x_t) = -ax - bx + f, \] (24)

where \( a, b, \) and \( f \) are the given constants satisfying the conditions \( a > b > 0, f > 0 \). It is easy to see that \( \langle x \rangle = \bar{f}, \) where \( \bar{f} = f/(a+b) \). Since the force is linear, the system has a Gaussian steady-state distribution, which exists for arbitrary delay time \( \tau \). We introduce a new stochastic variable \( z \), defined as \( z = x - \bar{f} \). The FPE corresponding to \( z \) reads as

\[ \partial_t P(z, t) = -\partial_z \bar{G}(z) P(z, t) + \partial^2 z P(z, t), \] (25)

where \( \bar{G}(z) = \int dz (-az - by) P(y, t - \tau | z, t) \). At the steady state, the probability current vanishes, i.e., \( J^\text{ss}(z) = \bar{G}(z) P^\text{ss}(z) - \partial_z P^\text{ss}(z) = 0 \) vanishes. Here, \( P^\text{ss}(z) \) denotes the steady-state distribution. Let \( \phi(t) = \langle z(0) z(t) \rangle \) be the time-correlation function of \( z \), then it was shown that \( \phi(t) = A_+ e^{-c|t|} + A_- e^{c|t|} \) for all \( |t| \leq \tau \) [44, 50], where \( c = \sqrt{a^2 - b^2}, A_+ = 1/2 [\phi(0) + D/c], \) and \( \phi(0) = \langle z^2 \rangle \)

\[ \phi(0) = \langle z^2 \rangle = \frac{D c + b \sinh(c \tau)}{e + b \cosh(c \tau)} \] (26)

First, we consider the TUR for observables that are antisymmetric under time reversal. According to Eq. (23), the following inequality should be satisfied.

\[ \frac{\langle \eta \rangle^2}{\text{Var}[\eta]} \leq \frac{e^{\langle \sigma \rangle} - 1}{2}. \] (27)
Since evaluating $\langle \sigma_t \rangle$ for $T > \tau$ becomes complicated calculations, we consider the only case of $T \leq \tau$, in which the path probability $P(X_{[0,T]})$ can be calculated analytically as [42]

$$P(X_{[0,T]}) \propto \exp \left[ -\frac{1}{4D} \int_0^T dt \left( \dot{x} + cx - cT \right)^2 \right] \times$$

$$\exp \left( -\frac{c}{2D} \frac{A_+ e^{-cT} (x(0) - \bar{f}) - A_- (x(T) - \bar{f})}{A_+^2 e^{-2cT} - A_-^2} \right).$$

It can be confirmed that $P(X) = P(X^\dagger)$, thus $\langle \sigma_t \rangle = 0$. Consequently, Eq. (27) implies that an arbitrary observable that is antisymmetric under time reversal vanishes in the average, i.e., $\langle \gamma \rangle = 0$. For the current-type observable defined by $j(X) = \int_0^\infty dt \Lambda(x) \dot{x}(t)$, where $\Lambda(x)$ is an arbitrary projection function, one can easily check that $\langle \gamma \rangle = T \int_{-\infty}^\infty dz \Lambda(z + \bar{f}) J_x^s(z) = 0$. Generally, this can be proven as

$$\langle \gamma \rangle = \int D\mathbf{X} j(X) P(X)$$

$$= \frac{1}{2} \left( \int D\mathbf{X} j(X) P(X) - \int D\mathbf{X}^\dagger j(X^\dagger) P(X^\dagger) \right) = 0. \quad (29)$$

Next, let us consider the TUR for non-current observables that are antisymmetric under position reversal. Specifically, we validate the TUR for the observable $j(X) = \int_0^T dt \Lambda(x) \dot{x}(t)$, which represents the area under the trajectory. The average of the observable is $\langle \gamma \rangle = T \dot{x}(T)$. For $T \leq \tau$, using the path integral, $\sigma_p$ can be calculated as

$$\sigma_p = \ln P(X_{[0,T]}) / P(X_{[0,T]})^\dagger = \frac{cT}{D} \int_0^T dt \left( \dot{x} + cx \right)$$

$$= 2cT \frac{A_+ e^{-cT} x(0) - A_- x(T)}{A_+ e^{-cT} + A_-}.$$  \quad (30)

Since the system is in the steady state, we obtain

$$\langle \sigma_p \rangle = \left( cT + 2 \frac{A_+ e^{-cT} - A_-}{A_+ e^{-cT} + A_-} \right) \frac{cT^2}{D}. \quad (31)$$

The variance of the observable can also be obtained analytically as follows:

$$\text{Var}[j] = \left( \int_0^T dt \int_0^T ds \left( x(t) - \bar{f} \right) \left( x(s) - \bar{f} \right) \right)$$

$$= \int_0^T dt \int_0^T ds \phi(t-s)$$

$$= \int_0^T dt \int_0^T ds \left( A_+ e^{-c(t-s)} + A_- e^{c(t-s)} \right)$$

$$= \frac{2}{c^2} \left( A_+ (e^{-cT} + cT - 1) + A_- (e^{cT} - cT - 1) \right). \quad (32)$$

We define

$$E_p = \frac{2 \langle \gamma \rangle^2}{\text{Var}[j](e^{\langle \sigma_p \rangle} - 1)}.$$ \quad (33)

which should satisfy $E_p \leq 1$. Using Eq. (31) and Eq. (32), one can numerically evaluate $E_p$ and verify the TUR for $T \leq \tau$. For the $T > \tau$ case, one can calculate $\langle \sigma_p \rangle$ via Eq. (15) and obtain

$$\langle \sigma_p \rangle = \frac{TF^2}{D} + \left( cT + 2 \frac{A_+ e^{-cT} - A_-}{A_+ e^{-cT} + A_-} \right) \frac{cT^2}{D}. \quad (34)$$

From Eq. (34), it can be concluded that decreasing the force $f$ or increasing the noise intensity $D$ both result in higher current fluctuation, which is consistent with the intuition. In the long-time limit $T \to \infty$, we have $\lim_{T \to \infty} T^{-1} \text{Var}[j] = \chi_f^2(0)$, where $\chi_f(k)$ is the scaled cumulant generating function defined by $\chi_f(k) = \lim_{T \to \infty} T^{-1} \ln \langle e^{cT} \rangle$. Using discrete Fourier series, one can obtain $\chi_f(k) = k^2 + Dk^2/(a + b)^2$ (see Appendix A). Therefore, the derived TUR is can be confirmed for $T \to \infty$ as

$$\frac{\text{Var}[j]}{\langle \sigma_p \rangle^2} = \frac{2D}{TF^2} \geq \frac{2}{\langle \sigma_p \rangle} \geq \frac{2}{e^{\langle \sigma_p \rangle} - 1}. \quad (35)$$

Finally, we run numerical simulations to calculate $\text{Var}[j]$ for $T > \tau$ and verify the derived TUR. We randomly select parameters $(a, b, f, D, \tau, T)$ and repeat the simulations $2 \times 10^6$ times for each of the selected parameter settings with time step $\Delta t = 10^{-3}$. We plot $E_p$ as a function of $\langle \sigma_p \rangle$ with triangle points in Fig. 2. The
ranges of parameters are given in the corresponding caption. As can be seen, all triangle points are located below the dashed line, which corresponds to the saturated TUR; thus, the derived TUR is empirically validated in this system.

B. Two-dimensional system

Here, we consider a simple two-dimensional system with the drift force

$$ F(x, x_\tau) = \begin{bmatrix} -ax_1 + bx_2, \tau \\ -ax_2 - bx_1, \tau \end{bmatrix}, $$

(36)

where $a > b > 0$ are the given constants and $x_{i,\tau} \equiv x_i(t - \tau)$. This system is manipulated under a parabolic potential with linear delay feedback. Since the force is linear, the steady-state distribution $P^{\infty}(x)$ of the system is Gaussian, i.e., $P^{\infty}(x) \propto \exp(-1/2x^\top \Phi^{-1}x)$. Here, $\Phi$ is the covariance matrix with elements $\Phi_{ij} = \phi_{ij}(0)$, and $\phi_{ij}(z) = (x_i(t)x_j(t+z))$ denotes the time-correlation function. The analytical form of the time-correlation function can be obtained for $|z| \leq \tau$, (see Appendix B 1). When $T \leq \tau$, $\langle \sigma_i \rangle$ can be calculated by using a path integral (see Appendix B 2)

$$ \langle \sigma_i \rangle = \frac{4A_{12}^2(1 - e^{-2cT})}{[(A_{11})^2 + A_{12}^2]e^{-2cT} - [(A_{11})^2 + A_{12}^2]}, $$

(37)

where $c = \sqrt{a^2 - b^2}$ and

$$ A_{11}^\pm = \frac{D}{2c} \times \frac{e^{\pm c\tau}}{a \cosh(c\tau) + c \sinh(c\tau)}, $$

$$ A_{12} = \frac{D}{2c} \times \frac{b}{a \cosh(c\tau) + c \sinh(c\tau)} $$

As can be seen, due to the time delay, $\langle \sigma_i \rangle$ is positive; thus, implying the time-reversal symmetry in the system is broken. Now, we validate the TUR for the following current-type observable

$$ j(X) = \int_0^T dt \left[ (-ax_1 + bx_2) \circ \dot{x}_1 + (-ax_2 - bx_1) \circ \dot{x}_2 \right]. $$

(39)

We consider only the $T \leq \tau$ case, where $\langle \sigma_i \rangle$ can be obtained analytically. The effective forces are also linear and can be calculated explicitly (see Appendix B 3)

$$ \overline{F}_1(x) = -\overline{a}x_1 + \overline{b}x_2, \overline{F}_2(x) = -\overline{a}x_2 - \overline{b}x_1, $$

(40)

where

$$ \overline{a} = \frac{c(a \cosh(c\tau) + c \sinh(c\tau))}{a \sinh(c\tau) + c \cosh(c\tau)}, $$

$$ \overline{b} = \frac{bc}{a \sinh(c\tau) + c \cosh(c\tau)}. $$

The average of the observable is then obtained

$$ \langle j \rangle = T \int dx \left[ (-ax_1 + bx_2)J_1^w(x) + (-ax_2 - bx_1)J_2^w(x) \right] = \frac{2DT\overline{b}^2}{a \cosh(c\tau) + c \sinh(c\tau)}, $$

(42)

which is always positive for arbitrary delay time $\tau$. Equation (42) reveals that increasing $D$, $T$, or $b$ all leads to a higher average of the current. We also consider a non-current observable $\tilde{j}(X) = \text{sign}[j(X)]$, which represents the sign of the observable $j$. This observable is obviously antisymmetric under time reversal. We define $\mathcal{E}_i = 2\langle \sigma_i \rangle^2 / [\text{Var}[j] (e^{\langle \sigma_i \rangle} - 1)]$ and $\mathcal{E}_i = 2\langle \tilde{j} \rangle^2 / [\text{Var}[\tilde{j}] (e^{\langle \sigma_i \rangle} - 1)]$, which should satisfy $\mathcal{E}_i \leq 1$ and $\mathcal{E}_i \leq 1$. We run numerical simulation with the same setting as in one-dimensional system. We plot $\mathcal{E}_i$ and $\mathcal{E}_i$ as functions of $\langle \sigma_i \rangle$ with circle and square points, respectively, in Fig. 2. As can be seen, all circle and square points lie below the dashed line; thus, implying that the derived TUR is empirically verified.

V. CONCLUSION

In summary, we derived the TUR for the time-delayed systems being in the steady state. We provided two bounds on the relative fluctuation of general dynamical observables that are antisymmetric under conjugate operations. For observables that are antisymmetric under time reversal, the fluctuation is lower bounded by $2/(e^{\langle \sigma_i \rangle} - 1)$, where $\langle \sigma_i \rangle$ can be considered a generalization of the total entropy production. On the other hand, the fluctuation of observables that are odd under position reversal is constrained by the same bound, in which $\langle \sigma_i \rangle$ is replaced by $\langle \sigma_p \rangle$, which reflects the position-symmetry breaking in the system. The results hold for an arbitrary observation time. Since the underlying dynamics of the systems is not required, the derived TUR holds for a large class of time-delayed systems, including both continuous-time and discrete-time systems. This generalization of the TUR for delayed systems can be used as a tool to estimate a hidden thermodynamic quantity in real-world systems that involve time delay from finite-time experimental data.

ACKNOWLEDGMENTS

This work was supported by MEXT KAKENHI Grant No. JP16K00325.

Appendix A: Scaled cumulant generating function of observables

Here we calculate the scaled cumulant generating function (SCGF) of the observable $j(X) = \int_0^T dt x$ in the
long-time limit $T \to \infty$. Note that $j = T \mathcal{J} + \int_0^T dt \, z$. By imposing periodic boundary conditions on the trajectories, $z(t)$ can be expanded in discrete Fourier series [51] as

$$z(t) = \sum_{n=-\infty}^{\infty} z_n e^{-i\omega_n t}, \quad (A1)$$

where the coefficient $z_n$ can be calculated via inverse transforms

$$z_n = \frac{1}{T} \int_0^T dt \, z(t) e^{i\omega_n t}, \quad (A2)$$

where $\omega_n = 2\pi n/T$. By substituting Eq. (A1) into the Langevin equation, we obtain

$$(a + be^{i\omega_n}) z_n = \xi_n, \quad (A3)$$

with $\langle \xi_n \xi_m \rangle = 2D/T \delta_{n,-m}$. The current $j$ then can be expressed as $j = T \mathcal{J} + T \mathcal{Z}_0 = T \mathcal{J} + T \mathcal{X}_0/(a+b)$. Substituting $j$ into the definition of the SCGF, we obtain

$$\chi_j(k) = \lim_{T \to \infty} T^{-1} \ln \langle \exp \{ kT (\mathcal{J} + \xi_0/(a+b)) \} \rangle$$

$$= kT + \lim_{T \to \infty} T^{-1} \ln \left( \int_{-\infty}^{\infty} \exp \{ kT \mathcal{X}_0/(a+b) \} \right), \quad (A4)$$

where

$$P(\xi_0) = \sqrt{\frac{T}{4\pi D}} \exp \left( -\frac{T \xi_0^2}{4D} \right). \quad (A5)$$

Taking the integration in Eq. (A4), we get $\chi_j(k) = kT + Dk^2/(a+b)^2$.

Appendix B: Detailed derivations in the two-dimensional system

1. Time-correlation function

Here we calculate the stationary time-correlation function $\phi_{ij}(z) = \langle x_i(t)x_j(t+z) \rangle$. Using the method in Ref. [44], for arbitrary $z \geq 0$, we have

$$\frac{d}{dz} \phi_{11}(z) = -a\phi_{11}(z) + b\phi_{21}(\tau - z) + \langle x_1(t)\xi_1(t+z) \rangle, \quad (B1)$$

$$\frac{d}{dz} \phi_{12}(z) = -a\phi_{12}(z) - b\phi_{11}(\tau - z) + \langle x_1(t)\xi_2(t+z) \rangle, \quad (B1)$$

$$\frac{d}{dz} \phi_{21}(z) = -a\phi_{21}(z) + b\phi_{22}(\tau - z) + \langle x_2(t)\xi_1(t+z) \rangle, \quad (B1)$$

$$\frac{d}{dz} \phi_{22}(z) = -a\phi_{22}(z) - b\phi_{12}(\tau - z) + \langle x_2(t)\xi_2(t+z) \rangle. \quad (B1)$$

From the Fokker-Planck equation, we have

$$0 = \frac{d}{dt} \langle x_1^2 \rangle = -2a\phi_{11}(0) + 2b\phi_{21}(\tau) + 2D. \quad (B2)$$

On the other hand, from Langevin equation we also obtain

$$0 = \frac{d}{dt} \langle x_1^2 \rangle = -2a\phi_{11}(0) + 2b\phi_{21}(\tau) + 2\langle x_1(t)\xi_1(t) \rangle. \quad (B3)$$

Comparing Eq. (B2) and Eq. (B3), we obtain the relation $\langle x_1(t)\xi_1(t) \rangle = D$. Similarly, we also get $\langle x_2(t)\xi_2(t) \rangle = D$. Using Fourier transform $g(t) = \int_{-\infty}^{\infty} dt e^{i\omega t} g(t)$ for an arbitrary function $g(t)$, we obtain the relation that $\mathbf{x}(\omega) = \mathbf{H}(\omega)\mathbf{\xi}(\omega)$. Here, $\mathbf{H}(\omega)$ is a response function matrix in the frequency domain, given by

$$\mathbf{H}(\omega) = \frac{1}{(a - i\omega)^2 + b^2 e^{2i\omega T}} \begin{pmatrix} a - i\omega & b e^{i\omega T} \\ -b e^{i\omega T} & a - i\omega \end{pmatrix}. \quad (B4)$$

The time-correlation function can be calculated via inverse Fourier transform of the spectral density $S(\omega)$, given by

$$S(\omega) = 2\mathbf{H}(\omega)D\mathbf{H}^*(\omega), \quad (B5)$$

where $D = \text{diag}(D, D) \in \mathbb{R}^{2 \times 2}$. Since $S_{11}(\omega) = S_{22}(\omega)$, $S_{12}(\omega) = S_{21}(\omega) = 0$, we readily obtain

$$\phi_{11}(z) = \phi_{22}(z), \phi_{12}(z) + \phi_{21}(z) = 0. \quad (B6)$$

Using the relations in Eq. (B6), we obtain that for $0 \leq z \leq \tau$

$$\frac{d^2}{dz^2} \phi_{11}(z) = \frac{d}{dz} \left[ -a\phi_{11}(z) + b\phi_{21}(\tau - z) \right] = (a^2 - b^2)\phi_{11}(z). \quad (B7)$$

The solution of time-correlation function $\phi_{11}(z)$ in Eq. (B7) has the following form:

$$\phi_{11}(z) = \alpha \cosh(cz) + \beta \sinh(cz), \quad (B8)$$

where $c = \sqrt{a^2 - b^2}$ and $\alpha, \beta$ are constants that determined via the conditions:

$$\frac{d}{dz} \phi_{11}(z) \bigg|_{z \to 0} = -D, \quad \phi_{11}(z) \bigg|_{z \to 0} = 0. \quad (B9)$$

Finally, we obtain that for $0 \leq z \leq \tau$

$$\phi_{11}(z) = \phi_{22}(z) = A_{11} e^{-cz} + A_{12} e^{cz}, \quad (B10)$$

$$\phi_{12}(z) = -\phi_{21}(z) = A_{12} (e^{-cz} - e^{cz}), \quad (B11)$$

where

$$A_{11} = \frac{D}{2c} \frac{e^{\pm a \cosh(\sigma T)}}{a \cosh(\sigma T) + c \sinh(\sigma T)}, \quad (B12)$$

$$A_{12} = \frac{D}{2c} \frac{1}{a \cosh(\sigma T) + c \sinh(\sigma T)}. \quad (B12)$$

Since $\phi_{11}(z)$ is an even function and $\phi_{12}(z)$ is an odd function, we have that $\phi_{11}(z) = \phi_{22}(z) = A_{11} e^{-|z|} + A_{12} e^{|z|}$ and $\phi_{12}(z) = -\phi_{21}(z) = A_{12} (e^{-|z|} - e^{|z|})$ for $|z| \leq \tau$. 

2. Path integral

Since the process is Gaussian, the path probability is given by

\[ \mathcal{P}(X) \propto \exp \left(-\frac{1}{2} \int_0^T dt \int_0^T dt' [x_1(t) x_2(t)] \begin{bmatrix} \Gamma_{11}(t, t') & \Gamma_{12}(t, t') \\ \Gamma_{21}(t, t') & \Gamma_{22}(t, t') \end{bmatrix} \begin{bmatrix} x_1(t') \\ x_2(t') \end{bmatrix} \right), \tag{B13} \]

where \( \Gamma_{ij}(t, t') \) is the inverse of the stationary time-correlation function \( \phi_{ij}(z) \) defined via the following relation:

\[ \int_0^T ds \begin{bmatrix} \phi_{11}(t-s) & \phi_{12}(t-s) \\ \phi_{21}(t-s) & \phi_{22}(t-s) \end{bmatrix} \begin{bmatrix} \Gamma_{11}(s, t') & \Gamma_{12}(s, t') \\ \Gamma_{21}(s, t') & \Gamma_{22}(s, t') \end{bmatrix} = \begin{bmatrix} \delta(t-t') & 0 \\ 0 & \delta(t-t') \end{bmatrix}. \tag{B14} \]

Now, we discretize the problem and take the continuum limit at the end. We divide the time interval \([0, T]\) into \( N \) equipartitioned intervals with a time step \( \epsilon = T/N \), where \( t_k = k\epsilon \) (\( k = 0, \ldots, N \)) and \( x^i_k = x_1(t_k) \), \( x^k_k = x_2(t_k) \) (superscripts denote points in a temporal sequence). Equation (B13) then reads

\[ \mathcal{P}(x^0_1, x^0_2, t_0; \ldots; x^N_1, x^N_2, t_N) \propto \exp \left(-\frac{1}{2} \sum_{i,j} \left[ x^i_1 \Gamma_{11}^{ij} x^j_1 + x^i_2 \Gamma_{12}^{ij} x^j_2 + x^{i+1}_1 \Gamma_{21}^{ij} x^j_1 + x^{i+1}_2 \Gamma_{22}^{ij} x^j_2 \right] \right), \tag{B15} \]

and Eq. (B14) corresponds to the following equation:

\[ \sum_{p=1}^2 \sum_{j=0}^N \phi_{mp}^{ij} \Gamma_{pm}^{jk} = \delta_{mn}\delta_{ik}, \tag{B16} \]

where \( \phi_{mp}^{ij} \equiv \phi_{mp}(t_j - t_i) \). The matrices \( \Gamma_{mn} \) (\( 1 \leq m, n \leq 2 \)) can be analytically calculated and has the following form:

\[ \begin{align*}
\Gamma_{11} &= \Gamma_{22}, \quad \Gamma_{12} = -\Gamma_{21}, \\
\Gamma_{11}^{0N} &= \Gamma_{11}^N = (A_{11}^- A_{11}^+ + A_{12}^2) e^{-N\epsilon e}, \\
\Gamma_{12}^{ij} &= 0, \quad \forall \ 1 < |i - j| < N, \\
\Gamma_{21}^{ij} &= (A_{11}^- A_{11}^+) [1 - e^{-2\epsilon e}], \quad \forall \ 1 < |i - j| = 1, \\
\Gamma_{11}^{i} &= (1 + e^{-2\epsilon e}) A_{11}^+ (A_{11}^- A_{11}^+), \quad \forall \ 0 < i < N, \\
\Gamma_{12}^{0} &= \Gamma_{12}^N = (A_{12}^+ A_{12}^-) e^{-N\epsilon e}, \\
\Gamma_{12}^{ij} &= 0, \quad \forall \ |i - j| \neq N.
\end{align*} \tag{B17} \]

Using the result in Eq. (B17), the quadratic form in Eq. (B15) can be obtained explicitly

\[ \begin{align*}
\sum_{i,j} &\left[ x^i_1 \Gamma_{11}^{ij} x^j_1 + x^i_2 \Gamma_{12}^{ij} x^j_2 + x^{i+1}_1 \Gamma_{21}^{ij} x^j_1 + x^{i+1}_2 \Gamma_{22}^{ij} x^j_2 \right] \\
&= \frac{1}{A_{11}^- - A_{11}^+} \left( \sum_{i=1}^N \sum_{k=1}^N \left( \frac{x^k_i e^{-\epsilon e x^k_i} - x^{k+1}_i}{1 - e^{-2\epsilon e}} \right)^2 - \frac{1}{\Omega_T} \sum_{i=1}^N \left[ A_{12} \left( e^{-N\epsilon e} x^0_i - x^N_i \right)^2 + \left( A_{11}^- e^{-N\epsilon e} x^0_i - A_{11}^+ x^N_i \right)^2 \right] \right) \\
&\quad - \frac{2A_{12} e^{-\epsilon N e}}{\Omega_T} \left( x^0_i x^N_i - x^0_i x^0_2 \right),
\end{align*} \tag{B18} \]

where \( \Omega_T = (A_{11}^+)^2 + A_{12}^2 - ((A_{11}^-)^2 + A_{12}^2) e^{-2\epsilon T} \). Taking the continuum limit \( \epsilon \to 0 \), \( N \to \infty \), with \( N\epsilon = T \) gives \[ \lim_{\epsilon \to 0} \frac{1}{\Omega_T} \sum_{k=1}^N \left( \frac{x^k_i - x^{k-1}_i}{1 - e^{-2\epsilon e}} \right)^2 = \frac{1}{2\epsilon} \int_0^T dt \left( \dot{x}_i(t) + cx_i(t) \right)^2. \tag{B19} \]
Finally, we obtain the expression of the path probability for \( T \leq \tau \):

\[
P(X) \propto \exp \left( -\sum_{i=1}^{2} \int_{0}^{T} dt \left[ \dot{x}_i(t) + c x_i(t) \right]^2 \frac{1}{4D} \right) \times \exp \left( \frac{c}{2D\Omega_T} \sum_{i=1}^{2} \left\{ A_{i2}^2 \left[ e^{-cT} x_i(0) - x_i(T) \right]^2 + \left[ \frac{A_{11}^T e^{-cT} x_i(0) - A_{1T} x_i(T)}{\Omega_T} \right]^2 \right\} \right) \times \exp \left( \frac{A_{12} e^{-cT}}{\Omega_T} \left[ x_1(0) x_2(T) - x_1(T) x_2(0) \right] \right) .
\]  

(20)

3. Analytical form of effective forces

We calculate the analytical form of the effective force \( \overline{F}_i(x) \) from its definition. We note that \( \overline{F}_i(x) \) cannot be completely determined from the steady-state FPE, i.e., \( \sum_{i=1}^{2} \partial_{x_i} \left[ \overline{F}_i(x) P(x, t) - D \partial_{x_i} P(x, t) \right] = 0 \). Specifically, if the effective force takes the form \( \overline{F}_i(x) = \sum_{j=1}^{2} \gamma_{ij} x_j \), then one obtains \( \gamma_{11} = \gamma_{22} = -D/\phi_{11}(0) \), \( \gamma_{12} + \gamma_{21} = 0 \). Here, we use the path integral to calculate \( \overline{F}_i(x) \). From the definition, we have

\[
\overline{F}_i(v) = \int du F_i(v, u) P(u, t - \tau | v, t) = \int du F_i(v, u) P(v, t; u, t - \tau) / P(v, t)
\]

(21)

where the integration is taken over all trajectories \( X \) that starts from \( u \) at time \( t - \tau \) and ends in \( v \) at time \( t \). The first term in the path probability can be simplified further by using the well-known expression of the transition probability for the Smoluchowski processes [52]

\[
\int_{x(0)}^{x(\tau)} D X \exp \left( -\sum_{i=1}^{2} \int_{0}^{T} dt \left[ \dot{x}_i(t) + c x_i(t) \right]^2 \frac{1}{4D} \right) \propto \exp \left( -c \frac{\left[ x(\tau) - x(0) e^{-c \tau} \right]^2}{2D} \right).
\]  

(22)

Consequently, we obtain

\[
\overline{F}_i(v) = \int du \frac{F_i(v, u)}{P(v, t)} G(u, v),
\]  

(23)

where

\[
G(u, v) \propto \exp \left( -\frac{c}{2D} \frac{\left\| v - uc e^{-c \tau} \right\|^2}{1 - e^{-2c \tau}} + \frac{c}{2D\Omega_T} \left[ A_{12}^2 \left\| e^{-c \tau} u - v \right\|^2 + \left\| A_{11}^T e^{-c \tau} u - A_{1T} v \right\|^2 \right] - \frac{A_{12} e^{-c \tau}}{\Omega_T} \left[ u_1 v_2 - u_2 v_1 \right] \right).
\]  

(24)

Taking the integration in Eq. (23), we obtain

\[
\overline{F}_1(x) = -\frac{c}{a \sinh(c \tau) + c \cosh(c \tau)} x_1 + \frac{bc}{a \sinh(c \tau) + c \cosh(c \tau)} x_2,
\]

\[
\overline{F}_2(x) = -\frac{c}{a \sinh(c \tau) + c \cosh(c \tau)} x_2 - \frac{bc}{a \sinh(c \tau) + c \cosh(c \tau)} x_1.
\]  

(25)

[1] K. Sekimoto, Prog. Theor. Phys. Supp. 130, 17 (1998).
[2] D. J. Evans and D. J. Searles, Adv. Phys. 51, 1529 (2002).
[3] U. Seifert, Phys. Rev. Lett. 95, 040602 (2005).
[4] U. Seifert, Rep. Prog. Phys. 75, 126001 (2012).
[5] Y. D. Decker, Physica A 428, 178 (2015).
[6] J. Fuchs, S. Goldt, and U. Seifert, EPL 113, 60009 (2016).
[7] Z. Gong, Y. Lan, and H. T. Quan, Phys. Rev. Lett. 117, 180603 (2016).
[8] S. Goldt and U. Seifert, Phys. Rev. Lett. 118, 010601 (2017).
[9] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995).
[10] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997).
[11] A. C. Barato and U. Seifert, Phys. Rev. Lett. **114**, 158101 (2015).
[12] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Phys. Rev. Lett. **116**, 120601 (2016).
[13] P. Pietzonka, A. C. Barato, and U. Seifert, Phys. Rev. E **93**, 052145 (2016).
[14] M. Polettini, A. Lazarescu, and M. Esposito, Phys. Rev. E **94**, 052104 (2016).
[15] J. M. Horowitz and T. R. Gingrich, Phys. Rev. E **96**, 020103 (2017).
[16] A. C. Barato, R. Chetrite, A. Faggionato, and D. Gabrielli, J. Phys. Chem. B **119**, 6555 (2015).
[17] P. Pietzonka, A. C. Barato, and U. Seifert, J. Stat. Mech. **2016**, 124004 (2016).
[18] Y. Hasegawa, Phys. Rev. E **98**, 032405 (2018).
[19] A. C. Barato and U. Seifert, J. Phys. Chem. B **119**, 6555 (2015).
[20] G. Falasco, R. Pfaller, A. P. Bregulla, F. Cichos, and K. Kroy, Phys. Rev. E **94**, 030602 (2016).
[21] C. Gupta, J. M. L´ opez, W. Ott, K. c. v. Josi´ c, and M. R. Bennett, Phys. Rev. Lett. **111**, 058104 (2013).
[22] B. Novák and J. J. Tyson, Nat. Rev. Mol. Cell Biol. **9**, 981 (2008).
[23] S. Kim, S. H. Park, and H.-B. Pyo, Phys. Rev. Lett. **82**, 1620 (1999).
[24] T. D. Frank, Phys. Rev. E **72**, 011112 (2005).
[25] T. D. Frank, P. J. Beek, and R. Friedrich, Phys. Rev. E **68**, 021912 (2003).
[26] N. Merhav and Y. Kafri, J. Stat. Mech: Theory Exp. **2010**, P12022 (2010).
[27] G. M. Rotskoff, Phys. Rev. E **95**, 030101 (2017).
[28] J. P. Garrahan, Phys. Rev. E **95**, 032134 (2017).
[29] T. R. Gingrich and J. M. Horowitz, Phys. Rev. Lett. **119**, 170601 (2017).
[30] K. Proesmans and C. V. den Broeck, EPL **119**, 20001 (2017).
[31] D. Chiuchi` u and S. Pigolotti, Phys. Rev. E **97**, 032100 (2018).
[32] K. Brandner, T. Hanazato, and K. Saito, Phys. Rev. Lett. **120**, 090601 (2018).
[33] W. Hwang and C. Hyeon, J. Phys. Chem. Lett. **9**, 513 (2018).
[34] A. C. Barato, R. Chetrite, A. Faggionato, and D. Gabrielli, New J. Phys. **20**, 103023 (2018).