Dyonic Kerr-Newman black holes, complex scalar field and Cosmic Censorship

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Abstract

We construct a gedanken experiment, in which a weak wave packet of the complex massive scalar field interacts with a four-parameter (mass, angular momentum, electric and magnetic charges) Kerr-Newman black hole. We show that this interaction cannot convert an extreme the black hole into a naked singularity for any black hole parameters and any generic wave packet configuration. The analysis therefore provides support for the weak cosmic censorship conjecture.

1 Introduction

A black hole is an object surrounded by an event horizon, i.e. a surface with the property that observers in the exterior cannot receive signals from events in the interior. This is the essential feature of black holes, i.e. what makes them “black”.

Black holes also harbor spacetime singularities; in some sense, this is what makes them ”holes”. The singularity of any black hole is inside —more precisely, to the future of— its event horizon.

If a spacetime features a singularity that is not hidden behind an event horizon, i.e. far-away observers can receive signals from it, the singularity
is said to be “naked”. Since initial conditions cannot be specified at a singularity —put another way, anything can come out of the singularity—a naked singularity would prevent predictability in a spacetime. Therefore, it is conjectured that all singularities are hidden behind event horizons; more precisely, “naked singularities” cannot be produced from regular initial conditions and reasonable matter properties—the “cosmic censorship” conjecture (CCC) of Penrose \cite{1}. This form was later renamed as the weak cosmic censorship conjecture (WCCC)\cite{1}.

In the absence of a general proof, gedanken- and numerical experiments have been devised to check the validity of the cosmic censorship conjecture (CCC) under different limited circumstances, by studying the evolution of various initially regular systems to see if a naked singularity develops \cite{3,4,5}. Black holes are obvious subjects for such investigations.

According to the well-known “no-hair” theorem \cite{6,7}, of classical General Relativity, stationary black holes in asymptotically flat space can be described uniquely by three variables (Mass $M$, charge $Q$ and angular momentum per unit mass, $a$), hence by the Kerr-Newman metric \cite{8}. This spacetime has a horizon —therefore describes a black hole— if and only if

$$M^2 \geq Q^2 + a^2.$$ \hfill (1)

If (1) is not satisfied, the metric describes a naked singularity. Therefore one check for the CCC would be to ask if a Kerr-Newman spacetime can somehow evolve from a form satisfying (1) to one that does not. The first such thought experiment was constructed by Wald, who showed \cite{9} that one cannot overcharge and/or overspin an extreme black hole [i.e. one saturating the inequality (1)] by throwing in test point particles with electric charge and/or spin.

However, black holes could also have a magnetic charge, if such existed. A particle with electric charge $Q_e$ and magnetic charge $Q_m$ is called a dyon, and the interest in the possibility of dyonic black holes has grown since magnetic monopoles have been predicted in various extensions of the standard model of particle physics. The metric of such a black hole is identical to the Kerr-Newman metric with the replacement $Q^2 \rightarrow Q_e^2 + Q_m^2$, as can be seen by noting that the energy-momentum tensor is invariant under the

\footnote{Penrose later introduced the strong CCC, that all singularities should be spacelike or null \cite{2}. The terminology is somewhat misleading, however, since neither statement implies the other. In this work, we do not deal with the strong CCC.}
electromagnetic duality transformation, which also gives the electromagnetic field/potential of the dyonic black hole [10]. Wald’s argument was generalized (for spinless test particles) to the case of the dyonic black hole by Hiscock [11] and independently, by Semiz [10].

The problem of trying to overcharge a maximally charged black hole with another kind of charge was revisited by Bekenstein & Rosenzweig [12], where nonzero size of the particle — classical and quantum — is pointed out and argued to save CCC. Hod [13] considered lowering the differently-charged particle slowly towards the horizon (to minimize its energy), and argued that the finite size of the particle helps preserving the horizon in this case too, aided by the polarization of the black hole.

Thought experiments that do not use the electric black hole-magnetic test particle combination and duality-rotated cases include that of Ford & Roman [14], who considered sending a flux of negative energy down an extreme Reissner-Nordström (R-N) black hole by using a quantized massless scalar field. These negative energy fluxes are allowed in Quantum Field Theory, but the authors find that the exposure of the naked singularity can be only temporary (“Cosmic Flashing”) and that the effect probably gets swamped by quantum fluctuations. Jensen [15], analyzed the effect of the collapse of a spherical domain-wall with negative energy onto an extreme Reissner-Nordström (R-N) black hole and decided that the WCCC remains valid in this case. Hubeny [16] argued that one can overcharge a R-N black hole by starting not from extreme, but slightly sub-extreme condition, i.e. one may “jump over” the extreme condition. Hod [17] made the same claim for general black holes while pointing out that curved-space modifications to particle self-energy may be relevant. The concept of jumping over the extreme condition was also treated by Jacobson & Sotiriou [18]. Hod & Piran [19] suggested that the WCCC can be violated classically, but may be saved by Quantum Gravity. de Felice & Yu [20] calculated that a naked singularity can be produced by throwing a fast-spinning flat disk into an extreme R-N black hole. More recently, Matsas et. al. [21, 22] made the same claim for quantum tunneling of low-energy, high-angular-momentum massless scalar particles; and the claim was disputed by Hod [23, 24]. Bouhmadi-López et. al. [25] generalized Wald’s thought experiment to higher-dimensional black

\footnote{However, [11] claims that the test particle approximation is not valid everywhere in the parameter space, and that inequality (1), therefore the CCC can be violated, e.g. for the interaction of an electric test particle and an extreme magnetic black hole.}


holes and black rings.

It has also been argued [26] that the extreme black hole, which was the starting point of some of the above-mentioned works, can not be reached starting from a non-extreme black hole, which seems to be in accord with the claims [27, 28] that the extreme black holes are not the limit of normal black holes; in particular, that the entropy of the extreme black hole is zero, contrary to the Bekenstein-Hawking formula.

The status of the CCC is currently unresolved. As mentioned above, no general proof exists. Some numerical experiments, e.g. [29], and some of the above-mentioned references suggest its violation, but these violations are thought to be “non-generic” or possibly “of measure zero” [3], especially since what constitutes “reasonable matter properties” is not clear.

In the present work, we consider another thought experiment to check the validity of WCCC: We try to see if a fully general (dyonic Kerr-Newman) classical extreme black hole can be pushed beyond extremality by absorbing electric charge and/or angular momentum from a fully general (complex, massive) classical scalar test field.

We require the scalar field to have finite energy (e.g. constitute a wave packet). Initially \((t \rightarrow -\infty)\), there is just the black hole, and no scalar field. Then the field comes in from infinity, part of it gets swallowed by the black hole, part of it gets reflected back to infinity; energy, charge and angular momentum get transferred. As \(t \rightarrow \infty\), the scalar field decays away, and by the no-hair theorem, the geometry becomes again a Kerr-Newman geometry, but with new values of \(M\), \(Q\), and \(a\).

We calculate the changes of the black hole parameters by first constructing expressions for fluxes of charge, angular momentum and energy carried by the scalar field directly and by its accompanying electromagnetic field, then evaluating them to the lowest nontrivial order in a perturbation expansion around the initial Kerr-Newman configuration. We ask if –in this thought experiment and to this order– inequality (1), therefore the Weak Cosmic Censorship Conjecture can be violated for any combination of black hole parameters.
2 Changes in the mass, electric charge and angular momentum of the black hole

The Kerr-Newman metric describing the four-parameter black hole is given by

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{a^2 \sin^2 \theta - \Delta}{\rho^2} dt^2 + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\rho^2} \sin^2 \theta d\phi^2 \]
\[ + 2 \frac{\Delta - (r^2 + a^2)}{\rho^2} a \sin^2 \theta dt d\phi + \rho^2 \Delta^{-1} dr^2 + \rho^2 d\theta^2 \]  

(2a)

where

\[ \rho^2 = r^2 + a^2 \cos^2 \theta \]  
\[ \Delta = r^2 - 2Mr + a^2 + Q_e^2 + Q_m^2 \]  

(2b)
(2c)

and we are using MTW [30] notation and conventions.

The vector potential describing the electromagnetic field of the black hole is

\[ A_t = -Q_e \frac{r}{\rho^2} + Q_m \frac{a \cos \theta}{\rho^2} \]  
\[ A_r = A_\theta = 0 \]  
\[ A_\phi = Q_e \frac{ar \sin^2 \theta}{\rho^2} + Q_m [\pm 1 - \cos \theta \frac{r^2 + a^2}{\rho^2}] \]  

(3a)
(3b)
(3c)

The two signs in the last term correspond to the two gauges used to avoid the “Dirac string” in the magnetic part of the vector potential, as described in [10]. A wavefunction in this formalism is called a section [32, 31].

To calculate the changes in the black hole parameters, we start from the associated conservation laws. As is well known, the energy and angular momentum conservation laws are consequences of spacetime symmetries and locally take the form of continuity equations

\[ (T^{\mu\nu} X_\nu)_{;\mu} = 0 \]  

(4)

where \( T^{\mu\nu} \) is the energy-momentum tensor for test particles or fields and \( X^\nu \) is the Killing vector generating the symmetry.
We also have the current continuity equation
\[ (\sqrt{-g} j^\nu)_{,\nu} = 0 \] (5)
where \( j^\nu \) is the current density four-vector.

The Kerr-Newman metric is independent of \( t \) and \( \phi \), therefore its Killing vectors are \( \frac{\partial}{\partial x^0} \) and \( \frac{\partial}{\partial x^3} \). For the first of these, integrating (4) over a spacelike volume we get
\[
\frac{d}{dx^0} \int_V (\sqrt{-g} T^0_0) \, d^3x = -\int_V (\sqrt{-g} T^0_0)_{,4} \, d^3x = -\int_S \sqrt{-g} T^0_i \, dS_i \] (6)
where \( S \) is the surface (boundary) of the volume \( V \).

The mass of a physical system is defined only in the asymptotically flat part of space at infinity, if it exists \[30, \S 19.3, \S 20.2\]; and the energy density is \(-T^0_0\) \[30,\text{eq.}(5.11)\]. Therefore we have
\[
\left( \frac{dM}{dt} \right)_{b.h.} = \int_{S_\infty} \sqrt{-g} T^0_0 \, d\theta d\phi \] (7)
where the label b.h. stands for black hole, \( S_\infty \) is the spherical surface as \( r \to \infty \) (cf. \[30, \text{eq.}(20.27)\)). Similarly, eqs. (4) and (5) lead to
\[
\left( \frac{dL}{dt} \right)_{b.h.} = -\int_{S_\infty} \sqrt{-g} T^3_1 \, d\theta d\phi \] (8)
\[
\left( \frac{dQ}{dt} \right)_{b.h.} = -\int_{S_\infty} \sqrt{-g} j^1 \, d\theta d\phi \] (9)

To evaluate \( T_{\mu\nu} \) for our problem, we start from the Lagrangian of the combined scalar and electromagnetic fields:
\[
\mathcal{L} = \frac{1}{8\pi} [ -g^{\alpha\beta} (\partial_\alpha - ieA_\alpha) \psi^*(\partial_\beta + ieA_\beta) \psi - \mu^2 \psi^* \psi ] - \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \] (10)
The energy-momentum tensor is defined by \[30, \S 21.3\]
\[
T_{\mu\nu} = -2 \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L} \] (11)
hence
\[
T^\nu_\mu = \frac{1}{8\pi} [ (\partial_\mu - ieA_\mu) \psi^*(\partial^\nu + ieA^\nu) \psi + (\partial^\nu - ieA^\nu) \psi^*(\partial_\mu + ieA_\mu) \psi ] + \frac{1}{4\pi} F_{\mu\beta} F^{\nu\beta} + \delta^\nu_\mu \mathcal{L}. \] (12)
Also, the current density \( j^1 \) is given by

\[
j^1 = \frac{ie}{8\pi}[\psi^* (\partial^1 + ieA^1)\psi - \psi (\partial^1 - ieA^1)\psi^*].
\] (13)

We assume deviations from background are small, (‘test field’) and expand the fields in some small parameter \( \epsilon \). The background is the dyonic Kerr-Newman spacetime, therefore for the zeroth order fields we have \( \psi^{(0)} = 0, \quad F_{\mu\nu}^{(0)} \neq 0 \implies A_{\mu}^{(0)} \neq 0 \).

From the Lagrangian (10) we derive the equations of motion for the fields. The equation of motion for \( \Psi \) gives to first order (the zeroth order part has no content, \( 0 = 0 \))

\[
\frac{1}{\sqrt{-g}}(\partial_{\mu} + ieA_{\mu}^{(0)})[\sqrt{-g}(\partial^{\mu} + ieA^{(0)\mu})]\psi^{(1)} = \mu^2 \psi^{(1)}
\] (14)

Similarly, the equation of motion for \( F^{\mu\nu} \) gives to zeroth and first orders

\[
F^{(0)\alpha\beta} = 0, \quad F^{(1)\alpha\beta} = 0
\] (15)

and to second order

\[
F^{(2)\alpha\beta}_{\ : \beta} = 4\pi \left\{ \frac{ie}{8\pi} [\psi^{* (1)} (\partial^\alpha + ieA^{(0)\alpha})\psi^{(1)} - \psi^{(1)} (\partial^\alpha - ieA^{(0)\alpha})\psi^{* (1)}] \right\} = 4\pi j^{(2)\alpha}
\] (16)

We require there to be no photons. Since there is no source for the electromagnetic field to zeroth or first order, this requirement, together with boundary conditions at infinity, makes \( F^{(0)\alpha\beta} + F^{(1)\alpha\beta} \) the dyonic Kerr-Newman solution. Since \( F^{(0)\alpha\beta} \) is the background, and therefore already is Kerr-Newman, we have \( F^{(1)\alpha\beta} = 0 \implies A_{\mu}^{(1)} = 0 \) and \( F^{(2)\alpha\beta}_{\ : \beta} = 4\pi j^{(2)\alpha} \), with \( j^{(2)\alpha} \) is made up of \( \psi^{(1)} \)'s, as above.

The lowest-order contributions to \( T_0^1, T_3^1 \), and \( j^1 \) are second order in \( \epsilon \): Terms with \( \psi, \psi^* \) and their derivatives are obvious. \( FF \) terms also vanish to zeroth order, a result one can verify by calculation and understand physically: There are no energy, or angular momentum flows on the exact Kerr-Newman metric. \( FF \) terms also vanish to first order, because \( F^{(1)\alpha\beta} = 0 \). Therefore the lowest order contributions to \( dM/dt, dL/dt, \) and \( dQ_\epsilon/dt \) [given in eqs. (7-9)] are second order in \( \epsilon \).
3 Treating separated equations

As shown in [32], equation (14) can be separated in Boyer-Lindquist coordinates by

\[ \psi^{(1)} = R(r)\Theta(\theta)e^{-i\omega t}e^{i(m\mp eQ_m)\phi}. \]  (17)

The angular equation

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left[ a^2(\omega^2 - \mu^2) \cos^2 \theta - \frac{(m - eQ_m \cos \theta)^2}{\sin^2 \theta} - 2\omega eQ_m \cos \theta + \lambda \right] \Theta = 0 \]  (18)

constitutes a Sturm-Liouville problem, therefore the solutions form a complete set and can be orthonormalized. Labeling the eigenvalue \( \lambda \) by \( l \), we choose

\[ \int Y_{qlm'}(a, \mu, \omega, \theta, \phi)Y_{qlm}(a, \mu, \omega, \theta, \phi)d\Omega = \delta_{ll'}\delta_{mm'} \]  (19)

where

\[ Y_{qlm}(a, \mu, \omega, \theta, \phi) = \Theta_{qlm}(a, \mu, \omega, \theta)e^{i(m\mp eQ_m)\phi} \]  (20)

are functions we had decided to call monopole spheroidal harmonics [32].

The radial equation is

\[ \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + \left\{ \frac{1}{\Delta} [(r^2 + a^2)\omega + eQ_\epsilon r - am]^2 - \mu^2 r^2 + 2\omega eQ_m \cos \theta + 2\omega eQ_m \cos \theta + \lambda \right\} R = 0 \]  (21)

Switching the independent variable to the ‘tortoise’ coordinate \( r^* \) defined by

\[ \frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta} \Rightarrow \Delta \frac{d}{dr} = (r^2 + a^2) \frac{d}{dr^*} \]  (22)

and substituting

\[ R(r) = \frac{U(r)}{\sqrt{r^2 + a^2}} \]  (23)

for the dependent variable, we get after extensive manipulation

\[ \frac{d^2}{dr^{*2}} U + \left\{ \frac{(r^2 + a^2)\omega + eQ_\epsilon r - am}{r^2 + a^2} \right\}^2 \right. \]

\[ - \frac{\Delta(\mu^2 r^2 - 2a\omega \mu + a^2 \omega^2 + \lambda)}{(r^2 + a^2)^2} - \frac{\Delta^2(a^2 - 2r^2)}{(r^2 + a^2)^4} - \frac{2\Delta(r - M)}{(r^2 + a^2)^3} \} U = 0 \]  (24)
which near the horizon \((r \to r_+, r^* \to -\infty)\) reduces to (easily seen, since \(r_+\) is the larger root of \(\Delta\))

\[
\frac{d^2 U}{dr^{*2}} + \left[ \frac{(r_+^2 + a^2)\omega + eQe_r - am}{r_+^2 + a^2} \right]^2 U = 0 = \frac{d^2 U}{dr^{*2}} + \bar{\omega}^2 U = 0 \tag{25}
\]

(where \(\bar{\omega}\) has been defined in the equation) and near infinity (as \(r^* \to r \to \infty\)) becomes

\[
\frac{d^2 U}{dr^{*2}} + (\omega^2 - \mu^2) U = 0 \tag{26}
\]

That is, in terms of \(U\) and \(r^*\), the radial problem at a given frequency looks like a one-dimensional Schrödinger scattering problem with two (different) constant potentials at two ends and a complicated potential well in the middle.

The solutions for \(U\) near the horizon are, from eq.\((25)\),

\[
U = B e^{\pm i\bar{\omega}r^*} \tag{27}
\]

and near infinity, from eq.\((26)\),

\[
U = e^{\pm ikr^*} \text{ for } \omega^2 > \mu^2 , \text{ where } k^2 = \omega^2 - \mu^2 \tag{28}
\]

and

\[
U = e^{\pm \kappa r^*} \text{ for } \omega^2 < \mu^2 , \text{ where } \kappa^2 = \mu^2 - \omega^2 \tag{29}
\]

For \(\omega^2 < \mu^2\), the exponentially growing solution \(e^{\kappa r^*}\) is unphysical, and the solution \(e^{-\kappa r^*}\) corresponds to a bound particle and vanishes exponentially at infinity, and therefore will not contribute to our integrals for \(dM/dt\), \(dQ/dt\), \(dL/dt\).

For \(\omega^2 > \mu^2\), the outgoing solution is unphysical at the future event horizon: It varies infinitely fast as seen by an ingoing, future-directed observer. We are left with the ingoing solution:

\[
\lim_{r \to r_+} U_{\omega ml}(r^*) = B_{\omega ml} e^{-i\bar{\omega}r^*} \tag{30}
\]

and for the same mode,

\[
\lim_{r \to \infty} U_{\omega ml}(r^*) = e^{-ikr^*} + A_{\omega ml} e^{ikr^*} \tag{31}
\]

where the solution has been normalized such that the coefficient of the first term on the right-hand side of the last equation is unity. In other words, this
corresponds to a wave of unit amplitude coming in from infinity, being transmitted into the black hole with amplitude $B$ and reflected back to infinity with amplitude $A$.

Since eq. (24) is real, the complex conjugate of any solution is also a solution, and since (24) has no first derivative, the Wronskian of the two solutions $U$ and $U^*$

$$W = U \frac{dU^*}{dr^*} - U^* \frac{dU}{dr^*}$$  \tag{32}$$
is constant; therefore its values at the two limits can be put equal to each other, giving after a few lines

$$\lim_{r \to r_+} W = \bar{\omega} B B^* = \lim_{r \to \infty} W = k (1 - AA^*)$$  \tag{33}$$

where the labels $\omega lm$ are implied for the relevant variables in the last two equations.

### 4 Testing the Cosmic Censorship Conjecture

To test the inequality (1), let us define a cosmic censorship indicator,

$$CCC = M^2 - (Q^2 + a^2)$$  \tag{34}$$

Then, using $dQ_m/dt = 0$ and $a = L/M$, and implicitly defining $dC/dt$,

$$\delta (CCC) = \int \frac{dC}{dt} dt = \frac{1}{M} \left\{ (M^2 + a^2) \frac{dM}{dt} - MQ_e \frac{dQ_e}{dt} - a \frac{dL}{dt} \right\} dt.$$  \tag{35}$$

Recalling the results of section 2 we have to second order in $\epsilon$

$$\frac{dC}{dt} = \int_{S_\infty} \sqrt{-g} [(M^2 + a^2) T^{1\theta} + MQ_e j^1 + a T^{1\phi}] d\theta d\phi$$  \tag{36}$$

In the pure electromagnetic contribution to $dC/dt$ (the part coming from the $F_{\mu\beta} F^{\nu\beta}$ term in $T_{\mu \nu}$, see eq. (12)) we write $F_{\mu\beta}$ in terms of $A_{\mu}$:

$$\left( \frac{dC}{dt} \right)_{em} = \frac{1}{2\pi M} \int_{S_\infty} \sqrt{-g} \{ (M^2 + a^2) (A_{\alpha,0} - A_{0,\alpha}) F^{1\alpha} + a [A_{\alpha,3} - (A_3 + Q_m)_{,\alpha}] F^{1\alpha} \} d\theta d\phi$$  \tag{37}$$
where the $\mp Q_m$ was added to $A_3$ for later convenience. Rewriting the $\alpha$-derivative terms as integrals by part, we get

$$\left(\frac{dC}{dt}\right)_{em} = \frac{1}{2\pi M} \int_{S_{\infty}} \left\{ (M^2 + a^2)[\sqrt{-g}A_{\alpha,0}F^{1\alpha} - (\sqrt{-g}A_0F^{1\alpha})_{,\alpha} + A_0(\sqrt{-g}F^{1\alpha})_{,\alpha}] \right. $$
$$+ a \left[\sqrt{-g}A_{\alpha,3}F^{1\alpha} - [(A_3 \mp Q_m)\sqrt{-g}F^{1\alpha}]_{,\alpha} + (A_3 \mp Q_m)(\sqrt{-g}F^{1\alpha})_{,\alpha}\right\} d\theta d\phi (38)$$

In the second and fifth terms, the $\alpha = 2$ parts vanish when integrated over $\theta$, because of $\sin \theta$ in

$$\lim_{r \to \infty} \sqrt{-g} = r^2 \sin \theta, \quad (39)$$

the $\alpha = 3$ parts vanish when integrated over $\phi$, because $\phi$ is cyclic. In the third and sixth terms, $(\sqrt{-g}F^{1\alpha})_{,\alpha} = 4\pi \sqrt{-g} j^1$. Also, the lowest nonvanishing contributions to $A_{\alpha,0}$, $A_{\alpha,3}$ and $j^1$ are of second order, so we get for the pure electromagnetic contribution to $\delta(CCC)$

$$\delta(CCC)_{em} = \frac{1}{2\pi M} \int_{-\infty}^{\infty} \int_{S_{\infty}} \left\{ (M^2 + a^2)[\sqrt{-g}F^{(0)1\alpha}A^{(2)}_{\alpha,0} - (\sqrt{-g}A_0F^{10})_{,0} + 4\pi A_0^{(0)}(\sqrt{-g}F^{10})_{,0}] \right. $$
$$+ a \left[\sqrt{-g}F^{(0)1\alpha}A^{(2)}_{\alpha,3} - [(A_3 \mp Q_m)\sqrt{-g}F^{10}]_{,0} + 4\pi \sqrt{-g}(A_3^{(0)} \mp Q_m)j^{(2)1}\right\} d\theta d\phi dt (40)$$

$F^{(0)1\alpha}$ is a part of the background field, therefore independent of time, so the time integral can be done for the first term. The second and fifth terms are now total time derivatives, so those time integrals can be done also. On $S_{\infty}$, $A_0^{(0)}$ goes to $-Q_e/r$, therefore becomes angle-independent, so the angular integral can be done in the third term, giving $dQ_e/dt$ via eq.(37). $F^{(0)1\alpha}$ is also $\phi$-independent, so the $\phi$-integral in the fourth term applies to $A^{(2)}_{\alpha,3}$ and vanishes because $\phi$ is cyclic. So $\delta(CCC)_{em}$ becomes

$$\frac{1}{2\pi M} \left\{ \int_{S_{\infty}} \sqrt{-g}(M^2 + a^2)^1_{\alpha}A_{\alpha,0}^{(2)} \big|_{t = -\infty} d\theta d\phi - \int_{S_{\infty}} \sqrt{-g}(M^2 + a^2)A_0F^{10} \big|_{t = -\infty} d\theta d\phi \right.$$  
$$- 4\pi(M^2 + a^2)\frac{Q_e}{r} \int_{-\infty}^{\infty} dQ_e \frac{dt}{dt} - \int_{S_{\infty}} a[\sqrt{-g}(A_3 \mp Q_m)F^{10}]_{t = -\infty} d\theta d\phi$$
$$+ 4\pi a \int_{-\infty}^{\infty} \int_{S_{\infty}} \sqrt{-g}(A_3^{(0)} \mp Q_m)j^{(2)1} d\theta d\phi d\phi \right\} (41)$$

At $t = -\infty$, all second order terms are zero. As $t \to \infty$, they reduce to Kerr-Newman type fields, with $Q_m^{(2)} \to 0$, $Q_e^{(2)} \to \delta Q_e$. Therefore as $t \to \infty$, $F^{(0)10} \propto 1/r^2$, $A_0^{(2)} \propto 1/r$; $F^{(0)13} \propto 1/r^4$, $A_3^{(2)} \propto r^0$ on $S_{\infty}$, which makes the first two time-integrated terms vanish. The angle-integrated term is simply
$4\pi (M^2 + a^2) \frac{2}{r} \delta Q e$ and therefore also vanishes on $S_\infty$. The fourth term becomes

$$-\frac{a}{2\pi M} \int r^2 \sin \theta \ Q_m (-\cos \theta) (-\frac{\delta Q_e}{r^2}) d\theta \ d\phi$$

and vanishes by $\theta$-integration. Therefore

$$\delta(CCC)_{em} = \frac{2a}{M} \int_{-\infty}^{\infty} \int_{S_\infty} \sqrt{-g} (A_3^{(0)} \mp Q_m) j^{(2)} d\theta \ d\phi \ dt \quad (42)$$

Now substituting eqs (12) and (13) into the remaining terms of $\delta(CCC)$, and combining (42) with the second term of (36),

$$\delta(CCC) = \frac{1}{4\pi M} \int_{-\infty}^{\infty} \int_{S_\infty} \sqrt{-g} \left\{ \left[ (M^2 + a^2) (\partial_0 - ieA_0) \psi^* (\partial^1 + ieA^1) \psi \right. \right.$$

$$+ a \left. \frac{8\pi}{\partial_3 - ieA_3} \psi^* (\partial^1 + ieA^1) \psi + \frac{M \pi}{8\pi} \frac{aQ_e + a(A_3 \mp Q_m)}{ie\psi^* (\partial^1 + ieA^1) \psi} \right\} \right. \left. d\theta \ d\phi \ dt \quad (43)$$

where c.c. denotes the complex conjugate of the expression that precedes it and the $A$'s are all zeroth order now. Then we observe that $A^1 = 0$; factor out $\partial^1 \psi$ and note that the $A_3$ terms cancel. So

$$\delta(CCC) = \frac{1}{4\pi M} \int_{-\infty}^{\infty} \int_{S_\infty} \sqrt{-g} \left\{ \left[ (M^2 + a^2) (\partial_0 - ieA_0) + a\partial_3 + ie(MQ_e + aQ_m) \right] \psi^* \partial^1 \psi \right.$$

$$+ \left. c.c. \right\} d\theta \ d\phi \ dt \quad (44)$$

Since $e^{-i\omega t}$ and the monopole spheroidal harmonics $Y_{qlm}(a, \mu, \omega, \theta, \phi)$ form complete sets, we can expand the wavefunction $\psi$ in terms of the eigenfunctions:

$$\psi = \sum_{l,m} d\omega \ f_{lm}(\omega) e^{-i\omega t} Y_{qlm}(a, \mu, \theta, \phi) R_{lm}(a, \mu, \omega, r) \quad (45)$$

where $f_{lm}(\omega)$ are arbitrary coefficients.

Since the integrals are taken on $S_\infty$, we can use the limiting form of $\psi$:

$$\lim_{r \to \infty} \psi = \sum_{l,m} d\omega \ f_{lm}(\omega) \left\{ e^{-i\omega t} \Theta_{qlm}(a, \mu, \omega, \theta) e^{i(m+eQ_m)\phi} \frac{1}{r} (e^{-ikr} + A_{wlme^{ikr}}) \right\} \quad (46)$$

where the integration range is understood to exclude $-\mu < \omega < \mu$ and we have also separated the $Y_{qlm}$’s into $\theta$- and $\phi$-dependent parts. Substituting
the above into $\delta(\text{CCC})$, and also noting (39),

$$\delta(\text{CCC}) = \frac{1}{4\pi M} \int_{-\infty}^{\infty} \int_{\infty} \frac{r^2 \sin \theta \sum_{l,m} \sum_{l',m'} \int d\omega d\omega' f_{lm}(\omega) f_{l'm'}^{*}(\omega')}$$

$$\left\{ \left[ (M^2 + a^2)(i\omega' - i\omega A) + i\omega (m' \mp eQ_m) + i\epsilon (MQ_e \mp aQ_m) \right] \right\}$$

$$\frac{1}{r} \left( e^{i\omega r} + A_{\omega'M'}^{*} e^{-i\omega r} \right) \left[ \frac{1}{r} (-i\epsilon e^{-i\omega r} + \epsilon A_{\omega'M'} e^{i\omega r}) + O(\frac{1}{r^2}) \right] + \text{c.c.}$$

$$\Theta_{lm}(\omega, \theta) \Theta_{l'm'}^{*}(\omega', \theta) e^{i(\omega' - \omega)t} e^{i(m - m')\phi} d\theta d\phi dt$$

The contribution of the $A_0$ terms can be neglected, since they are one order in $r$ lower than $\omega$ or $\omega'$; similarly the $O(\frac{1}{r^2})$ terms coming from the derivatives of the radial wavefunctions (in the square brackets) have no contribution; also the $\mp eQ_m$ terms cancel.

We next take the time-integral, which gives a $\delta$-function in $\omega$ and $\omega'$, allowing the $\omega'$-integral to be done, too:

$$\delta(\text{CCC}) = \frac{1}{2\pi M} \int_{\infty} \frac{r^2 \sin \theta \sum_{l,m} \sum_{l',m'} \int d\omega f_{lm}(\omega) f_{l'm'}^{*}(\omega) \Theta_{lm}(\omega, \theta) \Theta_{l'm'}^{*}(\omega, \theta) e^{i(m - m')\phi}$$

$$\left\{ \left[ (M^2 + a^2)\omega - am' + eQ_e M \right] \frac{1}{r} \left( e^{i\omega r} + A_{\omega'M'}^{*} e^{-i\omega r} \right) \frac{k}{r} \left( e^{-i\omega r} - A_{\omega'M'} e^{i\omega r} \right) + \text{c.c.} \right\} d\theta d\phi$$

The $\phi$ and $\theta$-integrals can now be done (They could not be done previously, because the orthogonality relation (19) for the angular wavefunctions is only valid if the frequencies in the functions $\Theta_{qlm}(a, \mu, \omega, \theta)$ and $\Theta_{ql'm'}^{*}(\omega, \mu, \omega', \theta)$ are the same), giving $\delta_{ll'}\delta_{mm'}$ and thereby allowing the $l'$ and $m'$ sums to be done:

$$\delta(\text{CCC}) = \frac{1}{2\pi M} \int d\omega \sum_{l,m} f_{lm}(\omega) f_{lm}^{*}(\omega) k [ (M^2 + a^2)\omega - am + eQ_e M ] [ 1 - A_{\omega lm} A_{\omega lm}^{*} ]$$

Using the Wronskian relation (33),

$$\delta(\text{CCC}) = \frac{1}{2\pi M} \int d\omega \sum_{l,m} f_{lm}(\omega) f_{lm}^{*}(\omega) [ (M^2 + a^2)\omega - am + eQ_e M ] \tilde{\omega} B_{\omega lm} B_{\omega lm}^{*}$$

$$= \frac{M^2 + a^2}{2\pi M} \int d\omega \sum_{l,m} f_{lm}(\omega) f_{lm}^{*}(\omega) [ \omega + \frac{eQ_e M - am}{M^2 + a^2} ] [ \omega + \frac{eQ_e r_+ - am}{r_+^2 + a^2} ] B_{\omega lm} B_{\omega lm}^{*}$$

For the extreme black hole, $r_+ \to M$, therefore

$$\delta(\text{CCC})_+ = \frac{M^2 + a^2}{2\pi M} \int d\omega \sum_{l,m} f_{lm}(\omega) f_{lm}^{*}(\omega) \tilde{\omega}^2 B_{\omega lm} B_{\omega lm}^{*}.$$
where $\delta(CCC)_+$ and $\bar{\omega}_+$ are $\delta(CCC)$ and $\bar{\omega}$, when $r_+ \to M$.

Expression (51) is strictly nonnegative, but expression (50) is not. Therefore by judicious choice of frequency, $\delta(CCC)$ can be made negative for non-extreme black holes, although the frequency interval we can choose from shrinks monotonically to zero as the black hole approaches extremality. Still, in contradiction to the assertions of [26], it seems that extremality can be reached by sending appropriate Klein-Gordon wave packets into a black hole.

Only if all $\bar{\omega}_+ = 0$ vanish can expression (51) be zero, therefore, the second order not dominant. Except for this very limited case the expression (51) for $\delta(CCC)_+$ will be the dominant one, and its strict positivity means that $M^2$ increases faster than $(Q^2 + a^2)$ to the dominant (second) order in the field, meaning that to this order and with the very limited, inconclusive exception noted, the Cosmic Censorship Conjecture cannot be violated by adding charge and/or angular momentum to an extreme black hole via a classical Klein-Gordon test field.

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3 For this case to be relevant, the wave must contain only a set of discrete frequencies, one for each $m$, if $a \neq 0$, only one frequency if $a = 0$. In fact, the case discussed by [21, 22, 23, 24] fits in this category: they take $e = 0$, $a = 0$ and $\omega \to 0$. Let us recall that expressions (50) and (51) are valid to second order in $\epsilon$, that is, to first order in $\delta M$, $\delta Q e$ and $\delta L$. Another way to see the vanishing of $\delta(CCC)_+$ in [21, 22, 23, 24] to this order is noting that in the case they discuss, $\delta(M^2)$, $\delta(Q^2)$ and $\delta(a^2)$ all vanish, although $\delta Q$ and $\delta a$ do not vanish. Of course, it is possible to find other subcases where $\delta M$, $\delta Q$ and/or $\delta a$ do not vanish individually, but $\delta(CCC)_+$ still vanishes.

Then, the right-hand-side of (51) does not represent any more the contribution to lowest nonvanishing order in $\epsilon$, one has to go to higher order and/or analyze backreaction; and in this very limited case the present analysis is therefore inconclusive.
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