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CHAPTER 1

SUPPLY AND PRODUCTION NETWORKS: FROM THE BULLWHIP EFFECT TO BUSINESS CYCLES

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Network theory is rapidly changing our understanding of complex systems, but the relevance of topological features for the dynamic behavior of metabolic networks, food webs, production systems, information networks, or cascade failures of power grids remains to be explored. Based on a simple model of supply networks, we offer an interpretation of instabilities and oscillations observed in biological, ecological, economic, and engineering systems.

We find that most supply networks display damped oscillations, even when their units – and linear chains of these units – behave in a non-oscillatory way. Moreover, networks of damped oscillators tend to produce growing oscillations. This surprising behavior offers, for example, a new interpretation of business cycles and of oscillating or pulsating processes. The network structure of material flows itself turns out to be a source of instability, and cyclical variations are an inherent feature of decentralized adjustments. In particular, we show how to treat production and supply networks as transport problems governed by balance equations and equations for the adaptation of production speeds.

The stability and dynamic behavior of supply networks is investigated for different topologies, including sequential supply chains, “supply circles”, “supply ladders”, and “supply hierarchies”. Moreover, analytical conditions for absolute and convective instabilities are derived. The empirically observed bullwhip effect in supply chains is explained as a form of convective instability based on resonance effects. An application of this theory to the optimization of production networks has large optimization potentials.
1. Introduction

Supply chain management is a major subject in economics, as it significantly influences the efficiency of production processes. Many related studies focus on subjects such as optimum buffer sizes and stock levels.\(^1\),\(^2\),\(^3\),\(^4\) However, the optimal structure of supply and distribution networks is also an important topic,\(^5\),\(^6\) which connects this scientific field with the statistical physics of networks.\(^7\),\(^8\),\(^9\),\(^10\),\(^11\),\(^12\),\(^13\),\(^14\) Problems like the adaptivity and robustness of supply networks have not yet been well covered and are certainly an important field for future research.

In this chapter, we will focus on the dynamical properties and linear stability of supply networks in dependence of the network topology or, in other words, the supply matrix (input matrix). Presently, there are only a few results on this subject, since the response of supply networks to varying demand is not a trivial problem, as we will see. Some “fluid models” have, however, been proposed to study this subject. Daganzo,\(^15\) for example, applies the method of cumulative counts\(^16\) and a variant of the Lighthill-Whitham traffic model to study the so-called bullwhip effect.\(^17\),\(^18\),\(^19\),\(^20\),\(^21\),\(^22\),\(^23\),\(^24\),\(^25\),\(^26\),\(^27\),\(^28\) This effect has, for example, been reported for beer distribution,\(^29\),\(^30\) but similar dynamical effects are also known for other distribution or transportation chains. It describes increasing oscillations in the delivery rates and stock levels from one supplier to the next upstream supplier delivering to him.

Similar models are studied by Armbruster \textit{et al.}.\(^31\),\(^32\),\(^33\),\(^34\) Most publications, however, do not investigate the impact of the network topology, but focus on sequential supply chains or re-entrant problems.\(^34\),\(^35\) An exception is the transfer function approach by Dejonckheere \textit{et al.}\(^24\) as well as Disney and Towill.\(^26\) Ponzi, Yasutomi, and Kaneko\(^36\) have coupled a supply chain model with a model of price dynamics, while Witt and Sun\(^37\) have suggested to model business cycles analogously to stop-and-go traffic. Similar suggestions have been made by Daganzo\(^15\) and Helbing.\(^38\) We should also note the relationship with driven many-particle models.\(^39\) This is the basis of event-driven simulations of production systems, which are often used in logistics software.

In a previous publication, one of the authors has proposed a rather general model for supply networks, and has connected it to queueing theory and macroeconomics.\(^38\) That paper also presents numerical studies of the bullwhip effect in sequential supply chains and the effect of heterogeneous production units on the dynamics of supply networks. Moreover, the de-
dependence of the maximum oscillation amplitude on the model parameters has been numerically studied, in particular the phase transition from stable to bullwhip behavior. The underlying simulation models are of non-linear nature, so that phenomena such as wave selection and synchronization have been observed. Finally, a large variety of management strategies including the effect of forecasting inventories has been studied together with Nagatani. The latter has also investigated the dynamical effect of multiple production lines.

In the following review, we will mainly focus on a linearized model of supply networks, which could be viewed as a dynamical version of Leontief’s classical input-output model. It is expected to be valid close to the stationary state of the supply system or, in other words, close to the (commonly assumed) economic equilibrium. This model allows to study many properties of supply networks in an analytical way, in particular the effects of delayed adaptation of production rates or of a forecasting of inventories. In this way, we will be able to understand many numerically and empirically observed effects, such as resonance effects and the stability properties of supply networks in dependence of the network topology.

Our review is structured as follows: Section 2 introduces our model of supply networks and linearizes it around the stationary state. Section 2.2 discusses the bullwhip effect for sequential supply chains. Surprisingly, the amplification of amplitudes does not require unstable eigenvalues, but it is based on a resonance effect. Sec. 3.1 presents general methods of solution for arbitrary topologies of supply networks. It reveals some useful properties of the eigenvalues characterizing the stability of supply systems. Interestingly enough, it turns out that all supply networks can be mapped on formulas for linear (serial) supply chains. Sec. 3.2 will investigate, how the dynamic behavior of supply networks depends on their network structure, while Sec. 4 gives an interpretation of business cycles based on material flow networks. Sec. 6 finally summarizes our results and points to some future research directions.

2. Input-Output Model of Supply Networks

Our production model assumes $u$ production units or suppliers $j \in \{1, \ldots, u\}$ which deliver $d_{ij}$ products of kind $i \in \{1, \ldots, p\}$ per production cycle to other suppliers and consume $c_{kj}$ goods of kind $k$ per production cycle. The coefficients $c_{kj}$ and $d_{ij}$ are determined by the respective production process, and the number of production cycles per unit time (e.g. per
day) is given by the production speed $Q_j(t)$. That is, supplier $j$ requires an average time interval of $1/Q_j(t)$ to produce and deliver $d_{ij}$ units of good $i$. The temporal change in the number $N_i(t)$ of goods of kind $i$ available in the system is given by the difference between the inflow

$$Q_{i\text{in}}(t) = \sum_{j=0}^{u} d_{ij}Q_j(t)$$

and the outflow

$$Q_{i\text{out}}(t) = \sum_{i=1}^{u+1} c_{ij}Q_j(t).$$

In other words, it is determined by the overall production rates $d_{ij}Q_j(t)$ of all suppliers $j$ minus their overall consumption rates $c_{ij}Q_j(t)$:

$$\frac{dN_i}{dt} = Q_{i\text{in}}(t) - Q_{i\text{out}}(t) = \sum_{j=1}^{u} d_{ij}Q_j(t) - \sum_{j=1}^{u} c_{ij}Q_j(t) - Y_i(t).$$

In this dynamic variant of Leontief’s classical input-output model, the quantity

$$Y_i(t) = \frac{c_{i,u+1}Q_{u+1}(t)}{\text{consumption and losses}} - \frac{d_{i0}Q_0(t)}{\text{inflow of resources}}$$

comprises the consumption rate of goods $i$, losses, and waste (the “export” of material), minus the inflows into the considered system (the “imports”). In the following, we will assume that the quantities are measured in a way that $0 \leq c_{ij}, d_{ij} \leq 1$ (for $1 \leq i \leq p, 1 \leq j \leq u$) and that the “normalization conditions”

$$d_{i0} = 1 - \sum_{j=1}^{u} d_{ij} \geq 0, \quad c_{i,u+1} = 1 - \sum_{j=1}^{u} c_{ij} \geq 0$$

are fulfilled. Equations (3) can then be interpreted as conservation equations for the flows of goods.

### 2.1. Adaptation of Production Speeds

In addition, we have formulated an equation for the production or delivery rate $Q_j(t)$. Changes in the consumption rate $Y_i(t)$ sooner or later require
an adaptation of $Q_j(t)$. For the temporal change $dQ_j/dt$ of the delivery rate we will assume:

$$\frac{dQ_j}{dt} = F_j(\{N_i(t)\}, \{dN_i/dt\}, \{Q_l(t)\}) \, .$$

Herein, the curly brackets indicate that the so-called management or control function $F_j(\ldots)$ may depend on all inventories $N_i(t)$ with $i \in \{1, \ldots, p\}$, their derivatives $dN_i/dt$, and/or all production speeds $Q_l(t)$ with $l \in \{1, \ldots, u\}$. Some reasonable specifications will be discussed later on.

### 2.2. Modelling Sequential Supply Chains

![Diagram of linear supply chain](image)

**Fig. 1.** Illustration of the linear supply chain treated in this chapter, including the key variables of the model. Circles represent different suppliers $i$, $N_i$ their respective stock levels, and $Q_i$ the delivery rate to supplier $i$ or the production speed of this supplier. $i = 0$ corresponds to the resource sector generating the basic products and $i = u + 1$ to the consumers.

For simplicity, let us first investigate a model of sequential supply chains (see Fig. 1), which corresponds to $d_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i+1,j}$. In other words, the products $i$ are directly associated with producers $j$ and we have an input matrix of the form

$$C = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \, .$$

This implies $Q_{in}^i(t) = Q_i(t) = Q_{out}^{i-1}(t)$ and $Q_{out}^i(t) = Q_{i+1}(t) = Q_{i+1}(t)$, so that the inventory of goods of kind $i$ changes in time $t$ according to

$$\frac{dN_i}{dt} = Q_{in}^i(t) - Q_{out}^i(t) = Q_i(t) - Q_{i+1}(t) \, .$$

The assumed model consists of a series of $u$ suppliers $i$, which receive products of kind $i - 1$ from the next “upstream” supplier $i - 1$ and generate products of kind $i$ for the next “downstream” supplier $i + 1$ at a rate $Q_i(t)$.45,31
The final products are delivered at the rate $Q_u(t)$ and removed from the system with the consumption rate $Y_u(t) = Q_{u+1}(t)$.

In the following, we give some possible specifications of equation (6) for the production rates:

- The delivery rate $Q_i(t)$ may, for example, be adapted to a certain desired rate $\hat{W}_i(N_i, dN_i/dt)$ according to the equation
  \[
  \frac{dQ_i}{dt} = \frac{1}{T_i} \left[ \hat{W}_i \left( N_i, \frac{dN_i}{dt} \right) - Q_i(t) \right],
  \]  
  where $T_i$ denotes the relaxation time.

- A special case of this is the equation
  \[
  \frac{dQ_i}{dt} = \frac{1}{T_i} \left[ \hat{W}_i \left( N_i(t) + \Delta t \right) - \hat{W}_i \left( N_i(t) \right) \right].
  \]  
  Herein, the desired production speed $\hat{W}_i(N_i)$ is monotonously falling with increasing inventories $N_i$, which are forecasted over a time period $\Delta t$.

- A further specification is given by the equation
  \[
  \frac{dQ_i}{dt} = \frac{1}{T_i} \left\{ \frac{N_i^0 - N_i(t)}{\tau_i} - \beta_i \frac{dN_i}{dt} + \epsilon_i [Q_i^0 - Q_i(t)] \right\},
  \]  
  which assumes that the production rate is controlled in a way that tries to reach some optimal inventory $N_i^0$ and production rate $Q_i^0$, and attempts to minimize changes $dN_i/dt$ in the inventory to reach a constant work in progress (CONWIP strategy). Note that at least one of the parameters $T_i$, $\tau_i$, $\beta_i$ and $\epsilon_i$ could be dropped.

- In other cases, it can be reasonable to work with a model focusing on relative changes in the variables:
  \[
  \frac{dQ_i}{dt} = \hat{\nu}_i \left( \frac{N_i^0}{N_i(t)} - 1 \right) - \hat{\mu}_i \frac{dN_i}{dt} + \hat{\epsilon}_i \left( \frac{Q_i^0}{Q_i(t)} - 1 \right).
  \]  
  This model assumes large production rates when the inventory is low and prevents that $Q_i(t)$ can fall below zero. Apart from this, $N_i(t)$ and $Q_i(t)$ are adjusted to some values $N_i^0$ and $Q_i^0$, which are desirable from a production perspective (i.e. in order to cope with the breakdown of machines, variations in the consumption rate, etc.) Moreover, the control strategy (12) counteracts temporal
changes $dN_i/dt$ in the inventory. For comparison with the previous strategies, we set $T_i = 1/Q_i^0$, $\tau_i = N_i^0/\hat{\nu}_i$, $\beta_i = \hat{\mu}_i/N_i^0$, and $\epsilon_i = \hat{\epsilon}_i T_i$.

The above control strategies appear to be appropriate to keep the inventories $N_i(t)$ stationary, to maintain a certain optimal inventory $N_i^0$ (in order to cope with stochastic variations due to machine breakdowns etc.), and to operate with the equilibrium production rates $Q_i^0 = Y_u^0$, where $Y_u^0$ denotes the average consumption rate. However, the consumption rate is typically subject to perturbations, which may cause a bullwhip effect, i.e. growing variations in the stock levels and deliveries of upstream suppliers (see Sec. 2.4).

### 2.3. More Detailed Derivation of the Production Dynamics

*(with Dieter Armbruster)*

Let us assume a sequential supply chain and let $Q_i(t)$ be the influx in the last time period $T = 24$ hours from the supplier $i - 1$ to producer $i$. This influx is equal to the release $R_{i-1}(t)$ over the last 24 hours:

$$Q_i(t) = R_{i-1}(t). \quad (13)$$

Based on the current state (the inventory and fluxes at time $t$ and before), a price $P_i(t)$ is set according to some pricing policy and instantaneously communicated downstream to the customer $i + 1$. Then, based on price, fluxes, and inventory, the order quantity $D_{i+1}(t)$ is decided at the end of day $t$ according to some order policy $W_{i+1}$:

$$D_{i+1}(t) = W_{i+1}\left(N_{i+1}(t), P_i(t), \ldots \right). \quad (14)$$

It determines the upstream release $R_i(t + T)$ on the next day $t + T = t + 1$:

$$R_i(t + T) = R_i(t + 1) = D_{i+1}(t). \quad (15)$$

Altogether, this implies

$$Q_i(t + T) = Q_i(t + 1) = R_{i-1}(t + 1) = D_i(t) = W_i\left(N_i(t), \ldots \right) \quad (16)$$

and the following equation for the change in the production rate:

$$\frac{dQ_i}{dt} \approx \frac{\Delta Q_i(t)}{T} = \frac{1}{T}[Q_i(t + T) - Q_i(t)] = \frac{1}{T}\left[W_i\left(N_i(t), \ldots \right) - Q_i(t)\right]. \quad (17)$$
This is consistent with Eq. (9). Moreover, we obtain the usual balance equation for the change of inventory in time:

\[
\frac{dN_i}{dt} \approx \frac{\Delta N_i}{T} = \frac{N_i(t + T) - N_i(t)}{T} = N_i(t) - N_i(t) = R_i(t) - R_i(t) = Q_i(t) - Q_{i+1}(t).
\]  

(18)

2.4. Dynamic Solution and Resonance Effects (with Tadeusz P/suppress latkowski)

Let us now calculate the dynamic solution of the sequential supply chain model for cases where the values \(N_i^0\) and \(Q_i^0\) correspond to the stationary state of the production system.\(^6,39\) Then, the linearized model equations for the control approaches (9) to (12) are exactly the same. Representing the deviation of the inventory from the stationary one by \(n_i(t) = N_i(t) - N_i^0\) and the deviation of the delivery rate by \(q_i(t) = Q_i(t) - Q_i^0\), they read

\[
\frac{dq_i}{dt} = \frac{1}{T_i} \left[ -\frac{n_i(t)}{\tau_i} - \beta_i \frac{dn_i}{dt} - \epsilon_i q_i(t) \right].
\]  

(19)

Moreover, the linearized equations for the inventories are given by

\[
\frac{dn_i}{dt} = q_i(t) - q_{i+1}(t).
\]  

(20)

Deriving Eq. (19) with respect to \(t\) and inserting Eq. (20) results in the following set of second-order differential equations:

\[
\frac{d^2q_i}{dt^2} + \frac{(\beta_i + \epsilon_i)}{T_i} \frac{dq_i}{dt} + \frac{1}{T_i \tau_i} q_i(t) = \frac{1}{T_i} \left[ \frac{q_{i+1}(t)}{\tau_i} + \beta_i \frac{dq_{i+1}}{dt} \right] = f_i(t).
\]  

(21)

This corresponds to the differential equation of a damped harmonic oscillator with damping constant \(\gamma\), eigenfrequency \(\omega\), and driving term \(f_i(t)\).

The eigenvalues of these equations are

\[
\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2} = -\frac{1}{2T_i} \left[ (\beta_i + \epsilon_i) \mp \sqrt{(\beta_i + \epsilon_i)^2 - 4T_i/\tau_i} \right].
\]  

(22)

For \((\beta_i + \epsilon_i) > 0\) their real parts are always negative, corresponding to a stable behavior in time. Nevertheless, we will find a convective instability, i.e. the oscillation amplitude can grow from one supplier to the next one upstream.

Assuming periodic oscillations of the form \(f_i(t) = f_i^0 \cos(\alpha t)\), the general solution of Eq. (21) is of the form

\[
q_i(t) = f_i^0 F_i \cos(\alpha t + \varphi_i) + D_i^0 e^{-\gamma t} \cos(\Omega_i t + \theta_i),
\]  

(23)
where the parameters $D_i^0$ and $\theta_i$ depend on the initial conditions. The other parameters are given by

$$\tan \varphi_i = \frac{2\gamma_i \alpha}{\alpha^2 - \omega_i^2} = \frac{\alpha (\beta_i + \epsilon_i)}{\alpha^2 T_i - 1/\tau_i},$$

(24)

$$\Omega_i = \sqrt{\omega_i^2 - \gamma_i^2} = \frac{1}{T_i} \sqrt{1/\tau_i^2 - (\beta_i + \epsilon_i)^2/4},$$

(25)

and

$$F_i = \frac{1}{\sqrt{(\alpha^2 - \omega_i^2)^2 + 4\gamma_i^2 \alpha^2}} = \frac{T_i}{\sqrt{(\alpha^2 T_i - 1/\tau_i)^2 + \alpha^2 (\beta_i + \epsilon_i)^2}}.$$  

(26)

The dependence on the eigenfrequency $\omega_i$ is important for understanding the occurring resonance effect, which is particularly likely to appear, if the oscillation frequency $\alpha$ of the consumption rate is close to one of the resonance frequencies $\omega_i$. After a transient time much longer than $1/\gamma_i$ we find

$$q_i(t) = f_i^0 F_i \cos(\alpha t + \varphi_i).$$

(27)

Equations (21) and (27) imply

$$f_{i-1}(t) = \frac{1}{T_i} \left[ \frac{q_i(t)}{\tau_i} + \beta_i \frac{dq_i}{dt} \right] = f_{i-1}^0 \cos(\alpha t + \varphi_i + \delta_i)$$

(28)

with

$$\tan \delta_i = \alpha \beta_i \tau_i \quad \text{and} \quad f_{i-1}^0 = f_i^0 \frac{F_i}{T_i} \sqrt{(1/\tau_i)^2 + (\alpha \beta_i)^2}.$$  

(29)

Therefore, the set of equations (21) can be solved successively, starting with $i = u$ and progressing to lower values of $i$.

### 2.5. The Bullwhip Effect

The oscillation amplitude increases from one supplier to the next upstream one, if

$$\frac{f_{i-1}^0}{f_i^0} = \left\{ 1 + \frac{\alpha^2 [\epsilon_i (\epsilon_i + 2\beta_i) - 2T_i/\tau_i] + \alpha^4 T_i^2}{(1/\tau_i)^2 + (\alpha \beta_i)^2} \right\}^{-1/2} > 1.$$  

(30)

One can see that this resonance effect can occur for $0 < \alpha^2 < 2/(T_i \tau_i) - \epsilon_i (\epsilon_i + 2\beta_i)/T_i^2$. Therefore, variations in the consumption rate are magnified under the instability condition

$$T_i > \epsilon_i \tau_i (\beta_i + \epsilon_i/2).$$  

(31)
Fig. 2. Frequency response for different $\beta_i$ and $T_i = \tau_i = \epsilon_i = 1$. For small $\beta_i$, corresponding to a small prognosis time horizon $\Delta t$, a resonance effect with an amplification factor greater than 1 can be observed. Perturbations with a frequency $\alpha$ close to the eigenfrequencies $\omega_i = 1/\sqrt{T_i \tau_i}$ are amplified and cause variations in stock levels and deliveries to grow along the supply chain. This is responsible for the bullwhip effect.

Supply chains show the bullwhip effect (which corresponds to the phenomenon of convective, i.e. upstream amplification), if the adaptation time $T_i$ is too large, if there is no adaptation to some equilibrium production speed (corresponding to $\epsilon_i = 0$), or if the production management reacts too strong to deviations of the actual stock level $N_i$ from the desired one $N^0_i$ (corresponding to a small value of $\tau_i$), see Fig. 3. The latter is very surprising, as it implies that the strategy

$$\frac{dQ_i}{dt} = \frac{1}{T_i \tau_i} [N^0_i - N_i(t)],$$

(32)

which tries to maintain a constant work in progress $N_i(t) = N^0_i$, would ultimately lead to an undesirable bullwhip effect, given that production units are adjusted individually, i.e. in a decentralized way. In contrast, the management strategy

$$\frac{dQ_i}{dt} = \frac{1}{T_i} \left\{ - \beta_i \frac{dN_i}{dt} + \epsilon_i [Q^0_i - Q_i(t)] \right\}$$

(33)
would avoid this problem, but it would not maintain a constant work in progress.

Fig. 3. (a) Plot of the maximum amplitude of oscillation in the inventories as a function of the adaptation time $T$. (b) Plot of the maximum amplitude of the inventories as a function of the time horizon $\Delta t$ for an adaptation time of $T = 2$. (After Ref.41)

The control strategy (32) with a sufficiently large value of $\tau$ would fulfill both requirements. Having the approximate relation (10) in mind, a forecast with a sufficiently long prognosis time horizon $\Delta t$ (implying large values of $\beta_i$) is favorable for production stability. Delays in the determination of the inventories $N_i$, corresponding to negative values of $\Delta t$ and $\beta_i$, are destabilizing. Note that the values of $\Delta t$ which are sufficient to stabilize the system are often much smaller than the adaptation time $T$.

3. Network Effects (with Péter Šeba)

The question is now, whether and how the bullwhip effect can be generalized to supply networks and how the dynamic behavior depends on the respective network structure. Linearization of the adaptation equation (6) leads to

$$\frac{d q_j}{dt} = - \sum_i V_{ji} n_i(t) - \sum_i W_{ji} \frac{dn_j}{dt} - \sum_l X_{jl} q_l(t).$$

However, in order to avoid a vast number of parameters $V_{ji}$, $W_{ji}$, $X_{jl}$ and to gain analytical results, we will assume product- and sector-independent parameters $V_{jk} = V \delta_{jk}$, $W_{jk} = W \delta_{jk}$, and $X_{jl} = \delta_{jl}$ in the following. This
case corresponds to the situation that each production unit \( j \) is characterized by one dominating product \( i = j \), which also dominates production control. For this reason, we additionally set \( d_{ij} = \delta_{ij} \). Generalizations are discussed later (see Sec. 7.4).

Using vector notation, we can write the resulting system of differential equations as

\[
\frac{d\vec{n}}{dt} = \mathbf{S}\vec{q}(t) - \vec{y}(t) \quad (35)
\]

and

\[
\frac{d\vec{q}}{dt} = -V\vec{n}(t) - W\frac{d\vec{n}}{dt} - \vec{q}(t) . \quad (36)
\]

Inserting Eq. (36) into Eq. (35) gives

\[
\frac{d}{dt} \begin{pmatrix} \vec{n} \\ \vec{q} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \vec{n} \\ \vec{q} \end{pmatrix} + \begin{pmatrix} -\vec{y} \\ -W\vec{y} \end{pmatrix} \quad (37)
\]

with

\[
\mathbf{M} = \begin{pmatrix} 0 & \mathbf{S} \\ -VE & -E - WS \end{pmatrix} \quad (38)
\]

and the supply matrix \( \mathbf{S} = \mathbf{D} - \mathbf{C} = (d_{ij} - c_{ij}) \). For comparison of Eq. (19) the one above, one has to scale the time by introducing the unit time \( T/\epsilon \) and to set \( V = 1/(\tau\epsilon) \), \( W = \beta/\epsilon \).

### 3.1. General Methods of Solution

It is possible to rewrite the system of 2\( u \) first order differential equations (37) in the form of a system of \( u \) differential equations of second order:

\[
\frac{d^2\vec{q}}{dt^2} + (\mathbf{E} + WS)\frac{d\vec{q}}{dt} + V\mathbf{S}\vec{q}(t) = \vec{g}(t) , \quad (39)
\]

where

\[
\vec{g}(t) = V\vec{y}(t) + W\frac{d\vec{q}}{dt} . \quad (40)
\]

Introducing the Fourier transforms

\[
\hat{F}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\alpha t} \vec{q}(t) \quad (41)
\]
and

\[ \vec{G}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-i\alpha t} \vec{g}(t) \] (42)

reduces the problem to solving

\[ \begin{bmatrix} -\alpha^2 + i\alpha(\mathbf{S} + \mathbf{W}) + \mathbf{V} \end{bmatrix} \vec{F}(\alpha) = \vec{G}(\alpha) \] (43)

with given \( \vec{G}(\alpha) \). The variable \( \alpha \) represents the perturbation frequencies, and the general solution of Eq. (39) is given by

\[ \vec{q}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \, e^{i\alpha t} \vec{F}(\alpha) . \] (44)

Note that there exists a matrix \( \mathbf{T} \) which allows one to transform the matrix \( \mathbf{E} - \mathbf{S} \) via \( \mathbf{T}^{-1}(\mathbf{E} - \mathbf{S})\mathbf{T} = \mathbf{J} \) into either a diagonal or a Jordan normal form \( \mathbf{J} \). Defining \( \vec{x}(\tau) = \mathbf{T}^{-1} \vec{q}(\tau) \) and \( \vec{h}(t) = \mathbf{T}^{-1} \vec{g}(t) \), we obtain the coupled set of second-order differential equations

\[ \frac{d^2 x_i}{dt^2} + 2\gamma_i \frac{dx_i}{dt} + \omega_i^2 x_i(t) = b_i \left[ W x_{i+1}(t) + V \frac{dx_{i+1}}{dt} \right] + h_i(t) , \] (45)

where

\[ \gamma_i = \frac{[1 + W(1 - J_{ii})]}{2}, \quad \omega_i = \left[ V(1 - J_{ii}) \right]^{1/2} \quad \text{and} \quad b_i = J_{i,i+1} . \] (46)

This can be interpreted as a set of equations for linearly coupled damped oscillators with damping constants \( \gamma_i \), eigenfrequencies \( \omega_i \), and external forcing \( h_i(t) \). The other forcing terms on the right-hand side are due to interactions of suppliers. They appear only if \( \mathbf{J} \) is not of diagonal, but of Jordan normal form with some \( J_{i,i+1} \neq 0 \). Because of \( b_i = J_{i,i+1} \), Eqs. (45) can again be analytically solved in a recursive way, starting with the highest index \( i = u \). Note that, in the case \( \mathbf{D} = \mathbf{E} \) (i.e. \( d_{ij} = \delta_{ij} \)), \( J_{ii} \) are the eigenvalues of the input matrix \( \mathbf{C} \) and \( 0 \leq |J_{ii}| \leq 1 \). Equation (45) has a special periodic solution of the form

\[ x_i(t) = x_i^0 e^{i(\omega t - \chi_i)} , \] (47)

\[ h_i(t) = h_i^0 e^{i\omega t} , \] (48)

where \( i = \sqrt{-1} \) denotes the imaginary unit. Inserting this into (45) and dividing by \( e^{i\omega t} \) immediately gives

\[ (-\alpha^2 + 2i\alpha \gamma_i + \omega^2_i) x_i^0 e^{-i\chi_i} = b_i (V + i\alpha W) x_{i+1}^0 e^{-i\chi_{i+1}} + h_i^0 . \] (49)
With $e^{\pm i\phi} = \cos(\phi) \pm i \sin(\phi)$ this implies

$$x_i^0 e^{-i\chi_i} = \frac{b_i (V + i\alpha W)x_{i+1}^0 e^{-i\chi_{i+1}} + h_i^0}{-\alpha^2 + 2i\alpha \gamma_i + \omega_i^2} = \frac{\sqrt{\text{Re}^2 + \text{Im}^2} e^{i\rho_i}}{\sqrt{(\omega_i^2 - \alpha^2)^2 + (2\alpha \gamma_i)^2}} e^{i\phi_i},$$

(50)

where

$$\text{Re} = b_i x_{i+1}^0 (V \cos(\chi_{i+1}) + \alpha W \sin(\chi_{i+1})) + h_i^0,$$

$$\text{Im} = b_i x_{i+1}^0 (\alpha W \cos(\chi_{i+1}) - V \sin(\chi_{i+1})),\quad (51)$$

and

$$x_i^0 = \sqrt{[V^2 + (\alpha W)^2][b_i x_{i+1}^0]^2 + h_i^0} H_i + (h_i^0)^2 \frac{(\omega_i^2 - \alpha^2)^2 + (2\alpha \gamma_i)^2}{(\omega_i^2 - \alpha^2)^2 + (2\alpha \gamma_i)^2}$$

(52)

with

$$H_i = 2b_i x_{i+1}^0 (V \cos(\chi_{i+1}) + \alpha W \sin(\chi_{i+1})).$$

(53)

Finally, we have

$$\chi_i = \phi_i - \rho_i \quad \text{with} \quad \tan \phi_i = 2\alpha \gamma_i / (\omega_i^2 - \alpha^2)$$

(54)

and

$$\tan \rho_i = \frac{b_i x_{i+1}^0 (\alpha W \cos(\chi_{i+1}) - V \sin(\chi_{i+1}))}{b_i x_{i+1}^0 (V \cos(\chi_{i+1}) + \alpha W \sin(\chi_{i+1})) + h_i^0}.$$\quad (55)

For $h_i^0 = 0$, we obtain

$$\tan(\phi_i - \chi_i) = \tan(\delta - \chi_{i+1})$$

(56)

with

$$\tan \delta = \alpha W / V,$$

(57)

i.e. the phase shift between $i$ and $i+1$ is just

$$\chi_i - \chi_{i+1} = \phi_i - \delta.$$\quad (58)

According to Eq. (45), the dynamics of our supply network model can be reduced to the dynamics of a sequential supply chain. However, the eigenvalues are now potentially complex and the new entities $i$ have the meaning of “quasi-suppliers” (analogously to “quasi-species”\textsuperscript{46,47}) defined by the linear combination $\vec{x}(\tau) = T^{-1} \vec{q}(\tau)$. This transformation makes it possible to define the bullwhip effect for arbitrary supply networks: It occurs if the amplitude $x_i^0$ is greater than the oscillation amplitude $x_{i+1}^0$ of the
next downstream supplier $i + 1$ and greater than the amplitude $h_i^0$ of the
external forcing, i.e. if $x_i^0 / \max(x_{i+1}^0, h_i^0) > 1$.

Note that in contrast to sequential supply chains, the oscillation amplitude
of $\bar{x}(t)$ may be amplified in the course of time, depending on the
network structure. This case of absolute instability can occur if at least
one of the eigenvalues $\lambda_{i, \pm}$ of the homogeneous equation (45) resulting for
$h_i = b_i = 0$ has a positive real part, which may be true when some complex
eigenvalues $J_{ii}$ exist. The (up to) $2u$ eigenvalues

$$\lambda_{i, \pm} = -\gamma_i \pm \sqrt{\gamma_i^2 - \omega_i^2}$$

$$= -\left[1 + W(1 - J_{ii})\right]/2 \pm \sqrt{\left[1 + W(1 - J_{ii})\right]^2/4 - V(1 - J_{ii})}$$

depend on the (quasi-)supplier $i$ and determine the temporal evolution of
the amplitude of deviations from the stationary solution.

3.2. Examples of Supply Networks

It is useful to distinguish the following cases:

**Symmetric supply networks:** If the supply matrix $S$ is symmetric, as
for most mechanical or electrical oscillator networks, all eigenvalues $J_{ii}$ are
real. Consequently, if $\omega_i < \gamma_i$ (i.e. if $V$ is small enough), the eigenvalues
$\lambda_{i, \pm}$ of $M$ are real and negative, corresponding to an overdamped behavior.
However, if $W$ is too small, the system behavior may be characterized by
damped oscillations.

**Irregular supply networks:** Most natural and man-made supply net-
works have directed links, and $S$ is not symmetric. Therefore, some of the
eigenvalues $J_{ii}$ will normally be complex, and an overdamped behavior is
untypical. The characteristic behavior is rather of oscillatory nature (al-
though asymmetry does not always imply complex eigenvalues$^5$, see Fig. 9).
For small values of $W$, it can even happen that the real part of an eigen-
value $\lambda_{i, \pm}$ becomes positive. This implies an amplification of oscillations in
time (until the oscillation amplitude is limited by non-linear terms). Sur-
prisingly, this also applies to most upper triangular matrices, i.e. when no
loops in the material flows exist.

**Regular supply networks:** Another relevant case are regular supply
networks. These are mostly characterized by degenerate zero eigenvalues
$J_{ii} = 0$ and Jordan normal forms $J$, i.e. the existence of non-vanishing
upper-diagonal elements $J_{i,i+1}$. Not only sequential supply chains, but also
fully connected graphs, regular supply ladders, and regular distribution sys-
tems belong to this case$^6$ (see Fig. 4a, c, d). This is characterized by the
two $u$-fold degenerate eigenvalues

$$
\lambda_{\pm} = -(1 + W)/2 \pm \sqrt{(1 + W)^2/4 - V},
$$

independently of the suppliers $i$. For small enough values $V < (1 + W)^2/4$, the corresponding supply systems show overdamped behavior, otherwise damped oscillations.

Fig. 4. Different examples of supply networks: (a) Sequential supply chain, (b) closed supply chain or supply circle, (c) regular supply ladder, (d) regular hierarchical distribution network.

For the purpose of illustration, the following equations display some regular input matrices $C$ and their corresponding Jordan matrices $J$: For a fully connected network we have

$$
C = \begin{pmatrix}
\frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \cdots & \frac{1}{u} \\
\frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \cdots & \frac{1}{u} \\
\frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \cdots & \frac{1}{u} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \frac{1}{u} & \cdots & \frac{1}{u}
\end{pmatrix}
\quad \text{and} \quad
J = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
$$

The Jordan normal matrix $J$ of a sequential supply chain corresponds to the input matrix $C$ itself, i.e. $J = C$. This, however, is quite exceptional.
For the supply ladder displayed in Fig. 4c we have
\[
C = \begin{pmatrix}
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
and
\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\],
(62)
where the number of ones corresponds to the number of levels of the supply ladder. For the hierarchical distribution network shown in Fig. 4d, but with 3 levels only, we have
\[
C = \begin{pmatrix}
0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
and
\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\],
(63)
where the number of ones corresponds to the number of levels of the supply ladder.

Randomized regular supply networks belong to the class of irregular supply networks, but they can be viewed as slightly perturbed regular supply networks. For this reason, there exist approximate analytical results for their eigenvalues. Even very small perturbations of the regular matrices S discussed in the previous paragraph can change the eigenvalue spectrum qualitatively. Instead of the two multiply degenerate eigenvalues \( \lambda_{\pm} \) of Eq. (59), we find a scattering of eigenvalues around these values. The question is: why?

In order to assess the behavior of randomized regular supply networks, we apply Gersgorin’s theorem on the location of eigenvalues. According to this, the \( n \) complex eigenvalues \( \lambda_k \in \mathbb{C} \) of some \( n \times n \)-matrix N are
located in the union of \( n \) disks:

\[
\lambda_k \in \bigcup_i \left\{ z \in \mathbb{C} : |z - N_{ii}| \leq \sum_{j \neq i} |N_{ij}| \right\} \tag{64}
\]

Furthermore, if a union of \( l \) of these \( n \) discs form a connected region that is disjoint from all the remaining \( n - l \) discs, then there are precisely \( l \) eigenvalues of \( N \) in this region.

Let us now apply this theorem to the perturbed matrix

\[
C_\eta = C + \eta P \tag{65}
\]

For small enough values of \( \eta \), the corresponding eigenvalues \( J_{ii} \) should be located within discs of radius

\[
R_i(\eta) = \eta \sum_j |P_{ij}| \tag{66}
\]

around the (possibly degenerated) eigenvalues \( J_{ii} \) of the original matrix \( C \) (see Fig. 5). This radius grows monotonously, but not necessarily linearly in the parameter \( \eta \) with \( 0 < \eta \leq 1 \), which allows to control the size of the perturbation. Moreover, \( P^{(\eta)} = R_\eta^{-1}PR_\eta \), where \( R_\eta \) is the orthogonal matrix which transforms \( C_\eta \) to a diagonal matrix \( D^{(\eta)} \), i.e. \( R_\eta^{-1}C_\eta R_\eta = D^{(\eta)} \).

(This assumes a perturbed matrix \( C_\eta \) with no degenerate eigenvalues.) Similar discs as for the eigenvalues of \( C_\eta \) can be determined for the associated eigenvalues \( \lambda_{i,\pm}^{(\eta)} \) of the perturbed \( (2u \times 2u) \)-matrix \( M_\eta \) belonging to the perturbed \( u \times u \)-matrix \( C_\eta \), see Eq. (38) and Fig. 5.

![Fig. 5](image-url)

(a) Example of a supply network (regular distribution network) with two degenerated real eigenvalues (see squares in the right subfigure). (b) The eigenvalues \( \lambda_{i,\pm}^{(\eta)} \) of the randomly perturbed supply network (crosses) are mostly complex and located within Gersgorin’s discs (large circles). (After Ref. 6)

Let us now discuss the example of a structural perturbation of a sequential supply chain (with \( \eta = 0 \)) towards a supply circle (with \( \eta = 1 \),
see Fig. 4b. For this, we set
\[ C = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ \eta & 0 & 0 & \ldots & 0 \end{pmatrix}, \] (67)

While the normal form for \( \eta = 0 \) is given by a Jordan matrix \( J \) which agrees with \( C \), for any \( \eta > 0 \) we find the diagonal matrix
\[ J = \begin{pmatrix} J_{11} & 0 & \ldots & 0 \\ 0 & J_{22} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & J_{nn} \end{pmatrix}, \] (68)

where the diagonal elements \( J_{ii} \) are complex and equally distributed on a circle of radius \( \sqrt{\eta} \) around the origin of the complex plane. Therefore, even an arbitrarily small perturbation can change the eigenvalues qualitatively and remove the degeneration of the eigenvalues.

4. Network-Induced Business Cycles (with Ulrich Witt and Thomas Brenner)

In order to investigate the macroeconomic dynamics of national economies, it is useful to identify the production units with economic sectors (see Fig. 6). It is also necessary to extend the previous supply network model by price-related equations. For example, increased prices \( P_i(t) \) of products of sector \( i \) have a negative impact on the consumption rate \( Y_i(t) \) and vice versa. We will describe this by a standard demand function \( L_i \) with a negative derivative \( L_i'(P_i) = dL_i(P_i)/dP_i \):
\[ Y_i(t) = |Y_i^0 + \xi_i(t)|L_i(P_i(t)). \] (69)

This formula takes into account random fluctuations \( \xi_i(t) \) over time around a certain average consumption rate \( Y_i^0 \) and assumes that the average value of \( L_i(P_i(t)) \) is normalized to one. The fluctuation term \( \xi_i(t) \) is introduced here in order to indicate that the variation of the consumption rate is a potentially relevant source of fluctuations.

Inserting (69) into (3) results in
\[ \frac{dN_i(t)}{dt} = Q_i(t) - \sum_{j=1}^{m} c_{ij} Q_j(t) - \frac{[Y_i^0 + \xi_i(t)]L_i(P_i(t))}{Y_i(t)} = Y_i(t). \] (70)
Fig. 6. Main service and commodity flows among different economic sectors according to averaged input-output data of France, Germany, Japan, UK, and USA. For clarity of the network structure, we have omitted the sector ‘wholesale and retail trade’, which is basically connected with all other sectors.

Herein, we have assumed $d_{ij} = \delta_{ij}$ as in Leontief’s classical input-output model. Moreover, in our simulations we have applied the common linear demand function

$$L_i(P_i) = \max(0, L^0_i - L^1_i P_i),$$

(71)

where $L^0_i$ and $L^1_i$ are non-negative parameters.

Due to the price dependence of consumption, economic systems have the important equilibration mechanism of price adjustment. These can compensate for undesired inventory levels and inventory changes. A mathematical equation reflecting this is

$$\frac{1}{P_i(t)} \frac{dP_i}{dt} = \nu_i \left( \frac{N^0_i}{N_i(t)} - 1 \right) - \mu_i \frac{dN_i}{dt}.$$  

(72)

The use of relative changes guarantees the required non-negativity of prices $P_i(t) \geq 0$. $\nu_i$ is an adaptation rate describing the sensitivity to relative deviations of the actual inventory $N_i(t)$ from the desired one $N^0_i$, and $\mu_i$ is a dimensionless parameter reflecting the responsiveness to relative deviations $(dN_i/dt)/N_i(t)$ from the stationary equilibrium state.

If the same criteria are applied to adjustments of the production rates $Q_i(t)$, we have the equation

$$\frac{1}{Q_i(t)} \frac{dQ_i}{dt} = \hat{\nu}_i \left( \frac{N^0_i}{N_i(t)} - 1 \right) - \hat{\mu}_i \frac{dN_i}{dt}.$$  

(73)
$\hat{\alpha}_i = \hat{\nu}_i / \nu_i$ is the ratio between the adjustment rate of the output flow and the adjustment rate of the price in sector $i$. For simplicity, the same ratio $\hat{\alpha}_i = \hat{\mu}_i / \mu_i$ will be assumed for the responsiveness.

4.1. Treating Producers Analogous to Consumers (with Dieter Armbruster)

According to Eqs. (69) and (71), the change of the consumption rate in time is basically given by

$$\frac{dY_i}{dt} = [Y_i^0 + \xi_i(t)] \frac{dL_i}{dP_i} \frac{dP_i}{dt} = -[Y_i^0 + \xi_i(t)] L_i \frac{dP_i}{dt} ,$$

i.e. it is basically proportional to the price change $dP_i/dt$, but with a negative and potentially fluctuating prefactor. However,

$$\frac{1}{Q_i(t)} \frac{dQ_i}{dt} \propto \frac{1}{P_i(t)} \frac{dP_i}{dt} ,$$

i.e. according to Eqs. (72) and (73), the change $dQ_i/dt$ of the production rate close to the equilibrium state with $Q_i(t) \approx Q_i^0$ and $P_i(t) \approx P_i^0$ is basically proportional to the change $dP_i/dt$ in the price with a positive prefactor. Is this an inconsistency of the model? Shouldn’t we better treat producers analogous to consumers?

In order to do so, we have to introduce the delivery flows $D_{ij}$ of products $i$ to producers $j$. As in Eq. (74), we will assume that their change $dD_{ij}/dt$ in time is proportional to the change $dP_i/dt$ in the price, with a negative (and potentially fluctuating) prefactor. We have now to introduce the stock level $O_i(t)$ in the output buffer of producer $i$ and the stock levels $I_{ij}(t)$ of product $i$ in the input buffers of producer $j$. For the output buffer, we find the balance equation

$$\frac{dO_i}{dt} = Q_i(t) - \sum_j D_{ij}(t) - Y_i(t) \tag{76}$$

analogous to Eq. (70), as the buffer is filled with the production rate of producer $i$, but is emptied by the consumption rate and delivery flows. For the input buffers we have the balance equations

$$\frac{dI_{ij}}{dt} = D_{ij}(t) - c_{ij} Q_j(t) , \tag{77}$$

as these are filled by the delivery flows, but emptied with a rate proportional to the production rate of producer $j$, where $c_{ij}$ are again the input coefficients specifying the relative quantities of required inputs $i$ for production. Generalizations of these equations are discussed elsewhere.38
Let us now investigate the differential equation for the change $dN_i/dt$ of the overall stock level of product $i$ in all input and output buffers. We easily obtain
\[
\frac{dN_i}{dt} = \frac{dQ_i}{dt} + \sum_j \frac{dI_{ij}}{dt} = Q_i(t) - \sum_j c_{ij}Q_j(t) - Y_i(t), \quad (78)
\]
as before. That is, the equations for the delivery flows drop out, and we stay with the previous set of equations for $N_i$, $Q_i$, $P_i$, and $Y_i$. As a consequence, we will focus on Eqs. (69) to (73) in the following.

5. Reproduction of Some Empirically Observed Features of Business Cycles

Our simulations of the above dis-aggregate (i.e. sector-wise) model of macroeconomic dynamics typically shows asynchronous oscillations, which seems to be characteristic for economic systems. Due to phase shifts between sectoral oscillations, the aggregate behavior displays slow variations of small amplitude (see Fig. 7). If the function $L_i(P_i)$ and the parameters $\nu_i/\mu_i^2$ are suitably specified, the non-linearities in Eqs. (70) to (69) will additionally limit the oscillation amplitudes.

Our business cycle theory differs from the dominating one\textsuperscript{49,50} in several favourable aspects:

(i) Our theory explains irregular, i.e. non-periodic oscillations in a natural way (see Fig. 7). For example, $w$-shaped oscillations result as superposition of the asynchronous oscillations in the different economic sectors, while other theories have to explain this observation by assuming external perturbations (e.g. due to technological innovations).

(ii) Although our model may be extended by variables such as the labor market, interest rates, etc., we consider it as a potential advantage that we did not have to couple variables in our model which are qualitatively that different. Our model rather focusses on the material flows among different sectors.

(iii) Moreover, we will see that our model can explain emergent oscillations, which are not triggered by external shocks.
Fig. 7. Typical simulation result of the time-dependent gross domestic product $P_i(t)$ in percent, i.e. relative to the initial value. The input matrix was chosen as in Figs. 6 and 9a–d, but $Y^0_i$ was determined from averaged input-output data. $Q^0_i$ was obtained from the equilibrium condition, and the fluctuations $\xi_i(t)$ were specified as a Gaussian white noise with mean value 0 and variance $\sigma_0 = 10^3$ (about 10% of the average final consumption). The initial prices $P_i(0)$ were selected from the interval $[0.9;1.1]$. Moreover, in this example we have assumed $L_i(P) = \max[0, 1 + d(P - P^0_i)]$ with $d = f'(P^0_i) = -L'_{i1} = -10$ and the parameters $\nu_i = 0.1$, $\mu_i = 0.0001$, $\alpha_i = 1 = P^0_i$, and $N^0_i = Y^0_i$. Although this implies a growth of small oscillations (cf. Fig. 1d), the oscillation amplitudes are rather limited. This is due to the non-linearity of model equations (70) to (69) and due to the phase shifts between oscillations of different economic sectors $i$. Note that irregular oscillations with frequencies between 4 to 6 years and amplitudes of about 2.5% are qualitatively well compatible with empirical business cycles. Our material flow model can explain v-shaped, non-periodic oscillations without having to assume technological shocks or externally induced perturbations. The long-term growth of national economies was intentionally not included in the model in order to separate this effect from network-induced instability effects. (After Ref. 5)

5.1. Dynamic Behaviors and Stability Thresholds

The possible dynamic behaviors of the resulting dis-aggregate macroeconomic model can be studied by analytical investigation of limiting cases and by means of a linear stability analysis around the equilibrium state (in which we have $N_i(t) = N^0_i$, $Y_i(t) = Y^0_i$, $Q^0_i - \sum_j c_{ij}Q^0_j = Y^0_i$, and $P_i(t) = P^0_i$). The linearized equations for the deviations $n_i(t) = N_i(t) - N^0_i$, $Q^0_i$,
Fig. 8. The displayed phase diagram shows which dynamic behavior is expected by our dis-aggregate model of macroeconomic dynamics depending on the respective parameter combinations. The individual curves for different countries are a result of the different structures of their respective input matrices $C$. As a consequence, structural policies can influence the stability and dynamics of economic systems.

$p_i(t) = P_i(t) - P_i^0$, and $q_i(t) = Q_i(t) - Q_i^0$ from the equilibrium state read

$$\frac{dn_i}{dt} = q_i - \sum_j c_{ij} q_j - Y_i^0 f_i'(P_i^0) p_i - \xi_i(t),$$

(79)

$$\frac{dp_i}{dt} = \frac{P_i^0}{N_i^0} \left( -\nu_i n_i - \mu_i \frac{dn_i}{dt} \right),$$

(80)

$$\frac{dq_i}{dt} = \frac{\hat{\alpha}_i Q_i^0}{N_i^0} \left( -\nu_i n_i - \mu_i \frac{dn_i}{dt} \right).$$

(81)

This system of coupled differential equations describes the response of the inventories, prices, and production rates to variations $\xi_i(t)$ in the demand. Denoting the $m$ eigenvalues of the input matrix $C = (c_{ij})$ by $J_{ii}$ with $|J_{ii}| < 1$, the $3m$ eigenvalues of the linearized model equations are 0 ($m$
times) and
\[ \lambda_{i,\pm} \approx \frac{1}{2} \left( -A_i \pm \sqrt{(A_i)^2 - 4B_i} \right), \]
(82)
where
\[ A_i = \mu_i [C_i + \hat{\alpha}_i D_i (1 - J_{ii})], \]
\[ B_i = \nu_i [C_i + \hat{\alpha}_i D_i (1 - J_{ii})], \]
\[ C_i = P^0_i Y^0_i \left| f'(P^0_i) \right| / N^0_i, \]
\[ D_i = Q^0_i / N^0_i, \]
(83)
see Fig. 8. Formula (82) becomes exact when the matrix \( C \) is diagonal or the parameters \( \mu_i C_i, \hat{\alpha}_i \mu_i D_i, \nu_i C_i \) and \( \hat{\alpha}_i \nu_i D_i \) are sector-independent constants, otherwise the eigenvalues must be numerically determined.

It turns out that the dynamic behavior mainly depends on the parameters \( \hat{\alpha}_i, \nu_i / \mu_i^2 \), and the eigenvalues \( J_{ii} \) of the input matrix \( C \) (see Fig. 9): In the case \( \hat{\alpha}_i \to 0 \) of fast price adjustment, the eigenvalues \( \lambda_{i,\pm} \) are given by
\[ 2\lambda_{i,\pm} = -\mu_i C_i \pm \sqrt{(\mu_i C_i)^2 - 4\nu_i C_i}, \]
(84)
i.e. the network structure does not matter at all. We expect an exponential relaxation to the stationary equilibrium for \( 0 < \nu_i / \mu_i^2 < C_i / 4 \), otherwise damped oscillations. Therefore, immediate price adjustments or similar mechanisms are an efficient way to stabilize economic and other supply systems. However, any delay \( \hat{\alpha}_i > 0 \) will cause damped or growing oscillations, if complex eigenvalues \( J_{ii} = \text{Re}(J_{ii}) + i \text{Im}(J_{ii}) \) exist. Note that this is the normal case, as typical supply networks in natural and man-made systems are characterised by complex eigenvalues (see top of Fig. 9).

Damped oscillations can be shown to result if all values
\[ \nu_i / \mu_i^2 = \hat{\alpha}_i \nu_i / \mu_i^2 \]
lie below the instability lines
\[ \nu_i / \mu_i^2 \approx \left\{ C_i + \hat{\alpha}_i D_i [1 - \text{Re}(J_{ii})] \right\} \times \left( 1 + \frac{[C_i + \hat{\alpha}_i D_i [1 - \text{Re}(J_{ii})]^2]}{[\hat{\alpha}_i D_i \text{Im}(J_{ii})]^2} \right) \]
(86)
given by the condition \( \text{Re}(\lambda_{i,\pm}) \leq 0 \). For identical parameters \( \nu_i / \mu_i^2 = \nu / \mu^2 \) and \( \hat{\alpha}_i = \hat{\alpha} \), the minimum of these lines agrees exactly with the numerically obtained curve in Fig. 9. Values above this line cause small oscillations to grow over time (see Appendix A).
Fig. 9. Properties of our dynamic model of supply networks for a characteristic input matrix specified as average input matrix of macroeconomic commodity flows of several countries (top) and for a synthetic input matrix generated by random changes of input matrix entries until the number of complex eigenvalues was eventually reduced to zero (bottom). Subfigures (a), (e) illustrate the color-coded input matrices $A$, (b), (f) the corresponding network structures, when only the strongest links (commodity flows) are shown, (c), (g) the eigenvalues $J_{ii} = \text{Re}(J_{ii}) + i \text{Im}(J_{ii})$ of the respective input matrix $A$, and (d), (h) the phase diagrams indicating the stability behavior of the model equations (70) to (69) on a double-logarithmic scale as a function of the model parameters $\hat{\alpha}_i = \hat{\alpha}$ and $\nu_i/\mu_i^2 = \nu/\mu^2 = V/M^2$. The other model parameters were set to $\nu_i = C_i = D_i = P_0^i = N_0^i = Y_0^i = 1$. Surprisingly, for empirical input matrices $A$, one never finds an overdamped, exponential relaxation to the stationary equilibrium state, but network-induced oscillations due to complex eigenvalues $J_{ii}$. (After Ref. 5)

In some cases, all eigenvalues $J_{ii}$ of the input matrix $C$ are real. This again applies to symmetric matrices $C$ and matrices equivalent to Jordan normal forms. Hence, the existence of loops in supply networks is no sufficient condition for complex eigenvalues $J_{ii}$ (see also Fig. 1f). It is also no necessary condition. For cases with real eigenvalues only, Eq. (82) predicts a stable, overdamped behavior if all values $\nu_i/\mu_i^2 = \hat{\nu}_i/\hat{\mu}_i^2$ lie below the lines

$$\nu_i/\mu_i^2 \approx \frac{[C_i + \hat{\alpha}_i D_i (1 - J_{ii})]}{4}$$

defined by $\min_i (A_i^2 - 4B_i) > 0$ (see Appendix B). For identical parameters $\nu_i/\mu_i^2 = \nu/\mu^2$ and $\hat{\alpha}_i = \hat{\alpha}$, the minimum of these lines corresponds exactly to the numerically determined curve in Fig. 9. Above it, one observes damped oscillations around the equilibrium state, but growing oscillations...
are not possible. In supply systems with a slow or non-existent price adjustment mechanism (i.e. for $\hat{\alpha}_i \gg 1$ or $C_i = 0$), Eq. (87) predicts an overdamped behavior for real eigenvalues $J_{ii}$ and

$$\hat{\nu}_i/\hat{\mu}_i^2 < D_i(1 - J_{ii})/4$$

(88)

for all $i$, while Eq. (86) implies the stability condition

$$\hat{\nu}_i/\hat{\mu}_i^2 < D_i[1 - \Re(J_{ii})][1 + [1 - \Re(J_{ii})]^2/\Im(J_{ii})^2]$$

(89)

for all $i$, given that some eigenvalues $J_{ii}$ are complex. Moreover, for the case of sector-independent constants $V = \hat{\alpha}_i\nu_i D_i$ and $W = \hat{\alpha}_i\mu_i D_i$, the eigenvalues $\lambda_{i,\pm}$ can be calculated as

$$\lambda_{i,\pm} = -W(1 - J_{ii})/2 \pm \sqrt{[W(1 - J_{ii})]^2/4 - V(1 - J_{ii})}.$$  

(90)

Considering that Eq. (36) for the adjustment of production speeds is slightly more general than specification (81), this result is fully consistent with Eq. (59).

6. Summary

In this contribution, we have investigated a model of supply networks. It is based on conservation equations describing the storage and flow of inventories by a dynamic variant of Leontief’s classical input-output model. A second set of equations reflects the delayed adaptation of the production rate to some inventory-dependent desired production rate. Increasing values of the adaptation time $T$ tend to destabilize the system, while increasing values of the time horizon $\Delta t$ over which inventories are forecasted have a stabilizing effect.

A linear stability analysis shows that supply networks are expected to show an overdamped or damped oscillatory behavior, when the supply matrix is symmetrical. Some regular supply networks such as sequential supply chains or regular supply hierarchies with identical parameters show stable system behavior for all values of the adaptation time $T$. Nevertheless, one can find the bullwhip effect under certain circumstances, i.e. the oscillations in the production rates are larger than the variations in the consumption rate (demand). The underlying mechanism is a resonance effect.

Regular supply networks are characterized by multiply degenerate eigenvalues. When the related supply matrices are slightly perturbed, the degeneration is broken up. Instead of two degenerate eigenvalues, one finds complex eigenvalues which approximately lie on a circle around them in the complex plane. The related heterogeneity in the model parameters can
have a stabilizing effect\cite{38} when they manage to reduce the resonance effect. Moreover, a randomly perturbed regular supply network may become linearly unstable, if the perturbation is large. If the supply matrix is irregular, the supply network is generally expected to show either damped or growing oscillations. In any case, supply networks can be mapped onto a generalized sequential supply chain, which allows one to define the bullwhip effect for supply networks.

While previous studies have focussed on the synchronization of oscillators in different network topologies\cite{8,51}, we have found that many supply networks display damped oscillations, even when their units—and linear chains of these units—behave in an overdamped way. Furthermore, networks of damped oscillators tend to produce growing (and mostly asynchronous) oscillations. Based on these findings, it is promising to explain business cycles as a result of material flows among economic sectors, which adjust their production rates in a decentralized way. Such a model can be also extended by aspects like labor force, money flows, information flows, etc.

7. Future Research Directions

7.1. Network Engineering

Due to the sensitivity of supply systems to their network structure, network theory\cite{7,8,9,10,11,12,13,14,52} can make useful contributions: On the basis of Eqs. (86) and (87) one can design stable, robust, and adaptive supply networks ("network engineering"). For this reason, our present studies explore the characteristic features of certain types of networks: random networks, hierarchical networks, small world networks, preferential attachment networks, and others. In this way, we want to identify structural measures which can increase the robustness and reliability of supply networks, also their failure and attack tolerance. These results should, for example, be relevant for the optimization of disaster management\cite{53}.

7.2. Cyclic Dynamics in Biological Systems

Oscillations are probably not always a bad feature of supply systems. For example, in systems with competing goals (such as pedestrian counterflows at bottlenecks\cite{54} or intersecting traffic streams), oscillatory solutions can be favourable. Oscillations are also found in ecological systems\cite{55} and some metabolic processes\cite{56,57,58,59}, which could be also treated by a generalized model of supply networks. Examples are oscillations of nutrient flows in
slime molds, the citrate cycle, the glycolytic oscillations in yeast, and the Calcium cycle. Considering the millions of years of evolution, it is highly unlikely that these oscillations do not serve certain biological functions. Apart from metabolic flows, however, we should also mention protein networks and metabolic expression networks as interesting areas of application of (generalized) supply network models.

### 7.3. Heterogeneity in Production Networks

Another interesting subject of investigation would be heterogeneity. The relevance of this issue is known from traffic flows and multi-agent systems in economics. Preliminary results indicate that, depending on the network structure, heterogeneous parameters of production units can reduce the bullwhip effect in supply networks. That is, identical production processes all over the world may destabilize economic systems (as monocultures do in agriculture). However, in heterogeneous systems a lot more frequencies are influencing a single production step, which may also have undesirable effects. These issues deserve closer investigation.

### 7.4. Multi-Goal Control

Let us go back to Eq. (34) for the control of supply or production rates. This equation is to be applied to cases where a production unit produces several products and the production rate is not only adapted to the inventory of one dominating product, but also to the inventories of some other products. This leads to the problem of multi-goal control. It is obvious that the different goals do not always have to be in harmony with each other: A low inventory of one product may call for an increase in the production rate, but the warehouse for another product, which is generated at the same time, may be already full. A good example is the production of chemicals. We conjecture that multi-goal control of production networks may imply additional sources of dynamical instabilities and that some of the problems may be solved by price adjustments, which can influence the consumption rates. Again, a closer investigation of these questions is needed.

### 7.5. Non-Linear Dynamics and Scarcity of Resources

Finally, we should note that coupled equations for damped harmonic oscillators approximate the dynamic behavior of supply networks only close to their stationary state. When the oscillation amplitudes become too large,
non-linearities will start to dominate the dynamic behavior. For example, they may select wave modes or influence their phases in favour of a synchronized behavior, etc. Deterministically chaotic behavior seems to be possible as well.

Additional non-linearities come into play, if there is a scarcity of resources required to complete certain products. Let’s assume that the rate of transferring products \( i \) to \( j \) is \( Z_{ji} \). If \( N_i \) is the number of products that may be delivered, the maximum delivery rate is therefore limited by \( Z_{ji}N_i(t) \). The maximum production rate is given by the minimum of the delivery rate of all components \( i \), divided by the respective number \( c_{ij} \) of parts required to complete one unit of product \( i \). In mathematical terms, we have to make the replacement

\[
Q_j(t) \rightarrow Q_j(t) \min_i \left( 1, \frac{Z_{ji}N_i(t)}{c_{ij}} \right)
\]

in Eq. (70) and corresponding replacements in all derived formulas. Therefore, the generalized relationship for systems in which resources may run short is

\[
\frac{dN_i}{dt} = \sum_{j=1}^{u} (d_{ij} - c_{ij})Q_j(t) \min_k \left( 1, \frac{Z_{jk}N_k(t)}{c_{jk}} \right) - Y_i(t).
\]

These coupled non-linear equations are expected to result in a complex dynamics, if the transportation rate \( Z_{jk} \) or inventory \( N_k \) are too small. Note that this non-linearity may be even relevant close to the stationary state. In such cases, results of linear stability analyses for systems with scarce resources would be potentially misleading.

In situations where scarce resources may be partially substituted by other available resources, one may use the replacement

\[
Q_j(t) \rightarrow \left[ 1 + \sum_i \left( \frac{Z_{ji}N_i(t)}{c_{ij}} \right)^z_j \right]^{1/z_j}
\]

instead of (91). Such an approach is, for example, reasonable for disaster management. \( z_j \) is a parameter allowing to fit the ease of substitution of resources. The minimum function results in the limit \( z_j \rightarrow -\infty \).

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Appendix A. Boundary between Damped and Growing Oscillations

Starting with Eq. (82), stability requires the real parts $\text{Re}(\lambda_i)$ of all eigenvalues $\lambda_i$ to be non-positive. Therefore, the stability boundary is given by $\max_i \text{Re}(\lambda_i) = 0$. Writing

$$C_i + \hat{\alpha}_i D_i (1 - J_{ii}) = \hat{\theta}_i + i \hat{\beta}_i$$

(A.1)

with $C_i = P_0^i Y_0^i |f'(P_0^i)| / N_0^i$ and defining

$$\hat{\theta}_i = C_i + \hat{\alpha}_i D_i [1 - \text{Re}(J_{ii})]$$,
$$\hat{\beta}_i = \mp \hat{\alpha}_i D_i \text{Im}(J_{ii})$$ (complex conjugate eigenvalues),
$$\hat{\gamma}_i = 4 \nu_i / \mu_i^2$$

(A.2)

we find

$$2 \lambda_i / \mu_i = - \hat{\theta}_i - i \hat{\beta}_i + \sqrt{R_i + i I_i}$$

(A.3)

with

$$R_i = \hat{\theta}_i^2 - \hat{\beta}_i^2 - \hat{\gamma}_i \hat{\theta}_i$$ and
$$I_i = 2 \hat{\theta}_i \hat{\beta}_i - \hat{\gamma}_i \hat{\beta}_i$$

(A.4)

The real part of (A.3) can be calculated via the relation

$$\text{Re} \left( \sqrt{R_i \pm i I_i} \right) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{R_i^2 + I_i^2} + R_i}$$

(A.5)

The condition $\text{Re}(2 \lambda_i / \mu_i) = 0$ is fulfilled by $\hat{\gamma}_i = 0$ and

$$\hat{\gamma}_i = 4 \nu_i (1 + \hat{\theta}_i^2 / \hat{\beta}_i^2)$$

(A.6)

i.e. the stable regime is given by

$$\frac{\hat{\gamma}_i}{4} = \frac{\nu_i}{\mu_i^2} = \frac{\hat{\alpha}_i \nu_i}{\mu_i^2} \leq \hat{\theta}_i \left( 1 + \frac{\hat{\theta}_i^2}{\hat{\beta}_i^2} \right)$$

(A.7)

for all $i$, corresponding to Eq. (86).\textsuperscript{5}
Appendix B. Boundary between Damped Oscillations and Overdamped Behavior

For $\hat{\alpha}_i > 0$, the imaginary parts of all eigenvalues $\lambda_i$ vanish if $\text{Im}(J_{ii}) = 0$ (i.e. $\hat{\beta}_i = 0$) and if $R_i \geq 0$. This requires

\[
\frac{4\hat{\alpha}_i}{\mu_i^2} = \hat{\gamma}_i \leq \frac{\hat{\beta}_i^2}{\tilde{\theta}_i} = \tilde{\theta}_i = C_i + \hat{\alpha}_i D_i (1 - J_{ii})
\]

(B.1)

for all $i$, corresponding to Eq. (87).

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