Lukewarm black holes in quadratic gravity.

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I. INTRODUCTION

A complete description of the gravitational phenomena should be given by the quantum gravity, which, in turn, may be a part of the even more fundamental theory. Unfortunately, at the present stage we have no clear idea how such a theory should be constructed. It is expected, however, that the action functional describing its low-energy approximation should consist of the higher order terms constructed from the curvature and its covariant derivatives to some required order. Such generalizations of the Einstein-Hilbert action functional necessarily introduce a number of coupling constants, which, at the present state of affairs, should be determined empirically.

For traces of possible breakdown of the classical General Relativity one, quite naturally, turns to cosmology. Indeed, it is possible that such concepts as dark matter or/and dark energy proposed in order to explain observational data should be abandoned in favour of modifications of the classical gravitational Lagrangian (see e.g. Ref. 1 and references cited therein).

Among various modifications of the general relativity proposed so far, a prominent role is played by the quadratic gravity. Various aspects of such theories have been discussed extensively in the literature. (See for example Refs. 2-12). As is well-known the motivations for introducing linear combination of $R^2$, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ into the gravitational action functional are numerous. For example, when invented, the equations of quadratic gravity have been treated as an exact formulation of the theory of gravitation. For historic informations and important references the reader is referred to Ref. 13. It may be considered, quite naturally, as truncation of series expansion of the action of the more general theory. Moreover, from the point of view of the semi-classical gravity it might be treated as some sort of a poor man’s stress-energy tensor, allowing in a relatively simple way to mimic, especially when the application of the full stress-energy tensor would produce extremely complicated result, the fairly more complex source term of the field equations. This is especially true when the right-hand-side of the semiclassical Einstein field equations is taken to be the renormalized stress-energy tensor of the quantized fields in a large mass limit constructed within the Schwinger-DeWitt framework. Such tensors comprise the linear combination of the purely geometric terms and the spin of the field enters through the numeric coefficients [14–18]. As compared with the General Relativity the Lagrangian of the quadratic gravity in four dimensions requires two additional terms

$$\alpha R + \beta \alpha^2,$$

where $\alpha$ and $\beta$ are the coupling constants. The possible third term constructed form the Kretschmann scalar can be relegated by means of the Gauss-Bonnet term.

In general, the equations derived form the higher-order action functional are very hard to solve. Fortunately, since the coupling constants are expected to be small one can easily employ a perturbative approach to the problem treating the classical solution of the Einstein field equations as the zeroth-order of the approximation. Successive perturbations are therefore solutions of the differential equations of ascending complexity. It should be noted that although the method is clear the calculations beyond the first-order may by intractable. Such perturbative approach is in concord with the philosophy of the effective theories.

Exact and approximate solutions to the equations of the quadratic gravity have been studied in a number of papers. Of particular interest are the spherically-symmetric configurations which, potentially, may describe black holes. The black hole solutions have been studied for various sources in Refs. 19-24. Here we shall analyze a particular class of solutions of the equations of the quadratic gravity with the cosmological term describing lukewarm black holes. For the Reissner-Nordström-de Sitter class of solutions it is possible to find configurations in which the event and cosmological horizons have the same temperature. These solutions are known as lukewarm black holes and the interest in them stems from the fact that if $T_H = T_c$ it is possible to construct a regular thermal state for a two dimensional models. Moreover, analyses of the field fluctuation indicate that this is the case in four dimensions. (It should be noted that such configurations are prohibited in the geometry of the Schwarzschild-deSitter black hole.) The natural question that arises in connection with the foregoing discussion is whether or not it is possible to construct
the lukewarm black hole in the quadratic gravity. And although the full, detailed answer is beyond our capabilities, it is possible to provide an affirmative answer to the restricted problem. Indeed, since the complexity of the coupled equations of the quadratic gravity and electrodynamics, even in the simplest case of spherically-symmetric and static geometries, hinders construction of the exact solution, one has to refer to the analytical approximations or numerical methods. Here we shall employ the perturbative approach. The obtained results can also be viewed as a first step towards incorporation of the quantum effects into the picture. It is because the renormalized stress-energy tensor of the quantized massive field may be approximated by the object constructed from the curvature tensor, its covariant derivatives and contractions.

II. EQUATIONS OF THE QUADRATIC GRAVITY

The coupled system of the classical electrodynamics and the quadratic gravity is described by the action

\[ S = \frac{1}{16\pi G} S_g + S_m \]  

(1)

where

\[ S_g = \int (R + 2\Lambda + \alpha R^2 + \beta R_{ab}R^{ab}) \sqrt{-g} \, d^4x, \]  

(2)

and

\[ S_m = -\frac{1}{16\pi} \int F_{ab}F^{ab} \sqrt{-g} \, d^4x, \]  

(3)

where all symbols have their usual meaning. The third possible term constructed form the Kretschmann scalar, \( R_{abcd}R^{abcd} \), may be removed from the Lagrangian with the help of the Gauss-Bonnet invariant

\[ R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2. \]  

(4)

Of numerical parameters \( \alpha \) and \( \beta \) we assume, as usual, that they are small and of comparable order, otherwise they would lead to the observational consequences within our solar system. Their ultimate values should be determined from observations of light deflection, binary pulsars and cosmological data [31–33]. Moreover, following Ref. 19, we shall restrict ourselves to spacetimes of small curvatures, for which the conditions

\[ |\alpha R| \ll 1, \quad |\beta R_{ab}| \ll 1 \]  

(5)

hold. Although the additional constraints could be obtained from non-tachyon conditions [12, 21], which, in turn, are simple consequences of demanding the linearized equations to posses a real mass, we shall treat the parameters \( \alpha \) and \( \beta \) as small but arbitrary.

Differentiating functionally the action \( S \) with respect to the metric tensor one has

\[ L^{ab} = G^{ab} + \Lambda g_{ab} - \alpha I^{ab} - \beta J^{ab} = 8\pi T^{ab}, \]  

(6)

where

\[ I^{ab} = 2R^{-ab} - 2RR^{ab} + \frac{1}{2}g^{ab}(R^2 - 4\Box R) \]  

(7)

and

\[ J^{ab} = R^{;ab} - \Box R^{ab} - 2R_{cd}R^{cd} + \frac{1}{2}g^{ab}(R_{cd}R^{cd} - \Box R). \]  

(8)

Let us consider the spherically symmetric and static configuration described by the line element of the form

\[ ds^2 = -e^{2\psi(r)} f(r) dt^2 + \frac{ds^2}{f(r)} + r^2 d\Omega^2, \]  

(9)

where

\[ f(r) = 1 - \frac{2M(r)}{r}. \]  

(10)
The spherical symmetry places restrictions on the components of $F_{ab}$ tensor and its only nonvanishing components compatible with the assumed symmetry are $F_{01}$ and $F_{23}$. Simple calculations yield the stress-energy tensor in the form

$$T^t_t = T^r_r = \frac{Q^2 + P^2}{8\pi r^4} = -\frac{Z^2}{8\pi r^4}$$  \hspace{1cm} (11)$$
and

$$T^\theta_\theta = T^\phi_\phi = \frac{Q^2 + P^2}{8\pi r^4} = \frac{Z^2}{8\pi r^4},$$  \hspace{1cm} (12)$$
where the integration constants $Q$ and $P$ are interpreted as the electric and magnetic charge, respectively.

### III. PERTURBATIVE SOLUTION

#### A. General Case

To simplify calculations and to keep control of the order of terms in complicated series expansions we shall introduce a dimensionless parameter $\varepsilon$ substituting $\alpha \rightarrow \varepsilon \alpha$ and $\beta \rightarrow \varepsilon \beta$. We shall put $\varepsilon = 1$ at the final stage of calculations. Of functions $M(r)$ and $\psi(r)$ we assume that they can be expanded as

$$M(r) = M_0(r) + \varepsilon M_1(r) + \mathcal{O}(\varepsilon^2)$$  \hspace{1cm} (13)$$
and

$$\psi(r) = \varepsilon \psi_1(r) + \mathcal{O}(\varepsilon^2).$$  \hspace{1cm} (14)$$
Consider the left hand side of Eq. (11) calculated for the line element (9) first. Making use of the above expansions and collecting the terms with the like power one obtains

$$L^t_t = \Lambda - \frac{2}{r^2} (M_0' + \varepsilon M_1' - \varepsilon S^t_t),$$  \hspace{1cm} (15)$$
where

$$S^t_t = \beta \left( \frac{2 M_0'}{r^2} - \frac{8 M_0 M_0'}{r^3} + \frac{2 M_0^{'2}}{r^2} - \frac{2 M_0''}{r} + \frac{5 M_0 M_0''}{r^2} - \frac{M_0' M_0''}{r} \right.$$
$$+ \frac{M_0''^2}{2} + M_0^{(3)} - \frac{M_0 M_0^{(3)}}{r} - M_0' M_0^{(3)} + r M_0^{(4)} - 2 M_0 M_0^{(4)} \left. \right)$$
$$- \alpha \left( \frac{24 M_0 M_0'}{r^3} - \frac{8 M_0'}{r^2} - \frac{4 M_0^{'2}}{r^2} + \frac{8 M_0''}{r} - \frac{18 M_0 M_0''}{r^2} - \frac{M_0''^2}{r} \right.$$
$$+ \frac{2 M_0' M_0''}{r} - 4 M_0^{(3)} + \frac{6 M_0 M_0^{(3)}}{r} + 2 M_0' M_0^{(3)} - 2 r M_0^{(4)} + 4 M_0 M_0^{(4)} \left. \right)$$  \hspace{1cm} (16)$$
and primes as well as $M_0^{(i)}$ for $i \geq 3$ denote derivatives with respect to the radial coordinate. It should be noted that Eq. (15) does not depend on the function $\psi(r)$. The zeroth-order and the first-order equations integrated with the conditions $M_0(r_+) = r_+/2$ and $M_1(r_+) = 0$, respectively, give

$$f(r) = 1 - \frac{r_+}{r} - \frac{Z^2}{rr_+} + \frac{\Lambda r^3}{3 r} + \frac{Z^2}{r^2} - \frac{\Lambda r^2}{3} - \left( \frac{8}{r^2} - 8 \frac{\Lambda Z^2}{r r_+} \right) \alpha$$
$$- \left( \frac{12}{5} \frac{Z^4}{r^6} + 4 \frac{Z^2}{r^4} + \frac{Z^2 r_+^3 \Lambda}{r^5} - \frac{\Lambda Z^2}{rr_+} - 3 \frac{Z^2 r_+}{r^5} - 3 \frac{Z^2}{r^5 r_+} + 3 \frac{Z^2}{5 r r_+} - \frac{Z^2}{r r_+^3} \right) \beta.$$  \hspace{1cm} (17)
It should be noted that a correct choice of \( r_+ \) in the zeroth-order solution requires some prescience. Indeed, for a given \( Z^2 \) and \( \Lambda \) it should allow for a positive root, say \( c \), interpreted as the cosmological horizon which satisfies \( r_+ \leq c \). Remaining roots, say \( a \) and \( b \), should satisfy \( a \leq b \leq r_+ \leq c \).

On the other hand, the difference between the radial and time component of the tensor \( L^b_a \) can be easily integrated to yield

\[
\psi_1(r) = (2\alpha + \beta)M_0^{(3)} - \frac{4}{r^2}(3\alpha + \beta)M_0 + C_1,
\]

where \( C_1 \) is the integration constant. To determine \( C_1 \) we shall adopt the natural condition

\[
g_{tt}(r_\infty)g_{rr}(r_\infty) = -1,
\]

or, equivalently,

\[
\psi(r_\infty) = 0,
\]

where \( r_\infty \) is either infinity or the cosmological horizon or the radius of a cavity in which the system is placed (the latter case is not considered in this paper). Simple integration gives

\[
\psi_1(r) = \beta Z^2 \left( \frac{1}{r^4} - \frac{1}{r_\infty^4} \right)
\]

and the equations (17) and (21) provide complete first-order solution to the problem. In what follows, for simplicity, we shall choose \( r_\infty = \infty \). The other choices do not influence location of \( r_c \) nor the relations describing equality of temperatures of the cosmological and the event horizons.

The zeroth-order solution can be expressed in a more familiar form using, for example, the Abbott-Deser mass. Indeed, if the coefficient that stands in front of \( r^{-1} \) is small it is possible to relate it with the Abbott-Deser mass.

On the other hand, one can introduce the parameter

\[
\mathcal{M} = \frac{r_+}{2} + \frac{Z^2}{2r_+} - \frac{\Lambda r_+^3}{6}
\]

to get

\[
f(r) = 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}.
\]

Simple manipulations in Eq. (22) give

\[
1 - \frac{2\mathcal{M}}{r_+} + \frac{Q^2}{r_+^2} - \frac{\Lambda r_+^2}{3} = 0
\]

and the parameter \( \mathcal{M} \) can be referred to as the horizon defined mass.

Returning to the first order solution one concludes that the cosmological horizon is given by

\[
r_c = \hat{r}_c + \varepsilon r^{(1)}_c,
\]

where

\[
r^{(1)}_c = -\frac{4A Q^2 (\hat{r}_c - r_+)}{\kappa_c r_+^2 r_+} + \frac{3(\hat{r}_c^5 - 5r_+^4\hat{r}_c + 4r_+^5)}{10\kappa_c r_+^9} \beta
\]

\[
- \frac{(-4\hat{r}_c r_+^3 + \Lambda r_+^4) Q^2}{2\kappa_c r_+^{12} + r_+^{12}} \beta
\]

and \( \kappa_c = \hat{f}'(\hat{r}_c)/2 \), provided \( \hat{r}_c \) is the greatest positive root of \( \hat{f}(r) = 0 \). The function \( \hat{f}(r) \) is the unperturbed part of \( f(r) \). Equations (17), (21) and (26) constitute the full (first-order) solution solution to the problem.
Now, we shall restrict ourselves to the particular class of solutions describing black holes for which temperature of the event horizon equals the temperature of the cosmological horizon. Consider the zeroth-order solution ($\varepsilon = 0$) first. Solving the system
\[ f(\tilde{r}_c) = 0 \quad \text{and} \quad f'(r_p) + f'(\tilde{r}_c) = 0 \]
with respect to $\Lambda$ and $Z^2$ one has
\[ \Lambda = \frac{3}{(r_+ + \tilde{r}_c)^2} \quad \text{and} \quad Z^2 = \left( \frac{r_+ \tilde{r}_c}{r_+ + \tilde{r}_c} \right)^2. \tag{28} \]
In the $(r_+, \tilde{r}_c)$—parametrization the line element assumes the form with $\psi(r) = 0$ and
\[ \tilde{f}(r) = \left( 1 - \frac{r_+ \tilde{r}_c}{(r_+ + \tilde{r}_c)r} \right)^2 - \frac{r^2}{(r_+ + \tilde{r}_c)^2} \]
whereas in the $(l, z)$—parametrization one has
\[ \tilde{f}(r) = \left( 1 - \frac{z}{r} \right)^2 - \frac{r^2}{l^2} \tag{30} \]
where $z = \sqrt{Q^2 + P^2}$ and $l = \sqrt{3/\Lambda}$. For $4z < l$ one has two pair of roots:
\[ r_+ = \frac{l}{2} \left[ 1 - \sqrt{1 - \frac{4z}{l}} \right] \quad \text{and} \quad \tilde{r}_c = \frac{l}{2} \left[ 1 + \sqrt{1 - \frac{4z}{l}} \right], \tag{31} \]
which are interpreted as the event and cosmological horizons, and
\[ r_{-\text{}} = \frac{l}{2} \left[ 1 + \sqrt{1 + \frac{4z}{l}} \right] \quad \text{and} \quad r_{-\text{}} = \frac{l}{2} \left[ -1 + \sqrt{1 + \frac{4z}{l}} \right]. \tag{32} \]
The positive root $r_{-\text{}}$ is interpreted as the inner horizon whereas $r_{-\text{}}$ is a negative root which has no physical meaning.

By the construction the temperature of the event horizon equals that of the cosmological horizon
\[ T_H = T_c = \frac{\tilde{r}_c - r_+}{2\pi(r_+ + \tilde{r}_c)^2}. \tag{33} \]
Such configurations are usually referred to as the lukewarm black holes $^{23-27, 29, 30}$. From the point of view of the quantum field theory in curved background the lukewarm black holes are special. It has been shown that for the two-dimensional models it is possible to construct a regular thermal state $^{28}$. Moreover, recent calculations of the vacuum polarization indicate that it is regular on both the event and cosmological horizons of the D=4 lukewarm RN-dS black holes $^{29, 30}$.

The lukewarm configuration in the quadratic gravity should simultaneously satisfy
\[ f(r_c) = 0 \quad \text{and} \quad T_H = T_c. \tag{34} \]
The Euclidean section is regular at $r_+$ and $r_c$ if $\tau = it$ is periodic with a period $2\pi/\kappa$, where $\kappa$ is the surface gravity. The equality of the temperatures is equivalent to
\[ \left( \frac{1}{\sqrt{g_{\tau\tau}g_{rr}}} \frac{dg_{\tau\tau}}{dr} \right)_{r=r_+} + \left( \frac{1}{\sqrt{g_{\tau\tau}g_{rr}}} \frac{dg_{\tau\tau}}{dr} \right)_{r=r_c} = 0, \tag{35} \]
and, consequently, the lukewarm configuration is described by the above equation and the first equation of $^{33}$. Since $r_+$ always denotes the exact location of the event horizon, mathematically, we have two relations for three parameters and to describe the lukewarm configuration completely Eqs. $^{33}$ should be supplemented by additional condition. Here, following Ref. $^{32}$, we shall adopt the point of view that the cosmological constant is not a parameter of the
space of solutions, but, rather, the parameter in the space of theories. To construct the perturbed lukewarm black hole let us substitute (25) and
\[ Z^2 = \left( \frac{r_+ \tilde{r}_c}{r_+ + \tilde{r}_c} \right)^2 + \varepsilon \Delta \] into the line element. Solving the system (34) one obtains
\[ \Delta = \frac{24 \tilde{r}_c^2 r_+^2 \alpha}{(r_+ + \tilde{r}_c)^3} + \frac{2 r_+ \left( 5 \tilde{r}_c^4 + 20 \tilde{r}_c^2 r_+^2 + 28 r_+^3 r_+ + 6 \tilde{r}_c r_+^3 + r_+^4 \right) \beta}{5 (r_+ + \tilde{r}_c)^3} \] and
\[ r_c^{(1)} = \frac{\beta (\tilde{r}_c - r_+)^2 \left( \tilde{r}_c^2 + 4 \tilde{r}_c r_+ + r_+^2 \right)}{5 \tilde{r}_c (r_+ + \tilde{r}_c)^3 r_+} \] The higher-order curvature corrections to the geometry of the classical lukewarm black holes can readily be obtained by substituting (37) and (38) into the line element (9), and, after expansion, retaining the terms linear in \( \varepsilon \).

It is of some interest to examine a few special cases of the results (36-38). In the limit \( \tilde{r}_c \to \infty \) the correction \( \Delta \) tends to zero and one obtains the first order corrections to the extreme Reissner-Nordström solution with the near-horizon geometry of the type \( AdS_2 \times S^2 \). If \( \beta = 0 \) and \( \alpha \neq 0 \) the correction to the cosmological horizon vanishes and
\[ Z^2 = \left( \frac{r_+ \tilde{r}_c}{r_+ + \tilde{r}_c} \right)^2 + \frac{24 \tilde{r}_c^2 r_+^2 \alpha}{(r_+ + \tilde{r}_c)^3} \] On the other hand, if \( \alpha = 0 \) and \( \beta \neq 0 \) both \( \Delta \) and \( r_c \) are always of the same sign as \( \beta \). Finally, if \( 3 \alpha + \beta = 0 \) both \( \Delta \) and \( r_c \) are always of the same sign as \( -\alpha \). Additional constraints can be obtained from the non-tachyon conditions.

**IV. FINAL REMARKS**

In this paper we have constructed the first order-correction to the geometry of the Reissner-Nordström-deSitter black holes within the framework of the quadratic gravity. Special emphasis has been put on the lukewarm configurations which are characterized by the equality of the temperature of the event and cosmological horizon. Though we have studied the consequences of the inclusion of the simplest higher-order correction to the gravitational action our results can shed some light on the important issue of the semi-classical lukewarm black holes. Since the approximate renormalized effective action of the quantized massive fields in the large mass limit is constructed solely form the curvature invariants (the type of the field is encoded in the numerical coefficients) the analysis should follow the same lines as the analysis presented here. This strongly support the hypothesis that solutions describing the quantum-corrected lukewarm black holes do exist. We intend to return to this group of problems elsewhere.

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