ANALYTIC CONTINUATION OF WEIGHTED \( q \)-GENOCCHI NUMBERS AND POLYNOMIALS

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ABSTRACT. In the present paper, we analyse analytic continuation of weighted \( q \)-Genocchi numbers and polynomials. A novel formula for weighted \( q \)-Genocchi-Zeta function \( \tilde{\zeta}_{G,q}(s | \alpha) \) in terms of nested series of \( \tilde{\zeta}_{G,q}(n | \alpha) \) is derived. Moreover, we introduce a novel concept of dynamics of the zeros of analytically continued weighted \( q \)-Genocchi polynomials.

1. INTRODUCTION

In this paper, we use notations like \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \), where \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{R} \) denotes the field of real numbers and \( \mathbb{C} \) also denotes the set of complex numbers. When one talks of \( q \)-extension, \( q \) is variously considered as an indeterminate, a complex number or a \( p \)-adic number.

Throughout this work, we will assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). The \( q \)-integer symbol \( [x : q] \) denotes as

\[
[x : q] = \frac{q^x - 1}{q - 1}.
\]

Firstly, analytic continuation of \( q \)-Euler numbers and polynomials was investigated by Kim in \([1]\). He gave a new concept of dynamics of the zeros of analytically continued \( q \)-Euler polynomials. Actually, we were motivated from his excellent paper which is "Analytic continuation of \( q \)-Euler numbers and polynomials, Applied Mathematics Letters 21 (2008) 1320-1323." We also procure to analytic continuation of weighted \( q \)-Genocchi numbers and polynomials as parallel to his article. Also, we give some interesting identities by using generating function of weighted \( q \)-Genocchi polynomials.

2. PROPERTIES OF THE WEIGHTED \( q \)-GENOCCHI NUMBERS AND POLYNOMIALS

For \( \alpha \in \mathbb{N} \cup \{0\} \), the weighted \( q \)-Genocchi polynomials are defined by means of the following generating function:

For \( x \in \mathbb{C} \),

\[
\sum_{n=0}^{\infty} \tilde{G}_{n,q}(x | \alpha) t^n / n! = [2 : q] t \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x:q^n]}. \tag{2.1}
\]

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As a special case $x = 0$ into (2.1), $\tilde{G}_{n,q}(0 | \alpha) := \tilde{G}_{n,q}(\alpha)$ are called weighted $q$-Genocchi numbers. By (2.1), we readily derive the following

\begin{equation}
\frac{\tilde{G}_{n+1,q}(x | \alpha)}{n+1} = \binom{2 : q}{\alpha : q}^{n} (1 - q)^{n} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} \frac{q^{\alpha l x}}{1 + q^{\alpha l + 1}},
\end{equation}

where \((\cdot)^{n}\) is the binomial coefficient. By expression (2.1), we see that

\begin{equation}
\tilde{G}_{n,q}(x | \alpha) = \left( q^{-\alpha x} \tilde{G}_{q}(\alpha) + [x : q^{\alpha}] \right)^{n},
\end{equation}

with the usual convention of replacing \(\left( \tilde{G}_{q}(\alpha) \right)^{n}\) by \(\tilde{G}_{n,q}(\alpha)\) is used (for details, see [7], [8]).

Let $\tilde{T}_{q}^{(\alpha)}(x, t)$ be the generating function of weighted $q$-Genocchi polynomials as follows:

\begin{equation}
\tilde{T}_{q}^{(\alpha)}(x, t) = \sum_{n=0}^{\infty} \tilde{G}_{n,q}(x | \alpha) \frac{t^{n}}{n!}.
\end{equation}

Then, we easily notice that

\begin{equation}
\tilde{T}_{q}^{(\alpha)}(x, t) = [2 : q] t \sum_{n=0}^{\infty} (-1)^{n} q^{n} e^{t[n+x : q^{\alpha}]}.
\end{equation}

From expressions (2.4) and (2.5), we procure the followings:

For $k (=\text{even})$ and $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

\begin{equation}
q^{k} \frac{\tilde{G}_{n+1,q}(k | \alpha)}{n+1} - \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} = [2 : q] \sum_{l=0}^{k-1} (-1)^{l} q^{k-l-1} \left[ l : q^{\alpha} \right]^{n}.
\end{equation}

For $k (=\text{odd})$ and $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

\begin{equation}
q^{k} \frac{\tilde{G}_{n+1,q}(k | \alpha)}{n+1} + \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} = [2 : q] \sum_{l=0}^{k-1} (-1)^{l} q^{k-l-1} \left[ l : q^{\alpha} \right]^{n}.
\end{equation}

Via Eq. (2.6), we easily obtain the following:

\begin{equation}
\tilde{G}_{n,q}(x | \alpha) = q^{-\alpha x} \sum_{k=0}^{n} \binom{n}{k} q^{\alpha k x} \tilde{G}_{k,q}(\alpha) [x : q^{\alpha}]^{n-k}.
\end{equation}

From (2.6) - (2.8), we get the following:

\begin{equation}
[2 : q] \sum_{l=0}^{k-1} (-1)^{l} q^{k-l-1} \left[ l : q^{\alpha} \right]^{n} = (q^{\alpha kn} - 1) \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} + q^{-\alpha k} \sum_{j=0}^{n} \frac{1}{n+1} \binom{n+1}{j} q^{\alpha j k} \tilde{G}_{k,q}(\alpha) [k : q^{\alpha}]^{n+1-k},
\end{equation}
here \( k \) is an even positive integer. If \( k \) is an odd positive integer. Then, we can derive the following equality:

\[
(2.10) \quad \left[2 : q\right] \sum_{l=0}^{k-1} (-1)^l q^{k-1} \left[ l : q^\alpha \right]^n
= \left(q^{\alpha k n} + 1\right) \frac{G_{n+1,q} (\alpha)}{n+1} + q^{-\alpha k} \sum_{j=0}^{n} \frac{1}{n+1} \binom{n+1}{j} q^{\alpha j k} \tilde{G}_{k,q} (\alpha) \left[ k : q^\alpha \right]^{n+1-k}.
\]

### 3. WEIGHTED \( q \)-GENOCCHI-ZETA FUNCTION

The famous Genocchi polynomials were defined as

\[
(3.1) \quad \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n (x) \frac{t^n}{n!}, \quad |t| < \pi \text{ cf. [4].}
\]

For \( s \in \mathbb{C}, x \in \mathbb{R} \) with \( 0 \leq x < 1 \), Genocchi-Zeta function are given by

\[
(3.2) \quad \zeta_G (s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s},
\]

and

\[
(3.3) \quad \zeta_G (s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.
\]

By (3.1), (3.2) and (3.3), Genocchi-Zeta functions are related to the Genocchi numbers as follows:

\[
\zeta_G (n) = \frac{G_{n+1}}{n+1}.
\]

Moreover, it is simple to see

\[
\zeta_G (n, x) = \frac{G_{n+1} (x)}{n+1}.
\]

The weighted \( q \)-Genocchi Hurwitz-Zeta type function are defined by

\[
\tilde{\zeta}_{G,q} (s, x | \alpha) = \left[2 : q\right] \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{[m + x : q^\alpha]^s}.
\]

Similarly, weighted \( q \)-Genocchi-Zeta function are given by

\[
\tilde{\zeta}_{G,q} (s | \alpha) = \left[2 : q\right] \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^s}.
\]

For \( n, \alpha \in \mathbb{N} \cup \{0\} \), we have

\[
\tilde{\zeta}_{G,q} (n | \alpha) = \frac{\tilde{G}_{n+1,q} (\alpha)}{n+1}.
\]

We now consider the function \( \tilde{G}_q (n : \alpha) \) as the analytic continuation of weighted \( q \)-Genocchi numbers. All the weighted \( q \)-Genocchi numbers agree with \( \tilde{G}_q (n : \alpha) \), the analytic continuation of weighted \( q \)-Genocchi numbers evaluated at \( n \). For \( n \geq 0, \tilde{G}_q (n : \alpha) = \tilde{G}_{n,q} (\alpha) \).
We can now state \( \tilde{G}_q(s : \alpha) \) in terms of \( \tilde{\zeta}_{G,q}(s \mid \alpha) \), the derivative of \( \tilde{\zeta}_{G,q}(s : \alpha) \)

\[
\frac{\tilde{G}_q(s + 1 : \alpha)}{s + 1} = \tilde{\zeta}_{G,q}(-s \mid \alpha), \quad \frac{\tilde{G}_q(s + 1 : \alpha)}{s + 1} = \tilde{\zeta}_{G,q}(-s \mid \alpha).
\]

For \( n, \alpha \in \mathbb{N} \cup \{0\} \)

\[
\frac{\tilde{G}_q(2n + 1 : \alpha)}{2n + 1} = \tilde{\zeta}_{G,q}(-2n \mid \alpha).
\]

This is suitable for the differential of the functional equation and so supports the coherence of \( \tilde{G}_q(s : \alpha) \) and \( \tilde{\zeta}_{G,q}(s : \alpha) \) with \( \tilde{G}_n,q(\alpha) \) and \( \tilde{\zeta}_{G,q}(s \mid \alpha) \). From the analytic continuation of weighted q-Genocchi numbers, we derive as follows:

\[
\frac{\tilde{G}_q(s + 1 : \alpha)}{s + 1} = \tilde{\zeta}_{G,q}(-s \mid \alpha) \quad \text{and} \quad \frac{\tilde{G}_q(-s + 1 : \alpha)}{-s + 1} = \tilde{\zeta}_{G,q}(s \mid \alpha).
\]

Moreover, we derive the following:

For \( n \in \mathbb{N} - \{1\} \)

\[
\frac{\tilde{G}_{-n+1,q}(\alpha)}{-n + 1} \rightarrow \frac{\tilde{G}_q(-n + 1 : \alpha)}{-n + 1} = \tilde{\zeta}_{G,q}(n \mid \alpha).
\]

The curve \( \tilde{G}_q(s : a) \) review quickly the points \( \tilde{G}_{-s,q}(\alpha) \) and grows \( \sim n \) asymptotically \( (-n) \rightarrow -\infty \). The curve \( \tilde{G}_q(s : a) \) review quickly the point \( \tilde{G}_q(-s : a) \). Then, we procure the following:

\[
\lim_{n \to \infty} \frac{\tilde{G}_q(-n + 1 : \alpha)}{-n + 1} = \lim_{n \to \infty} \tilde{\zeta}_{G,q}(n \mid \alpha) = \lim_{n \to \infty} \left[ 2 : q \right] \sum_{m=1}^{\infty} \left( \frac{-1}{m : q^m} \right)^n \left[ m : q^m \right] = -q^2 \left[ 2 : q^{-1} \right].
\]

From this, we easily note that

\[
\frac{\tilde{G}_q(-n + 1 : \alpha)}{-n + 1} = \tilde{\zeta}_{G,q}(n \mid \alpha) \rightarrow \frac{\tilde{G}_q(-s + 1 : \alpha)}{-s + 1} = \tilde{\zeta}_{G,q}(s \mid \alpha).
\]

4. ANALYTIC CONTINUATION OF WEIGHTED q-GENOCCHI POLYNOMIALS

For coherence with the redefinition of \( \tilde{G}_{n,q}(\alpha) = \tilde{G}_q(n : \alpha) \), we have

\[
\tilde{G}_{n,q}(x \mid \alpha) = q^{-\alpha x} \sum_{k=0}^{n} \left( \frac{n}{k} \right) q^{\alpha k x} \tilde{G}_{k,q}(\alpha) [x : q^x]^{n-k}.
\]
Let $\Gamma (s)$ be Euler-gamma function. Then the analytic continuation can be get as

$$n \mapsto s \in \mathbb{R}, \; x \mapsto w \in \mathbb{C},$$

$$\tilde{G}_{n,q}(\alpha) \mapsto \tilde{G}_q(k + s - [s] : \alpha) = \tilde{\zeta}_{G,q}(- (k + s - \lfloor s \rfloor) | \alpha),$$

$$\begin{align*}
{n \choose k} &= \frac{\Gamma(n + 1)}{\Gamma(n + 1 - k) \Gamma(k + 1)} \mapsto \frac{\Gamma(s + 1)}{\Gamma(1 + k + (s - \lfloor s \rfloor)) \Gamma(1 + \lfloor s \rfloor - k)}
\end{align*}$$

$$\tilde{G}_{s,q}(w | \alpha) \mapsto \tilde{G}_q(s, w : \alpha) = q^{-\alpha w} \sum_{k=0}^{[s]} \frac{\Gamma(s + 1) \tilde{G}_q(k + (s - \lfloor s \rfloor) : \alpha)}{\Gamma(1 + k + (s - \lfloor s \rfloor)) \Gamma(1 + \lfloor s \rfloor - k)} [w : q^\alpha]^{[s] - k}$$

$$= q^{-\alpha w} \sum_{k=0}^{[s]+1} \frac{\Gamma(s + 1) \tilde{G}_q(-1 + k + (s - \lfloor s \rfloor) : \alpha)}{\Gamma(k + (s - \lfloor s \rfloor)) \Gamma(2 + \lfloor s \rfloor - k)} [w : q^\alpha]^{[s] + 1 - k}.$$