A Global Analog of Cheshire Charge

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Abstract

It is shown that a model with a spontaneously broken global symmetry can support defects analogous to Alice strings, and a process analogous to Cheshire charge exchange can take place. A possible realization in superfluid He-3 is pointed out.

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1 Introduction

March-Russell, Preskill, and Wilczek[1] showed that vortices of a theory with a global $U(1)$ symmetry broken to $Z_2$ can scatter quanta of $Z_2$ charge with a cross section almost equal to to the maximal Aharonov-Bohm cross-section, due to a frame-dragging of local mass eigenstates. Here we demonstrate the realization of a non-abelian Aharonov-Bohm phenomenon (Cheshire charge) in the context of a global model.

In the first section, we describe a relativistic field theory that supports a global analog of Alice strings[2] and then describe how the process of charge exchange occurs by means of quantum interference. Some differences as well as similarities to the parallel phenomenon of Cheshire charge in gauge theories are mentioned, as well as the manner in which the global phenomenon can be viewed as a limit of the gauge case at very weak gauge coupling.

One of the motivations for studying global vortices and global Aharonov-Bohm scattering is that many more global than local symmetry-breaking transitions are available for manipulation in condensed matter systems. The possibility arises of finding condensed-matter systems which can serve as laboratory analogs of otherwise observationally inaccessible gauge string phenomena. In section 2, we consider the possibility of finding a laboratory analog of Cheshire charge in the superfluid A phase of helium-3. The group-theoretic properties necessary for the existence of Cheshire charge are present in He-3 A, although in practice it may be difficult to devise an experiment to observe it.

2 Cheshire Charge in a Theory With Broken Global Symmetry

The Model

Consider a theory with a global $SO(3)$ symmetry containing a Higgs scalar $\Phi$ transforming as the 5-dimensional symmetric tensor representation, which we will write as a $3 \times 3$ $SO(3)$ matrix, and another scalar field $\Psi$ transforming as a 3-dimensional vector. $\Psi$ will serve as the test particle that scatters from vortices. The
fields transform under $SO(3)$ according to:

$$\Psi \rightarrow \Omega \Psi, \quad \Phi \rightarrow \Omega \Phi \Omega^{-1}. \quad (1)$$

where $\Omega$ is $3 \times 3$ $SO(3)$ matrix. We will denote the generators of $G = SO(3)$ as:

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2)$$

We also introduce a bilinear coupling of the $\Psi$ fields to the Higgs:

$$\Delta \mathcal{L} = \lambda \Psi^T \Phi \Psi. \quad (3)$$

Now let the Higgs field acquire a vacuum expectation value $\Phi_0 = \nu \text{diag}(1, 1, -2)$. This breaks the symmetry group down to $H = U(1) \times S.D. Z_2$. This is the same symmetry breaking pattern previously considered in the case of Alice strings [2]; the difference is that we are considering a global, rather than a gauge, symmetry. The VEV $\Phi_0$ induces a mass splitting among the the members of the multiplet $\Psi$, much as in Reference [1]. The first two components of $\Psi$ are degenerate and are mixed by the unbroken $U(1)$ generator $T_3$, while the third component is an H singlet. From the first two we can form basis eigenstates of opposite $U(1)$ charges:

$$u_+ = (1, i, 0), \quad u_- = (1, -i, 0). \quad (4)$$

The VEV in this theory can be thought of as taking values on the surface of a sphere with antipodal points identified. A visual analogy for the symmetry-breaking pattern is the director field of a nematic liquid crystal (NLC). The order parameter of the NLC, like that of our theory, can be thought of as an undirected line segment at each point in space. The group of transformations which leave this segment invariant include continuous rotations about the director’s axis (the $U(1)$ component) as well
as a discrete 180° rotation about an axis perpendicular to the segment. This 180° flip generates the $Z_2$ component. Since the discrete 180° rotation does not commute with continuous rotations about the preferred axis, the full unbroken group is a *semidirect* product. This visual analogy will be useful later in explorations of condensed-matter systems. Our model can be reformulated in terms of a director field $\vec{d}$, a vector in internal space, rather than a tensor, by defining

$$\Phi_{ab} = d_ada_b - d^2\delta_{ab}$$

and making the identification $\vec{d} \equiv -\vec{d}$.

**Construction of Alice Strings and Vortices**

The above model can form topologically stable $\pi_1$ type defects. For simplicity we consider the model in two spatial dimensions, so that the defects are vortices. All arguments can be generalized straightforwardly to strings in three spatial dimensions. A vortex with core at the origin could have an asymptotic field configuration far from the origin given in polar coordinates by:

$$\Phi(x) = \exp[-i\varphi T_1/2]\Phi_0\exp[i\varphi T_1/2].$$

The Higgs field is single valued, but the mass eigenstates of the $\Psi$ field are not well-defined globally. One can define local (frame-dragged) mass eigenstates $\rho_i$ at any point outside the core by $\rho = \exp[-i\varphi T_1/2]\Psi$.

As in reference [1], these local eigenstates define a frame at each point outside the core, and a state is adiabatically transported if its components in the local basis remain unchanged at each point. Notice that $\rho_2(0) = -\rho_2(2\pi)$. This means that when a state $\rho_2$ is adiabatically transported through a loop that encloses the vortex core once, it acquires an Aharonov-Bohm-like phase of $-1$, whereas $\rho_1$ acquires no such phase. In terms of the fields of definite $U(1)$ charge, $\rho_+ = \rho_1 + i\rho_2$ and $\rho_- = \rho_1 - i\rho_2$, the boundary condition can be written as $\rho_+(2\pi) = \rho_-(0), \rho_-(2\pi) = \rho_+(0)$. 

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This means that the sign of the global U(1) charge is reversed when a ρ particle is adiabatically transported once around the string. Thus we have an global analog of the Alice string that occurs in the corresponding gauge theory.

In the gauge theory case it is known that a vortex-antivortex pair can acquire a “Cheshire charge” which compensates the charge gained by the particle upon threading the pair, so that the total charge is actually conserved.\[^{[3]}\] This charge is a global property of the vortex pair; it cannot gauge-invariantly be localized to either one of the vortices or to any region of space near them. To see how charge conservation is maintained in this global model, consider a vortex-antivortex pair with the two cores separated along the x-axis by a distance D which is large compared to the core radius. Outside the cores, the vortex-antivortex solution to the field equations can be written:

\[
\Phi_0(x) = \exp\left[-i \frac{\Delta \varphi T_1}{2}\right]\Phi_0 \exp\left[i \frac{\Delta \varphi T_1}{2}\right]
\]

(7)

where \(\Phi_0 = \nu \text{diag}(1, 1, -2)\) and \(\Delta \varphi = \varphi_1 - \varphi_2\) as defined in figure 1. (Since the fluxes of the two vortices lie in the same U(1) subgroup, it is easy to show that when the Higgs field is constrained to lie in the vacuum manifold, the static 2-d field equations reduce to Laplace’s equation. Thus \(\vec{\nabla}\Phi\) is dual to the electric field of two opposite charges in two dimensions, and the solution (7) is obtained.)

At points far away from both cores \((r \gg D)\), the Higgs field approaches a single asymptotic value \(\Phi_0\). Thus the embedding of the unbroken group H is well-defined on any large circle outside the the two cores: its connected component is the U(1) generated by \(T_3\).

The local mass eigenstates are given by \(\rho = \exp[-i \Delta \varphi T_1/2]\Psi\) and are thus unchanged under adiabatic transport along paths that remain far away from the pair where \(\Delta \varphi \approx 0\). However, if a \(\Psi_2\) state is transported from \(y = -\infty\) to \(y = +\infty\) along the y-axis (or along any path that passes between the two cores), the angle \(\Delta \varphi\) winds through \(2\pi\), and the state \(\Psi_2\) acquires an Aharonov-Bohm phase of -1, whereas \(\Psi_1\) acquires no such phase. None of the triplet components acquires a phase if it is
transported from $y = -\infty$ to $y = +\infty$ on a path which does not pass between the two cores. Thus an eigenstate of $U(1)$ charge changes the sign of its charge when it passes through the pair.

In order to understand how charge conservation is maintained, we must realize that there is an infinite family of vortex pair solutions related by $U(1)$ rotations. The solutions

$$\Phi_\alpha(x) = \exp(i\alpha T_3)\Phi_0(x) \exp(-i\alpha T_3), \quad (8)$$

where $0 < \alpha < 2\pi$ and $\Phi_0(x)$ is given by (7), all have the same energy because they are related by a global symmetry transformation. Since they all have the same asymptotic value of the Higgs field, they can be continuously deformed into each other; thus there is a charge rotor zero mode. Figure 2 shows the action of the symmetry on the field of a vortex-antivortex pair. The order parameter is drawn as an undirected line segment as discussed above. Figure 2a shows one representative of a class of flux eigenstates. Other states degenerate with this one are obtained by rotating each of the directors through arbitrary angle $\alpha$ about the x-axis: Fig. 2b shows the result when $\alpha = \pi/2$. (Note that unlike our model, physical NLC’s do not in general possess a continuous degeneracy of this type, but only a twofold degeneracy, because the free energy is not invariant under purely internal rotations of the director, but only under rotations of the whole coordinate frame. The broken symmetry in NLC’s is not truly an internal one of the type that occurs in relativistic field theories.)

The pair states which transform as irreducible representations of the asymptotically unbroken $U(1)$ group are coherent superpositions of the solutions (8): they are the quantized energy levels of the zero mode. A state with charge $n$ is given by:

$$|n> = \int_0^{2\pi} d\alpha \exp(-in\alpha)|\Phi_\alpha>. \quad (9)$$
The Charge Transfer Process. To understand the process of charge transfer by which a test particle reverses its charge and the vortex pair acquires a compensating charge, consider first the case where the vortex pair is originally in the state $|\alpha\rangle$ described above. For each value of $\alpha$, there is one component of $\Psi$ that acquires a phase upon passing between the vortices, and another which does not. However, these states depend on the value of $\alpha$. Let $u_{1\alpha}$ be the state which acquires no phase, and $u_{2\alpha}$ be the state which acquires a phase of $-1$. Then:

$$u_{1\alpha} = (\cos \alpha, -\sin \alpha, 0), \quad u_{2\alpha} = (\sin \alpha, \cos \alpha, 0).$$

(10)

The $U(1)$ charge eigenstates $u_+ = (1, i, 0)$ and $u_- = (1, -i, 0)$ are expressed in terms of $u_{1\alpha}$ and $u_{2\alpha}$ as:

$$u_+ = \exp(-i\alpha)(u_{1\alpha} + iu_{2\alpha}), \quad u_- = \exp(i\alpha)(u_{1\alpha} - iu_{2\alpha}).$$

(11)

Thus, when the state $u_+$ is adiabatically transported along a path that threads the pair, it is turned into the state $\exp(-2i\alpha)u_-$, whereas $u_-$ becomes $\exp(2i\alpha)u_+$. These relations may be expressed simply in terms of a monodromy matrix, written in the charge eigenstate basis as follows:

$$\rho(2\pi) = \mathcal{M}(\alpha)\rho(0),$$

where

$$\mathcal{M}(\alpha) = \begin{bmatrix} 0 & e^{2i\alpha} \\ e^{-2i\alpha} & 0 \end{bmatrix}. \quad (12)$$

Now take an initial state in which the vortex pair is in the charge-zero eigenstate $|0\rangle$ and the test particle is in the state $u_+$:

$$|u_+\rangle \otimes |0\rangle = \int_0^{2\pi} d\alpha |u_+\rangle \otimes |\Phi_\alpha\rangle. \quad (13)$$
After the particle is dragged through the loop, the final state will be:

$$
\int_0^{2\pi} d\alpha \exp(-2i\alpha)|u_- > \otimes |\Phi_\alpha >= |u_- > \otimes |2 > .
$$

(14)

The state has evolved into one in which the vortex pair has charge +2, because of the different phases acquired by the wavefunction in the different $\alpha$ sectors. The zero mode has been excited by means of a quantum-mechanical interference process, which is the usual means for the transfer of Cheshire charge, except that in this case it has occurred in a model with no gauge symmetry and does not have any topological interpretation \cite{5} in terms of lines of electric flux being trapped between the vortices.

It may be noticed that even though the charged states $|n >$ exist for all integers, only those with $n$ even can be produced by this process from an initially uncharged vortex. This is not necessarily the case in gauge Alice models. We can, in the case of a gauge model, take the initial gauge group $G$ to be $SU(2)$ rather than $SO(3)$, and include matter fields transforming in the spinor representation. After the symmetry is broken to $U(1) \times S.D. Z_2$, the spinor components become two oppositely charged states which interchange under the action of the $Z_2$ flip. The smallest electric charge in the theory is that carried by these spinor particles, and it is by passing these through a loop of Alice string (or pair of Alice vortices) that the odd-numbered Cheshire charge states are excited. However, the frame-dragging effect considered in Ref. \cite{1} and in the present paper requires a matter field bilinearly coupled to the symmetry-breaking order parameter. Since the Higgs field in our model transforms in the 5-dimensional representation, no singlet can be formed from the Higgs field with only two spinors, and we are forced to consider matter fields $\Psi$ lying in a vector representation. Thus, in comparing our global Alice system to the corresponding gauge model, the states which we have called $\rho_\pm$ should be thought of as doubly charged. The monodromy matrix (12), for example, has the property $\mathcal{M}^2 = 1$, rather than $\mathcal{M}^2 = -1$ as in the case of singly charged objects.\cite{6}
Comparison With Ordinary Gauge Cheshire Charge

It is interesting that, although the existence of Cheshire charge is a consequence of the symmetry breaking pattern only, and occurs in this global model for the same reason as in a gauge theory, the nature of the charge is rather different. As we see in the following paragraphs, global Cheshire charge actually is localizable and is carried by the scalar fields rather than the vector fields.

In the global theory we are considering, the zero mode is simply a “rigid” $U(1)$ rotation of the entire Higgs field configuration: thus it is a subgroup of the global $SO(3)$ transformations. $\alpha$ is the coordinate of the zero mode, and classically the excitations of this mode are states where $d\alpha/dt \neq 0$. Quantum mechanically, $d\alpha/dt$ will be replaced by a canonical momentum with discrete eigenvalues. This is to be contrasted with the case of a gauge model, where a rigid rotation of the Higgs fields alone fails to satisfy the equations of motion. In temporal gauge, the zero mode can be written:

$$\Phi = \Omega \bar{\Phi} \Omega^{-1}, \quad A_\mu = \Omega \bar{A}_\mu \Omega^{-1} + \delta^i_\mu \partial_i \Omega \Omega^{-1},$$

(15)

where $\bar{\Phi}$ and $\bar{A}_\mu$ are static solutions and $\Omega(x, t)$ is a spatially varying $SO(3)$ transformation that tends to an element of the unbroken group $H$ (namely $\exp(i\alpha T_3)$) at infinity. (Notice that if the term $\delta^i_\mu \partial_i \Omega \Omega^{-1}$ were replaced by $\partial_\mu \Omega \Omega^{-1}$ then this would be a physically irrelevant gauge transformation.) By transforming to another gauge, one can view the zero mode as purely an excitation of the gauge fields,

$$\Phi = \bar{\Phi}, \quad A_\mu = \bar{A}_\mu - \delta^0_\mu \partial_0 \Omega \Omega^{-1},$$

(16)

whereas in the global case it is purely an excitation of the scalar fields.

In our global case one can see that there is a nonzero charge density (i.e. 0 component of the global current) which is localized in the region of space surrounding the vortices:

$$J_{0(3)} = 2\text{Tr} \Phi T_3 \partial_0 \Phi = 36\nu \sin^2(\Delta \varphi/2)(d\alpha/dt).$$

(17)

This definitely localizable charge carried by the scalar fields contrasts with the usual
case, where the charge is carried by the gauge fields and its apparent location can be moved by performing gauge transformations. Since the global charge density is locally measurable, it must be the case that the charge is gradually transferred as the charged particle moves between the vortex cores. One must suppose that the charge density propagates away from the particle as the particle moves, and spreads itself through the region of space surrounding the cores. The transfer of gauge Cheshire charge, on the other hand, cannot gauge-invariantly be said to happen at a particular time and place, or even incrementally at a well-defined rate.

The relation between charge and energy is also different in the global case compared to the gauge case. This can be demonstrated by quantizing the zero mode. First treat $\alpha$ as a classical coordinate. Assume that the Higgs field configuration is given by

$$\Phi(x, t) = \exp[i\alpha(t)T_3]\Phi_0(x)\exp[-i\alpha(t)T_3]$$

(18)
i.e., a time-varying rigid rotation of the angle $\alpha$. Then $\partial_t \Phi = i\dot{\alpha}[T_3, \Phi_0]$ and the gradient part of the Lagrangian gives

$$L = \int dt d^3x \partial_0 \Phi \cdot \partial^0 \Phi = -\int dt \int d^3x \text{Tr}([T_3, \Phi_0] \dot{\alpha})^2,$$

(19)
(Since the static configurations of different $\alpha$ are degenerate, all other terms in the Lagrangian are independent of $\alpha$ and do not enter in that coordinate’s equations of motion.) Letting

$$I = \int d^3x \text{Tr}[[T_3, \Phi_0]^2],$$

(20)
we can define a momentum $\Pi_{\alpha}$ conjugate to $\alpha$ and write a Hamiltonian

$$H = \frac{\Pi_{\alpha}^2}{2I}.$$  

(21)
Now the commutator $[T_3, \Phi_0]$ is of order $|\Phi|$ within some volume surrounding the vortex cores. Since there is no domain wall connecting the cores, $D$ is the only
relevant length scale, and dimensional analysis then dictates that the “moment of inertia” \( I \) scales as \( D^d \), where \( d \) is the number of spatial dimensions. Accordingly, the charge rotor Hamiltonian has eigenvalues \( E_n = \frac{n^2 \hbar^2}{2m} \approx \frac{n^2 \hbar^2}{\langle |\Phi|^2 \rangle \times \text{Volume}} \), so the energy splittings among the global charge eigenstates scale as \( 1/D^d \). One may contrast this with the Coulomb energy (logarithmic in 2 space dimensions, \( 1/D \) in 3) of the usual gauge Cheshire charge.

The global vortex pair can be considered as a limit of the gauge model where the gauge coupling is so small that the Compton wavelength of the massive vector bosons is much larger than the separation \( D \) of the two vortex centers. In this limit, the Higgs field cores (regions where the Higgs leaves the vacuum manifold) can remain small while the gauge field cores (regions of nonzero magnetic flux) of the vortices become much larger than \( D \) so that there is actually no winding of the gauge field near the pair. In this case all the fields are defined on a trivial bundle and the Higgs VEV is covariantly non-constant just as in the global model. Presumably the Cheshire charge states will behave as in the global case, with a gauge invariantly localizable charge density carried by the scalars near the string.

**Scattering From a Global Alice String**

It is worth noting briefly that a single global Alice string of the type we have constructed will scatter incoming quanta of the \( \Psi \) fields. In fact, if we limit our attention to the \( \rho_2 \) and \( \rho_3 \) components, the calculation of the scattering amplitude proceeds precisely as in reference [1]. \( \rho_1 \), on the other hand, does not scatter at all (except perhaps off the vortex core itself). In other words, \( \rho_1 \) and \( \rho_2 \) are the monodromy eigenstates, and \( \rho_1 \) has eigenvalue unity. The result is that, if an incoming plane wave consists of either of the charge eigenstates \( \rho_+ \) and \( \rho_- \), the scattered wave will be pure \( \rho_2 \), which is a superposition \(-i(\rho_+ - \rho_-)/2\) of the two charge eigenstates. Consider the scattering of a \( \rho_+ \) incident at momentum below the threshold for \( \rho_3 \) production. We expect the \( \rho_1 \) and \( \rho_2 \) to behave asymptotically as an incoming and a
scattered wave. We may write this asymptotic behavior as follows:

\[
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & e^{i\varphi/2} \end{pmatrix} \left\{ \begin{pmatrix} 1 \\
i \end{pmatrix} e^{-ikx} + \begin{pmatrix} 1 \\
i \end{pmatrix} f_+ (\varphi) \frac{e^{ikr}}{r^{1/2}} + \begin{pmatrix} 1 \\
-i \end{pmatrix} f_- (\varphi) \frac{e^{ikr}}{r^{1/2}} \right\}.
\]

(22)

\(f_+\) and \(f_-\) are the charge-preserving and charge-reversing amplitudes for the scattered particle. The diagonal matrix in front enforces the boundary conditions on the frame-dragged states. For simplicity we are considering a vortex in the flux eigenstate with \(\alpha = 0\). Proceeding by analogy with [1], the equations of motion for the second component lead to

\[
f_+ (\varphi) - f_- (\varphi) = \frac{e^{-i\varphi/2}}{(2\pi ik)^{1/2}} \left( \frac{1}{\cos(\varphi/2)} + 2 \sum_{n=0}^{\infty} (-1)^n (e^{i\Delta_n} - 1) \cos \left[ (n + \frac{1}{2}) \varphi \right] \right),
\]

\[\Delta_n = \pi \left( n + \frac{1}{2} - \sqrt{(n + \frac{1}{2})^2 + \frac{1}{4}} \right). \tag{23}\]

Since the first component does not scatter, we also have \(f_+ (\varphi) + f_- (\varphi) = 0\) and the amplitudes are uniquely determined. The differential cross-sections for charge-preserving and charge-flipped scattering, \(d\sigma_+/d\theta\) and \(d\sigma_-/d\theta\), are identical to each other and equal to one fourth the scattering cross-section derived in reference [1] for an abelian global vortex:

\[
\frac{d\sigma_+}{d\theta} = \frac{d\sigma_-}{d\theta} = \frac{1}{8\pi k \sin^2 (\theta/2)} \left( 1 + C(\theta) \right) \tag{24}
\]

where \(\theta = \pi - \varphi\) is the scattering angle and \(C(\theta)\), which vanishes at \(\theta = 0\), is a function obtained by summing (and squaring) the series in (23). The inclusive cross section \(d\sigma_+/d\theta + \sigma_-/d\theta\) is half that of the abelian case considered in [1], because the \(\rho_1\) state is, so to speak, “filtered out” of the scattered wave, just as a linear polarizer halves the intensity of a circularly polarized light beam.
Setting $C(\theta) = 0$ in (24) would give the cross section for doubly charged projectiles scattering from a gauge Alice string. $C(\theta)$, a correction present only in the global analog, results from diagonal $1/r^2$ potential terms appearing in the equations of motion for the $\rho$ fields. Evidently, these corrections modify the inclusive cross section but they do not affect the ratio $\sigma_+ / \sigma_-$, which depends on the monodromy properties of the scattered particles and not on the local or global nature of the vortices.

3 A Condensed-Matter Example

Cheshire charge is a generic phenomenon that occurs when a theory has vortices whose winding fails to commute with a generator of the unbroken symmetry group of the vacuum. The results of the previous section show that Cheshire charge also arises if the theory has only global and not gauge symmetries. The essential group-theoretic concept is the same although the mechanism is different.

In this section, we discuss a physical system which exhibits the type of symmetry-breaking necessary for the existence of Cheshire charge: one which allows mixing of flux eigenstates within a conjugacy class. The system is the superfluid A-phase of liquid helium-3. While the right symmetry-breaking pattern is present, it may be difficult to observe Cheshire charge phenomena experimentally. There are many complications in dealing with a real condensed-matter system rather than a relativistic field theory. Some of these difficulties will be pointed out.

The Order Parameter in Superfluid He-3 A

He-3 atoms are fermions. A condensation of Cooper pairs of atoms is thought to be responsible for superfluidity in this system. Unlike the electrons in BCS superconductors, however, the helium atoms tend to pair in p-wave, rather than s-wave states, so they have a net orbital angular momentum of 1. In order for the two-atom wavefunction to be symmetric, therefore, members of the pair must also have their

* This cross-section was derived by neglecting off-diagonal terms that cause mixing of $\rho_2$ and $\rho_3$ near the vortex. Navin’s analysis suggests that the corrections to the standard Aharonov-Bohm cross section may disappear when the scattering problem is solved exactly.
spins aligned in a triplet state with total spin 1. In addition to the overall phase of the condensate wavefunction, there are thus two separate angular momentum vectors which can a priori rotate independently. The full internal symmetry group of the pair wavefunction is

\[ SO(3)^{(L)} \times SO(3)^{(S)} \times U(1)^{\phi}. \]

This richness in degrees of freedom leads to a wealth of interesting phenomena associated with He-3 superfluidity. Let us denote the generators of these three factors by \( \hat{L}, \hat{S}, \) and \( \hat{I} \), respectively. The symmetry of the superfluid ground state depends on temperature and pressure: there are at least two phases which are stable in bulk, unmagnetized fluid, characterized by different ground state configurations. The A phase is described by a spin state \(|10\rangle\) along some preferred direction, and an orbital state \(|11\rangle\) along some other axis. The object corresponding to the Higgs field is a \( 3 \times 3 \) complex matrix of two-particle correlation functions, \( A_{ai} \), with each entry representing a particular spin state and orbital harmonic. Rotations in spin space act on the first index, \( a \), and rotations in ordinary space act on the second index: \( A_{ai} \) transforms as a vector under each of the two \( SO(3) \) factors of the symmetry group. The \( U(1) \) factor acts on the overall phase of the matrix. In the A-phase, the matrix takes a value of the form:

\[
A_{ai} = \Delta_A(T) d_a(e_{1i} + ie_{2i}) e^{i\phi}
\]

Here \( \Delta_A(T) \), a temperature-dependent gap parameter, can be thought of as the magnitude of the superfluid wave function, much like the magnitude of the higgs vev in a field theory. The vector \( \vec{d} \) is the axis along which the projection of the spin angular momentum is zero. \( \vec{e}_1 \) and \( \vec{e}_2 \), together with the vector \( \vec{\ell} = \vec{e}_1 \times \vec{e}_2 \), define a local orthonormal frame such that the projection of the pair’s mutual orbital angular momentum onto \( \vec{\ell} \) is +1. The phase \( \phi \) represents the overall phase of the pair wavefunction.

In order to see what the pattern of symmetry-breaking is, consider what transformations leave the order parameter invariant. Continuous rotations in spin space
about the axis $\vec{d}$ leave $A_{ai}$ unchanged. These form an unbroken $U(1)$ subgroup with generator $\hat{S}_z$ (The state $|10\rangle$ is invariant under rotations about the z axis.) Rotations in orbital space about $\vec{l}$ result in a phase (corresponding to the phase gained in rotations of the state $|11\rangle$ about the z axis.) However, this can be compensated by a change in $\phi$. Thus the $\hat{L}_z - \hat{I}$ generates another unbroken U(1) subgroup. These are the only two elements of the Lie algebra which annihilate $A_{ai}$, but the discrete transformation $\vec{d} \rightarrow -\vec{d}$, $\phi \rightarrow \phi + \pi$ leaves the matrix invariant. The $\vec{d}$ vector can be flipped 180 degrees if the phase $\phi$ is simultaneously shifted by $\pi$. This makes the unbroken group $H = U(1) \times U(1) \times S.D. Z_2$.

Because of the presence of this discrete 180° rotation in the little group, the spin quantization axis $\vec{d}$ acts like the director field in a NLC: there are configurations in which $\vec{d}$ can be rotated continuously through 180° along a closed path which winds once around the core of a vortex. Such a configuration is the “half-quantum vortex,” so called because the phase $\phi$ winds only halfway around the unit circle and the vortex carries only half of the conventional quantum of circulation. Half-quantum vortices are analogous to the Alice vortices of the previous section. A similar zero mode should in principle exist. In the next section, configurations with such a zero mode are described.

**Half-quantum Vortices in He-3 A**

Static configurations of the superfluid order parameter are extrema of the Landau-Ginzburg free energy functional$^{[10]}$, which takes the place of the field Hamiltonian. The free energy density includes a gradient energy term:

$$F_G = \gamma_1 \partial_i A_{aij} \partial_i A^*_{aij} + \gamma_2 \partial_i A_{aij} \partial_j A^*_{aij} + \gamma_3 \partial_i A_{aij} \partial_j A^*_{aij}$$

(26)

where $\gamma_i$ are constants. For general values of $\gamma_i$, this term is not invariant under rotations of the orbital frame $(\vec{e}_1, \vec{e}_2, \vec{l})$ unless the external coordinates are simultaneously rotated. However, since the spin indices $\alpha$ are never contracted with any of the differentiation indices, the gradient energy is invariant under all global rotations.
of \( \mathbf{d} \), regardless of the values of \( \gamma_i \). \( \text{SO}(3)_{\text{spin}} \) is truly an internal symmetry if only the gradient energy is included.

Only the spin-orbit, or dipole, interaction couples spin with orbital indices. This term has no analog in the model of section 2:

\[
F_D = g_D (A_{ii} A_{kk}^* + A_{ik} A_{kl}^*) = -2g_D \Delta^2 \mathbf{A} (\mathbf{d} \cdot \mathbf{\ell})^2. \tag{27}
\]

The dipole force is weak compared to the other interactions, but it has the consequence that in the bulk fluid, \( \mathbf{d} \) tends to line up parallel to \( \mathbf{\ell} \). This is known as dipole locking.

In the presence of a pair of HQV’s, the dipole energy must depart from minimum over some region between the cores (dipole unlocking). This is because \( \mathbf{\ell} \), unlike \( \mathbf{d} \), cannot wind by odd multiples of \( \pi \), so \( \mathbf{\ell} \) will tend instead to remain constant. When the cores are widely separated, the consequence is that the winding of the \( \mathbf{d} \) vector occurs within a domain wall, or soliton, whose width is of order

\[
\xi_D \sim \sqrt{\frac{-\gamma_i}{g_D}}. \tag{28}
\]

\( \xi_D \), known as the dipole length, is the scale at which the dipole energy becomes comparable to the gradient energy. The presence of a domain wall causes the half-quantum vortices to be confined linearly rather than merely logarithmically in two dimensions. Figure 3 shows \( \mathbf{\ell} \) and \( \mathbf{d} \) for such a configuration. Since the dipole energy depends only on the angle between \( \mathbf{\ell} \) and \( \mathbf{d} \), a global rotation of all the \( \mathbf{d} \) vectors in figure 3 about the x axis will still leave the Landau-Ginzburg free energy invariant. This is the zero mode which gives rise to Cheshire charge in this case: a global rotation which belongs to the subgroup unbroken at infinity.

The Observability of Cheshire Charge

In the case of He-3 vortices, the “charge” that can be transferred is a form of angular momentum. The momentum conjugate to \( \mathbf{d} \) is the spin density, or net nuclear magnetization, \( \mathbf{\tilde{S}} \). In the A-phase equilibrium, the spin density has expectation value
zero. Dynamics slower than the gap frequency characterizing the symmetry-breaking scale but faster than $\sim 1\text{Hz}$ is governed by an effective Hamiltonian

$$
\mathcal{H} = \frac{1}{2} \gamma^2 S_\alpha (\chi^{-1})_{\alpha\beta} S_\beta - \vec{H} \cdot \vec{S} + F_G + F_D,
$$

where $\vec{H}$ is an externally applied magnetic field and $\gamma$ is the gyromagnetic ratio of the atomic spins. Unless otherwise stated, we will assume no external field. $\chi$, the magnetic susceptibility, is a symmetric tensor with two distinct eigenvalues $\chi_\parallel$ and $\chi_\perp$ reflecting the greater polarizability of the fluid in directions perpendicular to $d$:

$$
\chi_{ij} = \chi_\parallel d_i d_j + \chi_\perp (\delta_{ij} - d_i d_j).
$$

In spite of the coupling of $\vec{\ell}$ and $\vec{d}$ through the dipole interaction, (29) ignores the dynamics of the orbital axis $\vec{\ell}$ is because the motion of $\vec{\ell}$ is so strongly damped that all but very slow motions of $\vec{d}$ can be treated as occurring on a background of fixed $\vec{\ell}$.

The Hamiltonian (29), together with the commutation relations $\{S_i, d_j\} = \epsilon_{ijk} d_k$, $\{S_i, S_j\} = \epsilon_{ijk} S_k$, leads to the Leggett\cite{12} equations of motion. $\vec{d}$ precesses about $\vec{S}$ according to

$$
\frac{\partial \vec{d}}{\partial t} = \gamma \vec{d} \times (-\frac{\gamma S}{\chi_\perp}).
$$

Thus the first term of (29) plays the role of a kinetic term for motions of $\vec{d}$, with $\vec{S}$ being the momentum. In particular, for the pair configuration shown in Figure 3, the $x$ component of $\vec{S}$ will be nonzero when the charge rotor is excited, and the quantized zero mode will have excitations where the total angular momentum $\int d^d x S_x = n\hbar$. These excitations will carry an energy of order

$$
E_{ZM} \sim \frac{\gamma^2 n^2 \hbar^2}{\chi_\perp V_{\text{Soliton}}},
$$

where $V_{\text{Soliton}}$ is the volume of the dipole-unlocked region near the cores. For large $n$, one can identify a classical precession frequency $\omega = (1/\hbar) dE/dn$, and the kinetic
energy is given by:

\[ E_{ZM} \sim \frac{\omega^2 V_{\text{Sol}} \chi_{\perp}}{\gamma^2}. \] (33)

As in the model of Section 2, this energy has, for fixed \( n \), an inverse dependence on the soliton volume, and thus on the separation of the cores. In a two-dimensional geometry where string tension does not operate, it is possible that a sufficiently excited pair could be stabilized against the attractive force due to the dipole interaction, provided the “Cheshire charge” cannot be radiated away rapidly. This would occur if the energy stored in the zero mode was comparable to the dipole energy of the soliton connecting the two cores:

\[ E_D \sim g_D \Delta^2_A V_{\text{Sol}} \sim E_{ZM} \sim \gamma^2 \frac{n^2 \hbar^2}{V_{\text{Sol}} \chi_{\perp}}. \] (34)

Using formulas and numbers that can be found in Reference[10], one can estimate the volume at which this occurs (more details of this estimate are found in the Appendix):

\[ V \sim n \times (1 - T/T_c)^{-1/2} \times 10^{-16} \text{cm}^3. \] (35)

The classical precession frequency of \( \vec{d} \) corresponding to this energy level is approximately \( 10^4 \text{s}^{-1} \). By comparison, the gap frequency below \( T_c \) is typically \( \sim \frac{k T_C}{\hbar} \sim 10^7 \text{s}^{-1} \).

The stability of a charged excited state of a vortex loop or pair is also uncertain. It may depend on details of the spin relaxation behavior of the fluid and such factors as coupling between the superfluid and the normal fluid component[13]. However, since the precession frequency found above to be sufficient to cancel the attractive force is lower than the gap frequency, one would imagine that at least the radiation of “Higgs” modes would be suppressed. Also, since the “Cheshire charge” consists of a nonzero spin density in the direction of \( \vec{\ell} \) (assuming that \( \vec{\ell} \) maintains a uniform value everywhere which is parallel to the asymptotic value of \( \vec{\ell} \)), no torque should be exerted on this component of \( \vec{S} \) by the dipole force. This renders one of the usual means[11] for relaxation of \( \vec{S} \) ineffective: namely its damping by coupling to \( \vec{\ell} \).
A Charge Exchange Process

The spin wave excitations of the He-3 A order parameter carry quantum numbers which allow the possibility of an Aharonov-Bohm interaction with half-quantum vortices. In general, spin waves consist of a coupled oscillation of $\vec{d}$ and $\vec{S}$. Consider the form

$$\vec{d} = \vec{d}_0(\vec{r}) + \vec{\psi}(\vec{r}, t).$$

Oscillations of $\psi$ parallel to $\vec{d}_0$ have a gap characterized by the symmetry-breaking scale. On the other hand, oscillations of $\psi$ perpendicular to $\vec{d}_0$ only have a frequency shift $\Omega_A$ proportional to the dipole energy. In particular, the two propagating low-frequency modes obey a wave equation of the form

$$-\frac{\partial^2 \psi}{\partial t^2} = \Omega_A^2 \psi + \Omega_A^2 (U\psi + D\psi), \quad (36)$$

where

$$\Omega_A^2 = \frac{\gamma_2^2 \Delta_A^2 g_D}{\chi_\perp}, \quad (37)$$

$U$ is a potential which is zero in the bulk fluid, and $D$ is a kinetic operator:

$$D\psi = -\xi_d^2 [\Delta \psi + \frac{\rho_{sp}^\parallel}{\rho_{sp}} \cdot (\vec{\ell} \cdot \vec{\nabla}) \psi]. \quad (38)$$

$\rho_{sp}^\perp \propto (2\gamma_1 + \gamma_2 + \gamma_3)$ and $\rho_{sp}^\parallel \propto 2\gamma_1$ are the spin rigidity coefficients describing the energy of gradients in $\vec{d}$. The lower cutoff frequency $\Omega_A$ arises because an oscillation of $\vec{d}$ about $\vec{\ell}$ is an oscillation in the potential well formed by the dipole coupling $g_D(\vec{\ell} \cdot \vec{d})^2$. (As mentioned previously, $\vec{\ell}$ can effectively be regarded as fixed on the time scales of these oscillations. The fluctuations of $\vec{\ell}$ are diffusive or overdamped.)

The potential $U$ becomes nonzero when $\vec{\ell}$ is not parallel to $\vec{d}$ (as inside domain walls) or when nonuniform textures of the order parameter are present. These oscillations of $\psi$ perpendicular to $\vec{d}_0$ would be Goldstone modes if the dipole energy were neglected,
and they are degenerate with each other as long as $\vec{d}_0 \parallel \vec{\ell}$. There are thus three modes which have the pattern of splitting analogous to the splitting of the $\Psi$ modes in section 2.

Figure 4 demonstrates the “frame dragging” of the spin wave modes. We assume a frequency lower than the gap frequency, which corresponds to the assumption in Section 2 or in reference [1] that scattering experiments are done at an energy such that only the light components of the split multiplet $\Psi$ are excited. Then the two propagating oscillations (corresponding to $\Psi_1$ and $\Psi_2$) are the two different polarizations of $\psi$ perpendicular to $\vec{d}_0$. In accordance with eqn. (31), the fluctuation of $\vec{d}$ is accompanied by a fluctuation of $\vec{S}$ in the $\vec{\psi} \times \vec{d}_0$ direction. In the region far from the vortex cores, $\vec{\ell}$ and $\vec{d}$ are taken to lie along the x-axis, so that one of the propagating modes, labeled 1, involves $\psi_y$ and $S_z$, while mode 2 involves $\psi_z$ and $S_y$. As one follows a path around one of the vortices, however, mode 2 experiences a frame-dragging which causes it to mix with $S_z$, acquiring an Aharonov-Bohm minus sign when transported around a loop. Mode 1 remains unaffected. It is therefore conceivable that a similar charge exchange process could occur in the scattering of spin waves off pairs of half-quantum vortices. A circularly polarized spin wave (an eigenstate of the unbroken subgroup of rotations) could scatter from the pair of vortices, changing to the opposite circular polarization and depositing angular momentum (Cheshire charge) in the vicinity of the vortices.

The theoretical possibility of an Aharonov-Bohm type scattering from HQV’s using collective excitations of the fluid as the projectiles has been mentioned previously by Khazan and others[^15,10]. However, the context studied by these authors was that of NMR experiments in which either spin waves or an orbital “Higgs” excitation called the clapping mode are excited by means of a fluctuating magnetic field. The high steady-state magnetic field which is used in NMR breaks the degeneracy between the two “light” spin wave modes, leaving us with an abelian situation like that of reference [1]. Situations in which the non-abelian Aharonov-Bohm effect might be seen were not discussed.

[^15]: Reference to a specific page number or section number in the text.
[^10]: Reference to a specific page number or section number in the text.
It may in practice be difficult to devise a Cheshire charge experiment without having the $SO(3)$ symmetry destroyed by an external field. An additional difficulty arises from the potentials $U$ in equation (36). In addition to the Aharonov-Bohm effect, one expects spin waves to be scattered by these non-topological potentials which are nonzero within a soliton where $\mathbf{d}_0$ is not parallel to $\mathbf{\ell}$. These potentials arise because of the change in the dipole restoring force as $\mathbf{d}_0$ leaves the bottom of the dipole potential well, and because of other anisotropies associated with the orbital state. In fact, the two linear polarization states of the spin wave experience different potentials inside the soliton, which could give rise to phase shifts between them in addition to the Aharonov-Bohm phase.

APPENDIX A

Estimate of Charge Necessary to Stabilize Pair of HQV’s.

This appendix contains a derivation of the order-of-magnitude estimate (35) for the level of excitation of the charge rotor mode (and corresponding classical precession frequency) at which its energy becomes comparable to the dipole energy of the soliton connecting a pair of half-quantum vortices. The data and formulas used here can be found in references [10] and [11].

We begin with equation (34), equating the dipole energy with the zero-mode energy:

$$E_D \sim g_D \Delta^2_{\mathbf{X}} V_{\text{Sol}} \sim E_{ZM} \sim \gamma^2 \frac{n^2 \hbar^2 \chi_{\perp}}{V_{\text{Sol}}}. \quad (A1)$$

$\chi_{\perp}$ is the larger eigenvalue of the magnetic susceptibility tensor. It differs from the susceptibility $\chi^0_N = \gamma^2 \hbar^2 N(0)$ of a noninteracting degenerate Fermi gas only by a factor of order unity, so we may use this value as an estimate of $\chi_{\perp}$. In the previous expression, $N(0)$ is the density of states at the Fermi surface, given by $N(0) = m^* k_f / 2\pi^2 \hbar^2$ where $k_f \hbar$ is the Fermi momentum.
We also rewrite the dipole coupling constant in terms of measurable length scales as follows:

\[ g_D = \frac{\gamma_0}{\xi_D^2} = \frac{N(0)\xi_0^2}{\xi_D^2}. \]  
(A2)

\( \gamma_0 \) is a typical coefficient of the gradient energy: in the weak-coupling, or small-interaction limit, the coefficients \( \gamma_i \) in the gradient energy (26) are all equal to \( \gamma_0 \). We have in turn related this coefficient to the coherence length \( \xi_0(1 - T/T_C)^{-1/2} \).

The two length scales are quoted by [10] as of order \( \xi_D \sim 10^{-3} \) cm and \( \xi_0 \sim 10^{-6} \) cm.

Finally, we use the relation \( \Delta_A \sim kT_C(1 - T/T_C)^{1/2} \) for the gap parameter. We can substitute the estimates for \( \chi_\perp, g_D, \) and \( \Delta_A \) into (A1). Assuming temperatures in the millikelvin range, molar volumes of a few tens of \( cm^3 \), and quasiparticle mass \( m^* \) approximately equal to the atomic weight of helium, we obtain:

\[ V_{\text{Sol}} \sim \frac{n\hbar^2\xi_D}{\xi_0 m^* k_f kT_C(1 - T/T_C)^{1/2}} \sim n \times 10^{-16} cm^3. \]  
(A3)

We may also express the answer in terms of a classical precession frequency. Expressed in terms of \( \omega \), (A1) becomes:

\[ \frac{\omega^2 V_{\text{Sol}} \chi_\perp}{\gamma^2} \sim g_D \Delta_A^2 V_{\text{Sol}}. \]  
(A4)

Using the same estimates as above, we find:

\[ \omega \hbar \sim \frac{\xi_0}{\xi_D} \Delta_A. \]  
(A5)

This shows that the frequency is of order \( 10^{-3} \) times the gap frequency, or about 10 kHz if \( (1 - T/T_C) \sim 1 \). Not surprisingly, this is also of the same order as the cutoff frequency \( \Omega_A \) for spin waves.
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FIGURE CAPTIONS

1) Definition of angles $\varphi_1$, $\varphi_2$, and $\Delta \varphi$ in vortex pair geometry.

2) Degenerate configurations of vortex pair. This figure shows two order parameter configurations for a vortex-antivortex pair, related by a global symmetry operation. The order parameter is represented by an undirected line segment. In Fig.2A, the directors all lie within the plane of the page. In Fig.2B, they rotate outward, and only their projection in the plane of the page is shown.

3) Domain wall of dipole energy. The spin axis $\vec{d}$ is represented by the undirected line segments, while the thick arrows represent $\vec{\ell}$. The region in which $\vec{\ell}$ and $\vec{d}$ are not parallel has a width of order $\xi_D$.

4) Parallel transport of orthogonal spin wave modes. The effect of parallel transport about an HQV core on the two degenerate spin-wave modes is shown. $\vec{d}$ is indicated by the undirected line segments. The amplitude of oscillation of the spin density $\vec{S}$ is shown by the thick arrows. The thin arrows show the corresponding motion of $\vec{d}$. In mode 1 (upper figure) the spin density amplitude points out of the page and remains the same on transport around the core. For mode 2, (lower figure) the spin density amplitude is within the page and experiences a sign change.
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FIGURE 1
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This figure "fig1-3.png" is available in "png" format from:

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FIGURE 3
Spin density amplitude, $S$

Motion of spin quantization axis, $\psi$

FIGURE 4