Quantum criticality and formation of a singular Fermi liquid in the attractive $SU(N > 2)$ Anderson model

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While much is known about repulsive quantum impurity models, significantly less attention has been devoted to their attractive counterparts. This motivated us to study the attractive SU($N$) Anderson impurity model. While for the repulsive case, the phase diagram features mild $N$ dependence and the ground state is always a Fermi liquid, in the attractive case a Kosterlitz-Thouless charge localization phase transition is revealed for $N > 2$. Beyond a critical value of attractive interaction an abrupt jump appears in the number of particles at the impurity site, and a singular Fermi liquid state emerges, where the scattering of quasiparticles is found to exhibit power law behavior with fractional power. The capacity diverges exponentially at the quantum critical point, signaling the Kosterlitz-Thouless transition.

Introduction – Quantum impurity models (QIMs) such as Kondo [1] or Anderson models [2] play the fundamental role of gold standards in the field of correlated systems. In spite of their deceiving simplicity, these models capture the nature of interactions between localized degrees of freedom and a continuum of extended states, and display a plethora of appealing effects, including Kosterlitz-Thouless transitions [3], Fermi liquid vs. non-Fermi liquid behavior [4], asymptotic freedom and quantum criticality [5], just to list a few. They emerge in all kinds of correlated systems including Majorana systems coupled to electrodes [6–8], Josephson junction arrays [9], or in the context of rotating molecules in a quantum liquid [10], and they also provide the first step towards understanding bulk correlated materials [11].

In recent years, it became possible to engineer various quantum impurity models using the artillery of nanotechnology, and to investigate correlated states and quantum phase transitions in a controlled way in these model systems [12–15]. Ultracold atoms provide further media to develop QIMs with a tunable interaction strength [16]. Loading ultracold atoms with large hyperfine spins into optical lattices, one can design exotic SU($N$) spins with $N \geq 2$, predicted to give rise to exotic SU($N$) Kondo states [17–21].

Most research so far focused on repulsive quantum impurity models. However, in ultracold settings, not only the strength but also the sign of the interspecies interaction can be tuned by using Feshbach resonances to reach the attractive $U < 0$ regime [20]. Attractive interactions may also emerge in nanostructures: an attractive electron-electron interaction mediated by Coulomb repulsion has been demonstrated in carbon nanotubes [22], but attractive interactions can also be engineered in superconducting single electron transistors [23]. Recent experiment in heterostructures reveals the presence of 2, 3 and even 4 particle bound states [24].

In this work, we shall focus on the poorly studied $U < 0$ regime of the SU($N$) Anderson model, and show that it conceals an unexpected quantum phase transition, where charge degrees of freedom become localized. In this novel charge localized phase, electrons display singular Fermi liquid properties with power law anomalies [25]. The phase transition we find is the impurity analogue of the "baryonic" transition found in attractive SU($N$) Hubbard models, discussed intensively in the context of cold atoms [26, 27].

The SU($N$) Anderson model, we study here, describes local fermions of $N$ different flavors, $d_{\alpha}$ ($\alpha = 1, \ldots, N$), interacting with each other on a level of energy $\varepsilon$, and immersed in a sea of conduction electrons. Its Hamiltonian is defined as

$$H = \frac{U}{2} : Q :^2 + \varepsilon : \hat{Q} : + t \sum_{\alpha=1}^{N} (\psi_\alpha^\dagger d_{\alpha} + h.c.) + H_{\text{bath}},$$

with $Q := \sum_{\alpha=1}^{N} (d_{\alpha}^\dagger d_{\alpha} - 1/2) = \hat{Q} - N/2$ the normal ordered occupation number of level $\varepsilon$, and $U$ the strength of the interaction. The term $t$ describes the hybridization of the level with $N$ channels of conduction electrons, $\psi_{\alpha}$, with the last term $H_{\text{bath}}$ generating the dynamics of the Fermion field at the origin, $\psi_{\alpha} = \psi_{\alpha}(0)$.

The standard SU(2) version of this model has extensively been investigated in all possible regimes [28] and its phase diagram is well known by now. The characteristic behaviour of the model depends on the position of the level $\varepsilon$, and the interaction strength $U$ compared to the width of the level, $\Delta = \pi t^2 \vartheta_0$, generated by quantum fluctuations to the electron bath, characterised by the density of states of the electrons at the Fermi energy,
Kondo valence physics at charge $\langle Q \rangle \approx 0$ for $\rho$ lines. For $U < 0$ a charge jump transition appears at $U < U_C$ where $\langle Q \rangle$ is discontinuous at $\varepsilon = 0$.

$$\vartheta_0 = (2\pi v_F)^{-1},$$
with $v_F$ the Fermi velocity. For large repulsive interactions, it displays the famous spin Kondo effect [28–30] for $-U/2 \leq \varepsilon \leq U/2$, and gives rise to mixed valence physics at $\varepsilon \approx \pm U/2$. For $U < 0$, on the other hand, a charge Kondo effect appears at around $\varepsilon \approx 0$, where charge fluctuations between the states $\hat{Q} = 0$ and $\hat{Q} = 2$ lead to the emergence of a Kondo resonance [31–39]. Apart from these interesting features, however, the SU(2) Anderson model displays regular behavior for any finite $U$ and $\varepsilon$, and assumes a local Fermi liquid state with well-defined quasiparticles at low temperatures [40–42].

The SU(N) variant of the Anderson model has also attracted a lot of attention recently due to its relevance in various nanostructures [43, 44] and in ultracold settings [45]. Its behavior has been extensively analyzed in the repulsive regime [46–49], where it displays properties analogous to the SU(2) case: it exhibits mixed valence fluctuations at energies $\varepsilon \approx U(k - \frac{N-1}{2})$ with $k = 0, \ldots, N - 1$, and hosts various types of SU(N) Kondo effect between these (see Fig. 1). For any $\varepsilon$ and $U > 0$ one finds, however, a generalized Fermi liquid state, similar to the one appearing in the SU(2) Anderson model [42].

A naive expectation would thus be that the attractive SU(N) Anderson model is also a simple Fermi liquid. This expectation is, however, completely wrong, and, as demonstrated here, the $N > 2$ Anderson model displays a dissipative quantum phase transition at some critical interaction value $(U/\Delta)_c < 0$, and a singular behavior for $U/\Delta < (U/\Delta)_c$ (see Fig. 1).

Before engaging in a detailed analysis, let us give some strong and robust arguments for the existence of this phase transition. We first observe that, similar to the repulsive case [50], for $|U| \ll \Delta$ perturbation theory in $U$ is governed by the expansion parameter $|U|/\Delta \ll 1$, and therefore all properties are analytical functions of $U$ and $\varepsilon$. Thus the Fermi liquid state at $U > 0$ naturally extends to the regime of small negative $U$’s. The situation is, however, dramatically different for $|U| \gg \Delta$. Focusing on the regime, $\varepsilon \approx 0$, we may want to attempt there to perform perturbation theory in $t$. For $\varepsilon = 0$ and $t = 0$, however, the lowest lying $\hat{Q} = 0$ and $\hat{Q} = N$ states of the isolated level are degenerate. As depicted in Fig. 2.a, these two states are connected through high order virtual processes yielding a ”tunneling” term $\sim (D^\dagger \psi_1 \ldots \psi_N + h.c.)$ with $D^\dagger$ defined as $D^\dagger = d_{N}^\dagger \ldots d_1^\dagger$ (see Fig. 2 (a)). Simple power counting shows that this tunneling term is irrelevant in the renormalization group sense. As a consequence, charge fluctuations between the states $\hat{Q} = 0$ and $\hat{Q} = N$ must be suppressed at zero temperature. This leads us to the conclusion that for large attractive interactions, a first order ”charge jump” quantum phase transition must take place, as depicted in Fig. 1.

To substantiate this claim and describe this phase transition in detail, we first identify SU(N) invariant terms generated by quantum fluctuations of the charge, $\hat{Q}$. In addition to the tunneling term discussed above, charge fluctuations $\hat{Q} = 0 \leftrightarrow 1$ and $\hat{Q} = N \leftrightarrow N - 1$ generate a charge state dependent potential of the form $\sim (D^\dagger D - 1/2)\sum_\alpha \psi_\alpha^\dagger \psi_\alpha + H_{\text{bath}}$. This leads us to the effective Hamiltonian

$$H_{\text{eff}} = N\varepsilon (D^\dagger D - \frac{1}{2}) + j a^{N/2-1}(\psi_1^\dagger \ldots \psi_N^\dagger D + h.c.) + u (D^\dagger D - \frac{1}{2}) \sum_{\alpha=1}^N v_F \psi_\alpha^\dagger \psi_\alpha + H_{\text{bath}}$$

with $j \sim t^N/|U|^{N-1}$ denoting a dimensionless amplitude of ”charge exchange”, $a \sim v_F/U$ the natural cut-off parameter of the problem, and $u \approx 4\Delta/(|U|N - 1)$ the dimensionless strength of potential scattering. The fermionic bath in Eq. (2) can be expressed in terms of chiral one-dimensional fermions, $H_{\text{bath}} = v_F \sum_{\alpha=1}^N \int \psi_\alpha^\dagger(x) i\partial_x \psi_\alpha(x) dx/(2\pi)$, with $\psi(x > 0)$ and $\psi(x < 0)$ representing incoming and outgoing spherical waves [51], respectively [52].

A straightforward renormalization group analysis of $H_{\text{eff}}$ then yields the scaling equations (see the Supplemental material)

$$\frac{dj}{d\ln(a)} = \left(1 - \frac{N}{2}\right)j + Nju + \ldots,$$

$$\frac{du}{d\ln(a)} = 4j^2 + \ldots,$$

and the flow diagram presented in Fig. 2 (b). Clearly, below a critical value, $u < u_C$, (i.e., $\Delta/|U| < (\Delta/|U|)_C$) a small charge exchange $j$ is irrelevant, while for $u > u_C$, i.e., $\Delta/|U| > (\Delta/|U|)_C$, one always finds a flow to strong
coupling, \(j \to \infty\), signaling the dominance of quantum fluctuations and the formation of a charge Kondo state. The phase transition is of Kosterlitz-Thouless type, implying a Fermi liquid scale vanishing as

\[
T_{\text{FL}} \sim \exp \left( -C \sqrt{\frac{|U|}{\Delta - D_c}} \right). \tag{4}
\]

Mapping to the dissipative two-state system. – The nature of the phase transition can further be clarified by means of Abelian bosonization, which we use to map our model (2) to an Ohmic dissipative two state system [53]. To this end, we express the fermionic fields in terms of chiral bosons, \(\psi_\alpha(x) \equiv (\gamma_\alpha/\sqrt{a}) e^{-i\phi_\alpha(x)}\), with the Klein factors \(\gamma_\alpha\) – now represented as Majorana fermions – assuring the correct anticommutation relations between different Fermion fields. Introducing now the charge field, \(\varphi \equiv (\varphi_1 + \ldots + \varphi_N)/\sqrt{N}\), we express the Hamiltonian as

\[
H_{\text{eff}} = H_{\text{bath}} + j \frac{v_F}{a} \left( \gamma D e^{i\sqrt{N}\varphi_c} + \text{h.c.} \right) + u v_F \sqrt{N} (D^\dagger D - \frac{1}{2}) \frac{\partial_x \varphi_c}{N} , \tag{5}
\]

with \(\gamma \equiv \gamma_1 \ldots \gamma_N\). The fermionic bath can be represented here as

\[
H_{\text{bath}} = \frac{v_F}{4a} \int : (\partial_x \varphi_c(x))^2 : dx + \ldots ,
\]

with the dots referring to \(\text{SU}(N)\) spin excitations, decoupled from the local charge degree of freedom.

\[
H_{\text{eff}} = H_{\text{bath}} + j \frac{v_F}{a} \left( \gamma D e^{i\sqrt{N}\varphi_c} + \text{h.c.} \right) + u v_F \sqrt{N} (D^\dagger D - \frac{1}{2}) \frac{\partial_x \varphi_c}{N} , \tag{5}
\]

Here \(\Delta = 2jv_F/a\) denotes the tunneling amplitude of the two-state system, while \(\alpha = \frac{N}{2}(1 - u)^2\) stands for Leggett’s famous dissipation parameter [54]. The latter parameter governs the dissipative phase transition between a charge delocalized phase for \(\alpha < 1\) with coherent charge oscillations for \(\alpha < 1/2\), and a charge localized state for \(\alpha > 1\). We thus conclude that the charge localization transition in the \(U < 0\) Anderson model is essentially identical to Leggett’s famous dissipative quantum phase transition, with the critical value, \(\alpha = 1\) corresponding to \(u_c = 1 - \sqrt{2/N}\).

\[\text{NRG approach.} – \text{Our discussion above follows from an effective model, strictly justified only in the limit } |U| \gg \Delta, \text{ a condition certainly violated in the vicinity of the predicted quantum phase transition. To substantiate our claim and to demonstrate the Kosterlitz-Thouless character of the phase transition, we resort to (density matrix) numerical renormalization group (NRG), which we apply directly on the Anderson model (1) in the first non-trivial cases, } N = 3 \text{ and } N = 4. \text{ Then the Hamiltonian (1) has a global } U(1) \times \text{SU}(N) \text{ symmetry corresponding to charge conservation and SU(3) spin rotations, which we both exploit in our computations [47, 55].}

The average occupation \(\langle Q \rangle = \sum \alpha (d^\dagger_\alpha d_\alpha)\) is presented in Fig. 3 for \(N = 3\) as a function of \(\varepsilon\) for a fixed \(U < 0\) but different values of \(\Delta\). As expected, for \(\varepsilon \to \infty\), the local level becomes empty, while for \(\varepsilon \to -\infty\), it becomes completely occupied. For small attractive interactions, the occupation \(\langle Q \rangle\) gently crosses over between these two values as \(\varepsilon\) is varied, and is exactly \(\langle Q \rangle = 3/2\) at the particle-hole symmetric point, \(\varepsilon = 0\). The Kosterlitz-Thouless phase transition is signaled by the appearance of a sudden jump in \(\langle Q \rangle\) once the ratio \(U/\Delta\) exceeds a critical value, \(\langle U/\Delta \rangle_c \approx 2.43\) [56].

The capacity, \(\chi = \partial_\varepsilon \langle Q \rangle |_{\varepsilon=0}\) diverges as one approaches the transition from the Fermi liquid side, \(|U|/\Delta < \langle U/\Delta \rangle_c\), and its inverse defines the Fermi liquid scale, \(T_{\text{FL}} \equiv \chi^{-1}\). The scale \(T_{\text{FL}}\) is found to vanish exponentially as one approaches the transition from the weak coupling side, in agreement with Eq. (4) (see the inset of Fig. 3).

\[\text{Scattering and singular Fermi liquid.} – \text{The Fermi liquid scale vanishes on the first order transition line, } \varepsilon = 0 \text{ and } |U/\Delta | > \langle U/\Delta \rangle_c, \text{ where a free charge degree of freedom appears. It is this residual charge degree of freedom, which is responsible for the singular Fermi liquid properties.}

\[\text{FIG. 2. (a) Schematic picture of the } Q = 0 \text{ and } N \text{ states, connected through high order virtual processes through } 0 < Q < N \text{ states. (b) Leading order scaling trajectories obtained from Eqs. (3) signaling a Kosterlitz-Thouless phase transition at } u = u_c \text{ and } j = 0. \text{ Arrows indicate the upon decreasing energy scale. The shaded region marks charge localized states.}\]
The charge localization transition is also clearly observable in the scattering properties of the carriers. We have determined numerically the energy dependence of the total scattering cross section \( \sigma(\omega) \) of the conduction electrons as a function of their energy, \( \omega \). By the optical theorem, \( \sigma(\omega) \) is just proportional to the imaginary part of the T-matrix, \( \sigma(\omega) \propto -\text{Im} T(\omega) \), which is directly proportional to the d-level’s propagator, \( T(\omega) = 2\pi v_F \Delta G_{d\uparrow d\downarrow}(\omega) \). Fig. 4 displays the dimensionless cross section \( A(\omega) = -\text{Im} T(\omega) / (2\pi v_F) \) below and above the quantum phase transition.

In the Fermi liquid phase, scattering becomes maximally strong at energies below the Fermi liquid scale, \( T_{FL} \), and the dimensionless cross section exhibits an analytical behavior, \( A(\omega) = 1 - C \omega^2 / T^2_{FL} + \ldots \). In contrast, in the charge localized phase \( A(0) \) takes on a small value, \( A_0 = \sin^2(\delta/\pi) \), practically independent of the energy of the electrons. Here \( \delta \) denotes the state dependent residual phase shifts, characterizing the scattering of electrons at the Fermi energy. These are directly related to the interaction parameter \( u \) in the bosonization approach, \( \delta \approx \pm u \pi/2 \). This residual phase shift, extracted from the finite size spectrum of our NRG calculations, is displayed in Fig. 4. It assumes a universal value \( \delta_c = (1 - \sqrt{2/\xi}) \pi/2 \) at the transition point, and tends to \( \delta \rightarrow 0 \) in the limit of strong attractive interactions, \( |U|/\Delta \rightarrow \infty \). Notice that the critical value, \( A_c(0) \approx 0.081 \) (also confirmed by our numerics) is substantially reduced compared to the maximally strong scattering characterizing the Fermi liquid phase, implying that charge localization reduces the scattering cross section by \( \sim 92\% \).

In the charge localized phase, the scattering cross section displays an additional, slow power law dependence, \( A(\omega) = A_0 + A_1 |\omega|^{\beta} + \ldots \). The exponent \( \beta \) here can be determined from our effective model, (2), by means of Abelian bosonization. For \( N = 3 \) this yields \( \beta = \beta_c = 2(1 - \sqrt{2/3}) \) at the transition point, with \( \beta \) gradually increasing to \( \beta = 1 \) in the limit of strong attractive interaction (for the precise relation, see the Supplemental material [57]). This dependence is indeed verified by our NRG computations, shown in the inset of Fig. 4b. 

**Conclusions.** In this work, we have demonstrated that one of the most paradigmatic models, the attractive SU\((N)\) Anderson model displays a surprising charge localization transition for \( N > 2 \), characterized by an exponentially divergent Fermi liquid scale, a jump in the zero-temperature scattering cross section, and an emerging singular Fermi liquid state. The phase transition analyzed here is thus the impurity analogue of the baryon - color superfluid phase transition predicted in the attractive SU(3) Hubbard model [26], realized by trapped \(^6\text{Li}\) atoms. The charge localized phase of the Anderson model corresponds simply to the baryonic state, where heavy particles are bound together to trions, while the attractive heavy Fermi liquid formed at \( 0 > U/\Delta > -(|U|/\Delta)c \) is the mirror image of the color superfluid.

The present results signal a similar transition in the attractive SU\((4)\) Hubbard model, where quadruplons (4 particle bound states) must form at strong attractive interactions, while at weak couplings, a superfluid state is the natural candidate. These quadruplons may form a strongly interacting Bose condensate or display charge localization. We remark that bound \( N\)-ons become very heavy, since their effective hopping rate on a lattice is renormalized as \( t_{\text{eff}} \sim t^N / |U|^{N-1} \), and is much smaller than the nearest neighbor repulsion \( V \sim t^2 / |U| \) generated by quantum fluctuations. Therefore, attractive strongly interacting SU\((N)\) systems are susceptible to charge ordering, in close analogy with valence skipping.
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SUPPLEMENTAL INFORMATION

Effective model and the perturbative RG

We can treat our effective model in Eq. (2) by using the Wilson’s perturbative renormalization group (RG) approach [59]. For that we use $G(t)$ and $G_D(t)$ as the corresponding non-interacting Green's functions for the conduction and localized channels. The propagator of the electrons $G(t)$ decays as

$$G(t) = \frac{1}{v_F t + i a s g u(t)},$$

with $a$ the short distance cut-off.

From the RG point of view, $j$ is an irrelevant coupling that scales to zero at the tree level. Still, along the RG procedure it generates marginal terms $\propto u$, which renormalize $j$ itself, and can ultimately drive $j$ to strong coupling. In Wilson’s RG scheme we expand the action in the interaction part $S_{\text{int}}$, evaluate these corrections using Wick’s theorem, and then generate the RG by rescaling the cut-off $a$ and the couplings $j$ and $u$. To capture the quantum phase transition it is sufficient to restrict ourselves to second order perturbation theory, i.e. to one loop order. The relevant second order diagrams are presented in Fig. (5). The diagram in panel (a) in Fig. 5

renormalizes the exchange coupling $j$ while the one in panel (b) renormalizes $u$. For example, by rescaling $a$, the diagram depicted in Fig. 5(a) generates a term in the renormalized effective action

$$\delta S_{\text{int}} = N \sqrt{a(u,j)} \int dt \prod_{\alpha=1}^{N} \tilde{\psi}_\alpha(t) D(t) \int dt' \delta G(t'),$$

were the prefactor $N$ comes from the internal contractions, and $\delta G$ denotes the change in the propagator induced by the rescaling of $a \to a'$. The time integral yields $\int dt' \delta G_a(t') = -2 \delta a/a$. Clearly, the term generated this way can be incorporated in the original action by simply renormalizing the couplings. In terms of the scaling variable, $1 = \ln(a/a_0)$, this gives us the differential equations presented in the main text, Eqs. (3).

Low frequency properties of the T-matrix in the charge localized phase

The asymptotic behavior of the scattering cross-section can be determined by means of Abelian bosonization. By investigating the equation of motion of the propagator $G_\alpha(t) = -i(T \psi_\alpha(t) \psi_\alpha^{\dagger}(0))$ and the fields $\psi_\alpha$ and $\psi_\alpha^{\dagger}$, we find that the self-energy of the propagator $G_\alpha(t) = -i(T \psi_\alpha(t) \psi_\alpha^{\dagger}(0))$ is simply the correlator

$$\Sigma_\alpha(t) = -i(T_t O_\alpha(t) O_\alpha^{\dagger}(0)),$$

where $O_\alpha$ stands for the composite operator, $O_\alpha^{\dagger} = \sum_{n} a^{-N/2} D^{\dagger} \psi_n \cdots \psi_{n+1} \psi_{n-1} \cdots \psi_1$. Within the effective field theory, Eq. (2), this operator corresponds to the operator $d_\alpha^{\dagger}$. For simplicity, let us focus on $O_1^{\dagger}$. In the bosonized form, this operator can be expressed as $j a^{-1/2} D^{\dagger} \gamma_N \cdots \gamma_2 e^{-i(\varphi_1 + \cdots + \varphi_N)}$. Next we introduce the charge field $\varphi_\alpha \equiv \varphi \cdot \tau_\alpha$ with $\tau_\alpha = \{1, ... , 1\}/\sqrt{N}$ and $N - 1$ orthogonal and properly normalized spin fields, $\varphi^{(k)} \equiv \varphi \cdot \vec{\tau}^{(k)}$ with the $\vec{\tau}^{(k)}$ denoting unit vectors orthogonal to $\tau_\alpha$ and to each other.

We can then rewrite the operator $O_1^{\dagger}$ as

$$O_1^{\dagger} = \sum a^{-1/2} D^{\dagger} \gamma_N \cdots \gamma_2 \exp\{-i(\varphi_{c} + q \varphi_{s})\} \{9\}

with $q^2 = \frac{N-1}{N}$ and $\varphi_{c}$, a properly normalized combination of the spin fields $\varphi^{(k)}$. Clearly, only the charge part of this operator is affected by the interaction terms in Eq. (5).

By the optical theorem, the scattering cross section at energy $\omega$ is directly proportional to the imaginary part of the retarded self energy at that frequency. Therefore, we need to determine the low frequency behavior of $\Sigma_\alpha(\omega)$, related to the long time behavior of $\Sigma_\alpha(t)$. Since the coupling $j$ is irrelevant along the charge transition line,
in leading order, we can set it to zero in Eq. (5), and then eliminate the term \( \sim u \) by the unitary transformation 
\[
\tilde{U} = e^{iu\hat{\phi}_c(0)\sigma_z/2},
\]
transforming \( O_1^\dagger \) into
\[
\hat{O}_1^\dagger \sim D^\dagger \gamma_N \ldots \gamma_2 e^{-i\left(\frac{N-3}{\sqrt{N}} - \sqrt{N}u\right)\hat{\phi}_c - iq\hat{\phi}_s}.
\] (10)
with \( u \) related to the residual phase shift as \( u = 2\delta/\pi \).

Since, at this point, we have eliminated all non-trivial terms of the Hamiltonian, the asymptotic behavior is just determined by the vertex operator in (10), yielding
\[
\Sigma_1(t) \sim t^{-(N-1-2u(N-1)+u^2N)}.
\]
Fourier transformation then leads straightforwardly to the expression,
\[
\beta = N - 2 - 2u(N-1) + u^2N.
\]