PEARSON EQUATIONS FOR DISCRETE ORTHOGONAL POLYNOMIALS: II. GENERALIZED CHARLIER, MEIXNER AND HAHN OF TYPE I CASES

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ABSTRACT. The Cholesky factorization of the moment matrix is considered for the generalized Charlier, generalized Meixner and generalized Hahn of type I discrete orthogonal polynomials. For the generalized Charlier we present an alternative derivation of the Laguerre–Freud relations found by Smet and Van Assche. Third order and second order nonlinear ordinary differential equations are found for the recursion coefficient \( \gamma_n \). Laguerre–Freud relations are also found for the generalized Meixner case, which are compared with those of Smet and Van Assche. Finally, the generalized Hahn of type I discrete orthogonal polynomials are studied as well, and Laguerre-Freud equations are found and the differences with the equations found by Dominici and by Filipuk and Van Assche are given.

1. Introduction
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1. Introduction

Discrete orthogonal polynomials is a distinguished and well established part of the theory of orthogonal polynomials and classical monographs have been devoted to this type of orthogonal polynomials. For example, the classical case is discussed in detail in [45] and the Riemann–Hilbert problem was considered in order to study asymptotics and applications in [15]. For a authoritative discussion of the subject see [32, 33, 16, 49]. Semiclassical discrete orthogonal polynomials, where a discrete Pearson equation is fulfilled by the weight, has been treated extensively in the literature, see [23, 24, 21, 22] and references therein for an comprehensive account. For some specific type of weights of generalized Charlier and Meixner types, the corresponding Freud–Laguerre type equations for the coefficients of the three term recurrence has been studied, see for example [19, 26, 27, 28, 48].
This paper is a sequel of [42] in which we used the Cholesky factorization of the moment matrix to study discrete orthogonal polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) on the homogeneous lattice. We considered semiclassical discrete orthogonal polynomials with weights subject to a discrete Pearson equation and, consequently, with moments constructed in terms of generalized hypergeometric functions. A banded semi-infinite matrix \( \Psi \), named as Laguerre–Freud structure matrix, that models the shifts by \( \pm 1 \) in the independent variable of the sequence of orthogonal polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) was given. We also show in that paper that the contiguous relations for the generalized hypergeometric functions translate into symmetries for the corresponding moment matrix, and that the 3D Nijhoff–Capel discrete Toda lattice [44, 31] describes the corresponding contiguous shifts for the squared norms of the orthogonal polynomials. In [40] we gave an interpretation for the contiguous transformations for the generalized hypergeometric functions in terms of simple Christoffel and Geronimus transformations. Using the Geronimus–Uvarov perturbations we got determinantal expressions for the shifted orthogonal polynomials.

In this paper we consider the generalized Charlier, Meixner and type I Hahn discrete orthogonal polynomials, analyze the Laguerre–Freud structure matrix \( \Psi \), and derive from its banded structure and its compatibility with the Toda equation and the Jacobi matrix, a number of nonlinear equations for the coefficients \( \{\beta_n, \gamma_n\} \) of the three term recursion relations \( zP_n(z) = P_{n+1}(z) + \beta_n P_n(z) + \gamma_n P_{n-1}(z) \) satisfied by the orthogonal polynomial sequence. These non linear recurrences for the recursion coefficients are of the type \( \gamma_{n+1} = F_1(n, \gamma_n, \gamma_{n-1}, \ldots, \beta_n, \beta_{n-1}) \) and \( \beta_{n+1} = F_2(n, \gamma_{n+1}, \gamma_n, \ldots, \beta_n, \beta_{n-1}, \ldots) \), for some functions \( F_1, F_2 \). Magnus [35, 36, 37, 38] named these type of relations, attending to [34, 29], as Laguerre–Freud equations. A comparative discussion with the Laguerre–Freud given by Smet & Van Assche [48] is also performed. Finally, in §4 we discuss the generalized Hahn of type I orthogonal polynomials, see [21, 23], giving in Theorem 8 the pentadiagonal Laguerre–Freud structure matrix and in Theorem 9 we discuss the corresponding Laguerre–Freud relations and its comparison with the results of Dominici [21] and Filipuk & Van Assche [28] is given, see also [25].

Now, to complete this Introduction we give a brief resume of the basic facts regarding discrete orthogonal polynomials and then a briefing of the relevant facts of [42].

1.1. Basics on orthogonal polynomials. Given a linear functional \( \rho_z \in \mathbb{C}^*[z] \), the corresponding moment matrix is

\[
G = (G_{n,m}), \quad G_{n,m} = \rho_{n+m}, \quad \rho_n = \langle \rho_z, z^n \rangle, \quad n, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots \},
\]

with \( \rho_n \) the \( n \)-th moment of the linear functional \( \rho_z \). If the moment matrix is such that all its truncations, which are Hankel matrices, \( G_{i+1,j} = G_{i,j+1} \),

\[
G^{[k]} = \begin{pmatrix}
G_{0,0} & \cdots & G_{0,k-1} \\
\vdots & \ddots & \vdots \\
G_{k-1,0} & \cdots & G_{k-1,k-1}
\end{pmatrix}
= \begin{pmatrix}
\rho_0 & \rho_1 & \cdots & \rho_{k-1} \\
\rho_1 & \rho_2 & \cdots & \rho_k \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{k-1} & \rho_k & \cdots & \rho_{2k-2}
\end{pmatrix}
\]
are nonsingular; i.e. the Hankel determinants $\Delta_k := \det G^{[k]}$ do not cancel, $\Delta_k \neq 0$, $k \in \mathbb{N}_0$. If this is the case, we have monic polynomials

$$P_n(z) = z^n + p_n^1 z^{n-1} + \cdots + p_n^n, \quad n \in \mathbb{N}_0,$$

with $p_n^1 = 0$, fulfilling the orthogonality relations

$$\langle \rho, P_n(z)z^k \rangle = 0, \quad k \in \{0, \ldots, n-1\}, \quad \langle \rho, P_n(z)z^n \rangle = H_n \neq 0,$$

and $\{P_n(z)\}_{n \in \mathbb{N}_0}$ is a sequence of orthogonal polynomials, i.e., $\langle \rho, P_n(z)P_m(z) \rangle = \delta_{n,m} H_n$ for $n, m \in \mathbb{N}_0$. The symmetric bilinear form $\langle F, G \rangle_\rho := \langle \rho, FG \rangle$, is such that the moment matrix is the Gram matrix of this bilinear form and $\langle P_n, P_m \rangle_\rho := \delta_{n,m} H_n$.

Introducing $\chi(z) := (1 \ z \ z^2 \cdots)^\top$ the moment matrix is $G = \langle \rho, \chi\chi^\top \rangle$, and $\chi$ is an eigenvector of the shift matrix, $\Lambda \chi = x \chi$, where

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & \cdots \\ & & & \\ & & & \end{pmatrix}.$$

Hence $\Lambda G = GA^\top$, and the moment matrix is a Hankel matrix. As the moment matrix symmetric its Borel–Gauss factorization is a Cholesky factorization

$$G = S^{-1}HS^{-\top},$$

where $S$ is a lower unitriangular matrix that can be written as

$$S = \begin{pmatrix} 1 & 0 & \cdots \\ & S_{1,0} & 1 \\ & S_{2,0} & S_{2,1} \\ & \vdots & \ddots \end{pmatrix},$$

and $H = \text{diag}(H_0, H_1, \ldots)$ is a diagonal matrix, with $H_k \neq 0$, for $k \in \mathbb{N}_0$. The Cholesky factorization does hold whenever the principal minors of the moment matrix; i.e., the Hankel determinants $\Delta_k$, do not cancel.

The components $P_n(z)$ of

$$P(z) := S\chi(z),$$

are the monic orthogonal polynomials of the functional $\rho$.

**Proposition 1.** We have the determinantal expressions

$$H_k = \frac{\Delta_{k+1}}{\Delta_k}, \quad p_n^1 = -\frac{\Delta_k}{\Delta_k},$$

with the Hankel determinants given by

$$\Delta_k := \det \begin{pmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_{k-2} & \cdots & \rho_{2k-3} & \rho_{2k-2} \\ \rho_{k-1} & \cdots & \rho_{2k-3} & \rho_{2k-2} \end{pmatrix}, \quad \tilde{\Delta}_k := \det \begin{pmatrix} \rho_0 & \cdots & \rho_{k-2} & \rho_k \\ \rho_{k-2} & \cdots & \rho_{2k-2} & \rho_{2k-1} \\ \rho_{k-1} & \cdots & \rho_{2k-2} & \rho_{2k-1} \end{pmatrix}.$$
We introduce the lower Hessenberg semi-infinite matrix
\[ J = SΛS^{-1} \] (4)
that has the vector \( P(z) \) as eigenvector with eigenvalue \( zJP(z) = zP(z) \). The Hankel condition \( ΛG = GA^T \) and the Cholesky factorization gives
\[ JH = (JH)^T = HJ^T. \] (5)
As the Hessenberg matrix \( JH \) is symmetric the Jacobi matrix \( J \) is tridiagonal. The Jacobi matrix \( J \) given in (4) reads
\[
J = \begin{pmatrix}
β_0 & 1 & 0 & \cdots \\
γ_1 & β_1 & 1 & \cdots \\
0 & γ_2 & β_2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}.
\]
The eigenvalue equation \( JP = zP \) is a three term recursion relation \( zP_n(z) = P_{n+1}(z) + \beta_nP_n(z) + \gamma_nP_{n-1}(z) \), that with the initial conditions \( P_{-1} = 0 \) and \( P_0 = 1 \) completely determines the sequence of orthogonal polynomials \( \{P_n(z)\}_{n∈\mathbb{N}_0} \) in terms of the recursion coefficients \( \beta_n, γ_n \). The recursion coefficients, in terms of the Hankel determinants, are given by
\[ \beta_n = p_n^1 - p_{n+1}^1 = \frac{\tilde{Δ}_n}{Δ_n} + \frac{\tilde{Δ}_{n+1}}{Δ_{n+1}}, \quad γ_{n+1} = \frac{H_{n+1}}{H_n} = \frac{Δ_{n+1}Δ_{n-1}}{Δ_n^2}, \quad n ∈ \mathbb{N}_0, \] (6)
For future use we introduce the following diagonal matrices \( \gamma := \text{diag}(γ_1, γ_2, \ldots) \) and \( \beta := \text{diag}(β_0, β_1, \ldots) \) and \( J_− := Λ^Tγ \) and \( J_+ := β + Λ \), so that we have the splitting \( J = Λ^Tγ + β + Λ = J_− + J_+ \). In general, given any semi-infinite matrix \( A \), we will write \( A = A_− + A_+ \), where \( A_− \) is a strictly lower triangular matrix and \( A_+ \) an upper triangular matrix. Moreover, \( A_0 \) will denote the diagonal part of \( A \).

The lower Pascal matrix is defined by
\[ B = (B_{n,m}), \quad B_{n,m} := \begin{cases} \binom{n}{m}, & n ≥ m, \\
0, & n < m. \end{cases} \] (7)
so that
\[ χ(z + 1) = Bχ(z). \] (8)
Moreover,
\[ B^{-1} = (\tilde{B}_{n,m}), \quad \tilde{B}_{n,m} := \begin{cases} (-1)^{n+m}\binom{n}{m}, & n ≥ m, \\
0, & n < m. \end{cases} \]
and \( χ(z + 1) = B^{-1}\chi(z) \). The lower Pascal matrix and its inverse are explicitly given by
\[
B = \begin{pmatrix}
1 & 0 & \cdots & \cdots \\
1 & 1 & 0 & \cdots & \cdots \\
1 & 2 & 1 & 0 & \cdots \\
1 & 3 & 3 & 1 & 0 & \cdots \\
1 & 4 & 6 & 4 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}, \quad B^{-1} = \begin{pmatrix}
1 & 0 & \cdots & \cdots \\
-1 & 1 & 0 & \cdots & \cdots \\
-1 & -2 & 1 & 0 & \cdots \\
-1 & -3 & -3 & 1 & 0 & \cdots \\
-1 & -4 & -6 & -4 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]
in terms of which we introduce the dressed Pascal matrices, \( \Pi := SBS^{-1} \) and \( \Pi^{-1} := SB^{-1}S^{-1} \), which are connection matrices; i.e.,

\[
P(z + 1) = \Pi P(z), \quad P(z - 1) = \Pi^{-1} P(z).
\]

The lower Pascal matrix can be expressed in terms of its subdiagonal structure as follows

\[
B^{\pm 1} = I \pm \Lambda^\top D + (\Lambda^\top)^2 D^{[2]} \pm (\Lambda^\top)^3 D^{[3]} + \cdots,
\]

where \( D = \text{diag}(1, 2, 3, \ldots) \) and \( D^{[k]} = \frac{1}{k} \text{diag}(k^{(k)}, (k+1)^{(k)}, (k+2)^{(k)} \cdots) \), in terms of the falling factorials \( x^{(k)} = x(x-1)(x-2) \cdots (x-k+1) \). That is,

\[
D^{[k]} = \frac{(n+k) \cdots (n+1)}{k}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0.
\]

The lower unitriangular factor can be also written in terms of its subdiagonals \( S = I + \Lambda^\top S^{[1]} + (\Lambda^\top)^2 S^{[2]} + \cdots \) with \( S^{[k]} = \text{diag}(S_0^{[k]}, S_1^{[k]}, \ldots) \). From (3) is clear the following connection between these subdiagonals coefficients and the coefficients of the orthogonal polynomials given in (1)

\[
S_k^{[k]} = p_{n+k}^k.
\]

We will use the shift operators \( T_{\pm} \) acting over the diagonal matrices as follows

\[
T_{\pm} \text{diag}(a_0, a_1, \ldots) := \text{diag}(a_1, a_2, \ldots), \quad T_{\pm} \text{diag}(a_0, a_1, \ldots) := \text{diag}(0, a_0, a_1, \ldots).
\]

These shift operators have the following important properties, for any diagonal matrix \( A = \text{diag}(A_0, A_1, \ldots) \)

\[
\Lambda A = (T_- A) \Lambda, \quad AA = \Lambda (T_+ A), \quad AA^\top = \Lambda^\top (T_- A), \quad \Lambda^\top A = (T_+ A) \Lambda^\top.
\]

**Remark 1.** Notice that the standard notation, see [45], for the differences of a sequence \( \{f_n\}_{n \in \mathbb{N}_0} \),

\[
\Delta f_n := f_{n+1} - f_n, \quad n \in \mathbb{N}_0,
\]

\[
\nabla f_n = f_n - f_{n-1}, \quad n \in \mathbb{N},
\]

and \( \nabla f_0 = f_0 \), connects with the shift operators by means of

\[
T_- = I + \Delta, \quad T_+ = I - \nabla.
\]

In terms of these shift operators we find

\[
2D^{[2]} = (T_- D) D, \quad 3D^{[3]} = (T_- D) (T_- D) D = 2(T_- D^{[2]}) D = 2D^{[2]} (T_- D).
\]

**Proposition 2.** The inverse matrix \( S^{-1} \) of the matrix \( S \) expands as follows

\[
S^{-1} = I + \Lambda^\top S^{-[1]} + (\Lambda^\top)^2 S^{-[2]} + \cdots.
\]

The subdiagonals \( S^{-[k]} \) are given in terms of the subdiagonals of \( S \). In particular,

\[
S^{-[1]} = -S^{[1]},
\]

\[
S^{-[2]} = -S^{[2]} + (T_- S^{[1]}) S^{[1]},
\]

\[
S^{-[3]} = -S^{[3]} + (T_- S^{[2]}) S^{[1]} + (T_- S^{[1]}) S^{[2]} - (T_- S^{[1]}) (T_- S^{[1]}) S^{[1]}.
\]

**Remark 2.** Corresponding expansions for the dressed Pascal matrices are

\[
\Pi^{\pm 1} = I + \Lambda^\top \pi^{\pm 1} + (\Lambda^\top)^2 \pi^{[\pm 2]} + \cdots
\]

with \( \pi^{[\pm n]} = \text{diag}(\pi_0^{[\pm n]}, \pi_1^{[\pm n]}, \ldots) \).
Proposition 3 (The dressed Pascal matrix coefficients). We have
\begin{align}
\pi_n^{[\pm 1]} &= \pm(n+1), \quad \pi_n^{[\pm 2]} = \frac{(n+2)(n+1)}{2} \pm p_{n+2}^1(n+1) \mp (n+2)p_{n+1}^1 \\
\pi_n^{[\pm 3]} &= \pm \frac{(n+3)(n+2)(n+1)}{3} + \frac{(n+2)(n+1)}{2}p_{n+3}^1 - \frac{(n+3)(n+2)}{2}p_{n+1}^1 \\
&\quad \pm (n+1)p_{n+3}^2 \mp (n+3)p_{n+2}^1 \pm (n+3)p_{n+2}^1p_{n+1}^1 \mp (n+2)p_{n+3}^1p_{n+1}^1.
\end{align}
Moreover, the following relations are fulfilled
\begin{equation}
\pi^{[1]} + \pi^{[-1]} = 0, \quad \pi^{[2]} + \pi^{[-2]} = 2D^2, \quad \pi^{[3]} + \pi^{[-3]} = 2((T^2 S^{[1]})D^2 - (T_-. D^2) S^{[1]}).
\end{equation}

1.2. Discrete orthogonal polynomials and Pearson equation. We are interested in measures with support on the homogeneous lattice \(\mathbb{N}_0\) as follows \(\rho = \sum_{k=0}^{\infty} \delta(z - k)w(k)\), with moments given by
\begin{equation}
\rho_n = \sum_{k=0}^{\infty} k^n w(k),
\end{equation}
and, in particular, with 0-th moment given by
\begin{equation}
\rho_0 = \sum_{k=0}^{\infty} w(k).
\end{equation}
The weights we consider in this paper satisfy the following discrete Pearson equation
\begin{equation}
\nabla(\sigma w) = \tau w,
\end{equation}
that is \(\sigma(k)w(k) - \sigma(k-1)w(k-1) = \tau(k)w(k),\) for \(k \in \{1, 2, \ldots\}\), with \(\sigma(z), \tau(z) \in \mathbb{R}[z].\) If we write \(\theta := \tau - \sigma,\) the previous Pearson equation reads
\begin{equation}
\theta(k+1)w(k+1) = \sigma(k)w(k), \quad k \in \mathbb{N}_0.
\end{equation}
Theorem 1 (Hypergeometric symmetries). Let the weight \(w\) be subject to a discrete Pearson equation of the type \([19]\), where the functions \(\theta, \sigma\) are polynomials, with \(\theta(0) = 0.\) Then,
\begin{enumerate}
\item The moment matrix fulfills
\begin{equation}
\theta(\Lambda) G = B\sigma(\Lambda) G B^\top.
\end{equation}
\item The Jacobi matrix satisfies
\begin{equation}
\Pi^{-1} H \theta(J^\top) = \sigma(J) H \Pi^\top,
\end{equation}
and the matrices \(H \theta(J^\top)\) and \(\sigma(J)H\) are symmetric.
\end{enumerate}
If \(N+1 := \deg \theta(z)\) and \(M := \deg \sigma(z),\) and zeros of these polynomials are \(\{-b_i + 1\}_{i=1}^{N}\) and \(\{-a_i\}_{i=1}^{M}\), we write \(\theta(z) = z(z + b_1 - 1) \cdots (z + b_N - 1)\) and \(\sigma(z) = \eta(z + a_1) \cdots (z + a_M).\) According to \([17]\) the 0-th moment
\begin{equation}
\rho_0 = \sum_{k=0}^{\infty} w(k) = \sum_{k=0}^{\infty} \frac{(a_1)_{k} \cdots (a_M)_{k}}{(b_1 + 1)_{k} \cdots (b_N + 1)_{k}} \frac{\eta^k}{k!} = M F_N [a_1, \ldots, a_M; b_1, \ldots, b_N; \eta] = M F_N \begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix} \eta,
\end{equation}
is the generalized hypergeometric function, where we are using the two standard notations, see \([14]\). Then, according to \([16]\), for \(n \in \mathbb{N},\) the corresponding higher moments \(\rho_n = \sum_{k=0}^{\infty} k^n w(k),\) are
\begin{equation}
\rho_n = \partial^n_\eta \rho_0 = \partial^n_\eta \left( M F_N \begin{bmatrix} a_1 & \cdots & a_M \\ b_1 & \cdots & b_N \end{bmatrix} \eta \right), \quad \partial^n_\eta := \eta \frac{\partial}{\partial \eta}.
\end{equation}
Given a function $f(\eta)$, we consider the Wronskian

$$W_n(f) = \det \begin{pmatrix} f & \partial_\eta f & \partial_\eta^2 f & \ldots & \partial_\eta^k f \\ \partial_\eta f & \partial_\eta^2 f & \ldots & \partial_\eta^k f \\ \vdots & \vdots & \ddots & \vdots \\ \partial_\eta^k f & \partial_\eta^{k+1} f & \ldots & \partial_\eta^{2k} f \end{pmatrix}.$$  

Then, we have that the Hankel determinants $\Delta_k = \det G[k]$ determined by the truncations of the corresponding moment matrix are Wronskians of generalized hypergeometric functions,

$$\Delta_k = \tau_k,$$

$$\tau_k := W_k \left( M F_{\eta} \frac{a_1 \ldots a_M}{b_1 \ldots b_N \eta} \right),$$

Moreover, using Proposition [1] we get

$$H_k = \frac{\tau_{k+1}}{\tau_k},$$

$$p_k = -\partial_\eta \log \tau_k.$$  

The functions $\tau_k$ are known in the literature on integrable systems as tau functions.

**Theorem 2** (Laguerre–Freud structure matrix). Let us assume that the weight $w$ solves the discrete Pearson equation [19] with $\theta, \sigma$ polynomials such that $\theta(0) = 0$, deg $\theta(z) = N + 1$, deg $\sigma(z) = M$. Then, the Laguerre–Freud structure matrix

$$\Psi := \Pi^{-1} H \theta(J^\top) = \sigma(J) H \Pi^\top = \Pi^{-1} \theta(J) H = H \sigma(J^\top) \Pi^\top$$

$$\Psi = (\Lambda^\top)^M \psi^{(-M)} + \ldots + \Lambda^\top \psi^{(-1)} + \psi^{(0)} + \Lambda + \ldots + \psi^{(N+1)} \Lambda^{N+1},$$

for some diagonal matrices $\psi^{(k)}$. In particular, the lowest subdiagonal and highest superdiagonal are given by

$$\psi^{(-M)} = \eta (J_-)^M H, \quad \psi^{(-M)} = \eta H \prod_{k=0}^{M-1} T_k^\gamma = \eta \text{diag} \left( H_0 \prod_{k=1}^{M} \gamma_k, H_1 \prod_{k=2}^{M+1} \gamma_k, \ldots \right),$$

$$\psi^{(N+1)} \Lambda^{N+1} = H(J^\top)^{N+1}, \quad \psi^{(N+1)} = H \prod_{k=0}^{N} T_k^\gamma = \text{diag} \left( H_0 \prod_{k=1}^{N+1} \gamma_k, H_1 \prod_{k=2}^{N+2} \gamma_k, \ldots \right).$$

The vector $P(z)$ of orthogonal polynomials fulfill the following structure equations

$$\theta(z) P(z - 1) = \Psi H^{-1} P(z), \quad \sigma(z) P(z + 1) = \Psi^\top H^{-1} P(z).$$

The compatibility of the recursion relation, i.e. eigenfunctions of the Jacobi matrix, and the recursion matrix leads to some interesting equations:

**Proposition 4.** The following compatibility conditions for the Laguerre–Freud and Jacobi matrices hold

$$[\Psi H^{-1}, J] = \Psi H^{-1},$$

$$[J, \Psi^\top H^{-1}] = \Psi^\top H^{-1}.$$
1.3. The Toda flows. Let us define the strictly lower triangular matrix
\[ \Phi := (\vartheta S)S^{-1}. \]

**Proposition 5.**

i) The semi-infinite vector \( P \) fulfills
\[ \vartheta P = \Phi P. \]

ii) The Sato–Wilson equations holds
\[ -\Phi H + \vartheta H - H\Phi^\top = JH. \]

Consequently, \( \Phi = -J \) and \( n \in \mathbb{N} \), we have \( \vartheta \log H_n = J_n \).

Moreover,

**Proposition 6** (Toda). The following equations hold
\[ \Phi = (\vartheta S)S^{-1} = -\Lambda^\top \gamma, \]
\[ (\vartheta H)H^{-1} = \beta. \]

In particular, for \( n, k - 1 \in \mathbb{N} \), we have
\[ \vartheta P_n^1 = -\gamma_n, \quad \vartheta P_{n+k}^k = -\gamma_{n+k}P_{n+k-1}^k, \]
\[ \vartheta \log H_n = \beta_n. \]

The functions \( q_n := \log H_n, n \in \mathbb{N} \), satisfy the Toda equations
\[ \vartheta^2 q_n = e^{q_{n+1} - q_n} - e^{q_n - q_n - 1}. \]

For \( n \in \mathbb{N} \), we also have \( \vartheta P_n(z) = -\gamma_n P_{n-1}(z) \).

**Proposition 7.** The following Lax equation holds \( \vartheta J = [J, J] \). The recursion coefficients satisfy the following Toda system
\[ \vartheta \beta_n = \gamma_{n+1} - \gamma_n, \]
\[ \vartheta \log \gamma_n = \beta_n - \beta_{n-1}, \]
for \( n \in \mathbb{N}_0 \) and \( \beta_1 = 0 \). Consequently, we get
\[ \vartheta^2 \log \gamma_n + 2\gamma_n = \gamma_{n+1} + \gamma_{n-1}. \]

For the compatibility of \( \Phi \) and \( \Phi^\top \), that is, for the compatibility of the systems
\[ \{ \begin{align*}
P(z + 1) &= \Pi P(z), \\
\vartheta \log P(z) &= \Phi P(z).
\end{align*} \]

we obtain \( \vartheta(\Pi) = [\Phi, \Pi] \). In the general case the dressed Pascal matrix \( \Pi \) is a lower unitriangular semi-infinite matrix, that possibly has an infinite number of subdiagonals. However, for the case when the weight \( w(z) = v(z)\eta^z \) satisfies the Pearson equation \( (19) \), with \( v \) independent of \( \eta \), that is \( \theta(k + 1)v(k + 1)\eta = \sigma(k)v(k) \), the situation improves as we have the banded semi-infinite matrix \( \Psi \) that models the shift in the \( z \) variable as in \( (28) \). From the previous discrete Pearson equation we see that \( \sigma(z) = \eta \kappa(z) \) with \( \kappa, \theta \eta \)-independent polynomials in \( z \)
\[ \theta(k + 1)v(k + 1) = \eta \kappa(k)v(k). \]

**Proposition 8.** Let us assume a weight \( w \) satisfying the Pearson equation \( (19) \). Then, the Laguerre–Freud structure matrix \( \Psi \) given in \( (25) \) satisfies
\[ \vartheta (\eta^{-1}\Psi^\top H^{-1}) = [\Phi, \eta^{-1}\Psi^\top H^{-1}], \]
\[ \vartheta (\Psi H^{-1}) = [\Phi, \Psi H^{-1}]. \]

Relations \( (38a) \) and \( (38b) \) are gauge equivalent.
2. Generalized Charlier weights

A particular simple case is when \( \sigma \) does not depend upon \( z \), that we name as extended Charlier weights. The generalized Charlier, which is the main example of these extended Charlier weights, corresponds to the choice \( \sigma(z) = \eta \) and \( \theta(z) = z(z + b) \). In fact, a weight that satisfies the Pearson equation

\[(k + 1)(k + 1 + b)w(k + 1) = \eta w(k)\]

is proportional to the generalized Charlier weight

\[w(z) = \frac{\Gamma(b + 1)\eta^z}{\Gamma(z + b + 1)\Gamma(z + 1)} = \frac{1}{(b + 1)z!} \eta^z.\]

The finiteness of the moments requires \( b > -1 \) and \( \eta > 0 \), see [23]. The generalized Charlier 0-th moment, as we have discussed, is the confluent hypergeometric limit function \( \rho_0(\eta) = {}_0F_1(b + 1; \eta) \).

**Remark 3.** Is well known the relation of this function with the Bessel functions \( J_\nu \) and \( I_{\nu} \):

\[
\frac{(\eta^2/4)^b}{\Gamma(b + 1)} \rho_0(\eta) = J_0(\eta), \quad \frac{(\eta^2/4)^b}{\Gamma(b + 1)} \rho_0(\eta) = I_0(\eta).
\]

Here \( J_\nu \) is the \( \nu \)-th Bessel function and \( I_{\nu} \) the \( \nu \)-th modified Bessel, both are connected: \( e^{\pm i \pi \nu} I_{\nu}(x) = J_{\nu}(e^{\pm i \pi \nu} x) \).

Thus, in terms of these modified Bessel functions, we can write, see [23]

\[\rho_0(\eta) = \frac{\Gamma(b + 1)}{\sqrt{\eta^2}} I_0(2\sqrt{\eta}).\]

**Remark 4.** The non generalized Charlier polynomials or Poisson–Charlier polynomials were discussed by Charlier in [18].

**Proposition 9.** Let us consider an extended Charlier type weight \( w \), i.e. \( \theta(k + 1)w(k + 1) = \eta w(k) \) with \( \theta(0) = 0 \). Then, the semi-infinite matrix \( H\theta(J)H \) admits the Cholesky factorization

\[(39) \quad \theta(J)H = H\theta(J^T) = \Theta^{-1}H\Theta^{-T} = \Pi H\Pi^T,
\]

with the dressed Pascal semi-infinite matrix \( \Pi = \Theta^{-1} \) being a lower unitriangular semi-infinite matrix with only its first degree \( \theta \) subdiagonals different from zero.

**Proof.** From (25) we have \( \Pi^{-1}H\theta(J^T) = H\Pi^T \); i.e. \( H\theta(J^T) = \Pi H\Pi^T \) and given the uniqueness of the Cholesky factorization and its band structure we deduce the result. \( \square \)

**Proposition 10.** For an extended Charlier type weight, with the choice \( \sigma = \eta \) and \( \theta(z) = z(z^{N-1} + \cdots + \theta_1) \), we have the following subdiagonal structure for the dressed Pascal matrix \( \Pi = \sum_{k=0}^{N} (\Lambda^T)^k \pi[k] \) with the main diagonal and first subdiagonals given by \( \pi[0] = I \) and \( \pi[1] = D = \text{diag}(1, 2, 3, \cdots) \), and the lowest subdiagonal

\[(\Lambda^T)^N \pi \parallel [N] = (J_\rho)^N, \quad \pi \parallel [N] = \prod_{k=0}^{N-1} T^k_{\gamma} = \text{diag} \left( \prod_{k=1}^{N} \gamma_k, \prod_{k=2}^{N+1} \gamma_k, \cdots \right) = \text{diag} \left( \frac{H^N}{H_0}, \frac{H_{N+1}}{H_1}, \cdots \right).\]

**Proof.** The only new fact to prove is the explicit expression for the lowest subdiagonal, that obviously will come from the lowest diagonal of \( \theta(J) \) that is \( J^N \).

**Theorem 3** (The generalized Charlier Laguerre–Freud structure matrix). For a generalized Charlier weight, i.e. \( \sigma = \eta \) and \( \theta = z(z + b) \), the Laguerre–Freud structure matrix is

\[
\Psi = \begin{pmatrix}
\eta H_0 & \eta H_0 & H_2 & 0 & 0 & \cdots \\
0 & \eta H_1 & 2\eta H_1 & H_3 & 0 \\
0 & 0 & \eta H_2 & 3\eta H_2 & H_4 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.
\]
Proof. For the structure matrix $\Psi$ we have that, see (27), $\Psi = \psi^{(0)} + \psi^{(1)} \Lambda + \psi^{(2)} \Lambda^2$. Let us find these diagonal matrix coefficients. In the one hand, as

(40) \[ \Psi = H \Pi^\top = \eta H + \eta H \Lambda + \eta H \pi^2 \Lambda^2 + \cdots, \]

we get from the main diagonal that $\psi^{(0)} = \eta H$, and from the first superdiagonal that $\psi^{(1)} = \eta H \Lambda$. In the other hand, we observe that

\[
\Psi = \Pi^{-1} H \left( (J^\top)^2 + b J^\top \right) \\
= (I - \Lambda^\top D + (\Lambda^\top)^2 \pi^{-2} + \cdots) H \\
\times ((\Lambda^\top)^2 + \Lambda^\top (\beta + T_\gamma + \beta^2 + b \beta) + \gamma + T_\gamma + \beta^2 + b \beta) + (\beta + T_\gamma + b I) \gamma \Lambda + (T_\gamma) \gamma \Lambda^2
\]

so that

(41) \[ \Psi = \cdots + H (\gamma + T_\gamma + \beta^2 + b \beta) - \Lambda^\top D H (\beta + T_\gamma + b I) \gamma \Lambda + (\Lambda^\top)^2 \pi^{-2} H (T_\gamma) \gamma \Lambda^2 \\
\times \left( \eta \gamma + \eta \gamma^{-1} D \Lambda + \Lambda^2 \right) \\
\times (\Lambda^\top)^2 + \Lambda^\top (\beta + T_\gamma + \beta^2 + b \beta) + \gamma + T_\gamma + \beta^2 + b \beta + (\beta + T_\gamma + b I) \gamma \Lambda + (T_\gamma) \gamma \Lambda^2
\]

and the result is proven. \qed

Proposition 11 (Compatibility). For a generalized Charlier weight the recursion coefficients fulfill

(42) \[ n \eta (\beta_n - \beta_{n-1} - 1) = (-\gamma_{n+1} + \gamma_{n-1}) \gamma_n, \]

for $n \in \mathbb{N}$ and $\gamma_0 = 0$. Alternative forms of this equation are

(43a) \[ n \delta_n \left( \frac{\eta}{\gamma_n} \right) = \gamma_{n+1} - \gamma_{n-1}, \]

(43b) \[ \delta_n \left( \frac{\gamma_n \gamma_{n+1}}{\eta} \right) = (n+1) \gamma_n - n \gamma_{n+1}, \]

for $n \in \mathbb{N}$ and $\gamma_0 = 0$. We also have

(44) \[ \beta_n - \beta_{n-2} - 1 = \eta (n \gamma_n^{-1} - (n-1) \gamma_{n-1}^{-1}), \]

$n \geq 2$.

Proof. Recalling $\gamma = (T_- H)^{-1}$ we write

\[
\Psi H^{-1} = \eta + \eta H D \Lambda H^{-1} + H (T_- \gamma) \gamma \Lambda^2 H^{-1} = \eta + \eta H (T_- H)^{-1} D \Lambda + H (T_-^2 H)^{-1} (T_- \gamma) \gamma \Lambda^2
\]

\[
= \eta + \eta \gamma^{-1} D \Lambda + \Lambda^2
\]

\[
= \left( \begin{array}{cccc}
\eta & \frac{\eta}{\gamma_1} & 1 & 0 & \ldots \\
0 & \eta & \frac{\eta}{\gamma_2} & 1 & \ldots \\
0 & 0 & \eta & \frac{\eta}{\gamma_3} & \ldots \\
& & & & \ddots
\end{array} \right).
\]
The compatibility equation (29b), \( [\Psi^{-1}, J] = \Psi^{-1} \), reads

\[
\begin{pmatrix}
\frac{n}{\gamma_1} & 1 & 0 & \ldots \\
0 & \frac{2n}{\gamma_2} & 1 & \ldots \\
& & & \ddots \\
\end{pmatrix}
= -
\begin{pmatrix}
\beta_0 & 1 & 0 & \ldots \\
\gamma_1 & \beta_1 & 1 & \ldots \\
0 & \gamma_2 & \beta_2 & \ldots \\
& & & \ddots \\
\end{pmatrix}
\begin{pmatrix}
\frac{n}{\gamma_1} & 1 & 0 & \ldots \\
0 & \frac{2n}{\gamma_2} & 1 & \ldots \\
& & & \ddots \\
\end{pmatrix}

+ \begin{pmatrix}
\frac{n}{\gamma_1} & 1 & 0 & \ldots \\
0 & \frac{2n}{\gamma_2} & 1 & \ldots \\
& & & \ddots \\
\end{pmatrix}
\begin{pmatrix}
\beta_0 & 1 & 0 & \ldots \\
\gamma_1 & \beta_1 & 1 & \ldots \\
0 & \gamma_2 & \beta_2 & \ldots \\
& & & \ddots \\
\end{pmatrix}
\begin{pmatrix}
\frac{n}{\gamma_1} & 1 & 0 & \ldots \\
0 & \frac{2n}{\gamma_2} & 1 & \ldots \\
& & & \ddots \\
\end{pmatrix}
\]

and, consequently, we get (42) on the first superdiagonal and from the second superdiagonal we get (44).

Let us look to the alternative expression (43a). First, using the Toda system (35b) we see that (43a) is equivalent to (42). Indeed, from (35b) we get

\[
\eta \frac{\partial}{\partial \eta} \left( \gamma_n \eta \log \gamma_n + 2 \gamma_n \right) + n \gamma_n \eta = -n \beta_n - \beta_{n-1} - 1,
\]

and the statement follows. An alternative proof for (43a) is obtained from the compatibility condition (38b), i.e.,

\[
\frac{\partial}{\partial \eta} \left( \gamma_n \eta \log \gamma_n + 2 \gamma_n \right) + n \gamma_n \eta = -n \beta_n - \beta_{n-1} - 1,
\]

Equation (38a) with

\[
\Pi = \eta^{-1} \Psi^{-1} H^{-1} =
\]

gives (43b).

\[\square\]

**Theorem 4 (Third order ODE for generalized Charlier).** The recursion coefficient \( \gamma_n \) of the generalized Charlier polynomials is subject to the following third order nonlinear ODE

\[
\frac{\gamma_n (\partial^2 \eta^{-1} \log \gamma_n + 2 \gamma_n) + n^2 \eta}{\gamma_n} = 2 \gamma_n
\]
Proof. In the one hand, Equation (43b) can be written as follows
\[ \pm \vartheta_\eta \left( \frac{\gamma_n \gamma_{n+1}}{\eta} \right) = (n \pm 1) \gamma_n - n \gamma_{n+1}, \]
and, consequently, we find
\[
\vartheta_\eta \left( \frac{\gamma_n \gamma_{n+1} + \gamma_{n-1} \gamma_n}{\eta} \right) = (n + 1) \gamma_n - n \gamma_{n+1} - (n - 1) \gamma_n + n \gamma_{n-1} = 2 \gamma_n - n (\gamma_{n+1} - \gamma_{n-1}) = 2 \gamma_n - n^2 \vartheta_\eta \left( \frac{\eta}{\gamma_n} \right),
\]
where (43a) has been used. In the other hand,
\[
\vartheta_\eta \left( \frac{\gamma_n \gamma_{n+1} + \gamma_{n-1} \gamma_n}{\eta} \right) = \vartheta_\eta \left( \frac{\gamma_n}{\eta} (\gamma_{n+1} + \gamma_{n-1}) \right) = \vartheta_\eta \left( \frac{\gamma_n (\vartheta^2 \log \gamma_n + 2 \gamma_n)}{\eta} \right),
\]
where the Toda equation (36) for the \( \gamma \)'s has been used. Hence, comparing the previous equations we get Equation (45).

\[ \square \]

Remark 5. Using the relations
\[
\vartheta^2 \log \gamma_n = \vartheta_\eta \left( \frac{\eta}{\gamma_n} \right) \frac{d \gamma_n}{d \eta} + \frac{\vartheta^2}{\eta} \frac{d^2 \gamma_n}{d \eta^2}, \quad \vartheta_\eta \left( \frac{\vartheta^2 \gamma_n}{\eta} \right) = \frac{\vartheta^2}{\eta} \frac{d \gamma_n}{d \eta}
\]
we write (45) as follows
\[
\frac{d}{d \eta} \left( \eta P'_{\text{III},n}(\gamma_n) \right) = 2 \frac{\gamma_n}{\eta}, \quad P'_{\text{III},n}(\gamma_n) := \frac{d^2 \gamma_n}{d \eta^2} + \frac{1}{\gamma_n} \left( \frac{d \gamma_n}{d \eta} \right)^2 + \frac{2 \vartheta^2}{\eta} + \frac{n^2}{\gamma_n}.
\]

The notation here is motivated by the Okamoto’s, see [47], alternative form \( P_{\text{III}} \) of the Painlevé III equation \( (P'_{\text{III},n}(\eta) = 0) \), listed as [32.2.9] in the Digital Library of Mathematical Functions (DLMF) at NIST, with the following choice of parameters given there: \( \alpha = 8, \gamma = \beta = 0 \) and \( \delta = 4n^2 \). The Okamoto’s notation in [47] is \( P_{\text{III}} \). In fact, the corresponding Painlevé equation is, after suitable rescaling of dependent and independent variable, the \( P_{\text{III}} \) equation (9) with \( \beta = 0 \) in [46]. See Theorem I (iv) in [46] and the comment immediately after. For the connection with \( P_{\text{III}} \) see [48][49] and [49]. In particular, Remark 4.5 in [49] gives the functions \( p, q \) of the Hamiltonian system \( H_{\text{III}} \), see [47], in terms of \( S_n = -p^4_n = \vartheta_\eta \gamma_n^2 \)

\[
q = \frac{\eta \vartheta^2 S_n - 2(n + b) \frac{d S_n}{d \eta} + 2n}{4 \frac{d S_n}{d \eta} (1 - \frac{d S_n}{d \eta})}, \quad p = \frac{d S_n}{d \eta}
\]
with parameters \( \theta_0 = n + b \) and \( \theta_\infty = n - b \). In terms of \( \gamma_n = \eta \frac{d S_n}{d \eta} \) we have

\[
q = \frac{\eta^2 \vartheta \gamma_n^2 - (2n + 2b + 1) \eta \vartheta \gamma_n + 2n \eta^2}{4 \gamma_n (\vartheta^2 - \gamma_n)}, \quad p = \frac{\gamma_n}{\eta}
\]

From [49] §2.4 we conclude that \( q \) satisfies \( P_{\text{III}} \) with \( \alpha = -4(n - b), \beta = 4(n + b + 1), \gamma = 4 \) and \( \delta = -4 \), in [32.2.9] at DLMF, or Equation (2.4) in [49] with \( A = -2 \theta_\infty = -2(n - b) \) and \( B = 2(\theta_0 + 1) = 2(n + b + 1) \),

\[
\frac{d^2 q}{d \eta^2} = \frac{1}{q} \left( \frac{d q}{d \eta} \right)^2 - \frac{1}{\eta} \frac{d q}{d \eta} - (n - b) \frac{\vartheta^2}{\eta^2} + \frac{n + b + 1}{\eta} + \frac{q^3}{2} - \frac{1}{q},
\]

Notice also [49][47] that \( p^3_n \) is a solution to the \( P_{\text{III}} \) \( \sigma \)-equation. After all these observations, one is lead to conjecture that Equation (45) should have the Painlevé property, and probably is solved in terms of the \( P_{\text{III}} \) transcendent. In terms of \( p = \frac{\gamma_n}{\eta} = -\frac{d p^3_n}{d \eta} \), Equation (45) can be written as follows

\[
\vartheta_\eta \left( \vartheta^2 p^2 - \frac{(\vartheta \eta p)^2}{p} + 2 \eta p^2 + \frac{\eta^2}{p} \right) = 2 \eta p
\]
that expands to
\[ p^2 \partial_y^3 p - 2p(\partial_y p)(\partial_y^2 p) + (\partial_y p)^3 + (2\eta p^3 - n^2) \partial_y p + 2p^3 (p - \eta) = 0. \]

In [19] it is shown that \( \frac{\eta}{p} \) satisfies an instance of Painlevé \( V \).

**Proposition 12** (Laguerre–Freud relations for the generalized Charlier case). For \( n \in \mathbb{N}_0 \), we find that the recursion coefficients satisfy the following Laguerre–Freud relations

\[
\begin{align*}
\beta_{n+1} &= \frac{\gamma n(n+1)}{\gamma_{n+1}} - \beta_n + n - b, \\
\gamma_{n+1} &= \eta - \gamma_n - \beta_n^2 - b\beta_n + \frac{\gamma_n - 1}{\gamma} + \frac{\eta n^2}{\gamma_n}, \quad \gamma_{-1} = \gamma_0 = 0.
\end{align*}
\]

For \( n \in \mathbb{N} \), the following expression for the coefficient of the subleading term

\[ p_n^1 = \frac{n(n+1)}{2} - n\beta_n - \frac{\gamma_{n+1}}{\gamma} \]

holds, as a function of near neighbors recursion coefficients.

**Proof.** From (40) and (41), using (11), we get two different expressions for the first superdiagonal involving only the recursion coefficients that we must equate, i.e.

\[ \eta HD = H(\beta + T_- \beta + bI) \gamma - T_+(DH(T_- \gamma) \gamma) \]

so that

\[ \eta D \gamma^{-1} = \beta + T_- \beta + bI - (T_+ D)(T_+ T_-^2 H)(T_- H)^{-1} = \beta + T_- \beta + bI - T_+ D, \]

where we used \( \gamma = (T_- H)H^{-1} \), that component wise is (40a). Alternatively, notice that Equation (44), after summing and dealing with a telescopic series on the RHS, gives (46a).

From the main diagonal and the second superdiagonal, again using (11), we get the following two expressions,

\[
\begin{align*}
\eta H &= H(\gamma + T_+ \gamma + \beta^2 + b\beta) - T_+(DH(\beta + T_- \beta + bI) \gamma) + T^2_+(\pi^{-2} H(T_- \gamma) \gamma), \\
\eta H \pi^{[2]} &= H(T_- \gamma) \gamma.
\end{align*}
\]

From (51) and (13) we obtain (47). Again, using \( \gamma = (T_- H)H^{-1} \) we get

\[
\eta = \gamma + T_+ \gamma + \beta^2 + b\beta - (T_+ D)(T_+ H)H^{-1} T_+(\beta + T_- \beta + bI)(T_+ T_- H)T_+(H^{-1}) + T^2_+(\pi^{-2} T_-^2 H)H^{-1}
\]

\[
= \gamma + T_+ \gamma + \beta^2 + b\beta - (T_+ D)(T_+(\eta D \gamma^{-1} + T_+ D) + T^2_+(\pi^{-2}))
\]

\[
= \gamma + T_+ \gamma + \beta^2 + b\beta - (T_+ D)(T^2_+ D - \eta(T_+ D)^2 T_+(\gamma^{-1}) + T^2_+ \pi^{-2}.
\]

Noticing that \( 2D[2] = DT_- D \), we see that (15) implies \( \pi^{-2} = D(T_- D) - \pi^{[2]} \), and (51) gives \( \pi^{-2} = D(T_- D) - \eta^{-1}(T_-^2 H)H^{-1} \). Hence, \( T^2_+ \pi^{-2} = (T^2_+ D)(T_+ D) - \eta^{-1} HT_+(H^{-1}) \) and we obtain

\[
\eta = \gamma + T_+ \gamma + \beta^2 + b\beta - (T_+ D)(T^2_+ D - \eta(T_+ D)^2 T_+(\gamma^{-1}) + (T^2_+ D)(T_+ D) - \eta^{-1} HT_+(H^{-1})
\]

\[
= \gamma + T_+ \gamma + \beta^2 + b\beta - \eta(T_+ D)^2 T_+(\gamma^{-1}) - \eta^{-1} HT^2_+(H^{-1}).
\]

We finally find

\[
\gamma + T_+ \gamma + \beta^2 + b\beta - \eta = \eta(T_+ D)^2 T_+(\gamma^{-1}) + \eta^{-1} HT^2_+(H^{-1}),
\]

that component wise gives (46b).
and we finally get a relation involving only $\gamma$

Notice that we can write (46a) as follows

The recursion coefficients

Lemma 1.

previously found in [26],

Corollary 1.

we can go backwards in the chain of equalities leading to (55) and get the stated result.

□

that is (52). The inverse statement is easily proven to hold. If we assume (52), (46a) and (42) to be true,

(55)

Hence, for $n \geq 1$ and taking $\gamma_{-1} = \gamma_0 = 0$ we can write (46b) as follows

\[
\gamma_{n+1} = \eta - \gamma_n - \frac{\beta_n^2}{\eta} - b\beta_n + \gamma_n \left( \frac{n\eta(\beta_n - \beta_{n-1} - 1)}{\gamma_n} + \frac{\eta^2}{\gamma_n} \right)
\]

\[
= \eta + \frac{\gamma_n \gamma_{n+1}}{\eta} - \gamma_n + \frac{\eta^2}{\gamma_n} - \beta_n^2 - b\beta_n + n(\beta_n - \beta_{n-1} - 1)
\]

\[
= \eta + \frac{\gamma_n \gamma_{n+1}}{\eta} - \gamma_n + \frac{\eta^2}{\gamma_n} - \beta_n^2 - b\beta_n + n(2\beta_n - \frac{\eta}{\gamma_n} - n + b)
\]

\[
= \eta + \frac{\gamma_n \gamma_{n+1}}{\eta} - \gamma_n - \beta_n^2 - b\beta_n + 2n\beta_n - n^2 + bn.
\]

that is (52). The inverse statement is easily proven to hold. If we assume (52), (46a) and (42) to be true, we can go backwards in the chain of equalities leading to (55) and get the stated result.

□

Corollary 1. The $\beta$’s are subject to the nonlinear recursion

\[
\frac{\eta(n+1)}{\beta_{n+1} + \beta_n - n + b} = \eta + \frac{n - 1}{\beta_{n-1} + \beta_n - n + 2 + b} - 1 \frac{\eta n}{\beta_n + \beta_{n-1} - n + 1 + b - \beta_n^2 + b\beta_n + n(\beta_n + \beta_{n-1} - n + 1 + b)}.
\]

Proof. Notice that we can write (46a) as follows

\[
\gamma_{n+1} = \frac{\eta(n+1)}{\beta_{n+1} + \beta_n - n + b},
\]

that we can introduce in (46b) to get (56).

□

We now seek for a second order ODE for $\gamma_n$. From the above results it easily follows that (this was previously found in [26]),

Lemma 1. The recursion coefficients $\beta_n$ and $\gamma_n$ satisfy the following system of first order nonlinear ODEs

\[
\partial_\eta \beta_n = \eta + (b - n)n + (2n - b - \beta_n)\beta_n - \gamma_n - \gamma_n,
\]

(58a)

\[
\partial_\eta \gamma_n = (b - n + 1 + 2\beta_n)\gamma_n - n\eta.
\]

(58b)
Proof. Equation (55), after some cleaning, reads
\[
\gamma_{n+1} = \frac{\eta + (b - n)n + (2n - b - \beta_n)\beta_n - \gamma_n}{\eta - \gamma_n},
\]
which is \([26\text{ Equation (16)}\]. In the one hand, from (59) and the Toda equation (35a) we find (58a). That is Equation (18) in \([26\]. On the other hand, from (46a) and the Toda equation (35b) we get (58b). □

Remark 7. Differential system (58) was used in \([26\] to get a second order nonlinear ODE for \(\beta_n\) and then showed \([26\text{ Theorem 2.1}\] that an auxiliary function \(y\), see \([26\text{ Equation (20)}\], satisfies an instance of the PV related to the PIII. Notice that \(y\) is a solution to a Ricatti equation in where \(\beta_n\) appears in the coefficients.

Theorem 5 (Second order ODE for generalized Charlier). The recursion coefficient \(\gamma_n\) satisfies the second order nonlinear ODE
\[
(1 - \frac{\gamma_n}{\eta}) \left( \partial_\eta \left( \frac{\partial_\eta \gamma_n}{\gamma_n} + \frac{\eta \eta}{\gamma_n} \right) + 2\gamma_n \right) + 2(\gamma_n - \eta + (n - b)n) = -\frac{1}{2} \left( \frac{\partial_\eta \gamma_n}{\gamma_n} + \frac{\eta \eta}{\gamma_n} \right)^2 + (n + 1) \left( \frac{\partial_\eta \gamma_n}{\gamma_n} + \frac{\eta \eta}{\gamma_n} \right) + (b - n - 1)(-b + 3n + 1).
\]

Proof. Observe that Equation (58b) leads to
\[
\beta_n = \frac{1}{2} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} - b + n - 1 \right),
\]
so that
\[
\partial_\eta \beta_n = \frac{1}{2} \partial_\eta \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} \right).
\]

From (58a) and (62) we get
\[
(1 - \frac{\gamma_n}{\eta}) \left( \frac{1}{2} \partial_\eta \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} \right) + \gamma_n \right) + \gamma_n - \eta + (n - b)n = (2n - b)\beta_n - \beta_n^2,
\]
and replacing \(\beta_n\) by the expression provided in (61) we get (60). To check it let us elaborate on the RHS of the equation
\[
(2n - b)\beta_n - \beta_n^2 = \frac{2n - b}{2} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} - b + n - 1 \right) - \frac{1}{4} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} - b + n - 1 \right)^2
\]
\[
= \frac{2n - b}{2} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} - b + n - 1 \right) - \frac{1}{4} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} \right)^2
\]
\[
- \frac{(b - n + 1)^2}{4} + \frac{b - n + 1}{2} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} \right)
\]
\[
= -\frac{1}{4} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} \right)^2 + \frac{n + 1}{2} \left( \partial_\eta \log \gamma_n + \frac{\eta \eta}{\gamma_n} \right) + \frac{(b - n - 1)(-b + 3n + 1)}{2},
\]
and the statement is proven. □

Some additional properties of this generalized Charlier case follow.

Proposition 14. For the generalized Charlier case, \(\sigma = \eta\) and \(\theta = z(z + b)\), the following holds:
i) The dressed Pascal matrix is

\[
\Pi = I + \Lambda^\top D + \eta^{-1}(\Lambda^\top)^2(T_{-}\gamma)\gamma = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
1 & 1 & 0 & \\
\frac{\gamma_{1,2}}{\eta} & 2 & 1 & \\
0 & \frac{\gamma_{2,1}}{\eta} & 3 & \\
& & & \ddots
\end{pmatrix}.
\]

Moreover, the Jacobi and dressed Pascal matrices are linked by

\[
J^2 + bJ = \eta\Pi H\Pi^\top H^{-1}.
\]

ii) The corresponding orthogonal polynomials satisfy

\[
\eta = \frac{P_{n+1}(0)P_n(-b) - P_{n+1}(-b)P_n(0)}{P_{n+2}(0)P_{n+1}(-b) - P_{n+2}(-b)P_{n+1}(0)}, \quad \frac{n + 1}{\gamma_{n+1}} = \frac{P_{n+2}(0)P_n(-b) - P_{n+2}(-b)P_n(0)}{P_{n+2}(0)P_{n+1}(-b) - P_{n+2}(-b)P_{n+1}(0)}.
\]

**Proof.** ii) We apply the ideas that leads to Christoffel formula for a Christoffel perturbation. From (28) we get \( \theta(z)P(z-1) = \Psi H^{-1}P(z) = \eta H\Pi^\top H^{-1}P(z) \), where the last equation holds due the generalized Charlier restrictions to the weight. Therefore,

\[
\theta(z)H^{-1}P(z-1) = \Pi^\top H^{-1}P(z).
\]

As \( \theta(0) = \theta(-b) = 0 \) we have

\[
\Pi^\top H^{-1}P(0) = 0, \quad \Pi^\top H^{-1}P(-b) = 0.
\]

Both equations can be simplified to

\[
(\eta^{-1}\gamma_{n+2}\gamma_{n+1}, n + 1) = -(H_{n+1}^{-1}P_n(0), H_{n+1}^{-1}P_n(-b)) \left( H_{n+1}^{-1}P_{n+2}(0), H_{n+1}^{-1}P_{n+2}(-b) \right)^{-1},
\]

from where the result follows.

\[\square\]

3. Generalized Meixner weights

Another interesting case appears when one takes \( \sigma = z + a \), that we call extended Meixner type. The generalized Meixner, which is the main example of these extended Meixner weights, corresponds to the choice \( \sigma(z) = \eta(z + a) \) and \( \theta(z) = z(z + b) \). A weight that satisfies the Pearson equation

\[
(k + 1)(k + 1 + b)w(k + 1) = \eta(k + a)w(k)
\]

is proportional to the generalized Meixner weight

\[
w(z) = \frac{\Gamma(b + 1)\Gamma(z + a)\eta^z}{\Gamma(a)\Gamma(z + b + 1)\Gamma(z + 1)} = \frac{(a)_z}{(b + 1)_z} \eta^z.
\]

The finiteness of the moments requires \( a(b + 1) > 0 \) and \( \eta > 0 \). In this case the moments are expressed in terms of \( \rho_0 = {}_1F_1(a; b + 1; \eta) = M(a, b + 1, \eta) \), also known as the confluent hypergeometric function or Kummer function.

**Remark 8.** Meixner introduced the non generalized version with \( w = (a)_z \frac{\eta^z}{z!} \) in [43].
Remark 9. For an extended Meixner type weight, as $M = \deg \sigma = 1$ we have that the Laguerre–Freud matrix 
$\Psi = \Pi^{-1} H \theta (J^\top) = \eta (J + a) H \Pi^T$ is a banded matrix with only one subdiagonal and $N$ superdiagonals. Now we
do not have, as in the extended Charlier case that $\Psi = H \Pi^T$, which implied that the dressed Pascal matrix $\Pi$ had
only three nonzero subdiagonals. In the extended Meixner case the dressed Pascal matrix will possibly have an infinite
number of nonzero subdiagonals.

Theorem 6 (The generalized Meixner Laguerre–Freud structure matrix). For a generalized Meixner weight;
i.e. $\sigma = \eta (z + a)$ and $\theta = z (z + b)$, the structure matrix is

\[ \Psi = \begin{pmatrix} \eta(\beta_0 + a) H_0 & (\beta_0 + \beta_1 + b) H_1 & H_2 & 0 & \cdots \\ \eta H_1 & \eta(\beta_1 + a + 1) H_1 & (\beta_1 + \beta_2 + b - 1) H_2 & H_3 & \cdots \\ 0 & \eta H_2 & \eta(\beta_2 + a + 2) H_2 & (\beta_2 + \beta_3 + b - 2) H_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \]

Proof. The structure matrix has a diagonal structure, see (27), $\Psi = \Lambda^T \psi(-1) + \psi(0) + \psi(1) \Lambda + \psi(2) \Lambda^2$. As
$\Psi = \eta (J + a I) H \Pi^T$ we find

\[ \Psi = \eta(\Lambda^T \gamma + \beta + a I + \Lambda) H (I + D \Lambda + \pi[2] \Lambda^2 + \pi[3] \Lambda^3 + \cdots) \]

\[ = \frac{\eta \Lambda^T \gamma H}{\text{first subdiagonal}} + \eta(\Lambda^T \gamma H D \Lambda + (\beta + a I) H) + \eta(\Lambda^T \gamma H \pi[2] \Lambda^2 + (\beta + a I) H D \Lambda + \Lambda H) \]

\[ + \eta(\Lambda^T \gamma H \pi[3] \Lambda^3 + (\beta + a I) H \pi[2] \Lambda^2 + \Lambda H D \Lambda) + \cdots \]

\[ \text{second superdiagonal} \]

we get $\psi^{(0)} = \eta T_+ (\gamma H D) + \eta(\beta + a I) H$. Now, as

\[ J^2 + b J = (\Lambda^T)^2 (T_- \gamma) + \Lambda^T (\beta + T_- \beta + b I) \gamma + \gamma + T_+ \gamma + \beta^2 + b \beta + (\beta + T_- \beta + b I) \Lambda + \Lambda^2, \]

from the alternative expression $\Psi = \Pi^{-1} H ((J^\top)^2 + b J^\top)$, we find

\[ \Psi = (I - \Lambda^T D + (\Lambda^T)^2 \pi[-2] - (\Lambda^T)^3 \pi[-3] + \cdots) H \]

\[ = \frac{H (T_- \gamma) \gamma \Lambda^2}{\text{second superdiagonal}} - \Lambda^T D H (T_- \gamma) \gamma \Lambda^2 + H (\beta + T_- \beta + b I) \gamma \Lambda \]

\[ + \Lambda^T D H (\beta + T_- \beta + b I) \gamma \Lambda + (\Lambda^T)^2 \pi[-2] H (T_- \gamma) \gamma \Lambda^2 \]

\[ \text{first superdiagonal} \]

\[ + \Lambda^T D H (\gamma + T_+ \gamma + \beta^2 + b \beta) - \Lambda^T D H (\beta + T_- \beta + b I) \gamma \Lambda + (\Lambda^T)^2 \pi[-2] H (T_- \gamma) \gamma \Lambda^2 \]

\[ \text{main diagonal} \]

\[ + H \Lambda^T (\beta + T_- \beta + b I) - \Lambda^T D H (\gamma + T_+ \gamma + \beta^2 + b \beta) + (\Lambda^T)^2 \pi[-2] H (\beta + T_- \beta + b I) \gamma \Lambda - (\Lambda^T)^3 \pi[-3] H (T_- \gamma) \gamma \Lambda^2. \]

Thus, $\psi^{(1)} = -T_+ (D H (T_- \gamma) H) + H (\beta + T_- \beta + b I) \gamma$. Finally, the Laguerre–Freud matrix has the form

\[ \Psi = \eta \Lambda^T T_- H + \eta(\beta + a I + T_+ D) H + (\beta + T_- \beta + b I - T_+ D) T_- H \Lambda + T_-^2 H \Lambda^2. \]

When expressed component wise we get the given form in (65).
Proposition 15 (Compatibility). For the generalized Meixner case the recursion coefficients fulfill
\[ (68a) \quad (\beta_n + \beta_{n+1} + b - n - \eta)\gamma_{n+1} - (\beta_{n-1} + \beta_n + b - n + 1 - \eta)\gamma_n = \eta(\beta_n + a + n), \]
\[ (68b) \quad \gamma_{n+2} - \gamma_n - 2\eta + (\beta_n + \beta_{n+1} + b - n - \eta)(\beta_{n+1} - \beta_n - 1) = 0. \]

Proof. The matrix
\[
\Psi H^{-1} = \begin{pmatrix}
\eta(\beta_0 + a) & \beta_0 + \beta_1 + b & 1 & 0 & \cdots \\
\eta \gamma_1 & \eta(\beta_1 + a + 1) & \beta_1 + \beta_2 + b - 1 & 1 & \cdots \\
0 & \eta \gamma_2 & \eta(\beta_2 + a + 2) & \beta_2 + \beta_3 + b - 2 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
satisfies the compatibility condition (29a). As we have
\[
[\Psi H^{-1}, J] = \begin{pmatrix}
(\beta_0 + \beta_1 + b - \eta)\gamma_1 & (\beta_0 + 1 - \eta)\gamma_2 & 1 & 0 & \cdots \\
\eta \gamma_1 & (\beta_1 + \beta_2 + b - 1 - \eta)\gamma_2 & (\beta_2 + \beta_3 + b - 2 - \eta)\gamma_3 & 1 & \cdots \\
0 & \eta \gamma_2 & (\beta_2 + \beta_3 + b - 1 - \eta)\gamma_3 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
we get for the first row
\[
\gamma_1 = \frac{\eta(\beta_0 + a)}{\beta_0 + \beta_1 + b - \eta}, \quad \gamma_2 = \beta_0 + \beta_1 + b - (\beta_0 + \beta_1 + b - \eta)(\beta_1 - \beta_0) + \eta,
\]
while for the next rows we obtain
\[
(\beta_n + \beta_{n+1} + b - n - \eta)\gamma_{n+1} - (\beta_{n-1} + \beta_n + b - n + 1 - \eta)\gamma_n = \eta(\beta_n + a + n),
\]
\[
\gamma_{n+2} - \gamma_n - \eta + (\beta_n + \beta_{n+1} + b - n - \eta)(\beta_{n+1} - \beta_n) = \beta_n + \beta_{n+1} + b - n.
\]

Which are (68a) and (68b) (in disguise the last one, some clearing is required).

An alternative manner to get (68a) follows. The matrix \(\Psi H^{-1}\) fulfills (38b), where \(\Phi = -J_-\), and we have \(\vartheta_\eta(\Psi H^{-1}) = [\Psi H^{-1}, J_-]\). The explicit expression of the above commutator is,
\[
[\Psi H^{-1}, J_-] = \begin{pmatrix}
(\beta_0 + \beta_1 + b)\gamma_1 & (\beta_1 + \beta_2 + 1 - \eta)\gamma_2 & 0 & 0 & \cdots \\
\eta \gamma_1 & (\beta_1 + \beta_2 + b - 1)\gamma_2 & \beta_2 + \beta_3 + b - 2 & 1 & \cdots \\
0 & \eta \gamma_2 & (\beta_2 + \beta_3 + b - 1)\gamma_2 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
and, consequently, we find the following equations
\[ (69) \quad \vartheta_\eta(\eta \gamma_n) = \eta \gamma_n(\beta_n - \beta_{n-1} + 1), \]
\[ (70) \quad \vartheta_\eta(\beta_n + \beta_{n+1} + b - n) = \gamma_{n+2} - \gamma_n, \]
\[ (71) \quad \vartheta_\eta(\eta(\beta_n + a + n)) = \gamma_{n+1}(\beta_n + \beta_{n+1} + b - n) - \gamma_n(\beta_{n-1} + \beta_n + b - (n - 1)). \]

Notice that (69) and (70) follow from the Toda equations for the recursion coefficients (33). The Toda equation (33a) allows us to write (71) as follows
\[
\eta(\beta_n + a + n) + \eta \gamma_{n+1} - \eta \gamma_n = \gamma_{n+1}(\beta_n + \beta_{n+1} + b - n) - \gamma_n(\beta_{n-1} + \beta_n + b - (n - 1)),
\]
and we get (68a). □
Proposition 16 (Laguerre–Freud relations). For \( n \in \mathbb{N} \), we find the following Laguerre–Freud equations

\[
\gamma_{n+1} = -\gamma_n - (\beta_n + b - n)\beta_n + n(b - a - n + 1) + \eta(\beta_n + a + n) + \eta^{-1}(\beta_{n-1} + \beta_n + b - n + 1 - \eta)\gamma_n,
\]

\[
0 = (n + 3)(n + 1)(\beta_n - \beta_{n+2} + 1) - (\eta^{-1}\gamma_{n+3} - n - 3)\gamma_{n+2}
\]

Moreover, from \( \beta \) we get

\[
\pi_n = -(\beta_{n+1} + \beta_{n+2} + b - n - 1 - \eta)\gamma_{n+2} + \beta_{n+1} + a(n + 2) + \frac{(n + 2)(n + 1)}{2}.
\]

Proof. Comparing (66) with (67) and recalling (11) we obtain

\[
\eta(T_H - H) = (\beta + T_+\beta + bI)T_H - DH(\gamma + T_+\gamma + \beta^2 + b\beta) + T_+(\pi^{-2}[\beta + T_+\beta + bI]H\gamma) + T_2^2(\pi^{-3}][H(T_\gamma)\gamma),
\]

\[
\eta(T_+(\gamma HD) + (\beta + aI)H) = H(\gamma + T_+\gamma + \beta^2 + b\beta) - T_+(DH(\beta + T_+\beta + bI)\gamma) + T_2^2(\pi^{-2}[H(T_\gamma)\gamma),
\]

Comparing (66) with (67) and recalling (11) we obtain

Let us study relations (76) and (77). In the one hand, if we choose to solve (77) for \( \pi^2 \), we get (recall that  

\[
\pi^2 = -H^{-1}\gamma^{-1}T_-(\beta + aI)HD + T_-H) - \eta^{-1}DT_+_\gamma + \gamma^{-1}H^{-1}\gamma^{-1}T_-(H\gamma(\beta + T_+\beta + bI))
\]

so that

\[
\pi^2_n = -(\beta_{n+1} + \beta_{n+2} + b - n - 1 - \eta)\gamma_{n+2} - (n + 2)(\beta_{n+1} + a).
\]

Moreover, from \( \beta = T_+S[1] - S[1] \) and \( \pi^2 = D[2] + (T_-S[1] - S[1])D - S[1] \) we get

\[
S[1] = -\pi^2 + D[2] - DT_\beta
\]

that, component wise, is (74). Now, recalling (15), we write (76) as follows

\[
\eta(\beta + aI + T_+D) = \gamma + T_+\gamma + \beta^2 + b\beta - T_+(D(\beta + T_+\beta + bI)) + T_2^2(-\pi^2 + 2D[2]),
\]

that is, using (12),

\[
\eta(T_2\beta + aI + T_-D) = T_2\gamma + T_-\gamma + T^2\beta^2 + bT^2\beta - (T_-D(T_\beta + T^2\beta + bI - D)) - \pi^2
\]

\[
= T_2\gamma + T_-\gamma + T^2\beta^2 + bT^2\beta - (T_-D(T_\beta + T^2\beta + bI - D) - \eta^{-1}(T_\beta + T^2\beta + bI - D - \eta)T_-\gamma + (T_-D)(T_\beta + aI)
\]
Hence, component wise we get (73).

Notice that the long and cumbersome expression in

Thus, we get the following equation

that component wise reads

\[ \eta(\beta_{n+2} + a + n + 2) = \gamma_{n+3} + \gamma_{n+2} + \beta_{n+2}^2 + b\beta_{n+2} - (n + 2)(\beta_{n+2} + b - a - n - 1) \]

\[ - \eta^{-1}(\beta_{n+1} + \beta_{n+2} + b - n - 1 - \eta)\gamma_{n+2}, \]

which is (72).

Now, (78) can be written as follows

(82)

\[ \pi^{[3]} = -(T_\gamma + aI)(T_\gamma \pi^{[2]} + (\eta^{-1}T_\gamma - T_\gamma D)T_\gamma \]

\[ = -(T_\gamma + aI)(\eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)) + (\eta^{-1}T_\gamma \gamma - T_\gamma D)T_\gamma \gamma, \]

where (79) has been used. From (15), \[\pi^{[3]} + \pi^{[-3]} = 2((T_\gamma S^{[1]})(D^{[2]} - (T_\gamma D^{[2]}))S^{[1]}),\] and (80) (noticing that \[3D^{[3]} = 2D^{[2]}(T_\gamma D^{[2]} - T_\gamma D^{[2]})) \]

we get

\[ \pi^{[-3]} = 2D^{[2]}(T_\gamma \pi^{[2]} + (\eta^{-1}T_\gamma - T_\gamma D)T_\gamma \gamma \]

\[ = 3D^{[3]} + 2(T_\gamma D^{[2]})(\pi^{[2]} + DT_\gamma) - 2D^{[2]}(T_\gamma \pi^{[2]} + (T_\gamma D)(T_\gamma \beta + aI)) \]

\[ = 3D^{[3]}T_\gamma(\beta - T_\gamma \beta + I) + 2(T_\gamma D^{[2]})(\pi^{[2]} + T_\gamma D^{[2]} - \pi^{[2]} - 3D^{[3]}T_\gamma(\beta - T_\gamma \beta + I) \]

\[ - 2D^{[2]}(\eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)) \]

\[ + (T_\gamma + aI)(\eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)). \]

We now write (75) as follows

\[ \pi^{[-3]} = -(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma + (T_\gamma D)(T_\gamma \gamma + T_\gamma \beta + bT_\gamma \beta - (T_\gamma + T_\gamma \beta + bI)T_\gamma(2D^{[2]} - \pi^{[2]})) \]

\[ = -(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma + (T_\gamma D)(T_\gamma \gamma + T_\gamma \beta + bT_\gamma \beta) \]

\[ -(T_\gamma + T_\gamma \beta + bI)T_\gamma(2D^{[2]} - \eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)). \]

Thus, we get the following equation

\[ 0 = 3D^{[3]}(T_\gamma \beta - T_\gamma \beta + I) - (\eta^{-1}T_\gamma \gamma - T_\gamma D)T_\gamma \gamma \]

\[ + 2(T_\gamma D^{[2]})(\eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)) \]

\[ - 2D^{[2]}(\eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)) \]

\[ + (T_\gamma + aI)(\eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - T_\gamma D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)) \]

\[ + (T_\gamma + T_\gamma \beta + bI)T_\gamma(2D^{[2]} - \eta^{-1}(T_\gamma \beta + T_\gamma \beta + bI - D - \eta)T_\gamma \gamma - (T_\gamma D)(T_\gamma \beta + aI)). \]

Hence, component wise we get (73).

\[ \square \]

**Remark 10.** Notice that the long and cumbersome expression in (73) is a ligature for the recursion coefficients

\[(\beta_{n+4}, \beta_{n+2}, \beta_{n+1}, \beta_n, \gamma_{n+4}, \gamma_{n+3}, \gamma_{n+2}).\]
We just wanted to show that the method gives these equations. Indeed, (72) is a better Laguerre–Freud equation for the β’s. However, notice that (72) implies an expression of β_{n+1} in terms \((β_{n-1}, β_n, γ_{n+1}, γ_n)\) while (68a) involves and expression for β_n in terms of \((β_n, β_{n-1}, γ_n, γ_{n+1})\). In both cases we need to go two steps down, and we say that we have length two relations. However, we can mix both Laguerre–Freud equations to find an improved expression for β_{n+1} involving only one step down, i.e. \((β_n, γ_n, γ_{n+1})\), a length one relation. A similar reasoning holds for (68b).

**Proposition 17.** The generalized Meixner recursion coefficients are subject to the following length one Laguerre–Freud equation

\[
β_{n+1} = \frac{1}{γ_{n+1}} \left( η(γ_n + (β_n + b - n)β_n - n(b - a - n + 1)) + η(η + 1)(β_n + a + n) \right) - β_n - b + n + 1 + 2η.
\]

**Proof.** We write (72) as follows

\[(β_{n-1} + β_n + b - n + 1 - η)γ_n = η(γ_{n+1} + γ_n + (β_n + b - n)β_n - n(b - a - n + 1)) + η^2(β_n + a + n)\]

that, when introduced in (68a), delivers

\[(β_n + β_{n+1} + b - n - η)γ_{n+1} = (β_{n-1} + β_n + b - n + 1 - η)γ_n + η(β_n + a + n)\]

\[= η(γ_{n+1} + γ_n + (β_n + b - n)β_n - n(b - a - n + 1)) + η(η + 1)(β_n + a + n),\]

and we obtain the Laguerre–Freud equation (83).

Let us shift by \(n → n + 1\) the relation (72)

\[
γ_{n+2} = -γ_{n+1} - (β_{n+1} + b - n - 1)β_{n+1} + n(b - a - n) + η(β_{n+1} + a + n + 1) + η^{-1}(β_n + β_{n+1} + b - n - η)γ_{n+1},
\]

and introduce the expression for γ_{n+2} into (68b). In doing so we obtain

\[
γ_n + 2η - (β_n + β_{n+1} + b - n - η)(β_{n+1} - β_n - 1) = -(β_{n+1} + b - n - 1)β_{n+1} + n(b - a - n) + η(β_{n+1} + a + n + 1) + η^{-1}(β_n + β_{n+1} + b - n - 2η)γ_{n+1},
\]

but (83) can be written

\[
γ_n + (β_n + β_{n+1} + b - n - 2η) = η(γ_n + (β_n + b - n)β_n - n(b - a - n + 1)) + η(η + 1)(β_n + a + n) + γ_{n+1},
\]

so that

\[
γ_n + 2η - (β_n + β_{n+1} + b - n - η)(β_{n+1} - β_n - 1) = -(β_{n+1} + b - n - 1)β_{n+1} + n(b - a - n) + η(β_{n+1} + a + n + 1) + η^{-1}(β_n + β_{n+1} + b - n - 1 + η)γ_{n+1}.
\]

Our best Laguerre–Freud equations are (83) and (72), having lengths one and two, respectively. For the reader convenience we reproduce them again as a Theorem

**Theorem 7** (Laguerre–Freud equations for generalized Meixner). The generalized Meixner recursion coefficients satisfy the following Laguerre–Freud relations

\[
β_{n+1} = \frac{1}{γ_{n+1}} \left( η(γ_n + (β_n + b - n)β_n - n(b - a - n + 1)) + η(η + 1)(β_n + a + n) \right) - β_n - b + n + 1 + 2η,
\]

\[
γ_{n+1} = -γ_n - (β_n + b - n)β_n + n(b - a - n + 1) + η(β_n + a + n) + η^{-1}(β_{n-1} + β_n + b - n + 1 - η)γ_n.
\]
Remark 11. In [48] a system of Laguerre–Freud equations was presented. There are two functions \( u_n, v_n \) that are linked with \( \beta_n, \gamma_n \) by
\[
\gamma_n = \eta \gamma_n - (a - 1) u_n, \quad \beta_n = n + a - b - 1 + \eta - \frac{(a - 1) v_n}{\eta},
\]
and the nonlinear system, see equations (3.2) and (3.3) in [48], is
\[
(u_n + v_n)(u_{n+1} + v_n) = \frac{a - 1}{\eta^2} v_n(v_n - \eta)(v_n - \eta - \frac{a - b - 1}{a - 1}),
\]
\[
(u_n + v_n)(u_n + v_{n-1}) = \frac{u_n}{u_n - \frac{m}{a - 1}}(u_n + \eta)(u_n + \eta - \frac{a - b - 1}{a - 1}).
\]
Observe that the first relation, (3.2) in [48], gives \( \gamma_{n+1} \) as a rational function of \((\gamma_n, \beta_n)\), a length one relation. The rational function is a cubic polynomial divided by a linear function on the recursion coefficients. Instead, our (72) is a length two relation, but is quadratic polynomial in the recursion coefficients, not rational and involving cubic polynomials as does (3.1). The second relation (3.3) in [48] gives \( \beta_{n+1} \) in terms of rational function of \((\gamma_{n+1}, \beta_n)\) a length one relation. Now the rational function is a cubic polynomial divided by a quadratic one. Relation (83) is length one as well, but our rational function is quadratic polynomial divided by a linear one.

Remark 12. A nice feature of the Smet–Van Assche system is that for the particular value \( a = 1 \) provides the explicit expression \( \beta_n = n - b + \eta \) and \( \gamma_n = \eta n \). Indeed, one we check using SageMath 9.0 that (68a), (68b), (72) and (73) are satisfied. For \( a = 1 \), the weight is \( w(z) = \frac{\eta}{(b + 1)z} \) and the 0-th moment is given by the Kummer function \( \rho_0 = M(1, b + 1, \eta) = \frac{b}{\eta} e^{\eta} (\Gamma(b) - \Gamma(b, \eta)) \), where \( \Gamma(b, \eta) = \int_{\eta}^{\infty} t^{b-1} e^{-t} \, dt \) is the incomplete Gamma function. According to [27], it corresponds to the Charlier case on the shifted lattice \( \mathbb{N} - b \).

4. Generalized Hahn of type I weights

The choice \( \sigma(z) = \eta(z + a)(z + b) \) and \( \theta(z) = z(z + c) \) leads to the following Pearson equation
\[
(k + 1)(k + 1 + c)w(k + 1) = \eta(k + a)(k + b)w(k)
\]
whose solutions are proportional to \( w(z) = \frac{(a z + b z + \eta)}{(c + 1)z} \). According to [21] this is the generalized Hahn weight of type I. The first moment is \( \rho_0 = _2F_1 \left[ \frac{a, b}{c + 1} ; \eta \right] \), i.e., the Gauss hypergeometric function. For \( \eta = 1 \), Hahn introduced these discrete orthogonal polynomials in [30]. The standard Hahn polynomials considered in the literature take \( a = \alpha + 1, b = -N \) and \( c = -N - 1 - \beta \), with \( N \) a positive integer.

Theorem 8 (The generalized Hahn Laguerre–Freud structure matrix). For a generalized Hahn of type I weight; i.e. \( \sigma = \eta(z + a)(z + b) \) and \( \theta = z(z + c) \), we find
\[
p_{n+1} = (n + 2) \beta_{n+1} + \beta_{n+1} + \left( \frac{\eta - 1}{\eta + 1} \right) \gamma_{n+3} + \gamma_{n+2} + \beta_{n+2}^2 + \frac{(n + 2)(n + 1)}{2}
\]
\[
+ \frac{1}{\eta + 1} \gamma_{n+3} + \gamma_{n+2} + \beta_{n+2}^2 + \eta (a + b + c)(n + 2),
\]
\[
\pi_n^{[2]} = \frac{1}{1 + \eta} \left( (n + 2)(n + 1) - \eta ab - \eta (a + b + c) \beta_{n+2} + (\eta (a + b + c)(n + 2))
\]
\[
+ \frac{1 - \eta}{1 + \eta} \gamma_{n+3} + \gamma_{n+2} + \beta_{n+2}^2 - (n + 2)(\beta_{n+2} + \beta_{n+1}).
\]
The Laguerre–Freud structure matrix is

\[
\Psi = \begin{bmatrix}
\eta(\gamma + \beta + a)(\beta + b)H_0 & \eta(\beta + c)H_1 & \cdots & \eta(\beta + c - 1)H_2 \\
\eta(\beta + b + a + b)H_1 & \eta(\gamma + \beta + a)(\beta + c) + \frac{1}{\gamma} & \cdots & \eta(\beta + c - 1)H_2 \\
\eta(\beta + c)H_2 & \eta(2\beta + a + b + c)H_1 & \cdots & \eta(\beta + c - 1)H_2 \\
\eta(\beta + c)H_3 & \eta(\beta + c - 1)H_2 & \cdots & \eta(\beta + c - 1)H_2 \\
\end{bmatrix}
\]

**Proof.** In this case, since the polynomials \(\sigma \gamma \theta\), the Freud–Laguerre matrix has the following diagonal structure

\[
\Psi = (\Lambda^\top)^2 \psi(-2) + \Lambda^\top \psi(-1) + \psi(0) + \psi(1) \Lambda + \psi(2) \Lambda^2.
\]

That is, it has two subdiagonals, two superdiagonals and the main diagonal.

In the one hand, from \(\Psi = \sigma(J)H\Pi^\top\), we get for the Laguerre–Freud structure matrix

\[
(86) \quad \Psi = \eta^\Lambda^\top \gamma \Lambda^\top \gamma H + \eta^2 \Lambda^\top \gamma HDA + \eta^3 \Lambda^\top \gamma (\beta + b)H + (\beta + a)^\Lambda^\top \gamma H
\]

\[
+ \eta^\Lambda^\top \gamma \Lambda^\top \gamma H + (\beta + a)^2 \Lambda^\top \gamma H + (\beta + a)^3 \Lambda^\top \gamma H
\]

\[
+ \eta^\Lambda^\top (\beta + b)H \pi[2] \Lambda^2 + (\beta + a)^\Lambda^\top \gamma \pi [2] \Lambda^2 + \Lambda^\top \gamma \Lambda^\top \gamma H \pi[3] \Lambda^3
\]

In the other hand, from \(\Psi = \Pi^{-1}H\theta(J)\top\) we deduce

\[
\Psi = \left(H(\Lambda^\top)^2 + \Lambda^\top DHA^\top (\beta + T_\beta + c) + (\Lambda^\top)^2\pi^{-2}H(\gamma + T_\gamma + \beta^2 + c\beta)
\]

\[
- (\Lambda^\top)^3\pi^{-3}H(\beta + T_\beta + c)\gamma\Lambda + (\Lambda^\top)^4\pi^{-4}H(T_\gamma)\gamma\Lambda^2
\]

\[
+ H\Lambda^\top (\beta + T_\beta + c) - \Lambda^\top DH(\gamma + T_\gamma + \beta^2 + c\beta) + (\Lambda^\top)^2\pi^{-2}H(\beta + T_\beta + c)\gamma\Lambda - (\Lambda^\top)^3\pi^{-3}H(T_\gamma)\gamma\Lambda^2
\]

\[
+ (H(\gamma + T_\gamma + \beta^2 + c\beta) - \Lambda^\top DH(\beta + T_\beta + c)\gamma\Lambda + (\Lambda^\top)^2\pi^{-2}H(T_\gamma)\gamma\Lambda^2
\]

\[
+ H(\beta + T_\gamma + \beta^2 + c\beta) - \Lambda^\top DH(\beta + T_\gamma + \beta^2 + c\beta)\gamma\Lambda + (\Lambda^\top)^2\pi^{-2}H(T_\gamma)\gamma\Lambda^2
\]

\[
+ (H(\beta + T_\gamma + \beta^2 + c\beta) - \Lambda^\top DH(T_\gamma)\gamma\Lambda^2 + H(T_\gamma)\gamma\Lambda^2
\]

\[
H(T_\gamma)\gamma\Lambda^2
\]
From (87) we get the first two subdiagonals of \( \Psi \), namely
\[
\psi^{(-2)} = \eta H T_{-} \gamma, \quad \psi^{(-1)} = \eta \gamma T_{+} \gamma T_{+} H + \gamma (\beta + b) H + (T_{-} \beta + a) \gamma H,
\]
and from (88) we get the first two superdiagonals of \( \Psi \), namely
\[
\psi^{(1)} = (\beta + T_{-} \beta + c) \gamma - (T_{+} D)(T_{+} H) \gamma T_{+} \gamma, \quad \psi^{(2)} = H (T_{-} \gamma) \gamma.
\]
We can obtain an expression for the main diagonal that does not depend on \( \pi^{[+2]} \) o de \( \pi^{[-2]} \) by equating the non-zero diagonals of both matrices, two identities for the second matrices whose only non-zero diagonals are the main diagonal, the first and second subdiagonals and the first second superdiagonals. Equating the non-zero diagonals of both matrices, two identities for the second coefficients satisfy the following Laguerre–Freud relations
\[
\begin{aligned}
\text{Theorem 9} & \quad \text{(Laguerre–Freud equations for generalized type I Hahn).} & \quad \text{The generalized Hahn of type I recursion coefficients satisfy the following Laguerre–Freud relations} \\
(89a) & \quad (\eta^2 - 1)(\beta_{n+1} + \beta_n) \gamma_{n+1} - (\beta_{n-1} + \beta_n) \gamma_n & \quad + \eta(\beta_n(2\beta_n + a + b + c) + 2(\gamma_{n+1} + \gamma_n) + n(a + b - c + n - 1) + ab) \\
& \quad + (\eta + 1)((\eta(a + b) - c - (\eta + 1)n)(\gamma_{n+1} - \gamma_n) + (\eta + 1)\gamma_n) & = 0, \\
(89b) & \quad (\eta + 1)((n - 1)\beta_n + (n + 1)\beta_{n+1}) + (\eta - 1)(\gamma_{n+2} - \gamma_n + \beta_{n+1}^2 - \beta_n^2 + n) & \quad + (\eta(a + b) - c)(\beta_{n+1} - \beta_n) + \eta(a + b) + c = 0.
\end{aligned}
\]
Proof. We analyze the compatibility \([\Psi H^{-1}, J] = \Psi H^{-1}\) by diagonals. In both sides of the equation we find matrices whose only non-zero diagonals are the main diagonal, the first and second subdiagonals and the first and second superdiagonals. Equating the non-zero diagonals of both matrices, two identities for the second
superdiagonal and subdiagonal are obtained. From the remaining diagonals we obtain the two Laguerre–Freud equations (we obtain the same equality from the first subdiagonal and from the first superdiagonal).

Firstly, by simplifying we obtain that:

\[
\Psi H^{-1} = \eta(\Lambda^\top)^2(T_\gamma \gamma + \eta\Lambda^\top \gamma(T_+(D) + (\beta + b) + T_- (\beta + a))
+ \frac{\eta}{\eta + 1}(2(T_+ \gamma + \gamma + \beta^2) + (a + b + c)\beta + ab + T_+ D(a - b - c) + T_+ DT_+^2 D)
+ (\beta + T_- \beta - c - T_+(D))\Lambda + \Lambda^2.
\]

From the main diagonal, clearing, we obtain:

\[
(1 - \eta)\beta_{n-1}\gamma_n + (\eta - 1)\beta_{n+1}\gamma_{n+1} + \beta_n(\frac{\eta}{\eta + 1}(2\beta_n + a + b + c) + (\eta - 1)(\gamma_{n+1} - \gamma_n)) + \frac{\eta}{\eta + 1}(2(\gamma_{n+1} + \gamma_n) - nc + n(a + b + n - 1) + ab) - (\eta + 1)(n(\gamma_n - \gamma_{n+1} - \gamma_n) + (\eta(a + b) - c)(\gamma_{n+1} - \gamma_n)) = 0
\]

and we get Equation (89a).

From the first superdiagonal and from the first subdiagonal we get:

\[
((\eta + 1)\beta_n + \beta_{n+1} + n(\beta_{n+1} - \beta_n)) + (\eta - 1)(\gamma_{n+2} - \gamma_n + \beta^2_{n+1} - \beta^2_n + n) + c(1 + \beta_n - \beta_{n+1})
+ \eta(a + b)(1 + \beta_{n+1} - \beta_n) = 0
\]

that leads to (89b).

We now proceed with the compatibility \([\Psi H^{-1}, J_-] = \partial_\eta(\Psi H^{-1})\), recall that \(J_- := \Lambda^\top \gamma\) and \(\partial_\eta = \eta \frac{d}{d\eta}\).

As we will see we get no further equations than those already obtained in Theorem 9.

**Proposition 18.** The recursion coefficients for the generalized Hahn of type I orthogonal polynomials satisfy

\[
(90a) \quad \partial_\eta(\beta_n + \beta_{n+1} + c - n) = \gamma_{n+2} - \gamma_n,
(90b) \quad \partial_\eta(\frac{\eta}{\eta + 1}(2(\gamma_{n+1} + \gamma_n + \beta^2_n) + c(\beta_n - n) + n(n - 1) + (a + b)(\beta_n + n) + ab)) =
\gamma_{n+1}(\beta_n + \beta_{n+1} + c - n) - \gamma_n(\beta_{n-1} + \beta_n + c - (n - 1)),
(90c) \quad \partial_\eta(\eta\gamma_{n+1}(n + a + b + \beta_n + \beta_{n+1})) =
\frac{\eta}{\eta + 1} \gamma_{n+1}(2(\gamma_n + \beta^2_n + \beta^2_{n+1} + (a + b + c)(\beta_{n+1} - \beta_n) + 2n + (a + b - c))
\partial_\eta(\eta\gamma_{n+1}\gamma_{n+2}) = \eta\gamma_{n+1}\gamma_{n+2}(\beta_{n+2} - \beta_n + 1).
\]

**Proof.** From the diagonals of \([\Psi H^{-1}, J_-] = \partial_\eta(\Psi H^{-1})\) we get

i) From the first superdiagonal we obtain (90a).
ii) From the main diagonal cleaning up we get (90b).
iii) From the first subdiagonal we get, symplifying (90c).
iv) Finally, from the second subdiagonal we get (90d).

\[
\square
\]

**Remark 13.** We see that (90a) follow from the Toda equation (35a) and (90d) follow from Toda equation (35b). Moreover the two remaining equations (from the main diagonal and the first subdiagonal) are those of Theorem 9.

**Remark 14.** Dominici in [21 Theorem 4] found the following Laguerre–Freud equations

\[
(1 - \eta)\nabla (\gamma_{n+1} + \gamma_n) = \eta v_n \nabla (\beta_n + n) - u_n \nabla (\beta_n - n),
\]

\[
\Delta \nabla (u_n - \eta v_n) \gamma_n = u_n \nabla (\beta_n - n) + \nabla (\gamma_{n+1} + \gamma_n),
\]

where
with $u_n := \beta_n + \beta_{n+1} - n + c + 1$ and $v_n := \beta_n + \beta_{n+1} - n + 1 + a + b$. Therefore, the first one is of type $\gamma_{n+1} = F_1(n, \gamma_n, \gamma_{n-1}, \beta_n, \beta_{n-1})$, of length two, and the second of the form $\beta_{n+1} = F_2(n, \gamma_{n+1}, \gamma_n, \gamma_{n-1}, \beta_n, \beta_{n-1}, \beta_{n-2})$, is of length three.

**Remark 15.** Filipuk and Van Assche in [28, Equations (3.6) and (3.9)] introduce new non local variables $(x_n, y_n)$, 

$$\beta_n = x_n + \frac{n + (n + a + b)\eta - c - 1}{1 - \eta},$$

$$\frac{1 - \eta}{\eta} \gamma_n = y_n + \sum_{k=0}^{n-1} x_k + \frac{n(n + a + b - c - 2)}{1 - \eta}.$$

Then, in [28, Theorem 3.1] Equations (3.13) and (3.14) for $(x_n, y_n)$ are found, of length 0 and 1 respectively, in the new variables. Recall, that these new variables are non-local and involve all the previous recursion coefficients. In this respect, it is not so clear the meaning of length. The nice feature in this case, is that [28, Equations (3.13) and (3.14)] is a discrete Painlevé equation, that combined with the Toda equations lead to a differential system for the new variables $x_n$ and $y_n$ that after suitable transformation can be reduced to Painlevé VI $\sigma$-equation. Very recently, [25] it has been shown that this system is equivalent to $dP(D_4^{(1)}/D_4^{(1)})$, known as the difference Painlevé V.

**Conclusions and outlook**

In their studies of integrable systems and orthogonal polynomials, Adler and van Moerbeke have thoroughly used the Gauss–Borel factorization of the moment matrix, see [1, 2, 3]. This strategy has been extended and applied by us in different contexts, CMV orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials and multivariate orthogonal, see [5, 4, 6, 7, 8, 9, 11, 12]. For a general overview see [39].

Recently, see [42], we extended those ideas to the discrete world. In particular, we applied that approach to the study of the consequences of the Pearson equation on the moment matrix and Jacobi matrices. For that description a new banded matrix is required, the Laguerre–Freud structure matrix that encodes the Laguerre–Freud relations for the recurrence coefficients. We have also found that the contiguous relations fulfilled generalized hypergeometric functions determining the moments of the weight described for the squared norms of the orthogonal polynomials a discrete Toda hierarchy known as Nijhoff–Capel equation, see [44]. In [40] we study the role of Christoffel and Geronimus transformations for the description of the mentioned contiguous relations, and the use of the Geronimus–Christoffel transformations to characterize the shifts in the spectral independent variable of the orthogonal polynomials.

In this paper we have deepen in that program and search further in the discrete semi-classical cases, finding Laguerre–Freud relations for the recursion coefficients for three types of discrete orthogonal polynomials of generalized Charlier, generalized Meixner and generalized Hahn of type I cases.

For the future, we will study the generalized Hahn of type II polynomials, and extend these techniques to multiple discrete orthogonal polynomials [13] and its relations with the transformations presented in [17] and quadrilateral lattices [20, 41].

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