FORMALITY OF KAPRANOV’S BRACKETS ON PRE-LIE ALGEBRAS

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Abstract. We construct a $L_\infty$ structure on the suspension of a graded pre-Lie algebra, induced by a dg Lie algebra structure on the associated graded Lie algebra: moreover, using a simple criterion we prove that this $L_\infty$ structure is homotopy abelian. As an example we recover the $L_\infty$ structure introduced by Kapranov on the suspended Dolbeault complex of a Kähler manifold.

1. Introduction

In a remarkable paper [7] by M. Kapranov it is proved, among several other results, the existence of a $L_\infty$ structure on the suspended Dolbeault complex $A^0_*(T_X)[-1]$ of the tangent bundle $T_X$ of a Kähler manifold $X$: the linear and higher brackets are induced by the Dolbeault differential and the contraction with the curvature and its higher covariant derivatives. An important motivation for this construction is to unravel a form of the Jacobi identity satisfied by the Atiyah class of the tangent bundle $T_X$ (of which the curvature is a Dolbeault representative), which roughly says it turns $T_X[-1]$ into a Lie algebra object in the derived category of bounded below complexes of sheaves of $\mathcal{O}_X$-modules, cf. [7] and for instance the introduction to [3] (together with the references therein). As already proved in [7], Proposition 2.3.2, the induced graded Lie algebra structure on the suspended Dolbeault cohomology $H(A^0_*(T_X),\bar{\partial})[-1]$ is abelian, i.e., all brackets vanish: in this paper we strengthen this result by showing that in fact Kapranov’s $L_\infty$ structure is homotopy abelian, that is, the induced higher Massey brackets on $H(A^0_*(T_X),\bar{\partial})[-1]$ vanish as well.

This is accomplished in stages: in Section 3 we introduce what we call the splitting property for $L_\infty$ algebras. This can be nicely stated in Kontsevich and Soibelman’s geometric language [9], where a $L_\infty$ structure is thought of as a system of coordinates on a formal pointed differential graded (dg) manifold $M$: then $M$ has the splitting property if the natural evaluation morphism from the dg Lie algebra of vector fields on $M$ to the tangent complex at the base point admits a dg right inverse. The main result of the section is Theorem 3.6, where we prove that a $L_\infty$ algebra has the splitting property if and only if it is homotopy abelian.

In Section 4 we construct a $L_\infty$ structure on the suspension $L[-1]$ of a graded pre-Lie algebra, induced by a dg Lie algebra structure $(L,d,[\cdot,\cdot])$ on the associated graded Lie algebra. The linear bracket is $d$ itself, while the quadratic one measures how far is $d$ from satisfying the Leibniz rule with respect to the pre-Lie product: the higher brackets are defined recursively, by mimicking the operation of taking higher covariant derivatives, cf. Proposition 4.3. We call this hierarchy of brackets the Kapranov’s brackets on $L[-1]$ associated to $d$. We show that the $L_\infty$ algebras arising from this construction have the splitting property, thus are homotopy abelian.

Finally, in the last section we recover Kapranov’s $L_\infty$ algebra as a particular case of the above construction, completing the proof that it is homotopy abelian.

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Recently there have been many efforts \cite{2, 3, 5, 11, 15} to extend Kapranov’s construction in several directions. We mention in particular the two papers \cite{3, 11}, in which the authors recover Kapranov’s $L_\infty$ algebra as a particular case of a more general construction, associating, under an additional torsion freeness assumption (cf. \cite{3}, Theorem 62), a $L_\infty$ algebra to a matched pair of Lie algebroids. It seems possible, but we didn’t make any attempt in this direction, that these $L_\infty$ algebras could be also obtained as particular cases of our construction, and in particular they would be homotopy abelian.

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Preliminaries. We work over a field $\mathbb{K}$ of characteristic zero, graded means $\mathbb{Z}$-graded. For a graded space $V = \bigoplus_{i \in \mathbb{Z}} V^i$ we denote by $V^\otimes n$ the $n$-th tensor power of $V$, i.e., the tensor product of $n$ copies of $V$, and by $V^\odot n$ the $n$-th symmetric tensor power, i.e., the space of coinvariants of the symmetric group $S_n$, with the usual Koszul rule for twisting signs. For graded spaces $V$ and $W$, we denote by $\text{Hom}(V, W) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i(V, W)$ the internal mapping space in the category of graded spaces.

2. Review of $L_\infty[1]$ algebras

In this section we review some standard material on $L_\infty$ algebras. We actually prefer to work in the isomorphic category of $L_\infty[1]$ algebras, the two are related by the so called décalage isomorphisms, cf. Remark 2.2.

Let $V$ be a graded space, we denote by $\overline{SV} = \bigoplus_{n \geq 1} V^\odot n$ the reduced symmetric coalgebra over $V$: recall that this is the free locally conilpotent graded commutative coalgebra over $V$, that is, the functor $\overline{\mathbf{G}} : \mathbf{G} \to \mathbf{GCC} : V \to \overline{SV}$, from graded spaces to locally conilpotent graded commutative coalgebras, is a right adjoint to the forgetful functor. If $p : \overline{SV} \to V^\odot 1 = V$ denotes the natural projection, then the required identification $\text{Mor}_{\mathbf{GCC}}(C, \overline{SV}) \cong \text{Mor}_{\mathbf{G}}(C, V) = \text{Hom}^0(C, V)$ is given by corestriction $F \mapsto pF$. If $C = \overline{SW}$, then $\text{Hom}^0(\overline{SW}, V) = \prod_{n \geq 1} \text{Hom}^0(W^\odot n, V)$: when $F \in \text{Mor}_{\mathbf{GCC}}(\overline{SW}, \overline{SV})$ goes in $pF = (f_1, \ldots, f_n, \ldots)$ under corestriction, we call $f_n : W^\odot n \to V$ the $n$-th Taylor coefficient of $F$. It is well known (\cite{9}) that $F \in \text{Mor}_{\mathbf{GCC}}(\overline{SW}, \overline{SV})$ is an isomorphism (resp.: monomorphism, epimorphism) if and only if the linear Taylor coefficient $f_1 : W \to V$ is. Finally, a morphism $F \in \text{Mor}_{\mathbf{GCC}}(\overline{SW}, \overline{SV})$ is called linear if $f_n = 0$ for $n \neq 1$.

We denote by $\text{Coder}(\overline{SV})$ the graded Lie algebra of coderivations of $\overline{SV}$, then corestriction induces an isomorphism of graded spaces

$$\text{Coder}(\overline{SV}) \cong \text{Hom}(\overline{SV}, V) = \prod_{n \geq 1} \text{Hom}(V^\odot n, V) : Q \longrightarrow pQ = (q_1, \ldots, q_n, \ldots).$$

We call $q_n : V^\odot n \to V$ the $n$-th Taylor coefficient of $Q$, a coderivation is linear if $q_n = 0$ for $n \neq 1$. An inverse to the above isomorphisms sends $(q_1, \ldots, q_n, \ldots)$ to the coderivation defined by

$$(2.1) \quad Q(v_1 \odot \cdots \odot v_n) = \sum_{i=1}^n \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) q_{\sigma(i)} (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \odot v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)},$$

where $S(p, q)$ is the set of $(p, q)$-unshuffles, i.e., permutations $\sigma \in S_{p+q}$ such that $\sigma(k) < \sigma(k+1)$ for $k \neq p$, and $\varepsilon(\sigma) = \varepsilon(\sigma; v_1, \ldots, v_n)$ is the Koszul sign.

As usual, the graded Lie algebra structure on $\text{Coder}(\overline{SV})$ is given by taking the commutator of coderivations: this is induced by a right pre-Lie product (cf. Remark 4.2) on $\text{Coder}(\overline{SV})$, which we call the Nijenhuis-Richardson product and denote by $\bullet$. If $Q, R \in \text{Coder}(\overline{SV})$ then $Q \bullet R$
is the coderivation which restricts to $pQR$. More explicitly: if $pQ = q = (q_1, \ldots, q_n, \ldots)$ and $pR = r = (r_1, \ldots, r_n, \ldots)$, then $p(Q \bullet R) = q \bullet r = ((q \bullet r)_1, \ldots, (q \bullet r)_n, \ldots)$ is given by

$$
(q \bullet r)_n(v_1 \odot \cdots \odot v_n) = \sum_{i=1}^{n} \sum_{\sigma \in S(i, n-i)} \varepsilon(\sigma) q_{n-i+1}(r_1(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \odot v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)})
$$

Definition 2.1. A $L_\infty[1]$ algebra is a pair $(V, Q)$, where $V$ is a graded space and $Q \in \text{Coder}^{1}(\overline{SV})$ is a degree one coderivation squaring to zero, i.e., $Q^2 = Q \bullet Q = \frac{1}{2}[Q, Q] = 0$. If $Q$ is linear, then $(V, Q = q_1)$ is called an abelian $L_\infty[1]$ algebra. A $L_\infty[1]$ morphism $F : (V, Q) \to (W, R)$ of $L_\infty[1]$ algebras is a morphism of coalgebras $F : \overline{SV} \to \overline{SW}$ such that $FQ = FR$ if $L_\infty[1]$ morphism is also called strict.

Remark 2.2. The category of $L_\infty[1]$ algebras is isomorphic to the usual category of $L_\infty$ algebras, as defined for instance by Lada and Stasheff in [10], via the family of décalage isomorphisms $\text{Hom}(V[1]^n, W[1]) \cong \text{Hom}(V^{\otimes n}, W)[1-n], n \geq 1$ (where we denote by $V^\otimes n$ the $n$th exterior power of $V$). For our purposes, we may just define a $L_\infty[1]$ structure on $V$ to be a $L_\infty[1]$ structure on $V[1]$, and similarly a $L_\infty$ morphisms $V \to W$ to be a $L_\infty[1]$ morphism $V[1] \to W[1]$.

Example 2.3. As a particular case of the above remark, every dg Lie algebra $(L, d, [\cdot, \cdot])$ can be considered as a $L_\infty[1]$ algebra, where the $L_\infty[1]$ structure $Q$ on $L[1]$ is defined by $q_1(l) = -dl$, $q_2(l_1 \odot l_2) = (-1)^{|l_1|} l_1 l_2$ and $q_n = 0$ for $n \geq 3$. This is part of a faithful (not full) functor, sending a morphism of dg Lie algebras to the corresponding strict morphism of $L_\infty[1]$ algebras.

Definition 2.4. Let $(V, Q)$ be a $L_\infty[1]$ algebra, then in particular $(q_1)^2 = 0$, that is, $q_1 : V \to V$ is a differential on $V$: the dg space $(V, q_1)$ is called the tangent complex of the $L_\infty[1]$ algebra $(V, Q)$. Given a $L_\infty[1]$ morphism $F : (V, Q) \to (W, R)$ of $L_\infty[1]$ algebras, its linear Taylor coefficient $f_1 : (V, q_1) \to (W, r_1)$ is a dg morphism between the tangent complexes: $F$ is a weak equivalence if $f_1$ is a quasi-isomorphism of complexes.

Given a graded space $V$, the free locally conilpotent graded coalgebra over $V$ is the reduced tensor coalgebra $\overline{TV} = \sum_{n \geq 1} V^{\otimes n}$. Again, costriction induces an isomorphism of graded spaces $\text{Coder}(\overline{TV}) \cong \prod_{n \geq 1} \text{Hom}(V^{\otimes n}, V)$. The graded Lie algebra structure on $\text{Coder}(\overline{TV})$ is induced by a right pre-Lie product, which we call the Gerstenhaber product (and the associated bracket the Gerstenhaber bracket) and denote by $\circ$. Explicitly: if $f \in \text{Hom}(V^{\otimes i}, V)$ and $g \in \text{Hom}(V^{\otimes j}, V)$, then $f \circ g \in \text{Hom}(V^{\otimes i+j-1}, V)$ is given by

$$
f \circ g(v_1 \otimes \cdots \otimes v_{i+j-1}) = \sum_{k=0}^{i-1} \pm K f(v_1 \otimes \cdots \otimes v_k \odot g(v_{k+1} \otimes \cdots \otimes v_{k+j}) \odot \cdots \odot v_{i+j-1}),
$$

where $\pm K$ denotes the correct Koszul sign, in this case $(-1)^{|g|(|v_1|+\cdots+|v_k|)}$.

Remark 2.5. An $A_\infty[1]$ algebra is the non commutative analogous of a $L_\infty[1]$ algebra, that is, a graded space $V$ together with $M \in \text{Coder}^1(\overline{TV})$ such that $M^2 = 0$. For instance: there is an $A_\infty[1]$ structure on the desuspension $A[1]$ of an associative dg algebra $A$, in a similar way to Example 2.3.

3. Homotopy abelian $L_\infty[1]$ algebras

Definition 3.1. A $L_\infty[1]$ algebra is homotopy abelian if it is weakly equivalent to an abelian one.

The aim of this section is to prove Theorem 3.6, where we give several necessary and sufficient conditions for a $L_\infty[1]$ algebra to be homotopy abelian. We start by noticing that the previous definition is actually equivalent to the seemingly stronger one in the next lemma.
Lemma 3.2. A $L_\infty[1]$ algebra $(V, Q)$ is homotopy abelian if and only if it is isomorphic to the abelian $L_\infty[1]$ algebra $(V, q_1)$.

Proof. The if part is clear. For the only if part: recall that a $L_\infty[1]$ algebra $(W, R)$ is minimal if $r_1 = 0$, and that a minimal model for $(V, Q)$ is a minimal $L_\infty[1]$ algebra $(W, R)$ together with a weak equivalence $(W, R) \to (V, Q)$. It follows from structure theory of $L_\infty[1]$ algebras [9] that a minimal model always exists and it is well defined up to isomorphism over $(V, Q)$: moreover, $(V, Q)$ is isomorphic to the direct product of a minimal model and an abelian $L_\infty[1]$ algebra (with trivial tangent cohomology). If $(V, Q)$ is homotopy abelian, then the minimal model has a trivial, in particular abelian, $L_\infty[1]$ structure: in fact, the statement is homotopy invariant and true for abelian $L_\infty[1]$ algebras. This implies the existence of an isomorphism $F$ (not just a weak equivalence!) between $(V, Q)$ and an abelian $L_\infty[1]$ algebra, then $f_1$ is a dg isomorphism between the tangent complexes: composing $F$ with the inverse $(f_1)^{-1}$, seen as a strict $L_\infty[1]$ morphism between abelian $L_\infty[1]$ algebras, gives the desired isomorphism. □

For a graded space $V$, we denote by $SV = \oplus_{n \geq 0} V \odot^n$ ($V^{\odot 0} := \mathbb{K}$) the non reduced symmetric coalgebra over $V$, and by $p : SV \to V$ the natural projection: again, corestriction induces isomorphisms of graded spaces

$$
\text{Coder}(SV) \xrightarrow{\cong} \text{Hom}(SV, V) = \prod_{n \geq 0} \text{Hom}(V^{\odot n}, V) : Q \longrightarrow pQ = (q_0, \ldots, q_n, \ldots).
$$

The graded Lie algebra structure on $\text{Coder}(SV)$ is induced by a right pre-Lie product, which we still call the Nijenhuis-Richardson product and denote by $\bullet$: explicit formulas for $\bullet$, as well as for an inverse to the above isomorphism, can be given as in (2.2) and (2.1), taking the sum over $0 \leq i \leq n$ and with the convention that $q_0(\partial) := q_0(1)$. In particular $Q(1) = q_0(1) \in V^{\odot 1} \subset SV$ for every $Q \in \text{Coder}(SV)$.

The natural embedding $\text{Coder}(SV) \to \text{Coder}(SV) : (q_1, \ldots, q_n, \ldots) \mapsto (0, q_1, \ldots, q_n, \ldots)$ (respects the pre-Lie structures and) fits into an exact sequence of graded spaces

$$(3.1) \quad 0 \longrightarrow \text{Coder}(SV) \xrightarrow{i} \text{Coder}(SV) \xrightarrow{\text{ev}_1} V \longrightarrow 0$$

where $\text{ev}_1 : \text{Coder}(SV) \to V : Q \to q_0(1) = Q(1)$ is evaluation at the identity. If we denote by $\sigma_v \in \text{Coder}(SV)$ the coderivation given in Taylor coefficients by $p\sigma_v = (j_v, 0, \ldots, 0, \ldots)$, where $j_v : \mathbb{K} \to V : 1 \to v$, then $\sigma : V \to \text{Coder}(SV)$ splits the above exact sequence. More explicitly:

$$(3.2) \quad \sigma_v(1) = v, \quad \text{and for } n \geq 1 \quad \sigma_v(v_1 \odot \cdots \odot v_n) = v \odot v_1 \odot \cdots \odot v_n.$$

The following lemma is an immediate consequence of the explicit formula (2.2).

Lemma 3.3. Given $v \in V$ and $Q \in \text{Coder}(SV)$, with $pQ = (q_0, \ldots, q_n, \ldots)$, then $\sigma_v \bullet Q = 0$, and $[Q, \sigma_v] = Q \bullet \sigma_v$ is given in Taylor coefficients by $p[Q, \sigma_v] = [q, \sigma_v] = ([q, \sigma_v]_0, \ldots, [q, \sigma_v]_n, \ldots)$ by

$$
[q, \sigma_v]_0(1) = q_1(v), \quad \text{and for } n \geq 1 \quad [q, \sigma_v]_n(v_1 \odot \cdots \odot v_n) = q_{n+1}(v \odot v_1 \odot \cdots \odot v_n).
$$

Now we consider what happens if we have a $L_\infty[1]$ structure on $V$.

Definition 3.4. Let $(V, Q)$ be a $L_\infty[1]$ algebra, then the (resp.: reduced) Chevalley-Eilenberg complex of $(V, Q)$ with coefficients in itself is the dg space $(\text{Coder}(SV), [Q, \cdot])$ (resp.: $(\text{Coder}(SV), [Q, \cdot])$), its homology is called the (resp.: reduced) Chevalley-Eilenberg cohomology of $(V, Q)$ with coefficients in itself, and it is denoted by $H_{\text{CE}}(V, V)$ (resp.: $H^{\text{CE}}(V, V)$).
The exact sequence (3.1) enriches to an exact sequence of dg spaces

\[ 0 \xrightarrow{} \text{Coder}([SV, [\cdot, \cdot]]) \xrightarrow{i^{-1}} \text{Coder}([SV, [\cdot, \cdot]]) \xrightarrow{\text{ev}1} (V, q_1) \xrightarrow{} 0 \]

Thanks to Lemma 3.3, the splitting \( \sigma : v \to \sigma_v \) respects the differentials precisely when \( Q \) is an abelian \( L_{\infty}[1] \) structure. More in general, we give the following definition.

**Definition 3.5.** A \( L_{\infty}[1] \) algebra \( (V, Q) \) has the splitting property if the exact sequence (3.3) splits in the category of dg spaces.

**Theorem 3.6.** For a \( L_{\infty}[1] \) algebra \( (V, Q) \), the following are equivalent conditions:

1. \( (V, Q) \) is homotopy abelian;
2. \( (V, Q) \) has the splitting property;
3. \( H(i) : H_{CE}(V, V) \to H_{CE}(V, V) \) is injective, where \( i \) is the embedding in the exact sequence (3.3);
4. the natural spectral sequence computing \( H_{CE}(V, V) \) degenerates at \( E_1 \).

**Remark 3.7.** The equivalence (1) \( \Leftrightarrow \) (4) has recently been proven by Manetti [13], where it is also proven that if the same spectral sequence degenerates at \( E_2 \), then \( (V, Q) \) is formal\(^1\).

**Proof.** (of Theorem 3.6) (2) \( \Leftrightarrow \) (3) by elementary homological algebra, (4) \( \Rightarrow \) (3) by a standard spectral sequence argument, (1) \( \Rightarrow \) (4) by Lemma 3.2 and invariance of the spectral sequence under \( L_{\infty}[1] \) isomorphisms, cf. [13].

The (3) \( \Rightarrow \) (1) implication is proven in [1, 4]: namely, it is proven there that the \( L_{\infty} \) algebra \( (V[-1], Q) \) is weakly equivalent to the the homotopy fiber of the embedding of dg Lie algebras \( i : (\text{Coder}([SV, [\cdot, \cdot]]), [\cdot, \cdot]) \to (\text{Coder}([SV, [\cdot, \cdot]], [\cdot, \cdot])) \), then one applies a criterion for homotopy abelianity due to Manetti, cf. [6], Lemma 2.1. It remains to show the (1) \( \Rightarrow \) (2) implication: suppose \( (V, Q) \) is homotopy abelian, then (the proof of) Lemma 3.2 implies the existence of a coalgebra automorphism \( F : SV \to SV \) such that \( FQ = q_1 F \), where we regard \( q_1 \) as a linear coderivation on \( SV \), and such that moreover \( f_1 = \text{id}_V \). \( F \) extends to an automorphism \( F : SV \to SV \) by \( F(1) = 1 \).

A dg right inverse to the evaluation is given by \( V \to \text{Coder}([SV]) : v \to F^{-1} \sigma_v F \). In fact: \( F^{-1} \sigma_v F(1) = f_1^{-1}(v) = v \), since \( f_1 = \text{id}_V \), moreover, \( \text{Coder}([SV]) \to \text{Coder}([SV]) : R \to F^{-1} RF \) is clearly bracket preserving, thus we see that

\[ [Q, F^{-1} \sigma_v F] = [F^{-1} q_1 F, F^{-1} \sigma_v F] = F^{-1} [q_1, \sigma_v] F = F^{-1} \sigma_{q_1(v)} F. \]

The equivalence \( (1) \Leftrightarrow (4) \) has recently been proven by Manetti [13], where it is also proven that if the same spectral sequence degenerates at \( E_2 \), then \( (V, Q) \) is formal\(^1\).

**4. Kapranov’s brackets on pre-Lie algebras**

**Definition 4.1.** A graded left pre-Lie algebra \( (L, \triangleright) \) is a graded space \( L \) together with a bilinear product \( \triangleright : L^\otimes 2 \to L \), such that the associator, defined by

\[ A : L^\otimes 3 \to L : x \otimes y \otimes z \to A(x, y, z) = (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z), \]

is graded symmetric in the first two arguments, that is, \( A(x, y, z) = (-1)^{|x||y|}A(y, x, z) \), \( \forall x, y, z \). We denote, motivated by geometric examples, by \( \nabla : L \to \text{End}(L) : x \to \{\nabla_x : y \to x \triangleright y\} \) the left adjoint morphism: then the left pre-Lie identity can be rewritten as

\[ [\nabla_x, \nabla_y] = \nabla_{[x, y]} \quad \forall x, y \in L, \]

\(^1\)A \( L_{\infty} \) algebra \( (L, l_1, \ldots, l_n, \ldots) \) is formal if the graded Lie algebra \( (H(L, l_1), 0, [\cdot, \cdot]) \) is a minimal model, where the bracket is induced from \( l_2 \).
where \([x, y]\) in the right hand side is the commutator
\[
[x, y] := x \triangleright y - (-1)^{|x||y|} y \triangleright x = \nabla_x(y) - (-1)^{|x||y|}\nabla_y(x).
\]
The above defined bracket \([\cdot, \cdot] : L^{\otimes 2} \to L\) induces an associated graded Lie algebra structure on \(L\); the Jacobi identity follows directly from the pre-Lie identity. We denote by \(\text{Der}(L, [\cdot, \cdot])\) the graded Lie algebra of derivations of the grade Lie algebra \((L, [\cdot, \cdot])\).

**Remark 4.2.** A graded right pre-Lie algebra \((L, \triangleright)\) is a graded space \(L\) together with a bilinear product \(\cdot : L^{\otimes 2} \to L\), such that the associator is graded symmetric in the last two arguments: this implies the Jacobi identity for the commutator bracket \([\cdot, \cdot] : L^{\otimes 2} \to L\), inducing on \(L\) an associated graded Lie algebra structure. The functor sending a graded right pre-Lie algebra \((L, \triangleright)\) to the graded left pre-Lie algebra \((L, \triangleright)\), with \(\triangleright\) defined by \(x \triangleright y := (-1)^{|x||y|+1} y \triangleright x\), is an isomorphism over the category of graded Lie algebras.

**Proposition 4.3.** Let \((L, \triangleright)\) be a graded left pre-Lie algebra. For every \(d \in \text{Der}(L, [\cdot, \cdot])\), consider the following recursive definition of maps \(\Phi(d)_n : L^{\otimes n} \to L, n \geq 1,\)

\[
\begin{cases}
\Phi(d)_1 = d \\
\Phi(d)_2(x \otimes y) = \nabla_{dx}(y) - [d, \nabla_x](y) \\
\Phi(d)_{n+1}(x \otimes y_1 \otimes \cdots \otimes y_n) = -\Phi(d)_n, \nabla_x(y_1 \otimes \cdots \otimes y_n) & \text{for } n \geq 2,
\end{cases}
\]

where the bracket is the Gerstenhaber bracket in \(\text{Coder}({\overline{TL}})\), cf. (2.3), and for every \(x \in L\) we identify \(\nabla_x : L \to L\) with the corresponding linear coderivation \(\overline{TL} \to \overline{TL}\). Then the above recursion actually defines a hierarchy of graded symmetric brackets \(\Phi(d)_n : L^{\otimes n} \to L\), which together assemble to a coderivation \(\Phi(d) = (\Phi(d)_1, \Phi(d)_2, \ldots)\) of the reduced symmetric coalgebra \(\overline{SL}\).

Finally, the correspondence \(\Phi : \text{Der}(L, [\cdot, \cdot]) \to \text{Coder}(\overline{SL}) : d \to \Phi(d)\) is a morphism of graded Lie algebras.

**Definition 4.4.** Given a left pre-Lie algebra \((L, \triangleright)\) and \(d \in \text{Der}(L, [\cdot, \cdot])\), we call \(\Phi(d)_n : L^{\otimes n} \to L\), defined as in the previous proposition, the \(n\)-th Kapranov’s bracket on \(L\) associated to \(d\). If \((L, \triangleright)\) is a graded right pre-Lie algebra and \(d \in \text{Der}(L, [\cdot, \cdot])\), then the Kapranov’s brackets \(\Phi(d)_n : L^{\otimes n} \to L\), \(n \geq 1\), can be defined in the same way, after turning \(L\) in left pre-Lie as in Remark 4.2.

**Proof.** (of Proposition 4.3) We rewrite \(\Phi(d)_2\) as

\[
\Phi(d)_2(x \otimes y) = \nabla_{dx}(y) - [d, \nabla_x](y) = dx \triangleright y + (-1)^{|x||d|} x \triangleright dy - d(x \triangleright y).
\]

In other words, \(\Phi(d)_2\) measures how far \(d\) from satisfying the Leibniz rule with respect to the pre-Lie product \(\triangleright\). A straightforward computation shows

\[
\Phi(d)_2(x \otimes y) - (-1)^{|x||y|}\Phi(d)_2(y \otimes x) = [dx, y] + (-1)^{|x||d|}[x, dy] - d[x, y] = 0,
\]

since \(d \in \text{Der}(L, [\cdot, \cdot])\). We can therefore identify \(\Phi(d)_2\) with a quadratic coderivation in \(\text{Coder}(SL)\).

It follows from the recursive definition that \(\Phi(d)_3\) is graded symmetric in the last two arguments, so it suffices to show that it is also graded symmetric in the first two. We notice that

\[
\Phi(d)_3(x \otimes y \otimes z) = -[\Phi(d)_2, \nabla_x](y \otimes z) = -[[\Phi(d)_2, \nabla_x], \sigma_{y}](z),
\]

cf. Lemma 3.3, where the bracket is now the Nijenhuis–Richardson one in \(\text{Coder}(SL)\). Graded symmetry of \(\Phi(d)_3\) follows from the following computation in \(\text{Coder}(SL)\):

\[
[[\Phi(d)_2, \nabla_x], \sigma_{y}] = (-1)^{|x||y|}[\Phi(d)_2, \nabla_{dx}, \sigma_{y}] = \quad = \quad = \quad = \quad = \quad = \quad = \quad = \quad = \quad =
\]

\[
\begin{aligned}
&= [\Phi(d)_2, \nabla_x, \sigma_{y}] + (-1)^{|x||y|}[[\Phi(d)_2, \sigma_{y}], \nabla_x] - (-1)^{|x||y|}[[\Phi(d)_2, \nabla_{dy}], \sigma_{x}] - [[\Phi(d)_2, \sigma_{x}], \nabla_{dy}] = \\
&= [\Phi(d)_2, \nabla_x, \sigma_{x}] + (-1)^{|x||y|}[[\Phi(d)_2, \nabla_{dy}], \sigma_{x}] - [[\Phi(d)_2, \nabla_{dy}], \sigma_{x}] - [[\Phi(d)_2, \nabla_{dy}], \nabla_{x}] = \\
&= \nabla_d\nabla_x(y) - (-1)^{|x||y|}\nabla_d\nabla_y(x) - [\nabla_{dx}, \nabla_y] + (-1)^{|x||y|}[\nabla_{dy}, \nabla_{x}] = \\
&= \quad = \quad = \quad =
\end{aligned}
\]
4.1 \( \Phi(\sigma) \) is the composition of \( \Phi \) and the \( \text{SL} \):

\[
[\Phi(d), \sigma] = [\Phi(d), \sigma_x] = [\Phi(d), \sigma_y] = 0, \quad \forall x, y \in L.
\]

Thus graded symmetry of \( \Phi(d)_{n+1} \) follows by the same reasoning as before.

In other words, what we have proven is that

\[
(4.3) \quad \forall d \in \text{Der}(L, [\cdot, \cdot]), \quad \exists \Phi(d) \in \text{Coder}(\text{SL}) \text{ s.t. } [\Phi(d), \sigma_x + \nabla_x] = \sigma_{dx} + \nabla_{dx} \quad \forall x \in L.
\]

In fact, expanding the above equation in Taylor coefficients

\[
[[\Phi(d_1), \Phi(d_2)], \sigma_x + \nabla_x] = \sigma_{[d_1, d_2]x} + \nabla_{[d_1, d_2]x} \quad \forall x \in L,
\]

by another application of (4.3) this implies \( [\Phi(d_1), \Phi(d_2)] = \Phi([d_1, d_2]) \), as desired.

**Remark 4.5.** In some situations it turns out that it is more natural to consider the hierarchy of brackets \( \overline{\Phi}(d)_n := (-1)^n + 1 \Phi(d)_n \) on \( L \); then it is not hard to prove that the resulting coderivation \( \overline{\Phi}(d) = (\overline{\Phi}(d)_1, \overline{\Phi}(d)_2, \ldots) \) is characterized by

\[
\forall d \in \text{Der}(L, [\cdot, \cdot]), \quad \exists \overline{\Phi}(d) \in \text{Coder}(\text{SL}) \text{ s.t. } [\overline{\Phi}(d), \sigma_x - \nabla_x] = \sigma_{dx} - \nabla_{dx} \quad \forall x \in L,
\]

which leads to a recursive definition of the brackets \( \overline{\Phi}(d)_n \) similar to the one in (4.2). As in the proof of the previous proposition, it follows that the correspondence \( \overline{\Phi} : \text{Der}(L, [\cdot, \cdot]) \to \text{Coder}(\text{SL}) \) is a morphism of graded Lie algebras (this is also clear since \( \overline{\Phi} \) is the composition of \( \Phi \) and the automorphism \( \text{Coder}(\text{SL}) \to \text{Coder}(\overline{\text{SL}}) : (q_1, \ldots, q_n, \ldots) \mapsto (q_1, \ldots, (-1)^{n+1} q_n, \ldots) \)).

**Theorem 4.6.** Let \( (L, \nabla) \) be a graded left pre-Lie algebra, together with \( d \in \text{Der}^1(L, [\cdot, \cdot]) \) such that \( d^2 = 0 \); then the hierarchy of Kapranov’s brackets associated to \( d \) defines a homotopy abelian \( L_{\infty}[1] \) structure \( \Phi(d) \) on \( L \).

**Proof.** Since \( [\Phi(d), \Phi(d)] = \Phi([d, d]) = 0 \), we see that \( \Phi(d) \) is a \( L_{\infty}[1] \) structure on \( L \). Moreover, the \( L_{\infty}[1] \) algebra \( (L, \Phi(d)) \) has the splitting property (Definition 3.5), so it is homotopy abelian by Theorem 3.6: in fact, Equation (4.3) exhibits \( s : (L, d) \to \text{Coder}(\text{SL}) : \Phi(d), [], \cdot \) : \( x \to \sigma_x + \nabla_x \) as a dg right inverse to the evaluation.

5. Kapranov’s brackets in Kähler geometry

Let \( X \) be a hermitian manifold, we denote by \( \mathcal{A}_X \) the de Rham algebra of complex valued forms on \( X \), and by \( \mathcal{A}(T_X) \) the \( \mathcal{A}_X \)-module of smooth forms with coefficients in the tangent bundle \( T_X \). We denote by \( D = \nabla + \overline{\nabla} : \mathcal{A}^{p, q}(T_X) \to \mathcal{A}^{p+1, q}(T_X) \oplus \mathcal{A}^{p, q+1}(T_X) \) the Chern connection on \( \mathcal{A}(T_X) \)

(4.6) (that is, the only connection compatible with both the metric and the complex structure on \( T_X \)). Finally, let \( (z^1, \ldots, z^d) \) be a local system of holomorphic coordinates on some open \( U \subset X \), together with the corresponding local frame \( (\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^d}) \) of \( T_X \).

For \( \alpha \in \mathcal{A}^{p, q}(T_X) \), the contraction operator \( \iota_\alpha \in \text{End}^{p-1, q}(\mathcal{A}(T_X)) \) is defined as follows: if locally \( \alpha = \sum_i \alpha^i \otimes \frac{\partial}{\partial z^i} \) and \( \beta = \sum_j \beta^j \otimes \frac{\partial}{\partial z^j} \), then locally \( \iota_\alpha(\beta) = \sum_j \left( \sum_i \alpha^i \wedge (\frac{\partial}{\partial z^i}, \beta^j) \right) \otimes \frac{\partial}{\partial z^j} \).
where we denote by \( \cdot \) the contraction of forms with vector fields. An easy computation shows that
\[
[\partial, i_\alpha] = i_{\partial \alpha}.
\]

We denote by \( D_\alpha := [i_\alpha, D] \in \text{End}^p(A(T_X)) \) and by \( \nabla_\alpha := [i_\alpha, \nabla] \in \text{End}^q(A(T_X)) \). Recall that \( A(T_X) \) carries a natural structure of (bi)graded Lie algebra induced by the bracket of vector fields, cf. for instance [12]: under the additional hypothesis \( i_\alpha(\beta) = i_\beta(\alpha) = 0 \), the usual Cartan identities \( [i_\alpha, i_\beta] = 0 \) and \( [D_\alpha, i_\beta] = i_{[\alpha, \beta]} \) hold. This hypothesis is verified in particular for \( \alpha, \beta \in A^{0,*}(T_X) \): since \( D_\alpha = [i_\alpha, \nabla + \overline{\partial}] = \nabla_\alpha + (-1)^{|\alpha|}i_{\partial \alpha} \), in this case we also see that
\[
[\nabla_\alpha, i_\beta] = i_{[\alpha, \beta]} \quad \forall \alpha, \beta \in A^{0,*}(T_X).
\]

We define \( \triangleright : A^{0,*}(T_X) \otimes A^{0,*}(T_X) \to A^{0,*}(T_X) \) by
\[
\alpha \triangleright \beta := \nabla_\alpha(\beta) = D_\alpha(\beta) \quad \forall \alpha, \beta \in A^{0,*}(T_X),
\]
then \( \triangleright \) is a graded left pre-Lie product on \( A^{0,*}(T_X) \) precisely when the hermitian metric on \( X \) is Kähler. This can be seen as follows.

As well known [8], the curvature \( D^2 \in \text{End}^2(A(T_X)) \) is \( A_X \)-linear: this implies that if locally \( \beta = \sum_j \beta^j \otimes \overline{\partial}_j \), then locally \( D^2(\beta) = \sum_i [\sum_j (\beta^j \wedge \Omega^i_j) \otimes \overline{\partial}_j] \), where the forms \( \Omega^i_j \in A^{1,1}_X \) are locally defined by
\[
D^2(\overline{\partial}_j) = \sum_i \Omega^i_j \otimes \overline{\partial}_j.
\]
For the Chern connection we moreover that \( \Omega^i_j \in A^{1,1}_X \), \( \forall i, j \) [8]: this implies \( D^2 = \frac{i}{2}[\nabla + \overline{\partial}, \nabla + \overline{\partial}] \in \text{End}^{1,1}(A(T_X)) \), thus
\[
D^2 = [\overline{\partial}, [\nabla, \nabla]] = 0.
\]
By the Jacobi identity also \( [\nabla_\alpha, [\nabla_\beta, \nabla]] = [[\nabla_\alpha, \nabla], \nabla] = 0, \forall \alpha \in A(T_X). \) For \( \alpha, \beta \in A^{0,*}(T_X) \) we see (by the Jacobi and Cartan identities) that
\[
[\nabla_\alpha, \nabla_\beta] = [\nabla_\alpha, [\nabla_\beta, \nabla]] = [[\nabla_\alpha, \nabla], \nabla] = [i_{[\alpha, \beta]}, \nabla] = [\nabla_{[\alpha, \beta]}, \nabla].
\]
Then the pre-Lie condition (4.1) holds if and only if the bracket associated to \( \triangleright \) is the natural one on \( A^{0,*}(T_X) \), i.e., \( [\alpha, \beta] = \nabla_\alpha(\beta) - (-1)^{|\alpha||\beta|} \nabla_\beta(\alpha) = D_\alpha(\beta) - (-1)^{|\alpha||\beta|} D_\beta(\alpha), \forall \alpha, \beta \in A^{0,*}(T_X) \):
\[
\text{in other words } \triangleright \text{ is a left pre-Lie product on } A^{0,*}(T_X) \text{ if and only if } D \text{ is torsion free, but as well known [8] this is equivalent to the hermitian metric on } X \text{ being Kähler.}
\]

We assume in the remainder that \( X \) is a Kähler manifold. The Dolbeault differential \( \overline{\partial} \) induces a dg Lie algebra structure on the graded Lie algebra associated to \( \langle A^{0,*}(T_X), \triangleright \rangle \): in fact, \( (A^{0,*}(T_X), \overline{\partial}, [\cdot, \cdot]) \) is the Kodaira-Spencer dg Lie algebra controlling the infinitesimal deformations of the complex structure on \( X \), cf. [12]. We are in the setup of Theorem 4.6, thus \( \Phi(\overline{\partial}) \) defines a homotopy abelian \( L_\infty[1] \) structure on \( A^{0,*}(T_X) \).

Next we recall the construction of the \( L_\infty[1] \) structure on \( A^{0,*}(T_X) \) by Kapranov [7]. We can form a bundle of commutative coalgebras \( S(T_X) = \oplus_{n \geq 1} T_X^{\otimes n} \) over \( X \) and a bundle of Lie algebras \( \text{Coder}(S(T_X)) \) over \( X \), together with their non reduced versions \( S(T_X) \) and \( \text{Coder}(S(T_X)) \), as in sections 2 and 3 (but notice that now everything is in an ungraded setting), working in the symmetric monoidal category of holomorphic vector bundles over \( X \) (possibly of infinite rank). As a holomorphic vector bundle \( \text{Coder}(S(T_X)) = \prod_{n \geq 1} \text{Hom}(T_X^{\otimes n}, T_X) \), similarly for \( \text{Coder}(S(T_X)) \). The bundle of Lie algebras structure on \( \text{Coder}(S(T_X)) \) induces a dg Lie algebra structure on the Dolbeault complex \( A^{0,*}(\text{Coder}(S(T_X))) = \prod_{n \geq 1} A^{0,*}(\text{Hom}(T_X^{\otimes n}, T_X)) \). Finally, there is a morphism of dg Lie algebras
\[
\Psi : (A^{0,*}(\text{Coder}(S(T_X))), \overline{\partial}, [\cdot, \cdot]) \to (\text{Coder}(\overline{\partial}(A^{0,*}(T_X))), \overline{\partial}, [\cdot, \cdot]),
\]
where in the right hand side we regard $\tilde{\partial}$ as a linear coderivation on $S(A^{0,*}(T_X))$. The morphism $\Psi$ sends $R_n \in A^{0,*}(\text{Hom}(T_X^\otimes n, T_X))$ to $\Psi(R_n) : A^{0,*}(T_X) \otimes \cdots \otimes A^{0,*}(T_X) \to A^{0,*}(T_X)$, defined by the composition

$$\Psi(R_n) : A^{0,*}(T_X) \otimes \cdots \otimes A^{0,*}(T_X) \xrightarrow{R_n} A^{0,*}(\text{Hom}(T_X^\otimes n, T_X)) \otimes A^{0,*}(T_X) \to A^{0,*}(T_X),$$

where the first map is induced from the wedge product of forms, and the second one from the natural contraction. We leave to the reader the easy verification that $\Psi$ is indeed a morphism of dg Lie algebras. We notice that the brackets $\Psi(R_n)$ are $A^{0,*}$-multilinear in the following graded sense:

$$\forall \alpha_1, \ldots, \alpha_n \in A^{0,*}(T_X), \omega \in A^{0,*}. \quad \Psi(R_n)(\alpha_1 \otimes \cdots \otimes (\omega \wedge \alpha_k) \otimes \cdots \otimes \alpha_n) = (-1)^{|\omega||R_n|_1 + \sum_{j=1}^{k-1} |\alpha_j|} \omega \wedge \Psi(R_n)(\alpha_1 \otimes \cdots \otimes \alpha_n),$$

where the first map is induced from the wedge product of forms, and the second one from the natural contraction. We leave to the reader the easy verification that $\Psi$ is indeed a morphism of dg Lie algebras. We notice that the brackets $\Psi(R_n)$ are $A^{0,*}$-multilinear in the following graded sense:

$$\forall \alpha_1, \ldots, \alpha_n \in A^{0,*}(T_X), \omega \in A^{0,*}. \quad \Psi(R_n)(\alpha_1 \otimes \cdots \otimes (\omega \wedge \alpha_k) \otimes \cdots \otimes \alpha_n) = (-1)^{|\omega||R_n|_1 + \sum_{j=1}^{k-1} |\alpha_j|} \omega \wedge \Psi(R_n)(\alpha_1 \otimes \cdots \otimes \alpha_n),$$

Recall that the Chern connection $D = \nabla + \tilde{\partial}$ on $T_X$ induces the Chern connection on each one of the associated bundles $\text{Hom}(T_X^\otimes n, T_X)$, $n \geq 1$, which we still denote by the same symbol $D = \nabla + \tilde{\partial} \in \text{End}^{1,0}(A(\text{Hom}(T_X^\otimes n, T_X))) \otimes \text{End}^{0,1}(A(\text{Hom}(T_X^\otimes n, T_X))).$ Following Kapranov [7], we define recursively a hierarchy of tensors $R_n \in A^{0,1}(\text{Hom}(T_X^\otimes n, T_X))$, $n \geq 2$, starting with the curvature form $R_2 = \Omega = \sum_{i,j} \Omega_{ij} \partial z_i \otimes \frac{\partial}{\partial z_j} \in A^{1,1}(\text{End}(T_X)) \cong A^{0,1}(\text{Hom}(T_X^{\otimes 2}, T_X))$, and then by

$$R_{n+1} = \nabla(R_n) \in A^{1,1}(\text{Hom}(T_X^\otimes n, T_X)) \cong A^{0,1}(\text{Hom}(T_X^{\otimes n+1}, T_X)).$$

It turns out ([7], it will also follow from Theorem 5.1 and Proposition 4.3), by torsion freeness of $D$, that the tensors $R_n$ are symmetric in their holomorphic covariant indices: in other words, the above Recursion (5.2) actually defines a hierarchy $R_n \in A^{0,1}(\text{Hom}(T_X^\otimes n, T_X))$, $n \geq 2$, which we can assemble to $R = (0, R_2, \ldots, R_n, \ldots) \in A^{0,1}(\text{Coder}(S(T_X)))$. Finally, Kapranov proves (cf. [7], Theorem 2.6) that $R$ is a Maurer-Cartan element in the dg Lie algebra $(A^{0,*}(\text{Coder}(S(T_X))), \tilde{\partial}, [\cdot, \cdot])$, that is,

$$\tilde{\partial} R + \frac{1}{2}[R, R] = 0 \quad \text{(5.3)}.$$

It follows that $\tilde{\partial} + \Psi(R)$ is a $L_\infty[1]$ structure on $A^{0,*}(T_X)$, where again we are regarding $\tilde{\partial}$ as a linear coderivation on $S(A^{0,*}(T_X))$, in fact, $\tfrac{1}{2}(\tilde{\partial} + \Psi(R)) \cdot \Psi(R) = \Psi(\tilde{\partial} R + \frac{1}{2}[R, R]) = 0$: this is the $L_\infty[1]$ structure on $A^{0,*}(T_X)$ considered in [7]. Since $\Psi$ is clearly injective, the next theorem will conversely imply Equation (5.3).

**Theorem 5.1.** With the previous notations, $\tilde{\partial} + \Psi(R) = \Phi(\tilde{\partial})$: in particular, Kapranov’s $L_\infty[1]$ structure on $A^{0,*}(T_X)$ is homotopy abelian.

**Proof.** $\Phi(\tilde{\partial}) = \tilde{\Phi}(\tilde{\partial})_1$ by definition, then we have to prove $\Psi(R_n) = \tilde{\Phi}(\tilde{\partial})_n$, $\forall n \geq 2$. We start by computing $\tilde{\Phi}(\tilde{\partial})_2$, which is given by

$$\Phi(\tilde{\partial})_2(\alpha \otimes \beta) = \nabla \pi_\alpha(\beta) - [\tilde{\partial}, \nabla \pi_\alpha](\beta) = \nabla \pi_\alpha(\beta) - [\tilde{\partial}, \nabla](\beta) = (-1)^{|\alpha|} [i_\beta, \nabla](\beta) = (-1)^{|\alpha|} i_\beta D(\beta).$$

If locally $\alpha = \sum_i \alpha^i \otimes \frac{\partial}{\partial z^i}$ and $\beta = \sum_j \beta^j \otimes \frac{\partial}{\partial z^j}$, then locally

$$\Phi(\tilde{\partial})_2(\alpha \otimes \beta) = \sum_k \left( \sum_{i,j} (-1)^{|\alpha|} \alpha^i \wedge \left( \frac{\partial}{\partial z^i} \wedge \Omega^k_{ij} \right) \right) \otimes \frac{\partial}{\partial z^k} =$$

$$= \sum_k \left( \sum_{i,j} (-1)^{|\alpha| + |\beta|} \alpha^i \wedge \beta^j \wedge \left( \frac{\partial}{\partial z^i} \wedge \Omega^k_{ij} \right) \right) \otimes \frac{\partial}{\partial z^k}.$$

Graded symmetry of this expression also follows from the identity $\frac{\partial}{\partial z^i} \wedge \Omega^k_{ij} = \frac{\partial}{\partial z^j} \wedge \Omega^k_{ij}$, $\forall i, j, k$, which itself is a consequence of torsion freeness of $D$. Obviously, the above computation shows
that $\Psi(R_n) = \Phi(\partial)_{2n}$, in particular it implies that $\Phi(\partial)$ is $A_X^{0,*}$-bilinear. More in general, every $\Phi(\partial)_{n,k}, n \geq 2$, is $A_X^{0,*}$-multilinear in the sense of (5.1): by graded symmetry it suffices to consider the case $k = 1$, where it follows by induction, using the recursive definition and the easily established identity 
$\sum_{\omega \wedge \alpha}(*) = \omega \wedge \nabla_*(\beta), \forall \alpha, \beta \in A^{0,*}(T_X), \omega \in A_X^{0,*}$. Finally, in order to prove in general that $\Psi(R_n)(\alpha_1 \cdots \alpha_n) = \Phi(\partial)_{n}(\alpha_1 \cdots \alpha_n), \forall \alpha_1, \ldots, \alpha_n \in A^{0,*}(T_X)$, we have to reduce the case $\alpha_k = \frac{\partial}{\partial z_k}, k = 1, \ldots, n$. Proceeding by induction, we see that for all $n \geq 3$

$$\Psi(R_n)\left(\frac{\partial}{\partial z_1} \odot \cdots \odot \frac{\partial}{\partial z_n}\right) = \sum_k \left(\sum_h (R_n)_{1_{1,2} \cdots n_k} \omega \odot \cdots \odot \omega\right) \odot \frac{\partial}{\partial z_k} =$$

$$= \sum_k \left(\sum_h \left(\nabla_{\frac{\partial}{\partial z_{k'}}} R_{n-1}\right)_{1_{k'} \cdots n_k} \omega \odot \cdots \odot \omega\right) \odot \frac{\partial}{\partial z_k} = [\nabla_{\frac{\partial}{\partial z_{k}}} \Psi(R_{n-1})] \left(\frac{\partial}{\partial z_{k}} \odot \cdots \odot \frac{\partial}{\partial z_{n}}\right) =$$

$$= -[\Phi(\partial)_{n-1}, \nabla_{\frac{\partial}{\partial z_{k}}} \Psi(R_{n-1})] \left(\frac{\partial}{\partial z_{k}} \odot \cdots \odot \frac{\partial}{\partial z_{n}}\right) = \Phi(\partial)_n \left(\frac{\partial}{\partial z_{k}} \odot \cdots \odot \frac{\partial}{\partial z_{n}}\right)$$

Together with Theorem 4.6, this proves the claim of the theorem. 

\[\square\]  

**Remark 5.2.** Since the brackets $\Phi(\partial)_{n,k}, n \geq 1$, are all $O_X$-multilinear, we see that $(A_X^{0,*}(T_X), \Phi(\partial))$ is an $O_X$-multilinear $L_{\infty}$-algebra, for instance in the sense of [14]. In the previous theorem, homotopy abelianness has to be intended over the field $\mathbb{C}$ of complex numbers, and not in this more specific sense.

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