Algebraic geometry
Parabolic subgroups and automorphism groups of Schubert varieties

Sous-groupes paraboliques et groupes d’automorphismes des variétés de Schubert

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ARTICLE INFO

Article history:
Received 22 November 2017
Accepted after revision 3 April 2018
Available online 10 April 2018
Presented by Claire Voisin

ABSTRACT

Let G be a simple algebraic group of adjoint type over the field C of complex numbers, B be a Borel subgroup of G containing a maximal torus T of G. Let w be an element of the Weyl group W and X(w) be the Schubert variety in G/B corresponding to w. In this article we show that given any parabolic subgroup P of G containing B properly, there is an element w ∈ W such that P is the connected component, containing the identity element of the group of all algebraic automorphisms of X(w).

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RÉSUMÉ

Soit G un groupe algébrique du type adjoint sur le corps des nombres complexes C et B un sous-groupe de Borel de G contenant un tore maximal T. Soit w un élément du groupe de Weil W et X(w) la variété de Schubert dans G/B correspondant à w. Dans cet article, nous montrons que, pour tout sous-groupe parabolique P de G contenant B, il existe un élément w dans W tel que P est la composante connexe contenant l'élément identité du groupe des automorphismes algébriques de X(w).

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1. Introduction

Recall that if X is a projective variety over C, the connected component containing the identity element of the group of all algebraic automorphisms of X is an algebraic group (see [12, Theorem 3.7, p. 17]). Let G be a simple algebraic group of adjoint type over C. Let T be a maximal torus of G, and let R be the set of roots with respect to T. Let R+ ⊂ R be a set of positive roots. Let B+ be the Borel subgroup of G containing T, corresponding to R+. Let B be the Borel subgroup of G opposite to B+ determined by T. For w ∈ W, let X(w) := BwB/B denote the Schubert variety in G/B.
corresponding to $w$. Let $\text{Aut}^0(X(w))$ denote the connected component containing the identity element of the group of all algebraic automorphisms of $X(w)$. Let $\alpha_0$ denote the highest root of $G$ with respect to $T$ and $B^+$. For the left action of $G$ on $G/B$, let $P_w$ denote the stabilizer of $X(w)$ in $G$. If $G$ is simply laced and $X(w)$ is smooth, then we have $P_w = \text{Aut}^0(X(w))$ if and only if $w^{-1}(\alpha_0) < 0$ (see [10, Theorem 4.2(2), p. 772]). Therefore, it is a natural question to ask whether, given any parabolic subgroup $P$ of $G$ containing $B$ properly, there is an element $w \in W$ such that $P = \text{Aut}^0(X(w))$. In this article, we show that this question has an affirmative answer (see Theorem 2.1). If $P = B$, there is no such Schubert variety in $G/B$. We prove some partial results for Schubert varieties in partial flag varieties of type $A_n$. If $P^+$ is the maximal parabolic subgroup of $\text{PSL}(n+1, \mathbb{C})$ corresponding to the simple root $\alpha_1$ or $\alpha_n$, then $G/P^+$ is the projective space $\mathbb{P}^n$. The Schubert varieties in $\mathbb{P}^n$ are $\mathbb{P}^j (0 \leq i \leq n)$. $\mathbb{P}^n$ is the only Schubert variety in $\mathbb{P}^n$ for which the action of $B$ is faithful. Further, we have $\text{Aut}^0(\mathbb{P}^n) = \text{PSL}(n+1, \mathbb{C})$ (see Corollary 6.4). Therefore, the answer to the above question is negative if we consider partial flag varieties.

2. Notation and result

In this section, we set up some notation and preliminaries. We refer to [5], [7], [8], [9] for preliminaries in algebraic groups and Lie algebras.

Let $G$ be a simple algebraic group of adjoint type over $\mathbb{C}$ and $T$ be a maximal torus of $G$. Let $W = N_G(T)/T$ denote the Weyl group of $G$ with respect to $T$ and we denote the set of roots of $G$ with respect to $T$ by $R$. Let $B^+$ be a Borel subgroup of $G$ containing $T$. Let $B$ be the Borel subgroup of $G$ opposite to $B^+$ determined by $T$. That is, $B = n_0B^+n_0^{-1}$, where $n_0$ is a representative in $N_C(T)$ of the longest element $w_0$ of $W$. Let $R^+ \subset R$ be the set of positive roots of $G$ with respect to the Borel subgroup $B^+$. Note that the set of roots of $B$ is equal to the set $R^+ = -R^+$ of negative roots.

Let $S = \{\alpha_1, \ldots , \alpha_n\}$ denote the set of simple roots in $R^+$. For $\beta \in R^+$, we also use the notation $\beta > 0$. The simple reflection in $W$ corresponding to $\alpha_i$ is denoted by $s_{\alpha_i}$. Let $g$ be the Lie algebra of $G$. Let $h \subset g$ be the Lie algebra of $T$ and $b \subset g$ be the Lie algebra of $B$. Let $X(T)$ denote the group of all characters of $T$. We have $X(T) \otimes \mathbb{R} = \text{Hom}_\mathbb{R}(h_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $h$. The positive definite $W$-invariant form on $\text{Hom}_\mathbb{R}(h_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of $g$ is denoted by $(\cdot, \cdot)$. We use the notation $(\cdot \cdot)$ to denote $(\mu, \alpha) = \frac{2(\mu, \alpha)}{(|\alpha|)}$, for every $\mu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$. We denote by $X(T)^+$ the set of dominant characters of $T$ with respect to $B^+$. Let $\rho$ denote the half sum of all positive roots of $G$ with respect to $T$ and $B^+$. For any simple root $\alpha$, we denote the fundamental weight corresponding to $\alpha$ by $\omega_\alpha$. For $1 \leq i \leq n$, let $\lambda(\alpha_i) \in h$ be the fundamental coweight corresponding to $\alpha_i$. That is, $\lambda(\alpha_i)(\omega(\alpha_j)) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta.

For $w \in W$, let $l(w)$ denote the length of $w$. We define the dot action of $w$ on $X(T) \otimes \mathbb{R}$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \text{ where } w \in W \text{ and } \lambda \in X(T) \otimes \mathbb{R}.$$ 

We set $R^+(w) := \{\beta \in R^+ : w(\beta) \in -R^+\}$. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in $G/B$ corresponding to $w$.

For a simple root $\alpha$, we denote by $P_\alpha$ the minimal parabolic subgroup of $G$ generated by $B$ and $n_\alpha$, where $n_\alpha$ is a representative of $s_\alpha$ in $N_C(T)$, and we denote by $P_{\hat{\alpha}}$ the maximal parabolic subgroup of $G$ generated by $B$ and $[n_\beta : \beta \in S \setminus \{\alpha\}]$, where $n_\beta$ is a representative of $s_\beta$ in $N_C(T)$. For a subset $J$ of $S$, we denote by $W_J$ the subgroup of $W$ generated by $\{s_\alpha : \alpha \in J\}$. Let $W^J := \{w \in W : w(\alpha) \in R^+ \text{ for all } \alpha \in J\}$. For each $w \in W_J$, choose a representative element $n_w \in N_C(T)$. Let $N_J := \{n_w : w \in W_J\}$. Let $P_J := BN_J B$.

Our main result in this article is the following.

Theorem 2.1. Let $G$ be a simple algebraic group of adjoint type over $\mathbb{C}$ and $P$ be a parabolic subgroup of $G$ containing $B$ properly. Then there is an element $w \in W$ such that $P = \text{Aut}^0(X(w))$.

Let $G = \text{PSL}(n+1, \mathbb{C})$. For $1 \leq r \leq n$ and $w \in W \setminus \{e\}$, we denote the Schubert variety corresponding to $w$ in the Grassmannian $G/P_{\alpha_r}$ by $X_{P_{\alpha_r}}(w)$.

Proposition 2.2. Let $w = (s_{\alpha_1} \cdots s_{\alpha_r})(s_{\alpha_{r+1}} \cdots s_{\alpha_n}) \in W(r)$. Let $J^r(w) := \{i \in \{1, 2, \ldots , r-1\} : a_{i+1} - a_i \geq 2\}$, $J''(w) = \{1 + a_1 : i \in J^r(w)\}$ and $J(w) = \{\alpha_j : j \in \{1, \ldots , n\} \setminus J''(w)\}$. Then we have $P_{J(w)} = \text{Aut}^0(X_{P_{\alpha_1}}(w))$.

For more precise statement, see Proposition 6.2.

3. Proof of Theorem 2.1 except in three cases

In this section, we prove Theorem 2.1 in all cases except in three cases. The three cases left will be treated by Proposition 5.1.

Proof. Let $P$ be a parabolic subgroup of $G$ containing $B$ properly. If $P = G$, then we take $w = w_0$, the longest element $w_0$ of $W$. In this case, we have the following:

$$\text{Aut}^0(X(w_0)) = \text{Aut}^0(G/B) = G \text{ (see [1, Theorem 2, p. 75]).}$$
Now we assume that $P$ is any proper parabolic subgroup of $G$ such that $B \subseteq P \subseteq G$. Since $B \subseteq P \subseteq G$, there is a subset $\emptyset \neq I \subseteq S$ such that $P = P_I$. Consider $J = S \setminus I$. Hence, there exist unique elements $w_0^I \in W^I$ and $w_0 \in W_J$ such that $w_0 = w_0^I \cdot w_{0,J}$. Consider the natural left action of $G$ on $G/B$. Take $w = (w_0^I)^{-1}$. Then $P$ is the stabiliser of $X(w)$, since $R^+(w^{-1}) \cap S = I$. The natural action of $P$ on $X(w)$ induces a homomorphism, 

$$
\phi_w : P \longrightarrow \text{Aut}^0(X(w))
$$

of algebraic groups.

We note that $\phi_w : P \longrightarrow \text{Aut}^0(X(w))$ is injective, since $w^{-1}(\alpha_0) < 0$ (see [10, Theorem 4.2(2), p. 772]).

Let $J' = -w_0(J)$, and $P' = P_{J'}$. Consider the natural morphism $\pi : G/B \longrightarrow G/P'$. We denote the restriction of $\pi$ to $X(w)$ also by $\pi$. Then $\pi : X(w) \longrightarrow G/P'$ is a birational morphism. Therefore, by [5, Theorem 3.3.4(a), p. 96] and [5, Lemma 3.3.3(b), p. 95], we have:

$$
\pi_*(\mathcal{O}(X(w))) = \mathcal{O}_{G/P'}
$$

Thus, from [4, Corollary 2.2., p. 45], $\pi$ induces a homomorphism of algebraic groups:

$$
\pi_* : \text{Aut}^0(X(w)) \longrightarrow \text{Aut}^0(G/P').
$$

Since $\pi$ is birational, $\pi_* : \text{Aut}^0(X(w)) \longrightarrow \text{Aut}^0(G/P')$ is injective.

If $G$ is of type $B_n, C_n$ or $G_2$, then $w_0 = -id$ (see [3, p. 216, 217, p. 233]). If $G$ is of type $B_n$ and $P = P_{\alpha_0},$ then $I = \{\alpha_0\}$. Therefore, $J' = -w_0(J) = J = S \setminus \{\alpha_0\}$ and $P' = P_{\alpha_0}$. Thus, $(G, P')$ is one of the three types as in the statement of [1, Theorem 2, p. 75]. If $G$ is of the type $C_n$ and $P = P_{\alpha_0}$, then $(G, P') = (G, P_{\alpha_0})$ is one of the three types as in the statement of [1, Theorem 2, p. 75]. If $G$ is of type $G_2$ and $P = P_{\alpha_0}$, then $(G, P') = (G, P_{\alpha_0}) = (G, P_{\alpha_2})$ is one of the three types as in the statement of [1, Theorem 2, p. 75]. Similarly, we can see that if $(G, P')$ is one of the three types as in [1, Theorem 2, p. 75], then $(G, P)$ is one of the three types as in the statement of Proposition 5.1.

Case 1: $G$ is not of type $B_n, C_n$ and $G_2$. Then, for any parabolic subgroup $P$ of $G$, $(G, P)$ is not one of the three types as in Proposition 5.1. Therefore, $(G, P')$ is not one of the three exceptional types as in the statement of [1, Theorem 2, p. 75].

Case 2: $G = B_n, C_n$ or $G_2$ and $(G, P)$ is not one of the three types as in the statement of Proposition 5.1. In these cases, $w_0 = -id$ and $J' = -w_0(J) = J = S \setminus I$. Therefore, $(G, P')$ is not one of the three exceptional types as in the statement of [1, Theorem 2, p. 75]. Thus $(G, P)$ is not one of the three types as in the statement of Proposition 5.1 if and only if $(G, P')$ is not one of the three exceptional types as in the statement of [1, Theorem 2, p. 75]. Hence, we have $\text{Aut}^0(G/P') = G$. Therefore, $\text{Aut}^0(X(w))$ is a parabolic subgroup of $G$ containing $P$. Since $P$ is the stabiliser of $X(w)$, we have $P = \text{Aut}^0(X(w))$. Now, the proof follows from Cases 1 and 2.

4. Preliminaries for three left cases

Let $V$ be a rational $B$-module. Let $\phi : B \longrightarrow GL(V)$ be the corresponding homomorphism of algebraic groups. The total space of the vector bundle $\mathcal{L}(V)$ on $G/B$ is defined by the set of equivalence classes $\mathcal{L}(V) = G \times_B V$ corresponding to the following equivalence relation on $G \times V$:

$$(g, v) \sim (gb, \phi(b^{-1}) \cdot v) \text{ for } g \in G, b \in B, v \in V.$$ 

We denote the restriction of $\mathcal{L}(V)$ to $X(w)$ also by $\mathcal{L}(V)$. We denote the cohomology modules $H^i(\mathcal{L}(W), \mathcal{L}(V))$ by $H^i(w, V)$ ($i \in \mathbb{Z}_{\geq 0}$). If $V = C_l$ is the one-dimensional representation $\lambda : B \longrightarrow \mathbb{C}^\times$ of $B$, then we denote $H^i(w, V)$ by $H^i(w, \lambda)$.

Let $L_\alpha$ denote the Levi subgroup of $P_\alpha$ containing $T$. Note that $L_\alpha$ is the product of $T$ and the homomorphic image $G_\alpha$ of $SL(2, \mathbb{C})$ via a homomorphism $\psi : SL(2, \mathbb{C}) \longrightarrow L_\alpha$ (see [7, II, 1.3]). We denote the intersection of $L_\alpha$ and $B$ by $B_\alpha$. We note that the morphism $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B \text{ induced by the inclusion } L_\alpha \hookrightarrow P_\alpha$ is an isomorphism. Therefore, to compute the cohomology modules $H^i(P_\alpha/B, \mathcal{L}(V)) (0 \leq i \leq 1)$ for any $B$-module $V$, we treat $V$ as a $B_\alpha$-module, and we compute $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$.

We use the following lemma to compute cohomology groups. The following lemma is due to Demazure (see [6, p. 1]). He used this lemma to prove Borel–Weil–Bott’s theorem.

**Lemma 4.1.** Let $w = \tau s_\alpha$, $l(w) = l(\tau) + 1$, and $\lambda$ be a character of $B$. Then we have:

1. if $(\lambda, \alpha) \geq 0$, then $H^j(w, \lambda) = H^j(\tau, H^{0}(s_\alpha, \lambda))$ for all $j \geq 0$;
2. if $(\lambda, \alpha) \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$;
3. if $(\lambda, \alpha) \leq -2$, then $H^j(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$;
4. if $(\lambda, \alpha) = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.

Let $\pi : \tilde{G} \longrightarrow G$ be the simply connected covering of $G$. Let $L_\alpha$ (respectively, $B_\alpha$) be the inverse image of $L_\alpha$ (respectively, of $B_\alpha$) in $\tilde{G}$. Note that $L_\alpha/B_\alpha$ is isomorphic to $L_\alpha/B_\alpha$. We make use of this isomorphism to use the same notation for the vector bundle on $L_\alpha/B_\alpha$ associated with a $B_\alpha$-module. Let $V$ be an irreducible $L_\alpha$-module and $\lambda$ be a character of $B_\alpha$. 


Then, we have the following lemma.

**Lemma 4.2.**

(1) If \( \langle \lambda, \alpha \rangle \geq 0 \), then, the \( L_{\alpha} \)-module \( H^0(L_{\alpha}/B_{\alpha}, V \otimes C_{\lambda}) \) is isomorphic to the tensor product of \( V \) and \( H^0(L_{\alpha}/B_{\alpha}, C_{\lambda}) \). Further, we have \( H^1(L_{\alpha}/B_{\alpha}, V \otimes C_{\lambda}) = 0 \) for every \( j \geq 1 \).

(2) If \( \langle \lambda, \alpha \rangle \leq -2 \), then, we have \( H^0(L_{\alpha}/B_{\alpha}, V \otimes C_{\lambda}) = 0 \). Further, the \( L_{\alpha} \)-module \( H^1(L_{\alpha}/B_{\alpha}, V \otimes C_{\lambda}) \) is isomorphic to the tensor product of \( V \) and \( H^0(L_{\alpha}/B_{\alpha}, C_{\lambda}) \).

(3) If \( \langle \lambda, \alpha \rangle = -1 \), then \( H^1(L_{\alpha}/B_{\alpha}, V \otimes C_{\lambda}) = 0 \) for every \( j \geq 0 \).

**Proof.** By [9, I, Proposition 4.8, p. 53] and [9, I, Proposition 5.12, p. 77] for \( j \geq 0 \), we have the following isomorphism as \( L_{\alpha} \)-modules:

\[
H^1(L_{\alpha}/B_{\alpha}, V \otimes C_{\lambda}) \cong V \otimes H^1(L_{\alpha}/B_{\alpha}, C_{\lambda}).
\]

Now, the proof of the lemma follows from Lemma 4.1 by taking \( w = s_{\alpha} \) and the fact that \( L_{\alpha}/B_{\alpha} \cong P_{\alpha}/B \). \( \square \)

We now state the following Lemma on indecomposable \( \hat{B}_{\alpha} \) (respectively, \( B_{\alpha} \)) modules that will be used in computing the cohomology modules (see [2, Corollary 9.1, p. 30]).

**Lemma 4.3.**

(1) Any finite-dimensional indecomposable \( \hat{B}_{\alpha} \)-module \( V \) is isomorphic to \( V' \otimes C_{\lambda} \) for some irreducible representation \( V' \) of \( L_{\alpha} \), and some character \( \lambda \) of \( \hat{B}_{\alpha} \).

(2) Any finite dimensional indecomposable \( B_{\alpha} \)-module \( V \) is isomorphic to \( V' \otimes C_{\lambda} \) for some irreducible representation \( V' \) of \( L_{\alpha} \), and some character \( \lambda \) of \( B_{\alpha} \).

**Proof.** Proof of part (1) follows from [2, Corollary 9.1, p. 30].

Proof of part (2) follows from the fact that every \( B_{\alpha} \)-module can be viewed as a \( \hat{B}_{\alpha} \)-module via the natural homomorphism. \( \square \)

5. Proof of Theorem 2.1 in three left cases

To complete the proof of Theorem 2.1, it is sufficient to prove the following proposition. By \((G, P)\), we mean that \( G \) is a simple algebraic group of adjoint type over \( C \) and \( P \) is a parabolic subgroup of \( G \) containing \( B \).

**Proposition 5.1.** Let \((G, P)\) be one of the following types:

(1) \( G \) is of type \( B_n \) and \( P = P_{\alpha_1} \) is the minimal parabolic subgroup of \( G \) corresponding to \( \alpha_1 \);

(2) \( G \) is of type \( C_n \) and \( P = P_{\alpha_1} \) is the minimal parabolic subgroup of \( G \) corresponding to \( \alpha_1 \);

(3) \( G \) is of type \( G_2 \) and \( P = P_{\alpha_1} \) is the minimal parabolic subgroup of \( G \) corresponding to \( \alpha_1 \).

Then, there exists an element \( w \in W \) such that \( P = \text{Aut}^0(X(w)) \).

**Proof.** Let \( X(w) \) be the tangent sheaf of \( X(w) \). Let \( T_{G/B} \) be the restriction of the tangent bundle to \( X(w) \). Then \( TX(w) \) is a subsheaf of \( T_{G/B} \) on \( X(w) \). By [12, Lemma 3.4, p. 13], we have \( \text{Lie}(\text{Aut}^0(X(w))) = H^0(X(w), TX(w)) \subseteq H^0(X(w), T_{G/B}) = H^0(w, g/b) \).

As in the strategy of proof in Section 3, it is sufficient to prove that, for all the three types \((G, P)\) as above, there is an element \( w \in W \) such that

(i) \( P \) is the stabiliser of \( X(w) \) in \( G \);

(ii) \( w^{-1}(\alpha_0) < 0 \);

(iii) \( H^0(w, g/b) = g \).

For instance, let \( \phi_w : P \to \text{Aut}^0(X(w)) \) be the natural homomorphism induced by the action of \( P \) on \( X(w) \).

Since \( w^{-1}(\alpha_0) < 0 \), \( \phi_w : P \to \text{Aut}^0(X(w)) \) is injective. Since \( H^0(w, g/b) = g \), we have \( H^0(X(w), TX(w)) \subseteq g \). Therefore, \( \text{Aut}^0(X(w)) \) is a closed subgroup of \( G \) containing \( P \). Since \( P \) is the stabiliser of \( X(w) \) in \( G \), we have \( P = \text{Aut}^0(X(w)) \).

We first make a note about statement (ii) and statement (iii). Let \( w \in W \) be such that \( w^{-1}(\alpha_0) < 0 \). To prove that \( H^0(w, g/b) = g \), it is sufficient to prove that for any negative root \( \beta \), the dimension of the weight space \( H^0(w, g/b)_\beta \) is one.

The proof of this note is as follows.
The restriction of the natural map \( g \to g/b \to \bigoplus_{\alpha \in \mathbb{R}^+} g_{\alpha} \) is an isomorphism of \( T \)-modules and, hence, we have \( g/b = \bigoplus_{\alpha \in \mathbb{R}^+} C_{\alpha} \). Since \( s_i \) permutes all positive roots other than \( \alpha_i \) for every \( 1 \leq i \leq n \), every indecomposable \( B_{\alpha_i} \)-summand \( V \) of \( g/b \) with highest weight, a positive root different from \( \alpha_i \) is indeed an \( \hat{L}_{\alpha_i} \)-module, and hence, for every \( \alpha \in \mathbb{R}^+ \setminus S \), the dimension of the weight space \( H^0(s_i, g/b)_{\alpha} \) is one. Using this argument and by induction on the length of \( w \), we see that the dimension of the weight space \( H^0(w, g/b)_{\alpha} \) is one for every \( \alpha \in \mathbb{R}^+ \setminus S \). Further, since \( (g/b)_{\alpha} \) is one dimensional for every simple root \( \alpha \), each fundamental coweight \( h(\alpha_i) \) \((1 \leq i \leq n)\) appears exactly once. Hence, it is sufficient to prove that, for any negative root \( \beta \), the dimension of the weight space \( H^0(w, g/b)_{\beta} \) is one.

We prove the existence of an element \( w \in W \) satisfying the first two conditions and that the dimension of the weight space \( H^0(w, g/b)_{\beta} \) is one for any negative root \( \beta \) in all the three cases, separately.

Case 1: assume that \( G \) is of type \( B_n \) and \( P = P_n \). For every \( 1 \leq r \leq n - 1 \), let \( v_r = s_3s_5 \cdots s_r \). Take \( w = v_1v_2 \cdots v_{n-1} \). It is easy to see that \( P_n \) is the stabiliser of \( X(w) \).

In this case, \( \alpha_0 = \alpha_2 \). So, we have \( v_1^{-1}(\alpha_0) = \alpha_2 + 2(\sum_{i=3}^{n} \alpha_i) \). This is the highest root of type \( B_{n-1} \) corresponding to the root system whose set of simple roots is \( S \setminus \{\alpha_1\} \). By induction on the rank of \( G \), we have \( w^{-1}(\alpha_0) = (v_2 \cdots v_{n-1})^{-1}(\alpha_2 + 2(\sum_{i=3}^{n} \alpha_i)) < 0 \).

Now, if \( v \in W \) is of minimal length such that the dimension of \( H^0(v, g/b)_{\beta} \) is at least two for some negative root \( \beta \), then \( \beta = -1(\sum_{j=1}^{n} \alpha_j) \) for some \( 1 \leq i \leq n - 1 \).

The justification of the above statement is as follows. Clearly, for any such \( v \), \( l(v) > 1 \). Choose \( y \in S \) such that \( l(s_yv) = l(v) - 1 \). Let \( u = s_yv \).

Then, we have \( \dim H^0(s_y, H^0(u, g/b)) \geq 2 \).

If \( \langle \beta, \gamma \rangle = 1 \), then there exists an indecomposable \( B_{\gamma} \)-summand \( V \) of \( H^0(u, g/b) \) such that \( H^0(u, V)_{\beta} \neq 0 \). In this case, either \( V = C_{\beta} \otimes C_{\beta+\gamma} \) or \( V = C_{\beta} \).

So we have \( \dim H^0(s_y, H^0(u, g/b)) \beta = 1 \).

If \( \langle \beta, \gamma \rangle = -1 \), we have either \( V = C_{\beta} \otimes C_{\beta+\gamma} \) or \( V = C_{\beta+\gamma} \).

So we have \( \dim H^0(s_y, H^0(u, g/b)) \beta = 1 \).

If \( \langle \beta, \gamma \rangle = 2 \), then there exists a unique indecomposable \( B_{\gamma} \)-summand \( V \) of \( H^0(u, g/b) \) with highest weight \( \beta \).

Therefore, \( \dim H^0(s_y, H^0(u, g/b)) \beta = 1 \).

If \( \langle \beta, \gamma \rangle = -2 \), then there exists a unique indecomposable \( B_{\gamma} \)-summand of \( H^0(u, g/b) \) with highest weight \( \beta + 2\gamma \).

Therefore, \( \dim H^0(s_y, H^0(u, g/b)) \beta = 1 \).

Following the case-by-case analysis as above, we conclude that \( \langle \beta, \gamma \rangle = 0 \) and that there is a unique indecomposable \( B_{\gamma} \)-summand \( V \) of \( H^0(u, g/b) \) such that \( V = C_{\beta+\gamma} \otimes C_{\beta} \).

In particular, we have \( \beta + \gamma \in \mathbb{R}^+ \). Since \( G \) is of type \( B_n \), we have \( \gamma = \alpha_n \) and \( \beta = -1(\sum_{j=1}^{n} \alpha_j) \) for some \( 1 \leq i \leq n - 1 \).

By induction on the rank of \( G \), we may assume that \( H^0(v_2v_3 \cdots v_{n-1}, g/b) \) is one dimensional for every \( 2 \leq i \leq n - 1 \). Also \( H^0(v_2v_3 \cdots v_{n-1}, g/b)_{-(\sum_{j=1}^{n} \alpha_j)} = 0 \).

Since \( (\sum_{j=1}^{n} \alpha_j, \alpha_1) = 0 \) for every \( 3 \leq i \leq n - 1 \), the restriction of the evaluation map
\[
H^0(s_1v_2v_3 \cdots v_{n-1}, g/b)_{-(\sum_{j=1}^{n} \alpha_j)} \to H^0(v_2v_3 \cdots v_{n-1}, g/b)_{-(\sum_{j=1}^{n} \alpha_j)}
\]
is an isomorphism for every \( 3 \leq i \leq n - 1 \) (see Lemma 4.1 and Lemma 4.2).

Since \( -(\sum_{j=2}^{n} \alpha_j, \alpha_1) = 1 \), we have
\[
H^0(s_1v_2v_3 \cdots v_{n-1}, g/b)_{-(\sum_{j=1}^{n} \alpha_j)} = H^0(s_1, H^0(v_2v_3 \cdots v_{n-1}, g/b))_{-(\sum_{j=1}^{n} \alpha_j)}
\]
is one dimensional for every \( i = 1, 2 \) (see Lemma 4.1 and Lemma 4.2).

Now, it is easy to see that, for every \( 2 \leq r \leq n \), the evaluation map
\[
H^0(s_1v_2v_3 \cdots v_{n-1}, g/b)_{-(\sum_{j=1}^{n} \alpha_j)} \to H^0(s_1v_2v_3 \cdots v_{n-1}, g/b)_{-(\sum_{j=1}^{n} \alpha_j)}
\]
is an isomorphism for every $1 \leq i \leq n$ by induction on $r$ and using Lemma 4.1, Lemma 4.2. Thus, the space $H^0(w, g/b)_{\alpha}$ is one dimensional for every negative root $\alpha$.

Case 2: assume that $G$ is of type $C_n$ ($n \geq 3$) and $P = P_1$. Take $w = s_1 s_2 \cdots s_n$. In this case we have $\alpha_0 = 2\omega_1$, and $w^{-1}(\alpha_0) = -\alpha_0$. Further, the stabiliser of $X(w)$ in $G$ is $P_1$.

First, note that

$$H^0(s_n, g/b) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha)} \oplus C_{-\alpha} \quad (\text{see Lemma 4.1 and Lemma 4.2}).$$

Further, we have:

$$H^0(s_{n-1}s_n, g/b) = H^0(s_{n-1}, H^0(s_n, g/b))$$

$$= \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha)} \oplus C_{-\alpha} + C_{h(\alpha_{n-1})} \oplus C_{-\alpha_{n-1}}$$

$$+ C_{-(\alpha_{n-1} + \alpha_n)} + C_{-(2\alpha_{n-1} + \alpha_n)} \quad (\text{see Lemma 4.1 and Lemma 4.2}).$$

By using Lemma 4.1, Lemma 4.2 and the descending induction on $1 \leq r \leq n - 1$, we see that

$$H^0(s_r \cdots s_{n-1}s_n, g/b) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus \bigoplus_{i=r}^{n} C_{h(\alpha_i)} \oplus C_{-\mu},$$

where $\mu$ runs over all positive roots in $\sum_{i=1}^{n} Z_{\geq 0}\alpha_i$. Thus, we have $H^0(w, g/b) = g$.

Case 3: assume that $G$ is of type $G_2$ and $P = P_1$. Take $w = s_1 s_2 s_1 s_2$. Here, we follow the convention in [7]. In this case, we have $\alpha_0 = 3\alpha_1 + 2\alpha_2$. Further, $w^{-1}(\alpha_0) = -\alpha_2$.

First note that

$$H^0(s_2, g/b) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha_2)} \oplus C_{-\alpha_2} \quad (\text{see Lemma 4.1 and Lemma 4.2}),$$

$$H^0(s_1, H^0(s_2, g/b)) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha_2)} \oplus C_{-\alpha_2} \oplus C_{h(\alpha_1)} \oplus C_{-\alpha_1} \oplus \bigoplus_{i=1}^{3} C_{-(\alpha_2 + i\alpha_1)}$$

(see Lemma 4.1 and Lemma 4.2).

Therefore, we have:

$$H^0(s_1 s_2, g/b) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha_2)} \oplus C_{-\alpha_2} \oplus C_{h(\alpha_1)} \oplus C_{-\alpha_1} \oplus \bigoplus_{i=1}^{3} C_{-(\alpha_2 + i\alpha_1)}.$$

$$H^0(s_2, H^0(s_1 s_2, g/b)) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha_2)} \oplus C_{-\alpha_2} \oplus C_{h(\alpha_1)} \oplus C_{-\alpha_1} \oplus \bigoplus_{i=1}^{3} C_{-(\alpha_2 + i\alpha_1)} \oplus C_{-(3\alpha_1 + 2\alpha_2)} = g$$

(see Lemma 4.1 and Lemma 4.2).

Therefore, we have:

$$H^0(s_2 s_1 s_2, g/b) = \bigoplus_{\alpha \in R^+} C_{\alpha} \oplus C_{h(\alpha_2)} \oplus C_{-\alpha_2} \oplus C_{h(\alpha_1)} \oplus C_{-\alpha_1} \oplus \bigoplus_{i=1}^{3} C_{-(\alpha_2 + i\alpha_1)} \oplus C_{-(3\alpha_1 + 2\alpha_2)}.$$

Thus, we have $H^0(w, g/b) = H^0(s_1, g) = g$. 

Example 5.2. Let $G = PSL(3, \mathbb{C})$. In this case, $B$ is the set of invertible lower triangular matrices, $P_{\alpha_1} = Aut^0(X(s_1 s_2))$ and $X(s_1 s_2)$ is smooth.

Remark 5.3. In Theorem 2.1, for a given parabolic subgroup $P$ of $G$ containing $B$ properly, the Schubert variety $X(w)$ for which $P = Aut^0(X(w))$ is not necessarily smooth. For example, take $G = PSL(4, \mathbb{C})$, and $P_{\alpha_2} = Aut^0(X(s_2 s_1 s_3 s_2))$. Note that $X(s_2 s_1 s_3 s_2)$ is not smooth (see [11, Theorem 2.2, p. 48]).
6. Automorphism groups of Schubert varieties in partial flag varieties of type $A_n$

In this section, we discuss about parabolic subgroups of $G = PSL(n + 1, \mathbb{C})$ and the connected component containing the identity element of the group of all algebraic automorphisms of Schubert varieties in the Grassmannian $G/P_{\alpha_j}$, where $1 \leq r \leq n$ and $P_{\alpha_j} = P_{S(\alpha_j)}$.

Lemma 6.1. Let $G = PSL(n + 1, \mathbb{C})$. Let $1 \leq r \leq n$ and $w \in W^{S(\alpha_j)}$. Then $w^{-1}(\alpha_0) < 0$ if and only if there exists an increasing sequence $1 \leq a_1 < a_2 < \cdots < a_r = n$ of positive integers such that $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$. Now, it is easy to see that $w^{-1}(\alpha_0) < 0$ if and only if $i = 1$ and $a_1 = n$. $\square$

Proof. Note that $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Let $w \in W^{S(\alpha_j)}$ be such that $w \neq id$. Then there exists an integer $1 \leq i \leq n$ and an increasing sequence of positive integers $1 = a_1 < a_2 < \cdots < a_r \leq n$ such that $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$. Now, let $W(r) = \{w \in W^{S(\alpha_j)} : w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r), \text{ where } 1 \leq a_1 < a_2 < \cdots < a_r = n \}$. For $w \in W^{S(\alpha_j)}$, we denote the Schubert variety in the Grassmannian $G/P_{\alpha_j}$ corresponding to $w$ by $X_{P_{\alpha_j}}(w)$.

Proposition 6.2. Let $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W(r)$. Let $J'(w) := [i \in \{1, 2, \ldots, r-1\} : a_{i+1} - a_i \geq 2], \quad J''(w) = [1 + a_i : i \in J'(w)]$ and $J(w) = \{a_i : i \in \{1, \ldots, n\} \setminus J'(w)\}$. Then we have $P_{J(w)} = Aut^0(X_{P_{\alpha_j}}(w))$.

Proof. Let $P_w$ be the stabiliser of $X_{P_{\alpha_j}}(w)$ in $G$. First, we show that $P_w = P_{J(w)}$. If $a_{i+1} - a_i \geq 2$ for some $1 \leq i \leq r - 1$ then $s_{a_i}P_w > w$, and $s_{a_i+1}P_w \in W^{S(\alpha_j)}$. Hence, $s_{a_i+1}$ is not in the Weyl group of $P_w$. Therefore, $P_w$ is a subgroup of $P_{J(w)}$. Let $R(P_{\alpha_j}) = R \cap (\bigcup_{\alpha \in S(\alpha_j)} \mathbb{Z}_{\alpha})$. Further, it is easy to see that, for $\alpha \in J(w)$, we have either $w^{-1}(\alpha) < 0$ or $w^{-1}(\alpha) \in R(P_{\alpha_j})$.

Therefore, $P_{J(w)} \subseteq P_w$.

Let $\psi_w : P_{J(w)} \rightarrow Aut^0(X_{P_{\alpha_j}}(w))$ be the natural homomorphism induced by the action of $P_{J(w)}$ on $X_{P_{\alpha_j}}(w)$.

Since $w \in W(r)$, $w^{-1}(\alpha_0) < 0$ (see Lemma 6.1). Therefore, $\psi_w : P_{J(w)} \rightarrow Aut^0(X_{P_{\alpha_j}}(w))$ is injective.

Let $p_{\alpha_j}$ be the Lie algebra of $P_{\alpha_j}$. Since $G$ is simply laced, the restriction map $H^0(w_0, g/p_{\alpha_j}) \rightarrow H^0(w, g/p_{\alpha_j})$ is surjective, where $w_0, r \in W^{S(\alpha_j)}$ is the minimal representative of $w_0$ (see [10, Lemma 3.5(3), p. 770]).

Further, since $w^{-1}(\alpha_0) < 0$, $H^0(w_0, g/p_{\alpha_j}) = g \rightarrow H^0(w, g/p_{\alpha_j})$ is an isomorphism.

Therefore, we have $H^0(X_{P_{\alpha_j}}(w), T_{X_{P_{\alpha_j}}}(w)) \subseteq g$. Hence $Aut^0(X_{P_{\alpha_j}}(w))$ is a closed subgroup of $G$ containing $P_{J(w)}$. Thus we have $P_{J(w)} = Aut^0(X_{P_{\alpha_j}}(w))$. $\square$

Corollary 6.3. Let $B \subseteq P$ be a parabolic subgroup of $G$ and $w \in W^{S(\alpha_j)}$ such that $P = Aut^0(X_{P_{\alpha_j}}(w))$. Then we have $P = P_{J(w)}$.

Corollary 6.4.

(1) If $P \neq G$, then there is no element $w \in W^{S(\alpha_j)}$ such that $P = Aut^0(X_{P_{\alpha_j}}(w))$.

(2) If $P \neq G$, then there is no element $w \in W^{S(\alpha_j)}$ such that $P = Aut^0(X_{P_{\alpha_j}}(w))$.

Proof. Proof of (1): the Schubert varieties in $G/P_{\alpha_j}$ are projective space $\mathbb{P}^1$ $(0 \leq i \leq n)$. Therefore the automorphism groups of these Schubert varieties are $PSL(i + 1, \mathbb{C})$ $(0 \leq i \leq n)$. Further, the map $\phi_w$ is injective for only one $w$.

Proof of (2): it is similar to that of (1). $\square$

Acknowledgements

We are grateful to the Infosys Foundation for the partial financial support. We are grateful to the referee for suggesting the reference of the book D.N. Akhiezer, Lie group actions in complex analysis, which helps to improve the exposition of this article.

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