The geometric and topological interpretation of Berry phase on a torus

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Abstract

Illustration of the geometric and topological properties of Berry phase is often in an obscure and abstract language of fiber bundles. In this article, we demonstrate these properties with a lucid and concrete system whose parameter space is a torus. The instantaneous eigenstate is regarded as a basis. And the corresponding connection and curvature are calculated respectively. Furthermore, we find the magnitude of curvature is exactly the Gaussian curvature, which shows its local property. The topological property is reflected by the integral over the torus due to Gauss-Bonnet theorem. When we study the property of parallel transportation of a vector over a loop, we make a conclusion that the Berry phase is just the angle between the final and initial vectors. And we also illuminate the geometric meaning of gauge transformation, which is just a rotation of basis.

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I. INTRODUCTION

The study of Berry’s phase was first introduced by his famous paper in the context of the cyclic adiabatic evolution and nondegenerate case [1]. And its connection with fiber bundles was built by Simon [2]. It was generalized to degenerate case by Wilczek and Zee [3] as well as the nonadiabatic case by Aharonov and Anandan [4]. And it was also generalized in the mixed state under unitary evolution by Sjöqvist et. al. [5]. The elaboration about this field is covered by two monographs [6, 7]. Moreover many mathematical skills of differential geometry used in this field could be find in the book written by Flanders [8].

At the same time, the topological phenomenon in integer quantum Hall effect was explained by Thouless et. al. [9], who proposed the famous TKNN invariant. Haldane predicted that quantum Hall effect can also be realized in graphene under the background of applied staggered magnetic field, which broke time reversal symmetry [10]. Nearly two decades later, taking account of spin-orbit coupling, Kane and Mele began to study a quantum spin Hall insulator in the context of two layers graphene without breaking time reversal symmetry [11, 12]. However, the intrinsic spin orbital coupling is too tiny to be observed in experiments. Soon, Bernevig, Hughes and Zhang proposed that the spin Hall effect might be realized in HgTe/CdTe quantum wells [13], which was verified by König et. al. [14]. There are also some excellent reviewed articles [15, 16] and monograph [17].

In 2006, Qi, Wu and Zhang proposed a square lattice as a simple model of topological insulator, whose Brillouin zone is a two dimensional torus [18]. So the geometric and topological properties whose base is torus should be elucidated in detail. It is organised as follows. In the next section, Berry’s phases will be reviewed together with connection and curvature. In Sec. III, in order to illustrate properties vividly, we will treat a basis as the instantaneous eigenstate. And the corresponding connection and curvature will be calculated respectively. Furthermore, we will also study the topological property. In Sec. IV, the analogy between Berry phase and parallel transportation will be drawn. And we also elucidate the geometric meaning of gauge transformation as well. Finally, a conclusion is drawn in the last section.
II. REVIEW OF BERRY’S PHASES

Suppose there exists a quantum system under periodic evolution whose Hamiltonian depends on some slowly varying parameters, which can be described as

$$H(\mathbf{R}(t)) = H(R_1(t), R_2(t), \cdots R_n(t)),$$

with $\mathbf{R}(T) = \mathbf{R}(0)$, where $T$ is the period. Its evolution is described by Schrödinger equation, which is

$$i\hbar \frac{d}{dt} \vert \Psi(t) \rangle = H(\mathbf{R}(t)) \vert \Psi(t) \rangle. \quad (1)$$

Under adiabatic approximation, the solution to the above equation can have this form

$$\vert \Psi(t) \rangle = c(t) \vert n, \mathbf{R}(t) \rangle, \quad (2)$$

where $\vert n, \mathbf{R}(t) \rangle$ is an instantaneous eigenstate of $H(\mathbf{R}(t))$, which reads

$$H(\mathbf{R}(t)) \vert n, \mathbf{R}(t) \rangle = E_n \vert n, \mathbf{R}(t) \rangle. \quad (3)$$

The evolution of this system will trace a curve in the parameter space $\mathbf{R}(t)$. Moreover, if the system undergoes periodic motion, it will trace a closed curve. Suppose the energy eigenstate is nondegenerate, by substituting Eq. (2) into Eq. (1) and multiplying $\langle n, \mathbf{R}(t) \vert$ on the left hand side at both sides of the equation, one can obtain

$$\vert \Psi(t) \rangle = e^{i\delta} e^{i\gamma} \vert n, \mathbf{R}(t) \rangle$$

where $\delta$ is called dynamical phase

$$\delta = -\frac{1}{\hbar} \int_0^T E(\mathbf{R}(t)) dt$$

and $\gamma$ is called Berry’s phase or geometric phase $[1, 6]$

$$\gamma = i \int \langle n, \mathbf{R}(t) \vert \frac{\partial}{\partial \mathbf{R}} \vert n, \mathbf{R}(t) \rangle \cdot d\mathbf{R}. \quad (4)$$

The integrand is called the vector potential, which is

$$A = i \langle n, \mathbf{R}(t) \vert \frac{\partial}{\partial \mathbf{R}} \vert n, \mathbf{R}(t) \rangle. \quad (5)$$

And it can be also written as differential one form

$$A = A \cdot d\mathbf{R} = i \langle n, \mathbf{R}(t) \vert d \vert n, \mathbf{R}(t) \rangle \quad (6)$$
By use of Stokes’s theorem, the loop integral can be converted to a surface integral,

\[ \gamma = \oint_{c=\partial s} A = \int_A dA. \]

So we can obtain Berry’s curvature

\[ F = dA, \]

which can also be expressed in vector notation,

\[ \mathbf{F} = \nabla \times \mathbf{A}. \]

In the following section, we will elucidate the geometric meaning of vector potential and Berry phase in a concrete parameter space, i.e. torus.

III. BERRY’S CONNECTION, CURVATURE AND TOPOLOGY

At first, let’s build the parametric equations of torus step by step. Let’s draw a circle on \( x - z \) plane whose center is at \((b, 0)\) and radius is \(a\), where \(b > a\). Its equation can be written as

\[ r = (b + a \cos \psi) \mathbf{i} + a \sin \psi \mathbf{k}, \]

where \(\mathbf{i}, \mathbf{j}\) and \(\mathbf{k}\) are unite vectors about \(x, y\) and \(z\) axis respectively. And the corresponding parametric equations read

\[
\begin{align*}
x &= (b + a \cos \psi) \\
z &= a \sin \psi
\end{align*}
\]

When we rotate this circle along \(z\)–axis by \(SO(2)\) transformation, a torus can be obtained, whose equation can be expressed as

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
b + a \cos \psi \\
0 \\
a \sin \psi
\end{bmatrix}.
\]

Thus, the final equation of torus can be written as [19]

\[
\begin{align*}
x &= (b + a \cos \psi) \cos \theta \\
y &= (b + a \cos \psi) \sin \theta \\
z &= a \sin \psi
\end{align*}
\]
For convenient, we want to transform it to a vector notation, which is

\[ \mathbf{r} = x(\theta, \psi)\mathbf{i} + y(\theta, \psi)\mathbf{j} + z(\theta, \psi)\mathbf{k}. \]

(7)

For physical realization, we may contrive a spin subject to a magnetic field, whose Hamiltonian takes this form

\[ H = -\mathbf{\mu} \cdot \mathbf{B} \]

where \( \mathbf{\mu} \) is the magnetic moment and \( \mathbf{B} \) is the magnetic field. And the components of \( \mathbf{B} \) are

\[
\begin{align*}
B_x &= (\beta + \alpha \cos \psi) \cos \theta \\
B_y &= (\beta + \alpha \cos \psi) \sin \theta \\
B_z &= \alpha \sin \psi,
\end{align*}
\]

where \( \beta > \alpha \). We are not intend to calculate its connection nor curvature in this article, but to uncover their underlying geometric meaning. At first, We can differentiate \( \mathbf{r}(\theta, \psi) \) (7) to obtain the basis vector of tangent plane of torus, which is

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta} d\theta + \frac{\partial \mathbf{r}}{\partial \psi} d\psi. \]

(8)

By an explicit calculation, we can obtain

\[ \varepsilon_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -(b + a \cos \psi) \sin \theta \mathbf{i} + (b + a \cos \psi) \cos \theta \mathbf{j} \]

and

\[ \varepsilon_\psi = \frac{\partial \mathbf{r}}{\partial \psi} = -a \sin \psi \cos \theta \mathbf{i} - a \sin \psi \sin \theta \mathbf{j} + a \cos \psi \mathbf{k}. \]

And the lengths of \( \varepsilon_\theta \) and \( \varepsilon_\psi \) are

\[ h_\theta = b + a \cos \psi \]

and

\[ h_\psi = a \]

respectively. From the above equations, the orthonormal basis vectors of the tangent plane can be get after an observation,

\[ \mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \]

(9)
and

\[ e_\psi = -\sin \psi \cos \theta \mathbf{i} - \sin \psi \sin \theta \mathbf{j} + \cos \psi \mathbf{k} \]  \hspace{1cm} (10)

So by substituting Eq. (9) and (10) into Eq. (8), we can obtain

\[ dr = (b + a \cos \psi) d\theta e_\theta + a d\psi e_\psi. \]

Furthermore, the area element on the torus can be calculated by multiplying the two coefficients before \( e_\theta \) and \( e_\psi \) directly, which is

\[ d\sigma = a(b + a \cos \psi) d\theta d\psi. \]  \hspace{1cm} (11)

In order to comparison with instantaneous eigenstate (3), we define a new orthogonal basis on the surface of torus, which takes the following form [7]

\[ e_+ = \frac{1}{\sqrt{2}} (e_\theta + i e_\psi) \]  \hspace{1cm} (12)

and

\[ e_- = \frac{1}{\sqrt{2}} (e_\theta - i e_\psi). \]  \hspace{1cm} (13)

The connection 1-form can be evaluated easily as

\[ A = i e_+ ^* \cdot d e_+ = \sin \psi d\theta, \]  \hspace{1cm} (14)

which has a similar form as Berry’s connection one form (6). And it can be translated into the language of vector potential, which is

\[ A = \frac{\sin \psi}{b + a \cos \theta} e_\theta \]  \hspace{1cm} (15)

Next, we calculate the curvature two-form

\[ F = dA = -\cos \psi d\theta \wedge d\psi, \]

which can also has vector form

\[ F = -\frac{\cos \psi}{a(b + a \cos \psi)} e_\perp, \]

Where \( e_\perp \) is the normal vector of the torus. Next, we want to calculate Gaussian curvature of torus, which will uncover the geometric meaning of Berry’s curvature. At first, the normal vector of torus is evaluated by
\[ e_\perp = e_\theta \times e_\psi \]
\[ = \cos \theta \cos \psi \hat{i} + \sin \theta \cos \psi \hat{j} + \sin \psi \hat{k}. \]

In order to obtain the corresponding area element of surface of a unit sphere, we differentiate \( e_\perp \) with respect to \( \theta \) and \( \psi \) respectively, which is

\[ d(e_\perp) = \cos \psi d\theta e_\theta + d\psi e_\psi. \]

Hence, the area element on the unit sphere is obtained by multiplying the two coefficients before \( e_\theta \) and \( e_\psi \) directly, which reads

\[ d\sigma' = \cos \psi d\theta d\psi. \quad (16) \]

By use of Eq. (11) and Eq. (16), the Gaussian curvature can be calculated as

\[ K = \frac{d\sigma'}{d\sigma} = \frac{\cos \psi}{a(b + a \cos \theta)}, \]

which is exactly the magnitude of Berry’s curvature \( F \). So Berry’s curvature reflects the local geometric property of torus.

The direct calculation of integral of Gaussian curvature over the torus reflects its topological property, which is

\[ \iiint K d\sigma = - \iiint F = 0. \]

It is certainly in accordance with Gauss-Bonnet theorem, which takes this form

\[ \iiint K d\sigma = 4\pi (1 - g), \]

where \( g \) is called genus, which is equal to the number of holes of the manifold. In the case of torus, \( g = 1 \).

IV. PARALLEL TRANSPORT AND GAUGE TRANSFORMATION

Suppose there exists a tangential vector on the torus, which takes this form

\[ \mathbf{v} = v_+ e_+ + v_- e_- . \]

Moreover, the above vector can also represented in the original basis as

\[ \mathbf{v} = v_\theta e_\theta + v_\psi e_\psi. \]
Due to the basis transformation between them, i.e. Eq. (12) and (13), the transformations between components can also be built, which are

\[ v_+ = \frac{1}{\sqrt{2}}(v_\theta - iv_\psi) \]  

and

\[ v_- = \frac{1}{\sqrt{2}}(v_\theta + iv_\psi) \]

This vector will be transported parallel along a curve on the surface of torus. Due to the parallel transportation, it is natural to require the vector is invariant, which is

\[ dv = 0. \]

By substituting Eq. (12) and Eq. (13) into the above equation, we can obtain

\[ (dv_+ - i \sin \psi v_+ d\theta) e_+ + (dv_- + i \sin \psi v_- d\theta) e_- = 0, \]

where the normal component is discarded in accordance with the requirement of tangential vector. Because \( e_+ \) and \( e_- \) are orthonormal basis, the following results can be get directly, which are

\[ dv_+ - i \sin \psi v_+ d\theta = 0 \]  

(18)

and

\[ dv_- + i \sin \psi v_- d\theta = 0. \]

In comparison with Berry phase, let’s imagine our vector is along the \( e_+ \) axis initially, hence the component along \( e_- \) vanishes. So the vector takes the following form

\[ v_0 = v_0 e_+. \]

The parallel transportation along a closed curve can be described by the Eq. (18), so this equation can be solved as

\[ v_+ = v_0 \exp \left( i \oint \sin \psi d\theta \right). \]

By substituting the vector potential (15) into the above equation, it can be converted to

\[ v_+ = v_0 \exp \left( i \oint \mathbf{A} \cdot d\mathbf{r} \right), \]  

(19)

where \( d\mathbf{r} = (b + a \cos \psi) d\theta \mathbf{e}_\theta + a d\psi \mathbf{e}_\psi \) is the infinitesimal displacement vector along the surface of this torus. From the above Eq. (19), we can make a conclusion that the direction
of the final vector maybe different from its original one, where this phenomenon is called holonomy in the realm of differential geometry. However, when $\psi = 0$ or $\psi = \pi$, which are two geodesics on the torus, the loop integral in Eq. (19) vanishes. This is due to the fact that when the vector is transported parallel along a geodesic, the direction will not change after a closed contour. In addition, we would illuminate the geometric feature of the loop integral that is equivalent to Berry phase. For simplicity, we denote

$$\gamma = \oint A \cdot dr.$$  \tag{20}$$

Substituting Eq. (17) and Eq. (20) into Eq. (19),

$$\frac{1}{\sqrt{2}}(v_{+\theta} - iv_{+\psi}) = \frac{1}{\sqrt{2}}(v_{0\theta} - iv_{0\psi})(\cos \gamma + i \sin \gamma).$$

After comparing the real and imaginary parts of the above equation, one can obtain that

$$\begin{pmatrix} v_{+\theta} \\ v_{+\psi} \end{pmatrix} = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} v_{0\theta} \\ v_{0\psi} \end{pmatrix},$$

which means that the vector is rotated clockwise by angle $\gamma$ after parallel transportation along a closed curve.

Next, we want to study the geometric meaning of gauge transformation [1, 6, 7]. First of all, let’s do a gauge transformation on $e_+$ (12), which is

$$e'_+ = \exp(-i\chi)e_+,$$  \tag{21}$$

where in general $\chi$ is a function of $\theta$ and $\phi$. By plugging the above Eq. (21) into connection (14), we can obtain the new connection which takes the following form

$$A' = ie'_+ \cdot de'_+ = A + d\chi,$$

which is exactly the same as the gauge transformation of Berry’s connection. In order to illustrate the geometric meaning of gauge transformation, we go back to original basis $e_\theta$ and $e_\phi$. By substituting Eq. (9) and Eq. (10) into Eq. (21) and comparing the real and imaginary parts at the both side of equations, we can obtain

$$\begin{pmatrix} e'_{\theta} \\ e'_{\psi} \end{pmatrix} = \begin{pmatrix} e_{\theta} \\ e_{\psi} \end{pmatrix} \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix},$$

is just the basis transformation. The primed basis vectors are obtained by rotating the unprimed ones counterclockwise by angle $\chi$. 
V. CONCLUSIONS AND ACKNOWLEDGEMENTS

In this paper, we illustrate the geometric and topological property of Berry phase whose parameter space is a torus. The instantaneous eigenstate is like the basis $e_+ (12)$ and $e_- (13)$. And the corresponding connection and curvature are calculated respectively. Furthermore, we find the magnitude of curvature is exactly the Gaussian curvature, which demonstrate its local property. The topological property is reflected by the integral over the torus due to Gauss-Bonnet theorem. When we study the property of parallel transportation of a vector over a loop, we make a conclusion that the Berry phase is just the angle between the final and initial vectors. And we also elucidate the geometric meaning of gauge transformation, which is just a rotation of basis.

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