The Shannon Lower Bound is Asymptotically Tight for Sources with Finite Rényi Information Dimension

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Abstract

The Shannon lower bound is one of the few lower bounds on the rate-distortion function that holds for a large class of sources. In this paper, it is demonstrated that its gap to the rate-distortion function vanishes as the allowed distortion tends to zero for all source vectors having a finite differential entropy and a finite Rényi information dimension. Conversely, it is demonstrated that the rate-distortion function of a source with infinite Rényi information dimension is infinite for any finite distortion.

1 Introduction

Suppose that we wish to quantize a memoryless, $k$-dimensional source with a distortion not larger than $D$. Specifically, suppose the source produces the sequence of independent and identically distributed (i.i.d.), $k$-dimensional, real-valued, random vectors $\{X_k, k \in \mathbb{Z}\}$ according to the distribution $P_X$, and suppose that we employ a vector quantizer that produces a sequence of quantized vectors $\{\hat{X}_k, k \in \mathbb{Z}\}$ satisfying

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E \left[ \|X_k - \hat{X}_k\|^r \right] \leq D$$

for some norm $\| \cdot \|$ and some $r > 0$. (We use $\limsup$ to denote the limit superior and $\liminf$ to denote the limit inferior.) Rate-distortion theory tells us that if for every blocklength $n$ and distortion constraint $D$ we quantize the sequence of source symbols $X_1, \ldots, X_n$ to one of $e^{nR(D)}$ possible sequences of quantized symbols $\tilde{X}_1, \ldots, \tilde{X}_n$, then the smallest rate $R(D)$ (in nats per channel use) for which there exists a vector quantizer satisfying (1) is given by \cite{1, 2}

$$R(D) = \inf_{P_{\hat{X}|X}} I(\hat{X}; \hat{X})$$

where the infimum is over all conditional distributions of $\hat{X}$ given $X$ for which

$$E \left[ \|X - \hat{X}\|^r \right] \leq D$$

and where the expectation is computed with respect to the joint distribution $P_X P_{\hat{X}|X}$. Here and throughout the paper we omit the time indices where they are immaterial. The rate $R(D)$ as a function of $D$ is referred to as the rate-distortion function.

Unfortunately, the rate-distortion function is unknown except in very few special cases, hence, it needs to be assessed by means of upper and lower bounds. Arguably, for sources with a finite differential entropy, the most important lower bound is the Shannon lower bound \cite{2}, which for a $k$-dimensional, real-valued source and the distortion constraint (3) is given by \cite{3}

$$R_{\text{SLB}}(D) = h(X) + \frac{k}{r} \log \frac{1}{D} - \frac{k}{r} + \log \frac{1}{kV_k} + \left( \frac{k}{r} - 1 \right) \log \frac{k}{r}$$

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Here $V_k$ denotes the volume of the unit sphere $\{x \in \mathbb{R}^k : \|x\| \leq 1\}$ and $\Gamma(\cdot)$ denotes the Gamma function. While this lower bound is only tight for some special sources, it converges to the rate-distortion function as the allowed distortion $D$ tends to zero, provided that the source satisfies some regularity conditions [4, 5, 6, 7]. Thus, in this case the Shannon lower bound provides a good approximation of the rate-distortion function for small distortions.

To the best of our knowledge, the most general proof of the asymptotic tightness of the Shannon lower bound is due to Linder and Zamir [7]. While Linder and Zamir considered more general distortion measures, specialized to the norm-based distortion (3), they showed the following.

**Theorem 1** (Linder and Zamir [7, Cor. 1]). Suppose that $X$ has a probability density function (pdf) and that $h(X) > -\infty$. Further assume that there exists an $\alpha > 0$ such that $\mathbb{E}[\|X\|^\alpha] < \infty$. Then the Shannon lower bound is asymptotically tight, i.e.,

$$
\lim_{D \downarrow 0} \{R(D) - R_{\text{SLB}}(D)\} = 0.
$$

(5)

The theorem’s conditions are very mild and satisfied by the most common source distributions. In fact, Theorem 1 demonstrates that the Shannon lower bound is asymptotically tight even if there exists no quantizer with a finite number of codevectors and of finite distortion, i.e., when $\mathbb{E}[\|X\|^\alpha] = \infty$. That said, the conditions are more stringent than the ones sometimes required to analyze the rate and distortion redundancies of high-resolution quantizers. For example, in [8] Gray et al. analyze the asymptotic distortion of entropy-constrained vector quantization in the limit as the rate tends to infinity, thereby rigorously proving a theorem by Zador [9]. In their work, they consider source vectors $X$ that have a density, whose differential entropy is finite, and that satisfy

$$
H([X]) < \infty.
$$

(6)

Here we use $\lfloor a \rfloor$ to denote the largest integer not larger than $a$ and $|a|$, $a \in \mathbb{R}$ denotes the vector with components $\{a_1, \ldots, a_k\}$. In words, condition (6) demands that quantizing the source with a cubic lattice quantizer with unit-volume cells gives rise to a discrete random vector of finite entropy, thereby ensuring that the output of the lattice quantizer can be further compressed using a lossless variable-length code of finite expected length.

The quantity $H([X])$ is intimately related with the Rényi information dimension [10], defined as

$$
d(X) \triangleq \lim_{m \to \infty} \frac{H([mX]/m)}{\log m},
$$

(7)

which in turn coincides with the rate-distortion dimension introduced by Kawabata and Dembo [11]; see also [12]. Indeed, generalizing Proposition 1 in [12] to the vector case, it can be shown that the Rényi information dimension is finite if, and only if, condition (6) is satisfied. It can be further shown that a sufficient condition for finite Rényi information dimension is $\mathbb{E}[\log(1 + \|X\|)] < \infty$, which in turn holds for any source vector for which $\mathbb{E}[\|X\|^\alpha] < \infty$ for some $\alpha > 0$. Thus, condition (6) is weaker than the assumption that $\mathbb{E}[\|X\|^\alpha] < \infty$.

It is common to assume that the differential entropy of the source vector is finite, since otherwise the Shannon lower bound (4) is uninteresting. We next briefly discuss how (6) and the assumption of a finite differential entropy are related. Indeed, as demonstrated for example in the proof of Theorem 3 in [13], the condition (6) implies that $h(X) < \infty$. In fact, one can show that if (6) holds and the random vector $X$ has a pdf, then $h(X) \leq H([X])$ [14, Cor. 1]. Conversely, one can find source vectors for which the differential entropy is finite but $H([X])$ is infinite. For example, consider a one-dimensional source with pdf

$$
f_X(x) = \sum_{m=2}^{\infty} p_m m \mathbb{1}\left\{m \leq x < m + \frac{1}{m}\right\}, \quad x \in \mathbb{R}
$$

(8)

where

$$
p_m = \frac{1}{K m \log^2 m}, \quad m = 2, 3, \ldots
$$

(9a)

$$
K = \sum_{m=2}^{\infty} \frac{1}{m \log^2 m}
$$

(9b)
and \( \mathbb{1}\{\cdot\} \) denotes the indicator function. It is easy to check that for such a source

\[
H(\{X\}) = \sum_{m=2}^{\infty} p_m \log \frac{1}{p_m} = \sum_{m=2}^{\infty} \frac{\log k + \log m + 2 \log \log m}{k \log^2 m} = \infty
\]

and

\[
h(X) = - \int_{\mathbb{R}} f_X(x) \log f_X(x) \, dx = \sum_{m=2}^{\infty} \frac{\log m + 2 \log \log m}{km \log^2 m} < \infty.
\]

(See remark after Theorem 1 in [10, pp. 197–198].)

In this paper, we demonstrate that for sources that have a pdf and whose differential entropy is finite, the Shannon lower bound (4) is asymptotically tight if, and only if, (6) is satisfied. This ensures the asymptotic tightness of the Shannon lower bound under the most general conditions required in the analysis of high-resolution quantizers.

## 2 Problem Setup and Main Result

We consider a \( k \)-dimensional, real-valued source \( X \) with support \( \mathcal{X} \subseteq \mathbb{R}^k \) whose distribution is absolutely continuous with respect to the Lebesgue measure, and we denote its pdf by \( f_X \). We further assume that \( x \mapsto f_X(x) \log f_X(x) \) is integrable, ensuring that

\[
h(X) \triangleq - \int_{\mathcal{X}} f_X(x) \log f_X(x) \, dx
\]

is well-defined and finite. We have the following result.

**Theorem 2** (Main Result). Suppose that the \( k \)-dimensional, real-valued source \( X \) has a pdf and that \( h(X) > -\infty \). If \( H(\{X\}) < \infty \), then the Shannon lower bound is asymptotically tight, i.e.,

\[
\lim_{D \to 0} \left\{ R(D) - R_{\text{SLB}}(D) \right\} = 0.
\]

Conversely, if \( H(\{X\}) = \infty \), then \( R(D) = \infty \) for any \( D > 0 \). Thus, the Shannon lower bound is asymptotically tight if, and only if, \( H(\{X\}) \) is finite.

**Proof.** See Section 3. \( \square \)

In all fairness, we should mention that Linder and Zamir derived conditions for the asymptotic tightness of the Shannon lower bound that are weaker than the ones presented in Theorem 1. Specifically, they showed that the Shannon lower bound is asymptotically tight if \( X \) has a pdf, if \( h(X) > -\infty \), and if there exists a function \( \delta : \mathbb{R}^k \to [0, \infty) \) satisfying the following [7, Th. 1]:

(a) The equations

\[
a(D) \int_{\mathbb{R}^k} e^{-s(D)\delta(x)} \, dx = 1 \quad \text{(14a)}
\]

\[
a(D) \int_{\mathbb{R}^k} \delta(x)e^{-s(D)\delta(x)} \, dx = D \quad \text{(14b)}
\]

have a unique pair of solutions \((a(D), s(D))\) for all \( D > 0 \). Moreover, \( a(D) \) and \( s(D) \) are continuous functions of \( D \).

(b) Let \( Z_D \) be a random vector independent of \( X \) with pdf

\[
f_{Z_D}(z) = \left( \frac{k}{r} \right)^{\frac{k}{r}-1} \frac{1}{V_k \Gamma (k/r) D^{\frac{k}{r}}} e^{-\frac{r}{k} ||z||^2}, \quad z \in \mathbb{R}^k.
\]

We have \( 0 < \mathbb{E}[\delta(X)] < \infty \) and \( \mathbb{E}[\delta(X + Z_D)] \to \mathbb{E}[\delta(X)] \) as \( D \) tends to zero.

It is unclear whether there exists a function \( \delta(\cdot) \) with the above properties that allows us to prove the asymptotic tightness of the Shannon lower bound for all source vectors \( X \) having a finite differential entropy and satisfying (6). In fact, even if there existed such a function, proving that it satisfies the required conditions may be involved. Fortunately, the existence of such a function is not essential. Indeed, the proof of Theorem 2 follows closely the proof of Theorem 1 in [7] but avoids the use of \( \delta(\cdot) \).
3 Proof of Theorem 2

The proof consists of two parts. In the first part, we show that if $H(\lfloor X \rfloor) < \infty$, then the Shannon lower bound is asymptotically tight (Section 3.1). In the second part, we show that if $H(\lfloor X \rfloor) = \infty$, then $R(D) = \infty$ for any $D > 0$ (Section 3.2).

3.1 Asymptotic Tightness

In this section, we demonstrate that if $H(\lfloor X \rfloor) < \infty$ and $h(X) > -\infty$, then the Shannon lower bound is asymptotically tight. The first steps in our proof are identical to the ones in the proof of Theorem 1 in [7]. To keep this article self-contained, we reproduce all the proof steps.

To prove asymptotic tightness of the Shannon lower bound $R_{SLB}(D)$, we derive an upper bound on $R(D)$ whose gap to $R_{SLB}(D)$ vanishes as $D$ tends to zero. In view of (2), an upper bound on $R(D)$ follows by choosing $\hat{X} = X + Z_D$, where $Z_D$ is a $k$-dimensional, real-valued, random vector that is independent of $X$ and has pdf (15), namely,

$$f_{Z_D}(z) = a(D)e^{-s(D)\|z\|^r}, \quad z \in \mathbb{R}^k$$

where

$$s(D) \triangleq \frac{k}{rD}$$
$$a(D) \triangleq \frac{r}{kV_k \Gamma(k/r)}.$$  \hspace{1cm} (17a, 17b)

It can be shown that the random vector $Z_D$ satisfies $E[\|Z_D\|^r] = D$; see, e.g., [3, Sec. VI]. It follows that

$$R(D) \leq I(X; X + Z_D) = h(X + Z_D) - h(X + Z_D|Z_D) = h(X + Z_D) - h(Z_D).$$ \hspace{1cm} (18)

Furthermore, by evaluating $h(Z_D)$ and comparing the result with (4),

$$R_{SLB}(D) = h(X) - h(Z_D).$$ \hspace{1cm} (19)

Combining (18) and (19) gives

$$0 \leq R(D) - R_{SLB}(D) \leq h(X + Z_D) - h(X).$$ \hspace{1cm} (20)

Thus, asymptotic tightness of the Shannon lower bound follows by proving that

$$\lim_{D \uparrow 0} h(X + Z_D) \leq h(X).$$ \hspace{1cm} (21)

To this end, we follow (17)–(21) in [7] but with $Y_{\Delta(D)}$ and $Y_{\Delta(0)}$ replaced by the random vectors $Y_D$ and $Y_0$ with respective pdfs

$$f_{Y_D}(y) = \sum_{i \in \mathbb{Z}^k} \Pr(X + Z_D \in V(i)) \mathbb{1}\{y \in V(i)\}$$
$$f_{Y_0}(y) = \sum_{i \in \mathbb{Z}^k} \Pr(X \in V(i)) \mathbb{1}\{y \in V(i)\}$$ \hspace{1cm} (22a, 22b)

where $V(i)$ denotes the $k$-dimensional, length-1 cube whose lower-most corner is located at $i = (i_1, \ldots, i_k)$, i.e.,

$$V(i) \triangleq [i_1, i_1 + 1) \times \cdots \times [i_k, i_k + 1).$$ \hspace{1cm} (23)

It follows immediately that

$$D(f_{X+Z_D}||f_{Y_D}) = H(\lfloor X + Z_D \rfloor) - h(X + Z_D).$$ \hspace{1cm} (24)
and
\[ D(f_X\|f_{Y_D}) = H(|X|) - h(X). \] \hfill (25)

The random vector \( Z_D \) has the same pdf as \( D^{1/2} Z_1 \), where \( Z_1 \) denotes \( Z_D \) for \( D = 1 \). Consequently, \( Z_D \to 0 \) almost surely as \( D \) tends to zero (where \( 0 \) denotes the all-zero vector) and, hence, also in distribution. Since \( X \) and \( Z_D \) are independent, it follows that \( X + Z_D \to X \) in distribution as \( D \) tends to zero. Furthermore, since the distribution of \( X \) is absolutely continuous with respect to the Lebesgue measure, for every \( i \in \mathbb{Z}^k \) the set \( V(i) \) is a continuity set of \( X \), so
\[ \lim_{D \downarrow 0} \Pr(X + Z_D \in V(i)) = \Pr(X \in V(i)), \quad i \in \mathbb{R}^k. \] \hfill (26)

We thus conclude that the pdf of \( Y_D \) converges pointwise to the pdf of \( Y_0 \), which by Scheffe’s lemma [15, Th. 16.12] implies that \( Y_D \to Y_0 \) in distribution as \( D \) tends to zero.

By the lower semicontinuity of relative entropy (see, e.g., proof of Lemma 4 in [16] and references therein), it follows that
\[ \lim_{D \downarrow 0} D(f_X + Z_D \| f_{Y_D}) \geq D(f_X \| f_{Y_0}). \] \hfill (27)

Combining (27) with (24) and (25) yields
\[ \lim_{D \downarrow 0} \{ H(|X + Z_D|) - h(X + Z_D) \} \geq H(|X|) - h(X). \] \hfill (28)

Since \( h(X) > -\infty \) and \( H(|X|) < \infty \), the claim (21) follows by showing that \( H(|X + Z_D|) \) tends to \( H(|X|) \) as \( D \) tends to zero. We shall present this result in the following lemma, which is proven in Appendix A.

**Lemma 1.** Suppose the random vector \( X \) has a pdf and satisfies \( H(|X|) < \infty \). Let \( Z_D \) have pdf (16). Then
\[ \lim_{D \downarrow 0} H(|X + Z_D|) = H(|X|). \] \hfill (29)

**Proof.** See Appendix A. \( \square \)

Combining Lemma 1 with (28) implies (21), which in turn demonstrates that the Shannon lower bound is tight if \( H(|X|) < \infty \) and \( h(X) > -\infty \).

### 3.2 Infinite Rate-Distortion Function

To prove that if \( H(|X|) = \infty \) then \( R(D) = \infty \) for every \( D > 0 \), we follow along the lines of the proof of Theorem 6 in [14, App. A]. Indeed, by the Data Processing Inequality [17, Lemma 7.16], we have
\[ I(X; \hat{X}) \geq I(|X|; |\hat{X}|). \] \hfill (30)

Furthermore, by Lemma 7.20 in [17], it follows that the mutual information on the right-hand side (RHS) of (30) can be written as
\[ I(|X|; |\hat{X}|) = H(|X|) - H(|X| \mid |\hat{X}|). \] \hfill (31)

Since \( H(|X|) = \infty \) by assumption, the claim follows by showing that the second entropy on the RHS of (31) is bounded for all \((X, \hat{X})\) satisfying (3). Indeed, we have
\[ H(|X| \mid |\hat{X}|) \leq H(|X - \hat{X}|) + H(|X| \mid \hat{X}, |X - \hat{X}|). \] \hfill (32)

\footnote{Strictly speaking, Lemma 7.16 in [17] requires that \( H(|X|) \) be finite which, by assumption, is not true. However, it can be shown that (31) continues to hold for infinite \( H(|X|) \) as long as \( H(|X| \mid |X|) \) is finite which, as we show in (32)-(36), is indeed true. Alternatively, one could replace \(|X|\) by the same vector clipped to the hypercube \([-\Upsilon, \Upsilon]^k\). For every \( \Upsilon > 0 \), such a vector has a finite entropy, thereby satisfying the condition of Lemma 7.16 in [17]. The desired result would then follow by letting \( \Upsilon \) tend to infinity.}
Since for all \((X, \hat{X})\) satisfying (3) we have \(E\left[\log(1 + \|X - \hat{X}\|)\right] < \infty\), generalizing Proposition 1 in [12] to the vector case directly yields that

\[
H(|X - \hat{X}|) < \infty. \tag{33}
\]

Furthermore, denoting \(Y = X - \hat{X}\), we obtain

\[
H([X] \mid [\hat{X}], [X - \hat{X}]) = H([Y + \hat{X}] \mid [\hat{X}], [Y]). \tag{34}
\]

Since for each component of \(\hat{X}\) and \(Y\) we have

\[
|\hat{X}_\ell| + |Y_\ell| \leq |\hat{X}_\ell + Y_\ell| \leq |\hat{X}_\ell| + |Y_\ell| + 1 \tag{35}
\]

it follows that

\[
H([\hat{X} + Y] \mid [\hat{X}], [Y]) \leq k \log 2 \tag{36}
\]

(see also proof of Proposition 8 in [14]). Consequently, combining (32), (33), and (36) yields

\[
H([X] \mid [\hat{X}]) < \infty. \tag{37}
\]

Summing up, if \(H(|X|) = \infty\), then (30), (31), and (37) demonstrate that \(I(X; \hat{X}) = \infty\) for every pair of vectors \((X, \hat{X})\) satisfying (3). Hence, in this case the rate-distortion function \(R(D)\) is infinite for every finite \(D\). This proves the second part of Theorem 2.

### 4 Conclusions

The Shannon lower bound is one of the few lower bounds on the rate-distortion function that hold for a large class of sources. We have demonstrated that this lower bound is asymptotically tight as the allowed distortion vanishes for all source vectors having a finite differential entropy and a finite Rényi information dimension. Conversely, we have demonstrated that if the source vector has an infinite Rényi information dimension, then the rate-distortion function is infinite for any finite distortion. Thus, for source vectors of finite differential entropy, the Shannon lower bound is asymptotically tight if, and only if, the source has finite Rényi information dimension.

Assuming a finite Rényi information dimension is tantamount to assuming that quantizing the source vector with a cubic lattice quantizer with unit-volume cells gives rise to a discrete random vector of finite entropy. The latter assumption seems very natural in rate-distortion theory and is indeed often encountered. To this effect, we have demonstrated that this assumption is not only natural, but it is also sufficient for the asymptotic tightness of the Shannon lower bound. Conversely, we have demonstrated that all source vectors that do not satisfy this condition are uninteresting, since their rate-distortion function is infinite. One may thus argue that the Shannon lower bound is asymptotically tight for all sources that are of interest to us.

For ease of exposition, we have only considered norm-based difference distortion measures, which is slightly less general than the distortion measures studied by Linder and Zamir [7]. While our analysis could probably be generalized to more general distortion measures, we have refrained from doing so, because we believe that it would obscure the analysis without offering much more insight.

### A Proof of Lemma 1

We first note that, as demonstrated in Section 3.1 (cf. (26)), we have

\[
\lim_{D \downarrow 0} \Pr(X + Z_D \in V(i)) = \Pr(X \in V(i)), \quad i \in \mathbb{R}^k. \tag{38}
\]
Thus, by Fatou’s lemma [18, Th. 1.6.8, p. 50] and the continuity of $x \mapsto x \log(1/x)$,

$$\lim_{D \downarrow 0} H([X + Z_D]) = \lim_{D \downarrow 0} \sum_{i \in \mathbb{Z}^k} \Pr(X + Z_D \in \mathcal{V}(i)) \log \frac{1}{\Pr(X + Z_D \in \mathcal{V}(i))} \geq \sum_{i \in \mathbb{Z}^k} \lim_{D \downarrow 0} \Pr(X + Z_D \in \mathcal{V}(i)) \log \frac{1}{\Pr(X + Z_D \in \mathcal{V}(i))} = \sum_{i \in \mathbb{Z}^k} \Pr(X \in \mathcal{V}(i)) \log \frac{1}{\Pr(X \in \mathcal{V}(i))} = \sum_{i \in \mathbb{Z}^k} \Pr(X \in \mathcal{V}(i)) \log \frac{1}{\Pr(X \in \mathcal{V}(i))} \geq \sum_{i \in \mathbb{Z}^k} \lim_{D \downarrow 0} \Pr(X + Z_D \in \mathcal{V}(i)) \log \frac{1}{\Pr(X \in \mathcal{V}(i))} = H([X]).$$

(39)

To prove Lemma 1, it remains to show that

$$\lim_{D \downarrow 0} H([X + Z_D]) \leq H([X]).$$

(40)

To this end, we upper-bound

$$H([X + Z_D]) \leq H([X]) + H([X + Z_D] \mid [X]) \leq H([X]) + H(V_D)$$

(41)

where we define $V_D \triangleq [X + Z_D] - [X]$. We then prove (40) by showing that $H(V_D)$ vanishes as $D$ tends to zero. We next need the following two lemmas, whose proofs are deferred to Appendices A.1 and A.2.

**Lemma 2.** The random vector $V_D$ satisfies

$$\lim_{D \downarrow 0} \Pr(V_D = i) = \mathbb{1}\{i = 0\}.$$  

(42)

**Proof.** See Appendix A.1.  

**Lemma 3.** The random vector $V_D$ satisfies

$$\lim_{D \downarrow 0} \sum_{i \not\in \{-1,0,1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)} = 0.$$  

(43)

**Proof.** See Appendix A.2.

We continue by expressing the entropy as

$$H(V_D) = \sum_{i \in \{-1,0,1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)} + \sum_{i \not\in \{-1,0,1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)}$$

(44)

and evaluating the terms on the RHS of (44) using Lemmas 2 and 3. Indeed, by Lemma 3 the second term on the RHS of (44) vanishes as $D$ tends to zero, so

$$\lim_{D \downarrow 0} H(V_D) = \lim_{D \downarrow 0} \sum_{i \in \{-1,0,1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)}.$$  

(45)

Since the sum on the RHS of (45) consists of finitely many terms, it further follows that

$$\lim_{D \downarrow 0} \sum_{i \in \{-1,0,1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)} = \sum_{i \in \{-1,0,1\}^k} \lim_{D \downarrow 0} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)} = 0,$$  

(46)

where the last step follows from Lemma 2 and the continuity of $x \mapsto x \log(1/x)$. Hence, (45) and (46) combine to show that

$$\lim_{D \downarrow 0} H(V_D) = 0$$

(47)

which together with (41) proves (40). Combining (40) with (39) proves Lemma 1.

\[\square\]
A.1 Proof of Lemma 2

Let $X \triangleq X - |X|$. Recall that $Z_D \to 0$ in distribution as $D$ tends to zero. Since $X$ and $Z_D$ are independent, this implies that $X + Z_D \to X$ in distribution as $D$ tends to zero.

Recalling that $V_D = |X + Z_D| - |X|$, the probability mass function of $V_D$ can be written as

$$\Pr(V_D = i) = \Pr(\bar{X} + Z_D \in \mathcal{V}(i)), \quad i \in \mathbb{Z}^k. \quad (48)$$

Since the distribution of $X$ is absolutely continuous with respect to the Lebesgue measure, so is the distribution of $X$. Consequently, for every $i \in \mathbb{Z}^k$ the set $\mathcal{V}(i)$ is a continuity set of $X$ and

$$\lim_{D \downarrow 0} \Pr(\bar{X} + Z_D \in \mathcal{V}(i)) = \Pr(\bar{X} \in \mathcal{V}(i)) = 1 \{i = 0\} \quad (49)$$

where the last step follows since, by definition of $\bar{X}$, the support of $\bar{X}$ is $\mathcal{V}(0)$. This proves Lemma 2.

A.2 Proof of Lemma 3

We first note that the left-hand side (LHS) of (43) is nonnegative. It thus remains to show that

$$\lim_{D \downarrow 0} \sum_{i \notin \{-1, 0, 1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)} \leq 0. \quad (50)$$

To this end, we first use that on a finite-dimensional vector space any two norms are within a constant factor of one another [19, p. 273]. Consequently, we have

$$c_U \|z\|_1 \leq \|z\| \leq c_L \|z\|_1 \quad (51)$$

for some finite constants $c_U \geq c_L > 0$, where $\|z\|_1 \triangleq |z_1| + \ldots + |z_k|$ denotes the $L^1$-norm. We can thus lower-bound the pdf of $Z_D$ by

$$f_{Z_D}(z) \geq a(D) e^{-s(D)c_U \|z\|_1^r}, \quad z \in \mathbb{R}^k. \quad (52)$$

We then note that every vector in $\{z \in \mathbb{R}^k : \bar{X} + z \in \mathcal{V}(i), \bar{X} \in \mathcal{V}(0)\}$ satisfies $\|z\|_1 \leq \|i\| + 1_1$, where $\|i\|$ denotes the component-wise absolute value of $i$ and $1_1$ denotes the all-one vector. For every $z \in \mathcal{V}(i)$, we can thus further lower-bound (52) by

$$f_{Z_D}(z) \geq a(D) e^{-s(D)c_U \|i\| + 1_1 \|z\|_1^r}, \quad z \in \mathcal{V}(i), i \in \mathbb{Z}^k. \quad (53)$$

Together with (48) and the fact that $\mathcal{V}(i)$ has volume 1, this yields

$$\Pr(V_D = i) = \Pr(\bar{X} + Z_D \in \mathcal{V}(i)) \geq a(D) e^{-s(D)c_U \|i\| + 1_1 \|z\|_1^r}, \quad i \in \mathbb{Z}^k. \quad (54)$$

Applying (54) to the LHS of (50) gives

$$\sum_{i \notin \{-1, 0, 1\}^k} \Pr(V_D = i) \log \frac{1}{\Pr(V_D = i)} \leq \sum_{i \notin \{-1, 0, 1\}^k} \Pr(V_D = i) \log a(D)$$

$$+ c_U s(D) \sum_{i \notin \{-1, 0, 1\}^k} \Pr(V_D = i) \|i\| + 1_1 \|z\|_1^r. \quad (55)$$

To upper-bound the first term on the RHS of (55), we use that any vector in

$$\bigcup_{i \notin \{-1, 0, 1\}^k} \{z \in \mathbb{R}^k : \bar{X} + z \in \mathcal{V}(i), \bar{X} \in \mathcal{V}(0)\}$$

satisfies $\|z\|_1 \geq 1$. Consequently,

$$\sum_{i \notin \{-1, 0, 1\}^k} \Pr(V_D = i) = \sum_{i \notin \{-1, 0, 1\}^k} \Pr(\bar{X} + Z_D \in \mathcal{V}(i))$$

$$\leq \Pr(\|Z_D\|_1 \geq 1) \leq \mathbb{E}\|Z_D\|_1^r \leq \frac{D}{c_L}. \quad (56)$$
where the second-to-last inequality follows by Chebyshev’s inequality \[18, \text{Th. 4.10.7, p. 192}\], and the last inequality follows by \((51)\) and by noting that, by definition, \(E[\|Z_D\|^r] = D\). Together with the definitions of \(s(D)\) and \(a(D)\), i.e., \((17a)\) and \((17b)\), this yields

\[
\sum_{i \notin \{-1,0,1\}^k} \Pr(V_D = i) \log a(D) \leq \frac{D}{c_L} \left| \log \left(k/r\right)^{k/r - 1} - \frac{k}{r} \log D \right| \tag{57}
\]

which vanishes as \(D\) tends to zero.

We next consider the second term on the RHS of \((55)\). To this end, we write

\[
\Pr(V_D = i) \|i\| + 1 \|^r \geq \int_{\mathcal{V}(i)} f_{\bar{X} + Z_D}(\bar{X}) \|i\| + 1 \|^r \, d\bar{X}
\]

where \(f_{\bar{X} + Z_D}\) denotes the pdf of \(\bar{X} + Z_D\). Noting that \(\bar{X}\) and \(Z_D\) are independent and that \(\bar{X}\) has a pdf (cf. proof of Lemma 2), which we shall denote by \(f_{\bar{X}}\), it follows that \([20, \text{Th. 4.10, p. 29}]\)

\[
f_{\bar{X} + Z_D}(\xi) = \int_{\mathcal{V}(0)} f_{\bar{X}}(\bar{x}) f_{Z_D}(\xi - \bar{x}) \, d\bar{x}, \quad \xi \in \mathbb{R}^k.
\tag{59}
\]

By Fubini’s theorem \([18, \text{Th. 2.6.4, p. 105}]\), we thus obtain from \((58)\) and \((59)\) that

\[
\sum_{i \notin \{-1,0,1\}^k} \Pr(V_D = i) \|i\| + 1 \|^r = \int_{\mathcal{V}(i)} \int_{\mathcal{V}(0)} f_{\bar{X}}(\bar{x}) f_{Z_D}(\xi - \bar{x}) \|i\| + 1 \|^r \, d\bar{x} \, d\xi
\]

\[
= \int_{\mathcal{V}(i)} f_{\bar{X}}(\bar{x}) \sum_{i \notin \{-1,0,1\}^k} \int_{\mathcal{V}(i - \bar{x})} f_{Z_D}(z) \|i\| + 1 \|^r \, dz \, d\bar{x} \tag{60}
\]

where \(\mathcal{V}(i - \bar{x})\) denotes the \(k\)-dimensional, length-1 cube whose lower-most corner is located at \(i - \bar{x}\).

Since for every \(i \notin \{-1,0,1\}^k\), \(z \in \mathcal{V}(i - \bar{x})\), and \(\bar{x} \in \mathcal{V}(0)\), we can upper-bound \(\|i\| + 1 \|_1\) by \(3\|z\|_1\), it follows that

\[
\sum_{i \notin \{-1,0,1\}^k} \int_{\mathcal{V}(i - \bar{x})} f_{Z_D}(z) \|i\| + 1 \|^r \, dz \leq \sum_{i \notin \{-1,0,1\}^k} \int_{\mathcal{V}(i - \bar{x})} f_{Z_D}(z) 3^r |z|^r \, dz
\]

\[
\leq 3^r \int_{\|z\|_1 \geq 1} f_{Z_D}(z) |z|^r \, dz \tag{61}
\]

where the last inequality follows because any vector in \(\bigcup_{i \notin \{-1,0,1\}^k} \mathcal{V}(i - \bar{x})\) satisfies \(\|z\|_1 \geq 1\). Noting that the RHS of \((61)\) does not depend on \(\bar{x}\), so it follows from \((60)\) and \((61)\) that

\[
\sum_{i \notin \{-1,0,1\}^k} \Pr(V_D = i) \|i\| + 1 \|^r \leq 3^r E[\|Z_D\|^r \mathbb{I}\{\|Z_D\|_1 \geq 1\}] \tag{62}
\]

Using \((51)\) and that \(Z_D = D^{1/r} Z_1\), the RHS of \((62)\) can be further upper-bounded by

\[
3^r E[\|Z_D\|^r \mathbb{I}\{\|Z_D\|_1 \geq 1\}] \leq 3^r \frac{D}{c_L} E \left[\|Z_1\|^r \mathbb{I}\{\|Z_1\| \geq \frac{1}{D^r c_L}\}\right] \tag{63}
\]

Combining \((60)\)–\((63)\) with the definition of \(s(D)\), the second term on the RHS of \((55)\) can be upper-bounded by

\[
\sum_{i \notin \{-1,0,1\}^k} \Pr(V_D = i) \|i\| + 1 \|^r \leq \left(\frac{3c_L}{c_L}\right)^r \frac{k}{r} E \left[\|Z_1\|^r \mathbb{I}\{\|Z_1\| \geq \frac{1}{D^r c_L}\}\right]. \tag{64}
\]

Since the function \(z \mapsto \|z\|^r \mathbb{I}\{\|z\| \geq 1/(D^{1/r} c_L)\}\) is dominated by \(z \mapsto \|z\|^r\), and since \(E[\|Z_1\|^r] = 1\), it follows from the Dominated Convergence Theorem \([18, \text{Th. 1.6.9, p. 50}]\) that

\[
\lim_{D \downarrow 0} E \left[\|Z_1\|^r \mathbb{I}\{\|Z_1\| \geq \frac{1}{D^r c_L}\}\right] = 0. \tag{65}
\]

Thus, \((55)\), \((57)\), \((64)\), and \((65)\) combine to prove \((50)\), which in turn proves Lemma 3.
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