Adjoint characteristics for Eulerian two-dimensional supersonic flow

Jacques Peter*\textsuperscript{a,b}, Jean-Antoine Désidéri\textsuperscript{b}

\textsuperscript{a}DAAA, ONERA, Université Paris Saclay, F-92322 Châtillon, France
\textsuperscript{b}Centre Inria Université Côte d’azur, Inria, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis, France

Abstract
The formal expressions, in terms of local flow variables, defining the adjoint characteristic curves, and the associated compatibility relationships satisfied along them, are formally derived in the case of a supersonic flow governed by the compressible Euler equations in two dimensions. These findings extend their well-known counterparts for the direct system, and should serve analytical and possibly numerical studies of the perfect-flow model with respect to adjoint fields or sensitivity questions. Beside the analytical theory, the results are demonstrated by the numerical integration over a very fine grid of the compatibility relationships for discrete flow-fields and dual-consistent adjoint fields.

Key words: continuous adjoint method, compressible Euler equations, supersonic flow, characteristic curves

1. Introduction

In 1988, Jameson derived the continuous adjoint equations associated with the 2D and 3D Euler equations using general curvilinear coordinates \cite{1}. With this landmark article, the fluid dynamics and aeronautical communities became better aware of the potential of the adjoint approach for design, that is, the possibility to calculate gradient information at a cost scaling with the number of functions to be differentiated, independently of the number of design parameters. The equations in \cite{1} appeared to be a natural starting point for local optimizations involving a large number of design variables by using adjoint gradients. However, in that setting, the flow and the dual fields had to be calculated over a structured mesh.

Nine years later, Anderson and Venkatakrishnan \cite{2, 3} and also Giles and Pierce derived the corresponding equations in Cartesian coordinates \cite{4} thus allowing the application of the continuous approach (sometimes referred to as the differentiate-then-discretize approach) on all types of meshes and, in particular, on unstructured meshes. Note that in the context of unstructured-grid fluid-flow computations, gradient-based optimum design has also been discussed in different textbooks, in particular in \cite{5}.

For the sake of simplicity, we present here the two-dimensional case only in which the adjoint equations read

\[-A^T \frac{\partial \psi}{\partial x} - B^T \frac{\partial \psi}{\partial y} = 0, \quad \text{in } \Omega \text{ the fluid domain} \]

(1)

where \(A\) and \(B\) are the Jacobian matrices of the flux vectors \(F_x\) and \(F_y\) of the Euler equations in the \(x\) and \(y\) directions respectively. In the most common case where the function interest is a line integral along the solid wall \(\Gamma_w\), it can be shown easily that the adjoint wall boundary condition is well-posed provided that the function of interest depends only on the static pressure \cite{3}. In the classical case where the functional output of interest is the force on \(\Gamma_w\) projected in direction \(\vec{d}\), \(J = \int_{\Gamma_w} p(\vec{n} \cdot \vec{d}) ds\), the wall boundary condition reads

\[\vec{n} \cdot \vec{d} + \psi_2 n_x + \psi_3 n_y = 0 \quad \text{on } \Gamma_w.\]

(2)
For the farfield of an external flow, as well as for the inlet and outlet of an internal flow, the boundary conditions are derived from the theory of local one-dimensional characteristic decomposition [2], [5]. Here, the continuous adjoint Euler equations and the associated boundary conditions are abbreviated as (AE). Along with the growing use of the adjoint method for shape optimization, goal oriented mesh adaptation and also meta-modelling, stability or control, great effort is being devoted to gain understanding in the mathematical properties of the (AE) solutions. The main results are summarized here before discussing the characteristic relations for the (AE) system.

After the derivation of the (AE) equations, the first demonstrated property was also due to Giles and Pierce [4]: in the common case where the function of interest is an integral along the wall, the authors proved that the first and last components of the adjoint vector \( \psi \), associated with mass and energy conservation, satisfy \( \psi_1 = H \psi_4 \) in which \( H \) is the total enthalpy.

Besides, the integration by parts yielding (1) is not valid in the entire fluid domain in the presence of flow discontinuities. After a series of works dealing with the quasi-1D Euler equations – see [6, 7] and references therein – Baeza et al. presented the equations complementing (1) along a shock line [7] (denoted here \( \Sigma \) as in the original reference). The new equations are derived by introducing a complementary set of Lagrange multipliers, multiplying the Rankine-Hugoniot conditions, viewed as constraints on \( \Sigma \). Finally, the continuity of the adjoint field \( \psi \) along \( \Sigma \) is established, although \( \nabla \psi \) may be discontinuous across \( \Sigma \), as well as \( \psi \) over \( \Gamma_w \cap \Sigma \), and a so-called internal boundary condition is derived:

\[
(\partial \psi / \partial t) (F_{1,t_x} + F_{1,t_y}) = 0 \quad \text{on} \quad \Sigma \quad \text{(with t the unit vector tangent to} \Sigma). \quad (3)
\]

Lozano [8] and Renac [9] have derived additional relationships by using (1), the jump operator applied to (1) across \( \Sigma \) and the Rankine-Hugoniot equations.

The fact that \( \psi_1 = H \psi_4 \) can be proven simply by forming the linear combination the first three lines of system (1) with coefficients \((1, u/2, v/2)\). This yields \( U \cdot \nabla \psi_1 - H U \cdot \nabla \psi_4 = 0 \) (with \( U = (u, v) \) the velocity vector). Note that this was also derived in [4] by an approach based on physical source terms, constituting an important analysis technique for the adjoint field of usual functional outputs. In particular, this method proved to be very fruitful to identify the zones where numerical divergence of the adjoint vector is observed and mathematical divergence of the solutions of (AE) is suspected. For the sake of clarity and brevity, we restrict the present discussion to 2D flows about lifting airfoils, and to two of these zones, namely the stagnation streamline and the wall, and to the lift and drag as functions of interest.

More precisely, Giles and Pierce [4] introduced four physical punctual source terms (or Green’s functions in the classical mathematical vocabulary) denoted here \( \delta R^1, \delta R^2, \delta R^3, \delta R^4 \). These terms are added to the right hand-side of the linearised Euler equations and correspond respectively to (i) a mass source at fixed stagnation pressure \( p_0 \) and enthalpy \( H \); (ii) a normal force ; (iii) a change in \( H \) at fixed static pressure \( p \) and \( p_0 \); and (iv) a change in \( p_0 \) at fixed \( p \) and \( H \). They are linearly independent (We refer to the original reference for the detailed expression of these source terms.) The resulting changes in the output function \( J, \delta J \), can be expressed as the integral over the domain of \( \psi \delta R^l \) that is, the value at the source location since \( \delta R^l \) is a Green’s function. These source terms also admit a physical interpretation and their influence on the flow can be understood in terms of mechanical principles, and sometimes even quantified. Combining the two points of view can be useful: for example, since \( \delta R^3 \) does not perturb the static pressure field, regardless the source location, \( \delta J^3 = \psi \delta R^3 = 0 \) and that is equivalent to \( \psi_1 = H \psi_4 \); in the vicinity of the stagnation streamline, since \( \delta R^1, \delta R^2 \) do not involve convective phenomena with perturbations transported exclusively to the pressure or suction side, \( \psi \delta R^1 \) and \( \psi \delta R^2 \) are continuous across the line whereas the individual adjoint components may exhibit numerical divergence with opposite signs across the stagnation streamline.

It has been observed that the lift adjoint exhibits numerical divergence along the stagnation streamline and at the wall at subcritical flow conditions. Also the drag and lift adjoint of a transonic airfoil exhibit numerical divergence at the same locations if the foot of at least one shock wave is located strictly upwind the trailing edge – see [10] and references therein. The reference [9] includes a careful verification of this physical perturbations approach applied to the discrete adjoint with a preliminary assessment of the consistency between the linear (discrete adjoint) and the non-linear (flow perturbation) evaluations of \( \delta J \). After this verification step, the non-linear perturbed flow approach has been used (considering the physical source terms point of view prior to the classical adjoint) and it appeared that: (a) \( \delta R^3 \) is the only source term causing a numerical divergence of \( \delta J \) in the vicinity of the wall and stagnation streamline ; (b) in transonic condition, the numerical divergence of \( \delta C_l p^3 \) and \( \delta C_D p^3 \) in these zones is...
mainly due to the displacement of the shockfoot (or shock-feet if two shocks are not based at the trailing edge); (c) this numerical divergence is transferred to the adjoint components via the inverse matrix of the source terms; (d) this does not necessarily prevent the numerical satisfaction of the adjoint lift- (resp. drag-) boundary condition at the wall [12] as the calculation of \( \psi n_x + \psi n_y \) in this approach involves the product of \( \delta CLp^* \) (resp. \( \delta CDP^* \)) by \( (un_x + vn_y) \).

The method of characteristics for 2D inviscid supersonic flows is a classical method for deriving ordinary differential equations and, eventually, explicit algebraic relations satisfied along two families of curves, denoted \( \mathcal{C}^+ \) (left running with respect to a streamline) and \( \mathcal{C}^- \) (right running). These equations may be discretized to compute a supersonic flow using the transport of physical variables along the streamlines, the \( \mathcal{C}^+ \) and the \( \mathcal{C}^- \). Here, we recall the derivation of the continuous equations and study their counterparts for the (AE) equations.

The earliest presentation of the direct problem seems to be attached to irrotational flows for which a 3 × 3 linear system is posed involving one mechanical equation and two Taylor expansions for the calculation of the velocity derivatives between two neighboring points (denoted here \( a \) and \( b \) of the fluid domain ([12] eq. (11.5) to (11.7) [13]). An extension to rotational flow is also possible (see [14] chap. 1) with three mechanical equations and three Taylor formulas for the derivatives of the static pressure, entropy and streamline angle. In both cases the derivatives calculation problem appears to be singular if \( b \) is on the streamline of \( a \) or if the angle of \( ab = (dx, dy) \) with respect to streamline passing through \( a \) is \( \pm \arcsin(1/M) \) (\( M \) being the local Mach number). The physical existence and boundedness of the vector of unknowns allows to conclude from the nullity of the determinant in the denominator of Cramers formulas, to the nullity of the determinants appearing in the numerators, and this, for all the variables. This permits to prove the well-known equations

\[
k^- = \phi + v(M) \text{ is constant along a } \mathcal{C}^- \quad \quad k^+ = \phi - v(M) \text{ is constant along a } \mathcal{C}^+ \tag{4}
\]

in which \( \phi \) is the streamline angle, \( v(M) \) the Prandtl-Meyer function,

\[
\phi = \tan^{-1}(v/u), \quad v(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1}\left( \sqrt{\frac{\gamma - 1}{\gamma + 1}(M^2 - 1)} \right) - \tan^{-1}\left( \sqrt{M^2 - 1} \right),
\]

and \( \gamma \) the ratio of specific heats (\( \gamma = \frac{7}{5} \) for diatomic perfect gas). In the more general presentation where rotational flow can be dealt with [14], the constant-entropy property of the streamtrace is part of the 6x6 system posed to derive the characteristic relations whereas the constant total enthalpy property is supposed to be used besides, to recover the velocity magnitude from the static pressure ([14] §1.3). However, note that the 2D characteristic equations could also be calculated in an inexpert way, without taking advantage of the known properties of the streamtraces. Then the following the 8 × 8 linear system relating the derivatives of the conservative variables would be solved:

\[
\begin{bmatrix}
dx & 0 & 0 & 0 & dy & 0 & 0 & 0 \\
0 & dx & 0 & 0 & dy & 0 & 0 & 0 \\
0 & 0 & dx & 0 & 0 & dy & 0 & 0 \\
0 & 0 & 0 & dx & 0 & 0 & dy & 0 \\
\end{bmatrix}
\begin{bmatrix}
(\partial \rho / \partial x) \\
(\partial \rho u / \partial x) \\
(\partial \rho v / \partial x) \\
(\partial \rho / \partial y) \\
(\partial \rho u / \partial y) \\
(\partial \rho v / \partial y) \\
(\partial \rho E / \partial y) \\
\end{bmatrix}
= \begin{bmatrix}
\rho^b - \rho^a \\
p u^b - p u^a \\
p v^b - p v^a \\
p E^b - p E^a \\
0. \\
0. \\
0. \\
0. \\
\end{bmatrix}
\tag{5}
\]

The starting point of our analytical development resides in the observation that (5) and the corresponding linear system for the (AE) equations, ([6], have the same determinant. From this observation, the adjoint Euler characteristics equations are established in §2. The theoretical findings are illustrated by numerical computational solutions over a very fine grid in §3. Conclusions are drawn in §4.

2. Adjoint characteristic equations for 2D supersonic flow

The method exposed in [12] (resp. [14]) for potential (resp. general) inviscid flow has served as a guideline to our derivation for the adjoint system.
2.1. Problem statement

Given two fixed close points in the supersonic zone, $a$ and $b$, is it possible to estimate $(\partial \psi / \partial x), (\partial \psi / \partial y)$ from the local value of the flow field and $(\psi_a, \psi_b)$? This question is the starting point of the method of characteristics in which specific lines are identified along which this problem is ill-posed, and particular ordinary differential equations are satisfied. Let us denote $\vec{ab} = (dx, dy)$ and first assume that $dx \neq 0$. By definition of differential forms, and in view of the adjoint system \(1\), the following holds

\[
\begin{bmatrix}
dx & 0 & 0 & 0 \\
0 & dx & 0 & 0 \\
0 & 0 & dx & 0 \\
0 & 0 & 0 & dy \\
\end{bmatrix}
\begin{bmatrix}
(\partial \psi_i / \partial x) \\
(\partial \psi_i / \partial x) \\
(\partial \psi_i / \partial x) \\
(\partial \psi_i / \partial y) \\
\end{bmatrix} =
\begin{bmatrix}
d\psi_1 \\
d\psi_2 \\
d\psi_3 \\
d\psi_4 \\
\end{bmatrix}
\]

\(6\)

in which by neglecting second-order terms in space: $(d\psi_1, d\psi_2, d\psi_3, d\psi_4) = (\psi^0_i - \psi^1_i, \psi^0_j - \psi^2_j, \psi^0_k - \psi^3_k, \psi^0_l - \psi^4_l)$. The determinant of the linear system is evidently

\[
\begin{vmatrix}
dxI & dyI \\
-A^T & -B^T \\
\end{vmatrix} =
\begin{vmatrix}
dxI & 0 \\
-A^T & -B^T + dy/dxA^T \\
\end{vmatrix} =
\begin{vmatrix}
dx^4 | -B^T + dy/dxA^T | = | -dxB + dyA | \\
\end{vmatrix}
\]

Of course, $| -dxB + dyA |$ is equal to $| -dxB + dyA |$ and the value of this determinant is known from the eigenvalues of the matrix:

\[
D = | -dxB + dyA | = (-v dx + u dy)^2 (-v dx + u dy + c ds)(-v dx + u dy - c ds),
\]

in which

\[
c = \sqrt{\frac{TP}{\rho}}, \quad ds = \sqrt{dx^2 + dy^2}.
\]

Similarly to the flow derivatives reconstruction \[12, 14\], the problem of adjoint derivatives reconstruction in a supersonic zone is ill-posed along the same three families of curves

\[
\begin{align*}
-v dx + u dy &= 0 \quad \text{(trajectory)} \quad (7) \\
-v dx + u dy + c ds &= 0 \quad \text{($C^-$ characteristics)} \quad (8) \\
-v dx + u dy - c ds &= 0 \quad \text{($C^+$ characteristics)} \quad (9)
\end{align*}
\]

Classically, the method of characteristics uses the ill-posedness of \(6\) in the following way: along the curves defined by equations \(7\), \(8\) or \(9\), not only the denominator appearing in the Cramer formulas applied to the linear equations \(6\) is equal to zero, but the numerators giving the eight components of $(\partial \psi / \partial x)$ and $(\partial \psi / \partial y)$ must also be equal to zero for the fractions not be singular. This (somehow paradoxical) technique allows the derivation of equations \(4\). It is derived here for the (AE) system by analysing the \(6\) set of linear equations.

2.2. Numerators in Cramer’s formula for linear system \(6\)

The transposed of the Euler flux Jacobian matrices in $x$ and $y$ direction read

\[
A^T = \begin{bmatrix}
0 & \gamma_i E_v - u^2 - uv (\gamma_i E_v - H)u \\
1 & (3 - \gamma)u & v & H - \gamma_i u^2 \\
0 & -\gamma_i v & u & -\gamma_i uv \\
0 & \gamma_i 0 & \gamma_i u
\end{bmatrix}, \quad
B^T = \begin{bmatrix}
0 & -uv & \gamma_i E_v - v^2 (\gamma_i E_v - H)v \\
0 & v & -\gamma_i u & -\gamma_i uv \\
1 & u & (3 - \gamma)v & H - \gamma_i v^2 \\
0 & 0 & \gamma_i & \gamma_i v
\end{bmatrix}
\]
in the usual notations in aerodynamics and $\gamma_1 = \gamma -1$. Let

$$ t = \frac{dy}{dx}, \quad w = ut - v, $$

and also introduce the following notations for the column vectors of the transposed Jacobian matrices: $A^T = [A_1 | A_2 | A_3 | A_4]$, $B^T = [B_1 | B_2 | B_3 | B_4]$. Before presenting the results, the principle of the calculation is recalled in one of the cases that leads to the simplest calculations: the definition of $(\partial \psi_i / \partial x)$ along the curves where (7), (8) or (9) is satisfied, that is, the streamtraces, the \'C\' or \'C\' characteristic, requires that, along these curves

$$
\begin{vmatrix}
 dx & 0 & 0 & dy & 0 & 0 & 0 & 0 \\
 0 & dx & 0 & dy & 0 & 0 & 0 & 0 \\
 0 & 0 & dx & dy & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & dy & 0 & 0 & 0 & 0 \\
 -A_1 & -A_2 & -A_3 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\
 -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -B_4 \\
 -A_1 & -A_2 & -A_3 & -B_1 & -B_2 + tA_2 & -B_3 & -B_4 & -B_4 \\
 -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -B_4 \\
\end{vmatrix} = 0
$$

The determinant is expanded along the fourth column and the following notations are used

$$ -C_{4x}^1 d\psi_1 + C_{4x}^2 d\psi_2 - C_{4x}^3 d\psi_3 + C_{4x}^4 d\psi_4 = 0 \quad (10) $$

in which, for example

$$
\begin{vmatrix}
 0 & dx & 0 & 0 & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & dx & 0 & 0 & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & dx & 0 & 0 & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & dx & 0 & 0 & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -A_1 & -A_2 & -A_3 & 0 & -B_1 & -B_2 & -B_3 & -B_4 & -A_1 & -A_2 & -A_3 & -B_1 & -B_2 + tA_2 & -B_3 & -B_4 & -B_4 \\
 -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -B_4 & -B_4 \\
 -A_1 & -A_2 & -A_3 & -B_1 & -B_2 + tA_2 & -B_3 & -B_4 & -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -B_4 & -B_4 \\
 -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -A_1 & -A_2 & -A_3 & -B_1 & -B_2 & -B_3 & -B_4 & -B_4 & -B_4 \\
\end{vmatrix}
$$

Finally

$$ C_{4x}^1 = dx^2 dy \quad \gamma_1 w (u + vt) $$

The determinant of this $4 \times 4$ matrix is easily calculated thanks to the simplicity of the first two columns $A_1$ and $B_1$. The final result is

$$ C_{4x}^1 = dx^2 dy \quad \gamma_1 w (u + vt) $$

We emphasise that at this stage no assumption is made on the value of $t = dy/dx$ w.r.t. the velocity vector $(u, v)$. In particular $t$ is not assumed to be the tangent of the angle of the velocity w.r.t. the x axis and $w$ is not assumed to be zero. This is mandatory to derive relations that can be used for all three types of specific curves and also to account for the multiplicity of the eigenvalue $(-v \ dx + u \ dy)$ along streamtraces. The other terms of the differential form of interest read

$$
\begin{align*}
C_{4x}^2 &= -dx^2 dy \ | \ -A_2 \ (-B_1 + tA_1) & -B_2 \ (-B_3 + tA_3) \ | = -dx^2 dy \ \gamma_1 w (u^2 + v^2) \\
C_{4x}^3 &= dx^2 dy \ | \ -A_1 \ (-B_1 + tA_1) & (-B_2 + tA_2) & -B_3 \ | = dx^2 dy \ \gamma_1 w t (u^2 + v^2) \\
C_{4x}^4 &= dx^3 \ | \ (-B_1 + tA_1) & (-B_2 + tA_2) & (-B_3 + tA_3) & -B_4 \ | = dx^3 w (\gamma_1 + \gamma^2) u^2 v - 2uv^2 t + \gamma_1 H w + (\gamma + \gamma t^2) v^3 - \gamma_1 v (1 + t^2) E c
\end{align*}
$$

Two or three of the vectors $(-B_1 + tA_1)$, $(-B_2 + tA_2)$, $(-B_3 + tA_3)$ and $(-B_4 + tA_4)$ appear in the formulas of the $C_{ij}^k$ coefficients expressed as the determinant of a $4 \times 4$ matrix. They may be precalculated as

$$
\begin{align*}
-B_1 + tA_1 = & \begin{bmatrix} 0 & t \\ t & -1 \\ 1 & 0 \end{bmatrix} \\
-B_2 + tA_2 = & \begin{bmatrix} t & \gamma_1 E_c - uv \\ -v + t(3 - \gamma)u \\ -u - t\gamma v/n \end{bmatrix} \\
-B_3 + tA_3 = & \begin{bmatrix} -\gamma_1 E_c - v w \\ \gamma_1 u + tv \\ -(3 - \gamma) v + tu \end{bmatrix} \\
-B_4 + tA_4 = & \begin{bmatrix} \gamma_1 E_c - H w \\ \gamma w \\ H - \gamma uv \end{bmatrix}
\end{align*}
$$
2.3. Conditions for existence of \( \partial \psi_1 \partial x \), \( \partial \psi_2 \partial x \), \( \partial \psi_3 \partial x \), \( \partial \psi_4 \partial x \) along the characteristics

- The \( C^1_{1x} \) coefficients read

\[
C^1_{1x} = -dx^3 w ((2tu + (t^2 - 1)v)(\gamma H + \gamma Ec + \gamma w) - (u + v)(\gamma tH + \gamma uw + \gamma wt))
\]

\[
C^2_{1x} = dx^2 dy \gamma w (u^2 + v^2) H
\]

\[
C^3_{1x} = -dx^2 dy \gamma w t (u^2 + v^2) H
\]

\[
C^4_{1x} = dx^2 dy \gamma w (u + vt) H^2
\]

The differential form associated with the existence of \( \partial \psi_2 \partial x \) along the characteristic curves reads

\[
dx^3 w ((2tu + (t^2 - 1)v)(\gamma H + \gamma Ec + \gamma w) - (u + v)(\gamma tH + \gamma uw + \gamma wt)) d\psi_1
+ dx^2 dy \gamma w (u^2 + v^2) H d\psi_2 + dx^2 dy \gamma w t (u^2 + v^2) H d\psi_3 + dx^2 dy \gamma w (u + vt) H^2 d\psi_4
\]  
\tag{11}
\]

in which \( dx, dy \) is a differential displacement. Hence, along a characteristic, this form is set equal to zero

\[
dx^3 w ((2tu + (t^2 - 1)v)(\gamma H + \gamma Ec + \gamma w) - (u + v)(\gamma tH + \gamma uw + \gamma wt)) d\psi_1
+ dx^2 dy \gamma w (u^2 + v^2) H d\psi_2 + dx^2 dy \gamma w t (u^2 + v^2) H d\psi_3 + dx^2 dy \gamma w (u + vt) H^2 d\psi_4 = 0
\]  
\tag{12}
\]

since it is a necessary condition for \( \partial \psi_2 \partial x \) to be bounded. We emphasise that this equation is valid for all \( w \) values. Hence it may be simplified by the common \( w \) factor wherever \( w \) is not zero and, by continuity of all terms, we expect the resulting equation to be also valid where \( w = 0 \) (i.e. where \( dx, dy \) is proportional to the local velocities), along trajectories. Simplifying previous equation by the \( w dx^2 \) factor (after using \( dy = tdx \)) yields

\[
((2tu + (t^2 - 1)v)(\gamma H + \gamma Ec + \gamma w) - (u + vt)(\gamma tH + \gamma uw + \gamma wt)) d\psi_1
+ \gamma t (u^2 + v^2) H d\psi_2 + \gamma t (u + vt) H^2 d\psi_4 = 0
\]  
\tag{13}
\]

In the remaining of this section, we do not discuss anymore the simplification by this factor.

- The \( C^2_{1x} \) coefficients read

\[
C^1_{2x} = -dx^2 dy w (\gamma H + \gamma Ec + \gamma w)
\]

\[
C^2_{2x} = -dx^2 w (\gamma u^2 v - uv^2 t + \gamma^3 - \gamma (ut + v)(Ec + H))
\]

\[
C^3_{2x} = dx^2 dy w (\gamma u^2 v - uv^2 t + \gamma^3 - \gamma (ut + v)(Ec + H))
\]

\[
C^4_{2x} = dx^2 dy H w (\gamma H + \gamma Ec + \gamma w)
\]

Equating to zero the differential form associated with the existence of \( \partial \psi_2 \partial x \) along the characteristic curves reads

\[
dx^2 dy w(\gamma H + \gamma Ec + \gamma w) d\psi_1 - dx^3 w(\gamma u^2 v - uv^2 t + \gamma^3 - \gamma (ut + v)(Ec + H)) d\psi_2
- dx^2 dy w(\gamma u^2 v - uv^2 t + \gamma^3 - \gamma (ut + v)(Ec + H)) d\psi_3 + dx^2 dy H w (\gamma H + \gamma Ec + \gamma w) d\psi_4 = 0
\]  
\tag{14}
\]

or

\[
\gamma t H + \gamma Ec + \gamma w (d\psi_1 + Hd\psi_4)
- \gamma t u^2 v - uv^2 t + \gamma^3 - \gamma (ut + v)(Ec + H)) (d\psi_2 + t d\psi_3) = 0
\]  
\tag{15}
\]

- The \( C^3_{1x} \) coefficients read

\[
C^1_{3x} = dx^2 dy w (t\gamma H + t\gamma Ec - \gamma w)
\]

\[
C^2_{3x} = -dx^2 dy w ((\gamma + 1)u^2 v - t\gamma uv^2 + (v + ut)(\gamma Ec + \gamma H - \gamma^2))
\]

\[
C^3_{3x} = dx^2 w ((tv^2 - uvw)(-\gamma u + (\gamma + 1)tv) - (ut - (1 + 2v^2)\gamma)(\gamma Ec + \gamma H - \gamma^2))
\]

\[
C^4_{3x} = -dx^2 dy H w (t\gamma H + t\gamma Ec - \gamma w)
\]
Equating to zero the differential form associated with the existence of \( (\partial \psi_2 / \partial x) \) along the characteristic curves reads

\[
\begin{align*}
&dx^2 \gamma_1 w (u + vt) d\psi_1 - dx^2 \gamma_1 w (u^2 + v^2) d\psi_2 - dx^2 \gamma_1 w t (u^2 + v^2) d\psi_3 \\
&dx^3 w ((\gamma + \gamma^2) u^3 - 2u^2 v t - \gamma twH + (\gamma + \gamma^2) u v^2 - \gamma u (1 + t^2) E c)
\end{align*}
\]

Equating to zero the differential form associated with the existence of \( (\partial \psi_3 / \partial y) \) along the characteristic curves reads

\[
\begin{align*}
&dx^2 \gamma_1 w (u^2 + v^2) d\psi_2 + dx^3 \gamma_1 w t (u^2 + v^2) H d\psi_3 + dx^3 \gamma_1 w (u + vt) H^2 d\psi_4 = 0
\end{align*}
\]

This equation may be simplified by the common w factor wherever w is not zero. As in the previous subsection, by continuity of all terms, we expect the resulting equation to be valid also along the trajectories (where \( w = 0 \)). Simplifying previous equation by a \( w dx^3 \) factor yields

\[
\begin{align*}
&((\gamma + \gamma^2) u^3 - 2u^2 v t - \gamma twH + (\gamma + \gamma^2) u v^2 - \gamma u (1 + t^2) E c) d\psi_1 \\
&+ \gamma_1 (u^2 + v^2) H (d\psi_2 + td\psi_3) + \gamma_1 (u + vt) H^2 d\psi_4 = 0
\end{align*}
\]

This simplification is not further discussed below.

The \( C_{1y} \) coefficients read

\[
\begin{align*}
C_{1y}^1 &= dx^3 w ((\gamma + \gamma^2) u^3 - 2u^2 v t - \gamma twH + (\gamma + \gamma^2) u v^2 - \gamma u (1 + t^2) E c)
\end{align*}
\]

The \( C_{2y} \) coefficients read

\[
\begin{align*}
C_{2y}^1 &= -dx^3 \gamma_1 w (u^2 + v^2) H \\
C_{2y}^2 &= dx^3 \gamma_1 w t (u^2 + v^2) H \\
C_{2y}^3 &= -dx^3 \gamma_1 w (u^2 + v^2) d\psi_2 + dx^3 \gamma_1 w t (u^2 + v^2) H d\psi_3 + dx^3 \gamma_1 w (u + vt) H^2 d\psi_4 = 0
\end{align*}
\]

Equating to zero the differential form associated with the existence of \( (\partial \psi_2 / \partial y) \) along the characteristic curves reads

\[
\begin{align*}
&dx^3 w((\gamma H + \gamma E_c + \gamma w) d\psi_1 + dx^3 w ((vw + u^2)(vt - (\gamma + \gamma^2) u) - (vt - (2 + t^2) u)(\gamma E c + \gamma H + \gamma w)) \end{align*}
\]
Removing a \(w dx^3\) factor as before yields

\[
(\gamma H + \gamma H + \gamma w) (d\psi_1 + H d\psi_2) + ((v w + u^2)(v t - (\gamma + \gamma^2) u t) - (v t - (2 + i^2) u) (\gamma H + \gamma H + \gamma w) d\psi_2
\]

\[
- (\gamma u^2v - uv^2t + \gamma^2 - \gamma (u t + v)(Ec + H) d\psi_3 = 0
\]

(23)

- The \(C_{3y}\) coefficients read

\[
\begin{align*}
C_{3y} &= -d x^3 w (\gamma t Ec + \gamma t H - \gamma w) \\
C_{3y} &= d x^3 w ((\gamma + 1) u^2 v - t \gamma w u^2 + (v + u t)(\gamma Ec + \gamma H - \gamma u^2)) \\
C_{3y} &= -d x^3 w ((\gamma + 1) u^2 v - t \gamma w u^2 + (v + u t)(\gamma Ec + \gamma H - \gamma u^2)) \\
C_{3y} &= d x^3 w H (\gamma t Ec + \gamma t H - \gamma w)
\end{align*}
\]

Equating to zero the differential form associated with the existence of \((\partial \psi_1 / \partial y)\), equation (24), reads

\[
\begin{align*}
&dx^3 d (\gamma H + t \gamma H - \gamma w d)(d\psi_1 + d x^3 d (\gamma + 1) u^2 v - t \gamma w u^2 + (v + u t)(\gamma Ec + \gamma H - \gamma u^2)) d\psi_2 \\
+ &d x^3 d w t (\gamma Ec + \gamma H - \gamma u^2) d\psi_3 = 0
\end{align*}
\]

or

\[
(\gamma H + t \gamma H - \gamma w d)(d\psi_1 + H d\psi_4) \\
+ ((\gamma + 1) u^2 v - t \gamma w u^2 + (v + u t)(\gamma Ec + \gamma H - \gamma u^2)) (d\psi_2 + t d\psi_3) = 0
\]

(25)

- The \(C_{4y}\) coefficients read

\[
\begin{align*}
C_{4y} &= -d x^3 \gamma w (u + v t) \\
C_{4y} &= d x^3 \gamma w (u^2 + v^2) \\
C_{4y} &= -d x^3 \gamma w t (u^2 + v^2) \\
C_{4y} &= d x^3 w ((\gamma + \gamma^2) u^2 - 2 u^2 v t - \gamma t w H + (\gamma + \gamma^2) u^2 - \gamma u (1 + t^2) E c)
\end{align*}
\]

Equating to zero the differential form associated with the existence of \((\partial \psi_4 / \partial y)\), equation (26), reads

\[
\begin{align*}
&dx^3 \gamma w (u + v t) d\psi_1 - d x^3 \gamma w (u^2 + v^2) d\psi_2 - d x^3 \gamma w t (u^2 + v^2) d\psi_3 \\
&- d x^3 w ((\gamma + \gamma^2) u^2 - 2 u^2 v t - t w H + (\gamma + \gamma^2) u^2 - t E c (1 + t^2)) d\psi_4 = 0
\end{align*}
\]

or

\[
\gamma (u + v t) d\psi_1 + \gamma (u^2 + v^2) (d\psi_2 + t d\psi_3) \\
+ ((\gamma + \gamma^2) u^2 - 2 u^2 v t - t w H + (\gamma + \gamma^2) u^2 - \gamma u E c (1 + t^2)) d\psi_4 = 0
\]

(27)

2.5. General properties of the \(C_{1x}\) and \(C_{1y}\) coefficients

Equation (23) refers to the limit of small space steps and the search of characteristic curves; nevertheless, the expressions of the \(C_{1x}\) and \(C_{1y}\) coefficients may be considered for an arbitrary direction of vector \((dx, dy)\). Without any assumption linking \((dx, dy)\) and \((u, v)\), the relations between the coefficients of the same differential forms are:

\[
\begin{align*}
C_{1x} &= HC_{1x}, \quad C_{1x} = tC_{2x} \quad ; \quad C_{2x} = -HC_{2x}, \quad C_{3x} = tC_{2x} \quad ; \quad C_{3x} = -HC_{3x}, \quad C_{4x} = tC_{4x} \\
C_{1y} &= HC_{1y}, \quad C_{1y} = tC_{2y} \quad ; \quad C_{2y} = -HC_{2y}, \quad C_{3y} = -HC_{3y}, \quad C_{3y} = tC_{2y} \quad ; \quad C_{3y} = -HC_{3y}
\end{align*}
\]

(28)

(29)

Besides, twelve of the sixteen coefficients of the differential forms for the \(x\) and \(y\) derivatives are proportional by a \((-t)\) factor:

\[
\begin{align*}
C_{1x} &= -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \\
C_{1x} &= t C_{1y} \quad ; \quad C_{1x} = t C_{1y} \quad ; \quad C_{1x} = t C_{1y} \quad ; \quad C_{1x} = t C_{1y} \\
C_{1x} &= -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \\
C_{1x} &= -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \quad ; \quad C_{1x} = -t C_{1y} \\
\end{align*}
\]

(30)

(31)

(32)

(33)
Finally, the $C_{1l}^4$ and $C_{4l}^4$ coefficients are equal

$$C_{1x}^1 = C_{4x}^4$$
$$C_{1y}^1 = C_{4y}^4$$

(34)

2.6. Ordinary differential equations for the adjoint along the streamtraces

The trajectories are one of the families of specific curves for the gradient calculation problem (6). Along these curves $u \ dy - v \ dx = 0$ is a zero of the denominator of Cramer’s formulas with multiplicity two. Let us first assume that point $a$ is fixed and point $b$ is very close to $\mathcal{S}(a)$, the streamtrace passing through $a$ but not on this curve. The first-order expression of $\partial \psi_l / \partial x$ is

$$\frac{\partial \psi_l}{\partial x} = \frac{C_{1l}^1 d\psi_l - C_{2l}^2 d\psi_2 + C_{3l}^3 d\psi_3 - C_{4l}^4 d\psi_4}{(-v \ dx + u \ dy)(-v \ dx + u \ dy + c \ ds)(-v \ dx + u \ dy - c \ ds)}$$

Actually $wdx = u \ dy - v \ dx$ is a factor of all four coefficients $C_{1l}^1, C_{2l}^2, C_{3l}^3$ and $C_{4l}^4$. We denote by $\bar{C}_{mx}^l$ the coefficients obtained by removing the ($wdx$) factor from the corresponding $C_{mx}^l$. Obviously

$$\frac{\partial \psi_l}{\partial x} = \frac{\bar{C}_{1l}^1 d\psi_l - \bar{C}_{2l}^2 d\psi_2 + \bar{C}_{3l}^3 d\psi_3 - \bar{C}_{4l}^4 d\psi_4}{(-v \ dx + u \ dy)(-v \ dx + u \ dy + c \ ds)(-v \ dx + u \ dy - c \ ds)}$$

If point $b$ is moved closer and closer to $\mathcal{S}(a)$ ($-v \ dx + u \ dy$) → 0, so that the boundedness of ($\partial \psi_l / \partial x$) requires that

$$C_{1l}^1 d\psi_l - C_{2l}^2 d\psi_2 + C_{3l}^3 d\psi_3 - C_{4l}^4 d\psi_4 = 0 \quad \text{on} \quad \mathcal{S}(a)$$

This expression and its counterparts for the other derivatives ($\partial \psi_l / \partial y$) ... ($\partial \psi_l / \partial y$) have to be satisfied for all trajectories. How many of these eight differential forms are independant? If $w \ dx = 0$, then

$$\bar{C}_{1x}^1 = -t \bar{C}_{1y}^1 \quad \bar{C}_{2x}^2 = -t \bar{C}_{2y}^2 \quad \bar{C}_{3x}^3 = -t \bar{C}_{3y}^3 \quad \bar{C}_{4x}^4 = -t \bar{C}_{4y}^4$$

(35)

as in this case

$$\bar{C}_{1x}^1 = \bar{C}_{4x}^4 = -0.5 \gamma d^2 x^2 (1 + t^2)^2 u^3 \quad \bar{C}_{1y}^1 = \bar{C}_{4y}^4 = 0.5 \gamma d^2 x^2 (1 + t^2)^2 u^3$$

$$\bar{C}_{2x}^2 = 2 d x^2 \gamma t u H \quad \bar{C}_{2y}^2 = -2 d x^2 \gamma t u H \quad \bar{C}_{3x}^3 = 2 d x^2 \gamma t u H \quad \bar{C}_{3y}^3 = -2 d x^2 \gamma t u H.$$

Relations (30) to (33) are valid for the $\bar{C}$ coefficients (as they stand whatever the values of $w$ and $dx$, they may be simplified by $wdx$). In the specific case where $w = 0$, they are completed by (35). Equations (27), (29), (31), (33) (necessary for the boundedness of the $\partial \psi_l / \partial y$) and (13), (15), (17), (19) (same for the $\partial \psi_l / \partial x$) are then proportional by a ($-t$) factor. Considering the range of the set of the eight differential forms, it appears that one of these two sets of four equations need be accounted for.

The relations stemming from the existence of the $\partial x$ partial derivative are retained. Equation (15), required for the definition of ($\partial \psi_l / \partial x$), is further simplified using the specific properties of a trajectory ($w = 0 \quad v = u$):

$$0.5t \gamma_1 (1 + t^2)^2 u^3 \psi_1 + \gamma_1 t (1 + t^2) u^2 H (d \psi_2 + t d \psi_3) + \gamma_1 t (1 + t^2) u \gamma_2 d \psi_4 = 0$$
$$0.5 (1 + t^2) u^2 d \psi_1 + u H (d \psi_2 + t d \psi_3) + H^2 d \psi_4 = 0$$

For the streamtraces, the equation finally derived from the existence of ($\partial \psi_l / \partial x$) is

$$E_c \ d\psi_1 + H (u \ d\psi_2 + v \ d\psi_3) + H^2 d\psi_4 = 0$$

(36)

Note that we have assumed that $u \neq 0$ and $dx \neq 0$ to perform the calculations but finally obtained an expression that is also well-defined in this specific case. Equation (15), is further simplified for the motion along a trajectory:

$$t (\gamma_1 H + \gamma_1 E_c) (d \psi_1 + H d \psi_4) + 2 \gamma_1 u t H (d \psi_2 + t d \psi_3) = 0$$
$$(H + E_c) (d \psi_1 + H d \psi_4) + 2 H (u \ d\psi_2 + v \ d\psi_3) = 0$$
Using the first relation and simplifying by $H$, we get

\[ H(d\psi_1 + Hd\psi_4) + 2H (u\, d\psi_2 + v\, d\psi_3) + EcHd\psi_4 - H(u\, d\psi_2 + v\, d\psi_3) - H^2d\psi_4 = 0 \]

and finally

\[ d\psi_1 + u\, d\psi_2 + v\, d\psi_3 + E_c\, d\psi_4 = 0 \] (37)

Simplifying equation (17) for trajectories yields

\[ (H + E_c)(d\psi_1 + Hd\psi_4) + 2H(u\, d\psi_2 + v\, d\psi_3) = 0, \]

that had already been derived. Using $w = 0$ and $v = ut$, the fourth relation (first simplified by a $wdx$ factor) gives

\[ \gamma_1\, tu(1 + r^2)\, d\psi_1 + \gamma_1\, tu^2(1 + r^2)\, d\psi_2 + \gamma_1\, tu^2(1 + r^2)\, d\psi_3 + 0.5\gamma_1(1 + r^2)^2\, u^3\, d\psi_4 = 0 \]

or

\[ d\psi_1 + u\, d\psi_2 + tu\, d\psi_3 + 0.5(1 + r^2)^2\, u^3\, d\psi_4 = 0 \]

and finally

\[ d\psi_1 + u\, d\psi_2 + v\, d\psi_3 + E_c\, d\psi_4 = 0 \]

This equation is similar to (37). These differential forms along the trajectories $\gamma$ may be turned into differential equations. The natural variable w.r.t. which differentiate, is the curvilinear abscissa along the streamtraces, $s$, increasing in the direction opposite to the motion (as this is the direction of the adjoint transport of information from the support of the function of interest). The final equations along the $\gamma$ curves are then

\[
\begin{align*}
E_c \frac{d\psi_1}{ds} + H(u \frac{d\psi_2}{ds} + v \frac{d\psi_3}{ds}) + H^2 \frac{d\psi_4}{ds} &= 0 \\
\frac{d\psi_1}{ds} + u \frac{d\psi_2}{ds} + v \frac{d\psi_3}{ds} + E_c \frac{d\psi_4}{ds} &= 0
\end{align*}
\] (38) (39)

Although the calculations of this section are not very complex, they have been checked with the computer algebra software Maple. The independent variables of the formal calculations are $(u, v)$ also $M$ the Mach number (that allows the calculation of the speed of sound $c$ and then the total enthalpy $H$) and $\gamma$. The set of four differential forms \{(21), (23), (25), (27)\} was associated with \{(36), (37)\} in a $6 \times 4$ matrix that was again found to have rank two.

2.7. Ordinary differential equation for the adjoint along the $C^+$ and $C^-$ characteristics

Let us first note that the determinant in the denominator of the Cramer formulas for (9) may be calculated as

\[ D = dx\, C^1_{1x} + dy\, C^1_{1y} \]

developing $D$ along the first line. Doing the same along the second, third and fourth lines yields

\[ D = dx\, C^2_{2x} + dy\, C^2_{2y} = dx\, C^3_{3x} + dy\, C^3_{3y} = dx\, C^4_{4x} + dy\, C^4_{4y} \]

As $D = 0$ along the $C^+$ and $C^-$ characteristics, equations (30) to (33) are completed by

\[ C^1_{1x} = -t\, C^1_{1y} \quad C^2_{2x} = -t\, C^2_{2y} \quad C^3_{3x} = -t\, C^3_{3y} \quad C^4_{4x} = -t\, C^4_{4y} \]

so that the differential forms stemming from the boundedness of $(\partial\psi/\partial x)$ and their counterparts for $(\partial\psi/\partial x)$ are proportional. (Incidentally, note that this argument may have been used also in the previous subsection.)

Numerical tests indicate that the four differential forms associated with the existence of (choosing the second set $(\partial\psi_1/\partial y), (\partial\psi_2/\partial y), (\partial\psi_3/\partial y), (\partial\psi_4/\partial y)$) are proportional but the corresponding calculations are much more complex than in the previous subsection as the expression of $t$ is now:

\[ t = \tan(\phi + \beta) \quad \text{with} \quad \tan\phi = \frac{v}{u} \quad \sin\beta = \frac{1}{M} \] (40)

with $\epsilon = 1$ for the $C^+$ and $\epsilon = -1$ for the $C^-$ curves.

With this expression of $t$ and $w$ not being equal to zero, searching the rank of \{(21), (23), (25), (27)\} is much more
difficult. However, the task was successfully accomplished with the assistance of Maple, using once again \((u, v, M, \gamma)\) as independent formal variables. Rank one was indicated by Maple and the correspondence with the counterpart characteristics for the flow seemed very sound. Knowing this results from formal calculation, its demonstration by hand was searched for.

First the value of \(t\) is calculated along the \(\mathcal{C}^+\) and \(\mathcal{C}^-\) curves is calculated. For \(\beta\) in \([-\pi/2, \pi/2]\) \[40\] yields

\[
t^\varepsilon = \frac{v}{u} + \frac{\varepsilon}{\sqrt{M^2 - 1}} = \frac{uvM^2 + \varepsilon(u^2 + v^2)\sqrt{M^2 - 1}}{u^2(M^2 - 1) - v^2} = \frac{uv + \varepsilon c^2\sqrt{M^2 - 1}}{u^2 - c^2} = \frac{uv + \varepsilon c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}
\]

\[41\]

We then note that if the differential forms \((21)\) and \((27)\) were proportional, from the expressions of \(C_{1y}^2, C_{1y}^3, C_{4y}^2\) and \(C_{4y}^3\), the ratio between their terms would be \((-H)\). It is then easily checked that the two non-trivial conditions for proportionality, \(C_{1y}^2 = -HC_{1y}^4\) and \(C_{4y}^1 = -HC_{4y}^3\), are equivalent to a single equality

\[
\gamma_1 (u + vt)H = ((\gamma_1 + \gamma_2)u^3 - 2u^2vt - \gamma_1wH + (\gamma + \gamma_1^2)u\gamma^2 - \gamma_1u (1 + t^2)Ec)
\]

\[42\]

Wherever \(u \neq 0\), this condition is equivalent to

\[
\gamma_1 (1 + t^2) H = \gamma_1 (1 + t^2) Ec + (u - v)^2
\]

that is, precisely, the degree two equation which roots are the values of \(t\) along the \(\mathcal{C}^+\) and \(\mathcal{C}^-\) curves \[41\]. Along these curves, using \[42\] that is now an established property along the \(\mathcal{C}^+\) and \(\mathcal{C}^-\), these two differential forms may be simplified as

\[
(u + vt^\varepsilon)^2 \frac{d\psi_1 + (u^2 + v^2)(d\psi_2 + t^\varepsilon d\psi_3) + H(u + vt^\varepsilon) d\psi_4 = 0,
\]

\[43\]

or, under the form of an ordinary differential equation,

\[
(u + vt^\varepsilon)^2 \frac{d\psi_1 + (u^2 + v^2)(d\psi_2 + t^\varepsilon d\psi_3) + H(u + vt^\varepsilon) d\psi_4 = 0.
\]

\[44\]

Comparing equations \[25\] - expressing the boundedness of \((\partial\psi_1/\partial y)\) - and \[43\] - its counterpart for \((\partial\psi_2/\partial y)\) and \((\partial\psi_3/\partial y)\) - it is easily derived that these equations are proportional on the \(\mathcal{C}^+\) and \(\mathcal{C}^-\) curves if and only if for \(t = t^\varepsilon\)

\[
(t\gamma_1 H + t\gamma_1 E_c - \gamma uv)(u^2 + v^2) = ((\gamma + 1)u^2v - t\gamma uv^2 + (v + ut)(\gamma Ec + \gamma_1 H - \gamma_2^2))(u + vt)
\]

Wherever \(uv \neq 0\), this equation is equivalent to

\[
\gamma(u^2 + v^2) = ((\gamma + 1)u - t\gamma v)(u + vt) + (\gamma Ec + \gamma_1 H - \gamma_2^2)(1 + t^2)
\]

that is found to be equivalent to equation \[41\], the degree two equation \(t\) which roots are the slope coefficients of the \(\mathcal{C}^+\) and \(\mathcal{C}^-\). Equation \[25\] hence also reduces to \[43\] along the \(\mathcal{C}^+\) and \(\mathcal{C}^-\) curves.

Finally, we consider the last differential form \[23\], expressing the boundedness of \((\partial\psi_2/\partial y)\). Whether it is proportional to \[43\] is not straightforward in particular due to complex expression of \(C_{2y}^2\) and the ratio \(C_{3y}^3/C_{2y}^2\) that is not obviously equal to \(t\). Nevertheless, we have proven that, along the characteristics curves the differential form expressing the boundedness of \((\partial\psi_2/\partial x)\) and \((\partial\psi_3/\partial y)\) (equations \[15\] and \[23\]) are proportional by a minus \(t\) factor. So we may use this property to derive a simpler expression of \(C_{2y}^2\) or prove the proportionality of \[15\] with \[43\] along the \(\mathcal{C}^+\) and \(\mathcal{C}^-\) curves. Whatever the approach the condition for proportionality reads

\[
t (\gamma H + \gamma E_c + \gamma uv) (d\psi_1 + H d\psi_3) + (\gamma_1 (u + v) (Ec + H) + u^2 - t - \gamma_1 u^2 v - \gamma_3^3) (d\psi_2 + t d\psi_3) = 0
\]

or, equivalently,

\[
t (\gamma H + \gamma E_c + \gamma uv)(u^2 + v^2) = (\gamma_1 (u + v) (Ec + H) + u^2 - t - \gamma_1 u^2 v - \gamma_3^3)(u + vt),
\]

11
that is found to be equivalent to (41).

In a final formal calculation verification, it was checked that, on the \( \mathcal{C}^+ \) and \( \mathcal{C}^- \) curves, (21), is proportional to (43) (and we already know that equations (25), (25), (27) are proportional to (21) along these curves).

In conclusion, we have found two differential equations (38) (39) valid along the streamtraces for the adjoint system. They are the counterpart of constant total enthalpy and constant entropy properties for the flow. We have found one differential equation for the \( \mathcal{C}^+ \) and one differential equation for the \( \mathcal{C}^- \), equation (44) with relevant value of \( t^c \) for each curve. They are the counterpart of the characteristic angular equations (4) satisfied for the flow.

3. Numerical assessment of the adjoint ODE along the characteristics

3.1. Assessment method

The numerical assessment method consists in computing flow and adjoint fields over a very fine mesh, and calculating the following integrals:

\[
K_{OS}^{S1} = \int_{\mathcal{C}} \left( E_c \frac{d\psi_1}{ds} + H(u\frac{d\psi_2}{ds} + v\frac{d\psi_3}{ds}) + H^2\frac{d\psi_4}{ds} \right) ds
\]

\[
K_{OS}^{S2} = \int_{\mathcal{C}} \left( \frac{d\psi_1}{ds} + u\frac{d\psi_2}{ds} + v\frac{d\psi_3}{ds} + Ec\frac{d\psi_4}{ds} \right) ds
\]

\[
K_{OS}^{C+} = \int_{\mathcal{C}^+} \left( (u + vt^+)\frac{d\psi_1}{ds} + (u^2 + v^2)(\frac{d\psi_2}{ds} + t^+\frac{d\psi_3}{ds}) + H(u + vt^+)\frac{d\psi_4}{ds} \right) ds
\]

\[
K_{OS}^{C-} = \int_{\mathcal{C}^-} \left( (u + vt^-)\frac{d\psi_1}{ds} + (u^2 + v^2)(\frac{d\psi_2}{ds} + t^-\frac{d\psi_3}{ds}) + H(u + vt^-)\frac{d\psi_4}{ds} \right) ds.
\]

Here, \( t^+ \) and \( t^- \) denote values of \( t \) on the \( \mathcal{C}^+ \) and \( \mathcal{C}^- \) (equation 41) with respectively \( \varepsilon = 1 \) and \( \varepsilon = -1 \) and the intermediate subscript \( O \) stands for the output functional of interest; it is subsequently replaced by \( L \) (for the lift, \( CDp \)) and \( D \) (for drag, \( CDp \)), consistently with the adjoint vector placed on the right-hand side.

The integration is performed in the forward sense for the adjoint, that is, backwards w.r.t. the direction of the flow information propagation. The integration domain for the above line integrals extends to the interior of the disk of radius 3c centred at (0.5c, 0), chosen for plotting readability, while the flow computational domain itself extends to 150c. It may be shorter, in particular in the transonic case where the \( \mathcal{C}^+ \) and \( \mathcal{C}^- \) curves are limited to the supersonic bubble(s). The four quantities \( K_{OS}^{S1}, \ldots, K_{OS}^{C-} \) are expected to be close to zero and, to avoid any error in scale, also calculated and plotted as the corresponding subparts, that is, for \( K_{OS}^{S1} \) for example,

\[
K_{OS}^{S1} = \int_{\mathcal{C}} \left( E_c \frac{d\psi_1}{ds} \right) ds
\]

The sum of the four terms is expected to be much smaller than each one of them individually. All the integrals are calculated backwards, along a finely discretized characteristic curve, simply by the trapezoidal rule.

The discrete flows were available from former computations [9] in which the Jameson-Schmidt-Turkel scheme [15] was applied, and using the adjoint module of the elsA code [16]. In that publication, it was demonstrated for structured meshes how to slightly modify the scheme’s Jacobian (in the derivative of the dissipation flux, for the next to wall faces) to get a dual consistent linearization. This slight modification of the exact schemes’s Jacobian is retained here to work with adjoint fields that are consistent with the continuous equations discussed in §2. Only the solutions calculated over the finest mesh defined in the reference [17] (structured 4097×4097 mesh) are used here.

3.2. Supersonic flow about the NACA0012 airfoil

The retained flow conditions are \( M_{ao} = 1.5 \), \( AoA = 1^\circ \). We first assess the streamtraces equations (38) (39). The \( K_{OS}^{S1}, K_{OS}^{S2}, K_{OS}^{C1}, K_{OS}^{C2} \) integrals and their subparts are calculated along the trajectory passing through \((c, 0.1c)\). The integration indeed leads to very small values of \( K_{OS}^{S1}, K_{OS}^{S2}, K_{OS}^{C1}, K_{OS}^{C2} \) along the curve with respect to their subparts. It is well-known for this kind of flow that the exact lift- and drag- adjoint is equal to zero downstream the backwards flow-characteristics emanating from the trailing edge (since no perturbation downstream those two
lines can affect the pressure on the aerofoil and, consequently, the lift or the drag – see for example fig. 6 and A21 in [9]). This property is well satisfied by discrete adjoint fields and, as the integration is performed backwards along the streamtrace, null values of $K_D S^1$, $K_D S^2$, $K_L S^1$, $K_L S^2$ and all their subparts are observed above a specific $x$ corresponding to the intersection of the streamtrace with this trailing-edge $C^-$. The results are equivalently accurate for lift and drag. They are presented for the drag in figure 1. A $C^+$ and a $C^-$ curve are then extracted using equation (4). The selected $C^+$ is initiated at $x \simeq 0.3$ upperside and the

![Figure 1: $M_\infty = 1.50, \alpha = 1^\circ$, (4097×4097 mesh [17]) Numerical assessment of equation (38) (left) and (39) (right) for the lift. Method of verification: the black curve should ideally be coincided with the y axis](image)

retained $C^-$ starts at the same abscissa but on the lower side. This choice was guided by the extraction method and the observation that $k^+$ (resp. $k^-$) is almost constant on the lower (resp. upper) side. The $K_D C^+, K_L C^+, K_D C^-, K_L C^-$ terms and their subparts have been computed. The results appear to be very satisfactory. Also observed is the equality of $K_D C^+_1$ and $K_D C^+_4$, $K_L C^-_1$ and $K_L C^-_4$ and along the $C^+$ and correspondingly along the $C^-$ curves. This is due to the fact that $\psi_1 = H \psi_4$ (for Euler flows and for pressure-based integrals along the wall that is well satisfied at the discrete level – see for example [9] fig. A21, A22, A23) and to the expression of the $d \psi_1$ and $d \psi_4$ terms in (43). Figure 2 presents the verification plots for the two functions along the selected $C^+$.

![Figure 2: $M_\infty = 1.50, \alpha = 1^\circ$, (4097×4097 mesh [17]) Numerical assessment of equation (43) for the lift (left) and for the drag (right). Method of verification: the black curve should ideally be coincided with the y axis](image)
3.3. Transonic flow about the NACA0012 airfoil

The flow conditions are \( M_\infty = 0.85 \), \( \alpha = 2^\circ \). Careful verification of the streamtraces equations \((38)\) \((39)\) has been performed for the streamtrace passing through \((c, 0.1c)\) and very satisfactory results have been found. As similar results have been presented in the previous and the next subsections, we focus here on the \( C^+ \) and \( C^- \) curves. A \( C^+ \) and \( C^- \) curves have been extracted taking care to select the longest possible curves (and to avoid, for the \( C^- \) the curve passing by the shockfoot where numerical divergence of the adjoint may be observed). The selected \( C^+ \) (resp. \( C^- \)) is passing approximately through the point \((0.197, 0.057)\) (resp. \((0.954, 0.141)\)). The verification of the consistency of the numerical solutions w.r.t. \((43)\) is satisfactory although the largest observed errors appear in this case, for the \( C^- \) curve in the vicinity of the inlet of the supersonic bubble – see figure 3.

![Figure 3](image-url)

Figure 3: \( M_\infty = 1.50, \alpha = 1^\circ \), (4097×4097 mesh \(17\)). Numerical assessment of equation \((43)\) for the lift (left) and for the drag (right). Method of verification: the black curve should ideally be confounded with the y-axis.

3.4. Subsonic flow about the NACA0012 airfoil

We expect relations \((38)\) \((39)\) to be valid along trajectories for subsonic flows also. The retained flow conditions have been \( M_\infty = 0.4 \), \( \alpha = 5^\circ \). The \( K_D^1, K_D^2, K_L^1, K_L^2 \) integrals and their subparts are calculated along the trajectory passing through \((c, 0.1c)\). The integration indeed leads to very small values of \( K_D^1, K_D^2, K_L^1, K_L^2 \) along the curve with respect to their subparts. The results are equivalently good for lift and drag and are presented in figure 4 for the lift.
4. Conclusion

The characteristic relationships associated with the adjoint compressible Euler equations have been established analytically for two-dimensional supersonic flows. The most technical derivations were conducted with the assistance of the formal calculation software Maple, and verified by numerical computations.

Unsurprisingly, the characteristic curves were found to be the same for the adjoint system as for the direct system, although meant to be described inversely. The theoretical findings have been illustrated with flows, lift-adjoints and drag-adjoints over the classical NACA0012 airfoil using a very fine mesh and a dual-consistent adjoint method. All the results apply to the strict supersonic regime, but those related to the streamtraces to the subsonic or transonic flow cases as well, and several verifications were made in this sense.

The purpose for establishing these formal expressions was to bring an additional insight to better understand the Euler adjoint fields, and thus also to provide a verification tool for discrete solutions.

Lastly, the strong link observed between the characteristic relationships of the direct and adjoint systems raises an open question: are the streamtrace relations (38) (39) verified also in three dimensions as their counterpart for the flow?

Acknowledgments

The authors express their warm gratitude to J.C. Vassberg and A. Jameson for allowing the co-workers of D. Destarac to use their hierarchy of O-grids around the NACA0012 airfoil, as well as E. Hubert and B. Mourrain (Inria Aromath Project Team) for their helpful guidance for the analytical developments using the computer algebra software Maple.

This research did not receive any specific grant from funding agencies in the public, commercial or not-for-profit sector.

Bibliography

References

[1] Jameson, A. Aerodynamic design via control theory. *Journal of Scientific Computing* 3(3), 233–260 (1988).
[2] Anderson, W. and Venkatakrishnan, V. Adjoint design optimization on unstructured grid using a continuous adjoint formulation. In *AIAA Paper Series, Paper 1997-0643*. (1997).
[3] Anderson, W., and Venkatakrishnan, V. Adjoint design optimization on unstructured grid using a continuous adjoint formulation. *Computers & Fluids* 28, 443–480 (1999).
[4] Giles, M. and Pierce, N. Adjoint equations in CFD: Duality, boundary conditions and solution behaviour. In AIAA Paper Series, Paper 97-1850. (1997).

[5] Mohammadi, B. and Pironneau, O. Applied Shape Optimization for Fluids. Oxford Univ. Press, May (2001).

[6] Giles, M. B. and Pierce, N. A. Analytic adjoint solutions for the quasi-one-dimensional Euler equations. *J. Fluid Mech.* **426**, 327–345 (2001).

[7] Baeza, A., Castro, C., Palacios, F., and Zuazua, E. 2d Euler shape design on non-regular flows using adjoint Rankine-Hugoniot relations. *AIAA Journal* **47**(3), 552–562 (2009).

[8] Lozano, C. Singular and discontinuous solutions of the adjoint Euler equations. *AIAA Journal* **56**(11), 4437–4451 (2018).

[9] Peter, J., Renac, F., and Labbé, C. Analysis of finite-volume discrete adjoint fields for two-dimensional compressible euler flows. *Journal of Computational Physics* **449**, 110811 (2022).

[10] Lozano, C. Watch your adjoints! lack of mesh convergence in inviscid adjoint solutions. *AIAA Journal* **57**(9), 3991–4006 (2019).

[11] Peter, J. Contributions to discrete adjoint method in aerodynamics for shape optimization and goal-oriented mesh adaptation. University of Nantes. Mémoire pour Habilitation à Diriger des Recherches, (2020).

[12] Anderson, J. *Modern Compressible Flow (third edition).* McGraw-Hill series in Aernautical and Aerospace Engineering, (2003).

[13] Gas dynamics & supersonic flows – method of characteristics. [http://www.aerodynamics4students.com/gas-dynamics-and-supersonic-flow/gasdynamics_w.php?page=92](http://www.aerodynamics4students.com/gas-dynamics-and-supersonic-flow/gasdynamics_w.php?page=92) Accessed: 2022-03-06.

[14] Délery, J. *Traité d’aérodynamique compressible. Volume 3.* Collection mécanique des fluides. Lavoisier – Hermès Science, (2008).

[15] Jameson, A., Schmidt, W., and Turkel, E. Numerical solutions of the Euler equations by finite volume methods using Runge-Kutta time-stepping schemes. In *AIAA Paper Series, Paper 1981-1259*. (1981).

[16] Peter, J., Renac, F., Dumont, A., and Méheut, M. Discrete adjoint method for shape optimization and mesh adaptation in the elsA code. status and challenges. In *Proceedings of 50th 3AF Symposium on Applied Aerodynamics, Toulouse*, (2015).

[17] Vassberg, J. and Jameson, A. In pursuit of grid convergence for two-dimensional Euler solutions. *Journal of Aircraft* **47**(4), 1152–1166 (2010).