GLAZMAN-KREIN-NAIMARK THEORY, LEFT-DEFINITE THEORY AND THE SQUARE OF THE LEGENDRE POLYNOMIALS DIFFERENTIAL OPERATOR

LANCE L. LITTLEJOHN AND QUINN WICKS

We dedicate this paper to the memory of W. N. (Norrie) Everitt (1924-2011).

Abstract. As an application of a general left-definite spectral theory, Everitt, Littlejohn and Wellman, in 2002, developed the left-definite theory associated with the classical Legendre self-adjoint second-order differential operator $A$ in $L^2(-1,1)$ which has the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions. As a consequence, they explicitly determined the domain $D(A^2)$ of the self-adjoint operator $A^2$. However, this domain, in their characterization, does not contain boundary conditions. In fact, this is a general feature of the left-definite approach developed by Littlejohn and Wellman. Yet, the square of the second-order Legendre expression is in the limit-4 case at each end point $x = \pm 1$ in $L^2(-1,1)$ so $D(A^2)$ should exhibit four boundary conditions. In this paper, we show that this domain can, in fact, be expressed using four separated boundary conditions using the classical GKN (Glazman-Krein-Naimark) theory. In addition, we determine a new characterization of $D(A^2)$ that involves four non-GKN boundary conditions. These new boundary conditions are surprisingly simple - and natural - and are equivalent to the boundary conditions obtained from the GKN theory.

1. Introduction

The analytical study of the classical second-order Legendre differential expression

$$\ell[y](x) = -((1 - x^2)y'(x))'$$

has a long and rich history stretching back to the seminal work of H. Weyl in 1910 [23] and E. C. Titchmarsh in 1940 [22]. Part, if not most, of the reason for the importance of this second-order expression lies in the fact that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are solutions. More specifically, the Legendre polynomial $y = P_n(x)$, for $n \in \mathbb{N}_0$, is a solution of the eigenvalue equation

$$\ell[y](x) = n(n+1)y(x).$$

In the Hilbert space $L^2(-1,1)$, there is a continuum of self-adjoint operators generated by $\ell[\cdot]$. One such operator $A$ stands out from the rest: this is the Legendre polynomials operator, so named because the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are eigenfunctions of $A$. We review properties of this operator in Section 2.

In the mid 1970’s, Å. Pleijel wrote two papers (see [18] and [19]) on the Legendre expression from a left-definite spectral point of view. W. N. Everitt’s contribution [8] continued this left-definite study in addition to detailing an in-depth analysis of the Legendre expression in the right-definite setting $L^2(-1,1)$ where he discovered new properties of functions in the domain $\mathcal{D}(A)$ of $A$. In [14], A. M. Krall and Littlejohn considered properties of the Legendre expression under the left-definite energy norm. In 2000, R. Vonhoff extended Everitt’s results in [20] with an extensive study of $\ell[\cdot]$ in its (first) left-definite setting. In 2002, Everitt, Littlejohn and Marić [10] published further results in which they gave several equivalent conditions for functions to belong to $\mathcal{D}(A)$; this result is given below in Theorem 1. We also refer the reader to the

\textit{Date:} September 26, 2015 (Square of Legendre Operator-LW. tex).

\textit{2000 Mathematics Subject Classification.} Primary 33C45, 34B24; Secondary 34B30.

\textit{Key words and phrases.} Legendre polynomials, self-adjoint operator, left-definite theory, Glazman-Krein-Naimark theory, boundary conditions.
paper [16] by Littlejohn and Zettl where the authors determine all self-adjoint operators, generated by the Legendre expression $f[\cdot]$, in the Hilbert spaces $L^2(-1, 1)$, $L^2(-\infty, -1)$, $L^2(1, \infty)$ and $L^2(\mathbb{R})$.

Littlejohn and Wellman [15], in 2002, developed a general left-definite theory for an unbounded self-adjoint operator $T$ bounded below by a positive constant in a Hilbert space $H = (V, \langle \cdot, \cdot \rangle)$, where $V$ denotes the underlying (algebraic) vector space and $H$ is the resulting topological space induced by the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. In a nutshell, the authors construct a continuum of Hilbert spaces $\{H_r = (V_r, \langle \cdot, \cdot \rangle_r)\}_{r>0}$, forming a Hilbert scale, generated by positive powers of $T$. The authors called these Hilbert spaces left-definite spaces; they are constructed using the Hilbert space spectral theorem (see [21]) for self-adjoint operators.

It is a difficult problem, in general, to explicitly determine the domain of a power of an unbounded operator. However, the authors in [15] prove that $V_r = D(T^{r/2})$ and $\langle f, g \rangle_r = (T^{r/2}f, T^{r/2}g)$. Furthermore, in many practical applications, as the authors demonstrate in [15], the computation of the vector spaces $V_r$ and inner products $\langle \cdot, \cdot \rangle_r$ is surprisingly not difficult. In a subsequent paper, Everitt, Littlejohn and Wellman [11] applied this theory to the Legendre polynomials operator $A$. Among other results, the authors explicitly compute the domains of $D(A^{n/2})$ for each $n \in \mathbb{N}$. Specifically, they proved

\begin{equation}
D(A^{n/2}) = \{ f : (-1, 1) \to \mathbb{C} \mid f', f'', \ldots, f^{(n-1)} \in AC_{\text{loc}}(-1, 1); (1-x^2)^{n/2}f^{(n)} \in L^2(-1, 1) \} \quad (n \in \mathbb{N}).
\end{equation}

In particular, we see that $D(A^2)$ is explicitly given by

\begin{equation}
B = \{ f : (-1, 1) \to \mathbb{C} \mid f', f'', f''' \in AC_{\text{loc}}(-1, 1); (1-x^2)^2f^{(4)} \in L^2(-1, 1) \};
\end{equation}

the reason for using the notation $B$, instead of $D(A^2)$, will be made clear shortly. Of course, for $f \in B$, we have $A^2f = \ell^2[f]$, where $\ell^2[\cdot]$ is the square of the Legendre differential expression given by

\begin{equation}
\ell^2[y](x) = ((1-x^2)^2y'''(x))'' - 2 ((1-x^2)y''(x))' + (1-x^2)y'(x)).
\end{equation}

Notice that, curiously, there are no ‘boundary conditions’ given in (1.2). From the Glazman-Krein-Naimark (GKN) theory [17, Theorem 4, Section 18.1], there should be four such boundary conditions. This begs an obvious question: how can we ‘extract’ boundary conditions from the representation of $D(A^2)$ in (1.2)? In this paper, we will answer this question. It is interesting that the condition $(1-x^2)^2f^{(4)} \in L^2(-1, 1)$ seems to ‘encode’ these boundary conditions. In fact, along the way, we will characterize $D(A^2)$ in four different ways. Of course, we have the algebraic definition

\begin{equation}
D(A^2) := \{ f \in D(A) \mid Af \in D(A) \}
\end{equation}

(we will show that $D(A^2)$, given in (1.3), is equal to $B$, defined in (1.2)). We will also prove that $D(A^2)$ is characterized by GKN boundary conditions associated with a self-adjoint operator $S$, generated by $\ell^2[\cdot]$, in $L^2(-1, 1)$. Specifically, we prove that $D(A^2)$ is equal to

\begin{equation}
D(S) := \{ f : (-1, 1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); f, \ell^2[f] \in L^2(-1, 1); f, \ell^2[f] \in L^2(-1, 1); \lim_{x \to \pm 1} f(1, 1) = 0; \lim_{x \to \pm 1} f(1, 1) = 0 \},
\end{equation}

where $[\cdot, \cdot]_2$ is the sesquilinear form associated with Green’s formula and $\ell^2[\cdot]$ in $L^2(-1, 1)$; this form will be defined in Section 4. In this paper, we also show that $D(A^2)$ is equal to

\begin{equation}
D := \{ f : (-1, 1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1, 1); f, \ell^2[f] \in L^2(-1, 1); f, \ell^2[f] \in L^2(-1, 1); \lim_{x \to \pm 1} (1-x^2)f'(x) = 0; \lim_{x \to \pm 1} ((1-x^2)^2f'''(x))' = 0 \}.
\end{equation}

This characterization of $D(A^2)$ is surprising since the boundary conditions in (1.6) are not GKN boundary conditions; we say that $D$ is a GKN-like domain. The boundary conditions in (1.6) are remarkably simple; indeed, they are obtained as limits from each of the two terms in (1.3) minus one derivative.
As a consequence of our results in this paper, we are able to generalize (1.7) by proving

\[ f \in D(A^2) \Rightarrow f'' \in L^2(-1,1) \]

where \( \Delta \) (2.2) is given by

\[ \Delta \] adjoints of each other.

and the Legendre polynomials operator \( A \) see Corollary 1 below.

A key and indispensable analytic tool - the Chisholm-Everitt Theorem - used in the proofs of these theorems is discussed in Section 6. The proof that \( B = D(S) \) is given in Section 7. Section 8 establishes the proof that \( D = D(S) \). In Section 9 we show that \( D(S) = D \). The proofs of the theorems in these last three sections establish our main result, Theorem 6, which we state in Section 5. Lastly, in Section 10 we conjecture a generalization of our main results. Further details on all of the results contained in this manuscript can be found in the Ph.D. thesis [24] of Quinn Wicks.

One final remark: to summarize, in this paper we show that our left-definite characterization (1.2) of \( D(A^2) \) can be rewritten as a GKN domain (Theorem 4) and as a GKN-like domain (Theorem 5). Presumably, techniques developed in this paper will establish, for odd, positive integers \( n \),

that the left-definite theory also explicitly determines the domains \( D(A^n/2) \) of \( A^n/2 \) for odd, positive integers \( n \). The GKN theory was not built to handle these operators or domains.

The contents of this paper are as follows. In Section 2, we discuss properties of the Legendre expression and the Legendre polynomials operator \( A \) in \( L^2(-1,1) \). Section 3 deals briefly with the algebraic definition of the square \( A^2 \) of \( A \). In Section 4 we define a self-adjoint operator \( S \) using the GKN Theory; this operator \( S \) will ultimately be shown to be \( A^2 \). The main theorems proven in this paper are stated in Section 5. A key and indispensable analytic tool - the Chisholm-Everitt Theorem - used in the proofs of these theorems is discussed in Section 6. The proof that \( D(A^2) = D(S) \) is given in Section 7. Section 8 establishes the proof that \( B = D(S) \). In Section 9 we show that \( D(S) = D \). The proofs of the theorems in these last three sections establish our main result, Theorem 6, which we state in Section 5. Lastly, in Section 10 we conjecture a generalization of our main results. Further details on all of the results contained in this manuscript can be found in the Ph.D. thesis [24] of Quinn Wicks.

The classic second-order Legendre differential expression is defined by

\[ \ell[y](x) := -((1 - x^2)y' (x))' \quad \text{(a.e.) } x \in (-1,1). \]

The maximal operator, associated with \( \ell[] \) in \( L^2(-1,1) \), is defined by

\[ T_{1,\text{max}} f = \ell[f] \]

\[ f \in \Delta_{1,\text{max}}, \]

where \( \Delta_{1,\text{max}} \) is the maximal domain, defined by

\[ \Delta_{1,\text{max}} := \{ f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell[f] \in L^2(-1,1) \}. \]

The corresponding minimal operator \( T_{1,\text{min}} \) is defined to be

\[ T_{1,\text{min}} f = \ell[f] \]

\[ f \in D(T_{1,\text{min}}), \]

where \( D(T_{1,\text{min}}) \) is the minimal domain given by

\[ D(T_{1,\text{min}}) := \{ f \in \Delta_{1,\text{max}} \mid | [f, g]_1(x) \}_{\alpha}^\beta = 0 \text{ for all } g \in \Delta_{1,\text{max}} \}. \]

We note that this operator \( T_{1,\text{min}} \) is a closed, symmetric operator. Furthermore, \( T_{1,\text{max}} \) and \( T_{1,\text{min}} \) are adjoints of each other.

Green’s formula, for an arbitrary compact subinterval \( [\alpha, \beta] \) of \(( -1,1 ) \) and \( f, g \in \Delta_{1,\text{max}} \), is given by

\[ \int_{\alpha}^{\beta} \ell[f](x)g(x)dx - \int_{\alpha}^{\beta} f(x)\ell[g](x)dx = [f, g]_{1}(x)\big|_{\alpha}^{\beta}, \]
where the sesquilinear form $[\cdot, \cdot]_1$ is defined by
\begin{equation}
[f, g]_1(x) := -(1 - x^2)(f'(x)\overline{g}(x) - f(x)\overline{g}'(x)) \quad (f, g \in \Delta_{1, \text{max}}).
\end{equation}

By definition of $\Delta_{1, \text{max}}$ and Hölder’s inequality, we see that the limits
\[\lim_{x \to \pm 1} [f, g](x)\]
exist and are finite for all $f, g \in \Delta_{1, \text{max}}$.

The endpoints $x = \pm 1$ are both regular singular endpoints, in the sense of Frobenius, of $\ell[\cdot]$ and it is well-known that this expression is in the limit-circle case at each endpoint. Consequently, the deficiency index of the minimal operator $T_{1, \text{min}}$ is $(2, 2)$. This implies that there is a continuum of self-adjoint restrictions of $T_{1, \text{max}}$. The GKN Theorem [17, Theorem 4, Section 18.1] (see also [1, Volume II, Chapter 8] and [6, Chapter XIII]) provides a ‘recipe’ for determining each of these operators. We are interested in that particular self-adjoint restriction $A$ which has the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions.

This Legendre polynomials operator $A : D(A) \subset L^2(-1, 1) \to L^2(-1, 1)$ is specifically given by
\begin{equation}
Af = \ell[f], \quad f \in D(A),
\end{equation}
where
\begin{equation}
D(A) := \{ f \in \Delta_{1, \text{max}} \mid \lim_{x \to \pm 1} (1 - x^2)f'(x) = 0 \}.
\end{equation}

We note that the boundary conditions expressed in (2.5) are equivalent to
\[ [f, 1]_1(\pm 1) = 0 \quad (f \in \Delta_{1, \text{max}}). \]
Furthermore, it is well known that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ form a complete (orthogonal) set of eigenfunctions of $A$ and the spectrum $\sigma(A)$ is discrete and given explicitly by
\[ \sigma(A) := \{ n(n+1) \mid n \in \mathbb{N}_0 \}. \]

For our purposes, it is the case that
\[ (Af, f) = \int_{-1}^{1} \ell[f](x)\overline{f}(x)dx = \int_{-1}^{1} (1 - x^2)|f'(x)|^2 \, dx \geq 0 \quad (f \in D(A)); \]
that is to say, $A$ is a positive operator. The positivity of $A$ implies that the left-definite theory developed by Littlejohn and Wellman in [15] can be used to determine $D(A^n)$ for each $n \in \mathbb{N}$; indeed, see [14].

The following theorem, shown by Everitt, Littlejohn and Marić in [11], lists several equivalent conditions for a function $f$ to belong to $D(A)$. Note the surprising, and remarkable, equivalence of conditions (ii) and (iii) (and (ii) and (v)) below; parts (ii) and (v) will be of particular use to us in this paper.

**Theorem 1.** Let $f \in \Delta_{1, \text{max}}$, where $\Delta_{1, \text{max}}$ is given in (2.2). The following conditions are equivalent:

1. $f \in D(A)$;
2. $f' \in L^2(-1, 1)$;
3. $f' \in L^1(-1, 1)$;
4. $f$ is bounded on $(-1, 1)$;
5. $f \in AC[-1, 1]$;
6. $(1 - x^2)^{1/2}f' \in L^2(-1, 1)$;
7. $(1 - x^2)f'' \in L^2(-1, 1)$. 

3. The Square of the Legendre Polynomials Operator

The square \( A^2 : D(A^2) \subset L^2(-1,1) \to L^2(-1,1) \) of the Legendre polynomials operator \( A \) in \( L^2(-1,1) \) is algebraically defined by

\[
A^2 f := \ell^2[f]
\]

for \( f \in D(A^2) \), where \( D(A^2) \) is defined in (3.1), and where

\[
\ell^2[y](x) := ((1-x^2)^2 y''(x))'' - 2 ((1-x^2)y'(x))' = (1-x^2)^2 y^{(4)}(x) - 8x(1-x^2)y''(x) + (14x^2 - 6)y''(x) + 4xy'(x).
\]

By standard results from functional analysis (specifically, the Hilbert space spectral theorem), it can be shown that \( A^2 \) is a self-adjoint operator in \( L^2(-1,1) \), the spectrum of \( A^2 \) is given by \( \sigma(A^2) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0 \} \) and the Legendre polynomials \( \{P_n\}_{n=0}^\infty \) are eigenfunctions of \( A^2 \).

It is natural to ask whether we can explicitly describe the functions in the domain \( D(A) \) similar to how we characterize elements in \( D(A) \) as in (2.5) (or by Theorem 1). In the next section, we identify \( A^2 \) with a self-adjoint operator \( S \) obtained through an application of the GKN theory.

4. A GKN Self-Adjoint Operator Generated by the Square of the Legendre Differential Expression

The maximal domain \( \Delta_{2,\text{max}} \) in \( L^2(-1,1) \) associated with the square of the Legendre expression \( \ell^2[\cdot] \), defined in (3.2), is given by

\[
\Delta_{2,\text{max}} := \{ f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1,1); f, \ell^2[f] \in L^2(-1,1) \}.
\]

The sesquilinear form \([\cdot, \cdot]_2(\cdot) : \Delta_{2,\text{max}} \times \Delta_{2,\text{max}} \times (-1,1)\), associated with \( \ell^2[\cdot] \), is defined by

\[
[f, g]_2(x) := ((1-x^2)^2 f''(x))' \overline{g}(x) - ((1-x^2)^2 g''(x))' f(x)
\]

\[
- (1-x^2)^2 f''(x) \overline{g}(x) + (1-x^2)^2 g''(x) f(x)
\]

\[
- 2(1-x^2)f'(x) \overline{g}(x) + 2(1-x^2)f(x) \overline{g}'(x) \quad (x \in (-1,1)).
\]

For \( f, g \in \Delta_{2,\text{max}} \) and \([\alpha, \beta] \subset (-1,1)\), Green’s formula for \( \ell^2[\cdot] \) is given by

\[
\int_{\alpha}^{\beta} \ell^2[f](x) \overline{g}(x) dx - \int_{\alpha}^{\beta} f(x) \overline{\ell^2[g](x)} dx = [f, g]_2(x) |_{\alpha}^{\beta}.
\]

By definition of \( \Delta_{2,\text{max}} \) and Hölder’s inequality, we see that the limits

\[
[f, g]_2(\pm 1) := \lim_{x \to \pm 1} [f, g]_2(x)
\]

exist and are finite for all \( f, g \in \Delta_{2,\text{max}} \). Clearly

\[
P_n \in \Delta_{2,\text{max}} \quad (n \in \mathbb{N}_0),
\]

where \( P_n(x) \) is the \( n^{\text{th}} \) degree Legendre polynomial. In particular, the functions 1 and \( x \) belong to \( \Delta_{2,\text{max}} \).

The endpoints \( x = \pm 1 \) are both regular singular points, in the sense of Frobenius, of \( \ell^2[\cdot] \). The Frobenius indicial equation, at either endpoint, is given by

\[
r^2(r-1)^2 = 0.
\]

It follows, from the general Weyl theory, that each endpoint is in the limit-4 case so the deficiency index of the minimal operator \( T_{2,\text{min}} \), generated by \( \ell^2[\cdot] \), in \( L^2(-1,1) \) is (4.4). Consequently, each self-adjoint operator, generated by \( \ell^2[\cdot] \), in \( L^2(-1,1) \) is determined by restricting \( \Delta_{2,\text{max}} \) to four boundary conditions of the form

\[
[f, f_j](1) - [f, f_j](-1) = 0,
\]

\[
[f, f_j](1) - [f, f_j](-1) = 0,
\]

\[
[f, f_j](1) - [f, f_j](-1) = 0,
\]

\[
[f, f_j](1) - [f, f_j](-1) = 0,
\]

\[
[f, f_j](1) - [f, f_j](-1) = 0.
\]
where \([\cdot, \cdot]_2\) is given in (1.2) and where \(\{f_1, f_2, f_3, f_4\} \subset \Delta_{2, \text{max}}\) is linearly independent modulo the minimal domain \(\Delta_{2, \text{min}}\) defined by

\[
\Delta_{2, \text{min}} := \{f \in \Delta_{2, \text{max}} \mid [f, g]_2|_{-1} = 0 \text{ for all } g \in \Delta_{2, \text{max}}\}.
\]

We now identify a particular self-adjoint operator restriction \(S\) of \(T_{\text{max}}\), generated by \(\ell^2[\cdot, \cdot]\), having the Legendre polynomials \(\{P_n\}_{n=0}^\infty\) as a complete set of eigenfunctions.

For \(j = 1, 2, 3, 4\), define \(f_j \in \Delta_{2, \text{max}} \cap C^4[-1, 1]\) by

\[
f_1(x) = \begin{cases} 1 & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} \quad f_2(x) = \begin{cases} 0 & \text{near } x = 1 \\ 1 & \text{near } x = -1, \end{cases}
\]

\[
f_3(x) = \begin{cases} x & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} \quad f_4(x) = \begin{cases} 0 & \text{near } x = 1 \\ x & \text{near } x = -1, \end{cases}
\]

Proposition 1. The functions \(\{f_j\}_{j=1}^4\), defined in (4.6), are linearly independent modulo \(\Delta_{2, \text{min}}\).

Proof. Calculations show that the functions \(\ln(1 \pm x)\) and \((1 \pm x) \ln(1 \pm x)\) belong to \(\Delta_{2, \text{max}}\). We modify these functions by defining the four functions \(g_j \in \Delta_{2, \text{max}} \cap C^4(-1, 1) \ (j = 1, 2, 3, 4)\)

\[
g_1(x) = \begin{cases} \ln(1 - x) & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} \quad g_2(x) = \begin{cases} 0 & \text{near } x = 1 \\ \ln(1 + x) & \text{near } x = -1, \end{cases}
\]

\[
g_3(x) = \begin{cases} (1 - x) \ln(1 - x) & \text{near } x = 1 \\ 0 & \text{near } x = -1, \end{cases} \quad g_4(x) = \begin{cases} 0 & \text{near } x = 1 \\ (1 + x) \ln(1 + x) & \text{near } x = -1, \end{cases}
\]

Suppose that

\[
\sum_{j=1}^4 \alpha_j f_j \in \Delta_{2, \text{min}};
\]

then, by definition of \(\Delta_{2, \text{min}}\), we see that

\[
\left[\sum_{j=1}^4 \alpha_j f_j, g\right]_1 = 0 \quad (g \in \Delta_{2, \text{max}}),
\]

where \([\cdot, \cdot]_2\) is the sesquilinear form defined in (1.2). A calculation shows that

\[
0 = \left[\sum_{j=1}^4 \alpha_j f_j, g_1\right] = -4\alpha_3
\]

so \(\alpha_3 = 0\). Similarly, we find that \(\alpha_1 = \alpha_2 = \alpha_4 = 0\) after substituting \(g = g_2, g_3, g_4\) into (4.7). This completes the proof. \(\square\)

It is clear that the boundary conditions

\[
[f, f_1]_2(1) = [f, f_3]_2(1) = [f, f_2]_2(-1) = [f, f_4]_2(-1) = 0
\]

are equivalent to the boundary conditions

\[
[f, 1]_2(\pm 1) = [f, x]_2(\pm 1) = 0.
\]

We are now in position to define the operator \(S\) which we shall later (see Section 7) to be equal to the operator \(A^2\), given in (3.1) and (1.4). Indeed, let \(S : D(S) \subset L^2(-1, 1) \to L^2(-1, 1)\) be defined by

\[
Sf = \ell[\ell[f]] := \ell[\ell[f]]
\]

\(f \in D(S)\).
where the domain $\mathcal{D}(S)$ of $S$ is defined in (1.3). By the GKN Theorem [17] Theorem 4, Section 18.1, $S$ is self-adjoint in $L^2(-1,1)$. Moreover, notice that for $f \in \Delta_{2,\text{max}}$, 

\begin{equation}
[f,1)_2(x) = ((1 - x^2)^2 f^{''}(x))' - 2(1 - x^2) f^{'}(x)
\end{equation}

and 

\begin{equation}
[f,x]_2(x) = ((1 - x^2)^2 f^{''}(x))^x - (1 - x^2)^2 f^{''}(x) - 2x(1 - x^2)f^{'}(x) + 2(1 - x^2)f(x)
\end{equation}

From (4.9) and (4.10), it is easy to see that the Legendre polynomials $\{P_n\}_{n=0}^\infty$ satisfy 

\[ [P_n,1]_2(\pm 1) = [P_n,x]_2(\pm 1) = 0. \]

That is to say, the Legendre polynomials $\{P_n\}_{n=0}^\infty \subset \mathcal{D}(S)$. Moreover $\ell^2[P_n] = \ell[\ell[P_n]] = n(n+1)\ell[P_n] = n^2(n+1)^2P_n \quad (n \in \mathbb{N}_0)$. From [17] and standard results in spectral theory, the following result holds.

**Theorem 2.** The operator $S$, defined in (1.3) and (1.4), is an unbounded self-adjoint operator in $L^2(-1,1)$. The Legendre polynomials $\{P_n\}_{n=0}^\infty$ form a complete set of (orthogonal) eigenfunctions of $S$ in $L^2(-1,1)$. The spectrum $\sigma(S)$ of $S$ is discrete and given explicitly by

\[ \sigma(S) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0 \}. \]

5. Statements of the Main Theorems

There are four main theorems that we prove in this paper.

**Theorem 3.** Let $\mathcal{D}(A^2)$ and $\mathcal{D}(S)$ be given, respectively, as in (1.4) and (1.5). Then

\[ \mathcal{D}(A^2) = \mathcal{D}(S). \]

**Proof.** see Section 7

**Theorem 4.** Let $B$ and $\mathcal{D}(S)$ be given, respectively, as in (1.1) and (1.6). Then

\[ B = \mathcal{D}(S). \]

**Proof.** see Section 8

**Theorem 5.** Let $\mathcal{D}(S)$ and $D$ be given, respectively, as in (1.5) and (1.6). Then

\[ D = \mathcal{D}(S). \]

**Proof.** see Section 9

From these three theorems, we obtain our main result, namely

**Theorem 6.** Let $\Delta_{2,\text{max}}$, given in (1.4), be the maximal domain of the formal square $\ell^2[\cdot]$ of the Legendre differential expression defined by

\[ \ell^2[y](x) = ((1 - x^2)^2 y^{''}(x))^{''} - 2((1 - x^2)y^{'}(x))' \quad (x \in (-1,1)) \]

and let $[\cdot,\cdot]_2$ be the associated sesquilinear form for $\ell^2[\cdot]$ given in (1.2). Define the operator $T : \mathcal{D}(T) \subset L^2(-1,1) \rightarrow L^2(-1,1)$ by

\[ (T f)(x) = \ell^2[f](x) \quad (a.e. \ x \in (-1,1)) \]

\[ f \in \mathcal{D}(T) := \mathcal{D}(A^2), \]

where $\mathcal{D}(A^2)$, algebraically defined in (1.4), is the domain of the square of the Legendre polynomials operator $A$ defined in (2.2). That is to say, $T$ is the square of the classical Legendre polynomials operator $A$, given in (2.2) and (2.3). Then the following statements are equivalent:
Moreover, the following inequalities hold.

Then a necessary and sufficient condition that \( T \) is a self-adjoint operator in \( L^2(-1, 1) \) having the Legendre polynomials \( \{ P_n \}_{n=0}^\infty \) as a complete set of eigenfunctions in \( L^2(-1, 1) \) and having discrete spectrum \( \sigma(T^2) \) explicitly given by

\[
\sigma(T^2) = \{ n^2(n + 1)^2 \mid n \in \mathbb{N}_0 \}.
\]

6. A Key Integral Inequality

A key result in our analysis below is the following operator inequality established by Chisholm and Everitt (CE) in [5].

**Theorem 7. (The CE Theorem)** Let \((a, b)\) be an open interval of the real line (bounded or unbounded) and let \( w \) be a Lebesgue measurable function that is positive a.e. \( x \in (a, b) \). Suppose \( \varphi, \psi : (a, b) \to \mathbb{C} \) satisfy the conditions

(i) \( \varphi, \psi \in L^2_{\text{loc}}((a, b); w); \)
(ii) there exists \( c \in (a, b) \) such that \( \varphi \in L^2((a, c]; w) \) and \( \psi \in L^2((c, b]; w); \)
(iii) for all \([a, \beta] \subset (a, b)\)

\[
\int_{\alpha}^{\beta} |\varphi(x)|^2 w(x)dx > 0 \text{ and } \int_{\alpha}^{\beta} |\psi(x)|^2 w(x)dx > 0.
\]

Define the linear operators \( A, B : L^2((a, b); w) \to L^2_{\text{loc}}((a, b); w) \) by

\[
(Af)(x) = \varphi(x) \int_{x}^{b} \psi(t)f(t)w(t)dt \quad (t \in (a, b); f \in L^2((a, b); w)),
\]
and

\[
(Bf)(x) = \psi(x) \int_{a}^{x} \varphi(t)f(t)w(t)dt \quad (t \in (a, b); f \in L^2((a, b); w)).
\]

Let \( K : (a, b) \to (0, \infty) \) be given by

\[
K(x) := \left( \int_{a}^{x} |\varphi(t)|^2 w(t)dt \right)^{1/2} \left( \int_{x}^{b} |\psi(t)|^2 w(t)dt \right)^{1/2} \quad (t \in (a, b)),
\]
and define \( K \in [0, \infty] \) by

\[
K := \sup \{ K(x) \mid x \in (a, b) \}.
\]

Then a necessary and sufficient condition that \( A \) and \( B \) are both bounded operators from \( L^2((a, b); w) \) into \( L^2((a, b); w) \) is that

\[
0 < K < \infty.
\]

Moreover, the following inequalities hold

\[
\|Af\| \leq 2K \|f\| \quad (f \in L^2((a, b); w)) \tag{6.3}
\]
\[
\|Bg\| \leq 2K \|g\| \quad (g \in L^2((a, b); w)) \tag{6.4}
\]
where the number \( K \) is defined by \( \| \). In general, the number \( 2K \) appearing in both \( (6.3) \) and \( (6.4) \) is best possible for these inequalities to hold.
Remark 1. Theorem [7] proven by Chisholm and Everitt in 1970, was extended in 1999 by Chisholm, Everitt and Littlejohn to the spaces $L^p((a,b);w)$ and $L^q((a,b);w)$ where $p,q > 1$ are conjugate indices; see [9]. Both Theorem [4] and its generalization in [9] have seen several applications including a new proof of the classical Hardy integral inequality [13] Section 9.8, Theorem 327 (see [9] Example 1) and numerous applications to orthogonal polynomials (for example, see [9] Section 6). Several more applications of the CE Theorem will be given in this paper. Indeed, Theorem [7] proves to be an indispensable tool in our analysis below.

7. PROOF OF THEOREM 3

We now prove Theorem 3 namely that $D(A^2) = D(S)$, where $D(A^2)$ is defined in [1,4] and $D(S)$ is given in [1,4]. Throughout this section, we assume that $f$ is a real-valued function on $(-1,1)$.

Proof. $D(S) \subset D(A^2)$: Let $f \in D(S)$. We know that

(i) $f, f', f'', f''' \in AC_{\text{loc}}(-1,1)$;
(ii) $f \in L^2(-1,1)$;
(iii) $\ell^2[f] \in L^2(-1,1)$ where $\ell^2[\cdot]$ is defined by (5.2);
(iv) $[f, 1]_2(\pm 1) = 0$, where $[\cdot, 1]_2(\cdot)$ is given in (4.9);
(v) $[f, x]_2(\pm 1) = 0$, where $[\cdot, x]_2(\cdot)$ is given in (4.10).

Taking into account the definition of $D(A)$ in (2.5) and $D(A^2)$ in (1.4), we need to show that

(a) $f, f' \in AC_{\text{loc}}(-1,1)$;
(b) $f \in L^2(-1,1)$;
(c) $\ell[f] = -(1 - x^2)f'' = -(1 - x^2)f'' + 2xf' \in L^2(-1,1)$; in fact we will show that $\ell[f] \in AC[-1,1]$;
(d) $\lim_{x \to \pm 1} (1 - x^2)f'(x) = 0$;
(e) $\ell[f], \ell'[f] \in AC_{\text{loc}}(-1,1)$;
(f) $\ell^2[f] \in L^2(-1,1)$;
(g) $\lim_{x \to \pm 1} (1 - x^2)\ell'[f] = \lim_{x \to \pm 1} (1 - x^2)((1 - x^2)f''(x) - 4xf''(x) - 2f'(x)) = 0$.

Clearly, (a), (b) and (f) are satisfied. As for (g), note that

$$-\ell'(f)(x) = \ell[\ell(f)](x) = -(1 - x^2)(1 - x^2)f''(x) - 2(1 - x^2)f'(x)$$

which implies

$$\ell^2[f](x) = \ell[\ell(f)](x) = -(1 - x^2)f''(x)$$

(7.1)

so (g) follows from (iv) above. Moreover, by (i) and the fact that the product of a polynomial and a function $g \in AC_{\text{loc}}(-1,1)$ also belongs to $AC_{\text{loc}}(-1,1)$, we see that (e) follows. To show (c) note that, by (iii),

$$\ell^2[f](x) = \ell[\ell(f)](x) = -(1 - x^2)f''(x)$$

(7.2)

We now apply the CE Theorem on the interval $[0,1]$ with $\psi(x) = 1$, $\varphi(x) = 1/(1 - x^2)$ and $w(x) = 1$; note that $\varphi \in L^2(0,1/2)$ and $\psi \in L^2[1/2,1)$. A calculation shows that

$$K^2(x) = \int_0^x \frac{dt}{(1 - t^2)^2} \cdot \int_x^1 dt \quad (x \in (0,1))$$

is bounded on $(0,1)$. Hence we see, from Theorem [7] that

$$\varphi(x) \int_x^1 \psi(t)\ell^2[f](t)w(t)dt = \frac{1}{1 - x^2} \int_x^1 \ell^2[f](t)dt \in L^2[0,1].$$
That is to say, by (7.2),
\[(7.3) \quad \frac{1}{1-x^2} \left( (1-x^2)\ell'[f](x) - \lim_{x \to 1} (1-x^2)\ell'[f](x) \right) \in L^2(0,1). \]
By (iv) and (7.1), we know
\[\lim_{x \to 1} (1-x^2)\ell'[f](x) = 0. \]
Hence, (7.3) simplifies to
\[\ell'[f] \in L^2(0,1). \]
A similar application of the CE Theorem on \((-1,0]\) reveals that \(\ell'[f] \in L^2(-1,0]\) and thus we see that
\[\ell'[f] \in L^2(-1,1). \]
It follows that
\[\ell[f] \in AC[-1,1] \subset L^2(-1,1), \]
establishing (c). It remains to show that (d) holds. To this end, observe, from (2.1) and (3.2) that
\[(1-x^2)^2 f''(x) \in L^2(-1,1) \]
from which we see that
\[(1-x^2)^2 f''(x) \in AC[-1,1]. \]
In particular, we see that the limits
\[(7.4) \quad \lim_{x \to \pm 1} (1-x^2)^2 f'(x) \]
and
\[(7.5) \quad \lim_{x \to \pm 1} (1-x^2)^2 f''(x) \]
exist and are finite. Moreover, from (iv), (v) and (4.10), we see that
\[(7.6) \quad 0 = \lim_{x \to \pm 1} (x[f,1]|_{-1}^x - z[f,2](x)) = \lim_{x \to \pm 1} ((1-x^2)^2 f''(x) - 2(1-x^2) f(x)) \]
in concert with (7.5), we can say that
\[\lim_{x \to \pm 1} (1-x^2) f(x) = 0 \]
exists and is finite. We claim that \(r = 0\); to show this, we deal with the limit as \(x \to 1\); a similar proof can be made as \(x \to -1\). Suppose, to the contrary, that \(r \neq 0\); without loss of generality, suppose \(r > 0\). Then there exists \(x^* > 0\) such that
\[(1-x^2)f(x) \geq \frac{r}{2} \text{ for } x \in [x^*,1]. \]
However, in this case, we see that
\[\infty > \int_{-1}^{1} |f(x)|^2 \, dx \geq \int_{x^*}^{1} |f(x)|^2 \, dx \geq \left( \frac{r}{2} \right)^2 \int_{x^*}^{1} \frac{dx}{(1-x^2)^2} = \infty, \]
contradicting (ii). Hence it follows that
\[\lim_{x \to \pm 1} (1-x^2) f(x) = 0. \]
Consequently, we see from (7.6), that
\[\lim_{x \to \pm 1} (1-x^2)^2 f''(x) = 0. \]
and, hence
\[(7.7) \quad \lim_{x \to \pm 1} (1 - x)^2 f''(x) = 0.\]

We are now in position to prove part (d). We show that
\[(7.8) \quad \lim_{x \to 1} (1 - x^2) f'(x) = 0;\]
a similar argument establishes the limit as \(x \to -1\). Let \(\varepsilon > 0\). From \((7.7)\), there exists \(x^* \in (0, 1)\) such that
\[|(1 - x)^2 f''(x)| < \frac{\varepsilon}{2} \text{ for } x \in [x^*, 1).\]

Integrating this inequality over \([x^*, x] \subset [x^*, 1)\) yields
\[\frac{\varepsilon}{2(1 - x^*)} + f'(x^*) - \frac{\varepsilon}{2(1 - x)} < f'(x) < \frac{\varepsilon}{2(1 - x)} + f'(x^*) - \frac{\varepsilon}{2(1 - x^*)} \text{ for } x \in [x^*, 1).\]

Multiplying this inequality by \((1 - x^2)\) yields
\[(7.9) \quad (1 - x^2) \left( f'(x^*) + \frac{\varepsilon}{2(1 - x^*)} \right) - \frac{\varepsilon(1 + x)}{2} < (1 - x^2) f'(x) < \frac{\varepsilon(1 + x)}{2} + (1 - x^2) \left( f'(x^*) - \frac{\varepsilon}{2(1 - x^*)} \right).\]

Letting \(x \to 1\), we obtain
\[-\varepsilon \leq \lim_{x \to 1} (1 - x^2) f'(x) \leq \varepsilon\]
and this establishes \((7.8)\). This completes the proof that \(D(S) \subset D(A^2)\).

Let \(f \in D(A^2)\). Then \(f \in D(A)\) so
\[(7.10) \quad f, f' \in AC_{\text{loc}}(-1, 1)\]
and
\[(7.11) \quad f \in L^2(-1, 1).\]

Moreover, since \(\ell[f] \in D(A)\), it follows that
\[(7.12) \quad \ell^2[f] = \ell[\ell[f]] \in L^2(-1, 1),\]

\[(7.13) \quad \ell[f] = -(1 - x^2) f'' + 2x f' \in AC_{\text{loc}}(-1, 1)\]
and
\[(7.14) \quad \ell'[f] = -(1 - x^2) f''' + 4x f'' + 2f' \in AC_{\text{loc}}(-1, 1).\]

It is clear that if \(f, g \in AC_{\text{loc}}(-1, 1)\) then
- (a') \(f + g \in AC_{\text{loc}}(-1, 1)\);
- (b') \(fg \in AC_{\text{loc}}(-1, 1)\);
- (c') \(\text{If } g > 0 \text{ on } (-1, 1) \text{ then } f/g \in AC_{\text{loc}}(-1, 1)\).

In particular, from \((7.10)\) and (b'), we see that \(2xf' \in AC_{\text{loc}}(-1, 1)\). Combining this with (a') and \((7.13)\), we obtain \((1 - x^2)f'' \in AC_{\text{loc}}(-1, 1)\). Since \(1 - x^2 > 0\) on \((-1, 1)\) we infer from (c') that
\[(7.15) \quad f'' \in AC_{\text{loc}}(-1, 1).\]

Continuing, \(-4xf'' - 2f' \in AC_{\text{loc}}(-1, 1)\) so from (a') and \((7.14)\), we have \((1 - x^2)f''' \in AC_{\text{loc}}(-1, 1)\) and it then follows that
\[(7.16) \quad f''' \in AC_{\text{loc}}(-1, 1).\]
By definition of $D(A)$ and the fact that $\ell[f] \in D(A)$, we see that
\[
\lim_{x \to \pm 1} (1 - x^2)\ell'[f](x) = 0;
\]
consequently, in view of (7.1), we see that
\[
(7.17) 0 = \lim_{x \to \pm 1} [f, 1]_2(x) = \lim_{x \to \pm 1} \left( (1 - x^2)^2 f''(x) \right)' - 2(1 - x^2) f'(x). \]
Furthermore since $f \in D(A)$, we have
\[
(7.18) \lim_{x \to \pm 1} (1 - x^2) f'(x) = 0
\]
so, from (7.17), we see that
\[
(7.19) \lim_{x \to \pm 1} ((1 - x^2)^2 f''(x))'' = 0.
\]
To finish the proof, we need to show that
\[
(7.20) 0 = [f, x]_2(\pm 1) = \lim_{x \to \pm 1} (x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2) f(x))
\]
\[
= \lim_{x \to \pm 1} (-(1 - x^2)^2 f''(x) + 2(1 - x^2) f(x)) \text{ by (7.17)}.
\]
We note again, from Green’s formula (4.3), that the limits in (7.20) exist and are finite. Since $f \in D(A)$, we see from Theorem 1 part (v) that $f \in AC[-1, 1]$ and hence
\[
(7.21) \lim_{x \to \pm 1} (1 - x^2) f(x) = 0.
\]
Thus, proving (7.20) reduces to showing
\[
(7.22) \lim_{x \to \pm 1} (1 - x^2)^2 f''(x) = 0.
\]
We show that
\[
(7.23) \lim_{x \to 1} (1 - x^2)^2 f''(x) = 0;
\]
a similar argument will show
\[
\lim_{x \to -1} (1 - x^2)^2 f''(x) = 0.
\]
Suppose, to the contrary, that
\[
\lim_{x \to 1} (1 - x^2)^2 f''(x) = c \neq 0;
\]
without loss of any generality, we can suppose that $c > 0$. Then there exists $x^* \in (0, 1)$ such that
\[
(1 - x^2)^2 f''(x) \geq r := \frac{c}{2} \text{ on } [x^*, 1);
\]
that is,
\[
f''(x) \geq \frac{R}{(1 - x)^2} \text{ on } [x^*, 1)
\]
for some $R > 0$. Integrating this inequality over $[x^*, x] \subset [x^*, 1)$ yields
\[
f'(x) \geq R \int_{x^*}^{x} \frac{dt}{(1 - t)^2} + f'(x^*)
\]
\[
= \frac{R}{1 - x} + f'(x^*) - \frac{R}{1 - x^*}.
\]
Consequently,

\[(1 - x^2)f'(x) \geq R(1 + x) + (1 - x^2) \left( f'(x) - \frac{R}{1 - x^2} \right) \]

\[\rightarrow 2R > 0 \quad \text{(as } x \to 1)\]

contradicting (7.18). It follows that (7.20) holds and this proves (7.20). Combining (7.10), (7.11), (7.12), (7.15), (7.16) and (7.20), we see that \(f \in \mathcal{D}(A^2)\) implies \(f \in \mathcal{D}(S)\). This completes the proof of the theorem. \(\square\)

8. Proof of Theorem \(\text{[4]}\)

In order to prove Theorem \(\text{[4]}\) we first need to establish three preliminary facts, the first of which is the following result.

**Lemma 1.** If \(f \in \mathcal{D}(S)\), then

\[(8.1) \quad \frac{1}{1 - x^2} \left( (1 - x^2)^2 f''(x) \right)' \in L^2(-1, 1).\]

**Proof.** Let \(f \in \mathcal{D}(S) = \mathcal{D}(A^2)\) so \(f' \in L^2(-1, 1), \ [f, 1]_{\mathcal{L}^2}(\pm 1) = 0\) and \(\ell^2[f] \in L^2(-1, 1)\). We apply the CE Theorem on \([0, 1]\) with \(\psi(x) = 1, \varphi(x) = -1/(1 - x^2)\) and \(w(x) = 1\). These functions satisfy the conditions of this theorem on \([0, 1]\) so

\[\frac{-1}{1 - x^2} \int_x^1 \ell^2[f](t)dt \in L^2(0, 1).\]

However, using (4.19), a calculation shows

\[\frac{-1}{1 - x^2} \int_x^1 \ell^2[f](t)dt = \frac{-1}{1 - x^2} \int_x^1 \left[ ((1 - t^2)^2 f''(t))'' - 2(1 - t^2)f'(t) \right] dt\]

\[= \frac{-1}{1 - x^2} \left[ \lim_{x \to 1} \left( ((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x) \right) \right] + \frac{1}{1 - x^2} \left[ ((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x) \right]\]

\[= \frac{-1}{1 - x^2} \left[ \lim_{x \to 1} \left[ f, 1 \right]_{\mathcal{L}^2}(x) - ((1 - x^2)^2 f''(x))' + 2(1 - x^2)f'(x) \right] \]

\[= \frac{1}{1 - x^2} \left( (1 - x^2)^2 f''(x) \right)' - 2f'(x).\]

A similar calculation shows that

\[\frac{1}{1 - x^2} \left( (1 - x^2)^2 f''(x) \right)' - 2f'(x) \in L^2(-1, 0)\]

and hence

\[\frac{1}{1 - x^2} \left( (1 - x^2)^2 f''(x) \right)' - 2f'(x) \in L^2(-1, 1).\]

Since \(f' \in L^2(-1, 1)\), we see, by linearity, that

\[\frac{1}{1 - x^2} \left( (1 - x^2)^2 f''(x) \right)' \in L^2(-1, 1).\]

\(\square\)

**Lemma 2.** For \(f \in \mathcal{D}(S)\), we have

\[(8.2) \quad \lim_{x \to \pm 1} (1 - x^2)^2 f''(x) = 0.\]
Proof. Let \( f \in D(S) = D(A^2) \). Since \( f \in D(A) \), we have \( f \in AC[-1,1] \) so
\[
\lim_{x \to \pm 1} (1 - x^2)f(x) = 0.
\]
Furthermore, we have
\[
0 = \lim_{x \to \pm 1} [f,1]_2(x) = \lim_{x \to \pm 1} \left( (1 - x^2)^2 f''(x) \right) - 2(1 - x^2)f'(x).
\]
Consequently, from (8.3) and (8.4), we find that
\[
0 = \lim_{x \to \pm 1} [f,x]_2(x) = \lim_{x \to \pm 1} \left( (1 - x^2)^2 f''(x) + 2(1 - x^2)f(x) \right) = -\lim_{x \to \pm 1} (1 - x^2)^2 f''(x).
\]
\( \blacksquare \)

The last preliminary result is the following theorem. Since \( D(S) = D(A^2) \), this next result generalizes the well-known result for \( D(A) \) established in Theorem 11 part (v).

**Theorem 8.** If \( f \in D(S) \), then
\[
f'' \in L^2(-1,1).
\]
Moreover,
\[
pf'' \in L^2(-1,1)
\]
for any bounded, Lebesgue measurable function \( p \), including any polynomial.

Proof. Once we establish \( f'' \in L^2(-1,1) \), the statement in (8.5), for any bounded measurable function, follows clearly. Let \( f \in D(S) \). We prove that \( f'' \in L^2(0,1) \); a similar proof will establish \( f'' \in L^2(-1,0) \) and prove the theorem. We again use the CE Theorem with \( \psi(x) = 1 - x^2 \), \( \varphi(x) = 1/(1 - x^2)^2 \) and \( w(x) = 1 \) on [0,1). Indeed, from the CE Theorem and (8.1), we find that
\[
-\frac{1}{(1 - x^2)^2} \int_x^1 (1 - t^2) \left( 1 - t^2 \right) \left( (1 - t^2)^2 f''(t) \right) \, dt \in L^2(0,1).
\]
However, from Lemma 2
\[
-\frac{1}{(1 - x^2)^2} \int_x^1 (1 - t^2) \left( 1 - t^2 \right) \left( (1 - t^2)^2 f''(t) \right) \, dt = -\frac{1}{(1 - x^2)^2} \left( \lim_{x \to 1} (1 - x^2)^2 f''(x) - (1 - x^2)^2 f''(x) \right) = f''(x).
\]
\( \blacksquare \)

We are now in position to prove Theorem 4 specifically \( B = D(S) \), where \( B \) is defined in (1.2) and \( D(S) \) is given in (1.1).

**Proof.** \( B \subset D(S) \):

Let \( f \in B \). We assume that \( f \) is real-valued on \((-1,1)\). We begin by showing, using the CE Theorem, that the condition
\[
(1 - x^2)^2 f^{(4)} \in L^2(-1,1)
\]
implies the two conditions
\[
(1 - x^2) f''' \in L^2(-1,1)
\]
and
\[
f'' \in L^2(-1,1).
\]
Consequently, we will show
\[(8.8) \quad (1 - x^2)f''' \in L^2(0, 1); \]
a similar proof will yield
\[(8.9) \quad (1 - x^2)f'''' \in L^2(-1, 0) \]
and, together, they establish \[(8.6)\]. Since \((1 - x^2)^2 f^{(4)} \in L^2(0, 1)\), we use the CE Theorem on \([0, 1]\) with
\[\varphi(x) = (1 - x^2)^{-2}, \quad \psi(x) = 1 - x^2 \quad \text{and} \quad w(x) = 1 \quad (x \in [0, 1]).\]

It follows that
\[(1 - x^2)f'''(x) = (1 - x^2) \int_0^x \frac{1}{(1 - t^2)^2} (1 - t^2)^2 f^{(4)}(t) dt + f'''(0)(1 - x^2) \in L^2(0, 1).\]

To see (8.7), we apply the CE Theorem once again on \([0, 1]\) to prove that
\[f''' \in L^2(0, 1);\]
a similar argument will show that \(f''' \in L^2(-1, 0)\). To this end, let
\[\varphi(x) = (1 - x^2)^{-1}, \quad \psi(x) = 1 \quad \text{and} \quad w(x) = 1 \quad (x \in [0, 1]).\]

In this case, we see that
\[f''(x) = \int_0^x \frac{1}{1 - t^2} \left((1 - t^2)f'''(t)\right) dt + f''(0) \in L^2(0, 1).\]

Consequently, we see that
\[f, f' \in AC[-1, 1] \subset L^2(-1, 1).\]

Moreover, it is clear that \(g(x)(1 - x^2)f''(x), g(x)f''(x)\) and \(g(x)f'(x)\) all belong to \(L^2(-1, 1)\) for any bounded, measurable function \(g\) on \((-1, 1)\). Hence
\[L^2[f](x) = (1 - x^2)^2 f^{(4)}(x) - 8x(1 - x^2)f'''(x) + (14x^2 - 6)f''(x) + 4xf'(x) \in L^2(-1, 1)\]

and, in particular,
\[(8.10) \quad 4xf' \in L^2(-1, 1).\]

It remains to show that
\[(8.11) \quad \lim_{x \to \pm 1} [f, 1]_2(x) = \lim_{x \to \pm 1} [f, x]_2(x) = 0.\]

Since \(1, x \in \Delta_{2,\text{max}}\), we see from Green’s formula in (4.3) that the limits in (8.11) both exist and are finite. Now \(f' \in AC[-1, 1]\) so
\[\lim_{x \to \pm 1} (1 - x^2)f'(x) = 0.\]

Consequently,
\[\lim_{x \to \pm 1} [f, 1]_2(x) = \lim_{x \to \pm 1} (((1 - x^2)^2 f''(x))' - 2(1 - x^2)f'(x))
\[= \lim_{x \to \pm 1} ((1 - x^2)^2 f''(x))'.\]

We claim that
\[(8.12) \quad \lim_{x \to \pm 1} ((1 - x^2)^2 f''(x))' = 0;\]
a similar proof will establish
\[ \lim_{x \to -1} ((1 - x^2)^2 f''(x))' = 0. \]
Suppose to the contrary that
\[ \lim_{x \to 1} ((1 - x^2)^2 f''(x))' = c \neq 0; \]
we can assume that \( c > 0 \). It follows that there exists \( x^* \in (0, 1) \) such that
\[ ((1 - x^2)^2 f''(x))' \geq r := \frac{c}{2} > 0 \quad (x \in [x^*, 1]). \]
Note that since
\[ (1 - x^2)^2 f''(x) \geq 4x(1 - x^2)f''(x), \]
we see that the inequality in (8.13) can be rewritten as
\[ (1 - x^2)f'''(x) - 4xf''(x) \geq \frac{r}{1 - x^2} \quad \text{on } [x^*, 1). \]
However, from (8.5) and (8.6), we see that
\[ (1 - x^2)f'''(x) - 4xf''(x) \in L^2(-1, 1) \]
so the inequality in (8.14) is not possible. Hence (8.12) is established and thus
\[ \lim_{x \to 1} [f, 1]_2(x) = 0. \]
We now show that
\[ \lim_{x \to 1} [f, x]_2(x) = 0. \]
Since the argument for \( x \to -1 \) mirrors the proof for \( x \to 1 \), we will only show that
\[ \lim_{x \to 1} [f, x]_2(x) = 0. \]
Now, since \( f \in AC[-1, 1] \), we see that \( \lim_{x \to 1} (1 - x^2)f(x) = 0 \); moreover, using (8.12),
\[ \lim_{x \to 1} [f, x]_2(x) = \lim_{x \to 1} \left( x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2)f(x) \right) = -\lim_{x \to 1} (1 - x^2)^2 f''(x). \]
Suppose that
\[ \lim_{x \to 1} (1 - x^2)^2 f''(x) = d \neq 0; \]
we can assume that \( d > 0 \). Then, with possibly different \( x^* \) as given in the above argument, there exists an \( x^* \in (0, 1) \) with
\[ (1 - x^2)^2 f''(x) \geq d' := \frac{d}{2} \quad (x \in [x^*, 1]). \]
Hence
\[ f''(x) \geq \frac{d'}{(1 - x^2)^2} \quad (x \in [x^*, 1]). \]
However, this implies that \( f'' \notin L^2(0, 1) \), contradicting (8.7). Thus (8.16) is established and this completes the proof that \( B \subset D(S) \).
\[ D(S) \subset B: \]
Let \( f \in D(S) \). We need only to show that
\[ (1 - x^2)^2 f^{(4)}(x) \in L^2(-1, 1). \]
Since, by Theorem 8, \( f'' \in L^2(-1, 1) \), we see that \( gf'' \in L^2(-1, 1) \) for any bounded, measurable function \( g \) on \((-1, 1)\). In particular, it is the case that
\[ 4xf'' \in L^2(-1, 1) \]
and
\[(8.18) \quad (14x^2 - 6)f'' \in L^2(-1, 1).\]
By (8.17),
\[(8.19) \quad (1 - x^2)f'''(x) - 4xf''(x) = 1 - x^2 \left( (1 - x^2)^2 f''(x) \right)' \in L^2(-1, 1).\]
By linearity, it follows from (8.17) and (8.19) that
\[(1 - x^2)f''' \in L^2(-1, 1).\]
Consequently, \(g(1 - x^2)f''' \in L^2(-1, 1)\) for every bounded, measurable function \(g\) on \((-1, 1)\); in particular,
\[(8.20) \quad 8xf'(x) \in L^2(-1, 1).\]
Furthermore, since \(f' \in L^2(-1, 1)\), it follows that
\[(8.21) \quad 4xf'(x) \in L^2(-1, 1).\]
Finally, since \(\ell^2[f] \in L^2(-1, 1)\), we see from (8.20) and (8.21) that
\[(1 - x^2)^2 f^{(4)} = \ell^2[f] + 8x(1 - x^2)f'' - (14x^2 - 6)f'' - 4xf' \in L^2(-1, 1).\]
This establishes (8.16) and proves \(\mathcal{D}(S) \subset B\). This completes the proof of Theorem 4.

9. Proof of Theorem 5

We now prove Theorem 5 namely \(\mathcal{D}(S) = D\), where \(\mathcal{D}(S)\) is given in (1.5) and \(D\) is defined in (1.6).

Proof: Since functions \(f\) in both \(\mathcal{D}(S)\) and \(D\) satisfy the ‘maximal domain’ conditions \(f^{(j)} \in AC_{\text{loc}}(-1, 1)\) \((j = 0, 1, 2, 3)\), \(f \in L^2(-1, 1)\) and \(\ell^2[f] \in L^2(-1, 1)\), we need only to prove that the other properties in their definitions hold.

\(\mathcal{D}(S) \subset D\): Let \(f \in \mathcal{D}(S) = \mathcal{D}(A^2)\). Then \(f \in \mathcal{D}(A)\) so
\[(9.1) \quad \lim_{x \to \pm 1} (1 - x^2)f'(x) = 0.\]
Moreover,
\[(9.2) \quad 0 = [f, 1]_2(\pm 1) = \lim_{x \to \pm 1} \left( (1 - x^2)^2 f''(x) \right)' - 2(1 - x^2)f'(x).\]
The identities in (9.1) and (9.2) prove that \(\mathcal{D}(S) \subset D\).

\(D \subset \mathcal{D}(S)\): Let \(f \in D\). Clearly,
\[(9.3) \quad [f, 1]_2(\pm 1) = \lim_{x \to \pm 1} \left( (1 - x^2)^2 f''(x) \right)' - 2(1 - x^2)f'(x) = 0\]
so we need to show that
\[(9.4) \quad \lim_{x \to \pm 1} [f, x]_2(\pm 1) = 0.\]
We remark that the limits in (9.4) exist (by Green’s formula) and are finite.

Claim: \( \ell'[f] \in L^2(-1,1) \).

To see this, recall the two representations of \( \ell^2[f] \): the one given in (3.2) and the one given in (7.2). Since \( \ell^2[f] \in L^2(-1,1) \), we apply the CE Theorem on \([0,1]\) with \( \varphi(x) = (1 - x^2)^{-1} \), \( \psi(x) = 1 \) and \( w(x) = 1 \) to obtain

\[
\frac{1}{1 - x^2} \int_x^1 \ell^2[f](t)dt \in L^2(0,1).
\]

However, from (3.2) and (7.2), we see that

\[
\frac{1}{1 - x^2} \int_x^1 \ell^2[f](t)dt = \frac{1}{1 - x^2} \left( \lim_{x \to 1} \left( ((1 - x^2)^2 f'(x) )' - 2(1 - x^2) f'(x) + (1 - x^2) \ell'[f](x) \right) \right)
\]

\[
= \ell'[f](x) \text{ by definition of } D;
\]

a similar calculation shows that \( \ell'[f] \in L^2(-1,0) \). It follows that \( \ell[f] \in AC[-1,1] \subset L^2(-1,1) \). We again apply the CE Theorem on \([0,1]\) with \( \varphi(x) = (1 - x^2)^{-1} \), \( \psi(x) = 1 \) and \( w(x) = 1 \) to obtain

\[
\frac{1}{1 - x^2} \int_x^1 \ell[f](t)dt \in L^2(0,1).
\]

Another calculation shows that

\[
\frac{1}{1 - x^2} \int_x^1 \ell[f](t)dt = \frac{-1}{1 - x^2} \int_x^1 ((1 - t^2) f'(t) )' dt
\]

\[
= \frac{-1}{1 - x^2} \left( \lim_{x \to 1} (1 - x^2) f'(x) - (1 - x^2) f'(x) \right)
\]

\[
= f'(x) \text{ by definition of } D;
\]

a similar argument shows that \( f' \in L^2(-1,0) \). Hence

(9.5) \( f' \in L^2(-1,1) \).

Thus, \( f \in AC[-1,1] \) and

(9.6) \( \lim_{x \to \pm 1} (1 - x^2) f(x) = 0 \).

From (9.3) and (9.6), we see that

\[
\lim_{x \to \pm 1} [f, x]_2(x) = \lim_{x \to \pm 1} (x[f, x]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2)^2 f(x))
\]

\[
= - \lim_{x \to \pm 1} (1 - x^2)^2 f''(x).
\]

To establish (9.4), it now suffices to prove that

(9.7) \( \lim_{x \to \pm 1} (1 - x^2)^2 f''(x) = 0 \).

Since the proof as \( x \to -1 \) is similar to the proof that \( x \to 1 \), we will only show that

\[
\lim_{x \to 1} (1 - x^2)^2 f''(x) = 0;
\]

By way of contradiction, suppose that

\[
\lim_{x \to 1} (1 - x^2)^2 f''(x) = c \neq 0;
\]
without loss of generality, we may assume that \( c > 0 \). Then there exists \( x^* \in (0, 1) \) such that

\[
 f''(x) \geq \frac{c}{2(1-x^2)^2} \geq \frac{c}{8(1-x)^2} \quad (x \in [x^*, 1]).
\]

Integrating this inequality over \([x^*, x] \subset [x^*, 1] \) yields

\[
 f'(x) \geq \frac{c}{8(1-x)} + f'(x^*) - \frac{c}{8(1-x^*)} \quad (x \in [x^*, 1]).
\]

But this contradicts (9.5)\(^3\). It follows that (9.7) holds and this, in turn, establishes (9.4)\(^3\). Consequently, \( D \subset \mathcal{D}(S) \) and this completes the proof of the theorem. \( \square \)

As revealed in the proofs of Theorems 3, 4, 5 and 8 we have the following interesting result.

**Corollary 1.** If \( f \in \mathcal{D}(A^2) = \mathcal{D}(S) = B = D \), then

(i) \( f'' \in L^2(-1, 1) \) so \( f, f' \in AC[-1, 1] \);
(ii) \( \ell'[f] \in L^2(-1, 1) \) and \( \ell[f] \in AC[-1, 1] \).

**Remark 2.** As discussed in Section 4\(^4\), the minimal operator \( T_{2,\min} \) in \( L^2(-1, 1) \) generated by \( \ell[\cdot] \) has deficiency index \((4, 4)\). From the GKN Theorem (see [17, Theorem 4, Section 18.1]), GKN boundary conditions for any self-adjoint extension of \( T_{2,\min} \) in \( L^2(-1, 1) \) are restrictions of the maximal domain \( \Delta_{2,\max} \) and have the appearance (see (4.25))

\[
 [f, f]_2(1) - [f, f]_2(-1) = 0 \quad (f \in \Delta_{2,\max}, j = 1, 2, 3, 4),
\]

where \( \{f_j\}_{j=1}^4 \subset \Delta_{2,\max} \) are linearly independent modulo the minimal domain \( \Delta_{2,\min} \). Taking into account \([\cdot, \cdot]_2\), defined in (4.25), it is clear that the boundary conditions given in (10.1)\(^4\) are not GKN boundary conditions.

10. Concluding Remarks

In [11], the authors showed that, for \( n \in \mathbb{N} \), the \( n^{th} \) composite power of the Legendre differential expression \( \ell[\cdot] \) is explicitly given by

\[
 (10.1) \quad \ell^n[y](x) = \sum_{j=1}^n (-1)^j \binom{n}{j} \left( (1-x^2)^j y^{(j)}(x) \right)^{(j)},
\]

where the numbers

\[
 \binom{n}{j} := \sum_{r=0}^j (-1)^{r+j} \frac{(2r + 1) (r^2 + r)^n}{(j-r)! (j+r+1)!}
\]

are the so-called Legendre-Stirling numbers, a subject of current study in combinatorics (for example, see [2, 3, 4, 7, 10] and [12]). The expression in (10.1)\(^4\) is the key in generating the domain \( \mathcal{D}(A^n) \) of \( A^n \) given in (1.1)\(^4\).

We conjecture:

**Conjecture** Let \( A \) denote the Legendre polynomials self-adjoint operator defined in (2.4) and (2.5). For \( n \in \mathbb{N} \), let \( \ell^n[\cdot] \) be given as in (10.1)\(^5\) and let \([\cdot, \cdot]_n\) be the sequilinear form associated with the maximal domain \( \Delta_{n,\max} \) of \( \ell^n[\cdot] \) in \( L^2(-1, 1) \). Then \( A_n = B_n = C_n = D_n \), where

\( \square \)
(i) \( A_n := D(A^n) \),

(ii) \( B_n := \{ f : (-1, 1) \to \mathbb{C} | f, f', \ldots, f^{(2n-1)} \in AC_{\text{loc}}(-1, 1); (1 - x^2)^n f^{(2n)} \in L^2(-1, 1) \} \),

(iii) \( C_n := \{ f : (-1, 1) \to \mathbb{C} | f, f', \ldots, f^{(2n-1)} \in AC_{\text{loc}}(-1, 1); f, f^n[f] \in L^2(-1, 1); [f, x^n]_{\text{loc} j = 1} = 0 \text{ for } j = 0, 1, 2, \ldots, n-1 \} \),

(iv) \( D_n := \{ f : (-1, 1) \to \mathbb{C} | f, f', \ldots, f^{(2n-1)} \in AC_{\text{loc}}(-1, 1); f, f^n[f] \in L^2(-1, 1); \lim_{x \to \pm 1} (1 - x^2)^j y^{(j)}(x) \) for } j = 1, 2, \ldots, n}.\)

By repeated applications of the CE Theorem, it is not difficult to establish that if \( f \in B_n \), then \( f^{(n)} \in L^2(-1, 1) \); this result generalizes Theorem \( \text{[1]} \) part (iii) \( n = 1 \) and Corollary \( \text{[1]} \) part (i) \( n = 2 \).

We remark that, in (iii) above, we can replace the monomials \( \{ x^j \}_{j=0}^{n-1} \) by the Legendre polynomials \( \{ P_j \}_{j=0}^{n-1} \). One of the difficulties in our efforts to try and prove this conjecture lies in the fact that the corresponding sesquilinear form \( [,]_n \), associated with the \( n \)th power \( \ell^n[\cdot] \), is unwieldy at the present time.

Acknowledgement The author LLL would like to recognize the many years of mentoring from W. N. (Norrie) Everitt, who passed away on July 17, 2011 at the age of 87. The Legendre differential expression and Legendre polynomials were among the favorites of Norrie’s many mathematical interests. Quinn and I both felt we were guided by the hand of Norrie in the writing of this paper.

References

[1] N. I. Akhiezer and I. M. Glazman, Theory of linear operators in Hilbert space, Vol. I and II, Pitman Advanced Publishing Program, London, 1981.

[2] G. E. Andrews, E. S. Egge, W. Gawronski and L. L. Littlejohn, The Jacobi-Stirling numbers, J. Combinatorial Theory Ser. A, 120 (2013), 288-303.

[3] G. E. Andrews, W. Gawronski and L. L. Littlejohn, The Legendre-Stirling numbers, Discrete Math. 311 (2011), no. 14, 1255-1272.

[4] G. E. Andrews and L. L. Littlejohn, A combinatorial interpretation of the Legendre-Stirling numbers, Proc. Amer. Math. Soc. 137 (2009), no. 8, 288-303.

[5] R. S. Chisholm and W. N. Everitt, On bounded integral operators in the space of integrable-square functions, Proc. Roy. Soc. Edinb. (A), 69 (1970/71), 199-204.

[6] N. Dunford and J. T. Schwartz, Linear Operators II: Spectral Theory, John Wiley Interscience Publishers, 1963.

[7] G. E. Andrews, Legendre-Stirling permutations, European J. Combin. 31 (2010), 1735-1750.

[8] W. N. Everitt, Legendre polynomials and singular differential operators, Lecture Notes in Mathematics, Volume 827, Springer-Verlag, New York, 1980, 83-106.

[9] W. N. Everitt, R. S. Chisholm and L. L. Littlejohn, An Integral Operator Inequality with Applications, J. of Inequal. & Applications 3(1999), 245-266.

[10] W. N. Everitt, L. L. Littlejohn and V. Marić, On properties of the Legendre differential expression, Result. Math. 42 (2002), 42-68.

[11] W. N. Everitt, L. L. Littlejohn and R. Wellman, Legendre polynomials, Legendre-Stirling numbers, and the left-definite spectral analysis of the Legendre differential expression, J. Comput. Appl. Math., 148(2002), 213-238.

[12] W. Gawronski, L. L. Littlejohn and T. Neuschem, On the asymptotic normality of the Legendre-Stirling numbers of the second kind, European J. Combin. 49 (2015), 218-231.

[13] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, 1952.

[14] A. M. Krall and L. L. Littlejohn, The Legendre polynomials under a left-definite energy norm, Quaestiones Math., 16(4), 1993, 393-403.

[15] L. L. Littlejohn and R. Wellman, A general left-definite theory for certain self-adjoint operators with applications to differential equations, J. Differential Equations, 181(2), 2002, 280-339.

[16] L. L. Littlejohn and A. Zettl, The Legendre equation and its self-adjoint operators, Electron. J. Diff. Equ., Vol. 2011 (2011), No. 69, pp. 1-43.

[17] M. A. Naimark, Linear differential operators II, Frederick Ungar Publishing Co., New York.

[18] Á. Pleijel, On Legendre’s Polynomials, Mathematical Studies 21, 1975, 175-180, North Holland Publishing Co.

[19] Á. Pleijel, On the boundary conditions for the Legendre polynomials, Ann. Acad. Sci. Fenn. Ser. A1 Math. 2, 1976, 397-408.

[20] R. Vonhoff, A left-definite study of Legendre’s differential equation and of the fourth-order Legendre type differential equation, Result. Math. 37, 2000, 155-196.
[21] W. Rudin, *Functional Analysis*, McGraw-Hill Publishers, New York, 1973.

[22] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Clarendon Press, Oxford, England, 1946.

[23] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Mathematische Annalen, Vol. 68, 1910, 220-269.

[24] Q. Wicks, *The Square of the Legendre Differential Operator*, Ph.D. thesis, Baylor University, Waco, TX, May 2016.

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328

E-mail address: Lance_Littlejohn@baylor.edu

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328

E-mail address: Quinn_Wicks@baylor.edu