Nonspending wave packets in a general potential $V(x,t)$ in one dimension

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We discuss nonspending wave packets in one dimensional Schrödinger equation. We derive general rules for constructing nonspending wave packets from a general potential $V(x,t)$. The essential ingredients of a nonspending wave packet, the shape function $f(x)$, the motion $d(t)$, the phase function $\phi(x,t)$ are derived. Since the form of the shape of a nonspending wave packet does not change in time, the shape equation should be time independent. We show that the shape function $f(x)$ is the eigenfunction of the time independent Schrödinger equation with an effective potential $V_{\text{eff}}$ and an energy $E_{\text{eff}}$. We derive nonspending wave packets found by Schrödinger, Senitzky, and Berry and Balazs as examples. We show that most stationary potentials can only support stationary nonspending wave packets. We show how to construct moving nonspending wave packets from time dependent potentials, which drive nonspending wave packets into an arbitrary motion.

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INTRODUCTION

Nonspreading wave packets are those packets whose form of the probability density function $|\Psi(x,t)|^2$ under time evolution do not change. The shapes of energy eigenstates of a time independent potential are stationary and are therefore nonspreading in the trivial case. The first interesting nonspreading wave packets was found by Schrödinger [1]. This packet is normalizable, and is the shifted ground state of a harmonic oscillator. The motion of the packet behaves just like a classical particle shifted from its equilibrium position, executing the simple harmonic motion. After that, Senitzky extended the result showing that shifted higher energy eigenstates are also nonspreading [2]. Senitzky also showed that the expectation value of energy of a nonspreading wave packet is $E_n + E_{cl}$, where $E_n$ is the quantum mechanical energy of a wave packet when it is stationary, and $E_{cl}$ the classical energy of the particle. Thus, though the motion of a classical particle and a nonspreading wave packet is the same, their energy contents are different. We may view $E_n$ the structure energy of the stationary wave packet besides the classical energy.

The extension of nonspreading packets from shifted ground state to shifted higher energy eigenstates were also found by Roy and Singh [3], Yan [4], and Mentrup et al. [5], etc.

Much later than Schrödinger, Berry and Balazs found another interesting nonspreading Airy packet $\text{Ai}[x]$ in free space [6]. Airy packets $\text{Ai}[x]$ though bounded are not square integrable. An Airy packet does not move in uniform velocity in free space, instead, it accelerates. Thus, in contrast to Schrödinger packets, the motion of an Airy wave packet is different from that of a classical particle in free space. This is due to that Airy packet is not normalizable, and therefore does not really describe a particle. Nonspreading Airy packets even exist in time-dependent uniform force [6]. It had been shown that no other potentials except those described above will support nonspreading Airy packets [7]. There are beginning a lot of investigation on the interesting optical Airy beams. The Airy optical beams was recently first observed by Christodoulides el. [8]. Nonspreading wave packets in an imaginary potential was also observed by Stüzle el. [9]. These packets are called Michelangelo wave packets, which though reach a stationary width, the probability density however decays in time, and hence are not really form invariant. In second section, we show that it needs a real potential to have a form invariant packet.

Nonspreading wave packets is an interesting and important subject. The essential ingredients of a nonspreading wave packet are its shape function $f(x)$, its motion $d(t)$, and its phase function $\phi(x,t)$. From the previous works of many people, we can have a general discussion on this subject. We derive the general rule for constructing nonspreading wave packets in the second section. We derive Berry and Balazs’s result in free space and a linear potential in the third section. We derive Schrödinger and Senitzky’s result in the fourth section. We discuss results in other potentials in the fifth section showing that most stationary potentials can only support stationary wave packets. We discuss nonspreading wave packets in time dependent potentials in the final section.

GENERAL DERIVATION

We start with Schrödinger equation,

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = H\Psi(x,t).$$ (1)

The Hamiltonian $H$ is

$$H = -\frac{\hbar^2}{2m} \partial_x \partial_x + V(x,t).$$ (2)

The potential $V(x,t)$ is allowed to be complex and is written as

$$V(x,t) = V_r(x,t) + i V_i(x,t),$$ (3)
where \( V_r(x, t) \) and \( V_i(x, t) \) are two real functions. We are looking for non-spreading wave packets in a general potential \( V(x, t) \). Let the initial wave packet be

\[
\Psi(x, 0) = f(x) e^{i\theta(x)},
\]

where \( f(x) \) and \( \theta(x) \) are real functions. We require the time evolved wave function \( \Psi(x, t) \) being of the form

\[
\Psi(x, t) = f(q) e^{i\phi(x, t)},
\]

where

\[
q = x - d(t).
\]

The two functions \( d(t) \) and \( \phi(x, t) \) are real, and satisfy the boundary conditions

\[
d(0) = 0, \\
\phi(x, 0) = \theta(x).
\]

The function \( d(t) \) describes the motion of the packet, and \( \dot{d}(t) \) is its group velocity. We have \( |\Psi(x, 0)|^2 = f(x)^2 \) and \( |\Psi(x, t)|^2 = f(q)^2 \), so \( |\Psi(x, t)|^2 \) has the same form as \( |\Psi(x, 0)|^2 \); therefore, \( \Psi(x, t) \) is a nonspreading wave packet.

We need to determine the functional form of \( f(q) \) and the related phase function \( \phi(x, t) \). We rewrite the Schrödinger equation as

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} + \frac{\hbar^2}{2m} \partial_x \partial_x \Psi(x, t) - V(x, t)\Psi(x, t) = 0.
\]

Substituting (5) into (8) and separating the equation into real and imaginary parts, we have then two equations. The real part gives the equation:

\[
\frac{\hbar^2}{2m} f''(q) = [ -V_r(x, t) + \hbar \partial_t \phi(x, t) + \frac{\hbar^2}{2m} (\partial_x \phi(x, t))^2 ] f(q).
\]

The imaginary part gives the equation:

\[
[-\hbar \dot{d}(t) + \frac{\hbar^2}{m} \partial_x \phi(x, t)] f'(q) = [ V_i(x, t) - \frac{\hbar^2}{2m} \partial_x \partial_x \phi(x, t) ] f(q).
\]

We have used the notation: \( f'(q) \) represents the derivative of \( f(q) \) with respect to \( q \), and \( \dot{d}(t) \) the derivative of \( d(t) \) with respect to \( t \). The real part equation relates \( f''(q) \) and \( f(q) \). The imaginary part equation relates \( f'(q) \) and \( f(q) \). Eq. (10), when multiplied by \( 2mf(q) \), can be rearranged into a form as a flux equation

\[
\partial_x j(x, t) = 2m V_i(x, t)f(q)^2,
\]

where \( j(x, t) \) is the flux.
where
\[ j(x, t) = [-m \hbar \dot{d}(t) + \hbar^2 \partial_x \phi(x, t)] f(q)^2. \] (12)

Taking integration of \( x \) of both sides of (11) from \(-\infty \) to \( \infty \), we have
\[ j(\infty, t) - j(-\infty, t) = \int_{-\infty}^{\infty} 2mV(x, t)f(q)^2 dx. \] (13)

We consider bounded \( f(x) \), such that \( f(x) \to 0 \) for \( |x| \to \infty \). Hence the left hand side is zero. The right hand side, if \( V_i \) is not a zero function, in general is not zero. This means a complex potential in general does not support nonspreading packets. It is then to require that
\[ V_i(x, t) = 0. \] (14)

From (11), we have
\[ \partial_x j(x, t) = 0. \] (15)

(15) shows that \( j(x, t) \) can only be a function of time. Then
\[ j(x, t) = [-m \hbar \dot{d}(t) + \hbar^2 \partial_x \phi(x, t)] f^2(q) \equiv b(t). \] (16)

Using again the boundary condition of \( f(x) \) at infinity, we have
\[ b(t) = 0. \] (17)

From (16) and (17), we then have the important result
\[ -m \hbar \dot{d}(t) + \hbar^2 \partial_x \phi(x, t) = 0. \] (18)

From (18), the form of \( \phi(x, t) \) is determined to be
\[ \phi(x, t) = \phi_1(t) x + \phi_0(t). \] (19)
\[ \phi_1(t) = \frac{m\dot{d}(t)}{\hbar}. \] (20)
Thus the phase function, \( \phi(x, t) \), is restricted to be a function linear in \( x \). From (14), the potential is real, we then simply write the potential as \( V(x, t) \).

We next need to determine the functional form of \( f(x) \). Substituting (19), (20) into (9), we have the equation determining the shape of the function \( f(q) \).

\[
\frac{\hbar^2}{2m} f''(q) = \left[ V(x, t) + \frac{m\ddot{d}(t)^2}{2} + m\dot{d}(t) \dot{x} + \hbar \dot{\phi}_0(t) \right] f(q).
\]  

(21)

The left-hand side of (21) is a function of \( q \), so should be the right-hand side. We replace the variable \( x \) by \( q + d(t) \), and then expand the terms in the bracket of the right hand side in terms of the powers of \( q \). We have

\[
\frac{\hbar^2}{2m} f''(q) = \left( \sum_{n=0}^{\infty} C_n q^n \right) f(q).
\]  

(22)

Intuitively, each of the coefficient \( C_n \) is time dependent constant, and should be written as \( C_n(t) \). However, in order to be consistent with the left-hand side being a function of \( q \) only, \( C_n \) can only be time independent constant. We call the requirement that \( C_n \) should be time independent, the \textbf{consistency} condition. The constant \( C_0 \) contains terms \( m\dot{d}(t)^2/2 + m\ddot{d}(t) \dot{d}(t) + \hbar \dot{\phi}_0(t) \), and also the constant term from \( V(x, t) \), which is, \( V(d(t), t) \). We denote the coefficient \( C_0 \) by \( -E_{\text{eff}} \) and denote all the rest terms in the summation by \( V_{\text{eff}} \). Then

\[
-E_{\text{eff}} \equiv V(d(t), t) + \frac{m\ddot{d}(t)^2}{2} + m\dot{d}(t) \ddot{d}(t) + \hbar \dot{\phi}_0(t).
\]  

(23)

and

\[
V_{\text{eff}}(q) \equiv V(q + d(t), t) - V(d(t), t) + m\dot{d}(t) q.
\]  

(24)

Eq. (21) can then be rewritten as a form similar to that of the time independent Schrödinger equation. That is

\[
-\frac{\hbar^2}{2m} f''(q) + V_{\text{eff}}(q) f(q) = E_{\text{eff}} f(q).
\]  

(25)

We call (25) the \textbf{shape} equation. We see that the function \( f(q) \) is the eigenfunction of a Hamiltonian with a potential \( V_{\text{eff}}(q) \). Since the functional form of a nonspreading wave packet does not change in time, the shape equation being related to the time independent Schrodinger equation is quite interesting. Finally, by setting \( E_{\text{eff}} \) to one of the energy eigenvalues, we can solve \( \phi_0(t) \) from (23).

We have obtained the general rule for constructing nonspreading wave packets from a general potential \( V(x, t) \). We list the following formulas for a conclusion:
\[ \Psi(x,0) = f(x) \, e^{i\phi(x)}. \]  
\[ \Psi(x,t) = f(q) \, e^{i\phi(x,t)}. \]  
\[ q = x - d(t). \]  
\[ \phi(x,t) = \phi_1(t) \, x + \phi_0(t). \]

\[ \phi_1(t) = \frac{m\ddot{d}(t)}{\hbar}. \]

\[ -\frac{\hbar^2}{2m} \, f''(q) + V_{\text{eff}}(q) \, f(q) = E_{\text{eff}} \, f(q). \]

\[ V_{\text{eff}}(q) = V(q + d(t), t) - V(d(t), t) + m\ddot{d}(t) \, q. \]

\[ E_{\text{eff}} = -\left[ V(d(t), t) + \frac{m\ddot{d}(t)^2}{2} + m\dot{d}(t) \, d(t) + \hbar \dot{\phi}_0(t) \right]. \]

Together with the consistency condition, and the boundary condition: \( f(x) \to 0 \) for \( |x| \to \infty \).

In what follows, we give applications to these formulas.

**BERRY AND BALAZS’S RESULTS**

We start from a real potential \( V(x, t) = 0 \). To determine the shapes of the nonspreading wave packets in this free space, we first calculate the effective potential \( V_{\text{eff}}(q) \). We have from \( (32) \)

\[ V_{\text{eff}}(q) = m\ddot{d}(t) \, q. \]  
\[ (34) \]

Since \( V_{\text{eff}} \) can only be a function of \( q \), it needs that \( \ddot{d}(t) \) be a constant. We set

\[ \ddot{d}(t) = \frac{B^3}{(2m^2)}. \]  
\[ (35) \]

Then \( V_{\text{eff}} = B^3 q/(2m) \), which is a linear potential. Substituting this into \( (31) \), we obtain the shape equation

\[ f''(q) - \frac{B^3}{\hbar^2} \, q \, f(q) = -\frac{2mE_{\text{eff}}}{\hbar^2} \, f(q). \]  
\[ (36) \]

The solution of \( (36) \) is the Airy function. Thus the nonspreading wave packet in free space is determined to be Airy functions. We consider here the simplest solution by taking \( E_{\text{eff}} = 0 \). The solution of Eq. \( (36) \) is then \( A_i[(B/\hbar^{2/3}) \, q] \), where we neglect the divergent solution \( B_i[(B/\hbar^{2/3}) \, q] \).

The motion of the packet, \( d(t) \), can be solved from \( (35) \). Choosing the boundary conditions as \( d(0) = 0 \), and \( \dot{d}(0) = 0 \), we have the simplest solution:

\[ d(t) = \frac{B^3 t^2}{4m^2}. \]  
\[ (37) \]
We obtain \( \phi_1(t) \) from (30). Then
\[
\phi_1(t) = \frac{B^3t}{2mh}.
\] (38)

We obtain \( \phi_0(t) \) from (33) with \( E_{\text{eff}}=0 \), and the boundary condition \( \phi_0(0) = 0 \), we have
\[
\phi_0(t) = -\frac{B^6t^3}{12m^3\hbar}.
\] (39)

We compare these results with Berry’s result:
\[
\Psi(x, t) = A[i\frac{B}{\hbar}2^{3/2}(x - \frac{B^3t^2}{4m^2})]e^{i\frac{B^3t}{2m\hbar}(x - \frac{B^3t^2}{6m^2})}.
\] (40)

We see they are the same. From (36), we also see that the only nonspreading wave packets in free space is the Airy function [6], [10]. The trivial plane wave solution corresponds to taking \( B = 0 \) in (36).

We next consider the real potential
\[
V(x, t) = -F(t)x.
\] (41)

From (32), we have
\[
V_{\text{eff}}(q) = [-F(t) + m\ddot{d}(t)]q.
\] (42)

For consistency, it needs that \([-F(t) + m\ddot{d}(t)]\) be a constant. We set
\[
-F(t) + m\ddot{d}(t) = B^3/(2m).
\] (43)

Then \( V_{\text{eff}} = B^3/(2m)q \). Thus the nonspreading wave packets in the potential \( V(x, t) = -F(t)x \) are the Airy functions again, as found by Berry and Balazs. Again, we calculate

The rest of the complete results are easily shown to be
\[
d(t) = \frac{B^3t^2}{4m^2} + \frac{1}{m} \int_0^t \int_0^\tau F(x) dx d\tau.
\] (44)

\[
\phi_1(t) = \frac{B^3t}{2m\hbar} + \frac{1}{\hbar} \int_0^t F(\tau)d\tau.
\] (45)

\[
\phi_0(t) = -\frac{B^6t^3}{12m^3\hbar} - \frac{1}{2m\hbar} \int_0^t \int_0^\tau [\int_0^\tau F(x) dx]^2 d\tau
- \frac{B^3}{2m^2\hbar} [\int_0^t \int_0^\tau F(x) dx d\tau + \int_0^t \int_0^\tau \int_0^z F(y) dy dx d\tau].
\] (46)
Using the identity

$$\int_0^t (t - \tau) F(\tau) d\tau = \int_0^t \int_0^\tau F(x) dx d\tau. \quad (47)$$

we see these results are the same as those of Berry and Balazs’s [6].

**SCHRÖDINGER AND SENITZKY’S RESULTS**

We consider the potential

$$V(x) = \frac{1}{2} m \omega^2 x^2. \quad (48)$$

From (32), we have

$$V_{\text{eff}}(q) = \frac{1}{2} m \omega^2 q^2 + m[ \ddot{d}(t) + \omega^2 d(t) ] q. \quad (49)$$

It needs that $\ddot{d} + \omega^2 d(t)$ be a constant. For a simplest solution, we set the constant to be zero. Hence

$$\ddot{d} + \omega^2 d(t) = 0. \quad (50)$$

Then $d(t)$ executes simple harmonic motion. This is the same as the classical solution of $x(t)$. Taking (50) into (49), we have

$$V_{\text{eff}}(q) = \frac{1}{2} m \omega^2 q^2. \quad (51)$$

The shape equation then is of the form

$$-\frac{\hbar^2}{2m} f''(q) + \frac{1}{2} m \omega^2 q^2 f(q) = E_{\text{eff}} f(q) \quad (52)$$

We see $f(q) = \psi_n(q)$, the energy eigenfunctions of the simple harmonic oscillator. Each eigenfunction $\psi_n(q)$ then offers a nonspreading wave packet. These packets though with the shape of stationary energy eigenfunction, however, they are moving and are in fact executing the simple harmonic motion. We have then reproduced Schrödinger and Senitzky’s result.

Starting from (50), we have the following results

$$d(t) = x_0 \sin(\omega t). \quad (53)$$
\[ \phi_1(t) = \frac{m\omega x_0}{\hbar} \cos(\omega t). \]  
(54)

\[ \phi_0(t) = -\frac{m\omega x_0^2}{4\hbar} \sin(2\omega t) - \frac{E_n t}{\hbar}. \]  
(55)

We also easily see that if the frequency is time dependent, that is \( \omega = \omega(t) \), then the effective potential from (49) is

\[ V_{\text{eff}}(q) = \frac{1}{2} m\omega(t)^2 q^2 + m \left[ \ddot{d}(t) + \omega(t)^2 \dot{d}(t) \right] q. \]  
(56)

From the consistency requirement, we need the coefficients of \( q \) and \( q^2 \) be constant, hence \( \omega \) should be a constant. Thus a harmonic oscillator with a time dependent frequency can not support a nonspreading wave packet, as described in Yan’s paper [4].

**POTENTIAL** \( V(x) = \lambda x^n, n \geq 3 \)

For potentials of the form

\[ V(x) = \lambda x^n, n \geq 3, \]  
(57)

it is trivial to see that from \( V_{\text{eff}} \) and the consistency equation, the motion \( d(t) \) must be a constant. function

From (32), we have

\[ V_{\text{eff}}(q) = \lambda q^n + \lambda nd(t)^n - \lambda d(t)^n + m \ddot{d} q. \]  
(58)

Expanding \( \lambda[q + d(t)]^n \) in terms of powers of \( q \), we have the first few terms

\[ \lambda q^n + \lambda nd(t)q^{n-1} + \cdots. \]  
(59)

The second term of (59) shows that \( d(t) \) needs to be a constant. Thus the packet does not move. This implies that the nonspreading packets can only be stationary packets. We set this constant as \( d \), then we have

\[ d(t) = d. \]  
(60)

And then we have

\[ V_{\text{eff}}(q) = \lambda[q + d]^n - \lambda d^n. \]  
(61)
Substituting this to \((31)\), we have the shape equation of the form

\[
-\frac{\hbar^2}{2m} f''(q) + \lambda[q + d]^n f(q) = E f(q),
\]

(62)

where \(E = E_{\text{eff}} + \lambda d^n\). As \(q + d = x\), formula \((62)\) shows that the function \(f(q) = \psi_n(x)\), the energy eigenfunction of the Hamiltonian with potential \(V(x) = \lambda x^n\). Since the packet is the energy eigenfunction, it is a stationary wave; and therefore it does not move. We conclude that there is no moving nonspreading packet in a potential of the form \(V(x,t) = \lambda x^n, n \geq 3\). The same argument applies also to showing that a potential, such as \(V(x) = \lambda/x\), does not support moving nonspreading wave packets. We see most stationary potentials can only support stationary wave packets.

**TIME DEPENDENT POTENTIALS WITH MOVING NONSPREADING WAVE PACKETS**

From above results, hence, in order to have moving nonspreading wave packets, we need to consider time dependent potentials. The simplest way is to move the potential. That is to consider \(V(x - d(t))\). In this way, there are in fact many nonspreading wave packets with arbitrary motion that can be constructed. For instance, we start from a given motion, such as \(d(t) = t^2\), etc., and then consider the time dependent potential

\[
V(x, t) = \frac{1}{2} m \omega^2 (x - d(t))^2 - m x \ddot{d}(t) - \frac{1}{2} m \omega^2 x_0^2 \sin(\omega t)^2.
\]

(63)

The effective potential calculated from \((63)\) and \((62)\) is

\[
V_{\text{eff}}(q) = \frac{1}{2} m \omega^2 q^2.
\]

(64)

Thus, the potential in Eq. \((63)\) offers nonspreading wave packets, with the shapes of the eigenfunctions of SHO, moving with the arbitrary function \(d(t)\). If we set \(d(t) = x_0 \sin(\omega t)\), and substituting this to \((63)\), we find the needed potential for this motion is \(V(x, t) = \frac{1}{4} m \omega^2 x^2\), as expected.

Another example is considering the time dependent potential

\[
V(x, t) = \lambda(x - d(t))^4 - m x \ddot{d}(t).
\]

(65)

The effective potential can be calculated to be

\[
V_{\text{eff}}(q) = \lambda q^4.
\]

(66)

The potential in Eq. \((65)\) then offers nonspreading wave packets with the shapes of the eigenfunctions of the potential \(V(x) = \lambda x^4\). These nonspreading wave packets move with the arbitrary function \(d(t)\) as the moving potential does. We see that in order to move a nonspreading wave packet, we need a moving potential, and also a linear potential \(-m x \ddot{d}(t)\), which offers a force \(m d(t)\); this is just the force needed for a particle with a motion \(d(t)\). In conclusion, a moving potential \(\lambda(x - d(t))^4\), together with a time dependent linear potential \(-m x \ddot{d}(t)\), will drive stationary-shaped packets going around with a journey \(d(t)\). It is like to have water moving in space, we put it in a bowl, and carry the bowl going around.
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