More on the Non-Solvable Graphs and Solvabilizers

B. Akbari

Department of Mathematics, Sahand University of Technology, P.O. Box 51335-1996, Tabriz, IRAN.

E-mails: b.akbari@sut.ac.ir and b.akbari@dena.kntu.ac.ir

June 5, 2018

Abstract

The non-solvable graph of a finite group $G$, denoted by $S_G$, is a simple graph whose vertices are the elements of $G$ and there is an edge between $x, y \in G$ if and only if $\langle x, y \rangle$ is not solvable. If $R(G)$ is the solvable radical of $G$, the isolated vertices in $S_G$ are exactly the elements of $R(G)$. So in the case where $G$ is a non-solvable group, it is wise to consider the induced subgraph of $S_G$ with respect to $G \setminus R(G)$. This graph is denoted by $\hat{S}_G$. Let $G$ be a finite group and $x \in G$. The solvabilizer of $x$ with respect to $G$, denoted by $\text{Sol}_G(x)$, is the set $\{y \in G \mid \langle x, y \rangle \text{ is solvable}\}$. The purpose of this paper is to study some properties of the non-solvable graph $\hat{S}_G$ and the structure of $\text{Sol}_G(x)$ for every $x \in G$. We also show that there is no finite group in which some vertices in its non-solvable graph have the degree $n - 2$ where $n = |G| - |R(G)|$.

1 Introduction

All groups appearing here are assumed to be finite. One of the most important methods and interesting subjects is to study finite groups by algebraic properties associated with them. There are a lot of ways to relate an algebraic property to a finite group. One of them is to consider some properties of the graphs associated with it. Each property of the graph can learn us a property of the group. Let $G$ be a finite group. The non-solvable graph $S_G$ is a simple graph that constructs as follows. The vertex set is $G$ and two distinct elements $x$ and $y$ are adjacent if and only if the subgroup $\langle x, y \rangle$ is not solvable. In fact, Thompson’s Theorem asserts that a group $G$ is solvable if and only if $\langle x, y \rangle$ is solvable for every $x, y \in G$. Hence, $S_G$ is an empty graph if and only if $G$ is solvable. Therefore, we only study $S_G$ if $G$ is not solvable.

In fact, Thompson’s Theorem asserts that a group $G$ is solvable if and only if $\langle x, y \rangle$ is solvable for every $x, y \in G$. Hence, $S_G$ is an empty graph if and only if $G$ is solvable. Therefore, we only study $S_G$ if $G$ is not solvable.

For two non-empty subsets $A, B$ of $G$, we call $\text{Sol}_A(B)$ the solvabilizer of $B$ with respect to $A$ which is the subset

$$\{a \in A \mid \langle a, b \rangle \text{ is solvable } \forall b \in B\}.$$ 

Note that $\text{Sol}_A(B)$ is not necessarily a subgroup of $G$. We put $\text{Sol}_A(x) := \text{Sol}_A(\{x\})$ and $\text{Sol}(G) := \text{Sol}_G(G)$. Let $R(G)$ be the solvable radical of $G$. In [6], it was obtained that $\text{Sol}(G) = R(G)$. It is also clear that $\text{Sol}_A(x) = \text{Sol}_A(\langle x \rangle)$. We focus our attention on $\text{Sol}_G(x)$.

It was shown in [7] that $\text{Sol}_G(x)$ is the union of all solvable subgroups of $G$ containing $x$. It was also proved that $\text{Sol}_G(x)$ is a disjoint union of some cosets of $\langle x \rangle$.

According to above, for every $x \in G$ we have

$$\text{deg}(x) = |G| - |\text{Sol}_G(x)|,$$
where \( \text{deg}(x) \) is the degree of vertex \( x \) in \( S_G \).

It is obvious that every two elements of \( \text{Sol}(G) \) are not adjacent in \( S_G \). On the other hand, as mentioned before, \( \text{Sol}(G) = R(G) \) where \( R(G) \) is the solvable radical of \( G \), which means that if \( x \) is an element of \( G \) such that for every \( y \in G \), \( \langle x, y \rangle \) is solvable, then \( x \in R(G) \). Therefore, for all \( x \in G \setminus R(G) \), there exists an element \( y \in G \setminus R(G) \) such that \( \langle x, y \rangle \) is not solvable. So we can conclude that the elements of \( R(G) \) are exactly the isolated vertices in \( S_G \). Hence, if \( G \) is a non-solvable group, then it is logical to consider the induced graph of \( S_G \) with respect to \( G \setminus \text{Sol}(G) \) which is denoted by \( \tilde{S}_G \). It is seen that the degree of vertex \( x \in G \setminus \text{Sol}(G) \) in \( \tilde{S}_G \) is equal to its degree in \( S_G \).

The non-solvable graph of a group can be generalized in the following way (see [1]).

Let \( G \) be a finite group. The non-nilpotent graph of \( G \), which is denoted by \( N_G \), is a simple graph whose vertices are the elements of \( G \) and two vertices \( x, y \) are adjacent by an edge if and only if \( \langle x, y \rangle \) is not nilpotent. The induced subgraph of \( N_G \) on \( G \setminus \text{nil}(G) \), where \( \text{nil}(G) = \{ x \in G \mid \langle x, y \rangle \text{ is nilpotent for all } y \in G \} \), was introduced as \( \tilde{N}_G \). This graph was completely verified in [1]. Clearly, \( S_G \) (resp. \( \tilde{S}_G \)) is a subgraph of \( N_G \) (resp. \( \tilde{N}_G \)).

We are going to focus on non-solvable graph. In fact, we are interested in finding the structure of a group through some properties of its non-solvable graph. Many properties of this graph were studied in [7].

Our terminology and notation for groups will be standard. Thus, we only introduce some notation for graphs used in this paper.

**Notation for Graphs.** A simple graph \( \Gamma \) with vertex set \( V = V(\Gamma) \) and edge set \( E = E(\Gamma) \) is a graph with no loops or multiple edges. A graph \( \Gamma \) is \( k \)-regular if the degrees of all vertices in \( \Gamma \) is \( k \). A regular graph is one that is \( k \)-regular for some \( k \). A \((n - 1)\)-regular graph with \( n \) vertices is said a complete graph. A complete graph with \( n \) vertices is denoted by \( K_n \). A set of vertices of a graph is independent if the vertices are pairwise nonadjacent. The independence number \( \alpha(\Gamma) \) of a graph \( \Gamma \) is the cardinality of a largest independent set of \( \Gamma \). The distance between two vertices of a graph is the minimum length of the paths connecting them. The diameter of a graph is the greatest distance between two vertices of the graph. An acyclic graph is one that contains no cycles. A connected acyclic graph is called a tree. A graph is bipartite if its vertex set can be partitioned into two subsets \( X \) and \( Y \) so that every edge has one end in \( X \) and one end in \( Y \), such a partition \( (X, Y) \) is called a bipartition of the graph, and \( X \) and \( Y \) its parts. We recall that a graph is bipartite if and only if it contains no odd cycle (a cycle of odd length). A bipartite graph with bipartition \((X, Y)\) in which every two vertices from \( X \) and \( Y \) are adjacent is called a complete bipartite graph and denoted by \( K_{|X|,|Y|} \). An edge subdivision operation for an edge \( e \in E \) with two endpoints \( u,v \in V \), is the deletion of \( e = uv \) from \( \Gamma \) and the addition of two edges \( uw \) and \( vw \) along with the new vertex \( w \). A graph which has been derived from \( \Gamma \) by a sequence of edge subdivision operations is called a subdivision of \( \Gamma \). A graph is planar if it can be drawn on the plane without edges crossing except at endpoints. In fact, a graph is planar if and only if it does not contain a subdivision of \( K_{3,3} \) and \( K_5 \)(Kuratowski’s Theorem).

In this paper, we are interested in characterizing certain properties of a group in terms of some properties of non-solvable graph and solvabilizers.

In section 3, we study some properties of the induced subgraph \( S_G \) with respect to \( G \setminus \text{Sol}(G) \), for a group \( G \).

In section 4, we investigate the structure of the solvabilizer of \( x \in G \) with respect to \( G \). We prove that \( N_G(\langle x \rangle) \subseteq \text{Sol}_G(x) \) where \( N_G(\langle x \rangle) \) is the normalizer of \( \langle x \rangle \) in \( G \). In general, we show that if \( H \) is a solvable subgroup of \( G \), then for all \( x \in H \), \( N_G(H) \subseteq \text{Sol}_G(x) \) (Lemma 4.1). Furthermore, in [7], it was shown that for \( x \in G \), \( \text{deg}(x) \leq n - 2 \) where \( n = |G| - |\text{Sol}(G)| \). In this paper, we prove that \( \text{deg}(x) \neq n - 2 \) and so \( \text{deg}(x) \leq n - 3 \) for every \( x \in G \).
2 Preliminary Results

In this section, we express some results obtained in [7] which will help us for further investigations. We begin with a Theorem taken from [6].

**Theorem 2.1** Let $G$ be a non-solvable group and $x, y \in G$ such that $x, y \notin \text{Sol}(G)$. Then there exists $z \in G$ such that $\langle x, z \rangle$ and $\langle y, z \rangle$ are not solvable.

Note that in Theorem 2.1, if $G$ is a non-solvable group in which any proper subgroup is solvable (equivalently every maximal subgroup is solvable), then it has the property that $G$ is simple modulo the Frattini subgroup (trivial) and is generated by two elements. The simple groups occurring are classified by Thompson (This is a famous result of John Thompson in series of papers on N-groups). So the Theorem still holds. For example, let $G$ be the alternating group $A_5$ whose the maximal subgroups are as follows: the alternating group $A_4$, the dihedral group $D_{10}$ and the symmetric group $S_3$ (see [5]). Then using the fact that any finite simple group can be generated by two elements (Steinberg in [9] proved this for a Chevalley group and Aschbacher and Guralnick in [2] verified it for the sporadic groups), any one nontrivial element is part of a generating pair and the Theorem still holds.

As a straightforward result of Theorem 2.1, we have the non-solvable graph $\hat{S}_G$ is connected and its diameter is at most 2. More precisely, it was shown in [7] that the diameter of $\hat{S}_G$ can not be equal to 1. So we can state the following Lemma.

**Lemma 2.1** ([7]) Let $G$ be a non-solvable group. Then $\text{diam}(\hat{S}_G) = 2$.

If $x, y$ are two elements of a group $G$ with order 2, then $\langle x, y \rangle$ is a dihedral group. Indeed, we have the following Lemma.

**Lemma 2.2** ([7]) Let $G$ be a group and $x, y \in G$ such that $o(x) = o(y) = 2$. Then $\langle x, y \rangle$ is solvable.

Now, we collect some results on solvabilizer of an element of $G$ with respect to $G$ obtained in [7].

**Lemma 2.3** ([7]) Let $G$ be a group. Suppose that $N \triangleleft G$ such that $N \subseteq \text{Sol}(G)$ and $x, g \in G$. Then the following statements hold:

1. $\text{Sol}_{G/N}(xN) = \text{Sol}_G(x)/N$;
2. $\text{Sol}_G(gxg^{-1}) = g\text{Sol}_G(x)g^{-1}$;
3. If $A, B \subseteq G$ are two subsets such that $A \subseteq B$ and $x \in A$ is an element, then $\text{Sol}_A(x) \subseteq \text{Sol}_B(x)$.

**Lemma 2.4** ([7]) Let $G$ be a group and $x \in G$. Then we have:

1. $|\text{Sol}_G(x)|$ is divisible by $|\text{Sol}(G)|$;
2. $|\text{Sol}_G(x)|$ is divisible by $o(x)$ and $|C_G(x)|$.

As mentioned before, $\deg(x) = |G| - |\text{Sol}_G(x)|$ for every $x \in G$. Thus, it is found from Lemma 2.4 (2) that $|C_G(x)|$ divides $\deg(x)$.

**Lemma 2.5** Let $G$ be a non-solvable group and $x \in G$. Moreover, let $H$ be a solvable subgroup of $G$. Then the following statements hold:
(1) $\text{Sol}_G(x) = \text{Sol}_G(x^i)$ where $1 \leq i \leq o(x)$ and $(i, o(x)) = 1$. In particular, $\text{deg}(x) = \text{deg}(x^i)$.

(2) $H \leq \text{Sol}_G(x)$, for every $x \in H$.

Proof. Part (1) is a straightforward result of Lemma 2.11 in [7]. Also, part (2) is a conclusion of Thompson’s Theorem. □

In [7], the degrees of vertices in the non-solvable graph were investigated and the following results were found.

Lemma 2.6 ([7]) Let $G$ be a non-solvable group and $x \in G \setminus \text{Sol}(G)$. Moreover, assume that $n = |G| - |\text{Sol}(G)|$. Then the following hold:

(1) $2o(x) \leq \text{deg}(x)$;
(2) $5 < \text{deg}(x) < n - 1$;
(3) $\text{deg}(x)$ is not a prime.

In section 4, we will show that if $G$ is a non-solvable group, then for all $x \in G$, $\text{deg}(x) \neq n - 2$. Thus, we can conclude that $\text{deg}(x) \leq n - 3$ for all $x \in G$.

As mentioned before, for an element $x \in G$, $\text{Sol}_G(x)$ needs not to be a subgroup of $G$ in general. If $G$ is a group in which $\text{Sol}_G(x) \leq G$ for all $x \in G$, then $G$ is called an $S$-group. The structure of an $S$-group was studied in [7]. In fact, the following Lemma was proved.

Lemma 2.7 ([7]) Let $G$ be a group. Then $G$ is solvable if and only if $G$ is an $S$-group.

The result obtained in Lemma 2.7, states an equivalent condition for solvability. In other words, $G$ is solvable if and only if $G$ has the following property: For every $x, y, z \in G$, if $\langle x, y \rangle$ and $\langle x, z \rangle$ are solvable, then $\langle x, yz \rangle$ is solvable.

We recall that if a graph contains the complete bipartite graph $K_{3,3}$, then it is not planar. In [7], the following Lemma was proved.

Lemma 2.8 ([7]) Let $G$ be a non-solvable group. Then $\hat{\text{S}}_G$ contains $K_{4,4}$ as a subgraph.

Now as a conclusion of Lemma 2.8 and Kuratowski’s Theorem, we can see that $\hat{\text{S}}_G$ is not planar.

Lemma 2.9 ([7]) Let $G$ be a non-solvable group. Then $\hat{\text{S}}_G$ is irregular.

3 Some Properties of Non-solvable Graphs

In this section, we are going to investigate some graphic properties of $\hat{\text{S}}_G$.

Lemma 3.1 Let $G$ be a non-solvable group. Then $\hat{\text{S}}_G$ is not a tree.

Proof. Assume to the contrary that $\hat{\text{S}}_G$ is a tree. Then it contains at least two vertices having degree one (see [4]) which contradicts Lemma 2.6 (2). So, $\hat{\text{S}}_G$ is not a tree. □

The elements of $\text{Sol}(G)$ are exactly the isolated vertices in $\text{S}_G$. Thus $\text{Sol}(G)$ is an independent set of $\text{S}_G$ and so the independence number of $\text{S}_G$ is greater or equal than $|\text{Sol}(G)|$. We can also state the following Lemma.

Lemma 3.2 Let $G$ be a non-solvable group. Then $\alpha(\text{S}_G) \geq \max\{o(x)\mid x \in G\}$. 

Proof. For every element \( x \in G \), the set \( \langle x \rangle \) is an independent set because it is clear that for all \( 1 \leq i, j \leq o(x) \), \( \langle x^i, x^j \rangle \subseteq \langle x \rangle \) is a solvable subgroup of \( G \) and thus \( x^i \) and \( x^j \) are not adjacent in \( \hat{S}_G \). So the proof is complete. \( \square \)

Let \( A \) be an independent set of \( \hat{S}_G \). Then it is easy to see that \( A \cup \text{Sol}(G) \subseteq \text{Sol}_G(x) \) for all \( x \in A \). Moreover, if \( A \cup \{1\} \) is a subgroup of \( G \), then it is a solvable subgroup.

Now, we consider the non-solvable graphs of subgroups and quotient groups of a finite group.

**Lemma 3.3** Let \( G \) be a non-solvable group. Let \( H \) and \( N \) be two subgroups of \( G \) such that \( N \trianglelefteq G \), \( N \subseteq \text{Sol}(G) \) and \( \text{Sol}(G) \subseteq H \). Then the following statements hold:

1. If \( x \) and \( y \) are joined in \( \hat{S}_H \) for every \( x, y \in H \), then \( x \) and \( y \) are joined in \( \hat{S}_G \). In other words, \( \hat{S}_H \) is a subgraph of \( \hat{S}_G \).

2. For two elements \( x, y \notin \text{Sol}(G) \), \( xN \) and \( yN \) are adjacent in \( \hat{S}_{G/N} \) if and only if \( x \) and \( y \) are adjacent in \( \hat{S}_G \).

**Proof.** (1) Since \( \text{Sol}(G) \subseteq H \), so it is seen that \( \text{Sol}(G) \subseteq \text{Sol}(H) \). Thus if \( x, y \notin \text{Sol}(H) \), then \( x, y \notin \text{Sol}(G) \). The rest of proof is clear.

(2) We only prove the sufficiency. The necessity is similar. Assume that \( xN \) and \( yN \) are adjacent in \( \hat{S}_{G/N} \). It follows that \( yN \notin \text{Sol}_{G/N}(xN) \). Then considering Lemma 2.3 (1), we obtain that

\[
yN \notin \text{Sol}_G(x)/N.
\]

Hence \( y \notin \text{Sol}_G(x) \) which implies that \( x \) and \( y \) are adjacent in \( \hat{S}_G \). \( \square \)

In a view of Lemma 3.3, if \( \text{Sol}(G) = 1 \), then the non-solvable graph of each subgroup of \( G \) is a subgraph of \( \hat{S}_G \).

**Corollary 3.1** Let \( G \) be a non-abelian simple group and \( H \) a subgroup of \( G \). Then \( \hat{S}_H \) is a subgraph of \( \hat{S}_G \).

**Proof.** The proof is obvious. \( \square \)

**Lemma 3.4** Let \( G \) be a non-solvable group. Let \( H \) be a proper subgroup of \( G \) and \( N \) a normal subgroup of \( G \) such that \( N \subseteq \text{Sol}(G) \) and \( \text{Sol}(G) \subseteq H \). Then the following statements hold:

1. \( \hat{S}_H \) is not isomorphic to \( \hat{S}_G \).

2. \( \hat{S}_{G/N} \) is not isomorphic to \( \hat{S}_G \).

**Proof.** (1) By contrast, assume that \( \hat{S}_H \cong \hat{S}_G \). Thus the vertex set of \( \hat{S}_H \) coincides with one of \( \hat{S}_G \). It follows that

\[
|H| - |\text{Sol}(H)| = |G| - |\text{Sol}(G)|.
\]

We observe that \( \text{Sol}(G) \trianglelefteq \text{Sol}(H) \). If \( \text{Sol}(G) = \text{Sol}(H) \), then \( |G| = |H| \) which is impossible. Hence, \( \text{Sol}(G) \trianglelefteq \text{Sol}(H) \). It implies that

\[
|\text{Sol}(G)| \leq \frac{1}{2}|\text{Sol}(H)|.
\]

So we can conclude that

\[
|G| \leq |H| - 2|\text{Sol}(G)| + |\text{Sol}(G)| = |H| - |\text{Sol}(G)|,
\]
which is false. Therefore, \( \hat{S}_H \) is not isomorphic to \( \hat{S}_G \).

(2) By contrast, suppose that \( \hat{S}_{G/N} \cong \hat{S}_G \). It forces that the vertex set of \( \hat{S}_{G/N} \) coincides with one of \( \hat{S}_G \). Thus we have

\[
|G| - |\text{Sol}(G)| = \frac{|G|}{|N|} - \frac{|\text{Sol}(G)|}{|N|},
\]

which is a contradiction. So the proof is complete. \( \square \)

4 The Structure of Solvabilizers and Non-solvable Graphs with Certain Degrees of Vertices

In this section, we consider the structure of solvabilizer \( \text{Sol}_G(x) \) for every \( x \in G \). We also study non-solvable graphs whose some vertices have certain degree. Finally, we state a problem on characterization of finite groups by solvabilizers.

**Theorem 4.1** Let \( G \) be a non-solvable group and \( x \) an element of \( G \). Then \( N_G(\langle x \rangle) \subseteq \text{Sol}_G(x) \). In particular, if \( x, y \in G \setminus \text{Sol}(G) \) are two elements such that \( y \in N_G(\langle x \rangle) \), then \( y \) is not adjacent to \( x \) in \( \hat{S}_G \).

**Proof.** Suppose that \( y \in N_G(\langle x \rangle) \). Thus \( \langle y \rangle \subseteq N_G(\langle x \rangle) \) which yields that \( \langle x \rangle \langle y \rangle \) is a subgroup of \( G \). It follows that \( \langle x, y \rangle = \langle x \rangle \langle y \rangle \). Moreover, it is easy to see that \( \langle x \rangle \triangleleft \langle x, y \rangle \). We observe that \( \langle x \rangle \) and \( \langle x, y \rangle / \langle x \rangle \cong \langle y \rangle \) are solvable. So we can conclude that \( \langle x, y \rangle \) is solvable. Therefore, \( y \in \text{Sol}_G(x) \). The rest of proof is obvious. \( \square \)

As an important result on solvable groups, Thompson’s Theorem states that a group \( G \) is solvable if and only if \( \langle x, y \rangle \) is solvable for every \( x, y \in G \). Indeed, Theorem 4.1 confirms the following fact.

**Corollary 4.1** Let \( G \) be a group. If all cyclic subgroups of \( G \) are normal subgroups in \( G \), then \( G \) is solvable.

Before proceeding our study, we define certain subgroups of a group.

**Definition 4.1** Let \( G \) be a group. A local subgroup of \( G \) is a subgroup \( K \) of \( G \) if there is a nontrivial solvable subgroup \( H \) of \( G \) such that \( K = N_G(H) \).

When we are considering the solvibilizers of the elements belonging to the solvable subgroups of a finite group, we can generalize Theorem 4.1 to the following Lemma.

**Lemma 4.1** Let \( G \) be a group and \( K = N_G(H) \) a local subgroup of \( G \) for some solvable subgroup \( H \) of \( G \). Then for every \( x \in H \), we have \( K \subseteq \text{Sol}_G(x) \).

**Proof.** Suppose that \( y \in K \). It is seen that \( \langle y \rangle H \) is a subgroup of \( G \). We also have

\[
\frac{\langle y \rangle H}{H} \cong \frac{\langle y \rangle}{H \cap \langle y \rangle}.
\]

Since \( \frac{\langle y \rangle}{H \cap \langle y \rangle} \) and \( H \) are solvable, so \( \langle y \rangle H \) is solvable. On the other hand, we observe that for every \( x \in H \), \( \langle x, y \rangle \leq \langle y \rangle H \). It follows that \( \langle x, y \rangle \) is solvable and hence \( y \in \text{Sol}_G(x) \). Therefore, the proof is complete. \( \square \)

**Theorem 4.2** Let \( G \) be a non-solvable group and \( n = |G| - |\text{Sol}(G)| \). Then there is no element \( x \in G \setminus \text{Sol}(G) \) such that \( \text{deg}(x) = n - 2 \).
Proof. Suppose to the contrary that \( x \) an element of \( G \) with \( \text{deg}(x) = n - 2 \).

It is good to mention that \( \text{deg}(x) = |G| - |\text{Sol}_G(x)| \). Thus

\[
|\text{Sol}_G(x)| = |\text{Sol}(G)| + 2.
\]

According to Lemma 2.4 (1), \( |\text{Sol}_G(x)| \) is divisible by \( |\text{Sol}(G)| \) which forces that \( |\text{Sol}(G)| = 1 \) or 2. First of all, we claim that \( |\text{Sol}(G)| \neq 2 \).

Assume that \( |\text{Sol}(G)| = 2 \). Then we can see that \( |\text{Sol}_G(x)| = 4 \). We find from Lemma 2.4 (2) that \( o(x) | |\text{Sol}_G(x)| \) which follows that \( o(x) = 2 \) or 4. Now, we examine these cases separately.

Case 1. First let \( o(x) = 2 \). If \( x \) is the only element of \( G \) with \( o(x) = 2 \), then for every \( g \in G \), \( o(g^{-1}xg) = o(x) \) which yields that \( x = g^{-1}xg \). It implies that \( x \in Z(G) \subseteq \text{Sol}(G) \) which is false. Therefore, there exists \( y \in G, y \neq x \), with \( o(y) = 2 \). According to Lemma 2.2, \( \langle x, y \rangle \) is solvable and thus \( y \in \text{Sol}_G(x) \). So we can conclude that \( G \) has at most three elements of order 2, namely, \( x, y_1 \) and \( y_2 \).

We claim that there exists two elements \( g_1, g_2 \in G \) such that \( y_i = g_i^{-1}xg_i \). For this purpose, we assume to the contrary that for every \( g_1, g_2 \in G \), \( g_1^{-1}xg_1 = g_2^{-1}xg_2 \). It implies that \( g_1g_2^{-1} \in C_G(x) \).

On the other hand, we have

\[
C_G(x) \trianglelefteq C_G(\langle x \rangle) \subseteq N_G(\langle x \rangle) \subseteq \text{Sol}_G(x).
\]

Therefore, for any \( g_1, g_2 \in G \), \( o(g_1g_2^{-1}) = 2 \) or 4. If \( o(g_1g_2^{-1}) = 2 \) for all \( g_1, g_2 \in G \), then \( G \) is an elementary group which forces that \( G \) is nilpotent. This is a contradiction. Assume now that \( o(g_1g_2^{-1}) = 4 \) for all \( g_1, g_2 \in G \). Then every nontrivial element of \( G \) has order 2 or 4. In [8], the structure of a group with elements of order at most 4 was completely determined. In fact, we use Theorem 1 in [8] and gain a contradiction. Consequently, there exist two elements \( g_1, g_2 \in G \) such that \( y_i = g_i^{-1}xg_i \).

Since \( |\text{Sol}(G)| = 2 \), thus \( \text{Sol}(G) = \langle y_1 \rangle \) or \( \text{Sol}(G) = \langle y_2 \rangle \). On the other hand, for every \( g \in G \), we have \( g^{-1}\text{Sol}(G)g \subseteq \text{Sol}(G) \) because \( \text{Sol}(G) \trianglelefteq G \). It follows that \( x \in \text{Sol}(G) \) which is false.

Case 2. Let \( o(x) = 4 \). We obtain from Lemma 4.1 that

\[
\langle x \rangle \subseteq N_G(\langle x \rangle) \subseteq \text{Sol}_G(x).
\]

Since \( |\langle x \rangle| = |\text{Sol}_G(x)| \), hence

\[
\text{Sol}_G(x) = \langle x \rangle = \{1, x, x^2, x^3\}.
\]

clearly, \( x, x^3 \notin \text{Sol}(G) \).

We claim that \( x^2 \notin \text{Sol}(G) \). Suppose to the contrary that \( x^2 \in \text{Sol}(G) \). Then, according to the order of \( \text{Sol}(G) \), we have \( \text{Sol}(G) = \langle x^2 \rangle \). In the sequel, for the sake of simplicity of the notation, we put \( K := \text{Sol}(G) \). Now, it follows from Lemma 2.3 that

\[
\text{Sol}_{G/K}(xK) = \frac{\text{Sol}_G(x)}{K}.
\]

On the other hand, \( \text{Sol}_G(x)/K \) is a subgroup of \( G/K \) with order 2. It implies that \( \text{Sol}_{G/K}(xK) = \langle xK \rangle \). We show that \( G/K \) is an abelian simple group. To do this, we suppose that \( G/K \) is not simple. Therefore, there exists a nontrivial normal subgroup \( N/K \) of \( G/K \). Assume first that \( xK \in N/K \). Since \( \text{Sol}_{G/K}(xK) \) is the union of all solvable subgroups of \( G/K \) containing \( xK \) and \( \text{Sol}_{G/K}(xK) = \langle xK \rangle \), so it is seen that \( xK \) is a sylow 2-subgroup of \( G/K \). Now, we use Frattini’s argument and obtain that \( G/K = \text{N}_{G/K}(\langle xK \rangle) N/K \). Moreover,

\[
\langle xK \rangle \subseteq \text{N}_{G/K}(\langle xK \rangle) \subseteq \text{Sol}_{G/K}(xK)
\]
which yields that $N_{G/K}(\langle xK \rangle) = \langle xK \rangle$. Thus $G/K = N/K$, that is impossible. Therefore, we may suppose that $xK / xK$ is not a solvable subgroup of $G/K$. It is clear that there exists a prime $r$ dividing $N/K$. If $R/K$ is a solvable $r$-subgroup of $N/K$, then we can see from Frattini’s argument that

$$G/K = N_{G/K}(R/K)N/K.$$ 

By assumption, we have $xK / xK$. Note that $o(xK) = 2$ and so we can not write $xK$ as product of two nontrivial elements $g_1 K \in N_{G/K}(R/K)$ and $g_2 K \in N/K$. It forces that $xK / xK$ is a solvable subgroup of $G/K$ containing $\langle xK \rangle$ while $\langle xK \rangle$ is the largest solvable subgroup of $G/K$ having element $xK$. So we derive a contradiction. It follows that $G/K$ is a simple group.

As before, $\langle xK \rangle$ is a solv-able 2-subgroup of $G/K$ with order 2. It forces that $G/K$ is not a non-abelian simple group. It follows that $|G/K| = 2$ and so $|G| = 4$ which is false.

We conclude that $x^2 \notin Sol(G)$. Consequently, $x^2, x^3$ are not adjacent to $x$. Hence, $deg(x) \leq n - 3$ that is impossible.

We deduce that $|Sol(G)| = 1$. As mentioned above, $|Sol_G(x)| = |Sol(G)| + 2$ and thus $|Sol_G(x)| = 3$. In a view of Lemma 2.4 (2), $|Sol_G(x)|$ is divisible by $o(x)$ and hence $o(x) = 3$. We observe that

$$\langle x \rangle \subseteq N_G(\langle x \rangle) \subseteq Sol_G(x).$$

By a similar way, we get that $Sol_G(x) = \langle x \rangle$. So we conclude

$$\langle x \rangle = N_G(\langle x \rangle) = Sol_G(x).$$

It follows that there is no solvable subgroup of $G$ containing $x$ except for $\langle x \rangle$. Clearly, if $R$ is a solvable 3-subgroup of $G$, then $|R| = 3$. To gain a contradiction, we will try to find a solvable subgroup of $G$ containing $x$ distinct from $\langle x \rangle$.

Assume first that $G$ is not a simple group. Then it has a nontrivial normal subgroup, say $N$. Suppose that $x \in N$. Thus, we obtain from Frattini’s argument that

$$G = N_G(\langle x \rangle)N = \langle x \rangle N = N,$$

which is false. It implies that $x \notin N$. Since $N \neq 1$, so there exists a prime $p$ dividing $|N|$. Let $P$ be a solvable $p$-subgroup of $N$. Again, by Frattini’s argument, we find $G = N_G(P)N$. According to assumption, we have $x \notin N$. Since $o(x) = 3$, hence we can not write $x$ as product of two nontrivial elements $g_1 \in N_G(P)$ and $g_2 \in N$. It forces that $x \in N_G(P)$. Therefore, $P\langle x \rangle$ is a solvable subgroup which is desired.

Next, suppose that $G$ is a non-abelian simple group. Considering the classification of finite groups, the possibilities for simple group $G$ are as follows:

1. an alternating group $A_n$ on $n \geq 5$ letters;
2. one of the 26 sporadic groups;
3. a simple group of Lie type.

It is worth to mention that the order of a solvable 3-subgroup of $G$ is 3.

If $G$ is an alternating or sporadic group, then according to the order of these groups, $G$ is one of groups $A_5$ and $J_1$. It is seen from [5] that if $x \in A_5$ (resp. $J_1$) with $o(x) = 3$, then $x$ is included in some solvable subgroups of $A_5$ distinct from $\langle x \rangle$ (resp. $J_1$).

Let now $G$ be a simple group of Lie type. Using the orders of Lie type groups, it is enough to examine the following groups:

- the projective special linear groups $A_1(3)$ and $A_1(q)$ defined over a field of characteristic $p$;
• $A_2(q)$ where $3 \mid q + 1$ and $9 \nmid q + 1$;
• the unitary group $^2A_2(q)$ where $3 \mid q - 1$ and $9 \nmid q - 1$.

It is good to note that the structure of all subgroups of $A_1(q)$ are determined in [10]. Moreover, using Tables 8.3, 8.5 in [3], we can find the maximal subgroups of $A_2(q)$ and $^2A_2(q)$. Thus, it is easily seen that if $x$ is an element of one of these groups with $o(x) = 3$, then $x$ is included in some solvable subgroups distinct from $\langle x \rangle$.

Therefore, the proof is complete. □

For a finite group $G$, we define $\text{Ord} (\text{Sol}_G) = \{ |\text{Sol}_G(x)| \mid x \in G \}$. Now, it can be asked the following question.

**Problem 4.1** Let $G$ and $H$ be two finite groups. If $\text{Ord} (\text{Sol}_G)$ coincides with $\text{Ord} (\text{Sol}_H)$, then is $G$ isomorphic to $H$?

**References**

[1] A. R. Abdollahi and M. Zarrin, *Non-nilpotent graph of a group*, Comm. Algebra, 38(12)(2010), 4390-4403.

[2] M. Aschbacher and R. Guralnick, *Some applications of the first cohomology group*, J. Algebra, 90(2)(1984), 446-460.

[3] J. Bray, D. Holt and C. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, London Mathematical Society Lecture Note Series, 407. Cambridge University Press, 2013.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, New York, 2008.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, 1985.

[6] R. Guralnick, B. Kunyavskii, E. Plotkin and A. Shalev, *Thompson-like characterization of the solvable radical*, J. Algebra, 300(1)(2006), 363-375.

[7] D. Hai-Reuven, *Non-solvable graph of a finite group and solvabilizers*, arXiv:1307.2924 [math.GR].

[8] D. V. Lytkina, *Structure of a group with elements of order at most 4*, Sib. Math. J., 48(2)(2007), 283–287.

[9] R. Steinberg, *Generators for simple groups*, Canad. J. Math., 14 (1962), 277-283.

[10] M. Suzuki, *Group Theory I*, Springer-Verlag, Berlin-New York, 1982.