Semiglobal Oblique Projection Exponential Dynamical Observers for Nonautonomous Semilinear Parabolic-Like Equations

Sérgio S. Rodrigues

Abstract
The estimation of the full state of a nonautonomous semilinear parabolic equation is achieved by a Luenberger-type dynamical observer. The estimation is derived from an output given by a finite number of average measurements of the state on small regions. The state estimate given by the observer converges exponentially to the real state, as time increases. The result is semiglobal in the sense that the error dynamics can be made stable for an arbitrary given initial condition, provided a large enough number of measurements, depending on the norm of the initial condition, are taken. The output injection operator is explicit and involves a suitable oblique projection. The results of numerical simulations are presented showing the exponential stability of the error dynamics.

Keywords Exponential observer · State estimation · Nonautonomous semilinear parabolic equations · Finite-dimensional output · Oblique projection output injection · Continuous data assimilation

Mathematics Subject Classification 93C20 · 93C50 · 93B51 · 93E10

Communicated by Eliot Fried.

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1 Introduction

We consider evolutionary nonlinear parabolic-like equations, for time \( t \geq 0 \), as

\[
\dot{y} + Ay + A_{\text{rc}} y + \mathcal{N}(y) = f, \quad w = Z_S y,
\]

evolving in a real separable Hilbert space \( V \), where \( A \) and \( A_{\text{rc}} = A_{\text{rc}}(t) \) are, respectively, a time-independent symmetric linear diffusion-like operator and a time-dependent linear reaction–convection-like operator. Further, \( \mathcal{N}(y) = \mathcal{N}(t, y) \) is a time-dependent nonlinear operator and \( f = f(t) \) is a time-dependent external forcing. The tuple \( (A, A_{\text{rc}}, \mathcal{N}, f) \), defining the dynamics, is assumed to be known.

The unknown state of the equation is the variable \( y = y(t) \in V \). The (column) vector output \( w = w(t) = Z_S y(t) \in \mathbb{R}^{S_\sigma \times 1} \sim \mathbb{R}^{S_\sigma} \) consists of a finite number of measurements, where \( S_\sigma \) is a positive integer. The output operator \( Z_S : V \to \mathbb{R}^{S_\sigma \times 1} \) is linear.

The initial state \( y(0) \in V \), at time \( t = 0 \), is assumed to be unknown. Our task is to estimate the state \( y \) from the output \( w \), which is assumed to be given in the form of “averages” as

\[
w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_{S_\sigma}(t) \end{bmatrix}, \quad w_i(t) := (w_i, y(t))_H, \quad 1 \leq i \leq S_\sigma,
\]

where \((\cdot, \cdot)_H\) is the scalar product in a pivot Hilbert space \( H \supset V \). Each \( w_i \in H \) will be referred to as a sensor, and we assume that

the family of sensors, \( W_S := \{w_i | 1 \leq i \leq S_\sigma \} \subset H \), is linearly independent. (1.2b)

We consider the case where we can place the sensors, depending on their number \( S_\sigma \), so that we will actually have

\[
w_i(t) = w_{i,S}(t) \in \mathbb{R}, \quad w_i = w_{i,S} \in H, \quad 1 \leq i \leq S_\sigma.
\]

1.1 Motivation

State estimates are demanded in applications, for example, in the implementation of output-based stabilizing controls. Suppose we have a feedback operator \( K(t) \) such that the associated feedback control \( f(t) = K(t)y(t) \) stabilizes system (1.1); see Rodrigues (2020) for such a stabilizing feedback control. We are particularly interested in the case where the state \( y \) is modeled by partial differential equations as parabolic equations; in this case, the state is infinite-dimensional, and it is not realistic to expect that we will be able to know/measure the entire state \( y(t) \) at each instant of time \( t \). However, we can expect that, with a good enough estimate \( \hat{y}(t) \) for \( y(t) \), the approximated control \( f(t) = K(t)\hat{y}(t) \) will be able to stabilize (1.1).
We cannot expect that an infinite-dimensional state \( y(t) \) can be reconstructed from the finite set \( w(t) = \mathcal{Z}_S y(t) \) at a fixed time \( t \). Hence, using the knowledge of the dynamics of (1.1), we shall look for a Luenberger-type dynamical observer in order to construct an estimate \( \hat{y}(t) \) for \( y(t) \), which will be improving as time increases.

In applications to concrete parabolic equations evolving in a spatial domain \( \Omega_1 \subset \mathbb{R}^d \) we would like to take average-like measurements of the state in a finite number of small regions \( \omega_i \subset \Omega_1 \), \( 1 \leq i \leq S_\sigma \). The estimation result we derive in this manuscript can be applied to this situation where, moreover, the total volume \( \text{vol} \left( \bigcup_{i=1}^{S_\sigma} \omega_i \right) \) covered by the measurement regions can be taken arbitrarily small. We shall illustrate this application in Sects. 1.4, 4, and 5.

**Remark 1.1** For simplicity, we may think of \( S_\sigma = S \). In the application of the result to concrete examples, it is convenient to have a particular subsequence \( (S_\sigma)_{S \in \mathbb{N}_0} \) of positive integer numbers, as we shall see in Sect. 4, where we shall take \( S_\sigma = \sigma(S):=2^S \), for scalar parabolic equations evolving in rectangular spatial domains \( \Omega_1 \subset \mathbb{R}^d \).

### 1.2 The Main Result

Together with the family of sensors, we will need also a family of auxiliary functions

\[
\tilde{W}_S := \{ \tilde{w}_i = \tilde{w}_{i,S} | 1 \leq i \leq S_\sigma \} \subset D(A), \quad \text{which is linearly independent,}
\]

where \( D(A) \subset V \) is another Hilbert space, to be precise later on, namely as the domain of the symmetric invertible diffusion operator \( A \). We will also consider the corresponding linear spans

\[
\mathcal{W}_S := \text{span} W_S \subset H \quad \text{and} \quad \tilde{\mathcal{W}}_S := \text{span} \tilde{W}_S \subset D(A).
\]

**Remark 1.2** In Azouani et al. (2014) and Olson and Titi (2003), this problem of constructing a dynamic state estimate is referred to as “continuous data assimilation.”

**Definition 1.3** Let \( F \) and \( G \) be closed subspaces of the Hilbert space \( H \). If \( H = F + G \) and \( F \cap G = \{0\} \) algebraically, we say that \( F + G \) is a direct sum and write \( H = F \oplus G \); in this case, the operator \( P_F^G \in \mathcal{L}(H, F) \) denotes the oblique projection in \( H \) onto \( F \) along \( G \), which is defined as follows: We can write every \( h \in H \) as \( h = h_F + h_G \) for one, and only one, pair \((h_F, h_G) \in F \times G\), and then, we set \( P_F^G h := h_F \). The orthogonal projection in \( H \) onto \( F \) is denoted by \( P_F := P_F^\perp \in \mathcal{L}(H, F) \).

We shall take sensors and auxiliary functions such that we have the direct sums

\[
\tilde{W}_S^\perp \oplus \mathcal{W}_S = H = \mathcal{W}_S \oplus \tilde{\mathcal{W}}_S^\perp
\]

and show that full-dimensional Luenberger observers as

\[
\dot{\tilde{y}} + A \tilde{y} + A_{rc} \tilde{y} + \mathcal{N}(\tilde{y}) = f + \mathcal{Z}_S^\perp(\mathcal{Z}_S \tilde{y} - w), \quad \tilde{y}(0) = \tilde{y}_0 \in V,
\]

(1.3a)
(cf. Luenberger 1964, Sect. III, Luenberger 1971, Sect. II.B) with the output injection operator given by

\[ I_{\lambda, \ell} S := -\lambda A^{-1} P_{\tilde{W}_S} A^\ell P_{\tilde{W}_S}^+ Z W_S, \quad \lambda > 0, \quad 0 \leq \ell \leq 2, \]  

(1.3b)

are able to estimate the state $y$ of system (1.1), for any given $\ell \in [0, 2]$ and suitable tuples $(\lambda, W_S, \tilde{W}_S)$. Here, $Z W_S : \mathbb{R}^{S_0 \times 1} \to W_S$ is the linear operator defined by

\[ Z W_S z := \sum_{i=1}^{S_0} \left( [V_S]^{-1} z \right)_i w_{i,S}, \quad z \in \mathbb{R}^{S_0 \times 1}, \]  

(1.3c)

where $[V_S] \in \mathbb{R}^{S_0 \times S_0}$ is the generalized Vandermonde matrix, whose entries in the $i$th row and $j$th column are

\[ [V_S]_{i,j} = (w_{i,S}, w_{j,S})_H. \]  

(1.3d)

For suitable $\varrho \geq 1$ and $\mu > 0$, we will have the inequality

\[ |\hat{y}(t) - y(t)|_V \leq \varrho e^{-\mu(t-s)} |\hat{y}(s) - y(s)|_V, \quad \text{for all } t \geq s \geq 0. \]  

(1.4)

Note that, from (1.1) and (1.3), the error $z = \hat{y} - y$ satisfies

\[ \dot{z} + Az + A_r z + N_y(z) = J^{[\lambda, \ell]} S z, \quad z(0) = z_0 \in V, \]  

(1.5a)

where

\[ N_y(z) := N(y + z) - N(y) = N(\hat{y}) - N(y), \]  

(1.5b)

and $z_0 = \hat{y}_0 - y(0)$. Our goal (1.4) reads now

\[ |z(t)|_V \leq \varrho e^{-\mu(t-s)} |z(s)|_V, \quad \text{for all } t \geq s \geq 0. \]  

(1.6)

**Remark 1.4** Observe that $z_0$ in (1.5) is unknown for us, because so is $y(0)$. On the other hand, the choice of $\hat{y}_0 = \hat{y}(0)$ is at our disposal; for example, we can choose $\hat{y}_0$ as an initial guess we might have for $y(0)$.

**Remark 1.5** We can see that in (1.3b), we have that $J^{[\lambda, \ell]} S z = -\lambda A^{-1} P_{\tilde{W}_S} A^\ell q$ with $q \in \tilde{W}_S \subset D(A)$. Hence, in case $\ell > 1$ we may have that $p := A^\ell q \in D(A^{1-\ell}) \setminus D(A^0)$. In this case, $P_{\tilde{W}_S}^+$ must be understood as an extension to $D(A^{1-\ell})$ of the oblique projection in $H = D(A^0)$. Such extensions are well defined, as we shall see later in Proposition 3.9.

Omitting the details, which shall be given in Theorem 3.1, the main result of this paper is as follows.
Main Result. Under general conditions on the tuple \((\tilde{y}, A, A_{rc}, N, y)\) and under particular conditions on the tuple \((\tilde{W}_S, W_S)\), it holds the following. For any given \(\ell \in [0, 2]\), \(R > 0\), \(\varrho > 1\), and \(\mu > 0\), there are \(S \in \mathbb{N}_0\) and \(\lambda > 0\), both large enough, such that: for all initial error \(z_0\) satisfying \(|z_0|_V \leq R\), it follows that the corresponding solution of (1.5), with the output injection operator as in (1.3b), satisfies (1.6).

Definition 1.6. If (1.6) holds true, we say that the error dynamics is exponentially stable with rate \(-\mu < 0\) and transient bound \(\varrho \geq 1\). In this case, we say that system (1.3) is an exponential observer for system (1.1), that system (1.1) is detectable, and that \(I^{[\lambda, \ell]}_S\) is a detecting output injection operator.

Note that Main Result says that we can achieve an exponentially stable nonlinear error dynamics for arbitrary large initial errors \(|z_0|_V \leq R\), with an arbitrary small exponential rate \(-\mu < 0\), and arbitrary small transient bound \(\varrho > 1\). For that, we simply have to take a large enough number of suitable sensors \(S_\sigma\) and a large enough \(\lambda\). In general, the “optimal” transient bound \(\varrho = 1\) cannot be taken in Main Result. However, later on, in Sect. 6 we shall give classes of systems where we can indeed take \(\varrho = 1\). Such classes include linear and suitable semilinear systems. The case \(\varrho = 1\) is interesting simply because it means that the error norm is strictly decreasing. Observe also that \(\varrho = 1\) is the smallest value possible for \(\varrho\) in (1.6) (e.g., by taking \(t = s\)).

In the particular case where our system is linear (i.e., \(N = 0\)), it is not difficult to show that the observer proposed here is a global observer. By global observer, we mean that the output injection operator \(I^{[\lambda, \ell]}_S\) can be taken independent of the norm of the initial error.

1.3 On Previous Related Works in the Literature

For partial differential equations, the results in the literature on state estimation concern mainly the autonomous case. For example, we refer to Ahmed-Ali et al. (2016), Feng and Guo (2017), Jadachowski et al. (2011), Kang and Fridman (2019), Ramdani et al. (2016), Buchot et al. (2015), Fujii (1980), Orlov et al. (2017) and Zhang and Wu (2020). Exceptions are Meurer and Kugi (2009), Meurer (2013) and Jadachowski et al. (2013) for one-dimensional parabolic equations, \(d = 1\), by using the nontrivial backstepping and Cole–Hopf transformations. In Meurer (2013), reaction-type Lipschitz nonlinearities are considered, while in Jadachowski et al. (2013) convection nonlinearities are also included, where some details are omitted concerning the stability of the semilinear error dynamics, as also referred by Jadachowski et al. (2013, Sect. C). See also the auxiliary nonautonomous heat equation in Meurer and Kugi (2009, Eq. (17)).

In the investigation of the autonomous case, as in Ramdani et al. (2016), the spectral properties of the time-independent operator dynamics play a crucial role in the derivation of the results. The (un)stability results in Wu (1974) suggest that such spectral properties in the nonautonomous case (at each fixed time \(t > 0\)) are not an appropriate tool to deal with the nonautonomous case. The recent work Astrovskii and Gaishun...
(2019) also shows that, in general, the state estimation problem in the nonautonomous case is not an easy task even for the case of finite-dimensional systems.

The approach in Azouani et al. (2014) is applicable to state estimation of parabolic-like systems for which we can derive the existence of a finite set of so-called determining parameters. For this purpose, it is also important in Azouani et al. (2014) that solutions do exist for all time $t \geq 0$. This includes, for example, the 2D Navier–Stokes equations. The method in Azouani et al. (2014) is quite interesting because, depending on the nature of the “chosen” determining parameters, it can be applied to several types of measurements, including average-like measurements as we are particularly interested in. However, in this manuscript we consider a class of nonlinear equations whose free dynamics solutions may blow up in finite time, and hence, the results in Azouani et al. (2014) are not directly applicable. The possible blow up of the solutions is also the reason why we cannot derive a detectability result independent of the target state $y$, but rather a semiglobal result depending on a suitable persistent bound for $y$ (Assumption 2.5). For a class of dissipative systems, where solutions are defined globally in time, we can derive such a persistent bound as in Azouani et al. (2014, Thm. 4).

In the case of concrete parabolic-like equations evolving in a spatial domain $\Omega \subset \mathbb{R}^d$, the results in this manuscript can be compared to those in Azouani et al. (2014), in the case of average-like measurements. While in Azouani et al. (2014) the measurements are taken in domains forming a partition of the entire spatial domain $\Omega$, here we show that we can take measurements in domains whose union covers a region with arbitrarily small volume $r \operatorname{vol}(\Omega), r \in (0, 1)$.

In Rodrigues (2021a), a global observer was presented to estimate the state of linear parabolic equations, where the placement of the actuators plays an important role. The results in this manuscript are also derived under the assumption that we are allowed to suitably place the sensors. Such assumption seems to be natural and to reflect common sense: It matters (or, may matter) where we take our measurements in. Again, the observer in Rodrigues (2021a) provides an estimate for the weak solution and the exponential convergence is derived in the pivot $H$ norm. Weak solutions (for initial states in $H \setminus V$) do not necessarily exist for the class of nonlinear systems we consider, and this is one reason we will (need to) use a different output injection operator in this manuscript, to deal with strong solutions and derive the exponential convergence in the stronger $V$ norm.

Finally, we must say that some of the above mentioned works, as Kang and Fridman (2019) and Buchot et al. (2015), do not consider the observer design problem alone, but (already) coupled with a stabilization problem (output-based feedback control). Also, some of the above works deal with boundary measurements, while here we deal with internal measurements.

### 1.4 Illustrating Example: Scalar Parabolic Equations

The results will follow under general assumptions on the operators corresponding to the plant dynamics and on the targeted real state. We shall need also a particular assumption involving the set of sensors. Such assumptions will be presented later on
and will be satisfied, in particular, for a general class of semilinear parabolic equations, under either Dirichlet or Neumann boundary conditions, including

\[
\begin{align*}
\frac{\partial}{\partial t} y + (-\Delta + 1)y + ay + b \cdot \nabla y + \tilde{a}|y|^{s-1}y + (\tilde{b} \cdot \nabla y)|y|^{r-1}y &= f, \quad (1.7a) \\
\mathcal{G}y|_{\Gamma} &= g, \quad w = \mathcal{Z}_S y, \quad (1.7b)
\end{align*}
\]

with \( r \in (1, 5) \) and \( s \in [1, 2) \), defined in a bounded connected open spatial subset \( \Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\} \), with boundary \( \Gamma = \partial \Omega \). \( \Omega \) is assumed to be either smooth or a convex polygon. The state is a function \( y = y(x, t) \), defined for \((x, t) \in \Omega \times (0, +\infty) \). The operator \( \mathcal{G} \) imposes the boundary conditions,

\[
\begin{align*}
\mathcal{G} &= 1, \quad \text{for Dirichlet boundary conditions;} \\
\mathcal{G} &= \mathbf{n} \cdot \nabla = \frac{\partial}{\partial \mathbf{n}}, \quad \text{for Neumann boundary conditions;}
\end{align*}
\]

where \( \mathbf{n} = \mathbf{n}(\tilde{x}) \) stands for the outward unit normal vector to \( \Gamma \), at \( \tilde{x} \in \Gamma \).

The functions \( a = a(x, t), b = b(x, t), \tilde{a} = \tilde{a}(x, t), \tilde{b} = \tilde{b}(x, t) \), and \( f = f(x, t) \) are defined in \( \Omega \times (0, +\infty) \), and the function \( g = g(\tilde{x}, t) \) is defined for \((\tilde{x}, t) \in \Gamma \times (0, +\infty) \). Thus, the data tuple \((a, b, \tilde{a}, \tilde{b}, f, g)\) is allowed to depend on both space and time variables. We assume that

\[
\begin{align*}
a \text{ and } \tilde{a} \text{ are in } L^\infty(\Omega \times (0, +\infty)), \quad (1.8a) \\
b \text{ and } \tilde{b} \text{ are in } L^\infty(\Omega \times (0, +\infty))^d, \quad (1.8b) \\
\text{There exists } \tau_y > 0 \text{ such that } \sup_{t \geq 0} \left( |y(t)|_{H^1(\Omega)} + |y|_{L^2((t, t+\tau_y), H^2(\Omega))} \right) < +\infty. \quad (1.8c)
\end{align*}
\]

**Remark 1.7** In (1.8c), we assume, in particular, that the real state \( y \) must be a globally defined strong solution \( y \in L^\infty((0, +\infty), H^1(\Omega)) \cap L^2_{\text{loc}}((0, +\infty), H^2(\Omega)) \). In general, for regular enough external force \( f \) (e.g., for \( f = 0 \)) we will only have the local existence in time: for a suitable \( \tau_\ast > 0, y \in L^\infty((0, \tau), H^1(\Omega)) \cap L^2((0, \tau), H^2(\Omega)) \), for \( \tau < \tau_\ast \). There are, however, cases where (1.8c) will hold true, for example, for the case where \( g = 0 \) and \( f = Ky \) is a stabilizing feedback control. See Sect. 1.1. Another example is the case of time-periodic systems having time-periodic solutions. A third example is Lyapunov stable (not necessarily asymptotic stable) systems.

**Remark 1.8** In (1.7), we consider nonlinearities involving the absolute value of the state (cf. Chipot and Weissler 1989; Weinan 1994; Grishakov et al. 2012) because we allow fractional powers polynomial nonlinearities and are interested in real-valued states, \( y(x, t) \in \mathbb{R} \). For polynomial nonlinearities with positive integer powers, \( r \in \{2, 3, 4\} \) and \( s = 1 \), we can omit the absolute value. In other words, our results also cover nonlinearities as \( a_4y^4 + a_3y^3 + a_2y^2 \) with \( a_i \in \mathbb{R} \), appearing in models for population dynamics and chemical reactions (Fisher 1937; Olmos and Shizgal 2006; Schlögl 1972; Gugat and Tröltzsch 2015; Lacey 1998).
As output, we take the averages of the solution in subdomains \( \omega_i = \omega_{i,S} \subset \Omega \), as

\[
\bar{w}_i(t) = \left(1_{\omega_i}, y(\cdot, t)\right)_{L^2(\Omega)} = \int_{\omega_i} y(x, t) \, dx, \quad 1 \leq i \leq S_\sigma. \tag{1.9}
\]

We will be interested in the case the regions \( \omega_i \), where we take the measurements in, are constrained to cover not more than an a priori fixed volume, namely \( \text{vol}\left(\bigcup_{i=1}^{S_\sigma} \omega_{i,S}\right) \leq r \text{vol}(\Omega) \) with \( 0 < r < 1 \) independent of \( S \). In other words, we allow ourselves to take/place as many sensors as we want/need, but we are allowed to perform measurements only in (at most) a fixed percentage of the spatial domain \( \Omega \), namely \( 100r\% \).

**Remark 1.9** The usual average over \( \omega_i \) is \( \tilde{w}_i := \frac{\int_{\omega_i} y(x, t) \, dx}{\int_{\omega_i} \, dx} \). However, we assume that we know our sensors; that is, we know the regions \( \omega_{i,S} \) where we take the measurements in. In this case, knowing/measuring \( \tilde{w}_i \) is equivalent to knowing/measuring \( w_i \).

In order to apply our results to system (1.7), we have just to rewrite (1.7) as an evolutionary equation (1.1). To this purpose, we define for both Dirichlet and Neumann boundary conditions the spaces

\[
D(A) = H^2_\mathcal{G}(\Omega) := \{h \in H^2(\Omega) | \mathcal{G}h |_{\Gamma} = 0\}, \quad \text{for} \quad \mathcal{G} \in \left\{1, \frac{\partial}{\partial n}\right\},
\]

and

\[
V = H_1^1(\Omega) := H^1_0(\Omega) = \{h \in H^1(\Omega) | h |_{\Gamma} = 0\}, \quad H^1_{\frac{\partial}{\partial n}}(\Omega) := H^1(\Omega),
\]

with the operators

\[
A := -\nu \Delta + 1, \quad A_{\text{RC}} := a 1 + b \cdot \nabla, \quad \text{and} \quad N(t, y) := \tilde{a}(t) |y|^{r-1} y + \tilde{b}(t) \cdot \nabla y |y|^{s-1} y.
\]

Then, we just construct the Luenberger observer as in (1.3) and apply the Main Result.

### 1.5 Contents and Notation

In Sect. 2, we present the assumptions we require for the operators corresponding to the plant dynamics and for all the “parameters” involved in the output injection operator. In Sect. 3, we prove that under such assumptions, the error of the observer estimate decreases exponentially to zero. In Sect. 4, we show that the required assumptions are satisfiable for standard parabolic equations evolving in rectangular domains. In Sect. 5, we present the results of numerical simulations showing the exponential stability of the error dynamics, for a rectangular domain, namely the unit square. In Sect. 6, we comment on the derived results. Finally, “Appendix” gathers the proofs of auxiliary results needed to derive the main result.
Concerning the notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we set $\mathbb{R}_r := (r, +\infty)$, $r \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

Given two Hilbert spaces $X$ and $Y$, if the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \overset{d}{\hookrightarrow} Y$ and $X \overset{c}{\hookrightarrow} Y$, if the inclusion is also dense and compact.

The space of continuous linear mappings from $X$ into $Y$ is denoted by $\mathcal{L}(X, Y)$. In case $X = Y$, we write $\mathcal{L}(X) := \mathcal{L}(X, X)$. The continuous dual of $X$ is denoted $X' := \mathcal{L}(X, \mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X, Y)$ will be denoted $L^* \in \mathcal{L}(Y', X')$.

For a given interval $I \subseteq \mathbb{R}_0$, we denote $W(I, X, Y) := \{y \in L^2(I, X) | \dot{y} \in L^2(I, Y)\}$ and $W_{\text{loc}}(\mathbb{R}_0, X, Y) := \{y | y \in W((0, T), X, Y)\}$ for all $T > 0$.

The space of continuous functions from $X$ into $Y$ is denoted by $C(X, Y)$. The space of real-valued increasing functions defined in $\mathbb{R}_0$ and vanishing at 0 is denoted by:

$$C_{0, t}(\mathbb{R}_0, \mathbb{R}) := \{i \in C(\mathbb{R}_0, \mathbb{R}) | i(0) = 0, \text{ and } i(\tau_2) \geq i(\tau_1) \text{ if } \tau_2 \geq \tau_1 \geq 0\}.$$

We also denote the vector subspace $C_{b, t}(X, Y) \subset C(X, Y)$ by

$$C_{b, t}(X, Y) := \{f \in C(X, Y) | \exists i \in C_{0, t}(\mathbb{R}_0, \mathbb{R}) \forall x \in X : |f(x)|_Y \leq i(|x|_X)\}.$$

Given a sequence $(a_j)_{j \in \{1, 2, ..., n\}}$ of real numbers, $n \in \mathbb{N}_0$, we denote $||a|| := \max_{1 \leq j \leq n} |a_j|$.

By $\overline{C}_{[a_1, ..., a_n]}$, we denote a nonnegative function that increases in each of its nonnegative arguments $a_i$, $1 \leq i \leq n$. To shorten the notation, for given tuples $a^j = (a^j_1, ..., a^j_n)$, $j = 1, ..., m$, with nonnegative arguments $a^j$, we denote $\overline{C}_{[a^1, ..., a^m]} := \overline{C}_{[a^1_1, ..., a^1_n, ..., a^m_1, ..., a^m_n]}$.

Finally, $C$, $C_1$, $C_2$, ..., stand for unessential nonnegative constants, which may take different values at different places in the manuscript.

### 2 Assumptions

The results will follow under general assumptions on the plant dynamics operators $A$, $\Delta$, $N$, and on our targeted real state $y$. We will also need a particular assumption on the triple $(\mathcal{V}^S, \mathcal{W}^S, \lambda)$.

Throughout this manuscript, all Hilbert spaces are assumed to be real and separable. Our system (1.5) is evolving in a Hilbert space $V \subseteq H$, which is a subspace of a pivot Hilbert space $H = H'$.

**Assumption 2.1** $A \in \mathcal{L}(V, V')$ is symmetric, that is, $\langle Ay, z \rangle_{V', V} = \langle Az, y \rangle_{V', V}$ for all $(y, z) \in V \times V$, and $(y, z) \mapsto \langle Ay, z \rangle_{V', V}$ is a complete scalar product in $V$.

From now on, we suppose that $V$ is endowed with the scalar product $(y, z)_V := \langle Ay, z \rangle_{V', V}$, which still makes $V$ a Hilbert space. Necessarily, $A : V \rightarrow V'$ is an isometry.

**Assumption 2.2** The inclusion $V \subseteq H$ is dense, continuous, and compact.
Necessarily, we have that

\[ \langle y, z \rangle_{V'} = (y, z)_H, \quad \text{for all } (y, z) \in H \times V, \]

and that the operator \( A \) is densely defined in \( H \), with domain \( D(A) \) satisfying

\[ D(A) \overset{d,c}{\hookrightarrow} V \overset{d,c}{\hookrightarrow} H \overset{d,c}{\hookrightarrow} V' \overset{d,c}{\hookrightarrow} D(A'). \]

Further, \( A \) has a compact inverse \( A^{-1}: H \to H \), and we can find a nondecreasing system of (repeated accordingly to their multiplicity) eigenvalues \( (\alpha_n)_{n \in \mathbb{N}_0} \) and a corresponding complete basis of eigenfunctions \( (e_n)_{n \in \mathbb{N}_0} \):

\[ 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \alpha_{n+1} \to +\infty \quad \text{and} \quad Ae_n = \alpha_n e_n. \]

We can define, for every \( \xi \in \mathbb{R} \), the fractional powers \( A^{\xi} \), of \( A \), by

\[ y = \sum_{n=1}^{+\infty} y_n e_n, \quad A^{\xi} y = A^{\xi} \sum_{n=1}^{+\infty} y_n e_n := \sum_{n=1}^{+\infty} \alpha_n^{\xi} y_n e_n, \]

and the corresponding domains \( D(A^{\xi}) := \{ y \in H | A^{\xi} y \in H \} \), and \( D(A^{-\xi}) := D(A^{\xi})' \). We have that \( D(A^{\xi}) \overset{d,c}{\hookrightarrow} D(A^{\xi}) \), for all \( \xi > \xi_1 \), and we can see that \( D(A^0) = H, D(A^1) = D(A), D(A^{\frac{1}{2}}) = V \). Throughout this manuscript, \( D(A^{\xi}) \) is assumed endowed with the scalar product \( \langle y, z \rangle_{D(A^{\xi})} := \langle A^{\xi} y, A^{\xi} z \rangle_H \).

For the time-dependent operators and targeted state, we assume the following.

**Assumption 2.3** For almost every \( t > 0 \), we have \( A_{rc}(t) \in \mathcal{L}(V, H) \), and we have a uniform bound as \( |A_{rc}|_{L^\infty(\mathbb{R}_0, \mathcal{L}(V, H))} =: C_{rc} < +\infty \).

**Assumption 2.4** We have \( \mathcal{N}(t, \cdot, \cdot) \in \mathcal{C}_{b,c}(D(A), H) \), and there exist constants \( C_N \geq 0 \), \( n \in \mathbb{N}_0 \), \( \xi_{1j} \geq 0 \), \( \xi_{2j} \geq 0 \), \( \delta_{1j} \geq 0 \), \( \delta_{2j} \geq 0 \), with \( j \in \{1, 2, \ldots, n\} \), such that for all \( t > 0 \) and all \( (y_1, y_2) \in D(A) \times D(A) \), we have

\[ |\mathcal{N}(t, y_1) - \mathcal{N}(t, y_2)|_H \leq C_N \sum_{j=1}^{n} \left( |y_1|_{V}^{\xi_{1j}} |y_1|_{D(A)}^{\xi_{2j}} + |y_2|_{V}^{\xi_{1j}} |y_2|_{D(A)}^{\xi_{2j}} \right) |d|_{V}^{\delta_{1j}} |d|_{D(A)}^{\delta_{2j}}, \]

with \( d := y_1 - y_2 \), \( \xi_{2j} + \delta_{2j} < 1 \) and \( \delta_{1j} + \delta_{2j} = 1 \). Further, we have that the function \( t \mapsto \mathcal{N}(t, y) \in H \) is strongly measurable, for every \( y \in D(A) \).

**Assumption 2.5** The targeted state \( y \), satisfying (1.1), satisfies the uniform persistent boundedness estimate as follows. There are constants \( C_y \geq 0 \) and \( \tau_y > 0 \) such that

\[ \sup_{s \geq 0} |y(s)|_V \leq C_y \quad \text{and} \quad \sup_{s \geq 0} |y|_{L^2((s, s+\tau_y), D(A))} < C_y. \]

We denote by \( B^\perp := \{ h \in H | (h, s)_H = 0 \, \text{for all} \, s \in B \} \) the orthogonal complement to a given subset \( B \subset H \) of the pivot Hilbert space \( H \).
Assumption 2.6 The pair \((\sigma, (\tilde{W}_S, W_S))_{S \in \mathbb{N}_0}\) satisfies:
\[
\sigma : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \text{ is strictly increasing}
\]
and, with \(S_\sigma := \sigma(S)\), \(\tilde{W}_S = \text{span} \tilde{W}_S\), and \(W_S = \text{span} W_S\),
\[
W_S := \{w_{j, S} | 1 \leq j \leq S_\sigma\} \subset H,
\]
\[
\tilde{W}_S := \{\tilde{w}_{j, S} | 1 \leq j \leq S_\sigma\} \subset D(A) \subset H.
\]
\[
\dim W_S = S_\sigma = \dim \tilde{W}_S \text{ and } H = \tilde{W}_S \oplus W_S^\perp.
\]

The key assumption concerns the following Poincaré-like constant
\[
\beta_S := \inf_{h \in D(A) \cap W_S^\perp \setminus \{0\}} \frac{|h|^2_{D(A)}}{|h|^2_V}. \tag{2.1}
\]

Assumption 2.7 The sequence \((\beta_S)_{S \in \mathbb{N}_0}\) in (2.1) is divergent, \(\lim_{S \to +\infty} \beta_S = +\infty\).

The last assumption concerns the type of outputs (cf (1.2)).

Assumption 2.8 The output \(w = Z y \in \mathbb{R}^{S_\sigma} \times 1\) is of the form \(w_j(t) = (w_{j, S}, y(t))_H\), with \(w_{j, S} \in W_S\).

Assumptions 2.1–2.6 are satisfiable for parabolic systems as (1.7). Assumptions 2.1–2.3 are usually not hard to check for such systems. Assumption 2.4 is satisfied by a general class of polynomial nonlinearities as in (1.7). Assumption 2.5 is a requirement on our targeted state, which simply says that the real state to be estimated is a strong solution which is bounded in a general appropriate way. It is also not difficult to construct spaces satisfying Assumption 2.6, and then, in Assumption 2.8 we are simply requiring the form of the output.

The satisfiability of Assumption 2.7 is nontrivial. We shall prove in Sect. 4 that it is satisfied for scalar parabolic equations evolving in rectangular spatial domains \(\Omega \subset \mathbb{R}^d\), for suitable placement of the sensors (as indicator functions). The proof can be adapted to general convex polygonal domains. The satisfiability of Assumption 2.7 for general smooth domains is an open question. See the discussion in Rodrigues (2021a, Sect. 8.2).

Remark 2.9 Note that Assumption 2.3 is stronger than the one taken in Rodrigues (2021a, Assum. 2.3) in the linear setting. We need extra regularity for \(A_{rc}\) because weak solutions, as considered in Rodrigues (2021a), living in \(W_{loc}(\mathbb{R}_0^a, V, V')\), are not regular enough to deal with the entire class of nonlinear systems we shall consider here. We need strong solutions, living in \(W_{loc}(\mathbb{R}_0^a, D(A), H)\), to guarantee the existence and uniqueness of solutions for all systems involved in our analysis. The existence of such solutions (in a small time interval) can be proven for an arbitrary initial state \(y(0) \in V\) and an arbitrary external force \(f \in L_{loc}^2(\mathbb{R}_0^a, H)\), for example, via the Galerkin method by analogous arguments as in Sect. 3.4; see also Rodrigues (2020, Sect. 4.3), Temam (1995, Sect. 3.3, 2018, Sects. 1.3, 3.2, and 3.7). In order to have solutions satisfying Assumption 2.5, we may need particular external forces; recall Remark 1.7.
3 Exponential Stability of the Error Dynamics

For given \( S \in \mathbb{N}_0 \) and \( \ell \in \mathbb{R} \), we define another Poincaré-like constant as follows

\[
0 < \alpha_{S, \ell} := \inf_{q \in V_S \setminus \{0\}} \frac{|q|^2_{D(\lambda^\ell)}}{|q|^2_{D(A)}}, \quad (3.1)
\]

We prove the following more precise statement of Main Result in Sect. 1.2.

**Theorem 3.1** Let Assumptions 2.1–2.8 hold true, and let us be given \( \ell \in [0, 2] \), \( R > 0 \), \( \varrho > 1 \), and \( \mu > 0 \). Then, there exists a pair \((S^*, \lambda^*)\) \( \in \mathbb{N}_0 \times \mathbb{R}_0 \) such that: for all pairs \((S, \lambda)\) satisfying \( S \geq S^* \) and \( \lambda \geq \lambda^*(S, \ell) \), the error dynamics

\[
\dot{z} + Az + A_{rc}(t)z + \mathcal{N}_y(t, z) = \mathcal{J}^\ell_S z, \quad z(0) = z_0 \in V, \quad t \geq 0, \quad (3.2a)
\]

where \( \mathcal{J}^\ell_S \) as in (1.3b) and \( \mathcal{N}_y(t, z) := \mathcal{N}(t, z + y) - \mathcal{N}(t, y) \), is exponentially stable with rate \(-\mu\) and transient bound \( \varrho \). For all \( z_0 \in V \) with \( |z_0|_V \leq R \), it holds that

\[
|z(t)|_V \leq \varrho e^{-\mu(t-s)} |z(s)|_V, \quad \text{for all} \quad t \geq s \geq 0. \quad (3.2b)
\]

Furthermore, the constants \( S^* \) and \( \lambda^*(S, \ell) \) can be taken of the form

\[
S^* = \overline{C}[a^1, a^2, a^3] \quad \text{and} \quad \lambda^*(S, \ell) = \overline{C}[a^1, a^2, a^3, (1)_{\mathcal{L}(D(A), V), g^*}^{-1}], \quad (3.2c)
\]

where

\[
a^1 := \left( R, \mu, \varrho, \frac{1}{\log(\varrho)} \right), \quad a^2 := \left( C_{rc}, \tau_y, (\tau_y)^{-1}, C_y \right),
\]

\[
a^3 := \left( n, C_N, \|\xi_1\|, \|\xi_2\|, \frac{1}{\|\delta_1\|}, \frac{1}{\|\delta_1 - \xi_2\|}, \frac{1}{\|\delta_1 - \xi_2\|}, \frac{1}{1 - \|\delta_2\|}, \frac{1}{1 - \|\xi_2\|} \right).
\]

and \((C_{rc}, C_y, \tau_y, \delta, \xi, n, C_N)\) is the data in Assumptions 2.3–2.5.

**Remark 3.2** Recall that \( \|a\| := \max_{1 \leq j \leq n} |a_j|_{\mathbb{R}} \), for example, \( \|\xi_j\|_{\delta_i} = \max_{1 \leq j \leq n} |\xi_j\|_{\delta_i} \). Note that \( \delta_{1j} \geq \delta_{1j} - \xi_{2j} = \delta_{1j} + \delta_{2j} - (\delta_{2j} + \xi_{2j}) > 0 \), by Assumption 2.4.

**Remark 3.3** Observe that from (3.2c), if we can show that for a given \( \ell \in [0, 2] \), we have that \( \alpha_{S, \ell} \geq \alpha > 0 \) with \( \alpha \) independent of \( S \), then we can conclude that the lower bound \( \lambda^*(S, \ell) \) can be taken independent of \( S \). This is always the case for \( \ell = 2 \) because \( \alpha_{S, 2} = 1 \). For \( 0 \leq \ell < 2 \), the existence of such \( \alpha > 0 \) is not clear and may depend on \( \mathcal{W}_V \). We will come back to this point in Sect. 4; see Proposition 4.7, where we give an example where such strictly positive lower bound \( \alpha \) does not exist for \( \ell \in [0, 1] \).
From Rodrigues (2021a, Sect. 3.2), we know that \( Z^W S = P_{W_S} \), where \( P_{W_S} = P_{W_S}^\perp \) is the orthogonal projection in \( H \) onto \( W_S \). Thus, we find that (cf. Rodrigues 2021b, Sect. 1.2),

\[
\tilde{\gamma}^{[\lambda,\ell]}_{S} Z_S z = -\lambda A^{-1} P_{W_S} \tilde{W}_S^\perp A^\ell P_{W_S} \tilde{W}_S z = -\lambda A^{-1} P_{W_S} \tilde{W}_S^\perp A^\ell P_{W_S} \tilde{W}_S z. \tag{3.3}
\]

### 3.1 Auxiliary Results

In the proof of Theorem 3.1, given in Sect. 3.2, we will use some auxiliary results, which are gathered in this section.

We start with the following estimate for the nonlinear term.

**Lemma 3.4** Let Assumptions 2.1, 2.2, and 2.4 hold true, and let \( P \in \mathcal{L}(H) \). Then, there is a constant \( C_{N_1} > 0 \) such that: for all \( \hat{\gamma}_0 > 0 \), all \( t > 0 \), and all \( (y_1, y_2) \in D(A) \times D(A) \), we have

\[
2 \left( P (\mathcal{N}(t, y_1) - \mathcal{N}(t, y_2)) , A(y_1 - y_2) \right)_H \\
\leq \hat{\gamma}_0 |y_1 - y_2|_{D(A)}^2 + \left( 1 + \frac{1+|\Delta|_2^2}{\hat{\gamma}_0^2} \right) C_{N_1} \sum_{j=1}^n |y_j - y_k|_V \sum_{k=1}^2 |y_k|_{P_{\mathcal{L}(H)}}. 
\]

Further, the constant \( C_{N_1} \) is of the form \( C_{N_1} = \tilde{C}_{n,1} \), where \( C_{N_1} = C_{N_1,1} \).

The proof of the lemma is given in Rodrigues (2020, Sect. A.1) for operators as \( P = P_{W_S} \); however, the steps of such proof can be repeated for a general operator \( P \in \mathcal{L}(H) \). See Rodrigues (2020, Proposition 3.5).

Next, we have the following property for oblique projections in \( H \).

**Lemma 3.5** Let \( F \) and \( G \) be closed subspaces of \( H \). Then, we have that

\[
H = F \oplus G \iff H = F^\perp \oplus G^\perp \text{ and } (P_F^G)^* = P_{G^\perp}^F.
\]

The proof can be found in Kunisch et al. (2021, Lem. 3.4).

The proof of next result illustrates the roles played by the number \( S_\sigma \) of sensors (through the Poincaré constant \( \beta_S \)) and by the output injection gain \( \lambda \).

**Lemma 3.6** Let Assumptions 2.1, 2.2, 2.6, and 2.7 hold true. Let \( (\Xi_S)_{S \in \mathbb{N}_0} \) be a sequence of positive real numbers and \( \ell \in [0, 2] \). Then, for every constant \( \zeta > 0 \) we can find \( S^* \in \mathbb{N}_0 \) such that for every \( S \geq S^* \), we can find \( \lambda^* = \lambda^*(S, l) \) such that

\[
|z|_{D(A)}^2 + \lambda^* \Xi_S \left| A^\frac{l}{2} P_{W_S} \tilde{W}_S z \right|_H^2 \geq \zeta |z|_V , \quad \text{for all } z \in D(A).
\]

Moreover, \( S^* = \tilde{C}_{\zeta} \) and \( \lambda^*(S, \ell) = \tilde{C}_{\zeta,1} \Xi_{(D(A), V)} a_{S,\ell}^{-1} \), where \( a_{S,\ell} \) is as in (3.1).
Proof By Assumption 2.6, we can write

\[ z = \Theta + \theta, \quad \text{with } \theta := P_{\mathcal{W}_S} W_S^\perp z \quad \text{and} \quad \Theta := P_{\tilde{W}_S} W_S^\perp z, \]

where \( \theta \in \tilde{W}_S \subset D(A) \) and \( \Theta = z - \theta \in \mathcal{W}_S^\perp \cap D(A) \), for all \( z \in D(A) \). Proceeding as in the proof of Kunisch et al. (2021, Lem. 3.5), using (2.1) and by choosing \( \lambda^* > \mathcal{g}_{S,I}^{-1} \mathcal{g}_{S}^{-1} \), it follows that

\[
|z|^2_{D(A)} + \lambda^* \mathcal{E}_S \left| A^{\ell} P_{\mathcal{W}_S} W_S^\perp z \right|^2_H \geq \frac{1}{2} |\Theta|^2_{D(A)} + (\lambda^* \mathcal{g}_{S,I} \mathcal{E}_S - 1) |\Theta|^2_{D(A)}
\]

\[
\geq \frac{1}{2} \beta_S |\Theta|^2_V + |1|_{L_2(D(A), V)}^2 (\lambda^* \mathcal{g}_{S,I} \mathcal{E}_S - 1) |\Theta|^2_V.
\]

Note that Assumption 2.7 implies that

\[ \beta^*_S := \min_{S \geq S^*} \beta_S \to +\infty \quad \text{as} \quad \overline{S} \to +\infty. \]

For every given \( \zeta > 0 \), we choose \( S^* := \min \{ \overline{S} \in \mathbb{N} \mid \beta^*_S \geq 4 \zeta \} \).

Then, for every integer \( S \geq S^* \) we choose

\[ \mathcal{g}^* = \mathcal{g}^*(S, \ell) := (2 \zeta |1|_{L_2(D(A), V)}^2 + 1) \mathcal{g}_{S,I}^{-1} \mathcal{g}_{S}^{-1} > \mathcal{g}_{S,I}^{-1} \mathcal{g}_{S}^{-1}, \] (3.4)

which leads us to

\[
|z|^2_{D(A)} + \lambda^* \mathcal{E}_S \left| A^{\ell} P_{\mathcal{W}_S} W_S^\perp z \right|^2_H \geq 2 \zeta \left( |\Theta|^2_V + |\Theta|^2_V \right) \geq \zeta |\theta + \Theta|^2_V = \zeta |z|^2_V,
\]

which ends the proof. \( \square \)

Finally, we present a sequence of auxiliary results as the following propositions. The corresponding proofs are presented later in “Appendix.”

An estimate for \( \mathcal{N}_y(t, \hat{y} - y) = \mathcal{N}(t, \hat{y}) - \mathcal{N}(t, y) \) is as follows.

**Proposition 3.7** Let Assumptions 2.1, 2.2, 2.4, and 2.5 hold true. Then, there are constants \( \mathcal{C}_{\Omega 1} > 0 \), and \( \mathcal{C}_{\Omega 2} > 0 \) such that: for all \( \gamma_0 > 0 \), all \( t > 0 \), all \( (z_1, z_2) \in D(A) \times D(A) \), we have

\[
2 \left( \mathcal{N}_y(t, z_1) - \mathcal{N}_y(t, z_2), A(z_1 - z_2) \right)_H \\
\leq \mathcal{Y}_0 |z_1 - z_2|^2_{D(A)} \\
+ \left( 1 + \frac{\gamma_0^{1+\gamma_0/2}}{1+\gamma_0/2} \right) \mathcal{C}_{\Omega 1} \sum_{j=1}^{n} |z_1 - z_2|^2_V \sum_{k=1}^{2} |y + z_k|^2_{D(A)} |y + z_k|^2_{D(A)}.
\] (3.5)
and

\[ 2\left(\mathcal{M}_W(t, z), AZ_1\right)_H \]
\[ \leq \tilde{\gamma}_0 |z_1|^2_{L(D(A))} \]
\[ + \tilde{C}_{\mathcal{M}_W} \left(1 + \tilde{\gamma}_0^{-x_3}\right) \left(1 + \tilde{\gamma}_0^{-x_3}\right) \left(1 + |y|^2_{D(A)}\right) \left(1 + |z_1|^2_{D(A)}\right) |z_1|^2_{D(A)} . \]

(3.6)

with \( \tilde{C}_{\mathcal{M}_W} = \tilde{C} \left[n, \frac{1}{2}, \varepsilon, n, C_{\mathcal{N}}\right] \), \( \tilde{C}_{\mathcal{M}_W} = \tilde{C} \left[n, C_{\mathcal{N}}\varepsilon, \varepsilon, n, C_{\mathcal{N}}\varepsilon\right] \), and

\[ \chi_1 := 2 \left\| \frac{\xi_1}{\delta_1 - \xi_2} \right\| \geq 0, \quad \chi_2 := \left\| \frac{2\xi_2}{\delta_1} \right\| \in [0, 2), \]
\[ \chi_3 := 2 \left\| \frac{\xi_1 + \xi_2}{\delta_1 - \xi_2} \right\| \geq 0, \quad \chi_4 := \left\| \frac{\delta_1}{\delta_1 - \xi_2} \right\| \geq 1, \quad \chi_5 := \frac{1 + \|\delta_2\|}{1 - \|\delta_2\|} \geq 1. \] (3.7)

The next auxiliary results concern further properties of oblique projections. Recall that \( \mathcal{W}_S \subset H = D(A^0) \) and \( \mathcal{W}_S \subset D(A) = D(A^1) \), due to Assumption 2.6.

**Proposition 3.8** Let \( \xi \in [0, 1] \). The restriction of the oblique projection \( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \in \mathcal{L}(H) \) to \( D(A^\xi) \subseteq H \) is the oblique projection in \( D(A^\xi) \) onto \( \mathcal{W}_S^\perp \) along \( \mathcal{W}_S^\perp \cap D(A^\xi) \). That is, \( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} = P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp \cap D(A^\xi)} \in \mathcal{L}(D(A^\xi)) \).

For given \( \xi \in [0, 1] \), let us define the mapping \( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} : D(A^{-\xi}) \to D(A^{-\xi}) \) by

\[ \left( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} \right) : z, w \mapsto \begin{pmatrix} P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} \end{pmatrix}_z, w \big|_{D(A^{-\xi}), D(A^\xi)} , \]

(3.9)

for all \( (z, w) \in D(A^{-\xi}) \times D(A^\xi) \).

**Proposition 3.9** Let \( \xi \in [0, 1] \). The mapping \( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} \) is an extension of the oblique projection \( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \in \mathcal{L}(H) \) to \( D(A^{-\xi}) \subseteq H \), and we have the adjoint and norm identities as

\[ P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} = \left( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} \right)^* \quad \text{and} \quad P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^{-\xi})} = P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} \big|_{D(A^{-\xi})} \big|_{D(A^\xi)} , \]

where \( P_{\mathcal{W}_S^\perp}^{\mathcal{W}_S^\perp} \big|_{D(A^\xi)} \) is the restriction in Proposition 3.8.

Finally, we present auxiliary results that we use to analyze the stability of the nonlinear error dynamics.
Proposition 3.10 Let \( \eta_1 > 0, \eta_2 > 0 \) and \( s \in (0, 1) \). Then,

\[
\max_{\tau \geq 0} \{-\eta_1 \tau + \eta_2 \tau^s\} = (1-s)\delta \frac{\eta_2^s}{s} \eta_1^{\frac{1}{1-s}} \eta_1^{-s}.
\]

Proposition 3.11 Let \( T > 0, C_h > 0, r > 1, \) and \( h \in L^r_{\text{loc}}(\mathbb{R}_0, \mathbb{R}) \) satisfying

\[
\sup_{s \geq 0} |h|_{L^r((s,s+T), \mathbb{R})} = C_h \leq +\infty. \tag{3.10}
\]

Let also, \( \mu > 0, \) and \( \varrho > 1. \) Then, for every scalar \( \mu > 0 \) satisfying

\[
\mu \geq \max \left\{ 2 \frac{r-1}{r} \left( \frac{C_h}{r \log(\varrho)} \right)^{\frac{1}{r-1}}, 2\mu \right\} + T \frac{1}{r} C_h, \tag{3.11}
\]

we have that the scalar ODE system

\[
\dot{v} = -(\overline{\mu} - |h|_{\mathbb{R}}) v, \quad v(0) = v_0, \tag{3.12}
\]

is exponentially stable with rate \(-\mu\) and transient bound \( \varrho. \) For every \( v_0 \in \mathbb{R}, \)

\[
|v(t)|_{\mathbb{R}} = \varrho e^{-\mu(t-s)} |v(s)|_{\mathbb{R}}, \quad t \geq s \geq 0.
\]

Proposition 3.12 Let \( T > 0, C_h > 0, r > 1, \) and \( h \in L^r_{\text{loc}}(\mathbb{R}_0, \mathbb{R}) \) satisfy (3.10). Let also \( R > 0, p > 0, \mu > 0, \varrho > 1, \) and \( c > 1. \) Then, the scalar ODE

\[
\dot{\sigma} = -\overline{\mu} - |h|_{\mathbb{R}} (1 + |\sigma|_{\mathbb{R}}^p) \sigma, \quad \sigma(0) = \sigma_0, \tag{3.13}
\]

is exponentially stable with transient bound \( \varrho \) and rate \(-\mu_0 \leq -\mu\) as

\[
\mu_0 := \max \left\{ \mu, \frac{\log(2)p}{pT} \left( \frac{e^{2p+1}R^p C_h}{\frac{1}{\theta^2} - 1} \right)^{\frac{1}{r-1}} \left( \frac{r-1}{r} \right)^{\frac{1}{r-1}}, \frac{2^{p+1}}{p} \left( \frac{2C_h^{\frac{1}{p}}}{r \log(\varrho)} \right)^{\frac{1}{r-1}}, \frac{4}{p} \right\} + T \frac{1}{r} C_h, \tag{3.14}
\]

if

\[
|\sigma_0| \leq R \quad \text{and} \quad \overline{\mu} \geq \overline{\mu}_+ := \max \left\{ 2 \frac{r-1}{r} \left( \frac{2C_h^{\frac{1}{p}}}{r \log(\varrho)} \right)^{\frac{1}{r-1}}, 4\mu_0 \right\} + T \frac{1}{r} C_h. \tag{3.15}
\]

That is, the solution satisfies

\[
|\sigma(t)|_{\mathbb{R}} \leq \varrho e^{-\mu_0(t-s)} |\sigma(s)|_{\mathbb{R}}, \quad \text{for all} \quad t \geq s \geq 0, \quad \text{if} \quad |\sigma_0| < R. \tag{3.16}
\]
3.2 Proof of the Main Theorem 3.1

Using (3.3), we find

\[ \dot{z} = -Az - A_{rc}z - \mathcal{N}(y) - \lambda A^{-1} P_{\mathcal{W}_S} W^\perp_{\mathcal{W}_S} A^\ell z \]

\[ = -Az - A_{rc}z - \mathcal{N}_y(z) - \lambda A^{-1} P_{\mathcal{W}_S} W^\perp_{\mathcal{W}_S} A^\ell \theta \]

with \( \theta := P_{\mathcal{W}_S} W^\perp_{\mathcal{W}_S} z \), from which we obtain

\[ \frac{d}{dt} |z|_V^2 = 2 \left( -Az - A_{rc}z - \mathcal{N}_y(z) - \lambda A^{-1} P_{\mathcal{W}_S} W^\perp_{\mathcal{W}_S} A^\ell \theta, Az \right)_H. \quad (3.17) \]

Observe that, by direct computations, using Assumptions 2.1–2.3 and the Young inequality, we find for all \( \gamma_1 > 0 \),

\[ 2 (-Az - A_{rc}z, Az)_H \leq -(2 - \gamma_1) |z|_{D(A)}^2 + \gamma_1^{-1} C_{rc}^2 |z|_V^2. \quad (3.18) \]

Using the symmetry of \( A \) (cf. Assumption 2.1) and Lemma 3.5, we also find

\[ 2 \left( -\lambda A^{-1} P_{\mathcal{W}_S} W^\perp_{\mathcal{W}_S} A^\ell \theta, Az \right)_H = -2\lambda \left( P_{\mathcal{W}_S} W^\perp_{\mathcal{W}_S} A^\ell \theta, z \right)_H = -2\lambda \left| A^\ell \theta \right|_H^2. \quad (3.19) \]

For the nonlinear term, from (3.6) and the Young inequality, we find for all \( \gamma_2 > 0 \),

\[ 2 \left( \mathcal{N}_y(t, z), Az \right)_H \leq \gamma_2 |z|_{D(A)}^2 \]

\[ + \tilde{C} \eta_1 \left( 1 + y_2^{-\chi_1} \right) \left( 1 + y_2^{-\frac{(\chi_2 + 1)\chi_3 \chi_4}{2}} \right) \left( 1 + |y|^{X_1}_V \right) \left( 1 + |y|^{X_2}_{D(A)} \right) \left( 1 + |z|^{X_3}_V \right) |z|_V^2 \]

which implies

\[ 2 \left( \mathcal{N}_y(t, z), Az \right)_H \leq \gamma_2 |z|_{D(A)}^2 + \tilde{C} \Psi(y) \left( 1 + |z|^{X_3}_V \right) |z|_V^2, \quad (3.20a) \]

with

\[ \tilde{C} = \tilde{C}_{\eta_1, C_{X_1}, \eta_2, C_{X_2}} \left[ \eta_1^{-\frac{\chi_3 \chi_4}{2}} \left( 1 + |y|^{X_1}_V \right) \left( 1 + |y|^{X_2}_{D(A)} \right) \right], \quad (3.20b) \]

\[ \Psi(y) := (1 + |y|^{X_1}_V) \left( 1 + |y|^{X_2}_{D(A)} \right). \quad (3.20c) \]
Combining (3.17), (3.18), (3.19), and (3.20), it follows that
\[
\frac{d}{dt} |z|_V^2 \leq -(2 - \gamma_1 - \gamma_2) |z|_{D(A)}^2 - 2\lambda |\theta|_{D(A)}^2 + \gamma_1^{-1} C_{rc}^2 |z|^2_V + \widehat{C} \Psi(y) \left(1 + |z|_V^{X_1}\right) |z|^2_V.
\]

Next, for simplicity, we choose \((\gamma_1, \gamma_2) = (\frac{1}{2}, \frac{1}{2})\) and obtain
\[
\frac{d}{dt} |z|_V^2 \leq -|z|_{D(A)}^2 - 2\lambda |\theta|_{D(A)}^2 + 2C_{rc}^2 |z|^2_V + \widehat{C} \Psi(y) \left(1 + |z|_V^{X_1}\right) |z|^2_V.
\]

Let us fix an arbitrary \(\mu > 0\). By Lemma 3.6 (with \(\zeta = \mu + 2C_{rc}^2\) and the constant sequence \(\Xi_S = 1\)), there exist \(S^* = \overline{C}[\mu, C_{rc}]\) and \(\lambda^* = \lambda^*(S, \ell) = \overline{C}[\mu, C_{rc}, 1, \Xi_S, \ell]\) such that we will have
\[
\frac{d}{dt} |z|_V^2 \leq -\mu |z|_V^2 + \widehat{C} \Psi(y) \left(1 + |z|_V^{X_1}\right) |z|^2_V, \quad \text{for all } S \geq S^* \text{ and } \lambda \geq \lambda^*.
\]

Using Assumption 2.5, we arrive at
\[
\frac{d}{dt} |z|_V^2 \leq -(\mu - |h(y)| \left(1 + |z|_V^{X_1}\right)) |z|^2_V, \quad |z(0)| = |z_0|, \quad (3.21a)
\]
with, in the case \(\chi_2 \in (0, 2)\), see (3.7),
\[
|h(y)| = h(y) := \widehat{C} \Psi(y) \in L^r_{\text{loc}}(\mathbb{R}_0, \mathbb{R}), \quad r := \frac{2}{\chi_2} > 1, \quad (3.21b)
\]
\[
|h(y)|_{L^r((s, s+\tau_y), \mathbb{R})} = \widehat{C} \Psi(y)_{L^r((s, s+\tau_y), \mathbb{R})} \leq \widehat{C} \left|1 + |y|_V^{X_1}\right|_{L^{\infty}((s, s+\tau_y), \mathbb{R})} \left|1 + |y|_V^{X_2}\right|_{L^r((s, s+\tau_y), \mathbb{R})} \leq \widehat{C}(1 + C^{X_1}) \left(\tau_y^\frac{1}{r} + |y|_V^{X_2} \right)_{L^r((s, s+\tau_y), D(A))} =: C_{h}. \quad (3.21c)
\]

In the case \(\chi_2 = 0\), (3.21) holds with arbitrary \(r > 1\). Therefore, in either case, the norm \(\sigma = |z|_V^2\) satisfies system (3.13), with \(h = h(y)\) and \(p = \frac{\chi_2}{r} \geq 0\).

In the case \(p > 0\), we use Proposition 3.12 to conclude that, for any given \(\rho > 1\) and \(\mu > 0\), the norm satisfies
\[
|z(t)|_V^2 \leq \rho e^{-\mu(t-s)} |z(s)|_V^2, \quad \text{for } t \geq s \geq 0, \quad \text{and } |z(0)|_V^2 < R, \quad p > 0, \quad (3.22)
\]
provided we take \(\mu\) large enough. In the case \(p = 0\), we use Proposition 3.11 to conclude that, for any given \(\rho > 1\) and \(\mu > 0\), the norm satisfies
\[
|z(t)|_V^2 \leq \rho e^{-\mu(t-s)} |z(s)|_V^2, \quad \text{for } t \geq s \geq 0, \quad \text{and } z(0) \in V, \quad p = 0,
\]
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provided we take \( \overline{\mu} \) large enough. In particular, (3.22) actually holds for all \( p \geq 0 \):
\[
|z(t)|_V^2 \leq \varrho e^{-\mu(t-s)}|z(s)|_V^2, \quad \text{for all } t \geq s \geq 0, \quad \text{and all } |z(0)|_V^2 < R, \tag{3.23}
\]
provided we take \( \overline{\mu} \) large enough, namely recalling Propositions 3.11 and 3.12, as
\[
\overline{\mu} = \overline{C} \left[ \mu, \frac{1}{\gamma}, \varrho, \frac{1}{e^{z - 1}}, \frac{1}{\log(\varrho)}, \frac{r + 1}{r - 1}, R, Ch, \chi_3 \right].
\]
That is, provided we take \( S \geq S^* \) and \( \lambda \geq \lambda^*(S, \ell) > 0 \) large enough, namely as
\[
S^* = \overline{C} \left[ \mu, \frac{1}{\gamma}, \varrho, \frac{1}{e^{z - 1}}, \frac{1}{\log(\varrho)}, \frac{r + 1}{r - 1}, R, Ch, \chi_3, C_{\text{IC}} \right],
\]
and
\[
\lambda^*(S, \ell) = \overline{C} \left[ \mu, \frac{1}{\gamma}, \varrho, \frac{1}{e^{z - 1}}, \frac{1}{\log(\varrho)}, \frac{r + 1}{r - 1}, R, Ch, \chi_3, C_{\text{IC}}, |H^2_{L(D(A), V)}|^\chi_3^{-1} \right],
\]
with arbitrary \( \tau > 1 \) in case \( \chi_2 = 0 \). Hence, with the tuples \( a^i \) defined as in Theorem 3.1, recalling (3.20) and (3.21), and using the fact that \( \frac{1}{e^{z - 1}} < 2 \frac{1}{\log(\varrho)} \), due to \( \varrho > 1 \), we can conclude that we may choose
\[
S^* = \overline{C} \left[ a^1, a^2, a^3, (\chi_1, \frac{2 + \chi_2}{2 - \chi_2}, \chi_3) \right] \quad \tag{3.24a}
\]
and
\[
\lambda^*(S, \ell) = \overline{C} \left[ a^1, a^2, a^3, (\chi_1, \frac{2 + \chi_2}{2 - \chi_2}, \chi_3), (\vartheta_{S, \ell}^{-1}, |H^2_{L(D(A), V)}|) \right] \quad \tag{3.24b}
\]
We can finish the proof by recalling (3.7) and (3.8), and observing that \( \chi_3 \geq \chi_1 \). □

**Remark 3.13** We see that the injection operator gain parameter \( \lambda \geq \lambda^*(S, \ell) \) depends (or, may depend) on \( S \) and may increase with \( \alpha_{S, \ell}^{-1} \). Thus, it is possible that we may need larger \( \lambda \) for a larger number of sensors \( S_\sigma \). In Azouani et al. (2014, Thms. 1 and 2, and Props. 1 and 2) (see also Jones and Titi 1993; Lunasin and Titi 2017, Sect. 2§2) for Navier–Stokes equations, for average measurements taken in regions covering the entire spatial domain we can see that the number of sensors and the output injection gain can be taken satisfying, roughly speaking,
\[
S_\sigma \geq C_1, \quad \lambda \geq C_2, \quad \text{and} \quad \frac{\lambda}{S_\sigma} \leq C_3
\]
where the positive constants \( C_1, C_2, \) and \( C_3 \) depend only on the free dynamics. Therefore, we also have a result for large enough \( \lambda \geq \lambda^* \) and \( S_\sigma \geq (S_\sigma)^* \), but the quotient
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relation implies that we have to choose $\lambda^* = C_2$ first and then $(S_\sigma)^* \equiv (S_\sigma)^*_\ast (\lambda)$, contrary to our approach where we must choose $S^*$ first and then $\lambda^* = \lambda^*(S)$. In this manuscript, in the case $\ell = 2$, we can actually choose $\lambda^*$ and $S^*$ independently of each other, because we have that, in (3.1), $\alpha_{S,2} = 1$ is independent of $S$, and hence, $\lambda^*(S, 2)$ in (3.2c) is independent of $S$; note that with $\Xi_S = 1$ (as used in the proof of Theorem 3.1) we have that (3.4) reads $\lambda^* = \lambda^*(S, 2) := 2\zeta |1|_D^{\frac{2}{\lambda}} + 1 > 1$, hence independent of $S$.

**Remark 3.14** We have used Lemma 3.6, in the proof of Theorem 3.1, with the particular sequence $\Xi_S = 1$. The reason we present the lemma for a general sequence (cf. (Kunisch et al. 2021, Lem. 3.1)) is that: firstly, the proof is similar and, secondly, considering such general sequence can be useful to show that the result also holds for more general output injection operators $-\lambda \widehat{I} \circ Z_S$. Essentially, recalling that we use Lemma 3.6 where $z = \hat{y} - y$ is the estimation error, we will need an analog of Lemma 3.6 for the inequality

$$|\hat{y} - y|^2_{D(A)} + \lambda^* (\hat{y}(Z_S y) - \hat{y}(Z_S y)), A(\hat{y} - y))^2_H \geq \xi |\hat{y} - y|^2_1.$$

In the particular case $\hat{I} = A^{-1} Z^{W_S}$, we would need

$$|z|^2_{D(A)} + \lambda^* |P_W z|^2_H \geq \xi |z|^2_1.$$

To derive the latter, we can use Lemma 3.6 with an appropriate sequence $\Xi_S$ following the arguments in Kunisch et al. (2021, Lem.3.6, Cor. 3.1).

### 3.3 Boundedness of the Output Injection Operator

Here, we present estimates on the norm of the linear injection operator

$$J^{[\lambda, \ell]}_{S} = -\lambda A^{-1} P_{W_S}^{-1} A^\ell P_{W_S^{-1}}^{\ell} Z^{W_S} \in \mathcal{L}(\mathbb{R}^{S_\sigma \times 1}, H), \quad \ell \in [0, 2].$$

Due to (1.3c), we have that $Z^{W_S} \in \mathcal{L}(\mathbb{R}^{S_\sigma \times 1}, W_S)$ and we show now that we can write

$$\left| J^{[\lambda, \ell]}_{S} \right|_{\mathcal{L}(\mathbb{R}^{S_\sigma \times 1}, H)} \leq \lambda \tilde{C}^{[\ell]}_{2S} \left| Z^{W_S} \right|_{\mathcal{L}(\mathbb{R}^{S_\sigma \times 1}, H)},$$

with

$$\tilde{C}^{[\ell]}_{2S} := \left| A^{-1} P_{W_S}^{-1} A^\ell P_{W_S^{-1}}^{\ell} \right|_{\mathcal{L}(H)} < +\infty.$$

To show such boundedness, we consider the cases $\ell \in [1, 2]$ and $\ell \in [0, 1]$ separately.
In the case $\ell \in [1, 2]$, we have $1 - \ell \in [-1, 0]$ and $P_{\mathcal{Y}^\perp_S} \in \mathcal{L}(D(A^{1-\ell}))$, due to Proposition 3.9, and we find

$$\tilde{C}^{[\ell]}_{\mathcal{Y}^\perp_S} \leq \left| 1 \right|_{\mathcal{L}(D(A^{1-\ell}), D(A^{-1}))} \left| P_{\mathcal{Y}^\perp_S} \right|_{\mathcal{L}(D(A^{1-\ell}))} \left| 1 \right|_{\mathcal{L}(H, D(A))} \left| P_{\mathcal{Y}^\perp_S} \right|_{\mathcal{L}(H)},$$

where we have also used $|A^{-1}|_{\mathcal{L}(D(A^{-1}), H)} = 1 = |A^\ell|_{\mathcal{L}(D(A^\ell), D(A^{1-\ell}))}$.

In the case $\ell \in [0, 1]$, we have $1 - \ell \in [0, 1]$ and

$$\tilde{C}^{[\ell]}_{\mathcal{Y}^\perp_S} \leq \left| 1 \right|_{\mathcal{L}(D(A), H)} \left| P_{\mathcal{Y}^\perp_S} \right|_{\mathcal{L}(H)} \left| 1 \right|_{\mathcal{L}(D(A^{1-\ell}), H)} \left| 1 \right|_{\mathcal{L}(H, D(A))} \left| P_{\mathcal{Y}^\perp_S} \right|_{\mathcal{L}(H)},$$

where we have also used $|A^{-1}|_{\mathcal{L}(H, D(A))} = 1$.

Next, we show that the total “energy” spent by the injection operator is bounded, in case (3.2b) holds true. Indeed, recalling (3.3) we find that

$$\left| \gamma_S^{[k, \ell]}(\mathcal{Z}\mathcal{Y} - w) \right|_{L^2(\mathbb{R}_0, H)} \leq \left| \gamma_S^{[k, \ell]} \right|_{L^2(\mathbb{R}_0, H)} \leq \lambda \tilde{C}^{[\ell]}_{\mathcal{Y}^\perp_S} \left| z \right|_{L^2(\mathbb{R}_0, H)} \leq \lambda \rho \tilde{C}^{[\ell]}_{\mathcal{Y}^\perp_S} \left| 1 \right|_{\mathcal{L}(V, H)} \left| z(0) \right|_{V} \left( \int_0^{+\infty} e^{-2\mu t} \, dt \right)^{\frac{1}{2}},$$

which leads us to

$$\left| \gamma_S^{[k, \ell]}(\mathcal{Z}\mathcal{Y} - w) \right|_{L^2(\mathbb{R}_0, H)} \leq \lambda \rho (2\mu)^{-\frac{1}{2}} \tilde{C}^{[\ell]}_{\mathcal{Y}^\perp_S} \left| 1 \right|_{\mathcal{L}(V, H)} \left| z(0) \right|_{V}.$$

### 3.4 On the Existence and Uniqueness of Solutions for the Error

The existence and uniqueness of solutions are derived via the Galerkin method. The existence of a solution $z$ is proven as a weak limit of a sequence $(z^N)_{N \in \mathbb{N}_0}$ of Galerkin approximations, due to uniformity of estimates on $N$. The uniqueness follows by an energy-like estimate and Gronwall inequality. Let us consider the Galerkin approximation

$$\dot{z}^N + A z^N + P_{E_N} A \mathcal{Y}(t) z^N + P_{E_N} \mathcal{Y}(t, z^N) = P_{E_N} \gamma_S^{[k, \ell]} \mathcal{Z} z^N, \quad t \geq 0, \quad (3.25a)$$

$$z^N(0) = P_{E_N} z_0 \in V, \quad (3.25b)$$

where $P_{E_N} \in \mathcal{L}(H)$ is the orthogonal projection in $H$ onto the space $E_N := \text{span}\{e_n|1 \leq n \leq N\}$ spanned by the first eigenfunctions of $A$. 
Let us fix $\varrho > 1$, $\mu > 0$, and $s > 0$. We may repeat the estimates in Sect. 3.2 and arrive to the analogs of (3.21) and (3.23),

\[
\frac{d}{dt} \left| z^N \right|^2_V \leq -\left( \overline{\mu} - |h(y)| \left( 1 + \left| z^N \right|^{X_3} \right) \right) \left| z^N \right|^2_V ,
\]

(3.26a)

\[
\left| z^N (t) \right|^2_V \leq \varrho e^{-\mu(t-s)} \left| z^N (s) \right|^2_V , \quad \text{for all} \quad t \geq s \geq 0, \quad |z_0|^2_V < R .
\]

(3.26b)

provided we take a large enough $S$ and a large enough $\lambda > 0$.

Note that $\overline{\mu}$, $h(y)$, and $X_3$ are independent of $N$, and that $|P_{E_N} z_0|^2_V \leq |z_0|^2_V < R$. Hence, $S$ and $\lambda$ can be taken independent of $N$. From (3.26b) and (3.25a), it follows

\[
\left| z^N \right|^2_{W((0,s),D(A),H)} = \left| z^N \right|^2_{L^2((0,s),D(A))} + \left| z^N \right|^2_{L^2((0,s),H)} \leq C
\]

with $C$ independent of $N$. Indeed, proceeding as in Rodrigues (2020, Sect. 4.3), multiplying the equation (3.25a) by $2A z^N$, and using suitable Young inequalities,

\[
\frac{d}{dt} \left| z^N \right|^2_V \leq -2 \left| z^N \right|^2_{D(A)} + 2C_{\text{re}} \left| z^N \right|_V \left| z^N \right|_{D(A)} + 2 \left| \mathcal{S}^{[\lambda,\ell]} \mathcal{Z}_S z^N \right|_H \left| z^N \right|_{D(A)}
\]

\[
+ 2 \left. \mathcal{M}_y (t, z^N) \right|_H \left| z^N \right|_{D(A)}
\]

\[
\leq - \left| z^N \right|^2_{D(A)} + 3C_{\text{re}} \left| z^N \right|^2_V + 3 \left| \mathcal{S}^{[\lambda,\ell]} \mathcal{Z}_S \right|^2_{\mathcal{L}(H)} \left| z^N \right|^2_H
\]

\[
+ \tilde{\mathcal{C}} \Phi (y) \left( 1 + \left| z^N \right|^{X_3}_V \right) \left| z^N \right|^2_V ,
\]

where we used (3.20) with $\gamma_2 = \frac{1}{3}$. By (3.26b), and recalling $\Phi (y)$ defined in (3.20),

\[
\frac{d}{dt} \left| z^N \right|^2_V \leq - \left| z^N \right|^2_{D(A)} + C_1 (1 + |y|^{X_3}_{D(A)}) .
\]

After integration, recalling that $0 \leq X_2 < 2$ and using Assumption 2.5,

\[
\left| z^N (s) \right|^2_V + \left| z^N \right|^2_{L^2((0,s),D(A))} \leq \left| z^N (0) \right|^2_V + C_1 \left( s + \left[ \frac{s}{\tau_y} \right] \frac{2^{-X_2}}{\tau_y} C^{X_2}_y \right),
\]

where, for $r \geq 0$, $\left[ r \right] \in \mathbb{N}$ stands for the nonnegative integer defined by

\[
r \leq \left[ r \right] < r + 1.
\]

(3.27)

Therefore, $\left| z^N \right|^2_{L^2((0,s),D(A))} \leq C_2$, with $C_2$ independent of $N$. Using now (3.25a), it follows that $\left| \dot{z}^N \right|^2_{L^2((0,s),H)} \leq C_3$, with $C_3$ independent of $N$. Hence, there exists a weak limit $z^\infty \in W((0,s), D(A), H)$ so that

\[
z^N \xrightarrow{L^2((0,s),D(A))} z^\infty \quad \text{and} \quad \dot{z}^N \xrightarrow{L^2((0,s),H)} \dot{z}^\infty .
\]
Clearly, for the linear terms we have
\[
\Phi N := Az^N + A_{\text{rec}}(t)z^N - J_S^{[h, \ell]}Z_S z^N \xrightarrow{L^2((0,s), H)} Az^{\infty} + A_{\text{rec}}(t)z^{\infty} - J_S^{[h, \ell]}Z_S z^{\infty} =: \Phi^\infty,
\]
from which we can derive
\[
Az^N + P_{\text{EN}} A_{\text{rec}}(t)z^N - P_{\text{EN}} J_S^{[h, \ell]}Z_S z^N \xrightarrow{L^2((0,s), H)} Az^{\infty} + A_{\text{rec}}(t)z^{\infty} - J_S^{[h, \ell]}Z_S z^{\infty},
\]
due to the facts that \(Az^N = P_{\text{EN}} Az^N\), and that for all \(h \in L^2((0, s), H)\),
\[
\left( P_{\text{EN}} \Phi^N, h \right)_{L^2((0,s),H)} = \left( \Phi^N, h \right)_{L^2((0,s),H)} - \left( \Phi^N, (1 - P_{\text{EN}})h \right)_{L^2((0,s),H)},
\]
which gives us
\[
\lim_{N \to +\infty} \left| \left( P_{\text{EN}} \Phi^N, h \right)_{L^2((0,s),H)} \right|_{\mathbb{R}} \leq \lim_{N \to +\infty} \left| \Phi^N \right|_{L^2((0,s),H)} \left| (1 - P_{\text{EN}})h \right|_{L^2((0,s),H)},
\]
since \(\left| \Phi^N \right|_{L^2((0,s),H)}\) is bounded and \(\left| (1 - P_{\text{EN}})h \right|_{\mathcal{L}(H)} \to 0\). Concerning the existence, it remains to prove that the nonlinear term also converges weakly. Actually, we can show that it converges strongly
\[
P_{\text{EN}} \mathfrak{N}_y(t, z^N) \xrightarrow{L^2((0,s), H)} \mathfrak{N}_y(t, z^{\infty}). \tag{3.28}
\]
In order to show (3.28), we follow arguments from Rodrigues (2020, Sect. 4.3). From Assumption 2.4, we have that
\[
\left| \mathfrak{N}_y(t, z^N) - \mathfrak{N}_y(t, z^{\infty}) \right|_H = \left| \mathcal{N}(t, y + z^N) - \mathcal{N}(t, y + z^{\infty}) \right|_H \\
\leq C_N \sum_{j=1}^n \left( \left| y + z^N \right|_{\mathfrak{L}^{\xi_j}} \left| y + z^N \right|_{\mathcal{D}(A)} + \left| y + z^{\infty} \right|_{\mathfrak{L}^{\xi_j}} \left| y + z^{\infty} \right|_{\mathcal{D}(A)} \right) \left| d^N \right|_{\mathfrak{L}^{\xi_j}} \left| d^N \right|_{\mathcal{D}(A)} \\
= C_N \sum_{j=1}^n \sum_{k=1}^2 \left| w_k \right|_{\mathfrak{L}^{\xi_j}} \left| w_k \right|_{\mathcal{D}(A)} \left| d^N \right|_{\mathfrak{L}^{\xi_j}} \left| d^N \right|_{\mathcal{D}(A)}
\]
with \(d^N := z^N - z^{\infty}, \ w_1 := y + z^N, \) and \(w_2 := y + z^{\infty}. \) Hence, we arrive at
\[
\left| \mathfrak{N}_y(t, z^N) - \mathfrak{N}_y(t, z^{\infty}) \right|_H \\
\leq C_N \sum_{j=1}^n \left( \left( \sum_{k=1}^2 \left| w_k \right|_{\mathfrak{L}^{\xi_j}} \left| w_k \right|_{\mathcal{D}(A)} \right) \left| d^N \right|_{\mathcal{D}(A)} \right) \left| d^N \right|_{\mathfrak{L}^{\xi_j}} \left| d^N \right|_{\mathcal{D}(A)}^{2j} \tag{3.28}
\]
whose right-hand side is similar to an expression we find in Rodrigues (2020, Sect. 4.3). Thus, we can repeat the arguments in Rodrigues (2020) to conclude that

\[ \mathcal{N}_y(t, z^N) \rightarrow \mathcal{N}_y(t, z^\infty), \]

from which we can derive (3.28), due to

\[
\left| P_{EN} \mathcal{N}_y(t, z^N) - \mathcal{N}_y(t, z^\infty) \right|_{L^2((0,s), H)}^2 \\
= \left| P_{EN} \left( \mathcal{N}_y(t, z^N) - \mathcal{N}_y(t, z^\infty) \right) \right|_{L^2((0,s), H)}^2 + \left| (1 - P_{EN}) \mathcal{N}_y(t, z^\infty) \right|_{L^2((0,s), H)}^2 \\
+ 2 \left( P_{EN} \left( \mathcal{N}_y(t, z^N) - \mathcal{N}_y(t, z^\infty) \right), (1 - P_{EN}) \mathcal{N}_y(t, z^\infty) \right)_{L^2((0,s), H)},
\]

which imply

\[
\lim_{N \to +\infty} \left| P_{EN} \mathcal{N}_y(t, z^N) - \mathcal{N}_y(t, z^\infty) \right|_{L^2((0,s), H)}^2 \\
= \lim_{N \to +\infty} 2 \left( P_{EN} \left( \mathcal{N}_y(t, z^N) - \mathcal{N}_y(t, z^\infty) \right), (1 - P_{EN}) \mathcal{N}_y(t, z^\infty) \right)_{L^2((0,s), H)} \\
\leq \lim_{N \to +\infty} 2 \left( \mathcal{N}_y(t, z^N) - \mathcal{N}_y(t, z^\infty) \right)_{L^2((0,s), H)} \left( (1 - P_{EN}) \mathcal{N}_y(t, z^\infty) \right)_{L^2((0,s), H)} \\
= 0.
\]

Therefore, \( z^\infty \) solves system (3.2).

Finally, we show the uniqueness of the solution of system (3.2) in \( W((0,s), D(A), H) \). For an arbitrary solution \( z \) in \( W((0,s), D(A), H) \), \( z(0) = z_0 \), for \( G := z - z^\infty \) we find

\[
\dot{G} + AG + A_{rc}G + \mathcal{N}_y(z) - \mathcal{N}_y(z^\infty) = \mathcal{I}^{(k,\ell)}_S z S G, \quad G(0) = 0.
\]

Observe also that \( \mathcal{N}_y(z) - \mathcal{N}_y(z^\infty) = \mathcal{N}(t, y + z) - \mathcal{N}(t, y + z^\infty) \). Again we can repeat the argument in Rodrigues (2020, Sect. 4.3), by Assumption 2.4 to conclude that, with \( z_1 = y + z \) and \( z_2 = y + z^\infty \),

\[
2 \left( \left( \mathcal{N}(t, z_1) - \mathcal{N}(t, z_2) \right), AG \right)_H \leq |G|_{D(A)}^2 + \Phi(t) |G|_{V}^2,
\]

\[
\Phi(t) := \mathcal{C}_1 \sum_{j=1}^{n} \left( |z_1|_{V}^{\xi j_{1j} - n_{1j}} + |z_2|_{V}^{\xi j_{1j} - n_{1j}} + |z_1|_{D(A)}^2 + |z_2|_{D(A)}^2 \right).
\]
By using Assumption 2.3 and the Young inequality, we find
\[
\frac{d}{dt} |G|^2_V \leq -2 |G|^2_{D(A)} + \Phi(t) |G|^2_V + 2C^2_{rc} |G|^2_V + 2 \left| \mathcal{S}^{[\lambda, \ell]}_{LS} \right|^2_{L(H)} |G|^2_V + 2 |G|^2_{D(A)} \leq \Phi_2(t) |G|^2_V.
\]

with \( \Phi_2(t) := 2C^2_{rc} + 2 \left| \mathcal{S}^{[\lambda, \ell]}_{LS} \right|^2_{L(H)} + \Phi(t) \). From \( z_1 = y + z \) and \( z_2 = y + z^\infty \), Assumption 2.5, and \( \{z_1, z_2\} \subset \mathcal{C}([0, s], V) \bigcap L^2((0, s), D(A)) \), we see that \( \Phi_2 \) is integrable on \((0, s)\). Hence, by the Gronwall inequality,
\[
|G(t)|^2_V \leq e^{\int_0^t \Phi_2(\tau) d\tau} |G(0)|^2_V = 0, \quad \text{for all} \quad t \in [0, s].
\]

That is, \( G = 0 \) and \( \dot{z} = z^\infty + G = z^\infty \). We have shown the uniqueness of the solution for (3.2) in \( W((0, s), D(A), H) \), for arbitrary \( s > 0 \). In other words, the solution for (3.2) is unique in \( W_{loc}(\mathbb{R}_0, D(A), H) \).

### 3.5 On the Existence and Uniqueness of Solutions for Systems (1.1) and (1.3)

Proceeding as in Sect. 3.4, see also Rodrigues (2020, Sect. 4.3), we can show that the solution \( y \) for system (1.1), assumed in Assumption 2.5 to exist in \( W_{loc}(\mathbb{R}_0, D(A), H) \), is unique. From Sect. 3.4, the solution \( z \), given by Theorem 3.1 for the error dynamics is also unique. Consequently, the solution \( \hat{y} = y + z \in W_{loc}(\mathbb{R}_0, D(A), H) \) for (1.3) exists and is unique.

### 4 Parabolic Equations Evolving in Rectangular Domains

In order to apply Theorem 3.1 to the case of scalar parabolic equations, it is enough to show that our Assumptions 2.1–2.6 are satisfied, for the operators defined as in Sect. 1.4. Assumptions 2.1–2.2 are satisfied with \( A = -\nu \Delta + 1 \). Assumption 2.3 is satisfied with \( A_{rc} = a \mathbf{1} + b \cdot \nabla \mathbf{1} \in L^\infty(\mathbb{R}_0, \mathcal{L}(V, H)) \), because \( a \) and \( b \) are both essentially bounded, see (1.8). Assumption 2.4 is proven in Rodrigues (2020, Sect. 5.2). Assumption 2.5 will follow for suitable external forces \( f \); see discussion in Sect. 1.1 and Remark 1.7. Assumption 2.8 is satisfied for outputs as in (1.9).

It remains to show the satisfiability of Assumptions 2.6–2.7. For this purpose, we borrow arguments from Rodrigues (2021a, Sect. 5, 2021b, Sect. 6). We restrict ourselves to the case of rectangular domains \( \Omega^\infty = \times_{j=1}^d (0, L_j) \in \mathbb{R}^d \).

As set of sensors we take the set of indicators functions
\[
W_S := \{1_{\omega_i} | 1 \leq i \leq S \sigma \equiv (2S)^d \}, \quad (4.1a)
\]
where the $\omega_i$s are subrectangles

$$\omega_i = \omega_{i,S} = \prod_{j=1}^{d} \left( p_j^{i,S}, p_j^{i,S} + \frac{r L_j}{2S} \right), \quad p_j^{i,S} = \frac{(2j-1)L_i}{4S} - \frac{r L_i}{4S},$$

(4.1b)
as Rodrigues (2021a, Sect. 5), these regions are illustrated in Fig. 1, for a planar rectangle $\Omega^x = (0, L_2) \times (0, L_2) \subset \mathbb{R}^2$, where the total volume (area) covered by the sensors is independent of $S$. In the figure, such volume is given by $\frac{1}{16} \text{vol}(\Omega^x)$, which is 6.25% of the volume of $\Omega^x$, $r = \frac{1}{4}$.

The choice of the auxiliary set $\tilde{W}_S \subset D(A)$ is at our disposal. For example, we can take the Cartesian product eigenfunctions of $A$ as Rodrigues (2021a, Sect. 5),

$$\tilde{W}_S = E_S := \{ e_i | i \in \mathbb{S}^d \}, \quad e_i(x) := \prod_{j=1}^{d} e_{j_i}(x_j), \quad \mathbb{S} := \{1, 2, \ldots, 2S\};$$

(4.2a)
or, the more ad hoc bump-like functions as in Rodrigues (2021b, Sect. 6)

$$\tilde{W}_S = \Phi_S := \{ \Phi_i | 1 \leq i \leq (2S)^d \}, \quad \Phi_i(x) := 1_{\omega_i}(x) \prod_{j=1}^{d} \sin^2 \left( 2S \pi \frac{x_j - p_j^{i,S}}{r L_j} \right),$$

(4.2b)
or, we could construct and take the functions

$$\tilde{W}_S = \mathcal{A}_S := \{ A^{-1} 1_{\omega_i} | 1 \leq i \leq (2S)^d \}.$$  

(4.2c)

From Rodrigues (2021a, Sect. 5, 2021b, Sect. 6), we know that Assumption 2.6 is satisfied for the first two choices in (4.2), with $\sigma(S) := (2S)^d$. For the third choice, we simply observe that the matrix $\mathcal{M} := \{ (A^{-1} 1_{\omega_i}, 1_{\omega_j}) \}_{L^2(\Omega^x)} \in \mathbb{R}^{S_\sigma \times S_\sigma}$ with entry $(A^{-1} 1_{\omega_i}, 1_{\omega_j})^{L^2(\Omega^x)}$ in the $i$th row and $j$th column is invertible (cf. Kunisch and Rodrigues 2019, Lem. 2.7); indeed, if $\mathcal{M} v = 0$, $v \in \mathbb{R}^{S_\sigma \times 1}$, we have that $p := \sum_{j=1}^{S_\sigma} v_j 1_{\omega_j} \in \mathcal{W}_S$ satisfies $(A^{-1} 1_{\omega_i}, p)^{L^2(\Omega^x)} = 0$ for all $1 \leq i \leq S_\sigma$, which leads us to $|p|_{\mathcal{W}_S}^2 = (A^{-1} p, p)^{L^2(\Omega^x)} = 0$, hence $p = 0$; consequently, $v = 0$ because the family $\mathcal{W}_S$ is linearly independent.

It remains to show the satisfiability of Assumption 2.7.
4.1 Previous Related Work

In Rodrigues (2021a, Sect. 5), for sensors as indicator functions with supports $\omega_i$ as in Fig. 2,

$$\omega_i = \omega_{i,N} =: \bigotimes_{j=1}^d \left( p_{i,j} + rL_j/N \right), \quad p_{i,j} = \frac{(2j-1)L_i}{2N} - \frac{rL_i}{2N}, \quad (4.3)$$

it has been shown that a Poincaré-like condition as

$$\lim_{N \to +\infty} \inf_{Q \in (V \cap O_1^d) \setminus \{0\}} \frac{|Q|_V^2}{|Q|_H^2} = +\infty. \quad (4.4)$$

is satisfied, where $O_{Nd}: = \text{span} \{1, \omega_{i,N} | 1 \leq i \leq N^d \}$. Here, we prove that the condition in Assumption 2.7 is also satisfied for the subsequence of sets of sensors as in (4.1b).

In particular, note that $O_{(2S)^d} = \mathcal{W}_S = \text{span} \mathcal{W}_S$.

The proof in Rodrigues (2021a, Sect. 5) takes the case of $N = 1$, corresponding to one sensor, as a reference and is based on the observation that the positioning of the actuators in (4.3) gives us a partition of $\Omega^x = \bigcup_{i \in \hat{N}} \mathcal{R}_i$, $\hat{N} = \{1, 2, \ldots, N\}$ into rectangles $\mathcal{R}_i$ which are rescaled copies of the rectangle corresponding to the case of $1^d = 1$ sensor $1_{\omega_1^x} = 1_{\omega_{1,1,\ldots,1}}$, with the rescaling factor $N^{-1}$; see one of these copies highlighted in Fig. 2, at the bottom-right corner of the case $N = 6$. Then, the Poincaré constant in (4.4) is shown to satisfy, for $N > 1$,

$$\inf_{Q \in (V \cap O_1^d) \setminus \{0\}} \frac{|Q|_V^2}{|Q|_H^2} \geq \left( vN^2D_0C_0 + 1 \right), \quad D_0 := \inf_{Q \in (V \cap O_1^d) \setminus \{0\}} \frac{|Q|_V^2}{|Q|_H^2}, \quad (4.5a)$$
where $D_0$ is the Poincaré constant in (4.4), in $\Omega^\times$, for the case of 1 sensor. Further $C_0$ is a constant satisfying, in the case $N = 1$,

$$C_0 |h|^2_{\mathcal{V}} \leq |\nabla_x (h)|^2_{L^2(\Omega^\times)^d} + |(h, 1_{\omega \times})|^2_{\mathbb{R}}, \text{ for all } h \in H^1(\Omega). \tag{4.5b}$$

### 4.2 Satisfiability of Assumption 2.7

We have mentioned that the proof in Rodrigues (2021a, Sect. 5) uses the case $N = 1$ as a reference to derive (4.5b). Here, we use the case $S = 1$, corresponding to $2^d$ sensors, as a reference to derive the analogous estimate required in Assumption 2.7.

**Lemma 4.1** For $S = 1$, we have an analogous version of (4.5b) as

$$C_0 |h|^2_{D(A)} \leq \left( \sum_{j \in \{1, 2\}^d} |(h, 1_{\omega \times})|^2_{\mathbb{R}} \right) + |\nabla^2_x h|^2_{L^2(\Omega^\times)^d}, \text{ for all } h \in H^2(\Omega), \tag{4.6}$$

where $\{\omega_{1, j}\}_{j \in \{1, 2\}^d} = \{\omega_i, s = 1, 2, \ldots, 2^d\}$.

For the proof, we will need some auxiliary results.

Note that the number of sensors is given by $S_\sigma = (2S)^d$, thus $2^d$ for $S = 1$.

Above, $\nabla^2_x$ stands for second-order derivatives,

$$|\nabla^2_x h|^2_{L^2(\Omega^\times)^d} := \left( \sum_{k \in K_{d, 2}} \left| \frac{\partial^{k_1} \partial^{k_2} \ldots \partial^{k_d} h}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_d^{k_d}} \right|^2_{L^2(\Omega^\times)} \right)^{\frac{1}{2}},$$

$$K_{d, 2} := \{ k \in \{0, 1, 2\}^d \mid \sum_{s=1}^d k_s = 2 \}.$$

Note that the locations as in (4.1b) induce a partition of $\Omega^\times$ with $S^d$ rescaled copies of the case $S = 1$. See Fig. 1, case $S = 3$, where a rescaled copy of the case $S = 1$ is highlighted at the bottom-right corner.

The following lemma can be found in Nečas (2012, Ch. 1, Sect. 1.1.7, Thm. 1.6), written in a slightly different way.

**Lemma 4.2** Let $\mathbb{P}_{\times, 1} := \left\{ c_0 + \sum_{j=1}^d a_j x_j \mid c_0 \in \mathbb{R}, a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d \right\}$ be the set of polynomials of degree at most 1 defined in $\Omega^\times$, and consider its orthogonal in $H^2(\Omega^\times)$, $\mathbb{P}_{\times, 1}^\perp H^2 := \{ h \in H^2(\Omega^\times) \mid (h, p)_{H^2(\Omega^\times)} = 0, \text{ for all } p \in \mathbb{P}_{\times, 1} \}$. Then, there exists a constant $C > 0$ such that

$$|h|^2_{H^2(\Omega^\times)} \leq C \left| \nabla^2_x h \right|^2_{L^2(\Omega^\times)^d}, \text{ for all } h \in \mathbb{P}_{\times, 1}^\perp H^2.$$
Proposition 4.3  Let $J_{d,2} := \{ j \in \{ 1, 2 \}^d \mid \sum_{j=1}^d j_j \leq d + 1 \}$. Then, the semi-norm $G(\cdot) := \left( \sum_{j \in J_{d,2}} (\cdot, 1_{\omega_{1,j,1}^x})^2_{L^2(\Omega^x)} \right)^{1/2}$ is a norm in $\mathbb{P}_{\times,1}$.

The proof is given in Sect. A.7. Note that $J_{d,2} \subset \mathbb{S}^d$ has cardinality $\#J_{d,2} = d + 1 = \dim \mathbb{P}_{\times,1}$.

Corollary 4.4  The usual norm $\| \cdot \|_{H^2(\Omega^x)}$, in $H^2(\Omega^x)$, is equivalent to the norm

$$
\left( \left\| \nabla_x^2 h \right\|_{L^2(\Omega^x)}^2 + \sum_{j \in J_{d,2}} \left| (h, 1_{\omega_{1,j,1}^x}) \right|_{L^2(\Omega^x)}^2 \right)^{1/2}.
$$

Proof  The proof is standard and can be done by repeating the arguments from the proofs in Nečas (2012, Ch. 1, Sect. 1.7, Thms. 1.8 and 1.10). Indeed, it is enough to observe that we have $H^2(\Omega^x) = \mathbb{P}_{\times,1}^1 \oplus \mathbb{P}_{\times,1}$ and use Proposition 4.3. □

Proof of Lemma 4.1  Let $h \in H^2(\Omega^x)$. By Corollary 4.4, there exists $C_1 > 0$ such that

$$
\left\| \nabla_x^2 h \right\|_{L^2(\Omega^x)}^2 + \sum_{j \in \{1,2\}^2} \left| (h, 1_{\omega_{1,j,1}^x}) \right|_{L^2(\Omega^x)}^2 \geq \left\| \nabla_x^2 h \right\|_{H^2(\Omega^x)}^2 + \sum_{j \in J_{d,2}} \left| (h, 1_{\omega_{1,j,1}^x}) \right|_{L^2(\Omega^x)}^2 \geq C_1 \| h \|_{H^2(\Omega^x)}^2 \geq C_1 C_2 \| h \|_{D(A)}^2,
$$

where $C_2$ is a constant satisfying $\| h \|_{D(A)}^2 \leq C_2^{-1} \| h \|_{H^2(\Omega^x)}^2$. That is, we may take $C_0 = C_1 C_2$ in (4.6). Note that such $C_2$ can be found as $\| h \|_{D(A)}^2 = \| -\nu \Delta h + h \|_{L^2(\Omega^x)}^2 \leq 2(\| -\nu \Delta h \|_{L^2(\Omega^x)}^2 + \| h \|_{L^2(\Omega^x)}^2) \leq 2(\nu^2 d \| h \|_{H^2(\Omega^x)}^2 + \| h \|_{H^2(\Omega^x)}^2)$; that is, we may take $C_2^{-1} = 2(\nu^2 d + 1)$. □

Proceeding as in Rodrigues (2021a, Sect. 5), we observe that for a suitable translation $\mathcal{T}_i$, the injective affine transformation

$$
\phi_1: \Omega^x \rightarrow \mathcal{R}_i, \quad x \mapsto z^i := \frac{x}{S} + \mathcal{T}_i, \quad i \in \mathbb{S}^d,
$$

maps $\Omega^x$ onto $\mathcal{R}_i$, and the sensor regions $\{ \omega_{1,j}^x \mid j \in \{1, 2\}^d \}$ onto rescaled sensor regions $\{ \omega_{i,j}^x \subset \mathcal{R}_i \mid j \in \{1, 2\}^d \}$ in the corresponding copy $\mathcal{R}_i$.

$$
\phi_1(\Omega^x) = \mathcal{R}_i, \quad \phi(\omega_{i,j}^x) = \omega_{i,j}^x.
$$
Further, denoting

**Lemma 4.5**

For a suitable constant $C > 0$, we have

$$\frac{\partial^{k_1} \partial^{k_2} \ldots \partial^{k_d} Q(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_d^{k_d}} = \frac{1}{S} \frac{\partial^{k_1} \partial^{k_2} \ldots \partial^{k_d} Q(z)}{\partial z_1^{k_1} \partial z_2^{k_2} \ldots \partial z_d^{k_d}}.$$  Further for $Q \in D(A) \cap \mathcal{C}^1_{2d}$, we find

$$\int_{\mathcal{O}_2} \sum_{k \in K_{d,2}} \left( \frac{1}{S} d \right) \frac{\partial^{k_1} \partial^{k_2} \ldots \partial^{k_d} Q(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_d^{k_d}} x = \int_{\mathcal{O}_4} \sum_{k \in K_{d,2}} \left( \frac{1}{S} Q \right)^2 d x,$$

and, for all $g \in L^2(\omega)$, we have

$$|\nabla^2 Q|_{L^2(\mathcal{O}_x)^{2d}} = S^d |\nabla^2 Q \circ \Phi_i^{-1}|_{L^2(\mathcal{O}_x)^{2d}},$$

which give us,

$$\frac{2}{S} |\nabla^2 Q |_{L^2(\mathcal{O}_x)^{2d}} = S^d |\nabla^2 Q \circ \Phi_i^{-1}|_{L^2(\mathcal{O}_x)^{2d}},$$

Further, denoting $[\omega_1^x]_2 := \{[\omega_1^x]_j \mid j \in \{1, 2\} \}$ and $[\omega_1^x]_2 := \{[\omega_1^x]_j \mid j \in \{1, 2\} \}$ and, choosing $g \in [\omega_1^x]_2$, we also find

$$Q \circ \Phi_i^{-1} \in [\omega_1^x]_2 \iff Q \in [\omega_1^x]_2.$$

**Lemma 4.5** *For a suitable constant $C > 0$, we have*

$$\inf_{Q \in D(A) \cap \mathcal{C}^1_{2d}} \frac{|Q|^2_{D(A)}}{|Q|^2_{V}} \geq C D_0 S^2, \quad D_0 := \inf_{Q \in D(A) \cap \mathcal{C}^1_{2d}} \frac{|Q|^2_{H^2(\Omega)}}{|Q|^2_{H^1(\Omega)}},$$

*Proof* For arbitrary given $Q \in W^1_S \cap D(A)$, since the norms $|\cdot|_{D(A)}$ and $|\cdot|_{H^2(\mathcal{O}_x)}$ are equivalent (for both Dirichlet and Neumann boundary conditions), and since $\mathcal{G}(Q) = 0$, we find

$$C_4 |\nabla^2 Q |_{L^2(\mathcal{O}_x)^{2d}}^2 \leq |Q|^2_{D(A)} \geq C_3 |\nabla^2 Q |_{L^2(\mathcal{O}_x)^{2d}}^2.$$
for suitable constants $C_3 > 0, C_4 > 0$. Furthermore,

$$|Q|^2_{D(A)} \geq C_3 \left| \nabla_x Q \right|^2_{L^2(\Omega^x)d} = C_3 \sum_{i \in \mathcal{S}^d} \left| \nabla_x Q \right|^2_{L^2(\mathcal{D}_i)^d}$$

$$= C_3 \sum_{i \in \mathcal{S}^d} S^{d-4} \left| \nabla_{\Phi_i^{-1}(x)} (Q \circ \Phi_i) \right|^2_{L^2(\Omega^x)^d} \geq C_3 C_1 S^{d-4} \sum_{i \in \mathcal{S}^d} |Q \circ \Phi_i|^2_{H^2(\Omega^x)}$$

$$\geq C_3 C_1 D_0 S^{d-4} \sum_{i \in \mathcal{S}^d} |Q \circ \Phi_i|^2_{H^1(\Omega^x)}$$

$$= C_3 C_1 D_0 S^{d-4} \sum_{i \in \mathcal{S}^d} \left( |\nabla_x Q \circ \Phi_i|^2_{L^2(\Omega^x)} + |Q \circ \Phi_i|^2_{L^2(\Omega^x)} \right)$$

with $D_0$ as in (4.7). By using the relation

$$\left| \nabla_{\Phi_i^{-1}(x)} (Q) \right|^2_{L^2(\Omega^x)d} = S^{d-2} \left| \nabla_x Q \circ \Phi^{-1} \right|^2_{L^2(\mathcal{D}_i)^d},$$

which we can find in Rodrigues (2021a), we arrive at

$$|Q|^2_{D(A)} \geq C_3 C_1 D_0 S^{d-4} \sum_{i \in \mathcal{S}^d} \left( S^{d-2} \left| \nabla_x Q \right|^2_{L^2(\mathcal{D}_i)} + S^d \left| Q \right|^2_{L^2(\mathcal{D}_i)} \right)$$

$$= C_3 C_1 D_0 S^2 \sum_{i \in \mathcal{S}^d} \left( |\nabla_x Q|^2_{L^2(\mathcal{D}_i)} + S^2 \left| Q \right|^2_{L^2(\mathcal{D}_i)} \right) \geq C_3 C_1 D_0 S^2 \left| Q \right|^2_{H^1(\Omega^x)}$$

$$\geq C_3 C_1 C_5 D_0 S^2 \left| Q \right|^2_{\mathcal{W}_\mathcal{V}},$$

which gives us (4.7), with $C = C_3 C_1 C_5$, and $C_5 := \inf_{Q \in \mathcal{V}^d \setminus \{0\}} \frac{|Q|^2_{H^1(\Omega^x)}}{|Q|^2_{\mathcal{W}_\mathcal{V}}}$. \hfill $\Box$

Note that (4.7) implies the satisfiability of Assumption 2.7. Recall that $\mathcal{O}(2\mathcal{S})^d = \mathcal{W}_\mathcal{S}$.

We have proven the satisfiability of Assumption 2.7 for rectangular domains.

**Conjecture 4.6.** Assumption 2.7 can be satisfied for smooth domains.

An analogous conjecture has been stated in Rodrigues (2021a, Sect. 8.2), where we can also find arguments supporting the conjecture. Here, it would be convenient/important for applications that all the sensors (for a fixed $S$) have the same profile/shape (up to a translation and rotation, e.g., as the indicator functions of small balls or rectangles as in Fig. 1). This will be possible for convex polygonal domains; see the discussion in Rodrigues (2021a, Sect. 8.2). In real-world applications, we cannot expect to have sensors with arbitrary shapes at our disposal, and it is probably not very reasonable to build sensors for each application (e.g., with shapes depending on the spatial domain).

Finally, we end this section with the following result concerning Remark 3.3.
Proposition 4.7 For pairwise disjoint sensor rectangular regions $\omega_i \subset D(A)$, and auxiliary functions $\tilde{W}_S$ as in (4.2b), we have that, for any $\theta \in \tilde{W}_S \subset D(A)$,

$$|\theta|_V^2 = (C_1 S^2 + 1) |\theta|_H^2 \quad \text{and} \quad |\theta|_{D(A)}^2 = (C_2 S^4 + 2C_1 S^2 + 1) |\theta|_H^2,$$

with $C_1 = \frac{16}{3} \nu \pi^2 r^{-2} \sum_{i=1}^d L_i^{-2}$ and $C_2 = \frac{\nu^2 28 \pi^4}{3} r^{-4} \sum_{i=1}^d L_i^{-4}$.

The proof is given in “Appendix,” Sect. A.8.

As a consequence, it follows that the Poincaré-like constant $\bar{\alpha}_{S, \ell}$ in (3.1) satisfies

$$\lim_{S \to +\infty} \bar{\alpha}_{S, \ell} = 0,$$

for $\ell \in \{0, 1\}$. In these cases, it may be necessary to choose firstly $S$ and subsequently $\lambda$ (depending on $S$) in Theorem 3.1; see Remark 3.3.

5 Numerical Simulations

We show results of simulations illustrating the achievable stability of the error dynamics stated in Main Result in Introduction; see main Theorem 3.1. We consider the following scalar parabolic system as a toy model for the error dynamics; see (1.5).

$$\frac{\partial}{\partial t} z + (-\nu A + 1) z + az + b \cdot \nabla z - |z|^3 R z + \left( \frac{\partial}{\partial x_1} z - 2 \frac{\partial}{\partial x_2} z \right) z = -\lambda \Lambda^{-1} P_{\tilde{W}_S} A \tilde{Z}_S z,$$

$$z(0) = z_0 \in H^1(\Omega), \quad \frac{\partial z}{\partial \mathbf{n}} |_{\partial \Omega} = 0$$

evolving in $V = H^1(\Omega)$ under Neumann boundary conditions, where $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ is the unit square. As parameters, we set

$$\ell = 2, \quad \nu = 0.1, \quad a = -2 + x_1 - |\sin(t + x_1)| \mathbb{R}, \quad b = \begin{bmatrix} x_1 + x_2 \\ \cos(t) x_1 x_2 \end{bmatrix}.$$

As sensors, we take indicator functions $1_{\omega_i} = 1_{\omega_i}(x)$ of rectangular subdomains $\omega_i \subset \Omega$ as in Fig. 1. Hence, the output $\mathcal{Z}_S y(t) \in \mathbb{R}^S$ consists of the “averages” of the solution over the same subdomains,

$$(\mathcal{Z}_S y(t))_i = (1_{\omega_i}, y(t))_{L^2(\Omega)} = \int_{\omega_i} y(t), \ d\Omega$$

and the output error is $\mathcal{Z}_S z(t) = \mathcal{Z}_S \hat{y}(t) - \mathcal{Z}_S y(t) \in \mathbb{R}^S$, 

$$\begin{aligned}(\mathcal{Z}_S z(t))_i &= (1_{\omega_i}, z(t))_{L^2(\Omega)} = (1_{\omega_i}, \hat{y}(t))_{L^2(\Omega)} - (1_{\omega_i}, y(t))_{L^2(\Omega)}.\end{aligned}$$
Finally, we set the normalized initial condition as

$$z_0 = \frac{2 - x_1x_2}{|2 - x_1x_2|_V} \in V.$$  

The number of sensors $S$ and the parameter $\lambda$, which we know should be both large enough (cf. Main Result in Introduction), will be set later on.

As auxiliary functions, we will take the functions in (4.2b).

### 5.1 Discretization

The following simulations have been performed in MATLAB and correspond to a piecewise linear (hat functions based) finite element discretization of the equation in the spatial variable. Subsequently, for the time variable and for discrete time instants $t_j = kj$, $j \in \mathbb{N}$ and with time step $k > 0$, we use the standard linear approximation for the time derivative, a Crank–Nicolson scheme to approximate the symmetric operator $A = (-v\Delta + 1 + a\mathbf{1})$, and an Adams–Bashforth scheme for the remaining terms $R$; that is, denoting $t_j^* := \frac{t_j + t_{j+1}}{2}$ we take $\frac{\partial}{\partial t} z(t_j^*) \approx \frac{z(t_{j+1}) - z(t_j)}{k}$, $A(t_j^*)z(t_j^*) \approx \frac{A(t_j)z(t_j) + A(t_{j+1})z(t_{j+1})}{2}$ and $R(t_j^*, z(t_j^*)) \approx \frac{3R(t_j, z(t_j)) - R(t_{j-1}, z(t_{j-1}))}{2}$.

We will consider the cases where the number $S$ of sensors belongs to $\{4, 9, 16\}$.

The corresponding triangulations of the spatial domain, used in the simulations, are shown in Fig. 3.

In the figures below, the symbol “npts$\Omega$” stands for the number of mesh points in the triangulation of the spatial domain $\Omega$, the symbol “$k$” stands for the time step, and $T$ stands for the end point of the time interval $[0, T]$ where the simulations have been run. If the plots in the figures do not include the entire interval $[0, T]$, then it means that the norm $|z|_V$ of the error blows up at time $t_{bu} < T$ near the last plotted time instant.

### 5.2 Necessity of Large $S$ for Error Stability

In Fig. 4, we see that the free error dynamics (i.e., under no output injection) is blowing up in finite time, namely at time $t_{bu} \approx 0.11$. The simulations correspond to the mesh corresponding to 4 sensors. Also in Fig. 4, we see that the error norm, for the
output injection corresponding to the case of 4 sensors, blows up at time $t_{bu} \approx 9.5$ for $\lambda = 0.004$, while it blows up at time $t_{bu} \approx 10.5$ for larger $\lambda \in \{0.02, 0.1, 0.5\}$. That is, the blow up time increases due to the output injection, but such injection is not able to achieve the stability of the error dynamics. In particular, we see that the blow-up time seems to converge to a value in the interval $[10.5, 11]$ as $\lambda$ increases. Therefore, we can conclude that 4 sensors are likely not able to achieve the stability of the estimation error norm. This confirms the statement of Theorem 3.1 on the necessity of a large enough number of sensors.

In Fig. 5, we see that 9 and 16 sensors are able to achieve the stability of the error norm for the considered values of $\lambda$. We also see that, for a fixed number $S_\sigma$ of sensors, the exponential stability rate increases with $\lambda$ and converges to a bounded value. This means that if we want to achieve a larger stability rate, it is not enough to increase $\lambda$; we will need to increase also the number of sensors as stated in Theorem 3.1. This is confirmed in Fig. 5 where we see that with 16 sensors, we obtain a faster decrease in the error norm $|z|_V$; namely, for $\lambda = 0.02$ we find the rates $\mu \approx 5 = \frac{150}{15}$ for $S_\sigma = 9$, and $\mu \approx 11.5 \approx \frac{1}{2} \frac{150}{15}$ for $S_\sigma = 16$. 

Fig. 4 The free dynamics and the case of 4 sensors

Fig. 5 The case of 9 and 16 sensors
5.3 Necessity of Large $\lambda$ for Error Stability

We know that the free error dynamics, with $\lambda = 0$, is not stable. Here, we show that $\lambda > 0$ must be large enough in order to achieve stability for the error dynamics. Indeed, in Fig. 6 we see that, for small $\lambda$, neither 9 nor 16 sensors are able to achieve such stability.

6 Final Remarks

We have presented explicit output injection operators $\mathcal{I}^{[\lambda, \ell]}_{S}$ for a class of semilinear parabolic-like equations. We have presented numerical results showing that both $S_\sigma$ and $\lambda$ must be taken large enough in order to achieve the stability of the error dynamics, which agree with the theoretical results. Though the “best” choice of all the parameters involved in the output injection operator $\mathcal{I}^{[\lambda, \ell]}_{S}$ is not the main focus of this paper, such choice is (or, may be) important for applications (e.g., numerical simulations). Here, we just discuss briefly several points related to the semiglobal detectability result (cf. Def. 1.6) presented in this manuscript, from practicability viewpoints, and mention related problems which could be the subject of further investigation.

6.1 On the Choice of $S$ and $\lambda$

Let us fix $(\mu, R)$. For a given $\ell$, the detectability property of the output injection operator $\mathcal{I}^{[\lambda, \ell]}_{S}$, in Theorem 3.1, depends on the desired exponential decreasing rate $\mu$ and on the upper bound $R$ for the norm of the initial error, simply because the pair $(S, \lambda)$ depends on, and “increases with” $(\mu, R)$. In practice, the initial error $z_0$ is unknown for us, and thus, we will not be able to surely choose an appropriate detecting operator $\mathcal{I}^{[\lambda, \ell]}_{S}$. However, on the other hand, we are sure that it is enough to increase both $S$ and $\lambda$ to find a detecting $\mathcal{I}^{[\lambda, \ell]}_{S}$. Furthermore, the fact that the transient bound $\varrho > 1$ will get smaller, for large $S$ and $\lambda$, can be used in applications to decide whether we
should (still) increase $S$ and $\lambda$. Namely, we increase $S$ and/or $\lambda$ if (e.g., in simulations) we realize that the error norm is not starting decreasing after a suitable amount of time.

If we knew that the transient bound is $Q = 1$, then we would know that for large enough $S$ and $\lambda$, the error norm must be strictly decreasing. Hence, we would increase $S$ and $\lambda$ if we realize that the norm is not strictly decreasing.

### 6.2 Strictly Decreasing Estimate Error Norm

Let us fix again $(\mu, R)$. We shall see now that we can achieve the optimal transient bound constant $Q = 1$ when we have $\xi_{2i} = 0$ for all $i \in \{1, 2, \ldots, n\}$ in Assumption 2.4. This case holds for parabolic equations (1.7) for reaction terms in the cases $d \in \{1, 2\}$ with $r > 1$, and in the case $d = 3$ with $r \in (1, 3]$; these facts have been proven in Rodrigues (2020, Sects. 5.2 and 5.3). We will also have $\xi_{2j} = 0$ for $d \in \{1, 2, 3\}$ and convection terms with $s = 1$; indeed, following a slightly different argument as in Rodrigues (2020, Sects. 5.2.2), using suitable Sobolev embeddings, Agmon inequalities, and interpolation inequalities, we find that, for $\mathcal{N}_i(t, y) := (\tilde{b} \cdot \nabla) y$,

$$
|\mathcal{N}_i(t, y_1) - \mathcal{N}_i(t, y_2)|_H = |(\tilde{b} \cdot \nabla y_1)(y_1 - y_2) - (\tilde{b} \cdot \nabla (y_2 - y_1))y_2|_{L^2} \\
\leq C_1 |\tilde{b}|_{L^\infty} (|\nabla y_1|_{L^2} y_1 |y_1 - y_2|_{L^\infty} + |\nabla (y_1 - y_2)|_{L^2} |y_2|_{L^6}) \\
\leq C_2 \left( |y_1|_{H^1} |y_1 - y_2|_{H^1}^{3/2} + |\nabla (y_1 - y_2)|_{H^1} \right) \\
\leq C_3 \left( |y_1|_{V} |y_1 - y_2|_{V}^{3/2} + |\nabla (y_1 - y_2)|_{H^1} \right) \\
\leq C_4 \left( |y_1|_{V} + |y_2|_{V} \right) |y_1 - y_2|_{V}^{3/2}.
$$

Hence, for such convection term we can take $(\xi_{1i}, \xi_{2i}, \delta_{1i}, \delta_{2i}) = (1, 0, \frac{1}{2}, \frac{1}{2})$. Note that our simulations have been performed in a two-dimensional domain, and in Fig. 5, the estimation error norm is strictly decreasing.

To show that we can take $Q = 1$ if $\xi_{2i} = 0$, $i \in \{1, 2, \ldots, n\}$, we observe that in such case, we have $\chi_2 = 0$, due to (3.7). Thus, we obtain $\Psi(y) = 2 \left( 1 + |y|^3 \right)$ in (3.20), with $|h(y)|_{L^\infty(\mathbb{R}_0, R)} \leq \tilde{C}$ and

$$
\frac{d}{dt} |z|^2_V \leq -\left( |\mu - h(y)| \left( 1 + |z|^3 \right) \right) |z|^2_V \leq -\left( \tilde{C}_1 - \tilde{C}_2 |z|^3 \right) |z|^2_V
$$

in (3.21), with $\tilde{C}_1 := |\mu - \tilde{C}| \left( 1 + |z|^3 \right)$ and $\tilde{C}_2 = \tilde{C}$. Recalling (3.24) and the fact that $\mu$ can be made arbitrarily large by choosing both $S$ and $\lambda$ large enough, it is clear that for any given $R > 0$ and $\mu > 0$, we can set both $S$ and $\lambda$ large enough so that $\tilde{\mu} := \tilde{C}_1 - \tilde{C}_2 R^{3/2} \geq \mu$. Then, by Proposition 4.3 in Rodrigues (2020), we find the following estimate, with transient bound $Q = 1$,

$$
|z(t)|^2_V \leq e^{-\tilde{\mu}(t-s)} |z(s)|^2_V \leq e^{-\mu(t-s)} |z(s)|^2_V, \quad \text{if} \quad |z(0)|^2_V \leq R.
$$
Table 1 Norm of output injection at initial time

| $(S_\sigma, \lambda)$ | (4, 0.5) | (9, 0.02) | (16, 0.02) |
|------------------------|----------|-----------|-----------|
| $|I_{[\lambda, \ell]}^{[S_{z,0}]}|_H$ | 3537.9599 | 747.3875 | 2594.0443 |

6.3 On the Choice of the Set of Auxiliary Functions $\tilde{W}_S$ and $\ell$

The choice of the auxiliary set $\tilde{W}_S \subset D(A)$ is at our disposal. In Sect. 4, we have suggested three possible choices, namely those in (4.2). In Sect. 5, we have taken only the choice in (4.2b). We did not compare with other possible choices because the “optimal” choice for $\tilde{W}_S$ is not the main goal of this paper. However, we must say that though the operator norm of the oblique projection $P_{\tilde{W}_S}^{\perp}$ does not play any crucial role in the detectability result, it plays a role in the norm of the injection operator, as we have seen in Sect. 3.3. A large operator norm of the oblique projection can influence negatively the practicability of the observer in applications (e.g., leading to the need of taking a very small time step $k$ in simulations), as shown/discussed through numerical results presented in Rodrigues (2021a, Sect. 7) (in there, for choices as spans of eigenfunctions, cf. (4.2a)). By this reason, it could be interesting to investigate the performance of the feedback for different choices of $\tilde{W}_S$ (e.g., those in (4.2)), or even try to define and investigate the “optimal choice.”

In our simulations, we have taken only the border case $\ell = 2$ for the power $\ell$ of the diffusion taken in the injection operator $I_{[\lambda, \ell]}^{[S_{z,0}]}$. Another point that could be investigated is the performance of the observer for different values of $\ell$.

6.4 On the Time Step

In Table 1, we see the $H$-norm, $H = L^2(\Omega^\times)$, of the output injection at initial time, for some pairs $(S_\sigma, \lambda)$. This is the reason we took a small time step as $k = 10^{-4}$. Note that such norm increases with $\lambda$, so for larger $\lambda$ we may need to take a smaller time step to capture (or, accurately approximate) the effect of the output injection on the dynamics. A very small time step may be impracticable for real-world applications, and thus, it could be interesting to investigate, in a future work, whether an appropriate choice of $\tilde{W}_S$ and/or $\ell$ allows us to take larger $k$.

6.5 Robustness Against Noisy Measurements and Disturbances

In applications, our sensor measurements will often be subject to small errors (noisy measurements). Hence, it is important that observers are robust against such measurement errors. The state estimate, given by the observer, can be used to construct an output based feedback control stabilizing the real state to a targeted solution. In this case, we can consider also the robustness against model uncertainty (disturbances), because the mathematical model for the state (which we also use to construct the observer) is an approximation of the real-world dynamics.
It could be also interesting to compare (e.g., numerically) the performance and robustness of the explicit Luenberger-like observers (Luenberger 1964, 1966; Rodrigues 2021b) we propose in this manuscript with other approaches. An example could be to compare the performances with classical Riccati-based observers (Kalman–Bucy filters) (Kalman 1960; Kalman and Bucy 1961; Zhuk et al. 2021; Afshar et al. 2017).

### 6.6 Further Remarks

We have presented a class of output injection operators which allow us to exponentially detect/estimate the state of parabolic-like equations. It could be interesting to investigate whether the presented method can be extended to other types of evolution processes, for example, those modeled by damped wave-like equations Azmi and Rodrigues (2020), Kalantarov and Titi (2018), Ammari et al. (2016) and time-fractional evolution equations (Chen et al. 2020; Jin et al. 2015; Antil et al. 2016; Li and Li 2021).

**Acknowledgements** The author thanks the anonymous Referees for their comments and suggestions, which led to an improvement of the exposition in the manuscript. The author is supported by ERC advanced Grant 668998 (OCLOC) under the EU’s H2020 research program. The author acknowledges partial support from the Austrian Science Fund (FWF): P 33432-NBL.

**Appendix**

### A.1 Proof of Proposition 3.7

With \( \hat{y}_1 := y + z_1 \) and \( \hat{y}_2 := y + z_2 \), we write

\[
\mathcal{N}_y(t, z_1) - \mathcal{N}_y(t, z_2) = \mathcal{N}(t, \hat{y}_1) - \mathcal{N}(t, y) - (\mathcal{N}(t, \hat{y}_2) - \mathcal{N}(t, y)) \\
= \mathcal{N}(t, \hat{y}_1) - \mathcal{N}(t, \hat{y}_2) = \mathcal{N}(t, y + z_1) - \mathcal{N}(t, y + z_2),
\]

which leads us to \( \hat{y}_1 - \hat{y}_2 = z_1 - z_2 =: d \) and, using Lemma 3.4,

\[
2\left( \mathcal{N}_y(t, z_1) - \mathcal{N}_y(t, z_2), A(z_1 - z_2) \right)_H = 2\left( \mathcal{N}(t, \hat{y}_1) - \mathcal{N}(t, \hat{y}_2), A(\hat{y}_1 - \hat{y}_2) \right)_H \\
\leq \gamma_0 |d|^2_{D(A)} + \left( 1 + \frac{1+|\delta_0|}{1-|\delta_2|} \right) \tilde{C}_\mathcal{N} \sum_{j=1}^n |\hat{y}|^2 \sum_{k=1}^2 |\hat{y}_k|_{D_j}^\frac{2\gamma_{1j}}{\eta_j} |\hat{y}_k|_{D_j}^\frac{2\gamma_{2j}}{\eta_j}.
\]

Therefore, (3.5) follows with \( \tilde{C}_{\mathcal{M}} = \tilde{C}_\mathcal{N} = \tilde{C} \left[ n, \frac{1}{1-|\delta_2|}, |\mathcal{N}|, \frac{2\gamma_{1j}}{\eta_j}, \frac{2\gamma_{2j}}{\eta_j} \right] \).
By setting \( z_2 = 0 \) in (3.5), we obtain for each \( \tilde{\gamma}_0 > 0 \),

\[
2\left( \mathfrak{H}_y(t, z_1), Az_1 \right) \leq \tilde{\gamma}_0 |z_1|^2_{D(A)} + \left( 1 + \tilde{\gamma}_0^{-\frac{1+|\tau_{1j}|}{1-|\tau_{1j}|}} \right) \tilde{C}_\mathfrak{H}_1 \sum_{j=1}^n |z_1|^2_V \sum_{l=0}^1 y + lz_1 |y + lz_1|^2_{D(A)}.
\]

(A.1)

Now, for simplicity we fix \( j \), and set

\[
p = p_j := \frac{2\zeta_{1j}}{\delta_{1j}} \quad \text{and} \quad q = q_j := \frac{2\zeta_{2j}}{\delta_{1j}} < 2.
\]

Note that \( p \geq 0 \) and \( q \in [0, 2) \) due to the relations \( \delta_{1j} + \delta_{2j} = 1 \) and \( \zeta_{2j} + \delta_{2j} < 1 \), in Assumption 2.4.

We consider first the case \( q \neq 0 \). By the triangle inequality and Phan and Rodrigues (2017, Prop. 2.6), we obtain

\[
\Upsilon_j := |z_1|^2_V \sum_{l=0}^1 |y + lz_1|^p_V |y + lz_1|^q_{D(A)}
= |z_1|^2_V |y|^p_V |y|^q_{D(A)} + |z_1|^2_V |y + z_1|^p_V |y + z_1|^q_{D(A)}
\leq |z_1|^2_V |y|^p_V |y|^q_{D(A)}
+ |z_1|^2_V (1 + 2^{p-1})(1 + 2^{q-1}) (|y|^p_V + |z_1|^p_V) (|y|^q_{D(A)} + |z_1|^q_{D(A)}).
\]

(A.2)

Setting \( D_{p,q} = (1 + 2^{p-1})(1 + 2^{q-1}) \) and using the Young inequality, the last term satisfies for each \( \gamma_2 > 0 \),

\[
D_{p,q}^{-1} \Upsilon_j := |z_1|^2_V (|y|^p_V + |z_1|^p_V) (|y|^q_{D(A)} + |z_1|^q_{D(A)})
\leq |z_1|^2_V (|y|^p_V + |z_1|^p_V |z_1|^q_{D(A)} + |z_1|^2_V (|y|^p_V + |z_1|^p_V) |y|^q_{D(A)}
\leq \gamma_2 \frac{2}{\gamma_2} |z_1|^2_{D(A)} + \gamma_2^{-1-\frac{q}{2}} (|z_1|^2_V (|y|^p_V + |z_1|^p_V)^{1-\frac{q}{2}} + |z_1|^2_V (|y|^p_V + |z_1|^p_V) (1 + |y|^q_{D(A)}).
\]
which implies, since $1 - \frac{q}{2} = \frac{2 - q}{2}$,

\[
D_{p,q}^{-1} T_j \leq \gamma_2^2 |z_1|_{D(A)}^2 + \gamma_2^{-2} |z_1|_{D(A)}^4 (|y|_V^p + |z_1|_{V}^p)^{\frac{2}{2 - q}} \\
+ |z_1|_{V}^2 (|y|_V^p + |z_1|_{V}^p) \left( 1 + |y|_{D(A)}^{\frac{2q}{2 - q}} \right) \\
\leq \gamma_2^2 |z_1|_{D(A)}^2 + \gamma_2^{-2} |z_1|_{D(A)}^4 (1 + 2^{\frac{2}{2 - q} - 1}) |z_1|_{V}^4 (|y|_V^p + |z_1|_{V}^p)^{\frac{2}{2 - q}} \\
+ |z_1|_{V}^2 (|y|_V^p + |z_1|_{V}^p) \left( 1 + |y|_{D(A)}^{\frac{2q}{2 - q}} \right) \\
\leq \gamma_2^2 |z_1|_{D(A)}^2 \\
+ D_{q,\gamma_2} \left( |z_1|_{V}^4 (|y|_V^p + |z_1|_{V}^p)^{\frac{2}{2 - q}} + |z_1|_{V}^2 (|y|_V^p + |z_1|_{V}^p) \right) \left( 1 + |y|_{D(A)}^{\frac{2q}{2 - q}} \right)
\]

with

\[
D_{q,\gamma_2} := 1 + \gamma_2^{-2} (1 + 2^{\frac{2}{2 - q} - 1}),
\]

and then

\[
D_{q,\gamma_2}^{-1} \left( D_{p,q}^{-1} T_j - \gamma_2^2 |z_1|_{D(A)}^2 \right) \left( 1 + |y|_{D(A)}^{\frac{2q}{2 - q}} \right)^{-1} \\
\leq \left( |z_1|_{V}^4 (|y|_V^p + |z_1|_{V}^p)^{\frac{2}{2 - q}} + |z_1|_{V}^2 (|y|_V^p + |z_1|_{V}^p) \right) |z_1|_{V}^2 . \tag{A.3a}
\]

Observe that

\[
\frac{2p}{2 - q} = \frac{4\xi_1}{2\delta_1 - 2\xi_2} \leq 2 \frac{\xi_1}{\delta_1 - \xi_2} \tag{A.4a},
\]

\[
\frac{2(2 + q + p) - 4}{2 - q} = \frac{4(\xi_1 + \xi_2)}{2\delta_1 - 2\xi_2} \leq 2 \frac{\xi_1 + \xi_2}{\delta_1 - \xi_2} \tag{A.4b}.
\]

and that $\frac{2c}{2 - q} \geq c \iff 0 \geq -qc$. Thus, we obtain that $\frac{2c}{2 - q} \geq c$ for all $c \geq 0$, and hence,

\[
\frac{2(2 + q + p) - 4}{2 - q} \geq \frac{2p}{2 - q} \geq p \quad \text{and} \quad \frac{2(2 + q + p) - 4}{2 - q} \geq \frac{2(2 + q) - 4}{2 - q} = \frac{2q}{2 - q} \geq q.
\]
which together with (A.3) give us

\[ D_{q,y_2}^{-1} \left( D_{p,q}^{-1} T_j - \gamma_2^2 |z_1|^2_{D(A)} \right) \left( 1 + |y|^{2\gamma}_{D(A)} \right)^{-1} \]
\[ \leq \left( |z_1|_V^{2(2+q)-4} + |z_1|_V^{2(2+q+p)-4} + 1 + |z_1|^p \right) \left( 2 + |y|^{2p}_{D(A)} + |y|^{p}_{D(A)} \right) |z_1|^2_V \]
\[ \leq \left( 4 + 3 |z_1|_V^{2\left(\frac{\gamma_1\gamma_2}{\gamma_1^2 - \gamma_2^2}\right)} \right) \left( 4 + 2 |y|_V^{2\left(\frac{\gamma_1}{\gamma_1^2 - \gamma_2^2}\right)} \right) |z_1|^2_V , \]

which implies

\[ T_j \leq D_{p,q} \gamma_2^2 |z_1|^2_{D(A)} + \tilde{D} \left( 1 + |y|^{\gamma}_V \right) \left( 1 + |y|^{\gamma}_{D(A)} \right) \left( 1 + |z_1|^\gamma_V \right) |z_1|^2_V , \quad \text{(A.5a)} \]

\[ D_{p,q} := (1 + 2^{q-1})(1 + 2^{q-1}), \quad \tilde{D} := 16 D_{p,q} D_{q,y_2} , \quad \text{(A.5b)} \]

\[ \chi_1 := 2 \left\| \frac{\gamma_1}{\gamma_1 - \gamma_2} \right\| \geq 0, \quad \chi_2 := \left\| \frac{2\gamma_2}{\gamma_1} \right\| \geq 0, \quad \chi_3 := 2 \left\| \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \right\| \geq 0. \quad \text{(A.5c)} \]

For the first term on the right-hand side of (A.2), we also obtain

\[ F_j := |z_1|^2_{D(A)} \left| y \right|^{\gamma}_{D(A)} \left( 1 + |y|^{\gamma}_V \right) \left( 1 + |y|^{\gamma}_{D(A)} \right) |z_1|^2_V \quad \text{(A.6)} \]

because \( p \leq \frac{2p}{2^{q-1}} \leq \chi_1 \) and \( 0 < q \leq \chi_2 \); see (A.4).

Therefore, by (A.2), (A.5), and (A.6), it follows that for all \( \gamma_2 \in (0, 1] \),

\[ Y_j \leq F_j + T_j \]
\[ \leq D_{p,q} \gamma_2^2 |z_1|^2_{D(A)} + (1 + \tilde{D}) \left( 1 + |y|^{\gamma}_V \right) \left( 1 + |y|^{\gamma}_{D(A)} \right) \left( 1 + |z_1|^\gamma_V \right) |z_1|^2_V , \]
\[ \leq D_{p,q} \gamma_2^2 |z_1|^2_{D(A)} + (1 + \tilde{D}) \left( 1 + |y|^{\gamma}_V \right) \left( 1 + |y|^{\gamma}_{D(A)} \right) \left( 1 + |z_1|^\gamma_V \right) |z_1|^2_V , \quad \text{(A.7a)} \]

for \( q \neq 0 \), with \( \gamma_2 \leq 1 \).

(A.7b)

Note that \( \gamma_2^\frac{2}{q} \leq \gamma_2^\frac{2}{\gamma_2} \) because \( \gamma_2 \leq 1 \) and \( 0 < q \leq \chi_2 \).
Finally, we consider the case $q = 0$. We find

$$
\gamma_j := |z_1|^2 \sum_{l=0}^1 |y + lz_1|^p = |z_1|^2 |y|^p + |z_1|^2 |y + z_1|^p
$$

$$
\leq |z_1|^2 \left( (2 + 2^{p-1}) |y|^p + (1 + 2^{p-1}) |z_1|^p \right)
\leq (2 + 2^{p-1}) \left( 1 + |y|^p \right) \left( 1 + |z_1|^p \right)
\leq (2 + 2^{p-1}) (2 + |y|^{X_1}) \left( 2 + |z_1|^{X_3} \right)
\leq 8 \left( 1 + 2^{p-1} \right) \left( 1 + |y|^{X_1} \right) \left( 1 + |z_1|^{X_3} \right) |z_1|_V^2, \quad q = 0.
$$

(A.8a)

for $q = 0$, with $\gamma_2 \leq 1$,

(A.8b)

with $\chi_1$ and $\chi_3$ as in (A.5), where we have used (A.4).

Now, we observe that

$$
(1 + 2^{p-1}) \leq D_{p,q} = (1 + 2^{p-1})(1 + 2^q - 1)
\leq \left( 1 + 2^{\left\| \frac{2\chi_1 - \delta_1}{\delta_1 - \delta_2} \right\|} \right) \left( 1 + 2^{\left\| \frac{2\chi_2 - \delta_1}{\delta_1 - \delta_2} \right\|} \right) =: \tilde{D}_1
$$

(A.9a)

and, since $0 < \frac{2}{2^q} \leq \left\| \frac{\delta_1}{\delta_1 - \delta_2} \right\|$,

$$
\tilde{D} = 16D_{p,q} \tilde{D}_{q,\gamma_2} \leq 16\tilde{D}_1 (1 + 2^{\frac{2q}{2-q}}) \left( 1 + \gamma_2 \frac{2}{2-q} \right)
\leq 16\tilde{D}_1 (2 + 2^{X_3}) \left( 2 + \gamma_2 \left\| \frac{\delta_1}{\delta_1 - \delta_2} \right\| \right) \leq \tilde{D}_2 \left( 1 + \gamma_2^{-X_4} \right),
$$

(A.9b)

$$
\tilde{D}_2 := 32 \tilde{D}_1 (2 + 2^{X_3}),
$$

(A.9c)

$$
\chi_4 := \left\| \frac{\delta_1}{\delta_1 - \delta_2} \right\| \geq 1.
$$

(A.9d)

Further, from $\tilde{D}_2 > 64\tilde{D}_1 \geq 8 \left( 1 + 2^{p-1} \right)$ and from (A.7), (A.8), and (A.9), we conclude that for both cases, $q > 0$ and $q = 0$, we have

$$
\gamma_j \leq \tilde{\theta} \left| z_1 \right|^2 \tilde{D}_j (A) + \tilde{D}_2 \left( 1 + \gamma_2^{-X_4} \right) \left( 1 + \left| y \right|^{X_1} \right) \left( 1 + \left| y \right|^{X_2} \right) \left( 1 + \left| z_1 \right|^{X_3} \right) \left| z_1 \right|_V^2,
$$

(A.10a)

for all $\gamma_2 \in (0, 1]$, with $\tilde{\theta} :=
\begin{cases}
\tilde{D}_1 \gamma_2^{\frac{2}{X_2}}, & \text{for } \chi_2 > 0, \\
0, & \text{for } \chi_2 = 0,
\end{cases}

(A.10b)

and $\tilde{D}_1 = \mathcal{C} \left[ \left\| \xi_1 \right\|, \left\| \xi_2 \right\|, \left\| \frac{1}{\chi_1} \right\| \right]$, $\tilde{D}_2 = \mathcal{C} \left[ \left\| \xi_1 \right\|, \left\| \xi_2 \right\|, \left\| \frac{1}{\chi_1} \right\|, \left\| \frac{\chi_1 + \chi_2}{\chi_1 - \chi_2} \right\| \right]$.

(A.10c)
Now, from (A.1) and (A.10), for all \( \tilde{\gamma}_0 > 0 \) and \( \gamma_2 \in (0, 1] \), and with

\[
\chi_S^{\gamma_2} = \frac{1 + \| \delta_2 \|}{1 - \| \delta_2 \|} \geq 1,
\]
we derive that

\[
\begin{align*}
2 \left( \mathfrak{g}_1(t, z_1), A z_1 \right)_H & \leq \tilde{\gamma}_0 |z_1|^2_{D(A)} + \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \tilde{c}_{\gamma_1} \sum_{j=1}^n \gamma_j, \\
& \leq \left( \tilde{\gamma}_0 + n \vartheta \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \tilde{c}_{\gamma_1} |z_1|^2_{D(A)} \\
& + n \tilde{D}_2 \left( 1 + \gamma_2^{-x_4} \right) \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \tilde{c}_{\gamma_1} (1 + |y|_{\psi}^2) \left( 1 + |y|_{D(A)}^2 \right) (1 + |z_1|^2_{\psi}) |z_1|^2_{\psi}. \right. \\
& \left. \quad \text{for } \chi_2 > 0. \right)
\end{align*}
\]

(A.11)

For an arbitrary \( \tilde{\gamma}_0 > 0 \), we can choose

\[
\gamma_2 = \left( \frac{\tilde{\gamma}_0}{n + 1} \tilde{D}_1 \tilde{c}_{\gamma_1} \left( 1 + \tilde{\gamma}_0^{-x_S} \right) + \tilde{\gamma}_0 \right) \frac{\chi_2}{2} \leq 1,
\]
and \( \tilde{\gamma}_0 = \frac{\tilde{\gamma}_0}{n + 1} \). Note that, in particular,

\[
\tilde{D}_1 \gamma_2^{1/2} \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \tilde{c}_{\gamma_1} < \gamma_2^{1/2} \left( \tilde{D}_1 \tilde{c}_{\gamma_1} \left( 1 + \tilde{\gamma}_0^{-x_S} \right) + \frac{\tilde{\gamma}_0}{n + 1} \right) = \frac{\tilde{\gamma}_0}{n + 1}, \quad \text{for } \chi_2 > 0.
\]

and thus for the coefficient of \( |z_1|^2_{D(A)} \) in (A.11), we find

\[
\begin{align*}
\tilde{\gamma}_0 + n \vartheta \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \tilde{c}_{\gamma_1} & < \frac{\tilde{\gamma}_0}{n + 1} + n \frac{\tilde{\gamma}_0}{n + 1} = \tilde{\gamma}_0, \quad \text{if } \chi_2 > 0; \\
\tilde{\gamma}_0 + n \vartheta \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \tilde{c}_{\gamma_1} & = \frac{\tilde{\gamma}_0}{n + 1} < \tilde{\gamma}_0, \quad \text{if } \chi_2 = 0.
\end{align*}
\]

(A.12)

Observe, next, that

\[
1 + \tilde{\gamma}_0^{-x_S} = 1 + (n + 1)x_S \tilde{\gamma}_0^{-x_S}, \quad \text{(A.13)}
\]

and

\[
1 + \gamma_2^{-x_4} = 1, \quad \text{if } \chi_2 = 0,
\]

\[
1 + \gamma_2^{-x_4} \leq 1 + \left( (n + 1) \tilde{D}_1 \tilde{c}_{\gamma_1} \left( 1 + \tilde{\gamma}_0^{-x_S} \right) + \tilde{\gamma}_0 \right) \frac{\chi_2}{x_4} \frac{\chi_2}{\tilde{\gamma}_0} \frac{\tilde{\gamma}_2}{\gamma_0} \frac{\gamma_2}{x_4} \leq 1 + \left( 1 + 2 \frac{\chi_2}{x_4} \right) \left( (n + 1) \tilde{D}_1 \tilde{c}_{\gamma_1} \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \right) \frac{\chi_2}{\gamma_0} \frac{\gamma_2}{x_4} \leq \tilde{C}_1 + \tilde{C}_2 \left( 1 + \tilde{\gamma}_0^{-x_S} \right) \frac{\chi_2}{\gamma_0} \frac{\gamma_2}{x_4}, \quad \text{if } \chi_2 > 0;
\]
with
\[
\hat{C}_1 := 1 + (1 + 2^{\frac{X_2}{X_4}} - 1) = \mathcal{C} \left[ \|\xi_2\|, \left\| \frac{1}{X_1^{1/2}} \right\|, \frac{s_1}{s_1 - s_2} \right],
\]
\[
\hat{C}_2 := (1 + 2^{\frac{X_2}{X_4}} - 1) \left( (n + 1) \tilde{D}_1 \tilde{C}_{m1} \right)^{\frac{X_2}{X_4}} = \mathcal{C} \left[ n, C_{N'}, \left\| \xi_1 \right\|, \left\| \frac{1}{X_1^{1/2}} \right\|, \frac{s_1}{s_1 - s_2} \right].
\]

Since \( \hat{C}_1 \geq 2 \) holds for \( X_2 \geq 0 \) we can write
\[
1 + y_2^{-X_4} \leq \hat{C}_1 + \hat{C}_2 \left( 1 + \tilde{y}_0^{-X_5} \right)^{\frac{X_2}{X_4}} \tilde{y}_0^{-\frac{X_2}{X_4}}, \quad \text{for } X_2 \geq 0.
\]

Further, we see that
\[
1 + y_2^{-X_4} \leq \hat{C}_1 + \hat{C}_2 \left( 1 + 2^{\frac{X_2}{X_4}} - 1 \right) \left( \tilde{y}_0^{-\frac{X_2}{X_4}} + \tilde{y}_0^{-\frac{X_2+1}{X_4}} \right)
\]
and
\[
\tilde{y}_0^{-\frac{X_2}{X_4}} \leq 1 + \tilde{y}_0^{-\frac{(X_2+1)X_2}{X_4}},
\]
from which we obtain
\[
1 + y_2^{-X_4} \leq \hat{C}_1 + \hat{C}_2 \left( 1 + 2^{\frac{X_2}{X_4}} - 1 \right) \left( 1 + (1 + (n + 1)^{\frac{X_2+1}{X_4}}) \tilde{y}_0^{-\frac{(X_2+1)X_2}{X_4}} \right)
\]
\[
\leq \hat{C}_3 + \hat{C}_4 \tilde{y}_0^{-\frac{(X_2+1)X_2}{X_4}}, \quad (A.14a)
\]
with
\[
\hat{C}_3 := \hat{C}_1 + \hat{C}_2 (1 + 2^{\frac{X_2}{X_4}} - 1), \quad \hat{C}_4 := \hat{C}_2 (1 + 2^{\frac{X_2}{X_4}} - 1) (1 + (n + 1)^{\frac{X_2+1}{X_4}}).
\]

Therefore, (A.13) and (A.14) lead us to
\[
\left( 1 + y_2^{-X_4} \right) \left( 1 + \tilde{y}_0^{-X_5} \right) \leq \hat{C}_5 \left( 1 + \tilde{y}_0^{-X_5} \right) \left( 1 + \tilde{y}_0^{-\frac{(X_2+1)X_2}{X_4}} \right), \quad (A.15a)
\]
with, recalling that \( X_2, X_4, \) and \( X_5 \) are nonnegative,
\[
\hat{C}_5 := (n + 1)^{X_5} \left( \hat{C}_3 + \hat{C}_4 \right) = \mathcal{C} \left[ n, C_{N'}, \left\| \xi_1 \right\|, \left\| \xi_2 \right\|, \left\| \frac{1}{X_1^{1/2}} \right\|, \frac{s_1}{s_1 - s_2}, \frac{1}{1 - \left\| X_2 \right\|} \right]. \quad (A.15b)
\]
Hence, (A.11), (A.12), and (A.15) give us
\[
2\left( \mathfrak{m}_\nu(t, z_1), Az_1 \right)_H \\
\leq \tilde{\gamma}_0 |z_1|_{D(A)} + \tilde{C}_\eta_2 \left( 1 + \tilde{\gamma}_0 y_{\mathfrak{m}_\nu} \right) \left( 1 + |y_{\mathfrak{m}_\nu}| \right) \left( 1 + |z_1 y_{\mathfrak{m}_\nu}| \right),
\]
with \( \tilde{C}_\eta_2 := n \tilde{D}_2 \tilde{C}_5 \tilde{C}_\eta_1 = C \left[ n, C_{\mathcal{N}}, \|\xi_1\|, \|\xi_2\|, \frac{1}{\delta_1}, \frac{1}{\delta_2}, \frac{\|\xi_1 + \xi_2\|}{\delta_1 - \delta_2}, \frac{1}{\delta_2} \right] \). This ends the proof of Proposition 3.7. \( \square \)

### A.2 Proof of Proposition 3.8

Recall that \( D(A^\xi) \hookrightarrow H \), for \( \xi \geq 0 \), and \( H = \tilde{W}_S \oplus W^\perp_S \). We prove firstly that \( \tilde{W}_S \) and \( W^\perp_S \cap D(A^\xi) \) are closed subspaces of \( D(A^\xi) \). Clearly, \( \tilde{W}_S \) is closed, because it is finite-dimensional. Let now \( (h_n)_{n \in \mathbb{N}_0} \) be an arbitrary sequence in \( W^\perp_S \cap D(A^\xi) \) and a vector \( \tilde{h} \in D(A^\xi) \), so that \( |h_n - \tilde{h}|_{D(A^\xi)} \to 0 \), as \( n \to +\infty \). Since \( |h_n - \tilde{h}|_H \leq C |h_n - \tilde{h}|_{D(A^\xi)} \), for a suitable constant \( C > 0 \), it follows that \( |h_n - \tilde{h}|_H \to 0 \), and since \( W^\perp_S \) is closed in \( H \), it follows that \( \tilde{h} \in W^\perp_S \). Thus, \( \tilde{h} \in W^\perp_S \cap D(A^\xi) \), and we can conclude that \( W^\perp_S \cap D(A^\xi) \) is a closed subspace of \( D(A^\xi) \). Next, we observe that \( D(A^\xi) = \tilde{W}_S \oplus (W^\perp_S \cap D(A^\xi)) \), which is a straightforward consequence of \( H = \tilde{W}_S \oplus W^\perp_S \). To show that the oblique projection \( P_{\tilde{W}_S}^{W^\perp_S \cap D(A^\xi)} \) in \( D(A^\xi) \) coincides with the restriction \( P_{\tilde{W}_S}^{W^\perp_S \cap D(A^\xi)} \big|_{D(A^\xi)} \) of the oblique projection \( P_{\tilde{W}_S}^{W^\perp_S} \) in \( H \), it is enough to observe that by definition of a projection we have that

\[
P_{\tilde{W}_S}^{W^\perp_S \cap D(A^\xi)} \big|_{D(A^\xi)} w_1 = w_1 = P_{\tilde{W}_S}^{W^\perp_S} w_1, \quad \text{for all } w_1 \in \tilde{W}_S,
\]

\[
P_{\tilde{W}_S}^{W^\perp_S \cap D(A^\xi)} \big|_{D(A^\xi)} w_2 = 0 = P_{\tilde{W}_S}^{W^\perp_S} w_2, \quad \text{for all } w_2 \in W^\perp_S \cap D(A^\xi).
\]

Finally, we have \( P_{\tilde{W}_S}^{W^\perp_S \cap D(A^\xi)} \in \mathcal{L}(D(A^\xi)) \) because (oblique) projections are continuous; see Brezis (2011, Sect. 2.4, Thm. 2.10). \( \square \)

### A.3 Proof of Proposition 3.9

It is clear that \( P_{\tilde{W}_S}^{W^\perp_S} \big|_{D(A^{-\xi})} \) is an extension of the oblique projection \( P_{\tilde{W}_S}^{W^\perp_S} \in \mathcal{L}(H) \) to \( D(A^{-\xi}) \supseteq H \), because for \( z \in H \) we have that

\[
(z, P_{\tilde{W}_S}^{W^\perp_S} w)_H = (P_{\tilde{W}_S}^{W^\perp_S} z, w)_H, \quad \text{where for the last identity we have used Lemma 3.5.}
\]
By relation (3.9) and Proposition 3.8, the inequalities
\[ \left|\tilde{P}_{\tilde{W}^S_t} D(A^{-\xi})\right|_{\mathcal{L}(D(A^{-\xi}))} \leq \left|P_{\tilde{W}^S_t} D(A^\xi)\right|_{\mathcal{L}(D(A^\xi))} \]
< +\infty follow. Afterward, the same relation (3.9) gives us the converse inequality. Hence, we obtain the stated norm identity. Finally, by definition of the adjoint operator we also have the stated adjoint identity. \(\square\)

A.4 Proof of Proposition 3.10

Since \(s \in (0, 1)\), we have that \(g(\tau):= -\eta_1 \tau + \eta_2 \tau^s\) satisfies \(g(0) = 0\), \(\lim_{\tau \to +\infty} g(\tau) = -\infty\), and \(\frac{d}{d\tau}|_{\tau=\tau_0} g(\tau) = -\eta_1 + \sigma \eta_2 \tau_0^{s-1}\), for \(\tau_0 > 0\). In particular, \(g\) is differentiable at each \(\tau_0 > 0\). Furthermore,
\[
\frac{d}{d\tau}|_{\tau=\tau_0} g(\tau) > 0 \iff \tau_0^{s-1} > \eta_1 (\sigma \eta_2)^{-1} \iff \tau_0^{1-s} < (\sigma \eta_2)^{-1}
\]
\[
\iff \tau_0 < (\sigma \eta_2)^{1-s} \eta_1^{1-s}.
\]

Thus, \(g(\tau)\) strictly increases if \(\tau \in (0, \bar{\tau})\) with \(\bar{\tau} := (\sigma \eta_2)^{1-s} \eta_1^{1-s} = (\sigma \eta_2)^{1-s} \eta_1^{1-s}\). Analogously, we find that \(\frac{d}{d\tau}|_{\tau=\tau_0} g(\tau) < 0 \iff \tau_0 > \bar{\tau}\). Necessarily, the maximum is attained at \(\bar{\tau} > 0\) and can be computed as
\[
-\eta_1 \bar{\tau} + \eta_2 \bar{\tau}^s = -\eta_1 (\sigma \eta_2)^{1-s} \eta_1^{1-s} + \eta_2 \left( (\sigma \eta_2)^{1-s} \eta_1^{1-s} \right)^{\frac{s}{s-1}}
\]
\[
= \eta_2^{\frac{1}{s-1}} \eta_1^{\frac{s}{s-1}} \left( -\frac{1}{s-1} \eta_2^{-s} + \eta_2^{-s} \right).
\]

Thus, 
\[-\eta_1 \bar{\tau} + \eta_2 \bar{\tau}^s = (1-s)\eta_1^{\frac{1}{s-1}} \eta_2^{\frac{s}{s-1}} \eta_1^{\frac{s}{s-1}},\]
which finishes the proof. \(\square\)

A.5 Proof of Proposition 3.11

For the sake of simplicity, we shall omit the subscript in the usual norm in \(\mathbb{R}\), that is, \(|\cdot| := |\cdot|_{\mathbb{R}}\). The solution of (3.12) is given by
\[
v(t) = e^{-\mu(t-s) + \int_s^t |h(\tau)| \, d\tau} v(s), \quad t \geq s \geq 0, \quad v(0) = v_0. \quad (A.16)
\]

Observe that the exponent satisfies, using (3.10),
\[
-\mu(t-s) + \int_s^t |h(\tau)| \, d\tau \leq -\mu(t-s) + (t-s)^{\frac{r-1}{r}} \left( \int_s^t |h(\tau)|^r \, d\tau \right)^{\frac{1}{r}}
\]
\[
\leq -\mu(t-s) + (t-s)^{\frac{r-1}{r}} \left( \int_s^{s+\frac{t-s}{T}} |h(\tau)|^r \, d\tau \right)^{\frac{1}{r}}
\]
\[
\leq -\mu(t-s) + (t-s)^{\frac{r-1}{r}} \left( \frac{t-s}{T} |C_h|^r \right)^{\frac{1}{r}},
\]

where $\lceil \frac{l-s}{r} \rceil \in \mathbb{N}$ is the nonnegative integer defined in (3.27). Hence,

$$-\overline{\mu}(t-s) + \int_s^t |h(\tau)| \, d\tau \leq -\overline{\mu}(t-s) + (t-s)^{-1} \left( \frac{t-s}{T} + 1 \right)^{\frac{1}{r}} C_h$$

$$\leq T^{-\frac{1}{r}} (-\overline{\mu} \frac{1}{t} + C_h)(t-s) + (t-s)^{-1} r C_h, \quad (A.17)$$

where we have used $\left( \frac{l-s}{r} + 1 \right)^{\frac{1}{r}} \leq \left( \frac{l-s}{r} \right)^{\frac{1}{r}} + 1$, since $r > 1$; see Phan and Rodrigues (2017, Proposition 2.6).

By (3.11), we have that

$$\hat{\mu} := T^{-\frac{1}{r}} (\overline{\mu} \frac{1}{t} - C_h) \geq \max \left\{ \frac{2}{r} \left( \frac{C^r_h}{r \log(\varrho)} \right)^{\frac{1}{r}} , 2\mu \right\} > 0, \quad (A.18)$$

from which, together with (A.17) and Proposition 3.10, we obtain

$$-\overline{\mu}(t-s) + \int_s^t |h(\tau)| \, d\tau \leq -\frac{1}{2} \hat{\mu}(t-s) - \frac{1}{2} \hat{\mu}(t-s) + (t-s)^{-1} r C_h$$

$$\leq -\frac{\hat{\mu}}{2} (t-s) + \frac{1}{r} (\frac{r-1}{r})^{r-1} C^r_h \left( \frac{\hat{\mu}}{2} \right)^{1-r}, \quad (A.19)$$

because by Proposition 3.10, with $s = \frac{r-1}{r}$ and $\eta_1 = \frac{\hat{\mu}}{2}, \eta_2 = C_h$,

$$\max_{t-s \geq 0} \left\{ -\eta_1 (t-s) + (t-s)^{-1} \eta_2 \right\} = (1-s) s \frac{\eta_1}{r} \eta_2^{\frac{s-1}{s}} \eta_1^{\frac{1}{s}} = \frac{1}{r} (\frac{r-1}{r})^{r-1} \eta_2 \eta_1^{1-r}.$$ 

Therefore, from (A.16), (A.18), and (A.19), we derive that

$$|v(t)| \leq e^{\frac{C^r_h}{r} \left( \frac{2(r-1)}{r} \right)^{r-1} \hat{\mu}^{1-r}} e^{-\frac{\hat{\mu}}{2} (t-s)} |v(s)| \leq \varrho e^{-\mu(t-s)} |v(s)|.$$ 

Indeed, observe that

$$e^{\frac{C^r_h}{r} \left( \frac{2(r-1)}{r} \right)^{r-1} \hat{\mu}^{1-r}} \leq \varrho \iff \frac{C^r_h}{r} \left( \frac{2(r-1)}{r} \right)^{r-1} \hat{\mu}^{1-r} \leq \log(\varrho)$$

$$\iff \frac{C^r_h}{r \log(\varrho)} \left( \frac{2(r-1)}{r} \right)^{r-1} \hat{\mu}^{r-1} \leq \hat{\mu} \iff \hat{\mu} \geq \left( \frac{C^r_h}{r \log(\varrho)} \right)^{\frac{1}{r-1}} 2(r-1),$$

and the last inequality follows from (A.18), which also gives us $\frac{\hat{\mu}}{2} > \mu.$  \qed
A.6 Proof of Proposition 3.12

We shall use a fixed point argument, through the contraction principle, in the closed subset

$$\mathcal{Z}_{\mu_0}^{\mu_0} := \left\{ g \in L^\infty(\mathbb{R}_0, \mathbb{R}) \mid \left| e^{\mu_0 t} g(t) \right| \leq \varrho \left| \varpi_0 \right| \right\}$$

of the Banach space

$$\mathcal{Z}_{\mu_0}^{\mu_0} := \left\{ g \in L^\infty(\mathbb{R}_0, \mathbb{R}) \mid \left| e^{\mu_0 t} g(t) \right| \leq \varrho \left| \varpi_0 \right| \right\} , \quad \left| g \right|_{\mathcal{Z}_{\mu_0}^{\mu_0}} := \sup_{t \geq 0} \left| e^{\mu_0 t} g(t) \right| .$$

We show now that since (3.15) holds true, the mapping

$$\Psi : \mathcal{Z}_{\mu_0}^{\mu_0} \rightarrow \mathcal{Z}_{\mu_0}^{\mu_0}, \quad \tilde{\varpi} \mapsto \varpi,$$

where \( \varpi \) solves

$$\dot{\varpi} = -(\mu - |h|)\varpi + |\varpi| |\tilde{\varpi}|^p \tilde{\varpi}, \quad \varpi(0) = \varpi_0, \quad (A.20)$$

is well defined and is a contraction in \( \mathcal{Z}_{\mu_0}^{\mu_0} \).

We look at (A.20) as a perturbation of the nominal linear system

$$\dot{v} = -(\mu - |h|)v, \quad v(0) = v_0 = \varpi_0 \in \mathbb{R}. \quad (A.21)$$

Note that (3.15) implies that

$$\bar{\mu} \geq \max \left\{ 2 \frac{r - 1}{r} \left( \frac{C_h^r}{r \log \left( \frac{1}{\varrho_1^2} \right)} \right)^{\frac{1}{r+1}}, 4\mu_0 \right\} + T^{-\frac{1}{r}} C_h$$

which we use together with Proposition 3.11 to conclude that the solution

$$v(t) =: S(t, s)v(s)$$

of (A.21) satisfies

$$\left| v(t) \right| = \left| S(t, s)v(s) \right| \leq \varrho \frac{1}{2} e^{-2\mu_0 (t-s)} |v(s)| , \quad t \geq s \geq 0, \quad v(0) = v_0. \quad (A.22)$$

By the Duhamel formula, we have that the solution \( w \) of (A.20) is given as

$$\varpi(t) = S(t, s)\varpi(s) + \int_s^t S(t, \tau) \left| h(\tau) \right| |\tilde{\varpi}(\tau)|^p \tilde{\varpi}(\tau) \, d\tau, \quad \varpi = \Psi(\tilde{\varpi}). \quad (A.23)$$
Step 1: $\Psi$ maps $Z_{\varrho_0}^{\mu_0}$ into itself, if $|\varpi_0| < \varrho R$. We observe that (A.22) and (A.23) give us the estimate

$$|\varpi(t)| \leq \varrho \frac{1}{2} e^{-2\mu_0 t} |\varpi_0| + \int_0^t \varrho \frac{1}{2} e^{-2\mu_0 (t-\tau)} |h(\tau)| |\tilde{\varpi}(\tau)|^{p+1} d\tau. \quad (A.24)$$

Next, we also find, since $\tilde{\varpi} \in Z_{\varrho_0}^{\mu_0}$,

$$\int_0^t e^{-2\mu_0 (t-\tau)} |h(\tau)| |\tilde{\varpi}(\tau)|^{p+1} d\tau \leq \varrho^{p+1} |\varpi_0|^{p+1} \int_0^t e^{-2\mu_0 (t-\tau)} e^{-\mu_0 pT} |h(\tau)| d\tau$$

$$\leq \varrho^{p+1} |\varpi_0|^{p+1} e^{-\mu_0 t} \int_0^t e^{-\tau \mu_0 (t-\tau)} d\tau \left( \int_0^t e^{-\tau \mu_0 pT} |h(\tau)| d\tau \right)^{\frac{p-1}{p}}$$

$$\leq \varrho^{p+1} |\varpi_0|^{p+1} e^{-\mu_0 t} \left( \sum_{i=1}^{[\frac{T}{T}]} e^{-\tau \mu_0 p(i-1)T} \int_{(i-1)T}^{iT} |h(\tau)| d\tau \right)^{\frac{p-1}{p}}$$

$$\leq \varrho^{p+1} |\varpi_0|^{p+1} e^{-\mu_0 t} \left( \frac{1}{1 - e^{-\tau \mu_0 pT}} \right)^{\frac{1}{p}} \left( \frac{1}{\tau \mu_0} \right)^{\frac{p-1}{p}} e^{-\mu_0 t}. \quad (A.25)$$

By combining (A.24) with (A.25), we arrive at

$$e^{\mu_0 t} |\varpi(t)| \leq \varrho \frac{1}{2} e^{-\mu_0 t} |\varpi_0| + \varrho^{p+1} |\varpi_0|^{p+1} \left( \frac{1}{1 - e^{-\tau \mu_0 pT}} \right)^{\frac{1}{p}} \left( \frac{1}{\tau} \right)^{\frac{p-1}{p}} \mu_0^{\frac{1}{p-1}} |\varpi_0| \mu_0^{-\frac{1}{p-1}}$$

$$\leq \varrho \left( 1 + \varrho^{p+1} |\varpi_0|^{p} C_h \left( \frac{1}{1 - e^{-\tau \mu_0 pT}} \right)^{\frac{1}{p}} \left( \frac{1}{\tau} \right)^{\frac{p-1}{p}} \mu_0^{\frac{1}{p-1}} \right) |\varpi_0|. \quad (A.26)$$

Next, we use (3.14) and $|\varpi_0| \leq \varrho R$ to obtain

$$\frac{1}{1 - e^{-\tau \mu_0 pT}} \leq \frac{1}{1 - e^{-\mu_0 pT}} \leq 2, \quad (A.27a)$$
and

\[
1 + \varrho^{p+1} |\varpi_0|^p C_h \left( \frac{1}{1 - e^{-\mu_0 \rho T}} \right)^{\frac{1}{r}} \left( \frac{r - 1}{r} \right)^{\frac{r-1}{\bar{r}}} \mu_0^{\frac{1-r}{\bar{r}}}
\]

\[
\leq 1 + \varrho^{2p+1} R^p C_h \left( \frac{r - 1}{r} \right)^{\frac{r-1}{\bar{r}}} \mu_0^{\frac{1-r}{\bar{r}}} 2^{\frac{1}{r}}
\]

\[
\leq 1 + \varrho^{2p+1} R^p C_h \left( \frac{r - 1}{r} \right)^{\frac{r-1}{\bar{r}}} 2^{\frac{1}{r}} \left( \frac{\varrho^{2p+1} R^p C_h}{\varrho^{2} - 1} \right)^{-1} 2^{-\frac{1}{r}} \left( \frac{r - 1}{r} \right)^{\frac{1-r}{\bar{r}}}
\]

\[
= 1 + \left( \frac{1}{\varrho^{2} - 1} \right)^{-1}
\]

\[
= \varrho^{\frac{1}{2}}.
\]

(A.27b)

From (A.26) and (A.27), we find \( e^{\mu_0 t} |\varpi (t) | \leq \varrho |\varpi_0| \), and hence, \( \varpi = \Psi (\tilde{\varpi}) \in Z^{\mu_0}_{\varrho, |\varpi_0|} \).

\[\text{Step 2: } \Psi \text{ is a contraction in } Z^{\mu_0}_{\varrho, |\varpi_0|}, \text{ if } |\varpi_0| < \varrho R. \text{ For an arbitrary given } (\tilde{\varpi}_1, \tilde{\varpi}_2) \in Z^{\mu_0}_{\varrho, |\varpi_0|} \times Z^{\mu_0}_{\varrho, |\varpi_0|}, \text{ we have that the difference}
\]

\[D := \Psi (\tilde{\varpi}_1) - \Psi (\tilde{\varpi}_2)
\]

solves

\[
\dot{D} = -(\overline{p} - |h|) D + |h| \left( |\tilde{\varpi}_1|^p \tilde{\varpi}_1 - |\tilde{\varpi}_2|^p \tilde{\varpi}_2 \right), \quad D(0) = 0.
\]

By the Duhamel formula and the mean value theorem, we obtain

\[
|D(t)| = |S(t, 0)D(0)| + \left| \int_0^t \left| S(t, \tau) |h(\tau) | (|\tilde{\varpi}_1|^p \tilde{\varpi}_1 - |\tilde{\varpi}_2|^p \tilde{\varpi}_2) \right| d\tau \right|
\]

\[
\leq \varrho^{\frac{1}{2}} (p + 1) \int_0^t e^{-\mu_0 (t - \tau)} |h(\tau)| \left( |\tilde{\varpi}_1|^p + |\tilde{\varpi}_2|^p \right) |\tilde{\varpi}_1(\tau) - \tilde{\varpi}_2(\tau) | d\tau
\]

\[
\leq \varrho^{\frac{1}{2}} (p + 1) |\tilde{\varpi}_1 - \tilde{\varpi}_2| Z^{\mu_0}_{\varrho, |\varpi_0|} e^{-\mu_0 t} \int_0^t |h(\tau)| \left( |\tilde{\varpi}_1(\tau)|^p + |\tilde{\varpi}_2(\tau)|^p \right) d\tau
\]

\[
\leq 2 \varrho^{p+\frac{1}{2}} (p + 1) |\varpi_0|^p |\tilde{\varpi}_1 - \tilde{\varpi}_2| Z^{\mu_0}_{\varrho, |\varpi_0|} e^{-\mu_0 t} \int_0^t e^{-\mu_0 \tau^p} |h(\tau)| d\tau. \quad (A.28)
\]
Note that
\[
\int_0^t e^{-\mu_0 \tau} |h(\tau)| \, d\tau = \int_0^t e^{-\frac{r}{T} \mu_0 \tau} e^{-\frac{1}{T} \mu_0 \tau} |h(\tau)| \, d\tau
\]
\[
\leq \left( \int_0^t e^{-\mu_0 \tau} d\tau \right) \left( \int_0^t e^{-\mu_0 \tau} |h(\tau)| \, d\tau \right)^{\frac{1}{2}}
\]
\[
\leq (\mu_0 p)^{\frac{1-t}{t}} \left( \sum_{i=1}^T e^{-\mu_0 p(i-1)T} \int_{(i-1)T}^{iT} |h(\tau)| \, d\tau \right)^{\frac{1}{2}}
\]
\[
\leq (\mu_0 p)^{\frac{1-t}{t}} C_h \left( \sum_{i=1}^T e^{-\mu_0 p(i-1)T} \right)^{\frac{1}{2}} \leq (\mu_0 p)^{\frac{1-t}{t}} C_h \left( \frac{1}{1 - e^{-\mu_0 pT}} \right)^{\frac{1}{t}}. \quad (A.29)
\]

From (A.28) and (A.29),
\[
e^{\mu_0 t} |D(t)| \leq 2\varepsilon^{p+\frac{r}{T}} (p + 1) p^{\frac{1-t}{t}} C_h \left( \frac{1}{1 - e^{-\mu_0 pT}} \right)^{\frac{1}{t}} |\sigma_0|^p \mu_0^{\frac{1-t}{t}} |\tilde{\sigma}_1 - \tilde{\sigma}_2|_{\tilde{Z}^{\mu_0}_{\sigma_0}},
\]
which together \(|\sigma_0| \leq \varepsilon R\) and \(\mu_0 \geq \frac{\log(2)}{p T}\), see (3.14), give us \(\frac{1}{1 - e^{-\mu_0 pT}} \leq 2\) and
\[
e^{\mu_0 t} |D(t)|
\]
\[
\leq 2^{\frac{r-t}{r}} \varepsilon^{2p+\frac{r}{T}} (p + 1) p^{\frac{1-t}{t}} C_h R^p \mu_0^{\frac{1-t}{t}} |\tilde{\sigma}_1 - \tilde{\sigma}_2|_{\tilde{Z}^{\mu_0}_{\sigma_0}}
\]
\[
\leq 2^{\frac{r-t}{r}} \varepsilon^{2p+\frac{r}{T}} (p + 1) p^{\frac{1-t}{t}} C_h R^p \left( 2^{\frac{r-t}{r}} \varepsilon^{2p+\frac{r}{T}} C_h R^p \frac{1}{p} \right)^{\frac{r}{p+1}} \left( \frac{1}{2^{\frac{r-t}{r}} \varepsilon^{2p+\frac{r}{T}} C_h R^p} \right)^{\frac{r}{p+1}} |\tilde{\sigma}_1 - \tilde{\sigma}_2|_{\tilde{Z}^{\mu_0}_{\sigma_0}}
\]
\[
\leq c^{-1} p^{\frac{1-t}{r}} \left( \frac{1}{p} \right)^{\frac{r}{p+1}} |\tilde{\sigma}_1 - \tilde{\sigma}_2|_{\tilde{Z}^{\mu_0}_{\sigma_0}} = c^{-1} |\tilde{\sigma}_1 - \tilde{\sigma}_2|_{\tilde{Z}^{\mu_0}_{\sigma_0}} \quad (A.30)
\]
with \(c > 1\) as in (3.14). Therefore, (A.30) implies that
\[
|\Psi(\tilde{\sigma}_1) - \Psi(\tilde{\sigma}_2)|_{\tilde{Z}^{\mu_0}_{\sigma_0}} = |D|_{\tilde{Z}^{\mu_0}_{\sigma_0}} \leq c^{-1} |d|_{\tilde{Z}^{\mu_0}_{\sigma_0}} = c^{-1} |\tilde{\sigma}_1 - \tilde{\sigma}_2|_{\tilde{Z}^{\mu_0}_{\sigma_0}},
\]
which shows that \(\Psi\) is a contraction.

\(\circ\) Step 3: Existence of a solution in \(\tilde{Z}^{\mu_0}_{\sigma_0}\), if \(|\sigma_0| < \varepsilon R\). By the contraction mapping principle, there exists a fixed point for \(\Psi\) in \(\tilde{Z}^{\mu_0}_{\sigma_0}\). Such fixed point is a solution for (3.13).

\(\circ\) Step 4: Uniqueness of the solution in \(L^\infty(\mathbb{R}_0, \mathbb{R})\). The uniqueness follows from the fact that the right-hand side of (3.13) is locally Lipschitz.

\(\circ\) Step 5: Estimate (3.16) holds true. Fix \(s \geq 0\) and note that \(\tilde{h}(\tau) := h(\tau + s)\) also satisfies (3.10), with \(C_{\tilde{h}} \leq C_h\).
Let $\omega_s := \omega \mid_{\mathbb{R}_s}$ be the restriction to $\mathbb{R}_s = [s, +\infty)$ of the solution $\omega \in Z_{\mu, |\sigma_0|}$ of (3.13), and observe that $z(\tau) := \omega_s(\tau + s)$ solves

$$
\frac{d}{d\tau} z = -(\mu - |h|)z + |h| |z|^p z, \quad z(0) = z_0, \quad \tau \geq 0.
$$

If $|\sigma_0| < R$, it follows that $|z_0| = |\sigma(s)| \leq \varrho e^{-\mu s} |\sigma_0| \leq \varrho R$. Then, by Step 3 we have that $z \in Z_{\mu, |z_0|}$, which implies that for $t \geq s$,

$$
|\sigma(t)| = |\omega_s(s + t - s)| = |z(t - s)| \leq \varrho e^{-\mu_0(t-s)} |z(0)| = \varrho e^{-\mu_0(t-s)} |\sigma(s)|,
$$

which gives us (3.16).

The proof is finished. \hfill \Box

### A.7 Proof of Proposition 4.3

Let us denote by $\tau^i = (\tau^i_1, \tau^i_2, \ldots, \tau^i_d) \in \mathbb{R}^d$ the unit vector whose coordinates are $\tau^i_j = 1$ and $\tau^i_j = 0$ for $j \neq i$. Observe that $J_{d,2}$ has exactly $d + 1$ vectors. The only element in $J_{d,2}$ with $\sum_{j=1}^d \tau^i_j = d$ is $1^d := (1, 1, \ldots, 1)$. All the other elements in $J_{d,2}$ are of the form $1^d + \tau^i, i = 1, 2, \ldots, d$.

Let now $p \in \mathbb{P}_{\times,1}$ such that $\mathcal{S}(p) = 0$, which implies that

$$
\left( p, 1_{\omega_{1^d,1}} \right)^2_{\mathbb{R}} = 0, \quad \text{and} \quad \left( p, 1_{\omega_{1^d+\tau^i,1}} \right)^2_{\mathbb{R}} = 0, \quad \text{for all } i = \{1, 2, \ldots, d\},
$$

that is, with $\omega_s := \omega_{1^d,1}$

$$
\int_{\omega_s} p(x) \, dx = 0, \quad \text{and} \quad \int_{\omega_s} p(x - \tau^i) \, dx = 0, \quad 1 \leq i \leq d,
$$

Denoting $\mathcal{L}_a x := \sum_{i=1}^d a_i x_i$, and $p(x) := a_0 + \mathcal{L}_a x$, we obtain

$$
\int_{\omega_s} c_0 + \mathcal{L}_a x \, dx = 0, \quad \text{and} \quad \int_{\omega_s} c_0 + \mathcal{L}_a (x - \tau^i) \, dx = 0, \quad 1 \leq i \leq d,
$$

which implies

$$
\int_{\omega_s} c_0 + \mathcal{L}_a x \, dx = 0, \quad \text{and} \quad \int_{\omega_s} \mathcal{L}_a \tau^i \, dx = 0, \quad 1 \leq i \leq d. \quad (A.31)
$$

Note that for fixed $i$, we have

$$
\int_{\omega_s} \mathcal{L}_a \tau^i \, dx = 0 \iff \int_{\omega_s} a_i \, dx = 0 \iff a_i = 0,
$$

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which together with (A.31) leads us to $a_i = 0, 1 \leq i \leq d$, and $c_0 = 0$.

We have just shown that $p \in \mathbb{P}_{x,1}$ and $\mathcal{G}(p) = 0$ imply that $p = 0$. Therefore, we can conclude that $\mathcal{G}(\cdot)$ is a norm on $\mathbb{P}_{x,1}$. □

A.8 Proof of Proposition 4.7

Let $\theta = \sum_{k=1}^{\alpha} \theta_k \Phi_k \in \tilde{W}_S$, with the auxiliary functions $\Phi_i$ as in (4.2b). Then, after a translation and denoting $\hat{L}_i := \frac{rL_i}{2S}$, for the $H$-norm we find that

$$\|\Phi_k\|_H^2 = \prod_{j=1}^{d} \left| \sin^2 \left( \frac{\pi x_j}{\hat{L}_j} \right) \right|_{L^2(0, \hat{L}_j), \mathbb{R}}}^2 = \left( \frac{3}{8} \right)^d \prod_{j=1}^{d} \hat{L}_j$$

and, with $\hat{L}^\times := \prod_{j=1}^{d} \hat{L}_j$, since the $\Phi_i$s are pairwise orthogonal, we arrive at

$$|\theta|_H^2 = \sum_{k=1}^{\alpha} \theta_k^2 \|\Phi_k\|_H^2 = \left( \frac{3}{8} \right)^d \hat{L}^\times \sum_{k=1}^{\alpha} \theta_k^2.$$

Next, for the $V$-norm we find

$$|\theta|_V^2 = \sum_{k=1}^{\alpha} \theta_k^2 \|\Phi_k\|_V^2 = v \sum_{k=1}^{\alpha} \theta_k^2 \|\nabla \Phi_k\|_{L^2(\Omega)}^2 + |\theta|_H^2$$

and, due to

$$\|\nabla \Phi_k\|_{L^2(\Omega)}^2 = \sum_{i=1}^{d} \left| \frac{\pi}{\hat{L}_i} \sin \left( \frac{2\pi x_i}{\hat{L}_i} \right) \right|_{L^2((0, \hat{L}_i), \mathbb{R})}^2 \prod_{i \neq j=1}^{d} \left| \sin^2 \left( \frac{\pi x_j}{\hat{L}_j} \right) \right|_{L^2(0, \hat{L}_j), \mathbb{R}}}^2 = \sum_{i=1}^{d} \left( \frac{\pi}{\hat{L}_i} \right)^2 \hat{L}_i \prod_{i \neq j=1}^{d} \left( \frac{\pi}{\hat{L}_j} \right)^2 \hat{L}_j \frac{4}{3} \prod_{j=1}^{d} \frac{3\hat{L}_j}{8}$$

we obtain

$$|\theta|_V^2 = \left( v \frac{4\pi^2}{3} \sum_{i=1}^{d} \frac{1}{\hat{L}_i^2} + 1 \right) |\theta|_H^2 = \left( v \frac{4\pi^2}{3} \left( \frac{2S}{r} \right)^2 \sum_{i=1}^{d} \frac{1}{\hat{L}_i^2} + 1 \right) |\theta|_H^2.$$
That is,

\[ |\theta|^2_V = (C_1 S^2 + 1) |\theta|^2_H, \]

with \( C_1 := \frac{16}{3} \nu \pi^2 r^{-2} \sum_{i=1}^{d} L_i^{-2} \).

Finally, for the \( D(A) \)-norm we find

\[
|\theta|^2_{D(A)} = \frac{-v \Delta \theta + \theta}{H} = v^2 |\Delta \theta|^2_H + 2v |\nabla \Phi_k|^2_{L^2(\Omega)^d} + |\theta|^2_H
\]

\[
= v^2 |\Delta \theta|^2_H + 2 |\theta|^2_V - |\theta|^2_H = v^2 |\Delta \theta|^2_H + \left( 2C_1 S^2 + 1 \right) |\theta|^2_H
\]

and from

\[
|\Delta \Phi_k|^2_H = \sum_{i=1}^{d} 2 \left( \frac{\pi}{L_i} \right)^2 \cos \left( \frac{2\pi x_i}{L_i} \right) \left| \frac{\pi x_i}{L_i} \right|^2 \cdot \prod_{i \neq j=1}^{d} \left| \sin \left( \frac{\pi x_j}{L_j} \right) \right|^2 L^2((0, L_i), \mathbb{R})
\]

\[
= \sum_{i=1}^{d} 4 \left( \frac{\pi}{L_i} \right)^4 \frac{L_i}{8} \times 3 \frac{L_j}{8} = \sum_{i=1}^{d} 4 \left( \frac{\pi}{L_i} \right)^4 \frac{1}{3} \times 3 \frac{L_j}{8}
\]

\[
= \left( \frac{3}{8} \right)^d \frac{L \times 16 \pi^4}{3} \sum_{i=1}^{d} \frac{1}{L_i^4},
\]

we obtain

\[
|\Delta \theta|^2_H = \sum_{k=1}^{S_0} \theta_k^2 |\Delta \Phi_k|^2_H = \frac{16 \pi^4}{3} \sum_{i=1}^{d} \frac{1}{L_i^4} |\theta|^2_H = \left( \frac{2S}{r} \right)^4 \frac{16 \pi^4}{3} \sum_{i=1}^{d} \frac{1}{L_i^4} |\theta|^2_H,
\]

and hence,

\[
|\theta|^2_{D(A)} = \left( C_2 S^4 + 2C_1 S^2 + 1 \right) |\theta|^2_H, \quad \text{with} \quad C_2 := \frac{v^2 2^8 \pi^4}{3} r^{-4} \sum_{i=1}^{d} L_i^{-4},
\]

which finishes the proof. \( \square \)

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