QCD Mass Inequalities in the Heavy Quark Limit

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Abstract

QCD inequalities are derived for the masses of mesons and baryons containing a single heavy quark using the heavy quark effective field theory. A rigorous lower bound is obtained for the $\Lambda$ parameters of the heavy quark effective theory that parameterize $1/m$ corrections, $\Lambda \geq 237$ MeV for mesons, and $\Lambda \geq 657$ MeV for baryons. The inequalities on $\Lambda$ imply the inequalities $m_c < 1627$ MeV and $m_b < 5068$ MeV for the mass parameters of the heavy quark effective field theory.
Rigorous inequalities between hadron masses in QCD can be derived using the Euclidean functional integral formulation of the theory\cite{1\,2\,3\,4\,5}. The basic idea is that the functional integral measure for a vector-like theory such as QCD is real and positive, so that the Cauchy-Schwarz inequality can be used to derive inequalities on Euclidean correlation functions. The inequalities among correlation functions imply inequalities among hadron masses. The usual QCD inequalities hold for arbitrary quark masses, and so apply also to the heavy quark case e.g., \(m(\rho) \geq m(\pi)\) \cite{1} has the heavy quark analog \(m(B^*) \geq m(B),\) etc. These will not be discussed further here. QCD inequalities are derived in this paper for the heavy quark effective field theory\cite{6\,7\,8}, which is a systematic expansion about the infinite quark mass limit. The quark propagator in the infinite mass limit is the path ordered integral of the exponential of the vector potential, which is a unitary matrix. This implies certain inequalities which would not necessarily hold for light quarks.

In this paper, mass inequalities will be derived for hadrons containing only a single heavy quark \(Q\). The heavy quark effective theory has the leading order fermion Lagrangian

\[
\mathcal{L}_v = i \overline{Q}_v (v \cdot D) Q_v, \tag{1}
\]

where \(Q_v\) is a Dirac spinor field that annihilates a heavy quark with velocity \(v\), and \(D\) is the gauge covariant derivative. The total Lagrangian is the sum of the heavy quark Lagrangian and the usual QCD Lagrangian for the light quarks and gluons. (In this paper, light quarks will refer to quarks with mass \(m_q\) that is finite, but not necessarily small compared to \(\Lambda_{\text{QCD}}\).) The field \(Q_v\) satisfies the constraint

\[
Q_v = \frac{1 + \gamma^1}{2} Q_v. \tag{2}
\]

The masses of hadrons in the heavy quark effective theory containing a single heavy quark \(Q\) are \(m - m_Q\), where \(m\) is the mass of the hadron, and \(m_Q\) is the mass of the heavy quark. The heavy quark mass \(m_Q\), which is defined so that the effective Lagrangian eq. (1) has no residual mass term, is a well defined quantity which is a parameter of the heavy quark effective theory. A detailed discussion of this issue can be found in Ref. \cite{9}.

\footnote{A good discussion of QCD inequalities can be found in the Caltech lecture notes for Ph230 of J. Preskill (unpublished).}
The mass inequalities can be easily derived using the continuum formulation of the theory, provided one uses certain “obvious” properties of the products of Dirac delta functions. The same results can also be obtained using a lattice regulated version of the theory which avoids “problems” with Dirac delta functions, and is the method used here. The Lorentz frame for the heavy quark theory can be chosen so that the velocity vector \( v \) is \((1, 0, 0, 0)\), before analytically continuing the theory to Euclidean space. The hypercubic Euclidean lattice is chosen with edges that are parallel or perpendicular to \( v \), for simplicity. The Euclidean coordinate parallel to \( v \) will be denoted by \( t \), and the transverse coordinates will be denoted by \( x \). The lattice spacing is \( a \), with \( t \equiv n_0 a \) and \( x \equiv n a \). The Euclidean propagator for the heavy quark theory is then

\[
W(x, t; y, s) = a^{-3} \frac{1 + \gamma^0}{2} U(x, t; x, t - a)U(x, t - a; x, t - 2a) \ldots
\]

\[
\ldots U(x, s + 2a; x, s + a)U(x, s + a; x, s) \text{ if } x = y, \ t > s
\]

\[
= 0 \quad \text{otherwise.}
\]

Here \( U \) is a unitary matrix in color space which is defined on the links of lattice, and can be thought of as the path-ordered exponential of the gluon field,

\[
U \sim P \exp ig \int A_\mu dx^\mu.
\]

\( W \) satisfies the discretized version of the Green’s function equation

\[
(v \cdot D) W(x, t; y, s) = \frac{1 + \gamma^0}{2} \delta(x - y)\delta(t - s),
\]

\[
\frac{1}{a} [W(x, t; y, s) - U(x, t; x, t - a)W(x, t - a; y, s)] = a^{-4} \frac{1 + \gamma^0}{2} \delta_{xy} \delta_{ts},
\]

where \( \delta_{ab} \) is a Kronecker delta. To avoid complicating the notation in the correlation functions to be computed below, \((x, t)\) will be denoted by \( x \).

Consider the meson correlation function (at \( y = x, \ y^0 > x^0 \))

\[
\langle \bar{q} \Gamma^\dagger Q_v(y) \, \bar{Q}_v \Gamma q(x) \rangle = (-1) \int d\mu \text{Tr} W(y, x) \Gamma S(x, y; m_q) \Gamma^\dagger,
\]

where the trace is over spinor and color indices, the minus sign is due to Fermi statistics, and the fermion propagator for the quark \( q \) is

\[
S(x, y; m_q) = \langle x | (\not{\partial} + m_q)^{-1} | y \rangle.
\]
The Euclidean measure $d\mu$ in the functional integral is defined using the QCD action of the light quarks and gluons, and angle brackets are used to denote the functional integral over the light quark and gluon fields,

$$\langle f \rangle \equiv \int d\mu \ f = \frac{1}{Z} \int DA \ e^{-S_{\text{eff}}(A)} \ f,$$

where

$$S_{\text{eff}} = \frac{1}{2} \text{Tr} \ G^2 - \sum_i \ln \det (D + m_i),$$

and the sum is over the different light quark flavors. The QCD Euclidean path integral measure is real and positive for $\theta_{\text{QCD}} = 0$, since QCD is a vector-like gauge theory, so one can use the Cauchy-Schwarz inequality on eq. (5),

$$\left| \langle \bar{q} \Gamma^\dagger Q_v(x) Q_v^\dagger \Gamma q(x) \rangle \right|^2 \leq \int d\mu \ \text{Tr} \ \Gamma \Gamma^\dagger \ \text{Tr} \ W(y,x) W(y,x)^\dagger \ \times \ \int d\mu \ \text{Tr} \ S(x,y;m_q) S^\dagger(x,y;m_q).$$

The term on the r.h.s. proportional to the light quark propagator can be reexpressed as a meson correlation function (neglecting annihilation graphs), since

$$\langle \bar{q} i\gamma_5 q(x) \rangle = \int d\mu \ \text{Tr} \ S(x,y;m_q) S^\dagger(x,y;m_q),$$

using

$$\gamma_5 \ S(y,x;m_q) \gamma_5 = S(x,y;m_q)^\dagger.$$ Combining eqs. (7) and (8), and using ($N_c$ is the number of colors)

$$\int d\mu \ \text{Tr} \ \Gamma \Gamma^\dagger \ \text{Tr} \ W(x,y) W(x,y)^\dagger \ \leq 2N_c \ \text{Tr} \ \Gamma \Gamma^\dagger \ a^{-6} \ \int d\mu$$

which follows from eq. (8) and the unitarity of $U$ gives

$$\left| \langle \bar{q} \Gamma^\dagger Q_v(y) Q_v^\dagger \Gamma q(x) \rangle \right|^2 \leq C \ \langle \bar{q} i\gamma_5 q(x) \rangle \langle \bar{q} i\gamma_5 q(y) \rangle$$

where $C$ is a (cutoff dependent) constant which is independent of $x$ and $y$. At large distances, the Euclidean correlation function is dominated by the lightest particle in a given channel. For the heavy quark effective theory, this is true for the correlation function in eq. (7) provided $x = y$ (since we are working in the frame where $v = 0$), otherwise the
correlation function vanishes. The long distance behavior of the correlation function of any operator is
\[
\langle O(t, x)O^\dagger(s, x) \rangle \sim A e^{-M(t-s)},
\]
where \(M\) is the mass of the lightest state that can be created from the vacuum by \(O\), and \(t > s\). The lightest state that can couple to the l.h.s. of eq. (10) is a heavy quark meson with the spin quantum numbers of \(\Gamma\), e.g. for \(\Gamma = i\gamma_5\), it is the lightest pseudoscalar, etc. On the r.h.s., the state is the lightest pseudoscalar \(q\bar{q}\) state (neglecting annihilation diagrams), so we get the inequality
\[
m(\overline{Q}\Gamma q) - m_Q \geq \frac{1}{2} m'(\overline{q}i\gamma_5 q),
\]
where we have replaced the mass of the heavy meson in the effective theory by the difference of the mass of the heavy meson and heavy quark. The prime on \(m\) is a reminder that annihilation graphs are neglected.

The inequality eq. (11) is stronger than the inequality
\[
m(\overline{Q}\Gamma q) \geq \frac{1}{2} m'(\overline{q}i\gamma_5 q) + \frac{1}{2} m'(\overline{q}i\gamma_5 q),
\]
which can be derived in QCD without assuming that \(Q\) is infinitely heavy. The inequality eq. (11) is an inequality between two quantities which are finite in the infinite mass limit. Eq. (12) can be rewritten in the form
\[
m(\overline{Q}\Gamma q) - m_Q \geq \frac{1}{2} m'(\overline{Q}i\gamma_5 Q) - m_Q + \frac{1}{2} m'(\overline{q}i\gamma_5 q).
\]
In the limit \(m_Q \to \infty\), the mass of the lightest \(\overline{Q}i\gamma_5 Q\) bound state can be computed using QCD perturbation theory to be \(2m_Q - E_b\), where the binding energy \(E_b\) is \(E_b = \frac{1}{2} \left[\frac{4}{3} \alpha_s (m_Q)\right]^2 m_Q\) to lowest order in perturbation theory. \(E_b \to \infty\) in the heavy quark limit, so eq. (12) becomes the trivial inequality \(\text{finite} > -\infty\) in the heavy quark limit.

The binding energy is missing in the inequality eq. (11), which makes it a stronger inequality than eq. (12) in the heavy quark limit. The reason the term is missing in eq. (11) is that the heavy quark propagator in the effective theory is a straight line in space-time with \(v=(1,0,0,0)\), and no quantum fluctuations. The r.h.s. of the Cauchy-Schwarz inequality in eq. (7) contains the heavy quark propagator and its hermitian conjugate, which is the propagator for an anti-quark. The two propagators represent the interaction of a quark-antiquark pair at the same point in space, since the heavy quark does not
move. This static $Q\overline{Q}$ pair is color neutral, and does not interact with the gauge field (equivalently $WW^\dagger = 1$ is independent of the gluon vector potential). On the other hand, quantum fluctuations are included if one works with finite quark masses in the full QCD theory, and the propagator $S$ is obtained by summing over all paths that start at the initial point $x$, and end at the final point $y$. Thus $|S|^2$ is obtained by summing over all paths for a quark-antiquark pair to start at $x$ and end at $y$, with independent paths for the quark and antiquark. There are terms in the sum where the quark and antiquark paths are not identical. These paths can exchange a gluon between them, and this gives the binding energy $E_b$. The divergent binding energy $E_b$ is the reason the heavy quark effective theory breaks down for hadrons containing more than one heavy quark. The QCD inequalities follow from studying the square of the original meson correlation function, and thus have the particle content of a meson-antimeson pair. In deriving eq. (11), the correlation function was computed in the effective theory before squaring. In eq. (12), the square of the correlation function was computed in the full theory before taking the $m_Q \to \infty$ limit. This requires the study of the correlation function for $Q\overline{Q}q\overline{q}$ which contains two heavy quarks, so the heavy quark effective theory breaks down. The order of limits does not commute, which is why we obtain a stronger inequality by computing correlation functions directly in the effective theory, and then using the Cauchy-Schwarz inequality.

The l.h.s. of eq. (11) (in the pseudoscalar channel) is the meson $\overline{\Lambda}$ parameter of the heavy quark effective theory, which is a physical parameter that occurs in the corrections to meson form factors [11][12][9], so the inequality can be rewritten as

$$2\overline{\Lambda} (\overline{Q} \Gamma q) \geq m' (\overline{q} i \gamma_5 q).$$

The inequality determines the sign of $\overline{\Lambda}$ to be positive. The r.h.s. of the inequality can also be determined, since the $\overline{q} i \gamma_5 q$ mass neglecting annihilation graphs is the same as the $\overline{q}_1 i \gamma_5 q_2$ mass in the limit that $q_1$ and $q_2$ have equal masses. The masses of these states can be derived using chiral perturbation theory, and the known values of the pion and kaon masses,

$$m'(\overline{u} i \gamma_5 u) = \sqrt{\frac{2m_u}{m_u + m_d}} m_{\pi^0} = 114 \text{ MeV},$$

$$m'(\overline{d} i \gamma_5 d) = \sqrt{\frac{2m_d}{m_u + m_d}} m_{\pi^0} = 153 \text{ MeV},$$

$$m'(\overline{s} i \gamma_5 s) = \sqrt{\frac{2m_s}{m_u + m_d}} m_{\pi^0} = 685 \text{ MeV},$$

(14)
using the lowest order values for the light quark mass ratios, $m_u/m_d = 0.56$, and $m_s/m_d = 20.1$. One can improve the values of the meson masses by using the second order formulæ for the pseudoscalar masses. This will be discussed elsewhere. Eqs. (11) and (14) gives the inequalities for the meson $\Lambda$ parameters,

$$
\Lambda(Q\bar{u}) \geq 57 \text{ MeV}, \\
\Lambda(Q\bar{d}) \geq 76 \text{ MeV}, \\
\Lambda(Q\bar{s}) \geq 343 \text{ MeV}.
$$

Mass inequalities for baryon containing one heavy quark are obtained by studying the $Qq_1q_2$ correlation function. There are no new subtleties which occur in this case, so we will not present the details here. The QCD inequality is

$$
|\left\langle Qq_1q_2(y)\bar{Q}q_1\bar{q}_2(x)\right\rangle|
\leq \left\langle \left[\text{Tr } S(x, y, m_1) S^\dagger(x, y, m_1) \text{ Tr } S(x, y, m_2) S^\dagger(x, y, m_2)\right]^{1/2}\right\rangle
\leq \left\langle \text{Tr } S(x, y, m_1) S^\dagger(x, y, m_1)\right\rangle^{1/2} \left\langle \text{Tr } S(x, y, m_2) S^\dagger(x, y, m_2)\right\rangle^{1/2},
$$

where the $Qq_1q_2$ baryon on the l.h.s. can be in any spin channel, and $m_1$ and $m_2$ are the masses of $q_1$ and $q_2$ respectively. Comparing the exponential fall-off of both sides gives the inequality for the $\bar{\Lambda}$ parameter for baryons containing a heavy quark,

$$
\bar{\Lambda}(Qq_1q_2) = m(Qq_1q_2) - m_Q \geq \frac{1}{2} m' (\bar{q}_1 i\gamma_5 q_1) + \frac{1}{2} m' (\bar{q}_2 i\gamma_5 q_2),
$$

where the prime signifies that annihilation graphs are neglected for the mesons on the r.h.s. of the inequality. This gives the inequalities

$$
\bar{\Lambda}(Qss) \geq 685 \text{ MeV}, \\
\bar{\Lambda}(Quu) \geq 114 \text{ MeV}, \\
\bar{\Lambda}(Qus) \geq 400 \text{ MeV}, \\
\bar{\Lambda}(Qud) \geq 133 \text{ MeV}, \\
\bar{\Lambda}(Qds) \geq 419 \text{ MeV}, \\
\bar{\Lambda}(Qdd) \geq 153 \text{ MeV},
$$

using eq. (14).

The l.h.s. of the inequalities eqs. (13) and (17) are finite in the limit that the light quark mass $m_q \to 0$. $SU(3)$ symmetry breaking corrections to the $\bar{\Lambda}$ parameter are of order $\Delta m_q/\Lambda_\chi$, where $\Delta m_q$ is the difference of two light quark masses, and $\Lambda_\chi$ is the chiral symmetry breaking scale which is of order 1 GeV. The r.h.s. of eqs. (13) and (17) vanish as $m_q^{1/2}$ as $m_q \to 0$. Thus the strongest inequality for hadrons containing $u$ and
$d$ quarks is obtained by using $SU(3)$ symmetry and the inequality for hadrons containing $s$ quarks. This procedure has the disadvantage that the resulting inequalities are only approximate, since $SU(3)$ breaking corrections can be either positive or negative. A better method is to use the inequalities for hadrons containing a $s$ quark and the experimental values for the hadron mass differences to obtain inequalities for hadrons containing $u$ and $d$ quarks. The inequalities on $\Lambda$ obtained by this method are stronger than the inequalities in eq. (15) because we have included some extra information from experiment. For the $c$ quark system, one has the inequalities

$$m_c = m(D_s) - \Lambda(D_s) \leq m(D_s) - 343 \text{ MeV} \leq 1627 \text{ MeV},$$

(19)

using $\Lambda(D_s) \geq 343 \text{ MeV}$ from eq. (15), and

$$\begin{align*}
\Lambda(D^0) &= \Lambda(D_s) + m(D^0) - m(D_s) \geq 237 \text{ MeV},
\Lambda(D^+) &= \Lambda(D_s) + m(D^+) - m(D_s) \geq 243 \text{ MeV},
\Lambda(\Lambda^+_c) &= \Lambda(D_s) + m(\Lambda^+_c) - m(D_s) \geq 657 \text{ MeV},
\end{align*}$$

(20)

and similarly for the other hadrons containing a $c$ quark. The experimental numbers have been taken from the 1992 Particle Data Book [15], and we have used $1\sigma$ experimental errors. A similar analysis for hadrons containing a $b$ quark gives

$$m_b \leq m(B_s) - 343 \text{ MeV} \leq 5068 \text{ MeV},$$

(21)

using the experimental value $80 \leq m(B_s) - m(B) \leq 130 \text{ MeV}$ from [15], and $\Lambda(B_s) \geq 343 \text{ MeV}$ from eq. (15). There are also inequalities for the $\Lambda$ parameters,

$$\begin{align*}
\Lambda(B^{0,+,+}_s) &= \Lambda(B_s) + m(B^{0,+,+}_s) - m(B_s) \geq 213 \text{ MeV},
\Lambda(\Lambda^0_b) &= \Lambda(B_s) + m(\Lambda^0_b) - m(B_s) \geq 523 \text{ MeV}.
\end{align*}$$

(22)

The inequalities for the $\Lambda$ parameters for the $b$ system are weaker than those for the $c$ system because of the larger experimental uncertainties in the masses of the $B_s$ and $\Lambda^0_b$ hadrons. In the heavy quark limit, the $\Lambda$ parameters for the $c$ and $b$ hadrons are the same, and one can use the inequalities eq. (21) for the $b$ system. Finally, the inequalities eqs. (19)–(22) imply that $\Lambda/m_Q$ which characterizes the size of $1/m_Q$ corrections is greater than 0.15 for $D$ mesons, 0.04 for $B$ mesons, 0.4 for the $\Lambda^+_c$, and 0.1 for the $\Lambda^0_b$.

We would like to thank U. Wolff for helpful discussions. This work was supported in part by the U.S. Department of Energy under Grant No. DOE-FG03-90ER40546 and by a NSF Presidential Young Investigator Award PHY-8958081.
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