Some remarks on Oscillating Inflation

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In a recent paper Damour and Mukhanov describe a scenario where inflation may continue during the oscillatory phase. This effect is possible because the scalar field spends a significant fraction of each period of oscillation on the upper part of the potential. Such additional period of inflation could push perturbations after the slow roll regime to observable scales. Although in this work we show that the small region of the Damour-Mukhanov parameter $q$ gives the main contribution to oscillating inflation, it was not satisfactory understood until now. Furthermore, it gives an expression for the energy density spectrum of perturbations, which is well behaved in the whole physical range of $q$.

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I. INTRODUCTION

Nowadays inflation is a widely accepted element of the early cosmology [1]. It gives the possibility of solving many of the shortcomings of the standard hot big bang model and provides the source for the early energy density fluctuations responsible of the large scale structure of the universe observed today. Although there are many models of inflation, the underlying physical ideas are well established. These are characterized by a period of “slow roll” evolution of a scalar field (called inflaton) toward the vacuum potential. During this period the field changes very slowly, so that the kinetic energy $\dot{\varphi}^2/2$ remains smaller than its potential energy $V(\varphi)$. The energy density associated to the scalar field acts as a “cosmological constant” term, allowing a period of quasi exponential expansion of the scale factor. When the period of inflation ends, the scalar field $\varphi$ start a phase of rapid coherent oscillations around the vacuum.

Very recently [2,3] it has been pointed out that inflation can persist during the coherent oscillations of the inflation field phase. This exciting result is possible when the inflaton potential verifies a simple constrain of curvature far from the core convex part, where the inflaton field can roll slowly. The efficiency of this phenomena could have important implications for GUT scale baryogenesis [4]. In fact, as suggested by Damour and Mukhanov (2), it can be expected that due to the increase of the oscillation frequency, there is the possibility to generate massive particles heavier than $\sim 10^{16} GeV$.

In ref. [3] Damour and Mukhanov estimated the amount of inflation to be $\sim 10$ e-fold (powers of the scale factor). They argue that this effect can be more efficient than the parametric resonance effect [3] for the amplification of cosmological perturbations [2]. In ref. [2] Liddle and Mazumdar showed that Mukhanov et al. overestimated the number of e-fold because they have used a slow-roll definition of this object. In their paper, Liddle and Mazumdar found an analytical expression for the number of e-fold of inflation using the appropriate definition finding a number of $\sim 3$ e-fold concluding that this effect is not very efficient. The study of adiabatic perturbations in this phase has been made by Taruya [5]. He found a poor amplification in the case of a single scalar field model but anticipated an enormous amplification for multi-field systems.

In this letter we review the problem. In particular we find that the analytical expressions used to compare with the numerical estimation are not well defined in the $q \sim 0$ region and propose a way to correct these analytical estimations. Furthermore, with this result we study the evolution of the scalar field finding total agreement with the conclusions of ref. [2] for $q > 0.2$, but a remarkable different result for small $q$. For this region, the initial conditions are very important. We find that $q \sim 0$ gives the leading contribution for oscillating inflation and the dominant part in the amplification of the fluctuations.

The letter is organized as follow; first we describe briefly the Damour-Mukhanov model. Then, we make some comments about the initial conditions for this phenomenon and later we propose an improved expression, valid for the leading region of $q$, which is our main contribution.

II. BASIC EQUATIONS

Now we shall restrict ourselves to models of inflation driven by a single scalar field. The equations are

$$\dot{\varphi} + 3H \ddot{\varphi} + V_{,\varphi} = 0,$$  \hspace{1cm} (1)

$$H^2 = \kappa^2 \left(\frac{1}{2} \dot{\varphi}^2 + V\right),$$  \hspace{1cm} (2)

Here $H = \dot{a}/a$ is the Hubble parameter, $a$ is the scale factor of the universe and $\kappa^2 = 8\pi/3M_p^2$ with $M_p = 1.2 \cdot 10^{19} GeV$ the Planck mass. During the oscillatory
phase of \( \varphi \) we have two time scales; the inverse of the
frequency \( \omega^{-1} \) of oscillations of \( \varphi \) and the inverse of the
rate of expansion \( H^{-1} \). If the limit \( \omega \gg H \) is taken we
can neglect terms proportional to \( H \) in the equations. So
from (1) we can integrate to obtain,

\[
\rho = \frac{1}{2} \dot{\varphi}^2 + V = \text{cte} = V_m, \tag{3}
\]

where \( V_m = V(\dot{\varphi}_m) \) is the maximum value of \( V(\varphi) \) in
each oscillation when the field reaches the maximum
value \( \varphi_m \). From this relation we obtain the period of
a single oscillation, \( T = 4 \int_0^{\varphi_m} d\varphi \left[ 2(V_m - V(\varphi)) \right]^{1/2} \).
When \( \omega \ll H \) we can define an adiabatic average index \( \gamma \) by
\( \gamma = \langle (\rho + p)/\rho \rangle \), where the bracket means \( \langle ... \rangle = T^{-1} \int_0^T ... \, dt \).
Equations (1,2) can be re-written in the fluid form

\[
\dot{\rho} = -3H(p + \rho), \tag{4}
\]

\[
\dot{a} = -\frac{1}{3}(\rho + 3p), \tag{5}
\]

then from the definition of \( \gamma \) and eqns. (3,4) we have several
ways to compute the adiabatic index

\[
\gamma = \frac{\langle \dot{\varphi}^2 \rangle}{V_m} = \frac{\langle \dot{\varphi}V_{\varphi} \rangle}{V_m} = 2(1 - \frac{\langle V \rangle}{V_m}). \tag{6}
\]

Because \( p = (\gamma - 1)\rho \) and (3) we have a superluminal ex-
pansion \( \dot{a} > 0 \) when \( \gamma < 2/3 \). From the last two relations
in eqn.(6) the inequality \( \gamma < 2/3 \) leads to

\[
\langle V - \varphi V_{\varphi} \rangle > 0. \tag{7}
\]

### III. THE DAMOUR-MUKHANOV MODEL

Until now everything has been done for an arbitrary
potential, but from now on we shall consider the potential

\[
V(\varphi) = \frac{A}{q} \left[ \left( \frac{\varphi^2}{\varphi_c^2} + 1 \right) \right]^{q/2} - 1, \tag{8}
\]

where \( q \) is a dimensionless parameter, \( A = \text{[mass]}^4 \) is a constant and \( \varphi_c = \text{[mass]} \) determines the size of the
convex core of \( V(\varphi) \). We assume for a while that \( \varphi_c \)
marks the end of oscillating inflation. The analysis made
in ref. [3] works well far from the core of the potential.
Further, the limit \( \varphi \gg \varphi_c \) of eqn.(8) was written as:

\[
V(\varphi) \simeq \frac{A}{q} \left( \frac{\varphi}{\varphi_c} \right)^q. \tag{9}
\]

In this case, the adiabatic index can be computed exactly
given (8) by

\[
\gamma = \frac{2q}{q + 2}, \tag{10}
\]

so, from the inequality \( \gamma < 2/3 \) we note that to hold
inflation during the oscillatory phase we must have \( q < 1 \).
By using eqn.(10) in eqn.(1) we obtain \( \dot{\rho} = -3H\gamma\rho \) and
together with eqn.(3) we have

\[
a \propto t^{2/3\gamma} = e^{(q+2)/3q}, \tag{11}
\]

\[
\varphi_m \propto t^{-2/q} \propto a^{-6/(q+2)}, \tag{12}
\]

\[
\rho = V(\varphi_m) \propto t^{-2} \propto a^{-6q/(q+2)}, \tag{13}
\]

where \( \varphi_m \) is the amplitude of the oscillations, \( \varphi_c < \varphi_m < \varphi_s \) and \( \varphi_s \) is a typical value of \( \varphi \) at the end of slow-roll
inflation and the beginning of oscillating inflation. To compute the number of e-fold of inflation during oscil-
лating inflation we cannot use the standard expression
\( N = \ln(a_f/a_i) \), appropriate for the slow-roll stage, but
the improved expression proposed in ref. [4]

\[
\tilde{N} = \ln \frac{a_f H_f}{a_i H_i}, \tag{14}
\]

because in each oscillation, while the field spends time in
the core region, the universe continue their expansion so
\( H \) can vary. Then from (2) and (13) \( H \propto a^{-3q/(q+2)} \) the
product \( aH \propto \varphi_m^{(1-q)/3} \) and from (14) we obtain

\[
\tilde{N} \simeq \frac{1 - q}{3} \left( \frac{\ln \varphi_c}{\varphi_c} - 2 \right), \tag{15}
\]

where we have used \( \varphi_s \approx qM_p/\sqrt{16\pi} \). In [3] the num-
erical curves for \( \varphi_c = 10^{-6}M_p \) show that \( N \lesssim 3 \). Using
the analytical expression (15) we do not find agreement
for small values of \( q \). However there is not a compelling
reason to believe in (15) for small \( q \).

### IV. THE SMALL Q-REGION

To study the small \( q \) region, we must use the correct
limit \( q \rightarrow 0 \) of (5) which leads to

\[
V(\varphi) \simeq \frac{A}{2} \ln \left( \frac{\varphi}{\varphi_c} + 1 \right), \tag{16}
\]

so, if now we take the limit \( \varphi \gg \varphi_c \) we obtain the loga-
rithmic potential \( V(\varphi) \simeq A \ln(\varphi/\varphi_c) \). A very important
fact to note from (3) is that the limit \( q \rightarrow 0 \) does not
exist. Of course, the expression (3) is wrong around the
\( q \approx 0 \) region and the expressions derived from this are
ill-defined. But, some work has been done in this regard
[2]. For the logarithmic potential the adiabatic index is
\( \gamma = 1/\ln(\varphi_m/\varphi_c) \), so from (3) and (2) we obtain

\[
a(t) \propto \exp \left[ -\frac{A}{2}(t_{\text{end}} - t)^2 \right], \tag{17}
\]
but this form does not permit us to write an explicit expression for \( \dot{N} \) (see ref. [3]). Let us make some comments about this result. Because \( \gamma = 1/\ln(\varphi_m/\varphi_c) \) from [3] we obtain \( \dot{\rho} = -3H\rho \), then \( a \sim (\varphi_m/\varphi_c)^{-1/3} \). Moreover, from [3] we have \( H \propto \rho^{1/2} \propto (\ln(\varphi_m/\varphi_c))^{1/2} \), then to compute \( \dot{N} \) we should evaluate the factor \( (\ln(\varphi_m/\varphi_c))^{1/2}(\varphi_m/\varphi_c)^{-1/3} \) at the extremes \( \varphi_s \) and \( \varphi_c \), but this is not possible. The problem arises when \( \varphi_m \) is chosen close to \( \varphi_c \) in an expression valid for \( \varphi \gg \varphi_c \). Because [3] is valid for \( \varphi \gg \varphi_c \), too, the same problem should appear in the calculation of \( \dot{N} \). In fact this is the case but, to see that, we must include the constant term \( A/q \) in [3] or equivalently, write the limit correctly.

When we use the potential

\[
V(\varphi) \simeq A \left[ \left( \frac{\varphi}{\varphi_c} \right)^q - 1 \right],
\]

the adiabatic index become time-dependent, satisfying the equation

\[
\gamma_\rho = \frac{2q}{q + 2} \left( \rho + \frac{A}{q} \right),
\]

where \( \rho(t) = V(\varphi_m(t)) = A[(\varphi_m(t)/\varphi_c)^q - 1]/q \). Replacing this in [3] we obtain the same behavior as in eqn.[12]. But when we calculate \( H \) we obtain \( H \propto \rho^{1/2} \propto q^{-1/2}[(\varphi_m/\varphi_c)^q - 1]^{1/2} \), which is not well defined at the point \( \varphi_m = \varphi_c \). The same happens to the e-fold number, as mentioned before.

The authors of [3] found \( \varphi_s \approx qM_p/\sqrt{16\pi} \) using the potential [3]. From this result they found a decrease of \( \dot{N} \) at decreasing values of \( q \). This seems very strange because, for smaller values of \( q \) we obtain flatter potentials at \( \varphi \gg \varphi_c \), then \( \varphi \) rolls slowly most of the time increasing the amount of inflation. Furthermore, we are not safe of how they set the initial conditions for \( \varphi \) in their numerical analysis. In general, the initial and final field configuration depends on the potential.

For \( q \) close to zero, \( \varphi_s \) is not proportional to \( q \). In fact, we know that \( \varphi_s \) came from the saturation of the slow-roll inequality \( |V'|/V| < \sqrt{48\pi}/M_p \) using the expression [3] for the potential

\[
\left( \frac{\varphi_s}{\varphi_c} \right)^q = \frac{\sqrt{48\pi\varphi_c}}{qM_p} \left[ \left( \frac{\varphi_s}{\varphi_c} \right)^q - 1 \right].
\]

for the \( \varphi \gg \varphi_c \) region. In Figure 1 we see the behavior of \( \varphi_s \) in terms of \( q \) given by eqn.[20]. Of course, in the large \( q \)-region both curves agree. In order to illustrate this point and compare with ref. [3] we use its expression given by eqn.[14] but instead of using \( \varphi_s \approx qM_p/\sqrt{16\pi} \), we use the numerical values of \( \varphi_s \) obtained from eqn.[20]. The results are plotted in Figure 2. In the small \( q \)-region the field \( \varphi_s \) grows preventing the fall of \( \dot{N} \) predicted in ref. [3].

Moreover, as we have anticipated before, the value of the field at the end of oscillating inflation will have a \( q \)-dependence too. We know from ref. [3] that the intercept \( U(\varphi) = V(\varphi) - \varphi V_{,\varphi} \), must be positive to hold oscillating inflation. Let us define \( \varphi_f \) to be the value of the inflaton field \( \varphi \) at which \( U(\varphi_f) = 0 \). This condition represents the end of inflation due to oscillation. We need thus to compare \( \varphi_f \) for different values of \( q \).

If we take the potential [3] and define \( x = \varphi/\varphi_c \), we obtain

\[
U(x) = A q \left[ x^q (1 - q) - 1 \right].
\]

From this equation we can extract an explicit expression for \( \varphi_f \). If we impose \( U(x_f) = 0 \) we obtain the value of the scalar inflaton field at the end of this phase

\[
\varphi_f = \varphi_c (1 - q)^{-1/q}.
\]

Using the improved expression eqn.[14] for the e-fold number (ref. [3]), but inserting the corrected values for \( \varphi_s \) and \( \varphi_f \) given by eqns. (20) and (22) we obtain

\[
\dot{N} \simeq \ln \left\{ \frac{\left( \frac{\varphi_s}{\varphi_f} \right)^{(2+q)/6} \left[ (\varphi_f/\varphi_c)^q - 1 \right]^{1/2}}{(\varphi_s/\varphi_c)^q - 1} \right\}.
\]

The corrected value for \( \varphi_f \) leads to a even smaller amount of inflation when comparing with the value obtained in ref. [3]. In figure 3, we plot eqn.[23] and show the behavior of both effects combined. Because \( \varphi_f \) is greater than \( \varphi_c \), the amount of inflation is smaller than one obtained by Liddle et al. [3] in the whole range of \( q \in (0, 1) \). Moreover, the correct values of \( \varphi_s \) produce a positive contribution to \( \dot{N} \) in the small \( q \)-range, which avoids the fall of \( \dot{N} \), as \( q \) goes to smaller values, predicted by Liddle et al. [3]. Again, it is not possible to show the whole range of \( q \) because \( q > \varphi_c/\varphi \) (see comments below eqn.[20]).

Because oscillating inflation adds e-foils of inflation after the slow-roll regime, where the observed perturbations are generated, is possible that this additional period of inflation could push perturbations to observable scales. In order to obtain the required amplitude of density perturbations and without imposing unphysical constraint on the potential, we should compute the density perturbation spectrum for the model being studied.

In reference [3] an expression for this object was derived

\[
\delta_H^2 = \frac{512\pi}{75} \frac{A}{q^3 M_p^6 \varphi_c^8} \varphi^{q+2}.
\]

However, this expression is not well defined close enough to zero. Using the primordial density perturbation spectrum \( \delta_H^2 \) as was define in [3] we obtain for eqn.[3]

\[
\delta_H^2 = \frac{512\pi}{75} \frac{A}{q^3 M_p^6} \left\{ (\varphi/\varphi_c)^q - 1 \right\}^{3/2},
\]

which is well defined even for the small values of \( q \):

\[
\delta_H^2 \approx \frac{512\pi}{75} \frac{A}{M_p^3} \ln^3 \left( \frac{\varphi}{\varphi_c} \right).
\]
Because the COBE satellite require $\delta_H \approx 2 \cdot 10^{-5}$, in the $q = 0$ case the amplitude of the potential for $\varphi_c = 10^{-6}M_p$ gives $A^{1/4} \sim 2 \cdot 10^{-3}M_p$, which is a typical number for inflationary models.

V. SUMMARY

As a summary, we have made some corrections about how to compute the e-fold number, which accounts for the amount of inflation during the oscillatory phase. In particular we note that previous studies are not accurate because they are not valid close to the core of the potential $\varphi \sim \varphi_c$. Thus, we make the analysis for the small $q$ region of the potential, which has not been considered until now. Finally we find that, in order to extract the correct amount of inflation during this phase, a very careful definition of initial and final field configuration is needed. Our results show that near $q \sim 0$ the e-fold number is maximal but it is still not enough to be more efficient than the parametric resonant effect discussed in [5].

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VI. FIGURES CAPTION

Figure 1: We plot the $q$ dependence of $\varphi_s$. We see that for $q > 0.2$ both curves agree but for smaller values of $q$ the field $\varphi_s$ grows, preventing the fall of $\tilde{N}$.

Figure 2: The e-fold number $\tilde{N}$ is shown as a function of $q$, taking into account the behavior of $\varphi_s$ described for the eqn. (20).

Figure 3: The e-fold number $\tilde{N}$ is plotted vs $q$, taking into account the combined effects: the behavior of $\varphi_s$ described by eqn. (20) and the definition of $\varphi_f$ discussed in the text.

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This work

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$\phi_s / \phi_c$

$10^4$

$q$
This work

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e-fold number

q
This work

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q

e-fold number

0,0 0,1 0,2 0,3 0,4 0,5 0,6 0,7 0,8 0,9 1,0

0,0 0,05 0,1 0,15 0,2 0,25 0,3 0,35 0,4 0,45 0,5 0,55 0,6 0,65 0,7 0,75 0,8 0,85 0,9 0,95 1,0 1,05