A note on the extreme points of the cone of quasiconvex quadratic forms with orthotropic symmetry

Davit Harutyunyan

May 23, 2018

Abstract

We study the extreme points of the cone of quasiconvex quadratic forms with linear elastic orthotropic symmetry. We prove that if the determinant of the acoustic matrix of the associated fourth order tensor of the quadratic form is an extremal polynomial, then the quadratic form is an extreme point of the cone in the same symmetry class. The extremality of polynomials and quadratic forms here is understood in the classical convex analysis sense. The first example of a quadratic form that is an extreme point of the bigger cone of quasiconvex quadratic forms has been previously found by the author and Milton in [8].

Keywords: Extremal quasiconvex quadratic forms, rank-one convexity

1 Introduction

Quasiconvexity has been a central concept in applied mathematics since the work of Morrey [19,20]. It is tied with the existence of minimizers of integral functionals with the Lagrangian satisfying certain type of growth conditions [27]. One of the equivalent definitions of quasiconvexity reads as follows: Assume $n, N \in \mathbb{N}$ and $f : \mathbb{R}^{N \times n} \to \mathbb{R}$. If the function $f$ is Borel measurable and locally bounded, then it is said to be quasiconvex if

$$f(\xi) \leq \int_{[0,1]^n} f(\xi + \nabla \varphi(x))dx, \quad (1.1)$$

for all matrices $\xi \in \mathbb{R}^{N \times n}$ and all functions $\varphi \in W^{1,\infty}_0([0,1]^n, \mathbb{R}^N)$. The condition of rank-one convexity occurs naturally in the second variation of an integral functional

$$\int_D L(\nabla u(x))dx,$$

reducing to a pointwise condition on the Hessian of the Lagrangian $L$; the condition is known as the Legendre-Hadamard condition [26,27,23,7]. The condition of rank-one-convexity reads

---

*University of California Santa Barbara, harutyunyan@ucsb.edu
as follows: Assume \( n, N \in \mathbb{N} \) and \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \). If the function \( f \) is Borel measurable and locally bounded, then it is said to be rank-one-convex, if

\[
f(\lambda A + (1 - \lambda) B) \leq \lambda f(A) + (1 - \lambda) f(B)
\]

for all \( \lambda \in [0, 1] \) and all matrices \( A, B \in \mathbb{R}^{N \times n} \) such that \( \text{rank}(A - B) \leq 1 \). For \( C^2 \) functions \( f \) the condition (1.2) is equivalent to the Legandre-Hadamard condition mentioned above and is given by

\[
\sum_{0 \leq \alpha, \gamma \leq N} \sum_{0 \leq \beta, \delta \leq n} \frac{\partial^2 f(\xi)}{\partial \xi_{\alpha \beta} \partial \xi_{\gamma \delta}} x_{\alpha} y_{\beta} x_{\gamma} y_{\delta} \geq 0,
\]

for all \( x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \) and \( y = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). A special choice of the test function \( \varphi \) in (1.1) proves that in fact quasiconvexity implies rank-one-convexity [7].

In the present work we continue the study of the so called extremal quasiconvex quadratic forms introduced by Milton in [15]. It is know that when the function \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) is a quadratic form, then the quasiconvexity and rank-one-convexity of \( f \) become equivalent conditions [7]. An extremal form mentioned above is a quasiconvex quadratic form that loses the quasiconvexity property whenever a convex quadratic form is subtracted from it. Milton showed in [17], that such forms may be used to derive new bounds on the effective properties of composites, improving the previous ones derived by the translation method using Null-Lagrangians [21, 24, 1, 5, 6, 14, 16, 18, 11, 13] in different cases. In the work [12] the authors use special type of extremals to obtain bounds on the volume fraction of two materials in three dimensions, which is not possible to do using only Null-Lagrangians in the translation method. Here, Null-Lagrangians are the linear combinations of second order minors of the argument matrix \( \xi \) of the function \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \).

Our study of extremal quasiconvex quadratic forms started in [8], where an explicit example of such a form was given for the first time in the literature, and it was also shown that there is no such candidates in the class of quadratic forms with cubic symmetry other than the trivial Null-Lagrangians. Of course, in the mentioned problem, a quadratic form \( Q(\xi), \xi \in \mathbb{R}^{N \times n} \), is considered modulo Null-Lagrangians, as the rank-one-convexity condition of a quadratic form has been shown by Van Hove [26, 27] to be equivalent to the condition

\[
Q(x \otimes y) \geq 0, \quad \text{for all} \quad x \in \mathbb{R}^N, \ y \in \mathbb{R}^n,
\]

where any Null-Lagrangian vanishes and does not play a role. Considering quadratic forms with linear elastic orthotropic symmetry, our next work [9] reveals a link between such extremal forms and extremal polynomials, suggesting that the question of extremality of a quadratic form \( Q(\xi) \) is closely related with the extremality of the determinant of the so called acoustic matrix (or the \( y \)-matrix) of the biquadratic form \( Q(x \otimes y) \), which is the matrix \( T(y) \), where \( Q(x \otimes y) = x T(y) x^T \). Recall that in classical convex analysis a homogeneous nonnegative polynomial \( P \) is called an extremal, if it loses the non-negativity property when another nonnegative homogeneous polynomial of the same degree and other than a scalar multiple of \( P \) is subtracted from it. The later work [10] goes deeper and proves that the condition found in [9] applies to any quadratic forms (not necessarily with
orthotropic symmetry), see [10, Theorems 2.4–2.7]. For the convenience of the reader we recall/formulate Theorem 2.5 of [10].

**Theorem 1.1.** Assume $Q(\xi): \mathbb{R}^{3 \times 3} \to \mathbb{R}$ is a quasiconvex quadratic form such that $\det(T(y))$ is an extremal polynomial that is not a perfect square, where $T(x \otimes y) = xT(y)x^T$. Then $Q(\xi)$ is an extremal (in the sense of Milton, mentioned above).

Theorems 2.5 and 2.6 in [10] study the case when $\det(T(y))$ is a perfect square, including the case when it is identically zero, proving that then the form $Q$ must be an extremal or a polyconvex form, or a sum of those with some additional specific properties. In the present work, in contrast to the works [9,10], we will not impose any additional constraint on $\det(T(y))$ other than extremality. We will solely work with quadratic forms with orthotropic symmetry due to the interest in elasticity, however our analysis applies to other classes of quasiconvex quadratic forms too as it will become clear later in Section 4. In the meantime we will study a deeper problem here, namely not the extremals in the sense of Milton, that lose the quasiconvexity property when a convex form is subtracted from them, but the extreme points of the convex cone of the appropriate class of quasiconvex quadratic forms in the classical convex analysis sense. We will prove that the same link found in [9] between quadratic forms and their $y$–matrix determinant is still present, see Theorem 3.1. Let us also mention that whether the classical extreme points and the extremals in the sense of Milton are or are not the same is currently not known. Also, another related problem concerning sixth order homogeneous polynomials in three variables and determinants of $y$–matrices of quadratic forms, namely whether or not any such polynomial, in particular the well known Robinson’s polynomial is a determinant of a $y$–matrix is open as well [22,25]. In [3] the authors construct the first examples of nonnegative biquadratic forms with a tensor in $(\mathbb{R}^3)^4$, that have maximal number of nontrivial zeroes, namely ten of them. Note that the later can be regarded as quasiconvex quadratic forms that we believe are likely to be extreme points of the appropriate cone as their $y$–matrix determinants are scalar multiples of the generalized Robinson’s polynomial [3,22]; this is a subject for future study. In conclusion we mention that while in the papers [22,25] and in the references therein the approach to the problem of extremals and extreme points is algebraic and algebraic geometric, we regard it in the context of applied mathematics and adopt a somewhat simple and elementary approach to it.

## 2 Quadratic forms with orthotropic symmetry

In this section we recall the definition of orthotropic materials, i.e., quadratic forms that have orthotropic symmetry in linear elasticity. A homogeneous orthotropic elastic material has three mutually orthogonal planes such that the material properties are symmetric under reflection about each plane. If cartesian coordinate axes are chosen orthogonal to these planes, then the properties are invariant under the transformations $x_a \to -x_a$, $x_b \to x_b$, and $x_c \to x_c$, where $abc$ is a permutation of 123. Recall that in linear elasticity the stress tensor

\footnote{It is not difficult to prove that the square of a third order homogeneous polynomial in three variables is an extremal polynomial.}
\( \sigma \) is related to the strain vector \( \epsilon \) linearly via a forth order constant tensor (the elasticity tensor) \( C \):

\[
\sigma = C \cdot \epsilon. 
\] (2.1)

Due to the symmetry of the strain tensor, the relation (2.1) is then reduced to a similar one with \( C \) being a six by six matrix and \( \sigma \) and \( \epsilon \) being six-vectors (Voigt notation). In the case of orthotropic materials, the elements of the elasticity tensor such as \( C_{abcc} \) and \( C_{abb} \), change sign under a reflection about a symmetry plane mentioned above, thus those must be zero. Thus the elements \( C_{ijkl} \) of the elasticity tensor must be zero unless the indices \( ijk\ell \) contain an even number of repetitions of the indices 1, 2 or 3. The Voigt notation takes the form

\[
\sigma = C \cdot \epsilon, \quad \text{where} \quad \sigma = \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix}, \quad \epsilon = \begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{23} \\
2\epsilon_{31} \\
2\epsilon_{12}
\end{bmatrix}, \quad C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix},
\] (2.2)

The related quadratic form then has the form

\[
Q(\xi) = \sum_{i,j=1}^{3} C_{ij} \xi_{ii} \xi_{jj} + 4C_{44} \xi_{23}^2 + 4C_{55} \xi_{31}^2 + C_{66} \xi_{12}^2 
\] (2.3)

\[
= \sum_{i,j=1}^{3} C_{ij} \xi_{ii} \xi_{jj} + C_{44}(\xi_{23} + \xi_{32})^2 + C_{55}(\xi_{31} + \xi_{13})^2 + C_{66}(\xi_{12} + \xi_{21})^2.
\]

The mechanical properties are in general different along each axis. Orthotropic materials require 9 elastic constants and have as subclasses isotropic materials (with 2 elastic constants), cubic materials (with 3 elastic constants), and transversely isotropic materials (with 5 elastic constants). The wood in a tree trunk is an example of a material which is locally orthotropic: the material properties in three perpendicular directions, axial, radial, and circumferential, are different. Many crystals and rolled metals are also examples of orthotropic materials.

3 Main Results

the below theorem is the main result of the manuscript.

**Theorem 3.1.** Denote the convex cone of \( 3 \times 3 \) quasiconvex quadratic forms with linear elastic orthotropic symmetry by \( \mathcal{C} \). Assume \( Q(\xi) \in \mathcal{C} \), where \( Q \) having the form shown in (2.3):

\[
Q(\xi) = \sum_{i,j=1}^{n} C_{ij} \xi_{ii} \xi_{jj} + C_{44}(\xi_{23} + \xi_{32})^2 + C_{55}(\xi_{31} + \xi_{13})^2 + C_{66}(\xi_{12} + \xi_{21})^2
\] (3.1)
satisfies the strict inequalities

\[ C_{ii} > 0, \quad \text{for} \quad i = 1, 2, \ldots, 6. \] (3.2)

If the determinant of the acoustic tensor of \( Q \) is an extremal polynomial, then \( Q \) is an extreme point of \( C \).

**Remark 3.2.** It has been shown in [9, Theorem 5.1], that under the condition \( C_{11}, C_{22}, C_{33} > 0 \) and that the determinant of the acoustic tensor of \( Q \) is an extremal polynomial that is not a perfect square, then \( Q \) is an extremal in the sense of Milton, i.e., it loses the quasiconvexity property when a rank one form \(^3\) is subtracted from it. We claim that given the additional conditions \( C_{44}, C_{55}, C_{66} > 0 \), i.e., the condition \((3.2)\), the mentioned determinant automatically can not be a perfect square. Indeed, the form of \( \det(T(y)) \) in \((4.13)\) suggests that if it is a perfect square, then it has the form \( (a_1y_1^3 + a_2y_2^3 + a_3y_3^3 + P(y))^2 \), where \( P(y) \) is a homogeneous polynomial of degree 3 in the variable \( y \), free of any of the monomials \( y_1^3, y_2^3, \) and \( y_3^3 \). If then a monomial \( y_i^2y_j \) with \( i \neq j \) occurs in \( P(y) \), then the monomial \( y_i^2y_j \) will occur in the determinant, which is not the case due to \((4.13)\). Thus \( P(y) \) may only involve \( y_1y_2y_3 \), which results in the determinant involving \( y_1^2y_2y_3 \), which is again not possible. This being said, due to [9, Theorem 5.1], we automatically obtain that under the assumptions of Theorem 3.1, the quadratic form \( Q(\xi) \) is also an extremal in the sense of Milton.

4 Proof of the main result

4.1 A lemma from linear algebra

Here we prove a lemma from linear algebra, that will be a key factor in the proof of the main results. Although the result of the lemma will be needed only in 3D, we will formulate and prove it for a general \( n \in \mathbb{N} \) as it may be of separate interest.

**Lemma 4.1.** Let \( n \in \mathbb{N} \) be such that \( n \geq 2 \). Assume \( A, B \in M_{\text{sym}}^{n\times n} \) be symmetric positive semi-definite matrices such that \( A \geq B \) in the sense of quadratic forms. Then for any integers \( 1 \leq k < m \leq n \) one has the inequality

\[
\frac{1}{\binom{n}{m}} \sum_{M_m(B)} M_m(B) \text{cof}_A(M_m(B)) \leq \frac{1}{\binom{n}{k}} \sum_{M_k(B)} M_k(B) \text{cof}_A(M_k(B)),
\] (4.1)

where the number \( \binom{n}{m} \) is the binomial coefficient, and the sum \( \sum_{M_m(B)} \) is taken over all \( m \)-th order minors \( M_m(B) \) of \( B \), and \( \text{cof}_A(M_m(B)) \) denotes the cofactor of the minor in the matrix \( A \), obtained choosing the same rows and columns as to get the minor \( M_m(B) \) in \( B \).

**Proof.** We may assume without loss of generality that \( A \) and \( B \) are positive definite as we can prove the statement for the matrices \( A + \epsilon I \) and \( B + \epsilon I \), where the parameter \( \epsilon \) is

\(^2\) Note that due to the quasiconvexity of \( Q \), the bounds \( a_{ii} \geq 0 \) for \( i = 1, 2, \ldots, 6 \) must be fulfilled.
\(^3\) A rank-one form is the square of a linear form.
positive and then send $\epsilon$ to zero and recover the estimate (4.1) for $A$ and $B$. Consider next the polynomial

$$P(t) = \det(A - tB), \quad t \in \mathbb{R}. \quad (4.2)$$

Denoting

$$S_m = \sum_{M_m(B)} M_m(B) \text{cof}_A(M_m(B)), \quad m = 0, 1, 2, \ldots, n$$

we clearly have that

$$P(t) = \sum_{m=0}^{n} (-1)^m S_m t^m, \quad t \in \mathbb{R}. \quad (4.3)$$

We have on one hand due to the fact $\det(B) > 0$, that

$$P(t) = \det(A - tB) = \det(B) \det(B^{-1}A - tI),$$

thus the roots of $P$ are real as $P$ is a scalar multiple of the characteristic polynomial of the symmetric matrix $B^{-1}A$. On the other hand the inequalities $A \geq B$ and $\det(B) > 0$ imply that $P(t) > 0$ for $t \in (0,1)$. Consequently we obtain that the roots of $P$ are real and belong to the interval $[1, \infty)$. Denoting them by $t_1, t_2, \ldots, t_n$ we have

$$P(t) = (-1)^n \det(B)(t - t_1)(t - t_2) \cdots (t - t_n),$$

which together with (4.3) and Vieta’s theorem gives the formulae

$$S_m = \det(B) \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} t_{i_1} t_{i_2} \cdots t_{i_m}. \quad (4.4)$$

Finally, (4.1) follows from (4.4) and from the fact that $t_i \geq 1$, for all $i = 1, 2, \ldots, n$. \hfill $\square$

### 4.2 Proof of Theorem 3.1

**Proof.** Assume in contradiction that the assertion of Theorem 3.1 fails to hold, thus there exists a form $Q_1 \in \mathcal{C}$ such that $Q_1 \neq \alpha Q$ for any $\alpha \in [0,1]$ such that

$$0 \leq Q_1(x \otimes y) \leq Q(x \otimes y), \quad \text{for all} \quad x, y \in \mathbb{R}^3. \quad (4.5)$$

It is clear that the form $Q$ is equivalent modulo Null-Lagrangians to the quadratic form

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j + b(x_i^2 + x_j^2) + c(x_{12}^2 + x_{21}^2) + d(x_{13}^2 + x_{31}^2) + e(x_{23}^2 + x_{32}^2), \quad (4.6)$$

where we have $a_{ii} = C_{ii} > 0$ for $i = 1, 2, 3$ and $b = C_{66} > 0, c = C_{55} > 0$ and $d = C_{44} > 0$. Therefore we will assume that $Q$ has the form in (4.6) in the sequel. Assume $Q(x \otimes y) = x^T(y) x^T$ and $Q_1(x \otimes y) = x^T(y) x^T$, where
\[ Q_1(\xi) = \sum_{i,j=1}^{n} a_{ij}^1 \xi_{ii} \xi_{jj} + b^1 (\xi_{12}^2 + \xi_{31}^2) + c^1 (\xi_{13}^2 + \xi_{31}^2) + d^1 (\xi_{23}^2 + \xi_{32}^2). \] (4.7)

The following polynomial in the variable \( \lambda \) is going to be a key factor of the proof:

\[
P(\lambda) = \det(T(y) - \lambda T^1(y))
= \det(T(y)) - \lambda \sum_{i,j=1}^{3} t_{ij}(y)(\text{cof}(T(y)))_{ij} + \lambda^2 \sum_{i,j=1}^{3} t_{ij}(y)(\text{cof}(T^1(y)))_{ij} - \lambda^3 \det(T^1(y)).
\] (4.8)

It is clear that due to Lemma 4.1 we have the estimates

\[
0 \leq 3 \det(T_1(y)) \leq \sum_{i,j=1}^{3} t_{ij}(y)(\text{cof}(T^1(y)))_{ij} \leq \sum_{i,j=1}^{3} t_{ij}^1(y)(\text{cof}(T^1(y)))_{ij} \leq 3 \det(T(y)),
\] (4.9)

for all \( y \in \mathbb{R}^3 \).

Therefore all the polynomials \( 3 \det(T^1(y)) \), \( \sum_{i,j=1}^{3} t_{ij}(y)(\text{cof}(T^1(y)))_{ij} \) and \( \sum_{i,j=1}^{3} t_{ij}(y)(\text{cof}(T(y)))_{ij} \), being between zero and the extremal polynomial \( 3 \det(T(y)) \), must be a scalar multiple of \( \det(T(y)) \), i.e., we have

\[
\det(T^1(y)) = \alpha \det(T(y)),
\] (4.10)

\[
\sum_{i,j=1}^{3} t_{ij}(y)(\text{cof}(T^1(y)))_{ij} = \beta \det(T(y)),
\]

\[
\sum_{i,j=1}^{3} t_{ij}^1(y)(\text{cof}(T(y)))_{ij} = \gamma \det(T(y)),
\]

for some \( \alpha, \beta, \gamma \geq 0 \).

Consequently we get from (4.8) and (4.10) the identity

\[
det(T(y) - \lambda T^1(y)) = (1 - \gamma \lambda + \beta \lambda^2 - \alpha \lambda^3) \det(T(y)), \quad \text{for all } y \in \mathbb{R}^3, \lambda \in \mathbb{R}. \] (4.11)

Owing to the form (4.6) of the quadratic for \( Q(\xi) \), we have that

\[
T(y) = \begin{bmatrix}
a_{11}y_1^2 + by_2^2 + cy_3^2 & a_{12}y_1y_2 & a_{13}y_1y_3 \\
a_{12}y_1y_2 & a_{22}y_2^2 + by_1^2 + dy_3^2 & a_{23}y_2y_3 \\
a_{13}y_1y_3 & a_{23}y_2y_3 & a_{33}y_3^2 + cy_1^2 + dy_2^2
\end{bmatrix}.
\] (4.12)
Next we have by direct calculation (or using maple), that
\[
\det(T(y)) = (a_{11}bc)y_1^6 + (a_{22}bd)y_2^6 + (a_{33}cd)y_3^6
\]
\[
+ (a_{11}bd + a_{11}a_{22}c + b^2c - a_{12}^2c)y_1^2y_2^2
\]
\[
+ (a_{22}bc + a_{11}a_{22}d + b^2d - a_{12}^2d)y_2^2y_1^2
\]
\[
+ (a_{11}cd + a_{11}a_{33}b + c^2b - a_{13}^2b)y_1^2y_3^2
\]
\[
+ (a_{33}bc + a_{11}a_{33}d + c^2d - a_{13}^2d)y_3^2y_1^2
\]
\[
+ (a_{22}cd + a_{22}a_{33}b + d^2b - a_{23}^2b)y_2^2y_3^2
\]
\[
+ (a_{33}bd + a_{22}a_{33}c + d^2c - a_{23}^2c)y_3^2y_2^2
\]
\[
+ (a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 + a_{11}d^2 + a_{22}c^2 + a_{33}b^2 + 2bcd)y_1^2y_2^2y_3^2.
\]

Due to the strict inequalities (3.2), we can denote \(\frac{a_{ij}}{a_{ij}} = q_{ij}, i, j = 1, 2, 3, \) and \(\frac{b_i}{b} = q_b, \) \(c_i = q_c, \) \(d_i = q_d.\) The identity (4.11) tells us that the quotient of the coefficients of any monomials \(y_1^{s_1}y_2^{s_2}y_3^{s_3}\) and \(y_1^{s_1'}y_2^{s_2'}y_3^{s_3'}\) in \(\det(T(y) - \lambda T^1(y))\) is exactly that of in \(\det(T(y))\), thus we consider the coefficients of \(y_1^6, y_2^6\) and \(y_3^6\) we get the following set of identities:

\[
(1 - q_{11}\lambda)(1 - q_c\lambda) = (1 - q_{22}\lambda)(1 - q_d\lambda),
\]
\[
(1 - q_{11}\lambda)(1 - q_b\lambda) = (1 - q_{33}\lambda)(1 - q_d\lambda),
\]
\[
(1 - q_{22}\lambda)(1 - q_b\lambda) = (1 - q_{33}\lambda)(1 - q_c\lambda),
\]
for all \(\lambda \in \mathbb{R}.\)

The conditions in (4.14) imply that the roots of the polynomials on the right and the left are the same, thus we get set equalities

\[
\{q_{11}, q_c\} = \{q_{22}, q_d\}, \quad \{q_{11}, q_b\} = \{q_{33}, q_d\}, \quad \{q_{22}, q_b\} = \{q_{33}, q_c\}.
\]

The are three cases possible.

**Case1.** \(q_{11} = q_{22} = q_{33} = q = s,\) and \(q_b = q_c = q_d = t.\)

**Case2.** \(q_{11} = q_{22} = q_c = q_d = s,\) and \(q_{33} = q_b = t.\)

**Case3.** \(q_{11} = q_d = s, q_{22} = q_c = t,\) and \(q_{33} = q_b = u.\)

The goal is to prove that \(s = t\) in the first two cases and \(s = t = u\) in the third case.

**Case1.** The coefficient of \(y_1^6\) in \(\det(T(y) - \lambda T^1(y))\) is divisible by \((1 - s\lambda)\) and the only summand in the coefficient of \(y_1^2y_2^2y_3^2\) that does not have the multiplier \((1 - s\lambda)\) is \(2bcd(1-t\lambda)^3,\) thus we clearly get \(s = t.\)

**Case2.** In this case the coefficient of \(y_1^6\) in \(\det(T(y) - \lambda T^1(y))\) is a scalar multiple of \((1 - s\lambda)^2(1 - t\lambda),\) thus also the coefficient of \(y_2^2y_3^2\) in \(\det(T(y) - \Lambda T^1(y))\) has to have the same property. The first three summands in the coefficient of \(y_2^2y_3^2\) in \(\det(T(y) - \lambda T^1(y))\) have the factor \(1 - s\lambda,\) thus if we assume that \(s \neq t,\) then the polynomial \((1 - q_{23}\lambda)^2\) must have the factor \(1 - s\lambda,\) which gives \(q_{23} = s.\) Consequently all the three summands in the coefficient of \(y_2^2y_3^2\) in \(\det(T(y) - \lambda T^1(y))\) are scalar multiples of \((1 - s\lambda)^2(1 - t\lambda)\) except the second one, which is a scalar multiple of \((1 - s\lambda)(1 - t\lambda)^2,\) which again yields \(s = t.\)
Case 3. In this case the coefficient of $y_4^0$ in \( \det(T(y) - \lambda T^1(y)) \) is a scalar multiple of \((1-s\lambda)(1-t\lambda)(1-u\lambda)\). Considering the coefficient of $y_3^1 y_1^2$ in $\det(T(y) - \lambda T^1(y))$, we have that the summands in it except one equal to a scalar multiple of \((1-t\lambda)(1-u\lambda)^2\), thus we obtain that either $t = s$ or $u = s$. It is clear that both cases reduce to Case 2, thus we finally end up with $s = t = u$.

Having now that $q_{11} = q_{22} = q_{33} = q_6 = q_9 = q_4 = s$, we can prove in the analogy of Case 2 that $q_{12} = q_{13} = q_{23} = s$, which yields the conclusion $T^1(y) = s \cdot T(y)$, or equivalently $Q_1(\xi) = s \cdot Q(\xi)$. The proof is complete now.

Remark 4.2. The tools of the proof of Theorem 3.1 are quite robust and it can be easily checked that quadratic forms having for instance the form

$$Q(\xi) = \sum_{i,j=1}^{n} a_{ij} \xi_{ii} \xi_{jj} + b \xi_{12}^2 + c \xi_{23}^2 + d \xi_{31}^2$$

(4.16)

can also be treated by the same technique. It has been proven in [3], that there is an extreme point of form (4.16), namely the quadratic form

$$F(\xi) = \xi_{11}^2 + \xi_{22}^2 + \xi_{33}^2 + \xi_{12}^2 + \xi_{23}^2 + \xi_{31}^2 - 2(\xi_{11} \xi_{22} + \xi_{22} \xi_{33} + \xi_{33} \xi_{11}).$$

(4.17)

which is not only an extreme point of the cone of quadratic forms that have the same form (4.16), but also it is an extreme point of the cone of all quasiconvex quadratic forms. Example (4.17) is the first one as such in the literature. Also, quadratics forms having the more general form

$$Q(\xi) \sum_{i,j=1}^{n} a_{ij} \xi_{ii} \xi_{jj} + b_1 \xi_{12}^2 + c_1 \xi_{23}^2 + d_1 \xi_{31}^2 + b_2 \xi_{21}^2 + c_2 \xi_{32}^2 + d_2 \xi_{13}^2$$

(4.18)

that involves orthotropic ones, forms looks possible to fit into the above analysis, but due to the bigger number of parameters involved, and thus possible cases to consider, we prefer not to present a detailed analysis here.

References

[1] G. Allaire and R.V. Kohn. Optimal lower bounds on the elastic energy of a composite made from two non-well-ordered isotropic materials, Quarterly of applied mathematics, vol. LII, 331-333 (1994)

[2] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal, 63, 337-403 (1977).

[3] A. Buckley and K. Šivic. Nonnegative biquadratic forms with maximal number of zeros, preprint, https://arxiv.org/pdf/1611.09513.pdf

[4] A. Cherkaev. Variational Methods for Structural Optimization, Springer Applied Mathematical Sciences, Vol. 140 (2000).
[5] A. V. Cherkaev and L. V. Gibiansky. The exact coupled bounds for effective tensors of electrical and magnetic properties of two-component two-dimensional composites, *Proceedings of the Royal Society of Edinburgh. Section A, Mathematical and Physical Sciences*, 122 (1992), pp. 93–125.

[6] A. V. Cherkaev and L. V. Gibiansky. Coupled estimates for the bulk and shear moduli of a two-dimensional isotropic elastic composite, *Journal of the Mechanics and Physics of Solids*, 41 (1993), pp. 937–980.

[7] B. Dacorogna. *Direct methods in the calculus of variations*. Springer Applied Mathematical Sciences, Vol. 78, 2nd Edition (2008).

[8] D. Harutyunyan and G. W. Milton. Explicit examples of extremal quasiconvex quadratic forms that are not polyconvex, *Calculus of Variations and Partial Differential Equations*, October 2015, Vol. 54, Iss. 2, pp. 1575-1589.

[9] D. Harutyunyan and G. W. Milton. On the relation between extremal elasticity tensors with orthotropic symmetry and extremal polynomials, *Archive for Rational Mechanics and Analysis*, Vol. 223, Iss. 1, pp 199-212, 2017

[10] D. Harutyunyan and G. W. Milton. Towards characterization of all $3 \times 3$ extremal quasiconvex quadratic forms, *Communications of Pure and Applied Mathematics*, Vol. 70, Iss. 11, Nov. 2017, pp. 2164-2190.

[11] H. Kang, E. Kim, and G. W. Milton. Sharp bounds on the volume fractions of two materials in a two-dimensional body from electrical boundary measurements: the translation method, *Calculus of Variations and Partial Differential Equations*, 45, 367-401 (2012).

[12] H. Kang and G. W. Milton. Bounds on the volume fractions of two materials in a three dimensional body from boundary measurements by the translation method, *SIAM Journal on Applied Mathematics*, 73, 475–492 (2013).

[13] H. Kang, G. W. Milton, and J.-N. Wang. Bounds on the Volume Fraction of the Two-Phase Shallow Shell Using One Measurement, *Journal of Elasticity*, 114, 41-53 (2014).

[14] R. V. Kohn and R. Lipton. Optimal bounds for the effective energy of a mixture of isotropic, incompressible, elastic materials, *Archive for Rational Mechanics and Analysis*, 102, 331–350 (1988).

[15] G. W. Milton. On characterizing the set of positive effective tensors of composites: The variational method and the translation method, *Communications on Pure and Applied Mathematics*, Vol. XLIII, 63-125 (1990).

[16] G. W. Milton. *The Theory of Composites*, vol. 6 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, United Kingdom, 2002
[17] G. W. Milton. Sharp inequalities which generalize the divergence theorem: an extension of the notion of quasi-convexity, *Proceedings Royal Society A* 469, 20130075 (2013).

[18] G. W. Milton and L. H. Nguyen. Bounds on the volume fraction of 2-phase, 2-dimensional elastic bodies and on (stress, strain) pairs in composites. *Comptes Rendus Mécanique*, 340, 193-204 (2012).

[19] C. B. Morrey. Quasiconvexity and the lower semicontinuity of multiple integrals, *Pacific Journal of Mathematics* 2, 25-53 (1952)

[20] C. B. Morrey. *Multiple integrals in the calculus of variations*, Springer–Verlag, Berlin, 1966.

[21] F. Murat and L. Tartar. Calcul des variations et homogénéisation. (French) [Calculus of variation and homogenization], in *Les méthodes de l’homogénéisation: théorie et applications en physique*, volume 57 of Collection de la Direction des études et recherches d’Electricité de France, pages 319-369, Paris, 1985, Eyrolles, English translation in *Topics in the Mathematical Modelling of Composite Materials*, pages 139-173, ed. by A. Cherkaev and R. Kohn, ISBN 0-8176-3662-5.

[22] B. Reznick. On Hilbert’s construction of positive polynomials, *preprint*, available at: https://arxiv.org/pdf/0707.2156v1.pdf.

[23] D. Serre. Condition de Legendre-Hadamard: Espaces de matrices de rang≠ 1. (French) [Legendre-Hadamard condition: Space of matrices of rank≠ 1], *Comptes Rendus de l’Académie des sciences*, Paris 293, 23-26 (1981).

[24] L. Tartar. Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics*, Heriot-Watt Symposium, Volume IV, edited by R. J. Knops, volume 39 of Research Notes in Mathematics, pages 136-212, London, 1979, Pitman Publishing Ltd.

[25] R. Quarez. On the real zeros of positive semidefinite biquadratic forms, *Comm. Algebra*, 43 (2015), 1317–1353.

[26] L. Van Hove. Sur l’extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusieurs functions inconnues, *Nederl. Akad. Wetensch. Proc.* 50 (1947), 18-23.

[27] L. Van Hove. Sur le signe de la variation seconde des intégrales multiples à plusieurs functions inconnues, *Acad. Roy. Belgique Cl. Sci. Mém. Coll.* 24 (1949), 68.