A GROTHENDIECK-LEFSCHETZ THEOREM FOR EQUIVARIANT PICARD GROUPS

CHARANYA RAVI

Abstract. We prove a Grothendieck-Lefschetz theorem for equivariant Picard groups of non-singular varieties with finite group actions.

1. Introduction

The geometry and $K$-theory of schemes with group scheme actions have been extensively studied by various authors in recent years (e.g., see [7], [9], [8]). The generalization of some of the fundamental theorems of algebraic geometry to the equivariant setting has played an important role in the development of this subject. The classical Lefschetz-type theorems compare the various algebraic invariants of non-singular projective varieties and their hyperplane sections. Let $X$ be a non-singular projective variety over a field $k$ of characteristic zero and let $Y$ be a non-singular subvariety of $X$, of dimension $\geq 3$, which is a scheme-theoretic complete intersection in $X$. The Grothendieck-Lefschetz theorem for Picard groups (see [4, Théorème XI.3.1], [6, Corollary IV.3.3]) states that the Picard groups of $X$ and $Y$ are isomorphic. The purpose of this article is to prove an analogous result for varieties with finite group actions.

For a variety $X$ with $G$-action, let $\text{Pic}^G(X)$ denote the equivariant Picard group of $X$ (see [10, 1.3, page 32]). Our main result is the following.

Theorem 1.1. Let $k$ be a field of characteristic zero and let $G$ be a finite group. Let $X$ be a non-singular projective variety over $k$ with $G$-action and let $Y$ be a non-singular $G$-invariant subvariety of dimension $\geq 3$, which is a scheme-theoretic complete intersection in $X$. Then the natural map $\text{Pic}^G(X) \to \text{Pic}^G(Y)$ is an isomorphism.

In view of the Kodaira-Akizuki-Nakano vanishing theorem, Theorem 1.1 is a straightforward consequence of the technical result Theorem 3.3, which is proved by closely following the proof of the Grothendieck-Lefschetz theorems given in [6, Chapter IV]. The main idea is to use the formal completion of $X$ along $Y$ and a suitable equivariant generalization of the Lefschetz conditions, which is discussed in (2.2). As a corollary to Theorem 3.3, we also deduce that if $G$ acts on a projective space $X$ over $k$ (a field of arbitrary characteristic) and $Y$ is a $G$-invariant scheme-theoretic complete intersection in $X$ such that $\dim(Y) \geq 3$, then the equivariant Picard groups of $X$ and $Y$ are isomorphic (see Corollary 3.4, [6, Corollary IV.3.2] for the non-equivariant case).

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2. Preliminaries

We will work over a base field $k$ of arbitrary characteristic. All schemes are assumed to be separated and of finite type over $k$. The term variety will refer to an integral scheme over $k$. In this note, $G$ will always denote a finite group.

2.1. Group action on formal schemes. In this section, we recall briefly the notion of $G$-action on a locally ringed space and equivariant sheaves. In the process we set up notations and terminologies for the rest of the paper.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. A $G$-action on $(X, \mathcal{O}_X)$ is a group homomorphism from $G$ to the group of automorphisms of $(X, \mathcal{O}_X)$. A morphism $\theta : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of locally ringed spaces with $G$-actions is said to be $G$-equivariant if it is compatible with the $G$-actions on $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$.

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space with a given $G$-action.

1. A $G$-sheaf of abelian groups on $X$ is a sheaf of abelian groups $\mathcal{F}$ together with a collection of sheaf isomorphisms $\phi_g : \mathcal{F} \cong g_*\mathcal{F}$, for each $g \in G$, which are subject to the conditions $\phi_e = id$ and $\phi_g = h_*(\phi_{g'}) \circ \phi_h$, for all $h \in G$. We shall denote a $G$-sheaf in the sequel by $(\mathcal{F}, \{\phi_g\})$.

2. A $G$-module is a $G$-sheaf $(\mathcal{F}, \{\phi_g\})$ such that $\mathcal{F}$ is an $\mathcal{O}_X$-module and each $\phi_g$ is an $\mathcal{O}_X$-module isomorphism. A locally free (resp. invertible) $G$-sheaf is a $G$-module $(\mathcal{F}, \{\phi_g\})$, where $\mathcal{F}$ is a locally free (resp. invertible) sheaf of $\mathcal{O}_X$-module.

3. A $G$-equivariant morphism of $G$-sheaves $f : (\mathcal{F}, \{\phi_g\}) \to (\mathcal{G}, \{\phi_g\})$ is a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ such that $\phi'_{g'} \circ f = g_*(f) \circ \phi_g$, for all $g \in G$. The set of $G$-equivariant morphisms from $\mathcal{F}$ to $\mathcal{G}$ is denoted by $\text{Hom}_G(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ and $\mathcal{G}$ are $G$-modules, the set of $G$-equivariant $\mathcal{O}_X$-module homomorphisms is denoted by $\text{Hom}_G(\mathcal{F}, \mathcal{G})$.

Example 2.2. When $X$ is a $k$-scheme, a $G$-action on the locally ringed space $X$ defined as above coincides with the usual notion of group scheme action on schemes, where $G$ is viewed as a finite constant group scheme over $k$. Let $\sigma : G \times X \to X$ denote the action map. It is easy to verify that a $G$-module structure on a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules as above is equivalent to giving an isomorphism of $\mathcal{O}_{G \times X}$-modules, $\phi : \sigma^*\mathcal{F} \to p_1^*\mathcal{F}$, over $G \times X$. Therefore $\mathcal{F}$ is a $G$-module in the sense of [10].

Example 2.3. Let $X$ be a noetherian scheme with $G$-action and let $Y$ be a $G$-invariant closed subscheme, defined by a sheaf of ideals $\mathcal{I}$ (which is a $G$-submodule of $\mathcal{O}_X$). Then $(\hat{X}, \mathcal{O}_{\hat{X}})$, the formal completion of $X$ along $Y$, has
an induced $G$-action, as the direct image functor commutes with inverse limits. The canonical morphism $i : \hat{X} \to X$ is then $G$-equivariant. Given a $G$-equivariant coherent $\mathcal{O}_X$-module $F$, the completion $\hat{F}$ of $F$ along $Y$, has a natural $G$-equivariant $\mathcal{O}_{\hat{X}}$-module structure. Furthermore, the functor $F \mapsto \hat{F}$ from the category of coherent $\mathcal{O}_X$-modules to the category of coherent $\mathcal{O}_{\hat{X}}$-modules is exact (see [5, Corollary II.9.8]) and therefore it is easy to see that it induces an exact functor on the category of coherent $G$-modules.

Let $(X, \mathcal{O}_X)$ be a locally ringed space with $G$-action. Let $\text{Sh}^G(X)$ denote the category of $G$-sheaves, which is an abelian category with enough injectives. Given a $G$-sheaf $\mathcal{F}$ on $X$, the group $G$ acts on the global sections $\Gamma(X, \mathcal{F})$. Let $\Gamma(X, \mathcal{F})^G$ denote the $G$-invariant global sections, and let $H^p(G; X, -)$ denote the right derived functors of the functor $\Gamma(X, -)$.

**Lemma 2.4.** Let $(X, \mathcal{O}_X)$ be a locally ringed space with $G$-action and let $(\mathcal{F}, \{\phi_g\}), (\mathcal{G}, \{\phi'_g\})$ be $G$-modules. The sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has an induced $G$-module structure such that $\text{Hom}^G_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^G$.

**Proof.** For each $g \in G$, let $\rho_g : \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \text{Hom}_{\mathcal{O}_X}(g_*\mathcal{F}, g_*\mathcal{G})$ be the $\mathcal{O}_X$-module homomorphism defined as follows. Given an open subset $U$ of $X$ and $f \in \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, let $\rho_g|_U(f) := (\phi'_{g^{-1}}|_U) \circ f \circ (\phi_g^{-1}|_U)$. Let $\bar{\rho}_g = \theta_g \circ \rho_g$, where $\theta_g : \text{Hom}_{\mathcal{O}_X}(g_*\mathcal{F}, g_*\mathcal{G}) \to g_*\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are the canonical isomorphisms. Then $(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}), \{\rho_g\})$ is a $G$-module. Now,

\[
f \in \text{Hom}^G_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \iff \rho_g(X)(f) = \phi'_g \circ f \circ (\phi_g^{-1}) = g_*(f), \forall g \in G
\]

\[
\iff \bar{\rho}_g(X)(f) = f, \forall g \in G
\]

\[
\iff f \in \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^G.
\]

**Remark 2.5.** If $\mathcal{F}$ and $\mathcal{G}$ are $G$-sheaves then one can show using the same argument as above that $\text{Hom}(\mathcal{F}, \mathcal{G})$ is a $G$-sheaf and $\text{Hom}^G(\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{G})^G$.

**Corollary 2.6.** Let $(X, \mathcal{O}_X)$ be a locally ringed space with $G$-action and let $\mathcal{F}$ be an invertible $G$-sheaf on $X$. There is a $G$-equivariant isomorphism $\mathcal{O}_X \cong \mathcal{F}$, where $\mathcal{O}_X$ has the canonical $G$-action, if and only if $\Gamma(X, \mathcal{F})^G$ has a nowhere vanishing section.

**Proof.** The proof follows from Lemma 2.4, since $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ as $G$-sets and isomorphisms $\mathcal{O}_X \to \mathcal{F}$ correspond to nowhere vanishing sections in $\Gamma(X, \mathcal{F})$. \qed

### 2.2. The equivariant Lefschetz Conditions

In [4, Section X.2], Grothendieck introduced the Lefschetz conditions for pairs $(X, Y)$, inspired by Lefschetz, where $X$ is a locally noetherian scheme and $Y$ is a closed subscheme of $X$. These were essential in the proof of Grothendieck’s theorems comparing the Picard groups and the fundamental groups of a projective variety $X$ with a complete intersection subvariety $Y$. For schemes with action of a finite group
Definition 2.7. Let \( X \) be a noetherian scheme with \( G \)-action, and let \( Y \subseteq X \) be a \( G \)-invariant closed subscheme. Let \( \hat{X} \) be the formal completion of \( X \) along \( Y \). Then \( \hat{X} \) is a locally ringed space with \( G \)-action as discussed in Example 2.3.

(1) The pair \((X, Y)\) satisfies the **equivariant Lefschetz condition**, written \( L^G(X, Y) \), if for every \( G \)-invariant open set \( U \supseteq Y \), and every \( G \)-equivariant locally free sheaf \( E \) on \( U \), there exists a \( G \)-invariant open set \( U' \) with \( Y \subseteq U' \subseteq U \) such that the natural map \( \Gamma(U', E|_{U'})^G \xrightarrow{\sim} \Gamma(\hat{X}, \hat{E})^G \) is an isomorphism.

(2) The pair \((X, Y)\) satisfies the **equivariant effective Lefschetz condition**, written \( eL^G(X, Y) \), if \( L^G(X, Y) \) holds, and in addition, for every \( G \)-equivariant locally free sheaf \( E \) on \( \hat{X} \), there exists a \( G \)-invariant open set \( U \supseteq Y \) and a \( G \)-equivariant locally free sheaf \( E \) on \( U \) such that \( \hat{E} \cong E \) as \( G \)-modules.

With \((X, Y)\) as above, let \( E \) and \( F \) be locally free \( G \)-sheaves defined on \( G \)-invariant open neighbourhoods \( U \) and \( V \) of \( Y \), respectively. We write \( E \sim F \) if there exists a \( G \)-invariant open set \( W \) with \( Y \subseteq W \subseteq U \cap V \) such that \( E|_W \cong F|_W \) as \( G \)-sheaves. We define the category \( LF^0_G(Y) \) of germs of locally free \( G \)-sheaves around \( Y \) as follows. An object of this category is a class of locally free \( G \)-sheaves defined on \( G \)-invariant open neighbourhoods of \( Y \) under the equivalence relation \( \sim \). For any two objects \([E]\) and \([F]\) in \( LF^0_G(Y) \), the set of homomorphisms from \([E]\) to \([F]\) is defined to be the set \( \lim_{\rightarrow U} \text{Hom}_{L^G(U)}(E, F) \), where the colimit is taken over all \( G \)-invariant open neighbourhoods \( U \) of \( Y \) such that both \( E \) and \( F \) are defined over \( U \). Let \( LF_G(\hat{X}) \) denote the category of locally free \( G \)-sheaves on \( \hat{X} \).

Lemma 2.8. Let \( \wedge : LF^0_G(Y) \rightarrow LF_G(\hat{X}) \) be the functor sending \( E \mapsto \hat{E} \).

(1) If \( L^G(X, Y) \) holds, then \( \wedge \) is fully faithful.

(2) If \( eL^G(X, Y) \) holds, then \( \wedge \) is an equivalence of categories.

Proof. Suppose \( L^G(X, Y) \) holds. Let \( E, F \in LF^0_G(Y) \). Without loss of generality, we may assume that \( E, F \) are \( G \)-equivariant locally free \( O_Y \)-modules for some \( G \)-invariant open neighbourhood \( V \) of \( Y \). Let \( U \) be any \( G \)-invariant open neighbourhood of \( Y \) such that \( U \subseteq V \). \( \text{Hom}_{L^G(U)}(E, F) \) is a \( G \)-equivariant locally free \( O_U \)-module, by Lemma 2.4. Since \( L^G(X, Y) \) holds, there exists a \( G \)-invariant open set \( U' \) with \( Y \subseteq U' \subseteq U \) such that the natural map \( \Gamma(U', \text{Hom}_{L^G(U)}(E, F))^G \cong \Gamma(\hat{X}, \text{Hom}_{L^G(U)}(E, F))^G \) is an isomorphism. Since \( \text{Hom}_{L^G(U)}(E, F)^\wedge \cong \text{Hom}_{L^G(\hat{X})}(\hat{E}, \hat{F}) \) as \( G \)-sheaves, \( \Gamma(\hat{X}, \text{Hom}_{L^G(U)}(E, F)^\wedge)^G \cong \Gamma(\hat{X}, \text{Hom}_{L^G(\hat{X})}(\hat{E}, \hat{F}))^G \) is an isomorphism and hence \( \Gamma(U', \text{Hom}_{L^G(U)}(E, F))^G \cong \Gamma(\hat{X}, \text{Hom}_{L^G(\hat{X})}(\hat{E}, \hat{F}))^G \) is an isomorphism. By Lemma 2.4, the above isomorphism can be rewritten as \( \text{Hom}_{L^G(U')}^G(E, F) \cong \text{Hom}_{L^G(\hat{X})}^G(\hat{E}, \hat{F}) \). This proves that the functor \( \wedge \) is fully faithful. If \( eL^G(X, Y) \) holds, \( \wedge \) is further essentially surjective (by definition), and therefore an equivalence of categories. \( \square \)
Proposition 2.9. Let \( X \) be a non-singular projective variety with \( G \)-action. Let \( Y \subseteq X \) be a \( G \)-invariant closed subscheme, which is a scheme-theoretic complete intersection in \( X \). If \( \dim(Y) \geq 2 \), then \( eL^G(X, Y) \) holds.

Proof. Let \( U \supseteq Y \) be any \( G \)-invariant open set, and let \( E \) be a locally free \( G \)-sheaf on \( U \). Since \( Y \) is a complete intersection, by \([6, \text{Corollary IV.1.2}]\) and the proof of \([6, \text{Proposition IV.1.1}]\), the \( G \)-equivariant restriction map \( \Gamma(U, E) \xrightarrow{\sim} \Gamma(\tilde{X}, \tilde{E}) \) is an isomorphism. This induces an isomorphism \( \Gamma(U, E)^G \xrightarrow{\sim} \Gamma(\tilde{X}, \tilde{E})^G \). Therefore \( L^G(X, Y) \) holds.

Let \( \tilde{X} \) be the formal completion of \( X \) along \( Y \), and let \( (E, \{\phi_g\}) \) be a locally free \( G \)-sheaf on \( \tilde{X} \). Since \( Y \) is a scheme-theoretic local complete intersection, by \([6, \text{Theorem IV.1.5}]\), we can find an open set \( U \supseteq Y \) (not necessarily \( G \)-invariant) and a locally free sheaf \( E \) on \( U \) such that \( \theta : \tilde{E} \xrightarrow{\sim} E \) non-equivariantly. We may assume that \( U \) is \( G \)-invariant by replacing \( U \) by the open set \( \bigcap_{g \in G} gU \). For each \( g \in G \), \( g_*E \) is then a locally free sheaf on \( U \) such that we have induced isomorphisms \( g_*E \cong g_*E \), since direct image functor commutes with inverse limits. Since \( E, g_*E \) are locally free sheaves on \( U \), \( \text{Hom}_{\mathcal{O}_U}(E, g_*E) \) is a locally free \( \mathcal{O}_U \)-module. Again as above, we have isomorphisms \( \text{Hom}_{\mathcal{O}_U}(E, g_*E) \cong \text{Hom}_{\mathcal{O}_{\tilde{X}}}(\tilde{E}, g_*\tilde{E}) \) for each \( g \in G \). Therefore, \( \phi_g \in \text{Hom}_{\mathcal{O}_U}(E, g_*E) \) can be uniquely lifted to a morphism \( \tilde{\phi}_g \in \text{Hom}_{\mathcal{O}_{\tilde{X}}}(\tilde{E}, g_*\tilde{E}) \). Since the lifts are unique and \( \{\tilde{\phi}_g\}_{g \in G} \) defines a \( G \)-module structure on \( E \), \( \{\tilde{\phi}_g\}_{g \in G} \) defines a \( G \)-module structure on \( E \). Further \( \theta : \tilde{E} \to E \) is a \( G \)-equivariant morphism, by definition of the \( G \)-action on \( E \). Therefore, \( eL^G(X, Y) \) holds. \( \square \)

3. Equivariant Grothendieck-Lefschetz theorem

We prove Theorem 1.1 in this section. The following Lemma identifying the equivariant Picard groups of a variety \( X \) and its formal completion \( \tilde{X} \) will be crucial for proving our main result.

Lemma 3.1. Let \( X \) be a non-singular variety with \( G \)-action and let \( Y \subseteq X \) be a \( G \)-invariant closed subscheme such that \( Y \) meets every effective divisor on \( X \). Let \( \tilde{X} \) denote the completion of \( X \) along \( Y \) with the induced \( G \)-action. Assume that \( \dim(X) \geq 2 \) and \( eL^G(X, Y) \) holds. Then the canonical map \( \text{Pic}^G(X) \to \text{Pic}^G(\tilde{X}) \) is an isomorphism.

Proof. Since \( eL^G(X, Y) \) holds, every invertible \( G \)-sheaf on \( \tilde{X} \) extends uniquely to an invertible \( G \)-sheaf on some \( G \)-invariant open neighbourhood \( U \) of \( Y \) by Lemma 2.8. Since \( Y \) meets every effective divisor on \( X \), we have \( \text{codim}(X - U, X) \geq 2 \). Therefore by \([2, \text{Lemma 2(1)}]\), \( \text{Pic}^G(X) \to \text{Pic}^G(U) \) is an isomorphism. The canonical morphism \( \text{Pic}^G(X) \to \text{Pic}^G(\tilde{X}) \) factors through \( \text{Pic}^G(U) \) for every \( G \)-invariant open \( U \) such that \( Y \subseteq U \). Hence we conclude that \( \text{Pic}^G(X) \to \text{Pic}^G(\tilde{X}) \) is an isomorphism. \( \square \)

Lemma 3.2. Let \( X \) be a proper scheme with \( G \)-action and let \( Y \subseteq X \) be a \( G \)-invariant closed subscheme defined by a \( G \)-sheaf of ideals \( \mathcal{I} \). For \( n \geq 1 \), let
$Y_n$ denote the $G$-invariant closed subscheme defined by the sheaf of ideals $\mathcal{I}^n$. Then $\text{Pic}^G(\hat{X}) \cong \varprojlim_n \text{Pic}^G(Y_n)$.

**Proof.** If $\mathcal{F}$ is an invertible $G$-sheaf on $\hat{X}$, then $F_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n}$ is an invertible $G$-sheaf on $Y_n$. This defines a map $f : \text{Pic}^G(\hat{X}) \to \varprojlim_n \text{Pic}^G(Y_n)$.

An element of $\varprojlim_n \text{Pic}^G(Y_n)$ is given by a collection of invertible $G$-sheaves $F_n$ on $\text{Pic}^G(Y_n)$ along with $G$-equivariant isomorphisms $F_{n+1} \otimes_{\mathcal{O}_{Y_{n+1}}} \mathcal{O}_{Y_n} \cong F_n$. Composing with the natural $G$-equivariant map $F_{n+1} \to F_{n+1} \otimes_{\mathcal{O}_{Y_{n+1}}} \mathcal{O}_{Y_n}$, we get a projective system of invertible $G$-sheaves of $\mathcal{O}_X^*$-modules. Then $\mathcal{F} = \varprojlim_n F_n$ is an invertible $G$-sheaf on $\hat{X}$ with $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n} \cong F_n$. Therefore $f$ is surjective. To see that $f$ is injective, let $\mathcal{F}$ be an invertible $G$-sheaf on $\hat{X}$ such that for each $n$, there is a $G$-equivariant isomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n} \cong \mathcal{O}_{Y_n}$, where $\mathcal{O}_{Y_n}$ has the canonical $G$-action. By [5, Proposition II.9.2] and since $(-)^G$ is an additive left exact functor preserving products, it follows that the function $\Gamma(Y, -)^G$ preserves inverse limits. Therefore $\Gamma(\hat{X}, \mathcal{F})^G = \varprojlim_n \Gamma(Y_n, F_n)^G$, where $F_n := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_n}$ and the inverse system $\Gamma(Y_n, F_n)^G$ satisfies the Mittag-Leffler condition [3, Chapter 0, 13.1.2] (since $Y_n$ is proper, $\Gamma(Y_n, F_n)^G$ is a finite-dimensional $k$-vector space). By Corollary 2.6, each $F_n$ has a nowhere vanishing $G$-invariant section. Therefore the stable images in the inverse system have nowhere vanishing sections, so we can find a nowhere vanishing section $s \in \Gamma(\hat{X}, \mathcal{F})^G$. Therefore, again by Corollary 2.6, $\mathcal{F} \cong \mathcal{O}_X^*$ is trivial. \hfill \Box

**Theorem 3.3.** Let $k$ be a field and let $G$ be a finite group. Let $X$ be a proper non-singular variety over $k$ with $G$-action and let $Y \subseteq X$ be a $G$-invariant closed subscheme defined by a $G$-sheaf of ideals $\mathcal{I}$. Suppose that

1. $eL^G(X, Y)$ holds (see Definition 2.7(2));
2. $Y$ meets every effective divisor on $X$; and
3. $H^i(G; Y, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$ for $i = 1, 2$ for all $n \geq 1$.

Then the natural map $\text{Pic}^G(X) \to \text{Pic}^G(Y)$ is an isomorphism.

**Proof.** The natural map in question factors as $\text{Pic}^G(X) \xrightarrow{\alpha} \text{Pic}^G(\hat{X}) \xrightarrow{\beta} \text{Pic}^G(Y)$, where $\alpha$ and $\beta$ are the natural restriction maps. The map $\alpha$ is an isomorphism by Lemma 3.1. Factorise the map $\beta$ further as follows. For $n \geq 1$, let $Y_n$ denote the $G$-invariant closed subscheme defined by the sheaf of ideals $\mathcal{I}^n$. We have natural maps:

$$\text{Pic}^G(\hat{X}) \to \varprojlim_n \text{Pic}^G(Y_n) \to \cdots \to \text{Pic}^G(Y_{n+1}) \to \text{Pic}^G(Y_n) \to \cdots \to \text{Pic}^G(Y).$$

We will show that all the above maps are isomorphisms. The first map is an isomorphism by Lemma 3.2. Let $n \geq 1$ and consider the exact sequence of $G$-sheaves $0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{Y_n}^* \to \mathcal{O}_{Y_{n+1}}^* \to 0$, where $\mathcal{O}^*$ denotes the multiplicative group of units and the first map is given by $x \mapsto 1 + x$. This gives a long exact sequence of $G$-cohomology groups:

$$\cdots \to H^1(G; Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \to H^1(G; Y_{n+1}, \mathcal{O}_{Y_n}^*) \to H^1(G; Y_n, \mathcal{O}_{Y_n}^*) \to H^2(G; Y, \mathcal{I}^n/\mathcal{I}^{n+1}) \to \cdots.$$
By hypothesis (3), we conclude that $H^1(G; Y_{n+1}, \mathcal{O}_{Y_{n+1}}^*) \cong H^1(G; \mathcal{O}_{Y_n}^*)$. By [7, Theorem 2.7], this shows that Pic$^G(Y_{n+1}) \to$ Pic$^G(Y_n)$ is an isomorphism. Consequently, $\lim_{\to n}$ Pic$^G(Y_n)$ is isomorphic to Pic$^G(Y_n)$ for every $n \geq 1$. This completes the proof of the theorem. $\Box$

**Corollary 3.4.** Suppose $G$ acts on $\mathbb{P}^n_k$ and $Y$ is a $G$-invariant closed subscheme of dimension $\geq 3$ which is a scheme-theoretic complete intersection in $\mathbb{P}^n_k$. Then the natural map Pic$^G(\mathbb{P}^n_k) \to$ Pic$^G(Y)$ is an isomorphism.

**Proof.** Since $Y$ is a $G$-invariant scheme-theoretic complete intersection and $\dim(Y) \geq 3$, $eL^G(X, Y)$ holds by Proposition 2.9 and $Y$ meets every effective divisor on $\mathbb{P}^n_k$ by [6, Theorem III.5.1, Proposition IV.1.1]. Further if $Y$ is an intersection of hypersurfaces of degree $d_1, \cdots, d_r$ then $I/I^2 \cong \mathcal{O}_Y(-d_1) \oplus \cdots \oplus \mathcal{O}_Y(-d_r)$. Hence for all $n \geq 1$, $I^n/I^{n+1}$ is a direct sum of sheaves of the form $\mathcal{O}_Y(-m)$ for suitable integers $m < 0$. By [11, Proposition 5], $H^i(Y, \mathcal{O}_Y(m)) = 0$ for all $0 \leq i < \dim(Y)$ for $m < 0$. Since $\dim(Y) \geq 3$, $H^i(Y, I^n/I^{n+1}) = 0$ for $0 \leq i \leq 2$. Therefore by [7, (2.5)], $H^i(G; Y, I^n/I^{n+1}) = 0$ for $i = 1, 2$. This shows that the hypotheses of Theorem 3.3 are satisfied. $\Box$

**Proof of Theorem 1.1.** It is enough to check as in the above corollary that $H^i(Y, \mathcal{O}_Y(m)) = 0$ for $0 \leq i \leq 2$ and all $m < 0$. This follows from the Kodaira-Akizuki-Nakano vanishing theorem (see [1, Corollary 2.11]) as $\dim(Y) \geq 3$. $\Box$

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Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway

E-mail address: charanyr@math.uio.no