Two Timescale Stochastic Approximation with Controlled Markov noise

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Abstract

We present for the first time an asymptotic convergence analysis of two-timescale stochastic approximation driven by controlled Markov noise. In particular, both the faster and slower recursions have non-additive Markov noise components in addition to martingale difference noise. We analyze the asymptotic behavior of our framework by relating it to limiting differential inclusions in both timescales that are defined in terms of the invariant probability measures associated with the controlled Markov processes. Finally, we present a solution to the off-policy convergence problem for temporal difference learning with linear function approximation, using our results.

Keywords: Markov noise, two-timescale stochastic approximation, asymptotic convergence, temporal difference learning

1. Introduction

Stochastic approximation algorithms are sequential non-parametric methods for finding a zero or minimum of a function in the situation where only the noisy observations of the function values are available. Two time-scale stochastic approximation algorithms represent one of the most general subclasses of stochastic approximation methods. These algorithms consist of two sub-recursions

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which are updated with different (one is considerably smaller than the other) step sizes which in turn facilitate convergence for such algorithms.

Two time-scale stochastic approximation algorithms [1] have successfully been applied to several complex problems arising in the areas of reinforcement learning, signal processing and admission control in communication networks. There are many reinforcement learning applications where non-additive Markov noise is present in one or both iterates thus requiring the current two-time scale framework to be extended to include Markov noise (for example, in [2, p. 5] it is mentioned that in order to generalize the analysis to Markov noise, the theory of two-timescale stochastic approximation needs to include the latter).

Here we present a more general framework of two time-scale stochastic approximation with “controlled” Markov noise, i.e., the noise is not simply Markov; rather it is driven by the iterates as well. As two time-scale problems are generally analyzed by reducing them to two single time-scale problems, we look into the single time-scale controlled Markov noise stochastic approximation framework [3] and find that some extra assumptions are required there. In [3] it is assumed that the state space where the controlled Markov Process takes values is Polish. This space is then compactified using the fact that a Polish space can be homeomorphically embedded into a dense subset of a compact metric space. The vector field $h(\cdot, \cdot) : \mathbb{R}^d \times S \to \mathbb{R}^d$ is considered bounded when the first component lies in a compact set. This would, however, require a continuous extension of $h'(x, s') : \mathbb{R}^d \times \phi(S) \to \mathbb{R}^d$ defined by $h'(x, s') = h(x, \phi^{-1}(s'))$ to $\mathbb{R}^d \times \phi(S)$. Here $\phi(\cdot)$ is the homeomorphism defined by $\phi(s) = (\rho(s, s_1), \rho(s, s_2), \ldots) \in [0, 1]^\infty$, and $\{s_i\}$ and $\rho$ is a countable dense subset and metric of the Polish space respectively. A sufficient condition for the above is $h'$ to be uniformly continuous [13, Ex:13, p. 99]. However, this is hard to verify. This motivates us to take the range of the Markov process as compact for our problem.

We analyze the asymptotic behaviour of our framework by relating it to limiting differential inclusions in both timescales that are defined in terms of the invariant probability measures associated with the controlled Markov processes. Next, using these results for the special case of our framework where
the random processes are irreducible Markov chains, we present a solution to
the off-policy convergence problem for temporal difference learning with linear
function approximation. While the off-policy convergence problem for reinforce-
ment learning (RL) with linear function approximation has been one of the most
interesting problems, there are very few solutions available in the current liter-
ature. One such work [4] shows the convergence of the least squares temporal
difference learning algorithm with eligibility traces (LSTD(λ)) as well as the
TD(λ) algorithm. While the LSTD methods are not feasible when the dimen-
sion of the feature vector is large, off-policy TD(λ) is shown to converge only
when the eligibility function \( \lambda \in [0, 1] \) is very close to 1. In [5, 6, 7] the gradi-
ent temporal difference learning (GTD) algorithms were proposed to solve this
problem. However, the authors make the assumption that the data is available
in the “off-policy” setting whereas, in reality, one has only the “on-policy” tra-
jectory corresponding to a given behaviour policy. We use one of the algorithms
from [7] called TDC with “importance-weighting” which takes the “on-policy”
data as input and show its convergence using the results we develop. Our con-
vergence analysis can also be extended for the same algorithm with eligibility
traces for a sufficiently large range of values of \( \lambda \).

To the best of our knowledge there is only one related work [8] where two
time-scale stochastic approximation algorithms with algorithm state dependent
non-additive Markov noise is analyzed. However, the assumptions made there
are not verifiable as the problem lies in the way non-additive Markov noise is
handled. For example, consider the single time-scale controlled Markov noise
recursion:

\[
\theta_{n+1} = \theta_n + a(n)F(\theta_n, \eta_{n+1}),
\]

where \( \{\eta_n\} \) is controlled Markov with the control process being \( \{\theta_n\} \). It is
assumed that there exists an \( \tilde{F}(\cdot, \cdot) \) and \( f(\cdot) \) such that

\[
\tilde{F}(\theta_n, \eta_{n+1}) - E[\tilde{F}(\theta_n, \eta_{n+1})|F_n] = F(\theta_n, \eta_{n+1}) - f(\theta_n)
\]

where \( F_n = \sigma(\theta_m, \eta_m, m \leq n) \). The above iteration can then be cast in the usual
stochastic approximation setting with \( M_{n+1} = F(\theta_n, \eta_{n+1}) - f(\theta_n) \) being the
martingale difference sequence with filtration $F_n = \sigma(\theta_m, \eta_m, m \leq n)$. However, it is not clear how to find such an $f$ and $\tilde{F}(\ldots)$ in general. Thus, the Markov noise problem is only done away with by recasting the same in the usual stochastic approximation framework by imposing extra assumptions.

Section 2 formally defines the problem and provide background and assumptions. Section 3 shows the main results. Section 4 shows how our results can be used to solve the off-policy convergence problem for temporal difference learning with linear function approximation. Finally, we conclude by providing some future research directions.

2. Background, Problem Definition, and Assumptions

In the following we describe the preliminaries and notation that is used in our proofs. Most of the definitions and notation are from [9, 10, 11].

2.1. Definition and Notation

Let $F$ denote a set-valued function mapping each point $\theta \in \mathbb{R}^m$ to a set $F(\theta) \subset \mathbb{R}^m$. $F$ is called a Marchaud map if the following holds:

(i) $F$ is upper-semicontinuous in the sense that if $\theta_n \to \theta$ and $w_n \to w$ with $w_n \in F(\theta_n)$ $\forall n \geq 1$, then $w \in F(\theta)$. In order words, the graph of $F$ defined as \{(\theta, w) : w \in F(\theta)\} is closed.

(ii) $F$ is a non-empty compact convex subset of $\mathbb{R}^m$ for all $\theta \in \mathbb{R}^m$.

(iii) $\exists c > 0$ such that for all $\theta \in \mathbb{R}^m$,

$$\sup_{z \in F(\theta)} \|z\| \leq c(1 + \|\theta\|),$$

where $\|\cdot\|$ denotes any norm on $\mathbb{R}^m$.

A solution for the differential inclusion (D.I.)

$$\dot{\theta}(t) \in F(\theta(t))$$

(1)
with initial point $\theta_0 \in \mathbb{R}^m$ is an absolutely continuous (on compacts) mapping $\theta : \mathbb{R} \to \mathbb{R}^m$ such that $\theta(0) = \theta_0$ and

$$\dot{\theta}(t) \in F(\theta(t))$$

for almost every $t \in \mathbb{R}$. If $F$ is a Marchaud map, it is well-known that (1) has solutions (possibly non-unique) through every initial point. The differential inclusion (1) induces a set-valued dynamical system $\{\Phi_t\}_{t \in \mathbb{R}}$ defined by

$$\Phi_t(\theta_0) = \{\theta(t) : \theta(\cdot) \text{ is a solution to (1) with } \theta(0) = \theta_0\}.$$  

Consider the autonomous ordinary differential equation (o.d.e.)

$$\dot{\theta}(t) = h(\theta(t)),$$  

where $h$ is Lipschitz continuous. One can write (2) in the format of (1) by taking $F(\theta) = \{h(\theta)\}$. It is well-known that (2) is well-posed, i.e., it has a unique solution for every initial point. Hence the set-valued dynamical system induced by the o.d.e. or flow is $\{\Phi_t\}_{t \in \mathbb{R}}$ with

$$\Phi_t(\theta_0) = \{\theta(t)\},$$

where $\theta(\cdot)$ is the solution to (2) with $\theta(0) = \theta_0$. It is also well-known that $\Phi_t(\cdot)$ is a continuous function $\forall t \in \mathbb{R}$.

A set $A \subset \mathbb{R}^m$ is said to be invariant (for $F$) if for all $\theta_0 \in A$ there exists a solution $\theta(\cdot)$ of (1) with $\theta(0) = \theta_0$ such that $\theta(\mathbb{R}) \subset A$.

Given a set $A \subset \mathbb{R}^m$ and $\hat{\theta}, \hat{w} \in A$, we write $\hat{\theta} \hookrightarrow_A \hat{w}$ if for every $\epsilon > 0$ and $T > 0$ there exists $n \in \mathbb{N}$, solutions $\theta_1(\cdot), \ldots, \theta_n(\cdot)$ to (1) and real numbers $t_1, t_2, \ldots, t_n$ greater than $T$ such that

(i) $\theta_i(s) \in A$ for all $0 \leq s \leq t_i$ and for all $i = 1, \ldots, n$,

(ii) $\|\theta_i(t_i) - \theta_{i+1}(0)\| \leq \epsilon$ for all $i = 1, \ldots, n - 1$,

(iii) $\|\theta_1(0) - \hat{\theta}\| \leq \epsilon$ and $\|\theta_n(t_n) - \hat{w}\| \leq \epsilon$.  

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The sequence \((\theta_1(\cdot), \ldots, \theta_n(\cdot))\) is called an \((\epsilon, T)\) chain (in \(F\) from \(\hat{\theta}\) to \(\hat{w}\)) for \(F\). A set \(A \subset \mathbb{R}^m\) is said to be **internally chain transitive**, provided that \(A\) is compact and \(\hat{\theta} \mapsto_A \hat{w} \ \forall \hat{\theta}, \hat{w} \in A\). It can be proved that in the above case, \(A\) is an invariant set.

A compact invariant set \(A\) is called an **attractor** for \(\Phi\), provided that there is a neighbourhood \(U\) of \(A\) (i.e., for the induced topology) with the property that \(d(\Phi_t(\hat{\theta}), A) \to 0\) as \(t \to \infty\) uniformly in \(\hat{\theta} \in U\). Here \(d(X, Y) = \sup_{\hat{\theta} \in X} \inf_{\hat{w} \in Y} \|\hat{\theta} - \hat{w}\|\) for \(X, Y \subset \mathbb{R}^m\). Such a \(U\) is called a **fundamental neighbourhood** of the attractor \(A\). An attractor of a well-posed o.d.e. is an attractor for the set-valued dynamical system induced by the o.d.e.

The set

\[ \omega_\Phi(\hat{\theta}) = \bigcap_{t \geq 0} \Phi_{[t, \infty)}(\hat{\theta}) \]

is called the \(\omega\)-limit set of a point \(\hat{\theta} \in \mathbb{R}^m\). If \(A\) is a set, then

\[ B(A) = \{ \hat{\theta} \in \mathbb{R}^m : \omega_\Phi(\hat{\theta}) \subset A \} \]

denotes its **basin of attraction**. A **global attractor** for \(\Phi\) is an attractor whose basin of attraction consists of all \(\mathbb{R}^m\). Then the following lemma will be useful for our proofs, see [9] for a proof.

**Lemma 2.1.** Suppose \(\Phi\) has a global attractor \(A\). Then every internally chain transitive set lies in \(A\).

We also require another result which will be useful to apply our results to the RL application we mention. Before stating that we recall some definitions from Appendix 11.2.3 of [10]:

A point \(\theta^* \in \mathbb{R}^m\) is called **Lyapunov stable** if \(\forall \epsilon > 0\), there exists a \(\delta > 0\) such that every trajectory initiated in the \(\delta\)-neighbourhood of \(\theta^*\) remains in its \(\epsilon\)-neighbourhood. \(\theta^*\) is called **globally asymptotically stable** if \(\theta^*\) is Lyapunov stable and all trajectories of the o.d.e. converge to it.

**Lemma 2.2.** Consider the autonomous o.d.e. \(\dot{\theta}(t) = h(\theta(t))\) where \(h\) is Lipschitz continuous. Let \(\theta^*\) be globally asymptotically stable. Then \(\theta^*\) is the global attractor of the o.d.e.
Proof. As all trajectories of the o.d.e. converge to $\theta^*$, its basin of attraction is $\mathbb{R}^m$. To prove that $\theta^*$ is an attractor, we use the following facts:

(i) all trajectories of the o.d.e. converge to $\theta^*$.

(ii) $\theta^*$ is Lyapunov stable.

(iii) continuity of flow around the initial point.

Therefore any compact neighbourhood around $\theta^*$ works as a fundamental neighbourhood for $\theta^*$. We refer the readers to Lemma 1 of [10, Chapter 3] where a similar proof technique is used.

2.2. Problem Definition

Our goal is to perform an asymptotic analysis of the following coupled iterations:

\[ \theta_{n+1} = \theta_n + a(n) \left[ h(\theta_n, w_n, Z_n^{(1)}) + M_n^{(1)} \right], \]  
\[ w_{n+1} = w_n + b(n) \left[ g(\theta_n, w_n, Z_n^{(2)}) + M_n^{(2)} \right], \]

where $\{Z_n^{(i)}\}, \{M_n^{(i)}\}, i = 1, 2$ are random processes that we describe below.

We make the following assumptions:

(A1) $\{Z_n^{(i)}\}$ takes values in a compact metric space $S^{(i)}, i = 1, 2$.

(A2) $h : \mathbb{R}^{d+k} \times S^{(1)} \rightarrow \mathbb{R}^d$ is jointly continuous as well as Lipschitz in its first two arguments uniformly w.r.t the third. The latter condition means that

\[ \forall z^{(1)} \in S^{(1)}, \|h(\theta, w, z^{(1)}) - h(\theta', w', z^{(1)})\| \leq L^{(1)}(\|\theta - \theta'\| + \|w - w'\|). \]

Note that the Lipschitz constant $L^{(1)}$ does not depend on $z^{(1)}$.

(A3) $g : \mathbb{R}^{d+k} \times S^{(2)} \rightarrow \mathbb{R}^k$ is jointly continuous as well as Lipschitz in its first two arguments uniformly w.r.t the third. The latter condition similarly means that

\[ \forall z^{(2)} \in S^{(2)}, \|g(\theta, w, z^{(2)}) - g(\theta', w', z^{(2)})\| \leq L^{(2)}(\|\theta - \theta'\| + \|w - w'\|). \]

Note that the Lipschitz constant $L^{(2)}$ does not depend on $z^{(2)}$. 

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(A4) \( \{M_n^{(i)}\}, i = 1, 2 \) are martingale difference sequences w.r.t increasing \( \sigma \)-fields

\[ F_n = \sigma(\theta_m, w_m, M_m^{(i)}, Z_m^{(i)}, m \leq n, i = 1, 2), n \geq 0, \]

satisfying

\[ E[\|M_{n+1}^{(i)}\|^2 | F_n] \leq K (1 + \|\theta_n\|^2 + \|w_n\|^2), i = 1, 2, \]

for \( n \geq 0. \)

(A5) The stepsizes \( \{a(n)\}, \{b(n)\} \) are positive scalars satisfying

\[ \sum_n a(n) = \sum_n b(n) = \infty, \sum_n (a(n)^2 + b(n)^2) < \infty, \frac{a(n)}{b(n)} \to 0. \]

Moreover, \( a(n), b(n), n \geq 0 \) are nonincreasing.

(A6) The processes \( \{Z_n^{(i)}\}, i = 1, 2 \) are \( S^{(i)} \)-valued, \( i = 1, 2 \), controlled Markov processes respectively with their individual dynamics specified by

\[ P(Z_{n+1}^{(i)} \in A^{(i)} | Z_m^{(i)}, \theta_m, w_m, m \leq n) = \int_{A^{(i)}} p^{(i)}(dy | Z_n^{(i)}, \theta_n, w_n), n \geq 0 \]

for \( A^{(i)} \) Borel in \( S^{(i)}, i = 1, 2 \) respectively.

Before stating the assumption on the transition kernel \( p^{(i)}, i = 1, 2 \) we need to define the metric in the space of probability measures \( P(S) \). Here we mention the definitions and main theorems on the spaces of probability measures that we use in our proofs (details can be found in Chapter 2 of [12]). We denote the metric by \( d \) and is defined as

\[ d(\mu, \nu) = \sum_j 2^{-j} | \int f_j d\mu - \int f_j d\nu |, \mu, \nu \in P(S), \]

where \( \{f_j\} \) are countable dense in the unit ball of \( C(S) \). Then the following are equivalent:

(i) \( d(\mu_n, \mu) \to 0, \)

(ii) \( \forall f \in C_b(S), \)

\[ \int_S f d\mu_n \to \int_S f d\mu, \tag{5} \]
(iii) \( \forall f \) bounded and uniformly continuous,
\[
\int_S f \, d\mu_n \to \int_S f \, d\mu.
\]
Hence we see that \( d(\mu_n, \mu) \to 0 \) iff \( \int_S f_j \, d\mu_n \to \int_S f_j \, d\mu \forall j \). Any such sequence of functions \( \{f_j\} \) is called convergence determining class in \( \mathcal{P}(S) \). Sometimes we also denote \( d(\mu_n, \mu) \to 0 \) using the notation \( \mu_n \Rightarrow \mu \).

Also, we recall the characterization of relative compactness in \( \mathcal{P}(S) \) that relies on the definition of tightness. \( \mathcal{A} \subset \mathcal{P}(S) \) is a tight set if for any \( \epsilon > 0 \), there exists a compact \( K_\epsilon \subset S \) such that \( \mu(K_\epsilon) > 1 - \epsilon \forall \mu \in \mathcal{A} \). Clearly, if \( S \) is compact then any \( \mathcal{A} \subset \mathcal{P}(S) \) is tight. By Prohorov’s theorem, \( \mathcal{A} \subset \mathcal{P}(S) \) is relatively compact if and only if it is tight.

With the above definitions we assume the following:

(A7) The map \( S^{(i)} \times \mathbb{R}^{d+k} \ni (z^{(i)}, \theta, w) \to p^{(i)}(dy|z^{(i)}, \theta, w) \in \mathcal{P}(S^{(i)}) \) is a continuous map.

(A8) For \( \theta_n = \theta, w_n = w \forall n \) with a fixed deterministic \( (\theta, w) \in \mathbb{R}^{d+k} \), the time-homogeneous Markov processes \( Z_n^{(i)}, i = 1, 2 \) have invariant distributions \( \eta_{\theta, w}^{(i)}, i = 1, 2 \) (possibly non-unique) which satisfy the following:
\[
\int_{S^{(i)}} f^{(i)}(z) \eta_{\theta, w}^{(i)}(dz) = \int_{S^{(i)}} \int_{S^{(i)}} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(dz) \quad (6)
\]
for \( f^{(i)} : S^{(i)} \to \mathcal{R} \subset C_b(S^{(i)}) \).

We denote by \( D^{(i)}(\theta, w), i = 1, 2 \) the set of all such invariant distributions for the prescribed \( \theta \) and \( w \). In the following we prove some properties of the map \( (\theta, w) \to D^{(i)}(\theta, w) \).

Lemma 2.3. The map \( (\theta, w) \to D^{(i)}(\theta, w) \) is convex and closed.

Proof. Let \( \eta_{\theta, w}^{(i)}(j) \in D^{(i)}(\theta, w), j = 1, \ldots n \) and \( \eta_{\theta, w}^{(i)} = \sum_{j=1}^{n} \alpha_j \eta_{\theta, w}^{(i)}(j) \), \( \alpha_j \geq 0 \forall j \) and \( \sum_{j=1}^{n} \alpha_j = 1 \). Clearly, \( \eta_{\theta, w}^{(i)} \in \mathcal{P}(S^{(i)}) \). Now, from (6) we get,
\[
\int_{S^{(i)}} f^{(i)}(z) \eta_{\theta, w}^{(i)}(j)(dz) = \int_{S^{(i)}} \int_{S^{(i)}} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(j)(dz)
\]
for \( j = 1, \ldots, n \). Now,

\[
\int_{S(i)} f^{(i)}(z)\eta_{\theta, w}^{(i)}(dz) = \sum_{j=1}^{n} \alpha_j \int_{S(i)} f^{(i)}(z)\eta_{\theta, w}^{(i)}(j)(dz)
\]

\[
= \sum_{j=1}^{n} \alpha_j \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(j)(dz)
\]

\[
= \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\sum_{j=1}^{n} \alpha_j \eta_{\theta, w}^{(i)}(j)(dz)
\]

\[
= \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(dz).
\]

Hence, the map is convex. Next, we prove that the map is closed. It is sufficient to prove that \( \Box \) is closed under convergence in \( \mathcal{P}(S^{(i)}) \). Let \( D^{(i)}(\theta, w) \ni \eta_{\theta, w}^{(i)}(n) \Rightarrow \eta_{\theta, w}^{(i)} \in \mathcal{P}(S^{(i)}) \). We now show that \( \eta_{\theta, w}^{(i)} \in D^{(i)}(\theta, w) \):

\[
\int_{S(i)} f^{(i)}(z)\eta_{\theta, w}^{(i)}(dz) = \lim_{n \to \infty} \int_{S(i)} f^{(i)}(z)\eta_{\theta, w}^{(i)}(n)(dz)
\]

\[
= \lim_{n \to \infty} \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(n)(dz).
\]

It is enough to prove that \( g^{(i)}(z) = \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w) \in C_b(S^{(i)}) \). Boundedness follows from the fact that \( f^{(i)}(\cdot) \) is bounded. Let \( z_n \to z \). Then \( p^{(i)}(dy|z_n, \theta, w) \Rightarrow p^{(i)}(dy|z, \theta, w) \in \mathcal{P}(S^{(i)}) \). Then using the implication \((i) \Rightarrow (ii)\) in \( \Box \) we get that \( g^{(i)}(z_n) \to g^{(i)}(z) \) as \( n \to \infty \).

**Lemma 2.4.** The map \((\theta, w) \rightarrow D^{(i)}(\theta, w)\) is compact.

**Proof.** First we prove that \( \mathcal{P}(S^{(i)}) \) is compact, i.e., sequentially compact (as the space is a metric space). This follows from the fact that any sequence of probability measures in \( \mathcal{P}(S^{(i)}) \) is tight due to the compactness of \( S^{(i)} \). Because \( S^{(i)} \) is compact, it is separable. Then from an application of Prohorov’s theorem, it follows that \( \mathcal{P}(S^{(i)}) \) is compact. Now Lemma 2.3 shows that the map is closed and a closed subset of a compact set is compact.

**Lemma 2.5.** The map \((\theta, w) \rightarrow D^{(i)}(\theta, w)\) is upper-semi-continuous.

**Proof.** Let \( \theta_n \to \theta, w_n \to w \) and \( \eta^{(i)}_n \Rightarrow \eta^{(i)} \in \mathcal{P}(S^{(i)}) \) such that \( \eta^{(i)}_n \in D^{(i)}(\theta_n, w_n) \). Let \( g^{(i)}_n(z) = \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta_n, w_n) \) and \( g^{(i)}(z) = \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w) \) for \( j = 1, \ldots, n \). Now,

\[
\int_{S(i)} f^{(i)}(z)\eta_{\theta, w}^{(i)}(dz) = \sum_{j=1}^{n} \alpha_j \int_{S(i)} f^{(i)}(z)\eta_{\theta, w}^{(i)}(j)(dz)
\]

\[
= \sum_{j=1}^{n} \alpha_j \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(j)(dz)
\]

\[
= \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\sum_{j=1}^{n} \alpha_j \eta_{\theta, w}^{(i)}(j)(dz)
\]

\[
= \int_{S(i)} \int_{S(i)} f^{(i)}(y)p^{(i)}(dy|z, \theta, w)\eta_{\theta, w}^{(i)}(dz).
\]
\[ \int_{S^{(i)}} f^{(i)}(y)p^{(i)}(dy|z, \theta, w). \text{ From (1)} \text{ we get that} \]
\[
\int_{S^{(i)}} f^{(i)}(z)\eta^{(i)}(dz) = \lim_{n \to \infty} \int_{S^{(i)}} f^{(i)}(z)\eta_n^{(i)}(dz)
\]
\[
= \lim_{n \to \infty} \int_{S^{(i)}} \int_{S^{(i)}} f^{(i)}(y)p^{(i)}(dy|z, \theta_n, w_n)\eta_n^{(i)}(dz)
\]
\[
= \lim_{n \to \infty} \int_{S^{(i)}} g_n^{(i)}(z)\eta_n^{(i)}(dz).
\]

Now, \( p^{(i)}(dy|z, \theta_n, w_n) \Rightarrow p^{(i)}(dy|z, \theta, w) \) implies \( g_n^{(i)}(\cdot) \to g^{(i)}(\cdot) \) pointwise. We prove that the convergence is indeed uniform. It is enough to prove that this sequence of functions is equicontinuous. Then along with pointwise convergence it will imply uniform convergence on compacts \[13, \text{ p. 168, Ex: 16}].

Define \( g' : S^{(i)} \times \mathbb{R}^{d+k} \to \mathbb{R} \) by \( g'(z', \theta', w') = \int_{S^{(i)}} f^{(i)}(y)p^{(i)}(dy|z', \theta', w') \). Then \( g' \) is continuous. Let \( A = S^{(i)} \times (\{\theta_n\} \cup \theta) \times (\{w_n\} \cup w) \). So, \( A \) is compact and \( g'|_A \) is uniformly continuous. This implies that \( \forall \epsilon > 0, \exists \delta > 0 \) such that if \( \rho'(s_1, s_2) < \delta, \|\theta_1 - \theta_2\| < \delta, \|w_1 - w_2\| < \delta \), then \( |g'(s_1,\theta_1,w_1) - g'(s_2,\theta_2,w_2)| < \epsilon \) where \( s_1, s_2 \in S^{(i)}, \theta_1, \theta_2 \in (\{\theta_n\} \cup \theta), w_1, w_2 \in (\{w_n\} \cup w) \) and \( \rho' \) denotes the metric in \( S^{(i)} \). Now use this same \( \delta \) for the \( \{g_n^{(i)}(\cdot)\} \) to get \( \forall n \) the following for \( \rho'(z_1, z_2) < \delta \):
\[
|g_n^{(i)}(z_1) - g_n^{(i)}(z_2)| = |g'(z_1, \theta_n, w_n) - g'(z_2, \theta_n, w_n)| < \epsilon.
\]

Hence \( \{g_n^{(i)}(\cdot)\} \) is equicontinuous. For large \( n \), \( \sup_{z \in S^{(i)}} |g_n^{(i)}(z) - g^{(i)}(z)| < \epsilon/2 \) because of uniform convergence of \( \{g_n^{(i)}(\cdot)\} \), hence \( \int_{S^{(i)}} |g_n^{(i)}(z) - g^{(i)}(z)|\eta_n^{(i)}(dz) < \epsilon/2 \) . Now (for \( n \) large):
\[
|\int_{S^{(i)}} g_n^{(i)}(z)\eta_n^{(i)}(dz) - \int_{S^{(i)}} g^{(i)}(z)\eta^{(i)}(dz)|
\]
\[
= |\int_{S^{(i)}} g_n^{(i)}(z) - g^{(i)}(z)|\eta_n^{(i)}(dz) + \int_{S^{(i)}} g^{(i)}(z)\eta_n^{(i)}(dz) - \int_{S^{(i)}} g^{(i)}(z)\eta^{(i)}(dz)|
\]
\[
< \epsilon/2 + |\int_{S^{(i)}} g^{(i)}(z)\eta_n^{(i)}(dz) - \int_{S^{(i)}} g^{(i)}(z)\eta^{(i)}(dz)|
\]
\[
< \epsilon.
\]

(7)
The last inequality is due to the fact that \( \eta_n^{(i)} \Rightarrow \eta^{(i)} \). Hence from (7) we get,

\[
\int_{S^{(i)}} f^{(i)}(z) \eta^{(i)}(dz) = \int_{S^{(i)}} \int_{S^{(i)}} f^{(i)}(y)p^{(i)}(dy \mid \theta, w) \eta^{(i)}(dz)
\]

proving that the map is upper-semi-continuous. □

Define \( \tilde{g}(\theta, w, \nu) = \int g(\theta, w, z)\nu(dz) \) and \( \hat{g}_\theta(w) = \{ \tilde{g}(\theta, w, \nu) : \nu \in D^{(2)}(\theta, w) \} \).

**Lemma 2.6.** \( \forall \theta \in \mathbb{R}^d, \hat{g}_\theta(\cdot) \) is a Marchaud map.

**Proof.** (i) Convexity and compactness follow trivially from the same for the map \( (\theta, w) \rightarrow D^{(2)}(\theta, w) \).

(ii)

\[
\| \tilde{g}(\theta, w, \nu) \| = \| \int g(\theta, w, z)\nu(dz) \| \\
\leq \int \| g(\theta, w, z) \| \nu(dz) \\
\leq \int L^{(2)}(\|w\| + \|g(\theta, 0, z)\|)\nu(dz) \\
\leq \max(L^{(2)}, L^{(2)} \int \|g(\theta, 0, z)\|\nu(dz))(1 + \|w\|).
\]

Clearly, \( K(\theta) = \max(L^{(2)}, L^{(2)} \int \|g(\theta, 0, z)\|\nu(dz)) > 0 \).

(iii) Let \( w_n \rightarrow w, \tilde{g}(\theta, w_n, \nu_n) \rightarrow m, \nu_n \in D^{(2)}(\theta, w_n) \). Now, \( \{\nu_n\} \) is tight, hence has a convergent sub-sequence \( \{\nu_{n_k}\} \) with \( \nu \) being the limit. Then using the arguments similar to the proof of Lemma 2.5 one can show that \( m = \tilde{g}(\theta, w, \nu) \) whereas \( \nu \in D^{(2)}(\theta, w) \) follows directly from the upper-semi-continuity of the map \( w \rightarrow D^{(2)}(\theta, w) \forall \theta \).

□
2.3. Other assumptions needed for two-timescale convergence analysis

We now list the other assumptions required for two-timescale convergence analysis:

**(A9)** \( \forall \theta \in \mathbb{R}^d \), the differential inclusion

\[ \dot{w}(t) \in \hat{g}_\theta(w(t)) \quad (8) \]

has a singleton global attractor \( \lambda(\theta) \) where \( \lambda : \mathbb{R}^d \to \mathbb{R}^k \) is a Lipschitz map with constant \( K \).

**(A10)** The inclusion

\[ \dot{\theta}(t) \in \hat{h}(\theta(t)), \quad (9) \]

has a global attractor set \( A_0 \). Here \( \hat{h}(\theta) = \{ \hat{h}(\theta, \lambda(\theta), \nu) : \nu \in D^{(1)}(\theta, \lambda(\theta)) \} \) and \( \hat{h}(\theta, w, \nu) = \int h(\theta, w, z) \nu(dz) \). Note that the map \( \hat{h} \) can be shown to be a Marchaud map using the exact same technique as Lemma 2.6.

**(A11)** \( \sup_n(\|\theta_n\| + \|w_n\|) < \infty \) a.s.

We call (8) and (9) as the faster and slower D.I. to correspond with faster and slower recursions, respectively.

3. Main Results

We first discuss an extension of the single time-scale controlled Markov noise framework to prove our main results. We begin by describing the intuition behind the proof techniques in [3].

The space \( C([0, \infty); \mathbb{R}^d) \) of continuous functions from \([0, \infty)\) to \( \mathbb{R}^d \) is topologized with the coarsest topology such that the map that takes \( f \in C([0, \infty); \mathbb{R}^d) \) to its restriction to \([0, T] \) when viewed as an element of the space \( C([0, T]; \mathbb{R}^d) \), is continuous for all \( T > 0 \). In other words, \( f_n \to f \) in this space iff \( f_n|_{[0,T]} \to f|_{[0,T]} \). The other notations used below are same as those in [3, 10]. We present a few for easy reference.

Consider the single time-scale controlled Markov noise recursion:

\[ \theta_{n+1} = \theta_n + a(n) [h(\theta_n, Y_n) + M_{n+1}] \quad (10) \]
Define time instants $t(0) = 0, t(n) = \sum_{m=0}^{n-1} a(m), n \geq 1$. Let $\tilde{\theta}(t), t \geq 0$ be the continuous, piecewise linear trajectory defined by $\tilde{\theta}(t(n)) = \theta_n, n \geq 0$, with linear interpolation on each interval $[t(n), t(n+1))$, i.e.,

$$\tilde{\theta}(t) = \theta_n + (\theta_{n+1} - \theta_n) \frac{t - t(n)}{t(n+1) - t(n)}, t \in [t(n), t(n+1)).$$

Now, define $\tilde{h}(\theta, \nu) = \int h(\theta, z) \nu(dz)$ for $\nu \in \mathcal{P}(S)$. Let $\mu(t), t \geq 0$ be the random process defined by $\mu(t) = \delta_Y$ for $t \in [t(n), t(n+1)), n \geq 0$, where $\delta_Y$ is the Dirac measure corresponding to $y$. Consider the non-autonomous o.d.e.

$$\tilde{\theta}(t) = \tilde{h}(\tilde{\theta}(t), \mu(t)). \quad (11)$$

Let $\theta^s(t), t \geq s$, denote the solution to (11) with $\theta^s(s) = \tilde{\theta}(s)$, for $s \geq 0$. Note that $\theta^s(t), t \in [s, s+T]$ and $\theta^s(t), t \geq s$ can be viewed as elements of $C([0, T], \mathbb{R}^d)$ and $C([0, \infty), \mathbb{R}^d)$ respectively. With this abuse of notation, it is easy to see that $\{\theta^s(\cdot)|_{[s, s+T]}, s \geq 0\}$ is a pointwise bounded and equicontinuous family of functions in $C([0, T]; \mathbb{R}^d) \forall T > 0$. By Arzela-Ascoli theorem, it is relatively compact. From Lemma 2.2 of [3] one can see that $\forall s(n) \uparrow \infty, \{\tilde{\theta}(s(n) + \cdot)|_{[s(n), s(n)+T]}, n \geq 1\}$ has a limit point in $C([0, T]; \mathbb{R}^d) \forall T > 0$. With the above topology for $C([0, \infty); \mathbb{R}^d)$, $\{\theta^s(\cdot), s \geq 0\}$ is also relatively compact in $C([0, \infty); \mathbb{R}^d)$ and $\forall s(n) \uparrow \infty, \{\tilde{\theta}(s(n) + \cdot), n \geq 1\}$ has a limit point in $C([0, \infty); \mathbb{R}^d)$.

One can write from (11) the following:

$$\tilde{\theta}(u(n) + t) = \tilde{\theta}(u(n)) + \int_0^t \tilde{h}(\tilde{\theta}(u(n) + \tau), \nu(u(n) + \tau))d\tau + W^n(t),$$

where $u(n) \uparrow \infty$, $\tilde{\theta}(u(n) + \cdot) \rightarrow \tilde{\theta}(\cdot), \nu(t) = Y_n$ for $t \in [t_n, t_{n+1}), n \geq 0$ and $W^n(t) = W(t + u(n)) - W(u(n)), W(t) = W_n + (W_{n+1} - W_n)\frac{t - t(n)}{t(n+1) - t(n)}, W_n = \sum_{k=0}^{n-1} a_k M_{k+1}, n \geq 0$. From here one cannot directly take limit on both sides as limit points of $\nu(s + \cdot)$ as $s \rightarrow \infty$ is not meaningful. Now, $h(\theta, y) = \int h(\theta, z)\delta_y(dz)$. Hence by defining $\tilde{h}(\theta, \rho) = \int h(\theta, z)\rho(dz)$ and $\mu(t) = \delta_{\nu(t)}$ one can write the above as

$$\theta(u(n) + t) = \tilde{\theta}(u(n)) + \int_0^t \tilde{h}(\tilde{\theta}(u(n) + \tau), \mu(u(n) + \tau))d\tau + W^n(t). \quad (12)$$

The advantage is that the space $\mathcal{U}$ of measurable functions from $[0, \infty)$ to $\mathcal{P}(S)$ is compact metrizable, so subsequential limits exist. Note that $\mu(\cdot)$ is not a
member of \( \mathcal{U} \), rather we need to fix a sample point, i.e., \( \mu(\cdot, \omega) \in \mathcal{U} \). For ease of understanding, we abuse the terminology and talk about the limit points \( \tilde{\mu}(\cdot) \) of \( \mu(s + \cdot) \).

From (12) one can infer that the limit \( \tilde{\theta}(\cdot) \) of \( \tilde{\theta}(u(n) + \cdot) \) satisfies the o.d.e. \( \dot{\theta}(t) = h(\theta(t), \mu(t)) \) with \( \mu(\cdot) \) replaced by \( \tilde{\mu}(\cdot) \). Here each \( \tilde{\mu}(t), t \in \mathbb{R} \) in \( \tilde{\mu}(\cdot) \) is generated through different limiting processes each one associated with the compact metrizable space \( U_t = \) space of measurable functions from \([0, t]\) to \( \mathcal{P}(S) \).

This will be problematic if we want to further explore the process \( \tilde{\mu}(\cdot) \) and convert the non-autonomous o.d.e. into an autonomous one.

Hence it is proved using one auxiliary lemma [3, Lemma 2.3] other than the tracking lemma (Lemma 2.2 of [3]). Let \( u(n(k)) \uparrow \infty \) be such that \( \tilde{\theta}(u(n(k)) + \cdot) \rightarrow \tilde{\theta}(\cdot) \) and \( \mu(u(n(k)) + \cdot) \rightarrow \tilde{\mu}(\cdot) \), then using Lemma 2.2 of [3] one can show that \( \theta^{u(n(k))}(\cdot) \rightarrow \tilde{\theta}(\cdot) \). Then the auxiliary lemma shows that the o.d.e. trajectory \( \theta^{u(n(k))}(\cdot) \) associated with \( \mu(u(n(k)) + \cdot) \) tracks (in the limit) the o.d.e. trajectory associated with \( \tilde{\mu}(\cdot) \). Hence Lemma 2.3 of [3] links the two limiting processes \( \tilde{\theta}(\cdot) \) and \( \tilde{\mu}(\cdot) \) in some sense. Note that Lemma 2.3 of [3] involves only the o.d.e. trajectories, not the interpolated trajectory of the algorithm.

Consider the iteration
\[
\theta_{n+1} = \theta_n + a(n) \left[ h(\theta_n, Y_n) + \epsilon_n + M_{n+1} \right],
\]
(13)
where \( \epsilon_n \rightarrow 0 \) and the rest of the notations are same as [3]. Specifically, \( \{Y_n\} \) is the controlled Markov process driven by \( \{\theta_n\} \) and \( M_{n+1}, n \geq 0 \) is a martingale difference sequence.

The convergence analysis of (13) requires some changes in Lemma 2.2 and 3.1 of [3]. The modified versions of them are precisely the following two lemmas.

**Lemma 3.1.** For any \( T > 0 \), \( \sup_{t \in [s, s+T]} \| \tilde{\theta}(t) - \theta^*(t) \| \rightarrow 0 \), a.s. as \( s \rightarrow \infty \).

**Proof.** Let \( t(n + m) \) be in \([t(n), t(n) + T]\). Then by construction,
\[
\tilde{\theta}(t(n+m)) = \tilde{\theta}(t(n)) + \sum_{k=0}^{m-1} a(n+k)h(\tilde{\theta}(t(n+k)), Y_{n+k}) + \sum_{k=0}^{m-1} a(n+k)\epsilon_{n+k} + \delta_{n,n+m},
\]
where $\delta_{n,n+m} = \zeta_{n+m} - \zeta_n$ with $\zeta_n = \sum_{m=0}^{n-1} a(m) M_{m+1}$, $n \geq 1$. Then the proof goes along the same lines as Lemma 2.2 of [3] except that there is an extra term in the R.H.S of the below inequality.

\[
\| \bar{\theta}(t(n+m)) - \theta^{t(n)}(t(n+m)) \|
\leq L \sum_{k=0}^{m-1} a(n+k) \| \bar{\theta}(t(n+k)) - \theta^{t(n)}(t(n+k)) \|
+ C_T L \sum_{k \geq 0} a(n+k)^2 + \sum_{k=0}^{m-1} a(n+k) \| \epsilon_{n+k} \|
+ \sup_{k \geq 0} \| \delta_{n,n+k} \|
\leq L \sum_{k=0}^{m-1} a(n+k) \| \bar{\theta}(t(n+k)) - \theta^{t(n)}(t(n+k)) \|
+ C_T L \sum_{k \geq 0} a(n+k)^2 + T \sup_{k \geq 0} \| \epsilon_{n+k} \|
+ \sup_{k \geq 0} \| \delta_{n,n+k} \|, \text{ a.s.}
\]

Define

\[ K_{T,n} = C_T L \sum_{k \geq 0} a(n+k)^2 + T \sup_{k \geq 0} \| \epsilon_{n+k} \| + \sup_{k \geq 0} \| \delta_{n,n+k} \|. \]

So, $K_{T,n} \to 0$ a.s. as $n \to \infty$ and the rest of the proof follows tracking lemma (Lemma 2.1 of [10, Chapter 2]) of the usual stochastic approximation framework.

Now, $\mu$ can be viewed as a random variable taking values in $\mathcal{U} = \text{the space of measurable functions from } [0, \infty) \to \mathcal{P}(S)$. This space is topologized with the coarsest topology such that the map

\[ \nu(\cdot) \in \mathcal{U} \to \int_0^T g(t) \int f d\nu(t) dt \in \mathbb{R} \]

is continuous for all $f \in C(S), T > 0, g \in L_2[0,T]$. Note that $\mathcal{U}$ is compact metrizable.

**Lemma 3.2.** Almost surely every limit point of $(\mu(s+.), \bar{\theta}(s+.)$) as $s \to \infty$ is of the form $(\tilde{\mu}(\cdot), \tilde{\theta}(\cdot))$ where $\tilde{\mu}(\cdot)$ satisfies $\tilde{\mu}(t) \in D(\tilde{\theta}(t)) \text{ a.e. } t$. 

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Proof. Suppose that \( u(n) \uparrow \infty, \mu(u(n) +.) \rightarrow \tilde{\mu}(\cdot) \) and \( \tilde{\theta}(u(n) +.) \rightarrow \tilde{\theta}(\cdot) \). Let \( \{f_i\} \) be countable dense in the unit ball of \( C(S) \), hence a separating class, i.e.,
\[
\forall i, \int f_i d\mu = \int f_i d\nu \implies \mu = \nu.
\]
For each \( i \),
\[
\zeta_i^n = \sum_{m=1}^{n-1} a(m)(f_i(Y_{m+1}) - \int f_i(y)p(dy|Y_m, \theta_m)),
\]
is a zero-mean martingale with \( \mathcal{F}_n = \sigma(\theta_m, Y_m, m \leq n) \). Moreover, it is a square integrable martingale due to the fact that \( f_i \)'s are bounded and the martingale is a finite sum. Its quadratic variation process
\[
A_n = \sum_{m=0}^{n-1} a(m)^2 E[(f_i(Y_{m+1}) - \int f_i(y)p(dy|Y_m, \theta_m))^2|\mathcal{F}_m] + E[(\zeta_i^0)^2]
\]
is almost surely convergent. By the martingale convergence theorem, \( \zeta_i^n, n \geq 0 \) converges a.s. \( \forall i \). As before let \( \tau(n, t) = \min\{m \geq n : t(m) \geq t(n) + t\} \) for \( t \geq 0, n \geq 0 \). Then as \( n \rightarrow \infty \),
\[
\sum_{m=n}^{\tau(n,t)} a(m)(f_i(Y_{m+1}) - \int f_i(y)p(dy|Y_m, \theta_m)) \rightarrow 0, \text{ a.s.}
\]
for \( t > 0 \). By our choice of \( \{f_i\} \) and the fact that \( \{a(n)\} \) is an eventually non-increasing sequence (this is the only time the latter property is used), we have
\[
\sum_{m=n}^{\tau(n,t)} (a(m) - a(m + 1))f_i(Y_{m+1}) \rightarrow 0, \text{ a.s.}
\]
From the foregoing,
\[
\sum_{m=n}^{\tau(n,t)} (a(m + 1) f_i(Y_{m+1}) - a(m) \int f_i(y)p(dy|Y_m, \theta_m)) \rightarrow 0, \text{ a.s.}
\]
\( \forall t > 0 \), which implies
\[
\sum_{m=n}^{\tau(n,t)} a(m)(f_i(Y_m) - \int f_i(y)p(dy|Y_m, \theta_m)) \rightarrow 0, \text{ a.s.}
\]
\( \forall t > 0 \) due to the fact that \( a(n) \rightarrow 0 \) and \( f_i(\cdot) \) are bounded. This implies
\[
\int_{\tau(n)}^{\tau(n) + t} \left[ \int (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s)))\mu(s, dz) \right] ds \rightarrow 0, \text{ a.s.}
\]
and that in turn implies
\[
\int_{u(n)}^{u(n)+t} (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s)))\mu(s, dz)ds \to 0, \text{ a.s.}
\]
(this is true because \(a(n) \to 0\) and \(f_i(\cdot)\) is bounded) where \(\tilde{\theta}(s) = \theta_n\) when \(s \in [t(n), t(n+1))\) for \(n \geq 0\). Now, one can claim from the above that
\[
\int_{u(n)}^{u(n)+t} (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s)))\mu(s, dz)ds \to 0, \text{ a.s.}
\]
This is due to the fact that the map \(S \times \mathbb{R}^d \ni (z, \theta) \to \int f_i(y)p(dy|z, \theta)\) is continuous and hence uniformly continuous on the compact set \(A = S \times M\) where \(M\) is the compact set s.t. \(\theta_n \in M\ \forall n\). Here we also use the fact that \(\|\tilde{\theta}(s) - \theta_m\| = \|h(\theta_m, Y_m) + \epsilon_m + M_{m+1}\|\) is bounded and \(s - s_m\) is bounded as the first two terms inside the norm in the R.H.S are bounded. The above convergence is equivalent to
\[
\int_0^t (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s)))\mu(s + u(n), dz)ds \to 0, \text{ a.s.}
\]
Fix a sample point in the probability one set on which the convergence above holds for all \(i\). Then the convergence above leads to
\[
\int_0^t (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s)))\mu(s, dz)ds = 0 \forall i. \quad (14)
\]
Here we use one part of the proof from Lemma 2.3 of [3] that if \(\mu^n(\cdot) \to \mu^\infty(\cdot)\) in \(\mathcal{U}\) then for any \(t > 0\),
\[
\int_0^t \int \tilde{f}(s, z)\mu^n(s, dz)ds - \int_0^t \int \tilde{f}(s, z)\mu^\infty(s, dz)ds \to 0,
\]
for all \(\tilde{f} \in C([0, t] \times S)\) and the fact that \(\tilde{f}_n(s, z) = \int f_i(y)p(dy|z, \tilde{\theta}(s + u(n)))\) converges uniformly to \(\tilde{f}(s, z) = \int f_i(y)p(dy|z, \tilde{\theta}(s)).\) To prove the latter, define \(g : C([0, t] \times [0, t] \times S \to \mathbb{R}\) by \(g(\theta(\cdot), s, z) = \int f_i(y)p(dy|z, \theta(s))).\) To see that \(g\) is continuous we need to check that if \(\theta_n(\cdot) \to \theta(\cdot)\) uniformly and \(s(n) \to s\), then \(\theta_n(s(n)) \to \theta(s).\) This is because \(\|\theta_n(s(n)) - \theta(s)\| = \|\theta_n(s(n)) - \theta(s(n)) + \theta(s(n)) - \theta(s)\| \leq ||\theta_n(s(n)) - \theta(s(n))|| + ||\theta(s(n)) - \theta(s)||.\) The first and second terms go to zero due to the uniform convergence of \(\theta_n(\cdot), n \geq 0\) and continuity of
\(\theta(\cdot)\) respectively. Let \(A = \{\tilde{\theta}(u(n))|_{u(n),u(n) + t}], n \geq 1\} \cup \tilde{\theta}(\cdot)|_{[0,t]}\). \(A\) is compact as it is the union of a sequence of functions and their limit. So, \(g|_{(A \times [0,t] \times S)}\) is uniformly continuous. Then using the same arguments as in Lemma 2.5 we can show equicontinuity of \(\{\tilde{f}_n(\cdot, \cdot)\}\), that results in uniform convergence and thereby (14). An application of Lebesgue’s theorem in conjunction with (14) shows that

\[
\int (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(t)))\tilde{\mu}(t, dz) = 0 \quad \forall i
\]

for a.e. \(t\). By our choice of \(\{f_i\}\), this leads to

\[
\tilde{\mu}(t, dy) = \int p(dy|z, \tilde{\theta}(t))\tilde{\mu}(t, dz)
\]

a.e. \(t\).

\(\square\)

**Remark 1.** Note that the above invariant distribution does not come “naturally”; rather it arises from the assumption made to match the natural timescale intuition for the controlled Markov noise component, i.e., the slower iterate should see the average effect of the Markov component.

The proof of the following lemma, in this case, will be unchanged from its original version, so we just mention it for completeness and refer the reader to Lemma 2.3 of [10] for its proof.

**Lemma 3.3.** Let \(\mu^\infty(\cdot) \in U\). Let \(\theta^n(\cdot), n = 1, 2, \ldots, \infty\) denote solutions to (11) corresponding to the case where \(\mu(\cdot)\) is replaced by \(\mu^n(\cdot)\), for \(n = 1, 2, \ldots, \infty\). Suppose \(\theta^n(0) \to \theta^\infty(0)\). Then

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} ||\theta^n(t) - \theta^\infty(t)|| = 0
\]

for every \(T > 0\).

**Lemma 3.4.** \(\{\theta_n\}\) converges a.s. to an internally chain transitive invariant set of the differential inclusion

\[
\dot{\theta}(t) \in \hat{h}(\theta(t)),
\]

(15)

where \(\hat{h}(\theta) = \{\hat{h}(\theta, \nu) : \nu \in D(\theta)\}\).
Lemma 3.3 shows that every limit point \((\tilde{\mu}(\cdot), \tilde{\theta}(\cdot))\) of \((\mu(s+), \bar{\theta}(s+))\) as \(s \to \infty\) is such that \(\tilde{\theta}(\cdot)\) satisfies (11) with \(\mu(\cdot) = \tilde{\mu}(\cdot)\). Hence, \(\tilde{\theta}(\cdot)\) is absolutely continuous. Moreover, using Lemma 3.2, one can see that it satisfies (15) a.e. \(t\), hence is a solution to the differential inclusion (15). Hence the proof follows.

\[\square\]

Lemma 3.5 (Faster timescale result). \((\theta_n, w_n) \to \{\langle \theta, \lambda(\theta) \rangle : \theta \in \mathbb{R}^d\}\) a.s.

**Proof.** We first rewrite (4) as

\[\theta_{n+1} = \theta_n + b(n) \left[ \epsilon_n + M_{n+1}^{(3)} \right],\]

where \(\epsilon_n = \frac{a(n)}{b(n)} h(\theta_n, w_n, Z^{(1)}_n) \to 0\) as \(n \to \infty\) a.s. and \(M_{n+1}^{(3)} = \frac{a(n)}{b(n)} M_{n+1}^{(1)}\) for \(n \geq 0\). Let \(\alpha_n = (\theta_n, w_n), \alpha = (\theta, w) \in \mathbb{R}^{d+k}, G(\alpha, z) = (0, g(\alpha, z)), \epsilon'_n = (\epsilon_n, 0), M_{n+1}^{(4)} = (M_{n+1}^{(3)}, M_{n+1}^{(2)}).\) Then one can write (3) and (4) in the framework of Lemma 13 as

\[\alpha_{n+1} = \alpha_n + b(n) \left[ G(\alpha_n, Z^{(2)}_n) + \epsilon'_n + M_{n+1}^{(4)} \right],\]

with \(\epsilon'_n \to 0\) as \(n \to \infty\). \(\alpha_n, n \geq 0\) converges almost surely to an internally chain transitive invariant set of the differential inclusion

\[\dot{\alpha}(t) \in \hat{G}(\alpha(t)),\]

where \(\hat{G}(\alpha) = \{\hat{G}(\alpha, \nu) : \nu \in D^{(2)}(\theta, w)\}.\) In other words, \((\theta_n, w_n), n \geq 0\) converges to an internally chain transitive invariant set of the differential inclusion

\[\dot{w}(t) \in \hat{g}_{\theta(t)}(w(t)), \dot{\theta}(t) = 0.\]

Hence the result follows using (A9) and Lemma 2.1.

\[\square\]

In other words, \(\|w_n - \lambda(\theta_n)\| \to 0\) a.s., i.e, \(\{w_n\}\) asymptotically tracks \(\{\theta_n\}\) a.s.

Now, consider the non-autonomous o.d.e.

\[\dot{\theta}(t) = \tilde{h}(\theta(t), \lambda(\theta(t)), \mu(t)),\]

where \(\mu(t) = \delta_{Z^{(1)}_n}\) when \(t \in [t(n), t(n + 1))\) for \(n \geq 0\) and \(\tilde{h}(\theta, w, \nu) = \int h(\theta, w, z) \nu(dz).\) Let \(\theta^s(t), t \geq s\) denote the solution to (16) with \(\theta^s(s) = \tilde{\theta}(s),\) for \(s \geq 0.\) Then
Lemma 3.6. For any $T > 0$, $\sup_{t \in [s,s+T]} \| \tilde{\theta}(t) - \theta^*(t) \| \to 0$, a.s.

PROOF. The slower recursion corresponds to

$$\theta_{n+1} = \theta_n + a(n) \left[ h(\theta_n, w_n, Z_n^{(1)}) + M_n^{(1)} \right]. \quad (17)$$

Let $t(n + m) \in [t(n), t(n) + T]$. Let $[t] = \max\{ t(k) : t(k) \leq t \}$. Then by construction,

$$\tilde{\theta}(t(n + m)) = \tilde{\theta}(t(n)) + \sum_{k=0}^{m-1} a(n + k)h(\tilde{\theta}(t(n + k)), w_{n+k}, Z_{n+k}^{(1)}) + \delta_{n,n+m} \nu$$

$$= \tilde{\theta}(t(n)) + \sum_{k=0}^{m-1} a(n + k)h(\tilde{\theta}(t(n + k)), \lambda(\tilde{\theta}(t(n + k))), Z_{n+k}^{(1)})$$

$$+ \sum_{k=0}^{m-1} a(n + k)(h(\tilde{\theta}(t(n + k)), w_{n+k}, Z_{n+k}^{(1)}) - h(\tilde{\theta}(t(n + k)), \lambda(\theta_{n+k}), Z_{n+k}^{(1)}))$$

$$+ \delta_{n,n+m},$$

where $\delta_{n,n+m} = \zeta_{n+m} - \zeta_n$ with $\zeta_n = \sum_{m=0}^{n-1} a(m)M_m^{(1)}$, $n \geq 1$.

$$\theta^{(n)}(t(n + m)) = \tilde{\theta}(t(n)) + \int_{t(n)}^{t(n+m)} \tilde{h}(\theta^{(n)}(t), \lambda(\theta^{(n)}(t)), \mu(t))dt$$

$$= \tilde{\theta}(t(n)) + \sum_{k=0}^{m-1} a(n + k)h(\theta^{(n)}(t(n + k)), \lambda(\theta^{(n)}(t(n + k))), Z_{n+k}^{(1)})$$

$$+ \int_{t(n)}^{t(n+m)} (h(\theta^{(n)}(t), \lambda(\theta^{(n)}(t)), \mu(t)) - h(\theta^{(n)}([t]), \lambda(\theta^{(n)}([t]), \mu([t]))))dt.$$  

Let $t(n) \leq t \leq t(n + m)$. Now, if $0 \leq k \leq (m-1)$ and $t \in (t(n+k), t(n+k+1)]$,

$$\| \theta^{(n)}(t) \| \leq \| \tilde{\theta}(t(n)) \| + \int_{t(n)}^{t} \| \tilde{h}(\theta^{(n)}(\tau), \lambda(\theta^{(n)}(\tau)), \mu(\tau)) \| d\tau$$

$$\leq \| \theta_n \| + \sum_{k=0}^{k-1} \int_{t(n+k)}^{t(n+k+1)} \| h(0,0, Z_{n+k}^{(1)}) \| + L^{(1)}(\| \lambda(0) \| + (K+1)\| \theta^{(n)}(\tau) \| )d\tau$$

$$+ \int_{t(n+k)}^{t} \| h(0,0, Z_{n+k}^{(1)}) \| + L^{(1)}(\| \lambda(0) \| + (K+1)\| \theta^{(n)}(\tau) \| )d\tau$$

$$\leq C_0 + (M + L^{(1)}\| \lambda(0) \| )T + L^{(1)}(K+1) \int_{t(n)}^{t} \| \theta^{(n)}(\tau) \| d\tau,$$

where $C_0 = \sup_n \| \theta_n \| < \infty$, $\sup_{z \in S^{(1)}} \| h(0,0,z) \| = M$. By Gronwall’s inequality, it follows that

$$\| \theta^{(n)}(t) \| \leq (C_0 + (M + L^{(1)}\| \lambda(0) \| )T)e^{L^{(1)}(K+1)T}.$$
\[
\|\theta^{(n)}(t) - \theta^{(n)}(t(n + k))\| \leq \int_{t(n+k)}^{t} \|h(\theta^{(n)}(s), \lambda(\theta^{(n)}(s)), Z^{(1)}_{n+k})\|ds \\
\leq (\|h(0, 0, Z^{(1)}_{n+k})\| + L^{(1)}\|\lambda(0)\|)(t - t(n + k)) \\
+ L^{(1)}(K + 1) \int_{t(n+k)}^{t} \|\theta^{(n)}(s)\|ds \\
\leq C_T a(n + k),
\]

where \(C_T = (M + L^{(1)}\|\lambda(0)\|) + L^{(1)}(K + 1)(C_0 + (M + L^{(1)}\|\lambda(0)\|)T)e^{L^{(1)}(K + 1)T} \).

Thus,
\[
\left\| \int_{t(n)}^{t(n+m)} h(\theta^{(n)}(t), \lambda(\theta^{(n)}(t)), \mu(t)) - h(\theta^{(n)}([t]), \lambda(\theta^{(n)}([t])), \mu([t]))dt \right\|
\leq \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} \|h(\theta^{(n)}(t), \lambda(\theta^{(n)}(t)), Z^{(1)}_{n+k}) - h(\theta^{(n)}([t]), \lambda(\theta^{(n)}([t])), Z^{(1)}_{n+k})\|dt \\
\leq L \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} \|\theta^{(n)}(t) - \theta^{(n)}(t(n + k))\|dt \\
\leq C_T L \sum_{k=0}^{m-1} a(n + k)^2 \\
\leq C_T L \sum_{k=0}^{\infty} a(n + k)^2 \rightarrow 0, \text{ where } L = L^{(1)}(K + 1).
\]

Hence
\[
\|\tilde{\theta}(t(n + m)) - \theta^{(n)}(t(n + m))\| \leq \sum_{k=0}^{m-1} \|\tilde{\theta}(t(n + k)) - \theta^{(n)}(t(n + k))\| \\
+ C_T L \sum_{k=0}^{\infty} a(n + k)^2 + \sup_{k \geq 0} \|\delta_{n,n+k}\| \\
+ L^{(1)} \sum_{k=0}^{m-1} a(n + k)\|w_{n+k} - \lambda(\theta_{n+k})\| \\
\leq L \sum_{k=0}^{m-1} a(n + k)\|\tilde{\theta}(t(n + k)) - \theta^{(n)}(t(n + k))\| \\
+ C_T L \sum_{k=0}^{\infty} a(n + k)^2 + \sup_{k \geq 0} \|\delta_{n,n+k}\| \\
+ L^{(1)}T \sup_{k \geq 0} \|w_{n+k} - \lambda(\theta_{n+k})\|, \text{ a.s.}
\]
Define
\[ K_{T,n} = C_T \sum_{k=0}^{\infty} a(n+k)^2 + \sup_{k \geq 0} \| \delta_{n,n+k} \| + L^{(1)} T \sup_{k \geq 0} \| w_{n+k} - \lambda(\theta_{n+k}) \|. \]

Note that \( K_{T,n} \to 0 \) a.s. The remainder of the proof follows in the exact same manner as the tracking lemma, see Lemma 1, Chapter 2 of [10]. \( \square \)

**Lemma 3.7.** Suppose, \( \mu^n(\cdot) \to \mu^\infty(\cdot) \in U^{(1)} \). Let \( \theta^n(\cdot), n = 1, 2, \ldots, \infty \) denote solutions to (16) corresponding to the case where \( \mu(\cdot) \) is replaced by \( \mu^n(\cdot), \) for \( n = 1, 2, \ldots, \infty \). Suppose \( \theta^n(0) \to \theta^\infty(0) \). Then
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \| \theta^n(t) - \theta^\infty(t) \| \to 0
\]
for every \( T > 0 \).

**Proof.** It is shown in Lemma 2.3 of [3] that
\[
\int_0^t \int \tilde{f}(s,z) \mu^n(s,dz)ds - \int_0^t \int \tilde{f}(s,z) \mu^\infty(s,dz)ds \to 0
\]
for any \( \tilde{f} \in C([0,T] \times S) \). Using this, one can see that
\[
\| \int_0^t (\tilde{h}(\theta^\infty(s), \lambda(\theta^\infty(s)), \mu^n(s)) - \tilde{h}(\theta^\infty(s), \lambda(\theta^\infty(s)), \mu^\infty(s)))ds \| \to 0.
\]
This follows because \( \lambda \) is continuous and \( h \) is jointly continuous in its arguments. As a function of \( t \), the integral on the left is equicontinuous and pointwise bounded. By the Arzela-Ascoli theorem, this convergence must in fact be uniform for \( t \) in a compact set. Now for \( t > 0 \),
\[
\| \theta^n(t) - \theta^\infty(t) \|
\leq \| \theta^n(0) - \theta^\infty(0) \| + \int_0^t \| \tilde{h}(\theta^n(s), \lambda(\theta^n(s)), \mu^n(s)) - \tilde{h}(\theta^\infty(s), \lambda(\theta^\infty(s)), \mu^\infty(s)) \| ds
\leq \| \theta^n(0) - \theta^\infty(0) \| + \int_0^t (|| \tilde{h}(\theta^n(s), \lambda(\theta^n(s)), \mu^n(s)) - \tilde{h}(\theta^\infty(s), \lambda(\theta^\infty(s)), \mu^\infty(s)) ||) ds
\leq \int_0^t (|| \tilde{h}(\theta^\infty(s), \lambda(\theta^\infty(s)), \mu^n(s)) - \tilde{h}(\theta^\infty(s), \lambda(\theta^\infty(s)), \mu^\infty(s)) ||) ds.
\]
Now, using the fact that \( \lambda \) is Lipschitz with constant \( K \) the remaining part of the proof follows in the same manner as Lemma 2.3 of [3]. \( \square \)
Note that Lemma 3.7 shows that every limit point \((\tilde{\mu}(\cdot), \tilde{\theta}(\cdot))\) of \((\mu(s+), \tilde{\theta}(s+))\) as \(s \to \infty\) is such that \(\tilde{\theta}(\cdot)\) satisfies (10) with \(\mu(\cdot) = \tilde{\mu}(\cdot)\).

**Lemma 3.8.** Almost surely every limit point of \((\mu(s+), \tilde{\theta}(s+))\) as \(s \to \infty\) is of the form \((\tilde{\mu}(\cdot), \tilde{\theta}(\cdot))\), where \(\tilde{\mu}(\cdot)\) satisfies \(\tilde{\mu}(t) \in D^{(1)}(\tilde{\theta}(t), \lambda(\tilde{\theta}(t)))\).

**Proof.** Suppose that \(u(n) \uparrow \infty, \mu(u(n)+) \to \tilde{\mu}(\cdot)\) and \(\tilde{\theta}(u(n)+) \to \tilde{\theta}(\cdot)\). Let \(\{f_i\}\) be countable dense in the unit ball of \(C(S)\), hence it is a separating class, i.e., \(\forall \int f_id\mu = \int f_id\nu\) implies \(\mu = \nu\). For each \(i\),

\[
\zeta^i_n = \sum_{m=1}^{n-1} a(m)(f_i(Z^{(1)}_{m+1}) - \int f_i(y)p(dy|Z^{(1)}_m, \theta_m, w_m)),
\]

is a zero-mean martingale with \(F_n = \sigma(\theta_m, w_m, Z^{(1)}_m, m \leq n), n \geq 1\). Moreover, it is a square-integrable martingale due to the fact that \(f_i\)'s are bounded and the martingale is a finite sum. Its quadratic variation process

\[
A_n = \sum_{m=0}^{n-1} a(m)^2 E[(f_i(Z^{(1)}_{m+1}) - \int f_i(y)p(dy|Z^{(1)}_m, \theta_m, w_m))^2 | F_m] + E[(\zeta^i_0)^2]
\]

is almost surely convergent. By the martingale convergence theorem, \(\{\zeta^i_n\}\) converges a.s. Let \(\tau(n,t) = \min\{m \geq n : t(m) \geq t(n) + t\}\) for \(t \geq 0, n \geq 0\). Then as \(n \to \infty\),

\[
\sum_{m=n}^{\tau(n,t)} a(m)(f_i(Z^{(1)}_{m+1}) - \int f_i(y)p(dy|Z^{(1)}_m, \theta_m, w_m)) \to 0, \text{ a.s.,}
\]

for \(t > 0\). By our choice of \(\{f_i\}\) and the fact that \(\{a_n\}\) are eventually non-increasing (this is the only time the latter property is used),

\[
\sum_{m=n}^{\tau(n,t)} (a(m) - a(m+1)) f_i(Z^{(1)}_{m+1}) \to 0, \text{ a.s.}
\]

Thus,

\[
\sum_{m=n}^{\tau(n,t)} a(m)(f_i(Z^{(1)}_m) - \int f_i(y)p(dy|Z^{(1)}_m, \theta_m, w_m)) \to 0, \text{ a.s.}
\]

which implies

\[
\int_{t(n)}^{t(n)+t} \left( \int f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s), \tilde{w}(s)) \mu(s, dz) \right) ds \to 0, \text{ a.s.}
\]

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Recall that $u(n)$ can be any general sequence other than $t(n)$. Therefore

$$\int_{u(n)}^{u(n)+t} \left( \int (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s), \tilde{\theta}(s)))\mu(s, dz) \right) ds \to 0, \text{ a.s.}$$

(this follows from the fact that $a(n) \to 0$ and $f_i$’s are bounded) where $\tilde{\theta}(s) = \theta_n$ and $\tilde{w}(s) = w_n$ when $s \in [t(n), t(n + 1)), n \geq 0$. Now, one can claim from the above that

$$\int_{u(n)}^{u(n)+t} \left( \int (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s), \lambda(\tilde{\theta}(s))))\mu(s, dz) \right) ds \to 0, \text{ a.s.}$$

This is due to the fact that the map $S^{(1)} \times \mathbb{R}^{d+k} \ni (z, \theta, w) \mapsto \int f_i(y)p(dy|z, \theta, w)$ is continuous and hence uniformly continuous on the compact set $A = S^{(1)} \times M_1 \times M_2$ where $M_1$ is the compact set s.t. $\theta_n \in M_1 \forall n$ and $M_2 = \{w : \|w\| \leq \max(\sup\|u_n\|, K')\}$ where $K'$ is the bound for the compact set $\lambda(M_1)$. Here we also use the fact that $\|w_m - \lambda(\tilde{\theta}(s))\| \to 0$ for $s \in [t_m, t_{m+1})$ as $\lambda$ is Lipschitz and $\|w_m - \lambda(\theta_m)\| \to 0$. The above convergence is equivalent to

$$\int_0^t \left( \int (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s+u(n)), \lambda(\tilde{\theta}(s+u(n))))\mu(s+u(n), dz) \right) ds \to 0 \text{ a.s.}$$

Fix a sample point in the probability one set on which the convergence above holds for all $i$. Then the convergence above leads to

$$\int_0^t \left( \int f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(s), \lambda(\tilde{\theta}(s))))\tilde{\mu}(s, dz) \right) ds = 0 \forall i. \quad (18)$$

For showing the above, we use one part of the proof from Lemma 2.3 of [3] that if $\mu^n(\cdot) \to \mu^\infty(\cdot) \in \mathcal{U}$ then for any $t$,

$$\int_0^t \int \tilde{f}(s, z)\mu^n(s, dz) ds - \int_0^t \int \tilde{f}(s, z)\mu^\infty(s, dz) ds \to 0$$

for all $\tilde{f} \in C([0, t] \times S)$. In addition, we make use of the fact that $\tilde{f}_n(s, z) = \int f_i(y)p(dy|z, \tilde{\theta}(s+u(n)), \lambda(\tilde{\theta}(s+u(n))))$ converges uniformly to $\tilde{f}(s, z) = \int f_i(y)p(dy|z, \tilde{\theta}(s), \lambda(\tilde{\theta}(s)))$. To prove this, define $g : C([0, t]) \times [0, t] \times S^{(1)} \to \mathbb{R}$ by $g(\theta(\cdot), s, z) = \int f_i(y)p(dy|z, \theta(s), \lambda(\theta(s))))$. Let $A' = \{\tilde{\theta}(u(n)+.)|[u(n), u(n)+t], n \geq 1\} \cup \tilde{\theta}(\cdot)|_{[0, t]}$. Using the same technique as Lemma 8.32 and (A9), i.e., $\lambda$ is Lipschitz (the latter helps to claim that if $\theta_n(\cdot) \to \theta(\cdot)$ uniformly then $\lambda(\theta_n(\cdot)) \to$
\( \lambda(\theta(\cdot)) \) uniformly, it can be seen that \( g \) is continuous. Then \( A' \) is compact as it is a union of a sequence of functions and its limit. So, \( g_{|(A' \times [0,t] \times S(\tau))} \) is uniformly continuous. Then a similar argument as in Lemma 2.3 shows equicontinuity of \( \{\tilde{f}_n(\cdot,..)\} \) that results in uniform convergence and thereby (18). An application of Lebesgue’s theorem in conjunction with (18) shows that
\[
\int (f_i(z) - \int f_i(y)p(dy|z, \tilde{\theta}(t), \lambda(\tilde{\theta}(t)))\tilde{\mu}(t, dz) = 0 \quad \forall i
\]
for a.e. \( t \). By our choice of \( \{f_i\} \), this leads to
\[
\tilde{\mu}(t, dy) = \int p(dy|z, \tilde{\theta}(t), \lambda(\tilde{\theta}(t)))\tilde{\mu}(t, dz),
\]
a.e. \( t \). \( \square \)

Lemma 3.7 shows that every limit point \((\tilde{\mu}(\cdot), \tilde{\theta}(\cdot))\) of \((\mu(s + \cdot), \tilde{\theta}(s + \cdot))\) as \( s \to \infty \) is such that \( \tilde{\theta}(\cdot) \) satisfies (16) with \( \mu(\cdot) = \tilde{\mu}(\cdot) \). Hence, \( \tilde{\theta}(\cdot) \) is absolutely continuous. Moreover, using Lemma 3.8, one can see that it satisfies (9) a.e. \( t \), hence is a solution to the differential inclusion (9).

The following theorem is our main result:

**Theorem 3.9 (Slower timescale result).** Under Assumptions (A1)-(A11),
\[
(\theta_n, w_n) \to \cup_{\theta^* \in A_0(\theta^*, \lambda(\theta^*))} a.s. \text{ as } n \to \infty.
\]

**Proof.** From the previous three lemmas we can see that \( \theta_n \) converges almost surely to an internally chain transitive invariant set of the differential inclusion
\[
\dot{\theta}(t) \in \hat{h}(\theta(t)),
\]
where \( \hat{h}(\theta) = \{\tilde{h}(\theta, \lambda(\theta), \nu) : \nu \in D^{(1)}(\theta, \lambda(\theta))\} \) and hence to \( A_0 \) (using Lemma 2.1 and (A10)). \( \square \)

4. Application : Off-policy temporal difference learning with linear function approximation

In this section, we present an application of our results in the setting of off-policy temporal difference learning with linear function approximation. In this
In the framework, we need to estimate the value function for a target policy $\pi$ given the continuing evolution of the underlying MDP (with finite state and action spaces $S$ and $A$ respectively, specified by expected reward $r(\cdot, \cdot, \cdot)$ and transition probability kernel $p(\cdot|\cdot, \cdot)$) for a behaviour policy $\pi_b$ with $\pi \neq \pi_b$. The authors of [5, 6, 7] have proposed two approaches to solve the problem:

(i) Sub-sampling: In this approach, the transitions which are relevant to deterministic target policy are kept and the rest of the data is discarded from the given “on-policy” trajectory. We use the triplet $(S, R, S')$ to represent (current state, reward, next state). Therefore one has “off-policy” data $(X'_n, R_n, W_n), n \geq 0$ where $E[R_n|X'_n = s, W_n = s'] = r(s, a, s')$, $P(W_n = s'|X'_n = s) = p(s'|s, a)$ with $\pi(s) = a$, $\pi$ being the target policy and $X'_n, n \geq 0$ is a random process generated by sampling the “on-policy” trajectory at increasing stopping times.

(ii) Importance-weighting: In this approach, unlike sub-sampling, all the data from the given “on-policy” trajectory is used. One advantage of this method is that we can allow the policy to be randomized in case of both behaviour and target policies unlike the sub-sampling scenario where one can use only deterministic policy as a target policy.

Then they introduce gradient temporal difference learning algorithms (GTD) [5, 6, 7] for both the approaches.

Currently, all GTD algorithms make the assumption that data are available in the “off-policy” setting i.e. of the form $(X'_n, R_n, W_n), n \geq 0$ where $\{X'_n\}$ are i.i.d, $E[R_n|X'_n = s, W_n = s'] = r(s, a, s')$ and $P(W_n = s'|X'_n = s) = p(s'|s, a)$ with $\pi(s) = a$, $\pi$ being the deterministic target policy. However, such data cannot be generated from sub-sampling given only the “on-policy” trajectory. The reason is that a Markov chain sampled at increasing stopping times cannot be i.i.d and even not Markov in general. In the following, we show how gradient temporal-difference learning along with importance weighting can be used to solve the off-policy convergence problem stated above for TD when only the “on-policy” trajectory is available.
Remark 2. Note that the reason for introducing importance weighting in [7] was to use all the data from the given “on-policy” trajectory, i.e., some kind of efficiency gain compared to sub-sampling. But, here this property is useful to us for a different reason: the on-policy trajectory is Markov unlike the “off-policy” data generated from sub-sampling, allowing us to analyze the convergence of GTD algorithms with the theory developed in the previous sections.

4.1. Problem Definition

Suppose we are given an on-policy trajectory \((X_n, A_n, R_n, X_{n+1}), n \geq 0\) where \(\{X_n\}\) is a time-homogeneous irreducible Markov chain with unique stationary distribution \(\nu\) and generated from a behavior policy \(\pi_b \neq \pi\). Here the quadruplet \((S, A, R, S')\) represents (current state, action, reward, next state). Also, assume that \(\pi_b(a|s) > 0 \ \forall s \in S, a \in A\). We need to find the solution \(\theta^*\) for the following:

\[
0 = \sum_{s,a,s'} \nu(s)\pi(a|s)p(s'|s,a)\delta(\theta; s, a, s')\phi(s)
\]

\[
= E[R_{X_n}\delta_{X_n,R_n,X_{n+1}}(\theta)\phi(X)]
\]

\[
= b - A\theta,
\]

where

(i) \(\theta \in \mathbb{R}^d\) is the parameter for value function.

(ii) \(\phi : S \to \mathbb{R}^d\) is a vector of state features.

(iii) \(X \sim \nu\).

(iv) \(0 < \gamma < 1\) is the discount factor.

(v) \(E[R_n|X_n = s, X_{n+1} = s'] = \sum_{a \in A} \pi_b(a|s)r(s,a,s')\).

(vi) \(P(X_{n+1} = s'|X = s) = \sum_{a \in A} \pi_b(a|s)p(s'|s,a)\).

(vii) \(\delta(\theta; s, a, s') = r(s,a,s') + \gamma \theta^T\phi(s') - \theta^T\phi(s)\) is the temporal difference term with expected reward.

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(viii) \( \rho_{X,A_n} = \frac{\pi(A_n|X)}{\pi(A_n|X)} \).

(ix) \( \delta_{X,R_n,X_{n+1}} = R_n + \gamma \theta^T \phi(X_{n+1}) - \theta^T \phi(X) \) is the online temporal difference.

(x) \( A = E[\rho_{X,A_n} \phi(X)(\phi(X) - \gamma \phi(X_{n+1}))^T] \).

(xi) \( b = E[\rho_{X,A_n} R_n \phi(X)] \).

Hence the desired approximate value function under the target policy \( \pi \) is \( V_\pi^* = \theta^* \).

Let \( V_\theta = \theta^T \phi \). It is well-known (2) that \( \theta^* \) satisfies the projected fixed point equation namely

\[ V_\theta = \Pi_{\mathcal{G},\nu} T^\pi V_\theta, \]

where

\[ \Pi_{\mathcal{G},\nu} \tilde{V} = \arg\min_{f \in \mathcal{G}} \| \tilde{V} - f \|_\nu, \]

with \( \mathcal{G} = \{ V_\theta | \theta \in \mathbb{R}^d \} \) and the Bellman operator

\[ T^\pi V_\theta(i) = \sum_{j \in S} \sum_{a \in A} \pi(a|s)p(j|i,a) [\gamma V_\theta(i) + r(i,a,j)]. \]

Therefore to find \( \theta^* \), the idea is to minimize the mean square projected Bellman error \( J(\theta) = \| V_\theta - \Pi_{\mathcal{G},\nu} T^\pi V_\theta \|_\nu^2 \) using stochastic gradient descent. It can be shown that the expression of gradient contains product of multiple expectations. Such framework can be modelled by two-timescale stochastic approximation where one iterate stores the quasi-stationary estimates of some of the expectations and the other iterate is used for sampling.

4.2. The TDC Algorithm with importance-weighting

We consider the TDC (Temporal Difference with Correction) algorithm with importance-weighting from Sections 4.2 and 5.2 of [2]. The gradient in this case can be shown to satisfy

\[ -\frac{1}{2} \nabla J(\theta) = E[\rho_{X,R_n} \delta_{X,R_n,X_{n+1}}(\theta) \phi(X)] - \gamma E[\rho_{X,R_n} \phi(X_{n+1}) \phi(X)^T] w(\theta), \]

\[ w(\theta) = E[\phi(X)\phi(X)^T]^{-1} E[\rho_{X,R_n} \delta_{X,R_n,X_{n+1}}(\theta) \phi(X)]. \]
Define $\phi_n = \phi(X_n)$, $\phi'_n = \phi(X_{n+1})$, $\delta_n(\theta) = \delta_{X_n,R_n,X_{n+1}}(\theta)$ and $\rho_n = \rho_{X_n,A_n}$. Therefore the associated iterations in this algorithm are:

\[ \theta_{n+1} = \theta_n + a(n) \rho_n \left[ \delta_n(\theta_n) \phi_n - \gamma \phi'_n \phi^T_n w_n \right], \]
\[ w_{n+1} = w_n + b(n) \left[ (\rho_n \delta_n(\theta_n) - \phi^T_n w_n) \phi_n \right], \]

(19)

with $\frac{a(n)}{b(n)} \to 0$ as $n \to \infty$.

4.3. Convergence Proof

Theorem 4.1 (Convergence of TDC with importance-weighting). Consider the iterations (19) of the TDC. Assume the following:

(i) $a(n) > 0, b(n) > 0, n \geq 0$ are non-increasing,
(ii) $\sum_n a(n) = \sum_n b(n) = \infty$,
(iii) $\sum_n a(n)^2, \sum_n b(n)^2 < \infty$,
(iv) $\{(X_n, R_n, X_{n+1}), n \geq 0\}$ is such that $\{X_n\}$ is a time-homogeneous finite state irreducible Markov chain generated from the behavior policy $\pi_b$ with unique stationary distribution $\nu$. $E[R_n|X_n = s, X_{n+1} = s'] = \sum_{a \in A} \pi_b(a|s)r(s,a,s')$ and $P(X_{n+1} = s'|X_n = s) = \sum_{a \in A} \pi_b(a|s)p(s'|s,a)$ where $\pi_b$ is the behaviour policy. $\pi \neq \pi_b$. Also, $E[R^2_n|X_n, X_{n+1}] < \infty \ \forall n$ almost surely, and
(v) $E[\phi(X)^T \phi(X)] > 0$ and $E[\rho_{X,R_n} \phi(X)(\phi(X) - \gamma \phi(X_{n+1}))^T] > 0$ where $X \sim \nu$.
(vi) $\pi_b(a|s) > 0 \ \forall s \in S, a \in A$.
(vii) $\sup_n (\|\theta_n\| + \|w_n\|) < \infty \ \text{w.p. 1}$.

Then the parameter vector $\theta_n$ converges with probability one as $n \to \infty$ to the TD(0) solution (12).

Proof. The iterations (19) can be cast into the framework of Section 2 with
(i) \( Z_n^{(i)} = X_{n-1} \),

(ii) \( h(\theta, w, z) = E[(\rho_n(\delta_n(\theta) - \gamma\phi_n'\phi_n^T w))|X_{n-1} = z] \),

(iii) \( g(\theta, w, z) = E[(\rho_n(\delta_n(\theta) - \phi_n^T w)|X_{n-1} = z] \),

(iv) \( M^{(1)}_{n+1} = \rho_n(\delta_n(\theta_n)\phi_n - \gamma\phi_n'\phi_n^T w_n) - E[\rho_n(\delta_n(\theta_n)\phi_n - \gamma\phi_n'\phi_n^T w_n)|X_{n-1}, \theta_n, w_n] \),

(v) \( M^{(2)}_{n+1} = (\rho_n\delta_n(\theta_n) - \phi_n^T w_n)\phi_n - E[(\rho_n\delta_n(\theta_n) - \phi_n^T w_n)|X_{n-1}, \theta_n, w_n] \),

(vi) \( F_n = \sigma(\theta_m, w_m, R_{m-1}, X_{m-1}, A_{m-1}, m \leq n, i = 1, 2), n \geq 0 \).

Note that in (ii) and (iii) we can define \( h \) and \( g \) independent of \( n \) due to time-homogeneity of \( \{X_n\} \).

Now, we verify the assumptions \((A1)-(A11)\) (mentioned in Sections 2.2 and 2.3) for our application:

(i) \((A1)\) and \((A6)\): \( Z_n^{(i)}, \forall n, i = 1, 2 \) takes values in compact metric space as \( \{X_n\} \) is a finite state Markov chain.

(ii) \((A7)\): Continuity of transition kernel follows trivially from the fact that we have a finite state MDP.

(iii) \((A2)\):

\[
\|h(\theta, w, z) - h(\theta', w', z)\| = \|E[\rho_n(\theta - \theta')^T(\gamma\phi(W_n) - \phi(X_n))\phi(X_n) - \gamma\rho_n\phi(W_n)\phi(X_n)^T(w - w')|X_{n-1} = z]|| \\
\leq L(2\|\theta - \theta'\|M^2 + \|w - w'\|M^2),
\]

where \( M = \max_{s \in S} \|\phi(s)\| \) with \( S \) being the state space of the MDP and \( L = \max_{(s,a) \in (S \times A)} \frac{p(a|s)}{p(a|s)} \). Hence \( h \) is Lipschitz continuous in the first two arguments uniformly w.r.t the third. In the last inequality above, we use the Cauchy-Schwarz inequality.

(iv) \((A3)\): As with the case of \( h \)(see (iii)), \( g \) can be shown to be Lipschitz continuous in the first two arguments uniformly w.r.t the third.
(v) **(A2)-(A3):** Joint continuity of $h$ and $g$ follows from (iii) and (iv) respectively as well as the finiteness of $S$.

(vi) **(A4):** Clearly, $\{M^{(i)}_{n+1}\}, i = 1, 2$ are martingale difference sequences w.r.t. increasing $\sigma$-fields $F_n$. Note that $E[\|M^{(i)}_{n+1}\|^2|F_n] \leq K(1 + \|\theta_n\|^2 + \|w_n\|^2)$ a.s., $n \geq 0$ since $E[R_{n+1}^2|X_n, X_{n+1}] < \infty \forall n$ almost surely and $S$ is finite.

(vii) **(A8):** $\{X_n\}$ is an irreducible time-homogeneous Markov chain with unique stationary distribution.

(viii) **(A5):** This follows from the conditions (i)-(iii) in the statement of Theorem 4.1.

Now, one can see that the faster o.d.e. becomes

$$\dot{w}(t) = E[\rho X, A_n \delta X, R_n, X_{n+1}(\theta)\phi(X)] - E[\phi(X)^T \phi(X)]w(t).$$

Clearly, $\frac{E[\rho X, A_n \delta X, R_n, X_{n+1}(\theta)\phi(X)]}{E[\phi(X)^T \phi(X)]}$ is the globally asymptotically stable equilibrium of the o.d.e. which turns out to be the global attractor for the O.D.E using Lemma 2.2. Hence $\lambda(\theta) = \frac{E[\rho X, A_n \delta X, R_n, X_{n+1}(\theta)\phi(X)]}{E[\phi(X)^T \phi(X)]}$ and it is Lipschitz continuous in $\theta$, verifying **(A9).** For the slower o.d.e., the global attractor is $\frac{E[\rho X, A_n \delta X, R_n, \phi(X)]}{E[\rho X, A_n \phi(X)]}$ verifying **(A10).** The attractor set here is a singleton. Therefore the assumptions **(A1) – (A11)** are verified. The proof would then follow from Theorem 3.9. Note that the second equality in (19) follows from the assumption (vi) in the statement of Theorem 4.1. □

**Remark 3.** Here we assume the stability of the iterates (19). Certain sufficient conditions without proof have been given for showing stability of single timescale stochastic recursions with controlled Markov noise [10, p. 75, Theorem 9]. This subsequently needs to be extended to the case of two-timescale recursions. Then one can put assumptions on $\phi(\cdot)$ so that the sufficient conditions to ensure stability are satisfied and then we may use Theorem 4.1.

Another way to ensure boundedness of the iterates is to use a projection operator. However, projection may introduce spurious fixed points on the boundary of the projection region. Therefore we do not use projection in our algorithm.
Remark 4. Convergence analysis for TDC with importance weighting along with eligibility traces cf. [3, p. 74] where it is called GTD(\(\lambda\)) can be done similarly using our results. The main advantage is that it works for \(\lambda < \frac{1}{L\gamma}\) (\(\lambda \in [0, 1]\) being the eligibility function) whereas the analysis in [4] is shown only for \(\lambda\) very close to 1.

5. Conclusion

We presented a general framework for two-timescale stochastic approximation with controlled Markov noise. Moreover, using a special case of our results, i.e., when the random process is a finite state irreducible time-homogeneous Markov chain (hence has a unique stationary distribution) and uncontrolled (i.e., does not depend on iterates), we provided a rigorous proof of convergence for off-policy temporal difference learning algorithm that is also extendible to eligibility traces for a sufficiently large range of \(\lambda\) with linear function approximation under the assumption that the “on-policy” trajectory for a behaviour policy is only available. This has previously not been done to our knowledge.

Note that here \(\lambda\) is a single-valued map which turns out to be Lipschitz continuous. But, in reality, this can be a set-valued map (for example, for the RL application shown, if the underlying Markov chain is not irreducible then, the limit is a differential inclusion of the form \(\dot{\theta}(t) \in h(\theta(t))\) with \(h(\theta) = \{A(\nu) - C\theta : \nu \in D\}, C > 0\). Then \(\{\frac{A(\nu)}{C} : \nu \in D\}\) would lie in the global attractor set for the inclusion), thus requiring our results to be extended for set-valued \(\lambda\). Another future direction would be to investigate the case where vector fields are set-valued maps for which there is no result yet for the single-timescale framework. This then can be extended for the case of two-timescale recursions as well.

References

[1] V.S.Borkar. Stochastic approximation with two time scales. Systems and Control Letters 1997;29(5):291–4.
[2] T.Degis , M.White , R.S.Sutton . Off-policy actor-critic. International Conference on Machine Learning; Scotland, UK; 2012.

[3] V.S.Borkar . Stochastic approximation with ‘controlled Markov noise’. Systems and Control Letters 2006;55(2):139–45.

[4] H.Yu . Least squares temporal difference methods: an analysis under general conditions. SIAM Journal on Control and Optimization 2012;50(6):3310–43.

[5] R.S.Sutton , H.R.Maei , C.Szepesvári . A convergent O(n) algorithm for off-policy temporal-difference learning with linear function approximation. Advances in Neural Information Processing Systems; Vancouver, B.C., Canada; 2008.

[6] R.S.Sutton , H.R.Maei , D.Precup , S.Bhatnagar , D.Silver , E.Wiewiora . Fast gradient-descent methods for temporal-difference learning with linear function approximation. International Conference on Machine Learning; Montreal, Canada; 2009.

[7] H.R.Maei . Gradient temporal-difference learning algorithms. Ph.D. thesis; University of Alberta; 2011. URL: webdocs.cs.ualberta.ca/~sutton/papers/maei-thesis-2011.pdf

[8] V.B.Tadic . Almost sure convergence of two time-scale stochastic approximation algorithms. American Control Conference; Boston; 2004.

[9] M.Benaim , J.Hofbauer , S.Sorin . Stochastic approximations and differential inclusions. SIAM Journal of Control and Optimization 2005;44(1):328–48.

[10] V.S.Borkar . Stochastic Approximation : A Dynamic Systems Viewpoint. Cambridge University Press; 2008.

[11] J.Aubin , A.Cellina . Differential Inclusions: Set-Valued Maps and Viability Theory. Springer; 1984.
[12] V.S.Borkar . Probability Theory : An Advanced Course. Springer; 1995.

[13] W.Rudin . Principles of Mathematical Analysis. 3rd ed.; McGraw-Hill Science/Engineering/Math; 1976.