A SINGULAR PERTURBATION LIMIT OF DIFFUSED INTERFACE ENERGY WITH A FIXED CONTACT ANGLE CONDITION

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Abstract. We study a general asymptotic behavior of critical points of a diffused interface energy with a fixed contact angle condition defined on a domain $\Omega \subset \mathbb{R}^n$. We show that the limit varifold derived from the diffused energy satisfies a generalized contact angle condition on the boundary under a set of assumptions.

1. Introduction

In this paper, we consider a general asymptotic behavior of critical points of the energy functional

$$E_\varepsilon(u) = \int_{\Omega} \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \, dx + \int_{\partial\Omega} \sigma(u) \, d\mathcal{H}^{n-1}$$

under the restriction

$$\int_{\Omega} u \, dx = m,$$

where $\varepsilon \in (0, 1)$ is a small parameter, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u$ is a function defined on $\bar{\Omega}$, $W$ is a double well potential with strict minima at $\pm 1$, $\sigma$ is a function on $\mathbb{R}$ and $m \in (-|\Omega|, |\Omega|)$ is a fixed constant. $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure. According to the van der Waals-Cahn-Hilliard theory [3] and Cahn’s approach [2], the energy (1.1) is a typical energy modeling separation phenomena for capillary surfaces (see [12]). The function $u$, the strict minima of $W$ and the function $\sigma$ correspond to the normalized density of a multi-phase fluid, stable fluid phases and a contact energy density between the fluid and the container wall $\partial\Omega$, respectively. The condition (1.2) corresponds to fixing the total mass of the fluid in $\Omega$. If $E_\varepsilon(u_\varepsilon)$ is uniformly bounded with respect to $\varepsilon \in (0, 1)$ for critical points $u_\varepsilon$ of $E_\varepsilon$, we may expect that the domain $\Omega$ is mostly divided into two regions $\{u_\varepsilon \approx 1\}$ and $\{u_\varepsilon \approx -1\}$ for sufficiently small $\varepsilon$.

For energy minimizer of (1.1), Modica studied the contact angle condition in [12] within the framework of $\Gamma$-convergence. He showed the existence of energy minimizers $\{u_\varepsilon\}_{\varepsilon \in (0, 1)}$ and the subsequential limit $u$ in $L^1$ as $\varepsilon \to 0$, and proved that $u = \pm 1$ a.e. on $\Omega$. Furthermore, in a weak sense, he showed under a suitable assumption on $\sigma$ that the contact angle $\theta$ formed

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by the boundary $\partial \Omega$ and the reduced boundary of $\{u = 1\}$ in $\Omega$ is equal to
\begin{equation}
\theta = \arccos \left( \frac{\sigma(1) - \sigma(-1)}{c_0} \right),
\end{equation}
where
\begin{equation}
c_0 = \int_{-1}^{1} \sqrt{2W(s)} \, ds.
\end{equation}

The characterization of the contact angle condition is through the energy minimality of the $\Gamma$-limit functional and it is essential that $u_\varepsilon$'s are global energy minimizers for the $\Gamma$-convergence argument. In view of the corresponding dynamical problem, however, it is interesting to analyze the problem under a weaker assumption of being critical points. Our aim is to study the rigorous characterization of the contact angle condition due to the presence of the second term of (1.1) as $\varepsilon \to 0$.

This line of research has been carried out by introducing a natural varifold associated with $u_\varepsilon$ (cf. [5, 6, 13, 14]). Heuristically, the weight measure of the varifold behaves more or less like a surface measure of phase interface. One of the key tools to analyze a behavior of the varifold is the first variation. In this paper, we focus on a behavior of the first variation of the associated varifolds up to the boundary and characterize the contact angle condition for the limit varifold along the line studied in [7], as described in Theorem 3.2. Roughly speaking, we give a characterization of the tangential component of the first variation on $\partial \Omega$ which reduces to an appropriate contact angle condition if all relevant quantities are smooth.

Very closely related is the case of Neumann boundary condition, namely, the case of $\sigma \equiv 0$. Mizuno and the second author [10] studied the gradient flow of (1.1) in the case of $\sigma \equiv 0$ and analyzed a behavior of the first variation of the moving varifolds up to the boundary to derive a suitable Neumann boundary condition for the limit Brakke flow.

This paper is organized as follows. In Section 2 we state known characterizations of limit varifold in the interior of the domain due to [5, 14] along with setting our notation. Section 3 describes main results of the present paper, which are the characterization of boundary behavior of the limit varifold. In Section 4 we prove the main results and we give final remarks in Section 5.

## 2. Preliminaries and interior behavior

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We first describe the interior behavior of general critical points of $E_\varepsilon$ under the following assumptions. Here we ignore the boundary conditions until the next section.

(A1) $W \in C^\infty(\mathbb{R})$ satisfies $W \geq 0$; $W(\pm 1) = 0$; for some $\gamma \in (0, 1)$, $W''(s) > 0$ for all $|s| \geq \gamma$; $W$ has a unique local maximum in $(-1, 1)$. 

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\[ \theta = \arccos \left( \frac{\sigma(1) - \sigma(-1)}{c_0} \right), \]

where

\[ c_0 = \int_{-1}^{1} \sqrt{2W(s)} \, ds. \]
(A2) For a sequence \( \{\varepsilon_i\}_{i=1}^{\infty} \subset (0, 1) \) with \( \lim_{i \to \infty} \varepsilon_i = 0 \), \( \{u_{\varepsilon_i}\}_{i=1}^{\infty} \subset C^\infty(\Omega) \) satisfy
\[
-\varepsilon_i \Delta u_{\varepsilon_i} + \frac{W'(u_{\varepsilon_i})}{\varepsilon_i} = \lambda_{\varepsilon_i} \quad \text{on } \Omega
\]
for some \( \lambda_{\varepsilon_i} \in \mathbb{R} \).

(A3) There exist constants \( C > 0 \) and \( E_0 > 0 \) such that
\[
\sup_i \|u_{\varepsilon_i}\|_{L^\infty(\Omega)} \leq C, \quad \sup_i |\lambda_{\varepsilon_i}| \leq C
\]
and
\[
\sup_i E_{\varepsilon_i}(u_{\varepsilon_i}) \leq E_0.
\]

Remark 2.1. Assumption (A1) says that \( W \) is a W-shaped function with two non-degenerate minima \( \pm 1 \). The equation \((2.1)\) means that \( u_{\varepsilon_i} \) is a critical point of \( E_{\varepsilon_i} \) with the volume constraint \((1.2)\). Since we are primarily interested in \( u_{\varepsilon} \) whose values are not far from \([-1, 1]\) and whose energy remains \( O(1) \), \((2.2)\) and \((2.3)\) are reasonable assumptions. They are the same set of assumptions in [5, 14].

We next summarize the direct consequences of (A1)-(A3) due to [5, 14] which give a fairly complete characterization of the limiting behavior in the interior of \( \Omega \). We introduce notation and definitions related to varifolds to describe the results. We refer to [1, 15] for more information on varifold.

Let \( G(n, n-1) \) denote the space of \((n-1)\)-dimensional subspaces of \( \mathbb{R}^n \). We also regard \( S \in G(n, n-1) \) as the orthogonal projection of \( \mathbb{R}^n \) onto \( S \), and write \( S_1 \cdot S_2 = \text{trace}(S_1 \circ S_2) \). For open \( U \subset \mathbb{R}^n \), we say \( V \) is an \((n-1)\)-dimensional varifold in \( U \) if \( V \) is a Radon measure on \( G_{n-1}(U) = U \times G(n, n-1) \). Let \( V_{n-1}(U) \) denote the set of all \((n-1)\)-dimensional varifolds. Convergence in the varifold sense means convergence in the usual sense of measures. For \( V \in V_{n-1}(U) \), we let \( \|V\| \) be the weight measure of \( V \). Let \( \text{spt}\|V\| \) be the support of \( \|V\| \). For \( V \in V_{n-1}(U) \), we define the first variation of \( V \) by
\[
\delta V(g) := \int_{G_{n-1}(U)} \nabla g(x) \cdot S \, dV(x, S)
\]
for any vector field \( g \in C_c^1(U; \mathbb{R}^n) \). We also write the total variation of \( \delta V \) by \( \|\delta V\| \). If \( \|\delta V\| \) is a Radon measure, we may apply the Radon-Nikodym theorem to \( \delta V \) with respect to \( \|V\| \). Writing the singular part of \( \|\delta V\| \) with respect to \( \|V\| \) as \( \|\delta V\|_{\text{sing}} \), we have \( \|V\| \) measurable vector field \( h \), \( \|\delta V\| \) measurable \( \nu_{\text{sing}} \) with \( |\nu_{\text{sing}}| = 1 \|\delta V\| \)-a.e., and a Borel set \( Z \subset U \) such that \( \|V\|(Z) = 0 \) with
\[
\delta V(g) = -\int_U \langle g, h \rangle \, d\|V\| + \int_Z \langle \nu_{\text{sing}}, g \rangle \, d\|\delta V\|_{\text{sing}}
\]
for all $g \in C_c^1(U; \mathbb{R}^n)$. We recall that $h$ is the generalized mean curvature vector of $V$, $\nu_{\text{sing}}$ is the (outer-pointing) generalized co-normal of $V$ and $Z$ is the generalized boundary of $V$.

If $V \in \mathbf{V}_{n-1}(U)$ satisfies

$$(2.5) \quad V(\phi) = \int_M \phi(x, \Tan_x M) \Theta(x) \, d\mathcal{H}^{n-1}(x)$$

for all $\phi \in C_c(G_{n-1}(U))$, where $M$ is an $\mathcal{H}^{n-1}$ measurable, countably $n-1$ rectifiable set, $\Tan_x M$ is the approximate tangent space which exists for $\mathcal{H}^{n-1}$ a.e. on $M$, $\Theta : M \to \mathbb{N}$ is an integer-valued $\mathcal{H}^{n-1}$ measurable function, $V$ is said to be integral. $\mathbf{IV}_{n-1}(U)$ denotes the set of all integral varifolds. Note that the $n-1$ dimensional density of $\|V\|$ (denoted by $\Theta(\|V\|, x)$) exists $\|V\|$ a.e. and is equal to $\Theta(x)$ in (2.5).

Let $u_{\varepsilon_i}$ be the functions defined on $\overline{\Omega}$ satisfying (A1)-(A3). For each $u_{\varepsilon_i}$, we define a varifold $V_{\varepsilon_i} \in \mathbf{V}_{n-1}(\mathbb{R}^n)$ as follows. Define a Radon measure $\mu_{\varepsilon_i}$ on $\mathbb{R}^n$ by

$$d\mu_{\varepsilon_i} := \frac{1}{c_0} \left( \frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} + \frac{W(u_{\varepsilon_i})}{\varepsilon_i} \right) dL^n|_\Omega,$$

where $L^n$ is the Lebesgue measure on $\mathbb{R}^n$ and $c_0$ is as in (1.4). Define $V_{\varepsilon_i} \in \mathbf{V}_{n-1}(\mathbb{R}^n)$ by

$$V_{\varepsilon_i}(\phi) := \int_{\{ \nabla u_{\varepsilon_i} \neq 0 \}} \phi \left( x, I - \frac{\nabla u_{\varepsilon_i}}{|\nabla u_{\varepsilon_i}|} \otimes \frac{\nabla u_{\varepsilon_i}}{|\nabla u_{\varepsilon_i}|} \right) d\mu_{\varepsilon_i},$$

for $\phi \in C_c(G_{n-1}(\mathbb{R}^n))$, where $I$ is the $n \times n$ identity matrix. Then by the definition, we have

$$(2.6) \quad \delta V_{\varepsilon_i}(g) = \int_{\{ \nabla u_{\varepsilon_i} \neq 0 \}} \nabla g \cdot \left( I - \frac{\nabla u_{\varepsilon_i}}{|\nabla u_{\varepsilon_i}|} \otimes \frac{\nabla u_{\varepsilon_i}}{|\nabla u_{\varepsilon_i}|} \right) d\mu_{\varepsilon_i},$$

for each $g \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$. In addition, we define a function

$$\xi_{\varepsilon_i} := \frac{1}{c_0} \left( \frac{\varepsilon_i |\nabla u_{\varepsilon_i}|^2}{2} - \frac{W(u_{\varepsilon_i})}{\varepsilon_i} \right)$$

on $\overline{\Omega}$ and $\xi_{\varepsilon_i} := 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$. This is called a discrepancy in the literature. The following two theorems are direct consequences of [5, 14].

**Theorem 2.2.** ([5 Theorem 1]) Under the assumptions (A1)-(A3), there exists a subsequence (denoted by the same index) such that

$$\lambda_{\varepsilon_i} \to \lambda, \quad u_{\varepsilon_i} \to u \text{ in } L^1(\Omega), \quad u \in BV(\Omega), \quad V_{\varepsilon_i} \to V \text{ in the varifold sense of } \mathbf{V}_{n-1}(\mathbb{R}^n),$$

$$|\xi_{\varepsilon_i}| dL^n \to d\xi \text{ in the sense of Radon measures on } \mathbb{R}^n.$$

Moreover,

1. $u(x) = \pm 1$ for $L^n$ a.e. on $\Omega$,
2. $V|_{G_{n-1}(\Omega)} \in \mathbf{IV}_{n-1}(\Omega),$
3. $\text{spt } \xi \subset \partial \Omega$ and $\xi \leq \|V\| |_{\partial \Omega},$
4. $\Omega \cap \text{spt } \partial^* \{ u = 1 \} \subset \text{spt } \|V\|$ and $u_{\varepsilon_i} \to \pm 1$ locally uniformly on $\Omega \setminus \text{spt } \|V\|$. 
By the well-known property of $BV$ functions (see for example [4]), away from the reduced boundary

$$M := \Omega \cap \partial^* \{ u = 1 \}$$

of $\{ u = 1 \}$ in $\Omega$, we may define $u(x) \in \{ \pm 1 \}$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Omega \setminus M$. We also write $\nabla u / |\nabla u|$ which exists for $\mathcal{H}^{n-1}$ a.e. on $M$ as the inward-pointing unit normal to $\partial^* \{ u = 1 \}$.

**Theorem 2.3.** ([14, Theorem 3.2]) Let $\lambda, u, V, M$ be as above. Then we have the following.

(a) $V|_{G_{n-1}(\Omega)}$ (as an element of $V_{n-1}(\Omega)$) has a generalized mean curvature $h$ with $\| \delta V \|_{\text{sing}} = 0$ in $\Omega$. We have $\mathcal{H}^{n-1}(M \setminus \text{spt} \| V \|) = 0$.

(b) $V$ has a locally constant mean curvature in $\Omega$, namely,

$$h = \begin{cases} \frac{2\lambda}{c_0} \frac{\nabla u}{|\nabla u|} \mathcal{H}^{n-1} \text{a.e. on } M, \\ 0 \mathcal{H}^{n-1} \text{a.e. on spt } \| V \| \cap \Omega \setminus M \end{cases}$$

and

$$\Theta(\| V \|, x) = \begin{cases} \text{odd } \mathcal{H}^{n-1} \text{a.e. on } M, \\ \text{even } \mathcal{H}^{n-1} \text{a.e. on spt } \| V \| \cap \Omega \setminus M. \end{cases}$$

(c) If $\lambda \neq 0$, then “odd” in (b) is replaced by “1”.

(d) If $\lambda > 0$, then $\mathcal{H}^{n-1}(\{ u = 1 \} \cap \text{spt } \| V \| \cap \Omega \setminus M) = 0$. If $\lambda < 0$, then $\mathcal{H}^{n-1}(\{ u = -1 \} \cap \text{spt } \| V \| \cap \Omega \setminus M) = 0$.

The portion of “even multiplicity part” $\text{spt } \| V \| \cap \Omega \setminus M$ may be regarded as a hidden boundary, in the sense that it does not appear as a boundary of $\{ u = 1 \}$. Just to clarify the point of above claim, consider the case when $\lambda = 0$. Then (b) says that $V$ is stationary in $\Omega$ with the density parity as described. If $\lambda > 0$, then the even multiplicity part which has 0 mean curvature only appears (if it does exist non-trivially) in the region of $\{ u = -1 \}$ due to (d). In the following, Theorem 2.3 is not used and it is presented for the convenience of the reader.

**Remark 2.4.** It is important to note for the following section that [5, Theorem 1] proves $|\xi_{\varepsilon_i}| \to 0$ on $\Omega$. This leaves the possibility of having non-trivial measure $\xi$ living only on $\partial \Omega$. When $\Omega$ is strictly convex and $\sigma = 0$, it is proved that $\xi = 0$ in [10]. We conjecture that $\xi = 0$ also for non-trivial $\sigma$ and under some geometric condition (such as convexity) on $\Omega$. Due to the trivial inequality $\xi \leq \| V \|$, if $\| V \|_{|\partial \Omega|} = 0$, then we have $\xi = 0$. Thus, if the measures $\mu_{\varepsilon_i}$ do not concentrate on $\partial \Omega$, we have $\xi = 0$ in particular.

### 3. Boundary Behavior

In addition to (A1)-(A3) in the previous section, we now consider the following three assumptions.
(A4) A given function $\sigma \in C^\infty(\mathbb{R})$ satisfies

\[
|\sigma'(s)| \leq C_1 \sqrt{2W(s)}
\]
for some $C_1 \in [0, 1)$ and for all $s \in \mathbb{R}$.

(A5) The functions $\{u_{\varepsilon_i}\}$ as in (A2) satisfy

\[
\varepsilon_i \langle \nabla u_{\varepsilon_i}, \nu \rangle = -\sigma'(u_{\varepsilon_i}) \text{ on } \partial \Omega,
\]
where $\nu$ is the outer unit normal vector field on $\partial \Omega$.

(A6) $\xi = 0$, where $\xi$ is as in Theorem 2.2 (3).

From a heuristic argument as well as the $\Gamma$-convergence result of [12], note that we expect the energy $E_{\varepsilon}$ should behave like

\[
E_{\varepsilon}(u_{\varepsilon}) \approx c_0 \mathcal{H}^{n-1}(\Omega \cap \partial \{u = 1\}) + (\sigma(1) - \sigma(-1)) \mathcal{H}^{n-1}(\partial \Omega \cap \{u = 1\}) + \text{Constant}.
\]

Imposing (A4) ensures that $|\sigma(1) - \sigma(-1)| \leq \int_{-1}^{1} |\sigma'(s)| \, ds \leq C_1 \int_{-1}^{1} \sqrt{2W(s)} \, ds < c_0$. Physically, this ensures that the contact energy density $|\sigma(1) - \sigma(-1)|$ of the interface $\{u_{\varepsilon} \approx 1\}$ with $\partial \Omega$ is strictly smaller than the surface tension density $c_0$ of the interface inside of $\Omega$. As $|\sigma(1) - \sigma(-1)| \nearrow c_0$, we expect to have a “perfect wetting” (see [21]) of the interface. The equality (3.2) is satisfied for critical points of (1.1) with the volume constraint (1.2), as one can check easily by taking the first variation of $E_{\varepsilon}$.

For (A6), as mentioned in Remark 2.4, we do not know in general that this is satisfied under the assumptions (A1)-(A5). However, it is a reasonable assumption since we expect $\|V\|_{\partial \Omega} = 0$ (and thus $\xi \leq \|V\|_{\partial \Omega} = 0$) unless the situation is somewhat pathological. We also note that adding the stability assumption (that is, the second variation of $E_{\varepsilon}$ is non-negative) does not appear helpful to show $\xi = 0$ on $\partial \Omega$, despite the result of $\Gamma$-convergence of [12].

In the following, we first describe the behavior of $u_{\varepsilon_i}|_{\partial \Omega}$.

**Theorem 3.1.** Under the assumptions (A1)-(A5) (thus leaving out (A6)), there exist a subsequence (denoted by the same index) and a function $\tilde{u} \in BV(\partial \Omega)$ such that

\[
u_{\varepsilon_i}|_{\partial \Omega} \to \tilde{u} \quad \mathcal{H}^{n-1} \text{ a.e. on } \partial \Omega,
\]

\[\tilde{u} = \pm 1 \quad \mathcal{H}^{n-1} \text{ a.e. on } \partial \Omega,
\]

where $u_{\varepsilon_i}|_{\partial \Omega}$ is the restriction of $u_{\varepsilon_i}$ to $\partial \Omega$.

In general, the trace of $u$ (obtained in Theorem 2.2) on $\partial \Omega$ may not coincide with $\tilde{u}$, as one can construct a sequence of critical points of $E_{\varepsilon}$ with $\sigma = 0$ which converge to $u = 1$ on $\Omega$ while $u_{\varepsilon}|_{\partial \Omega} \approx -1$ (see [10, Section 8]). The next result is the main theorem of the paper.

**Theorem 3.2.** Under the assumptions (A1)-(A6), let $V$ be as in Theorem 2.2 and let $\tilde{u}$ be as in Theorem 3.1. Then we have the following. Let $\theta$ be defined as in (1.3).
1. The total variation \( \| \delta V \| (\mathbb{R}^n) = \| \delta V \| (\Omega) \) (as an element of \( V_{n-1}(\mathbb{R}^n) \)) is finite.

2. For any vector field \( g \in C(\partial \Omega; \mathbb{R}^n) \) such that \( (g, \nu) = 0 \) on \( \partial \Omega \), we have

\[
(3.3) \quad \delta V_{|\partial \Omega}(g) = \cos \theta \int_{\partial^* \{x \in \partial \Omega : \tilde{u}(x) = 1\}} \langle g, \tau \rangle \, d\mathcal{H}^{n-2},
\]

where \( \tau(x) \in \text{Tan}_x(\partial \Omega) \) is the \( \mathcal{H}^{n-2} \) measurable unit inward-pointing normal to \( \partial^* \{x \in \partial \Omega : \tilde{u}(x) = 1\} \) which exists \( \mathcal{H}^{n-2} \) a.e. on \( \partial^* \{x \in \partial \Omega : \tilde{u}(x) = 1\} \).

The equality (3.3) gives a complete description of the tangential component of the first variation on the boundary. Also, (3.3) may be considered as a generalized contact angle condition satisfied for a pair of varifold \( V \) and \( \tilde{u} \). To see this, consider a case that \( \| V \| = \mathcal{H}^{n-1}|_M \) and \( M \) is a smooth hypersurface having a smooth boundary \( \partial M \subset \partial \Omega \). Then the first variation \( \delta V_{|\partial \Omega}(g) \) is represented as

\[
\int_{\partial M} \langle g, \tilde{\nu} \rangle \, d\mathcal{H}^{n-2},
\]

where \( \tilde{\nu} \) is the unit outward-pointing co-normal to \( \partial M \). Then (3.3) shows that \( \partial M \cap \{ \tilde{\nu} \neq \nu \} = \partial^* \{ \tilde{u} = 1 \} \) and the angle formed by \( \tilde{\nu} \) and \( \tau \) is \( \theta \). Away from \( \partial^* \{ \tilde{u} = 1 \}, \partial M \) (if such set is non-empty) intersects with \( \partial \Omega \) orthogonally. Hence, more precisely, we should say that the contact angle condition with angle \( \theta \) is satisfied on \( \partial^* \{ \tilde{u} = 1 \} \). For further remark on the implication of (3.3), see Section 5.

4. Proof of Theorem 3.1 and 3.2

Throughout this section, we will replace the notation \( \varepsilon_i \) by \( \varepsilon \). First, we derive a formula for the first variation \( \delta V_{\varepsilon} \).

Lemma 4.1. For \( u_\varepsilon \) satisfying (2.4) and (3.2) and for \( g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n) \), we have

\[
(4.1) \quad c_0 \delta V_{\varepsilon}(g) = \int_{\Omega \cap \{ \nabla u_\varepsilon \neq 0 \}} \nabla g \cdot \nabla u_\varepsilon \otimes \nabla u_\varepsilon \left( \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon} \right) \, dx \quad - \int_{\Omega \cap \{ \nabla u_\varepsilon = 0 \}} \nabla g \cdot \frac{W(u_\varepsilon)}{\varepsilon} \, dx + \int_{\Omega} \lambda_\varepsilon u_\varepsilon \, \text{div} \, g \, dx \quad + \int_{\partial \Omega} \left( \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} - \lambda_\varepsilon \right) \langle g, \nu \rangle \, d\mathcal{H}^{n-1} + \int_{\partial \Omega} \sigma'(u_\varepsilon) \langle \nabla u_\varepsilon, g \rangle \, d\mathcal{H}^{n-1} =: I_1^{\varepsilon}(g) + I_2^{\varepsilon}(g) + I_3^{\varepsilon}(g) + I_4^{\varepsilon}(g) + I_5^{\varepsilon}(g).
\]
Proof. We fix a vector field $g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ and calculate the right-hand side of (2.6). Using the boundary condition \((3.2)\) and by integration by parts, we have

\[
\int_{\partial \Omega \setminus \{\nabla u_\varepsilon \neq 0\}} \nabla g \cdot I \varepsilon \frac{\nabla |u_\varepsilon|^2}{2} \, dx = \int_{\Omega} \nabla g \cdot I \varepsilon \frac{\nabla |u_\varepsilon|^2}{2} \, dx
\]

Equal to

\[
\int_{\partial \Omega} \varepsilon \frac{\nabla |u_\varepsilon|^2}{2} \langle g, \nu \rangle \, d\mathcal{H}^{n-1} - \varepsilon \int_{\Omega} \nabla^2 u_\varepsilon \cdot \nabla u_\varepsilon \otimes g \, dx
\]

Again

\[
\int_{\partial \Omega} \varepsilon \frac{\nabla |u_\varepsilon|^2}{2} \langle g, \nu \rangle \, d\mathcal{H}^{n-1} + \varepsilon \int_{\Omega} \nabla g \cdot \nabla u_\varepsilon \otimes \nabla u_\varepsilon - \langle \nabla u_\varepsilon, \nabla (\nabla u_\varepsilon, g) \rangle \, dx
\]

Further

\[
\int_{\partial \Omega} \varepsilon \frac{\nabla |u_\varepsilon|^2}{2} \langle g, \nu \rangle + \sigma'(u_\varepsilon) \langle \nabla u_\varepsilon, g \rangle \, d\mathcal{H}^{n-1} + \varepsilon \int_{\Omega} \Delta u_\varepsilon \langle \nabla u_\varepsilon, g \rangle + \nabla g \cdot \nabla u_\varepsilon \otimes \nabla u_\varepsilon \, dx.
\]

Also by integration by parts, we obtain

\[
\int_{\Omega \cap \{\nabla u_\varepsilon \neq 0\}} \frac{W(u_\varepsilon)}{\varepsilon} \nabla g \cdot I \, dx = \int_{\Omega} \frac{W(u_\varepsilon)}{\varepsilon} \nabla g \cdot I \, dx - \int_{\Omega \cap \{\nabla u_\varepsilon = 0\}} \frac{W(u_\varepsilon)}{\varepsilon} \nabla g \cdot I \, dx
\]

Substituting \((4.2)\) and \((4.3)\) into (2.6), we have by the interior equation (2.1)

\[
c_0 \delta V_\varepsilon \langle g \rangle = \int_{\Omega \cap \{\nabla u_\varepsilon \neq 0\}} \nabla g \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \otimes \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \left( \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} - \frac{W(u_\varepsilon)}{\varepsilon} \right) \, dx
\]

\[
- \int_{\Omega \cap \{\nabla u_\varepsilon = 0\}} \nabla g \cdot I \frac{W(u_\varepsilon)}{\varepsilon} \, dx - \int_{\Omega} \lambda \langle \nabla u_\varepsilon, g \rangle \, dx
\]

\[
+ \int_{\partial \Omega} \left( \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) \langle g, \nu \rangle \, d\mathcal{H}^{n-1} + \int_{\partial \Omega} \sigma'(u_\varepsilon) \langle \nabla u_\varepsilon, g \rangle \, d\mathcal{H}^{n-1}.
\]

By integration by parts for the third term of right-hand side, we obtain \((4.1)\).

\[
\int_{\partial \Omega} \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \, d\mathcal{H}^{n-1} \leq C_2.
\]

Lemma 4.2. Under the assumption of (A1)-(A5), there exists a constant $C_2 > 0$ depending only on $\Omega, C, E_0, C_1$ such that

\[
\int_{\partial \Omega} \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \, d\mathcal{H}^{n-1} \leq C_2.
\]

Proof. We choose a smooth function $f : \overline{\Omega} \to \mathbb{R}$ which satisfies $\nabla f = \nu$ on $\partial \Omega$. For example, $f(x) = -\operatorname{dist}(x, \partial \Omega)$ near $\partial \Omega$ with a suitable truncation away from $\partial \Omega$ suffices. We then use $g = \nabla f$ in (4.1). By the definition (2.6) and (2.3), we have $c_0 \delta V_\varepsilon(\nabla f) \leq E_0 \sup \|f\|_{C^2}$ so the left-hand side of (4.1) is bounded depending only on $E_0$ and $\Omega$. The terms $I^1_f(\nabla f), I^2_f(\nabla f)$ and $I^3_f(\nabla f)$ are also bounded by a constant depending only on $C, E_0, \Omega$. Thus we have

\[
\int_{\partial \Omega} \left( \frac{\varepsilon |\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon} \right) \, d\mathcal{H}^{n-1} \leq - \int_{\partial \Omega} \sigma'(u_\varepsilon) \langle \nabla u_\varepsilon, \nu \rangle \, d\mathcal{H}^{n-1} + c(C, E_0, \Omega)
\]
where $\nabla f|_{\partial \Omega} = \nu$ is used. By Young’s inequality and the assumption (3.1),

$$
\left| \int_{\partial \Omega} \sigma'(u_\varepsilon) \langle \nabla u_\varepsilon, \nu \rangle \, d\mathcal{H}^{n-1} \right| \leq \int_{\partial \Omega} \varepsilon C_1 |\nabla u_\varepsilon|^2 + \frac{(\sigma'(u_\varepsilon))^2}{2C_1\varepsilon} \, d\mathcal{H}^{n-1} \leq C_1 \int_{\partial \Omega} \varepsilon |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} \, d\mathcal{H}^{n-1},
$$

(4.5)

Since $C_1 \in [0, 1)$, we have the conclusion by setting $C_2 = c(C, E_0, \Omega)/(1 - C_1)$. \hfill \Box

Proof of Theorem 3.2 (1). Fixing $g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, we have $\lim_{i \to \infty} \delta V_{\varepsilon_i}(g) = \delta V(g)$ due to the varifold convergence. In (4.1), due to (A6), we have $\lim_{i \to \infty} |I_{\varepsilon_i}^3(g)| + |I_{\varepsilon_i}^4(g)| = 0$. By Theorem 2.2, we have

$$
\lim_{i \to \infty} I_{\varepsilon_i}^3(g) = \lambda \int_{\Omega} u \, \text{div} \, g \, dx = -2\lambda \int_{M} \langle g, \frac{\nabla u}{|\nabla u|} \rangle \, d\mathcal{H}^{n-1} + \lambda \int_{\partial \Omega} u \langle g, \nu \rangle \, d\mathcal{H}^{n-1},
$$

(4.6)

where $M = \Omega \cap \partial^* \{ u = 1 \}$. Using (4.4) and a similar argument as in (4.5), we can show $|I_{\varepsilon_i}^3(g)| + |I_{\varepsilon_i}^4(g)| \leq c \sup |g|$, where $c$ is independent of $g$ or $i$. Combined all these estimates, we show that $|\delta V(g)| \leq c \sup |g|$ and $\|\delta V\|(\bar{\Omega})$ is finite. \hfill \Box

Proof of Theorem 3.2 (2). It suffices to prove the claim for $g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ with $\langle g, \nu \rangle = 0$ on $\partial \Omega$, since the general $C_c(\mathbb{R}^n; \mathbb{R}^n)$ case can be proved by approximation. For such $g$, in (4.6), the last term vanishes and also $I_3^i(g) = 0$ in (4.1). For $I_4^i(g)$, we have $\langle \nabla u_\varepsilon, g \rangle = \langle \nabla_{\partial \Omega} u_\varepsilon, g \rangle$ due to $\langle g, \nu \rangle = 0$. Thus, by the divergence theorem on $\partial \Omega$, we have

$$
I_{\varepsilon_i}^5(g) = \int_{\partial \Omega} \sigma'(u_\varepsilon) \langle \nabla_{\partial \Omega} u_\varepsilon, g \rangle \, d\mathcal{H}^{n-1} = -\int_{\partial \Omega} \sigma(u_\varepsilon) \text{div}_{\partial \Omega} g \, d\mathcal{H}^{n-1}.
$$

These lead to the conclusion that

$$
c_0 \delta V(g) = -2\lambda \int_{M} \langle g, \frac{\nabla u}{|\nabla u|} \rangle \, d\mathcal{H}^{n-1} - \int_{\partial \Omega} \sigma(\tilde{u}) \text{div}_{\partial \Omega} g \, d\mathcal{H}^{n-1}.
$$

(4.7)
Since $\tilde{u} \in BV(\partial \Omega)$ with values in $\{\pm 1\}$, $\partial^*\{\tilde{u} = 1\}$ and the inward-pointing unit normal $\tau$ are well-defined, and

$$-\int_{\partial \Omega} \sigma(\tilde{u}) \text{div}_{\partial \Omega} g \, d\mathcal{H}^{n-1} = (\sigma(1) - \sigma(-1)) \int_{\partial^*\{\tilde{u} = 1\}} \langle \tau, g \rangle \, d\mathcal{H}^{n-2}. \quad (4.8)$$

Since we are interested in obtaining $\delta V|_{\partial \Omega}$, and since $M \subset \Omega$, we obtain (3.3) from (4.7) and (4.8).

□

5. Additional remarks

5.1. The case $\|V\|(\partial \Omega) = 0$. If we further assume that $\|V\|(\partial \Omega) = 0$, then, non-trivial $\delta V|_{\partial \Omega}$ is necessarily singular with respect to $\|V\|_{\partial \Omega}$. Thus using the notation of (2.4), we conclude from (3.3) that

$$\int_Z \langle \nu_{\text{sing}}, g \rangle \, d\|\delta V\|_{\text{sing}} = \cos \theta \int_{\partial^*\{\tilde{u} = 1\}} \langle g, \tau \rangle \, d\mathcal{H}^{n-2}$$

for $g \in C(\partial \Omega, \mathbb{R}^n)$ with $\langle g, \nu \rangle = 0$ on $\partial \Omega$. If $Z = \partial^*\{\tilde{u} = 1\}$ and $\|\delta V\|_{\text{sing}}|_Z = \mathcal{H}^{n-2}|_Z$, then we have a clear-cut statement that $\nu_{\text{sing}} - \langle \nu_{\text{sing}}, \nu \rangle \nu = (\cos \theta) \tau$ on $Z$, which says that the generalized co-normal of $V$ satisfies the contact angle condition with angle $\theta$. Unfortunately, even in this case, we can only conclude that $\partial^*\{\tilde{u} = 1\} \subset Z$. Also we do not know in general if $\|\delta V\|_{\text{sing}}|_{\partial^*\{\tilde{u} = 1\}} = \mathcal{H}^{n-2}|_{\partial^*\{\tilde{u} = 1\}}$. On the other hand, on $Z \setminus \partial^*\{\tilde{u} = 1\}$, even though we equally do not know what $\|\delta V\|_{\text{sing}}$ is in general, we may conclude $\nu_{\text{sing}} = \nu$, $\|\delta V\|_{\text{sing}}$ a.e. since the right-hand side is 0 away from $\partial^*\{\tilde{u} = 1\}$. Thus the right-angle condition is simpler to describe than other non-right-angle conditions.

5.2. The case $\|V\|(\partial \Omega) > 0$. It may be somewhat counter-intuitive to imagine that the measures $\mu_\varepsilon$ may “pile-up” on the boundary as $\varepsilon \to 0$, resulting in $\|V\|(\partial \Omega) > 0$. For $\sigma = 0$ and $\Omega = B_1(0)$, it is not difficult to construct such example, however, as described in [10] (see also [8, 9] for examples for more general domains and of higher-multiplicity concentration). Interestingly, even if $\|V\|(\partial \Omega) > 0$, as long as $\xi = 0$, results in the paper still hold true. We expect that the presence of non-trivial $\|V\|$ in $\partial \Omega$ affects the normal component of the first variation, but not the tangential one. In all known examples where boundary concentration of $\|V\|$ occurs, $\xi$ is zero.

5.3. Monotonicity formula. In [7], motivated by the present paper, we introduce a notion of generalized contact angle condition for varifold and derive a monotonicity formula valid up to the boundary. The condition in [7] is even weaker than the one obtained in Theorem 3.2 in that we do not need to have a bounded first variation up to the boundary. Thus the result of [7] applies to $V$ in this paper and up to the boundary monotonicity formula can be obtained. For $\sigma = 0$ and convex $\Omega$, in [17], the similar up to the boundary monotonicity
formula was obtained even for the diffused energy (i.e. before letting $\varepsilon \to 0$). To gain a better understanding on $V$ obtained in this paper, it is desirable to establish such monotonicity formula for diffused energy since one can conclude a better convergence of interface to $\text{spt} \|V\|$. This is ultimately connected to getting a good estimate on the discrepancy up to the boundary and showing $\xi = 0$, along the line of logics in [3, 6, 10].

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