A Quantum Logic Gate Representation of Quantum Measurement: Reversing and Unifying the Two Steps of von Neumann’s Model

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(March 2, 2022)

Abstract

In former work, quantum computation has been shown to be a problem solving process essentially affected by both the reversible dynamics leading to the state before measurement, and the logical-mathematical constraints introduced by quantum measurement (in particular, the constraint that there is only one measurement outcome). This dual influence, originated by independent initial and final conditions, justifies the quantum computation speed-up and is not representable inside dynamics, namely as a one-way propagation.

In this work, we reformulate von Neumann’s model of quantum measurement at the light of above findings. We embed it in a broader representation based on the quantum logic gate formalism and capable of describing the interplay between dynamical and non-dynamical constraints. The two steps of the original model, namely (1) dynamically reaching a complete entanglement between pointer and quantum object and (2) enforcing the one-outcome-constraint, are unified and reversed. By representing step (2) right from the start, the same dynamics of step (1) yields a probability distribution of mutually exclusive measurement outcomes. This appears to be a more accurate and complete representation of quantum measurement.

PACS: 03.67.-a, 03.67.Lx, 03.65.Bz
I. INTRODUCTION

A former research\cite{1} on the character of the quantum speed-up (the fact that quantum problem solving can be more efficient than all known classical algorithms) has identified some very special features concerning the non-dynamical character of quantum measurement.

In the current work, these features will be used to reformulate von Neumann’s quantum measurement model in what appears to be a more accurate and complete way.

It should be noted that the approach pursued in \cite{1} is in contrast with the dominant notion that “quantum algorithms” are in fact algorithms, namely sequential Turing-machine computation and thus dynamical processes (see also \cite{2-5}). We believe that this very common notion might obfuscate understanding why there is the speed-up. In the interpretation given in \cite{1}, “quantum algorithms” are not algorithms and are not dynamical processes. In fact, the speed-up is shown to violate dynamics, namely the topical assumption that all time evolutions can be described as one-way propagations\footnote{By this we mean that the evolving state at time $t + dt$ is either a deterministic or a stochastic function of the state at time $t$.}.

In the following Section, we will review the conceptual points of the approach in question, which is common to the current work (we refer to \cite{1} for the mathematics involved). We apologize if we will have to re-examine some very basic notions of computer science. This is needed to introduce, so to speak, from scratch a computation paradigm that justifies the speed-up.

A. Quantum algebraic computation

We shall re-examine the notions of definition and computation in the framework of the arithmetical problems we are dealing with. Of course, these problems implicitly (algebraically) define their solutions. Factorization is an example: given the known product $c$ of

\begin{equation}
1\text{By this we mean that the evolving state at time } t + dt \text{ is either a deterministic or a stochastic function of the state at time } t.
\end{equation}
two unknown prime numbers $x$ and $y$, the numerical algebraic equation $c = x \cdot y$ implicitly defines the values of $x$ and $y$ which satisfy it. In order to compute these values, the algebraic definition must be substituted by an equivalent constructive definition, namely by an algorithm which prescribes how to compute the solution. The notion of algorithm must thus include an abstraction of the way things are constructed in reality. The Turing machine is an example, the Boolean network representation of a computation process is another.

For historical reasons, the “reality” we are dealing with is the classical non-relativistic one. Here any construction process (and moreover, any physical change in time) is generally believed to be representable as a dynamical process.

Consequently, there is a tight parallelism between algorithms and dynamics. An algorithm originates a propagation of one-way conditional logical implications (i.e. the logical state at step $k + 1$ is a Boolean function of the state at step $k$), which starts from a completely defined input to end up in an output. Similarly, dynamics originates a one-way time-propagation (i.e. the physical state at time $t + dt$ is a function of the state at time $t$), which starts from a completely defined initial state to end up in a final state. The same definition can hold for non-deterministic algorithms and stochastic dynamics, provided that the above functions are considered to be stochastic in character.

We should emphasize the fact that these propagations must be one-way – we exclude dynamics on closed time-like lines. Consequently, it is never the case that the Boolean network representation of a computation process has undetermined inputs, and outputs either constrained to fixed values or connected to upstream inputs participating in the determination of those same outputs (i.e. belonging to “feedback loops”).

Interestingly, Boolean networks, besides representing the process of computing the solution of a numerical problem, can represent the problem itself, namely the algebraic, implicit

\footnote{With respect to the direction of logical implication which, in a computation process, is the direction of time.}
definition of the solution. For example, the former equation $x \cdot y = c$ can be represented as a network of Boolean gates with the undetermined inputs $x$ and $y$ and the constrained output $c$. Evidently, these latter Boolean networks are not subject to the constraints applying to the Boolean network representation of computation.

This revisitation of the notion of algorithm will serve to show that the “quantum algorithms” do not fit it. On the contrary, they do operations that are forbidden to an algorithm, in fact they violate the notions of both algorithm and dynamics.

This is particularly clear in the “quantum algorithms” yielding an exponential speed-up, e.g. in Simon’s[6] and Shor’s[7] algorithms. In such algorithms, at some stage of the computation process quantum measurement selects, out of the parallel outputs produced that far, all and only those outputs that correspond to the same eigenvalue of some observable. This is of course an immediate consequence of the one-outcome constraint. The mathematics and logics inherent in that selection (see [1] for details) are equivalent to creating and solving an algebraic system of numerical (i.e. Boolean) equations representing the problem to be solved or the hard part thereof – i.e. the implicit definition of the solution.

In fact, an essential point is that the Boolean network representation of this system contains feedback loops. It is well known in engineering and computer science that, because of such loops, these networks are exponentially hard to solve (the equivalent computational networks are exponentially larger). The point is that quantum measurement, by selecting only one outcome, solves them at the same speed as there were no loops – loops are completely transparent to the measurement process.

This is an evident justification of the speed-up and hints at the non-algorithmic, non-dynamical character of the quantum measurement stage of quantum computation.

As a matter of fact, [1] shows that the outcome of a computation yielding the speed-up is dually affected by independent initial and final conditions⁴, thus it is not originated by a

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³The one-outcome constraint is independent of the former evolution since it holds unaltered for
one-way propagation, namely by dynamics. It is originated by a mathematical interplay between dynamics (yielding the state before measurement) and the non-dynamical constraints introduced by quantum measurement (in particular, the constraint that there is only one measurement outcome – the fact that this is randomly chosen is irrelevant for what concerns the speed-up).

In this sense, “quantum algorithms” are not algorithms at all. They belong to a new non-algorithmic and non-dynamical computation paradigm, where there is identity between algebraic, implicit definition of a solution and its physical determination. It is natural to call this paradigm “quantum algebraic computation”.

We have already noted that the speed-up has nothing to do with the fact that the measurement outcome is randomly chosen. In an evolution affected by both the initial and final conditions, quantum non-determinism assures that no information is sent back in time.

Noticeably, the concrete fruit of algebraic computation, namely the speed-up, violates dynamics (i.e. the assumption that all time evolutions are one-way propagations). One can see a precedent in the fact that quantum theory violates Bell’s inequalities, i.e. locality.

B. von Neumann’s model revisited

We shall now go to re-thinking the representation of quantum measurement at the light of the above results. As anticipated, we will use as a baseline von Neumann’s model, whose essential features are reviewed in the following.

Let the preparation be the state $\alpha |0\rangle_q + \beta |1\rangle_q$ of a qubit $q$, represented in the basis of an immediately subsequent measurement. This state will be thought of as the result of a former measurement in a different basis. By $[q]$ we designate the binary number stored in qubit $q$. Measuring $[q]$ in the above preparation yields the eigenvalue $\gamma = 0$ or $\gamma = 1$ in a mutually exclusive way. Correspondingly, the state of the “pointer” $p$ of the measurement all evolutions.
apparatus goes to $|0\rangle_p$ or $|1\rangle_p$ and, we assume, the quantum state goes to $|0\rangle_q$ or $|1\rangle_q$.

von Neumann’s model consists of two separate steps (i.e. not unified in a common representation). In the first step, the measurement process is represented by a unitary evolution yielding the following transition from the state before to the state after measurement:

$$|0\rangle_p (\alpha |0\rangle_q + \beta |1\rangle_q) \xrightarrow{U} \alpha |0\rangle_p |0\rangle_q + \beta |1\rangle_p |1\rangle_q, \quad (1)$$

where $|0\rangle_p$ is the pointer initial state and $U$ is a unitary transformation. The second step amounts to randomly selecting, as the measurement outcome, either $|0\rangle_p |0\rangle_q$ or $|1\rangle_p |1\rangle_q$ in a mutually exclusive way and with probabilities respectively $|\alpha|^2$ or $|\beta|^2$.

At the light of the results of ref. [1], we assume of course the non-dynamical character of the one-outcome constraint. An immediate consequence is that (1), being a purely dynamical description, in no way can represent this constraint. Therefore, given that this constraint yields relevant information for the purpose of predicting (i.e. a-priori describing) the quantum measurement process, (1) cannot be considered a maximal description of this process.

von Neumann’s dynamical model will be completed by embedding it in a “broader” model capable of representing the interplay between dynamical and non-dynamical constraints. The broader model is very congenially based on the quantum logic gate formalism.

II. QUANTUM GATE REPRESENTATION OF MEASUREMENT

In the next Section II.A we will develop the logical-mathematical representation of a very simple measurement. In Section II.B we will show that this representation can scale and be applied to more complex measurement situations.

A. One qubit in a coherent superposition

Let $\alpha |0\rangle_q + \beta |1\rangle_q$ be the state before measurement, at time $t_0$, of a qubit $q$. We want to a-priori describe the measurement process by taking into account the one-outcome constraint.
We shall list the “tools of the trade” first (see Fig. 1).

The measurement process will be represented by means of the unitary input-output transformation undergone by a four qubits register. This transformation is modeled as a quantum Boolean gate. While qubits $p$ and $q$ are supposed to be physical objects – their meaning is the same as in Section I – the ancillary qubits $e$ and $a$ are (unconventionally) abstract objects required for the description of the physical process. Qubit $e$ will be used in the description of the one-outcome constraint and qubit $a$ will serve to ensure gate reversibility, as clarified further below.

At time $t_0$, we want to represent the a-priori available knowledge that measurement will choose one value of $[q]$, either $\gamma = 0$ or $\gamma = 1$ in a mutually exclusive way and with probability amplitudes respectively $|\alpha|^2$ or $|\beta|^2$. Correspondingly, the state of the pointer $p$ and qubit $q$ will either go to $|0\rangle_p \! \! |0\rangle_q$ or $|1\rangle_p \! \! |1\rangle_q$. Which eigenvalue is not known at time $t_0$, therefore we must represent it as a stochastic Boolean variable $\gamma (\gamma = 0, 1)$ with probability distribution

$$p(\gamma = 0) = |\alpha|^2, \quad p(\gamma = 1) = |\beta|^2.$$  

(2)

We are dealing with a final constraint which, in current assumptions, is not completely representable inside gate dynamics. This constraint is introduced in the description of the measurement process by means of the ancillary qubit $e$. We define the input state of qubit $e$ as follows:

$$|\psi\rangle_e \equiv \bar{\gamma} |0\rangle_e + \gamma |1\rangle_e,$$

(3)

where $\bar{\gamma}$ is the negation of $\gamma$.\footnote{State (3) does not need to have the form $\bar{\gamma} |0\rangle_e + e^{i\delta} \gamma |1\rangle_e$, where $e^{i\delta}$ is a phase factor, since in any case (i.e. for $\gamma = 0, 1$) there is only one element in this superposition; this can also be seen}
measurement. Gate dynamics will have the task of transferring to the measurement outcome the constraint applied to the input state of qubit $e$, thus satisfying the prediction. It is worth discussing the character of $|\psi\rangle_e$:

- $|\psi\rangle_e$ is the most general solution of the (algebraic) projection equation $P |\psi\rangle = \gamma |\psi\rangle$, where $P = |1\rangle_e \langle 1|_e$ and $|\psi\rangle$ is a normalized ket variable belonging to $\mathcal{H}_e \equiv \text{span} \{|0\rangle_e, |1\rangle_e\}$ (it is the “unknown” of the algebraic equation – see [1] for the relationship between this and quantum algebraic computation);

- this “emulates” in $\mathcal{H}_e$ the projection equation that generates in $\mathcal{H}_q \equiv \text{span} \{|0\rangle_q, |1\rangle_q\}$ the eigenvalues/eigenstates of the measurement basis, namely the possible measurement outcomes; from the standpoint of determining the eigenvalues of $\gamma$, the result is the same;

- since $\gamma$ is a stochastic variable, (3) is an incompletely defined state (which eigenvalue will be sorted out is not a-priori known), represented with the method of random phases\[5\];

- we should not think that qubit $e$ is physically implementable. This constrained qubit serves to represent a physical principle (in fact, the one-outcome constraint) and, like all such principles, is abstract in character.

Now we need to define the gate truth table in such a way that the constraint applied to the input state of qubit $e$ becomes a constraint to be satisfied by the measurement outcome. \[6\]

\[6\] formally: $e^{i\delta}$ disappears in the density matrix representation of this state (we should keep in mind that $\bar{\gamma} \gamma = 0$).

\[5\] This method is used in place of the usual density matrix representation. The density matrix of qubit $e$ is obtained by taking the average over $\gamma$ of $|\psi\rangle_e \langle \psi|_e$, which yields $\langle |\psi\rangle_e \langle \psi|_e \rangle_\gamma = |\alpha|^2 |0\rangle_e \langle 0|_e + |\beta|^2 |1\rangle_e \langle 1|_e$ as expected. We should note that the mutual exclusivity between eigenstates is lost in the density matrix representation.
This requires that, if the input state of qubit $e$ is $|0\rangle_e \ (|1\rangle_e)$, then the output states of qubits $p$ and $q$ are $|0\rangle_p \ (|0\rangle_q \ (|1\rangle_p \ (|1\rangle_q))$. Moreover, it is convenient to assume that the input $|0\rangle_e \ (|1\rangle_e)$ goes identically into the output $|0\rangle_e \ (|1\rangle_e)$; thus the meaning of the state of qubit $e$ remains unchanged. This completely defines the gate truth table but for the output column $a$, which is defined by the requirement (justified further below) of ensuring gate reversibility (see table I).

\[
\begin{array}{cccc|cccc}
\text{Input} & \text{Output} \\
 e & p & q & a & e & p & q & a \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Table I

We can disregard the remaining 12 rows of the truth table, which can always be compiled in a way that ensures gate reversibility, thus the unitarity of the corresponding transformation.

Why we require gate reversibility is justified as follows. Before randomly choosing the value of $\gamma$, we are dealing with the evolution of a closed system of four qubits. The fact that two qubits are abstract in character (they encode "knowledge" concerning the evolution itself), does not alter the fact that the evolution of the four qubits cannot loose information, and should therefore be reversible-unitary in character.

We are now in the position of writing the gate input-output transition:

\[
\text{Input} \equiv \left( \gamma |0\rangle_e + \gamma |1\rangle_e \right) |0\rangle_p (\alpha |0\rangle_q + \beta |1\rangle_q) |0\rangle_a \xrightarrow{\text{U}} \\
\text{Output} \equiv \left( \gamma |0\rangle_e |0\rangle_p |0\rangle_q + \gamma |1\rangle_e |1\rangle_p |1\rangle_q \right) (\alpha |0\rangle_a + \beta |1\rangle_a) ,
\]

where $U$ is a unitary transformation whose matrix elements are immediately derivable from the gate truth table. We can see that the measurement outcome comprises either $|0\rangle_p |0\rangle_q$
or $|1\rangle_p |1\rangle_q$ in a mutually exclusive way with probabilities respectively $|\alpha|^2$ and $|\beta|^2$ (keeping in mind the definition of $\gamma$).

A difficulty of the original von Neumann’s model has disappeared. The usual entanglement between pointer and quantum system has changed form in a way which constrains the measurement outcome to be a single one.

The output state of qubit $a$ keeps memory of the preparation of qubit $q$, as it should be. Otherwise, the information about the phase of this preparation would be lost in the output, and we are dealing with the evolution of a closed system which cannot loose information. By means of qubit $a$, further background information is put in the description of the measurement process. In fact, measurement theory requires that an ensemble of identical preparations is compared with the corresponding measurement outcomes. This implies that the description of each individual measurement process includes, after measurement, the memory of the preparation (for example, this could mean putting the memory of the observer in the description).

As a final step, we can sort out the value of $\gamma$ according to its probability distribution. Say it comes out $\gamma = 1$. Although this is an irreversible operation, transition (4) remains reversible (we have to substitute $\gamma = 1$ everywhere):

Input $\equiv |1\rangle_e |0\rangle_p (\alpha |0\rangle_q + \beta |1\rangle_q) |0\rangle_a \xrightarrow{U} $

Output $\equiv |1\rangle_e |1\rangle_p |1\rangle_q (\alpha |0\rangle_a + \beta |1\rangle_a)$.

In this idealized representation, irreversibility resides in the random choice between two reversible histories. By the way, writing the value of $\gamma$ at previous times does not mean sending information back in time; it means making the report of an event after the event has occurred. It is like saying that, yesterday, the lottery number 123... came out. We are dealing with a model that faces uncertainty; by fixing the value of $\gamma$, the same model changes from being an a-priori description to being a report.

Let us test the model in the case the preparation is itself an eigenstate of the measurement basis, say it is $|1\rangle_q$ (thus, $\alpha = 0$, $\beta = 1$). This implies that the probability distribution is
\( p(\gamma = 0) = 0, \ p(\gamma = 1) = 1 \), which yields \( \gamma = 1 \) (thus \(|\psi\rangle_e = |1\rangle_e\), from eq. 3). By substituting these values in (4) (or, alternatively, by using the truth table row of input \( e = 1 \) and \( q = 1 \)), we obtain:

\[
\text{Input} \equiv |1\rangle_e |0\rangle_p |1\rangle_q |0\rangle_a \xrightarrow{U}
\]

\[
\text{Output} \equiv |1\rangle_e |1\rangle_p |1\rangle_q |1\rangle_a,
\]
as expected.

We should like to make a few comments on the model developed so far:

i) The method of representing together dynamical and non-dynamical constraints yields a peculiar form of entanglement. From the one hand, one can say that the first factor in the measurement outcome (4) is not entangled since, if \( \gamma = 1 \) (0), then \( \gamma = 0 \) (1), anyhow there is only one element in the superposition. From the other, we do not know whether \( \gamma = 0 \) or 1 before having performed measurement. In the face of this uncertainty, we should consider this factor an entangled state. Naturally, this kind of entanglement yields the appropriate correlation between the final state of the quantum system and that of the classical pointer.

ii) It is interesting to examine in more detail how the non-dynamical constraint on the ancilla \( e \) interplays with von Neumann’s dynamics. For this purpose, it is convenient to introduce a more suitable von Neumann’s model, which is directly comparable to transition (4), as follows:

\[
(\alpha |0\rangle_e + \beta |1\rangle_e) |0\rangle_p (\alpha |0\rangle_q + \beta |1\rangle_q) |0\rangle_a \xrightarrow{U}
\]

\[
(\alpha |0\rangle_e |0\rangle_p |0\rangle_q + \beta |1\rangle_e |1\rangle_p |1\rangle_q) (\alpha |0\rangle_a + \beta |1\rangle_a)
\]

where \( U \) is the same as in (4). Now we should think that all qubits are “real”, whereas qubits \( e \) and \( q \) have been prepared in the same state. We shall assume that transition (5) is the dynamical step of von Neumann’s model in the case of an ad-hoc designed measurement.\(^6\)

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\(^6\)Keeping in mind truth table I, the prescription would be: measure \([e]\) in preparation (5), “write” the measurement outcome in both the state of the pointer \( p \) and that of qubit \( q \), keep memory of
Transition (5) can be directly related to transition (4). In fact, by multiplying the two end-states of (5) by the operator $S \equiv \frac{5}{a} |0\rangle_e \langle 0|_e + \frac{2}{b} |1\rangle_e \langle 1|_e$, we obtain the two end-states of (4). This change does not affect gate dynamics ($S$ commutes with the gate transformation), therefore we can assume that transition (4) follows the same dynamics of von Neumann’s model (5).

We can thus state that reversibly reaching a state of maximal entanglement between qubits $e, p, q$ (as in transition 5), in the case that one qubit ($p$) describes the state of a classical object, actually amounts to having sorted out one measurement outcome (as in transition 4). This fundamental feature appears to be represented in the current model (see also points iii and iv).

iii) In a problem solving context, ref. [1] shows that the action of measurement at the same time introduces and solves an algebraic system of Boolean equations; solving this system in the classical framework could be computationally hard (see Section I). Therefore, the above statement (point ii) tells that the measurement process is unaffected by the computational hardness of the logical operations involved by it. A feature which is essential in order to achieve the quantum speed-up is thus represented in the current model. The same feature is of course present in von Neumann’s model, but in that case it is postulated, so to speak, from outside the model.

iv) Until now quantum measurement has been seen as the discrete transition from the state before to the state after measurement, while von Neumann’s model is continuous in time. It may be interesting to see the continuous form of the current model.

We should first find the continuous, reversible von Neumann’s transformation corresponding to transition (5) (let us think that this is routine), then apply the operator $S$ (point ii) to the evolving ket of this transformation. The result would be a continuous transformation corresponding to transition (4). For the sake of exemplification, we shall figure the input state of qubit $q$ in the output state of qubit $a$. 

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out a possible form of this latter transformation:

\[ \psi, t \rangle_{e,p,q,a} = \bar{\gamma} |0\rangle_e (|0\rangle_p |0\rangle_q |0\rangle_a + \beta \cos \omega t |0\rangle_p |1\rangle_q |0\rangle_a + \beta \sin \omega t |0\rangle_p |0\rangle_q |1\rangle_a) + \gamma |1\rangle_e (|\alpha\rangle_p |0\rangle_q |0\rangle_a + \beta \cos \omega t |0\rangle_p |1\rangle_q |0\rangle_a + \alpha \sin \omega t |1\rangle_p |1\rangle_q |1\rangle_a), \]

where \( t \) ranges from 0 to \( \frac{\pi}{2\omega} \). For \( t = 0, \frac{\pi}{2\omega} \), (6) overlaps on the two end-states of (4). As can be seen, \( \bar{\gamma} |0\rangle_e \) labels a unitary transformation of the preparation into one measurement outcome, as it should be (this is already a property of transition 4, as readily checked); similarly, \( \gamma |1\rangle_e \) labels a unitary transformation of the same preparation into the other outcome. The measurement process is thus represented by a probability distribution of mutually exclusive unitary transformations affected by both ends, each leading from the preparation to one of the possible measurement outcomes.

By taking the partial trace over \( e,p,a \) of \( \langle \psi, t |_{e,p,q,a} \langle \psi, t |_{e,p,q,a} \), we obtain the density matrix of qubit \( q \) as a function of \( \gamma \) (averaging over \( \gamma \) would yield the conventional density matrix):

\[ \rho_q(t, \gamma) = \bar{\gamma} \begin{pmatrix} |\alpha|^2 + |\beta|^2 \sin^2 \omega t, & \alpha \beta^* \cos \omega t \\ \alpha^* \beta \cos \omega t, & |\beta|^2 \cos^2 \omega t \end{pmatrix} + \gamma \begin{pmatrix} |\alpha|^2 \cos^2 \omega t, & \alpha \beta^* \cos^2 \omega t \\ \alpha^* \beta \cos^2 \omega t, & |\beta|^2 \cos^2 \omega t + \sin^2 \omega t \end{pmatrix} \]

Of particular interest are the following end-values (keeping in mind that \( \gamma + \bar{\gamma} = 1 \)):

\[ \rho_q(0, \gamma) = \begin{pmatrix} |\alpha|^2, & \alpha \beta^* \\ \alpha^* \beta, & |\beta|^2 \end{pmatrix}, \rho_q(\frac{\pi}{2\omega}, \gamma) = \begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{pmatrix}. \]

We can see that the eigenstates corresponding to the population elements of the density matrix become mutually exclusive when the coherence elements vanish. It is natural to think that this model should also be applicable to the decoherence representation of quantum measurement.

**B. Two qubits in a singlet state**

We shall apply the quantum gate representation of measurement to a more complex situation. Let us consider two qubits, \( q_1 \) and \( q_2 \), in a singlet state:
Here both qubits are represented in the same measurement reference. By rotating one reference by $\varphi$, state (7) changes into

$$|\psi_{\varphi}\rangle_{q_1,q_2} \equiv \frac{1}{\sqrt{2}} \left( -\sin \varphi |0\rangle_{q_1} |0\rangle_{q_2} + \cos \varphi |0\rangle_{q_1} |1\rangle_{q_2} - \cos \varphi |1\rangle_{q_1} |0\rangle_{q_2} - \sin \varphi |1\rangle_{q_1} |1\rangle_{q_2} \right).$$  (8)

State (8) will be the preparation of qubits $q_1$ and $q_2$ at the input of a quantum gate representing measurement of $[q_1]$ and $[q_2]$. Now the register is 8 qubits: $e_1, e_2, p_1, p_2, q_1, q_2, a_1, a_2$. The gate truth table is just the combination of two independent truth tables of the form I, labeled by subfixes respectively 1 and 2. Without going into detail, let us develop the input-output transformation.

The state of input $e$ (Section II.A) is substituted by the product

$$|\psi\rangle_{e_1,e_2} = \left( \tilde{\gamma}_1 |0\rangle_{e_1} + \gamma_1 |1\rangle_{e_1} \right) \left( \tilde{\gamma}_2 |0\rangle_{e_2} + \gamma_2 |1\rangle_{e_2} \right) = \tilde{\gamma}_1 \gamma_2 |0\rangle_{e_1} |0\rangle_{e_2} + \tilde{\gamma}_1 \gamma_2 |0\rangle_{e_1} |1\rangle_{e_2} + \gamma_1 \tilde{\gamma}_2 |1\rangle_{e_1} |0\rangle_{e_2} + \gamma_1 \gamma_2 |1\rangle_{e_1} |1\rangle_{e_2}.  \hspace{1cm} \text{(9)}$$

By comparing (9) with (8), we see that the joint probability distribution of the two Boolean variables is

$$p \left( \tilde{\gamma}_1 \tilde{\gamma}_2 = 1 \right) = p \left( \gamma_1 \gamma_2 = 1 \right) = \frac{1}{2} \sin^2 \varphi, \quad p \left( \tilde{\gamma}_1 \gamma_2 = 1 \right) = p \left( \gamma_1 \tilde{\gamma}_2 = 1 \right) = \frac{1}{2} \cos^2 \varphi.  \hspace{1cm} \text{(10)}$$

Eq. (9), with probability distribution (10), represents the principle that measuring $[q_1]$ and $[q_2]$ in state (8) sorts out a single valuation of the eigenvalues $\gamma_1$ and $\gamma_2$ with probability equal to the square module of the corresponding amplitude.

It should be noted that (10) is a complete description of the probability distribution of $\gamma_1$ and $\gamma_2$. From it, we can derive any auxiliary probability distribution; for example: $P \left( \gamma_i = 0 \right) = P \left( \gamma_i = 1 \right) = \frac{1}{2}$ (for $i = 1, 2$), $P \left( \gamma_1 = 1/\gamma_2 = 1 \right) = \sin^2 \varphi$ (the sign / means: conditioned to), etc.. These distributions are useful in the case that measurement of the two qubits is performed in two successive steps.

The input-output transformation is

$$\frac{1}{\sqrt{2}} \left( |0\rangle_{q_1} |1\rangle_{q_2} - |1\rangle_{q_1} |0\rangle_{q_2} \right).  \hspace{1cm} \text{(7)}$$
Input \equiv |\psi_{e_1,e_2}\rangle_0 |0\rangle_{p_1} |0\rangle_{p_2} |\psi_{\phi}\rangle_{q_1,q_2} |0\rangle_{a_1} |0\rangle_{a_2} \xrightarrow{U} (11)

Output \equiv (\gamma_1 \gamma_2 |0\rangle_{e_1} |0\rangle_{e_2} |0\rangle_{p_1} |0\rangle_{p_2} |0\rangle_{q_1} |0\rangle_{q_2} + \tilde{\gamma}_1 \gamma_2 |0\rangle_{e_1} |1\rangle_{e_2} |0\rangle_{p_1} |1\rangle_{p_2} |0\rangle_{q_1} |1\rangle_{q_2} + 
\gamma_1 \gamma_2 |0\rangle_{e_1} |0\rangle_{e_2} |1\rangle_{p_1} |0\rangle_{p_2} |1\rangle_{q_1} |0\rangle_{q_2} + \gamma_1 \gamma_2 |0\rangle_{e_1} |0\rangle_{e_2} |0\rangle_{p_1} |1\rangle_{p_2} |1\rangle_{q_1} |1\rangle_{q_2} |\psi_{\phi}\rangle_{a_1,a_2}

where |\psi_{\phi}\rangle_{a_1,a_2} is obtained from eq. (8) by substituting subfix q with a.

We can see from (11) that both in the case of simultaneous and subsequent measurement of the two qubits, each pointer is in a sharp state and the probabilities of the different outcomes agree with theory.

As the current model can be applied to quantum measurement in an entangled state, it can evidently be applied to all “quantum algorithms”.

III. CONCLUSIONS

Quantum measurement has been represented as a probability distribution of mutually exclusive unitary transformations affected by both ends, each leading from the preparation to one of the possible measurement outcomes. This two-way influence appears to be “richer” than (one-way causality) dynamics, as it can give the quantum speed-up (ref.[1] and Section I).

Of course the quantum speed-up is not a mathematical curiosity but a concrete thing, which has the potential of giving huge benefits in practical computation. Consequently, the fact it implies the violation of dynamics, appears to be a significant feature.

There is a precedent in the violation of Bell’s inequalities, thus of locality. As well known, this latter violation is experimentally verified. It can be argued that the experimental verification of a quantum speed-up, thus of dynamics violation, should be as interesting. This should constitute a further motivation for implementing a quantum computer.

The fact that the transition from the quantum to the classical world has non-dynamical aspects and, for this reason, can give tangible benefits, apparently unachievable in the
dynamical processes of either world, gives unforeseen grounds to Bohr’s idea that this transi-
tion is a central thing in quantum mechanics. This finding should renovate interest in the
quantum measurement problem, while giving a new perspective for investigating it.

Thanks are due to A. Ekert for stimulating discussions and valuable suggestions.
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Fig. 1