Gravitational Pressure, apparent horizon and thermodynamics of FLRW universe in the teleparallel gravity

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Abstract

In the context of the teleparallel equivalent of general relativity the concept of gravitational pressure and gravitational energy-momentum arisen in a natural way. In the case of a Friedmann-Lemaitre-Robertson-Walker space FLRW we obtain the total energy contained inside the apparent horizon and the radial pressure over the apparent horizon area. We use these definitions to written a thermodynamics relation \( T_A dS_A = dE_A + P_A dV_A \) at the apparent horizon, where \( E_A \) is the total energy inside the apparent horizon, \( V_A \) is the areal volume of the apparent horizon, \( P_A \) is the radial pressure over the apparent horizon area, \( S_A \) is the entropy which can be assumed as one quarter of the apparent horizon area only for a non stationary apparent horizon. We identify \( T_A \) as the temperature at the surface of the apparent horizon. We shown that for all expanding accelerated FLRW model of universe the radial pressure is positive.

Keywords: *teleparallel gravity, general relativity, gravitational pressure, apparent horizon, thermodynamic*

PACS numbers: 04.20.-q, 04.20.Cv, 04.70.Dy

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1 Introduction

Using quantum mechanics together with general relativity Bekenstein[1] and Hawking[2] showed that black hole horizons have entropy and temperature associated with its horizon and that black holes behaves like a black body emitting thermal radiation. This discovery led to the formulation of thermodynamics of black holes with an entropy proportional to its horizon area and a temperature proportional to its surface gravity at the horizon. It is widely known in the literature that the Hawking temperature and horizon entropy together with the mass of the black hole, which is identified with its energy, satisfy the first law of thermodynamics. Since the relations for black hole entropy and temperature are geometrical quantities associated with the black holes geometry, the thermodynamics analysis of black holes are in general based on geometrical aspects of black holes. In addition, it is believed that there exist some connection between black hole thermodynamic and Einstein’s equations[3]. After the discovery by Hawking that black holes emit radiation with a temperature proportional to its surface gravity at the event horizon[2], Gibbons and Hawking showed that from a thermodynamic point of view the de Sitter cosmological event horizon have the same properties, behaving like a Schwarzschild horizon[4].

The initial literature on thermodynamic of black holes was focused on static and stationary black holes. The thermodynamic analysis of black holes evolving with time where we may have: black hole collapse, black hole evaporation or black hole interacting with another black hole, requires the generalization of the concept of event horizon to dynamical horizon. A dynamical horizon define a marginally trapping surface on which no timelike Killing vector is available. With this generalization, thermodynamics properties of black holes could be generalized to others type of horizons other than black holes horizon event [5, 6, 7]. In fact, in the Friedman–Lemaitre–Robertson–Walker FLRW space always exist a dynamical apparent horizon on which it is possible to formulation thermodynamics analysis (see [8] and references therein). Recently many authors using different approaches have shown that from the first law of thermodynamics at the apparent horizon of FLRW space it is possible to derive the Friedman’s equations[9].

For a static spherically symmetric black hole of mass $M$, the first law of thermodynamic on the event horizon is in general written as $dE = TdS$. Where $E = M$ is the energy, $T$ the temperature and $S = A/4$ is the entropy, $A$ is the horizon area. Looking at the relationship that states the first law
of thermodynamics for black holes we note the absence of a work term due to the pressure $P$ at the event horizon. The pressure $P$ can be due the source of matter and the gravitational field. Trying to solve this problem in Refs. [3,10], the authors consider a spherically symmetric space time to show that it is possible to derive the Einstein’s equations at the event horizon from the thermodynamic identity $dE + PdV = TdS$. Where in Refs. [3,10] $P$ is strictly due the term of source in Einstein’s equations. In addition, the term of energy present in the first law of thermodynamic for black holes takes into account only the mass of the black hole, leaving out the contribution of energy due the gravitational field. For instance Brown and York[11] using a quasi-local energy expression showed that the energy inside the event horizon of a Schwarzchild black hole is given by $2M$.

In this work using the teleparallel equivalent of general relativity TEGR we obtain an equation for the thermodynamic relation $T_AdS_A = dE_A + P_A dV_A$ at the apparent horizon of a general FLRW space. Where $E_A$ represent the total energy (gravitational plus matter) defined in the context of the TEGR. The definition of gravitational radial pressure $P_A$ obtained in this analysis follows from the fields equations and from the gravitational energy-momentum four-vector. The definition of pressure arises naturally from the time derivative of the spatial components of the energy-momentum four-vector. In the case of FLRW universes in accelerated expansion we will show that the radial pressure at the apparent horizon is positive. The infinitesimal element of volume $dV_A$ is obtained out of the areal volume of the apparent horizon.

In the next section we present a brief summary of the Lagrangian formulation of the TEGR and show how we can get the definitions of the total energy-momentum four-vector $P^a$, of the gravitational energy-momentum tensor $t_{\mu\nu}$ and the gravitational radial pressure $P_A$ that naturally arise from the definition of the energy-momentum four-vector and the fields equations of the formalism. In Section III we apply these definitions to the FLRW models of space to obtain the total energy inside the apparent horizon as well as the radial pressure over the apparent horizon. In Section IV we write a thermodynamic relation $T_A dS_A = dE_A + P_A dV_A$ at the apparent horizon and, for a non stationary apparent horizon, we obtain a expression for the temperature $T_A$. In Section V we present our final considerations.

We use the following notation: space-time indexes $\mu, \nu, ...$ and $SO(3,1) a, b, ...$ run from 0 to 3. Time and spaces indexes are indicated as $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted by $e^a_{\mu}$, and the torsion tensor as
\[ T^a_{\mu\nu} = \partial\mu e^a_{\nu} - \partial\nu e^a_{\mu}. \] The flat Minkowski metric is denoted by \( \eta_{ab} = \epsilon_{a\mu}\epsilon_{b\nu} = (-1,1,1,1) \) and \( c = G = 1. \)

2 The energy-momentum four-vector and pressure in the TEGR

Before we present the definitions of energy-momentum and pressure in the TEGR, let us present a brief summary of the teleparallel equivalent of general relativity. In this framework the gravitational field is represented in terms of the tetrad field \( e^a_{\mu} \), and the Lagrangian density is constructed out of a quadratic combination of the torsion tensor \( T_{a\mu\nu} = \partial\mu e_{a\nu} - \partial\nu e_{a\mu} \) that is related to the antisymmetric part of the Weitzenböck connection \( \Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_{a\nu} \). To show the equivalence of the TEGR with general relativity constructed in the usual metric formulation, let us consider the Christoffel symbols

\[ \hat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}), \quad (1) \]

using the relation between the metric tensor and the tetrad field, \( g_{\mu\nu} = e_{a \mu} e_{a \nu} \), it is possible to write it in terms of the Weitzenböck and the torsion-free Levi-Civita connections as

\[ \hat{\Gamma}^\lambda_{\mu\nu} = e^a_{\lambda}(\hat{\omega}_{\mu ab}) e^b_{\nu} + e^a_{\lambda}\partial_\mu e_{a\nu}, \quad (2) \]

with the Levi-Civita connection \( \hat{\omega}_{\mu ab} \) given by

\[ \hat{\omega}_{\mu ab} = -K_{\mu ab}, \quad (3) \]

where \( K_{\mu ab} = \frac{1}{2} e_a^\lambda e_b^\nu (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda}) \) is the contorsion tensor and \( T_{\lambda\mu\nu} = e_a^\mu e^b_{\nu} b_{\lambda ab} \). With the torsion-free Levi-Civita connection it is possible to write a curvature tensor \( R^a_{b\mu\nu}(\hat{\omega}) \), and with the identity given in Eq. (3) its scalar curvature may be written identically as \[ eR(\hat{\omega}) = -e(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T_a T_a) + 2\partial_\mu(eT^\mu), \quad (4) \]

where \( e \) is the determinant of the tetrad field. Therefore in the context of the TEGR the Lagrangian density for the gravitational and matter fields is
written as

\[ L = -\kappa e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - \frac{1}{c} L_m \]

\[ \equiv -\kappa e \Sigma^{abc} T_{abc} - \frac{1}{c} L_m , \tag{5} \]

in which \( \kappa = c^3 / 16\pi G \), and \( \Sigma^{abc} \) is defined by

\[ \Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^{b} - \eta^{ab} T^{c}) , \tag{6} \]

and \( L_m \) is the Lagrangian density for the matter fields. The Lagrangian density \( L \) in Eq. (5) is invariant under the global \( \text{SO}(3,1) \) transformations. The absence of the divergence term on the right-hand side of Eq. (5) preclude the invariance of \( L \) under arbitrary local \( \text{SO}(3,1) \) transformations.

The variation of \( L \) with respect to \( e^a_{\mu} \) gives the field equations

\[ e_{a \lambda} e_{b \mu} \partial_\nu (e \Sigma^{b \lambda \nu}) - e (\Sigma^{b \nu \alpha} T_{\alpha \mu} - \frac{1}{4} e_{a \mu} T_{bcd} \Sigma^{bcd}) = \frac{1}{4\kappa} e T_{a \mu} , \tag{7} \]

where \( e T_{a \mu} = \delta L_m / \delta e^{a \mu} \). In the following we will make \( c = G = 1 \). The Eq. (7) can be rewritten as

\[ \partial_\nu (e \Sigma^{a \lambda \nu}) = \frac{1}{4\kappa} e e^a_{\mu} (t^{\lambda \mu} + T^{\lambda \mu}) , \tag{8} \]

where \( T^{\lambda \mu} = e^a_{\lambda} T^{a \mu} \) is the energy-momentum tensor of the matter fields and \( t^{\lambda \mu} \) defined by

\[ t^{\lambda \mu} = \kappa (4 \Sigma^{b \lambda \nu} T_{bc \mu} - g^{\lambda \mu} \Sigma^{bcd} T_{bcd}) . \tag{9} \]

is identified as the energy-momentum tensor of the gravitational field \[16\].

Due the antisymmetry in the last two indexes of \( \Sigma^{a \mu \nu} \), it follows from Eq. (8) that

\[ \partial_\lambda \left[ e e^a_{\mu} (t^{\lambda \mu} + T^{\lambda \mu}) \right] = 0 . \tag{10} \]

Since in the last equation we have a four-divergence of a contra variant four-vector density, using the Guass's theorem (see for instance section 7.4 of Ref. [17]) it is possible to write a continuity equation, that is

\[ \frac{d}{dt} \int_V d^3 x e e^a_{\mu} (t^{0 \mu} + T^{0 \mu}) = - \int_S dS_j \left[ e e^a_{\mu} (t^{j \mu} + T^{j \mu}) \right] , \tag{11} \]
where $S$ is the surface that enclose the arbitrary volume $V$ of the three-

\[ dS_j = \frac{1}{2!} \epsilon_{jkl} dS^{kl}, \quad dS^{kl} = \det \left( \frac{dx^k}{dx^i} \frac{dx^l}{dx^r} \right), \]

where $\epsilon_{ijk}$ is the Levi-Civita alternating symbol with $\epsilon_{123} = 1$.

On the left-hand side of Eq. (11) we have the time derivative of the total 

\[ P^a = \int_V d^3 x e^a_\mu (t^0_\mu + T^0_\mu), \quad (12) \]

and on the right-hand side we identify the quantities

\[ \Phi^a_g = \oint_S dS_j (ee^a_\mu t^j_\mu), \quad (13) \]

\[ \Phi^a_m = \oint_S dS_j (ee^a_\mu T^j_\mu), \quad (14) \]

as the fluxes of energy-momentum of the gravitational field and matter per 

unit of time, respectively. $S$ represent the spatial surface that enclose the 

volume $V$. From Eq. (11) and the definitions in Eqs. (12), (13) and (14) we have

\[ \frac{dP^a}{dt} = -\Phi^a_g - \Phi^a_m. \quad (15) \]

Using Eq. (8), Eq. (12) may be written in terms of $\Pi^{ai} = -4\kappa e \Sigma^{a0i}$, which is the density of momentum canonically conjugate to $e_{ai}$

\[ P^a = - \int_V d^3 x \partial_i \Pi^{ai} = - \oint_S dS_i \Pi^{ai}. \quad (16) \]

The definition of energy momentum four-vector $P^a$ in the above equation 

was obtained for the first time in the context of Hamiltonian formulation of 

the TEGR in the vacuum [18]. In non-empty space times $P^a$ represents the 

total energy momentum four-vector of the gravitational and matter fields. 

This definition is invariant under coordinate transformations of the three-

dimensional space and under time reparametrization. The component $a = (0)$ of $P^a$ give us the total energy contained inside the surface $S$ that ensure the volume $V$. 

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Substituting Eq. (16) on the left-hand side of Eq. (11) and using Eq. (8), the Eq. (11) is written as
\[
\frac{dP^a}{dt} = - \oint_S dS_j \phi^{aj},
\]  
where
\[
\phi^{aj} = 4\kappa \partial_a (e^a_{\mu} \Sigma^{aj\mu}) = ee^a_{\mu} (t^{j\mu} + T^{j\mu}).
\]  

If we now restrict the Lorentz index \( a \) to be \( a = (i) \), with \( i = 1, 2, 3 \), the Eq. (17) can be written as
\[
\frac{dP^{(i)}}{dt} = - \oint_S dS_j \phi^{(ij)}. 
\]  

The left-hand side of Eq. (19) has dimension of force and hence the density \( \phi^{(ij)} \) on its right-hand side in dimensional coordinates has dimension of force per unit of area and represents a pressure density along the \( (i) \)-direction over an element of area oriented along the \( j \)-direction [16]. In cartesian coordinates
\( j = 1, 2, 3 \) represents the directions \( \hat{x}, \hat{y}, \hat{z} \), respectively. In spherical type coordinates we fix \( j = r, \theta, \phi \), as consequence \( j = 1 \) is associated with the radial direction therefore, to obtain the radial pressure we need consider only the index \( j = 1 \). Thus in spherical type coordinates we define the radial density of force as
\[
-\phi^{(r)} = - (\sin \theta \cos \phi \phi^{(1)} + \sin \theta \sin \phi \phi^{(2)} + \cos \theta \phi^{(3)}),
\]
therefore, from the expression above we define the radial force as
\[
f(r) = \int_0^{2\pi} d\phi \int_0^\pi d\theta [-\phi^{(r)}].
\]  

This expression was applied recently in the study of the thermodynamics of Kerr [19] and Reissner-Nordström [20] black holes, respectively. In Ref. [19], it is shown that the efficiency of the Penrose process for a Kerr black hole is lower than in the thermodynamic formulation in general relativity. While in Ref. [20] the authors show that for a Reissner-Nordström black hole \( TdS \geq (\kappa/8\pi)dA = TdS_{BH} \), where \( S_{BH} \) is the standard Bekeinstein-Hawking entropy, and the equality is valid only for a Schwarzschild black hole. In the next section we will obtain expressions for the energy and the radial pressure associated with the apparent horizon of the FLRW models of
universe to write a thermodynamic relation $T_A dS_A$ entirely in the context of the TEGR, with no *a priori* identification between $T_A dS_A$ and the variation $dA_A$ the area of the apparent horizon.

3 Energy and Radial Pressure in a FLRW Universe

In comoving coordinates $(r, t, \theta, \phi)$ the FLRW line element is given by

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\}, \quad (22)$$

where $k = 0, 1, -1$ is the curvature index and $a(t)$ is the scale factor. If the FLRW model of universe contains a perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p) u_\mu u_\nu, \quad (23)$$

where $\rho$, $p$ and $u^\mu$ are the energy density, pressure and the four-velocity field of the fluid, respectively. The Einstein’s equations read as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (24)$$

From the two equations above one has

$$2\ddot{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi p, \quad (25)$$

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi \rho}{3}, \quad (26)$$

here the over dot represent the derivative with respect to the cosmological time $t$. An important equation following from the two above equations is

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\ddot{a}}{a} = 4\pi(\rho + p), \quad (27)$$

The dynamical apparent horizon of the FLRW universe is determined by the relation $h^{AB}\partial_A R(r, t)\partial_B R(r, t) = 0$, where $R(r, t) = a(t)r$ is the areal radius and $h_{AB} = \left[-1, a^2(t)/(1 - kr^2)\right]$ represent the transverse two metric spanned by $x^A = (t, r)$ where the indexes $A$ and $B$ can take the values $(0, 1)$.
This condition implies that the gradient $\nabla R(r, t)$ is a null vector on the surface of the apparent horizon. Out of the explicit evaluation of apparent horizon for the FLRW universe we obtain the apparent horizon radius as

$$R_A = \frac{1}{\sqrt{H^2 + k/a^2}},$$

(28)

where $H = \dot{a}/a$ is the Hubble parameter. Note also that due the Eq. (26), the argument of the square root in (28) is positive for positive densities $\rho$ what ensures that the apparent horizon radius is real.

In order to obtain the tetrad field related to a metric tensor we consider the relation $e^a_\mu e_{a\nu} = g_{\mu\nu}$. To analyze the physical properties of a metric tensor, we chose a tetrad field adapted to a field of static observers, whose trajectories and velocities in space-time are given by $x^\mu(s)$ and $u^\mu(s) = dx^\mu/ds$, respectively. Where $s$ is the proper time and we identify $u^\mu = e_{(0)}^\mu$. A set of tetrad field adapted to static observers is achieved by imposing the following conditions: (1) $e_{(0)}^i = 0$, which implies that the three spatial translational velocities of the adapted observer are zero; (2) $e_{(0)}^i = 0$ which implies that the three spatial axes of the adapted observer are not rotating with respect to a non rotating frame [19].

A set of tetrad field related to the line element given in (22) and satisfying the conditions (1) and (2) above is given by [1]

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \alpha \sin \theta \cos \phi & ar \cos \theta \cos \phi & -ar \sin \theta \sin \phi \\ 0 & \alpha \sin \theta \sin \phi & ar \cos \theta \sin \phi & ar \sin \theta \cos \phi \\ 0 & \alpha \cos \theta & -ar \sin \theta & 0 \end{pmatrix},$$

(29)

where

$$\alpha = \frac{a(t)}{\sqrt{1 - kr^2}},$$

(30)

and its determinant is $e = \alpha a^2 r^2 \sin \theta$. We emphasize that due to symmetry

\[1\] If we define the tetrad field as $e^a_\mu = \frac{\partial x^a}{\partial x^\mu}$, we can immediately shows that all the components of the torsion tensor will be nulls i.e, $T^a_{\mu\nu} = 0$. In the context of the TEGR the gravitational field is described by configuration of $e^a_\mu$ such that $T_{\mu\nu} \neq 0$. In addition if we consider a tetrad field such as that given in Eq. (14) of the Ref. [21], some drawbacks may appear. For example, the emergence of non-vanishing torsion components in Minkowski space-time. These drawbacks may be eliminated by means of the regularization procedures presented in Ref. [12].
in the line element in Eq. (22), it is more convenient to write the tetrad field $e_{a\mu}$ in spherical coordinates.

We are now in position to evaluate all necessary quantities to obtain the energy contained inside the apparent horizon and the radial pressure over the apparent horizon. We emphasize that only after doing all the calculations is that we apply the results on the surface of apparent horizon. The calculations are lengthy, but otherwise straightforward. Therefore we find important to show some intermediate steps. First we need to calculate the necessary quantities $\Sigma^{\alpha\mu\nu}$ related to Eq. (6). Using the definition of torsion tensor $T_{a\mu\nu} = \partial_{\mu}e_{a\nu} - \partial_{\nu}e_{a\mu}$, they are given by

$$
\begin{align*}
\Sigma^{001} &= \frac{1}{ra\alpha^2} (\alpha - a), \\
\Sigma^{110} &= -\frac{\dot{a}}{a\alpha^2}, \\
\Sigma^{220} &= -\frac{\dot{a}}{r^2a^3}, \\
\Sigma^{330} &= -\frac{\dot{a}}{a^3r^2\sin^2 \theta}, \\
\Sigma^{212} &= \frac{(\alpha - a)}{2r^3a^3\alpha^2}, \\
\Sigma^{313} &= \frac{(\alpha - a)}{2r^3a^3\alpha^2 \sin^2 \theta}.
\end{align*}
$$

(31)

All the others components of $\Sigma^{\alpha\mu\nu}$ are zero. Since $e^{(0)}_i = 0$ and $e^{(0)}_0 = 1$, we have that $\Sigma^{(0)01} = e^{(0)}_0 \Sigma^{001} = \Sigma^{001}$. Finally, from the four-divergence in Eq. (18) and using the results presented in Eq. (31), the quantities $\phi^{(i)}$ are given by

$$
\begin{align*}
\phi^{(1)1} &= -4\kappa \left[ \partial_0(\dot{a}a)r^2 + 1 - \sqrt{1 - kr^2} \right] \sin^2 \theta \cos \phi, \\
\phi^{(2)1} &= -4\kappa \left[ \partial_0(\ddot{a}a)r^2 + 1 - \sqrt{1 - kr^2} \right] \sin^2 \theta \sin \phi, \\
\phi^{(3)1} &= -4\kappa \left[ \partial_0(\dot{a}a)r^2 + 1 - \sqrt{1 - kr^2} \right] \sin \theta \cos \theta.
\end{align*}
$$

(32)

Note for example that from Eq. (18) to calculate $\phi^{(1)1}$ we should start from

$$
\phi^{(1)1} = 4\kappa [\partial_0(ee^{(1)}_1 \Sigma^{110}) + \partial_2(ee^{(1)}_2 \Sigma^{212}) + \partial_3(ee^{(1)}_3 \Sigma^{313})],
$$

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where we use that $\Sigma^{\alpha\mu} = 0$. The other quantities $\phi^{(r)1}$ are obtained similarly. Now with help of Eqs. (32) and after some calculations we arrive at an expression for $\phi^{(r)1}$ given by Eq. (20), it is

$$-\phi^{(r)1} = 4\kappa \left[ \partial_0 (\dot{a} a) r^2 + (1 - \sqrt{1 - kr^2}) \right] \sin \theta,$$

(33)

this expression will be considered later.

Let us now evaluation the energy contained inside the apparent horizon. Remember that the density of energy is given by $\Pi^{(0)1} = -4\kappa e \Sigma^{(0)01}$, from the first equation in (31) and Eq. (16), the energy enclosed by a spherical surface of radius $r$ is given by

$$E \equiv P^{(0)} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \Pi^{(0)1} = ar(1 - \sqrt{1 - kr^2}),$$

(34)

where we have used $\kappa = 1/16\pi$. Now if we assume $r = r'$ such that $R_A = a(t)r'$, and with the definition of $R_A$ in Eq. (28), the energy contained inside the apparent horizon is given by

$$E_A = R_A(1 - \sqrt{1 - kR_A^2/a^2}).$$

(35)

With Eqs. (20) and (28), $E_A$ may be rewritten as

$$E_A = 2M_A - HR_A^2,$$

(36)

where

$$M_A = \frac{4\pi R_A^3}{3} \rho,$$

is the Minsner-Sharp-Hernandez mass [8], which is defined only for spherically symmetric space-times. Here we would like to remind the readers that in Eq. (35), the energy $E_A$ inside the apparent horizon, is not only the Misner-Sharp-Hernandez mass. The reason is that $E_A$ represent the total energy of matter plus the energy of the gravitational field, respectively. From the Eq. (35), we conclude that $E_A$ is positive for $k = 1$, negative for $k = -1$ and zero for $k = 0$, respectively. In this latter case, the negative binding gravitational energy and the positive matter energy inside the apparent horizon exactly cancel out resulting in an universe that has flat spatial section [22].

To obtain the radial pressure over the apparent horizon, firstly we inserting $-\phi^{(r)1}$ given by Eq. (33) into Eq. (21) and making the integration of the angular variables. After the angular integration we obtain

$$f(r) = [\partial_0 (\dot{a} a) r^2 + (1 - \sqrt{1 - kr^2})].$$

(37)
Again assuming \( r = r' \) such that \( R_A = a(t)r' \), and with the definition of \( R_A \) in Eq. (28), the radial force over the surface of the apparent horizon read

\[
f_A = \left[ \partial_0(\dot{a}a) \frac{R_A^2}{a^2} + 1 - R_A H \right].
\]  

(38)

Since the FLRW models of universe are homogeneous and isotropic, dividing \( f_A \) by the area of the apparent horizon we obtain the radial pressure over the surface of the apparent horizon. It is given by

\[
P_A = \frac{1}{4\pi R_A^2} \left[ \left( \frac{\ddot{a}}{a} + \frac{a^2}{a^2} \right) R_A^2 + 1 - R_A H \right].
\]  

(39)

Here we note that according Eqs. (13) and (14), \( P_A \) is due the fluxes of gravitational field and matter, respectively. Taking into consideration only accelerated expanding FLRW models of universe (\( \ddot{a} > 0 \)), using the definition of \( R_A \), for \( k = 1 \) or \( k = 0 \), \( 1 - HR_A \geq 0 \) and for \( k = -1 \), \( H^2 R_A^2 + 1 - HR_A > 0 \), so it is not difficult to see that the radial pressure over the apparent horizon is positive, directed outward over it, like a tension. Since \( H^2 R_A^2 + 1 - HR_A > 0 \) for all models of universe, the radial pressure in (39) is negative only for decelerated models of universe with \( \ddot{a}R_A^2/a < -(H^2 R_A^2 + 1 - HR_A) \) and in this case the pressure produces a force on the surface of the apparent horizon that points on the outside to the inside. We note also that even for flat \((k = 0)\) FLRW model of universe where \( E_A = 0 \), the radial pressure over the surface of the apparent horizon is not necessarily zero.

Before we close this section let us evaluate the consistency of Eq. (17). Since in this case \( e_{(0)i} = e_{(i)0} = 0 \), for \( a = (0) \) only \( \phi^{(0)1} = -4\kappa\partial_0(e\Sigma^{(0)01}) \) in Eq. (18) is not zero so from Eq. (17), at the apparent horizon, we have

\[
\dot{P}^{(0)}_A = -\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \phi^{(0)1}(R_A) = \partial_0 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta [4\kappa\Sigma^{(0)01}(R_A)] = \partial_0 E_A.
\]  

(40)

To obtain the radial quantities \( \Phi^{(r)}_g \) and \( \Phi^{(r)}_m \) in Eqs. (13) and (14) on the surface of a sphere of radius \( r \), we note that the radial unit vector \( \hat{r} \) is given in term of the component of the tetrad field \( e_{(i)1} \), i.e \( \hat{r} = e_{(i)1}/\alpha \), so projecting the Eq. (18) along the radial direction we have

\[
\phi^{(r)1} = \frac{1}{\alpha} e_{(i)1} \phi^{(i)1} = \frac{1}{4\kappa\alpha} e g_{11}(4^{11} + T^{11}) = \phi^{(r)1}_g + \phi^{(r)1}_m,
\]
where from Eq. (9) and using the quantities in Eq. (31) after some calculations we obtain

\[ \phi^{(r)1}_g = 2\kappa \sin \theta \left[ -(Hra)^2 - \left( 2(1 - \sqrt{1 - kr^2}) - kr^2 \right) \right], \]

and

\[ \phi^{(r)1}_m = pr^2a^2 \sin \theta. \]

Now performing the integrating of the two quantities above as in (13) and (14) on the surface of the apparent horizon \( r = R_A/a(t) \), we obtain firstly the radial flow of momentum per unit time due the matter

\[ -\Phi^{(r)}_m(R_A) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \phi^{(r)1}_m = -4\pi pR_A^2, \]

and using the definition of \( R_A \) in (28) it is not difficult to shown that the radial flow per unit of time due the gravitational field is given by

\[ -\Phi^{(r)}_g(R_A) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \phi^{(r)1}_g = \left( H^2R_A^2 + \frac{1}{2} - HR_A \right). \]

To our surprise it is always positive for any model of universe expanding or contracting. Using the Eq. (25), we have that 

\[ -4\pi p = \frac{\ddot{a}}{a} + \frac{1}{(2R_A^2)} \]

therefore, from the two equations above the total radial flow of momentum per unit time can be written as

\[ \dot{P}^{(r)}_A = -\Phi^{(r)}_g(R_A) - \Phi^{(r)}_m(R_A) = \left( \frac{\ddot{a}}{a} + H^2 \right) R_A^2 + 1 - R_A H = f_A, \quad (41) \]

that is the same result obtained in (38).

4 Thermodynamic of apparent horizon

In this section, we will use the definitions of energy and radial pressure obtained in the latter section to write the thermodynamic relation \( T_A dS_A = dE_A + P_A dV_A \). In order we identify \( T_A \) with the temperature at apparent horizon. \( S_A \) and \( V_A \) are the entropy and the areal volume of the apparent horizon, respectively. To compare our results with those obtained in the context of general relativity, in the variation \( dE_A \) we consider that in a infinitesimal interval of cosmological time \( dt \), we have \( dR_A = \dot{R}_A dt \) and
\[ dH = \dot{H}dt. \] Where here we are assuming that \( \dot{R}_A \neq 0 \), the case \( \dot{R}_A = 0 \) (a stationary apparent horizon) will be analyzed in the end of this section. From Eq. (35) we have

\[ dE_A = \left( 1 - 2HR_A - \frac{\dot{H}R_A^2}{R_A} \right) \frac{dA_A}{8\pi R_A}, \]

where \( A_A = 4\pi R_A^2 \) is the areal area of the apparent horizon and \( \dot{R}_A = R_A^3 H (H^2 + k/a^2 - \ddot{a}/a) \). The work term due the radial pressure is given by

\[ P_A dV_A = \left[ \left( \frac{\ddot{a}}{a} + H^2 \right) R_A^2 + 1 - H R_A \right] \frac{dA_A}{8\pi R_A}, \]

where \( V_A = 4\pi R_A^3 / 3 \) is the spherical volume defined by \( R_A \).

We are now in position to write an expression for the first law of thermodynamics, i.e

\[ T_A dS_A = dE_A + P_A dV_A, \]

With the help of Eq. (28) the right side of this equation can be simplified and we obtain

\[ T_A dS_A = \frac{1}{8\pi R_A} \left[ -2\kappa' R_A + (1 - HR_A)^2 - \left( HR_A + \frac{\dot{H}R_A^2}{R_A} \right) \right] dA_A, \]

where

\[ \kappa' = \frac{1}{2} \square_{(h)} R = -\frac{R_A}{2} \left( \frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} \right), \]

is the Kodama-Hayward surface gravity\(^{[23]}\) and \( \square_{(h)} R \) is the d’Lambertian of the areal radius \( R(t, r) = a(t)r \)

\[ \square_{(h)} R = \frac{1}{2\sqrt{-h}} \partial_A [\sqrt{-h} h^{AB} \partial_B (a(t)r)], \]

where \( h_{AB} = [-1, a^2(t)/(1 - kr^2)] \). In the expression (45) the last term in parentheses can be simplified and written as

\[ -\frac{k\ddot{a}}{\dot{a}} \frac{R_A}{(\dot{a}^2 + k - a\ddot{a})}, \]
and \( T_A dS_A \) can be rewritten in terms of the variation of the Bekeinstein-Hawking entropy, i.e.
\[
dS_{BH} = dA_A/4.
\]

\[
T_A dS_A = \frac{1}{2\pi R_A} \left[ -2\kappa' R_A + (1 - HR_A)^2 - \frac{k \ddot{a}}{a} \frac{R_A}{(\dot{a}^2 + k - a\ddot{a})} \right] dS_{BH}. \tag{48}
\]

The first term in \( T_A dS_A \) is exactly twice the Kodama-Hayward temperature at the apparent horizon, which is given by
\[
T_{KH} = \frac{-\kappa'}{2\pi} = \frac{R_A}{4\pi} \left( \frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} \right). \tag{49}
\]

The Kodama-Hayward temperature written as, \( T_{KH} = |\kappa'|/2\pi \), is usually employed in the literature to write the first law of thermodynamics at the apparent horizon of the FLRW model of universe. However, as pointed out in Refs. [24, 25], there are several nonequivalent prescriptions for \( \kappa' \) and as consequence there will be several nonequivalent expressions for \( T_{KH} \). In the following we summarize the behavior of \( T_A \) for \( k = 0, -1, 1 \) when we assume \( S_A = A_A/4 = S_{BH} \).

1) If we consider a flat model of universe \( k = 0 \) (\( E_A = 0 = 1 - HR_A \)), the temperature at the apparent horizon reduce to twice the Kodama-Hayward temperature and is positive if and only if the Ricci scalar is, which correspond to equations of state satisfying \( \rho > 3p \), and is negative if and only if the Ricci scalar is, this is a result similar to that obtained in Ref. [8] in the context of general relativity. Note also that if we assume \( R_A > 0 \), the Ricci scalar in Eq. (46) is negative if and only if, \( \dot{a}R_A^2/a < -1 \), that correspond to decelerated universe with a radial pressure \( P_A \) negative. If the Ricci scalar vanishes we have a cold apparent horizon with \( T_A = 0 \). Note that in this case \( P_A = 0 \) and \( E_A = 0 \). The reason of \( T_A = 2T_{KH} \) for \( k = 0 \), can be related to the fact that in the context of the TEGR the radial pressure over the apparent horizon is due the fluxes of matter and gravitational field such that the two contributions became somehow equal, giving twice \( T_{KH} \), one from matter the other from gravity.

2) In the case of an open, \( k = -1 \), FLRW model of universe Eq. (48) can be written as
\[
T_A = \frac{1}{2\pi R_A} \left[ R_A^2 \frac{\ddot{a}}{a} + (1 - HR_A)^2 + 1 + \frac{\ddot{a}}{\dot{a}\sqrt{\dot{a}^2 - 1}} \frac{1}{(\dot{a}^2 - 1 - \ddot{a}a)} \right], \tag{50}
\]
where here we use the definition of \( \kappa' \) in Eq. (46). Since in this case \( \dot{a} \geq 1 \), for an accelerated expanding universe \( \ddot{a} \geq 0 \), without violation the weak energy condition, \( \rho + p > 0 \ (\dot{a}^2 - 1 - \ddot{a}a > 0) \), the temperature \( T_A \) above is positive.
3) For a closed model of universe, $k = 1$, the situation is more complicated and unfortunately we do not have a general physical criterion to show that $T_A$ is positive. However, in this case with the help of the definition of $\kappa'$, in Eq. (46) the temperature $T_A$ can be written as

$$T_A = \frac{1}{2\pi R_A} \left[ R_A^2 \frac{\ddot{a}}{a} + (1 - HR_A)^2 + 1 - \frac{\ddot{a}a}{\dot{a}\sqrt{a^2 + 1}} \frac{1}{\dot{a}^2 + 1 - \ddot{a}a} \right].$$  (51)

If we assume the weak energy condition as $\rho + p \geq 0$, the boundary $\rho + p = 0$ with $H \neq 0$ correspond to a stationary apparent horizon i.e

$$\dot{R}_A = HR_A^3 \left( H^2 + \frac{k}{a^2} - \frac{\ddot{a}}{a} \right) = 4\pi HR_A^3 (\rho + p) = 0,$$  (52)

in this case the work term $P_A dV_A$ vanishes in Eq. (44). For a stationary apparent horizon from Eq. (36) the time evolution of the energy inside the apparent horizon is given by $\dot{E}_A = -(k R_A^2)/a^2$ and from Eq. (44) follows that

$$T_A \dot{S}_A = -k \frac{R_A^2}{a^2}.$$  (53)

The equation above is an important result of the paper. This equation states that: if, $k = 0$, the time evolution of $E_A$ ($E_A = 0$) inside the apparent horizon is zero and $T_A \dot{S}_A = 0$, a result consistent with the fact that in this case $dE_A = P_A dV_A = 0$. If $k = -1$, $E_A$ inside the apparent horizon is negative and increase with $t$ and $T_A \dot{S}_A > 0$. For a closed, $k = 1$, model of universe, the result in equation above may seem unphysical however, in this case $\dot{E}_A$ is always negative and physically this means that there is a flow of energy from the inside to the outside of the apparent horizon and according to the second law of thermodynamics it will result in a decrease of entropy $S_A$ i.e, $\dot{S}_A < 0$, and so the result shown in Eq. (53) is consistent whenever $T_A > 0$. In this sense the apparent horizon is not an thermodynamically isolated system. Note that for a stationary apparent horizon with $H \neq 0$, from Eqs. (25), (26) and (27) we have $\rho = -p = constant$, $\ddot{a}/a = 1/R_A^2 = constant > 0$. Therefore it is not obvious a priori that a constant entropy for stationary apparent horizon with, $\rho = -p$, $\ddot{a}/a$ and $R_A$ constants, should occur when $a$ is varying with cosmological time $t$.

For example, if we consider a cosmological constant $\Lambda > 0$ as the only source of gravity ($\rho = p = 0$), the solution of FLRW equations in the presence of the cosmological constant for $k = -1$ give the scale factor

$$a(t) = a_0 \sinh(t/a_0),$$
and the apparent horizon radius has the constant value \( R_A = \sqrt{3/\Lambda} = a_0 \), and Eq. (53) give us

\[
T_A \dot{S}_A = \frac{R_A^2}{a^2(t)}.
\]  

(54)

For \( k = 1 \) with a cosmological constant \( \Lambda > 0 \) as the only source of gravity \((\rho = p = 0)\), the scale factor is given by

\[
a(t) = a_0 \cosh(t/a_0),
\]

again the constant apparent horizon radius is \( R_A = \sqrt{3/\Lambda} = a_0 \), and from Eq. (53) we have

\[
T_A \dot{S}_A = -\frac{R_A^2}{a^2(t)}.
\]  

(55)

The results in Eqs. (54) and (55) are physically equivalent to those given just in Eq. (53) when \( \rho + p = 0 \), \( k = -1 \) and \( k = 1 \), respectively. The results presented in Eqs. (54) and (55) imply that in the context of the TEGR, in general we cannot write the entropy as proportional to the area of the apparent horizon.

5 Conclusions

In this work we have presented the definitions of energy contained within an arbitrary volume enclosed by an arbitrary surface as well as the total pressure (due the fluxes of matter and gravitational field) on the surface in question. These definitions arise in a natural way in the context of the teleparallel equivalent of general relativity. Considering a FLRW model of universe, we calculation the total energy within the apparent horizon and shown that it is negative, positive or zero for \( k = -1, 1, 0 \), respectively. The energy \( E_A \) is different of Misner-Sharp-Ernandez mass which is given by \( M_A = (4\pi/3)\rho R_A^3 = R_A/2 \). This difference is not surprise because in the TEGR the total energy enclosed by the apparent horizon is ascribed to the energy due the matter and others possible form of energy. This difference implies that in general we cannot write the variation \( dE_A \) as being proportional to the variation \( dA_A \) the area of the apparent horizon, and as consequence this has implications on the thermodynamic relation for the apparent horizon i.e, in general we cannot write \( dE_A + P_A dV_A \) as proportional to \( dA_A \).
We also computation the total radial pressure on the surface of apparent horizon and show that this is not only the pressure due the fluid inside the apparent horizon. The pressure computed in the context of the TEGR is due the fluxes of matter and gravitational field, respectively. The radial pressure $P_A$ given in Eq. (42) is positive whenever $\ddot{a} > 0$. In particular for a flat model of universe, $1 - H R_A = 0$, and with the equations (25) and (26), the radial pressure $P_A$ can be written as $P_A = (\rho - 3p)/3$. Therefore, for a flat model of universe with $\rho > 3p$, assuming that the density of the fluid is positive, the lower is the pressure of the fluid higher is the radial pressure $P_A$. The especial case $k = 0$ of non relativistic matter dominated universe is modeled by dust approximation with a pressureless matter $p = 0$, and in this case the radial pressure can be written as, $P_A = \rho/3 = (1/4\pi a)[\ddot{a} + \dot{a}^2/a] > 0$, therefore the positive term, $[\ddot{a} + \dot{a}^2/a]$ appear in $P_A$ as an effective acceleration, which may be responsible for the accelerated expansion of the universe.

With the expressions of energy and pressure for the apparent horizon of FLRW space, for a non stationary apparent horizon, we have obtained a thermodynamic relation $T_A dS_A = dE_A + P_A dV_A$ (see Eq. (48)) entirely within the framework of the TEGR without a priori identify $dS_A$ with the variation $dA_A$ of the area of the apparent horizon. To compare our result obtained in the framework of the TEGR given in Eq. (45) with the standard result $T_{KH} dS_{BH}$, we have rewritten the right-hand side of Eq. (45) in terms of $dS_{BH} = dA_A/4$. The result is presented in the Eq. (48) which implies that in the context of the TEGR $T_A dS_A \neq T_{KH} dS_{BH}$. In particular for $k = 0$ and $k = -1$ models of universe that no violation the weak energy condition $\rho + p > 0$ and in accelerated expansion, $T_A dS_A > T_{KH} dS_{BH}$, which implies that if we assume $dS_A = dS_{BH}$, the temperature obtained in the framework of the TEGR is greater than the Kodama-Hayward temperature, i.e $T_A > T_{KH}$. The reason that the thermodynamic relation given by Eq. (48), obtained in the framework of the TEGR is different from that obtained in the context of general relativity is because the energy $E_A$ take into account the energy of matter and the gravitational field and that the pressure $P_A$ is due the fluxes of matter and gravitational field, respectively.

Finally, in the case of a closed, $(k = 1)$, model of universe when we assume $\dot{R}_A = 0$, (a stationary apparent horizon), we show that $\dot{E}_A < 0$. This means that there are a flow of energy from inside to outside of the apparent horizon and as consequence the entropy $S_A$ always decreases with the cosmological time $t$ which implies that the relationship in Eqs. (53) and (55) do not violate the second law of thermodynamics whenever $T_A > 0$. So, in the framework
of the TEGR the apparent horizon does not constitute a thermodynamically isolated system.

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