Diagonalization of infinite transfer matrix of boundary $U_{q,p}(A^{(1)}_{N-1})$ face model

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Dedicated to Professor Etsuro Date on the occasion of the 60th birthday

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Abstract

We study infinitely many commuting operators $T_B(z)$, which we call infinite transfer matrix of boundary $U_{q,p}(A^{(1)}_{N-1})$ face model. We diagonalize the infinite transfer matrix $T_B(z)$ by using free field realizations of the vertex operators of the elliptic quantum group $U_{q,p}(A^{(1)}_{N-1})$.

1 Introduction

There have been many developments in the field of exactly solvable models. Various models were found to be exactly solvable and various methods were invented to solve these models [1, 2, 3, 4, 5, 6]. The vertex operator approach [6, 7] provides a powerful method to study exactly solvable lattice models. This paper is devoted to boundary problem of the vertex operator approach to exactly solvable lattice model.
Solvability of lattice models was understood by means of the method of commuting transfer matrix. Two-dimensional solvable lattice models can be related to one-dimensional solvable Hamiltonians of one-dimensional quantum mechanics. The transfer matrix of two-dimensional lattice models is a generating function of these Hamiltonians. The commuting transfer matrix methods are classified to four ways of doing this: (A) Clifford algebras and fermion operators [1], (B) Bethe ansatz and quantum inverse scattering methods [3, 4], (C) Finite-lattice matrix functional relations (Baxter’s $T$-$Q$ relations) [2], and (D) Vertex operator approach [6]. The first three methods (A), (B), (C) work for finite systems. Meanwhile, the vertex operator approach (D) works only for infinite systems. In infinite systems, the transfer matrix is written by using vertex operators, this can be related to the corner transfer matrix method [2]. The corner transfer matrix method suggested a link between solvable lattice models and representation theory of infinite algebra. The vertex operator approach arose from interplay between exactly solvable lattice models and representation theory of infinite algebra. The vertex operator approach (D) is completely different from the first three methods (A), (B), (C). The first paper on boundary problem of the vertex operator approach, is devoted to the XXZ spin chain [12], in which vertex operators act on the highest weight representation of the quantum affine group $U_q(A_1^{(1)})$. In [12] the authors diagonalize infinitely many commuting transfer matrices $T_B(z)$, using free field realizations of the vertex operators. There have been two generalizations of this theory [12]. The $U_q(A_{N-1}^{(1)})$ analogue of XXZ spin chain, which give higher rank generalization of [12], was studied in [14]. The boundary ABF model, which give elliptic deformation of [12], was studied in [13]. The boundary ABF is related to the elliptic quantum group $U_{q,p}(A_1^{(1)})$. This paper is devoted to a generalization of the above papers [12, 13, 14].

Exactly solvable lattice model with open boundary is defined by both Boltzmann weight functions and boundary Boltzmann weight functions, which satisfy Yang-Baxter equation and boundary Yang-Baxter equation [8] respectively. In this paper we are going to study the $U_{q,p}(sl_N)$ face model defined by elliptic solution of Yang-Baxter equation [10] and boundary Yang-Baxter equation [15], associated with the elliptic quantum group $U_{q,p}(sl_N)$ [20, 21]. We are going to study the infinite transfer matrix $T_B(z)$ associated with the boundary $U_{q,p}(A_{N-1}^{(1)})$ face model. The boundary $U_{q,p}(A_{N-1}^{(1)})$ face model gives a generalization of both the $U_q(A_{N-1}^{(1)})$ analogue of XXZ spin chain [14], and the boundary ABF model [13]. In this paper we diagonalize infinitely many commuting transfer matrices $T_B(z)$ acting on the bosonic Fock space, in which the free field realization of the elliptic quantum group $U_{q,p}(A_{N-1}^{(1)})$ is constructed. We construct free field realization of the boundary state $|k\rangle_B$, which satisfies $T_B(z)|k\rangle_B = |k\rangle_B$, by using free field realizations of the vertex operators [17]. Multiplying the type-II vertex operators [19], we
diagonalize infinitely many commuting transfer matrices \( T_B(z) \). We give complete proof of the eigenvalue problem \( T_B(z)|k\rangle_B = |k\rangle_B \), this was only conjectured for \( U_{q,p}(A^{(1)}_2) \) case \([22]\), by using identities of the elliptic theta functions. We calculate the norm \( B\langle k|k\rangle_B \) by using the coherent state. We discuss physical and graphical interpretation of the boundary state \( |k\rangle_B \) for the boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model.

The plan of the paper is as follows. In section 2 we prepare the Boltzmann weight function, the boundary Boltzmann weight function, and the vertex operators. We introduce infinite transfer matrix of the boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model, and formulate the problem. In section 3 we prepare the free field realization of the vertex operators. Section 4 is main part of this paper. In section 4, we construct the free field realization of the boundary state \( |k\rangle_B \). Multiplying the type-II vertex operators we diagonalize the infinite transfer matrix \( T_B(z) \) of the boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model. In section 5, we calculate the norm \( B\langle k|k\rangle_B \) by using coherent state. In appendix we consider physical meaning of our problem. We discuss the relation between our diagonalization problem and the boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model.

### 2 Infinite transfer matrix

In this section we introduce infinitely many commuting operators \( T_B(z) \), which we call infinite transfer matrix of boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model.

#### 2.1 Notation

We prepare some notations. Let us set integer \( N = 2, 3, \cdots \). We assume that \( 0 < x < 1 \) and \( r \geq N + 2 \) (\( r \in \mathbb{Z} \)). We use parameterizations \( z, u \) and \( \tau \).

\[
z = x^{2u}, \quad x = e^{-\pi i/2r}.
\]

We set the elliptic theta function \([u]\) by

\[
[u] = x^{-u^2/2} \Theta_{x^{2^{u}}}(x^{2^{u}}), \quad \Theta_q(z) = (q, q)_\infty(z; q)_\infty(q/z; q)_\infty.
\]

Here we have used

\[
(z; q, q_1, q_2, \cdots, q_m)_{\infty} = \prod_{j_1, j_2, \cdots, j_m = 0}^{\infty} (1 - q_1^{j_1} q_2^{j_2} \cdots q_m^{j_m} z).
\]

The elliptic theta function \([u]\) satisfies the following quasi-periodicities.

\[
[u + r] = -[u], \quad [u + r\tau] = -e^{-\pi i r - \frac{2\pi i u}{r}}[u].
\]
Let $\epsilon_\mu (1 \leq \mu \leq N)$ be the orthonormal basis of $\mathbb{R}^N$ with the inner product $(\epsilon_\mu | \epsilon_\nu) = \delta_{\mu,\nu}$. Let us set $\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon$ where $\epsilon = \frac{1}{N} \sum_{\nu=1}^{N} \epsilon_\nu$. Note that $\sum_{\mu=1}^{N} \bar{\epsilon}_\mu = 0$. Let $\alpha_\mu (1 \leq \mu \leq N - 1)$ the simple root:

$$
\alpha_\mu = \bar{\epsilon}_\mu - \bar{\epsilon}_{\mu+1}.
$$

Let $\omega_\mu (1 \leq \mu \leq N - 1)$ be the fundamental weights, which satisfy

$$
(\alpha_\mu | \omega_\nu) = \delta_{\mu,\nu}, \quad (1 \leq \mu, \nu \leq N - 1).
$$

Explicitly we set $\omega_\mu = \sum_{\nu=1}^{\mu} \bar{\epsilon}_\nu$. The type $A_{N-1}$ weight lattice is the linear span of $\bar{\epsilon}_\mu$ or $\omega_\mu$.

$$
P = \sum_{\mu=1}^{N-1} \mathbb{Z} \bar{\epsilon}_\mu = \sum_{\mu=1}^{N-1} \mathbb{Z} \omega_\mu.
$$

For $a \in P$ we set $a_\mu$ and $a_{\mu,\nu}$ by

$$
a_{\mu,\nu} = a_\mu - a_\nu, \quad a_\mu = (a + \rho | \epsilon_\mu), \quad (\mu, \nu \in P).
$$

Here we set $\rho = \sum_{\mu=1}^{N-1} \omega_\mu$. Let us set the restricted path $P_{r-N}^+$ by

$$
P_{r-N}^+ = \{ a = \sum_{\mu=1}^{N-1} c_\mu \omega_\mu \in P | c_\mu \in \mathbb{Z}, c_\mu \geq 0, \sum_{\mu=1}^{N-1} c_\mu \leq r - N \}.
$$

For $a \in P_{r-N}^+$, condition $0 < a_{\mu,\nu} < r$, $(1 \leq \mu < \nu \leq N - 1)$ holds.

### 2.2 Yang-Baxter equation

We recall elliptic solutions of the Yang-Baxter equation of face type. An ordered pair $(b, a) \in P^2$ is called admissible if and only if there exists $\mu (1 \leq \mu \leq N)$ such that

$$
b - a = \bar{\epsilon}_\mu.
$$

An ordered set of four weights $(a, b, c, d) \in P^4$ is called an admissible configuration around a face if and only if the ordered pairs $(b, a)$, $(c, b)$, $(d, a)$ and $(c, d)$ are admissible. Let us set the Boltzmann weight functions of the face model [10],

$$
W \left( \begin{array}{cc} c & d \\ b & a \end{array} \right) = \mathbb{R}(u),
$$

associated with admissible configuration $(a, b, c, d) \in P^4$. For $a \in P_{r-N}^+$, we set

$$
W \left( \begin{array}{cc} a + 2\bar{\epsilon}_\mu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu & a \end{array} \right) = \mathbb{R}(u),
$$

(2.8)
\[
W \left( \begin{array}{c|c}
 a + \bar{\epsilon}_\mu + \bar{\epsilon}_\nu & a + \bar{\epsilon}_\mu \\
 a + \bar{\epsilon}_\nu & a \\
\end{array} \right) u = R(u) \frac{[u][a_{\mu,\nu} - 1]}{[u - 1][a_{\mu,\nu}]}, \quad (2.9)
\]
\[
W \left( \begin{array}{c|c}
 a + \bar{\epsilon}_\mu + \bar{\epsilon}_\nu & a + \bar{\epsilon}_\mu \\
 a + \bar{\epsilon}_\nu & a \\
\end{array} \right) u = R(u) \frac{[u - a_{\mu,\nu}][1]}{[u - 1][a_{\mu,\nu}]}. \quad (2.10)
\]

The normalizing function \( R(u) \) is given by
\[
R(u) = \frac{z N \varphi(z^{-1})}{\varphi(z)}, \quad \varphi(z) = \frac{(x^2 z; x^2, x^{2N})_\infty (x^{2r+2N-2} z; x^2, x^{2N})_\infty}{(x^{2r} z; x^2, x^{2N})_\infty (x^{2N} z; x^2, x^{2N})_\infty}. \quad (2.11)
\]

Because \( 0 < a_{\mu,\nu} < r \) (1 \( \leq \mu < \nu \leq N - 1 \)) holds for \( a \in P^+_{r-N} \), the Boltzmann weight functions are well defined.

The Boltzmann weight functions satisfy the following relations (1), (2), (3) and (4). \([10, 11]\)

(1) Yang-Baxter equation:
\[
\sum_g W \left( \begin{array}{c|c}
 d & e \\
 c & g \\
\end{array} \right) u_1 W \left( \begin{array}{c|c}
 c & g \\
 b & a \\
\end{array} \right) u_2 W \left( \begin{array}{c|c}
 e & f \\
 g & a \\
\end{array} \right) u_1 - u_2 = \sum_g W \left( \begin{array}{c|c}
 g & f \\
 b & a \\
\end{array} \right) u_1 W \left( \begin{array}{c|c}
 d & e \\
 g & f \\
\end{array} \right) u_2 W \left( \begin{array}{c|c}
 d & g \\
 c & b \\
\end{array} \right) u_1 - u_2 . \quad (2.12)
\]

(2) The first inversion relation:
\[
\sum_g W \left( \begin{array}{c|c}
 c & g \\
 b & a \\
\end{array} \right) - u W \left( \begin{array}{c|c}
 c & d \\
 g & a \\
\end{array} \right) u = \delta_{b,d}. \quad (2.13)
\]

(3) The second inversion relation:
\[
\sum_g G_g W \left( \begin{array}{c|c}
 g & b \\
 d & c \\
\end{array} \right) N - u W \left( \begin{array}{c|c}
 g & d \\
 b & a \\
\end{array} \right) u = \delta_{a,c} \frac{G_b G_d}{G_a}, \quad (2.14)
\]

where we have set \( G_a = \prod_{1 \leq j < k \leq N} [a_{j,k}] \).

(4) Initial conditions:
\[
W \left( \begin{array}{c|c}
 g & b \\
 d & c \\
\end{array} \right) 0 = \delta_{b,d}. \quad (2.15)
\]

The normalization function \( R(u) \) (2.11) is chosen to satisfy conditions (2.13), (2.14) and (2.15). Upon this normalization \( R(u) \), the corner transfer matrix method works well, and the maximum eigenvalue of the corner transfer matrix becomes 1.
2.3 Boundary Yang-Baxter equation

An order set of three weights \((a, b, g) \in P^3\) is called an admissible configuration at a boundary if and only if the ordered pairs \((g, a)\) and \((g, b)\) are admissible. Let us set the boundary Boltzmann weight functions \(K\left(\begin{array}{c|c} a & g \\ \hline b & u \end{array}\right)\) for admissible weights \((a, b, g)\) by

\[
K\left(\begin{array}{c|c} a + \epsilon \mu & g \\ \hline u & b \end{array}\right) = z^{r-1} h(z^{-1}) \left[\frac{c - u}{c + u}\right]^{\beta a_1} \delta_{a,b}.
\]

In this paper, we consider the case that continuous parameter \(0 < c < 1\). In what follows we need the case for \(a \in P^+_{r-N}\). The normalizing function \(h(z)\) is given by

\[
h(z) = \frac{(2N - 2c - a_1, j, j)_{\infty}(2N - 2c - a_1, j, j)_{\infty}}{(2N - 2c - a_1, j, j)_{\infty}(2N - 2c - a_1, j, j)_{\infty}}.
\]

The boundary Boltzmann weight functions and the Boltzmann weight functions satisfy the following relations.

(1) Boundary Yang-Baxter equation:

\[
\sum_{f,g} W\left(\begin{array}{c|c} c & f \\ \hline b & a \end{array}\right) W\left(\begin{array}{c|c} c & d \\ \hline f & g \end{array}\right) K\left(\begin{array}{c|c} g & u \\ \hline f & a \end{array}\right) K\left(\begin{array}{c|c} e & v \\ \hline d & g \end{array}\right) = \sum_{f,g} W\left(\begin{array}{c|c} c & d \\ \hline f & e \end{array}\right) W\left(\begin{array}{c|c} c & f \\ \hline b & g \end{array}\right) K\left(\begin{array}{c|c} g & u \\ \hline f & a \end{array}\right) K\left(\begin{array}{c|c} e & v \\ \hline b & g \end{array}\right).
\]

(2) Boundary unitary condition:

\[
K\left(\begin{array}{c|c} a & u \\ \hline b & a \end{array}\right) K\left(\begin{array}{c|c} a & -u \\ \hline b & a \end{array}\right) = 1.
\]
(3) Initial conditions:

\[
K \begin{pmatrix} a & 0 \\ c & 0 \\ b & 0 \end{pmatrix} = \delta_{a,b}.
\] (2.20)

Here we comment on the normalization \( h(z) \). The boundary Boltzmann weight functions associated with \( A_{N}^{(1)} \), \( B_{N}^{(1)} \), \( C_{N}^{(1)} \), \( D_{N}^{(1)} \), \( A_{N}^{(2)} \) satisfy the boundary crossing relations (except \( A_{N \geq 2}^{(1)} \)) like following [15].

\[
\sum_{g} \left( \frac{G_{g}}{G_{b}} \right)^{1/2} W \begin{pmatrix} a & g & b \\ c & 2u + \lambda \end{pmatrix} K \begin{pmatrix} a & u + \lambda \\ g & -u \end{pmatrix} = K \begin{pmatrix} a & u - \lambda \\ b & -u \end{pmatrix}.
\] (2.21)

In paper on \( U_{q,p}(A_{1}^{(1)}) \)-face model [13], the authors choose the normalization \( h(z) \) of boundary Boltzmann weights, from the equations (2.19), (2.20) and (2.21). This normalization makes the maximum eigenvalue of the boundary state \( |k\rangle_{B} \) becomes 1 [13]. However there does not exist boundary crossing symmetry for higher-rank case \( A_{N \geq 2}^{(1)} \). It is nontrivial to find the normalization function \( h(z) \) for higher-rank case \( A_{N \geq 2}^{(1)} \). In this paper the author choose the normalization function \( h(z) \) such that there exists the boundary state \( |k\rangle_{B} \) whose eigenvalue is 1, without using the boundary crossing relations (2.21).

2.4 Vertex operator

In this section we introduce the vertex operator \( \Phi^{(b,a)}(z) \) for the \( U_{q,p}(A_{N-1}^{(1)}) \) face model in the Regime III. In appendix, we explain the physical interpretation of the vertex operators \( \Phi^{(b,a)}(z) \) for the elliptic quantum group \( U_{q,p}(A_{N-1}^{(1)}) \). In this section we treat them symbolically. In what follows we consider the so-called Regime III : \( 0 < u < 1 \) [9, 10, 16]. The vertex operator \( \Phi^{(b,a)}(z) \) and the dual vertex operator \( \Phi^{*(a,b)}(z) \) of admissible pair \( (b, a) \in P^{2} \), for the \( U_{q,p}(A_{N-1}^{(1)}) \) face model, are the operators which satisfy the following functional relations (1) and (2), (resp. (1) and (2')).

(1) Commutation relation:

\[
\Phi^{(a,b)}(z_{1})\Phi^{(b,c)}(z_{2}) = \sum_{g} W \begin{pmatrix} a & g & b \\ c & u_{2} - u_{1} \\ a & u_{1} \end{pmatrix} \Phi^{(a,g)}(z_{2})\Phi^{(g,c)}(z_{1}),
\] (2.22)

\[
\Phi^{*(a,b)}(z_{1})\Phi^{*(b,c)}(z_{2}) = \sum_{g} W \begin{pmatrix} c & b & g \\ a & u_{2} - u_{1} \\ c & u_{1} \end{pmatrix} \Phi^{*(a,g)}(z_{2})\Phi^{*(g,c)}(z_{1}),
\] (2.23)
\[
\Phi^{(a,b)}(z_1)\Phi^{*(b,c)}(z_2) = \sum_g W \begin{pmatrix} g & c \\ a & b \end{pmatrix} \left| u_1 - u_2 \right| \Phi^{*(a,g)}(z_2)\Phi^{(g,c)}(z_1).
\] (2.24)

(2) Inversion relation :

\[
\Phi^{(a,g)}(z)\Phi^{*(g,b)}(z) = \delta_{a,b}.
\] (2.25)

(2') Inversion relation :

\[
\sum_g \Phi^{*(a,g)}(z)\Phi^{(g,b)}(z) = \delta_{a,b}.
\] (2.26)

We have used parameterization \( z = x^{2u} \). The relations (1) and (2) is equivalent with the relations (1) and (2'). In this section we don’t mention the space which these operators act. There exist the operators which satisfy the functional relations (1) and (2) (resp. (1) and (2')). The free field realization of the vertex operators \( \Phi^{(b,a)}(z) \) and \( \Phi^{*(a,b)}(z) \) are given in [17]. In this paper we treat the problem in the Fock space \( \mathcal{F}_{l,k} \), which are introduced in [17]. We review free field realization of vertex operators in the next section.

### 2.5 Boundary transfer matrix

We define the infinite transfer matrix \( T_B(z) \) of the boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model, by using the vertex operators.

\[
T_B(z) = \sum_{\mu=1}^N \Phi^{*(a,a+\bar{\epsilon}_\mu)}(z^{-1})K \begin{pmatrix} a + \bar{\epsilon}_\mu & a \\ a & u \end{pmatrix} \Phi^{(a+\bar{\epsilon}_\mu,a)}(z).
\] (2.27)

Later, in appendix, we explain physical and graphical interpretation of the infinite transfer matrix \( T_B(z) \).

**Proposition 2.1** The infinite transfer matrix \( T_B(z) \) commute with each other.

\[
[T_B(z_1), T_B(z_2)] = 0.
\] (2.28)

**Proof.** Let’s start from the product \( T_B(z_1)T_B(z_2) \).

\[
\sum_{g_1 \cdot g_2} \Phi^{*(a,g_1)}(z_1^{-1})\Phi^{(g_1,a)}(z_1)\Phi^{*(a,g_2)}(z_2^{-1})\Phi^{(g_2,a)}(z_2)K \begin{pmatrix} g_1 & a \\ u_1 & a \end{pmatrix} K \begin{pmatrix} g_2 & a \\ u_2 & a \end{pmatrix}.
\]
Exchange the ordering of \( \Phi^{(g_1,a)}(z_1)\Phi^{*(a,g_2)}(z_2^{-1})\Phi^{(g_2,a)}(z_2) \) by using the commutation relations of vertex operators (2.22), (2.23), and use the boundary Yang-Baxter equation (2.18). We have

\[
\sum_{g_1,g_2,g_3,g_4} \Phi^{*(a,g_1)}(z_1^{-1})\Phi^{*(g_1,g_3)}(z_2^{-1})\Phi^{(g_3,g_4)}(z_2)\Phi^{(g_4,a)}(z_1)
\times W\left(\begin{array}{c|c}
g_3 & g_4 \\
g_2 & a \\
\end{array}\middle| u_2 + u_1\right)W\left(\begin{array}{c|c}
g_3 & g_2 \\
g_1 & a \\
\end{array}\middle| u_2 - u_1\right)K\left(\begin{array}{c|c}
g_4 & a \\
a & u_1 \\
\end{array}\right)K\left(\begin{array}{c|c}
g_2 & a \\
a & u_2 \\
\end{array}\right).
\]

Exchanging the ordering of the vertex operators \( \Phi^{*(a,g_1)}(z_1^{-1})\Phi^{*(g_1,g_3)}(z_2^{-1})\Phi^{(g_3,g_4)}(z_2) \) by using (2.23), (2.24), we get \( T_B(z_2)T_B(z_1) \).

Q.E.D.

### 2.6 Boundary state

The commutativity of the transfer matrix \([T_B(z_1),T_B(z_2)] = 0\) ensues that, if the transfer matrices \( T_B(z) \) are diagonalizable, the transfer matrices \( T_B(z) \) are diagonalized by the basis which is independent of the spectral parameter \( z \).

**Definition 2.2** We call eigenvector \( |k\rangle_B \) with eigenvalue 1 the boundary state.

\[ T_B(z)|k\rangle_B = |k\rangle_B. \] (2.29)

In this paper we construct the free field realization of the boundary state \( |k\rangle_B \). The construction of the boundary state \( |k\rangle_B \) is main result of this paper.

### 2.7 Type-II vertex operator

In this section we introduce the type-II vertex operator \( \Psi^{*(b,a)}(z) \) for the \( U_{q,p}(A_{N-1}^{(1)}) \) face model, which we use for diagonalization of the transfer matrix \( T_B(z) \). For this purpose we prepare some functions. Let us set \( r^* = r - 1 \). Let us set the elliptic theta function \([u]^*\) by

\[ [u]^* = x^{\frac{u^2}{2}}u^{r^*}(x^{2u}). \] (2.30)

Let us set the Boltzmann weight functions

\[ W^*\left(\begin{array}{c|c}
a & b \\
c & d \\
\end{array}\middle| u\right)\]
for admissible configuration \((a, b, c, d) \in P^4\). For \(a \in P^{r-1-N}_r\), we set

\[
W^* \begin{pmatrix} a + 2\bar{\epsilon}_\mu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu & a \end{pmatrix} u = R^*(u),
\]

(2.31)

\[
W^* \begin{pmatrix} a + \bar{\epsilon}_\mu + \bar{\epsilon}_\nu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\nu & a \end{pmatrix} u = R^*(u) \frac{[u]^* [a_{\mu,\nu} - 1]^*}{[u - 1]^* [a_{\mu,\nu}]^*},
\]

(2.32)

\[
W^* \begin{pmatrix} a + \bar{\epsilon}_\mu + \bar{\epsilon}_\nu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\nu & a \end{pmatrix} u = R^*(u) \frac{[u - a_{\mu,\nu}]^*[1]^*}{[u - 1]^* [a_{\mu,\nu}]^*}.
\]

(2.33)

The function \(R^*(u)\) is given by

\[
R^*(u) = z^{-\sum_{r-N-1}^{N-1} \phi^*(z^{-1})} \varphi^*(z), \quad \varphi^*(z) = (z; x^{2r-1}, x^{2N})_\infty (x^{2N+2r-2} z; x^{2r}, x^{2N})_\infty.
\]

(2.34)

The Boltzmann weight function \(W^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} u\) satisfies (1) the Yang-Baxter equation, (2) the first inversion relation, (3) the second inversion relation, and (4) initial condition, like (2.12), (2.13), (2.14) and (2.15).

The type-II vertex operator \(\Psi^{(b,a)}(z)\) of admissible pair \((b, a) \in P^2\), for the \(U_q(A^{(1)}_{N-1})\) face model, are the operators which satisfy the following functional relations (1) and (2).

(1) Commutation relation :

\[
\Psi^{(a,b)}(z_1) \Psi^{(b,c)}(z_2) = \sum_g W^* \begin{pmatrix} a & g \\ b & c \end{pmatrix} u_1 - u_2 \Psi^{(a,g)}(z_2) \Psi^{(g,c)}(z_1),
\]

(2.35)

(2) Commutation relation with vertex operator:

\[
\Phi^{(d,e)}(z_1) \Psi^{(b,a)}(z_2) = \chi(z_2/z_1) \Psi^{(b,a)}(z_2) \Phi^{(d,e)}(z_1),
\]

\[
\Phi^{(c,d)}(z_1) \Psi^{(b,a)}(z_2) = \chi(z_1/z_2) \Psi^{(b,a)}(z_2) \Phi^{(c,d)}(z_1).
\]

(2.36)

(2.37)

where we have set

\[
\chi(z) = z^{-\sum_{r-N-1}^{N-1} \frac{(-z^{2N-1} z^{-1}; x^{2N})_\infty (z z^{2N}; x^{2N})_\infty}{(-z^{-1}; x^{2N})_\infty (-z^{2N-1} z^{-1}; x^{2N})_\infty}}.
\]

(2.38)

The free field realization of the type-II vertex operators \(\Psi^{(b,a)}(z)\) are given in [19].

**Definition 2.3** We set the vectors \(|\xi_1, \xi_2, \cdots, \xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}\) \((1 \leq \mu_1, \mu_2, \cdots, \mu_M N)\).

\[
|\xi_1, \xi_2, \cdots, \xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M} = \Psi^{(b+\bar{\epsilon}_1+\bar{\epsilon}_{\mu_1}+\cdots+\bar{\epsilon}_{\mu_M} + b+\bar{\epsilon}_{\mu_2}+\cdots+\bar{\epsilon}_{\mu_M})}(\xi_1) \cdots \Psi^{(b+\bar{\epsilon}_{\mu_{M-1}}+\bar{\epsilon}_{\mu_M} + b+\bar{\epsilon}_{\mu_M})}(\xi_{M-1}) \Psi^{(b+\bar{\epsilon}_{\mu_M} + b)}(\xi_M) |k\rangle_B.
\]

We call the vectors \(|\xi_1, \xi_2, \cdots, \xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}\) the excitation of the boundary state \(|k\rangle_B\).
Proposition 2.4  The excitations $|\xi_1, \xi_2, \cdots, \xi_M\rangle_{\mu_1, \mu_2, \cdots, \mu_M}$ are eigenvectors of the transfer matrix $T_B(z)$.

$$T_B(z)|\xi_1, \xi_2, \cdots, \xi_M\rangle_{\mu_1, \mu_2, \cdots, \mu_M} = \prod_{j=1}^{M} \chi(\xi_j/z)\chi(1/\xi_jz)|\xi_1, \xi_2, \cdots, \xi_M\rangle_{\mu_1, \mu_2, \cdots, \mu_M}. \quad (2.40)$$

3 Free field realization

In this section we give free field realization of the vertex operators $\Phi^{(b,a)}(z)$ and the type-II vertex operators $\Psi^{(d,c)}(z)$.

3.1 Boson

We set the bosonic oscillators $\beta^j_m, (i = 1, 2, \cdots, N - 1; m \in \mathbb{Z})$ by

$$[\beta^j_m, \beta^k_n] = \begin{cases} m[(r - 1)m]_x [(N - 1)m]_x \delta_{m+n,0} & (j = k) \\ -m_x N^m_{(j-k)} [(r - 1)m]_x [m]_x \delta_{m+n,0} & (j \neq k). \end{cases} \quad (3.1)$$

Here the symbol $[a]_x = \frac{x^a - x^{-a}}{x - x^{-1}}$. Let us set $\beta^N_m$ by $\sum_{j=1}^{N} x^{-2jm} \beta^j_m = 0$. The above commutation relations are valid for all $1 \leq j, k \leq N$. We also introduce the zero-mode operators $P_\alpha, Q_\alpha, (\alpha \in P)$ by

$$[iP_\alpha, Q_\beta] = (\alpha | \beta), \quad (\alpha, \beta \in P). \quad (3.2)$$

In what follows we deal with the bosonic Fock space $\mathcal{F}_{l,k}$, generated by $\beta^j_m (m > 0)$ over the vacuum vector $|l,k\rangle$, where

$$l = b + \rho, \quad k = a + \rho, \quad (3.3)$$

for $a \in P^+_{r-N}$ and $b \in P^+_{r-1-N}$.

$$\mathcal{F}_{l,k} = \mathbb{C}[\{\beta^j_{-1}, \beta^j_{-2}, \cdots \}_{j=1,\cdots,N-1}] |l,k\rangle, \quad (3.4)$$

where

$$\beta^j_m |l,k\rangle = 0, \quad (m > 0),$$

$$P_\alpha |l,k\rangle = \left( \alpha \sqrt{\frac{r}{r-1} - \sqrt{\frac{r-1}{r}}} \right) |l,k\rangle, \quad (3.5)$$

$$|l,k\rangle = e^{i \sqrt{Q_l - \sqrt{Q_k}}} |0,0\rangle.$$
3.2 Vertex operator

We give a free field realization of the vertex operators \( \Phi^{(b,a)}(z) \) [17], associated with the elliptic algebra \( U_{q,p}(A^{(1)}_{N−1}) \) [20, 21]. Let us set basic operators \( P_−(z), Q_−(z), R_−^i(z), S_−^i(z) \), \( 1 \leq j \leq N−1 \) by

\[
P_−(z) = \sum_{m>0} \frac{1}{m} \beta_{−m}z^m, \quad \Phi_−(z)
\]

\[
Q_−(z) = -\sum_{m>0} \frac{1}{m} \beta_{−m}z^m, \quad \Phi_+(z)
\]

\[
R_−^i(z) = -\sum_{m>0} \frac{1}{m} (\beta_{−m}^i - \beta_{−m}^{i+1})x^{jm}z^m, \quad \Phi_−^i(z)
\]

\[
S_−^i(z) = \sum_{m>0} \frac{1}{m} (\beta_{−m}^i - \beta_{−m}^{i+1})x^{jm}z^m. \quad \Phi_+^i(z)
\]

Let us set the basic operators \( U(z), F_{α_j}(z) \), \( 1 \leq j \leq N−1 \) on the Fock space \( F_{l,k} \).

\[
U(z) = z^{r−1} e^{-i\sqrt{Q_1}z} \sqrt{P_1} e^{P_−(z)}e^{Q_−(z)}, \quad (3.10)
\]

\[
F_{α_j}(z) = z^{r−1} e^{i\sqrt{Q_α_j}z} \sqrt{P_{α_j}} e^{R_−^i(z)}e^{S_−^i(z)}. \quad (3.11)
\]

In what follows we set

\[
π_{μ} = √(r−1)P_μ, \quad π_{μ,ν} = π_μ - π_ν. \quad (3.12)
\]

Then \( π_{μ,ν} \) acts on \( F_{l,k} \) as an integer \( (ε_μ - ε_ν)rl - (r−1)k \). We give the free field realization of the vertex operators \( \Phi^{(k+iμ,k)}(z) \), \( 1 \leq μ \leq N−1 \) by

\[
\Phi^{(k+iμ,k)}(z_{0}^{−1}) = U(z_{0}),
\]

\[
\Phi^{(k+iμ,k)}(z_{0}^{−1}) = \oint \cdots \oint \prod_{j=1}^{μ−1} \frac{dz_j}{2πiz_j} U(z_0)F_{α_1}(z_1)F_{α_2}(z_2) \cdots F_{α_{μ−1}}(z_{μ−1}) \times \prod_{j=1}^{μ−1} \frac{[u_j - u_{j−1} + 1/2 - π_{j,μ}]}{[u_j - u_{j−1} - 1/2]} \quad (3.13)
\]

Here we set \( z_j = x^{2u_j} \). We take the integration contour to be simple closed curve that encircles \( z_j = 0, x^{1+2r_s}z_{j−1}, (s \in \mathbb{N}) \) but not \( z_j = x^{−1−2r_s}z_{j−1}, (s \in \mathbb{N}) \) for \( 1 \leq j \leq μ−1 \). The \( \Phi^{(k+iμ,k)}(z) \) is an operator such that \( \Phi^{(k+iμ,k)}(z) : F_{l,k} \to F_{l,k+iμ} \). The vertex operators \( \Phi^{(b,a)}(z) \) satisfy the commutation relation (2.22). The free field realization of the dual vertex operator \( \Phi^*(a,b)(z) \) is given by similar way [17]. In this paper we omit the free field realization of the dual vertex operator \( \Phi^*(a,b)(z) \), because we don’t use the explicit formulae of the dual vertex operators in this paper. We only use the inversion relation \( \Phi^{(a,g)}(z)\Phi^*(g,b)(z) = δ_{a,b}, (2.25) \).
3.3 Type-II Vertex operator

In this section we give the free field realization of the type-II vertex operators $\Psi^{(b,a)}(z)$ [19], associated with the elliptic algebra $U_{q,p}(A^{(1)}_{N-1})$. Let us set $P^*_+(z), Q^*_+(z), R^j_+(z), S^j_+(z), (1 \leq j \leq N - 1)$ by

$$
P^*_+(z) = \sum_{m>0} \frac{[rm]_x}{m[r^m]_x} \beta^1_{m} z^m,
$$

(3.14)

$$
Q^*_+(z) = \sum_{m>0} \frac{[rm]_x}{m[r^m]_x} \beta^1_{-m} z^-m,
$$

(3.15)

$$
R^j_+(z) = \sum_{m>0} \frac{[rm]_x}{m[r^m]_x} (\beta^j_{m} - \beta^{j+1}_{m}) z^m,
$$

(3.16)

$$
S^j_+(z) = - \sum_{m>0} \frac{[rm]_x}{m[r^m]_x} (\beta^j_{m} - \beta^{j+1}_{m}) z^{-m}.
$$

(3.17)

Let us set the basic operators $V(z), E_{\alpha_j}(z)$ acting on the Fock space $F_{l,k}$.

$$
V^*(z) = z^{\frac{N-1}{2N}} e^{i \sqrt{P} \zeta_2 \sqrt{P} \zeta_1} e^{P^*_+(z) Q^*_+(z)},
$$

(3.18)

$$
E_{\alpha_j}(z) = z^{\frac{N-1}{2N}} e^{-i \sqrt{P} \zeta_2 \sqrt{P} \zeta_1} e^{P^*_+(z) Q^*_+(z)}. \frac{1}{(2i)^{2l}}.
$$

(3.19)

We give a free field realization of the type-II vertex operators $\Psi^{(l+\mu,j)}(z)$ for $1 \leq \mu \leq N - 1$.

$$
\Psi^{(l+\mu,j)}(-z^1_{0-1}) = V^*(z_0),
$$

$$
\Psi^{(l+\mu,j)}(-z^1_{0-1}) = \oint \cdots \oint \prod_{j=1}^{\mu-1} \frac{dz_j}{z_j} V^*(z_0) E_{\alpha_1}(z_1) E_{\alpha_2}(z_2) \cdots E_{\alpha_{\mu-1}}(z_{\mu-1})
$$

$$
\times \frac{\prod_{j=1}^{\mu-1} \left[ u_j - u_{j-1} - \frac{1}{2} + \pi_{j,\mu} \right]^*}{\prod_{j=1}^{\mu-1} \left[ u_j - u_{j-1} + \frac{1}{2} \right]^*}.
$$

(3.20)

The integration should be carried out in the order $z_{\mu-1}, \cdots, z_2, z_1$ along the contour for $z_j$-integration which encircles $z_j = x^{1-2s} r^{1}, (s = 0, 1, 2, \ldots)$, but $z_j = x^{1-2s} r^{1} z_{j-1}, (s = 0, 1, 2, \cdots)$. The operator $\Psi^{(l+\mu,j)}(z)$ is an operator such that $\Psi^{(l+\mu,j)}(z) : F_{l,k} \rightarrow F_{l+\mu,k}$.

4 Boundary state

In this section we give a free field realization of the boundary state $|k\rangle_B$ in the bosonic Fock space $F_{k,k}$.

$$
T_B(z)|k\rangle_B = |k\rangle_B.
$$

In this paper, we consider the case $l = k$, which represents the ground state of the boundary $U_{q,p}(A^{(1)}_{N-1})$ face model. In the appendix, we explain the physical meaning of $(l,k)$. The label
k represents the condition at the origin and the label l represents the asymptotic boundary condition at \( \infty \). The condition \( l = k \in P^{+}_{r-1-N} \) represents the ground state for \( 0 < c < 1 \) in the Regime III, i.e., the case \( 0 < u < 1 \).

Using the inversion relation of the vertex operator \( \Phi^{(a,g)}(z)\Phi^{(g,b)}(z) = \delta_{a,b} \), we have the following proposition.

**Proposition 4.1** The relation (2.29) is equivalent to the following relation.

\[
K \begin{pmatrix} k + \bar{\epsilon}_j & k \end{pmatrix} u \Phi^{(k+\bar{\epsilon}_j,k)}(z) |k\rangle_B = \Phi^{(k+\bar{\epsilon}_j,k)}(z^{-1})|k\rangle_B. \tag{4.1}
\]

In what follows we will show the relation (4.1).

### 4.1 Main result

In this section we explain the main result of this paper. The commutation relation of bosons \( \beta^j_m \) is not symmetric. It is convenient to introduce new generators of bosons \( \alpha^j_m \), whose commutation relation is symmetric. Let us set \( \alpha^j_m \) \( (m \in \mathbb{Z} \neq 0; 1 \leq j \leq N - 1) \) by

\[
\alpha^j_m = x^{-jm}(\beta^j_m - \beta^{j+1}_m). \tag{4.2}
\]

They satisfy the following commutation relations.

\[
[\alpha^j_m, \alpha^k_n] = m \begin{pmatrix} (r-1)m \end{pmatrix} x \begin{pmatrix} A_{j,k}m \end{pmatrix} x \begin{pmatrix} m \end{pmatrix} x \delta_{m+n,0}, \tag{4.3}
\]

where \( A_{j,k} \) is a matrix element of the Cartan matrix of \( A_{N-1} \) type.

\[
(A_{j,k})_{1 \leq j, k \leq N-1} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{4.4}
\]

Let us set \( I_{j,k}(m) \) \( (m \in \mathbb{Z} \neq 0; 1 \leq j, k \leq N - 1) \) by

\[
I_{j,k}(m) = \begin{pmatrix} jm \end{pmatrix} x \begin{pmatrix} (N-k)m \end{pmatrix} x \begin{pmatrix} m \end{pmatrix} x \begin{pmatrix} Nm \end{pmatrix} x = I_{k,j}(m) \quad (1 \leq j \leq k \leq N - 1). \tag{4.5}
\]
The matrix \((I_{j,k}(m))_{1 \leq j,k \leq N-1}\) gives the inverse matrix of \(\left(\frac{[A_{j,k}(m)]}{[m]}\right)_{1 \leq j,k \leq N-1}\).

Let us set

\[ [a]_x^+ = x^a + x^{-a}, \quad \theta_m(x) = \begin{cases} x & m \text{ even}, \\ 0 & m \text{ odd}. \end{cases} \quad (4.6) \]

**Theorem 4.2** The free field realization of the boundary state \(|k\rangle_B\) is given by

\[ |k\rangle_B = e^F |k, k\rangle. \quad (4.7) \]

This bosonic vector satisfies

\[ T_B(z)|k\rangle_B = |k\rangle_B. \quad (4.8) \]

Here we have set

\[ F = -\frac{1}{2} \sum_{m>0} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \frac{1}{m \left[(r-1)m\right]_x} I_{j,k}(m) \alpha_m^j \alpha_m^k + \sum_{m>0} \sum_{j=1}^{N-1} \frac{1}{m} D_j(m) \beta_m, \quad (4.9) \]

where we have set

\[ D_j(m) = -\theta_m \left( \frac{(N-j)m/2_x [rm/2]_x^+(3j-N-1)m}{(r-1)m/2_x} \right) \]

\[ + x^{(j-1)m} \left[(r-2\pi_1 j + 2c - j + 2)m\right]_x \]

\[ + \frac{[m]_x x^{(r-2c+2j-2)m}}{(r-1)m]_x} \left( \sum_{k=j+1}^{N-1} x^{-2m\pi_1,k} \right) \]

\[ + x^{(2j-N)m} \left[(r-2\pi_1,N - 2c + N - 1)m\right]_x. \quad (4.10) \]

**Theorem 4.3** The free field realization of the boundary state \(B|k\rangle\) is given by

\[ B|k\rangle = \langle k, k\rangle e^G. \quad (4.11) \]

This bosonic vector satisfies

\[ B|k\rangle T_B(z) = B|k\rangle. \quad (4.12) \]

Here we have set the bosonic operator \(G\) by

\[ G = -\frac{1}{2} \sum_{m>0} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \frac{x^{2Nm}}{m \left[(r-1)m\right]_x} I_{j,k}(m) \alpha_m^j \alpha_m^k + \sum_{m>0} \sum_{j=1}^{N-1} \frac{1}{m} E_j(m) \beta_m, \quad (4.13) \]
where we have set

\[
E_j(m) = \theta_m \left( \frac{[(N-j)m/2]_x [rm/2]_x \left( \frac{(3N+1-3j)m}{2} \right)}{[(r-1)m/2]_x} \right) \nonumber - \frac{x^{(N+1-j)m}[r-2c+N-j-2\pi_{1,j}m]_x}{[(r-1)m]_x} \nonumber - \frac{[m]_x x^{(r-2c+2N-2)j}m}{[(r-1)m]_x} \sum_{k=j+1}^{N-1} x^{-2m\pi_{1,k}} \nonumber - \frac{x^{2(N-j)m}[-r+2c+2\pi_{1,N}m]_x}{[(r-1)m]_x}. \tag{4.14}
\]

We construct diagonalization of the transfer matrix \( T_B(z) \) on the subspace of \( \mathcal{F}_{k,k} \), which is spanned by the excitations \( |\xi_1 \cdots \xi_M\rangle_{\mu_1 \cdots \mu_M} \). See proposition 2.4. This subspace is expected to become the physical space of the boundary \( \mathcal{U}_{q,p}(A^{(1)}_{N-1}) \) face model explained in appendix.

### 4.2 Proof

In this section we give a proof of theorem 4.2. At first we prepare some propositions, which are derived by direct calculations.

**Proposition 4.4** \hspace{1em} The adjoint action of \( e^F \) has the effect of a Bogoliubov transformation.

\[
e^{-F} \alpha_m^j e^F = \alpha_m^j - \alpha_{-m}^j \tag{4.15}
\]

\[
+ \frac{[(r-1)m]_x}{[rm]_x} (x^{(-j+1)m}D_j(m) - x^{(-j-1)m}D_{j+1}(m)), \quad (m > 0; 1 \leq j \leq N - 2),
\]

\[
e^{-F} \alpha_m^{-N-1} e^F = \alpha_m^{-N-1} - \alpha_{-m}^{-N-1} + \frac{[(r-1)m]_x}{[rm]_x} x^{(-N+2)m}D_{N-1}(m), \quad (m > 0), \tag{4.16}
\]

\[
e^{-F} \beta_m^1 e^F = \beta_m^1 - \beta_{-m}^1 \nonumber + \frac{[(r-1)m]_x}{[rm]_x [Nm]_x} (\langle N-1 \rangle_m x D_1(m) - [m]_x x^{-Nm} \sum_{j=2}^{N-2} D_j(m)). \tag{4.17}
\]

**Proposition 4.5** \hspace{1em} The adjoint action of \( e^G \) has the effect of a Bogoliubov transformation.

\[
e^G \alpha_m^j e^{-G} = \alpha_m^j - x^{2Nm} \alpha_m^j \tag{4.18}
\]

\[
+ \frac{[(r-1)m]_x}{[rm]_x} (x^{(j-1)m}E_j(m) - x^{(j+1)m}E_{j+1}(m)), \quad (m > 0; 1 \leq j \leq N - 2),
\]

\[
e^G \alpha_m^{-N-1} e^{-G} = \alpha_m^{-N-1} - x^{2Nm} \alpha_m^{-N-1} + \frac{[(r-1)m]_x}{[rm]_x} x^{(N-2)m}E_{N-1}(m), \quad (m > 0), \tag{4.19}
\]

\[
e^G \beta_m^{-N} e^{-G} = \beta_m^{-N} - \beta_{-m}^{-N} - \frac{[(r-1)m]_x [m]_x x^{-Nm}}{[rm]_x [Nm]_x} \sum_{j=1}^{N-1} E_j(m). \tag{4.20}
\]
Proposition 4.6  The actions of the basic operators on the boundary state are given by

\[ e^{Q_-(z)} |k\rangle_B = h(z) e^{P_-(1/z)} |k\rangle_B, \quad (4.21) \]
\[ e^{S_+^j(w)} |k\rangle_B = g_j(w) e^{R_+^j(1/w)} |k\rangle_B, \quad (1 \leq j \leq N - 1). \quad (4.22) \]

Here we have set

\[ h(z) = \frac{(x^{2r+2N-2/z^2}; x^{2r}, x^{4N})_\infty}{(x^{2r/z^2}; x^{2r}, x^{4N})_\infty} \frac{(x^{4N/z^2}; x^{2r}, x^{4N})_\infty}{(x^{2r-2c/z}; x^{2r}, x^{4N})_\infty} \]
\[ \times \frac{(x^{2r+2c/z}; x^{2r}, x^{4N})_\infty}{(x^{2r-2c-z^2}; x^{2r}, x^{4N})_\infty}, \quad (4.23) \]
\[ g_j(w) = \frac{(1 - w^{j}) (x^{2c+2+2+j/w}; x^{2r})_\infty}{(x^{2r-2+j/w}; x^{2r})_\infty}, \quad (4.24) \]

Proposition 4.7  The actions of the basic operators on the dual boundary state are given by

\[ B \langle k | e^{P_+^*}(z/x^N) = h^*(z) B \langle k | e^{Q_+^*(1/x^N z)}, \quad (4.25) \]
\[ B \langle k | e^{R_+^j}(w/x^N) = g^*_j(w) B \langle k | e^{S_+^j(1/x^N w)}, \quad (1 \leq j \leq N - 1). \quad (4.26) \]

Here we have set

\[ h^*(z) = \frac{(x^{2r+2N-2z^2}; x^{2r}, x^{4N})_\infty}{(x^{2r-z^2}; x^{2r}, x^{4N})_\infty} \frac{(x^{4N/z}; x^{2r}, x^{4N})_\infty}{(x^{2r-2c-2z^2}; x^{2r}, x^{4N})_\infty} \]
\[ \times \frac{(x^{2r+2c+2z}; x^{2r}, x^{4N})_\infty}{(x^{2r-2c-z^2}; x^{2r}, x^{4N})_\infty} \frac{(x^{2c+2+2z}; x^{2r}, x^{4N})_\infty}{(x^{2c+2+2z}; x^{2r}, x^{4N})_\infty} \]
\[ \times N \prod_{j=1}^{N-1} \frac{(x^{2r+2N-2c-2z^2}; x^{2r}, x^{4N})_\infty}{(x^{2r-2c-2z^2}; x^{2r}, x^{4N})_\infty} \frac{(x^{2r+2c+2z}; x^{2r}, x^{4N})_\infty}{(x^{2r-2c-z^2}; x^{2r}, x^{4N})_\infty}, \quad (4.27) \]
\[ g^*_j(w) = \frac{(1 - w^{j}) (x^{2c+2+2z^2+j}; x^{2r})_\infty}{(x^{2r-2c+2z}; x^{2r})_\infty}, \quad (4.28) \]

In order to write proof compactly, we introduce "weak equality" in the following sense.
Definition 4.8  When functions $F(z_1, z_2, \cdots, z_L)$ and $G(z_1, z_2, \cdots, z_L)$ satisfy
\[
\sum_{\epsilon_1=\pm 1} \cdots \sum_{\epsilon_L=\pm 1} F(z_1^{\epsilon_1}, z_2^{\epsilon_2}, \cdots, z_L^{\epsilon_L}) = \sum_{\epsilon_1=\pm 1} \cdots \sum_{\epsilon_L=\pm 1} G(z_1^{\epsilon_1}, z_2^{\epsilon_2}, \cdots, z_L^{\epsilon_L}),
\]
we write
\[
F(z_1, z_2, \cdots, z_L) \sim_{(z_1, \cdots, z_L)} G(z_1, z_2, \cdots, z_L)
\]
showing the weak equality.

We note that the meanings of weak equality,
\[
F(z_1, z_2, \cdots, z_L) \sim_{(z_1, \cdots, z_L)} G(z_1, z_2, \cdots, z_L)
\]
and weak equality,
\[
F(z_1, z_2, \cdots, z_L) \sim_{(z_1, \cdots, z_{L-1})} G(z_1, z_2, \cdots, z_L)
\]
are different. We prepare propositions in terms of weakly equality.

Proposition 4.9  The function $g_j(w)$, $(1 \leq j \leq N - 1)$ satisfy the following.
\[
g_j(w)w^{\Theta_{x,2r}(x^{2c+2\pi_1,j-1-j^2}w)\Theta_{x,2r}(x^{2c+2\pi_1,j+1-j^2}/w)} \sim_w 0.
\]

Proposition 4.10  The following theta identity holds.
\[
\Theta_{x,2r}(x^{2c+2k-1}w)\Theta_{x,2r}(x^{2c+1}/w)
\times \left(z\Theta_{x,2r}(x^{2c}/z)\Theta_{x,2r}(x^{2k+2c}z)\Theta_{x,2r}(x^{2k-1}/wz)\Theta_{x,2r}(xw/z) - z^{-1}\Theta_{x,2r}(x^{2c}z)\Theta_{x,2r}(x^{2k+2c}/z)\Theta_{x,2r}(x^{2k-1}/w)\Theta_{x,2r}(xwz) \right) \sim_w 0.
\]

Proof of main theorem  Now let us start a proof of main theorem 4.2. We would like to show
\[
T_B(z)|k\rangle_B = |k\rangle_B.
\]
Multiplying the vertex operators $\Phi^{(k+\epsilon_{\mu},k)}(z)$ from the left, and using the inversion relation of vertex operators (2.25), we get a necessary and sufficient condition for $\mu = 1, 2, \cdots, N - 1$,
\[
\Phi^{(k+\epsilon_{\mu},k)}(z)z^{-\frac{\pi_1}{2r}}N^{-\frac{1}{2}}\pi_1h(z)\rho_{1,\mu} + c + u|k\rangle_B
\]
\[
= \Phi^{(k+\epsilon_{\mu},k)}(z^{-1})z^{-\frac{\pi_1}{2r}}N^{-\frac{1}{2}}\pi_1h(z^{-1})\rho_{1,\mu} + c + u|k\rangle_B.
\]
When we change variable $z \rightarrow z^{-1}$ in LHS, we get RHS. We will show (4.34), which has good symmetry.
• The case of $\mu = 1$. Using the free field realization of the vertex operators, we get LHS of (4.34) as following.

$$e^{-i\sqrt{1-\tau}Q_{\tau}^2}h(z)h(z^{-1})[c - u][c + u]e^{P_{-}(z) + P_{-}(1/z)}|k\rangle_B.$$  

(4.35)

This is invariant under $z \to z^{-1}$. Hence LHS and RHS coincide.

• The case of $\mu = 2$. In this case the two theta relations (4.31) and (4.32) play important roles. Using the free field realization of the vertex operators, we get LHS of (4.34) for $\mu = 2$ as following.

$$e^{-i\sqrt{1-\tau}Q_{\tau}^2}c(\pi_{1,2})h(z)h(z^{-1})\Theta_{x^{2r}}(x^{2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} + 2c}z)$$

$$ \times \int \frac{dw}{w} g_1(w)w \Theta_{x^{2r}}(xw/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} - 1}/zw) \frac{e^{P_{-}(z) + P_{-}(1/z) + R_{-}(w) + R_{-}(1/w)}}{D(z,w)} |k\rangle_B.$$  

(4.36)

where $c(\pi_{1,2})$ is independent of $w, z$ and we have set

$$D(z, w) = (xzw; x^{2r})_\infty (xz/w; x^{2r})_\infty (xw/z; x^{2r})_\infty (x/wz; x^{2r})_\infty.$$  

(4.37)

The integration contour encircles $w = 0, x^{1+2rs}z^{1+}, (s \in \mathbb{N})$ but not $x^{-1-2rs}z^{1+}, (s \in \mathbb{N})$. Note that the operator part,

$$e^{P_{-}(z) + P_{-}(1/z) + R_{-}(w) + R_{-}(1/w)} |k\rangle_B,$$

and the function $D(z, w)$ are invariant under $(z, w) \to (1/z, w), (z, 1/w), (1/z, 1/w)$. We have LHS-RHS of (4.34) for $\mu = 2$ as following.

$$e^{-i\sqrt{1-\tau}Q_{\tau}^2}c(\pi_{1,2})h(z)h(z^{-1}) \int \frac{dw}{w} g_1(w)w$$

$$ \times \left( z\Theta_{x^{2r}}(x^{2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} + 2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} - 1}/zw) \right.$$  

$$ \left. - z^{-1}\Theta_{x^{2r}}(x^{2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} + 2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} - 1}/zw) \right) \frac{e^{P_{-}(z) + P_{-}(1/z) + R_{-}(w) + R_{-}(1/w)}}{D(z,w)} |k\rangle_B,$$

(4.38)

where the integration contour encircles $w = 0, x^{1+2rs}z^{1+}, (s \in \mathbb{N})$ but not $w = x^{-1-2rs}z^{1+}, (s \in \mathbb{N})$. The contour of integral is invariant under $w \to w^{-1}$. Hence sufficient condition of LHS−RHS= 0 becomes the following weakly sense relation.

$$g_1(w)w \times \left( z\Theta_{x^{2r}}(x^{2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} + 2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} - 1}/zw)\Theta_{x^{2r}}(xw/z) \right.$$

$$ \left. - z^{-1}\Theta_{x^{2r}}(x^{2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} + 2c}/z)\Theta_{x^{2r}}(x^{2\pi_{1,2} - 1}/zw)\Theta_{x^{2r}}(xzw) \right) \sim_{w} 0.$$  

(4.39)

Using theta identity (4.31) for $g_1(w)$, we have the exactly the same relation as (4.32) for $k = \pi_{1,2}$.

We have shown the relation (4.34) for $\mu = 2$.  

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• The case of $\mu = 3$. Using the free field realization of the vertex operators, we get LHS of (4.34) for $\mu = 3$ as following.

$$
\begin{align*}
& e^{-i\sqrt{\frac{1}{\tau+1}}Q_3}c(\pi_{1,3}, \pi_{2,3}) h(z) h(z^{-1}) \Theta_{x,z}(x^{2c}/z) \Theta_{x,z}(x^{2\pi_{1,3}+2c} z) \\
\times & \int \frac{dw_1}{w_1} g_1(w_1) \int \frac{dw_2}{w_2} g_2(w_2) w_2 \Theta_{x,z}(xw_1/z) \Theta_{x,z}(x^{2\pi_{1,3}-1}/z) \Theta_{x,z}(xw_1w_2) \Theta_{x,z}(x^{2\pi_{2,3}-1}w_1/w_2) D(z, w_1) D(w_1, w_2) \\
\times & e^{P_-(z)+P_-(1/z)+R^1_T(1/w_1)+R^1_T(1/w_2)+R^2_T(1/w_2)+R^2(1/w_2)}|k\rangle_B,
\end{align*}
$$

(4.40)

where $c(\pi_{1,3}, \pi_{2,3})$ is independent of $w_1, w_2, z$. The integration contour encircles $w_1 = 0, x^{1+2rs}z^1, (s \in \mathbb{N})$ but not $w_1 = x^{-1-2rs}z^1, (s \in \mathbb{N})$. The integration contour encircles $w_2 = 0, x^{1+2rs}w_1^1, (s \in \mathbb{N})$ but not $w_2 = x^{-1-2rs}w_1^1, (s \in \mathbb{N})$. Note that the operator part,

$$
e^{P_-(z)+P_-(1/z)+R^1_T(1/w_1)+R^1_T(1/w_2)+R^2_T(1/w_2)+R^2(1/w_2)}|k\rangle_B,$$

and the function $D(z, w_1) D(w_1, w_2)$ are invariant under

$$(z, w_1, w_2) \rightarrow (z^{\pm 1}, w_1^{\pm 1}, w_2^{\pm 1}), (z^{\pm 1}, w_1^{\pm 1}, w_2^{\mp 1}), (z^{\pm 1}, w_1^{\mp 1}, w_2^{\pm 1}), (z^{\mp 1}, w_1^{\mp 1}, w_2^{\mp 1}).$$

We have LHS-RHS of (4.34) for $\mu = 3$ as following.

$$
\begin{align*}
& e^{-i\sqrt{\frac{1}{\tau+1}}Q_3}c(\pi_{1,3}, \pi_{2,3}) h(z) h(z^{-1}) \int \frac{dw_1}{w_1} g_1(w_1) \int \frac{dw_2}{w_2} g_2(w_2) w_2 \Theta_{x,z}(xw_1/z) \Theta_{x,z}(x^{2\pi_{1,3}+2c} z) \\
\times & (z \Theta_{x,z}(x^{2c}/z) \Theta_{x,z}(x^{2\pi_{1,3}+2c} z)) \Theta_{x,z}(xw_1/z) \Theta_{x,z}(x^{2\pi_{1,3}-1}/z) \Theta_{x,z}(xw_1w_2) \Theta_{x,z}(x^{2\pi_{2,3}-1}w_1/w_2) D(z, w_1) D(w_1, w_2) \\
\times & e^{P_-(z)+P_-(1/z)+R^1_T(1/w_1)+R^1_T(1/w_2)+R^2_T(1/w_2)+R^2(1/w_2)}|k\rangle_B.
\end{align*}
$$

(4.41)

Here the integration contour encircles $w_1 = 0, x^{1+2rs}z^1, (s \in \mathbb{N})$ but not $w_1 = x^{-1-2rs}z^1, (s \in \mathbb{N})$. The integration contour encircles $w_2 = 0, x^{1+2rs}w_1^1, (s \in \mathbb{N})$ but not $w_2 = x^{-1-2rs}w_1^1, (s \in \mathbb{N})$. The integration contour is invariant under $(w_1, w_2) \rightarrow (w_1, w_2), (1/w_1, w_2), (1/w_1, 1/w_2)$. Hence sufficient condition of LHS--RHS= 0 becomes the following weakly sense relation.

$$
\begin{align*}
g_1(w_1) g_2(w_2) w_2 \Theta_{x,z}(xw_1w_2) \Theta_{x,z}(x^{2\pi_{2,3}-1}w_1/w_2) \\
\times (z \Theta_{x,z}(x^{2c}/z) \Theta_{x,z}(x^{2\pi_{1,3}+2c} z)) \Theta_{x,z}(xw_1/z) \Theta_{x,z}(x^{2\pi_{1,3}-1}/z) \Theta_{x,z}(xw_1w_2) \Theta_{x,z}(x^{2\pi_{2,3}-1}w_1/w_2) \\
\times e^{P_-(z)+P_-(1/z)+R^1_T(1/w_1)+R^1_T(1/w_2)+R^2_T(1/w_2)+R^2(1/w_2)}|k\rangle_B.
\end{align*}
$$

(4.42)

Using theta identity (4.32) for $k = \pi_{1,3}$, the sufficient condition is reduced to the following.

$$
\begin{align*}
g_1(w_1) g_2(w_2) w_2 \Theta_{x,z}(x^{2\pi_{2,3}+2c-1}/w_1) \Theta_{x,z}(x^{2\pi_{1,3}+2c-1}/w_1) \Theta_{x,z}(x^{2\pi_{1,3}+2c-1}/w_1) \Theta_{x,z}(x^{2\pi_{2,3}+2c-1}/w_1) \Theta_{x,z}(x^{2\pi_{1,3}+2c-1}/w_1) \Theta_{x,z}(x^{2\pi_{2,3}+2c-1}/w_1) \sim (w_1, w_2).
\end{align*}
$$

(4.42)
Using theta identity (4.31) for \( g_1(w) \) the sufficient condition is reduced to the following.

\[
g_2(w_2,w_2(w_1^{-1}\Theta_{x_2r}(x^{2c+2\pi_1\mu^{-1}w_1})\Theta_{x_2r}(x^{2\pi_2\mu^{-1}w_1}w_2))w_2-w_1\Theta_{x_2r}(x^{2\pi_1\mu^{-1}w_1})\Theta_{x_2r}(x^{2\pi_2\mu^{-1}w_1}w_2))w_2 \sim (4.43)
\]

Using relation (4.31) for \( j = 2 \), we get exactly the same relation (4.32) with substitutions \( z \to w_1^{-1}, w \to w_2, c \to c + \pi_{1,2,} - \frac{1}{2} \) and \( k \to \pi_{2,3} \). We have shown the relation (4.34) for \( \mu = 3 \).

**The case of general \( \mu \).** Using the free field realization of the vertex operators, we get LHS of (4.34) for \( \mu \) as following.

\[
e^{-i\sqrt{-1}Q_{\mu}}c(\pi_{1,\mu}, \pi_{2,\mu}, \ldots, \pi_{\mu-1,\mu})
\times zh(z)h(z^{-1})\Theta_{x_2r}(x^{2c/z})\Theta_{x_2r}(x^{2\pi_1\mu^{-1}+2c}z) \oint \cdots \oint \prod_{j=1}^{\mu-1} \frac{dw_j}{w_j} g_j(w_j)w_{\mu-1}^1
\Theta_{x_2r}(xw_1/z)\Theta_{x_2r}(x^{2\pi_1\mu^{-1}z}w_1) \prod_{j=2}^{\mu-1} \Theta_{x_2r}(xw_jz)\Theta_{x_2r}(x^{2\pi_1\mu^{-1}z}w_j)
\times D(z,w_1) \prod_{j=2}^{\mu-1} D(w_{j-1}, w_j)
\times e^{P_+(z) + P_-(1/z) + \sum_{j=1}^{\mu-1} (R^1(w_j) + R^1(1/w_j))}[k]_B.
\]

(4.44)

where \( c(\pi_{1,\mu}, \ldots, \pi_{\mu-1,\mu}) \) is independent of \( w_1, w_2, \ldots, w_{\mu-1}, z \). Here the integration contour encircles \( w_1 = 0,x^{1+2rs}z^{1},(s \in \mathbb{N}) \) but not \( w_1 = x^{-1-2rs}z^{1},(s \in \mathbb{N}) \). For \( j = 1,2,\ldots, \mu-1 \), the integration contour encircles \( w_{j+1} = 0,x^{1+2rs}z^{1},(s \in \mathbb{N}) \) but not \( w_{j+1} = x^{-1-2rs}z^{1},(s \in \mathbb{N}) \). Note that the operator part,

\[
e^{P_+(z) + P_-(1/z) + \sum_{j=1}^{\mu-1} (R^1(w_j) + R^1(1/w_j))}[k]_B,
\]

and the function \( D(z,w_1) \prod_{j=2}^{\mu-1} D(w_{j-1}, w_j) \) are invariant under \( z \to z^{-1} \) or \( w_j \to w_j^{-1},(j = 1,2,\ldots, \mu-1) \). Hence we have LHS-RHS of (4.34) for general \( \mu \) as following.

\[
e^{-i\sqrt{-1}Q_{\mu}}c(\pi_{1,\mu}, \pi_{2,\mu}, \ldots, \pi_{\mu-1,\mu})h(z)h(z^{-1}) \oint \cdots \oint \prod_{j=1}^{\mu-1} \frac{dw_j}{w_j} g_j(w_j)w_{\mu-1}^1
\times (z\Theta_{x_2r}(x^{2c/z})\Theta_{x_2r}(x^{2\pi_1\mu^{-1}+2c}z)\Theta_{x_2r}(x^{2\pi_1\mu^{-1}z}w_1)
\times \frac{-z^{-1}\Theta_{x_2r}(x^{2c}z)\Theta_{x_2r}(x^{2\pi_1\mu^{-1}+2c}z)\Theta_{x_2r}(x^{2\pi_1\mu^{-1}z}w_1)}{w_j}
\times \prod_{j=2}^{\mu-1} \Theta_{x_2r}(xw_{j-1}z)\Theta_{x_2r}(x^{2\pi_1\mu^{-1}z}w_{j-1})
\times D(z,w_1) \prod_{j=2}^{\mu-1} D(w_{j-1}, w_j)
\times e^{P_+(z) + P_-(1/z) + \sum_{j=1}^{\mu-1} (R^1(w_j) + R^1(1/w_j))}[k]_B.
\]

(4.45)
Here the integration contour encircles $w_1 = 0, x^{1+2rs} z^\pm 1, (s \in \mathbb{N})$ but not $w_1 = x^{-1-2rs} z^\pm 1, (s \in \mathbb{N})$. For $j = 1, 2, \cdots, \mu - 1$, the contour of integral encircles $w_{j+1} = 0, x^{1+2rs} w_j^{\pm 1}, (s \in \mathbb{N})$ but not $w_{j+1} = x^{-1-2rs} w_j^{\pm 1}, (s \in \mathbb{N})$. The contour of integral is invariant under $w_j \to w_j^{\pm 1}, (j = 1, 2, \cdots, \mu - 1)$. Hence sufficient condition of LHS–RHS= 0 becomes the following weakly sense relation.

$$
\prod_{j=1}^{\mu-1} g_j(w_j) w_{\mu-1} \prod_{j=\nu}^{\mu-1} \Theta_{x^{2r}}(xw_{j-1}w_j) \Theta_{x^{2r}}(x^{\pi,\mu-1} w_{j-1}/w_j)
\times (z\Theta_{x^{2r}}(x^{2c}/z) \Theta_{x^{2r}}(x^{2\pi,\mu+2c} z) \Theta_{x^{2r}}(x^{2\pi,\mu-1}/zw_1) \Theta_{x^{2r}}(xw_1/z) \\
- z^{-1} \Theta_{x^{2r}}(x^{2c} z) \Theta_{x^{2r}}(x^{2\pi,\mu+2c}/z) \Theta_{x^{2r}}(x^{2\pi,\mu-1} z/w_1) \Theta_{x^{2r}}(xzw_1)) \sim_{(w_1, w_2, \cdots, w_{\mu-1})} 0.
$$

(4.46)

The next step is to show theta identity (4.46) of $(\mu - 1)$ variables $w_1, w_2, \cdots, w_{\mu-1}$. We show it by induction. Using theta identity (4.31) and (4.32), the relation (4.46) is reduced to the weakly sense relation (4.47) for $\mu = 2$. Using theta identity (4.31) and (4.32) repeatedly, the relation (4.47) for $\mu = 2$ is reduced to the weakly sense relation (4.47) for $2 \leq \nu \leq \mu - 1$.

$$
\prod_{j=\nu}^{\mu-1} g_j(w_j) w_{\mu-1} \prod_{j=\nu}^{\mu-1} \Theta_{x^{2r}}(xw_{j-1}w_j) \Theta_{x^{2r}}(x^{\pi,\mu-1} w_{j-1}/w_j)
\times (w_{\nu-1}^{-1} \Theta_{x^{2r}}(x^{2\pi,\mu+2c-\nu+1} w_{\nu-1}) \Theta_{x^{2r}}(x^{2\pi,\mu+2c-\nu+1}/w_{\nu-1}) \Theta_{x^{2r}}(x^{2\pi,\mu-1} w_{\nu-1}/w_\nu) \Theta_{x^{2r}}(xw_{\nu-1}w_\nu) \\
- w_{\nu-1} \Theta_{x^{2r}}(x^{2\pi,\mu+2c-\nu+1}/w_{\nu-1}) \Theta_{x^{2r}}(x^{2\pi,\mu+2c-\nu+1} w_{\nu-1}) \Theta_{x^{2r}}(x^{2\pi,\mu-1}/w_{\nu-1} w_\nu) \Theta_{x^{2r}}(xw_{\nu}/w_{\nu-1}) \\
\sim_{(w_{\nu-1}, w_\nu, \cdots, w_{\mu-1})} 0, \quad (2 \leq \nu \leq \mu - 1).
$$

(4.47)

The relation (4.47) for $\nu$ is reduced to those for $\nu + 1$. The relation (4.47) for $\nu = \mu - 1$ is the same as (4.32). Now we have shown the theta identity (4.46). Hence we have shown the identity (4.34) for general $\mu$.

Q.E.D.

We give a comment on a proof of the dual boundary state,

$$
B\langle k| T_B^{(k)}(z) = B\langle k |.
$$

(4.48)

Multiplying the vertex operators $\Phi^*(k, k + \vec{\nu})(z)$ from the right, and using the inversion relation of vertex operators (2.25), we get a necessary and sufficient condition for $1 \leq \mu \leq N$.

$$
B\langle k | z^{N-1} \frac{N-1}{N} - \frac{1}{2} \pi_1 h(z)[c - u][\pi_1, \mu + c + u] \Phi_{\mu}^*(z^{-1}) \\
= B\langle k | z^{N-1} \frac{N-1}{N} + \frac{1}{2} \pi_1 h(z^{-1})[c + u][\pi_1, \mu + c - u] \Phi_{\mu}^*(z).
$$

(4.49)
As the same manner the above we can give a proof for the dual boundary state. We omit details.

Here we note two useful relations of the dual boundary state.

**Proposition 4.11**

\[ g_j^* (1/w) w \Theta_{x^{2r}} (x^{2e-N+j+2\pi_1,j}/w) \Theta_{x^{2r}} (x^{2e+2-N+j+2\pi_1,j+1}/w) \sim \Theta_{x^{2r}} (x^{2e+2-N+j+2\pi_1,j+1}/w) \sim 0, \quad (1 \leq j \leq N - 1). \]  (4.50)

**Proposition 4.12**

\[ \frac{h(z)h^*(z)}{h(1/z)h^*(1/z)} = \frac{\Theta_{x^{2r}} (x^{2e-z}) \Theta_{x^{2r}} (x^{2e+2\pi_1,N}/z)}{\Theta_{x^{2r}} (x^{2c/z}) \Theta_{x^{2r}} (x^{2c+2\pi_1,N}/z)} \]  (4.51)

The relation between the functions \( h(z) \) and \( h^*(z) \) gives self consistency between the boundary state and the dual boundary state.

## 5 Norm

In this section we calculate the norm of the boundary state, \( B \langle k|k \rangle_B \).

The free field realization of \( e^F \) and \( e^G \) are quadratic. Hence evaluation of the norm is reduced to the Gaussian integrals, which is possible to calculate. Let us set

\[ \{z\}_\infty = (z; x^{4N}, x^{2r})_\infty, \quad \{z\}^*_\infty = (z; x^{4N}, x^{2r^*})_\infty, \]

\[ [z]_\infty = (z; x^{2N}, x^{2r})_\infty, \quad [z]^*_\infty = (z; x^{2N}, x^{2r^*})_\infty. \]

(5.1)

**Theorem 5.1** *The norm \( B \langle k|k \rangle_B \) is evaluated by double infinite product.*

\[
B \langle k|k \rangle_B = \frac{1}{(x^{2N}; x^{2N})^{N(N-1)/2}} \prod_{i=1}^{N-1} \left( \frac{\sqrt{x^{4N-2-2i}; x^{4N}}}{(x^{4N-2i}; x^{4N})_\infty} \right)^{(N-i)} \prod_{i=1}^{N-1} \left( \frac{x^{4N-2+2i}; x^{4N}}{(x^{4N}; x^{4N})_\infty} \right) \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \left[ x^{2r-4i+2+2j+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2j+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2j+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2j+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2j+2\pi_1,j} \right]_\infty
\]

\[
\times \prod_{1 \leq i < j \leq N} \left[ x^{4i+2+2\pi_1,i+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2\pi_1,i+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2\pi_1,i+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2\pi_1,i+2\pi_1,j} \right]_\infty \left[ x^{2r-4i+2+2\pi_1,i+2\pi_1,j} \right]_\infty
\]

\[
\times \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty
\]

\[
\times \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty \left[ x^{2r-4i+2+2j+4\pi_1,i} \right]_\infty
\]
\[
\times \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \sqrt{\frac{\{x^{4N-2r+4c+2} - 22j - 2k - 4\pi_{1,i}\}_\infty}{\{x^{2N-2r+4c+2} + 2i - 2j - 2k + 4\pi_{1,i}\}_\infty}} \times \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \sqrt{\frac{\{x^{4N-2r+4c+2} - 22j - 2k + 4\pi_{1,i+1}\}_\infty}{\{x^{2N-2r+4c+2} + 2i - 2j - 2k + 4\pi_{1,i+1}\}_\infty}} \times \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \sqrt{\frac{\{x^{4N-2r+4c+2} - 22j - 2k + 4\pi_{1,i}\}_\infty}{\{x^{2N-2r+4c+2} + 2i - 2j - 2k + 4\pi_{1,i}\}_\infty}} \times \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} \sqrt{\frac{\{x^{4N-2r+4c+2} - 22j - 2k + 4\pi_{1,i+1}\}_\infty}{\{x^{2N-2r+4c+2} + 2i - 2j - 2k + 4\pi_{1,i+1}\}_\infty}} \times \frac{\{x^{4N-2r+4c+2} - 22j - 2k + 4\pi_{1,i}\}_\infty}{\{x^{2N-2r+4c+2} + 2i - 2j - 2k + 4\pi_{1,i}\}_\infty},
\]

(5.2)

In what follows we explain how to calculate the norm \(B\langle k|k\rangle_B\). We calculate it by using a decomposition of the identity on \(F_{j,k}\) which employs coherent state. We define the coherent state
\[
|\xi_1 \cdots \xi_{N-1}\rangle = \exp \left( \sum_{m>0}^{N-1} \sum_{j=1}^{m} \frac{[rm]_x}{(r-1)m_x} \xi_j(m) \alpha^j_m \right) |l, k\rangle,
\]
(5.3)
\[
\langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}| = |l, k| \exp \left( \sum_{m>0}^{N-1} \sum_{j=1}^{m} \frac{[rm]_x}{(r-1)m_x} \bar{\xi}_j(m) \alpha^j_m \right).
\]
(5.4)

The actions of the boson \(\alpha^i_m\) are given by
\[
\alpha^i_m |\xi_1 \cdots \xi_{N-1}\rangle = \sum_{j=1}^{N-1} \frac{[A_{i,j}m]_x}{[m]_x} \xi_j(m) |\xi_1 \cdots \xi_{N-1}\rangle,
\]
(5.5)
\[
\langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}| \alpha^i_m = \langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}| \sum_{j=1}^{N-1} \frac{[A_{i,j}m]_x}{[m]_x} \bar{\xi}_j(m).
\]
(5.6)

The action of the bosons \(\beta^i_m\) on the coherent state are given by
\[
\beta^i_m |\xi_1 \cdots \xi_{N-1}\rangle = \xi_1(m) |\xi_1 \cdots \xi_{N-1}\rangle,
\]
(5.7)
\[
\beta^i_m |\xi_1 \cdots \xi_{N-1}\rangle = (-x^{im} \bar{\xi}_{i-1}(m) + x^{-i}m \xi_i(m)) |\xi_1 \cdots \xi_{N-1}\rangle, (2 \leq i \leq N-1),
\]
(5.8)
\[
\langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}| \beta^i_m = \bar{\xi}_1(m) \langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}|,
\]
(5.9)
\[
\langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}| \beta^i_m = (x^{-i}m \bar{\xi}_i(m) - x^{-i}m \bar{\xi}_{i-1}(m)) \langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1}|, (2 \leq i \leq N-1).
\]
(5.10)

Using the following Gaussian integral,
\[
\int_{-\infty}^{\infty} \prod_{m>0} \mu_m d\xi_m d\bar{\xi}_m \exp \left( -\frac{1}{2} \sum_{m>0} \mu_m (\bar{\xi}_m, \xi_m) A_m \left( \frac{\bar{\xi}_m}{\xi_m} \right) + \sum_{m>0} (\xi_m, \xi_m) B_m \right)
= \frac{1}{\prod_{m>0} \sqrt{-\det(A_m)}} \exp \left( \frac{1}{2} \sum_{m>0} \frac{1}{\mu_m} tB_m A_m^{-1} B_m \right),
\]
(5.11)
we conclude that the identity on $\mathcal{F}_{l,k}$ is decomposed by following.

\[
\begin{align*}
\text{id} &= \int_{\infty}^\infty \prod_{m>0} \prod_{j=1}^{N-1} \frac{1}{m} \frac{[(j+1)m]_x[rm]_x}{(r-1)m'_x} d\xi_j(m) d\bar{\xi}_j(m) \\
&\times \exp \left( -\sum_{m>0} \sum_{j,k=1}^{N-1} \frac{1}{m} \frac{[A_{j,k,m}]_x[rm]_x}{(r-1)m'_x} \xi_j(m) \bar{\xi}_k(m) \right) \langle \xi_1 \cdots \xi_{N-1} \rangle \langle \bar{\xi}_1 \cdots \bar{\xi}_{N-1} \rangle. \tag{5.12}
\end{align*}
\]

Inserting this decomposition of the identity between $e^G$ and $e^F$, we find

\[
B(k|k)_B = \langle k, k | e^G e^F | k, k \rangle
\]

\[
= \int_{\infty}^\infty \prod_{m>0} \prod_{j=1}^{N-1} \frac{1}{m} \frac{[(j+1)m]_x[rm]_x}{(r-1)m'_x} d\xi_j(m) d\bar{\xi}_j(m) \\
\times \exp \left( -\frac{1}{2} \sum_{m>0} \sum_{j,k=1}^{N-1} \frac{1}{m} \frac{[A_{j,k,m}]_x[rm]_x}{(r-1)m'_x} (2\xi_j(m) \bar{\xi}_k(m) - x^{2Nm} \xi_j(m) \xi_k(m) - \bar{\xi}_j(m) \bar{\xi}_k(m)) \right) \\
\times \sum_{m>0} \sum_{j=1}^{N-1} \frac{1}{m} (Y_j(m) \xi_j(m) + X_j(m) \bar{\xi}_j(m)) \tag{5.13}
\]

where we have set

\[
Y_j(m) = x^{jm} (x^{-m} E_j(m) - x^{-m} E_{j+1}(m))
\]

\[
= x^{Nm} \frac{[(r-2c+N-j-2-2\pi_{1,j+1})m]_x + [(-r+2c-N+j+2\pi_{1,j})m]_x}{[(r-1)m]_x} + \theta_m \left( \frac{x^{Nm}[m/2]_x[rm/2]_x^+}{[(r-1)m/2]_x} \right), \tag{5.14}
\]

\[
X_j(m) = x^{-jm} (-x^{-m} D_{j+1}(m) + x^{-m} D_j(m))
\]

\[
= \frac{-1}{[(r-1)m]_x} \frac{[(-r-2c+j+2\pi_{1,j+1})m]_x + [(r+2c-j-2-2\pi_{1,j})m]_x}{[(r-1)m/2]_x} - \theta_m \left( \frac{[m/2]_x[rm/2]_x^+}{[(r-1)m/2]_x} \right). \tag{5.15}
\]

Here we read $D_N(m) = E_N(m) = 0$.

Let us repeat Gaussian integral (5.11), we get formulae without integrals in proposition.

**Proposition 5.2** The norm of boundary states is written by infinite product.

\[
B(k|k)_B
\]

\[
= \frac{1}{(x^{2N}; x^{2N})_{\infty}^2} \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \frac{[r-1)m]_x}{[Nm]_x[m]_x} \right) \\
\times \sum_{j=1}^{N-1} [(N-j)m]_x \frac{[x^{2Nm}/2]_xX_j(m)^2 + \frac{1}{2} Y_j(m)^2 - X_j(m)Y_j(m)}{} \tag{5.16}
\]

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\[ + \sum_{1 \leq j < l \leq N-1} \left[ \left. jm \right| (N-l)m \right| x^{2N_m} X_j(m) X_l(m) + Y_j(m) Y_l(m) - X_j(m) Y_l(m) - Y_j(m) X_l(m) \right]. \]

Here \( X_j(m) \) and \( Y_j(m) \) are given by (5.15) and (5.14).

At first glance, the norm \( B\langle k|k\rangle_B \) is evaluated by

\[ (z; x^{2N_1}, x^{2r}, x^{2r})_\infty. \]

After some calculations, cancellations occur. In order to write the norm \( B\langle k|k\rangle_B \), we need only

\[ (z; x^{2N_1}, x^{2r})_\infty, \ (z; x^{2N_1}, x^{2r})_\infty. \]

Here we omit detailed calculations.

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### A Some formulae

In this appendix we summarize some formulae.

\[ \log h(z) = - \sum_{m>0} \frac{1}{2m} \frac{[\left( r-1 \right)m]_x [(N-1)m]_x}{[rm]_x [Nm]_x} z^{-2m} \]
\[ - \sum_{m>0} \frac{1}{m} \frac{\left[ \left( r-1 \right)m \right]_x}{[rm]_x [Nm]_x} \left( [\left( N-1 \right)m]_x D_1(m) - [m]_x x^{-N_m} \sum_{k=2}^{N-1} D_k(m) \right) z^{-m}, \]
\[ \log g_j(z) = - \sum_{m>0} \frac{1}{2m} \frac{\left[ \left( r-1 \right)m \right]_x (x^m + x^{-m})}{[rm]_x} z^{-2m} \]
\[ + \sum_{m>0} \frac{1}{m} \frac{\left[ \left( r-1 \right)m \right]_x x^{-jm} (x^m D_j(m) - x^{-m} D_{j+1}(m))}{[rm]_x} z^{-m}, \quad (j = 1, 2, \ldots, N-2), \]
\[ \log g_{N-1}(z) = - \sum_{m>0} \frac{1}{2m} \frac{\left[ \left( r-1 \right)m \right]_x (x^m + x^{-m})}{[rm]_x} z^{-2m} \]
\[ + \sum_{m>0} \frac{1}{m} \frac{\left[ \left( r-1 \right)m \right]_x x^{-(N+2)m} D_{N-1}(m)}{[rm]_x} z^{-m}. \]
\[ \log h^*(z) = - \sum_{m>0} \frac{1}{2m} \frac{\left[ \left( r-1 \right)m \right]_x [(N-1)m]_x}{[rm]_x [Nm]_x} z^{-2m} \]

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\[
\log g^*_j(z) = - \sum_{m>0} \frac{1}{2m} \frac{[(r-1)m]_x (x^m + x^{-m})}{[rm]_x} z^{-2m} \\
+ \sum_{m>0} \frac{1}{m} \frac{[(r-1)m]_x x^{(j-N)m}(x^m E_{j+1}(m) - x^{-m} E_j(m))}{[rm]_x} z^{-m}, \quad (j = 1, 2, \cdots, N - 2),
\]

\[
\log g^*_{N-1}(z) = - \sum_{m>0} \frac{1}{2m} \frac{[(r-1)m]_x (x^m + x^{-m})}{[rm]_x} z^{-2m} \\
- \sum_{m>0} \frac{1}{m} \frac{[(r-1)m]_x x^{-2m} E_{N-1}(m)}{[rm]_x} z^{-m}.
\]

**B Physical Interpretation**

In this appendix we explain a physical interpretation of our problem. In main text of this paper we use no picture. In this appendix we give graphical definition of various quantities of our problem. We present an ordered pair \((b, a) \in P^2\) as FIG.1.

![FIG.1. Ordered pair](image1)

For admissible pair \((a, g, h, f) \in P^4\) we present the Boltzmann weight functions \(W\left( \begin{array}{c} h \\ f \\ g \\ a \\ u \end{array} \right)\) by FIG.2.

![FIG.2. Boltzmann weight](image2)
In what follows we consider the restricted path \( a \in P^+_{r-N} \).

For admissible pair \((a, b, g) \in P^3\) we present boundary Boltzmann weight function \(K\) by FIG.3.

\[
\begin{pmatrix}
  a \\
  g \\
  b \\
\end{pmatrix}
\]

FIG.3. Boundary Boltzmann weight

In what follows we consider \( a \in P^+_{r-N} \). Using the Boltzmann weight functions \( W \) and the boundary Boltzmann weight functions \( K \), we introduce two dimensional solvable lattice model, which we call the boundary \( U_{q,p}(A_{N-1}^{(1)}) \) face model. We fix parameters \( N = 2, 3, 4, \ldots, 0 < x < 1 \) and \( r \geq N + 2, (r \in \mathbb{N}) \). We fix a continuous parameter \( c \) in the boundary Boltzmann weight function as \( 0 < c < 1 \). In what follows we consider infinite product of the Boltzmann weight functions \( W \) and the boundary Boltzmann weight functions \( K \) in the Regime III : \( 0 < u < 1 \).

The boundary \( U_{q,p}(A_{N-1}^{(1)}) \) face model defined by FIG.4.
In order to consider the boundary $U_{q,p}(A^{(1)}_{N-1})$ face model, we divide it into some parts. For this purpose we prepare the space $H_{l,k}$, on which the operators act. Following the general scheme of algebraic approach in solvable lattice models, we consider the corner transfer matrix $A^{(a)}(u)$ which represent north-west quadrant as FIG.5.

In the large lattice limit $M \to \infty$, apart from a divergence scalar, the corner transfer matrix $A^{(a)}(u)$, $(a \in P^{+}_{r-N})$ is of the form

$$A^{(a)}(u) \sim x^{2uH},$$

where the operator $H$ is independent of $u$. The spectrum of $H$ is given in [10]. Let us set the space $H_{l,k}$, where

$$l = b + \rho, \quad k = a + \rho,$$

the space spanned by the eigenvectors of $A^{(a)}(u)$ with the asymptotic boundary condition $(b, b + \omega, b + \omega_2, \ldots, b + \omega_{N-1})$ given by the choice of $b \in P^{+}_{r-1-N}$.

Let us set the vertex operator $\Phi^{(a_1,a_2)}_{N}(-u)$ by FIG.6.
The vertex operator $\Phi^{(a,a+\epsilon_{\mu})}_N(u)$ is an operator

$$\Phi^{(a,a+\epsilon_{\mu})}_N(u) : \mathcal{H}_{l,k+\epsilon_{\mu}} \rightarrow \mathcal{H}_{l,k}.$$ 

Let us set the vertex operator $\Phi^{(a_1,a_2)}_W(u)$ by FIG. 7.

The vertex operator $\Phi^{(a_1,a_2)}_W(u)$ is an operator

$$\Phi^{(a_1,a_2)}_W(u) : \mathcal{H}_{l,k} \rightarrow \mathcal{H}_{l,k+\epsilon_{\mu}}.$$ 

The vertex operators $\Phi^{(a_1,a_2)}_N(u)$ and $\Phi^{(a_1,a_2)}_W(u)$ satisfy the following commutation relations (1), (2) and (2').

(1) Commutation relation:

$$\Phi^{(a,b)}_W(u_1)\Phi^{(b,d)}_W(u_2) = \sum_g W \begin{pmatrix} a & g \\ b & d \end{pmatrix} \left( \Phi^{(a,g)}_W(u_2)\Phi^{(g,d)}_W(u_1), u_2 - u_1 \right),$$

$$\Phi^{(a,b)}_N(u_1)\Phi^{(b,d)}_N(u_2) = \sum_g W \begin{pmatrix} d & b \\ g & a \end{pmatrix} \left( \Phi^{(a,g)}_N(u_2)\Phi^{(g,d)}_N(u_1), u_2 - u_1 \right),$$

$$\Phi^{(a,b)}_W(u_1)\Phi^{(b,d)}_N(u_2) = \sum_g W \begin{pmatrix} g & d \\ a & b \end{pmatrix} \left( \Phi^{(a,g)}_N(u_2)\Phi^{(g,d)}_W(u_1), u_1 - u_2 \right).$$
(2) Inversion relation:
\[ \Phi^{(a,g)}_W(z) \Phi^{(g,b)}_N(z) = \delta_{a,b}. \]

(2') Inversion relation:
\[ \sum_g \Phi^{(a,g)}_N(z) \Phi^{(g,b)}_W(z) = \delta_{a,b}. \]

Let us set the infinite transfer matrix \( T^{(a)}_B(u) \) by
\[
T^{(a)}_B(u) = \sum_{\mu=1}^N \Phi^{(a,a+\bar{\epsilon}_\mu)}_N(-u) K \left( a + \bar{\epsilon}_\mu \left| \begin{array}{c} a \\ u \end{array} \right) \Phi^{(a+\bar{\epsilon}_\mu,a)}_W(u). \right.
\]

The infinite transfer matrix \( T^{(a)}_B(u) \) is an operator,
\[ T^{(a)}_B(u) : \mathcal{H}_{l,k} \rightarrow \mathcal{H}_{l,k}. \]

The infinite transfer matrix \( T^{(a)}_B(u) \) is given by FIG.8 at the same time.

![FIG.8. Boundary transfer matrix](image)

Because of the commutation relations of the vertex operators and the boundary Yang-Baxter equation, we get the commutativity of the infinite transfer matrix \( T^{(a)}_B(u) \).
\[ [T^{(a)}_B(u), T^{(a)}_B(v)] = 0. \]

In this paper, we are interested in the ground state. The boundary \( U_{q,p}(A^{(1)}_{N-1}) \) face model is given by infinite product of the infinite transfer matrix \( T^{(a)}_B(u) \). We figure out the ground
state of the problem in the infinite transfer matrix $T_{B}^{(a)}(u)$. Note that we consider the case $0 < c < 1$, $0 < a_{1,\mu} < r$. In the limit $x \to 1$ we have

$$
\left| K \left( \begin{array}{c|c} a + \bar{\epsilon}_1 & a \\ \hline a & u \end{array} \right) \right| > \left| K \left( \begin{array}{c|c} a + \bar{\epsilon}_\mu & a \\ \hline a & u \end{array} \right) \right| (\mu \neq 1).
$$

Therefore, the ground state configuration is given by following figure. Asymptotic boundary condition should be $b = a$. The ground state is given by FIG.9.

![FIG.9. Ground state](image)

Hence, in what follows, we consider the space $\mathcal{H}_{l,k}$ for $l = k = a + \rho$.

$$\mathcal{H}_{k,k}.$$  

Let us set the boundary state $|k\rangle_B$ by the half infinite plane figure : FIG.10.
The boundary state $|k\rangle_B$ and the infinite transfer matrix $T_B^{(a)}(u)$ satisfy the following relations.

$$T_B^{(a)}(u)|k\rangle_B = |k\rangle_B.$$ 

Let us set the dual boundary state $B\langle k|$ by the half infinite plane figure: FIG.11.
FIG.11. Boundary state

The dual boundary state $B\langle k \rvert$ and the infinite transfer matrix $T_B^{(a)}(u)$ satisfy the following relations.

$$B\langle k \rvert T_B^{(a)}(u) = B\langle k \rvert.$$ 

For $k = a + \rho, l = b + \rho$, if we set

$$\Phi^{(a,a+\ell_\mu)}_N(u) = \Phi^{*(k,k+\ell_\mu)}(z),$$
$$\Phi^{(a+\ell_\mu,a)}_W(u) = \Phi^{(k+\ell_\mu,k)}(z),$$
$$T_B^{(a)}(u) = T_B(z),$$

the above relations in the appendix are satisfied by the free field realizations on the Fock space $\mathcal{F}_{k,k}$. However this is not perfect identification because the space $\mathcal{H}_{k,k}$ and the space $\mathcal{F}_{k,k}$ have different characters. We expect BRST cohomology of the certain complex consisting of the space $\mathcal{F}_{l,k}$ provides the correct identification of the space $\mathcal{H}_{l,k}$ [18, 13, 16]. Upon this assumption, this appendix gives the physical interpretation of the main text.

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