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Impulsive vaccination and dispersal on dynamics of an SIR epidemic model with restricting infected individuals boarding transports

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\begin{abstract}
To understand the effect of impulsive vaccination and restricting infected individuals boarding transports on disease spread, we establish an SIR model with impulsive vaccination, impulsive dispersal and restricting infected individuals boarding transports. This SIR epidemic model for two regions, which are connected by transportation of non-infected individuals, portrays the evolvement of diseases. We prove that all solutions of the investigated system are uniformly ultimately bounded. We also prove that there exists globally asymptotically stable infection-free boundary periodic solution. The condition for permanence is discussed. It is concluded that the approach of impulsive vaccination and restricting infected individuals boarding transports provides reliable tactic basis for preventing disease spread.
\end{abstract}

1. Introduction

The work of Kermack and McKendrick \cite{1} was the fundamental study of epidemic models described by nonlinear differential equations. In this field, epidemic models have recently attracted much attention of mathematical epidemiologists and are perceived as significant \cite{2–7}. Wang, Takeuchi and Liu \cite{8} studied a multi-group SVEIR epidemic model with distributed delay and vaccination. Sun and Shi \cite{9} considered a multigroup SEIR model with nonlinear incidence of infection and nonlinear removal functions between compartments. To understand the effect of transport-related infection on disease spread, Cui, Takeuchi and Saito \cite{10} proposed the spreading disease with transport-related infection. Takeuchi, Liu and Cui \cite{11} investigated the global dynamics of SIS models with transport-related infection, their conclusions implied that transport-related infection on disease can make the disease endemic even if all the isolated regions are disease free. Yan and Zou \cite{12} considered two control variables representing the quarantine and isolation strategies for SARS epidemics, they gave a theoretical interpretation to the practical experiences that the early quarantine and isolation strategies are critically important.

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to control the outbreaks of epidemic. Chowell and Castillo-Chavez [13] used the uncertainly and sensitivity analysis of the basic reproductive number \( R_0 \) to assess the role that the model parameters play in outbreak control. Quarantine and isolation measures have been widely used to control the spread of diseases such as yellow fever, smallpox, measles, ebola, pandemic influenza, diphtheria, plague, cholera, and, more recently, severe acute respiratory syndrome (SARS) [14–19]. Xie, et al. [20] simultaneously use two kinds of measures: expand the treatment ranges of suspected case and limit population flows freely to suppress the diffusion of SARS effectively. Gong et al. [21] showed that the SARS may fluctuate with import of SARS infectiousness from outside Beijing, weakness of quarantine, more social activities and so on.

Different types of vaccination policies and strategies combining pulse vaccination policy, treatment, pre-outbreak vaccination or isolation have already been introduced by many referees [22–31]. The pulse vaccination strategy (PVS) consists of repeated application of vaccine at discrete time with equal interval in a population in contrast to the constant vaccination [22,23]. At each vaccination time a constant fraction of the susceptible population is vaccinated successfully. Since 1993, attempts have been made to develop mathematical theory to control infectious diseases using pulse vaccination [22]. Compared to the proportional vaccination models, the study of pulse vaccination models is in its infancy [23]. The control of childhood viral infections by pulse vaccination strategy is discussed by Nokes and Swinton [24,25]. Stone et al. [26] presented a theoretical examination of the pulse vaccination strategy in the SIR epidemic model. d’Onofrio [27] investigated the application of the pulse vaccination policy to eradicate infectious disease for SIR and SEIR epidemic models.

Theories of impulsive differential equations have been introduced into population dynamics lately [32–38]. Impulsive equations are found in almost every domain of applied science and have been studied in many investigation [38–41], they generally describe phenomena which are subject to steep or instantaneous changes. The theories of population dynamical system and its application have been achieved many good results. However, the oasis vegetation degradation combining with dynamical system has been considered very little. In this paper, we will investigate an impulsive dispersal on SIR model with restricting infected individuals boarding transports. We expect to obtain some dynamical properties of the investigated system. We also expect that impulsive dispersal will provides reliable tactic for controlling epidemic.

The organization of this paper is as follows. In Section 2, we introduce the model and background concepts. In Section 3, some important lemmas are presented. We give the globally asymptotically stable conditions of the infection-free boundary periodic solution of System (2.2), and the permanent condition of System (2.2). In Section 4, a brief discussion is given to conclude this work.

2. The model

Inspired by the above discussion, we establish an SIR model with impulsive vaccination, impulsive dispersal and restricting infected individuals boarding transports.

\[
\begin{align*}
\frac{dS_1(t)}{dt} &= \lambda_1 - d_1S_1(t) - \frac{\beta_1S_1(t)I_1(t)}{1 + \alpha I_1(t)}, \\
\frac{dI_1(t)}{dt} &= \frac{\beta_1S_1(t)I_1(t)}{1 + \alpha I_1(t)} - (r_1 + d_1 + b_1)I_1(t), \\
\frac{dR_1(t)}{dt} &= r_1I_1(t) - d_1R_1(t), \\
\frac{dS_2(t)}{dt} &= \lambda_2 - d_2S_2(t) - \frac{\beta_2S_2(t)I_2(t)}{1 + \alpha I_2(t)}, \\
\frac{dI_2(t)}{dt} &= \frac{\beta_2S_2(t)I_2(t)}{1 + \alpha I_2(t)} - (r_2 + d_2 + b_2)I_2(t), \\
\frac{dR_2(t)}{dt} &= r_2I_2(t) - d_2R_2(t), \\
\Delta S_1(t) &= D(S_2(t) - S_1(t)), \\
\Delta I_1(t) &= 0, \\
\Delta R_1(t) &= D(R_2(t) - R_1(t)), \\
\Delta S_2(t) &= D(S_1(t) - S_2(t)), \\
\Delta I_2(t) &= 0, \\
\Delta R_2(t) &= D(R_1(t) - R_2(t)), \\
\Delta S_1(t) &= -\mu_1S_1(t), \\
\Delta I_1(t) &= 0, \\
\Delta R_1(t) &= \mu_1S_1(t), \\
\Delta S_2(t) &= -\mu_2S_2(t), \\
\Delta I_2(t) &= 0, \\
\Delta R_2(t) &= \mu_2S_2(t),
\end{align*}
\]

\( t \neq (n + l)\tau, \quad t \neq (n + 1)\tau, \quad t = (n + l)\tau, \quad n \in \mathbb{Z}^+ \)

\( t = (n + 1)\tau, \quad n \in \mathbb{Z}^+ \)
here $S_i(t), I_i(t)$ and $R_i(t)$ represent the number of susceptible, infected, recovered individuals in city $i$ ($i = 1, 2$) at time $t$. It is assumed that we adopt the fixed number of offspring, denoted by $\lambda_i$ ($i = 1, 2$), joins into the susceptible class per unit time in city $i$ ($i = 1, 2$). The natural death rate is assumed as the same constant $d_i$ ($i = 1, 2$) for the susceptible, infected, recovered individuals in city $i$ ($i = 1, 2$). Disease is transmitted with the incidence rate, that is, the number of new cases of infection per unit time

$$\frac{\beta_i S_i(t) I_i(t)}{1 + \alpha_i I_i(t)}, \quad i = 1, 2,$$

with city $i = 1, 2$. The transmission rate with city $i$ is a constant $\beta_i$ ($i = 1, 2$). $\alpha_i$ is a nonnegative constant representing the half saturation constant with city $i$ ($i = 1, 2$). The infected individuals in city $i$ ($i = 1, 2$) suffer an extra disease-related death with constant rate $b_i$ ($i = 1, 2$). $r_i$ ($i = 1, 2$) is the recovery rate of the infected individuals in city $i$ ($i = 1, 2$). By boarding transports, the susceptible and recovered individuals of city $i$ leave to city $j$ ($i \neq j, i, j = 1, 2$) with a dispersal rate $0 < D < 1$ at moment $t = (n + l)\tau, n \in \mathbb{Z}_+$. The susceptible is successfully vaccinated with $\mu_i$ ($i = 1, 2$) in city $i$ ($i = 1, 2$) at moment $t = (n + 1)\tau, n \in \mathbb{Z}_+$.

Because $R_i(t)$ ($i = 1, 2$) do not affect the other equations of (2.1), we can simplify system (2.1) and restrict our attention to the following system

$$\begin{align*}
\frac{dS_1(t)}{dt} &= \lambda_1 - d_1 S_1(t) - \frac{\beta_1 S_1(t) I_1(t)}{1 + \alpha_1 I_1(t)}, \\
\frac{dI_1(t)}{dt} &= \frac{\beta_1 S_1(t) I_1(t)}{1 + \alpha_1 I_1(t)} - (r_1 + d_1 + b_1) I_1(t), \\
\frac{dS_2(t)}{dt} &= \lambda_2 - d_2 S_2(t) - \frac{\beta_2 S_2(t) I_2(t)}{1 + \alpha_2 I_2(t)}, \\
\frac{dI_2(t)}{dt} &= \frac{\beta_2 S_2(t) I_2(t)}{1 + \alpha_2 I_2(t)} - (r_2 + d_2 + b_2) I_2(t), \\
\Delta S_1(t) &= D(S_2(t) - S_1(t)), \\
\Delta I_1(t) &= 0, \\
\Delta S_2(t) &= D(S_1(t) - S_2(t)), \\
\Delta I_2(t) &= 0, \\
\Delta S_1(t) &= -\mu_1 S_1(t), \\
\Delta I_1(t) &= 0, \\
\Delta S_2(t) &= -\mu_2 S_2(t), \\
\Delta I_2(t) &= 0,
\end{align*}
$$

(2.2)

3. The lemmas

The solution of (2.1), denote by $X(t) = (S_1(t), I_1(t), R_1(t), S_2(t), I_2(t), R_2(t))^T$, is a piecewise continuous function $X : \mathbb{R}_+ \to \mathbb{R}_+^6$. $X(t)$ is continuous on $(n\tau, (n + l)\tau]$ and $((n + l)\tau, (n + 1)\tau], n \in \mathbb{Z}_+$ and $X((n + l)\tau^+) = \lim_{t \to (n + l)\tau^+} X(t)$, $X((n + l)\tau^+) = \lim_{t \to (n + l)\tau^+} X(t)$ exist. Obviously the global existence and uniqueness of solutions of (2.1) is guaranteed by the smoothness properties of $f$, which denotes the mapping defined by right-side of system (2.1) (see Lakshmikantham [39]).

Let $V : \mathbb{R}_+ \times \mathbb{R}_+^6 \to \mathbb{R}_+$, then $V$ is said to belong to class $V_0$, if

(i) $V$ is continuous in $(n\tau, (n + l)\tau] \times \mathbb{R}_+^6$ and $((n + l)\tau, (n + 1)\tau] \times \mathbb{R}_+^6$, for each $z \in \mathbb{R}_+^6, n \in \mathbb{Z}_+, V(n\tau^+, z) = \lim_{(t, y) \to ((n + l)\tau^+, y)} V(t, y), V((n + l)\tau^+, z) = \lim_{(t, y) \to ((n + 1)\tau^+, y)} V(t, y)$ exist.

(ii) $V$ is locally Lipschitzian in $z$.

Definition 3.1. $V \in V_0$, then, for $(t, z) \in (n\tau, (n + l)\tau] \times \mathbb{R}_+^6$ and $((n + l)\tau, (n + 1)\tau] \times \mathbb{R}_+^6$, the upper right derivative of $V(t, z)$ with respect to the impulsive differential system (2.1) is defined as

$$D^+ V(t, z) = \lim_{h \to 0} \sup_{h} \frac{1}{h} [V(t + h, z + h\Delta(t, z)) - V(t, z)].$$

It can easily be obtained from the following lemma.

Lemma 3.2. Suppose $X(t)$ is a solution of (2.1) with $X(0^+) \geq 0$, then $X(t) \geq 0$ for $t \geq 0$ and further $X(t) > 0$ ($t \geq 0$) for $X(0^+) > 0$. 

Lemma 3.3 ([39]). Let the function \( m \in PC^*[R^+, R] \) satisfy the inequalities

\[
\begin{align*}
  &m(t) \leq p(t)m(t) + q(t), \\
  &t \geq t_0, \quad t \neq t_k, \quad k = 1, 2, \ldots, \\
  &m(t_k^+) \leq d_k m(t_k) + b_k, \quad t = t_k,
\end{align*}
\]

(3.1)

where \( p, q \in PC^*[R^+, R] \) and \( d_k \geq 0, b_k \) are constants, then

\[
m(t) \leq m(t_0) \prod_{t_0 < t < t} d_k \exp \left( \int_{t_0}^t p(s)ds \right) + \sum_{t_0 < t < t} \left( \prod_{t_0 < t < t} d_k \exp \left( \int_{t_0}^t p(s)ds \right) \right) b_k \\
+ \int_{t_0}^t \prod_{t_0 < t < t} d_k \exp \left( \int_{s}^{t} p(\sigma)d\sigma \right) q(s)ds, \quad t \geq t_0.
\]

Now, we show that all solutions of (2.1) are uniformly ultimately bounded.

Lemma 3.4. There exists a constant \( M > 0 \) such that \( S_i(t) \leq M, I_i(t) \leq M, R_i(t) \leq M \) (\( i = 1, 2 \)) for each solution \( (S_1(t), I_1(t), R_1(t), S_2(t), I_2(t), R_2(t)) \) of (2.1) with all \( t \) large enough.

Proof. Define

\[
V(t) = S_1(t) + I_1(t) + R_1(t) + S_2(t) + I_2(t) + R_2(t),
\]

and \( d = \min(d_1, d_2) \). Then \( t \neq n\tau, t \neq (n + l)\tau \), we have

\[
D^+ V(t) + dV(t) = \lambda_1 + \lambda_2 - \sum_{i=1}^{2} [(d_i - d)S_i(t) + (d_i - d)I_i(t) + (d_i - d)R_i(t)] - \sum_{i=1}^{2} b_i I_i(t) \\
\leq \lambda_1 + \lambda_2.
\]

When \( t = n\tau \), \( V(n\tau^+) = S_1(n\tau^+) + I_1(n\tau^+) + R_1(n\tau^+) + S_2(n\tau^+) + I_2(n\tau^+) + R_2(n\tau^+) = S_1(n\tau) + I_1(n\tau) + R_1(n\tau) + S_2(n\tau) + I_2(n\tau) + R_2(n\tau) = V(n\tau) \). When \( t = (n + l)\tau \), \( V((n + l)\tau^+) = S_1((n + l)\tau^+) + I_1((n + l)\tau^+) + R_1((n + l)\tau^+) + S_2((n + l)\tau^+) + I_2((n + l)\tau^+) + R_2((n + l)\tau^+) = V((n + l)\tau) \). By Lemma 3.3, for \( t \in (n\tau, (n + 1)\tau) \), we have

\[
V(t) \leq V(0) \exp(-dt) + \int_{0}^{t} (\lambda_1 + \lambda_2) \exp(-d(t - s))ds \\
= V(0) \exp(-dt) + \frac{\lambda_1 + \lambda_2}{d} (1 - \exp(-dt)) \\
\rightarrow \frac{\lambda_1 + \lambda_2}{d}, \quad \text{as} \quad t \rightarrow \infty.
\]

So \( V(t) \) is uniformly ultimately bounded. Hence, by the definition of \( V(t) \), we have there exists a constant \( M > 0 \) such that \( S_i(t) \leq M, I_i(t) \leq M, R_i(t) \leq M \) (\( i = 1, 2 \)) for \( t \) large enough. The proof is complete.

If \( I_i(t) = 0 \) (\( i = 1, 2 \)), we have the following subsystem of (2.2)

\[
\begin{align*}
  \frac{dS_1(t)}{dt} = \lambda_1 - d_1 S_1(t), \quad &t \neq (n + l)\tau, \quad t \neq (n + 1)\tau, \\
  \frac{dS_2(t)}{dt} = \lambda_2 - d_2 S_2(t), \quad &t \neq (n + l)\tau, \\
  \Delta S_1(t) = D(S_2(t) - S_1(t)), \quad &t = (n + l)\tau, \\
  \Delta S_2(t) = D(S_1(t) - S_2(t)), \quad &t = (n + l)\tau, \\
  \Delta S_1(t) = -\mu_1 S_1(t), \quad &t = (n + 1)\tau, \quad n = 1, 2, \ldots,
\end{align*}
\]

(3.2)
We can easily obtain the analytic solution of (3.2) between pulses as follows

\[
S_1(t) = \begin{cases} 
\frac{1}{d_1}[\lambda_1 - (\lambda_1 - d_1S_1(\tau t^+))e^{-d_1(\tau t^-)}], & t \in [\tau t, (n + l)\tau), \\
\frac{1}{d_1}[\lambda_1 - (\lambda_1 - d_1S_1((n + l)\tau^+))e^{-d_1((n + l)\tau^-)}], & t \in [(n + l)\tau, (n + 1)\tau), 
\end{cases} 
\]

\[
S_2(t) = \begin{cases} 
\frac{1}{d_2}[\lambda_2 - (\lambda_2 - d_2S_2(\tau t^+))e^{-d_2(\tau t^-)}], & t \in [\tau t, (n + l)\tau), \\
\frac{1}{d_2}[\lambda_2 - (\lambda_2 - d_2S_2((n + l)\tau^+))e^{-d_2((n + l)\tau^-)}], & t \in [(n + l)\tau, (n + 1)\tau). 
\end{cases} 
\]

Considering the third and fourth equations of (3.2), we have

\[
\begin{align*}
S_1((n + l)\tau^+) &= \frac{1 - D}{d_1}[\lambda_1 - (\lambda_1 - d_1S_1(\tau t^+))e^{-d_1(\tau t^-)}] + \frac{D}{d_2}[\lambda_2 - (\lambda_2 - d_2S_2(\tau t^+))e^{-d_2(\tau t^-)}], \\
S_2((n + l)\tau^+) &= \frac{1 - D}{d_1}[\lambda_1 - (\lambda_1 - d_1S_1((n + l)\tau^+))e^{-d_1((n + l)\tau^-)}] + \frac{1 - D}{d_2}[\lambda_2 - (\lambda_2 - d_2S_2((n + l)\tau^+))e^{-d_2((n + l)\tau^-)}].
\end{align*}
\]

Considering the fifth and sixth equations of (3.2), we also have

\[
\begin{align*}
S_1((n + 1)\tau^+) &= \frac{1 - \mu_1}{d_1}[\lambda_1 - (\lambda_1 - d_1S_1((n + l)\tau^+))e^{-d_1((n + l)\tau^-)}], \\
S_2((n + 1)\tau^+) &= \frac{1 - \mu_2}{d_2}[\lambda_2 - (\lambda_2 - d_2S_2((n + l)\tau^+))e^{-d_2((n + l)\tau^-)}].
\end{align*}
\]

Substituting (3.4) into (3.5), we have the stroboscopic map of (3.2)

\[
\begin{align*}
S_1((n + 1)\tau^+) &= (1 - \mu_1)(1 - D)e^{-d_1(\tau t^+)}S_1(\tau t^+) + (1 - \mu_1)De^{-d_1((n + l)\tau^-)}S_2(\tau t^+) + \frac{D}{d_1}[\lambda_1(1 - e^{-d_1(\tau t^-)}) + D\lambda_1(1 - e^{-d_1((n + l)\tau^-)})e^{-d_1(\tau t^-)}] \\
S_2((n + 1)\tau^+) &= (1 - \mu_2)De^{-d_2(\tau t^+)}S_1(\tau t^+) + (1 - \mu_2)(1 - D)e^{-d_2((n + l)\tau^-)}S_2(\tau t^+) + \frac{d_2}{d_1}[\lambda_1(1 - e^{-d_2(\tau t^-)}) + \lambda_2(1 - e^{-d_2((n + l)\tau^-)})e^{-d_2(\tau t^-)}].
\end{align*}
\]

(3.5) has one fixed point as

\[
\begin{align*}
S_1 &= \frac{(1 - A_1)B - AA_2}{(1 - A_1)(1 - B_2) - A_2B_1} > 0, \\
S_2 &= \frac{B_2B - A(1 - B_2)}{(1 - A_1)(1 - B_2) - A_2B_1} > 0,
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= (1 - \mu_1)(1 - D)e^{-d_1(\tau t^-)} < 1, \\
B_1 &= (1 - \mu_1)De^{-d_1((n + l)\tau^-)} < 1, \\
A_2 &= (1 - \mu_2)De^{-d_2((n + l)\tau^-)} < 1, \\
B_2 &= (1 - \mu_2)(1 - D)e^{-d_2(\tau t^-)} < 1, \\
A &= (1 - \mu_1)\times \frac{\lambda_1(1 - e^{-d_1(\tau t^-)})(1 - (1 - D)e^{-d_1(\tau t^-)})}{d_1} + \frac{D\lambda_1(1 - e^{-d_1((n + l)\tau^-)})e^{-d_1(\tau t^-)}}{d_2} > 0, \\
B &= (1 - \mu_2)\times \frac{D\lambda_1(1 - e^{-d_1((n + l)\tau^-)})e^{-d_2(\tau t^-)}}{d_1} + \frac{\lambda_2(1 - e^{-d_2((n + l)\tau^-)})(1 - (1 - D)e^{-d_2(\tau t^-)})}{d_2} > 0.
\end{align*}
\]

**Lemma 3.5.** The unique fixed point \((S_1^n, S_2^n)\) of (3.6) is globally asymptotically stable.

**Proof.** For convenience, we make a notation as \((S_1^n, S_2^n) = (S_1(n\tau^+), S_2(n\tau^+))\). The linear form of (3.6) can be written as

\[
\begin{pmatrix}
S_1^{n+1} \\
S_2^{n+1}
\end{pmatrix} = M \begin{pmatrix}
S_1^n \\
S_2^n
\end{pmatrix},
\]

(3.8)
Obviously, the near dynamics of \((S_1^*, S_2^*)\) is determined by linear system (3.6). The stabilities of \((S_1^*, S_2^*)\) are determined by the eigenvalue of \(M\) less than 1. If \(M\) satisfies the Jury criteria [22], we can know the eigenvalue of \(M\) less than 1,

\[ 1 - trM + detM > 0. \tag{3.9} \]

We can easily know that \((S_1^*, S_2^*)\) is unique fixed point of (3.6), and

\[ M = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}. \tag{3.10} \]

For

\[ 1 - trM + detM = 1 - (A_1 + B_2) + (A_1B_2 - A_2B_1) \]

\[ = (1 - A_1)(1 - B_2) - A_2B_1 \]

\[ = [(1 - (1 - \mu_1))e^{-d_1\tau} + (1 - \mu_1)De^{-d_1\tau}][(1 - (1 - \mu_2))e^{-d_2\tau} + (1 - \mu_2)De^{-d_2\tau}] \]

\[ = [1 - (1 - \mu_1)e^{-d_1\tau}][1 - (1 - \mu_2)e^{-d_2\tau}] \]

\[ + [1 - (1 - \mu_1)e^{-d_1\tau}](1 - \mu_2)De^{-d_2\tau} + [1 - (1 - \mu_2)e^{-d_2\tau}](1 - \mu_1)De^{-d_1\tau} > 0. \]

From Jury criteria, \((S_1^*, S_2^*)\) is locally stable. Because the fixed point \((S_1^*, S_2^*)\) of (3.6) is unique, then, it is globally asymptotically stable. This completes the proof.

**Lemma 3.6.** The periodic solution \((\hat{S}_1(t), \hat{S}_2(t))\) of System (3.2) is globally asymptotically stable, where

\[
\frac{1}{d_1}\left[ \lambda_1 - (1 - d_1S_1^*)e^{-d_1(t-n\tau)} \right], \quad t \in [n\tau, (n + l)\tau),
\]

\[
\frac{1}{d_1}\left[ \lambda_1 - (1 - d_1S_1^*)e^{-d_1((n+1)\tau-t)} \right], \quad t \in [(n + l)\tau, (n + 1)\tau),
\]

\[
\frac{1}{d_2}\left[ \lambda_2 - (1 - d_2S_2^*)e^{-d_2(t-n\tau)} \right], \quad t \in [n\tau, (n + l)\tau),
\]

\[
\frac{1}{d_2}\left[ \lambda_2 - (1 - d_2S_2^*)e^{-d_2((n+1)\tau-t)} \right], \quad t \in [(n + l)\tau, (n + 1)\tau),
\]

here \(S_1^*\) and \(S_2^*\) are determined as (3.7), \(S_1^{**}\) and \(S_2^{**}\) are defined as

\[
S_1^{**} = \frac{1 - D}{d_1}\left[ \lambda_1 - (1 - d_1S_1^*)e^{-d_1\tau} \right] + \frac{D}{d_2}\left[ \lambda_2 - (1 - d_2S_2^*)e^{-d_2\tau} \right],
\]

\[
S_2^{**} = \frac{D}{d_1}\left[ \lambda_1 - (1 - d_1S_1^*)e^{-d_1\tau} \right] + \frac{1 - D}{d_2}\left[ \lambda_2 - (1 - d_2S_2^*)e^{-d_2\tau} \right]. \tag{3.12}
\]

**4. The dynamics**

**Theorem 4.1.** If

\[
D < \frac{1}{2}, \tag{4.1}
\]

and

\[
\max_{i=1,2} \frac{\beta_i}{d_i} \left[ \lambda_i\tau + \frac{\lambda_i - d_iS_i^*}{d_i}(e^{-d_i\tau} - 1) + \frac{\lambda_i - d_iS_i^{**}}{d_i}(e^{-d_i\tau} - e^{-d_i\tau}) \right] - (r_i + d_i + b_i)\tau < 0. \tag{4.2}
\]

hold, the infection-free boundary periodic solution \((\hat{S}_1(t), 0, \hat{S}_2(t), 0)\) of (2.2) is globally asymptotically stable, where \(S_i^*(i = 1, 2)\) and \(S_i^{**}(i = 1, 2)\) are defined as (3.7) and (3.12).

**Proof.** First, we prove the local stability of the infection-free boundary periodic solution \((\hat{S}_1(t), 0, \hat{S}_2(t), 0)\) of (2.2). Defining

\[ S_{11}(t) = S_1(t) - \hat{S}_1(t), S_{12}(t) = S_2(t) - \hat{S}_2(t), I_1(t) = I_1(t), I_2(t) = I_2(t), \]

then we have the following linearly similar system
for (2.2) which is concerning one periodic solution $(\tilde{S}_1(t), 0, \tilde{S}_2(t), 0)$

\[
\begin{pmatrix}
\frac{dS_{11}(t)}{dt} \\
\frac{dS_{12}(t)}{dt} \\
\frac{dS_{1}(t)}{dt} \\
\frac{dS_{2}(t)}{dt}
\end{pmatrix}
= \begin{pmatrix}
-d_1 & -\beta_1 \tilde{S}_1(t) & 0 & 0 \\
0 & \beta_1 \tilde{S}_1(t) - (r_1 + d_1 + b_1) & 0 & 0 \\
0 & 0 & -d_2 & -\beta_2 \tilde{S}_2(t) \\
0 & 0 & 0 & \beta_2 \tilde{S}_2(t) - (r_2 + d_2 + b_2)
\end{pmatrix}
\begin{pmatrix}
S_{11}(t) \\
S_{12}(t) \\
S_1(t) \\
S_2(t)
\end{pmatrix}.
\]

It is easy to obtain the fundamental solution matrix

\[
\Phi(t) = \begin{pmatrix}
\exp(-d_1 t) & *_1 & 0 & 0 \\
0 & \exp\left(\int_0^t (\beta_1 \tilde{S}_1(s) - (r_1 + d_1 + b_1)) ds\right) & 0 & 0 \\
0 & 0 & \exp(-d_2 t) & *_2 \\
0 & 0 & 0 & \exp\left(\int_0^t (\beta_2 \tilde{S}_2(s) - (r_2 + d_2 + b_2)) ds\right)
\end{pmatrix}.
\]

There is no need to calculate the exact form of $*_i$ ($i = 1, 2$), as they are not required in the analysis that follows. The linearization of the fifth, sixth, seventh and eighth equations of (2.2) is

\[
\begin{pmatrix}
S_{11}(n\tau^+) \\
S_{12}(n\tau^+) \\
I_1(n\tau^+) \\
I_2(n\tau^+)
\end{pmatrix} = \begin{pmatrix}
1 - D & 0 & D & 0 \\
0 & 1 & 0 & 0 \\
D & 0 & 1 - D & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
S_{11}(n\tau) \\
S_{12}(n\tau) \\
I_1(n\tau) \\
I_2(n\tau)
\end{pmatrix}.
\]

The linearization of the ninth, tenth, eleventh and twelfth equations of (2.2) is

\[
\begin{pmatrix}
S_{11}(n\tau^+) \\
S_{12}(n\tau^+) \\
I_1(n\tau^+) \\
I_2(n\tau^+)
\end{pmatrix} = \begin{pmatrix}
1 - \mu_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 - \mu_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
S_{11}(n\tau) \\
S_{12}(n\tau) \\
I_1(n\tau) \\
I_2(n\tau)
\end{pmatrix}.
\]

The stability of the periodic solution $(\tilde{S}_1(t), 0, \tilde{S}_2(t), 0)$ is determined by the eigenvalues of

\[
M = \begin{pmatrix}
1 - D & 0 & D & 0 \\
0 & 1 & 0 & 0 \\
D & 0 & 1 - D & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 - \mu_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 - \mu_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \Phi(\tau),
\]

where are

\[
\lambda_1 = (1 - \mu_1)(1 - D)e^{-d_1 \tau} < 1,
\]

and

\[
\lambda_2 = \frac{(1 - D)(K_1 + K_3) + \sqrt{(1 - D)^2(K_1 + K_3)^2 - 4(1 - 2D)K_1 K_3}}{2} \leq \frac{(1 - D)(K_1 + K_3) + (1 + D)^2(K_1 + K_3)^2}{2} \leq \frac{(K_1 + K_3)}{2},
\]

and

\[
\lambda_3 = (1 - \mu_2)(1 - D)e^{-d_2 \tau} < 1,
\]
and

\[
\lambda_4 = \left| \frac{(1-D)(K_1 + K_2) - \sqrt{(1-D)^2(K_1 + K_2)^2 - 4(1-2D)K_1K_2}}{2} \right|
\]

\[
\leq \left| \frac{(1-D)(K_1 + K_3) - \sqrt{(1-D)^2(K_1 - K_3)^2}}{2} \right|
\]

\[
= \left| \frac{(1-D)(K_1 + K_3) - \sqrt{(1-D)^2(K_1 - K_3)^2}}{2} \right|
\]

\[
\leq (1-D) \max[K_1, K_3],
\]

where \( K_1 = e^{\int_0^t \beta_i S_1(t^-) \, dt} \) and \( K_3 = e^{\int_0^t \beta_i S_2(t^-) \, dt} \). According to condition (4.1), (4.2) and the Floquet theory [39], i.e.

\[
\exp \left[ \int_0^\tau (\beta_i \widehat{S}(t) - (r_i + d_i + b_i)) \, dt \right] < 1 \quad (i = 1, 2),
\]

then,

\[
\lambda_2 \leq \left| \frac{(K_1 + K_3)}{2} \right| < 1,
\]

and

\[
\lambda_4 \leq (1-D) \max[K_1, K_3] < 1,
\]

hold, the infection-free boundary periodic solution \((\widehat{S}_1(t), 0, \widehat{S}_2(t), 0)\) of (2.2) is locally stable.

In the following, we will prove the global attraction. Choose \( \varepsilon > 0 \) such that

\[
\rho_i = \exp \left[ \int_0^\tau (\beta_i (\widehat{S}(t) + e) - (r_i + d_i + b_i)) \, dt \right] < 1 \quad (i = 1, 2).
\]

From the first and third equations of (2.2), we notice that \( \frac{dS_i(t)}{dt} \leq \lambda_i - d_i \hat{S}_i(t) \) \( (i = 1, 2) \). Then, we consider the following impulsive comparative differential equation

\[
\begin{align*}
\frac{dS_{21}(t)}{dt} &= \lambda_1 - d_i S_{21}(t), & t \neq (n + 1)\tau, \\
\frac{dS_{22}(t)}{dt} &= \lambda_2 - d_2 S_{22}(t), & t \neq (n + 1)\tau, \\
\Delta S_{21}(t) &= D(S_{22}(t) - S_{21}(t)), & t = (n + 1)\tau, \\
\Delta S_{22}(t) &= D(S_{21}(t) - S_{22}(t)), & t = (n + 1)\tau.
\end{align*}
\]  

(4.3)

From Lemma 3.6 and comparison theorem of impulsive equation (see theorem 3.1.1 in Ref. [39]), we have \( S_1(t) \leq S_{21}(t), S_2(t) \leq S_{22}(t) \) and \( S_{21}(t) \rightarrow \widehat{S}_1(t), S_{22}(t) \rightarrow \widehat{S}_2(t) \), as \( t \rightarrow \infty \). Then

\[
\begin{align*}
S_1(t) &\leq S_{21}(t) \leq S_1(t) + \varepsilon, \\
S_2(t) &\leq S_{22}(t) \leq S_2(t) + \varepsilon,
\end{align*}
\]

(4.4)

for all \( t \) large enough. For convenience, we may assume (4.4) holds for all \( t \geq 0 \). From (2.2) and (4.4), we get

\[
\frac{dI_i(t)}{dt} \leq [\beta_i (\widehat{S}_i(t) + \varepsilon) - (r_i + d_i + b_i)]I_i(t) \quad (i = 1, 2).
\]

(4.5)

So \( I_i((n + 1)\tau) \leq I_i(n\tau^+) \exp [\int_{n\tau}^{(n + 1)\tau} (\beta_i (\widehat{S}(t) + \varepsilon) - (r_i + d_i + b_i)) \, dt] \) \( (i = 1, 2) \). Hence \( I_i(n\tau) \leq I_i(0^+) \rho_i^n (i = 1, 2) \) and \( I_i(n\tau) \rightarrow 0 \) \( (i = 1, 2) \) as \( n \rightarrow \infty \), therefore \( I_i(t) \rightarrow 0 \) \( (i = 1, 2) \) as \( t \rightarrow \infty \).

Next, we will prove that \( S_i(t) \rightarrow \hat{S}_i(t) \) \( (i = 1, 2) \) as \( t \rightarrow \infty \). For \( \varepsilon_1 > 0 \), there must exist a \( t_0 > 0 \) such that \( 0 < I_i(t) < \varepsilon_1 \) \( (i = 1, 2) \) for all \( t \geq t_0 \). Without loss of generality, we may assume that \( 0 < I_i(t) < \varepsilon_1 \) for all \( t \geq 0 \). For system (2.2) we have

\[
\lambda_1 - \left( d_i + \frac{\beta_i \varepsilon_1}{1 + \alpha_i \varepsilon_1} \right) S_i(t) \leq \frac{dS_i(t)}{dt} \leq \lambda_i - d_i \hat{S}_i(t) \quad (i = 1, 2),
\]

(4.6)
then we have $S_{31}(t) \leq S_1(t) \leq S_{21}(t) , S_{32}(t) \leq S_2(t) \leq S_{22}(t)$ and $S_{31}(t) \rightarrow \overline{S_{31}}(t), S_{32}(t) \rightarrow \overline{S_{32}}(t), S_{21}(t) \rightarrow \overline{S_1}(t), S_{22}(t) \rightarrow \overline{S_2}(t)$ as $t \rightarrow \infty$. While $(S_{31}(t), S_{32}(t))$ and $(S_{21}(t), S_{22}(t))$ are the solutions of

\[
\begin{align*}
\frac{dS_{31}(t)}{dt} &= \lambda_1 - \left( d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1} \right) S_{31}(t), \\
\frac{dS_{32}(t)}{dt} &= \lambda_2 - \left( d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1} \right) S_{32}(t), \\
\Delta S_{31}(t) &= D(S_{32}(t) - S_{31}(t)), \\
\Delta S_{32}(t) &= D(S_{31}(t) - S_{32}(t)), \\
\Delta S_{31}(t) &= -\mu_1 S_{31}(t), \\
\Delta S_{32}(t) &= -\mu_2 S_{32}(t),
\end{align*}
\]

(4.7)

and

\[
\begin{align*}
\frac{dS_{21}(t)}{dt} &= \lambda_1 - d_1 S_{21}(t), \\
\frac{dS_{22}(t)}{dt} &= \lambda_2 - d_2 S_{22}(t), \\
\Delta S_{21}(t) &= D(S_{22}(t) - S_{21}(t)), \\
\Delta S_{22}(t) &= D(S_{21}(t) - S_{22}(t)), \\
\Delta S_{21}(t) &= -\mu_1 S_{21}(t), \\
\Delta S_{22}(t) &= -\mu_2 S_{22}(t),
\end{align*}
\]

(4.8)

respectively.

\[
\begin{align*}
\overline{S_{31}}(t) &= \left\{ \begin{array}{ll}
\frac{1}{d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1}} & t \in [\pi, (n+1)\tau), \\
\frac{1}{d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1}} & t \in [(n+1)\tau, (n+1)\tau), \\
\frac{1}{d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1}} & t \in [(n+1)\tau, (n+1)\tau), \\
\frac{1}{d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1}} & t \in [(n+1)\tau, (n+1)\tau), 
\end{array} \right.
\end{align*}
\]

(4.9)

Here $S^*_1$ and $S^*_2$ are determined as

\[
\begin{align*}
S^*_1 &= \frac{(1 - A_{31})B_3 - A_3A_{32}}{(1 - A_{31})(1 - B_{32}) - A_{32}B_{31}} > 0, \\
S^*_2 &= \frac{B_{31}B_3 - A_3(1 - B_{32})}{(1 - A_{31})(1 - B_{32}) - A_{32}B_{31}} > 0,
\end{align*}
\]

(4.10)

and $S'^{\ast}_{31}$ and $S'^{\ast}_{32}$ are defined as

\[
\begin{align*}
S'^{\ast}_{31} &= \frac{1 - D}{d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1}} \left[ \lambda_1 - \left( \lambda_1 - \left( d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1} \right) S^*_1 \right) e^{-\left( d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1} \right) t} \right] + \frac{D}{d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1}} \left[ \lambda_2 - \left( \lambda_2 - \left( d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1} \right) S^*_2 \right) e^{-\left( d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1} \right) t} \right], \\
S'^{\ast}_{32} &= \frac{1 - D}{d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1}} \left[ \lambda_1 - \left( \lambda_1 - \left( d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1} \right) S^*_1 \right) e^{-\left( d_1 + \frac{\beta_1\epsilon_1}{1 + \alpha_1\epsilon_1} \right) t} \right] + \frac{D}{d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1}} \left[ \lambda_2 - \left( \lambda_2 - \left( d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1} \right) S^*_2 \right) e^{-\left( d_2 + \frac{\beta_2\epsilon_1}{1 + \alpha_2\epsilon_1} \right) t} \right],
\end{align*}
\]

(4.11)
where
\[ A_{31} = (1 - \mu_1)(1 - D)e^{-\left(d_1 + \frac{\rho_1}{\tau + \alpha_1}\right)\tau} < 1, \]
\[ B_{31} = (1 - \mu_1)De^{-\left[d_1 + \frac{\rho_1}{\tau + \alpha_1}\right](1-\epsilon)} < 1, \]
\[ A_{32} = (1 - \mu_2)De^{-\left[d_2 + \frac{\rho_2}{\tau + \alpha_2}\right](1-\tau)} < 1, \]
\[ B_{32} = (1 - \mu_2)(1 - D)e^{-\left(d_2 + \frac{\rho_2}{\tau + \alpha_2}\right)\tau} < 1, \]
\[ A_3 = (1 - \mu_1) \times \left[ \frac{\lambda_1 e^{-\left(d_1 + \frac{\rho_1}{\tau + \alpha_1}\right)\tau}}{\left(d_1 + \frac{\rho_1}{\tau + \alpha_1}\right)} \right] - \frac{\alpha_1 e^{-\left(d_2 + \frac{\rho_2}{\tau + \alpha_2}\right)(1-\tau)}}{\left(d_2 + \frac{\rho_2}{\tau + \alpha_2}\right)} > 0, \]
\[ B_3 = (1 - \mu_2) \times \left[ \frac{\lambda_2 e^{-\left(d_2 + \frac{\rho_2}{\tau + \alpha_2}\right)(1-\tau)}}{\left(d_2 + \frac{\rho_2}{\tau + \alpha_2}\right)} \right] - \frac{\alpha_2 e^{-\left(d_1 + \frac{\rho_1}{\tau + \alpha_1}\right)\tau}}{\left(d_1 + \frac{\rho_1}{\tau + \alpha_1}\right)} > 0. \]

For any \( \epsilon_2 > 0 \), there exists a \( t_1, t > t_1 \) such that
\[ S_{31}(t) - \epsilon_2 < S_1(t) < S_{31}(t) + \epsilon_2, \]
and
\[ S_{32}(t) - \epsilon_2 < S_2(t) < S_{32}(t) + \epsilon_2. \]

Let \( \epsilon_1 \to 0 \), so we have
\[ S_{31}(t) - \epsilon_2 < S_1(t) < S_{31}(t) + \epsilon_2, \]
and
\[ S_{32}(t) - \epsilon_2 < S_2(t) < S_{32}(t) + \epsilon_2, \]
for \( t \) large enough, which implies \( S_1(t) \to S_{11}(t) \) and \( S_2(t) \to S_{22}(t) \) as \( t \to \infty \). This completes the proof.

The next work is to investigate the permanence of system (2.2).

**Definition 4.2.** System (2.2) is said to be permanent if there are constants \( m, M > 0 \) (independent of initial value) and a finite time \( T_0 \) such that for all solutions \( (S_1(t), I_1(t), S_2(t), I_2(t)) \) with all initial values \( S_1(0^+) > 0, I_1(0^+) > 0, S_2(0^+) > 0, I_2(0^+) > 0, m \leq S_1(t) \leq M, m \leq I_1(t) \leq M, m \leq S_2(t) \leq M, m \leq I_2(t) \leq M, \) hold for all \( t \geq T_0 \). Here \( T_0 \) may depend on the initial values \( (S_1(0^+), I_1(0^+), S_2(0^+), I_2(0^+)) \).

**Theorem 4.3.** If
\[ \min_{i=1,2} \frac{\rho_i}{\tau + \alpha_i}, \frac{\rho_i}{\tau + \alpha_2}, \frac{\rho_i}{\tau + \alpha_2} > 1, \]
holds, system (2.2) is permanent. Where \( S_i^* \) (i = 1, 2) and \( S_{i1}^* \) (i = 1, 2) are defined as (3.7) and (3.12).

**Proof.** Suppose \((S_1(t), I_1(t), S_2(t), I_2(t))\) is a solution of (2.2) with \( S_1(0) > 0, I_1(0) > 0, S_2(0) > 0, I_2(0) > 0 \). By Lemma 3.4, we have proved there exists a constant \( M > 0 \) such that \( S_1(t) \leq M, I_1(t) \leq M, S_2(t) \leq M, I_2(t) \leq M, \) for \( t \) large enough. From system (2.2), we know \( I_1(t) > I_1(0^+) \) for all \( t \) large enough. Thus, we only need to
find \( m_1 > 0 \) and \( \varepsilon_3 \) such that \( l_i(t) \geq m_1 \) (\( i = 1, 2 \)) for \( t \) large enough. Otherwise, we can select \( m_2 > 0 \) small enough, and prove \( l_i(t) < m_2 \) (\( i = 1, 2 \)) cannot hold for \( t \geq 0 \). By the condition (4.12), we can obtain

\[
\sigma_i = \frac{\beta_i}{d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2}} \left[ \lambda_i \tau + \frac{\lambda_i - \left( d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2} \right) S^*_t}{d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2}} \left( e^{-\left( d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2} \right) \tau} - 1 \right) \right. \\
+ \left. \left( d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2} \right) S^*_t \left( e^{-\left( d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2} \right) \tau} - e^{-\left( d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2} \right) \tau} \right) \right] \\
- \left( r_i + \left( d_i + \frac{\beta_i m_2}{1 + \alpha_i m_2} \right) + b_i \right) \tau - \beta_i \varepsilon_3 \tau > 0,
\]

with \( S^*_t \) (\( i = 1, 2 \)) and \( S^*_t \) (\( i = 1, 2 \)) are defined as (4.16) and (4.17). Then,

\[
\begin{align*}
\frac{dS_1(t)}{dt} &> \lambda_1 - \left( d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2} \right) S_1(t), \quad t \neq (n + l) \tau, \quad t \neq (n + 1) \tau, \\
\frac{dS_2(t)}{dt} &> \lambda_2 - \left( d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2} \right) S_2(t), \\
\Delta S_1(t) & = D(S_2(t) - S_1(t)), \\
\Delta S_2(t) & = D(S_1(t) - S_2(t)), \\
\Delta S_1(t) & = -\mu_1 S_1(t), \\
\Delta S_2(t) & = -\mu_2 S_2(t).
\end{align*}
\]

By Lemmas 3.6, we have \( S_1(t) \geq S_{41}(t), S_2(t) \geq S_{42}(t) \) and \( S_{41}(t) \rightarrow \overline{S_{41}(t)}, S_{42}(t) \rightarrow \overline{S_{42}(t)}, t \rightarrow \infty \), where \((S_{41}(t), S_{42}(t))\) is the solution of

\[
\begin{align*}
\frac{dS_{41}(t)}{dt} & = \lambda_1 - \left( d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2} \right) S_{41}(t), \quad t \neq (n + l) \tau, \quad t \neq (n + 1) \tau, \\
\frac{dS_{42}(t)}{dt} & = \lambda_2 - \left( d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2} \right) S_{42}(t), \\
\Delta S_{41}(t) & = D(S_{42}(t) - S_{41}(t)), \\
\Delta S_{42}(t) & = D(S_{41}(t) - S_{42}(t)), \\
\Delta S_{41}(t) & = -\mu_1 S_{41}(t), \\
\Delta S_{42}(t) & = -\mu_2 S_{42}(t).
\end{align*}
\]

with

\[
S_{41}(t) = \begin{cases} 
\frac{1}{d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2}} \left[ \lambda_1 - \left( \lambda_1 - \left( d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2} \right) S^*_t \right) \right] e^{-\left( d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2} \right) \tau}, & t \in [n \tau, (n + l) \tau), \\
\frac{1}{d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2}} \left[ \lambda_1 - \left( \lambda_1 - \left( d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2} \right) S^*_t \right) \right] e^{-\left( d_1 + \frac{\beta_1 m_2}{1 + \alpha_1 m_2} \right) \tau}, & t \in [(n + l) \tau, (n + 1) \tau),
\end{cases}
\]

\[
S_{42}(t) = \begin{cases} 
\frac{1}{d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2}} \left[ \lambda_2 - \left( \lambda_2 - \left( d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2} \right) S^*_t \right) \right] e^{-\left( d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2} \right) \tau}, & t \in [n \tau, (n + l) \tau), \\
\frac{1}{d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2}} \left[ \lambda_2 - \left( \lambda_2 - \left( d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2} \right) S^*_t \right) \right] e^{-\left( d_2 + \frac{\beta_2 m_2}{1 + \alpha_2 m_2} \right) \tau}, & t \in [(n + l) \tau, (n + 1) \tau),
\end{cases}
\]
here $S^*_4$ and $S^*_4$ are determined as

$$\begin{cases}
S^*_4 = \frac{(1 - A_{41})B_4 - A_4A_{42}}{(1 - A_{41})(1 - B_{42}) - A_{42}B_{41}} > 0, \\
S^*_4 = \frac{B_4A_4B_4 - A_4(1 - B_{42})}{(1 - A_{41})(1 - B_{42}) - A_{42}B_{41}} > 0,
\end{cases} \tag{4.16}$$

and $S^{**}_{41}$ and $S^{**}_{42}$ are defined as

$$\begin{cases}
S^{**}_{41} = \frac{1 - D}{(d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2})} \left[ \lambda_1 - \left( \lambda_1 - \left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) S^*_4 \right) e^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) t} \right] \\
+ \frac{D}{(d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2})} \left[ \lambda_2 - \left( \lambda_2 - \left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) S^*_3 \right) e^{-\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) t} \right], \\
S^{**}_{42} = \frac{d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2}}{(d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2})} \left[ \lambda_1 - \left( \lambda_1 - \left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) S^*_4 \right) e^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) t} \right] \\
+ \frac{1 - D}{d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2}} \left[ \lambda_2 - \left( \lambda_2 - \left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) S^*_3 \right) e^{-\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) t} \right],
\end{cases} \tag{4.17}$$

where

$$A_{41} = (1 - \mu_1)(1 - D)e^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) t} < 1,$$

$$B_{41} = (1 - \mu_1)De^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) t} < 1,$$

$$A_{42} = (1 - \mu_2)De^{-\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) t} < 1,$$

$$B_{42} = (1 - \mu_2)(1 - D)e^{-\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) t} < 1,$$

$$A_4 = (1 - \mu_1) \times \left[ \lambda_1 \left( 1 - e^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) t} \right) \left( 1 - (1 - D)e^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) (1-\alpha) t} \right) \right] \frac{\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right)}{\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right)} > 0,$$

$$B_4 = (1 - \mu_2) \times \left[ D\lambda_1 \left( 1 - e^{-\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right) t} \right) e^{-\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right) (1-\alpha) t} \right] \frac{\left( d_1 + \frac{\beta_1m_2}{1+\alpha_1m_2} \right)}{\left( d_2 + \frac{\beta_2m_2}{1+\alpha_2m_2} \right)} > 0.$$

Therefore, there exist $T_1 > 0$ and $\varepsilon_3 > 0$ such that

$$S_1(t) \geq S_{41}(t) \geq \overline{S}_{41}(t) - \varepsilon_3,$$

and

$$S_2(t) \geq S_{42}(t) \geq \overline{S}_{42}(t) - \varepsilon_3.$$

Then,

$$\frac{dI_i(t)}{dt} \geq [\beta_i(S_{4i}(t) - \varepsilon_3) - (r_i + d_i + b_i)]I_i(t) \quad (i = 1, 2), \tag{4.18}$$
for $t \geq T_1$. Let $N_1 \in N$ and $N_1 \tau > T_1$. Integrating (4.18) on $(n \tau, (n + 1) \tau)$, $n \geq N_1$, we have

$$I_i((n + 1)\tau) \geq I_i(n\tau^+) \exp \left( \int_{n\tau}^{(n+1)\tau} \left[ \beta_i(S_i(t) - \varepsilon_3) - (r_i + d_i + b_i) \right] dt \right)$$

$$= I_i(n\tau) e^{\alpha_i} \quad (i = 1, 2),$$

then $I_i((N_1 + k)\tau) \geq I_i(N_1\tau^+) e^{\alpha \cdot k} \rightarrow \infty$, as $k \rightarrow \infty$, which is a contradiction to the boundedness of $I_i(t)$. Hence, there exists a $t_1 > 0$ such that $I_i(t) \geq m_1 (i = 1, 2)$.

Thus, we can obtain $\frac{dS_i(t)}{dt} > \lambda_i - (d_i + \beta_i M) S_i(t) \quad (i = 1, 2)$. Then, the following comparatively impulsive differential equation is

$$\left\{ \begin{array}{ll}
\frac{dS_{51}(t)}{dt} = \lambda_1 - \left( d_i + \frac{\beta_1 M}{1 + \alpha_1 M} \right) S_{51}(t), & t \neq (n + l)\tau, \quad t \neq (n + 1)\tau, \\
\frac{dS_{52}(t)}{dt} = \lambda_2 - \left( d_2 + \frac{\beta_2 M}{1 + \alpha_2 M} \right) S_{52}(t), & t \neq (n + l)\tau, \quad t \neq (n + 1)\tau, \\
\Delta S_{51}(t) = D(S_{52}(t) - S_{51}(t)), & t = (n + l)\tau, \\
\Delta S_{52}(t) = D(S_{51}(t) - S_{52}(t)), & t = (n + 1)\tau.
\end{array} \right.$$  

(4.19)

Similar to Lemma 3.6, we have

$$\tilde{S}_{51}(t) = \left\{ \begin{array}{ll}
\frac{1}{d_i + \frac{\beta_i M}{1 + \alpha_i M}} \left[ \lambda_1 - \left( \lambda_1 - (d_i + \frac{\beta_1 M}{1 + \alpha_1 M}) \right) S_{51}^* \right] e^{\left( d_i + \frac{\beta_i M}{1 + \alpha_i M} \right) (t - n\tau)}, & t \in [n\tau, (n + l)\tau), \\
\frac{1}{d_i + \frac{\beta_i M}{1 + \alpha_i M}} \left[ \lambda_1 - \left( \lambda_1 - (d_i + \frac{\beta_1 M}{1 + \alpha_1 M}) \right) S_{51}^* \right] e^{\left( d_i + \frac{\beta_i M}{1 + \alpha_i M} \right) (t - (n + l)\tau)}, & t \in [(n + l)\tau, (n + 1)\tau), \\
\end{array} \right.$$  

(4.20)

$$\tilde{S}_{52}(t) = \left\{ \begin{array}{ll}
\frac{1}{d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}} \left[ \lambda_2 - \left( \lambda_2 - (d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}) \right) S_{52}^* \right] e^{\left( d_2 + \frac{\beta_2 M}{1 + \alpha_2 M} \right) (t - n\tau)}, & t \in [n\tau, (n + l)\tau), \\
\frac{1}{d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}} \left[ \lambda_2 - \left( \lambda_2 - (d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}) \right) S_{52}^* \right] e^{\left( d_2 + \frac{\beta_2 M}{1 + \alpha_2 M} \right) (t - (n + l)\tau)}, & t \in [(n + l)\tau, (n + 1)\tau),
\end{array} \right.$$  

here $S_{51}^*$ and $S_{52}^*$ are determined as

$$S_{51}^* = \frac{(1 - A_{51}) B_5 - A_{51} A_{52}}{(1 - A_{51})(1 - B_{52}) - A_{52} B_{51}} > 0,$$

$$S_{52}^* = \frac{B_{51} B_{54} - A_{51} (1 - B_{52})}{(1 - A_{51})(1 - B_{52}) - A_{52} B_{51}} > 0,$$

(4.21)

and $S_{51}^{**}$ and $S_{52}^{**}$ are defined as

$$\tilde{S}_{51}^{**} = \left\{ \begin{array}{ll}
\frac{1 - D}{d_i + \frac{\beta_i M}{1 + \alpha_i M}} \left[ \lambda_1 - \left( \lambda_1 - (d_i + \frac{\beta_1 M}{1 + \alpha_1 M}) S_{51}^* \right) e^{\left( d_i + \frac{\beta_i M}{1 + \alpha_i M} \right) \tau} \right] \\
+ \frac{D}{d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}} \left[ \lambda_2 - \left( \lambda_2 - (d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}) S_{52}^* \right) e^{\left( d_2 + \frac{\beta_2 M}{1 + \alpha_2 M} \right) \tau} \right], & t \neq (n + l)\tau, \quad t \neq (n + 1)\tau, \\
\end{array} \right.$$  

(4.22)
where
\[ A_{51} = (1 - \mu_1)(1 - D)e^{-\left(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M}\right)\tau} < 1, \]
\[ B_{51} = (1 - \mu_1)De^{-\left[(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M})(1 - h) + (d_2 + \frac{\beta_2 M}{1 + \alpha_2 M})\right]\tau} < 1, \]
\[ A_{52} = (1 - \mu_2)De^{-\left[(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M}) + (d_2 + \frac{\beta_2 M}{1 + \alpha_2 M})\right]\tau} < 1, \]
\[ B_{52} = (1 - \mu_2)(1 - D)e^{-\left(d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}\right)\tau} < 1, \]
\[ A_5 = (1 - \mu_1) \times \left[ \frac{\lambda_1 \left(1 - e^{-\left(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M}\right)\tau}\right)}{\left(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M}\right)} \right] \frac{\lambda_2 \left(1 - e^{-\left(d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}\right)\tau}\right)}{\left(d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}\right)} > 0, \]
\[ B_5 = (1 - \mu_2) \times \left[ \frac{D\lambda_1 \left(1 - e^{-\left(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M}\right)\tau}\right)}{\left(d_1 + \frac{\beta_1 M}{1 + \alpha_1 M}\right)} \right] \frac{\lambda_2 \left(1 - e^{-\left(d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}\right)\tau}\right)}{\left(d_2 + \frac{\beta_2 M}{1 + \alpha_2 M}\right)} > 0. \]

For any \( \varepsilon_4 \) small enough, we obtain
\[ S_{51}(t) > S_{51}(t) - \varepsilon_4, \]
and
\[ S_{52}(t) > S_{52}(t) - \varepsilon_4. \]

From the comparison theorem of impulsive differential equations, we have
\[ S_1(t) > S_{51}(t) > S_{52}(t) - \varepsilon_4 > (S_{51}(t)^ + + S_{51}(t)^ +) - \varepsilon_4 = m_{51}, \]
and
\[ S_2(t) > S_{52}(t) > S_{52}(t) - \varepsilon_4 > (S_{52}(t)^ + + S_{52}(t)^ +) - \varepsilon_4 = m_{52}, \]
i.e., \( S_1(t) > m_{51} \) and \( S_2(t) > m_{52} \). This completes the proof.

**Corollary 4.4.** If
\[ \min_{i=1,2} \frac{\beta_i}{d_i} \left[ \lambda_i \tau + \frac{\lambda_i - d_i S_i^*}{d_i} (e^{-d_i \tau_{i_1}} - 1) + \frac{\lambda_i - d_i S_{i_1}^*}{d_i} (e^{-d_i \tau_{i_2}} - e^{-d_i \tau_{i_1}}) \right] - (r_i + d_i + b_i) \tau > 0, \]
holds, system (2.1) is permanent. Where \( S_i^* \) (\( i = 1, 2 \)) and \( S_{i_1}^* \) (\( i = 1, 2 \)) are defined as (3.7) and (3.12).

5. Discussion

In this paper, we establish an SIR model with impulsive dispersal, vaccination and restricting infected individuals boarding transports. This SIR epidemic model for two regions, which are connected by transportation of non-infected individuals, portrays the evolution of diseases. We prove that all solutions of the investigated system are uniformly ultimately bounded. From (4.1) and (4.2), if \( \max_{i=1,2} \frac{\beta_i}{d_i} \left[ \lambda_i \tau + \frac{\lambda_i - d_i S_i^*}{d_i} (e^{-d_i \tau_{i_1}} - 1) + \frac{\lambda_i - d_i S_{i_1}^*}{d_i} (e^{-d_i \tau_{i_2}} - e^{-d_i \tau_{i_1}}) \right] - (r_i + d_i + b_i) \tau < 0 \) holds, the infection-free boundary periodic solution \( (S_1(t), 0, 0, S_2(t), 0, 0) \) of system (2.1) is globally asymptotically stable. From (4.12) or (4.23), if \( \min_{i=1,2} \frac{\beta_i}{d_i} \left[ \lambda_i \tau + \frac{\lambda_i - d_i S_i^*}{d_i} (e^{-d_i \tau_{i_1}} - 1) + \frac{\lambda_i - d_i S_{i_1}^*}{d_i} (e^{-d_i \tau_{i_2}} - e^{-d_i \tau_{i_1}}) \right] - (r_i + d_i + b_i) \tau > 0 \) holds, system (2.1) is permanent. It is concluded that the approach of impulsive vaccination and restricting infected individuals boarding transports provides reliable tactic basis for preventing disease spread. In the real world, after the etiology the
The authors declare that they have no competing interest. All authors have read and approved the final manuscript.

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