FOURIER-BESSEL TRANSFORMS OF DINI-LIPSCHITZ FUNCTIONS ON LEBESGUE SPACES $L_{p,\gamma}(\mathbb{R}^n_+)$

ISMAIL EKINCIOGLU, ESRA KAYA, AND S. ELIFNUR EKINCIOGLU

Abstract. In this paper, we prove a generalization of Titchmarsh’s theorem for the Laplace-Bessel differential operator in the space $L_{p,\gamma}(\mathbb{R}^n_+)$ for functions satisfying the $(\psi, p)$-Laplace-Bessel Lipschitz condition for $1 < p \leq 2$ and $\gamma > 0$.

1. INTRODUCTION

Integral transforms and their inverse transforms are widely used to solve various problems in calculus, Fourier analysis, mechanics, mathematical physics, and computational mathematics. Fourier transform is one of the most important integral transforms. Since it was introduced by Fourier in the early 1880s, it has become an important mathematical concept that is at the centre of the highly developed branch of mathematics called Fourier Analysis. It has many application areas. The Fourier transform of the kernel of singular integral operator is very important in applications of singular integral operator theory. The properties of the Fourier transform of the kernel give information about the existence of the solution of singular integral equations. Since singular integrals are convolution type operators, their Fourier transforms are the product of the Fourier transforms of two functions.

As it is well known that if Lipschitz conditions are applied on a function $f(x)$, then these conditions greatly affect the absolute convergence of the Fourier-Bessel series and behaviour of $F_* f$ Fourier-Bessel transforms of $f$. In general, if $f(x)$ belongs to a certain function class, then the Lipschitz conditions have bearing as to the dual space to which the Fourier coefficients and Fourier-Bessel transforms of $f(x)$ belong. Younis (see [12]) worked the same phenomena for the wider Dini Lipschitz class for some classes of functions. Daher, El Quadih, Daher and El Hamma proved an analog Younis (see [12, Theorem 2.5]) in for the Fourier-Bessel transform for functions satisfies the Fourier-Bessel Dini Lipschitz condition in the

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Lebesgue space $L^2_{\alpha, n}$ (see [10]). El Hamma and Daher proved a generalization of Titchmarsh’s theorem for the Bessel transform in the space $L_{2, \gamma}(\mathbb{R}_+^n)$ (see [1]).

In this paper we prove a generalization of Titchmarsh’s theorem for the Laplace-Bessel transform in the space $L^p; \gamma(\mathbb{R}_+^n)$, where $1 < p \leq 2$ and $\gamma > 0$.

2. Preliminaries

Let $\mathbb{R}_+^n$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, \ldots, x_n)$, defined by the inequality $x_n > 0$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. $S^n_+$ denote the unit sphere on $\mathbb{R}_+^n$, which can be defined as $S^n_+ = \{x \in \mathbb{R}^n_+ : |x| = 1\}$. $S_+ = S(\mathbb{R}^n_+)$ be the space of functions which are the restrictions to $\mathbb{R}^n_+$ of the test functions of the Schwartz that are even with respect to $x_n$, decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$D^n_\gamma = D^\alpha_1 B^n_1 \cdots D^\alpha_{n-1}_1 B^n_{n-1} B^n_n = \frac{\partial^{\alpha_1}}{\partial x^n_1} \cdots \frac{\partial^{\alpha_{n-1}}}{\partial x^n_{n-1}} \frac{\partial}{\partial x^n_n},$$

i.e., for all $\varphi \in S_+$, $\sup_{x \in \mathbb{R}^n_+} |x^\beta D^n_\gamma \varphi| < \infty$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indexes, and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $B^n_n = \frac{\partial^2}{\partial x^n_n} + x^n_1 \frac{\partial}{\partial x^n_n}$ is the Bessel differential expansion. For $\gamma \geq 0$, we introduce the Bessel normalized function of the first kind $j_\gamma$ defined by

$$j_\gamma(z) = \Gamma(\gamma + 1) \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n + \gamma + 1)} \left( \frac{z}{2} \right)^{2n},$$

(1)

where $\Gamma$ is the gamma-function (see [9]). Moreover, from [1] we see that

$$\lim_{z \to 0} \frac{j_{\gamma+1}(z) - 1}{z^2} \neq 0$$

by consequence, there exist $C > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow \left| j_{\gamma+1}(z) - 1 \right| \geq C |z|^2.$$  

(2)

The function $u = j_{\gamma+1}(z)$ satisfies the differential equation

$$B^n_n u(x, y) = B^n_y u(x, y)$$

with the initial conditions $u(x, 0) = f(x)$ and $u_y(x, 0) = 0$ is function infinitely differentiable, even, and, moreover entire analytic.

The Fourier-Bessel transformation and its inverse on $S_+$ are defined by

$$F_\gamma f(x) = \int_{\mathbb{R}^n} f(y) e^{-i(x', y')} j_{\gamma+1}(x_n y_n) y_n dy,$$

$$F^{-1}_\gamma f(x) = C_{\gamma, n} F_\gamma f(-x', x_n),$$

where $C_{\gamma, n}$ is a constant that depends on $\gamma$ and $n$. The Fourier-Bessel transformation is a unitary operator on $L^2_{\gamma}(\mathbb{R}_+^n)$.
where \((x', y') = x_1y_1 + \ldots + x_{n-1}y_{n-1}, j, \gamma > 0,\) is the normalized Bessel function, and
\[
C_{n, \gamma} = (2\pi)^{n-1/2} 2^{\gamma-1} \Gamma^2((\gamma + 1)/2),
\]
(see [4] [9] [11]). This transform is associated to the Laplace-Bessel differential operator
\[
\Delta_\gamma = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma}{x_n} \frac{\partial}{\partial x_n}, \gamma > 0.
\]
The expression (3) is a hybrid of the Hankel transform. For a fixed parameter \(\gamma > 0,\) let \(L_{p, \gamma} = L_{p, \gamma}(\mathbb{R}_+^n)\) be the space of measurable functions with a finite norm
\[
\|f\|_{L_{p, \gamma}} = \left( \int_{\mathbb{R}_+^n} |f(x)|^p x_n^\gamma dx \right)^{1/p}, \ 1 \leq p < \infty.
\]
The space of the essentially bounded measurable function on \(\mathbb{R}_+^n\) is denoted by \(L_{1, \gamma}(\mathbb{R}_+^n).\)

Given \(1 < p \leq 2, \frac{1}{p} + \frac{1}{q} = 1\) and \(f \in L_{p, \gamma},\) we have the Hausdorff-Young inequality
\[
\|F_{\gamma}f\|_{q, \gamma} \leq C_q \|f\|_{p, \gamma},
\]
where and \(C_q\) is a positive constant.

### 3. Fourier-Bessel Transforms of Dini-Lipschitz Functions

In this section we give the main result of this paper. We need first to define \((\psi, p)\)-Laplace Bessel Lipschitz class.

**Definition 1.** A function \(f \in L_{p, \gamma}(\mathbb{R}_+^n)\) is said to be in the \((\psi, p)\)-Laplace Bessel Lipschitz class, denoted by \(\text{Lip}(\psi, \gamma, p),\) if
\[
\|T^y f(x) - f(x)\|_{p, \gamma} = O(\psi(y)) \quad \text{as} \quad y \to 0,
\]
where $\psi(x)$ is a continuous increasing function on $\mathbb{R}^n_+$, $\psi(0) = 0$, and $\psi(xs) = \psi(x)\psi(s)$ for all $x, s \in \mathbb{R}^n_+$.

**Theorem 2.** Let $f(x)$ belong to $\text{Lip}(\psi, \gamma, p)$. Then
\[
\int_{|\xi| \geq \tau} |F_{\gamma} f(\xi)|^q \xi_n^\gamma d\xi = O(\psi(\tau^{-q})), \quad \text{as} \quad \tau \to +\infty.
\]

**Proof.** Let $f \in \text{Lip}(\psi, \gamma, p)$. Then we have
\[
\|T^y f(x) - f(x)\|_{p, \gamma} = O(\psi(y)) \quad \text{as} \quad y \to 0.
\]

Now we consider Fourier-Bessel transform of generalized shift operator. We get
\[
F_{\gamma}[T^y f(x)](\xi) = \int_{\mathbb{R}^n_+} T^y f(x) j_{\gamma - 1}(x_n \xi_n) x_n^\gamma dx
\]
\[
= \int_{\mathbb{R}^n_+} T^y [j_{\gamma - 1}(x_n \xi_n)] f(x) x_n^\gamma dx
\]
\[
= \int_{\mathbb{R}^n_+} j_{\gamma - 1}(x_n \xi_n) j_{\gamma - 1}(y_n \xi_n) f(x) x_n^\gamma dx
\]
\[
= j_{\gamma - 1}(y_n \xi_n) \int_{\mathbb{R}^n} f(x) j_{\gamma - 1}(x_n \xi_n) x_n^\gamma dx
\]
\[
= j_{\gamma - 1}(y_n \xi_n) F_{\gamma}(f)(\xi),
\]
where $T^y(j_p(\sqrt{x} \xi_n)) = j_p(\sqrt{y} \xi_n)j_p(\sqrt{x} \xi_n)$. From formulas (4) and (5), we obtain
\[
\int_{\mathbb{R}^n_+} F_{\gamma}[T^y f(x) - f(x)]^q x_n^\gamma dx = \int_{\mathbb{R}^n_+} |F_{\gamma} T^y f(x) - F_{\gamma} f(x)|^q x_n^\gamma dx
\]
\[
= \int_{\mathbb{R}^n_+} |j_{\gamma - 1}(\xi f(\xi) - F_{\gamma} f(\xi))|^q \xi_n^\gamma d\xi
\]
\[
= \int_{\mathbb{R}^n_+} |F_{\gamma} f(\xi)[1 - j_{\gamma - 1}(\xi f(\xi))]|^q \xi_n^\gamma d\xi
\]
\[
= \int_{\mathbb{R}^n_+} |1 - j_{\gamma - 1}(\xi f(\xi))|^q |F_{\gamma} f(\xi)|^q \xi_n^\gamma d\xi
\]
\[
\leq C_q \int_{\mathbb{R}^n_+} |T^y f(x) - f(x)|^q \xi_n^\gamma d\xi
\]
\[
\leq C_q \|T^y f(x) - f(x)\|_{p, \gamma}^q.
\]
From (2), we have
\[
\int_{|\xi| \leq \frac{\xi}{\xi}} |F_{\gamma} f(\xi)|^q \xi_n^\gamma d\xi = C_q \int_{|\xi| \leq \frac{\xi}{\xi}} |1 - j_{\gamma - 1}(\xi h)|^q |F_{\gamma} f(\xi)|^q \xi_n^\gamma d\xi
\]
\[
\geq C_q |h|^{-1} \int_{|\xi| \leq \frac{\xi}{\xi}} |F_{\gamma} f(\xi)|^q \xi_n^\gamma d\xi,
\]
0 < h ≤ 1. It follows from the above consideration that there exists a positive constant C such that
\[
\int_{1/h \leq |\xi| \leq h} |F_{\gamma} f(\xi)|^q \xi_1^q d\xi \leq C \psi(h).
\]
Therefore, we get
\[
\int_{|\xi| \leq 2\tau} |F_{\gamma} f(\xi)|^q \xi_1^q d\xi \leq C \psi(\tau^{-q}).
\]
In fact, we have
\[
\int_{|\xi| \leq \infty} |F_{\gamma} f(\xi)|^q \xi_1^q d\xi = \sum_{k=1}^{\infty} \int_{2^{k-1} \leq |\xi| < 2^k \tau} |F_{\gamma} f(\xi)|^q \xi_1^q d\xi
\]
\[
\leq C_q \psi((2^k \tau)^{-q}) + C_q \psi((2\tau)^{-q}) + C_q \psi((2^2 \tau)^{-q}) + \ldots
\]
\[
\leq C_q \psi((2\tau)^{-q}) (1 + \psi(2^{-q}) + \psi^2(2^{-q}) + \psi^3(2^{-q}) + \ldots).
\]
Thus, we can write
\[
\int_{|\xi| \geq \tau} |F_{\gamma} f(\xi)|^q \xi_1^q d\xi = O(\psi((2\tau)^{-q})) \quad \text{as} \quad \tau \to \infty.
\]
Thus, the proof of theorem is completed.

We can give the following result which is used for many the theorem given above.

It is well known that
\[
F_{\gamma} (B_n^\alpha f) (x) = (-x_1^2)^\alpha F_{\gamma} f(x),
\]
\[
F_{\gamma} (D_i^{2\alpha} f) (x) = (-x_i^2)^\alpha F_{\gamma} f(x), \quad i = 1, \ldots, n-1,
\]
\[
F_{\gamma} (\Delta f) (x) = -|x|^2 F_{\gamma} f(x) \quad \text{and} \quad F_{\gamma} (f \otimes g) = F_{\gamma} f F_{\gamma} g,
\]
\[
F_{\gamma} \left( D^{2\alpha}_{x} B_n^\alpha f \right) (x) = (-1)^{[\alpha]} x_1^{2\alpha} F_{\gamma} f(x)
\]
We can use the mathematical induction method for k = 1, we get
\[
F_{\gamma} (\Delta f) (x) = C_{n,\gamma} \int_{\mathbb{R}^n} \Delta f(y) e^{-ix'y'} j_{\gamma,1}(x, y) y_n^\gamma dy
\]
\[
= C_{n,\gamma} \int_{\mathbb{R}^n} \left( \sum_{k=1}^{n} \frac{\partial^2 f(y)}{\partial y_k^2} + \frac{\gamma \partial f(y)}{y_n} \right) e^{-ix'y'} j_{\gamma,1}(x, y) y_n^\gamma dy
\]
\[ C_{n, \gamma} \int_{\mathbb{R}^n_+} \left( \sum_{k=1}^{n} \frac{\partial^2 f(y)}{\partial y_k^2} e^{-ix'y'} j_{\gamma-1}(x_n y_n) y_n^\gamma dy \right) \]

\[ - C_{n, \gamma} \int_{\mathbb{R}^n_+} \left( \sum_{k=1}^{n} \frac{\gamma}{y_n} \frac{\partial f(y)}{\partial y_n} e^{-ix'y'} j_{\gamma-1}(x_n y_n) y_n^\gamma dy \right) = I_1 + I_2. \]

If we apply partial integration to the second term of \( I_1 \) and \( I_2 \), then we have

\[ F_\gamma(\Delta, u)(x) = C_{n, \gamma} \int_{\mathbb{R}^n_+} f(y) e^{-ix'y'} (\Delta \psi_{\gamma-1}(x_n y_n)) y_n^\gamma dy. \]

Here, if we use the following equality \[8\],

\[ \int_{0}^{\infty} f(y) \Delta \psi_{\gamma-1}(xy)y^\gamma dy = -|x|^2 \int_{0}^{\infty} f(y) j_{\gamma-1}(xy)y^\gamma dy \]

then we have

\[ F_\gamma(\Delta \psi f)(x) = -|x|^2 F_\gamma f(x). \]

Since \( f \in Lip(\psi, \gamma, p) \), it is clear that

\[ \|F_\gamma(\Delta \psi f)\|_{L^p_{\gamma, \gamma}(\{\xi \geq \tau\})} \leq C_{n, \gamma} O(\psi(\tau^{-\eta})) \]

as \( \tau \to +\infty \).

There are many examples. Here is one of them and a simple method to produce many more: \( f(x) = |x|^p \) for \( 1 < p < \infty \), where \( f(0) = 0 \) is understood. These functions are uniformly continuous on all of \( \mathbb{R}^n_+ \). If \( p = 2 \), \( f \) belongs to the Lipschitz class at \( \mathbb{R}^n_+ \).

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*Current address*: Ismail Ekincioglu: Department of Mathematics, Dumlupinar University, Kutahya, Turkey.
*E-mail address*: ismail.ekincioglu@dpu.edu.tr
ORCID Address: https://orcid.org/0000-0002-5636-1214

*Current address*: Esra Kaya: Department of Mathematics, Dumlupinar University, Kutahya, Turkey.
*E-mail address*: kayaesra.e.k@gmail.com
ORCID Address: https://orcid.org/0000-0002-6971-0452

*Current address*: S. Elifnur Ekincioglu: Department of Mathematics, Dumlupinar University, Kutahya, Turkey.
*E-mail address*: ekinciogluelifnur@gmail.com
ORCID Address: https://orcid.org/0000-0001-6700-8232