Mean-field approach in the multi-component gas of interacting particles applied to relativistic heavy-ion collisions

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Abstract

Generalized mean-field approach for thermodynamic description of relativistic single- and multi-component gas in the grand canonical ensemble is formulated. In the framework of the proposed approach different phenomenological excluded-volume procedures are presented and compared to the existing ones. The mean-field approach is then used to effectively include hard-core repulsion in hadron-resonance gas model for description of chemical freeze-out in heavy-ion collisions. We calculate the collision energy dependence of several quantities for different values of hard-core hadron radius and for different excluded-volume procedures such as van der Waals and Carnahan-Starling models. It is shown that a choice of the excluded-volume model becomes important for large particle densities, and for large enough values of hadron radii ($r \gtrsim 0.9$ fm) there can be a sizable difference between different excluded-volume procedures used to describe the chemical freeze-out in heavy-ion collisions.

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I. INTRODUCTION

Thermodynamic models of description of properties of the strongly interacting matter are among the most valuable tools in modern high-energy physics. The hadron-resonance gas model and its modifications have been successfully used to extract thermodynamic parameters of matter created in heavy-ion collisions, by fitting the rich data on mean hadron multiplicities in various experiments ranging from low energies at SchwerIonenSynchrotron (SIS) to highest energy of the Large Hadron Collider (LHC) [1–8]. Various equations-of-state of the strongly interacting matter (either phenomenological or based on lattice QCD) are used as an input into fluid dynamical models, which describe the dynamics of nucleus-nucleus collisions. In the simplest case, the hadronic phase is described by the multi-component ideal gas of point-like hadrons. In more realistic model one needs to take into account the attractive and repulsive interactions between hadrons. According to the arguments of Dashen, Ma, and Bernstein [9], the inclusion into the model of all known resonances as free particles allows to effectively model the attraction between hadrons. In order to describe the repulsive part of hadronic interaction, various phenomenological excluded-volume procedures have been proposed [10–13]. Another way of modeling the attractive and repulsive interactions is the relativistic mean-field theory such as Walecka model [14, 15] and its generalizations. The excluded-volume effects were also treated in the framework of generalized mean-field approach [16–19] where the resulting temperature-dependent mean-field allows to perform a van der Waals kind of excluded-volume procedure. In the present work we extend this approach by formulating the generalized mean-field theory for single- and multi-component gases of particles from basic thermodynamic considerations. We show that the presented approach allows one to conveniently formulate various excluded-volume procedures, such as van der Waals or Carnahan-Starling models, in the grand canonical ensemble and in a thermodynamically consistent way. The importance of thermodynamic consistency constraint in phenomenological excluded-volume models has recently been discussed in Ref. [20].

The paper is organized as follows. In Sec. II we formulate the mean-field theory from thermodynamic considerations. In Sec. III we connect our approach to the virial expansion of the interacting classical gas. In Sec. IV we generalize the mean-field theory for a multi-component system. In Sec. V different excluded-volume procedures for systems with repulsive interaction are formulated in terms of mean fields. In Sec. VI we perform calcu-
lations in the mean-field version of the hadron-resonance gas model for the description of
chemical freeze-out in heavy-ion collisions. Sec. VII closes the article with conclusions.

II. THE THERMODYNAMIC MEAN-FIELD THEORY FOR SINGLE-COMPONENT GAS

Let us consider the system of interacting particles from general thermodynamic point of
view. We provide this consideration in terms of the density of free energy \( \phi(n, T) \) (we reserve
the letter "\( f \)" for the distribution function), which depends on the density of particles \( n \) and
temperature \( T \), where throughout the text we adopt the system of units \( k_B = c = \hbar = 1 \). The
density of free energy (DFE) relates to the main thermodynamic quantities in the following
way

\[
\phi(n, T) = \varepsilon(n, T) - T s(n, T),
\]

\[
\phi(n, T) = n \mu(n, T) - p(n, T),
\]

where \( \varepsilon(n, T) \) is the energy density, \( p(n, T) \) is the pressure. Two quantities \( \mu(n, T) \) (the
chemical potential) and \( s(n, T) \) (the entropy density) are given as partial derivatives with
respect to independent variables \( (n, T) \)

\[
\mu = \left( \frac{\partial \phi}{\partial n} \right)_T, \quad s = - \left( \frac{\partial \phi}{\partial T} \right)_n.
\]  

(3)

For a system of interacting particles the DFE can be written as a sum of free and inter-
acting contributions

\[
\phi(n, T) = \phi_0(n, T) + \phi_{\text{int}}(n, T),
\]  

(4)

where \( \phi_0 \) is the DFE of the ideal system (without interaction). The chemical potential can
be split also into “free” and “interacting” pieces. In accordance with (3) we obtain

\[
\mu = \mu_0 + \left( \frac{\partial \phi_{\text{int}}}{\partial n} \right)_T, \quad \text{where} \quad \mu_0 = \left( \frac{\partial \phi_0}{\partial n} \right)_T.
\]  

(5)

Taking into account Eqs. (2), (4) and (5) for the system of interacting particles one can
represent the pressure in the following form

\[
p = n \mu(n, T) - \phi(n, T) = p_0(n, T) + n \left( \frac{\partial \phi_{\text{int}}}{\partial n} \right)_T - \phi_{\text{int}}.
\]  

(6)
where
\[ p_0(n, T) = n \mu_0(n, T) - \phi_0(n, T). \] (7)

Here the independent variables \( n \) and \( T \) correspond to the interacting system. One can put in correspondence to free pressure \( p_0 \) from the last equation (7) the pressure of the ideal gas \( \tilde{p}_0 \) calculated in the grand canonical ensemble for the same values \( T \) and \( \mu_0 \) as they are taken in (7)
\[ \tilde{p}_0(T, \mu_0) = g \frac{1}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}} f_0(k; T, \mu_0), \] (8)
where \( g \) is the degeneracy factor, \( f_0(k; T, \mu_0) \) is the ideal gas distribution function (the Boltzmann, Fermi-Dirac or Bose-Einstein one).

By definition we introduce now the following important notations:
\[ U(n, T) = \left[ \frac{\partial \phi_{\text{int}}(n, T)}{\partial n} \right]_T, \] (9)
\[ P_{\text{ex}}(n, T) = n \left[ \frac{\partial \phi_{\text{int}}(n, T)}{\partial n} \right]_T - \phi_{\text{int}}(n, T). \] (10)
We immediately fix that these two quantities are related to one another by equality
\[ n \frac{\partial U(n, T)}{\partial n} = \frac{\partial P_{\text{ex}}(n, T)}{\partial n}. \] (11)

As a next step we use these quantities \( U(n, T) \) and \( P_{\text{ex}}(n, T) \) to determine the pressure. First we substitute \( P_{\text{ex}}(n, T) \) from (10) to Eq. (6) and obtain
\[ p = p_0(n, T) + P_{\text{ex}}(n, T). \] (12)
Evidently, if in this equation one regards \( p_0(n, T) \) as the pressure of the ideal gas, then, the quantity \( P_{\text{ex}}(n, T) \) should be treated as the *excess pressure*.

Next, in our evaluations of the thermodynamic quantities of the interacting system we would like to use formula (8) for the pressure of the ideal gas. Meanwhile, for the interacting gas the independent variables are, for instance, \( n \) and \( T \) in the canonical ensemble or \( \mu \) and \( T \) in the grand canonical ensemble. Hence, in Eq. (8) we need to express the free chemical potential \( \mu_0 \) through these variables. One can do this by substituting notation (9) into Eq. (5) for the free chemical potential \( \mu_0 \) and thus obtain
\[ \mu_0 = \mu - U(n, T). \] (13)
Here we can treat the particle density \( n \) in the grand canonical ensemble as \( n(\mu, T) \) and consequently then treat the field \( U \) as \( U[n(\mu, T), T] \). From definition \( (9) \) it is evident that the quantity \( U(n, T) \) represents the interaction energy per particle. So, we calculate the pressure of the interacting gas using Eqs. \( (12) \) and \( (8) \), where we substitute in \( (8) \) the free chemical potential \( \mu_0 \) from \( (13) \). As a result, in the grand canonical ensemble for the pressure of the gas of interacting particles we obtain the following expression

\[
p(T, \mu) = \frac{g}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}} f(k; T, \mu, n) + P^{ex}(n, T),
\]

where it is implied that \( n = n(T, \mu) \) and

\[
f(k; T, \mu) = \left\{ \exp \left[ \frac{\sqrt{m^2 + k^2} + U(n, T) - \mu T}{T} \right] + a \right\}^{-1}
\]

with \( a = +1 \) for fermions, \( a = -1 \) for bosons and \( a = 0 \) for the Boltzmann approximation.

It is an easy task to go to the nonrelativistic sector, where \( |k| \ll m \). Indeed, we just make an alteration in the dispersion law

\[
\sqrt{m^2 + k^2} \rightarrow m + \frac{k^2}{2m},
\]

and take into account that the “nonrelativistic” chemical potential, \( \tilde{\mu} \), relates to “relativistic” one, \( \mu \), as \( \tilde{\mu} = \mu - m \).

It is reasonable to check the standard relation

\[
N(T, \mu, V) = - \left[ \frac{\partial \Omega(T, \mu, V)}{\partial \mu} \right]_{T, V},
\]

which is appropriate in the grand canonical ensemble. Let us note that we already made transfer from independent variables \((T, n)\) of canonical ensemble to the independent variables \((T, \mu)\) in the grand canonical one. Taking into account that in homogeneous system \( \Omega(T, \mu, V) = -p(T, \mu) V \) we have

\[
n(T, \mu) = \frac{\partial p(T, \mu)}{\partial \mu}.
\]

For this calculation it is convenient to use expression of the pressure \( (14) \) in the equivalent form

\[
p(T, \mu) = \frac{gT}{a} \int \frac{d^3k}{(2\pi)^3} \ln \left\{ 1 + a \exp \left[ -\sqrt{m^2 + k^2} - U(n, T) + \mu T \right] \right\} + P^{ex}(n, T).
\]

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Hence, for the derivative on the r.h.s. of (18) we have
\[
\frac{\partial p(T, \mu)}{\partial \mu} = \frac{g}{\sqrt{m^2 + k^2}} \int \frac{d^3k}{(2\pi)^3} f(k; T, \mu) \left[ 1 - \frac{\partial U(n, T)}{\partial n} \frac{\partial n(T, \mu)}{\partial \mu} \right] + \frac{\partial P_{\text{ex}}^*(n, T)}{\partial n} \frac{\partial n(T, \mu)}{\partial \mu},
\]
where we set
\[
n = g \int \frac{d^3k}{(2\pi)^3} f(k; T, \mu).
\]
The expression in square brackets on the r.h.s. of the second line in (20) equals to zero due to the relation (11). Hence, we obtain the standard formula, \(\frac{\partial p(T, \mu)}{\partial \mu} = n\), for calculation of the particle density in the grand canonical ensemble and, therefore, confirm validity of (18). We confirm as well that the function \(f(k; T, \mu)\) given in (15) is indeed the distribution function which contains the mean field \(U(n, T)\) accounting for interaction of the particles in the gas as a correction to the free single-particle energy \(\sqrt{m^2 + k^2}\).

For the energy density \(\varepsilon\), using the Euler relation, \(\varepsilon + p = Ts + \mu n\), and \(s = \frac{\partial p}{\partial T}\) one obtains
\[
\varepsilon(T, \mu) = \frac{g}{\sqrt{m^2 + k^2}} \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} f(k; T, \mu) + n U(n, T) - P_{\text{ex}}^*(n, T) +
+ T \left\{ \left[ \frac{\partial P_{\text{ex}}^*(n, T)}{\partial T} \right]_n - n \left[ \frac{\partial U(n, T)}{\partial T} \right]_n \right\}.
\]

The self-consistency of the thermodynamic picture that we obtain above is due to relation (11) between the mean field \(U(n, T)\) and the excess pressure \(P_{\text{ex}}^*(n, T)\). That is why it is reasonable to regard relation (11) as a \textit{condition of thermodynamic consistency} when one tries to define \(U(n, T)\) and \(P_{\text{ex}}^*(n, T)\) from phenomenological considerations. Indeed, if, for instance, we adopt relation (11) by definition, then taking the partial derivative of thermodynamic potential \(\Omega(T, \mu, V)\) we obtain a correct expression for the particle density, i.e. \(-[\partial \Omega/\partial \mu]_{T, V} = nV\). On the other hand, if one is aimed to obtain correct expression of the particle density \(n\) using the mean-field approach, then it is necessary to postulate the relation (11), saying that we accept it to provide the thermodynamic consistency in the mean-field description of the interacting gas. Meanwhile, in our approach we obtain relation (11) as intrinsic property of the mean field \(U(n, T)\) and the excess pressure \(P_{\text{ex}}^*(n, T)\) introduced in (9) and (10).

Attention should be drawn to the fact that for a given function \(\phi_{\text{nl}}(n, T)\) (as the starting point of the problem) or, which is the same, for a given mean field \(U(n, T)\), Eq. (21) is not
a function but an equation for the particle density $n$. This equation has to be solved in a self-consistent way for any given point in the $(T, \mu)$ plane. The solution will result in the explicit dependence $n = n(T, \mu)$, which in general will differ from the ideal gas dependence, $n_0(T, \mu_0)$.

This approach has been of wide use in relativistic mean field theories \cite{14, 15, 21} where the particles interact with a scalar field $\phi$ (attraction) and a vector field $V_\mu$ (repulsion). Due to the rotation invariance only the “time” component of the vector field, i.e. $V_0$, survives. In our notations it corresponds to the mean field $U(n, T)$.

Equations (21), (14) and (22) provide us with a self-consistent procedure for including the interaction to our description in a phenomenological way.

III. SINGLE PARTICLE REPRESENTATION OF INTERACTING CLASSICAL GAS

To illustrate the above considerations we consider a classical gas with particle repulsion. The grand partition function of the classical gas is

$$\Xi(T, \mu, V) = \sum_{N=0}^{\infty} V^N g^N \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_N}{(2\pi)^3} \exp \left\{ -\frac{1}{T} \left[ \sum_{j=1}^{N} e_0(k_j) - \mu N \right] \right\} \times \frac{1}{V^N N!} \int d^3r_1 \cdots d^3r_N \exp \left( -\frac{U_N}{T} \right),$$

where $e_0(k) = k^2/2m$ in the nonrelativistic case and $e_0(k) = \sqrt{k^2 + m^2}$ in the relativistic one. Here the potential energy of the $N$-particle system reads as

$$U_N = \sum_{i<j}^{N} \Phi(|r_i - r_j|),$$

where $\Phi(|r|)$ is the two-particle potential.

On the next step we use the first and second Mayer’s theorems \cite{22, 23} which are the crucial points of our consideration. Due to these theorems the virial expansion of the grand partition function $\Xi(T, n)$ for the Boltzmann gas with two-particle interaction reads

$$\ln \Xi = V \left[ n + \sum_{i=2}^{\infty} B_i(T)n^i \right],$$

$$\ln z_0 = \ln n + \sum_{i=2}^{\infty} \frac{i}{i-1} B_i(T)n^{i-1},$$

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where $z_0$ is the ideal single-particle partition function

$$z_0(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} \exp \left( -\frac{e_0(k) - \mu}{T} \right).$$

(27)

The virial coefficients $B_i(T)$ depend on temperature $T$, e.g., the second virial coefficient for particles interacting through the potential $\Phi(r)$ is given by

$$B_2(T) = \frac{1}{2} \int d^3r \left\{ 1 - \exp \left[ -\frac{\Phi(r)}{T} \right] \right\}.$$ 

(28)

The expansions entering on the r.h.s. of Eqs. (25) and (26), can be incorporated into the quantities

$$\Delta P_{cl}(n, T) = T \sum_{i=2}^{\infty} B_i(T)n^i,$$

(29)

$$U_{cl}(n, T) = T \sum_{i=2}^{\infty} \frac{i}{i-1} B_i(T)n^{i-1},$$

(30)

which obey a relation analogous to (11),

$$n \frac{\partial U_{cl}(n, T)}{\partial n} = \frac{\partial \Delta P_{cl}(n, T)}{\partial n}.$$ 

(31)

Equation (31) is valid for every pair of the correspondent terms in expansions (29) and (30) separately. Thus it will still be valid if one truncates the series at any order. With the use of these notations, Eqs. (25) and (26) can be rewritten as

$$p(T, \mu) = \frac{T}{V} \ln \Xi = Tn(T, \mu) + \Delta P_{cl}(n, T),$$

(32)

$$n(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} \exp \left[ -\frac{e_0(k) - \mu + U_{cl}(n, T)}{T} \right].$$

(33)

These expressions, which determine the thermodynamic behaviour of the classical (Boltzmann) gas, have been obtained here without any approximations and can be regarded as the single particle representation of the original partition function (23). On the other hand, in view of the last two expressions, one can regard the quantities $U(n, T)$ and $\Delta P(n, T)$ as “mean” field and excess pressure respectively, but also as rigorous quantities in the framework of classical statistics. And in addition, if one describes many-particle system by expressions (32) and (33) then the thermodynamic quantities $U(n, T)$ and $\Delta P(n, T)$ possess the virial expansions (29) and (30).
For the energy density $\varepsilon$, using the Euler relation, $\varepsilon + p = Ts + \mu n$, and $s = \partial p/\partial T$ we obtain
\[
\varepsilon(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} e_0(k) f(k; T, \mu, n) + n U_{cl}(n, T) - \Delta P_{cl}(n, T) +
+ T \left\{ \left[ \frac{\partial \Delta P_{cl}(n, T)}{\partial T} \right]_n - n \left[ \frac{\partial U_{cl}(n, T)}{\partial T} \right]_n \right\}.
\] (34)

A. Relation to mean field hamiltonian

Let us consider here how the thermodynamic mean-field from our approach can be connected to the microscopic mean-field. The hamiltonian of a classical interacting Boltzmann gas in a self-consistent mean field $U(n, T)$ which depends on density and temperature has the following form
\[
H = \sum_i [e_0(k_i) + U_{cl}(n, T)] - V \Delta P_{cl}(n, T),
\] (35)
where term $V \Delta P_{cl}(n, T)$ is necessary to avoid double counting of interaction energy. The grand partition function can then be expressed as
\[
\Xi(\mu, T) = \sum_{N=0}^{\infty} \exp \left( \frac{\mu N}{T} \right) \frac{V^N}{N!} \exp \left\{ -NU_{cl}(n, T) - V\Delta P_{cl}(n, T) \right\} \times
\times \prod_{i=1}^{N} \frac{g}{2\pi^2} \int dk_i k_i^2 \exp \left( -\frac{e_0(k_i)}{T} \right).
\] (36)

Calculation of this sum gives
\[
\Xi(\mu, T) = \exp \left\{ \exp \left( -\frac{U_{cl}}{T} \right) V z_0(\mu, T) + \frac{V\Delta P_{cl}^{ex}}{T} \right\}.
\] (37)

Consequently, we can write
\[
\ln \Xi(\mu, T) = V \left[ \exp \left( -\frac{U_{cl}}{T} \right) z_0(\mu, T) + \frac{\Delta P_{cl}^{ex}}{T} \right].
\] (38)

Taking into account expression for density (33) we see that this expression coincides with the virial expansion of the grand partition function (25) provided that $\Delta P_{cl}(n, T)$ and $U(n, T)$ are defined by (29) and (30) respectively. However, when starting from the effective Hamiltonian, it is necessary to take into account conditions of thermodynamic consistency, i.e. that thermodynamical averages coincide with statistical averages. Thermodynamic averages regarding the particle and energy densities are given by Eqs. (33) and (34) respectively.
Statistical average for particle density can be written as

\[ n_{\text{stat}} = \frac{\langle N \rangle}{V} = \frac{1}{V \Xi(\mu, T)} \sum_{N=0}^{\infty} \exp \left( \frac{\mu N}{T} \right) \frac{V^N}{N!} \exp \left\{ -\frac{NU(n, T) - V \Delta P_{\text{cl}}(n, T)}{T} \right\} \times \]
\[ \times \prod_{i=1}^{N} \frac{g}{2\pi^2} \int dk_i k_i^2 \exp \left( -\frac{e_0(k_i)}{T} \right). \]  

(39)

This quantity can be directly evaluated giving

\[ n_{\text{stat}} = z_0(T, \mu) \exp \left\{ -\frac{U_{\text{cl}}[n(T, \mu)]}{T} \right\}, \]  

(40)

which coincides with Eq. (33). Quantity \( z_0(T, \mu) \) is given by (27). For energy density we have

\[ \varepsilon_{\text{stat}} = \frac{\langle E \rangle}{V} = \frac{1}{V \Xi(\mu, T)} \sum_{N=0}^{\infty} \exp \left( \frac{\mu N}{T} \right) \frac{V^N}{N!} \exp \left\{ -\frac{NU_{\text{cl}}(n, T) - V \Delta P_{\text{cl}}(n, T)}{T} \right\} \times \]
\[ \times \prod_{i=1}^{N} \frac{g}{2\pi^2} \int dk_i k_i^2 \exp \left( -\frac{e_0(k_i)}{T} \right) \] .  

(41)

Calculation results in

\[ \varepsilon_{\text{stat}} = \frac{g}{2\pi^2} \int dk k^2 \exp \left[ \frac{\mu - e_0(p) - U_{\text{cl}}(n, T)}{T} \right] + n U_{\text{cl}}(n, T) - \Delta P_{\text{cl}}(n, T). \]  

(42)

Comparing this expression to thermodynamical average (34) we see that the following condition is necessary for description of system with effective Hamiltonian (35) to be consistent at finite temperatures

\[ \frac{\partial \Delta P_{\text{cl}}(n, T)}{\partial T} = n \frac{\partial U_{\text{cl}}(n, T)}{\partial T}. \]  

(43)

Together with relation (31) it means that microscopic mean field \( U_{\text{cl}}(n, T) \) and corresponding excess pressure \( P_{\text{ex}}(n, T) \) can only be functions of particle density and not temperature. Only in this case Hamiltonian (35) can describe the system consistently. More generally, functions \( U_{\text{cl}} \) and \( \Delta P_{\text{cl}} \) from (35) must satisfy conditions (31) and (43) in order for thermodynamic consistency to be satisfied also in case of quantum Fermi or Bose gas of quasi-particles in microscopic mean field \( U \) (see Ref. [24]). Nevertheless, the approach presented in previous subsection, where these quantities may explicitly depend on temperature, is thermodynamically consistent as it does not assume that Hamiltonian has a specific structure given by
Eq. (35), but rather starts directly from the expression for the thermodynamical potential. The only assumption is that virial expansion of the grand partition function is convergent. The presented result indicates that, in cases when the excess pressure explicitly depends on temperature, as in case of Van-der-Waals gas, the system can not be consistently described by the mean-field Hamiltonian given by (35) and, therefore, one cannot regard the temperature-dependent quantity $U_{cl}(n, T)$ (30) as a microscopic mean field.

B. Correspondence to thermodynamic mean-field theory

We adjust now the correspondence between the approach which was elaborated in Sec. II and the present one. For this purpose we consider the first approach in region where classical statistics can be used. Distribution function (15) then reduces to

$$f(k; T, \mu, n) = \exp \left[ - \frac{e_0(k) + U(n, T) - \mu}{T} \right].$$

The type of statistical description (quantum or classical) of the thermodynamic system does not influence on the existing quantum background of the many-particle system, which means that the energy spectrum has to be same for both types of statistics. Thus, the mean field $U(n, T)$ does not change at all and the thermodynamic description can be obtained from Eqs. (14) and (21) using distribution function (44). So, we compare two descriptions for the same system: The first one is based on grand partition function (23), in the form of Eqs. (32), (33), while the second one uses the distribution function (44) for pressure (14) and particle density (21). If these two approaches give indeed the same result then we obtain

$$U(n, T) = U_{cl}(n, T),$$

and consequently

$$P^{ex}(n, T) = \Delta P_{cl}(n, T).$$

This conclusion yields us a constructive algorithm to develop a model of the interacting system in terms of a mean field. We can extract the mean field from the known classical equation of state.
IV. MEAN-FIELD MODEL FOR MULTI-COMPONENT GAS

The proposed phenomenological mean field model can be generalized for the case of a mixture of \( f \) different particle species. Let us denote \( \mathbf{n} = (n_1, \ldots, n_f) \) as a set of particle densities and \( \mathbf{\mu} = (\mu_1, \ldots, \mu_f) \) as a set of chemical potentials corresponding to each of the particle species. In case of mixture of different particles there will be \( f \) different mean fields \( U_i(\mathbf{n}, T) \) corresponding to each particle species and a common excess pressure \( P^{\text{ex}}(\mathbf{n}, T) \). It is natural that \( U_i(\mathbf{n}, T) \) and \( P^{\text{ex}}(\mathbf{n}, T) \) can depend on any of the particle densities of different species. The density of free energy in this case reads as

\[
\phi(\mathbf{n}, T) = \sum_{i=1}^{f} n_i \mu_i(\mathbf{n}, T) - p(\mathbf{n}, T). \tag{47}
\]

This density can be expressed as sum of free and interacting parts as

\[
\phi(\mathbf{n}, T) = \sum_{i=1}^{f} \phi_0(n_i, T) + \phi_{\text{int}}(\mathbf{n}, T). \tag{48}
\]

Using the same considerations as in Section II one can switch to the grand canonical ensemble and express the total pressure \( p(T, \mathbf{\mu}) \) and density \( n_i(T, \mathbf{\mu}) \) of particle species \( i \) as

\[
p(T, \mathbf{\mu}) = \sum_{i=1}^{f} \frac{g_i}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m_i^2 + k^2}} f_i(k; T, \mathbf{\mu}) + P^{\text{ex}}(\mathbf{n}, T), \tag{49}
\]

\[
n_i(T, \mathbf{\mu}) = g_i \int \frac{d^3k}{(2\pi)^3} f_i(k; T, \mathbf{\mu}) \tag{50}
\]

where \( f_i(k; T, \mathbf{\mu}) \) is the distribution function for particle species \( i \)

\[
f_i(k; T, \mathbf{\mu}) = \left\{ \exp \left[ \sqrt{m_i^2 + k^2 + U_i(\mathbf{n}(T, \mathbf{\mu}), T)} - \mu_i \right] + a_i \right\}^{-1}. \tag{51}
\]

The excess pressure \( P^{\text{ex}}(\mathbf{n}, T) \) and mean fields \( U_i(\mathbf{n}, T) \) are defined as

\[
P^{\text{ex}}(\mathbf{n}, T) = \sum_{i=1}^{f} n_i \left[ \frac{\partial \phi_{\text{int}}(\mathbf{n}, T)}{\partial n_i} \right]_{T, n_j \neq i} - \phi_{\text{int}}(\mathbf{n}, T), \tag{52}
\]

\[
U_i(\mathbf{n}, T) = \left[ \frac{\partial \phi_{\text{int}}(\mathbf{n}, T)}{\partial n_i} \right]_{T, n_j \neq i} \tag{53}
\]

and related to each other through a set of equations corresponding to thermodynamic consistency:

\[
\sum_{j=1}^{f} n_j \frac{\partial U_j}{\partial n_i} = \frac{\partial P^{\text{ex}}}{\partial n_i}, \quad i = 1 \ldots f. \tag{54}
\]
From the relation \( \frac{\partial^2 P^{\text{ex}}}{\partial n_i \partial n_k} = \frac{\partial^2 P^{\text{ex}}}{\partial n_k \partial n_i} \) one can see that mean fields \( U_i \) are not independent but are related to each other via

\[
\frac{\partial U_i}{\partial n_j} = \frac{\partial U_j}{\partial n_i}, \quad i, j = 1 \ldots f. \tag{55}
\]

This relation can be useful when defining mean fields for multi-component gases from phenomenological considerations.

Equation (51) represents system of \( f \) equations (usually transcendental) for particle densities \( n_i \). This system can be solved numerically, for instance, using iteration method or Broyden’s method. Energy density can be evaluated from thermodynamical relation

\[
\varepsilon(T, \mu) = T \frac{\partial p}{\partial T} + \sum_{i=1}^{f} \mu_i \frac{\partial p}{\partial \mu_i} - p. \tag{56}
\]

It results in

\[
\varepsilon(T, \mu) = \sum_{i=1}^{f} g_i \int \frac{d^3 k}{(2\pi)^3} \sqrt{m_i^2 + k^2} f_i(k; T, \mu) + \sum_{i=1}^{f} n_i U_i(n, T) - P^{\text{ex}}(n, T) + T \left[ \frac{\partial P^{\text{ex}}(n, T)}{\partial T} - \sum_{i=1}^{f} n_i \frac{\partial U_i(n, T)}{\partial T} \right]. \tag{57}
\]

A. Conserved charges

In relativistic mechanics it is commonly the case that number of particles in closed system can change, for instance due to creation of pairs of particles and anti-particles, and instead of particle number conservation there are conservation laws for the charges. As an example it can be baryon charge, electric charge and strangeness in strongly interacting systems. Consequently, the grand canonical treatment of relativistic many particle species systems is usually formulated in terms of independent chemical potentials which correspond to conserved charges rather than particle numbers. Let us have \( s \) independent conserved charges and let \( \tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_s) \) be corresponding independent chemical potentials.

If we denote the \( j \)-th charge of the \( i \)-th particle species as \( Q_i^j \) then chemical potential \( \mu_i \) of \( i \)-th particle species can be expressed as

\[
\mu_i(\tilde{\mu}) = \sum_{j=1}^{s} Q_i^j \tilde{\mu}_j. \tag{58}
\]
Density $\rho_j$ of $j$-th charge can be expressed via a set of particle densities $\{n_i\}$ as

$$\rho_j(T, \tilde{\mu}) = \sum_{i=1}^{f} Q_j^i n_i[T, \mu(\tilde{\mu})].$$

There will be a total of $f$ mean fields $U_i(T, \tilde{\mu})$, one for each of the particle species, also in the case of just $s < f$ independent chemical potentials. Let us show that we can satisfy the conditions of thermodynamic consistency, $\rho_j = \partial p(T, \tilde{\mu})/\partial \tilde{\mu}_j$, provided that the excess pressure and mean fields satisfy (54) and that pressure and particles densities are given by Eqs. (49) and (50), but where $\mu$ depend on $\tilde{\mu}$ via (58). Indeed, we can calculate $\partial p(T, \tilde{\mu})/\partial \tilde{\mu}_j$ as

$$\frac{\partial p(T, \tilde{\mu})}{\partial \tilde{\mu}_j} = \sum_{i=1}^{f} g_i \int \frac{d^3k}{(2\pi)^3} f_i(k; T, \tilde{\mu}) \left[ Q_j^i - \sum_{k=1}^{f} \frac{\partial U_i(n, T)}{\partial n_k} \frac{\partial n_k(T, \tilde{\mu})}{\partial \tilde{\mu}_j} \right]$$

$$+ \sum_{k=1}^{f} \frac{\partial P^\text{ex}(n, T)}{\partial n_k} \frac{\partial n_k(T, \tilde{\mu})}{\partial \tilde{\mu}_j}$$

$$= \sum_{i=1}^{f} Q_j^i n_i - \sum_{k=1}^{f} \left[ \sum_{i=1}^{f} n_i \frac{\partial U_i(n, T)}{\partial n_k} - \frac{\partial P^\text{ex}(n, T)}{\partial n_k} \right] \frac{\partial n_k(T, \mu)}{\partial \tilde{\mu}_j},$$

where we have used that $\partial \mu_i/\partial \tilde{\mu}_j = Q_j^i$. Taking into account Eq. (54) we have

$$\frac{\partial p(T, \tilde{\mu})}{\partial \tilde{\mu}_j} = \rho_j(T, \tilde{\mu}),$$

in accordance with the thermodynamic relations. In such a way, one can use the formulation given by Eqs. (49)-(51) also in the case of multi-component systems with conserved charges.

V. EXCLUDED-VOLUME PROCEDURE

In this section we consider systems with a short-range repulsion and how they can be effectively formulated as an excluded-volume procedure in the framework of the mean field model.

A. Mean fields which are proportional to temperature

Let us consider firstly a single-component classical gas. In classical systems where repulsive interactions are modelled as a hard-core repulsion, the virial coefficients $B_i$ are independent of temperature, see e.g. (28). It follows from (30) and (29) that, in this case, the mean
field and the excess pressure describing such a system are linear functions of temperature, i.e.

\[ U(n, T) \propto T, \quad P^{\text{ex}}(n, T) \propto T, \]  

(62)

More generally, let us consider a class of mean fields which are linear in \( T \). One can see from (22) that in this case the energy density is determined only by the integral term

\[ \varepsilon(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} f(k; T, \mu). \]  

(63)

In case of the Boltzmann statistics for this class of mean fields we find

\[ n(T, \mu) = e^{-\tilde{U}(n)} n_0(T, \mu), \]  

(64)

\[ \varepsilon(T, \mu) = e^{-\tilde{U}(n)} \varepsilon_0(T, \mu), \]  

(65)

where \( \tilde{U}(n) = U(T, n)/T \) in accordance with (62) and we introduce notations

\[ n_0(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} f_0(k; T, \mu), \]  

(66)

\[ \varepsilon_0(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} f_0(k; T, \mu). \]  

(67)

Here

\[ f_0(k; T, \mu) = \exp \left[ -\frac{\sqrt{m^2 + k^2} + \mu}{T} \right] \]  

(68)

is the distribution function of ideal gas at temperature \( T \) and chemical potential \( \mu \).

One can interpret the factor

\[ e^{-\tilde{U}(n)} \equiv \vartheta(n), \]  

(69)

on the r.h.s. of equations (64) and (65) as a suppression-factor of the free volume \( V \) of the system. Indeed, if we introduce notation

\[ \tilde{V}(n) = \vartheta(n) V, \]  

(70)

we can rewrite Eqs. (64), (65) in the form

\[ N(T, \mu) = \tilde{V} n_0(T, \mu), \]  

(71)

\[ E(T, \mu) = \tilde{V} \varepsilon_0(T, \mu). \]  

(72)

Obviously, in case of a repulsive mean field \( \tilde{U}(n) = U(T, n)/T \) we have

\[ 0 < \vartheta(n) \leq 1 \quad \Rightarrow \quad \tilde{V} \leq V. \]  

(73)
This means that a repulsive mean field generates an effective proper volume \( \tilde{v}_0 \) of every particle, which belongs to the system, resulting in the reduction of the total volume \( V \) of the system of these \( N \) particles, i.e. \( V \to V - N\tilde{v}_0 = \tilde{V} \). One can then evaluate the effective proper volume \( \tilde{v}_0(n) \) as

\[
\tilde{v}_0(n) = \frac{1}{N} \left( V - \tilde{V} \right) = v \left[ 1 - \vartheta(n) \right].
\]  
(74)

where \( v = V/N \) is the mean classical volume per particle. The interpretation of \( \tilde{v}_0 \) in accordance with (74) is evident enough. Indeed, the quantity \( V - \tilde{V} \) represents the total effective self-volume of \( N \) particles of the system which is due to repulsive interaction between them. Then, the effective single particle self-volume is in one-to-one correspondence with the original repulsive mean field

\[
\tilde{v}_0(n) = \frac{1}{n} \left\{ 1 - \exp \left[ -\tilde{U}(n) \right] \right\},
\]  
(75)

where the r.h.s. of equation depends on the particle density only.

So, we prove a theorem which says: if the mean field is the repulsive one \( (U(n,T) > 0) \) and it is proportional to the temperature, i.e. \( U(n,T) \propto T \), then, for the Boltzmann statistics, the presence of the interaction in the system appears just as reduction of the total volume of the system \( \tilde{V} \to \vartheta(n) V \), where \( \vartheta(n) = \exp \left[ -U(n,T)/T \right] \). The number of particles and total energy are calculated by using the ideal gas expressions in grand canonical ensemble in the following way: \( N(T,\mu) = \tilde{V} n_0(T,\mu) \) and \( E(T,\mu) = \tilde{V} \varepsilon_0(T,\mu) \).

In case of the quantum statistics we have for densities

\[
n(T,\mu) = \vartheta(n) \tilde{n}_0(T,\mu),
\]  
(76)

\[
\varepsilon(T,\mu) = \vartheta(n) \tilde{\varepsilon}_0(T,\mu),
\]  
(77)

where \( \tilde{U}(n) = U(T,n)/T \) in accordance with (62) in the same way as in the previous case and

\[
\tilde{n}_0(T,\mu) = g \int \frac{d^3k}{(2\pi)^3} \tilde{f}_0(k;T,\mu),
\]  
(78)

\[
\tilde{\varepsilon}_0(T,\mu) = g \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} \tilde{f}_0(k;T,\mu).
\]  
(79)

Here \( \tilde{f}_0(k;T,\mu) \) is the distribution function of ideal Fermi or Bose gas

\[
\tilde{f}_0(k;T,\mu) = \left[ \exp \left( \frac{\sqrt{m^2 + k^2} - \mu}{T} \right) + a \vartheta(n) \right]^{-1}.
\]  
(80)
It turns out that, in addition to volume suppression, here we also effectively obtain intermediate quantum statistics because the second term in the denominator is in the range $0 < \vartheta(n) \leq 1$. We remind that $\vartheta(n) = e^{-\tilde{v}(n)}$, and $a = 1$ for fermions or $a = -1$ for bosons. It is interesting to note that in this case of repulsive mean field, when one increases the particle density $n$ the distribution function (80) shifts closer to the classical Boltzmann one. Hence, in case of the quantum statistics we can conclude that the repulsive mean fields which are linear in $T$ reduce the total volume of the system ($V \rightarrow \tilde{V}$), and effectively result in reduction of the effects connected to quantum statistics. These effects are dependent just on the particle density.

B. Models for single-component gas

Let us now consider some particular examples of excluded-volume models and how they can be expressed in the framework of mean field approach. As the first example we consider a single-component gas where an effective proper volume of single particle $\tilde{v}_0$ (75) is independent of particle density $n$. It follows from (75) that in this case mean field is given by

$$U_{de}(n, T) = -T \ln(1 - \tilde{v}_0 n) .$$

(81)

We call this case as model of directly excluded-volume. The equation (21) for particle density of the classical gas can then be solved explicitly

$$n_{de}(T, \mu) = \frac{n^0(T, \mu)}{1 + \tilde{v}_0 n^0(T, \mu)} ,$$

(82)

where $n^0 = z_0$ is the density of point-like particles (Eq. (21) with $U = 0$). This formula demonstrates explicitly our general statement: switching the repulsive interaction on, keeping $T$ and $\mu$ constant, results in a decrease of the particle density of the ideal gas $n^0$. This phenomenon is valid qualitatively in quantum statistics as well.

For the pressure in the Boltzmann sector we obtain using Eq. (32)

$$p_{de}(T, \mu) = -\frac{T}{\tilde{v}_0} \ln \left[ 1 - \tilde{v}_0 n_{de}(T, \mu) \right] = \frac{T}{\tilde{v}_0} \ln \left[ 1 + \tilde{v}_0 n^0(T, \mu) \right] ,$$

(83)

and the energy density is given by

$$\varepsilon_{de}(T, \mu) = \frac{\varepsilon^0(T, \mu)}{1 + \tilde{v}_0 n^0(T, \mu)} .$$

(84)
Pressure, Eq. (83), is the same as in the lattice gas model \[25\] where the lattice has been constructed from the cells with the volume $\tilde{v}_0$ and there are $N_{\text{cell}} = V/\tilde{v}_0$ cells in the lattice. The pressure and energy density in this model are due to the number of possible combinations to place $N$ particles in $N_{\text{cell}}$ cells (combination entropy).

Next, we consider the 2nd order virial expansion for classical gas of hard spheres of radius $r$

$$p = Tn + T v_0 n^2,$$  \hspace{1cm} (85)

where virial coefficient $v_0 = 4 (4\pi) r^3 / 3$ is four times the intrinsic volume of a particle. The excess pressure and mean field are then given by

$$P_{\text{ex}}(n, T) = T v_0 n^2,$$ \hspace{1cm} (86)

$$U(n, T) = 2 T v_0 n.$$ \hspace{1cm} (87)

Eqs. (86) and (87) are general expressions for a second order virial expansion with virial coefficient $v_0$. We note that the model of directly excluded-volume is consistent with the 2nd order virial expansion (85) if we let $\tilde{v}_0 = 2v_0$.

Perhaps the most conventional example of a system with excluded volume is the van der Waals equation of state

$$p_{\text{vdW}} = \frac{T n}{1 - v_0 n},$$ \hspace{1cm} (88)

where attractive interactions have been neglected. Eq. (88) represents extrapolation of Eq. (85) to higher values of $v_0 n$. The excess pressure ($P_{\text{ex}} = p - nT$) is now

$$P_{\text{vdW}}^{\text{ex}}(n, T) = T n \frac{v_0 n}{1 - v_0 n}.$$ \hspace{1cm} (89)

Using Eq. (31) in the form

$$U(n, T) = \int_0^n 1 \frac{\partial P_{\text{ex}}(n', T)}{\partial n'} dn',$$

and setting $U(n = 0, T) = 0$, one obtains

$$U_{\text{vdW}}(n, T) = T \frac{v_0 n}{1 - v_0 n} - T \ln(1 - v_0 n).$$ \hspace{1cm} (90)

Density can be calculated from (21) and can be written as

$$n_{\text{vdW}} = (1 - v_0 n_{\text{vdW}}) n^0(T, \mu) \exp \left(-\frac{v_0 n_{\text{vdW}}}{1 - v_0 n_{\text{vdW}}} \right).$$ \hspace{1cm} (91)

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Here $n^0(T, \mu)$ is the density of a corresponding ideal gas and henceforth the superscript 0 will denote the ideal gas quantities. The solution is obtained by solving the transcendental equation (91) for particle density $n_{\text{VdW}}$.

A thermodynamically consistent procedure to incorporate the excluded-volume correction in the hadron gas models was formulated in Ref. [12]. Let us show that, in the Boltzmann approximation, this procedure gives equivalent result to the presented Van der Waals procedure in the mean field approach given by Eqs. (89) and (91). Pressure and density in approach from Ref. [12] in the Boltzmann approximation are given by

$$P_{\text{excl}} = n^0(T, \mu) T e^{-v_0 P_{\text{excl}}/T},$$  \hspace{1cm} (92)

$$n_{\text{excl}} = \frac{n^0(T, \mu) e^{-v_0 P_{\text{excl}}/T}}{1 + v_0 n^0(T, \mu) e^{-v_0 P_{\text{excl}}/T}}.$$ \hspace{1cm} (93)

The solution usually proceeds by solving the transcendental equation (92) for pressure. From (93) one gets

$$n^0(T, \mu) e^{-v_0 P_{\text{excl}}/T} = \frac{n_{\text{excl}}}{1 - v_0 n_{\text{excl}}}.$$ \hspace{1cm} (94)

Substituting this expression into Eq. (92) we get

$$P_{\text{excl}} = \frac{n_{\text{excl}} T}{1 - v_0 n_{\text{excl}}}.$$ \hspace{1cm} (95)

which coincides with (89). Similarly, for $n_{\text{excl}}$ we get

$$n_{\text{excl}} = (1 - v_0 n_{\text{excl}}) n^0(T, \mu) \exp \left( - \frac{v_0 n_{\text{excl}}}{1 - v_0 n_{\text{excl}}} \right).$$ \hspace{1cm} (96)

The last example of an excluded-volume procedure for a single-component gas that we consider is an approach based on a well-known Carnahan-Starling equation of state [26], which goes beyond van der Waals and is known to describe the fluid phase of the hard sphere model more accurately. The pressure is given in terms of the packing fraction $\eta$ by the Carnahan-Starling formula

$$P_{\text{CS}} = T n \frac{1 + \eta + \eta^2 - \eta^3}{(1 - \eta)^3}.$$ \hspace{1cm} (97)

Packing fraction $\eta$ can be expressed in terms of second virial coefficient $v_0$ and density $n$ as $\eta = v_0 n/4$. The excess pressure reads as

$$P_{\text{CS}}^e(n, T) = T n \frac{v_0 n - (v_0 n/4)^2/8}{(1 - v_0 n/4)^3}.$$ \hspace{1cm} (98)

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and mean field \(U_{CS}(n, T)\) is

\[
U_{CS}(n, T) = -3T \left[ 1 - \frac{1 - v_0n/12}{(1 - v_0n/4)^3} \right].
\]  

(99)

In Fig. 1 we plot the dependence of the excess pressure normalized by the ideal gas pressure, \(P_{ex}/nT\), on the packing fraction, \(\eta = v_0n/4\), for four different hard-sphere models: 2nd order virial expansion (dashed green line), model of directly excluded-volume (dotted blue line), van der Waals equation of state (dash-dotted red line), and Carnahan-Starling equation of state (solid black line). The Monte Carlo data from Ref. [27] on the compressibility of hard spheres is depicted by open triangles. We note that all four models coincide up to 2nd order virial expansion. We also note that \(P_{ex}/nT \equiv Z - 1\), where \(Z\) is the compressibility factor. It is evident that the packing fraction is strictly bounded from above in the latter three approaches: \(\eta < 0.125\) in the model of directly excluded-volume, \(\eta < 0.25\) in the van der Waals model, and \(\eta < 1\) in the Carnahan-Starling model. We note that the dense packing limit for hard spheres is \(\eta = \pi/(3\sqrt{2}) \approx 0.76\). All the approaches coincide at small values of \(\eta\) but start to diverge significantly for \(\eta > 0.1\). Comparison with Monte Carlo data indicates that the Carnahan-Starling equation-of-state is the most accurate of the considered.

The presented mean-field approach allows one to conveniently implement even more accurate and elaborate equations of state for a gas of rigid spheres, such as one from Ref. [28] or one given as an appropriate Padé approximant, but we do not consider them in the present study.

C. Virial expansion of multi-component gas

Let us now consider a case of multi-component gas. The virial expansion for classical multi-component gas of \(f\) particle species can written as

\[
p(n, T) = T \sum_{i=1}^{f} n_i + T \sum_{i,j} b_{ij}(T)n_in_j + \sum_{i,j,k} b_{ijk}(T)n_in_jn_k + \ldots = T \sum_{i=1}^{f} n_i + P_{ex}(n, T). 
\]

(100)

Here \(b_{ij}, b_{ijk}, \ldots\) are virial coefficients. To get the expressions for mean fields \(U_m(n, T)\) we express them as power series

\[
U_i(n, T) = T \sum_{j} a_{ij}(T)n_j + \sum_{j,k} a_{ijk}(T)n_jn_k + \ldots
\]

(101)
Figure 1: (Color online) The dependence of the quantity $\frac{P_{\text{ex}}}{nT}$ on the packing fraction $\eta$ for four different hard-sphere models: 2nd order virial expansion (dashed green line), model of directly excluded-volume (dotted blue line), van der Waals equation of state (dash-dotted red line), and Carnahan-Starling equation of state (solid black line). The Monte Carlo data from Ref. [27] is depicted by open triangles.

Substituting $U_m$ in form (101) into expressions given by (54) for conditions of thermodynamic consistency we get a closed system of linear equations for coefficients $a_{ij}, a_{ijk}, \ldots$. By solving this system we can recover mean fields $U_i$ in an unambiguous way from the virial expansion (100).

In case of second order virial expansion we have $P_{\text{ex}}(n, T) = T \sum_{i,j} b_{ij}(T)n_i n_j$. Mean fields can then be expressed as $U_i(n, T) = T \sum_j a_{ij}(T)n_j$. System of equations for coefficients $a_{ij}$ can be solved explicitly and yield

$$a_{ij}(T) = 2b_{ij}(T), \quad (102)$$
$$U_i(n, T) = 2T \sum_j b_{ij}(T)n_j. \quad (103)$$

D. Excluded-volume in multi-component gas: available volume approach

One way to extend the van der Waals excluded-volume procedure to multi-component system of particles with different proper volumes $v_i$ is to replace the total volume $V$ of ideal gas by the total available volume $V - \sum_i v_i N_i$. This procedure was done in thermodynamically
cally consistent way in Ref. [29], by inserting the total available volume into grand partition function. As the result, one has a transcendental equation for the system pressure $p(T, \mu)$

$$p(T, \mu) = \sum_{i=1}^{f} p_i^0[T, \mu_i - v_i p(T, \mu)].$$

(104)

Density $n_i$ of particle species $i$ is then expressed as

$$n_i(T, \mu) = \frac{n_i^0[T, \mu_i - v_i p(T, \mu)]}{1 + \sum_{j=1}^{f} v_j n_j^0[T, \mu_j - v_j p(T, \mu)]}.$$  

(105)

Here $p_i^0$ and $n_i^0$ are the ideal gas relations for pressure and particle density of species $i$. In the Boltzmann limit, the expression for pressure (104) reads as

$$p(T, \mu) = \sum_{i=1}^{f} p_i^0(T, \mu_i) \exp \left[ -v_i p(T, \mu) / T \right].$$

(106)

Let us show how mean-field approach can be used to formulate a thermodynamically consistent procedure for multi-component gas which is equivalent to the above procedure in the Boltzmann limit by using the similar considerations regarding the available volume. To do that we substitute the excluded-volume $v_0 n$ in the expression [29] for single-component van der Waals excess pressure by the total excluded-volume $\sum_i v_i n_i$ of multi-component gas. We also treat $n$ as a total density of all particle species. The excess pressure then reads as

$$P_{\text{VdW}}^\text{ex}(n, T) = T \left( \sum_{i=1}^{f} n_i \right) \frac{\sum_{i=1}^{f} v_i n_i}{1 - \sum_{i=1}^{f} v_i n_i}.$$  

(107)

In order to perform calculations of thermodynamic variables one then needs to solve the system of equations (54) for mean fields $U_i$. We remind that this system manifests the conditions of thermodynamic consistency. In this particular case, solution for $U_i$ can be obtained in closed form and it reads as

$$U_i(n, T) = T \frac{v_i \sum_{j=1}^{f} n_j}{1 - \sum_{j=1}^{f} v_j n_j} - T \ln \left( 1 - \sum_{j=1}^{f} v_j n_j \right).$$

(108)

With this expression for mean fields $U_i$ we can now express the particle densities $n_i$ in the Boltzmann approximation as

$$n_i(T, \mu) = \left( 1 - \sum_{j=1}^{f} v_j n_j \right) n_i^0(T, \mu_i) \exp \left( -\frac{v_i \sum_{j=1}^{f} n_j}{1 - \sum_{j=1}^{f} v_j n_j} \right).$$

(109)
We recall that the total pressure in the system can be written as

\[ p(T, \mu) = T \left( \sum_{i=1}^{f} n_i \right) + P_{VdW}^{ex}[n(T, \mu), T] = T \left( \sum_{i=1}^{f} n_i \right) \left( 1 + \frac{\sum_{i=1}^{f} v_i n_i}{1 - \sum_{i=1}^{f} v_i n_i} \right) \]

Substituting (109) into numerator of (110) we get

\[ p(T, \mu) = \sum_{i=1}^{f} p_i^0(T, \mu_i) \exp \left( -\frac{v_i \sum_{j=1}^{f} n_j}{1 - \sum_{j=1}^{f} v_j n_j} \right) \]

where we use that \( p_i^0(T, \mu_i) = T n_i^0(T, \mu_i) \). Using (110) one can rewrite exponent in (111) to get

\[ p(T, \mu) = \sum_{i=1}^{f} p_i^0(T, \mu_i) \exp \left[ -v_i p(T, \mu)/T \right] \]

which coincides with (106).

E. Excluded-volume in multi-component gas: van der Waals procedure

Let us now formulate a different excluded-volume procedure for multi-component gas: one which is consistent with second-order virial expansion. Let \( b_{ij}, i, j = 1 \ldots f \) denote the virial coefficients. For hard spheres model they are

\[ b_{ij} = \frac{2}{3} \pi (r_i + r_j)^3, \]

where \( r_i \) are hard core radii. The pressure in second order virial expansion reads as

\[ P = T \sum_{i=1}^{f} n_i + T \sum_{i,j=1}^{f} b_{ij} n_i n_j. \]

Performing a similar van der Waals extrapolation procedure as in case of single-component gas we can write the van der Waals pressure of multi-component gas as (see e.g. [30])

\[ P = T \sum_{i=1}^{f} \left( \frac{n_i}{1 - \sum_{j=1}^{f} \tilde{b}_{ij} n_j} \right). \]

In order for (115) to be consistent with second-order virial expansion (114) it is necessary that \( \tilde{b}_{ii} = b_i \) and \( \tilde{b}_{ij} + \tilde{b}_{ji} = 2b_{ij} \). The choice of cross-terms \( \tilde{b}_{ij} \) is not unique. For instance, one may let \( \tilde{b}_{ij} = b_{ij} \). And in Ref. [30] these coefficients are defined as

\[ \tilde{b}_{ij} = \frac{2b_{ii} b_{ij}}{b_{ii} + b_{jj}}. \]
The excess pressure reads as

\[ P^{\text{ex}}(n, T) = T \sum_{i=1}^{f} \left( n_i \sum_{j=1}^{f} \tilde{b}_{ji}n_j \right) \frac{1}{1 - \sum_{j=1}^{f} \tilde{b}_{ji}n_j}. \] (117)

The solution to the equations (54) of thermodynamic consistency can be obtained in closed form and for mean fields \( U_i \) we get the following expressions

\[ U_i(n, T) = T \sum_{j=1}^{f} \frac{\tilde{b}_{ji}n_j}{1 - \sum_{k=1}^{f} \tilde{b}_{kj}n_k} - T \ln \left( 1 - \sum_{j=1}^{f} \tilde{b}_{ji}n_j \right). \] (118)

With this expression for mean fields \( U_i \) we can now express the particle densities \( n_i \) in the Boltzmann approximation as

\[ n_i(T, \mu) = \left( 1 - \sum_{j=1}^{f} \tilde{b}_{ji}n_j \right) n_i^0(T, \mu_i) \exp \left( -\sum_{j=1}^{f} \frac{\tilde{b}_{ji}n_j}{1 - \sum_{k=1}^{f} \tilde{b}_{kj}n_k} \right), \quad i = 1 \ldots f. \] (119)

Eq. (119) represents system of transcendental equations for particle densities \( n_i \) which can be solved numerically.

Let us show now that our mean field approach is equivalent to the van der Waals procedure for multi-component gas developed in Ref. [30]. There the total pressure can be expressed as a sum \( p(T, \mu) = \sum_{i=1}^{f} p_i(T, \mu) \), where \( p_i(T, \mu) \) are solutions to the system of transcendental equations

\[ p_i(T, \mu) = p_i^0(T, \mu_i) \exp \left( -\sum_{j=1}^{f} \frac{\tilde{b}_{ij}n_j}{T} \right). \] (120)

We can partition pressure (115) in our approach into sum of “partial” pressures as

\[ p(T, \mu) = \sum_{i=1}^{f} p_i(T, \mu), \] (121)

\[ p_i(T, \mu) = \frac{Tn_i}{1 - \sum_{j=1}^{f} \tilde{b}_{ji}n_j}. \] (122)

Substituting (119) into numerator of (122) we get

\[ p_i(T, \mu) = p_i^0(T, \mu_i) \exp \left( -\sum_{j=1}^{f} \frac{\tilde{b}_{ij}n_j}{T} \right), \] (123)

where we use that \( p_i^0(T, \mu_i) = T n_i^0(T, \mu_i) \). Using (122) one can rewrite exponent in (123) to get

\[ p_i(T, \mu) = p_i^0(T, \mu_i) \exp \left[ -\sum_{j=1}^{f} \tilde{b}_{ij}p_j(T, \mu)/T \right] \] (124)

which coincides with (120).
VI. MEAN FIELD APPROACH IN HADRON-RESONANCE GAS MODEL

Statistical models are very successful in description of wide range of data on mean hadron multiplicities in various heavy-ion collision experiments. The most well-known such statistical model is the ideal hadron-resonance gas (I-HRG) model formulated in the grand canonical ensemble which represents a statistical system of non-interacting hadrons and resonances. The pressure in such system is given by

$$p = \sum_{i=1}^{f} p_i(T, \mu_i) = \sum_{i=1}^{f} \frac{g_i}{6\pi^2} \int_{0}^{\infty} \frac{k^4 \, dk}{\sqrt{k^2 + m_i^2}} \left[ \exp \left( \frac{\sqrt{k^2 + m_i^2} - \mu_i}{T} \right) + a_i \right]^{-1}, \quad (125)$$

where $g_i$ is the degeneracy of hadron species $i$, $T$ is temperature, $a_i = -1$ for bosons, $+1$ for fermions and 0 for Boltzmann approximation. The chemical potential $\mu_i$ consists of contributions from baryon, strangeness and electric charge chemical potentials:

$$\mu_i = B_i \mu_B + S_i \mu_S + Q_i \mu_Q. \quad (126)$$

The particle density of hadron species $i$ reads as

$$n_i(T, \mu_i) = \frac{g_i}{2\pi^2} \int_{0}^{\infty} k^2 \, dk \left[ \exp \left( \frac{\sqrt{k^2 + m_i^2} - \mu_i + U_i(n, T)}{T} \right) + a_i \right]^{-1}. \quad (127)$$

Considering $T$ and $\mu_B$ as free parameters and properly taking into account decays of unstable hadrons one can fit multiplicity ratios measured in various heavy-ion collision experiments. Chemical potentials $\mu_S$ and $\mu_Q$ are usually expressed as functions of $T$ and $\mu_B$ by fixing the total net strangeness $\langle S \rangle = 0$ and electric to baryon charge ratio $\langle Q \rangle / \langle B \rangle = Z/A \approx 0.4$.

In the mean field hadron resonance gas (MF-HRG) such quantities as excess pressure $P^{\text{ex}}(n, T)$ and mean fields $U_i(n, T)$ are introduced and expressions (125) and (127) are modified as

$$p = \sum_{i=1}^{f} \frac{g_i}{6\pi^2} \int_{0}^{\infty} \frac{k^4 \, dk}{\sqrt{k^2 + m_i^2}} \left[ \exp \left( \frac{\sqrt{k^2 + m_i^2} - \mu_i + U_i(n, T)}{T} \right) + a_i \right]^{-1} + P^{\text{ex}}(n, T),$$

$$n_i = \frac{g_i}{2\pi^2} \int_{0}^{\infty} k^2 \, dk \left[ \exp \left( \frac{\sqrt{k^2 + m_i^2} - \mu_i + U_i(n, T)}{T} \right) + a_i \right]^{-1} = \frac{g_i}{2\pi^2} \int_{0}^{\infty} k^2 \, dk \, f_i(k; n, T, \mu_i). \quad (128)$$

In our calculations we include all known hadrons and resonances (along with antiparticles) with masses up to 2.3 GeV and consisting of u, d or s quarks. The finite width of
resonances and their Breit-Wigner form is taken into account in the similar way as in THERMUS package [31], by making the following modification of the integration of distribution function:

\[
\int d^3 k f_i(k; n, T, \mu_i) \rightarrow \int d\tilde{m} \int d^3 k \frac{1}{\pi} \frac{2 \tilde{m} m_i \Gamma_i}{(\tilde{m}^2 - m_i^2)^2 + m_i^2 \Gamma_i^2} f_i(k; n, T, \mu_i).
\] (129)

In order to perform calculations one needs to solve self-consistently a system of transcendental equations for hadron densities (128). In the present study we include effects of excluded-volume in the framework of the mean-field approach and we consider only the case when the repulsive interaction between all kinds of hadrons is the same, i.e. all hadrons have the same hard-core radius \( r \). In this case all mean fields coincide \( U_i \equiv U \) and depend only on the total density of all hadrons, \( n = \sum_i n_i \). Therefore, only single transcendental equation for total density \( n \) needs to be solved, which is done using the secant method.

We consider three different excluded-volume parametrizations for mean field: a) The model of directly excluded-volume:

\[
P^{\text{ex}}(n, T) = -\frac{T}{2v_0} \ln(1 - 2v_0 n) - nT, \quad U(n, T) = -T \ln(1 - 2v_0 n),
\] (130) (131)

where \( v_0 \) is four times the intrinsic volume of hadron.

b) The van der Waals excluded-volume procedure:

\[
P^{\text{ex}}(n, T) = Tn \frac{v_0 n}{1 - v_0 n},
\] (132)

\[
U(n, T) = T \frac{v_0 n}{1 - v_0 n} - T \ln(1 - v_0 n).
\] (133)

c) The Carnahan-Starling model:

\[
P^{\text{ex}}(n, T) = Tn \frac{v_0 n - (v_0 n)^2/8}{(1 - v_0 n/4)^3},
\] (134)

\[
U(n, T) = -3T \left[ 1 - \frac{1 - v_0 n/12}{(1 - v_0 n/4)^3} \right].
\] (135)

We note that all three cases are consistent with second-order virial expansion for a gas of hard spheres.

We would like to study properties of a hadron-resonance gas present at the stage of a chemical freeze-out in heavy-ion collisions. As a result of analysis of particle multiplicities
at various heavy-ion collision experiments the collision energy dependence of temperature and baryochemical potential at the chemical freeze-out can be parametrized as \[32\]

\[
T(\sqrt{s_{NN}}) = 0.166 \text{GeV} - 0.139 \text{GeV}^{-1} \mu_B^2 - 0.053 \text{GeV}^{-3} \mu_B^4, \tag{136}
\]

\[
\mu_B(\sqrt{s_{NN}}) = \frac{1.308 \text{GeV}}{1 + 0.273 \text{GeV}^{-1} \sqrt{s_{NN}}}. \tag{137}
\]

In order to account for incomplete equilibration in the strangeness sector a suppression factor $\gamma_S$ is usually introduced in the distribution function and its energy dependence parametrization reads as \[33\]

\[
\gamma_S(\sqrt{s_{NN}}) = 1 - 0.396 \exp\left(-1.23 \frac{T}{\mu_B}\right). \tag{138}
\]

This factor is introduced into $f_i(k; n, T, \mu_i)$ in expression for density and pressure as

\[
f_i(k; n, T, \mu_i) = \left[\gamma_S^{-|s_i|} \exp\left(\frac{\sqrt{k^2 + m_i^2} - \mu_i + U_i(n, T)}{T}\right) + a_i\right]^{-1}, \tag{139}
\]

where $|s_i|$ is the absolute strangeness content for hadron species $i$, i.e. $|s_i|$ is the total number of strange quarks and anti-quarks in $i$-th hadron. The results of the calculations for energy dependence of the net-baryon density on the center-of-mass collision energy, $\sqrt{s_{NN}}$, is presented in Fig. 2. We note that results for the van der Waals excluded model are consistent with those in Ref. \[34\] obtained with the help of a THERMUS package. The inclusion of a hard-core repulsion results in a shift of a maximum in net-baryon density energy dependence to the region of lower collision energies. For instance, for a hard-core radius of 0.5 fm the maximum is shifted from $\sqrt{s_{NN}} \simeq 8$ GeV to $\sqrt{s_{NN}} \simeq 6$ GeV, which is firmly in the range of a future Compressed Baryonic Matter (CBM) experiment at the Facility for Antiproton and Ion Research (FAIR).

The energy dependence of the mean field and excess pressure is depicted in Fig. 3. It is seen that they both increase with collision energy, and the difference between the considered excluded-volume procedures increases with collision energy.

The effective single particle proper volume $\tilde{v}_0$ generated by the repulsive mean field can be calculated using Eq. (75) and is plotted in Fig. 4. We normalize the value of $\tilde{v}_0$ by the intrinsic particle volume $v_{in} = 4\pi r^3/3$ and we use $r = 0.5$ fm. It is seen that for the model of directly excluded-volume the value of $\tilde{v}_0$ is fixed at $\tilde{v}_0 = 8v_{in}$, which can be interpreted as the volume around the center of a spherical particle, inside of which no other center of a spherical particle can be located if the spheres are non-penetrable. For the more realistic
Figure 2: (Color online) Energy dependence of the net-baryon density calculated in the mean field approach for different values of hadron hard core radii using three different excluded-volume procedures: model of directly excluded-volume (dotted line), van der Waals equation of state (dash-dotted line), and Carnahan-Starling equation of state (solid line).

Figure 3: (Color online) Energy dependence of (a) mean field $U$ and (b) excess pressure $P_{ex}$ calculated for different hadron hard core radii in the mean field approach using three different excluded-volume procedures: model of directly excluded-volume (dotted line), van der Waals equation of state (dash-dotted line), and Carnahan-Starling equation of state (solid line).

As was discussed in Sec. IV, the factor $\vartheta(n) = e^{-U(n,T)/T}$ quantifies the suppression of hard sphere equations of state such as van der Waals and Carnahan-Starling we have that $\tilde{v}_0 < 8v_{in}$.
densities and of quantum statistics compared to the case of ideal gas. We plot the energy dependence of this quantity for the hard-core radius of 0.5 fm in Fig. 5. It is seen that, even at a moderate value of $r = 0.5$ fm the densities and also the quantum statistics become significantly suppressed, especially at higher collisions energies. These results further indicate that, in most cases, using just Boltzmann approximation is sufficient for the studies of chemical freeze-out in heavy-ion collisions.

We would like to investigate the effects of the size of the hard-core hadron radii $r$ that we use in our study. In Fig. 6 we plot the dependence of the net-baryon density on the value of $r$ for a fixed collision energy of $\sqrt{s_{NN}} = 8$ GeV, which corresponds to the following values of parameters: $T = 141$ MeV, $\mu_B = 411$ MeV and $\gamma_S = 0.74$. It is seen that all three considered procedures give very similar results for moderate values of hadron radii ($r \leq 0.5$ fm) but the relative difference for the net-baryon density given by different excluded-volume approaches increases with value of $r$ and reaches about $20 - 25\%$ at $r \gtrsim 0.9$ fm. These results show that the choice of excluded-volume procedure becomes important in the studies of chemical freeze-out if one considers large values of hard-core hadron radii and in this case the usual
van der Waals procedure may become quite inaccurate. Whether a choice of hadrons of large radii is appropriate is another matter though. If we are to interpret our system as a gas of hard spheres then, evidently, Carnahan-Starling model gives the most accurate results among the presented approaches.

VII. SUMMARY AND CONCLUSIONS

In the presented mean-field theory we express interactions in a multi-component gas in terms of mean fields, $U_i(n, T)$, and the excess pressure, $P^{ex}(n, T)$, and this approach provides a prescription to conveniently generalize known phenomenological models of equation-of-state to the case of a system with variable number of particles in the grand canonical formulation, which is of vital significance for relativistic systems.

In the present work we formulated three different excluded-volume procedures for the mixture of particles with the same hard-core radii in terms of mean-fields: the directly excluded-volume, the van der Waals equation-of-state, and the Carnahan-Starling equation-of-state. The mean fields in an excluded-volume models are linearly dependent on temperature and we show that such kind of fields generate an effective self-volume of a particle. As
Figure 6: (Color online) Dependence of the net-baryon density on the value of the hard-core hadron radius. Calculations are done for values $T = 141$ MeV, $\mu_B = 411$ MeV and $\gamma_S = 0.74$, which correspond to the collision energy of $\sqrt{s_{NN}} = 8$ GeV.

As a result, the expressions for the total number of particles and the total energy are reduced to the case of ideal gas, but with total volume $V$ and the quantum statistics parameter $a$ ($a = +1$ for Fermi and $a = -1$ for Bose statistics) are suppressed by a density-dependent factor of $\vartheta(n) = \exp(-U(T, n)/T)$.

The three different excluded-volume procedures are then used in the hadron-resonance gas model for studies of the collision energy dependence of various parameters at chemical freeze-out in heavy-ion collisions. In all three cases, the inclusion of hard-core repulsion results in a shift of the net-baryon maximum location to lower collision energies, a result also reported earlier [34]. The calculations also show that the differences between the three procedures stay rather small at moderate values of hadron radii ($r \lesssim 0.5$ fm) but may become significant at large enough values of hadron radii ($r \gtrsim 0.9$ fm), where the relative differences for net-baryon density may reach about 20 − 25%. This indicates that a choice of the model for repulsive interactions may be significant when one considers hadrons of large size, and if one interprets hadrons as rigid spheres, then the Carnahan-Starling procedure is the most accurate of the three considered in our work. The effects of relativistic contraction would inevitably distort the picture of rigid spheres, however, this effect is expected to be small for a hadron gas. Since the temperature of a hadron gas is significantly smaller than masses
of all hadrons except pions, than this effect may be significant only for thermal pions. Since
the majority of pions come from resonance decays, which again have a large mass compared
to temperature, than this effect can be expected to stay small even for pions in general.

The presented mean-field approach allows one to consider as well a multi-component sys-
tem of particles of different sizes, and, consequently, different interactions between different
kinds of particles. We formulate two different van-der-Waals-like excluded-volume proce-
dures for this case, which are shown to be consistent with previously proposed formulations
in the Boltzmann limit. In this multi-component case one then needs to solve numerically
a system of transcendental equations for densities of all considered particle species for any
given values of temperature and chemical potentials.

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