WEAK SOLUTIONS FOR SINGULAR MULTIPLICATIVE SDES VIA REGULARIZATION BY NOISE

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Abstract. We study multiplicative SDEs perturbed by an additive fractional Brownian motion on another probability space. Provided the Hurst parameter is chosen in a specified regime, we establish existence of probabilistically weak solutions to the SDE if the measurable diffusion coefficient merely satisfies an integrability condition. In particular, this allows to consider certain singular diffusion coefficients.

1. Introduction

Consider the classical problem
\[ dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0 \in \mathbb{R}^d, \]
where \( B \) is an \( n \)-dimensional Brownian motion. As it is well-known, provided the coefficients \( b: \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma: \mathbb{R}^d \to \mathbb{M}^{d \times n} \) are Lipschitz continuous the above stochastic differential equation admits a unique strong solution [RY99], [Oks03]. If \( b, \sigma \) are only continuous and satisfy a growth condition [Sko61], [HS12] or a Lyapunov condition [HS13], it is still possible to obtain weak solutions. This can be done in passing by a tightness argument: Mollifying \( b \) and \( \sigma \), we obtain in a first step a sequence \( (X_\epsilon)_\epsilon \) of associated solutions for which we can show tightness in appropriate spaces. By martingale arguments, it is then possible to identify all the terms with the corresponding limit in a second step, concluding the consideration.

In the present work, we establish that the conditions imposed in particular on the diffusion coefficient \( \sigma \) that ensure the existence of weak solutions can be further relaxed in the presence of an additional additive fractional Brownian motion. Our approach mainly relies on a robustification of the above two step strategy harnessing the regularizing effects due to local times of fractional Brownian motion and averaging operators [HP21], [CG16], [GG21]. For the reader’s convenience, let us briefly recall the definition of the latter and how they are employed to establish regularization by noise results for ordinary differential equations.

For a \( d \)-dimensional fractional Brownian motion \( w^H \) and some smooth nonlinearity \( b: \mathbb{R}^d \to \mathbb{R}^d \), consider the problem
\[ Y_t = y_0 + \int_0^t b(Y_s)ds - w^H_t. \]
By a simple transformation, we see that \( Y \) solves the above if and only if \( X = Y + w^H \) solves
\[ X_t = y_0 + \int_0^t b(X_s - w^H_s)ds. \tag{1.1} \]
Heuristically, as one expects the oscillations of the fractional Brownian motion to dominate the oscillations of $X$, we may conjecture that locally, i.e. for $0 < t - s \ll 1$, the above Lebesgue integral should behave as
\[
\int_s^t b(X_r - w_r^H)dr \simeq \int_s^t b(X_s - w_s^H)dr =: (T_{s,t}^{-w^H} b)(X_s),
\]
where $T_{s,t}^{-w^H} b$ denotes the averaging operator, which averages out the function $b$ by integrating along the highly fluctuating paths of $-w^H$. As it was remarked in [CG16], [GG21], this averaging gives rise to a quantifiable regularization effect in the case of fractional Brownian motion. Alternatively to studying the regularization effect on the averaging operator directly, Harang and Perkowski [HP21] observed that if $w^H$ admits a local time $L$, the above averaging operator may be rewritten thanks to the occupation times formula as
\[
(T_{s,t}^{-w^H} b)(X_s) = \int_s^t b(X_s - w_r^H)dr = (b * L_{s,t})(X_s).
\]
By Young’s inequality in Besov spaces [KS21], the spatial regularity of the averaging operator is therefore the sum of the regularity of $b$ and $L$, yielding also a quantifiable regularization. Let us remark at this point that while the results in [GG21] are sharper, allowing even to consider non-autonomous equations, the approach in [HP21] has the merit of being pathwise stable in the sense Theorem 2.1 (see also Remark 2.4). In our analysis, we will make use of both approaches and their respective merits.

In both approaches this gain of regularity on the level of averaging operators can subsequently be exploited to reconstruct the Lebesgue integral in (1.1) by “sewing together” the local approximations in (1.2) by means of the Sewing Lemma. Eventually, this allows for the study of an abstract non-linear Young problem
\[
X_t = y_0 + (IA^{X})_t,
\]
where $I$ denotes the Sewing operator and $A^X$ the germ
\[
A^X_{s,t} = (T_{s,t}^{-w^H} b)(X_s).
\]
Remark that due to the quantifiable regularization of the averaging operator, (1.3) might be well posed even in the case of continuous or merely distributional $b$, therefore establishing a regularization by noise phenomenon with respect to the ordinary differential equation.

In the following, we intend to study
\[
Y_t = x_0 + \int_0^t \sigma(Y_s)dB_s - w_t^H,
\]
or rather, again via the above transformation,
\[
X_t = x_0 + \int_0^t \sigma(X_s - w_s^H)dB_s.
\]
Since pathwise regularization by noise for singular drifts as sketched above is by now quite well understood, we set $b = 0$ for readability focusing on the diffusion term. However upon combining it with results from [GG21], [HP21], the same approach applies mutatis mutandis to (1.4) with a general, distributional drift $b$.

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1We refer to Appendix for the basic definitions for local times and the occupation times formula.

2For the readers convenience, we state the Sewing Lemma in the corresponding section of the Appendix.
Regularization by noise for multiplicative SDEs where the above Brownian motion $B$ is replaced by another fractional Brownian motion $\beta^{H'}$ of Hurst parameter $H' > 1/2$ was recently studied by Galeati and Harang [GH20]. The main idea in [GH20] is that due to a probabilistic result in [HL20], one can introduce an averaged multiplicative field

$$(\Gamma_t^{-wH})(x) := \int_0^t \sigma(x - w^H_s)dB^{H'}_s$$

whose regularity properties are inherited from the regularity properties of the averaged field $T^{-wH}\sigma$ introduced above. While very powerful allowing in particular to consider distributional $\sigma$, the above strategy does not apply in the present setting of classical Brownian motion where $H' = 1/2$. Refer to Remark 1.4 for a further comparison of our results with the ones obtained in [GH20].

In contrast with [GH20], our considerations are mainly based on classical Itô-calculus, which we further robustify thanks to the regularizing effect of the averaging operator $(T^{-wH}\text{Tr}(\sigma^2)) = \|\sigma\|_{HS}^2 * L_t$: Essentially, in going through the Itô-isometry, we exploit a regularization effect on the second moment of the diffusion term. A natural broad function space to consider that is also well behaved under the operator of “taking the square” is therefore $\sigma \in L^p_x$. Notice however that as $\text{Tr}(\sigma^*)$ has to be well-defined for this approach, we are unable to study regularization effects on distributional scales as in [GH20]. We obtain the following main result.

**Theorem 1.1.** Let $w^H$ be a $d$-dimensional fractional Brownian motion of Hurst parameter $H$ on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$. For $p \in [2, \infty)$, let $\sigma \in L^p_x$. Suppose that $d/p < 1$. Let $H$ be such that

$$2H < (1 + \frac{d}{(p/2) \wedge 4/3})^{-1}.$$ 

Then there exists a $\mathcal{F}^H$ measurable set $\Gamma(\sigma) \subset \Omega^H$ such that $\mathbb{P}^H(\Gamma(\sigma)) = 1$ and such that for any $\omega^H \in \Gamma(\sigma)$ the stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(X_s - w^H_s(\omega^H))dB_s$$

admits a weak solution on any bounded interval $[0, T]$.

**Remark 1.2.** Our result relies on the regularizing of the square of the Hilbert-Schmidt norm $\|\sigma\|_{HS}^2$. Thus, the same proof applies also to an infinite dimensional cylindrical Wiener process $B$ under an appropriate Hilbert-Schmidt assumption on $\sigma$, while keeping the dimension $d$ of the solution and the fractional Brownian motion finite.

Let us illustrate the above result with an example.

**Example 1.3.** Let $n = d$ and consider for $\gamma > 0$ and $K > 0$

$$\sigma(x) = \text{Id} * \frac{1}{|x|^\gamma} 1_{|x| \leq K}.$$ 

Note that then $\sigma \in L^p_x$ provided $\gamma < d/p$. The condition $d/p < 1$ appearing above means that we need to impose $\gamma < 1$. In order to ensure that we may choose $p \geq 2$, we need to
demand further $2 < d/\gamma$, which represents a constraint only in the case $d = 1$. Overall, we obtain that for

$$2H < \left(1 + \frac{d}{(d/(2\gamma)) \wedge 4/3}\right)^{-1}$$

the SDE

$$X_t = x_0 + \int_0^t \left(\frac{1}{|X_s - w^H_s|^{\gamma/2}} \mathbb{1}_{|X_s - w^H_s| \leq K}\right) dB_s$$

admits a weak solution.

**Remark 1.4** (Comparison with results in [GH20]). Let $H' \in (1/2, 1)$ and let $\beta^{H'}$ be a fractional Brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}')_t)$. Note that for $\sigma \in L^p_x$, combining [GH20, Theorem 1] with Theorem 2.2 below, one obtains that provided

$$H < H' - 1/2 \quad \text{or} \quad 2 + d/p,$$

the problem

$$X_t = x_0 + \int_0^t \sigma(X_s - w^{H'}_s) dB^{H'}_s$$

admits a unique solution in the sense that for any $x_0 \in \mathbb{R}^d$ any two pathwise solutions defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ are indistinguishable. There holds moreover path-by-path uniqueness (refer to [GH20, Definition 40]).

**Notation.** We endow the space of $d \times n$ matrices $\mathbb{M}^{d \times n}$ with the Hilbert-Schmidt norm

$$\|A\|_{HS} := \sqrt{\text{Tr}(A^*A)}.$$

For a function $\sigma : \mathbb{R}^d \to \mathbb{M}^{d \times n}$, we denote by $L^p_\mathcal{F}$ the Bochner space equipped with the norm

$$\|\sigma\|_{L^p_\mathcal{F}} = \left(\int_{\mathbb{R}^d} \|\sigma(x)\|_{HS}^p \, dx\right)^{1/p}.$$

2. Tightness

For the readers convenience we begin by citing two results on the regularity of averaging operators associated with fractional Brownian motion $T^{-w^H} f$. As we assume throughout $H < 1/d$, $\mathbb{P}^H$-almost any realization of the fractional Brownian motion admits a local time $L$ allowing to write

$$T^{-w^H} f : (t, x) \to \int_0^t f(x - w^H_s) ds = (f \ast L_t)(x).$$

The results cited represent a slight adaptation to our purposes. In particular, Theorem 2.2 provides a more quantitative statement on the Hölder-continuity of the averaging operator than provided in the reference, which is however immediate from the proof of [GG21, Theorem 4].

**Theorem 2.1** (Regularity of averaging operators I, [HP21, Theorem 3.4]). Let $w^H$ be $d$-dimensional fractional Brownian motion of Hurst parameter $H$ on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$ such that $H < 1/d$ and let $p \in [1, \infty)$. Then there exists a $\mathcal{F}^H$ measurable set $\Gamma \subset \Omega^H$ such that $\mathbb{P}^H(\Gamma) = 1$ and such that for any $\omega^H \in \Gamma$, $w^H(\omega^H)$ admits a local time $L$ and for any

$$\lambda < 1/(2H) - d/(p \wedge 2), \quad \gamma < 1 - (\lambda + d/2)H$$

...
we have $T^{-w^H} f \in C^\lambda_x$ provided $\lambda \in \mathbb{R}^+ \backslash \mathbb{N}$ and $f \in L^p_x$. Moreover we have the stability property

$$
\left\| T^{-w^H} (f_1 - f_2) \right\|_{C^\lambda_x} \lesssim \| f_1 - f_2 \|_{L^p_x}.
$$

(2.1)

**Theorem 2.2** (Regularity of averaging operators II, [GG21, Theorem 4]). Let $w^H$ be $d$-dimensional fractional Brownian motion of Hurst parameter $H$ on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$. For $p \in [1, \infty)$ let $f \in L^p_x$. Then there exists a $\mathcal{F}^H$ measurable set $\Gamma(f) \subset \Omega^H$ such that

$$
\mathbb{P}^H(\Gamma(f)) = 1 \quad \text{and such that for any } \omega^H \in \Gamma(f) \text{ and }
$$

$$
\lambda < 1/(2H) - d/p, \quad \gamma < 1 - (\lambda + d/p)H
$$

we have $T^{-w^H} f \in C^\lambda_x$ provided $\lambda \in \mathbb{R}^+ \backslash \mathbb{N}$. Moreover we have the stability property

$$
\mathbb{E}^H \left[ \left\| T^{-w^H} (f_1 - f_2) \right\|_{C^\lambda_x}^{2k} \right] \lesssim \| f_1 - f_2 \|_{L^p_x}^{2k}.
$$

(2.2)

**Remark 2.3.** Since we use a similar argumentation throughout the paper to get various bounds on the parameters, let us outline how the conditions on parameters in Theorem 2.2 are obtained from [GG21, Theorem 4]. We apply [GG21, Theorem 4] with $s = 0$, $q = \infty$ and $p = \lambda + m$ and use the Sobolev embedding for Bessel potential spaces $L^{\lambda + m, p} \subset C^\lambda_x$ which requires $m = d/p + \varepsilon$ for some $\varepsilon > 0$. Then the condition in [GG21, Theorem 4] reads

$$
H(\lambda + m) < 1/2
$$

hence

$$
\lambda < 1/(2H) - d/p - \varepsilon.
$$

(2.3)

Therefore, whenever $\lambda$ satisfies the condition in Theorem 2.2, there is an $\varepsilon$ so that the condition (2.3) is satisfied for some $\varepsilon$.

**Remark 2.4.** Note that Theorem 2.1 is weaker than Theorem 2.2 in terms of the regularization effect. However, it allows for the pathwise stability property (2.1), compared to the stability result (2.2) that employs moments and thus implicitly necessitates some measurability in $\omega^H$. In particular, the zero-set outside of which Theorem 2.1 holds is independent of the considered function $f$, whereas in Theorem 2.2 it depends on $f$. As we will lose measurability in $\omega^H$ by going through a tightness argument, we resort to Theorem 2.2 for the most part of our analysis.

Throughout the remainder of the paper, let us fix some realization $w^H(\omega^H)$ of a fractional Brownian motion on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$ that permits to employ the regularity statements of Theorems 2.1 and 2.2 for an accordingly prescribed Hurst parameter. More precisely, by Theorems 2.1 and 2.2 we find a $\Gamma(\sigma) \subset \Omega^H$ such that $\mathbb{P}^H(\Gamma(\sigma)) = 1$ and such that for any $\omega^H \in \Gamma(\sigma)$ the averaging operator enjoys the cited regularity.

**Lemma 2.5** (A priori bounds for solutions to mollified problem). Let $p \in [2, \infty)$ and let $\sigma : \mathbb{R}^d \to \mathbb{M}^{d \times n}$ satisfy $\sigma \in L^p_x$. Let $\sigma_\varepsilon$ be a mollification of some smooth cut-off of $\sigma$, i.e. for a sequence of mollifiers $(\rho^\varepsilon)_\varepsilon$ and for $\varphi^\varepsilon \in C_0^\infty$ with $\text{supp}(\varphi^\varepsilon) \subset B_0(1/\varepsilon)$, set $\sigma_\varepsilon = \rho^\varepsilon \ast (\varphi^\varepsilon \cdot \sigma)$. Let $B^\varepsilon$ be an $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon, (\mathcal{F}_t^\varepsilon))$ Brownian motion. Suppose that

$$
2H < (1 + \frac{d}{(p/2)^{\lambda/(4\lambda)}})^{-1}
$$

Let $X^\varepsilon$ be the unique strong solution to the problem

$$
X_t^\varepsilon = x_0 + \int_0^t \sigma_\varepsilon(X_s^\varepsilon - w^H(\omega^H))dB^\varepsilon_s.
$$
Then for any \( m \geq 4 \) and \( \gamma_0 < 1 - Hd/2 \) we have

\[
\sup_{s \neq t \in [0, T]} \mathbb{E}|X^\epsilon_{s,t}|^m \leq \frac{\mathbb{E}|X^\epsilon_{s,t}|^m}{|t-s|^{m\gamma_0/2}} < \infty
\]

and

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |X^\epsilon_t|^m \right] < \infty
\]

uniformly in \( \epsilon > 0 \).

**Proof.** Let \( m \geq 4 \). Note that by the Burkholder-Davis-Gundy inequality, we have that for \( \epsilon > 0 \) fixed

\[
\mathbb{E}[|X^\epsilon_{s,t}|^m] \lesssim \mathbb{E}\left[ \left( \int_s^t \|\sigma^\epsilon(X^\epsilon_r - w^H_r)\|_{HS}^2 \, dr \right)^{m/2} \right]
\]

\[
\lesssim \max_{1 \leq i,j \leq d} \|\sigma^{ij}_\epsilon\|_{L^\infty}|t-s|^{m/2},
\]

i.e. the constant

\[
c^m_{m,\epsilon,\gamma_0} := \sup_{s \neq t \in [0, T]} \mathbb{E}[|X^\epsilon_{s,t}|^m] \leq \frac{\mathbb{E}|X^\epsilon_{s,t}|^m}{|t-s|^{m\gamma_0/2}} < \infty
\]

is finite for any fixed \( \epsilon > 0 \) and \( \gamma_0 < 1 \) but a priori diverging in \( \epsilon \). For \( \gamma_0 < 1 - Hd/2 \) we now use the stochastic Sewing Lemma [Le20] (see also Lemma 4.4) to give an alternative definition to the above expression that allows to show that \( c^m_{m,\epsilon,\gamma_0} \) is uniformly bounded in \( \epsilon > 0 \). Define the germ

\[
A^\epsilon_{s,t} := (\|\sigma^\epsilon\|_{HS}^2 \ast L_{s,t})(X^\epsilon_s) = (T_{s,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2)(X^\epsilon_s),
\]

where by \( \|\sigma^\epsilon\|_{HS}^2 \) we understand the mapping \( x \to \|\sigma^\epsilon(x)\|_{HS}^2 \), which by assumption is in \( L^{P/2}_{x,\epsilon} \) uniformly in \( \epsilon \). We note that by Theorem 2.1 and the condition imposed on the Hurst parameter \( H \), we have for some small \( \delta > 0 \) and \( \gamma_0, \gamma_1 \in (1/2, 1) \) such that \( \gamma_0/2 + \gamma_1 > 1 \) and

\[
\left( (T_{s,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2) \right)_{C^{1+\delta}} \lesssim |t-s|^\gamma_0, \quad \left( (T_{s,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2) \right)_{C^{1+\delta}} \lesssim |t-s|^{\gamma_1}
\]

uniformly in \( \epsilon > 0 \). Indeed, remark that the condition \( 2H < (1 + \frac{d}{(\eta/2)^{1/2}})^{-1} \) ensures \( A^\epsilon_t \in C^{1+\delta} \) whereas \( 2H < (1 + \frac{d}{3^{1/3}})^{-1} \) ensures that \( \gamma_0, \gamma_1 \) may be chosen such that \( \gamma_0/2 + \gamma_1 > 1 \).

The maximal \( \gamma_0 \) we may choose for such \( H \) is precisely the one we give in the condition. Using the notation of the sewing Lemma introduced in the Appendix, we thus have

\[
|A^\epsilon_{s,t}| \lesssim \left( (T_{s,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2) \right)_{L^{2\epsilon}_{x,\epsilon}} \lesssim |t-s|^\gamma_0,
\]

as well as

\[
|(\delta A^\epsilon)_{s,u,t}| = \left( (T_{u,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2)(X^\epsilon_s) - (T_{u,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2)(X^\epsilon_u) \right)
\]

\[
\lesssim \left( (T_{u,t}^{-w^H} \|\sigma^\epsilon\|_{HS}^2) \right)_{C^1_{\epsilon}} |X^\epsilon_u - X^\epsilon_s|.
\]
We have by Jensens’ inequality that
\[
\left\| \mathbb{E}\left[(\delta A^s_t)^{m/2}(\mathcal{F}_s^\epsilon) \right] \right\|_{L^{m/2}(\Omega^\epsilon)} \leq \left( \mathbb{E}\left[(\delta A^s_t)^m(\mathcal{F}_s^\epsilon) \right] \right)^{1/m} \\
\leq \left( \mathbb{E}\left[|t-u|^\gamma_1 |X^u_t - X^s_t|^m \right] \right)^{1/m} \\
\leq |t-u|^\gamma_1 \mathbb{E}[|X^u_t - X^s_t|^m]^{1/m} \\
\leq c m, \epsilon, \gamma_1 |t-u|^\gamma_1 |u-s|^\gamma_0/2
\]
As by our assumption \(\gamma_1 + \gamma_0/2 > 1\), the above shows that \(A^\epsilon\) admits a stochastic sewing \(A^\epsilon\). The stochastic sewing Lemma [Le20] furthermore implies that
\[
\left\| A^\epsilon_{s,t} \right\|_{L^{m/2}(\Omega^\epsilon)} \leq \left\| A^\epsilon_{s,t} \right\|_{L^{m/2}(\Omega^\epsilon)} + c m, \epsilon, \gamma_0 |t-s|^\gamma_1 + \gamma_0/2 \lesssim |t-s|^\gamma_0 (1 + c m, \epsilon, \gamma_0 T^{\gamma_1 - \gamma_0/2}).
\]
Moreover, due to the regularity of \(\sigma_c\), we may identify the sewing with the Lebesgue integral in (2.4) in the sense that
\[
\left\| A^\epsilon_{s,t} \right\|_{L^{m/2}(\Omega^\epsilon)} = \mathbb{E}\left[ \int_s^t \left\| \sigma_c(X^\epsilon_r - w^H_r) \right\|_{H^2} dr \right]^{m/2}.
\]
We refer to Lemma [4.6] the Appendix for a proof of this equality. Overall, going back to (2.4), this yields
\[
\mathbb{E}[|X^\epsilon_{s,t}|^m] \lesssim \left\| A^\epsilon_{s,t} \right\|_{L^{m/2}(\Omega^\epsilon)} \lesssim |t-s|^m \mathbb{E}[|X^\epsilon_{s,t}|^m]
\]
and therefore
\[
c m, \epsilon, \gamma_0 \lesssim 1 + c m, \epsilon, \gamma_0
\]
yielding that indeed \(c m, \epsilon, \gamma_0\) is uniformly bounded in \(\epsilon\). \(\square\)

Note that by the Kolmogorov continuity theorem, we obtain the following.

**Corollary 2.6** (Tightness). We have for any \(\epsilon > 0\) that \(X^\epsilon \in C^{\gamma_0/2}([0,T], \mathbb{R}^d)\) and moreover, for any \(m \geq 1\) we have
\[
\mathbb{E}\left[ \sup_{t \neq s \in [0,T]} \left| \frac{X^\epsilon_{s,t}}{|t-s|^\gamma_0/2} \right|^m \right] < \infty
\]
uniformly in \(\epsilon > 0\). Hence the laws of and \((X^\epsilon, B^\epsilon)\) are tight\(^3\) on \(C^{\gamma_0/2}([0,T], \mathbb{R}^d) \times C^{\gamma_0/2}([0,T], \mathbb{R}^n)\).

By Prokhorov’s and Skorokhod’s theorem, we may therefore conclude that:

**Corollary 2.7** (Extraction of a convergent subsequence). There exists a probabilistic basis \((\Omega, \mathcal{F}, \mathbb{P})\), processes \((X^\epsilon, B^\epsilon)\) on the said basis whose laws coincide with those of \((X^\epsilon, B^\epsilon)\), and a process \((\bar{X}, \bar{B})\) such that
\[
(X^\epsilon, B^\epsilon) \rightarrow (\bar{X}, \bar{B})
\]

\(^3\)Remark that strictly speaking, we loose an (arbitrarily small) amount of regularity in applying the Kolmogorov continuity theorem and the compact embedding \(C^{\gamma_0/2} \rightarrow C^{\gamma_0/2-\delta}\) for \(\delta > 0\). However, as \(\gamma_0\) needs to satisfy a strict upper bound, we can incorporate these small losses of regularity into the strict inequality satisfied by \(\gamma_0\).
almost surely and that from which we conclude by Kolmogorov’s continuity theorem again that \( \bar{X} \) belongs to \( C^{7/2} \), \( \mathbb{P} \)-almost surely and that

\[
\mathbb{E}[|\bar{X}_{s,t}|^m] \lesssim \liminf_{\epsilon \to 0} \mathbb{E}[|\bar{X}_{s,t}^\epsilon|^m] \lesssim |t-s|^{m\gamma_0/2} \tag{2.6}
\]

from which we conclude by Kolmogorov’s continuity theorem again that \( \bar{X} \in C^{7/2} \), \( \mathbb{P} \)-almost surely and that

\[
\mathbb{E} \left[ \sup_{t \neq s \in [0,T]} \frac{|\bar{X}_{s,t}|}{|t-s|^{7/2}}^m \right] < \infty. \tag{2.7}
\]

### 3. Identification of the Limit

Assuming the conditions in Lemma 2.5 hold, we proceed by a stochastic compactness argument [BFH18], adapted to the present context. Throughout this section, let \( (e_j)_{j=1,...,d} \) be the canonical basis of \( \mathbb{R}^d \) and \( (f_i)_{i=1,...,n} \) be the canonical basis of \( \mathbb{R}^n \). We know that the processes

\[
t \to M_t^{j,\epsilon} := \langle \bar{X}^\epsilon_t, e_j \rangle - \langle x_0, e_j \rangle = \langle e_j, \int_0^t \sigma_\epsilon(\bar{X}^\epsilon_r - w^H_r)dB^\epsilon_r \rangle,
\]

\[
t \to (M_t^{j,\epsilon})^2 - \int_0^t |\sigma_\epsilon^*(\bar{X}^\epsilon_r - w^H_r)e_j|^2dr,
\]

\[
t \to M_t^{j,\epsilon}(\bar{B}^\epsilon_t, f_i) - \int_0^t \langle f_i, \sigma_\epsilon^*(\bar{X}^\epsilon_r - w^H_r)e_j \rangle dr
\]

are martingales with respect to \( (\bar{F}_t)_t \). Note that since \( \sigma_\epsilon \) is smooth, we have by Lemma 4.6 that

\[
\int_0^t |\sigma_\epsilon^*(\bar{X}_r^\epsilon - w^H_r)e_j|^2dr = (LA^{i,j})_t,
\]

\[
\int_0^t \langle f_i, \sigma_\epsilon^*(\bar{X}_r^\epsilon - w^H_r)e_j \rangle dr = (La^{i,j})_t,
\]

where

\[
A^{j,\epsilon}_{s,t} = (|\sigma_\epsilon^*e_j|^2 * L_a)(X^\epsilon_s)
\]

\[
a^{i,j,\epsilon}_{s,t} = (\sigma_\epsilon^{*i} * L_a)(X^\epsilon_s).
\]

Note that (3.1) being martingales is equivalent to having that for any bounded continuous functional \( \phi \) on \( C([0,s],\mathbb{R}^d) \times C([0,s],\mathbb{R}^n) \):

\[
\mathbb{E}[\phi(\bar{X}^\epsilon_t|_{[0,s]}, \bar{B}^\epsilon_t|_{[0,s]})(M_t^{j,\epsilon} - M_s^{j,\epsilon})] = 0,
\]

\[
\mathbb{E}[\phi(\bar{X}^\epsilon_t|_{[0,s]}, \bar{B}^\epsilon_t|_{[0,s]})((M_t^{j,\epsilon})^2 - (M_s^{j,\epsilon})^2 - (LA^{i,j})_{s,t})] = 0, \tag{3.2}
\]

\[
\mathbb{E}[\phi(\bar{X}^\epsilon_t|_{[0,s]}, \bar{B}^\epsilon_t|_{[0,s]})(M_t^{j,\epsilon}(\bar{B}^\epsilon_t, f_i) - M_s^{j,\epsilon}(\bar{B}^\epsilon_s, f_i) - (LA^{i,j})_{s,t})] = 0.
\]

We next intend to pass to the limit in (3.2). Note that by almost sure convergence and (2.5) all the terms with the exception of the appearing sewings converge due to Vitali’s convergence theorem. For the sewings \( LA^{j,\epsilon} \) and \( La^{i,j,\epsilon} \) we shall employ Lemma 4.5 adapted to the stochastic sewing setting.
Lemma 3.1. For \( s < t \in [0,T] \) and \( m \geq 1 \), we
\[
\| (\mathcal{I}A^{i,j})_{s,t} \|_{L^m(\Omega)} \to 0
\]
\[
\| (\mathcal{I}a^{i,j})_{s,t} \|_{L^m(\Omega)} \to 0,
\]
where
\[
A^{i,j}_{s,t} = (|\sigma^* e_j|^2 * L_{s,t})(\bar{X}_s)
\]
\[
a^{i,j}_{s,t} = (\sigma^* L_{s,t})(\bar{X}_s).
\]

Proof. The assertion follows from the stability of averaging operators cited in (2.1) and the fact that \( \sigma_\epsilon \to \sigma \) in \( L^p_\infty \) and \( \bar{X}_\epsilon \to \bar{X} \) in \( C([0,T],\mathbb{R}^d) \) as well as Lemma 4.5.

Indeed, observe that due to (2.5) we have that
\[
\| (\delta a^{i,j})_{s,u,t} \|_{L^m(\Omega)} = \| (\sigma_\epsilon^{i,j} * L_{u,t})(\bar{X}_u^\epsilon) - (\sigma_\epsilon^{i,j} * L_{u,t})(\bar{X}_u) \|_{L^m(\Omega)}
\]
\[
\leq \mathbb{E} \left[ \sup_{t \neq s \in [0,T]} \left| \frac{\bar{X}_{s,t}}{|t-s|^{\gamma_0/2}} \right|^m \right]^{1/m} \| \sigma_\epsilon \|_{L^p_t} \| t-s \|^{\gamma_0/2 + \gamma_1}
\]
\[
\leq \| \sigma \|_{L^p_t} \| t-s \|^{\gamma_0/2 + \gamma_1}
\]
uniformly in \( \epsilon > 0 \). Moreover, also by (2.5) and Vitali’s theorem, we have that actually \( X_\epsilon \to X \) in \( L^m(\Omega, C([0,T],\mathbb{R}^d)) \). We therefore observe that
\[
\| a^{i,j}_{s,t} - a^{i,j,\epsilon}_{s,t} \|_{L^m(\Omega)} \leq \| (\sigma^{i,j} * L_{s,t})(\bar{X}_s) - (\sigma^{i,j} * L_{s,t})(\bar{X}_s^\epsilon) \|_{L^m(\Omega)}
\]
\[
+ \| (\sigma^{i,j} * L_{s,t})(\bar{X}_s^\epsilon) - (\sigma^{i,j} * L_{s,t})(\bar{X}_s^\epsilon) \|_{L^m(\Omega)}
\]
\[
\leq |t-s|^{\gamma_0} \mathbb{E} \left[ \| \bar{X}_\epsilon - \bar{X} \|_{\infty}^m \right]^{1/m} + \| \sigma - \sigma_\epsilon \|_{L^p_t} |t-s|^{\gamma_0/2}.
\]
By Lemma 4.5, this implies that indeed \( \mathcal{I}a^{i,j,\epsilon} \to \mathcal{I}a^{i,j} \) in \( C_t^{\gamma_0/2,\gamma_1} L^m(\bar{\Omega}) \) and thus in particular the claim. Similar considerations hold for \( A^{i,j} \) by remarking that
\[
\left\| |\sigma^* e_j|^2 - |\sigma^* e_j|^2 \right\|_{L^p_{t/2}} \to 0.
\]

By the above Lemma, we may now pass to the limit in (3.2), obtaining:

Corollary 3.2. For \( i = 1, \ldots, n \) and \( j = 1, \ldots, d \), the processes
\[
t \to M_i^j := (e_j, X_t - x_0),
\]
\[
t \to (M_i^j)^2 - (\mathcal{I}A^j)_t,
\]
\[
t \to M_i^j (\tilde{B}_t, f_i) - (\mathcal{I}a^{i,j})_t,
\]
are martingales with respect to the filtration \( (\tilde{\mathcal{F}}_t)_t \).

In order to conclude that indeed \( M \) coincides with the corresponding stochastic integral in the limit, we need to extend [Hof13] Proposition A.1 to our sewing setting. That is precisely the content of the next Lemma 5.3, a technical part of which we moved into the subsequent Lemma 3.4 for the sake of readability.
Lemma 3.3. Suppose that for $i = 1, \ldots, n$ and $j = 1, \ldots, d$, the processes in $(3.3)$ are martingales. Then we have

$$M_t = \int_0^t \sigma(X_s - w_s^H)dB_s.$$ 

Proof. We show that for any $j = 1, \ldots, d$

$$\mathbb{E}[\langle M_t - \int_0^t \sigma(X_t - w_t^H)dB_t, e_j \rangle^2] = 0.$$ 

Let $\sigma_\epsilon$ be again mollifications multiplied with smooth cut off functions. Note that by definition (refer to Lemma 4.7),

$$\lim_{\epsilon \to 0} \mathbb{E}[\langle \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s - \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s, e_j \rangle^2] = 0.$$ 

Hence, it suffices to show

$$\mathbb{E}[\langle M_t - \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s, e_j \rangle^2] \to 0.$$ 

Note that by our previous findings, we have

$$\mathbb{E}[\langle M_t - \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s, e_j \rangle^2] = \mathbb{E}[\langle M_t, e_j \rangle^2] - \mathbb{E}[\langle \int_0^t \sigma_\epsilon(\bar{X}_s - w_s^H)dB_s, e_j \rangle^2] - 2\mathbb{E}[\langle M_t, e_j \rangle \langle \int_0^t \sigma_\epsilon(\bar{X}_s - w_s^H)dB_s, e_j \rangle].$$ 

where we recall that

$$A_{s,t} = (|\sigma^*e_j|^2 * L_{s,t})(\bar{X}_s) = ((\sigma^*)^{jj} * L_{s,t})(\bar{X}_s).$$ 

Note moreover that we have, again by identification of the Lebesgue integral with the Sewing in the smooth setting

$$\mathbb{E}[\langle M_t, e_j \rangle \langle \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s, e_j \rangle] = \mathbb{E}[\langle IA^j_t \rangle],$$ 

where

$$G_{s,t} = ((\sigma^*)^{jj} * L_{s,t})(\bar{X}_s).$$ 

By Lemma 3.4, we have

$$\mathbb{E}[\langle M_t, e_j \rangle \langle \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s, e_j \rangle] = \mathbb{E}[\langle IG^{j,\epsilon} \rangle],$$ 

where

$$G_{s,t} = ((\sigma^*)^{jj} * L_{s,t})(\bar{X}_s).$$ 

We therefore conclude that

$$\mathbb{E}[\langle M_t - \int_0^t \sigma(\bar{X}_s - w_s^H)dB_s, e_j \rangle^2] = \mathbb{E}[\langle IA^j_t \rangle] + \mathbb{E}[\langle IF^{j,\epsilon} \rangle, t] - 2\mathbb{E}[\langle IG^{j,\epsilon} \rangle, t]$$ 

where

$$H = \mathbb{E}[\langle IA^j_t \rangle] + \mathbb{E}[\langle IF^{j,\epsilon} \rangle, t] - 2\mathbb{E}[\langle IG^{j,\epsilon} \rangle, t].$$
and since the Sewings are stable under $\epsilon \to 0$ thanks to the stability property in Lemma 2.1 and Lemma 4.5, we may indeed conclude our claim that

$$M_t = \int_0^t \sigma(X_s - w^H_s)dB_s.$$ 

\[ \square \]

In summary, this concludes the proof of Theorem 1.1.

**Lemma 3.4.** Suppose the conditions of Lemma 2.5. Let $(e_j)_{j=1,...,d}$ be a canonical basis of $\mathbb{R}^d$. For $\epsilon > 0$ fixed and $d/p < 1$ we have for any $j = 1, \ldots, d$

$$\mathbb{E}[\langle M_t, e_j \rangle \langle \int_0^t \sigma_t(X_s - w^H_s)dB_s, e_j \rangle] = \mathbb{E}[\langle IG^{j,\epsilon}_s \rangle],$$

where

$$G^{j,\epsilon}_{s,t} = \langle (\sigma \sigma^*_\epsilon)^{ij} * L_{s,t} \rangle(X_s).$$

**Proof.** Note that $\mathbb{P}^H$ almost surely the process

$$t \to \sigma_t(X_t - w^H_t)$$

is progressively measurable and in $L^2(\Omega \times [0,T])$. Hence we can approximate it by elementary processes, i.e. take

$$\sigma_{\epsilon,N}(s) := \sum_{i=1}^N \sigma_{\epsilon}(X_{t_i} - w^H_{t_i})1_{[t_i,t_{i+1})}(s)$$

where $s = t_1 < t_2 < \cdots < t_{N+1} = t$. Let $(f_k)_{k=1,...,n}$ be the canonical basis of $\mathbb{R}^n$. Then

$$\mathbb{E}[\langle M_t - M_s, e_j \rangle \langle \int_s^t \sigma_{\epsilon,N}(r)dB_r, e_j \rangle | \mathcal{F}_s]$$

$$= \mathbb{E}\left[ \sum_i \langle M_{t_{i+1}} - M_{t_i}, e_j \rangle \langle (B_{t_{i+1}} - B_{t_i}), \sigma^*_\epsilon(X_{t_i} - w^H_{t_i})e_j \rangle | \mathcal{F}_s \right]$$

$$= \sum_i \sum_{k=1}^n \mathbb{E}[\langle f_k, \sigma^*_\epsilon(X_{t_i} - w^H_{t_i})e_j \rangle \mathbb{E}[\langle M_{t_{i+1}} + e_j \rangle \langle B_{t_{i+1}} - B_{t_i}, f_k \rangle] - \langle M_{t_i}, e_j \rangle \langle B_{t_i} + e_j, f_k \rangle | \mathcal{F}_{t_i} | \mathcal{F}_s]$$

$$= \sum_i \sum_{k=1}^n \mathbb{E}[\langle f_k, \sigma^*_\epsilon(X_{t_i} - w^H_{t_i})e_j \rangle \mathbb{E}[(Ia^{k,j})_{t_i,t_{i+1}} | \mathcal{F}_{t_i}] | \mathcal{F}_s]$$

$$= \mathbb{E}\left[ \sum_i \sum_{k=1}^n \langle f_k, \sigma^*_\epsilon(X_{t_i} - w^H_{t_i})e_j \rangle (Ia^{k,j})_{t_i,t_{i+1}} | \mathcal{F}_s \right].$$

Upon taking expectation we obtain

$$\mathbb{E}[\langle M_t - M_s, e_j \rangle \langle \int_s^t \sigma_{\epsilon,N}(r)dB_r, e_j \rangle]$$

$$= \sum_i \mathbb{E}\left[ \sum_{k=1}^n \langle f_k, \sigma^*_\epsilon(X_{t_i} - w^H_{t_i})e_j \rangle (\sigma^{jk} * L_{t_i,t_{i+1}})(X_{t_i}) \right]$$

$$+ \sum_i \mathbb{E}\left[ \sum_{k=1}^n \langle f_k, \sigma^*_\epsilon(X_{t_i} - w^H_{t_i})e_j \rangle \left((Ia^{k,j})_{t_i,t_{i+1}} - (\sigma^{jk} * L_{t_i,t_{i+1}})(X_{t_i}) \right) \right].$$
Remark that by the (classical) Sewing Lemma in combination with (2.7), the second sum above vanishes in the limit $N \to \infty$. Notice moreover that

$$
\lim_{N \to \infty} N \sum_{i=1}^{n} \langle f_k, \sigma^*_\epsilon(\bar{X}_{t_i} - w^H_{t_i})e_j \rangle (\sigma^{j_k} * L_{t_i, t_{i+1}})(\bar{X}_{t_i})
$$

if convergent, is by definition nothing but the sewing of the germ

$$
Z^{j, \epsilon}_{s,t} := \mathbb{E} \sum_{k=1}^{n} \langle f_k, \sigma^*_\epsilon(\bar{X}_s - w^H_s) e_j \rangle (\sigma^{j_k} * L_{s,t})(\bar{X}_s)
$$

Indeed, $Z^{j, \epsilon}$ admits a sewing as

$$
\delta Z^{j, \epsilon}_{s,u,t} = \mathbb{E} \sum_{k=1}^{n} \langle f_k, \sigma^*_\epsilon(\bar{X}_s - w^H_s) - \sigma^*_\epsilon(\bar{X}_u - w^H_u) e_j \rangle (\sigma^{j_k} * L_{u,t})(\bar{X}_s)
$$

$$
\leq |t - s|^{\gamma_0/2 + \gamma_1} + \sum_{k=1}^{n} |t - s|^{H \wedge \gamma_0/2} \left\| \sigma^{j_k} * L_{u,t} \right\|_{L^\infty}
$$

where in the above inequality, we exploited (2.6) and again $T^{-w^H_{u,t}} \sigma^{j_k} \in C^1_t C^1_x$, as well as the Lipschitz-continuity of $\sigma_\epsilon$ (this is where the mollification is required). Moreover, a competition between the term $\sigma_\epsilon(\bar{X}_s - w^H)$ - which is more regular in time, provided $H$ large - and $\left\| T^{-w^H_{u,t}} \sigma^{j_k} \right\|_{L^\infty}$ - which is more regular, provided $H$ small - occurred. By Theorem 2.2, we know that

$$
\left\| T^{-w^H_{u,t}} \sigma^{j_k} \right\|_{L^\infty} \lesssim |t - u|^{1 - d/p H - \eta}
$$

for any $\eta > 0$, as $\sigma^{j_k}$ is time independent. Note that it is at this point that we need to employ Theorem 2.2 in order to not lose too much time regularity of the averaging operator that would be otherwise problematic in the ‘regularity competition’ above. This is also the reason the set $\Gamma$ in our main result 1.1 depends on $\sigma$. As by assumption $d/p < 1$, we have that

$$
H + 1 - d/p H - \eta > 1
$$

for $\eta > 0$ sufficiently small. Moreover, by choosing the maximal $\gamma_0$ available in Lemma 2.5, namely $\gamma_0 = 1 - H d/2 - \eta$ for any $\eta > 0$, we have

$$
\frac{1}{2} (1 - H d/2 - \eta) + 1 - H d/p > 1
$$

which is satisfied under the condition

$$
2 H < (d/4 + d/p)^{-1}
$$

Remark however that for $d/p < 1$, this condition on the Hurst parameter is already satisfied under the assumptions of Lemma 2.5. Overall, we conclude that $Z^{j, \epsilon}_{s,t}$ admits a
sowing. Hence, we have
\[
\lim_{N \to \infty} \sum_{i} \mathbb{E} \left[ \sum_{k=1}^{N} (f_k, \sigma_k^* (X_{t_i} - w_{t_i}^H) e_j) (\sigma^{jk} \ast L_{t_i, t_{i+1}}) (X_{t_i}) \right] = (IZ^{j, \epsilon})_t.
\]
Finally, again by a mollification argument on \( \sigma \) similar to Lemma 4.6 one verifies that
\[
(IZ^{j, \epsilon})_t = (IG^{j, \epsilon})_t,
\]
concluding the statement.

4. Appendix

Local time and occupation times formula. We recall for the reader the basic concepts of occupation measures, local times and the occupation times formula. A comprehensive review paper on these topics is [GH80].

Definition 4.1. Let \( w : [0, T] \to \mathbb{R}^d \) be a measurable path. Then the occupation measure at time \( t \in [0, T] \), written \( \mu^w_t \) is the Borel measure on \( \mathbb{R}^d \) defined by
\[
\mu^w_t(A) := \lambda(\{s \in [0, t] : w_s \in A\}), \quad A \in \mathcal{B}(\mathbb{R}^d),
\]
where \( \lambda \) denotes the standard Lebesgue measure.

The occupation measure thus measures how much time the process \( w \) spends in certain Borel sets. Provided for any \( t \in [0, T] \), the measure is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^d \), we call the corresponding Radon-Nikodym derivative local time of the process \( w \):

Definition 4.2. Let \( w : [0, T] \to \mathbb{R}^d \) be a measurable path. Assume that there exists a measurable function \( L^w : [0, T] \times \mathbb{R}^d \to \mathbb{R}_+ \) such that
\[
\mu^w_t(A) = \int_A L^w_t(z) dz,
\]
for any \( A \in \mathcal{B}(\mathbb{R}^d) \) and \( t \in [0, T] \). Then we call \( L^w \) local time of \( w \).

Note that by the definition of the occupation measure, we have for any bounded measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) that
\[
\int_0^t f(w_s) ds = \int_{\mathbb{R}^d} f(z) \mu^w_t(dz).
\]
(4.1)
The above equation (4.1) is called occupation times formula. Remark that in particular, provided \( w \) admits a local time, we also have for any \( x \in \mathbb{R}^d \)
\[
\int_0^t f(x - w_s) ds = \int_{\mathbb{R}^d} f(x - z) \mu^w_t(dz) = \int_{\mathbb{R}^d} f(x - z) L^w_t(z) dz = (f \ast L^w_t)(z).
\]
(4.2)

The Sewing Lemma and its stochastic version. We recall the Sewing Lemma due to [Gnh04] (see also [FH14, Lemma 4.2]) as well as its stochastic version due to [Le20]. Let \( E \) be a Banach space, \([0, T]\) a given interval. Let \( \Delta_n \) denote the \( n \)-th simplex of \([0, T]\), i.e. \( \Delta_n := \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq T\} \). For a function \( A : \Delta_2 \to E \) define the mapping \( \delta A : \Delta_3 \to E \) via
\[
(\delta A)_{s, u, t} := A_{s, t} - A_{s, u} - A_{u, t}
\]
Provided $A_{s,t} = 0$ we say that for $\alpha, \beta > 0$ we have $A \in C^{\alpha,\beta}_2(E)$ if $\|A\|_{\alpha,\beta} < \infty$ where

$$\|A\|_{\alpha} := \sup_{(s,t) \in \Delta_2} \|A_{s,t}\|_{L^\alpha}, \quad \|\delta A\|_{\beta} := \sup_{(s,u,t) \in \Delta_3} \|\delta A_{s,u,t}\|_{L^\beta}, \quad \|A\|_{\alpha,\beta} := \|A\|_{\alpha} + \|\delta A\|_{\beta}$$

For a function $f : [0, T] \to E$, we note $f_{s,t} := f_t - f_s$

Moreover, if for any sequence $(P^n([s, t]))_n$ of partitions of $[s, t]$ whose mesh size goes to zero, the quantity

$$\lim_{n \to \infty} \sum_{[u,v] \in P^n([s, t])} A_{u,v}$$

converges to the same limit, we note

$$(\mathcal{I}A)_{s,t} := \lim_{n \to \infty} \sum_{[u,v] \in P^n([s, t])} A_{u,v}.$$

**Lemma 4.3** (Sewing). Let $0 < \alpha \leq 1 < \beta$. Then for any $A \in C^{\alpha,\beta}_2(E)$, $(\mathcal{I}A)$ is well defined. Moreover, denoting $(\mathcal{I}A)_t := (\mathcal{I}A)_{0,t}$, we have $(\mathcal{I}A) \in C^{\alpha}([0, T], E)$ and $(\mathcal{I}A)_0 = 0$ and for some constant $c > 0$ depending only on $\beta$ we have

$$\| (\mathcal{I}A)_t - (\mathcal{I}A)_s - A_{s,t} \|_E \leq c \|\delta A\|_{\beta} |t - s|^\beta.$$

We moreover cite the stochastic version of the Sewing Lemma due to [Le20]:

**Lemma 4.4** (Stochastic Sewing). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}$ be a complete filtered probability space. Let $p \geq 2$ and let $A : \Delta_2 \to \mathbb{R}^d$ be such that for any $s \leq t$, $A_{s,t}$ is $\mathcal{F}_t$ measurable and $A_{s,t} \in L^p(\Omega)$. Suppose moreover that for some $\epsilon_1, \epsilon_2 > 0$, we have

$$\|E[(\delta A)_{s,u,t} | \mathcal{F}_s]\|_{L^p(\Omega)} \leq \Gamma_1 (t-s)^{1+\epsilon_1}$$

$$\|\delta A_{s,u,t}\|_{L^p(\Omega)} \leq \Gamma_2 (t-s)^{1/2+\epsilon_2}.$$

Then there exists a unique (up to a modification) $(\mathcal{F}_t)_{t \in [0, T]}$-adapted and $L^p(\Omega)$ integrable stochastic process $(A_t)_{t \in [0, T]}$ with values in $\mathbb{R}^d$ such that $A_0 = 0$ and such that

$$\|A_{s,t} - A_{s,u} \|_{L^p(\Omega)} \lesssim \Gamma_1 (t-s)^{1+\epsilon_1} + \Gamma_2 (t-s)^{1/2+\epsilon_2}$$

$$\|E [A_{s,t} - A_{s,u} | \mathcal{F}_u]\|_{L^p(\Omega)} \lesssim \Gamma_2 |t-s|^{1+\epsilon_1}.$$  

Finally, for any sequence $(P^n([s, t]))_n$ of partitions of $[s, t]$, the sequence $(A^n_{s,t})_n$ defined by

$$A^n_{s,t} := \sum_{[u,v] \in P^n([s, t])} A_{u,v}$$

converges in $L^p(\Omega)$ to $A_{s,t}$.

Let us finally cite a result allowing to commute limits and sewings.

**Lemma 4.5** (Lemma A.2 [Gal21]). For $0 < \alpha \leq 1 < \beta$ and $E$ a Banach space, let $A \in C^{\alpha,\beta}_2(E)$ and $(A^n)_n \subset C^{\alpha,\beta}_2(E)$ such that for some $R > 0$ sup$_{n \in \mathbb{N}} \|\delta A^n\|_{\beta} \leq R$ and such that $\|A^n - A\|_{\alpha} \to 0$. Then

$$\|I(A - A^n)\|_{\alpha} \to 0.$$

Moreover, adapting the proof in [Gal21] in conjunction with the a priori bounds [14], the result canonically extends to the Stochastic Sewing setting: Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a
complete filtered probability space. Let \((A^n)\) be a sequence of \(\tilde{F}_t\) adapted processes \(A^n : \Delta_2 \to \mathbb{R}^d\) such that \(A_{s,t} \in L^p(\Omega)\). Suppose that the inequalities \((4.3)\) hold uniformly in \(n \in \mathbb{N}\) and suppose that \(\|A^n\|_{C^1_t L^p(\Omega)} \to 0\) for some \(\gamma \in (0, 1)\). Then, if \(A^n\) denotes the stochastic sewing of \(A^n\), we have that \(\|A^n\|_{C^\gamma_t L^p(\Omega)} \to 0\).

**Identifications of Riemann integrals with sewings.** In the following, we establish the identity of Riemann integrals with corresponding sewings in our setting for a particular example. Other identifications follow similarly and their proofs are thus omitted.

**Lemma 4.6** (Identification). Let \(\sigma : \mathbb{R}^d \to \mathbb{M}^{d \times n}\) be smooth and bounded. Let \(X\) be a stochastic process satisfying

\[
\sup_{s \neq t \in [0, T]} \mathbb{E}[|X_{s,t}|^m] < \infty
\]

for some \(\gamma > 1/2\). Moreover suppose \(2H < (1 + \frac{d}{(p/2)^{(4/3)}})^{-1}\). Then \(\mathbb{P}^H\) almost surely, the germ

\[
A^\epsilon_{s,t} = (\|\sigma\|_{HS}^2 * L_{s,t})(X_s)
\]

admits a stochastic sewing \(A^\epsilon\) and moreover, we have

\[
\left\|A^\epsilon_{0,t} - \left( \int_0^t \|\sigma(r, X_r - w^H_r)\|_{HS}^2 \, dr \right) \right\|_{L^{m/2}(\Omega)} = 0.
\]

**Proof.** As seen in the above proof to Lemma 2.5, the germ \(A^\epsilon\) does admit a stochastic sewing. Moreover, it can be easily seen that for fixed \(s \in [0, T]\), the germ \(B^\epsilon_{s,v} = \|\sigma(u, X_s - w^H_u)\|_{HS}^2 (v-u)\) admits a stochastic sewing, for which we have by definition

\[
(IB^\epsilon)_{s,t} = A^\epsilon_{s,t}
\]

understood as an equality in \(L^{m/2}\). Similarly, the germ \(C_{u,v} = \|\sigma(u, X_u - w^H_u)\|_{HS}^2 (v-u)\) admits a stochastic sewing, for which by definition, we have

\[
(IC)_{s,t} = \int_s^t \|\sigma(u, X_r - w^H_r)\|_{HS}^2 \, dr
\]

and moreover \(B^\epsilon_{s,t} = C_{s,t}\). We therefore conclude that

\[
\left\| \int_s^t \|\sigma(r, X_r - w^H_r)\|_{HS}^2 \, dr - A^\epsilon_{s,t} \right\|_{L^{m/2}(\Omega)} \lesssim \|(IC)_{s,t} - C_{s,t}\|_{L^{m/2}(\Omega)} + \|B^\epsilon_{s,t} - (IB^\epsilon)_{s,t}\|_{L^{m/2}(\Omega)} + \|A^\epsilon_{s,t} - A^\epsilon_{s,t}\|_{L^{m/2}(\Omega)} = O(|t-s|^{1+\delta}).
\]

Hence, the function

\[
t \to \left\| \int_0^t \|\sigma(r, X_r - w^H_r)\|_{HS}^2 \, dr - A^\epsilon_t \right\|_{L^{m/2}(\Omega)}
\]

is a constant allowing to conclude. \(\square\)
Extending the Ito integral in the presence of a perturbative fractional Brownian motion. In the following, we demonstrate how to extend the definition of the stochastic integral in the presence of a perturbative fractional Brownian motion, i.e., we define the stochastic integral
\[
\int_0^t \sigma(X_r - w_r^H) dB_r
\]
for \(\sigma \in L^p_x\). Note that for progressively measurable processes \(X\) such that
\[
\sup_{s \neq t \in [0,T]} E[|X_{s,t}|^m] |t - s|^{m\gamma_0/2} < \infty
\]
it is a priori not obvious why this stochastic integral should be well defined as the integrand is not an element of \(L^2(\Omega \times [0,T])\). However, note that due to the previous Lemma 4.6, we can pass by mollifications of \(\sigma\) and subsequently exploit the regularizing effect due to the associated averaging operator on the level of the Ito isometry.

Lemma 4.7. Let \(\sigma \in L^p_x\) and \(\sigma\) be any cut-off mollification. Let \(X\) and \(A^\epsilon\) be as in Lemma 4.6 above. There hold the robustified Ito isometry
\[
E \left[ \left( \int_0^t \sigma_{\epsilon}(X_r - w_r^H) dB_r \right)^2 \right] = \| A^\epsilon_T \|_{L^1(\Omega)}
\]
and for \(m \geq 4\) the Burkholder-Davis-Gundy inequality
\[
E \left[ \sup_{t \in [0,T]} \left( \int_0^t \sigma_{\epsilon}(X_r - w_r^H) dB_r \right)^m \right] \lesssim \| A^\epsilon_T \|_{L^{m/2}(\Omega)}
\]
\(\mathbb{P}^H\)-almost surely. In particular, the sequence \(\left( \int_0^t \sigma_{\epsilon}(X_r - w_r^H) dB_r \right)_{\epsilon}\) is \(\mathbb{P}^H\)-almost surely Cauchy in \(L^{m/2}(\Omega, C([0,T]))\), whose limit we denote by
\[
t \to I_t \sigma = \int_0^t \sigma(X_r - w_r^H) dB_r.
\]
The construction is independent of the cut-off mollification chosen and adapted to the filtration generated by \(B\). It is a martingale with respect to that filtration. Moreover, the so constructed integral is linear in the sense that for \(\sigma_1, \sigma_2\), we have
\[
I_t(\sigma_1 + \sigma_2) = I_t \sigma_1 + I_t \sigma_2.
\]

Proof. The above (4.5) and (4.6) are immediate consequences of the classical Ito isometry and Burkholder-Davis-Gundy inequality available in this setting as well as the previous Lemma 4.6. Moreover, for \(\epsilon, \epsilon' > 0\), we have similarly
\[
E \left[ \sup_{t \in [0,T]} \left( \int_0^t (\sigma_{\epsilon}(X_r - w_r^H) - \sigma_{\epsilon'}(X_r - w_r^H)) dB_r \right)^m \right] \lesssim \| A^{\epsilon, \epsilon'}_T \|_{L^{m/2}(\Omega)}
\]
where \(A^{\epsilon, \epsilon'}\) denotes the stochastic Sewing of the germ
\[
A^{\epsilon, \epsilon'}_{s,t} = (||\sigma_\epsilon - \sigma_{\epsilon'}||_{HS} + L_{s,t})(X_s)
\]
Again similarly to Lemma 2.5 one shows that
\[
\| A^{\epsilon, \epsilon'}_{s,t} \|_{L^{m/2}(\Omega)} \lesssim \| \sigma_\epsilon - \sigma_{\epsilon'} \|_{L^p} |t - s|^{\gamma_0/2}
\]
as well as
\[
\left\| \left( \delta A^{\epsilon, \epsilon'} \right)_{s,u,t} \right\|_{L^{m/2}(\Omega)} \lesssim \| \sigma_{\epsilon} - \sigma_{\epsilon'} \|_{L^p}(t-s)^{\gamma_0/2 + \gamma_1}.
\]

From the a priori bound \[4.4\] available in the stochastic sewing lemma and the fact that \( A^{\epsilon, \epsilon'}_0 = 0 \), we infer that
\[
\left\| A^{\epsilon, \epsilon'}_T \right\|_{L^{m/2}(\Omega)} \lesssim \left\| A^{\epsilon, \epsilon'} \right\|_{C^{\gamma_0/2}L^{m/2}(\Omega)} + \left\| \left( \delta A^{\epsilon, \epsilon'} \right) \right\|_{C^{\gamma_0/2+\gamma_1}L^{m/2}(\Omega)} \lesssim \| \sigma_{\epsilon} - \sigma_{\epsilon'} \|_{L^p}.
\]

We conclude thus that indeed the sequence \( \left( \int_0^\cdot \sigma_{\epsilon}(X_r - w^H_r)dB_r \right)_{\epsilon} \) is \( \mathbb{P}^H \)-almost surely Cauchy in \( L^{m/2}(\Omega, C([0, T])) \), allowing to define the corresponding limit as the extended stochastic integral. Remark moreover that this construction is independent of the sequence of cut-off mollifications chosen, which is immediate by replacing \( \sigma_{\epsilon'} \) by \( \sigma \) in the above considerations. Adaptedness follows from the fact that the sequence of approximations is adapted by classical Ito theory. Linearity follows from the fact that cut-off mollifications are linear, i.e. \( (\sigma_1 + \sigma_2)^\epsilon = \sigma_1^\epsilon + \sigma_2^\epsilon \) as well as the fact that the classical Ito integral is linear. Finally, the martingale property follows from Lemma 3.1 and Corollary 3.2 above.

\[\square\]

ACKNOWLEDGEMENT

The authors acknowledge support from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 949981)

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