Multi-time, multi-scale correlation functions in turbulence and in turbulent models

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Abstract

A multifractal-like representation for multi-time multi-scale velocity correlation in turbulence and dynamical turbulent models is proposed. The importance of subleading contributions to time correlations is highlighted. The fulfillment of the dynamical constraints due to the equations of motion is thoroughly discussed. The prediction stemming from this representation are tested within the framework of shell models for turbulence.

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1 Introduction

Turbulent flows are characterized by a highly chaotic and intermittent transfer of fluctuations from the stirring length, outer scale, $L_0$, down to the viscous dissipation length, inner scale, $l_d$. The Reynolds number defines the ratio between the outer and the inner scales: $L_0/l_d = Re^{3/4}$. We talk about fully developed turbulent flows in the limit $Re \to \infty$, in this limit it is safely assumed that there exists an inertial range of scales, $l_d \ll r \ll L_0$, where the time evolution feels only the non-linear terms of the Navier-Stokes eqs.

The highly chaotic and intermittent transfer of energy leads to non-trivial correlation among fluctuations of the velocity fields at different scales and at different time-delays [1].

The natural set of observable which one would like to control are the following:

$$C^{p,q}(r, R|t) = \langle \delta v_p^r(t) \cdot \delta v_q^R(0) \rangle$$

(1)

where $\delta v_r(t) = v(x + r, t) - v(x, t)$ and $l_d \ll r \ll R \ll L_0$. In (1) we have, for sake of simplicity, neglected the vectorial and tensorial dependencies in the velocity fields and velocity correlations respectively.

Some subclasses of the Multi-Scale Multi-Time (MSMT) correlation functions (1) have recently attracted the attention of many scientists [2, 3, 8, 5, 6]. By evaluating (1) with $r = R$, at changing $R$, and at zero-time delay, $t = 0$, we have the celebrated Structure Functions (SF) of order $q + p$. Further, we may also investigate Multi-Scale (MS) correlation functions when we have different lengths involved $r \neq R$ at zero delay, $t = 0$ as well as single-scale correlation functions (CF) by fixing $r = R$ at varying time delay $t$ etc...

Structure functions have been, so far, the most studied turbulent quantities (see [1] for a recent theoretical and experimental review). On the other hand, only recently some theoretical and experimental efforts have been done in order to understand the time properties of single scale (CF) correlations, $C^{p,q}(r, r|t)$ [3, 5] and the scaling properties of multi-scale correlations (MS) at zero time delay, $C^{p,q}(r, R|0)$ [2, 3, 5].

In this Paper, we propose and check a general phenomenological framework capable to capture all the above mentioned correlation functions and in agreement with the typical structure of non-linear terms of Navier-Stokes
In Section 2 the framework of the multifractal description of correlations is briefly recalled and critically examined. In Section 3 a representation for single scale time correlations is introduced, and its predictions are tested within the framework of shell models of turbulence. In Section 4 we deal with the most generic two-scales time correlation.

2 Background: the multifractal description of time correlations

One of the most important outcomes of experimental and theoretical analysis of turbulent flows is the spectacular ability of simple multifractal phenomenology [1, 7] to capture the leading behavior of structure functions and of multi-scale correlation functions at zero-time delays [2, 3, 5]. This may appear not surprising because, as far as time-delays are not concerned, one may expect that (many) different phenomenological descriptions may well reproduce scaling laws typical of (SF) and of (MS) functions: multifractals being just one of these descriptions. More striking were the recent findings [3] that multifractal phenomenology may easily be extended to the time-domain such as to give a precise prediction on the behavior of the time properties of single scale correlations. As soon as time enters in the game, one must ask consistency with the equation of motion: the major break-through was that one may write a time-multifractal description in agreement with the equation of motion

Let us now quickly enter in the details of previous findings [3] in order to clarify both the phenomenological framework and the notation that we will use in the following.

Let us remind that the multifractal (Parisi-Frisch) description of single-time correlation functions is based on the assumptions that inertial range statistics is fully determined by a cascade process conditioned to some large scale configuration:

\[ \delta v_r = W(r, R) \cdot \delta v_R \]  

An important remark is now in order: when we refer to time-properties of turbulent flows we always mean the time-properties of the velocity fields once the trivial sweeping effects of large scale on small scales is removed, for example by choosing to work in a quasi-Lagrangian reference frame [3].
where the fluctuating function $W(r, R)$ can be expressed in terms of a superposition of local scaling solution $W(r, R) \sim \left( \frac{r}{R} \right)^{h(x)}$ with a scaling exponent $h(x)$ which assumes different values $h$ in a class of interwoven fractal sets with fractal codimension $Z(h) = 3 - D(h)$. From this assumption one can write the expression for any structure functions of order $m$, which in our notation ($m = p + q$) becomes:

$$S^m(R) \equiv C^{p,q}(R, R|0) \sim \langle W(R, L_0)^m \rangle \langle U_0^m \rangle \equiv \langle U_0^m \rangle \int d\mu_{R,L_0}(h) \left( \frac{R}{L_0} \right)^{h m} \sim \left( \frac{R}{L_0} \right)^{\zeta(m)}$$

where we have introduced the shorthand notation $d\mu_{R,L_0}(h) \equiv dh \left( \frac{R}{L_0} \right)^{Z(h)}$ to define the probability of having a local exponent $h$ connecting fluctuations between scales $R$ and $L_0$. In (4) we have used a steepest descent estimate, in the limit $R/L_0 \to 0$, in order to define the intermittent scaling exponents $\zeta(m)$. Intensity of intermittency depends on the departure of the $\zeta(m)$ exponents from a linear behavior in $m$.

In order to extend this description to the time domain, Procaccia and coll. have proposed to consider that two velocity fluctuations, both at scale $R$ but separated by a time delay $t$, can be thought to be characterized by the same fragmentation process $W_{R,L_0}(t) \sim W_{R,L_0}(0)$ as long as the time separation $t$ is smaller than the “instantaneous” eddy-turn-over time of that scale, $\tau_R$, while they must be almost uncorrelated for times larger than $\tau_R$. Considering that the eddy-turn-over time at scale $R$ is itself a fluctuating quantity $\tau_R \sim R/(\delta v_R) \sim R^{1-h}$ we may write down (3):

$$C^{p,q}(R, R|t) \sim \int d\mu_{R,L_0}(h) \left( \frac{R}{L_0} \right)^{h(p+q)} F_{p,q} \left( \frac{t}{\tau_R} \right)$$

where the time-dependency is hidden in the function $f(x)$ which must be a smooth function of its argument (for example a decreasing exponential).

From (3) is it straightforward to realize that at zero-time separation we recover the usual SF representation. It is much more interesting to notice that (3) is also in agreement with the constraints imposed by the non-linear part of the Navier-Stokes eqs. Indeed, to make short a long story (see for a rigorous discussion) we may say that under the only hypothesis that non-linear terms are dominated by local interactions in the Fourier space...
we can safely assume that as far as power-law counting is concerned the
inertial terms of Navier-Stokes equations for the velocity difference \( \delta v_R \) can be estimated to be of the form:

\[
\partial_t \delta v_R(t) \sim O \left( \frac{(\delta v_R(t))^2}{R} \right)
\]  

and therefore we may check that:

\[
\partial_t C_{p,q}(R, R|t) \sim \int d\mu_{R,L_0}(h) \left( \frac{R}{L_0} \right)^{h(p+q)} (\tau_R)^{-1} F_{p,q}^t \left( \frac{t}{\tau_R} \right) \sim \frac{C_{p+1,q}(R, R|t)}{R}
\]  

where of course in the last relation there is hidden the famous closure-problem
of turbulence, now restated in term of the relation : \( \frac{d}{dt} F_{p,q}(t) \sim F_{p+1,q}(t) \). Let us therefore stress that we are “not solving turbulence” but just building up
a phenomenological framework where all the leading (and sub-leading, see
below) scaling properties are consistent with the constraints imposed by the
equation of motion.

In the following we shall show how the representation (5) must be im-
proved to encompass the most general multi-time multi-scale correlation
\( C_{p,q}(r, R|t) \).

3 Single scale time correlations

We shall first show in which respect the expression (3) may not be con-
sidered a satisfactory representation of single scale time correlation. The
first comment that can be raised about (3) is that it misses important
sub-leading terms which may completely spoil the long-time scaling be-
behaviour: indeed, the main hypothesis that correlation \( C_{p,q}(R, R|t) \) feels only
the eddy-turn-over time of the scale \( R \) is too strong. It is actually cor-
rect only when the correlation function has zero disconnected part, i.e. when
\( \lim_{t \to \infty} \langle \delta v^p_r(0) \rangle \langle \delta v^q_R(t) \rangle \equiv 0 \) which is certainly false in the most general case.
The problem is not only limited to the necessity of taking into account the
asymptotic mismatch to zero given by the disconnected terms – which would

\footnote{In order to really attack the NS equations in this framework one should dive into the
structure of the \( F_{pq} \)-functions in great detail: a problem which seems still to be far from
convergence.}
be a trivial modification of (5) – because as soon as the disconnected part is present the whole hierarchy of fluctuating eddy-turn-over times from the shortest, \( t_R \), up to the largest, \( t_{L_0} \), must be felt by the correlation.

Let us, for the sake of simplicity, introduce a hierarchical set of scales, \( l_n = 2^n L_0 \) with \( n = 0, \ldots, n_d \), which span the whole inertial range such that \( l_d = l_{n_d} = 2^{-n_d} L_0 \), and let us simplify the notation by taking \( L_0 = 1 \) and by writing \( u_n = \delta v_r \) in order to refer to a velocity fluctuation at scale \( r = l_n \).

The picture which will allow us to generalize the time-multifractal representation to the multi-time multi-scale case goes as follows.

For time-delays, \( t \sim t_m \), typical of the eddy-turn-over time of the \( m \)-th scale we may safely say that the two velocity fluctuations follow the same fragmentation process from the integral scale \( L_0 \) down to scale \( m \) while they follow two uncorrelated processes from scale \( m \) down to the smallest scale in the game, \( n \). In the multifractal language we must write that for time \( t = t_m \pm O(t_m) \) we have:

\[
\begin{align*}
    u_n(0) & \sim W_{n,m}^r(0) W_{m,0}(0) u_0(0) \sim \left( \frac{l_n}{l_m} \right)^{h'} \left( \frac{l_m}{l_n} \right)^h \\
    u_n(t) & \sim W_{n,m}^r(t) W_{m,0}(t) \sim W_{n,m}^r(t) W_{m,0}(0) u_0(0) \sim \left( \frac{l_n}{l_m} \right)^{h''} \left( \frac{l_m}{l_n} \right)^h 
\end{align*}
\]

where the exponents \( h, h', h'' \) are independent outcomes of the same probability distribution functions and where we have used the fact that in this time-window \( W_{m,0}(t) \sim \text{const.} \)

Apart from subtle further-time dependencies (see below) we should therefore conclude that for time \( t \sim t_m \) the correlation functions may be approximated as:

\[
    C_{p,q}^{n,n}(t_m) \sim \left\langle W_{n,m}^p \right\rangle \left\langle W_{n,m}^q \right\rangle \left\langle W_{m,0}^{p+q} \right\rangle 
\]

which must be considered the fusion-rules prediction for the time-dependent fragmentation process [2, 3, 4]. Let us notice that this proposal has already been presented in [5] and considered to express the leading term in the limit of large time delays \( t_m \to \infty \); here we want to refine the proposal made in [5] showing that by adding the proper time-dependencies it is possible to obtain a coherent description of the correlation functions for all time-delays. The expression (10) summarizes the idea that for time delay larger than \( t_m \)
but smaller then \( t_{m-1} \), velocity components with support on wavenumbers \( k < k_{m-1} \) did not have enough time to relax and therefore the local exponent, \( h \), which describes fluctuations on those scales must be the same for both fields. On the other hands, components with support on wavenumbers \( k > k_{m-1} \) have already decorrelated for \( t > t_{m-1} \) and therefore we must consider two independent scaling exponents \( h', h'' \) for describing fluctuations on these scales.

Adding up all this fluctuations, centered at different time-delays, we end with the following multifractal representation for \( C_{n,n}^{p,q}(l_{n}, l_{n}|t) \equiv C_{n,n}^{p,q}(t) \):

\[
C_{n,n}^{p,q}(t) = \int d\mu_{m,0}(h)d\mu_{m,m}(h_{1})d\mu_{n,m}(h_{2})l_{(q+p)\cdot h}F_{p,q}\left(\frac{t}{t_{n}}\right) +
\sum_{m=1}^{n-1} \int d\mu_{m,0}(h)d\mu_{m,m}(h_{1})d\mu_{n,m}(h_{2})l_{(q+p)\cdot h}\left(\frac{l_{n}}{t_{m}}\right)^{q_{h_{1}}}\left(\frac{l_{m}}{t_{m}}\right)^{p_{h_{2}}}f_{p,q}\left(\frac{t}{t_{m}}, \frac{t_{n}}{t_{m}}\right)
\]

Let us now spend a few words in order to motivate the previous expression. In the first row of (11) we have explicitly separated the only contribution we would have in the case of vanishing disconnected part. This term remains the leading contribution in the static limit \( (t = 0) \) also when disconnected parts are non-zero. About the new terms controlling the behavior of the correlation functions for larger time we still have at this stage the most general dependency from all times entering in the game \( t, t_{n}, t_{m} \). In practice one may guess a very simple functional form which is in agreement with all the above mentioned phenomenological requirements. In particular, we may simply assume that \( f_{p,q}\left(\frac{t}{t_{m}}, \frac{t_{n}}{t_{m}}\right) \equiv f_{p,q}\left(\frac{t}{t_{m}}\right) \) with \( f_{p,q}(x) \) being a function peaked at its argument \( x \sim 0(1) \) which must be exactly zero for \( x = 0 \) and different from zero only in a interval of width \( \delta x \sim O(1) \).

Let us now face the consistency of the representation (11) with the constraint imposed by the equations of motion. By applying a time-derivative to a correlation \( C_{n,n}^{p,q}(t) \) you produce a new correlation with by-definition zero-disconnected part, whose representation has thus no subleading term \( (f_{p,q} \equiv 0) \). When performing the time derivative on both sides of (11) it is evident that – in order to accomplish the dynamical constraints – all the derivatives of the subleading terms must sum to a zero contribution.

This is the first non-trivial result we have reached until now. If our representation (11) is correct, we claim that all eddy-turn-over times must be present in the general single-scale correlation functions but strong cancellations of
all sub-leading terms must take place whenever disconnected contributions
vanish.

Let us finally notice that for time delays larger then the eddy-turn-over
time of the integral scale we should add to the RHS of (11) the final expo-
ential decay toward the full disconnected term $\langle u_p^n \rangle \langle u_q^n \rangle$.
In order to check this representation we have performed some numerical in-
vestigation in a class of dynamical models of turbulence (shell models) [9].
Without entering in the details of this popular dynamical model for the tur-
bulent energy cascade let us only say that within this modeling the approx-
imation of local-interactions among velocity fluctuations at different scales
is exact and therefore no-sweeping effects are presents. This fact makes of
shell models the ideal framework where non-trivial temporal properties can
be investigated.
In order to test the dependency of (11) from the whole set of eddy-turn-
over times we show in Fig. 1 the correlation $C_{p,q}^{nn}(t)$ for two cases with and
without disconnected part. As it is evident, the correlation with a non-zero
disconnected part decays in a time-interval much longer then the charac-
tistic time of the shell $\tau_n$. This shows that it is not possible to associate a single
time-scale $\tau_n$ to the correlation functions of the form (11). In Fig. 2 we also
compare the correlation when one of the two observable is a time-de-
ervative with the correlation chosen such as to have the same dimensional properties
but without being an exact time-derivative. Also in this case the difference
is completely due to the absence (presence) of all sub-leading terms in the
former (latter).

3.1 Intermittent integral time-scales
In the case when the disconnected part of the time correlation is absent, in
the representation (11) all the subleading terms mutually cancel, leaving the
fully-connected contribution alone.
Under this condition and in presence of intermittency one expects anomalous
scaling behavior for the integral time-scales, $s^{p,q}(R)$, characterizing the mean
decorrelation time of fluctuations at scale $R$, defined as [10]:

$$s^{(p,q)}(R) = \int_0^\infty dt \frac{C_{p,q}(R, R|t)}{C_{p,q}(R, R|0)}$$  (12)
exploiting the multifractal representation \( s^{(p,q)}(R) \) it is easy to show that:

\[
s^{(p,q)}(R) \simeq \left( \frac{R}{L_0} \right)^{z(p+q)}
\]

where the exponents \( z(m) \) are fully determined in terms of the intermittent spatial scaling exponents: \( z(m) = 1 + \zeta(m - 1) - \zeta(m) \).

This prediction is in practice always very difficult to check: indeed full cancellation of the subleading terms requires an extremely long time span, and since the cancellations affect dramatically the convergence of the time integral, there is no chance of measuring with sufficient precision the \( z(m) \) exponents.

In order to bypass this problem we devised an alternative way to extract the integral times.

We introduce fluctuating decorrelation times at a scale \( R \), defined as the time interval \( T_i \) in which the instantaneous value of the correlation has changed by a fixed factor \( \lambda \), i.e. in our octave notation:

\[
u_n(t_i)u_n(t_i + T_i) = \lambda^\pm 1 |u_n(t_i)|^2.
\]

At time \( t_{i+1} = t_i + T_i \) we repeat this procedure and we record the new decorrelation time \( T_{i+1} \) and so forth for an overall number of trials \( N \). The averaged decorrelation times can be thus defined as

\[
\tau_n^{(m)} = \frac{\langle T^2 |u_n|^m \rangle_e}{\langle |u_n|^m \rangle_e} = \frac{\langle T |u_n|^m \rangle_t}{\langle |u_n|^m \rangle_t},
\]

where \( \langle \cdots \rangle_e \) stands for ensemble averaging over the \( N \) trials and \( \langle \cdots \rangle_t \) represents the usual time average. Since the multifractal description applies to time averages the averaged decorrelation times scale as \( \tau_n^{(m)} \sim l_n^{z(m)} \) with the same scaling exponents of the integral times \( s_n^m \).

In Table 1 we report the observed numerical \( \zeta(m) \) along with the observed and expected scaling exponents for \( \tau_n^{(m)} \), showing a very good agreement.

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\[\text{The relation between the } e\text{-average and the } t\text{-average is simply derived by observing that } \langle |u_n|^m T \rangle_t = \frac{\langle \int_0^T T |u_n|^m dt \rangle_e}{\langle T \rangle_e} \simeq \frac{\langle \sum_i T_i^2 |u_n(t_i)|^m \rangle/(\sum_i T_i) \rangle_e}{\langle |u_n|^m \rangle_e/(\langle T \rangle_e)}.\]
4 Two-scales time correlations

Let us now jump to the most general multi-scale multi-time correlation functions:

$$C_{p,q}^{N,n}(t) = \langle u_n^q(0) \cdot u_N^p(t) \rangle$$

(15)

where from now on we will always suppose that $u_N$ describes the velocity fluctuation at the smallest of the two scales considered, i.e. $N > n$. It is clear that now we have to consider the joint statistics of two fields: the first, the slower, at large scale, $u_n(0)$ and the second, the faster, at small scale and at a time delay $t$, $u_N(t)$.

Following the same reasonings as before we may safely assume that from zero time delays up to time delays of the order of the slower velocity field, $t = t_n$, the velocity field at small scale feels the same transfer process of $u_n$ up to scale $n$ and then from scale $n$ to scale $N$ an uncorrelated transfer mechanism:

$$u_N(t) = W_{N,n}(t)u_n(t) \sim W_{N,n}(t)u_n(0) \sim W_{N,n}(t)W'_{n,0}(0)u_0 \quad \text{for } 0 \leq t \leq t_n$$

(16)

Similarly, for time delays within $t_n \leq t_m < t < t_{m-1} \leq t_0$ also the field at large scale $n$ will start to see different transfer processes:

$$u_n(0) \sim W''_{n,m}(0)u_m(0) \sim W''_{n,m}(0)W'_m(0)u_0$$

$$u_N(t) \sim W_{N,m}(t)u_m(t) \sim W_{N,m}(t)W'_m(0)u_0$$

(17)

(18)

It is clear now, how we may write down the correlation for any time:

$$C_{p,q}^{N,n}(t) =$$

(19)

$$\sum_{m=1}^n \int d\mu_{m,0}(h)d\mu_{n,m}(h_1)d\mu_{N,m}(h_2) \quad l^{(q+p)}h\left(\frac{t}{t_m}\right)^{qh_1}\left(\frac{t_m}{t_m}\right)^{ph_2} f_{p,q}\left(\frac{t}{t_m}, \frac{t_n}{t_m}, \frac{t_N}{t_m}\right)$$

where we want to stress that the sum in the above expression goes only up to the index of the largest scale $n$ (see below). In order to understand which would be a reasonable functional shape for the $f_{p,q}$ function we need to point out a few preliminary remarks. Once we have to cope with two-scale correlation functions is natural to suppose that the the time-delay, $t_{n,N} = t_n - t_N$, needed for an energy burst to travel from shell $n$ to shell $N$ will play an important rôle. Furthermore, as far as the time-decaying properties are concerned
we must require that only eddy-turn-over times from the slower time $t_n$ up to the large-scale eddy-turn-over time $t_0$ enter in the game. This is because only for time larger than $t_n$ the correlation is a true multi-time correlation. Indeed, for time-delay shorter than $t_n$ only the field at small scale, $u_N$, is changing but always under the same large scale configuration, $u_n$. The final $f$-shape may be therefore guessed as: $f_{p,q} \left( \frac{t}{t_m}, \frac{t_n}{t_m}, \frac{t_N}{t_m} \right) = f_{p,q} \left( \frac{(t-t_n)}{t_m} \right)$. Where again the function $f(x)$ must be a function peaked for $x \sim 1$ and with a width $\delta x \sim O(1)$. The matching of representation (19) with the equation of motion reveals some important dynamical properties. From simple time-differentiation we should have

$$\partial_t C_{p,q}^{n,N}(t) \sim O \left[ \frac{C_{p,q}^{n+1,N}(t)}{l_N} \right]$$

which seems to be in disagreement with the time-representation proposed (19) because in the RHS of (20) does appear explicitly the fast eddy-turn-over time $t_N$ (through the dependency from $l_N$). Actually, the representation (19) is still in agreement with the equation of motion because the dependency of (20) from $t_N$ is false: again, exact cancellations must take place in the RHS. The explanation goes as follows: in the multifractal language we may write $u_N(t) = W_{N,n}(t) u_n(t)$ and therefore

$$\frac{d}{dt} u_N(t) = \left( \frac{d}{dt} W_{N,n}(t) \right) u_n(t) + W_{N,n}(t) \left( \frac{d}{dt} u_n(t) \right)$$

but for time shorter than the eddy-turn-over, $t_n$, of the large scale $u_n$, the term $W_{N,n}(t) \left( \frac{d}{dt} \right) u_n(t)$ is zero because the shell $u_n$ did not move at all, while, once averaged, the first term of the RHS of (21) becomes $\langle \left( \frac{d}{dt} W_{N,n} \right) \rangle \langle u_n \rangle$ which also vanishes because of the total time derivative. The time derivative, $\partial_t C_{p,q}^{n,N}(t)$ will therefore be a function which scales as $\frac{C_{p,q}^{n+1,N}(t)}{l_N}$ instead of $\frac{C_{p,q}^{n+1,N}(t)}{l_N}$ as simple power counting would predict. This may even be shown rigorously by evaluating the following averages:

$$\partial_t \langle u_n^q(t) u_N^p(t) \rangle \equiv 0 \equiv \langle (\partial_t u_n^q) u_N^p \rangle + \langle u_n^q \partial_t u_N^p \rangle.$$
be contradictory:

\[ \langle (\partial_t u^q_n) u^p_N \rangle \sim \frac{\mathcal{C}_{p,q}^{N,n+1}(0)}{l_n} \neq \langle u^q_n (\partial_t u^p_N) \rangle \sim \frac{\mathcal{C}_{N,n}^{p+1,q}(0)}{l_N} \]  

(23)

Now, in view of the previous discussion, we know that it is the correlation with the time derivative at small scale, \( \langle u^q_n (\partial_t u^p_N) \rangle \), which does not satisfy the multifractal power law, but has the same scaling of \( \langle (\partial_t u^q_n) u^p_N \rangle \) as our representation correctly reproduces.

In order to test all these properties, we plot in Fig. 3 the typical multi-time multiscale velocity correlation \( \mathcal{C}_{p,q}^{N,n}(t) \) for \( p = q = 1 \), \( n = 6 \), \( N = 6 - 13 \). As one can see the correlation has a peak which is in agreement with the delay predicted by (19), which saturates at the value \( \tau_{nN} \approx \tau_n \) for \( N \ll n \). Let us also notice that due to the dynamical delay, \( t_{m,N} \), the simultaneous multiscale correlation functions \( \mathcal{C}_{N,n}^{p,q}(0) \) do not show the fusion-rules prediction, i.e. pure power laws behaviors at all scales:

\[ \mathcal{C}_{N,n}^{p,q}(0) \sim \left( \frac{l_N}{l_n} \right)^{\zeta(p)} \left( \frac{l_n}{l_m} \right)^{\zeta(p+q)} \]

(24)

Indeed, for \( t \to 0 \), the term \( m = n \) dominates in (19) (because it makes \( t_{nN} \) minimum) and \( f_{p,q}\left( -\frac{t_{nN}}{t_{m,m}} \right) \) can be considered a constant only in the limit of large scale separation, \( n \ll N \), while otherwise we will see finite-size corrections.

The effect of the delay in multi-scale correlations is shown in Fig. 4 where we compare \( \mathcal{C}_{N,n}^{1,1}(0) \) and \( \mathcal{C}_{N,n}^{1,1}(T_{nN}) \) (for \( N > n = 6 \)) rescaled with the Fusion Rule prediction (24). The time delay \( T_{nN} \) is the time of the maximum of \( \mathcal{C}_{N,n}^{1,1}(t) \) computed from Fig. 3. We see that without delay, the prediction (24) is recovered only for \( N \gg n \) with a scaling factor \( f_{1,1}(-1) \approx 0.83 \), while including the average delay \( T_{nN} \) the Fusion Rule prediction is almost verified over all the inertial range.

Of course the delay \( \tau_{nN} \) is a fluctuating quantity and one should compute the average (19) with fluctuating delays. In this case the dimensional estimate \( \tau_{nN} \approx l_n u_n^{-1} - l_N u_N^{-1} \) is somehow ill-defined, first of all being not positive definite. To find a correct definition for the fluctuating time delays is a subtle point which lays beyond the scope of the present Paper.
5 Conclusions

In conclusion, we have proposed a multifractal-like representation for the multi-time multi-scale velocity correlation which should take into account all possible subtle time-dependencies and scale-dependencies. The proposal can be seen as a merging of the proposal made in [3] – valid only for cases when the disconnected part is vanishing – and the proposal made in [6] – valid only in the asymptotic regime of large time delays and large scale separation. Our proposal is phenomenologically realistic and consistent with the dynamical constraints imposed by the equation of motion. We have numerically tested our proposal within the framework of shell models for turbulence.

A new way to measure intermittent integral-time scales, $s^{p,q}(R)$, has also been proposed and tested.

Further tests on the true Navier-Stokes eqs. would be of first-order importance. Furthermore, the building of a synthetic signals which would reproduce the correct dynamical properties of the energy cascade would also be of primary importance [11, 12].

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TABLE CAPTIONS

TABLE 1: Comparison between the integral time-scales intermittency exponents, $z_m$, estimated from the measured spatial intermittent exponents, $z^{(th)}_m = 1 + \zeta(m - 1) - \zeta(m)$, and from direct measuring via the “doubling-time” $T$, $z^{(num)}_m$. 

FIGURE CAPTIONS

FIGURE 1:
The time dependency of single-scale correlation functions, $C_{nn}(t)$, in two different cases. The continuous line is the case with a non-zero disconnected part, $C_{nn}(t) = \langle |u_n(0)||u_n(t)| \rangle - \langle |u_n| \rangle^2$, while the dashed line represents a case with vanishing disconnected part $\tilde{C}_{nn}(t) = \Re(\langle u_n(0)u_n^*(t) \rangle)$. Both correlations are rescaled to their value at $t = 0$. The scale is fixed in the middle of the inertial range, $n = 12$, and the eddy turn over time of the reference scale was $\tau_{12} \simeq 0.29$. The average has been performed over approximately $500\tau_{12}$, about 10 large eddy turn over times. The presence of subleading terms in $C_{nn}$ is apparent. The remnant anticorrelation in $\tilde{C}_{nn}$, for $t > \tau_{12}$ reveals a partial cancellation of subleading terms: full cancellation requires averaging over a time interval of many more large eddy turn over times.

FIGURE 2:
Comparison between the two correlations $C_{nn}(t) = k_n\langle |u_n|^2(0)|u_n|^3(t) \rangle$, continuous line. $D_{nn}(t) = -\langle |u_n|^2(0)\frac{d|u_n|^2}{dt}(t) \rangle$, dashed line. The two correlations have the same dimensional properties, but $D_{nn}(t)$ decays faster due to cancellations of subleading terms. $D_{nn}$ vanishes at zero delay because of stationarity and smoothness of the process $u_n(t)$. Scale and characteristic times as in Figure 1.

FIGURE 3:
Multi-time multi-scale correlation functions, $C_{n,N}(t)$, for $n = 6$ and $N = 6, \cdots, 13$ (from bottom curve to top curve). Observe the saturation in the time-delays, $\tau_{n,N} = \tau_n - \tau_N \rightarrow \tau_n \simeq O(1)$ when $N$ increases.

FIGURE 4:
Lin-log plot of Multi-scale correlation $C_{n,N}^{1,1}(t) = \langle |u_n(0)||u_N(t)| \rangle$ rescaled with the Fusion Rule prediction: $C_{n}^{1,1}N(t)/(S_n^1S_n^2/S_n^3)$ at fixed $n = 6$ and at changing $N \geq n$. The lower line represent the zero-delay correlation ($t = 0$),
the upper line is for the average delay $t = T_{6,N}$. 
| $m$ | $\zeta_m$ | $z_m^{(num)}$ | $z_m^{(th)}$ |
|-----|---------|-----------|-----------|
| 1   | 0.39    | -0.61     | -0.61     |
| 2   | 0.72    | -0.68     | -0.67     |
| 3   | 1.00    | -0.72     | -0.72     |
| 4   | 1.26    | -0.75     | -0.74     |
| 5   | 1.49    | -0.77     | -0.78     |
| 6   | 1.71    | -0.78     | -0.78     |
| 7   | 1.93    | -0.80     | -0.80     |
| 8   | 2.13    | -0.80     | -0.80     |

Table 1:
