A PROJECTIVE MANIFOLD WHERE ENTIRE AND BRODY CURVES BEHAVE VERY DIFFERENTLY.

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Abstract. We give an example of a projective manifold with dense entire curves such that every Brody curve is degenerate.

1. Introduction

Let $X$ be a complex manifold. An entire curve is a non-constant holomorphic map from $\mathbb{C}$ to $X$. Assume $X$ to be endowed with a hermitian metric. Then an entire curve $f : \mathbb{C} \to X$ is called a "Brody curve" iff its derivative is bounded with respect to the euclidean metric on $\mathbb{C}$ and the given hermitian metric on $X$. If $X$ is compact, the notion of a "Brody curve" is independent of the choice of the metric.

The famous result of Brody ([2]) implies: A compact complex manifold $X$ admits an entire curve if and only if it admits a Brody curve.

This is an important result which, for example, allows an easy characterization of those submanifolds of abelian varieties which are hyperbolic in the sense of S. Kobayashi.

Given the above result, it is natural to ask the following question: "Let $X$ be a compact complex manifold and $p \in X$. Assume that there exists an entire curve $f : \mathbb{C} \to X$ with $p \in f(\mathbb{C})$. Does this implies that there is a Brody curve $f$ with $p \in f(\mathbb{C})$?"

In [11] we gave an example of a non-compact manifold (actually a domain in an abelian variety) where the behaviour of entire curves and Brody’s curves differ.

However this was not really a negative answer, since the question really concerns compact manifold.

Now we are able to give an example of a compact complex manifold where the above formulated question has a negative answer.

We prove the following theorem:
Main theorem. There exists a projective manifold $X$ with a hypersurface $Z$ such that for every point $x \in X$ there exists an entire curve $f : \mathbb{C} \to X$ with $f(\mathbb{C}) = X$ and $x \in f(\mathbb{C})$, but $Z$ contains the image of every Brody curve.

2. Proof of the Main theorem

The main theorem is a consequence of the more specific theorem below.

Theorem 1. There exists an abelian threefold $A$ with a smooth curve $C$ such that the smooth projective variety $\hat{A}$ obtained by blowing up $A$ along $C$ has the following properties:

1. For every point $p \in \hat{A}$ there exists a non-constant entire curve $\gamma : \mathbb{C} \to \hat{A}$ with $p \in \gamma(\mathbb{C})$ and $\gamma(\mathbb{C}) = A$.

2. For a given point $p \in \hat{A}$ there exists a non-constant Brody curve $\gamma : \mathbb{C} \to \hat{A}$ with $p \in \gamma(\mathbb{C})$ if and only if $p$ is contained in the exceptional divisor $E = \pi^{-1}(C)$ of the blow-up $\pi : \hat{A} \to A$.

The key idea is that the hermitian metric will explode in some directions due to the blow up and that this will create an obstruction against lifting Brody curves. As a consequence there will be no Brody curves outside the exceptional divisor. To realize this idea it is necessary to ensure that the center of the blow-up intersects the closure of the image of each brody curve. To achieve this it will be necessary to blow up a center of positive dimension. Furthermore, since the center of a blow-up has real codimension at least four, it is necessary to choose the abelian variety in such a way that for every Brody curve the closure is at least real 4-dimensional.

Proof. (1). For every $x \in A$ and $v \in T_xA$ there is an affine-linear curve $\gamma : \mathbb{C} \to A$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Recall that $\pi : \hat{A} \to A$ is an isomorphism outside $C$ and that each point of $x \in C$ is replaced by $\mathbb{P}(T_x/T_xC)$. Observe further that each entire curve $\gamma : \mathbb{C} \to \hat{A}$ lifts to $\hat{A}$ unless $\gamma(\mathbb{C}) \subset C$. Combined, these facts yield statement (1).

(2). We have $\pi^{-1}(x) \simeq \mathbb{P}_1$ for every $x \in C$. This implies that there is a Brody curve through every point in $E = \pi^{-1}(C)$. Conversely, let $\hat{f} : \mathbb{C} \to \hat{A}$ be a Brody curve. We will see that $\hat{f}(\mathbb{C}) \subset E = \pi^{-1}(C)$ if we choose $A$ and $C$ according to prop. below. Now $\hat{f}$ being a Brody curve implies that $f = \pi \circ \hat{f} : \mathbb{C} \to A$ is a Brody curve or constant. Let us assume that $f$ is not constant. If $A = \mathbb{C}^3/\Gamma$, then $f$ lifts to an affine-linear map $F : \mathbb{C} \to \mathbb{C}^3$. $f(\mathbb{C})$ is thus the orbit of a complex one-parameter subgroup $P$ of $A$. Let $H$ denote the (real) closure of
$P$ in $A$. Thanks to prop. 6 we may assume that $H$ and $C$ intersect transversally in some point $p$. But now we arrive at a contradiction because according to prop. 2 under these circumstances $f : \mathbb{C} \to A$ cannot be induced by a Brody curve $\hat{f} : \mathbb{C} \to \hat{A}$. Thus $f = \pi \circ \hat{f}$ must be constant. Since $\hat{f}$ is non-constant and $\pi$ is an isomorphism outside of $C$, it follows that $\hat{f}(\mathbb{C}) \subset E = \pi^{-1}(C)$. □

3. Brody curves

We recall some basic facts on Brody curves.

Let $X$ be a complex manifold endowed with some hermitian metric. Then an entire curve is a non-constant holomorphic map from $\mathbb{C}$ to $X$ and a Brody curve is a non-constant holomorphic map $f : \mathbb{C} \to X$ for which the derivative $f'$ is bounded (with respect to the euclidean metric on $\mathbb{C}$ and the given hermitian metric on $X$).

If $X$ is compact, the notion of a “Brody curve” is independent of the choice of the hermitian metric.

If $\phi : X \to Y$ is a holomorphic map between compact complex manifolds and $f : \mathbb{C} \to X$ is a Brody curve, then $\phi \circ f : \mathbb{C} \to Y$ is a Brody curve, too. (But not necessarily conversely.)

If $X = \mathbb{C}^g/\Gamma$ is a compact complex torus (e.g. an abelian variety), then an entire curve $f : \mathbb{C} \to X$ is a Brody curve if and only if it lifts to an affine linear map $\tilde{f} : \mathbb{C} \to \mathbb{C}^g$.

4. Local model of blow-up

The idea we use is: If we blow up something, the hermitian metric will explode somewhere. We will now make this precise.

**Proposition 1.** Let $A$ be a three-dimension complex manifold, $C$ a smooth curve, $\pi : \hat{A} \to A$ the corresponding blow-up with center $C$, $p \in C$ and $L_n$ a sequence of curves converging to a curve $L_0$ such that

1. $L_0$ intersects $C$ transversally in $p$.
2. The intersection $L_n \cap C$ is empty for all $n \neq 0$.

Furthermore assume $A$ and $\hat{A}$ endowed with hermitian metrics.

Then there exists sequences $p_n \in L_n$ and $v_n \in T_{p_n}(L_n)$ such that

$$\lim p_n = p$$

and

$$\limsup \frac{||\pi^{-1}(v_n)||_{\hat{A}}}{||v_n||_A} = \infty$$

(note that $L_n \subset A \setminus C$ and that $\pi$ is an isomorphism on $A \setminus C$.)
Proof. Let \( \hat{p} \) denote the point in \( \pi^{-1}(p) \) which points in the direction of \( L \) (using the isomorphism between \( \pi^{-1}(p) \) and the projectivization of the normal tangent space \( T_pA/T_pC \)).

We fix local holomorphic coordinates on \( A \) and \( \hat{A} \) around \( p \) resp. \( \hat{p} \) such that the defining equations for \( C \) and \( L_0 \) become as simply as possible. Doing this we get local holomorphic coordinates such that

\[
C = \{ (z_1, z_2, z_3) : z_1 = z_2 = 0 \},
\]

\[
L_0 = \{ (z_1, z_2, z_3) : z_2 = z_3 = 0 \},
\]

Now the projection:

\[
\pi(x_1, x_2, x_3) = (x_1 x_2, x_2, x_3)
\]

Since \( \lim L_n = L_0 \), the curves \( L_n \) can be parametrized as

\[
L_n = \{ \gamma_n(t) = (t, \alpha_n(t), \beta_n(t)) \}
\]

where \( t \) runs through an appropriate small neighbourhood of 0 and where \( \alpha_n, \beta_n \) are sequences of holomorphic functions converging uniformly to the constant function zero on this small neighbourhood.

Since all the calculations happen in some small neighbourhood of \( p \) resp. \( \hat{p} \), we may replace the given hermitian metrics by the euclidean metric with respect to our coordinate systems.

Our next step is to define the auxiliary function

\[
\phi_n(t) = t + \alpha_n(t)\alpha_n'(t)
\]

We observe that \( \phi_n \) converges to the identity map \( \phi(t) = t \). Therefore the theorem of Rouche allows us to choose a sequence \( s_n \) with \( \lim_n s_n = 0 \) and \( \phi_n(s_n) = 0 \) for all \( n \).

We claim: \( \alpha_n(s_n) \neq 0 \). Indeed, assume \( \alpha_n(s_n) = 0 \). Then

\[
\alpha_n(s_n)\alpha_n'(s_n) = 0
\]

and consequently

\[
0 = \phi_n(s_n) = s_n + 0 \Rightarrow s_n = 0
\]

and therefore

\[
0 = \alpha_n(s_n) = \alpha_n(0).
\]

But \( \alpha_n(0) = 0 \) is impossible, because \( L_n \cap C \) is empty. Thus the assumption \( \alpha_n(s_n) = 0 \) leads to a contradiction, i.e. \( \alpha_n(s_n) \) must be non-zero.

Hence we may divide by \( \alpha_n(s_n) \) and thereby deduce that \( \phi_n(s_n) = 0 \) implies \( \alpha_n'(s_n) = -s_n/\alpha_n(s_n) \). If \( \hat{\gamma}(t) \) denotes the point in \( \hat{A} \) lying
above \( \gamma_n(t) \in A \setminus C \), we obtain
\[
\hat{\gamma}_n(s_n) = \left( \frac{s_n}{\alpha_n(s_n)}, \alpha_n(s_n), \beta_n(s_n) \right) = (-\alpha'_n(s_n), \alpha_n(s_n), \beta_n(s_n))
\]
which converges to \((0, 0, 0) = \hat{p} \in \hat{A}\) if \(n\) goes to infinity.

Now
\[
\hat{\gamma}_n'(s_n) = \left( \frac{1 - \alpha'_n(s_n)}{(\alpha_n(s_n))^2}, \alpha_n'(s_n), \beta'_n(s_n) \right) \Rightarrow \lim_n ||\hat{\gamma}_n'(s_n)|| = +\infty
\]

\[
\sup_{t \in C} \frac{||\hat{\gamma}'(t)||_{\hat{A}}}{||\gamma'(t)||_A} = +\infty
\]

where \(\hat{\gamma} : \mathbb{C} \to \hat{A}\) is the natural lift of \(\gamma\).

Since \(\gamma\) is induced by an affine-linear map, the norm \(||\gamma'(t)||\) is a positive constant and in particular bounded from below by a number
greater than zero. Together with the above equation this implies that cannot be Brody curve.

6. Excluding Real subtori of dimension three

In this section we deduce the following statement:

Proposition 3. There exists an abelian three-fold $A$ such that every real subtorus of real dimension three is totally real in $A$.

We will prove this assertion by showing that every very general abelian three-fold has this property, i.e. we demonstrate:

Proposition 4. Let $U \to D$ be a locally complete family of abelian varieties of dimension three.

Then there exists a countable family of nowhere dense closed analytic subsets $Z_i \subset D$ such that every abelian threefold $A$ corresponding to a point outside the union $\cup_i Z_i$ has the property “Every real subtorus of real dimension three is totally real in $A$”

Before proving the proposition, we need some lemmata.

Lemma 1. Let $A = \mathbb{C}^3 / \Gamma$ be an complex abelian 3-fold, $S$ a real subtorus of dimension three.

Then there is a joint deformation of $S \subset A$ over the unit disc such that $A_t$ is an abelian variety for all $t$ and $S_t$ is totally real for all $t \neq 0$.

Proof. Let $\Lambda \subset \Gamma$ be the $\mathbb{Z}$-submodule corresponding to $S$. Since $A$ is an abelian variety, $\mathbb{C}^3$ admits a hermitian form $H$ such that $B = \Re H$ has integer values on $\Gamma \times \Gamma$. Now $B$ is alternating and $3 = \text{rank}_\mathbb{Z}(\Lambda)$ is odd, hence there is an element $v \in \Lambda$ for which $B(v, \cdot)$ vanishes identically on $\Lambda$. Let $\Lambda_\mathbb{R}$ resp. $\Lambda_\mathbb{C}$ be the real resp. complex vector subspace of $\mathbb{C}^3$ generated by $\Lambda$. We may assume that $\Lambda_\mathbb{R}$ is not totally real. Then $\dim_\mathbb{C}(\Lambda_\mathbb{C}) = 2$ and $L = \Lambda_\mathbb{R} \cap i\Lambda_\mathbb{R}$ is a complex line. Now we choose an element $w \in \Gamma$ such that

1. $B(v, w) \neq 0$,
2. $B(w, \cdot)$ does not vanish identically on $L$ and
3. $w \notin \Lambda_\mathbb{C}$.

We define $\mathbb{R}$-linear self-maps $\phi_t$ of $\mathbb{C}^3$ as follows: First we observe that $\mathbb{C}^3$ is the direct sum of $\mathbb{R} \cdot v$ and $K = \{ x : B(x, w) = 0 \}$. Second we set $\phi_t(v) = v + tw$ and $\phi_t(x) = x$ for all $x \in K$. It is easy to check that $\phi_t$ is always bijective and moreover an isometry for $B$. Hence $\Gamma_t = \phi_t(\Gamma)$ is a lattice for which the assertion $B(\Gamma_t, \Gamma_t) \subset \mathbb{Z}$ holds. Thus $A_t = \mathbb{C}^3 / \Gamma_t$ is an abelian variety.

Now let us look at $\phi_t(\Lambda_\mathbb{R})$. First we consider the real vector subspace $V = \Lambda_\mathbb{R} \oplus \mathbb{R}w$. Let $K = \{ x : B(x, w) = 0 \}$. Then $V = (V \cap K) \oplus \mathbb{R}v$. 
Now \( \phi_t \) acts trivially on \( K \) and \( \phi_t(v) = v + tw \in V \). Hence \( \phi_t \) stabilizes \( V \). We note that \( \dim_{\mathbb{R}}(V) = 4 \) and \( V \varsubsetneq \mathbb{C}^3 \), because \( w \not\in \Lambda_\mathbb{C} \). Therefore \( V \) contains a unique complex line, which must be \( L \). Since \( \phi_t(\Lambda_{\mathbb{R}}) \subset V \), we may deduce that for each \( t \) either \( \phi_t(\Lambda_{\mathbb{R}}) \) is totally real or contains \( L \). Now, by the construction of \( \phi_t \) it is clear that 

\[ \phi_t(\Lambda_{\mathbb{R}}) \cap \phi_s(\Lambda_{\mathbb{R}}) = \Lambda_{\mathbb{R}} \cap K \]

for any \( s \neq t \). Since \( L \neq \Lambda_{\mathbb{R}} \cap K \) due to condition (2) for the choice of \( w \), we may deduce that \( L \not\subseteq \phi_t(\Lambda_{\mathbb{R}}) \) for \( t \neq 0 \). As a consequence, \( \phi_t(\Lambda_{\mathbb{R}}) \) is totally real for \( t \neq 0 \). \( \square \)

**Lemma 2.** Let \( \pi : U \to D \) be a family of three-dimensional complex abelian varieties, parametrized by \( D \) which we assume to be the unit ball in some \( \mathbb{C}^N \).

Let \( S_0 \) be a real three-dimensional subtorus of the abelian variety \( U_0 = \pi^{-1}(0) \). Then there is natural deformation \( S_t \) of \( S_0 \) \( (t \in D) \) such that 

\[ Z = \{ t \in D : S_t \text{ is not totally real} \} \]

is a closed complex analytic subset of \( D \).

**Proof.** The family \( U \) can be described as a quotient \( \mathbb{C}^3 \times D \) by a \( \mathbb{Z}^6 \)-action which is given as 

\[ (m_1, \ldots, m_3, n_1, \ldots, n_3) : (v; t) \mapsto \left( v + (m_1, m_2, m_3) + \sum_i n_i f_i(t); t \right) \]

where \( v \in \mathbb{C}^3, t \in D \) and where the \( f_i \) are holomorphic maps from \( D \) to \( \mathbb{C}^3 \).

We may assume that \( S_0 \) is the subtorus fro which the corresponding subgroup of \( \mathbb{Z} \)-rank 3 is generated by \( f_1(0), f_2(0) \) and \( f_3(0) \). Then \( S_t \) corresponds to the subgroup generated by the \( f_i(t) \) and \( S_t \) is totally real if and only if this group spans \( \mathbb{C}^3 \) as a complex vector space. Therefore the set of all \( t \in D \) for which \( S_t \) fails to be totally real is the zero locus of \( \det(f_1(t), f_2(t), f_3(t)) \) and thus a closed complex analytic set. \( \square \)

Now we can prove the proposition

**Proof.** There are only countably many different real subtori of real dimension three for a given abelian 3-fold \( A \), each corresponding to a \( \mathbb{Z} \)-submodule of rank three of \( \mathbb{Z}^6 = H_1(A, \mathbb{Z}) \).

Inside the family \( U \to D \) there are canonical isomorphisms 

\[ H^1(U_0, \mathbb{Z}) \simeq H^1(U_t, \mathbb{Z}) \]

which we may therefore identify.
Now for each fixed $\mathbb{Z}$-submodule of rank three the set of all $t \in D$ for which the corresponding subtorus $S_t$ fails to be totally real is a closed analytic subset (lemma 2) which is not all of $D$ (lemma 1). This proves the proposition 4 and thereby prop. 3. □

Remark. 1.) Since every subtorus of dimension smaller than three can be embedded into a subtorus of dimension three, the property “All real subtori of real dimension three are totally real” is equivalent to the property “All real subtori of real dimension up to three are totally real”. 2.) An abelian threefold $A$ is a simple abelian variety iff it contains no elliptic curve. The latter property is equivalent to the statement “All real subtori of real dimension up to two are totally real”. Hence the property “All real subtori of real dimension three are totally real” implies that the abelian 3-fold under discussion is simple.

7. Dealing with real subtori of dimension four

The main goal of this section is to to verify that we can a choose a curve $C$ in a 3-dimensional abelian variety $A$ such that $C$ intersects the closure of every translate of every real subtorus of real dimension four.

Lemma 3. Let $A$ be an abelian threefold, $x \in A$ and let $L$ be a complex line in $T_x A$.

Then there exist smooth curves $C \subset A$ with $x \in C$ such that $T_x C$ is arbitrarily close to $L$.

Proof. We construct curves by embedding $A$ into a projective space and taking the intersection of $A$ with linear subspaces of codimension two containing $x$. Then the statement follows from Bertini’s theorem. □

Lemma 4. Let $A$ be an abelian threefold with smooth curves $C$ and $C'$. Then there is a dense open subset $U \subset A$ such that $C \cup \lambda_t^* C'$ is smooth for $t \in U$ where $\lambda_t$ denotes translation by $t$.

Proof. $C \cup \lambda_t^* C'$ is smooth iff $C$ and $\lambda_t^* C'$ are disjoint. Hence $U = A \setminus \{x - y : x \in C, y \in C'\}$. □

Lemma 5. Let $V$ be a complex three-dimensional vector space equipped with a hermitian inner product $H$ and let $W$ be a real four-dimensional real subspace. Then there exists a complex line $L \subset V$ such that the angle between $W$ and $L$ is at least $\pi/4$. i.e.,

$$| < v, w > | \leq \cos(\pi/4)||v|| \cdot ||w||$$

for all $v \in L, w \in W$. 
Remark. If $H(\ ,\ )$ is an hermitian inner product, its real part is the associated euclidean inner product and thus the angle between two vectors $v$ and $w$ is the number $\phi \in [0, \pi/2]$ for which $\cos \phi = \Re H(v, w)$.

Proof. We may choose vectors $A, B, C$ such that $(A, iA, B, iB, C, iC)$ is an orthonormal basis for $\Re H$ and 

$$\langle A, iA, B, C + \lambda iB \rangle = W$$

for some $\lambda \in \mathbb{R}$.

Then we choose

- $L = \langle C \rangle_C$ if $|\lambda| > 1$,
- $L = \langle B + iC \rangle_C$ if $0 \leq \lambda \leq 1$ and
- $L = \langle B - iC \rangle_C$ if $-1 \leq \lambda < 0$.

It is easy to check that in each case the angle is at least $\pi/4$. □

Proposition 5. Let $A$ be an abelian threefold (i.e. an abelian variety of dimension three). Then there exists a smooth complex curve $C \subset A$ such that for every real 4-dimensional subtorus $S \subset A$ and every point $a \in A$ there exists a point $p \in C$ where $C$ and $S(a)$ (the $S$-orbit in $A$ through $a$) intersect transversally.

Proof. We have to consider all 4-subtori. Since the set of all such tori lacks good geometric properties, we instead consider the larger set of all connected real Lie subgroups of real dimension 4, or, equivalently, the real Grassmann variety $M$ which parametrizes all real vector subspaces of dimension 4 of the Lie algebra $\text{Lie}(A) \simeq \mathbb{C}^3$. This is a real compact variety.

Now we fix an hermitian inner product on $\text{Lie}(A) \simeq \mathbb{C}^3$ (e.g. the standard one for $\mathbb{C}^3$ or the one corresponding to the Riemann condition). For each element $H \in M$ we define a closed neighbourhood $B_H$ as follows: An element $H'$ belongs to $B_H$ iff for every vector $v'$ in $H'$ there is a vector $v \in H$ such that the angle between $v$ and $v'$ is at least $\pi/16$. Due to compactness of $M$ there is a finite collection of elements $H_i \in M$, $i \in I$ such that $M = \cup_{i \in I} B_{H_i}$.

Next we will choose a smooth complex curve $C_i \subset A$ for each $i \in I$. Fix an index $i$. Let $H = H_i$ and $B = B_{H_i}$. Choose a complex line $L$ in $T_eA \simeq \text{Lie}(A)$ such that the angle between $L$ and $H$ is at least $\pi/8$ (which is possible due to lemma 3). Then we choose a smooth complex curve $S = S_i$ through $e$ such that for each $v \in T_eS$ there is a vector $v' \in L$ such that the angle between $v$ and $v'$ is at most $\pi/16$ (lemma 3). By the definition of $B$, the angle between $T_eS$ and $H'$ is at least $\pi/8$ for every $H' \in B$. 


Now let \( \pi : \mathbb{C}^3 \to A \) denote the universal covering. Let \( F \) denote a fundamental region, i.e. a compact subset of \( \mathbb{C}^3 \) with \( \pi(F) = A \). Let \( W \) be an open neighbourhood of \( e \) in \( S \) which is small enough such that the embedding of \( W \) in \( A \) lifts to an embedding into \( \mathbb{C}^3 \), taking \( e \) to \( 0 \). In addition, we require that \( W \) is small enough such that for every \( w \in W, v \in T_wS \setminus \{0\} \) and \( v' \in T_eS \setminus \{0\} \) the angle between \( v \) and \( v' \) is at most \( \pi/16 \).

For each \( H' \in B \) we define \( Z(H') = \{ c + h : c \in W, h \in H' \} \).

We claim: There exists a number \( \rho > 0 \) such that \( Z(H') \) contains the ball with radius \( \rho \) and center 0 for every \( H' \in B \). Indeed, assume the contrary. Then there are sequences \( v^{(k)} \in \mathbb{C}^3 \) and \( H^{(k)} \in B \) such that:

1. \( H^{(k)} \) converges to an element \( H'' \in B \) (recall that \( B \) is compact),
2. \( \lim v^{(k)} = 0 \),
3. \( v^{(k)} \not\in Z(H^{(k)}) \).

But this would contradict the fact that \( H'' \) and \( T_eW \) are transversal. Thus we can find such a number \( \rho \). Next, using compactness of \( F \), we choose a finite set \( \Sigma \subset \mathbb{C}^3 \) such that for every \( x \in F \) there is an element \( s \in \Sigma \) with \( ||x - s|| < \rho/3 \).

Using this fact and lemma \( \square \) we can find a map \( \xi : \Sigma \to \mathbb{C}^3 \) such that:

- \( \cup_{s \in \Sigma}(\pi(\xi(s)) + S) \) is smooth and
- \( ||\xi(s)|| < \rho/3 \) for all \( s \in \Sigma \)

Then we have constructed a smooth curve in \( A \), namely

\[ S' = \cup_{s \in \Sigma}(\pi(\xi(s)) + S) \]

with the following property:

\( \langle T \rangle \) For every vector \( u \in \mathbb{C}^3 \) with \( ||u|| < \rho/3 \) and every real 4-dimensional subtorus \( H \) of \( A \) with \( \text{Lie}(H) \in B \) every \( H \)-orbit in \( A \) intersects \( \pi(u) + S' \) in some point transversally.

We found this curve \( S' \) after fixing an element \( i \in I \). We can do the same for every element \( i \in I \), obtaining a family of curves \( S'_i \) and a family of positive real numbers \( \rho_i \).

Then by lemma \( \square \) we can choose vectors \( u_i \) such that \( ||u_i|| < \rho_i/3 \) and such that \( C = \cup_{i \in I} (\pi(u_i) + S'_i) \) is a smooth curve.

By construction this curve has the property that it intersects each translate of each real 4-dimensional subtorus of \( A \) in at least one point transversally.

\( \square \)

**Proposition 6.** There exists an abelian threefold \( A \) with a complex curve \( C \) such that the following property holds:
For every complex one-parameter subgroup $P$ of $A$ and every point $a$ in $A$ there is a point $p$ in $C$ where $C$ and the (real) closure of $P \cdot a$ intersect transversally.

Proof. We may choose $A$ such that every real three-dimensional real subtorus is totally real (prop. 3). Then evidently real subtori of smaller dimension are totally real as well. Now let $P$ be a complex one-parameter subgroup of $A$. The closure of $P$ is again a subgroup, and therefore in fact a real subtorus. This subtorus does not need to be complex, but it can not be totally real, since it contains $P$. Therefore for every complex one-parameter subgroup $P$ of $A$ the real dimension of its closure is at least 4. Now it suffices to choose the curve $C$ according to thm. 5. □

8. Brody curves and sets of rational points

Conjecturally entire curves or Brody curves with values in projective varieties defined over some number field behave somewhat analogously to sets of rational points (admitting finite field extensions).

As we have seen, Brody curves and arbitrary entire curves behave differently. So which are the right analogue for rational point sets? In our construction at one point we made a “very generic” choice. For this reason it is not clear whether one can find such an example which is defined over a number field.

If such an example can be defined over a number field, it would suggest that complex-analytic concept corresponding to infinite rational point sets are arbitrary entire curves and not Brody curves: For every abelian variety $A$ defined over a number field $k$ there is a finite field extension $K/k$ such that $A(K)$ is Zariski dense. Then also $X(K)$ is Zariski dense in $X$ for every projective manifold $X$ obtained from $A$ by blowing up something. Thus if our construction can be realized over a number field, it would yield a projective variety defined over some number field $K$ such that every Brody curve is degenerate, but there is a Zariski dense subset of $K$-rational points.

In any case, dense sets of rational points as well as dense entire curves behave nicely under birational transformations while our example shows that the behaviour of Brody curves may change dramatically.

This suggests that the right complex-analytic analogue to infinite sets of rational points should be arbitrary entire curves rather than Brody curves.
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