ON THE EXTENDABILITY OF SOME CLASSES OF MAPS ON HILBERT \( C^* \)-MODULES

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Abstract. In this paper, we show that every completely semi-\( \phi \)-map on a submodule of a Hilbert \( C^* \)-module has a completely semi-\( \phi \)-map extension on the whole of module. We also investigate the extendability of \( \phi \)-maps and provide examples of \( \phi \)-maps which has no \( \phi \)-map extension. Finally, we introduce a category of Hilbert \( C^* \)-module and determine injective objects in this category.

1. INTRODUCTION

One of the most fundamental theorems in the theory of operator spaces is the Wittsock’s extension theorem for completely bounded maps which is the noncommutative counterpart of the celebrated Hann-Banach’s Extension theorem. The authors in [3], introduced the concept of completely semi-\( \phi \)-maps as a generalization of \( \phi \)-maps. Also, they shown that every operator valued completely bounded linear map on a Hilbert \( C^* \)-module is a completely semi-\( \phi \)-map for some completely positive map \( \phi \) on the underlying \( C^* \)-algebra of the Hilbert \( C^* \)-module and vice versa [4]. Thus it is natural to seeking for an analogue of Wittsock’s extension theorem for completely semi-\( \phi \)-maps.

In this note, we show that every \( \phi \)-map or completely semi-\( \phi \)-map on a submodule of a Hilbert \( C^* \)-module has a completely semi-\( \phi \)-map extension on the whole of the Hilbert \( C^* \)-module. Furthermore, we provide examples of some \( \phi \)-map which has no \( \phi \)-map extension on the whole of module. However, for some special case of \( \phi \)-maps, we will show to how construct a completely semi-\( \phi \)-map extension of a \( \phi \)-map which is close to being a \( \phi \)-map.

In the last section, we introduce a category of Hilbert \( C^* \)-module and determine injective objects in this category.

For every Hilbert spaces \( \mathcal{H}, \mathcal{K} \), the set of all bounded operators \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is a right Hilbert \( \mathcal{B}(\mathcal{H}) \)-module, where the module action is the composition of operators and the \( \mathcal{B}(\mathcal{H}) \)-inner product on \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is given by \( \langle T, S \rangle = T^*S \) for every \( S,T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \).

Assume \( \mathcal{A} \) and \( \mathcal{B} \) are \( C^* \)-algebras, \( \mathcal{E} \) and \( \mathcal{G} \) are right Hilbert \( C^* \)-modules over \( \mathcal{A} \) and \( \mathcal{B} \) respectively, \( \phi : \mathcal{A} \to \mathcal{B} \) is a completely positive map and \( \Phi : \mathcal{E} \to \mathcal{G} \) is a map, we say

1. \( \Phi \) is a \( \phi \)-map, if \( \langle \Phi(x), \Phi(y) \rangle = \phi(\langle x, y \rangle) \), for all \( x, y \in \mathcal{E} \).
2. \( \Phi \) is a \( \phi \)-morphism, if \( \Phi \) is a \( \phi \)-map and \( \phi \) is a \( * \)-homomorphism.
3. \( \Phi \) is a completely semi-\( \phi \)-map, if \( \langle \Phi_n(x), \Phi_n(x) \rangle \leq \phi_n(\langle x, x \rangle) \) for every \( x \in \mathcal{M}_n(\mathcal{E}) \) and \( n \in \mathbb{N} \).

When \( \mathcal{G} = \mathcal{B}(\mathcal{H}, \mathcal{K}) \) and \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \) for some Hilbert spaces \( \mathcal{H}, \mathcal{K} \), a \( \phi \)-morphism is called \( \phi \)-representation and in this case

4. \( \Phi \) is non-degenerate, if \( [\Phi(\mathcal{E})\mathcal{H}] = \mathcal{K} \).

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Note that a \( \phi \)-morphism \( \Phi \) is linear and satisfies \( \Phi(xa) = \Phi(x)\phi(a) \) for every \( x \in E \) and \( a \in A \), therefore \( \Phi \) is a ternary morphism (triple morphism), that is \( \Phi(xy, z) = \Phi(x)\langle \Phi(y), \Phi(z) \rangle \) for all \( x, y, z \in E \). For more information on representation theory of Hilbert \( C^* \)-modules, \( \phi \)-maps and their dilation theory refer to [1, 2, 4, 6, 8] and [9].

2. EXTENDABILITY OF COMPLETELY \( \phi \)-MAPS

Throughout this section, we assume that \( \mathcal{H}, \mathcal{H}_1, \mathcal{H}_2 \) are Hilbert spaces, \( A \) is a \( C^* \)-algebra and \( F \) is a non-trivial closed submodule of a Hilbert \( A \)-module \( E \). Also we note that the set \( \mathcal{F}^\perp = \{ x \in E \mid \langle x, y \rangle = 0 \text{ for all } y \in F \} \) is a closed submodule of \( E \).

In this section, we concentrate on operator valued maps on Hilbert \( C^* \)-modules. In fact, if \( \phi : A \to \mathcal{B}(\mathcal{H}_1) \) is a completely positive map, then an operator valued \( \phi \)-map on \( F \) as \( \Phi \), means that \( \Phi \) is a \( \phi \)-map from \( F \) into \( \mathcal{B}(\mathcal{H}_1) \)-module \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \).

We first discuss the extension problem for \( \phi \)-maps. In the following, we provide an example of some \( \phi \)-map on a submodule of a Hilbert \( C^* \)-module which can not be extended to any \( \phi \)-map on the whole of the Hilbert \( C^* \)-module.

**Example 2.1.** Suppose that \( K(\mathcal{H}) \) is the set of all compact operators on \( \mathcal{H} \). Clearly, \( E = K(\mathcal{H}) \oplus K(\mathcal{H}) \) is a full Hilbert \( K(\mathcal{H}) \)-module (by \( K(\mathcal{H}) \)-valued inner product \( ((T_1, S_1), (T_2, S_2)) = T_1^*T_2 + S_1^*S_2 \) and \( F = K(\mathcal{H}) \oplus 0 \) is a non-trivial Hilbert submodule of \( E \). Consider the inclusion map \( \phi = \text{id} : K(\mathcal{H}) \to K(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}) \). The map \( \Phi : F \to \mathcal{B}(\mathcal{H}, \mathcal{H}) \) defined by \( \Phi((T, 0)) = T \) is a \( \phi \)-map which doesn’t have any \( \phi \)-map extension on \( E \).

In fact, if \( \Phi' : E \to \mathcal{B}(\mathcal{H}, \mathcal{H}) \) is a \( \phi \)-map extension of \( \Phi \), then

\[
\Phi'((T_1, S_1))^{\ast} \Phi'(T_2, S_2) = T_1^{\ast}T_2 + S_1^{\ast}S_2 \text{ and } \Phi'((T_1, 0)) = T_1,
\]

for all \( T_1, T_2, S_1, S_2 \in K(\mathcal{H}) \). A directly calculation shows that \( \Phi'((0, S)) = 0 \) for all \( S \in K(\mathcal{H}) \). Consequently, \( \Phi' = \Phi \oplus 0 \) and so \( \Phi' \) is not a \( \phi \)-map on \( E \).

The following lemma provides a necessary condition for a completely positive map \( \phi \), such that every \( \phi \)-map on a submodule has a \( \phi \)-map extension on the whole of the Hilbert \( C^* \)-module.

**Lemma 2.2.** If \( \phi : A \to \mathcal{B}(\mathcal{H}_1) \) is a completely positive map and there exists a non-degenerate \( \phi \)-map \( \Phi : F \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) which has a \( \phi \)-map extension on \( E \), then

(i) \( \phi((\mathcal{F}^\perp, E)) = \{0\} \),

(ii) every operator valued \( \phi \)-map on \( F \) has a \( \phi \)-map extension on \( E \).

**Proof.** (i) Assume there is a \( \phi \)-map \( \Psi : E \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \), extending \( \Phi \). For every \( h, h' \in \mathcal{H}_1 \) and \( x \in F, \ z \in \mathcal{F}^\perp \) one has

\[
\langle \Psi(z)h, \Phi(x)h' \rangle = \langle \Psi(z)h, \Psi(x)h' \rangle = \langle \Psi(x)^*\Psi(z)h, h' \rangle = \langle \phi((x, z))h, h' \rangle = 0.
\]

By the assumption, \( \Phi \) is a non-degenerate map and therefore \( \Psi(z)h = 0 \), thus \( \Psi(\mathcal{F}^\perp) = \{0\} \). On the other hand, \( \Psi \) is a \( \phi \)-map, so for every \( x \in E, y \in \mathcal{F}^\perp \)

\[
\phi(\langle y, x \rangle) = \Psi(y)^*\Psi(x) = 0.
\]

Then \( \phi(\mathcal{F}^\perp, E) = \{0\} \).

(ii) Assume \( \Theta : F \to \mathcal{B}(\mathcal{H}_1, K) \) is a \( \phi \)-map. By [4] Theorem 2.2, there is an isometry \( S : \mathcal{H}_2 \to K \) such that \( S\Phi(x) = \Theta(x) \), for every \( x \in F \). Define \( \Theta' : E \to \mathcal{B}(\mathcal{H}_1, K) \) by \( \Theta'(x) := S\Psi(x) \) for each \( x \in E \). Since \( \Psi \) is a \( \phi \)-map extension of \( \Phi \) and \( S \) is an isometry, \( \Theta' \) is a \( \phi \)-map extension of \( \Theta \).

By Kolmogorov’s decomposition theorem, for every completely positive map \( \phi : A \to \mathcal{B}(\mathcal{H}_1) \), there is at least a non-degenerate operator valued \( \phi \)-map on \( F \). Therefore,

**Corollary 2.3.** Assume \( \phi : A \to \mathcal{B}(\mathcal{H}_1) \) is a completely positive map. If every operator valued \( \phi \)-map on \( F \) has a \( \phi \)-map extension on \( E \), then \( \phi((\mathcal{F}^\perp, E)) = \{0\} \).
Theorem 2.4. Suppose that \( \phi : A \to \mathcal{B}(H_1) \) is a completely positive map and \( \Phi : \mathcal{F} \to \mathcal{B}(H_1, H_2) \) is a completely semi-\( \phi \)-map. Then \( \Phi \) has a completely semi-\( \phi \)-map extension \( \Phi' : \mathcal{E} \to \mathcal{B}(H_1, H_2) \). Furthermore, if \( \Phi \) is a \( \phi \)-map and \( \phi(\mathcal{F}^\perp, \mathcal{E}) = 0 \), then

(i) \( \Phi'(\mathcal{F}^\perp) = \{0\} \),

(ii) \( \Phi'(x)^*\Phi'(y) = \phi(\langle x, y \rangle) \) and \( \Phi'(y)^*\Phi'(x) = \phi(\langle y, x \rangle) \), for all \( x \in \mathcal{E}, y \in \mathcal{F} \oplus \mathcal{F}^\perp \).

Proof. Let \( (\rho, \mathcal{K}, V) \) be a minimal Stinespring’s dilation triple for \( \phi \). There is a triple \( ((\Phi_\rho, \mathcal{H}_\rho), (\Psi_\rho, \mathcal{K}_\rho, W_\rho)) \) consisting of a non-degenerate \( \phi \)-map \( \Phi_\rho : \mathcal{E} \to \mathcal{B}(H_1, H_\rho) \), a non-degenerate \( \rho \)-representation \( \Psi_\rho : \mathcal{E} \to \mathcal{B}(\mathcal{K}, \mathcal{K}_\rho) \) and a unitary operator \( W_\rho : \mathcal{H}_\rho \to \mathcal{K}_\rho \) such that satisfies \( W_\rho \Phi_\rho(x) = \Psi_\rho(x)V \) for every \( x \in \mathcal{E} \) by [4] Theorem 2.2 part (i). Since \( \Phi \) is a completely semi-\( \phi \)-map, we have \( [\Phi(x_i)^*\Phi(x_j)]_{i,j} \leq [\phi((x_i, x_j))]_{i,j} \), for every \( x_1, \ldots, x_n \in \mathcal{F} \). Therefore, for every \( h_1, \ldots, h_n \in H_1 \) we have

\[
\left\| \sum_{i=1}^{n} \Phi(x_i)h_i \right\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi(x_j, x_i)h_i, h_j \rangle = \left\| \sum_{i=1}^{n} \Phi(x_i)h_i \right\|^2.
\]

Thus there is a unique contractive linear operator \( S_0 : [\Phi_\rho(\mathcal{F})]H_1 \to H_2 \) such that \( S_0\Phi_\rho(x)h = \Phi(x)h \) for every \( x \in \mathcal{F} \) and \( h \in H_2 \). Let \( P \in \mathcal{B}(H_\rho) \) be the orthogonal projection onto \( [\Phi_\rho(\mathcal{F})]H_1 \).

Put \( S := S_0P : \mathcal{H}_\rho \to H_2 \). Therefore \( \Phi(x) = SW_\rho\Psi_\rho(x)V \) for every \( x \in \mathcal{F} \). Put \( W := W_\rhoS^* \) and define \( \Phi' : \mathcal{E} \to \mathcal{B}(H_1, H_2) \) by

\[
\Phi'(x) := W^*\Psi_\rho(x)V
\]

for all \( x \in \mathcal{E} \). Since \( W \) is a contraction, \( \Phi' \) is a completely semi-\( \phi \)-map by [4] Theorem 2.2 part (iii). Obviously, \( \Phi' \) is an extension for \( \Phi \).

Now, let \( \Phi \) be a \( \phi \)-map and \( \phi(\mathcal{F}^\perp, \mathcal{E}) = 0 \). In this case, the above inequality becomes equality and so \( S_0 \) becomes an isometry. Also, \( 0 \leq \Phi'(x)^*\Phi'(x) \leq \phi(\langle x, x \rangle) = 0 \) satisfies for every \( x \in \mathcal{F}^\perp \). Therefore \( \Phi'(\mathcal{F}^\perp) = \{0\} \).

Finally it must be shown that \( \Phi' \) satisfies \( \Phi'(x)^*\Phi'(y) = \phi(\langle x, y \rangle) \) for all \( x \in \mathcal{E} \), \( y \in \mathcal{F} \oplus \mathcal{F}^\perp \). For this, assume \( x \in \mathcal{E}, y \in \mathcal{F} \) and \( h \in H_1 \), we have

\[
WW^*\Psi_\rho(y)Vh = W_\rhoS^*SW_\rho^*\Psi_\rho(y)Vh = W_\rhoP\Phi_\rho(y)h = W_\rho\Phi_\rho(y)h = \Psi_\rho(y)Vh,
\]

therefore

\[
\Phi'(x)^*\Phi'(y)h = V^*\Psi_\rho(x)^*WW^*\Psi_\rho(y)Vh = V^*\Psi_\rho(x)^*\Psi_\rho(y)Vh = V^*\rho(x, y)Vh = \phi(\langle x, y \rangle)h.
\]

Thus \( \Phi'(x)^*\Phi'(y) = \phi(\langle x, y \rangle) \) for every \( x \in \mathcal{E} \) and \( y \in \mathcal{F} \). Since \( \Phi'(\mathcal{F}^\perp) = \{0\} \) and \( \phi(\mathcal{E}, \mathcal{F}^\perp) = \{0\} \), for every \( x \in \mathcal{E}, y \in \mathcal{F}, z \in \mathcal{F}^\perp \) we have \( \Phi'(x)^*\Phi'(z) = 0 = \phi(\langle x, z \rangle) \), and therefore

\[
\Phi'(x)^*\Phi'(y + z) = \Phi'(x)^*\Phi'(y) + \Phi'(x)^*\Phi'(z) = \phi(\langle x, y \rangle) + \phi(\langle x, y \rangle) = \phi(\langle x, y + z \rangle).
\]

The following corollary says that if a non-degenerate operator valued \( \phi \)-map has a \( \phi \)-map extension, then the extension is unique.

Corollary 2.5. Let \( \phi : A \to \mathcal{B}(H_1) \) be a completely positive map, \( \Phi : \mathcal{F} \to \mathcal{B}(H_1, H_2) \) a non-degenerate \( \phi \)-map and \( \Gamma : \mathcal{E} \to \mathcal{B}(H_1, H_2) \) a \( \phi \)-map such that \( \Gamma|_{\mathcal{F}} = \Phi \). Then \( \Gamma = \Phi' \), where \( \Phi' \) is as in the proof of Theorem 2.4.

Proof. We use the notions of the proof of Theorem 2.4. Since \( \Gamma \) is an \( \phi \)-extension of \( \Phi \) and \( \Phi \) is a non-degenerate map, then \( \Gamma \) is non-degenerate and \( \mathcal{H}_2 = [\Phi(\mathcal{F})]H_1 = [\Gamma(\mathcal{E})]H_1 \). By [4] Theorem 2.2, there is a unitary operator \( W' : \mathcal{H}_2 \to \mathcal{K}_\rho \) such that \( W'\Gamma(e)h := \Psi_\rho(e)Vh \) for all \( e \in \mathcal{E}, h \in \mathcal{H}_1 \). Thus

\[
W'\Phi(f)h = W'T(f)h = \Psi_\rho(f)Vh = W\Phi'(f)h = W\Phi(f)h
\]

for all \( f \in \mathcal{F}, h \in \mathcal{H}_1 \). Therefore \( W' = W \).
Now, if $x, y \in E$ and $h \in H_1$, then we have
\[
\Gamma(x)^* \Gamma(y) h = \phi((x, y)) h = V^* \rho((x, y)) V h = V^* \Psi^\rho(x)^* \Psi^\rho(y) V h = V^* \Psi^\rho(x)^* W \Gamma(y) h = V^* \Psi^\rho(x)^* W \Gamma(y) h = (\Phi(x)^*) \Gamma(y) h.
\]
Since $\Gamma(x)^*$ and $\Phi'(x)^*$ are bounded operators and $|\Gamma(E)H_1| = H_2$, we have $\Gamma(x)^* = \Phi'(x)^*$. \hfill \Box

If $A$ is a C*-subalgebra of $K(H)$, then it is well known that, every closed submodule $F$ of $E$ satisfies the equations $F^⊥⊥ = F$ and $F \oplus F^⊥ = E$. Therefore, by Theorem 2.4 and Corollary 2.5 we can conclude that the necessary condition $\phi((F^⊥, E)) = 0$ in Lemma 2.2 is a sufficient condition for existence of a map extension, in this case.

**Corollary 2.6.** If $A$ is a C*-algebra of compact operators and $\phi : A \to B(H_1)$ is a completely positive map and $E$ is a full Hilbert $A$-module. Then the following statements are equivalent:

(i) $\phi((F^⊥, E)) = 0$,

(ii) there is a non-degenerate operator valued $\phi$-map on $F$ which has a $\phi$-map extension on $E$,

(iii) every operator valued $\phi$-map on $F$ has a $\phi$-map extension on $E$.

(iv) for every $\phi$-map $\Phi : F \to B(H_1, H_2)$, the map $\Phi' = \Phi \oplus 0 : E = F \oplus F^⊥ \to B(H_1, H_2)$ is a $\phi$-map and also $\Phi'$ is the unique $\phi$-map extension of $\Phi$ on $E$.

Since the C*-algebra $K(H)$ is simple, every nonzero Hilbert $K(H)$-module is full. In particular, $(F^⊥, E) = (F^⊥, F^⊥) = K(H)$. Therefore, $\phi((F^⊥, E)) \neq 0$, for every nonzero completely positive map $\phi : K(H) \to B(H_1)$. Hence we have

**Corollary 2.7.** If $A = K(H)$ and $\phi : A \to B(H_1)$ is a nonzero completely positive map, then any operator valued $\phi$-map on $F$ has no $\phi$-map extension on $E$.

### 3. Category of Hilbert C*-modules and Completely semi-$\phi$-maps

In the following, we define the category $C_{H,C^*}$ as a category whose objects are pairs $(E, A)$ where $A$ is a C*-algebra and $E$ is a right Hilbert $A$-module and a morphism from $(E_1, A_1)$ to $(E_2, A_2)$ is a pair $(\Phi, \phi)$ consists of a completely positive map $\phi : A \to B$ and a completely semi-$\phi$-map $\Phi : E_1 \to E_2$ and the composition of two morphisms $(\Phi, \phi)$ and $(\Psi, \psi)$ is $(\Phi, \phi) \circ (\Psi, \psi) := (\Phi \circ \Psi, \phi \circ \psi)$. If we restrict ourselves to the case of full Hilbert C*-modules over unital C*-algebras and unital completely positive maps, we obtain a subcategory of $C_{H,C^*}$ which we denote by $C^1_{H,C^*}$. In the following we generalize some results on the characterization of completely semi-$\phi$-maps and use it to better understanding $C^1_{H,C^*}$ as a subcategory of operator systems $C_{OS}$, and characterize its injective objects. Finally, we compare this new category with the category of Hilbert C*-modules when its morphisms are $\phi$-maps, completely bounded maps or Hilbert modules morphisms.

For a Hilbert C*-module $E$ over a C*-algebra $A$, the smallest operator system which contains $\mathcal{A}$ and $E$ is denoted by $S_A(E)$ and is defined as follow $S_A(E) := \bigg[ \begin{bmatrix} C I_E & E \\ \text{adj} & A \end{bmatrix} \bigg] = \bigg\{ \begin{bmatrix} \lambda \ x \\ y^* \ a \end{bmatrix} | a \in \mathcal{A}, \lambda \in \C, \ x, y \in E \bigg\}$. The following theorem is a generalization of [3] Lemma 3.2 which is useful in the study of $C_{H,C^*}$.

**Proposition 3.1.** Suppose that $E$ and $F$ are right Hilbert C*-modules over the C*-algebras $A, B$, respectively, and also $\phi : A \to B$ is a completely positive map and $\Phi : E \to F$ is a linear map. Then, $\Phi$ is a completely semi-$\phi$-map if and only if

\[
\begin{bmatrix} \text{id} & \Phi \\ \Phi^* & \phi \end{bmatrix} : S_A(E) \to S_B(F) \quad (\text{given by} \quad \begin{bmatrix} \lambda \ x \\ y^* \ a \end{bmatrix} \mapsto \begin{bmatrix} \lambda \Phi(x) \\ \phi(a) \end{bmatrix})
\]

is a completely positive map.

**Proof.** The same argument as in the proof of [3] Lemma 3.2, works here. \hfill \Box

Hence we have the following result.
Theorem 3.2. \( C_{H,C^*}^1 \) is (up to isomorphism) a subcategory of \( C_{OS} \), the category of operator systems.

Proof. Define the map \( \Sigma : C_{H,C^*}^1 \to C_{OS} \) which corresponds to every object \((E,A)\) of \( C_{H,C^*}^1 \), the operator system \( S_A(E) \), and corresponds to every morphism \((\Phi,\phi)\) between two objects of \( C_{H,C^*}^1 \) such as \((E,A)\) and \((F,B)\), the unital completely positive map \( \begin{bmatrix} \text{id} & \Phi \\ \Phi^* & \phi \end{bmatrix} : S_A(E) \to S_B(F) \).

It is easy to check that for \((\Phi_1,\phi_1) : (E_1,A_1) \to (E_2,A_2)\) and \((\Phi_2,\phi_2) : (E_2,A_2) \to (E_3,A_3)\) \( \Sigma((\Phi_2,\phi_2) \circ (\Phi_1,\phi_1)) = \Sigma((\Phi_2,\phi_2)) \circ \Sigma((\Phi_1,\phi_1)). \) Thus \( \Sigma \) is a one-to-one covariant functor. \( \square \)

Therefore, we can consider \( C_{H,C^*}^1 \) as a category consists of block-wise operator systems \( \begin{bmatrix} \mathcal{C}_E & E \\ E^* & A \end{bmatrix} \), where \( A \) is a unital \( C^* \)-algebra and \( E \) is a full right Hilbert \( A \)-module, and morphisms are corner preserving unital completely positive maps.

We remark that there is some completely positive map between operator systems \( S_A(E) \) and \( S_B(F) \) which is not corner preserving.

Example 3.3. For a given Hilbert space \( H \) and every bounded operators \( T_1, T_2, T_3, T_4 \) on it, by elementary row and column operations we have the following unitary equivalence in \( B(H^4) \)

\[
\begin{bmatrix}
T_1 & 0 & 0 & T_2 \\
0 & T_4 & 0 & 0 \\
0 & 0 & T_1 & 0 \\
T_3 & 0 & 0 & T_3
\end{bmatrix}
\cong
\begin{bmatrix}
T_1 & T_2 & 0 & 0 \\
T_3 & T_4 & 0 & 0 \\
0 & 0 & T_1 & 0 \\
0 & 0 & 0 & T_4
\end{bmatrix}
\]

Therefore the map \( \varphi : B(H^2) \to B(H^4) \) defined by

\[
\varphi(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}) := \\
\begin{bmatrix} T_1 & 0 & 0 & T_2 \\
0 & T_4 & 0 & 0 \\
0 & 0 & T_1 & 0 \\
T_3 & 0 & 0 & T_3
\end{bmatrix}
\]

is a unital completely positive map which is not corner-preserving. Now considering \( B(H^2) \) as Hilbert \( C^* \)-module over itself, for \( i = 2, 4 \), and restriction of \( \varphi \) on \( S_{B(H^2)}(B(H^2)) \) provides an example of a unital completely positive map \( \varphi : S_{B(H^2)}(B(H^2)) \to S_{B(H^4)}(B(H^4)) \) which is not corner preserving, thus it is not a morphism in \( C_{H,C^*}^1 \).

Definition 3.4. Let \((E,A)\) and \((F,B)\) be two objects of \( C_{H,C^*}^1 \). We say that \((E,A)\) contained in \((F,B)\) (or \((F,B)\) contains \((E,A)\)), and denote it by \((E,A) \subseteq (F,B)\), when \( A \) is a \( C^* \)-subalgebra of \( B \) and \( E \subseteq F \) and \( \langle x,y \rangle_E = \langle x,y \rangle_F \) for every \( x,y \in E \).

Definition 3.5. An object \((E,A) \in C_{H,C^*}^1 \) is an injective object in \( C_{H,C^*}^1 \) when for every pair of elements of \( C_{H,C^*}^1 \) such as \((F,B)\) and \((G,C)\) contained in \((F,B)\) if there exists a morphism \((\Phi,\phi) : (G,C) \to (E,A)\), then there exists a morphism \((\Psi,\psi) : (F,B) \to (E,A)\) such that \( \psi \) is an extension of \( \phi \) and \( \Psi \) is an extension for \( \Phi \).

We are going to give a characterization of injective objects of \( C_{H,C^*}^1 \). In fact, the next theorem is a generalization of Theorem 2.4. Before proving the theorem, we recall some results on injectivity. For an operator space \( W \), its injective envelope is denoted by \( I(W) \) and is the operator space which contains \( W \) such that for every operator space \( V \) and every completely bounded map \( \Phi : W \to V \) there exists a completely bounded map \( \Psi : I(W) \to V \) such that \( \Psi|_W = \Phi \). The Paulson operator system associated to \( W \) is \( \begin{bmatrix} \mathcal{C}_{id} & W \\ W^* & \mathcal{C}_{id} \end{bmatrix} \) and denoted by \( S(W) \).

First, we prove the following lemma.

Lemma 3.6. Let \( E \) be a full right Hilbert \( C^* \)-module over a unital \( C^* \)-algebra \( A \). Then \( I(S(E)) = I(S_A(E)) \).

Proof. There exists a Hilbert space \( H \) such that \( E \) and \( A \) be contained in \( B(H) \) and therefore \( S(E) \) and \( S_A(E) \) can be considered as subsets of \( B(H^2) \). Since \( B(H) \) is a unital injective \( C^* \)-algebra, there is an injective envelope \( I(S(E)) \) of \( S(E) \) such that \( S(E) \subseteq I(S(E)) \subseteq B(H^2) \) and a
completely contractive idempotent \( \Phi : B(H^2) \to B(H^2) \) which is completely positive and its image is \( I(S(E)) \) and act identically on \( S(E) \), see [5, 4.2.7]. Since \( \Phi \) is a unital completely positive map and idempotent, there exist unital completely positive maps \( \varphi_i : B(H) \to B(H) \) for \( 1 \leq i \leq 2 \), and an idempotent \( \varphi : B(H) \to B(H) \) such that \( \Phi = \begin{bmatrix} \varphi_1 & \varphi \\ \varphi^* & \varphi_2 \end{bmatrix} \). Apply [7, Corollary 5.2.2] or [5, 2.6.16] for \( \Phi \) and projections \( p = \begin{bmatrix} id_H & 0 \\ 0 & 0 \end{bmatrix} \) and \( id_{H^2} - p \). The injective envelope \( I(S(E)) \) is a unital \( C^* \)-algebra by the product \( \circ_\phi \) defined by \( u_1 \circ_\phi u_2 := \Phi(u_1 u_2) \) for every \( u_1, u_2 \in I(S(E)) \) and it has the following block-wise structure \( \begin{bmatrix} I_{11}(E) & I(E) \\ I(E) & I_{22}(E) \end{bmatrix} \) where \( I(E) = \varphi(B(H)) \) is the injective envelope of \( E \) (by [5, 4.4.3] see [5, 4.4.2]) and \( I_{ii}(E) = \varphi_i(B(H)) \) for \( i = 1, 2 \) are injective \( C^* \)-algebras. Since \( I(S(E)) \) is a unital \( C^* \)-algebra, \( I_{11}(E) \) and \( I_{22}(E) \) are unital \( C^* \)-algebras and \( I(E) \) is a Hilbert \( I_{11}(E) \)-\( I_{22}(E) \)-bimodule. By the assumption \( E \) is full and for every \( u_1, u_2 \in E \) we have

\[
\begin{bmatrix} 0 & 0 \\ u_2^* & 0 \end{bmatrix} \circ_\Phi \begin{bmatrix} 0 & u_1 \\ 0 & 0 \end{bmatrix} = \Phi \begin{bmatrix} 0 & 0 \\ u_2^* & 0 \end{bmatrix} \begin{bmatrix} 0 & u_1 \\ 0 & 0 \end{bmatrix} = \Phi \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \varphi_2((u_2, u_1)) \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & 0 \\ 0 & I_{22}(E) \end{bmatrix},
\]

thus \( \varphi_2(A) \subset I_{22}(E) \). Note that \( \varphi_2 \) is a unital completely positive map on \( B(H) \), which is not necessarily multiplicative, but its restriction on \( A \) is an isometric and multiplicative map from \( A \) into \( I_{22}(E) \). To show this, note that \( \Phi \) is a completely contractive unital idempotent map which acts identically on \( S(E) \), thus for every \( u \in E \) and \( a \in A \) by [5, Theorem 4.4.9 (Youngson)] or [7, Lemma 6.1.2] we have

\[
\Phi(\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}) = \Phi(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = \Phi(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

therefore, if \( \varphi_2(a) = 0 \) for some \( a \in A \), then \( ua = 0 \) for every \( u \in E \), thus \( a = 0 \). Thus \( \varphi_2 \) is one to one. Let \( u \in E \) and \( a, b \in A \). Put \( T_1 := \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}, T_2 := \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \) and \( T_3 := \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \). Since \( T_1 T_2 T_3 \) and \( T_1 T_2 \) belongs to \( S(E) \) and \( \Phi \) is identity on \( S(E) \) then [5, Theorem 4.4.9 (Youngson)] or [7, Lemma 6.1.2] implies that

\[
T_1 \circ_\Phi (T_2 T_3) = \Phi(T_1 \Phi(T_2 T_3)) = \Phi(T_1 T_2 T_3) = \Phi(T_1 T_2 T_3) = T_1 T_2 T_3.
\]

Therefore for every \( u \in E \) we have \( u \circ_\Phi (\varphi_2(ab) - \varphi_2(a) \circ_\Phi \varphi_2(b)) = 0 \) which implies that \( \varphi_2(ab) = \varphi_2(a) \circ_\Phi \varphi_2(b) \), because \( E^* \circ_\Phi E \) is an essential ideal in \( I_{22}(E) \). Therefore the restriction of \( \varphi_2 \) on \( A \) is an one to one *-homomorphism and therefore it is an isometry from \( A \) into \( I_{22}(E) \), thus \( A \subset I_{22}(E) \). Thus \( S_A(E) \subset I(S(E)) \) which implies that \( I(S_A(E)) = I(S(E)) \). \( \square \)

**Theorem 3.7.** A given object \((E, A) \in C_{H,C}^1 \) is injective if and only if \( A \) and \( E \) are injective objects in the category of operator spaces.

**Proof.** Let \((E, A) \) be an injective element in the category \( C_{H,C}^1 \). By Lemma [5,6] \((E, A) \) is contained in \( (I(E), I_{22}(E)) \). Thus the identity morphism \((id, id) : (E, A) \to (E, A) \) has an extension to a morphism \((\Phi, \phi) : (I(E), I_{22}(E)) \to (E, A) \). Thus \( \phi : I_{22}(E) \to A \) is a completely positive map which extends the identity map \( \Phi \) thus it is unital and \( \Phi : I(E) \to E \) is a completely semi-\( \phi \)-map and therefore it is completely contractive. On the other hand, the inclusion of \( E \) in \( I(E) \) is rigid and \( \Phi|_E = id_E \), therefore \( \Phi = id_{I(E)} \), by [7, Theorem 6.1.2]. Thus \( E = I(E) \), and \( E \) is injective.

Similarly, using the fact that \((E, A) \) is contained in \((E, I(A)) \) we can show that \( A = I(A) \) and hence \( A \) is injective.

Conversely, assume \( E \) and \( A \) are injective operator spaces and \( E \) is a full right Hilbert \( A \)-module. We show that \((E, A) \) is an injective object in \( C_{H,C}^1 \). Let \((W, B), (V, C) \in C_{H,C}^1 \) and \((W, B) \) contained in \((V, C) \). Assume \((\Phi, \phi) \) is a morphism from \((W, B) \) into \((E, A) \). Since \( A \) is an injective \( C^* \)-algebra, there is a unital completely positive map \( \psi : C \to A \) extending \( \phi \). Note that
\[ B \subset C, \text{ thus we can consider } \mathcal{W} \text{ as a Hilbert } C\text{-module and } \Phi \text{ is a completely semi-}\psi\text{-map. Thus the map } \\
\Lambda := \begin{bmatrix} \text{id} & \Phi \\ \Phi^* & \psi \end{bmatrix} : S_C(\mathcal{W}) \rightarrow S_A(\mathcal{E}) \text{ is a unital completely positive map (by Proposition 3.1).} \]

On the other hand \( S_A(\mathcal{E}) \subset I(S_A(\mathcal{E})) = I(S(\mathcal{E})) = \begin{bmatrix} I_{11}(\mathcal{E}) & I(\mathcal{E}) \\ I(\mathcal{E}) & I_{22}(\mathcal{E}) \end{bmatrix}. \) Note that \( S_C(\mathcal{W}) \subset S_C(\mathcal{V}) \) and \( I(S(\mathcal{E})) \) is injective, thus there is a unital completely positive map \( \Theta : S_C(\mathcal{V}) \rightarrow I(S(\mathcal{E})) \) which extends \( \Lambda \). It is obvious that \( \Theta \) has the matrix decomposition form \( \begin{bmatrix} \text{id} & \Psi \\ \Psi^* & \psi \end{bmatrix} \) for some linear map \( \Psi : \mathcal{V} \rightarrow I(\mathcal{E}) \), but \( \mathcal{E} \) is injective, thus \( I(\mathcal{E}) = \mathcal{E} \) and, there exists a linear map \( \Psi : \mathcal{V} \rightarrow \mathcal{E} \) such that \( \begin{bmatrix} \text{id} & \Psi \\ \Psi^* & \psi \end{bmatrix} : S_C(\mathcal{V}) \rightarrow S_A(\mathcal{E}) \) is a unital completely positive map. Now Proposition 3.1 implies that \( \Psi \) is a completely semi-\( \psi \)-map, extending \( \Phi \). Therefore \( (\mathcal{E}, A) \) is an injective object in \( \mathcal{C}^1_{H,C^*} \). \( \square \)

Note that \( \mathcal{C}^1_{H,C^*} \) is different from the category of operator spaces, we show this by an example of an injective object in \( \mathcal{C}^1_{H,C^*} \) which its corresponding object is not a injective operator space.

**Example 3.8.** Assume \( \mathcal{H} \) is a infinite dimensional Hilbert space. Put \( \mathcal{E} = \mathcal{B}(\mathcal{C}, \mathcal{H}) \). By Theorem 2.4 or Theorem 3.7 \( (\mathcal{B}(\mathcal{C}, \mathcal{H}), \mathcal{C}) \) is an injective object in \( \mathcal{C}^1_{H,C^*} \) and \( \mathcal{C}^1_{H,C^*} \). But \( S_C(\mathcal{E}) = S(\mathcal{E}) \) and \( I(S(\mathcal{E})) = \begin{bmatrix} \mathcal{B}(\mathcal{H}) & \mathcal{E} \\ \mathcal{E}^* & \mathcal{C} \end{bmatrix}. \) Thus \( S(\mathcal{E}) \) is not an injective operator system.

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