REGULARITY OF GLOBAL ATTRACTORS FOR
REACTION-DIFFUSION SYSTEMS WITH NO MORE THAN
QUADRATIC GROWTH

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Abstract. We consider reaction-diffusion systems in a three-dimensional bounded domain under standard dissipativity conditions and quadratic growth conditions. No smoothness or monotonicity conditions are assumed. We prove that every weak solution is regular and use this fact to show that the global attractor of the corresponding multi-valued semiflow is compact in the space $(H^1_0(\Omega))^N$.

1. Introduction. In this paper we consider reaction-diffusion systems in a three-dimensional bounded domain under dissipativity and growth conditions which guarantee global resolvability but not uniqueness of the Cauchy problem. In this general case it is known [8], [4] that all weak solutions generate a multi-valued semiflow, which has in the phase space $H = (L^2(\Omega))^N$ an invariant, compact, connected, stable global attractor, which consists of bounded complete trajectories.

Some additional information becomes available if we consider the global attractor as a section of a trajectory attractor [1], [9], because a trajectory attractor is compact in the topologies of certain functional spaces [17], [8]. But more detailed information about the structure and regularity of the global attractor in the non-uniqueness case can be obtained only under additional assumptions on the system. In particular, the boundedness in $H^1_0(\Omega)$ for the global attractor of the multivalued

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seminiflow generated by regular solutions was obtained in [4] for scalar reaction-diffusion equations. Under an additional assumption on the external force it was shown in [4] that the global attractor of the multivalued semiflow generated by all weak solutions is bounded in \((L^\infty(\Omega))^N\). In [10] the regularity of the global attractor was considered for a parabolic inclusion with linear growth of the multi-valued right-hand part. For the nonautonomous case see [6].

The compactness of the global attractor in \(H^1_0(\Omega)\) was obtained recently in [7] for a scalar reaction-diffusion equation with no more than quadratic growth. In this paper we extend the results from [7] for general reaction-diffusion systems with no more than quadratic growth.

The key point in order to prove this result is to obtain that any weak solution is regular. The question about the regularity of solutions for reaction-diffusion equations and inclusions without uniqueness is an important and interesting task, which has been studied recently in several works (see e.g. [2], [7], [13]).

2. Setting of the problem. In a bounded domain \(\Omega \subset \mathbb{R}^n\), \(1 \leq n \leq 3\), with sufficiently smooth boundary \(\partial \Omega\), we consider the following reaction-diffusion system (RD-system for short)

\[
\begin{cases}
    u_t = d \Delta u - f(u) + h(x), \\
    u|_{\partial \Omega} = 0,
\end{cases}
\]

where \(u = u(t, x) = (u^1(t, x), ..., u^N(t, x))\) is an unknown vector-function, \(a = \text{diag}[a_1, ..., a_N]\) is a diagonal \(N \times N\) matrix with \(a \geq \beta I\), \(\beta > 0\), \(h = (h_1, ..., h_N)\), \(f = (f_1, ..., f_N)\) are given functions,

\[ h \in (L^2(\Omega))^N, \quad f \in C(\mathbb{R}^N; \mathbb{R}^N), \]

and for given numbers \(C_1, C_2 \geq 0\), \(\gamma > 0\), \(p_i \geq 2\), \(i = 1, N\) the following conditions hold:

\[
\sum_{i=1}^N |f_i(v)|^{p_i} \leq C_1(1 + \sum_{i=1}^N |v^i|^{p_i}), \quad \forall v \in \mathbb{R}^N,
\]

\[
\sum_{i=1}^N f_i(v)v^i \geq \gamma \sum_{i=1}^N |v^i|^{p_i} - C_2, \quad \forall v \in \mathbb{R}^N, \tag{2}
\]

where \(\frac{1}{p_i} + \frac{1}{q_i} = 1\), \(i = 1, N\).

Further, we shall use the notation \(H = (L^2(\Omega))^N\), \(V = (H^1_0(\Omega))^N\).

It is well-known [1] that under conditions (2) for every initial data \(u_0\) from the phase space \(H\) there exists at least one weak solution of (1) such that

\[ u(0) = u_0 \]

and every weak solution of (1) belongs to \(C([0, +\infty); H]\), the function \(t \mapsto ||u(t)||_H^2\) is absolutely continuous, for a.a. \(t \geq 0\) the following energy equality holds

\[
\frac{1}{2} \frac{d}{dt} ||u(t)||_H^2 + (a\nabla u(t), \nabla u(t)) + (f(u(t)), u(t)) = (h, u(t)), \tag{3}
\]

and for all \(t \geq s \geq 0\) the following estimates hold

\[
||u(t)||_H^2 + \beta \int_s^t ||u(\tau)||_V^2 d\tau + 2\gamma \sum_{i=1}^N ||u_i(\tau)||_{L^{p_i}}^2 d\tau \leq ||u(s)||_H^2 + C_3(t-s), \tag{4}
\]
\[ \| u(t) \|_H^2 \leq e^{-\delta(t-s)} \| u(s) \|_H^2 + C_4, \quad (5) \]

where the positive constants \( C_3, C_4, \delta \) depend only on the parameters of the problem \( (1) \).

Moreover, the multivalued mapping \( G : \mathbb{R}_+ \times H \to 2^H \)

\[ G(t, u_0) = \{ u(t) \mid u(\cdot) \text{ is a weak solution of } (1), \ u(0) = u_0 \} \quad (6) \]

is a multivalued semiflow (m-semiflow) \([8], [9]\), which has the compact, invariant global attractor \( A \subset H \), that is, a compact (in \( H \)) set \( A \) such that

\[ A = G(t, A) \quad \forall \ t \geq 0, \]

\[ \text{dist}_H(G(t, B), A) \to 0, \ t \to +\infty \quad \text{for every bounded } B \subset H. \quad (7) \]

The main goal of this paper is to prove the compactness of \( A \) in the space \( V = (H_0^1(\Omega))^N \) and the attraction property \((7)\) in the topology of \( V \).

3. **Main results.** Our additional assumptions on the vector-function \( f \) are the following: the numbers \( p_i \) from \((2)\) are equal to 2 or 3 and there exists \( C > 0 \) such that for all \( v \in \mathbb{R}^N \) one has

\[ |f_1(v)| \leq C(1 + |v|^2 + |v_1|^2 \sum_{i=2}^N |v^i| + \sum_{i=1}^N |v^i|), \quad \text{if } p_1 = 3, \]

\[ |f_1(v)| \leq C(1 + \sum_{i=1}^N |v^i|), \quad \text{if } p_1 = 2, \]

\[ |f_2(v)| \leq C(1 + |v| \sum_{i=1}^N |v_i| + |v^i| \sum_{i=3}^N |v^i| + \sum_{i=3}^N |v^i|), \quad \text{if } p_2 = 3, \]

\[ |f_2(v)| \leq C(1 + \sum_{i=1}^N |v^i|), \quad \text{if } p_2 = 2, \]

\[ |f_3(v)| \leq C(1 + |v|^2 + |v_3|^2 + |v_1|^2 \sum_{i=4}^N |v^i| + |v^4| \sum_{i=4}^N |v^i| + \sum_{i=4}^N |v^i|), \quad \text{if } p_2 = 3, \]

\[ |f_3(v)| \leq C(1 + \sum_{i=1}^N |v^i|), \quad \text{if } p_3 = 2, \]

\[ \ldots \]

\[ |f_N(v)| \leq C(1 + \sum_{i=1}^N |v^i|^2), \quad \text{if } p_N = 3, \]

\[ |f_N(v)| \leq C(1 + \sum_{i=1}^N |v^i|), \quad \text{if } p_N = 2. \quad (8) \]

So we will consider the case of no more than quadratic growth of the non-linear term \( f \). Let us consider some examples \([14], [1]\), which show that assumptions \((8)\) are not restrictive.
Example 3.1. (Feld-Noyes equations in chemical kinetics)
\[
\begin{aligned}
\frac{\partial u^1}{\partial t} &= d_1 \Delta u^1 + \alpha (u^2 - u^1 u^2 + u^1 - \beta (u^1)^2), \\
\frac{\partial u^2}{\partial t} &= d_2 \Delta u^2 + \alpha^{-1} (\gamma u^3 - u^2 - u^1 u^2), \\
\frac{\partial u^3}{\partial t} &= d_3 \Delta u^3 + \delta (u^1 - u^3).
\end{aligned}
\]

In this case \( p_1 = p_2 = 3 \), \( p_3 = 2 \) and conditions (3) are satisfied. In this example, the second condition in (2) is not satisfied. However, as shown in [14] the set \( D = [0, a] \times [0, b] \times [0, c] \),

for some \( a, b, c \) which depend on the constants of the problem, is an invariant region. Then restricting the solutions to this set we can prove that (2) holds.

Example 3.2. (Lotka-Volterra system)
\[
\begin{aligned}
\frac{\partial u^1}{\partial t} &= d_1 \Delta u^1 + u^1 (a_1 - u^1 - a_{12} u^2 - a_{13} u^3), \\
\frac{\partial u^2}{\partial t} &= d_2 \Delta u^2 + u^2 (a_2 - u^2 - a_{21} u^1 - a_{23} u^3), \\
\frac{\partial u^3}{\partial t} &= d_3 \Delta u^3 + u^3 (a_3 - u^3 - a_{31} u^1 - a_{32} u^2).
\end{aligned}
\]

In this case \( p_1 = p_2 = p_3 = 3 \) and conditions (3) are satisfied. It is clear that conditions (2) are also satisfied.

We denote by \( A \) the operator \(-\Delta\) with Dirichlet boundary conditions, so that \( D(A) = H^2(\Omega) \cap H_0^1(\Omega) \). As usual, denote the eigenvalues and the eigenfunctions of \( A \) by \( \lambda_i, e_i, i = 1, 2, \ldots \). We define the usual sequence of spaces

\[
V^{2\alpha} = D(A^{\alpha}) = \{u \in L^2(\Omega) : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} |(u, e_i)|^2 < \infty\},
\]

where \( \alpha \geq 0 \). It is well known that \( V^s \subset H^s(\Omega) \) for all \( s \geq 0 \) (see [16, Chapter IV] or [11]). We note also that \( V^1 = H_0^1(\Omega), V^0 = L^2(\Omega) \).

The main result is the following.

**Theorem 3.3.** Let conditions (3), (8) take place. Then the global attractor \( A \) of the m-semiflow (9) is compact in \( V \) and

\[
\text{dist}_V(G(t, B), A) \to 0, \quad t \to +\infty \quad \text{for every bounded set } B \subset H.
\]

**Proof.** The key idea is to prove that under condition (8) every weak solution \( u(t) \) is regular, that is

\[
u \in L^\infty(\epsilon, T; V), \quad \frac{\partial u}{\partial t} \in L^2(\epsilon, T; H) \quad \forall \ 0 < \epsilon < T.
\]

Let us fix a weak solution \( u = (u^1, \ldots, u^N) \), \( u(0) = u_0 \) and consider the first equation from (1) as an equation with unknown function \( v = u^1 \) and known fixed functions \( u^2, \ldots, u^N \):

\[
\begin{aligned}
\nu_t &= a_1 \Delta v + g(t, x, v), \quad x \in \Omega, \ t > 0, \\
\nu|_{\partial \Omega} &= 0, \quad \nu|_{t=0} = u^1(0),
\end{aligned}
\]

where \( g(t, x, v) = f_1(v, u^2(t, x), \ldots, u^N(t, x)) \).

It should be noted that the problem (9) can have more than one solution and that a solution of (9) is not necessarily the first component of some solution of (1).
But $u^1$ is a solution of (9) and all facts which we will be able to prove for solutions of (9) will be true for $u^1$. Therefore, in further arguments by a solution of (9) we always mean the function $v = u^1$.

We will use the following well-known result.

**Lemma 3.4. (Regularity Lemma).** [13] p.163 If $z$ is a weak solution of the problem

\[
\begin{align*}
    z_t &= a \Delta z + d(t, x), \quad x \in \Omega, \ t > 0, \\
    z|_{t=0} &= 0, \ z|_{\partial \Omega} = z_0, \\
\end{align*}
\]

where $a \geq \beta > 0$, $d \in L^2(0, T; V^*)$, $z_0 \in V^{s+1}$, then

\[ z \in L^\infty(0, T; V^{s+1}) \cap L^2(0, T; V^{s+2}). \]

Moreover, $z \in C([0, T]; V^{s+1})$ and the following inequality takes place: for $T \geq t \geq s \geq 0$

\[ \|u(t)\|_{V^{s+1}}^2 + \beta \int_s^t \|u(\tau)\|_{V^{s+2}}^2 d\tau \leq \|u(s)\|_{V^{s+1}}^2 + \frac{1}{\beta} \int_s^t \|d(\tau)\|_{V^*}^2 d\tau. \]  

(10)

So, if $p_1 = 2$, then

\[ |g(t, x, v)| \leq C(1 + |v| + \sum_{i=2}^N |u^i(t, x)|), \]

and

\[ \|g\|_{L^2(0, T; L^2(\Omega))} \leq C_5(1 + \|v\|_{L^2(0, T; L^2(\Omega))}^2 + \sum_{i=2}^N \|u^i\|_{L^2(0, T; L^2(\Omega))}^2) \leq C_6, \]  

(11)

where from (5) the positive constants $C_5, C_6$ depend only on $T$ and $R$, $\|u_0\|_H \leq R$, but do not depend on $u(\cdot)$ (the same situation will be for all constants $C_i$ in this proof).

If we fix a weak solution $u = (u^1, ..., u^N)$, then from the inclusion

\[ u(\cdot) \in L^2_{loc}(0, +\infty; V) \]

we deduce that for arbitrary small $\delta > 0$ we can find $\epsilon \in (0, \delta)$ such that $u^1(\epsilon) \in H^1_0(\Omega)$. This fact allows us to consider the Cauchy problem

\[
\begin{align*}
    z_t &= a \Delta z + d(t, x), \quad x \in \Omega, \ t > 0, \\
    z|_{t=0} &= 0, \ z|_{\partial \Omega} = u^1(\epsilon), \\
\end{align*}
\]

for arbitrary small $\epsilon > 0$. After that we put $d(t, x) = g(t, x, u^1(t, x))$. This problem has a unique weak solution $z = z(t, x)$ on $(\epsilon, T)$, which coincides with $u^1(t, x)$ due to (9). Applying the Regularity Lemma on $[\epsilon, T]$, we obtain

\[ z = u^1 \in C([\epsilon, T]; H^1_0(\Omega)) \cap L^2(\epsilon, T; H^2(\Omega)), \ z_t = u^1_t \in L^2(\epsilon, T; L^2(\Omega)). \]

Note that from inequality (9) for arbitrary small $\epsilon > 0$ we can always choose $\theta_\epsilon \in (0, \epsilon)$ such that $u(\theta_\epsilon) \in V$ and

\[ \|u(\theta_\epsilon)\|_V^2 \leq \frac{R^2}{\beta \epsilon} + \frac{C_5}{\beta}. \]  

(12)

So, from (10)

\[ \|u^1\|_{L^\infty(\epsilon, T; H^1_0(\Omega))} \leq C_7(1 + \frac{1}{\epsilon}). \]  

(13)
Now let us consider the case \( p_1 = 3 \). We have
\[
|g(t, x, v)| \leq C(1 + |v(t, x)|^2 + |v(t, x)| \sum_{i=2}^{N} |u^i(t, x)|),
\]
\[
v = u^1 \in L^3(0, T; L^3(\Omega)).
\]
Moreover, for a.a. \( t \in (\epsilon, T) \) we obtain
\[
\int_{\Omega} |g(t, x, v(t, x))|^2 dx \leq C_0(1 + \|v(t)\|_{L^3(\Omega)}^2 + \sum_{i=2}^{N} \|u^i(t)\|_{L^3(\Omega)}^2).
\]
From the Sobolev embedding \( H^{\frac{3}{2}}(\Omega) \subset L^3(\Omega) \), the interpolation inequality
\[
\|v\|_{L^3(\Omega)} \leq C_0 \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}
\]
and the inclusion \( u \in C([0, T]; H) \) we deduce
\[
\|g(v)\|^2_{L^2(0, T; H^{-\frac{1}{2}}(\Omega))} \leq \|g(v)\|^2_{L^2(0, T; H^{\frac{3}{2}}(\Omega))}
\]
\[
\leq C_{10}(1 + \int_0^T \|v(t)\|^2_{L^2(\Omega)} \|v(t)\|_{H^1(\Omega)}^2 dt + \sum_{i=2}^{N} \int_0^T \|u^i(t)\|^2_{L^2(\Omega)} \|u^i(t)\|_{H^1(\Omega)}^2 dt) \quad (14)
\]
\[
\leq C_{11}(1 + \int_0^T \|v(t)\|_{H^1(\Omega)}^2 dt + \sum_{i=2}^{N} \int_0^T \|u^i(t)\|_{H^1(\Omega)}^2 dt) \leq C_{12}.
\]
Then from the Regularity Lemma with \( d(t, x) = g(t, x, v(t, x)) \in L^2(\epsilon, T; V^{-\frac{1}{2}}) \) we have
\[
v \in C([\epsilon, T]; H^{\frac{3}{2}}(\Omega)) \cap L^2(\epsilon, T; H^{\frac{3}{2}}(\Omega)). \quad (15)
\]
So \( v = u^1 \in C([\epsilon, T]; L^3(\Omega)), \)
\[
\|v\|^2_{L^3(\epsilon, T; L^3(\Omega))} \leq C_{13}(1 + \frac{1}{\epsilon}),
\]
\[
\|g(v)\|^2_{L^2(\epsilon, T; L^2(\Omega))}
\]
\[
\leq C_{14}(1 + \int_{\epsilon}^T \int_{\Omega} |v(t, x)|^2 dx dt + \sum_{i=2}^{N} \int_{\epsilon}^T \int_{\Omega} |v(t, x)|^2 |u^i(t, x)|^2 dx dt) \quad (17)
\]
\[
\leq C_{15}(1 + \int_{\epsilon}^T \int_{\Omega} |v(t)|^2_{L^2(\Omega)} |v(t)|^2_{H^2(\Omega)} dt + \sum_{i=2}^{N} \int_{\epsilon}^T \int_{\Omega} |v(t)|^2_{L^2(\Omega)} |u^i(t)|^2_{H^2(\Omega)} dt) \leq C_{16}(1 + \frac{1}{\epsilon}).
\]
Thus \( d(t, x) = g(t, x, v(t, x)) \in L^2(\epsilon, T; L^2(\Omega)) \) and the Regularity Lemma implies
\[
v = u^1 \in C([\epsilon, T]; H^1(\Omega)) \cap L^2(\epsilon, T; H^2(\Omega)), \quad \frac{\partial v}{\partial t} \in L^2(\epsilon, T; L^2(\Omega)), \quad (18)
\]
\[
\|v\|^2_{L^2(\epsilon, T; H^1(\Omega))} \leq C_{17}(1 + \frac{1}{\epsilon}).
\]
In the same way using conditions \( (8) \) we can prove that for any \( i = 1, N \) and for arbitrary small \( \epsilon > 0 \)
\[
u_i \in C([\epsilon, T]; H^1(\Omega)) \cap L^2(\epsilon, T; H^2(\Omega)), \quad \frac{\partial u_i}{\partial t} \in L^2(\epsilon, T; L^2(\Omega)).
\]
Now let us prove that for every closed bounded ball $B_R \subset H$ the set $G(2, B_R)$ is compact in $V$. Let $u(\cdot)$ be a solution of (1), $\|u(0)\|_H \leq R$, a point $\theta \in (0, \epsilon)$ is taken from [12], $T > 2$. As the function $u(\cdot)$ is continuous from $[\epsilon, T]$ to $V$, from [19] we obtain the estimate

$$\sup_{t \in [\epsilon, T]} \|u(t)\|^2_V \leq C_{18} (1 + \frac{1}{\epsilon}).$$

(20)

In particular, the set $G(2, B_R)$ is bounded in $V$. Let us take an arbitrary sequence $\{\xi_n\} \subset G(2, B_R)$. Then $\xi_n = u_n(2)$, $\{u_n(0)\} \subset B_R$, where $u_n(\cdot)$ is a solution of (1). Then up to subsequence $u_n(0) \rightarrow u_0$ weakly in $H$, $\xi_n \rightarrow \xi$ weakly in $V$. From [8, Lemma 1] up to a subsequence

$$u_n \rightarrow u \text{ in } L^2(0, T; H),$$

$$u_n(t, x) \rightarrow u(t, x) \text{ for a.a. } (t, x) \in (0, T) \times \Omega,$$  

(21)

where $u(\cdot)$ is a solution of (1), $u(0) = u_0$. Moreover, from [17, Theorem 2]

$$u_n \rightarrow u \text{ in } L^2(\epsilon, T; V)$$

(22)

and, therefore,

$$u_n(t) \rightarrow u(t) \text{ in } V \text{ for a.a. } t \in (\epsilon, T).$$

(23)

Multiplying (1) by $u_t$ in $H$ and using the regularity of the function $u(\cdot)$, we obtain that for $T \geq t \geq s \geq \epsilon$ the functions $u(\cdot)$, $u_n(\cdot)$, $n \geq 1$ satisfy the inequality

$$\|u(t)\|^2_V \leq \|u(s)\|^2_V + \int_s^t \|f(u(\tau))\|^2_H d\tau + (t - s)\|h\|^2_H.$$  

(24)

Let us consider for $t \in [1, T]$ the functions

$$J(t) = \|u(t)\|^2_V - \int_1^t \|f(u(\tau))\|^2_H d\tau - (t - 1)\|h\|^2_H,$$

$$J_n(t) = \|u_n(t)\|^2_V - \int_1^t \|f(u_n(\tau))\|^2_H d\tau - (t - 1)\|h\|^2_H.$$  

(21)

From

$$f(u_n(t, x)) \rightarrow f(u(t, x)) \text{ for a.a. } (t, x) \in (1, T) \times \Omega.$$  

Moreover, from the estimate

$$\int_1^T \int_\Omega |f(u_n(t, x))|^4 dx dt \leq C_{19} (1 + \int_1^T \int_\Omega |u_n(t, x)|^6 dx dt),$$

the embedding $H^1_0(\Omega) \subset L^6(\Omega)$ and [20] we deduce that

$$\{f^2(u_n)\} \text{ is bounded in } L^{\frac{4}{3}}(1, T; L^2(\Omega)).$$

So from a standard Lemma [11, Lemma 1.2]

$$f^2(u_n) \rightarrow f^2(u) \text{ weakly in } L^{\frac{4}{3}}(1, T; L^2(\Omega)).$$

Then, using [23] we deduce that

$$J_n(t) \rightarrow J(t) \text{ for a.a. } t \in (1, T).$$
Since the functions $J_n, J$ are continuous and monotone, we obtain in an standard way that
\[ \limsup J_n(t) \leq J(t) \quad \forall \, t \in (1, T). \]
In particular,
\[ \limsup \|u_n(2)\|_V \leq \|u(2)\|_V. \]
The last inequality together with the weak convergence implies that
\[ \xi_n = u_n(2) \to \xi = u(2) \text{ in } V, \]
so the set $G(2, B_R)$ is compact in $V$. As the global attractor $A$ is invariant and bounded in $H$, then
\[ A = G(2, A) \subset G(2, B_R) \]
for sufficiently large $R > 0$ and, therefore, $A$ is precompact in $V$. But $A$ is closed in $H$, so it is closed in $V$ and, finally, it is compact in $V$.

Let us prove the attraction property (7) in the topology of $V$. If it is not true, then there exist $\delta > 0$, $R > 0$ and sequences $t_n \to \infty$, $y_n \in G(t_n, B_R)$ such that
\[ \text{dist}_V(y_n, A) \geq \delta. \tag{25} \]
From the dissipation property (5)
\[ y_n \in G(2, G(t_n - 2, B_R)) \subset G(2, B_{\sqrt{\pi + c_4}}), \forall \ n \geq n_0. \]
So $\{y_n\}$ is precompact in $V$. But the global attractor consists of all cluster points (in $H$) of sequences $\xi_n \in G(t_n, B_R)$ for all $t_n \to \infty$ and $R > 0$. Thus up to subsequence $y_n \to y \in A$ in $H$ and, therefore, in $V$, which is a contradiction with (25). The theorem is proved.

**Lemma 3.5.** The statements of Theorem 3.3 remain true, if the matrix $a$ is lower triangular with $a \geq \beta I$.

**Proof.** Let $a = ((a_{ij}))$, where $a_{ij} = 0$, if $i < j$. Then the first equation in (1) is analyzed without any changes, the second one contains a summand $a_{21} \Delta u^1$, which belongs to $L^2(\varepsilon, T; L^2(\Omega))$ from (18), and so on. \hfill \Box

**Lemma 3.6.** In Examples 3.1-3.2 we can prove that $A \subset (C^\infty(\Omega))^N$.

**Proof.** For every complete trajectory $u(\cdot)$, lying in $A$, we have that $u \in C(\mathbb{R}; V) \cap L^2_{loc}(\mathbb{R}; (H^2(\Omega))^N)$. Let us prove that $f_i(u(\cdot, \cdot)) \in L^2(\tau, T; H^1_0(\Omega))$, for any $T \geq \tau$, where $f_i$ is the $i$-th component of the right-hand side in Examples 3.1-3.2. It is easy to see that $\nabla f_i(u)$ consists of summands like $v \nabla w$, where $v$ and $w$ are components of $u$. Then from the Gagliardo-Nirenberg inequality
\[
\int_\tau^T \int_\Omega v^2(\nabla w)^2\, dx\, dt \leq C_{20} \int_\tau^T \|v\|^2_{L^4(\Omega)}\|w\|^2_{L^6(\Omega)}\|w\|^2_{H^2(\Omega)}\, dt.
\]
Therefore, $\nabla f_i(u) \in L^2(\tau, T; L^2(\Omega))$ and $f_i(0) = 0$ imply that $f_i(u(\cdot, \cdot)) \in L^2(\tau, T; H^1_0(\Omega)) = L^2(\tau, T; V^1)$. Hence, from the Regularity Lemma
\[ u \in C([\tau, T]; (H^2(\Omega))^N) \cap L^2(\tau, T; (H^3(\Omega))^N), \]
so $A \subset (H^2(\Omega))^N$. Repeating these arguments we prove that $A \subset (H^m(\Omega))^N$ \quad \forall \ m \geq 1. \hfill \Box
Remark 1. If $\Omega \subset \mathbb{R}^2$, then we can omit conditions (8) and claim that Theorem 3.3 is true only under conditions (2). Indeed, from estimate (4) and the interpolation inequality
\[ \|v\|_{L^4(\Omega)} \leq C \|v\|_{L^2(\Omega)}^2 \|v\|_{H^1(\Omega)}, \]
we deduce that for every $i = 1, \ldots, N$ one has $f_i \in L^2(0,T;L^2(\Omega))$, so every weak solution is regular and estimation (19) holds.

If $\Omega \subset \mathbb{R}$, then Theorem 3.3 is true only under conditions (2) even in the case $p_i = 4$. Indeed, using the estimate (4) and the interpolation inequality
\[ \|v\|_{L^6(\Omega)} \leq C \|v\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}, \]
we deduce that for every $i = 1, \ldots, N$ we have $f_i \in L^2(0,T;L^2(\Omega))$, so every weak solution is regular and estimation (19) holds.

In particular, for the Fitz-Hugh-Nagumo system\[ \begin{cases} 
  u_t = d_1 \Delta u - f_1(u) - v + h_1(x), \\
  v_t = d_2 \Delta v + \delta u - \gamma v + h_2(x), \\
  u|_{\partial \Omega} = v|_{\partial \Omega} = 0,
\end{cases} \]
where $\Omega = (0,L)$, $d_1, d_2, \delta, \gamma$ are positive constants, $h_1, h_2 \in L^2(0,L)$, $f_1 \in C(R)$,\[ |f_1(u)| \leq C_1 (1 + |u|^3); \quad f_1(u)u \geq \alpha |u|^4 - C_2, \]
the global attractor is compact in $V$ and, therefore, in $C([0,L])$.

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REFERENCES

[1] V. V. Chepyzhov and M. I. Vishik, Attractors for Equations of Mathematical Physics, American Mathematical Society, Providence, Rhode Island, 2002.
[2] N. V. Gorban and P. O. Kasyanov, On regularity of all weak solutions and their attractors for reaction-diffusion inclusions in unbounded domains, in Continuous and distributed systems, Solid Mechanics and its Applications, (M.Z. Zgurovsky and V.A. Sadovnichiy eds.), Springer International Publishing, Switzerland, 211 (2013), 205–220.
[3] N. V. Gorban, O. V. Kapustyan and P. O. Kasyanov, Uniform trajectory attractor for non-autonomous reaction-diffusion equations with Caratheodory’s nonlinearity, Nonlinear Anal., 98 (2014), 13–26.
[4] O. V. Kapustyan, P. O. Kasyanov and J. Valero, Structure and regularity of the global attractor of a reaction-diffusion equation with non-smooth nonlinear term, Discrete Contin. Dyn. Syst., 34 (2014), 4155–4182.
[5] O. V. Kapustyan, P. O. Kasyanov and J. Valero, Regular solutions and global attractors for reaction-diffusion systems without uniqueness, Communications on Pure and Applied Analysis, 13 (2014), 1891–1906.
[6] O. V. Kapustyan, P. O. Kasyanov and J. Valero, Structure of the uniform global attractor for general non-autonomous reaction-diffusion systems, Solid Mechanics and its Applications 211 (M.Z. Zgurovsky and V.A. Sadovnichiy eds.), Springer International Publishing Switzerland, 2014, 163–180.
[7] O. V. Kapustyan, P. O. Kasyanov and J. Valero, Structure of the global attractor for weak solutions of a reaction-diffusion equation, Appl. Math. Inf. Sci., 9 (2015), 2257–2264.
[8] O. V. Kapustyan and J. Valero, On the Kneser property for the complex Ginzburg-Landau equation and the Lotka-Volterra system with diffusion, J. Math. Anal. Appl., 357 (2009), 254–272.
[9] O. V. Kapustyan and J. Valero, Comparison between trajectory and global attractors for evolution systems without uniqueness of solutions, Internat. J. Bifur. Chaos, 20 (2010), 2723–2734.
[10] P. O. Kasyanov, L. Toscano and N. V. Zadoianchuk, Regularity of weak solutions and their attractors for a parabolic feedback control problem, *Set-Valued Var. Anal.*, 21 (2013), 271–282.

[11] J. L. Lions and E. Magenes, *Problèmes Aux Limites Non-homogènes et Applications*, Dunod, Paris, 1968.

[12] V. S. Melnik and J. Valero, On attractors of multi-valued semi-flows and differential inclusions, *Set-Valued Anal.*, 6 (1998), 83–111.

[13] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations* Springer, 2002.

[14] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer, New-York, 1983.

[15] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics* Springer-Verlag, New York, 1997.

[16] M. I. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer Academic Publishers, Dordrecht, 1988.

[17] M. I. Vishik, S. V. Zelik and V. V. Chepyzhov, Strong trajectory attractor of dissipative reaction-diffusion system *Doklady RAN*, 435 (2010), 155–159.

[18] M. Z. Zgurovsky and P. O. Kasyanov, Multivalued dynamics of solutions for autonomous operator differential equations in strongest topologies, in *Continuous and distributed systems*, Solid Mechanics and its Applications 211 (M.Z. Zgurovsky and V.A. Sadovnichiy eds.), Springer International Publishing, Switzerland, 2014, 149–162.

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