Weak Continuity of the Flow Map for the Benjamin-Ono Equation on the Line

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Abstract

In this paper we show that the flow map of the Benjamin-Ono equation on the line is weakly continuous in $L^2(\mathbb{R})$, using “local smoothing” estimates. $L^2(\mathbb{R})$ is believed to be a borderline space for the local well-posedness theory of this equation. In the periodic case, Molinet [27] has recently proved that the flow map of the Benjamin-Ono equation is not weakly continuous in $L^2(\mathbb{T})$. Our results are in line with previous work on the cubic nonlinear Schrödinger equation, where Goubet and Molinet [11] showed weak continuity in $L^2(\mathbb{R})$ and Molinet [28] showed lack of weak continuity in $L^2(\mathbb{T})$.

1 Introduction

In this paper we study the weak continuity of the solution operator of the initial value problem for the Benjamin-Ono equation:

$$\begin{cases}
\partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0, & x \in \mathbb{R}, \ t \in \mathbb{R}, \\
u(x, 0) = \phi(x), & x \in \mathbb{R},
\end{cases}$$

(1.1)

where $\mathcal{H}$ represents the Hilbert transformation.

The Benjamin-Ono equation (1.1) is a model for one-dimensional long waves in deep water (cf. [2] and [33]) and is completely integrable. Well-posedness of the problem (1.1) has been extensively studied by many authors, cf. [3], [7], [10], [13], [14], [16], [28].
In particular, in [13], it was proved that this problem is globally well-posed in $L^2(\mathbb{R})$. Thus, for any given $T > 0$ there exists a mapping $S : L^2(\mathbb{R}) \to C([-T,T],L^2(\mathbb{R}))$, which is Lipschitz continuous when restricted in any bounded sets in $L^2(\mathbb{R})$, such that for any $\phi \in L^2(\mathbb{R})$, the function $u(\cdot,t) = (S\phi)(t) =: S(t)\phi$ is a solution of the problem (1.1) in the time interval $[-T,T]$. In this paper we study the following problem: Is the operator $S(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ weakly continuous (for fixed $t$)? Note that since $S(t)$ is a nonlinear operator, we cannot give this question a positive answer by merely using the continuity of $S(t)$ in norm.

Our motivation to study the above problem is inspired by the important series of works of Martel and Merle [23]–[26], which studied finite time blow-up and instability of solitary waves for the generalized Korteweg-de Vries equations, in critical and subcritical cases. One key step in their strategy in these works is a reduction to a nonlinear Liouville type theorem. Martel-Merle then reduce this nonlinear Liouville theorem to a corresponding linear one, involving the linearized operator around the solitary wave. It is in both these steps that the weak continuity of the flow map for generalized KdV in suitable Sobolev spaces plays a central role. Recently, by using a similar strategy, Kenig and Martel [20] established the asymptotic stability of solitons for the Benjamin-Ono equation (1.1) in the energy space $H^{1/2}(\mathbb{R})$. Thus, the weak continuity of the flow map in the energy space for the equation (1.1) is needed and it is established by these authors. The proof is very simple and reduces matters to the uniform continuity of the flow map for the Benjamin-Ono equation for data whose small frequencies coincide in a Sobolev space of strictly smaller index than $H^{1/2}(\mathbb{R})$, which depends on local well-posedness of the initial value problem (1.1) in $L^2(\mathbb{R})$ proved in the above mentioned work of Ionescu and Kenig [13]. Naturally, it would be desirable to prove the asymptotic stability of solitons for the Benjamin-Ono equation in $L^2(\mathbb{R})$. However, since no local well-posedness theory for this equation is available in Sobolev spaces of negative indices and it is strongly suspected that, in fact, uniform continuity of the flow map even restricted to data whose small frequencies coincide, must fail for Sobolev spaces of negative indices, the approach used in [20] does not work in the $L^2(\mathbb{R})$.

Another interesting result which motivates this study is a recent work of Molinet [30], in which the periodic initial-boundary value problem of the Benjamin-Ono equation was studied, and it was proved that the flow map of the periodic initial-boundary value problem of the Benjamin-Ono equation is not weakly continuous in $L^2(\mathbb{T})$, despite that such a problem is globally well-posed in $L^2(\mathbb{T})$, by another work of Molinet [28].

We would also like to mention a recent work of Goubet and Molinet [11], where a similar problem for the cubic nonlinear Schrödinger equation on the line was studied. For this equation the global well-posedness in $L^2(\mathbb{R})$ was established in [36], while in [17]
(focusing case) and [6] (defocusing case) it was shown that the flow map is not uniformly continuous in any Sobolev space of negative index. Thus, the weak continuity in $L^2(\mathbb{R})$ of the flow map cannot be treated by the approach used in the works of Martel and Merle [23–26] and Kenig and Martel [20]. Goubet and Molinet [11] affirmatively settled this problem by taking advantage of the “local smoothing” effect estimates together with a suitable uniqueness result.

In this paper we establish the weak continuity in $L^2(\mathbb{R})$ of the flow map for the Benjamin-Ono equation (1.1). The main idea of the proof of this result is similar to that used in [11], i.e., we shall prove that the desired weak continuity is ensured by certain local compactness results coupled with suitable uniqueness. However, unlike the cubic nonlinear Schrödinger case where local compactness is obtained from “local smoothing” effect estimates of the equation, in the present Benjamin-Ono case this will be derived from the properties of general functions in the space $F^\sigma$ in which local solutions of the problem (1.1) are constructed. Another interesting difference lies in the fact that, unlike the cubic nonlinear Schrödinger case, the uniqueness for (1.1) is only established in [13] for limits of smooth solutions.

To state our main result, we recall that $(p,q)$ is called an admissible pair for the operator $\partial_t + H\partial_x^2$ if it satisfies the following conditions: $2 \leq p \leq \infty$, $4 \leq q \leq \infty$, and $2/q = 1/2 - 1/p$. The main result of this paper reads as follows:

Theorem 1.1 Assume that $\phi_n$ weakly converges to $\phi$ in $L^2(\mathbb{R})$. Let $u_n$ and $u$ be the solutions of the problem (1.1) with initial data $\phi_n$ and $\phi$, respectively, i.e., $u_n(\cdot, t) = S(t)\phi_n$ and $u(\cdot, t) = S(t)\phi$. Then given $T > 0$, we have the following assertions:

(i) For any admissible pair $(q,p)$, $u_n$ weakly converges to $u$ in $L^q([-T,T], L^p(\mathbb{R}))$ (in case either $q = \infty$ or $p = \infty$, weak convergence here refers to $\ast$-weak convergence).

(ii) For any $|t| \leq T$, $u_n(t)$ weakly converges to $u(t)$ in $L^2(\mathbb{R})$. Moreover, this weak convergence is uniform for $|t| \leq T$ in the following sense: For any $\varphi \in L^2(\mathbb{R})$ we have

$$\lim_{n \to \infty} \sup_{|t| \leq T} \|(u_n(t) - u(t), \varphi)\| = 0,$$

(1.2)

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\mathbb{R})$.

The arrangement of this paper is as follows: In Section 2 we give a review of the well-posedness result established in [13] and introduce the spaces used in the proof of this well-posedness result. In Section 3 we derive some preliminary estimates. The proof of Theorem 1.1 will be given in Section 4 after these preparations.

Finally, we would like to give a remark on the modified Benjamin-Ono equation:

$$\partial_t u + H\partial_x^2 u + u^2 \partial_x u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.$$  

(1.3)
For this equation, it has been proved by Kenig and Takaoka in [18] that its initial value problem is globally well-posed in the Sobolev space $H^{1/2}(\mathbb{R})$, whereas the solution operator of a such problem is not uniformly continuous in any Sobolev space $H^s(\mathbb{R})$ of index $s < 1/2$ (so that $H^{1/2}(\mathbb{R})$ is a borderline space for the local well-posedness theory of this equation).

It is thus natural to ask if the flow map of this equation in $H^{1/2}(\mathbb{R})$ is weakly continuous. The answer to this question is affirmative and its proof is relatively easier, due to a priori regularities possessed by functions in the space $C([-T, T], H^{1/2}(\mathbb{R}))$. See the remark at the end of the paper.

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This second version of the manuscript (note the new title) deletes the section on the weak continuity of the flow map of the cubic nonlinear Schrodinger equation in $L^2(\mathbb{R})$ and gives a simpler proof of the weak continuity of the flow map of the modified Benjamin-Ono equation in $H^{1/2}(\mathbb{R})$. The authors are very indebted to an anonymous referee who pointed out to them that the work of Goubet and Molinet [11] already contained a proof for the cubic nonlinear Schrodinger and that a (simplified) version of the Goubet-Molinet proof could be used to give a very short proof of our mBO weak continuity result, which is presented in the last remarks of this paper. The authors would also like to thank Professor L.Molinet for helpful correspondence on these issues.

**Notations:**

- For $1 \leq p \leq \infty$, $\| \cdot \|_p$ denotes the norm in the Lebesgue space $L^p(\mathbb{R})$.

- For $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and a function $u = u(x, t)$ ($x \in \mathbb{R}$, $t \in \mathbb{R}$), $\|u\|_{L^q_t L^p_x}$ and $\|u\|_{L^q_t L^p_x}$ denote norms of the mappings $t \mapsto u(\cdot, t)$ and $x \mapsto u(x, \cdot)$ in the spaces $L^q_t(\mathbb{R}_t, L^p_x(\mathbb{R}_x))$ and $L^p_x(\mathbb{R}_x, L^q_t(\mathbb{R}_t))$, respectively. In case that $\mathbb{R}_t$ is replaced by $[-T, T]$ or $\mathbb{R}_x$ by $[-R, R]$, the corresponding notation $L^q_T$ or $L^p_R$ in the norm notation will be replaced by $L^q_T$ or $L^p_R$, respectively, so that $\|u\|_{L^q_T L^p_R}$ denotes the norm of the mapping $t \mapsto u(\cdot, t)$ in the space $L^q([-T, T], L^p([-R, R]))$, etc. .

- $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ denote Fourier transformations in the variables $(x, t)$, $x$ and $t$, respectively; they will also be denoted as $\sim$, $\sim^1$ and $\sim^2$, respectively. The dual variables of $x$ and $t$ are denoted as $\xi$ and $\tau$, respectively. Thus $\tilde{u}(\xi, \tau) = \mathcal{F}(u)(\xi, \tau)$,
\( \hat{u}^1(\xi,t) = \mathcal{F}_1(u)(\xi,t) \), and \( \hat{u}^2(x,\tau) = \mathcal{F}_2(u)(x,\tau) \). In case no confusion may occur we often omit 1 and 2 in the notations \( \sim^1 \) and \( \sim^2 \), so that \( \hat{\varphi}(\xi) = \mathcal{F}_1(\varphi)(\xi) \) for \( \varphi = \varphi(x) \), and \( \hat{\psi}(\tau) = \mathcal{F}_2(\psi)(\tau) \) for \( \psi = \psi(t) \). The inverses of \( \mathcal{F}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are denoted by \( \mathcal{F}^{-1}, \mathcal{F}_1^{-1} \) and \( \mathcal{F}_2^{-1} \), respectively.

- \( \mathcal{H} \) denotes the Hilbert transformation, i.e., \( \mathcal{H} \varphi = \mathcal{F}_\xi^{-1}[-i \text{sgn } \xi \cdot \hat{\varphi}(\xi)] \) for \( \varphi \in S'(\mathbb{R}) \) such that \( \text{sgn } \xi \cdot \hat{\varphi}(\xi) \) makes sense and belongs to \( S'(\mathbb{R}) \). If \( \varphi \) is a locally integrable function they we have
  \[
  \mathcal{H} \varphi = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x-y} dy
  \]
  in case the right-hand side makes sense.

- For a real \( s \), \( D^s_x \) and \( D^s_t \) denote absolute derivatives of order \( s \) in the \( x \) and \( t \) variables, respectively, i.e., \( \mathcal{D}_x^s \varphi(x) = \mathcal{F}_1^{-1}[|\xi|^s \hat{\varphi}(\xi)] \) for \( \varphi \in S'(\mathbb{R}) \) such that \( |\xi|^s \hat{\varphi}(\xi) \in S'(\mathbb{R}) \), and similarly for \( \mathcal{D}_t^s \). \( \langle D_x \rangle^s \) denotes the Fourier multiplier operator with symbol \( (\langle \xi \rangle^s)^{1/2} \), i.e., \( \langle D_x \rangle^s \varphi(x) = \mathcal{F}_1^{-1}[\langle \xi \rangle^s \hat{\varphi}(\xi)] \) for \( \varphi \in S'(\mathbb{R}) \).

- For a real \( s \), \( \hat{H}^s \) and \( H^s \) respectively denote the homogeneous and inhomogeneous \( L^2 \)-type Sobolev spaces on \( \mathbb{R} \) of index \( s \).

## 2 Review of \( L^2 \) well-posedness

Before giving the proof of Theorem 1.1, let us first make a short review to the well-posedness result established in [13]. In some previous work (cf. [14] [34] [35] for instance) it has been proved that the problem (1.1) is globally well-posed in \( H^\sigma(\mathbb{R}) \) for large \( s \), and the best result is \( \sigma \geq 1 \) obtained by Tao in [35]. By these results, there exists a continuous mapping \( S^\infty : H^\sigma(\mathbb{R}) := \cap_{\sigma > 0} H^\sigma(\mathbb{R}) \to C(\mathbb{R}, H^\infty(\mathbb{R})) \), such that for every \( \phi \in H^\infty(\mathbb{R}), u = S^\infty \phi \in C(\mathbb{R}, H^\infty(\mathbb{R})) \) is a solution of (1.1). For \( T > 0 \) let \( S^T : H^\infty(\mathbb{R}) \to C([-T, T], H^\infty(\mathbb{R})) \) be the restriction of the mapping \( S^\infty \) to the time interval \([-T, T] \). The result of [13] shows that the restriction \( \sigma \geq 1 \) can be weakened to \( \sigma \geq 0 \).

We copy the main result of [13] (see Theorem 1.1 there) as follows:

**Theorem 2.1**  
(a) For any \( T > 0 \), the mapping \( S^T_\sigma : H^\infty(\mathbb{R}) \to C([-T, T], H^\infty(\mathbb{R})) \) extends uniquely to a continuous mapping \( S^T_\sigma : L^2(\mathbb{R}) \to C([-T, T], L^2(\mathbb{R})) \) and \( \| S^T_\sigma(\phi)(\cdot, t) \|_2 = \| \phi \|_2 \) for any \( t \in [-T, T] \) and \( \phi \in L^2(\mathbb{R}) \). Moreover, for any \( \phi \in L^2(\mathbb{R}) \), the function \( u = S^T_\sigma(\phi) \) solves the initial-value problem (1.1) in \( C([-T, T], H^2(\mathbb{R})) \).

(b) In addition, for any \( \sigma \geq 0 \), \( S^T_\sigma(H^\sigma(\mathbb{R})) \subseteq C([-T, T], H^\sigma(\mathbb{R})) \), \( \| S^T_\sigma(\phi)(\cdot, t) \|_{C([-T, T], H^\sigma)} \leq C(T, \sigma, \| \phi \|_{H^\sigma}) \), and the mapping \( S^T_\sigma = S^T_\sigma|_{H^\sigma(\mathbb{R})} : H^\sigma(\mathbb{R}) \to C([-T, T], H^\sigma(\mathbb{R})) \) is continuous. □
Remark From the discussion of [13] we see that for any \( \phi \in L^2(\mathbb{R}) \), the solution \( u = S^2_t(\phi) \) has more regularity than merely being in \( C([-T,T],L^2(\mathbb{R})) \); for instance, we have \( u \in L^8([-T,T],L^4(\mathbb{R})) \) (cf. Lemma 3.6 in Section 3 below and note that (4.8) is an admissible pair). Thus by inhomogeneous Strichartz estimates we see that \( \int_0^T e^{-i(t-t')\mathcal{H}\partial_x^2}u^2(\cdot,t)dt' \in C([-T,T],L^2(\mathbb{R})) \cap L^4([-T,T],L^4(\mathbb{R})) \) for any admissible pair \((p,q)\). Noticing this fact, it can be easily seen that for any \( \phi \in L^2(\mathbb{R}) \), \( u = S^2_T(\phi) \) also solves the initial-value problem (1.1) in the sense that it satisfies the integral equation

\[
u(\cdot,t) = e^{-t\mathcal{H}\partial_x^2}u_0 + \frac{1}{2} \partial_x \int_0^t e^{-i(t-t')\mathcal{H}\partial_x^2}u^2(\cdot,t')dt'
\]

for \((x,t) \in \mathbb{R} \times (-T,T)\) in distribution sense. Conversely, it can also be easily seen that if a solution \( u \) (in distribution sense) of this integral equation has certain regularity, for instance, \( u \in C([-T,T],L^2(\mathbb{R})) \cap L^4([-T,T],L^4(\mathbb{R})) \), then \( u \) also solves the initial-value problem (1.1) in \( C([-T,T],H^{-2}(\mathbb{R})) \).

The main ingredients in proving the above result are a gauge transformation and the spaces \( F^\sigma \) \((\sigma \geq 0)\). For our purpose we review these ingredients in the following paragraphs.

Let \( P_{\text{low}}, P_{\text{high}} \) and \( P \) be projection operators on \( L^2(\mathbb{R}) \) defined respectively by

\[
P_{\text{low}}(\phi) = \mathcal{F}^{-1}_1(\hat{\phi}\chi_{[-2^{10},2^{10}]}), \quad P_{\text{high}}(\phi) = \mathcal{F}^{-1}_1(\hat{\phi}\chi_{[2^{10},\infty]}),
\]

\[
P(\phi) = \mathcal{F}^{-1}_1(\hat{\phi}\chi_{[0,\infty]}),
\]

where \( \chi_E \) (for given subset \( E \) of \( \mathbb{R} \)) denotes the characteristic function of the subset \( E \). Let \( \phi \in H^\infty(\mathbb{R}) \) and set

\[
\phi_{\text{low}} = P_{\text{low}}(\phi), \quad \phi_{\text{high}} = P_{\text{high}}(\phi).
\]

It can be easily verified that for real-valued \( \phi \), the function \( \phi_{\text{low}} \) is also real-valued. Let \( u_0 = S^\infty(\phi_{\text{low}}) \) be the solution of the following problem:

\[
\begin{cases}
\partial_t u_0 + \mathcal{H}\partial_x^2 u_0 + \partial_x (u_0^2/2) = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
u_0(x,0) = \phi_{\text{low}}(x), & x \in \mathbb{R}.
\end{cases}
\]

(2.1)

Note that since \( \phi_{\text{low}} \) is real-valued, we have that \( u_0 \) is also real-valued. Besides, since \( \|\phi_{\text{low}}\|_{H^\sigma} \leq C_{\sigma} \|\phi\|_{L^2} \) for any \( \sigma \geq 0 \), it follows from the equation of \( u_0 \) that

\[
\sup_{|t| \leq T} \|\partial_t^\sigma \partial_x^2 u_0(\cdot,t)\|_{L^2_x} \leq C_{\sigma_1,\sigma_2}(\|\phi\|_{L^2}) \|\phi\|_{L^2}, \quad \sigma_1, \sigma_2 \in \mathbb{Z} \cap [0, \infty). \quad (2.2)
\]
We define a gauge $U_0$ as follows: First let $U_0(0, t)$ be the solution of the following problem:

$$
\partial_t U_0(0, t) + \frac{1}{2} \mathcal{H} \partial_x u_0(0, t) + \frac{1}{4} u_0^2(0, t) = 0 \quad \text{for } t \in \mathbb{R}, \quad \text{and } U_0(0, 0) = 0,
$$

and next extend $U_0(x, t)$ to all $x \in \mathbb{R}$ (for fixed $t \in \mathbb{R}$) by using the following equation:

$$
\partial_x U_0(x, t) = \frac{1}{2} u_0(x, t).
$$

Note that since $u_0$ is real-valued, we see that $U_0$ is also real-valued. Besides, for any integers $\sigma_1, \sigma_2 \geq 0, (\sigma_1, \sigma_2) \neq (0, 0),$

$$
\sup_{|t| \leq T} \|\partial_t^\sigma_1 \partial_x^\sigma_2 U_0(\cdot, t)\|_{L^2_x} \leq C_{\sigma_1, \sigma_2}(\|\phi\|_{L^2_x})\|\phi\|_{L^2_x}. \quad (2.3)
$$

We now define

$$
\begin{cases}
  w_+ = e^{iU_0} P_{\text{high}}(u - u_0), \\
  w_- = e^{-iU_0} P_{\text{low}}(u - u_0), \\
  w_0 = P_{\text{low}}(u - u_0).
\end{cases}
$$

Then $(w_+, w_-, w_0)$ satisfies the following system of equations (see (2.10), (2.12) and (2.14) of [13]):

$$
\begin{aligned}
  &\partial_t w_+ + \mathcal{H} \partial_x^2 w_+ = E_+(w_+, w_-, w_0), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
  &\partial_t w_- + \mathcal{H} \partial_x^2 w_- = E_-(w_+, w_-, w_0), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
  &\partial_t w_0 + \mathcal{H} \partial_x^2 w_0 = E_0(w_+, w_-, w_0), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
  &\|w_+, w_-, w_0\|_{L^2} = (e^{iU_0(x, 0)} \phi_{\text{high}}, e^{-iU_0(x, 0)} \phi_{\text{low}}, 0),
\end{aligned}
$$

where (see (2.11), (2.13) and (2.15) of [13])

$$
\begin{aligned}
  &E_+(w_+, w_-, w_0) = -e^{iU_0} P_{\text{high}}[\partial_x(e^{-iU_0} w_+ + e^{iU_0} w_- + w_0)^2/2] \\
  &\quad -e^{iU_0} P_{\text{high}}\{\partial_x[u_0 \cdot P_{\text{high}}(e^{iU_0} w_-) + u_0 \cdot P_{\text{low}}(w_0)]\} \\
  &\quad +e^{iU_0}(P_{\text{high}} + P_{\text{low}})\{\partial_x[u_0 \cdot P_{\text{high}}(e^{-iU_0} w_+)]\} \\
  &\quad +2iP_+\{\partial_x^2[e^{iU_0} P_{\text{high}}(e^{iU_0} w_+)]\} \\
  &\quad -P_+(\partial_x u_0) \cdot w_+,
\end{aligned}
$$

$$
\begin{aligned}
  &E_-(w_+, w_-, w_0) = -e^{-iU_0} P_{\text{low}}[\partial_x(e^{-iU_0} w_+ + e^{iU_0} w_- + w_0)^2/2] \\
  &\quad -e^{-iU_0} P_{\text{low}}\{\partial_x[u_0 \cdot P_{\text{high}}(e^{-iU_0} w_+) + u_0 \cdot P_{\text{low}}(w_0)]\} \\
  &\quad +e^{-iU_0}(P_{\text{high}} + P_{\text{low}})\{\partial_x[u_0 \cdot P_{\text{low}}(e^{iU_0} w_-)]\} \\
  &\quad -2iP_+\{\partial_x^2[e^{-iU_0} P_{\text{low}}(e^{iU_0} w_-)]\} \\
  &\quad -P_-(\partial_x u_0) \cdot w_-,
\end{aligned}
$$

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\[ E_0(w_+, w_-, w_0) = -\frac{1}{2} P_{low} \{ \partial_x \left[ (e^{-iu_0}w_+ + e^{iu_0}w_- + w_0 + u_0)^2 - u_0^2 \right] \}. \]

It is immediate to see that the following relation holds (see Lemma 2.1 of [13]):

\[ u = e^{-iu_0}w_+ + e^{iu_0}w_- + w_0 + u_0. \] (2.5)

The mapping \( u \rightarrow (w_+, w_-, w_0) \) is called gauge transform (in more precise sense the components \( u_0 \) and \( U_0 \) should also be comprised into this notion; but for simplicity of the notation we omit them). The above deduction shows that if \( u \) is a smooth solution of (1.1) (or more precisely, a solution of (1.1) whose initial data belong to \( H^\infty(\mathbb{R}) \) ) then \( (w_+, w_-, w_0) \) is a solution of (2.4). The converse assertion cannot be directly verified. The proof (for smooth \( \phi \) ) that if \( (w_+, w_-, w_0) \) is a solution of (2.4) then the expression \( u \) given by (2.5) is a solution of (1.1) is given in Section 10 of [13]; see (10.38) in [13]. The main idea in the proof of Theorem 3.1 is as follows: First one proves that for small initial data the problem (2.4) is well-posed in suitable function spaces; in particular it has a solution in \( C([-T, T], (L^2(\mathbb{R}))^3) \) depending continuously on the initial data. Using this fact and the relation (2.5) established for smooth solutions, one then proves that the solution operator \( S_T^\phi \) defined for smooth data can be extended into a continuous mapping from \( L^2(\mathbb{R}) \) to \( C([-T, T], L^2(\mathbb{R})) \). Since \( H^\infty(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \), the extension is unique, and is denoted by \( S_T^0 \). For any given \( T > 0 \) and \( \phi \in L^2(\mathbb{R}) \) with sufficiently small norm, \( u = S_T^0(\phi) \) then defines a solution of the problem (1.1) for \( |t| \leq T \). A standard scaling argument enables one to convert this small-data assertion into a local well-posedness result for (1.1) for arbitrary initial data in \( L^2(\mathbb{R}) \), and the \( L^2 \) conservation law then yields the desired global well-posedness result.

Well-posedness of the problem (2.4) is established in a class of spaces \( F^\sigma (\sigma \geq 0) \), whose definition is given below. Let \( \eta_0 : \mathbb{R} \rightarrow [0,1] \) denote an even function supported in \([-8/5, 8/5]\) and equal to 1 in \([-5/4, 5/4]\). For \( k \in \mathbb{Z}, k \geq 1 \), let \( \eta_k(\xi) = \eta_0(2^{-k}\xi) - \eta_0(2^{-k+1}\xi) \). We also denote, for all \( k \in \mathbb{Z}, \chi_k(\xi) = \eta_0(2^{-k}\xi) - \eta_0(2^{-k+1}\xi) \). It follows that

\[ \sum_{k=0}^{\infty} \eta_k(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}, \]

and

\[ \sum_{k=-\infty}^{\infty} \chi_k(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}\setminus\{0\}. \]

Note that \( \text{supp} \chi_k \subseteq [(8/5)2^k, -(8/5)2^k] \cup [5/8)2^k, (5/8)2^k]\) for all \( k \in \mathbb{Z} \), and \( \text{supp} \eta_k \subseteq [-(8/5)2^k, -(5/8)2^k] \cup [(5/8)2^k, (8/5)2^k] \) for \( k \geq 1 \). For \( k \in \mathbb{Z} \) we denote \( I_k = [-2^{-k+1}, -2^{-k-1}] \cup [2^{-k-1}, 2^{-k+1}] \), and for \( k \in \mathbb{Z}, k \geq 1 \), we also denote \( \tilde{I}_k = [-2, 2] \) if \( k = 0 \) and \( \tilde{I}_k = I_k \) if
Let $\sigma \geq 1$. Next, we denote

$$\omega(\xi) = -\xi|\xi| \quad (\xi \in \mathbb{R}) \quad \text{and} \quad \beta_{k,j} = 1 + 2^{(j-2k)/2} \quad (j, k \in \mathbb{Z}),$$

and for $k \in \mathbb{Z}$ and $j \geq 0$ we let

$$D_{k,j} := \begin{cases} 
\{ (\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \, \tau - \omega(\xi) \in \tilde{I}_j \} & \text{if } k \geq 1; \\
\{ (\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \, \tau \in \tilde{I}_j \} & \text{if } k \leq 0.
\end{cases}$$

We now define spaces $\{Z_k\}_{k=0}^{\infty}$ as follows:

$$Z_k = \begin{cases} 
X_k + Y_k & \text{if } k = 0 \text{ or } k \geq 100, \\
X_k & \text{if } 1 \leq k \leq 99,
\end{cases}$$

where

$$X_k = \{ f \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ supported in } I_k \times \mathbb{R} \text{ and } \}

\|f\|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - \omega(\xi)) f(\xi, \tau)\|_{L_{\xi,\tau}^2} < \infty \quad \text{for } k \geq 1,$n

$$X_0 = \{ f \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ supported in } \tilde{I}_0 \times \mathbb{R} \text{ and } \}

\|f\|_{X_0} := \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} 2^{j-l} \|\eta_j(\tau) \chi_l(\xi) f(\xi, \tau)\|_{L_{\xi,\tau}^2} < \infty,$n

and

$$Y_k = \{ f \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ supported in } \bigcup_{j=0}^{k-1} D_{k,j} \text{ and } \}

\|f\|_{Y_k} := 2^{-k/2} \|\mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)]\|_{L^1_{\xi}L^2_{\tau}} < \infty \quad \text{for } k \geq 1,$n

$$Y_0 = \{ f \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ supported in } \tilde{I}_0 \times \mathbb{R} \text{ and } \}

\|f\|_{Y_0} := \sum_{j=0}^{\infty} 2^{j} \|\mathcal{F}^{-1}[\eta_j(\tau)f(\xi, \tau)]\|_{L^1_{\xi}L^2_{\tau}} < \infty.$n

Let $\sigma \geq 0$. The space $F^\sigma$ is defined as follows:

$$F^\sigma = \{ u \in S'(\mathbb{R} \times \mathbb{R}) : \|u\|^2_{F^\sigma} := \sum_{k=0}^{\infty} 2^{2\sigma k} \|\eta_k(\xi)(I - \partial_\tau^2) \tilde{u}(\xi, \tau)\|_{Z_k}^2 < \infty \}. $$

$F^\sigma$ is the space which plays a role in the study of well-posedness of the problem (2.4) similar to the role of the Bourgain space $X^{\sigma,b}$ in the study of well-posedness of the KdV equation. However, the corresponding space in the space variable is not $H^\sigma$, but instead $\tilde{H}^\sigma$, which is defined as follows. First we define

$$B_0 = \{ f \in L^2(\mathbb{R}) : f \text{ supported in } \tilde{I}_0 \text{ and } \}

\|f\|_{B_0} := \inf_{g \in h} \left\{ \|F_{\xi}^{-1}(g)\|_{L^1_{\xi}} + \sum_{l=-\infty}^{1} 2^{-l} \|\chi_l h\|_{L^2_{\xi}} \right\} < \infty.$$
It is clear that \( \| f \|_{L^2} \leq 2 \| f \|_{B_0} \). Then we define

\[
\tilde{H}^\sigma = \{ \phi \in L^2(\mathbb{R}) : \| \phi \|_{H^\sigma}^2 := \| \eta_0 \hat{\phi} \|_{B_0}^2 + \sum_{k=1}^{\infty} 2^{2\sigma k} \| \eta_k \hat{\phi} \|_{L^2}^2 < \infty \}.
\]

Since \( B_0 \hookrightarrow L^2 \), we see that \( \tilde{H}^\sigma \hookrightarrow H^\sigma \), and \( \| \phi \|_{H^\sigma} \leq \sqrt{2} \| \phi \|_{\tilde{H}^\sigma} \). By Lemma 4.2 of [13] we know that

\[
F^\sigma \subseteq C(\mathbb{R}, \tilde{H}^\sigma)
\]

for any \( \sigma \geq 0 \), and the embedding is continuous.

Given \( T > 0 \), we denote by \( F_T^\sigma \) the restriction of the space \( F^\sigma \) in \( \mathbb{R} \times [-T, T] \). From the discussion in Section 10 of [13] we have:

**Theorem 2.2** Given \( T > 0 \), there exists corresponding \( \varepsilon > 0 \) such that for any \( \psi_+ \), \( \psi_- \), \( \psi_0 \in \tilde{H}^0 \) satisfying

\[
\| \psi_+ \|_{\tilde{H}^0} + \| \psi_- \|_{\tilde{H}^0} + \| \psi_0 \|_{\tilde{H}^0} \leq \varepsilon,
\]

the initial value problem

\[
\begin{align*}
&\partial_t v_+ + \mathcal{H} \partial_x^2 v_+ = E_+(v_+, v_-, v_0), \quad x \in \mathbb{R}, \quad t \in [-T, T], \\
&\partial_t v_- + \mathcal{H} \partial_x^2 v_- = E_-(v_+, v_-, v_0), \quad x \in \mathbb{R}, \quad t \in [-T, T], \\
&\partial_t v_0 + \mathcal{H} \partial_x^2 v_0 = E_0(v_+, v_-, v_0), \quad x \in \mathbb{R}, \quad t \in [-T, T], \\
&(v_+, v_-, v_0)|_{t=0} = (\psi_+, \psi_-, \psi_0)
\end{align*}
\]

has a solution \( (v_+, v_-, v_0) \in (F_T^0)^3 \), which lies in a small ball \( B_{\varepsilon'} \) of \( (F_T^0)^3 \) and is the unique solution of (2.7) in this ball, where \( \varepsilon' = \varepsilon'(\varepsilon) > 0 \) is such that \( \varepsilon' \to 0 \) as \( \varepsilon \to 0 \), and the mapping \( (\psi_+, \psi_-, \psi_0) \to (v_+, v_-, v_0) \) from \( (\tilde{H}^0)^3 \) to \( (F_T^0)^3 \) is continuous. Moreover, if \( \psi_+ \), \( \psi_- \), \( \psi_0 \in \tilde{H}^\sigma \) for some \( \sigma > 0 \) then \( (v_+, v_-, v_0) \in (F_T^\sigma)^3 \), and the mapping \( (\psi_+, \psi_- \psi_0) \to (v_+, v_-, v_0) \) from \( (\tilde{H}^\sigma)^3 \) to \( (F_T^\sigma)^3 \) is continuous. \( \square \)

As we saw before, the relation (2.5) connecting the solution \( u \) of (1.1) with the solution \( (w_+, w_-, w_0) \) of (2.4) was only established for smooth initial data. With the aid of Theorem 2.2, we can extend it to all solutions with \( L^2 \) data, i.e., we have the following result:

**Theorem 2.3** Given \( T > 0 \), there exists corresponding \( \varepsilon > 0 \) such that for any \( \phi \in L^2(\mathbb{R}) \) satisfying \( \| \phi \|_{L^2} \leq \varepsilon \), the solution \( u = S^0_T(\phi) \) of the problem (1.1) has the expression (2.5), with \( u_0 \) and \( U_0 \) as in (2.1)–(2.3), and \( (w_+, w_-, w_0) \) being the unique solution of (2.4) in (a small neighborhood of the origin of) the space \( (F_T^0)^3 \) with norm \( \leq \varepsilon'(\varepsilon) \).
Proof: By (2.9) and Lemma 10.1 of [13] we see that for any $\phi \in H^\sigma (\sigma \geq 0)$ we have
\[(e^{it_0\cdot 0})f_{\phi+\text{high}}, e^{-it_0\cdot 0})f_{\phi-\text{high}}, 0) \in (\widehat{H}^\sigma)^3,
\]
and the mapping $\phi \rightarrow (\psi_+ (\phi), \psi_- (\phi), \psi_0 (\phi)) := (e^{it_0\cdot 0})f_{\phi+\text{high}}, e^{-it_0\cdot 0})f_{\phi-\text{high}}, 0)$ from $H^\sigma$ to $(\widehat{H}^\sigma)^3$ is continuous. Using this assertion particularly to $\sigma = 0$, we see that for $\varepsilon > 0$ as in (2.6), there exists corresponding $\varepsilon'$ such that if $\|\phi\|_{L^2} \leq \varepsilon'$ then
\[
\|\psi_+ (\phi)\|_{\widehat{H}^0} + \|\psi_- (\phi)\|_{\widehat{H}^0} + \|\psi_0 (\phi)\|_{\widehat{H}^0} \leq \varepsilon.
\]
By Theorem 2.2, for such $\phi \in L^2 (\mathbb{R})$ the problem (2.4) has a unique solution $(w_+, w_-, w_0) \in (F_T^0)^3$. We now assume that $\phi \in L^2 (\mathbb{R})$ is a such function, i.e., $\|\phi\|_{L^2} \leq \varepsilon'$, and let $u = S_T^0(\phi)$. Let $\phi_n = F_T^{-1}(\sigma_X[-2^{0n}, 2^{0n}])$, $n = 1, 2, \ldots$. Then we have $\phi_n \in H^\infty (\mathbb{R})$,
\[
\|\phi_n\|_{L^2} \leq \|\phi\|_{L^2} \leq \varepsilon', \quad n = 1, 2, \ldots, \quad \text{and} \quad \lim_{n \to \infty} \|\phi_n - \phi\|_{L^2} = 0.
\]
Let $u_n = S_T^\infty(\phi_n)$, and let $u_{n0}, U_{n0}, w_{n+}, w_{n-}, w_{n0}$ be the corresponding counterparts of $u_0, U_0, w_+, w_-, w_0$ defined before when $\phi$ is replaced by $\phi_n, n = 1, 2, \ldots$. Then we have
\[
u_n = e^{-it_0}w_{n+} + e^{it_0}w_{n-} + w_{n0} + u_{n0}, \quad n = 1, 2, \ldots.
\]
From the special construction of the function $\phi_n$ we see that $P_{\text{low}}(\phi_n) = P_{\text{low}}(\phi)$ for all $n \in \mathbb{N}$, so that $u_{n0} = u_0$ for all $n \in \mathbb{N}$ and, consequently, $U_{n0} = U_0$ for all $n \in \mathbb{N}$. Thus, the above relations can be rewritten as follows:
\[
u_n = e^{-it_0}w_{n+} + e^{it_0}w_{n-} + w_{n0} + u_0, \quad n = 1, 2, \ldots.
\]
Note that $(w_{n+}, w_{n-}, w_{n0})$ are in a small ball in $(F_T^0)^3$. Using Lemma 10.1 in [13] and the facts that $U_{n0} = U_0$ for all $n \in \mathbb{N}$ and $\phi_n \to \phi$ strongly in $L^2 (\mathbb{R})$, we see that
\[
\|\psi_+ (\phi_n) - \psi_+ (\phi)\|_{\widehat{H}^0} + \|\psi_- (\phi_n) - \psi_- (\phi)\|_{\widehat{H}^0} + \|\psi_0 (\phi_n) - \psi_0 (\phi)\|_{\widehat{H}^0} \to 0 \quad \text{as} \quad n \to \infty.
\]
Thus, by the continuity assertion in Theorem 2.2 we conclude that
\[
\|w_{n+} - w_+\|_{F_T^0} + \|w_{n-} - w_-\|_{F_T^0} + \|w_{n0} - w_0\|_{F_T^0} \to 0 \quad \text{as} \quad n \to \infty.
\]
Since $F_T^0$ is continuously embedded into $C([-T, T], L^2 (\mathbb{R}))$, this implies that
\[
\sup_{|t| \leq T} \|w_{n+}(\cdot, t) - w_+(\cdot, t)\|_2 + \sup_{|t| \leq T} \|w_{n-}(\cdot, t) - w_-(\cdot, t)\|_2 + \sup_{|t| \leq T} \|w_{n0}(\cdot, t) - w_0(\cdot, t)\|_2 \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence, by letting $n \to \infty$ in (2.8) and using the facts that $u = S_T^0(\phi) = \lim_{n \to \infty} S_T^\infty(\phi_n)$ (in $C([-T, T], L^2 (\mathbb{R}))$ norm) and $u_n = S_T^0(\phi_n)$, we see that (2.5) follows. To get the desired assertion we only need to re-deny $\varepsilon'$ as $\varepsilon$. This completes the proof. \qed
3 Preliminary estimates

Lemma 3.1 For any \( k \geq 0 \), if \( f_k \in Z_k \) then
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^\infty_r L^1} \leq C 2^{-k/2} \|f_k\|_{Z_k}. \tag{3.1}
\]

Proof: For \( k \geq 1 \), this assertion has been proved in [13] (see Lemma 4.2 (c) of [13]). Hence, in the sequel we only consider the case \( k = 0 \).

Let \( \phi_0 \in C_0^\infty(\mathbb{R}) \) such that \( \phi_0(\xi) = 1 \) for \( |\xi| \leq 2 \). Next let \( \psi \in C_0^\infty(\mathbb{R}) \) such that \( \text{supp} \psi \subseteq [-5/2, 5/2] \), \( \psi(\xi) = 1 \) for \( |\xi| \leq 2/5 \), and define \( \phi_k(\tau) = \psi(2^{-k-2}\tau) - \psi(2^{-k+2}\tau) = \psi_0(2^{-k}\tau) \) for \( k \geq 1 \), where \( \psi_0(\tau) = \psi(\tau/4) - \psi(4\tau) \). Then \( \phi_k \in C_0^\infty(\mathbb{R}) \) and \( \phi_k(\tau) = 1 \) if \( (5/8)2^k \leq |\tau| \leq (8/5)2^k \) \( (k \geq 1) \). Hence, since \( \text{supp} \eta_k \subseteq [-2(8/5)2^k, -(8/5)2^k] \cup [(5/8)2^k, (8/5)2^k] \) for \( k \geq 1 \), we have \( \phi_k(\tau) \eta_k(\tau) = \eta_k(\tau) \) for all \( k \geq 1 \).

We first assume that \( f_0 \in X_0 \). Then, since \( f_0 \) is supported in \( \tilde{I} \times \mathbb{R} \), we have
\[
f_0(\xi, \tau) = \phi_0(\xi) f_0(\xi, \tau) = \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \eta_j(\tau) \chi_l(\xi) \phi_0(\xi) f_0(\xi, \tau)
= \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \phi_j(\tau) \eta_j(\tau) \chi_l(\xi) \phi_0(\xi) f_0(\xi, \tau) = \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \phi_0(\xi) \phi_j(\tau) \cdot f_{0j}(\xi, \tau),
\]
where \( f_{0j}(\xi, \tau) = \eta_j(\tau) \chi_l(\xi) f_0(\xi, \tau) \). Hence,
\[
\mathcal{F}^{-1}(f_0(\xi, \tau)) = \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \mathcal{F}^{-1}(\phi_0(\xi) \phi_j(\tau) \cdot f_{0j}(\xi, \tau))
= \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \mathcal{F}^{-1}(\phi_0(\xi) \phi_j(\tau)) \ast \mathcal{F}^{-1}(f_{0j}(\xi, \tau)), \tag{3.2}
\]
which yields
\[
\|\mathcal{F}^{-1}(f_0)\|_{L^\infty_r L^1_t} \leq \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \|\mathcal{F}^{-1}(\phi_0(\xi) \phi_j(\tau))\|_{L^\infty_r L^1_t} \|\mathcal{F}^{-1}(f_{0j}(\xi, \tau))\|_{L^2_{r,t}}
\leq C \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \|\phi_0(\xi)\|_{L^2_{\xi}} \|2^j \tilde{\psi}_0(2^j t)\|_{L^1_t} \|f_{0j}(\xi, \tau)\|_{L^2_{\xi,t}} \quad (\tilde{\psi}_0 = \mathcal{F}_2^{-1}(\bar{\psi}_0))
\leq C \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \|\eta_j(\tau) \chi_l(\xi) f_0(\xi, \tau)\|_{L^2_{\xi,t}} \leq C \|f_0\|_{X_0}.
\]

We next assume that \( f_0 \in Y_0 \). Then as before we have
\[
f_0(\xi, \tau) = \phi_0(\xi) f_0(\xi, \tau) = \sum_{j=0}^{\infty} \eta_j(\tau) \phi_0(\xi) f_0(\xi, \tau) = \sum_{j=0}^{\infty} \phi_0(\xi) \phi_j(\tau) \cdot f_{0j}(\xi, \tau),
\]

where \( f_{0j}(\xi, \tau) = \eta_j(\tau) f_0(\xi, \tau) \). Hence,

\[
\mathcal{F}^{-1}[f_0(\xi, \tau)] = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\phi_0(\xi) \phi_j(\tau) \cdot f_{0j}(\xi, \tau)] = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\phi_0(\xi) \phi_j(\tau)] \ast \mathcal{F}^{-1}[f_{0j}(\xi, \tau)], \quad (3.3)
\]

which yields

\[
\|\mathcal{F}^{-1}(f_0)\|_{L^\infty_x L^2_t} \leq \sum_{j=0}^{\infty} \|\mathcal{F}^{-1}[\phi_0(\xi) \phi_j(\tau)]\|_{L^\infty_x L^1_t} \|\mathcal{F}^{-1}[f_{0j}(\xi, \tau)]\|_{L^1_x L^2_t} \\
\leq C \sum_{j=0}^{\infty} \|\phi_0(\xi)\|_{L^1_x} \|2^j \tilde{\psi}_0(2^j t)\|_{L^1_t} \|\mathcal{F}^{-1}[\eta_j(\tau) f_0(\xi, \tau)]\|_{L^1_x L^2_t} \quad (\tilde{\psi}_0 = \mathcal{F}^{-1}_2(\psi_0)) \\
\leq C \sum_{j=0}^{\infty} \|\mathcal{F}^{-1}[\eta_j(\tau) f_0(\xi, \tau)]\|_{L^1_x L^2_t} \leq C \|f_0\|_{Y_0}.
\]

Now let \( f_0 \in Z_0 \). Then there exists \( g_0 \in X_0 \) and \( h_0 \in Y_0 \) such that

\[
f_0 = g_0 + h_0 \quad \text{and} \quad \|g_0\|_{X_0} + \|h_0\|_{Y_0} \leq 2 \|f_0\|_{Z_0}.
\]

Thus

\[
\|\mathcal{F}^{-1}(f_0)\|_{L^\infty_x L^2_t} \leq \|\mathcal{F}^{-1}(g_0)\|_{L^\infty_x L^2_t} + \|\mathcal{F}^{-1}(h_0)\|_{L^\infty_x L^2_t} \leq C(\|g_0\|_{X_0} + \|h_0\|_{Y_0}) \leq C \|f_0\|_{Z_0}.
\]

This completes the proof of Lemma 3.1. \( \square \)

In the proof of the following lemma we shall use the following fact: If \( f_k \in Z_k \ (k \geq 0) \) then for any \( \alpha \geq 0 \),

\[
\| \xi^\alpha f_k(\xi, \tau)\|_{Z_k} \leq C 2^{\alpha k} \|f_k\|_{Z_k}.
\]

This is an immediate consequence of Lemma 4.1 a) of [13].

**Lemma 3.2** If \( w \in F^0 \) then for any \( 0 \leq \theta < 1 \) we have \( D^\theta_x w \in L^\infty_x L^2_t \), and

\[
\|D^\theta_x w\|_{L^\infty_x L^2_t} \leq C \|w\|_{F^0}. \quad (3.4)
\]

**Proof:** Let \( f_k(\xi, \tau) = \eta_k(\xi) \bar{w}(\xi, \tau) \), \( k = 0, 1, 2, \ldots \), where \( \bar{w} = F(w) \). Then \( w \in F^0 \) implies that \( (I - \partial^2_\tau)f_k \in Z_k \), \( k = 0, 1, 2, \ldots \), and

\[
\|w\|_{F^0} = \left( \sum_{k=0}^{\infty} \|(I - \partial^2_\tau)f_k\|_{Z_k}^2 \right)^{\frac{1}{2}} < \infty.
\]
Since \( \tilde{w}(\xi, \tau) = \sum_{k=0}^{\infty} f_k(\xi, \tau) \), for any \( 0 \leq \theta < 1 \) we have

\[
(1 + t^2)D_{x}^{\theta}w(x, t) = \sum_{k=0}^{\infty} \mathcal{F}^{-1}[[\xi]^{\theta}(I - \partial_{\tau}^{2})f_k(\xi, \tau)].
\]

Hence, by Lemma 3.1 we have

\[
\| (1 + t^2)D_{x}^{\theta}w(x, t) \|_{L^\infty_{\xi}L_{x}^{2}} \leq \sum_{k=0}^{\infty} \| \mathcal{F}^{-1}[[\xi]^{\theta}(I - \partial_{\tau}^{2})f_k(\xi, \tau)] \|_{L^\infty_{\xi}L_{x}^{2}}
\]

\[
\leq C \sum_{k=0}^{\infty} 2^{-\frac{k}{2}} \| \xi \|^{\theta}(I - \partial_{\tau}^{2})f_k(\xi, \tau) \|_{Z_k} \leq C \sum_{k=0}^{\infty} 2^{-\frac{k}{2}} \| (I - \partial_{\tau}^{2})f_k(\xi, \tau) \|_{Z_k}
\]

\[
\leq C \left( \sum_{k=0}^{\infty} 2^{-(1-\theta)k} \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} \| (I - \partial_{\tau}^{2})f_k(\xi, \tau) \|_{Z_k}^2 \right)^{\frac{1}{2}} \leq C \| w \|_{L^0}.
\]

From this estimate (3.4) follows immediately. \( \square \)

**Lemma 3.3** Let \( 0 \leq \theta < 1 \). For any \( k \geq 0 \), if \( f_k \in Z_k \) then

\[
\| D_{t}^{\theta}\mathcal{F}^{-1}(f_k) \|_{L^\infty_{\xi}L_{t}^{2}} \leq C 2^{-(1-\theta)k/2} \| f_k \|_{Z_k}. \tag{3.5}
\]

**Proof**: We first assume that \( k \geq 1 \) and \( f_k \in X_k \). Let \( f_{k,j}(\xi, \tau) = \eta_j(\tau - \omega(\xi))f_k(\xi, \tau), \) \( j \in \mathbb{Z}, j \geq 0 \). Then \( \text{supp} f_{k,j} \subseteq D_{k,j}, \)

\[
f_k(\xi, \tau) = \sum_{j=0}^{\infty} f_{k,j}(\xi, \tau), \tag{3.6}
\]

and

\[
\| f_k \|_{X_k} = \sum_{j=0}^{\infty} 2^{j/2}(1 + 2^{(j-2k)/2}) \| f_{k,j}(\xi, \tau) \|_{L^{2}_{\xi,\tau}}. \tag{3.7}
\]

From (3.6) we have

\[
D_{t}^{\theta}\mathcal{F}^{-1}(f_k) = \mathcal{F}^{-1}[[\tau]^{\theta}f_k(\xi, \tau)] = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[[\tau]^{\theta}f_{k,j}(\xi, \tau)]
\]

From the proof of (3.1) (see Line 3, Page 763 of [13]) we know that

\[
\| \mathcal{F}^{-1}[[\tau]^{\theta}f_{k,j}(\xi, \tau)] \|_{L^\infty_{\xi}L_{t}^{2}} \leq C 2^{-k/2}2^{j/2} \| \tau \|^{\theta} \| f_{k,j}(\xi, \tau) \|_{L^{2}_{\xi,\tau}}.
\]
Since $f_{k,j}$ is supported in $D_{k,j}$, we have $|\xi| \leq C 2^k$ and $|\tau - \omega(\xi)| \leq C 2^j$ for $(\xi, \tau) \in \text{supp}(f_{k,j})$. If $j \leq 2k$ then we have

$$|\tau| \leq |\tau - \omega(\xi)| + |\omega(\xi)| \leq C 2^j + C 2^{2k} \leq C 2^j,$$

so that

$$\|\mathcal{F}^{-1}[\tau f_{k,j}(\xi, \tau)]\|_{L^\infty L^2_t} \leq C 2^{-k/2} 2^{j/2} 2^{\theta k/2} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}} = C 2^{-(1-\theta)k/2} 2^{j/2} 2^{\theta(j-2k)/4} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}} \leq C 2^{-(1-\theta)k/2} 2^{j/2} \beta_{k,j} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}}.$$ 

If $j \geq 2k + 1$ then we have

$$|\tau| \leq |\tau - \omega(\xi)| + |\omega(\xi)| \leq C 2^j + C 2^{2k} \leq C 2^j,$$

so that

$$\|\mathcal{F}^{-1}[\tau f_{k,j}(\xi, \tau)]\|_{L^\infty L^2_t} \leq C 2^{-k/2} 2^{j/2} 2^{\theta j/4} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}} = C 2^{-(1-\theta)k/2} 2^{j/2} 2^{\theta(j-2k)/4} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}} \leq C 2^{-(1-\theta)k/2} 2^{j/2} \beta_{k,j} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}}.$$ 

Hence

$$\|D^j_{\xi} \mathcal{F}^{-1}(f_k)\|_{L^\infty L^2_t} \leq \sum_{j=0}^\infty \|\mathcal{F}^{-1}[\tau f_{k,j}(\xi, \tau)]\|_{L^\infty L^2_t} \leq C 2^{-(1-\theta)k/2} \sum_{j=0}^\infty 2^{j/2} \beta_{k,j} \|f_{k,j}(\xi, \tau)\|_{L^2_{\xi, \tau}} = C 2^{-(1-\theta)k/2} \|f_k\|_{X_k}. \tag{3.8}$$

We next assume that $k \geq 100$ and $f_k \in Y_k$. Then $\text{supp} f_k \subseteq \bigcup_{j=0}^{k-1} D_{k,j}$, and

$$\|f_k\|_{Y_k} = 2^{-k/2} \|\mathcal{F}^{-1}[(\tau - \omega(\xi) + i) f_k(\xi, \tau)]\|_{L^1 L^2_t} = 2^{-k/2} \|F^{-1}_1[(\tau - \omega(\xi) + i) f_k(\xi, \tau)]\|_{L^1 L^2_t}. \tag{3.9}$$

Let $g_k(x, \tau) = 2^{-k/2} F^{-1}_1[(\tau - \omega(\xi) + i) f_k(\xi, \tau)] = e^{i \tau \xi} (\tau - \omega(\xi) + i) f_k(\xi, \tau) d\xi$. Then $f_k(\xi, \tau) = 2^{k/2} (\tau - \omega(\xi) + i)^{-1} \int_{-\infty}^{\infty} e^{-i \tau \xi} g_k(x, \tau) dx$ and, by (3.9), $\|f_k\|_{Y_k} = \|g_k\|_{L^1 L^2_t}$. By the fact that $\text{supp} f_k \subseteq \bigcup_{j=0}^{k-1} D_{k,j}$ we have $f_k(\xi, \tau) = \psi_k(\xi) \eta_0(2^{-k}(\tau - \omega(\xi))) f_k(\xi, \tau)$, where $\psi_k(\xi) = \eta_0(2^{-(k+1)} \xi) - \eta_0(2^{-(k-2)} \xi)$, so that

$$f_k(\xi, \tau) = 2^{k/2} \psi_k(\xi) \eta_0(2^{-k}(\tau - \omega(\xi))) (\tau - \omega(\xi) + i)^{-1} \int_{-\infty}^{\infty} e^{-i y \xi} g_k(y, \tau) dy.$$
Let
\[ h_k(y, \xi, \tau) = 2^{k/2} \psi_k(\xi) \eta_0(2^{-k}(\tau - \omega(\xi)))(\tau - \omega(\xi) + i)^{-1} e^{-iy\xi} g_k(y, \tau). \] (3.10)
Then the above calculation shows that
\[ f_k(\xi, \tau) = \int_{-\infty}^{\infty} h_k(y, \xi, \tau) dy, \] (3.11)
so that
\[ D_t^\theta \mathcal{F}^{-1}(f_k)(x, t) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{iy\tau} \tau h_k(y, \xi, \tau) d\xi d\tau \right) dy. \] (3.12)
In what follows we prove that
\[ \| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{iy\tau} \tau h_k(y, \xi, \tau) d\xi d\tau \|_{L^\infty_t L^2_x} \leq C 2^{-(1-\theta)k/2} \| g_k(y, \cdot) \|_2, \] (3.13)
where \( C \) is independent of \( k \) and \( y \). If this inequality is proved, then by (3.11) we have
\[ \| D_t^\theta \mathcal{F}^{-1}(f_k) \|_{L^\infty_t L^2_x} = \left\| \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{iy\tau} \tau h_k(y, \xi, \tau) d\xi d\tau \right) dy \right\|_{L^\infty_t L^2_x} \]
\[ \leq \int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix\xi} e^{iy\tau} \tau h_k(y, \xi, \tau) d\xi d\tau \right\|_{L^\infty_t L^2_x} dy \]
\[ \leq C 2^{-(1-\theta)k/2} \int_{-\infty}^{\infty} \| g_k(y, \cdot) \|_2 dy = C 2^{-(1-\theta)k/2} \| g_k \|_{L^1_t L^2_x}, \]
which, combined with the fact that \( \| f_k \|_{y_k} = \| g_k \|_{L^1_t L^2_x} \), yields the following estimate:
\[ \| D_t^\theta \mathcal{F}^{-1}(f_k) \|_{L^\infty_t L^2_x} \leq C 2^{-(1-\theta)k/2} \| f_k \|_{y_k}. \] (3.14)
We neglect the parameter \( y \) in (3.10) and (3.13). By the Plancherel’s theorem, (3.13) follows if we prove that
\[ \left\| \int_{-\infty}^{\infty} e^{ix\xi} \tau h_k(\xi, \tau) d\xi \right\|_{L^\infty_t L^2_x} \leq C 2^{-(1-\theta)k/2} \| g_k \|_2. \] (3.15)
To prove this estimate, we first recall that for \( k \geq 100 \) (see (4.22) in [13]),
\[ \left| \int_{-\infty}^{\infty} e^{ix\xi} \psi_k(\xi) \eta_0(2^{-k}(\tau - \omega(\xi)))(\tau - \omega(\xi) + i)^{-1} d\xi \right| \leq C 2^{-k} \] (3.16)
uniformly for \( x \) and \( \tau \). Next, we note that on the support of \( h_k \) we have \( |\xi| \leq C 2^k \) and \( |\tau - \omega(\xi)| \leq C 2^k \), which implies that \( |\tau| \leq C 2^k \). Hence, the left-hand side of (3.15) is dominated by
\[ 2^{k/2} \sup_{x, \tau} \left| \int_{-\infty}^{\infty} e^{ix\xi} \psi_k(\xi) \eta_0(2^{-k}(\tau - \omega(\xi)))(\tau - \omega(\xi) + i)^{-1} d\xi \right| \cdot \| \tau h_k(\tau) \|_{L^2_x} \]
\[ \leq 2^{k/2} \cdot C 2^{-k} \cdot C 2^{\theta k/2} \| g_k \|_2 = C 2^{-(1-\theta)k/2} \| g_k \|_2, \]
as desired.

By (3.8) and (3.14), we see that (3.5) holds for \( k \geq 1 \). We now consider the case \( k = 0 \). If \( f_0 \in X_0 \) then by (3.2) we have

\[
D_\tau^\theta \mathcal{F}^{-1} [f_0(\xi, \tau)] = \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} D_\tau^\theta \mathcal{F}^{-1} [\phi_0(\xi) \phi_j(\tau)] * \mathcal{F}^{-1} [f_{0j}(\xi, \tau)],
\]

so that

\[
\| D_\tau^\theta \mathcal{F}^{-1} (f_0) \|_{L^\infty_t L^2_x} \leq \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \| D_\tau^\theta \mathcal{F}^{-1} [\phi_0(\xi) \phi_j(\tau)] \|_{L^2_x L^1_t} \| \mathcal{F}^{-1} [f_{0j}(\xi, \tau)] \|_{L^2_t}
\]

\[
\leq C \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \| \phi_0(\xi) \|_{L^2_x} \| 2^{j(1+\frac{\theta}{2})} \tilde{\psi}_0(2^j t) \|_{L^1_t} \| f_{0j}(\xi, \tau) \|_{L^2_t} \quad (\tilde{\psi}_0 = \mathcal{F}^{-1}(\psi_0))
\]

\[
\leq C \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} 2^{j\theta} \| f_{0j}(\xi, \tau) \|_{L^2_t} \leq C \| f_0 \|_{X_0}.
\]

If \( f_0 \in Y_0 \) then by (3.3) we have

\[
D_\tau^\theta \mathcal{F}^{-1} [f_0(\xi, \tau)] = \sum_{j=0}^{\infty} D_\tau^\theta \mathcal{F}^{-1} [\phi_0(\xi) \phi_j(\tau)] * \mathcal{F}^{-1} [f_{0j}(\xi, \tau)],
\]

so that

\[
\| D_\tau^\theta \mathcal{F}^{-1} (f_0) \|_{L^\infty_t L^2_x} \leq \sum_{j=0}^{\infty} \| D_\tau^\theta \mathcal{F}^{-1} [\phi_0(\xi) \phi_j(\tau)] \|_{L^\infty_t L^1_x} \| \mathcal{F}^{-1} [f_{0j}(\xi, \tau)] \|_{L^1_t}
\]

\[
\leq C \sum_{j=0}^{\infty} \| \phi_0(\xi) \|_{L^2_x} \| 2^{j(1+\frac{\theta}{2})} \tilde{\psi}_0(2^j t) \|_{L^1_t} \| \mathcal{F}^{-1} [\eta_j(\tau) f_0(\xi, \tau)] \|_{L^1_t} \quad (\tilde{\psi}_0 = \mathcal{F}^{-1}(\psi_0))
\]

\[
\leq C \sum_{j=0}^{\infty} 2^{j\theta} \| \mathcal{F}^{-1} [\eta_j(\tau) f_0(\xi, \tau)] \|_{L^1_t L^2_x} \leq C \| f_0 \|_{Y_0}.
\]

Hence (3.5) also holds for \( k = 0 \). The proof is complete.  

\[ \square \]

**Lemma 3.4** If \( w \in F^0 \) then for any \( 0 \leq \theta < 1 \) we have \( D_\tau^\theta w \in L^\infty_t L^2_x \), and

\[
\| D_\tau^\theta w \|_{L^\infty_t L^2_x} \leq C \| w \|_{F^0}.
\]

**Proof:** Let \( f_k(\xi, \tau) = \eta_k(\xi) \tilde{w}(\xi, \tau) \), \( k = 0, 1, 2, \ldots \), where \( \tilde{w} = F(w) \). Then \( w \in F^0 \) implies that \( (I - \partial^2_\tau) f_k \in Z_k \), \( k = 0, 1, 2, \ldots \), and

\[
\| w \|_{F^0} = \left( \sum_{k=0}^{\infty} \| (I - \partial^2_\tau) f_k \|_{Z_k}^2 \right)^{\frac{1}{2}} < \infty.
\]
Since $\tilde{w}(\xi, \tau) = \sum_{k=0}^{\infty} f_k(\xi, \tau)$, for any $0 \leq \theta < 1$ we have
\[
D_t^\theta [(1 + t^2)w(x, t)] = \sum_{k=0}^{\infty} D_t^\theta F^{-1} [(I - \partial_{\tau}^2) f_k(\xi, \tau)].
\]
Hence, by Lemma 3.3 we have
\[
\| D_t^\theta [(1 + t^2)w(x, t)] \|_{L^\infty L^2} \leq \sum_{k=0}^{\infty} \| D_t^\theta F^{-1} [(I - \partial_{\tau}^2) f_k(\xi, \tau)] \|_{L^\infty L^2}
\]
\[
\leq C \sum_{k=0}^{\infty} 2^{-(1-\theta)k/2} \| (I - \partial_{\tau}^2) f_k(\xi, \tau) \|_{Z_k}
\]
\[
\leq C \left( \sum_{k=0}^{\infty} \| (I - \partial_{\tau}^2) f_k(\xi, \tau) \|^2_{Z_k} \right)^{\frac{1}{2}} = C \| w \|_{F^0}. \tag{3.18}
\]
Since $w(x, t) = (1 + t^2)^{-1} \cdot (1 + t^2)w(x, t)$, by Theorem A.12 in [15] we have
\[
\| D_t^\theta w - (1 + t^2)^{-1} D_t^\theta [(1 + t^2)w(x, t)] - D_t^\theta [(1 + t^2)^{-1} \cdot (1 + t^2)w(x, t)] \|_{L^\infty L^2}
\]
\[
\leq C \| (1 + t^2)^{-1} \|_{\infty} \| D_t^\theta [(1 + t^2)w(x, t)] \|_{L^\infty L^2}
\]
From this estimate and (3.18), we see that (3.17) follows. \qed

Lemma 3.5 Let $f_k \in Z_k$, $k \geq 0$. Then for any admissible pair $(p, q)$ we have
\[
\| F^{-1}(f_k) \|_{L^p L^q} \leq C(p, q) \| f_k \|_{Z_k}. \tag{3.19}
\]

Proof: Assume first that $k \geq 1$ and $f_k \in X_k$. Let $f_{k,j}(\xi, \tau) = \eta_j(\tau - \omega(\xi)) f_k(\xi, \tau)$, $j \in \mathbb{Z}$, $j \geq 0$. Then supp$f_{k,j} \subseteq D_{k,j}$, and (3.6), (3.7) hold. Let $f_{k,j}^\#(\xi, \tau) = f_{k,j}(\xi, \tau + \omega(\xi))$. Then supp$f_{k,j}^\# \subseteq I_k \times I_j$. We have
\[
F^{-1}(f_{k,j}) = c^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it\tau} e^{i\xi \tau} f_{k,j}(\xi, \tau) d\xi d\tau
\]
\[
= c^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it\tau} e^{i\xi \tau} e^{i\omega(\xi)} f_{k,j}(\xi, \tau + \omega(\xi)) d\xi d\tau
\]
\[
= c^2 \int_{I_j} \int_{-\infty}^{\infty} e^{it\tau} (e^{i\xi \tau} e^{i\omega(\xi)} f_{k,j}^\#(\xi, \tau) d\xi) d\tau
\]
Let $g_{k,j}^\tau(x) = c \int_{-\infty}^{\infty} e^{i\xi \tau} f_{k,j}^\#(\xi, \tau) d\xi = F_1^{-1}(f_{k,j}^\#(\cdot, \tau))$. Then $f_{k,j}^\#(\xi, \tau) = F_1(g_{k,j}^\tau)$, so that
\[
F^{-1}(f_{k,j}) = c \int_{I_j} e^{it\tau} F_1^{-1} \left( e^{i\omega(\xi)} F_1(g_{k,j}^\tau) \right) d\tau = c \int_{I_j} e^{it\tau} W(t) g_{k,j}^\tau(x) d\tau.
\]

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It follows that
\[
\|F^{-1}(f_{k,j})\|_{L^p_t L^p_x} \leq c \int_{I_j} \|W(t)g_{k,j}^\tau(x)\|_{L^p_t L^p_x} d\tau \leq C \int_{I_j} \|g_{k,j}^\tau(x)\|_2 d\tau
\]
(by Strichartz estimate)
\[
= C \int_{I_j} \left[ \int_{-\infty}^\infty |f_{k,j}^\tau(\xi, \tau)|^2 d\xi \right]^{\frac{1}{2}} d\tau \leq C 2^{j/2} \|f_{k,j}\|_{L^2_{\xi,\tau}}. \tag{3.20}
\]

By (3.6), (3.7) and (3.20) we have
\[
\|F^{-1}(f_k)\|_{L^p_t L^p_x} \leq \sum_{j=0}^{\infty} \|F^{-1}(f_{k,j})\|_{L^p_t L^p_x} \leq C \sum_{j=0}^{\infty} 2^{j/2} \|f_{k,j}\|_{L^2_{\xi,\tau}} \leq C \|f_k\|_{X_k}. \tag{3.21}
\]

Next assume that \( k \geq 1 \) and \( f_k \in Y_k \). Then \( \text{supp} f_k \subseteq \cup_{j=0}^{k-1} D_{k,j} \), and (3.9) holds. Let \( g_k(y, \xi) \) and \( h_k(y, \xi, \tau) \) be as in the proof of Lemma 3.3. In what follows we prove that
\[
\left\| \int_{-\infty}^\infty \int_{-\infty}^\infty e^{ix\xi} e^{itr} h_k(y, \xi, \tau) d\xi d\tau \right\|_{L^2_t L^p_x} \leq C \|g_k(y, \cdot)\|_2, \tag{3.22}
\]
where \( C \) is independent of \( k \) and \( y \). If this inequality is proved, then by (3.11) we have
\[
\|F^{-1}(f_k)\|_{L^p_t L^p_x} = \left\| \int_{-\infty}^\infty \left( \int_{-\infty}^\infty e^{ix\xi} e^{itr} h_k(y, \xi, \tau) d\xi d\tau \right) dy \right\|_{L^2_t L^p_x}
\]
\[
\leq \int_{-\infty}^\infty \left\| \int_{-\infty}^\infty e^{ix\xi} e^{itr} h_k(y, \xi, \tau) d\xi d\tau \right\|_{L^2_t L^p_x} dy
\]
\[
\leq C \int_{-\infty}^\infty \|g_k(y, \cdot)\|_2 dy = C \|g_k\|_{L^1_y L^p_x},
\]
which, combined with the fact that \( \|g_k\|_{L^1_y L^p_x} = \|f_k\|_{Y_k} \), gives the desired assertion.

We neglect the parameter \( y \) in (3.10) and (3.22). Since \( k \geq 100 \) and \( |\xi| \in [2^{k-2}, 2^{k+2}] \), we may assume that the function \( g_k = g_k(\tau) \) in (3.10) is supported in the set \( \{ \tau : |\tau| \in [2^{2k-10}, 2^{2k+10}] \} \). Let \( g_k^+ = g_k \cdot \chi_{[0,\infty)} \), \( g_k^- = g_k \cdot \chi_{(-\infty,0]} \), and define the corresponding function \( h_k^+ \) and \( h_k^- \) as in (3.10). By symmetry, it suffices to prove (2.22) for \( h_k^+ \), which is supported in \( \{ (\xi, \tau) : \xi \in [-2^{k-2}, -2^{k+2}], \tau \in [2^{2k-10}, 2^{2k+10}] \} \). Since \( \omega(\xi) = -\xi |\xi| \), we have \( \tau - \omega(\xi) = \tau - \xi^2 \) on the support of \( h_k^+ \), and \( h_k^+ (\xi, \tau) = 0 \) unless \( |\sqrt{\tau} + |\xi| \leq C \). Let
\[
h_k^+(\xi, \tau) = 2^{k/2} \psi^\tau_k(-\sqrt{\tau}) \eta_0(\sqrt{\tau} + \xi)[\tau - \xi^2 + (\sqrt{\tau} + \xi) + i \sqrt{\tau} 2^{-k}]^{-1} g_k(\tau).
\]
By the argument in Lines 25–28 in Page 762 of [13], we know that
\[
\|h_k^+ - h_k^-\|_{X_k} \leq C \|g_k^+\|_2.
\]
Hence, by (3.21) we have
\[
\|\mathcal{F}^{-1}(h^+_k - h'^+_k)\|_{L^q_t L^p_x} \leq C\|g^+_k\|_2.
\] (3.23)

To estimate \(\|\mathcal{F}^{-1}(h'^+_k)\|_{L^q_t L^p_x}\), we make the change of variables \(\xi \to \xi'\) by letting \(\xi = \xi' - \sqrt{\tau}\).
Then we have
\[
\mathcal{F}^{-1}(h'^+_k)(x, t) = 2^{k/2} \int_{-\infty}^{\infty} e^{it\tau} e^{-ix\sqrt{\tau}} \psi_k(-\sqrt{\tau}) g^+_k(\tau)(2\sqrt{\tau})^{-1} d\tau \\
\times \int_{-\infty}^{-\infty} e^{i\xi'\eta_0(\xi')(\xi' + i2^{-k-1})^{-1}} d\xi'.
\]

It can be easily seen that the second integral is bounded by a constant independent of \(x\) and \(k\). Next we compute
\[
\| \int_{-\infty}^{\infty} e^{it\tau} e^{-ix\sqrt{\tau}} \psi_k(-\sqrt{\tau}) g^+_k(\tau)(2\sqrt{\tau})^{-1} d\tau \|_{L^q_t L^p_x} \\
= \| \int_{-\infty}^{\infty} e^{i\xi^2} e^{-ix\xi} \psi_k(-\xi) g^+_k(\xi^2)^2 d\xi \|_{L^q_t L^p_x} \quad \text{(by letting \(\tau = \xi^2\))}
\]
\[
\leq C \left[ \int_{-\infty}^{\infty} |\psi_k(-\xi) g^+_k(\xi^2)^2 d\xi \right]^{1/2} \quad \text{(by using Strichartz and Plancherel)}
\]
\[
= C \left[ \int_{-\infty}^{\infty} |\psi_k(-\sqrt{\tau}) g^+_k(\tau)|^2 (2\sqrt{\tau})^{-1} d\tau \right]^{1/2}
\]
\[
\leq C 2^{-k/2} \|g^+_k\|_2 \quad \text{(because \(\sqrt{\tau} \sim 2^k\))}
\]

Hence
\[
\|\mathcal{F}^{-1}(h'^+_k)\|_{L^q_t L^p_x} \leq C\|g^+_k\|_2.
\]

Combining this estimate with (3.23), we see that the desired assertion follows.

From the above deduction we see that (3.19) holds for \(k \geq 1\). For the case \(k = 0\), the argument is similar to that in the proof of Lemma 3.1. Indeed, if \(f_0 \in X_0\) then from (3.2) we have
\[
\|\mathcal{F}^{-1}(f_0)\|_{L^q_t L^p_x} \leq \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \|\mathcal{F}^{-1}[\phi_0(\xi) \phi_j(\tau)]\|_{L^q_t L^p_x} \|\mathcal{F}^{-1}[f_{0jt}(\xi, \tau)]\|_{L^2_{\xi, \tau}}
\]
\[
(1/r = 1/p + 1/2, \quad 1/s = 1/q + 1/2)
\]
\[
\leq C \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} \|\mathcal{F}^{-1}(\phi_0)\|_{L^q_\xi} |2^j \tilde{\psi}_0(2^j t)| \|f_{0jt}(\xi, \tau)\|_{L^2_{\xi, \tau}} \quad (\tilde{\psi}_0 = \mathcal{F}^{-1}(\psi_0))
\]
\[
\leq C \sum_{j=0}^{\infty} \sum_{l=-\infty}^{1} 2^{j(1-1/s)} \|\eta_j(\tau) \chi(\xi) f_0(\xi, \tau)\|_{L^2_{\xi, \tau}} \leq C \|f_0\|_{X_0}.
\]

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If \( f_0 \in Y_0 \) then from (3.3) we have
\[
\| \mathcal{F}^{-1}(f_0) \|_{L^q_t L^p_x} \leq \sum_{j=0}^{\infty} \| \mathcal{F}^{-1}[\phi_0(\xi)\phi_j(\tau)] \|_{L^q_t L^p_x} \| \mathcal{F}^{-1}[f_0(\xi, \tau)] \|_{L^q_t L^p_x} \quad (1/s = 1/q + 1/2)
\]
\[
\leq C \sum_{j=0}^{\infty} \| \mathcal{F}^{-1}(\phi_0) \|_{L^q_t} \| 2^j \tilde{\psi}_0(2^j t) \|_{L^q_t} \| \mathcal{F}^{-1}[\eta_j(\tau)f_0(\xi, \tau)] \|_{L^q_t L^p_x}
\]
\[
\leq C \sum_{j=0}^{\infty} 2^{j(1-1/s)} \| \mathcal{F}^{-1}[\eta_j(\tau)f_0(\xi, \tau)] \|_{L^q_t L^p_x} \leq C \| f_0 \|_{Y_0}.
\]
Hence the desired assertion also holds for \( k = 0 \).  \( \square \)

Using the above lemma and a similar argument to the one in the proof of Lemma 3.2 we have

**Lemma 3.6** Assume that \( w \in F^0 \). Then for any admissible pair \((p, q)\) we have \( w \in L^q_t(L^p_x(\mathbb{R})) \), and
\[
\| w \|_{L^q_t L^p_x} \leq C_{pq} \| w \|_{F^0}.
\]
(3.24)

Since \((6, 6)\) is an admissible pair, by the above lemma we have

**Corollary 3.7** Assume that \( w \in F^0 \). Then \( w \in L^6(\mathbb{R}^2) \), and
\[
\| w \|_{L^6_x L^2_t} \leq C \| w \|_{F^0}.
\]
(3.25)

Using the expression (2.5) and Lemma 3.6 we have

**Corollary 3.8** Let \( \phi \in L^2(\mathbb{R}) \) and let \( u \) be the global solution of the problem (1.1) ensured by Theorem 2.2. Then for any \( T > 0 \) and any admissible pair \((p, q)\) we have \( u \in L^q([-T, T], L^p_x(\mathbb{R})) \). Moreover, the mapping \( \phi \mapsto u \) from \( L^2(\mathbb{R}) \) to \( L^q([-T, T], L^p_x(\mathbb{R})) \) defined in this way is continuous, and there exists corresponding function \( c_{pqT} : [0, \infty) \to [0, \infty) \) such that
\[
\| u \|_{L^q_t L^p_x} \leq c_{pqT}(\| \phi \|_2).
\]
(3.26)

**Proof:** Choose \( M \geq \| \phi \|_2 \) and fix it. With \( M \) fixed in this way, let \( \delta \) be as in Corollary 3.4. By dividing the interval \([-T, T]\) into subintervals \([-\delta, \delta], [\delta, 2\delta], [2\delta, 3\delta], \ldots, [((N-1)\delta, T]\), where \( N \) is the smallest integer such that \( T \leq N\delta \), and using the \( L^2 \)-conservation law, we only need to prove the assertion holds for \( T = \delta \). For \( T = \delta \) the expression (2.5) holds, from which the desired assertion easily follows. Indeed, from (2.2) it is clear that for any admissible pair \((p, q)\) we have \( u_0 \in L^q([-T, T], L^p_x(\mathbb{R})) \),
\[
\| u_0 \|_{L^q_t L^p_x} \leq c_{pqT}(\| \phi \|_2),
\]
(3.27)
and the mapping $\phi \to u_0$ from $L^2(\mathbb{R})$ to $L^q([-T, T], L^p_x(\mathbb{R}))$ is continuous. Secondly, since $U_0$ is real, we see that $e^{\pm itU_0}$ are uniformly bounded, and it is clear that the mappings $\phi \to e^{\pm iTU_0}$ from $L^2(\mathbb{R})$ to $L^\infty(\mathbb{R} \times [-T, T])$ are continuous. Finally, by Theorem 2.1, Lemma 3.6, and the continuity of the mapping $\phi \to (e^{iU_0(\cdot,\cdot)}\phi_{\text{high}}, e^{-iU_0(\cdot,\cdot)}\phi_{\text{high}}, 0)$ from $L^2(\mathbb{R})$ to $(\tilde{H}^0)^3$, we see that for any admissible pair $(q, p)$ we have $w_+, w_-, w_0 \in L^q([-T, T], L^p_x(\mathbb{R}))$,

$$\|w_+\|_{L^q_t L^p_x} + \|w_-\|_{L^q_t L^p_x} + \|w_0\|_{L^q_t L^p_x} \leq c_{pqT}(\|\phi\|_2),$$

and the mapping $\phi \to (w_+, w_-, w_0)$ from $L^2(\mathbb{R})$ to $[L^q([-T, T], L^p_x(\mathbb{R}))]^3$ is continuous. By (2.5), (3.27), (3.28) and the uniform boundedness of $e^{\pm iTU_0}$ we have

$$\|u\|_{L^q_t L^p_x} \leq \|w_+\|_{L^q_t L^p_x} + \|w_-\|_{L^q_t L^p_x} + \|w_0\|_{L^q_t L^p_x} + \|u_0\|_{L^q_t L^p_x} \leq c_{pqT}(\|\phi\|_2).$$

Hence $u \in L^q([-T, T], L^p_x(\mathbb{R}))$ and (2.26) holds. Moreover, the above argument also shows that the mapping $\phi \to u$ from $L^2(\mathbb{R})$ to $L^q([-T, T], L^p_x(\mathbb{R}))$ is continuous. The proof is complete. \qed

4 The proof of Theorem 1.1

Since we are not clear if the space $F^0$ in which uniqueness of the solution of (2.4) is ensured is reflexive, we cannot use functional analysis to get the assertion that any bounded sequence in $F^0$ has a weakly convergent subsequence. To overcome this difficulty, we shall appeal to the following preliminary result:

**Lemma 4.1** Let $w_n \in F^0 \cap L^2(\mathbb{R}^3)$, $n = 1, 2, \cdots$. Assume that $\|w_n\|_{F^0} \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$, and there exists $T > 0$ such that $w_n(t) = 0$ for all $|t| \geq T$ and $n \in \mathbb{N}$. Assume further that as $n \to \infty$, $w_n \rightharpoonup w$ weakly in $L^2(\mathbb{R}^2)$. Then $w \in F^0$, and $\|w\|_{F^0} \leq M$, or more precisely,

$$\|w\|_{F^0} \leq \liminf_{n \to \infty} \|w_n\|_{F^0}.$$  \hspace{1cm} (4.1)

**Proof:** We fulfill the proof in three steps.

**Step 1:** We first prove that similar results hold for the spaces $Y_k$ and $X_k$. That is, taking $Y_k$ as an example and assuming that $f_n \in Y_k \cap L^2(\mathbb{R}^2)$, $n = 1, 2, \cdots$, $\|f_n\|_{Y_k} \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$, and as $n \to \infty$, $f_n \rightharpoonup f$ weakly in $L^2(\mathbb{R}^2)$, we have that $f \in Y_k$, and $\|f\|_{Y_k} \leq M$. Note that if this assertion is proved, then it follows immediately that

$$\|f\|_{Y_k} \leq \liminf_{n \to \infty} \|f_n\|_{Y_k}.$$
Consider first the case \( k \geq 1 \). Let \( \psi \in C^\infty_0(\mathbb{R}^2) \) be such that

\[
0 \leq \psi \leq 1, \quad \text{and} \quad \psi(x, t) = 1 \text{ for } |x| \leq 1, \ |t| \leq 1.
\]

Let \( \psi_R(x, t) = \psi(x/R, t/R), \ R > 1 \). Since \( \|\psi_R\|_{L^\infty_{x,t}} = 1 \), we have, for any \( R > 1 \) and \( n \in \mathbb{N} \),

\[
\|\psi_R \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f_n(\xi, \tau)]\|_{L^2_x L^2_t} \leq \|\psi_R\|_{L^\infty_{x,t}} \cdot 2^{k/2}\|f_n\|_{Y_k} \leq 2^{k/2}M. \tag{4.2}
\]

We first assume that as \( n \to \infty, f_n \to f \) strongly in \( L^2(\mathbb{R}^2) \). Let \( \varphi_k \in C^\infty_0(\mathbb{R}^2) \) be such that \( \varphi_k(\xi, \tau) = 1 \) for \( (\xi, \tau) \in \cup_{j=1}^{k-1} D_{k,j} \). Since \( \text{supp}f_n \subseteq \cup_{j=1}^{k-1} D_{k,j} \) for all \( n \in \mathbb{N} \), for any \( m, n \in \mathbb{N} \) we have

\[
\|\psi_R \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f_n(\xi, \tau)] - \|\psi_R\|_{L^\infty_{x,t}} \cdot 2^{k/2}\|f_n - f_m\|_{Y_k} \leq 2^{k/2}M.
\]

From this we see that for any \( R > 1, \psi_R \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f_n(\xi, \tau)] \) is convergent in \( L^1_x L^2_t \), so that \( \psi_R \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f_n(\xi, \tau)] \in L^1_x L^2_t \) and, by letting \( n \to \infty \) in (4.2) we get, for any \( R > 0 \),

\[
\|\psi_R \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)]\|_{L^1_x L^2_t} \leq 2^{k/2}M. \tag{4.3}
\]

Clearly, \( \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)] \) is a measurable function and, as \( R \to \infty, \psi_R \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)] \) pointwise converges to \( \mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)] \). Hence, by letting \( R \to \infty \) in (2.3) and using Fatou’s lemma we get

\[
\|\mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)]\|_{L^1_x L^2_t} \leq 2^{k/2}M,
\]

so that \( f \in Y_k \) and \( \|f\|_{Y_k} \leq M \). We next assume that as \( n \to \infty, f_n \to f \) weakly in \( L^2(\mathbb{R}^2) \). By a well-known theorem in functional analysis, we know that there is another sequence \( f'_n, n = 1, 2, \cdots \), with each \( f'_n \) being a convex combination of finite elements in \{\( f_n \}\), such that as \( n \to \infty, f'_n \to f \) strongly in \( L^2(\mathbb{R}^2) \). Clearly, \( \|f'_n\|_{Y_k} \leq M \) for all \( n \in \mathbb{N} \). Hence, by the assertion we have just proved it follows that \( f \in Y_k \) and \( \|f\|_{Y_k} \leq M \). This proves the desired assertion for the case \( k \geq 1 \).

Consider next the case \( k = 0 \). For any \( N \in \mathbb{N} \) we have

\[
\sum_{j=0}^{N} 2^j \|\mathcal{F}^{-1}[h_j(\tau)f_n(\xi, \tau)]\|_{L^2_x L^2_t} \leq \|f_n\|_{Y_0} \leq M, \quad n = 1, 2, \cdots.
\]
Since for every $0 \leq j \leq N$, $\eta_j(\tau)f_n(\xi, \tau)$ have supports contained in a common compact set, the argument for the case $k \geq 1$ applies to the sequences $\{\eta_j(\tau)f_n(\xi, \tau)\}$, $j = 0, 1, \cdots, N$, so that
\[
\|F^{-1}[\eta_j(\tau)f(\xi, \tau)]\|_{L^2_{\xi, \tau}} \leq \liminf_{n \to \infty} \|F^{-1}[\eta_j(\tau)f_n(\xi, \tau)]\|_{L^2_{\xi, \tau}}, \quad j = 0, 1, \cdots, N.
\]
Hence
\[
\sum_{j=0}^{N} 2^j \|F^{-1}[\eta_j(\tau)f(\xi, \tau)]\|_{L^2_{\xi, \tau}} \leq \liminf_{n \to \infty} \sum_{j=0}^{N} 2^j \|F^{-1}[\eta_j(\tau)f_n(\xi, \tau)]\|_{L^2_{\xi, \tau}} \leq M.
\]
By the arbitrariness of $N$, we conclude that $f \in Y_0$ and $\|f\|_{Y_0} \leq M$, as desired.

The proof for $X_k$ ($k \geq 0$) follows from a similar argument as in the proof for $Y_0$.

Step 2: We next prove that a similar result holds for the space $Y_k$, namely, assuming that $f_n \in Z_k \cap L^2(\mathbb{R}^2)$, $n = 1, 2, \cdots$, $\|f_n\|_{Z_k} \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$, and as $n \to \infty$, $f_n \rightharpoonup f$ weakly in $L^2(\mathbb{R}^2)$, then we have $f \in Z_k$, and $\|f\|_{Z_k} \leq M$, or more precisely,
\[
\|f\|_{Z_k} \leq \liminf_{n \to \infty} \|f_n\|_{Z_k}.
\]
To prove this assertion, we only need to prove that $f \in Z_k$ and, for any $\varepsilon > 0$, we have $\|f\|_{Z_k} \leq M + \varepsilon$. Assume that either $k \geq 100$ or $k = 0$ (the case $1 \leq k \leq 99$ is obvious). Given $\varepsilon > 0$, by the definition of $Z_k$ we can find for each $n \in \mathbb{N}$ two functions $g_n$ and $h_n$, $g_n \in X_k$, $h_n \in Y_k$, and
\[
\|g_n\|_{X_k} + \|h_n\|_{Y_k} \leq \|f_n\|_{Z_k} + \varepsilon \leq M + \varepsilon. \tag{4.4}
\]
Let $\varphi_k$ be as before. Then we have
\[
h_n = \varphi_k h_n = \varphi_k f_n - \varphi_k g_n. \tag{4.5}
\]
From the definition of $X_k$ and the fact that $g_n \in X_k$ it can be easily seen that $\varphi_k g_n \in L^2(\mathbb{R}^2)$, and there exists constant $C_k > 0$ such that
\[
\|\varphi_k g_n\|_{L^2(\mathbb{R}^2)} \leq C_k \|g_n\|_{X_k} \leq C_k (M + \varepsilon).
\]
Hence, there exists a subsequence of $\{g_n\}$, for simplicity of the notation we assume that this subsequence is the whole sequence $\{g_n\}$, and a function $h_0 \in L^2(\mathbb{R}^2)$, such that as $n \to \infty$, $\varphi_k g_n \rightharpoonup h_0$ weakly in $L^2(\mathbb{R}^2)$. Let $h = \varphi_k f - h_0$. Then $h \in L^2(\mathbb{R}^2)$, and by (2.5) and the fact that $f_n \rightharpoonup f$ weakly in $L^2(\mathbb{R}^2)$ we see that $h_n \rightharpoonup h$ weakly in $L^2(\mathbb{R}^2)$. Since
\[ \|h_n\|_{Y_k} \leq M + \varepsilon, \quad n = 1, 2, \ldots, \]
by using the assertion in Step 1 we conclude that \( h \in Y_k \), and
\[ \|h\|_{Y_k} \leq \lim_{n \to \infty} \inf \|h_n\|_{Y_k} < \infty. \tag{4.6} \]

Now, since both \( f_n \to f \) and \( h_n \to h \) weakly in \( L^2(\mathbb{R}^2) \), it follows that \( g_n \to g \equiv f - h \) weakly in \( L^2(\mathbb{R}^2) \), which further implies that for any \( j \in \mathbb{Z} \cap [0, \infty) \), \( \eta_j(\tau - \omega(\xi))g_n(\xi, \tau) \to \eta_j(\tau - \omega(\xi))g(\xi, \tau) \) weakly in \( L^2(\mathbb{R}^2) \). Since for any \( N \in \mathbb{N} \) we have
\[ \sum_{j=0}^{N} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - \omega(\xi))g_n(\xi, \tau)\|_{L_{\xi, \tau}^2} \leq \|g_n\|_{X_k} \leq M + \varepsilon, \quad n = 1, 2, \ldots, \]
letting \( n \to \infty \) we get
\[ \sum_{j=0}^{N} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - \omega(\xi))g(\xi, \tau)\|_{L_{\xi, \tau}^2} \leq \lim_{n \to \infty} \inf \sum_{j=0}^{N} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - \omega(\xi))g_n(\xi, \tau)\|_{L_{\xi, \tau}^2} \]
\[ \leq \lim_{n \to \infty} \|g_n\|_{X_k} < \infty. \]

By arbitrariness of \( N \) we conclude that \( g \in X_k \), and
\[ \|g\|_{X_k} \leq \lim_{n \to \infty} \inf \|g_n\|_{X_k} < \infty. \tag{4.7} \]

Hence, \( f = g + h \in Z_k \), and by (4.4), (4.6) and (4.7) we have
\[ \|f\|_{Z_k} \leq \|g\|_{X_k} + \|h\|_{Y_k} \leq \lim_{n \to \infty} \inf (\|g_n\|_{X_k} + \|h_n\|_{Y_k}) \leq M + \varepsilon. \]

This proves the desired assertion.

**Step 3:** We now arrive at the last step of the proof of Lemma 4.1. Let \( f_{nk}(\xi, \tau) = \eta_k(\xi)(I - \partial_2^2)\tilde{w}_n(\xi, \tau) \), \( k = 0, 1, 2, \ldots, \) and \( f_k(\xi, \tau) = \eta_k(\xi)(I - \partial_2^2)\tilde{w}(\xi, \tau) \), where \( \tilde{w}_n = F(w_n) \) and \( \tilde{w} = F(w) \). Then
\[ \|w_n\|_{F_0} = \left( \sum_{k=0}^{\infty} \|f_{nk}\|_{Z_k}^2 \right)^{1/2} \leq M, \quad n = 1, 2, \ldots, \]
so that for any \( N \in \mathbb{N} \) we have
\[ \sum_{k=0}^{N} \|f_{nk}\|^2_{Z_k} \leq M^2, \quad n = 1, 2, \ldots, \tag{4.8} \]

Since \( w_n \) weakly converges to \( w \) in \( L^2(\mathbb{R}^2) \) and \( w_n(t) = 0 \) for \( |t| \geq T \), we have that also \( (1 + t^2)w_n \) weakly converges to \( (1 + t^2)w \) in \( L^2(\mathbb{R}^2) \). By the Parseval formula
\[ \int \int_{\mathbb{R}^2} \tilde{f}(\xi, \tau)\varphi(\xi, \tau) d\xi d\tau = \int \int_{\mathbb{R}^2} f(x, t)\tilde{\varphi}(x, t) dx dt, \quad f, \varphi \in L^2(\mathbb{R}^2), \]
25
it follows immediately that \((I - \partial^2_t)\hat{w}_n(\xi, \tau)\) weakly converges to \((I - \partial^2_t)\hat{w}(\xi, \tau)\) in \(L^2(\mathbb{R}^2)\), which further implies that for every \(k \in \mathbb{Z} \cap [0, \infty)\), \(f_{nk}\) weakly converges to \(f_k\) in \(L^2(\mathbb{R}^2)\). Hence, by the assertion we proved in Step 2 and (2.7) we get

\[
\sum_{k=0}^{N} \|f_k\|_{Z_k}^2 \leq \sum_{k=0}^{N} \liminf_{n \to \infty} \|f_{nk}\|_{Z_k}^2 \leq \liminf_{n \to \infty} \sum_{k=0}^{N} \|f_{nk}\|_{Z_k}^2 \leq M^2.
\]

Letting \(N \to \infty\), we get the desired assertion. \(\square\)

We are now ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1:** We split the proof into four steps.

**Step 1:** We prove that if the assertion of Theorem 3.1 holds for \(T = \delta\) for some small quantity \(\delta > 0\), then it also holds for any given \(T > 0\). This follows from a division of the time interval \([-T, T]\) and an induction argument. Indeed, let \(m = T/\delta\) if \(T/\delta\) is an integer and \(m = \lfloor T/\delta \rfloor + 1\) otherwise. Let \(I_j = [(j-1)\delta, j\delta]\), \(j = 1, 2, \ldots, m-1\), \(I_m = [(m-1)\delta, T]\), and \(I_{-j} = -I_j\), \(j = 1, 2, \ldots, m\). Since the length of each time interval \(I_{\pm j}\) is not larger than \(\delta\), by assumption we see that the assertions \((i)\) and \((ii)\) of Theorem 1.1 applies to each of these intervals provided \(u_n(\cdot, t)\) weakly converges to \(u(\cdot, t)\) in \(L^2(\mathbb{R})\) for \(t\) equal to one of the two endpoints of this interval, but which follows from induction. Hence, the assertions \((i)\) and \((ii)\) of Theorem 1.1 holds for each of these intervals. Now, since for any \(f \in L^q([-T, T], L^p(\mathbb{R}))\) \((p, q\) are as in \((i)\) of Theorem 1.1\) we have

\[
\int_{-T}^{T} \int_{-\infty}^{\infty} [u_n(x, t) - u(x, t)] f(x, t) dx dt = \sum_{|j| = 1}^{m} \int_{I_j}^{\infty} \int_{-\infty}^{\infty} [u_n(x, t) - u(x, t)] f(x, t) dx dt,
\]

the assertion \((i)\) follows immediately. Similarly, since for any \(\varphi \in L^2(\mathbb{R})\),

\[
\sup_{|t| \leq T} |(u_n(\cdot, t) - u(\cdot, t), \varphi)|_{L^2} = \sup_{1 \leq |j| \leq m} \sup_{t \in I_j} |(u_n(\cdot, t) - u(\cdot, t), \varphi)|_{L^2}
\]

the assertion \((ii)\) also follows immediately.

**Step 2:** By the result of Step 1 combined with a standard scaling argument, we see that we only need to prove Theorem 1.1 under the additional assumption that for \(\varepsilon\) as in Theorem 2.3,

\[
\|\phi_n\|_{L^2} \leq \varepsilon, \quad n = 1, 2, \ldots \quad \text{and} \quad \|\phi\|_{L^2} \leq \varepsilon.
\]

Thus, in what follows we always assume that this assumption is satisfied. Moreover, by density of \(C([-T, T], C_0^\infty(\mathbb{R}))\) in \(L^q([-T, T], L^p(\mathbb{R}))\) for any admissible pair \((p, q)\) and
boundedness of the sequence \( \{u_n\} \) in \( L^q([-T,T], L^p(\mathbb{R})) \) (ensured by Corollary 3.8), it can be easily seen that the assertion (i) follows if we prove that

\[
\text{for any } f \in C([-T,T], C_0^\infty(\mathbb{R})), \quad \lim_{n \to \infty} \int_{-T}^{T} \int_{-\infty}^{\infty} [u_n(x, t) - u(x, t)] f(x, t) dx dt = 0. \tag{4.10}
\]

Similarly, by density of \( C_0^\infty(\mathbb{R}) \) in \( L^2(\mathbb{R}) \) and uniform boundedness of \( \{u_n(\cdot, t)\} \) in \( L^2(\mathbb{R}) \) ensured by the \( L^2 \) conservation law, we see that the assertion (ii) follows if we prove that

\[
\text{for any } \varphi \in C_0^\infty(\mathbb{R}), \quad \lim_{n \to \infty} \sup_{|t| \leq T} |(u_n(\cdot, t) - u(\cdot, t), \varphi)_{L^2}| = 0. \tag{4.11}
\]

**Step 3:** Due to (4.9), we have, by Theorem 2.3, the following expressions:

\[
u_n = e^{-it\nu_0} u_{n^+} + e^{it\nu_0} u_{n^-} + w_{n0} + u_{n0}, \quad n = 1, 2, \ldots, \tag{4.12}
\]

\[u = e^{-it\nu_0} u_{+} + e^{it\nu_0} u_{-} + w_{0} + u_{0}. \tag{4.13}
\]

In what follows we prove that

\[
\begin{cases}
\text{for any } R > 0 \text{ and } k,l \in \mathbb{Z}_+, \quad \partial_t^k \partial_x^l u_n \to \partial_t^k \partial_x^l u_0 \\
\text{uniformly on } [-R, R] \times [-T, T] \text{ as } n \to \infty.
\end{cases} \tag{4.14}
\]

Note that if this assertion is proved, then it follows immediately that also

\[
\begin{cases}
\text{for any } R > 0 \text{ and } k,l \in \mathbb{Z}_+, \quad \partial_t^k \partial_x^l U_n \to \partial_t^k \partial_x^l U_0 \\
\text{uniformly on } [-R, R] \times [-T, T] \text{ as } n \to \infty.
\end{cases} \tag{4.15}
\]

Let \( \phi_{low} \) be as before, i.e., \( \phi_{low} = P_{low}(\phi) \), and let \( \phi_{nlow} = P_{low}(\phi_n), n = 1, 2, \ldots. \) Then \( u_{n0} \) and \( u_{0} \) are respectively solutions of the following problems:

\[
\begin{cases}
\partial_t u_{n0} + \mathcal{H} \partial_x^2 u_{n0} + \partial_x (u_{n0}^2/2) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
u_{n0}(x, 0) = \phi_{nlow}(x), \quad x \in \mathbb{R},
\end{cases} \tag{4.16}
\]

\[
\begin{cases}
\partial_t u_{0} + \mathcal{H} \partial_x^2 u_{0} + \partial_x (u_{0}^2/2) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
u_{0}(x, 0) = \phi_{low}(x), \quad x \in \mathbb{R}.
\end{cases} \tag{4.17}
\]

Since the sequence \( \{\phi_{nlow}\} \) is bounded in \( L^2(\mathbb{R}) \), by (2.2) we see that for any \( k,l \in \mathbb{Z}_+ \) there exists corresponding constant \( C_{kl}(T) > 0 \) such that

\[
\|\partial_t^k \partial_x^l u_{n0}\|_{L^2(\mathbb{R} \times [-T,T])} \leq \sqrt{2T} \sup_{|t| \leq T} \|\partial_t^k \partial_x^l \phi_{nlow}()\|_{L^2} \leq C_{kl}(T), \quad n = 1, 2, \ldots.
\]
Hence, by using the compact embedding $H^{m+2}([-R, R] \times [-T, T]) \hookrightarrow C^m([-R, R] \times [-T, T])$ for any $R > 0$ and $m \in \mathbb{Z}_+$ and a diagonalisation argument we see that there exists a subsequence $\{u_{n0}\}$ of $\{u_{n}\}$, such that for any $R > 0$ and $k, l \in \mathbb{Z}_+$, $\partial_t^k \partial_x^lu_{n0}$ is uniformly convergent in $[-R, R] \times [-T, T]$. Replacing $n$ with $n_k$ in (4.16) and then letting $k \to \infty$, we see that the limit function of $u_{n_k0}$ is a smooth solution of the problem (4.17). Since $\phi_{low} \in H^\infty(\mathbb{R})$, we know that the solution of (4.17) in $C([-T, T], H^\infty(\mathbb{R}))$ is unique, so that the limit function of $u_{n_k0}$ is $u_0$. Thus we have shown that

$$\{(\text{for any } R > 0 \text{ and } k, l \in \mathbb{Z}_+, \partial_t^k \partial_x^lu_{n0} \to \partial_t^k \partial_x^lu_0)\}$$

uniformly on $[-R, R] \times [-T, T]$ as $n \to \infty$.

Since the above argument is also valid when $\{u_{n0}\}$ is replaced by any of its subsequence, we see that the assertion (4.14) follows.

**Step 4:** By the assertions (4.14), (4.15) and the expressions (4.12), (4.13), it follows immediately that (4.10) and (4.11) will follow if we prove that

$$w_{na} \to w_\alpha \text{ weakly in } L^q([-T, T], L^p(\mathbb{R})) \text{ for any admissible pair } (p, q) \quad (4.18)$$

(in case either $p = \infty$ or $q = \infty$, weakly here refers to $*$-weakly), and

$$\text{for any } \varphi \in C^\infty_0(\mathbb{R}), \lim_{n \to \infty} \sup_{|t| \leq T} |(w_{na}(\cdot, t) - w_\alpha(\cdot, t), \varphi)_{L^2}| = 0, \quad (4.19)$$

where $\alpha = +, -, 0$.

To prove the assertion (4.18), we note that since $\phi_n \to \phi$ weakly in $L^2(\mathbb{R})$, using the assertion (4.15) we easily see that also $\psi_\alpha(\phi_n) \to \psi_\alpha(\phi)$ weakly in $L^2(\mathbb{R})$ for $\alpha = +, -, 0$. Moreover, by Lemma 10.1 of [13] we see that $\{\psi_\alpha(\phi_n)\} \ (\alpha = +, -, 0)$ are bounded in $H^0$. The latter assertion implies that the sequences $\{w_{na}\} \ (\alpha = +, -, 0)$ are bounded in $F^0_T$, which further implies, by Lemma 3.6, that for any admissible pair $(p, q)$ the sequences $\{w_{na}\} \ (\alpha = +, -, 0)$ are also bounded in $L^q([-T, T], L^p(\mathbb{R}))$. If the assertion (4.18) does not hold for some admissible pair $(p, q)$ then it follows that there exist subsequences $\{w_{n_\alpha}\} \ (\alpha = +, -, 0)$ and functions $w'_\alpha \in L^q([-T, T], L^p(\mathbb{R})) \ (\alpha = +, -, 0), \ (w'_+, w'_-, w'_0) \neq (w_+, w_-, w_0)$, such that $w_{n_\alpha} \to w'_\alpha$ weakly in $L^q([-T, T], L^p(\mathbb{R}))$ (in case either $p = \infty$ or $q = \infty$ then weakly here refers to $*$-weakly, which will not be repeated later on). Since $\{w_{n_\alpha}\} \ (\alpha = +, -, 0)$ are bounded in $L^\infty([-T, T], L^2(\mathbb{R}))$ (by Lemma 3.6), by replacing them with subsequences of them when necessary, we may assume that also $w_{n_\alpha} \to w'_\alpha \ (\alpha = +, -, 0)$ weakly in $L^\infty([-T, T], L^2(\mathbb{R}))$, so that also $w'_\alpha \in L^\infty([-T, T], L^2(\mathbb{R})) \ (\alpha = +, -, 0)$. Using Lemmas 3.2, 3.4 and a well-known compact embedding result, we easily deduce that there exists a subsequence, which we still denote as $\{w_{n_\alpha}\}$, such that for any
$R > 0$, $w_{n,k} \to w'_\alpha$ strongly in $L^2([-R,R] \times [-T,T])$ ($\alpha = +, -, 0$). Thus, by replacing 
$\{w_{n,k}\}$ with a subsequence when necessary, we may assume that $w_{n,k} \to w'_\alpha$ ($\alpha = +, -, 0$) 
almost everywhere in $\mathbb{R} \times [-T,T]$. Now, From (2.4) we have 
\[
w_{n,k}(t) = W(t)\psi_{n,k}(\phi_{n,k}) + \int_0^t W(t-t')E_\alpha(w_{n,k}^+(t'), w_{n,k}^-(t'), w_{n,k^0}(t'))dt', \quad \alpha = +, -, 0. 
\] (4.20)

Here the second term on the right-hand sides of the above equations should be understood 
in the following sense: All partial derivatives in $x$ included in $E_\alpha$’s acting on any terms 
containing $w_{n,k}^+$, $w_{n,k}^-$ or $w_{n,k^0}$ should be either taken outside of the integral or moved to 
other terms containing only $u_{n,k^0}$ and $U_{n,k^0}$ by using integration by parts. For instance, 
recalling that \( (2.11) \) of [13] 
\[
E_+(w_+, w_-, w_0) = -e^{it\theta}P_{\text{high}}[\partial_x(e^{-it\theta}w_+ + e^{it\theta}w_- + w_0)^2/2]
\]
\[
- e^{it\theta}P_{\text{high}}\{\partial_x[u_0 \cdot P_{\text{high}}(e^{it\theta}w_-) + u_0 \cdot P_{\text{low}}(w_0)]\}
\]
\[
+ e^{it\theta}(P_{\text{high}} + P_{\text{low}})\{\partial_x[u_0 \cdot P_{\text{high}}(e^{-it\theta}w_+)]\}
\]
\[
+ 2iP_{\text{low}}[\partial_x^2[e^{it\theta}P_{\text{high}}(e^{-it\theta}w_+)]
\]
\[
- P_{\text{high}}(\partial_xu_0) \cdot w_+, 
\]
the equation in (4.20) for $\alpha = +$ should be understood to represent the following equation:
\[
w_{n,k^0}(t) = W(t)\psi_{n,k^0}(\phi_{n,k}) - \partial_x \int_0^t W(t-t')e^{it\theta}P_{\text{high}}[(e^{-it\theta}w_{n,k^0}^+ + e^{it\theta}w_{n,k^0}^- + w_{n,k^0})^2/2]dt'
\]
\[
+ \int_0^t W(t-t')\partial_x(e^{it\theta}w_{n,k^0})P_{\text{high}}[(e^{-it\theta}w_{n,k^0}^+ + e^{it\theta}w_{n,k^0}^- + w_{n,k^0})^2/2]dt'
\]
\[
- \partial_x \int_0^t W(t-t')e^{it\theta}P_{\text{high}}[u_{n,k^0} \cdot P_{\text{high}}(e^{it\theta}w_{n,k^-}) + u_{n,k^0} \cdot P_{\text{low}}(w_{n,k^0})]dt'
\]
\[
+ \int_0^t W(t-t')\partial_x(e^{it\theta}w_{n,k^0})P_{\text{high}}[u_{n,k^0} \cdot P_{\text{high}}(e^{it\theta}w_{n,k^-}) + u_{n,k^0} \cdot P_{\text{low}}(w_{n,k^0})]dt'
\]
\[
+ \partial_x \int_0^t W(t-t')e^{it\theta}(P_{\text{high}} + P_{\text{low}})[u_{n,k^0} \cdot P_{\text{high}}(e^{-it\theta}w_{n,k^+})]dt'
\]
\[
- \int_0^t W(t-t')\partial_x(e^{it\theta}w_{n,k^0})(P_{\text{high}} + P_{\text{low}})[u_{n,k^0} \cdot P_{\text{high}}(e^{-it\theta}w_{n,k^+})]dt'
\]
\[
+ 2i\partial_x^2 \int_0^t W(t-t')P_{\text{low}}[e^{it\theta}P_{\text{high}}(e^{-it\theta}w_{n,k^+})]dt'
\]
\[
- \int_0^t W(t-t')P_{\text{high}}(\partial_xu_{n,k^0} \cdot w_{n,k^+})dt'; 
\]
moreover, the equations in (4.20) for \( \alpha = - \) and \( \alpha = 0 \) should be understood similarly. Thus, letting \( k \to \infty \) and using the Vitali convergence theorem (see Corollary A.2 in the appendix B), we see that \((w'_+, w'_-, w'_0)\) satisfies the integral equations

\[
w'_\alpha(t) = W(t)\psi_\alpha(\phi) + \int_0^t W(t-t')E_\alpha(w'_+(t'), w'_-(t'), w'_0(t'))dt', \quad \alpha = +, -, 0.
\]

Note that these equations should be understood as (4.20) in the meaning explained above. Since by Lemma 4.1 we have \( w'_\alpha \in F_\alpha^0 \) (\( \alpha = +, -, 0 \)) and both \((w'_+, w'_-, w'_0)\) and \((w_+, w_-, w_0)\) are in a small ball of \((F_\alpha^0)^3\), by uniqueness of the solution of the above equation in a small ball of \((F_\alpha^0)^3\) we conclude that \((w'_+, w'_-, w'_0) = (w_+, w_-, w_0)\), which is a contradiction.

The argument for the proof of (4.19) is similar. Indeed, let \( v^1_{n_k\alpha} \) and \( v^2_{n_k\alpha} \) denote the first and the second terms on the right-hand side of (4.20), respectively, and by \( v^1 \) and \( v^2 \) the corresponding terms in \( w'_\alpha \). It can be easily seen that

\[
\lim_{k \to \infty} \sup_{|t| \leq T} |(v^1_{n_k\alpha}(-, t), \varphi)_{L^2_x} - (v^1(-, t), \varphi)_{L^2_x}| = 0. \tag{4.21}
\]

To treat \((v^2_{n_k\alpha}(-, t), \varphi)_{L^2_x}\) we only need to move all partial derivatives in \( x \) contained in \( E_\alpha \)'s either to terms expressed in \( u_{n_0} \) and \( U_{n_0} \) by using integration by parts, or to the test function \( \varphi \), also by using integration by parts. With this trick in mind, we can also prove that

\[
\lim_{k \to \infty} \sup_{|t| \leq T} |(v^2_{n_k\alpha}(-, t), \varphi)_{L^2_x} - (v^2(-, t), \varphi)_{L^2_x}| = 0. \tag{4.22}
\]

We omit the details. Combining (4.21) and (4.22), we see that the assertion (4.19) follows. This completes the proof of Theorem 1.1. \( \Box \)

**Remark** For the modified Benjamin-Ono equation (1.3), it has been proved by Kenig and Takaoka in [18] that its initial value problem is globally well-posed in the Sobolev space \( H^{1/2}(\mathbb{R}) \), whereas the solution operator of a such problem is not uniformly continuous in any Sobolev spaces \( H^s(\mathbb{R}) \) of index \( s < 1/2 \) (so that \( H^{1/2}(\mathbb{R}) \) is a borderline space for the local well-posedness theory of this equation). It is thus natural to ask if the flow map of this equation in \( H^{1/2}(\mathbb{R}) \) is weakly continuous. The answer to this question is affirmative. The proof is as follows: Let \( \phi_n \) \((n = 1, 2, \cdots)\) be a sequence of functions in \( H^{1/2}(\mathbb{R}) \) which is weakly convergent, and let \( \phi \) be its limit. Let \( u_n \) and \( u \) be the solutions of the equation (1.3) in \( C(\mathbb{R}, H^{1/2}(\mathbb{R})) \) such that \( u_n|_{t=0} = \phi_n \) \((n = 1, 2, \cdots)\) and \( u|_{t=0} = \phi \). Then for any \( T > 0 \), \( \{u_n\} \) is bounded in \( L^\infty([-T,T], H^{1/2}(\mathbb{R})) \). Using the equation (1.3), we then deduce that \( \{\partial_t u_n\} \) is bounded in \( L^\infty([-T,T], H^{-3/2}(\mathbb{R})) \). It follows that there exists a subsequence \( \{u_{n_k}\} \) such that for any \( R > 0 \), \( \{u_{n_k}\} \) is strongly convergent in
$L^2([−R,R] \times [−T,T])$ and, consequently, by replacing $\{u_{nk}\}$ with a suitable subsequence of it, we may assume that $\{u_{nk}\}$ converges almost everywhere in $\mathbb{R} \times [−T,T]$. Thus by following the approach developed in [11] we obtain the desired assertion. (One needs to, in addition, observe that the uniqueness in [18] easily extends to solutions of the integral equation in $C([−T,T], H^{1/2}(\mathbb{R})) \cap X^{1/2}$, where $X^{1/2}$ is the space in [18]). We are grateful to one of the anonymous referees for pointing to us this proof.

Appendix: Vitali convergence theorem

**Theorem A.1** (Vitali convergence theorem, cf. [12]) Let $X$ be a measurable set. Let $u_n \in L^1(X)$, $n = 1, 2, \cdots$. Assume that the following three conditions are satisfied:

(a) $u_n$ converges to $u$ in measure.

(b) For any $\varepsilon > 0$ there exists corresponding $M > 0$ such that

$$\int_{\{|u_n(x)| > M\}} |u_n(x)| dx < \varepsilon \text{ for all } n \in \mathbb{N}.$$

(c) For any $\varepsilon > 0$ there exists corresponding measurable subset $E$ of $X$ with $\text{meas}(E) < \infty$,

such that

$$\int_{X \setminus E} |u_n(x)| dx < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Then $u \in L^1(X)$ and

$$\lim_{n \to \infty} \int_E u_n(x) dx = \int_E u(x) dx.$$

**Remark** If $\text{meas}(X) < \infty$, then the condition (c) is clearly satisfied by any sequence of measurable functions on $X$: We may choose $E = X$.

What we used in the proof of Theorem 1.1 is the following corollary of the above theorem:

**Corollary A.2** Let $E$ be a measurable set, $\text{meas}(E) < \infty$. Let $1 < p < \infty$ and $u_n \in L^p(E)$, $n = 1, 2, \cdots$. Assume that (i) $u_n$ converges to $u$ in measure, and (ii) $\{u_n\}$ is bounded in $L^p(E)$. Then $u \in L^p(E)$, and for any $1 \leq q < p$ we have

$$\lim_{n \to \infty} \|u_n - u\|_q = 0. \quad (A.1)$$
Proof: The assertion that $u \in L^p(E)$ follows from Fatou’s lemma. To prove (A.1) we assume that $\|u_n\|_p \leq C$ for all $n \in \mathbb{N}$. Then we also have $\|u\|_p \leq C$, by Fatou’s lemma. Thus, for any $M > 0$ we have

$$\int_{\{|u_n(x) - u(x)| > M\}} |u_n(x) - u(x)|^q dx \leq M^{-(p-q)} \int_E |u_n(x) - u(x)|^p dx \leq (2C)^p M^{-(p-q)},$$

which implies that

$$\lim_{M \to \infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n(x) - u(x)| > M\}} |u_n(x) - u(x)|^q dx = 0.$$ 

Hence, the desired assertion follows from Theorem A.1. \qed

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