ON SOME PROPERTIES OF Lie-CENTROIDS OF LEIBNIZ ALGEBRAS

J.M. CASAS, X. GARCÍA-MARTÍNEZ, AND N. PACHECO-REGO

Abstract. We study some properties on Lie-centroids related to central Lie-derivations, generalized Lie-derivations and almost inner Lie-derivations. We also determine the Lie-centroid of the tensor product of a commutative associative algebra and a Leibniz algebra.

1. Introduction

In the semi-abelian categories context, classical commutatory theory developed by Higgins and Huq [19, 20] studies how far are objects of being abelian. Replacing abelian by any Birkhoff subcategory, we obtain relative commutator theory. This was first investigated following the lines of relative central extensions of Fröhlich [13] and Janelidze and Kelly’s categorical Galois theory [21, 22] (see also [11, 14, 26]), and further developed by Everaert and Van der Linden, among others [8, 12, 10].

The category of Lie algebras forms a Birkhoff subcategory of the category of Leibniz algebras. This means that we can study relative commutator theory on Leibniz algebras respect to Lie, giving rise to interesting developments in the comprehension of both algebraic structures [2, 3, 5, 7, 33]. Moreover, the interplay between these two categories can be somehow tricky, since it is also possible to find Leibniz algebras as a subcategory of a certain type of Lie algebras [25], although many interesting categorical properties are not preserved [15, 16, 17, 18].

The study of the properties of the centroid of a Lie algebra has been a key step in the classification of finite-dimensional extended affine Lie algebras [11, 28]. For finite-dimensional simple associative algebras, centroids are essential in the investigation of Brauer groups, division algebras and derivations. The theory of centroids in other algebraic structures such as Jordan algebras, superalgebras, n-Lie algebras, Zinbiel algebras, among others, can be found, for instance in [27, 29, 32, 34, 35].

In the recent manuscript [4], it was introduced the concept of centroid of a Leibniz algebra with respect to the Liesation functor, named Lie-centroid, together with the study of the interplay between Lie-central derivations, Lie-centroids and Lie-stem Leibniz algebras.

Our goal in this paper is to continue with the analysis of properties of the Lie-centroid introduced in [4] and study its interaction with Lie-central derivations, generalised Lie-derivations, quasi-Lie-centroids and almost inner Lie-derivations, establishing a parallelism between the absolute results (classical properties for Lie algebras) with the results obtained. We note that not all classical results are immediately translated into the relative case, so new requirements are needed.
The manuscript is organised as follows: Section 2 contains the necessary notions relative to the Liesation functor. In Section 3 we review some properties of Lie-central derivations and Lie-centroids and we obtain some new results on Lie-centroids. In Section 4 we introduce the notions of generalised derivations and quasi-centroids relative to the Liesation functor and we analyse their connections with Lie-centroids. An important problem in the absolute context tries to determine the conditions under which the central deviations coincide with the almost inner derivations. Our goal in Section 5 is the study of this problem in the relative context. The most relevant fact is that we need additional conditions to characterise these conditions. We conclude our study analysing the Lie-centroid of the tensor product of a commutative associative algebra and a Leibniz algebra in Section 6.

2. Preliminaries on Leibniz algebras

Let $K$ be a fixed ground field such that $\frac{1}{2} \in K$. Throughout the paper, all vector spaces and tensor products are considered over $K$.

A Leibniz algebra $G$ is a vector space equipped with a bilinear map $[-, -]: G \times G \to G$, satisfying the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [x, [y, z]], \quad x, y, z \in G.$$  

A subalgebra $h$ of a Leibniz algebra $g$ is said to be a left (resp. right) ideal of $g$ if $[h, g] \in h$ (resp. $[g, h] \in h$), for all $h \in h$, $g \in g$. If $h$ is both left and right ideal, then $h$ is called a two-sided ideal of $g$. In this case $g/h$ naturally inherits a Leibniz algebra structure.

Given a Leibniz algebra $g$, we denote by $g^{\text{ann}}$ the subspace of $g$ spanned by all elements of the form $[x, x]$, $x \in g$. It is clear that the quotient $g_{\text{Lie}} = g/g^{\text{ann}}$ is a Lie algebra. This procedure defines the so-called Liesation functor $(-)_{\text{Lie}}: \text{Leib} \to \text{Lie}$, which assigns to a Leibniz algebra $g$ the Lie algebra $g_{\text{Lie}}$. Moreover, the canonical surjective homomorphism $g \to g_{\text{Lie}}$ is universal among all homomorphisms from $g$ to a Lie algebra, implying that the Liesation functor is left adjoint to the inclusion functor $\text{Lie} \hookrightarrow \text{Leib}$.

Given a Leibniz algebra $g$, we define the bracket $[-, -]_{\text{Lie}}: g \to g$ by $[x, y]_{\text{Lie}} = [x, y] + [y, x]$, for $x, y \in g$.

Let $m, n$ be two-sided ideals of a Leibniz algebra $g$. The following notions come from [2], which were derived from [8].

The Lie-commutator of $m$ and $n$ is the two-sided ideal of $g$

$$[m, n]_{\text{Lie}} = \{[m, n]_{\text{Lie}}, m \in m, n \in n\}.$$  

The Lie-centre of the Leibniz algebra $g$ is the two-sided ideal $Z_{\text{Lie}}(g) = \{z \in g \mid [x, z]_{\text{Lie}} = 0 \text{ for all } x \in g\}$.

The Lie-centraliser of $m$ and $n$ over $g$ is

$$C^g_{\text{Lie}}(m, n) = \{x \in g \mid [x, m]_{\text{Lie}} \in n, \text{ for all } m \in m\}.$$  

When $n = 0$, we denote it by $C^g_{\text{Lie}}(m)$. Obviously, $C^g_{\text{Lie}}(g) = Z^g_{\text{Lie}}(g)$.

The right-centre of $g$ is the two-sided ideal $Z^r(g) = \{z \in g \mid [x, z] = 0 \text{ for all } x \in g\}$, whereas the left-centre of a Leibniz algebra $g$ is the set $Z^l(g) = \{z \in g \mid [z, x] = 0 \text{ for all } x \in g\}$, which might not even be a subalgebra. The centre of $g$ is the two-sided ideal obtained by $Z(g) = Z^l(g) \cap Z^r(g)$. 


Let \( n \) be a two-sided ideal of a Leibniz algebra \( g \). The lower Lie-central series of \( g \) relative to \( n \) is the sequence
\[
\cdots \leq \gamma^\text{Lie}_1(g, n) \leq \cdots \leq \gamma^\text{Lie}_2(g, n) \leq \gamma^\text{Lie}_3(g, n)
\]
of two-sided ideals of \( g \) defined inductively by
\[
\gamma^\text{Lie}_1(g, n) = n \quad \text{and} \quad \gamma^\text{Lie}_i(g, n) = \gamma^\text{Lie}_{i-1}(g, n) \cdot g, \quad i \geq 2.
\]
We use the notation \( \gamma^\text{Lie}_i(g) \) instead of \( \gamma^\text{Lie}_i(g, g) \), \( 1 \leq i \leq n \). The Leibniz algebra \( g \) is said to be Lie-nilpotent relative to \( n \) of class \( c \) if \( \gamma^\text{Lie}_{c+1}(g, n) = 0 \) and \( \gamma^\text{Lie}_c(g, n) \neq 0 \).

**Remark 2.1.** Note that from the Leibniz identity we can deduce that \([x, [y, z]]_\text{Lie} = 0\) is also an identity. This means that it is not interesting at all to study Lie-solvability, since the third ideal will always be trivial.

3. **Lie-Central derivations and Lie-centroids**

In this section we recall some notions and results from [4] and we provide some new results concerning Lie-central derivations and Lie-centroids.

**Definition 3.1.** A linear map \( d : g \to g \) of a Leibniz algebra \( g \) is said to be a Lie-derivation if for all \( x, y \in g \), the following condition holds:
\[
d([x, y]_\text{Lie}) = [d(x), y]_\text{Lie} + [x, d(y)]_\text{Lie}
\]
We denote by \( \text{Der}^\text{Lie}(g) \) the set of all Lie-derivations of \( g \) which can be equipped with a structure of Lie algebra by means of the usual bracket
\[
[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1, \quad \text{for all } d_1, d_2 \in \text{Der}^\text{Lie}(g).
\]

**Example 3.2.**

(a) Absolute derivations of a Leibniz algebra \( g \), i.e., linear maps \( d : g \to g \) such that \( d([x, y]) = [d(x), y] + [x, d(y)] \), for \( x, y \in g \), are also Lie-derivations.

(b) If \( g \) is a Lie algebra, then every linear map \( d : g \to g \) is a Lie-derivation.

(c) Let \( g \) be the two-dimensional Leibniz algebra with basis \( \{e, f\} \) and bracket operation \( [e, f] = e \) and zero elsewhere (see [9]). The linear maps \( d : g \to g \) represented by a matrix of the form
\[
\begin{pmatrix}
a & 0 \\
0 & 0 \\
\end{pmatrix}
\]
are Lie-derivations.

(d) Let \( g \) be the two-dimensional Leibniz algebra with basis \( \{e, f\} \) and bracket operation \( [f, f] = \lambda e, \lambda \in \mathbb{K} \), and zero elsewhere (see [9]). The linear maps \( d : g \to g \) represented by a matrix of the form
\[
\begin{pmatrix}
2a & b \\
0 & a \\
\end{pmatrix}
\]
are Lie-derivations, but it is not an absolute derivation.

**Definition 3.3** ([4]). A Lie-derivation \( d : g \to g \) of a Leibniz algebra \( g \) is said to be a Lie-central derivation if its image is contained in the Lie-centre of \( g \).

We denote the set of all Lie-central derivations of a Leibniz algebra \( g \) by \( \text{Der}_c^\text{Lie}(g) \). Obviously \( \text{Der}_c^\text{Lie}(g) \) is a subalgebra of \( \text{Der}^\text{Lie}(g) \) and every element of \( \text{Der}_c^\text{Lie}(g) \) annihilates \( \gamma_2^\text{Lie}(g) \). Moreover, \( \text{Der}_c^\text{Lie}(g) = C_{\text{Der}^\text{Lie}(g)}((\text{RR} + \text{LL})(g)) \), where \( \text{LL}(g) \) is formed by \( L_x \), the left multiplication operators \( L_x(y) = [x, y] \); and \( \text{RR}(g) \) by its right counterparts \( R_x \).

**Example 3.4.** The Lie-derivation given in Example 3.2 (c) is not a Lie-central derivation, except in the case \( a = 0 \). The Lie-derivation given in Example 3.2 (d) is a Lie-central derivation when \( a = 0 \).
Definition 3.5 ([4]). The Lie-centroid of a Leibniz algebra \( g \) is the set of all linear maps \( d: g \rightarrow g \) satisfying the identities
\[
d([x, y]_{\text{Lie}}) = [d(x), y]_{\text{Lie}} = [x, d(y)]_{\text{Lie}},
\]
for all \( x, y \in g \). We denote this set by \( \Gamma_{\text{Lie}}(g) \).

Example 3.6.
(a) If \( g \) is a Lie algebra, then every linear map \( d: g \rightarrow g \) is a Lie-centroid.
(b) Let \( g \) be the two-dimensional Leibniz algebra with basis \( \{ e, f \} \) and bracket operation \( [e, f] = e \) and zero elsewhere (see [9]). The Lie-centroid of \( g \) are the linear maps \( d: g \rightarrow g \) represented by the matrix of the form \( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \).
(c) Let \( g \) be the two-dimensional Leibniz algebra with basis \( \{ e, f \} \) and bracket operation \( [f, f] = \lambda e, \lambda \in \mathbb{K}/\mathbb{K}^2 \), and zero elsewhere (see [9]). The linear maps \( d: g \rightarrow g \) represented by a matrix of the form \( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \) are Lie-centroids.

Proposition 3.7 ([4] Proposition 4.2). For any Leibniz algebra \( g \), \( \Gamma_{\text{Lie}}(g) \) is a subalgebra of \( \text{End}_K(g) \) such that \( \text{Der}_{\mathbb{Z}}^e(g) = \text{Der}_{\mathbb{Z}}^g(g) \cap \Gamma_{\text{Lie}}(g) \).

Theorem 3.8. Let \( \{ e_k \} \) be a basis of \( \text{Der}_{\mathbb{Z}}^g(g) \) and \( \{ \varphi_j \} \) be a maximal subset of \( \Gamma_{\text{Lie}}(g) \) such that \( \{ \varphi_j \}_{j \in J} \cap \{ e_k \}_{k \in K} \) is linearly independent and let \( \Psi \) denote the subspace spanned by \( \{ \varphi_j \} \). Then \( \{ e_k, \varphi_j \} \) is a basis of \( \Gamma_{\text{Lie}}(g) \) and \( \Gamma_{\text{Lie}}(g) = \text{Der}_{\mathbb{Z}}^g(g) \oplus \Psi \).

Proof. Since \( \{ \varphi_j \}_{j \in J} \cap \{ e_k \}_{k \in K} \) is linearly independent, then \( \{ \varphi_j \} \) is linearly independent in \( \Gamma_{\text{Lie}}(g) \). Moreover, by construction and [4] Proposition 4.2, \( \{ e_k, \varphi_j \} \) is a linearly independent set in \( \Gamma_{\text{Lie}}(g) \). For \( \varphi \in \Gamma_{\text{Lie}}(g) \), since \( \{ \varphi_j \}_{j \in J} \cap \{ e_k \}_{k \in K} \) is a basis of the vector space \( \{ \varphi_j \}_{j \in J} \cap \{ e_k \}_{k \in K} \), then \( \varphi = \sum_{j \in J} e_j \varphi_j + \sum_{k \in K} e_k \), \( J \) denotes a finite set of indexes and \( e_j, e_k \in \mathbb{K}, j, k \in J \). Thus, for any \( x, y \in g \), we have:
\[
\varphi(x, y)_{\text{Lie}} = \sum_{j \in J} e_j \varphi_j(x, y)_{\text{Lie}} = \sum_{j \in J} e_j \varphi_j(x)_{\text{Lie}} \varphi_j(y)_{\text{Lie}}.
\]
i.e., \( \varphi = \sum_{j \in J} e_j \varphi_j \). Moreover, it is easy a routine computation that \( \varphi = \sum_{j \in J} e_j \varphi_j, for all \( \varphi \in \text{Der}_{\mathbb{Z}}^g(g) \). Consequently, \( \varphi = \sum_{j \in J} e_j \varphi_j + \sum_{k \in K} e_k \).

Theorem 3.9. Let \( \pi: g \rightarrow h \) be a surjective homomorphism of Leibniz algebras. For any \( f \in \text{End}(g) \) \( \Rightarrow \) \( g(\text{Ker}(\pi)) \) \( \subseteq \text{Ker}(\pi) \) there exists a unique \( \overline{f} \in \text{End}(h) \) such that \( \pi \circ f = \overline{f} \circ \pi \).

Moreover, the following statements hold:
(i) The homomorphism \( \pi^*: \text{End}(g) \rightarrow \text{End}(h), f \mapsto \overline{f} \), satisfies the following properties:
(a) \( \pi^*(\text{RR} + LL)(g) = \text{RR} + LL)(h) \).
(b) \( \pi^*(\Gamma_{\text{Lie}}(g) \cap \text{End}(g)) \subseteq \Gamma_{\text{Lie}}(h) \).
(c) There is a homomorphism \( \pi_{\text{Lie}}: \Gamma_{\text{Lie}}(g) \cap \text{End}(g) \rightarrow \Gamma_{\text{Lie}}(h), \varphi \mapsto \overline{\varphi} \).
(d) If \( \text{Ker}(\pi) = Z_{\text{Lie}}(g) \), then every \( \varphi \in \Gamma_{\text{Lie}}(g) \) leaves \( \text{Ker}(\pi) \) invariant, i.e., \( \pi_{\text{Lie}} \) is defined on all of \( \Gamma_{\text{Lie}}(g) \).
(ii) If \( \text{Ker}(\pi) \subseteq Z_{\text{Lie}}(g) \), for any \( \varphi \in \Gamma_{\text{Lie}}(g) \cap \text{End}(g) \) such that \( \pi_{\text{Lie}}(\varphi) = 0 \), then \( \varphi(Z_{\text{Lie}}(g)) = 0 \).
Proof. The existence of $\mathcal{T}$ is a consequence of the following commutative diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{\pi} & \text{Ker}(\pi) \\
\downarrow{f} & & \downarrow{\mathcal{T}} \\
0 & \xrightarrow{\pi} & \text{Ker}(\pi) \\
\end{array}
$$

(i) Let $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$ such that $\pi(x) = y$. Then,

$$
\pi^*((RR + LL)(x)) = \pi^*(RR_x + LL_x) = RR_{y_x} + LL_{y_x} = RR_y + LL_y \in (RR + LL)(y).
$$

(b) Since we have the isomorphism $\mathfrak{h} \cong \mathfrak{g}/\text{Ker}(\pi)$, for any $f \in \Gamma_{\text{Lie}}(\mathfrak{g}) \cap \text{End}(\mathfrak{g})$ we have that $\mathcal{T}(y) = f(x) + \text{Ker}(\pi)$ where $y = x + \text{Ker}(\pi)$. Then, an easy computation shows that $\mathcal{T} \in \Gamma_{\text{Lie}}(\mathfrak{h})$.

(c) Direct checking.

(d) For any $x \in \text{Ker}(\pi) = Z_{\text{Lie}}(\mathfrak{g})$, we have that $[\varphi(x), y]_{\text{Lie}} = \varphi([x, y]_{\text{Lie}}) = 0$, for all $y \in \mathfrak{g}$, i.e., $\varphi(x) \in Z_{\text{Lie}}(\mathfrak{g})$.

(ii) Let $\varphi \in \Gamma_{\text{Lie}}(\mathfrak{g}) \cap \text{End}(\mathfrak{g})$ be such that $\pi_{\text{Lie}}(\varphi) = 0$. Then, we know that $\varphi(\mathfrak{g}) \subseteq \text{Ker}(\pi) \subseteq Z_{\text{Lie}}(\mathfrak{g})$. For every $x \in \gamma_{\text{Lie}}^2(\mathfrak{g})$, $x = \sum_i k_i[x_i', x_i'']_{\text{Lie}}$, with $k_i \in \mathbb{K}, x_i', x_i'' \in \mathfrak{g}$, we have

$$
\varphi(x) = \sum_i k_i \varphi([x_i', x_i'']_{\text{Lie}}) = \sum_i k_i \varphi(x_i') \varphi(x_i'')_{\text{Lie}} = 0.
$$

□

Proposition 3.10. For a Leibniz algebra $\mathfrak{g}$, the elements of its Lie-centroid commute when applied to $\gamma_{\text{Lie}}^2(\mathfrak{g})$.

Proof. Let $x = \sum_i k_i[x_i', x_i''] \in \gamma_{\text{Lie}}^2(\mathfrak{g})$ and let $\varphi, \psi \in \Gamma_{\text{Lie}}(\mathfrak{g})$. Then,

$$
\varphi \circ \psi(x) = \varphi \circ \psi(\sum_i k_i[x_i', x_i'']) = \sum_i k_i(\varphi \circ \psi([x_i', x_i'']))
$$

$$
= \sum_i k_i(\varphi(x_i'), \psi(x_i'')) = \varphi(\psi(x)), \quad \psi(\varphi(x)) = \psi(\varphi(x)).
$$

□

Proposition 3.11. Let $\mathfrak{g}$ be a Leibniz algebra. Then:

(i) $\gamma_{\text{Lie}}^2(\mathfrak{g})$ is indecomposable if and only if the only idempotents of $\Gamma_{\text{Lie}}(\gamma_{\text{Lie}}^2(\mathfrak{g}))$ are $0$ and $\text{Id}$.

(ii) For every $\varphi \in \Gamma_{\text{Lie}}(\mathfrak{g})$ and any Lie-invariant $\mathbb{K}$-bilinear form $f$ of $\mathfrak{g}$ (i.e., $f$ is a $\mathbb{K}$-bilinear form on $\mathfrak{g}$ satisfying that $f([a, c]_{\text{Lie}}, b) + f(a, [b, c]_{\text{Lie}}) = 0$, for all $a, b, c \in \mathfrak{g}$) the following equality holds for any $x \in \gamma_{\text{Lie}}^2(\mathfrak{g}), b \in \mathfrak{g}$:

$$
f(\varphi(x), b) = f(\varphi(x), \varphi(b)).
$$

Proof. (i) Assume that $\gamma_{\text{Lie}}^2(\mathfrak{g})$ has a decomposition $\gamma_{\text{Lie}}^2(\mathfrak{g}) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We can take the idempotent $\varphi \in \Gamma_{\text{Lie}}(\gamma_{\text{Lie}}^2(\mathfrak{g}))$ such that $\varphi|\mathfrak{g}_1 = \text{Id}, \varphi|\mathfrak{g}_2 = 0$.

Let us assume now that $\varphi \in \Gamma_{\text{Lie}}(\gamma_{\text{Lie}}^2(\mathfrak{g}))$ is an idempotent such that $\varphi \neq 0$ and $\varphi \neq \text{Id}$. Then $\varphi^2(x) = \varphi(x)$, for all $y \in \gamma_{\text{Lie}}^2(\mathfrak{g})$. We claim that $\text{Ker}(\varphi)$ and $\text{Im}(\varphi)$ are two-sided ideals of $\gamma_{\text{Lie}}^2(\mathfrak{g})$. Indeed, for any $x \in \text{Ker}(\varphi)$, $y \in \gamma_{\text{Lie}}^2(\mathfrak{g})$,

$$
\varphi([x, y]) = \varphi([x, \sum_i k_i[y_i', y_i'']_{\text{Lie}}]) = \varphi(0) = 0,
$$

hence $[x, y] \in \text{Ker}(\varphi)$, for all $y \in \gamma_{\text{Lie}}^2(\mathfrak{g})$. On the other hand,

$$
\varphi([y, x]) = \varphi((\sum_i k_i[y_i', y_i'']_{\text{Lie}}, x)) = \varphi((\sum_i k_i[y_i', y_i'']_{\text{Lie}}, x)_{\text{Lie}})
$$

$$
= \sum_i k_i[y_i', y_i'']_{\text{Lie}, \varphi(x)]_{\text{Lie}} = 0.
$$
4. Generalised Lie-derivations and quasi-Lie-centroids

In this section, we introduce the notions of generalised Lie-derivations and quasi-Lie-centroids and analyse their interplay with Lie-centroids.

**Definition 4.1.** Let \( g \) be a Leibniz algebra. A linear map \( f : g \to g \) is called a *generalised Lie-derivation of* \( g \) if there exist linear maps \( f', f'' : g \to g \) such that

\[
[f(x), y]_\text{Lie} + [x, f''(y)]_\text{Lie} = f'(f(x), y)_\text{Lie},
\]

for all \( x, y \in g \). We denote by \( \text{GenDer}^{\text{Lie}}(g) \) the set of all generalised Lie-derivations of \( g \).

We say that \( f \) is a *quasi-Lie-derivation of* \( g \) if there exists a linear map \( f' : g \to g \) such that

\[
[f(x), y]_\text{Lie} + [x, f(y)]_\text{Lie} = f'([x, y]_\text{Lie}),
\]

for all \( x, y \in g \). We denote by \( \text{QDer}^{\text{Lie}}(g) \) the set of all quasi-Lie-derivations of \( g \).

It is easy to check that \( \text{GenDer}^{\text{Lie}}(g) \) and \( \text{QDer}^{\text{Lie}}(g) \) are subalgebras of \( \text{End}_k(g) \). In fact, we have the following inclusion tower:

\[
\text{Der}^{\text{Lie}}(g) \subseteq \text{Der}^{\text{Lie}}(g) \subseteq \text{QDer}^{\text{Lie}}(g) \subseteq \text{GenDer}^{\text{Lie}}(g) \subseteq \text{End}_k(g).
\]

**Example 4.2.** Let \( g \) be the two-dimensional Leibniz algebra in Example 3.2 (c).

(a) Let \( f, f' : g \to g \) be the linear maps represented by the matrices

\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\]

respectively. It can be checked that \( f \in \text{QDer}^{\text{Lie}}(g) \), although \( f \notin \text{Der}^{\text{Lie}}(g) \).
(b) Consider the linear maps \( f, f', f'': \mathfrak{g} \to \mathfrak{g} \) represented by the matrices 
\[
\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
respectively. It is easy to check that \( f \in \text{GenDer}^{\text{Lie}}(\mathfrak{g}) \), but \( f \not\in \text{QDer}^{\text{Lie}}(\mathfrak{g}) \).

**Definition 4.3.** The quasi-Lie-centroid of a Leibniz algebra \( \mathfrak{g} \) is the set of all linear maps \( d: \mathfrak{g} \to \mathfrak{g} \) satisfying
\[
[d(x), y]_{\text{Lie}} = [x, d(y)]_{\text{Lie}}
\]
for all \( x, y \in \mathfrak{g} \). We denote this set by \( \text{QDer}^{\text{Lie}}(\mathfrak{g}) \).

**Example 4.4.** Let \( \mathfrak{g} \) be the two-dimensional Leibniz algebra in Example 3.2 (c). The linear maps \( f: \mathfrak{g} \to \mathfrak{g} \) represented by the matrices \( \begin{pmatrix} a & b \\ c & a \end{pmatrix}, a, b, c \in \mathbb{C} \), are quasi-Lie-centroids.

**Lemma 4.5.** Let \( \mathfrak{g} \) be a Leibniz algebra. We have the following inclusions,

(i) \( [\text{Der}^{\text{Lie}}(\mathfrak{g}), \Gamma^{\text{Lie}}(\mathfrak{g})] \subseteq \Gamma^{\text{Lie}}(\mathfrak{g}) \),

(ii) \( [\text{QDer}^{\text{Lie}}(\mathfrak{g}), \text{QDer}^{\text{Lie}}(\mathfrak{g})] \subseteq \text{QDer}^{\text{Lie}}(\mathfrak{g}) \),

\( [\text{QDer}^{\text{Lie}}(\mathfrak{g}), \text{QDer}^{\text{Lie}}(\mathfrak{g})] \subseteq \text{QDer}^{\text{Lie}}(\mathfrak{g}) \),

(iii) \( \text{QDer}^{\text{Lie}}(\mathfrak{g}), \text{QDer}^{\text{Lie}}(\mathfrak{g}) \subseteq \text{QDer}^{\text{Lie}}(\mathfrak{g}) \),

(iv) \( \Gamma^{\text{Lie}}(\mathfrak{g}) \subseteq \text{Der}^{\text{Lie}}(\mathfrak{g}) \),

(v) \( \text{QDer}^{\text{Lie}}(\mathfrak{g}) + \text{QDer}^{\text{Lie}}(\mathfrak{g}) \subseteq \text{GenDer}^{\text{Lie}}(\mathfrak{g}) \).

**Proof.** (i) Let \( d, d' \in \text{Der}^{\text{Lie}}(\mathfrak{g}), d'' \in \Gamma^{\text{Lie}}(\mathfrak{g}) \), then for all \( x, y \in \mathfrak{g} \):
\[
[d, d'](x, y)_{\text{Lie}} = d[x, d(y)]_{\text{Lie}} = d[x, d(y)]_{\text{Lie}} - [x, [d, d'](x, y)_{\text{Lie}}]
\]
A similar computation shows the other equality.

(ii) Let \( d, d' \in \text{QDer}^{\text{Lie}}(\mathfrak{g}), d'' \in \text{QDer}^{\text{Lie}}(\mathfrak{g}) \), then for all \( x, y \in \mathfrak{g} \):
\[
[d, d'](x, y)_{\text{Lie}} = [dd' - d'd](x, y)_{\text{Lie}} = d'[x, d(y)]_{\text{Lie}} - [d', [d, d'](x, y)_{\text{Lie}}]
\]

The second inclusion can be checked analogously.

(iii) Let \( d, d' \in \text{QDer}^{\text{Lie}}(\mathfrak{g}) \). Since
\[
[d, d'](x, y)_{\text{Lie}} + [x, [d, d'](y)]_{\text{Lie}} = [d', x, d(y)]_{\text{Lie}} - [d', [x, d(y)]_{\text{Lie}}]
\]
then \( [d, d'], [x, [d, d'](y)]_{\text{Lie}} = 0 \).

(iv) Let \( d \in \Gamma^{\text{Lie}}(\mathfrak{g}) \), then \( [d(x), y]_{\text{Lie}} + [x, d(y)]_{\text{Lie}} = 2d([x, y]_{\text{Lie}}) \), and therefore \( d \in \text{GenDer}^{\text{Lie}}(\mathfrak{g}) \), where \( d' = 2d \).

(v) Let \( d \in \text{QDer}^{\text{Lie}}(\mathfrak{g}), d' \in \Gamma^{\text{Lie}}(\mathfrak{g}) \). Since
\[
((d + d')(x, y)]_{\text{Lie}} = d_1([x, y]_{\text{Lie}}) - [x, d(y)]_{\text{Lie}} + [x, d'(y)]_{\text{Lie}}
\]
then \( (d + d')(x, y)]_{\text{Lie}} + (d - d')(y)]_{\text{Lie}} = d_1([x, y]_{\text{Lie}}) \), i.e., \( d + d' \in \text{GenDer}^{\text{Lie}}(\mathfrak{g}) \). □

**Theorem 4.6.** Let \( \mathfrak{g} \) be a Leibniz algebra. Then
\[
[\Gamma^{\text{Lie}}(\mathfrak{g}), \text{QDer}^{\text{Lie}}(\mathfrak{g})] \subseteq \text{End}_{\mathbb{K}}(\mathfrak{g}, Z^{\text{Lie}}(\mathfrak{g})).
\]
Moreover, if \( Z^{\text{Lie}}(\mathfrak{g}) = 0 \), then \( [\Gamma^{\text{Lie}}(\mathfrak{g}), \text{QDer}^{\text{Lie}}(\mathfrak{g})] = 0 \).
Proof. Let $d \in \Gamma^{\text{Lie}}(g)$, $d' \in Q\Gamma^{\text{Lie}}(g)$. Then,
\[
[d, d'](x, y)_{\text{Lie}} = [dd'(x), y]_{\text{Lie}} - [d'd(x), y]_{\text{Lie}}
= d([d'(x), y]_{\text{Lie}}) - [d(x), d'(y)]_{\text{Lie}}
= d([x, d'(y)]_{\text{Lie}}) - [d(x), d'(y)]_{\text{Lie}} = 0,
\]
i.e., $[d, d'](x) \in Z^{\text{Lie}}(g)$. \hfill $\Box$

**Theorem 4.7.** Let $g$ be a Leibniz algebra. Then $\Gamma^{\text{Lie}}(g) = Q\text{Der}^{\text{Lie}}(g) \cap Q\Gamma^{\text{Lie}}(g)$.

**Proof.** Let $f \in Q\text{Der}^{\text{Lie}}(g) \cap Q\Gamma^{\text{Lie}}(g)$, then there exists a linear map $f' : g \to g$ such that $f'(x, y)_{\text{Lie}} = [f(x), y]_{\text{Lie}} + [x, f(y)]_{\text{Lie}}$.

We claim that $f \in \Gamma^{\text{Lie}}(g)$. Indeed, $[f(x), y]_{\text{Lie}} = [x, f(y)]_{\text{Lie}}$ since $f \in Q\Gamma^{\text{Lie}}(g)$. On the other hand, $f((x, y)_{\text{Lie}}) = [f(x), y]_{\text{Lie}}$ since $f'(x, y)_{\text{Lie}} = 2[f(x), y]_{\text{Lie}}$, therefore $f' = 2f$ provides the needed equality.

Conversely, let $f \in \Gamma^{\text{Lie}}(g)$ implies that $f((x, y)_{\text{Lie}}) = [f(x), y]_{\text{Lie}} = [x, f(y)]_{\text{Lie}}$, hence $f \in Q\Gamma^{\text{Lie}}(g)$. Moreover $f((x, y)_{\text{Lie}}) = 2f((x, y)_{\text{Lie}})$, so just taking $f' = 2f$ we conclude the proof. \hfill $\Box$

**Lemma 4.8.** Let $m$ and $n$ be two two-sided ideals of a Leibniz algebra $g$, such that $g = m \oplus n$. Then:

(i) $Z^{\text{Lie}}(g) = Z^{\text{Lie}}(m) \oplus Z^{\text{Lie}}(n),$
(ii) If $Z^{\text{Lie}}(g) = 0$ then,
(a) $\text{Der}^{\text{Lie}}(g) = \text{Der}^{\text{Lie}}(m) \oplus \text{Der}^{\text{Lie}}(n).$
(b) $\text{GenDer}^{\text{Lie}}(g) = \text{GenDer}^{\text{Lie}}(m) \oplus \text{GenDer}^{\text{Lie}}(n).$
(c) $Q\text{Der}^{\text{Lie}}(g) = Q\text{Der}^{\text{Lie}}(m) \oplus Q\text{Der}^{\text{Lie}}(n).$
(d) $\Gamma^{\text{Lie}}(g) = \Gamma^{\text{Lie}}(m) \oplus \Gamma^{\text{Lie}}(n).$
(c) $Q\Gamma^{\text{Lie}}(g) = Q\Gamma^{\text{Lie}}(m) \oplus Q\Gamma^{\text{Lie}}(n).$

**Proof.**

(i) Let $m \in Z^{\text{Lie}}(m), n \in Z^{\text{Lie}}(n)$, then for any $y = m' + n' \in g$, we have:

$[m + n, y]_{\text{Lie}} = [m, m']_{\text{Lie}} + [m, n']_{\text{Lie}} + [n, m']_{\text{Lie}} + [n, n']_{\text{Lie}} = 0,$
i.e., $m + n \in Z^{\text{Lie}}(g).$

Conversely, for any $x = m + n \in Z^{\text{Lie}}(g)$ and for all $y = m' + n' \in g$, we have $0 = [x, y]_{\text{Lie}} = [m, m']_{\text{Lie}} + [n, n']_{\text{Lie}}$, hence $[m, m']_{\text{Lie}} = 0, [n, n']_{\text{Lie}} = 0$ for all $m' \in m, n' \in n$, i.e., $m \in Z^{\text{Lie}}(m), n \in Z^{\text{Lie}}(n).$

(ii) (a) Let $d \in \text{Der}^{\text{Lie}}(g)$. For any $m \in m, n \in n$, we have

$[d(m), n]_{\text{Lie}} = d([m, n]_{\text{Lie}}) - [m, d(n)]_{\text{Lie}} = -[m, m]_{\text{Lie}},$
for some $m_1 \in m$. Assume that $d(m) = m' + n'$, then

$-[m, m]_{\text{Lie}} = [d(m), n]_{\text{Lie}} = [m', n']_{\text{Lie}},$
hence $[n', n]_{\text{Lie}} = 0$ for all $n \in n$, so $n' \in Z^{\text{Lie}}(n) = 0$. Consequently $d(m) = m'$, i.e., $d(m) \subseteq m$. Similarly, $d(n) \subseteq n$.

Conversely, let $d_1 \in \text{Der}^{\text{Lie}}(m), d_2 \in \text{Der}^{\text{Lie}}(n)$, then the derivation $d : g \to g$, defined by $d(g) = d_1(m) + d_2(n)$, for all $g = m + n \in g$, is a Lie-derivation of $g$.

The other statements are obtained in a similar way. \hfill $\Box$

**Proposition 4.9.** Let $g$ be a Leibniz algebra. Then, $Q\Gamma^{\text{Lie}}(g) + [Q\Gamma^{\text{Lie}}(g), Q\Gamma^{\text{Lie}}(g)]$ is a subalgebra of $\text{GenDer}^{\text{Lie}}(g)$.

**Proof.** By Lemma 4.5 (iii) and (v), we have:

$Q\Gamma^{\text{Lie}}(g) + [Q\Gamma^{\text{Lie}}(g), Q\Gamma^{\text{Lie}}(g)] \subseteq Q\Gamma^{\text{Lie}}(g) + Q\text{Der}^{\text{Lie}}(g) \subseteq \text{GenDer}^{\text{Lie}}(g).$
Let us denote by $\text{Der}$ if and only if the following identity was true:

$$\text{Der} = \text{Der}$$

which expanded together with Remark 2.1 would mean that

$$\text{Der}$$

Therefore, it only makes sense to study inner $\text{Lie}$-symmetry of the $\text{Lie}$.

$$\text{Lie}$$

for all $x, y, z \in \mathfrak{g}$, which is the same as saying that $\gamma_2^{\text{Lie}}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$. Note that by the symmetry of the Lie-bracket it does not matter to consider left or right adjoint maps.

5. Almost inner Lie-derivations

In this section we study the context under which almost inner Lie-derivations and central derivations coincide. It is interesting to note that the relative setting requires an additional condition with respect to the absolute context.

**Definition 5.1.** An almost inner Lie-derivation of a Leibniz algebra $\mathfrak{g}$ is a Lie-derivation $d: \mathfrak{g} \to \mathfrak{g}$ such that $d(x) \in [x, \mathfrak{g}]_{\text{Lie}}$ for all $x \in \mathfrak{g}$. We denote by $\text{Der}_{\text{Lie}}(\mathfrak{g})$ the subspace of $\text{Der}^{\text{Lie}}(\mathfrak{g})$ consisting in all almost inner Lie-derivations of $\mathfrak{g}$, that is

$$\text{Der}_{\text{Lie}}(\mathfrak{g}) = \{ d \in \text{Der}^{\text{Lie}}(\mathfrak{g}) \mid d(x) \in [x, \mathfrak{g}]_{\text{Lie}}, \text{ for all } x \in \mathfrak{g} \}.$$

Let us denote by $T_c\left(\frac{\mathfrak{g}}{\mathfrak{z}^{\text{Lie}}(\mathfrak{g})}, \gamma_2^{\text{Lie}}(\mathfrak{g})\right)$ the following vector space:

$$\left\{ f \in \text{Hom}(\mathfrak{g}, \mathfrak{g}) \mid f(x + \mathfrak{z}^{\text{Lie}}(\mathfrak{g})) \in [x, \mathfrak{g}]_{\text{Lie}}, \text{ for all } x \in \mathfrak{g} \right\}.$$

**Proposition 5.2.** Let $\mathfrak{g}$ be a Lie-nilpotent Leibniz algebra with class of Lie-nilpotency 2. Then $\text{Der}_{\text{Lie}}(\mathfrak{g}) \cong T_c\left(\frac{\mathfrak{g}}{\mathfrak{z}^{\text{Lie}}(\mathfrak{g})}, \gamma_2^{\text{Lie}}(\mathfrak{g})\right)$.

**Proof.** Let $d \in \text{Der}_{\text{Lie}}(\mathfrak{g})$, the map $\psi_d: \frac{\mathfrak{g}}{\mathfrak{z}^{\text{Lie}}(\mathfrak{g})} \to \gamma_2^{\text{Lie}}(\mathfrak{g})$, $\psi_d(x + \mathfrak{z}^{\text{Lie}}(\mathfrak{g})) = d(x)$, is a linear map in $T_c\left(\frac{\mathfrak{g}}{\mathfrak{z}^{\text{Lie}}(\mathfrak{g})}, \gamma_2^{\text{Lie}}(\mathfrak{g})\right)$.

Since $\gamma_2^{\text{Lie}}(\mathfrak{g}) = 0$, then $\text{Der}_{\text{Lie}}(\mathfrak{g})$ is an abelian Leibniz algebra and the vector space $T_c\left(\frac{\mathfrak{g}}{\mathfrak{z}^{\text{Lie}}(\mathfrak{g})}, \gamma_2^{\text{Lie}}(\mathfrak{g})\right)$ is also an abelian Leibniz algebra.

The map $\Psi: \text{Der}_{\text{Lie}}(\mathfrak{g}) \to T_c\left(\frac{\mathfrak{g}}{\mathfrak{z}^{\text{Lie}}(\mathfrak{g})}, \gamma_2^{\text{Lie}}(\mathfrak{g})\right)$, $\Psi(d) = \psi_d$, is an isomorphism of abelian Leibniz algebras. \qed
Let $g$ be a Leibniz algebra satisfying $\gamma_2^{\text{Lie}}(g) \subseteq Z(g)$. By $\text{IDer}^{\text{Lie}} = \{d_x : x \in g\}$ we denote the set of all inner Lie-derivations of $g$.

**Example 5.3.** The three-dimensional Leibniz algebra with basis $\{a_1, a_2, a_3\}$ and bracket operation $[a_3, a_3] = a_1$ and zero elsewhere (class 2 (b) in [5]) is a Leibniz algebra satisfying $\gamma_2^{\text{Lie}}(g) \subseteq Z(g)$.

Now we are going to explore the similarities between $\text{Der}_c^{\text{Lie}}(g)$ and $\text{Der}_z^{\text{Lie}}(g)$.

**Theorem 5.4.** Let $g$ be a finite-dimensional non-abelian Leibniz algebra satisfying $\gamma_2^{\text{Lie}}(g) \subseteq Z(g)$. Then $\text{Der}_c^{\text{Lie}}(g) = \text{Der}_z^{\text{Lie}}(g)$ if and only if $Z^{\text{Lie}}(g) = \gamma_2^{\text{Lie}}(g)$ and $\text{Der}_z^{\text{Lie}}(g) \cong T\left(\frac{\mathfrak{g}}{Z^{\text{Lie}}(g)}, \gamma_2^{\text{Lie}}(g)\right)$.

**Proof.** Assume that $\text{Der}_c^{\text{Lie}}(g) = \text{Der}_z^{\text{Lie}}(g)$. For any $d \in \text{Der}_c^{\text{Lie}}(g)$, the homomorphism

$$\psi_d : \frac{\mathfrak{g}}{Z^{\text{Lie}}(g)} \to \gamma_2^{\text{Lie}}(g), \quad \psi_d(x + Z^{\text{Lie}}(g)) = d(x),$$

is a linear map. Now we define

$$\Psi : \text{Der}_c^{\text{Lie}}(g) \to T\left(\frac{\mathfrak{g}}{Z^{\text{Lie}}(g)}, \gamma_2^{\text{Lie}}(g)\right), \quad \Psi(d) = \psi_d.$$

Clearly $\Psi$ is an isomorphism of Leibniz algebras.

Now, by the isomorphism of $K$-vector spaces given in [3] Lemma 3.5, we have

$$\dim \left(T\left(\frac{\mathfrak{g}}{Z^{\text{Lie}}(g)}, \gamma_2^{\text{Lie}}(g)\right)\right) = \dim \left(\text{Der}_c^{\text{Lie}}(g)\right) = \dim \left(\text{Der}_z^{\text{Lie}}(g)\right)$$

$$= \dim \left(T\left(\frac{\mathfrak{g}}{\gamma_2^{\text{Lie}}(g)}, Z^{\text{Lie}}(g)\right)\right).$$

Therefore, since $\gamma_2^{\text{Lie}}(g) \subseteq Z^{\text{Lie}}(g)$, by a linear algebra argument concerning dimensions we know that $\gamma_2^{\text{Lie}}(g) = Z^{\text{Lie}}(g)$.

Conversely, assume that $\gamma_2^{\text{Lie}}(g) = Z^{\text{Lie}}(g)$ and $\text{Der}_z^{\text{Lie}}(g) \cong T\left(\frac{\mathfrak{g}}{Z^{\text{Lie}}(g)}, \gamma_2^{\text{Lie}}(g)\right)$. It is clear that $\text{Der}_c^{\text{Lie}}(g) \subseteq \text{Der}_z^{\text{Lie}}(g)$. On the other hand, due to [3] Lemma 3.5, we have

$$\dim \left(\text{Der}_c^{\text{Lie}}(g)\right) = \dim \left(T\left(\frac{\mathfrak{g}}{\gamma_2^{\text{Lie}}(g)}, Z^{\text{Lie}}(g)\right)\right) = \dim \left(T\left(\frac{\mathfrak{g}}{Z^{\text{Lie}}(g)}, \gamma_2^{\text{Lie}}(g)\right)\right)$$

which completes the proof. □

**Corollary 5.5.** Let $g$ be a finite-dimensional non-abelian Leibniz algebra satisfying $\gamma_2^{\text{Lie}}(g) \subseteq Z(g)$ such that $\dim \left(Z^{\text{Lie}}(g)\right) = 1$. Then $\text{Der}_c^{\text{Lie}}(g) = \text{Der}_z^{\text{Lie}}(g)$ if and only if $Z^{\text{Lie}}(g) = \gamma_2^{\text{Lie}}(g)$.

**Proof.** If $Z^{\text{Lie}}(g) = \gamma_2^{\text{Lie}}(g)$, then $\text{IDer}^{\text{Lie}}(g)$ and $\text{Der}_c^{\text{Lie}}(g)$ are subalgebras of $\text{Der}_z^{\text{Lie}}(g)$. Since $\text{IDer}^{\text{Lie}}(g) \cong \text{IDer}^{\text{Lie}}(g)$ and by [3] Lemma 3.5, we have

$$\dim \left(\text{Der}_z^{\text{Lie}}(g)\right) = \dim \left(T\left(\frac{\mathfrak{g}}{\gamma_2^{\text{Lie}}(g)}, Z^{\text{Lie}}(g)\right)\right),$$

hence

$$\dim \left(\text{Der}_z^{\text{Lie}}(g)\right) = \dim \left(\frac{\mathfrak{g}}{\gamma_2^{\text{Lie}}(g)}\right) \cdot \dim \left(Z^{\text{Lie}}(g)\right)$$

$$= \dim \left(\frac{\mathfrak{g}}{Z^{\text{Lie}}(g)}\right) = \dim \left(\text{IDer}^{\text{Lie}}(g)\right).$$
Consequently, \( \text{Der}_c^\text{Lie}(g) \subseteq \text{Der}_z^\text{Lie}(g) = I\text{Der}^\text{Lie}(g) \subseteq \text{Der}_e^\text{Lie}(g) \).

The converse is just an immediate consequence of Theorem 5.3.

**Example 5.6.** The three-dimensional Leibniz algebra with basis \( \{a_1, a_2, a_3\} \) and bracket operation given by \([a_2, a_2] = [a_3, a_3] = a_1 \) and zero elsewhere (class 2 (c) in [9]), is a Leibniz algebra satisfying the requirements of Corollary 5.5.

6. Lie-Centroid of a Tensor Product

In this section we analyse some properties concerning the Lie-centroid of the Leibniz algebra tensor product of a commutative associative algebra and a Leibniz algebra.

Let \( A \) be a commutative associative algebra over \( \mathbb{K} \). The centroid of \( A \) is the associative subalgebra of \( \text{End}_{\mathbb{K}}(A) \)

\[ \Gamma(A) = \{ f \in \text{End}_{\mathbb{K}}(A) \mid f(ab) = fa(b) \} . \]

For a Leibniz algebra \( g \), let \( A \otimes g \) be the tensor product over \( \mathbb{K} \) of the underlying vector spaces of \( A \) and \( g \). Then \( A \otimes g \) can be endowed with a structure of Leibniz algebra with the operation

\[ [a_1 \otimes g_1, a_2 \otimes g_2] = (a_1 a_2) \otimes [g_1, g_2] \]

for all \( a_1, a_2 \in A, g_1, g_2 \in g \). This Leibniz algebra is called the tensor product Leibniz algebra of \( A \) and \( g \).

Given \( f \in \text{End}_{\mathbb{K}}(A), \varphi \in \text{End}_{\mathbb{K}}(g) \), the map \( f \otimes \varphi: A \otimes g \to A \otimes g \) defined by \( f \otimes \varphi(a \otimes g) = f(a) \otimes \varphi(g) \) is an endomorphism of the Leibniz algebras.

**Corollary 6.1.** Let \( g \) be a finite-dimensional non-abelian Leibniz algebra satisfying \( \gamma_2^\text{Lie}(g) \subseteq Z(g) \) such that \( \dim(Z^\text{Lie}(g)) = 1 \). Then \( \text{Der}_c^\text{Lie}(g) = \text{Der}_z^\text{Lie}(g) \cap \Gamma^\text{Lie}(g) \)

if and only if \( Z^\text{Lie}(g) = \gamma_2^\text{Lie}(g) \).

**Proof.** By Corollary 5.3 we know that \( \text{Der}_c^\text{Lie}(g) = \text{Der}_z^\text{Lie}(g) \). Then [3] Proposition 4.2 concludes the proof.

**Lemma 6.2.** We have the following inclusion:

\[ \Gamma(A) \otimes \Gamma^\text{Lie}(g) \subseteq \Gamma^\text{Lie}(A \otimes g) . \]

**Proof.** For any \( f \in \Gamma(A) \) and \( \varphi \in \Gamma^\text{Lie}(g) \) we have:

\[
\hat{f} \otimes \hat{\varphi}([a_1 \otimes g_1, a_2 \otimes g_2]_{\text{Lie}}) = \hat{f} \otimes \hat{\varphi}((a_1 a_2) \otimes [g_1, g_2] + (a_2 a_1) \otimes [g_2, g_1]) \\
= f(a_1 a_2) \otimes \varphi([g_1, g_2]|_{\text{Lie}}) = (f(a_1) a_2) \otimes \varphi(g_1, g_2)_{\text{Lie}} \\
= [f(a_1) \otimes \varphi(g_1), a_2 \otimes g_2]_{\text{Lie}} \\
= [\hat{f} \otimes \hat{\varphi}(a_1 \otimes g_1), a_2 \otimes g_2]|_{\text{Lie}} .
\]

The other equality can be checked in a similar way.

**Example 6.3.** Let \( A \) be the two-dimensional commutative associative algebra with basis \( \{e_1, e_2\} \) and product \( e_1 e_1 = e_1, e_1 e_2 = e_2, e_2 e_1 = e_1, e_2 e_2 = 0 \) (see class \( A^4_2 \) in [21]) and let \( g \) be the two-dimensional Leibniz algebra with basis \( \{a_1, a_2\} \) and bracket operation given by \([a_1, a_2] = a_1 \) and zero elsewhere [2]. The Lie-centroid of \( A \otimes g \) are the linear maps represented by a matrix of the form

\[
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
\mu & 0 & \lambda & 0 \\
0 & \mu & 0 & \lambda
\end{pmatrix}
\]
The element of \((A \otimes g)\) represented by the matrix \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}
\] cannot be obtained as the tensor product of an element \(\Gamma(A)\), whose associated matrix has the form \[
\begin{pmatrix}
\lambda & 0 \\
\mu & \lambda
\end{pmatrix},
\]
and an element of \(\Gamma^{\text{Lie}}(g)\), whose associated matrix is of the form \[
\begin{pmatrix}
0 & 0 \\
0 & \delta
\end{pmatrix}.
\]

If \(A\) is unital, then the isomorphism \(\sigma: \Gamma(A) \to A\) given by \(\sigma(f) = f(1)\), implies that \(\Gamma(A) \cong A\).

From now on we assume that \(g\) is a Leibniz algebra over an algebraically closed field \(K\) of characteristic zero and \(A\) is a unital commutative associative algebra over \(K\).

Let \(\Psi\) denote the subspace of \(\Gamma^{\text{Lie}}(g)\) spanned by \(\{\varphi_j\}_{j \in J}\), a maximal subset of \(\Gamma^{\text{Lie}}(g)\) such that \(\{\varphi_j\}_{j \in J}^{\text{Lie}(g)}\) is linearly independent.

**Proposition 6.4.** We have the following inclusion:

\[A \otimes \Psi + \text{End}_K(A) \otimes \text{Der}^{\text{Lie}}_K(g) \subseteq \Gamma^{\text{Lie}}(A \otimes g)\]

**Proof.** We just need to show that \(\text{End}_K(A) \otimes \text{Der}^{\text{Lie}}_K(g) \subseteq \Gamma^{\text{Lie}}(A \otimes g)\), since the other part is implied by Lemma 6.2. For any \(f \in \text{End}_K(A)\), and \(\varphi \in \text{Der}^{\text{Lie}}_K(g)\), we have:

\[
f \otimes \varphi([a_1 \otimes g_1, a_2 \otimes g_2]_{\text{Lie}}) = f(a_1, a_2) \otimes \varphi([g_1, g_2]_{\text{Lie}})
\]

\[
= f(a_1, a_2) \otimes ([\varphi(g_1), g_2]_{\text{Lie}} + [g_1, \varphi(g_2)]_{\text{Lie}}) = 0,
\]

\[
[f \otimes \varphi(a_1 \otimes g_1), a_2 \otimes g_2]_{\text{Lie}} = f(a_1) a_2 \otimes [\varphi(g_1), g_2]_{\text{Lie}} = 0,
\]

\[
[a_1 \otimes g_1, f \otimes \varphi(a_2 \otimes g_2)]_{\text{Lie}} = a_1 f(a_2) \otimes [g_1, \varphi(g_2)]_{\text{Lie}} = 0.
\]

\(\square\)

**Theorem 6.5.** Let \(g\) be a Leibniz algebra such that \(\varphi_2^{\text{Lie}}(g) \neq 0\) and \(\Gamma^{\text{Lie}}(g) = K \cdot \text{Id}\), then \(\Gamma^{\text{Lie}}(A \otimes g) = \Gamma(A) \otimes \Gamma^{\text{Lie}}(g) \cong A\).

**Proof.** By Lemma 6.2 it is enough to prove that \(\Gamma^{\text{Lie}}(A \otimes g) \subseteq \Gamma(A) \otimes \Gamma^{\text{Lie}}(g) \cong A\). Assume that \(\{A_i\}\) is a basis of \(A\). Then for any \(\varphi \in \Gamma^{\text{Lie}}(A \otimes g)\), \(a \in A\), there exists a set of linear transformations \(\{\eta(a, -) : g \to g\}\) such that for all \(g \in g\),

\[
\varphi(a \otimes g) = \sum_i A_i \otimes \eta_i(a, g) \tag{1}
\]

where the sum has only a finite number of non-zero summands.

We claim that \(\eta_i(a, -)\) are elements of \(\Gamma^{\text{Lie}}(g)\), for all \(i\). Indeed, on one hand

\[
\varphi([a_1 \otimes g_1, 1 \otimes g_2]_{\text{Lie}}) = [\varphi(a_1 \otimes g_1), 1 \otimes g_2]_{\text{Lie}}
\]

\[
= \left[ \sum_i A_i \otimes \eta_i(a_1, g_1), 1 \otimes g_2 \right]_{\text{Lie}}
\]

\[
= \sum_i A_i \otimes \eta_i(a_1, g_1)_{\text{Lie}},
\]

and on the other hand

\[
\varphi([a_1 \otimes g_1, 1 \otimes g_2]_{\text{Lie}}) = \varphi(a_1 \otimes [g_1, g_2]_{\text{Lie}})
\]

\[
= \sum_i A_i \otimes \eta_i(a_1, [g_1, g_2]_{\text{Lie}}).
\]
Therefore \([\eta(a_1, g_1), g_2]_{\text{Lie}} = \eta(a_1, [g_1, g_2]_{\text{Lie}})\). A similar computation shows the other equality.

Since \(\Gamma_{\text{Lie}}(g) = \mathbb{K}.\text{Id}\), then \(\eta(a, g) = \lambda_i(a)g, \lambda_i(a) \in \mathbb{K}\). By identity (1), for any \(\varphi \in \Gamma_{\text{Lie}}(A \otimes g)\), \(a \in A\), there exists a finite set of indices \(J\) such that if \(i \notin J\), then \(\eta_i(a, -) = 0\). Thus we have that for all \(g \in g\)

\[
\varphi(a \otimes g) = \sum_{i \in J} A_i \otimes \eta_i(a, g) = \sum_{i \in J} A_i \otimes \lambda_i(a)g = \sum_{i \in J} \lambda_i(a)(A_i \otimes g).
\]

Let \(\rho : A \rightarrow A\) be the map defined by \(\rho(a) = \sum_{i \in J} \lambda_i(a)A_i\), for all \(a \in A\). Then, \(\varphi(a \otimes g) = \rho(a) \otimes g\). Since

\[
\varphi([a_1 \otimes g_1, l \otimes g_2]_{\text{Lie}}) = \varphi(a_1 \otimes [g_1, g_2]_{\text{Lie}}) = \rho(a_1) \otimes [g_1, g_2]_{\text{Lie}},
\]

\[
[a_1 \otimes g_1, \varphi(l \otimes g_2)]_{\text{Lie}} = [a_1 \otimes g_1, \rho(l) \otimes g_2]_{\text{Lie}} = \rho(l)a_1 \otimes [g_1, g_2]_{\text{Lie}},
\]

then we get \(\rho(a_1) \otimes [g_1, g_2]_{\text{Lie}} = \rho(1)a_1 \otimes [g_1, g_2]_{\text{Lie}}\). Moreover, since \(\gamma_1^{\text{Lie}}(g) \neq 0\), we have that \(\rho(a_1) = \rho(1)a_1, a_1 \in A\). Consequently,

\[
\varphi(a \otimes g) = (\rho(1)a) \otimes g = (\rho(1) \otimes \text{Id}_g)(a \otimes g),
\]

for all \(a \in A, g \in g\). This means that \(\varphi\) belongs to \(\Gamma(A) \otimes \mathbb{K}.\text{Id} = \Gamma(A) \otimes \Gamma_{\text{Lie}}(g)\). \(\square\)

**Example 6.6.** Example \(\mathfrak{L}_3\) (b) satisfies the requirements of Theorem 6.5.

**Remark 6.7.** Theorem \(\mathfrak{L}_3\) does not hold if \(A\) is not a unital algebra, as the following example shows. Let \(g\) be the complex five-dimensional Leibniz algebra with the basis \(\{a_1, a_2, a_3, a_4, a_5\}\) and bracket operation

\[
\begin{align*}
[a_2, a_1] &= -a_3, & [a_1, a_2] &= a_3, & [a_1, a_3] &= -2a_1, \\
[a_3, a_1] &= 2a_1, & [a_3, a_2] &= -2a_2, & [a_2, a_3] &= 2a_2, \\
[a_5, a_1] &= a_4, & [a_4, a_2] &= a_5, & [a_4, a_3] &= -a_4, \\
& & [a_5, a_3] &= a_5,
\end{align*}
\]

and zero elsewhere \[\mathfrak{L}_3\] Example 3.2]. It can be checked that \(\gamma_1^{\text{Lie}}(g) = \langle \{a_4, a_5\} \rangle\) and \(\Gamma_{\text{Lie}}(g) = \mathbb{K}.\text{Id}\).

Let \(\mathbb{C}[t]\) be the polynomial ring in the variable \(t\) with coefficients in the field of the complex numbers. Let \(B = t^m \mathbb{C}[t], m > 0\), be the subalgebra of \(\mathbb{C}[t]\), which does not contain the unit element. It is easy to show that \(f(t).\text{Id}_B \otimes \text{Id}_g \in \Gamma_{\text{Lie}}(B \otimes g)\), for any \(f(t) \in \mathbb{C}[t]\). This fact implies that \(\Gamma_{\text{Lie}}(B \otimes g) \neq B\), since the elements \(f(t).\text{Id}_B \otimes \text{Id}_g\) are identified with \(f(t) \in \mathbb{C}[t]\), which is not an element of \(B\).

**Definition 6.8.** A linear map \(\phi \in \Gamma_{\text{Lie}}(A \otimes g)\) is said to have a finite \(g\)-image if, for any \(a \in A\) there exist finitely many \(a_1, \ldots, a_n \in A\) such that

\[
\phi(\mathbb{K}a \otimes g) \subseteq (\mathbb{K}a_1 \otimes g) + \cdots + (\mathbb{K}a_n \otimes g).
\]

From now on we denote by \(\Gamma(A) \otimes \Gamma_{\text{Lie}}(g)\) the \(\mathbb{K}\)-span of all endomorphisms \(f \otimes \varphi\) of \(A \otimes g\). Due to Lemma 6.2, we know that \(\Gamma(A) \otimes \Gamma_{\text{Lie}}(g) \subseteq \Gamma_{\text{Lie}}(A \otimes g)\). Moreover,

\[
\Gamma(A) \otimes \Gamma_{\text{Lie}}(g) \subseteq \{\phi \in \Gamma_{\text{Lie}}(A \otimes g) \mid \phi\text{ has finite }g\text{-image}\}.
\]

**Lemma 6.9.** Let \(\{a_i\}_{i \in I}\) be a basis of \(A\) and \(\varphi \in \Gamma_{\text{Lie}}(A \otimes g)\). Let \(\varphi_i : g \rightarrow g\) be the linear maps defined by \(\varphi(1 \otimes g) = \sum_{i \in I} a_i \otimes \varphi_i(g)\). Then \(\varphi_i \in \Gamma_{\text{Lie}}(g)\).
Proof.
\[ \varphi(1 \otimes [g_1, g_2]_{\text{Lie}}) = \sum_{i \in I} a_i \otimes \varphi_i([g_1, g_2]_{\text{Lie}}), \]
\[ \varphi(1 \otimes [g_1, g_2]_{\text{Lie}}) = \varphi([1 \otimes g_1, 1 \otimes g_2]_{\text{Lie}}) = [\varphi(1 \otimes g_1), 1 \otimes g_2]_{\text{Lie}} \]
\[ = \sum_{i \in I} a_i \otimes \varphi_i(g_1), 1 \otimes g_2]_{\text{Lie}} = \sum_{i \in I} a_i \otimes [\varphi_i(g_1), g_2]_{\text{Lie}}, \]
\[ \varphi(1 \otimes [g_1, g_2]_{\text{Lie}}) = \sum_{i \in I} a_i \otimes [\varphi_i(g_1), g_2]_{\text{Lie}}. \]

Hence \( \varphi_i([g_1, g_2]_{\text{Lie}}) = [\varphi_i(g_1), g_2]_{\text{Lie}} = [g_1, \varphi_i(g_2)]_{\text{Lie}}, \) for all \( g_1, g_2 \in g, \) and therefore \( \varphi_i, \) \( i \in I \).

\[ \Box \]

**Proposition 6.10.** Let \( g \) be Liebniz algebra. Then,
\begin{enumerate}
  \item \( \gamma^\text{Lie}_{1/2}(A \otimes g) = A \otimes \gamma^\text{Lie}_{1/2}(g). \)
  \item If \( \gamma^\text{Lie}_{1/2}(g) \) is finite generated as an \( \Gamma^\text{Lie}(g) \)-module, then every \( \varphi \in \Gamma^\text{Lie}(A \otimes \gamma^\text{Lie}_{1/2}(g)) \) has finite \( \gamma^\text{Lie}_{1/2}(g) \)-image.
  \item If \( \gamma^\text{Lie}_{1/2}(g) \neq 0 \) and \( \Gamma^\text{Lie}(g) = K.\text{Id}_g \), then \( \varphi \in \Gamma^\text{Lie}(A \otimes g) \) has finite \( g \)-image.
\end{enumerate}

Proof. (i) For any \( a \otimes g \in A \otimes \gamma^\text{Lie}_{1/2}(g) \), we have
\[ a \otimes g = a.1 \otimes \sum_{i} k_i [g_{1i}, g_{2i}]_{\text{Lie}} = \sum_{i} k_i [a \otimes g, 1 \otimes g_{2i}]_{\text{Lie}} \in \gamma^\text{Lie}_{1/2}(A \otimes g). \]

(ii) Let \( x_1, \ldots, x_n \in \gamma^\text{Lie}_{1/2}(g) \) such that \( \gamma^\text{Lie}_{1/2}(g) = \psi_1 x_1 + \cdots + \psi_n x_n, \) where \( \psi_i \in \Gamma^\text{Lie}(\gamma^\text{Lie}_{1/2}(g)) \). Let \( \varphi \in \Gamma^\text{Lie}(A \otimes g) \) and \( a \in A \). There exist finite families \( \{x_{ij}\}_{j \in J} \) in \( \gamma^\text{Lie}_{1/2}(g) \) and \( \{a_{ij}\}_{j \in J} \) in \( A \), such that \( \varphi(a \otimes x_i) = \sum_j a_{ij} \otimes x_{ij}, \) for \( 1 \leq i \leq n \). Hence, by Lemma 6.2, statement (i) and Proposition 6.10, we have
\[ \varphi(a \otimes g) = \varphi \left( a \otimes \sum_i \psi_i x_i \right) = \sum_i \varphi(a \otimes \psi_i x_i) = \sum_i \varphi \circ (\text{Id} \otimes \psi_i)(a \otimes x_i) = \sum_i (\text{Id} \otimes \psi_i) \circ \varphi(a \otimes x_i) = \sum_i \sum_j (\text{Id} \otimes \psi_i)(a_{ij} \otimes x_{ij}) \subseteq \sum_i \sum_j (a_{ij} \otimes g). \]

(iii) Assume that \( \Gamma_i^\text{Lie}(g) = K.\text{Id}_g \) and \( \varphi \in \Gamma^\text{Lie}(A \otimes g) \), then by Theorem 6.5, we have
\[ \varphi(1 \otimes g) = \sum_{i \in I} a_i \otimes \varphi_i(g) = \sum_{i \in I} a_i \otimes (k_i \text{Id})(g) = \sum_{i \in I} a_i \otimes (k_i g). \]

Fixed \( g \in g, \) then almost all \( k_i g = 0, \) and almost \( k_i = 0, \) which in turn means that \( \varphi \) has finite \( g \)-image.

\( \Box \)

**Theorem 6.11.** Let \( g \) be a Leibniz algebra such that \( \gamma^\text{Lie}_{1/2}(g) \neq 0, \) \( \Gamma^\text{Lie}(g) = K.\text{Id} \) and \( R = \mathbb{K}[x_1, \ldots, x_n]. \) Then \( \Gamma^\text{Lie}(R \otimes g) = R \otimes \Gamma^\text{Lie}(g). \)

**Proof.** The inclusion \( R \otimes \Gamma^\text{Lie}(g) \subseteq \Gamma^\text{Lie}(R \otimes g) \) is provided by Lemma 6.2.

Let us consider \( \{m_i\} \) a basis of \( R, \) \( \varphi \in \Gamma^\text{Lie}(R \otimes g), \) a polynomial \( p \in R \) and \( g_1, g_2 \in g. \) Let us denote by \( \eta_i(p, \cdot) \) the suitable maps in \( \text{End}_g(g) \) such that \( \varphi(p \otimes g) = \sum_i m_i \otimes \eta_i(p, g), \) as in (1). Then,
\[ \varphi([p \otimes g_1, 1 \otimes g_2]_{\text{Lie}}) = [\varphi(p \otimes g_1), 1 \otimes g_2]_{\text{Lie}} = \left[ \sum_{i} m_i \otimes \eta_i(p, g_1), 1 \otimes g_2 \right]_{\text{Lie}}, \]
\[ = \sum_{i} m_i \otimes [\eta_i(p, g_1), g_2]_{\text{Lie}}. \]
but
\[ \varphi(p \otimes g_1, g \otimes g_2) = \varphi(p \otimes [g_1, g_2]) = \sum m_i \otimes \eta_i(p, [g_1, g_2]_{\text{Lie}}). \]

Consequently, for every \( i \) and \( p \), we have \( \eta_i(p, g_1), g_2)_{\text{Lie}} = \eta_i(p, [g_1, g_2]_{\text{Lie}}) \). With a similar computation, we have that
\[ [\eta_i(p, g_1), g_2]_{\text{Lie}} = [g_1, \eta_i(p, g_2)]_{\text{Lie}}, \]
i.e., \( \eta_i(p, -) \in \Gamma_{\text{Lie}}(g) = K \text{Id} \). Therefore, \( \eta_i(p, g) = k_i(p)g \), for all \( g \in g \) and suitable scalars \( k_i(p) \). Hence, \( \varphi(p \otimes g) = \sum m_i \otimes k_i(p)g = \sum k_i(p)m_i \otimes g \), which is an element in \( R \otimes g \) for every \( p \), so \( k_i(p) \) is non-zero for a finite number of \( i \) (it is enough to take any \( g \in g \), \( g \neq 0 \), to see this), that is
\[ \varphi(p \otimes g) = \sum m_i \otimes k_i(p)g = \sum k_i(p)m_i \otimes g. \]

Then the map \( \rho: p \mapsto \sum k_i(p)m_i \) is well-defined. Thus we have \( \varphi(p \otimes g) = \rho(p) \otimes p \), and hence,
\[ [\varphi(p \otimes g), 1 \otimes g']_{\text{Lie}} = [\rho(p) \otimes g, 1 \otimes g']_{\text{Lie}} = \rho(p) \otimes [g, g']_{\text{Lie}}. \]
\[ [p \otimes g, \varphi(1 \otimes g')]_{\text{Lie}} = [p \otimes g, \rho(1) \otimes g']_{\text{Lie}} = \rho(1) \otimes [g, g']_{\text{Lie}}. \]
Choosing \( g, g' \in g \) such that \([g, g']_{\text{Lie}} \neq 0\), we conclude that \( \rho(p) = \rho(1) \), for all \( p \in R \). This means that \( \rho \) is determined by the action on \( 1 \), and therefore \( \rho \in R \).

Thus, \( \varphi(p \otimes g) = \rho(p) \otimes g = (\rho \otimes \text{Id})(p \otimes g) \), which means exactly that \( \varphi \in R \otimes \text{Id} \cong R \otimes \Gamma_{\text{Lie}}(g) \).

\[ \square \]

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Email address: jmcasas@uvigo.es
Email address: xabier.garcia.martinez@uvigo.gal
Email address: natarego@gmail.com