Higher Chow cycles on Jacobian of Fermat curves and hypergeometric functions

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Abstract

In [Col97], Collino constructed a higher Chow cycle which lies in $K_1$ of the Jacobian of a hyperelliptic curve $C$. Colombo showed in [Col02] that the regulator image of such a cycle is obtained as an iterated integral. In [SS14], the hyperelliptic assumption is generalised. In this article we specialize $C$ to Fermat curves of degree $N$. The regulator map in this case is expressed as a special value of the hypergeometric functions.

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1 Introduction

Let $\mathcal{V}_k$ be the category of smooth projective varieties defined over a number field $k$. For $X \in \mathcal{V}_k$, there is the Abel-Jacobi map

$$AJ_X : CH^i_{\text{hom}}(X) \to J^i(X)$$

where $J^i(X) = \frac{H^{2i-1}(X, \mathcal{C})^*}{F^i + H^{2i-1}(X, \mathbb{Z}(i))}$ is the $i$th intermediate Jacobian.

The regulator map is a generalisation of this map. It is a map from a motivic cohomology group to a generalised torus, namely the Deligne cohomology. Beilinson [Bei84] defined the motivic cohomology groups $H^i_M(X, \mathbb{Q}(n)) := K^{n}_{2n-i}(X)_\mathbb{Q}$, where $K^n_{2n-i}(X)_\mathbb{Q}$ is the $n$th graded piece, with respect to the Adams filtration, on the $(2n - i)^{th}$ rational higher $K$-group. He formulated a set of conjectures to explain the relationship between special values of motivic $L$-functions of $X$ and the $\mathbb{Q}$ structure induced by the regulator map. A particular case is Dirichlet’s class number formula for number fields. Beilinson defined a regulator map from

$$\text{reg}_\mathbb{Q} : H^i_M(X, \mathbb{Q}(n)) \to H^i_D(X_{\mathbb{R}}, \mathbb{Q}(n))$$

for $2n - i \geq 1$ and made conjectures relating special values with the determinant of a period matrix.

An alternative approach to define the regulator map is the following. The motivic cohomology groups $H^i_M(X, \mathbb{Q}(n))$ can be interpreted as the group of extensions in the conjectured category of mixed motives $\mathcal{M}_M \mathbb{Q}$

$$H^i_M(X, \mathbb{Q}(n)) \simeq \text{Ext}^1_{\mathcal{M}_M \mathbb{Q}}(\mathbb{Q}(-n), h^{i-1}(X))$$

where $\mathbb{Q}(-n)$ is a twist of the Tate motive and $h^i(X)$ is the motive associated to $H^i(X)$ and $2n-i > 0$. Further, the Real Deligne cohomology is

$$H^i_D(X_{\mathbb{R}}, \mathbb{R}(n)) = \text{Ext}^1_{\mathcal{M}_{MHS}}(\mathbb{R}(-n), H^{i-1}(X))$$
where $\text{R} = \text{MHS}$ denotes the category of Real mixed Hodge structures. From this point of view, the regulator map can be understood as the map induced by the realisation functors from $\mathcal{M}_\text{Q}$ to $\text{R} - \text{MHS}$,

$$\text{reg}_R : \text{Ext}^1_{\mathcal{M}_\text{Q}}(\mathbb{Q}(-n), h^{i-1}(X)) \rightarrow \text{Ext}^1_{\text{R} - \text{MHS}}(\mathbb{R}(-n), H^{i-1}(X))$$

Similarly, all other regulator maps – for instance, to integral Deligne cohomology – can simply be interpreted as realisations of extensions.

Beilinson [Beil84] proved most of the conjecture in the case when $X$ is a product of modular curves. He showed that there are at least as many elements in the motivic cohomology group as predicted and the determinant of the regulator map with respect certain basis is, up to a non-zero rational number, the special value of the $L$-function of $H^2$. The method of proof is to decompose the motive of $X_0(N)$ into motives of modular forms of weight 2 and then use some classical results to conclude the result. The conjecture has been proved in very few other cases.

In [Ots11], Otsubo computed the regulator of an element of $H^2_{\text{M}}(F_N, \mathbb{Q}(2))$ which is second graded piece of $K_2(F_N)_\text{Q}$. He expressed the regulator map as a special values of a hypergeometric functions. Also in [Ots11] he expressed Abel-Jacobi image of Ceresa cycle in Jacobian of a Fermat’s curve as a special value of a hypergeometric functions. The domain of Abel Jacobi map is a graded piece of $K_0(\text{Jac}(F_N))_{\mathbb{Q}}^{g-1}$. We consider the case of $K_1(\text{Jac}(F_N))_{\mathbb{Q}}^{g}$. We have the following theorems.

**Theorem 1.1.** Let $X = \text{Jac}(F_N)$ where $F_N$ is the Fermat curve of degree $N$ defined over $\mathbb{Q}$. Let $Z_{QR,P} \in H_{\text{M}}^{2g-1}(X, \mathbb{Q}(g))$ be the element constructed in §3.1.1, (2). Let $\omega^a_b = x^a y^b - N \frac{dx}{x}$ be a differential 1 form on $F_N$, where $a, b \in \mathbb{Z}$. Then one has

$$\text{reg}_R(Z_{QR,P})(\Omega^a_b)_{N} = \delta_{a,c} \left[ \tilde{F}_1 \left( \frac{b}{N}, \frac{d-b}{N}, \frac{a}{N} \right) - \tilde{F}_1 \left( \frac{-b}{N}, \frac{b-d}{N}, \frac{-a}{N} \right) \right]$$

$$+ \tilde{F}_1 \left( \frac{b}{N}, \frac{b-d}{N}, \frac{-a}{N} \right) - \tilde{F}_1 \left( \frac{-b}{N}, \frac{b-d}{N}, \frac{-a}{N} \right)$$

$$+ \delta_{a,c} \left[ \tilde{F}_1 \left( \frac{d}{N}, \frac{d-b}{N}, \frac{a}{N} \right) - \tilde{F}_1 \left( \frac{-d}{N}, \frac{d-b}{N}, \frac{-a}{N} \right) \right]$$

$$+ \tilde{F}_1 \left( \frac{d}{N}, \frac{d-b}{N}, \frac{-a}{N} \right) - \tilde{F}_1 \left( \frac{-d}{N}, \frac{d-b}{N}, \frac{-a}{N} \right)$$

$$- \delta_{b,d} \left[ \tilde{F}_1 \left( \frac{a}{N}, \frac{a-c}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{-a}{N}, \frac{a-c}{N}, \frac{-b}{N} \right) \right]$$

$$+ \tilde{F}_1 \left( \frac{a}{N}, \frac{a-c}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{-a}{N}, \frac{a-c}{N}, \frac{-b}{N} \right)$$

$$- \delta_{b,d} \left[ \tilde{F}_1 \left( \frac{c}{N}, \frac{c-a}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{-c}{N}, \frac{c-a}{N}, \frac{-b}{N} \right) \right]$$

$$+ \tilde{F}_1 \left( \frac{c}{N}, \frac{c-a}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{-c}{N}, \frac{c-a}{N}, \frac{-b}{N} \right) \right] .$$

(1)

Here $\delta_{a,a}$ is the Kronecker delta function, $\Omega^a_b = \omega^a_b \cap \omega^c_d - \omega^a_b \cap \omega^c_d \in F^1(\wedge^2 H^1(F_N, R(1)))^{n-1}$, $a, b, c, d > 0$ and $a + b, c + d < N$. Let $(a)$ denote the integer which represent $a$ in the set $\{1, 2, N\}$.
and

\[
\mathcal{F}_1 \left( \frac{(a)}{N}, \frac{(i)}{N}, \frac{(b)}{N} \right) = \frac{1}{\frac{(i)}{N}} \beta \left( \frac{(a)}{N} + \frac{(i)}{N}, \frac{(b)}{N} \right)_3 \mathcal{F}_2 \left( \frac{(a)}{N} + \frac{(i)}{N}, \frac{(i)}{N}, 1 \right),
\]

is a special value of a hypergeometric function. \( \beta(x, y) \) is the beta function.

Let \( B^{g-1} \) be the free group generated by the subvariety of \( \text{Jac}(F_N) \) of codimension \( g - 1 \) modulo homological equivalence. Poincaré duality identifies \( B^{g-1} \) with the image of Neron-Severi group of \( F_N \times F_N \) in \( \otimes^2 H^1(F_N) \). If \( N \) is prime to 6, by Theorem of [KR78] one knows that the image of \( NS(F_N \times F_N) \) is generated by \( \omega_{a,b}^N \otimes \omega_{-c,-d}^N \), where \( a + b < N, c + d < N \) and \( \{a, b, N - (a + b)\} = \{c, d, N - (c + d)\} \) up to a permutation. In particular

**Theorem 1.2.** For \( \text{Jac}(F_N) \in \mathcal{V}_Q \) and \( N = 7 \), the cycle \( Z_{QR,P} \in H^{2g-1}_M(X, \mathbb{Z}(g))_{\text{indec}} \).

This article is organised as follows. In §3 we recall some necessary definitions like- Motivic cohomology, Deligne cohomology and the regulator map. Here we also recall Theorem 4.16 of [SS14], which is a key formula used in this article. In §4 we discuss some facts about Fermat curves and Fermat motives. These facts are available from several sources. We mainly follow [Ots11]. The main theorem of this article will be proved in §5. We discuss some applications of main theorem.

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## 2 Notation

- \( \zeta_N \) — a primitive \( N^{th} \) root of unity.
- \( E_N = \mathbb{Q}(\zeta_N) \).
- \( H_N = \text{Gal}(E_N/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^* \).
- \( \tau : \mathbb{Q}(\zeta_N) \to \mathbb{C} \) an embedding such that \( \tau(\zeta_N) = e^{2\pi i/N} \).
- \( G_N = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} \).
- \( I_N = \{(a, b) \in G_N| a, b + a \neq 0(\text{mod } N)\} \subset G_N \).
- \( [a, b] = H_N \) orbit of \( (a, b) \) in \( G_N \).
- \( \mathcal{V}_k \) — the category of smooth projective varieties defined over a number field \( k \).
- \( \mathcal{M}_Q \) — the category of motives over \( \mathbb{Q} \) with coefficients in \( \mathbb{Q} \).
- \( H^i_B(X, \mathbb{Z}) \) — the Singular (Betti) cohomology group of \( X \).
- \( H^i_{dR}(X, \mathbb{Z}) \) — the (de Rham) cohomology group of \( X \).
• $\mathbb{L} = (\text{Spec } k, \Delta_{\text{Spec } k}, -1)$ — the Lefschetz motive.
• $\mathbb{I} = (\text{Spec } k, \Delta_{\text{Spec } k}, 0)$ — Unit motive or Tate motive.
• $H^i_M(X, \mathbb{Q}(n))$ — the Motivic cohomology group of $X$.
• $H^i_D(X, \mathbb{R}(n))$ — the Deligne cohomology group with coefficients $\mathbb{R}$.
• $H^i_D(X/\mathbb{R}, \mathbb{R}(n))$ — the ‘Real’ Deligne cohomology group.
• $F_N$ — the Fermat curve of degree $N$ given by the equation $X^N + Y^N - Z^N = 0$ considered as a variety over $\mathbb{Q}$.
• $F_{N,E}$ — $F_{N,E} := F_N \otimes \text{Spec}(E_N) \in \mathcal{E}_N$
• $\langle a \rangle$ denote the integer which represent $a$ in the set $\{1, 2, \ldots, N\}$.
• $F_N^{[a,b]} \wedge F_N^{[c,d]} = \left( F_N \times F_N, P_N^{[a,b]} \times P_N^{[c,d]} - P_N^{[a,b]} \times P_N^{[c,d]}, 0 \right) \in \mathcal{M}_Q$

3 Prelimineries

Let $X$ be a smooth projective variety defined over $\mathbb{Q}$.

3.1 Motivic cohomology

Let $K_i(X)$ be the $i^{th}$ higher algebraic $K$-group introduced by Quillen. The motivic cohomology groups of $X$ are defined to be, for integer $n \geq 0$,

$$H^{2n-1}_M(X, \mathbb{Q}(n)) := K_1^{(n)}(X)_\mathbb{Q},$$

where $K_1^{(n)}(X)_\mathbb{Q}$ is the Adams eigenspace of weight $n$ [Sch88], a $\mathbb{Q}$ vector space. This has the following representation. Let $Z_i(X)$ be the free abelian group generated by irreducible subvariety of $X$ of codimension $i$ and $Z$ be a irreducible subvariety of codimension $n-1$ in $X$. Let $j: \tilde{Z} \to Z$ be a normalization of $Z$. We denote $\text{div}_Z(f) := j_* \text{div}_{\tilde{Z}}(f) \in Z^n(X)$. Then one has

$$H^{2n-1}_M(X, \mathbb{Q}(n)) := \left( \bigoplus_{Z \in Z^{n-1}(X)} k_Z^* \otimes_{Z^{n-1}(X)} Z^n(X) \right) \otimes \mathbb{Q},$$

where $k_Z$ is the field of rational functions on $Z$ and $k_Z^*$ is the set of all nonzero elements of $k_Z$. An element of the above group can be represented by $Z = \sum_{i=1}^{i=t} (Z_i, f_i)$ such that $Z_i \in Z^{n-1}(X)$ and $f_i \in k(Z_i)^*$ such that $\sum_{i=1}^{i=t} \text{div}_{Z_i}(f_i) = 0$. 


Elements in the motivic cohomology group $H^2_{M}(X,\mathbb{Q}(g))$.

Using the cup product on motivic cohomology, define

$$H^2_{M}(X,\mathbb{Q}(g))_{dec} := \bigoplus_{[L:Q]<\infty} Nm_{L/Q} \text{Im} \left( H^{2g-2}_{M}(X,\mathbb{Q}(g-1)) \otimes H^1_{M}(X_L,\mathbb{Q}(1)) \to H^{2g-1}_{M}(X_L,\mathbb{Q}(g)) \right).$$

The group of indecomposable cycles is defined by

$$H^2_{M}(X,\mathbb{Q}(g))_{indec} := H^2_{M}(X,\mathbb{Q}(g))/H^2_{M}(X,\mathbb{Q}(g))_{dec}.$$ 

An element $Z = \sum_{i=1}^{t} (Z_i, f_i)$ is said to be indecomposable if $reg_R(Z)$ does not lie in the image of $H^{2g-2}_{M}(X,\mathbb{Q}(g-1)) \otimes H^1_{M}(X_L,\mathbb{Q}(1)) \cong \text{CH}^{g-1}(X)_{\mathbb{Q}} \otimes \mathbb{C}^*$, for any finite extension $L$ of $\mathbb{Q}$, where $\mathbb{C}^*$ be the set of nonzero complex numbers.

### 3.1.1 Elements in the motivic cohomology group $H^2_{M}(X,\mathbb{Q}(g))$

The following construction of $Z \in H^2_{M}(X,\mathbb{Q}(g))$ was done by Collino [Col97] in the case when $C$ is a hyperelliptic curve and $Q$, $R$ and $P$ are Weierstrass points. We assume the following hypothesis on $C$. Let $C$ be a smooth projective curve of genus $g > 0$ and $C$ satisfy the following condition. There exist two $\mathbb{Q}$-rational points $Q, R \in C$ be such that $Q - R$ is torsion in the Jacobian of $C$.

It means there exists a function $f_{QR} : C \to \mathbb{P}^1$ with

$$\text{div}(f_{QR}) = N(Q - R)$$

for some integer $N$. To determine the function precisely we have to choose another distinct point $P \in C$ and add a requirement that $f_{QR}(P) = 1$. Example of such curves are modular curve $X_0(N)$ and Fermat curves of degree $N$.

Let $C_Q$ be the image of the map $C \to \text{Jac}(C)$ given by $x \to x - Q$. Similarly, let $C_R$ be the image of the map $C \to \text{Jac}(C)$ given by $x \to R - x$ and $f_Q, f_R$ denote the function $f_{QR}$ considered as a function on $C_Q$ and $C_R$ respectively.

Consider the cycle

$$Z_{QR,P} := (C_Q, f_Q) + (C_R, f_R) \quad (2)$$

As

$$\text{div}_{C_Q}(f_Q) + \text{div}_{C_R}(f_R) = N(0) - N(R - Q) - N(0) + N(R - Q) = 0,$$

$Z_{QR,P}$ lies in $H^2_{M}(X,\mathbb{Q}(g))$. Let

$$\eta : C \times C \to \text{Jac}(C) \quad (x, y) \mapsto (x - y).$$

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It induces a functorial homomorphism
\[ \eta_* : H^2_d(C \times C, \mathbb{Q}(2)) \to H^{2g-1}_M(Jac(C), \mathbb{Q}(g)). \]

Let us define
\[ Z_{\Delta QR, P} := (C \times Q, 1/f^Q) + (\Delta, f^\Delta) + (R \times C, 1/f^R) \in H^3_M(C \times C, \mathbb{Q}(2)), \]

Bloch constructed \( Z_{\Delta QR, P} \) when \( C = X_0(37) \) and it maps to Collino cycle –
\[ \eta_*(Z_{\Delta QR, P}) = Z_{QR, P} \in H^{2g-2}_M(X, \mathbb{Q}(g)). \]

### 3.2 Deligne Cohomology and the regulator maps

For \( X \in \mathcal{M}_g \) the cohomology group of \( X \) is denoted by \( H^i(X) := \{ H^i_B(X), H^i_{dR}(X) \} \). It admits a Mixed Hodge structure \((H^i(X), W_\bullet, F^\bullet)\), where \( W_\bullet \) is a weight filtration on \( H^i_B(X)_\mathbb{Q} \) and \( F^\bullet \) is the Hodge filtration on \( H^i_{dR}(X)_\mathbb{C} \). The Deligne cohomology group of \( X \) with \( \mathbb{Q} \) coefficients is defined to be the hypercohomology of a complex \( Z_D \otimes \mathbb{Q}(g) \), known as Deligne complex. Deligne cohomology group can be identified with the group of extension of MHS. As \( X = Jac(C) \), one has

\[
H^{2g-1}_d(X, \mathbb{Q}(g)) := \text{Ext}^1_{MHS}(\mathbb{Q}(-g), H^{2g-2}_d(X)) \\
\cong \frac{H^{2g-2}(Jac(C), \mathbb{C})}{H^2(\mathbb{Q}(g))} \\
\cong \frac{F^g H^{2g-2}(X, \mathbb{C}) + H^{2g-2}(Jac(C), \mathbb{Z}(g))}{H^2(Jac(C), \mathbb{C})} \\
\cong \frac{F^2 H^2(Jac(C)), \mathbb{C}) + H^2(Jac(C), \mathbb{Z}(2))}{F^2 H^2_{dR}(Jac(C))} \\
\cong \frac{(F^1 H^2(Jac(C), \mathbb{C}))^*}{H_2(Jac(C), \mathbb{Z}(2))}
\]

where the first isomorphism follows from Proposition 2, in [Car80] and the second one is induce by the Poincaré duality. Since \( X = \text{Jac}(C) \), \( F^1 H^2(X) = F^1 \wedge^2 H^1(C) \). Thus \( H^{2g-1}_d(X, \mathbb{Q}(g)) \) is a vector space of linear functionals on a cohomology group \( F^1(\wedge^2 H^1(C)) \).

Let \( c_\infty \) be the complex conjugation acting on the coefficients of the \( \mathbb{C} \) vector spaces and \( F_\infty \) be the map induced by complex conjugation acting on \( X(\mathbb{C}) \). The ‘Real’ Deligne cohomology group of \( X \) is defined to be
\[
H^{2g-1}_d(X/\mathbb{R}, \mathbb{R}(g)) \cong H^{2g-1}_d(X_\mathbb{C}, \mathbb{R}(g)) F_\infty \otimes c_\infty = 1 \cong \frac{(F^1 H^2(X, \mathbb{C}) F_\infty \otimes c_\infty = 1)^*}{H_2(X, \mathbb{Q}(2))}. \]

#### 3.2.1 Regulator Maps

The regulator map is a map from motivic cohomology group to Deligne cohomology group. Conjectural any element of \( Z \in H^{2g-1}_M(X, \mathbb{Q}(g)) \) corresponds to an extension in \( \mathcal{M}_g \mathcal{M}_Q \). Regulator map is realisation of such extension in the category of MHSs. Carlson isomorphism of such extension class corresponds to a equivalence class of functionals on the cohomology group \( F^1 \wedge^2 H^1(C) \). An example
of such extension class of MHS was discussed in §6 in [KLMS06]. The map is defined as follows. Suppose
\[ Z = \sum_{i=1}^{i=t} (C_i, f_i) \in H^{2g-1}_M(X, \mathbb{Q}(g)) \]
where \( f_i : C_i \to \mathbb{P}^1 \) are functions satisfying the co-cycle condition
\[ \sum_{i=1}^{i=t} \text{div}_{C_i}(f_i) = 0 \]
and \( C_i \) are curves in \( \text{Jac}(C) \). Let \([0, \infty]\) be the path from 0 to \( \infty \) along the real axis in \( \mathbb{P}^1 \). Let \( \mu_i : \tilde{C}_i \to C_i \) be a resolution of singularities. We can think of \( f_i \) as a function on \( \tilde{C}_i \). Let \( \gamma_i = \mu_i^{-1}([0, \infty]) \).

From the co-cycle condition we have
\[ \sum_{i=1}^{i=t} \gamma_i = \partial(D) \]
where \( D \) is 2-chain in \( X \) because \( H_2(X) \) does not have torsion. The regulator map to Deligne cohomology with \( \mathbb{Q} \) coefficients, \( H^{2g-1}_D(X, \mathbb{Q}(g)) \), is defined to be
\[ \langle \text{reg}_{\mathbb{Q}}(Z), \omega \rangle = \frac{1}{2\pi i} \left( \sum_{i=1}^{i=t} \int_{C_i-\gamma_i} \log(f_i)\omega + 2\pi i \int_D \omega \right). \tag{6} \]
where \( \omega \in F^1 \wedge H^1(C, C) \).

The real regulator map \( \text{reg}_{\mathbb{R}} \) is a current on the subspace \( F^1(\wedge^2(H^1(C))^F_{\infty \otimes c_{\infty}^{-1}}. \) Hence it is given by
\[ \langle \text{reg}_{\mathbb{R}}(Z), \omega \rangle = \frac{1}{2\pi i} \sum_{i=1}^{i=t} \int_{C_i-\gamma_i} \log|f_i|\omega. \tag{7} \]

In the case \( C \), a hyperelliptic curve with Weierstrass point \( \{Q, R, P\} \), [Col02] Colombo obtains an extension which is the image of Collino cycle under the regulator map. Theorem 4.19 in [SS14] is a generalisation to the case when \( C \) is smooth projective curve of genus \( g > 0 \) and there exist two distinct \( \mathbb{Q} \) rational point \( \{Q, R\} \) such that \( Q - R \) is torsion in \( \text{Jac}(C) \).

**Theorem 3.1.** [SS14] Let \( C \) be a smooth projective curve of genus \( g > 0 \) with the property that there exist two points \( Q, R \) in \( C \) such that \( Q - R \) is torsion in \( \text{Jac}(C) \). Let us fix a point \( P \in C \setminus \{Q, R\} \) and choose a function \( f_{QR} \) such that \( \text{div}(f_{QR}) = N(Q - R) \) and \( f_{QR}(P) = 1 \). Let \( Z_{QR,P} \) be the element of \( H^{2g-1}_M(\text{Jac}(C), \mathbb{Q}(g)) \) constructed in (2), §3.1.1. Then there is an extension class \( \epsilon_{QR,P} \in \text{Ext}_{MHS}(\mathbb{Q}(-2), \wedge^2 H^1(C)) \) constructed from the mixed Hodge structure on the fundamental groups of \( C \setminus \{Q\} \) and \( C \setminus \{R\} \) such that
\[ \epsilon_{QR,P} = (2g + 1) \text{reg}_{\mathbb{Q}}(Z_{QR,P}) \in \text{Ext}_{MHS}(\mathbb{Q}(-2), \wedge^2 H^1(C)). \]

**Proof.** Proof is given Theorem 4.18 in [SS14].

The following theorem gives the formula of the real regulator map \( \text{reg}_{\mathbb{R}} \).
Theorem 3.2. Assumptions are as in Theorem 3.1 and \( f_{\text{QR}} = f \). Let \( dz_i \) and \( dz_j \) be two holomorphic form on \( C \) then
\[
\omega_{i,j} = 2\pi i (dz_i \wedge d\bar{z}_j - d\bar{z}_i \wedge dz_j) \in (H^1(C) \wedge H^1(C)(1))^{F_\infty = -1}
\]
Then the real regulator map is given by
\[
\langle \text{reg}_R(Z_{\text{QR}}), \omega_{i,j} \rangle = (\int_{\gamma_j} \log |f|dz_i - \int_{\gamma_j} \log |f|d\bar{z}_i)
+ (\int_{\gamma_i} \log |f|dz_j - \int_{\gamma_i} \log |f|d\bar{z}_j)
\]
where \( \gamma_j \) and \( \gamma_j \) are Poincaré duals of \( dz_j \) and \( d\bar{z}_j \) respectively.

Proof. From preceding theorem 3.1 we have
\[
\langle \text{reg}_Q(Z_{\text{QR}}), \omega \rangle = \int_{\text{P.D}} \log fdz_i - \int_{\text{P.D}} \log fd\bar{z}_j,
\]
where \( \omega = dz_i \wedge d\bar{z}_j \in F^1 \wedge^2 H^1(C) \). Since \( F_\infty(dz_i \wedge d\bar{z}_j) = d\bar{z}_i \wedge dz_j \), one has
\[
\omega_{i,j} = 2\pi i (dz_i \wedge d\bar{z}_j - d\bar{z}_i \wedge dz_j) \in (H^1(C) \wedge H^1(C))^{F_\infty \otimes c_\infty = 1}.
\]
Hence from equation (7),
\[
\langle \text{reg}_R(Z_{\text{QR}}), \omega_{i,j} \rangle = (\int_{\gamma_j} \log |f|dz_i - \int_{\gamma_j} \log |f|d\bar{z}_i)
+ (\int_{\gamma_i} \log |f|dz_j - \int_{\gamma_i} \log |f|d\bar{z}_j).
\]
\( \square \)

3.3 Hypergeometric functions

Hypergeometric functions were introduced by Euler and generalized by Gauss, Appell and Thomae. This section we discuss certain properties of hypergeometric functions. In a later section we relate certain values of hypergeometric functions to the regulator image of some cycles.

Let \( \alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \) and \( j \) be a non-negative integer. Define the Pochhammer Symbol to be
\[
(\alpha)_j = \alpha(\alpha + 1) \cdots (\alpha + j - 1) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)}.
\]
A generalised hypergeometric function of type \((n, n-1)\) is a function defined by the series expansions
\[
\genfrac{[}{]}{0pt}{}{a_1, a_2 \ldots a_n}{b_1, b_2 \ldots b_{n-1}}; z = \sum_{j=0}^{\infty} \frac{(a_1)_j(a_2)_j \cdots (a_n)_j}{(b_1)_j(b_2)_j \cdots (b_{n-1})_j(1)_j} z^j.
\]
It converges absolutely for \(|z| < 1\) and converge at \( z = 1 \) if \( \sum_{j=1}^{n-1} b_j - \sum_{j=1}^{n} a_j > 0 \). It has an analytic continuation in the entire complex plane. Good references for this is [CDS01],[BW88],[BH89].

We are mainly interested in hypergeometric functions of type \((3, 2)\). A hypergeometric function of type \((3, 2)\) has the following integral representation which allows one to analytically continue it to the entire complex plane \( \mathbb{C} \). Let \( [a] \) denote the congruence class of \( a \) modulo \( N \) in the set \( \{1, \ldots, N\} \).
Lemma 3.3.

\[ 3F_2 \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} ; z \right) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_1 - a_1)\Gamma(a_3)\Gamma(b_2 - a_3)} \times \]

\[ \int\int_{\Delta} u^{a_1-1}(1-zu)^{-b}(1-u(1-v))^{b_1-a_1}u^{b_2-a_3}(1-v)^{a_3-a_1}dudv, \quad (8) \]

where \( \Delta = \{(u,v)|u,v,1-u-v \geq 0\} \) and branches of logarithm is chosen as follows-
\[ \arg(u) = 0, \arg(1-u) = 0 \text{ and } \arg(1-zu) \leq \pi/2. \]

Proof. See §4.10, [Ots11].

Another interesting property of type (3, 2) hypergeometric function is Dixon’s Formula [Sla66], which is the following

\[ 3F_2 \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} ; 1 \right) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(s)}{\Gamma(a_1)\Gamma(a_2 + s)\Gamma(a_3 + s)} 3F_2 \left( \begin{array}{c} b_1 - a_1, b_2 - a_1, s \\ a_2 + s, a_3 + s \end{array} ; 1 \right) \]

where \( s = b_1 + b_2 - a_1 - a_2 - a_3. \)

Define

\[ \tilde{F}_1 \left( \frac{\langle a \rangle}{N}, \frac{\langle i \rangle}{N}, \frac{\langle b \rangle}{N} \right) = \frac{1}{\beta(\frac{a}{N}, \frac{b}{N})} 3F_2 \left( \begin{array}{c} \langle a \rangle \frac{N}{N} + \frac{\langle i \rangle}{N}, \frac{\langle b \rangle}{N} \langle \frac{N}{N} \\ \frac{\langle a \rangle}{N} + \frac{\langle b \rangle}{N} + 1 \end{array} ; 1 \right), \]

Then from lemma (3.3) we have

\[ \tilde{F}_1 \left( \frac{\langle a \rangle}{N}, \frac{\langle i \rangle}{N}, \frac{\langle b \rangle}{N} \right) = \frac{1}{\beta(\frac{a}{N}, \frac{b}{N})} \int\int_{\Delta} u^{-\langle a \rangle}(1-u)^{-\langle b \rangle}v^{-\langle i \rangle}(1-v)^{-\langle i \rangle}dudv. \]

4 Specializing to Fermat Curves

Here we shall introduce some facts on Fermat curves and their Jacobian varieties. Let \( F_N \) be the Fermat curve of degree \( N \) over \( \mathbb{Q} \) defined by the homogeneous equation

\[ F_N : x^N + y^N = z^N. \]

The set of complex points \( F_N(\mathbb{C}) \) forms a Riemann surface of genus \( \frac{(N-1)(N-2)}{2} \). Let \( \zeta_N \) be a primitive \( N \) th roots of unity such that \( \zeta_N^d = \zeta_N \). The set of points with one of \( x, y \) or \( z \) being 0 are \( (\zeta_N^r : 0 : 1), (0 : \zeta_N^s : 1) \) and \( (\zeta_N^r : 1 : 0) \), where \( r \in \mathbb{Z} \) are called cusps or the points at infinity. There are \( 3N \) such points over \( E_N = \mathbb{Q}(\zeta_N) \). Let \( \tau : E_N \hookrightarrow \mathbb{C} \) be an embedding such that \( \tau(\zeta_N) = e^{2\pi i/N} \). By abuse of notations we consider \( \tau(\zeta_N) = e^{2\pi i/N} = \zeta_N \).

Let \( G_N = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} \). We think of an element as \( g_N^{r,s} \) with the group action being given multiplicatively

\[ g_N^{r,s} \cdot g_N^{t,u} = g_N^{r+t,s+u}. \]

The action of \( G_N \) on \( F_{N,E_N} := F_N \otimes \text{Spec}(E_N) \) is defined by

\[ G_N \times F_{N,E_N} \rightarrow F_{N,E_N} \]

\[ (g_N^{r,s}, [x : y : z]) \rightarrow [\zeta_N^r x : \zeta_N^s y : z]. \]
4.1 Motives of Fermat curves

For \((a, b) \in G_N\) we consider the group homomorphism \(\theta_{a,b}^N\) defined by

\[
\theta_{a,b}^N : G_N \rightarrow E_N^* \\
g_N \rightarrow \zeta_{a+b}^N.
\]

Define

\[
P_{a,b}^N = \frac{1}{N^2} \sum_{g \in G_N} \theta_{a,b}^N(g^{-1})g \in E_N[G_N]
\]

The Galois group \(H_N = (\mathbb{Z}/N\mathbb{Z})^*\) acts on \(G_N\) by multiplication on each component so we extend linearly the action of \(H_N\) to \(E_N[G_N]\). Let \(O_{H_N}(G_N)\) denote the set of \(H_N\) orbits in \(E_N[G_N]\). For an orbit \([a, b]\) in \(O_{H_N}(G_N)\) we define

\[
P_{a,b}^N = \sum_{(c, d) \in [a, b]} P_{c,d}^N.
\]

**Lemma 4.1.** Let \([a, b] \in O_{H_N}(G_N)\) then we have

\[
P_{a,b}^N \circ P_{c,d}^N = \begin{cases} 
0 & \text{for } [a, b] \neq [c, d] \text{ (mod } N) \\
P_{a,b}^N & \text{for } [a, b] = [c, d] \text{ (mod } N),
\end{cases}
\]

\[
\sum_{[a, b] \in O_{H_N}(G_N)} P_{a,b}^N = 1.
\]

**Proof.** See §2, [Ots11].

Let \(\mathcal{M}_Q\) be the category of Chow motives defined over a number field \(Q\) and coefficients in \(Q\). We recall this part from [Ots11]. Objects consist of triples \(M = (X, P, m)\) where \(X \in \mathcal{V}_Q\), \(m \in \mathbb{Z}\) and \(P\) be a projector (an algebraic correspondence \(P \in \text{CH}^1(X \times X) \otimes Q\) which is an idempotent). Let \(C\) be a smooth projective curve of genus \(g > 0\). Then from Chow Kunneth decomposition one obtains

\[
h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C).
\]

where

\[
h^0(C) = (C, \{R\} \times C, 0) \cong \mathbb{I} \\
h^2(C) = (C, C \times \{R\}, 0) \cong \mathbb{L} \\
h^1(C) = (C, \Delta_C - \{R\} \times C - C \times \{R\}, 0).
\]

Chow Kunneth decomposition also holds for \(X = \text{Jac}(C)\). Then \(h^4(X) = \wedge^4 h^1(C)\). Such decomposition is not known in general.

A Fermat motive is an object of the form \(F_{a,b}^N = (F_N, P_{a,b}^N, 0) \in \mathcal{M}_Q\). Using lemma 4.1 one obtains

**Lemma 4.2.** Let \(I_N = \{(a, b) \in G_N : a, b, a + b \not\equiv 0 \text{ (mod } N)\}\) and \(O_{H_N}(I_N)\) be \(H_N\) orbit of the set \(I_N \subset G_N\). Then one has
(i) \( h^1(F_N) = \bigoplus_{[a,b] \in \mathcal{O}_{HN}(I_N)} F_N^{[a,b]} \) in \( \mathcal{M}_Q \)

(ii) \( F_N^{[a,b]} = 0 \) if one of \( a, b, a+b = 0 \) (mod \( N \))

(iii) \( F_N^{[0,0]} = \mathbb{I} + \mathbb{L} \)

where \( \mathbb{I}, \mathbb{L} \) are Tate motive and Lefchetz motive.

Proof. A proof is given in [Ots11] Lemma 2.11.

Let us consider
\[
F_N^{[a,b]} \wedge F_N^{[c,d]} := \left( F_N \times F_N, \left( P_N^{[a,b]} \otimes P_N^{[c,d]} - P_N^{[c,d]} \otimes P_N^{[a,b]} \right), 0 \right) \in \mathcal{M}_Q.
\]

4.2 Differentials on Fermat curves

For \( a, b \geq 1 \) define
\[
\omega_{N,\tau}^{a,b} = x^a y^b \frac{dx}{x} = -x^{-N} y^b \frac{dy}{y}.
\]

Then \( \omega_{N,\tau}^{a,b} \) is a differential form of second kind of the scheme \( F_{N,\tau} = F_N \times_\tau \mathbb{C} \) or it can be thought of as a differential form on the manifold \( F_{N,\tau}(\mathbb{C}) \). \( \omega_{N,\tau}^{a,b} \) may have poles at the points at infinity. For \( (a, b) \in I_N \) we consider \( \omega_{N,\tau}^{a,b} = \omega_{N,\tau}^{(a),(b)} \). The action of \( G_N \) on \( F_{N,E_N} \) induces an action on \( H^1_{dR}(F_{N,\tau}) \).

Since we fixed an embedding \( \tau : E_N \hookrightarrow \mathbb{C} \), we use \( \omega_{N,\tau}^{a,b} \) instead of \( \omega_{N}^{a,b} \).

Lemma 4.3. 1. \( \{ \omega_{N}^{a,b} : (a, b) \in I_N, a + b < N \} \) forms a basis of the vector space \( H^{1,0}(F_N) \).

2. \( \{ \omega_{N}^{a,b} : (a, b) \in I_N \} \) forms a basis for \( H^1(F_N(\mathbb{C}), \mathbb{C}) \).

3. Let \( a, b \in \mathbb{Z} \) and \( 1 \leq i \leq N \). Then for \( j \geq 0 \),
\[
\omega_{N}^{a+i,jN,b} = \frac{(\frac{a+i}{N})^j}{(\frac{a+i+b}{N})^j} \omega_{N}^{a+i,b}
\]
in \( H^1(F_N(\mathbb{C}), \mathbb{C}) \).

Proof. Proof of statements 1 of 2 is given in Chapter 3, Theorem 2.1 and Theorem 2.2 respectively in [Lan82] and proof of 3 is given in Lemma 4.19 in [Ots11].

4.3 Computation of the Poincaré dual

Let \( \delta_N \) be the path in \( F_N(\mathbb{C}) \) defined by
\[
\delta_N : [0, 1] \longrightarrow F_N(\mathbb{C})
\]
\[
t \rightarrow (\sqrt{t}, \sqrt{1-t})
\]
and the branches are taken in \( \mathbb{R}_{\geq 0} \). Let \( \gamma_N \) be the path
\[
\gamma_N = \frac{1}{N^2} \sum_{(r,s) \in G_N} (1 - g_{N}^{r,0})(1 - g_{N}^{0,s}) \delta_N.
\]
\( \gamma_N \) is a closed path and independent of choice of \( \zeta_N \). Hence it represents a homology class in \( H_1(F_N(C), Z) \). The pairing

\[
\langle \cdot, \cdot \rangle : H_1(F_N(C), Z) \times H^1(F_N(C), C) \rightarrow C
\]

\[
\langle \sigma, \omega_{a,b}^N \rangle = \int_{\sigma} \omega_{a,b}^N
\]

induces de Rham isomorphism \( \text{Hom}(H_1(F_N(C), Z), C) \simeq H^1(F_N(C), C) \). In particular for \((a, b) \in I_N\) we obtain

\[
\int_{\gamma_N} \omega_{a,b}^N = \frac{1}{N} \beta\left(\frac{a}{N}, \frac{b}{N}\right),
\]

where

\[
\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m + n)}
\]

and \( m, n > 0 \), \( \beta(m, n) \) is the Beta function. We choose the following normalisation

\[
\omega_{a,b}^N = \frac{\omega_{a,b}^N}{1 \frac{\beta\left(\frac{a}{N}, \frac{b}{N}\right)}{N}}.
\]

g\(r,s\) acts on \( F_N(C) \), so acts on \( H_1(F_N(C), C) \). Let us Consider

\[
P_{a,b}^c,d \gamma_N := \frac{1}{N^2} \sum_{(r,s) \in G_N} \zeta_{N}^{-\langle ar+bs \rangle} (g_{r,s}^N)^{r,s} \gamma_N \in E_N[G_N].
\]

Therefore

\[
\langle g_{r,s}^N \gamma_N, \omega_{a,b}^N \rangle = \langle \gamma_N, g_{r,s}^N \omega_{a,b}^N \rangle
\]

\[
= \zeta_{N}^{-\langle ar+bs \rangle} \int_{\gamma_N} \omega_{a,b}^N
\]

one has

\[
\int_{P_{c,d}^N \gamma_N} \omega_{a,b}^N = \int_{\gamma_N} (P_{c,d}^N)^* \omega_{a,b}^N = \frac{1}{N^2} \sum_{(r,s) \in G_N} \zeta_{N}^{-\langle a-c\rangle r+(b-d)s}.
\]

\[
= \begin{cases} 1 & \text{if } (c, d) = (a, b) \\ 0 & \text{otherwise.} \end{cases}
\]

**Poincaré duality** is defined by the following pairing

\[
\langle \cdot, \cdot \rangle_{PD} : H^1(F_N(C), C) \times H^1(F_N(C), C) \rightarrow C
\]

\[
\langle \omega_{c,d}^N, \omega_{a,b}^N \rangle_{PD} = \int_X \omega_{c,d}^N \wedge \omega_{a,b}^N.
\]

**Lemma 4.4.**

(i) \( H_1(F_N(C), Q) \) is a cyclic \( E_N[G_N] \) module generated by \( \gamma_N \).
(ii) For \((a, b) \in I_N, P_N^{−a,b}\gamma_N \in H^1(F_N, \mathbb{C})\) is the Poincaré dual of \(\omega_N^{a,b}\).

**Proof.** Proof (i) is given in [Ots11].

(ii) \(\langle \omega_N^{a,b}, \omega_N^{c,d}\rangle_{PD}\) is nonzero if and only if \(\omega_N^{a,b} \wedge \omega_N^{c,d}\) is locally a volume form which is true if and only if \((c, d) = (-a, -b)\). Therefore the Poincaré dual of \(\omega_N^{a,b}\) is \(\omega_N^{-a,-b}\). Thus \(P_N^{-a,-b}\gamma_N\) is the homology class that corresponds to Poincaré dual of \(\omega_N^{a,b}\). \(\square\)

The action of \(G_N \times G_N\) on \(F_{N,E_N} \times F_{N,E_N} \in \mathcal{Y}_{E_N}\) induces an action of \(E_N[G_N] \otimes E_N[G_N]\) on \(H^i_\bullet(F_N \times F_N)\), where \(\bullet \in \{\text{Betti, de Rham}\}\). Thus cohomology of the motive \(F_N^{[a,b]} \otimes F_N^{[c,d]}\) is defined to be the image of the map induced by the correspondences

\[ P_N^{[a,b]} \otimes P_N^{[c,d]} : H^\bullet(F_N \times F_N) \to H^\bullet(F_N \times F_N). \]

If \(H_\bullet\) is given by de Rham cohomology, one obtains

\[ H^2_{dR}(F_N^{[a,b]} \wedge F_N^{[c,d]}, \mathbb{C}) := (P_N^{[a,b]} \otimes P_N^{[c,d]} - P_N^{[c,d]} \otimes P_N^{[a,b]}) \otimes H^1_{dR}(F_N, \mathbb{C}) \]

\[ \cong \bigoplus_{h,k \in H_N} (\omega_N^{ah,bh} \otimes \omega_N^{ck,dk} - \omega_N^{ck,dk} \otimes \omega_N^{ah,bh}), \mathbb{C} \]

**Remark 4.5.** For \([a, b] \in \mathcal{O}_{H_N}(I_N), F_N^{[a,b]}\) is the motive corresponding to \(h^1(C_N^{a,b})\), where \(C_N^{a,b}\) is a Fermat quotient defined by normalisation of the curve \(y^N = x^a(1 - x)^b\).

### 4.4 Deligne cohomology of \(X = \text{Jac}(F_N)\)

For \(F_N \in \mathcal{Y}_Q\), we consider \(X = \text{Jac}(F_N)\). From §3.3 we have

\[ H^{2g-1}_{D}(\text{Jac}(F_N), \mathbb{Q}(g)) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))). \]

To obtain real regulator map one has to apply \(\mathbb{R}\)-MHS. From (5) one has

\[ H^{2g-1}_{D}(\text{Jac}(F_N)_{\mathbb{R}}, \mathbb{R}(g)) \cong \frac{(F^1H^2(X, \mathbb{C})^e \otimes \mathbb{C}_e = 1)^\ast}{H_2(X, \mathbb{Q}(2))}, \tag{9} \]

where \(F^1 \wedge^2 H^1(C)^\ast\) is the dual of \(\mathbb{C}\) vector space of \(F^1 \wedge^2 H^1(C)\). As \(c_\infty\) acts as multiplication by \(-1\) on \(\mathbb{R}(1)\) and \(\mathbb{C} = \mathbb{R} \oplus \mathbb{R}(1)\), one obtains

\[ (\wedge^2 H^1_{D}(F_N) \otimes \mathbb{R}(1))^e \otimes c_\infty = 1 = 2\pi i (\wedge^2 H^1_{D}(F_N, \mathbb{R}))^e = -1. \]

Let \(F_N \in \mathcal{Y}_{E_N}\). From lemma 4.4 we know that \(H^1(F_N, \mathbb{Z})\) is a cyclic \(E_N[G_N]\) module generated by \(\gamma_N\). Thus \(\int_\gamma \omega_N^{a,b} \in E_N\), for all \(\gamma \in H_1(F_N, \mathbb{Z})\). Then \(H^1(F_N(C), \mathbb{C}) = H^1(F_N(C), E_N) \otimes \mathbb{C}\) is spanned by \(\omega_N^{a,b}\). By abuse of notations, \(F_\infty = F_\infty \otimes 1\), \(c_\infty = 1 \otimes c_\infty\) acts on \(H^1(F_N(C), \mathbb{C}) \otimes_\mathbb{C}\). We have

\[ F_\infty(\omega_N^{a,b}) = \omega_N^{-a,-b} = c_\infty(\omega_N^{a,b}). \tag{10} \]

**Proposition 4.6.**

i. For \((a, b), (c, d) \in I_N\) and \(a + b, c + d < N\), let

\[ \Omega_N^{a,b,c,d} := 2\pi i(\tilde{\omega}_N^{a,b} \wedge \tilde{\omega}_N^{-c,d} - \tilde{\omega}_N^{-a,-b} \wedge \tilde{\omega}_N^{c,d}). \]

The set \(\{\Omega_N^{a,b,c,d} : a + b, c + d < N\}\) generate the real vector space \((F^1 \wedge^2 H^1(F_N))^e = -1\).
ii. For \([a, b], [c, d] \in \mathcal{O}_{H_N}(I_N)\). Then \((F^1H^2(F_N^{[a,b]} \wedge F_N^{[c,d]}))_{F_\infty=-1}\) is generated by the \(E_N\) basis \(\{\Omega_N^{h,b,h,c,d,k} : h \in H_{a,b}, k \in H_{c,d}\}\), where \(H_{a,b} := \{h \in (\mathbb{Z}/N\mathbb{Z})^* : \langle ha \rangle + \langle hb \rangle < N\}\).

Proof. Part (i) follows from Lemma 4.3 and (10). Let \([a, b], [c, d] \in I_N\). \(F_N^{[a,b]} \in \mathcal{M}_Q\) and \(F_N^{[a,b]} = \bigoplus_{h \in H_N} F_N^{a,b,h} \). From Lemma 4.3 one knows \(F^1(\wedge^2 H^1(F_N))\) is generated by the set \(\{\omega_N^{a,b} \wedge \omega_N^{a-c,-d} : (a, b), (c, d) \in I_N, a + b, c + d < N\}\). Since

\[
F^1H^2(F_N^{[a,b]} \wedge F_N^{[c,d]}, C)_{F_\infty=-1} = (P_N^{[a,b]} \otimes P_N^{[c,d]} - P_N^{[c,d]} \otimes P_N^{[a,b]}) \left(\otimes^2 H^1(F_N, C)\right)_{F_\infty=-1}.
\]

\(F^1H^2(F_N^{[a,b]} \wedge F_N^{[c,d]}, C)_{F_\infty=-1}\) is generated by \(\{\Omega_N^{a,b,h,c,d,k} : h \in H_{a,b}, k \in H_{c,d}\}\). Hence part ii follows.

5 Main Theorem

Here we evaluate the real regulator map. Let \(Q = (1 : 0 : 1), R = (0 : 1 : 1)\) be two points on \(F_N(\mathbb{C})\). Let \(P = (\frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 1)\). The function \(f_{QR} = \frac{x^2 - y}{x - y}\) satisfies the property that \(\text{div}(f_{QR}) = NQ - NR\) and \(f_{QR}(P) = 1\). From (2) one have \(\mathcal{Z}_{QR,P} \in H^2_{\mathcal{M}}(\text{Jac}(F_N), Q(g))\). Let \(\omega_N^{a,b} = x^a y^b N dx \wedge dy\) be a differential 1 form on \(F_N\). The set \(\{\omega_N^{a,b} : a, b \in \mathbb{Z}/N\mathbb{Z} \text{ and } a, b, a + b \neq 0\}\) is linearly independent and spans \(H^1(F_N, C)\). Recall that from (2) one has

\[
\mathcal{Z}_{QR,P} := (C_Q, f_Q) + (C^R, f^R).
\]

Let

\[
\Omega_N^{a,b,c,d} = 2\pi i \left(\tilde{\omega}_N^{a,b} \wedge \tilde{\omega}_N^{a-c,-d} - \tilde{\omega}_N^{a,-b} \wedge \tilde{\omega}_N^{c,d}\right) \in \left(F^1 \wedge^2 H^1(F_N, C)\right)_{F_\infty=-1},
\]

where \(a + b < N\) and \(c + d < N\). Lemma 4.3 asserts that \(\tilde{\omega}_N^{a,b}\) (a normalisation of \(\omega_N^{a,b}\)) is a holomorphic 1 form if and only if \(a + b < N\) and from Lemma 4.4 \(P_{N\gamma_N}^{a,-b}N\) is the Poincaré-dual of \(\tilde{\omega}_N^{a,b}\). As the regulator map is a linear functional on the subspace of \((1, 1)\) forms, one has

\[
\text{reg}_R(\mathcal{Z}_{QR})(\Omega_N^{a,b,c,d}) = \left(\int_{P_N^{c,d} \gamma_N} \log \left| \frac{1 - x}{1 - y} \right| \omega_N^{a,b} - \int_{P_N^{c,d} \gamma_N} \log \left| \frac{1 - x}{1 - y} \right| \tilde{\omega}_N^{a,b}\right) + \left(\int_{P_N^{a,b} \gamma_N} \log \left| \frac{1 - x}{1 - y} \right| \tilde{\omega}_N^{c,d} - \int_{P_N^{a,-b} \gamma_N} \log \left| \frac{1 - x}{1 - y} \right| \tilde{\omega}_N^{c,d}\right). \tag{11}
\]

Lemma 5.1. Let \(\omega_N^{a,b}\) be a holomorphic 1 form on \(F_N(\mathbb{C})\) and \(\gamma_N\) be the path in \(F_N(\mathbb{C})\) (See Section 4.3). For any element \(g_N^{r,s} \in G_N\), one has

\[
\int_{g_N^{r,s} \gamma_N} \log(1 - x)\omega_N^{a,b} = -\sum_{i=1}^{N} \left(\frac{S_N^{(a+i)r+bs}}{i/N} 3F_2\left(\begin{array}{c}
a + i, i \\
N, 1 \\
N + 1
\end{array} ; 1\right)\right) \int_{\delta_N^{a+i,b}} \omega_N^{a+i,b}
\]

\[
\int_{g_N^{r,s} \gamma_N} \log(1 - \bar{x})\omega_N^{a,b} = -\sum_{i=1}^{N} \left(\frac{S_N^{(a+i)r+bs}}{i/N} 3F_2\left(\begin{array}{c}
a + i, i \\
N, 1 \\
N + 1
\end{array} ; 1\right)\right) \int_{\delta_N^{a+i,b}} \omega_N^{a+i,b}
\]

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\[ \int_{g_N^r \gamma_N} \log(1 - y) \omega_N^{a,b} = - \sum_{i=1}^{N} \frac{\zeta_N^{ar+(b+i)s}}{i/N} 3F_2 \left( \frac{a+i}{N}, \frac{i}{N}, \frac{1}{N} ; 1 \right) \int_{\delta_N} \omega_N^{a,b+i} \]

\[ \int_{g_N^r \gamma_N} \log(1 - \bar{y}) \omega_N^{a,b} = - \sum_{i=1}^{N} \frac{\zeta_N^{ar+(b-i)s}}{i/N} 3F_2 \left( \frac{a+i}{N}, \frac{i}{N}, \frac{1}{N} ; 1 \right) \int_{\delta_N} \omega_N^{a,b+i}. \]

**Proof.** Recall from Section 4.3, \( \gamma_N \) is the path in \( F_N(\mathbb{C}) \)

\[ \gamma_N = \frac{1}{N^2} \sum_{(r,s) \in G_N} (1 - g_N^{r,0})(1 - g_N^{0,s}) \delta_N, \]

where

\[ \delta_N : [0, 1] \to F_N(\mathbb{C}) \]

\[ t \to (t^{1/N}, (1 - t)^{1/N}) . \]

Let \( \delta_N(\omega_N^{a,b}) \) and \( \delta_N f \) be the pullback of the differential form \( \omega_N^{a,b} \) and rational function \( f \) respectively. In particular for \( f = 1 - x \) and \( g_N^{r,s} \in G_N \), one has

\[ g_N^{r,s} \delta_N^r(1 - x) = 1 - \zeta_N^r t^{1/N} \]

\[ g_N^{r,s} \delta_N^r(1 - y) = (1 - \zeta_N^r(1 - t)^{1/N}) \]

\[ g_N^{r,s} \delta_N^r(\omega_N^{a,b}) = \frac{1}{N} \zeta_N^{ar+bs} t^{a/N-1}(1 - t)^{b/N-1} dt. \]

Since, \( |\zeta_N^r t^{1/N}| \leq 1 \) and \( |\zeta_N^r(1 - t)^{1/N}| \leq 1 \) for \( t \in [0, 1] \), we have

\[ \log g_N^{r,s} \delta_N^r(1 - x) = \log(1 - \zeta_N^r t^{1/N}) = - \sum_{i \geq 1} \frac{\zeta_N^{ri} t^{i/N}}{i} \]

\[ \log g_N^{r,s} \delta_N^r(1 - y) = \log(1 - \zeta_N^r(1 - t)^{1/N}) = - \sum_{i \geq 1} \frac{\zeta_N^{r}(1 - t)^{i/N}}{i}. \]

From Lemma 5.1 (3) \( \omega_N^{a+i+jN,b} = \frac{\zeta_N^{(a+i)(jN)}}{N^{i+jN}} \omega_N^{a+i,b} \). Therefore

\[ \int_{g_N^{r,s} \gamma_N} \log(1 - x) \omega_N^{a,b} = \frac{1}{N^2} \sum_{(l,m) \in G_N} \left( \int_{g_N^{l,s} \delta_N} \log(1 - x) \omega_N^{a,b} - \int_{g_N^{r+s,l} \delta_N} \log(1 - x) \omega_N^{a,b} \right) \]

\[ + \int_{g_N^{r+s,m}} \log(1 - x) \omega_N^{a,b} - \int_{g_N^{r+s,m} \delta_N} \log(1 - x) \omega_N^{a,b} \]

\[ = \int_0^1 \log(g_N^{r,s} \delta_N^r(1 - x)) g_N^{r,s} \delta_N^r(\omega_N^{a,b}) \]

\[ = - \int_0^1 \sum_{i \geq 1} \frac{\zeta_N^{(a+i)(jN)}}{iN} t^{a+i-1}(1 - t)^{b+1} dt \]
Therefore the first expression in the above lemma follows. Replacing $x$ by $y$ we get the third expression. Now consider

$$
\int_{g_N^* \gamma_N} \log (1 - \bar{x}) \omega_{N, b}^{a, b} = \int_0^1 \log(g_{N}^{r, s} \delta_N(1 - \bar{x}))g_{N}^{r, s} \delta_N(\omega_{N, b}^{a, b})
$$

$$
= - \frac{1}{N} \int_0^1 \sum_{i \geq 1} \left( a^{i+N} \right) \frac{1}{i} \int_{\delta_n} \left( \frac{a + i}{N}, \frac{i}{N}, 1 \right) \frac{1}{i + 1} \int_{\delta_n} \omega_{N, b}^{a, b}
$$

Using an argument similar to preceding case we obtain

$$
\int_{g_N^* \gamma_N} \log(1 - \bar{x}) \omega_{N, b}^{a, b} = \sum_{i \geq 1} \left( a^{i+N} \right) \frac{1}{i} \int_{\delta_n} \left( \frac{a + i}{N}, \frac{i}{N}, 1 \right) \frac{1}{i + 1} \int_{\delta_n} \omega_{N, b}^{a, b}
$$

Therefore expressions 2 and 4 follows.

**Theorem 5.2.** Let $X = \text{Jac}(F_N)$ where $F_N$ is the Fermat curve of degree $N$ defined over $\mathbb{Q}$. Let $Z_{Q, P} \in H^{2g-1}_M(X, \mathbb{Q}(g))$ be the element constructed in §3.1.1, (2). Then one has

$$
\text{reg}_R(Z_{Q, P})(\Omega_{N}^{a, b, c, d, e}) = \delta_{a, c} \left[ \tilde{F}_1 \left( \frac{b}{N}, \frac{a - b}{N}, \frac{a}{N} \right) - \tilde{F}_1 \left( \frac{b}{N}, \frac{a - b}{N}, \frac{a}{N} \right) \right] + \delta_{a, c} \left[ \tilde{F}_1 \left( \frac{a}{N}, \frac{c - a}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{a}{N}, \frac{c - a}{N}, \frac{b}{N} \right) \right] - \delta_{b, d} \left[ \tilde{F}_1 \left( \frac{c}{N}, \frac{a}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{c}{N}, \frac{a}{N}, \frac{b}{N} \right) \right] + \delta_{b, d} \left[ \tilde{F}_1 \left( \frac{c}{N}, \frac{a}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{c}{N}, \frac{a}{N}, \frac{b}{N} \right) \right] - \delta_{b, d} \left[ \tilde{F}_1 \left( \frac{c}{N}, \frac{a}{N}, \frac{b}{N} \right) - \tilde{F}_1 \left( \frac{c}{N}, \frac{a}{N}, \frac{b}{N} \right) \right]
$$
Here $\delta_{*,*}$ is the Kronecker delta function, $\Omega_{a,b,c,d}^N = \tilde{\omega}_N^{a,b} \land \tilde{\omega}_N^{c,d} - \tilde{\omega}_N^{a,-b} \land \omega_N^{c,d} \in F^1(\land^2 H^1(F_N, \mathbb{R}(1)^{F_N} = -1)$, $a + b, c + d < N$, $\langle a \rangle$ denote the integer which represent $a$ in the set $\{1, 2, \ldots, N\}$ and

$$\tilde{\mathcal{F}}_1 \left( \frac{\langle a \rangle}{N}, \frac{\langle i \rangle}{N}, \frac{\langle b \rangle}{N} \right) = \frac{1}{\langle i/N \rangle} \sum_{(r,s) \in G_N} \zeta_N^{\langle cr + ds \rangle} g_N^{r,s} \gamma_N.$$

is a special value of a hypergeometric function.

Proof. Recall that

$$P_N^{c,d,\gamma_N} = \frac{1}{N^2} \sum_{(r,s) \in G_N} \zeta_N^{\langle cr + ds \rangle} g_N^{r,s} \gamma_N.$$

From Lemma (5.1) we have

$$\int_{P_N^{c,d,\gamma_N}} \log(1 - x) \tilde{\omega}_N^{a,b} = -\frac{1}{N^2} \sum_{i=1}^{N} \frac{1}{i/N} \sum_{(r,s) \in G_N} \zeta_N^{\langle a+i \rangle} g_N^{r,s} \gamma_N$$

$$= -\frac{\delta_{b,d}}{\langle c-a \rangle} \left( \frac{a + \langle c-a \rangle}{N}, \frac{\langle c-a \rangle}{N}, 1 \right) \sum_{(r,s) \in G_N} \zeta_N^{(a+i-c) r + (b-d) s} \int_{\delta_N} \frac{\omega_N^{a+i,b}}{\beta(\frac{a}{N}, \frac{b}{N})}.$$

where $\delta_{a,*,*}$ is the Kronecker Delta function. Note that equation (13) is 0 unless $b \equiv d (\text{mod} \ N)$.

Similarly for the other expressions

$$\int_{P_N^{c,d,\gamma_N}} \log(1 - x) \tilde{\omega}_N^{a,b} = -\frac{\delta_{b,d}}{\langle a-c \rangle} \left( \frac{a + \langle a-c \rangle}{N}, \frac{\langle a-c \rangle}{N}, 1 \right) \sum_{(r,s) \in G_N} \zeta_N^{(a+i-c) r + (b-d) s} \int_{\delta_N} \frac{\omega_N^{a+(c-a)b}}{\beta(\frac{a}{N}, \frac{b}{N})}.$$ (14)

$$\int_{P_N^{c,d,\gamma_N}} \log(1 - y) \tilde{\omega}_N^{a,b} = -\frac{\delta_{a,c}}{\langle d-b \rangle} \left( \frac{b + \langle d-b \rangle}{N}, \frac{\langle d-b \rangle}{N}, 1 \right) \sum_{(r,s) \in G_N} \zeta_N^{a,b + (d-b) s} \int_{\delta_N} \frac{\omega_N^{a,b+(d-b)}}{\beta(\frac{a}{N}, \frac{b}{N})}.$$ (15)

$$\int_{P_N^{c,d,\gamma_N}} \log(1 - \bar{y}) \tilde{\omega}_N^{a,b} = -\frac{\delta_{a,c}}{\langle b-d \rangle} \left( \frac{b + \langle b-d \rangle}{N}, \frac{\langle b-d \rangle}{N}, 1 \right) \sum_{(r,s) \in G_N} \zeta_N^{a,b + (b-d) s} \int_{\delta_N} \frac{\omega_N^{a,b+(b-d)}}{\beta(\frac{a}{N}, \frac{b}{N})}.$$ (16)

The above expressions are true when $a + b < N$ and $c + d < N$. However we extend these calculations to the case $a + b > N$ and $c + d > N$. Let

$$\tilde{\mathcal{F}}_1 \left( \frac{\langle a \rangle}{N}, \frac{\langle i \rangle}{N}, \frac{\langle b \rangle}{N} \right) = \frac{1}{\langle i/N \rangle} \sum_{(r,s) \in G_N} \zeta_N^{\langle cr + ds \rangle} g_N^{r,s} \gamma_N$$

$$= \frac{1}{\langle i/N \rangle} \sum_{(r,s) \in G_N} \zeta_N^{\langle cr + ds \rangle} g_N^{r,s} \gamma_N,$$

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Thus expression (11) becomes

\[
\text{reg}_R(\mathcal{Z}_{QR,P})(\Omega^0_{N},c,d) = \delta_{a,c} \left[ F_1 \left( \frac{b}{N}, \frac{d-a}{N}, \frac{a}{N} \right) - F_1 \left( \frac{b-a}{N}, \frac{d-b}{N}, \frac{-a}{N} \right) \right] \\
+ \delta_{a,d} \left[ F_1 \left( \frac{d}{N}, \frac{c-a}{N}, \frac{b}{N} \right) - F_1 \left( \frac{c-a}{N}, \frac{d-b}{N}, \frac{-b}{N} \right) \right] \\
+ \delta_{b,d} \left[ F_1 \left( \frac{c}{N}, \frac{a-c}{N}, \frac{b}{N} \right) - F_1 \left( \frac{a-c}{N}, \frac{c-b}{N}, \frac{-b}{N} \right) \right].
\]

(17)

For $F_N \in \mathcal{F}_Q$, $H^{2g-1}_M(Jac(F_N), Q(g))$ is a $Q$ vector space. The action of $G^2_N$ on $F_{N,E_N} \times F_{N,E_N}$ induces an action $H^3_M(F_{N,E_N} \times F_{N,E_N}, Q(2))$. Let

\[
H^3_M(F_N^{[a,b]} \otimes F_N^{[c,d]}, Q(2)) := \mathcal{P}_N^{[a,b]} \otimes P_N^{[c,d]}(H^3_M(F_N \times F_N, Q(2)) \otimes E_N).
\]

Let $\eta_*$ is the map induced by $\eta$ defined in (3). It induces a functorial map between Deligne cohomology groups. One has the following commutative diagram

\[
\begin{array}{ccc}
H^3_M(F_N \times F_N, Q(2)) & \longrightarrow & \text{Ext}^1_{MHS}(R(-2), \otimes H^1(F_{N,R})) \\
\downarrow \eta_* & & \downarrow \\
H^3_M(Jac(F_N), Q(2)) & \longrightarrow & \text{Ext}^1_{MHS}(R(-2), \wedge^2 H^1(F_{N,R})),
\end{array}
\]

where the right vertical map is the functorial map from $\otimes^2 H^1(F_N) \rightarrow \wedge^2 H^1(F_N)$ induced by $\eta_*$ and horizontal maps are given by regulator. Thus projection by $P_N^{[a,b]} \otimes P_N^{[c,d]} \in E_N[G_N] \otimes E_N[G_N]$, we obtain

\[
\begin{array}{ccc}
H^3_M(F_N^{[a,b]} \otimes F_N^{[c,d]}, Q(2)) & \longrightarrow & \text{Ext}^1_{MHS}(R(-2), H^1(F_N^{[a,b]})) \wedge H^1(F_N^{[c,d]}) \\
\downarrow \eta_* & & \downarrow \\
H^3_M(F_N^{[a,b]} \otimes F_N^{[c,d]}, Q(2)) & \longrightarrow & \text{Ext}^1_{MHS}(R(-2), H^1(F_N^{[a,b]}) \wedge H^1(F_N^{[c,d]})),
\end{array}
\]

where $H^1(F_N^{[a,b]}) \wedge H^1(F_N^{[c,d]})$ is the image of the vector space $(H^1(F_N^{[a,b]}) \otimes H^1(F_N^{[c,d]}))^{F_\infty \otimes e_\infty = 1}$ in the vector space $(\wedge^2 H^1(F_N))^{F_\infty \otimes e_\infty = 1}$ and $H^3_M(F_N^{[a,b]} \otimes F_N^{[c,d]}, Q(2))$ is defined to be the image of the following homomorphism

\[
\eta_* \circ \left( P_N^{[a,b]} \otimes P_N^{[c,d]} - P_N^{[a,b]} \otimes P_N^{[a,b]} \right): H^3_M(F_N \times F_N, Q(2)) \otimes E_N \rightarrow H^{2g-1}_M(Jac(F_N), Q(g)) \otimes E_N
\]
Theorem 5.3. Let \([a, b], [c, d] \in \mathcal{O}_{H_N}(I_N)\) and
\[
Z_{QR,P}^{a,b,c,d} := \eta_* \left( P_N^{[a,b]} \otimes P_N^{[c,d]} - P_N^{[a,b]} \otimes P_N^{[c,d]} \right) Z_{QR,P} \in H^3_M(F_N^{[a,b]} \wedge F_N^{[c,d]}, \mathbb{Q}(2)).
\]
Let
\[
\Omega_N^{ah,bh,ck,dk} = 2\pi i \left( \tilde{\omega}_N^{ah,bh} \wedge \tilde{\omega}_N^{-ck,-dk} - \tilde{\omega}_N^{-ah,-bh} \wedge \tilde{\omega}_N^{ck,dk} \right) \in F^1 H^2_{dR}(F_N^{[a,b]} \wedge F_N^{[c,d]}) \otimes \mathbb{C} = 1, \]
where \(h \in H_{a,b}, k \in H_{c,d}\). Then the regulator map is defined by
\[
\text{reg}_{\mathbb{R}} : H^3_M(F_N^{[a,b]} \wedge F_N^{[c,d]}, \mathbb{Q}(2)) \longrightarrow H^3_D(F_N^{[a,b]} \wedge F_N^{[c,d]}, \mathbb{R}(2)).
\]
where
\[
\text{reg}_{\mathbb{R}}(Z_{QR,P}^{a,b,c,d})(\Omega_N^{ah,bh,ck,dk}) = \delta_{ah,ck} \left[ \tilde{F}_1 \left( \frac{\langle bh \rangle}{N}, \frac{\langle dk - bh \rangle}{N}, \frac{\langle ah \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -bh \rangle}{N}, \frac{\langle bh - dk \rangle}{N}, \frac{\langle -ah \rangle}{N} \right) \right]
+ \tilde{F}_1 \left( \frac{\langle bh \rangle}{N}, \frac{\langle bk - dk \rangle}{N}, \frac{\langle ah \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -dk \rangle}{N}, \frac{\langle dk - bh \rangle}{N}, \frac{\langle -ah \rangle}{N} \right),
+ \delta_{ah,ck} \left[ \tilde{F}_1 \left( \frac{\langle bk \rangle}{N}, \frac{\langle bk - dk \rangle}{N}, \frac{\langle ah \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -dk \rangle}{N}, \frac{\langle dk - bh \rangle}{N}, \frac{\langle -ah \rangle}{N} \right) \right].
- \delta_{bh,dk} \left[ \tilde{F}_1 \left( \frac{\langle ah \rangle}{N}, \frac{\langle ah - bk \rangle}{N}, \frac{\langle bh \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -ah \rangle}{N}, \frac{\langle ah - ck \rangle}{N}, \frac{\langle -bh \rangle}{N} \right) \right],
+ \tilde{F}_1 \left( \frac{\langle ah \rangle}{N}, \frac{\langle ah - bk \rangle}{N}, \frac{\langle bh \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -ah \rangle}{N}, \frac{\langle ck - ah \rangle}{N}, \frac{\langle -bh \rangle}{N} \right) \right],
- \delta_{bh,dk} \left[ \tilde{F}_1 \left( \frac{\langle bk \rangle}{N}, \frac{\langle bk - ck \rangle}{N}, \frac{\langle bh \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -ck \rangle}{N}, \frac{\langle ck - ah \rangle}{N}, \frac{\langle -bh \rangle}{N} \right) \right],
+ \tilde{F}_1 \left( \frac{\langle bk \rangle}{N}, \frac{\langle bk - ck \rangle}{N}, \frac{\langle bh \rangle}{N} \right) - \tilde{F}_1 \left( \frac{\langle -ck \rangle}{N}, \frac{\langle ck - ah \rangle}{N}, \frac{\langle -bh \rangle}{N} \right) \right]. \tag{18}
\]
Proof. From 4.6(ii.), \(F^1 \wedge^2 H^2_{dR}(F_N)\) is generated by \(\Omega_N^{ah,bh,ck,dk}\) such that \(ah + bh, ck + dk < N\). Hence from Theorem 5.2, the proof follows.

5.1 Indecomposable cycles and application to Fermat quotients

Recall from 3.3 that a cycle \(Z \in H^2_{M}(X, \mathbb{Q}(g))\) is indecomposable if its image under the regulator map does not lie in the image of decomposable cycles \(H^2_{M}(X, \mathbb{Q}(g))_{\text{dec}}\). Thus to check \(Z\) indecomposable, it is suffices to show that regulator of \(Z\) does not lies in the image of decomposable cycles. One of the aim of this section is to show that the cycle \(Z_{QR,P}\) we constructed in (2) is indecomposable in \(H^2_{M}(\text{Jac}(F_N), \mathbb{Q}(g))\). The image of \(\text{CH}^1(X)\) lies in the vector space generated by Hodge classes of \(\wedge^2 H^1(F_N)\). Hence to show that the element \(Z_{QR,P}\) is indecomposable it suffices to show that the regulator evaluated at a certain \((1,1)\) form which does not lie in the image of Hodge class, is nonzero. Here we consider a closed 2 form \(\Omega\) which is orthogonal to \(NS(X)_{\mathbb{Q}}\) and we shall show \(\text{reg}_{\mathbb{R}}(Z_{QR,P})(\Omega) \neq 0\). In [KR78], one has an explicit description of Hodge classes in \(\otimes^2 H^1(C)\) and in particular in \(\wedge^2 H^1(C)\). It follows from Shioda ([Shi79]), and Remark 1.5 of [Aok91] that \(F_N^{[a,b]}\) is isogenous to \(F_N^{[c,d]}\) if and only if \(\Omega_N^{ah,bh,ck,dk}\) is generated by Hodge cycles for \(h \in H_{a,b}, k \in H_{c,d}\).
Lemma 5.4. [KR78] For $a + b < N$ and $c + d < N$ and $N$ is prime to 6, $\Omega_{N}^{a,b,c,d} := \tilde{\omega}_{N}^{a,b} \wedge \tilde{\omega}_{N}^{-c,-d}$ is corresponding to a Hodge cycle if and only if the triples \( \{a, b, N - (a + b)\} = \{c, d, N - (c + d)\} \) upt to a permutation.

Proof. Theorem 0.3 [Aok91] and see [KR78].

Theorem 5.5. For $\text{Jac}(F_{N}) \in \mathcal{Y}_{Q}$ and $N = 7$, the cycle $Z_{QR,P}$ is regulator-indecomposable, in particular indecomposable.

Proof. From Theorem (5.2) $\text{reg}_{R}(Z_{QR,P})(\Omega_{N}^{a,b,c,d}) = 0$ unless $\Omega_{N}^{a,b,c,d}$ is $\mathbb{R}$ linear combination of elements of the forms

$$
\tilde{\omega}_{N}^{a,b} \wedge \tilde{\omega}_{N}^{N-a,N-b-i} - \tilde{\omega}_{N}^{N-a,N-b} \wedge \tilde{\omega}_{N}^{a,b+i}, \quad \tilde{\omega}_{N}^{a,b} \wedge \tilde{\omega}_{N}^{N-a-i,N-b} - \tilde{\omega}_{N}^{N-a,N-b} \wedge \tilde{\omega}_{N}^{a+i,b}.
$$

Therefore, either $[c, d] = [a + i, b]$ or $[a, b + i]$. We assume $[c, d] = [1, j]$, $[a, b] = [1, i]$ and $i \neq j \neq 0$. From Lemma (5.4) $\Omega_{N}^{a,b,c,d}$ is a Hodge cycle if and only if either $N - i = j + 1$ or $N - j = i + 1$ modulo $N$ if and only if $i + j + 1 \neq 0 (\text{mod } N)$. At $N = 7$, $\Omega_{N}^{1,2,1,3}$ is not supported on Hodge classes. We obtain $\text{reg}_{R}(Z_{QR,P})(\Omega_{N}^{1,2,1,3}) = -11.257$. Hence $Z_{QR,P}$ is regulator indecomposable.

Remark 5.6. Let us choose $j = 2i$ where $0 < i \leq N/4$. Then theorem (5.2) shows

$$
\text{reg}(Z_{QR,P})(\Omega_{N}^{1,i,1,2i}) = \left( \tilde{F}_{1}(\frac{i}{N}, \frac{i}{N}) - \tilde{F}_{1}(-\frac{i}{N}, -\frac{i}{N}) \right)
+ \left( \tilde{F}_{1}(\frac{i}{N}, -\frac{i}{N}) \frac{1}{N} - \tilde{F}_{1}(-\frac{i}{N}, \frac{i}{N}) \frac{1}{N} \right)
+ \left( \tilde{F}_{1}(\frac{2i}{N}, -\frac{i}{N}) \frac{1}{N} - \tilde{F}_{1}(-\frac{2i}{N}, \frac{i}{N}) \frac{1}{N} \right)
+ \left( \tilde{F}_{1}(\frac{2i}{N}, \frac{i}{N}) \frac{1}{N} - \tilde{F}_{1}(-\frac{2i}{N}, \frac{i}{N}) \frac{1}{N} \right).
$$

In the following table, values are obtained by the software Mathematica. For integers $i$ and $N$, without the hypothesis $i \leq N/4$, we define

$$
f(i, N) := \text{reg}_{R}(Z_{QR,P})(\Omega_{N}^{1,i,1,2i}).
$$

The nonvanishing of $f(i, N)$ conclude that the cycle $Z_{QR,P}$ is regulator indecomposable. Using theorem 5.3, one obtains the formula of the regulator for Chow motive $F_{N}^{[1,i]} \wedge F_{N}^{[1,2i]}$. Thus following tables will be useful to determine indecomposability of the higher Chow cycle $Z_{QR,P}^{1,i,1,2i} \in H_{M}^{3}(F_{N}^{[1,i]} \wedge F_{N}^{[1,2i]}, \mathbb{Q}(2))$.

|   | N | f(i,N)   |
|---|---|---------|
| 1 | 11| 28.1775 |
| 2 | 13| 33.3368 |
| 2 | 17| 43.7499 |
| i | N  | f(i,N)     |
|---|-----|-----------|
| 3 | 17  | 45.7937   |
| 4 | 17  | 47.2211   |
| 5 | 17  | 48.3664   |
| 9 | 17  | 52.9933   |
| 11 | 17  | 69.8063   |
| 12 | 17  | 44.0733   |
| 13 | 17  | 58.8503   |

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