2 × 2 block representations of the Moore–Penrose inverse and orthogonal projection matrices

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In this paper, new block representations of Moore–Penrose inverses for arbitrary complex 2 × 2 block matrices are given. The approach is based on block representations of orthogonal projection matrices.

Keywords Moore–Penrose inverse, generalized inverses of matrices, block representations, orthogonal projection matrices

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1 Introduction

The aim of this paper is the following: Given an arbitrary complex \((p + q) \times (s + t)\) block matrix

\[
E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

(1.1)

with \(p \times s\) block \(a\), we give new block representations \(E^\dagger = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\) with \(s \times p\) block \(\alpha\) of the Moore–Penrose inverse \(E^\dagger\) of \(E\) as well as new block representations \(P_{R(E)} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}\) with \(p \times p\) block \(e_{11}\) of the orthogonal projection matrix \(P_{R(E)}\) onto the column space \(R(E)\) of \(E\). The block entries should be given by expressions involving the blocks \(a, b, c,\) and \(d\) of \(E\), where generalized inverses are build only of matrices of the block sizes, i.e., with number of rows and columns from the set \(\{p, q, s, t\}\). We will see that this goal can be realized (without additional assumptions) by computation of \(\{1\}\)-inverses of four (non-negative Hermitian) matrices of the block sizes.

To the best of our knowledge, except the above mentioned authors Hung/Markham [12] only Miao [13], Groß [9], and Yan [27] describe explicit block representations of the Moore–Penrose inverse of arbitrary complex 2 × 2 partitioned matrices without making additional assumptions. In Miao [13], a certain weighted Moore–Penrose inverse is used in the formulas for the block entries of \(E^\dagger\). In [9], Groß gives a block representation of the Moore–Penrose inverse of non-negative Hermitian 2 × 2 block matrices. Thus, the well-known formulas \(E^\dagger = E^*(EE^*)^\dagger\) and \(E^\dagger = (E^*E)^\dagger E^*\) can then easily be used to derive a block representation for the Moore–Penrose inverse \(E^\dagger\) of an arbitrary block matrix \(E\). In Yan [27], a full rank factorization of \(E\)
derived from full rank factorizations of the block entries $a, b, c, d$ is utilized to obtain a block representation of $E^\dagger$.

Assuming certain additional conditions, e.g., on column spaces or ranks, several authors derived block representations of the Moore–Penrose inverse of matrices, see, e.g. [4, 10, 14]. The existence of a Banachiewicz–Schur form for $E^\dagger$ is studied e.g. in [2, 21]. Furthermore, special classes of matrices were considered in this context, see, e.g. [18, 19] for so-called block $k$-circulant matrices. Block representations of $E^\dagger$ involving regular transformations, e.g., permutations, have also been considered, see e.g. [11]. A representation of the Moore–Penrose inverse of a block column or block row in terms of the block entries is given in [1]. Under a certain rank additivity condition, a block representation of $m \times n$ partitioned matrices can be found in [20] as well. Several results on block representations of partitioned operators are obtained, e.g., in [6, 7, 24–26]. The here mentioned list of references on this topic is not exhaustive.

2 Notation

Throughout this paper let $m, n, p, q, s, t$ be positive integers. We denote by $\mathbb{C}^{m \times n}$ the set of all complex $m \times n$ matrices and by $\mathbb{C}^n := \mathbb{C}^{n \times 1}$ the set of all column vectors with $n$ complex entries. We write $0_{n \times m}$ for the zero matrix in $\mathbb{C}^{m \times n}$ and $I_n$ for the identity matrix in $\mathbb{C}^{n \times n}$. Let $\mathcal{U}$ and $\mathcal{V}$ be linear subspaces of $\mathbb{C}^n$. If $\mathcal{U} \cap \mathcal{V} = \{0_{n \times 1}\}$, then we write $\mathcal{U} \oplus \mathcal{V}$ for the direct sum of $\mathcal{U}$ and $\mathcal{V}$. Let $\mathcal{U}^\perp$ be the orthogonal complement of $\mathcal{U}$. If $\mathcal{U} \subseteq \mathcal{V}$, we use the notation $\mathcal{V} \ominus \mathcal{U} := \mathcal{V} \cap (\mathcal{U}^\perp)$. We write $\mathcal{R}(M)$ and $\mathcal{N}(M)$ for the column space and the null space of a complex matrix $M$. Let $M^*$ be the conjugate transpose of a complex matrix $M$. If $M$ is an arbitrary complex $m \times n$ matrix, then there exists a unique complex $n \times m$ matrix $X$ such that the four equations

\begin{align}
(1) & \quad MXM = M , & (2) & \quad XMX = X , & (3) & \quad (MX)^* = MX , & (4) & \quad (XM)^* = XM \quad (2.1)
\end{align}

are fulfilled (see [16]). This matrix $X$ is called the Moore–Penrose inverse of $M$ and is designated usually by the notation $M^\dagger$. Following [3] Ch. 1, Sec. 1, Def. 1, for each $M \in \mathbb{C}^{m \times n}$ we denote by $M \{j, k, \ldots, \ell\}$ the set of all $X \in \mathbb{C}^{n \times m}$ which satisfy equations $(j), (k), \ldots, (\ell)$ from the equations (1)–(4) in (2.1). Each matrix belonging to $M \{j, k, \ldots, \ell\}$ is said to be a $\{j, k, \ldots, \ell\}$-inverse of $M$.

Remark 2.1. Let $\mathcal{U}$ be a linear subspace of $\mathbb{C}^n$. Then there exists a unique complex $n \times n$ matrix $P_\mathcal{U}$ such that $P_\mathcal{U}x \in \mathcal{U}$ and $x - P_\mathcal{U}x \in \mathcal{U}^\perp$ for all $x \in \mathbb{C}^n$. This matrix $P_\mathcal{U}$ is called the orthogonal projection matrix onto $\mathcal{U}$. If $P \in \mathbb{C}^{n \times n}$, then $P = P_\mathcal{U}$ if and only if the three conditions $P^2 = P$ and $P^* = P$ as well as $\mathcal{R}(P) = \mathcal{U}$ are fulfilled. Furthermore, the equation $P_{\mathcal{U^\perp}} = I_n - P_{\mathcal{U}}$ holds true.

Our strategy to give a block representation of the Moore–Penrose inverse $E^\dagger$ of the block matrix $E$ given in [11] consists of three elementary steps:

**Step (I)** We consider the following factorization problem for orthogonal projection matrices: Find a complex $(s + t) \times (p + q)$ matrix $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$ fulfilling $P_{\mathcal{R}(E)} = E R$ with block entries $r_{11} \in \mathbb{C}^{s \times p}$, $r_{12} \in \mathbb{C}^{s \times q}$, $r_{21} \in \mathbb{C}^{t \times p}$, and $r_{22} \in \mathbb{C}^{t \times q}$ expressible explicitly only using the block entries $a, b, c, d$ of $E$.

Remark 2.2. Let $M \in \mathbb{C}^{m \times n}$ and let $X \in \mathbb{C}^{n \times m}$. In view of Remark [2.1] then:
We consider a complex \( (p + q) \times (s + t) \) matrix \( E \). Let (1.1) be the block representation of \( E \) with \( p \times s \) block \( a \). Setting

\[
Y \coloneqq [a, b], \\
Z \coloneqq [c, d], \\
S \coloneqq \begin{bmatrix} a \\ c \end{bmatrix}, \\
T \coloneqq \begin{bmatrix} b \\ d \end{bmatrix}
\]

Our investigations involve an analogous decomposition, namely \( E = \mathbb{P}_\mathcal{S} E + (I_{p+q} - \mathbb{P}_\mathcal{S}) E \) with the orthogonal projection matrix \( \mathbb{P}_\mathcal{S} \) onto the linear subspace \( \mathcal{S} \) spanned by the first \( s \) columns of \( E \) occurring on the left-hand side (see Lemma 3.2).
then
\[
E = \begin{bmatrix} Y \\ Z \end{bmatrix}, \quad E = [S, T].
\] (3.2)

Let
\[
\begin{align*}
\mu &:= a a^* + b b^*, \\
\zeta &:= c c^* + d d^*, \\
\rho &:= c a^* + d b^*,
\end{align*}
\] (3.3)
\[
\begin{align*}
\sigma &:= a^* a + c^* c, \\
\tau &:= b^* b + d^* d, \\
\lambda &:= a^* b + c^* d.
\end{align*}
\] (3.4)

In view of (3.1), then
\[
\begin{align*}
\mu &= YY^*, \\
\zeta &= ZZ^*, \\
\rho &= ZY^*,
\end{align*}
\] (3.5)
\[
\begin{align*}
\sigma &= S^* S, \\
\tau &= T^* T, \\
\lambda &= S^* T.
\end{align*}
\] (3.6)

Choose \(\mu^{(1)}\) \(\in \mu\{1\}\) and \(\sigma^{(1)} \in \sigma\{1\}\). Regarding (3.5), then Theorem 2.3 shows that
\[
Y^{(1,2,4)} := Y^* \mu^{(1)}, \quad S^{(1,2,3)} := \sigma^{(1)} S^*
\] (3.8)

fulfill
\[
Y^{(1,2,4)} \in Y\{1, 2, 4\}, \quad S^{(1,2,3)} \in S\{1, 2, 3\}.
\] (3.9)

Let
\[
\begin{align*}
\phi &:= c - (c a^* + d b^*) \mu^{(1)} a, \\
\psi &:= d - (c a^* + d b^*) \mu^{(1)} b, \\
\eta &:= b - a \sigma^{(1)} (a^* b + c^* d), \\
\theta &:= d - c \sigma^{(1)} (a^* b + c^* d).
\end{align*}
\] (3.10)
\[
\begin{align*}
\eta &= \theta = [\eta, \theta]
\end{align*}
\] (3.12)

admit the representations
\[
Z(I_{s+t} - Y^{(1,2,4)} Y) = Z - Z Y^* \mu^{(1)} Y = Z - \rho \mu^{(1)} Y
\] (3.13)
\[
= [c, d] - \rho \mu^{(1)} [a, b] = [\phi, \psi] = V
\]

and
\[
(I_{p+q} - SS^{(1,2,3)} T) = T - S \sigma^{(1)} S^* T = T - S \sigma^{(1)} \lambda
\] (3.14)
\[
= \begin{bmatrix} b \\ d \end{bmatrix} - \begin{bmatrix} a \\ c \end{bmatrix} \sigma^{(1)} \lambda = \begin{bmatrix} \eta \\ \theta \end{bmatrix} = W.
\]

Using (3.13), (3.14), (3.9), Remarks 2.2 and 2.1, and \(\mathcal{R}(Y^*)\)^⊥ = \(\mathcal{N}(Y)\), we can infer
\[
V = Z P_{\mathcal{N}(Y)}, \quad W = P_{\mathcal{R}(S)^\perp} T.
\] (3.15)
Let 
\[ \nu := \phi^* + \psi^* \], \quad \omega := \eta^* \eta + \theta^* \theta. \] (3.16)

In view of (3.12), (3.15), and Remark 2.1 then
\[ \nu = V V^* = Z^* \mathcal{N}(Y) Z^* = Z^* Z, \quad \omega = W^* W = T^* \mathbb{P}_{[\mathcal{R}(S)]} T = T^* W. \] (3.17)

Choose \( \nu^{(1)} \in \nu \{1 \} \) and \( \omega^{(1)} \in \omega \{1 \} \). Regarding (3.20), (3.18), (3.8), (3.7), (3.1), and (3.12), then Theorem 2.3 shows that
\[ V^{(1,2,4)} := V^* \nu^{(1)} \quad \text{and} \quad W^{(1,2,3)} := \omega^{(1)} W^* \] (3.18)
fulfill
\[ V^{(1,2,4)} \in V \{1, 2, 4 \} \quad \text{and} \quad W^{(1,2,3)} \in W \{1, 2, 3 \}. \] (3.19)

Obviously, we have \( \mu \in \mathbb{C}_{\geq}^{p \times p}, \sigma \in \mathbb{C}_{\geq}^{s \times s}, \zeta \in \mathbb{C}_{\geq}^{q \times q}, \tau \in \mathbb{C}_{\geq}^{t \times t}, \nu \in \mathbb{C}_{\geq}^{q \times q}, \omega \in \mathbb{C}_{\geq}^{t \times t}, \rho \in \mathbb{C}_{\geq}^{q \times p}, \) and \( \lambda \in \mathbb{C}_{\geq}^{t \times t} \), where \( \mathbb{C}_{\geq}^{n \times n} \) denotes the set of all non-negative Hermitian complex \( n \times n \) matrices.

**Remark 3.1.** Let
\[ L := [(I_{s+t} - V^{(1,2,4)} Z)Y^{(1,2,4)}, V^{(1,2,4)}], \quad R := [S^{(1,2,3)}(I_{p+q} - TW^{(1,2,3)})/W^{(1,2,3)}]. \] (3.20)

Regarding (3.20), (3.18), (3.8), (3.7), (3.1), and (3.12), then
\[ L = [(I_{s+t} - V^* \nu^{(1)} Z)Y^* \mu^{(1)}, V^* \nu^{(1)}] = \left[ (Y^* - V^* \nu^{(1)} Z Y^*) \mu^{(1)}, V^* \nu^{(1)} \right] = Y^* \mu^{(1)} I_p + V^* \nu^{(1)} [-\rho \mu^{(1)}, I_q] = \left[ a^* \mu^{(1)} I_p, 0_{p \times q} \right] + \left[ \phi^* \right] \nu^{(1)} [-\rho \mu^{(1)}, I_q] = \left[ \ell_{11}, \ell_{12} \right], \] where
\[ \ell_{12} := \phi^* \nu^{(1)}, \quad \ell_{11} := (a^* - \ell_{12} \rho) \mu^{(1)}, \quad \ell_{22} := \psi^* \nu^{(1)}, \quad \ell_{21} := (b^* - \ell_{22} \rho) \mu^{(1)}, \] (3.21)

and
\[ R = \left[ \begin{array}{c} \sigma^{(1)} S^*(I_{p+q} - T \omega^{(1)} W^*) \\ \omega^{(1)} W^* \end{array} \right] = \left[ \begin{array}{c} \sigma^{(1)} S^* - \sigma^{(1)} T \omega^{(1)} W^* \\ \omega^{(1)} W^* \end{array} \right] = \left[ \begin{array}{c} \sigma^{(1)} S^* - \sigma^{(1)} \lambda \omega^{(1)} W^* \\ \omega^{(1)} W^* \end{array} \right] \]
\[ = \left[ \begin{array}{c} I_s \\ 0_{t \times s} \end{array} \right] \sigma^{(1)} S^* + \left[ \begin{array}{c} -\sigma^{(1)} \lambda \\ I_t \end{array} \right] \omega^{(1)} W^* \]
\[ = \left[ \begin{array}{c} I_s \\ 0_{t \times s} \end{array} \right] \sigma^{(1)} [a^*, c^*] + \left[ \begin{array}{c} -\sigma^{(1)} \lambda \\ I_t \end{array} \right] \omega^{(1)} [\eta^*, \theta^*] = \left[ \begin{array}{c} r_{11} \\ r_{21} \end{array} \right] \left[ \begin{array}{c} r_{12} \\ r_{22} \end{array} \right], \]
where
\[ r_{21} := \omega^{(1)} \eta^*, \quad r_{11} := \sigma^{(1)} (a^* - \lambda r_{21}), \quad r_{22} := \omega^{(1)} \theta^*, \quad r_{12} := \sigma^{(1)} (c^* - \lambda r_{22}). \] (3.22)
Lemma 3.2. Let $\mathcal{E} := \mathcal{R}(E)$, let $\mathcal{I} := \mathcal{R}(S)$, and let $W := \mathcal{R}(W)$. Then $\mathcal{W} = \mathcal{E} \cap \mathcal{I}$ and $P_{\mathcal{W}} = P_{\mathcal{E}} + P_{\mathcal{I}} = ER$.

Proof. We first check $\mathcal{W} = \mathcal{E} \cap (\mathcal{I}^\perp)$. Because of (3.19), (3.19), and Remark 2.2, we have $P_{\mathcal{W}} = WW^{(1,2,3)}$ and $P_{\mathcal{I}} = SS^{(1,2,3)}$. According to Remark 2.1, then $I_{p+q} - SS^{(1,2,3)} = P_{\mathcal{I}}$. By virtue of (3.14), then $W = P_{\mathcal{I}}T$ follows. From Remark 2.1 we know $R(P_{\mathcal{I}}) = \mathcal{I}^\perp$. Consequently, $\mathcal{W} \subseteq \mathcal{E}$ and $P_{\mathcal{W}} = W(T - SS^{(1,2,3)}T) = [S,T] \begin{bmatrix} -S^{(1,2,3)}T & I \\ I_t \end{bmatrix} = E \begin{bmatrix} -S^{(1,2,3)}T & I \\ I_t \end{bmatrix}$, implying $\mathcal{W} \subseteq \mathcal{W}$. Thus, $\mathcal{W} \subseteq \mathcal{E} \cap (\mathcal{I}^\perp)$ is proved. Now we consider an arbitrary $w \in \mathcal{W} \cap (\mathcal{I}^\perp)$. Then $w \in \mathcal{E}$; so there exists some $v \in \mathbb{C}^{s+t}$ with $w = Ev$. Let $v = \begin{bmatrix} v_s \\ v_t \end{bmatrix}$ be the block representation of $v$ with $x \in \mathbb{C}^s$. Regarding (3.12), we have $Sx + Ty = Wy$. In view of $w \in \mathcal{I}^\perp$ and $Sx \in \mathcal{I}$, moreover $P_{\mathcal{I}}w = w$ and $P_{\mathcal{I}}Sx = 0_{p+q}$. Taking additionally into account $W = P_{\mathcal{I}}T$, we obtain then $w = P_{\mathcal{I}}w = P_{\mathcal{I}}(Sx + Ty) = P_{\mathcal{I}}Ty = Wy$, implying $w \in \mathcal{W}$. Thus, we have also shown $\mathcal{E} \cap (\mathcal{I}^\perp) \subseteq \mathcal{W}$ holds true. Since $\mathcal{I} \subseteq \mathcal{E}$, hence $\mathcal{W} = \mathcal{E} \cap \mathcal{I}$ and $P_{\mathcal{W}} = SS^{(1,2,3)} + WW^{(1,2,3)} = S^{(1,2,3)} + (I_{p+q} - SS^{(1,2,3)})TW^{(1,2,3)}$ $= S^{(1,2,3)}(I_{p+q} - TW^{(1,2,3)}) + TW^{(1,2,3)} = [S,T] \begin{bmatrix} S^{(1,2,3)}(I_{p+q} - TW^{(1,2,3)}) & I \\ I_t \end{bmatrix} = ER$. □

The following result can be proved analogously. We omit the details.

Lemma 3.3. Let $\mathcal{E} := \mathcal{R}(E^*)$, let $\mathcal{I} := \mathcal{R}(Y^*)$, and let $\mathcal{W} := \mathcal{R}(W^*)$. Then $\mathcal{W} = \mathcal{E} \cap \mathcal{I}$ and $P_{\mathcal{W}} = P_{\mathcal{E}} + P_{\mathcal{I}} = LE$.

Remark 3.4. From Lemma 3.2 and Remark 2.2, we can infer $R \in E \{1,3\}$, whereas Lemma 3.3 and Remark 2.2 yield $L \in E \{1,4\}$.

Now we obtain the announced block representations of orthogonal projection matrices.

Proposition 3.5. Let $E$ be a complex $(p+q) \times (s+t)$ matrix and let $\begin{bmatrix} 1 & 1 \end{bmatrix}$ be the block representation of $E$ with $p \times s$ block a. Let $S$ be given by (3.1). Let $\alpha$ and $\lambda$ be given by (3.3) and (3.4). Let $\sigma(1) \in \sigma(1)$. Let $\eta, \theta$ and $W$ be given by (3.11) and (3.12). Let $\omega$ be given by (3.16) and let $\omega(1) \in \omega(1)$. Then $P_{R(E)} = S\sigma(1)S^* + W\omega(1)W^*$.

Proof. In the proof of Lemma 3.2, we have already shown $P_{R(E)} = SS^{(1,2,3)} + WW^{(1,2,3)}$. Taking additionally into account (3.8), (3.18), (3.21), and (3.12), the assertions follow. □

Now we are able to prove a $2 \times 2$ block representation of the Moore–Penrose inverse.
Theorem 3.6. Let $E$ be a complex $(p + q) \times (s + t)$ matrix and let (3.1) be the block representation of $E$ with $p \times s$ block $a$. Let $Y, S$ be given by (3.1). Let $\mu, \sigma$ and $\rho, \lambda$ be given by (3.3) and (3.4). Let $\mu^{(1)} \in \mu \{1\}$ and let $\sigma^{(1)} \in \sigma \{1\}$. Let $\phi, \psi$ and $\eta, \theta$ be given by (3.10) and (3.11). Let $V$ and $W$ be given by (3.12). Let $\nu$ and $\omega$ be given by (3.10). Let $\nu^{(1)} \in \nu \{1\}$ and let $\omega^{(1)} \in \omega \{1\}$. Then

$$E^\dagger = LER = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = (Y^* - V^* \nu^{(1)} \rho) \mu^{(1)} a \sigma^{(1)} (S^* - \lambda \omega^{(1)} W^*) + (Y^* - V^* \nu^{(1)} \rho) \mu^{(1)} b \omega^{(1)} W^*$$

$$+ V^* \nu^{(1)} c \sigma^{(1)} (S^* - \lambda \omega^{(1)} W^*) + V^* \nu^{(1)} d \omega^{(1)} W^*$$

with

$$\alpha := \ell_{11} a_r + \ell_{12} b_r + \ell_{11} c_r + \ell_{12} d_r, \quad \beta := \ell_{11} a_r + \ell_{12} b_r + \ell_{11} c_r + \ell_{12} d_r$$

$$\gamma := \ell_{21} a_r + \ell_{22} b_r + \ell_{21} c_r + \ell_{22} d_r, \quad \delta := \ell_{21} a_r + \ell_{22} b_r + \ell_{21} c_r + \ell_{22} d_r$$

where, for each $j, k \in \{1, 2\}$, the matrices $\ell_{jk}$ and $r_{jk}$ are given by (3.21) and (3.22), resp.

Proof. According to Lemmas 3.3 and 3.2, we have $P_{R(E^*)} = LE$ and $P_{R(E)} = ER$. Thus, we can apply Remark 3.5 to obtain $E^\dagger = LER$. Using Remark 3.1 and (1.1), we furthermore obtain

$$LER = \begin{bmatrix} (Y^* - V^* \nu^{(1)} \rho) \mu^{(1)} (V^* S^{(1)} - \lambda \omega^{(1)} W^*) + (Y^* - V^* \nu^{(1)} \rho) \mu^{(1)} b \omega^{(1)} W^* \\ V^* \nu^{(1)} c \sigma^{(1)} (S^* - \lambda \omega^{(1)} W^*) + V^* \nu^{(1)} d \omega^{(1)} W^* \end{bmatrix}$$

as well as

$$LER = \begin{bmatrix} \ell_{11} & \ell_{12} & a & b & r_{11} & r_{12} \\ \ell_{21} & \ell_{22} & c & d & r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

\[\square\]

4 Examples for consequences

In this section, we give some examples of applications of the block representations of orthogonal projection matrices and Moore–Penrose inverses given in Proposition 3.5 and Theorem 3.6. In order to avoid lengthy formulas, we give only hints for computations.

Example 4.1. Let $N$ be a positive integer and let $\mathcal{U}$ and $\mathcal{V}$ be two complementary linear subspaces of $\mathbb{C}^N$, i.e., the subspaces $\mathcal{U}$ and $\mathcal{V}$ fulfill $\mathcal{U} \oplus \mathcal{V} = \mathbb{C}^N$. Then there exists a unique complex $N \times N$ matrix $P_{\mathcal{U}, \mathcal{V}}$ such that $P_{\mathcal{U}, \mathcal{V}} x = u$ for all $x \in \mathbb{C}^N$, where $x = u + v$ is the unique representation of $x$ with $u \in \mathcal{U}$ and $v \in \mathcal{V}$. This matrix $P_{\mathcal{U}, \mathcal{V}}$ is called the oblique projection matrix on $\mathcal{U}$ along $\mathcal{V}$ and admits the representations

$$P_{\mathcal{U}, \mathcal{V}} = (P_{\mathcal{V} \perp \mathcal{U}}) \dagger = [(I_N - P_{\mathcal{V}}) P_{\mathcal{U}}] \dagger,$$

see [8] or also [3] Ch. 2, Sec. 7, Ex. 60, formula (80)]. Assume that $N = p + q$ and that $\mathcal{U} = R(E)$ and $\mathcal{V} = R(F)$ for two matrices $E \in \mathbb{C}^{(p+q) \times (s+t)}$ and $F \in \mathbb{C}^{(p+q) \times (m+n)}$ with block representations (1.1) and $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a$ is a $p \times s$ block and $e$ is a $p \times m$ block. Then (1.1) together with Proposition 3.5 and Theorem 3.6 could be used to obtain a block representation of $P_{\mathcal{U}, \mathcal{V}}$ in terms of $a, b, c, d$ and $e, f, g, h$. 


Example 4.2. Because \( \mathcal{U} \oplus (\mathcal{V}^\perp) = \mathbb{C}^n \) holds true for every linear subspace \( \mathcal{U} \) of \( \mathbb{C}^n \), the Moore–Penrose inverse of matrices is a special case of the uniquely determined \( 1,2 \)-inverse with simultaneously prescribed column space and null space. More precisely, if \( M \in \mathbb{C}^{m \times n} \) and linear subspaces \( \mathcal{U} \) of \( \mathbb{C}^m \) and \( \mathcal{V} \) of \( \mathbb{C}^n \) with \( R(M) \oplus \mathcal{U} = \mathbb{C}^m \) and \( N(M) \oplus \mathcal{V} = \mathbb{C}^n \) are given, then there exists a unique complex \( n \times m \) matrix \( X \) such that the four conditions

\[
MXM = M, \quad XMX = X, \quad R(X) = \mathcal{V}, \quad \text{and} \quad N(X) = \mathcal{U}
\]

are fulfilled. This matrix \( X \) is denoted by \( M_{\mathcal{U},\mathcal{V}}^{(1,2)} \) and admits the representations

\[
M_{\mathcal{U},\mathcal{V}}^{(1,2)} = P_{\mathcal{V},N(M)}M^{(1)}P_{R(M),\mathcal{U}} = (P_{\mathcal{V}}P_{\mathcal{V},N(M)})^\dagger M^{(1)}(P_{\mathcal{R}(M)}P_{R(M),\mathcal{V}})^\dagger \tag{4.2}
\]

with every \( M^{(1)} \in M\{1\} \) (see, e.g. [8, Ch. 2, Sec. 6, Thm. 12]). In particular, (4.2) is valid for \( M^{(1)} = M^\dagger \). Furthermore, \( M^\dagger = M_{[\mathcal{U},\mathcal{V}],\mathcal{W}}^{(1,2)} \) of \( M_{[\mathcal{U},\mathcal{V}],\mathcal{W}}^{(1,2)} \), if the subspaces \( \mathcal{U} \) and \( \mathcal{V} \) are given as column spaces of certain matrices, partitioned accordingly to a block representation of \( M \), and if a matrix \( M^{(1)} \in M\{1\} \) is known. (In particular, for \( M^{(1)} = M^\dagger \) one can use Theorem 3.6)

5 An alternative approach

In this final section, we give alternative representations of the matrices \( L \) and \( R \) occurring in Theorem 3.6 and Lemmas 3.3 and 3.2. Utilizing these representations, further block representations of the Moore–Penrose inverse \( E^\dagger \) could possibly be obtained, in particular, in the case of \( E \) satisfying additional conditions. We will not pursue this direction any further here. We continue to use the notations given above.

Lemma 5.1 (Rohde [17], see, e.g. [8, Ch. 5, Sec. 2, Ex. 10(a)]). Let \( M \in \mathbb{C}^{(p+q)\times(p+q)} \) and let \( M = [m_{11} \  m_{12}] \)

be the block representation of \( M \) with \( p \times p \) block \( m_{11} \). Let \( m_{11}^{(1)} \in m_{11}\{1\} \) and let \( \varsigma := m_{22} - m_{21}m_{11}^{(1)}m_{12} \). Let \( \varsigma^{(1)} \in \varsigma \{1\} \). Then

\[
M^{(1)} := \begin{bmatrix}
m_{11}^{(1)} + m_{11}^{(1)}m_{12}\varsigma^{(1)}m_{21}m_{11}^{(1)} & -m_{41}^{(1)}m_{12}\varsigma^{(1)} \\
-m_{41}^{(1)}m_{21}m_{11}^{(1)} & \varsigma^{(1)}
\end{bmatrix}
\]

belongs to \( M^{(1)} \in M\{1\} \).

Remark 5.2. Regarding (3.17), (3.13), (3.14), (3.3), (3.6), and, (3.7), we can infer

\[
\nu = VZ^* = Z(I_{s+t} - Y^{(1,2,4)}Y)^* = Z(I_{s+t} - Y^*\mu^{(1)}Y)Z^* = \varsigma - \rho\mu^{(1)}\rho^* \tag{5.1}
\]

and

\[
\omega = T^*W = T^*(I_{p+q} - SS^{(1,2,3)})T = T^*(I_{p+q} - S\sigma^{(1)}S^*)T = \tau - \lambda^*\sigma^{(1)}\lambda. \tag{5.2}
\]

Lemma 5.3. Let

\[
G := EE^* \quad \text{and} \quad H := E^*E. \tag{5.3}
\]
Let $\mu^{(1)} \in \mu\{1\}$ and $\nu^{(1)} \in \nu\{1\}$ and let $\sigma^{(1)} \in \sigma\{1\}$ and $\omega^{(1)} \in \omega\{1\}$. Then the matrices

\[
G^{(1)} := \begin{bmatrix}
\mu^{(1)} + \mu^{(1)} \rho^{*} \nu^{(1)} \rho^{(1)} \\
-\nu^{(1)} \rho^{(1)} \\
-\nu^{(1)} \\
\end{bmatrix}
\quad (5.4)
\]

and

\[
H^{(1)} := \begin{bmatrix}
\sigma^{(1)} + \sigma^{(1)} \lambda \omega^{(1)} \lambda^{*} \sigma^{(1)} \\
-\omega^{(1)} \lambda^{*} \sigma^{(1)} \\
-\omega^{(1)} \\
\end{bmatrix}
\quad (5.5)
\]

fulfill $G^{(1)} \in G\{1\}$ and $H^{(1)} \in H\{1\}$.

**Proof.** Clearly, $G \in \mathbb{C}^{(p+q) \times (p+q)}$ and $H \in \mathbb{C}^{(s+t) \times (s+t)}$. Regarding (3.2), (3.3), and (3.4), we have $G = [Y^* Y^*] = \begin{bmatrix} \mu^* \sigma^* \end{bmatrix}$ and $H = [Z^* Z^*] = \begin{bmatrix} \sigma^* \lambda^* \end{bmatrix}$. Taking additionally into account Remark 5.2, thus from Lemma 5.1, the assertions immediately follow. \qed

Finally, in the following result, we not only list new representations for the matrices $L$ and $R$ occurring in Theorem 3.6 and Lemmas 3.3 and 3.2 but also obtain their belonging to the set $E\{1, 2, 4\}$ and $E\{1, 2, 3\}$, resp., thereby improving Remark 3.4.

**Lemma 5.4.** The matrices $L$ and $R$ admit the representations $L = E^* G^{(1)}$ and $R = H^{(1)} E^*$ and fulfill $L \in E\{1, 2, 4\}$ and $R \in E\{1, 2, 3\}$.

**Proof.** Using (3.9) and Remarks 2.2 and 2.1 we have $(Y^{(1,2,4)} Y^*)^* = F^{r_2} R(Y^*) = F^{r_2} R(Y^*) = Y^{(1,2,4)} Y^*$ and, analogously, $(S S^{(1,2,3)})^* = S S^{(1,2,3)}$. Taking additionally into account (3.13), (3.14), (3.8), and (3.7), we thus can conclude

\[
V^* = (I_{s+t} - Y^{(1,2,4)} Y^*) Z^* = Z^* - Y^* \mu^{(1)} Y Z^* = Z^* - Y^* \mu^{(1)} \rho^*
\quad (5.6)
\]

and

\[
W^* = T^* (I_{p+q} - S S^{(1,2,3)}) T^* - T^* S \sigma^{(1)} S^* = T^* - \lambda^* \sigma^{(1)} S^*.
\quad (5.7)
\]

Regarding (3.2), (5.4), (5.5), (5.11), (5.2), (3.6), (3.4), and Remark 3.1, we obtain

\[
E^* G^{(1)} = [Y^*, Z^*] \begin{bmatrix}
I_p \\
0_{q \times p}
\end{bmatrix} \mu^{(1)} [I_p, 0_{p \times q}] + \begin{bmatrix}
-\mu^{(1)} \rho^* \\
I_q
\end{bmatrix} \nu^{(1)} [-\rho^{(1)}, I_q] = Y^* \mu^{(1)} [I_p, 0_{p \times q}] + (Z^* - Y^* \mu^{(1)} \rho^*) \nu^{(1)} [-\rho^{(1)}, I_q] = L
\quad (5.8)
\]

and

\[
H^{(1)} E^* = \begin{bmatrix}
I_t \\
0_{t \times s}
\end{bmatrix} \begin{bmatrix}
-\sigma^{(1)} \lambda \\
I_t
\end{bmatrix} \omega^{(1)} [-\lambda^* \sigma^{(1)}, I_t] \begin{bmatrix}
S^* \\
T^*
\end{bmatrix} = [I_t] \sigma^{(1)} S^* + \begin{bmatrix}
-\sigma^{(1)} \lambda \\
I_t
\end{bmatrix} \omega^{(1)} (T^* - \lambda^* \sigma^{(1)} S^*) = R.
\quad (5.9)
\]

According to Lemma 5.3, we have $G^{(1)} \in G\{1\}$ and $H^{(1)} \in H\{1\}$. Taking additionally into account (5.3), (5.8), and (5.9), then $L \in E\{1, 2, 4\}$ and $R \in E\{1, 2, 3\}$ follow from Theorem 2.3. \qed

Observe that Lemma 5.4 in connection with Theorem 3.6 yields a factorization $E^! = \text{LER}$ with particular matrices $L \in E\{1, 2, 4\}$ and $R \in E\{1, 2, 3\}$. This gives a special factorization of the kind mentioned in Urquhart’s result (Theorem 2.4), whereby all matrices can be expressed explicitly in terms of the block entries $a, b, c, d$ of the given matrix $E$. 

9
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