The Trace Map and Integrability of the Multifunctions

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Abstract. In this paper we construct the quotient map (with respect to the special equivalence relation) for the restriction of the trace map $F(x, y) = (xy, (x - 2)^2)$ on the invariant subset of the plane that consists of unbounded curves. There is the unique curve in every point of which (with the exception of the unique point) the quotient map is two-valued and upper semicontinuous (but not lower semicontinuous). In other points this map is single-valued and continuous. We introduce also the concept of integrability for multifunctions of the above type, formulate and prove the (necessary) conditions for integrability. These conditions are based on the reduction of the above multifunctions to the multivalued skew products.

1. Introduction
In 90-s years of XX-th century the papers were published where investigation of discrete Schrödinger equation was reduced to investigation of special trace maps (see, e.g., [1, 2, 3, 4]). For example, as it is shown in [2], study of passing and reflection coefficients of the plane wave with a given impulse in the field of the crystal lattice with knots formed Thue-Morse chain, can be reduced to study of the trace map

$$F(x, y) = (xy, (x - 2)^2). \quad (1)$$

Geometric properties and the unbounded subset of the set of wandering points of the quadratic trace map given by formula (1), were described in [5]. Different dynamical properties of maps of the one-parameter family $F_\mu(x, y) = (xy, (x - \mu)^2) \ (\mu \in [0, 2])$ were investigated in [6, 7, 8]. Point out the papers [9, 10] devoted to symbolic aspects of dynamics of the trace map for the weakly coupled Fibonacci hamiltonian.

This paper is the direct continuation of [5]. We need the main result of the work [5] here. For this goal we consider the following subsets of the closure $\overline{K}_1$ of first quadrant $K_1$:

$$\Delta_2 = \{(x; y) : x; y \geq 0; x + y \leq 4\};$$

$$G_{\Delta_2} = \{(x; y) \in \overline{K}_1 : x + y \geq 4\};$$

$$G_{\Delta_2}^2 = \{(x; y) \in \overline{K}_1 : x + y > 4\};$$

$$D_{\Delta, \infty} = \{(x; y) : x \geq 3, \ y \geq 1\};$$
\[ \tilde{G} = G_{\Delta_2}^{\infty} \cap \bigcup_{i=0}^{\infty} F^{-i}(D_{2,\infty}), \]

where \( F^{-i}(\cdot) \) is \( i \)-th complete preimage of a set;

\[ G' = G_{\Delta_2} \setminus \tilde{G}. \]

**Theorem.** The set \( \tilde{G} \) is open and everywhere dense in \( G_{\Delta_2} \) and consists of \( F \)-wandering points.

The set \( G' \) is the union of continuous unbounded curves (homeomorphic to an unbounded interval); the cardinality of the set of these curves equals continuum. Moreover, these curves satisfy the following conditions:

- every two curves either do not intersect each other in \( G_{\Delta_2} \), or intersect each other in a unique point on the hypothenuse of the triangle \( \Delta_2 \); in addition, second property holds iff these two curves form a boundary of a connected component of \( \tilde{G} \);
- every tube neighborhood of a curve from \( G' \) contains a set of curves from \( G' \) of continuum cardinality;
- each curve from \( G' \) intersects a closed interval \( x + y = C, 0 \leq x, y \leq C \), where \( C > 4 \), in a unique point.

The set \( G' \) is \( F \)-invariant and nowhere dense in \( G_{\Delta_2} \) [5].

In this paper we construct the quotient map for the restriction of the trace map \( F \) on \( G' \) with respect to the special equivalence relation. There is the unique curve of the set \( G' \) that consists of points (with the exception of the unique point on the hypothenuse of the triangle \( \Delta_2 \)) where the quotient map is two-valued and upper semicontinuous (but not lower semicontinuous), in other points this map is single-valued and continuous. We introduce also the concept of integrability for multifunctions of the above type, formulate and prove the (necessary) conditions for integrability.

2. Construction of the Quotient Map

2.1. Preliminaries

To construct the quotient map for the trace map \( F \) (with respect to the special equivalence relation) we need detailed information about the connected components of the set \( \tilde{G} \) and their boundaries. Let \( A_2 = (3, 1) \) be \( F \)-fixed point (\( A_2 \) is a source), and \( A_{k,2} \) be a homoclinic point of \( F \) on the hypothenuse of the triangle \( \Delta_2 \) of an order \( k \geq 1 \) \((i = 1, 2, \ldots, 2^k - 1)\) to the fixed point \( A_2 \). (Remind that by the order of a homoclinic point \((x, y)\) of the trace map \( F \) on the hypothenuse of the triangle \( \Delta_2 \) to a fixed point \((x^*, y^*)\) we mean the minimal natural number \( k \) such that \( F^k(x, y) = (x^*, y^*) \).)

In [5] the following unbounded closed domains were correctly defined:

(i) \( H_{0,\text{max}}(A_2) \) satisfying \( D_{2,\infty} \subset H_{0,\text{max}}(A_2) \) and \( \partial H_{0,\text{max}}(A_2) = \eta^{A_2} \cup \eta^{A_2}_r \), where \( \partial(\cdot) \) is the boundary of a set; \( \eta^{A_2} \) is the graph of the unbounded \( C^1 \)-smooth strictly decreasing function \( y = \eta^{A_2}(x) \), which is defined on the interval \([2, 3]\) and has the vertical asymptote \( x = 2 \); \( \eta^{A_2}_r \) is the graph of \( C^1 \)-smooth strictly decreasing function \( y = \eta^{A_2}_r(x) \), which is defined on the unbounded interval \([3, +\infty)\) and has the horizontal asymptote \( y = 0 \);

(ii) \( H_{k,\text{max}}(A_{k,2}) \), where \( H_{k,\text{max}}(A_{k,2}) \) is a connected component of \( k \)-th complete preimage of \( H_{0,\text{max}} \) with respect to \( F_{|G_{\Delta_2}} \) containing a homoclinic point \( A_{k,2} \). Moreover, \( \partial H_{k,\text{max}}(A_{k,2}) = \eta^{A_{k,2}}_l \cup \eta^{A_{k,2}}_r \), where \( \eta^{A_{k,2}}_l, \eta^{A_{k,2}}_r \) are \( C^1 \)-smooth curves (with the unique common point \( A_{k,2} \)) which have asymptotes such that every of them belongs to the union of the set of connected components of \( m \)-th \((0 \leq m \leq k) \) complete preimage (with respect to the map \( F_{|G_{\Delta_2}} \)) of the critical lines \( x = 2 \), \( x = 0 \); and also the axis \( y = 0 \). (Note that there are curves of the set
\[ \{ \eta^{A_i}_l, \eta^{A_i}_r \} \text{ for } k \geq 1, 1 \leq i \leq 2^k - 1, \text{ which are not graphs of a function of the variable } x \text{ or a function of the variable } y. \]

The unbounded closed domain \( H_0^{max}(A_2) \) is called the principal maximal generative domain, and the unbounded closed domains \( H_k^{max}(A_i^k) \) \( (k \geq 1, 1 \leq i \leq 2^k - 1) \) are called the maximal generative domains of k-th order.

Consider, for example, the maximal generative domain \( H_1^{max}(A_i^1) \). Let \( \eta^{A_i}_l, \eta^{A_i}_r \) be the space (endowed with Hausdorff metric \( \text{dist} \)) of compact rectifiable arcs of the set \( G_{\Delta_2} \), which start from all points of the hypotenuse \( x + y = 4, x \in [0, 4] \), of the triangle \( \Delta_2 \); in addition, these arcs are parametrized with the natural parameter \( s \in [0, s_*] \).

Remark 1. The trace map \( F \) is \( C^1 \)-diffeomorphism on the set of interior points \( (int() \) of every maximal generative domain \( H_0^{max}(A_2), H_k^{max}(A_i^k) \) \( (k \geq 1, 1 \leq i \leq 2^k - 1) \), and \( F \) is homeomorphism on every maximal generative domain. Hence, the following equalities are valid (see [5]):

\[
\begin{align*}
F(int(H_0^{max}(A_2))) &= int(H_0^{max}(A_2)), \quad F(\eta^{A_2}_l) = \eta^{A_2}_r, \quad F(\eta^{A_2}_r) = \eta^{A_2}_l; \\
F(int(H_1^{max}(A_i^1))) &= int(H_0^{max}(A_2)), \quad F(\partial(H_1^{max}(A_i^1))) = \partial(H_0^{max}(A_2)); \\
F(int(H_k^{max}(A_i^k))) &= int(H_{k-1}^{max}(A_i^{k-1})), \quad F(\partial(H_k^{max}(A_i^k))) = \partial(H_{k-1}^{max}(A_i^{k-1})), \quad k \geq 2.
\end{align*}
\]

The map \( F \) preserves the order of points on the curves \( \eta^{A_2}_l, \eta^{A_2}_r; \eta^{A_i}_l, \eta^{A_i}_r \) \( (k \geq 1, 1 \leq i \leq 2^k - 1) \).

Remark 2. Since curves \( \eta^{A_2}_l, \eta^{A_2}_r; \eta^{A_i}_l, \eta^{A_i}_r \) are \( C^1 \)-smooth then, bounded arcs of these curves are rectifiable [5].

By previous Remark 2 all curves \( \eta^{A_2}_l, \eta^{A_2}_r; \eta^{A_i}_l, \eta^{A_i}_r \) admit parametrization with use of the natural parameter \( s \in [0, +\infty) \). Take a number \( s_* \in (0, +\infty) \). Let \( 2^{k_*} \) be the space (endowed with Hausdorff metric \( \text{dist} \)) of compact rectifiable arcs of the set \( G_{\Delta_2} \), which start from all points of the hypotenuse \( x + y = 4, x \in [0, 4] \), of the triangle \( \Delta_2 \); in addition, these arcs are parametrized with the natural parameter \( s \in [0, s_*] \).

Remark that Hausdorff metric \( \text{dist} \) (see [11, ch. 2, § 21, VII]) for arcs \( \gamma|[0, s_*] \), \( \gamma"|[0, s_*] \in 2^{k_*} \) is defined by the equality

\[
\text{dist}(\gamma|[0, s_*]), \gamma"|[0, s_*]) = \max \{ \sup_{(x, y) \in \gamma|[0, s_*]} \rho((x, y), \gamma"|[0, s_*]), \sup_{(x, y) \in \gamma|[0, s_*]} \rho(\gamma"|[0, s_*], (x, y)) \}.
\]

Here \( \rho \) is a metric in the plane.

Proposition 1. The metric space \( (2^{k_*}, \text{dist}) \) is complete for any \( s_* \in (0, +\infty] \) [5].

Proposition 2. The set of compact arcs

\[
\eta^{A_2, s_*}_l = \eta^{A_2}_l |[0, s_*], \eta^{A_2, s_*}_r = \eta^{A_2}_r |[0, s_*], \eta^{A_i, s_*}_l = \eta^{A_i}_l |[0, s_*], \eta^{A_i, s_*}_r = \eta^{A_i}_r |[0, s_*],
\]

where \( k \geq 1, 1 \leq i \leq 2^k - 1 \), is dense in itself with respect to Hausdorff metric \( \text{dist} \) for any \( s_* \in (0, +\infty) \); moreover, every set \( \{ \eta^{A_2, s_*}_l, \eta^{A_2, s_*}_r \} \) and \( \{ \eta^{A_i, s_*}_l, \eta^{A_i, s_*}_r \} \) is dense in itself too [5].

Proposition 3. Let \( \{(x_n, y_n)\}_{n \geq 1} \) be a convergent sequence of points in the plane which belong to compact arcs of the set \( \{ \eta^{A_2, s_*}_l, \eta^{A_2, s_*}_r, \eta^{A_2, s_*}_r, \eta^{A_i, s_*}_r \} \) \( (k \geq 1, 1 \leq i \leq 2^k - 1) \). Then corresponding sequence of arcs (containing the above points) converges in Hausdorff metric \( \text{dist} \) [5].
Proposition 4. Between any two different curves of the set \( G' \) which do not belong to the boundary of the same maximal generative domain there exist a maximal generative domain and a curve of \( G' \) which does not belong to the boundary of a maximal generative domain \([5]\).

Proposition 5. Functions \( y = \eta_i^{A_2}(x) \) and \( y = \eta_r^{A_2}(x) \) are convex down. The straight line \( x + y = 4 \) is the tangent to the graphs \( \eta_i^{A_2} \) and \( \eta_r^{A_2} \) of these functions respectively at the point \( A_2 \); moreover, \( x + y = 4 \) is the tangent to curves \( \eta_i^{A_k}, \eta_r^{A_k} \) at the point \( A_k^i \) (\( k \geq 1, 1 \leq i < 2^k - 1 \)) \([5]\).

Proposition 6. The following equality is valid for the open set \( \tilde{G} \):

\[
\tilde{G} = \text{int}(H_0^{\max}(A_2)) \cup \left( \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{2^k-1} \text{int}(H_k^{\max}(A_k^i)) \right) \ [5].
\]

2.2. The Equivalence Relation and the Quotient Map

We use the mathematical induction principle to define the equivalence relation on the maximal generative domains.

1. Begin to define the equivalence relation from the principal maximal generative domain \( H_0^{\max}(A_2) \). Let \( M_l(A_2) \) and \( M_r(A_2) \) be arbitrary points of the curves \( \eta_i^{A_2} \) and \( \eta_r^{A_2} \), respectively, where \( M_l(A_2) \neq A_2 \), \( M_r(A_2) \neq A_2 \).

We identify points \( M_l(A_2) \) and \( M_r(A_2) \) iff

\[
s([A_2, M_l(A_2)]) = s([A_2, M_r(A_2)]), \tag{2}
\]

where \( s([\cdot]) \) is the length of an arc.

Let points \( M_l(A_2) \) and \( M_r(A_2) \) be identified by the equality (2). As it follows from Proposition 5, the closed interval \([M_l(A_2), M_r(A_2)]\) belongs to the domain \( H_0^{\max} \). Let a point \( M_l'(A_2) \in \eta_i^{A_2} \) be so that \( s([A_2, M_l'(A_2)]) > s([A_2, M_l(A_2)]) \), and a point \( M_r'(A_2) \in \eta_r^{A_2} \) be identified with \( M_l'(A_2) \). Then, the inequality \( s([A_2, M_r'(A_2)]) > s([A_2, M_r(A_2)]) \) holds. It means that two closed intervals \([M_l(A_2), M_r(A_2)]\) and \([M_l'(A_2), M_r'(A_2)]\), where \( M_l(A_2) \in \eta_i^{A_2} \) and \( M_r(A_2) \in \eta_r^{A_2} \) are identified, either do not intersect each other (in \( H_0^{\max} \)) or coincide. Therefore, it is correct to identify all points of the closed interval \([M_l(A_2), M_r(A_2)]\) with the point \( M_l(A_2) \).

2. Define the equivalence relation on the maximal generative domain \( H_1^{\max}(A_1^i) \) of first order. By item (ii) of the Subsection 2.1 the domain \( H_1^{\max}(A_1^i) \) is the connected component of first complete preimage of the principal maximal generative domain \( H_0^{\max}(A_2) \) with respect to \( F|_{G_2} \), moreover, \( H_1^{\max}(A_1^i) \) contains the homoclinic point \( A_1^1 \). By Remark 1 the map \( F|_{H_1^{\max}(A_1^i)} \) is homeomorphism of \( H_1^{\max}(A_1^i) \) on \( H_0^{\max}(A_2) \), moreover, \( F|_{\text{int}(H_1^{\max}(A_1^i))} \) is diffeomorphism of \( \text{int}(H_1^{\max}(A_1^i)) \) on \( \text{int}(H_0^{\max}(A_2)) \). Hence, there exists a regular arc \( \gamma[M_l(A_1^i), M_r(A_1^i)] \subset H_1^{\max}(A_1^i) \) satisfying

\[
F(\gamma[M_l(A_1^i), M_r(A_1^i)]) = [M_l(A_2), M_r(A_2)]. \tag{3}
\]

Identify all points of the arc \( \gamma[M_l(A_1^1, M_r(A_1^i)] \) with the point \( M_l(A_1^i) \in \eta_i^{A_1^i} \).

3. Suppose that we have defined the equivalence relation on maximal generative domains \( H_m^{\max}(A^i_m) \) of \( m \)-th order, where \( 1 \leq m \leq k, k \geq 1, 1 \leq i \leq 2^m - 1 \). It means that we have identified all points of a regular arc \( \gamma[M_l(A^i_m), M_r(A^i_m)] \) with the point \( M_l(A^i_m) \in \eta_i^{A^i_m} \). Here

\[
F(\gamma[M_l(A^i_m), M_r(A^i_m)]) = \gamma[M_l(A^i_{m-1}), M_r(A^i_{m-1})]
\]

for \( k \geq 2, 2 \leq m \leq k, 1 \leq i' \leq 2^{m-1} - 1 \).

4
Define the equivalence relation on a maximal generative domain $H^\text{max}(A_{k+1}^j)$ of $(k+1)$-st order, $k \geq 1$, $1 \leq j \leq 2^{k+1} - 1$. By Remark 1 the map $F|_{H^\text{max}(A_{k+1}^j)}$ is homeomorphism of $H^\text{max}(A_{k+1}^j)$ on $H^\text{max}(A_{k+1}^{j'})$, and $F|_{\text{int}(H^\text{max}(A_{k+1}^j))}$ is diffeomorphism of $\text{int}(H^\text{max}(A_{k+1}^j))$ on $\text{int}(H^\text{max}(A_{k+1}^{j'}))$, where a maximal generative domain $H^\text{max}(A_{k+1}^j)$ is the connected component of first complete preimage of $H^\text{max}(A_{k+1}^{j'})(1 \leq j' \leq 2^k - 1)$ with respect to $F|_{G\Delta_2}$, moreover, the homoclinic point $A_{k+1}^j$ belongs to $H^\text{max}(A_{k+1}^{j'})$. Therefore, there exists a regular arc $\gamma[M_l(A_{k+1}^j), M_r(A_{k+1}^j)] \subset H^\text{max}(A_{k+1}^j)$ satisfying the equality

$$F(\gamma[M_l(A_{k+1}^j), M_r(A_{k+1}^j)]) = \gamma[M_l(A_{k+1}^j), M_r(A_{k+1}^j)].$$

Identify all points of the arc $\gamma[M_l(A_{k+1}^j), M_r(A_{k+1}^j)]$ with the point $M_l(A_{k+1}^j) \in \eta_l^{A_{k+1}}$.

Thus, using equalities (2) – (4) we constructed the equivalence relation on the set

$$H^\text{max}(A_2) \bigcup \left( \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{2^k - 1} H^\text{max}(A_{k}^i) \right)$$

and, in particular, on the set of curves

$$L_H = \{ \eta_l^{A_2}, \eta_r^{A_2} \} \bigcup \left( \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{2^k - 1} \{ \eta_l^{A_k^i}, \eta_r^{A_k^i} \} \right).$$

By the above, in the both cases (see Remark 1 and Proposition 6) the set of equivalence classes (with respect to the introduced equivalence relation) coincides with the set of curves

$$L_{H, l} = \eta_l^{A_2} \bigcup \left( \bigcup_{k=1}^{+\infty} \bigcup_{i=1}^{2^k - 1} \eta_l^{A_k^i} \right).$$

Further, we consider the equivalence relation only on the set $L_H \subset G'$. Denote by $\pi^{L_H}$ the constructed equivalence relation on the set of curves $L_H$.

4. Extend the equivalence relation $\pi^{L_H}$ to the equivalence relation $\pi$ defined on all curves of the set $G'$. Let $\gamma(A)$ be an unbounded curve from the set $G' \setminus L_H$ (here $A$ is the original point of $\gamma(A)$ on the hypothenuse of the triangle $\Delta_2$) and $M_{\gamma(A)}$ be a point of $\gamma(A)$. By the Theorem and Proposition 3 there is a sequence of points $M_n \in \eta_p^{A_{l,n}}(p = l \text{ or } r; n \geq 1)$ on curves of the set $L_H$ satisfying

$$M_{\gamma(A)} = \lim_{n \to +\infty} M_n.$$

Let $M_{l,n}(A_{k,n}) \in \eta_l^{A_{l,n}}$ be a point identified with $M_n$ under equivalence relation $\pi^{L_H}$. Using Proposition 1 and the equality (5) we find a positive number $s_*$ satisfying

$$M_{\gamma(A)} \in \gamma^{s_*}(A), \text{ where } \gamma^{s_*}(A) = \gamma(A)_{[0, s_*]}; \ M_n \in \eta_p^{A_{l,n}, s_*}, \ M_{l,n}(A_{k,n}) \in \eta_l^{A_{l,n}, s_*}.$$

By Proposition 3 the sequence of arcs $\{\eta_l^{A_{l,n}, s_*}\}_{n \geq 1}$ converges in Hausdorff metric $\text{dist}$ to the arc $\gamma^{s_*}(A)$. By Proposition 4 the sequence of arcs $\{\eta_l^{A_{l,n}, s_*}\}_{n \geq 1}$ also converges in Hausdorff metric to the arc $\gamma^{s_*}(A)$.  

5
Prove that the sequence of points \( \{M_{l,n}(A_{k,n}^i)\}_{n \geq 1} \) converges (to some point \( M_{\gamma(A)}^1 \in \gamma^{s*}(A) \)). Note that the proof of convergence of the sequence \( \{M_{l,n}(A_{k,n}^i)\}_{n \geq 1} \) does not depend on belonging of the curve \( \gamma(A) \) to the set \( G' \setminus L_H \) or \( L_H \). Suppose that the sequence of points \( \{M_{l,n}(A_{k,n}^i)\}_{n \geq 1} \) diverges. Then, this sequence has at least, two different limit points \( M_{\gamma(A)}^1, M_{\gamma(A)}^2 \in \gamma^{s*}(A) \). Suppose that
\[
A \prec M_{\gamma(A)}^1 \prec M_{\gamma(A)}^2,
\]
where " \( \prec \) " is the natural order on the curve \( \gamma(A) \) corresponding to growth of the parameter \( s \) (the value of \( s = 0 \) corresponds to the point \( A \)). Select two different subsequences of points \( \{M_{l,n}(A_{k,n}^i)\}_{n \geq 1} \) and \( \{M_{l,n}(A_{k,n}^i)\}_{n \geq 1} \) such that
\[
M_{\gamma(A)}^1 = \lim_{n(1) \to +\infty} M_{l,n}(A_{k,n}^i), \quad M_{\gamma(A)}^2 = \lim_{n(2) \to +\infty} M_{l,n}(A_{k,n}^i),
\]
where \( M_{l,n}(A_{k,n}^i) \in \eta^{n(1),s*}, M_{l,n}(A_{k,n}^i) \in \eta^{n(2),s*} \). Since the map \( F \) preserves the order of points, in particular, on every curve \( \eta_i^{n(1)} \) (see Remark 1) then, by (7) the sequence of arcs \( \gamma[A_{k,n}^i, M_{l,n}(A_{k,n}^i)] \) converges in Hausdorff metric \( \text{dist} \) to the arc \( \gamma[A_{k,n}^i, M_{\gamma(A)}^1, M_{\gamma(A)}^2] \).

Using (6) we obtain from here the contradiction with convergence of the sequence \( \eta_i^{n(1),s*} \) to the arc \( \gamma^{s*}(A) \) in Hausdorff metric. Thus, the equality \( M_{\gamma(A)}^1 = M_{\gamma(A)}^2 \) holds. Set \( M_{\gamma(A)}' = M_{\gamma(A)}^1 = M_{\gamma(A)}^2 \). Define \( \pi(M_{\gamma(A)}) \) by the equalities
\[
\pi(M_{\gamma(A)}) = M_{\gamma(A)}' = \lim_{n \to +\infty} M_{l,n}(A_{k,n}^i) = \lim_{n \to +\infty} \pi^{L_H}(M_n).
\]
Thus, the equivalence class of points of a curve \( \gamma(A) \in G' \setminus L_H \) coincides with \( \gamma(A) \). Construction of the equivalence relation is finished.

Described construction of the equivalence relation (see items 1 – 4) shows that the quotient space with respect to \( \pi \) coincides with the curves that form the set \( L_{H,1} \bigcup (G' \setminus L_H) \). Moreover, every curve of the set \( L_{H,1} \) consists of double points.

The following Lemma 1 proves correctness of the above construction of the equivalence relation \( \pi \) in the points of curves of the set \( G' \setminus L_H \).

**Lemma 1.** The value \( \pi(M_{\gamma(A)}) \) in any point \( M_{\gamma(A)} \) of a curve \( \gamma(A) \in G' \setminus L_H \) does not depend on the choice of a sequence of points satisfying (5).

**Proof.** Let a curve \( \gamma(A) \) and a point \( M_{\gamma(A)} \) be as in the above item 4. Let different sequences of points \( M_n \) and \( M_n' \) \( (n \geq 1) \) on curves of the set \( G' \) converge to \( M_{\gamma(A)} \). Using Proposition 2 we can suppose (without loss of generality) that points \( M_n \) and \( M_n' \) belong to curves of the set \( L_H \).

Form a new sequence \( \{\overline{M}_n\}_{n \geq 1} \) with the set of values obtained as the union of the sets of values of the above sequences \( \{M_n\}_{n \geq 1} \) and \( \{M_n'\}_{n \geq 1} \). Then, the sequence \( \{\overline{M}_n\}_{n \geq 1} \) converges to \( M_{\gamma(A)} \). Therefore, by equalities (8) we have:
\[
\pi(M_{\gamma(A)}) = \lim_{n \to +\infty} \pi^{L_H}(\overline{M}_n).
\]
The choice of the sequence \( \{\overline{M}_n\}_{n \geq 1} \) and the last equality imply
\[
\lim_{n \to +\infty} \pi^{L_H}(M_n) = \lim_{n \to +\infty} \pi^{L_H}(M_n') = \pi(M_{\gamma(A)}).
\]
Lemma 1 is proved.
Lemma 2. Equivalence relation $\pi$ is continuous.

Proof. 1. Continuity of the equivalence relation $\pi$ in points of curves of the set $G' \setminus L_H$ follows from formulas (8) and Lemma 1.

2. Note that the equivalence relation $\pi$ is continuous in points of curves of the set $L_H$. In fact, let $\gamma(A)$ here be a curve of the set $\{\eta^{A_1}, \eta^{A_2}; A_i^k, \eta^{A_i^k}\}$ $(k \geq 1, 1 \leq i \leq 2^k - 1)$. Let $M(\gamma(A))$ be a point of this curve, and $\{M_n\}_{n>1}$ be a convergent sequence of points on curves of the set $G'$. By Proposition 2 we assume (without loss of generality) that points $M_n$ belong to curves of the set $L_H$. Let the point sequence $\{M_n(A_i^k)\}_{n>1}$ be chosen analogously to the point sequence from item 4 of construction of the equivalence relation $\pi$. Then, this sequence converges to some point $M'_n(\gamma(A)) \in \gamma(A)$.

Consider, for determination, the case $\gamma(A) = \eta^{A_2}$. So, $\gamma(\eta^{A_2}) \in \eta^{A_2}$. To prove the continuity of $\pi (\text{in a point } M_n^{A_2})$ we must prove the equality

$$M_n^{A_2}(\eta^{A_2}) = M_{\eta^{A_2}}.$$

In fact, let $M_n^{A_2}$ on the curve $\eta^{A_2}$ be the equivalent point for $M_n^{A_2} \in \eta^{A_2}$. Then,

$$s(\{A_2, M_n^{A_2}\}) = s(\{A_2, M_n^{A_2}\}).$$

Therefore, if we carry the point $M_n^{A_2}$ on the curve $\eta^{A_2}$ preserving the length of the arc $[A_2, M_n^{A_2}]$ then, we obtain the point $M_n^{A_2}$. It means that (9) is equivalent to the equality

$$M'_{\eta^{A_2}} = \pi(\eta^{A_2}).$$

Correctness of the equalities (9) – (10) follows immediately from Proposition 4 and the inequalities

$$\|DF_{\eta^{A_2}}\|_{C^0} \geq K; \|DF_{\eta^{A_2}}\| \geq K$$

for some $K > 1$. Here $\|\cdot\|_{C^0}$ is the $C^0$-norm of the differential of a map:

$$\|DF_{\eta^{A_2}}\|_{C^0} = \max\left\{\sup_{y \in \eta^{A_2}} |\eta^{A_2}(x)| + |x|, \sup_{x \in \eta^{A_2}} \{2(x - 2)\}\right\}.$$ (11)

$y = \eta^{A_2}(x)$ is a function with the graph $\eta^{A_2}$, $D(\eta^{A_2})$ is the domain of definition, and $E(\eta^{A_2})$ is the set of values of $\eta^{A_2}$ ($p = l$ or $r$). Remind that

$$D(\eta^{A_2}) = (2, 3), \ D(\eta^{A_2}) = [3, +\infty), \ E(\eta^{A_2}) = [1, +\infty), \ E(\eta^{A_2}) = (0, 1].$$

Moreover, derivative $(\eta^{A_2}(x))'$ increases from $-\infty$ to $-1$, when $x$ changes on $(2, 3]$; and derivative $(\eta^{A_2}(x))'$ increases from $-1$ to $0$ when $x$ changes on $[3, +\infty)$.

As it follows from (11), one can assume $K = 2$ (see the above), Lemma 2 is proved.

By Lemmas 1, 2 the trace map $F$ induces the quotient map $F^*$ defined on the set of points of curves that form the set $L_H \cup (G' \setminus L_H)$. Moreover, $F^*$ satisfies the equality

$$\pi \circ F = F^* \circ \pi.$$ (12)

By the equality (12) second coordinate function of the map $F^*$ is double valued and upper semicontinuous (but not lower semicontinuous) in every point $(x, y)$, where $(x, y) \in \eta^{A_2}, (x, y) \neq (3, 1)$; and second coordinate function of the map $(F^*)^k (k \geq 2)$ is double valued and upper semicontinuous (but not lower semicontinuous) in every point $(x, y)$, where $(x, y) \in \eta^{A_i^q}, (x, y) \neq A_i^q (1 \leq q \leq k, 1 \leq i \leq 2^q - 1)$. By the above construction of the equivalence relation $\pi$ every set $L_H, G' \setminus L_H$ is invariant under $F^*$. Using invariance of the closed interval $x + y = 4$, $x \in [0, 4]$ under $F$ we obtain from here that $F^*$ is semiconjugate to the map $f(x) = x(4 - x)$ of the closed interval $[0, 4]$. 

7
3. Reducing to a Skew Product

We begin this part of the work from the following words by Birkhoff: “If we try to formulate the exact definition of integrability, then we see that many definitions are possible, and every of them is of some theoretical interest” [12].

Definition 1 of integrability proposed here, develops the idea by Grigorchuk of reduction of an integrable map in the plane to a skew product of maps of an interval (see [13]) and generalizes the definition of integrability from the paper [5] (see also [14]) on the case of upper semicontinuous multivalued functions defined on noncompact sets. In comparison with the definition by Grigorchuk, our definition is based on the semiconjugacy property of an integrable map in a plane with a map of an interval.

**Definition 1.** Let $\Pi \subseteq \mathbb{R}^2$ be (open or closed; bounded or not) domain in the plain $\mathbb{R}^2$. We say that a map $G : \Pi \to \Pi$ (single-valued or multi-valued) is integrable if there exists a self-map $\psi$ of some interval $J$ of the real line $\mathbb{R}$ so that $\psi$ is semiconjugate with $G$ by means a continuous surjection $\tilde{H} : \Pi \to J$, so that

$$\tilde{H} \circ G = \psi \circ \tilde{H}.$$  \hspace{1cm} (13)

The criterion for integrability of a continuous map $G$ defined on a compact curvilinear trapezoid $\Pi$ is given in [5]. In this paper we give necessary conditions for integrability in the unbounded domain.

Discuss Definition 1 for $\Pi = G_{\Delta_2}$, $J = \{(x, y) : x + y = 4, x \in [0, 4]\}$. Assume that $\tilde{H}$ is one-to-one with respect to $x$. Then, we have:

1. (1) for every $C \geq 4$ the section of $\Pi$ by a straight line $x + y = C$ is a non-degenerate compact interval corresponding $x \in [0, C]$;
2. (2) for all points $x', x'' \in [0, 4]$, where $x' \neq x''$, the equality $\tilde{H}^{-1}(x', 4 - x') \cap \tilde{H}^{-1}(x'', 4 - x'') = \emptyset$ is valid;
3. (3) the equality $\Pi = \bigcup_{x' \in [0, 4]} \tilde{H}^{-1}(x', 4 - x')$ holds.

Since $\tilde{H}$ is continuous and injective with respect to $x$, then for every $C > 4$ the map $\tilde{H}$ is bijection of the closed interval $J_C = \{(x, y) : x + y = C, x \in [0, C]\}$ on $\tilde{H}(J_C)$. Moreover, $\tilde{H}(J_C)$ is a nondegenerate closed interval in $J$. Property (1.3) and injectivity of $\tilde{H}$ with respect to $x$ imply the equality $\tilde{H}(J_C) = J$. Thus, for every $x' \in [0, 4]$ the set $\tilde{H}^{-1}(x', 4 - x')$ is the graph (denote it by $\gamma_{x'}$) of a continuous function $x = \gamma_{x'}(C)$. It means that $\tilde{H}^{-1}(x', 4 - x')$ is a connected set. Moreover, the unique curve $\gamma_{x'}$ passes through every point $(x', C - x') \in \Pi$, and $\gamma_{x'}$ intersects $J_C$ in a unique point for every $C > 4$. Hence, by properties (1.2), (1.3) the one-parameter family of graphs $\{\gamma_{x'}\}_{x' \in [0, 4]}$ of the above functions defines the continuous foliation in $\Pi$ (see, e.g., [15, ch. 4, § 16]). By (13) and $G$-invariance property of the closed interval $J$ the foliation $\{\gamma_{x'}\}_{x' \in [0, 4]}$ is invariant. It means that inclusion $G(\gamma_{x'}) \subseteq \gamma_{\psi(x')}$ is valid for every $x' \in [0, 4]$.

**Theorem 1.** Let $\Pi = G_{\Delta_2}$, and $G : \Pi \to \Pi$ be a continuous single-valued map or a map with continuous single-valued first coordinate function and double-valued upper semicontinuous second coordinate function. Moreover, in the last case there is a nonempty set of pairwise disjoint curves $\{\gamma_{x'_1}, \gamma_{x'_2}, \ldots, \gamma_{x'_r}\}$ $(1 \leq r < +\infty)$, where second coordinate function of the map $G$ is double-valued and discontinuous in every point which does not coincide with $(x'_i, 4 - x'_i)$ $(1 \leq i \leq r)$ (in all other points this function is single-valued and continuous).

Let, also, the section of $\Pi$ by a straight line $x + y = 4$ be $G$-invariant, and $G$ be integrable in the sense of Definition 1 by means of a continuous surjection $\tilde{H} : \Pi \to J$, which is one-to-one with respect to $x$ (here $J$ is defined as above).

Then there exists a homeomorphism that reduces $G$ to a skew product of interval maps (i.e., a map $\tilde{G}(u; v) = (\varphi(u); \psi(u; v))$) defined on an unbounded with respect to $v$ planar rectangle.
Theorem 1.
Let the map \( \theta : \Pi_{\sigma OC} \to \Pi'_{uO'v} \) (here \( \Pi_{\sigma OC} = \Pi = G_{\Delta_2} \)) be defined as follows:

\[
\begin{aligned}
  u &= \tilde{H}(x, C - x), \\
  v &= C - 4.
\end{aligned}
\]  

Then, \( \theta \) is a continuous bijection of \( \Pi_{\sigma OC} \) on \( \Pi'_{uO'v} \).

Note that \( \theta : \Pi_{\sigma OC} \to \Pi'_{uO'v} \) is homeomorphism. In fact, by (14) the map \( \theta \) is closed. Hence, \( \theta \) is mutually continuous map. Remind that a continuous map \( \theta : X \to Y \) is called mutually continuous, if \( \theta \) is surjection of \( X \) on \( Y \), and a set \( A \subset Y \) is closed (open) iff its complete preimage \( \theta^{-1}(A) \) is closed (open) [11, ch. 1, § XIII, XV]. Since \( \theta \) is mutually continuous and bijective map then \( \theta \) is homeomorphism [11, ch. 1, § XIII, XV].

By (14) a set \( \theta(\gamma_x) \) is a ray \( u = x', v \geq 0 \), for every \( x' \in [0, 4] \). Hence,

\[
\Pi'_{uO'v} = [0, 4] \times [0, +\infty).
\]

Let \( \tilde{G} \) be the map in the plane of variables \( u \) and \( v \) corresponding to the map \( G \) in the plane of variables \( x \) and \( y \). Then, \( \tilde{G} : \Pi'_{uO'v} \to \Pi'_{uO'v}, \tilde{G} \) is topologically conjugate with \( G \) under homeomorphism \( \theta \). It means that the following equality holds:

\[
\tilde{G} = \theta \circ G \circ \theta^{-1}.
\]

Denote by \( \tilde{g}_1 \) and \( \tilde{g}_2 \) first and second coordinate functions of the map \( \tilde{G} \) respectively. Since \( \theta \) is homeomorphism then, \( \tilde{g}_1 \) and \( \tilde{g}_2 \) inherit properties of first and second coordinate functions of the map \( G \) respectively. It means, in particular, that \( \tilde{g}_1 \) is continuous function.

2. Prove that \( \tilde{g}_1 \) does not depend on variable \( v \). In fact, \( \tilde{G} \) maps every vertical ray

\[
\{(x'; v) : v \geq 0\}
\]

into the vertical ray

\[
\{(\psi(x')) : v \geq 0\},
\]

where \( \psi(x') = G(x', 4 - x') \).

Prove that partial derivative \( \frac{\partial}{\partial v'} \tilde{g}_1(u, v) \) exists in every point \((u, v)\) of the rectangle \( \Pi'_{uO'v} \) and equals 0. In fact, let \((x'; v'), (x'; v)\) be arbitrary points of a vertical ray \( \Pi'_{uO'v} \cap \{(x'; v) : v \geq 0\} \). Then, we have:

\[
\frac{\partial}{\partial v'} \tilde{g}_1(u, v) = \lim_{v \to v'} \frac{\tilde{g}_1(x'; v) - \tilde{g}_1(x', v')}{{v' - v}} = \lim_{v \to v'} \frac{\psi(x') - \psi(x')}{{v' - v'}} = 0. \tag{15}
\]

Since \( \Pi'_{uO'v} \) is the convex connected set then, by (15) the coordinate function \( \tilde{g}_1 \) of the map \( \tilde{G} \) does not depend on the variable \( v \). Thus, \( \tilde{G} \) is the skew product in the plane.

Theorem 1 is proved.

Remark 3. We mean direct products as elements of the set of skew products dynamical systems.

Remark 4. Assumption on convexity of the unbounded domain \( \Pi'_{uO'v} \) can not be omitted.

Remark 5. Note that in considered case we have:

\[
\tilde{G}([0, 4] \times \{0\}) = [0, 4] \times \{0\}.
\]

It means that upper semicontinuity of the map \( \tilde{g}_2(u, v) \) in the points \((x', v) (1 \leq i \leq r, v > 0)\) does not imply double connectivity at the set values of \( \tilde{g}_2(u, v) \) in a vertical fiber \( u = \psi(x') \). Properties of \( \tilde{g}_2(u, v) \) do not influence on the proof of Theorem 1.

Using Definition 1 and results of the Subsection 1.2 we obtain the following result.

Theorem 2. The quotient map \( F^* \) for the trace map given by formula (1) is integrable in the sense of Definition 1.

By Theorems 1 and 2 the quotient map \( F^* \) for the trace map \( F \) is reducible to the upper semicontinuous skew product defined on the unbounded rectangle \([0, 4] \times [0, +\infty)\).
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