Galois Closure of Essentially Finite Morphisms

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Abstract. Let $X$ be a reduced connected $k$-scheme pointed at a rational point $x \in X(k)$. By using tannakian techniques we construct the Galois closure of an essentially finite $k$-morphism $f : Y \to X$ satisfying the condition $H^0(Y, \mathcal{O}_Y) = k$; it is a torsor $p : \hat{X}_Y \to X$ dominating $f$ by an $X$-morphism $\lambda : \hat{X}_Y \to Y$ and universal for this property. Moreover we show that $\lambda : \hat{X}_Y \to Y$ is a torsor under some finite group scheme we describe. Furthermore we prove that the direct image of an essentially finite vector bundle over $Y$ is still an essentially finite vector bundle over $X$. We develop for torsors and essentially finite morphisms a Galois correspondence similar to the usual one. As an application we show that for any pointed torsor under a finite group scheme $f : Y \to X$ satisfying the condition $H^0(Y, \mathcal{O}_Y) = k$, $Y$ has a fundamental group scheme $\pi_1(Y, y)$ fitting in a short exact sequence with $\pi_1(X, x)$.

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1 Introduction

Let $k$ be a field and $X$ a proper reduced $k$-scheme satisfying the condition $H^0(X, \mathcal{O}_X) = k$ (then it is in particular connected)\footnote{In Nori’s papers $k$ is a perfect field. The only point where this hypothesis is used is to ensure that the ring of endomorphisms $\text{End}(\mathcal{O}_X)$ of the unit object $\mathcal{O}_X$ of the tannakian category $\text{EF}(X)$ is a field. We do not assume here that $k$ is perfect, but we make the hypothesis $H^0(X, \mathcal{O}_X) = k$.}. The Nori fundamental group scheme was defined as the Galois tannakian group scheme of the category of essentially finite vector bundles on $X$\cite{11}. We define the notion of essentially finite morphism: these are finite faithfully flat morphims $f : Y \to X$ such that $f_*(\mathcal{O}_Y)$ is essentially finite. For instance torsors under finite group schemes, or more generally towers of torsors are essentially finite morphisms. On the other hand finite étale morphisms are essentially finite, and they are the only ones if $\text{ch}(k) = 0$.

The aim of this article is to define the Galois closure of an essentially finite morphism $f : Y \to X$. It is a torsor under a finite group scheme which dominates $f$ and satisfies obvious universal properties (theorem \ref{galois_closure}). Moreover given a torsor $T \to X$ under a finite group scheme $G$, there is a Galois correspondence between subgroups of $G$ and intermediate essentially finite morphisms, correspondence similar to the usual Galois correspondence. Then we prove that the direct image of an essentially finite vector bundle over $Y$ is still an essentially finite vector bundle over $X$ (theorem \ref{direct_image}). This allows us to give positive answers to natural questions: the composite of two essentially finite morphisms is essentially finite and being essentially finite is a property stable after base change.

The point of view chosen here is to use the tannakian techniques introduced by Nori in his definition of the fundamental group scheme. Another approach discussed also by Nori is to see the fundamental group scheme as the projective limit of the finite group schemes occurring in torsors over $X$. This last point of view has been generalized by Gasbarri\cite{6} allowing him to define the fundamental group schemes of relative schemes over Dedekind schemes.

We show that if $f : Y \to X$ is a torsor under a finite group scheme satisfying the condition $H^0(Y, \mathcal{O}_Y) = k$ then $Y$ has a fundamental group scheme which fits in a short exact sequence of fundamental group schemes (theorem \ref{exact_sequence}). Using Gasbarri’s approach M. Garuti addressed the question of the Galois closure of a tower of torsors\cite{5}. His Galois closure is not, however, the smallest possible, moreover he obtained a short exact sequence similar to ours in the more general setting of relative schemes over Dedekind schemes.
2 The fundamental group scheme.

In this section we briefly recall the tannakian construction of the Nori fundamental group scheme (cf. [10] and [11]). The situation is the following:

**Notation 2.1.** From now on $k$ will denote a field. Let $\theta : X \to \text{Spec}(k)$ be a reduced and proper $k$-scheme such that $H^0(X, \mathcal{O}_X) = k$ endowed with a section $x : \text{Spec}(k) \to X$.

We will consider in this article torsors over $X$ under finite $k$-group schemes $G$. They are finite, faithfully flat and $G$-invariant morphisms $p : Y \to X$, locally trivial for the fpqc topology, where $Y$ is a $k$-scheme endowed with a right action of $G$ and $X$ with the trivial action.

Nori defines in [11] the fundamental group scheme of $X$ as the Galois group of a neutral tannakian category generated by the so called “finite vector bundles”.

**Definition 2.2.** A vector bundle $F$ over $X$ is said to be finite if there exist two polynomials $p(x), q(x) \in \mathbb{Z}[x]$, where $p(x) \neq q(x)$ have nonnegative coefficients such that

$$p(F) \simeq q(F),$$

where the sum is the direct sum of vector bundles over $X$ and the product is the tensor product over $\mathcal{O}_X$.

It is a fact that finite vector bundles are semi-stable in the sense of Nori ([11], Ch. I, §2.3) and Nori considers the subcategory generated by the finite vector bundles in the tannakian category $\text{SS}(X)$ of semi-stable vector bundles on $X$. More precisely:

**Definition 2.3.** Let $S$ be a set of objects of $\text{SS}(X)$. We denote $\text{SS}(X)(S)$ the full abelian subcategory of $\text{SS}(X)$ generated by $S$. Since finite vector bundles over $X$ are semi-stables ([10] Corollary 3.5), if $F$ is the set of finite vector bundles over $X$, we denote $EF(X) = \text{SS}(X)(F)$. Objects of $EF(X)$ are called essentially finite vector bundles over $X$.

More generally take any subset $U \subseteq \text{Ob}(EF(X))$ and let $U^\vee$ be the set of duals of objects of $U$; let $U_1 := U \cup U^\vee$ and $S$ be the set of all possible tensor products of elements in $U_1$. We denote $\text{SS}(X)(S)$ by $EF(X,U)$ the full abelian subcategory of $\text{SS}(X)$ generated by $U$. Let

$$i_U : EF(X,U) \to \text{Qcoh}(X)$$

be the inclusion functor and let

$$x^* : \text{Qcoh}(X) \to \text{k-mod},$$

be the fiber functor associated to the section $x \in X(k)$.

In what follows, we will use the neutral fiber functor

$$\omega_U = x^* \circ i_U : EF(X,U) \to \text{k-mod}.$$ 

When $EF(X,U) = EF(X)$, we will use the notations $\omega = \omega_U$ and $i_X = i_U$. 

3
Nori proves the following result ([11], Chapter I, Proposition 3.7)

**Theorem 2.4.** For any \( U \in \text{Ob}(\text{EF}(X)) \), the category \((\text{EF}(X,U), \otimes, \omega_U, \mathcal{O}_X)\)
is a neutral tannakian category.

Let \((\mathcal{C}, \otimes, \gamma, 1_{\mathcal{C}})\) be any neutral tannakian category, where \( \gamma \) is a neutral fiber functor, let \( S \) be any \( k \)-scheme and \( \alpha, \beta : \mathcal{C} \to \mathcal{Qcoh}(S) \) be two fiber functors with values in the category of quasi-coherent sheaves on \( S \). We denote the functor

\[
\text{Isom}_S^\otimes(\alpha, \beta) : S\text{-Sch} \to \text{Set}
\]

\[
T \mapsto \text{Isom}_T^\otimes(\phi^\ast \circ \alpha, \phi^\ast \circ \beta)
\]

where \( \phi : T \to S \) is a morphism of schemes, \( S\text{-Sch} \) is the category of relative schemes over \( S \), \( \text{Set} \) is the category of sets and \( \text{Isom}_\otimes(\cdot, \cdot) \) stands for (iso)morphisms commuting with the tensor product.

The general Tannaka duality insures that \( \text{Aut}_k^\otimes(\gamma) := \text{Isom}_S^\otimes(\gamma, \gamma) \) is represented by an affine group scheme and that the category \( \mathcal{C} \) is equivalent to the category of representations of this group scheme (cf. for instance [3], Theorem 2.11). We will refer to \( G := \text{Aut}_k^\otimes(\gamma) \) as the tannakian Galois group scheme of the tannakian category \( \mathcal{C} \) attached to the neutral fiber functor \( \gamma \).

Let \( p : S \to \text{Spec}(k) \) be a scheme and \( \text{Fibs}(\mathcal{C}) \) the category of fiber functors \( \mathcal{C} \to \mathcal{Qcoh}(S) \). Let \( G\text{-Tors}_S \) be the category of (right) \( G \)-torsors over \( S \). We have the following fundamental result ([2] and [3]):

**Theorem 2.5.** The functor

\[
\text{Fibs}(\mathcal{C}) \to G\text{-Tors}_S
\]

\[
\eta \mapsto \text{Isom}_S^\otimes(p^\ast \circ \gamma, \eta)
\]

is an equivalence of gerbes.

**Definition 2.6.** Let \( k \) be a field, \( \theta : X \to \text{Spec}(k) \) a reduced and proper \( k \)-scheme such that \( H^0(X, \mathcal{O}_X) = k \) and let \( x \in X(k) \). Let \( \text{EF}(X) \) be the category of essentially finite vector bundles over \( X \) and \( \omega := x^\ast \circ i_X : \text{EF}(X) \to \text{k-mod} \) the fiber functor defined above. Then \( \pi_1(X,x) := \text{Aut}_k^\otimes(\omega) = \text{Isom}_X^\otimes(\omega, \omega) \) is the fundamental group scheme of \( X \) in \( x \).

The natural inclusion fiber functor \( i_U : \text{EF}(X,U) \to \mathcal{Qcoh}(X) \) gives rise to a torsor over \( X \) given by

\[
\hat{X}_U := \text{Isom}_X^\otimes(\theta^\ast \circ \omega_U, i_U)
\]

under the (right) action of the affine \( k \)-group scheme \( \pi_1(X,U,x) := \text{Aut}_k^\otimes(\omega_U) \). Moreover the fiber at \( x \) is equipped with a rational point \( \hat{x}_U \). Indeed we have the following canonical isomorphisms (cf. lemma 3.15)

\[
x^\ast(\hat{X}_U) \simeq x^\ast \text{Isom}_X^\otimes(\theta^\ast \circ \omega_U, i_U) \simeq
\]
$\simeq \text{Isom}_k^\otimes (x^* \circ \theta^* \circ \omega_U, x^* \circ i_U) \simeq \text{Isom}_k^\otimes (\omega_U, \omega_U) = \pi_1 (X, U, x)$

The image of the neutral element of $\pi_1 (X, U, x)$ in $x^* (\hat{X}_U)$ is $\hat{x}_U$ by definition. A torsor over $X$ under a finite $k$-group scheme endowed with a $k$-rational point in the fiber of $X$ will be referred to as a pointed torsor.

**Definition 2.7.** We will call $(\hat{X}_U, \hat{x}_U)$ the universal $\pi_1 (X, U, x)$-torsor over $X$ (associated to the tannakian category $EF(X, U)$). When $EF(X, U) = EF(X)$, the corresponding universal torsor will be denoted $(\hat{X}, \hat{x})$.

When $|U|$ is finite, the fundamental group scheme $\pi_1 (X, U, x)$ is finite. Thus $\pi_1 (X, x)$ is the projective limit of finite group schemes, and the universal torsor $\hat{X}$ is the projective limit of torsors under finite group schemes.

**Definition 2.8.** We will consider the category whose objects are pointed torsors under finite group schemes, i.e. triples $(Y, G, y)$ where $f : Y \to X$ is a $G$-torsor under some finite $k$-group scheme $G$ endowed with a section $y : \text{Spec}(k) \to Y$ such that $f(y) = x$.

A morphism $\varphi : (Y_1, G_1, y_1) \to (Y_2, G_2, y_2)$ between two pointed torsors is the datum of two morphisms $\alpha : Y_1 \to Y_2$ and $\beta : G_1 \to G_2$ where $\beta$ is a group scheme morphism, $\alpha(y_1) = y_2$ and such that the following diagram

\[
\begin{array}{ccc}
G_1 \times Y_1 & \to & Y_1 \\
\downarrow & \circ & \downarrow \\
G_2 \times Y_2 & \to & Y_2
\end{array}
\]

commutes (horizontal arrows being the actions of the concerned group schemes).

**Definition 2.9.** A pointed torsor $(Y, G, y)$, as in definition 2.8, is said to be a quotient torsor if for any pointed torsor $(Y', G', y')$ and any morphism $\varphi = (\alpha, \beta) : (Y', G', y') \to (Y, G, y)$, $\beta$ is a faithfully flat morphism$^2$.

**Remark 2.10.** A pointed torsor $(Y, G, y)$ is a quotient torsor if and only if $G$ is a quotient of the fundamental group scheme $\pi_1 (X, x)$, that is the canonical morphism $\pi_1 (X, x) \to G$ is faithfully flat (cf. for instance [1], Corollary 2.8).

## 3 Galois closure of essentially finite morphisms.

We keep the notations introduced in paragraph 2.

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$^2$In [1], Ch. II, Nori says “reduced” instead of “quotient”. Because of the possible confusion with the usual notion of reduced scheme, we changed the terminology.
3.1 Statement of the results

**Definition 3.1.** Let \( f : Y \to X \) be a finite and flat morphism (then faithfully flat since finite and flat imply that \( f \) is closed and open and as \( X \) is connected, \( f \) is surjective). The morphism \( f \) is said to be essentially finite if and only if \( f_* (O_Y) \) is an essentially finite vector bundle (cf. def. [2.3]).

Let \( f : Y \to X \) be an essentially finite morphism. Let us consider the full tannakian subcategory of \( EF(X) \) generated by \( f_* (O_Y) \) that is

\[ EF(X, \{ f_* (O_Y) \}) \]

provided with the fiber functor \( \omega_U : x^* \circ i_U : EF(X, U) \to k\text{-mod} \), where \( U := \{ f_* (O_Y) \} \) and \( i_U : EF(X, U) \to Qcoh(X) \) is as in paragraph 2. From these data we get the fundamental group scheme \( G = \pi_1(X, U, x) \) and the universal torsor \((\hat{X}_U, \hat{x}_U)\). Denote by \( p : \hat{X}_U \to X \) the structural morphism of the universal torsor. The main result of this section is the following theorem.

**Theorem 3.2.** Assume that \( \theta : X \to \text{Spec}(k) \) is a proper reduced locally noetherian \( k \)-scheme over a field \( k \) such that \( H^0(X, O_X) = k \), endowed with a rational point \( x : \text{Spec}(k) \to X \). Let \( f : Y \to X \) be an essentially finite morphism such that \( H^0(X, f_* (O_Y)) = k \) given with a rational point \( y \) in the fiber \( Y_x \) of \( f \) at \( x \).

1. Under these conditions there exists a unique faithfully flat morphism \( \lambda : \hat{X}_U \to Y \) sending \( \hat{x}_U \) to \( y \) and satisfying \( f \circ \lambda = p \).

2. Moreover \( \lambda : (\hat{X}_U, \hat{x}_U) \to (Y, y) \) has the structure of a pointed right torsor under the stabiliser \( G_y \) of \( y \) under the action of \( G = \pi_1(X, U, x) \) on the fiber \( Y_x \).

3. Finally the universal torsor \((\hat{X}_U, \hat{x}_U)\) satisfies the following universal property: for any quotient pointed torsor \((T, H, t)\), and any faithfully flat morphism \( \mu : T \to Y \) such that \( \mu(t) = y \), there exists a unique morphism of pointed torsors \((u, \varphi)\) where \( \varphi : H \to G = \pi_1(X, U, x) \) and \( u : T \to \hat{X}_U \) sending \( t \) to \( \hat{x}_U \) making the following diagram commutative

\[
\begin{array}{ccc}
T & \xrightarrow{\mu} & Y \\
\downarrow{\gamma} & & \downarrow{f} \\
\hat{X}_U & \xrightarrow{\lambda} & X \\
\end{array}
\]

**Notation 3.3.** In the situation of theorem 3.2, the group \( H \) acts, via the morphism \( \varphi : H \to \pi_1(X, U, x) \) on the fiber \( Y_x \). We will denote by \( H_y \) the stabilizer of \( y \) in this action.
Definition 3.4. Let \( f : Y \to X \) be an essentially finite morphism endowed with a rational point \( y \) in the fiber of \( x \), and \( g : T \to X \) a quotient torsor pointed by a rational point \( t \in T(k) \). We will say that the pointed torsor dominates the morphism \( f \) if there exists a faithfully flat morphism \( \lambda : T \to Y \) such that \( g = f \circ \lambda \) and \( \lambda(t) = y \).

Corollary 3.5. Let \( g : T \to X \) be a quotient torsor under a finite group scheme \( H \) over \( k \) pointed by a rational point \( t \in T(k) \).

1. The correspondence which associates to any essentially finite morphism \( f : Y \to X \) pointed by \( y \in Y(k) \) dominated by the given torsor, the stabilizer \( H_y < H \) is a bijection between pointed essentially finite morphisms \( f : Y \to X \) dominated by the given torsor, up to isomorphism, and closed \( k \)-subgroup schemes of \( H \), up to conjugation.

2. Moreover, \( f : Y \to X \) is a torsor if and only if \( H_y \) is normal in \( H \); in this case it is a torsor under the group scheme \( H/H_y \).

Remark 3.6. In the characteristic 0 case, essentially finite morphisms are just finite étale morphisms. Let indeed \( f : Y \to X \) be an essentially finite morphism. After extension of scalars we may assume that there are points \( x \in X(k) \) and \( y \in Y(k) \) such that \( f(y) = x \). The theorem \( \text{[3.2]} \) insures the existence of a Galois closure \( \hat{f} : \hat{X}_U \to X \) which is a torsor under a finite group scheme \( G \). As \( \text{ch}(k) = 0 \), \( G \) is an étale group scheme, and then \( \hat{f} : \hat{X}_U \to X \) is étale, which implies that \( f : Y \to X \) is itself étale.

We will see in paragraph 4 other examples of essentially finite morphisms, namely towers of torsors under finite group schemes.

Remark 3.7. In [12], Nori shows that the fundamental group scheme of an abelian variety is abelian. It then follows from the Galois correspondence that any essentially finite morphism \( f : Y \to X \) satisfying \( H^0(Y, \mathcal{O}_Y) = k \), where \( X \) is an abelian variety defined over a field \( k \), is itself a torsor under a finite abelian group scheme.

Let \( f : Y \to X \) be an essentially finite morphism. A natural question that arises is whether the direct image of an essentially finite vector bundle over \( Y \) is still essentially finite over \( X \). Theorem \([3.8]\) gives a positive answer in a little more general setting. As an application we will show that the composite morphism \( f' \circ f \) of two torsors under finite group schemes \( f : Y \to X \) and \( f' : Y' \to Y \), which in general is not a torsor itself, is an essentially finite morphism; therefore we will be able to apply to this morphism the construction of theorem \([3.2]\) and to construct a sharp Galois closure for towers of torsor, i.e. the smallest torsor dominating (in the sense of definition \([3.4]\)) the morphism \( f' \circ f \). We use a recent result of Garuti (cf. [5]) who constructs a “Galois closure” of towers of torsors (which is not the smallest possible torsor dominating the tower).
Proof. Let \( X \) be a finite \( k \)-scheme such that \( H^0(X, \mathcal{O}_X) = k \), endowed with a rational point \( x \in X(k) \). Let \( Y \) be a \( k \)-scheme and \( f : Y \to X \) an essentially finite morphism. Let \( F \) be a vector bundle over \( Y \) trivialised by a torsor over \( Y \) under a finite \( k \)-group scheme. Then the sheaf \( f_* (F) \) is an essentially finite vector bundle over \( X \).

From this theorem follows immediately that the composition of two essentially finite morphisms is essentially finite:

**Corollary 3.9.** Let \( X \) be a reduced, proper \( k \)-scheme such that \( H^0(X, \mathcal{O}_X) = k \), endowed with a rational point \( x \in X(k) \). Let \( Y \) be a reduced, proper \( k \)-scheme such that \( H^0(Y, \mathcal{O}_Y) = k \), endowed with a rational point \( y \in Y(k) \). Let \( Z \) be a \( k \)-scheme and let \( f : Y \to X \) and \( g : Z \to Y \) be two essentially finite morphisms. Then \( f \circ g \) is essentially finite.

Theorem 3.8 can now be applied to towers of torsors:

**Corollary 3.10.** Let \( k \) be a field and \( X \) a proper, reduced \( k \)-scheme, such that \( H^0(X, \mathcal{O}_X) = k \), provided with a point \( x : \text{Spec}(k) \to X \). Suppose we are given two finite \( k \)-group schemes \( G \) and \( G' \), a \( G \)-torsor \( f : Y \to X \) and a \( G' \)-torsor \( f' : Y' \to Y \).

Then \( (f \circ f')_* (\mathcal{O}_{Y'}) \) is an essentially finite vector bundle over \( X \).

**Proof.** As \( f' : Y' \to Y \) is a torsor under \( G' \), then \( Y' \times_Y Y' \simeq Y' \times G' \). Therefore \( f_* (\mathcal{O}_{Y'}) \) is trivialised by the torsor \( f' : Y' \to Y \). According to Theorem 3.8 \( (f \circ f')_* (\mathcal{O}_{Y'}) \) is an essentially finite vector bundle.

From this and theorem 3.9, we finally obtain the Galois closure of towers of torsors:

**Corollary 3.11.** Let \( k \) be a field and \( X \) a reduced proper \( k \)-scheme, such that \( H^0(X, \mathcal{O}_X) = k \), provided with a point \( x : \text{Spec}(k) \to X \). Let \( G \) and \( G' \) be two finite \( k \)-group schemes, \( f : Y \to X \) a \( G \)-torsor and \( f' : Y' \to Y \) a \( G' \)-torsor. We assume the existence of a point \( y : \text{Spec}(k) \to Y \) lying over \( x \) and of a point \( y' : \text{Spec}(k) \to Y' \) lying over \( y \). We assume that \( H^0(Y, \mathcal{O}_Y) = H^0(Y', \mathcal{O}_{Y'}) = k \).

1. Then it exists a finite \( k \)-group scheme \( \tilde{G} \), a \( \tilde{G} \)-torsor \( p : U \to X \) pointed by a rational point \( u \in U(k) \) above \( x \) and a faithfully flat morphism \( \lambda : U \to Y' \) with a right torsor structure such that \( f \circ f' \circ \lambda = p \).

2. Moreover the torsor \( p : U \to X \) satisfies the following universal property: for any quotient triple \((T, H, t)\), where \( g : T \to X \) is a torsor under a finite \( k \)-group scheme \( H \) and \( t \) a \( k \)-rational point over \( x \), and a faithfully flat morphism \( \mu : T \to Y' \) such that \( \mu(t) = y' \), there exists a unique morphism of pointed torsors \((h, \varphi)\) where \( \varphi : H \to \tilde{G} \) and \( h : T \to U \) sending \( t \) to \( u \) making the following diagram commutative.

\[
\begin{array}{ccc}
T & \xrightarrow{T} & U \\
\mu & \downarrow & \lambda \\
Y' & \to & Y'
\end{array}
\]
3. Denote by $Y'_{x}$ the fiber of $x$ in the morphism $f^{'}$, and $Y'_{y} \subset Y'_{x}$ the fiber of $y$ in the morphism $f^{'}$. Let $\hat{G}_{y'}$ (resp. $\hat{G}_{y}$) be the stabilizer of $y'$ (resp. of $Y'_{y}$) in the action of $\hat{G}$ on $Y'_{x}$.

(a) Then $\hat{G}_{y}$ is a normal subgroup of $\hat{G}$; $f' \circ \lambda : \hat{X} \rightarrow Y$ is a right torsor under $\hat{G}_{y}$; and $G \simeq \hat{G}/\hat{G}_{y}$ in the Galois correspondence.

(b) Also $\hat{G}_{y'}$ is normal in $\hat{G}_{y}$; $\lambda : \hat{X} \rightarrow Y'$ is a torsor under $\hat{G}_{y'}$; and $G' \simeq \hat{G}_{y}/\hat{G}_{y'}$ in the Galois correspondence.

Proof. According to corollary 3.13 the sheaf $(f \circ f^{'})_{*}(O_{Y'})$ is a finite vector bundle. Theorem 3.2 insures the existence of a Galois closure $U \rightarrow X$ of the essentially finite morphism $f \circ f^{'}$ and it says that $\lambda : U \rightarrow Y'$ is a $\hat{G}_{y}$-torsor where $\hat{G}_{y}$ is the stabilizer of $y'$ in the action of $\hat{G}$ on the fiber $Y'_{x}$. So conclusions 1 and 2 of the theorem are immediate consequences of theorem 3.2 applied to the essentially finite morphism $f \circ f^{'}$. To prove 3 first remark that $f' : Y' \rightarrow Y$ induces a morphism $Y'_{x} \rightarrow Y_{x}$ which is compatible with the actions of $\hat{G}$. Thus the stabilizer $\hat{G}_{y}$ of $Y'_{x}$ is also the stabilizer of $y$ in the action of $\hat{G}$ on $Y'_{x}$. Thus the point 3 (a) is a consequence of the point 2 of corollary 3.5 applied to the torsor $p : U \rightarrow X$ and the torsor $f : Y \rightarrow X$. Finally by the point 1 of corollary 3.5 $U \rightarrow Y$ is a torsor under $\hat{G}_{y}$ and the point 3 (b) is a consequence again of second point of corollary 3.5 applied to this torsor and the $G'$-torsor $f' : Y' \rightarrow Y$.

In corollary 3.13 we will prove that for a morphism the property of being essentially finite is stable after base change:

**Lemma 3.12.** Let $X$ and $X'$ be two reduced and proper $k$-schemes such that $k = H^{0}(X, O_{X}) = H^{0}(X', O_{X'})$. Let $i : X' \rightarrow X$ be any $k$-morphism. Assume the existence of a $k$-rational point in $X'$. Let $\mathcal{F}$ be an essentially finite vector bundle over $X$. Then $i^{*}(\mathcal{F})$ is an essentially finite vector bundle over $X'$.

**Proof.** Let $p : T \rightarrow X$ be a torsor under a finite $k$-group scheme trivializing $\mathcal{F}$, set $T' := T \times_{X} X'$ with projections $p^{'} : T' \rightarrow X'$ and $i' : T' \rightarrow T$. It follows that $p'$ trivialize $i^{'*}(\mathcal{F})$, indeed $O_{T'}^{\mathcal{F}'} \simeq i'^{*}p^{'*}(\mathcal{F}) \simeq p'^{*}i^{'*}(\mathcal{F})$. Let $C$ be
a proper and normal (then integral and regular) \( k \)-curve and \( j : C \to X' \) any non constant morphism. Then \( \deg(j^*i^*(\mathcal{F})) = 0 \) by assumption, since \( \mathcal{F} \) is essentially finite, when \( i \circ j \) is not constant and trivially when it is constant. Now, the inclusion \( \mathcal{O}_{X'} \to p'_*(\mathcal{O}_T') \) states that \( \mathcal{O}_{X'} \) is a subbundle of \( p'_*(\mathcal{O}_T') \) (i.e. the resulting quotient is a vector bundle) since \( p' \) is finite and faithfully flat\(^3\). Tensoring by \( i^*(\mathcal{F}) \) we deduce that \( i^*(\mathcal{F}) \) is a subbundle of \( i^*F \otimes p'_*(\mathcal{O}_T') \); the latter being isomorphic to \( p'_*p'^*i^*(\mathcal{F}) \simeq p'_*\mathcal{O}_{T'}^{\oplus r} \), we finally obtain that \( i^*(\mathcal{F}) \) is a subbundle of \( p'_*\mathcal{O}_{T'}^{\oplus r} \). So in particular \( i^*(\mathcal{F}) \) is essentially finite (cf. [10] proposition 3.7), \( p'_*\mathcal{O}_{T'}^{\oplus r} \) being (essentially) finite. 

\begin{corollary}
Let \( X \) and \( X' \) be two reduced and proper \( k \)-schemes such that \( H^0(X, \mathcal{O}_X) = H^0(X', \mathcal{O}_{X'}) = k \). Let \( i : X' \to X \) be any \( k \)-morphism. Assume the existence of a \( k \)-rational point in \( X' \). Let \( f : Y \to X \) be an essentially finite morphism. Then \( f' : Y' = Y \times_X X' \to X' \) is essentially finite too.

Proof. \( f'_*(\mathcal{O}_{Y'}) \simeq i^*f_*(\mathcal{O}_Y) \) is essentially finite according to previous lemma.
\end{corollary}

\section{Preliminary tools}

\begin{lemma}
Let \( C \) and \( C' \) be two tannakian categories, \( \gamma, \eta : C \to \text{Qcoh}(S) \) two fiber functors over a \( k \)-scheme \( S \) and \( F : C' \to C \) an exact tensor functor. We have the following relation between torsors

\[ \text{Isom}^\otimes_S(\gamma \circ F, \eta \circ F) \simeq \text{Isom}^\otimes_S(\gamma, \eta) \times \text{Aut}^\otimes_S(\gamma \circ F) \]

the second term being the contracted product (see for instance [3], III, §4, 3.2.).

Proof. Left to the reader.
\end{lemma}

\begin{lemma}
Let \( C \) be a tannakian category, \( j : S' \to S \) a flat morphism of \( k \)-schemes, \( \eta, \gamma \) two fiber functors over \( S \), then there is a canonical morphism of right torsors

\[ \text{Isom}^\otimes_S(j^* \circ \eta, j^* \circ \gamma) \simeq j^* \text{Isom}^\otimes_S(\eta, \gamma). \]

Proof. Left to the reader.
\end{lemma}

\[^3\text{The property of being locally free is local for the flat topology. And locally for the flat topology, a finite and faithfully flat morphism } V \to U \text{ has a section ([3], proof of proposition 2.18, p. 17).}\]
Let $G$ be a finite $k$-group scheme. There is a one-to-one correspondence between right torsors $f : T \to X$ under the group $G$ over $X$ and tensor functors $\gamma : \text{Rep}_k(G) \to \text{Qcoh}(X)$ given by the following relation:

$$T \simeq \text{Isom}_X^\otimes (\theta^* \circ \text{forget}_{kG}, \gamma)$$

where $\text{forget}_{kG} : \text{Rep}_k(G) \to k\text{-mod}$ is the forgetful functor (cf. theorem 2.5). The fiber functor $\gamma$ factors through a tensor functor $\tilde{\gamma} : \text{Rep}_k(G) \to \text{EF}(X)$, i.e. $\gamma = i_X \circ \tilde{\gamma}$, where $i_X$ is the inclusion of $\text{EF}(X)$ into the category $\text{Qcoh}(X)$ (11, Chapter I, Prop. 3.8).

If one composes $\tilde{\gamma}$ with the inverse of $\tilde{x}$, one gets a tensor functor $(\tilde{x})^{-1} \circ \tilde{\gamma} : \text{Rep}_k(G) \to \text{Rep}_k(\pi_1(X, x))$ which is equivalent to a morphism $\varphi : \pi_1(X, x) \to G$. We consider the contracted product

$$\hat{X} \times_{\pi_1(X, x)} G$$

for the morphism $\varphi$. This is a right $G$-torsor.

**Proposition 3.16.** If $T$ has a $k$-rational point over $x$, then $T \simeq \hat{X} \times_{\pi_1(X, x)} G$.

**Proof.** Recall that $\hat{X} = \text{Isom}_X^\otimes (\theta^* \circ \omega, i_X)$ where $\omega = x^* \circ i_X$. Using lemma 3.14 one has

$$\uparrow \hat{X} \times_{\pi_1(X, x)} G \simeq \text{Isom}_X^\otimes (\theta^* \circ \omega \circ \tilde{\gamma}, i_X \circ \tilde{\gamma}) \simeq \text{Isom}_X^\otimes (\theta^* \circ x^* \circ \gamma, \gamma)$$

Using lemma 3.15 and the definition of $T$, one gets

$$x^*T \simeq \text{Isom}_X^\otimes (x^* \circ \theta^* \circ \text{forget}_{kG}, x^* \circ \gamma) = \text{Isom}_X^\otimes (\text{forget}_{kG}, x^* \circ \gamma)$$

The fact that $T$ has a $k$-point over $x$ means that $x^*T$ is trivial, and then the functors $\text{forget}_{kG}$ and $x^* \circ \gamma$ are equivalent.

Replacing in the formula $\uparrow$, we get

$$\hat{X} \times_{\pi_1(X, x)} G \simeq \text{Isom}_X^\otimes (\theta^* \circ \text{forget}_{kG}, \gamma) \simeq T$$

which completes the proof of the proposition.

**Proposition 3.17.** Under the hypothesis of proposition 3.16, there is an isomorphism of $\mathcal{O}_X$-algebras

$$f_* (\mathcal{O}_T) \simeq \tilde{\gamma}(kG)$$

where $kG$ denotes the regular representation of $G$ ($G = \text{Spec}(kG)$).
Proof. Recall the following commutative diagrams of functors:

\[
\begin{align*}
EF(X) \xrightarrow{x^*} & \ k\text{-mod} \\
\xrightarrow{\tilde{x}} & \ Rep_k(\pi_1(X, x)) \\
\gamma \xrightarrow{\tilde{\gamma}} & \ Qcoh(X) \\
\xrightarrow{\pi} & \ EF(X)
\end{align*}
\]

So \(x^*T = Isom_X^\otimes(\text{forget}_{kG}, \text{Forget}_{k\pi_1(X, x)} \circ \tilde{x} \circ \tilde{\gamma}) \simeq G\) viewed with the left action of \(\pi_1(X, x)\) on \(G\) defined by the morphism \(\varphi : \pi_1(X, x) \to G\) induced by the functor \(\tilde{x} \circ \tilde{\gamma} : Rep_k(G) \to Rep_k(\pi_1(X, x))\). As \(\tilde{x}\) is an equivalence of categories, \(f_*(\mathcal{O}_T) = (\tilde{x})^{-1}(V)\), where \(V\) is the regular representation \(kG\) viewed as a representation of \(\pi_1(X, x)\) through \(\varphi\), i.e. \(V = \tilde{x} \circ \tilde{\gamma}(kG)\). One concludes that \(f_*(\mathcal{O}_T) \simeq (\tilde{x})^{-1} \circ \tilde{x} \circ \tilde{\gamma}(kG) \simeq \gamma(kG)\). As the functors involved in the proof are tensor functors, they make correspond \(k\)-algebras and \(\mathcal{O}_X\)-algebras, and the isomorphism \(f_*(\mathcal{O}_T) \simeq \gamma(kG)\) is thus an isomorphism of \(\mathcal{O}_X\)-algebras.

Proposition 3.18. Under the hypothesis of proposition \([3.10]\) the essential image of \(\tilde{\gamma}\) is constituted by the objects of \(EF(X)\) trivialized by the torsor \(f : T \to X\).

Proof. In one direction it is obvious: as \(T = Isom_X^\otimes(\theta^* \circ \text{forget}_{kG}, i_X \circ \tilde{\gamma})\) and \(f^*T = Isom_X^\otimes(f^* \circ \theta^* \circ \text{forget}_{kG}, f^* \circ i_X \circ \tilde{\gamma})\) is trivial, for any representation \(V\) of \(G\), \(f^* \circ i_X \circ \tilde{\gamma}(V)\) is isomorphic to \(f^* \circ \theta^* \circ \text{forget}_{kG}(V)\) which is a trivial vector bundle.

In the other direction, if \(F\) is an essentially finite vector bundle on \(X\) which is trivialized by \(f : T \to X\), then \(f^*F \simeq \mathcal{O}_T \oplus \cdots \oplus \mathcal{O}_T\), and then \(f_*f^*F \simeq f_*(\mathcal{O}_T) \oplus \cdots \oplus f_*(\mathcal{O}_T)\). Moreover, by proposition \([3.14]\) \(f_*(\mathcal{O}_T)\) is the image by \(\tilde{\gamma}\) of the regular representation of \(G\) and then \(f_*(\mathcal{O}_T) \oplus \cdots \oplus f_*(\mathcal{O}_T)\) is in the essential image of \(\tilde{\gamma}\). As \(f\) is faithfully flat, \(F \simeq f_*f^*F\), and then \(F\) is a sub-object of an object of the tannakian category generated by \(f_*(\mathcal{O}_T)\) and thus is an object of the essential image of \(\tilde{\gamma}\).

In conclusion there is a one-to-one correspondence between the following objects:

1. torsors \(T \to X\) under a finite group scheme \(G\) endowed with a rational point \(t \in T(k)\) over \(x \in X(k)\)
2. morphisms \(\varphi : \pi_1(X, x) \to G\)
3. exact tensor functors \(\gamma : Rep_k(G) \to Qcoh(X)\) satisfying the relation \(x^* \circ \gamma \simeq \text{forget}_{kG}\).
Proposition 3.19. With the previous notations, the following statements are equivalent

1. $H^0(T, \mathcal{O}_T) = k$
2. $\varphi$ is faithfully flat
3. $\gamma$ is fully faithful

The proof, for which we refer the reader to [11], Chapter II, Proposition 3, relies on the following lemma, that will be used later:

Lemma 3.20. Let $G$ be an affine group scheme and $\Phi : \text{Rep}_k(G) \to EF(X)$ a fully faithful tensor functor. Then for any representation $V$ of $G$, $H^0(X, \Phi(V)) \simeq V^G$.

Proof. We have the following equalities:

$$V^G \simeq \text{Hom}_G(V^\vee, k) \simeq \text{Hom}(\Phi(V)^\vee, \mathcal{O}_X) \simeq H^0(X, \text{Hom}(\Phi(V)^\vee, \mathcal{O}_X)) \simeq H^0(X, \Phi(V)).$$

\[\square\]

Corollary 3.21. Let $f : T \to X$ be a pointed torsor under a finite group scheme $G$. Then it is a quotient torsor if and only if $H^0(T, \mathcal{O}_T) = k$.

Corollary 3.22. Let $f : T \to X$ be a $G$-torsor pointed on $t \in T_x(k)$, where $G$ is a finite $k$-group scheme, and $\varphi : \pi_1(X, x) \to G$ the corresponding morphism. Consider the tannakian category $EF(X, \{f_*(\mathcal{O}_T)\})$. Then the fundamental group scheme $\pi_1(X, \{f_*(\mathcal{O}_T)\}, x)$ is isomorphic to the image $H$ of $\varphi$. It is a closed subgroup of $G$, and is equal to $G$ if and only if the pointed torsor $(T, t)$ is a quotient torsor, and in this case the universal torsor of the category $EF(X, \{f_*(\mathcal{O}_T)\})$ based at $x$ is isomorphic to $f : T \to X$.

Proof. Examine the proof of the preceding proposition. \[\square\]
3.3 Proof of Theorem 3.2

We keep the notations introduced in the statement of theorem 3.2. We will need the following lemma.

Lemma 3.23. If $H^0(Y, \mathcal{O}_Y) \cong k$, the morphism $\rho : G \to Y_x$ defined by $g \to g \cdot y$ is faithfully flat and induces an isomorphism $G/G_y \cong Y_x$.

Proof. First of all we observe that for any $k$-linear representation $(V, \sigma)$ of $G = \text{Spec}(A)$ the set of fixed elements i.e. the set of all those $v \in V$ such that $g \cdot (v \otimes 1_R) = (v \otimes 1_R)$ (for any $k$-algebra $R$ and any $g \in G(R)$) coincides with the set of all $v \in V$ such that $\gamma(v) = v \otimes 1_A$ where $\gamma : V \to V \otimes A$ is the comodule structure associated to $(V, \sigma)$. Now set $Y_x := \text{Spec}(V)$ where $V$ is a $k$-linear representation of $G$ with an additional $k$-algebra structure. The quotient $Y_x/G$ is the cokernel of the double arrow

$$G \times Y_x \xrightarrow{q_1} Y_x$$

where $q_1 : G \times Y_x \to Y_x$ maps $(g, z) \mapsto z$ and $q_2 : G \times Y_x \to Y_x$ (the action) maps $(g, z) \mapsto g \cdot z$ in the category of sheaves for the fppf topology. It is represented by $V^G$, the kernel of the double arrow

$$V \xrightarrow{n} V \otimes A$$

where $\gamma : V \to V \otimes A$ is the coaction and $u : V \to V \otimes A$ maps $v \mapsto v \otimes 1$ (cf. the proof of the affine case in Th. 3.2, III, paragraph 2, 4.1 to 4.4 of [4]). According to lemma 3.20 $V^G \cong k$: apply indeed the lemma to the equivalence $F : \text{Rep}_k(G) \to EF(X, U)$ associated to the universal torsor $p : X_U \to X$; $F(V) \cong f_*(\mathcal{O}_Y)$, so in particular $H^0(X, f_*\mathcal{O}_Y) \cong H^0(Y, \mathcal{O}_Y) \cong k$ by assumption. So $Y_x/G \cong \text{Spec}(k)$. It follows from this that $G \times Y_x \to Y_x \times Y_x$ is surjective for the fppf topology (Th. 3.2, b, III, paragraph 2 of [4]) and thus the morphism $\rho : G \to Y_x$ is surjective for the fppf topology.

On the other hand, $\rho$ induces a monomorphism of fppf sheaves $G/G_y \to Y_x$, where $G_y$ denotes the stabilizer of $y$ ([4], III, paragraph 3, 1.6). And then $G/G_y \cong Y_x$.

Finally, according to [4], III, paragraph 3, 2.5, the morphism $G \to G/G_y$ is faithfully flat. This concludes the proof of lemma.

Proof of the Theorem 3.2

1. Nori defines the functor $F : \text{Rep}_k(G) \to EF(X, U)$, already mentioned in the proof of lemma 3.23 which is an equivalence between the category of representations of $G = \pi_1(X, U, x)$ and $EF(X, U)$. As described in [2], 7.5-7.12 the functor $F$ induces an equivalence between finite $G$-schemes over $k$ and finite morphisms $g : Z \to X$ such that $g_*(\mathcal{O}_Z) \in EF(X, U)$. In this equivalence $G$ corresponds to the universal torsor $X_U$ and $Y_x$ corresponds
to $Y$. Then the $k$-morphism $\rho$ corresponds to an $X$-morphism $\lambda : \hat{X}_U \to Y$:

\[
\begin{array}{ccc}
\hat{X}_U & \xrightarrow{\lambda} & Y \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

whose fiber at $x : \text{Spec}(k) \to X$ is precisely $\rho$.

2. The facts that $G/G_y \simeq Y_X$ and that the action $m$ of $G_y$ on $G$ is free imply that the morphism

\[
(1) \quad G \times G_y \xrightarrow{pr_1 \times m} G \times_{Y_X} G
\]

is an isomorphism. This is an isomorphism of $G$-schemes, where $G$ is endowed with the left action of $G$ on itself and $G_y$ with the trivial action of $G$. The image of this diagram by the equivalence of tannakian categories $F : \text{Rep}_k(G) \to EF(X, U)$ is the following isomorphism

\[
(2) \quad \hat{X}_U \times G_y \to \hat{X}_U \times_Y \hat{X}_U
\]

whose fiber at $x$ is given by the isomorphism (1).

To prove the second part of the statement, the only thing to check is that $\lambda : \hat{X}_U \to Y$ is faithfully flat. Consider again the commutative diagram

\[
\begin{array}{ccc}
\hat{X}_U & \xrightarrow{\lambda} & Y \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

and pull it back by $\hat{X}_U \to X$. As we have seen in the proof of proposition 8.16 the functors $p^*$ and $p^* \circ \theta^* \circ \chi^*$ from $EF(X, U)$ to $\text{Qcoh}(\hat{X}_U)$ are equivalent. Applying the two functors to the preceding diagram one gets that the pull back by $p$ is

\[
\begin{array}{ccc}
\hat{X}_U \times G \xrightarrow{1_{\hat{X}_U} \times \rho} \hat{X}_U \times Y_x \\
\downarrow_{pr_1} & & \downarrow \\
\hat{X}_U & & \\
\end{array}
\]

As $p$ and $\rho$ are faithfully flat, $\lambda$ is also faithfully flat.
3. Since $f$ is affine, $f_*$ is exact; so from the inclusion $\mathcal{O}_Y \hookrightarrow \mu_*(\mathcal{O}_T)$ we get the inclusion $f_*(\mathcal{O}_Y) \hookrightarrow f_*(\mu_*(\mathcal{O}_T)) \simeq p_*(\mathcal{O}_T)$. Being semi-stable, $f_*(\mathcal{O}_Y)$ is a sub-object of $p_*(\mathcal{O}_T)$ and then an object of the tannakian category $EF(X, \{p_*(\mathcal{O}_T)\})$. Thus the inclusion is a fully faithful functor of tannakian categories

$$EF(X, \{f_*(\mathcal{O}_Y)\}) \hookrightarrow EF(X, \{p_*(\mathcal{O}_T)\})$$

which induces a faithfully flat morphism

$$H \to \pi_1(X, \{f_*(\mathcal{O}_Y)\}, x)$$

between their tannakian Galois group schemes. As $T$ is the universal torsor associated to the tannakian category $EF(X, \{p_*(\mathcal{O}_T)\})$, from this morphism one gets $u : T \to \hat{X}_U$ commuting with the actions of $H$ and $\pi_1(X, \{f_*(\mathcal{O}_Y)\}, x)$. The same kind of arguments used in the second part of the proof shows that this morphism is also faithfully flat.

**Proof of the Corollary**

If $f : Y \to X$ is an essentially finite morphism pointed at $y \in Y_x(k)$, call $U = \{f_*(\mathcal{O}_Y)\}$ and $G = \pi_1(X, U, x)$.

Suppose that $f : Y \to X$ is dominated by the pointed torsor $g : T \to X$: there exists a faithfully flat morphism $\mu : T \to Y$ such that $f \circ \mu = g$ and $\mu(t) = y$. By Theorem 3.2 there exists a unique morphism of torsors $(u, \varphi)$, where $\varphi : H \to G$ is a morphism of groups and $u : T \to \hat{X}_U$ is a morphism of torsors making the following diagram commutative

$$\begin{tikzcd}
T \arrow[r, u] \arrow[r, \mu] & \hat{X}_U \arrow[r, \lambda] & Y \\
\downarrow \downarrow \downarrow \varphi \downarrow \downarrow \downarrow f \downarrow \downarrow \downarrow & p \downarrow \downarrow \downarrow & X \downarrow \downarrow \downarrow f \\
H \arrow[r, \pi_1(X, x)] & G
\end{tikzcd}$$

To these morphisms of torsors correspond morphisms of fundamental groups

$$\pi_1(X, x) \to H \to G$$

One gets an action of $H$ on $Y_x$ and the stabilizer $H_y$ of $y$ under this action.

Conversely if $H' \subset H$ is a subgroup of $H$, the quotient $H/H'$ is endowed with an action of the fundamental group scheme $\pi_1(X, x)$ trough the morphism $\pi_1(X, x) \to H$ attached to the torsor $p : T \to X$. To the $\pi_1(X, x)$-$k$-scheme $H/H'$ corresponds a $X$-scheme $f : Y \to X$ such that
f_*(\mathcal{O}_X) is essentially finite and Y_x \simeq H/H'. Moreover Y is pointed at y \in Y_x(k) corresponding to the image of the unit element in H/H' and H' = H_y.

These correspondences are inverse of each other as in the situation considered above Y_x \simeq G/G_y \simeq H/H_y as \pi_1(X,x)-k-schemes.

If the morphism f : Y \to X is a torsor, then it is a quotient torsor under the group G. Then G_y = 1 and H_y which is the inverse image of G_y in the morphism H \to G is a normal subgroup of H. In this case G \simeq H/H_y and thus f : Y \to X is a torsor under the quotient group H/H_y.

Conversely if H_y is normal in H, G_y is normal in G, and as Y_x \simeq G/G_y as representations of \pi_1(X,x), the fundamental group G = \pi_1(X_U,x) which is the image of \pi_1(X,x) in this representation is isomorphic to the group G/G_y. Thus G_y = 1 and f : Y \to X is a torsor under G.

Remark 3.24. In the situation of theorem 3.24 (with the only difference that we do not need the assumption H^0(X,f_*(\mathcal{O}_Y)) = k), Y_x = Spec(V) where V is naturally a representation of \pi_1(X,x) which factors through the morphism \varphi : \pi_1(X,x) \to G associated to the universal torsor (X_U, \tilde{x}_U). Then Ker(\varphi) is the kernel of the representation of \pi_1(X,x) on V.

Let indeed K be this kernel. The inclusion Ker(\varphi) \subset K is obvious. In the other direction, V is a representation of \pi_1(X,x)/K, and as V generates the tannakian category Rep_k(G), one has the inclusion

\text{Rep}_k(G) \hookrightarrow \text{Rep}_k(\pi_1(X,x)/K)

which induces a surjective morphism \pi_1(X,x)/K \to \pi_1(X,x)/Ker(\varphi) = G. Thus K = Ker(\varphi) and G \simeq \pi_1(X,x)/K.

3.4 Proof of Theorem 3.8

Throughout this section k will be any field. We first recall that for an integral and projective curve C over k and any coherent sheaf F over C we define, respectively, the rank and the degree of F as follows:

\text{rk}(F) := \dim_k(\xi)(F_\xi) \quad \text{deg}(F) := \chi(F) - \text{rk}(F) \cdot \chi(\mathcal{O}_C)

where \xi is the generic point of C and \chi(F) is the Euler-Poincaré characteristic of F. Assume moreover that C is normal and let 0 \to F' \to F \to F'' \to 0 be an exact sequence of coherent sheaves over C then clearly \text{rk}(F) = \text{rk}(F') + \text{rk}(F'') and consequently \text{deg}(F) = \text{deg}(F') + \text{deg}(F''). If V and W are locally free sheaves over an integral curve C over k then one can compute the degree of V \otimes_{\mathcal{O}_C} W as follows (cf. [14]), Ch 6, §7, ex. 9):

\text{deg}(V \otimes_{\mathcal{O}_C} W) = \text{rk}(V)\text{deg}(W) + \text{rk}(W)\text{deg}(V).

If F is a coherent \mathcal{O}_C-sheaf then the formula still holds:
(*) \( \deg(F \otimes_{O_C} W) = \text{rk}(F) \deg(W) + \text{rk}(W) \deg(F) \).

Indeed \( F \) has a locally free resolution of length 1 (cf. [7], III, Example 6.5.1, Proposition 6.11 A and Exercise 6.5) thus there exist two locally free \( O_C \)-modules \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) and an exact sequence \( 0 \to \mathcal{L}_1 \to \mathcal{L}_0 \to F \to 0 \) that we tensor by \( W \) thus obtaining another exact sequence

\[
0 \to \mathcal{L}_1 \otimes_{O_C} W \to \mathcal{L}_0 \otimes_{O_C} W \to F \otimes_{O_C} W \to 0
\]

whence the formula, by the additivity of the degree. We need the following lemma, whose easy proof is left to the reader (otherwise compare with [8], Lemma 3.2.1):

**Lemma 3.25.** Let \( C \) and \( C' \) be integral and projective curves over \( k \) and \( f : C' \to C \) a finite morphism of degree \( d \). Let \( F \) be a coherent \( O_{C'} \)-module and \( G \) a coherent \( O_C \)-module then \( \deg(f_*(F)) = \deg(F) + \deg(f_*(O_{C'})) \cdot \text{rk}(F) \); if moreover \( f \) is flat or \( G \) locally free then \( \deg(f^*(G)) = d \cdot \deg(G) \).

**Lemma 3.26.** Let \( C \) be a normal and proper curve over \( k \), \( C' \) a curve over \( k \) and \( f' : C' \to C \) a finite and flat morphism. Denote by \( d = \text{rk}_{O_{C'}}(f'_*(O_{C'})) \) the degree of \( f' \). Let \( A \) and \( B \) be, respectively, a locally free and a coherent sheaf over \( C' \). Let us denote the coherent \( O_C \)-module \( f'_*(A \otimes_{O_{C'}} B) \) by \( F \). Then

\[
\deg(F) = d^{-1} \left( \text{rk}(f'_*(A)) \deg(f'_*(B)) + \text{rk}(f'_*(B)) \deg(f'_*(A)) - \deg(f'_*(O_{C'})) \text{rk}(F) \right)
\]

**Proof.** We let \( \mathcal{M} \) denote \( F \otimes_{O_C} f'_*(O_{C'}) \) and we compute its degree:

\[
\deg(\mathcal{M}) = \deg(F) \text{rk}(f'_*(O_{C'})) + \deg(f'_*(O_{C'})) \text{rk}(F).
\]

We will prove below the formula \( \mathcal{M} \cong f'_*(A) \otimes_{O_C} f'_*(B) \), thus, as \( f' \) is flat, \( f'_*(A) \) is locally free and the formula (*) gives

\[
\deg(\mathcal{M}) = \text{rk}(f'_*(B)) \deg(f'_*(A)) + \text{rk}(f'_*(A)) \deg(f'_*(B))
\]

which is enough to conclude. It only remains to prove the isomorphism \( \mathcal{M} \cong f'_*(A) \otimes_{O_C} f'_*(B) \) as \( O_C \)-sheaves: we have the following isomorphisms

\[
f'_*(B) \otimes_{O_C} f'_*(A) \cong f'_*(B \otimes_{O_{C'}} f'' f'_*(A)) \cong f'_*(B \otimes_{O_{C'}} A \otimes_{O_{C'}} f'' f'_*(O_{C'})) \cong
\]

\[
\cong f'_*(B \otimes_{O_{C'}} A) \otimes_{O_C} f'_*(O_{C'})
\]

where the first and third isomorphisms hold by the projection formula (cf. [7], II, Exercise 5.1) and the second is a consequence of the isomorphism

\[
f'' f'_* \mathcal{G} \cong \mathcal{G} \otimes_{O_{C'}} f'' f'_*(O_{C'})
\]

which holds for any coherent \( O_{C'} \)-module \( \mathcal{G} \) since \( f' \) is affine. \( \square \)
Definition 3.27. Let $\mathcal{F}$ be a vector bundle over a $k$-scheme $T$ such that $\deg(i^*(\mathcal{F})) = 0$ for any proper and normal $k$-curve $D$ and any non constant morphism $i : D \to T$; we will say that $\mathcal{F}$ has restricted degree $0$.

Lemma 3.28. Let $X$ and $Y$ be two $k$-schemes and let $f : Y \to X$ be a finite and flat morphism such that $f_* (\mathcal{O}_Y)$ has restricted degree $0$. Let $\mathcal{F}$ be a vector bundle over $Y$ which has restricted degree $0$. Then the vector bundle $f_* (\mathcal{F})$ has restricted degree $0$.

Proof. We denote $rk(f_* (\mathcal{O}_Y))$ by $d$. Let $C$ be a normal and proper curve over $k$, $j : C \to X$ a non constant morphism and $C' := Y \times_X C$; we consider the following diagram:

$$
\begin{array}{ccc}
\tilde{C} & \xrightarrow{s} & C' \\
\downarrow & & \downarrow \scriptstyle{j'} \\
Y & \xrightarrow{f} & X
\end{array}
$$

where $\tilde{C}$ is the normalization of an irreducible component (surjective over $C$) of the curve $C_{\text{red}}$ obtained by $C'$ after reduction. We want to show that $\deg(j^* f_*(\mathcal{F})) = 0$. We know that

1. $\deg(s^* j'^*(\mathcal{F})) = 0$ by assumption since $j' \circ s$ is not constant because $f' \circ s$ is surjective;
2. $f'_* j'^*(\mathcal{F}) \cong j^* f_*(\mathcal{F})$;
3. for any quasi coherent $\mathcal{O}_{C'}$-module $\mathcal{G}$ we have $s_* s^* (\mathcal{G}) \cong \mathcal{G} \otimes_{\mathcal{O}_{C'}} s_* (\mathcal{O}_{\tilde{C}})$.

We denote $f'_* s_* s^* j'^*(\mathcal{F})$ by $\mathcal{P}$. According to lemma 3.26 and point 1, we know that

$$
\deg(\mathcal{P}) = \deg(f'_* s_* (\mathcal{O}_{\tilde{C}})) \cdot rk(\mathcal{F}).
$$

Using point 3, we obtain

$$
\mathcal{P} = f'_* s_* s^* j'^*(\mathcal{F}) \cong f'_*(j'^*(\mathcal{F}) \otimes_{\mathcal{O}_{C'}} s_* (\mathcal{O}_{\tilde{C}})).
$$

We observe that $\deg(f'_*(\mathcal{O}_{C'})) = 0$: indeed by point 2 we have $f'_*(\mathcal{O}_{C'}) \cong f'_* j'^*(\mathcal{O}_Y) \cong j^* f_*(\mathcal{O}_Y)$ then $\deg(f'_*(\mathcal{O}_{C'})) = 0$ by assumption. Hence, according to lemma 3.26 $\deg(\mathcal{P}) =

$$
= rk(f'_*(\mathcal{O}_{C'}))^{-1} \left( rk(f'_* s_* (\mathcal{O}_{\tilde{C}})) \cdot \deg(f'_* j'^*(\mathcal{F})) + rk(f'_* j'^*(\mathcal{F})) \cdot \deg(f'_* s_* (\mathcal{O}_{\tilde{C}})) \right)
$$

$$
= d^{-1} \left( rk(f'_* s_* (\mathcal{O}_{\tilde{C}})) \cdot \deg(f'_* j'^*(\mathcal{F})) \right) + d^{-1} \left( d \cdot rk(\mathcal{F}) \cdot \deg(f'_* s_* (\mathcal{O}_{\tilde{C}})) \right);
$$

but we already know that $\deg(\mathcal{P}) = \deg(f'_* s_* (\mathcal{O}_{\tilde{C}})) \cdot rk(\mathcal{F})$ thus $\deg(f'_* j'^*(\mathcal{F})) = 0$ and by point 2 we finally obtain $\deg(j^* f_*(\mathcal{F})) = 0$. \qed
Proof of Theorem 3.8. We are given a proper reduced $k$-scheme $X$ such that $H^0(X, \mathcal{O}_X) = k$ and an essentially finite morphism $f : Y \to X$. Let $g : Z \to Y$ be a torsor under a finite $k$-group scheme $G_2$ trivialising $\mathcal{F}$ (i.e. $g^*(\mathcal{F}) \simeq \mathcal{O}_Z^{\oplus r}$ where $r = \text{rk}(\mathcal{F})$). First we need to observe that $\mathcal{F}$ has restricted degree $0$. So let $C$ be an integral, proper and normal $k$-curve, $j : C \to Y$ a non constant morphism and $C' := C \times_Y Z$. Let us consider the following diagram:

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{s} & Z \\
\downarrow & & \downarrow \\
C' & \xrightarrow{j'} & Z \\
\downarrow & & \downarrow \\
C & \xrightarrow{j} & Y \\
\end{array}
\]

where $\tilde{C}$ is an irreducible component of the curve $C'_{\text{red}}$ obtained by $C'$ after reduction then

$$\text{deg}(s^*g'^*j'^*\mathcal{F}) = \text{deg}(s^*j'^*g^*\mathcal{F}) = \text{deg}(\mathcal{O}_C^{\oplus r}) = 0$$

and by means of lemma 3.25 $\text{deg}(j^*(\mathcal{F})) = 0$. By theorem 3.2 it exists a torsor $T_1$ over $X$ under a finite $k$-group scheme $G_1$ dominating $f$. Set $T_2 := T_1 \times_Y Z$. It is a $G_2$-torsor over $T_1$. According to [5] there exist two torsors under finite $k$-group schemes $u : T \to X$ and $r : T \to T_2$ such that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{r} & T_2 \\
\downarrow & & \downarrow \\
T_1 & \xrightarrow{t} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

commutes. Set $h := t \circ r : T \to Z$; it is a faithfully flat morphism. Then $h^*g^*(\mathcal{F}) \simeq \mathcal{O}_T^{\oplus r}$ thus $u_*h^*g^*(\mathcal{F})$ is a finite vector bundle because $u_*(\mathcal{O}_T)$ is. Since $g \circ h$ is faithfully flat then $\mathcal{O}_Y \leftarrow (g \circ h)_*(\mathcal{O}_T)$ which implies $\mathcal{F} \simeq \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \leftarrow \mathcal{F} \otimes_{\mathcal{O}_Y} (g \circ h)_*(\mathcal{O}_T) \simeq (g \circ h)_*(g \circ h)^*(\mathcal{F})$. This means (see the proof of lemma 3.12) that $\mathcal{F}$ is a subbundle of $(g \circ h)_*(g \circ h)^*(\mathcal{F})$. The morphism $f$ being affine and flat we deduce, from last inclusion, that $f_*(\mathcal{F})$ is a subbundle of $f_*(g \circ h)_*(g \circ h)^*(\mathcal{F}) = u_*h^*g^*(\mathcal{F}) \simeq (u_*\mathcal{O}_T)^{\oplus r}$ but $(u_*\mathcal{O}_T)^{\oplus r}$ is finite, then in particular essentially finite. According to [10], Proposition 3.7, we just need to verify that $f_*\mathcal{F}$ has restricted degree $0$ in order to prove it is essentially finite, but this is a consequence of lemma 3.28 since $\mathcal{F}$ has restricted degree $0$. 

20
The short exact sequence of fundamental group schemes

Nori showed in [10] that under the hypothesis of section 2, the category of torsors under finite group schemes over \( X \) pointed above \( x \) is filtered, and that the fundamental group scheme \( \pi_1(X, x) \) is the projective limit of the groups occurring in these torsors. This led Nori to approach the construction of fundamental group scheme in a different manner: a \( k \)-scheme pointed at \( x \in X(k) \) has a fundamental group scheme based at \( x \) if there exists a universal torsor pointed above \( x \) that dominates every torsor pointed above \( x \) under the action of a finite group scheme (cf. [11], Chapter II, Definition 1). Then he proves that this is equivalent as saying that the category of torsors under finite group schemes over \( X \) pointed above \( x \) is filtered (cf. [11], Chapter II, Proposition 1). This point of view has been generalized by Gasbarri in [6] to schemes over Dedekind rings.

As a consequence of the previous paragraph, we show here that if \( X \) is a proper reduced scheme satisfying the condition \( H^0(X, \mathcal{O}_X) = k \) endowed with a rational point \( x \in X(k) \) and \( Y \to X \) is a quotient torsor under a finite group scheme \( G \) pointed on \( y \in Y(k) \) above \( x \), then \( Y \) has a fundamental group scheme. Moreover \( \pi_1(X, x) \) and \( \pi_1(Y, y) \) fit in an short exact sequence. This result was obtained independently by Garuti [5] by another method in the more general situation of relative schemes over Dedekind schemes.

**Theorem 4.1.** Let \( \theta : X \to \text{Spec}(k) \) be as before a locally noetherian proper reduced \( k \)-scheme endowed with a rational point \( x \in X(k) \) such that \( H^0(X, \mathcal{O}_X) = k \) and \( f : Y \to X \) a quotient torsor under a finite group scheme \( G \), pointed on \( y \in Y(k) \) over \( x \), corresponding to a faithfully flat morphism \( \varphi : \pi_1(X, x) \to G \).

Then \( Y \) has a fundamental group scheme based at \( y \), \( \pi_1(Y, y) \simeq \text{Ker}(\varphi) \) and we have the following short exact sequence of group schemes:

\[
1 \longrightarrow \pi_1(Y, y) \longrightarrow \pi_1(X, x) \xrightarrow{\varphi} G \longrightarrow 1
\]

**Proof.** Let \( f : Y \to X \) be as in the statement of the theorem. It follows from theorem 3.2 that there is a unique morphism of torsors \( \lambda : \hat{X} \to Y \) with respect to the morphism \( \varphi : \pi_1(X, x) \to G \) such that \( f \circ \lambda = p \) and \( \lambda(\hat{x}) = y \). Moreover \( \lambda : \hat{X} \to Y \) is a torsor under \( \text{Ker}(\varphi) \).

On the other hand let \( g : Z \to Y \) be a torsor under a finite group scheme, pointed by \( z \in Z(k) \) above \( y \). According to corollary 3.10, \( h = f \circ g \) is an essentially finite morphism, and thus is dominated by its Galois closure \( \hat{Z} \to X \), which is a quotient torsor itself dominated by \( p : \hat{X} \to X \), where \( p : \hat{X} \to X \) is the universal torsor of \( X \) based at \( x \).

Thus \( \lambda : \hat{X} \to Y \) is the projective limit of the pointed torsors \( g : Z \to Y \); it is the universal torsor of \( Y \) based at \( y \). \( \square \)
Corollary 4.2. Let $\theta : X \to \text{Spec}(k)$ be as before a locally noetherian proper reduced connected $k$-scheme endowed with a rational point $x \in X(k)$ and $f : Y \to X$ an essentially finite morphism pointed at $y \in Y(k)$ above $x$. Assume $H^0(Y, \mathcal{O}_Y) = k$. Then $Y$ has a fundamental group scheme based at $y$ and $\pi_1(Y,y) \simeq \pi_1(X,x)_y$ where $\pi_1(X,x)_y$ denotes the stabiliser of $y$ in the natural action of $\pi_1(X,x)$ on $Y_x$.

Proof. Let $f' : Y' \to Y$ be a pointed $G'$-torsor on $Y$ under a finite group scheme $G'$. Denote $\tilde{f} : \tilde{Y} \to X$ the Galois closure of $f$ and consider the following diagram, where the upper square is cartesian:

\[
\begin{array}{ccc}
Z = Y' \times_Y \tilde{Y} & \xrightarrow{g'} & Y' \\
\downarrow & & \downarrow \quad f' \\
\tilde{Y} & \xrightarrow{g} & Y \\
\downarrow \quad \tilde{f} & & \downarrow f \\
X & & X
\end{array}
\]

Then $Z \to \tilde{Y}$ is a pointed $G'$-torsor, and according to theorem 4.1 there are unique morphisms of pointed torsors $\lambda : \tilde{X} \to Z$ and $\mu : \tilde{X} \to \tilde{Y}$ making the following diagram commutative:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\lambda} & Z = Y' \times_Y \tilde{Y} \\
\downarrow \mu & & \downarrow g' \\
\tilde{Y} & \xrightarrow{g} & Y \\
\downarrow \tilde{f} & & \downarrow f \\
X & & X
\end{array}
\]

where $\tilde{X}$ denotes the universal torsor on $X$ based at $x$.

According to theorem 4.1 $g \circ \mu$ is a pointed torsor over $Y$ under the stabiliser $\pi_1(X,x)_y$ of $y$ in the action of $\pi_1(X,x)$ on $Y_x$. According to 3.2, lemma 1, a $k$-scheme morphism between two torsors $P$ and $P'$ under affine group schemes $G$ and $G'$ on a $k$-scheme $Y$ is a morphism of torsors relative to some unique morphism of groups $\varphi : \pi_1(X,x)_y \to G'$ such that $g' \circ \lambda$ is a morphism of pointed torsors relative to $\varphi$. This proves that $g \circ \mu : \tilde{X} \to Y$ is an universal object in the category of pointed torsors under finite group schemes on $Y$. One concludes that $Y$ has a fundamental group scheme and that $\pi_1(Y,y) \simeq \pi_1(X,x)_y$. 

\[\square\]
5 An example

We will restrict ourselves to the case of a characteristic 0 field \( k \). With the hypothesis of section 2, we have the classical short exact sequence of Grothendieck étale fundamental groups:

\[
1 \to \pi_1^{\text{ét}}(X, \bar{x}) \to \pi_1^{\text{ét}}(\overline{\mathbb{A}^1}_k, \bar{x}) \to \text{Gal}(\overline{k}/k) \to 1
\]

where \( \overline{k} \) is an algebraic closure of \( k \) and \( \bar{x} \) is the geometric point corresponding to \( x \). The rational point \( x \) gives rise to a section \( s : \text{Gal}(\overline{k}/k) \to \pi_1^{\text{ét}}(X, \bar{x}) \) and \( \pi_1^{\text{ét}}(X, \bar{x}) \) is in this way the semi-direct product of the geometric fundamental group \( \pi_1^{\text{ét}}(X_\overline{k}, \bar{x}) \) by the absolute Galois group of \( k \). The section defines an action of \( \text{Gal}(\overline{k}/k) \) on \( \pi_1^{\text{ét}}(X_\overline{k}, \bar{x}) \) and this group endowed with this action can be viewed as a pro-\( k \)-group scheme. This is the Nori’s fundamental group of \( X \) based at \( x \). In the case of an algebraically closed field \( k \) of characteristic 0, it suffices to remark that finite \( k \)-group schemes are just finite abstract groups, and that torsors under such finite group schemes are just Galois étale coverings (cf. [14], Corollary 6.7.20). The general case is explained in Appendix 1.

We would like to translate in terms of étale fundamental groups the Galois closure constructed in section 3. Let \( f : Y \to X \) be an essentially finite morphism, which is just in this context a finite étale morphism. We assume as usual that \( H^0(Y, \mathcal{O}_Y) = k \), which means that \( Y \) is geometrically connected. If we enumerate the geometric fiber at \( \bar{x} \) as \( \{1, \ldots, d\} \), then the data of the degree \( d \) étale covering \( Y \to X \) is equivalent to a morphism \( \psi : \pi_1^{\text{ét}}(X, \bar{x}) \to S_d \). This morphism factors as \( \psi = \theta \circ \Phi \), where \( \Phi : \pi_1^{\text{ét}}(X, \bar{x}) \to G, G \) is the image of \( \psi \) and \( \theta \) the inclusion \( G \subset S_d \). The surjective morphism \( \Phi \) corresponds to the Galois closure \( Z \to X \) of \( f : Y \to X \) in the sense of Galois theory. Its restriction to \( \pi_1^{\text{ét}}(X_\overline{k}, \bar{x}) \) factors through a surjective morphism \( \phi : \pi_1^{\text{ét}}(X_\overline{k}, \bar{x}) \to H \).

We have the following commutative diagram where all lines and the first and third columns are exact:

\[
\begin{array}{ccccccccc}
1 & \to & \pi_1^{\text{ét}}(X_\overline{k}, \bar{x}) & \overset{\varphi}{\longrightarrow} & \pi_1^{\text{ét}}(X, \bar{x}) & \overset{\Phi}{\longrightarrow} & \text{Gal}(\overline{k}/k) & \to & 1 \\
1 & \downarrow{\varphi} & \downarrow{\Phi} & & \downarrow{\text{Gal}(\overline{k}/k)} & & & & \\
1 & \to & H & \overset{\theta}{\longrightarrow} & G & \overset{\text{Gal}(L/k)}{\longrightarrow} & 1 & & \\
1 & \downarrow{\theta} & & \downarrow{S_d} & & & \downarrow{1} & & \\
& & 1 & & & 1 & & & &
\end{array}
\]

where \( L = H^0(Z, \mathcal{O}_Z) \) is a finite Galois extension of \( k \). Geometrically it corresponds to the following diagram:
where $Z$ is a Galois geometrically connected étale cover of $X_L$ of group $H$. The section $s$ induces a section $s_L : Gal(L/k) \to G$ of the second exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow Gal(L/k) \longrightarrow 1$$

and descent data from $L$ to $k$ of the Galois cover $Z \to X_L$ in a cover $p : T \to X$ as well as of the constant group $H$ in a $k$-group scheme $H_k$ (the Hopf algebra of the constant group $H$ over $L$ is $L^H$, whereas the Hopf algebra of $H_k$ is the sub-algebra $(L^H)^{Gal(L/k)}$ of fixed elements under the natural action of $Gal(L/k)$). Moreover these descent data are compatible with the right multiplication of $H$ on itself identified with the fiber at $\bar{x}$ which commutes with the action of the fundamental group and thus induces a right action of $H_k$ on $T$.

One can describe explicitly these actions in terms of the preceding diagrams. First the fiber of $Z \to X_L$ at $\bar{x}$ is identified to $H$ and the action of an element $\gamma$ of $\pi_1(X, \bar{x})$ on this fiber is the left multiplication of $\varphi(\gamma) \in H$ on $H$ through this identification, whereas the fiber of $Z \to X$ at $x$ is identified with $G$. Secondly the action of an element $\sigma \in Gal(\bar{k}/k)$ on the fibers at $x$ - that of $Z \to X$ identified with $G$ and that of $Y \to X$ identified with $\{1, \ldots, d\}$ - is the conjugation by $\Phi \circ s(\sigma)$ (resp. $\theta \circ \Phi \circ s(\sigma)$).

The étale cover $p : T \to X$, which is in general not Galois, corresponds to the morphism $\Psi : \pi_1(X, \bar{x}) \to S_H$ defined in the following way:

$$\forall \gamma \in \pi_1(X_{\bar{k}}, \bar{x}), \forall \sigma \in Gal(\bar{k}/k), \forall h \in H \quad \Psi(\gamma s(\sigma)), h = \varphi(\gamma)\Phi(s(\sigma))h\Phi(s(\sigma))^{-1}$$

which is an action clearly extending the action of $\pi_1^{et}(X_{\bar{k}}, \bar{x})$ on $H$ via $\varphi$. The étale cover $p : T \to X$ which is a torsor under $H_k$ is the Galois closure of $f : Y \to X$ defined in paragraph 3.

With the tools introduced here one can check that $p : T \to X$ factors through $f : Y \to X$ if $Y$ has a $k$-rational point $y$. Indeed one can index the points of the geometric fiber at $x$ such that $y$ corresponds to 1, which will be fixed under the action of $Gal(\bar{k}/k)$, i.e.

$$\forall \sigma \in Gal(\bar{k}/k) \quad (\theta \circ \Phi \circ s(\sigma)).1 = 1$$

Let us define a map $H \to \{1, \ldots, d\}$ by the formula

$$F : h \to (\theta(h)).1$$

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As $Y \to X$ is geometrically connected, the action of $H$ on $\{1, \ldots, d\}$ is transitive, and thus $F$ is a surjective map. It is compatible with the actions of $\pi_1(X, \bar{x})$ on $H$ and $\{1, \ldots, d\}$, and thus defines an unique morphism of covers $g : T \to Y$ over $X$. This is indeed a consequence of the following computation:

$$F((\gamma s(\sigma)).\Phi) = F(\varphi(\gamma) \Phi(s(\sigma)) \Phi^{-1}) =$$

$$= (\theta \circ \varphi(\gamma) \theta \circ \Phi(s(\sigma)) \theta(h)) = (\theta \circ \varphi(\gamma) \theta \circ \Phi(s(\sigma)) \theta(h)).1$$

In the other hand

$$\gamma s(\sigma).F(h) = \gamma s(\sigma).\theta(h).1 = \theta(\varphi(\gamma) \Phi(s(\sigma)) \Phi^{-1}).1 =$$

$$= (\theta \circ \varphi(\gamma) \theta \circ \Phi(s(\sigma)) \theta(h)).1$$

Finally remark that the torsor $p : T \to X$ is a Galois cover of $X$ if and only if the action of $\text{Gal}(\bar{k}/k)$ on $H$ is trivial or equivalently the Nori Galois group $H_k$ is the constant group $H$. In this case the the group $G$ is isomorphic to the direct product

$$G \simeq H \times \text{Gal}(L/k)$$

Consider the particular case where $f : Y \to X$ is itself a geometrically connected torsor under a finite group scheme $H_0$ and suppose that $Y(k) \neq \emptyset$. According to corollary 3.22 the group-scheme $H_k$ defined above is isomorphic to $H_0$ and the morphism $g : T \to Y$ defined above is an isomorphism of torsors. We have the following cartesian diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\simeq} & Y \\
\downarrow & & \downarrow f \\
X_L & \xrightarrow{\simeq} & X \\
\downarrow & & \downarrow \\
\text{Spec}(L) & \xrightarrow{\simeq} & \text{Spec}(k)
\end{array}
$$

Let us consider another model $f' : Y' \to X$ over $k$ of the Galois cover $Z \simeq Y_L \to X_L$. It corresponds to another section $s'$ of the short exact sequence

$$1 \to H \to G \to \text{Gal}(L/k) \to 1$$

where $H = H_0 \times \text{Spec}(L)$ is a constant group and $G = \text{Gal}(Z/X)$. The group $H$ with the action of $\text{Gal}(L/k)$ defined by $s'$ defines a $k$-group scheme $H'$ and $Y' \to X$ is a torsor under $H'$.

If $Y'(k) \neq \emptyset$, the argument used for $Y \to X$ applies to $Y' \to X$, and one concludes that the torsor $Y' \to X$ is isomorphic to $T \to X$, and thus to $Y \to X$. We have shown the following statement:
Proposition 5.1. Let $f : Y \to X$ be a geometrically connected torsor defined over $k$ under a $k$-group scheme $H_0$, $L/k$ the smallest Galois extension of $k$ such that $H = H_0 \times_{\text{Spec}(k)} \text{Spec}(L)$ is constant. Suppose that $Y_s(k) \neq \emptyset$. Then there is a one to one correspondence between classes of conjugation of sections of the short exact sequence

$$1 \to H \to G \to \text{Gal}(L/k) \to 1$$

and classes of isomorphism of $k$-models $Y' \to X$ of $Y_L \to X_L$. Both sets are parametrised by $H^1(k, H_0)$ pointed by the section $s$ attached to the rational point $x \in X(k)$. This section corresponds to the unique (up to $k$-isomorphism) model $Y \to X$ such that $Y_s(k) \neq \emptyset$.

6 Appendix 1

6.1 Gerbes, groupoïds, and short exact sequences

The general tannakian duality statement says that a tannakian category is equivalent to the category of representations of the gerbe of its fiber functors, or equivalently to the category of representations of some $k$-groupoïd. The correspondence between gerbes over $k$ and $k$-groupoïds acting transitively on a $k$-scheme is explained in [2]. Let us recall it briefly.

Given a gerbe $G$ over a field $k$ and a section $\omega$ over some $k$-scheme $X$, one defines the groupoïd $\Gamma_{X,G,\omega} = \text{Aut}(\omega) \to X \times_k X$, representing the functor associating to any morphism $(b, a) : T \to X \times_k X$ the set $\text{Isom}_{\Gamma}(a^*\omega, b^*\omega)$. If $u : Y \to X$ is a $k$-morphism, one has the following formula

$$u^*\Gamma_{X,G,\omega} = \Gamma_{Y,G,u^*\omega}$$

In the other direction, given a groupoïd $\Gamma \to X \times_k X$ acting transitively on the $k$-scheme $X$, one defines a fiber category $G_{X,\Gamma}^0$ whose objects are $k$-morphisms $T \to X$ and where the morphisms from $a : T \to X$ to $b : T \to X$ are $(a, b)^*\Gamma$. The gerbe $G_{X,\Gamma}$ corresponding to the groupoïd $\Gamma$ is the stack associated to $G_{X,\Gamma}^0$.

Any morphism $u : Y \to X$, where $Y$ is a non-empty $k$-scheme induces an equivalence

$$G_{Y,u^*\Gamma} \simeq G_{X,\Gamma}$$

We are considering here the case of the étale topology on $\text{Spec}(k)$ and we are going to explain that a gerbe $G$ over $\text{Spec}(k)_{\text{ét}}$ (or equivalently a $k$-groupoïd acting on $\text{Spec}(k)$) is equivalent to a short exact sequence built from a section $\omega \in G(k)$:

$$1 \to \text{Aut}_{[k]}^G(\omega) \to \Pi \to \text{Gal}(\bar{k}/k) \to 1$$
where \( \tilde{k} \) denotes the separable closure of \( k \) and \( \Pi \) is some profinite group we are defining as follows.

As explained above, from \( \omega \in \mathcal{G}(\tilde{k}) \) one can define the groupoid \((s,t): \Gamma = Aut(\omega) \to \text{Spec}(\tilde{k}) \times_k \text{Spec}(\tilde{k}) \) acting transitively on \( \text{Spec}(\tilde{k}) \). Call \( \Delta = \text{Spec}(k) \times_k \text{Spec}(\tilde{k}) \) the trivial groupoid acting on \( \text{Spec}(k) \). And denote by \( s : \Gamma_1 \to \text{Spec}(\tilde{k}) \) (resp. \( pr_1 : \Delta_1 \to \text{Spec}(\tilde{k}) \)) the schemes \( \Gamma \) (resp. \( \Delta \)) endowed with the map \( s \) (resp. \( pr_1 \)) over \( \text{Spec}(k) \).

The set \( \Delta_1(\tilde{k}) \) is canonically in bijection with \( \text{Gal}(\tilde{k}/k) \) and an element \( \gamma \in \Gamma_1(\tilde{k}) \) which maps to \( \sigma \in \text{Gal}(\tilde{k}/k) \) is an element of \( \text{Isom}^\tilde{k}_k(\omega, \sigma^\ast \omega) \). There is a natural group structure on \( \Gamma_1(\tilde{k}) \) compatible with the map \( \Gamma_1(\tilde{k}) \to \Delta_1(\tilde{k}) \) and the group structure on \( \Delta_1(\tilde{k}) \cong \text{Gal}(\tilde{k}/k) \): it is defined in the following manner: if \( \sigma, \tau \in \text{Gal}(\tilde{k}/k) \), \( \gamma \in \text{Isom}^\tilde{k}_k(\omega, \sigma^\ast \omega) \), \( \delta \in \text{Isom}^\tilde{k}_k(\omega, \tau^\ast \omega) \), then
\[
\gamma \ast \delta = \sigma \delta \gamma \in \text{Isom}^\tilde{k}_k(\omega, (\tau \sigma)^\ast \omega)
\]

thus the map \( \Gamma_1(\tilde{k}) \to \Delta_1(\tilde{k}) \) is a group homomorphism whose kernel is \( Aut^\tilde{k}_k(\omega) \).

Thus one gets the following short exact sequence
\[
(1) \quad 1 \to Aut^\tilde{k}_k(\omega) \to \Gamma_1(\tilde{k}) \to \text{Gal}(\tilde{k}/k) \to 1.
\]

Let \( k \subset K \subset \tilde{k} \) be an algebraic extension of \( k \); one can pull the short exact sequence \((1)\) by the morphism \( \text{Gal}(\tilde{k}/K) \to \text{Gal}(\tilde{k}/k) \). A section \( s \) of this exact sequence is the data for all \( \sigma \in \text{Gal}(\tilde{k}/K) \) of an isomorphism \( \varphi_\sigma : \omega \to \omega^\sigma \) satisfying Weil cocycle conditions or equivalently descent data from \( \tilde{k} \) to \( K \) for \( \omega \). It is thus a section \( \tilde{s} \) of the gerbe \( \mathcal{G} \) on \( \text{Spec}(K) \).

Given two sections \( s \) and \( t \) of the short exact sequence \((1)\) on \( \text{Spec}(K) \), the morphisms from \( \tilde{s} \) to \( \tilde{t} \) in \( \mathcal{G}_L \), where \( K \subset L \subset \tilde{k} \) is a finite extension of \( K \), are the automorphisms of \( \omega \) which are compatible with the descent data, that is the elements \( \gamma \in Aut^\tilde{k}_k(\omega) \) satisfying
\[
\forall \sigma \in \text{Gal}(\tilde{k}/L) \quad \gamma \ast s(\sigma) = t(\sigma) \ast \gamma
\]

which expresses the commutativity of the following diagrams
\[
\begin{array}{ccc}
\tilde{x}^* & \xrightarrow{\gamma} & \tilde{x}^* \\
\downarrow{s(\sigma)} & & \downarrow{t(\sigma)} \\
(\sigma \tilde{x})^* & \xrightarrow{\sigma \gamma} & (\sigma \tilde{x})^*
\end{array}
\]
We have defined a fully faithful functor \( s \to \tilde{s} \) from the category of sections of the short exact sequence \((1)\) to the gerbe \( \mathcal{G} \). This functor is essentially surjective: let \( \rho \) be a section of \( \mathcal{G} \) on an extension \( K \) of \( k \), \( k \subset K \subset \tilde{k} \), then \( \rho _{\tilde{k}} \) is isomorphic over \( \tilde{k} \) to the section \( \omega \). And then \( \rho \) is defined from \( \omega \) through descent data.

As we have seen above one can associate to these descent data a section of the short exact sequence \((1)\). We have proved the following statement:
Proposition 6.1. Any section \( s \) of the short exact sequence (1) on a finite extension \( K \) of \( k \) gives rise to descent data from \( \bar{k} \) to \( K \) for \( \omega \) and then to a section \( \tilde{s} \) of the gerbe \( \mathcal{G} \) on \( K \). The functor \( s \mapsto \tilde{s} \) is an equivalence of gerbes between the gerbe of sections of the short exact sequence (1) and the gerbe \( \mathcal{G} \).

When the gerbe \( \mathcal{G} \) is neutral, one can choose a section \( \xi \in \mathcal{G} (\text{Spec}(k)) \) and \( \omega = \xi_{\bar{k}} \). Then the descent data from \( \bar{k} \) to \( k \) defining \( \xi \) from \( \omega \) give rise to a section \( \bar{s} \) of the short exact sequence:

\[
\begin{align*}
\sigma & : \xi_{\bar{k}} \simeq \sigma \xi_{\bar{k}} = \xi_{\bar{k}} \\
\lambda & \in \text{Aut}_{\bar{k}}(\xi_{\bar{k}}), \quad \sigma \lambda = s(\sigma) \ast \lambda \ast s(\sigma)^{-1}
\end{align*}
\]

One gets the following statement:

Proposition 6.2. Suppose the gerbe \( \mathcal{G} \) being neutral and choose a section \( \xi \in \mathcal{G} (\text{Spec}(k)) \). The \( k \)-group \( \text{Aut}_{\bar{k}}(\xi) \) is then defined by the abstract group \( \text{Aut}_{\bar{k}}(\xi_{\bar{k}}) \) endowed with the action of \( \text{Gal}(\bar{k}/k) \) by conjugation through \( s \).

6.2 Nori’s fundamental group and Grothendieck’s fundamental group

We assume here that \( \text{ch}(k) = 0 \). Let \( X \) be as usual a proper reduced and connected \( k \)-scheme with a rational point \( x \in X(k) \). Denote by \( \bar{x} \) the geometric point corresponding to \( x \) and by \( k \to \bar{k} \) the embedding of fields. This geometric point \( \bar{x} \) gives rise to a fibre functor \( \bar{x}^* \) from the Galois category of étale covering of \( X \) to the category of sets (identified to the category of finite étale \( \bar{k} \)-schemes). On the other hand \( x \) gives rise to a neutral fiber functor \( \omega_x \) of the tannakian category \( EF(X) \) to the category of finite \( k \)-vector spaces.

In this situation, the arithmetic and geometric étale fundamental groups are well defined and fit in the fundamental short exact sequence

\[
\begin{align*}
1 & \to \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to \text{Gal}(\bar{k}/k) \to 1.
\end{align*}
\]

The rational point \( x \in X(k) \) defines a splitting of this short exact sequence, and thus an action of \( \text{Gal}(\bar{k}/k) \) over the abstract profinite group \( \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \).

In the gerbe \( \mathcal{G} \) of fibre functors of the tannakian category \( EF(X) \), for any fibre functor \( \omega, \text{Aut}^G(\omega) = \text{Aut}^G(\omega) \), where this notation represents the group of automorphisms of \( \omega \) compatible with the tensor product. To show that Nori’s fundamental group \( \text{Aut}^G(\omega_x) \) is the \( k \)-group defined by \( \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \) endowed with the action defined above, in view of Proposition 6.2 it suffices to show the following proposition:

Proposition 6.3. There are isomorphisms making the following diagram commutative:
$$1 \longrightarrow \text{Aut}^\mathcal{G}(\omega_x) \longrightarrow \Gamma_1(\bar{k}) \longrightarrow \text{Gal}(\bar{k}/k) \longrightarrow 1$$

$$1 \longrightarrow \pi_1^{\text{et}}(X_{\bar{k}, \bar{x}}) \longrightarrow \pi_1^{\text{et}}(X, \bar{x}) \longrightarrow \text{Gal}(\bar{k}/k) \longrightarrow 1$$

where the first row is the exact sequence (1).

\textit{Proof.} For all $\sigma \in \text{Gal}(\bar{k}/k)$ and for any étale covering $h : Y \rightarrow X$, we have a cartesian diagram

$$\begin{array}{cccc}
Y_{\bar{k}} = \sigma Y_{\bar{k}} & \xrightarrow{\beta \sigma} & Y_{\bar{k}} \\
\downarrow \sigma h_{\bar{k}} & & \downarrow h_{\bar{k}} \\
X_{\bar{k}} & \xrightarrow{\alpha \sigma} & X_{\bar{k}} \\
\downarrow Spec(\bar{k}) & \xrightarrow{\bar{\sigma}} & \downarrow Spec(\bar{k}) \\
\end{array}$$

The restrictions of $\beta \sigma$ to the fibers of $\bar{x}$ and $\sigma \bar{x}$ induce maps between finite sets

$$\beta \sigma : (\sigma \bar{x})^*(\sigma Y) \rightarrow \bar{x}^*(Y)$$

They define a natural transformation that we still denote $\alpha_\sigma : \sigma \bar{x}^* \Rightarrow \bar{x}^*$. To $\gamma \in \text{Isom}^\mathcal{G}(\omega_{\bar{x}}, \sigma \omega_{\bar{x}}) \subset \Gamma_1(\bar{k})$ let us associate $\bar{\gamma} : \bar{x}^* \Rightarrow \sigma \bar{x}^*$ and define $\Phi(\gamma) = \alpha_\sigma \circ \bar{\gamma} \in \pi_1(X, \bar{x})$. The following commutative diagram proves that $\Phi$ is a group homomorphism:

$$\begin{array}{cccc}
(\sigma^* \bar{x})^*(\sigma Y) & \xrightarrow{\sigma \beta \sigma} & (\sigma \bar{x})^*(\sigma Y) & \xrightarrow{\beta \sigma} & (\bar{x})^*(Y) \\
\downarrow \sigma \Phi(\delta) & & \downarrow \Phi(\delta) & & \downarrow \Phi(\gamma) \\
(\sigma \bar{x})^*(\sigma Y) & \xrightarrow{\beta \sigma} & (\bar{x})^*(Y) & & (\bar{x})^*(Y) \\
\downarrow \bar{\gamma} & & & & \downarrow \Phi(\gamma) \\
\end{array}$$

To verify that $\Phi$ is an isomorphism, it suffices to check that the diagram

$$\begin{array}{cccc}
1 & \longrightarrow & \text{Aut}^\mathcal{G}(\omega_x) & \longrightarrow \Gamma_1(\bar{k}) & \longrightarrow & \text{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
\Phi |_{\text{Aut}^\mathcal{G}(\omega_x)} & & \Phi & & = & & = & \\
1 & \longrightarrow & \pi_1^{\text{et}}(X_{\bar{k}, \bar{x}}) & \longrightarrow \pi_1^{\text{et}}(X, \bar{x}) & \longrightarrow & \text{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
\end{array}$$

The category of étale finite covering of $X$ can be identified to a subcategory of $\text{EF}(X)$ by the functor which sends a finite étale covering $f : Y \rightarrow X$ to $f_*\mathcal{O}_Y$. We are identifying the restriction to this subcategory of $\omega_{\bar{x}, \bar{k}}$ with $\bar{x}^*$

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4The category of étale finite covering of $X$ can be identified to a subcategory of $\text{EF}(X)$ by the functor which sends a finite étale covering $f : Y \rightarrow X$ to $f_*\mathcal{O}_Y$. We are identifying the restriction to this subcategory of $\omega_{\bar{x}, \bar{k}}$ with $\bar{x}^*$
is commutative and that the two maps $\Phi_{\text{Aut}^G(\omega_x)} : \text{Aut}^G(\omega_x) \to \pi_1(X, \bar{x})$ and $\Delta_1(k) \to \text{Gal}(\bar{k}/k)$ are isomorphisms. It is obvious for the second one. As for the first one it suffices to notice that $\text{Aut}^G(\omega_x)$ is the Nori’s fundamental group of $X_{\bar{k}}$ based at $\bar{x}$, which is known to be isomorphic to the projective limit of finite $k$-group schemes occurring in finite torsors $Y \to X_{\bar{k}}$, which is equivalent to finite étale Galois covering of $X_{\bar{k}}$. Thus $\text{Aut}^G(\omega_x)$ is isomorphic to $\pi_1(X_{\bar{k}}, \bar{x})$.

To check that the diagram is commutative we only have to check that the right square is commutative. The morphism $\pi_1(X, \bar{x}) \to \text{Gal}(\bar{k}/k)$ is associated in the Galois theory with the functor which sends any finite étale $k$-algebra $k \subset K$ to the purely arithmetic covering $X \times_{\text{Spec}(k)} \text{Spec}(K) \to X$.

Let $k \subset K$ be a finite étale $k$-algebra. The structural morphism $\bar{x}^*(X_K) \to \text{Spec}(\bar{k})$ can be identified canonically to $\text{Spec}(K \otimes_k \bar{k}) \simeq \text{Spec}(\bar{k}^{S_K}) \to \text{Spec}(\bar{k})$, where $S_K$ is the set of $k$-embeddings of $K$ in $\bar{k}$, corresponding to the diagonal morphism $\bar{k} \hookrightarrow \bar{k}^{S_K}$. In particular it does not depend on the $\bar{k}$-point $\bar{x}$.

Let $\gamma$ be in $\text{Isom}^G(\omega_{\bar{x}}, \sigma^*\omega_{\bar{x}}) \subset \Gamma_1(\bar{k})$ where $\sigma \in \text{Gal}(\bar{k}/k)$. When we restrict $\gamma$ to the full subcategory $\mathcal{T}$ of $\mathcal{EF}(X)$ whose objects are $\mathcal{O}_{X_K}$, where $k \hookrightarrow K$ runs among finite étale $k$-algebras (or more generally finite $k$-vector spaces), we get a tensor automorphism of the trivial fibre functor extended to $\bar{k}$ from the category $\mathcal{EF}(\text{Spec}(k))$. It is easy to check that the Nori fundamental group of $\text{Spec}(k)$ is trivial, and thus, the restriction of $\gamma$ to $\mathcal{T}$ is trivial.

On the other hand, when we restrict the natural transformation $\alpha_\sigma$ to objects of the form $X_K \to X$, where $K$ is a finite étale $k$-algebra, $\sigma$ induces $1_K \otimes \sigma : K \otimes_k \bar{k} \to K \otimes_k \bar{k}$, and modulo the isomorphism $K \otimes_k \bar{k} \simeq \bar{k}^{S_K}$, the isomorphism $\bar{k}^{S_K} \to \bar{k}^{S_K}$ given by the following formula:

\[ (*) \quad \lambda_\varphi \in S_K \to (\sigma(\lambda_{\sigma^{-1}\varphi}))_{\varphi \in S_K} \]

Finally, the restriction of $\Phi(\gamma) = \alpha_\sigma \circ \tilde{\gamma}$ to objects of the form $X_K \to X$ is given by the formula $(*)$, which corresponds on the set $S_K$ of $\bar{k}$ points of $\bar{k}^{S_K}$ to the map

\[ \varphi \to \sigma \circ \varphi \]

We have checked that the image of $\Phi(\gamma) \in \pi_1(X, \bar{x})$ in $\text{Gal}(\bar{k}/k)$ is $\sigma \in \text{Gal}(\bar{k}/k)$ as expected. \square

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**References**

[1] M. Antei, *Comparison Between the Fundamental Group Scheme of a Relative Scheme $X$ and that of its Generic Fiber*, arXiv:0807.2286v2, (2008).
[2] P. Deligne, *Catégories Tannakiennes*, in *The Grothendieck Festschrift*, Vol II, Birkhäuser, (1990), 111-195.

[3] P. Deligne, J.S. Milne, *Tannakian Categories*, in *Hodge Cycles, Motives, and Shimura Varieties*, Lectures Notes in Mathematics 900, Springer-Verlag, (1982), 101-227.

[4] M. Demazure, P. Gabriel, *Groupes Algébriques*, North-Holland Publ. Co., Amsterdam, (1970).

[5] M. A. Garuti, *On the “Galois closure” for Torsors*, Proc. Amer. Math. Soc. 137, 3575-3583 (2009).

[6] C. Gasbarri, *Heights Of Vector Bundles And The Fundamental Group Scheme Of A Curve*, Duke Mathematical Journal, Vol. 117, No. 2, (2003) 287-311.

[7] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer, (1977).

[8] D. Huybrechts, M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Aspects of Mathematics E 31, Vieweg (1997).

[9] J.S. Milne, *Étale Cohomology*, Princeton University Press, 33 (1980).

[10] M.V. Nori, *On The Representations Of The Fundamental Group*, Compositio Mathematica, Vol. 33, Fasc. 1, (1976). p. 29-42.

[11] M.V. Nori, *The Fundamental Group-Scheme*, Proc. Indian Acad. Sci. (Math. Sci.), Vol. 91, Number 2, (1982), p. 73-122.

[12] M.V. Nori, *The Fundamental Group Scheme of an Abelian Variety*, Math. Annalen, Vol. 263, (1983), p. 263-266.

[13] M. Raynaud., *Passage au quotient par une relation d’ équivalence plate*, Proceedings of a Conference on Local Fields, Springer-Verlag (1967), p. 78-85.

[14] T. Szamuely, *Galois Groups and Fundamental Groups*, Cambridge studies in advanced mathematics, no. 117, (2009).