INTEGRAL SOLUTIONS OF CERTAIN DIOPHANTINE EQUATION IN QUADRATIC FIELDS

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Abstract. Let \( K = \mathbb{Q}(\sqrt{d}) \) be a quadratic field and \( \mathcal{O}_K \) be its ring of integers. We study the solvability of the Diophantine equation \( r + s + t = rst = 2 \) in \( \mathcal{O}_K \). We prove that except for \( d = -7, -1, 17 \) and 101 this system is not solvable in the ring of integers of other quadratic fields.

1. Introduction

In 1960, Cassels [4] proved that the system of equations

\[
 r + s + t = rst = 1,
\]

is not solvable in rationals \( r, s \) and \( t \). Later in 1982, Small [3] studied the solutions of (1.1) in the rings \( \mathbb{Z}/m\mathbb{Z} \) and in the finite fields \( F_q \) where \( q = p^n \) with \( p \) a prime and \( n \geq 1 \). Further in 1987, Mollin et al. [8] considered (1.1) in the ring of integers of \( K = \mathbb{Q}(\sqrt{d}) \) and proved that solutions exist if and only if \( d = -1, 2 \) or 5, where \( x, y \) and \( z \) are units in \( \mathcal{O}_K \). Bremner [1, 2] in a series of two papers determined all cubic and quartic fields whose ring of integers contain a solution to (1.1). Later in 1999, Chakraborty et al. [6] also studied (1.1) in the ring of integers of quadratic fields reproducing the findings of Mollin et al. [8] for the original system by adopting a different technique.

Extending the study further, we consider the equation

\[
 r + s + t = rst = 2.
\]

The sum and product of numbers equals 1 has natural interest where as sum and product equals other naturals is a curious question. The method adopted here may not be suitable to consider a general \( n \) instead of 2 as for each particular \( n \) the system give rise to a particular elliptic curve which may have different ‘torsion’ and ‘rank’ respectively. The next case, i.e. when the sum and product equals to 3 is discussed in the last section.

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To begin with we perform suitable change of variables and transform (1.2) to an elliptic curve with the Weierstrass form

$$E_{297} : Y^2 = X^3 + 135X + 297 \quad (1.3)$$

and then study $E_{297}$ in the ring of integers of $K = \mathbb{Q}(\sqrt{d})$.

Remark 1. We transform (1.2) into an elliptic curve (1.3) to show that one of the $(r, s, t)$ has to belong to $\mathbb{Q}$ (shown in §3).

System (1.2) give rise to the quadratic equation

$$x^2 - (2 - r)x + \frac{2}{r} = 0, \quad r \neq 0,$$

with discriminant

$$\Delta = \frac{r(r^3 - 4r^2 + 4r - 8)}{r}. \quad (1.4)$$

At hindsight there are infinitely many choices for the quadratic fields contributed by each $r$ of the above form where the system could have solutions. The main result of this article is that the only possibilities are $r = \pm 1, 2$ and $-8$. Thus (1.2) is solvable only in $K = \mathbb{Q}(\sqrt{d})$ with $d = -7, -1, 17$ and $101$. Also the solutions are explicitly given. Throughout this article we denote ‘the point at infinity’ of an elliptic curve by $O$. Now we state the main result of the paper.

**Theorem 1.1.** Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field and $\mathcal{O}_K$ denote its ring of integers. Then the system

$$r + s + t = rst = 2$$

has no solution in $\mathcal{O}_K$ except for $d = -7, -1, 17$ and $101$.

In §4 we discuss the rank of $E_{297}$ in $\mathbb{Q}$ and in quadratic fields of our interest.

2. **Preliminaries**

In this section we mention some results which are needed for the proof of the Theorem 1.1. First we state a basic result from algebraic number theory.

**Theorem 2.1.** Let $K = \mathbb{Q}(\sqrt{d})$ with $d$ a square-free integer, then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \pmod{4}, \\ \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2,3 \pmod{4}. \end{cases}$$
We study the solutions of a family of elliptic curves defined over \( \mathbb{Q} \) in the ring of integers of a quadratic field.

Let \( K = \mathbb{Q}(\sqrt{d}) \) and \( s' \) be the conjugate of an element \( s \in K \) over \( \mathbb{Q} \). Further \( R = \mathcal{O}_K[S^{-1}] \), where \( S \) is some finite set of primes in \( \mathcal{O}_K \). Thus \( \mathcal{O}_K \subset \mathcal{O}_K[S^{-1}] \subset K \).

Laska [7] considered the equation (for \( r \in \mathbb{Z} \) and \( r \neq 0 \))

\[
\Gamma_r : y^2 = x^3 - r,
\]

which defines the Weierstrass form of an elliptic curve (call it \( E_r \)) over \( \mathbb{Q} \). For an elliptic curve \( E \) over \( \mathbb{Q} \) the “trace map”

\[
\sigma : E(K) \longrightarrow E(\mathbb{Q})
\]

is given by

\[
\sigma(\mathcal{P}) = \mathcal{P} \oplus \mathcal{P}'.
\]

Here \( \mathcal{P}' \) is the conjugate of the element \( \mathcal{P} \in E(K) \) arising from the conjugation in \( K \) and \( \oplus \) is the usual elliptic curve addition. Laska considered the “trace map” for \( \Gamma_r \) and calls it

\[
\sigma_{r,R} : \Gamma_r(R) \rightarrow \Gamma_r(\mathbb{Q}) \cup \{0\}.
\]

If \( r, R \) are fixed we simply write \( \sigma \) instead of \( \sigma_{r,R} \). The aim was to study \( \sigma^{-1}(\mathcal{O}) \) for a given \( \mathcal{O} \in \Gamma_r(\mathbb{Q}) \cup \{0\} \).

If \( \mathcal{P} = (s, t) \in \Gamma_r(R) \) is a non-exceptional solution then \( s = s' \) and from the Weierstrass equation that implies \( t = \pm t' \). Thus if \( \mathcal{P} = 0 \), then all candidates in \( \sigma^{-1}(\mathcal{O}) \) are non-exceptional. More precisely, one can show that,

\[
\sigma^{-1}(\mathcal{O}) = \{(z^{-1}p, z^{-2}q\theta) : (p, q) \in \Gamma_{z^3}(R \cap \mathbb{Q}), p \in z(R \cap \mathbb{Q}), q \in z^2(R \cap \mathbb{Q})\} \tag{2.1}
\]

where for simplicity it is assumed that \( \theta^{-1} \in R \).

If \( \mathcal{P} \neq 0 \), then the non-exceptional solutions \( \mathcal{P} \) contained in \( \sigma^{-1}(\mathcal{O}) \) are exactly given by the conditions

\[
\mathcal{P} \in \Gamma_r(R \cap \mathbb{Q}), \mathcal{P} \oplus \mathcal{P} = \mathcal{P}.
\]

Thus the non-exceptional solutions of \( \Gamma_r \) in \( R \) which are contained in \( \sigma^{-1}(\mathcal{O}) \) are obtained by solving the equation \( \Gamma_{z^3} \) respectively \( \Gamma_r \) in \( R \cap \mathbb{Q} \). Hence either \( \mathcal{P} \in \Gamma_r(\mathbb{Q}) \) or \( \mathcal{P} \oplus \mathcal{P} = 0 \).
Here we substitute $\Gamma_r$ by $E_{297}$ and $R$ by $O_K$ to study the solutions of (1.2) in $O_K$ by pulling back the elements of $E_{297}(Q)$ using the above mentioned technique.

3. Proof of Theorem 1.1

Proof. Let us first transform the system (1.2) to the desired Weierstrass form $E_{297}$. The system is

$$r + s + t = rst = 2.$$  

Putting the value of $t$ upon simplification it becomes

$$s + r + \frac{2}{rs} = 2.$$  

Now substituting $r = -2/x$ and $s = -y/x$ in the last equation give

$$y^2 + 2y + 2xy = x^3. \quad (3.1)$$

Putting $x = x_1 - 1/2$ and $y = \frac{y_1}{2} - x_1 - \frac{1}{2}$ in (3.1) rids it from the $xy$ term

$$y_1^2 = 4x_1^3 - 2x_1^2 + 7x_1 + \frac{1}{2}. \quad (3.2)$$

Further substituting $x_1 = x_2 + \frac{1}{6}$ and $y_1 = y_2$ rids (3.2) the $x_1^2$ term

$$y_2^2 = 4x_2^3 + \frac{20}{3}x_2 + \frac{44}{27}. \quad (3.3)$$

Now again putting $y_2 = Y_1/27$ and $x_2 = X_1/9$ give

$$Y_1^2 = 4X_1^3 + 540X_1 + 1188. \quad (3.4)$$

Finally substituting $Y_1 = 2Y$ and $X_1 = X$ get us the required Weierstrass form

$$E_{297} : Y^2 = X^3 + 135X + 297. \quad (3.5)$$

Here with the help of SAGE [9] we can conclude that

$$E_{297}(Q)_{\text{tors}} \cong \mathbb{Z}_3 = \{O, (3, \pm 27)\}$$

and the $\mathbb{Q}$-rank of $E_{297}$ is zero (we give a mathematical proof of this fact in §4).

Thus $O$ and $(3, \pm 27)$ are the only $\mathbb{Q}$ rational points of (3.5). It is not difficult to see that the inverse transformation

$$r = 18/(3 - X) \text{ and } s = (Y - 3X - 18)/(3(3 - X))$$

allows us to pass from (3.5) to (1.2). Before proceeding to final leg of the proof of Theorem 1.1 we make a couple of claims and give their proofs.

Claim 1: One of the $(r, s, t)$ satisfying (1.2) must belong to $\mathbb{Q}.$
Proof. (Proof of claim 1) Two cases needed to be considered. 

Case I: \( P \) is non-exceptional. In this case either \( P \in E_{297}(\mathbb{Q}) \) or \( P \oplus \overline{P} = \mathcal{O} \). If \( P \in E_{297}(\mathbb{Q}) \) then this implies \( r \in \mathbb{Q} \).

Let \( P \oplus \overline{P} = \mathcal{O} \) and \( P = (a+b\sqrt{d}, k+l\sqrt{d}) \). As \( P \) is non-exceptional, \( b = 0 \) and \( k = 0 \). Thus \( P = (a, l\sqrt{d}) \) and in this case too

\[ r = \frac{18}{3-a} \in \mathbb{Q}. \]

Case II: Now if \( P \) is exceptional then \( P + \overline{P} = (3, \pm 27) \). The curve \((3.5)\) has exactly three elements over \( \mathbb{Q} \) and the non-trivial elements are of order 3. Lets call them \( \mathcal{O}, \Omega \) and \( 2\Omega \). If \( P + \overline{P} = \Omega \), then clearly \( P + \Omega \) is non-exceptional, since

\[ (P + \Omega) + P + \Omega = \mathcal{P} + P + 2\Omega = 3\Omega = \mathcal{O}. \]

Now if \( P + \overline{P} = 2\Omega \), as before we can show that \( P + 2\Omega \) is also non-exceptional. As the claim is valid for non-exceptional elements, it is true for \( P + \Omega \) and \( P + 2\Omega \). Hence it is true for \( P \) itself.

Hence without loss of generality we assume that \( r \in \mathbb{Q} \).

Claim 2 The only possibilities for \( r \) satisfying the system

\[ r + s + t = rst = 2 \]

in \( \mathcal{O}_K \) are \( \pm 1, 2 \) and \( -8 \).

Proof. (Proof of claim 2) As \( r \in \mathbb{Q} \) and we are looking for solutions in \( \mathcal{O}_K \), this would imply that \( r \in \mathbb{Z} \). Three possibilities needed to be considered:

- If \( r = \pm 1 \). In this case solutions exist.
- If \( r \) is odd. In this case the denominator of \((1.4)\) will be multiple of an odd number but in \( \mathcal{O}_K \) denominator is only 2 or 1 by Theorem 2.1. So in this case there do not exist any solution in \( \mathcal{O}_K \).
- If \( r \) is even. In this case save for \( r = 2 \) and \(-8 \), the denominator of \((1.4)\) will be multiple of 2 and in some other cases denominator is 1 but \( d \equiv 1 \mod 4 \). Thus again by Theorem 2.1 in this case too we don’t (except for \( r = 2, -8 \)) get any solution in \( \mathcal{O}_K \). Thus except these values of \( r = \pm 1, 2 \) and \(-8 \), this system of equation is not solvable in the ring of integers of other quadratic fields.

We are now in a position to complete the proof of Theorem 1.1. We deal with all the four possibilities of \( r \) separately.
When \( r = 1 \), from (3) we have \( s + t = 1 \) and \( st = 2 \). Thus we get
\[
(s, t) = \left( \frac{1 - \sqrt{-7}}{2}, \frac{1 - \sqrt{-7}}{2} \right) \quad \text{and} \quad \left( \frac{1 + \sqrt{-7}}{2}, \frac{1 - \sqrt{-7}}{2} \right).
\]

When \( r = -1 \), we have \( s + t = 3 \) and \( st = -2 \). In this case
\[
(s, t) = \left( \frac{3 - \sqrt{17}}{2}, \frac{3 + \sqrt{17}}{2} \right) \quad \text{and} \quad \left( \frac{3 + \sqrt{17}}{2}, \frac{3 - \sqrt{17}}{2} \right).
\]

Similarly when \( r = 2 \) and \(-8\), we get
\[
(s, t) = (i, -i)(-i, i), \left( \frac{10 + \sqrt{101}}{2}, \frac{10 - \sqrt{101}}{2} \right)
\]
and \( \left( \frac{10 - \sqrt{101}}{2}, \frac{10 + \sqrt{101}}{2} \right) \).

To conclude
\[
(1, \frac{1 - \sqrt{-7}}{2}, \frac{1 + \sqrt{-7}}{2}), (1, \frac{1 + \sqrt{-7}}{2}, \frac{1 - \sqrt{-7}}{2}),
\]
\[
(-1, \frac{3 - \sqrt{17}}{2}, \frac{3 + \sqrt{17}}{2}), (-1, \frac{3 + \sqrt{17}}{2}, \frac{3 - \sqrt{17}}{2}),
\]
\[
(-8, \frac{10 + \sqrt{101}}{2}, \frac{10 - \sqrt{101}}{2})(-8, \frac{10 - \sqrt{101}}{2}, \frac{10 + \sqrt{101}}{2})
\]
are the only solutions of (3) in \( \mathcal{O}_K \). \( \square \)

4. RANK OF \( E_{297} \)

In this section we discuss the rank of \( E_{297} \) over \( \mathbb{Q} \) and over \( \mathbb{Q}(\sqrt{d}) \) for \( d = -7, -1, 17 \) and 101.

Lemma 4.1. The rank of \( E_{297}(\mathbb{Q}) \) is zero.

Proof. If possible let rank \( E_{297}(\mathbb{Q}) \neq 0 \). This would imply that (3.5) have rational solutions. Let \((X, Y) = (\frac{x_1}{x_2}, \frac{y_1}{y_2}) \) with \((x_1, x_2) = (y_1, y_2) = 1\), be one such solution. Now putting the values of \( X \) and \( Y \) in (3.5), we obtain
\[
y_1^2x_2^3 = x_1^3y_2^2 + 135x_1y_2^2x_2^2 + 297y_2^2x_2^3. \quad (4.1)
\]
Let \( p \) be a prime such that \( y_2 = p^\alpha y_{22} \) where \( \alpha \in \mathbb{Z}, \alpha \geq 1 \) and \((p, y_{22}) = 1\). Putting the value of \( y_2 \) in (4.1), gives that right hand side of (4.1) is divisible by \( p \). Therefore \( p \mid y_1x_2 \). Which implies either \( p \mid y_1 \) or \( p \mid x_2 \). Suppose \( p \mid y_1 \) then since \( y_2 = p^\alpha y_{22} \), we get a contradiction to the fact that \((y_1, y_2) = 1\). Thus \( p \mid x_2 \) and we write
\[ x_2 = p^\beta x_{22} \text{ where } \beta \in \mathbb{Z}, \beta \geq 1 \text{ and } (p, x_{22}) = 1. \]

Now putting this \( y_2 \) and \( x_2 \) in (4.1), gives
\[
p^{3\beta} x_{22}^3 y_1^2 = p^{2\alpha} y_{22}^2 x_1^3 + 135 x_1 p^{2\alpha+2\beta} y_{22}^2 x_{22}^2 + 297 p^{2\alpha+3\beta} y_{22}^2 x_{22}^3. \tag{4.2}
\]

Three cases can occur.

Case I. Suppose \( 3\beta > 2\alpha \). In this case,
\[
p^{3\beta-2\alpha} x_{22}^3 y_1^2 = y_{22}^2 x_1^3 + 135 x_1 p^{2\beta} y_{22}^2 x_{22}^2 + 297 p^{3\beta} y_{22}^2 x_{22}^3. \tag{4.3}
\]

Further from (4.3) \( p \mid y_{22} x_1 \) and forces \( p \mid x_1 \) as \( (p, y_{22}) = 1 \), which is a contradiction to the fact that \( (x_1, x_{22}) = 1 \).

Case II. Suppose \( 3\beta > 2\alpha \). In that case \( p \mid y_1 \), which is also contradiction to the fact that \( (y_1, y_2) = 1 \).

Case III. This case is analogous to Case I.

Hence (4.3) has no solution in \( \mathbb{Z} \) and that in turn implies (5.5) has no solution in \( \mathbb{Q} \). Thus the rank of \( E_{297}(\mathbb{Q}) \) defined by the equation (3.5) is zero. \( \square \)

**Lemma 4.2.** The rank of \( E_{297}(\mathbb{Q}(\sqrt{d})) \) for \( d = -7, 17 \) is 1 and for \( d = 101 \) it is 2.

**Proof.** Let \( E/K \) be an elliptic curve and \( d \in K^* \) be such that \( L = K(\sqrt{d}) \) is a quadratic extension. Let \( E_d/K \) be the twist of \( E/K \), then by [5],
\[
\text{rank } E(L) = \text{rank } E(K) + \text{rank } E_d(K). \tag{4.4}
\]

We conclude using SAGE [9] that (already we have noted that rank \( E_{297}(\mathbb{Q}) = 0 \) rank \( E_{-7,297}(\mathbb{Q}) = 1 \), rank \( E_{-1,297}(\mathbb{Q}) = 1 \), rank \( E_{17,297}(\mathbb{Q}) = 1 \) and rank \( E_{101,297}(\mathbb{Q}) = 2 \). Now using (4.4) we have,
\[
\text{rank } (E(\mathbb{Q}(\sqrt{-7}))) = \text{rank } (E(\mathbb{Q}(\sqrt{-1}))) = \text{rank } (E(\mathbb{Q}(\sqrt{17}))) = 1.
\]

and rank \( (E(\mathbb{Q}(\sqrt{101}))) = 2 \) \( \square \)

5. Concluding remarks

We showed that the system
\[ r + s + t = rst = 2 \]
has no solution in \( \mathcal{O}_K \) except for \( d = -7, -1, 17 \) and 101 and the solutions are explicitly given.

It would of interest to consider the next case, i.e,
\[ r + s + t = rst = 3. \tag{5.1} \]
Suitable change of variables transform (5.1) to an elliptic curve with Weierstrass form

\[ E_{13122} : y^2 = x^3 + 3645x - 13122. \]

Torsion of \( E_{13122} \) over \( \mathbb{Q} \) is isomorphic to \( \mathbb{Z}_3 \) and it’s rank is zero (using SAGE [9]).

We can conclude that except for \( d = -2, -1, 7, 10 \) and 13, the system (5.1) has no solutions in the ring of integers of \( K \) and the solutions can be explicitly given (following analogous argument). Analogously other individual cases can be treated.

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