Moduli of (1,7)-polarized abelian surfaces via syzygies

Dedicated to the memory of Alf B. Aure

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Abstract. We prove that the moduli space \(X(1,7)\) of (1,7)-polarized abelian surfaces with canonical level-structure is birational to the Fano 3-fold \(V_{22}\) of polar hexagons of the Klein quartic \(\mathcal{X}(7)\). In particular \(X(1,7)\) is rational and the birational map to \(\mathbb{P}^3\) is defined over \(\mathbb{Q}\). As a byproduct we obtain explicitly the equations of the (1,7)–very-ample-polarized abelian surfaces embedded in \(\mathbb{P}^6\).

0. Introduction

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0. Introduction.

Moduli spaces of polarized abelian varieties is a much studied subject. The common approach to their construction is as arithmetic quotient of the Siegel upper half space. Their study involves then the beautiful subject of modular forms. In this paper we follow a different approach. We construct them as Heisenberg invariant part of a Hilbert scheme.

In the particular case of (1,7)-polarized abelian surfaces we can obtain in this way only a birational model of the moduli space, because not every polarization is very ample. However our method goes quite far: We obtain a birational parametrization of the moduli space defined over \(\mathbb{Q}\).

Moreover we discovered a new relation between the moduli space \(X(1,7)\) and the modular curve \(X(7)\) of elliptic curves with a level 7 structure. The variety of sums of 6 powers of the Klein quartic \(\mathcal{X}(7)\), i.e. the variety of polar hexagons of the Klein quartic, is our model of \(X(1,7)\).

The paper is organised as follows, each section devoted to one basic idea. In section 1 we review the construction of the moduli space as invariant part of the Hilbert scheme and collect some basic notations.

In section 2 we recall that a very ample line bundle of class \((1,7)\) embeds an abelian surface projectively normal and study its syzygies. Due to \(H^1(A, \mathcal{O}) \neq 0\) the minimal free resolution is longer than the codimension. However if we allow a locally free resolution, there is a rather natural self-dual resolution \(F\). A result like this should hold quite generally for Gorenstein subvarieties of smooth manifolds, whose canonical bundle is induced, cf. [EPW], [W] for some results in this direction.

Section 3 brings in the action of the Heisenberg group. With this, the middle syzygy map boils down to a \(3 \times 2\) matrix, which can be interpreted as the Hilbert-Burch matrix (cf. [E] Thm 20.15) of a twisted cubic in a certain \(\mathbb{P}^3\). The complex condition on \(F\) gives certain linear relations among the coefficients of the defining quadratic equations of this twisted cubic. This is enough to determine our model of the moduli space: It is a particular Fano 3-fold \(V_{22} \subset \mathbb{P}^{13}\) of degree 22.

In section 4 we recall the various descriptions of a \(V_{22}\), one of them being the variety of sums of powers of a plane quartic curve. For our \(V_{22}\) this is the Klein quartic. We finish with the birational map \(\mathbb{P}^3 \rightarrow X(1,7)\) induced by the triple projection from a particular point of the \(V_{22}\).

The Appendix contains some formulas of the representation theory of the Heisenberg group \(H_7\) and \(SL_2(\mathbb{Z}_7)\).

In several aspects we are not yet completely satisfied with our results here.

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1. A detailed study of the geometry of the surfaces (over boundary points) comparable to the study of Barth, Hulek and Moore [BHM] of $X(1,5)$ via the Horrocks-Mumford bundle is missing.

2. Points on our $V_{22}$ parametrize a family of 10-nodal Kummer quartics in $\mathbb{P}^3$. Could it be that every $V_{22}$ parametrizes some 10-nodal quartics?

However the paper had already some fruits: As observed first by Alf Aure and Kristian Ranestad, conics on the $V_{22}$ correspond to pencils of abelian surfaces which sweep out a Calabi-Yau 3-fold $Y$ of degree $14$. In some sense the study of pairs $\{(A,Y) \mid A \subset Y\}$ of abelian surfaces contained in some Calabi-Yau is easier than studying $A$’s alone. This point of view turned out to be the key in the solution of Gross and Popescu [GP] of the problem posed by Gritsenko [Gr] to decide, which Siegel modular 3-folds are rational.

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Notation. Most of the notation will be introduced directly in the text. We recall here only some of it and also some notation which will be used tacitly:

- $A$ an abelian variety
- $\mathcal{L}$ an ample line bundle of type $(1,7)$ on $A$
- $\hat{A} = \text{Pic}^0(A)$
- $t_x$ is the automorphism of translation by $x \in A$
- $V^*$ for the dual of a vector space $V$
- $UV$ or $U \cdot V$ for $U \otimes V$, where $U$, $V$ are vector spaces
- $nV$ for $\bigoplus^n V$
- $G(k,V)$ for the Grassmann variety of $k$ dimensional subspaces of the vector space $V$
- $\mathbb{P}(V)$ the projective space of lines in $V$.

We shall use the Macaulay short hand notation for numerical data of a free resolutions over the graded polynomial ring $R$ and for its sheafified version. A table like

\[
\begin{array}{ccc}
1 & - & - & - \\
- & 7 & 8 & - \\
- & - & 3 & 8 & 3 \\
\end{array}
\]

stands for a complex

\[ R \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow 0 \]

with 5 terms $F_0 = R$ and $F_i = \bigoplus_{k=1}^{r_i} R(-a_{ik})$ for $i = 1, \ldots, 4$ where the number of generators of $F_i$ in a given degree are encoded by the numbers in the $i^{th}$ column. More precisely the number in position $(i,-j)$ in the table is the number of generators of degree $i+j$ of $F_i$. In the example above, the image of $F_1 = 7R(-2)$ is an ideal generated by 7 quadrics, which have 8 linear syzygies and further 3 quadratic syzygies, corresponding to $F_2 = 8R(-3) \oplus 3R(-4); F_3 = 8R(-5), F_4 = 3R(-6)$.

Notice that the entries of a block in a syzygy matrix above corresponding to two consecutive numbers in the same line are linear, while the maps to the upper left and from the lower right corners of a square are quadratic. Examples: The syzygies of the twisted cubic in $\mathbb{P}^3$ have shape

\[
\begin{array}{ccc}
1 & - & - \\
- & 3 & 2 \\
\end{array}
\]

The syzygies of a plane cubic in $\mathbb{P}^3$ union a point, which is not in that plane, look like

\[
\begin{array}{ccc}
1 & - & - \\
- & 3 & 3 & 1 \\
- & 1 & 1 \\
\end{array}
\]

1. Review. We review some well known facts about abelian varieties following [Mum66] and [Mum70] (see also [LB]).
(1.1) Let $A$ be an abelian variety with an ample line bundle $\mathcal{L}$, 
\[ \phi_{\mathcal{L}} : A \to Pic^0(A), \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \]
the associated group homomorphism, and let $K_{\mathcal{L}}$ be the kernel of $\Phi_{\mathcal{L}}$. $K_{\mathcal{L}}$ is finite, since $\mathcal{L}$ is ample (cf. [Mum70], II.6. Application 1). Moreover, by the theorem of the cube, $\Phi_{\mathcal{L}}$ is a skew bilinear map $L \to L = \mathbb{C}^*$. Since $K_{\mathcal{L}}$ is finite, we may replace $\mathbb{C}^*$ by the finite group $\mu_{K_{\mathcal{L}}}$ generated by the image of $e^L$. Moreover $| \mu_{K_{\mathcal{L}}} |$ is a divisor of the exponent of $K_{\mathcal{L}}$. The pushout of a certain central extension group $G_{\mathcal{L}}$ is called the (infinite) Heisenberg group of $\mathcal{L}$:
\[ 1 \to \mathbb{C}^* \to G_{\mathcal{L}} \to K_{\mathcal{L}} \to 0 \]
We will work mainly with a finite version of $G_{\mathcal{L}}$: Since $G_{\mathcal{L}}$ is a central extension of an abelian group, taking commutators induces a skew bilinear map 
\[ e^L : K_{\mathcal{L}} \times K_{\mathcal{L}} \to \mathbb{C}^*, \quad e^L(x,y) = \tilde{x} \cdot \tilde{y} \cdot \tilde{x}^{-1} \cdot \tilde{y}^{-1} \quad (\tilde{x}, \tilde{y} \in G_{\mathcal{L}} \text{ preimages of } x, y) . \]
Since $K_{\mathcal{L}}$ is finite, we may replace $\mathbb{C}^*$ by the finite group $\mu_{K_{\mathcal{L}}}$ generated by the image of $e^L$. Moreover $| \mu_{K_{\mathcal{L}}} |$ is a divisor of the exponent of $K_{\mathcal{L}}$. The (infinite) Heisenberg group $H_{\mathcal{L}}$ is the pushout of a certain central extension:
\[ 1 \to \mu_{K_{\mathcal{L}}} \to H_{\mathcal{L}} \to K_{\mathcal{L}} \to 0 \]
\[ 1 \to \mathbb{C}^* \to G_{\mathcal{L}} \to K_{\mathcal{L}} \to 0 . \]
(1.2) An explicit description of $H_{\mathcal{L}}$ uses more notation. In the analytic context $c_1(\mathcal{L}) \in H^2(A^{an}, \mathbb{Z}) \cong \Lambda^2 Hom(\Gamma, \mathbb{Z})$, where $\Gamma$ is a lattice in $\mathbb{C}^g$ such that $A \cong \mathbb{C}^g / \Gamma$. Hence $c_1(\mathcal{L})$ is an alternating nondegenerated 2-form on $\Gamma$ with values in $\mathbb{Z}$, (cf. [Mum70] p. 16). In a convenient basis of $\Gamma$ this alternating form can be written as a matrix:
\[ c_1(\mathcal{L}) = \left( \begin{array}{cc} 0 & \Delta \\ -\Delta & 0 \end{array} \right) \]
where $\Delta$ is a diagonal matrix $(d_1, \ldots, d_g)$ of positive integers such that $d_1 | d_2 | \ldots | d_g$. The collection of elementary divisors $(d_1, \ldots, d_g) =: \delta$ is called the type of the line bundle $\mathcal{L}$.

The diagram
\[ 0 \to \Gamma \to \mathbb{C}^g \to A \to 0 \]
\[ c_1(\mathcal{L}) \quad \downarrow \cong \quad \downarrow \Phi_{\mathcal{L}} \]
\[ 0 \to \hat{\Gamma} \to \hat{\mathbb{C}}^g \to \hat{A} \to 0 \]
yields an isomorphism $K_{\mathcal{L}} \cong \hat{\Gamma} / \Gamma \cong (\mathbb{Z}/d_1 \mathbb{Z})^2 \oplus \ldots \oplus (\mathbb{Z}/d_g \mathbb{Z})^2$.

In the algebraic setting the elementary divisors of $K_{\mathcal{L}}$ define the type: $e^L$ is nondegenerated and produces a decomposition of $K_{\mathcal{L}}$ as a direct sum of two subgroups $K_1, K_2$, where $K_2 \cong K_1 \cong Hom(K_1, \mathbb{C}^*)$, (cf. [Mum66], p. 293). The most convenient description of $H_{\mathcal{L}}$, up to isomorphism, is as the following extension $H_\delta$:
\[ 1 \to \mu_{d_1} \to H_\delta \to ((\mathbb{Z}/d_1 \mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/d_g \mathbb{Z})) \oplus (\mu_{d_1} \times \ldots \times \mu_{d_g}) \to 0 , \]
where $\mu_{d} = Hom(\mathbb{Z}/d \mathbb{Z}, \mathbb{C}^*)$ is the group of the $d$-th roots of unity. If we write the last term in the above exact sequence as $K(\delta) \oplus \hat{K}(\delta)$, with $K(\delta) = Hom(K(\delta), \mathbb{C}^*)$, then the multiplication in $H_\delta = \mu_{d_1} \oplus K(\delta) \oplus \hat{K}(\delta)$ is defined by
\[(\alpha, x, \rho) \circ (\beta, y, \sigma) = (\alpha \cdot \beta \cdot x + y, \rho \cdot \sigma) .\]
On $K(\delta) \oplus \tilde{K}(\delta)$ one has also naturally an alternate multiplicative form $e^\delta$. If $e_1, \ldots, e_g$ and $e_1^*, \ldots, e_g^*$ are the canonical basis of $K(\delta)$ and $\tilde{K}(\delta)$ respectively then

$$e^\delta : (K(\delta) \oplus \tilde{K}(\delta)) \times (K(\delta) \oplus \tilde{K}(\delta)) \to \mu_{d_g}$$

is defined by:

$$e^\delta(e_r^*, e_r) = (e^\delta(e_r, e_r^*))^{-1} = \exp(2\pi i/d_r), \text{ for all } r = 1, \ldots, g \text{ and } 1 \text{ otherwise}$$

(1.3) $H^0(A, L)$ is an irreducible $H_\delta$-module. The argument uses that $K(\delta)$ lifts to a subgroup of $H_\delta$, and that $L$ descends to $A/K(\delta)$, (cf. [Mum66], pp. 290, 297). $H^0(A, L)$ is the unique irreducible representation of $H_\delta$, on which the center $\mu_\mathbb{C} \subseteq \mathbb{C}^*$ acts by scalar multiplication. This $(\prod_{i=1}^{g} d_i)$-dimensional representation $V$ is called the Schrödinger representation of $H_\delta$.

(1.4) Definition. ([Mum66]) A theta–structure on the pair $(A, \mathcal{L})$ is any isomorphism $\alpha : H_\mathcal{L} \to H_\delta$ which induces the identity on the centers $\mu_\mathcal{L}$ and $\mu_{d_g}$ viewed as subgroups of $\mathbb{C}^*$. A theta–structure induces a level-structure of canonical type i.e. a symplectic isomorphism $\alpha' : K_\mathcal{L} \to K_\delta$, symplectic with respect to $e^\mathcal{L}$ and $e^\delta$.

Since $\Phi_\mathcal{L}$ depends only on the numerical equivalence class of $\mathcal{L}$ we can speak of a level-structure for a polarized abelian variety.

The reason to consider level-structures is the following:

(1.5) Theorem. (Mumford) There exists a coarse moduli space $\mathcal{M}(\delta)$ for ample polarized abelian varieties with level-structure of type $\delta = (d_1, \ldots, d_g)$.

(1.6) If $d_1 \geq 3$ the proof of this is elementary: By Lefschetz's theorem an ample line bundle $\mathcal{L}$ of type $\delta = (d_1, \ldots, d_g)$ with $d_1 \geq 3$ is very ample. The irreducibility of the Schrödinger representation $V$ and the level-structure on $(A, \mathcal{L})$ gives a canonical identification $\mathcal{P}(V) = \mathbb{P}(H^0(A, \mathcal{L}))$ by Schurs lemma. Thus every pair $(A, L)$ with level structure of type $\delta$ occurs as a point in the Hilbert scheme $\text{Hilb}(\mathbb{P}(V))$. Moreover $A \subseteq \mathcal{P}(V)$ is $H_\delta$-invariant. So we can take as $\mathcal{M}(\delta)$ an open part of an irreducible component of the fixpoint set $\text{Hilb}(\mathbb{P}(V))^{H_\delta}$. To see that this has the right (in fact reduced) scheme structure we compare the tangent spaces:

$$T_{\text{Hilb}^{H_\delta}, A} = H^0(A, N_A)^{H_\delta} \cong \text{Im}(H^0(A, N_A) \to H^1(A, T_A)).$$

Indeed, from the exact sequences:

$$0 \to T_A \to T \to N_A \to 0$$

$$0 \to \mathcal{O}_A \to V \mathcal{O}_A(1) \to T \to 0$$

where $T$ is the restriction to $A$ of the tangent bundle of $\mathbb{P}(V)$, one deduces the exact sequence:

$$0 \to H^0(T_A) \to H^0(T) \to H^0(N_A) \to H^1(T_A) \to \ldots$$

$$\begin{array}{c|c|c}
| gI & gI \oplus Z & I \\
\hline
\end{array}$$

where $I$ is the trivial 1–representation of $H_\delta$ and $Z$ is the complement of $gI$ in $H^0(T)$. This shows that that the image of $H^0(N_A) = Z \oplus H^0(N_A)^{H_\delta}$ in $H^1(T_A)$ is $H^0(N_A)^{H_\delta}$. $\Box$

(1.7) Remarks. (1) Notice that the universal family over the Hilbert scheme is not necessarily a universal family in the sense of moduli, because, as subscheme, $A$ has no distinguished origin. Indeed picking the origin appropriate we obtain that $\mathcal{O}(1)$ restricts to any desired line bundle $\mathcal{L}$ within the polarization class $c_1(\mathcal{L})$.

(2) In general $\text{Hilb}(\mathbb{P}(V))^{H_\delta}$ has many components. For example in the elliptic curve case, only if $d_1$ is prime, there is a single component. For composed numbers, there are several components whose points correspond to union of $d_1/n$ elliptic curves of degree $n$, for every divisor $n$ of $d_1$, (cf. [P]).

4
2. Syzygies.

(2.1) According to a theorem of Reider, a line bundle of type \((1,d)\), \(d \geq 5\) on a general abelian surface is very ample (cf. [R] or [LB] p. 301-302). In this section we describe the syzygies of an abelian surface \(A \subset \mathbb{P}^6\) embedded by an very ample line bundle of type \((1,7)\). Although the canonical class \(\omega_A\) is induced from \(\mathbb{P}^6\) the minimal free resolution is not symmetric, since the coordinate ring is not projectively Cohen-Macaulay. However the only obstacle for this is \(H^1(A, \mathcal{O})\). Using also some locally (not globally) free sheaves in the resolution we obtain a nice self-dual resolution.

(2.2) Let \((A, \mathcal{L})\) be an abelian surface with a very ample line bundle of type \((1,7)\), and consider its image

\[ A \hookrightarrow \mathbb{P}^6 = \mathbb{P}(V), \]

as an \(H_7 = H_{(1,7)}\)-invariant subvariety.

(2.3) **Lemma.** \(A \subset \mathbb{P}^6\) is not contained in a quadric.

**Proof.** If a quadric would contain \(A\) then \(h^0(\mathbb{P}^6, \mathcal{I}_A(2)) \geq 7\), since \(H^0(\mathbb{P}^6, \mathcal{O}(2))\) is a direct sum of irreducible representations of dimension 7, cf. Appendix. This is too much: By Castelnuovo’s argument the 14 points \(Z = A \cap \mathbb{P}^4\) for a general \(\mathbb{P}^4 \subset \mathbb{P}^6\) impose at least 9 conditions on quadrics, i.e. \(h^0(\mathbb{P}^4, \mathcal{I}_Z(2)) \leq 6\).

On the other hand \(h^0(\mathbb{P}^4, \mathcal{I}_Z(2)) = h^0(\mathbb{P}^6, \mathcal{I}_A(2)) + 2 \geq 9\) by the exact sequence

\[ 0 \to \mathcal{I}_A \to 2\mathcal{I}_A(1) \to \mathcal{I}_A(2) \to \mathcal{I}_{Z, \mathbb{P}^6}(2) \to 0, \]

a contradiction. \(\Box\)

(2.4) **Corollary.** \(A \subset \mathbb{P}^6\) has syzygies:

\[
\begin{array}{cccccc}
1 & - & - & - & - & - \\
- & - & - & - & - & - \\
- & 21 & 49 & 42 & 14 & 2 \\
- & - & - & - & 1 & - \\
\end{array}
\]

**Proof.** The map \(H^0(\mathbb{P}^6, \mathcal{O}(2)) \to H^0(A, \mathcal{L} \otimes 2)\) is injective by the Lemma, hence an isomorphism, because both spaces are 28-dimensional. So \(H^1(\mathbb{P}^6, \mathcal{I}_A(2)) = 0\), i.e. \(A \subset \mathbb{P}^6\) is quadratically normal. By the above: \(h^0(\mathbb{P}^4, \mathcal{I}_Z(2)) = 2\). So \(Z \subset \mathbb{P}^4\) has the Hilbert function

\[
(1,5,13,14,14,\ldots),
\]

because its different function has no negative value, being the Hilbert function of an artinian ring. It follows \(H^1(\mathbb{P}^4, \mathcal{I}_Z(n)) = 0\) for \(n \geq 3\) and induction with the sequence (2.3.1) gives: \(A \subset \mathbb{P}^6\) is projectively normal.

Moreover \(\mathcal{I}_A\) is 4-regular in the sense of Castelnuovo-Mumford and nonzero syzygy numbers can only be in the indicated range, some of whose values are clear:

\[
\begin{array}{cccccc}
1 & - & - & - & - & - \\
- & - & - & - & - & - \\
- & 21 & ? & ? & ? & 2 \\
- & ? & ? & i & 1 & - \\
\end{array}
\]

Namely, the number of cubic generators of the ideal is \(h^0(\mathbb{P}^6, \mathcal{I}_A(3)) = h^0(\mathbb{P}^6, \mathcal{O}(3)) - h^0(A, \mathcal{O}_A(3)) = 21\). The last 2 represents \(h^1(\mathcal{O}_A)\) and the 1 comes from the facts that \(\omega_A \cong \mathcal{O}_A\) and that dualizing the above resolution one obtains \(\mathcal{E}xt^2_{\mathcal{O}_A}(\mathcal{O}_A, \omega_A) \cong \omega_A\).

The last argument gives also \(i = 0\), since \(\mathcal{E}xt^i_{\mathcal{O}_A}(\mathcal{O}_A, \omega_A) = 0\) for \(i \leq 3\) and \(A\) is non-degenerate. Now all the vanishing is clear, and the nonzero values can be computed from the Hilbert function. \(\Box\)

The resolution has \(\text{length} > \text{codim} A\), since \(A \subset \mathbb{P}^6\) is not arithmetically Cohen-Macaulay. In particular it is not symmetric. However, if we allow locally free sheaves instead of only direct sums of line bundles, then there is a nice self-dual resolution:
(2.5) **Theorem.** \( A \subset \mathbb{P}^6 \) has a self-dual resolution of type

\[
0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 21\mathcal{O}(-3) \xleftarrow{\alpha} 2\Omega^3 \leftarrow 21\mathcal{O}(-4) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0
\]

with \( \alpha' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \beta' = t\beta \).

**Proof.** The above resolution is obtained by a kind of subtracting a piece of the Koszul sequence multiplied with \( h^1(A, \mathcal{O}_A) \) from the resolution in Corollary (2.4). Let \( K \) be the kernel of the map \( \mathcal{O} \leftarrow 21\mathcal{O}(-3) \) in the resolution in (2.4). Then comparing the Koszul resolution of the \( \mathbb{C} \) vector space \( H^1(A, \mathcal{O}_A)^* \cong \text{Ext}_{S}^{5}(S, S(-7)) \) with the dual complex of (2.4), yields a commutative diagram with exact rows and columns:

\[
\begin{array}{c}
0 \\
\downarrow \\
21\mathcal{O}(-4) \\
\downarrow \\
0 \leftarrow 2 \cdot \Omega^3 \leftarrow 2 \cdot 35\mathcal{O}(-4) \leftarrow 2 \cdot 21\mathcal{O}(-5) \leftarrow 2 \cdot 7\mathcal{O}(-6) \leftarrow 2 \cdot \mathcal{O}(-7) \leftarrow 0 \\
\downarrow \\
0 \leftarrow K \leftarrow 49\mathcal{O}(-4) \leftarrow 42\mathcal{O}(-5) \leftarrow 14\mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2\mathcal{O}(-7) \leftarrow 0 \\
\downarrow \\
0 \leftarrow \mathcal{O}(-7) \\
\downarrow \\
0
\end{array}
\]

The map \( 2 \cdot 21\mathcal{O}(-5) \rightarrow 42\mathcal{O}(-5) \) is surjective, and \( \ker(2 \cdot 35\mathcal{O}(-4) \rightarrow 49\mathcal{O}(-4)) \cong 21\mathcal{O}(-4) \), because otherwise \( A \subset \mathbb{P}^6 \) would be contained in a quadric, or more then 21 cubics. A diagram chase gives the desired resolution. To see the assertions about the maps, we compare this complex with its dual:

\[
\begin{array}{c}
0 \\
\downarrow \phi \\
\downarrow \\
\downarrow u \\
0 \leftarrow \mathcal{O}_A \\
\downarrow \phi \\
\downarrow \\
\downarrow u \\
0 \leftarrow \omega_A \\
\downarrow \phi \\
\downarrow \\
\downarrow u \\
0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 21\mathcal{O}(-3) \xleftarrow{\alpha} H^1(\mathcal{O}_A) \otimes \Omega^3 \xleftarrow{\beta'} 21\mathcal{O}(-3) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0 \\
\downarrow \phi \\
\downarrow \\
\downarrow u \\
\downarrow \phi \\
\downarrow \\
\downarrow u \\
0 \leftarrow \omega_A \leftarrow \mathcal{O} \xleftarrow{\beta'} 21\mathcal{O}(-3) \xleftarrow{\alpha'} H^1(\mathcal{O}_A)^* \otimes \Omega^3 \xleftarrow{\beta} 21\mathcal{O}(-3) \xleftarrow{\beta} \mathcal{O}(-7) \leftarrow 0
\end{array}
\]

The isomorphism \( u \) is compatible with Serre duality. The diagrams:

\[
\begin{array}{c}
H^1(\mathcal{O}_A) \cong H^1(\mathcal{O}_A) \otimes \mathbb{C} \quad H^1(\mathcal{O}_A) \otimes H^1(\mathcal{O}_A) \twoheadrightarrow H^2(\mathcal{O}_A) \\
H^1(\phi) \downarrow \phi \quad H^1(\omega_A) \cong H^1(\mathcal{O}_A)^* \otimes \mathbb{C} \quad H^1(\mathcal{O}_A) \otimes H^1(\omega_A) \twoheadrightarrow H^2(\omega_A)
\end{array}
\]

commute. So the map in the first row of the last diagram is antisymmetric, as the one in the second row is. This gives the relations between \( \alpha', \beta' \) and \( \alpha \), respectively \( \beta \). \( \square \)
3. Symmetry. Taking into account the symmetries of \((A, L)\) we have the following \(H_7\) or \(G_7\)-invariant resolutions.

(3.1) Corollary. Taking the canonical \(H_7\)-invariant embedding of \(A\) corresponding to the taken polarization and the level structure, one gets:

\[
0 \leftarrow \mathcal{I}_A \leftarrow 3V_2\mathcal{O}(-3) \leftarrow 7V_1\mathcal{O}(-4) \leftarrow 6V_2\mathcal{O}(-5) \leftarrow 2V\mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2\mathcal{O}(-7) \leftarrow 0
\]

If one considers an \(G_7\)-invariant embedding one obtains:

\[
0 \leftarrow \mathcal{I}_A \leftarrow 3V_2\mathcal{O}(-3) \leftarrow (5V_1 \oplus 2V_1^2)\mathcal{O}(-4) \leftarrow 6V_2^2\mathcal{O}(-5) \leftarrow 2V^2\mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2S\mathcal{O}(-7) \leftarrow 0
\]

Proof. Everything follows using the tables from the appendix. \(\square\)

(3.2) Theorem. An abelian surfaces \(G_7\)-invariantly embedded in \(\mathbb{P}^6\) has an \(G_7\)-invariant resolution of the form:

\[
0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O}^\beta \leftarrow 3V_2\mathcal{O}(-3) \leftarrow \alpha \leftarrow 2S\Omega^3 \leftarrow 3V_1\mathcal{O}(-4) \leftarrow 2S\mathcal{O}(-7) \leftarrow 0
\]

with \(\alpha' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \beta' = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}.

Proof. The big diagram in the proof of theorem 2.5 reads as \(G_7\)-modules

\[
\begin{array}{ccc}
0 & \\
\downarrow & \\
3V_1\mathcal{O}(-4) & 0 \\
\downarrow & \\
0 & 2S \cdot 3V_2\mathcal{O}(-3) & 2S \cdot 3V_1\mathcal{O}(-4) & 2S \cdot \mathcal{O}(-7) & 0 \\
\downarrow & || & \downarrow & |
\end{array}
\]

\[
\begin{array}{ccc}
0 & \\
\downarrow & \\
K & 0 \\
\downarrow & \\
0 & \mathcal{O}(-7) \\
\downarrow & \\
0 & \\
\end{array}
\]

and the result is clear. \(\square\)

Next we view \(\alpha\) as a \(3 \times 2\)-matrix with entries in \(\text{Hom}(S\Omega^3, V_4\mathcal{O}(-3))\). Since \(\alpha\) defines a \(G_7\)-morphism the entries lie in the \(G_7\)-invariant part.

(3.3) Proposition.

\(\text{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3)) = 4I,\)

i.e. \(\alpha\) has entries in a 4-dimensional vector space.
Proof. Hom(Ω⁴, O(−3)) ≅ Λ³V = V₁ ⊕ 4V₁². Hence Hom(SΩ³, V₄O(−3)) ≅ V₄ ⊗ (V₁² ⊕ 4V₁) = 4I ⊕ S ⊕ 5Z and Hom_{G₇}(SΩ³, V₄O(−3)) = 4I.

(3.4) Remark. If F₁ and F₂ are two G₇-sheaves, Hom_{G₇}(F₁, F₂) is a N-module, because G₇ = H₇ × Z₂ is a normal subgroup of N ≅ H₇ × SL₂(ℤ₇), ɳ being central in SL₂(ℤ₇).

Using the character table of SL₂(ℤ₇), one sees that Hom_{G₇}(SΩ³, V₄O(−3)) ≅ U', with the notation from the appendix.

For the following considerations we choose a basis u₀, ..., u₃ of U', so that in the decomposition into irreducible G₇-modules of Λ³V = V₁ ⊕ (U' ⊗ V₁), u₀, ..., u₃ correspond to the V₁ pieces generated as a H₇-module by e₁ ∧ e₂ ∧ e₃, e₀ ∧ e₁ ∧ e₄, e₀ ∧ e₂ ∧ e₅, or e₀ ∧ e₄ ∧ e₃, respectively. Then, in the above decomposition of Λ³V, V₁ is generated as a H₇-module by e₁ ∧ e₂ ∧ e₃ + eₙ ∧ e₃ ∧ e₅. More precisely, the above elements correspond to u₀ ⊗ e₀, ..., u₃ ⊗ e₀ in U' ⊗ V₁ and the elements uₙ ⊗ eₙ are obtained permuting the indices of eₙ's via ɳ.

With these notations the matrix α = (aᵢₖ) will have entries (aᵢₖ) = ∑ₖ=₀ aₖᵢuₖ. We want to express more conveniently the condition αα' = 0, where α' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} α.

(3.5) Proposition. A matrix α as above satisfies αα' = 0 iff the three quadrics in P³ = P(U) given by its 2 × 2-minors are annihilated by each of the three operators:

\[ Δ₁ = \frac{∂^2}{∂u₀∂u₁} - \frac{1}{2} \frac{∂^2}{∂u₀∂u₂}, \quad Δ₂ = \frac{∂^2}{∂u₀∂u₂} - \frac{1}{2} \frac{∂^2}{∂u₀∂u₃}, \quad Δ₃ = \frac{∂²}{∂u₀∂u₃} - \frac{1}{2} \frac{∂²}{∂u₀∂u₁}. \]

Proof. We use the fact that under the identifications Hom(O(−4), Ω³) = Λ³V, Hom(Ω³, O(−3)) = Λ³V the composition of two maps O(−4) → Ω³, Ω³ → O(−3) is given by wedge product, if we identify canonically ∧₆V with V∗ = V₃ = H⁰(O(1)).

Observe now that u₀ interpreted as an element in Hom(V₁O(−4), SΩ³) is given by the following 1 × 7 matrix with entries in Λ³V:

\[ u₀ = (e₁₊ₖ ∧ e₄₊ₖ ∧ e₂₊ₖ - e₆₊ₖ ∧ e₃₊ₖ ∧ e₅₊ₖ) ∈ Z₇ \]

and similarly:

\[ u₁ = (eₖ ∧ e₁₊ₖ ∧ e₆₊ₖ), \quad u₂ = (eₖ ∧ e₂₊ₖ ∧ e₅₊ₖ), \quad u₃ = (eₖ ∧ e₄₊ₖ ∧ e₃₊ₖ). \]

The same elements, interpreted in Hom(SΩ³, V₄O(−3)), will be identified with the transpose of the above ones. Then the only compositions of two uᵢ's which are not 0 are:

\[ u₀u₁ = u₁u₀ = -u₂u₂ = B₁ := \begin{pmatrix} 0 & x₄ & 0 & 0 & 0 & 0 & -x₃ \\ -x₄ & 0 & x₅ & 0 & 0 & 0 & 0 \\ -x₅ & 0 & x₆ & 0 & 0 & 0 & 0 \\ 0 & 0 & -x₆ & 0 & x₀ & 0 & 0 \\ 0 & 0 & 0 & -x₀ & 0 & x₁ & 0 \\ 0 & 0 & 0 & 0 & -x₁ & 0 & x₂ \\ x₃ & 0 & 0 & 0 & 0 & -x₂ & 0 \end{pmatrix} \]

\[ u₀u₂ = u₂u₀ = -u₃u₃ = B₂ := \begin{pmatrix} 0 & 0 & x₁ & 0 & 0 & 0 & -x₆ \\ 0 & 0 & 0 & x₂ & 0 & 0 & -x₀ \\ -x₁ & 0 & 0 & 0 & x₃ & 0 & 0 \\ 0 & -x₂ & 0 & 0 & 0 & x₄ & 0 \\ 0 & 0 & -x₃ & 0 & 0 & 0 & x₅ \\ x₆ & 0 & 0 & -x₄ & 0 & 0 & 0 \\ 0 & x₀ & 0 & 0 & -x₅ & 0 & 0 \end{pmatrix} \]
identically zero, there is up to scalar a unique set of cubics whose relations they are: In each case the matrices have rank 6 in a general point of $l$.

So we get seven cubics with two skew symmetric matrices. Without any loss of generality, may assume $J = 7\times 3$ is a matrix as above, with three linearly independent matrices $B_j$ above. The condition that this composition is zero says that the quadrics defined as the $2 \times 2$-minors of the matrix $a$, considered as quadrics in the $u_j$'s, have the coefficients of $u_3$, $u_2$, $u_1$ respectively equal with the coefficients of $u_0 u_3$, $u_0 u_1$, $u_0 u_2$.

(3.6) Proposition. If a matrix $a$ with $\alpha \alpha' = 0$ comes from an exact complex, then the three quadrics given by its $2 \times 2$-minors are linearly independent.

Proof. Assume

$$\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

is a matrix as above, with three linearly dependent $2 \times 2$-minors. Without any loss of generality, may assume

$$\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{pmatrix} \text{ or } \alpha = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & a_{32} \end{pmatrix}.$$

Indeed, after a linear change of the rows we may assume that the minor corresponding of the $2^{nd}$ and $3^{rd}$ row is zero. A $2 \times 2$ determinant of linear forms is zero, iff either two rows or the two columns are linearly dependent. A further basis change gives $\alpha$ the shape above.

Case 1. If $\alpha$ has a zero row then the ideal contains a summand $V_4 O(-3)$, a contradiction.

Case 2. In the second case we first note that $a_{11} = l_0 u_0 + l_1 u_1 + l_2 u_2 + l_3 u_3$ is non-zero. Indeed, otherwise

$$0 \leftarrow I_A \leftarrow 3V_4 O(-3) \leftarrow SO^3 \leftarrow 0$$

would be exact, and $A$ could have codimension 2 at most.

Consider now

$$V_3 O(-3)_{a_{11}} \rightarrow SO^3(u_0, u_1, u_2, u_3) \rightarrow 4V_1 O(-4).$$

The composition is the concatenation of the four $7 \times 7$ block matrices $l_1 B_1 + l_2 B_2 + l_3 B_3 \mid l_0 B_1 - l_1 B_3 \mid l_0 B_2 - l_2 B_1 \mid l_0 B_3 - l_2 B_2$. Precisely 2 blocks are linearly independent, because on one hand $a_{11} \cdot a_{22} = a_{11} \cdot a_{23} = 0$, on the other hand the ideal $J$ in (4.1) has too few syzygies to allow a subideal of type $a_{11} \cdot (a, b, c)$. So we get seven cubics with two skew symmetric $7 \times 7$ matrices of relations. Since all 4 skew symmetric matrices have rang 6 in a general point of $P^6$ (eg. in the point $(1 : 2 : 3 : 4 : 5 : 6 : 7)$) unless they are identically zero, there is up to scalar a unique set of cubics whose relations they are: In each case the 7 principal pfaffians. These pfaffians are not proportional for two different blocks for any values $(l_0, \ldots, l_3)$.
unless the blocks themselves are proportional, as can be seen by a straight forward computation. This is the desired contradiction. \qed

4. Moduli. Denote by $X(1,7)$ the open set of abelian surfaces with a very ample polarization of class $(1,7)$. For each $A \in X(1,7)$ we choose a $G_7$-equivariant embedding $A \to \mathbb{P}^3$. Its syzygy determine a $3 \times 2$ matrix $\alpha = \alpha_A$ as in Theorem 3.2. $\alpha$ is determined by $A$ up to conjugation with $GL(3,\mathbb{C}) \times SL(2,\mathbb{C})$. The ideal $I = I_A \subset S = \mathbb{C}[u_0,u_1,u_2,u_3]$ of minors of $\alpha_A$ is uniquely determined by $A$. We denote by $C_A \subset \mathbb{P}(U)$ the zero loci of $I_A$.

(4.1) Proposition. $C_A \subset \mathbb{P}(U)$ is a projectively Cohen-Macaulay curve of degree 3 and arithmetic genus 0.

Proof. Denote by $S$ the graded ring $\mathbb{C}[u_0,u_1,u_2,u_3] = S(U') = \oplus_{t \geq 0} S^t U'$. By the Hilbert-Burch Theorem (cf. \cite{E}, Thm 20.15) the complex

$$
0 \leftarrow S/I \leftarrow S \leftarrow \bigoplus_1^3 S(-2) \otimes_{\mathbb{C}} \bigoplus_1^2 S(-3) \leftarrow 0
$$

is exact unless the three quadric minors of $\alpha$ have a common factor. Since the quadrics are linearly independent by Proposition 3.6, the second possibility occurs only if $S/I$ has syzygies

$$
1 \hspace{1cm} - \hspace{1cm} - \hspace{1cm} - \\
3 \hspace{1cm} 3 \hspace{1cm} 1
$$

However

$$
J := (\Delta_1, \Delta_2, \Delta_3)^\perp = (u_1u_2, u_2u_3, u_3u_1, u_1^2 + u_0u_3, u_3^2 + u_0u_2, u_2^2 + u_0u_1, u_0^2)
$$

has syzygies

$$
1 \hspace{1cm} - \hspace{1cm} - \hspace{1cm} - \\
7 \hspace{1cm} 8 \hspace{1cm} - \hspace{1cm} - \\
- \hspace{1cm} 3 \hspace{1cm} 8 \hspace{1cm} 3
$$

and $I \subset J$ by Proposition 3.5. So the second possibility cannot occur and the Hilbert-Burch complex is exact. \qed

(4.2) Corollary. $A$ is uniquely determined by $C_A$.

Proof. $C_A$ determines the Hilbert-Burch matrix $\alpha$ up to conjugation, which in turn determines $\alpha'$, $\beta'$ hence $I_A$. \qed

(4.3) The Hilbert scheme $Hilb_{3+1}(\mathbb{P}^3)$ has two components of dimension 12 and 15 (cf. \cite{PS}):

$$
Hilb_{3+1}(\mathbb{P}^3) = H_1 \cup H_2
$$

with general points of $H_1, H_2$ and the intersection $H_1 \cap H_2$ corresponding respectively to a twisted cubic, a plane cubic union a point or a plane nodal cubic with an embedded point at the node. For all $C \in H_1$, $h^0(\mathbb{P}^3, \mathcal{I}_C) = 3$. The morphism

$$
f : H_1 \leftarrow \mathbb{G}(3, h^0(\mathbb{P}^3, \mathcal{O}(2)))
\quad C \leftarrow H^0(\mathbb{P}^3, \mathcal{I}_C(2))
$$

is birational onto its image $H \subset \mathbb{G}(3,10)$, regular precisely on $H_1 - H_1 \cap H_2$, cf. \cite{EPS}. All varieties $H, H_1, H_2, H_1 \cap H_2, f(H_1 \cap H_2)$ are smooth.

Consider

$$
H(\Delta) := H \cap \mathbb{G}(3, J_2) \subset \mathbb{G}(3, h^0(\mathbb{P}^3, \mathcal{O}(2))).
$$

Since $\mathbb{G}(3, J_2)$ does not intersect $f(H_1 \cap H_2)$ we can regard $H(\Delta)$ as a subvariety of $H_1$ as well. $H(\Delta)$ has dimension at least 3 in every point by dimension count.

We are grateful to Geir Ellingsrud for pointing out to us, that such varieties were studied by Mukai.

(4.4) Theorem. $H(\Delta)$ is a smooth prime Fano 3-fold of genus 12.
Proof. Mukai [Muk89,92] proves that $H(\delta) = H \cap G(3, \delta^\perp) \subset G(3, H^0(\mathbb{P}^3, \mathcal{O}(2)))$ is a smooth prime Fano 3-fold for a general net of quadrics ($\delta_1, \delta_2, \delta_3$) in $\mathbb{P}(U)$. The proof that $\Delta_1, \Delta_2, \Delta_3$ is general in this sense, i.e. that $H(\Delta)$ is a smooth connected Fano 3-fold, is postponed until we have considered different models of $H(\Delta)$.

(4.5) Consider on $L = \mathbb{C}^7$ the net $\eta_{klein} : \Lambda^2 L \rightarrow W' = \mathbb{C}^3$ of alternating forms defined by the matrix

$$
\eta_{klein} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -y_1 & y_0 \\
0 & 0 & 0 & 0 & -y_2 & 0 & y_1 \\
0 & 0 & 0 & -y_0 & 0 & 0 & y_2 \\
0 & y_0 & 0 & y_1 & -y_2 & 0 & 0 \\
y_2 & 0 & -y_1 & 0 & y_0 & 0 & 0 \\
y_1 & 0 & 0 & y_2 & -y_0 & 0 & 0 \\
-y_0 & -y_1 & -y_2 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$G(3, L, \eta_{klein}) = \{ E \in G(3, L) \mid \Lambda^2 E \subset Ker(\eta_{klein} : \Lambda^2 L \rightarrow W') \}.$

As zero loci of a section of a homogenous bundle on the Grassmanian, $G(3, L, \eta_{klein})$ is a prime Fano 3-fold of genus 12, if it is smooth of expected dimension. Smoothness follows from the criterion in section 1 of [Muk89] by computation.

(4.6) Let $F = \{ f = 0 \} \subset \mathbb{P}^2$ be a plane quartic. The variety

$$VSP(F, 6) = \{ \{ l_1, \ldots, l_6 \} \in \text{Hilb}_6(\mathbb{P}^2) \mid f = l_1^4 + \ldots + l_6^4 \}$$

of sums powers presenting $f$ was studied by Rosanes [Ros] 1873, Scorza [Sc01,2] and more recently by Mukai [Muk89,92]. It is a prime Fano 3-fold of genus 12 for general $F$. Consider $f_{klein} = v_1^4 v_2 + v_2^4 v_3 + v_3^4 v_1$ the well-known equation of the modular curve $\mathcal{X}(7) \subset \mathbb{P}^2 = \mathbb{P}(W)$ due to Felix Klein, [K] §4.

(4.7) Theorem.

$H(\Delta) \cong G(3, L, \eta_{klein}) \cong VSP(\mathcal{X}(7), 6)$.

Proof. Every prime Fano 3-fold $V_{22}$ of genus 12 (hence degree 22) has these 3 descriptions [Muk89,92] over an algebraically closed field of characteristic 0. That these special ones correspond to each other follows from [Schr], where the relation between the defining data and the isomorphism between the different models is explained:

$\eta_{klein}$ can be identified with the Tor-multiplication

$$\Lambda^2 Tor^S_1(S/J, \mathcal{C})_2 \longrightarrow Tor^S_2(S/J, \mathcal{C})_4.$$

Note that these Tor-groups are 7 respectively 3-dimensional, cf. (4.1.1), and in fact $Tor^S_1(S/J, \mathcal{C})_2 = L$ and $Tor^S_2(S/J, \mathcal{C})_4 = W'$, because the minimal resolution of $S/J = \mathbb{C} \oplus U' \oplus W'$ over $S$ has the form:

$$0 \leftarrow S/J \leftarrow S \leftarrow LS(-2) \leftarrow M_1 S(-3) \oplus W'S(-4) \leftarrow M_1 S(-5) \leftarrow WS(-6) \leftarrow 0.$$

On the other hand the ideal $I_{paff}$ generated by the $6 \times 6$-Pfaffians of $\eta_{klein}$ gives a Gorenstein ring $A = \mathbb{C}[y_0, y_1, y_2]/I_{paff}$ of codimension 3 cf. [BE]. A is artinian and the dual socle generator is $f_{klein}$. This completes the proof of Theorem 4.7 and 4.4. \qed

(4.8) Remarks. (1) The discriminant of the net of quadrics $\delta$ is another quartic, which comes with a natural vanishing theta characteristic, cf. Scorza [1889,1899] and [DK]. In our case this is again the Klein quartic. For general $\delta$ this is a different quartic than the dual socle quartic, see [Schr] for more details. The fact that

$$\{ \text{quartics} \} \longrightarrow \{ \text{quartics with an odd theta characteristic} \}$$

is birational over $\mathbb{C}$ was discovered by Scorza. A more recent treatment is given in [DK], and with different point of view in [Schr].
(2) If we take Mukai’s results for granted, then it is clear that the quartic for the sum of powers has to coincide with the Klein quartic, because this curve is uniquely determined by its symmetry group: combine [ACGH] Ex. I F-17 with [H] Ex. IV 5.7 (b), or combine [H] Ex. IV 5.7 (a) and the appendix.

(4.9) Theorem. The Moduli space $X(1, 7)$ is birational to $VSP(\Omega(7), 6)$.

Proof. By 4.2 we have an immersion

$$X(1, 7) \overset{\psi}{\rightarrow} H(\Delta) \cong VSP(\Omega(7), 6).$$

Since both varieties are irreducible and 3-dimensional the result follows. $\square$

(4.10) Theorem. $X(1, 7)$ is rational with the rational map to $\mathbb{P}^3$ defined over $\mathbb{Q}$.

Proof. It suffices to prove that $G(3, L, \eta_{klein})$ is rational over $\mathbb{Q}$. For a general point $p \in G(3, L, \eta)$ the triple projection defined by $|H - 3p|$ defines a birational map

$$G(3, L, \eta) \dasharrow \mathbb{P}^3,$$

(oral communication of Mukai). Its base loci consists of the 6 conics passing through $p$. This map is defined over $\mathbb{Q}$, if the point is defined over $\mathbb{Q}$.

However the only readily visible rational point of $G(3, V, \eta_{klein})$ is the point $p_0$ corresponding to the curve $C_\epsilon \subset \mathbb{P}^3$ defined by $u_1u_2 = u_2u_3 = u_3u_1 = 0$. Interpreted as a sum of powers this corresponds to the degenerate presentation

$$(v_1 + \epsilon v_2)^4 - v_1^4 + (v_2 + \epsilon v_3)^4 - v_2^4 + (v_3 + \epsilon v_1)^4 - v_3^4 = 4\epsilon f_{klein},$$

viewed over $\mathbb{Q}[\epsilon]/(\epsilon^2)$. For this reason we call $p_0$ the equational point.

From the explicit form of $\eta = \eta_{klein}$ we see three lines $L_1, L_2, L_3 \subset G(3, L, \eta)$ passing through $p_0$. So $|H - 3p_0|$ has larger dimension than for a general point. We pass to a subsystem. $|H - 2L_1 - 2L_2 - 2L_3|$ has dimension 3. Its base loci consist of $L_1 \cup L_2 \cup L_3$ with each line with a 4-fold structure: The normal bundle of each line is $O_L \oplus O_L(-1)$. Hence $T_{LL}/T_{L}^{\perp}$ has a summand $O_L$ and this lies in the base loci since $|H - 2L_1 - 2L_2 - 2L_3| \subset |H - 3p_0|$. Note that the 4-fold structure is not generically a complete intersection.

Two general hyperplanes $H_1, H_2 \in |H - 2L_1 - 2L_2 - 2L_3|$ intersect along each line in a 5-fold structure. Hence the residual curve has degree 7. It intersects each line in one point. Hence a third general hyperplane $H_3 \in |H - 2L_1 - 2L_2 - 2L_3|$ intersects the residual curve in these 3 points with multiplicity 2 and a single further point, i.e. $|H - 2L_1 - 2L_2 - 2L_3|$ defines a birational map to $\mathbb{P}^3$.

Instead of giving all the details of the arguments above we prefer to describe the inverse map

$$\psi: \mathbb{P}^3 \rightarrow G(3, V, \eta) = G(3, V) \cap \mathbb{P}^{13} \subset \mathbb{P}^{34}$$

explicitly.

Consider the $3 \times 7$ matrix

$$\Psi = \begin{pmatrix}
-t_0t_3 & t_0t_1 + t_1^2 & -t_3^2 & 0 & t_1t_3 & -t_2t_3 & 0 \\
-t_1^3 & -t_2^3 & -t_0t_2 & -t_1t_2 & 0 & t_2t_3 & 0 \\
t_0t_1^2 + t_1t_2 + t_0t_3 & t_2t_3^2 & t_0t_2^2 + t_0t_3 & 0 & 0 & t_0t_3 & t_1t_2t_3
\end{pmatrix}$$

The rational map from $\mathbb{P}^3$ with coordinates $t_0, \ldots, t_3$ to the Grassmanian $G(3, 7)$ defined by $\Psi$ is for the given basis of $V$ the desired rational parametrization $\psi$. $\psi$ is bi-regular on $\mathbb{P}^3 - \{t_1t_2t_3 = 0\}$. $\square$

(4.11) Corollary. The rational universal family of $3 \times 2$ matrices on $\mathbb{P}^3 \times \mathbb{P}^3$ with coordinates $t, u$ is given by

$$\alpha(t) = \begin{pmatrix}
t_0u_1 + t_2u_2 & -t_2u_0 & -t_1u_1 \\
t_2u_2 & -t_0u_2 - t_3u_3 & -t_1u_1 \\
u_1 & u_2 & u_3 & t_3
\end{pmatrix}.$$
Proof. The 3x3 minors of \( \alpha(t) \) give 2 forms of bidegree (2,2) and two further forms of bidegree (3,2), (2,3) respectively. The 3 forms of degree 2 in \( u \)'s are simply the product of the matrix of basis elements of \( J \) with \( \Psi \). The form of degree 3 in the \( u \)'s is dependent unless \( t_1 = t_2 = t_3 = 0 \). Thus for any given point \( [t_0 : t_1 : t_2 : t_3] \neq [1 : 0 : 0 : 0] \) the minimal version of the matrix \( \alpha(t) \) is a 3 x 2 matrix of linear forms in the \( u \)'s, whose minors are the desired 3 quadrics. \( \square \)

Consider the vector
\[
D = (x_0 x_3 x_4, x_0 x_1 x_6, x_0 x_2 x_5, x_2^2 x_3 + x_4 x_5^2, x_1^2 x_5 + x_2 x_6^2, x_4^2 x_6 + x_1 x_2 x_4 + x_3 x_5 x_6 - x_0^3)
\]
of \( \tau \)-invariant cubics in \( S^3 V_4 \).

(4.12) Corollary. The rational family on \( \mathbb{P}^3 \times \mathbb{P}^6 \) defined by the \( H_7 \)-invariant subspace of cubics generated by the \( \tau \)-invariant forms
\[
(g_1, g_2, g_3) = D^4 \Psi
\]
has as its fibres a dense subfamily of the universal family of \( G_7 \)-invariant abelian surfaces of type \((1,7)\).

Proof. Observe that, with the notations from A5. one has:
\[
J = (f_3, f_1, f_2, f_4, f_5, f_6, f_0)
\]
and
\[
D = (f_3 e_0, f_1 e_0, f_2 e_0, f_4 e_0, f_5 e_0, f_6 e_0, f_0 e_0)
\]
The entries of \( J \) and \( D \) correspond to each other as elements in two isomorphic irreducible \( SL_2(\mathbb{Z}_7) \)-modules. \( \square \)

Appendix

A1. The Heisenberg Group \( H_7 = H_{(1,7)} \) and the extended variant \( G_7 = H_7 \rtimes \mathbb{Z}_2 \) We use notations similar to those from [HM] and [Man86], [Man89]

A1.1 The direct construction of \( H_{(1,7)} \) as a subgroup of \( SL_7(\mathbb{C}) \)

Let \( V = Map(\mathbb{Z}_7, \mathbb{C}) \). On \( V \) consider the automorphisms \( \sigma, \tau \) defined through:
\[
\sigma x(j) = x(j + 1), \quad \tau x(j) = \varepsilon^j x(j)
\]
where \( \varepsilon = e^{2\pi i/7} \in \mu_7 \) and let \( H_7 \) be the group generated by \( \sigma, \tau \) (the Heisenberg group \( H_{(1,7)} \)). Then \( H_7 \) is generated by matrices \( A_{ij} \) of the form:
\[
A_{ij} = (\varepsilon^{aj+bj+\delta_{i,j+\ell+c}}), \text{ where } a, b, c \in \mathbb{Z}_7
\]
and has order 343.

The Galois group \( \Theta \) of \( \mathbb{Q}(\varepsilon) \) over \( \mathbb{Q} \) acts on \( H \) and let \( \theta \) be the generator given by \( \theta(\varepsilon) = \varepsilon^3 \). Then \( \theta^3 = \text{complex conjugation} \). The group \( H \) is a central extension preserved by the action of \( \Theta \):
\[
1 \to \mu_7 \to H \to \mathbb{Z}_7 \times \mathbb{Z}_7 \to 0
\]
(we identified tacitly \( \mu_7 \) and \( \mathbb{Z}_7 \) in the last nonzero term and \( \sigma \mapsto (1,0), \tau \mapsto (0,1) \)).

The irreducible \( H \)-module \( V \) produces 5 more by the composition with the automorphisms \( \theta^i \in \Theta \); denote by \( V_i \) the representation \( H^\theta \to H \to \text{Aut} V \). These 6 representations are inequivalent, as one sees computing their characters, and together with the characters of \( \mathbb{Z}_7 \times \mathbb{Z}_7 \) are all irreducible characters of \( H \).

Let \( \Phi : \mathbb{Z}_7 \times \mathbb{Z}_7 \to H \) be the map
\[
\Phi(m, n) = \varepsilon^{4mn} \sigma^m \tau^n
\]
and consider
\[ \omega : \mu_7 \times (\mathbb{Z}_7 \times \mathbb{Z}_7) \to H \quad \text{given by} \quad \omega(\alpha, z) = \alpha \Phi(z) \]

Then \( \omega \) is a bijection and the product in \( H \) corresponds to
\[ (\alpha, z) \cdot (\alpha', z') = (\alpha \alpha' B(z, z'), z + z') \quad \text{where} \quad B(m, n; m', n') = \varepsilon^{3(mn' - m'n)} \]

Let \( N \) be the normaliser of \( H \) in \( SL_7(\mathbb{C}) \). Then \( N \) is a central extension:
\[ 1 \to H \to N \xrightarrow{\alpha} SL_2(\mathbb{Z}_7) \to 1 \]
where \( \alpha(x) = \) the automorphism of \( \mathbb{Z}_7 \times \mathbb{Z}_7 \) preserving \( B \), induced by conjugation by \( x \in Aut(H) \).

Note that for \( u \in SL_2(\mathbb{Z}_7) \) the action \( \gamma_u \) on \( H \) can be expressed:
\[ \gamma_u \omega(\alpha, z) = \omega(\alpha, u(z)). \]

We have seen that any polarized abelian surface \((A, L)\) with a polarization of type \((1, 7)\), \( L \) very ample and a fixed level structure is canonically embedded in \( \mathbb{P}^6 := \text{the projective space of lines in} V \). Moreover, we may suppose that \( L \) is symmetric with respect to the origin of \( A \), because, by a change of the origin, we can realize this situation. Then the map \( x \mapsto -x \) on \( A \) extends to an automorphism of order 2 of \( \mathbb{P}^6 \), induced by \( \iota \in SL(V) \), \( \omega(j) = -x(-j) \). Therefore we consider from now on abelian surfaces in \( \mathbb{P}^6 \) invariant under the action of \( G = H \rtimes \mathbb{Z}_2 \).

**Remark.** If \( \{e_i\}_i \) is the canonical basis of \( V = Map(\mathbb{Z}_7, \mathbb{C}) \), i.e. \( e_j(\ell) = \delta_{j\ell} \), and \( \{x_i\}_i \) is the dual basis of \( V^* = V_3 \), then the action of \( \sigma, \tau \) on \( V \) and on \( H^0(\mathbb{P}^6, \mathcal{O}(1)) = V^* = V_3 \) is given by:
\[
\begin{align*}
\sigma e_j &= e_{j-1} & \sigma x_j &= x_{j-1} \\
\tau e_j &= \varepsilon^2 e_j & \tau x_j &= \varepsilon^{-2} x_j 
\end{align*}
\]

(A1.2) **Character table of** \( G \) (cf. [Man86] for \( H_5 \rtimes \mathbb{Z}_2 \)):

| \( \{\alpha\} \) | \( C_{m,n} \) | \( C_\alpha \) |
|-----------------|-----------------|-----------------|
| 1               | 1               | \( \theta(\alpha) \) | \( V_i \) |
| 7\( \theta(\alpha) \) | 0               | -1              | \( S \) |
| 1               | 1               | -\( \theta(\alpha) \) | \( V_i^2 \) |
| 2               | \( \varepsilon^{sm+tn} + \varepsilon^{-sm-tn} \) | 0               | \( Z_{s,t} \) |

where:
\( \{\alpha\} \) is the conjugacy class containing only the central element \( \alpha \in \mu_7 \),
\( C_{m,n} = \{(\alpha, m, n), (\alpha, -m, -n) | \alpha \in \mu_7\} \) (with \( (m, n) \neq (0, 0) \))
and \( C_\alpha = \{(\alpha, m, n) | m, n \in \mathbb{Z}_7\} \).

Thus there are 7 classes \( \{\alpha\} \), 24 classes \( C_{m,n} \) (each with 14 elements) and 7 classes \( C_\alpha \) (each with 49 elements). We denote by \( Z \) the sum of all 24 \( Z_{s,t} \).

(A1.3) **Useful formulae**

We have the following formulae:
\[
\begin{align*}
V_i \otimes V_i &= 3V_{i+2} \oplus 4V_{i+3}^2 & \wedge^2 V_i &= 3V_{i+2} & \wedge^6 V_i &= V_{i+3} \\
V_i \otimes V_{i+1} &= 3V_{i+3} \oplus 4V_{i+4}^2 & \wedge^3 V_i &= V_{i+1} \oplus 4V_{i+1}^2 & \wedge^7 V_i &= I \\
V_i \otimes V_{i+2} &= 3V_{i+1} \oplus 4V_{i+1}^2 & \wedge^4 V_i &= V_{i+4} \oplus 4V_{i+4}^2 \\
V_i \otimes V_{i+3} &= I \oplus Z & \wedge^5 V_i &= 3V_{i+5}
\end{align*}
\]
and as a
Z
make some notations:

The Normaliser

The Table of Characters of the Group

Observing that the trace of

Then we have the following equalities, useful in computations:

A2 The Normaliser \( N \) of \( H \) in \( SL_7(\mathbb{C}) \). Via the map \( \alpha : N \to SL_2(\mathbb{Z}_7) \) we get, entirely like in [HM], that \( N \) is a semidirect product \( H \rtimes SL_2(\mathbb{Z}_7) \). Then, in fact \( N \subset SL_7(\mathbb{Q}(\varepsilon)) \). Thus \( \Theta \) acts on \( N \) and all \( V_i \) are also \( N \)-modules. One shows, like in [HM], that \( V_i \otimes V_i^* \cong I \oplus Z \), for all \( i \), where \( Z \) is the space of trace 0, and as a \( \mathbb{Z}_7 \times \mathbb{Z}_7 \)-module is the sum of all 48 nontrivial modules. In fact, as a \( N/\mu_7 \)-module it is irreducible and its character takes values in \( \mathbb{Q} \).

A3 The Table of Characters of the Group \( SL_2(\mathbb{Z}_7) \) and their Multiplication. First of all we make some notations:

Then we have the following equalities, useful in computations:

\[
\begin{align*}
\varepsilon + \varepsilon^2 + \varepsilon^4 &= -\alpha^- & \lambda_1 + \lambda_2 + \lambda_3 &= \alpha & \eta_1 + \eta_2 + \eta_3 &= -1 \\
\varepsilon^3 + \varepsilon^5 + \varepsilon^6 &= -\alpha^+ & \lambda_1 \lambda_2 \lambda_3 &= \alpha & \eta_1 \eta_2 \eta_3 &= 1
\end{align*}
\]
\[
\lambda_1^2 = \eta_3 - 2 \quad \lambda_1 \lambda_2 = \eta_3 - \eta_2 \quad \alpha \eta_1 = \lambda_1 - 2\lambda_2 \\
\lambda_2^2 = \eta_1 - 2 \quad \lambda_2 \lambda_3 = \eta_1 - \eta_3 \quad \alpha \eta_2 = \lambda_2 - 2\lambda_3 \\
\lambda_3^2 = \eta_2 - 2 \quad \lambda_3 \lambda_1 = \eta_2 - \eta_2 \quad \alpha \eta_3 = \lambda_3 - 2\lambda_1 
\]

The general shape of the elements in \( N \) is:
\[
A_{jk} = \pm \frac{1}{\sqrt{\eta}} \varepsilon^{aj+bj+ck+dk+e} \quad (a, b, \ldots, f \in \mathbb{Z}_7, \ b \neq 0)
\]
\[
A_{jk} = \pm \varepsilon^{aj+bj+ck} \quad (a, b, \ldots, e \in \mathbb{Z}_7, \ d \neq 0)
\]

where the signs are chosen to have \( \det(A_{jk}) = 1 \).

For the convenience of the computations, it is useful to identify some elements in \( N = H \cdot SL_2(\mathbb{Z}_7) \) and their images in \( SL_2(\mathbb{Z}_7) \):

\[
\mu x(j) = x(2j) \quad \text{(resp. } \mu e_j = e_{j/2} \text{) corresponding to } \overline{\nu} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \in SL_2(\mathbb{Z}_7)
\]
\[
\nu x(j) = \varepsilon^2 x(j) \quad \text{(resp. } \nu e_j = \varepsilon^2 e_j \text{) corresponding to } \overline{\nu} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}_7)
\]
\[
\delta x(j) = \frac{i}{\sqrt{\eta}} \sum_k \varepsilon^{kj} x(k) \quad \text{(resp. } \delta e_j = \frac{i}{\sqrt{\eta}} \sum_k \varepsilon^{kj} e_k \text{) corresponding to } \overline{\delta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}_7)
\]

Observe that \( \delta^2 = \iota \) and that the elements in \( SL_2(\mathbb{Z}_7) \) are given according to:

\[
\mu \sigma \mu^{-1} = \sigma^2 \quad \iota \sigma = \sigma^{-1} \quad \nu \sigma \nu^{-1} = \varepsilon^{1+2} \sigma \tau^2 \quad \delta \sigma \delta^{-1} = \tau \\
\mu \tau \mu^{-1} = \tau^4 \quad \iota \tau = \tau^{-1} \quad \nu \tau \nu^{-1} = \tau \quad \delta \tau \delta^{-1} = \sigma^{-1}
\]

Character Table of \( SL_2(\mathbb{Z}_7) \)

| \( \bar{v} \) | id | \( -id \) | \( \bar{\mu} \) | \( \bar{\nu} \) | \( \bar{\nu^3} \) | \( \bar{\nu}^3 \) | \( \bar{\nu}^5 \) | \( \bar{\delta} \) | \( (\bar{\nu})_{22} \) | \( (\bar{\nu})_{52} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| id | 1 | 1 | 56 | 56 | 24 | 24 | 24 | 24 | 42 | 42 |
| \( M \) | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| \( \bar{L} \) | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| \( \bar{U} \) | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| \( \bar{U}^* \) | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| \( T_1 \) | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| \( T_2 \) | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| \( \bar{W} \) | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| \( \bar{W}^* \) | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

We indicate here also the multiplication table of the characters of \( SL_2(\mathbb{Z}_7) \):
\[ M_1 \otimes M_1 = I \oplus 3M_2 \oplus 3L \oplus 2T \oplus W \oplus W' \]
\[ M_1 \otimes M_2 = 3M_1 \oplus 2U \oplus U' \oplus 2T_1 \oplus 2T_2 \]
\[ M_1 \otimes L = 3M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \]
\[ M_1 \otimes U = 2M_1 \oplus L \oplus T \oplus W \]
\[ M_1 \otimes U' = 2M_1 \oplus L \oplus T \oplus W' \]
\[ M_1 \otimes T_1 = 2M_1 \oplus 2L \oplus T \oplus W \oplus W' \]
\[ M_1 \otimes T_2 = 2M_1 \oplus 2L \oplus T \oplus W \oplus W' \]
\[ M_1 \otimes T = 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \]
\[ M_1 \otimes W = M_1 \oplus U \oplus T_1 \oplus T_2 \]
\[ M_1 \otimes W' = M_1 \oplus U' \oplus T_1 \oplus T_2 \]
\[ L \otimes L = I \oplus 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \]
\[ L \otimes U = M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \]
\[ L \otimes U' = M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \]
\[ L \otimes T_1 = 2M_1 \oplus U \oplus U' \oplus T_1 \oplus 2T_2 \]
\[ L \otimes T_2 = 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus T_2 \]
\[ L \otimes T = 2M_2 \oplus 2L \oplus T \oplus W \oplus W' \]
\[ L \otimes W = M_2 \oplus L \oplus T \]
\[ L \otimes W' = M_2 \oplus L \oplus T \]
\[ U \otimes U = L \oplus T \oplus W \]
\[ U \otimes U' = I \oplus M_2 \oplus L \]
\[ U' \otimes U' = L \oplus T \oplus W' \]
\[ U \otimes T_1 = M_2 \oplus L \oplus T \oplus W' \]
\[ U' \otimes T_1 = M_2 \oplus L \oplus T \oplus W \]
\[ U \otimes T_2 = M_2 \oplus L \oplus T \oplus W' \]
\[ U' \otimes T_2 = M_2 \oplus L \oplus T \oplus W \]
\[ U \otimes T = M_1 \oplus U' \oplus T_1 \oplus T_2 \]
\[ U' \otimes T = M_1 \oplus U \oplus T_1 \oplus T_2 \]
\[ U \otimes W = T_1 \oplus T_2 \]
\[ U' \otimes W = M_1 \oplus U \]
\[ U \otimes W' = M_1 \oplus U' \]
\[ U' \otimes W' = T_1 \oplus T_2 \]
\[ T_1 \otimes T_1 = I \oplus 2M_2 \oplus L \oplus T \oplus W \oplus W' \]
\[ T_1 \otimes T_2 = 2M_2 \oplus 2L \oplus T \]
\[ T_2 \otimes T_2 = I \oplus 2M_2 \oplus L \oplus T \oplus W \oplus W' \]
\[ T_1 \otimes T = 2M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \]
\[ T_2 \otimes T = 2M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \]
\[ T_1 \otimes W = M_1 \oplus U' \oplus T_1 \]
\[ T_2 \otimes W = M_1 \oplus U' \oplus T_2 \]
\[ T_1 \otimes W' = M_1 \oplus U \oplus T_1 \]
\[ T_2 \otimes W' = M_1 \oplus U \oplus T_2 \]
\[ T \otimes T = I \oplus 2M_2 \oplus L \oplus 2T \]
\[ T \otimes W = M_2 \oplus L \oplus W' \]
\[ T \otimes W' = M_2 \oplus L \oplus W \]
\[ W \otimes W' = T \oplus W' \]
\[ W' \otimes W' = T \oplus W \]

**A4 Multiplications, exterior, symmetric powers of \( N \) representations, some \( H^0(\Omega^3(j))'s \).**

\[
V_{2j} \otimes V_{2j} = (U' \oplus W') \otimes V_{2j+2} = V_{2j} \otimes V_{2j+2} = (U \oplus W) \otimes V_{2j+4}
\]
\[
V_{2j+1} \otimes V_{2j+1} = (U \oplus W) \otimes V_{2j+3} = V_{2j+1} \otimes V_{2j+2} = (U' \oplus W') \otimes V_{2j+5}
\]
\[
V_{2j} \otimes V_{2j+2} = (U \oplus W) \otimes V_{2j+1} = V_j \otimes V_{j+3} = I \oplus Z
\]
\[
V_{2j+1} \otimes V_{2j+3} = (U' \oplus W') \otimes V_{2j+2}
\]
\[
\Lambda^2 V_{2j} = W' \otimes V_{2j+2} = V_{2j} \otimes V_{2j+1} = (I \oplus U') \otimes V_{2j+1} = V_{2j+1} \otimes V_{2j+2} = (I \oplus U) \otimes V_{2j+4}
\]
\[
\Lambda^3 V_{2j} = (I \oplus U') \otimes V_{2j+1} = V_{2j+1} \otimes V_{2j+2} = (I \oplus U) \otimes V_{2j+4}
\]
\[
\Lambda^4 V_{2j} = (I \oplus U) \otimes V_{2j+4} = (I \oplus U') \otimes V_{2j+5}
\]
\[
\Lambda^5 V_{2j+1} = W \otimes V_{2j+3} = V_{2j+1} \otimes V_{2j+2} = (I \oplus U) \otimes V_{2j+2} = (I \oplus U') \otimes V_{2j+5}
\]
\[
\Lambda^6 V_{2j+1} = W' \otimes V_{2j+6} = V_{2j+1} \otimes V_{2j+2} = (I \oplus U) \otimes V_{2j+5}
\]
\[
\Lambda^7 V_{j+3} = V_{j+3} = I
\]

17
\[ S^2V_{2j} = U' \otimes V_{2j+2} \]
\[ S^3V_{2j} = (I \oplus L \oplus U) \otimes V_{2j+1} \]
\[ S^3V_{2j} = (L \oplus W \oplus U \oplus U' \oplus T_1 \oplus T_2) \otimes V_{2j+4} \]
\[ S^3V_{2j} = (I \oplus M_1 \oplus M_2 \oplus 2L \oplus U \oplus U' \oplus T_1 \oplus T_2 \oplus 2T \oplus W') \otimes V_{2j+5} \]

etc.

\[ H^0(\Omega^3(3)) = 0 \]
\[ H^0(\Omega^3(4)) = (I \oplus U') \otimes V_1 \]
\[ H^0(\Omega^3(5)) = (L \oplus U' \oplus W' \oplus T_1 \oplus T_2 \oplus T) \otimes V_2 \]
\[ H^0(\Omega^3(6)) = (M_1 \oplus M_2 \oplus 3L \oplus 2U \oplus 2W \oplus W' \oplus 4T_1 \oplus 4T_2 \oplus 3T) \otimes V \]
\[ H^0(\Omega^3(7)) = (I \oplus 2L \oplus U \oplus 2U' \oplus W' \oplus T_1 \oplus T_2 \oplus T)(I \oplus Z) \oplus Z \]

etc.

**A5 Concrete Decompositions of Certain \( N \) or \( SL_2(\mathbb{Z}_7) \) Representations.**

Consider now the decomposition of \( V \) into eigenspaces of \( \iota \):

\[ V = V^+ \oplus V^- \quad \text{where:} \]

\[ V^+ = \text{span} \{ e_{12} - e_{-12}, e_{2} - e_{-2}, e_{3} - e_{-3} \} = \text{span} \{ e_1 - e_6, e_4 - e_3, e_2 - e_5 \} \]

\[ V^- = \text{span} \{ 2e_0, e_1 + e_6, e_4 + e_3, e_2 + e_5 \} \]

Restricting \( \mu, \nu, \delta \) to \( V^+ \) and \( V^- \) respectively, one gets:

\[ \mu^+ = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \nu^+ = \text{diag} (\varepsilon, \varepsilon^2, \varepsilon^4), \quad \delta^+ = \frac{i}{\sqrt{2}} \begin{pmatrix} \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 \\ \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 \end{pmatrix} \]

\[ \mu^- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \nu^- = \text{diag} (1, \varepsilon, \varepsilon^2, \varepsilon^4), \quad \delta^- = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & \varepsilon + \varepsilon^6 & \varepsilon^4 + \varepsilon^3 & \varepsilon^2 + \varepsilon^5 \\ 2 & \varepsilon^4 + \varepsilon^3 & \varepsilon^2 + \varepsilon^5 & \varepsilon + \varepsilon^6 \\ 2 & \varepsilon^2 + \varepsilon^5 & \varepsilon + \varepsilon^6 & \varepsilon^4 + \varepsilon^3 \end{pmatrix} \]

From the character table of \( SL_2(\mathbb{Z}_7) \) one sees that, as a \( SL_2(\mathbb{Z}_7) \)-module, \( V = W' \oplus U' \) and from the above computations one gets concrete realizations of \( W', U' \), namely \( W' = V^+ \) and \( U' = V^- \).

\[ S^2W = T \quad S^2W' = T \]
\[ S^3W = L \oplus W' \quad S^3W' = L \oplus W \]
\[ S^4W = I \oplus M_2 \oplus T \quad S^4W' = I \oplus M_2 \oplus T \]

If we denote by

\[ v_1 = e_1 - e_6 \quad v_1 = e_4 - e_3 \quad v_1 = e_2 - e_5 \]

the chosen basis of \( W' \), then the only \( SL_2(\mathbb{Z}_7) \)-invariant quartic is the Klein quartic:

\[ f_{\text{klein}} = v_1^3v_2 + v_2^3v_3 + v_3^3v_1 \]

\[ S^2U' = L \oplus W' \]

We choose as basis for \( L \subset S^2U' \) the following elements:

\[ f_0 = u_0^2, \quad f_1 = u_2u_3, \quad f_2 = u_3u_1, \quad f_3 = u_1u_2, \quad f_4 = u_0u_3 + u_1^2, \quad f_5 = u_0u_1 + u_2^2, \quad f_6 = u_0u_2 + u_3^2. \]

and for \( W' \) the elements

\[ v_3 = u_0u_3 - u_1^2, \quad v_2 = u_0u_1 - u_2^2, \quad v_1 = u_0u_2 - u_3^2. \]
Then in the decomposition

\[ S^3V_3 = (I \oplus U' \oplus L)V_4 \]

the elements corresponding to \( f_j \varepsilon_0 \) are given by:

\[
\begin{align*}
 f_0 \varepsilon_0 &= x_1 x_2 x_4 + x_3 x_5 x_6 - x_0^2 \\
 f_1 \varepsilon_0 &= x_0 x_1 x_6 \\
 f_2 \varepsilon_0 &= x_0 x_2 x_5 \\
 f_3 \varepsilon_0 &= x_0 x_3 x_4 \\
 f_4 \varepsilon_0 &= x_2 x_3 + x_5^2 x_4 \\
 f_5 \varepsilon_0 &= x_3^2 x_5 + x_6^2 x_2 \\
 f_6 \varepsilon_0 &= x_4 x_6 + x_3^2 x_1
\end{align*}
\]

From here one obtains all \( f_j \varepsilon_k \) via cyclic permutation, in other words via the action of \( \sigma \).

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