RIGIDITY RESULTS ON GRADIENT SCHOUTEN SOLITONS

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Abstract. In this paper we consider \( \rho \)-Einstein solitons of type \( M = (B^n, g^*) \times (F^m, g_F) \), where \((B^n, g^*)\) is conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group, and \((F^m, g_F)\) is an Einstein manifold. We provide all the solutions for the gradient Schouten soliton case. Moreover, in the Riemannian case, we prove that if \( M = (B^n, g^*) \times (F^m, g_F) \) is a complete gradient Schouten soliton then \((B^n, g^*)\) is isometric to \( S^{n-1} \times \mathbb{R} \) and \( F^m \) is a compact Einstein manifold.

1. Introduction

In differential geometry, the Ricci flow is an intrinsic geometric flow. It is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat.

In 1982, R. Hamilton introduced a nonlinear evolution equation for Riemannian metrics with the aim of finding canonical metrics on manifolds (see [1] or [7]). This evolution equation is known as the Ricci flow, and it has since been used widely and with great success, most notably in Perelman’s solution of the Poincaré conjecture. Furthermore, several convergence theorems have been established.

Bryant [2] proved that there exists a complete, steady, gradient Ricci soliton that is spherically symmetric for any \( n \geq 3 \), which is known as Bryant’s soliton. In the bi-dimensional case, this solution was obtained explicitly and it is known as the Hamilton cigar.

Recently, Cao-Chen [3] showed that any complete, steady, gradient Ricci soliton that is locally conformally flat up to homothety, is either flat or isometric to Bryant’s soliton. Complete, conformally flat, shrinking gradient solitons have been characterized as being quotients of \( \mathbb{R}^n \), \( S^n \) or \( R \times S^{n-1} \) (see [8]).

Motivated by the notion of Ricci solitons on a semi-Riemannian manifold \((M^n, g)\), \( n \geq 3 \), it is natural to consider geometric flows of the following type:

\[
\frac{\partial}{\partial t} g(t) = -2(Ric - \rho Kg)
\]

for \( \rho \in \mathbb{R} \), \( \rho \neq 0 \), as in [4]. We call these the Ricci-Bourguignon flows. We notice that short time existence for the geometric flows described in (1.1) is provided in [5]. Associated to the flows, we have the following notion of gradient \( \rho \)-Einstein solitons, which generate self-similar solutions:

**Definition 1.1.** Let \((M^n, g), n \geq 3\), be a Riemannian manifold and let \( \rho \in \mathbb{R}, \rho \neq 0 \). We say that \((M^n, g)\) is a gradient \( \rho \)-Einstein soliton if there exists a smooth
function $h : M \rightarrow \mathbb{R}$, such that the metric $g$ satisfies the equation
\begin{equation}
Ric_g + Hess_g h = \rho K_g g + \lambda g
\end{equation}
for some constant $\lambda \in \mathbb{R}$, where $K_g$ is the scalar curvature of the metric $g$.

A $\rho$-Einstein soliton is said to be shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Furthermore, a $\rho$-Einstein soliton is said to be a gradient Einstein soliton, gradient traceless Ricci soliton, and gradient Schouten soliton if $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$, and $\rho = \frac{1}{2(n-1)}$, respectively.

In [1], the authors studied $\rho$--Einstein solitons and obtained important rigidity results, proving that every compact gradient Einstein, Schouten, or traceless Ricci soliton is trivial.

In [10], the authors considered $\rho$-Einstein solitons that are conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group. We proved all the solutions for the gradient Schouten soliton case. Moreover, we proved that if a gradient Schouten soliton is both complete, conformal to a Euclidean space and invariant under the action of the pseudo-orthogonal group, then it is isometric to $S^{n-1} \times \mathbb{R}$.

The objective of this paper is to generalize the results obtained in [10], considering a $\rho$-Einstein soliton as being product manifolds of type $M = (B^n, g^*) \times (F^m, g_F)$, where $(B^n, g^*)$ is conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group, $(F^m, g_F)$ is Einstein manifold. In addition, we will consider that the potential function is defined only in $B^n$.

More precisely, let $(\mathbb{R}^n, g)$ be the standard pseudo-Euclidean space with metric $g$ and coordinates $(x_1, \ldots, x_n)$, with $g_{ij} = \delta_{ij} \varepsilon_i$, $1 \leq i, j \leq n$, where $\delta_{ij}$ is the Kronecker delta, and $\varepsilon_i = \pm 1$, with at least one $\varepsilon_i$ equal to one. Let $r = \sum_{i=1}^n \varepsilon_i x_i^2$ be a basic invariant for an $(n-1)$--dimensional pseudo-orthogonal group. Let us found the necessary and sufficient conditions for the existence of differentiable functions $\psi(r)$ and $h(r)$ such that the metric $\tilde{g} = g^* + g_F$, where $g^* = \frac{\rho}{\psi^2}$ satisfies the equation
\begin{equation}
Ric_{\tilde{g}} + Hess_{\tilde{g}} h = \rho K_{\tilde{g}} \tilde{g} + \lambda \tilde{g}
\end{equation}
with $\rho = 1/2(n-1)$.

In the proof of the results, we will consider product manifolds of type $(\mathbb{R}^n, g^*) \times F^m$ as being a particular case of the warped product $(\mathbb{R}^n, g^*) \times_f F^m$ considering $f \equiv 1$. For more details, see [9].

We initially find a system of differential equations, which functions $h$ and $\psi$ must satisfy, so that the metric $\tilde{g} = g^* + g_F$ satisfies (1.3) (see Theorem 2.1).

Note that if the solutions are invariant under the action of the pseudo-orthogonal group, the system of partial differential equations given in Theorem 2.1 can be transformed into a system of ordinary differential equations (see Corollary 2.2).

Making $\rho = \frac{1}{2(n-1)}$, in Lemma 2.4 we found conditions necessary on $\psi(r)$ so that the problem allows for a solution. Finally, we obtain the necessary and sufficient conditions on $h(r)$ and $\psi(r)$ for the existence of one gradient Schouten soliton. In this case, all solutions are given explicitly. Moreover, in the Riemannian case, we prove that if $(B^n, g^*) \times F^m)$, $n \geq 3$ is a complete gradient Schouten soliton, where $(B^n, g^*)$ is conformal to a Euclidean metric and rotationally symmetric where $g^*_{ij} = \delta_{ij}/\psi^2$, and $F^m$ is an Einstein Riemannian manifold. Then $(B^n, g^*)$ is isometric to $S^{n-1} \times \mathbb{R}$, $F^m$ is a compact manifold.
In what follows, we state our main results. We denote the second order derivative of \( \psi \) and \( h \) by \( \psi_{x_i x_j} \) and \( h_{x_i x_j} \), respectively, with respect to \( x_i x_j \).

2. Main Results

**Theorem 2.1.** Let \( \mathbb{R}^n, g \) be a pseudo-Euclidean space, \( n \geq 3 \) with coordinates \( x = (x_1, \ldots, x_n) \) and \( g_{ij} = \delta_{ij} \varepsilon_i \). Consider the product manifold \( M = (\mathbb{R}^n, g^*) \times F^m \) with metric tensor \( \bar{g} = g^* + g_F \), where \( g^* = \frac{1}{\psi^2} g \). \( F \) is an Einstein semi-Riemannian manifold with constant Ricci curvature \( \lambda_F \), \( m \geq 1 \), \( \psi, h : \mathbb{R}^n \to \mathbb{R} \), are smooth functions. Then \( M \) is a gradient \( \rho \)-Einstein soliton with

\[
Ric_{\bar{g}} + Hess_{\bar{g}} h = \left( \rho K_{\bar{g}} + \lambda \delta \right) \bar{g}, \quad \rho \in \mathbb{R},
\]

if, and only if, the functions \( \psi \) and \( h \) satisfy

\[
- (n - 2) \psi_{x_i x_j} + \psi_{x_i} h_{x_j} + \psi_{x_j} h_{x_i} + \psi_{x_i} h_{x_j} = 0, \quad \forall i \neq j,
\]

and

\[
(2.5) \quad \sum_{k=1}^{n} \varepsilon_k \left( (n - 1) m \rho \psi_{x_k} \right) - 2(n - 1) \rho \psi_{x_k} = \lambda_F (m \rho - 1) + \lambda.
\]

**Proof.** Suppose initially that \( M = (\mathbb{R}^n, g^*) \times F^m \) is a gradient \( \rho \)-Einstein soliton with potential function \( h \), that is,

\[
Ric_{\bar{g}} + Hess_{\bar{g}} h = \left( \rho K_{\bar{g}} + \lambda \right) \bar{g}, \quad \rho \in \mathbb{R}.
\]

Let \( X_1, X_2, \ldots, X_n \in \mathcal{L}(\mathbb{R}^n) \) and \( Y_1, Y_2, \ldots, Y_m \in \mathcal{L}(F) \), where \( \mathcal{L}(\mathbb{R}^n) \) and \( \mathcal{L}(F) \) are respectively the spaces of lifts of a vector field on \( \mathbb{R}^n \) and \( F \) to \( (\mathbb{R}^n \times F^m) \), then

\[
Ric_{\bar{g}}(X_i, X_j) = \frac{(n - 2) \psi_{x_i x_j}}{\psi}, \quad \forall \ i \neq j, \ i, j = 1, \ldots, n
\]

\[
Ric_{\bar{g}}(X_i, X_i) = \frac{(n - 2) \psi_{x_i x_i} + \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \psi_{x_k x_i}}{\psi} - \left( n - 1 \right) \varepsilon_i \sum_{k=1}^{n} \left( \frac{\psi_{x_k}}{\psi} \right)^2, \quad \forall \ i = 1, \ldots, n
\]

\[
Ric_{\bar{g}}(Y_i, Y_j) = 0, \quad \forall i = 1, \ldots, n, \ j = 1, \ldots, m
\]

\[
Ric_{\bar{g}}(Y_i, Y_j) = \lambda_F \bar{g}(Y_i, Y_j), \quad \forall \ i, j = 1, \ldots, m.
\]

As the potential function \( h : \mathbb{R}^n \to \mathbb{R} \) is defined only on the basis, we have

\[
Hess_{\bar{g}} h(X_i, X_j) = Hess_{g^*}(h)(X_i, X_j) \quad \forall \ i, j = 1, \ldots, n.
\]

Therefore,

\[
\begin{align*}
Hess_{\bar{g}} h(X_i, X_j) &= h_{x_i x_j} + \frac{\psi_{x_i h_{x_j}}}{\psi} + \frac{\psi_{x_j h_{x_i}}}{\psi}, \quad \forall \ i \neq j, \ i, j = 1, \ldots, n \\
Hess_{\bar{g}} h(X_i, X_i) &= h_{x_i x_i} + \frac{2 \psi_{x_i h_{x_i}}}{\psi} - \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \psi_{x_k h_{x_i}}, \quad \forall \ i = 1, \ldots, n.
\end{align*}
\]
We split the proof into two cases. In the first case, we will consider the vector fields on the base and in the second one, on the fiber.

If \( X_1, X_2, \ldots, X_n \in \mathcal{L}(\mathbb{R}^n) \), then to \( i \neq j \), from equation (2.5) together with equations (2.6) and (2.7), follows that
\[
\frac{(n-2)\psi_{x_i x_j}}{\psi} + h_{x_i x_j} + \frac{\psi_{x_i} h_{x_j}}{\psi} + \frac{\psi_{x_j} h_{x_i}}{\psi} = 0.
\]

Therefore,
\[
(n-2)\psi_{x_i x_j} + \psi h_{x_i x_j} + \psi_{x_i} h_{x_j} + \psi_{x_j} h_{x_i} = 0.
\]

For \( i = j \) we will need the following facts. It is well known that (see, e.g., [6])
\[
K_{\tilde{g}} = K_{g^*} + K_F
\]
where \( K_{\tilde{g}}, K_{g^*} \) and \( K_F \) represent the scalar curvatures of \( M = (\mathbb{R}^n, g^*) \times F^n, \)
\( (\mathbb{R}^n, g^*) \) and \( F^n \), respectively. Note that
\[
K_{\tilde{g}} = \sum_{k=1}^{n} \varepsilon_k \left[ 2(n-1)\psi_{x_k x_k} - (n-1)n (\psi_{,x_k})^2 \right], \quad K_F = \lambda_F m.
\]

Substituting this expression into (2.8), we get
\[
K_{\tilde{g}} = \sum_{k=1}^{n} \varepsilon_k \left[ 2(n-1)\psi_{x_k x_k} - (n-1)n (\psi_{,x_k})^2 \right] + \lambda_F m.
\]

Multiplying by \( \rho \) on both sides of the above equality, we have
\[
\rho (K_{g^*} + K_F) = \rho \lambda_F m + \rho \sum_{k=1}^{n} \varepsilon_k \left[ 2(n-1)\psi_{x_k x_k} - (n-1)n (\psi_{,x_k})^2 \right],
\]

Replacing the expressions found in (2.6), (2.7) and (2.8) in (2.5), we have
\[
(n-2)\psi_{,x_i x_i} + \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \psi_{x_k x_k}
\]
\[
\frac{-((n-1)\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left( \psi_{,x_k} \psi \right)^2 + h_{,x_i x_i} + \frac{2\psi_{x_i} h_{x_i}}{\psi} - \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \psi_{x_k} h_{x_k}}{\psi} = \left( \rho (K_{g^*} + K_F) + \lambda \right) g^* (X_i, X_i).
\]

Substituting the equation (2.9) into (2.10) and multiplying by \( \psi^2 \), we yields
\[
\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[ \psi_{,x_k x_k} - (n-1) (\psi_{,x_k})^2 - \psi_{,x_k} h_{,x_k} - 2(n-1)\rho \psi_{,x_k x_k} + (n-1)n \rho (\psi_{,x_k})^2 \right]
\]
\[
+ [(n-2)\psi_{,x_i x_i} + \psi^2 h_{,x_i x_i} + 2\psi_{,x_i} h_{,x_i}] = \left( \lambda_F m \rho + \lambda \right) \varepsilon_i.
\]

If \( Y_1, Y_2, \ldots, Y_m \in \mathcal{L}(F) \), we have
\[
Hess_{\tilde{g}} (h) (Y_i, Y_j) = 0
\]

Substituting the equations given by (2.6), (2.9) and (2.11) into (2.5), we obtain
\[
\left( \rho \lambda_F m + \rho \sum_{k=1}^{n} \varepsilon_k \left( 2(n-1)\psi_{x_k x_k} - (n-1)n (\psi_{,x_k})^2 \right) + \lambda \right) g_F (Y_i, Y_j) = \lambda_F g_F (Y_i, Y_j),
\]
equivalently,

\[ \rho \lambda_F m + \rho \sum_{k=1}^{n} \varepsilon_k \left( 2(n-1)\psi_{x_k x_k} - (n-1)n (\psi_{x_k})^2 \right) + \tilde{\lambda} = \lambda_F, \]

which implies

\[ \sum_{k=1}^{n} \varepsilon_k \left( (n-1)n \rho (\psi_{x_k})^2 - 2(n-1)\rho \psi_{x_k x_k} \right) = \lambda_F (m \rho - 1) + \tilde{\lambda}. \]

We want to find solutions to the system of equations (2.2), (2.3) and (2.4) of the form \( \psi(r) \) and \( h(r) \), where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). The following theorem reduces the system of partial differential equations (2.2), (2.3) and (2.4) into a system of ordinary differential equations that must be satisfied by such solutions.

**Corollary 2.2.** Let \(( \mathbb{R}^n, g)\) be a pseudo-Euclidean space, \( n \geq 3 \), with coordinates \( x = (x_1, \ldots, x_n) \) and \( g_{ij} = \delta_{ij} \varepsilon_i \). Consider \( M = (\mathbb{R}^n, g^*) \times F^m \), where \( g^* = \frac{1}{\psi^2} g \), \( F^m \) an Einstein semi-Riemannian manifold with constant Ricci curvature \( \lambda_F \) and smooth functions \( \psi(r) \) and \( h(r) \), where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). Then \( M \) is a gradient \( \rho \)-Einstein soliton with \( h \) as a potential function if, and only if, the functions \( h \) and \( \psi \) satisfy:

\[ (n-2)\psi'' + \psi h'' + 2\psi' h' = 0, \]

\[ 2\psi [2(n-1)(1-n\rho)\psi' + \psi h'] + 4r \left[ (1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho - 1)(\psi')^2 - \psi' h' \right] = \left( \lambda_F m \rho + \tilde{\lambda} \right). \]

and

\[ -4n(n-1)\rho \psi h' + 4r \left[ (n-1)n \rho (\psi')^2 - 2(n-1)\rho \psi \psi'' \right] = \lambda_F (m \rho - 1) + \tilde{\lambda}. \]

**Proof.** Let \( g^* = \psi^{-2} g \) be a conformal metric of \( g \). We are assuming that \( \psi(r) \) and \( h(r) \) are functions of \( r \), where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). Hence, we have

\[ \psi_{,i} = 2 \varepsilon_i x_i \psi', \quad \psi_{,x_i} = 4 x_i^2 \psi'' + 2 \varepsilon_i \psi', \quad \psi_{,x_i x_j} = 4 \varepsilon_i \varepsilon_j x_i x_j \psi'', \quad \psi_{,x_i x_j} = 4 \varepsilon_i \varepsilon_j x_i x_j \psi''. \]

and

\[ h_{,i} = 2 \varepsilon_i x_i h', \quad h_{,x_i} = 4 x_i^2 h'' + 2 \varepsilon_i h', \quad h_{,x_i x_j} = 4 \varepsilon_i \varepsilon_j x_i x_j h''. \]

Substituting these expressions into the first equation of Theorem 2.11 we get

\[(n-2) (4 \varepsilon_i \varepsilon_j x_i x_j \psi'' + \psi (4 \varepsilon_i \varepsilon_j x_i x_j h'') + (2 \varepsilon_i x_i \psi') (2 \varepsilon_j x_j h') + (2 \varepsilon_i x_i \psi')^2 (2 \varepsilon_i x_i h') = 0, \]

equivalently,

\[ 4 \varepsilon_i \varepsilon_j [(n-2) \psi'' + \psi h'' + 2 \psi' h'] x_i x_j = 0. \]

Since there exists \( i \neq j \), such that \( x_i x_j \neq 0 \), it follows that

\[(n-2) \psi'' + \psi h'' + 2 \psi' h' = 0. \]

Similarly, considering the second equation of Theorem 2.11 we obtain

\[ \psi [(n-2) (4 x_i^2 \psi'' + 2 \varepsilon_i \psi') + \psi (4 x_i^2 h'' + 2 \varepsilon_i h') + 2 (2 \varepsilon_i x_i \psi') (2 \varepsilon_i x_i h')] + \]

\[ 2 \psi [2(n-1)(1-n\rho)\psi' + \psi h'] + 4r \left[ (1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho - 1)(\psi')^2 - \psi' h' \right] = \left( \lambda_F m \rho + \tilde{\lambda} \right). \]

and

\[ -4n(n-1)\rho \psi h' + 4r \left[ (n-1)n \rho (\psi')^2 - 2(n-1)\rho \psi \psi'' \right] = \lambda_F (m \rho - 1) + \tilde{\lambda}. \]
\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[(1 - 2(n-1)\rho) \psi \left(4x_k^2 \psi'' + 2x_k \psi'\right) + (n-1)(n\rho - 1) (2\varepsilon_k x_k \psi')^2 \right] + \\
\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[-\psi \left(2\varepsilon_k x_k \psi'\right) (2\varepsilon_k x_k h')\right] = \left(\lambda_F \rho \mu + \tilde{\lambda}\right) \varepsilon_i,
\varepsilon_i
\text{equivalently,}

\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[4 \left(1 - 2(n-1)\rho\right) \psi \psi'' + (n-1)(n\rho - 1) (\psi')^2 - \psi' h'\right] + \\
4 \psi \left[(n - 2) \psi'' + \psi h'' + 2\psi' h'\right] x_k^2 + 2 \psi \left[(n - 2) \psi' + \psi h'\right] \varepsilon_i + \\
2 \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[(1 - 2(n-1)\rho) \psi \varepsilon_k \right] = \left(\lambda_F \rho \mu + \tilde{\lambda}\right) \varepsilon_i.

From equation \textbf{(2.13)}, it follows that

\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[2 \left(1 - 2(n-1)\rho\right) \psi' h' + (1 - 2(n-1)\rho) n\psi'\right] + \\
+ 4r \left[(1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho - 1) (\psi')^2 - \psi' h'\right] = \left(\lambda_F \rho \mu + \tilde{\lambda}\right),
\text{which implies,}

\varepsilon_i \sum_{k=1}^{n} \varepsilon_k \left[2 \left(1 - 2(n-1)\rho\right) \psi' h' + \psi h''\right] + \\
4r \left[(1 - 2(n-1)\rho) \psi \psi'' + (n-1)(n\rho - 1) (\psi')^2 - \psi' h'\right] = \left(\lambda_F \rho \mu + \tilde{\lambda}\right).

Finally, from the last equation of Theorem \textbf{2.1} we have

\sum_{k=1}^{n} \varepsilon_k \left[-2(n-1)\rho \psi \left(4x_k^2 \psi'' + 2x_k \psi'\right) + (n-1)n\rho (2\varepsilon_k x_k \psi')^2 \right] = \lambda_F (m\rho - 1) + \tilde{\lambda},
\text{equivalently,}

-4(n-1)\rho \psi \psi' + 4r \left[(n-1)n\rho (\psi')^2 - 2(n-1)\rho \psi \psi''\right] = \lambda_F (m\rho - 1) + \tilde{\lambda},
\text{therefore, the proof is done.} \quad \square

When \rho = \frac{1}{2(n-1)}$, we obtain the following result:

\textbf{Corollary 2.3.} Let $(\mathbb{R}^n, g)$ be a pseudo-Euclidean space, \( n \geq 3 \), with coordinates \( x = (x_1, \ldots, x_n) \) and \( g_{ij} = \delta_{ij} \varepsilon_i \). Consider \( M = (\mathbb{R}^n, g^*) \times F^m \), where \( g^* = \frac{1}{\psi^2} g \), \( F^m \) is an Einstein semi–Riemannian manifold with constant Ricci curvature \( \lambda_F \) and smooth functions \( \psi(r) \) and \( h(r) \), where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). Then \( M \) is a gradient Schouten soliton with \( h \) as a potential function if, and only if, the functions \( h \) and \( \psi \) satisfy:

\begin{equation}
(2.16) \quad (n - 2)\psi'' + \psi h'' + 2\psi' h' = 0,
\end{equation}
and

\begin{equation}
(2.17) \quad \psi \left[(n - 2) \psi' + \psi h'\right] + r \left[(2 - n) (\psi')^2 - 2\psi \psi' h'\right] = \frac{\lambda_F m}{4(n-1)} + \frac{\tilde{\lambda}}{2},
\end{equation}
and
\[ -nψ' + 2r \left[ \frac{n}{2} (ψ')^2 - ψ'' \right] = \frac{λ_F}{4(n-1)} (m - 2n + 2) + \frac{λ}{2}. \]

**Proof.** By simply inserting \( ρ = \frac{1}{2(n-1)} \) into equations (2.17) and (2.18), the result follows. \( \square \)

The next lemma provides us with the necessary conditions for the existence of gradient Schouten solitons.

**Lemma 2.4.** Consider smooth functions \( ψ(r) \) and \( h(r) \) with \( r = \sum_{i=1}^{n} ε_i x_i^2 \). If \( ψ(r) \) and \( h(r) \) satisfy (2.16) and (2.17), then \( ψ(r) \) satisfies the following ordinary differential equation:

\[ rψ'' = r (ψ')^2 - ψ'. \]

**Proof.** Differentiating both sides of equation (2.17), we get

\[
ψ' [(n-2)ψ' + ψh'] + ψ [(n-2)ψ'' + ψ'h' + ψh''] + \left[ (2 - n) (ψ')^2 - 2ψψ' h' \right] + r \left\{ 2(2-n)ψ'ψ'' - 2 \left[ (ψ')^2 h' + ψψ'h' + ψψ'h'' \right] \right\} = 0.
\]

This is equivalent to

\[ (n-2)(ψ')^2 + ψψ'h' + (n-2)ψψ'' + ψψ'h' + ψ^2 h'' + (2 - n) (ψ')^2 - 2ψψ'h' + +2(2-n)rψψ'' - 2r (ψ')^2 h' - 2rψψ'h' - 2rψψ'h'' = 0. \]

From (2.16) we obtain

\[ ψh'' = -[(n-2)ψ'' + 2ψ'h']. \]

Substituting (2.21) into (2.20), we obtain

\[
(n-2)ψψ'' - ψ [(n-2)ψ'' + 2ψ'h'] + 2(2-n)rψψ'' - 2r (ψ')^2 h' - -2rψψ'h' + 2rψ' [(n-2)ψ'' + 2ψ'h'] = 0,
\]

which is equivalent to

\[ 2ψψ'h' - 2r (ψ')^2 h' + 2rψψ'h' = 0. \]

How \( h' \neq 0 \), we have that

\[ rψψ'' = r(ψ')^2 - ψ'. \]

This concludes the proof. \( \square \)

As a consequence of Lemma 2.4, we are going to prove a classification theorem for gradient Schouten solitons of type \( M = (\mathbb{R}^n, g^*) \times F^m, \) where \((\mathbb{R}^n, g^*)\) is conformal to a pseudo-Euclidean space and invariant by the action of the pseudo-orthogonal group and \( F^m \) is an Einstein manifold.

**Theorem 2.5.** Let \((\mathbb{R}^n, g)\) be a pseudo-Euclidean space, \( n \geq 3 \), with coordinates \( x = (x_1, \cdots, x_n) \) and \( g_{ij} = δ_{ij} ε_i. \) Consider \( M = (\mathbb{R}^n, g^*) \times F^m \) a semi-Riemannian manifold product, where \( g^* = \frac{1}{ψ^2} g, \) \( F^m \) is an Einstein semi-Riemannian manifold with \( m \geq 2 \), constant Ricci curvature \( λ_F \) and smooth functions \( ψ(r) \) and \( h(r) \),

...
where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). Then \( M \) is a gradient Schouten soliton with \( h \) as a potential function if, and only if, the functions \( h \) and \( \psi \) satisfy:

\[
(2.22) \quad \begin{cases} 
\psi(r) = k_2 r, \\
h(r) = \frac{\lambda_F}{2k_2^2} r^{-1} + k_1, \\
\tilde{\lambda} = -\frac{\lambda_F}{2(n-1)} (m - 2n + 2),
\end{cases}
\]

and

\[
(2.23) \quad \begin{cases} 
\psi(r) = k_2 r^{\frac{1}{s}}, \\
h(r) = \frac{(n-2)}{s} (\ln r)^2 + c \ln r + c_1, \\
\tilde{\lambda} = -(m-n+1)(n-2) k_2^2, \\
\lambda_F = (n-2)k_2^2,
\end{cases}
\]

where \( c, c_1 \) and \( k_1 \in \mathbb{R} \) and \( k_2 \in \mathbb{R}^+ \).

Proof. From Lemma 2.4 we know that \( \psi(r) = k_2 r^s \) with \( k_2 > 0 \). Since \( \psi(r) \) must satisfy equation (2.18), it follows that

\[-nk_2 r^s k_2 sr^{s-1} + nk_2^2 r^{s^2 s-2} - 2rk_2 r^s k_2 s(s-1)r^{s-2} = \frac{\lambda_F}{4(n-1)} (m - 2n + 2) + \frac{\tilde{\lambda}}{2},\]

equivalently,

\[-nk_2^2 sr^{2s-1} + nk_2^2 s^2 r^{2s-2} - 2k_2 s(s-1)r^{2s-1} = \frac{\lambda_F}{4(n-1)} (m - 2n + 2) + \frac{\tilde{\lambda}}{2},\]

which implies

\[
(2.23) \quad s(s-1)(n-2)k_2^2 r^{2s-1} = \frac{\lambda_F}{4(n-1)} (m - 2n + 2) + \frac{\tilde{\lambda}}{2}.
\]

Since the right side must be constant, we have either \( s = 1 \) or \( s = \frac{1}{2} \).

If \( s = 1 \), we have \( \lambda = -\frac{\lambda_F}{4(n-1)} (m - 2n + 2) \), by the fact that \( \psi(r) \) and \( h \) satisfy equation (2.17). Note that

\[
\psi(r) = k_2 r, \quad \psi'(r) = k_2, \quad \psi''(r) = 0.
\]

Consequently

\[
(n-2)k_2^2 r + k_2^2 r^2 h' + (2-n)k_2^2 r - 2k_2^2 r^2 h' = \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2},
\]

which implies,

\[
h(r) = \left( \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-1} + k_1,
\]

thus we have,

\[
h(r) = \frac{\lambda_F}{2k_2^2} r^{-1} + k_1.
\]

Reciprocally, the function \( \psi(r) \) and \( h(r) \) given in (2.22) satisfy (2.16), (2.17) and (2.18). Since

\[
h' = -\left( \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2}, \quad h'' = 2 \left( \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-3}.
\]
In this case
\[(n-2)\psi'' + \psi h'' + 2\psi' h' = 2 \left( \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-3} + 2 \left( -\frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2}\],
equivalently,
\[(n-2)\psi'' + \psi h'' + 2\psi' h' = 2 \left( \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2} - 2 \left( \frac{\lambda_F m}{4(n-1)k_2^2} + \frac{\tilde{\lambda}}{2k_2^2} \right) r^{-2}\].
Therefore \((n-2)\psi'' + \psi h'' + 2\psi' h' = 0\).

Clearly, in this case the second and third equation hold.

If \(s = \frac{3}{2}\), we have by (2.17) and (2.18) that 
\[\frac{\lambda_F}{4(n-1)} (m - 2n + 2) + \frac{\lambda}{2} = -\frac{(n-2)k_2^2}{4}\]
and \(\frac{\lambda_F}{4(n-1)} + \frac{1}{2} = \frac{(n-2)k_2^2}{4}\). Consequently we obtain 
\[\dot{\lambda} = -\frac{(m-n+1)(n-2)}{2(n-1)} k_2^2\] and \(\lambda_F = (n-2)k_2^2\). Substituting \(\psi(r) = k_2 r^2\) into (2.16), we obtain 
\[h(r) = \frac{(n-2)}{8} (\ln r)^2 + c_1 \ln r + c_1, \text{ where } c, c_1 \in \mathbb{R}^\ast\].

This concludes the proof of Theorem 2.5. \(\square\)

As a direct consequence of Theorem 2.5 we get, in the Riemannian case, the following result of rigidity.

**Corollary 2.6.** Let \((M^n, g^*) \times F^m\), \(n \geq 3\) and \(m \geq 2\) be a complete gradient Schouten soliton, where \((M^n, g^*)\) is conformal to an Euclidean metric and rotationally symmetric where \(g^*_{ij} = \delta_{ij}/\psi^2\) and \(F^m\) is an Einstein Riemannian manifold. Then \((M^n, g^*)\) is isometric to \(S^{n-1} \times \mathbb{R}\), \(F^m\) is a compact manifold with Ricci curvature equal to \(\lambda_F = (n-2)\) and \(\dot{\lambda} = -\frac{(m-n+1)(n-2)}{2(n-1)} k_2^2\).

**Proof.** Except for homothety we can consider \(k_2 = 1\) in Theorem 2.5. It has been proven in [10] that \((R^n, g^*), g^*_{ij} = \delta_{ij}/\psi^2\), with \(\psi(r) = r^s\), is complete only for \(s = \frac{3}{2}\). Moreover, \((\mathbb{R}^n \setminus \{0\}, g = \frac{1}{r^2} g_0)\) is isometric to \(\mathbb{R} \times S^{n-1}\). Since \(\lambda_F = (n-2) > 0\), it follows from the Bonnet–Myers Theorem that \(F\) is compact.

\(\square\)

Note that depending on the choice of \(m\) and \(n\), \(\dot{\lambda}\) can be negative, zero or positive. In this case, it is possible to build examples of complete gradient Schouten solitons, either expanding, steady or shrinking. We provide three examples of gradient Schouten solitons.

**Example 2.7.** The product manifold \((S^2 \times \mathbb{R}) \times (S^2 \times S^2)\), with the usual product metric, is a complete expanding gradient Schouten soliton with \(\dot{\lambda} = -\frac{3}{2}\). In fact, considering \(m = 4\) and \(n = 3\), and since \(S^2 \times S^2\) is an Einstein manifold with \(\text{com} \lambda_F = 1\), it meets the conditions of Theorem 2.5.

**Example 2.8.** The product manifold \((S^{n-1} \times \mathbb{R}) \times S^{n-1}\), with the usual product metric, is a complete steady gradient Schouten soliton with \(\dot{\lambda} = 0\). In fact, considering \(m = n - 1\) and since the unitary sphere \((n - 1) - \text{dimensional} S^{n-1}\) is an Einstein manifold with \(\lambda_F = n - 2\), it meets the conditions of Theorem 2.5.

**Example 2.9.** The product manifold \((S^3 \times \mathbb{R}) \times S^2\), with the usual product metric, is a complete shrinking gradient Schouten soliton with \(\dot{\lambda} = \frac{1}{2}\). In fact,
considering $m = 2$ and $n = 4$, and since the sphere $S^2$, with $R = \frac{\sqrt{2}}{2}$, is a Einstein manifold with $\lambda_F = 2$, it meets the conditions of Theorem 2.5.

**Example 2.10.** As a consequence of Corollary 2.6 we can build examples of complete gradient Schouten solitons that are not locally flat. To do this, just consider $((S^{n-1} \times \mathbb{R}) \times F^m)$ where $F^m$ is a non-trivial compact Einstein manifold with ricci curvature equal to $\lambda_F = (n - 2)$.

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