About homotopy perturbation method for solving heat–like and wave–like equations with variable coefficients

Francisco M. Fernández

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica, Diag. 113 y 64 (S/N), Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

Abstract

We analyze a recent application of homotopy perturbation method to some heat–like and wave–like models and show that its main results are merely the Taylor expansions of exponential and hyperbolic functions. Besides, the authors require more boundary conditions than those already necessary for the solution of the problem by means of power series.

1 Introduction

In the last years there has been great interest in the power–series solution of physical problems by means of homotopy perturbation method [1–5]. In a series of papers I have shown that the homotopy perturbation method, the Adomian decomposition method and the variational iteration method have produced the poorest and most laughable papers ever published [6–12]. Notice,
for example the application of homotopy perturbation method to obtain the Taylor expansion of exponential functions of the form $e^{iat}$ [4]. Curiously, some journals believe that such contributions may be of interest for the scientific community.

The purpose of this article is to discuss a recent paper by Öziş and Ağırseven [5] who have recently applied the homotopy perturbation method to heat–like and wave–like equations. We compare their approach with the straightforward power–series one.

\section{Heat–like models}

Öziş and Ağırseven studied particular cases of the equation [5]

$$\frac{\partial u(r, t)}{\partial t} = \hat{L}u(r, t) + f(r)$$

(1)

where $\hat{L}$ is a linear operator and $f(r)$ a differentiable function. As a first approximation we can naively try a time–power–series solution of the form

$$u(r, t) = \sum_{j=0}^{\infty} u_j(r) t^j$$

(2)

If we substitute equation (2) into equation (1) we obtain a recurrence relation for the coefficients of the expansion (2):

$$u_{n+1} = \frac{1}{n+1} \left( \hat{L}u_n + f \delta_{n0} \right), \ n = 0, 1, \ldots$$

(3)

where $\delta_{ij}$ is the Kronecker delta. Our first observation is that we only need the coefficient

$$u_0(r) = u(r, 0)$$

(4)

in order to obtain the remaining ones by means of the recurrence relation (3).
The first example studied by Öziş and Ağırseven [5] is a particular case of equation (1) with

\[ \hat{L} = \frac{x^2}{2} \frac{\partial^2}{\partial x^2}, \quad f = 0 \]  

(5)

If we apply the recurrence relation (3) with \( u_0(x) = u(x, 0) = x^2 \) we obtain the coefficients of the time–power series for the exact solution

\[ u(x, t) = x^2 e^t \]  

(6)

that one easily obtains by the method of separation of variables. It is amazing that Öziş and Ağırseven [5] require the additional boundary conditions \( u(0, t) = 0 \) and \( u(1, t) = e^t \). On the other hand, the power–series method reveals that the initial condition completely determines the solution obtained by those authors and that the remaining boundary conditions are redundant. But the most important fact is that those authors were allowed to publish an elaborated perturbation method for obtaining the Taylor series of \( e^t \) [5].

The reader may easily verify that the straightforward application of the recurrence relation (3) enables one to solve all the other heat–like equations chosen by Öziş and Ağırseven [5], and in all the cases one needs only the initial condition (4) in order to obtain the solutions. On the other hand, Öziş and Ağırseven [5] require some additional boundary conditions. For completeness we outline the results in what follows:

Example 2:

\[ \hat{L} = \frac{1}{2} \left( y^2 \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2} \right), \quad f = 0 \]  

(7)

The authors require four Neumann boundary conditions and the initial one \( u(x, y, 0) = y^2 \) [5]. The power series method completely determines the exact solution \( u(x, y, t) = y^2 \cosh t + x^2 \sinh t \) from only the initial condition. Notice
that in this case the authors were able to obtain the Taylor series for \( \sinh t \) and \( \cosh t \) which are far more difficult than the preceding feat.

Example 3:

\[
\dot{L} = \frac{1}{36} \left( x^2 \frac{\partial^2}{\partial x^2} + y^2 \frac{\partial^2}{\partial y^2} + z^2 \frac{\partial^2}{\partial z^2} \right), \quad f = x^4 y^4 z^4
\]  

(8)

The authors require six Neumann boundary conditions and the initial one \( u(x, y, z, 0) = 0 \) [5]. The time–power series completely determined by the initial condition clearly converges towards the exact solution \( u(x, y, z, t) = x^4 y^4 z^4 (e^t - 1) \). In this case the authors clearly show that the homotopy perturbation method is able to provide the Taylor series for \( e^t - 1 \) about \( t = 0 \).

3 Wave–like models

Öziş and Ağırseven [5] also studied some wave–like equations that are particular cases of

\[
\frac{\partial^2 u(r, t)}{\partial t^2} = \dot{L} u(r, t) + f(r)
\]  

(9)

If we substitute the time–power series (2) we realize that one can obtain the coefficients from the recurrence relation

\[
u_{n+2} = \frac{1}{(n+1)(n+2)} \left( \dot{L} u_n + f \delta_{n0} \right), \quad n = 0, 1, \ldots
\]  

(10)

In this case if we have the first two coefficients

\[
u_0(r) = u(r, 0), \quad u_1(r) = \left. \frac{\partial u(r, t)}{\partial t} \right|_{t=0}
\]  

(11)

we can obtain the remaining ones. The example 4 studied by Öziş and Ağırseven [5] is a particular case of equation (9) with the linear operator and function
given by equation (5). If we choose the coefficients $u_0(x) = x$ and $u_1(x) = x^2$ proposed by Öziş and Ağırseven [5] we easily obtain the remaining coefficients of the time–power series of the exact solution

$$u(x, t) = x + x^2 \sinh(t)$$  \hspace{1cm} (12)

Once again we see that we do not need the other boundary conditions $u(0, t) = 0$, and $u(1, t) = 1 + \sinh(t)$ already required by Öziş and Ağırseven [5]. In this case the authors were able to produce the Taylor series for $\sinh(t)$.

We draw the same conclusion regarding the other wave–like models studied by Öziş and Ağırseven [5] as shown in what follows:

Example 5:

$$\hat{L} = \frac{1}{12} \left( x^2 \frac{\partial^2}{\partial x^2} + y^2 \frac{\partial^2}{\partial y^2} \right), \ f = 0$$  \hspace{1cm} (13)

The authors require four Neumann conditions in addition to the initial ones $u_0(x, y) = x^4$, and $u_1(x, y) = y^4$ [5]. One easily verifies that the straightforward power–series method yields the exact solution $u(x, y, t) = x^4 \cosh t + y^4 \sinh t$ from only the two initial conditions already indicated. Once again, the authors were able to obtain the Taylor series for $\cosh t$ and $\sinh t$.

Example 6:

$$\hat{L} = \frac{1}{2} \left( x^2 \frac{\partial^2}{\partial x^2} + y^2 \frac{\partial^2}{\partial y^2} + z^2 \frac{\partial^2}{\partial z^2} \right), \ f = x^2 + y^2 + z^2$$  \hspace{1cm} (14)

The authors made use of six boundary conditions and the two initial ones $u_0(x, y, z) = 0$, and $u_1(x, y, z) = x^2 + y^2 - z^2$ [5]. In this case we verify that the power–series method with the two initial conditions completely determine the exact solution $u(x, y, z, t) = (x^2 + y^2)(e^t - 1) + z^2(e^{-t} - 1)$. As the reader may have already guessed, Öziş and Ağırseven [5] were able to derive the Taylor expansions for $e^t - 1$ and $e^{-t} - 1$ about $t = 0$. 

5
4 Conclusions

We have seen that the homotopy perturbation method proposed by Öziş and Ağırseven [5] produce the Taylor series about $t = 0$ of the solutions of some differential equations. Curiously, those authors require more boundary conditions than the ones already necessary for the straightforward application of the power-series approach. The main results in their paper are Taylor expansions of exponential and hyperbolic functions of time about $t = 0$. It is amazing that a supposedly respectable journal accepts such contribution. However, I may be mistaken and in a near future all the courses on calculus will be teaching homotopy perturbation theory instead of Taylor series.

References

[1] M. S. H. Chowdhury, I. Hashim, and O. Abdulaziz, Phys. Lett. A 368 (2007) 251-258.

[2] A. Yıldırım and T. Öziş, Phys. Lett. A 369 (2007) 70-76.

[3] M. Rafei, H. Daniali, D. D. Ganji, and H. Pashaei, Appl. Math. Comput. 188 (2007) 1419-1425.

[4] A. Sadighi and D. D. Ganji, Phys. Lett. A 372 (2008) 465-469.

[5] T. Öziş and D. Ağırseven, Phys. Lett. A 372 (2008) 5944-5950.

[6] F. M. Fernández, Perturbation Theory for Population Dynamics, arXiv:0712.3376v1

[7] F. M. Fernández, On Some Perturbation Approaches to Population Dynamics, arXiv:0806.0263v1

[8] F. M. Fernández, Phys. Lett. A 372 (2008) 5258-5260.
[9] F. M. Fernández, On the application of homotopy-perturbation and Adomian
decomposition methods to the linear and nonlinear Schrödinger equations,
arXiv:0808.1515v1

[10] F. M. Fernández, On the application of the variational iteration method to a
prey and predator model with variable coefficients, arXiv.0808.1875v2

[11] F. M. Fernández, On the application of homotopy perturbation method to
differential equations, arXiv:0808.2078v2

[12] F. M. Fernández, Homotopy perturbation method: when infinity equals five,
0810.3318v1