String Diagram Rewriting
Modulo Commutative Monoid Structure

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Abstract

We characterise freely generated props with a chosen commutative monoid structure as certain categories of hypergraphs with interfaces. We use this result to give a sound and complete interpretation of rewriting modulo commutative monoid equations in a prop in terms of double-pushout rewriting of hypergraphs.

1 From Monoids to Cospans

We recall some basic definitions on props, following the same notation and terminology as in [2]. We then proceed to formalize the interpretation of commutative monoids as cospans with certain restrictions in Corollary [11], which is the main aim of this section.

Definition 1. A symmetric monoidal theory is a pair $(\Sigma, \varepsilon)$, where $\Sigma$ is a monoidal signature i.e. a set of operations $o : m \to n$ with a fixed arity $m$ and coarity $n$, and $\varepsilon$ is a set of equations, i.e. pairs $\langle l, r \rangle$ of $\Sigma$-terms $l, r : v \to w$ with the same arity and coarity.

Definition 2. A prop is a symmetric strict monoidal category $(\mathcal{C}, +, 0)$ for which $\text{Ob}(\mathcal{C}) = \mathbb{N}$, the monoidal unit is $0 \in \mathbb{N}$, and the monoidal product on objects is given by addition.

The prop freely generated from a symmetric monoidal theory $(\Sigma, \varepsilon)$ has as morphisms the $\Sigma$-terms modulo $\varepsilon$ and the laws of symmetric monoidal categories (Figure 1).
Given two props $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is called a prop-morphism from $\mathcal{C}$ to $\mathcal{D}$ if it is identity-on-objects strict symmetric monoidal functor. props and prop-morphisms form a category $\text{PROP}$.

**Example 3.** The prop $\text{CMon}$ of commutative monoids is the prop freely generated by the signature consisting of $\mu : 2 \to 1$ (multiplication) and $\eta : 0 \to 1$ (unit), and equations expressing commutativity, unitality and associativity of $\mu$.

**Example 4.** The prop $\mathbb{F}$ of functions has morphisms $f : m \to n$ the functions from the finite set $[m]$ to $[n]$. Furthermore, we define the monoidal product on functions to be their disjoint union, i.e. given $f : m_1 \to m_2$ and $g : n_1 \to n_2$, then $f + g : m_1 + n_1 \to m_2 + n_2$ is defined to be a function from $[m_1 + n_1]$ to $[m_2 + n_2]$ s.t.:

$$(f + g)(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq m \\ m_1 + g(i) & \text{if } m_1 + 1 \leq i \leq m_1 + n_1 \end{cases}$$

The theory of commutative monoids presents the prop of functions. A proof of this result may be found for instance in [5].

**Proposition 5.** $\text{CMon} \cong \mathbb{F}$

We now move to consider cospans, and how they relate to the theory of monoids.
Definition 6. Let \( C \) be a category with all finite colimits. A cospan (in \( C \)) from \( X \) to \( Y \) is a pair of arrows \( X \to A \leftarrow Y \). Define \( \text{Csp}(C) \) as the category with:

- the same objects as \( C \) i.e. \( \text{Ob}(\text{Csp}(C)) = \text{Ob}(C) \)
- morphisms \( X \to Y \) being the isomorphism classes of cospans from \( X \) to \( Y \)
- composition is defined by pushout.

Furthermore, there exists a symmetric monoidal structure on \( \text{Csp}(C) \) with the monoidal unit being the initial object \( 0 \in C \) and the monoidal product given by a coproduct in \( C \).

Definition 7 (Right-monogamy in \( \text{Csp}(F) \)). Given \( m \xrightarrow{f} n \xleftarrow{g} t \) in \( \text{Csp}(F) \), we say that it is right-monogamous if \( n = t \) and \( g \) is an isomorphism. In other words, a cospan in \( \text{Csp}(F) \) is right-monogamous if it is of the form \( m \xrightarrow{f} n \xleftarrow{g} n \), for some natural numbers \( m \) and \( n \), a morphism \( f : m \to n \) in \( F \) and a morphism \( g : n \to n \) is an isomorphism in \( F \).

Note isomorphisms in \( F \) are simply bijections between finite sets. We use the more abstract terminology in view of the generalisation of this definition, that we will introduce in the next section.

Proposition 8. Given a right-monogamous cospan \( m \xrightarrow{f} n \xleftarrow{g} n \) in \( \text{Csp}(F) \), there exists a unique cospan \( m \xrightarrow{f^*} n \xleftarrow{id} n \) isomorphic to it.

Proof. Since \( g \) is an isomorphism, we can denote its inverse by \( g^{-1} \). If we define \( f^* \) in the following way \( f^* = f; g^{-1} = g^{-1} \circ f \), then the following diagram commute:

\[
\begin{array}{ccc}
  & n & \\
  m & \xrightarrow{f} & \xleftarrow{g} & n \\
  f^* & \downarrow{g^{-1}} & \searrow{id} & \downarrow{g} \\
  & n & \\
\end{array}
\]

commutes.

Since \( g^{-1} \) is also iso, the mentioned cospans are indeed isomorphic. Now, suppose there is another cospan \( m \xrightarrow{h} n \xleftarrow{id} n \) isomorphic to \( m \xrightarrow{f} n \xleftarrow{g} n \).

But, then, there must exist \( \psi \) s.t. the following diagram commutes:
This gives us \( \text{id}; \psi = \text{id} \Rightarrow \psi = \text{id} \Rightarrow h = f^*; \psi = f^*; \text{id} = f^* \)
Therefore, we have uniqueness.

Proposition 9. Let \( m \overset{f}{\to} n \xleftarrow{\text{id}} n \), \( n \overset{g}{\to} t \xleftarrow{\text{id}} t \), \( m \overset{f_1}{\to} n_1 \xleftarrow{\text{id}} n_1 \) and \( m_2 \overset{f_2}{\to} n_2 \xleftarrow{\text{id}} n_2 \) be right-monogamous cospans in \( \text{Csp}(\mathbb{F}) \). It holds:

1. Identities and symmetries are right-monogamous in \( \text{Csp}(\mathbb{F}) \) (or isomorphic to a right-monogamous cospan)
2. \((m \overset{f}{\to} n \xleftarrow{\text{id}} n); (n \overset{g}{\to} t \xleftarrow{\text{id}} t)\) is right-monogamous in \( \text{Csp}(\mathbb{F}) \)
3. \((m_1 \overset{f_1}{\to} n_1 \xleftarrow{\text{id}} n_1) \bigoplus (m_2 \overset{f_2}{\to} n_2 \xleftarrow{\text{id}} n_2)\) is right-monogamous in \( \text{Csp}(\mathbb{F}) \)

Proof. Firstly, identities in \( \text{Csp}(\mathbb{F}) \) are of the form \( m \overset{\text{id}}{\to} m \xleftarrow{\text{id}} m \) and symmetries are of the form \( m + n \overset{\sigma_{m,n}}{\Rightarrow} n + m \) \( \xleftarrow{\text{id}} n + m \) and they are clearly right-monogamous.
Secondly, \((m \overset{f}{\to} n \xleftarrow{\text{id}} n); (n \overset{g}{\to} t \xleftarrow{\text{id}} t)\) is obtained by taking a pushout and is \( m \overset{g \circ f}{\Rightarrow} t \xleftarrow{\text{id}} t \) which is right-monogamous.
Finally, \((m_1 \overset{f_1}{\to} n_1 \xleftarrow{\text{id}} n_1) \bigoplus (m_2 \overset{f_2}{\to} n_2 \xleftarrow{\text{id}} n_2)\) is obtained by taking a co-product which is simply a disjoint union in \( \mathbb{F} \), giving \( m_1 + m_2 \overset{f_1 + f_2}{\Rightarrow} n_1 + n_2 \xleftarrow{\text{id}} n_1 + n_2 \) 

The previous 2 propositions show us that right-monogamous cospans in \( \text{Csp}(\mathbb{F}) \) form a sub-prop of \( \text{Csp}(\mathbb{F}) \), which we denote by \( \text{RMCsp}(\mathbb{F}) \).

Proposition 10. There exists an isomorphism of props \( H : \mathbb{F} \to \text{RMCsp}(\mathbb{F}) \).

Proof. We define the functor \( H \) in the following way:

- \( H(m) = m \) for all \( m \in \text{Ob}(\mathbb{F}) \)
- \( H(f) = m \overset{f}{\to} n \xleftarrow{\text{id}} n \), for all morphisms \( f : m \to n \) in \( \mathbb{F} \)
Clearly, $H$ is identity-on-objects and maps identities and symmetries in $\mathbb{F}$ to identities and symmetries in $\text{RMCsp}(\mathbb{F})$, respectively. It’s also easy to check that it preserves composition and monoidal product as calculated in the proof of Proposition 9. Hence, $H$ is prop-morphism.

Now, suppose $H(f_1) = H(f_2)$ for $f_1 : m_1 \rightarrow n_1$ and $f_2 : m_2 \rightarrow n_2$. Then $m_1 \xymatrix{f_1} n_1 \xymatrix{\cong} n_1$ and $m_2 \xymatrix{f_2} n_2 \xymatrix{\cong} n_2$ are isomorphic and $m_1 = m_2$ and $n_1 = n_2$. Furthermore, we have a commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 m_1 \xymatrix{f_1} n_1 \xymatrix{\downarrow \psi} m_2 \xymatrix{f_2} n_1 \xymatrix{\downarrow \psi} n_1
\end{array}
\end{array}
\end{array}
\end{array}
$$

where $\psi$ is iso. But, this gives us $\text{id}; \psi = \text{id}$. Therefore, $\psi = \text{id}$ and $f_1 = f_2$. Hence, $H$ is faithful.

Finally, every $m \xymatrix{f} n \xymatrix{\leftarrow g} n$ in $\text{RMCsp}(\mathbb{F})$ is isomorphic to $m \xymatrix{f^*} n \xymatrix{\leftarrow \text{id}} n$, for some $f^*$ by Proposition 8, giving $H(f^*) = m \xymatrix{f^*} n \xymatrix{\leftarrow \text{id}} n$. Therefore, $H$ is full. \hfill $\square$

**Corollary 11.** There is an isomorphism of props $\text{CMon} \cong \text{RMCsp}(\mathbb{F})$.

**Proof.** By simply composing $H$ with the isomorphism in the Proposition 5, we get the isomorphism from $\text{CMon}$ to $\text{RMCsp}(\mathbb{F})$. \hfill $\square$

## 2 The Combinatorial Characterisation

In this section we recall the notion of hypergraph and prove that a freely generated prop with a chosen commutative monoid structure is isomorphic to a category of cospans of hypergraphs with certain restrictions. This is stated formally as Theorem 48 below.

**Remark 12.** Note the work [4], which appeared simultaneously to ours, provides a result dual to Theorem 48: instead of monoids, they consider props with a chosen commutative comonoid structure — called ‘CD-categories’. On the side of hypergraphs, instead of restricting the monogamicity of [3] to right-monogamicity, as we do below, they consider left-monogamicity. We will elaborate on the comparison with our work in a next version of this technical report.
2.1 The category of hypergraphs

Hypergraphs [1] generalise graphs by replacing edges with hyperedges: these not limited to connecting precisely two nodes, but are allowed to connect any number of nodes. In our work, we will be dealing with hyperedges that are ordered and directed, meaning that for each hyperedge we can define its source and target nodes as ordered lists.

**Definition 13.** A directed hypergraph $G$, consists of set of vertices $G_*$ and a set of hyperedges $G_{k,l}$ with an ordered list of $k$ source nodes and an ordered list of $l$ target nodes, for all $k, l \in \mathbb{N}$. In other words, for each $1 \leq i \leq k$ and for each $1 \leq j \leq l$, we can define $i^{th}$ source map $s_i : G_{k,l} \rightarrow G_*$ and $j^{th}$ target map $t_j : G_{k,l} \rightarrow G_*$. Hypergraphs without any hyperedges are called discrete hypergraphs.

Given two (directed) hypergraphs $G$ and $H$, we call a function $f$ a hypergraph homomorphism if it maps $G_*$ to $H_*$, and maps $G_{k,l}$ to $H_{k,l}$ for each $k, l \in \mathbb{N}$ such that for every hyperedge $h \in G_{k,l}$:

- $f(s_i(h)) = s_i(f(h))$, for all $1 \leq i \leq k$
- $f(t_j(h)) = t_j(f(h))$, for all $1 \leq j \leq l$

Directed hypergraphs and hypergraph homomorphisms form a category $\text{Hyp}$.

$\text{Hyp}$ may be also defined as a presheaf topos: this is particularly convenient for calculating (co)limits and to prove adhesivity.

**Proposition 14.** The category $\text{Hyp}$ with objects finite, directed hypergraphs and arrows hypergraph homomorphisms is isomorphic to the functor category $\mathcal{F}_I$, where $I$ is a category with:

- objects being pairs of natural numbers $(k, l) \in \mathbb{N} \times \mathbb{N}$ and an additional object $\star$
- $k + l$ arrows from $(k, l)$ to $\star$ for all $k, l \in \mathbb{N}$

A monoidal signature $\Sigma$ yields a directed hypergraph $G_\Sigma$ with only a single node and a hyperedge for every $\Sigma$-operation $o : k \rightarrow l$ with $k$ sources and $l$ targets (i.e. in $G_{k,l}$). We can use this observation to define a category of labelled hypergraphs.
Definition 15. The slice category $\text{Hyp} \downarrow G_\Sigma$ is called the category of $\Sigma$-labelled hypergraphs and denoted by $\text{Hyp}_\Sigma$.

As proven in [2], morphisms in props may be faithfully interpreted as discrete cospans of hypergraphs, where the cospan structure is meant to represent the ‘interfaces’ (left and right) of the morphism, once seen as a string diagram. Recalling Definition 6 we can define $\text{Csp}(\text{Hyp}_\Sigma)$ as the category with objects being the same as $\text{Hyp}_\Sigma$, arrows being cospans, with composition obtained by taking the pushout and monoidal product given by the coproduct. Following notation in [2], we introduce a faithful, coproduct-preserving functor $D : \mathbb{F} \to \text{Hyp}_\Sigma$ mapping every object $i \in \text{Ob}(\mathbb{F}) = \mathbb{N}$ to a discrete hypergraph with the set of nodes $[i]$ and mapping each function to the induced hypergraph homomorphism.

Definition 16. The category of discrete cospans of hypergraphs is the full subcategory of $\text{Csp}(\text{Hyp}_\Sigma)$ with objects being the discrete hypergraphs. We denote it by $\text{Csp}_D(\text{Hyp}_\Sigma)$.

2.2 Right-monogamous acyclic cospans

As shown in [3], the interpretation of string diagrams into cospans of hypergraphs is not full. In order to characterise the image of the interpretation, it is necessary to restrict to a certain class of cospans, for which the notion of monogamy is developed. In this section we recall such notion, and introduce the related notion of right-monogamy, as an extension of the one developed in Section 1.

Definition 17 (Degree of a node [3]). The in-degree of a node $v$ in hypergraph $G$ is the number of pairs $(h,i)$ where $h$ is an hyperedge with $v$ as its $i^{th}$ target. Similarly, the out-degree of $v$ is the number of pairs $(h,j)$ where $h$ is an hyperedge with $v$ as its $j^{th}$ source.

Definition 18 (Monogamy [3]). Given $m \xrightarrow{f} G \xleftarrow{g} n$ in $\text{Csp}_D(\text{Hyp}_\Sigma)$, let $\text{in}(G)$ be the image of $f$ and $\text{out}(G)$ the image of $g$. We say that the cospan is monogamous if $f$ and $g$ are mono and for all nodes $v$ of $G$:

the in-degree of $v$ is $\begin{cases} 0 & \text{if } v \in \text{in}(G) \\ 1 & \text{otherwise} \end{cases}$

the out-degree of $v$ is $\begin{cases} 0 & \text{if } v \in \text{out}(G) \\ 1 & \text{otherwise} \end{cases}$
We now relax the previous definition into right-monogamy. As we will see later, this is a straight generalisation of the right-monogamy introduced in Definition 7, whence the use of the same terminology.

**Definition 19 (Right-monogamy in $\text{Csp}_D(\text{Hyp}_\Sigma)$).** Given $m \xrightarrow{\ell} G \xleftarrow{\ell'} n$ in $\text{Csp}_D(\text{Hyp}_\Sigma)$, let $\text{in}(G)$ be the image of $f$ and $\text{out}(G)$ the image of $g$. We say that the cospan right-monogamous if $g$ is mono and for all nodes $v$ of $G$:

$$\text{the out-degree of } v \begin{cases} 0 & \text{if } v \in \text{out}(G) \\ 1 & \text{otherwise} \end{cases}$$

It is clear that monogamy is stronger condition than right-monogamy. In other words, monogamy implies right-monogamy.

**Lemma 20.** Identities and symmetries in $\text{Csp}_D(\text{Hyp}_\Sigma)$ are right-monogamous.

**Proof.** Identities and symmetries are monogamous as proven in [3]. Hence, by the previous remark, they are also right-monogamous. \qed

We will make use of the observation that $\text{Hyp}_\Sigma$ is adhesive [7], meaning pullbacks and pushouts are well-behaved for performing double-pushout rewriting. In particular, we use the following lemma.

**Proposition 21 ([6]).** In any adhesive category, the pushout of a monomorphism along any map is a monomorphism.

**Lemma 22.** Let $m \xrightarrow{i_m} G \xleftarrow{i_n} n$ and $n \xrightarrow{i_n} H \xleftarrow{i_k} k$ be arrows in $\text{Csp}_D(\text{Hyp}_\Sigma)$. If both are right-monogamous cospans, then $(m \xrightarrow{i_m} G \xleftarrow{i_n} n); (n \xrightarrow{i_n} H \xleftarrow{i_k} k)$ is also right-monogamous.

**Proof.** Firstly, let’s recall that the composition of cospans $(m \xrightarrow{i_m} G \xleftarrow{i_n} n); (n \xrightarrow{i_n} H \xleftarrow{i_k} k)$ is obtained by taking a pushout and is $m \xrightarrow{j_G \circ i_m} G + \aleph \circ i_n H \xleftarrow{j_H \circ i_k} k$, where $j_G$ is the pushout of $i_n$ along $i_m$, and similarly $j_H$ is the pushout of $i_n$ along $i_n$. By definition of right-monogamy, $i_n$ is mono, and by Propositions ?? and [21] so is $j_H$. Now, $i_k$ is mono, hence $j_H \circ i_k$ is also mono.

Furthermore, cospans $m \xrightarrow{i_m} G \xleftarrow{i_n} n$ and $n \xrightarrow{i_n} H \xleftarrow{i_k} k$ are composed by gluing $G$ with $H$ along the nodes in $n$. This means that each of the nodes in $\text{in}(H)$ can be identified by several nodes in $\text{out}(G)$. However, since $i_n$ is
mono, each of the nodes in \text{out}(G) is identified by exactly one node in \text{in}(H). Hence, two nodes in \text{in}(H) can never be identified with each other after gluing preserving the condition on out-degrees.

\[ \text{Lemma 23.} \] Let \( m_1 \to G_1 \leftarrow n_1 \) and \( m_2 \to G_2 \leftarrow n_2 \) be arrows in \( \text{Csp}_D(\text{Hyp}_\Sigma) \). If both are right-monogamous cospans, then \((m_1 \to G_1 \leftarrow n_1) \oplus (m_2 \to G_2 \leftarrow n_2)\) is also right-monogamous.

\[ \text{Proof.} \] Firstly, let’s recall that the monoidal product of cospans \((m_1 \to G_1 \leftarrow n_1) \oplus (m_2 \to G_2 \leftarrow n_2)\) is obtained by taking a coproduct and therefore the degree of each node is the same as in the original graphs \( G_1 \) and \( G_2 \). Moreover each node is an input iff it is an input in \( G_1 \) or in \( G_2 \) and it is an output iff it is an output in \( G_1 \) or \( G_2 \).

\[ \text{Definition 24.} \] Given a hypergraph \( G \) and two nodes or hyperedges \( a \) and \( b \), a path from \( a \) to \( b \) in \( G \) is an alternating list \( p = [p_1, \ldots, p_n], p_1 = a, p_n = b \) of hyperedges and nodes such that for all hyperedges \( p_i \) the nodes \( p_{i-1} \) and \( p_{i+1} \) are a source and target for \( p_i \) when they are defined (i.e. when \( i > 1 \) and \( i < n \), respectively). If there exists a path from \( a \) to \( b \), we’ll write \( a \leadsto b \). Otherwise, we will write \( a \not\leadsto b \).

A hypergraph is said to be acyclic if it has no path containing the same node twice. Similarly, we say that a cospan \( m \to G \leftarrow n \) is acyclic if \( G \) is acyclic.

\[ \text{Remark 25.} \] Equivalently to the previous definition, we can define acyclicity in terms of hyperedges, since the existence of a path containing the same node twice is equivalent to the the existence of a path containing the same hyperedge twice.

\[ \text{Lemma 26.} \] Identities and symmetries in \( \text{Csp}_D(\text{Hyp}_\Sigma) \) are acyclic.

\[ \text{Proof.} \] Identities and symmetries have no edges, so each path consists of only one node. Hence, identities and symmetries are trivially acyclic.

\[ \text{Lemma 27.} \] Let \( m_1 \to G_1 \leftarrow n_1 \) and \( m_2 \to G_2 \leftarrow n_2 \) be arrows in \( \text{Csp}_D(\text{Hyp}_\Sigma) \). If both are acyclic cospans, then \((m_1 \to G_1 \leftarrow n_1) \oplus (m_2 \to G_2 \leftarrow n_2)\) is also acyclic.

\[ \text{Proof.} \] Firstly, let’s recall that the monoidal product of cospans \((m_1 \to G_1 \leftarrow n_1) \oplus (m_2 \to G_2 \leftarrow n_2)\) is obtained by taking a coproduct \( G_1 + G_2 \), which
doesn’t connect any nodes and hyperedges of $G_1$ to nodes and hyperedges of $G_2$, hence we don’t get any new paths. Since $G_1$ and $G_2$ have no paths containing the same node twice, neither does $G_1 + G_2$. Therefore, $(m_1 \rightarrow G_1 \leftarrow n_1) \bigoplus (m_2 \rightarrow G_2 \leftarrow n_2)$ is acyclic.

Proposition 28. Let $m \rightarrow G \leftarrow n$, $n \rightarrow H \leftarrow o$, $m_1 \rightarrow G_1 \leftarrow n_1$ and $m_2 \rightarrow G_2 \leftarrow n_2$ be right-monogamous acyclic cospans in $\text{Csp}_D(\text{Hyp}_\Sigma)$. It holds:

1. Identities and symmetries are right-monogamous acyclic in $\text{Csp}_D(\text{Hyp}_\Sigma)$

2. $(m \rightarrow G \leftarrow n); (n \rightarrow H \leftarrow o)$ is right-monogamous acyclic in $\text{Csp}_D(\text{Hyp}_\Sigma)$

3. $(m_1 \rightarrow G_1 \leftarrow n_1) \bigoplus (m_2 \rightarrow G_2 \leftarrow n_2)$ is right-monogamous acyclic in $\text{Csp}_D(\text{Hyp}_\Sigma)$

Proof. Firstly, identities and symmetries are right-monogamous acyclic in $\text{Csp}_D(\text{Hyp}_\Sigma)$, by Lemmas 20 and 26.

Secondly, $(m \rightarrow G \leftarrow n); (n \rightarrow H \leftarrow o)$ is right-monogamous by Lemma 22 and is obtained by gluing $G$ with $H$ along the nodes in $n$. This means that each of the nodes in $\text{in}(H)$ can be identified by several nodes in $\text{out}(G)$, and each of the nodes in $\text{out}(G)$ is identified by exactly one node in $\text{in}(H)$. Now, let’s suppose there is a cycle in our newly composed hypergraph which we’ll denote by $K$ and observe that all of its hyperedges belong to either $G$ or $H$ and all of its nodes, except the glued ones, belong to either $G$ or $H$. Denote the shortest cycle by $q$. Now, since all of the nodes in $\text{out}(G)$ have out-degree 0, a cycle mustn’t arrive in any hyperedge from $G$, because, otherwise, either the whole cycle would consist of hyperedges in $G$, or there would exist a path from a hyperedge in $H$ to a hyperedge in $G$, which can’t be as all the glued nodes have out-degree 0 in $G$. This means that if a cycle contains a hyperedge from $G$, then it can be made shorter by removing that hyperedge. Hence, all of the hyperedges in $q$ must be from $H$. This further implies that none of the nodes from $G$ (except the glued ones) is in $q$. As we noted before, no node in $H$ can be identified with another one in $H$, hence every node in $q$ corresponds to exactly one node in $H$. Finally, the composition doesn’t add any new connections between nodes and hyperedges in $H$, which, combined with the previous observation, means that the whole path $q$ lays in $H$, making $H$ cyclic. We have a contradiction! Therefore, $(m \rightarrow G \leftarrow n); (n \rightarrow H \leftarrow o)$ is acyclic.
By Lemmas \[23\] and \[27\] \((m_1 \to G_1 \leftarrow n_1) \bigoplus (m_2 \to G_2 \leftarrow n_2)\) is also right-monogamous acyclic.

The previous proposition shows us that right-monogamous acyclic cospans in \(\text{Csp}_D(\text{Hyp}_\Sigma)\) form a sub-prop of \(\text{Csp}_D(\text{Hyp}_\Sigma)\), which we denote by \(\text{RMACsp}_D(\text{Hyp}_\Sigma)\).

### 2.3 Characterisation Theorem for \(S_\Sigma + \text{CMon}\)

The goal of this section is establishing the isomorphism between \(S_\Sigma + \text{CMon}\) and \(\text{RMACsp}_D(\text{Hyp}_\Sigma)\), where the former may be regarded as an algebraic theory of string diagrams with a chosen commutative monoid structure, and the latter as its combinatorial interpretation.

First, we establish the formal relationship between \(\text{CMon}\) and \(\text{RMACsp}_D(\text{Hyp}_\Sigma)\).

**Proposition 29.** There exists a faithful prop-homomorphism \(D^*: \text{RMCsp}(\mathbb{F}) \to \text{RMACsp}_D(\text{Hyp}_\Sigma)\)

**Proof.** In \[2\], it was shown that there exists a faithful prop-homomorphism \(\tilde{D}: \text{Csp}(\mathbb{F}) \to \text{Csp}_D(\text{Hyp}_\Sigma)\).

Let \(m \xrightarrow{f} n \xleftarrow{g} n\) be a right-monogamous cospan in \(\text{Csp}(\mathbb{F})\). Then, \(\tilde{D}(m \xrightarrow{f} n \xleftarrow{g} n) = Dm \xrightarrow{Df} Dn \xleftarrow{Dg} Dn\). Now, since \(g\) is an isomorphism in \(\mathbb{F}\), it follows that \(Dg\) is an isomorphism in \(\text{Hyp}_\Sigma\). Furthermore, \(Dn\) is discreet and all of its nodes are in \(\text{out}(Dn)\), and given that \(Dg\) is an isomorphism, they all have out-degree 0. Therefore, \(\tilde{D}(m \xrightarrow{f} n \xleftarrow{g} n)\) is right-monogamous. Acyclicity is trivial to show.

Hence, by simply restricting \(\tilde{D}\) to \(\text{RMCsp}(\mathbb{F})\), we get a faithful prop-morphism \(D^*: \text{RMCsp}(\mathbb{F}) \to \text{RMACsp}_D(\text{Hyp}_\Sigma)\).

The previous proposition establishes the correspondence between right-monogamous cospans in \(\text{Csp}(\mathbb{F})\) and right-monogamous acyclic cospans in \(\text{Csp}_D(\text{Hyp}_\Sigma)\) - in particular, discrete right-monogamous acyclic cospans in \(\text{Csp}_D(\text{Hyp}_\Sigma)\). All discrete cospans in \(\text{Csp}_D(\text{Hyp}_\Sigma)\) are trivially acyclic. But, for every discrete right-monogamous cospan \(m \xrightarrow{f} G \xleftarrow{g} n\) in \(\text{Csp}_D(\text{Hyp}_\Sigma)\), all nodes of \(G\) have out-degree of 0, hence every node in \(G\) is also in \(\text{out}(G)\) and mapping \(g\) is epi. By definition of right-monogamy in \(\text{Csp}_D(\text{Hyp}_\Sigma)\), \(g\) is also mono. Therefore, \(g\) is iso and \(G\) is a discrete hypergraph with \(n\) nodes. Furthermore, the morphisms between discrete hypergraphs are
simply mappings between sets of nodes and, thus, correspond perfectly to mappings between finite sets. This shows us that discrete right-monogamous (acyclic) cospans in \( \text{Csp}_D(\text{Hyp}_\Sigma) \) correspond perfectly to right-monogamous cospans in \( \text{Csp}(\mathbb{F}) \) and that the definitions of right-monogamy in \( \text{Csp}(\mathbb{F}) \) and \( \text{Csp}_D(\text{Hyp}_\Sigma) \) coincide.

**Definition 30.** Let \( |\cdot| : \text{CMon} \to \text{RMACsp}_D(\text{Hyp}_\Sigma) \) be the homomorphism obtained by composing the isomorphism of Corollary 11 with the homomorphism of Proposition 29.

In order to relate \( S_\Sigma + \text{CMon} \) and \( \text{RMACsp}_D(\text{Hyp}_\Sigma) \), we will use a strategy analogous to the one used in [3] for theories with just symmetric monoidal structure. In a nutshell, we want to show that \( \text{RMACsp}_D(\text{Hyp}_\Sigma) \) has the universal property of the coproduct.

\[
S_\Sigma \xrightarrow{[\cdot]} \text{RMACsp}_D(\text{Hyp}_\Sigma) \xleftarrow{|\cdot|} \text{CMon}
\]

where \([\cdot] : S_\Sigma \to \text{RMACsp}_D(\text{Hyp}_\Sigma)\) is defined as in [3].

To do so, the fundamental step is investigating how right-monogamous acyclic cospans can be decomposed into monogamous acyclic cospans that are in the image of \( S_\Sigma \) and discrete right-monogamous acyclic cospans that are in the image of \( |\cdot| : \text{CMon} \to \text{RMACsp}_D(\text{Hyp}_\Sigma) \). To this aim, we first introduce some notation and terminology.

**Definition 31.** Let \( v \) be a node of a hypergraph \( H \). We say that \( v \) is a terminal node if its out-degree is 0 i.e. there are no hyperedges of \( H \) with source \( v \).

We extend the definition of source and target maps to cospans of hypergraphs.

**Definition 32.** Let \( m \xrightarrow{f} G \xleftarrow{g} n \) be a cospan of hypergraph \( G \). Then, we define new source map \( s_0 \) as:

\[
s_0(v) = g(v) \text{ for any } v \in n
\]

Similarly, we define new target map \( t_0 \) as:

\[
t_0(v) = f(v) \text{ for any } v \in m
\]
Remark 33. If $G$ is right-monogamous, then terminal nodes are precisely the nodes in $\text{out}(G)$, and the number of terminal nodes is $n$.

Definition 34. Let $C_H = m \rightarrow H \leftarrow n$ be a cospan in $\text{Csp}_D(\text{Hyp}_\Sigma)$. Given a node $v$ of $H$, we call a pair $(z, i)$ an in-connection of $v$ (in $H$) if:

- $z$ is a hyperedge of $H$, such that $t_i(z) = v$, or
- $z \in m$, $i = 0$ and $t_0(z) = v$

We denote the set of in-connections of the first type by $\text{InConn}_H(v)$. We denote the set of all in-connections of $v$ (in $H$) by $\text{InConn}_{C_H}(v)$.

Similarly, we call a pair $(z, i)$ an out-connection of $v$ (in $H$) if:

- $z$ is a hyperedge of $H$, such that $s_i(z) = v$, or
- $z \in n$, $i = 0$ and $s_0(z) = v$

We denote the set of out-connections of the first type by $\text{OutConn}_H(v)$. We denote the set of all out-connections of $v$ (in $H$) by $\text{OutConn}_{C_H}(v)$.

Example 35. The in-connections of the terminal node of the cospan below are $\{(0, 0), (3, 0), (4, 0), (A, 1), (A, 2)\}$ and its only out-connection is $(5, 0)$. 
Definition 36. Let $C_H = m \to H \leftarrow n$ be a right-monogamous acyclic cospan and let $v$ be a terminal node of $H$. An ordered partition of $\text{InConn}_{C_H}(v)$ into $k$ pairwise disjoint sets is called a $k$-cut of $v$. Given a $k$-cut $\omega = (I_1, I_2, \ldots, I_k)$, we define a new cospan $\text{cut}_v(C_H, \omega) = m \to H_\omega \leftarrow n'$ such that:

- $H_\omega$ has additional nodes $u_1, u_2, \ldots, u_k$ which we call $\text{Added}(\omega)$, but doesn’t have $v$
- it preserves all the target maps that are not in $\text{InConn}_{C_H}(v)$
- for all in-connections of $(z, j)$ of $v$, if $(z, j) \in I_i$, then $t_j(z) = u_i$
- $n'$ consists of all the nodes in $n$, except, $l = s_0^{-1}(v)$ (in $C_H$), instead of which it has new nodes $v_1^\omega, \ldots, v_k^\omega$, which we denote $v^\omega$
- source maps are preserved except for $l$ and $s_0(v_i^\omega) = u_i$ in $\text{cut}_v(C_H, \omega)$.

Finally, denote by $\text{Reconnect}_\omega$ the mapping from $\{v_1^\omega, \ldots, v_k^\omega\}$ to $l$.

The number of nodes in the right discrete hypergraph increases by $k - 1$, when using a $k$-cut. Specifically, when performing 1-cut, the number of nodes in the right discrete hypergraph doesn’t change. In fact, 1-cut doesn’t change the cospan at all.

Example 37. By providing an ordered partition of the terminal node in Example 35 $\omega = (\{(0, 0), (A, 1)\}, \{(3, 0), (4, 0)\}, \{(A, 2)\}, \{\}\}$ we get its 4-cut and the cospan below. The mapping $\text{Reconnect}_\omega$ then maps $5_i \to 5$ for $i \in \{1, 2, 3, 4\}$.
The previous definition can be easily generalized by defining a cut on each terminal node in an intuitive way.

**Definition 38.** By performing the previous cutting on all the terminal nodes, we can define complete split of \( C_H \) writing \( \text{cut}(C_H, \Omega) = m \rightarrow H_\Omega \leftarrow n^\Omega \) where, \( \Omega = (\omega_1, \omega_2, ..., \omega_n) \) is the tuple of cuts for each terminal node which we call the complete cut. Furthermore, we denote a mapping from the new nodes to the original nodes by \( \text{Reconnect}_\Omega = \text{Reconnect}_{\omega_1} \cup ... \cup \text{Reconnect}_{\omega_n} \) the mapping from \( n^\Omega \) to \( n \). Here \( n^\Omega = \bigcup_{i=1}^n v_i^\omega \) for terminal nodes \( v_i \).

Now, we recall the definition of convex sub-hypergraphs.

**Definition 39 (3).** A sub-hypergraph \( H \subseteq G \) is convex if, for any nodes \( v, v' \) in \( H \) and any path \( p \) from \( v \) to \( v' \) in \( G \), every hyperedge in \( p \) is also in \( H \).

**Definition 40.** Let \( C_G = m \rightarrow G \leftarrow n \) be a right-monogamous acyclic cospan and let \( H \) be a convex sub-hypergraph of \( G \). We say that a node \( v \in H \) is left-shared if \( \text{InConn}_{C_G}(v) \setminus \text{InConn}_H(v) \neq \emptyset \). We call the partition \( \text{InConn}_{C_G}(v) \setminus \text{InConn}_H(v) \) into two disjoint sets, namely Upper(\( v \)) and Lower(\( v \)) the up-down split. The list of up-down splits for all the left-shared nodes is called an up-down signature.

With similar intuition as the Decomposition Lemma for monogamous acyclic cospans in [3], we now describe the process of decomposing right-monogamous acyclic cospans.
The first step to developing general enough decomposition is the following procedure.

**Procedure 41** (Weak decomposition). Let $C_G = m \rightarrow G \leftarrow n$ be a right-monogamous acyclic cospan and $L$ be a convex sub-hypergraph of $G$. We proceed to describe the ways to factor $G$ as

\[
\begin{align*}
    m \rightarrow \tilde{C}_1 \leftarrow i + k; \quad k \xrightarrow{id} k \quad \bigoplus \quad j + k \rightarrow C_2 \leftarrow n \\
    i \rightarrow L \leftarrow j
\end{align*}
\]

for $k \in \mathbb{N}$ and a cospan $i \rightarrow L \leftarrow j$ where all cospans are right-monogamous acyclic.

**Procedure.** Let $C_1$ be the smallest sub-hypergraph containing the inputs of $G$ and every hyperedge $h$ of $G$, that has a path to $L$, but is not in $L$. Let $C_2$ then be the smallest sub-hypergraph containing the outputs of $G$ such that $C_1 \cup L \cup C_2 = G$. If $C_2$ shared a hyperedge with $C_1$, then a smaller $C'_2$ would exist such that $C_1 \cup L \cup C'_2 = G$. Therefore, $C_2$ shares no hyperedges with $C_1$. Hence, $C_1$, $L$ and $C_2$ only overlap on nodes.

Now let

\[
\begin{align*}
    i' := (C_{1*} \cap L_*) \setminus C_{2*} \\
    j' := (C_{2*} \cap L_*) \setminus C_{1*} \\
    k' := (C_{1*} \cap C_{2*}) \setminus L_* \\
    t' := C_{1*} \cap C_{2*} \cap L_*
\end{align*}
\]

Note that the nodes in $t'$, namely, $\xi_1, \ldots, \xi_{t'}$ are precisely all the left-shared nodes and denote the up-down signature by

\[
\tau = ((\text{Upper}(\xi_1), \text{Lower}(\xi_1)), \ldots, (\text{Upper}(\xi_{t'}), \text{Lower}(\xi_{t'})))
\]

and let $\xi_{r_1}, \ldots, \xi_{r_u}$, be the nodes with non-empty $\text{Lower}(\xi_{r_i})$ sets, where $u$ is a number of nodes with non-empty sets.

Keeping the desired form of the decomposition from Equation (1) in mind, we need to find the decomposition satisfying the following:

\[
i' = \text{in}(L) \setminus \text{out}(L)
\]
$$j' \subseteq \text{out}(L) \setminus \text{in}(L)$$
\[ t' \subseteq (\text{in}(L) \cap \text{out}(L)) \cup (\text{out}(L) \setminus \text{in}(L)) \]
\[ i' \cup j' \cup t' = \text{in}(L) \cup \text{out}(L) \]

However, we need to decide whether we let a node $\xi_i \in t'$, be in $\text{in}(L) \cap \text{out}(L)$ or $\text{out}(L) \setminus \text{in}(L)$. If $\xi_i \in \text{in}(L)$, then there must exist some in-connection in $(\text{InConn}_{C_1}(v) \setminus \text{InConn}_{\tilde{L}}(v)) \cap \text{InConn}_{\tilde{L} \leftarrow L \leftarrow j}(v)$. Intuitively, for nodes shared among all $C_1, L, C_2$ that have in-connections not in $L$, we need to decide if we want to deal with those in-connections in upper part of the middle cospan i.e. $k \xrightarrow{id} k \xleftarrow{id} k$ or lower part of the middle cospan i.e. $i \rightarrow L \leftarrow j$. For that purpose, we can choose to ”cut ” some of those nodes into 2.

We will characterize this by using up-down signature $\tau$ in the following way:

- for each node $\xi \in t'$, if $\text{Lower}(\xi) = \emptyset$, we set $\xi \in \text{out}(L) \setminus \text{in}(L)$
- for nodes with non-empty Lower set $\xi_{r_1}, \ldots, \xi_{r_u}$, we set $\xi_{r_i} \in \text{in}(L) \cap \text{out}(L)$

Now, for every node $\xi \in t'$, we create a new terminal node $\xi'$ in $C_1$, namely $\xi'$ and we set $t_i(h) = \xi'$ for every in-connection $(h, i) \in \text{Upper}(\xi)$. We denote the set $\xi'_1, \ldots, \xi'_{u}$ by $t'_*$. Also, for every node $\xi_{r_j}$, we create a new terminal node in $C_1$, namely $\xi''_{r_j}$ and we set $t_i(h) = \xi''_{r_j}$ for every in-connection $(h, i) \in \text{Lower}(\xi_{r_j})$, and we denote our new hypergraph by $\tilde{C}_1$. We denote the set $\xi''_{r_1}, \ldots, \xi''_{r_u}$ by $u$ Finally, we remove old nodes $\xi_i$. Now, we define the cospans:

- $i' + u \rightarrow L \leftarrow j' + t'$
- $m \rightarrow \tilde{C}_1 \leftarrow k' + t'_* + i' + u$
- $k' + t'_* + j' + t' \rightarrow C_2 \leftarrow n$

where $u \rightarrow L$ maps $\xi''_{r_i}$ to $\xi_{r_i}$ for $1 \leq i \leq u$, $t'_* \rightarrow C_2$ maps $\xi'_{r_i}$ to $\xi_{r_i}$, and all the other maps are inclusion maps.

We denote:

- $i = i' + u, j = j' + t', k = k' + t'_*$

All these cospans are clearly acyclic since $G$ is acyclic. From the construction, $C_1$ (and hence $\tilde{C}_1$) is closed under predecessors and $C_2$ is closed under successors. Furthermore, the mapping from $k + i$ to $C_1$ is mono. Hence, it
follows that these cospans are right-monogamous. Then, (2) is computed as the colimit of:

\[ m \rightarrow \tilde{C}_1 \leftarrow k + i \rightarrow k + L \leftarrow k + j \rightarrow C_2 \leftarrow n \]

Therefore, we get the cospan \( m \rightarrow G \leftarrow n \). Finally, every terminal node \( v \) of the cospan \( i \rightarrow L \leftarrow j \) is either terminal in \( G \) (i.e. in \( \text{out}(G) \)), by Remark 33, or it’s a source of some hyperedge \( h \) of \( G \) not in \( H \). Now, \( h \) must be in \( C_2 \), because otherwise it would be in \( C_1 \), and by construction, it would have a path to some node \( v' \) of \( L \), which cannot happen since \( L \) is convex and must contain all the hyperedges on path from \( v \) to \( v' \). This gives us that \( v \) is in \( C_2 \), hence it’s in \( j \). Therefore, every terminal node in \( L \) is in \( j \), and \( i \rightarrow L \leftarrow j \) is right-monogamous.

\[ \square \]

The weak decomposition is a stepping stone towards the strong decomposition, which we introduce next.

**Definition 42 (Convex offcut).** Let \( H \) be a convex sub-hypergraph of \( G \), and let \( C_H = i \rightarrow H \leftarrow j \) be a cospan obtained using Procedure 41. Now, let \( \Omega \) be a complete cut of \( C_H \). Then \( \text{cut}(H, \Omega) = i \rightarrow H_{\Omega} \leftarrow j' \) is called a convex offcut of \( m \rightarrow G \leftarrow n \)

**Definition 43.** For the decomposition obtained via Procedure 41

\[ m \rightarrow \tilde{C}_1 \leftarrow k + i; \quad k + i \rightarrow k + L \leftarrow k + j; \quad k + j \rightarrow C_2 \leftarrow n \]

we say that the tuple \((\Omega_{\text{in}}, \Omega_{\text{out}})\) is its in-out signature if it satisfies the following:

- \( \Omega_{\text{in}} = (\omega_{\text{in},1}, \omega_{\text{in},2}, \ldots, \omega_{\text{in},k+i}) \) is a complete cut of \( m \rightarrow \tilde{C}_1 \leftarrow k + i \) s.t. the \( \omega_{\text{in},1}, \omega_{\text{in},2}, \ldots, \omega_{\text{in},k} \) are 1-cuts (i.e. the terminal nodes corresponding to \( k \) are not cut), and we’ll denote the obtained offcut by \( m \rightarrow (C_1)_{\Omega_{\text{in}}} \leftarrow k + i \Omega_{\text{in}} \)

- \( \Omega_{\text{out}} = (\omega_{\text{out},1}, \omega_{\text{out},2}, \ldots, \omega_{\text{out},j}) \) is a complete cut of \( i \rightarrow L \leftarrow j \) and we’ll denote the obtained offcut by \( i \rightarrow L_{\Omega_{\text{out}}} \leftarrow j \Omega_{\text{out}} \)

**Lemma 44 (Strong decomposition).** Let \( m \rightarrow \tilde{C}_1 \leftarrow k + i; \quad k + i \rightarrow [id, \beta] \rightarrow k + L \leftarrow k + j; \quad k + j \rightarrow [id, \delta] \rightarrow C_2 \leftarrow n \), be the decomposition of \( m \rightarrow G \leftarrow n \)
obtained via Procedure 41 and let \((\Omega_{\text{in}}, \Omega_{\text{out}})\) be its signature. Then, there exists a map \(\alpha\) and the decomposition of \(m \to G \leftarrow n\) into right-monogamous acyclic cospans as below:

\[
m \to (C_1)_{\Omega_{\text{in}}} \leftarrow k + i_{\Omega_{\text{in}}}^{\ell}; \quad k + i_{\Omega_{\text{in}}}^{\ell} \xrightarrow{id,\alpha} k + L_{\Omega_{\text{out}}} \xleftarrow{id,\Theta_{\Omega_{\text{out}}}} k + j_{\Omega_{\text{out}}} \xrightarrow{id,\delta \circ \text{Reconnect}_{\Omega_{\text{out}}}} C_2 \leftarrow n
\]

**Proof.** Firstly, note that the cut\((\cdot, \cdot)\) operation doesn’t violate right-monogamy or acyclicity, so all the cospans above are right-monogamous acyclic.

The first cospan is simple convex offcut and it is well-defined.

The third cospan is also well-defined from the definition of \(\text{Reconnect}(\cdot)\) map.

The only thing that remains to be defined is the mapping \(\alpha\).

Let a mapping \(\mu\) be such that if \(w\) is a left-shared node with non-empty \(\text{Lower}(w)\) in the up-down signature, and let denote its corresponding node by \(w'' = v \in \tilde{C}_1\). Then, \(\mu\) is any map from \(v^{\omega_{\text{in}}}\) (where \(\omega_{\text{in}}\) is a cut of \(v\) in \(\tilde{C}_1\)) to \(v^{\omega_{\text{out}}}\) (where \(\omega_{\text{out}}\) is a cut of \(v\) in \(L\)). Finally, let’s define \(\alpha\) in the following way. For \(v \in i_{\Omega_{\text{in}}}^{\ell}\) if \(\beta \circ \text{Reconnect}_{\Omega_{\text{in}}}\) is not in \(\text{out}(L_{\Omega_{\text{out}}})\) then \(\alpha(v) = \mu(v)\), otherwise \(\alpha(v) = \beta \circ \text{Reconnect}_{\Omega_{\text{in}}}(v)\).

This decomposition is indeed equal to \(m \to G \leftarrow n\), as the nodes that are cut via \(\Omega_{\text{out}}\) in \(L\) are identified with each other by definition of \(\text{Reconnect}(\cdot)\) map, nodes that are cut with \(\Omega_{\text{in}}\) in \(C_1\) are identified with each other by definition of \(\alpha\). The map \(\mu\) defines the way nodes shared among all \(C_1, C_2\) and \(L\) are ”merged”, and we’ll call it the inner-gluing map. \(\square\)

Note that the decomposition we introduced is fully specified by up-down signature, in-out signature and inner-gluing map. We now introduce a more specific version of the decomposition that will be used later on.

**Definition 45.** Let \(m \to G \leftarrow n\) be a right-monogamous acyclic cospan. We say that the node \(v\) in \(G\) is left-amonogamous if:

- it is in \(\text{in}(G)\) and its in-degree is not equal to 0, or
- it is not in \(\text{in}(G)\) and its in-degree is not equal to 1

We denote the total number of left-amonogamous nodes in \(G\) by \(\text{Lam}(m \to G \leftarrow n)\)

**Lemma 46** (Decomposition of left-amonogamous nodes). Let \(m \to G \leftarrow n\) be a right-monogamous acyclic cospan and \(v\) be a left-amonogamous node. Then there exists \(k \in \mathbb{N}\) and a unique cospan \(i \to v \leftarrow j\) such that \(G\) factors
\[
m \to C_1 \leftarrow i + k; \quad k \xrightarrow{id} k \xleftarrow{id} k \quad \bigoplus \quad j + k \to C_2 \leftarrow n
\]

where:

- cospans \( m \to C_1 \leftarrow i + k \) and \( j + k \to C_2 \leftarrow n \) are right-monogamous acyclic
- \( i \) is a discreet hypergraph with in-degree(v) nodes
- \( j \) is a discreet hypegraph with out-degree(v) nodes (i.e. only 1 node)
- \( \text{Lam}(m \to G \leftarrow n) > \text{Lam}(m \to C_1 \leftarrow i + k) + \text{Lam}(j + k \to C_2 \leftarrow n) \)

**Proof.** Let \( \tilde{C}_1 \) be the smallest sub-hypergraph containing the inputs of \( G \) and every hyperedge \( h \) that has a path to \( v \). Let \( C_2 \) then be the smallest sub-hypergraph containing the outputs of \( G \) such that \( \tilde{C}_1 \cup v \cup C_2 = G \). If \( C_2 \) shared a hyperedge with \( \tilde{C}_1 \), then a smaller \( C_2' \) would exist such that \( \tilde{C}_1 \cup v \cup C_2' = G \). Therefore, \( C_2 \) shares no hyperedges with \( \tilde{C}_1 \). Hence, \( \tilde{C}_1, v \) and \( C_2 \) only overlap on nodes.

Now let

\[
j := C_{2*} \cap v
\]

\[
k := (\tilde{C}_{1*} \cap C_{2*}) \setminus v
\]

Furthermore, let’s define new hypergraph \( C_1 \) that has all the same hyperedges as \( \tilde{C}_1 \) and the same nodes as \( \tilde{C}_1 \) with the exception of \( v \). Instead of \( v \), it has new nodes \( v_1, v_2, \ldots, v_{\text{in-degree}(v)} \), for each in-connection of \( v \). In other words, for each target equal to \( v \), we have a separate node in \( C_1 \). Denote a discreet hypergraph consisting of these nodes by \( i \).

Now, define the following cospans, where arrows are inclusion maps:

\[
m \to C_1 \leftarrow k + i
\]

\[
k + j \to C_2 \leftarrow n
\]

These cospans are clearly acyclic since \( G \) is acyclic and since none of the nodes in \( i \) adds any new paths. From the construction, \( C_1 \) is closed under predecessors and \( C_2 \) is closed under successors. Furthermore, the mapping
from \( k+i \) to \( C_1 \) is mono. Hence, it follows that the cospans are right-monogamous.

The cospan \( i \to v \leftarrow j \) is uniquely defined, with arrows mapping all the nodes in \( i \) and \( j \) to \( v \).

Then, (2) is computed as the colimit of:

\[
m \to C_1 \leftarrow k+i \to k+v \leftarrow k+j \to C_2 \leftarrow n
\]

Now, from the definitions of \( j \) and \( k \), it is clear the two spans identify the node \( v \) with a node in \( C_2 \), and the nodes that occur in both \( C_1 \) and \( C_2 \) (except \( v \)). They also identify the new nodes \( v_1, v_2, ..., v_{\text{in-degree}(v)} \) in \( C_1 \) with \( v \), because of the definition of \( i \) and cospan \( i \to v \leftarrow j \).

Therefore, we get the cospan \( m \to G \leftarrow n \). It only remains to check that \( \text{Lam}(m \to G \leftarrow n) > \text{Lam}(m \to C_1 \leftarrow i+k) + \text{Lam}(j+k \to C_2 \leftarrow n) \). By definition of \( j, k \) and \( k+j \to C_2 \leftarrow n \), we have that the map from \( k+j \) to \( C_2 \) is mono, hence none of the left-monogamous nodes in \( k+j \) is in \( C_2 \). Also, none of the new nodes \( v_1, v_2, ..., v_{\text{in-degree}(v)} \) in \( C_1 \) is left-monogamous. Finally, there cannot be any left-monogamous nodes in \( C_1 \setminus k \), \( C_2 \setminus k \) and \( k \) that aren’t left-monogamous in \( G \), hence:

\[
\text{Lam}(m \to G \leftarrow n) \geq \text{Lam}(m \to C_1 \leftarrow i+k) + \text{Lam}(j+k \to C_2 \leftarrow n) + \text{Lam}(i \to v \leftarrow j)
\]

\[
\Rightarrow \text{Lam}(m \to G \leftarrow n) > \text{Lam}(m \to C_1 \leftarrow i+k) + \text{Lam}(j+k \to C_2 \leftarrow n)
\]

\[
\square
\]

**Lemma 47.** Any right-monogamous acyclic cospan \( m \to G \leftarrow n \) can be decomposed into monogamous acyclic and discreet right-monogamous cospans.

**Proof.** We can prove this using induction on the number of left-monogamous nodes.

For the base case we note that any right-monogamous acyclic cospan without any left-monogamous nodes is simply monogamous acyclic, hence if \( \text{Lam}(m \to G \leftarrow n) = 0 \), then \( m \to G \leftarrow n \) is monogamous acyclic.

For the induction hypothesis, we assume that the statement holds for all the right-monogamous acyclic cospans with the number of left-monogamous nodes strictly less than \( k \) where \( k \) is a positive integer.

For the inductive case, suppose that \( \text{Lam}(m \to G \leftarrow n) = k \). Then, by Lemma 46, it can be decomposed as

\[
m \to C_1 \leftarrow k+i; k+i \to k+v \leftarrow k+j; k+j \to C_2 \leftarrow n
\]
where \( k = \text{Lam}(m \rightarrow G \leftarrow n) > \text{Lam}(m \rightarrow C \leftarrow i+k) + \text{Lam}(j+k \rightarrow C \leftarrow n) \Rightarrow k > \text{Lam}(m \rightarrow C \leftarrow i+k) \), By induction hypothesis, both \( m \rightarrow C \leftarrow i+k \) and \( j+k \rightarrow C \leftarrow n \) can be decomposed into decomposed into monogamous acyclic and discreet right-monogamous cospans. Since \( k+i \rightarrow k+v \leftarrow k+j \) is discreet right-monogamous, this concludes the proof.

We are now able to conclude with our characterisation theorem.

**Theorem 48.** There exists an isomorphism \( \Xi : S_\Sigma + \text{CMon} \rightarrow \text{RMACsp}_D(\text{Hyp}_\Sigma) \).

*Proof.* Let’s define \( \Xi \) as a copairing (in PROP) of the faithful functors 
\[
[[\cdot]] : S_\Sigma \rightarrow \text{RMACsp}_D(\text{Hyp}_\Sigma) \quad \text{and} \quad |\cdot| : \text{CMon} \rightarrow \text{RMACsp}_D(\text{Hyp}_\Sigma).
\]
It suffices to show that \( \text{RMACsp}_D(\text{Hyp}_\Sigma) \) satisfies the universal property of the coproduct \( S_\Sigma + \text{CMon} \) in prop.

Given a prop \( A \) and prop-morphisms \( \alpha : S_\Sigma \rightarrow A \), \( \alpha : \text{CMon} \rightarrow A \), we need to prove there exist a unique prop-morphism \( \gamma : \text{RMACsp}_D(\text{Hyp}_\Sigma) \rightarrow A \), such that the diagram above commutes. Now, since prop-morphisms are identity-on-objects functors, it is sufficient to consider what happens to the arrows of the above props.

Since the diagram needs to commute, for any arrow \( s \) in \( S_\Sigma \) and for any arrows \( c \) in \( \text{CMon} \) it holds:
\[
\gamma([[s]]) = \alpha(s), \quad \gamma(|c|) = \beta(c) \tag{3}
\]
But, by Lemma \[47\], any arrow \( h \) in \( \text{RMACsp}_D(\text{Hyp}_\Sigma) \) can be decomposed as \( h = h_1; h_2; \ldots; h_k \), where \( h_i \) is either \([s_i]\) for some arrow \( s_i \) in \( S_\Sigma \) or \(|c_i|\) for some arrow \( c_i \) in \( \text{CMon} \). In either case, \( \gamma(h_i) \) is uniquely defined by (3).

Now, as \( \gamma \) is a prop-morphism, it holds that:
\[
\gamma(h) = \gamma(h_1; h_2; \ldots; h_k) = \gamma(h_1); \gamma(h_2); \ldots; \gamma(h_k)
\]
This fully and uniquely defines \( \gamma \) by the previous observation. Finally, note that \( \Xi \) is also well-defined since any potential non-uniqueness of decomposition is irrelevant we are dealing with symmetric monoidal categories. \( \square \)
3 Characterisation of String Diagram Rewriting

Now that we have a characterisation theorem for $\Sigma + \text{CMon}$, we are ready to interpret rewriting modulo commutative monoid structure as DPO rewriting, and to show that such a correspondence is sound and complete.

We first recall some basic definitions and introduce our notions of boundary complement and convex matching, which are ‘weaker’ than the analogous ones introduced in [3].

**Definition 49 ([3]).** A rewriting system $\mathcal{R}$ in a prop $\mathcal{A}$ consists of a set of rewriting rules i.e. pairs $\langle l, r \rangle$ of arrows $l, r : i \to j$ in $\mathcal{A}$, with the same arities and coarities. Given $a, b : m \to n$ in $\mathcal{A}$, we say that $a$ rewrites into $b$ via $\mathcal{R}$, written $a \Rightarrow_{\mathcal{R}} b$, if they are decomposable as follows, for some rule $\langle l, r \rangle \in \mathcal{R}$.

\[
\begin{array}{c}
| a & n \\
\downarrow & \downarrow \\
| a_1 & a_2 \\
& k \\
| i & j \\
\end{array} = \begin{array}{c}
| a & n \\
\downarrow & \downarrow \\
| a_1 & a_2 \\
& k \\
| i & j \\
\end{array}
\]

In a similar fashion to [3], in order to preserve rewriting into right-monogamous acyclic cospans, we introduce notion of weak boundary complement.

**Definition 50 (Weak boundary complement).** For right-monogamous acyclic cospans $i \xrightarrow{a_1} L \xleftarrow{a_2} j$ and $n \xrightarrow{b_1} G \xleftarrow{b_2} m$ and a morphism $f : L \to G$, a pushout complement as below:

\[
\begin{array}{c}
L \xleftarrow{g} G \xrightarrow{h}
\end{array}
\]

is called a boundary complement if:

- if two nodes are mapped by $f$ to the same node in $G$, then they must be in the image of $j$ in $L$
• $c_1$ is mono

• no two nodes are both in the image of $c_1$ and $c_2$

and there exist $d_1 : n \rightarrow L^\perp$ and $d_2 : m \rightarrow L^\perp$ making the above diagram commute and such that:

$$j + n \xrightarrow{[c_2,d_1]} L^\perp \xleftarrow{[c_1,d_2]} i + m$$

is right-monogamous.

**Definition 51** (Convex offcut matching). We call $m : H \rightarrow G$ in $\text{Hyp}_\Sigma$ a convex offcut matching if it doesn’t merge any non-terminal nodes and its image is convex sub-hypergraph of $G$.

**Remark 52.** From the definition of convex offcut it is clear that $H$ corresponds perfectly to some offcut generated by cutting the mentioned convex sub-hypergraph (call it $L$), hence we can denote $H$ by $L_\Omega$ for some cut $\Omega$.

**Definition 53.** Given $D \leftarrow n + m$ and $E \leftarrow n + m$ in $\text{Hyp}_\Sigma$, we say that $D$ rewrites into $E$ with interface $n+m$ with notation $(D \leftarrow n + m) \hookrightarrow_R (E \leftarrow n + m)$ if there exists a rule $L \leftarrow i + j \rightarrow R$ in $\mathcal{R}$, object $C$ and cospan $i + j \rightarrow C \leftarrow n + m$ in $\text{Hyp}_\Sigma$ such that the diagram below commutes:

![Diagram]

and:

• $f : L \rightarrow D$ is a convex offcut matching

• $i + j \rightarrow C$ is a boundary complement in the left pushout

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3.1 From boundary complements to decomposition

Contrarily to boundary complements \[3\], weak boundary complements are not necessarily unique if they exist. However, if specified with a valid choice of target maps and mapping \(d_1\), they can be uniquely identified.

In order to illustrate this, first recall that \(\text{Hyp}_\Sigma\) is a presheaf category. That gives us an easy way to obtain a pushout of hypergraphs, by simply taking pushout of sets of nodes, pushout of sets of hyperedges and suitable choice of source and target maps. The two mentioned pushouts(and their complements) are in \(\mathbb{F}\):

\[
\begin{array}{ccc}
L_\star & \xleftarrow{\alpha_\star} & i + j \\
G_\star & \xleftarrow{g_\star} & L^\perp_\star \\
\end{array}
\quad
\begin{array}{ccc}
L_{k,l} & \xleftarrow{0} & 0 \\
G_{k,l} & \xleftarrow{g_{k,l}} & L^\perp_{k,l} \\
\end{array}
\]

where by \(\Upsilon_\star\) we denote the set of nodes of \(\Upsilon\), and by \(\Upsilon_{k,l}\) we denote the set of \((k,l)\)-hyperedges of \(\Upsilon\). \(G_{k,l}\) is a disjoint union, giving us \(L^\perp_{k,l} = G_{k,l} \setminus L_{k,l}\).

Considering the definition of \(f\), we use the following notation for the nodes of \(G\):

- \(i'\) consists of nodes that are in the image of \(\text{in}(L) \setminus \text{out}(L)\)
- \(t'\) consists of nodes that are in the image of \(\text{in}(L) \cap \text{out}(L)\)
- \(j'\) consists of nodes that are in the image of \(\text{out}(L) \setminus \text{in}(L)\) and not in the image of \(\text{in}(L) \cap \text{out}(L)\)

Let’s now use the following notation for the nodes in \(L\) considering the definition of \(f\):

- \(i'\) consists of nodes in \(\text{in}(L) \setminus \text{out}(L)\)
- \(t\) consists of nodes in \(\text{in}(L) \cap \text{out}(L)\)
- \(j_1\) consists of nodes in \(\text{out}(L) \setminus \text{in}(L)\) mapped to \(j'\)
- \(j_2\) consists of nodes in \(\text{out}(L) \setminus \text{in}(L)\) mapped to \(t'\)

Let’s now use the following notation for the nodes in \(i + j\) considering the definition of \(a\):
• \( i_0 \) consists of nodes in \( i \) mapped to \( \text{in}(L) \setminus \text{out}(L) \)
• \( T_0 \) consists of nodes in \( i \) mapped to \( \text{in}(L) \cap \text{out}(L) \)
• \( t \) consists of nodes in \( j \) mapped to \( \text{in}(L) \cap \text{out}(L) \)
• \( j_1 \) consists of nodes in \( j \) mapped to \( j_1 \)
• \( j_2 \) consists of nodes in \( j \) mapped to \( j_2 \)

Now, from the definition of \( c \) and \( f \), we have that the left pushout square can be rewritten as:

\[
\begin{array}{ccc}
  i' + t + j_1 + j_2 + x & \xrightarrow{a_*} & i_0 + T_0 \\
  & \downarrow f_* & \downarrow c_* \\
  i' + t' + j' + x + y & \xleftarrow{g_*} & i_0 + T_0 + j' + t' + z
\end{array}
\]

It’s clear that \( i_1 + t_1 + j_1 + z \) is also a pushout for the given span, giving us \( z \cong x + y \), so we can write the left pushout as:

\[
\begin{array}{ccc}
  i' + t + j_1 + j_2 + x & \xrightarrow{a_*} & i_0 + T_0 \\
  & \downarrow f_* & \downarrow c_* \\
  i' + t' + j' + z & \xleftarrow{g_*} & i_0 + T_0 + j' + t' + z
\end{array}
\]

As can be seen, when \( g_* \) is restricted to \( j' \cup t' \cup z \), it is mono. Now, we examine the source maps. Let \( h \) be a hyperedge in \( L_{k,l}^\perp \). Since \( g \) is a homomorphism, it holds:

\[
g_*(s_{k,l}(h)) = s_{k,l}(g_{k,l}(h)) \Rightarrow s_{k,l}(h) \in g_*^{-1}(s_{k,l}(g_{k,l}(h)))
\]

However, the right-monogamy of

\[
j + n \xrightarrow{[c_2,d_1]} L^\perp \xleftarrow{[c_1,d_2]} i + m
\]
tells us that no node in the image of $i$ can be a source of any hyperedge of $L^\perp$, hence $s_{k,i}(h) = v \in j' \cup t' \cup z$. Since $g_*$ is mono when restricted to $v \in j' \cup t' \cup z$, $s_{k,i}$ is uniquely defined.

Furthermore, right-monogamy shows us that any node $v$ in the image of $d_2 : m \to L$ is in $j' \cup t' \cup z$, hence $g_*^{-1}(g_*(v))$ is unique, hence the mapping $d_2$ is uniquely defined.

Now, let’s elaborate on what valid choice of target maps of $L^\perp$ and mapping $d_1$ might be. We’ll do this by considering $\text{InConn}_{C_\emptyset}(v)$, for all the nodes $v \in G$.

If $v \in j' \cup z$, then $g_*^{-1}(g_*(v))$ is unique.

Now, let $v \in i'$, then $g_*^{-1}(g_*(v))$ contains all the nodes in $i_0$ mapped to it and denote them by $v^1, ..., v^q$. Now, we partition the set of in-connections of $v$ in $C_G$ that are not already in $L$ into $q$ pairwise disjoint subsets, each corresponding to one of $v^1, ..., v^q$. Formally, given an ordered partition of $\text{InConn}_{C_\emptyset}(v) \setminus \text{InConn}_{L}(v)$, namely $\Omega_v = (I_1, ..., I_q)$, if $(h, p) \in I_r$, then $t_p(h) = v^r$ in $L^\perp$.

Finally, let $v \in t'$, then $g_*^{-1}(g_*(v))$ contains all the nodes in $T_0$ mapped to it and denote them by $V_{\text{down}} = \{v^1, ..., v^q\}$. It also contains a node $v_{up} \in t' \subseteq L^\perp$. Now, we partition the set of in-connections of $v$ in $C_G$ that are not already in $L$ into 2 pairwise disjoint subsets $\text{Upper}$ and $\text{Lower}$, each corresponding to one of $v^1, ..., v^q$. Formally, if $(h, i) \in \text{Upper}$ then $t_i(h) = v_{up}$ (in $L^\perp$) and otherwise $t_i(h) \in V_{\text{down}}$. We then further partition $V_{\text{down}}$ into $q$ pairwise disjoint subsets, each corresponding to one of $v^1, ..., v^q$. Formally, given an ordered partition of $V_{\text{down}}$, namely $\Omega_v = (I_1, ..., I_q)$, if $(h, p) \in I_r$, then $t_p(h) = v^r$ in $L^\perp$.

As we can see, by providing a partition of the hyperedges in $C_G$, for each of the nodes in $i'$, and by providing a partition into $\text{Upper}$ and $\text{Lower}$ and further partition of $\text{Lower}$ set for each of the nodes in $t'$, we fully determine the target maps. In fact, it also fully determines the mapping $d_1$ since $d_1$ is, in fact, $i_0$.

But this is precisely what we used in order to obtain target maps in $(C_1)_{\Omega_{in}}$ the decomposition of $C_G$ in the Procedure 41 and Lemma 44. In particular, up-down signature is provided via $\text{Upper}$ and $\text{Lower}$ sets as above. For the in-out signature $(\Omega_{in}, \Omega_{out})$, $\Omega_{in}$ is provided via the partitions of in-connections for nodes in $i'$, and the partition of in-connections in $\text{Lower}$ sets for nodes in $t'$. Finally, $\Omega_{out}$ and inner-gluing map are the property of the convex offset which, by definition $L$ is. This leads us to the following proposition.
Proposition 54. Given suitable target maps, mapping $d_1$ and a convex offcut mapping $f : L \rightarrow G$, there exists a decomposition of $n \rightarrow G \leftarrow m = m \rightarrow C_1 \leftarrow k + i; \ k + i \rightarrow k + L \leftarrow k + j; \ k + j \rightarrow C_2 \leftarrow n$, such that:

$$
(i + j \rightarrow C \leftarrow n + m) := \left\langle \begin{array}{c}
\text{i} \\
k \\
\text{j}
\end{array} \right\rangle
$$

where $\Xi(c_1) = n \rightarrow C_1 \leftarrow i + k$ and $\Xi(c_2) = j + k \rightarrow C_2 \leftarrow m$ is the unique weak boundary complement.

Proof. Firstly, note that for every decomposition the diagram above is a weak boundary complement, since $i \rightarrow C$ is mono, $m \rightarrow C$ is mono, and images of these maps don’t overlap, since nodes in the image of $i$ cannot be identified with the nodes in the image of $m$, as $i + k \rightarrow C_1$ is mono. It’s clearly also a pushout complement. Now, as we have seen in the discussion above, given the suitable choices of target maps, and mapping $d_1$, we can define a decomposition with those target maps and $d_1$ satisfied, by defining the decomposition signature as described. □

3.2 Sound and complete interpretation

We conclude the section with the following theorem.

Theorem 55. Let $\mathcal{R}$ be a rewriting system on $S_{\Sigma} + \text{CMon}$. Then,

$$
d \Rightarrow_{\mathcal{R}} e \iff \langle\langle [d] \rangle\rangle \leftrightarrow_{\mathcal{R}} \langle\langle [e] \rangle\rangle
$$

Proof. For the direction from left to right we proceed as follows. From the definition of rewriting, and given the assumption $d \Rightarrow_{\mathcal{R}} e$, we have that:
giving us the following in $\Sigma + \text{Frob}$:
This gives us:

\[
(0 \rightarrow D \leftarrow n + m) := \langle \lceil d \rceil \rangle = \langle \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array} \rangle
\]

\[
(0 \rightarrow E \leftarrow n + m) := \langle \lceil e \rceil \rangle = \langle \begin{array}{c}
\begin{array}{c}
E
\end{array}
\end{array} \rangle
\]

\[
(0 \rightarrow L \leftarrow i + j) := \langle \lceil l \rceil \rangle = \langle \begin{array}{c}
\begin{array}{c}
L
\end{array}
\end{array} \rangle
\]

\[
(0 \rightarrow R \leftarrow i + j) := \langle \lceil r \rceil \rangle = \langle \begin{array}{c}
\begin{array}{c}
R
\end{array}
\end{array} \rangle
\]

\[
(i + j \rightarrow C \leftarrow n + m) := \langle \begin{array}{c}
\begin{array}{c}
C
\end{array}
\end{array} \rangle.
\]

Recall that the composition of cospans is obtained by taking pushouts, hence we get the diagram as in definition of \( \leftarrow \). It’s also clear that the mapping from \( L \) to \( D \) is a convex offset matching, since \( l \) is simply a substring-diagram of \( d \). Furthermore, note that no two nodes from \( i + k \) can be identified with each other, hence \( i \rightarrow C \) is mono, and no node from \( i \) can be identified with any node in \( j \) or \( m \). As \( m \rightarrow C \) is trivially mono, we have that \( C \) is indeed a weak boundary complement, which we’ll call underlying boundary complement.

Let’s now examine the direction from right to left. Let

\[
\langle \lceil d \rceil \rangle =: \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array} \leftarrow n + m
\]

\[
\langle \lceil e \rceil \rangle =: \begin{array}{c}
\begin{array}{c}
E
\end{array}
\end{array} \leftarrow n + m
\]
Then, by Proposition 54, there exists a decomposition of $n \to D \leftarrow m$, namely:

$$
m \to C_1 \leftarrow k + i; \quad k + i \to k + L \leftarrow k + j; \quad k + j \to C_2 \leftarrow n
$$

such that the boundary complement rewriting $0 \to D \leftarrow n + m$ into $0 \to E \leftarrow n + m$ is given by:

$$
(i + j \to C \leftarrow n + m) = \langle\langle \langle d \rangle \rangle \rangle \mapsto \langle\langle \langle e' \rangle \rangle \rangle
$$

where $\Xi(c_1) = n \to C_1 \leftarrow i + k$ and $\Xi(c_2) = j + k \to C_2 \leftarrow m$.

This further gives us $\Xi(d) = \Xi(c_1; (id \oplus l); c_2)$ and since $\Xi$ is faithful (to $\text{Csp}_D(\text{Hyp}_\Sigma)$), we get $d = c_1; (id \oplus l); c_2$. But, now we have: $d \Rightarrow_{\Delta} e'$. But, by the previous direction, that means that $\langle\langle [d] \rangle \rangle \mapsto_{\Delta} \langle\langle [e'] \rangle \rangle$, with the underlying boundary complement exactly the picture above. That means that $\langle\langle [e] \rangle \rangle = \langle\langle [e'] \rangle \rangle$ hence $e' = e$.

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