On multi-soliton solutions to the Heisenberg ferromagnetic spin chain equation in (2+1)-dimensions*

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Abstract

This paper concentrates on the Heisenberg ferromagnetic spin chain (HFSC) equation in (2+1)-dimensions modelling nonlinear wave propagation in ferromagnetic spin chain. A variable transformation is first employed to reduce the studied equation. And then an associated matrix Riemann-Hilbert problem is built on the real line through analyzing spectral problem of the reduced equation. As a consequence, solving the obtained matrix Riemann-Hilbert problem with the identity jump matrix, corresponding to the reflectionless, the general multi-soliton solutions to the HFSC equation in (2+1)-dimensions are acquired. Specially, the one- and two-soliton solutions are worked out and analyzed graphically.

Keywords: Heisenberg ferromagnetic spin chain equation; Riemann-Hilbert problem; soliton solutions

1. Introduction

Wave phenomena appear in many fields of science and engineering, such as fluid mechanics, optics, solid mechanics, electromagnetism, structural mechanics and quantum mechanics. The waves for these applications are described by solutions to nonlinear partial differential equations (NLPDEs). To date, a number of methods have confirmed the effectiveness in investigating solutions to NLPDEs, some of which include the inverse scattering transformation[1], the Hirota bilinear method[2], the Darboux transformation[3] and the Bäcklund transformation[4]. In recent years, researchers have shown an increased interest in study of NLPDEs utilizing the Riemann-Hilbert technique, such as the coupled nonlinear Schrödinger equation[5], the Kundu–Eckhaus equation[6], the coupled mKdV system[7] and the 𝑁-coupled Hirota equation[8].

In this letter, we consider the Heisenberg ferromagnetic spin chain equation[9] in (2+1)-dimensions

\[ iu_t + \alpha_1 u_{xx} + \alpha_2 u_{yy} + \alpha_3 u_{xy} - \alpha_4 |u|^2 u = 0, \tag{1} \]

where \( \alpha_1 = \kappa^4 (\mu + \mu_2), \alpha_2 = \kappa^4 (\mu_1 + \mu_2), \alpha_3 = 2\kappa^4 \mu_2 \) and \( \alpha_4 = 2\kappa^4 \mu_3 \). In Eq. (1), \( \kappa \) is a lattice parameter, \( \mu \) and \( \mu_1 \) are the coefficients of bilinear exchange interactions along X- and Y-directions, \( \mu_2 \) means a neighboring interaction factor along the diagonal, and \( \mu_3 \) is an uniaxial crystal field anisotropy parameter. Equation (1) is integrable and model nonlinear wave propagation in ferromagnetic spin chain[10]. To date, many studies have been conducted on Eq. (1) and lots of results[10–24] have been achieved. For instance, Lax pair was presented,
based on which and Darboux transformation, a series of rogue wave solutions\cite{16} were attained. In a follow-up study, the mixed breather and rogue wave solution was explored by extended Darboux technique, and the interaction behaviors between the mixed waves were analyzed in detail\cite{20}. The main goal of the current study is to determine multi-soliton solutions to Eq. (1) in the framework of Riemann-Hilbert problem.

2. Matrix Riemann-Hilbert problem

Our starting point is the transformation \( \tilde{x} = x + ky \), which can reduce Eq. (1) to the following equation

\[
i u_t + (\alpha_1 + \alpha_2 k^2 + \alpha_3 k)u_{\tilde{x}\tilde{x}} - \alpha_4 |u|^2 u = 0. \tag{2}
\]

The Lax pair\cite{16} for Eq. (2) reads

\[
\varphi_{\tilde{x}} = U \varphi = (i\zeta \Lambda + iQ)\varphi, \tag{3}
\]

\[
\varphi_t = V \varphi = (-i\alpha_4 \zeta^2 \Lambda + V_1)\varphi, \tag{4}
\]

where \( \varphi = (\varphi_1, \varphi_2)^T \) is the spectral function, the superscript T stands for the vector transpose, and \( \zeta \in \mathbb{C} \) is a spectral parameter. Moreover, \( \alpha_4 = -2(\alpha_1 + \alpha_2 k^2 + \alpha_3 k), \lambda = \text{diag}(1, -1), \)

\[
Q = \begin{pmatrix}
0 & u^* \\
u & 0
\end{pmatrix}, \quad V_1 = \begin{pmatrix}
\frac{i}{2}i\alpha_4 uu^* & -i\alpha_4 \zeta u^* - \frac{1}{2}i\alpha_4 u^* \\
-i\alpha_4 \zeta u + \frac{1}{2}i\alpha_4 u^* & -\frac{i}{2}i\alpha_4 uu^*
\end{pmatrix}.
\]

In our analysis, the potential \( u \) is assumed to rapidly vanish at very large distances. It is known from (3) and (4) that \( \varphi \propto e^{i\zeta \tilde{x} - i\alpha_4 \zeta^2 \tilde{t}} \). Thus we introduce the transformation \( \varphi = \psi e^{i\zeta \tilde{x} - i\alpha_4 \zeta^2 \tilde{t}} \), which helps to turn the Lax pair (3) and (4) into

\[
\psi_{\tilde{x}} = i\zeta [\Lambda, \psi] + U_1 \psi, \tag{5}
\]

\[
\psi_t = -i\alpha_4 \zeta^2 [\Lambda, \psi] + V_1 \psi, \tag{6}
\]

where \([\Lambda, \psi] = \Lambda \psi - \psi \Lambda \) and \( U_1 = iQ \).

Below, we concentrate on the spectral problem (5) to perform spectral analysis. And \( t \) will be viewed as a dummy variable. We express the matrix Jost solutions \( \psi_{\pm}(\tilde{x}, \zeta) \) as

\[
\psi_{\pm}(\tilde{x}, \zeta) = ([\psi_{\pm}]_1, [\psi_{\pm}]_2)(\tilde{x}, \zeta), \tag{7}
\]

with the asymptotics

\[
\psi_{\pm}(\tilde{x}, \zeta) \to I_2, \quad \tilde{x} \to \pm \infty, \tag{8}
\]

where the subscripts in \( \psi \) refer to which end of the \( \tilde{x} \)-axis the boundary conditions are set, and \( I_2 \) is the identity matrix of size 2. By use of the variation of parameters and (8), one can obtain Volterra integral equations that can be cast in \( \psi_{\pm} \) as

\[
\psi_{\pm}(\tilde{x}, \zeta) \equiv I_2 \mp \int_{-\infty}^{\tilde{x}} e^{i\zeta \tilde{x} - i\alpha_4 \zeta^2 \tilde{z}} U_1(z) \psi_{\mp}(z, \zeta) e^{-i\zeta \tilde{z}} dz, \tag{9}
\]

\[
\psi_{+}(\tilde{x}, \zeta) = I_2 - \int_{\tilde{x}}^{+\infty} e^{i\zeta \tilde{x} - i\alpha_4 \zeta^2 \tilde{z}} U_1(z) \psi_{-}(z, \zeta) e^{-i\zeta \tilde{z}} dz. \tag{10}
\]
Then, Eqs. (9) and (10) are analyzed to see that $[\psi_-]_1$ and $[\psi_+]_2$ allow analytical extensions to $\mathbb{C}_-$ and continuous for $\zeta \in \mathbb{C}_- \cup \mathbb{R}$, however, $[\psi_+]_1$ and $[\psi_-]_2$ are analytically extendible to $\mathbb{C}_+$ and continuous for $\zeta \in \mathbb{C}_+ \cup \mathbb{R}$, where $\mathbb{C}_-$ and $\mathbb{C}_+$ are the lower and upper half $\zeta$-plane.

Applying the Abel’s identity, it is revealed that $\psi_{\pm}$ are independent of $\tilde{x}$, since $\text{tr}Q = 0$. Evaluating $\det \psi_-$ at $\tilde{x} = -\infty$ and $\det \psi_+$ at $\tilde{x} = +\infty$, we know that $\det \psi_{\pm}(\tilde{x}, \zeta) = 1$ for $\forall \tilde{x}$ and $\zeta \in \mathbb{R}$. Since $\psi_- e^{i\zeta \tilde{x}}$ and $\psi_+ e^{i\zeta \tilde{x}}$ are matrix solutions of (3), they must be linearly related by the scattering matrix $S(\zeta)$

$$
\psi_- e^{i\zeta \tilde{x}} = \psi_+ e^{i\zeta \tilde{x}} S(\zeta), \quad S(\zeta) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad \zeta \in \mathbb{R}. \tag{11}
$$

We point out that $\det S(\zeta) = 1$ because of $\det \psi_{\pm}(\tilde{x}, \zeta) = 1$.

A matrix Riemann-Hilbert problem required is related to two matrix analytic functions. In consideration of the analytic properties of $\psi_{\pm}$, we define the analytic function in $\mathbb{C}_+$ as

$$
P_1(\tilde{x}, \zeta) = ([\psi_+]_1, [\psi_-]_2)(\tilde{x}, \zeta) = \psi_+ H_1 + \psi_- H_2, \tag{12}
$$

where

$$
H_1 = \text{diag}(1, 0), \quad H_2 = \text{diag}(0, 1). \tag{13}
$$

Now, we examine the large-$\zeta$ asymptotic behavior of $P_1$. Since $P_1$ solves (5), we make an asymptotic expansion for $P_1$ at large-$\zeta$

$$
P_1 = P^{(0)}_1 + \zeta^{-1} P^{(1)}_1 + \zeta^{-2} P^{(2)}_1 + O(\zeta^{-3}), \quad \zeta \to \infty,
$$

and plug the asymptotic expansion into (5). Comparing the coefficients of the same powers of $\zeta$ yields

$$
O(1) : P^{(0)}_{1\tilde{x}} = i[A, P^{(1)}_1] + \bar{U} P^{(0)}_1; \quad O(\zeta) : i[A, P^{(0)}_1] = 0.
$$

Therefore, we find that $P^{(0)}_1 = I_2$. This means that $P_1 \to I_2$ as $\zeta \in \mathbb{C}_+ \to \infty$.

To determine the analytic counterpart of $P_2$ in $\mathbb{C}_-$, we consider the adjoint equation of (6)

$$
\chi \tilde{x} = i\lambda [A, \chi] - \chi U_1. \tag{15}
$$

It can be verified that the inverse matrices

$$
\psi_{\pm}^{-1} = \begin{pmatrix} [\psi_{\pm}^{-1}]^1 \\ [\psi_{\pm}^{-1}]^2 \end{pmatrix} \tag{16}
$$

solve (15), where $[\psi_{\pm}^{-1}]^j (j = 1, 2)$ denote the $j$-th row of $\psi_{\pm}^{-1}$, and obey the boundary conditions $\psi_{\pm}^{-1}(\tilde{x}, \zeta) \to I_2$ as $\tilde{x} \to \pm \infty$. It follows from (11) that

$$
\psi_-^{-1} = e^{i\zeta \tilde{x}} S^{-1}(\zeta) e^{-i\zeta \tilde{x}} \psi_+^{-1}, \quad \zeta \in \mathbb{R}, \tag{17}
$$

where $S^{-1}(\zeta) = (r_{jk})_{2 \times 2}$. Thus, the analytic function $P_2$ in $\mathbb{C}_-$ is expressed as

$$
P_2(\tilde{x}, \zeta) = \begin{pmatrix} [\psi_+^{-1}]^1 \\ [\psi_+^{-1}]^2 \end{pmatrix} (\tilde{x}, \zeta) = H_1 \psi_+^{-1} + H_2 \psi_-^{-1}, \tag{18}
$$

with $H_1$ and $H_2$ being given by (13). And the large-$\zeta$ asymptotic behavior of $P_2$ is $P_2 \to I_2$ as $\zeta \to \infty$. 
Insertion of (7) into (11) yields

\[ [\psi_-]_2 = s_{12} e^{-2i\xi} [\psi_+]_1 + s_{22} [\psi_+]_2. \]

Carrying (16) into (17) leads to

\[ [\psi_-]^2 = r_{21} e^{-2i\xi} [\psi_+]^1_1 + r_{22} [\psi_+]^1_2. \]

Consequently, \( P_1 \) and \( P_2 \) can be given as

\[
P_1 = ([\psi_+]_1, [\psi_-]_2) = ([\psi_+]_1, [\psi_+]_2) \begin{pmatrix} 1 & s_{12} e^{2i\xi} \\ 0 & s_{22} \end{pmatrix}, \quad P_2 = \begin{pmatrix} [\psi_+]^1_1 \\ [\psi_+]^1_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r_{21} e^{-2i\xi} & r_{22} \end{pmatrix} \begin{pmatrix} [\psi_+]^1_1 \\ [\psi_+]^1_2 \end{pmatrix}.
\]

Up to now, we have presented the analytic functions \( P_1 \) in \( \mathbb{C}_+ \) and \( P_2 \) in \( \mathbb{C}_- \), respectively. Through denoting that \( P_1 \to P^+ \) as \( \zeta \in \mathbb{C}_+ \to \mathbb{R} \) and \( P_2 \to P^- \) as \( \zeta \in \mathbb{C}_- \to \mathbb{R} \), a matrix Riemann-Hilbert problem required can be stated on the real line as follows

\[
P^-(\tilde{x}, \zeta) P^+(\tilde{x}, \zeta) = G(\tilde{x}, \zeta) = \begin{pmatrix} 1 & s_{12} e^{2i\xi} \\ r_{21} e^{-2i\xi} & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}, \]

with canonical normalization conditions \( P_1(\tilde{x}, \zeta) \to \mathbf{I}_2 \) as \( \zeta \in \mathbb{C}_+ \to \infty \) and \( P_2(\tilde{x}, \zeta) \to \mathbf{I}_2 \) as \( \zeta \in \mathbb{C}_- \to \infty \).

In what follows, we present reconstruction formula of the potential. Since \( P_1(\tilde{x}, \zeta) \) solves (5), expanding \( P_1(\tilde{x}, \zeta) \) at large-\( \zeta \) as

\[
P_1(\tilde{x}, \zeta) = \mathbf{I}_2 + \zeta^{-1} P_1^{(1)} + \zeta^{-2} P_1^{(2)} + O(\zeta^{-3}), \quad \zeta \to \infty,
\]

and plugging this expansion into (5), we see that

\[
Q = -[\Lambda, P_1^{(1)}] = \begin{pmatrix} 0 & -2(P_1^{(1)})_{12} \\ 2(P_1^{(1)})_{21} & 0 \end{pmatrix} \implies u = 2(P_1^{(1)})_{21}.
\]

Here \((P_1^{(1)})_{21}\) stands for the (2,1)-element of \( P_1^{(1)} \). Hence, the reconstruction for the potential is completed.

### 3. Soliton solutions

Our focus in this section will be on generating soliton solutions to Eq. (1) on basis of the established matrix Riemann-Hilbert problem. Now suppose that the Riemann-Hilbert problem (19) is irregular, i.e., \( \det P_1(\zeta) \) and \( \det P_2(\zeta) \) can be zeros at certain discrete locations in analytic domains. From \( \det \psi_{\pm} = 1, (12) \) and (18), and the scattering relation between \( \psi_+ \) and \( \psi_- \), we derive \( \det P_1(\zeta) = s_{22}(\zeta) \) and \( \det P_2(\zeta) = r_{22}(\zeta) \). Therefore, \( \det P_1(\zeta) \) and \( \det P_2(\zeta) \) are in possession of the same zeros as \( s_{22}(\zeta) \) and \( r_{22}(\zeta) \).

With above analysis, we now specify the locations of zeros. Manifestly, the matrix \( U_1 \) possesses the property \( U_1^\dagger = -U_1 \), (the symbol \( \dagger \) here means the matrix Hermitian). Using this property, we deduce

\[
\psi_{\pm}^\dagger(\tilde{x}, \zeta^*) = \psi_{\pm}^{-1}(\tilde{x}, \zeta).
\]

Taking the Hermitian to (12) and using (18), we find that

\[
P_1^\dagger(\zeta^*) = P_2(\zeta), \quad \zeta \in \mathbb{C}_-,
\]
and the involution property $S^t(\varsigma^*) = S^{-1}(\varsigma)$. It follows at once that $s^t_{12}(\varsigma^*) = r_{22}(\varsigma)$, which tells us that each zero $\varsigma_j$ of $\det P_1$ can generate each zero $\varsigma_j^*$ of $\det P_2$. Let $N \in \mathbb{N}$ be arbitrary. In the generic case, assume that $\det P_1$ and $\det P_2$ respectively possess simple zeros at $\varsigma_j \in \mathbb{C}_+$ and $\varsigma_j \in \mathbb{C}_-$, where $\varsigma_j = \varsigma_j^*, 1 \leq j \leq N$. In this case, each of the kernel of $P_1(\varsigma_j)$ comprises only a single basis column vector $\omega_j$, and each of the kernel of $P_2(\varsigma_j)$ comprises only a single basis row vector $\hat{\omega}_j$:

$$P_1(\varsigma_j)\omega_j = 0, \quad \hat{\omega}_j P_2(\varsigma_j) = 0,$$

where $\omega_j$ and $\hat{\omega}_j$ are column and row vectors. Taking the Hermitian to (22) and utilizing (21), we see that

$$\hat{\omega}_j = \omega_j^t, \quad 1 \leq j \leq N.$$

Then taking $\bar{x}$-derivative and $t$-derivative in (22) respectively, and using (5) and (6), we have

$$P_1(\varsigma_j) \left( \frac{\partial \omega_j}{\partial \bar{x}} - i \varsigma_j \Lambda \omega_j \right) = 0, \quad P_1(\varsigma_j) \left( \frac{\partial \omega_j}{\partial t} + i \alpha_4 \omega_j^2 \Lambda \omega_j \right) = 0.$$

Through computing, we attain $\omega_j = e^{(\varsigma_j \bar{x} - i \alpha_4 \varsigma_j^2 t)} \Lambda \omega_j$, with $\omega_j$ being independent of $\bar{x}$ and $t$. Recalling the relation (24), we obtain $\hat{\omega}_j = \omega_j^* e^{(-i \varsigma_j^* \bar{x} + i \alpha_4 \varsigma_j^2 t)} \Lambda, 1 \leq j \leq N$.

For derivation of soliton solutions, we take $G = I_2$ in (19), corresponding to the reflectionless. Hence, the solutions to the special Riemann-Hilbert problem [25] can be derived as

$$P_1(\varsigma) = I_2 - \sum_{k=1}^{N} \sum_{j=1}^{N} \omega_k \hat{\omega}_j (M^{-1})_{kj} / (\varsigma - \varsigma_j), \quad P_2(\varsigma) = I_2 + \sum_{k=1}^{N} \sum_{j=1}^{N} \omega_k \hat{\omega}_j (M^{-1})_{kj},$$

with $M$ being an $N \times N$ matrix determined by

$$m_{kj} = \frac{\hat{\omega}_k \omega_j}{\varsigma_j - \varsigma_k}, \quad 1 \leq k, j \leq N.$$

Consequently, by incorporating the established formulae after involution properties with $\omega_{j0} = (a_j, b_j)^T$ and $\vartheta_j = i \varsigma_j (x + ky) - i \alpha_4 \varsigma_j^2 t$, we acquire the explicit expression of general $N$-soliton solution to Eq. (1):

$$u(x, y, t) = -2 \sum_{k=1}^{N} \sum_{i=1}^{N} b_k a^*_i e^{\vartheta_j - \vartheta_k} (M^{-1})_{kj}, \quad m_{kj} = \frac{1}{\varsigma_j - \varsigma_k} (a^*_i a_k e^{\vartheta_j + \vartheta_k} + b^*_k b_j e^{-\vartheta_j - \vartheta_k}).$$

Our main concern in the rest of this section is to compute the one- and two-soliton solutions.

(i) When $N = 1$, a direct computation generates the following one-soliton solution

$$u(x, y, t) = -\frac{2a^*_1 b_1 (\varsigma_1 - \varsigma^*_1) e^{\vartheta_1 - \vartheta_1}}{|a_1|^2 e^{\vartheta_1 + \vartheta_1} + |b_1|^2 e^{-\vartheta_1 - \vartheta_1}},$$

where $\vartheta_1 = i \varsigma_1 (x + ky) - i \alpha_4 \varsigma_1^2 t$. Upon setting $b_1 = 1, |a_1|^2 = e^{2\varsigma_1}$ and $\varsigma_1 = \varsigma_{11} + i \varsigma_{12}$, the solution (27) becomes

$$u(x, y, t) = -2i a^*_1 \varsigma_{12} e^{-\xi_1} e^{\vartheta_1^* - \vartheta_1^*} \text{sech}(\vartheta_1^* + \vartheta_1 + \xi_1),$$

where $\vartheta_1^* + \vartheta_1 = -2\varsigma_{12} (x + ky) + 4i \alpha_4 \varsigma_{11} \varsigma_{12}$ and $\vartheta_1^* - \vartheta_1 = -2i \varsigma_{11} (x + ky) + 2i \alpha_4 \varsigma_{11}^2 t - 2i \alpha_4 \varsigma_{12}^2 t$. Equivalently, the solution (28) reads

$$u(x, y, t) = -2i a^*_1 \varsigma_{12} e^{-\xi_1} e^{-2i \varsigma_{11} (x + ky) + 2i \alpha_4 \varsigma_{11}^2 t - 2i \alpha_4 \varsigma_{12}^2 t} \text{sech}(-2i \varsigma_{12} (x + ky) + 4i \alpha_4 \varsigma_{11} \varsigma_{12} + \xi_1).$$

(29)
Figure 1: Profiles of one-soliton solution (29) with \(a_1 = 1, b_1 = 0.5, \varsigma_{11} = 0.2, \varsigma_{12} = 0.3, \xi_1 = 0, k = 1, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, y = 0\). (a) three-dimentional plot; (b) x-curves.

It is pointed out from (29) that the amplitude function \(|u|\) is a sech-shaped solitary wave with peak amplitude \(2|a_1|\varsigma_{12}e^{-\xi_1}\). The phase linearly relies on the spatial variables \(x, y\) and temporal variable \(t\). This wave propagates at velocity \(2\alpha_4\varsigma_{11}\) after setting \(y = 0\), which is merely dependant on the real part of the spectral parameter \(\varsigma_1\). By choosing parameters as \(a_1 = 1, b_1 = 0.5, \varsigma_{11} = 0.2, \varsigma_{12} = 0.3, \xi_1 = 0, k = 1, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, y = 0\), then the localization of this solution is plotted in \((x, t)\)-plane. From Fig. 1(b), it is seen that the wave travels towards the negative direction of the \(x\)-axis as time evolves.

(ii) When \(N = 2\), two-soliton solution is gained as

\[
u(x, y, t) = \frac{-2(b_1^* a_1^* m_{22} e^{\sigma_{11}^* - \sigma_1} + b_1 a_2 m_{12} e^{\sigma_{11}^* - \sigma_1} - b_2 a_1^* m_{21} e^{\sigma_{12}^*-\sigma_2} + b_2 a_2^* m_{11} e^{\sigma_{12}^*-\sigma_2})}{m_{11} m_{22} - m_{12} m_{21}},
\]

where

\[
m_{11} = \frac{1}{\varsigma_1 - \varsigma_1} \left(|a_1|^2 e^{\sigma_{11}^* + \sigma_1} + |b_1|^2 e^{-\sigma_{11}^* - \sigma_1}\right), \quad m_{12} = \frac{1}{\varsigma_2 - \varsigma_1} \left(a_1^* a_2 e^{\sigma_{11}^* + \sigma_2} + b_1^* b_2 e^{-\sigma_{11}^* - \sigma_2}\right),
\]

\[
m_{21} = \frac{1}{\varsigma_1 - \varsigma_2} \left(b_1^* a_2^* e^{\sigma_{12}^* + \sigma_1} + b_2^* a_1^* e^{-\sigma_{12}^* - \sigma_1}\right), \quad m_{22} = \frac{1}{\varsigma_2 - \varsigma_2} \left(|a_2|^2 e^{\sigma_{12}^* + \sigma_2} + |b_2|^2 e^{-\sigma_{12}^* - \sigma_2}\right),
\]

and \(\sigma_\epsilon = i\varsigma_\epsilon (x + ky) - i\alpha_4\varsigma_\epsilon^2 t, \varsigma_\epsilon = \varsigma_{1\epsilon} + i\varsigma_{2\epsilon}, \epsilon = 1, 2\). Under assumptions of \(a_1 = a_2, b_1 = b_2 = 1\) and \(|a_1|^2 = e^{2\xi_1}\), then the solution (30) is of the form

\[
u(x, y, t) = \frac{-2(a_1^* m_{22} e^{\sigma_{11}^* - \sigma_1} - a_2^* m_{12} e^{\sigma_{11}^* - \sigma_1} - a_1^* m_{21} e^{\sigma_{12}^*-\sigma_2} + a_2^* m_{11} e^{\sigma_{12}^*-\sigma_2})}{m_{11} m_{22} - m_{12} m_{21}},
\]

where

\[
m_{11} = \frac{-i}{\varsigma_{12}} e^{\xi_1} \cos(\sigma_{\epsilon}^* + \sigma_1 + \xi_1), \quad m_{12} = \frac{2 e^{\xi_1}}{\varsigma_{21} - \varsigma_{11} + i(\varsigma_{12} + \varsigma_{22})} \cos(\sigma_{\epsilon}^* + \sigma_2 + \xi_1),
\]

\[
m_{21} = \frac{-i}{\varsigma_{22}} e^{\xi_1} \cos(\sigma_{\epsilon}^* + \sigma_2 + \xi_1), \quad m_{22} = \frac{2 e^{\xi_1}}{\varsigma_{11} - \varsigma_{21} + i(\varsigma_{12} + \varsigma_{22})} \cos(\sigma_{\epsilon}^* + \sigma_1 + \xi_1).
\]

Below, we would like to examine two types of behaviors between two solitons:

Case 1. Two solitons propagate at the different speeds. The parameters in (31) are selected as \(a_1 = 1, a_2 = 1, \varsigma_{11} = 0.1, \varsigma_{12} = 0.3, \varsigma_{21} = 0.3, \varsigma_{22} = 0.5, \xi_1 = 0, k = 1, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, y = 0\), which can guarantee the different velocities of two solitons. The corresponding solution can be computed directly and displayed by figure. Fig. 2(a) exhibits the localized structure of this solution in three-dimensions. The collision between two
solitons occurs as shown in Fig. 2(b), where two solitons propagate together towards the negative direction of the $x$-axis. The soliton with a larger amplitude moves much faster than the soliton with a smaller amplitude, and the larger soliton gradually overtakes the smaller as time goes on. When $t = 0$, the amplitude superposition for two solitons reaches maximum value. And their interaction is elastic. The spatial structure of two solitons will be changed accordingly if we take other values for the parameters.

Case 2. Two solitons move at the equal speeds. Specifying the solution parameters in (31) as $a_1 = 1, a_2 = 1, \varsigma_{12} = 0.3, \varsigma_{21} = 0.3, \varsigma_{22} = 0.5, \xi_1 = 0, k = 1, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, y = 0$ as well as $\rho_{11} = \rho_{21} = 0$ or $\rho_{11} = \rho_{21} = 0.1$, the corresponding solution can be worked out and demonstrated in Fig. 3. Concerning this case, two solitons are spatial localization and stay together in traveling, namely, they are in a bound state. According to these parameter values, the wave speed for two solitons moving from right to left along the $x$-axis is -1.2 in Fig. 3(c). It can be observed that when the solitons propagate, the amplitude function is periodic in oscillating as time goes on. And this solution represents a breather.

4. Conclusion

In summary, this paper mainly presents an application of the Riemann-Hilbert technique to the Heisenberg ferromagnetic spin chain equation in (2+1)-dimensions which models nonlinear wave propagation in ferromagnetic spin chain. As a result, the general multi-soliton solutions to the studied equation were attained. Through choosing suitable values for the relevant parameters, the localization in three-dimensions and dynamics in two-dimensions of one- and two-soliton solutions were depicted with the Maple plot tool.

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Figure 3: Profiles of two-soliton solution (31) with \( a_1 = 1, a_2 = 1, \xi_{12} = 0.3, \xi_{22} = 0.5, \xi_1 = 0, k = 1, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, y = 0 \). (a) three-dimensional plot with \( \zeta_{11} = \zeta_{21} = 0 \); (b) \( x \)-curves in Fig. 3(a); (c) three-dimensional plot with \( \zeta_{11} = \zeta_{21} = 0.1 \); (d) \( x \)-curves in Fig. 3(c).
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