On Convergence of Tracking Differentiator with Multiple Stochastic Disturbances

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Abstract—In this paper, the convergence and noise-tolerant performance of a tracking differentiator in the presence of multiple stochastic disturbances are investigated for the first time. We consider a quite general case where the input signal is corrupted by additive colored noise, and the tracking differentiator itself is disturbed by additive colored noise and white noise. It is shown that the tracking differentiator tracks the input signal and its generalized derivatives in mean square and even in almost sure sense when the stochastic noise affecting the input signal is vanishing. Some numerical simulations are performed to validate the theoretical results.

Index Terms—Tracking differentiator, convergence, noise-tolerant performance, multiple stochastic disturbances.

I. INTRODUCTION

It is generally known that the powerful yet primitive proportional-integral-derivative (PID) control law developed during the period of the 1920s-1940s has been dominating control engineering for one century. However, the derivative control may be not practically feasible because the classical differentiation is sensitive to and may amplify the noise. A noise-tolerant tracking differentiator (TD) which is also the first part of the powerful active disturbance rejection control (ADRC) technology [11], was first proposed by Han in [2]. A detailed comparison with other differentiators aforementioned was made in [3]. The effectiveness of TD has been validated by numerous numerical experiments and engineering applications, see, for instance [4], [5], [6], [7]. The convergence of a simple linear TD was first presented in [8] with application for online estimation of the frequency of sinusoidal signals. Some convergence analyses of the nonlinear TD for both two-dimensional and high-dimensional cases under some weak assumptions were given in [9]. The weak convergence of a nonlinear TD based on finite-time stable system was presented in [10]. The more comprehensive introduction including the convergence analysis of linear, nonlinear and finite-time stable TD can be found in Chapter 2 of the monograph [11] without considering input noises. However, in practical implementations, stochastic disturbances are inevitable and the stochastic systems are modelled in many situations, see, for instance [12], [13], [14], [15]. Motivated from this consideration, in this paper, we investigate for the first time, the convergence and noise-tolerant performance of TD when the input signal is corrupted by additive colored noise, and the TD itself is disturbed by additive colored and white noises.

The main contributions and novelty of this paper are twofold. Firstly, from a theoretical perspective, the convergence and noise-tolerant performance of TD are firstly analyzed rigorously in the presence of multiple stochastic disturbances which include both additive colored noise and white noise. Secondly, the theoretical results reveal that the states of TD track both the input signal and its generalized derivatives in mean square and even in almost sure sense in the case that the stochastic noise corrupting the input signal is vanishing.

We proceed as follows. In the next section, section II the problem is formulated and some preliminaries are presented. In Section III the main results are presented with proofs in Appendices. Some numerical simulations are presented in section IV followed up concluding remarks in section V.

II. PROBLEM FORMULATION AND PRELIMINARIES

The following notations are used throughout the paper. The $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space; $\mathbb{E}X$ or $\mathbb{E}(X)$ denotes the mathematical expectation of a random variable $X$; For a vector or matrix $X$, $X^\top$ represents its transpose; $|X|$ represents the absolute value of a scalar $X$, and $\|X\|$ represents the Euclidean norm of a vector $X$; $a \wedge b$ denotes the minimum of reals $a$ and $b$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ on which three mutually independent one-dimensional standard Brownian motions $B_i(t) \ (i = 1, 2, 3)$ are defined. In many cases, the stochastic disturbances are modeled by white noise which is a stationary stochastic process that has zero mean and constant spectral density and is the generalized derivative of the Brownian motion (see, e.g., [16, p.51, Theorem 3.14]). Nevertheless, the white noise does not always well describe the stochastic disturbances occurring in nature because its $\delta$-function correlation is an idealization of the correlations of real processes which often have finite, or even long, correlation time [17]. A more realistic description could be given by an exponentially correlated process, which is known as colored noise or Ornstein-Uhlenbeck process [17], [18]. Let $\omega_i(t) \ (i = 1, 2)$ denote the
colored noise. They are the solutions of the Itô-type stochastic differential equations (see, e.g., [17] p.426, [19] p.101):
\[ dw_i(t) = -\alpha_i w_i(t) dt + \alpha_i \sqrt{2\beta_i} dB_i(t), \]
where \( \alpha_i > 0 \) and \( \beta_i \) are given constants describing the correlation time and the noise intensity, respectively, and the initial values \( w_i(0) \in L^2(\Omega; \mathbb{R}) \) are independent of \( B_i(t) \). In other words, the parameters \( \alpha_i \) describe the bandwidth of the noise, while \( \beta_i \) denote its spectral height, and the correlation functions of the processes \( w_i(t) \) are more realistic exponential functions yet not the \( \delta \)-ones (see, e.g., [17]). In what follows, \( \alpha_i \) and \( \beta_i \) can be unknown constants.

Let \( v(t) \) be a time-varying input signal which is supposed to be contaminated by additive colored noise. Therefore, the input signal is actually
\[ v^*(t) := v(t) + \sigma_1 w_1(t), \]
where \( \sigma_1 \) is a constant that could be unknown and represents the intensity of the colored noise. In addition, we consider a general case where the system constructing TD is disturbed by both additive colored noise and white noise as follows:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_2(t) dt, \\
\frac{dx_2(t)}{dt} &= x_3(t) dt, \\
\vdots \\
\frac{dx_{n-1}(t)}{dt} &= x_n(t) dt, \\
\frac{dx_n(t)}{dt} &= r^n f(x_1(t) - v^*(t), x_2(t), \ldots, x_n(t)) dt + \sigma_2 w_2(t) dt + \sigma_3 B_3(t),
\end{align*}
\]

where \( r > 0 \) is a tuning parameter, \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is an appropriate known function chosen to satisfy the following Assumption (A1), and \( \sigma_2 w_2(t) + \sigma_3 B_3(t) \) represents the multiple stochastic disturbances with \( \sigma_i \ (i = 2, 3) \) being constants that could be unknown and \( B_3(t) \) being the white noise which is the formal derivative of the Brownian motion. Han’s TD in [2] is a special case of (3) with \( \sigma_i = 0, i = 1, 2, 3 \). The consideration of such a TD is based on three aspects: First, such a TD itself in noisy environment is quite general whereas the TD without any noise corruption is just a special case of \( \sigma_2 = \sigma_3 = 0 \). Second, the quantization errors caused by the digital implementation of TD always exist and can be regarded as a kind of process noise. Finally, TD is the first part of the powerful ADRC which has been hardwired into the general purpose control chips made by industry giants such as Texas Instruments [20], where the hardware might work in noisy environment.

In addition, it should be noticed that the solution of (3) depends on the tuning parameter \( r \). Hereafter, we always drop \( r \) from solutions by abuse of notation without confusion.

The following Assumption (A1) is a prior assumption about the known function \( f(\cdot) \) chosen in constructing TD (3).

**Assumption (A1).** The \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a locally Lipschitz continuous function with respect to its arguments, \( f(0, \ldots, 0) \), and there exist known constants \( \lambda_i > 0 \ (i = 1, 2, 3, 4) \) and a twice continuously differentiable function \( V : \mathbb{R}^n \rightarrow [0, \infty) \) which is positive definite and radially unbounded such that
\[
\lambda_1 \| z \|^2 \leq V(z) \leq \lambda_2 \| z \|^2, \quad \lambda_3 \| z \|^2 \leq W(z) \leq \lambda_4 \| z \|^2,
\]
and
\[
\frac{\sum_{i=1}^{n-1} \partial V(z)}{\partial z_i} z_{i+1} + \frac{\partial V(z)}{\partial z_n} f(z) \leq -W(z),
\]
\[ \frac{\partial V(z)}{\partial z_j} \leq c_1 \| z \|, \quad \frac{\partial^2 V(z)}{\partial z_j \partial z_k} \leq c_2, \]
\[ \forall z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n, \quad j, n, \]
for some nonnegative continuous function \( W : \mathbb{R}^n \rightarrow [0, \infty) \) and some constants \( c_i > 0 \ (i = 1, 2) \).

**Remark II.1.** Generally speaking, the Assumption (A1) guarantees that the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is chosen so that the zero equilibrium state of the following system
\[ \dot{z}(t) = (z_2(t), z_3(t), \ldots, f(z(t))) \]
is globally exponentially stable with \( z = (z_1, z_2, \ldots, z_n) \). It is easy to verify that the simplest example to satisfy Assumption (A1) is the linear function
\[ f(z) = a_1 z_1 + \cdots + a_n z_n, \]
where the parameters \( a_i \ (i = 1, 2, \ldots, n) \) are chosen such that the matrix
\[ A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{pmatrix}_{n \times n}
\]
is Hurwitz. The TD with linear function \( f(\cdot) \) given by (6) is referred as linear TD in what follows.

The solution of (1) can be explicitly expressed as
\[ w_i(t) = e^{-\alpha_i t} w_i(0) + \int_0^t e^{-\alpha_i (t-s)} \alpha_i \sqrt{2\beta_i} dB_i(s). \]
Define
\[ \gamma_i = \mathbb{E}|w_i(0)|^2 + \alpha_i \beta_i, \quad i = 1, 2. \]
By the Itô isometric formula, it is easy to verify that the second moments of \( w_i(t) \ (i = 1, 2) \) are bounded:
\[ \mathbb{E}|w_i(t)|^2 \leq e^{-2\alpha_i t} \mathbb{E}|w_i(0)|^2 + \mathbb{E} \int_0^t e^{-\alpha_i (t-s)} \alpha_i \sqrt{2\beta_i} dB_i(s)^2 \leq \mathbb{E}|w_i(0)|^2 + 2\alpha_i^2 \beta_i \int_0^t e^{-2\alpha_i (t-s)} ds \leq \gamma_i, \quad \forall t \geq 0. \]
This is the reason behind that the TD may be feasible when the input signal is disturbed by additive colored noise.

### III. Main Results

Set \( \hat{B}_1(t) = \sqrt{r} B_1(t / r) \), \( \hat{B}_3(t) = \sqrt{r} B_3(t / r) \). Notice that for any \( r > 0 \), \( B_1(t) \) and \( \hat{B}_3(t) \) are still mutually independent one-dimensional standard Brownian motions. By definition of \( v^*(t) \) in (2), it follows that
\[ dv^*(t) = \dot{v}(t) dt - \sigma_1 \alpha_1 w_1(t) dt + \sigma_1 \alpha_1 \sqrt{2\beta_1} dB_1(t), \]
and then
\[ dv^*(t / r) = \dot{v}(t / r) dt - \frac{\sigma_1 \alpha_1}{r} w_1(t / r) dt + \frac{\sigma_1 \alpha_1 \sqrt{2\beta_1}}{\sqrt{r}} dB_1(t), \]
where \( i(\frac{\tau}{r}) \) denotes, in what follows, the derivative of \( \varphi(\frac{\tau}{r}) \) with respect to the time \( t \). For \( i = 2, \ldots, n \), set

\[
y_1(t) = x_1(\frac{t}{r}) - v^* \frac{t}{r}, \quad y_i(t) = \frac{1}{r^{i-1}} x_i(\frac{t}{r}). \tag{13}
\]

A direct computation shows that \( y(t) = (y_1(t), \ldots, y_n(t)) \) satisfies the following Itô-type stochastic differential equation:

\[
\begin{align*}
dy_1(t) &= y_2(t)dt - \frac{\gamma_1}{r} y_1(t)dt + \frac{\sigma_1}{r} w_1(t)dt \\
y_2(t) &= y_3(t)dt, \\
&\vdots \\
y_{n-1}(t) &= y_n(t)dt, \\
y_n(t) &= f(y(t))dt + \frac{\sigma_n}{\sqrt{n}} \sqrt{\pi} dt
\end{align*}
\tag{14}
\]

We first introduce Lemma III.1 below to present the existence and uniqueness of the global solution to system (14) and give an estimate of the second moment of the global solution.

**Lemma III.1.** Suppose that \( v : [0, \infty) \to \mathbb{R} \) is a continuously differentiable function satisfying \( \sup_{t \geq 0} (|v(t)| + |\dot{v}(t)|) \leq M \) for some constant \( M > 0 \) and Assumption (A1) holds, and the tuning parameter \( r \) is chosen so that \( r \geq 1 \). Then, for any initial value \( x(0) \in \mathbb{R}^n \), system (14) admits a unique global solution which satisfies

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} \|y(s)\|^2 \right) \leq \frac{1}{\lambda_1} \left( N_1 + \frac{N_2}{3} \right) e^{\lambda_2 t}, \quad \forall t \geq 0,
\tag{15}
\]

where the constants \( N_i (i = 1, 2, 3) \) are specified in (15).

**Proof.** See “Proof of Lemma III.1” in Appendix A.

In what follows, a value range of the tuning parameter to guarantee the convergence of TD (3) can be specified as

\[
R_0 := \left\{ r \geq 1 : \frac{1}{r} + \frac{\theta}{2^{2\theta-1}} \leq \frac{\theta}{2} \right\},
\tag{16}
\]

where \( \theta \in (0, 1) \) is any chosen parameter. Note that when \( \theta \in (0, 1) \) is increasing, the range \( R_0 \) will increase as well but the exponential decay rate \( \frac{\theta}{2} \) associated with the tracking error would be reduced.

The convergence result of TD (3) in the presence of multiple stochastic disturbances is summarized as the following Theorem III.1.

**Theorem III.1.** Suppose that \( v : [0, \infty) \to \mathbb{R} \) is a continuously differentiable function satisfying \( \sup_{t \geq 0} (|v(t)| + |\dot{v}(t)|) \leq M \) for some constant \( M > 0 \) and Assumption (A1) holds. Then, for any tuning parameter \( r \) \( \in R_0 \), initial value \( x(0) \in \mathbb{R}^n \) and \( T > 0 \), the TD (3) admits a unique global solution satisfying

\[
\begin{align*}
(i) \quad &\mathbb{E}|x_1(t) - v(t)|^2 \leq \frac{1 + \lambda_1}{\lambda_2} \Gamma + (1 + \mu) \sigma_1^2 \gamma_1, &\quad \text{uniformly in } t \in [T, \infty), \\
(ii) \quad &\lim_{r \rightarrow \infty} \sup_{t \geq 0} \mathbb{E}|x_1(t) - v(t)|^2 \leq \sigma_1^2 \gamma_1
\end{align*}
\tag{17}
\]

uniformly in \( t \in [T, \infty) \), where \( \mu \) is any positive constant and \( \Gamma \) specified in (16) is a positive constant independent of \( r \);

\[
(iii) \quad \lim_{r \rightarrow \infty} \sup_{t \geq 0} \mathbb{E}|x_1(t) - v(t)|^2 \leq \sigma_1^2 \gamma_1
\tag{18}
\]

uniformly in \( t \in [T, \infty) \) when \( \sigma_1 = 0 \).

**Proof.** See “Proof of Theorem III.1” in Appendix B. \( \square \)

**Remark III.1.** Note that the tracking error system (12) is an Itô-type stochastic system. Thus, the convergence in mean square is natural because the Itô integral terms are zero as martingales after taking mathematical expectation. In addition, the mean square sense denotes the convergence of an average level of the tracking error, which could be in line with engineering applications since the deviation of every sample path of the tracking error from the average level is often small. Finally, we can see from (17) that the upper bound of the tracking error in mean square can approach \( \sigma_1^2 \gamma_1 \) arbitrarily and quickly by tuning the parameter \( r \) to be sufficiently large since the convergence time \( T \) is any positive constant. This is what we mean by “TD is not sensitive to input noise”.

It seems impossible to make the tracking error as small as possible when \( \sigma_1 \neq 0 \) in (2).

**Remark III.2.** It is noteworthy that the selection of the function \( f(\cdot) \) guarantees that the “nominal part” of the tracking error system (12) defined in (5) is exponentially stable with the decay rate \( \frac{\lambda_1}{\lambda_2} \) which depends on \( f(\cdot) \). By definition of \( \Gamma \) from (16) in Appendix B, the constant \( \Gamma \) which is a part of the tracking error (depending on \( f(\cdot) \)) becomes smaller if the decay rate \( \frac{\lambda_1}{\lambda_2} \) becomes larger. In addition, another advantage could be mentioned is that when the decay rate \( \frac{\lambda_1}{\lambda_2} \) becomes larger, the value range of the tuning parameter \( r \) defined in (16) will increase.

Finally, we indicate an important fact that \( x_i(t) \) \( (i = 2, 3, \ldots, n) \) can always be regarded as an approximation of the corresponding \( (i - 1) \)-th derivative of \( v(t) \) in terms of generalized derivative whatever the classical derivatives of \( v(t) \) exist or not. In fact, for any \( a > 0 \), \( C_0^\infty(0,a) \) be the set that contains all infinitely differentiable functions with compact support on \( (0,a) \). Remember that for any locally integrable function \( h : (0,a) \to \mathbb{R} \), the usual \( (i - 1) \)-th generalized derivative of \( h \), still denoted by \( h^{i-1}(\cdot) \), always exists in the sense of distribution defined as a functional on \( C_0^\infty(0,a) \) that

\[
h^{i-1}(\phi) = (-1)^{i-1} \int_0^a h(t) \phi^{(i-1)}(t)dt,
\tag{20}
\]

for every test function \( \phi \in C_0^\infty(0,a) \) and \( 2 \leq i \leq n \) (see, e.g., [11] p.43). In addition, a generalized stochastic process \( \Phi \) is simply a random generalized function in the following sense:

For every test function \( \varphi \in C_0^\infty(0,a) \), a random variable \( \Phi(\varphi) \) is assigned such that the functional \( \Phi \) on \( C_0^\infty(0,a) \) is linear and continuous (see, e.g., [15] p.50). Thus, for each \( i = 2, 3, \ldots, n \), \( x_i \) itself can be regarded as a generalized stochastic process in the sense that

\[
x_i(\varphi) = \int_0^a x_i(t) \phi(t)dt, \quad \forall \varphi \in C_0^\infty(0,a).
\tag{21}
\]
For each \( i = 2, 3, \ldots, n \), the state \( x_i \) of the TD (3) is convergent to the \( (i-1) \)-th generalized derivative of the input signal \( v \) in mean square and almost sure sense, which is summarized in the following Theorem III.2.

**Theorem III.2.** Suppose that \( v : [0, a] \to \mathbb{R} \) is continuously differentiable and Assumption (A1) holds. Then, for any initial value \( v(0) \in \mathbb{R}^n \), the TD (3) admits a unique global solution, and for any \( \varphi \in C_0^\infty ((0, a)) \) and all \( i = 2, 3, \ldots, n \), there holds

\[
\lim_{r \to \infty} \sup_{t \in (0, a)} |v(i-1)(\varphi)| = 0\]  

(i) \( \lim_{r \to \infty} \mathbb{E} |x_i(\varphi) - v(i-1)(\varphi)|^2 \leq \sigma_i^2 \sup_{t \in (0, a)} |\varphi(i-1)(t)|^2 \sigma_1^2 \gamma_1; \)  

(ii) \( \lim_{r \to \infty} |x_i(\varphi) - v(i-1)(\varphi)| = 0 \) almost surely  

when the additive colored noise affecting the input signal is vanishing, i.e., \( \sigma_1 = 0 \).

**Proof.** See “Proof of Theorem III.2” in Appendix C. \( \square \)

**Remark III.3.** The convergence of linear TD without requiring Assumption (A1) can be concluded directly from Theorems III.1 and III.2. This is because the matrix \( A \) in (2) defined by the designed parameters \( a_i \ (i = 1, 2, \ldots, n) \) is Hurwitz so that there exists a unique positive definite matrix solution \( Q \) to the Lyapunov equation \( QA + A^T Q = -I_{n \times n} \) for \( n \)-dimensional identity matrix \( I_{n \times n} \). For this reason, we can define the Lyapunov functions \( V : \mathbb{R}^n \to [0, \infty) \) and \( W : \mathbb{R}^n \to [0, \infty) \) by \( V(z) = z^T Q z \) and \( W(z) = \|z\|^2 \) for \( z \in \mathbb{R}^n \), respectively. It is then easy to verify that all conditions in Assumption (A1) are satisfied, where the parameters in Assumptions (A1) are specified as \( \lambda_1 = \lambda_{\text{min}}(Q), \lambda_2 = \lambda_{\text{max}}(Q), \lambda_3 = \lambda_i = 1, c_1 = c_2 = 2 \lambda_{\text{max}}(Q), \) with \( \lambda_{\text{min}}(Q) \) and \( \lambda_{\text{max}}(Q) \) being respectively the minimal and maximal eigenvalues of the matrix \( Q \).

**Remark III.4.** The present paper focuses only on the convergence and noise-tolerant performance for TD in the presence of multiple stochastic disturbances. However, in practical applications, there may exist phase lags because of using the integration of TD to estimate the derivatives of the input signal, which can be overcome by introducing feedforward in the design of TD (5).

**IV. Numerical Simulations**

In this section, some numerical simulations are presented to verify the effectiveness of the main results. Let the input signal be \( v(t) = \sin(3t + 1) \). We design a second-order linear TD and a second-order nonlinear TD in the form of (3) in the presence of multiple stochastic disturbances. The linear TD is produced by a linear function given by

\[
f(z_1, z_2) = -2z_1 - 4z_2, \quad \forall (z_1, z_2) \in \mathbb{R}^2. \]  

Motivated by the nonlinear feedback controller design, a nonlinear function used for the construction of nonlinear TD can be the linear function (24) adding with a Lipschitz continuous function given by

\[
f(z_1, z_2) = -2z_1 - 4z_2 - \phi(z_1), \quad \forall (z_1, z_2) \in \mathbb{R}^2, \]  

where \( \phi(s) = \begin{cases} \frac{1}{4\pi}, & s \in (-\infty, -\frac{\pi}{2}], \\ \frac{1}{4\pi} \sin s, & s \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \frac{1}{4\pi}, & s \in (\frac{\pi}{2}, +\infty), \end{cases} \) and the Assumption (A1) holds for ([1], p.196)

\[
V(z_1, z_2) = 1.375z_1^2 + 0.1875z_2^2 + 0.5z_1z_2, \\
W(z_1, z_2) = 0.5z_1^2 + 0.5z_2^2, \lambda_1 = 0.13, \lambda_2 = 1.43, \lambda_3 = \lambda_4 = 0.5, c_1 = 3.91, c_2 = 2.75. \]  

In Figures 1-3, some relative parameters are chosen as \( \sigma_1 = 0.2, \sigma_2 = \sigma_3 = 2 \) and then the initial values are taken as

\[
x_1(0) = \sin(1), \quad x_2(0) = 0, \quad w_1(0) = 1, \quad w_2(0) = -1. \]  

the sampling period \( \Delta t = 0.001 \), and the diffusion terms \( dB_i(t) \ (i = 1, 2, 3) \) are simulated by \( \sqrt{\Delta t} \) multiplied by random sequences generated by the Matlab program command “randn”.

V. CONCLUDING REMARKS

The convergence and noise-tolerant performance of a tracking differentiator (TD) are investigated, where a general case is considered that the input signal and the TD itself are disturbed by additive colored noise and additive colored and white noises, respectively. The mathematical proofs are presented to show that the tracking errors of the states of TD to the input signal and its generalized derivatives are convergent in mean square and even in almost sure sense for the special vanishing input noise. Some numerical simulations are presented to demonstrate the validity of the proposed TD. Finally, it is
worth mentioning that the noise intensity maximum tolerance analysis of TD would be a potential interesting problem to be further investigated in the future study.

APPENDIX A: Proof of Lemma III.1

By (1), $w_i(t)$ ($i = 1, 2$) can be regarded as the augmented state variables of system (14). Since the function $f(·)$ satisfies the local Lipschitz condition, it follows from the existence-and-unique theorem for Itô-type stochastic systems (see, e.g., [19, p.58, Theorem 3.6]) that there exist a unique maximal and-unique theorem for Itô-type stochastic systems (see, the local Lipschitz condition, it follows from the existence-a

Thus, it follows from Assumption (A1), (10) and Young’s inequality that

$$
\begin{align*}
&\int_0^t \sigma_3 \frac{\partial V(y(s))}{\partial y_n} d\hat{B}_3(s). \\
&+ \int_0^t \frac{\sigma_3}{r^{n-\frac{3}{2}}} \frac{\partial V(y(s))}{\partial y_n} d\hat{B}_3(s).
\end{align*}
$$

By Assumption (A1), $\int_0^\tau \frac{\sigma_3}{r^{n-\frac{3}{2}}} \frac{\partial V(y(s))}{\partial y_n} d\hat{B}_3(s)$ is a

Fig. 1. The tracking effect of linear TD with $r = 30.$

Fig. 2. The tracking effect of nonlinear TD with $r = 30.$
Fig. 3. The tracking effect of nonlinear TD with \( r = 15 \).

By the Burkholder-Davis-Gundy inequality\(^\text{(19) Theorem 1.7.3)}\),

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} \int_0^t \frac{\sigma_3}{r^n} \frac{\partial V(y(s))}{\partial y_1} d\hat{B}_1(s) \right)^2 \\
\leq 4 \sqrt{\mathbb{E}} \left( \int_0^t \frac{\sigma_3}{r^n} \frac{\partial V(y(s \wedge \tau_m))}{\partial y_1} \right)^2 ds,
\]

and

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t} \int_0^t \frac{\sigma_3}{r^n} \frac{\partial V(y(s \wedge \tau_m))}{\partial y_n} d\hat{B}_3(s) \right)^2 \\
\leq 4 \sqrt{\mathbb{E}} \left( \int_0^t \frac{\sigma_3}{r^n} \frac{\partial V(y(s \wedge \tau_m))}{\partial y_n} \right)^2 ds,
\]

where we set

\[
N_1 = 2E \mathbb{V}(y(0)) + \frac{32c_1^2 \sigma_1^2 \gamma_1^2}{r}, \\
N_2 = \frac{c_2^2 M^2}{r} + \frac{c_2^2 \sigma_2^2 \gamma_1^2}{r} + \frac{2c_2 \sigma_2^2 \gamma_1}{r} + \frac{c_1 \sigma_1^2 \gamma_2}{r} + \frac{c_2 \sigma_2^2}{r^{2n+1}}, \\
N_3 = \frac{2}{\lambda_1 r} + \frac{1}{\lambda_1 r^n} + \frac{2}{\lambda_1}.
\]

Now, applying Gronwall’s inequality\(^\text{(19) Theorem 1.8.1)}\) yields

\[
\frac{N_2}{N_3} + \mathbb{E} \left( \sup_{0 \leq s \leq t} V(y(s \wedge \tau_m)) \right) \leq \left( N_1 + \frac{N_2}{N_3} \right) e^{N_3 t}.
\]

Thus,

\[
\mathbb{E} \left( \sup_{0 \leq s \leq t \wedge \tau_m} \|y(s)\|^2 \right) \\
\leq \frac{1}{\lambda_1} \left( N_1 + \frac{N_2}{N_3} \right) e^{N_3 t}.
\]

This implies that

\[
m^2 P(\tau_m \leq t) \leq \frac{1}{\lambda_1} \left( N_1 + \frac{N_2}{N_3} \right) e^{N_3 t}.
\]

Passing to the limit as \( m \to \infty \) gives \( P(\tau \leq t) = 0 \) which yields in turn \( P(\tau > t) = 1. \) Since \( t \geq 0 \) is arbitrary, we then have \( \tau = \infty \) almost surely, and \( y(t) \) exists globally. Furthermore, passing to the limit as \( m \to \infty \) for \( \beta \) again gives\(^\text{(15) Fatou’s Lemma.} \) This completes the proof of the Lemma.\(^\text{11}\)

**APPENDIX B: Proof of Theorem 11**

It follows from Lemma\(^\text{11} \) that the TD\(^\text{11} \) admits a unique global solution and \( \frac{\sigma_3}{r^n} \frac{\partial V(y(s))}{\partial y_1} d\hat{B}_1(s) \) is a martingale on \( t \geq 0 \), and so is for \( \frac{\sigma_3}{r^n} \frac{\partial V(y(s \wedge \tau_m))}{\partial y_n} d\hat{B}_3(s) \). Thus, for all \( t \geq 0 \), it follows that

\[
\mathbb{E} \int_0^t \frac{\sigma_3}{r^n} \frac{\partial V(y(s \wedge \tau_m))}{\partial y_n} d\hat{B}_3(s) = 0.
\]

Finding the derivative of \( \mathbb{E} V(y(t)) \) with respect to \( t \), it follows from Assumption (A1),\(^\text{(19), (30), (59)} \), \( r \in \mathbb{R}_0 \) and Young’s inequality that

\[
\frac{d}{dt} \mathbb{E} V(y(t)) \\
= \mathbb{E} \left( \sum_{i=t}^{n-1} \frac{\partial V(y(t))}{\partial y_i} y_{i+1}(t) + \frac{\partial V(y(t))}{\partial y_n} f(y(t)) \right) \\
+ \mathbb{E} \left( \frac{\partial V(y(t))}{\partial y_1} \left( - \frac{1}{r} \frac{du}{dt} \right) + \frac{\sigma_3}{r} w_{i+1}(t) \right) \\
+ \mathbb{E} \left( \frac{\sigma_3}{r^n} \frac{\partial V(y(t))}{\partial y_n} + \frac{\sigma_2}{r^n} \frac{\partial V(y(t))}{\partial y_2} \right) \\
+ \frac{\sigma_3}{2} \mathbb{E} \left( \frac{\partial V(y(t))}{\partial y_1} \right) \\
= \mathbb{E} \left( \sum_{i=t}^{n-1} \frac{\partial V(y(t))}{\partial y_i} y_{i+1}(t) + \frac{\partial V(y(t))}{\partial y_n} f(y(t)) \right) \\
+ \mathbb{E} \left( \frac{\partial V(y(t))}{\partial y_1} \left( - \frac{1}{r} \frac{du}{dt} \right) + \frac{\sigma_3}{r} w_{i+1}(t) \right) \\
+ \mathbb{E} \left( \frac{\sigma_3}{r^n} \frac{\partial V(y(t))}{\partial y_n} + \frac{\sigma_2}{r^n} \frac{\partial V(y(t))}{\partial y_2} \right) \\
+ \frac{\sigma_3}{2} \mathbb{E} \left( \frac{\partial V(y(t))}{\partial y_1} \right) \\
\]
is a positive constant independent of \( r \). Using the inequality \((a+b)^2 \leq (1 + \frac{1}{\mu})a^2 + (1 + \mu)b^2 \) for any \( \mu > 0 \) and \( a, b \in \mathbb{R} \), it is obtained by (10) and (45) that

\[
\mathbb{E}|x_1(t) - v(t)|^2
\leq (1 + \frac{1}{\mu})\mathbb{E}|x_1(t) - v^*(t)|^2 + (1 + \mu)\sigma_1^2\mathbb{E}|w_1(t)|^2
\leq (1 + \frac{1}{\mu})\Gamma + (1 + \mu)\sigma_1^2\gamma_1
\quad (47)
\]

uniformly in \( t \in [T, \infty) \). Since \( \mu > 0 \) is arbitrary, it follows from (47) that

\[
\limsup_{r \to \infty} \mathbb{E}|x_1(t) - v(t)|^2 \leq \sigma_1^2\gamma_1
\quad (48)
\]

uniformly in \( t \in [T, \infty) \). In addition, when \( \sigma_1 = 0 \), if follows from (48) that

\[
\lim_{r \to \infty} \mathbb{E}|x_1(t) - v(t)|^2 = 0
\quad (49)
\]

uniformly in \( t \in [T, \infty) \). Thus, for any \( \varepsilon := \frac{1}{m} > 0 \), \( m \in \mathbb{N}^+ \), there exists an \( m \)-dependent constant \( r^* = r^*(m) \) such that

\[
\mathbb{E}|x_1(t) - v(t)|^2 < \frac{1}{m}
\quad (50)
\]

uniformly in \( t \in [T, \infty) \) for all \( r \geq r^* \). By Chebyshev’s inequality ([19], p.5), it has

\[
P\left\{ \omega : |x_1(t) - v(t)| > \frac{1}{m} \right\} \leq \frac{1}{m^2}
\quad (51)
\]

uniformly in \( t \in [T, \infty) \) for all \( r \geq r^* \). By Borel-Cantelli’s lemma ([19], p.7]), it can be also obtained that for almost all \( \omega \in \Omega \), there exists an \( m_0 = m_0(\omega) \) such that

\[
|x_1(t) - v(t)| \leq \frac{1}{m}
\quad (52)
\]

uniformly in \( t \in [T, \infty) \) whenever \( m \geq m_0, r \geq r^* \). Therefore, for almost all \( \omega \in \Omega \),

\[
\limsup_{r \to \infty} |x_1(t) - v(t)| \leq \frac{1}{m}
\quad (53)
\]

whenever \( m \geq m_0 \). Setting \( m \to \infty \) gives

\[
\lim_{r \to \infty} |x_1(t) - v(t)| = 0, \text{ almost surely}
\quad (54)
\]

uniformly in \( t \in [T, \infty) \) when \( \sigma_1 = 0 \). This completes the proof of the Theorem III.1.

APPENDIX C: Proof of Theorem III.2

By (21) and performing the integration by parts, it can be easily obtained that for each \( i = 2, 3, \ldots, n \),

\[
x_i(\varphi) = (-1)^{(i-1)} \int_0^a x_i(t) \varphi^{(i-1)}(t) dt, \forall \varphi \in C_0^\infty(0, a).
\quad (55)
\]

From Theorem III.1 (55) and the definition of the generalized derivative in (20), for each \( i = 2, 3, \ldots, n \) and any \( 0 < \xi < a \), it follows that

\[
\mathbb{E}|x_i(\varphi) - v^{(i-1)}(\varphi)|^2
\leq \mathbb{E} \int_0^a \left( x_i(t)\varphi(t) - (1)^{(i-1)} \int_0^a x_i(t)\varphi^{(i-1)}(t) dt \right)^2 dt
\leq \mathbb{E} \int_0^a \left( x_i(t) - v(t) \right)^2 \varphi^{(i-1)}(t) dt^2
\quad (56)
\]
\[ \begin{align*}
&\leq a \int_0^\infty \mathbb{E}|x_1(t) - v(t)|^2 \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \\
&\leq a \int_0^\xi \mathbb{E}|x_1(t) - v(t)|^2 dt \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \\
&+ a \int_\xi^\infty \mathbb{E}|x_1(t) - v(t)|^2 dt \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \\
&\leq \xi a \max_{0 \leq t \leq \xi} \mathbb{E}|x_1(t) - v(t)|^2 \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \\
&+ a(a - \xi) \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \left(1 + \frac{1}{\mu}\right) \\
&+ (1 + \mu)\sigma_1^2 \gamma_1.
\end{align*} \]

Since \( \mu > 0 \) is arbitrary, passing to the limit as \( r \to \infty \) yields
\[ \limsup_{r \to \infty} \mathbb{E}|x_1(\varphi) - v^{(i-1)}(\varphi)|^2 \leq \xi a \max_{0 \leq t \leq \xi} \mathbb{E}|x_1(t) - v(t)|^2 \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \]
\[ + a(a - \xi) \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \sigma_1^2 \gamma_1. \] (57)

Setting \( \xi \to 0 \), we then have
\[ \limsup_{r \to \infty} \mathbb{E}|x_1(\varphi) - v^{(i-1)}(\varphi)|^2 \leq a^2 \sup_{t \in (0, a)} |\varphi^{(i-1)}(t)|^2 \sigma_1^2 \gamma_1. \] (58)

When \( \sigma_1 = 0 \), it follows from (58) that
\[ \lim_{r \to \infty} \mathbb{E}|x_1(\varphi) - v^{(i-1)}(\varphi)|^2 = 0. \] (59)

Similar to (50)-(54), it can be also obtained that
\[ \lim_{r \to \infty} |x_1(\varphi) - v^{(i-1)}(\varphi)| = 0, \text{ almost surely.} \] (60)

This completes the proof of the Theorem III.2

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