THE TT-GEOMETRY OF PERMUTATION MODULES
PART I: STRATIFICATION

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Abstract. We consider the derived category of permutation modules over a finite group, in positive characteristic. We stratify this tensor triangulated category using Brauer quotients. We describe the set underlying the tt-spectrum of compact objects, and discuss several examples.

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1. Introduction

1.1. Convention. We place ourselves in the setting of modular representation theory of finite groups. So unless mentioned otherwise, $G$ is a finite group and $k$ is a field of characteristic $p > 0$, with $p$ typically dividing the order of $G$.

Permutation modules. Among the easiest representations to construct, permutation modules are simply the $k$-linearizations $k(X)$ of $G$-sets $X$. And yet they play an important role in subjects as varied as derived equivalences [Ric96], Mackey functors [Yos83], or equivariant homotopy theory [MNN17], to name a few. The authors’ original interest stems from yet another connection, namely the one with Voevodsky’s theory of motives [Voe00], specifically Artin motives. For a gentle introduction to these ideas, we refer the reader to [BG21].

We consider a ‘small’ tensor triangulated category, the homotopy category

\[ \mathcal{K}(G) := \mathcal{K}_b \left( \text{perm}(G; k)^{\#} \right) \]
of bounded complexes of permutation $kG$-modules, idempotent-completed. It sits as the compact part $\mathcal{K}(G) = \mathcal{T}(G)^c$ of the ‘big’ tensor triangulated category
\begin{equation}
\mathcal{T}(G) := \mathcal{D}\text{Perm}(G;k)
\end{equation}
obtained for instance by closing $\mathcal{K}(G)$ under coproducts and triangles in the homotopy category $\mathcal{K}(\text{Mod}(kG))$ of all $kG$-modules. We call $\mathcal{T}(G)$ the derived category of permutation $kG$-modules. Details can be found in Recollection 2.2.

This derived category of permutation modules $\mathcal{T}(G)$ is amenable to techniques of tensor-triangular geometry [Bal10], a geometric approach that brings organization to otherwise bewildering tensor triangulated categories, in topology, algebraic geometry or representation theory. Tensor-triangular geometry has led to many new insights, for instance in equivariant homotopy theory [BS17, BHN+19, BHG20].

Our goal in the present paper and its follow-up [BG22c] is to understand the tensor-triangular geometry of the derived category of permutation modules, both in its big variant $\mathcal{T}(G)$ and its small variant $\mathcal{K}(G)$. In Part III of the series [BG22d] we extend our analysis to profinite groups and to Artin motives.

Having sketched the broad context and the aims of the series, let us now turn to the content of the present paper in more detail.

**Stratification.** In colloquial terms, one of our main results says that the big derived category of permutation modules is strongly controlled by its compact part:

1.4. **Theorem** (Theorem 8.11). The derived category of permutation modules $\mathcal{T}(G)$ is stratified by $\text{Spc}(\mathcal{K}(G))$ in the sense of Barthel-Heard-Sanders [BHS21a].

Let us remind the reader of BHS-stratification. What we establish in Theorem 8.11 is an inclusion-preserving bijection between the localizing $\otimes$-ideals of $\mathcal{T}(G)$ and the subsets of the spectrum $\text{Spc}(\mathcal{K}(G))$. This bijection is defined via a canonical support theory on $\mathcal{T}(G)$ that exists once we know that $\text{Spc}(\mathcal{K}(G))$ is a noetherian space (Proposition 8.1). Note that Theorem 1.4 cannot be obtained via ‘BIK-stratification’ as in Benson-Iyengar-Krause [BIK11], since the endomorphism ring of the unit $\text{Hom}(1,1) = k$ is too small. However, we shall see that [BIK11] plays an important role in our proof, albeit indirectly. An immediate consequence of stratification is the Telescope Property (Corollary 8.12):

1.5. **Corollary.** Every smashing $\otimes$-ideal of $\mathcal{T}(G)$ is generated by its compact part.

The key question is now to understand the spectrum $\text{Spc}(\mathcal{K}(G))$. For starters, recall from [BG20, Theorem 5.13] that the innocent-looking category $\mathcal{K}(G)$ actually captures much of the wilderness of modular representation theory. It admits as Verdier quotient the derived category $D_b(kG)$ of all finitely generated $kG$-modules. By Benson-Carlson-Rickard [BCR97], the spectrum of $D_b(kG)$ is the homogeneous spectrum of the cohomology ring $H^*(G,k)$. We deduce in Proposition 2.22 that $\text{Spc}(\mathcal{K}(G))$ contains an open piece $V_G$
\begin{equation}
\text{Spec}^c(H^*(G,k)) \cong \text{Spc}(D_b(kG)) =: V_G \hookrightarrow \text{Spc}(\mathcal{K}(G))
\end{equation}
that we call the cohomological open of $G$.

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1 Direct summands of finitely generated permutation modules are called *p-permutation* or *trivial source* $kG$-modules and form the category denoted $\text{perm}(G;k)$. It has only finitely many indecomposable objects up to isomorphism. If $G$ is a $p$-group, all $p$-permutation modules are permutation and the indecomposable ones are of the form $k(G/H)$ for subgroups $H \leq G$. 

In good logic, the closed complement of $V_G$ is the support
\begin{equation}
\text{Spc}(\mathcal{X}(G)) \setminus V_G = \text{Supp}(\mathcal{X}_{ac}(G))
\end{equation}
of the tt-ideal $\mathcal{X}_{ac}(G) = \text{Ker}(\mathcal{X}(G) \to D_0(kG))$ of acyclic objects in $\mathcal{X}(G)$. The problem becomes to understand this closed subset $\text{Supp}(\mathcal{X}_{ac}(G))$. To appreciate the issue, let us say a word of closed points. Corollary 6.31 gives the complete list: There is one closed point $M$ of the issue, let us say a word of closed points.

Let us return to Spc($\mathcal{X}(G)$). So let us return to Spc($\mathcal{X}(G)$).

**Modular fixed-points.** Let $H \leq G$ be a subgroup. We abbreviate by
\begin{equation}
G//H := W_G(H) = N_G(H)/H
\end{equation}
the Weyl group of $H$ in $G$. If $H \leq G$ is normal then of course $G//H = G/H$.

For every $G$-set $X$, its $H$-fixed-points $X^H$ is canonically a $(G//H)$-set. We also have a naive fixed-points functor $M \mapsto M^H$ on $kG$-modules but it does not ‘linearize’ fixed-points of $G$-sets, that is, $k(X)^H$ differs from $k(X^H)$ in general. And it does not preserve the tensor product. We would prefer a tensor-triangular functor
\begin{equation}
\Psi^H : \mathcal{T}(G) \to \mathcal{T}(G//H)
\end{equation}
such that $\Psi^H(k(X)) = k(X^H)$ for every $G$-set $X$.

A related problem was encountered long ago for the $G$-equivariant stable homotopy category $\text{SH}(G)$, see [LMSM86]: The naive fixed-points functor (a.k.a. the ‘genuine’ or ‘categorical’ fixed-points functor) is not compatible with taking suspension spectra, and it does not preserve the smash product. To solve both issues, topologists invented geometric fixed-points $\Phi^H$. Those functors already played an important role in tensor-triangular geometry [BS17, BGH20, PSW22] and it would be reasonable, if not very original, to try the same strategy for $\mathcal{T}(G)$. Such geometric fixed-points $\Phi^H$ can indeed be defined in our setting but unfortunately they do not give us the wanted $\Psi^H$ of (1.9), as we explain in Remark 3.11.

In summary, we need a third notion of fixed-points functor $\Psi^H$, which is neither the naive one $(-)^H$, nor the ‘geometric’ one $\Phi^H$ imported from topology. It turns out (see Warning 4.1) that it can only exist in characteristic $p$ when $H$ is a $p$-subgroup. The good news is that this is the only restriction (see Section 4):

**1.10. Proposition.** For every $p$-subgroup $H \leq G$ there exists a coproduct-preserving tensor-triangular functor on the big derived category of permutation modules
\begin{equation}
\Psi^H : \mathcal{T}(G) \to \mathcal{T}(G//H)
\end{equation}
such that $\Psi^H(k(X)) \cong k(X^H)$ for every $G$-set $X$. In particular, this functor preserves compacts and restricts to a tt-functor $\Psi^H : \mathcal{K}(G) \to \mathcal{K}(G//H)$ on (1.2).

We call the $\Psi^H$ the modular $H$-fixed-points functors. They already exist at the level of additive categories $\text{perm}(G; k)^\mathbb{F} \to \text{perm}(G//H; k)^\mathbb{F}$, where they agree with the classical Brauer quotient, although our construction is quite different. See Remark 4.8. These $\Psi^H$ also recover motivic functors considered by Bachmann in [Bac16, Corollary 5.48]. For us, modular fixed-points functors are only a tool that we want to use to prove theorems. So let us return to Spc($\mathcal{X}(G)$).
The spectrum. Each tt-functor $\Psi^H$ induces a continuous map on spectra

\[ \psi^H := \text{Spc}(\Psi^H) : \text{Spc}(\mathcal{K}(G//H)) \longrightarrow \text{Spc}(\mathcal{X}(G)). \]

In particular $\text{Spc}(\mathcal{X}(G))$ receives via this map $\psi^H$ the cohomological open $V_{G//H}$ of the Weyl group of $H$:

\[ V_{G//H} = \text{Spc}(\text{Db}(k(G//H))) \hookrightarrow \text{Spc}(\mathcal{K}(G//H)) \xrightarrow{\psi^H} \text{Spc}(\mathcal{X}(G)). \]

Using this, we describe the set underlying $\text{Spc}(\mathcal{X}(G))$ in Theorem 6.16:

1.13. Theorem. Every point of $\text{Spc}(\mathcal{X}(G))$ is the image $\psi^H(p)$ of a point $p \in V_{G//H}$ for some $p$-subgroup $H \leq G$, in a unique way up to $G$-conjugation, i.e. we have $\psi^H(p) = \psi^H(p')$ if and only if there exists $g \in G$ such that $H^g = H'$ and $p^g = p'$.

In this description, the trivial subgroup $H = 1$ contributes the cohomological open $V_G$ (since $\Psi^1 = \text{Id}$). Its closed complement $\text{Supp}(\mathcal{X}_{ac}(G))$, introduced in (1.7), is covered by images of the modular fixed-points maps (1.12), for $H$ running through all non-trivial $p$-subgroups of $G$. The main ingredient in proving Theorem 1.13 is our Conservativity Theorem 5.12 on the associated big categories: (2)

1.14. Theorem. The family of functors \[ \mathcal{I}(G) \xrightarrow{\Psi^H} \mathcal{I}(G//H) \rightarrow \text{KInj}(k(G//H)) \] indexed by the (conjugacy classes of) $p$-subgroups $H \leq G$, is conservative.

This determines the set $\text{Spc}(\mathcal{X}(G))$. The topology of $\text{Spc}(\mathcal{X}(G))$ is a separate plot, involving new characters. The reader will find them in Part II [BG22c].

Measuring progress by examples. Before the present work, we only knew the case of cyclic group $C_p$ of order $p = 2$, where $\text{Spc}(\mathcal{X}(C_2))$ is a 3-point space (3)

\[ \text{Supp}(\mathcal{X}_{ac}(C_2)) \]

This was the starting point of our study of real Artin-Tate motives [BG22b, Theorem 3.14]. It appears independently in Dugger-Hazel-May [DHM22, Theorem 5.4].

We now have a description of $\text{Spc}(\mathcal{X}(G))$ for arbitrary finite groups $G$. We gather several examples in Section 7 to illustrate the progress made since (1.15), and also for later use in [BG22d]. Let us highlight the case of the quaternion group $G = Q_8$ (Example 7.12). By Quillen, we know that the cohomological open $V_{Q_8}$ is the same as for its center $Z(Q_8) = C_2$, that is, the 2-point Sierpiński space displayed in green on the right-hand side of (1.15), and again below:

\[ \text{Supp}(\mathcal{X}_{ac}(Q_8)) = ? \]

If intuition was solely based on (1.15) one could believe that $\text{Spc}(\mathcal{X}(G))$ is just $V_G$ with some discrete decoration for the acyclics, like the single (brown) point on the left-hand side of (1.15). The quaternion group offers a stark rebuttal.

\[ \text{Supp}(\mathcal{X}_{ac}(Q_8)) = ? \]

\[ V_{Q_8} = V_{C_2} \]

\[ V_{Q_8} \approx V_{C_2} \]

2 Recall that Krause's homotopy category of injectives $\text{KInj}(k(G))$ is a compactly generated tensor triangulated category whose compact part identifies with $\text{Db}(k(G))$.

3 A line indicates specialization: The higher point is in the closure of the lower one.
Indeed, the spectrum \( \text{Spc}(\mathcal{X}(Q_8)) \) is the following space:

\[
(1.16)
\]

Its support of acyclics (in brown) is actually way more complicated than the cohomological open itself: It has Krull dimension two and contains a copy of the projective line \( \mathbb{P}^1_k \). In fact, the map \( \psi^C_2 \) given by modular fixed-points identifies the closed piece \( \text{Supp}(\mathcal{X}_{ac}(Q_8)) \sim \text{Spc}(\mathcal{X}(C_2^{2} \times C_2^{2})) \).

We hope that the outline of the paper is now clear from the above introduction and the table of contents.

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1.17. Terminology. A ‘tensor category’ is an additive category with a symmetric-monoidal product additive in each variable. We say ‘tt-category’ for ‘tensor triangulated category’ and ‘tt-ideal’ for ‘thick (triangulated) \( \otimes \)-ideal’. We say ‘big’ tt-category for a rigidly-compactly generated tt-category, as in [BF11].

We use a general notation \((-)^\ast\) to indicate everything related to graded rings. For instance, \( \text{Spec}^\ast(-) \) denotes the homogeneous spectrum.

For subgroups \( H, K \leq G \), we write \( H \preceq_G K \) to say that \( H \) is \( G \)-conjugate to a subgroup of \( K \), that is, \( H^g \preceq K \) for some \( g \in G \). We write \( \sim_G \) for \( G \)-conjugation. As always \( H^g = g^{-1}Hg \) and \( gH = gHg^{-1} \). We write \( N_G(H, K) \) for \( \{ g \in G \mid H^g \preceq K \} \) and \( N_G(H) = N_G(H, H) \) for the normalizer.

We write \( \text{Sub}_p(G) \) for the set of \( p \)-subgroups of \( G \). It is a \( G \)-set via conjugation.

1.18. Convention. When a notation involves a subgroup \( H \) of an ambient group \( G \), we drop the mention of \( G \) if no ambiguity can occur, like with \( \text{Res}_H \) for \( \text{Res}^G_H \).

Similarly, we sometimes drop the mention of the field \( k \) to lighten notation.

2. Recollections and Koszul objects

2.1. Recollection. We refer to [Bal10] for elements of tensor-triangular geometry. Recall simply that the spectrum of an essentially small tt-category \( \mathcal{X} \) is \( \text{Spc}(\mathcal{X}) = \{ \mathcal{P} \subseteq \mathcal{X} \mid \mathcal{P} \text{ is a prime tt-ideal} \} \). For every object \( x \in \mathcal{X} \), its support is \( \text{supp}(x) := \{ \mathcal{P} \in \text{Spc}(\mathcal{X}) \mid x \notin \mathcal{P} \} \). These form a basis of closed subsets for the topology.

2.2. Recollection. (Here \( k \) can be a commutative ring.) Recall our reference [BG21] for details on permutation modules. Linearizing a \( G \)-set \( X \), we let \( k(X) \) be the free \( k \)-module with basis \( X \) and \( G \)-action \( k \)-linearly extending the \( G \)-action on \( X \). A permutation \( kG \)-module is a \( kG \)-module isomorphic to one of the form \( k(X) \). These modules form an additive subcategory \( \text{Perm}(G; k) \) of \( \text{Mod}(kG) \), with all \( kG \)-linear maps. We write \( \text{perm}(G; k) \) for the full subcategory of finitely generated permutation \( kG \)-modules and \( \text{perm}(G; k)^\ast \) for its idempotent-completion.
We tensor $kG$-modules in the usual way, over $k$ with diagonal $G$-action. The linearization functor $k(-): \text{G-Sets} \rightarrow \text{Perm}(G;k)$ turns the cartesian product of $G$-sets into this tensor product. For every finite $X$, the module $k(X)$ is self-dual.

We consider the idempotent-completion $(-)^lat$ of the homotopy category of bounded complexes in the additive category $\text{perm}(G)$

$$\mathcal{K}(G) = \mathcal{K}(G;k) := \text{K}_b(\text{perm}(G;k)) \cong \text{K}_b(\text{perm}(G;k)^lat).$$

As $\text{perm}(G;k)$ is an essentially small tensor-additive category, $\mathcal{K}(G)$ becomes an essentially small tensor triangulated category. As $\text{perm}(G;k)$ is rigid so is $\mathcal{K}(G)$, with degreewise duals. Its tensor-unit $\mathbb{1} = k$ is the trivial $kG$-module $k = k(G/G)$.

The ‘big’ derived category of permutation $kG$-modules \cite[Definition 3.6]{BG21} is

$$\text{DPerm}(G;k) = \text{K}(\text{Perm}(G;k)) \left[\left\{\text{G-quasi-iso}\right\}^{-1}\right],$$

where a $\text{G}$-quasi-isomorphism $f: P \rightarrow Q$ is a morphism of complexes such that the induced morphism on $H$-fixed points $f^H$ is a quasi-isomorphism for every subgroup $H \leq G$. It is also the localization subcategory of $\text{K}(\text{Perm}(G;k))$ generated by $\mathcal{K}(G)$, and it follows that $\mathcal{K}(G) = \text{DPerm}(G;k)^e$.

2.3. Example. For $G$ trivial, the category $\mathcal{K}(1;k) = \text{D}_{\text{perf}}(k)$ is that of perfect complexes over $k$ (any ring) and $\text{DPerm}(1;k)$ is the derived category of $k$.

2.4. Remark. The tt-category $\mathcal{K}(G)$ depends functorially on $G$ and $k$. It is contravariant in the group. Namely if $\alpha: G \rightarrow G'$ is a homomorphism then restriction along $\alpha$ yields a tt-functor $\alpha^*: \mathcal{K}(G') \rightarrow \mathcal{K}(G)$. When $\alpha$ is the inclusion of a subgroup $G \leq G'$, we recover usual restriction

$$\text{Res}_G^{G'}: \mathcal{K}(G') \rightarrow \mathcal{K}(G).$$

When $\alpha$ is a quotient $G \rightarrow G' = G/N$ for $N \trianglelefteq G$, we get inflation, denoted here \footnote{We avoid the traditional $\text{Infl}_{G/N}^G$ notation which is not coherent with the restriction notation.} $\text{Infl}_{G/N}^G: \mathcal{K}(G/N;k) \rightarrow \mathcal{K}(G)$.

The covariance of $\mathcal{K}(G)$ in $k$ is simply obtained by extension-of-scalars. All these functors are the ‘compact parts’ of similarly defined functors on $\text{DPerm}$.

Let us say a word of $kG$-linear morphisms between permutation modules.

2.5. Recollection. Let $H, K \leq G$ be subgroups. Then $\text{Hom}_{kG}(k(G/H), k(G/K))$ admits a $k$-basis $\{f_g\}_{g}$ indexed by classes $[g] \in H\backslash G/K$. Namely, choosing a representative in each class $[g] \in H\backslash G/K$, one defines

$$f_g: \begin{array}{ccc} k(G/H) & \rightarrow & k(G/L) \\ \eta & \mapsto & c_g \cdot \eta \\ \epsilon & \mapsto & k(G/K) \end{array}$$

where we set $L := H \cap gK$, where $\eta$ and $\epsilon$ are the usual maps using that $L \leq H$ and $L \leq K$ (thus $\eta$ maps $[e]_H$ to $\sum_{\gamma \in H/L} \gamma$ and $\epsilon$ extends $k$-linearly the projection $G/L \rightarrow G/K$), and finally where the middle isomorphism $c_g$ is

$$c_g: \begin{array}{ccc} k(G/L) & \rightarrow & k(G/L^g) \\ [x]_L & \mapsto & [x \cdot g]_{L^g} \end{array}.$$
We can now begin our analysis of the spectrum of the tt-category $\mathcal{K}(G)$.

2.8. Proposition. Let $G \trianglelefteq G'$ be a subgroup of index invertible in $k$. Then the map $\text{Spc}(\text{Res}_G^{G'}): \text{Spc}(\mathcal{K}(G)) \rightarrow \text{Spc}(\mathcal{K}(G'))$ is surjective.

Proof. This is a standard argument. For a subgroup $G \trianglelefteq G'$, the restriction functor $\text{Res}_G^{G'}$ has a two-sided adjoint $\text{Ind}^{G'}_G : \mathcal{K}(G) \rightarrow \mathcal{K}(G')$ such that the composite of the unit and counit of these adjunctions $\text{Id} \rightarrow \text{Ind} \circ \text{Res} \rightarrow \text{Id}$ is multiplication by the index. If the latter is invertible, it follows that $\text{Res}_G^{G'}$ is a faithful functor. The result now follows from [Bal18, Theorem 1.3]. \hfill $\square$

2.9. Corollary. Let $k$ be a field of characteristic zero and $G$ be a finite group. Then $\text{Spc}(\mathcal{K}(G)) = *$ is a singleton.

Proof. Direct from Proposition 2.8 since $\text{Spc}(\mathcal{K}(1; k)) = \text{Spc}(\text{D}_{\text{perf}}(k)) = *$. \hfill $\square$

2.10. Remark. In view of these reductions, the fun happens with coefficients in a field $k$ of positive characteristic dividing the order of $G$. Hence Convention 1.1.

Let us now identify what the derived category tells us about $\text{Spc}(\mathcal{K}(G))$.

2.11. Notation. We can define a tt-ideal of $\mathcal{K}(G) = \text{K}_b(\text{perm}(G; k)^3)$ by

$$\mathcal{K}_{\text{ac}}(G) := \{ x \in \mathcal{K}(G) \mid x \text{ is acyclic as a complex of } kG\text{-modules} \}.$$  

It is the kernel of the tt-functor $\Upsilon_G : \mathcal{K}(G) \rightarrow \text{D}_b(kG) := \text{D}_b(\text{mod}(kG))$ induced by the inclusion $\text{perm}(G; k)^3 \hookrightarrow \text{mod}(kG)$ of the additive category of $p$-permutation $kG$-modules inside the abelian category of all finitely generated $kG$-modules.

2.12. Recollection. The canonical functor induced by $\Upsilon_G$ on the Verdier quotient

$$\frac{\mathcal{K}(G)}{\mathcal{K}_{\text{ac}}(G)} \rightarrow \text{D}_b(kG)$$

is an equivalence of tt-categories. This is [BG20, Theorem 5.13]. In other words, (2.13)

$$\Upsilon_G : \mathcal{K}(G) \rightarrow \text{D}_b(kG)$$

realizes the derived category of finitely generated $kG$-modules as a localization of our $\mathcal{K}(G)$, away from the Thomason subset $\text{Supp}(\mathcal{K}_{\text{ac}}(G))$ of (1.7). Following Neeman-Thomason, the above localization (2.13) is the compact part of a finite localization of the corresponding ‘big’ tt-categories $\mathcal{T}(G) \rightarrow \text{K}\text{Inj}(kG)$, the homotopy category of complexes of injectives. See [BG22a, Remark 4.21]. We return to this localization of big categories in Recollection 5.7.

We want to better understand the tt-ideal of acyclics $\mathcal{K}_{\text{ac}}(G)$ and in particular show that it has closed support.

2.14. Construction. Let $H \trianglelefteq G$ be a subgroup. We define a complex of $kG$-modules by tensor-induction (recall Convention 1.18)

$$\text{kos}(H) = \text{kos}_G(H) := \otimes \text{Ind}^G_H(0 \rightarrow k \xrightarrow{1} k \rightarrow 0)$$

where $0 \rightarrow k \xrightarrow{1} k \rightarrow 0$ is non-trivial in homological degrees 1 and 0; hence $\text{kos}(H)$ lives in degrees between $[G : H]$ and 0. Since $H$ acts trivially on $k$, the action of $G$ on $\text{kos}(H)$ is the action of $G$ by permutation of the factors $\otimes_{G/H}(0 \rightarrow k \xrightarrow{1} k \rightarrow 0)$. This can be described as a Koszul complex. For every $0 \leq d \leq [G : H]$, the complex $\text{kos}(H)$ in degree $d$ is the $k$-vector space $\Lambda^d(k(G/H))$ of dimension $\binom{[G : H]}{d}$. 
If we choose a numbering of the elements of $G/H = \{v_1, \ldots, v_{[G:H]}\}$ then $\text{kos}(H)_d$ has a $k$-basis $\{ v_i \wedge \cdots \wedge v_d | 1 \leq i_1 < \cdots < i_d \leq [G : H] \}$. The canonical diagonal action of $G$ permutes this basis but introduces signs when re-ordering the $v_i$’s so that indices increase. When $p = 2$ these signs are irrelevant. When $p > 2$, every such ‘sign-permutation’ $kG$-module is isomorphic to an actual permutation $kG$-module (by changing some signs in the basis, see [BG20, Lemma 3.8]).

2.15. Proposition. Let $H \leq G$ be a subgroup. Then $\text{kos}_G(H)$ is an acyclic complex of finitely generated permutation $kG$-modules which is concentrated in degrees between $[G:H]$ and 0 and such that it is $k$ in degree 0 and $k(G/H)$ in degree 1.

Proof. See Construction 2.14. Exactness is obvious since the underlying complex of $k$-modules is $(0 \to k \to k \to 0)^{\otimes [G:H]}$. The values in degrees 0, 1 are immediate. \Box

2.16. Example. We have $\text{kos}_G(G) = 0$ in $\mathcal{K}(G)$. The complex $\text{kos}_G(1)$ is an acyclic complex of permutation modules that was important in [BG20, §3]:

$$\text{kos}_G(1) = \cdots 0 \to P_n \to \cdots \to P_2 \to kG \to k \to 0 \cdots$$

2.17. Lemma. Let $H \triangleleft G$ be a normal subgroup and $H \leq K \leq G$. Then $\text{kos}_G(K) \cong \text{Infl}^G_{G/H}(\text{kos}_{G/H}(K/H))$. In particular, $\text{kos}_G(H) \cong \text{Infl}^G_{G/H}(\text{kos}_{G/H}(1))$.

Proof. The construction of $\text{kos}_G(K) = \otimes_{G/K}(0 \to k \to 0)$ depends only on the $G$-set $G/K$ which is inflated from the $G/H$-set $(G/H)/(K/H)$. \Box

In fact, $\text{kos}_G(H)$ is not only exact. It is split-exact on $H$. More generally:

2.18. Lemma. For every subgroups $H, K \leq G$ and every choice of representatives in $K \setminus G/H$, we have a non-canonical isomorphism of complexes of $kK$-modules

$$\text{Res}_K^G(\text{kos}_G(H)) \cong \bigotimes_{[g] \in K \setminus G/H} \text{kos}_K(K \cap {}^gH).$$

In particular, if $K \leq_G H$, we have $\text{Res}_K^G(\text{kos}_G(H)) = 0$ in $\mathcal{K}(K)$.

Proof. By the Mackey formula for tensor-induction, we have in $\text{Ch}_b(\text{perm}(K; k))$

$$\text{Res}_K^G(\text{kos}_G(H)) \cong \bigotimes_{[g] \in K \setminus G/H} \text{Ind}^K_{K \cap {}^gH}({}^g\text{Res}_K^H(0 \to k \to 0)).$$

The result follows since $\text{Res}(0 \to k \to 0) = (0 \to k \to 0)$. If $K \leq_G H$, the factor $\text{kos}_K(K)$ appears in the tensor product and $\text{kos}_K(K) = 0$ in $\mathcal{K}(K)$. \Box

We record a general technical argument that we shall use a couple of times.

2.19. Lemma. Let $\mathcal{A}$ be a rigid tensor category and $s = (\cdots s_2 \to s_1 \to s_0 \to 0 \cdots)$ a complex concentrated in non-negative degrees. Let $x \in \text{Ch}_b(\mathcal{A})$ be a bounded complex such that $s_1 \otimes x = 0$ in $K_b(\mathcal{A})$. Then there exists $n \gg 0$ such that $s_n^\otimes \otimes x$ belongs to the smallest thick subcategory $\langle s \rangle^\prime$ of $K(\mathcal{A})$ that contains $s$ and is closed under tensoring with $K_b(\mathcal{A}) \cup \{s\}$ in $K(\mathcal{A})$.

In particular, if $s \in K_b(\mathcal{A})$ is itself bounded, then $s_n^\otimes \otimes x$ belongs to the tt-ideal $\langle s \rangle$ generated by $s$ in $K_b(\mathcal{A})$.

Proof. Let $u := s_{\geq 1}[−1]$ be the truncation of $s$ such that $s = \text{cone}(d : u \to s_0)$. Similarly we have $u = \text{cone}(u_{\geq 1}[−1] \to s_1)$. Note that $u_{\geq 1}$ is concentrated in positive degrees. Since $x \otimes s_1 = 0$ we have $u \otimes x \cong u_{\geq 1} \otimes x$ in $K(\mathcal{A})$ and thus

$$u_n \otimes x \cong (u_{\geq 1})^\otimes \otimes x.$$
for all \( n \geq 0 \). For \( n \) large enough there are no non-zero maps of complexes from \( (u_{\geq 1}) \oplus^n \otimes x \) to \( s_0^n \otimes x \), simply because the former ‘moves’ further and further away to the left and \( x \) is bounded. So \( d_0^n \otimes x : u_{\geq 1} \oplus^n \otimes x \to s_0^n \otimes x \) is zero in \( \mathcal{X}(A) \).

Let \( \mathcal{L} \) be the tt-subcategory of \( \mathcal{K}(A) \) generated by \( K_b(A) \cup \{ s \} \); then \( \langle s \rangle' \) is a tt-ideal in \( \mathcal{L} \), and similarly we write \( \langle \text{cone}(d_0^n) \rangle' \) for the tt-ideal in \( \mathcal{L} \) generated by \( \text{cone}(d_0^n) \). By the argument above, we have \( s_0^\oplus n \otimes x \in \langle \text{cone}(d_0^n) \rangle' \subseteq \langle s \rangle' \). \( \square \)

**2.20. Corollary.** Let \( A \) be a rigid tensor category and \( J \subseteq K_b(A) \) a tt-ideal. Let \( s \in J \) be a (bounded) complex concentrated in non-negative degrees such that

1. \( \text{supp}(s_0) \supseteq \text{supp}(J) \) in \( \text{Spc}(K_b(A)) \) (for instance if \( s_0 = 1 \)), and
2. \( \text{supp}(s_1) \cap \text{supp}(J) = \emptyset \), meaning that \( s_1 \otimes x = 0 \) in \( K_b(A) \) for all \( x \in J \).

Then \( s \) generates \( J \) as a tt-ideal in \( K_b(A) \), that is, \( \text{supp}(J) = \text{supp}(s) \) in \( \text{Spc}(K_b(A)) \).

**Proof.** Let \( x \in J \). By (2), Lemma 2.19 gives us \( s_0^n \otimes x \in \langle s \rangle \) for \( n > 0 \). Hence \( \text{supp}(s_0) \cap \text{supp}(x) \subseteq \text{supp}(s) \). By (1) we have \( \text{supp}(x) \subseteq \text{supp}(s_0) \). Therefore \( \text{supp}(x) = \text{supp}(s_0) \cap \text{supp}(x) \subseteq \text{supp}(s) \). In short \( x \in \langle s \rangle \) for all \( x \in J \). \( \square \)

We apply this to the object \( s = \text{kos}_G(H) \) of Construction 2.14.

**2.21. Proposition.** For every subgroup \( H \leq G \), the object \( \text{kos}_G(H) \) generates the tt-ideal \( \text{Ker}(\text{Res}^G_H) \) of \( \mathcal{K}(G) \).

**Proof.** We apply Corollary 2.20 to \( J = \text{Ker}(\text{Res}^G_H) \) and \( s = \text{kos}_G(H) \). We have \( s \in J \) by Lemma 2.18. Conditions (1) and (2) hold since \( s_0 = k \) and \( s_1 = k(G/H) \) by Proposition 2.15 and Frobenius gives \( s_1 \otimes J = k(G/H) \otimes J = \text{Ind}_H^G \text{Res}_H^G(J) = 0 \). \( \square \)

We can apply the above discussion to \( H = 1 \) and \( J = \text{Ker}(\text{Res}^G_1) = \mathcal{K}_{ac}(G) \).

**2.22. Proposition.** The tt-functor \( \Upsilon_G : \mathcal{K}(G) \to \text{D}_b(kG) \) induces an open inclusion \( \nu_G : V_G \to \text{Spc}(\mathcal{K}(G)) \) where \( V_G = \text{Spc}(\text{D}_b(kG)) \cong \text{Spec}^*(\text{H}^*G, k) \). The closed complement of \( V_G \) is the support of \( \text{kos}_G(1) = \oplus \text{Ind}_1^G(0 \to k \to k \to 0) \).

**Proof.** The homeomorphism \( \text{Spc}(\text{D}_b(kG)) \cong \text{Spec}^*(\text{H}^*G, k) \) follows from the tt-classification [BCR97]; see [Bal10, Theorem 57]. By Recollection 2.12, the map \( \nu_G := \text{Spc}(\Upsilon_G) \) is a homeomorphism onto its image, and the complement of this image is \( \text{supp}(\mathcal{K}_{ac}(G)) = \text{supp}(\text{kos}_G(1)) \), by Proposition 2.21 applied to \( H = 1 \). In particular, \( \text{supp}(\mathcal{K}_{ac}(G)) \) is a closed subset, not just a Thomason. \( \square \)

**2.23. Remark.** The notation for the so-called cohomological open \( V_G \) has been chosen to evoke the classical projective support variety \( V_G(k) = \text{Proj}^*(\text{H}^*G, k) \cong \text{Spec}(\text{stmod}(kG)) \), which consists of \( V_G \) without its unique closed point, \( \text{H}^*G, k) \).

We can also describe the kernel of restriction for classes of subgroups.

**2.24. Corollary.** For every collection \( \mathcal{H} \) of subgroups of \( G \), we have an equality of tt-ideals in \( \mathcal{K}(G) \)

\[
\bigcap_{H \in \mathcal{H}} \text{Ker}(\text{Res}^G_H) = \langle \bigotimes_{H \in \mathcal{H}} \text{kos}_G(H) \rangle.
\]

**Proof.** This is direct from Proposition 2.21 and the general fact that \( \langle x \rangle \cap \langle y \rangle = \langle x \otimes y \rangle \). (In the case of \( \mathcal{H} = \emptyset \), the intersection is \( \mathcal{K}(G) \) and the tensor is \( 1 \).) \( \square \)
3. Restriction, Induction and Geometric Fixed-points

In the previous section, we saw how much of $\text{Spc}(\mathcal{X}(G))$ comes from $D_b(kG)$. We now want to discuss how much is controlled by restriction to subgroups, to see how far the ‘standard’ strategy of [BS17] gets us.

3.1. Remark. The tt-categories $\mathcal{X}(G)$ and $D_b(kG)$, as well as the Weyl groups $G//H$ are functorial in $G$. To keep track of this, we adopt the following notational system.

Let $\alpha: G \rightarrow G'$ be a group homomorphism. We write $\alpha^*: \mathcal{X}(G') \rightarrow \mathcal{X}(G)$ for restriction along $\alpha$, and similarly for $\alpha^*: D_b(kG') \rightarrow D_b(kG)$. When applying the contravariant $\text{Spc}(-)$, we simply denote $\text{Spc}(\alpha^*)$ by $\alpha_*: \text{Spc}(\mathcal{X}(G)) \rightarrow \text{Spc}(\mathcal{X}(G'))$ and similarly for $\alpha_*: V_G \rightarrow V_{G'}$ on the spectrum of derived categories.

As announced, Weyl groups $G//H = (N_G H)/H$ of subgroups $H \leq G$ will play a role. Since $\alpha(N_G H) \leq N_G(\alpha(H))$, every homomorphism $\alpha: G \rightarrow G'$ induces a homomorphism $\bar{\alpha}: G//H \rightarrow G'/\alpha(H)$. Combining with the above, these homomorphisms $\alpha$ define functors $\alpha^*$ and maps $\bar{\alpha}_*$. For instance, $\bar{\alpha}_*: V_{G//H} \rightarrow V_{G'/\alpha(H)}$ is the continuous map induced on $\text{Spc}(D_b(k(-)))$ by $\bar{\alpha}: G//H \rightarrow G'/\alpha(H)$.

Following tradition, we have special names when $\alpha$ is an inclusion, a quotient or a conjugation. For the latter, we choose the lightest notation possible.

(a) For conjugation, for a subgroup $G \leq G'$ and an element $x \in G'$, the isomorphism $c_x^*: G \xrightarrow{\sim} G^x$ induces a tt-functor $c^*_x: \mathcal{X}(G^x) \xrightarrow{\sim} \mathcal{X}(G)$ and a homeomorphism

\[
(-)^x := (c_x)_*: \text{Spc}(\mathcal{X}(G)) \xrightarrow{\sim} \text{Spc}(\mathcal{X}(G^x))
\]

Note that if $x = g \in G$ belongs to $G$ itself, the functor $c^*_x: \mathcal{X}(G) \rightarrow \mathcal{X}(G)$ is isomorphic to the identity and therefore we get the useful fact that

\[
g \in G \quad \Rightarrow \quad \mathcal{P}^g = \mathcal{P} \quad \text{for all } \mathcal{P} \in \text{Spc}(\mathcal{X}(G)).
\]

Similarly we have a conjugation homeomorphism $\mathcal{P} \mapsto \mathcal{P}^x$ on the cohomological opens $V_G \xrightarrow{\sim} V_{G^x}$, which is the identity if $x \in G$. When $H \leq G$ is a further subgroup then conjugation yields homeomorphisms $V_{G//H} \xrightarrow{\sim} V_{G^x//H}$, still denoted $\mathcal{P} \mapsto \mathcal{P}^x$. Again, if $x = g \in N_G H$, so $[g]_H$ defines an element in $G//H$, the equivalence $(c_g)_*: D_b(G//H) \xrightarrow{\sim} D_b(G//H)$ is isomorphic to the identity. Thus

\[
g \in N_G(H) \quad \Rightarrow \quad \mathcal{P}^g = \mathcal{P} \quad \text{for all } \mathcal{P} \in V_{G//H}.
\]

(b) For restriction, take $\alpha$ the inclusion $K \hookrightarrow G$ of a subgroup. We write

\[
\rho_K = \rho_K^G := \text{Spc}(\text{Res}_K^G) : \text{Spc}(\mathcal{X}(K)) \rightarrow \text{Spc}(\mathcal{X}(G))
\]

and similarly for derived categories. When $H \leq K$ is a subgroup, we write $\rho_K: V_{K//H} \rightarrow V_{G//H}$ for the map induced by restriction along $K//H \hookrightarrow G//H$. Beware that $\rho_K$ is not necessarily injective, already on $V_K \rightarrow V_G$, as ‘fusion’ phenomena can happen: If $g \in G$ normalizes $K$, then $\mathcal{Q}$ and $\mathcal{Q}^g$ in $V_K$ have the same image in $V_G$ by (3.2) but are in general different in $V_K$.

(c) For inflation, let $N \trianglelefteq G$ be a normal subgroup and let $\alpha = \text{proj}: G \rightarrow G/N$ be the quotient homomorphism. We write

\[
\pi^{G/N} = \pi^{G/N}_G := \text{Spc}(\text{Inf}_{G/N}^G) : \text{Spc}(\mathcal{X}(G)) \rightarrow \text{Spc}(\mathcal{X}(G/N))
\]
and similarly for derived categories. For $H \leq G$ a subgroup, we write $\bar{\pi}_G^{\text{G/H}} : V_{(G/H)} \to V(G/N)\backslash (HN/N)$ for the map induced by $\text{proj}: G\backslash H \to (G/N)\backslash (HN/N)$. (Note that this homomorphism is not always surjective, e.g. with $G = D_8$ and $N \cong C_2 \times C_2$.)

3.6. **Recollection.** One verifies that the Res$_H^G \dashv \text{Ind}_H^G$ adjunction is monadic, see for instance [Bal16, §4], and that the associated monad $A_H \otimes -$ is separable, where $A_H := k(G/H) = \text{Ind}_H^G k \in \text{perm}(G;k)$. The ring structure on $A_H$ is given by the usual unit $\eta : k \to k(G/H)$, mapping 1 to $\sum_{\gamma \in G/H} \gamma$, and the multiplication $\mu : A_H \otimes A_H \to A_H$ that is characterized by $\mu(\gamma \otimes \gamma') = \gamma$ and $\mu(\gamma \otimes \gamma') = 0$ for all $\gamma \neq \gamma'$ in $G/H$. The ring $A_H$ is separable and commutative. The tt-category Mod($A_H$) = Mod$_{\mathcal{K}(G)}(A_H)$ of $A_H$-modules in $\mathcal{K}(G)$ identifies with $\mathcal{K}(H)$, in such a way that extension-of-scalars to $A_H$ (i.e. along $\eta$) coincides with restriction Res$_H^G$. Similarly, extension-of-scalars along the isomorphism $c_{\gamma^{-1}} : A_H \otimes \gamma \to A_H$, being an equivalence, is the inverse of its adjoint, that is $(c_{\gamma^{-1}})^{-1} = c_{\gamma}^*$, hence is the conjugation tt-functor $c_{\gamma}^* : \mathcal{K}(H^g) \to \mathcal{K}(H)$ of Remark 3.1.

3.7. **Proposition.** The continuous map $\rho_H : \text{Spc}(\mathcal{K}(H)) \to \text{Spc}(\mathcal{K}(G))$ of (3.4) is a closed map and for every $y \in \mathcal{K}(H)$, we have $\rho_H(\text{supp}(y)) = \text{supp}(\text{Ind}_H^G(y))$ in $\text{Spc}(\mathcal{K}(G))$. In particular, $\text{Im}(\rho_H) = \text{supp}(k(G/H))$. Moreover, there is a coequalizer of topological spaces (independent of the choices of representatives $g$)

$$\bigoplus_{[g] \in H \backslash G/H} \text{Spc}(\mathcal{K}(H \cap \gamma H)) \cong \text{Spc}(\mathcal{K}(H)) \xrightarrow{\rho_H} \text{supp}(k(G/H))$$

where the two left horizontal maps are, on the $[g]$-component, induced by the restriction functor and by conjugation by $g$ followed by restriction, respectively.

**Proof.** We invoke [Bal16, Theorem 3.19]. In particular, we have a coequalizer (3.8)

$$\text{Spc}(\text{Mod}(A_H \otimes A_H)) \cong \text{Spc}(\text{Mod}(A_H)) \to \text{supp}(A_H)$$

where the two left horizontal maps are induced by the canonical ring morphisms $A_H \otimes \eta \otimes A_H : A_H \to A_H \otimes A_H$. For any choice of representatives $[g] \in H \backslash G/H$ the Mackey isomorphism

$$\bigoplus_{[g] \in H \backslash G/H} \text{Spc}(\mathcal{K}(H \cap \gamma H)) \cong \text{Spc}(\mathcal{K}(H)) \xrightarrow{\rho_H} \text{supp}(k(G/H))$$

maps $[x]_{H \cap \gamma H}$ to $[x]_H \otimes [x \cdot g]_H$. We can then plug this identification in (3.8). The second homomorphism $\eta \otimes A_H$ follows by the projection onto the factor indexed by $[g]$ becomes the composite $A_H \xrightarrow{c_{\gamma^{-1}}^*} A_H \xrightarrow{\eta} A_H \otimes A_H$. See Recollection 3.6. □

3.9. **Corollary.** For $\mathcal{P}, \mathcal{P}' \in \text{Spc}(\mathcal{K}(H))$ we have $\rho_H(\mathcal{P}) = \rho_H(\mathcal{P}')$ in $\text{Spc}(\mathcal{K}(G))$ if and only if there exists $g \in G$ and $\Omega \in \text{Spc}(\mathcal{K}(G))$ such that

$$\mathcal{P} = \rho_H^{\mathcal{P}}(\Omega) \quad \text{and} \quad \mathcal{P}' = (\rho_H^{\mathcal{P}}(\Omega))^g$$

using Remark 3.1 for the notation $(-)^g : \text{Spc}(\mathcal{K}(H)) \to \text{Spc}(\mathcal{K}(H))$.

**Proof.** This is [Bal16, Corollary 3.12], which implies the set-theoretic part of the coequalizer of Proposition 3.7. □

We single out a particular case.

3.10. **Corollary.** If $H \leq Z(G)$ is central in $G$ (for example, if $G$ is abelian) then restriction induces a closed immersion $\rho_H : \text{Spc}(\mathcal{K}(H)) \to \text{Spc}(\mathcal{K}(G))$. □
3.11. Remark. In view of Proposition 3.7, the image of the map induced by restriction \( \text{Im}(\rho_H) = \text{supp}(k(G/H)) \) coincides with the support of the tt-ideal generated by \( \text{Ind}^G_H(\mathcal{K}(H)) \). Following the construction of the geometric fixed-points functor \( \Phi^G : \text{SH}^e(G) \to \text{SH}^e \) in topology, we can consider the Verdier quotient
\[
\tilde{\mathcal{K}}(G) := \frac{\mathcal{K}(G)}{\langle \text{Ind}^G_H(\mathcal{K}(H)) \mid H \not\subseteq G \rangle}
\]
obtained by modding-out, in tensor-triangular fashion, everything induced from all proper subgroups \( H \). This tt-category \( \tilde{\mathcal{K}}(G) \) has a smaller spectrum than \( \mathcal{K}(G) \), namely the open complement in \( \text{Spc}(\mathcal{K}(G)) \) of the closed subset \( \cup_{H \not\subseteq G} \text{Im}(\rho_H) \) covered by proper subgroups. This method has worked nicely in [BS17, BGH20, PSW22] because, in those instances, this Verdier quotient is equivalent to the non-equivariant version of the tt-category under consideration. However, this fails for \( \tilde{\mathcal{K}}(G) \), for instance \( \tilde{\mathcal{K}}(C_2) \) is not equivalent to \( \mathcal{K}(1) = D_b(k) \):
\[
\frac{\text{SH}^e(G)}{\langle \text{Ind}^G_H(\text{SH}^e(H)) \mid H \not\subseteq G \rangle} \cong \text{SH}^e \quad \text{but} \quad \frac{\mathcal{K}(G)}{\langle \text{Ind}^G_H(\mathcal{K}(H)) \mid H \not\subseteq G \rangle} \not\cong \mathcal{K}(1).
\]
For small groups, for instance for cyclic \( p \)-groups \( C_p^n \), the tt-category \( \tilde{\mathcal{K}}(G) \) is reasonably complicated and one could still compute \( \text{Spc}(\mathcal{K}(G)) \) through an analysis of \( \tilde{\mathcal{K}}(G) \). However, the higher the \( p \)-rank, the harder it becomes to control \( \tilde{\mathcal{K}}(G) \).

One can already see the germ of the problem with \( G = C_2 \), see (1.15):
\[
\text{Spc}(\mathcal{K}(C_2)) = \frac{\mathcal{M}(C_2)}{\mathcal{M}(1)} \cong \langle \rangle.
\]
We have given names to the three primes. The only proper subgroup is \( H = 1 \) and the image of \( \rho_1 = \text{Spc}(\text{Res}_1) \) is simply the single closed point \( \{\mathcal{M}(1)\} = \text{supp}(kC_2) \). Chopping off this induced part, leaves us with the open \( \text{Spc}(\tilde{\mathcal{K}}(C_2)) = \{\mathcal{M}(C_2), \mathcal{P}\} \). So geometric fixed points \( \Phi^{C_2} : \mathcal{K}(C_2) \to \tilde{\mathcal{K}}(C_2) \) detects both of these points. (This also proves that \( \tilde{\mathcal{K}}(G) \not\cong \mathcal{K}(1) = D_b(k) \) since \( D_b(k) \) would have only one point in its spectrum.) However there is no need for a tt-functor detecting \( \mathcal{M}(C_2) \) and \( \mathcal{P} \) again, since \( \mathcal{P} \) is already in the cohomological open \( V_{C_2} \) detected by \( D_b(kC_2) \). In other words, geometric fixed points see too much, not too little: The target category \( \tilde{\mathcal{K}}(G) \) is too complicated in general. And as the group grows, the issue only gets worse, as the reader can check with Klein-four in Example 7.10.

In conclusion, we need tt-functors better tailored to the task, namely tt-functors that detect just what is missing from \( V_G \). In the case of \( C_2 \), we expect a tt-functor to \( D_b(k) \), to catch \( \mathcal{M}(C_2) \), but for larger groups the story gets more complicated and involves more complex subquotients of \( G \), as we explain in the next section.

4. Modular fixed-points functors

Motivated by Remark 3.11, we want to find a replacement for geometric fixed points in the setting of modular representation theory. In a nutshell, our construction amounts to taking classical Brauer quotients [Bro85, §1] on the level of permutation modules and then passing to the tt-categories \( \mathcal{K}(G) \) and \( \mathcal{F}(G) \). We follow a somewhat different route than [Bro85] though, more in line with the construction of the geometric fixed-points discussed in Remark 3.11. We hope some readers will benefit from our exposition.
It is here important that char\(k = p\) is positive.

4.1. Warning. A \(tt\)-functor \(\Psi^H : \mathcal{X}(G) \to \mathcal{X}(G//H)\) such that \(\Psi^H(k(X)) \cong k(X^H)\), as in (1.9), cannot exist unless \(H\) is a \(p\)-subgroup. Indeed, if \(P \subseteq G\) is a \(p\)-Sylow then since \([G : P]\) is invertible in \(k\), the unit \(1 = k\) is a direct summand of \(k(G/P)\) in \(\mathcal{X}(G)\). A \(tt\)-functor \(\Psi^H\) cannot map \(1\) to zero. Thus \(\Psi^H(k(G/P)) = k((G/P)^H)\) must be non-zero, forcing \((G/P)^H \neq \emptyset\). If \([g] \in G/P\) is fixed by \(H\) then \(H^g \leq P\) and therefore \(H\) must be a \(p\)-subgroup. (If char\(k = 0\) this would force \(H = 1\).)

4.2. Recollection. A collection \(\mathcal{F}\) of subgroups of \(G\) is called a family if it is closed under conjugation and subgroups. For instance, given \(H \leq G\), we have the family

\[
\mathcal{F}_H = \{K \leq G \mid (G/K)^H = \emptyset\} = \{K \leq G \mid H \nleq_G K\}.
\]

For \(N \leq G\) a normal subgroup, it is \(\mathcal{F}_N = \{K \leq G \mid N \nleq K\}\).

In view of Warning 4.1, we must focus attention on \(p\)-subgroups. The following standard lemma would not be true without the characteristic \(p\) hypothesis.

4.3. Lemma. Let \(N \leq G\) be a normal \(p\)-subgroup. Let \(H, K \leq G\) be subgroups such that \(N \leq H\) and \(N \nleq K\). Then every \(kG\)-linear homomorphism that factors as \(f : k(G/H) \xrightarrow{\ell} k(G/K) \xrightarrow{m} k\) must be zero.

Proof. By Recollection 2.5 and \(k\)-linearity, we can assume that \(m\) is the augmentation and that \(\ell = \epsilon \circ c_g \circ \eta\) as in (2.6), where \(g \in G\) is some element, where we set \(L = H \cap gK\) and where \(\epsilon : k(G/L^g) \to k(G/K)\), \(c_g : k(G/L) \to k(G/L^g)\) and \(\eta : k(G/H) \to k(G/L)\) are the usual maps, using \(L \leq H\) and \(L^g \leq K\). The composite \(m \circ \epsilon \circ c_g\) is an augmentation map again, hence our map \(f\) is the composite

\[
f : k(G/H) \xrightarrow{\eta} k(G/L) \xrightarrow{\epsilon} k.
\]

So \(f\) maps \([e]_H\) to \(\sum_{\gamma \in H/L} 1 = |H/L|\) in \(k\). Now, the \(p\)-group \(N \leq H\) acts on the set \(H/L\) by multiplication on the left. This action has no fixed point, for otherwise we would have \(N \leq_H L \leq_G K\) and thus \(N \leq K\), a contradiction. Therefore the \(N\)-set \(H/L\) has order divisible by \(p\). So \(|H/L| = 0\) in \(k\) and \(f = 0\) as claimed. \(\square\)

4.4. Proposition. Let \(N \leq G\) be a normal \(p\)-subgroup. Then the permutation category of the quotient \(G/N\) is an additive quotient of the permutation category of \(G\). More precisely, consider \(\text{proj}(\mathcal{F}_N) = \text{add}^? \{k(G/K) \mid K \in \mathcal{F}_N\}\), the closure of \(\{k(G/K) \mid N \nleq K\}\) under direct sum and summands in \(\text{perm}(G;k)\).

Consider the additive quotient of \(\text{perm}(G;k)^?\) by this \(\otimes\)-ideal. \(\tilde{\otimes}\) Then the composite

\[
\text{perm}(G/N;k)^? \xrightarrow{\text{Inf}^{G/N}} \text{perm}(G;k)^? \xrightarrow{\text{quot}} \text{perm}(G;k)^{\text{proj}(\mathcal{F}_N)}
\]

is an equivalence of tensor categories.

Proof. By the Mackey formula and since \(\mathcal{F}_N\) is a family, \(\text{proj}(\mathcal{F}_N)\) is a tensor ideal, hence quot is a tensor-functor. Inflation \(\text{Inf}^{G/N}_G\) : \(\text{perm}(G/N;k)^? \to \text{perm}(G;k)^?\) is

\(\tilde{\otimes}\) Keep the same objects as \(\text{perm}(G;k)^?\) and define Hom groups by modding out all maps that factor via objects of \(\text{proj}(\mathcal{F}_N)\), as in the ordinary construction of the stable module category.
also a tensor-functor. It is moreover fully faithful with essential image the subcategory \( \text{add}^3 \{ k(G/H) \mid N \leq H \} \). So we need to show that the composite
\[
\text{add}^3 \{ k(G/H) \mid N \leq H \} \rightarrow \text{perm}(G; k) \xrightarrow{\text{proj}(\mathcal{F}_N)} \text{perm}(G; k)^3 \rightarrow \text{add}^3 \{ k(G/K) \mid N \not\leq K \}
\]
is an equivalence. Both functors in the composite are full. The composite is faithful by Lemma 4.3, rigidity, additivity and the Mackey formula. Essential surjectivity is then easy (idempotent-completion is harmless since the functor is fully-faithful).

4.6. Construction. Let \( N \leq G \) be a normal \( p \)-subgroup. The composite of the additive quotient functor with the inverse of the equivalence of Proposition 4.4 yields a tensor-functor on the categories of \( p \)-permutation modules
\[
\Psi^N: \text{perm}(G; k)^3 \xrightarrow{\text{proj}(\mathcal{F}_N)} \text{perm}(G/N; k)^3.
\]
Applying the above degreewise, we get a tt-functor on homotopy categories \( K_b(\mathcal{K}(\mathcal{K})) \)
\[
\Psi^N = \Psi^{N:G}: \mathcal{K}(G) \rightarrow \mathcal{K}(G/N).
\]

4.8. Remark. Following up on Remark 3.11, we have constructed \( \Psi^N \) by modding-out in additive fashion this time, everything induced from subgroups not containing \( N \). We did it on the ‘core’ additive category and only then passed to homotopy categories. Such a construction would not make sense on bounded derived categories, as \( \Psi^N \) has no reason to preserve acyclic complexes.

The classical Brauer quotient seems different at first sight. It is typically defined at the level of individual \( kG \)-modules \( M \) by a formula like
\[
\text{coker} \left( \bigoplus_{Q \leq N} M^Q \xrightarrow{(\text{Tr}_N^Q)^N} M^N \right).
\]
A priori, this definition uses the ambient abelian category of modules and one then needs to verify that it preserves permutation modules, the tensor structure, etc. Our approach is a categorification of (4.9): Proposition 4.4 recovers the category \( \text{perm}(G/N; k)^3 \) as a tensor-additive quotient of \( \text{perm}(G; k)^3 \), at the categorical level, not at the individual module level. Amusingly, one can verify that it yields the same answer (Proposition 4.12) – a fact that we shall not use at all.

We relax the condition that the \( p \)-subgroup is normal in the standard way.

4.10. Definition. Let \( H \leq G \) be an arbitrary \( p \)-subgroup. We define the modular (or Brauer) \( H \)-fixed-points functor by the composite
\[
\Psi^{H:G}: \mathcal{K}(G) \xrightarrow{\text{Res}^G_{NGH}} \mathcal{K}(NGH) \xrightarrow{\Psi^{H:NGH}} \mathcal{K}(G/H)
\]
where \( NGH \) is the normalizer of \( H \) in \( G \) and \( G/H = (NGH)/H \) its Weyl group. The second functor comes from Construction 4.6. Note that \( \Psi^{H:G} \) is computed degreewise, applying the functors \( \text{Res}^G_{NGH} \) and \( \Psi^{H:NGH} \) at the level of \( \text{perm}(\mathcal{K})^3 \).

4.11. Remark. We prefer the phrase ‘modular fixed-points’ to ‘Brauer fixed-points’, out of respect for L. E. J. Brouwer and his fixed points. It also fits nicely in the flow: naive fixed-points, geometric fixed-points, modular fixed-points. Finally, the phrase ‘Brauer quotient’ would be unfortunate, as \( \Psi^H: \mathcal{K}(G) \rightarrow \mathcal{K}(G/H) \) is not a quotient of categories in any reasonable sense.

Let us verify that our \( \Psi^H \) linearize the \( H \)-fixed-points of \( G \)-sets, as promised.
4.12. Proposition. Let $H \leq G$ be a $p$-subgroup. The following square commutes up to isomorphism:

$$
\begin{array}{ccc}
G\text{-sets} & \xrightarrow{k(-)} & \text{perm}(G; k)^\sharp \\
\downarrow(-)^H & & \downarrow\Psi^H \\
(G//H)\text{-sets} & \xrightarrow{k(-)} & \text{perm}(G//H; k)^\sharp \\
\end{array}
$$

In particular, for every $K \leq G$, we have an isomorphism of $k(G//H)$-modules

$$(4.13) \quad \Psi^H (k(G/K)) \cong k((G/K)^H) = k(N_G(H, K)/K).$$

This module is non-zero if and only if $H$ is subconjugate to $K$ in $G$.

Proof. We only need to prove the commutativity of the left-hand square. As restriction to a subgroup commutes with linearization, we can assume that $H \leq G$ is normal. Let $X$ be a $G$-set. Consider its $G$-subset $X^H$ (which is truly inflated from $G/H$). Inclusion yields a morphism in $\text{perm}(G; k)$, natural in $X$,

$$(4.14) \quad f_X : k(X^H) \rightarrow k(X).$$

We claim that this morphism becomes an isomorphism in the quotient $\frac{\text{perm}(G; k)^\sharp}{\text{proj}(F_H)}$. By additivity, we can assume that $X = G/K$ for $K \leq G$. It is a well-known exercise that $(G/K)^H = N_G(H, K)/K$, which in the normal case $H \leq G$ boils down to $G/K$ or $\varnothing$, depending on whether $H \leq K$ or not, i.e. whether $K \not\in S_H$ or $K \in S_H$. In both cases, $f_X$ becomes an isomorphism (an equality or $0 \rightarrow k(G/K)$, respectively) in the quotient by $\text{proj}(F_H)$. Hence the claim.

Let us now discuss the commutativity of the following diagram

$$
\begin{array}{ccc}
G\text{-sets} & \xrightarrow{k(-)} & \text{perm}(G; k)^\sharp \\
\downarrow(-)^H & & \downarrow\Psi^H \\
G//H\text{-sets} & \xrightarrow{k(-)} & \text{perm}(G//H; k)^\sharp \\
\end{array}
$$

The module $k(X^H)$ in (4.14) can be written more precisely as $k(\text{Infl}^{G//H}_G(X^H)) \cong \text{Infl}^{G//H}_G k(X^H)$. So the first part of the proof shows that the left-hand ‘hexagon’ of the diagram commutes, i.e. the two functors $G\text{-sets} \rightarrow \frac{\text{perm}(G; k)^\sharp}{\text{proj}(F_H)}$ are isomorphic. The result follows by definition of $\Psi^H$, recalled on the right-hand side. \qed

Here is how modular fixed points act on restriction.

4.15. Proposition. Let $\alpha : G \rightarrow G'$ be a homomorphism and $H \leq G$ a $p$-subgroup. Set $H' = \alpha(H) \leq G'$. Then the following square commutes up to isomorphism

$$
\begin{array}{ccc}
\mathcal{K}(G') & \xrightarrow{\alpha^*} & \mathcal{K}(G) \\
\downarrow\Psi^{H'; G'} & & \downarrow\Psi^{H; G} \\
\mathcal{K}(G'//H') & \xrightarrow{\alpha^*} & \mathcal{K}(G//H).
\end{array}
$$
Proof. Exercise. This already holds at the level of perm\((-; k)\). □

4.16. Corollary. Let \( N \subseteq G \) be a normal \( p \)-subgroup. Then the composite functor 
\[ \Psi^N \circ \text{Infl}_G^{G/N} : \mathcal{X}(G/N) \to \mathcal{X}(G) \to \mathcal{X}(G/N) \] 
is isomorphic to the identity. Consequently, the map \( \text{Spc}(\Psi^H) \) is a split injection retracted by \( \text{Spc}(\text{Infl}_G^{G/H}) \).

Proof. Apply Proposition 4.15 to \( \alpha : G \to G/N \) and \( H = N \), and thus \( H' = 1 \). The second statement is just contravariance of \( \text{Spc}(\mathcal{X}) \). □

Composition of two ‘nested’ modular fixed-points functors almost gives another modular fixed-points functor. We only need to beware of Weyl groups.

4.17. Proposition. Let \( H \leq G \) be a \( p \)-subgroup and \( \bar{K} = K/H \) a \( p \)-subgroup of \( G//H \), for \( H \leq K \leq N_G H \). Then there is a canonical inclusion

\[ (G//H)//\bar{K} = (N_G//H\bar{K})//\bar{K} \to (N_G K)/K = G//K \]

and the following square commutes up to isomorphism

\[
\begin{array}{ccc}
\mathcal{X}(G) & \xrightarrow{\Psi^{H,G}} & \mathcal{X}(G//H) \\
\Psi^{K,G} \downarrow & & \downarrow \Psi^{K,G//H} \\
\mathcal{X}(G//K) & \xrightarrow{\text{Res}} & \mathcal{X}((G//H)//\bar{K})
\end{array}
\]

Proof. The inclusion comes from \( N_{N_G(H)K} \to N_G K \) and the rest is an exercise. Again, everything already holds at the level of perm\((-; k)\). □

4.18. Corollary. Let \( H \leq K \leq G \) be two \( p \)-subgroups with \( H \leq G \) normal. Then \( (G//H)//(K//H) \cong G//K \) and the following diagram commutes up to isomorphism

\[
\begin{array}{ccc}
\mathcal{X}(G) & \xrightarrow{\Psi^{H,G}} & \mathcal{X}(G//H) \\
\Psi^{K,G} \downarrow & & \downarrow \Psi^{K,G//H} \\
\mathcal{X}(G//K) & \xrightarrow{\text{Res}} & \mathcal{X}((G//H)//\bar{K})
\end{array}
\]

Proof. The surjectivity of the canonical inclusion \( G//H//(K//H) \to G//K \) of Proposition 4.17 holds since \( H \) is normal in \( G \). The result follows. □

4.19. Remark. We have essentially finished the proof of Proposition 1.10. It only remains to verify that there are variants of the constructions and results of this section for the big categories of Recollection 2.2. For a normal \( p \)-subgroup \( N \subseteq G \), the canonical functor on big additive categories

\[
(4.20) \quad \text{Add}^2(\{ k(G/H) \mid N \leq H \}) \to \frac{\text{Perm}(G; k) \cong \text{Proj}(\mathcal{F}_N)}{	ext{Proj}(\mathcal{F}_N)}
\]

is an equivalence of tensor categories, where

\[ \text{Proj}(\mathcal{F}_N) = \text{Add}^2(\{ k(G/K) \mid N \not\leq K \} \] is the closure of \( \text{proj}(\mathcal{F}_N) \) under coproducts and summands. Since the tensor product commutes with coproducts, \( \text{Proj}(\mathcal{F}_N) \) is again a \( \otimes \)-ideal in \( \text{Perm}(G; k) \). Fullness and essential surjectivity of \( (4.20) \) are easy, and faithfulness reduces to the finite case by compact generation. (A map \( f : P \to Q \) in \( \text{Perm}(G; k) \) is zero if and
As a consequence, the analogue of Proposition 4.4 also holds for big categories. Let us write $S(G)$ for $K(\operatorname{Perm}(G;k)) = K(\operatorname{Perm}(G;k))$, which is a compactly generated $\mathbb{T}$-category with compact unit. (Compactly generated is not obvious: see [BG21, Remark 5.12].) By the above discussion, the modular fixed-points functor with respect to a $p$-subgroup $H \leq G$ extends to the big categories $S(\_)$:

$$\Psi^H = \Psi^{H,G} : S(G) \xrightarrow{\operatorname{Res}^G_{NG}} S(N_G H) \to K \left( \frac{\operatorname{Perm}(N_G H;k)}{\operatorname{Proj}(\mathcal{F}_H)} \right) \xrightarrow{\inf_{N_G H}} S(G//H).$$

Note that $\Psi^H$ is a tensor triangulated functor that commutes with coproducts and maps $\mathcal{K}(G)$ into $\mathcal{K}(G//H)$. It follows that it restricts to $\Psi^H : \operatorname{DPerm}(G;k) \to \operatorname{DPerm}(G//H;k)$. The analogues of Propositions 4.12, 4.15 and 4.17 and Corollaries 4.16 and 4.18 all continue to hold for both $S(\_)$ and $\operatorname{DPerm}(\_;k)$.

This finishes our exposition of modular fixed-points functors $\Psi^H$ on derived categories of permutation modules. We now start using them to analyze the $\mathbb{T}$-geometry. First, we apply them to the Koszul complexes $\mathcal{K}(G)$ of Construction 2.14.

**4.21 Lemma.** Let $H, K \leq G$ be two subgroups, with $H$ a $p$-subgroup.

(a) If $H \nsubseteq G K$, then $\Psi^H(\mathcal{K}(G(K)))$ generates $\mathcal{K}(G//H)$ as a $\mathbb{T}$-ideal.

(b) If $H \leq G K$, then $\Psi^H(\mathcal{K}(G(K)))$ is acyclic in $\mathcal{K}(G//H)$.

(c) If $H \sim_G K$, then $\Psi^H(\mathcal{K}(G(K)))$ generates $\mathcal{K}_{\text{ac}}(G//H)$ as a $\mathbb{T}$-ideal.

**Proof.** For (a), we have $N_G(H,K) = \emptyset$ and thus $\Psi^H(k(G//K)) = 0$ by Proposition 4.12. It follows that $\Psi^H(\mathcal{K}(G(K))) = (\cdots \to * \to 0 \to k \to 0)$ by Proposition 2.15. Thus the $\otimes$-unit $1_{\mathcal{K}(G//H)} = k[0]$ is a direct summand of $\Psi^H(\mathcal{K}(G(K)))$.

For (b) and (c), by invariance under conjugation, we can assume that $H \leq K$. Let $N := N_G H$ be the normalizer of $H$. We have by Lemma 2.18 that

$$(4.22) \quad \Psi^{H,N}(\mathcal{K}(G(K))) = \Psi^{H,N} \operatorname{Res}^G_N(\mathcal{K}(G(K))) \simeq \bigotimes_{[g] \in N \setminus G/K} \Psi^{H,N}(\mathcal{K}(N\cap gK)).$$

For the index $g = e$ (or simply $g \in N_G K$), we can use $H \leq N \cap K$ and compute

$$\Psi^{H,N}(\mathcal{K}(N \cap K)) \cong \Psi^{H,N} \left( \operatorname{Inj}^N_{N/(N \cap K)} \mathcal{K}(N/(N \cap K)/H) \right)$$

by Lemma 2.17

$$\cong \mathcal{K}(N/(N \cap K)/H)$$

by Corollary 4.16.

As this object is acyclic in $\mathcal{K}(N/H)$ so is the tensor in (4.22). Hence (b). Continuing in the special case (c) with $H = K$, we have $(N \cap K)/H = 1$ and the above $\mathcal{K}(N/K)$ generates $\mathcal{K}_{\text{ac}}(N//H)$ by Proposition 2.21. It suffices to show that all the other factors in the tensor product (4.22) generate the whole $\mathcal{K}(G//H)$. This follows from Part (a) applied to the group $N$; indeed when $g \in G \setminus N$ we have $H \nsubseteq N \cap gH$ (as $H \leq N \cap gH$ and $H \leq N$ would imply $H = gH$).

**5. Conservativity via modular fixed-points**

In this section, we explain why the spectrum of $\mathcal{K}(G)$ is controlled by modular fixed-points functors $\Psi^H$ together with the localizations $\Upsilon_G : \mathcal{K}(G) \to \mathbb{D}_h(kG)$. It stems from a conservativity result on the `big' category $\mathcal{I}(G) = \operatorname{DPerm}(G;k)$, namely Theorem 5.12, for which we need some preparation.
5.1. Lemma. Suppose that $G$ is a $p$-group. Let $H \leq G$ be a subgroup and let $\bar{G} = G//H$ be its Weyl group. The modular $H$-fixed-points functor $\Psi^H : \mathrm{perm}(G;k)^2 \to \mathrm{perm}(\bar{G};k)^2$ induces a ring homomorphism

$$(5.2) \quad \Psi^H : \mathrm{End}_{kG}(k(G/H)) \longrightarrow \mathrm{End}_{k\bar{G}}(k(\bar{G})).$$

This homomorphism is surjective with nilpotent kernel: $(\ker(\Psi^H))^n = 0$ for $n \gg 1$. More precisely, it suffices to take $n \in \mathbb{N}$ such that $\mathrm{Rad}(kG)^n = 0$.

Proof. The reader can check this with Brauer quotients. We outline the argument. By (4.13) we have $\Psi^H(k(G/H)) \cong k(N_G(H)/H) = k(\bar{G})$, so the problem is well-stated. Recollection 2.5 provides $k$-bases for both vector spaces in (5.2), namely

$$\{ f_g = \epsilon \circ c_g \circ \eta \}_{[g] \in H\backslash G/H} \quad \text{and} \quad \{ c_g \}_{\bar{g} \in \bar{G}}$$

using the notation of (2.6) and (2.7). The homomorphism $\Psi^H$ in (5.2) respects those bases. Even better, it is a bijection from the part of the first basis indexed by $H\setminus (N_GH)/H$ onto the second basis, and it maps the rest of the first basis to zero. Indeed, when $g \in N_GH$, we have $f_g = c_g$ and $\Psi^H(f_g) = \Psi^H(c_g) = c_{\bar{g}}$ for $\bar{g} = [g]_H$. On the other hand, when $g \in G \setminus N_GH$ then $\Psi^H(f_g) = 0$, by the factorization (2.6) and the fact that $\Psi^H(k(G/L)) = 0$ for $L = H \cap {}^gH$ with $g \notin N_GH$, using again (4.13). Hence (5.2) is surjective and $\ker(\Psi^H)$ has basis $\{ f_g = \epsilon \circ c_g \circ \eta \}_{[g] \in H\backslash G/H, g \notin N_GH}$. One easily verifies that such an $f_g$ has image contained in $\mathrm{Rad}(kG) \cdot k(G/H)$, using that $H \cap {}^gH$ is strictly smaller than $H$. Composing $n$ such generators $f_{g_1} \circ \cdots \circ f_{g_n}$, then maps $k(G/H)$ into $\mathrm{Rad}(kG)^n \cdot k(G/H)$ which is eventually zero for $n \gg 1$, since $G$ is a $p$-group.

We now isolate a purely additive result that we shall of course apply to the case where $\Psi$ is a modular fixed-points functor.

5.3. Lemma. Let $\Psi : \mathcal{A} \to \mathcal{D}$ be an additive functor between additive categories. Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be full additive subcategories such that:

1. Every object of $A$ decomposes as $B \oplus C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$.
2. The functor $\Psi$ vanishes on $\mathcal{C}$, that is, $\Psi(\mathcal{C}) = 0$.
3. The restricted functor $\Psi|_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}$ is full with nilpotent kernel.\(^6\)

Let $X \in \mathrm{Ch}(\mathcal{A})$ be a complex such that $\Psi(X)$ is contractible in $\mathrm{Ch}(\mathcal{D})$. Then $X$ is homotopy equivalent to a complex in $\mathrm{Ch}(\mathcal{C})$.

Proof. Decompose every $X_i = B_i \oplus C_i$ in $\mathcal{A}$, using (1), for all $i \in \mathbb{Z}$. We are going to build a complex on the objects $C_i$ in such a way that $X_i$ becomes homotopy equivalent to $C_i$ in $\mathrm{Ch}(\mathcal{A})^2$, where $\mathcal{A}^2$ is the idempotent-completion of $\mathcal{A}$. As both $X_i$ and $C_i$ belong to $\mathrm{Ch}(\mathcal{A})$, this proves the result. By (2), the complex $\cdots \to \Psi(B_i) \to \Psi(B_{i-1}) \to \cdots$ is isomorphic to $\Psi(X)$, hence it is contractible. So each $\Psi(B_i)$ decomposes in $\mathcal{D}$ as $D_i \oplus D_{i-1}$ in such a way that the differential $\Psi(B_i) = D_i \oplus D_{i-1} \longrightarrow \Psi(B_{i-1}) = D_{i-1} \oplus D_{i-2}$ is just $\left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$. Since $\Psi|_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}$ is full with nilpotent kernel by (3), we can lift idempotents. In other words, we can decompose each $B_i$ in the idempotent-completion $\mathcal{B}^2$ (hence in $\mathcal{A}^2$) as

$$B_i \simeq B_i' \oplus B_i''$$

\(^6\)There exists $n \gg 1$ such that if $n$ composable morphisms $f_1, \ldots, f_n$ in $\mathcal{B}$ all go to zero in $\mathcal{D}$ under $\Psi$ then their composite $f_n \circ \cdots \circ f_1$ is zero in $\mathcal{B}$.
with \( \Psi(B'_i) \simeq D_i \) and \( \Psi(B''_i) \simeq D_{i-1} \) in a compatible way with the decomposition in \( \mathcal{D}^3 \). This means that when we write the differentials in \( X \) in components in \( \mathcal{A}^3 \)

\[
\cdots \rightarrow X_i = B'_i \oplus B''_i \oplus C_i \xrightarrow{\begin{pmatrix} * & b_i & * \\ * & 0 & * \\ * & 0 & * \end{pmatrix}} X_{i-1} = B'_{i-1} \oplus B''_{i-1} \oplus C_{i-1} \rightarrow \cdots
\]

the component \( b_i : B''_i \rightarrow B'_{i-1} \) maps to the isomorphism \( \Psi(B''_i) \simeq D_{i-1} \simeq \Psi(B'_{i-1}) \) in \( \mathcal{D}^3 \). Hence \( b_i \) is already an isomorphism in \( \mathcal{B}^3 \) by (3) again. (Note that (3) passes to \( \mathcal{B}^3 \rightarrow \mathcal{D}^3 \).) Using elementary operations on \( X_i \) and \( X_{i-1} \) we can replace \( X \) by an isomorphic complex in \( \mathcal{A}^3 \) of the form

\[
X_{i+1} = B'_i \oplus B''_i \oplus C_i \xrightarrow{\begin{pmatrix} 0 & b_i & 0 \\ 0 & 0 & 0 \\ * & 0 & * \end{pmatrix}} B'_{i-1} \oplus B''_{i-1} \oplus C_{i-1} \rightarrow X_{i-2} \rightarrow \cdots
\]

This being a complex forces the ‘previous’ differential \( X_{i+1} \rightarrow X_i \) to be of the form \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ * & 0 & * \end{pmatrix} \) and the ‘next’ differential \( X_{i-1} \rightarrow X_{i-2} \) to be of the form \( \begin{pmatrix} 0 & 0 & * \\ 0 & a & 0 \\ * & 0 & 0 \end{pmatrix} \).

We can then remove from \( X \) a direct summand in \( \text{Ch}(\mathcal{A}^3) \) that is homotopically trivial complex of the form \( \cdots 0 \rightarrow B'_i \xrightarrow{\sim} B'_{i-1} \rightarrow 0 \cdots \).

The reader might be concerned about how to perform this reduction in all degrees at once, since we do not put boundedness conditions on \( X \) (thus preventing the ‘obvious’ induction argument). The solution is simple. Do the above for all differentials in even indices \( i = 2j \). By elementary operations on \( X_{2j} \) and \( X_{2j-1} \) for all \( j \in \mathbb{Z} \), we can replace \( X \) up to isomorphism into a complex whose even differentials are of the form (5.4). We then remove the contractible complexes \( \cdots 0 \rightarrow B''_{2j} \xrightarrow{\sim} B'_{2j-1} \rightarrow 0 \cdots \). We obtain in this way a homotopy equivalent complex in \( \mathcal{A}^3 \) that we call \( \tilde{X} \), where \( B'_i', B''_i' \in \mathcal{B}^3 \) and \( C_i' \in \mathcal{C} \)

\[
\cdots \rightarrow B''_{2j+1} \oplus C_{2j+1} \xrightarrow{\begin{pmatrix} a_{2j+1} & * \\ * & * \\ * & * \end{pmatrix}} B'_{2j} \oplus C_{2j} \xrightarrow{\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix}} B''_{2j-1} \oplus C_{2j-1} \rightarrow \cdots
\]

in which the even differentials go to zero under \( \Psi \), by the above construction. In particular the homotopy trivial complex \( \Psi(\tilde{X}) \simeq \Psi(X) \) in \( \mathcal{D}^3 \) has the form

\[
\cdots \rightarrow \Psi(B''_{2j+1}) \xrightarrow{\Psi(a_{2j+1})} \Psi(B'_{2j}) \xrightarrow{\Psi(a_{2j})} \cdots \text{ hence its odd-degree differentials } \Psi(a_{2j+1}) \text{ are isomorphisms.}
\]

It follows that \( a_{2j+1} : B''_{2j+1} \rightarrow B'_{2j} \) is itself an isomorphism by (3) again. Using new elementary operations (again in all degrees), we change the odd-degree differentials of the complex \( \tilde{X} \) in (5.5) into diagonal ones and we remove the contractible summands \( 0 \rightarrow B''_{2j+1} \xrightarrow{\sim} B'_{2j} \rightarrow 0 \) as before, to get a complex consisting only of the \( C_i \) in each degree \( i \in \mathbb{Z} \). In summary, we have shown that \( X \) is homotopy equivalent to a complex \( C \in \text{Ch}(\mathcal{C}) \) inside \( \text{Ch}(\mathcal{A}^3) \), as announced.

\[ \square \]

5.6. Remark. Of course, it would be silly to discuss conservativity of the functors \( \{ \Psi^H \}_{H \leq G} \) since among them we find \( \Psi^1 = \text{Id} \). The interesting result appears when each \( \Psi^H \) is used in conjunction with the derived category of \( G//H \), or, in ‘big’ form, its homotopy category of injectives. Let us remind the reader.

5.7. Recollection. In [BG22a], we prove that the homotopy category of injective \( BG \)-modules, with coefficients in any regular ring \( R \) (e.g. our field \( k \)), is a localization of \( \text{DPerm}(G; R) \). In our case, we have an inclusion \( J_G : \text{KInj}(kG) \hookrightarrow \text{DPerm}(G; k) \),
inside $K(\text{Perm}(G;k))$, and this inclusion admits a left adjoint $\Upsilon_G$

$$\Upsilon_G \downarrow \uparrow J_G$$

(5.8) $K \text{Inj}(kG)$. 

This realizes the finite localization of $D\text{Perm}(G;k)$ with respect to the subcategory $\mathcal{K}_{\text{ac}}(G) \subseteq \mathcal{K}(G) = D\text{Perm}(G;k)^c$. In particular, $\Upsilon_G$ preserves compact objects and yields the equivalence $\Upsilon_G: K(G)/K_{\text{ac}}(G) \cong D_b(kG) \cong K\text{Inj}(kG)^r$ of (2.13), also denoted $\Upsilon_G$ for this reason. Note that $\Upsilon_G \circ J_G \cong \text{Id}$ as usual with localizations.

Let $P \subseteq G$ be a subgroup. Observe that induction $\text{Ind}^G_P$ preserves injectives so that $J_G \circ \text{Ind}^G_P \cong \text{Ind}^G_P \circ J_P$. Taking left adjoints, we see that

$$\text{Res}^G_P \circ \Upsilon_G \cong \Upsilon_P \circ \text{Res}^G_P.$$  

5.10. Notation. For each $p$-subgroup $H \subseteq G$, we are interested in the composite

$$\tilde{\Psi}^H = \tilde{\Psi}^{H;G}: D\text{Perm}(G;k) \overset{\Psi^{H;G}}{\longrightarrow} D\text{Perm}(G\!/H;k) \overset{\Upsilon_{G\!/H}}{\longrightarrow} K\text{Inj}(k(G\!/H))$$

of the modular $H$-fixed-points functor followed by localization to the homotopy category of injectives (5.8). We use the same notation on compacts

$$\tilde{\Psi}^H = \tilde{\Psi}^H_G: \mathcal{K}(G;k) \overset{\Psi^{H;G}}{\longrightarrow} \mathcal{K}(G\!/H;k) \overset{\Upsilon_{G\!/H}}{\longrightarrow} D_b(k(G\!/H)).$$

5.12. Theorem (Conservativity). Let $G$ be a finite group. The above family of functors $\tilde{\Psi}^H: \mathcal{I}(G) \to K\text{Inj}(k(G\!/H))$, indexed by all the (conjugacy classes of) $p$-subgroups $H \subseteq G$, collectively detects vanishing of objects of $D\text{Perm}(G;k)$.

Proof. Let $P \subseteq G$ be a $p$-Sylow subgroup. For every subgroup $H \subseteq P$, we have $P\!/H \hookrightarrow G\!/H$ and $\tilde{\Psi}^{H;P} \circ \text{Res}^G_P$ can be computed as $\text{Res}^G_{P\!/H} \circ \tilde{\Psi}^H_G$ thanks to Proposition 4.15 and (5.9). On the other hand, $\text{Res}^G_P$ is (split) faithful, as $\text{Ind}^G_P \circ \text{Res}^G_P$ admits the identity as a direct summand. Hence it suffices to prove the theorem for the group $P$, i.e. we can assume that $G$ is a $p$-group.

Let $G$ be a $p$-group and $\mathcal{F}$ be a family of subgroups (Recollection 4.2). We say that a complex $X$ in $\text{Ch}(\text{Perm}(G;k))$ is of type $\mathcal{F}$ if every $X_i$ is $\mathcal{F}$-free, i.e. a coproduct of $k(G/K)$ for $K \in \mathcal{F}$. So every complex is of type $\mathcal{F}_{\text{all}} = \{H \leq G\}$. Saying that $X$ is of type $\mathcal{F}_1 = \emptyset$ means $X = 0$. We want to prove that if $X$ defines an object in $D\text{Perm}(G;k)$ and $\tilde{\Psi}^H(X) = 0$ for all $H \leq G$ then $X$ is homotopy equivalent to a complex $X' \in \text{Ch}(\text{Perm}(G;k))$ of type $\mathcal{F}' \subsetneq \mathcal{F}$.

We proceed by a form of ‘descending induction’ on $\mathcal{F}$. Namely, we prove:

Claim: Let $X \in D\text{Perm}(G;k)$ be a complex of type $\mathcal{F}$ for some family $\mathcal{F}$ and let $H \in \mathcal{F}$ be a maximal element of $\mathcal{F}$ for inclusion. If $\tilde{\Psi}^H(X) = 0$ then $X \cong X'$ is homotopy equivalent to a complex $X' \in \text{Ch}(\text{Perm}(G;k))$ of type $\mathcal{F}' \subsetneq \mathcal{F}$.

By the above discussion, proving this claim proves the theorem. Explicitly, we are going to prove this claim for $\mathcal{F}' = \mathcal{F} \cap \mathcal{F}_H$, that is, $\mathcal{F}$ with all conjugates of $H$ removed. By maximality of $H$ in $\mathcal{F}$, every $K \in \mathcal{F}$ is either conjugate to $H$ or in $\mathcal{F}'$. Of course, for $H'$ conjugate to $H$ we have $k(G/H') \simeq k(G/H)$ in $\text{Perm}(G;k)$.

We apply Lemma 5.3 for $A = \text{Add} \{k(G/K) \mid K \in \mathcal{F}'\}$, $\mathcal{B} = \text{Add}(k(G/H))$, $\mathcal{C} = \text{Add} \{k(G/K) \mid K \in \mathcal{F}'\}$, $\mathcal{D} = \text{Perm}(G\!/H;k)$ and the functor $\Psi = \tilde{\Psi}^H$ naturally.
Let us check the hypotheses of Lemma 5.3. Regrouping the terms \(k(G/K)\) into those for which \(K\) is conjugate to \(H\) and those not conjugate to \(H\), we get Hypothesis (1). Hypothesis (2) follows immediately from (4.13) since \((G/K)^H = \emptyset\) for every \(K \in \mathcal{T}'\). Finally, Hypothesis (3) follows from Lemma 5.1 and additivity. So it remains to show that \(\Psi^H(X)\) is contractible. Since \(X\) is of type \(\mathcal{T}\) and \(H\) is maximal, we see that \(\Psi^H(X) \in \operatorname{Ch} \operatorname{Inj}(k(G//H))\) and applying \(\Upsilon_{G//H}\) gives the same complex (up to homotopy). In other words, \(\Phi^H(X) = 0\) forces \(\Phi^H(X)\) to be contractible and we can indeed get the above Claim from Lemma 5.3.

\[
6. \text{ The spectrum as a set}
\]

In this section, we deduce from the previous results the description of all points of \(\operatorname{Spc} (\mathcal{X}(G))\), as well as some elements of its topology. We start with a general fact, which is now folklore.

\[
6.1. \text{Proposition. Let } F : \mathcal{T} \rightarrow S \text{ be a coproduct-preserving tt-functor between ‘big’ tt-categories. Suppose that } F \text{ is conservative. Then } F \text{ detects } \otimes\text{-nilpotence of morphisms } f : x \rightarrow Y \text{ in } \mathcal{T}, \text{ whose source } x \in \mathcal{T}^e \text{ is compact, i.e. if } F(f) = 0 \text{ in } S \text{ then there exists } n \gg 1 \text{ such that } f^\otimes n = 0 \text{ in } \mathcal{S}. \text{ In particular, } F : \mathcal{T} \rightarrow \mathcal{S}^e \text{ detects nilpotence of morphisms and therefore } \operatorname{Spc}(F) : \operatorname{Spc}(\mathcal{S}^e) \rightarrow \operatorname{Spc}(\mathcal{F}^e) \text{ is surjective.}
\]

\[
\text{Proof. Using rigidity of compacts, we can assume that } x = 1. \text{ Given a morphism } f : 1 \rightarrow Y \text{ we can construct in } \mathcal{T} \text{ the homotopy colimit } Y^\infty := \operatorname{hocolim}_n Y^\otimes(n+1) \text{ under the transition maps } f \otimes \text{id} : Y^\otimes n \rightarrow Y^\otimes(n+1). \text{ Let } f^\infty : 1 \rightarrow Y^\infty \text{ be the resulting map. Now since } F(f) = 0 \text{ it follows that } F(Y^\infty) = 0 \text{ in } S, \text{ as it is a sequential homotopy colimit of zero maps. By conservativity of } F, \text{ we get } Y^\infty = 0 \text{ in } \mathcal{T}. \text{ Since } 1 \text{ is compact, the vanishing of } f^\infty : 1 \rightarrow \operatorname{hocolim} Y^\otimes n \text{ must already happen at a finite stage, that is, the map } f^\otimes n : 1 \rightarrow Y^\otimes n \text{ is zero for } n \gg 1, \text{ as claimed. The second statement follows from this, together with } [\text{Bal18, Theorem 1.4}].
\]

Combined with our Conservativity Theorem 5.12 we get:

\[
6.2. \text{Corollary. The family of functors } \Psi^H : \mathcal{X}(G) \rightarrow \operatorname{D}_b(k(G//H)), \text{ indexed by conjugacy classes of p-subgroups } H \leq G, \text{ detects } \otimes\text{-nilpotence. So the induced map }
\]

\[
\prod_{H \in \operatorname{Sub}_b(G)} \operatorname{Spc}(\operatorname{D}_b(k(G//H))) \rightarrow \operatorname{Spc}(\mathcal{X}(G))
\]

is surjective.

\[
6.3. \text{Definition. Let } H \leq G \text{ be a p-subgroup. We write (under Convention 1.18) }
\]

\[
\psi^H = \psi^{H;G} := \operatorname{Spc}(\Psi^H) : \operatorname{Spc}(\mathcal{X}(G//H)) \rightarrow \operatorname{Spc}(\mathcal{X}(G))
\]

for the map induced by the modular \(H\)-fixed-points functor. We write

\[
\psi^H = \psi^{H;G} := \operatorname{Spc}(\Psi^H) : \operatorname{Spc}(\operatorname{D}_b(G//H)) \rightarrow \operatorname{Spc}(\mathcal{X}(G))
\]

for the map induced by the tt-functor \(\Phi^H = \Upsilon_{G//H} \circ \Psi^H\) of (5.11). In other words, \(\psi^H\) is the composite of the inclusion of Proposition 2.22 with the above \(\psi^H\).

\[
\psi^H : \operatorname{V}_{G//H} = \operatorname{Spc}(\operatorname{D}_b(k(G//H))) \xrightarrow{\psi^{G//H}} \operatorname{Spc}(k(G//H)) \xrightarrow{\psi^H} \operatorname{Spc}(\mathcal{X}(G))
\]

6.4. Definition. Let \(H \leq G\) be a p-subgroup and \(p \in \operatorname{V}_{G//H}\) a ‘cohomological’ prime over the Weyl group of \(H\) in \(G\). We define a point in \(\operatorname{Spc}(\mathcal{X}(G))\) by

\[
\mathcal{P}(H, p) = \mathcal{P}_G(H, p) := (\psi^H)^{-1}(p).
\]
Corollary 6.2 tells us that every point of \( \text{Spc}(\mathcal{X}(G)) \) is of the form \( \mathcal{P}(H, p) \) for some \( p \)-subgroup \( H \leq G \) and some cohomological point \( p \in V_{G/H} \). Different subgroups and different cohomological points can give the same \( \mathcal{P}(H, p) \). See Theorem 6.16.

6.5. Remark. Although we shall not use it, we can unpack the definitions of \( \mathcal{P}_G(H, p) \) for the nostalgic reader. Let us start with the bijection \( V_G = \text{Spc}(D_b(kG)) \cong \text{Spec}^\ast(H^\ast(G, k)) \). Let \( p^\ast \subset H^\ast(G; k) = \text{End}^\ast_{D_b(kG)}(1) \) be a homogeneous prime ideal of the cohomology. The corresponding prime \( p \) in \( D_b(kG) \) can be described as

\[
p = \{ x \in D_b(kG) \mid \exists \zeta \in H^\ast(G; k) \text{ such that } \zeta \notin p^\ast \text{ and } \zeta \otimes x = 0 \}.
\]

Consequently, the prime \( \mathcal{P}_G(H, p) \) of Definition 6.4 is equal to

\[
\{ x \in \mathcal{X}(G) \mid \exists \zeta \in H^\ast(G/H; k) \setminus p^\ast \text{ such that } \zeta \otimes H^H(x) = 0 \text{ in } D_b(k(G/H)) \}\.
\]

6.6. Remark. By Proposition 4.15 and functoriality of \( \text{Spc}(-) \), the primes \( \mathcal{P}_G(H, p) \) are themselves functorial in \( G \). To wit, if \( \alpha : G \to G' \) is a group homomorphism and \( H \) is a \( p \)-subgroup of \( G \) then \( \alpha(H) \) is a \( p \)-subgroup of \( G' \) and we have

\[
\alpha_\ast(\mathcal{P}_G(H, p)) = \mathcal{P}_{G'}(\alpha(H), \alpha_\ast p)
\]

in \( \text{Spc}(\mathcal{X}(G')) \), where \( \alpha_\ast : \text{Spc}(\mathcal{X}(G)) \to \text{Spc}(\mathcal{X}(G')) \) and \( \alpha_\ast : V_{G/H} \to V_{G'/\alpha(H)} \) are as in Remark 3.1. We single out the usual suspects. Fix \( H \leq G \) a \( p \)-subgroup.

(a) For conjugation, let \( G \leq G' \) and \( x \in G' \). We get \( \mathcal{P}_G(H, p)^x = \mathcal{P}_{G'}(H^x, p^x) \) for every \( p \in V_{G/H} \). In particular, when \( x \) belongs to \( G \) itself, we get by (3.2)

\[
g \in G \implies \mathcal{P}_G(H, p) = \mathcal{P}_G(H^g, p^g).
\]

(b) For restriction, let \( K \leq G \) be a subgroup containing \( H \) and let \( p \in V_{K/H} \) be a cohomological point over the Weyl group of \( H \) in \( K \). Then we have

\[
\rho_K(\mathcal{P}_K(H, p)) = \mathcal{P}_G(H, \rho_K(p)),
\]

in \( \text{Spc}(\mathcal{X}(G)) \), where the maps \( \rho_K = (\text{Res}_K^G)_\ast : \text{Spc}(\mathcal{X}(K)) \to \text{Spc}(\mathcal{X}(G)) \) and \( \rho_K : V_{K/H} \to V_{G/H} \) are spelled out around (3.4).

(c) For inflation, let \( N \leq G \) be a normal subgroup. Set \( \bar{G} = G/N \) and \( \bar{H} = HN/N \). Then for every \( p \in V_{G/H} \), we have

\[
\bar{\pi}^\bar{G} (\mathcal{P}_G(H, p)) = \mathcal{P}_{\bar{G}}(\bar{H}, \bar{\pi}^\bar{G} p),
\]

in \( \text{Spc}(\mathcal{X}(\bar{G})) \), where the maps \( \bar{\pi}^\bar{G} = (\text{Inf}_N^G)_\ast : \text{Spc}(\mathcal{X}(G)) \to \text{Spc}(\mathcal{X}(\bar{G})) \) and \( \bar{\pi}^\bar{G} : V_{\bar{G}/\bar{H}} \to V_{\bar{G}/\bar{H}} \) are spelled out around (3.5).

Our primes also behave nicely under modular fixed-points maps:

6.11. Proposition. Let \( H \leq G \) be a \( p \)-subgroup and let \( H \leq K \leq N_G H \) defining a ‘further’ \( p \)-subgroup \( K/H \leq G/H \). Then for every \( p \in V_{(G/H)\langle K/H \rangle} \), we have

\[
\psi^{H,G}(\mathcal{P}_{G/H}(K/H, p)) = \mathcal{P}_{G}(K, \bar{p}(p))
\]

in \( \text{Spc}(\mathcal{X}(G)) \), where \( \bar{p} = \text{Spc}(\text{Res}_{(G/H)\langle K/H \rangle}^G K) : V_{(G/H)\langle (K/H) \rangle} \to V_{G\langle K \rangle} \). In particular, if \( H \leq G \) is normal, we have

\[
\psi^{H,G}(\mathcal{P}_{G/H}(K/H, p)) = \mathcal{P}_{G}(K, p)
\]

in \( \text{Spc}(\mathcal{X}(G)) \), using that \( p \in V_{(G/H)\langle (K/H) \rangle} = V_{G\langle K \rangle} \).

Proof. This is immediate from Proposition 4.17 and Corollary 4.18. □
The relation between Koszul objects and modular fixed-points functors, obtained in Lemma 4.21, can be reformulated in terms of the primes \( \mathcal{P}_G(H, p) \).

6.12. **Lemma.** Let \( H \leq G \) be a \( p \)-subgroup and \( p \in V_{G/H} \). Let \( K \leq G \) be a subgroup and \( \text{kos}_{G}(K) \) be the Koszul object of Construction 2.14. Then \( \text{kos}_{G}(K) \subseteq \mathcal{P}_G(H, p) \) if and only if \( H \leq G \). (Note that the latter condition does not depend on \( p \).)

**Proof.** We have seen in Lemma 4.21 (b) that if \( H \leq G \) then \( \tilde{\Psi}^G(\text{kos}_{G}(K)) = 0 \) in \( D_b(k(G/H)) \), in which case \( \text{kos}_{G}(K) \in (\tilde{\Psi}^G)^{-1}(0) \subseteq (\tilde{\Psi}^G)^{-1}(p) = \mathcal{P}_G(H, p) \) for every \( p \). Conversely, we have seen in Lemma 4.21 (a) that if \( H \not\leq G \) then \( \tilde{\Psi}^G(\text{kos}_{G}(K)) \) generates \( D_b(k(G/H)) \), hence is not contained in any cohomological point \( p \), in which case \( \text{kos}_{G}(K) \not\in (\tilde{\Psi}^G)^{-1}(p) = \mathcal{P}_G(H, p) \).

\( \square \)

6.13. **Corollary.** If \( \mathcal{P}_G(H, p) \subseteq \mathcal{P}_G(H', p') \) then \( H' \leq G \). Therefore if \( \mathcal{P}_G(H, p) = \mathcal{P}_G(H', p') \) then \( H \) and \( H' \) are conjugate in \( G \).

**Proof.** Apply Lemma 6.12 to \( K = H \) twice, for \( H \) being once \( H \) and once \( H' \).

\( \square \)

6.14. **Proposition.** Let \( H \leq G \) be a \( p \)-subgroup. Then the map \( \psi^H : V_{G/H} \rightarrow \text{Spec}(\mathcal{X}(G)) \) is injective, that is, \( \mathcal{P}_G(H, p) = \mathcal{P}_G(H, p') \) implies \( p = p' \).

**Proof.** Let \( N = N_G(H) \). By assumption we have \( \rho^N_G(\mathcal{P}_N(H, p)) = \rho^N_G(\mathcal{P}_N(H, p')) \). By Corollary 3.9, there exists \( g \in G \) and a prime \( \mathfrak{q} \in \text{Spec}(\mathcal{X}(N \cap gN)) \) such that

\[
\mathcal{P}_N(H, p) = \rho^N_G(\mathfrak{q}) \quad \text{and} \quad \mathcal{P}_N(H, p') = (\rho^N_G(\mathfrak{q}))^g.
\]

By Corollary 6.2 for the group \( N \cap gN \), there exists a \( p \)-subgroup \( L \leq N \cap gN \) and some \( q \in V_{N \cap gN} \) such that \( \mathfrak{q} = \mathcal{P}_{N \cap gN}(L, q) \). By (6.7) we know where such a prime \( \mathcal{P}_{N \cap gN}(L, q) \) goes under the maps \( \rho = \text{Spec}(\text{Res}) \) of (6.15) and, for the second one, we also know what happens under conjugation by Remark 6.6 (a). Applying these properties to the above relations (6.15) we get

\[
\mathcal{P}_N(H, p) = \mathcal{P}_N(L, q') \quad \text{and} \quad \mathcal{P}_N(H, p') = \mathcal{P}_N(L^g, q''')
\]

for suitable cohomological points \( q' \in V_{N/L} \) and \( q''' \in V_{N/L^g} \) that we do not need to unpack. By Corollary 6.13 applied to the group \( N \), we must have \( H \sim N L \) and \( H \sim N L^g \). But since \( H \leq N \), this forces \( H = L = L^g \) and therefore \( g \in N_G(H) = N \).

In that case, returning to (6.15), we have \( N \cap N^g = N = N^g \) and therefore

\[
\mathcal{P}_N(H, p) = \mathfrak{q} \quad \text{and} \quad \mathcal{P}_N(H, p') = \mathfrak{q}^g = \mathfrak{q}
\]

where the last equality uses \( g \in N \) and (3.2). Hence \( \mathcal{P}_N(H, p) = \mathfrak{q} = \mathcal{P}_N(H, p') \). As \( H \) is normal in \( N \) the map \( \psi^H : \text{Spec}(\mathcal{X}(N/H)) \rightarrow \text{Spec}(\mathcal{X}(N)) \) is split injective by Corollary 4.16, and we conclude that \( p = p' \).

We can now summarize our description of the set \( \text{Spec}(\mathcal{X}(G)) \).

6.16. **Theorem.** Every point in \( \text{Spec}(\mathcal{X}(G)) \) is of the form \( \mathcal{P}_G(H, p) \) as in Definition 6.4, for some \( p \)-subgroup \( H \leq G \) and some point \( p \in V_{G/H} \) of the cohomological open of the Weyl group of \( H \) in \( G \). Moreover, we have \( \mathcal{P}_G(H, p) = \mathcal{P}_G(H', p') \) if and only if there exists \( g \in G \) such that \( H = H^g \) and \( p = p^g \).

**Proof.** The first statement follows from Corollary 6.2. For the second statement, the “if”-direction follows from (6.8). For the “only if”-direction assume \( \mathcal{P}_G(H, p) = \mathcal{P}_G(H', p') \). By Corollary 6.13, this forces \( H \sim G \). Using (6.8), we can replace \( H' \) by \( H^g \) and assume that \( \mathcal{P}_G(H, p) = \mathcal{P}_G(H, p') \) for \( p, p' \in V_{G/H} \). We can then conclude by Proposition 6.14. \( \square \)
Here is an example of support, for the Koszul objects of Construction 2.14.

6.17. Corollary. Let $K \leq G$. Then $\text{supp}(\text{kos}_G(K)) = \{ \mathcal{P}(H, p) \mid H \not\leq_G K \}$.

Proof. Since all primes are of the form $\mathcal{P}(H, p)$, it is a simple contraposition on Lemma 6.12, for $\mathcal{P}(H, p) \in \text{supp}(\text{kos}_G(K)) \iff \text{kos}_G(K) \not\subseteq \mathcal{P}(H, p) \iff H \not\leq_G K$.  

We can use this result to identify the image of $\psi^H$. First, in the normal case:

6.18. Proposition. Let $H \leq G$ be a normal $p$-subgroup. Then the continuous map

$$\psi^H = \text{Spc}(\Psi^H): \text{Spc}(\mathcal{K}(G/H)) \to \text{Spc}(\mathcal{K}(G))$$

is a closed immersion, retracted by $\text{Spc}(\text{Infl}_G^{G/H})$. Its image is the closed subset

$$(6.19) \quad \text{Im}(\psi^H) = \{ \mathcal{P}_G(L, p) \mid H \leq L \in \text{Sub}_p G, \ p \in V_{G/L} \} = \bigcap_{K \in \mathcal{F}_H} \text{supp}(\text{kos}_G(K))$$

where we recall that $\mathcal{F}_H = \{ K \leq G \mid H \not\leq_K \}$. Furthermore, this image of $\psi^H$ is also the support of the object

$$(6.20) \quad \bigotimes_{K \in \mathcal{F}_H} \text{kos}_G(K)$$

and it is also the support of the tt-ideal $\cap_{K \in \mathcal{F}_H} \text{Ker}(\text{Res}^G_K)$.

Proof. By Corollary 4.16, the map $\psi^H$ has a continuous retraction hence is a closed immersion as soon as we know that its image is closed. So let us prove (6.19). By Proposition 6.11 and the fact that all points are of the form $\mathcal{P}(L, p)$, the image of $\psi^H$ is the subset $\{ \mathcal{P}_G(L, p) \mid H \leq L, \ p \in V_{G/L} \}$. Here we use $H \leq G$.

Corollary 6.17 tells us that every such point $\mathcal{P}(L, p)$ belongs to the support of $\text{kos}_G(K)$ as long as $L \not\leq_G K$, which clearly holds if $H \leq L$ and $H \not\leq_K$. Therefore $\text{Im}(\psi^H) \subseteq \cap_{K \in \mathcal{F}_H} \text{supp}(\text{kos}_G(K))$.

Conversely, let $\mathcal{P}(L, p) \in \cap_{K \in \mathcal{F}_H} \text{supp}(\text{kos}_G(K))$ and let us show that $H \leq L$. If $ab \text{ absurdo}$, $H \not\leq L$ then $L \in \mathcal{F}_H$ is one of the indices $K$ that appear in the intersection $\cap_{K \in \mathcal{F}_H} \text{supp}(\text{kos}_G(K))$. In other words, $\mathcal{P}(L, p) \in \text{supp}(\text{kos}_G(L))$. By Corollary 6.17, this means $L \not\leq_G L$, which is absurd. Hence the result.

The ‘furthermore part’ follows: The first claim is (6.19) since $\text{supp}(x) \cap \text{supp}(y) = \text{supp}(x \otimes y)$ and the second claim follows from Corollary 2.24. (For $H = 1$, the result does not tell us much, as $\psi^1 = \text{id}$ and $\otimes_{\emptyset} = \emptyset$.)  

Let us extend the above discussion to not necessarily normal subgroups $H$.

6.21. Notation. Let $H \leq G$ be an arbitrary subgroup. We define an object of $\mathcal{K}(G)$

$$(6.22) \quad \text{zul}_G(H) := \text{Ind}_N^G \left( \bigotimes_{K \in \mathcal{N}_G H} \text{kos}_{N_G H}(K) \right).$$

(Note that we use plain induction here, not tensor-induction as in Construction 2.14.) If $H \leq G$ is normal this $\text{zul}_G(H)$ is simply the object displayed in (6.20).

6.23. Corollary. Let $H \leq G$ be a $p$-subgroup. Then the continuous map

$$\psi^{H,G} = \text{Spc}(\Psi^{H,G}): \text{Spc}(\mathcal{K}(G/H)) \to \text{Spc}(\mathcal{K}(G))$$

is a closed map, whose image is $\text{supp}(\text{zul}_G(H))$ where $\text{zul}_G(H)$ is as in (6.22).
6.24. Corollary. The support of the tt-ideal of acyclics $\mathcal{K}_{\text{ac}}(G)$ is the union of the images of the modular $H$-fixed-points maps $\psi^H$, for non-trivial $p$-subgroups $H \leq G$.

Proof. By definition $\Psi^H:G = \Psi^{H:N_G H} \circ \text{Res}^G_{N_G H}$. We know the map induced on spectra by the second functor $\Psi^{H:N_G H}$ by Proposition 6.18 and we can describe what happens under the closed map $\text{Spc}(\text{Res})$ by Proposition 3.7. \hfill $\square$

We record the answer to a question stated in the Introduction:

6.25. Remark. Recall that in tt-geometry closed points $M \in \text{Spc}(\mathcal{K})$ are exactly the minimal primes for inclusion. Also every prime contains a minimal one.

For instance, the tt-category $D_b(kG)$ is local, with a unique closed point $0 = \text{Ker}(D_b(kG) \rightarrow D_b(k))$. (In terms of homogeneous primes in $\text{Spec}(H^\bullet(G,k))$ the zero tt-ideal $p = 0$ corresponds to the closed point $p^\bullet = H^0(G,k)$.)

6.26. Definition. Let $H \leq G$ be a $p$-subgroup. (This definition only depends on the conjugacy class of $H$ in $G$.) By Proposition 4.15, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{K}(G) & \xrightarrow{\text{Res}_H^G} & \mathcal{K}(H) \\
\downarrow_{\Psi^H:G} & & \downarrow_{\Psi^H:H}
\end{array}
$$

(6.27)

We baptize $F^H = F^{H,G}$ the diagonal. Its kernel is one of the primes of Definition 6.4

$$
\mathcal{M}(H) = \mathcal{M}_G(H) := \text{Ker}(F^H) = \mathcal{T}_G(H,0)
$$

where $0 \in \text{Spc}(D_b(k(G/H)))$ is the zero tt-ideal, i.e. the unique closed point of the cohomological open $V_{G,H}$ of the Weyl group. (See Remark 6.25.) We can think of $F^H: \mathcal{K}(G) \rightarrow D_b(k)$ as a tt-residue field functor at the (closed) point $\mathcal{M}(H)$.

6.29. Example. For $H = 1$, we have $\mathcal{M}(1) = \text{Ker}(\text{Res}_1^G: \mathcal{K}(G) \rightarrow D_b(k)) = \mathcal{K}_{\text{ac}}(G)$. In other words, $\mathcal{M}(1) = Y_G^{-1}(0)$ is the image under the open immersion $\nu_G: V_G \rightarrow \text{Spc}(\mathcal{K}(G))$ of Proposition 2.22 of the unique closed point $0 \in V_G$ of Remark 6.25. In general, a closed point of an open is not necessarily closed in the ambient space. Here $\mathcal{M}(1)$ is closed since by definition $\mathcal{M}(1) = \text{Im}(\rho_1^G)$ where $\rho_1^G = \text{Spc}(\text{Res}_1^G)$. By Proposition 3.7, we know that $\text{Im}(\rho_1^G) = \text{supp}(k(G))$ is closed.

6.30. Example. For $H = G$ a $p$-group, we can give generators of the closed point

$$
\mathcal{M}(G) = \{ k(G/K) \mid K \neq G \}.
$$

As $\mathcal{M}(G) = \ker(\Psi^G: \mathcal{K}(G) \rightarrow D_b(k))$, inclusion $\supseteq$ follows from Proposition 4.12. For $\subseteq$, let $X \in \mathcal{M}(G)$ be a complex that vanishes under $\Psi^G$. Splitting the modules $X_n$, in each homological degree $n$ into a trivial (i.e. a $k$-vector space with trivial action) and non-trivial permutation modules, Lemma 5.3 shows that $X$ is homotopy equivalent to a complex in the additive category generated by $k(G/K)$, $K \neq G$. 


6.31. **Corollary.** The closed points of \( \text{Spc}(\mathcal{K}(G)) \) are exactly the tt-primes \( M_G(H) \) of (6.28) for the \( p \)-subgroups \( H \leq G \). Furthermore, we have \( M_G(H) = M_G(H') \) if and only if \( H \) is conjugate to \( H' \) in \( G \).

**Proof.** Let us first verify that \( M_G(H) \) is closed for every \( H \leq G \). For \( H = 1 \), we checked it in Example 6.29. For \( H \neq 1 \), we have \( M_G(H) = \mathcal{P}_G(H,0) = \Psi^H(M_{G/H}(1)) \). This gives the result since \( M_{G/H}(1) \) is closed in \( \text{Spc}(\mathcal{K}(G/H)) \), by Example 6.29 again, and since \( \psi^H \) is a closed map by Corollary 6.23.

Now, every point \( p \in V_{G/H} \) admits 0 in its closure in \( \text{Spc}(\mathcal{D}_b(k(G/H))) = V_{G/H} \). (See Remark 6.25.) By continuity of \( \tilde{\psi}^V : V_{G/H} \to \text{Spc}(\mathcal{K}(G)) \), it follows that \( \tilde{\psi}^V(0) = M_G(H) \) belongs to the closure of \( \tilde{\psi}^V(p) = \mathcal{P}_G(H,p) \), which proves that the \( \mathcal{M}_G(H) \) are the only closed points.

We already saw that \( \mathcal{P}(H,0) = \mathcal{P}(H',0) \) implies \( H \sim_G H' \), in Theorem 6.16. \( \square \)

6.32. **Proposition.** For every \( p \)-subgroup \( H \leq G \), consider the subset
\[
V_G(H) := \text{Im}(\tilde{\psi}^V) = \tilde{\psi}^V(V_{G/H})
\]
of \( \text{Spc}(\mathcal{K}(G)) \). Then \( M_G(H) \) is the unique closed point of \( \text{Spc}(\mathcal{K}(G)) \) that belongs to \( V_G(H) \). We have a set-partition indexed by conjugacy classes of \( p \)-subgroups
\[
(6.33) \quad \text{Spc}(\mathcal{K}(G)) = \bigcup_{H \in (\text{Sub}_pG)/G} V_G(H)
\]
where each \( V_G(H) \) is open in its closure.

**Proof.** The partition is immediate from Theorem 6.16. Each subset \( V_G(H) = \{ \mathcal{P}(H,p) \mid p \in V_{G/H} \} \) is a subset of the closed set \( \text{Im}(\psi^H) \). By Corollary 6.24 and Proposition 6.11, the complement of \( V_G(H) \) in \( \text{Im}(\psi^H) \) consists of the images \( \text{Im}(\psi^K) \) for every ‘further’ \( p \)-group \( K \), i.e. such that \( H \leq K \leq N_G(H) \) and these are closed by Corollary 6.23. Thus \( V_G(H) \) is an open in the closed set \( \text{Im}(\psi^H) \). \( \square \)

6.34. **Remark.** We can use (6.33) to define a map \( \text{Spc}(\mathcal{K}(G)) \to (\text{Sub}_pG)/G \). Corollary 6.13 tells us that this map is continuous for the (opposite poset) topology on \( (\text{Sub}_pG)/G \) whose open subsets are the ones stable under subconjugacy.

Moreover, for \( H \leq G \) a \( p \)-subgroup, the square
\[
\begin{array}{ccc}
\text{Spc}(\mathcal{K}(G/H)) & \xrightarrow{\psi^H} & \text{Spc}(\mathcal{K}(G)) \\
\downarrow & & \downarrow \\
(\text{Sub}_p(G/H))/(G/H) & \xrightarrow{\sim} & (\text{Sub}_pG)/G
\end{array}
\]
commutes, where the bottom horizontal arrow is the canonical inclusion that sends \( H \leq K \leq N_G(H) \) to \( K \). This follows from Proposition 6.11.

An immediate consequence is that while \( \psi^H \) might not be injective in general, we still have \( (\psi^H)^{-1}(V_G(H)) = V_{G/H} \).

7. **Cyclic, Klein-four and quaternion groups**

Although the full treatment [BG22c] of the topology of \( \text{Spc}(\mathcal{K}(G)) \) will require more technology, we can already present the answer for small groups. Some of the most interesting phenomena are already visible once we reach \( p \)-rank two in Example 7.10. Let us start with the easy examples.
7.1. Notation. Fix an integer \( n \geq 0 \) and consider the following space \( \mathcal{W}^n \) consisting of \( 2n + 1 \) points, with specialization relations pointing upward as usual:

\[
\mathcal{W}^n = \begin{array}{c}
\bullet m_0 \\
\bullet m_1 \\
\bullet m_2 \\
\vdots \\
\bullet m_n \\
\end{array}
\begin{array}{c}
p_1 \\
p_2 \\
p_3 \\
\vdots \\
p_n \\
\end{array}
\]

The closed subsets of \( \mathcal{W}^n \) are simply the specialization-closed subsets, i.e. those that contain a \( p_i \) only if they contain \( m_{i-1} \) and \( m_i \). So the \( m_i \) are closed points and the \( p_i \) are generic points of the \( n \) irreducible V-shaped closed subsets \( \{m_{i-1}, p_i, m_i\} \).

7.3. Proposition. Let \( G = C_{p^n} \) be a cyclic \( p \)-group. Then \( \text{Spc}(\mathcal{K}(C_{p^n})) \) is homeomorphic to the space \( \mathcal{W}^n \) of (7.2).

More precisely, if we denote by \( 1 = N_n < N_{n-1} < \cdots < N_0 = G \) the \( n + 1 \) subgroups of \( C_{p^n} \), then the points \( p_i \) and \( m_i \) in \( \text{Spc}(\mathcal{K}(G)) \) are given by

\[
m_i = (\Psi N_i)^{-1}(0) \quad \text{and} \quad p_i = (\Psi N_i)^{-1}(D_{\text{perf}}(k(G/N_i)))
\]

where \( \Psi^N = \mathcal{T}_{G/N} \circ \Psi^N : \mathcal{K}(G) \to \mathcal{K}(G/N) \to D_b(k(G/N)) \) is the tt-functor (5.11).

Proof. By Proposition 6.32, we have a partition of the spectrum in subsets

\[
\text{Spc}(\mathcal{K}(G)) = \coprod_{i=0}^{n} V_G(N_i) = \coprod_{i=0}^{n} \text{Im}(\Psi N_i)
\]

and each \( V_G(N_i) \) is homeomorphic to \( \text{Spc}(D_b(kG/N_i)) = V_{G/N_i} \). For \( i > 0 \), each \( V_G(N_i) \) is a Sierpiński space \( \{p_i \sim m_i = M(N_i)\} \), while \( V_G(N_0) \) is a singleton set \( \{m_0 := M(G)\} \). In other words, we know the set \( \text{Spc}(\mathcal{K}(G)) \) has the announced \( 2n + 1 \) points and the unmarked specializations \( p_i \sim m_i \) below

\[
\begin{array}{c}
\bullet m_0 \\
\bullet m_1 \\
\vdots \\
\bullet m_{n-1} \\
\bullet m_n \\
\end{array}
\begin{array}{c}
p_1 \\
p_2 \\
\vdots \\
p_{n-1} \\
p_n \\
\end{array}
\]

We need to elucidate the topology. Since all \( m_i = M(N_i) \) are closed (Corollary 6.31), we only need to see where each \( p_i \) specializes for \( 1 \leq i \leq n \). By Corollary 6.13, the point \( p_i = \mathcal{P}(N_i, p) \) can only specialize to a \( \mathcal{P}(N_j, q) \) for \( N_j \supseteq N_i \), that is, to the points \( m_j \) or \( p_j \) for \( j \leq i \). On the other hand, direct inspection using (4.13) shows that \( \text{supp}(kG/N_{i-1}) = \{m_j | j \geq i-1\} \cup \{p_j | j \geq i\} \). This closed subset contains \( p_i \) hence its closure. Combining those two observations, we have

\[
\{p_i\} \subseteq \{m_j, p_j | j \leq i\} \cap (\{m_{i-1}\} \cup \{m_j, p_j | j \geq i\}) = \{m_{i-1}, p_i, m_i\}.
\]

If any of the \( \{p_i\} \) was smaller than \( \{m_{i-1}, p_i, m_i\} \), that is, if one of the specialization relations \( p_i \sim m_{i-1} \) marked with ‘?’ in (7.4) did not hold, then \( \text{Spc}(\mathcal{K}(G)) \) would be a disconnected space. This would force the rigid tt-category \( \mathcal{K}(G) \) to be the product of two tt-categories, which is clearly absurd, e.g. because \( \text{End}_{\mathcal{K}(G)}(\mathbb{1}) = k \).

With this identification, we can record the maps \( \psi^H \) of Definition 6.3 and the maps \( \rho_K \) and \( \pi^{G/N} \) of Remark 6.6, that relate different cyclic \( p \)-groups.

7.5. Lemma. Let \( n \geq 0 \). We identify \( \text{Spc}(\mathcal{K}(C_{p^n})) \) with \( \mathcal{W}^n \) as in Proposition 7.3.
(a) Let $0 \leq i \leq n$ and $H = N_i = C_{p^{n-i}} \leq C_{p^n}$, so that $C_{p^n}/H \cong C_{p^i}$. The map $\psi^H : \mathbb{W}^i \to \mathbb{W}^n$ induced by modular fixed points $\Psi^H$ is the inclusion $\psi : \mathbb{W}^i \hookrightarrow \mathbb{W}^n$ that catches the left-most points: $p_\ell \mapsto p_\ell$ and $m_\ell \mapsto m_\ell$.

(b) Let $0 \leq j \leq n$ and $K = C_{p^j} \leq C_{p^n}$. The map $\rho_K : \mathbb{W}^j \to \mathbb{W}^n$ induced by restriction $\text{Res}_K$ is the inclusion $\rho : \mathbb{W}^j \hookrightarrow \mathbb{W}^n$ that catches the right-most points: $m_\ell \mapsto m_\ell + n - j$ and $p_\ell \mapsto p_\ell + n - j$.

(c) Let $0 \leq m \leq n$. Inflation along $C_{p^n} \twoheadrightarrow C_{p^m}$ induces on spectra the map $\pi : \mathbb{W}^m \to \mathbb{W}^n$ that retracts $\psi$ and sends everything else to $m$, that is, for all $0 \leq \ell \leq n$ $\pi(p_\ell) = \begin{cases} p_\ell & \text{if } \ell \leq m \\ m & \text{otherwise} \end{cases}$ and $\pi(m_\ell) = \begin{cases} m_\ell & \text{if } \ell \leq m \\ m & \text{otherwise} \end{cases}$.

Proof. Part (a) follows from Proposition 6.11, while parts (b) and (c) follow from Remark 6.6.

Let us now move to higher $p$-rank.

7.6. Example. Let $E = (C_p)^{\times r}$ be the elementary abelian $p$-group of rank $r$. We know that $V_E = \text{Spc}(\text{D}_b(kE)) \cong \text{Spec}^*(\text{H}^*(E, k))$ is homeomorphic to the space $V^r := \text{Spec}^*(k[x_1, \ldots, x_r])$, that is, projective space $\mathbb{P}_k^{r-1}$ with one closed point ‘on top’. For instance, $V^0$ is a single point and $V^1$ is a 2-point Sierpiński space. The example of $r = 1$ (see Proposition 7.3 for $n = 1$) is not predictive of what happens in higher rank. Indeed, by Proposition 6.32, the closed complement $\text{Supp}(\text{Kac}(E))$ is far from discrete in general. It contains $\frac{p^{r-1}}{p-1}$ copies of $V^{r-1}$ and more generally $|\text{Gr}_p(d, r)|$ copies of the $d$-dimensional $V^d$ for $d = 0, \ldots, r-1$, where $|\text{Gr}_p(d, r)|$ is the number of rank-$d$ subgroups of $(C_p)^{\times r}$. Here is a ‘low-resolution’ picture for Klein-four $r = p = 2$:

(7.8)

The dashed lines indicate ‘partial’ specialization relations: Some points in the lower variety specialize to some points in the higher one; see Corollary 6.13. In rank 3, the similar ‘low-resolution’ picture of $\text{Spc}(\text{K}(C_2^{\times 3}))$, still for $p = 2$, looks as follows:

(7.9)

Each $V^d$ has Krull dimension $d \in \{0, 1, 2, 3\}$ and contains one of 16 closed points.
Let us now discuss the example of Klein-four and ‘zoom-in’ on (7.8) to display every point at its actual height, as well as all specialization relations.

7.10. Example. Let $G = C_2 \times C_2$ be the Klein four-group, in characteristic $p = 2$. The spectrum $\text{Spc}(\mathcal{X}(E))$ looks as follows, with colors matching those of (7.8):

\[ (7.11) \]

The green part on the right is the cohomological open $V_E \simeq \mathbb{V}^2$ as in (7.7), that is, a $\mathbb{P}^1$ with a closed point on top. We marked with $\bullet$ the closed point $M(1)$, the three $\mathbb{F}_2$-rational points $0, 1, \infty$ of $\mathbb{P}^1$ and its generic point $P_0$. The notation $\mathbb{P}_1^1$, and the dotted line indicate $\mathbb{P}^1 \setminus \{0, 1, \infty, P_0\}$. The specializations involving points of $\mathbb{P}_1^1$ are displayed with undulated lines. For instance, the gray undulated line indicates that all points of $\mathbb{P}_1^1$ specialize to $M(E)$.

The (brown) part on the left is the support of the acyclics. It contains the remaining four closed points, $M(E)$, and $M(N_0)$, $M(N_1)$, $M(N_\infty)$ for the rank-one subgroups that we denote $N_0, N_1, N_\infty < E$ to facilitate matching them with $0, 1, \infty \in \mathbb{P}^1(\mathbb{F}_2)$. The three Sierpiński subspaces $\{\mathcal{P}(N_i) \to \mathcal{M}(N_i)\}$ are images of $V_{E/N_i} \simeq \mathbb{V}^1$. The point $M(E)$ is the image of $V_{E/E} \simeq \mathbb{V}^0$. See Proposition 6.32.

The (gray) specializations require additional arguments and will be examined in detail in [BG22c]. Let us still say a few words. The specializations between $\mathcal{P}(N_i)$ and $M(E)$ are relatively easy, as one can show that $\{M(E), \mathcal{P}(N_i), M(N_i)\}$ is the image of $\psi^{N_i}$, using our description in the case of $C_2 = E/N_i$. Similarly, the specializations between the $\mathbb{F}_2$-rational points $0, 1, \infty$ and $M(N_0), M(N_1), M(N_\infty)$ can be verified using $\rho_{N_i} : \text{Spc}(\mathcal{X}(N_i)) \to \text{Spc}(\mathcal{X}(E))$. The image of the latter is $\text{supp}(E/N_i) = \{M(N_i), \mathcal{M}(N_i), M(1)\}$, for each $i \in \{0, 1, \infty\} = \mathbb{P}_1(\mathbb{F}_2)$.

The fact that $P_0$ is a generic point for the whole $\text{Spc}(\mathcal{X}(E))$ is not too hard either. The real difficulty is to prove that the non-$\mathbb{F}_2$-rational points specialize to the closed point $M(E)$, avoiding the $M(N_i)$ and $\mathcal{P}(N_i)$ entirely, as indicated by the undulated gray line between $\mathbb{P}_1^1$ and $M(E)$. These facts can be found in [BG22c].

7.12. Example. The spectrum of the quaternion group $Q_8$ is very similar to that of its quotient $E := Q_8/Z(Q_8) \cong C_2 \times C_2$, as we announced in (1.16). The center $Z := Z(Q_8) \cong C_2$ is the maximal elementary abelian 2-subgroup and it follows that $\text{Res}_{Q_8}^Z$ induces a homeomorphism $V_{C_2} \xrightarrow{\sim} V_{Q_8}$. In other words, $V_{Q_8}$ is again a Sierpiński space $\{\mathcal{P}, \mathcal{M}(1)\}$. On the other hand, the center $Z$ is also the unique minimal non-trivial subgroup. It follows from Corollary 6.2 and Proposition 6.18 that $\text{Supp}(\mathcal{X}_{ac}(Q_8))$ is the image under the closed immersion $\psi^Z$ of $\text{Spc}(\mathcal{X}(Q_8/Z))$. It only remains to describe the specialization relations between the cohomological open $V_{Q_8}$ and its closed complement $\text{Supp}(\mathcal{X}_{ac}(Q_8))$. Since $\mathcal{M}(1) \in V_{Q_8}$ is also a closed point in $\text{Spc}(\mathcal{X}(Q_8))$, we only need to decide where the generic point $\mathcal{P}$ of $V_{Q_8}$ specializes in $\text{Spc}(\mathcal{X}(Q_8))$. Interestingly, $\mathcal{P}$ will not be generic in the whole of $\text{Spc}(\mathcal{X}(Q_8))$. As $\mathcal{P}$ belongs to $\text{Im}(\rho_Z)$, it suffices to determine $\rho_Z(\mathcal{M}_{C_2}(C_2))$. The preimage of $\text{Im}(\rho_Z) = \text{supp}(k(Q_8/Z))$ under $\psi^Z$ is
supp_E(ψ^Z(k(Q_8/Z))) = supp_E(k(E)) = {M_E(1)}. It follows that \( \mathcal{P} \) specializes to exactly one point: \( \psi^Z(M_E(1)) = M_{Q_8}(Z) \) as depicted in (1.16).

8. Stratification

It is by now well-understood how to deduce stratification in the presence of a noetherian spectrum and a conservative theory of supports. We follow the general method of Barthel-Heard-Sanders [BHS21b, BHS21a].

8.1. Proposition. The spectrum \( \text{Spc}(\mathcal{K}(G)) \) is a noetherian topological space.

Proof. Recall that a space is noetherian if every open is quasi-compact. It follows that the continuous image of a noetherian space is noetherian. The claim now follows from Corollary 6.2. \( \square \)

We start with the key technical fact. Recall that coproduct-preserving exact functors between compactly-generated triangulated categories have right adjoints by Brown-Neeman Representability. We apply this to \( \Psi^H \).

8.2. Lemma. Let \( N \leq G \) be a normal \( p \)-subgroup and \( \Psi^N: \text{DPerm}(G/N; k) \rightarrow \text{DPerm}(G;k) \) the right adjoint of modular \( N \)-fixed points \( \Psi^N: \text{DPerm}(G;k) \rightarrow \text{DPerm}(G/N;k) \). Then \( \Psi^N(\mathbb{1}) \) is isomorphic to a complex \( s \) in \( \text{perm}(G;k) \), concentrated in non-negative degrees

\[
s = (\cdots \rightarrow s_n \rightarrow \cdots \rightarrow s_2 \rightarrow s_1 \rightarrow s_0 \rightarrow 0 \rightarrow \cdots)
\]

with \( s_0 = k \) and \( s_1 = \oplus_{H \in \mathcal{F}_N} k(G/H) \), where \( \mathcal{F}_N = \{ H \leq G \mid N \not{\leq} H \} \).

Proof. Following the recipe of Brown-Neeman Representability [Nee96], we give an explicit description of \( \Psi^N(\mathbb{1}) \) as the homotopy colimit in \( \mathcal{T}(G) \) of a sequence of objects \( x_0 = 1 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_{n+1} \xrightarrow{f_{n+1}} \cdots \) in \( \mathcal{K}(G) \). This sequence is built together with maps \( g_n : \Psi^N(x_n) \rightarrow 1 \) in \( \mathcal{K}(G/N) \) making the following commute

\[
\begin{array}{ccccccc}
\Psi^N(x_0) = 1 & \xrightarrow{\Psi^N(f_0)} & \cdots & \xrightarrow{\Psi^N(f_n)} & \Psi^N(x_n) & \xrightarrow{\Psi^N(f_{n+1})} & \Psi^N(x_{n+1}) & \cdots \\
& \downarrow{g_0 = \text{id}} & & & \downarrow{g_n} & & \downarrow{g_{n+1}} & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

Note that such \( g_n \) yield homomorphisms, natural in \( t \in \text{DPerm}(G;k) \), as follows

\[
\alpha_{n,t} : \text{Hom}_G(t,x_n) \xrightarrow{\Psi^N} \text{Hom}_{G/N}(\Psi^N(t),\Psi^N(x_n)) \xrightarrow{(g_n)_*} \text{Hom}_{G/N}(\Psi^N(t),1)
\]

where we abbreviate \( \text{Hom}_G \) for \( \text{Hom}_{\text{DPerm}(G;k)} \). We are going to build our sequence of objects \( x_0 \rightarrow x_1 \rightarrow \cdots \) and the maps \( g_n \) so that for each \( n \geq 0 \)

\[
\alpha_{n,t} \text{ is an isomorphism for every } t \in \{ \Sigma^i k(G/H) \mid i < n, \ H \leq G \}.
\]

It follows that, if we set \( x_\infty = \text{hocolim}_n x_n \) and \( g_\infty : \Psi^N(x_\infty) \cong \text{hocolim}_n \Psi^N(x_n) \rightarrow 1 \) the colimit of the \( g_n \), then the map

\[
\alpha_t : \text{Hom}_G(t,x_\infty) \xrightarrow{\Psi^N} \text{Hom}_{G/N}(\Psi^N(t),\Psi^N(x_\infty)) \xrightarrow{(g_\infty)_*} \text{Hom}_{G/N}(\Psi^N(t),1)
\]

is an isomorphism for all \( t \in \{ \Sigma^i k(G/H) \mid i \in \mathbb{Z}, \ H \leq G \} \). Since the \( k(G/H) \) generate \( \text{DPerm}(G;k) \), it follows that \( \alpha_t \) is an isomorphism for all \( t \in \text{DPerm}(G;k) \). Hence \( x_\infty = \text{hocolim}_n x_n \) is indeed the image of \( 1 \) by the right adjoint \( \Psi^N \).
Let us construct these sequences $x_n$, $f_n$ and $g_n$, for $n \geq 0$. In fact, every complex $x_n$ will be concentrated in degrees between zero and $n$, so that (8.5) is trivially true for $n = 0$ (that is, for $i < 0$), both source and target of $\alpha_{n,t}$ being zero in that case. Furthermore, $x_{n+1}$ will only differ from $x_n$ in degree $n+1$, with $f_n$ being the identity in degrees $\leq n$. So the verification of (8.5) for $n+1$ will boil down to checking the cases of $t = \sum k(G/H)$ for $i = n$.

As indicated, we set $x_0 = \mathbb{1}$ and $g_0 = \text{id}$. We define $x_1$ by the exact triangle

$$s_1 \mapsto \mathbb{1} \overset{f_0}{\to} x_1 \to \Sigma(s_1)$$

where $s_1 := \oplus_{H \in \mathcal{P}_N} k(G/H)$ and $\epsilon_H: k(G/H) \to k$ is the usual map. Note that $\Psi^N(s_1) = 0$ by (4.13), hence $\Psi^N(f_0): \mathbb{1} \to \Psi^N(x_1)$ is an isomorphism. We call $g_1$ its inverse. One verifies that (8.5) holds for $n = 1$: For $t = k(G/H)$ with $H \in \mathcal{P}_N$, both the source and target of $\alpha_{1,t}$ are zero thanks to the definition of $s_1$. For the case where $H \geq N$, there are no non-zero homotopies for maps $k(G/H) \to x_1$ thanks to Lemma 4.3.

Let us construct $x_{n+1}$ and $g_{n+1}$ for $n \geq 1$. For every $H \leq G$ let $t = \Sigma^h k(G/H)$ and choose generators $h_{H,1}, \ldots, h_{H,r_H}: t \to x_n$ of the $k$-module $\text{Hom}_G(t, x_n)$, source of $\alpha_{n,t}$. Define $s_{n+1} = \oplus_{H \leq G} \oplus_{r_H} k(G/H)$ in perm($G$; $k$), a sum of $r_H$ copies of $k(G/H)$ for every $H \leq G$, and define $h_n: \Sigma^n(s_{n+1}) \to x_n$ as being $h_{H,i}$ on the $i$-th summand $\Sigma^n k(G/H)$. Define $x_{n+1}$ as the cone of $h_n$ in $\mathcal{K}(G)$:

$$\Sigma^n(s_{n+1}) \overset{h_n}{\longrightarrow} x_n \overset{f_n}{\longrightarrow} x_{n+1} \to \Sigma^{n+1}(s_{n+1}).$$

Note that $x_{n+1}$ only differs from $x_n$ in homological degree $n+1$ as announced. Since $n \geq 1$, we get $\text{Hom}_{G/N}(\Psi^N(x_{n+1}), \mathbb{1}) \cong \text{Hom}_{G/N}(\Psi^N(x_n), \mathbb{1})$ and there exists a unique $g_{n+1}: \Psi^N(x_{n+1}) \to \mathbb{1}$ making (8.3) commute. It remains to verify that $\alpha_{n+1,t}$ is an isomorphism for $t \in \{ \Sigma^n k(G/H) \mid H \leq G \}$. Note that the target of this map is zero. Applying $\text{Hom}_G(\Sigma^n k(G/H), -)$ to the exact triangle (8.6) shows that the source of $\alpha_{n+1,t}$ is also zero, by construction. Hence (8.5) holds for $n+1$.

This realizes the wanted sequence and therefore $\Psi^N_{\rho}(\mathbb{1}) \simeq \text{hocolim}_{n}(x_n)$ has the following form:

$$\cdots \to s_n \to \cdots \to s_2 \to s_1 \to k \to 0 \to 0 \cdots$$

where $s_1 = \oplus_{H \in \mathcal{P}_N} k(G/H)$ and $s_n \in \text{perm}(G; k)$ for all $n$. \hfill \Box

8.7. Remark. The above description of $\Psi^N_{\rho}(\mathbb{1})$ gives a formula for the right adjoint $\Psi^N_{\rho}: \text{DPerm}(G/N; k) \to \text{DPerm}(G; k)$ on all objects. Indeed, for every $t \in \text{DPerm}(G/N; k)$, we have a canonical isomorphism in $\text{DPerm}(G; k)$

$$\Psi^N_{\rho}(t) \cong \Psi^N_{\rho}(\Psi^N_{\rho}(\text{Inf}_{G}^{N}(t) \otimes 1) \cong \text{Inf}_{G}^{N}(t) \otimes \Psi^N_{\rho}(1)$$

using that $\Psi^N_{\rho} \circ \text{Inf}_{G}^{N} \cong \text{id}$ and the projection formula. In other words, the right adjoint $\Psi^N_{\rho}$ is simply inflation tensored with the commutative ring object $\Psi^N_{\rho}(\mathbb{1})$.

8.8. Lemma. Let $H \leq G$ be a normal $p$-subgroup and $\Psi^H_{\rho}: \text{DPerm}(G/H; k) \to \text{DPerm}(G; k)$ the right adjoint of modular $H$-fixed points $\Psi^H: \text{DPerm}(G; k) \to \text{DPerm}(G/H; k)$. Then the object $\text{zul}_{G}(H)$ displayed in (6.20) belongs to the localizing $tt$-ideal of $\text{DPerm}(G; k)$ generated by $\Psi^H_{\rho}(\mathbb{1})$.

Proof. By Proposition 6.18, we know that the $tt$-ideal generated by $\text{zul}_{G}(H)$ is exactly $\cap_{K \in \mathcal{P}_H} \ker \text{Res}_{K}^{G}$. By Frobenius, the latter is the $tt$-ideal $\{ x \in \mathcal{K}(G) \mid s_1 \otimes x = 0 \}$ where $s_1 = \oplus_{K \in \mathcal{P}_H} k(G/K)$ is the degree one part of the complex $s \simeq \Psi^H_{\rho}(\mathbb{1})$.
of Lemma 8.2. We can now conclude by Lemma 2.19 applied to this complex and $x = zu|_{G}(H)$ that $x$ must belong to the localizing tensor-ideal of $\text{DPerm}(G;k)$ generated by $\Psi^H_{ρ}(1)$. (Note that $s_0 = 1$ here.)

Recall from Corollary 6.23 that the map $\psi^H$ has closed image in $\text{Spc}(\mathcal{X}(G))$.

8.9. Proposition. Let $H \leq G$ be a $p$-subgroup and let $\Psi^H_{ρ}: \text{DPerm}(G//H;k) \to \text{DPerm}(G;k)$ be the right adjoint of $\Psi^H: \text{DPerm}(G;k) \to \text{DPerm}(G//H;k)$. Then the tt-ideal of $\mathcal{X}(G)$ supported on the closed subset $\text{Im}(\psi^H)$ is contained in the localizing tt-ideal of $\text{DPerm}(G;k)$ generated by $\Psi^H_{ρ}(1)$.

Proof. Let $N = N_GH$. By definition, $\Psi^H_{ρ} = \Psi^H_{ρ} \circ \text{Res}^G_N$ and therefore the right adjoint is $\Psi^H_{ρ} = \text{Ind}^G_N \circ \Psi^H_{ρ}$. By Lemma 8.8, we can handle $H \leq N$ hence we know (see also Proposition 6.18) that the generator $zu|_{N}(H)$ of the tt-ideal supported on $\text{Im}(\psi^H_{N})$ belongs to $\text{Loc}_{(\Psi^H_{ρ}(1))}$ in $\text{DPerm}(N;k)$. Applying $\text{Ind}^G_N$ and using the fact that $\text{Res}^G_N$ is surjective up to direct summands (by separability), we see that $zu|_{N}(H) = \text{Ind}^G_N(zu|_{N}(H))$ belongs to $\text{Loc}_{(\Psi^H_{ρ}(1))} \subseteq \text{Loc}_{(\Psi^H_{ρ} \circ \text{Res}^G_N(1))}$ in $\text{DPerm}(G;k)$. □

Let us now turn to stratification. By noetherianity, we can define a support for possibly non-compact objects in the ‘big’ tt-category under consideration, here $\text{DPerm}(G;k)$, following Balmer-Favi [BF11, §7]. We remind the reader.

8.10. Recollection. Every Thomason subset $Y \subseteq \text{Spc}(\mathcal{X}(G))$ yields a so-called ‘idempotent triangle’ $e(Y) \to 1 \to f(Y) \to \Sigma e(Y)$ in $\mathcal{T}(G) = \text{DPerm}(G;k)$, meaning that $e(Y) \otimes f(Y) = 0$, hence $e(Y) \cong e(Y)^{\otimes 2}$ and $f(Y) \cong f(Y)^{\otimes 2}$. The left idempotent $e(Y)$ is the generator of $\text{Loc}_{(\mathcal{X}(G)_Y)}$, the localizing tt-ideal of $\mathcal{T}(G)$ ‘supported’ on $Y$. The right idempotent $f(Y)$ realizes localization of $\mathcal{T}(G)$ ‘away’ from $Y$, that is, the localization on the complement $Y^{c}$.

By noetherianity, for every point $P \in \text{Spc}(\mathcal{X}(G))$, the closed subset $\overline{\{P\}}$ is Thomason. Hence $\overline{\{P\}} \cap (Y_P)^c = \{P\}$, where $Y_P := \text{supp}(P) = \{Q \mid P \not\subseteq Q\}$ is always a Thomason subset. The idempotent $g(P)$ in $\mathcal{T}(G)$ is then defined as $g(P) = e(\overline{\{P\}}) \otimes f(Y_P)$.

It is built to capture the part of $\text{DPerm}(G;k)$ that lives both ‘over $\overline{\{P\}}$’ (thanks to $e(\overline{\{P\}})$) and ‘over $Y_P$’ (thanks to $f(Y_P)$); in other words, $g(P)$ lives exactly ‘at $P$’. This idea originates in [HPS97]. It explains why the support is defined as

$$\text{Supp}(t) = \{P \in \text{Spc}(\mathcal{X}(G)) \mid g(P) \otimes t \neq 0\}$$

for every (possibly non-compact) object $t \in \text{DPerm}(G;k)$.

8.11. Theorem. Let $G$ be a finite group and let $k$ be a field. Then the big tt-category $\mathcal{T}(G) = \text{DPerm}(G;k)$ is stratified, that is, we have an order-preserving bijection

$$\{\text{Localizing tt-ideals } \mathcal{L} \subseteq \mathcal{T}(G)\} \leftrightarrow \{\text{Subsets of } \text{Spc}(\mathcal{X}(G))\}$$

given by sending a subcategory $\mathcal{L}$ to the union of the supports of its objects; its inverse sends a subset $Y \subseteq \text{Spc}(\mathcal{X}(G))$ to $\mathcal{L}_Y := \{t \in \mathcal{T}(G) \mid \text{Supp}(t) \subseteq Y\}$.

Proof. By induction on the order of the group, we can assume that the result holds for every proper subquotient $G//H$ (with $H \neq 1$). By [BHS21a, Theorem 3.21], noetherianity of the spectrum of compacts reduces stratification to proving minimality of $\text{Loc}_{(g(P))}$ for every $P \in \text{Spc}(\mathcal{X}(G))$. This means that $\text{Loc}_{(g(P))}$
admits no non-trivial localizing tt-ideal subcategory. If \( \mathcal{P} \) belongs to the cohomological open \( V_G = \text{Sp}(D(G,kG)) \) then minimality at \( \mathcal{P} \) in \( \mathcal{T} = \mathcal{D}(G,k) \) is equivalent to minimality at \( \mathcal{P} \) in \( \mathcal{T}(V_G) \cong K \text{Inj}(kG) \) by [BHS21a, Proposition 5.2]. Since \( K \text{Inj}(kG) \) is stratified by [BIK11], we have the result in that case.

Let now \( \mathcal{P} \in \text{Supp}(\mathcal{K}_{ac}(G)) \). By Corollary 6.24, we know that \( \mathcal{P} = \mathcal{P}_G(H,p) \) for some non-trivial \( p \)-subgroup \( 1 \neq H \subseteq G \) and some cohomological point \( p \in V_G \).

(In the notation of Proposition 6.32, this means \( \mathcal{P} \in V_G(H) \).

Suppose that \( t \in \text{Loc}_G(g(\mathcal{P})) \) is non-zero. We need to show that \( \text{Loc}_G(t) = \text{Loc}_G(g(\mathcal{P})) \), that is, we need to show that \( g(\mathcal{P}) \in \text{Loc}_G(t) \).

Recall the tt-functor \( \Psi_H^G : \mathcal{D}(G;k) \to K \text{Inj}(kG/H) \) from Notation 5.10.

By general properties of BF-idempotents [BF11, Theorem 6.3], we have \( \Psi^K(g(\mathcal{P})) = g((\psi^K)^{-1}(\mathcal{P})) \) in \( K \text{Inj}(k(G//K)) \) for every \( K \in \text{Sub}_pG \). Since \( \psi^K \) is injective by Proposition 6.14, the fiber \( (\psi^K)^{-1}(\mathcal{P}) \) is a singleton (namely \( p \)) if \( K \sim H \) and is empty otherwise. It follows that for all \( K \sim H \) we have \( \Psi^K(g(\mathcal{P})) = 0 \) and therefore \( \Psi^K(t) = 0 \) as well.

Since \( t \) is non-zero, the Conservativity Theorem 5.12 forces the only remaining \( \Psi_H^G(t) \) to be non-zero in \( K \text{Inj}(k(G//H)) \). This forces \( \Psi_H^G(t) \) to be non-zero in \( \mathcal{T}(G/H) \) as well, since \( \Psi_H^G = \Upsilon_{G//H} \circ \Psi_H^G \).

This object \( \Psi_H^G(t) \) belongs to \( \text{Loc}_G(\Psi_H^G(g(\mathcal{P}))) \) = \( \text{Loc}_G(g((\psi_H^G)^{-1}(\mathcal{P}))) \). Note that \( \Upsilon_{G//H}(p) \) is the only preimage of \( \mathcal{P} = \mathcal{P}_G(H,p) \) under \( \psi_H^G \) (see Remark 6.34). By induction hypothesis, this localizing tt-ideal \( \text{Loc}_G(\Psi_H^G(g(\mathcal{P}))) \) is minimal. And it contains our non-zero object \( \Psi_H^G(t) \). Hence \( \Psi_H^G(g(\mathcal{P})) \in \text{Loc}_G(\Psi_H^G(t)) \).

Applying the right adjoint \( \Psi_H^G \), it follows that \( \Psi_H^G \Psi_H^G(g(\mathcal{P})) = \Psi_H^G(\text{Loc}_G(\Psi_H^G(t))) \subseteq \text{Loc}_G(t) \) where the last inclusion follows by the projection formula for \( \Psi_H^G \). Hence by the projection formula again we have in \( \mathcal{T}(G) \) that

\[
\Psi_H^G(\mathbb{1}) \otimes g(\mathcal{P}) \in \text{Loc}_G(t).
\]

But we proved in Proposition 8.9 that the localizing tt-ideal generated by \( \Psi_H^G(\mathbb{1}) \) contains \( \mathcal{K}(G)_{\text{loc}(\psi_H^G)} \) and in particular \( e(\{\mathcal{P}\}) \) and a fortiori \( g(\mathcal{P}) \). In short, we have \( g(\mathcal{P}) \cong g(\mathcal{P}) \otimes 2 \in \text{Loc}_G(\Psi_H^G(\mathbb{1}) \otimes g(\mathcal{P})) \subseteq \text{Loc}_G(t) \) as needed to be proved. \( \square \)

**8.12. Corollary.** The Telescope Conjecture holds for \( \mathcal{D}(G;k) \). Every smashing tt-ideal \( S \subseteq \mathcal{D}(G;k) \) is generated by its compact part: \( S = \text{Loc}_G(S^c) \).

**Proof.** This follows from noetherianity of \( \text{Sp}(\mathcal{K}(G)) \) and stratification by [BHS21a, Theorem 9.11]. \( \square \)

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