Spin Singularities: Clifford Kaleidoscopes and Particle Masses

Marcus S. Cohen
Department of Mathematical Sciences
New Mexico State University
marcus@NMSU.edu

ABSTRACT
Are particles singularities- vortex lines, tubes, or sheets in some global ocean of dark energy? We visit the zoo of Lagrangian singularities, or caustics in a \( \text{spin}(4, \mathbb{C}) \) phase flow over compactified Minkowsky space, \( M_\# \equiv S_1 \times S_3 \), and find that their varieties and energies parallel the families and masses of the elementary particles.

Singularities are classified by tensor products of Coxeter groups \( <p, q, r>^J \), generated by reflections. The multiplicity, \( s \), is the number reflections needed to close a cycle of null zigzags: nonlinear resonances of \( J \) chiral pairs of lightlike matter spinors with \( (4 - J) \) Clifford mirrors: dyads in the remaining unperturbed vacuum pairs. Using singular perturbations to "peel" phase-space singularities by orders in the vacuum intensity, we find that singular varieties with quantized mass, charge, and spin parallel the families of leptons \( (J = 1) \), mesons \( (J = 2) \), and hadrons \( (J = 3) \). Taking the symplectic 4 form, \( (\psi^I d\psi_I)^4 \), the volume element in the 8-spinor phase space as a natural Lagrangian, these singularities turn out to have rest energies of \( m = (s/2)^3 m_e \), within a few percent of the observed particle masses.

0.1 Spin Space: the Vacuum as a Nonlinear Medium

A nonlinear medium near a critical point typically displays dynamical symmetry breaking: the spontaneous formation of patterns. Microscopic perturbations may seed a wide variety of macroscopic forms in a nearly-homogeneous medium; for example, snowflakes, seeded by dust grains, and subject to slightly-different histories of vapor pressure and temperature.

The interactions of critical patterns - attraction or repulsion, merger or splitting, comes along "for free", through the same nonlinearity that created them. Examples include hydrodynamic vortices [Newell], scroll rings in reaction-diffusion systems [Winfree], limb-buds in morphogenesis. Neu showed that vortices in superconductors interact via an effective electromagnetic potential \( d\theta^0 \): the far-field \( u(1) \) phase differential, sourced in quantized charges; their winding numbers, \( \int d\theta^0 = 2\pi n \) [Neu]. Skyrme found \( SO(3) \) monopoles in a "vacuum" with a quartic nonlinearity, and showed they interacted via an effective \( so(3) \) potential [Sk]. Skyrmions were used to model hadrons by Witten,
Weiss and Jackson, and others [Witt1], [W+J], [],[]. The orbit space of interacting Skyrmions was worked out by Manton, Temple-Raston, and others. Witten used his version of the quartic Weiss-Zumino term to model particles with quantized baryon number and strong interactions [Wit]. This rekindled excitement in string theories, mostly of Kaluza-Klein type, which view the "internal" degrees of freedom of particles as compactified spacetime dimensions.

We take the opposite approach here. We view spinors as the ultimate reality, and spacetime as a neutral submanifold of spin space: an 8-spinor bundle, $\Psi$, with fiber group $\text{spin}(4, \mathbb{C})$. Particles are Lagrangian singularities in the local spinfluid flow, $(\Psi, d\Psi) \in T^*\Psi$. Their regular "tails" give effective electroweak gauge potentials in the $PT$--antisymmetric limit [MC1]:

$$\Psi^I d\Psi_I \rightarrow g^{-1} dg = d\theta^\alpha(x)\sigma_\alpha \in [u(1) \oplus su(2)]_{\text{loc}}.$$  

The $PT$--symmetric limit gives effective gravitostrong potentials [MC2]. The antiHermitan part of their spin curvatures gives effective $u(1) \oplus su(3)$ fields; the Hermitian part gives Clifford-algebra valued tetrads and metric tensors [MC3]. These pull back to inertial frames in spacetime via the spin map (see appendix). The gravitational field emerges as the curvature of the dilation-boost flow of the spinfluid [MC4]. This spin$(4, \mathbb{C})$ model realizes the vision of Anandan [Anan], in which the matter fields are the physical reality, and spacetime- with its gauge, curvature, and metric fields, emerges through stationerizing an action functional.

Now any admissible action must be invariant under the spin isometry group, or Einstein Group $\mathbf{E}$ of coordinated "external" transformations and their "internal" (spin- space) representations. This is general covariance. It demands that the external and internal frames, and their differentials, transform in complementary ways. For example, a rotation of the external frame by $2\pi$ radians demands rotations of the internal $l$ and $r$- chirality spin frames by $\pi$. This suggests that the that the matter spinors that span the internal spin frames and the geometric spinors that factor the external frames are different sections of some global spinor field that pervades all of space, including the supports of particles. Here, it is the vacuum spinors that mediate the interaction between the "distant masses" and localized matter spinors, giving a mechanism for Mach’s principle. Meanwhile, invariance indicates a unique Lagrangian: the volume $[\psi^I d\psi_I]_4$ in spin space $(\Psi, d\Psi)$- which gives the right mass ratios for the particles.

General covariance is automatic if our spacetime is a horizontal local section of an 8-spinor bundle, 8; a different section for an observer in a different frame. We treat the spinors here as the real, physical objects, and spacetime vector and tensor fields as horizontal projections. We call this the

**Spin principle, (P1).** The 4 spinors $\psi'^I \equiv \{l^+, r^+, l^-, r^-\}$,and 4 provisionally independent cospinors $\psi_1 = \{r_-, l_-, r_+, l_+\}$, are the real physical objects. The $\psi \equiv \{\psi'^I, \psi_1\}$, together with their differentials, live in spin space, $\{\psi, d\psi\} \in T^*\psi :$ the space of spin$(4, \mathbb{C})$ flows. The matter fields emerge as codimension-$J$ singularities in the projections of $J$ chiral pairs to a horizontal base space: our spacetime, $M$. This leaves $(4 - J)$ pairs of regular vacuum
spinors to factor the Clifford tetrads, the gauge fields, and the coupling constants.

Each spinor has spin weight ±1/2, and conformal weight (dimension) 1/2 [Pen]. It takes 4 spinors to make the pseudoscalar inner product, involving the Dirac conjugate,

\[ \overline{\psi}_j \psi_I = \psi_j^T [iq_2(x)] \psi_I = \psi^T \psi_I; \]
\[ i[q_2(x)] = i[l_1(x) \otimes r^1(x) - l_2(x) \otimes r^2(x)] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]

Two "invisible" vacuum spinors are hiding in the "metric spinor", \( i[q_2(x)] \), expressed as an antisymmetric matrix (spin-1) in the moving spin frames. It takes 4 spinors to make the metric tensor (spin 2),

\[ g_{\alpha\beta} = \frac{1}{2} [q_{\alpha} \otimes \overline{q}_{\beta} \pm q_{\beta} \otimes \overline{q}_{\alpha}], \quad (1) \]

and 8 spinors to make an \( E \)-invariant inner product, \( q^2 g_{\alpha\beta} q^{\beta} \).

It takes 4 spinors and 4 spinor differentials to make an \( E \)-invariant Lagrangian 4 form; this must be a C-scalar (\( \sim \sigma_0 \)) to be invariantly integrated. We choose the volume element in the complexified cotangent bundle, \( \psi \pm i \mathbf{d}\psi \in \mathbb{C}(T^*\Sigma) \) as a natural Lagrangian:

\[ S_g = \frac{1}{2} \int_M [(\psi^I - i\mathbf{d}\psi^I) \wedge (\psi_I + i\mathbf{d}\psi_I)]^4 \]

(sum on \( I = 1, 2, 3, 4 \)). Here \( \left[\right]_0 \) means "the scalar part of a complex-Clifford (CC) valued form". Its physical interpretation is the 4-form part of the free action, \( S \), representing flows in spin space. Physically, each spinor \( \psi_I \) interacts freely with each cospinor \( \psi^I \) and its differential, \( \mathbf{d}\psi^I \). Only the CC scalar part of the 8-spinor tensor product can contribute to the integral.

Action \( S_g \) is stationarized in either the PT-symmetric (\( PT_s \)) or PT-antisymmetric (\( PT_a \)) case, \( \psi^J \mathbf{d}\psi_I = \pm \mathbf{d}\psi^J \psi_I \) [SGGU]. Because summing over an orthonormal spin frame \( \psi^I \equiv \{l^+, r^+, l^-, r^-\} \) gives the Trace, the Lagrangian then reduces to the 8-spinor factorization \( \psi^J \mathbf{d}\psi_I \) of the Maurer-Cartan (M.C.) 4 form, \( Tr[g^{-1} \mathbf{d}g] \) for Spin(4, \( \mathbb{C} \)) = \( SU(2) \). The integral of the unity part \( U(1) \oplus SU(2) \) is quantized over compactified Minkowsky space, \( M_\# \equiv S_1 \times S_3 \):

\[ \int_{M_\#} Tr[g^{-1} \mathbf{d}g] = 16\pi^3 N. \]

This provides a bound for the stationery 8-spinor action, which is saturated in either the \( PT_a \) or \( PT_s \) limit,

\[ \psi^J \rightarrow \mp \psi_J^T [iq_2(x)] = \mp \psi_J^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \]

in the moving spin frame [SGGU]. In either case, the stationery action becomes

\[ S_g = \int_{M_\#} \mathcal{L}_{\psi} \rightarrow \int_{M_\#} [(\psi^1 \mathbf{d}\psi_1) \wedge (\psi^2 \mathbf{d}\psi_2) \wedge (\psi^3 \mathbf{d}\psi_3) \wedge (\psi^4 \mathbf{d}\psi_4)]_0 \equiv \int_{M_\#} [\psi^J \mathbf{d}\psi_I]^4_0 = 16\pi^3 N. \]
Stationarizing $S_g$ cuts out a minimal surface in spin space.

Note that invariance of $S_g$ under coupled external and internal $E$ transformations is automatic if the matter spinors, $\psi$, are envelope modulations riding on a background of vacuum spinors: $\psi(x) \equiv l^A(x)\psi_A(x)$ (sum on $A = (1, 2)$).

The differentials of the moving spin frames, $\ell^A(x)$ then appear as gauge fields in the covariant derivative:

$$d\psi_A \equiv d(\ell^A \psi_A) = l^A d\psi_A + dl^A \psi_A = l^A (\partial_\alpha + \Omega_\alpha) \psi_A e^\alpha \equiv l^A \nabla_\alpha \psi_A e^\alpha.$$  \hspace{1cm} (2)

Serendipitously, although $L_g$ is a natural 4 form with no coupling constants, the $(4-J)$ unperturbed vacuum spinors make effective coupling constants of the proper dimensions- and the right magnitudes- to couple $J$ matter or gauge-field envelopes $\chi_U$. In the regular, geometrical-optics regime, $S_g$ yields the proper effective actions for electroweak (PTa, or charge-separated) and gravitostrong (PTS, or neutral) fields, $d\psi = \kappa \psi$. Here it agrees with Witten's "Weiss-Zumino 4 form," action, $\int Tr(g^{-1}dg)^{A_4} g$, which is quantized over the boundary, $\gamma_4 \sim \partial B_5$, of a 5-manifold [Witten1], [Witten2]. We could find $B_5 \sim C \times S_3$ embedded in our position-world velocity phase space $z^\alpha = x^\alpha + y^\alpha \in C_4$, with complex time coordinate $z^0 = t + iT$; provided all particles were at rest, ($y^\alpha = 0$).

In the spinfluid regime, there is a unique flow world velocity, $y^\alpha(x^\alpha)$, at each spacetime point, that varies continuously from point to point $x^\alpha \in M$.

Geometrical optics breaks down on boundary caustics, $\gamma_4-J \in \partial B_5-J$, where, the spin map $S : TM \to T\Sigma$ becomes singular, and acquires a $J-$dimensional kernel. The domains these caustics enclose are branched covers, with $J$ extra bispinor sheets in spin space over each spacetime point. These accommodate the wave functions of $J-$bispinor particles.

Caustics arise in optics, hydrodynamics, chemical reactions, acoustics, etc. as loci of partial focusing, or shock fronts [Arnold]. Joe Keller, Alan Newell [Newell], and others have used a powerful tool to look inside these apparent singularities: singular perturbation theory or multiscaling; defining a short spacetime scale inside the shock, and matching the inner solution to the outer one on the shock boundaries. We apply it to give a system of coupled envelope-modulation equations [Newell] to nonlinear waves in the 8 spinor medium: the spinfluid, and find that their caustics are the elementary particles. We outline the results below; details of the calculations appear in Part III [M.C.4]. We previously showed how a homogeneous background of vacuum spinors could produce this chiral cross coupling: Mach’s principle for bispinor particles[I.M.,U]. We derived Einstein’s equations, using integration by parts to match the effective actions inside and outside the worldtube boundary: the inertial mass and the gravitational mass.

What we do here is

1) show how nonlinear interactions with the vacuum spinors "fold up" light-like spin rays inside a timelike worldtube, $B_4$.

2) Identify these folding patterns with the varieties of singularities- and the families of elementary particles.
3) Classify these by the Coxeter groups of multiplicity-s : the s-fold covers of the Weyl reflection groups . 
4) Derive the formula $m = (s/2)^3 m_c$ for the masses of the corresponding particles in terms of their Coxeter numbers, $s$.

1 Singularities and Stratification

In the geometrical-optics (g.o.) regime, $D^0$, regular phase flows are created by nonsingular active-local (perhaps, path-dependent) Einstein transformations, $(L(x), R(x)) \in E_A$, written as $GL(2, C)$ matrices in the complexified Clifford-algebra $\mathbb{C}C(R_+ \times S_3)$. These act on a fiducial spin frame of vacuum spinors, $(\hat{\ell}, \hat{r})$, written column wise and row wise respectively:

$$\ell(x) = \hat{\ell} \exp \left[ \frac{1}{2} \zeta^{\alpha} \zeta_{\alpha} (x) q_\alpha \right] \equiv \bar{\ell} L(x);$$
$$r(x) = \exp \left[ \frac{1}{2} \zeta^{\alpha} \zeta_{\alpha} (x) \bar{q}_\alpha \right] \bar{r} \equiv R(x) \bar{r},$$

(3)

In the regular, $PT$–antisymmetric ($PT\bar{a}$) case, $R(x) = L^{-1}(x)$, multiplying a spinor by the differential of the PT opposed spinor gives effective spin connections, or vector potentials: Clifford-algebra-valued 1 forms,

$$\Omega_L \equiv \ell^{-1} \left( d\ell (x) = d\zeta_L = [\partial_q \zeta^{\alpha} (x)] q_\alpha e^\alpha, \right.$$  
$$\Omega^R \equiv (d\bar{r})^{-1} (x) = d\zeta_R = [\partial_{\bar{q}} \zeta_{\alpha} (x)] \bar{q}_\alpha e^\alpha.$$  
(4)

However, even for a regular initial distribution of spinor fields, codimension-$J = (1, 2, 3, 4)$ phase singularities $\gamma (4 - J)$ will form, shift, merge, annihilate, and recombine, like the projections of folds in a sheet to the bed. In addition to the regular stratum, $\gamma$, where the projection $\pi$ from the Lagrangian submanifold of spin space solutions to the position-world velocity phase space,

$$\pi : (\psi_I + i d\psi_I) \in \mathbb{C}T^* \Sigma \rightarrow x^\alpha + iy^\alpha \in \mathbb{C}T^* M,$$

is 1 to 1, there will be codimension-$J = (1, 2, 3, 4)$ singular strata: branched covers, $D^J$, where $\pi$ is $J + 1$ to 1. Like the crisscrossing rays inside a kaleidoscope, there are $J + 1$ world-velocity sheets, $y^\alpha$, over each spacetime point, $x^\alpha \in B_{4-J}$, inside the support, $B_{4-J}$, of a $J$–bispinor particle. Each support is bounded by loci of partial focusing, boundary caustics, $\gamma_{4-J} \subset \partial B_{4-J}$: folds, cusps, tucks, swallow-tails and knots, where spin rays $\psi^I d\psi_I = d\zeta_I$ branch or converge [ref Arnold]. Each $(4 - J)$ brane, $B_{4-J}$, carries a $J$–form matter current, $*_J$ duality to the Clifford volume element contributed by the $(4 - J)$ vacuum pairs. We call this complex of branes and currents the Spin $(4, \mathbb{C})$ complex, or spinfoam.

A 3- dimensional example is a foam of soap bubbles, with the regular stratum, $B_3 = *D^3$ (the volumes), and singular strata, $\gamma_2 \subset \partial B_3, \gamma_1 \subset \partial B_2, \gamma_0 \subset \partial B_1$ : the surfaces, edges, and vertices. Each stratum, $D^J$, carries a $J$–form current: density in volumes $\gamma_3$ pressure on surfaces $\gamma_2$, tension in line segments $\gamma_1$, and force on nodes $\gamma_0$. 

5
A codimension-$J$ bifurcation occurs at the critical point, $x_c \in \gamma_{4-J}$, where the rank of the Jacobian matrix, $[d\zeta](x_c) \equiv [\partial_\alpha \zeta_\beta](x_c)$, drops by $J$. Here, $J + 1$ phase differentials become linearly dependent, to span only a $(4 - J)$-dimensional subspace. If the Hessian, $[d^2\zeta](x_c)$, is singular there too, $|\partial_\alpha \partial_\beta \zeta|(x_c) = 0$, $x_c$ is a degenerate critical point: a caustic, where rays $d\zeta_I$ merge or split, and there is a change in the topology of the orbits.

This is dynamical symmetry breaking. One tool to detect it is the Equivariant Branching lemma (Michel’s “theorem”): If the isotropy subgroup, $H \subset E$, that fixes a solution $\Psi_c$ contains just a single copy of the identity representation, then $\Psi_C$ is a possible direction for dynamical-symmetry breaking ref. [Sattinger].

Some corollaries are

1. the branched covers and boundary caustics stratify the base space, $M$, into orbits of $E$-group actions into isotropy subgroups, $H$:

$$M = \bigcup_{J=0}^4 B_{4-J} \oplus \gamma_{4-J}.$$ 

2. Generically, as you cross a boundary caustic $\gamma_{3-J} \equiv \partial B_{4-J}$, where $|dd\zeta| = 0$, ker $d\zeta$ picks up generators one at a time.

3. The boundary of each stratum consists of singular loci belonging to the next higher stratum, except where two caustics intersect. Here, their codimensions add:

$$\gamma_{4-J} \cap \gamma_{4-K} = \gamma_{4-M} : M = J + K.$$ 

A bifurcating pattern is a new identity representation if it is steady state; a stationary flow is an identity representation on the position-velocity phase space. What is flowing in this picture is the 8-spinor vacuum; the spinnfluid. What we are looking for are patterns that bifurcate locally as it expands, like snowflakes from a saturated cloud.

Localization involves gluing conditions for splicing a compact ”bubble”, $\Omega^J \equiv (\psi^J d\psi^J)$ of $J$ matter-spinor pairs into the vacuum distribution. These give constraints on their integrals; not only on spacelike surfaces, but on initial and final temporal boundaries. As the neck of the J-tube $\gamma_{4-J}$ joining the matter bubble and the vacuum background expands from a single point, the matter wave functions must match the vacuum spinors there. This gives quantized topological charges [Taubes], [Uhlenbeck]: integral periods for $J$-form matter currents over compactified spacetime cycles; e.g. Bohr orbits,

$$\int_{\gamma_1} pdq - Edt = (n + \frac{1}{2})\hbar.$$ 

The spacetime holonomy operator in the perturbed vacuum surrounding a charge is the electric field, $E_e \equiv E_{or}$.

For wavefunctions of definite spin (i.e. with only upper or a lower complex coordinates), Milnor’s Fibration Theorem [Milnor] guarantees a complete...
set of \((4 - J)\) parallelizable fiber coordinates bridging the perturbed vacuum between singular loci: the integral curves of the vacuum spin forms, \(\hat{\Omega}^{1-J}\) (Table I).
Table I: the vacuum spin forms.

Assuming the vacuum spinors all have the same amplitude, $k^\pm$, the vacuum spin forms are given by:

\[
\hat{\Omega} = \pm i \frac{k^\pm}{2a_0} q_0 e^\alpha
\]

\[
\hat{\Omega}^2 = \left( i k^\pm \frac{a_0}{2} \right)^2 q_0 \left[ \epsilon_{jk} e^j \wedge e^k \pm e^0 \wedge e^\ell \right]
\]

\[
\hat{\Omega}^3 = \pm \left( i k^\pm \frac{a_0}{2} \right)^3 q_0 \epsilon_{jk} e^j \wedge e^k \pm i \epsilon_{jk} q_0 e^j \wedge e^k \wedge e^\ell
\]

\[
\hat{\Omega}^4 = \left( i k^\pm \frac{a_0}{2} \right)^4 q_0 \left[ \epsilon_{\alpha\beta\gamma\delta} e^\alpha \wedge e^\beta \wedge e^\gamma \wedge e^\delta \right] = \frac{3}{2} \left( \frac{k^4}{a_0^4} \right) d^4 V,
\]

The constraint that the Lagrangian density must be a C scalar assures that only the parts of $\hat{\Omega}^{4-J}$ both Clifford and Hodge dual to the matter forms, $\tilde{\Omega}^J \equiv (\psi^I d\psi_I)^J$, to contribute to the action. These make the Clifford line, surface, and volume elements that multiply $\tilde{\Omega}^J$ to fill out the $E$-invariant (C-scalar) 4-volume element.

\[
| (d\zeta)^4 | \sigma_0 e^0 \wedge e^1 \wedge e^2 \wedge e^3 : \sim \gamma^4 (dx)^4.
\]

Any C-dual contribution to $S_g$ must therefore be Hodge dual, as well, effectively quantizing $\tilde{\Omega}^J$ against dual (perpendicular) cycles, $\gamma_{4-J}$, as well as over cycles $\gamma_J$ (e.g. quantization of electric, flux, $F_{\sigma\tau}$, over $S_2(\theta, \phi)$ [M.C. 2]).

These topological charges remain constant with cosmic expansion, while the vacuum spin forms, $\hat{\Omega}^{4-J}$ (table I) give a factor of $k^{4-J} \sim \gamma^{-J}$ to the action contributed by the $D^J$ stratum. Integrating in the comoving frame, $E^\alpha = \gamma e^\alpha$, results in a net action polynomial in the scale factor, $\gamma$: the effective potential,

\[
V(n, \gamma) = \int_{D_0} \hat{\Omega}^4 + \gamma \int_{D_1} \hat{\Omega}^3 \wedge (\psi^I d\psi_I) + \gamma^2 \int_{D_2} \hat{\Omega}^2 \wedge (\psi^I d\psi_I)^2
+ \gamma^3 \int_{D_3} \hat{\Omega} \wedge (\psi^I d\psi_I)^3 + \gamma^4 \int_{D_4} (\psi^I d\psi_I)^4 = 16\pi^3 \left[ n_0 + n_1 \gamma + n_2 \gamma^2 + n_3 \gamma^3 + n_4 \gamma^4 \right],
\]

where $n_J$ is the population of the $J$th stratum [M.C..3].

The polynomial $V(n, \gamma)$ can mimic the effect of the Higgs field by mixing positive-definite quadratic couplings in $\gamma^2$ with negative-definite quartic ones in $-\gamma^4$, to create a "Mexican hat" potential. But, unlike standard Q. F. T., the lepton, meson, hadron and atomic masses appear in a 4-term sequence, at $O(\gamma, \gamma^2, \gamma^3, \gamma^4)$, respectively.

The $\hat{\Omega}^3$ term contributes the 3-volume element in spin space to the Noether charge under complex-time ($z^0 \equiv t + iT$) translation, which includes the Jacobean determinant of the 3-space block of spin map, $S$:

\[
| (d\zeta)^3 | \sim s^3 e^1 \wedge e^2 \wedge e^3.
\]
This gives quantization of both mass and charge:
\[
\int_{B_3} \left[ (\partial_t \theta^0) - i(\partial_r \theta^0) \right] e^1 \wedge e^2 \wedge e^3 = M + iQ. \tag{6}
\]

It is the vacuum spinors, hiding the Clifford 3-volume element \( \hat{\Omega}^3 \), that endow frequency, \( \omega \equiv (\partial_t \theta^0) \), with mass: Mach’s principle in action. Both \( m \) and \( q \) come in integral units: particle numbers.

Continuity of the Gluing map [Taub] says that the matter spinors localized inside the compact world tube \( B_4 \) must match the vacuum distribution on its boundary, \( \gamma_3 \equiv \partial B_4 \). However, as you pass along a curve \( x \in \gamma_1 \) through a degenerate codimension- \( J = (1, 2, 3, 4) \) boundary singularity, \( x_c \in \gamma_{4-J} \subset \partial B_{5-J} \), both the Jacobean and the Hessian determinants vanish, and the rank of the spin map drops by \( J \):

\[
S \equiv [\partial_\alpha \zeta^\beta](x_c) \quad q_j e^\alpha : |\partial \zeta|(x_c) = |\partial^2 \zeta|(x_c) = 0 \Rightarrow r(x_c) = (4 - J). \tag{7}
\]

A point inside \( \gamma_{4-J} \) acquires \( 2J \) new preimages in the projection \( \pi : L \to M \) from the Lagrangian submanifold in spin space to spacetime. [Taub].

To look inside these singular loci, we use singular perturbation theory; what Don Cohen calls "two timing and double crossing". Following Joe Keller, Alan Newell [Newell], and others, we define a short spacetime scale, \( x = \gamma X \) inside the shock front, and match the inner solution to the outer one on the shock boundaries. We apply it here to caustics in the spinfluid. We outline the results here; details of calculations appear in Part III [M.C.4].

First, we express each spinor field as a vacuum field, \( \varphi_I \equiv (\hat{\ell}_\pm, \hat{r}_\pm) \) of amplitude \( k^{\frac{J}{2}} \sim \gamma^{-\frac{J}{2}} \), plus an envelope modulation:

\[
\ell_I(x, X) = k^{\frac{J}{2}} \hat{\ell}_\pm(X) + \psi^\pm_L(x) = \gamma^{-\frac{J}{2}} \hat{\ell}_\pm(X) + \psi_L(x) ; \tag{8}
\]
\[
r_\pm(x, X) = \gamma^{-\frac{J}{2}}(X) + \psi_{R_\pm}.
\]

In inflated regimes, like ours, \( \gamma \gg 1 \). In superdense regimes, \( \gamma \ll 1 \); the matter spinors are ripples riding on the vacuum: a deep ocean of dark energy. Since solutions are either symmetric or antisymmetric about the critical radius, \( a = \# a \); \( \gamma = 1 \), we can consider either case, and cover both [M.C.1]. Inserting ansatz (17), we obtain effective Lagrangians, \( \mathcal{L}' \), in which \( (4 - J) \) vacuum pairs couple \( J \) matter pairs. Varying with respect to \( \hat{\ell}_\pm \) or \( \hat{r}_\pm \) gives the massless Dirac equations. These say that the vacuum spinors are Clifford-analytic and conjugate-analytic respectively:

\[
\overline{D} \hat{\ell}_\pm \equiv q^a \left( \partial_\alpha + \hat{\Omega}^\mu_\alpha \right) \hat{\ell}_\pm(X) = O \tag{9}
\]
\[
D \hat{r}_\pm \equiv q^a \left( \partial_\alpha + \hat{\Omega}^\mu_\alpha \right) \hat{r}_\pm(X) = O.
\]
Covariantly constant (freely-falling) solutions, \((\partial_\alpha + \Omega_\alpha) \left( \hat{\ell}_\pm, \hat{r}_\pm \right) = 0\) define inertial spin frames. On \(M_\# \equiv S_1 x S_3(a_\#)\),

\[
\hat{\ell}_\pm (X) = \ell_\pm (0) \exp \left( \frac{i}{2a_\#} X^\alpha \sigma^\alpha_\pm \right); \quad \hat{r}_\pm (X) = \hat{r}_\pm (0) \exp \left( \frac{i}{2a_\#} X^\beta \sigma^\beta_\pm \right);
\]

\[
\hat{\Omega}_L^\pm = \frac{i}{2a_\#} \sigma^L_\pm e^\alpha, \quad \hat{\Omega}_R^\pm = \frac{i}{2a_\#} \sigma^R_\pm e^\beta.
\]  

(10)

For a given scale factor, \(\gamma\), the vacuum action is extremized when the inertial spinors span a hypercube in spin space.

Neutral combinations of vacuum spinors could be called "cosmological neutrinos", \(\nu_i = (\hat{\ell}_+ + \hat{r}_-)\); \(\nu_r = (\hat{\ell}_- + \hat{r}_+)\). More generally, left and right chirality moving spin frames, \(\ell_\pm\) and \(r_\pm\), are given by path-dependent, active-local \((E_\lambda)\) transformations on the inertial spinors [M.C. 1], [M.C. 2], [M.C. 3]. These vary on the cosmic scale, so \(\gamma\) beats the logic clock, \(\Delta X^0 = \gamma\), elapse for each beat, \(\Delta x^0 = 1\), of the local clock.

\[
\ell (X,x) \equiv \hat{\ell}_\pm (X) L^\pm (x);
\]

\[
r (X,x) = \hat{r}_\pm (X) R^\pm (x).
\]  

(11)

At \(O(\gamma)\), we obtain the massive Dirac system as our coupled-envelope equations. Dirac mass - chiral cross coupling - appears via a spin \((4, \mathbb{C})\) resonance; the 8-spinor analog of 4-wave mixing in nonlinear optics [M.C. 5].

To contribute a \(C\) scalar \(4\) form \(\sigma_0 e^0 \wedge e^1 \wedge e^2 \wedge e^3\) to the action integral, a chiral pair of matter spinors must find 3 other pairs of vacuum spinors whose product meets the Bragg (solvability) conditions; the massive Dirac equations,

\[
\mathcal{D} \psi^I \equiv q^\alpha \left( \partial_\alpha + \hat{\Omega}_L^I \right) \psi^I (X) = [2a_\#]^{-1} \psi^R_I;
\]

\[
D \psi^R_I \equiv q^\alpha \left( \partial_\alpha + \hat{\Omega}_R^I \right) \psi^R_I (X) = [2a_\#]^{-1} \psi^I_R.
\]  

(12)

The electron mass - the inverse of the critical diameter, \(2a_\#\) - comes from the product of the 3 unbroken vacuum pairs; the \(\hat{\Omega}^3\) in Table 1. If the vacuum spinors have different amplitudes, the scalar mass term is replaced by the term \(\psi^I [\hat{\Omega}^3]_I^I \psi_J\), in the lepton mass matrix. This is a rank-2 tensor product of the 6 remaining vacuum spinors \(C\) dual to \(\psi^I, \psi_J\); the ones needed to make the C-scalar \((\sigma_0)\) term, at \(O(\gamma)\) : \(\psi^I [\hat{\Omega}^3]_I^I \psi_J \in (2, \bar{6}) \in L^1\).

At \(O(\gamma^2)\), integration by parts gives wave equations in \((\mathcal{DD} + D\mathcal{D}) \equiv \Delta :\) Klein -Gordon (spin 1 or 0) equations, sourced in the current 3 form, \(J\), with charge quantized over 3 -cycles:

\[
(\Delta + \hat{\Omega}^2) \bar{\Omega} = J; \quad \int_{B_3} J = \int_{B_3} [\partial_\alpha \hat{\partial}_\alpha] d\zeta^1 \wedge d\zeta^2 \wedge d\zeta^3 = 8\pi^2 [2a_\#]^{-1} B.
\]  

(13)

Again, the mass term, \(\hat{\Omega}^2\), comes from the vacuum energy. For photons, \(B = 0\).
At $O(\gamma^3)$, the principal part is a system of 3 Euler equations, coupling each quark current, $Q_I$, to the 2 others through the vacuum spinors:

$$Q_I \in [l \otimes_I r]; \; Q^I \in [r \otimes_I l]: \; \begin{align*}
DQ_I & \sim [2a_{\#}]^{-1} T^I_K J^I Q_J Q^K, \\
DQ^I & \sim [2a_{\#}]^{-1} T^I_J K^I J^I Q^J Q^K.
\end{align*}$$

$T$ is the rank -3 "moment of inertia" tensor, with eigenvalues $I \equiv (p,q,r)$. Orbits lie on invariant tori or ellipsoids, and close for integer ratios $(p/q/r)$, with a frequency that is a common multiple, $s = CM(p,q,r)$. Pythagoras would like this; it is the condition for a harmonious 3-note chord.

At $O(\gamma^4)$, we obtain a class of exact solutions we call Spin$(4, C)$ vortices; "vortex atoms" with dense nuclei of matter currents flowing in the $+T$ direction, outward from the big bang, and diffuse shells of returning currents, with charges $+Z$ and $-Z$, respectively [M.C.5]. Kelvin would like this.

Behind all this algebraic structure lives a simple physical picture: each chiral pair, $q_I \equiv (l \otimes_I r)$, acts as a mirror for the other 3 chiral pairs, bootstrapping from noise a resonant $s$-cycle.

### 2 Reflection Varieties and their Masses

The Dirac operator, $D \equiv \sigma^\alpha \partial_\alpha : e^\beta \mapsto g^\beta$, assigns a spacetime differential to an infinitesimal displacements in the Clifford algebra [BDS]; [G- M].

But on boundary caustics, $\gamma_{3-J} \subset D \; B_{4-J}$ the spin map, $S^* = d\zeta = [\partial_\alpha \zeta^\beta] \sigma_\beta e^\alpha$, becomes singular, with rank $(4-J)$. Here, some steps in internal phase no longer pull back to spacetime increments. Meanwhile, inside there are $s$ bispinor sheets for each spin direction in the spin bundle over the particle support, $B_{4-J}$.

For a volume element, $e^4$, to contribute to the action, the product of the 4 reflection operators in it must be a scalar. Physically, each cycle of 4 "interference gratings" $\ell \otimes r$ — including the curved gratings involving matter spinors, must close to form a resonator, with a net loop transfer function proportional to the identity, $\sigma_\sigma$. This closure constraint admits only a few sets of integers $\{p,q,r,s\}$ characterizing the possible symmetry groups of singular loci and isophase contours for particle wave functions: the Coxeter groups, $R_s$, [Coxeter] with their invariant polynomials in 4 complex variables, the Breiskorn varieties [Milnor]:

$$R_s = \langle p, q, r \rangle_s; \quad B(x, y, z, T) \equiv (x^p + y^q + z^r + T^{-s}) = \text{const.} \quad (14)$$

These are fibred knots. For example, $(2, 2, 3)_3 \cap S^3(1)$ is a trefoil knot, with its isophase fibers, $f(x, y, z) = \text{const}$, wrapping 3 times vertically around the singular filament, $f(x, y, z) \equiv x^2 + y^2 + z^3 = 0$, over 2 horizontal circuits.

The isophase contours over position $x^\alpha \in M$ seem to cross in the projections,

$$\Pi : x^\alpha + iy^\alpha \longrightarrow x^\alpha, \quad \zeta^\alpha = \theta^\alpha + i \varphi^\alpha \longrightarrow \theta^\alpha,$$
from phase space to spacetime, like the crisscrossing rays in a 3D kaleidoscope. These apparent crossings are resolved by lifting via $\Pi^{-1}$: i.e. by separating overlapping C-algebra valued wave vectors in the "quiver" of spin waves over $x^\alpha$.

It turns out [Cox] that the Coxeter groups varieties $\langle p, q, r \rangle$ exhaust the topological types of resolvable singularites. This is just one aspect of "the profound connections between the critical points of functions, quivers, caustics, wavefronts, regular polyhedra, and the theory of groups generated by reflections" [Arnold 2]. The profound connections that are important here are

1. $L$ or $R$ multiplication by a spacelike C vector gives a $L$- or $R$-helicity twist about axis $\ell$ or $\bar{r}$ by angle $\frac{\pi}{2}$ or $\frac{\pi}{2}$:

$$\ell' = L(\lambda)\ell \equiv \exp\left(\frac{i}{2}\hat{r} \cdot \sigma\right)\ell; \quad \bar{r}' = r\bar{R}(\rho) \equiv r \exp\left(\frac{i}{2}\bar{r} \cdot \sigma\right) \Rightarrow q' = (\ell \otimes \bar{r})' = L(\ell \otimes \bar{r})\bar{R}.$$ 

$L$ or $R$ action generated by a null C vector gives an additional $U(1)$ twist:

$$\ell' = L\ell = \exp\frac{i}{2r} \left[\pm \sigma_0 \pm i\hat{r} \cdot \sigma\right]; \quad \bar{r}' = r\bar{R} = r \exp\left(\frac{i}{2}\bar{r} \cdot \sigma\right); \quad \Rightarrow q' \equiv (\ell \otimes \bar{r})' = L(\ell \otimes \bar{r})\bar{R}.$$ 

2. Conjugation by a spacelike C vector reflects a flag (a 3 vector, $q$, and its normal frame) in a mirror with unit normal $a$ (see appendix):

$$q' = -aq a^{-1} = [ia|q|ia^{-1}] = [ia](l \otimes r)[ia^{-1}] \Rightarrow l' = [ia]; \quad r' = r[ia^{-1}].$$

An ordinary (period -2) reflection reverses the flag $L \Rightarrow R$, preserves function values, but reverses differentials, creating a singularity on the mirror plane. A domain $B_3$ bounded by mirrors (like a laser cavity) becomes a resonator: it traps waves at its fundamental frequency or its harmonics to create a standing wave.

3. Reflections in mirror planes $P_\perp$ and $Q_\perp$ that intersect at dihedral angle $\frac{\pi}{4}$ give a rotation by $\theta$ around the (spacelike) axis. $a = P_\perp \cap Q_\perp$: $q' = \ell' \otimes \bar{r}' = L(\ell \otimes \bar{r})L^{-1}; \quad L = \exp\left(\frac{i}{2}\hat{r} \cdot \sigma\right)$.

4. $L$ or $R$ action by the complex Clifford ($\mathbb{C}C$) vector, $\exp\frac{i}{p}[\pm \sigma_0 \pm i\hat{r} \cdot \sigma]$, gives a period-p reflection. It takes $p$ repeated reflections to close a spatial cycle; this first happens for $\theta = \pi$, making an image with dihedral symmetry, $D_p$. Parallel mirrors a distance $\frac{\pi}{2}$ apart generate translations of $\Delta$.

5. On $[M_\#]_{\text{diag}} \equiv [S_1(a_\#) \times S_3(a_\#)]_{\text{diag}}$, multiple reflections in 3 planes that all intersect in one point form a 3D kaleidoscope in spin space. Its image is a discrete subgroup, $R_s \subset U(1) \times SU(2)$, provided that the three dihedral
angles, \( \left( \frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r} \right) \) and the multiplicity, \( s \), obey the closure constraint: to commute, all 4 arguments above must be multiples of \( \pi \):

\[
R_p \equiv \exp \left( i\pi sp^{-1} P \right) , \quad R_Q \equiv \exp \left( i\pi sq^{-1} Q \right) , \quad R_R \equiv \exp \left( i\pi sr^{-1} R \right) ; \quad R_S \equiv \exp \left( -i\pi s^{-1} \sigma_0 \right) ;
\]

\[
R_p^s = R_Q^s = R_R^s = R_S^s = R_pR_QR_RR_S = \pm 1 \quad \Rightarrow s \left( p^{-1} + q^{-1} + r^{-1} - 1 \right) = n ,
\]

where \( n \), and \( (p,q,r) \) are all integers. The integer \( s \), a common multiple \( s = \text{C.M.} \ (p,q,r) \), is the length of the string of reflections that reconstructs all of the images in the representation \( R_s \) of the Coxeter group, \( \langle p,q,r \rangle \) : the \( s \)-fold cover of the Rotation, Dihedral, Tetrahedral, Octahedral, or Icosahedral group, \( \langle A_p, D_p, T, E_6, E_8 \rangle \). The common multiple, \( s \), is called the multiplicity, or Coxeter number [Coxeter].

This brings us right to the main point:

The mass- the 3-volume in spin space spanned by the string of \( s \) reflections \( \langle p,q,r \rangle_s \), varies as the cube of the string length: \( m \sim (s/2)^3 \).

More precisely, the rest energy of the configuration \( \{ \psi_I, \psi^I \} \) i.e. its Noether charge under \( T \) translation, is

\[
\int_{M^\#} \left[ \frac{\partial L_g}{\partial T} \psi_I \right] \left[ \frac{\partial \psi^I}{\partial T} \right] = \int_{M^\#} [\psi^I d\psi_I]^3 = \left( \frac{s}{2} \right)^3 \left( \frac{1}{2a^\#} \right).
\]

For a periodic solution to match the vacuum fields on the boundary \( \gamma_3 = \partial B_4 \), the frequency, \( \omega(s,n) \) inside a particle’s world tube must be a harmonic of the vacuum frequency; \( \omega_0 = (2a^\#)^{-1} \). For the odd spin structure on \( M^\# \) [Geroch], it takes time \( \Delta t = 2\pi a^\# \) for a lightlike phase front, \( \theta^a = \text{const} \), to circumnavigate a ray on a closed light cone, \( \tilde{N} \in [S_1 \times S_3(a^\#)]_{\text{diag}} \); one circuit gives \( \Delta \theta^0 = \Delta \theta^j = \pi \), so \( \psi(t+2\pi a^\#) = \psi(t) \). A solution of period \( s \) contributes a mass increment inside its world tube of

\[
m = (s/2)^3 (2a^\#)^{-1} .
\]

The energy - the 3-volume in spin-space- is counted according to its multiplicity, \( s \): the number of spin-space sheets above the particle’s support. For \( s = 2 \), this is the mass of an electron, governed by the massive Dirac equations (21) inside its world tube, \( B_4 \), of radius \( a^\# \). For a free electron, \( e^- \equiv (1\oplus r) \in \{2, 2, 2\}_2 \), all 3 dihedral angles are \( \frac{\pi}{2} \). The 3 pairs of vacuum spinors which trap the matter pair inside a 3-cube form opposing pairs of corner-cube reflectors.
As we decrease one of the dihedral angles, we get a 3-cycle at $\frac{\pi}{3}$; 3 sheets bounded by a tuck caustic ref. [Arnold]. But the cycle generated by both reflections doesn’t close up again until we reach their least-common multiple (lcm), $2 \cdot 3 = 6$, giving a 6-fold cover of the reflection group: the Coxeter group, $<2, 2, 3 >_6$, with multiplicity 6. We identify this as the muon; and the next closed reflection cycle, $<2, 3, 4 >_{12}$ as the tauon. More generally,

A massive lepton, meson, or hadron is composed of $J = 1, 2, or 3$ pairs of oppositely-propagating bispinor pairs, trapped inside a timelike world tube by reflections off interference gratings with $(4 - J)$ vacuum pairs on its boundary.

What is new here is that the reflection groups $<p, q, r >_s$ of multiplicity $s = (2, 3, 4, 5, 6, 12, 30)$ not only classify the elementary particles, but give their mass ratios (table III),

$$\frac{m}{m_e} = \left(\frac{s}{2}\right)^3.$$ (18)

These agree with the observed mass ratios within a few percent (except for the $\pi$ mesons, which are off by $\sim 25\%$).

Table III: Spin-J Resonances:

| Particle | Binary Group | Coxeter Numbers | \( \frac{m}{m_e} \) |
|----------|--------------|-----------------|----------------------|
| $e^-$    | $D_2$        | $\langle 2, 2, 2 \rangle_2$ | 1 1 |
| $\mu^-$  | $O$          | $\langle 2, 3, 4 \rangle_{12}$ | 216 207 |
| $\tau^-$ | $I$          | $\langle 2, 3, 5 \rangle_{30}$ | 3375 3478 |
| $\pi^-$  | $D_3 \otimes \bar{D}_4$ | $\langle 2, 2, 3 \rangle_3 \otimes \langle 2, 2, 4 \rangle_4$ | $\bar{d}d$ 216 275 |
| $k^-$    | $D_3 \otimes \bar{D}_5$ | $\langle 2, 2, 5 \rangle_5 \otimes \langle 2, 2, 4 \rangle_4$ | $s\bar{u}$ 1000 975 |
| $D_{\pi}^+$ | $D_5 \otimes \bar{D}_6$ | $\langle 2, 2, 5 \rangle_5 \otimes \langle 2, 2, 6 \rangle_6$ | $s\bar{c}$ 3375 3647 |
| $p^+$    | $D_4 \otimes D_4 \otimes D_3$ | $\langle 2, 2, 4 \rangle_4 \otimes \langle 2, 2, 3 \rangle_3 \otimes \langle 2, 2, 4 \rangle_4$ | $[u, d]u\bar{u}$ 1728 1836 |

In the quantum calculation (III) we sum over histories in "imaginary time", $T$: all possible chains of null zigzags connecting the initial and final states [MC3].
Microscopically, it seem, the whole world, both outside and inside the world tubes of massive particles, resolves into a network of light-like spinors, and their scattering vertices: their multilinear interactions.

3 Conclusions and Open Question

Spin Principle P1 says that the 8-spinor bundle, \( S \), is the physical reality; and that the action is just its volume in spin space. Our spacetime 4-fold, \( M \), and the particle wave functions, \( \Psi \), are horizontal and vertical projections of a minimal-surface in spin space: the spinfoam. The regular stratum, or vacuum, \( D^0 \), can be combed parallel locally by path-dependent phase differentials, \( d\zeta_I = \Psi^I d\Psi_I \), by spin connections: the vector potentials. Their spin curvatures, \( \Psi^I d\Psi_I \), are the fields. If these carry a nontrivial flux (topological charge) over the boundary, it must enclose a singularity-at least, in the projection, \( \pi : S \rightarrow M \): a caustic. Caustics are characterized by their symmetry groups in spin space, and there are only a few admissible types: the Coxeter groups, \( \langle p,q,r \rangle_s \).

In the continuum picture, their representations are the wave functions of particles with definite spin. They look like the Brieskorn varieties: fibred knots, whose isophase contours and normal rays (“lines of force”) radiate and terminate on singularities. Their masses - i.e. their Noether time-translation charges, are \( m = (s/2)^2 \), in natural units of \( 2a_\#^{-1} \); the mass of the electron \( (s = 2) \).

In the discrete picture, a vertex where a \( r \)-chirality spinor reflects from a Bragg mirror \( l \otimes r \) into an oppositely-propagating \( l \)- chirality one is called a mass scattering: \( (l \otimes r) r \rightarrow l \) [Penrose]. A null zigzag is a pair of mass scatterings, \( L \rightarrow R \rightarrow L \); the discrete version of a fold. To close a cycle of null zigzags, each chiral component must return to its original value. This happens only after a common multiple (c. m.) of the three binary reflection degrees, \( s = cm(p,q,r) \). But it takes only \( \frac{s}{2} \) reflections to restore a bispinor state; \( R_s^2 : (l \oplus r) \rightarrow (r \oplus l) \), for \( \frac{s}{2} \) odd.

Why should the reflection groups -the same groups that classify resolvable singularities, regular polyhedra, Lie algebras, quivers, frieze patterns, honeycombs, crystals, and caustics-classify the elementary particles? Because they all arise from the generic structures of singularities in flows.

Like heat flow resolves into random walks, at the critical scale, \( a_\# \), the 8-spinor flow resolves into a microhistory of null zigzags. In each discrete history, the multiplicity, \( s \)- the number of null zig-zags it takes to close a cycle- must be a common multiple of the reflection degrees \( p,q \),and \( r \). This results in an image in spin space like that formed by light rays crisscrossing in a 3D kaleidoscope, with mirrors at angles \( \frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r} \). The nonlinear 8-spinor dynamics has appeared above as multilinear mode-mode coupling in spin space. This discrete picture of spin rays and Bragg reflections is only a skeleton of the quantum dynamics of the 8-spinor system. To get the quantum corrections to the particle masses, we must sum over all possible histories of spin rays and intermediate scattering vertices; just as we sum over random walks to get the heat propagator. The
"random walk" underlying the Dirac system - the Dirac propagator - is the sum over all null zigzag histories connecting the initial and final states [Feynman], [Penrose l], [Ord ].

What is subtle and beautiful about this picture is

1) how self-consistent cycles of $J$ chiral pairs of matter waves and $(4-J)$ vacuum pairs "bootstrap" each other into existence as the radius passes through $\gamma = 1$, where $T = a#$; the critical radius for the inflationary phase transition (III).

2) How the $J$-dimensional critical modes that "crystalize out" at $O(\gamma^{J+1})$, program the multilinear couplings of modes at the next shorter scale, much as a volume hologram couples input to output waves. This results in the ramification of patterns at smaller and smaller scales, much like the main sequence of wavenumber-doubling bifurcations leading to turbulence.

Is this what we’re seeing in the sequence of $l = (200, 400, 800, \ldots)$ modes in the Cosmic Microwave Background near the time of decoupling; or in the foam-like structure of incident $J = (1, 2, 3)$– branes in the large-scale distribution of galaxies?

Perhaps the regular background of vacuum spinors is the dark energy- the invisible Dirac sea, on which the wave functions of visible matter ride like waves on the surface of the ocean.

3) Since the Dirac mass term is created by products of vacuum spinors, these might be called dark matter. This picture not only shows how the "distant masses" endow particles with their rest masses, but closely approximates the measured particle masses.

4) To get the quantum corrections, we must sum over all null zigzag histories connecting initial and final states. Since particles are null zigzags climbing up a timelike worldtube, these intermediate histories could be interpreted as the creation and anihilation of "virtual" particles; but it is easier to sum zigzags on a null lattice than creations and aninhilations of all possible virtual particles.

We do this sum over histories for a Friedman universe and derive an effective potential for its dilation flow in the sequel [M. C. 3]. Since we are drifting along in this flow, like rafters on a river, we don’t see it directly. Instead, we see the scenery on the banks marching by in a sequence we call time.

4 Appendix: From Spin Space to Spacetime

Spinors live in the "square-root space" of Clifford (C) vectors, or Clifford tetrads, $q_{\alpha} = (l \otimes_\alpha r) \in C(R_4)$: the spin-1 representations of displacements in the Clifford algebra [BDS]. Their history goes back to Hamilton, who discovered the key to composing rotations in space: express each rotation as a product of two reflections, and conjugate the vector argument with the string of reflection operators. Pauli reexpressed these as $Sl(2, C)$ matrices, which decompose into tensor
products $q_α \in \text{column and row spinors: the fundamental representations of spin isometries (see appendix).}$

While Cartan and Clifford [Cliff] were developing the geometric role of spinors, Weyl discovered that a $l$ or $r$-chirality spinor by itself represents the wave function of a left or right-helicity neutrino, and Dirac discovered that their direct sum, $(l \oplus r) = e$, represents on electron. Meanwhile, Van der Waerden showed how to build any matter or geometric field from tensor products $(l \otimes r) = e$, of $l$− and $r$−chirality spinors; the fundamental representations of the spin isometry group, or Einstein Group, $E$, of rotations, translations, and boosts extended by $P$ (parity), and $T$ (cosmic) time reversal. Hamilton expressed reflection operators as quaternions. Pauli expressed these as spin matrices in a basis of $2 \times 2$ anti-Hermitian matrices, with an algebra isomorphic to the quaternion algebra.

Allowing independent spin transformations to act from the left and right, $V' = aVb$, gives spin 4, the spin representation of $SO(4)$. The reflection of a spacelike C vector in a mirror with normal C vector $a \equiv a_k q_k$ is expressed by

$$V = \begin{bmatrix} x_3 - ix_4 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 + ix_4 \end{bmatrix}; \quad V' = aVa = -aVa^{-1}$$

$$V \equiv V^1 q_1 + V^2 q_2 + V^3 q_3 \equiv V^j q_j .$$

A rotation, $r$, is composed of two reflections $a$ and $b$.

1) \[ V' = baVab \equiv rVr^{-1}; r = -ba, \]

\[ V' = r_2r_1Vr_1^{-1}r_2^{-1}. \]

Cartan defined the reflection operators $a$ and $a^{-1}$ in 1) as acting from the left and from the right on the left and right chirality spinors, with a basis of 2 lightlike column vectors $l \equiv \{l_1^T, l_2^T\}$ and 2 lightlike row vectors $r \equiv \{r^1, r^2\}. a \{2 \times 2\}$ Their dyads form a basis for all $2 \times 2$ spin matrices; in particular the position/velocity spin matrix for a particle a local inertial frame:

$$V = l \otimes r : l = [l_1^T, l_2^T]^T \in \mathbb{C}_2; \quad r = \{r^1, r^2\}.$$  

Allowing independent spin transformations to act from the left and right, $V' = aVb$, gives spin 4, the spin representation of $SO(4)$. This includes L and R-helicity screw translations. Adjoining a dilation generator $q_4 \equiv \frac{1}{2} \sigma_0$ gives translations on $SO(4)$. This includes L and R-helicity screw translations. Adjoining a dilation generator $q_4 \equiv \frac{1}{2} \sigma_0$ gives translations on $S_3 \times R_+ = R_4 \setminus \{0\}$.

Adjoining the $U(1)$ generator $q_0 \equiv \frac{1}{2} \sigma_0$ gives Spin$(1,3)$: translations and rotations on compactified Minkowsky space,

$M \equiv S_3 \times S_3$. Complexifying the timelike generator to $Q_0 \equiv q_0 \oplus iq_4$ gives Spin$^c4$, which covers both dilations and rotations.

But it is only by complexifying all 4 generators to $Q_α \equiv q_α \oplus ip_α \in CT^*M$, that we include Lorenz boosts, giving Spin $(4, \mathbb{C})$: the generalization of the
Poincaré’ group to $M_\# \,[\text{ref}].$ Allowing the position and world velocity $q_\alpha(x)$ and $p_\alpha(x)$ to vary locally on $M_\#$ paints a *dilation-boost flow, Spin $(4,\mathbb{C})_{\text{loc}}$ on a curved spacetime, $M_\# : S_3 \times R_+ = R_4 \setminus \{0\} :$ Wheeler’s "lumpy potato”.

Tensor products $q^A_B \equiv l_A \otimes r^B$ of 2 opposite-chirality spinors make null spin vectors: photons. The *null tetrads* are "vacuum photons", of helicity $\pm 1: \n\equiv l_+ \otimes r^-, q_+ \equiv l_+ \otimes r^+; q_- \equiv l_- \otimes r^-, \tag{19}\n$ Spin-1 sums of null tetrads make the *Clifford tetrads* of a moving Clifford algebra (C) frame:

\begin{align*}
q_0 & \equiv \ell \otimes \alpha r \in CTM: \quad q_0 \equiv (q_0 - q_1); \quad q_1 \equiv (q_+ + q_-); \\
q_2 & \equiv -i(q_+ - q_-); \quad q_3 \equiv (q_1 + q_1). \tag{20}
\end{align*}

The Clifford tetrads $q_\alpha$ are identified with the basis vectors, $e_\alpha$, of a spacetime frame via the infinitesimal form of the *spin map*, $S :$ the canonical isomorphism of compactified Minkowsky space, $M_\# = S_1 \times S_3(a_\#), \text{to the compact Lie group,}$ $U(1) \otimes SU(2) :$

\begin{equation}
S \equiv \exp\left(\frac{i}{2a_\#}x^\alpha \sigma_\alpha\right): M_\# = S_1 \times S_3(a_\#) \to U(1) \otimes SU(2) \equiv g;
\end{equation}

the Maurer-Cartan 1 form, valued in the Lie Algebra $u(1) \otimes su(2)$, generated by the Pauli spin matrices $\sigma_j$, along with $\sigma_0 = 1$, the 2x2 identity matrix. Here $d \equiv e^\alpha(x) \partial_\alpha(x)$ is the generalized (possibly path-dependent) exterior differential operator; the $e_\alpha$ are a moving frame of spacetime basis vectors, and the $q_\alpha(x) \equiv g \sigma_\alpha g^{-1}$ the isomorphic *moving frame in the Clifford algebra*. The pullback of the spin map is the *Dirac operator*, $D = S^*: -i(2a_\#)e^\beta \rightarrow q^\beta \equiv \ell \otimes \beta r$, which assigns a spacetime increment to chiral pairs of null spinors, $Dl = 0; Dr = 0 \,(3), \text{ where } D = \overline{\gamma}^\alpha \partial_\alpha : \overline{\gamma}^\alpha \equiv (q_0 - q_1)$. More generally, in moving frames in spin space and spacetime, the spin map reads

\begin{equation}
S_\ast(x) \equiv d\zeta(x) \equiv [\partial_\alpha \zeta^\beta](x)q_\beta(x)e^\alpha(x) : e_\alpha(x) \rightarrow [\partial_\alpha \zeta^\beta]q_\beta(x). \end{equation}

Its Jacobean determinant, \( |d\zeta| \,(x), \) is the 4- volume element in spin space; at a singular (critical) point, $x_c$, \( |d\zeta|(x_c) = 0 \). We use \( |d\zeta| \) as our Lagrangian density.

**References**

[1] \cite{Anandan} J Anandan Topological Phases and their Duality in Electromagnetic and Gravitational Fields. Arxiv: gr-qc 950-4002, 2007.

[2] \cite{Arnold} V.I. Arnold, *Catastrophe Theory;* Third Edition, Springer-Verlag, Berlin (1992).
[3] [BDS] Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman Advanced Publishing Program, Boston (1982).

[4] [M.C.1] M.S. Cohen, “Spin Geometry and Grand Unification,” *Advances in Applied Clifford Algebras*, 11, 1 (2001).

[5] [M.C.2] M.S. Cohen, “Clifford Residues and Charge Quantization,” *Advances in Applied Clifford Algebras*, 12, 1 (2002).

[6] [M.C.3] M.S. Cohen, “Clifford Tetrads, Null Zig zags and Quantum Gravity,” *Advances in Applied Clifford Algebras*, 13, 1 (2003).

[7] [M.C.4] M.S. Cohen, ”Spin (4,C) vortices”, preprint.

[8] [M.C.5] M.S. Cohen, ”Nonlinear Multiwave Resonances”, Journal of the Optical Society of America B 8, 106 (1991).

[9] [Coxeter] H.S.M. Coxeter, *Regular Complex Polytopes*, Cambridge University Press, Cambridge (1991).

[10] [Feynman] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill Book Company, New York (1965).

[11] [G,M] J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge (1991).

[12] [Neu] J Neu. Vortex Dynamics of the Nonlinear Wave Equation. Physica D, 43:407-420. 1990.

[13] [Newell] Alan C. Newell and J. V. Molony, *Nonlinear Optics*, Advanced Topics in Interdisciplinary Mathematical Sciences, Addison-Wesley, (1992), 2nd edition Westbrook, (2003).

[14] [Penrose1] R. Penrose and W. Rindler, *Spinors and spacetime*, Volume1; Cambridge University Press (1984).

[15] [Penrose2] R. Penrose and W. Rindler, *Spinors and spacetime*, Volume2; Cambridge University Press (1985).

[16] [Sattinger] D.H.Sattinger and 0. L. Weaver, *Lie Groups and Algebras with applications to physics, Geometry, and Mechanics*. Springer-Verlag, New York (1986).

[17] [Taub] C. H. Taubs, ”Gauge theory on asymptotically periodic 4-manifolds.” *Journal of Differential Geometry*, 25, 363-430 (1986).

[18] [Uhlenbeck] K. K. Uhlenbeck, ”Removable singularities in Yang-Mills fields.” *Communications in Mathematical Physics*, 83, 11-29 (1982a).

[2] [Witten1] E. Witten, ”Global Aspects of Current Algebra.” *Nuclear Physics*, B223, 422 (1983).
[20] [Witten2] E. Witten, *Monopoles and 4-Manifolds*, Math Research Letters 1, (1994).