CONGRUENCES FOR COEFFICIENTS OF LEVEL 2 MODULAR FUNCTIONS WITH POLES AT 0

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Abstract. We give congruences modulo powers of 2 for the Fourier coefficients of certain level 2 modular functions with poles only at 0, answering a question posed by Andersen and the first author. The congruences involve a modulus that depends on the binary expansion of the modular form’s order of vanishing at ∞.

1. Introduction

A modular form \( f(z) \) of level \( N \) and weight \( k \) is a function which is holomorphic on the upper half plane, satisfies the equation
\[
f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)
\]
for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \),
and is holomorphic at the cusps of \( \Gamma_0(N) \). Letting \( q = e^{2\pi i z} \), these functions have Fourier series representations of the form \( f(z) = \sum_{n=0}^{\infty} a(n) q^n \). A weakly holomorphic modular form is a modular form that is allowed to be meromorphic at the cusps. We define \( M_k^\flat(N) \) to be the space of weakly holomorphic modular forms of weight \( k \) and level \( N \) that are holomorphic away from the cusp at \( \infty \), and define \( M_k^\flat(N) \) similarly, but for forms holomorphic away from the cusp at 0.

The coefficients of many modular forms have interesting arithmetic properties; for instance, the coefficients \( c(n) \) of the \( j \)-invariant \( j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n \). A weakly holomorphic modular form is a modular form that is allowed to be meromorphic at the cusps. We define \( M_k^\flat(N) \) to be the space of weakly holomorphic modular forms of weight \( k \) and level \( N \) that are holomorphic away from the cusp at \( \infty \), and define \( M_k^\flat(N) \) similarly, but for forms holomorphic away from the cusp at 0.

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which vanishes at $\infty$ and has a pole only at $0$. Note that the functions $\phi(z)^m$ for $m \geq 0$ are a basis for $M_0^0(2)$. Andersen and the first author used powers of $\phi(z)$ to prove congruences involving $\psi = \frac{1}{\phi} = q^{-1} - 24 + \cdots \in M_0^0(2)$ in [3], and made the following remark: “Additionally, it appears that powers of the function $[\phi(z)]$ have Fourier coefficients with slightly weaker divisibility properties... It would be interesting to more fully understand these congruences.” In this paper, we prove congruences for these Fourier coefficients.

Write $\phi(z)^m$ as $\sum_{n=m}^{\infty} a(m, n)q^n$. The main result of this paper is the following theorem.

**Theorem 1.** Let $n = 2^\alpha n'$ where $2 \nmid n'$. Express the binary expansion of $m$ as $a_r \ldots a_2 a_1$, and consider the rightmost $\alpha$ digits $a_\alpha \ldots a_2 a_1$, letting $a_i = 0$ for $i > r$ if $\alpha > r$. Let $i'$ be the index of the rightmost 1, if it exists. Let

$$\gamma(m, \alpha) = \begin{cases} \# \{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m, \alpha)}}.$$ 

That the structure of the binary expansion of $m$ appears in the modulus of this congruence is a surprising result. We note that this congruence is not sharp. For $m = 1$, Allatt and Slater in [2] proved a stronger result that provides an exact congruence for many $n$.

As an example, the binary expansion for $m = 40$ is $m = \cdots 000101000$. As we increment $\alpha$, the $\gamma$ function gives the values in Table 1.

| $\alpha$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\cdots$ | $\alpha$ | $\cdots$ |
|----------|---|---|---|---|---|---|---|---|---|---|-----------|---------|--------|
| $\gamma(40, \alpha)$ | 0 | 0 | 0 | 0 | 1 | 2 | 2 | 3 | 4 | 5 | $\cdots$ | $\alpha - 4$ | $\cdots$ |

Table 1. Values of $\gamma(m, \alpha)$ for $m = 40$

Notice that once $\alpha$ surpasses 6—and the leftmost 1 in the binary expansion of $m$ occurs in the 6th place—$\gamma$ always increases by 1 as $\alpha$ increases by 1. This illustrates that $\gamma(m, \alpha)$ is unbounded for a fixed $m$.

We also prove the following result on the parity of $a(1, n)$.

**Theorem 2.** The $n$th coefficient $a(1, n)$ of $\phi(z)$ is odd if and only if $n$ is an odd square.

Section 2 contains the machinery and definitions we use in the proof of Theorem 1. The proof of Theorem 1 is in Section 3, and the proof of Theorem 2 is in Section 4.

2. **Preliminary Lemmas**

The operator $U_p$ on a function $f(z)$ is given by

$$U_p f(z) = \frac{1}{p} \sum_{j=0}^{p-1} f \left( \frac{z + j}{p} \right).$$

We have $U_p : M_k^0(N) \to M_k^0(N)$ if $p$ divides $N$. If $f(z)$ has the Fourier expansion $\sum_{n=n_0}^{\infty} a(n)q^n$, then the effect of $U_p$ is given by $U_p f(z) = \sum_{n=n_0}^{\infty} a(pn)q^n$. 
The following result describes how \( U_p \), applied to a modular function behaves under the Fricke involution. This will help us in Lemma 6 to write \( U_2 \phi^m \) as a polynomial in \( \phi \).

**Lemma 3.** [3] Theorem 4.6 | Let \( p \) be prime and let \( f(z) \) be a level \( p \) modular function. Then

\[
p(U_p f) \left( \frac{-1}{pz} \right) = p(U_p f)(pz) + f \left( \frac{-1}{p^2 z} \right) - f(z).
\]

The Fricke involution \( \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) swaps the cusps of \( \Gamma_0(2) \), which are 0 and \( \infty \). We will use this fact in the proof of Lemma 6 and the following relations between \( \phi(z) \) and \( \psi(z) \) will help us compute this involution.

**Lemma 4.** [3] Lemma 3 | The functions \( \phi(z) \) and \( \psi(z) \) satisfy the relations

\[
\phi \left( \frac{-1}{2z} \right) = 2^{-12} \psi(z),
\]

\[
\psi \left( \frac{-1}{2z} \right) = 2^{12} \phi(z).
\]

The following lemma is a special case of a result from one of Lehner’s papers [11]. It provides a polynomial whose roots are modular forms used in the proof of Theorem 5.

**Lemma 5.** [11] Theorem 2 | There exist integers \( b_j \) such that

\( U_2 \phi(z) = 2(b_1 \phi(z) + b_2 \phi(z)^2) \).

Furthermore, let \( h(z) = 2^{12} \phi(z/2) \), \( g_1(z) = 2^{14} (b_1 \phi(z) + b_2 \phi(z)^2) \), and \( g_2(z) = -2^{14} b_2 \phi(z) \). Then

\[
h(z)^2 - g_1(z) h(z) + g_2(z) = 0.
\]

In the following lemma, we extend the result from the first part of Lemma 5 writing \( U_2 \phi^m \) as an integer polynomial in \( \phi \). In particular, we give the least and greatest powers of the polynomial’s nonzero terms.

**Lemma 6.** For all \( m \geq 1 \), \( U_2 \phi^m \in \mathbb{Z}[\phi] \). In particular,

\[
U_2 \phi^m = \sum_{j=[m/2]}^{2m} d(m,j) \phi^j
\]

where \( d(m,j) \in \mathbb{Z} \), and \( d(m, \lfloor m/2 \rfloor) \) and \( d(m, 2m) \) are not 0.

**Proof.** Using Lemmas 3 and 4 we have that

\[
U_2 \phi(-1/2z)^m = U_2 \phi(2z)^m + 2^{-1} \phi(-1/4z)^m - 2^{-1} \phi(z)^m
\]

\[
= U_2 \phi(2z)^m + 2^{-1} - 12m \psi(2z)^m - 2^{-1} \phi(z)^m
\]

\[
= 2^{-1 - 12m} q^{-2m} + O(q^{-2m+2})
\]

\[
2^{1 + 12m} U_2 \phi(-1/2z)^m = q^{-2m} + O(q^{-2m+2}).
\]

Because \( \phi(z)^m \) is holomorphic at \( \infty \), \( U_2 \phi(z)^m \) is holomorphic at \( \infty \). So \( U_2 \phi(-1/2z)^m \) is holomorphic at 0 and, since it starts with \( q^{-2m} \), must be a polynomial of degree
2m in ψ(z). Let b(m, j) ∈ ℤ such that
\[ 2^{1+12m} U_2 φ(-1/2z)^m = \sum_{j=0}^{2m} b(m, j) ψ(z)^j, \]
and we note that b(m, 2m) is not 0. Now replace z with −1/2z and use Lemma 3 to get
\[ 2^{1+12m} U_2 φ(z)^m = \sum_{j=0}^{2m} b(m, j) 2^{12j} φ(z)^j, \]
which gives
\[ U_2 φ(z)^m = \sum_{j=0}^{2m} b(m, j) 2^{12j-1} φ(z)^j. \]
If m is even, the leading term of the above sum is \( q^{m/2} \), and if m is odd, the leading term is \( q^{(m+1)/2} \), so the sum starts with \( j = \lceil m/2 \rceil \) as desired. Notice that \( b(m, j) 2^{12j-1} \) is an integer because the coefficients of \( φ(z)^m \) are integers. □

We may repeatedly use Lemma 3 to write \( U_2^α φ^m \) as a polynomial in \( φ \). Let
\[ f(ℓ) = \lceil ℓ/2 \rceil, f^0(ℓ) = ℓ, \text{ and } f^k(ℓ) = f(f^{k-1}(ℓ)). \]
Using Lemma 3 the smallest power of \( q \) appearing in \( U_2^α φ^m \) is \( f^α(m) \). Lemma 7 provides a connection between \( γ(m, α) \) and the integers \( f^α(m) \).

**Lemma 7.** The function \( γ(m, α) \) as defined in Theorem 1 is equal to the number of odd integers in the list
\[ m, f(m), f^2(m), \ldots, f^{α-1}(m). \]

**Proof.** Write the binary expansion of \( m \) as \( a_r \ldots a_2 a_1 \), and consider its first \( α \) digits, \( a_α \ldots a_2 a_1 \), where \( a_i = 0 \) for \( i > r \) if \( α > r \). If all \( a_i = 0 \), then all of the integers in the list are even. Otherwise, suppose that \( a_i = 0 \) for \( 1 \leq i < i' \) and \( a_{i'} = 1 \). Apply \( f \) repeatedly to \( m \), which deletes the beginning 0s from the expansion, until \( a_{i'} \) is the rightmost remaining digit; that is, \( f^{i'-1}(m) = a_α \ldots a_{i'-1} a_{i'} \). In particular, this integer is odd. Having reduced to the odd case, we now treat only the case where \( m \) is odd.

If \( m \) in the list is odd, then \( a_1 = 1 \), which corresponds to the +1 in the definition of \( γ(m, α) \). Also, \( f(m) = \lceil m/2 \rceil = (m + 1)/2 \). Applied to the binary expansion of \( m \), this deletes \( a_1 \) and propagates a 1 leftward through the binary expansion, flipping 1s to 0s, and then terminating upon encountering the first 0 (if it exists), which changes to a 1. As in the even case, we apply \( f \) repeatedly to delete the new leading 0s, producing one more odd output in the list once all the 0s have been deleted. Thus, each 0 to the left of \( a_{i'} \) corresponds to one odd number in the list. □

### 3. Proof of the Main Theorem

Theorem 1 will follow from the following theorem.

**Theorem 8.** Let \( f(ℓ) \) be as in (1). Let \( γ(m, α) \) be as in Theorem 1, and let \( α \geq 1 \). Define
\[ c(m, j, α) = \begin{cases} -1 & \text{if } f^{α-1}(m) \text{ is even and is not } 2j, \\ 0 & \text{otherwise}. \end{cases} \]
Proof of Theorem 8. For the base case, we let \( m = 1 \), and seek to prove (3) by induction on the valuation of \( d \).

In particular, Lehner’s bound sometimes only gives the trivial result that the 2-adic valuation of \( d(m, j, \alpha) \) is greater than some negative integer.

We prove Theorem 8 by induction on \( \alpha \). The base case is similar to Lemma 6 from [3], which gives a subring of \( \mathbb{Z}[\phi] \) which is closed under the \( U_2 \) operator. The polynomials are useful because their coefficients are highly divisible by 2. Here, we employ a similar technique to prove divisibility properties of the polynomial coefficients in Lemma 6. We then induct to extend the divisibility results to the polynomials that arise from repeated application of \( U_2 \).

Proof of Theorem 8. For the base case, we let \( \alpha = 1 \), and seek to prove the statement

\[
U_2 \phi^m = \sum_{j=\lceil m/2 \rceil}^{2m} d(m, j, 1) \phi^j
\]

with

\[
\nu_2(d(m, j, 1)) \geq 8(j - \lceil m/2 \rceil) + c(m, j)
\]

where

\[
c(m, j) = \begin{cases} 
3 & m \text{ is odd,} \\
0 & m = 2j, \\
-1 & \text{otherwise.}
\end{cases}
\]

The term \( c(m, j) \) combines \( c(m, j, \alpha) \) and \( 3\gamma(m, \alpha) \) for notational convenience. We prove (3) by induction on \( m \).

We follow the proof techniques used in Lemmas 5 and 6 of [3]. From the definition of \( U_2 \), we have

\[
U_2 \phi^m = 2^{-1} \left( \phi \left( \frac{z}{2} \right)^m + \phi \left( \frac{z + 1}{2} \right)^m \right) = 2^{-1-12m} (h_0(z)^m + h_1(z)^m)
\]

where \( h_\ell(z) = 2^{12\phi(\frac{\ell-2}{2})} \). To understand this form, we construct a polynomial whose roots are \( h_0(z) \) and \( h_1(z) \). Let \( g_1(z) = 2^{16} \cdot 3\phi(z) + 2^{24}\phi(z)^2 \) and \( g_2(z) = -2^{24}\phi(z) \). Then by Lemma 5, the polynomial \( F(x) = x^2 - g_1(z)x + g_2(z) \) has \( h_0(z) \) as a root. It also has \( h_1(z) \) as a root because under \( z \mapsto z + 1 \), \( h_0(z) \mapsto h_1(z) \) and the \( g_\ell \) are fixed.

Recall Newton’s identities for the sum of powers of roots of a polynomial. For a polynomial \( \prod_{i=1}^{n}(x - x_i) \), let \( S_\ell = x_1^{\ell} + \cdots + x_n^{\ell} \) and let \( g_\ell \) be the \( \ell \)th symmetric polynomial in the \( x_1, \ldots, x_n \). Then

\[
S_\ell = g_1 S_{\ell-1} - g_2 S_{\ell-2} + \cdots + (-1)^{\ell+1} \ell g_\ell.
\]
We apply this to the polynomial $F(x)$, which has only two roots, to find that
\[ h_0(z)^m + h_1(z)^m = S_m = g_1S_{m-1} - g_2S_{m-2}. \]

Furthermore,
\[ U_2\phi^m = 2^{1-12m}S_m. \]

Lastly, let $R$ be the set of polynomials of the form $d(1)\phi(z) + \sum_{n=2}^N d(n)\phi(z)^n$ where for $n \geq 2$, $\nu_2(d(n)) \geq 8(n - 1)$. Now we rephrase the theorem statement in terms of $S_m$ and elements of $R$. When $m$ is odd, we wish to show that for some $r \in R$, $U_2\phi^m = 2^{-8([m/2]-1)+3r}$. Performing straightforward manipulations using (4), this is equivalent to $S_m = 2^{8(m+1)}r$ for some $r \in R$. Similarly, when $m$ is even and is not $2$, we wish to show that $U_2\phi^m = 2^{-8([m/2]-1)+1}r$ for some $r \in R$. This again reduces to showing that $S_m = 2^{8(m+1)}r$ for some $r \in R$. If $m = 2j$, then (3) gives $8(j - \lfloor j/2 \rfloor) + 0 = 0$, which means the polynomial has integer coefficients, which is true by Lemma 6.

When $m = 1$ or $2$, we have that $S_m = 2^{8(m+1)}r$ for some $r \in R$, as
\begin{align*}
S_1 &= g_1 = 2^{8(2)}(3\phi + 2^8\phi^3), \\
S_2 &= g_1S_1 - 2g_2 = 2^{8(3)}(2\phi + 2^83^2\phi^2 + 2^{17}\phi^3 + 2^{24}\phi^4).
\end{align*}

Now assume the equality is true for positive integers less than $m$ with $m$ at least 3. Then for some $r_1, r_2 \in R$,
\begin{align*}
S_m &= g_1S_{m-1} - g_2S_{m-2} \\
&= (2^{16}(3\phi + 2^8\phi^2))((2^{8m}r_1) + (2^{24}\phi)(2^{8(m-1)}r_2)) \\
&= 2^{8(m+1)}[(3 \cdot 2^8\phi + 2^8\phi^2)r_1 + 2^8\phi r_2],
\end{align*}
completing the proof where $\alpha = 1$.

Assume the theorem is true for $U_2^\alpha\phi^m = \sum_{j=s}^{2^m} d(j)\phi^j$, meaning
\[ \nu_2(d(j)) \geq 8(j - f^\alpha(m)) + 3\gamma(m, \alpha) + c(m, j, \alpha). \]

Note that $s = f^\alpha(m)$. Letting $s' = f(s)$ and $U_2\phi^j = \sum_{i=[j/2]}^{2j} b(j, i)\phi^j$, we define $d'(j)$ as the integers satisfying the following equation:
\[ U_2^{\alpha+1}\phi^m = U_2 \left( \sum_{j=s}^{2^m} d(j)\phi^j \right) \]
\[ = \sum_{j=s}^{2^m} d(j)U_2\phi^j \]
\[ = \sum_{j=s}^{2^m} \sum_{i=[j/2]}^{2j} d(j)b(j, i)\phi^j \]
\[ = \sum_{j=s'}^{2^{\alpha+1+m}} d'(j)\phi^j. \]

We wish to prove that
\[ \nu_2(d'(j)) \geq 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1). \]
We will prove inequalities that imply (7). Observe that
\[ c(m, j, \alpha + 1) = \begin{cases} 
-1 & \text{if } s \text{ is even and not } 2j, \\
0 & \text{if } s \text{ is odd or } s = 2j,
\end{cases} \]
and
\[ \gamma(m, \alpha + 1) = \begin{cases} 
\gamma(m, \alpha) & \text{if } s \text{ is even,} \\
\gamma(m, \alpha) + 1 & \text{if } s \text{ is odd.}
\end{cases} \]

Also, \( c(m, s, \alpha) = 0 \) because if \( f^{\alpha-1}(m) \) is even, then \( s = f^{\alpha-1}(m)/2 \) so \( f^{\alpha-1}(m) = 2s \). Therefore, \( \nu_2(d(s)) \geq 3\gamma(m, \alpha) \) by (5).

If \( s \) is even, we will show that
\[ \nu_2(d'(j)) \geq \max\{8(j - s') - 1 + \nu_2(d(s)), \nu_2(d(s))\}, \]
because then if \( j = s' \), we have
\[ \nu_2(d'(s')) \geq \nu_2(d(s)) \geq 8(s' - s') + 3\gamma(m, \alpha) + c(m, s', \alpha + 1), \]
and for all \( j \),
\[ \nu_2(d'(j)) \geq 8(j - s') + 3\gamma(m, \alpha) + c(m, j, \alpha + 1) \]
\[ = 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1), \]
so that (8) implies (7). If \( s \) is odd we will show that
\[ \nu_2(d'(j)) \geq 8(j - s') + 3 + \nu_2(d(s)), \]
because then
\[ \nu_2(d'(j)) \geq 8(j - s') + 3\gamma(m, \alpha) + 3 \]
\[ = 8(j - s') + 3(\gamma(m, \alpha) + 1) \]
\[ = 8(j - f^{\alpha+1}(m)) + 3\gamma(m, \alpha + 1) + c(m, j, \alpha + 1), \]
which is (7).

For the sake of brevity, we treat here only the case where \( s \) is odd. The case where \( s \) is even has a similar proof. This case breaks into subcases. We will only show the proof where \( j \leq 2s \), but the other cases are \( 2s < j \leq 2^{\alpha-1}m \) and \( 2^{\alpha-1}m < j \leq 2^{\alpha+1}m \), using the same subcases for when \( s \) is even. These subcases are natural to consider because in the first range of \( j \)-values, the \( d(s) \) term is included for computing \( d'(j) \), in the second range, there are no \( d(s) \) or \( d(2^a m) \) terms, and in the third range, there is a \( d(2^a m) \) term.

Let \( j \leq 2s \). Using (5), we know that \( d'(j) = \sum_{i=s}^{2j} d(i)b(i, j) \) by collecting the coefficients of \( \phi^j \). Let \( \delta(i) \) be given by
\[ \delta(i) = \nu_2(d(i)) + \nu_2(b(i, j)). \]
Let \( D = \{\delta(i) \mid s \leq i \leq 2j\} \). Therefore we have
\[ \nu_2(d'(j)) \geq \min\{\nu_2(d(i)) + \nu_2(b(i, j)) \mid s \leq i \leq 2j\} = \min D. \]
We claim that \( \delta(i) \) achieves its minimum with \( \delta(s) \), which proves (9). For that element of \( D \), we know by inequality (3) that
\[ \delta(s) \geq \nu_2(d(s)) + 8(j - s') + 3. \]
Now suppose $i > s$. Then every element of $D$ satisfies the following inequality:
\[
\delta(i) = \nu_2(d(i)) + 8(j - \lfloor i/2 \rfloor) + c(i, j) \\
\geq 8(i - s) - 1 + \nu_2(d(s)) + 8(j - \lfloor i/2 \rfloor) + c(i, j) \\
\geq 8(s + 1 - s + j - \lfloor (s + 1)/2 \rfloor) - 2 + \nu_2(d(s)) \\
= 8(j - s') + 6 + \nu_2(d(s)),
\]
but this is clearly greater than $\delta(s)$. Therefore, if $j \leq 2s$ and $s$ is odd, then $\nu_2(d^i(j)) \geq 8(\alpha - s') + 3 + \nu_2(d(s))$. The other cases are similar.  

Now Theorem 1 follows easily from Theorem 8.

**Theorem 1.** Let $n = 2^\alpha n'$ where $2 \nmid n'$. Express the binary expansion of $m$ as $a_r \ldots a_2 a_1$, and consider the rightmost $\alpha$ digits $a_\alpha \ldots a_2 a_1$, letting $a_i = 0$ for $i > r$ if $\alpha > r$. Let $i'$ be the index of the rightmost 1, if it exists. Let
\[
\gamma(m, \alpha) = \begin{cases} 
\# \{i \mid a_i = 0, i > i'\} + 1 & \text{if } i' \text{ exists}, \\
0 & \text{otherwise}.
\end{cases}
\]
Then
\[
a(m, 2^\alpha n') \equiv 0 \pmod{2^{3\gamma(m, \alpha)}}.
\]

**Proof.** Letting $j = f^\alpha(m)$ in (2), the right hand side reduces to
\[
3\gamma(m, \alpha) + c(m, f^\alpha(m), \alpha).
\]
Notice that $c(m, f^\alpha(m), \alpha) = 0$, because if $f^{\alpha-1}(m)$ is even, then $f^\alpha(m) = f^{\alpha-1}(m)/2$ so $f^{\alpha-1}(m) = 2f^\alpha(m)$. The right hand side of (2) is minimized when $j = f^\alpha(m)$, so we conclude that $\nu_2(a(m, 2^\alpha n')) \geq 3\gamma(m, \alpha)$. \hfill \qed

4. The Parity of $a(1, n)$

Table 2 contains all odd coefficients of $\phi(z) = \sum_{n=1}^{\infty} a(1, n) q^n$ up to $n = 225$. The table shows that, up to $n = 225$, the coefficient $a(1, n)$ is odd if and only if $n$ is an odd square. This holds in general.

**Theorem 2.** The $n$th coefficient $a(1, n)$ of $\phi(z)$ is odd if and only if $n$ is an odd square.

**Proof.** Substitute $\eta(z)$ into the definition of $\phi(z)$:
\[
\phi(z) = \left( \frac{\eta(2z)}{\eta(z)} \right)^{24} = \left( \frac{q^{2/24} \prod_{n=1}^{\infty} (1 - q^{2n})}{q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)} \right)^{24}.
\]
By recognizing that $(1 - q^{2n}) = (1 - q^n)(1 + q^n)$ and simplifying, it is easy to see that
\[
\phi(z) = q \prod_{n=1}^{\infty} (1 + q^n)^{24}.
\]
Reducing this mod 2, the odd coefficients will be the only nonzero terms. But $(24 \choose 1)$ is odd if and only if $i = 0, 8, 16, 24$. It follows that
\[
\phi(z) \equiv q \prod_{n=1}^{\infty} (1 + q^{8n} + q^{16n} + q^{24n}) \pmod{2}.
\]
Immediately, it is clear that the coefficient of \( q^n \) in the Fourier expansion of \( \phi(z) \) is even if \( n \neq 1 \) (mod 8).

Note that the coefficient of \( q^n \) in the product \( \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + q^{3n}) \) is odd if and only if the coefficient of \( q^{8n+1} \) is odd in the Fourier expansion of \( \phi(z) \). Furthermore, this product can be interpreted as the generating function for the number of partitions of \( n \) where each part is repeated at most 3 times. The \( n \)th coefficient of the generating function is equivalent mod 2 to \( T_n \) of [5]. Theorem 2.1 of [5] shows that \( n \) is a triangular number if and only if \( T_n \) is odd. Therefore, the coefficient of \( q^n \) is odd in the Fourier expansion of \( \phi(z) \) if and only if

\[
n = \frac{k(k+1)}{2} + 1 = 4k^2 + 4k + 1 = (2k + 1)^2,
\]

meaning that \( n \) is an odd square. \( \square \)

| \( n \) | \( a(1,n) \) |
|------|--------|
| 1    | 1      |
| 9    | 10400997 |
| 25   | 254038914924791 |
| 49   | 8032568516459357451913 |
| 81   | 288274504516836871723618295721 |
| 121  | 11156646861439805613118172199024038253 |
| 169  | 453988290543887189391963063089337222684840687 |
| 225  | 1914654794713295199068366112834958359726636848978587 |

Table 2. All odd coefficients of \( \phi(z) \) up to \( n = 225 \).

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