Hopf algebras, tetramodules, and $n$-fold monoidal categories

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Abstract

This paper is an extended version of my talk given in Zürich during the Conference “Quantization and Geometry”, March 2-6, 2009. The main results are the following.

1. We construct a 2-fold monoidal structure [BFSV] on the category $\text{Tetra}(A)$ of tetramodules (also known as Hopf bimodules) over an associative bialgebra $A$. According to an earlier result of R. Taillefer [Tai1,2], $\text{Ext}^q_{\text{Tetra}}(A, A)$ is equal to the Gerstenhaber-Schack cohomology [GS] of $A$, which governs the infinitesimal deformations of the bialgebra structure on $A$.

2. Given an $n$-fold monoidal abelian category $C$ with a common unit object $A$ and some mild property (*) formulated in the paper, we consider the graded vector space $W^* = \text{Ext}^*_C(A, A)$. We prove that $W^*$ has a natural $(n + 1)$-algebra structure whose product is the Yoneda product. As a conclusion, the Gerstenhaber-Schack cohomology of any Hopf algebra $A$ is a 3-algebra.

3. We find an operad of $\mathbb{Z}$-modules which acts on the Hochschild cohomology of any associative $\mathbb{Z}$-algebra $A$ flat over $\mathbb{Z}$. The $k$-th component of this operad is the graded space $\oplus_i \pi^{stable}_i(D^2_k)$ of stable homotopy groups of the space $D^2_k$, the $k$-th component of the little disks operad. We establish as well an $n$-monoidal version of this result.

4. We define a contravariant functor from the homotopical category of topological spaces with values in graded vector spaces, depending on an $n$-fold monoidal category (“Hochschild cohomology depending on topological space”). In particular, such a functor is assigned to any associative algebra, any bialgebra, etc.
Introduction

0.1

The author would like to warn the reader that this paper is a draft preliminary version; the proofs of some statements in Section 5 are just sketched or even omitted.

0.2

Let \( A \) be an associative algebra over a ground field \( k \) of characteristic 0, and denote by \( \text{HH}'(A, A) \) its Hochschild cohomology. It is known that the graded space \( \text{HH}'(A, A) \) is a 2-algebra, that is it has a commutative product of degree 0, a Lie bracket of degree -1, which are compatible as

\[
[a, b \cdot c] = [a, b] \cdot c \pm b \cdot [a, c]
\]

where the sign \( \pm \) in (0.1) is such that the bracket is odd, that is for homogeneous \( a, b, c \) the sign \( \pm = (-1)^{\deg a \deg b + 1} \).

These two structures exist also on the Hochschild complex of \( A \), where the product is the cup-product of cochains, and the bracket is the Gerstenhaber bracket. However, on the level of cochains the equation (0.1) fails.

Both the cup-product and the Gerstenhaber bracket are quite artificial constructions. The cohomology \( \text{HH}^*(A, A) \) is defined as

\[
\text{HH}^*(A, A) = \text{Ext}^*_{\text{Bimod}(A)}(A, A)
\]

where \( \text{Bimod}(A) \) is the abelian category of \( A \)-bimodules, and \( A \) is the tautological \( A \)-bimodule.
It is well-known that the product in (0.1) is the Yoneda product in the definition (0.2).

Here the following two questions arise: 1) why the Yoneda product is (graded) commutative (for a general abelian category it is not), and 2) how to define the bracket from the definition (0.2)?

The answer for the both questions uses the fact that $\text{Bimod}(A)$ is a monoidal category, and $A$ is the unit object in it. The monoidal structure is the tensor product over $A$, $M_1 \otimes_A M_2$. There is a general theorem which roughly says: given a monoidal category $\mathcal{C}$ and a unit object $A$ in it, $\text{Ext}_\mathcal{C}^* (A, A)$ is a 2-algebra. (More precisely, there is a condition (*) on $\mathcal{C}$ formulated later, which is required for this theorem). This principle was known to many people, although the author does not know whether any written proof existed before.

Here we prove this general theorem among other things.

As an example for the general theory, let us prove here that for the unit object $A$ in a monoidal category $\mathcal{C}$ (not necessarily abelian), the monoid $\text{Mor}_\mathcal{C}(A, A)$ is commutative.

For denote the monoidal bifunctor by $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Let $A$ be the unit object, and let $f, g \in \text{Mor}(A, A)$. Then $g \circ f = F ((id \times g) \circ (f \times id))$ and $f \circ g = F ((f \times id) \circ (id \times g))$. But the right-hand sides of the both expressions are equal to $F (f \times g)$, therefore, the left-hand sides are equal. We proved that $f \circ g = g \circ f \in \text{Mor}(A, A)$.

Our proof that the algebra $\text{Ext}_\mathcal{C}^*(A, A)$ is graded-commutative (in the case when $\mathcal{C}$ is a monoidal abelian category) can be considered as a generalization of the above proof, see Section 2.1.

0.3

Another question on Hochschild cohomology we address here is the following. Suppose the algebra $A$ is defined over $\mathbb{Z}$, and is flat (= torsion free) over $\mathbb{Z}$. What is the operad acting in this case on $HH^*(A, A)$?

Recall that the operad of 2-algebras can be defined as the homology operad of the little discs operad $\{D^2_n\}$. That is, for the operad $\mathcal{O}$ governing the 2-algebras, one has:

\[(0.3) \quad \mathcal{O}_n = H_*(D^2_n; k)\]

We prove here the following theorem.

**Theorem 0.1.** Let $A$ be an algebra over $\mathbb{Z}$ flat over $\mathbb{Z}$. There is a natural action of the operad $\{\mathcal{O}^\mathbb{Z}_n\}$ on $HH^*(A, A)$, where

\[(0.4) \quad \mathcal{O}^\mathbb{Z}_n = \pi^{\text{stab}}_*(D^2_n)\]

Here $\pi^{\text{stab}}_k$ stands for the stable homotopy groups,

\[(0.5) \quad \pi^{\text{stab}}_k(X) = \lim_{s \to \infty} \pi_{k+s}(\Sigma^s X)\]
(where $\Sigma$ is the suspension operator, and the limit is attained for any finite CW complex $X$ by the Freudenthal theorem).

The case when $A$ is defined over a field of characteristic 0 is obtained from this more general case of $\mathbb{Z}$-algebra $A$ by the following well-known theorem (see, e.g. [A], Part III, Lecture 4): for any CW complex $X$ one has a canonical (Hurewicz) isomorphism

$$
\pi_{\text{stab}}^k(X) \otimes \mathbb{Q} \xrightarrow{\sim} H_k(X, \mathbb{Q})
$$

Thus, we recover the 2-algebra operad.

0.4

Another subject of this paper, which originally motivated the overall project, is a proof of following conjecture due to Maxim Kontsevich: for any associative bialgebra $A$ its Gerstenhaber-Schack cohomology $H^\ast_{\text{GS}}(A, A)$ is a 3-algebra. The latter means that the graded space $H^\ast_{\text{GS}}(A, A)$ admits a graded commutative product of degree 0, a Lie bracket of degree -2 such that

$$
[a, b \cdot c] = [a, b] \cdot c \pm b \cdot [a, c]
$$

where the bracket is even, that is, for homogeneous $a, b, c$, the sign $\pm = (-1)^{\deg a \cdot \deg b}$.

The author tried to use the construction of Stefan Schwede [Sch], which uses the monoidal structure on the category of bimodules over an algebra $A$ in the case of Hochschild cohomology. There is a result of R.Taillefer [Tail1,2] stating that for any bialgebra $A$

$$
H^\ast_{\text{GS}}(A, A) = \text{Ext}^\ast_{\text{tetra}(A)}(A, A)
$$

where $\text{tetra}(A)$ is the category of tetramodules over $A$, an abelian category associated to $A$ which is parallel to the category of bimodules in the case when $A$ is algebra.

We construct on the category of tetramodules two monoidal structures $\otimes_1$ and $\otimes_2$, which are compatible in some rather complicated way. These compatibilities altogether give a 2-fold monoidal structure in the sense of [BFSV] on $\text{tetra}(A)$.

There is a concept of $n$-fold monoidal category, introduced in loc.cit.; it is a main technical tool of this paper. It is a category with $n$ ordered monoidal structures with some compatibilities. There is an operad of categories acting on any $n$-fold monoidal category; the classifying spaces operad of this operad is homotopy equivalent to the $n$-dimensional little discs operad (this fact is proven in [BFSV]).

We use the last fact to prove the following general theorem.

**Theorem 0.2.** Let $C$ be an abelian $n$-fold monoidal category satisfying some mild condition ($\ast$), see Section 2.1. Let $A$ be an object which is unit object for all $n$ monoidal structures. Then $\text{Ext}^\ast_C(A, A)$ is naturally an $(n + 1)$-algebra whose commutative product is the Yoneda product.
When $A$ is a Hopf algebra (that is, a bialgebra with an antipode), the category of tetramodules obeys the condition (*). As an immediate corollary, one has

**Corollary 0.3.** Let $A$ be a Hopf algebra. Then there is a natural $3$-algebra structure on the Gerstenhaber-Schack cohomology $H^r_{\text{GS}}(A, A)$.

### 0.5

The paper is organized as follows:

In Section 1 we recall the theory of Vladimir Retakh [R],[NR] of homotopy groups of the categories of extensions. The Schwede’s construction [Sch] is based on the Retakh’s theory, as well as all our generalizations of it. We give a rather detailed exposition with complete proofs, basically because the original note [R] is too concise, and [NR] works with a great generality of the Waldhausen categories;

In Section 2 we recall the mentioned above construction of Stefan Schwede. This construction gives a Lie bracket on $HH^r(A, A)$ in the intrinsic terms of the monoidal abelian category of $A$-bimodules. We discuss here the Schwede’s tensor product $\otimes_r$, the main ingredient of the construction, and give, following [Sch], an intrinsic construction of the Gerstenhaber bracket on the Hochschild cohomology;

Section 3 introduces $n$-fold monoidal categories. The main new result here is that the category of extensions $\bigwedge_k \mathcal{E}xt^k_c(A, A)$ in an $n$-fold monoidal abelian category $C$ with common unit object $A$, is naturally an $(n+1)$-fold monoidal category;

Section 4 contains our results on the category $\text{Tetra}(A)$ of tetramodules over a bialgebra $A$. We construct a 2-fold monoidal category structure on $\text{Tetra}(A)$. The structure maps $\eta_{MNPQ}$ in this category are not isomorphisms. To the best of our knowledge, it is the first example of $n$-monoidal categories with the unit object for $n > 1$ with this property. In particular, our 2-fold monoidal structure is not defined as a braided category. As well, we discuss the condition (*). It turns out that this condition holds automatically when our bialgebra is a Hopf algebra (that is, admits an antipode). In the case of Hopf algebras we give here a construction of Lie bracket of degree -2 on the Gerstenhaber-Schack cohomology.

In Section 5 we deal with spectra. All main results of the paper, in particular Theorems 0.1 and 0.2 above, are proven here. The main technical point here is to pass from a “spectrum” of categories with an action of operad of categories to a spectrum of topological spaces, preserving the action of the corresponding operad. A technical point appears: if we do it naively, the operad action does not admit the “base points”. The main effort in this Section is directed to introduce somehow the base points, preserving in the same time the homotopical type and the operad action. The basepoints are crucial when we use the smash-product in spectra, which was our initial way to think about the problem. The author tried many ways; finitely, he became successful with a construction imitating the free loop space. We do not achieve basepoints in a proper sense, but we replace them by “based
subcategories”. Here we are very brief sometimes. We hope to improve the exposition in the sequel version.

Section 6 is served as an Appendix. Here we expose the theory of Rachel Taillefer, claiming that $H_{GS}^q(A) = \text{Ext}^q_{\text{Tetra}}(A, A)$. This result is used throughout in Sections 4 and 5. Our exposition is very closed to the original exposition in [Ta1,2], but the presentation is a bit different. In particular, it is based on a concept of a $(P, Q)$-pair in an abelian category. As a new result, we give a proof of the well-known folklore statement about the Gerstenhaber-Schack cohomology of the free commutative cocommutative bialgebra $A = S(V)$.

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1 The categories of extensions

1.1

Let $\mathcal{A}$ be an abelian category, and let $M, N \in \text{Ob}(\mathcal{A})$, and $k \geq 1$ is integral number. Consider the following category $\mathcal{E}xt^k(M, N)$.

An object of $\mathcal{E}xt^k(M, N)$ is an exact sequence (an extension)

\begin{equation}
0 \to N \to F_1 \to \cdots \to F_k \to M \to 0
\end{equation}

and a morphism of two extensions is a map of complexes which is identity on the ends:

\begin{equation}
\begin{array}{ccc}
0 & \to & N \\
\downarrow & & \downarrow \\
E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \longrightarrow & 0 \\
\end{array}
\end{equation}
Each extension \((1.1)\) defines an element in \(\text{Ext}^k_A(M, N)\), let us recall the construction (see [M] for details). For a short exact sequence (extension of length 1) \(0 \to A \to B \to C \to 0\) the boundary map in the long exact sequence defines a map \(\delta: \text{Hom}(C, C) \to \text{Ext}^1(C, A)\), and the mentioned assignment is \(\delta(id)\). In general case, divide the long exact sequence into short exact sequences and take the composition of the maps above.

Thus, we have a map \(\varphi: \text{Ob}(\text{Ext}^k_A(M, N)) \to \text{Ext}^k(M, N)\). The natural question is: when two different extensions define the same elements in \(\text{Ext}^k(M, N)\)?

The answer goes back to Yoneda and is given e.g. in the MacLane’s book [M]. It is as follows.

**Lemma 1.1.** The map \(\varphi\) is surjective. Let \(\mathcal{E}, \mathcal{F} \in \text{Ob}(\text{Ext}^k(M, N))\) be two extensions. If there is any morphism \(q: \mathcal{E} \to \mathcal{F}\) in \(\text{Ext}^k(M, N)\), then \(\varphi(\mathcal{E}) = \varphi(\mathcal{F}) \in \text{Ext}^k(M, N)\).

Conversely, if \(\varphi(\mathcal{E}) = \varphi(\mathcal{F})\), the two extensions \(\mathcal{E}, \mathcal{F}\) can be connected by a zigzag of morphisms:

\[
\begin{array}{ccccccc}
& & X_1 & & X_3 & & X_L \\
\mathcal{E} & \downarrow & & \downarrow & & \downarrow & \mathcal{F} \\
& X_2 & & & & & \\
& & \cdots & & & & \\
& & & & & & \\
\end{array}
\]

The proof can be found in the MacLane’s book [M], Ch.3. □

1.2 V. Retakh’s theorem [R]

The starting point for the work of Vladimir Retakh [R] is just a reformulation of the previous Lemma in more contemporary terms.

Consider the nerve \(N\text{Ext}^k(M, N)\) (it is a simplicial set, see [May]), and its geometrical realization \(B\text{Ext}^k(M, N) = |N\text{Ext}^k(M, N)|\) (it is a topological set, see loc.cit.). The lemma above is clearly equivalent to the following statement:

**Corollary 1.2.** \(\pi_0(N\text{Ext}^k(M, N)) \simeq \text{Ext}^k(M, N)\) as abelian groups.

Here \(\pi_0\) is the 0-th homotopy group, that is, the set of linear connection components. In our case this set is an abelian group, as follows. The Baer sum of extensions (see [M], Ch.3) gives a functor \(B: \text{Ext}^k(M, N) \times \text{Ext}^k(M, N) \to \text{Ext}^k(M, N)\) which gives a map of simplicial sets \(N\text{Ext}^k(M, N) \times N\text{Ext}^k(M, N) \to N\text{Ext}^k(M, N)\). This gives the map of classifying spaces \(B\text{Ext}^k(M, N) \times B\text{Ext}^k(M, N) \to B\text{Ext}^k(M, N)\) because the finite limits commute with the geometric realization functor (see [May]). Finally, we get a map \(\pi_0(B\text{Ext}^k(M, N)) \times \pi_0(B\text{Ext}^k(M, N)) \to \pi_0(B\text{Ext}^k(M, N))\). The fact that the Baer sum becomes the usual sum in \(\text{Ext}^k(M, N)\) is proven in [M], Ch.3. □

Retakh [R] computed the higher homotopy groups of the space \(B\text{Ext}^k(M, N)\). He proves the following theorem:
Theorem 1.3. 1. For $\ell \leq k$,

\[
\pi_\ell(\operatorname{Ext}^k(M, N)) = \operatorname{Ext}^{k-\ell}(M, N)
\]

For $\ell > k$, $\pi_\ell(\operatorname{Ext}^k(M, N)) = 0$;

2. there are natural homotopy equivalences of topological spaces

\[
\Omega(B\operatorname{Ext}^{k-1}(M, N)) \to \Omega(B\operatorname{Ext}^k(M, N))
\]

where $\Omega$ is the loop space functor;

3. the terms of the spectrum of topological spaces in (2.) are direct products of the Eilenberg-MacLane spaces.

Remark 1.4. Note here that there is a paper of Alan Robinson [Rob1], which appeared 5 years before the Retakh’s paper, and where an analogous theorem for Tor groups was proven. We would like to mention subsequent papers of Robinson, especially the one on the extraordinary derived category [Rob2]. Probably Robinson was the first who tried to deal with “modules and algebras in spectra” in 80’s, while the category of symmetric spectra appeared in 90’s.

We overview the main ideas of the proof in Section 1.3.

Let us mention an immediate corollary of the Theorem above.

Corollary 1.5. Suppose the abelian category $A$ where we take the extensions is a $k$-linear abelian category for some field $k$. Then all homotopy groups $\pi_\ell(\operatorname{Ext}^k(M, N))$, which a priori are abelian groups, are in fact $k$-vector spaces.

In particular, if $\text{char}(k) = p$, all elements in $\pi_\ell(\operatorname{Ext}^k(M, N))$ are $p$-torsion, and if $\text{char}(k) = 0$, all elements in $\pi_\ell(\operatorname{Ext}^k(M, N))$ can be divided for any integral number.

The formulation of the Theorem above assumes the following lemma:

Lemma 1.6. All connection components in $\operatorname{Ext}^k(A, A)$, $k \geq 0$, are homotopically equivalent.

Proof. Denote by $\emptyset$ the zero extension with respect to the Baer sum in $\operatorname{Ext}^k(M, N)$. Suppose $X$ is an extension from another connected component. Denote the connected components of $\emptyset$ and $X$ by $\operatorname{Ext}^k(M, N)_{\emptyset}$ and $\operatorname{Ext}^k(M, N)_X$; we want to prove that these two categories are homotopy equivalent. There is a functor $F : \operatorname{Ext}^k(M, N)_{\emptyset} \to \operatorname{Ext}^k(M, N)_X$, $\alpha \mapsto \alpha + X$ (here $+$ is the Baer sum). Consider $Y \in \operatorname{Ext}^k(M, N)$ such that $X + Y$ is zero element on
Ext\(^k\)(M, N). Then we have a functor \(G : E \text{xt}\(^k\)(M, N)_X \to E \text{xt}\(^k\)(M, N)_\emptyset\), \(\beta \mapsto \beta + Y\). There is a zigzag of morphisms from \(X + Y\) to \(\emptyset\) in \(E \text{xt}\(^k\)(M, N)\), let it be

\[
\begin{array}{cccc}
\emptyset & C_1 & C_3 & C_{2s-1} \\
& \downarrow & \downarrow & \downarrow \\
C_2 & \ldots & \ldots & X + Y
\end{array}
\]

All these arrows can be considered as natural transformation of functors, say from the functor \(F_1 : E \text{xt}\(^k\)(M, N)_\emptyset \to E \text{xt}\(^k\)(M, N)_\emptyset\), \(\alpha \mapsto \alpha + C_t\), to the functor \(F_{t+1} : E \text{xt}\(^k\)(M, N)_\emptyset \to E \text{xt}\(^k\)(M, N)_\emptyset\), \(\alpha \mapsto \alpha + C_{t+1}\). Therefore, the maps of the classifying space of \(E \text{xt}\(^k\)(M, N)_\emptyset\) to itself, induced by these functors, are homotopic. Performing this for all arrows of the zigzag, we get that \(G \circ F\) is homotopic to the identity. Analogously one proves that \(F \circ G\) is homotopic to the identity.

\[\square\]

1.3 A proof of the Retakh’s theorem

Here we recall the proof [R] of Theorem 1.3. Some constructions in this proof will be used later in Section 5.

1.3.1 The idea

Let \(*\) be some fixed object in \(E \text{xt}\(^k\)(M, N)\). Consider the category of 1-pathes \(P_{\text{Ret}} E \text{xt}\(^k\)(M, N)\). An object of this category is a “path”

\[
\begin{array}{ccc}
\emptyset & \to & X \\
\downarrow & & \downarrow \\
X & \leftarrow & Y
\end{array}
\]

(1.7)

where \(X, Y \in E \text{xt}\(^k\)(M, N)\), and the arrows are morphisms of extensions. A morphism in the category \(P_{\text{Ret}} E \text{xt}\(^k\)(M, N)\) is defined as a commutative diagram

\[
\begin{array}{ccc}
\emptyset & \to & X & \leftarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
X' & \leftarrow & Y'
\end{array}
\]

(1.8)

There is a natural projection \(p : P_{\text{Ret}} E \text{xt}\(^k\)(M, N) \to E \text{xt}\(^k\)(M, N)\), which assigns \(Y\) to the object (1.7).

We prove the following three statements:

1. the category \(P_{\text{Ret}} E \text{xt}\(^k\)(M, N)\) is contractible;

2. the projection \(p : P_{\text{Ret}} E \text{xt}\(^k\)(M, N) \to E \text{xt}\(^k\)(M, N)\) satisfies the hypothesis of the Quillen’s Theorem B;
3. for some explicit choice of the based object *, the fiber \( p^{-1}(*) \) is homotopy equivalent to the category \( \mathcal{E}xt^{k-1}(M, N) \).

The second statement means that one can think on the projection \( p \) as on a fibration of topological spaces, by the first statement the total space of this fibration is contractible. These two statements together imply that the set-theoretical fiber \( p^{-1}(*) \) is homotopy equivalent to the homotopy fiber, which in the case of contractible total space is equal to the loop spaces on the base. Then, together with the third statement, we get that \( B\mathcal{E}xt^{k-1}(M, N) = \Omega(B\mathcal{E}xt^k(M, N)) \). This speculation proves the first two claims of Theorem 1.3. We refer the reader to [NR], Section 8 for a proof of the third claim of Theorem 1.3.

### 1.3.2 The category \( \mathcal{P}_R\mathcal{E}xt^k(M, N) \) is contractible

First of all, let us recall some general principles on the homotopy theory of categories (see [Q], Section 1).

Any functor \( F : C_1 \to C_2 \) defines a map \( F_B : BC_1 \to BC_2 \).

**Lemma 1.7.** (i) Any natural transformation between two functors \( F, G : C_1 \to C_2 \) defines a homotopy between the maps \( F_B, G_B : BC_1 \to BC_2 \);  

(ii) if a functor \( F : C_1 \to C_2 \) admits left or right adjoint, the map \( F_B \) is a homotopy equivalence;  

(iii) in particular, if a category \( C \) has initial or final object, the space \( BC \) is contractible.

All these statements are fairly simple, see [Q], Section 1. □

We say that a category \( C \) is contractible if the topological space \( BC \) is contractible. We prove

**Lemma 1.8.** For any (small) category \( C \), the category \( \mathcal{P}_R C \) is contractible.

**Proof.** Define the category \( C(*) \) as the category of objects of \( C \) under *. That is, an object of \( C(*) \) is a pair \((X, \varphi)\) where \( X \) is an object of \( C \) and \( \varphi : * \to X \) is a morphism. A morphism in \( C(*) \) is a commutative diagram

\[
\begin{array}{ccc}
* & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow \\
X' & \xleftarrow{\varphi'} & X'
\end{array}
\]
It is clear that the pair \((\ast, \text{id})\) is the initial object in \(\mathcal{C}(\ast)\). Therefore, the category \(\mathcal{C}(\ast)\) is contractible by Lemma 1.7(iii).

There is a natural functor \(F: \mathcal{C}(\ast) \to \mathcal{P}_R\mathcal{C}\), which assigns the 1-path \(\ast \xrightarrow{\varphi} X \xleftarrow{\text{id}} X\) to an object \(\ast \xrightarrow{\varphi} X\) of \(\mathcal{C}(\ast)\). This functor is the right adjoint to the functor which assigns to the path \(\ast \xrightarrow{\varphi} X \xleftarrow{Y}\) the object \(\ast \xrightarrow{\varphi} X\). Therefore, the categories \(\mathcal{P}_R\mathcal{C}\) and \(\mathcal{C}(\ast)\) are homotopy equivalent by Lemma 1.7(ii). \(\square\)

### 1.3.3 The Quillen’s Theorem B

We are going to prove that the natural projection functor \(p: \mathcal{P}_R\mathcal{E}\text{xt}^k(M, N) \to \mathcal{E}\text{xt}^k(M, N)\) satisfies the conditions of the Quillen’s Theorem B. Let us recall what it is. Actually we formulate the Corollary to Theorem B which we will only need.

Let \(g: E \to B\) be a map of topological spaces, \(b \in B\). The homotopical fiber \(F(g, b)\) of the map \(g\) over \(b\) is the spaces

\[
F(g, b) = \{ e \in E \text{ and a path } p: [0, 1] \to B | p(0) = g(e), p(1) = b \}
\]

In other words, we replace a map \(g: E \to B\) by a homotopical to it Serre fibration, and then compute the fiber. If the map \(g: E \to B\) is a Serre fibration itself, then up to homotopy the homotopical fiber coincides with the set-theoretical fiber.

**Example 1.9.** Let \(G\) be a topological group considered as a category \(\mathcal{G}\) with a single object, and let \(BG\) be its classifying space. There is a bundle \(p: EG \to BG\) which is a Serre fibration with fiber \(G\), and with \(EG\) homotopically trivial. Then \(G\) is homotopically equivalent to the homotopical fiber, which is \(\Omega BG\) since \(EG\) is contractible.

\[G \simeq \Omega BG\]

Let \(f: \mathcal{C}_1 \to \mathcal{C}_2\) be a functor. For any \(Y \in \text{Ob}(\mathcal{C}_2)\) define the *comma category* \(Y \setminus f\) whose objects are the pairs \(\{(X, v) | X \in \text{Ob}\mathcal{C}_1, v \in \text{Mor}_{\mathcal{C}_2}(Y, f(X))\}\). The morphisms are the maps \(X \to X'\) such that the corresponding triangle is commutative.

Morally, the comma category \(Y \setminus f\) has the role of the fiber category \(f^{-1}(Y)\) consisting from objects over \(Y\) and their morphisms which are mapped by \(f\) to the identity of \(Y\). Let us discuss a condition under which the categories \(Y \setminus f\) and \(f^{-1}(Y)\) are indeed homotopically equivalent.

One says that a functor \(f: \mathcal{C}_1 \to \mathcal{C}_2\) makes the category \(\mathcal{C}_1\) a *prefibred category* over \(\mathcal{C}_2\) if for every object \(Y \in \text{Ob}\mathcal{C}_2\) the functor

\[
f^{-1}(Y) \to Y \setminus f, \quad X \mapsto (X, \text{id}_Y)
\]

has a right adjoint.
It follows from Lemma 1.7(ii) that the functor above is a homotopically equivalence.

In the case above denote by \((X, v) \mapsto v^* X\) the right adjoint functor to (1.11). Then for any arrow \(u: Y \to Y'\) in \(C\) one has the following functor \(u^* : f^{-1}(Y') \to f^{-1}(Y)\):

\[
(1.12) \quad f^{-1}(Y') \to Y' \setminus f \xrightarrow{u} Y \setminus f \xrightarrow{v^*} f^{-1}(Y)
\]

This functor is called the base change functor.

Now we are ready to formulate the Corollary to the Quillen’s Theorem B:

**Theorem 1.10.** Suppose a functor \(f: C_1 \to C_2\) is prefibred and that for any arrow \(u: Y \to Y'\) in \(C_2\) the base change functor \(u': f^{-1}(Y') \to f^{-1}(Y)\) is a homotopy equivalence. Then for any \(Y \in \text{Ob} C_2\), the classifying space of the category \(f^{-1}(Y)\) is homotopy equivalent to the homotopy-fibre of \(f: BC_1 \to BC_2\) over \(Y\).

See [Q], Section 1 for a proof. □

1.3.4 The functor \(p: \mathcal{P}_H \mathcal{E}_{\mathcal{E}}^k(M, N) \to \mathcal{E}_{\mathcal{E}}^k(M, N)\) satisfies the hypothesis of the Quillen’s Theorem B

Let \(Y \in \text{Ob} \mathcal{E}_{\mathcal{E}}^k(M, N)\). Consider the category \(Y \setminus p\) (see previous Subsection). There is a natural inclusion of categories \(i: p^{-1}(Y) \to Y \setminus p\). We firstly prove

**Lemma 1.11.** The functor \(i\) has a right adjoint.

**Proof.** This is very easy. Let

\[
(1.13) \quad * \xrightarrow{v} X' \xleftarrow{\varphi} Y' \xrightarrow{v} Y
\]

be an element of the category \(Y \setminus p\). The right adjoint functor \(v^*\) associates to it the 1-path

\[
(1.14) \quad * \to X' \xleftarrow{\varphi v} Y
\]

Now for an arrow \(u: Y \to Y'\) the base change functor \(u': p^{-1}(Y') \to p^{-1}(Y)\) just maps

\[
(1.15) \quad * \to X' \xleftarrow{\varphi u} Y'
\]

to

\[
(1.16) \quad * \to X' \xleftarrow{\varphi u} Y
\]

We prove
Lemma 1.12. For any arrow \( u: Y \to Y' \), the base change functor \( u': p^{-1}(Y') \to p^{-1}(Y) \) is a homotopy equivalence.

Proof. To prove that it is a homotopy equivalence, it is enough to prove that it has a left adjoint. We use the following Lemma due to V.Retakh [R], Lemmata 1.2:

Lemma 1.13. 1. Every morphism in \( \mathcal{E}_{xt}^k(M, N) \) can be canonically factorized as the composition of two morphisms such that the first is (component-wise) monomorphic, and the second admits a section;

2. pushouts along component-wise monomorphic morphisms exist in \( \mathcal{E}_{xt}^k(M, N) \).

See [Sch], Lemma 4.4 for a proof.

Now the left adjoint functor assigns to the diagram

\[
\begin{array}{ccc}
* & \longrightarrow & X \\
\uparrow & & \uparrow \\
Y' & \longrightarrow & Y
\end{array}
\]

the upper line of the diagram

\[
\begin{array}{ccc}
* & \longrightarrow & X \quad \mathcal{V} \\ id & \downarrow \quad \downarrow & \downarrow \quad \downarrow \\
Y' & \longrightarrow & Y'
\end{array}
\]

if the morphism \( Y \to Y' \) is monomorphic, in this case \( X \sqcup_Y Y' \) is an extension (that is, is exact sequence) by Lemma 1.13(2). In the case of general morphism \( Y \to Y' \) we firstly use Lemma 1.13(1) to replace (1.18) by a diagram with monomorphic \( Y \to Y' \).

We have proved that the functor \( p: \mathcal{P}_{\mathcal{E}_{xt}^k}(M, N) \to \mathcal{E}_{xt}^k(M, N) \) satisfies the Quillen’s Theorem B.

1.3.5 The category \( \Omega_{R\mathcal{E}_{xt}^k}(M, N) = p^{-1}(*) \)

It remains to compute the fiber for some choice of the based object.

We choose for \( k \geq 2 \) the based object to be equal to

\[
0 \to N \xrightarrow{id} N \to 0 \to \cdots \to 0 \to M \xrightarrow{id} M \to 0
\]

and for \( k = 1 \) to be equal to

\[
0 \to N \to N \oplus M \to M \to 0
\]
with the natural inclusion and projection.

Denote the category $p^{-1}(\ast)$ by $\Omega_R\underline{\text{Ext}}^k(M, N)$. We prove

**Lemma 1.14.** The category $\Omega_R\underline{\text{Ext}}^k(M, N)$ is homotopy equivalent to the category $\underline{\text{Ext}}^{k-1}(M, N)$.

**Proof.** Consider firstly the case $k \geq 2$. Suppose $\ast \to X \leftarrow \ast$ be an object in $\Omega_R\underline{\text{Ext}}^k(M, N)$, where $\ast$ is given by (1.19), and $X$ is an extension

\[
0 \to N \to X_1 \to \cdots \to X_k \to M \to 0
\]

In particular, we have the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{id} & M \\
\downarrow{\alpha} & & \downarrow{id} \\
X_k & \xrightarrow{\delta_k} & M \\
\downarrow{\beta} & & \downarrow{id} \\
M & \xrightarrow{id} & M
\end{array}
\]

In particular, both $\alpha$ and $\beta$ give sections of $\delta_k$. We have an extension and a map

\[
0 \to N \to X_1 \to \cdots \to X_{k-1} \to \text{Ker}\delta_k \to 0
\]

The pull-back of this extension by $\alpha - \beta$ gives an element in $\underline{\text{Ext}}^{k-1}(M, N)$. We have constructed a functor $F : \Omega_R\underline{\text{Ext}}^k(M, N) \to \underline{\text{Ext}}^{k-1}(M, N)$. Let us prove that it is a homotopy equivalence.

By Lemma 1.7(ii), it is enough to construct a left adjoint functor $G$.

Suppose

\[
0 \to N \to Y_1 \to \cdots \to Y_{k-1} \xrightarrow{\delta_{k-1}} M \to 0
\]

be a $(k - 1)$-extension. Construct the extension

\[
0 \to N \to Y_1 \to \cdots \to Y_{k-1} \xrightarrow{\Delta \delta_{k-1}} M \oplus M \xrightarrow{(1, -1)} M \to 0
\]

where $\Delta : M \hookrightarrow M \oplus M$ is the diagonal imbedding. We can define by the last extension an element in $\Omega_R\underline{\text{Ext}}^k$ in a natural way. This functor $G : \underline{\text{Ext}}^{k-1}(M, N) \to \Omega_R\underline{\text{Ext}}^k(M, N)$ is the left adjoint to $F$. \qed
The first two claims of Theorem are proven. Concerning the third claim we refer the reader to [NR], Section 8.

The case \( k = 1 \) is trivial, because there is only one loop up to an isomorphism. On the other hand, \( \mathcal{E}xt^0(M, N) \) is a discrete category with only identity maps and the set of objects equal to \( \text{Hom}(M, N) \), by definition.

\[ \square \]

\section{A construction of S. Schwede}

Here we describe the construction of Stefan Schwede [Sch] which mainly motivated our work.

\subsection{2.1}

We are going to apply the results of Section 1 in the case when the abelian category \( \mathcal{A} \) (in which we consider extensions) is also \textit{monoidal}. This means that there is a bifunctor \( \otimes: \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) which is associative and distributive with respect to the direct sum. We always suppose that there is a two-sided unit object for the monoidal structure. A priori the monoidal structure is not exact, but we want to consider it as it would be. For this we impose the following condition:

\((*)\) There is a full additive subcategory \( \mathcal{A}_0 \subset \mathcal{A} \) such that

\begin{enumerate}
  \item the monoidal structure is exact on \( \mathcal{A}_0 \);
  \item the categories \( \mathcal{E}xt^k_{\mathcal{A}_0} \) and \( \mathcal{E}xt^k_{\mathcal{A}} \) are homotopically equivalent for any \( k \) (the inclusion of the nerves is a weak homotopy equivalence);
  \item the unit object belongs to \( \mathcal{A}_0 \);
  \item the category \( \mathcal{A}_0 \) is closed under the monoidal structure in \( \mathcal{A} \); hence, it is itself a monoidal additive category.
\end{enumerate}

\textbf{Example 2.1.} Let \( A \) be an associative algebra, and let \( \mathcal{A} \) be the category of \( A \)-bimodules. The monoidal structure is the tensor product of bimodules over \( A \), \( (M_1, M_2) \mapsto M_1 \otimes_A M_2 \). This product is not exact. Let \( \mathcal{A}_0 \) be the full additive subcategory of bimodules which are flat as right \( A \)-modules. The condition \((*)\) for \( \mathcal{A}_0 \) is proven in [Sch], Lemma 2.1. The unit object is the tautological bimodule \( A \), and it belongs to \( \mathcal{A}_0 \). It is clear that such \( \mathcal{A}_0 \) is closed under the monoidal structure.

\textbf{Example 2.2.} Suppose \( A \) is an associative bialgebra (see definition below in Section 4.1), and let \( \mathcal{A} \) be the category of left \( A \)-modules (as over algebra). For any two left modules \( M_1 \) and \( M_2 \), their tensor product \( M_1 \otimes_{k} M_2 \) over the ground field is naturally a module over the algebra \( A \otimes_{k} A \). Now the coproduct \( \Delta: A \to A \otimes_{k} A \) (which is a map of algebras) makes
$M_1 \otimes M_2$ a left $A$-module. This monoidal structure is exact, one can set $A_0 = A$. The unit object is the trivial module $k$; for existence of it one needs to have the counit.

The two ingredients of our game are the Yoneda product and the Schwede’s tensor product of extensions.

### 2.1.1 The Yoneda product

For extensions $E = \{0 \to M \to E_1 \to \cdots E_k \to N \to 0\}$ and $F = \{0 \to N \to F_1 \to \cdots \to F_\ell \to P \to 0\}$ their Yoneda product $E \# F$ is the extension

$$0 \to M \to E_1 \to \cdots E_k \to F_1 \to \cdots \to F_\ell \to P \to 0$$

where the “central” arrow $E_k \to F_1$ is the composition $E_k \to N \to F_1$. Clearly it is an extension.

**Lemma 2.3.** Under the map $\varphi$ of Lemma 1.1, the Yoneda product is corresponded to the natural product $\text{Ext}^k(N, M) \otimes \text{Ext}^\ell(P, N) \to \text{Ext}^{k+\ell}(P, M)$ (which also will be called the Yoneda product).

### 2.1.2 The Schwede’s tensor product

Let now $E$ and $F$ be extensions as above, with $M = N = P$ equal to $A$, the unit object of the monoidal category $\mathcal{A}$. Suppose also that the terms of these extensions belong to the subcategory $A_0$, on which $\otimes$ is exact.

The idea is to define an extension $E \otimes_\tau F$ of length $k + \ell$, such that there is a diagram of maps of extensions:

$$E \otimes_\tau F$$

Then it will follow from Lemma 1.1 that the extensions $E \sharp F$ and $(-1)^{kl}F \sharp E$ define the same element in $\text{Ext}_A^*(A, A)$ (in other words, the Yoneda product in $\text{Ext}^*(A, A)$ is graded commutative).

The most naive candidate for $E \otimes_\tau F$, the usual tensor product of the complexes $E$ and $F$, has the length for one more than $E \sharp F$ and $F \sharp E$. We modify this naive definition, as follows.
Denote by \( \tau(E) \) the following “truncated” complex: \( \tau(E) = \{ 0 \to A \to E_1 \to \ldots \to E_k \to 0 \} \) (what is truncated is the last term), and analogously \( \tau(F) = \{ 0 \to A \to F_1 \to \cdots \to F_\ell \to 0 \} \). We have: \( \tau(E) \) is quasiisomorphic to \( A[-k] \), and \( \tau(F) \) is quasiisomorphic to \( A[-\ell] \). As the elements of these complexes belong to \( \mathcal{A}_0 \), the usual tensor product \( \tau(E) \otimes \tau(F) \) is quasi-isomorphic to \( A[-k-\ell] \). This gives us the following extension of length \( k + \ell \): \( 0 \to \tau(E) \otimes \tau(F) \to A \to 0 \). We denote this extension by \( E \otimes \tau F \).

**Lemma 2.4.** There exists a diagram of extensions (2.2).

**Proof.** Represent \( \tau(E) \otimes \tau(F) \) as a rectangle, see Figure 1. The terms written down in the two marked on the Figure borders are the quotient-complexes isomorphic to \( E\sharp F \) and \( (−1)^{k\ell} F\sharp E \), correspondingly. \( \square \)

**Corollary 2.5.** Let \( \mathcal{A} \) be a monoidal category satisfying the condition \((*)\), and let \( A \) be the unit object in it. Then the Yoneda product on Ext\(^*(\mathcal{A}, A) \) is graded commutative.

It follows from the Lemma above, Lemma 2.1, and Lemma 1.1. \( \square \)

### 2.2 The Lie bracket

One can go one step further in these constructions, and consider the following (non-commutative in any sense) diagram:

![Figure 1:](image)
This diagram defines a loop in \( N_{\mathcal{E}_{\mathcal{T}}}^{k+\ell}(A, A) \), that is, an element in \( \pi_1(N_{\mathcal{E}_{\mathcal{T}}}^{k+\ell}(A, A)) \). By Retakh’s Theorem 1.3 this \( \pi_1 \) is canonically isomorphic to \( \text{Ext}^{k+\ell-1}_A(A, A) \). What we get is the bracket, induced by the Gerstenhaber bracket on Hochschild cohomology:

**Theorem 2.6. (Schwede)** The Lie bracket on \( HH^*(A, A) = \text{Ext}^*_{A_{\text{Bimod}}}(A, A) \) defined above is equal to the bracket induced from the Gerstenhaber bracket. In particular, it obeys the graded Jacobi identity, and together with the Yoneda product forms a 2-algebra.

**Remark 2.7.** In the above construction, one place should be treated more carefully. Namely, the Retakh’s theorem was proven for some fixed choice of the basepoint. The categories \( \mathcal{E}_{\mathcal{T}}^{k}(A, A) \) are not connected (\( \pi_0(\mathcal{E}_{\mathcal{T}}^{k}(A, A)) = \text{Ext}^k(A, A) \)), and the computation of the homotopy groups may depend on the choice of connection component. For this we use Lemma 1.6 which shows that all connection components are homotopically equivalent.

# 3 \( n \)-fold monoidal categories

## 3.1 Introduction

An \( n \)-fold monoidal category \( \mathcal{C} \) is a category with \( n \) monoidal structures \( \otimes_1, \ldots, \otimes_n : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) which obey some compatibility relations. They were introduced in [BFSV] with the following motivation.

The classifying space of a 1-monoidal category admits an action of the Stasheff operad (see [M2]). A connected space with an action of this operad has a homotopical type of loop spaces; the same is true for non-connected spaces whose linear connection components form a group (see loc.cit.). In [BFSV], the authors found a structure on a category (called \( n \)-monoidal), such that there is an operad of categories acting on it; the classifying space of this operad of categories is \( n \)-dimensional little disc operad. This is very close to say that the classifying space of an \( n \)-monoidal category is an \( n \)-fold loop space (see [M2]). The definition is iterative, like the definition of \( n \)-fold loop space.
3.2 Definition

**Definition 3.1.** A (strict) monoidal category is a category $\mathcal{C}$ together with a functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbb{I} \in \text{Ob}(\mathcal{C})$ such that

1. $\otimes$ is strictly associative;
2. $\mathbb{I}$ is a strict two-sided unit for $\otimes$.

A monoidal functor $(F, \eta): \mathcal{C} \to \mathcal{D}$ between monoidal categories is a functor $F$ such that $F(\mathbb{I}_\mathcal{C}) = \mathbb{I}_\mathcal{D}$ with a natural transformation

\[
\eta_{A,B}: F(A) \otimes F(B) \to F(A \otimes B)
\]

which satisfies the following conditions:

1. Internal associativity: the following diagram commutes

\[
\begin{array}{c}
F(A) \otimes F(B) \otimes F(C) \\
\downarrow \text{id}_{F(A)} \otimes \eta_{B,C} \\
F(A \otimes B) \otimes F(C) \\
\downarrow \eta_{A,B,C}
\end{array}
\begin{array}{c}
\eta_{A,B} \otimes \text{id}_{F(C)} \\
\quad
\end{array}
\begin{array}{c}
F(A \otimes B) \otimes F(C) \\
\downarrow \eta_{A,B,C}
\end{array}
\]

2. Internal unit conditions: $\eta_{A,\mathbb{I}} = \eta_{\mathbb{I},A} = \text{id}_{F(A)}$.

The crucial in this definition is that the map $\eta$ is not required to be an isomorphism.

Denote by $\textbf{MonCat}$ the category of (small) monoidal categories and monoidal functors.

3.2.1

**Definition 3.2.** A 2-fold monoidal category is a monoid in $\textbf{MonCat}$. This means, that we are given a monoidal category $(\mathcal{C}, \otimes_1, \mathbb{I})$, and a monoidal functor $(\otimes_2, \eta): \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfies the following axioms:

1. External associativity: the following diagram commutes in $\textbf{MonCat}$

\[
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow \text{id}_{\mathcal{C}} \times (\otimes_2, \eta) \\
\mathcal{C} \times \mathcal{C}
\end{array}
\begin{array}{c}
(\otimes_2, \eta) \times \text{id}_{\mathcal{C}} \\
\quad
\end{array}
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow (\otimes_2, \eta)
\end{array}
\]
2. External unit conditions: the following diagram commutes in \textbf{MonCat}

\[
\begin{array}{ccc}
C \times \mathcal{U} & \xrightarrow{\varepsilon} & C \\
\downarrow \cong & & \downarrow \cong \\
C & \xrightarrow{(\otimes, \eta)} & C \\
\end{array}
\]

Let us note that the role of the monoidal structures \(\otimes_1\) and \(\otimes_2\) in this definition is not symmetric.

Explicitly the definition above means that we have an operation \(\otimes_2\) with the two-sided unit \(\mathcal{U}\) (the same that for \(\otimes_1\)) and a natural transformation

\[
\eta_{A,B,C,D} : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)
\]

The internal unit conditions are: \(\eta_{A,B,i,i} = \eta_{i,i,A,B} = id_{A\otimes_2 B}\), and the external unit conditions are: \(\eta_{A,i,i,B} = \eta_{i,i,A,B} = id_{A\otimes_1 B}\). As well, one has the morphisms

\[
\eta_{A,i,i,B} : A \otimes_1 B \to A \otimes_2 B
\]

and

\[
\eta_{i,i,A,B} : A \otimes_1 B \to B \otimes_2 A
\]

which will be very essential in Section 3.

The internal associativity gives the commutative diagram:

\[
(U \otimes_2 V) \otimes_1 (W \otimes_2 X) \otimes_1 (Y \otimes_2 Z) \xrightarrow{\eta_{U,V,W,X,Y,Z}} ((U \otimes_1 W) \otimes_2 (V \otimes_1 X)) \otimes_1 (Y \otimes_2 Z)
\]

The external associativity condition gives the commutative diagram:

\[
(U \otimes_2 V \otimes_2 W) \otimes_1 (X \otimes_2 Y \otimes_2 Z) \xrightarrow{\eta_{U,V,W,X,Y,Z}} ((U \otimes_2 V) \otimes_1 (X \otimes_2 Y)) \otimes_2 (W \otimes_1 Z)
\]

Finally, [BFSV] gives
**Definition 3.3.** Denote by $\text{MonCat}_n$ the category of (small) $n$-fold monoidal categories. Then an $(n+1)$-fold monoidal category is a monoid in $\text{MonCat}_n$.

This gives the following compatibility axiom: for $1 \leq i < j < k \leq n$ the following diagram is commutative:

\begin{equation}
(3.10)
\end{equation}

\begin{diagram}
\text{((A}_1 \otimes_{k} A_2) \otimes_{j} (B_1 \otimes_{k} B_2)) \otimes_{i} ((C_1 \otimes_{k} C_2) \otimes_{j} (D_1 \otimes_{k} D_2)) \\
(A_1 \otimes_{j} B_1) \otimes_{k} ((A_2 \otimes_{j} B_2) \otimes_{i} (C_2 \otimes_{j} D_2)) \\
(A_1 \otimes_{i} C_1) \otimes_{k} ((A_2 \otimes_{i} C_2) \otimes_{j} (B_1 \otimes_{i} D_1) \otimes_{k} (B_2 \otimes_{i} D_2)) \\
((A_1 \otimes_{i} C_1) \otimes_{j} (B_1 \otimes_{i} D_1)) \otimes_{k} ((A_2 \otimes_{i} C_2) \otimes_{j} (B_2 \otimes_{i} D_2))
\end{diagram}

\[\eta_{i}^{(k)} \otimes \eta_{j}^{(k)} \quad \eta_{i}^{(j)} \quad \eta_{i}^{(k)} \otimes \eta_{j}^{(k)} \quad \eta_{i}^{(j)} \quad \eta_{i}^{(k)} \]

### 3.3 Examples

See examples of monoidal (=1-monoidal) categories given in Examples 2.1 and 2.2.

**Example 3.4.** According to a result of Joyal and Street [JS], there are just few examples for $n \geq 2$ when the map $\eta_{i,j}^{A,B,C,D}$ are isomorphisms for any $A, B, C, D$ and any $1 \leq i < j \leq n$ and when there is a common unit object for all $n$ monoidal structures. For $n = 2$ any such category is equivalent as a 2-fold monoidal category to a category with $A \otimes_{1} B = A \otimes_{2} B$ with a braiding $c_{A,B}: A \otimes B \to B \otimes A$ defining a structure of a braided category (see, e.g., [ES]) on $C$. Then we can construct a map $\eta_{A,B,C,D}: (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$ just as $\eta_{A,B,C,D} = id \otimes c_{23} \otimes id$. This construction gives a 2-fold monoidal category. For $n > 2$ and $\eta_{A,B,C,D}$ isomorphisms one necessarily has $A \otimes_{i} B = A \otimes_{j} B$ for any $i, j$ and all $\otimes_{i}$ are symmetric. We recommend the Fiedorowicz’s Obervolfach talk [F] (page 4 and thereafter) for a concise but clear overview of this result.

**Example 3.5.** Let $A$ be an associative bialgebra. We define a tetramodule over it as a $k$-vector space $M$ such that there is a bialgebra structure on $A \oplus \epsilon M$, where $\epsilon^2 = 0$ and
the restriction of the bialgebra structure to $A$ is the initial one (see Section 4 for details).
If we perform this definition replacing “bialgebra” by “associative algebra”, we recover the
concept of bimodule; thus, this Example is a generalization of Example 2.1. We construct
in Section 4 a 2-fold monoidal structure on the abelian category $\text{Tetra}(A)$ of tetramodules
over $A$.

Example 3.6. Examples 2.1 and 3.5 can be generalized as follows. Recall from Exam-
ple 2.2 that the left modules over an associative bialgebra form a monoidal category, with
the monoidal structure equal to the tensor product of the underlying vector spaces. De-
fine an $n$-fold monoidal bialgebra as an associative algebra with $n$ coassociative coproducts
$\Delta_1, \ldots, \Delta_n: A \rightarrow A \otimes_k A$ such that the corresponding $n$ monoidal structures on the cat-
egory of left $A$-modules form an $n$-fold monoidal category. Thus, 0-monoidal bialgebra is
just an associative algebra, and 1-monoidal bialgebra is a bialgebra. One can define the
category of tetramodules over an $n$-monoidal bialgebra analogously to the previous Exam-
ple. We claim that this category is an $(n + 1)$-fold monoidal category; a proof will appear
somewhere. The author believes that this concept of $n$-monoidal bialgebra is a conceptually
right $n$-categorical generalization of the concept of bialgebra.

3.4 The operad of categories governing the $n$-fold monoidal categories
Fix $n \geq 1$. For any $d \geq 0$ denote by $M_n(d)$ the full subcategory of the free $n$-fold monoidal
category generated by objects $x_1, \ldots, x_d$ consisting of objects which are monomials in $x_i$, where
each $x_i$ occurs exactly ones. For example, such monomials for $d = 3$ and $n = 2$
could be $(x_3 \otimes x_1 \otimes x_2)$ or $(x_2 \otimes x_3 \otimes x_1)$. For fixed $n$ and $d$ the category $M_n(d)$
has a finite number of objects. The morphisms in $M_n(d)$ are exactly those which can be
obtained as compositions of the associativities for a fixed $\otimes_i$, and $\eta_{ijkl}$, with exactly the
same commutative diagrams as in $n$-fold monoidal category.

When $n$ is fixed and $d$ is varied, the categories $M_n(d)$ form an operad of categories. The
following lemma follows from the definitions.

Lemma 3.7. A category is $n$-fold monoidal if and only if there is an action of the operad
$\{M_n(d)\}_{d \geq 0}$ of categories on it.

The following very deep theorem is in a sense the main result in [BFSV]:

Theorem 3.8. The classifying space of the operad of categories $\{M_n(d)\}$ is an operad of
topological space which is homotopically equivalent (as operad) to the $n$-dimensional little
discs operad.
3.5 The category of extensions of abelian \( n \)-fold monoidal category

Let \( C \) be an abelian \( n \)-fold monoidal category with common unit object \( A \). Consider the category of extensions \( \text{Ext}^k_C(A,A) \). The following Lemma is in a sense one of the main our observations.

**Key-lemma 3.9.** Under the above conditions, and the condition (*) in Section 2.1, the disjoint union of categories \( \bigoplus_{k \geq 1} \text{Ext}^k_C(A,A) \) is an \((n+1)\)-monoidal category.

**Proof.** Let \( \otimes_1, \ldots, \otimes_n \) be the (ordered) set of \( n \) monoidal structures in \( C \). Each of them, \( \otimes_s \), defines a monoidal structure \( \otimes_{s,\tau} \) (the Schwede’s tensor product), as is explained in Section 2.1.2. Set \( \otimes_i = \otimes_{i,\tau}, i = 1 \ldots n, \) and let \( \otimes_{n+1} \) be the Yoneda product of extensions (see (2.11)). The common identity object is the distinguished object in \( \text{Ext}^1_C(A,A) \) (that is, the extension \( 0 \rightarrow A \xrightarrow{id} A \rightarrow 0 \)). One needs to construct the maps \( \eta_{M,N,P,Q} \) for the pairs \( (\otimes_i, \otimes_{n+1}), 1 \leq i \leq n \), of monoidal structures. That is, we need to construct a map

\[
(3.11) \quad \eta_{M,N,P,Q}: (M\# N) \otimes_{i,\tau} (P\# Q) \rightarrow (M \otimes_{i,\tau} P)\# (N \otimes_{i,\tau} Q)
\]

The idea is shown in Figure 2. The idea is to project the whole rectangle (the l.h.s. of (3.11)) to the marked rectangles (the r.h.s. of (3.11)). Precisely, this map of complexes can be constructed as follows. Let \( M \) be an \( m \)-extension, \( N \) be an \( n \)-extension, etc. Let \( X(a) \otimes Y(b) \) be an element of the rectangle, where \( X \) is either \( M \) or \( N \), \( Y \) is either \( P \) or \( Q \). Note that \( M(1) = P(1) = A \), while \( N(1) = N_1 \) and \( Q(1) = Q_1 \). On the other hand, \( M(m+1) = M_m, P(p+1) = P_p, \) while \( N(n+1) = Q(q+1) = A \). Denote also by \( i_M: A \rightarrow M_1 \) the first arrow, and by \( p_M: M_m \rightarrow A \) the last arrow, analogously for \( N, P \) and \( Q \).

One should be especially careful with the elements in \( (M\# N) \otimes_{i,\tau} (P\# Q) \) shown by the bold points in Figure 3. The map \( \eta_{M,N,P,Q} \) is zero on the two white rectangles except these bold points.
Remark 3.10. In what follows we use two different indexations. The lower index $M_i$, $1 \leq i \leq m + 2$ denotes the $i$-th term of the extension $M$, and $M(i), 1 \leq i \leq m + 1$ denotes a coordinate of a point of the rectangle.

The map $\eta_{M,N,P,Q}$ is now defined, up to the total sign $(-1)^{np}$, as follows:

\begin{itemize}
  \item $M(a) \otimes P(b) \mapsto M(a) \otimes P(b)$ for $1 \leq a \leq m + 1, 1 \leq b \leq p + 1$
  \item $N(c) \otimes Q(d) \mapsto N(c) \otimes Q(d)$ for $1 \leq c \leq n, 1 \leq d \leq q$
  \item $N(n + 1) \otimes Q(d) \mapsto 0$ unless $d = q, q + 1$
  \item $N(c) \otimes Q(q + 1) \mapsto 0$ unless $c = n, n + 1$
  \item $N(n + 1) \otimes Q(q) \xrightarrow{id \otimes P_Q} A \otimes A \rightarrow A$
  \item $N(n) \otimes Q(q + 1) \xrightarrow{p_N \otimes id} A \otimes A \rightarrow A$
  \item $N(n + 1) \otimes Q(q + 1) \xrightarrow{id \otimes id} A \otimes A \rightarrow A$
  \item $M(a) \otimes Q(d) \mapsto 0$ unless $a = m + 1$
  \item $N(c) \otimes P(b) \mapsto 0$ unless $b = p + 1$
  \item $M(m + 1) \otimes Q(d) \xrightarrow{p_M \otimes id} A \otimes Q(d) = N_0 \otimes Q(d)$ for any $d$
  \item $N(c) \otimes P(p + 1) \xrightarrow{id \otimes P_P} N(c) \otimes A = N(c) \otimes Q_0$
\end{itemize}

\begin{align*}
(3.12)
\end{align*}

Remark 3.11. The last two lines of (3.12) are corresponded to the bold points on Figure 3.

Lemma 3.12. The map $\eta_{M,N,P,Q}$ defined as in (3.12) is a map of extensions.
Proof. The statement of Lemma means that the map \( \eta_{M,N,P,Q} \) is a map of complexes which is \( \text{id}_A \) on the both ends. This is clear.

Now we check that taking \( \sharp \) as \( \otimes_{n+1} \) as above, \( \eta_{M,N,P,Q} \) indeed satisfies the axioms (3.8), (3.9) and (3.10).

### 3.5.1 Check of (3.9)

We only need to check the commutativity of the diagram when the second monoidal structure \( \otimes_2 \) is the last one, the Yoneda product \( \sharp \). (If the both monoidal structures have numbers from 1 to \( n \), the commutativity follows automatically from the assumption that \( C \) is an \( n \)-monoidal category). Consider the case when \( \otimes_1 \) has any number from 1 to \( n \), and \( \otimes_2 = \sharp \).

The upper-right path of the diagram (3.9) is schematically drawn in Figure 4 below. The bold points are corresponded to the last two lines in the definition (3.12). Each of the two arrow “projects” the gray-color rectangle into two smaller gray rectangles. The bold points are the places outside the gray-color rectangles on which the “projection” is not zero.

![Figure 4](image1.png)

The lower-left path of the diagram (3.12) is shown in Figure 5. After the first map, we have some bold points which we did not have in Figure 4. However, they are mapped to the corresponding white points (see Figure 5), and therefore this difference is mapped to 0 under the second arrow.

We proved, that the diagram (3.9) is commutative in \( \prod_{k \geq 1} \text{Ext}_C^k(A, A) \).
3.5.2 Check of (3.8)

The check of (3.8) is analogous to the check of (3.9) above, where we should consider a $2 \times 2$ 3-dimensional cube instead of $3 \times 3$ 2-dimensional cube in (3.9).

3.5.3 Check of (3.10)

The check of (3.10) is a bit more tricky. At first, we suppose, as above, that $k = \sharp$. Here $i$ and $j$, $i < j$, may be arbitrary. We use the notation

\[(3.13) \quad (A_\alpha(\ell_1) \otimes_j B_\beta(\ell_2)) \otimes_i (C_\gamma(\ell_3) \otimes_j D_\delta(\ell_4))\]

where $\alpha, \beta, \gamma, \delta \in \{1, 2\}$ for a general element of the top of the diagram (3.10), $((A_1 \sharp A_2) \otimes_j (B_1 \sharp B_2)) \otimes_i ((C_1 \sharp C_2) \otimes_j (D_1 \sharp D_2))$.

Introduce some terminology. We say that an element (3.13) is \textit{regular}, if at least one of the maps in the (left-hand path of the) diagram belongs to the last two lines of (3.12) (the “bold points”), otherwise we call this element \textit{regular}. We prove the following

**Lemma 3.13.** Consider an element (3.13) on which the right-hand path of (3.10) is non-zero. If this element is regular, it is necessarily of the form

\[(3.14) \quad (A_\alpha(\ell_1) \otimes_j B_\alpha(\ell_2)) \otimes_i (C_\alpha(\ell_3) \otimes_j D_\alpha(\ell_4))\]

for $\alpha \in \{1, 2\}$. If this element is singular, it is necessarily of the form (3.13) such that some number $s \in \{1, 2, 3\}$ among the factors $A, B, C, D$ have the corresponding $\alpha$ equal to 1, and the corresponding value of $\ell_t$ at these factors is maximal possible (we denote it by max); the remaining $4 - s$ factors have necessarily the lower index $\beta = 2$, and their $\ell_q$ may be arbitrary. More precisely, these are elements of the following 3 groups:

\[(3.15)\]

\[\begin{align*}
(A_1(\max) \otimes_j B_2(\ell_1)) \otimes_i (C_2(\ell_2) \otimes_j D_2(\ell_3)) \\
(A_2(\ell_1) \otimes_j B_1(\max)) \otimes_i (C_2(\ell_2) \otimes_j D_2(\ell_3)) \\
(A_2(\ell_1) \otimes_j B_2(\ell_2)) \otimes_i (C_1(\max) \otimes_j D_2(\ell_3)) \\
(A_2(\ell_1) \otimes_j B_2(\ell_2)) \otimes_i (C_2(\ell_3) \otimes_j D_1(\max))
\end{align*}\]

\[(3.16)\]

\[\begin{align*}
(A_1(\max) \otimes_j B_1(\max)) \otimes_i (C_2(\ell_1) \otimes_j D_2(\ell_2)) \\
(A_1(\max) \otimes_j B_2(\ell_1)) \otimes_i (C_1(\max) \otimes_j D_2(\ell_2)) \\
(A_1(\max) \otimes_j B_2(\ell_1)) \otimes_i (C_2(\ell_2) \otimes_j D_1(\max)) \\
(A_2(\ell_1) \otimes_j B_1(\max)) \otimes_i (C_2(\ell_2) \otimes_j D_1(\max)) \\
(A_2(\ell_1) \otimes_j B_2(\ell_2)) \otimes_i (C_1(\max) \otimes_j D_1(\max))
\end{align*}\]
\[(A_2(\ell) \otimes_j B_1(\text{max})) \otimes_i (C_1(\text{max}) \otimes_j D_1(\text{max}))\]
\[(A_1(\text{max}) \otimes_j B_2(\ell)) \otimes_i (C_1(\text{max}) \otimes_j D_1(\text{max}))\]
\[(A_1(\text{max}) \otimes_j B_1(\text{max})) \otimes_i (C_2(\ell) \otimes_j D_1(\text{max}))\]
\[(A_1(\text{max}) \otimes_j B_1(\text{max})) \otimes_i (C_1(\text{max}) \otimes_j D_2(\ell))\]

(3.17)

The same statement is true for the left-hand path of (3.10).

**Proof.** Consider the right-hand path of diagram (3.10). We want to find all possible elements in the form \((A_{\alpha}(\ell_1) \otimes_j B_{\beta}(\ell_2)) \otimes_i (C_{\gamma}(\ell_3) \otimes_j D_{\delta}(\ell_4))\) which are mapped to non-zero elements by the right-hand path.

The right-hand path is the composition of 3 maps. The first map does not use (3.12), and, therefore, does not give any restrictions.

Consider the second map. Only the following expressions may map to nonzero elements.

Regular elements non-vanishing by the second map:

\[(A_{\alpha}(\ell_1) \otimes_i C_{\alpha}(\ell_2)) \otimes_j (B_{\beta}(\ell_3) \otimes_i D_{\beta}(\ell_4))\]

(3.18)

Here \(\alpha, \beta \in \{1, 2\}\), and \(\ell_s\) are arbitrary.

Singular elements non-vanishing by the second map:

\[(A_2(\ell_1) \otimes_i C_1(\text{max})) \otimes_j (B_{\alpha}(\ell_2) \otimes_i D_{\beta}(\ell_3))\]
\[(A_1(\text{max}) \otimes_i C_2(\ell_1)) \otimes_j (B_{\alpha}(\ell_2) \otimes_i D_{\beta}(\ell_3))\]
\[(A_{\alpha}(\ell_1) \otimes_i C_{\beta}(\ell_2)) \otimes_j (B_2(\ell_3) \otimes_i D_{1}(\text{max}))\]
\[(A_{\alpha}(\ell_1) \otimes_i C_{\beta}(\ell_2)) \otimes_j (B_1(\text{max}) \otimes_i D_{2}(\ell_3))\]

(3.19)

Here max is the maximal possible value of the current parameter. These elements are not necessarily non-vanishing, but for some \(\alpha, \beta, \ell_s\) they (and only they) may not vanish.

Now some of these elements will necessarily vanish by the third map.

Regular elements which may not vanish by the third map:

\[(A_{\alpha}(\ell_1) \otimes_i C_{\alpha}(\ell_2)) \otimes_j (B_{\alpha}(\ell_3) \otimes_i D_{\alpha}(\ell_4))\]

(3.20)

Singular elements which may not vanish by the third map:

\[(A_1(\text{max}) \otimes_i C_1(\text{max})) \otimes_j (B_2(\ell_1) \otimes_i D_2(\ell_2))\]
\[(A_2(\ell_1) \otimes_i C_2(\ell_2)) \otimes_j (B_1(\text{max}) \otimes_i D_1(\text{max}))\]

(3.21)
Conclusion:

We see from (3.20) that the only regular non-vanishing elements are exactly those for which all indices $\alpha, \beta, \gamma, \delta$ are equal (exactly as is stated by Lemma).

Now consider the non-vanishing singular elements.

We distinguish simple singular elements (those for which only one of the two last maps of the right-hand path of (3.10) are in singular position), and singular elements of the second order (which are singular for the both maps).

Simple singular elements:

Suppose we have an element (3.13) for which the second map is regular and the third is singular. Then clearly it is of the form (3.21), and only them (totally 2 elements). They are corresponded to the second and to the fifth lines in (3.16).

Suppose we have an element (3.13) for which the second map is singular and the third is regular. This case is corresponded to the elements (3.19) with either $\alpha = \beta = 2$, or one of indices $\alpha, \beta$ is equal to 1 and the corresponding $\ell_s = \max$, and the remaining index is 2. In the both cases the third map is regular. Indeed, the elements $X_1(\max) \ (X = A, B, C, D)$ with the lower index 1 will be mapped to $X_2(\ell)$ (for $\ell = 1$, but it is not essential). Then we get an element of the form (3.20) with $\alpha = 2$ which is regular for the third map. The first alternative gives the whole group (3.15), while the second alternative gives the remaining 4 elements (all lines except the second and the fifth) in (3.16).

Singular elements of the second order:

To get singular elements for the third maps (3.21) from the singular elements for the second map (3.19) we necessarily should have in (3.19) $\alpha = \beta = 1$, and the corresponding two $\ell_s = \max$. This gives the whole third group (3.17).

The Lemma is proven for the right-hand path.

The analysis of the left-hand path is fairly analogous, and we leave it to the reader. □

Now we can prove

**Lemma 3.14.** The diagram (3.10) is commutative.

**Proof.** We know from Lemma 3.13 the list of all elements on which the left-hand path and the right-hand path of (3.10) do not vanish, and we know that this list is precisely the same for the both pathes. Now the proof is a straightforward check that the both pathes coincide on any of these elements.

For the regular elements (which are the same for the both pathes) the statement is clear (it follows from the diagrams (3.8) and (3.9) in the underlying $n$-fold monoidal category $\mathcal{C}$). For the singular elements (3.15)-(3.17) it is a direct and a routine check; we leave it to the reader avoiding to make longer already a very long and technical proof of Key-Lemma 3.9. □
4 The category of tetramodules

4.1

Recall that an associative bialgebra is a vector space $A$ over a field $k$ equipped with two operations, the product $*: A^\otimes 2 \to A$ and the coproduct $\Delta: A \to A^\otimes 2$, which obey the axioms 1.-4. below:

1. Associativity: $a * (b * c) = (a * b) * c$;
2. Coassociativity: $(\Delta \otimes \text{id})\Delta(a) = (\text{id} \otimes \Delta)\Delta(a)$;
3. Compatibility: $\Delta(a * b) = \Delta(a) * \Delta(b)$.

We use the classical notation

$$\Delta(a) = \Delta^1(a) \otimes \Delta^2(a)$$

which is just a simplified form of the equation

$$\Delta(a) = \sum_i \Delta^1_i(a) \otimes \Delta^2_i(a)$$

We always assume that our bialgebras have a unit and a counit. A unit is a map $i: k \to A$ and the counit is a map $\varepsilon: A \to k$. We always assume

4. $i(k_1 \cdot k_2) = i(k_1) * i(k_2)$, $\varepsilon(a * b) = \varepsilon(a) \cdot \varepsilon(b)$.

We also denote the product $*$ by $m$.

A Hopf algebra is a bialgebra with antipode. An antipode is a $k$-linear map $S: A \to A$ which obeys

5. $m(1 \otimes S)\Delta(a) = m(S \otimes 1)\Delta(a) = i(\varepsilon(a))$

We need the existence of antipode in only one, but a crucial place, in Section 4.3. All results of Sections 4.1 and 4.2 do not need the existence of antipode.

As we already mentioned here, an “operadic” definition of a bimodule over an associative algebra $A$ is a $k$-vector space $M$ such that $A \oplus \varepsilon M$ is again associative algebra, where $\varepsilon^2 = 0$, and the restriction of the algebra structure to $A$ coincides with the initial one. Such $M$ is the same that an $A$-bimodule. We give an analogous definition in the case when $A$ is an associative bialgebra.
Definition 4.1. Let $A$ be an associative bialgebra. A Bernstein-Khovanova tetramodule $M$ over $A$ is a vector space such that $A \oplus \epsilon M$ is an associative bialgebra when $\epsilon^2 = 0$ and the restriction of the bialgebra structure to $A$ is the initial one. The category of tetramodules over a bialgebra $A$ is denoted $\text{Tetra}(A)$.

More precisely, one has maps $m_\ell : A \otimes M \to M$, $m_r : M \otimes A \to M$ (which make $M$ an $A$-bimodule), and maps $\Delta_\ell : M \to A \otimes M$ and $\Delta_r : M \to M \otimes A$ (which make $M$ an $A$-bicomodule), with some compatibility between these 4 maps. The compatibility written explicitly is the following:

\[(4.1)\quad \Delta_\ell(a \ast m) = (\Delta^1(a) \ast \Delta^1_\ell(m)) \otimes (\Delta^2(a) \ast \Delta^2_\ell(m)) \subset A \otimes_k M\]

\[(4.2)\quad \Delta_\ell(m \ast a) = (\Delta^1_\ell(m) \ast \Delta^1(a)) \otimes (\Delta^2(m) \ast \Delta^2_\ell(a)) \subset A \otimes_k M\]

\[(4.3)\quad \Delta_r(a \ast m) = (\Delta^1(a) \ast \Delta^1_r(m)) \otimes (\Delta^2(a) \ast \Delta^2_r(m)) \subset M \otimes_k A\]

\[(4.4)\quad \Delta_r(m \ast a) = (\Delta^1_\ell(m) \ast \Delta^1(a)) \otimes (\Delta^2(m) \ast \Delta^2_r(a)) \subset M \otimes_k A\]

Here we use the natural notation like $\Delta_\ell(m) = \Delta^1_\ell(m) \otimes \Delta^2_\ell(m)$ with $\Delta^1_\ell(m) \in A$, $\Delta^2_\ell(m) \in M$, etc. As well, we use the sign $\ast$ for the both product in $A$ and the module products $m_\ell$ and $m_r$.

The main example of a tetramodule over $A$ is a itself; it is called the tautological tetramodule.

When $A$ is finite-dimensional over $k$, a tetramodule is the same that a left module over some associative algebra $H(A)$. This algebra $H(A)$ is, as a vector space, the tensor product $H^0(A) = A \otimes_k A \otimes_k A^* \otimes A^*$, and the commutation relation are such that the equations (4.1)-(4.4) above agree. (This algebra $H(A)$ is the “double Heisenberg double” of the bialgebra $A$).

In particular, if $A$ is finite-dimensional over $k$, the category $\text{Tetra}(A)$ has enough projectives and enough injectives objects. For general $A$, R.Taillefer proved [Tai2] that the category $\text{Tetra}(A)$ has enough injectives.

The main relation with the deformation theory is the following theorem, proven by R.Taillefer [Tai1,2]:

Theorem 4.2. Let $H^r_{\text{GS}}(A, A)$ denote the Gerstenhaber-Schack cohomology of an associative bialgebra $A$. Then one has:

\[(4.5)\quad H_r^{\text{GS}}(A, A) = \text{Ext}_{\text{Tetra}(A)}^r(A, A)\]
The Gerstenhaber-Schack cohomology $H^q_{GS}(A, A)$ is known to control the infinitesimal deformations of the bialgebra $A$ [GS]. We overview the Gerstenhaber-Schack cohomology and prove Theorem 4.2 in the Appendix Section 6.

**Remark 4.3.** To control the global deformations of a bialgebra $A$ through the usual deformation theory (the Maurer-Cartan equation etc.) one needs to have an appropriate $L_\infty$ structure on the Gerstenhaber-Schack complex. (In this case it will be an $L_\infty$ structure with the components at least up to 4th, because the r.h.s. of the structure equation of bialgebra, $\Delta(a * b) = \Delta(a) * \Delta(b)$, is of 4th degree in the operations. This structure is not known yet. What we construct in Section 4.3 is a Lie bracket which presumably is induced on the cohomology from this structure.

**Example 4.4.** Consider the case when $A = S(V)$ is a free (co)commutative bialgebra, for simplicity suppose $V$ is finite-dimensional over $k$. We prove in Section 6 that

\begin{equation}
H^k_{GS}(A, A) = \oplus_{i+j=k} \Lambda^i V \otimes_k \Lambda^j V^*
\end{equation}

The wedge-product defines on the r.h.s. a super-commutative product of degree 0, and the contraction of $V$ and $V^*$ defines a Lie bracket of degree -2. Clearly the bracket is Poisson and even, it does not cause an additional sign. The product and the bracket obey the even Leibniz compatibility. Altogether it defines a 3-algebra structure on $H^*_{GS}(A, A)$ for $A = S(V)$. In this paper we construct such a structure for any bialgebra $A$.

### 4.2 The structure of a 2-fold monoidal category on Tetra($A$)

#### 4.2.1 Two “external” tensor products

Define firstly for a pair of $A$-tetramodules $M_1, M_2$ two their “external” tensor products $M_1 \boxtimes_1 M_2$ and $M_1 \boxtimes_2 M_2$ which are again $A$-tetramodules. In the both cases the underlying vector space is $M_1 \otimes_k M_2$.

The case of $M_1 \boxtimes_1 M_2$:

1. $m_\ell(a \otimes m_1 \boxtimes m_2) = (am_1) \boxtimes m_2$,
2. $m_r(m_1 \boxtimes m_2 \otimes a) = m_1 \boxtimes (m_2a)$,
3. $\Delta_\ell(m_1 \boxtimes m_2) = (\Delta^1_\ell(m_1) \otimes \Delta^1_\ell(m_2)) \otimes (\Delta^2_\ell(m_1) \boxtimes \Delta^2_\ell(m_2))$,
4. $\Delta_r(m_1 \boxtimes m_2) = (\Delta^1_\ell(m_1) \boxtimes \Delta^1_\ell(m_2)) \otimes (\Delta^2_\ell(m_1) \otimes \Delta^2_\ell(m_2))$.

The case of $M_1 \boxtimes_2 M_2$:
1. \( m_\ell(a \otimes m_1 \boxtimes m_2) = (\Delta^1(a)m_1) \boxtimes (\Delta^2(a)m_2) \),
2. \( m_r(m_1 \boxtimes m_2 \otimes a) = (m_1 \Delta^1(a)) \boxtimes (m_2 \Delta^2(a)) \),
3. \( \Delta_\ell(m_1 \boxtimes m_2) = \Delta^1_\ell(m_1) \otimes (\Delta^2_\ell(m_1) \boxtimes m_2) \),
4. \( \Delta_r(m_1 \boxtimes m_2) = (m_1 \boxtimes \Delta^1_r(m_2)) \otimes \Delta^2_r(m_2) \).

For the both definitions we do not use the whole tetramodule structures on \( M_1, M_2 \). In particular, for the first definition we never use the right multiplication \( m_r \) for \( M_1 \) and the left multiplication \( m_\ell \) for \( M_2 \). As well, for the second definition we do not use \( \Delta_r \) for \( M_1 \) and \( \Delta_\ell \) for \( M_2 \).

This gives us some additional possibilities, which we use to define the “internal” tensor products \( M_1 \otimes_1 M_2 \) and \( M_1 \otimes_2 M_2 \). For the both cases the tautological tetramodule is a unit object.

### 4.2.2 Two “internal” tensor products

**Definition 4.5.** Let \( M_1, M_2 \) be two tetramodules over a bialgebra \( A \). Their first tensor product \( M_1 \otimes_1 M_1 \) is defined as the quotient-tetramodule

\[
M_1 \otimes_1 M_2 = M_1 \boxtimes_1 M_2 \big/ ((m_1 a) \boxtimes_1 m_2 - m_1 \boxtimes_1 (am_2))
\]

One easily checks that this definition is correct. Analogously, the second tensor product \( M_1 \otimes_2 M_2 \) is defined as a sub-tetramodule

\[
M_1 \otimes_2 M_2 = \{ \sum m_{1i} \boxtimes_2 m_{2i} \subset M_1 \boxtimes_2 M_2 | \sum \Delta_r(m_{1i}) \otimes_k m_{2i} = \sum m_{1i} \otimes_k \Delta_\ell(m_{2i}) \}
\]

Again, one easily checks that this definition is correct.

**Lemma 4.6.** Suppose that the bialgebra \( A \) has a unit and a counit. Then the tautological tetramodule \( A \) is the unit object for the both monoidal structures.

**Proof.** Let \( M \) be a tetramodule. One can check that the following maps

\[
m_\ell: A \otimes_1 M \to M
\]

\[
m_r: M \otimes_1 A \to M
\]

and

\[
\Delta_\ell: M \to A \otimes_2 M
\]

\[
\Delta_r: M \to M \otimes_2 A
\]

are morphisms of tetramodules. If \( A \) has a unit, the first two maps are isomorphisms, while if \( A \) has a counit, the second ones two are. \( \square \)
4.2.3 The 2-fold monoidal structure

We construct the map \( \eta_{M,N,P,Q} : (M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \to (M \otimes_1 P) \otimes_2 (N \otimes_1 Q) \) in several steps. The first step is to check that the map \( \phi_0 : (M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \to (M \otimes_1 P) \otimes_2 (N \otimes_1 Q) \), \( \phi_0(m \otimes_k n \otimes_k p \otimes_k q) = m \otimes_k p \otimes_k n \otimes_k q \), is a map of tetramodules. We have:

\[
\begin{align*}
\Delta_1(a) * m & \otimes (\Delta_2(a) * n) \otimes p \otimes q \\
\Delta_1(a) * m & \otimes \Delta_2(a) * n \otimes p \otimes q \\
\end{align*}
\]

and

\[
\begin{align*}
\Delta_1(a) * (m \otimes_1 p) \otimes_2 (n \otimes_1 q) & = \\
(\Delta_1(a) * (m \otimes_1 p)) \otimes_2 (\Delta_2(a) * (n \otimes_1 q)) & = \\
(\Delta_1(a) * m) \otimes p \otimes (\Delta_2(a) * n) \otimes q \\
\end{align*}
\]

We see that

\[
\phi_0(\text{r.h.s. of (4.11)}) = \text{r.h.s. of (4.12)}
\]

That is, \( \phi_0 \) is a map of left modules; analogously it is a map of right modules.

Now prove that \( \phi_0 \) is a map of left comodules. We have:

\[
\begin{align*}
\Delta_\ell((m \otimes_2 n) \otimes_1 (p \otimes_2 q)) & = \\
\Delta_\ell(m \otimes_2 n) \otimes_1 \Delta_\ell(p \otimes_2 q) & \otimes_k \left( \Delta_\ell^2(m \otimes_2 n) \otimes_1 \Delta_\ell^2(p \otimes_2 q) \right) = \\
\end{align*}
\]

and

\[
\begin{align*}
\Delta_\ell((m \otimes_1 p) \otimes_2 (n \otimes_1 q)) & = \\
\Delta_\ell^1(m \otimes_1 p) \otimes_2 \left( \Delta_\ell^2(m \otimes_1 p) \otimes_2 (n \otimes_1 q) \right) & = \\
\end{align*}
\]

We see that

\[
\Delta_\ell \circ \phi_0 = id \otimes_k (\phi_0 \circ \Delta_\ell)
\]

that is, \( \phi_0 \) is a map of left comodules. It is proven analogously that \( \phi_0 \) is a map of right comodules.
At the second step we consider the natural projections of tetramodules \( p_{M,P} : M \boxtimes_1 P \to M \otimes_1 P \) and \( p_{N,Q} : N \boxtimes_1 Q \to N \otimes_1 Q \). We consider the composition

\[
\phi_1 = (p_{M,P} \boxtimes_2 p_{N,Q}) \circ \phi_0 : (M \boxtimes_2 N) \boxtimes_1 (P \boxtimes_2 Q) \to (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q)
\]

We want to check that the map \( \phi_1 \) defines naturally a map

\[
\phi_2 = \overline{\phi}_1 : (M \boxtimes_2 N) \otimes_1 (P \boxtimes_2 Q) \to (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q)
\]

that is, the elements of the form

\[
((m \boxtimes_2 n) \ast a) \otimes_k (p \boxtimes_2 q) - (m \boxtimes_2 n) \otimes_k (a \ast (p \boxtimes_2 q))
\]

are mapped to 0 by \( \phi_1 \).

Indeed,

\[
(4.19) \quad (m \ast \Delta^1 a) \otimes_k (n \ast \Delta^2 a) \otimes_k (p \otimes_k q) - (m \otimes_k n) \otimes_k (\Delta^1 a \ast p) \otimes_k (\Delta^2 a \ast q)
\]

which, after the permutation \( \phi_0 \) of the second and the third factors, is mapped to 0 in \( (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q) \). Therefore, the map \( \phi_2 \) is well-defined.

At the third step we restrict the map \( \phi_2 \) to \( (M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \subset (M \boxtimes_2 N) \otimes_1 (P \boxtimes_2 Q) \), and we need to check that the image of this restricted map belongs to \( (M \otimes_1 P) \otimes_2 (N \otimes_1 Q) \subset (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q) \).

Suppose \( m \otimes_k n \in M \otimes_2 N \) and \( p \otimes_k q \in P \otimes_2 Q \) (we assume the summation over several such monomials, but for simplicity we skip this summation). Then

\[
\Delta^1(m) \otimes_k \Delta^2(m) \otimes_k n = m \otimes_k \Delta^1(n) \otimes_k \Delta^2(n)
\]

with the middle factors in \( A \), and analogously

\[
\Delta^1(p) \otimes_k \Delta^2(p) \otimes_k q = p \otimes_k \Delta^1(q) \otimes_k \Delta^2(q)
\]

again, with the middle factors in \( A \).

One needs to prove that \((4.21)\) and \((4.22)\) together imply that

\[
(4.23) \quad (m \boxtimes_1 p) \boxtimes_2 (n \boxtimes_1 q) \in (M \otimes_1 P) \otimes_2 (N \otimes_1 Q) \subset (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q)
\]

that is,

\[
\Delta^1(m) \otimes_k \Delta^2(p) \otimes_k (\Delta^2(m) \ast \Delta^2(p)) \otimes_k n \otimes_k q = m \otimes_k p \otimes_k (\Delta^1(n) \ast \Delta^1(q)) \otimes_k \Delta^2(n) \otimes_k \Delta^2(q)
\]

To get \((4.24)\) from \((4.22)\) and \((4.23)\) we permute \((4.22)\) such that the factors in \( A \) are the most right, permute \((4.22)\) such that the factors in \( A \) are the most left, then take the equation
\[(\text{l.h.s. of (4.22)}) \otimes_k (\text{l.h.s. of (4.23)}) = (\text{r.h.s. of (4.22)}) \otimes_k (\text{r.h.s. of (4.23)}) \text{ (after they are permuted). Then for the two middle factor (in A) we apply the product } \ast: A \otimes_k A \rightarrow A, \text{ and then permute again.}\]

The map \(\eta_{M,N,P,Q}\) is constructed.

**Theorem 4.7.** The maps \(\eta_{M,N,P,Q}\) constructed above, together with the two tensor products \(\otimes_1\) and \(\otimes_2\), define a 2-fold monoidal structure on the category \(\text{Tetra}(A)\) of tetramodules over a bialgebra \(A\).

Proof. First of all, the two tensor products \(\boxtimes_1\) and \(\boxtimes_2\) with \(\bar{\eta}_{M,N,P,Q} = \phi_0\), clearly forms a 2-fold monoidal structure on the category \(\text{Tetra}(A)\) (but this 2-fold monoidal structure does not admit unit elements). In particular, the diagrams (3.8) and (3.9) are commutative for \(\bar{\eta}_{M,N,P,Q}\) (because \(\phi_0\) is just the permutation which switches the second and the third factors). Now the same diagrams for the actual structure \(\eta_{M,N,P,Q}\) are obtained from these simple ones by passing to subquotients. Therefore, they are commutative as well. □

### 4.3 The condition (*) for \(\text{Tetra}(A)\)

Recall the meaning of the condition (*): we want to restrict ourselves with a homotopy equivalent to \(\text{Ext}^k(A, A)\) subcategories on which the monoidal structures are exact. This condition is essential in the Schwede’s construction described in Section 2, and in its generalization described in the sequel Section 4.4.

We have two monoidal structures on \(\text{Tetra}(A)\), namely \(\otimes_1\) and \(\otimes_2\). We leave to the reader the following simple statement:

**Lemma 4.8.** The monoidal bifunctor \(\otimes_1\) is right exact, and the monoidal bifunctor \(\otimes_2\) is left exact. □

The author does not know, in this setting, how to construct a homotopically equivalent subcategory on which the both monoidal structures are exact, if we work with general bialgebras. It seems that the construction of [Sch], Lemma 2.1 can not be adopted to our situation.

But when we suppose that our bialgebra is a Hopf algebra, that is, it obeys an antipode, the situation is much better because of the following classical result:

**Lemma 4.9.** Let \(A\) be a Hopf algebra (that is, a bialgebra with an antipode), and let \(M\) be a \(k\)-vector space with structures of left \(A\)-module \(m_\ell: A \otimes M \rightarrow M\), and of left \(A\)-comodule \(\Delta_\ell: M \rightarrow A \otimes M\), which are compatible as

\[(4.25) \quad \Delta_\ell(a \ast m) = (\Delta^1(a) \ast \Delta^1_\ell(m)) \otimes (\Delta^2(a) \ast \Delta^2_\ell(m))\]
(this is just a $\frac{1}{3}$ of the structure of a tetramodule). Then $M$ is necessarily free as $A$-module and cofree as $A$-comodule.

Proof. See [Sw], Section 4.1 for a proof of the first claim; the second claim is dual to the first one. □

We see from the Lemma above that the both monoidal structures $\otimes_1$ and $\otimes_2$ are exact on the nose, so there is no necessity to restrict by a smaller category.

For this we should include the existence of antipode condition to all statements which use the condition (*). That is, in all such cases we should work not with general bialgebras, but with Hopf algebras.

4.4 A generalization of the Schwede’s construction

Let $A$ be a Hopf algebra. Here we construct a Lie bracket of degree -2 on the graded space $\text{Ext}_{\text{Tetra}(A)}^q(A, A)$ generalizing the Schwede’s construction of the Lie bracket of degree -1 on the Hochschild cohomology, see Section 2.2.

Let $M, N$ be two extensions of tetramodules, $M \in \text{Ext}_{\text{Tetra}(A)}^m(A, A)$, $N \in \text{Ext}_{\text{Tetra}(A)}^n(A, A)$. Consider the following “big octahedron” diagram. We prove just below in Lemma that the triangle 2-faces in (4.26) are commutative. That is, the diagram defines an element in $\pi_2(\text{Ext}_{\text{Tetra}(A)}^{m+n}(A, A))$. By Retakh’s theory (see Section 1) the latter is isomorphic to $\text{Ext}^{m+n-2}(A, A)$. This is the construction of the bracket. A priori it is not clear that this operation satisfies the Jacobi identity, it follows from much deeper results of Section 5.
Lemma 4.10. All triangle 2-faces in (4.26) are commutative diagrams.

Proof. We prove a more general statement. Given an $n$-fold monoidal category, let $1 \leq i < j < k \leq n$. Consider the maps $\varphi_{ij} : M \otimes_i N \to M \otimes_j N$, $\varphi_{jk} : M \otimes_j N \to M \otimes_k N$, and $\varphi_{ik} : M \otimes_i N \to M \otimes_k N$, defined in (3.6). Recall that $\varphi_{ij} = \eta^{ij}_{M,N,A,A}$ where $A$ is the common unit object. We prove that

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$$

This is an application of the commutative diagram (3.10) (this diagram is indeed commutative for $n = 3$ and $\otimes_3 = \sharp$ by the Key-Lemma 3.9). Consider the diagram (3.10) for $A_1 = M$, $A_2 = B_1 = B_2 = C_1 = C_2 = D_1 = A$, $D_2 = N$. Then the right-hand path of (3.10) gives $\varphi_{jk} \circ \varphi_{ij}$ while the left-hand path gives $\varphi_{ik}$, and (4.27) follows. This proves the commutativity of one of the 8 triangle faces in (4.26). The commutativity of the remaining 7 triangles is proven analogously, using $\theta_{ij} : M \otimes_i N \to N \otimes_j M$ defined in (3.7), and different substitutions to the diagram (3.10).

Remark 4.11. This lemma, and the commutativity of the triangle faces in (4.26), is a nice example on (a rather complicated) diagram (3.10), and a motivation for a rather technical proof of the Key-Lemma 3.9.
Remark 4.12. It follows from the existence of the big octahedron that the “bracket” of degree -1 defined as in Section 2.2 either from $\otimes_1$ or $\otimes_2$ separately, is equal to 0. In particular, the Lie bracket defined in [Ta2], Section 5, is identically 0. The idea is that in the presence of two monoidal structures compatible as in 2-fold monoidal category, the Schwede’s loop (2.3) is “divided” by 4 commutative triangles, and therefore is contractible in the nerve. Morally, to get non-trivial operations, we should take into account all possible monoidal structures.

5 Passage to Spectra

5.1 From “spectra” of categories to spectra of topological spaces

5.1.1 Appearance of a problem

There is an operad of categories $\{M_{n+1}(d)\}$ acting on each $(n + 1)$-fold monoidal category and, in particular, on $\bigsqcup_{k>0} \mathcal{E}_{xt}^k(A, A)$, where $\mathcal{C}$ is an $n$-fold abelian monoidal category satisfying the condition (*), and $A$ is a common unit object in it (see Section 3). Moreover, there are “spectrum structure maps” $G_k: \mathcal{E}_{xt}^k(A, A) \to \Omega_R \mathcal{E}_{xt}^{k+1}(A, A)$ (see Section 1.3). Our goal is to pass, by the classifying space functor, to an $\Omega$-spectrum $X$ of topological spaces with $X_k = |\mathcal{E}_{xt}^k(A, A)|$ such that the operad of spectra $\Sigma^\infty M_{n+1}(d)$ acts on this spectrum (after suitable reducing of the both spectra to $\Sigma$-spectra).

This compatibility of the operad action with the structure maps of spectra is more natural (and, seemingly, only possible) to prove on the level of categories. Here the problem we meet is the following. It is natural to consider the spectrum of based topological spaces. Therefore, the corresponding spectrum of categories should be also considered as based. But our operad is not compatible with any sense of the based objects, in the sense that the $n + 1$ monoidal structures do not give a based object if one of two its arguments is a based object.

More precisely, we would like to prove the commutativity of the following diagram of functors:

\[
\begin{array}{ccc}
\mathcal{E}_{xt}^k \times \mathcal{E}_{xt}^\ell & \xrightarrow{\otimes_1} & \mathcal{E}_{xt}^{k+\ell} \\
G_k \otimes id & \downarrow & G_k \\
(\Omega_R \mathcal{E}_{xt}^{k+1}) \times \mathcal{E}_{xt}^\ell & \xrightarrow{\otimes_1} & \Omega_R \mathcal{E}_{xt}^{k+\ell+1}
\end{array}
\]

However, this diagram does not make sense, because the lower horizontal arrow is ill-defined. Indeed, if $* \to X \leftarrow *$ is and element in $\Omega_R$, we would like to induce from any monoidal structure the component-wise product with $Y$, which should be $* \otimes Y \to X \otimes Y \leftarrow * \otimes Y$. The point is that this element is not an object of $\Omega_R$ because $* \otimes Y \neq *$. 

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The origin of this problem is that the \( n+1 \) monoidal structures \( \otimes_i : \mathcal{E}_{\text{xt}}^a(A, A) \times \mathcal{E}_{\text{xt}}^b(A, A) \to \mathcal{E}_{\text{xt}}^{a+b}(A, A) \) (defined for an \( n \)-fold monoidal abelian category \( \mathcal{C} \)), do not respect the based objects. Consequently, passing to the nerves we get a map \( \otimes_i : B(\mathcal{E}_{\text{xt}}^a(A, A)) \times B(\mathcal{E}_{\text{xt}}^b(A, A)) \to B(\mathcal{E}_{\text{xt}}^{a+b}(A, A)) \), and not a map \( \otimes_i : B(\mathcal{E}_{\text{xt}}^a(A, A)) \wedge B(\mathcal{E}_{\text{xt}}^b(A, A)) \to B(\mathcal{E}_{\text{xt}}^{a+b}(A, A)) \) (where \( \wedge \) is as usual the direct product in the category of based topological spaces). This circumstance makes impossible to use directly the smash-products of spectra, which we wish to use, in order to pass to the operad action on homotopy groups of the spectrum \( \{B(\mathcal{E}_{\text{xt}}^k(A, A))\} \).

### 5.1.2

The idea is to replace each category \( \mathcal{E}_{\text{xt}}^k(A, A) \) by a pair of a category and its subcategory, which has the same homotopy groups, as well as possess the \( (n+1) \)-fold monoidal structure on the corresponding disjoint unit. That is, we replace the based objects (which do not exist in a way compatible with monoidal structures) by based subcategories.

For any category \( \mathcal{E}_{\text{xt}}^k(A, A) \) define the category \( \Omega_{\text{free}} \mathcal{E}_{\text{xt}}^k(A, A) \) as follows. Its objects are “free loops” of length 1

\[
(5.2) \quad X \to Y \leftarrow X
\]

and the morphisms are the natural commutative diagrams.

This category has a full subcategory \( \Omega_{\text{free}} \mathcal{E}_{\text{xt}}^k(A, A) \) consisting of

\[
(5.3) \quad X \xrightarrow{id} X \xleftarrow{id} X
\]

clearly this category is isomorphic to \( \mathcal{E}_{\text{xt}}^k(A, A) \).

We claim the following:

**Key-lemma 5.1.** All homotopy groups of the pair \( \pi_i(\Omega_{\text{free}} \mathcal{E}_{\text{xt}}^k(A, A), \Omega_{\text{free}} \mathcal{E}_{\text{xt}}^k(A, A)) \) are isomorphic to \( \pi_{i+1}(\mathcal{E}_{\text{xt}}^k(A, A)) \).

We give a proof of this Lemma a bit later. Now let us explain how it can help us.

Define for that another category of free double loops, \( \Omega_{\text{free}}^2 \mathcal{E}_{\text{xt}}^k(A, A) \). Its object is a diagram

\[
(5.4) \quad \begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
Z & \rightarrow & W \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \\
\end{array}
\]
and the morphisms are natural commutative diagrams.

This category contains a full subcategory $\Omega_n^{free} Ext^k(A, A)$ forming from diagrams (5.4) such that either the vertical or the horizontal lines are of the form (5.3).

Fix now a monoidal structure on $\bigcup Ext^k(A, A)$. Then it defines a products on pairs of categories

$$\Omega_n^{free} Ext^k(A, A) \times \Omega_n^{free} Ext^b(A, A) \rightarrow (\Omega_n^{free} Ext^{a+b}(A, A), \Omega_n^{free} Ext^k(A, A))$$

Thus, at this point we achieved our goal for introduction of “based points” in the setting, which is necessary for the using of smash-products of spectra.

We can analogously define the categories $\Omega_n^{d} Ext^k(A, A)$ and $\Omega_n^{d} Ext^k(A, A)$. We have

**Key-lemma 5.2.** The homotopy groups of the pair $\pi_i(\Omega_n^{d} Ext^k(A, A), \Omega_n^{d} Ext^k(A, A))$ is isomorphic for any $d$ to $\pi_{i+d}(Ext^k(A, A))$.

### 5.2 A proof of Key-Lemma 5.1

#### 5.2.1 A topological counterpart

Before proving the Key-Lemma let us explain what does it mean topologically. Let $X$ be a topological space, and let $\Omega^{free} X$ be its free loop space. It has a complicated homotopical type. Nevertheless, one has

$$\pi_i(\Omega^{free} X) = \pi_i X \oplus \pi_i \Omega X$$

where $\Omega X$ is the based loop space.

Indeed, there is a fibration $p: \Omega^{free} X \rightarrow X$ with the fiber $\Omega X$. It has a canonical section, sending $x \in X$ to the constant loop based at $x$. Therefore, the long exact sequence of the fibration splits, which gives the result.

Now we prove

**Proposition 5.3.** $\pi_i(\Omega^{free} X, X) = \pi_i \Omega X$ for any $i$.

**Proof.** Consider the embedding $i: X \rightarrow \Omega^{free} X$. According to general principles of the Eckmann-Hilton duality,

$$\pi_i(\Omega^{free} X) = \pi_{i-1} P$$

where $P$ is the homotopy fiber of the embedding $i$. Let us compute this fiber. Its point is a path in $\Omega^{free} X$ from two fixed points belonging to the image $i(X)$. That is, it is a based 2-sphere in $X$, or an element of $\Omega^2 X$. We get from (5.7):

$$\pi_i(\Omega^{free} X) = \pi_{i-1} \Omega^2 X = \pi_i \Omega X = \pi_{i+1} X$$

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5.2.2 A categorical counterpart

5.3 Passage to spectra

Consider the following “spectrum of categories” $\mathcal{X}$:

$$
\mathcal{X}_0 = \mathsf{Ext}^0(A, A), \quad \mathcal{X}_1 = \Omega_{\text{free}}\mathsf{Ext}^2(A, A), \quad \ldots, \quad \mathcal{X}_n = \Omega_{\text{free}}^n\mathsf{Ext}^{2n}(A, A), \quad \ldots
$$

First of all, we should explain in which sense $\mathcal{X}$ is a spectrum.

One has the Retakh’s map $G: \mathsf{Ext}^k(A, A) \to \Omega_{\text{free}}^k\mathsf{Ext}^{k+1}(A, A)$, see Section 1.3.5. Applying this map twice, and the embedding $\Omega \hookrightarrow \Omega_{\text{free}}$, we get a map $\sigma: \mathcal{X}_k \to \Omega_{\text{free}}\mathcal{X}_{k+1}$. These are the structure maps of our spectrum $\mathcal{X}$.

The disjoint union of categories $\bigsqcup \mathcal{X}_k$ is an $(n+1)$-monoidal category in a natural way, when $\mathcal{C}$ is an abelian $n$-fold monoidal category satisfying the condition $(\ast)$.

Moreover, there is a subspectrum $\mathcal{X}^0 \subset \mathcal{X}$, where

$$
\mathcal{X}_n^0 = \Omega_{\text{free}}^n\mathsf{Ext}^{2n}(A, A)
$$

such that each monoidal structure is a map

$$
(5.9) \quad \otimes_i: (\mathcal{X}_k^0, \mathcal{X}_k^0) \times (\mathcal{X}_\ell^0, \mathcal{X}_\ell^0) \to (\mathcal{X}_{k+\ell}^0, \mathcal{X}_{k+\ell}^0)
$$

Now is the question: how these monoidal structures are compatible with the spectrum structure maps $\sigma$? More precisely, consider the diagram

$$
(5.10) \quad (\mathcal{X}_k^0, \mathcal{X}_\ell^0) \times (\mathcal{X}_\ell^0, \mathcal{X}_\ell^0) \xrightarrow{\otimes_i} (\mathcal{X}_{k+\ell}^0, \mathcal{X}_{k+\ell}^0)
$$

$$
\begin{array}{c}
\sigma \times \text{id} \\
\downarrow \\
(\Omega_{\text{free}}(\mathcal{X}_k), \Omega_{\text{free}}(\mathcal{X}_k^0)) \times (\mathcal{X}_\ell^0, \mathcal{X}_\ell^0) \otimes_i (\Omega_{\text{free}}(\mathcal{X}_{k+\ell}), \Omega_{\text{free}}(\mathcal{X}_{k+\ell}^0))
\end{array}
$$

**Lemma 5.4.** The diagram (5.10), and the analogous diagram for $\sigma$ applied to the right factor on the left arrow, is for any $i$ homotopically commutative. The latter means that after applying of the classifying space functor, the corresponding diagram of topological spaces is commutative up to homotopy.

The proof will be given in a later version of the paper. □

Denote by $B(-)$ the classifying space functor, and by $\Sigma^\infty Y$ the suspension spectrum of a topological space $Y$. Recall that we denote by $\{D^n_k\}_k$ the $n$-dimensional little disc operad.

Based on the Lemma above, we prove the following theorem:
Theorem 5.5. Let $C$ be an $n$-fold monoidal abelian category satisfying (*). Then there is a map in the homotopical category of spectra of topological spaces

$$\Sigma^\infty D_k^{n+1} \wedge (B(X), B(X^0)) \wedge \cdots \wedge (B(X), B(X^0)) \to (B(X, B(X^0)))$$

which obeys the operad action equations.

Here $- \wedge -$ is the smash-product on the homotopical category of spectra, see [A], Part III, Lecture 4.

Proof. The statement follows from the Lemma above and from the Theorem from [BFSV] that the classifying space operad of the operad $\mathcal{M}_n(d)$ (see Section 3.5) is homotopically equivalent to the $n$-dimensional little disc operad. \qed

5.4 Applications

All applications go the following line. Let $Y$ be a topological space; consider the suspension spectrum $\Sigma^\infty Y$. This spectrum is a “coalgebra object” in the homotopical category. Here we mean that there are ccoassociative maps

$$\Sigma^\infty Y \to \Sigma^\infty Y \wedge \cdots \wedge \Sigma^\infty Y$$

for any $k \geq 1$.

The maps (5.12), together with (5.11), give maps

$$[\Sigma^\infty Y, \Sigma^\infty D_k^{n+1}] \otimes \Sigma^\infty Y, (B(X), B(X^0))] \otimes k \to [\Sigma^\infty Y, (B(X), B(X^0))]$$

Here all maps of spectra are graded abelian groups, and $\{[\Sigma^\infty Y, \Sigma^\infty D_k^{n+1}]\}_{k \geq 1}$ is an operad of graded abelian groups.

We have proved

Theorem 5.6. Let $Y$ be a based topological space. Then the operad $\{[\Sigma^\infty Y, \Sigma^\infty D_k^{n+1}]\}_{k \geq 1}$ of graded abelian groups acts on the graded abelian group $[\Sigma^\infty Y, (B(X), B(X^0))]$.

When $Y$ is the disjoint union of two points, the maps from the spectrum $\Sigma^\infty Y$ are just stable homotopy groups. Thus, we get

Theorem 5.7. Let $C$ be an abelian $n$-fold monoidal category satisfying the condition (*). Then the operad $\{\bigotimes_i \pi_i^{stab}(D_k^{n+1})\}_{k \geq 1}$ of abelian groups acts on the graded space $\bigotimes_j \text{Ext}^j_C(A, A)$. 

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Proof. One only needs to compute the stable homotopy groups \( \pi_i^{\text{stab}}(B(X), B(X^0)) \) of the pair. But the latter is equal to \( \text{Ext}^i_C(A, A) \) by Key-Lemma 5.1 and by the Retakh’s theory, see Section 1.

The Hurewicz homomorphism gives an isomorphism
\[
\pi_i^{\text{stab}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_i(X, \mathbb{Q})
\]
where we get the link with the more usual statement that the homology operad of the little disc operad acts on the Hochschild cohomology of any associative algebra.

In particular, for any bialgebra \( A \) we constructed a 2-fold monoidal abelian category of tetramodules. We know from Section 4.3 that the category of tetramodules obeys the condition (*) when \( A \) is a Hopf algebra. By the result of R.Taillefer (see Section 6) we get:

**Corollary 5.8.** The Gerstenhaber-Schack cohomology of any Hopf algebra defined over \( \mathbb{Q} \) is naturally a 3-algebra.

6 Appendix: The Gerstenhaber-Schack cohomology, after R.Taillefer

Here we give an overview of the works of R.Taillefer on Gerstenhaber-Schack cohomology, from slightly different point of view. The main result is that the Gerstenhaber-Schack cohomology \( H^*_{\text{GS}}(A) \) (see Subsection 6.1) is equally equal to \( \text{Ext}^*_{\text{Tetra}(A)}(A, A) \) (Theorem 6.1 below). Let us notice that this result is true for any bialgebra, not necessarily a Hopf algebra.

6.1 The Gerstenhaber-Schack complex

Let \( A \) be a (co)associative bialgebra. Note that the bar-differential in \( \text{Bar}_{\mathbb{M}_1}(A) \) is given by maps of tetramodules; analogously, the cobar-differential in \( \text{Cobar}_{\mathbb{M}_2}(A) \) is given by maps of tetramodules.

Let us recall, that originally the Gerstenhaber-Schack complex was defined in [GS] as

\[
C^*_{\text{GS}}(A) = \text{Hom}_{\text{Tetra}(A)}(\text{Bar}_{\mathbb{M}_1}(A), \text{Cobar}_{\mathbb{M}_2}(A))
\]

Here \( \text{Bar}_{-}(B) \) and \( \text{Cobar}_{+}(C) \) are truncated complexes, which end (start) with \( B\mathbb{M}_1 B \) \( (C\mathbb{M}_2 C) \) correspondingly.

For convenience of the reader let us write down here the Gerstenhaber-Schack differential in \( C^*_{\text{GS}}(A) \) explicitly:
First of all, as a graded vector space,

\[ C^*_\text{GS}(A) = \oplus_{m,n \geq 0} \text{Hom}_k(A^\otimes m, A^\otimes n)[-m-n] \]

Now let \( \Psi: A^\otimes m \to A^\otimes n \in C^s_{\text{GS}}(A) \). We are going to define the Gerstenhaber-Schack differential \( d^s_{\text{GS}}(\Psi) \in \text{Hom}(A^\otimes (m+1), A^\otimes n) \oplus \text{Hom}(A^\otimes m, A^\otimes (n+1)) \). Denote the projection of \( d^s_{\text{GS}} \) to the first summand by \( (d^s_{\text{GS}})_1 \), and the projection to the second summand by \( (d^s_{\text{GS}})_2 \). The formulas for \( (d^s_{\text{GS}})_1 \) and \( (d^s_{\text{GS}})_2 \) are:

\[
(d^s_{\text{GS}})_1(\Psi)(a_0 \otimes \cdots \otimes a_m) = \Delta^{n-1}(a_0) * \Psi(a_1 \otimes \cdots \otimes a_m) \\
+ \sum_{i=0}^{m-1} (-1)^{i+1} \Psi(a_0 \otimes \cdots \otimes (a_i * a_{i+1}) \otimes \cdots \otimes a_m) \\
+ (-1)^{m-1} \Psi(a_0 \otimes \cdots \otimes a_{m-1}) * \Delta^{n-1}(a_m)
\]

and

\[
(d^s_{\text{GS}})_2(\Psi)(a_1 \otimes \cdots \otimes a_m) = \\
(\Delta^{(1)}(a_1) * \Delta^{(1)}(a_2) * \cdots * \Delta^{(1)}(a_m)) \otimes \Psi(\Delta^{(2)}(a_1) \otimes \cdots \otimes \Delta^{(2)}(a_m)) \\
+ \sum_{i=1}^{n} (-1)^i \Delta_i \Psi(a_1 \otimes \cdots \otimes a_m) \\
+ (-1)^{i+1} \Psi(\Delta^{(1)}(a_1) \otimes \Delta^{(1)}(a_2) \otimes \cdots \otimes \Delta^{(1)}(a_m)) \otimes (\Delta^{(2)}(a_1) * \Delta^{(2)}(a_2) * \cdots * \Delta^{(2)}(a_m))
\]

The goal of this Appendix is to prove the following Theorem due to R. Taillefer:

**Theorem 6.1.** ([Ta1,2]) Suppose a bialgebra \( A \) has unit and counit. Then one has:

\[ \text{Ext}^*_{\text{T} \text{etra}(A)}(A, A) = H^*(\text{Hom}_{\text{T} \text{etra}(A)}(\text{Bar}_{\text{op}}^\otimes(A), \text{Cobar}_+^\otimes(A))) \]

### 6.2 Two forgetful functors and their adjoint

#### 6.2.1

Let \( A \) be a (co)associative bialgebra. Besides the category \( \text{T} \text{etra}(A) \), we can consider the categories \( \text{Bimod}(A) \) of \( A \)-bimodules (when we consider \( A \) as an algebra) and \( \text{Bicomod}(A) \) of \( A \)-bicomodules (when we consider \( A \) as a coalgebra). Clearly there are two exact forgetful functors \( F_1: \text{T} \text{etra}(A) \to \text{Bicomod}(A) \) and \( F_2: \text{T} \text{etra}(A) \to \text{Bimod}(A) \). We have the following
Lemma 6.2. Let $A$ be a bialgebra which has unit and counit. Then the functor $F_1$ admits a left adjoint $L$ and the functor $F_2$ admits a right adjoint $R$. The functors $L$ and $R$ are exact.

Definition 6.3. Let $A$ be a bialgebra, and let $N$ be an $A$-bicomodule, and let $M$ be an $A$-bimodule. The tetramodule $L(N)$ is called the induced (from $N$) tetramodule, and the tetramodule $R(M)$ is called the coinduced (from $M$) tetramodule. The induced and coinduced tetramodules form full additive subcategories in the abelian category $\text{Tetra}(A)$. We denote them $\text{Tetra}_{\text{Ind}}(A)$ and $\text{Tetra}_{\text{Coind}}(A)$, respectively.

Proof of Lemma: we set

\[(6.5)\quad L(N) = A \boxtimes_1 N \boxtimes_1 A\]

and

\[(6.6)\quad R(M) = A \boxtimes_2 M \boxtimes_2 A\]

(see Section 4.2.1 for the definitions of $\boxtimes_1$ and $\boxtimes_2$). Strictly speaking, to write formulas like this, $M$ and $N$ should be tetramodules. But the reader probably have noticed that in the definition of $M_1 \boxtimes_1 M_2$ we do not use the right $A$-module structure in $M_1$ and the left $A$-module structure in $M_2$. Analogously, in the definition of $M_1 \boxtimes_2 M_2$ we do not use the right comodule structure in $M_1$ and the left comodule structure in $M_2$. Therefore, $(6.5)$ and $(6.6)$ make sense.

We should check the adjunction properties

\[(6.7)\quad \text{Hom}_{\text{Bicomod}}(A)(N, F_1(T)) = \text{Hom}_{\text{Tetra}}(A)(L(N), T)\]

and

\[(6.8)\quad \text{Hom}_{\text{Bimod}}(A)(F_2(T), M) = \text{Hom}_{\text{Tetra}}(A)(T, R(M))\]

as bifunctors.

Prove $(6.7)$. Any map of tetramodules $L(N) = A \boxtimes_1 N \boxtimes_1 A \rightarrow T$ is uniquely defined by its restriction to $1 \boxtimes_1 N \boxtimes_1 1$. This map clearly is a map of bicomodules $N \rightarrow F_1(T)$. Wise versa, any map of bicomodules $N \rightarrow F_1(T)$ can be uniquely extended to a map of bimodules $A \boxtimes_1 N \boxtimes_1 A \rightarrow T$ which is in fact a map of tetramodules. These two assignments are inverse to each other.

The proof of $(6.8)$ is analogous.

The exactness of $L$ and $R$ is clear from their constructions $(6.5)$ and $(6.6)$. $\square$
6.2.2

Lemma 6.4. Let $A$ be an associative bialgebra. Then any tetramodule $M \in \text{Tetra}(A)$ can be imbedded onto a coinduced tetramodule, and there is a surjection into $M$ from an induced tetramodule.

Proof. Let $M$ be an $A$-tetramodule. Consider $P(M) = A \boxtimes_1 M \boxtimes_1 A$, it is induced from the bicomodule $F_1(M)$. The map $p: A \boxtimes_1 M \boxtimes_1 A \to M$, $a \boxtimes_1 m \boxtimes_1 b \mapsto a \cdot m \cdot b$ is clearly a map (and an epimorphism, because $A$ contains a unit) of tetramodules. Analogously, the tetramodule $Q(M) = A \boxtimes_2 M \boxtimes_2 A$ is coinduced from the bimodule $F_2(M)$, and we have a monomorphism $j: M \to A \boxtimes_2 M \boxtimes_2 A$, $m \mapsto \Delta_l \circ \Delta_r(m)$.

\[\square\]

Corollary 6.5. ([Ta2]) For any bialgebra $A$ the category $\text{Tetra}(A)$ has enough injectives.

Proof. The functor $R$ is a right adjoint to an exact functor $F_2$, and, therefore, maps injective objects to injective (see [W], Prop. 2.3.10). Moreover, it is left exact ([W], Section 2.6). Therefore, it is sufficient to imbed $M$ as an $A$-bimodule onto an injective $A$-bimodule $I$ (see [W], Section 2.3) and apply the functor $R$ to this this imbedding of $A$-bimodules. This will give an imbedding $j: M \to Q(M)$ where $Q(M)$ is defined in the proof of Lemma above.

\[\square\]

The main fact about the induced and the coinduced tetramodules is the following

Proposition 6.6. Let $A$ be a bialgebra with unit and counit. Then the functor $X \mapsto \text{Hom}_{\text{Tetra}(A)}(X, Q)$ for fixed $Q \in \text{Tetra}_{\text{Coind}}(A)$ is an exact functor from $\text{Tetra}_{\text{Ind}}(A)$ opp to $\text{Ab}$. As well, the functor $Y \mapsto \text{Hom}_{\text{Tetra}(A)}(P, Y)$ for fixed $P \in \text{Tetra}_{\text{Ind}}(A)$ is an exact functor from $\text{Tetra}_{\text{Coind}}(A)$ to $\text{Ab}$.

Proof. Let us prove the first statement. Let

\[
0 \to LN' \to LN \to LN'' \to 0
\]

be an exact sequence of tetramodules, $N, N', N'' \in \text{Bicomod}(A)$. We should prove that the sequence

\[
0 \to \text{Hom}_{\text{Tetra}(A)}(LN'', RM) \to \text{Hom}_{\text{Tetra}(A)}(LN, RM) \to \text{Hom}_{\text{Tetra}(A)}(LN', RM) \to 0
\]

is exact for any $M \in \text{Bimod}(A)$.

By the adjunction, the exactness of (6.10) is equivalent to the exactness of the sequence

\[
0 \to \text{Hom}_{\text{Bimod}(A)}(F_2LN'', M) \to \text{Hom}_{\text{Bimod}(A)}(F_2LN, M) \to \text{Hom}_{\text{Bimod}(A)}(F_2LN', M) \to 0
\]

But this sequence is exact because for any $N \in \text{Bicomod}(A)$ the $A$-bimodule $F_2LN$ is free and, therefore, projective.

The second statement is proven analogously.

\[\square\]
6.3 Some homological algebra

In this Subsection we consider some homological algebra, which is useful for computation of $Ext$ functors.

6.3.1 A $(\mathcal{P}, \mathcal{Q})$-pair

Suppose $\mathcal{A}$ is an abelian category, and $\mathcal{P}$, $\mathcal{Q}$ are two additive subcategories. We say that they form a $(\mathcal{P}, \mathcal{Q})$-pair if the following conditions are satisfied:

1. the functor $\text{Hom}(?, Q)$ is exact on $\mathcal{P}^{\text{opp}}$ for any $Q \in \mathcal{Q}$;
2. the functor $\text{Hom}(P, ?)$ is exact on $\mathcal{Q}$ for any $P \in \mathcal{P}$;
3. for any object $M \in \mathcal{A}$ there is an epimorphism $P \to M$ for $P \in \mathcal{P}$ and there is a monomorphism $M \to Q$ for $Q \in \mathcal{Q}$;
4. a stronger version of 3: for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{A}$ the epimorphisms $P_i \to M_i$, $p_i : P_i \in \mathcal{P}$, can be chosen such that there is a map of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\
& & \downarrow{p_1} & & \downarrow{p_2} & & \downarrow{p_3} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0
\end{array}
$$

where the upper line is an exact sequence; also the dual condition for $\mathcal{Q}$ for monomorphisms $q_i : M_i \to Q_i$, $Q_i \in \mathcal{Q}$.

Note that the third condition guarantees that each object $M \in \mathcal{A}$ has a $\mathbb{Z}_{\leq 0}$-graded resolution in $\mathcal{P}$ and a $\mathbb{Z}_{\geq 0}$-graded resolution in $\mathcal{Q}$.

Example 6.7. If $\mathcal{A}$ has enough projectives and $\mathcal{P}$ is the additive subcategory of projective objects, $\mathcal{Q} = \mathcal{A}$ gives a $(\mathcal{P}, \mathcal{Q})$-pair. Analogously, if $\mathcal{A}$ has enough injectives and $\mathcal{Q}$ is the additive subcategory of injective objects, $\mathcal{P} = \mathcal{A}$ gives a $(\mathcal{P}, \mathcal{Q})$-pair.

Proposition 6.8. Suppose $\mathcal{A} = \text{Tetra}(A)$ for an associative unital (see Notations) bialgebra $A$. Then the pair $(\text{Tetra}_{\text{ind}}(A), \text{Tetra}_{\text{coind}}(A))$ is a $(\mathcal{P}, \mathcal{Q})$-pair.

Proof. The first two properties were proven in Proposition 6.7, and the third property follows from Lemma 6.5. Moreover, this construction in this Lemma gives the fourth assertion in the definition of a $(\mathcal{P}, \mathcal{Q})$-pair immediately. \qed
6.3.2 The Key-Lemma

The main fact about \((P, Q)\)-pairs is the following lemma:

**Key-lemma 6.9.** Let \(\mathcal{A}\) be an abelian category, having enough projective or injective objects. Suppose we are given a \((P, Q)\)-pair and let \(M, N \in \mathcal{A}\) be two objects. Suppose \(P' \to M\) be a resolution of \(M\) by objects in \(P\), and \(N \to Q'\) be a resolution of \(N\) by objects in \(Q\). Then

\[
\text{Ext}_A^q(M, N) = H^q(Hom_A(P', Q'))
\]

Proof. The proof consists from several steps. We give the proof for the case of enough injectives, the case of enough projectives is analogous. We recall the universal property which a derived functor obeys, and prove that the functor \((M, N) \mapsto H^q(Hom_A(P', Q'))\) has this universal property.

Let \(\mathcal{A}, \mathcal{B}\) be abelian categories, and \(F: \mathcal{A} \to \mathcal{B}\) be a left exact functor. Then the right derived functor \(L^qF\) enjoys the following universal property. We say that a collection of functors \(\{T_n: \mathcal{A} \to \mathcal{B}\}, n \geq 0\), is a (cohomological) \(\delta\)-functor if for any exact sequence

\[
0 \to M \to N \to L \to 0
\]

in \(\mathcal{A}\) one has a morphism \(\delta: T_n(L) \to T_{n+1}(M), n \geq 0\), in \(\mathcal{B}\) with the following long exact sequence

\[
\cdots \to T_{n-1}(L) \xrightarrow{\delta} T_n(M) \to T_n(N) \to T_n(L) \xrightarrow{\delta} T_{n+1}(M) \to \cdots
\]

\(n \geq 1\), depending functorially on the short exact sequence (6.14). Consider a \(\delta\)-functor \(\{T_n\}\). We say that this \(\delta\)-functor is *universal* if for any other \(\delta\)-functor \(\{S_n\}\) with the natural transformation \(f_0: T_0 \to S_0\) there is a unique morphism of \(\delta\)-functors \(\{f_n: T_n \to S_n\}\) extending \(f_0\). From this definition it follows that the universal \(\delta\)-functor with \(T_0 = F\), if it exists, is unique. This point of view, independent on existence of enough projective objects, due to Grothendieck [Tohoku] and is extracted by the author from [W], Chapter 2.

Now we consider the \(Hom_A(M, ?)\) as a functor of the second argument. If \(\mathcal{A}\) has enough injectives the functors \(T_n(M, N) = Ext^n_A(M, N)\) form a cohomological universal \(\delta\)-functor (see [W], Theorem 2.4.7).

Now our proof of the Key-Lemma will go as follows:

Step 1. \(T_k: (M, N) \mapsto H^k(Hom_A(P', Q'))\) with \(P' \in P, Q' \in Q\) is well-defined, that is does not depend on the choice of \(P'\) and \(Q'\);

Step 2. it is a homological \(\delta\)-functor with \(T_0 = Hom_A(M, N)\);

Step 3. it is a universal \(\delta\)-functor.
Clearly the Key-Lemma follows from these 3 statements. Let us prove them.

**Step 1:** it simply follows from the conditions 1) and 2) in the definition of a \((\mathcal{P}, \mathcal{Q})\)-pair.

**Step 2:** it follows easily from condition 4) in the definition of a \((\mathcal{P}, \mathcal{Q})\)-pair.

**Step 3:** this is a bit more tricky. An additive functor \(F : \mathcal{A} \to \mathcal{B}\) is called **effaceable** if for any object \(N \in \mathcal{A}\) there is a monomorphism \(j : M \to I\) such that \(F(j) = 0\). It is proven in [Tohoku] that a cohomological \(\delta\)-functor \(\{T_n\}\) for which all \(T_n\) for \(n \geq 1\) are effaceable, is universal. It remains to prove that our functors \(T_n(N) = H^n(\text{Hom}_\mathcal{A}(P^*(M), Q^*(N)))\), \(n \geq 1\), are effaceable. We can choose a monomorphism \(j : N \to I\) with \(I \in \mathcal{Q}\) by the condition 3) in the definition of a \((\mathcal{P}, \mathcal{Q})\)-pair. Now the effaceability follows from \(T_n(I) = 0\), \(n \geq 1\), which immediately follows from 1) and 2).

Thus, it is proven that the functors \(H^n(\text{Hom}_\mathcal{A}(P^*, Q^*))\) are universal \(\delta\)-functors with the same 0-component than \(\text{Ext}_\mathcal{A}^n(M, N)\). Therefore, these two functors coincide because the universal functor \(\{T_n\}\) with fixed \(T_0\) is unique.

\(\square\)

### 6.4 Example: computation of the Gerstenhaber-Schack cohomology for \(A = S(V)\)

Here we compute, as an application of the previous results of this Section, the Gerstenhaber-Schack cohomology for the (co)free commutative and cocommutative bialgebra \(A = S(V)\), where \(V\) is a finite-dimensional vector space. We prove

**Proposition 6.10.** Let \(A = S(V)\) be (co)free commutative cocommutative bialgebra, \(V\) finite-dimensional. Then the Gerstenhaber-Schack cohomology is \(H^k_{\text{GS}}(A) = \bigoplus_{i+j=k, i,j\geq 0} \Lambda^i V \otimes \Lambda^j (V^*)\).

**Proof.** We compute \(\text{Ext}^1_{\text{tetra}(A)}(A, A)\) for \(A = S(V)\) as \(H^1(\text{Hom}(P^*, Q^*))\), where \(P^*\) is a resolution of \(A\) by free modules, and \(Q^*\) is a resolution of \(A\) by cofree comodules. Then we take the usual Koszul resolutions of the diagonal for \(P^*\) and \(Q^*\).

As was noticed in Example 4.4, there is a canonical 3-algebra structure on the graded space \(\oplus_{i+j=k} \Lambda^i V \otimes \Lambda^j V^*\). It would be interesting to check explicitly that this 3-algebra structure coincides with the one defined in Section 4.4 via the 2-fold monoidal structure on the category of tetramodules.

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