Particle Motion with Hořava – Lifshitz type
Dispersion Relations

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ABSTRACT

Using super Hamiltonian formalism, we study the motion of particles whose dispersion relations are modified to incorporate Hořava – Lifshitz type anisotropic scaling symmetry. We find the following as consequences of this modified dispersion relation: (i) The speed of a charged particle under a constant electric field grows without bound and diverges. (ii) The speed, as measured by an observer locally at rest, of a particle falling towards the horizon also grows without bound and diverges as the particle approaches the horizon. (iii) This particle reaches the horizon in a finite coordinate time, in contrast to the standard case where it requires infinite time.
1. Introduction

Recently, Hořava has proposed [1] a candidate theory for gravity, symmetric under Lifshitz type anisotropic scaling of spacetime coordinates

\[ x^i \rightarrow lx^i \ , \ t \rightarrow lt \]  \hspace{1cm} (1)

where \( z \) is the scaling exponent. This theory is renormalisable if \( z = d \) and superrenormalisable if \( z > d \) where \( d \) is the number of spatial dimensions, and has many appealing features which are being actively explored. See [2, 3, 4, 5] for a sample of references.

Some of the features of Hořava’s theory are: it generically leads to a dispersion relation of the form

\[ \omega^2 - \left( c^2k^2 + \cdots + \alpha(k^2)^z \right) = 0 \]  \hspace{1cm} (2)

for massless excitations where \( c \) is the speed of light in vacuum in the IR and \( \alpha \), assumed to be \( > 0 \), is a marginal coupling parameter. The speed of propagation diverges in the UV. In early universe, such a dispersion relation leads to scale invariant spectrum of fluctuations, and also to matter equation of state \( p = \frac{z}{d} \rho \). Hořava’s theory may also lead to a bounce in the scale factor evolution if the spatial curvature is non zero. These and a variety of other cosmological consequences have been explored in [2].

Static spherically symmetric solutions of Hořava’s theory have also been studied. Such solutions generically have horizon(s) but differ from the standard Schwarzschild solution in other details. The thermodynamical properties of such solutions have also been studied. See [3] for a sample of references. Geodesic motion of probes in such spacetimes have also been studied [4], as also a host of other issues [5].

Given the generic modifications of dispersion relations, it is likely that those for classical particles, or semiclassical probe wave packets, in Hořava’s theory are also modified to a form of the type

\[ E^2 - \left( c^2p^2 + \cdots + \alpha(p^2)^z \right) = m^2c^4 \]  \hspace{1cm} (3)

which incorporates Hořava – Lifshitz type anisotropic scaling symmetry (1) where \( z = d \) for renormalisable case and \( z > d \) for superrenormalisable case. The particle dynamics will then differ from the standard one for high
momentum $p$, equivalently high energy $E$. Also, in the presence of gravitational field, probes obeying such dispersion relations are unlikely to follow the spacetime geodesics. It is clearly of interest to study the dynamics of such particles and the consequent differences from the standard case.

In this paper, we study the motion of particles whose dispersion relation is modified to incorporate Hořava–Lifshitz type anisotropic scaling symmetry (1). It turns out that the required modifications suggest themselves naturally not in the Lagrangian formalism, \(^1\) but in the super Hamiltonian formalism described in [7]. Equations of motion, with or without gravitational fields, may then be easily obtained. \(^2\)

Using this formalism, we first obtain the equations of motion for a charged particle without gravitational fields and study the consequences when only a constant electric field is present. This will illustrate the main feature of the dynamics, which one expects from the relation (3), that the speed of particle grows without bound and diverges for high momentum $p$, equivalently high energy $E$.

We then consider gravitational fields, obtain the equations of motion, specialise to spacetimes with horizon, and analyse the motion outside the horizon. Here again we find that the speed of the particle, as measured by an observer locally at rest \([10, 11]\), grows without bound and diverges as the particle approaches the horizon. Also, we find that the particle reaches the horizon in a finite coordinate time, in contrast to the standard case where it requires infinite coordinate time.

The plan of the paper is as follows. In section 2, we present relevant details of super Hamiltonian formalism, incorporate anisotropic scaling symmetry, and present the general equations of motion. In section 3, we study the motion of a charged particle without gravitational fields. In section 4, we study the motion with gravitational fields present. In section 5, we conclude with a few remarks.

2. Formalism

\(^1\) However, see [6] where anisotropic scaling transformations are implemented in the context of ‘conformal mechanics’ using Lagrangian formalism.

\(^2\) Other types of dispersion relations, which are typically of the form $F(E, p^2) = \text{constant}$ \([8, 9]\), may also be analysed similarly; the relevant equations may be obtained straightforwardly, but will not be presented here.
We first set up our notation to describe particle motion obeying a given modified dispersion relations. We will use the super Hamiltonian formalism given in [7] and write the equations of motion in a general form, which can then be analysed for specific cases.

Consider the motion of a particle in (3 + 1) – dimensional spacetime. Let \( x^\mu = (x^0, x^i) = (t, x^i) \), \( i = 1, 2, 3 \), denote spacetime coordinates, let \( \lambda \) parametrise the particle trajectory, and let \( u^\mu = \frac{dx^\mu}{d\lambda} \). Equations of motion can be obtained in a standard way by starting with a Lagrangian \( L(u^\mu, x^\nu) \). However, incorporating modified dispersion relations in the Lagrangian is not straightforward. Hence we follow an equivalent, alternate formalism [7] in which one starts with a ‘super Hamiltonian’ \( H(p^\mu, x^\nu) \) where \( p^\mu \) is the momentum, and obtains the equations of motion by extremising the action

\[
S = \int \left( p_\mu dx^\mu - H d\lambda \right)
\]  

with respect to \( x^\mu \) and \( p_\mu \). The resulting equations of motion are given by

\[
u^\mu \equiv \frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}.
\]  

These equations imply that \( \frac{dH}{d\lambda} = 0 \), hence that \( H = H_0 = \text{constant} \), along the particle trajectory. A straightforward exercise now shows that the standard geodesic equations can be obtained from

\[
H = H_{\text{std}} \equiv \frac{c^2}{2} G^{\mu\nu}(x) p_\mu p_\nu
\]  

where \( G^{\mu\nu} \) is the inverse of the spacetime metric \( G_{\mu\nu} \). In the case of charged particles, one first replaces \( p_\mu \) by \( (p_\mu + ea_\mu) \) in \( H \) where \( e \) is the particle charge, and \( a_\mu \) is the 4–vector potential, and then extremises the action \( S \). The constant of motion \( H_0 = 0 \) for massless particles and, in our notation, \( H_0 < 0 \) for particles with non zero rest mass. See [7] for further details. Also, in our notation, the dimensions of \( p_0 \), \( H \), and \( \lambda \) are (energy), (energy)\(^2\), and (time/energy) respectively as follows from \( x^0 = t \) and equations (4) and (6).

In Hořava’s theory, the gravitational fields \( (N, N^i, g_{ij}) \) are taken to be those of the metric \( G_{\mu\nu} \) given by

\[
ds^2 = G_{\mu\nu} dx^\mu dx^\nu = -c^2 N^2 dt^2 + g_{kl}(dx^k + N^k dt)(dx^l + N^l dt).
\]  

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The anisotropic scaling dimensions of various quantities in momentum units are given, taking $x^i$ to have length dimensions, by
\[
[x^i] = -1, \quad [t] = -z, \quad [p_0] = z, \\
[c] = z - 1, \quad [\mathcal{H}] = -[\lambda] = 2z, \\
[N] = [g_{ij}] = 0, \quad [N^i] = z - 1. \tag{8}
\]

Using equation (7), $\mathcal{H}_{std}$ may now be written as
\[
\mathcal{H}_{std} = \frac{1}{2} \left( -\frac{(p_0 - N^k p_k)^2}{N^2} + c^2 g^{kl} p_k p_l \right) \tag{9}
\]
which leads to the flat-space dispersion relation $E^2 - c^2 p^2 = \text{constant}$. The standard super Hamiltonian $\mathcal{H}_{std}$ can now be generalised to incorporate any modified dispersion relation of the form $F(E, p^2) = \text{constant}$ \cite{8, 9}. A natural proposal for such a generalised super Hamiltonian is given by
\[
\mathcal{H} = \frac{1}{2} F(\epsilon, X) ; \quad \epsilon = \sqrt{\frac{(p_0 - N^k p_k)^2}{N^2}} \quad X = g^{kl} p_k p_l. \tag{10}
\]

The modified equations of motion for particle trajectory, with or without gravitational fields, may now be obtained using equations (5) and (10). Those for charged particle may be obtained similarly upon replacing $p^\mu$ by $(p^\mu + e a^\mu)$ in $\mathcal{H}$.

Although it is straightforward to analyse the general case, we will consider here only Hořava – Lifshitz type modified dispersion relations as in equation (3). The corresponding super Hamiltonian is naturally given by
\[
\mathcal{H} = \frac{1}{2} \left( -\frac{(p_0 - N^k p_k)^2}{N^2} + f(X) \right) ; \quad f(X) = \sum_{n=1}^{z} \alpha_n X^n \tag{11}
\]
where $X = g^{kl} p_k p_l$, $z = 3$ for renormalisable case, and $z > 3$ for superrenormalisable case. The scaling dimensions of $\alpha_n$ are given by $[\alpha_n] = 2(z - n)$. Thus, $\alpha_z$ has vanishing scaling dimensions and is a marginal coupling parameter; other $\alpha_n$ have positive scaling dimensions and are relevant parameters. We set $\alpha_1 = c^2$ so as to obtain the standard case in the IR where $X$ is small. Determining other $\alpha_n$ requires detailed knowledge of the theory which describes the particles, or semiclassical probe wave packets. Here, for
simplicity, we just assume that $\alpha_n \geq 0$ and that $\alpha_z \equiv \alpha > 0$. This will be sufficient for our purposes here.

Before proceeding further, we now make a remark. In principle, given the super Hamiltonian $\mathcal{H}$, it is possible to obtain the corresponding Lagrangian $L$ by an appropriate Legendre transformation. However, the procedure involves inversion of a function which we are unable to carry out explicitly in a closed form except in very simple cases. The explicit form of the resulting $L$ is not illuminating, nor does it suggest a natural generalisation of $L$ which may incorporate a given modified dispersion relation.

3. Charged particle motion in flat space

For the purpose of illustrating the effects of modified dispersion relations, we first consider the motion of a charged particle in flat spacetime. Thus, $(N, N^i, g_{ij}) = (1, 0, \delta_{ij})$ in equation (7), and $p_\mu$ is to be replaced by $(p_\mu + ea_\mu)$ in $\mathcal{H}$ where $e$ is the particle charge and $a_\mu$ is the 4-vector potential. The super Hamiltonian in equation (11) thus becomes

$$
\mathcal{H} = \frac{1}{2} \left( -(p_0 + ea_0)^2 + f(X) \right), \quad f(X) = \sum_{n=1}^{z} \alpha_n X^n \quad (12)
$$

where $X = \delta^{kl}(p_k + ea_k)(p_l + ea_l)$. Let

$$
\eta_{\mu\nu} = \text{diag} (-c^2, 1, 1, 1), \quad F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu
$$

$$(\dot{u}^0, \dot{u}^i) = \left( \frac{u^0}{c^2}, \frac{u^i}{f'} \right), \quad f' = \frac{d f}{dX},
$$

and $\eta^{\mu\nu}$ be the inverse of $\eta_{\mu\nu}$. Then, after a straightforward algebra using equations (5) and $\mathcal{H} = \mathcal{H}_0 \equiv -\frac{1}{2}m^2c^4$ where $m$ is the rest mass of the particle, it follows that the charged particle motion is described by

$$
\dot{u}^\nu = \eta^{\mu\nu} (p_\mu + ea_\mu) \\
X = \delta_{kl} \dot{u}^k \dot{u}^l \\
(u^0)^2 = m^2c^4 + f(X) \\
\frac{d \dot{u}^\mu}{d \lambda} = e \eta^{\mu\alpha} u^\beta F_{\beta\alpha} \quad (13)
$$

The velocity $v^i$ with respect to the observer time $t$ is given by $v^i = \frac{dx^i}{dt} = \frac{u^i}{u^0}$. 

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It then follows from the above expressions that

\[ v^2 = \delta_{kl} v^k v^l = \frac{X (f')^2}{m^2 c^4 + f(X)} \rightarrow \alpha z^2 X z^{-1} \]  \hspace{1cm} (14)

in the limit \( X \rightarrow \infty \) for the function \( f(X) \) in equation (12).

To illustrate the salient features of the modified dispersion relation, let us consider the case where there is a constant electric field \( E \) along \( x^1 \)-direction, \( i.e. F_{01} = E \) and other independent components of \( F_{\mu\nu} = 0 \). Let \( u^i = 0 \), and hence \( X = f(X) = 0 \), initially at \( t = \lambda = 0 \). It is then clear that \( u^2 = u^3 = 0 \) for all \( t > 0 \). Also, let \( u^1 = u \) and \( \hat{u} = \frac{u}{\tau} \). Equations (13) now become

\[
\frac{d \hat{u}}{d \lambda} = e E u^0 \ , \quad (u^0)^2 = m^2 c^4 + f(X) \ , \quad X = \hat{u}^2 \ . \hspace{1cm} (15)
\]

Now \( \lambda(\hat{u}) \) and, thereby in principle, \( \hat{u}(\lambda) \) follows from

\[
e E \lambda(\hat{u}) = \int_0^{\hat{u}} \frac{d \hat{u}}{u^0} = \int_0^{\hat{u}} \frac{d w}{\sqrt{m^2 c^4 + f(w^2)}} \ . \hspace{1cm} (16)
\]

Then, \( X(\lambda) \) and \( u^0(\lambda) \) may be obtained using equations (15).

To proceed further, we need the explicit form of \( f(X) \). Let \( f(X) = c^2 X \). In this case, it can be shown easily that the standard answers are obtained. That is, \( e E t = m c \sinh (ec E \lambda) \), the characteristic time is \( \frac{m c}{e E} \), and

\[ u^2 = c^4 X = (ec^2 E t)^2 \ , \quad (u^0)^2 = m^2 c^4 + (ec E t)^2 \ . \hspace{1cm} (17)\]

From equation (14), and from the above expressions, it also follows that the velocity \( v = \frac{u}{u^0} \rightarrow c \) in the limit \( t \rightarrow \infty \), as expected of a particle acted upon by a constant force. Also, note that \( \tau = (mc^2) \lambda \) is the standard proper time of the particle since equation (15) for \( (u^0)^2 \), written in terms of \( \tau \), becomes

\[
\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2 \ .
\]

Now consider \( f(X) = \alpha X^z \) with \( z > 1 \). Then \( \lambda(\hat{u}) \) in equation (16) approaches a finite value \( \lambda_* \) from below as \( \hat{u} \rightarrow \infty \). Therefore, \( \hat{u}(\lambda) \) and hence \( X(\lambda) \) diverge to \( \infty \) as \( \lambda \rightarrow \lambda_*- \). Equation (14) then shows that, in this limit, the velocity \( v \rightarrow \infty \) since \( z > 1 \).
It is easy to obtain asymptotic expressions for various quantities in the limit \( \lambda \to \lambda_- \). For example,

\[
t \sim (\lambda_s - \lambda)^{-\frac{1}{z-1}} , \quad X \sim t^2 , \quad v^2 \sim t^{2(\alpha-1)} ,
\]

all diverging to \( \infty \) in the limit \( \lambda \to \lambda_- \) since \( z > 1 \). Thus we see that, as a consequence of the Hořava–Lifshitz type modified dispersion relation, the speed of a particle which is acted upon by a constant force increases without bound and diverges to \( \infty \) in the limit \( t \to \infty \). There is no inconsistency here because the particle is in the UV regime in this limit since \( X \) is large and, in Hořava’s theory, there is no Lorentz invariance in the UV.

### 4. Particle motion under gravity

We now consider the motion of a particle in spacetime with non trivial gravitational fields \((N, N^i, g_{ij})\) as in equation (7). For convenience, we set \( N^2 = B \) and assume that \( B > 0 \). Equations of motion are obtained using equations (5) and (11). With \( X = g^{kl}p_kp_l \) and \( f' = \frac{df}{dX} \), they are given by

\[
\begin{align*}
  u^0 &= -\frac{p_0 - N^k p_k}{B} \quad (18) \\
  u^i &= f' g^{ik} p_k + \frac{(p_0 - N^k p_k)}{B} N^i \quad (19) \\
  \mathcal{H} &= \frac{1}{2} \left( \frac{(p_0 - N^k p_k)^2}{B} + f(X) \right) = \mathcal{H}_0 \quad (20)
\end{align*}
\]

and

\[
\frac{dp_\mu}{d\lambda} = -\frac{(p_0 - N^k p_k)^2}{2B^2} \partial_\mu B - \frac{1}{2} f' p_k p_l \partial_\mu g^{kl} \quad (21)
\]

One can now express \( p_\mu \), thereby \( X \) and \( \mathcal{H} \), in terms of \( u^\mu \) using above equations. Thus,

\[
\begin{align*}
p_i &= \frac{1}{f'} g_{ik} (u^k + N^k u_0) \quad (22) \\
p_0 &= -Bu^0 + \frac{1}{f'} g_{kl} N^k (u^l + N^l u_0) \quad (23)
\end{align*}
\]
\begin{align}
X &= \frac{1}{f^2} g_{kl} (u^k + N^k u^0) (u^l + N^l u^0) \quad (24) \\
\mathcal{H} &= \frac{1}{2} \left( -B (u^0)^2 + f(X) \right) = \mathcal{H}_0. \quad (25)
\end{align}

It can be shown after some algebra that, for \( f(X) = c^2 X \), the above equations of motion reduce to the standard geodesic equations corresponding to the metric given in equation (7).

The initial conditions may be taken to be the values of \( x^\mu \) and \( u^\mu \) initially at \( t = \lambda = 0 \). Then, in principle, the initial values of \( X \) and \( p_\mu \), and the constant of motion \( \mathcal{H}_0 \) may all be obtained from equations (23) – (25). Further evolution is then determined by the equations of motion.

A general feature of the particle motion may now be seen directly from the above equations. First, using equations (7), (24) and (25), we write

\begin{equation}
G_{\mu\nu} u^\mu u^\nu = 2 c^2 \mathcal{H}_0 - c^2 f(X) + X f'^2. \quad (26)
\end{equation}

If \( f(X) = c^2 X \) then \( G_{\mu\nu} u^\mu u^\nu = 2c^2 \mathcal{H}_0 = \text{constant} \) along the particle trajectory. If \( f(X) \) is as given in equation (11) then, for \( z > 1 \) and in the limit \( X \to \infty \), we have

\[ G_{\mu\nu} u^\mu u^\nu \sim \alpha X^z (\alpha z^2 X^{z-1} - c^2) \to \infty. \]

Now, \( G_{\mu\nu} u^\mu u^\nu = 0 \) for null trajectories and, in our notation, is negative (positive) for timelike (spacelike) trajectories. Let the initial conditions be chosen to correspond to a timelike trajectory i.e. \( G_{\mu\nu} u^\mu u^\nu < 0 \) initially at \( t = \lambda = 0 \). If \( X \to \infty \) as the motion evolves then it follows that, for the dispersion relation in equation (11) with \( z > 1 \), \( G_{\mu\nu} u^\mu u^\nu \) switches sign, becomes positive and diverges to \( \infty \). This implies that as the motion evolves into the UV, the initially timelike particle trajectory becomes null and then spacelike.

This indeed happens, for example, in the spacetime described by equation (27) below where \( N^i = 0 \), the fields are time independent, \( B(r) > 0 \) for \( r > r_h \), and \( B(r) \to 0 \) as \( r \to r_h \). Because of time independence, it follows that \( p_0 = \text{constant} \). Equation (20) then gives

\[ f(X) = 2 \mathcal{H}_0 + \frac{p_0^2}{B}. \]
from which it follows that \( f(X) \) and, hence, \( X \) diverge to \( \infty \) as an infalling particle approaches \( r_h \). If such a particle obeys the modified dispersion relation given in equation (11) then we have \( z > 1 \), and that the initially timelike particle trajectory becomes null and then spacelike as the particle approaches \( r_h \).

**Static spherically symmetric black hole**

We now study the particle motion in the region outside the (outer) horizon \( r_h \) of a static spherically symmetric black hole. We take the line element \( ds \) to be given by

\[
d s^2 = -c^2 B d t^2 + \frac{d r^2}{B} + C \left( d \theta^2 + \sin^2 \theta \, d \phi^2 \right)
\]

(27)

where \( B \) and \( C \) are functions of \( r \) only, \( B(r_h) = 0 \), and \( B > 0 \) for \( r > r_h \). We study the motion outside the horizon, \( i.e. \) for \( r > r_h \).

Since the fields, and hence the super Hamiltonian \( \mathcal{H} \), are independent of \( t \) and \( \phi \) coordinates, it follows that \( p_0 \) and \( p_\phi \) are constants. Furthermore, let \( \theta = \frac{\pi}{2} \) and \( u^\theta = 0 \) initially. Then we obtain \( p_\theta = 0 \). The remaining equations may be written as

\[
B \, u^0 = -p_0 , \quad C \, u^\phi = f' J 
\]

(28)

\[
(u^r)^2 = B \, f'^2 \left( X - \frac{J^2}{C} \right) 
\]

(29)

\[
f(X) = 2 \, \mathcal{H}_0 + \frac{p_0^2}{B} 
\]

(30)

where \( p_0 \), \( J \), and \( \mathcal{H}_0 \) are constants of motion.

The parameter \( J \) describes orbital motions. The allowed range of \( J \) for stable orbits may be analysed. Although different in details, the orbital motion with modified dispersion relations is qualitatively similar to that in the standard case.

Consider a radially infalling motion. Then \( J = 0 \). With no loss of generality, we set \( r = R \) and \( u^r = 0 \), hence \( X = 0 \), initially at \( t = \lambda = 0 \). This corresponds to releasing a particle from rest at \( r = R \). This also ensures that \( \mathcal{H}_0 < 0 \), hence that \( G_{\mu \nu} u^\mu u^\nu < 0 \) and the trajectory is timelike initially. The particle will fall radially towards the horizon at \( r_h \) and, for the function \( f(X) \) given in equation (11) with \( z > 1 \), it is clear from the general
considerations given below equation (26) that the initially timelike particle trajectory will turn spacelike as \( r \to r_h^+ \).

To see this explicitly, let \( f = \alpha X^z \) with \( \alpha > 0 \), and study the motion near \( r_h \). The standard case may be seen by setting \( z = 1 \). The effects of modified dispersion relations may be seen by setting \( z > 1 \).

Let \( \rho = r - r_h \) and consider the limit \( \rho \to 0^+ \) which corresponds to approaching the horizon. In this limit, upto coefficients which are not important for present purposes, we have \( B \sim \rho \). Then \( X^z \sim \rho^{-1} \) and

\[
(u^r)^2 = BX^f^2 \sim \rho^{-\frac{z-1}{2}}, \quad (u^0)^2 \sim \rho^{-2}.
\]

Consider now the velocity, \( v_{\text{obs}} \), of the particle as measured by an observer who is at rest at \( r = r_h + \rho \). It is given by

\[
v_{\text{obs}} = c \sqrt{\frac{g_{rr}}{|g_{tt}|}} \frac{dr}{dt} = \frac{1}{B} \frac{u^r}{u^0}.
\]

See, for example, [10, 11] for more details on \( v_{\text{obs}} \). Thus, we have

\[
v_{\text{obs}}^2 = \frac{(u^r)^2}{P_0^2} \sim \rho^{-\frac{z-1}{z+1}}
\]

upto constant coefficients. Hence, in the limit \( \rho \to 0^+ \), it follows that the velocity \( v_{\text{obs}} \) of the particle, as measured by an observer who is at rest at \( r = r_h + \rho \), approaches a constant if \( z = 1 \) and diverges to \( \infty \) if \( z > 1 \).

Another consequence of the modified dispersion relations may be seen in the coordinate time required to reach the horizon, \( i.e. \) in the value of \( t \) as \( \rho \to 0^+ \). The time \( t \) required to reach \( r = \rho \) is given, upto a finite additive constant, by

\[
t = \int^\rho \frac{dr}{v^r}
\]

where \( v^r = \frac{dr}{dt} = \frac{u^r}{u^0} \). We have \( v^r \sim \rho^{\frac{z-1}{z+1}} \) in the limit \( \rho \to 0^+ \). From the above integral for \( t \) we obtain, upto finite additive constants, that

\[
t \sim -\ln \rho \quad \text{if} \quad z = 1
\]

and

\[
t \sim \rho^{\frac{z-1}{2z}} \quad \text{if} \quad z > 1.
\]
It now follows that the horizon is reached only in the limit \( t \rightarrow \infty \) if \( z = 1 \), which is well known, and that the horizon is reached in a finite time \( t \) if \( z > 1 \) which is a consequence of the modified dispersion relation in equation (11).

The above analysis is applicable also for the static spherically symmetric solutions of Hořava’s theory obtained in [3]. The above qualitative features remain valid in these cases also.

5. Summary and conclusion

We first summarise the present work. We have studied the motion of a particle obeying a modified dispersion relation. We used super Hamiltonian formalism which is better suited for this purpose. We considered the dispersion relation that incorporates Hořava – Lifshitz type anisotropic scaling symmetry, characterised by a scaling exponent \( z > 1 \). The standard case corresponds to \( z = 1 \).

We first studied the charged particle motion when a constant electric field is present and find that, for \( z > 1 \), the particle speed diverges in the UV, namely, at high momentum.

We then studied the particle motion outside the horizon with only gravitational fields present. For \( z > 1 \), we find that the particle speed, as measured by an observer locally at rest, diverges as the particle approaches the horizon. Also, the particle reaches the horizon in finite coordinate time \( t \), in contrast to the standard case where it requires infinite time.

Our method is also applicable in the analysis of other types of dispersion relations, postulated for various reasons [8, 9]. A class of those studied in [8], in particular, are likely to exhibit the same features as seen here.

We now conclude by mentioning several issues for further studies. For the classical particles, we simply postulated the dispersion relations and proposed a super Hamiltonian that incorporates it. It is desirable to derive them formally. One may consider a semiclassical probe wave packet, that may be expected to behave like a classical particle. The dispersion relation for such a wave packet may then be derived from a Lagrangian, of the type recently constructed [5], for its constituent fields having anisotropic scaling symmetry. In this context, see NOTE added.

Another issue is the further evolution of the particle approaching the horizon radially, or with small impact parameter. Such a particle has super-
luminal speed as it approaches the horizon. Does it enter the horizon? If it
does, can it then come out? It is not clear to us how best to address this
issue.

A related issue is whether a black hole forms in the near head-on collision
of two sufficiently high energy particles? In the standard case, a black hole is
expected to form once the two particles are within the Schwarzschild radius
of their center of mass energy. Here, because of the modified dispersion
relations, their speeds are likely to be superluminal and it is not obvious
that a black hole will result.

Another related issue is whether a black hole forms when matter collapses.
In Hořava’s theory, the equation of state for matter will become \( p = \frac{\dot{z}}{d} \rho \) in
the UV \([2]\). There seems to be no obstacle for such matter to collapse and
form a black hole. However, see NOTE added.

We find that, as a particle falls towards the horizon, its speed exceeds \( c \)
and grows without bound as it approaches the horizon. It will be interesting
to understand whether such a particle can tell the presence of the horizon.
This is not possible in the standard case, but may be possible now since the
particle is not following the geodesics.

A more general question is whether the description of Physical laws re-
main the same or not in the frame of a particle which starts with subluminal
speed and becomes superluminal due to its modified dispersion relation. We
find the question very interesting, but it is not clear to us how to find the
answer.

A recent paper \([12]\) has studied conformal structure and possible Penrose
type diagrams for spacetimes with anisotropic scaling symmetry. It will be
interesting to see if and how the particle motions presented here may fit in,
and/or may help elucidate, such a spacetime structure.

**NOTE added:**

Two recent papers \([13, 14]\) appeared while this paper was being writ-
ten. They start with a Lagrangian with anisotropic scaling symmetry, and
study the kinematics of the fields in the geometric optics approximation us-
ing WKB methods. Among other things, they also find a Hamiltonian and
the corresponding equations of motion, which agree with the present ones.

The paper \([13]\) also studies collapse of matter. It suggests that matter
(dust, with pressure \( p = 0 \)) will first collapse, but will bounce back near the
singularity. Although it is not proven yet, such a bounce may indicate that
black holes may not form in a collapse. However, in this paper, the same IR equation of state ($p = 0$) is used even in the UV near the singularity and it was found that matter has no effect on the bounce. But in Horava’s theory, the equation of state of matter changes in the UV, generically to $p = \frac{2}{3} \rho$. Such an equation of state will then have non trivial effect on the bounce, see S. K. Rama in [2]. So it is not clear to us if the bounce in [13] will still be present if the correct equation of state in the UV is taken into account.

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