Nuclear and neutron matter $G$-matrix calculations with Ch-EFT potential including effects of three-nucleon interaction

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Energies of symmetric nuclear matter and neutron matter are evaluated in the lowest order Brueckner theory using the Ch-EFT potential including effects of the three-nucleon force (3NF). The 3NF is first reduced to density-dependent nucleon-nucleon (NN) force by folding single-nucleon degrees of freedom in infinite matter. Adding the reduced NN force to the initial NN force and applying a partial-wave expansion, we perform $G$-matrix calculations in pure neutron matter as well as in symmetric nuclear. We obtain the saturation curve which is close to the empirical one. It is explicitly shown that the cutoff-energy dependence of the calculated energies is substantially reduced by including the 3NF. Characters of the 3NF contributions in separate spin and isospin channels are discussed. Calculated energies of the neutron matter are very similar to those used in the literature for considering neutron star properties.

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I. INTRODUCTION

One of the basic problems in nuclear physics is to understand characteristic properties of nuclei, especially the saturation and single-particle shell structure, starting from underlying nucleon-nucleon (NN) interactions. Various many-body theories have been developed since the 1950s. The Brueckner theory [1–3], which was initiated in a multiple-scattering viewpoint and later established as the linked-cluster expansion in terms of $G$-matrices, has served as an standard method to understand nuclei as the system of nucleons moving independently in a mean field in spite of the NN interactions having singularly strong repulsion in the short-ranged region. Another basic method for quantum many-body problems is a variational treatment [4], although shell structure is not intuitive in this framework. Two methods in a non-relativistic framework are now known to provide similar description of nuclear bulk properties [5]. These results unfortunately indicate that the saturation cannot be correctly reproduced in a non-relativistic framework when realistic NN potentials are used. Various elaborate many-body methods practiced in recent years, such as the coupled-cluster method, the unitary correlated method, and the no-core shell model with low-momentum interactions, confirm this situation.

Many attempts have been made to find other mechanisms to improve the description of the saturation properties, such as relativistic effects and three-nucleon force (3NF) contributions in the nuclear medium. It was demonstrated that relativistic Brueckner-Hartree-Fock calculations [6] can provide a satisfactory saturation curve. However, because contributions from higher-order correlations and three-nucleon forces have not been fully estimated in the relativistic treatment, the problem seems not to be settled yet. In the last decade, a new description of the NN potential has been developed [7–9]; that is, the interaction based on chiral effective field theory (Ch-EFT). The potential form is dictated by underlying chiral symmetry of the QCD, and potential parameters, low-energy constants, are adjusted to explain scattering data as in other realistic NN potentials. 3NFs are systematically introduced in this framework and most parameters for these terms are taken over from the NN sector.

The introduction of 3NFs has a long history since the 1950s, and various studies have been devoted for 3NF contributions in nuclear properties. Besides three-body correlations through ordinary NN forces, 3NFs may arise from excitations of virtual nucleon-antinucleon pairs as well as isobar A and other nucleon excited states in the medium [10–11]. The importance of 3NFs, in the standard non-relativistic description of nuclei, has been established by precise few-body calculations [12–13]. It has also been recognized that 3NFs are necessary to reproduce empirical saturation properties [3–14–16]. Although numerical calculations of reproducing nuclear saturation properties by including 3NF effects have been presented by many authors, the advantage of using the the Ch-EFT is that the contribution of the 3NF can be discussed in a way systematic and consistent with the initial NN interaction.

Some perturbative considerations for neutron matter properties with the Ch-EFT interaction including the 3NF have been reported in Refs. [17–21]. The present author gave, in Ref. [22], a brief report of the lowest-order Brueckner theory (LOBT) calculation in nuclear matter with using the reduced density-dependent NN force obtained from the Ch-EFT 3NF, in which a focus was put on the effective spin-orbit strength. Similar LOBT calculations also appeared in Ref. [22].

In this paper, we report, in details, nuclear and neutron matter calculations in the LOBT based on the N$^3$LO Ch-EFT potential including its N$^2$LO 3NF. Because the Ch-EFT is a definite way to organize the interaction be-
tween nucleons, it is important to study the implication of the interaction based on it to the nuclear many-body problem. However, it is currently impossible to consider full contributions of 3NFs together with many-body correlations, except for very light nuclei. Even for the NN force, it is already very difficult to take into account effects of more than three-nucleon correlations. Therefore, we introduce an approximation. First, reduced effective NN forces are constructed by averaging the 3NF over the third nucleon in the Fermi sea. Adding the reduced NN force to the initial Ch-EFT NN interaction, we carry out a partial-wave expansion of the reduced NN interaction. This is straightforward but somewhat intricate. Explicit expressions of the reduced NN interaction are given in Appendix A. Expressions after the partial-wave expansion are shown in Appendix B. Numerical results are presented first for nuclear matter in Sec. III, and then for neutron matter in Sec. IV. Cutoff-energy dependence of the calculated energies is demonstrated in these sections. Summary follows in Sec. V.

II. G-MATRIX INCLUDING REDUCED NN FORCE FROM 3NF

It is difficult to treat the 3NF $V_{123}$ directly in infinite matter. In this paper, we introduce an approximation of reducing the 3NF to an effective NN force by folding single-nucleon degrees of freedom, as has been often employed in the literature [11, 13, 24, 25]. That is, the density dependent NN interaction $V_{12(3)}$ is defined, in momentum space, by the following summation over the third nucleon in the Fermi sea of nuclear matter:

$$
\langle k'|\sigma_1'^{1/2}k_2'|[V_{123}]|k_1\sigma_1\tau_1,k_2\sigma_2\tau_2\rangle_A = \sum_{k_3,\sigma_3\tau_3} \langle k'_1\sigma_1'^1, k'_2\sigma_2'^1, k_3\sigma_3, \tau_3|V_{123}|k_1\sigma_1\tau_1, k_2\sigma_2\tau_2, k_3\sigma_3\tau_3\rangle_A. \quad (1)
$$

The suffix $A$ denotes an antisymmetrized matrix element, namely $|ab\rangle_A \equiv |ab - ba\rangle$ and $|abc\rangle_A \equiv |abc - acb + bca - bac\rangle$. The remaining two nucleons are supposed to be in a center-of-mass frame, $k_1 + k_2 = k_1 + k_2 = 0$. We do not include the three-body form factor in this folding procedure, but introduce it later in the reduced NN interaction. In this case, matrix elements and their partial wave expansion can be carried out analytically for the Ch-EFT 3NF, as presented in Appendix A.

The necessity of taking into account of correlations being neglected, contributions of the two- and three-nucleon forces, $V_{12}$ and $V_{123}$, to the energy are given by

$$
\frac{1}{2} \sum_{k_1,k_2} \langle k_1,k_2|V_{123}|k_1,k_2\rangle_A + \frac{1}{3!} \sum_{k_1,k_2,k_3} \langle k_1,k_2,k_3|V_{123}|k_1,k_2,k_3\rangle_A
= \frac{1}{2} \sum_{k_1,k_2} \langle k_1,k_2|V_{123}|k_1,k_2\rangle_A + \frac{1}{3} \sum_{k_1,k_2} \langle k_1,k_2|V_{123}|k_1,k_2\rangle_A \quad (2)
$$

This implies that the $G$-matrix may be defined by

$$
G_{12} = V_{12} + \frac{1}{3} V_{123} + (V_{12} + \frac{1}{3} V_{123}) \frac{Q}{\omega - H} G_{12}, \quad (3)
$$

where $Q$ stands for the Pauli exclusion operator and the denominator $\omega - H$ of the propagator is prescribed below.

The similar evaluation of the single-particle energy needs a different combination factor:

$$
\langle k|t|k'\rangle + \sum_{k''} \langle kk'|V_{12}|kk''\rangle_A + \frac{1}{2} \sum_{k''} \langle kk'k''|V_{12}|kk''\rangle_A
= \langle k|t|k\rangle + \sum_{k'} \langle kk'|V_{12}|kk\rangle + \frac{1}{2} \sum_{k'} V_{12}|kk\rangle_A, \quad (4)
$$

where $t$ is a kinetic-energy operator. It is reasonable to define the single-particle energy which is used in the denominator of the $G$-matrix equation, Eq. (3), employing the continuous prescription for intermediate states as

$$
\epsilon_k = \langle k|t|k\rangle + U_G(k) \quad (5)
$$

$$
U_G(k) \equiv \sum_{k'} \langle kk'|G_{12} + \frac{1}{6} V_{12}|kk\rangle \left(1 + \frac{Q}{\omega - H}\right) G_{12}|kk\rangle, \quad (6)
$$

supposing that effects of the NN correlation is approximated by that of the $G$-matrix equation. To be specific, the denominator $\omega - H$ in the $G$-matrix equation for $G|kk\rangle$ is given by $\epsilon_{k_1'} + \epsilon_{k_2'} - (t_1 + U_G(k'_1) + t_2 + U_G(k'_2))$, where $k'_1$ and $k'_2$ are momenta of intermediate nucleons.
Solving the $G$-matrix equation together with the denominator explained above, the total energy is evaluated by:

$$E = \sum_{k} \langle k | t | k \rangle + \frac{1}{2} \sum_{k} U_{E}(k) \quad (7)$$

$$U_{E}(k) = \sum_{k'} \langle kk' | G_{12} | kk' \rangle \quad (8)$$

The difference between $U_G(k)$ for the energy calculation and $U_{E}(k)$ appeared in the single-particle energy is a prototype of rearrangement energy. Naturally, the above treatment of the 3NF is heuristic. It is desirable to develop a more rigorous and systematic perturbative treatment. One possible framework may be a coupled cluster method, which was discussed in Ref. [20].

In actual calculations in nuclear matter, a partial wave expansion [27] is introduced with an angle-average approximation for the Pauli exclusion operator $Q$. The good quality of this approximation has been examined in the literature [28]. The partial wave expansion of the reduced NN interaction, Eq. (1), is carried out in a standard way, which may be found in the paper by Fujiwara et al. [29]. Partial waves up to the total angular momentum $J = 7$ and the orbital angular momentum $\ell = 7$ are included in numerical calculations.

For completeness, explicit expressions of the reduced NN interactions of $V_C$, $V_D$, and $V_E$ parts and their partial wave contributions are given in Appendices A and B. Similar calculations were presented by Holt, Kaiser and Weise [30]. We, however, do not use an approximation for the off-diagonal matrix elements assumed there. It is possible to obtain analytical expressions for the partial wave expansion by introducing several functions in a form of the integration of Legendre polynomials of the second kind, as given in Eqs. (B1)-(B6). All terms in $V_C$ and $V_D$ yield central and tensor interactions. Spin-orbit components appear only in the $c_1$ and $c_3$ terms of $V_C$. The $V_E$ interaction gives only an $\ell = 0$ central component; that is, in the $^1S_0$ and $^3S_1$ channels.

Low-energy constants of the Ch-EFT interaction used in numerical calculations in the following sections are those of the Jülich group [8]: $c_1 = -0.81$ GeV$^{-1}$, $c_3 = -3.4$ GeV$^{-1}$, and $c_4 = 3.4$ GeV$^{-1}$. Other constants are taken from the Ref. [18]: $c_D = -4.381$ and $c_E = -1.126$. As noted in Appendix A, the reduced effective interaction $V_{12(3)}$ is multiplied by a form factor $\text{exp}(-q^2/4\Lambda^2 - (q/\Lambda)^6)$. We assume the same cutoff $\Lambda$ as in the NN sector.

III. NUMERICAL CALCULATIONS IN SYMMETRIC NUCLEAR MATTER

First, we present results of LOBT calculations in symmetric nuclear matter, using only the initial NN part of the Ch-EFT potential. It is expected that the obtained saturation curve is not much different from those of other modern NN potentials. The Ch-EFT potential is regularized by a rather soft form factor as the interaction based on low-energy effective theory. The nuclear-matter energy may depend considerably on the cutoff energy $\Lambda$ of the regulator. We show in the beginning the results with $\Lambda = 550$ MeV, and later discuss the $\Lambda$-dependence. The obtained saturation curve in symmetric nuclear matter is shown by a dashed curve in Fig. 1, compared with results of other NN potentials: AV18 [30], NSC [31], and CD-Bonn [32] potentials. It is seen that the very similar saturation curve to those of AV18 and NSC is obtained. AV18 and NSC have comparatively stronger tensor component than CD-Bonn, which is reflected in the larger deuteron D-state probability. Although the Ch-EFT interaction shows a smaller deuteron D-state probability, the LOBT energy is similar to those of AV18 and NSC.

When the effect of the 3NF is included by the procedure explained in Sect. 2, we obtain the solid curve shown in Fig. 1. As a basis for comparison, the saturation curve expected from the Gogny D1 force [33] is included. The calculation at higher densities than $k_F = 1.6$ fm$^{-1}$ is unreliable and not shown, because calculated s.p. ener-
gies wobble badly at large momentum beyond the normal density where the Ch-EFT as the low-energy theory is not to be applied especially when the reduce NN force is included. The saturation property is much improved by including $V_{12(3)}$, although the energy at the saturation point is shallow by a few MeV. The deviation is probably within the uncertainty of the lowest-order calculation on the one hand, and the uncertainties of low-energy constants as well as the ambiguity of cut-off parameters on the other. Therefore, the long-standing problem of microscopic understanding of the nuclear saturation seems to be resolved by the inclusion of the 3NF. This recognition may not be new, because the role of the 3NF has been demonstrated repeatedly in the literature \[14 \text{–} 16\]. However, previous calculations inevitably include phenomenological adjustment. The advantage of the present calculation with using the Ch-EFT 3NF interaction is that the potential is systematically constructed and is consistent with the NN sector. The $c_3$ term of the Ch-EFT 3NF is found to give dominant repulsive contribution to the energy. This coupling constant is determined in the NN sector and therefore no room for an additional adjustment.

To see the contributions of the 3NF in more details, we show, in Fig. 2, partial wave decomposition of the calculated potential energy. The attractive contribution in the $^3S_1$ channel is seen to increase by including the 3NF. This is due to the enhancement of the tensor correlation by the supplemented tensor force. On the other hand, the $^1S_0$ contribution becomes less attractive. The $p$-wave contributions depend much on the total-angular momentum $J$. This is owing to the rather strong spin-orbit component. It has been known that the net effect of the triplet $p$-wave contribution is small, which is a rather remarkable character of the NN interaction. This property persists after including the 3NF, but the net $^3O$ contribution becomes repulsive when the 3NF is incorporated. The repulsion gradually grows as the density goes up, is important for improving the description of the nuclear saturation property. On the other hand, the singlet $p$ channel is affected little by the 3NF. These characteristics of the 3NF contributions may be utilized for improving the effective interactions for density-dependent Hartree-Fock calculations and/or density functionals for medium-heavy nuclei.

It has been recognized in nuclear structure calculations that the two-body spin-orbit force is not sufficient to explain a strong single-particle spin-orbit field which is essential to describe empirical nuclear shell structures characterized by nuclear magic numbers. As was shown in the separate paper \[22\], the additional spin-orbit strength from the 3NF is favorable to provide the empirical strength of the one-body spin-orbit field. The strength of the nuclear one-body spin-orbit potential from nucleon-nucleon interactions is represented by the Scheerbaum factor $B_S(\vec{q})$, the definition of which is found in Ref. \[22\]. $B_S(\vec{q})$ corresponds to the spin-orbit strength $W$ of the $\delta$-type two-body spin-orbit interaction $iW(\sigma_1 + \sigma_2) \cdot \left[ \nabla \times (r) \nabla \right]$ customarily used in nuclear Hartree-Fock calculations. The empirical value of $W$ is around 120 MeV-fm$^3$. Because those results in Ref. \[22\] were simply obtained by $G_{12}$ and not by $G_{12} + \frac{1}{6}V_{12(3)} \left(1 + \frac{Q}{\omega^2} \right) G_{12}$ explained in Sec. II, we show revised numbers in Table I. The additional term

![FIG. 2: $k_F$-dependence of partial wave contributions to the nuclear matter LOBT energy per nucleon for the Ch-EFT interaction with $\Lambda = 550$ MeV. Thick and thin curves are with and without the 3NF effects, respectively; (a) full decomposition, (b) different $J$ being summed.](image-url)
makes the value of $B_S(\tilde{q})$ slightly larger.

Now, we address the cutoff-energy dependence of calculated LOBT energies. We show, in Fig. 3, saturation curves with using $\Lambda = 450$ MeV and $\Lambda = 600$ MeV, in addition to the case of $\Lambda = 550$ MeV presented in Fig. 1. When only the NN interactions are employed, the calculated energies depend considerably on $\Lambda$. The smaller cutoff energy provides larger binding-energies. The result with $\Lambda = 450$ MeV is rather close to that of the CD-Bonn potential given in Fig. 1. It is impressive to observe that calculated energies with different $\Lambda$ become very close each other when the 3NF is added. That is, the cutoff-energy dependence is significantly reduced if the NN and 3NF which are constructed consistently in the Ch-EFT are simultaneously used in the LOBT calculation.

Finally in this section, we remark on the quantitative difference between $U_G(k)$ and $U_E(k)$ defined in Eqs. (6) and (8), respectively. Figure 4 compares $U_G(k)$ and $U_E(k)$ with the NN force and 3NF for three cases of the cutoff energy $\Lambda$. The s.p. potential $U_E(k)$ without the 3NF effects is also shown. The difference of $U_G(k)$ and $U_E(k)$, which is $\sum_{k} \frac{1}{2} (kk')V_{12(3)} \left(1 + \frac{Q_{12}}{\omega_{FF}}\right) G_{12} |kk'\rangle_A$, is on the order of 5 MeV for $|k| \lesssim 2$ fm$^{-1}$. That is, the s.p. energy is raised by around 5 MeV by the additional term. Through the starting energy dependence of the $G$-matrix, the total energy per nucleon is lowered by about 0.5 MeV. The large $\Lambda$-dependence of the s.p. potential beyond $|k| = 3$ fm$^{-1}$ has no physical significance. As the results in Fig. 3 suggest, $U_E(k)$ for $|k| \lesssim 2$ fm$^{-1}$ does not depend much on the cutoff energy, when the 3NF is included.

### IV. Numerical Calculations in Pure Neutron Matter

The energy per nucleon of neutron matter is fundamental to determine properties of neutron star matter. The $k_F$ dependence of calculated LOBT energies in pure neutron matter with and without including the 3NF is shown in Fig 5. Energies obtained with other modern NN potentials and results of the variational calculation by the Illinois group [15, 16] are also presented for comparison. The latter used the AV18 potential [30] and included the

| $k_F$ [fm$^{-1}$] | Nuclear Matter | Neutron Matter |
|------------------|----------------|----------------|
| $B_S(T = 0)$     | 2.5            | 7.3            |
| $B_S(T = 1)$     | 84.6           | 120.2          | 84.7 | 93.3          |

TABLE I: Scheerbaum factor $B_S(\tilde{q})$ in units of MeV-fm$^5$ with $\tilde{q} = 0.7$ fm$^{-1}$ for Jülich N$^3$LO [27] with and without 3NF. The $G$-matrix in Ref. [27] is replaced by $G_{12} + \frac{1}{2} V_{12(3)} \left(1 + \frac{Q_{12}}{\omega_{FF}}\right) G_{12}$ in this calculation.
3NF of the Fujita-Miyazawa [11] type supplemented by phenomenological terms. Because the strong tensor effect in the $^3S_1-^3D_1$ channel is absent, many-body correlations are relatively simple in neutron matter. Since the calculated saturation curve in symmetric nuclear matter already well corresponds to the empirical one, the present LOBT energy in neutron matter is expected to be trustful. In contrast to the symmetric nuclear matter, calculated energies with different NN potentials are very similar, as is seen in the $k_F$-dependence of neutron matter energies with Ch-EFT, AV18, NSC and CD-Bonn potentials in Fig. 5.

The Ch-EFT 3NF itself is more predictive for the application to neutron matter, because the contact $c_E$ term vanishes in pure neutron matter as the Pauli principle forbids three neutrons to assemble at the same place, and the $c_D$ term which gives null in the plane wave case gives a negligibly small contribution. In addition, the $c_4$ term does not contribute. Thus the contribution from the NNLO 3NF is determined by the $c_1$ and $c_3$ terms. These coupling constants are determined in the NN sector.

The results of the variational calculation in Ref. [10] shown in Fig. 5 have been utilized as the canonical equation of state (EoS) for discussing neutron star matter properties. It is interesting to see that the LOBT result obtained with including the 3NF, in which no phenomenological adjustment is introduced, is close to the EoS of Ref. [10].

As the Ch-EFT cannot be applied to high momentum region, it is not possible to discuss directly the EoS relevant to the core of high-density neutron stars. However, it is possible to provide the reference EoS at lower densities which should be smoothly matched to the EoS obtained by theories designed for the high density region. Such an attempt was recently reported in Ref. [34].

As was noted in Ref. [22], the magnitude of the spin-orbit component in $V_{12(3)}$ obtained in pure neutron matter is one third of that in symmetric nuclear matter. Although correlations somewhat modifies this number, as is given in Table I, the calculated additional contribution to the Scheerbaum factor from the 3NF in neutron matter is about $\frac{1}{3}$ of that in nuclear matter. Observing that the contribution of the genuine NN interaction to the s.p. spin-orbit strength is insensitive to the neutron-proton asymmetry $\alpha = \frac{N-Z}{N+Z}$, the 3NF can be the source of the asymmetry dependence of the strength of the s.p. spin-orbit potential. In a Woods-Saxon potential model, rather strong $\alpha$-dependence, such as $(1 - 0.54\alpha)$, was inferred, as in the textbook by Bohr-Mottelson [35]. Recent fitting [36] gives gentler $\alpha$-dependence, typically $1 - 0.25\alpha$. If we naively use the calculated numbers given in Table I and assume that $B_S(T = 1)$ in neutron matter depends little on the density, the $\alpha$-dependence of the s.p. spin-orbit strength is estimated as $(1 - 0.22\alpha)$, which is consistent with the value mentioned above.

Finally, Fig. 6 shows the variation in the neutron matter energy for a different choice of the cutoff energy $\Lambda$. The A-dependence of the calculated energies is already moderate in the case of the NN interaction only.
V. SUMMARY

We have calculated LOBT energies both in symmetric nuclear matter and pure neutron matter, using the Ch-EFT N^3LO NN interaction and NNLO 3NF of the Jülich group [8]. In the Ch-EFT, the 3NF is introduced in a systematic way along with the NN potential. Three of 5 coupling constants in the NNLO 3NF are fixed in a NN sector. The remaining two parameters are under control in the literature to reproduce properties of few-nucleon systems. The 3NF is treated by reducing it to density-dependent NN interactions by folding single-nucleon degrees of freedom in infinite matter. We have given, in the Appendices, explicit expressions of the reduced NN interactions and their partial-wave expanded forms.

Calculated results show that the empirical saturation property is well reproduced in nuclear matter. In a conventional understanding, effects of the Pauli blocking for the strong tensor coupling have been emphasized as the basic mechanism of causing the nuclear saturation property. Though this effect is fundamentally important, the sizable repulsive contribution of the 3NF is also crucial in the region around and above the normal nuclear matter density. This indicates that the Pauli blocking not only for the standard tensor correlation but also for other non-nucleonic excitations inherent in the two-nucleon process, such as isobar ∆ and anti-nucleon excitations, provides large repulsive effects.

It is noteworthy that the large cutoff-energy dependence of calculated energies obtained only with the Ch-EFT NN force reduces substantially when including the 3NF effects. This dependence arises predominantly in the triplet even channel; that is, the channel in which the tensor correlation is significant. Therefore, the cutoff-energy dependence is rather weak in neutron matter.

Contributions of the 3NF in the $^1E$ and $^3O$ channels are repulsive. Owing to the repulsion, the density-dependence of neutron matter energy per nucleon becomes very close to those favorable for describing neutron star properties, although the prediction of the Ch-EFT cannot be applied at high densities. The strength of the spin-orbit component in the $^3O$ channel increases by about 30%, which resolves the problem of the insufficiency of modern NN potentials to account for the empirical spin-orbit strength, as previously reported in Ref. [22]. The potential energy in the $^3S_1$ state turns out to become more attractive due to the enhancement of the tensor component. The knowledge of these specific properties of the 3NF contributions may be helpful for improving effective forces and/or energy-functionals for finite nuclei.

We conclude that although more rigorous treatment of the 3NF together with more than three-body correlations are required in future, the present calculations demonstrate that the 3NF constructed consistently with the NN part in the sense of effective field theory can reproduce basic nuclear properties, namely the saturation and strong spin-orbit field, without phenomenological adjustments.

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Appendix A: Effective NN forces from the 3NF in chiral effective field theory

In the leading order, NNLO, three-nucleon force $V_{123}$ consists of terms specified by five low-energy coupling constants $c_1$, $c_3$, $c_4$, $c_D$, and $c_E$: $V_{123} = V_C + V_D + V_E$. Each term is given as follows.

$$V_C = \frac{1}{2} \left( \frac{g_A}{2f_π} \right)^2 \sum_{i \neq j \neq k} \frac{(σ_i \cdot q_i)(σ_j \cdot q_j)}{(q_i^2 + m_π^2)(q_j^2 + m_π^2)^2} \left\{ \delta^{αβ} \left[ \frac{−4c_1m_π^2}{f_π^2} + \frac{2c_3}{f_π^2} q_i \cdot q_j \right] + \sum_γ \frac{c_4}{f_π^2} e^{αβγ} τ_k \cdot (q_i \times q_j) \right\}, \quad (A1)$$

$$V_D = -\frac{g_A}{8f_π^2 f_π^2 A_π} \sum_{i \neq j \neq k} \frac{(σ_j \cdot q_j)(σ_i \cdot q_i)}{q_j^2 + m_π^2} (τ_i \cdot τ_j), \quad (A2)$$

$$V_E = \frac{c_E}{2f_π^2 A_π} \sum_{j \neq k} (τ_j \cdot τ_k) = \frac{c_E}{f_π^2 A_π} \left( τ_1 \cdot τ_2 + τ_2 \cdot τ_3 + τ_3 \cdot τ_1 \right). \quad (A3)$$

The three coupling constants $c_1$, $c_3$, and $c_4$ are determined in the NN sector and the remaining $c_D$ and $c_E$ are adjusted in more than three nucleon systems. As is explained in Eq. (1), the three-nucleon force $V_{123}$ is reduced to an effective NN force $V_{12(3)}$ by summing over the third nucleon in the Fermi sea:

$$\langle k_1′σ_1′τ_1′, -k_1′σ_2′τ_2′|V_{12(3)}|k_1σ_1τ_1, -k_1σ_2τ_2⟩_a \equiv \sum_{k_3,σ_3τ_3} \langle k_1′σ_1′τ_1′, -k_1′σ_2′τ_2′, k_3σ_3τ_3|V_{123}|k_1σ_1τ_1, -k_1σ_2τ_2, k_3σ_3τ_3⟩_a. \quad (A4)$$

Form factors are not taken into account in this folding procedure. The obtained $V_{12(3)}$ is multiplied by a form factor in the form of $\exp\{−(q/Λ)^6 − (q/Λ)^6\}$.

In this appendix, we present details of the matrix elements of $\langle k_1′σ_1′τ_1′, -k_1′σ_2′τ_2′|V_{12(3)}|k_1σ_1τ_1, -k_1σ_2τ_2⟩$ from ChEFT 3NF forces $V_C$, $V_D$, and $V_E$. In the following, we use the standard notation for the tensor operator $S_{12}(k′, k)$ and their matrix elements between partial waves:

$$S_{12}(k′, k) = 3[(σ_1 × σ_2)^2] \cdot [k′ × k]^2, \quad (A5)$$

$$\langle S_{12}⟩_{J_1 J}^{ℓ′} = \frac{6\sqrt{J(J+1)}}{2J+1} \text{ for } ℓ′ = ℓ + 2 \text{ or } ℓ′ = ℓ − 2, \quad (A6)$$

$$\langle S_{12}⟩_{J J+1 J+1}^{J} = −\frac{2(J+2)}{2J+1}, \text{ and } \langle S_{12}⟩_{J J−1 J−1}^{J} = −\frac{2(J−1)}{2J+1}. \quad (A7)$$

Evaluating Eq. (A4), the $c_1$ term of $V_C$ provides

$$\frac{c_1g_A^2m_π^2ρ_0}{f_π^2} (σ_1 \cdot (k′_1 − k_1))(σ_2 \cdot (k′_1 − k_1)) \left( τ_1 \cdot τ_2 \right)$$

$$+ \frac{c_1g_A^2m_π^2}{f_π^2} \sum_{k_3} \left( \frac{2(σ_1 \cdot σ_2)(k′_1 − k_1)}{(k′_1 − k_1)^2 + m_π^2} \right) \left( k_3 + k_1 \right) + 2(σ_1 × σ_2)^2 \cdot ([k′_1 − k_1] × (k_3 + k_1))^2 \right)$$

$$+ \frac{2i(σ_1 × σ_2)(k′_1 − k_3) \cdot (k′_1 − k_1) + 2(σ_1 × σ_2)^2 \cdot ([k′_1 − k_3] × (k′_1 − k_1)]^2)}{(k′_1 − k_3)^2 + m_π^2} \left( τ_1 \cdot τ_2 \right)$$

$$+ \frac{6(k′_1 − k_3) \cdot (k_3 − k_1) + 3i(σ_1 + σ_2) \cdot ((k′_1 − k_3) × (k_3 − k_1))}{((k′_1 − k_3)^2 + m_π^2)((k′_1 − k_3)^2 + m_π^2)} \left( τ_1 \cdot τ_2 \right), \quad (A8)$$
the $c_3$ term of $V_C$

\[
\frac{c_3 g_A^2}{2 f_\pi^2} (\sigma_1 \cdot (k_1' - k_1)) (\sigma_2 \cdot (k_1' - k_1)) |k_1' - k_1|^2 (\tau_1 \cdot \tau_2)
\]

\[
- \frac{c_3 g_A^2}{2 f_\pi^2} \frac{2 (\tau_1 \cdot \tau_2)}{(k_1' - k_1)^2 + m_\pi^2} \{((k_1' - k_1) \cdot k_1)^2 (F_0(k_1) - 2 F_1(k_1)) + \frac{1}{3} (k_1' - k_1)^2 (k_1^2 F_2(k_1) + k_1^2 F_2(k_1'))
\]

\[
+ \frac{1}{k_1} \left( [(k_1' - k_1) \times (k_1' - k_1)]^2 \cdot [k_1 \times k_1]^2 \right) [F_3(k_1) + \frac{1}{k_1^2} ( [k_1' \times k_1')^2 k_1^2 F_3(k_1')]
\]

\[
+ \frac{c_3 g_A^2}{2 f_\pi^2} \frac{2 (\tau_1 \cdot \tau_2)}{(k_1' - k_1)^2 + m_\pi^2} \left\{ \frac{1}{3} S_{12}(k_1' - k_1, k_1)((k_1' - k_1) \cdot k_1) (F_0(k_1) - 2 F_1(k_1))
\]

\[
+ \frac{1}{3} S_{12}(k_1' - k_1, k_1)((k_1' - k_1) \cdot k_1)(F_0(k_1') - 2 F_1(k_1')) + \frac{1}{9} S_{12}(k_1' - k_1, k_1'(k_1^2 F_2(k_1) + k_1^2 F_2(k_1'))
\]

\[
- \frac{1}{9} S_{12}(k_1, k_1')((-2 k_1^2 + 3(k_1' - k_1)) + S_{12}(k_1', k_1)k_1^2 + S_{12}(k_1, k_1)(k_1^2 - 3(k_1' - k_1))) F_3(k_1)
\]

\[
- \frac{1}{9} S_{12}(k_1', k_1')((-2 k_1'^2 + 3(k_1 - k_1')) + S_{12}(k_1, k_1')k_1'^2 + S_{12}(k_1', k_1)(k_1^2 - 3(k_1' - k_1'))) F_3(k_1')
\]

\[
+ \frac{c_3 g_A^2}{2 f_\pi^2} \sum_{k_3} \frac{6(k_1' - k_3) \cdot (k_3 - k_1) + 3i(\sigma_1 + \sigma_2) \cdot ((k_1' - k_1) \times k_3 - k_1' \times k_1)}{((k_1' - k_3)^2 + m_\pi^2)((k_3 - k_1)^2 + m_\pi^2)} (\tau_1 \cdot \tau_2).
\]

and the $c_4$ term of $V_C$

\[
\frac{2 c_4 g_A^2}{4 f_\pi^2} \sum_{k_3} \left\{ \frac{(\sigma_1 \cdot (k_1' - k_1)) (\sigma_2 \cdot ((-k_1' - k_3) \times ((k_1' - k_1) \times (-k_1' - k_3))))}{((k_1' - k_1)^2 + m_\pi^2)((-k_1' - k_3)^2 + m_\pi^2)}}
\]

\[
+ \frac{(\sigma_1 \cdot (k_1' - k_1)) (\sigma_2 \cdot ((k_3 + k_1) \times (k_1' - k_1)) \times (k_3 + k_1))}{((k_1' - k_1)^2 + m_\pi^2)((k_3 + k_1)^2 + m_\pi^2)}}
\]

\[
+ \frac{(\sigma_1 \cdot (k_1' - k_3)) (\sigma_2 \cdot ((k_1' - k_3) \times (k_1 - k_3)))}{((k_1' - k_3)^2 + m_\pi^2)((k_1 - k_3)^2 + m_\pi^2)}}
\]

\[
- \frac{(\sigma_1 \cdot (k_1' - k_3)) (\sigma_2 \cdot ((k_1' - k_3) \times (-k_1' - k_3))) (\sigma_2 \cdot (-k_1' + k_1))}{((-k_1' + k_1)^2 + m_\pi^2)((k_1' - k_3)^2 + m_\pi^2)}}
\]

\[
- \frac{(\sigma_1 \cdot (k_1' - k_3)) (\sigma_2 \cdot ((k_3 - k_1) \times (-k_1' + k_3))) (\sigma_2 \cdot (-k_1' + k_1))}{((-k_1' + k_1)^2 + m_\pi^2)((k_3 - k_1)^2 + m_\pi^2)}}
\]

\[
+ \frac{(\sigma_1 \cdot (k_1' + k_1) \times (k_3 - k_1)) \times (k_3 - k_1)) (\sigma_2 \cdot (-k_1' + k_1))}{((-k_1' + k_1)^2 + m_\pi^2)((k_3 - k_1)^2 + m_\pi^2)}}
\]

\[
(\tau_1 \cdot \tau_2).
\]

The $V_D$ term is found to yield

\[
\frac{g_A}{8 f_\pi^2} \frac{c_D p_0}{f_\pi^2} \left\{ \frac{1}{2} (\sigma_1 \cdot \sigma_2) (k_1' - k_1)^2 + (\sigma_1 \times \sigma_2)^2 \cdot [k_1' - k_1) \times (k_1' - k_1)]^2 \right\} (\tau_1 \cdot \tau_2)
\]

\[
+ \frac{2 g_A}{8 f_\pi^2} \frac{c_D}{f_\pi^2} \Lambda_X \left\{ \frac{1}{3} (\sigma_1 \cdot \sigma_2) \left( \frac{1}{2} p_0 - m_\pi^2 F_0(k_1) - m_\pi^2 F_0(k_1') \right) \right\}
\]

\[
+ \left( (\sigma_1 \times \sigma_2)^2 \cdot [(k_1' - k_1) \times (k_1' - k_1)] \right) (F_0(k_1) - 2 F_1(k_1) + F_3(k_1) + F_0(k_1') - 2 F_1(k_1') + F_3(k_1')) (\tau_1 \cdot \tau_2)
\]

\[
+ \frac{6 g_A}{8 f_\pi^2} \frac{c_D}{f_\pi^2} \Lambda_X \left\{ \frac{1}{2} p_0 - m_\pi^2 F_0(k_1) - m_\pi^2 F_0(k_1') \right\}.
\]

Finally, the $V_E$ term gives a spin- and isospin-scalar interaction:

\[
- \frac{6 C_F}{f_\pi^2} \frac{1}{2} p_0
\]

(A12)
In the above expressions, (A9) and (A11), functions $F_0$, $F_1$, $F_2$, and $F_4$ are defined as follows.

\[
F_0(k) = \frac{1}{(2\pi)^3} \int \frac{dk'}{k'_{\leq k_F}} \frac{1}{(k - k')^2 + m_\pi^2} \left( k_F + \frac{k_F^2 + m_\pi^2 - k^2}{4k} \log \left( \frac{(k + k_F)^2 + m_\pi^2}{(k - k_F)^2 + m_\pi^2} \right) - m_\pi \left( \arctan \frac{k + k_F}{m_\pi} - \arctan \frac{k - k_F}{m_\pi} \right) \right),
\]
(A13)

\[
F_1(k) = \frac{1}{k} \frac{1}{(2\pi)^3} \int \frac{dk'}{k'_{\leq k_F}} \frac{k' \cos \theta}{(k - k')^2 + m_\pi^2} \left( \arctan \frac{k + k_F}{m_\pi} - \arctan \frac{k - k_F}{m_\pi} \right) + \frac{1}{16k^2} \left( m_\pi^2 + (3k^2 + k_F^2 - k^2) \log \left( \frac{(k + k_F)^2 + m_\pi^2}{(k - k_F)^2 + m_\pi^2} \right) \right),
\]
(A14)

\[
F_2(k) = \frac{2\pi}{(2\pi)^3} \frac{1}{k^3} \int_0^{k_F} dk' \cos Q_0 \left( \frac{k^2 + k'^2 + m_\pi^2}{2kk'} \right)
\]
\[
= \frac{1}{(2\pi)^2} \frac{1}{k^2} \left( \frac{1}{6} k_F(3k^2 + k_F - 9m_\pi^2) + \frac{(k_F^2 - k^4 - m_\pi^2 + 6k^2m_\pi^2)}{8k} \log \left( \frac{(k + k_F)^2 + m_\pi^2}{(k - k_F)^2 + m_\pi^2} \right) \right) + m_\pi(m_\pi^2 - k^2) \left( \arctan \frac{k + k_F}{m_\pi} - \arctan \frac{k - k_F}{m_\pi} \right),
\]
(A15)

\[
F_3(k) = \frac{2\pi}{(2\pi)^3} \frac{1}{k^3} \int_0^{k_F} dk' \cos Q_2 \left( \frac{k^2 + k'^2 + m_\pi^2}{2kk'} \right)
\]
\[
= \frac{1}{k^2} \frac{1}{(2\pi)^2} \left[ \frac{1}{32k^3} [(k_F^2 + k^2 + m_\pi^2)^3 + 2k^2m_\pi^2(m_\pi^2 + 6k^2) - 2k^2(k_F^4 + 3k^4)] \times \log \left( \frac{(k + k_F)^2 + m_\pi^2}{(k - k_F)^2 + m_\pi^2} \right) - k_F \left( m_\pi^2 + 4k^2m_\pi^2 + k_F^2 - 5k^4 + \frac{4}{3} k^2k_F^2 + 2m_\pi^2k_F^2 \right) \right). \]
(A16)

**Appendix B: Partial wave expansion**

The expressions of the Born kernel $\langle k'_1 \sigma_1^1 \tau_1^1, -k'_2 \sigma_2^2 \tau_2^2 | V_{12}(3)| k_1 \sigma_1 \tau_1, -k_1 \sigma_2 \tau_2 \rangle$ in the previous section need to be expanded into partial waves for standard nuclear-matter G-matrix calculations. The procedure may be found in Ref. 20. We use the abbreviated notations for integrals involving second kind Legendre functions $Q_\ell$'s.

\[
Q_{W_0}(k_1', k_1) \equiv \frac{1}{(2\pi)^2} \int_0^{k_F} dk_3 Q_\ell(x')Q_\ell(x),
\]
(B1)

\[
Q_{W_2}(k_1', k_1) \equiv \frac{1}{(2\pi)^2} \frac{1}{2k_3'} \int_0^{k_F} dk_3 Q_\ell(x')Q_\ell(x),
\]
(B2)

\[
Q_{W_4}(k_1', k_1) \equiv \frac{1}{(2\pi)^2} \frac{1}{2(k_3')^2} \int_0^{k_F} dk_3 Q_\ell(x')Q_\ell(x),
\]
(B3)

\[
Q_{W_1}(k_1', k_1) \equiv \frac{1}{(2\pi)^2} \frac{1}{2k_3'} \int_0^{k_F} dk_3 x' Q_\ell(x')Q_\ell(x),
\]
(B4)

\[
Q_{W_1}(k_1, k_1') \equiv \frac{1}{(2\pi)^2} \frac{1}{2k_3} \int_0^{k_F} dk_3 x Q_\ell(x')Q_\ell(x),
\]
(B5)

where $x \equiv \frac{k_1^2 + k_3^2 + m_\pi^2}{2k_1k_3}$ and $x' \equiv \frac{k_1^2 + k_3^2 + m_\pi^2}{2k_1'k_3'}$. 
The central component of the $c_1$ interaction of $V_C$, Eq. (A8), with an orbital angular momentum $\ell$ is

$$
\frac{c_1 g^2 m^2 \rho_0}{f^4 \pi^3} \left( \sigma_1 \cdot \sigma_2 \right) (\tau_1 \cdot \tau_2) \left( \frac{1}{2k_1 k_1} Q_\ell(z) + k_1^2 \frac{1}{2k_1 k_1} Q_{\ell'}(z) - k_1 k_1 Q_{\ell'}(z) - Q_\ell(z) \right)
$$

$$
- \left( \frac{2}{3} \frac{c_1 g^2 m^2 \rho_0}{f^4 \pi} (\sigma_1 \cdot \sigma_2) (\tau_1 \cdot \tau_2) \left( k_1^2 \frac{1}{2k_1 k_1} Q_\ell(z) + k_1^2 \frac{1}{2k_1 k_1} Q_{\ell'}(z) - \frac{1}{2} k_1 k_1 Q_{\ell'}(z) - \frac{1}{2} Q_\ell(z) \right) \right)
$$

$$
+ \frac{1}{2} \left( \frac{2\ell + 3}{2\ell + 1} Q_{\ell-1}(z) + \frac{2\ell - 1}{2\ell + 1} Q_{\ell+1}(z) \right) \right)
$$

for $\ell' = \ell \pm 1$. The spin-orbit component of the $c_1$ term of $V_C$, Eq. (A8), becomes

$$
\delta_{S1} \frac{c_1 g^2 m^2 \rho_0}{f^4 \pi^3} \frac{\ell(\ell + 1) + 2 - J(J + 1)}{2\ell + 1} \{ -Q_{W1}^{\ell-1}(k'_1, k_1) + Q_{W1}^{\ell+1}(k'_1, k_1) + W_{\ell\ell,0}^{\ell}(k'_1, k_1) \},
$$

where the function $W_{\ell\ell,0}^{\ell}(k'_1, k_1)$ is defined as

$$
W_{\ell\ell,0}^{\ell}(k'_1, k_1) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dk_3 k_3^3 4k_1' k_1 \left( k_1' Q_\ell(x)(Q_{\ell-1}(x') - Q_{\ell+1}(x')) + k_1 Q_\ell(x)(Q_{\ell-1}(x) - Q_{\ell+1}(x)) \right).
$$
The tensor components of the $c_3$ interaction of $V_C$, Eq. (A9), is

\[
\frac{c_3^2 g_0}{2 f_{\pi}^4} \frac{1}{3} (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) \left\{ \delta_{00} - \frac{m_\pi^2}{2k'_{1}k_1} Q_{\ell}(z) + \left( \frac{m_\pi^2}{2k'_{1}k_1} \right)^2 \left( \frac{\ell+1}{2} - 1 \right) (zQ_{\ell}(z) - Q_{\ell+1}(z)) \right\} 
- \frac{c_3^2 g_0}{2 f_{\pi}^4} \frac{2}{3} (\tau_1 \cdot \tau_2)(\sigma_1 \cdot \sigma_2) \left\{ \frac{1}{3} \left[ \delta_{00} - \frac{m_\pi^2}{2k'_{1}k_1} Q_{\ell}(z) \right] \left( k'_1 F_2(k_1) + k^2 F_2(k'_1) \right) 
+ \left[ \frac{1}{2} Q_{\ell}^{(1)}(z) + \frac{k^2}{2k'_{1}k_1} Q_{\ell}(z) \right] \left( F_0(k_1) - 2F_1(k_1) \right) 
+ \left[ \frac{1}{2} k'_1 \left( 1 - \frac{1}{2} k'_{1} - k_1 + \frac{1}{4} k_1 k'_{1} \right) + \frac{1}{2} (k_1^2 - k'_1)^2 + m_\pi^2 \right] \delta_{00} \right\} 
+ \left[ \left( \frac{k'_1}{2k'_{1}k_1} \right)^2 \left( k'_1^2 + k_1^2 + m_\pi^2 \right)^2 \left( k'_1^2 + k_1^2 + m_\pi^2 \right) - \frac{4}{3} k'_1 k_1 \right] \delta_{00} \right\} 
+ \left[ \left( \frac{k'_1}{2k'_{1}k_1} \right)^2 \left( k'_1^2 + k_1 + m_\pi^2 \right)^2 - k'_1^2 - \frac{2}{3} m_\pi^2 \right] \left( \frac{1}{2k_1' k_1} Q_{\ell}(z) \right] \left( k'_1 F_3(k_1) \right) 
+ \left[ \left( \frac{k'_1}{2k'_{1}k_1} \right)^2 \left( k'_1 + k_1 + m_\pi^2 \right)^2 - k'_1 - \frac{2}{3} m_\pi^2 \right] \left( \frac{1}{2k_1' k_1} Q_{\ell}(z) \right] \left( k'_1 F_3(k'_1) \right) 
- \frac{c_3^2 g_0}{2 f_{\pi}^4} \frac{1}{3} (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) \left\{ \delta_{00} - \frac{m_\pi^2}{2k'_{1}k_1} Q_{\ell}(z) + \left( \frac{m_\pi^2}{2k'_{1}k_1} \right)^2 \left( \frac{\ell+1}{2} - 1 \right) (zQ_{\ell}(z) - Q_{\ell+1}(z)) \right\} 
- \frac{c_3^2 g_0}{2 f_{\pi}^4} \frac{2}{3} (\tau_1 \cdot \tau_2)(\sigma_1 \cdot \sigma_2) \left\{ \frac{1}{3} \left[ \delta_{00} - \frac{m_\pi^2}{2k'_{1}k_1} Q_{\ell}(z) \right] \left( k'_1 F_2(k_1) + k^2 F_2(k'_1) \right) 
+ \left[ \frac{1}{2} Q_{\ell}^{(1)}(z) + \frac{k^2}{2k'_{1}k_1} Q_{\ell}(z) \right] \left( F_0(k_1) - 2F_1(k_1) \right) 
+ \left[ \frac{1}{2} k'_1 \left( 1 - \frac{1}{2} k'_{1} - k_1 + \frac{1}{4} k_1 k'_{1} \right) + \frac{1}{2} (k_1^2 - k'_1)^2 + m_\pi^2 \right] \delta_{00} \right\} 
+ \left[ \left( \frac{k'_1}{2k'_{1}k_1} \right)^2 \left( k'_1^2 + k_1^2 + m_\pi^2 \right)^2 \left( k'_1^2 + k_1^2 + m_\pi^2 \right) - \frac{4}{3} k'_1 k_1 \right] \delta_{00} \right\} 
+ \left[ \left( \frac{k'_1}{2k'_{1}k_1} \right)^2 \left( k'_1 + k_1^2 + m_\pi^2 \right)^2 - k'_1^2 - \frac{2}{3} m_\pi^2 \right] \left( \frac{1}{2k_1' k_1} Q_{\ell}(z) \right] \left( k'_1 F_3(k_1) \right) 
+ \left[ \left( \frac{k'_1}{2k'_{1}k_1} \right)^2 \left( k'_1 + k_1^2 + m_\pi^2 \right)^2 - k'_1^2 - \frac{2}{3} m_\pi^2 \right] \left( \frac{1}{2k_1' k_1} Q_{\ell}(z) \right] \left( k'_1 F_3(k'_1) \right) \right\} 
+ \left( \frac{1}{8} \rho_0 - \frac{3}{4} \rho_0 + \frac{1}{2} k'_1 + \frac{1}{4} k_1 \right) \left( F_0(k_1) - \frac{3}{4} m_\pi^2 + \frac{1}{2} k'_1 + \frac{1}{4} k_1 \right) F_0(k_1) 
+ \left( \frac{1}{4} \left( k^2 F_2(k'_1) + k^2 F_2(k_1) \right) \right] \delta_{00} \right\} 
+ \left( \frac{1}{4} \left( k^2 F_2(k'_1) + k^2 F_2(k_1) \right) \right] \delta_{00} \right\} 
+ \left( \frac{1}{4} \frac{k'_1}{k_1} \left( k'_1^2 + k_1 + m_\pi^2 \right)^2 Q_{W_0}(k'_1, k_1) - \left( k'_1^2 + k_1 + m_\pi^2 \right)^2 Q_{W_0}(k'_1, k_1) + \frac{\ell+1}{\ell} Q_{W_0}^{(1)}(k'_1, k_1) \right\} 
+ \left( \frac{1}{\ell} \frac{k'_1 k_1}{2k_1' k_1} \right) \left[ \frac{\ell+1}{\ell} Q_{W_0}^{(1)}(k'_1, k_1) + \frac{\ell}{\ell+1} (\ell+2) \right] \left( \frac{1}{2} \frac{Q_{W_0}^{(1)}(k'_1, k_1) + \frac{\ell+1}{\ell} Q_{W_0}^{(1)}(k'_1, k_1) }{2\ell+3} \right) \right\} \right\} (B11)

The tensor components of the $c_3$ interaction of $V_C$, Eq. (A8), are

\[
\frac{c_3^2 g_0}{2 f_{\pi}^4} \frac{1}{3} (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) \left\{ k^2 \frac{1}{2k'_{1}k_1} \left( Q_{\ell}(z) + \frac{m_\pi^2}{2k'_{1}k_1} Q_{\ell}(z) \right) 
+ \frac{k^2}{2k'_{1}k_1} \left( Q_{\ell}(z) + \frac{m_\pi^2}{2k'_{1}k_1} Q_{\ell}(z) \right) - \left( Q_J(z) + \frac{m_\pi^2}{2k'_{1}k_1} Q_J(z) \right) \right\} 
- \frac{c_3^2 g_0}{2 f_{\pi}^4} \frac{2}{3} (\tau_1 \cdot \tau_2)(\sigma_1 \cdot \sigma_2) \left\{ \left( F_0(k_1) - 2F_1(k_1) \right) \left( \frac{1}{2} Q_{\ell}^{(1)}(z) - \frac{k^2}{2k'_{1}k_1} Q_{\ell}(z) \right) \right\} 
- \frac{1}{3} \left( k^2 F_3(k_1) \right] \left( k^2 F_3(k'_1) \right] \left( k^2 F_3(k'_1) \right] \left( k^2 F_3(k'_1) \right] \left( \frac{1}{2} Q_{\ell}^{(1)}(z) + \frac{k^2}{2k'_{1}k_1} Q_{\ell}(z) \right) \right\} 
+ \left( \frac{1}{3} \left( k^2 F_3(k_1) \right] \left( k^2 F_3(k'_1) \right] \left( k^2 F_3(k'_1) \right] \left( k^2 F_3(k'_1) \right] \left( \frac{1}{2} Q_{\ell}^{(1)}(z) + \frac{k^2}{2k'_{1}k_1} Q_{\ell}(z) \right) \right\} (B12)
for \( \ell' = \ell \pm 1 \) (\( J = \ell \pm 1 \) and)

\[
\frac{c_3 g_A^2}{2 f_\pi^3} \left( S_{12} \right)_{\ell\ell'} \rho_0 \left( \tau_1 \cdot \tau_2 \right) \left[ \frac{k_{m,0}^2 + k_0^2}{2k_{1,k_1}} \left( Q_{\ell}(z) + \frac{m_\pi^2}{2k_{1,k_1}^2} Q_{\ell'}(z) \right) - \frac{1}{2} \left\{ \frac{2\ell + 3}{2\ell + 1} \left( Q_{\ell-1}(z) + \frac{m_\pi^2}{2k_{1,k_1}^2} Q_{\ell-1}^{(1)}(z) \right) + \frac{2\ell - 1}{2\ell + 1} \left( Q_{\ell+1}(z) + \frac{m_\pi^2}{2k_{1,k_1}^2} Q_{\ell+1}^{(1)}(z) \right) \right\} - \frac{c_3 g_A^2}{2 f_\pi^3} \left( \tau_1 \cdot \tau_2 \right) \left( F_0(k_1) - 2 F_1(k_1) \right) \left( -k_{1,k_1}^2 \left( \frac{1}{2} Q_{\ell}(z) - \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell}(z) \right) + \frac{1}{2} k_{1,k_1}^2 \left( \frac{2\ell + 3}{2\ell + 1} \left( \frac{1}{2} Q_{\ell-1}(z) - \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell-1}(z) \right) + \frac{2\ell - 1}{2\ell + 1} \left( \frac{1}{2} Q_{\ell+1}(z) - \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell+1}(z) \right) \right) \right) + \frac{1}{2} k_{1,k_1}^2 \left( \frac{2\ell + 3}{2\ell + 1} \left( \frac{1}{2} Q_{\ell-1}(z) - \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell-1}(z) \right) + \frac{2\ell - 1}{2\ell + 1} \left( \frac{1}{2} Q_{\ell+1}(z) - \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell+1}(z) \right) \right) \right) + \frac{1}{3} (2k_{1,k_1}^2 F_3(k_1) - k_{1,k_1}^2 F_3(k'_1)) \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell}(z) - \frac{1}{2} k_{1,k_1}^2 F_3(k_1) Q_{\ell}^{(1)}(z) + \frac{1}{3} (2k_{1,k_1}^2 F_3(k'_1) - k_{1,k_1}^2 F_3(k_1)) \frac{k_{1,k_1}^2}{2k_{1,k_1}^2} Q_{\ell}(z) - \frac{1}{2} k_{1,k_1}^2 F_3(k_1) Q_{\ell}^{(1)}(z) - \frac{1}{3} (k_{1,k_1}^2 F_3(k_1) + k_{1,k_1}^2 F_3(k'_1)) \frac{1}{4} \left( \frac{2\ell + 3}{2\ell + 1} Q_{\ell-1}(z) + \frac{2\ell - 1}{2\ell + 1} Q_{\ell+1}(z) \right) + k_{1,k_1}^2 (F_3(k'_1) + F_3(k_1)) \frac{1}{4} \left( \frac{2\ell + 3}{2\ell + 1} Q_{\ell-1}(z) + \frac{2\ell - 1}{2\ell + 1} Q_{\ell+1}(z) \right) \right) \right)
\]

(B13)

for \( \ell' = \ell \pm 1 \). The spin-orbit component of the \( c_3 \) term of \( V_C \), Eq. (A9), becomes

\[
\delta_{s_1} \frac{c_3 g_A^2}{2 f_\pi^3} \left( \frac{3}{2} \ell + 1 \right) \left( \frac{3}{2} \ell + 1 \right) \left[ \left( m_\pi^2 + \frac{1}{2} (k_{1,k_1}^2 + k_{1,k_1}^2) \right) \left( W_{W_0}^{(1)}(k_1) - W_{W_0}^{(1)}(k'_1, k_1) \right) - \frac{1}{2} k_{1,k_1}^2 F_3(k_1) \frac{1}{2} \left( -\frac{1}{2\ell + 1} W_{W_0}^{(1)}(k_1) + \frac{\ell + 2}{2\ell + 3} \right) \right) \]
The tensor components of the $c_4$ interaction of $V_C$, Eq. (A10), are

$$
2 \frac{c_A g_A^2}{4 f_A^2} \frac{2}{3} (S_2)_{l', l}(\tau_1 \cdot \tau_2) \left[ \frac{1}{2 k_1 k_1} (k_1^2 Q_{\ell}(z) + k_1^2 Q_{\ell}(z) - 2 k_1^2 k_1 Q_{l}(z)) \left( \frac{1}{2} \rho_0 - m_z^2 (F_0(k'_1) + F_0(k_1)) \right) \right.

+ \frac{k_1^4}{2 J + 1} (Q_{l+1}^j(k'_1, k_1) - Q_{l+1}^j(k'_1, k_1))

+ (F_0(k'_1) - 2 F_1(k'_1)) \left\{ \frac{k_1^2}{4 k_1^2} (-k_1^2 + k_1^2 + m_z^2) Q_{\ell}(z) \right.

- \frac{1}{4} (k_1^2 - k_1^2 + m_z^2) Q_{l}(z) - k_1^2 \delta_{l0} + \frac{1}{2} k_1^2 k_1 \delta_{l0} \}

+ (F_0(k_1) - 2 F_1(k_1)) \left\{ \frac{k_1^2}{4 k_1^2} (k_1^2 - k_1^2 + m_z^2) Q_{\ell}(z) \right.

- \frac{1}{4} (k_1^2 - k_1^2 + m_z^2) Q_{l}(z) - k_1^2 \delta_{l0} + \frac{1}{2} k_1^2 k_1 \delta_{l0} \}

+ \frac{1}{3} F_3(k'_1) \left\{ - \frac{3}{2} k_1^2 \delta_{l0} + \frac{3}{2} k_1^2 k_1 \delta_{l0} + \frac{3}{12} k_1^4 Q_{l}(z) \right.

- \frac{1}{4} (k_1^2 + 3 k_1^2 + 3 m_z^2) Q_{l}(z) - \frac{1}{4} (k_1^2 + 3 k_1^2 + 3 m_z^2) Q_{l}(z) + \frac{1}{2} k_1^2 k_1 Q_{l}(z) \}

+ \frac{1}{3} F_3(k_1) \left\{ - \frac{3}{2} k_1^2 \delta_{l0} + \frac{3}{2} k_1^2 k_1 \delta_{l0} + \frac{3}{12} k_1^4 Q_{l}(z) \right.

- \frac{1}{4} (k_1^2 + 3 k_1^2 + 3 m_z^2) Q_{l}(z) - \frac{1}{4} (k_1^2 + 3 k_1^2 + 3 m_z^2) Q_{l}(z) + \frac{1}{2} k_1^2 k_1 Q_{l}(z) \}

- \frac{1}{3} F_2(k'_1) \left\{ \frac{k_1^4}{2 k_1^2} Q_{l}(z) - \frac{1}{3} F_2(k_1) \right\} \frac{k_1^4}{2 k_1^2} Q_{l}(z)

$$

(B16)

for $\ell' = \ell \pm 1$ ($J = \ell \pm 1$) and

$$
2 \frac{c_A g_A^2}{4 f_A^2} \frac{2}{3} (S_2)_{l', l}(\tau_1 \cdot \tau_2) \left[ \left( \frac{1}{2} \rho_0 - m_z^2 (F_0(k'_1) + F_0(k_1)) \right) \left\{ \frac{k_1^2 + k_1^2}{2 k_1^2} Q_{l}(z) - \frac{1}{2} \ell + \frac{1}{2} Q_{l-1}(z) - \frac{1}{2} \ell + \frac{1}{2} Q_{l+1}(z) \right\} \right.

+ \frac{1}{2} k_1^2 \left\{ \left( (2 \ell + 1)^2 \right) \left( (2 \ell + 2) + (2 \ell + 3) \right) - 2 \right\} Q_{l+1}^j(k'_1, k_1) + \frac{(2 \ell + 1)(2 \ell + 2)}{(2 \ell + 1)(2 \ell + 3)} Q_{l+1}^j(k'_1, k_1)

+ (F_0(k'_1) - 2 F_1(k'_1)) \left\{ - \frac{1}{2} k_1^2 \delta_{l0} + \frac{5}{12} k_1^2 k_1 \delta_{l0} + \frac{1}{4} k_1^2 + k_1^2 + m_z^2 \right\} Q_{l}(z)

- \frac{1}{8 (2 \ell + 1)} (-k_1^2 + k_1^2 + m_z^2)((2 \ell + 3) Q_{l-1}(z) + (2 \ell + 1) Q_{l+1}(z)) \}

+ \frac{k_1^2}{4 k_1^2} (-k_1^2 + 3 k_1^2 + 3 m_z^2) Q_{l}(z) - \frac{1}{8 (2 \ell + 1)} (k_1^2 + 3 k_1^2 + 3 m_z^2)((2 \ell + 3) Q_{l-1}(z) + (2 \ell + 1) Q_{l+1}(z)) \}

- \frac{1}{3} F_2(k'_1) \left\{ \frac{k_1^4}{2 k_1^2} Q_{l}(z) + (F_0(k_1) - 2 F_1(k_1)) \left\{ - \frac{1}{2} k_1^2 \delta_{l0} + \frac{5}{12} k_1^2 k_1 \delta_{l0} + \frac{1}{4} k_1^2 + k_1^2 + m_z^2 \right\} Q_{l}(z)

- \frac{1}{8 (2 \ell + 1)} (k_1^2 - k_1^2 + m_z^2)((2 \ell + 3) Q_{l-1}(z) + (2 \ell + 1) Q_{l+1}(z)) \}

+ \frac{1}{3} F_3(k'_1) \left\{ - \frac{3}{2} k_1^2 \delta_{l0} + \frac{5}{4} k_1^2 k_1 \delta_{l0} \right\} \frac{k_1^4}{2 k_1^2} Q_{l}(z)

$$

(B17)

for $\ell' = \ell = J \pm 1$. There is no spin-orbit component from the $c_4$ interaction of $V_C$, Eq. (A10).

The central component of the $V_D$ interaction, Eq. (A11), is

$$
\frac{g_A}{8 f_A^2} \frac{c_D}{f_A^2} \frac{1}{3} \left( \sigma_1 \cdot \sigma_2 \right) (\tau_1 \cdot \tau_2) \left\{ \rho \rho_{\ell 0} + m_z^2 (F_0(k_1) + F_0(k'_1)) \right\}

+ \frac{3 g_A}{8 f_A^2} \frac{c_D}{f_A^2} \delta_{l0} \left( \rho_0 - 2 m_z^2 (F_0(k_1) + F_0(k'_1)) \right).

$$

(B18)

The tensor component for the initial $\ell$ and the final $\ell' = \ell \pm 1$ ($J = \ell \pm 1$) becomes

$$
2 \frac{g_A}{8 f_A^2} \frac{c_D}{f_A^2} \frac{1}{3} S_{12} \left( F_0(k_1) - 2 F_1(k_1) + F_3(k_1) + F_0(k'_1) - 2 F_1(k'_1) + F_3(k'_1) \right) (k_1^2 \delta_{l0} + k_1^2 \delta_{l0} - 2 k_1^2 k_1 \delta_{l0})

- \frac{g_A}{8 f_A^2} \frac{c_D}{f_A^2} \rho_{\ell 0} \frac{1}{3} S_{12} \left\{ \frac{k_1^2}{2 k_1^2} Q_{\ell}(x) + \frac{k_1^2}{2 k_1^2} Q_{\ell}(x) - Q_{J}(x) \right\},

$$

(B19)
and for $\ell' = \ell = J, J \pm 1$

\[
\frac{2g_A^C}{8f_\pi^2 f_\pi^2 L_x} (\tau_1 \cdot \tau_2) \left( F_0(k_1) - 2F_1(k_1) + F_3(k_1) + F_0(k_1') - 2F_1(k_1') + F_3(k_1') \right)
\times \left\{ k_1^2 \delta_{i0} + k_1^2 \delta_{i'0} - \frac{5}{3} k_1 k_1' \delta_{i1} \right\}
\]

\[-\frac{g_A^C}{8f_\pi^2 f_\pi^2 L_x} (\tau_1 \cdot \tau_2) \frac{1}{3} S_{12} \left( k_1^2 \frac{Q_\ell(x)}{2k_1} + k_1^2 \frac{Q_\ell(x)}{2k_1} - \frac{1}{2} \left( 2\ell + 3 \right) Q_{\ell-1}(z) + \frac{2\ell - 1}{2\ell + 1} Q_{\ell+1}(z) \right) \right\}. \tag{B20}
\]

There is no spin-orbit component from the $c_4$ interaction of $V_D$, Eq. (A11).

Finally, the $V_E$ interaction, Eq. (A12), gives only an $\ell = 0$ central component; namely, $-6g_A^C$ both for $^1S_0$ and $^3S_1$ channels.

[1] K. A. Bruckner, C. A. Levinson, and H. M. Mahmoud, Phys. Rev. 95, 217 (1954).
[2] B. D. Day, Rev. Mod. Phys. 39, 719 (1967).
[3] H. A. Bethe, Ann. Rev. Nucl. Sci. 21, 93 (1971).
[4] V. R. Pandharipande and R. B. Wiringa, Rev. Mod. Phys. 51, 821 (1979).
[5] M. Baldo and G. F. Burgio, Rep. Prog. Phys. 75, 026301 (2012).
[6] R. Brockmann and R. Machleidt, Phys. Rev. C 42, 1965 (1990).
[7] R. Machleidt and D. R. Entem, Phys. Rep. 503, 1 (2011).
[8] E. Epelbaum, W. Gockle, and U.-G. Meiβner, Nucl. Phys. A 747, 362 (2005).
[9] E. Epelbaum, H.-W. Hammer, and U.-G. Meiβner, Rev. Mod. Phys. 81, 1773 (2009).
[10] A. Klein, Phys. Rev. 90, 1101 (1953).
[11] J. Fujita and H. Miyazawa, Prog. Theor. Phys. 17, 366 (1957).
[12] R. B. Wiringa, S. C. Pieper, J. Carlson, and V. R. Pandharipande, Phys. Rev. C 62, 014001 (2000).
[13] N. Kalantar-Nayestanaki, E. Epelbaum, J. G. Messchendorp, and A. Nogga, Rep. Prog. Phys. 75, 016301 (2012).
[14] T. Kasahara, Y. Akahishi, and H. Tanaka, Prog. Theor. Phys. Suppl. 56, 96 (1974).
[15] B. Friedman and V. R. Pandharipande, Nucl. Phys. A 361, 361 (1981).
[16] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall, Phys. Rev. C 58, 1804 (1998).
[17] S. K. Bogner, A. Schwenk, R. J. Furnstahl, and A. Nogga, Nucl. Phys. A 763, 59 (2011).
[18] K. Hebeler, S. K. Bogner, R. J. Furnstahl, A. Nogga, and A. Schwenk, Phys. Rev. C 83, 031301(R) (2011).
[19] K. Hebeler and A. Schwenk, Phys. Rev. C 82, 014314 (2010).
[20] I. Tews, T. Krüger, K. Hebeler, and A. Schwenk, Phys. Rev. Lett. 110, 032504 (2013).
[21] L. Coraggio, J. W. Holt, N. Itaco, R. Machleidt, and F. Sammarruca, nucl-th/1209.5537.
[22] M. Kohno, Phys. Rev. C 86, 061301(R) (2012).
[23] F. Sammarruca, B. Chen, L. Coraggio, N. Itaco, and R. Machleidt, Phys. Rev. C 86, 054317 (2012).
[24] B. A. Loiseau, Y. Nogami, and C. K. Ross, Nucl. Phys. A 165, 601 (1971); Erratum A 176, 665 (1971).
[25] J. W. Holt, N. Kaiser, and W. Weise, Phys. Rev. C 81, 024002 (2010).
[26] M. Kohno and R. Okamoto, Phys. Rev. C 86, 014317 (2012).
[27] M. I. Haftel and F. Tabakin, Nucl. Phys. A 59, 1 (1970).
[28] K. Suzuki, R. Okamoto, M. Kohno, and S. Nagata, Nucl. Phys. A 665, 92 (2000).
[29] Y. Fujiwara, M. Kohno, T. Fujita, C. Nakamoto, and Y. Suzuki, Prog. Theor. Phys. 103, 755 (2000).
[30] R. B. Wiringa, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C 51, 38 (1995).
[31] T. A. Rijken, V. G. J. Stoks, and Y. Yamamoto, Phys. Rev. C 59, 21 (1999).
[32] R. Machleidt, Phys. Rev. C 63, 024001 (2001).
[33] D. Gogny and R. Padjen, Nucl. Phys. A 293, 365 (1977).
[34] T. Sasaki, N. Yasutake, M. Kohno, H. Kouno, and M. Yahiro, arXiv:1307.0681.
[35] A. Bohr and B. Mottelson, Nuclear structure, vol. I (Benjamin, Reading, Mass., 1969).
[36] Y. R. Shimizu, private communication.