GROMOV-HAUSDORFF ULTRAMETRIC

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Abstract. We show that there exists a natural counterpart of the Gromov-Hausdorff metric in the class of ultrametric spaces. It is proved, in particular, that the space of all ultrametric spaces whose metric take values in a fixed countable set is homeomorphic to the space of irrationals.

1. Introduction

Recall that the Hausdorff distance between two nonempty closed bounded subsets, $A$ and $B$, of a metric space is evaluated by the formula

$$d_H(A, B) = \inf \{ \varepsilon > 0 \mid A \subseteq O_\varepsilon(B), \ B \subseteq O_\varepsilon(A) \}.$$ 

Given two compact metric spaces, $(X, d_X)$ and $(Y, d_Y)$, the Gromov-Hausdorff distance between them is defined by the formula

$$\rho_{GH}(X, Y) = \inf \{ d_H(i(X), j(Y)) \mid i : X \to Z, \ j : Y \to Z \text{ are isometric embeddings} \}.$$ 

Recall that a metric $d$ on $X$ is called an ultrametric if it satisfies the following strong triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \ x, y, z \in X.$$ 

We are going to define a version of the Gromov-Hausdorff distance for ultrametric spaces.

Given two compact ultrametric spaces, $(X, d_X)$ and $(Y, d_Y)$, we define

$$\rho_{GHu}(X, Y) = \inf \{ d_H(i(X), j(Y)) \mid i : X \to Z, \ j : Y \to Z \text{ are isometric embeddings, where } Z \text{ is an ultrametric space} \}.$$ 

One can easily see that $\inf$ is well-defined as for every two ultrametric spaces there exists an ultrametric space in which they can be isometrically embedded.

Lemma 1.1. Let $X_1, X_2$ be ultrametric spaces, $X_1 \cap X_2 = A$ and the restrictions of ultrametrics in $X_1$ and $X_2$ onto $A$ coincide. Then the formula

$$d(x_1, x_2) = \inf \{ \max\{d_1(x_1, a), d_2(a, x_2)\} \mid a \in A \},$$

together with the initial ultrametrics on $X_1$ and $X_2$, determines an ultrametric on $X_1 \cup X_2$.

1991 Mathematics Subject Classification. 54B20, 54E35.
Proof. We are going to prove the strong triangle inequality. Let \( x, y, z \in X = X_1 \cup X_2 \). Without loss of generality, one may assume that \( x, y \in X_1 \setminus X_2, z \in X_2 \setminus X_1 \). There exist \( a, b \in A \) such that
\[
d(x, z) = \max\{d_1(x, a), d_2(a, z)\}, \quad d(y, z) = \max\{d_1(y, b), d_2(b, z)\}.
\]

For the sake of brevity, we introduce the following notations:
\[
\alpha = d_1(x, a), \quad \beta = d_1(b, y),
\gamma = d_2(a, z), \quad \delta = d_2(b, z),
\eta = d_1(x, y), \quad \epsilon = d(x, z), \quad \zeta = d(z, y).
\]

The rest of the proof consists in analyzing all possible cases.

First, we are going to show that \( d(x, y) \leq \max\{d(x, z), d(z, y)\} \), i.e. \( \eta \leq \max\{\epsilon, \zeta\} \).
1) \( \epsilon = \alpha, \ \zeta = \beta \). Suppose, on the contrary, that \( \eta > \max\{\epsilon, \zeta\} \), then \( d_1(x, b) = \eta \), \( d_1(a, b) = \eta \) and, therefore, \( \eta \leq \max\{\gamma, \delta\} \leq \max\{\alpha, \beta\} < \eta \), a contradiction.

2) \( \epsilon = \alpha, \ \zeta = \delta \). Suppose that \( \eta > \max\{\epsilon, \delta\} \), then \( d_1(a, y) = \eta \) and, since \( \eta > \delta \geq \beta \), we see that \( d_1(a, b) = \eta \). Thus \( \alpha \geq \gamma = \eta \), a contradiction.

3) \( \epsilon = \alpha, \ \zeta = \delta \). Suppose that \( \eta > \max\{\epsilon, \zeta\} \geq \max\{\gamma, \delta\} \). Since \( \alpha \leq \gamma < \eta \), we have \( d_1(a, y) = \eta \). In turn, since \( \beta \leq \zeta < \eta \), we have \( d_1(a, b) = d_2(a, b) = \eta \). But then
\[
\eta = d_2(a, b) \leq \max\{\gamma, \delta\} \leq \max\{\epsilon, \zeta\} < \eta
\]
and we come to a contradiction.

The case \( \epsilon = \gamma, \ \zeta = \beta \) is treated similarly to case 2).

Now we are going to show that \( d(z, y) \leq \max\{d(x, z), d(x, y)\} \), i.e. \( \zeta \leq \max\{\epsilon, \eta\} \).
1) \( \epsilon = \alpha, \ \zeta = \beta \). Suppose, on the contrary, that \( \zeta > \max\{\epsilon, \eta\} \), then \( \beta > \eta \) and \( d_1(x, b) = \beta = \zeta \). Since \( \zeta > \alpha \), we see that \( d_1(a, b) = \beta = \zeta \). Since \( \zeta > \alpha \geq \gamma \), we see that \( \delta = \zeta \). We have \( d_1(a, y) \leq \max\{\alpha, \eta\} < \zeta \). Also \( \gamma \leq \alpha < \zeta \) and therefore we obtain a contradiction \( \zeta \leq \max\{d_1(a, y), \gamma\} < \zeta \).

2) \( \epsilon = \alpha, \ \zeta = \delta \). Suppose that \( \zeta > \max\{\epsilon, \eta\} \geq \max\{\alpha, \eta\} \). We have \( \delta > \max\{\alpha, \eta\} \geq \gamma \) and, therefore, \( d_2(a, b) = d_1(a, b) = \delta \). Since \( d_1(a, b) = \delta > \alpha \), we see that \( d_1(x, b) = \delta \). We have
\[
d_1(a, y) \leq \max\{\eta, \alpha\} \leq \max\{\eta, \epsilon\} < \delta.
\]
Thus, \( \zeta \leq \max\{d_1(a, y), \gamma\} \leq \max\{d_1(a, y), \epsilon\} < \zeta \), a contradiction.

3) \( \epsilon = \gamma, \ \zeta = \beta \). Suppose that \( \zeta > \max\{\epsilon, \eta\} = \max\{\gamma, \eta\} \). We have \( d_1(a, y) \leq \max\{\alpha, \eta\} \leq \max\{\epsilon, \eta\} < \zeta \). Since \( \gamma \leq \epsilon < \zeta \), we obtain \( \zeta \leq \max\{d_1(a, y), \gamma\} < \zeta \), a contradiction.

4) \( \epsilon = \gamma, \ \zeta = \delta \). Suppose that \( \zeta > \max\{\epsilon, \eta\} \). Then \( d_2(a, b) = d_1(a, b) = \delta \). Since \( d_1(a, y) \leq \max\{\alpha, \eta\} \leq \max\{\epsilon, \eta\} < \zeta \), we obtain a contradiction \( \zeta \leq \max\{d_1(a, y), \gamma\} \leq \max\{d_1(a, y), \epsilon\} < \zeta \).

That \( d(z, x) \leq \max\{d(y, z), d(x, y)\} \) can be proven similarly. \( \square \)

**Theorem 1.2.** The function \( q_{GH_u} \) is an ultrametric on the set of isometry classes of ultrametric spaces.

Proof. The symmetry is obvious. Since \( q_{GH} \leq q_{GH_u} \), we see that \( q_{GH_u}(A, B) > 0 \) for nonisometric \( A \) and \( B \).
We are going to prove the strong triangle inequality. Let \( X_1, X_2, X_3 \) be ultrametric spaces and let \( \varepsilon > 0 \) be given. There exist ultrametric spaces \( Y \) and \( Z \) and isometric embeddings \( i_k : X_k \to Y \), \( k = 1, 2 \) and \( j_l : X_l \to Z \), \( l = 2, 3 \), such that
\[
 d_H(i_1(X_1), i_2(X_2)) \leq \varrho_{GHu}(X_1, X_2) + \varepsilon, \quad d_H(j_2(X_2), j_3(X_3)) \leq \varrho_{GHu}(X_2, X_3) + \varepsilon.
\]
Identify \( i_2(X_2) \) with \( j_2(X_2) \) along the map \( j_3i_2^{-1} \). We obtain the quotient set, which we denote by \( K \), of the disjoint union \( Y \sqcup Z \). For the sake of notational simplicity, we naturally identify \( Y \) and \( Z \) with the subspaces of \( K \). By Lemma 1.1, there exists an ultrametric, \( d \), on \( K \) which extends initial ultrametrics on \( Y \) and \( Z \). Since the Hausdorff metric on the space of nonempty compact subsets of an ultrametric space is an ultrametric, we see that, in \( K \),
\[
 d_H(i_1(X_1), j_3(X_3)) \leq \max\{d_H(i_1(X_1), i_2(X_2)), d_H(i_2(X_3), j_3(X_3))\}
\]
and therefore
\[
 \varrho_{GHu}(X_1, X_3) \leq d_H(i_1(X_1), j_3(X_3)) \leq \max\{d_H(i_1(X_1), i_2(X_2)), d_H(i_2(X_3), j_3(X_3))\} \leq \max\{\varrho_{GHu}(X_1, X_2) + \varepsilon, \varrho_{GHu}(X_2, X_3) + \varepsilon\}.
\]
Tending \( \varepsilon \) to 0, we are done. \( \square \)

2. Ultrametric Gromov-Hausdorff space

By \( U \) we denote the Gromov-Hausdorff space, i.e. the space of all isometry classes of compact ultrametric spaces endowed with the Gromov-Hausdorff ultrametric. For the sake of simplicity, we prefer to work with representatives of the isometry classes rather than with the classes themselves.

Denote by \( \exp X \) the set of all nonempty compact subsets in \( X \) endowed with the Hausdorff metric. It is well-known (see, e.g., [2]) that \( \exp X \) is complete if so is \( X \).

**Proposition 2.1.** The space \( U \) is complete.

*Proof.* Let \( (X_i)_{i=1}^{\infty} \) be a Cauchy sequence in \( U \). Without loss of generality, one may assume that \( X_i \) and \( X_{i+1} \) lie in the same ultrametric space, \( Y_i \). Let \( Y = \sqcup \{Y_i \mid i \in \mathbb{N}\} \). Similarly as in the proof of Lemma 1.1, we subsequently glue \( Y_2 \) to \( Y_1 \) along \( X_1 \), then glue the resulting space to \( Y_3 \) along \( X_2 \) etc. We obtain the expanding sequence of ultrametric spaces \( Y_1, Y_1 \sqcup X_2, Y_2, Y_1 \sqcup X_2, Y_2 \sqcup X_3, \ldots \). Let \( \tilde{Y} \) denote the union of this sequence. Obviously, \( \tilde{Y} \) is an ultrametric space and therefore so is its completion, which we denote by \( \tilde{Y} \). The spaces \( X_i \) are naturally embedded into \( \tilde{Y} \) and the sequence \( (X_i) \) is a Cauchy sequence in \( \tilde{Y} \). Since the space \( \exp \tilde{Y} \) is complete, there exists the limit of the sequence \( (X_i) \) in this space, which we denote by \( X \). It is evident that \( X \) is also the limit of the sequence \( (X_i) \) in the space \( U \). \( \square \)

**Proposition 2.2.** The space \( U \) is not separable.

*Proof.* For any \( c \in [1/2, 1] \), denote by \( X_c \) the two-point metric space with the nonzero distance equal to \( c \). We are going to prove that \( \varrho_{GHu}(X_{c_1}, X_{c_2}) \geq 1/4 \) whenever \( c_1 \neq c_2 \). Indeed, otherwise one can embed \( X_{c_1} \) and \( X_{c_2} \) in some ultrametric space so that the Hausdorff distance between the images is \( < 1/2 \). Without loss of generality
one may assume that there is an ultrametric, $d$, on the union $X_{c_1} \cup X_{c_2}$ extending the initial ultrametrics on $X_{c_1} = \{a_1, a_2\}$ and $X_{c_2} = \{b_1, b_2\}$ and $d(a_1, b_1) < 1/4$, $d(a_1, b_1) < 1/4$. It follows from the strong triangle inequality that $c_1 = d(a_1, a_2) = d(a_1, b_2) = d(b_1, b_2) = c_2$ and we obtain a contradiction. \hfill $\blacksquare$

*Lemma 2.3.* The space $U(K)$ is a closed subspace of $U$, for any $K \subset \mathbb{R}_+$ with $0 \in K$.

*Proof.* Let $(X_i)_{i=1}^\infty$ be a sequence in $U(K)$ converging to $X \in U$. Assume, on the contrary, that $X \notin U(K)$, then there exist $a, b \in X$ such that $d(a, b) \notin K$. There exists $i$ such that $d_{GH}(X, X_i) < \frac{1}{4}d(a, b)$. Without loss of generality, one may assume that $X, X_i$ are subsets of an ultrametric space $Z$ with $d_H(X, X_i) < \frac{1}{2}d(a, b)$. There exist $a'b' \in X_i$ such that $d(a, a') < \frac{1}{4}d(a, b)$, $d(b, b') < \frac{1}{4}d(a, b)$. It follows from the triangle $a, b, a'$ that $d(a', b) = d(a, b)$. Similarly, it follows from the triangle $a', b, b'$ that $d(a', b') = d(a', b) = d(a, b)$. We obtain a contradiction with the fact that $X_i \in U(K)$. \hfill $\blacksquare$

*Theorem 2.4.* Let $K$ be a countable subset of $\mathbb{R}_+$ with $0$ as its nonisolated point. Then the space $U(K)$ is homeomorphic to the space of irrationals.

*Proof.* First of all note that the space $U(K)$ is separable. To this end, we are going to demonstrate that the space $U_f(K) = \{Y \in U(K) \mid |Y| < \infty\}$, which is easily seen to be countable, is dense in $U(K)$.

Prove that $U(K)$ is nowhere locally compact. Let $X \in U(K)$ and $\varepsilon > 0$. Consider a finite $\varepsilon$-net $Y = \{x_1, \ldots, x_k\}$ in $X$. Without loss of generality, we may assume that $d(x_1, x_2) = \min\{d(x, y) \mid x, y \in X, x \neq y\}$. There exists a positive $c \in K$ such that $c < \min\{d(x_1, x_2), \varepsilon/2\}$. For every natural $n$, define a metric space $Y_n$ as follows. Let $Y_n = Y \cup \{1, \ldots, n\}$ and the metric $g$ on $Y$ is defined by the conditions $g((Y \times Y) = d(Y \times Y)$, $g(y, i) = d(y, x_1)$, for any $y \in Y$, $1 \leq i \leq n$, and $g(i, j) = c$, for every $i, j \in \{1, \ldots, n\}$, $i \neq j$.

An easy verification that $g$ is an ultrametric on $Y_n$ is left to the reader.

Next, we note that $g(Y_m, Y_n) \leq c$ for every $m, n$. In addition, from the pigeon hole principle it easily follows that $g(Y_m, Y_n) \geq c/2$, whenever $m \neq n$. Therefore, the set $\{Y_i \mid i \in \mathbb{N}\}$ is a countable discrete subset of a closed c-neighborhood of $X$ in $U(K)$. This demonstrates that the space $U(K)$ is nowhere locally compact.

Remark also that the space $U(K)$ being a closed subset of $U$ is complete.

It follows from [3] that the space $U(K)$ is homeomorphic to the space of irrationals. \hfill $\blacksquare$

3. Open problems

*Question 3.1.* Describe the topology of the space $U$.

A generalization of ultrametric spaces is introduced by David and Semmes [4]. A metric space $(X, d)$ is said to be *uniformly disconnected* if there exists $c > 0$ such that max\{$d(x_i, x_{i-1}) \mid i = 1, \ldots, N\} \leq cd(x, y)$ for all finite chains of points $x = x_0, x_1, \ldots, x_N = y$. In [4] it is proved that, for any metric space $(X, d)$, the metric $d$
is bi-Lipschitz equivalent with an ultrametric on \( X \) if and only if the space \((X, d)\) is uniformly disconnected. This result allows to find a counterpart of the notion of the Gromov-Hausdorff metric in the class of uniformly disconnected spaces.

**Question 3.2.** Is the obtained space of compact uniformly disconnected spaces separable?

It is proved in [5] that the space of (rooted) compact real trees is complete. Here it is assumed that the set of these trees is endowed with the Gromov-Hausdorff metric. Like in the case of ultrametric spaces, we obtain another metric if we restrict ourselves with embeddings in trees. We conjecture that the analogy between trees and ultrametric space (see, e.g., [6]) can be extended also to the case of the obtained hyperspaces).

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