Sequential convergence of AdaGrad algorithm for smooth convex optimization

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Abstract

We prove that the iterates produced by, either the scalar step size variant, or the coordinatewise variant of AdaGrad algorithm, are convergent sequences when applied to convex objective functions with Lipschitz gradient. The key insight is to remark that such AdaGrad sequences satisfy a variable metric quasi-Fejér monotonicity property, which allows to prove convergence.

Keywords: Convex optimization, adaptive algorithms, sequential convergence, Fejér monotonicity.

1. Introduction

We consider the problem of unconstrained minimization of a smooth convex function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ which gradient is globally Lipschitz. We will assume that the minimum of $F$ over $\mathbb{R}^n$, $F^*$, is attained. We are interested in the sequential convergence of a largely used adaptive gradient method called AdaGrad.

Sequential convergence. Continuous optimization algorithms are meant to converge if not to a global minimum at least to a local minimum of the cost function $F$, a necessary condition being, when the function is differentiable, Fermat rule, $\nabla F = 0$. Convergence of an iterative algorithm, producing a sequence of estimates in $\mathbb{R}^n$, $(x_k)_{k \in \mathbb{N}}$, can be measured in several ways: convergence of the norm of the gradients $\|\nabla F(x_k)\|_{\mathbb{R}^n}$, convergence of the suboptimality level $F(x_k) - F^*$, as $k$ grows to infinity. These measures of convergence do not translate directly into a characterization of the asymptotic behavior of the iterates $(x_k)_{k \in \mathbb{N}}$ themselves. In general, this needs not be true, without additional assumptions. For example, when $F$ is strongly or strictly convex, since the minimum is uniquely attained, convergence of the gradient or the suboptimality level to 0 implies convergence of the sequence.

Convergence of iterate sequences is an important measure of algorithmic stability. Indeed, in optimization applications (statistics [1], signal processing [3]) one may be concerned about the value of the argmin more than the minimum value. Sequential convergence ensures that the estimate of the argmin produced by the algorithm has some asymptotic stability property.

Adaptive gradient methods. First order methods are the most widespread methods for machine learning and signal processing applications [5]. We will focus on AdaGrad algorithm [12], which was initially developed in an online learning context, see also [18]. This is a simple gradient method for which the step size is tuned automatically, in a coordinatewise fashion, based on previous gradient observations, this is where the term “adaptive” comes from. Interestingly, this adaptivity property found a large interest in training of deep networks [13] with extensions and variants such as the widespread Adam algorithm [14]. The reason for this success is probably due to the adaptation between the composite structure of deep networks and the coordinatewise structure of the algorithm, as well as the empirical observation that adaptivity performs well in practice, without requiring much manual tuning, although this is not a consensus [23].

Getting back to the convex world, it was suggested that adaptive step sizes give the possibility to use a single step size strategy, independent of the class of problem at hand: smooth versus nonsmooth, deterministic versus noisy [15]. Indeed, it is known in convex optimization that constant step sizes can be used in the deterministic smooth case, while a decreasing schedule has to be used in the presence of nonsmoothness or noise. This idea was extended to adaptivity to strong convexity [2] and its extensions [24], as well as adaptivity in the context of variational inequalities [2].

In a more general nonconvex optimization context, there has been several recent attempts to develop a convergence theory for adaptive methods, with the application to deep network training in mind [16, 17, 5, 21, 22, 5]. These provide qualitative convergence guarantees toward critical point or complexity estimates on the norm of the gradient, which are also, of course valid in the convex setting.

Fejér monotonicity and extensions. In convex settings, the study of the convergence of the iterates of optimization algorithms has a long history. For many known first order algorithms, it turns out that the quantity $\|x_k - x^\ast\|_2^2$ is a Lyapunov function for the discrete dynamics for any solution $x^\ast$. This property is called Fejér monotonicity, it allows to obtain convergence rates [19] and also to prove convergence of iterate sequences in relation to Opial property. For example this property was used in [7], to prove convergence of proximal point algorithm, forward-backward splitting method, Douglas-Rashford
We use the notation \( \mathbf{L} \) for matrices in \( \mathbb{R}^{n \times n} \) and \( \preceq \) for a constrained minimization problem. In addition, we assume that \( F \) is convex and attains its minimum, that is, there exists \( x^* \in \mathbb{R}^n \) such that
\[
\forall x \in \mathbb{R}^n, \quad F(x) \geq F(x^*).
\]

The diagonal matrix with the \( i \)-th diagonal is represented by \( \text{diag}(v_i) \). Throughout this document we will consider the following unconstrained minimization problem
\[
\min_{x \in \mathbb{R}^n} F(x) \tag{1}
\]
where \( F : \mathbb{R}^n \to \mathbb{R} \) is differentiable and \( n \in \mathbb{N} \) is the ambient dimension. In addition, we assume that \( F \) is convex and attains its minimum, that is, there exists \( x^* \in \mathbb{R}^n \) such that
\[
\forall x \in \mathbb{R}^n, \quad F(x) \geq F(x^*). \tag{2}
\]

We finally assume that \( F \) has an \( L \)-Lipschitz gradient, for some \( L > 0 \). More explicitly, \( L \) is such that for any \( x, y \in \mathbb{R}^n \),
\[
\| \nabla F(x) - \nabla F(y) \| \leq L \| x - y \|. \tag{3}
\]

From this property, we can derive the classical Descent Lemma, which is a quantitative bound on the difference between \( f \) and its first order Taylor expansion, see for example in \cite[Lemma 1.2.3]{10}.

**Lemma 1** (Descent Lemma). Suppose that \( F : \mathbb{R}^n \to \mathbb{R} \) has \( L \)-Lipschitz gradient. Then for all \( x, y \in \mathbb{R}^n \), we have
\[
| f(y) - f(x) - \langle \nabla f(x), y - x \rangle | \leq \frac{L}{2} \| x - y \|^2. \tag{4}
\]

### 2. Technical preliminary

#### 2.1. Notations

In all this document we consider the set \( \mathbb{R}^n \) of real vectors of dimension \( n, n \in \mathbb{N} \). We denote by \( x_i \) the \( i \)-th component of the vector \( x \in \mathbb{R}^n \), with \( i \in \{1, 2, \cdots, n\} \). \( \nabla F \) is the gradient of a differentiable function \( F : \mathbb{R}^n \to \mathbb{R} \) and \( \nabla_i F \) its \( i \)-th component, corresponding to the \( i \)-th partial derivative. \( \parallel \cdot \parallel \) is the euclidean norm and \( \langle \cdot, \cdot \rangle \) its associated scalar product. Let \( (u_k)_{k \in \mathbb{N}} \) a sequence in \( \mathbb{R}^n \). We denote by \( u_{ij} \) the \( j \)-th component of the \( i \)-th vector of the sequence \( (u_k)_{k \in \mathbb{N}} \). The diagonal matrix with the \( v \)-th vector \( v \in \mathbb{R}^n \) as its diagonal is represented by \( \text{diag}(v) \in \mathbb{R}^{n \times n} \). We use the notation \( c_1 \) for the space of summable nonnegative sequences of real numbers. For a positive definite matrix \( W \in \mathbb{R}^{n \times n} \) we use the notation \( \|d\|_W^2 = (Wd, d) \) to denote the associated norm. We let \( \geq \) denote the partial order over symmetric matrices in \( \mathbb{R}^n \).

#### 2.2. Problem setting and assumptions

Throughout this document we will consider the following unconstrained minimization problem
\[
\min_{x \in \mathbb{R}^n} F(x) \tag{1}
\]
where \( F : \mathbb{R}^n \to \mathbb{R} \) is differentiable and \( n \in \mathbb{N} \) is the ambient dimension. In addition, we assume that \( F \) is convex and attains its minimum, that is, there exists \( x^* \in \mathbb{R}^n \) such that
\[
\forall x \in \mathbb{R}^n, \quad F(x) \geq F(x^*). \tag{2}
\]

We finally assume that \( F \) has an \( L \)-Lipschitz gradient, for some \( L > 0 \). More explicitly, \( L \) is such that for any \( x, y \in \mathbb{R}^n \),
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**Lemma 1** (Descent Lemma). Suppose that \( F : \mathbb{R}^n \to \mathbb{R} \) has \( L \)-Lipschitz gradient. Then for all \( x, y \in \mathbb{R}^n \), we have
\[
| f(y) - f(x) - \langle \nabla f(x), y - x \rangle | \leq \frac{L}{2} \| x - y \|^2. \tag{4}
\]

#### 2.3. Adaptive gradient algorithm (AdaGrad)

We study two versions of AdaGrad, the original algorithm performing adaptive gradient steps at a coordinate level and a simplified version which uses a scalar step size. The latter variant, was coined as AdaGrad-Norm in \cite{22}. It goes as follows:

**Algorithm 2.1** (AdaGrad-Norm). Given \( x_0 \in \mathbb{R}^n \), \( v_0 = 0 \), \( \delta > 0 \), iterate, for \( k \in \mathbb{N} \),
\[
v_{k+1} = v_k + \| \nabla F(x_k) \|^2 \tag{5}
\]
\[
x_{k+1} = x_k - \frac{1}{\sqrt{v_{k+1} + \delta}} \nabla F(x_k). \tag{6}
\]

The original version presented in \cite{12} applies the same idea combined with coordinate-wise updates using partial derivatives. The algorithm is as follows.

**Algorithm 2.2** (AdaGrad). Given \( x_0 \in \mathbb{R}^n \), \( v_0 = 0 \), \( \delta > 0 \), iterate, for \( k \in \mathbb{N} \) and \( j \in \{1, \cdots, n\} \),
\[
v_{k+1,j} = v_{k,j} + \left( \nabla_j F(x_k) \right)^2 \tag{7}
\]
\[
x_{k+1,j} = x_{k,j} - \frac{1}{\sqrt{v_{k+1,j} + \delta}} \nabla_j F(x_k). \tag{8}
\]

Our goal is, to prove that the sequences \( (x_k)_{k \in \mathbb{N}} \) generated by AdaGrad are convergent for both variants.

### 3. Results

Our main result is the following

**Theorem 3.1.** Let \( F : \mathbb{R}^n \to \mathbb{R} \) be convex with \( L \)-Lipschitz gradient and attain its minimum on \( \mathbb{R}^n \). Then any sequence \( (x_k)_{k \in \mathbb{N}} \) generated by AdaGrad-Norm (Algorithm 2.1) or AdaGrad (Algorithm 2.2) converges to a global minimum of \( F \) as \( k \) grows to infinity.

The coming section is dedicated to exposition of the proof arguments for this result.
3.1. Variable metric quasi Fejér monotonicity

The following definition is a simplification adapted from the more general exposition given in [10].

Definition 3.1. Let \((W_k)_{k \in \mathbb{N}}\) be a sequence of symmetric matrices such that \(W_k \succeq \alpha I_n\), \(\forall k \in \mathbb{N}\), for some \(\alpha > 0\). Let \(C\) be a nonempty, closed and convex subset of \(\mathbb{R}^n\), and let \((x_k)_{k \in \mathbb{N}}\) be a sequence in \(\mathbb{R}^n\). Then \((x_k)_{k \in \mathbb{N}}\) is variable metric quasi-Fejér monotone with respect to the target set \(C\) relative to \((W_k)_{k \in \mathbb{N}}\) if

\[
\left( \exists (\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N}) \right) (\forall \varepsilon > 0) \left( \exists (\epsilon_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N}) \right) \|x_{k+1} - \zeta \|_{w_{k+1}}^2 \leq (1 + \eta_k) \|x_k - \zeta \|_{w_k}^2 + \epsilon_k, \quad (\forall k \in \mathbb{N}). \tag{7}
\]

For variable metric quasi-Fejér sequences the following results have already been established [10, Proposition 3.2], we provide a proof in Appendix A.1 for completeness.

Proposition 3.2. Let \((u_k)_{k \in \mathbb{N}}\) be a variable metric quasi-Fejér sequence relative to a nonempty, convex and closed set \(C\) in \(\mathbb{R}^n\). These assertions hold.

(i) Let \(u \in C\). \(|u_k - u|_{W_k} \) converges.

(ii) \((u_k)_{k \in \mathbb{N}}\) is bounded.

Proposition 3.2 allows to prove sequential convergence of variable metric quasi-Fejér sequences. This result is again due to [10, Theorem 3.3] and a proof is given in Appendix A.2 for completeness.

Theorem 3.3. Let \((W_k)_{k \in \mathbb{N}}\) be a sequence of symmetric matrices such that \(W_k \succeq \alpha I_n\), \(\forall k \in \mathbb{N}\), for some \(\alpha > 0\). We suppose that the sequence \((W_k)_{k \in \mathbb{N}}\) converges to \(W\). Let \((x_k)_{k \in \mathbb{N}}\) be a variable metric quasi-Fejér sequence with respect to \(C\) and to a closed target set \(C \subset \mathbb{R}^n\). \((x_k)_{k \in \mathbb{N}}\) converges to a point in \(C\) if and only if every cluster point of \((x_k)_{k \in \mathbb{N}}\) is in \(C\).

Remark 3.2. If, for a variable metric quasi-Fejér sequence, \((W_k)_{k \in \mathbb{N}}\) is constant and \((\eta_k)_{k \in \mathbb{N}}\) is null for all \(k \in \mathbb{N}\), the sequence is simply called a quasi-Fejér sequence. Moreover if \((\epsilon_k)_{k \in \mathbb{N}}\) is null for all \(k \in \mathbb{N}\), it is called a Fejér monotone sequence, which provides a Lyapunov function. Of course, for these two special cases, the results stated above hold.

3.2. Convergence of AdaGrad-Norm

To prove convergence of sequences generated by AdaGrad-Norm, we start with the following lemma.

Lemma 2. Under the hypothesis of Theorem 3.3 suppose that \((x_k)_{k \in \mathbb{N}}\) is a sequence generated by AdaGrad-Norm in Algorithm 2.7. Then we have that \(\sum_{k=0}^{\infty} \|\nabla F(x_k)\|^2\) is finite.

Proof. This proof is inspired by the proof of Lemma 4.1 in [22]. Fix \(x^* \in \mathbb{R}^n\) such that \(F(x^*) = \inf_{x} F(x) > -\infty\). We split the proof into two cases.

- Suppose that there exists an index \(k_0 \in \mathbb{N}\) such that \(\sqrt{v_{k_0} + \delta} \geq L\). It follows using the descent Lemma 11 for any \(j \geq 1\)

\[
F(x_{k_0+j}) \leq F(x_{k_0+j-1}) + \left( \nabla F(x_{k_0+j-1}) \cdot x_{k_0+j} - x_{k_0+j-1} \right) + \frac{L}{2} \|x_{k_0+j} - x_{k_0+j-1}\|^2 \tag{8}
\]

where the transition from (8) to (9) is because \(\sqrt{v_{k_0+j} + \delta} \geq \sqrt{v_{k_0+j}} \geq L\), for all \(j \geq 0\), and (10) is a recursion. Fix any \(j \geq 1\), let \(Z = \sum_{k=k_0}^{k+j-1} \|\nabla F(x_k)\|^2\). It follows using \(k_0 + j = M\) in the preceding inequality

\[
2 \left( F(x_{k_0}) - F(x^*) \right) \geq 2 \left( F(x_{k_0}) - F(x_{k_0+j}) \right) \geq \frac{\sum_{k=k_0}^{k+j-1} \|\nabla F(x_k)\|^2}{\sqrt{v_{k_0+j} + \delta}} = \frac{Z}{\sqrt{Z + v_{k_0} + \delta}} \tag{10}
\]

By Lemma 3 it follows

\[
\sum_{k=k_0}^{k+j-1} \|\nabla F(x_k)\|^2 \leq 4 \left( F(x_{k_0}) - F(x^*) \right)^2 + 2 \left( F(x_{k_0}) - F(x^*) \right) \sqrt{v_{k_0} + \delta}. \tag{11}
\]

Since \(M\) was arbitrary, one may take the limit \(j \to \infty\), we have

\[
\sum_{k=k_0}^{\infty} \|\nabla F(x_k)\|^2 < \infty. \tag{12}
\]

That means \(\sum_{k=k_0}^{\infty} \|\nabla F(x_k)\|^2 < \infty\), which concludes the proof for this case.

- On the contrary, we have that \(\sqrt{v_k + \delta} < L\) for all \(k \in \mathbb{N}\), this means that \(\forall k \in \mathbb{N}\),

\[
\sum_{k=0}^{j} \|\nabla F(x_k)\|^2 < L^2 - \delta. \tag{13}
\]

Letting \(k\) goes to infinity gives

\[
\sum_{k=0}^{\infty} \|\nabla F(x_k)\|^2 < L^2 - \delta < \infty,
\]

which is the desired result.
We can now conclude the proof for AdaGrad-Norm.

**Proof.** Under the conditions of Theorem 3.1, assume that $x_k$ is a sequence generated by AdaGrad-Norm. Let $b_k = \sqrt{\delta} + v_k \geq \sqrt{\delta}$ for all $k \in \mathbb{N}$, which is a non decreasing sequence. Let $x^* \in \arg \min F$ be arbitrary. By assumption arg min $F$ is nonempty and it is convex and closed since $F$ is convex and continuous. We have for all $k \in \mathbb{N}$,
\[
\|x_{k+1} - x^*\|^2 \\
= \|x_k - x^* - \frac{1}{b_{k+1}} \nabla F(x_k)\|^2 \\
= \|x_k - x^*\|^2 + \frac{2}{b_{k+1}} (F(x^*) - F(x_k)) + \frac{1}{b_{k+1}^2} \|\nabla F(x_k)\|^2.
\]

Thanks to the convexity of $F$, the above equality gives
\[
\|x_{k+1} - x^*\|^2 \\
\leq \|x_k - x^*\|^2 + \frac{2}{b_{k+1}} (F(x^*) - F(x_k)) + \frac{1}{b_{k+1}^2} \|\nabla F(x_k)\|^2 \\
\leq \|x_k - x^*\|^2 + \frac{1}{\delta} \|\nabla F(x_k)\|^2. \tag{12}
\]

By Lemma 2 $\|\nabla F(x_k)\|^2$ is summable. Hence $(x_k)_{k \in \mathbb{N}}$ is a quasi-Fejér sequence relatively to $\arg \min F$. Proposition 3.2 says that $(x_k)_{k \in \mathbb{N}}$ is bounded. Thus it has at least an accumulation point. Then, thanks again to the Lemma 2, we have the set of accumulation points of $(x_k)_{k \in \mathbb{N}}$ included in $\arg \min F$. So using Theorem 3.3 and Remark 3.2, we conclude that $(x_k)_{k \in \mathbb{N}}$ is convergent and that its limit is a global minimum of $F$. □

### 3.3. Convergence of component-wise AdaGrad

We now consider the case of AdaGrad in Algorithm 2.2, taking into account the coordinatewise nature of the updates. The following corresponds to Lemma 2 for this situation.

**Lemma 3.** Under the hypothesis of Theorem 3.1 suppose that $(x_k)_{k \in \mathbb{N}}$ is a sequence generated by AdaGrad in Algorithm 2.2. We have that $\sum_{i=0}^{\infty} \|\nabla F(x_k)\|^2$ is finite.

**Proof.**

Let $I = \{i \in \{1, \cdots, n\} : \exists k_i \in \mathbb{N}, \sqrt{v_{k_i} + \delta} \geq L, \text{with } k_i \text{ the smallest possible} \}$. Set $k_0 = \max k_i, i \in I$. If $I$ is empty, we have, $\forall k \in \mathbb{N}$ and $\forall i \in \{1, \cdots, n\}$,
\[
\frac{1}{\sqrt{v_{k} + \delta}} \sum_{i=1}^{k} (\nabla F(x_i))^2 < L^2 - \delta.
\]

Making $k$ goes to infinity gives, $\forall i \in \{1, \cdots, n\}$,
\[
\sum_{i=0}^{\infty} (\nabla F(x_i))^2 < L^2 - \delta < \infty \text{ and } \sum_{i=0}^{\infty} \|\nabla F(x_i)\|^2 < \infty.
\]

So let us assume that $I$ is not empty. By Descent Lemma 1 for $j \geq 1$,
\[
F(x_{k_0+j}) \\
\leq F(x_{k_0+j-1}) + (\nabla F(x_{k_0+j-1}), x_{k_0+j} - x_{k_0+j-1}) + \frac{L}{2} \|x_{k_0+j} - x_{k_0+j-1}\|^2 \\
= F(x_{k_0+j-1}) + \sum_{i=1}^{n} (\nabla_i F(x_{k_0+j-1}))(x_{k_0+j} - x_{k_0+j-1})_i \\
+ \frac{L}{2} \sum_{i=1}^{n} (x_{k_0+j} - x_{k_0+j-1})_i^2 \\
= F(x_{k_0+j-1}) - \sum_{i=1}^{n} \frac{1}{\sqrt{v_{k_0+j_1} + \delta}} \left(1 - \frac{L}{2(\sqrt{v_{k_0+j_1} + \delta})}\right) (\nabla_i F(x_{k_0+j-1}))^2 \\
- \sum_{i=1}^{n} \frac{1}{\sqrt{v_{k_0+j_1} + \delta}} \left(1 - \frac{L}{2(\sqrt{v_{k_0+j_1} + \delta})}\right) \|\nabla F(x_{k_0+j-1})\|^2. \tag{13}
\]

For $i \notin I, \sqrt{v_i} + \delta < L$ for all $k \in \mathbb{N}$. Therefore
\[
0 \leq \sum_{k \in \mathbb{N}} \sum_{i \notin I} \frac{1}{\sqrt{v_{k_0+j} + \delta}} \left(1 - \frac{L}{2(\sqrt{v_{k_0+j} + \delta})}\right) (\nabla_i F(x_{k_0+j-1}))^2 \\
\leq \frac{1}{\sqrt{\delta}} \left(1 + \frac{L}{2\sqrt{\delta}}\right) \sum_{k \in \mathbb{N}} \|\nabla F(x_{k_0+j-1})\|^2 \\
\leq \frac{1}{\sqrt{\delta}} \left(1 + \frac{L}{2\sqrt{\delta}}\right) nL^2 = C < +\infty, \tag{14}
\]

where we let $C = \frac{1}{\sqrt{\delta}} \left(1 + \frac{L}{2\sqrt{\delta}}\right) nL^2$. Since for $i \in I, j \geq 1, \frac{L}{2(\sqrt{v_{k_0+j} + \delta})} > 1/2$, we have
\[
- \frac{1}{\sqrt{v_{k_0+j} + \delta}} \left(1 - \frac{L}{2(\sqrt{v_{k_0+j} + \delta})}\right) (1 - \frac{L}{2(\sqrt{v_{k_0+j} + \delta})}) \leq - \frac{1}{2} \frac{1}{\sqrt{v_{k_0+j} + \delta}}. \tag{15}
\]

By recurrence on (13), using (15) and (14), it follows that for all $j \geq 1$,
\[
F(x_{k_0+j}) \leq F(x_{k_0}) - \sum_{\ell=1}^{j} \frac{1}{2} \sum_{i \notin I} \frac{1}{\sqrt{v_{k_0+\ell} + \delta}} (\nabla_i F(x_{k_0+\ell-1}))^2 + C \\
\leq \sum_{i \notin I} \frac{1}{\sqrt{v_{k_0+\ell} + \delta}} \sum_{\ell=1}^{j} (\nabla_i F(x_{k_0+\ell-1}))^2.
\]

That is equivalent to
\[
2 \left( F(x_{k_0}) - F(x_{k_0+j}) + C \right) \geq \sum_{i \notin I} \frac{1}{\sqrt{v_{k_0+\ell} + \delta}} \sum_{\ell=1}^{j} (\nabla_i F(x_{k_0+\ell-1}))^2.
\]

4
Fix $p \in I$, we deduce that for all $j \geq 1$,
\[
\frac{1}{\sqrt{V_{b_0+j,p}+\delta}} \sum_{i=1}^{j} (\nabla_i F(x_{b_0+i-1}))^2
\leq \sum_{i \in I} \frac{1}{\sqrt{V_{b_0+i,j}+\delta}} \sum_{i=1}^{j} (\nabla_i F(x_{b_0+i-1}))^2
\leq 2 (F(x_{b_0}) - F(x^*) + C)
\leq 2 (F(x_{b_0}) - F(x^*) + C).
\]

By Lemma 5, we get
\[
\sum_{k=k_0}^{k_0+j-1} (\nabla_i F(x_k))^2 
\leq 4 (F(x_{b_0}) - F(x^*) + C)^2 + 2 (F(x_{b_0}) - F(x^*) + C) \sqrt{V_{b_0,j} + \delta}.
\]

We may let $j$ go to infinity and we obtain,
\[
\sum_{k=k_0}^{\infty} (\nabla_i F(x_k))^2 < \infty
\]
Since $p \in I$ was arbitrary, combining with (14), for all $i \in [1, \ldots, n]$,
\[
\sum_{k=0}^{\infty} (\nabla_i F(x_k))^2 < \infty,
\]
and the result follows
\[
\sum_{k=0}^{\infty} ||\nabla F(x_k)||^2 < \infty.
\]

We conclude this section with the convergence proof for AdaGrad.

**Proof.** Under the conditions of Theorem 3.1, assume that $x_k$ is a sequence generated by AdaGrad, Algorithm 2.2. Let $b_{k,i} = \sqrt{\delta + V_{k,i}}$, for $k \in \mathbb{N}$, $i \in [1, \ldots, n]$, all of them are increasing sequences. Fix any $x^* \in \text{arg min} F$, which is nonempty closed and convex since $F$ is convex, continuous and attains its minimum. Let $b_k = (b_{k,1}, \ldots, b_{k,n}) \in \mathbb{R}^n$. We have for all $k \in \mathbb{N}$ and $i = 1, \ldots, n$,
\[
\begin{align*}
  b_{k+1,i} (x_{k+1,i} - x^*_i)^2 \\
  = b_{k+1,i} \left( x_{k,i} - x^*_i + \frac{1}{b_{k+1,i}} \langle \nabla_i F(x_k) \rangle \right)^2 \\
  = b_{k+1,i} (x_k - x^*_i)^2 + 2 \langle \nabla_i F(x_k) \rangle (x^*_i - x_k) + \frac{1}{b_{k+1,i}} (\nabla_i F(x_k))^2.
\end{align*}
\]
By summing over $i = 1, \ldots, n$, we get for all $k \in \mathbb{N},$
\[
\begin{align*}
  \sum_{i=1}^{n} b_{k+1,i} (x_{k+1,i} - x^*_i)^2 \\
  = \sum_{i=1}^{n} b_{k+1,i} (x_k - x^*_i)^2 + 2 \sum_{i=1}^{n} \langle \nabla_i F(x_k) \rangle (x^*_i - x_k) + \frac{1}{b_{k+1,i}} (\nabla_i F(x_k))^2,
\end{align*}
\]
and hence,
\[
\begin{align*}
&& \sum_{i=1}^{n} b_{k+1,i} (x_k - x^*_i)^2 + 2 \langle \nabla F(x_k), x^* - x_k \rangle + \frac{1}{\sqrt{\delta}} ||\nabla F(x_k)||^2, \\
\leq && \sum_{i=1}^{n} b_{k+1,i} (x_k - x^*_i)^2 + 2 (F(x^*) - F(x_k)) + \frac{1}{\sqrt{\delta}} ||\nabla F(x_k)||^2.
\end{align*}
\]
\[
\text{It follows, for all } k \in \mathbb{N},
\begin{align*}
&& \sum_{i=1}^{n} b_{k+1,i} (x_k - x^*_i)^2 + 1 \frac{1}{\sqrt{\delta}} ||\nabla F(x_k)||^2 \\
\leq && \sum_{i=1}^{n} b_{k+1,i} (x_k - x^*_i)^2 + 1 \frac{1}{\sqrt{\delta}} ||\nabla F(x_k)||^2.
\end{align*}
\]
Let $M \in \mathbb{N}, M \geq 1$. For all $i \in [1, \ldots, n]$, we have,
\[
\sum_{i=1}^{M} \frac{b_{k+1,i}}{b_{k,i}} - 1 = \sum_{i=1}^{M} \frac{b_{k+1,i} - b_{k,i}}{b_{k,i}} \\
\leq \sum_{i=1}^{M} \frac{b_{k+1,i} - b_{k,i}}{\sqrt{\delta}} \\
= \frac{1}{\sqrt{\delta}} b_{M,i} \\
< \infty,
\]
where the boundedness follows from Lemma 5. So $\forall i \in [1, \ldots, n]$, the sequence $(\frac{b_{k,i}}{b_{k,i}} - 1)_{k \in \mathbb{N}}$ is summable, and since $(b_{k,i})_{k \in \mathbb{N}}$ is nondecreasing, it is also nonnegative. In particular
the sequence \((\max_{i \in \{1, \ldots, n\}} b_i) b_i^{-1}\) is summable and nonnegative.

Therefore \((x_k)_{k \in \mathbb{N}}\) is variable metric quasi-Fejér with target set \(C = \arg \min F\), metric \(W_k = B_k \geq \frac{1}{\delta_1} I\), \(\eta_k = \max_{i \in \{1, \ldots, n\}} \frac{b_i}{b_i^{-1}}\) and \(\varepsilon_k = \frac{1}{\|V F(x_k)\|}\), for all \(k \in \mathbb{N}\), using the notations of Definition 3.1. Note that \((\eta_k)_{k \in \mathbb{N}}\) does not depend on the choice of \(x^* \in C\) and is summable, and \((\varepsilon_k)_{k \in \mathbb{N}}\) is also summable by Lemma 3 so that the definition applies. By Lemma 3 \(C\) contains all the cluster points and \((W_k)_{k \in \mathbb{N}}\) converges. Thus Theorem 3.3 allows us to conclude that \((x_k)_{k \in \mathbb{N}}\) converges to a global minimum.

4. Discussion and future work

Sequential convergence of AdaGrad in the smooth convex case constitutes a further adaptivity property for this algorithm. Fejér monotonicity plays an important role here as one would expect. It is interesting to remark that our analysis does not require any assumption on the objective \(F\) beyond its Lipschitz gradient and the fact that it attains its minimum. Those are sufficient to ensure boundedness and convergence of any sequence. This is in contrast with analyses in more advanced, nonconvex, noisy settings where additional assumptions are required [22, 11]. Extensions of this analysis include the study of noise or non-smoothness in the convex case. It would also be interesting to see if the proposed approach allows to obtain better convergence bounds than the original regret analysis [12, 15].

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Appendix A. Remaining proofs and Lemmas

The following two proofs are simplification of the arguments in [10], we provide these details for completeness.

Appendix A.1. Proof of Proposition 3.2

We need the next lemma for the proof.

Lemma 4 (2). Let \((x_k)_{k \in \mathbb{N}}\) be a nonnegative sequence, let \((\eta_k)_{k \in \mathbb{N}} \in \ell^1\), and let \((\varepsilon_k)_{k \in \mathbb{N}} \in \ell^1\), such that, \(\forall k \in \mathbb{N}\), \(\alpha_k+1 \leq (1+\eta_k)\alpha_k + \varepsilon_k\). Then \((\alpha_k)_{k \in \mathbb{N}}\) converges.

Proof of the proposition.

(i) As an application of Lemma 4 to (7), with \(\alpha_k = \|u_k-u\|_{\lambda_k}\), we have that \((\|u_k-u\|_{\lambda_k})_{k \in \mathbb{N}}\) converges and hence \((\|u_k-u\|_{\lambda_k})_{k \in \mathbb{N}}\).

(ii) Let \(u \in C\). By assumption, \(W_k \geq \alpha I_n\), \(\forall k \in \mathbb{N}\), for some \(\alpha > 0\). So, \(\forall k \in \mathbb{N}\)

\[
\|u_k-u\|^2 \leq \frac{1}{\alpha} (u_k-u, W_k (u_k-u)) = \frac{\|u_k-u\|^2_{\lambda_k}}{\alpha}.
\]

Thanks to the previous point \((\|u_k-u\|_{\lambda_k})_{k \in \mathbb{N}}\) is bounded, then so is \((u_k)_{k \in \mathbb{N}}\).

Appendix A.2. Proof of Theorem 3.3

Proof of theorem. Let \(\mathcal{B}\) be the set of all accumulation points of \((x_k)_{k \in \mathbb{N}}\). First, suppose that \(x_k \to x \in C\), as \(k \to \infty\), then obviously, \(\mathcal{B} = \{x\} \subset C\).

Conversely suppose that \(\mathcal{B} \subset C\). \(\mathcal{B}\) is not empty since by the Proposition 3.2 \((x_k)_{k \in \mathbb{N}}\) is bounded. Fix \(x\) and \(x'\) in \(\mathcal{B}\) arbitrary. Say that \(x_m \to x\) and \(x_n \to x'\) along two subsequences. From Proposition 3.2 (i), we know that \((\|x_k-x\|_{\lambda_k})_{k \in \mathbb{N}}\) and \((\|x_k-x'\| W_k)_{k \in \mathbb{N}}\) converge since both \(x\) and \(x'\) are in \(C\). Also for any \(d \in \mathbb{R}^n\), \(\|d\|_{\lambda_k}^2 = (W_k d, d) \to (W d, d)\) as \(k \to \infty\). Since, for all \(k \in \mathbb{N}\)

\[
\langle W_k x_k, x-x' \rangle = \frac{1}{2} \left( \|x_k-x\|_{W_k}^2 - \|x_k-x\|_{W_k}^2 + \|x\|_{W_k}^2 + \|x'\|_{W_k}^2 \right),
\]

the sequence \(\langle W_k x_k, x-x' \rangle\) converges, let’s say to \(\lambda \in \mathbb{R}\), this means that

\[
\langle x_k, W_k(x-x') \rangle \to \lambda \in \mathbb{R}, \quad \text{as} \quad k \to \infty.
\]

(A.1)

Since, as \(k \to \infty\), \(x_m \to x\) and \(W_{m_k}(x-x') \to W(x-x')\), it follows from (A.1) that \(\langle x, W(x-x') \rangle = \lambda\). The same way we show that \(\langle x', W(x-x') \rangle = \lambda\). So

\[
0 = \langle x, W(x-x') \rangle - \langle x', W(x-x') \rangle = \langle x-x', W(x-x') \rangle \geq \alpha \|x-x'\|^2,
\]

and hence we obtain \(x = x'\). Since \(x\) and \(x'\) were arbitrary accumulation points, the set of accumulation points is reduced to a singleton, that is \(x_k \to x \in C\) as \(k \to \infty\). 

Appendix A.3. Technical lemmas

Lemma 5. Let \(a, b \geq 0\). If for \(Z \geq 0\), \(\frac{Z}{\sqrt{Z+a}} \leq b\), then

\[
Z \leq b^2 + b \sqrt{a}.
\]

Proof. We have

\[
Z \leq \frac{b^2 + \sqrt{b^4 + 4b^2 a}}{2} \leq \frac{b^2 + \sqrt{b^4 + 4b^2 a}}{2} = b^2 + b \sqrt{a}.
\]

For the equation of second order, \(\Delta = b^4 + 4b^2 a \geq 0\). We have two distinct real roots and the leading coefficient is positive, so for \(Z\) to satisfy the above inequality, we should have

\[
Z \leq b^2 + b \sqrt{a}.
\]