The non-existence, existence and uniqueness of limit cycles for quadratic polynomial differential systems

Jaume Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain (jllibre@mat.uab.cat)

Xiang Zhang
Department of Mathematics, MOE–LSC, Shanghai Jiao Tong University, Shanghai 200240, People’s Republic of China (xzhang@sjtu.edu.cn)

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We provide sufficient conditions for the non-existence, existence and uniqueness of limit cycles surrounding a focus of a quadratic polynomial differential system in the plane.

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1. Introduction and statement of the main results

One of the main problems in the qualitative theory of real planar differential systems is controlling the existence, non-existence and uniqueness of limit cycles for a given class of polynomial differential systems.

Limit cycles of planar differential systems were defined by Poincaré [15–18], and started to be studied intensively at the end of the 1920s by van der Pol [22], Liénard [11] and Andronow [1].

It is well known that if a quadratic polynomial differential system, or simply a quadratic system, has one limit cycle, this must surround a focus of the system (see, for example, [7, proposition 8.13]), and according to Bautin [2] such a system can be written in the form

\[
\begin{align*}
\dot{x} & = \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\
\dot{y} & = x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2.
\end{align*}
\]

(1.1)

In order to state our results we write the quadratic system (1.1) in polar coordinates \((r, \theta)\), defined by \(x = r \cos \theta\), \(y = r \sin \theta\), and we get

\[
\begin{align*}
\dot{r} & = \lambda_1 r + fr^2, \\
\dot{\theta} & = 1 + gr,
\end{align*}
\]

(1.2)
where

\[ f = f(\theta) = -\lambda_3 \cos^3 \theta + (3\lambda_2 + \lambda_5) \cos^2 \theta \sin \theta + (2\lambda_3 + \lambda_4 + \lambda_6) \cos \theta \sin^2 \theta - \lambda_2 \sin^3 \theta \]

and

\[ g = g(\theta) = \lambda_2 \cos^3 \theta + (3\lambda_3 + \lambda_4) \cos^2 \theta \sin \theta - (3\lambda_2 + \lambda_5) \cos \theta \sin^2 \theta - \lambda_6 \sin^3 \theta \]

are homogeneous polynomials of degree 3 in the variables \( \cos \theta \) and \( \sin \theta \). In the region

\[ C = \{(x, y) = (r \cos \theta, r \sin \theta) : r \geq 0 \text{ and } 1 + gr > 0\} \quad (1.3) \]

the differential system (1.2) is equivalent to the differential equation

\[ \frac{dr}{d\theta} = \frac{\lambda_1 r + fr^2}{1 + gr}. \quad (1.4) \]

We note that \( C \) is a simply connected region containing the origin of the coordinates. It has as its boundary the points that, in polar coordinates \((r, \theta)\), satisfy the equality \( r = -1/g \), i.e. the points of the curve \( \dot{\theta} = 0 \).

It is known that the periodic orbits surrounding the origin of system (1.2) do not intersect the curve \( \dot{\theta} = 1 + gr = 0 \) [4, appendix]. Therefore, these periodic orbits are contained in the region \( C \), and consequently are also periodic orbits of (1.4). Moreover, these periodic orbits can be studied via the change of variables [5]:

\[ \rho = \frac{r}{1 + g(\theta)r}. \]

In the new variable, \( \rho \), the differential equation (1.4) becomes

\[ \frac{d\rho}{d\theta} = g(\lambda_1 g - f)\rho^3 + (f - 2\lambda_1 g - g')\rho^2 + \lambda_1 \rho. \quad (1.5) \]

Quadratic systems have been investigated intensively; over a thousand articles have been published on such systems, and many on their limit cycles (see, for example, [19, 25]). However, we are interested in the results on the limit cycles of quadratic systems obtained using the trigonometric polynomials \( f \) and \( g \) that appear in (1.4). To the best of the authors’ knowledge, these are as follows.

(I) If \( g(\lambda_1 g - f) \geq 0 \) for all \( \theta \), then the quadratic system (1.1) has at most one limit cycle surrounding the origin (see [3, theorem 1.1(b)]).

(II) If \( f - 2\lambda_1 g - g' = 0 \), then the quadratic system (1.1) has at most one limit cycle surrounding the origin (see [9, theorem C(b)]).

Results (I) and (II) were proved for more general polynomial differential systems than the quadratic ones, and the versions presented here are their restriction to
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quadratic systems. Result (I) continues be optimal for quadratic systems, in the sense that, as we shall prove, there are quadratic systems satisfying these assumptions and having either no limit cycles or one limit cycle surrounding the origin of system (1.1). However, (II) is not optimal for quadratic systems, as we shall prove that all quadratic systems under the assumptions of (II) have no limit cycles.

Now we state our results on the limit cycles of the quadratic systems using the trigonometric polynomials $f$ and $g$. For completeness, we also include the known result (I).

**Theorem 1.1.** The quadratic system (1.1) has no limit cycles surrounding the origin if one of the following conditions holds:

(i) $f = 0$;
(ii) $f - \lambda_1 g = 0$;
(iii) $g = 0$;
(iv) $f - 2\lambda_1 g - g' = 0$;
(v) $\lambda_1 g - 9 f = 0$;
(vi) $9\lambda_1 g - f = 0$;
(vii) $(\lambda_1 g - f)(\lambda_1 g - 9 f) \leq 0$ for all $\theta$.

The quadratic system (1.1) has at most one limit cycle surrounding the origin if either of the following conditions holds:

(viii) $(\lambda_1 g - f)(9\lambda_1 g - f) \leq 0$ for all $\theta$ and $(\lambda_1 g - f)(9\lambda_1 g - f) \neq 0$;
(ix) $g(\lambda_1 g - f) \geq 0$ for all $\theta$ and $g(\lambda_1 g - f) \neq 0$.

Theorem 1.1 is proved in §3.

Note that condition (viii) cannot be obtained from (vii) by taking $9\lambda_1$ instead of $\lambda_1$, because for system (4.2) with $9\lambda_1$ instead of $\lambda_1$ we have

$$(\lambda_1 g - f)(9\lambda_1 g - f) = \frac{11(9\lambda_1 \cos \theta - 5 \sin \theta)^2}{81\lambda_1^2} \geq 0,$$

whereas

$$(\lambda_1 g - f)(\lambda_1 g - 9 f) = -\frac{(9\lambda_1 \cos \theta - 5 \sin \theta)^2}{81\lambda_1^2} \leq 0.$$ 

The next result follows easily from theorem 1.1.

**Corollary 1.2.** If the inequalities in (viii) and (ix) of theorem 1.1 are strict, then in the space there exists an open set of the twelve coefficients of the quadratic systems, where the corresponding quadratic systems have at most one limit cycle.

**Proposition 1.3.** There exist quadratic systems satisfying all the statements of theorem 1.1 except statement (viii).
Remark 1.4. Proposition 1.3 is proved in §4, in which we present examples of quadratic systems satisfying theorem 1.1, and examples of quadratic systems satisfying theorem 1.1(viii) with one limit cycle, and theorem 1.1(ix) with no limit cycle and one limit cycle. It remains an open question whether there are quadratic systems satisfying theorem 1.1(viii) without limit cycles.

Remark 1.5. In the appendix we present fourteen classes of quadratic systems for which it is known that at most one limit cycle surrounds the origin. We converted these classes to Bautin's normal form and checked that these quadratic systems do not satisfy conditions (viii) and (ix) of theorem 1.1. Consequently, the results on the uniqueness of limit cycles provided for conditions (viii) and (ix) of theorem 1.1 appear to be novel.

2. Preliminary results

In this section we recall some basic results that we shall need to prove theorem 1.1. The next two results correspond to [13, theorems 2 and 3].

Lemma 2.1. We have a differential system in polar coordinates

\[
\dot{r} = F(r, \theta), \quad \dot{\theta} = G(r, \theta),
\]

defined in a simply connected open set \( U \) containing the origin, where \( F \) and \( G \) are \( C^1 \) \( 2\pi \)-periodic functions such that \( F(0, \theta) = 0 \) for all \( \theta \) and \( G(r, \theta) > 0 \) in \( U \). Then, in \( U \) the differential system (2.1) is equivalent to the differential equation

\[
\frac{dr}{d\theta} = \frac{F(r, \theta)}{G(r, \theta)} = S(r, \theta).
\]

Therefore, if

\[
\frac{\partial S}{\partial r} \neq 0,
\]

and

either \( \frac{\partial S}{\partial r} \leq 0 \) or \( \frac{\partial S}{\partial r} \geq 0 \) in \( U \),

the differential system (2.1) has no limit cycles in \( U \).

Remark 2.2. Note that in [13] the inequalities (2.4) appear without the equals sign, but on checking the proof of [13, theorem 2] we see that it also works under the conditions (2.3) and (2.4).

Lemma 2.3. Consider the differential system (2.1) defined in an annular region \( A \) which encircles the origin and where \( G(r, \theta) > 0 \). Then, in \( A \) the differential system (2.1) is equivalent to the differential equation (2.2). If (2.3) and (2.4) hold in \( A \), then the differential system (2.1) has at most one limit cycle in \( A \).

Remark 2.2 applies to lemma 2.3 but now, using the proof of [13, theorem 3], we have the following.

Lemma 2.4. Under the assumptions of lemma 2.3, if \( \frac{\partial^3 S}{\partial r^3} \geq 0 \) in \( A \), then the differential system (2.1) has at most three limit cycles in \( A \).
Again, remark 2.2 applies to lemma 2.4 but now uses the proof of [13, theorem 8]. The following two results are well known. For a proof, see, for example, [8]. A proof of the next result can also be found in [12, theorem 1].

**Lemma 2.5.** The Riccati differential equation

\[ \frac{dr}{d\theta} = a(\theta)r^2 + b(\theta)r + c(\theta), \]

where \( a(\theta), b(\theta) \) and \( c(\theta) \) are continuous \( 2\pi \)-periodic functions, has at most two limit-cycle solutions.

The next result is [12, theorem 2] when \( a(\theta) > 0 \) (see also [14]). The case when \( a(\theta) \geq 0 \) but \( a(\theta) \neq 0 \) is proved in [6, theorem 8].

**Lemma 2.6.** The Abel differential equation

\[ \frac{dr}{d\theta} = a(\theta)r^3 + b(\theta)r^2 + c(\theta)r + d(\theta), \]

where \( a(\theta), b(\theta), c(\theta) \) and \( d(\theta) \) are continuous \( 2\pi \)-periodic functions and \( a(\theta) \geq 0 \) for all \( \theta \) but \( a(\theta) \neq 0 \), has at most three limit cycles.

The next result is due to Bautin [2].

**Lemma 2.7.** The Lyapunov constants of the quadratic differential system (1.1) when \( \lambda_1 = 0 \) are

\[
\begin{align*}
V_3 &= -\frac{1}{3} \pi \lambda_5 (\lambda_3 - \lambda_6), \\
V_5 &= \frac{1}{24} \pi \lambda_2 \lambda_4 (\lambda_3 - \lambda_6)(\lambda_4 + 5(\lambda_3 - \lambda_6)), \\
V_7 &= -\frac{1}{32} \pi \lambda_2 \lambda_4 (\lambda_3 - \lambda_6)^2 (\lambda_6 (\lambda_3 - 2\lambda_6) - \lambda_2),
\end{align*}
\]

where the expression for \( V_5 \) is given when \( V_3 = 0 \), and the expression for \( V_7 \) is given when \( V_3 = V_5 = 0 \). Moreover, the quadratic system (1.1) has a centre at the origin if and only if \( \lambda_1 = V_3 = V_5 = V_7 = 0 \).

3. **Proof of theorem 1.1**

We prove theorem 1.1 statement by statement.

(i) Since \( f = 0 \), \( dr/d\theta \) does not change sign. If \( \lambda_1 \neq 0 \), the solutions \( r(\theta) \) of (1.4) increase or decrease, so these solutions cannot be periodic in the region \( C \), and consequently the quadratic system (1.1) has no limit cycles surrounding the origin.

If \( \lambda_1 = 0 \), then \( dr/d\theta = 0 \) and all the solutions in the region \( C \) are periodic and circular (except the equilibrium point at the origin), so the system has no isolated periodic orbits surrounding the origin, i.e. no limit cycles surrounding the origin. So, statement (i) is proved.

(ii) Since \( f - \lambda_1 g = 0 \) we have that \( dr/d\theta = \lambda_1 r \). Now we can complete the proof by following the arguments in the proof of (i).
(iii) If \( g = 0 \), then the differential equation (1.4) becomes \( \frac{dr}{d\theta} = \lambda_1 r + fr^2 \), i.e. it is a Riccati differential equation. By lemma 2.5 it has at most two periodic solutions.

Since the differential equation (1.4) is invariant under the changes of variables \((r, \theta) \rightarrow (-r, \theta + \pi)\), it follows that if \( r(\theta) \) is a limit cycle of (1.4), then \(-r(\theta + \pi)\) is another limit cycle. Note that the differential equation (1.4) is defined on the annular region \( A = \{(r, \theta) : 1+gr > 0 \} \) of the cylinder \( \{(r, \theta) \in \mathbb{R} \times \mathbb{S}^1 \} \). Moreover, since \( r = 0 \) is always a periodic solution of (1.4), the differential equation (1.4) cannot have a periodic solution in the region of the half-cylinder \( r > 0 \) where it is defined, because then it would have at least three periodic solutions in \( A \), and we have proved that it has at most two periodic solutions. Consequently, the quadratic system (1.1) has no limit cycles surrounding the origin. This completes the proof of statement (iii).

(iv) Easy computations show that there are two classes of quadratic systems satisfying \( f - 2\lambda_1 g - g' = 0 \). These are

\[
\begin{align*}
\dot{x} &= -y + \frac{1}{4}\lambda_4 x^2 - 2\lambda_2 xy - \frac{1}{4}\lambda_4 y^2, \\
\dot{y} &= x + \lambda_2 x^2 + \frac{1}{2}\lambda_4 xy - \lambda_2 y^2,
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y + 2\lambda_1 \lambda_2 x^2 - 2\lambda_2 xy, \\
\dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + 2\lambda_1 \lambda_2 xy - \lambda_2 y^2.
\end{align*}
\]

Then, using lemma 2.7, it follows that all the Lyapunov constants of systems (3.1) and (3.2) with \( \lambda_1 = 0 \) are zero. So, these systems have a centre at the origin, and it is known that there are no limit cycles surrounding the origin.

It now remains to prove that system (3.2) with \( \lambda_1 \neq 0 \) has no limit cycles. Indeed, in addition to the focus at the origin, this system has a second equilibrium point, \((x, y) = (-1/\lambda_1, -\lambda_1/\lambda_2)\). We translate this equilibrium at the origin by a change of variables \( x = X - 1/\lambda_1 \) and \( y = Y - \lambda_1/\lambda_2 \), to obtain the system

\[
\begin{align*}
\dot{X} &= -\lambda_1 X + Y + 2\lambda_1 \lambda_2 X^2 - 2XY\lambda_2, \\
\dot{Y} &= -(1 + 2\lambda_1^2)X + \lambda_1 Y + \lambda_2 X^2 + 2\lambda_1 \lambda_2 XY - \lambda_2 Y^2.
\end{align*}
\]

Then, using lemma 2.7, it follows that all the Lyapunov constants of system (3.3) are \( \pm \sqrt{1 + \lambda_1^2} i \). We write the linear part of system (3.3) in its real Jordan normal form via the following change of variables

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sqrt{1 + \lambda_1^2}} & \lambda_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\]

In the variables \((u, v)\), system (3.3) becomes

\[
\begin{align*}
\dot{u} &= -\sqrt{\lambda_1^2 + 1} v - \sqrt{\lambda_1^2 + 1} u^2 + 2\lambda_1 u v + \sqrt{\lambda_1^2 + 1} v^2, \\
\dot{v} &= \sqrt{\lambda_1^2 + 1}(u - 2uv).
\end{align*}
\]

If we now compute the Lyapunov constants for this system, which is in the normal form of Bautin, using lemma 2.7 we obtain that the three are zero, so this system has a centre at the origin. Since quadratic systems having a centre have no limit cycles (see [20, 23]), this completes the proof of (iv).
Again, easy computations show that there are two classes of quadratic systems satisfying \( \lambda_1 g - 9f = 0 \):

\[
\begin{align*}
\dot{x} &= -y + y^2, \\
\dot{y} &= x - xy,
\end{align*}
\]  

(3.4)

and

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y + \frac{\lambda_1}{9} x^2 - 2yx + \frac{9}{9}\lambda_1 y^2, \\
\dot{y} &= x + \lambda_1 y + x^2 + \frac{\lambda_1^2 - 81}{9\lambda_1} xy - y^2,
\end{align*}
\]  

(3.5)

with \( \lambda_1 \neq 0 \).

By lemma 2.7 the quadratic system (3.4) has a centre at the origin, and consequently this system has no limit cycles.

If we write system (3.5) in polar coordinates and take \( \theta \) as the new independent variable, we obtain the differential equation

\[
\frac{dr}{d\theta} = \frac{\lambda_1 r(9\lambda_1 + \lambda_1 r \cos \theta - 9r \sin \theta)}{9(\lambda_1 + \lambda_1 r \cos \theta - 9r \sin \theta)} = S(r, \theta),
\]

(3.6)

defined in the simply connected region \( C \). Since

\[
\frac{\partial S}{\partial r} = \frac{\lambda_1}{9} \left( \frac{8\lambda_1^2}{(\lambda_1 + \lambda_1 r \cos \theta - 9r \sin \theta)^2 + 1} \right) \geq 0 \quad \text{if } \lambda_1 > 0
\]

or

\[
\frac{\partial S}{\partial r} = \frac{\lambda_1}{9} \left( \frac{8\lambda_1^2}{(\lambda_1 + \lambda_1 r \cos \theta - 9r \sin \theta)^2 + 1} \right) \leq 0 \quad \text{if } \lambda_1 < 0,
\]

we obtain that \( \partial S/\partial r \) satisfies (2.3) and (2.4) in the region \( C \), and thus we can apply lemma 2.1 to the differential equation (3.6), and the proof of (v) follows.

(vi) After easy computations, the quadratic systems satisfying \( 9\lambda_1 g - f = 0 \) yield system (3.4) and

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y + 9\lambda_1 x^2 - 2yx + \frac{1}{9\lambda_1} y^2, \\
\dot{y} &= x + \lambda_1 y + x^2 + \frac{81\lambda_1^2 - 1}{9\lambda_1} xy - y^2,
\end{align*}
\]  

(3.7)

with \( \lambda_1 \neq 0 \). Since we have proved that system (3.4) has no limit cycles, it remains only to prove that system (3.7) has no limit cycles.

Writing system (3.7) in polar coordinates and taking \( \theta \) as the new independent variable, we obtain the differential equation

\[
\frac{dr}{d\theta} = \frac{\lambda_1 r + r^2(9\lambda_1 \cos \theta - \sin \theta)}{1 + r(\cos \theta - \sin \theta/(9\lambda_1))} = S(\theta, r),
\]

(3.8)

Then

\[
\frac{\partial^3 S}{\partial r^3} = -\frac{3888\lambda_1^3 (9\lambda_1 \cos \theta - \sin \theta)^2}{(9\lambda_1 + r(9\lambda_1 \cos \theta - \sin \theta))^3}.
\]
We can assume that $\lambda_1 < 0$; otherwise, we reverse the sign of the time in the differential system (3.7). Therefore, we have that $\partial^3 S/\partial r^3 \geq 0$, and the equality holds at only finitely many points. Taking into account $C$, which is defined in (1.3), we know that (3.8) is defined on an annular region, say $A$, of the cylinder $\{ (\theta, r) \in S^1 \times \mathbb{R} \}$ (see figure 1).

Take $\theta = 0$ as a Poincaré section of (3.8) in $A$, and take $r_0 \in \mathbb{R}$ as an initial point of (3.8) at $\theta = 0$. Let $H(r_0) = r(2\pi, r_0) - r_0$ be the displacement function, with $r(\theta, r_0)$ being the solution of (3.8) such that $r(0, r_0) = r_0$ for the point $(0, r_0) \in A$. Since $\partial^3 S/\partial r^3 \geq 0$ and is not identically zero, from the proof of [13, theorem 8] we have that $H'''(r_0) > 0$ for all $r_0$, where the map $H$ is defined. There cannot be more than three zeros of $H(r_0)$, taking into account their multiplicities, otherwise $H'''(r_0)$ would have a zero, which is a contradiction. So, (3.8) has at most three different zeros with multiplicity 1.

We saw in the proof of (iii) that if $r(\theta)$ is a solution of the differential equation (1.4), then $-r(\theta + \pi)$ is another solution. Moreover, since $r = 0$ is always a periodic solution of (3.8) (see figure 1), when (3.8) has three simple zeros it has a unique positive zero. Consequently, the quadratic polynomial differential system (3.7) has at most one limit cycle in $r > 0$. Moreover, if the limit cycle exists, it is hyperbolic, because in this case the zeros of $H(x)$ are simple.

We shall now prove that the quadratic systems (3.7) have no limit cycles. Indeed, systems (3.7) with $\lambda_1 \neq 0$ have a unique finite equilibrium (a hyperbolic focus), and a unique pair of infinite equilibria that are semi-hyperbolic saddle nodes having only one separatrix $\gamma$ outside the circle at infinity, which, by the Poincaré–Bendixson theorem, must surround the unstable focus (see [7, §1.7] for more details). Moreover, using [7, theorem 2.19], it follows that if the origin is a stable (respectively, unstable) focus, then the separatrix $\gamma$ is unstable (respectively, stable). Therefore, since system (3.7) has at most one limit cycle, if such a limit cycle exists, it must be semi-stable, in contradiction with the fact that if the limit cycle exists it must be hyperbolic, which we proved in the previous paragraph. This completes the proof of (vi).

(vii) Under the change of variables $R = \sqrt{r}$ in the region $C$, the differential equation (1.4) becomes

\[
\frac{dR}{d\theta} = \frac{\lambda_1 R + fR^3}{2(1 + gR^2)} = S(R, \theta).
\]
Clearly, the image of the simply connected region $C$ under the map $r \to \sqrt{r} = R$ is another simply connected region $S$ containing the origin $R = 0$.

We have

$$\frac{\partial S}{\partial R} = \frac{\lambda_1 + (3f - \lambda_4)R^2 + fgR^4}{2(1 + gR^2)^2}.$$  

Then, clearly if $(3f - \lambda_4)^2 - 4\lambda_1 fg = (\lambda_4 - f)(\lambda_4 - 9f) \leq 0$, we have that $\partial S/\partial R$ satisfies (2.3) and (2.4) in the region $S$, and we can thus apply lemma 2.1 to the differential equation (3.9). The proof of (vii) follows.

(viii) Under the change of variables $R = 1/\sqrt{r}$ in the region $C$, the differential equation (1.4) becomes

$$\frac{dR}{d\theta} = -\frac{\lambda_1 R^3 + fR}{2(g + R^2)} = S(R, \theta). \quad (3.10)$$

The image of the region $C$ by the map $r \to 1/\sqrt{r} = R$ is now an annular region $A$, and one of the boundaries of this annulus is the circle at infinity.

We obtain

$$\frac{\partial S}{\partial R} = \frac{-fg - (f - 3\lambda_1 g)R^2 + \lambda_1 R^4}{2(g + R^2)^2}.$$  

Then, clearly if $(f - 3\lambda_1 g)^2 - 4\lambda_1 fg = (\lambda_4 - f)(\lambda_4 - 9f) \leq 0$, we have that $\partial S/\partial R$ satisfies conditions (2.3) and (2.4) in the annular region $A$, and thus we can apply lemma 2.3 to the differential equation (3.10). This completes the proof of (viii).

(ix) Note that $g(\lambda_4 - f)$ is the coefficient of $\rho^3$ in the Abel differential equation (1.5).

Assume that $g(\lambda_4 - f) \geq 0$ and $g(\lambda_4 - f) \neq 0$. Therefore, by lemma 2.6 the Abel differential equation (1.5) has at most three periodic solutions, and thus the differential equation (1.4) has at most three periodic solutions.

We saw in the proof of (iii) that if $r(\theta)$ is a solution of the differential equation (1.4), then $-r(\theta + \pi)$ is another solution. Moreover, since $r = 0$ is always a periodic solution of (1.4), we obtain that the differential equation (1.4) has at most one periodic solution in the region of the half-cylinder $r > 0$ where it is defined. Consequently, the quadratic system (1.1) has at most one limit cycle. This completes the proof of statement (ix). Thus, the proof of theorem 1.1 is complete.

4. Proof of proposition 1.3

We prove proposition 1.3 by giving examples of quadratic systems satisfying the statements of theorem 1.1.

Examples of statement (i). It is easy to compute that all quadratic systems (1.1) with $f = 0$ are

$$\dot{x} = \lambda_1 x - y - \lambda_4 y^2, \quad \dot{y} = x + \lambda_4 y + \lambda_4 xy. \quad (4.1)$$
Examples of statement (ii). Again, it is easy to verify that all quadratic systems (1.1) with \( f - \lambda_1 g = 0 \) are the systems (4.1) and also the systems

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y + \lambda_1 \lambda_2 x^2 - 2\lambda_2 xy + \frac{\lambda_2}{\lambda_1} y^2, \\
\dot{y} &= x + y\lambda_1 + \lambda_2 x^2 + \frac{(\lambda_1 - 1)(\lambda_1 + 1)\lambda_2}{\lambda_1} xy - \lambda_2 y^2.
\end{align*}
\]

Examples of statement (iii). The quadratic systems (1.1) having \( g = 0 \) are

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y - \lambda_3 x^2, \\
\dot{y} &= x + \lambda_1 y - \lambda_3 xy.
\end{align*}
\]

Examples of statements (iv)–(vi). These are given in the proofs of (iv)–(vi) of theorem 1.1.

Examples of statement (vii). For the quadratic systems

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y + \frac{5}{4\lambda_1^2} (\lambda_1 x - 5y)^2, \\
\dot{y} &= x + \lambda_1 y + \frac{5}{4\lambda_1^2} (\lambda_1 x - 5y)(5x + \lambda_1 y),
\end{align*}
\]

depending on the parameters \( \lambda_1 \), we obtain

\[
(\lambda_1 g - f)(\lambda_1 g - 9f) = -\frac{(\lambda_1 \cos \theta - 5 \sin \theta)^2}{\lambda_1^2} \leq 0.
\]

Therefore, there are quadratic systems satisfying the assumptions of statement (vii).

Now, the proof that quadratic systems (4.2) have no limit cycles is equivalent to showing the systems (3.7) have no limit cycles.

Examples of statement (viii). We consider the family of quadratic systems

\[
\begin{align*}
\dot{x} &= \lambda_1 x - y + \frac{5}{4\lambda_1} (5\lambda_1 x - y)^2, \\
\dot{y} &= x + \lambda_1 y + \frac{5}{4\lambda_1} (5\lambda_1 x - y)(x + 5\lambda_1 y),
\end{align*}
\]

depending on the parameters \( \lambda_1 \). We have that

\[
(\lambda_1 g - f)(9\lambda_1 g - f) = -\frac{16}{25}(\sin \theta - 5\lambda_1 \cos \theta)^2 \leq 0.
\]

Hence, this family of quadratic systems satisfies the hypotheses of statement (viii).

The quadratic system (4.3) with \( \lambda_1 \neq 0 \) has a unique finite equilibrium (a hyperbolic focus) and a unique pair of infinite equilibria that are semi-hyperbolic saddle nodes having only one separatrix \( \gamma \) outside the circle at infinity, which, by the Poincaré–Bendixson theorem, must surround the unstable focus. Moreover, using [7, theorem 2.19], it follows that if the origin is a stable (respectively, unstable) focus, then the separatrix \( \gamma \) is stable (respectively, unstable). So, again by the Poincaré–Bendixson theorem, at least one limit cycle surrounds such an unstable focus localized at the origin of coordinates (see figure 2). Then, by theorem 1.1(viii) this limit cycle is unique.
On the limit cycles of quadratic systems

Figure 2. Phase portrait of the quadratic system (4.3) with $\lambda_1 = \frac{1}{10}$.

In short, we have proved that there are quadratic systems satisfying the assumptions of theorem 1.1(viii), having one limit cycle surrounding the origin.

Examples of statement (ix). For the quadratic systems
\[
\dot{x} = \lambda x - y - bx^2 - \varepsilon xy, \quad \dot{y} = x + \lambda y - bxy,
\]
(4.4)
depending on the three parameters $\lambda_1$, $\varepsilon$ and $b$, we have that
\[
g(\lambda_1 g - f) = \varepsilon b \cos^2 \theta \sin^2 \theta + \varepsilon^2 \cos^2 \theta \sin^3 \theta (\cos \theta + \lambda_1 \sin \theta).
\]
So, if $\varepsilon$ is sufficiently small, and $\varepsilon b > 0$, we have that $g(\lambda_1 g - f) \geq 0$, and of course $g(\lambda_1 g - f) \neq 0$. Therefore, the quadratic system (4.4) satisfies the assumption of theorem 1.1(viii). Hence, it has at most one limit cycle.

The eigenvalues at the origin of system (4.4) are $\lambda_1 \pm i$. Therefore, when $\lambda_1 = 0$, if there is a non-zero Lyapunov constant, then a Hopf bifurcation of an infinitesimal periodic orbit occurs at the origin of the coordinates (see [10] for more details about Hopf bifurcations). The Lyapunov constants for the quadratic differential systems (1.1) with $\lambda_1 = 0$ were computed by Bautin [2]. The first one is
\[
V_3 = -\frac{1}{4} \pi \lambda_3 (\lambda_3 - \lambda_6)
\]
(see lemma 2.7). Then, for our system (4.4), when $\lambda_1 = 0$, we have $V_3 = \frac{1}{4} \varepsilon b \pi > 0$ because $\varepsilon b > 0$. Hence, there are systems (4.4) that exhibit no limit cycles or one limit cycle in a neighbourhood of $\lambda_1 = 0$ such that $\varepsilon b > 0$ and $\varepsilon$ are sufficiently small.

In short we have proved that there are quadratic systems satisfying the conditions of theorem 1.1(ix) that exhibit either zero, or one limit cycle surrounding the origin.

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Appendix A.

In the following we provide a list of fourteen classes of quadratic differential systems known to have at most one limit cycle surrounding the origin. The first twelve come from [24, §9]; the last two come from [26] and [21], respectively. There are some other results on the uniqueness of limit cycles of quadratic systems, but we do not list them here.

(i) The quadratic systems in Ye’s normal form (I) can be written as
\[ \dot{x} = -y + \delta x + lx^2 + xy + ny^2, \quad \dot{y} = x. \]

(ii) The quadratic systems in Ye’s normal form (III)\( n=0 \),
\[ \dot{x} = -y + lx^2 + mxy, \quad \dot{y} = x + ax^2 + bxy, \]
have a weak focus of order 2 at the origin, i.e. they satisfy the conditions
\[ ml - a(b + 2l) = 0, \quad ma(5a - m)(bl^2 - a^2(b + 2l)) \neq 0. \]

(iii) The quadratic systems in Ye’s normal form (III)\( n \neq 0 \),
\[ \dot{x} = -y + lx^2 + mxy + y^2, \quad \dot{y} = x + ax^2 + bxy, \]
have a weak focus of order 2 at the origin, i.e. they satisfy the conditions
\[ m(l + 1) - a(b + 2l) = 0, \quad ma(5a - m)((b + 1)(l + 1)^2 - a^2(b + 2l + 1)) \neq 0. \]

(iv) The quadratic systems in Ye’s normal form (III)\( a=0 \) can be written as
\[ \dot{x} = -y + \delta x + lx^2 + mxy + ny^2, \quad \dot{y} = x + bxy. \]

(v) A special quadratic system in Ye’s normal form (III) can be written as
\[ \dot{x} = -y + \delta x + ny^2, \quad \dot{y} = x + ax^2 - xy, \]
with \( 0 < n < 1 \).

(vi) Quadratic systems having a degenerate finite singularity can be written in Ye’s normal form (III) as
\[ \dot{x} = -y + \delta x + lx^2 - \delta xy + y^2, \quad |\delta| < 2, \quad \dot{y} = x + ax^2 - xy. \]

These have \((0, 1)\) as a degenerate singularity (i.e. the linearization of the system at \((0, 1)\) has two zero eigenvalues and is non-zero), and satisfy \(a\delta(2l - 1) \neq 0\) (otherwise, the origin is a centre).
(vii) Quadratic systems having a degenerate singularity at infinity can be written in Ye’s normal form (III) as
\[ \dot{x} = -y + \delta x + lx^2, \quad |\delta| < 2, \quad \dot{y} = x + ax^2 + bxy. \]

(viii) A quadratic system in Ye’s normal form (III) can be written as
\[ \dot{x} = -y + \delta x(y - 1) + lx^2 + ny^2, \quad \dot{y} = x + ax^2 - xy, \]
with \( \frac{1}{2} \leq n < 1 \) (otherwise the existence or non-existence of limit cycles is unknown).

(ix) A quadratic system in Ye’s normal form (III) can be written as
\[ \dot{x} = -y + \delta x(y - 1) + lx^2 + y^2, \quad \dot{y} = x + ax^2 + bxy, \]
with \( \delta \neq 0 \), \( a > 0 \) and \( b + 1 < 0 \).

(x) Quadratic systems having a finite singularity of multiplicity \( \geq 3 \) can be written in Ye’s normal form (III) as
\[ \dot{x} = -y + \delta x + lx^2 - (a + \delta)xy + y^2, \quad \dot{y} = x + ax^2 - xy. \]
These have \((0, 1)\) as a singularity of multiple 3, and satisfy \( \delta > 0 \) and \( 0 < a\delta / l < 1 \) (otherwise the systems have no limit cycles).

(xi) Quadratic systems having a singularity of multiplicity \( \geq 3 \) at infinity can be written in Ye’s normal form (III) as
\[ \dot{x} = -y + \delta x + lx^2 + mxy + ny^2, \quad \dot{y} = x + ax^2 + cxy, \]
with coefficients satisfying either \( b = l \neq 0 \), \( m = c \) or \( l = 0 \), \( m = b\delta \), \( b \neq 0 \).

(xii) The quadratic systems in Ye’s normal form (III) can be written as
\[ \dot{x} = -y + \delta x + (m - b\delta)x^2 + bxy, \quad \dot{y} = x + (n + 2\delta m - b\delta^2)x^2 + (b\delta - m)xy, \]
with coefficients satisfying either \( b = 0 \); or \( m^2 + bn = 0 \).

(xiii) The quadratic systems
\[ \dot{x} = -y + \delta x + lx^2 + ny^2, \quad \dot{y} = x + ax^2 - xy, \]
have at most one limit cycle under the conditions \( a < 0 \), \( l > \frac{1}{2} \), \( 0 < n < 1 \) and \( \delta < 0 \).

(xiv) The quadratic systems
\[ \dot{x} = -y + \delta x + lx^2 + ny^2, \quad \dot{y} = x + ax^2 - xy, \]
have at most one limit cycle under the conditions \( 0 < n \leq \frac{1}{2} \) \( (\frac{1}{2} \leq n < 1) \), \( -na^2 < l < \frac{1}{2} \), \( a\delta < 0 \) \( (a\delta > 0) \).
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