Braid groups and Artin groups

Luis Paris

Institut de Mathématiques de Bourgogne – UMR 5584 du CNRS
Université de Bourgogne, BP 47870
21078 Dijon cedex, France
email:lparis@u-bourgogne.fr

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1 Introduction

The braids go back to several centuries and were universally used for ornamental purposes or even practical ones, for example in the fashioning of ropes. Now, they are described by means of abstract models known under the name of “theory of braids”. The theory of braids studies the concept of braids (such as we imagine them) as well as various generalizations arising from various branches of the mathematics. The idea is that the braids form a group. The number of strands must be fixed so that the operation is well-defined. So, we have a braid group on two strands, a braid group on three strands, and so on. The braid group on one strand is trivial because a string cannot be braided (although it can be knotted).

We generally make the mathematical study of braids go back to an article of Emile Artin [7] dated from 1925, in which is described the notion of braids under various aspects, one being that obvious, like a “series of tended and interlaced strings”, and others more conceptual but equally deep, such as a presentation by generators and relations, or a presentation as the mapping class group of a punctured disk.

Since the 30s, a strong link between braids and links (and knots) were established by people such as Alexander and Markov (see [19]). This link is at the origin in the 80s of a deep revival in the theory of knots with the work of Jones and his invariant defined from the theory of braids (see [101], [102], [83], and [137]).

Later, interesting relations with the algebraic geometry and the theory of finite groups generated by reflections were established, in particular by Arnol’d [5], [3], [6] and Brieskorn [29], [31]. These relations become particularly interesting when we extend the notion of braid groups to that of Artin groups of spherical type, also called generalized braid groups. Although the Artin groups were introduced by Tits [145] as extensions of Coxeter groups, their study really began in the seventies with the works of Brieskorn [30], [31], Saito [32] and Deligne [71], where different aspects of these groups are studied, such as their combinatorics, as well as their link with the hyperplane arrangements and the singularities.

Some problems in group theory, often very close to the algorithmics, such as the word and conjugacy problems, have a renewal of interest not only through their applications in the other domains, but also because the notion of mathematical demonstration is changing. Indeed, we distinguish now the notion of demonstration from the notion of effective demonstration, the one which builds up the solution. Such a demonstration gives rise to an algorithm, and its complexity (calculation time) is of importance. The algorithmics in the braid groups is especially active. Problems of decision such as the conjugacy problem were solved by Garside [85] in 1969 with methods which are now the source of numerous works on the braid groups. In [70] is introduced a
more formal and more general framework to study algorithmic problems on the braid groups: the Garside groups. The idea is to isolate certain combinatorial properties of the braid groups, in particular these emphasized by Garside [85]. It is a less restrictive model which uses tools from the language theory (monoids, rewriting systems) and the combinatorics (ordered sets), tools that are especially adapted to treat algorithmic problems. Now, the major part of the algorithmic problems on the braid groups are studied within the framework of the Garside groups. Also, let us indicate that the Artin groups of spherical type are Garside groups.

This survey is written from these viewpoints but also maintaining two other objectives: (1) to make a survey understandable by non-specialists; (2) to make as often as possible the link with the mapping class groups.

The first section is about the “classical” theory of braid groups. Various aspects as well as some properties of them are presented. The second section is an introduction to the Artin groups, and the third is an introduction to the Garside groups. There, the reader will find algorithms to solve some decision problems such as the conjugacy one for the braid groups (and Garside groups).

The fifth section is about the cohomology of Artin groups, although the exposition goes beyond by explaining the Salvetti complexes. These are tools originally from the theory of hyperplane arrangements that turn out to be useful in the context of the braid groups.

The sixth section is about the linear representations of the braid groups studied by Bigelow [17] and Krammer [105], [106], as well as about its various generalizations (to the Artin groups). Both, the algebraic aspect and the topological aspect of these representations, are explained. Other linear representations of the braid groups have been studied and are also interesting but, for lack of place and for reason of coherence, these will not be treated in this text. We refer to [21] for a survey on the other linear representations.

The seventh section is about the geometric representations of the Artin groups. (By a geometric representation we simply mean a homomorphism in a mapping class group.) This subject is less popular than the previous ones but I strongly believe in its future. In particular, Subsection 7.3, where are explained the results of Castel [40], shows all the power of such a study.

Finally, I would like to indicate two aspects of the braid groups which are not in this survey and which “should be in any survey on the braid groups”.

The first aspect is the link of the braids with links and knots. This is very important in the theory but amply explained in all the books and almost all the surveys on the subject. So, I voluntarily ignore this aspect in order to be able to treat in a more detailed way the other ones. The reader will find in [19], [93], [127], [103] detailed expositions on this aspect and on the braid groups in general.

I would have wanted to make an eighth section to explain the second aspect: the orders in the braid groups. But, unfortunately, this article is long enough
and there is no more room for another section. Inspired by problems of set theory, Dehornoy [65] founded an explicit construction of a total ordering invariant by left multiplication in the braid group. The fact that the braid group is orderable is not maybe completely new, in the sense that it results from Nielsen theory [128], but Dehornoy’s ordering is interesting in itself. In my opinion, it is an important tool to understand the braid groups, and I augur numerous developments in this direction. The Artin groups of type $B_n$ and $\tilde{A}_n$ embed into braid groups (see Section 3) thus they are also orderable. The Artin groups of type $D_n$ embed into mapping class groups of surfaces with boundary (see Section 7), and, by [138], such a group is orderable. We do not know whether the other Artin groups are orderable or not. We encourage the reader to consult [68] for a detailed discussion on this subject.

2 Braid groups

2.1 Braids

Let $n \geq 1$ be an integer, and let $P_1, \ldots, P_n$ be $n$ distinct points in the plane $\mathbb{R}^2$ (except mention of the contrary, we will always assume $P_k = (k, 0)$ for all $1 \leq k \leq n$). Define a braid on $n$ strands to be a $n$-tuple $\beta = (b_1, \ldots, b_n)$ of paths, $b_k : [0, 1] \to \mathbb{R}^2$, such that

- $b_k(0) = P_k$ for all $1 \leq k \leq n$;
- there exists a permutation $\chi = \theta(\beta) \in \text{Sym}_n$ such that $b_k(1) = P_{\chi(k)}$ for all $1 \leq k \leq n$;
- $b_k(t) \neq b_l(t)$ for all $k \neq l$ and all $t \in [0, 1]$.

Two braids $\alpha$ and $\beta$ are said to be homotopic if there exists a continuous family $\{\gamma_s\}_{s \in [0, 1]}$ of braids such that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. Note that $\theta(\alpha) = \theta(\beta)$ if $\alpha$ and $\beta$ are homotopic.

We represent graphically a homotopy class of braids as follows. Let $I_k$ be a copy of the interval $[0, 1]$. Take a braid $\beta = (b_1, \ldots, b_n)$ and define the geometric braid $\beta^g : I_1 \sqcup \cdots \sqcup I_n \to \mathbb{R} \times [0, 1]$ by $\beta^g(t) = (b_k(t), t)$ for all $t \in I_k$ and all $1 \leq k \leq n$. Let $\text{proj} : \mathbb{R}^2 \times [0, 1] \to \mathbb{R} \times [0, 1]$ be the projection defined by $\text{proj}(x, y, t) = (x, t)$.

Up to homotopy, we can assume that $\text{proj} \circ \beta^g$ is a smooth immersion with only transversal double points that we call crossings. In each crossing we indicate graphically like in Figure 2.1 which strand goes over the other. Such
a representation of $\beta$ is called a \textit{braid diagram} of $\beta$. An example is illustrated in Figure 2.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{crossings.png}
\caption{Crossings in a braid diagram.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{braid_diagram.png}
\caption{A braid diagram.}
\end{figure}

The \textit{product} of two braids $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ is defined to be the braid

$$\alpha \cdot \beta = (a_1 b_{\chi(1)}, \ldots, a_n b_{\chi(n)}) ,$$

where $\chi = \theta(\alpha)$. An example is illustrated in Figure 2.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{product_braid.png}
\caption{Product of two braids.}
\end{figure}

Let $B_n$ denote the set of homotopy classes of braids on $n$ strands. It is easily seen that the above defined multiplication of braids induces an operation on $B_n$. Moreover, we have the following.

\textbf{Proposition 2.1.} The set $B_n$ endowed with this operation is a group.
From now on, except mention of the contrary, by a braid we will mean a homotopy class of braids. The group $B_n$ of Proposition 2.1 is called the braid group on $n$ strands. The identity is the constant braid $\text{Id} = (\text{Id}_1, \ldots, \text{Id}_n)$, where, for $1 \leq k \leq n$, $\text{Id}_k$ denotes the constant path on $P_k$. The inverse of a braid $\beta$ is its mirror as illustrated in Figure 2.4.

![Figure 2.4. Inverse of a braid.](image)

Recall that, if two braids $\alpha, \alpha'$ are homotopic, then $\theta(\alpha) = \theta(\alpha')$. Hence, the map $\theta$ from the set of braids on $n$ strands to $\text{Sym}_n$ induces a map $\theta : B_n \to \text{Sym}_n$. It is easily checked that this map is an epimorphism. Its kernel is called the pure braid group on $n$ strands and is denoted by $\mathcal{PB}_n$. It plays an important role in the theory.

Let $\sigma_k$ be the braid illustrated in Figure 2.5. One can easily verify that $\sigma_1, \ldots, \sigma_{n-1}$ generate the braid group $B_n$ and satisfy the relations

$$
\sigma_k \sigma_l = \sigma_l \sigma_k \quad \text{if } |k-l| \geq 2,
\sigma_k \sigma_l \sigma_k = \sigma_l \sigma_k \sigma_l \quad \text{if } |k-l| = 1.
$$

(See Figure 2.6.) These relations suffice to define the braid group, namely:

![Figure 2.5. The braid $\sigma_k$.](image)

**Theorem 2.2** (Artin [7], [8], Magnus [118]). The group $B_n$ has a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$
\sigma_k \sigma_l = \sigma_l \sigma_k \quad \text{if } |k-l| \geq 2,
\sigma_k \sigma_l \sigma_k = \sigma_l \sigma_k \sigma_l \quad \text{if } |k-l| = 1.
$$

**Theorem 2.3** (Burau [35], Markov [123]). For $1 \leq k < l \leq n$, let

$$
\delta_{kl} = \sigma_{l-1} \cdots \sigma_{k+1} \sigma_k^2 \sigma_{k+1}^{-1} \cdots \sigma_{l-1}^{-1}.
$$
Then the pure braid group \( \mathcal{P}\mathcal{B}_n \) has a presentation with generators

\[
\delta_{kl}, \quad 1 \leq k < l \leq n,
\]

and relations

\[
\begin{align*}
\delta_r \delta_k \delta_{kl}^{-1} &= \delta_{kl} \quad &\text{if } 1 \leq r < s < k < l \leq n \\
\delta_r \delta_k \delta_{kl}^{-1} &= \delta_{k-1} \delta_{k-1} \delta_k \delta_{kl} \delta_{kl} \quad &\text{if } 1 \leq r < k < l \leq n \\
\delta_r \delta_k \delta_{kl}^{-1} &= \delta_{k-1} \delta_{k-1} \delta_k \delta_{kl} \delta_{kl} \quad &\text{if } 1 \leq r < k < l \leq n \\
\delta_r \delta_k \delta_{kl}^{-1} &= \delta_{s-1} \delta_{s-1} \delta_k \delta_{kl} \delta_{kl} \delta_{kl} \delta_{kl} \quad &\text{if } 1 \leq r < k < l \leq n.
\end{align*}
\]

**Note.** Most of the proofs of Theorems 2.2 and 2.3 that can be found in the literature proceed as follows. Given an exact sequence

\[
1 \to K \to G \to H \to 1,
\]

there is a machinery to compute a presentation of \( G \) from presentations of \( K \) and \( H \). We start with the observation that \( \mathcal{P}\mathcal{B}_2 \simeq \mathbb{Z} \) and with the exact sequence

\[
1 \to F_n \to \mathcal{P}\mathcal{B}_{n+1} \to \mathcal{P}\mathcal{B}_n \to 1,
\]

(2.1)

where \( F_n \) is a free group of rank \( n \), to prove Theorem 2.3 by induction on \( n \). (The exact sequence (2.1) will be explained in Subsection 2.2.) Then we use the exact sequence

\[
1 \to \mathcal{P}\mathcal{B}_n \to \mathcal{B}_n \to \text{Sym}_n \to 1
\]

to prove Theorem 2.2 from Theorem 2.3. Another proof which, as far as I know, is not written in the literature but is known to experts, consists on extracting the presentation of Theorem 2.2 from the Salvetti complex of \( \mathcal{B}_n \). This is a cellular complex which is a \( K(\mathcal{B}_n, 1) \) (see Section 5).
2.2 Configuration spaces

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and $P_k$ with $k \in \mathbb{C}$ for all $1 \leq k \leq n$. For $1 \leq k < l \leq n$ we denote by $H_{k,l}$ the linear hyperplane of $\mathbb{C}^n$ defined by the equation $z_k = z_l$.

The big diagonal of $\mathbb{C}^n$ is defined to be

$$\text{Diag}_n = \bigcup_{1 \leq k < l \leq n} H_{k,l}. $$

The space of ordered configurations of $n$ points in $\mathbb{C}$ is defined to be

$$M_n = \mathbb{C}^n \setminus \text{Diag}_n. $$

This is the space of $n$-tuples $z = (z_1, \ldots, z_n)$ of complex numbers such that $z_k \neq z_l$ for $k \neq l$. The symmetric group $\text{Sym}_n$ acts freely on $M_n$. The quotient

$$N_n = M_n / \text{Sym}_n$$

is called the space of configurations of $n$ points in $\mathbb{C}$. This is the space of unordered $n$-tuples $z = \{z_1, \ldots, z_n\}$ of complex numbers such that $z_k \neq z_l$ for $k \neq l$.

**Proposition 2.4.** Let $P_0 = (1, 2, \ldots, n) \in M_n$. Then $\pi_1(M_n, P_0) = PB_n$.

**Proof.** For a pure braid $\beta = (b_1, \ldots, b_n)$ we set

$$\varphi(\beta) : [0, 1] \rightarrow M_n, \quad t \mapsto (b_1(t), \ldots, b_n(t)).$$

Clearly, $\varphi(\beta)$ is a loop based at $P_0$. Moreover, two pure braids $\alpha$ and $\alpha'$ are homotopic if and only if $\varphi(\alpha)$ and $\varphi(\alpha')$ are homotopic. Thus $\varphi$ induces a bijection $\varphi_* : PB_n \rightarrow \pi_1(M_n, P_0)$ which turns out to be a homomorphism. $\square$

For $z \in M_n$, we denote by $[z]$ the element of $N_n = M_n / \text{Sym}_n$ represented by $z$.

**Proposition 2.5.** $\pi_1(N_n, [P_0]) = B_n$.

**Proof.** For a braid $\beta = (b_1, \ldots, b_n)$ we set

$$\hat{\varphi}(\beta) : [0, 1] \rightarrow N_n, \quad t \mapsto [b_1(t), \ldots, b_n(t)].$$

Clearly, $\hat{\varphi}(\beta)$ is a loop based at $[P_0]$. It is easily checked that $\hat{\varphi}$ induces a homomorphism $\hat{\varphi}_* : B_n \rightarrow \pi_1(N_n, [P_0])$, and that the following diagram commutes

$$1 \rightarrow PB_n \rightarrow B_n \rightarrow \text{Sym}_n \rightarrow 1$$

$$\varphi \downarrow \approx \quad \hat{\varphi} \downarrow \hat{\varphi}_* \downarrow \text{Id}$$

$$1 \rightarrow \pi_1(M_n, P_0) \rightarrow \pi_1(N_n, [P_0]) \rightarrow \text{Sym}_n \rightarrow 1$$
The first row is exact by definition, and the second one is associated to the regular covering $M_n \to N_n = M_n/\text{Sym}_n$, so it is exact, too. We conclude by the five lemma that $\hat{\varphi}_*$ is an isomorphism.

Let $f, g \in \mathbb{C}[x]$ be two non-constant polynomials. Set
\[
    f = a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \quad a_0 \neq 0,
\]
\[
    g = b_0 x^n + b_1 x^{n-1} + \cdots + b_n, \quad b_0 \neq 0.
\]
The Sylvester matrix of $f$ and $g$ is defined to be
\[
    \text{Sylv}(f,g) = \begin{pmatrix}
        a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\
        a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\
        \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\
        a_m & \ddots & \vdots & a_0 & b_n & \ddots & b_0 \\
        0 & a_m & a_1 & 0 & b_n & b_0 & \ddots & \vdots \\
        \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
        0 & \cdots & 0 & a_m & 0 & \cdots & 0 & b_n
    \end{pmatrix}
\]

The resultant of $f$ and $g$ is defined to be
\[
    \text{Res}(f,g) = \det(\text{Sylv}(f,g)).
\]
The following is classical in algebraic geometry (see [54], for example).

**Theorem 2.6.** Let $f, g \in \mathbb{C}[x]$ be two non-constant polynomials. Then $f$ and $g$ have a common root if and only if $\text{Res}(f,g) = 0$.

**Corollary 2.7.** Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 2$. Then $f$ has a multiple root if and only if $\text{Res}(f,f') = 0$.

The number $\text{Res}(f,f')$ is called the discriminant of $f$ and is denoted by $\text{Disc}(f)$. For instance, if $f = ax^2 + bx + c$, then $\text{Disc}(f) = b^2 - 4ac$.

Let $n \geq 2$ and let $\mathbb{C}_n[x]$ be the set of monic polynomials of degree $n$. In particular, $\mathbb{C}_n[x]$ is isomorphic to $\mathbb{C}^n$. The map $\text{Disc} : \mathbb{C}_n[x] \to \mathbb{C}$ is clearly a polynomial function, thus
\[
    \mathcal{D} = \{ f \in \mathbb{C}_n[x] ; f \text{ has a multiple root} \} = \{ f \in \mathbb{C}_n[x] ; \text{Disc}(f) = 0 \}
\]
is an algebraic hypersurface called the $n$-th discriminant. It is related to the braid group by the following.

**Proposition 2.8.** $N_n = \mathbb{C}_n[x] \setminus \mathcal{D}$. 
Proof. Let $\Phi : M_n \to \mathbb{C}_n[x] \setminus D$ be the map defined by

$$\Phi(z_1, \ldots, z_n) = (x - z_1) \cdots (x - z_n).$$

Then $\Phi$ is surjective and we have $\Phi(u) = \Phi(v)$ if and only if there exists $\chi \in \text{Sym}_n$ such that $v = \chi(u)$. Thus $\mathbb{C}_n[x] \setminus D \simeq M_n/\text{Sym}_n = N_n$.

Now, recall the homotopy long exact sequence of a fiber bundle (see [96], for example).

**Theorem 2.9.** Let $p : M \to B$ be a locally trivial fiber bundle. Let $b_0 \in B$, let $F = p^{-1}(b_0)$, and let $P_0 \in F$. Assume that $F$ is connected. Then there is a long exact sequence of homotopy groups

$$\cdots \to \pi_{k+1}(B, b_0) \to \pi_k(F, P_0) \to \pi_k(M, P_0) \to \pi_k(B, b_0) \to \cdots$$

$$\cdots \to \pi_2(B, b_0) \to \pi_1(F, P_0) \to \pi_1(M, P_0) \to \pi_1(B, b_0) \to 1.$$

There are two cases where this long exact sequence becomes a short exact sequence: when $\pi_2(B, b_0) = \{0\}$, and when $p$ admits a cross-section $\kappa : B \to M$. In the latter case the short exact sequence splits. It turns out that both situations hold in the study of $M_n$.

**Theorem 2.10** (Fadell, Neuwirth [80]). Let $p : M_{n+1} \to M_n$ be defined by

$$p(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_n).$$

Then $p$ is a locally trivial fiber bundle which admits a cross-section $\kappa : M_n \to M_{n+1}$.

Let $b_0 = (1, 2, \ldots, n)$. Then the fiber $p^{-1}(b_0)$ is naturally homeomorphic to $\mathbb{C} \setminus \{1, 2, \ldots, n\}$ whose fundamental group is the free group $F_n$ of rank $n$. A cross-section of $p$ is the map $\kappa : M_n \to M_{n+1}$ defined by

$$\kappa(z_1, \ldots, z_n) = (z_1, \ldots, z_n, |z_1| + \cdots + |z_n| + 1).$$

**Corollary 2.11.** Let $n \geq 2$. Then there is a split exact sequence

$$1 \longrightarrow F_n \longrightarrow \mathcal{PB}_{n+1} \overset{p_*}{\longrightarrow} \mathcal{PB}_n \longrightarrow 1.$$

A connected CW-complex $X$ is called $K(\pi, 1)$ if its universal cover is contractible. Equivalently, $X$ is $K(\pi, 1)$ if $\pi_k(X) = \{0\}$ for all $k \geq 2$. In particular, a space $X$ is $K(\pi, 1)$ if an only if some of its connected cover $Y$ is $K(\pi, 1)$. The notion of $K(\pi, 1)$ spaces is of importance in the calculation of the (co)homology of groups. We refer to [34] for detailed explanations on the subject.

It is easily seen that $\mathbb{C} \setminus \{1, \ldots, n\}$ is $K(\pi, 1)$, thus, from Theorems 2.9 and 2.10 follows:
Corollary 2.12. The spaces $M_n$ and $N_n$ are $K( \pi, 1 )$.

It is also known that the fundamental group of a finite dimensional $K( \pi, 1 )$ space is torsion free (see [34]), thus:

Corollary 2.13. $E_n = \pi_1( N_n )$ is torsion free.

2.3 Mapping class groups

Let $\Sigma$ be an oriented compact surface, possibly with boundary, and let $\mathcal{P} = \{ P_1, \ldots, P_n \}$ be a collection of $n$ punctures in the interior of $\Sigma$. Let $\text{Homeo}^+( \Sigma, \mathcal{P} )$ denote the group of homeomorphisms $h : \Sigma \rightarrow \Sigma$ which preserve the orientation, which pointwise fix the boundary of $\Sigma$, and such that $h(\mathcal{P}) = \mathcal{P}$. Let $\text{Homeo}_{0}^{+}( \Sigma, \mathcal{P} )$ denote the connected component of the identity in $\text{Homeo}( \Sigma, \mathcal{P} )$. The mapping class group of the pair $(\Sigma, \mathcal{P})$ is defined to be

$$\mathcal{M}(\Sigma, \mathcal{P}) = \pi_0(\text{Homeo}^+( \Sigma, \mathcal{P} )) = \text{Homeo}^+( \Sigma, \mathcal{P} )/\text{Homeo}_{0}^{+}( \Sigma, \mathcal{P} ).$$

A braid of $\Sigma$ based at $\mathcal{P}$ is defined to be an $n$-tuple $\beta = (b_1, \ldots, b_n)$ of paths, $b_k : [0, 1] \rightarrow \Sigma$, such that

- $b_k(0) = P_k$ for all $1 \leq k \leq n$;
- there exists a permutation $\chi = \theta(\beta) \in \text{Sym}_n$ such that $b_k(1) = P_{\chi(k)}$ for all $1 \leq k \leq n$;
- $b_k(t) \neq b_l(t)$ for all $k \neq l$ and all $t \in [0, 1]$.

The homotopy classes of braids based at $\mathcal{P}$ form a group denote by $B_n(\Sigma, \mathcal{P})$ and called the braid group of $\Sigma$ on $n$ strands based at $\mathcal{P}$. It does not depend up to isomorphism on the choice of $\mathcal{P}$ but only on the cardinality $n = |\mathcal{P}|$. So, we may often write $B_n(\Sigma)$ in place of $B_n(\Sigma, \mathcal{P})$. If $\Sigma = D$ is a disk, then $B_n(\Sigma)$ is naturally isomorphic to the braid group $B_n$.

For $1 \leq k < l \leq n$, we denote by $H_{k,l}(\Sigma)$ the set of $n$-tuples $x = (x_1, \ldots, x_n) \in \Sigma^n$ such that $x_k = x_l$. The big diagonal of $\Sigma^n$ is defined to be

$$\text{Diag}_n(\Sigma) = \bigcup_{1 \leq k < l \leq n} H_{k,l}(\Sigma).$$

The space of ordered configurations of $n$ points in $\Sigma$ is defined to be

$$M_n(\Sigma) = \Sigma^n \setminus \text{Diag}_n(\Sigma).$$

This is the space of $n$-tuples $x = (x_1, \ldots, x_n)$ in $\Sigma^n$ such that $x_k \neq x_l$ for all $1 \leq k \neq l \leq n$. The symmetric group $\text{Sym}_n$ acts freely on $M_n(\Sigma)$, and the quotient

$$N_n(\Sigma) = M_n(\Sigma)/\text{Sym}_n.$$
is called the *space of configurations of* \( n \) *points in* \( \Sigma \). This is the space of unordered \( n \)-tuples \( \mathbf{x} = \{x_1, \ldots, x_n\} \) of elements of \( \Sigma \) such that \( x_k \neq x_l \) for all \( 1 \leq k \neq l \leq n \).

Set \( \mathbf{P}_0 = (P_1, \ldots, P_n) \in M_n(\Sigma) \). For \( \mathbf{x} \in M_n(\Sigma) \), we denote by \( [\mathbf{x}] \) the element of \( N_n(\Sigma) \) represented by \( \mathbf{x} \). The following can be proved in the same way as Proposition 2.5.

**Proposition 2.14.** \( \pi_1(N_n(\Sigma), [\mathbf{P}_0]) \simeq B_n(\Sigma) \).

Now, the surface braid groups and the mapping class groups are related by the following exact sequence.

**Theorem 2.15** (Birman [18]). *Suppose* \( \Sigma \) *is neither a sphere, nor a torus.*

Then we have the exact sequence

\[
1 \to B_n(\Sigma, \mathcal{P}) \to \mathcal{M}(\Sigma, \mathcal{P}) \to \mathcal{M}(\Sigma) \to 1.
\]

**Note.** Let

\[
\Phi : \text{Homeo}^+(\Sigma) \to N_n(\Sigma)
\]

\[
\varphi \mapsto \{\varphi(P_1), \ldots, \varphi(P_n)\}.
\]

Then \( \Phi \) is a locally trivial fiber bundle, and the fiber of \( \Phi \) over \( \mathcal{P} = [\mathbf{P}_0] \) is \( \text{Homeo}^+(\Sigma, \mathcal{P}) \). Furthermore, it is known that \( \pi_1(\text{Homeo}^+(\Sigma)) = \{1\} \) (see [92]), thus, by the homotopy long exact sequence of a fiber bundle (see [96]), we have the short exact sequence

\[
1 \to \pi_1(N_n(\Sigma, \mathcal{P})) \to \pi_0(\text{Homeo}^+(\Sigma, \mathcal{P})) \to \pi_0(\text{Homeo}^+(\Sigma)) \to 1,
\]

which is the same as the exact sequence of Theorem 2.15.

It is known that \( \mathcal{M}(\mathbb{D}) = \{1\} \) (see [1]), thus, by Theorem 2.15:

**Theorem 2.16** (Artin [7], [8]). *Let* \( \mathcal{P} = \{P_1, \ldots, P_n\} \) *be a collection of* \( n \) *punctures in the interior of the disk* \( \mathbb{D} \). *Then* \( \mathcal{M}(\mathbb{D}, \mathcal{P}) \simeq B_n. \)

The isomorphism \( \Phi : \mathcal{M}(\mathbb{D}, \mathcal{P}) \to B_n \) can be easily described as follows. Let \( \varphi \in \text{Homeo}^+(\mathbb{D}, \mathcal{P}) \). We know by [1] that \( \pi_0(\text{Homeo}^+(\mathbb{D})) = \{1\} \), thus there exists a continuous path \( \{\varphi_t\}_{t \in [0,1]} \) in \( \text{Homeo}^+(\mathbb{D}) \) such that \( \varphi_0 = \text{Id} \) and \( \varphi_1 = \varphi \). Let \( \beta = (b_1, \ldots, b_n) \) be the braid defined by

\[
b_k(t) = \varphi_t(P_k), \quad 1 \leq k \leq n \text{ and } t \in [0,1].
\]

Then \( \Phi(\varphi) \) is the homotopy class of \( \beta \).

The reverse isomorphism \( \Phi^{-1} : B_n \to \mathcal{M}(\mathbb{D}, \mathcal{P}) \) is more complicated to describe, but the images of the standard generators can be easily defined in terms of braid twists as follows.
We come back to the situation where $\Sigma$ is an oriented compact surface and $P = \{P_1, \ldots, P_n\}$ is a collection of $n$ punctures in the interior of $\Sigma$. Let $P_k, P_l \in P$, $k \neq l$. An essential arc joining $P_k$ to $P_l$ is defined to be an embedding $a : [0, 1] \to \Sigma$ such that $a(0) = P_k$, $a(1) = P_l$, $a((0, 1)) \cap P = \emptyset$, and $a([0, 1]) \cap \partial \Sigma = \emptyset$. Two essential arcs $a$ and $a'$ are said to be isotopic if there is a continuous family $\{a_t\}_{t \in [0, 1]}$ of essential arcs such that $a_0 = a$ and $a_1 = a'$. Isotopy of essential arcs is an equivalence relation that we denote by $a \sim a'$.

Let $a$ be an essential arc joining $P_k$ to $P_l$. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ be the standard disk, and let $A : D \to \Sigma$ be an embedding such that

- $a(t) = A(t - \frac{1}{2})$ for all $t \in [0, 1]$;
- $A(D) \cap P = \{P_k, P_l\}$.

Let $T \in \text{Homeo}^+(\Sigma, P)$ be defined by

$$(T \circ A)(z) = A(e^{2\pi iz}) , \quad z \in D,$$

and $T$ is the identity outside the image of $A$ (see Figure 2.7). The braid twist along $a$ is defined to be the element $\tau_a \in \mathcal{M}(\Sigma, P)$ represented by $T$, that is, the isotopy class of $T$. Note that:

- the definition of $\tau_a$ does not depend on the choice of $A : D \to \Sigma$;
- if $a$ is isotopic to $a'$, then $\tau_a = \tau_{a'}$.

![Figure 2.7. Braid twist.](image)

Now, we view the disk $D$ as the disk in $\mathbb{C}$ of radius $\frac{n+1}{2}$ centered at $\frac{n+1}{2}$, and we set $P_k = k$ for $1 \leq k \leq n$. Let $a_k : [0, 1] \to D$ be the arc defined by

$$a_k(t) = k + t , \quad t \in [0, 1].$$

(See Figure 2.8.) Then:

**Lemma 2.17.** The reverse isomorphism $\Phi^{-1} : B_n \to \mathcal{M}(\mathbb{D}, P)$ is defined by

$$\Phi^{-1}(\sigma_k) = \tau_{a_k} , \quad 1 \leq k \leq n - 1.$$
2.4 Automorphisms of free groups

For a group $G$, we denote by $\text{Aut}(G)$ the group of automorphisms of $G$, by $\text{Inn}(G)$ the group of inner automorphisms of $G$, and by $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ the group of outer automorphisms of $G$.

Let $F_n = F(x_1, \ldots, x_n)$ be the free group of rank $n$. For $1 \leq k \leq n - 1$, let $\tau_k : F_n \to F_n$ be the automorphism defined by

$$
\tau_k : \begin{cases} 
  x_k &\mapsto x_k^{-1}x_{k+1}x_k \\
  x_{k+1} &\mapsto x_k \\
  x_l &\mapsto x_l &\text{if } l \neq k, k+1
\end{cases}
$$

One can easily show the following.

**Proposition 2.18.** The mapping $\sigma_k \mapsto \tau_k$, $1 \leq k \leq n - 1$, determines a representation $\rho : B_n \to \text{Aut}(F_n)$.

The above representation $\rho : B_n \to \text{Aut}(F_n)$ is called the Artin representation. It is faithful, more precisely:

**Theorem 2.19** (Artin [7], [8]). (1) The Artin representation $\rho : B_n \to \text{Aut}(F_n)$ is faithful.

(2) An automorphism $\alpha \in \text{Aut}(F_n)$ belongs to $\text{Im} \rho$ if and only if $\alpha(x_n \cdots x_2x_1) = x_n \cdots x_2x_1$ and there exists a permutation $\chi \in \text{Sym}_n$ such that $\alpha(x_k)$ is conjugate to $x_{\chi(k)}$ for all $1 \leq k \leq n$.

In particular, $B_n$ can be viewed as a subgroup of $\text{Aut}(F_n)$. This has some consequences on $B_n$ itself such as the two properties defined below.

A group $G$ is called residually finite if for all $g \in G \setminus \{1\}$ there exists a homomorphism $\varphi : G \to H$ such that $H$ is finite and $\varphi(g) \neq 1$. A group $G$ is called Hopfian if every epimorphism $\varphi : G \to G$ is an isomorphism. It is known that the subgroups of $\text{Aut}(F_n)$ are both, residually finite and Hopfian (see [119]), thus, by Theorem 2.19:
Corollary 2.20. The braid group $B_n$ is residually finite and Hopfian.

There are several ways to describe geometrically the Artin representation. The first way is using the Fadell-Neuwirth fiber bundle $p : M_{n+1} \to M_n$ of Theorem 2.10. Let $\text{Sym}_n$ act on $M_n$ and on $M_{n+1}$. The second action is on the first $n$ coordinates, that is,

$$\chi(z_1, \ldots, z_n, z_{n+1}) = (z_{\chi^{-1}(1)}, \ldots, z_{\chi^{-1}(n)}, z_{n+1}), \quad \text{for } \chi \in \text{Sym}_n.$$  

The map $p : M_{n+1} \to M_n$ induces a map $\bar{p} : M_{n+1}/\text{Sym}_n \to M_n/\text{Sym}_n = N_n$ which turns out to be a locally trivial fiber bundle. The fiber is again homeomorphic to $\mathbb{C}\{1, 2, \ldots, n\}$, and $\bar{p} : M_{n+1}/\text{Sym}_n \to N_n$ has also a cross-section $\bar{\kappa} : N_n \to M_{n+1}/\text{Sym}_n$. So, from the homotopy long exact sequence of a fiber bundle (see Theorem 2.9) we obtain the following split exact sequence

$$1 \to F_n = \pi_1(\mathbb{C}\{1, \ldots, n\}) \longrightarrow \pi_1(M_{n+1}/\text{Sym}_n) \xrightarrow{\bar{p}_*} \pi_1(N_n) = B_n \to 1. \quad (2.2)$$

The action of $B_n = \pi_1(N_n)$ on $F_n = \pi_1(\mathbb{C}\{1, \ldots, n\})$ derived from the above split exact sequence is exactly the Artin representation.

Another way to represent the Artin representation is using the isomorphism $B_n \simeq M(\mathbb{D}, \{P_1, \ldots, P_n\})$. Fix a basepoint $P_0 \in \partial \mathbb{D}$. Then it is easily shown that $M(\mathbb{D}, \{P_1, \ldots, P_n\})$ acts on $\pi_1(\mathbb{D}\{P_1, \ldots, P_n\}, P_0) = F_n$, and that this action is the Artin representation.

The latter point of view of the Artin representations can be extended to all the mapping class groups. In this setting, it is known as the Dehn-Nielsen-Baer theorem. Here is a version of this theorem.

Theorem 2.21 (Dehn, Nielsen [128], Baer [9], Magnus [118]). Let $\Sigma$ be a closed oriented surface, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a collection of $n$ punctures in $\Sigma$. Then the natural homomorphism $\rho : M(\Sigma, \mathcal{P}) \to \text{Out}(\pi_1(\Sigma \setminus \mathcal{P}))$ is injective. Moreover, if $\mathcal{P} = \emptyset$, then the image of $\rho$ is an index 2 subgroup of $\text{Out}(\pi_1(\Sigma))$.

We refer to [99] for a detailed exposition on the Dehn-Nielsen-Baer theorem which include other versions of it.

Note. There are some variants of the Artin representations introduced in [150] and [59] that lead to invariants of links.
3 Artin groups

3.1 Definitions and examples

Let $S$ be a finite set. A Coxeter matrix over $S$ is a square matrix $M = (m_{st})_{s, t \in S}$ indexed by the elements of $S$ such that

- $m_{ss} = 1$ for all $s \in S$;
- $m_{st} = m_{ts} \in \{2, 3, 4, \ldots, +\infty\}$ for all $s, t \in S$, $s \neq t$.

A Coxeter matrix $M = (m_{st})_{s, t \in S}$ is usually represented by its Coxeter graph, $\Gamma = \Gamma(M)$. This is a labeled graph defined by the following data.

- $S$ is the set of vertices of $\Gamma$.
- Two vertices $s, t \in S$, $s \neq t$, are joined by an edge if $m_{st} \geq 3$. This edge is labeled by $m_{st}$ if $m_{st} \geq 4$.

Let $\Gamma$ be a Coxeter graph. Define the Coxeter system of type $\Gamma$ to be the pair $(W, S)$, where $W = W_\Gamma$ is the group presented by the generating set $S$ and the relations

\[ s^2 = 1 \quad \text{for all } s \in S, \]
\[ (st)^{m_{st}} = 1 \quad \text{for all } s, t \in S, s \neq t, \text{ and } m_{st} \neq +\infty, \]

where $M = (m_{st})_{s, t \in S}$ is the Coxeter matrix of $\Gamma$. The group $W = W_\Gamma$ is called the Coxeter group of type $\Gamma$.

If $a, b$ are two letters and $m \in \mathbb{N}$, then $\operatorname{prod}(a, b : m)$ denotes the word

\[ \operatorname{prod}(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even}, \\ (ab)^{\frac{m-1}{2}}a & \text{if } m \text{ is odd}. \end{cases} \]

Let $\Sigma = \{\sigma_s : s \in S\}$ be an abstract set in one-to-one correspondence with $S$. Define the Artin system of type $\Gamma$ to be the pair $(G, \Sigma)$, where $G = G_\Gamma$ is the group presented by the generating set $\Sigma$ and the relations

\[ \operatorname{prod}(\sigma_s, \sigma_t : m_{st}) = \operatorname{prod}(\sigma_t, \sigma_s : m_{st}) \quad \text{for } s, t \in S, s \neq t, \text{ and } m_{st} \neq +\infty. \]

The group $G$ is called the Artin group of type $\Gamma$.

It is easily checked that the group $W_\Gamma$ is also presented by the generating set $S$ and the relations

\[ s^2 = 1 \quad \text{for all } s \in S, \]
\[ \operatorname{prod}(s, t : m_{st}) = \operatorname{prod}(t, s : m_{st}) \quad \text{for all } s, t \in S, s \neq t \text{ and } m_{st} \neq +\infty. \]

This shows that the mapping $\Sigma \to \Gamma$, $\sigma_s \mapsto s$, induces a canonical epimorphism $\theta : G_\Gamma \to W_\Gamma$.

If $m_{st} = 2$, then

\[ \sigma_s \sigma_t = \operatorname{prod}(\sigma_s, \sigma_t : m_{st}) = \operatorname{prod}(\sigma_t, \sigma_s : m_{st}) = \sigma_t \sigma_s. \]
that is, $\sigma_s$ and $\sigma_t$ commute. So, if $\Gamma_1, \ldots, \Gamma_l$ are the connected components of $\Gamma$, then
\[ G_{\Gamma} = G_{\Gamma_1} \times G_{\Gamma_2} \times \cdots \times G_{\Gamma_l}. \]
Similarly, we have
\[ W_{\Gamma} = W_{\Gamma_1} \times W_{\Gamma_2} \times \cdots \times W_{\Gamma_l}. \]
We say that $G_{\Gamma}$ (or $W_{\Gamma}$) is irreducible if $\Gamma$ is connected. We say that $\Gamma$ (or $G_{\Gamma}$) is of spherical type if $W_{\Gamma}$ is finite.

**Example 1.** Suppose that $\Gamma$ is the graph $A_n$ of Figure 3.1. Then $W_{\Gamma} = Sym_{n+1}$ is the symmetric group of $\{1, \ldots, n, n+1\}$, and the Coxeter generators are the transpositions $s_1 = (1, 2), s_2 = (2, 3), \ldots, s_n = (n, n+1)$. The Artin group $G_{\Gamma}$ is the braid group $B_{n+1}$ on $n+1$ strands, and the Artin generators are the standard generators of $B_{n+1}$ given in Theorem 2.2. The canonical epimorphism coincides with the epimorphism described in Subsection 2.1.

**Example 2.** Suppose that $\Gamma$ is the Coxeter graph $B_n$ of Figure 3.1. Let $C_2 = \{\pm 1\}$ denote the cyclic group of order 2. Set $Cub_n = C_2^n \rtimes Sym_n$, where $Sym_n$ acts on $C_2^n$ by permutation of the coordinates. This is the group of isometries of a regular $n$-cube (see [94], for example). The group $Cub_n$ is the Coxeter group of type $B_n$, and the Coxeter generators are
\[ s_1 = (-1, 1, \ldots, 1) \in C_2^n, \quad s_i = (i-1, i) \in Sym_n \quad \text{for} \quad 2 \leq i \leq n. \]
Recall the Artin representation $\rho : B_n \to \text{Aut}(F_n)$ defined in Subsection 2.4. Set $G = F_n \rtimes B_n$. Recall also the action of $\text{Sym}_n$ on $M_{n+1}$ defined in Subsection 2.4. It follows from the exact sequence (2.2) that $G = \pi_1(M_{n+1}/\text{Sym}_n)$. In particular, $G$ is an index $n+1$ subgroup of $\pi_1(M_{n+1}) = \pi_1(N_{n+1}) = B_{n+1} = G_{A_n}$. Now, $G$ is the Artin group of type $B_n$, and the Artin generators are
\[ \tau_1 = x_1 \in F_n, \quad \tau_i = \sigma_{i-1} \in B_n \text{ for } 2 \leq i \leq n. \]
(See [60]).

**Example 3.** Suppose that $\Gamma$ is the Coxeter graph $D_n$ of Figure 3.1, where $n \geq 4$. Let $\text{sgn} : C_2^n \to C_2$ be the homomorphism defined by
\[ \text{sgn}(\varepsilon_1, \ldots, \varepsilon_n) = \prod_{i=1}^{n} \varepsilon_i, \]
and let $K$ be the kernel of $\text{sgn}$. The subgroup $K$ is invariant under the action of $\text{Sym}_n$, thus one can consider the subgroup $W = K \rtimes \text{Sym}_n$ of $\text{Cub}_n = C_2^n \rtimes \text{Sym}_n$. This is the Coxeter group of type $D_n$, and the Coxeter generators are
\[ s_1 = (-1, -1, 1, \ldots, 1) \cdot (1, 2), \quad s_i = (1, 1, 1, \ldots, 1) \cdot (i - 1, i) \text{ for } 2 \leq i \leq n. \]
(See [94], for example).

Let $F_{n-1} = F(y_1, \ldots, y_{n-1})$ be a free group of rank $n-1$. Let $\rho_{D,1} : F_{n-1} \to F_{n-1}$ be the automorphism defined by
\[ \rho_{D,1} : \begin{cases} y_1 &\mapsto y_1, \\ y_j &\mapsto y_1^{-1}y_j \text{ if } j \geq 2. \end{cases} \]
For $2 \leq i \leq n-1$, let $\rho_{D,i} : F_{n-1} \to F_{n-1}$ be the automorphism defined by
\[ \rho_{D,i} : \begin{cases} y_{i-1} &\mapsto y_i, \\ y_i &\mapsto y_{i-1}^{-1}y_i, \\ y_j &\mapsto y_j \text{ if } j \neq i - 1, i. \end{cases} \]
One can easily show the following.

**Lemma 3.1.** The mapping $\sigma_i \mapsto \rho_{D,i}, 1 \leq i \leq n-1$, determines a representation $\rho_D : B_n \to \text{Aut}(F_{n-1})$.

The following is implicit in [134] and explicit in [60].

**Theorem 3.2** (Perron, Vannier [134]). The representation $\rho_D : B_n \to \text{Aut}(F_{n-1})$ is faithful, and the semidirect product $F_{n-1} \rtimes \rho_D B_n$ is isomorphic to the Artin group $G_{D_n}$ of type $D_n$. 
Note. It was shown by Allcock [2] that the Artin group $G_{D_n}$ of type $D_n$ can be also presented as an index 2 subgroup of the $n$-strand braid group of a plane with a single orbifold point of degree 2.

Example 4. Suppose that $\Gamma$ is the graph $\tilde{A}_n$ of Figure 3.1. Let $\text{Sym}_{n+1}$ act on $\mathbb{Z}^{n+1}$ by permutations of the coordinates. Then $\mathbb{Z}^{n+1} \rtimes \text{Sym}_{n+1}$ is the Coxeter group of type $\Gamma$ (see [28]).

Let $\Phi : G_{B_{n+1}} \to \mathbb{Z}$ be the homomorphism defined by

$$\Phi(\sigma_1) = 1, \quad \Phi(\sigma_i) = 0 \text{ for } 2 \leq i \leq n.$$ 

It was observed by several authors [2], [47], [73], [104], that the kernel of $\Phi$ is isomorphic to the Artin group $G_{\tilde{A}_n}$ of type $\tilde{A}_n$. In particular, $G_{\tilde{A}_n}$ is a subgroup of $B_{n+2}$.

The Artin generators of $G_{\tilde{A}_n}$, viewed as a subgroup of $B_{n+2} = \mathcal{M}(\mathbb{D}, \{P_1, P_2, \ldots, P_{n+2}\})$, can be described in terms of braid twists as follows. We place $P_1, \ldots, P_{n+2}$ in the interior of $\mathbb{D}$ like in Figure 3.2. For $1 \leq i \leq n+1$, let $\tau_i$ denote the braid twist along the arc $a_i$. Then $\tau_1, \ldots, \tau_{n+1}$ are the Artin generators of $G_{\tilde{A}_n}$.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{fig32.png}
  \caption{Standard generators of $G_{\tilde{A}_n}$.}
\end{figure}

Note. For a group $G$ we denote by $Z(G)$ the center of $G$. If $\Gamma = A_n, B_n$, or $\tilde{A}_n$, then $G_\Gamma / Z(G_\Gamma)$ can be viewed as a finite index subgroup of the mapping class group of a punctured sphere. This has been cleverly exploited to study the group $G_\Gamma$ itself, in particular, to compute the group of automorphisms of $G_\Gamma$ (see [44], [10]). Note that the center of $G_{A_n}$ and $G_{B_n}$ is an infinite cyclic group (see [71], [32]), and the center of $G_{\tilde{A}_n}$ is trivial (see [100]).
3.2 Coxeter groups

The Coxeter groups were introduced by Tits [147] in a manuscript which was recently published, and whose results appeared in the seminal Bourbaki’s book [28]. The present subsection is a brief survey on these groups with a special emphasis on the results that are needed to study the Artin groups. Standard references for the subject are [28], [97].

Let $\Gamma$ be a Coxeter graph, let $M = (m_{st})_{s,t \in S}$ be its associated Coxeter matrix, and let $(W, S)$ be the Coxeter system of type $\Gamma$.

Let $\Pi = \{e_s; s \in S\}$ be an abstract set in one-to-one correspondence with $S$, whose elements are called simple roots. We denote by $V$ the real vector space having $\Pi$ as a basis, and by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ the symmetric bilinear form defined by

$$\langle e_s, e_t \rangle = \begin{cases} -\cos(\frac{\pi}{m_{st}}) & \text{if } m_{st} \neq +\infty, \\ -1 & \text{if } m_{st} = +\infty. \end{cases}$$

For $s \in S$ we define the reflection $r_s : V \to V$ by

$$r_s(x) = x - 2\langle x, e_s \rangle e_s, \quad x \in V.$$  

**Theorem 3.3** (Tits [147]). The mapping $s \mapsto r_s$, $s \in S$, determines a faithful linear representation $\rho : W \to GL(V)$.

The above linear representation is called the canonical representation of $(W, S)$. Note that the bilinear form $\langle \cdot, \cdot \rangle$ is invariant under the action of $W$.

The root system $\Phi$ of $(W, S)$ is defined to be the orbit of $\Pi$ under the action of $W$, that is,

$$\Phi = \{w \cdot e_s; w \in W, s \in S\}.$$

Let $f \in \Phi$. Write $f = \sum_{s \in S} \lambda_s e_s$, where $\lambda_s \in \mathbb{R}$ for all $s \in S$. We say that $f$ is a positive root (resp. a negative root) if $\lambda_s \geq 0$ (resp. $\lambda_s \leq 0$) for all $s \in S$.

The set of positive roots (resp. negative roots) is denoted by $\Phi_+$ (resp. by $\Phi_-$). The following is proved in [28] for the finite root systems, but the same proof works in general (see also [97], [72]).

**Proposition 3.4.** We have the disjoint union $\Phi = \Phi_+ \sqcup \Phi_-$. 

Let $A$ be a finite set that we call an alphabet. Let $A^*$ denote the set of finite sequences of elements of $A$ that we call words on $A$. We define an operation on $A^*$ by

$$(a_1, \ldots, a_p) \cdot (b_1, \ldots, b_q) = (a_1, \ldots, a_p, b_1, \ldots, b_q).$$

Clearly, $A^*$ endowed with this operation is a monoid which is called the free monoid on $A$. The unit in $A^*$ is the empty word $\epsilon = ()$. 


Each element \( w \) in the Coxeter group \( W \) can be written in the form \( w = s_1 s_2 \cdots s_l \), where \( s_1, s_2, \ldots, s_l \in S \). If \( l \) is as small as possible, then \( l \) is called the \textit{word length} of \( w \) and is denoted by \( l = lg_S(w) \). If \( w = s_1 s_2 \cdots s_l \), then the word \( \omega = (s_1, s_2, \ldots, s_l) \) is called an \textit{expression} of \( w \). If in addition \( l = lg_S(w) \), then \( \omega \) is called a \textit{reduced expression} of \( w \).

For \( w \in W \) we set \( \Phi_w = \{ f \in \Phi_+ ; w^{-1} f \in \Phi_- \} \).

Then the word length and the root systems are related by the following.

**Proposition 3.5** (Bourbaki [28]). We have \( |\Phi_w| = lg_S(w) \) for all \( w \in W \).

Let \( G \) be a group. A subset \( S \subset G \) is called a \textit{positive generating set} of \( G \) if it generates \( G \) as a monoid. Let \( S \) be a positive generating set of \( G \). for \( \omega \in S^* \), we denote by \( \bar{\omega} \) the element of \( G \) represented by \( \omega \). A solution to the \textit{word problem} for \( G \) is an algorithm which, given \( \omega \in S^* \), decides whether \( \bar{\omega} \) is trivial or not.

We turn now to describe Tits’ solution to the word problem for Coxeter groups.

Let \( \omega, \omega' \in S^* \). We say that \( \omega \) is \textit{transformable to} \( \omega' \) by an \textit{M-operation of type I} if there exist \( \omega_1, \omega_2 \in S^* \) and \( s \in S \) such that
\[
\omega = \omega_1 \cdot (s, s) \cdot \omega_2 \quad \text{and} \quad \omega' = \omega_1 \cdot \omega_2 .
\]
We say that \( \omega \) is \textit{transformable to} \( \omega' \) by an \textit{M-operation of type II} if there exist \( \omega_1, \omega_2 \in S^* \) and \( s, t \in S \) such that \( s \neq t \), \( m_{st} \neq +\infty \),
\[
\omega = \omega_1 \cdot \text{prod}(s, t : m_{st}) \cdot \omega_2 \quad \text{and} \quad \omega' = \omega_1 \cdot \text{prod}(t, s : m_{st}) \cdot \omega_2 .
\]
Note that an \textit{M-operation of type I} shortens the length of the word, but not an \textit{M-operation of type II}. An \textit{M-operation of type II} is reversible, but not an \textit{M-operation of type I}. If \( \omega \) is transformable to \( \omega' \) by an \textit{M-operation}, then \( \bar{\omega} = \bar{\omega}' \).

A word \( \omega \) is called \textit{M-reduced} if its length cannot be reduced by means of \textit{M-operations}.

**Theorem 3.6** (Tits [146]). (1) A word \( \omega \in S^* \) is reduced if and only if it is \textit{M-reduced}.

(2) Let \( \omega, \omega' \in S^* \) be two reduced words. We have \( \bar{\omega} = \bar{\omega}' \) if and only if one can pass from \( \omega \) to \( \omega' \) with a finite sequence of \textit{M-operations of type II}.

Now, we introduce a partial order on the Coxeter group \( W \) whose role is of importance in the study of the associated Artin group and monoid.

For \( u, v \in W \), we set \( u \leq_L v \) if there exists \( w \in W \) such that \( v = uw \) and \( lg_S(v) = lg_S(u) + lg_S(w) \).
Proposition 3.7 (Bourbaki [28]). (1) Let \( u, v \in W \). There exists a unique \( w^o \in W \) such that \( w^o \leq_L u \), \( w^o \leq_L v \), and \( w \leq_L w^o \) whenever \( w \leq_L u \) and \( w \leq_L v \).

(2) Suppose that \( W \) is finite. Let \( u, v \in W \). There exists a unique \( w_o \in W \) such that \( w \leq_L w_o \), \( v \leq_L w_o \), and \( w_o \leq_L w \) whenever \( u \leq_L w \) and \( v \leq_L w \).

The element \( w^o \) of Proposition 3.7 is denoted by \( w^o = u \wedge_L v \), and the element \( w_o \) is denoted by \( w_o = u \vee_L v \) (if it exists). Note that, by the above, \((W, \leq_L)\) is a lattice if \( W \) is finite. In that case, \( W \) has a greatest element which is often denoted by \( w_0 \).

We finish the subsection with the classification of the spherical type Coxeter graphs.

Recall that, if \( \Gamma_1, \ldots, \Gamma_l \) are the connected components of a Coxeter graph \( \Gamma \), then

\[
W_\Gamma = W_{\Gamma_1} \times W_{\Gamma_2} \times \cdots \times W_{\Gamma_l}.
\]

In particular, \( \Gamma \) is of spherical type if and only if all the components \( \Gamma_1, \ldots, \Gamma_l \) are of spherical type. So, we only need to classify the connected Coxeter graphs of spherical type.

Theorem 3.8 (Coxeter [55], [56]). (1) A Coxeter graph \( \Gamma \) is of spherical type if and only if the canonical bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) is positive definite.

(2) The connected spherical type Coxeter graphs are the Coxeter graphs listed in Figure 3.3.

3.3 Artin monoids

Let \( \Gamma \) be a Coxeter graph, let \((W,S)\) be the Coxeter system of type \( \Gamma \), and let \((G, \Sigma)\) be the Artin system of type \( \Gamma \). Define the Artin monoid of type \( \Gamma \) to be the monoid \( G^+ = G^+_\Gamma \) presented as a monoid by the generating set \( \Sigma = \{ \sigma_s \mid s \in S \} \) and the relations

\[
\text{prod}(\sigma_s, \sigma_t : m_{st}) = \text{prod}(\sigma_t, \sigma_s : m_{st}) \quad \text{for all} \quad s, t \in S,
\]

\[
s \neq t \quad \text{and} \quad m_{st} \neq +\infty.
\]

Theorem 3.9 (Paris [132]). The natural homomorphism \( G^+_{\Gamma} \to G_{\Gamma} \) is injective.

Recall the homomorphism \( \theta : G_{\Gamma} \to W_{\Gamma}, \sigma_s \mapsto s \). We denote by \( \theta^+ : G^+_{\Gamma} \to W_{\Gamma} \) the restriction of \( \theta \) to \( G^+_{\Gamma} \). We define a set-section \( \kappa : W_{\Gamma} \to G^+_{\Gamma} \) of \( \theta^+ \) as
follows. Let \( w \in W \), and let \( \omega = (s_1, s_2, \ldots, s_l) \) be a reduced expression of \( w \). Then

\[
\kappa(w) = \sigma_{s_1} \sigma_{s_2} \cdots \sigma_{s_l}.
\]

By Theorem 3.6, the definition of \( \kappa(w) \) does not depend on the choice of the reduced expression of \( w \).

Observe also that the defining relations of \( G^+_\Gamma \) are homogeneous, thus \( G^+_\Gamma \) has a well-defined word length \( \lg : G^+_\Gamma \to \mathbb{N} \), \( \sigma_{s_1} \cdots \sigma_{s_l} \mapsto l \). This word length satisfies the following properties:

- \( \lg(\alpha) = 0 \) if and only if \( \alpha = 1 \);
- \( \lg(\alpha\beta) = \lg(\alpha) + \lg(\beta) \) for all \( \alpha, \beta \in G^+_\Gamma \).

We define partial orders \( \leq_L \) and \( \leq_R \) on \( G^+_\Gamma \) by

- \( \alpha \leq_L \beta \) if there exists \( \gamma \in G^+_\Gamma \) such that \( \alpha\gamma = \beta \);
- \( \alpha \leq_R \beta \) if there exists \( \gamma \in G^+_\Gamma \) such that \( \gamma\alpha = \beta \).

The following is again a direct consequence of Theorem 3.6.
Lemma 3.10. Let \( u, v \in W \). We have \( u \leq_L v \) if and only if \( \kappa(u) \leq_L \kappa(v) \).

The set \( S = \{ \kappa(w); w \in W \} \) is called the set of simple elements of \( G_1^+ \). If \( W \) is finite and \( w_0 \) is the greatest element of \( W \), then \( \kappa(w_0) \) is called the Garside element of \( G_1^+ \) and is denoted by \( \Delta = \kappa(w_0) \).

The following theorems 3.11 and 3.12 are key results in the study of Artin monoids and groups. They are implicit in the work of Brieskorn and Saito [32], and explicit for the spherical type Artin groups in the work of Deligne [71]. Complete and detailed proofs of them can be found in [126].

Theorem 3.11. Let \( \alpha \in G_1^+ \). Set

\[
E(\alpha) = \{ a \in S; a \leq_L \alpha \}.
\]

Then \( E(\alpha) \) has a greatest element. That is, there exists \( a_0 \in E(\alpha) \) such that \( E(\alpha) = \{ a \in S; a \leq_L a_0 \} \).

For \( \alpha \in G_1^+ \) we denote by \( \delta(\alpha) \) the greatest element of \( E(\alpha) \).

Theorem 3.12. Let \( \alpha, \beta \in G_1^+ \). Then \( \delta(\alpha\beta) = \delta(\alpha\delta(\beta)) \).

Theorems 3.11 and 3.12 have the following consequences whose significance will become clear in the next section.

Theorem 3.13. (1) Let \( \alpha, \beta \in G_1^+ \). There exists a unique \( \gamma^o \in G_1^+ \) such that \( \gamma^o \leq_L \alpha \), \( \gamma^o \leq_L \beta \), and \( \gamma \leq_L \gamma^o \) whenever \( \gamma \leq_L \alpha \) and \( \gamma \leq_L \beta \).

(2) Suppose that \( \Gamma \) is of spherical type. Let \( \alpha, \beta \in G_1^+ \). There exists a unique \( \gamma_o \in G_1^+ \) such that \( \alpha \leq_L \gamma_o \), \( \beta \leq_L \gamma_o \), and \( \gamma_o \leq_L \gamma \) whenever \( \alpha \leq_L \gamma \) and \( \beta \leq_L \gamma \).

The element \( \gamma^o \) of Theorem 3.13 is denoted by \( \gamma^o = \alpha \wedge_L \beta \), and the element \( \gamma_o \) is denoted by \( \gamma_o = \alpha \vee_L \beta \) (if it exists). Note that the same result is valid if we replace \( \leq_L \) by \( \leq_R \).

Proof. We prove (1) by induction on \( \lg(\alpha) + \lg(\beta) \). By Proposition 3.7 and by Lemma 3.10, \( \alpha \wedge_L \beta \) exists if \( \alpha, \beta \in S \).

Let \( \alpha, \beta \in G_1^+ \). Set \( a = \delta(\alpha) \wedge_L \delta(\beta) \) ( \( a \) exists by the above observation). If \( a = 1 \), then we must have \( \gamma^o = \alpha \wedge_L \beta = 1 \). Suppose \( a \neq 1 \). Let \( \alpha', \beta' \in G_1^+ \) such that \( \alpha = a\alpha' \) and \( \beta = a\beta' \). The element \( \alpha' \wedge_L \beta' \) exists by induction. Then \( \gamma^o = a \cdot (\alpha' \wedge_L \beta') \) (the proof of this equality is left to the reader).

Now, we assume that \( \Gamma \) is of spherical type and turn to prove (2). Let \( w_0 \) be the greatest element of \( W \), and let \( \Delta = \kappa(w_0) \) be the Garside element of \( G_1^+ \). It is shown in [28] that \( w_0^{-1} = w_0 \) and \( w_0Sw_0 = S \). This implies that \( \Delta \cdot \Sigma \cdot \Delta^{-1} = \Sigma \), and, consequently, there exists a permutation \( \tau : S \rightarrow S \) such that \( \Delta \tau = \tau(\alpha) \Delta \) for all \( \alpha \in G_1^+ \).
Let \( \alpha \in G^+_\Gamma \). Set \( \alpha = a_1a_2 \cdots a_r \), where \( a_i = \delta(a_i a_{i+1} \cdots a_r) \in S \) for all \( 1 \leq i \leq r \). Using the above observation, it is easily shown that \( \alpha \leq L \Delta^r \).

Let \( \alpha, \beta \in G^+_\Gamma \). Set \( E = \\{ \gamma \in G^+_\Gamma ; \alpha \leq L \gamma \text{ and } \beta \leq L \gamma \} \). We have \( E \neq \emptyset \) since, by the above, it contains an element of the form \( \Delta^r \). Let \( \gamma_o \) be the smallest element of \( E \) (this element exists by (1)). Then \( \gamma_o = \alpha \lor L \beta \).

### 3.4 Artin groups

We turn now to present a geometrical interpretation of the Artin groups which extends the interpretation of the braid groups in term of configuration spaces. We focus our presentation on the spherical type Artin groups, but many of the results stated in this subsection can be extended in some sense to the other Artin groups.

Let \( \Gamma \) be a spherical type Coxeter graph, let \((W,S)\) be the Coxeter system of type \( \Gamma \), and let \((G,\Sigma)\) be the Artin system of type \( \Gamma \). Recall the set \( \Pi = \{ e_s ; s \in S \} \) of simple roots, the vector space \( V = \bigoplus_{s \in S} \mathbb{R}e_s \), and the canonical bilinear form \( \langle , \rangle : V \times V \to \mathbb{R} \), which, by Theorem 3.8, is positive definite. We assume that \( W \) is embedded in \( GL(V) \) via the canonical representation.

Let \( R \) be the set of reflections in \( W \). For each \( r \in R \), let \( H_r \) be the hyperplane of \( V \) fixed by \( r \). Then \( W \) acts freely on the complement of \( \bigcup_{r \in R} H_r \) (see [28]). Complexifying the action, we get an action of \( W \) on \( V_C = \mathbb{C} \otimes V \) which is free on the complement of \( \bigcup_{r \in R} \mathbb{C} \otimes H_r \). Set

\[
M_\Gamma = V_C \setminus \left( \bigcup_{r \in R} \mathbb{C} \otimes H_r \right), \quad N_\Gamma = M_\Gamma / W.
\]

By a theorem of Chevalley [49], Shephard, and Todd [143], \( V_C / W \) is isomorphic to \( \mathbb{C}^n \), thus \( N_\Gamma \) is the complement in \( \mathbb{C}^n \) of an algebraic set, \( (\bigcup_{r \in R} \mathbb{C} \otimes H_r) / W \), called the discriminant of type \( \Gamma \).

**Theorem 3.14** (Brieskorn [30]). \( \pi_1(N_\Gamma) \simeq G_\Gamma \).

**Note.** Infinite Coxeter groups also act as reflection groups on \( \mathbb{R}^n \). However, to extend Theorem 3.14 to these groups we should replace \( V \) by the Tits cone \( U \subset V \) (see [28]), and \( V_C \) by \( (U + iV) \subset V_C \). Then \( W \) acts freely on \( (U + iV) \setminus (\bigcup_{r \in R} \mathbb{C} \otimes H_r) = M_\Gamma \), and it was shown by Van der Lek [116] that \( \pi_1(N_\Gamma) \simeq G_\Gamma \), where \( N_\Gamma = M_\Gamma / W \).

An extension of Corollary 2.12 to the spherical type Artin groups is:

**Theorem 3.15** (Deligne [71]). Let \( \Gamma \) be a spherical type Coxeter graph. Then \( N_\Gamma \) and \( M_\Gamma \) are \( K(\pi,1) \).
Note. It is an open problem to know whether $N_\Gamma$ is $K(\pi, 1)$ if $\Gamma$ is not of spherical type. The answer is yes for the so-called FC-type Artin groups and 2-dimensional Artin groups [45], and also for few affine type Artin groups (see [47], [38]).

Note. We may replace $W$ by a finite complex reflection group acting on $\mathbb{C}^n$, and $M_\Gamma$ by $M(W) = \mathbb{C}^n \setminus (\cup_{r \in R} H_r)$, where $R$ is the set of reflections in $W$, and $H_r$ denotes the hyperplane fixed by $r$. Here again, the group $W$ acts freely on $M(W)$ and, by [49] and [143], $N(W) = M(W)/W$ is isomorphic to the complement in $\mathbb{C}^n$ of an algebraic set. It was recently proved by Bessis [15] that $N(W)$ is always $K(\pi, 1)$. A classification of the finite complex reflection groups was obtained by Shephard and Todd [143], and a nice presentation of $\pi_1(N(W))$ is known for all these groups but four exceptional cases (see [33], [16]).

4 Garside groups

4.1 Garside monoids

A monoid $M$ is called atomic if there exists a function $\nu : M \to \mathbb{N}$ such that
- $\nu(\alpha) = 0$ if and only if $\alpha = 1$;
- $\nu(\alpha \beta) \geq \nu(\alpha) + \nu(\beta)$ for all $\alpha, \beta \in M$.

Such a function $\nu$ is called a norm on $M$. An element $\alpha \in M$ is called an atom if it is indecomposable, that is, if $\alpha = \beta \gamma$, then either $\beta = 1$ or $\gamma = 1$.

The following is proved in [70].

Lemma 4.1. Let $M$ be an atomic monoid. A subset $S \subset M$ generates $M$ if and only if it contains all the atoms. In particular, $M$ is finitely generated if and only if it contains finitely many atoms.

Let $M$ be an atomic monoid. We define on $M$ two partial orders $\leq_L$ and $\leq_R$ as follows.
- Set $\alpha \leq_L \beta$ if there exists $\gamma \in M$ such that $\alpha \gamma = \beta$.
- Set $\alpha \leq_R \beta$ if there exists $\gamma \in M$ such that $\gamma \alpha = \beta$.

The orders $\leq_L$ and $\leq_R$ are called the left and right divisibility orders, respectively.

A monoid $M$ is called a Garside monoid if
- $M$ is atomic and finitely generated;
- $M$ is cancelative (that is, if $\alpha \beta \gamma = \alpha \beta' \gamma$, then $\beta = \beta'$, for all $\alpha, \beta, \beta', \gamma \in M$);
• $(M, \leq_L)$ and $(M, \leq_R)$ are lattices;

• there exists an element $\Delta \in M$, called a **Garside element**, such that the sets $L(\Delta) = \{\alpha \in M; \alpha \leq_L \Delta\}$ and $R(\Delta) = \{\alpha \in M; \alpha \leq_R \Delta\}$ are equal and generate $M$.

If $M$ is a Garside monoid, then the lattice operations of $(M, \leq_L)$ (resp. of $(M, \leq_R)$) are denoted by $\lor_L$ and $\land_L$ (resp. by $\lor_R$ and $\land_R$).

Let $M$ be a monoid. The **group of fractions** of $M$ is defined to be the group $G(M)$ presented with the generating set $M$ and the relations $\alpha \cdot \beta = \gamma$ if $\alpha \beta = \gamma$ in $M$. Such a group has the universal property that, if $\varphi : M \to H$ is a homomorphism and $H$ is a group, then there exists a unique homomorphism $\hat{\varphi} : G(M) \to H$ such that $\varphi = \hat{\varphi} \circ \iota$, where $\iota : M \to G(M)$ is the natural homomorphism. Note that the latter homomorphism $\iota : M \to G(M)$ is not injective in general.

A **Garside group** is defined to be the group of fractions of a Garside monoid.

**Note.** Garside monoids and groups were introduced in [70] in a slightly restricted sense, and in [67] in the larger sense which is now generally used. This notion was extended to the notion of quasi-Garside monoids [75], [13], to study some non-spherical Artin groups. Quasi-Garside monoids have the same definition as the Garside monoids except they are not required to be finitely generated. Recently, this notion was extended to the notion of Garside categories [107], [108], [76], [14], which, in some sense, has to be considered as a geometric object more than as an algebraic one. Garside categories are a central concept in Bessis’ solution to the $K(\pi, 1)$ problem for complex reflection arrangements (see [15]).

Motivating examples of Garside groups are the Artin groups of spherical type:

**Theorem 4.2.** Let $\Gamma$ be a spherical type Coxeter graph. Then $G^+_\Gamma$ is a Garside monoid. In particular, $G^+_{\Gamma}$ is a Garside group.

Note that Theorem 4.2 is essentially a restatement of Theorem 3.13.

Other interesting examples of Garside groups include all torus link groups (see [136]) and some generalized braid groups associated to complex reflection groups (see [15]).

**Note.** Two different Garside monoids can have the same group of fractions. In particular, the Artin groups of spherical type are groups of fractions of other Garside monoids, called **dual Artin monoids**, introduced by Birman, Ko, and Lee [27] for the braid groups, and by Bessis [12] for the other ones.

**Note.** A Garside element is not unique. For instance, if $\Delta$ is a Garside element, then $\Delta^k$ is a Garside element for all $k \geq 1$ (see [67]).
We say that a monoid $M$ satisfies the Ore conditions if
- $M$ is cancelative;
- for all $\alpha, \beta \in M$, there exist $\alpha', \beta' \in M$ such that $\alpha\alpha' = \beta\beta'$.

It is well-known that a monoid which satisfies the Ore conditions embeds in its group of fractions. On the other hand, a Garside monoid clearly satisfies the Ore conditions. Thus:

**Proposition 4.3.** Let $M$ be a Garside monoid. Then the natural homomorphism $\iota: M \to G(M)$ is injective.

Let $M$ be a Garside monoid and let $G = G(M)$ be the group of fractions of $M$. Then the partial orders $\leq_L$ and $\leq_R$ can be extended to $G$ as follows.
- Set $\alpha \leq_L \beta$ if $\alpha^{-1}\beta \in M$.
- Set $\alpha \leq_R \beta$ if $\beta\alpha^{-1} \in M$.

One can easily verify that $(G, \leq_L)$ and $(G, \leq_R)$ are lattices. This can be used, for example, to prove the following.

**Proposition 4.4.** A Garside group is torsion free.

**Proof.** Let $\alpha \in G$ such that $\alpha^n = 1$ for some $n \geq 1$. Set $\beta = 1 \vee_L \alpha \vee_L \cdots \vee_L \alpha^{n-1}$. It is easily seen that $\leq_L$ is invariant by left multiplication. This implies that $\alpha\beta = \beta$, hence $\alpha = 1$. \qed

**Note.** Let $G$ be a Garside group. Finite dimensional $K(G, 1)$ (that is, $K(\pi, 1)$ spaces having $G$ as fundamental group) were described in [69] and [46]. This implies that $G$ is torsion free, but also more.

### 4.2 Reversing processes and presentations

Let $\Sigma$ be a finite set. Let $\Sigma^*$ be the free monoid on $\Sigma$. Recall that the elements of $\Sigma^*$ are the finite sequences of elements of $\Sigma$ that are called words on $\Sigma$. If $\equiv$ is a congruence on $\Sigma^*$ and $M = (\Sigma^*/\equiv)$, then we denote by $\Sigma^* \to M$, $\omega \mapsto \bar{\omega}$ the natural epimorphism.

Define a complement on $\Sigma$ to be a map $f: \Sigma \times \Sigma \to \Sigma^*$ such that $f(x, x) = \epsilon$ for all $x \in \Sigma$, where $\epsilon$ denotes the empty word. To a complement $f$ we associate two monoids:

$$M^f_L = \langle \Sigma \mid xf(x, y) = yf(y, x) \text{ for all } x, y \in \Sigma \rangle^+;$$

$$M^f_R = \langle \Sigma \mid f(y, x)x = f(x, y)y \text{ for all } x, y \in \Sigma \rangle^+.$$

For $u, v \in \Sigma^*$, we use the notation $u \equiv^f_L v$ (resp. $u \equiv^f_R v$) to mean that $\bar{u} = \bar{v}$ in $M^f_L$ (resp. in $M^f_R$).
Example. Let $\Gamma$ be a Coxeter graph, and let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of $\Gamma$. Suppose that $m_{st} \neq +\infty$ for all $s, t \in S$, $s \neq t$. Let $\Sigma = \{\sigma_s; s \in S\}$. Let $f : \Sigma \times \Sigma \to \Sigma^*$ be the complement defined by

$$f(\sigma_s, \sigma_t) = \text{prod}(\sigma_s, \sigma_t : m_{st} - 1).$$

Then $G^+_f = M^+_f$.

Suppose given a complement $f : \Sigma \times \Sigma \to \Sigma^*$. Let $\Sigma^{-1} = \{x^{-1}; x \in \Sigma\}$ be the set of inverses of elements of $\Sigma$. Let $\omega, \omega' \in (\Sigma \cup \Sigma^{-1})^\ast$. We say that $\omega$ is $f$-reversible on the left in one step to $\omega'$ if there exist $\omega_1, \omega_2 \in (\Sigma \cup \Sigma^{-1})^\ast$ and $x, y \in \Sigma$ such that

$$\omega = \omega_1 x^{-1} y \omega_2 \quad \text{and} \quad \omega' = \omega_1 \cdot f(x, y) \cdot f(y, x)^{-1} \cdot \omega_2.$$

Note that $y$ can be equal to $x$ in the above definition. In that case we have $\omega = \omega_1 x^{-1} x \omega_2$ and $\omega' = \omega_1 \omega_2$. Note also that $\overline{\omega} = \overline{\omega}'$ in $G(M^+_f)$ if $\omega$ is $f$-reversible on the left in one step to $\omega'$.

Let $p \geq 0$. We say that $\omega$ is $f$-reversible on the left in $p$ steps to $\omega'$ if there exists a sequence $\omega = \omega_1, \omega_1, \ldots, \omega_p = \omega'$ in $(\Sigma \cup \Sigma^{-1})^\ast$ such that $\omega_i - 1$ is $f$-reversible on the left in one step to $\omega_i$ for all $1 \leq i \leq p$. The property that $\omega$ is $f$-reversible on the left to $\omega'$ is denoted by $\omega \vdash^f_L \omega'$.

We define the $f$-reversibility on the right in the same way, replacing subwords of the form $yx^{-1}$ by their corresponding words $f(x, y)^{-1} \cdot f(y, x)$. The property that $\omega$ is $f$-reversible on the right to $\omega'$ is denoted by $\omega \vdash^f_R \omega'$.

A word $\omega \in (\Sigma \cup \Sigma^{-1})^\ast$ is said to be $f$-reduced on the left (resp. $f$-reduced on the right) if it is of the form $\omega = vu^{-1}$ (resp. $\omega = u^{-1}v$) with $u, v \in \Sigma^\ast$.

It is shown in [66] that a reversing process is confluent, namely:

**Proposition 4.5** (Dehornoy [66]). Let $f : \Sigma \times \Sigma \to \Sigma^\ast$ be a complement, and let $\omega \in (\Sigma \cup \Sigma^{-1})^\ast$. Suppose that there exist $p \geq 0$ and a $f$-reduced word $vu^{-1}$ on the left such that $\omega$ is $f$-reversible on the left in $p$ steps to $vu^{-1}$. Then any sequence of left $f$-reversing transformations starting from $\omega$ converges to $vu^{-1}$ in $p$ steps.

Let $u, v \in \Sigma^\ast$. Suppose there exist $u', v' \in \Sigma^\ast$ such that $u^{-1} v \vdash^f_L v'(u')^{-1}$. By Proposition 4.5, the words $u'$ and $v'$ are unique. Moreover, it is easily checked that we also have $v^{-1} u \vdash^f_L u'(v')^{-1}$. In this case we set

$$u' = C^f_L(v, u) \quad \text{and} \quad v' = C^f_L(u, v).$$

Similarly, if there exist $u', v' \in \Sigma^\ast$ such that $vu^{-1} \vdash^f_R (u')^{-1}(v')$, then $uv^{-1} \vdash^f_R (v')^{-1}(u')$, $u'$ and $v'$ are unique, and we set

$$u' = C^f_R(u, v) \quad \text{and} \quad v' = C^f_R(v, u).$$
Lemma 4.6 (Dehornoy [66]). Let $f : \Sigma \times \Sigma \to \Sigma^*$ be a complement. Let $u,v \in \Sigma^*$. Suppose that $C^f_L(u,v)$ and $C^f_L(v,u)$ exist. Then
\[ u \cdot C^f_L(u,v) \equiv^f_L v \cdot C^f_L(v,u). \]

A complement $f : \Sigma \times \Sigma \to \Sigma^*$ is said to be coherent on the left if, for all $x,y,z \in \Sigma$, $C^f_L(f(x,y),f(x,z))$ and $C^f_L(f(y,x),f(y,z))$ exist and are $\equiv^f_L$-equivalent. Similarly, we say that $f$ is coherent on the right if, for all $x,y,z \in \Sigma$, $C^f_R(f(z,x),f(y,x))$ and $C^f_R(f(z,y),f(x,y))$ exist and are $\equiv^f_R$-equivalent.

Theorem 4.7 (Dehornoy, Paris [70], [67]). Let $M$ be a finitely generated monoid, and let $\Sigma$ be a finite generating set of $M$. Then $M$ is a Garside monoid if and only if it satisfies the following three conditions.

1. $M$ is atomic.
2. There exist a complement $f : \Sigma \times \Sigma \to \Sigma^*$ coherent on the left and a complement $g : \Sigma \times \Sigma \to \Sigma^*$ coherent on the right such that $M = M^f_L = M^g_R$.
3. There exists an element $\Delta \in M$ such that the sets $L(\Delta) = \{ \alpha \in M; \alpha \leq_L \Delta \}$ and $R(\Delta) = \{ \alpha \in M; \alpha \leq_R \Delta \}$ are equal and generate $M$.

We refer to [70] and [67] for more “algorithmic” conditions to detect a Garside monoid in terms of complements and presentations, and turn to explain some applications of the reversing processes.

Let $M$ be a Garside monoid, and let $G = G(M)$ be its group of fractions. Let $f : \Sigma \times \Sigma \to \Sigma^*$ and $g : \Sigma \times \Sigma \to \Sigma^*$ be complements such that $M = M^f_L = M^g_R$.

First, the complements $f$ and $g$ lead to algorithms:

Proposition 4.8 (Dehornoy, Paris [70], [67]). (1) The complement $f$ is coherent on the left, and the complement $g$ is coherent on the right.

2. Let $\omega \in (\Sigma \sqcup \Sigma^{-1})^*$. There exist a (unique) $f$-reduced word $vu^{-1}$ on the left, and a (unique) $g$-reduced word $(u')^{-1}(v')$ on the right, such that $\omega \mapsto^f_L vu^{-1}$ and $\omega \mapsto^g_R (u')^{-1}(v')$.

They can be used to solve the word problem:

Proposition 4.9 (Dehornoy, Paris [70], [67]). Let $\omega \in (\Sigma \sqcup \Sigma^{-1})^*$. Let $u,v \in \Sigma$ such that $\omega \mapsto^f_L vu^{-1}$ (see Proposition 4.8). Then $\omega = 1$ in $G = G(M)$ if and only if $u^{-1}v \mapsto^f_L \epsilon$, where $\epsilon$ denotes the empty word.

They can be also used to compute the lattice operations of $(M, \leq_L)$ and $(M, \leq_R)$.
Proposition 4.10 (Dehornoy, Paris [70], [67]). Let $u, v \in \Sigma^*$. Set $u' = C^f_L(u, v)$ and $v' = C^f_L(v, u)$. Then $\bar{u} \vee_L \bar{v}$ is represented by $uu' \equiv^f_L vv'$, and $\bar{u} \wedge_L \bar{v}$ is represented by $C^g_R(u, C^g_R(v', u')) \equiv^f_L C^g_R(v, C^g_R(u', v'))$.

4.3 Normal forms and automatic structures

Let $M$ be a Garside monoid, let $G = G(M)$ be the group of fractions of $M$, and let $\Delta$ be a fixed Garside element of $M$. Define the set of simple elements to be $S = \{a \in M : a \leq_L \Delta\} = \{a \in M : a \leq_R \Delta\}$.

By definition, $S$ is finite and generates $M$.

Let $\alpha \in M$. Then $\alpha$ can be uniquely written in the form

$$\alpha = a_1 a_2 \cdots a_l,$$

where $a_1, a_2, \ldots, a_l \in S$, and

$$a_i = \Delta \wedge_L (a_i a_{i+1} \cdots a_l) \quad \text{for all } 1 \leq i \leq l.$$

Such an expression of $\alpha$ is called the normal form of $\alpha$.

Let $\alpha \in G$. Then $\alpha$ can be written in the form $\alpha = \beta^{-1} \gamma$, where $\beta, \gamma \in M$ (see Proposition 4.8, for instance). Obviously, we can also assume that $\beta \wedge_L \gamma = 1$. In that case $\beta$ and $\gamma$ are unique. Let $\beta = b_1 b_2 \cdots b_p$ be the normal form of $\beta$ and let $\gamma = c_1 c_2 \cdots c_q$ be the normal form of $\gamma$. Then the expression

$$\alpha = b_p^{-1} \cdots b_2^{-1} b_1^{-1} c_1 c_2 \cdots c_q$$

is called the normal form of $\alpha$.

There is another notion of normal forms for the elements of $G$, called $\Delta$-normal forms, that are used, in particular, in several solutions to the conjugacy problem for $G$. They are defined as follows.

It is easily seen that there exists a permutation $\tau : S \to S$ such that $\Delta a \Delta^{-1} = \tau(a)$ for all $a \in S$. Moreover, for all $a \in S$, there exists $a^* \in S$ such that $a^* a = \Delta$ (i.e. $a^{-1} = \Delta^{-1} a^*$). These two observations show that every $\alpha \in G$ can be written in the form $\alpha = \Delta^p \beta$, where $p \in \mathbb{Z}$ and $\beta \in M$. One can choose $p$ to be maximal, and, in that case, $\beta$ is unique. Let $b_1 b_2 \cdots b_r$ be the normal form of $\beta$. Then the expression

$$\alpha = \Delta^p b_1 b_2 \cdots b_r$$

is called the $\Delta$-normal form of $\alpha$.

A finite state automaton is a quintuple $A = (Q, S, T, A, q_0)$, where
• $Q$ is a finite set, called the set of states;
• $S$ is a finite set, called the alphabet;
• $T$ is a map $T : Q \times S \to Q$, called the transition function;
• $A$ is a subset of $Q$, called the set of accepted states;
• $q_0$ is an element of $Q$, called the initial state.

The iterated transition function is the map $T^* : Q \times S^* \to Q$ defined by induction on the length of the second component as follows.

$$T^*(q, \epsilon) = q,$$
$$T^*(q, x_1x_2 \cdots x_l) = T(T^*(q, x_1 \cdots x_{l-1}), x_l).$$

The set

$$L_A = \{ \omega \in S^* : T^*(q_0, \omega) \in A \}$$

is called the language recognized by $A$. A regular language is a language recognized by a finite state automaton.

Let $G$ be a group generated by a finite set $S$. Define the word length of an element $\alpha \in G$, denoted by $\text{lg}_S(\alpha)$, to be the shortest length of a word in $(S \cup S^{-1})^*$ which represents $\alpha$. The distance between two elements $\alpha, \beta \in G$, denoted by $d_S(\alpha, \beta)$, is the length of $\alpha^{-1}\beta$.

Let $L \subset (S \cup S^{-1})^*$ be a language. We say that $L$ represents $G$ if every element of $G$ is represented by an element of $L$. We say, furthermore, that $L$ has the uniqueness property if every element of $G$ is represented by a unique element of $L$. We say that $L$ is symmetric if $L^{-1} = L$, where $L^{-1} = \{ \omega^{-1} ; \omega \in L \}$. We say that $L$ is geodesic if $\text{lg}(\omega) = \text{lg}_S(\bar{\omega})$ for all $\omega \in L$. Let $\omega = x_1^i \cdots x_l^i \in (S \cup S^{-1})^*$. For $t \in \mathbb{N}$ we set

$$\bar{\omega}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{x_1^i \cdots x_l^i}{\bar{\omega}} & \text{if } 1 \leq t \leq l \\ \bar{\omega} & \text{if } t \geq l \end{cases}$$

Let $c$ be a positive integer. We say that $L$ has the $c$-fellow traveler property if

$$d_S(\bar{u}(t), \bar{v}(t)) \leq c \cdot d_S(\bar{u}, \bar{v})$$

for all $u, v \in L$ and all $t \in \mathbb{N}$.

A group $G$ is said to be automatic if there exist a finite generating set $S \subset G$, a regular language $L \subset (S \cup S^{-1})^*$, and a constant $c > 0$, such that $L$ represents $G$ and has the $c$-fellow traveler property. If, in addition, $L^{-1}$ has also the $c$-fellow traveler property, then $G$ is said to be biautomatic. We say that $G$ is fully biautomatic if $L$ is symmetric, and that $G$ is geodesically automatic if $L$ is geodesic.

Biautomatic groups have many attractive properties. For instance, they have soluble word and conjugacy problems, and they have quadratic isoperimetric inequalities. We refer to [79] for a general exposition on the subject.
Theorem 4.11 (Charney [43], Dehornoy, Paris [70]). Let $M$ be a Garside monoid, and let $G = G(M)$ be the group of fractions of $M$. Let $\mathcal{L} \subset (S \sqcup S^{-1})^*$ be the language of normal forms. Then $\mathcal{L}$ is regular, represents $G$, has the uniqueness property, has the 5-fellow traveler property, is symmetric, and is geodesic.

Corollary 4.12. Garside groups are fully geodesically biautomatic.

Note. The language of $\Delta$-normal forms is also regular and satisfies some fellow traveler property, and the language of inverses of $\Delta$-normal forms satisfies the same fellow traveler property. So, $\Delta$-normal forms determine another biautomatic structure on $G$. This was proved by Thurston [79] for the braid groups and by Charney [42] for all the spherical type Artin groups, but the same proof works in general for all Garside groups.

4.4 Conjugacy problem

Let $G$ be a group and let $S$ be a finite generating set of $G$. A solution to the conjugacy problem for $G$ is an algorithm which, for given $u, v \in (S \sqcup S^{-1})^*$, decides whether $\bar{u}$ and $\bar{v}$ are conjugate or not.

The first solution to the conjugacy problem for the braid groups was obtained by Garside [85]. Garside’s algorithm was improved by El-Rifai and Morton [78], and this improvement was extended to the Garside groups by Picantin [135]. Picantin’s algorithm was improved by Franco and González-Meneses [81], then by Gebhardt [86], and now by Gebhardt and González-Meneses [87]. The algorithm that we present here is not the optimal one, but is probably the simplest one. It is based on the algorithm of [87].

Note. In addition to the above mentioned papers, there are several recent papers where the algorithms are analyzed, in particular to obtain the best possible complexity (see [22], [23], [24], [89], [115], [113], [114]). These analysis often lead to new and unexpected results on the braid groups and, more generally, on the Garside groups.

Let $M$ be a Garside monoid, let $G = G(M)$ be its group of fractions, let $\Delta$ be a fixed Garside element, and let $S = \{a \in M; a \leq L \Delta\}$ be the set of simple elements. Recall that, for every $a \in S$, there exists a unique $a^* \in S$ such that $aa^* = \Delta$. Recall also that there exists a permutation $\tau : S \to S$ such that $\Delta a \Delta^{-1} = \tau(a)$ for all $a \in S$.

Let $\alpha \in G$. Let $\alpha = \Delta^p a_1 a_2 \cdots a_r$ be the $\Delta$-normal form of $\alpha$. The number $p$ is called the infimum of $\alpha$ and is denoted by $\inf(\alpha)$, $p + r$ is called the supremum and is denoted by $\sup(\alpha)$, and $r$ is called the canonical length and is denoted by $\|\alpha\|$. The above terminology comes from the fact that $p$ is the
greatest number \( n \) such that \( \Delta^n \leq_L \alpha \), and \( p + r \) is the smallest number \( n \) such that \( \alpha \leq_L \Delta^n \). The (simple) element \( \tau^p(a_1) \) is called the initial factor of \( \alpha \) and is denoted by \( i(\alpha) \), and \( a_r \) is called the terminal factor and is denoted by \( t(\alpha) \). It is easily checked that \( i(\alpha^{-1}) = t(\alpha)^* \). Let
\[
\pi(\alpha) = i(\alpha) \cap_L t(\alpha)^* = i(\alpha) \cap_L i(\alpha^{-1}).
\]
Define the sliding of \( \alpha \) to be
\[
S(\alpha) = \pi(\alpha)^{-1} \cdot \alpha \cdot \pi(\alpha).
\]
Observe that \( \|S(\alpha)\| \leq \|\alpha\| \).

For \( \alpha, \beta \in G \), we use the notation \( \alpha \sim \beta \) to mean that \( \alpha \) is conjugate to \( \beta \).

Let \( \alpha \in G \). Define the sliding circuits of \( \alpha \) to be
\[
SC(\alpha) = \{ \beta \in G \mid \beta \sim \alpha \text{ and } S^m(\beta) = \beta \text{ for some } m \geq 1 \}.
\]
It is shown in [87] that the elements of \( SC(\alpha) \) have minimal canonical length in the conjugacy class of \( \alpha \), but not all the elements of the conjugacy class of minimal canonical length belong to \( SC(\alpha) \).

Clearly, if \( \alpha \sim \beta \), then \( SC(\alpha) = SC(\beta) \), and if \( \alpha \not\sim \beta \), then \( SC(\alpha) \cap SC(\beta) = \emptyset \). So, our solution to the conjugacy problem for \( G \) follows the following stages.

**Input.** Two elements \( \alpha, \beta \in G \).

**Stage 1.** Calculate an element \( \alpha_0 \in SC(\alpha) \) and an element \( \beta_0 \in SC(\beta) \).

**Stage 2.** Calculate the whole set \( SC(\alpha) = SC(\alpha_0) \) from \( \alpha_0 \).

**Output.** YES if \( \beta_0 \in SC(\alpha) \), and NO otherwise.

In order to find an element of \( SC(\alpha) \) we use the following which is easy to prove.

**Lemma 4.13.** Let \( \alpha \in G \). There exists \( m, k \geq 1 \) such that \( S^{m+k}(\alpha) = S^k(\alpha) \).
In particular, \( S^k(\alpha) \in SC(\alpha) \).

The key result for Stage 2 is the following.

**Theorem 4.14** (Gebhardt, González-Meneses [87]). Let \( \alpha, \beta \in G \) and let \( \gamma_1, \gamma_2 \in M \). If \( \beta, \gamma_1^{-1} \beta \gamma_1, \text{ and } \gamma_2^{-1} \beta \gamma_2 \) are elements of \( SC(\alpha) \), then \( (\gamma_1 \wedge_L \gamma_2)^{-1} \beta (\gamma_1 \wedge_L \gamma_2) \) is also an element of \( SC(\alpha) \).

**Corollary 4.15.** Let \( \alpha, \beta, \gamma \in G \) such that \( \beta \) and \( \gamma^{-1} \beta \gamma \) are elements of \( SC(\alpha) \). Let \( \gamma = \Delta^p c_1 c_2 \cdots c_r \) be the \( \Delta \)-normal form of \( \gamma \). Set \( \beta_0 = \Delta^{-p} \beta \Delta^p \), and \( \beta_i = c_i^{-1} \beta_{i-1} c_i \) for \( 1 \leq i \leq r \). Then \( \beta_i \in SC(\alpha) \) for all \( 0 \leq i \leq r \).
Proof. We prove that $\beta_i \in SC(\alpha)$ by induction on $i$. It is easily seen that, if $\beta \in SC(\alpha)$, then $\Delta^{-1}\beta\Delta \in SC(\alpha)$. In particular, we have $\beta_0 = \Delta^{-p}\beta\Delta^p \in SC(\alpha)$.

Let $i > 0$. By induction, $\beta_{i-1} \in SC(\alpha)$. By the above observation, we have $\Delta^{-1}\beta_{i-1}\Delta \in SC(\alpha)$. On the other hand, we have $\gamma^{-1}\beta \gamma = (c_1c_1+1\cdots c_r)^{-1} \beta_{i-1}(c_1c_1+1\cdots c_r) \in SC(\alpha)$. By definition of a normal form, we have $\Delta \wedge (c_1c_1+1\cdots c_r) = c_i$. We conclude by Theorem 4.14 that $\beta_i = c_i^{-1}\beta_{i-1}c_i \in SC(\alpha)$.

From Corollary 4.15 we obtain the following which, together with Lemma 4.13, provides an algorithm to compute $SC(\alpha)$.

Corollary 4.16. Let $\alpha \in G$. Let $\Omega_\alpha$ be the graph defined by the following data.

- The set of vertices of $\Omega_\alpha$ is $SC(\alpha)$.
- Two vertices $\beta, \beta' \in SC(\alpha)$ are joined by an edge if there exists $a \in S$ such that $\beta' = a^{-1}\beta a$.

Then $\Omega_\alpha$ is connected.

5 Cohomology and Salvetti complex

5.1 Cohomology

Let $\Gamma$ be a Coxeter graph, let $(W_\Gamma, S)$ be the Coxeter system of type $\Gamma$, and let $(G_\Gamma, \Sigma)$ be the Artin system of type $\Gamma$. Let $\Gamma_{ab}$ be the graph defined by the following data.

- $S$ is the set of vertices of $\Gamma$;
- two vertices $s, t \in S$ are joined by an edge if $m_{st} \neq +\infty$ and $m_{st}$ is odd.

The following is easy to prove from the presentation of $G_\Gamma$.

Proposition 5.1. Let $d$ be the number of connected components of $\Gamma_{ab}$. Then the abelianization of $G_\Gamma$ is a free abelian group of rank $d$. In particular, $H^1(G_\Gamma, \mathbb{Z}) \cong \mathbb{Z}^d$.

Now, assume that $\Gamma$ is of spherical type, and recall the space $N_\Gamma$ defined in Subsection 3.4. Except Proposition 5.1, all known results on the cohomology of $G_\Gamma$ use the fact that $\pi_1(N_\Gamma) = G_\Gamma$ (see Theorem 3.14), and $N_\Gamma$ is a $K(\pi, 1)$ space (see Theorem 3.15). Recall that these two results imply that $H^*(G_\Gamma, A) = H^*(N_\Gamma, A)$ for any $G_\Gamma$-module $A$.

In [4] Arnol’d established the following properties on the cohomology of the braid groups.
Theorem 5.2 (Arnol’d [4]). Let \( n \geq 2 \).

1. \( H^0(B_n, \mathbb{Z}) = H^1(B_n, \mathbb{Z}) = \mathbb{Z} \), \( H^q(B_n, \mathbb{Z}) \) is finite for all \( q \geq 2 \), and \( H^q(B_n, \mathbb{Z}) = 0 \) for all \( q \geq n \).

2. If \( n \) is even, then \( H^q(B_n, \mathbb{Z}) = H^q(B_{n+1}, \mathbb{Z}) \) for all \( q \geq 0 \).

3. \( H^q(B_n, \mathbb{Z}) = H^q(B_{2q-2}, \mathbb{Z}) \) for all \( q \leq \frac{1}{2}n + 1 \).

The study of the cohomology of the braid groups was continued by Fuchs [84] who calculated the cohomology of \( B_n \) with coefficients in \( \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} \). Let \( B_\infty = \lim B_n \), where the limit is taken relative to the natural embeddings \( B_n \hookrightarrow B_{n+1}, \; n \geq 2 \).

Theorem 5.3 (Fuchs [84]).

1. \( H^*(B_\infty, \mathbb{F}_2) \) is the exterior \( \mathbb{F}_2 \)-algebra generated by \( \{a_{m,k}; m \geq 1 \text{ and } k \geq 0\} \) where \( \deg a_{m,k} = 2^k(2^m - 1) \).

2. The natural embedding \( B_n \hookrightarrow B_\infty \) induces a surjective homomorphism \( H^*(B_\infty, \mathbb{F}_2) \to H^*(B_n, \mathbb{F}_2) \) whose kernel is generated by the monomials

\[
a_{m_1,k_1}a_{m_2,k_2} \cdots a_{m_t,k_t}
\]

such that

\[
2^{m_1 + \cdots + m_t + k_1 + \cdots + k_t} > n.
\]

Later, the cohomology with coefficients in \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \) (where \( p \) is an odd prime number) and the cohomology with coefficients in \( \mathbb{Z} \) were calculate by Cohen [53], Segal [142], and Vainstein [148].

Theorem 5.4 (Cohen [53], Segal [142], Vainstein [148]).

1. \( H^*(B_\infty, \mathbb{F}_p) \) is the tensor product of a polynomial algebra generated by \( \{x_i; i \geq 0\} \), where \( \deg x_i = 2p^{i+1} - 2 \), and an exterior algebra generated by \( \{y_j; j \geq 0\} \), where \( \deg y_j = 2p^j - 1 \).

2. The natural embedding \( B_n \hookrightarrow B_\infty \) induces a surjective homomorphism \( H^*(B_\infty, \mathbb{F}_p) \to H^*(B_n, \mathbb{F}_p) \), whose kernel is generated by the monomials

\[
x_{i_1}x_{i_2} \cdots x_{i_t}, y_{j_1}y_{j_2} \cdots y_{j_t}
\]

such that

\[
2(p^{i_1 + 1} + \cdots + p^{i_t + 1} + p^{j_1} + \cdots + p^{j_t}) > n.
\]

Let \( \beta_2 : H^*(B_n, \mathbb{F}_2) \to H^*(B_n, \mathbb{F}_2) \) be the homomorphism defined by

\[
\beta_2(a_{m,k}) = a_{m+1,0}a_{m,1} \cdots a_{m,k-1}.
\]

For an odd prime number \( p \), let \( \beta_p : H^*(B_n, \mathbb{F}_p) \to H^*(B_n, \mathbb{F}_p) \) be the homomorphism defined by

\[
\beta_p(x_i) = y_{i+1}, \quad \beta_p(y_j) = 0.
\]
Theorem 5.5 (Cohen [53], Vainstein [148]). Let \( q \geq 2 \). Then
\[
H^q(B_n, \mathbb{Z}) = \bigoplus_p \beta_p(H^{q-1}(B_n, \mathbb{F}_p)),
\]
where the sum is over all primes \( p \).

The integral cohomology of the Artin groups of type \( B \) and \( D \) were calculated by Goryunov [91] in terms of the cohomology groups of the braid groups.

Theorem 5.6 (Goryunov [91]).

1. Let \( n \geq 2 \), and let \( q \geq 2 \). Then
\[
H^q(G_{B_n}, \mathbb{Z}) = \bigoplus_{i=0}^n H^{q-i}(B_{n-i}, \mathbb{Z}).
\]

2. Let \( n \geq 4 \), and let \( q \geq 2 \). Then
\[
H^q(G_{D_n}, \mathbb{Z}) = H^q(B_n, \mathbb{Z}) \oplus \bigoplus_{i=0}^{+\infty} \ker \gamma_{n-2i}^{q-2i} \oplus \bigoplus_{j=0}^{+\infty} H^{q-2j-3}(B_{n-3j-3}, \mathbb{F}_2),
\]
where, for \( k \geq 2 \) and \( j \geq 0 \), \( \gamma_k^j : H^j(B_k, \mathbb{Z}) \to H^j(B_{k-1}, \mathbb{Z}) \) denotes the homomorphism induced by the inclusion \( B_{k-1} \hookrightarrow B_k \).

Finally, the integral cohomology of the remainder irreducible Artin groups of spherical type were calculated by Salvetti in [141].

Theorem 5.7 (Salvetti [141]). The integral cohomology of the Artin groups of type \( I_2(p) \) (\( p \geq 5 \)), \( E_6 \), \( E_7 \), \( E_8 \), \( F_4 \), \( H_3 \), and \( H_4 \) is given in Table 5.1.

Note. It is a direct consequence of [139] that \( N_\Gamma \) has the same homotopy type as a CW-complex of dimension \( n \), where \( n = |S| \). This implies that the cohomological dimension of \( G_\Gamma \) is \( \leq n \), and, therefore, that \( H^q(G_\Gamma, \mathbb{Z}) = 0 \) for all \( q > n \).

Note. Recall the space \( M_\Gamma \) of Subsection 3.4. The cohomology \( H^*(M_\Gamma, \mathbb{Z}) \) was calculated by Brieskorn in [31]. In particular, \( H^*(M_\Gamma, \mathbb{Z}) \) is torsion free and \( H^n(M_\Gamma, \mathbb{Z}) \neq 0 \). Let \( CG_\Gamma \) be the kernel of the canonical epimorphism \( \theta : G_\Gamma \to W_\Gamma \). By [71] we have \( H^n(M_\Gamma, \mathbb{Z}) = H^n(CG_\Gamma, \mathbb{Z}) \), thus, by the above, \( \text{cd}(G_\Gamma) = \text{cd}(CG_\Gamma) \geq n \), where \( \text{cd}(G_\Gamma) \) denotes the cohomological dimension of \( G_\Gamma \). We already know that \( \text{cd}(G_\Gamma) \leq n \), thus \( \text{cd}(G_\Gamma) = n \).
|     | $H^0$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ |
|-----|-------|-------|-------|-------|-------|
| $I_2(2q)$ | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ | $0$ | $0$ |
| $I_2(2q + 1)$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ | $0$ | $0$ |
| $H_3$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ |
| $H_4$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z} \times \mathbb{Z}_2$ | $\mathbb{Z}$ |
| $F_4$ | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |
| $E_6$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $E_7$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $E_8$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |

Table 5.1.(a). Cohomology of the spherical type Artin groups.

|     | $H^5$ | $H^6$ | $H^7$ | $H^8$ |
|-----|-------|-------|-------|-------|
| $E_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_3$ | $0$ | $0$ |
| $E_7$ | $\mathbb{Z}_6 \times \mathbb{Z}_6$ | $\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_2$ | $\mathbb{Z}$ | $0$ |
| $E_8$ | $\mathbb{Z}_2 \times \mathbb{Z}_6$ | $\mathbb{Z}_3 \times \mathbb{Z}_6$ | $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_2$ | $\mathbb{Z}$ |

Table 5.1.(b). Cohomology of the spherical type Artin groups.

**Note.** The ring structure of $H^*(G_\Gamma, \mathbb{Z})$, where $\Gamma$ is a Coxeter graph in the list of Theorem 5.7, was calculated in [111]. Some cohomologies with twisted coefficients were also considered. An interesting case is the cohomology over the module of Laurent polynomials $\mathbb{Q}[q^{\pm 1}]$ (resp. $\mathbb{Z}[q^{\pm 1}]$), because it determines the rational (resp. integral) cohomology of the Milnor fiber of the discriminant of type $\Gamma$ (see [36]). For the case $\Gamma = A_n$ (i.e. $G_\Gamma$ is the braid group $B_{n+1}$), the $\mathbb{Q}[q^{\pm 1}]$-cohomology was calculated by several people in several ways (see [82], [122], [52], [61]), and the $\mathbb{Z}[q^{\pm 1}]$-cohomology was calculated by Callegaro in [37]. The $\mathbb{Q}[q^{\pm 1}]$-cohomology for the other spherical type Artin groups was calculated in [62]. The $\mathbb{Z}[q^{\pm 1}]$-cohomology for the exceptional cases was calculated in [39], and the top $\mathbb{Z}[q^{\pm 1}]$-cohomology for all cases was calculated in [64].

**Note.** The cohomology of the non-spherical Artin groups is badly understood. Some calculations for the type $A_n$ were done in [38].

We refer to [149] for a more detailed exposition on the cohomology of the braid groups and the Artin groups of spherical type, and turn to present the Salvetti complex (of a real hyperplane arrangement). This is the main tool in Salvetti’s calculations of the cohomology of Artin groups (see [141]), but it can be used for other purposes. For instance, it can be also used to prove Theorems 3.14 and 3.15 (see [140] and [130]), and to produce a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G_\Gamma]$-modules (see Theorem 5.15).
5.2 Salvetti complex

Define a \textit{(real) hyperplane arrangement} to be a finite family $\mathcal{A}$ of linear hyperplanes of $\mathbb{R}^n$. For every $H \in \mathcal{A}$ we denote by $H_C$ the hyperplane of $\mathbb{C}^n$ having the same equation as $H$ (i.e. $H_C = \mathbb{C} \otimes H$), and we set

$$M(\mathcal{A}) = \mathbb{C}^n \setminus \left( \bigcup_{H \in \mathcal{A}} H_C \right).$$

Note that $M(\mathcal{A})$ is an open connected subvariety of $\mathbb{C}^n$.

The arrangement $\mathcal{A}$ subdivides $\mathbb{R}^n$ into \textit{facets}. We denote by $\mathcal{F}(\mathcal{A})$ the set of all facets. The \textit{support} of a facet $F \in \mathcal{F}(\mathcal{A})$ is the linear subspace $\langle F \rangle$ spanned by $F$. We denote by $\bar{F}$ the closure of a facet $F$. We order $\mathcal{F}(\mathcal{A})$ by $F \leq G$ if $F \subseteq \bar{G}$. The set $\mathcal{F}(\mathcal{A})$ has a unique minimal element: $\cap_{H \in \mathcal{A}} H$. The maximal elements of $\mathcal{F}(\mathcal{A})$ are the facets of codimension 0, and they are called \textit{chambers}. The set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$.

Set

$$\mathcal{X} = \{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) : F \leq C \}.$$ 

We partially order $\mathcal{X}$ as follows. For $F \in \mathcal{F}(\mathcal{A})$ we set $\mathcal{A}_F = \{ H \in \mathcal{A} : H \supset F \}$. For $F \in \mathcal{F}(\mathcal{A})$ and $C \in \mathcal{C}(\mathcal{A})$ we denote by $C_F$ the chamber of $\mathcal{A}_F$ which contains $C$. We set

$$(F_1, C_1) \leq (F_2, C_2) \text{ if } F_1 \leq F_2 \text{ and } (C_1)_{F_2} = (C_2)_{F_2}.$$ 

(See Figure 5.1.)

![Figure 5.1. Order in $\mathcal{X}$.](image)

Define the \textit{Salvetti complex} $\text{Sal}(\mathcal{A})$ of $\mathcal{A}$ to be the (geometric realization of the) flag complex of $(\mathcal{X}, \leq)$. That is, to every chain $X_0 < X_1 < \cdots < X_d$ in $\mathcal{X}$ corresponds a simplex $\Delta(X_0, X_1, \ldots, X_d)$ of $\text{Sal}(\mathcal{A})$, and every simplex of $\text{Sal}(\mathcal{A})$ is of this form.

\textbf{Theorem 5.8} (Salvetti [139]). \textit{The simplicial complex} $\text{Sal}(\mathcal{A})$ \textit{is homotopy equivalent to} $M(\mathcal{A})$. 

We turn now to describe a cellular decomposition of $\text{Sal}(\mathcal{A})$ which is the version which is usually used in the literature.

Without loss of generality, we can and do assume that $\mathcal{A}$ is essential, that is, $\cap_{H \in \mathcal{A}} H = \{0\}$. Consider the unit sphere $S^{n-1} = \{ x \in \mathbb{R}^n ; \| x \| = 1 \}$. The arrangement $\mathcal{A}$ determines a cellular decomposition of $S^{n-1}$: to each facet $F \in \mathcal{F}(\mathcal{A}) \setminus \{0\}$ corresponds the open cell $F \cap S^{n-1}$, and each cell is of this form. This cellular decomposition is regular in the sense that the closure of a cell is a closed disk. Hence, one can consider the barycentric subdivision. For each facet $F \in \mathcal{F}(\mathcal{A}) \setminus \{0\}$ we fix a point $x(F) \in F \cap S^{n-1}$. To each chain $\{0\} \neq F_0 < F_1 < \cdots < F_d$ in $\mathcal{F}(\mathcal{A}) \setminus \{0\}$ corresponds a simplex $\Delta(F_0, F_1, \ldots, F_d)$ whose vertices are $x(F_0), x(F_1), \ldots, x(F_d)$, and every simplex of $S^{n-1}$ is of this form. So, the simplicial decomposition of $S^{n-1}$ is the flag complex of $(\mathcal{F}(\mathcal{A}) \setminus \{0\}, \leq)$.

We extend the above simplicial decomposition of $S^{n-1}$ to a simplicial decomposition of the $n$-disk $B^n = \{ x \in \mathbb{R}^n ; \| x \| \leq 1 \}$, adding a single vertex $x(0) = 0$. That is, we view $B^n$ as the cone of $S^{n-1}$. Now, to any chain $F_0 < F_1 < \cdots < F_d$ in $\mathcal{F}(\mathcal{A})$ corresponds a simplex $\Delta(F_0, F_1, \ldots, F_d)$ of $B^n$ (here we may have $F_0 = 0$), and every simplex of $B^n$ is of this form. Note that this simplicial decomposition of $B^n$ is the flag complex of $(\mathcal{F}(\mathcal{A}), \leq)$.

Let $F_b \in \mathcal{F}(\mathcal{A})$ be a facet. It can be easily checked that the union of the simplices of the form $\Delta(F_0, F_1, \ldots, F_d)$ with $F_b = F_0 < F_1 < \cdots < F_d$ is a closed disk whose dimension is equal to $\text{codim} F_b$. Its interior is denoted by $U(F_b)$. So, the set $\{ U(F) ; F \in \mathcal{F}(\mathcal{A}) \}$ forms a cellular decomposition of $B^n$ called the dual decomposition.

**Example.** Let $\mathcal{A}$ be a collection of 3 lines in $\mathbb{R}^2$ (see Figure 5.2). The poset $\mathcal{F}(\mathcal{A})$ contains 6 chambers, 6 facets of dimension 1 (half-lines), and 0. The dual decomposition of $B^2 = \mathbb{D}$ has 6 vertices, 6 edges, and one 2-cell.

![Figure 5.2. A dual decomposition.](image-url)
\( C \in \mathcal{C}(A) \) such that \( F \leq C \) and \( (F_b, C_b) \leq (F, C) \). This implies that \( \bar{U}(X_b) \) is homeomorphic to \( \bar{U}(F_b) \) via the map \( (F, C) \mapsto \mathfrak{x}(F) \). Hence, \( \bar{U}(X_b) \) is a closed disk whose dimension is equal to \( \text{codim} F_b \). We denote by \( U(X_b) \) the interior of \( \bar{U}(X_b) \). So, \( \{U(X) ; X \in A \} \) forms a (regular) cell decomposition of \( \text{Sal}(A) \).

**0-skeleton.** For \( C \in \mathcal{C}(A) \), we set \( \omega(C) = U(C, C) = \bar{U}(C, C) \). Then the 0-skeleton of \( \text{Sal}(A) \) is

\[
\text{Sal}_0(A) = \{\omega(C) ; C \in \mathcal{C}(A)\}.
\]

**1-skeleton.** Let \( F \in \mathcal{F}(A) \) be a facet of codimension 1. There are exactly two chambers \( C, D \in \mathcal{C}(A) \) such that \( F \leq C \) and \( F \leq D \). Then there are two edges, \( U(F, C) \) and \( U(F, D) \), joining \( \omega(C) \) and \( \omega(D) \) in the 1-skeleton of \( \text{Sal}(A) \) (see Figure 5.3). We use the convention that \( U(F, C) \) is endowed with an orientation which goes from \( \omega(C) \) to \( \omega(D) \).

**2-skeleton.** Let \( F_b \in \mathcal{F}(A) \) be a facet of codimension 2, and let \( C_b \in \mathcal{C}(A) \) such that \( F_b \leq C_b \). Let \( C_0 = D_0 = C_b, C_1, \ldots, C_l = D_l, \ldots, D_1 \) be the chambers \( C \in \mathcal{C}(A) \) such that \( F_b \leq C \), arranged like in Figure 5.4. Let \( F_1, \ldots, F_i, G_1, \ldots, G_l \) be the facets \( F \in \mathcal{F}(A) \) of codimension 1 such that \( F_b \leq F_i \), arranged like in Figure 5.4. Set \( a_i = U(F_i, C_i-1) \) and \( b_i = U(G_i, D_i-1) \) for \( 1 \leq i \leq l \). Then \( U(F_b, C_b) \) is a 2-disk whose boundary is \((a_1a_2 \cdots a_i)(b_1b_2 \cdots b_i)^{-1}\).

Let \( \Gamma \) be a Coxeter graph of spherical type, let \( (W_\Gamma, S) \) be the Coxeter system of type \( \Gamma \), and let \( (G_\Gamma, \Sigma) \) be the Artin system of type \( \Gamma \). Recall the set \( S = \{s_i ; s \in S \} \) of simple roots, the linear space \( V = \oplus_{s \in S} \mathbb{R}e_s \), and the canonical bilinear form \( \langle , \rangle : V \times V \rightarrow \mathbb{R} \). Recall also from Theorem 3.8 that \( \langle , \rangle \) is positive definite, and that \( W = W_\Gamma \) can be viewed as a finite subgroup of \( O(V) = O(V, \langle , \rangle) \) generated by reflections.

Let \( A_\Gamma \) denote the set of reflecting hyperplanes of \( W \). Then \( M_\Gamma = M(A_\Gamma) \), the group \( W_\Gamma \) acts freely on \( M_\Gamma \), \( N_\Gamma = M_\Gamma / W_\Gamma \), and \( \pi_1(N_\Gamma) = G_\Gamma \) (see Subsection 3.4).

Fix a (base) chamber \( C_b \in \mathcal{C}(A_\Gamma) \). A hyperplane \( H \in A_\Gamma \) is called a wall of \( C_b \) if \( \text{codim}(C_b \cap H) = 1 \). The following is proved in [28].
Proposition 5.9. (1) $C_b$ is a simplicial cone.

(2) Let $H_1, \ldots, H_n$ be the walls of $C_b$, and, for $1 \leq i \leq n$, let $s_i$ be the orthogonal reflection with respect to $H_i$. Then, up to conjugation, $S = \{s_1, \ldots, s_n\}$ is the Coxeter generating set of $W$.

For $T \subset S$ we denote by $W_T$ the subgroup of $W$ generated by $T$, and by $\Gamma_T$ the full subgraph of $\Gamma$ spanned by $T$. It is a well-known fact (see [28], for example) that $(W_T, T)$ is the Coxeter system of type $\Gamma_T$. The Coxeter complex of $(W, S)$ is defined to be the set

$$\text{Cox}_\Gamma = \{wW_T : T \subset S \text{ and } w \in W\}$$

ordered by the reverse inclusion (i.e. $w_1W_{T_1} \leq w_2W_{T_2}$ if $w_1W_{T_1} \supset w_2W_{T_2}$).

We fix a base chamber $C_b$ and we take $S = \{s_1, \ldots, s_n\}$ like in Proposition 5.9. For each $s \in S$ we denote by $H_s$ the hyperplane fixed by $s$. So, $\{H_s : s \in S\}$ is the set of walls of $C_b$. Since $C_b$ is a simplicial cone, for every $T \subset S$ there exists a unique facet $F(T) \in \mathcal{F}(A_\Gamma)$ such that $F(T) \leq C_b$ and $\langle F(T) \rangle = \bigcap_{s \in T} H_s$. The proof of the following can be found in [28].

Proposition 5.10. The map

$$\psi : \text{Cox}_\Gamma \to \mathcal{F}(A_\Gamma), \quad wW_T \mapsto wF(T)$$

is well-defined and is an isomorphism of ordered sets.

Now, the following lemmas 5.11 and 5.12 are used to describe the poset $\mathcal{X}'$ in terms of Coxeter complexes.

Lemma 5.11 (Bourbaki [28]). Let $T \subset S$ and $w \in W$. Then $wW_T$ has a smallest element $u$ for the order $\leq_L$ (defined in Subsection 3.2). That is, for all $w' \in wW_T$ there exists a unique $v' \in W_T$ such that $w' = uv'$ and $l_S(w') = l_S(u) + l_S(v')$. 

Figure 5.4. 2-skeleton of $\text{Sal}(A)$. 


The smallest element of $wW_T$ is denoted by $u = \min_T(w)$, and such an element is called $T$-minimal. The set of $T$-minimal elements is denoted by $\Min(T)$. For $w \in W$, we denote by $\pi_T(w)$ the element $v \in W_T$ such that $w = \min_T(w) \cdot v$.

The proof of the following is left to the reader.

**Lemma 5.12.** Let $C_b$ be a base chamber, let $T \subset S$, and let $F = F(T)$. Let $w_1, w_2 \in W$. We have $(w_1C_b)_F = (w_2C_b)_F$ if and only if $\pi_T(w_1) = \pi_T(w_2)$.

Set

$$\hat{\Cox}_T = \{(T, w) ; w \in W \text{ and } T \subset S\}.$$ 

Let $\leq$ be the partial order on $\hat{\Cox}_T$ defined by

$$(T_1, w_1) \leq (T_2, w_2) \text{ if } T_1 \supset T_2, \min_{T_1}(w_1) = \min_{T_2}(w_2), \text{ and } \pi_{T_2}(w_1) = \pi_{T_2}(w_2).$$

Note that the conditions “$T_1 \supset T_2$ and $\min_{T_1}(w_1) = \min_{T_2}(w_2)$” are equivalent to the condition $w_1W_{T_1} \supset w_2W_{T_2}$, and, by Lemma 5.12, the condition $\pi_{T_2}(w_1) = \pi_{T_2}(w_2)$ is equivalent to the condition $(w_1C_b)_{F(T_2)} = (w_2C_b)_{F(T_2)}$.

So:

**Theorem 5.13.** The map

$$\hat{\psi} : \hat{\Cox}_T \to \mathcal{X}(A_T)$$

$$\hat{\psi}(T, w) \mapsto (wF(T), wC_b)$$

is well-defined and is an isomorphism of posets.

For $(T, w) \in \hat{\Cox}_T$ we set $U(T, w) = U(\hat{\psi}(T, w))$. So, $\{U(T, w) ; (T, w) \in \hat{\Cox}_T\}$ is a cellular decomposition of $\Sal(A_T)$. Moreover, the dimension of $U(T, w)$ is $|T|$ for all $(T, w) \in \hat{\Cox}_T$.

The Coxeter group $W$ acts on $\hat{\Cox}_T$ by

$$u \cdot (T, w) = (T, uw), \quad \text{for } (T, w) \in \hat{\Cox}_T \text{ and } u \in W.$$ 

It turns out that this action preserves the order of $\hat{\Cox}_T$ and induces a cellular action on $\Sal(A_T)$ defined by

$$u \cdot U(T, w) = U(T, uw) \quad \text{for } (T, w) \in \hat{\Cox}_T \text{ and } u \in W.$$ 

**Theorem 5.14** (Salvetti [141]). There exists an embedding $\Sal(A_T) \hookrightarrow M_T$ and a (strong) retracting deformation of $M_T$ onto $\Sal(A_T)$ that are equivariant under the action of $W$. In particular, there exists an embedding $\Sal(A_T)/W \hookrightarrow M_T/W = N_T$ and a (strong) retracting deformation of $N_T$ onto $\Sal(A_T)/W$. 


To each $T \subset S$ corresponds a unique cell $U_N(T)$ of $\text{Sal}(\mathcal{A}_T)/W$ of dimension $|T|$. This cell is the orbit of $U(T, w)$ for all $w \in W$. Every cell of $\text{Sal}(\mathcal{A}_T)/W$ is of this form.

The 0-skeleton of $\text{Sal}(\mathcal{A}_T)/W$ contains a unique vertex, $\omega_N = U_N(\emptyset)$. For every $s \in S$ there is an edge $U_N(s)$ in $\text{Sal}(\mathcal{A}_T)/W$ and each edge is of this form. For every pair $\{s, t\} \subset S$ there is a 2-cell $U_N(s, t)$ in $\text{Sal}(\mathcal{A}_T)/W$ whose boundary is

$$\text{prod}(U_N(s), U_N(t) : m_{st}) \cdot \text{prod}(U_N(t), U_N(s) : m_{st})^{-1},$$

and every 2-cell is of this form. Note that the 2-skeleton of $\text{Sal}(\mathcal{A}_T)/W$ is equal to the 2-cell complex associated to the standard presentation of $G_T$. This gives an alternative proof to Theorems 2.2 and 3.14.

For $0 \leq q \leq |S|$, set

$$C_q(G_T) = \bigoplus_{T \subset S \atop |T| = q} \mathbb{Z}[G_T] \cdot E_T,$$

the free $\mathbb{Z}[G_T]$-module freely spanned by $\{E_T; T \subset S$ and $|T| = q\}$. We fix a total order $S = \{s_1, \ldots, s_n\}$ on $S$ and we define $d : C_q(G_T) \to C_{q-1}(G_T)$ as follows. Let $T = \{s_{i_1}, \ldots, s_{i_q}\} \subset S$, $i_1 < \cdots < i_q$. Then

$$dE_T = \sum_{j=1}^{q} (-1)^{j-1} \left( \sum_{u \in W_T \atop u \in \text{Min}(T \setminus \{s_{i_j}\})} (-1)^{|u|} \kappa(u) \right) \cdot E_{T \setminus \{s_{i_j}\}},$$

where $\kappa : W \to G_T$ is the set-section of the canonical epimorphism $\theta : G_T \to W$ defined in Subsection 3.3.

**Theorem 5.15** (De Concini, Salvetti [63], Squier [144]). The complex $(C_\ast(G_T), d)$ is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G_T]$-modules.

**Note.** Squier’s proof of Theorem 5.15 does not use the Salvetti complexes at all and is independent from the proof of De Concini and Salvetti.

## 6 Linear representations

The existence (or non-existence) of faithful linear representations of the braid groups was one of the major problems in the field. This problem was solved by Bigelow [17] and Krammer [106] in 2000. They representation, which is known now as the LKB representation, was right afterwards extended to the Artin groups of type $D_n$ ($n \geq 4$) and $E_k$ ($k = 6, 7, 8$) by Digne [74], Cohen, and
Wales [51], and to all Artin groups of small type in [132]. The representations of Digne, Cohen and Wales are proved to be faithful. Hence, since any spherical type Artin group embeds in a direct product of Artin groups of type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), and $E_k$ ($k = 6, 7, 8$) (see [57]), any Artin group of spherical type is linear. The extension to the non-spherical type Artin groups gives rise to a linear representation over an infinite dimensional vector space, so it cannot be used for proving that these groups are linear. However, these representations are useful tools to study the non-spherical type Artin groups. In particular, they are the main tool in the proof of Theorem 3.9.

In Subsection 6.1 we present the algebraic approach to the LK B representations as constructed in [132] for the Artin groups of small type. Subsection 6.2 is dedicated to the topological construction of the LK B representations. Curiously, this topological point of view is known only for the braid groups.

### 6.1 Algebraic approach

Let $\Gamma$ be a Coxeter graph, let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of $\Gamma$, let $(W, S)$ be the Coxeter system of type $\Gamma$, let $(G, \Sigma)$ be the Artin system of type $\Gamma$, and let $G^+_\Gamma$ be the Artin monoid of type $\Gamma$.

We say that $\Gamma$ is of small type if $m_{st} \leq 3$ for all $s, t \in S$, $s \neq t$, and we say that $\Gamma$ is without triangle if there is no triple $\{s, t, r\}$ in $S$ such that $m_{st} = m_{tr} = m_{rs} = 3$. We assume from now on that $\Gamma$ is of small type and without triangle.

Recall from Subsection 3.2 the set $\Pi = \{e_s; s \in S\}$ of simple roots, the space $V = \bigoplus_{s \in S} \mathbb{R} e_s$, the canonical bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, and the root system $\Phi = \{w e_s; s \in S \text{ and } w \in W\}$. Recall also that we have the disjoint union $\Phi = \Phi_+ \sqcup \Phi_-$, where $\Phi_+$ is the set of positive roots and $\Phi_-$ is the set of negative roots (see Proposition 3.4).

Set $E = \{u_f; f \in \Phi_+\}$ an abstract set in one-to-one correspondence with $\Phi_+$, and $K = \mathbb{Q}(x, y)$. Note that $E$ is finite if and only if $\Gamma$ is of spherical type. We denote by $V$ the $K$-vector space having $E$ as a basis.

For all $s \in S$ we define a linear transformation $\varphi_s : V \to V$ by

$$
\varphi_s(u_f) = \begin{cases} 
0 & \text{if } f = e_s \\
u_f & \text{if } \langle e_s, f \rangle = 0 \\
y \cdot u_{f-ae_s} & \text{if } \langle e_s, f \rangle = a > 0 \text{ and } f \neq e_s \\
(1-y) \cdot u_f + u_{f+ae_s} & \text{if } \langle e_s, f \rangle = -a < 0
\end{cases}
$$

The following is easy to prove.

**Lemma 6.1.** The mapping $\sigma_s \mapsto \varphi_s$, $s \in S$, induces a homomorphism of monoids $\varphi : G^+_\Gamma \to \text{End}(V)$. 

For all $s \in S$ and all $f \in \Phi_+$ we choose a polynomial $T(s, f) \in \mathbb{Q}[y]$ and we define $\Phi_s : V \to V$ by

$$\Phi_s(u_f) = \varphi_s(u_f) + x \cdot T(s, f) \cdot u_e.$$ 

Now, we have:

**Theorem 6.2** (Paris [132]). There exists a choice of polynomials $T(s, f)$, $s \in S$ and $f \in \Phi_+$, such that the mapping $\sigma_s \mapsto \Phi_s$, $s \in S$, induces a homomorphism $\Phi : G_\Gamma^+ \to \text{GL}(V)$.

**Theorem 6.3** (Paris [132]). The above defined homomorphism $\Phi : G_\Gamma^+ \to \text{GL}(V)$ is injective.

**Corollary 6.4** (Paris [132]). The natural homomorphism $\iota : G_\Gamma^+ \to G_\Gamma$ is injective.

**Proof.** Since $G_\Gamma$ is the group of fractions of $G_\Gamma^+$, there exists a unique homomorphism $\hat{\Phi} : G_\Gamma \to \text{GL}(V)$ such that $\Phi = \hat{\Phi} \circ \iota$. Since $\Phi$ is injective, we conclude that $\iota$ is also injective. 

**Corollary 6.5** (Bigelow [17], Krammer [106], Digne [74], Cohen, Wales [51]). Suppose that $\Gamma$ is of spherical type. Let $\hat{\Phi} : G_\Gamma \to \text{GL}(V)$ be the homomorphism induced by $\Phi$. Then $\hat{\Phi}$ is injective.

**Proof.** Let $\alpha \in \ker \hat{\Phi}$. By Proposition 4.8, $\alpha$ can be written in the form $\alpha = \beta^{-1} \gamma$, with $\beta, \gamma \in G_\Gamma^+$. We have $1 = \hat{\Phi}(\alpha) = \Phi(\beta)^{-1} \Phi(\gamma)$, thus $\Phi(\beta) = \Phi(\gamma)$. Since $\Phi$ is injective, it follows that $\beta = \gamma$, thus $\alpha = \beta^{-1} \gamma = 1$. 

**Note.** It is shown in [132] that any Artin monoid $G_\Gamma^+$ can be embedded in an Artin monoid $G_\Omega^+$, where $\Omega$ is of small type without triangle. Moreover, if $\Gamma$ is of spherical type, then $\Omega$ can be chosen to be of spherical type (see also [57], [90], [58], [41]). So, Corollary 6.4 implies that $\iota : G_\Gamma^+ \to G_\Gamma$ is injective for all Coxeter graphs $\Gamma$, and Corollary 6.5 implies that all the Artin groups of spherical type are linear.

**Note.** It is shown in [120] that: if $\Gamma$ is of type $A_n$, $D_n$, $E_k$ ($k = 6, 7, 8$), then the image of $\Phi$ is Zariski dense in $\text{GL}(V)$. In particular, this shows that $\Phi$ is irreducible (see also [154], [121], [50]).

**Note.** The proof of Theorem 6.3 given in [132] is largely inspired by Krammer’s proof of the same theorem for the braid groups [106]. A new, short, and elegant proof can be found now in [95].
6.2 Topological approach

Now, we give a topological interpretation of the representation $\hat{\Phi} : G_\Gamma \to \text{GL}(V)$ in the case $\Gamma = A_{n-1}$, that is, when $G_\Gamma = B_n$ is the braid group on $n$ strands. Such an interpretation is unknown for the other Artin groups.

Let $M$ be a connected CW-complex, let $G = \pi_1(M)$, and let $R$ be a (right) $\mathbb{Z}[G]$-module. Let $M$ be the universal cover of $M$. The action of $G$ on $M$ induces an action of $G$ on the group $C_q(M)$ of (cellular) $q$-chains of $M$, and this action makes $C_q(M)$ a module over the group ring $\mathbb{Z}[G]$. It is also easily seen that the boundary maps $\partial : C_q(M) \to C_{q-1}(M)$ are $\mathbb{Z}[G]$-module homomorphisms. We define $C_q(M, R)$ to be $R \otimes_{\mathbb{Z}[G]} C_q(M)$. These groups form a chain complex with boundary map $\text{Id} \otimes \partial$. The homology groups $H_q(M, R)$ of this chain complex are the homology groups of $M$ with local coefficients $R$.

Now, for $n \geq 1$, $M_n$ denotes the space of ordered configurations of $n$ points in $\mathbb{C}$, and $N_n = M_n / \text{Sym}_n$ denotes the space of (unordered) configurations of $n$ points in $\mathbb{C}$ (see Section 2). Let $n, m \geq 2$. By [80], the map

$$p_{n,m} : M_{n+m} \ (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) \mapsto (z_1, \ldots, z_n)$$

is a locally trivial fiber bundle which admits a cross-section. The fiber of $p_{n,m}$ is as follows. Set

$$H_{ij} = \{ w \in \mathbb{C}^m ; w_i = w_j \} \quad \text{for} \ 1 \leq i < j \leq m$$
$$K_{ik} = \{ w \in \mathbb{C}^m ; w_i = k \} \quad \text{for} \ 1 \leq i \leq m \text{ and } 1 \leq k \leq n$$

Set

$$X_{n,m} = \mathbb{C}^m \setminus \left( \bigcup_{i<j} H_{ij} \cup \bigcup_{1 \leq i \leq m} K_{ik} \right).$$

Then

$$p_{n,m}^{-1}(1, 2, \ldots, n) = \{(1, 2, \ldots, n)\} \times X_{n,m}.$$

Let $\text{Sym}_n \times \text{Sym}_m$ act on $M_{n+m}$, $\text{Sym}_n$ acting by permutations on the $n$ first coordinates, and $\text{Sym}_m$ acting on the $m$ last ones. Set

$$N_{n,m} = M_{n+m} / (\text{Sym}_n \times \text{Sym}_m),$$
$$Y_{n,m} = X_{n,m} / \text{Sym}_m.$$

Then $p_{n,m}$ induces a locally trivial fiber bundle $\tilde{p}_{n,m} : N_{n,m} \to N_n$ whose fiber is $Y_{n,m}$.

For $z \in \mathbb{C}^n$ we set

$$\|z\|_\infty = \max\{|z_i| ; 1 \leq i \leq n\}.$$
It is easily checked that the map
\[ \kappa : M_n \to M_{n+m} \]
\[ z \mapsto (z, \|z\|_\infty + 1, \|z\|_\infty + 2, \ldots, \|z\|_\infty + m) \]
is a well-defined cross-section of \( p_{n,m} \) which is equivariant by the action of \( \text{Sym}_n \), thus it induces a cross-section \( \tilde{\kappa} : N_n \to N_{n,m} \) of \( \tilde{p}_{n,m} \). By the homotopy long exact sequence of a fiber bundle (see Theorem 2.9), we conclude that \( \pi_1(N_{n,m}) \) can be written as a semi-direct product \( \pi_1(N_{n,m}) = \pi_1(Y_{n,m}) \rtimes B_n \).

Set \( G_{n,m} = \pi_1(Y_{n,m}) \). We consider \( G_{n,m} \) as a subgroup of \( \pi_1(N_{n,m}) \) which, in its turn, is viewed as a subgroup of \( \pi_1(N_{n+m}) = B_{n+m} \). It is easily seen that \( G_{n,m} \) is generated by the set
\[ \{ \sigma_k : n + 1 \leq k \leq n + m \} \cup \{ \delta_{i,k} : 1 \leq i \leq n \text{ and } n + 1 \leq k \leq n + m \}, \]
where \( \delta_{i,k} \) is the pure braid defined in Theorem 2.3. Let \( b \) be the homology class of \( \sigma_{n+1} \) in \( H_1(G_{n,m}) = H_1(Y_{n,m}) \), and let \( a_i \) be the homology class of \( \delta_{i,n+1} \), \( 1 \leq i \leq n \). The proof of the following is left to the reader.

**Proposition 6.6.** \( H_1(Y_{n,m}) = H_1(G_{n,m}) \) is a free abelian group freely generated by \( \{ b, a_1, a_2, \ldots, a_n \} \).

Let \( \bar{\rho} : H_1(G_{n,m}) \to \mathbb{Q}(x,y)^* \) be the homomorphism which sends \( a_i \) to \( x \) for all \( 1 \leq i \leq n \), and sends \( b \) to \( y \). Let \( \rho : G_{n,m} \to \mathbb{Q}(x,y)^* \) be the composition of the natural projection \( G_{n,m} \to H_1(G_{n,m}) \) with \( \bar{\rho} \). This homomorphism makes \( \mathbb{Q}(x,y) \) a \( \mathbb{Z}[G_{n,m}] \)-module that we denote by \( \Gamma_\rho \).

The proof of the following is also left to the reader.

**Proposition 6.7.** The kernel of \( \rho \) is invariant under the action of \( B_n \), and \( B_n \) acts trivially on the quotient \( G_{n,m}/\ker \rho \cong \mathbb{Z} \times \mathbb{Z} \).

From Proposition 6.7 follows that the fibration \( \tilde{p}_{n,m} : N_{n,m} \to N_n \) induces a monodromy representation \( \Phi_{n,m} : B_n \to \text{Aut}_{\mathbb{Q}(x,y)}(H_2(Y_{n,m}, \Gamma_\rho)) \).

The following was announced by Krammer [105], [106], and proved in [17] (see also [129]).

**Theorem 6.8** (Bigelow [17]). The homomorphism \( \Phi_{n,2} : B_n \to \text{Aut}_{\mathbb{Q}(x,y)}(H_2(Y_{n,2}, \Gamma_\rho)) \) coincides with the representation \( \hat{\Phi} : \text{GL}(V) \) defined in Subsection 6.1.

**Note.** The representation \( \hat{\Phi} : \text{GL}(V) \) also coincides with the representation studied by Lawrence in [112]. Lawrence’s construction is also geometric. It slightly differs from the one presented above, but I do not know exactly how to relate them without the formulas.
Note. It is announced in [152] that $\Phi_{n,m}$ is faithful for all $m \geq 2$, and it is announced in [48] that $\Phi_{n,m} : B_n \to \text{Aut}_{\mathbb{Q}(x,y)}(H_m(Y_{n,m}, \Gamma_\rho))$ is irreducible for all $m \geq 2$.

7 Geometric representations

7.1 Definitions and examples

Let $\Sigma$ be an oriented compact surface, possibly with boundary, and let $\mathcal{P}$ be a finite collection of punctures in the interior of $\Sigma$. Let $\mathcal{M}(\Sigma, \mathcal{P})$ denote the mapping class group of the pair $(\Sigma, \mathcal{P})$, as defined in Subsection 2.3. Let $\Gamma$ be a Coxeter graph, and let $G_\Gamma$ be the Artin group of type $\Gamma$. Define a geometric representation of $G_\Gamma$ in $\mathcal{M}(\Sigma, \mathcal{P})$ to be a homomorphism from $G_\Gamma$ to $\mathcal{M}(\Sigma, \mathcal{P})$.

The main tools for constructing geometric representations of Artin groups are the Dehn twists and the braid twists. The braid twists are defined in Subsection 2.3, and the Dehn twists are defined as follows.

An essential circle is an embedding $a : S^1 \hookrightarrow \Sigma \setminus \mathcal{P}$ of the circle whose image is contained in the interior of $\Sigma$ and does not bound any disk in $\Sigma$ containing 0 or 1 puncture. Two essential circles $a, a'$ are isotopic if there exists a continuous family $\{a_t\}_{t \in [0,1]}$ of essential circles such that $a = a_0$ and $a' = a_1$. Isotopy of essential circles is an equivalence relation that we denote by $a \sim a'$.

Let $a : S^1 \to \Sigma \setminus \mathcal{P}$ be an essential circle. Take an embedding $A : [0,1] \times S^1 \to \Sigma \setminus \mathcal{P}$ of the annulus such that $A(t, z) = a(z)$ for all $z \in S^1$, and define $T \in \text{Homeo}^+(\Sigma, \mathcal{P})$ by

$$(T \circ A)(t, z) = A(t, e^{2\pi i t} z),$$

and $T$ is the identity outside the image of $A$ (see Figure 7.1). The Dehn twist along $a$, denoted by $\sigma_a$, is defined to be the element of $\mathcal{M}(\Sigma, \mathcal{P})$ represented by $T$. Note that

- the definition of $\sigma_a$ does not depend on the choice of the map $A$;
- if $a$ is isotopic to $a'$, then $\sigma_a = \sigma_{a'}$.

Recall that, for an essential arc $a$ of $(\Sigma, \mathcal{P})$, $\tau_a$ denotes the braid twist along $a$. The Dehn twists and the braid twists satisfy the following relations (see [20], [110]).

**Proposition 7.1.** (1) Let $a, b$ be two essential circles that intersect transversely. Then

- $\sigma_a \sigma_b = \sigma_b \sigma_a$ if $a \cap b = \emptyset$
- $\sigma_a \sigma_b \sigma_a = \sigma_b \sigma_a \sigma_b$ if $|a \cap b| = 1$
(2) Let \( a, b \) be two essential arcs of \((\Sigma, P)\). Then
\[
\tau_a \tau_b = \tau_b \tau_a \quad \text{if } a \cap b = \emptyset \\
\tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b \quad \text{if } a(0) = b(1) \text{ and } a \cap b = \{a(0)\}
\]

(3) Let \( a \) be an essential arc, and let \( b \) be an essential circle which intersects \( a \) transversely. Then
\[
\tau_a \sigma_b = \sigma_b \tau_a \quad \text{if } a \cap b = \emptyset \\
\tau_a \sigma_b \tau_a = \sigma_b \tau_a \tau_b \quad \text{if } |a \cap b| = 1
\]

**Example 1.** Suppose \( \Sigma = \mathbb{D} \) is a disk, and \( P_n = \{P_1, \ldots, P_n\} \) is a collection of \( n \) punctures in the interior of \( \Sigma \). Then the Artin isomorphism \( \Phi : B_n \to \mathcal{M}(\mathbb{D}, P_n) \) of Theorem 2.16 is a geometric representation of \( G_{A_{n-1}} = B_n \).

**Example 2.** Let \( n \geq 3 \). Suppose that, if \( n \) is odd, then \( \Sigma \) is a surface of genus \( \frac{n-1}{2} \) with one boundary component, and if \( n \) is even, then \( \Sigma \) is a surface of genus \( \frac{n-2}{2} \) with two boundary components. Let \( a_1, \ldots, a_{n-1} \) be the essential circles of \( \Sigma \) pictured in Figure 7.2. By Proposition 7.1, the mapping \( \sigma_i \mapsto \sigma_{a_i}, \quad 1 \leq i \leq n-1 \), induces a representation \( \rho_M : B_n \to \mathcal{M}(\Sigma) \) called the *monodromy representation* of \( B_n \). This geometric representation was introduced by Birman and Hilden in [25], where it is proved that \( \rho_M \) is faithful and its image consists on mapping classes arising from homeomorphisms symmetric with respect to a hyperelliptic involution (see also [26], [153], and [117]). It is also the geometric monodromy of the simple singularity of type \( A_{n-1} \) (see [134]). Let \( P_0 \in \partial \Sigma \) be a base-point. Then \( \rho_M \) induces a homomorphism \( \rho_M \sigma_i : B_n \to \text{Aut}(\pi_1(\Sigma, P_0)) \) which turns out to coincide with the homomorphism \( \rho_D : B_n \to \text{Aut}(F_{n-1}) \) defined in Subsection 3.1 (see [60]).

**Example 3.** Let \( \mathbb{D}^2 = \{z \in \mathbb{C}; |z| \leq 1\} \) be the standard disk. A *chord diagram* in \( \mathbb{D}^2 \) is defined to be a collection \( \{S_1, \ldots, S_n\} \) of segments in \( \mathbb{D}^2 \) such that
- the extremities of \( S_i \) belong to \( \partial \mathbb{D}^2 \) and its interior is contained in the interior of \( \mathbb{D}^2 \), for all \( 1 \leq i \leq n \);

![Figure 7.1. Dehn twist.](image-url)
Braid groups and Artin groups

either $S_i$ and $S_j$ are disjoint, or they intersect transversely in a unique point in the interior of $D^2$, for all $1 \leq i \neq j \leq n$.

From this data one can define a Coxeter matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ setting $m_{ij} = 2$ if $S_i$ and $S_j$ are disjoint, and $m_{ij} = 3$ if they intersect. The Coxeter graph $\Gamma$ of $M$ is called the intersection diagram of the chord diagram.

From this data one can also define a surface $\Sigma$ by attaching to $D^2$ a handle $H_i$ which joins both extremities of $S_i$, for all $1 \leq i \leq n$ (see Figure 7.3). Let $a_i$ be the essential circle of $\Sigma$ made with $S_i$ and the central arc of $H_i$. Then, by Proposition 7.1, the mapping $\sigma_i \mapsto \sigma_{a_i}, 1 \leq i \leq n$, induces a geometric representation $\rho_{PV} : G_\Gamma \to \mathcal{M}(\Sigma)$, called Perron-Vannier representation.

The Perron-Vannier representations were introduced in [134]. If $\Gamma = A_{n-1}$, then $\rho_{PV}$ is equal to the monodromy representation $\rho_M$ defined in Example 2.
More generally, if $\Gamma$ is $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), or $E_k$ ($k = 6, 7, 8$), then $\rho_{PV}$ is the geometric monodromy of the simple singularity of type $\Gamma$ (see [134]). For a connected graph $\Gamma$, the representation $\rho_{PV}$ is faithful if and only if either $\Gamma = A_n$ for some $n \geq 1$, or $\Gamma = D_n$ for some $n \geq 4$ (see [134], [109], [151]).

Example 4. This example comes from [58]. Recall that a Coxeter graph $\Gamma$ is of small type if $m_{s,t} \leq 3$ for all $s, t \in S$, where $M = (m_{s,t})_{s, t \in S}$ is the Coxeter matrix of $\Gamma$. Let $\Gamma$ be a small type Coxeter graph. We choose (arbitrarily) a total order $<$ on $S$. For $s \in S$, we set $S_t^s = \{t \in S; m_{s,t} = 3\} \cup \{s\}$. Write $S_t = \{t_1, t_2, \ldots, t_k\}$ such that $t_1 < t_2 < \cdots < t_k$, and suppose that $s = t_j$. For $1 \leq i \leq k$, the difference $i - j$ is called the relative position of $t_i$ with respect to $s$ and is denoted by $pos(t_i : s)$. In particular, $pos(s : s) = 0$.

Let $s \in S$ and let $k = |S_t|$. Let $A_n$ denote the annulus $A_n = (\mathbb{R}/2k\mathbb{Z}) \times [0, 1]$. We define the surface $\Sigma = \Sigma_\Gamma$ by

$$\Sigma = \left( \bigcup_{s \in S} A_n \right) / \sim,$$

where $\sim$ is the equivalence relation defined as follows. Let $s, t \in S$ such that $s < t$ and $m_{s,t} = 3$. Set $p = pos(t : s) > 0$ and $q = pos(s : t) < 0$. For all $(x, y) \in [0, 1] \times [0, 1]$ the relation $\sim$ identifies the point $(2p + x, y)$ of $A_n$ with the point $(2q + 1 - y, x)$ of $A_t$ (see Figure 7.4).

![Figure 7.4](image-url)

**Figure 7.4.** Identification of annuli.

We identify each annulus $A_n$ with its image in $\Sigma$, and we denote by $a_s$ its central curve. Note that $a_s$ is an essential circle, $a_s \cap a_t = \emptyset$ if $m_{s,t} = 2$, and $|a_s \cap a_t| = 1$ if $m_{s,t} = 3$. So, by Proposition 7.1, the mapping $\sigma_s \mapsto \sigma_{a_s}, s \in S,$ induces a geometric representation $\rho_{CP} : G_\Gamma \to M(\Sigma)$.

We have $\rho_{CP} = \rho_{PV}$ if $\Gamma$ is a tree. (Note that it may happen that $\rho_{PV}$ is not defined if $\Gamma$ is not a tree.) If $\Gamma = A_n$, then $\rho_{CP}$ is faithful (while, by [109], $\rho_{PV}$ is not faithful in this case).
7.2 Presentations

Let \( \Sigma_{g,r} \) be a surface of genus \( g \geq 1 \) with \( r \geq 0 \) boundary components, and let \( \mathcal{P}_n \) be a collection of \( n \) punctures in the interior of \( \Sigma_{g,r} \), where \( n \geq 0 \).

Assume first that \( r \geq 1 \). Consider the essential circles \( a_0, a_1, \ldots, a_r, b_1, b_2, \ldots, b_{2g-1}, c, d_1, \ldots, d_{r-1} \), and the essential arcs \( e_1, e_2, \ldots, e_{n-1} \) drawn in Figure 7.5. Note that there is no \( c \) if \( g = 1 \), there is no \( d_i \) if \( r = 1 \), there is no \( a_r \) if \( n = 0 \), and there is no \( e_i \) if \( n = 0 \) or 1. Let \( \Gamma(g, r, n) \) be the Coxeter graph drawn in Figure 7.6. One can show that the set

\[
\{ \sigma_{a_0}, \sigma_{a_1}, \ldots, \sigma_{a_r}, \sigma_{b_1}, \sigma_{b_2}, \ldots, \sigma_{b_{2g-1}}, \sigma_c, \sigma_{d_1}, \sigma_{d_2}, \ldots, \sigma_{d_{r-1}}, \tau_{e_1}, \ldots, \tau_{e_{n-1}} \}
\]

generates \( \mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n) \). On the other hand, by Proposition 7.1, the mapping

\[
\begin{align*}
x_i &\mapsto \sigma_{a_i} \quad (0 \leq i \leq r), \\
y_i &\mapsto \sigma_{b_i} \quad (1 \leq i \leq 2g-1), \\
z &\mapsto \sigma_c \\
u_i &\mapsto \sigma_{d_i} \quad (1 \leq i \leq r-1), \\
v_j &\mapsto \tau_{e_j} \quad (1 \leq j \leq n-1),
\end{align*}
\]

induces a homomorphism \( \rho: G_{\Gamma(g,r,n)} \to \mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n) \). So, in order to obtain a presentation for \( \mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n) \), it suffices to find normal generators for \( \text{Ker} \rho \).

This was done in [124] for \( r = 1 \) and \( n = 0 \), and in [110] for the other cases.

---

**Figure 7.5.** Generators of \( \mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n) \).

**Figure 7.6.** The Coxeter graph \( \Gamma(g, r, n) \).
One can use the same kind of arguments for the case \( r = 0 \). Consider the essential circles \( a_0, a_1, b_1, b_2, \ldots, b_{2g-1}, c \), and the essential arcs \( e_1, e_2, \ldots, e_{n-1} \) drawn in Figure 7.7. Then the set 
\[
\{ \sigma_{a_0}, \sigma_{a_1}, \sigma_{b_1}, \sigma_{b_2}, \ldots, \sigma_{b_{2g-1}}, \sigma_c, \tau_{e_1}, \tau_{e_2}, \ldots, \tau_{e_{n-1}} \}
\]
generates \( \mathcal{M}(\Sigma_g,0,\mathcal{P}_n) \), and the mapping
\[
\begin{align*}
x_i &\mapsto \sigma_{a_i} \, (i = 0, 1), \\
y_i &\mapsto \sigma_{b_i} \, (1 \leq i \leq 2g - 1), \\
z &\mapsto \sigma_c, \\
v_j &\mapsto \tau_{e_j} \, (1 \leq j \leq n - 1),
\end{align*}
\]
induces a homomorphism \( \rho : G_{\Gamma(g,1,n)} \to \mathcal{M}(\Sigma_g,0,\mathcal{P}_n) \). Here again, the kernel of \( \rho \) was calculated in [124] for \( n = 0 \), and in [110] for \( n \geq 1 \).

![Figure 7.7. Generators of \( \mathcal{M}(\Sigma_g,0,\mathcal{P}_n) \).](image)

In order to state the results of [124] and [110], we need the following notations. Let \( \Gamma \) be a Coxeter graph, let \( M = (m_{s,t})_{s,t \in S} \) be the Coxeter matrix of \( \Gamma \), and let \( (G, \Sigma) \) be the Artin system of type \( \Gamma \). For \( X \subset S \), we denote by \( \Gamma_X \) the full subgraph of \( \Gamma \) generated by \( X \), we set \( \Sigma_X = \{ \sigma_s ; s \in X \} \), and we denote by \( G_X \) the subgroup of \( G \) generated by \( \Sigma_X \). By [116], \( (G_X, \Sigma_X) \) is the Artin system of type \( \Gamma_X \) (see also [131]). If \( \Gamma_X \) is of spherical type, then we denote by \( \Delta(X) \) the Garside element of \( (G_X, \Sigma_X) \), viewed as an element of \( G \).

**Theorem 7.2** (Matsumoto [124]). (1) \( \mathcal{M}(\Sigma_{g,1}) \) is isomorphic with the quotient of \( G_{\Gamma(g,1,0)} \) by the following relations
\[
\begin{align*}
(R1) & \quad \Delta(y_1, y_2, y_3, z)^4 = \Delta(x_0, y_1, y_2, y_3, z)^2 \quad \text{if } g \geq 2 \\
(R2) & \quad \Delta(y_1, y_2, y_3, y_4, y_5, z)^2 = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) \quad \text{if } g \geq 3
\end{align*}
\]

(2) \( \mathcal{M}(\Sigma_{g,0}) \) is isomorphic with the quotient of \( G_{\Gamma(g,1,0)} \) by the above relations \( (R1) \) and \( (R2) \) together with
\[
(R3) \quad x_0^{2g-2} = 1 \quad (x_0y_1)^6 = 1 \quad \text{if } g = 1
\]

**Theorem 7.3** (Labruère, Paris [110]). Let \( g \geq 1, \ r \geq 1, \) and \( n \geq 0 \). Then \( \mathcal{M}(\Sigma_{g,r},\mathcal{P}_n) \) is isomorphic with the quotient of \( G_{\Gamma(g,r,n)} \) by the following relations.
• Relations from $\mathcal{M}(\Sigma_{g,1})$.

\begin{align*}
(R1) & \quad \Delta(y_1, y_2, y_3, z)^4 = \Delta(x_0, y_1, y_2, y_3, z)^2 & \text{if } g \geq 2 \\
(R2) & \quad \Delta(y_1, y_2, y_3, y_4, y_5, z)^2 = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) & \text{if } g \geq 3
\end{align*}

• Relations of commutation.

\begin{align*}
(R3) & \quad x_k \cdot \Delta(x_{i+1}, x_j, y_k)^{-1} x_k \Delta(x_{i+1}, x_j, y_k) \\
& \quad = \Delta(x_{i+1}, x_j, y_k)^{-1} x_k \Delta(x_{i+1}, x_j, y_k) \cdot x_k \\
& \quad \text{if } 0 \leq k < j \leq i \leq r - 1 \\
(R4) & \quad y_2 \cdot \Delta(x_{i+1}, x_j, y_2)^{-1} x_i \Delta(x_{i+1}, x_j, y_2) \\
& \quad = \Delta(x_{i+1}, x_j, y_2)^{-1} x_i \Delta(x_{i+1}, x_j, y_2) \cdot y_2 \\
& \quad \text{if } 0 \leq j < i \leq r - 1 \text{ and } g \geq 2
\end{align*}

• Expressions of the $u_i$’s.

\begin{align*}
(R5) & \quad u_1 = \Delta(x_0, x_1, y_1, y_2, y_3, z) \cdot \Delta(x_1, y_1, y_2, y_3, z)^{-2} & \text{if } g \geq 2 \\
(R6) & \quad u_{i+1} = \Delta(x_i, x_{i+1}, y_1, y_2, y_3, z) \cdot \Delta(x_{i+1}, y_1, y_2, y_3, z)^{-2} \\
& \quad \cdot \Delta(x_0, x_{i+1}, y_1)^2 \cdot \Delta(x_0, x_i, x_{i+1}, y_1)^{-1} & \text{if } 1 \leq i \leq r - 2 \text{ and } g \geq 2
\end{align*}

• Other relations.

\begin{align*}
(R7) & \quad \Delta(x_{r-1}, x_r, y_1, v_1) = \Delta(x_r, y_1, v_1)^2 & \text{if } n \geq 2 \\
(R8a) & \quad \Delta(x_0, x_1, y_1, y_2, y_3, z) = \Delta(x_1, y_1, y_2, y_3, z)^2 & \text{if } n \geq 1, \ g \geq 2, \text{ and } r = 1 \\
(R8b) & \quad \Delta(x_{r-1}, x_r, y_1, y_2, y_3, z) \cdot \Delta(x_r, y_1, y_2, y_3, z)^{-2} \\
& \quad = \Delta(x_0, x_{r-1}, x_r, y_1) \cdot \Delta(x_0, x_r, y_1)^{-2} & \text{if } n \geq 1, \ g \geq 2, \text{ and } r \geq 2
\end{align*}

Note that only the relations (R1), (R2), (R7), and (R8a) remain in the presentation if $r = 1$, and (R8a) must be replaced by (R8b) if $r \geq 2$. Note also that, if $g \geq 2$, then $u_1, \ldots, u_{r-1}$ can be removed from the generating set. However, to do so, one must add new long relations.

Theorem 7.4 (Labruère, Paris [110]). Let $g \geq 1$ and $n \geq 1$. Then $\mathcal{M}(\Sigma_{g,0}, \mathcal{P}_n)$ is isomorphic with the quotient of $G_{\Gamma(g,1,n)}$ by the following relations.
• Relations from $\mathcal{M}(\Sigma_{g,1}, \mathcal{P}_n)$.

(R1) \[ \Delta(y_1, y_2, y_3, z)^4 = \Delta(x_0, y_1, y_2, y_3, z)^2 \]
if $g \geq 2$

(R2) \[ \Delta(y_1, y_2, y_3, y_4, y_5, z)^2 = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) \]
if $g \geq 3$

(R7) \[ \Delta(x_0, x_1, y_1, v_1) = \Delta(x_1, y_1, v_1)^2 \]
if $n \geq 2$

(R8) \[ \Delta(x_0, x_1, y_1, y_2, y_3, z) = \Delta(x_1, y_1, y_2, y_3, z)^2 \]
if $n \geq 1$ and $g \geq 2$

• Other relations.

(R9a) \[ x_0^{2g-n-2} \cdot \Delta(x_1, v_1, \ldots, v_{n-1}) = \Delta(z, y_2, \ldots, y_{2g-1})^2 \]
if $g \geq 2$

(R9b) \[ x_0^n = \Delta(x_1, v_1, \ldots, v_{n-1}) \]
if $g = 1$

(R9c) \[ \Delta(x_0, y_1)^4 = \Delta(v_1, \ldots, v_{n-1})^2 \]
if $g = 1$

Note. Presentations of $\mathcal{M}(\Sigma_{g,r})$, also in terms of Artin groups, with more generators but simpler relations, were obtained by Gervais in [88]. On the other hand, a unified proof of all these presentations can be found in [11].

7.3 Classification

This subsection is an account of Castel’s results [40] on the geometric representations of the braid group $B_n$ on mapping class groups of surfaces of genus $g \leq \frac{n-1}{2}$.

Suppose first that $n$ is odd, $n \geq 5$. Write $n = 2k + 1$, where $k \geq 2$. Let $r \geq 0$. We present the surface $\Sigma_{k,r}$ as the union of three subsurfaces, $\Omega_0$, $A$, and $\Omega_1$, where $\Omega_0$ is a surface of genus $k$ with one boundary component, $c$, $\Omega_1$ is a surface of genus $0$ with $r + 1$ boundary components, $c', d_1, \ldots, d_r$, and $A$ is an annulus bounded by $c$ and $c'$ (see Figure 7.8). Consider the essential circles $a_1, a_2, \ldots, a_{2k}$ drawn in Figure 7.8. Then, by Proposition 7.1, there exists a homomorphism $\rho_M : B_n \to \mathcal{M}(\Sigma_{k,r})$ which sends $\sigma_i$ to $\sigma_{a_i}$, for all $1 \leq i \leq n - 1 = 2k$.

The statement of Castel’s classification of the geometric representations of $B_n$ in $\mathcal{M}(\Sigma_{k,r})$ involves the centralizer of $\text{Im} \rho_M$ in $\mathcal{M}(\Sigma_{k,r})$. That is why we start with a description of the latter.

The inclusion of $\Omega_1$ in $\Sigma_{k,r}$ induces a homomorphism $\mathcal{M}(\Omega_1) \to \mathcal{M}(\Sigma_{k,r})$ which is injective (see [133]). It is easily checked that the image of this homomorphism is contained in the centralizer of $\text{Im} \rho_M$. Another element of the
centralizer is the element \( u \in \mathcal{M}(\Sigma_{k,r}) \) represented by the homeomorphism \( U : \Sigma_{k,r} \to \Sigma_{k,r} \) which is the axial symmetry relative to the axis \( D \) on \( \Omega_0 \), a half-twist which pointwise fixes \( c' \) on the annulus \( A \), and the identity on \( \Omega_1 \).

**Proposition 7.5** (Castel [40]). The centralizer of \( \text{Im} \rho_M \) in \( \mathcal{M}(\Sigma_{k,r}) \) is generated by \( \mathcal{M}(\Omega_1) \cup \{ u \} \).

If \( r = 0 \), then \( \mathcal{M}(\Omega_1) = \{ 1 \} \), \( u \) is of order 2, and \( Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im} \rho_M) = \langle u \rangle \) is cyclic of order 2. If \( r = 1 \), then \( \mathcal{M}(\Omega_1) = \langle \tau \rangle \), \( u^2 = \tau \), and \( Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im} \rho_M) = \langle u \rangle \) is an infinite cyclic group. If \( r = 2 \), then \( Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im} \rho_M) \) is a free abelian group of rank 3 freely generated by \( \{ u, \sigma_{d_1}, \sigma_{d_2} \} \). If \( r \geq 3 \), then \( Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im} \rho_M) \) is more complicated.

For \( \varepsilon \in \{ \pm 1 \} \) and \( z \in Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im} \rho_M) \), the mapping \( \sigma_i \mapsto \sigma_i^\varepsilon z, 1 \leq i \leq n - 1 \), induces a homomorphism \( \rho_M(\varepsilon, z) : \mathcal{B}_n \to \mathcal{M}(\Sigma_{k,r}) \) called the transvection of \( \rho_M \) by \( (\varepsilon, z) \). On the other hand, a homomorphism \( \varphi : \mathcal{B}_n \to G \), where \( G \) is a group, is called cyclic if there exists \( \alpha \in G \) such that \( \varphi(\sigma_i) = \alpha \) for all \( 1 \leq i \leq n - 1 \).

**Theorem 7.6** (Castel [40]). Suppose \( n \) odd, \( n \geq 5 \), and set \( n = 2k + 1 \). Let \( g \geq 0 \) and \( r \geq 0 \).

1. If \( g < k \), then all the homomorphisms \( \varphi : \mathcal{B}_n \to \mathcal{M}(\Sigma_{g,r}) \) are cyclic.
2. All the non-cyclic homomorphisms \( \varphi : \mathcal{B}_n \to \mathcal{M}(\Sigma_{k,r}) \) are conjugate to transvections of \( \rho_M \).
3. The homomorphism \( \rho_M : \mathcal{B}_n \to \mathcal{M}(\Sigma_{k,r}) \) is injective if and only if \( r \geq 1 \).

Now, we suppose that \( n \) is even, \( n \geq 6 \), and we set \( n = 2k + 2 \). We choose \( r_1, r_2 \geq 0 \) such that \( r_1 + r_2 = r \) and we represent the surface \( \Sigma_{k,r} \).
as the union of three subsurfaces, a surface $\Omega_0$ of genus $k$ with two boundary components, $c_1$ and $c_2$, a surface $\Omega_1$ of genus 0 with $r_1 + 1$ boundary components $c_1, d_1, \ldots, d_{r_1}$, and a surface $\Omega_2$ of genus 0 with $r_2 + 1$ boundary components $c_2, d_{r_1 + 1}, \ldots, d_{r_1 + r_2}$ (see Figure 7.9). Consider the essential circles $a_1, \ldots, a_{n-1}$ drawn in Figure 7.9. Then, by Proposition 7.1, there exists a homomorphism $\rho_M(r_1, r_2) : B_n \to M(\Sigma_{k,r})$ which sends $\sigma_i$ to $\sigma_{a_i}$ for all $1 \leq i \leq n-1$.

![Figure 7.9. Decomposition of $\Sigma_{k,r}$ (n even).](image)

The inclusions $\Omega_1, \Omega_2 \subset \Sigma_{k,r}$ induce a homomorphism $M(\Omega_1) \times M(\Omega_2) \to M(\Sigma_{k,r})$ which is injective (see [133]), and we have:

**Proposition 7.7** (Castel [40]).

1. If $r > 0$, then the centralizer of $\text{Im} \rho_M(r_1, r_2)$ in $M(\Sigma_{k,r})$ is $M(\Omega_1) \times M(\Omega_2)$.
2. If $r = 0$, then the centralizer of $\text{Im} \rho_M(r_1, r_2)$ in $M(\Sigma_{k,r})$ is a cyclic group of order 2 generated by an element represented by the axial symmetry relative to the axis $D$ of Figure 7.9.

For $\varepsilon \in \{\pm 1\}$ and $z \in Z_{M(\Sigma_{k,r})}(\text{Im} \rho_M(r_1, r_2))$, the mapping $\sigma_i \mapsto \sigma_{a_i}^\varepsilon z$, $1 \leq i \leq n-1$, induces a homomorphism $\rho_M(r_1, r_2, \varepsilon, z) : B_n \to M(\Sigma_{k,r})$ called the transvection of $\rho_M(r_1, r_2)$ by $(\varepsilon, z)$.

**Theorem 7.8** (Castel [40]). Suppose $n$ even, $n \geq 6$, and set $n = 2k + 2$. Let $g \geq 0$ and $r \geq 0$.

1. If $g < k$, then all the homomorphisms $\varphi : B_n \to M(\Sigma_{g,r})$ are cyclic.
2. If $\varphi : B_n \to M(\Sigma_{k,r})$ is a non-cyclic homomorphism, then there exist $r_1, r_2 \geq 0$ such that $r_1 + r_2 = r$ and $\varphi$ is conjugate to a transvection of $\rho_M(r_1, r_2)$.
3. Let $r_1, r_2 \geq 0$ such that $r_1 + r_2 = r$. The homomorphism $\rho_M(r_1, r_2) : B_n \to M(\Sigma_{k,r})$ is injective if and only if $r_1 \geq 1$ and $r_2 \geq 1$. 


Recall that, for a group $G$, $\text{Out}(G)$ denotes the group of outer automorphisms of $G$. Now, Theorems 7.6 and 7.8 can be used for new proofs of the following two theorems.

**Theorem 7.9** (Dyer, Grossman [77]). We have $\text{Out}(B_n) = \mathbb{Z}/2\mathbb{Z}$ if $n \geq 5$.

**Theorem 7.10** (Ivanov [98], McCarthy [125]). Let $g \geq 2$ and $r \geq 0$. Then

$$\text{Out}(\mathcal{M}(\Sigma_{g,r})) = \begin{cases} \{1\} & \text{if } r \geq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } r = 0 \text{ and } g \geq 3 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } r = 0 \text{ and } g = 2 \end{cases}$$

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