A simple frequency approximation formula for a class of nonlinear oscillators

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Abstract
An astonishingly simple analytical frequency approximation formula for a class of nonlinear oscillators with large amplitudes is derived and applied to various example systems yielding useful quick first estimates.

1 Introduction by example

In addition to established methods like Harmonic Balance, Krylov Bogoliubov or Lindsted Poincare [13] many new approaches for approximating the limit cycle frequencies of strongly nonlinear oscillators have been introduced in recent years, e.g. the Energy Balance method [8], the Hamiltonian Approach [12], the Variational Iteration method [11], the Amplitude frequency formulation [10] or the Newton Harmonic Balance Method [20] and other methods [18, 10]. These new methods have been successfully applied to various systems (see e.g. [2, 5, 3, 6, 17, 7]). Here, we present an extremely simple straightforwardly applicable method for a class of strongly nonlinear oscillators which can be expressed in terms of a simple analytical formula and yields satisfactory results for a variety of systems and parameter ranges with a minimum of effort.

As a first example, let us consider the Duffing oscillator

\[ \ddot{x} + \alpha x + \epsilon x^3 = 0 \]  (1)

with the initial conditions \( x(t = 0) = A, \dot{x}(t = 0) = 0 \). The ansatz

\[ x(t) = A \cos(\omega t) \]  (2)

satisfies the initial conditions and becomes an exact solution in the linear case \( \epsilon = 0 \) if \( \omega^2 = \alpha \). Inserting our ansatz [2] into the differential equation [1] we obtain

\[ -\omega^2 A \cos(\omega t) + \alpha A \cos(\omega t) + \epsilon A^3 \cos^3(\omega t) = 0 \]  (3)
where the cubic cosine function can alternatively be written as \( \cos^3(\omega t) = \frac{1}{4}(3 \cos(\omega t) + \cos(3\omega t)) \) \[1\]. Due to the term proportional to \( \cos(3\omega t) \) our ansatz \( x = A \cos(\omega t) \) cannot be an exact solution of equation (1). We seek an approximate solution for the frequency \( \omega \) by means of a colocation method, i.e. by evaluating equation (3) at some time \( t \in [0,T/4] \) where \( T = 2\pi/\omega \), similar to the procedure used in [8] in the context of the Energy Balance method. In [9] He used an analogous approach in combination with a Galerkin method rather than colocation.

We want to choose our colocation time \( t \) such that the influence of the \( \cos(3\omega t) \)-term is small. We therefore evaluate the differential equation where \( \cos(3\omega t) = 0 \) for the first time which leads to the condition \( 3\omega t = \pi/2 \) or \( \omega t = \pi/6 \). (4)
The colocation point \( \omega t = \pi/6 \) was also successfully used, on a purely phenomenological basis, in the context of He’s Amplitude Frequency formulation [17]. Inserting condition (4) into equation (3) we obtain

\[ \omega = \sqrt{\alpha + \epsilon A^2 \cos^2\left(\frac{\pi}{6}\right)} = \sqrt{\alpha + \frac{3}{4} \epsilon A^2} \] (5)

with \( \cos(\pi/6) = \sqrt{3}/4 \). This approximate result coincides with other approaches like first order Harmonic Balance [13], first orders of He’s Energy Balance method [8] and his Hamiltonian approach [12] as well as other methods [13, 10, 20, 18]. In [8] it was shown that for \( \alpha = 1 \) the relative error of \( \omega \) is always less than 7.6% even in the extreme large amplitude limit \( \epsilon A^2 \to \infty \).

### 2 Simple frequency approximation formula

Now we consider a more general nonlinear oscillator of the type

\[ \ddot{x} + f(x) = 0 \] (6)

with the initial conditions \( x(t = 0) = A, \dot{x}(t = 0) = 0 \) and where \( f(x) \) is antisymmetric in \( x \), i.e. \( f(-x) = -f(x) \). Then the Fourier expansion \( x(t) = \sum_{k=1}^{\infty} a_{2k-1} \cos((2k-1)\omega t) \) contains only odd multiples of \( \omega t \) [13] (see also the discussion in [20] where the same class of systems is considered). Thus the leading and next to leading order terms are \( \cos(\omega t) \) and \( \cos(3\omega t) \) respectively. As in the introductory example we insert the ansatz \( x(t) = A \cos(\omega t) \) into our differential equation (6) arriving at

\[ -\omega^2 A \cos(\omega t) + f(A \cos(\omega t)) = 0. \] (7)

In analogy to the introductory example we colocate at \( \omega t = \pi/6 \), where the \( \cos(3\omega t) \)-terms are zero, which leads to the simple approximation formula

\[ \omega = \frac{\sqrt{f\left(\frac{A}{\sqrt{\frac{3}{4}}}\right)}}{A \sqrt{\frac{3}{4}}}. \] (8)
Table 1: The approximate frequency $\omega_{\text{approx}}$ from (10) for the cubic-quintic oscillator is compared with frequencies $\omega_{\text{RK}}$ from numerically exact Runge Kutta calculations [5] for different values of $\lambda$ and $\alpha = 1$, $\epsilon = 5$, $A = 1$.

| $\lambda$ | $\omega_{\text{RK}}$ [5] | $\omega_{\text{approx}}$ | Error (%) |
|-----------|-----------------|----------------|-----------|
| 1         | 2.2798          | 2.3049         | 1.1010    |
| 5         | 2.7318          | 2.7500         | 0.6662    |
| 10        | 3.2057          | 3.2210         | 0.4773    |
| 100       | 7.7762          | 7.8102         | 0.4372    |
| 1000      | 23.7999         | 23.8170        | 0.0718    |

3 Example applications

3.1 Example 1

The cubic quintic oscillator with the force function

$$f(x) = \alpha x + \epsilon x^3 + \lambda x^5$$

reduces to the Duffing Oscillator (1) in the limit case $\lambda = 0$. The simple approximation formula (8) yields the frequency

$$\omega = \sqrt{\alpha + \frac{3}{4} A^2 + \lambda \frac{9}{16} A^4}.$$  (10)

A comparison with numerically exact frequencies for different values of $\lambda$ (table 3.1) reveals a good agreement.

3.2 Example 2

The fractional strongly nonlinear oscillator described by

$$f(x) = x^{1/3},$$

(11)

has been considered in several articles [2, 4, 14, 15]. From equation (8) we obtain the approximate frequency

$$\omega = \left( \frac{4}{3} \right)^{1/6} A^{-1/3} \approx 1.0491 A^{-1/3}.$$  (12)

which coincides with the first order Harmonic Balance result [14]. A comparison with the exact frequency $\omega_{\text{ex}} = 1.070451 A^{-1/3}$ [4] reveals an error of approximately 2%.

3.3 Example 3

Next we consider the strongly nonlinear oscillator with

$$f(x) = x^{-1}$$

(13)
\lambda = 0.5:

| $A$ | $\omega_{ex}$ [21, 19] | $\omega_{approx}$ Error (%) |
|-----|-----------------|-----------------|
| 0.1 | 0.70842         | 0.70842 0.00014 |
| 1   | 0.78617         | 0.78869 0.32068 |
| 10  | 0.96810         | 0.97090 0.28895 |
| 100 | 0.99681         | 0.99711 0.02982 |

$\lambda = 0.9:

| $A$ | $\omega_{ex}$ [21, 19] | $\omega_{approx}$ Error (%) |
|-----|-----------------|-----------------|
| 0.1 | 0.32148         | 0.32149 0.00318 |
| 1   | 0.55668         | 0.56539 1.5641  |
| 10  | 0.94169         | 0.94698 0.56085 |
| 100 | 0.99425         | 0.99479 0.05406 |

Table 2: The approximate frequency $\omega_{approx}$ from (16) for Example 4 is compared with the corresponding exact frequencies $\omega_{ex}$ [21, 19] for different values of $A$ for $\lambda = 0.5$ and $\lambda = 0.9$.

Analyzed in [2, 16, 10]. Formula (8) yields the approximation

$$\omega = \frac{2}{\sqrt{3}} A^{-1} \approx 1.1547 A^{-1}$$

(14)

coinciding again with the first order Harmonic Balance result [16] and results from He’s homotopy perturbation method [10]. The exact frequency reads $\omega_{ex} = \sqrt{2A} A^{-1} \approx 1.2533141 A^{-1}$ [16]. The resulting error of 7.9% is acceptable considering the simplicity of our approach.

### 3.4 Example 4

An oscillator with the force function

$$f(x) = x - \lambda \frac{x}{\sqrt{1 + x^2}}$$

(15)

was used in [21, 19] to model the dynamics of a mass attached to a stretched wire. Using formula (8) we obtain the approximate frequency

$$\omega = \sqrt{1 - \frac{\lambda}{\sqrt{1 + (3/4)A^2}}}$$

(16)

which is in good agreement with the exact frequencies (see table 3.4).

### 4 Conclusion

A simple frequency approximation formula for strongly nonlinear oscillators with antisymmetric position-dependent force terms was derived and applied to several example systems. The formula yields decent to good results for various systems with a minimum of effort.

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