Lattice Chiral Schwinger Model: Selected Results

V. Bornyakov 1, G. Schierholz 2 3 and A. Thimm 2 4

1 Institute for High Energy Physics IHEP, RU-142284 Protvino, Russia
2 Deutsches Elektronen-Synchrotron DESY and HLRZ,
D-15735 Zeuthen, Germany
3 Deutsches Elektronen-Synchrotron DESY, D-22603 Hamburg, Germany
4 Institut für Theoretische Physik, Freie Universität Berlin,
D-14195 Berlin, Germany

Abstract: We discuss a method for regularizing chiral gauge theories. The idea is to formulate the gauge fields on the lattice, while the fermion determinant is regularized and computed in the continuum. A simple effective action emerges which lends itself to numerical simulations.

1 Introduction

The regularization of chiral fermions by means of a space-time lattice has well-known difficulties. To evade them it has been suggested [1, 2, 3] to discretize only the gauge fields and treat the fermions in the continuum by introducing an interpolation of the latter [4]. In the present talk we shall test the idea in the chiral Schwinger model. First results of this approach have been reported in [5]. For similar ideas see [6].

To compute the effective action we start from Wilson fermions. The action is

\begin{align*}
S_\pm &= \frac{1}{2a_f} \sum_{n,\mu} \left\{ \bar{\psi}(n)\gamma_\mu \left[ \left( P_+ + P_\pm U_\mu^f(n) \right) \psi(n+\mu) \right. \\
&- \left. \left( P_+ + P_\pm U_\mu^{f\dagger} (n-\mu) \right) \psi(n-\mu) \right] \right\} \\
&+ S_W(U^f),
\end{align*}

where \( P_\pm = (1 \pm \gamma_5)/2 \), and where we have denoted the lattice spacing of the fermionic lattice by \( a_f \). Later on we will take the limit \( a_f \to 0 \). In practice \( a_f = a/N \), \( N \) integer,

\footnote{Talk given by V. Bornyakov at 31st International Ahrenshoop Symposium on the Theory of Elementary Particles, Buckow, September 1997.}
where $a$ is the lattice spacing of the gauge field action. The link variables on the fine lattice, $U^f$, are obtained by a suitable interpolation from the links on the original lattice, $U$. The usual (gauged) Wilson term $S_W$ reads

$$S_W(U^f) = \frac{1}{2a_f} \sum_{n,\mu} \bar{\psi}(n)[2\psi(x) - U^f_\mu(n)\psi(n + \mu) - U^{f\dagger}_\mu(n - \mu)\psi(n - \mu)]. \quad (3)$$

We will consider ungauged, $S_W(U^f = 1)$, and partially gauged Wilson terms as well.

The effective action is obtained in three steps. First one defines

$$\exp(-W_\pm) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(-S_\pm). \quad (4)$$

Then one performs the limit $\lim_{a_f \to 0} W_\pm$. One finds that this action is not invariant under chiral gauge transformations, not even in the anomaly-free model. Gauge invariance can, however, be restored by adding a local counterterm. The counterterm can be identified analytically and its coefficient be calculated in one-loop perturbation theory. It is

$$c \sum_x A^2_\mu(x), \quad (5)$$

where $A_\mu = \lim_{a_f \to 0} (1/a_f) \text{Im} U^f_\mu$, and $c = -0.0202$ for both gauged and ungauged Wilson terms. We then arrive at the effective action

$$\tilde{W}_\pm = \lim_{a_f \to 0} W_\pm + c \sum_x A^2_\mu(x). \quad (6)$$

The effective action $W_\pm$ has been computed by means of the Lanczos method. Note that the action is non-hermitean. The Lanczos vectors were re-orthogonalized after every iteration. For smooth gauge fields it was shown that

$$\text{Re}\tilde{W}_\pm = \frac{1}{2}(W + W_0), \quad (7)$$

where $W$ ($W_0$) is the effective action of the vector model (free theory). This result was conjectured in for ‘perturbative’ gauge fields.

Because of lack of space we shall restrict ourselves in the written version of the talk to rough gauge fields. We assume that the reader is familiar with the problem and with previous results on the subject.

## 2 Effective action revisited

Before we turn to the problem of rough gauge fields, let us briefly mention some new results on the effective action.

---

\[ \text{It was shown in [3], for a rather general class of gauge fields, not including compact } U(1) \text{ fields though, that the effective action can be made gauge invariant and finite.} \]

\[ \text{This value can also be extracted from [8].} \]
On the $L^2$ torus the gauge field can be written

$$A_\mu(x) = \frac{2\pi}{L} t_\mu + \varepsilon_{\mu\nu} \partial_\nu \alpha(x) + ig^{-1}(x) \partial_\mu g(x), \quad g(x) = \exp(-i\beta(x))$$

(8)

assuming $A_\mu(x) \in [-\pi,\pi)$, where $t_\mu \in [-1,1)$ are the zero momentum modes (torons), $\Box \alpha(x) = -F_{12}(x)$ and $g(x) \in U(1)$ is a gauge transformation. We assume periodic boundary conditions for the gauge fields and antiperiodic boundary conditions for the fermions.

Writing $a_\mu = (2\pi/L) t_\mu$ and $\tilde{A}_\mu = A_\mu - a_\mu$, the real part of the effective action factorizes in the form

$$\text{Re}\tilde{W}_\pm(A) = \text{Re}\tilde{W}_\pm(\tilde{A}) + \text{Re}\tilde{W}_\pm(a) + W_0.$$  

(9)

Formally this relation follows from the property of the Dirac operator

$$\mathcal{D}(\tilde{A} + a) \exp(-i\beta + \gamma_5 \alpha) = \exp(-i\beta - \gamma_5 \alpha) \mathcal{D}(a).$$

(10)

A similar expression can be derived for the imaginary part. The toron part of the effective action, $\tilde{W}_\pm(a)$, is known analytically [9]. For the fluctuating part we obtain

$$\tilde{W}_\pm(A) = \sum_\alpha \frac{q_\alpha^2}{8\pi} \int d^2 x \tilde{A}_\mu \left[ \delta_{\mu\nu} - (\partial_\mu + i\epsilon_\alpha \partial_\mu) \frac{1}{\Box} (\partial_\nu + i\epsilon_\alpha \partial_\nu) \right] \tilde{A}_\nu,$$

(11)

where $\tilde{\partial}_\mu = \varepsilon_{\mu\nu} \partial_\nu$, and $q_\alpha$ and $\epsilon_\alpha$ are the fermion charge and chirality, respectively. This result is in agreement with the well known expression for the effective action in $\mathbb{R}^2$ [10]. Note that the imaginary part of (11) vanishes in the anomaly-free model.

To check the result (11) and our method of calculation, we discretized the continuum configuration

$$A_\mu(x) = c_\mu \cos(2\pi k x/L) + (2\pi/L) t_\mu$$

with $c_1 = c_2 = 0.32$ and $k_1 = 1$, $k_2 = 0$ (as chosen in [11]) and computed $W_\pm$. In Fig. 1 we show $\text{Im} W_\pm(A)$ as a function of $r = a f/a$ for two toron fields. For the extrapolated values we obtain $-0.004078$ and $0.002098$, respectively. This is to be compared with the analytical values $-0.004074$ and $0.002094$, respectively. We find the same good agreement with the analytic formulae for the real part.

In [7] we shall give an analytic proof of eqs. (9) and (11).

### 3 Vortex-antivortex configuration

In [5] we reported numerical evidence for gauge invariance of the anomaly-free effective action under a class of gauge transformations. This did not include ‘singular’ gauge transformations which create a vortex-antivortex pair.

Let us consider a lattice gauge field configuration

$$\theta_\mu(s) = \theta_\mu^v(s-v) - \theta_\mu^\bar{v}(s-\bar{v}) - \frac{2\pi}{L^2} \varepsilon_{\mu\nu} (v_\nu - \bar{v}_\nu),$$

(12)

as we shall see this assignment is not unique if we allow large gauge transformations.

Note that singular gauge transformations generally create a problem in the overlap approach [12].
where $\theta_\mu^\nu(s) = 2\pi \varepsilon_{\mu\nu} \partial_\nu G(s)$, $G(s)$ being the lattice inverse Laplacian. This configuration corresponds to a vortex-antivortex pair at positions $v$ and $\bar{v}$, respectively. It is gauge equivalent to the vacuum configuration $\theta_\mu(s) = 0$. This configuration gives rise to a non-zero toron field $t_\mu = -(1/L) \varepsilon_{\mu\nu}(v_\nu - \bar{v}_\nu)$. The imaginary part of the effective action is again given by the toron field contribution. It is non-zero. The real part of the effective action is found to be divergent as $a_f \to 0$, in agreement with the analytic result (11). In Fig. 2 we show $\Re W^\Sigma = \Re W_+ + c \sum_{s,\mu} 2(1 - \cos(\theta_\mu(s)))$ as a function of $r$ for two different distances of the vortices. The divergence is $\propto \log(1/a_f)$. Thus we conclude that these configurations have zero weight in the partition function. We also expect that they do not contribute to any observable.

### 4 Index theorem

The lattice action must fulfill the index theorem in order to be in the same universality class as the continuum action. In general, the index theorem states that the number of zero modes of positive chirality minus the number of zero modes for negative chirality is equal to the topological charge, $n_+ - n_- = Q$. In two dimensions it even holds that $n_+ = Q$ for $Q > 0$ and $n_- = |Q|$ for $Q < 0$. Accordingly, a right(left)-handed fermion has $Q \theta(Q)$ ($-Q \theta(-Q)$) zero modes.

We have checked that the vector model satisfies the index theorem. For the eigenvalues we find, both numerically and analytically, the asymptotic behavior $E_0 = \pi |Q|(a_f/aL)^2$. For the chiral model, and both gauged and ungauged Wilson terms, the index theorem is, however, broken. For the ungauged Wilson term there are no zero modes at all, while for the gauged Wilson term we find zero modes of both
chiralities like in the vector model.

The index theorem can be restored by considering the following Wilson term

\[ S_\pm^W(U_f) = \frac{1}{4a_f} \sum_{n,\mu} \bar{\psi}(n) P_\pm [2\psi(x) - U_f^\dagger(n)\psi(n + \mu) - U_f(n - \mu)\psi(n - \mu)] \]

\[ + \frac{1}{4a_f} \sum_{n,\mu} \bar{\psi}(n) P_\mp [2\psi(x) - \psi(n + \mu) - \psi(n - \mu)]. \]  

\[ \text{(13)} \]

The Wilson term (13) has the property \( S_+^W(U_f) + S_-^W(U_f) = S_W(U_f) + S_W(0) \). We have checked numerically that it fulfills the index theorem for \( |Q| = 1 \). The next step is to verify the index theorem also for higher topological charges.

In the \( Q = 0 \) sector the Wilson term (13) gives basically the same results as before. The coefficient \( c \) of the counterterm is different though. In this case we obtain \( c = -0.05971 \).

5 Conclusions

We may consider the problem of formulating the chiral Schwinger model on the lattice as being solved for the \( Q = 0 \) sector. Numerical simulations are now feasible. The real part of the effective action is effectively half the action of the corresponding vector model, while the imaginary part of the anomaly-free model can be computed analytically from the toron fields. We hope to be able to report results on the sectors with non-vanishing topological charge in the near future.
Acknowledgements

One of us (VB) would like to thank DESY-Zeuthen for its hospitality. This work has been supported in part by INTAS and the Russian Foundation for Fundamental Sciences through grants INTAS-96-370 and 96-02-17230a.

References

[1] L. Alvarez-Gaumé and S. Della-Pietra, in Recent Developments in Quantum Field Theory, eds. J.Ambjorn, B. J. Durhuus and J. L. Petersen (North-Holland, 1985).

[2] M. Gökceler and G. Schierholz, Nucl. Phys. B (Proc. Suppl.) 29B,C (1992) 114, ibid. 30 (1992) 609.

[3] G. ’t Hooft, Phys. Lett. B349 (1995) 491.

[4] M. Gökceler, A.S. Kronfeld, G. Schierholz and U.-J. Wiese, Nucl. Phys. B404 (1993) 839.

[5] V. Bornyakov, G. Schierholz and A. Thimm, Nucl. Phys. B (Proc. Suppl.) 63 (1998) 593.

[6] P. Hernandez and R. Sundrum, Nucl. Phys. B455 (1995) 287; G. T. Bodwin, Phys. Rev. D54 (1996) 6497; I. Montvay, CERN-TH-95-123 (1995) (hep-lat/9505015).

[7] V. Bornyakov, G. Schierholz and A. Thimm, in preparation.

[8] A. A. Slavnov and N. V. Zverev (hep-lat/9708022).

[9] L. Alvarez-Gaumé, G. Moore and C. Vafa, Commun. Math. Phys. 106 (1986) 1; M. Gökceler and G. Schierholz, unpublished (1994); R. Narayanan and H. Neuberger, Phys. Lett. B348 (1995) 549.

[10] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219.

[11] R. Narayanan and H. Neuberger, Nucl. Phys. B443 (1995) 305.

[12] R. Narayanan and H. Neuberger, Nucl. Phys. B477 (1996) 521; T. Aoyama and Y. Kikukawa, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 638.

[13] V. Mitrjushkin, Phys. Lett. B389 (1996)713.