Modified Stancu operators based on inverse Polya Eggenberger distribution

Sheetal Deshwal¹, PN Agrawal¹ and Serkan Araci²*

Abstract
In this paper, we construct a sequence of modified Stancu-Baskakov operators for a real valued function bounded on $[0, \infty)$, based on a function $\tau(x)$. This function $\tau(x)$ is infinite times continuously differentiable on $[0, \infty)$ and satisfy the conditions $\tau(0) = 0$, $\tau'(x) > 0$ and $\tau''(x)$ is bounded for all $x \in [0, \infty)$. We study the degree of approximation of these operators by means of the Peetre K-functional and the Ditzian-Totik modulus of smoothness. The quantitative Voronovskaja-type theorems are also established in terms of the first order Ditzian-Totik modulus of smoothness.

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1 Introduction
In 1923, Eggenberger and Pólya [1] were the first to introduce Pólya-Eggenberger distribution. After that, in 1969, Johnson and Kotz [2] gave a short discussion of Pólya-Eggenberger distribution.

The Pólya-Eggenberger distribution $X$ [2] is defined by

$$Pr(X = k) = \binom{n}{k} \frac{a(a + s) \cdots (a + (u - 1)s)b(b + s) \cdots (b + (n - u - 1)s)}{(a + b)(a + b + s) \cdots (a + b + (n - 1)s)},$$

$$k = 0, 1, 2, \ldots, n. \quad (1.1)$$

The inverse Pólya-Eggenberger distribution $N$ is defined by

$$Pr(N = n + k) = \binom{n + k - 1}{k} \frac{a(a + s) \cdots (a + (n - 1)s)b(b + s) \cdots (b + (k - 1)s)}{(a + b)(a + b + s) \cdots (a + b + (n + k - 1)s)},$$

$$k = 0, 1, 2, \ldots, n. \quad (1.2)$$

In 1970, Stancu [3] introduced a generalization of the Baskakov operators based on inverse Pólya-Eggenberger distribution for a real valued bounded function on $[0, \infty)$, defined by

$$V_n^{[\alpha]}(f; x) = \sum_{k=0}^{\infty} v_{n,k}(x, \alpha)f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n + k - 1}{k} \frac{1^{[n, \alpha]}x^{[\alpha]}\left(\frac{k}{n}\right)}{\left(1 + x\right)^{\alpha} k!},$$

$$k = 0, 1, 2, \ldots, n. \quad (1.3)$$

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where $\alpha$ is a non-negative parameter which may depend only on $n \in \mathbb{N}$ and $a^{[n,h]} = a(a - h)(a - 2h) \cdots (a - (n - 1)h), a^{[0,h]} = 1$ is known as a factorial power of $a$ with increment $h$.

For $\alpha = 0$, the operator (1.3) reduces to Baskakov operators [4].

In 1989, Razi [5] studied convergence properties of Stancu-Kantorovich operators based on Pólya-Eggenberger distribution. Very recently, Deo et al. [6] introduced a Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution and studied some of its convergence properties. For some other relevant research in this direction we refer the reader to [7–9].

Now, for $\alpha = \frac{1}{2}$, we get a special case of Stancu-Baskakov operators (1.3) defined as

$$V_n^{1/2}(f; x) = \frac{(2n - 1)!}{(n - 1)!} \sum_{k=0}^{\infty} \left( \begin{array}{c} n + k - 1 \\ k \end{array} \right) \frac{(nx)_k}{(n + nx)_{n+k}} f \left( \frac{k}{n} \right),$$

(1.4)

where $(a)_n := a^{[n-1]} = a(a + 1) \cdots (a + (n - 1))$ is called the Pochhammer symbol.

For the Lupas operator, given by

$$L_n(f; x) = \sum_{k=0}^{\infty} \left( \begin{array}{c} n + k - 1 \\ k \end{array} \right) \frac{t^k}{(1 + t)^{n+k}} f \left( \frac{k}{n} \right),$$

(1.5)

let $\mu_{n,m}(x) = L_n(t^m; x), m \in \mathbb{N} \cup \{0\}$ be the $m$th order moment.

Lemma 1 For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x) = 1$ and we have the recurrence relation

$$x(1 + x)\mu'_{n,m}(x) = n\mu_{n,m+1}(x) - nx\mu_{n,m}(x), \quad m \in \mathbb{N} \cup \{0\},$$

(1.6)

where $\mu'_{n,m}(x)$ is the derivative of $\mu_{n,m}(x)$.

Proof On differentiating $\mu_{n,m}(x)$ with respect to $x$, the proof of the recurrence relation easily follows; hence the details are omitted. \hfill \square

Remark 1 From Lemma 1, we have

$$\mu_{n,1}(x) = x, \quad \mu_{n,2}(x) = \frac{x + (n + 1)x^2}{n}, \quad \mu_{n,3}(x) = \frac{(n + 1)(n + 2)x^3 + 3(n + 1)x^2 + x}{n^2}.$$

The values of the Stancu-Baskakov operators (1.4) for the test functions $e_i(t) = t^i, \ i = 0, 1, 2,$ are given in the following lemma.

Lemma 2 ([10]) The Stancu-Baskakov operators (1.4) verify:

(i) $V_n^{1/2}(1; x) = 1$,
(ii) $V_n^{1/2}(t; x) = \frac{nx}{n + 1}$,
(iii) $V_n^{1/2}(t^2; x) = \frac{n^2}{[n^2 + 2n + 1]} x^2 + \frac{3(nx + 1)}{n} + \frac{1}{n} (1 - \frac{x}{n})$,
(iv) $V_n^{1/2}(t^3; x) = \frac{n^3}{[n^3 + 3n^2 + 3n + 1]} \left[ \frac{(n+1)(2n + 1)}{n^2} x^3 + \frac{3(2n^2 + n - 1)}{n} x^2 + \frac{(2n + 1)(3n - 1)}{n^2} x \right]$,
(v) $V_n^{1/2}(t^4; x) = \frac{n^4}{[n^4 + 4n^3 + 6n^2 + 4n + 1]} \left[ \frac{(n+1)(n+2)(n+3)}{n^3} x^4 + \frac{6(n+1)(n+2)(2n+1)}{n^2} x^3 + \frac{6(n+1)(n+2)(3n+1)}{n} x^2 + \frac{6(n+1)(n+2)(4n+1)}{n^2} x \right]$.
Proof: The identities (i)-(iii) are proved in [10], hence we give the proof of the identity (iv). The identity (v) can be proved in a similar manner.

We have

\[ V_{n}^{(\frac{1}{2})}(t^{3};x) = \frac{(2n-1)}{(n-1)!} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{n}{n+k} \right)^{k} \]

\[ = \frac{1}{B(nx,n)} \int_{0}^{\infty} \frac{\mu_{n,3}(t)}{(1+t)^{nx+n}} dt, \]

where \( B(nx,n) \) is the Beta function.

Therefore using Remark 1, we get

\[ V_{n}^{(\frac{1}{2})}(t^{3};x) = \frac{1}{B(nx,n)} \int_{0}^{\infty} \frac{\mu_{n,3}(t)}{(1+t)^{nx+n}} \left[ \frac{(n+1)(n+2)t^{3} + 3(n+1)t^{2} + t}{n^{2}} \right] dt. \]

Now, by a simple calculation, we get the required result. \( \square \)

As a consequence of Lemma 2, we obtain the following.

**Lemma 3**  For the Stancu-Baskakov operator (1.4), the following equalities hold:

(i) \( V_{n}^{(\frac{1}{2})}(t-x;x) = \frac{x}{n-t} \),

(ii) \( V_{n}^{(\frac{1}{2})}(t-x)^{2};x) = \frac{2nx(x+1)}{(n-1)(n-2)}, \)

(iii) \( V_{n}^{(\frac{1}{2})}(t-x)^{3};x) = \frac{12n(n^{2}-13n+2)x^{3}(x+1) + 12n(n^{2}+8n-13)x^{3}(x+1) + (26n^{2}+48n-22)x(x+1) + (29-75n)x}{n^{2}}. \)

Let \( 0 \leq r_{n}(x) \leq 1 \) be a sequence of continuous functions for each \( x \in [0,1] \) and \( n \in \mathbb{N} \). Using this sequence \( r_{n}(x) \), for any \( f \in C[0,1] \), King [11] proposed the following modification of the Bernstein polynomial for a better approximation:

\[ (B_{n}f) \circ r_{n}(x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} \left( r_{n}(x) \right)^{k} \left( 1 - r_{n}(x) \right)^{n-k}. \]

Gonska et al. [12] introduced a sequence of King-type operators \( D_{n}^{*}: C[0,1] \rightarrow C[0,1] \) defined as

\[ D_{n}^{*}f = (B_{n}f) \circ (B_{n}^{*}r)^{-1} \circ r, \]

where \( r \in C[0,1] \) such that \( r(0) = 0, r(1) = 1 \) and \( r'(x) > 0 \) for each \( x \in [0,1] \) and studied global smoothness preservation, the approximation of decreasing and convex functions, the validity of a Voronovskaja-type theorem and a recursion formula generalizing a corresponding result for the classical Bernstein operators.

Motivated by the above work, in the present paper we introduce modified Stancu-Baskakov operators based on a function \( \tau(x) \) and obtain the rate of approximation of these operators with the help of Peetre’s K-functional and the Ditzian-Totik modulus of smoothness. Also, we prove a quantitative Voronovskaja-type theorem by using the first order Ditzian-Totik modulus of smoothness.

Throughout this paper, we assume that \( C \) denotes a constant not necessarily the same at each occurrence.
2 Modified Stancu-Baskakov operators

Let $\tau(x)$ be continuously differentiable $\infty$ times on $[0, \infty)$, such that $\tau(0) = 0$, $\tau'(x) > 0$ and $\tau''(x)$ is bounded for all $x \in [0, \infty)$. We introduce a sequence of Stancu-Baskakov operators for $f \in C_B[0, \infty)$, the space of all continuous and bounded functions on $[0, \infty)$, endowed with the norm $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$, by

$$V_n^{(\frac{1}{2}, \tau)}(f; x) = \sum_{k=0}^{\infty} p_n^{(\frac{1}{2}, \tau)}(x) \binom{n}{k} \left( \frac{k}{n} \right)^{(n \tau(x))}, \quad x \in [0, \infty),$$  \hspace{1cm} (2.1)

where

$$p_n^{(\frac{1}{2}, \tau)}(x) = \frac{(2n-1)!}{(n-1)!} \binom{n+k-1}{k} \binom{n \tau(x)}{(n+1 \tau(x))}. $$

**Lemma 4** The operator defined by (2.1) satisfies the following equalities:

(i) $V_n^{(\frac{1}{2}, \tau)}(1; x) = 1$,

(ii) $V_n^{(\frac{1}{2}, \tau)}(t; x) = \frac{n \tau(x)}{(n-1)\tau(x)}$,

(iii) $V_n^{(\frac{1}{2}, \tau)}(t^2; x) = \frac{n^3}{(n-1)(n-2)} \left[ \frac{n(n+1)(n+2)}{n^2} \tau^2(x) + \frac{3(n^2-1)}{n^3} \tau^3(x) + \frac{6(n^3+1)}{n^4} \tau^4(x) \right]$,

(iv) $V_n^{(\frac{1}{2}, \tau)}(t^3; x) = \frac{n^4}{(n-1)(n-2)(n-3)} \left[ \frac{n(n+1)(n+2)(n+3)}{n^3} \tau^4(x) + \frac{8(n^4+1)(n^2+1)}{n^5} \tau^5(x) \right]$.

Proof The proof of lemma is straightforward on using Lemma 2. Hence we omit the details.

Let the $m$th order central moment for the operators given by (2.1) be defined as

$$\mu_{\tau, m}(x) = V_n^{(\frac{1}{2}, \tau)}((\tau(t) - \tau(x))^m; x).$$

**Lemma 5** For the central moment operator $\mu_{\tau, m}(x)$, the following equalities hold:

(i) $\mu_{\tau, 1}(x) = \frac{\tau(x)}{n}$,

(ii) $\mu_{\tau, 2}(x) = \frac{2n^2 \tau^2(x) + 2(2\tau(x)-1)\tau(x)}{(n-1)(n-2)}$,

(iii) $\mu_{\tau, 4}(x) = \frac{12n(n^2 - 13n + 2)\tau^2(x)\phi_2(x) + 12n(n^2 + 8n - 13)\tau(x)\phi_3(x) + (26n^2 + 48n - 22)\phi_2^2(x) + (29 - 75n)\tau(x)}{(n-1)(n-2)(n-3)(n-4)}$.

where $\phi_2^\tau(x) = \tau(x)(\tau(x) + 1)$.

Proof Using the definition (2.1) of the modified Stancu-Baskakov operators and Lemma 4, the proof of the lemma easily follows. Hence, the details are omitted.

Let

$$W^2 = \{g \in C_B[0, \infty) : g, g^" \in C_B[0, \infty)\}.$$ 

For $f \in C_B[0, \infty)$ and $\delta > 0$, the Peetre $K$-functional [13] is defined by

$$K(f; \delta) = \inf_{g \in W^2} \left\{ \|f - g\| + \delta\|g\|_{W^2} \right\},$$
where
\[ \|g\|_{w^2} = \|g\| + \|g'\| + \|g''\|. \]

From [14], Proposition 3.4.1, there exists a constant \( C > 0 \) independent of \( f \) and \( \delta \) such that
\[ K(f; \delta) \leq C(\omega_2(f; \sqrt{\delta}) + \min\{1, \delta\}\|f\|), \]  
(2.2)
where \( \omega_2 \) is the second order modulus of smoothness of \( f \in C_\mathbb{B}[0, \infty) \) and is defined as
\[ \omega_2(f; \delta) = \sup_{0 < |h| \leq \delta} \sup_{x, x + 2h \in [0, \infty)} \left| f(x + 2h) - 2f(x + h) + f(x) \right|. \]

In the following, we assume that \( \inf_{x \in (0, \infty)} \tau'(x) \geq a, a \in \mathbb{R}^+ := (0, \infty). \)

Next, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the \( K \)-functional [15]. Let \( \phi_\ast(x) := \sqrt{\tau(x)(1 + \tau(x))} \) and \( f \in C_\mathbb{B}[0, \infty) \). The first order modulus of smoothness is given by
\[ \omega_\phi(f; t) = \sup_{0 < h \leq t} \left\{ \left| f(x + \frac{h}{2}\phi_\ast(x)) - f(x - \frac{h}{2}\phi_\ast(x)) \right|, x \pm \frac{h\phi_\ast(x)}{2} \in [0, \infty) \right\}. \]

Further, the appropriate \( K \)-functional is defined by
\[ K_\phi(f; t) = \inf_{g \in W_\phi[0, \infty]} \left\{ \|f - g\| + t\|\phi_\ast g'\| \right\} \quad (t > 0), \]
where \( W_\phi[0, \infty] = \{ g : g \in AC_{\text{loc}}[0, \infty), \|\phi_\ast g'\| < \infty \} \) and \( g \in AC_{\text{loc}}[0, \infty) \) means that \( g \) is absolutely continuous on every interval \( [a, b] \subset [0, \infty) \). It is well known [15], p.11, that there exists a constant \( C > 0 \) such that
\[ K_\phi(f; t) \leq C\omega_\phi(f; t). \]  
(2.3)

**Theorem 1** If \( f \in C_\mathbb{B}[0, \infty) \), then
\[ \|V_n^{(\frac{1}{2}, \tau)}f\| \leq \|f\|. \]

**Proof** By the definition of the modified Stancu-Baskakov operators (2.1) and using Lemma 4 we have
\[ |V_n^{(\frac{1}{2}, \tau)}(f; x) - f(x)| \leq \sum_{k=0}^{n} p_{(\frac{1}{2}, \tau)}(\frac{k}{n}) \left| (f \circ \tau^{-1}) \left( \frac{k}{n} \right) \right| \leq \|f \circ \tau^{-1}\| \left| V_n^{(\frac{1}{2}, \tau)}(1; x) = \|f\| \right|, \]
for every \( x \in [0, \infty) \). Hence the required result is immediate. \( \square \)

**Theorem 2** Let \( f \in C_\mathbb{B}[0, \infty) \). Then, for \( n \geq 3 \), there exists a constant \( C > 0 \) such that
\[ |V_n^{(\frac{1}{2}, \tau)}(f; x) - f(x)| \leq C \left\{ \omega_2 \left( f; \frac{\phi_\ast(x)}{\sqrt{n-2}} \right) + \frac{\phi_\ast^2(x)}{n-2} \|f\| \right\} + \omega \left( f \circ \tau^{-1}; \frac{\tau(x)}{n-1} \right), \]
for each compact subset of \( [0, \infty) \).
Proof Let \( U \) be a compact subset of \([0, \infty)\). For each \( x \in U \), first we define an auxiliary operator as

\[
V_n^{(\frac{1}{n}; \tau)}(f; x) = V_n^{(\frac{1}{n}; \tau)}(f; x) - f \circ \tau^{-1}\left(\frac{n \tau(x)}{n - 1}\right) + f(x). \tag{2.4}
\]

Now, using Lemma 4, we have

\[
V_n^{(\frac{1}{n}; \tau)}(1; x) = 1 \quad \text{and} \quad V_n^{(\frac{1}{n}; \tau)}(\tau(t); x) = \tau(x) \quad \text{hence} \quad V_n^{(\frac{1}{n}; \tau)}(\tau(t) - \tau(x); x) = 0. \tag{2.5}
\]

Let \( g \in W^2, x \in U \) and \( t \in [0, \infty) \). Then by Taylor's expansion, and using results in [16], p.32, we get

\[
g(t) = (g \circ \tau^{-1})(\tau(t)) = (g \circ \tau^{-1})(\tau(x)) + (g \circ \tau^{-1})'((\tau(t) - \tau(x))) + \int_{\tau(x)}^{\tau(t)} (\tau(t) - u)(g \circ \tau^{-1})''(u) \, du
\]

\[
= g(x) + (g \circ \tau^{-1})'(\tau(x))(\tau(t) - \tau(x)) + \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} \, du
\]

\[- \int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} \, du. \tag{2.6}
\]

Now, applying the operator \( V_n^{(\frac{1}{n}; \tau)}(\cdot; x) \) to both sides of the above equality, we get

\[
V_n^{(\frac{1}{n}; \tau)}(g; x) - g(x)
\]

\[
= (g \circ \tau^{-1})'(\tau(x))V_n^{(\frac{1}{n}; \tau)}\left(\left(\tau(t) - \tau(x)\right); x\right)
\]

\[
+ V_n^{(\frac{1}{n}; \tau)}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} \, du; x\right)
\]

\[- V_n^{(\frac{1}{n}; \tau)}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} \, du; x\right)
\]

\[
= V_n^{(\frac{1}{n}; \tau)}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} \, du; x\right)
\]

\[- \int_{\tau(x)}^{\tau(t)} \left(\frac{n \tau(x)}{n - 1} - u\right) \frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2} \, du
\]

\[- V_n^{(\frac{1}{n}; \tau)}\left(\int_{\tau(x)}^{\tau(t)} (\tau(t) - u) \frac{g''(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} \, du; x\right)
\]

\[+ \int_{\tau(x)}^{\tau(t)} \left(\frac{n \tau(x)}{n - 1} - u\right) \frac{g''(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3} \, du. \tag{2.6}
\]

Again, for each \( x \in U \), we have

\[
\left|V_n^{(\frac{1}{n}; \tau)}(g; x) - g(x)\right|
\]

\[
\leq \frac{1}{2} \frac{g''}{a^2} V_n^{(\frac{1}{n}; \tau)}\left(((\tau(t) - \tau(x))^2; x\right) + \frac{1}{2} \frac{g''}{a^2} \left(\frac{n \tau(x)}{n - 1} - \tau(x)\right)^2
\]
\[
\begin{align*}
&\quad + \frac{1}{2}\frac{\|g\|\|\tau''\|}{a^2} V_n^{\frac{1}{2}, \tau} ((\tau(t) - \tau(x))^2; x) + \frac{1}{2}\frac{\|g'\|\|\tau''\|}{a^3} \left( \frac{n\tau(x)}{n-1} - \tau(x) \right)^2 \\
&= \frac{1}{2}\left( \frac{\|g\|}{a^2} + \frac{\|g'\|\|\tau''\|}{a^3} \right) \left[ V_n^{\frac{1}{2}, \tau} (\tau(t) - \tau(x))^2; x) + \left( \frac{\tau(x)}{n-1} \right)^2 \right]. \\
\end{align*}
\]

(2.7)

Now, using the definition of the auxiliary operators, Theorem 1 and inequality (2.7), for each \( x \in U \) we have

\[
\begin{align*}
&\quad |V_n^{\frac{1}{2}, \tau}(f; x) - f(x)| \\
&\quad \leq |V_n^{\frac{1}{2}, \tau}(f - g; x)| + |V_n^{\frac{1}{2}, \tau}(g; x) - g(x)| \\
&\quad + |g(x) - f(x)| + \left| f \circ \tau^{-1} \left( \frac{n\tau(x)}{n-1} \right) - f \circ \tau^{-1}(\tau(x)) \right| \\
&\quad \leq 4\|f - g\| + \frac{1}{2}\left( \frac{\|g\|}{a^2} + \frac{\|g'\|\|\tau''\|}{a^3} \right) \left[ V_n^{\frac{1}{2}, \tau} (\tau(t) - \tau(x))^2; x) + \left( \frac{\tau(x)}{n-1} \right)^2 \right] \\
&\quad + \omega(f \circ \tau^{-1}; \frac{\tau(x)}{n-1}).
\end{align*}
\]

(2.8)

Let \( C = \max(4, \frac{4}{a^2}, \frac{4}{a^3} \|\tau''\|) \), we get

\[
|V_n^{\frac{1}{2}, \tau}(f; x) - f(x)| \leq C \left( \|f - g\| + \|g\| \phi_2(x) \right) + \omega(f \circ \tau^{-1}; \frac{\tau(x)}{n-1}).
\]

(2.9)

Taking the infimum on the right side of the above inequality over all \( g \in W^2 \) and for all \( x \in U \), we have

\[
|V_n^{\frac{1}{2}, \tau}(f; x) - f(x)| \leq CK \left( f; \frac{\phi_2(x)}{n-2} \right) + \omega(f \circ \tau^{-1}; \frac{\tau(x)}{n-1}),
\]

(2.10)

using equation (2.2), we get the required result.

\( \square \)

**Theorem 3** Let \( f \in C_\beta[0, \infty) \). Then for every \( x \in [0, \infty) \), and \( n \geq 3 \) we have

\[
|V_n^{\frac{1}{2}, \tau}(f; x) - f(x)| \leq C \omega_b \left( f; \frac{\sqrt{n}c(x)}{a\sqrt{n-2}} \right).
\]

**Proof** For any \( g \in W_{\phi_r}[0, \infty) \), by Taylor’s expansion, we have

\[
g(t) = (g \circ \tau^{-1})(\tau(t)) = (g \circ \tau^{-1})(\tau(x)) + \int_{\tau(x)}^{\tau(t)} (g \circ \tau^{-1})'(u) \, du.
\]

Applying the operator \( V_n^{\frac{1}{2}, \tau}(; x) \) on both sides of the above equality, we get

\[
|V_n^{\frac{1}{2}, \tau}(g; x) - g(x)| = |V_n^{\frac{1}{2}, \tau} \left( \int_{\tau(x)}^{\tau(t)} (g \circ \tau^{-1})'(u) \, du \right)|.
\]

(2.11)
From [16], we have

\[
\left| \int_{\tau(x)}^{t} (g \circ \tau^{-1})(u) \, du \right| = \left| \int_{x}^{t} g'(y) \tau'(y) \, dy \right| = \left| \int_{x}^{t} \frac{\phi_{\tau}(y)}{\phi_{\tau}(x)} g'(y) \tau'(y) \, dy \right| \\
\leq \frac{\|\phi_{\tau}g'\|}{a} \left| \int_{x}^{t} \frac{\tau'(y)}{\phi_{\tau}(x)} \, dy \right| \\
(2.12)
\]

and

\[
\left| \int_{x}^{t} \frac{\tau'(y)}{\phi_{\tau}(x)} \, dy \right| \leq \left| \int_{x}^{t} \left( \frac{1}{\sqrt{\tau(y)}} + \frac{1}{\sqrt{1 + \tau(y)}} \right) \tau'(y) \, dy \right| \\
= \left| \int_{x}^{t} \frac{1}{\sqrt{\tau(y)}} \tau'(y) \, dy \right| + \left| \int_{x}^{t} \frac{1}{\sqrt{1 + \tau(y)}} \tau'(y) \, dy \right| \\
= 2 \left| \sqrt{\tau(t)} - \sqrt{\tau(x)} \right| + \left| \sqrt{1 + \tau(t)} - \sqrt{1 + \tau(x)} \right| \\
< 2 \left[ \tau(t) - \tau(x) \right] \left( \frac{1}{\sqrt{\tau(x)}} + \frac{1}{\sqrt{1 + \tau(x)}} \right) \\
= \frac{2[\tau(t) - \tau(x)]}{\sqrt{\tau(x)(1 + \tau(x))}} \left( \sqrt{1 + \tau(x)} + \sqrt{\tau(x)} \right) \\
= \frac{2[\tau(t) - \tau(x)]c(x)}{\sqrt{\tau(x)(1 + \tau(x))}} \\
= \frac{2c(x)[\tau(t) - \tau(x)]}{\phi_{\tau}(x)}. \\
(2.13)
\]

Now, from equations (2.12)-(2.13) and using the Cauchy-Schwarz inequality, we obtain

\[
\left| V_{n}^{\frac{1}{2};\tau}(g;x) - g(x) \right| \leq \frac{2c(x)\|\phi_{\tau}g'\|\sqrt{\frac{1}{n}}}{a \phi_{\tau}(x)} \left( \left| \tau(t) - \tau(x) \right|; x \right) \\
\leq \frac{2c(x)\|\phi_{\tau}g'\|\sqrt{\frac{1}{n}}}{a \phi_{\tau}(x)} \left( \left( \tau(t) - \tau(x) \right)^{2}; x \right)^{\frac{1}{2}} \\
= \frac{2c(x)\|\phi_{\tau}g'\|}{a \phi_{\tau}(x)} \left[ 2n\phi_{\tau}^{2}(x) + (2\tau(x) - 1)\tau(x) \right]^{\frac{1}{2}}. \\
(2.14)
\]

Thus, for \( f \in C_{b}[0, \infty) \) and any \( g \in W_{\phi_{\tau}}, [0, \infty) \), we have

\[
\left| V_{n}^{\frac{1}{2};\tau}(f;x) - f(x) \right| \leq \left| V_{n}^{\frac{1}{2};\tau}(f - g;x) \right| + \left| f(x) - g(x) \right| + \left| V_{n}^{\frac{1}{2};\tau}(g;x) - g(x) \right| \\
\leq 2\| f - g \| + \frac{2c(x)\|\phi_{\tau}g'\|}{a \phi_{\tau}(x)} \left[ 2n\phi_{\tau}^{2}(x) + (2\tau(x) - 1)\tau(x) \right]^{\frac{1}{2}} \\
= \frac{2c(x)\|\phi_{\tau}g'\|}{a} \left[ \frac{2(n + 1)}{(n - 1)(n - 2)} \right]^{\frac{1}{2}} + 2\| f - g \| \\
\leq 2 \left\{ \| f - g \| + \frac{\sqrt{6c(x)}}{a\sqrt{n - 2}} \| \phi_{\tau}g' \| \right\}. \\
(2.15)
\]
Taking the infimum on the right side of the above inequality over all $g \in W_{\phi_{\tau}}[0, \infty)$, we get
\[
\left| V_n^{(1, \tau)}(f; x) - f(x) \right| \leq 2K_{\phi_{\tau}} \left( f(x) \sqrt{\frac{c(x)}{a^2(n-2)}} \right).
\]

Finally, using equation (2.3), the theorem is immediate. \hfill \Box

**Theorem 4** For any $f \in C^2[0, \infty)$ and $x \in [0, \infty)$, the following inequality hold:
\[
\left| V_n^{(1, \tau)}(f; x) - f(x) \right| - \frac{f''(x)}{\tau''(x)} \mu_n^{r, 1}(x) - \frac{1}{2} \int \frac{f''(x)}{\tau''(x)} \left( \frac{x}{\tau''(x)} \right)^2 \mu_n^{r, 2}(x) \right| \\
\leq \left( \mu_n^{r, 2}(x) \right)^{\frac{1}{3}} \left( \int \left| f(x) \right| \mu_n^{r, 2}(x) \right)^{\frac{1}{3}} + \left\| \phi \right\|_{\mu_n^{r, 2}(x)} \left( \mu_n^{r, 4}(x) \right)^{\frac{1}{2}}.
\]

**Proof** Let $f \in C^2[0, \infty)$ and $x, t \in [0, \infty)$. Then by Taylor’s expansion, we have
\[
f(t) = (f \circ \tau^{-1})(\tau(t)) = (f \circ \tau^{-1})(\tau(x)) + (f \circ \tau^{-1})(\tau(x))(t - x) + \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau + (f \circ \tau^{-1})(u) d\tau.
\]

Hence,
\[
f(t) - f(x) = (f \circ \tau^{-1})(\tau(t)) - (f \circ \tau^{-1})(\tau(t)) = \left( f \circ \tau^{-1} \right)(\tau(x)) (t - x) - \frac{1}{2} \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau - \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau
\]
\[
= \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau - \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau.
\]

Applying $V_n^{(1, \tau)}$ to both sides of the above relation, we get
\[
\left| V_n^{(1, \tau)}(f; x) - f(x) \right| - \frac{f''(x)}{\tau''(x)} \mu_n^{r, 1}(x) - \frac{1}{2} \int \frac{f''(x)}{\tau''(x)} \left( \frac{x}{\tau''(x)} \right)^2 \mu_n^{r, 2}(x) \right| \\
= \left| V_n^{(1, \tau)} \left( \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau - (f \circ \tau^{-1})(\tau(x)) d\tau \right) \right| \\
\leq V_n^{(1, \tau)} \left( \left| \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau - (f \circ \tau^{-1})(\tau(x)) d\tau \right| \right).
\]

For $g \in W_{\phi_{\tau}}[0, \infty)$, we have
\[
\left| \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau - (f \circ \tau^{-1})(\tau(x)) d\tau \right| \\
\leq \left| \int_{\tau(x)}^{t} (f \circ \tau^{-1})(u) d\tau - (f \circ \tau^{-1})(\tau(x)) d\tau \right|
\]
\[
+ \left| \int_{\tau(x)}^{t} (g \circ \tau^{-1})(u) d\tau - (g \circ \tau^{-1})(\tau(x)) d\tau \right|
\]
Using the inequality

\[ \frac{|y - v|}{\nu(1 + \nu)} \leq \frac{|y - x|}{x(1 + x)}, \quad x < \nu < y, \]

we can write

\[ \frac{|\tau (y) - \tau (v)|}{\tau (v)(1 + \tau (v))} \leq \frac{|\tau (y) - \tau (x)|}{\tau (x)(1 + \tau (x))}. \]

Therefore,

\[ \int_{\tau (x)}^{\tau (t)} |\tau (t) - u| (f \circ \tau^{-1})''(u) - (f \circ \tau^{-1})'''(\tau (x)) \, du \]

\[ \leq \| (f \circ \tau^{-1})'' - g \| (\tau (t) - \tau (x))^2 \]

\[ + \| \phi_x g \| \left( \int_{\tau (x)}^{\tau (y)} \frac{|\tau (y) - \tau (x)|^{1/2}}{\tau (v)\phi_x (x)} \, dv \right) \left( \frac{\tau (v)}{|\tau (y) - \tau (v)|^{1/2}} \right) \left( \int_{\tau (x)}^{\tau (y)} |\tau (t) - \tau (y)| \, dv \right) |\tau (t) - \tau (y)| \, dv \]

\[ \leq \| (f \circ \tau^{-1})'' - g \| (\tau (t) - \tau (x))^2 \]

\[ + 2 \| \phi_x g \| \phi_x^{-1} \left( \int_{\tau (x)}^{\tau (y)} |\tau (y) - \tau (x)| \, dv \right) \left( \int_{\tau (x)}^{\tau (y)} (\tau (t) - \tau (x))^2 \, dv \right) |\tau (t) - \tau (y)| \, dv \]

\[ \leq \| (f \circ \tau^{-1})'' - g \| (\tau (t) - \tau (x))^2 + 2 \| \phi_x g \| \phi_x^{-1} \left( \int_{\tau (x)}^{\tau (y)} (\tau (t) - \tau (x))^2 \, dv \right) \frac{1}{\phi_x(x)} |\tau (t) - \tau (x)| \]

\[ \leq \| (f \circ \tau^{-1})'' - g \| (\tau (t) - \tau (x))^2 + 2 \| \phi_x g \| \phi_x^{-1} |\tau (t) - \tau (x)| \]

(2.17)

Now combining equations (2.16)-(2.17), applying Lemma 3 and the Cauchy-Schwarz inequality, we get

\[ V_{n,\tau}^{(\frac{1}{2}, \tau)}(f; x) - f(x) - \int_{\tau (x)}^{\tau (t)} \frac{f''(x) + x f''(x) \phi_x (x)}{\phi_x(x)} \, dv \]

\[ \leq \| (f \circ \tau^{-1})'' - g \| V_{n,\tau}^{(\frac{1}{2}, \tau)} \left( (\tau (t) - \tau (x))^2 ; x \right) + 2 \| \phi_x g \| \phi_x^{-1} \left( \frac{1}{\phi_x(x)} \right) V_{n,\tau}^{(\frac{1}{2}, \tau)} \left( (\tau (t) - \tau (x))^3 ; x \right) \]
\[ \leq \| (f \circ \tau^{-1})'' - g \| V_n^{(1/2)} \left( \left( (\tau(t) - \tau(x))^2; x \right) \right) \\
+ \frac{2\| \phi_1'g' \|}{a} \phi_1^{-1}(x) \left[ V_n^{(1/2)} \left( \left( (\tau(t) - \tau(x))^2; x \right) \right) \right]^{1/2} \left[ V_n^{(1/2)} \left( \left( (\tau(t) - \tau(x))^4; x \right) \right) \right]^{1/2} \\
= \| (f \circ \tau^{-1})'' - g \| \mu_{n,2}(x) \frac{2\| \phi_1'g' \|}{a} \phi_1^{-1}(x) \left( \mu_{n,4}(x) \right)^{1/2} \\
= \left( \mu_{n,2}(x) \right)^{1/2} \left[ \| (f \circ \tau^{-1})'' - g \| \left( \mu_{n,2}(x) \right)^{1/2} \frac{2\| \phi_1'g' \|}{a} \phi_1^{-1}(x) \left( \mu_{n,4}(x) \right)^{1/2} \right]. \]

This completes the proof of the theorem. \(\square\)

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors equally contributed to this work. All authors read and approved the final manuscript.

Author details
1 Indian Institute of Technology Roorkee, Roorkee, 247667, India. 2 Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, 27410, Turkey.

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