CODIMENSION BOUNDS FOR THE NOETHER-LEFSCHETZ 
COMPONENTS FOR TORIC VARIETIES

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Abstract

For a quasi-smooth hyper-surface \( X = \{ f = 0 \} \) in a projective simplicial toric variety \( \mathbb{P}_\Sigma \), the morphism \( i^* : H^p(\mathbb{P}_\Sigma) \to H^p(X) \) induced by the inclusion, is injective for \( p = d \) and an isomorphism for \( p < d - 1 \), where \( d = \dim \mathbb{P}_\Sigma \). This allows one to define the Noether-Lefschetz locus \( NL_\beta \) as the locus of quasi-smooth hypersurfaces of degree \( \beta \) such that \( i^* \) acting on the middle algebraic cohomology is not an isomorphism. In this paper we prove that, under some assumptions, if \( \dim \mathbb{P}_\Sigma = 2k + 1 \) and \( k\beta - \beta_0 = n\eta \) (\( n \in \mathbb{N} \)), where \( \eta \) is the class of an ample divisor, then every irreducible component \( V \) of the Noether-Lefschetz locus quasi-smooth hypersurfaces of degree \( \beta \) satisfies the bounds

\[
n + 1 \leq \text{codim} V \leq h^{k-1,k+1}(X).
\]

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1 Introduction

The classical Noether-Lefschetz theory is about the Picard number of surfaces in 3-dimensional projective space. Let $\mathcal{U}_d \subset \mathbb{P}^3 (O_{\mathbb{P}^3}(d))$ be the locus of smooth surfaces of degree $d$ in $\mathbb{P}^3$, with $d \geq 4$; then the very general surface in $\mathcal{U}_d$ has Picard number 1 (for an historical perspective of the Noether-Lefschetz problem, and exhaustive references the reader may consult [2]). Moreover, if $Z$ is a component of the locus in $\mathcal{U}_d$ whose points correspond to surfaces with Picard number greater than 1 (the Noether-Lefschetz locus), then

$$d - 3 \leq \text{codim}_{\mathcal{U}_d} Z \leq \left(\frac{d-1}{3}\right).$$

This result was generalized in [4, 10] to quasi-smooth surfaces in projective simplicial toric threefolds satisfying some conditions. The purpose of the present paper is to extend these bounds to the case of projective simplicial toric varieties of higher odd dimension, see Theorems 2.1 and 3.1 (when the ambient variety has even dimension the problem is trivial as the middle cohomology of hypersurfaces is controlled by the Lefschetz hyperplane theorem).

This short paper is a natural sequel to [6], where the definition of the Noether-Lefschetz locus was extended to simplicial projective toric varieties $\mathbb{P}_{\Sigma}^{2k+1}$ of arbitrary odd dimension.

Given an ample class $\beta$ in $\text{Pic}(\mathbb{P}_{\Sigma}^{2k+1})$, one considers sections $f \in \mathbb{P}(H^0(\mathbb{P}_{\Sigma}^{2k+1}(\beta)))$ such that $X_f = \{ f = 0 \}$ is a quasi-smooth hypersurface. Let $\mathcal{U}_\beta \subset \mathbb{P}(H^0(\mathbb{P}_{\Sigma}^{2k+1}(\beta)))$ be the open subset parameterizing quasi-smooth hypersurfaces and let $\pi : \chi_\beta \to \mathcal{U}_\beta$ be the tautological family. One considers the local system $\mathcal{H}^{2k} = R^{2k} \pi_* \mathcal{O}_{\mathcal{U}_\beta}$ over $\mathcal{U}_\beta$. The associated flat connection (the Gauss-Manin connection) will be denoted by $\nabla$.

Let $0 \neq \lambda_f \in H^{k,k}(X_f, \mathbb{Q})/i^* (H^{k,k}(\mathbb{P}_{\Sigma}^{2k+1}))$ and let $U$ be a contractible open subset around $f$. Finally, let $\lambda \in H^{2k}(U)$ be the section defined by $\lambda_f$ and let $\bar{\lambda}$ its image in $(\mathcal{H}^{2k}/F^k \mathcal{H}^{2k})(U)$, where $F^k \mathcal{H}^{2k} = \mathcal{H}^{2k,0} \oplus \mathcal{H}^{2k-1,1} \oplus \cdots \oplus \mathcal{H}^{k,k}$.

**Definition 1.1 (Local Noether-Lefschetz Locus).** $N_{\lambda,U}^{k,\beta} = \{ G \in U \mid \bar{\lambda}_G = 0 \}$.\footnote{A neat way to define the notion of quasi-smooth hypersurface $X$ in a toric variety $\mathbb{P}_{\Sigma}$ is to regard $\mathbb{P}_{\Sigma}$ as an orbifold: then $X$ is quasi-smooth if and only if it is a sub-orbifold of $\mathbb{P}_{\Sigma}$. Heuristically, $X$ is quasi-smooth if its only singularities are those “inherited” from $\mathbb{P}_{\Sigma}$.}

In this paper we continue the study of the Noether-Lefschetz locus and establish lower and upper bounds for the codimension of its components. In section 2 we obtain the lower bound, which, following the terminology in [4], we call the “explicit Noether-Lefschetz theorem for toric varieties.” In section 3, using the Hodge theory for hypersurfaces in complete simplicial toric varieties, and the orbifold structure of the quasi-smooth hyper-surfaces (see [1]), we establish the upper bound, extending the ideas in [4].

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2 Explicit Noether-Lefschetz theorem in toric varieties

This section is a natural extension to higher dimensions of the ideas developed in [4, 10] for the case of threefolds. To this end there are two points to consider:

1. Let $S = \oplus \beta S_\beta$ be the Cox ring of the toric variety $\mathbb{P}_\Sigma^3$ under consideration. In [4, 10] the following assumption was made. Let $\beta$ and $\eta$ be ample classes in Pic($\mathbb{P}_\Sigma^3$), with $\eta$ primitive and 0-regular (in the sense of Castelnuovo regularity), and $\beta - \beta_0 = n\eta$ for some $n \geq 0$, where $\beta_0$ is the anticanonical class of $\mathbb{P}_\Sigma^3$. Then one assumes that the multiplication map $S_\beta \otimes S_{n\eta} \rightarrow S_{\beta + n\eta}$ is surjective; this implies that a very general quasi-smooth surface of degree $\beta$ in $\mathbb{P}_\Sigma^3$ has the same Picard number as $\mathbb{P}_\Sigma^3$. In the higher dimensional case, if we assume again the surjectivity of the multiplication map, using Theorem 10.13 and Proposition 13.7 in [1], and Lemma 3.7 in [3], one proves that the primitive cohomology of degree $2k$ of a very general quasi-smooth hypersurface of degree $\beta$ is zero. Of course we recover the result of [8] when $k = 1$.

2. In [3, 10] it also was assumed that $H^1(O_{\mathbb{P}_\Sigma^3}(\beta - \eta)) = H^2(O_{\mathbb{P}_\Sigma^3}(\beta - 2\eta)) = 0$, which allowed one to conclude that a certain vector bundle is 1-regular with respect to $\eta$. We assume

$$H^q(O_{\mathbb{P}_\Sigma^{2k+1}}(\beta - q\eta)) = 0 \quad \text{for} \quad 1 \leq q \leq 2k$$

and will prove the same regularity for that vector bundle.

The next Theorem establishes the lower bound for the codimension of the components of the Noether-Lefschetz locus.

**Theorem 2.1.** Let $\mathbb{P}_\Sigma^{2k+1}$ be a Gorenstein projective simplicial toric variety, $\eta$ a 0-regular primitive ample Cartier class, and $\beta$ a Cartier class such that $k\beta - \beta_0 = n\eta$ ($n > 0$), where $\beta_0$ is the anticanonical class of $\mathbb{P}_\Sigma^{2k+1}$. Assume that the multiplication morphism $S_\beta \otimes S_{n\eta} \rightarrow S_{\beta + n\eta}$ is surjective, and that $H^q(O_{\mathbb{P}_\Sigma^{2k+1}}(\beta - q\eta)) = 0$ for $q = 1, \ldots, 2k$; then

$$\text{codim} \ Z \geq n + 1$$

for every irreducible component $Z$ of the Noether-Lefschetz locus $NL_{k,\beta}^{k,\alpha,U}$.

**Proof.** The proof is a higher dimensional generalization of that in [4] (which in turn largely mimics the proof of [7, 8] for the case of $\mathbb{P}^3$), with the modification proposed in [10]. We take a base point free linear system $W$ in $H^0(O_{\mathbb{P}_\Sigma^{2k+1}}(\beta))$ and a complete flag of linear subspaces

$$W = W_c \subset W_{c-1} \subset \cdots \subset W_1 \subset W_0 = H^0(O_{\mathbb{P}_\Sigma^{2k+1}}(\beta)).$$

4
Let $M_i$ be the kernel of the surjective map $W_i \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} \to \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} (\beta)$, which is locally free. We have to prove that $M_0$ is 1-regular with respect to $\eta$, i.e., that $H^q(M_0((1-q)\eta)) = 0$ for every positive $q$ (this is the regularity property we hinted at in the introduction). Taking cohomology from

$$0 \to M_0 \to W_0 \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} \to \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} (\beta) \to 0$$

we get

$$0 \to H^0(M_0) \to H^0(W_0 \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}) \to H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} (\beta)) \to H^1(M_0) \to 0;$$

as $\pi$ is surjective, $H^1(M_0) = 0$. The vanishing of $H^q(M_0((1-q)\eta)) = 0$ for $1 < q \leq 2k + 1$ is obtained by induction, tensoring the short exact sequence (1) by $\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} ((1-q)\eta)$, and considering the segment of the long exact sequence of cohomology

$$\cdots \to H^{q-1}(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} (\beta - (q-1)\eta)) \to H^q(M_0(-(q-1)\eta)) \to H^q(W_0 \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(-(q-1)\eta)) \to \cdots$$

where $H^{q-1}(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma} (\beta - (q-1)\eta)) = 0$ by the inductive assumption, while

$$H^q(W_0 \otimes \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(-(q-1)\eta)) = 0$$

as $\eta$ is 0-regular.

The rest of the proof follows as in [4, 10].

### 3 Upper bound for the Codimension of the Noether-Lefschetz Components in Toric Varieties

The Explicit Noether-Lefschetz Theorem has provided a lower bound for the codimension of the Noether-Lefschetz components. Hodge theory in toric varieties will give us the upper bound. For a class $\beta$ as in the previous Section, let $f$ be a point in the Noether-Lefschetz locus, let $X_f$ be the corresponding hypersurface in $\mathbb{P}^{2k+1}_\Sigma$, and let $\lambda$ be a class as in Definition 1.1.

**Theorem 3.1.** $\text{codim} Z \leq h^{k-1,k+1}(X_f)$ for every irreducible component $Z$ of the Noether-Lefschetz locus $NL_{k,\beta}^{k,\lambda, U}$.

This section is devoted to proving this theorem. Classically it is a consequence of Griffiths’ Transversality, which we want to extend to the context of projective simplicial toric varieties.

**Variations of Hodge Structure.** The tautological family $\pi : \mathcal{X}_{\beta} \subset \mathcal{U}_{\beta} \times \mathbb{P}_\Sigma \to \mathcal{U}_{\beta}$ is of finite type and separated since $\mathcal{X}_{\beta}$ and $\mathcal{U}_{\beta}$ are varieties. By Corollary 5.1 in [4] there exists a Zariski open set $\mathcal{U} \subset \mathcal{U}_{\beta}$ such that $\mathcal{X} = \pi^{-1}(\mathcal{U}) \to \mathcal{U}$ is a locally trivial fibration in the
classical topology, i.e., there exists an open cover of $U$ by contractible open sets such that for every element $U$ of the cover and every point $X_0 \in U$ we have $X_u \simeq \pi^{-1}(U) \simeq U \times X_0$, which implies that $X_u \simeq X_0$ for all $u \in U$ as $C^\infty$ orbifolds; moreover, $H^k(X_u) \simeq H^k(X_0)$. Thanks to the locally trivialization and as quasi-smooth hypersurfaces are orbifolds $[1]$, we can put an orbifold structure on $X = \pi^{-1}(U)$.

The Cartan-Lie formula. For every $k$, let $\mathcal{H}^k$ be the complex vector bundle on $U_\beta$ associated to the local system $R^k\pi_*\mathbb{C}$. Let $\Omega$ be a Zariski $k$-form on the orbifold $X$ such that $\Omega_u = \Omega_{\mid X_u}$ is closed for every $u \in U$; we can associate with it a local section $\omega$ of the vector bundle $\mathcal{H}^k$ by letting $\omega(u) = [\Omega_u] \in H^k(X_u, \mathbb{C})$.

The following result computes the Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_U$ in the direction $w$ restricted to $X_0$.

**Proposition 3.2 (Cartan-Lie Formula).** If $w \in T_{\mathcal{U},X_0}$ and $v \in \Gamma(T_{\mathcal{X}_{\mid X_0}})$ is such that $\phi_{\ast,x}(v) = w$ for all $x \in X_0$, one has

$$\nabla_w(\omega) = \left[\iota_v(d\Omega)_{\mid X_0}\right]$$

**Proof.** See [12]; actually the proof goes as in the classical case, see Proposition 9.2.2 in [16].

Again we take $U$ a contractible open set trivializing $\mathcal{X}_{\mid U} \simeq U \times X_0$.

**Definition 3.3.** The period map

$$\mathcal{P}^{p,k} : \mathcal{U} \to \text{Grass}(b^{p,k}, H^k(X, \mathbb{C}))$$

is the map which to $u \in U$ associates the term $F^p H^k(X_u, \mathbb{C})$ in the Hodge filtration of $H^k(X_u, \mathbb{C}) \simeq H^k(X_0, \mathbb{C})$.

Here $b^{p,k} = \dim F^p H^k(X_u, \mathbb{C})$. Note that $\mathcal{P}^{p,k}$ is a map of complex manifolds.

**Proposition 3.4.** The period map $\mathcal{P}^{p,k}$ is holomorphic.

**Proof.** For the reader’s convenience we sketch here a proof of this result, although it has been actually already proved in [12]. By Theorem 7.9 in [9] and the fact that Hodge theorem holds also in the orbifold case ([13] [18] and also section 2.1 in [11]) $\mathcal{P}^{p,k}$ is a $C^\infty$ map. The rest of the proof follows as in Theorem 10.9 in [16], whose strategy is to prove that the $\mathbb{C}$-linear extension of the differential to $T_u \mathcal{U} \otimes \mathbb{C}$ of $\mathcal{P}^{p,k}$ vanishes on the vectors of type $(0,1)$.

**Remark 3.5.** There is an intrinsic relation between the differential

$$d\mathcal{P}^{p,k}_u(w) : F^p H^k(X_u) \to H^k(X_0)/F^p H^k(X_u)$$
and the covariant derivative $\nabla_w : \mathcal{H}^k \to \mathcal{H}^k$, namely, given $\sigma \in F^p H^k(X_u)$ one can construct a local section of $\mathcal{H}^k$ over $U$

$$\tilde{\sigma} : U \to H^k(X_u)$$
$$u' \mapsto \tilde{\sigma}(u') \in F^p H(X_{u'})$$

such that $\tilde{\sigma}(u) = \sigma$. Hence,

$$d P^p_w (\sigma) = \nabla_w \tilde{\sigma} \mod F^p H^k(X_u)$$

Proposition 3.6 (Griffiths Transversality).

$$\nabla F^p H^k \subset F^{p-1} H^k$$

Proof. By the Cartan-Lie formula and the above remark

$$d P^p_w (\sigma) = [\iota_v d \Omega]_{X_0} \mod F^p H^k(X_u).$$

The fact that $P^p \mathcal{H}$ is holomorphic implies that that $\iota_v d \Omega|_{X_0} \in F^p H^k(X_u)$ if $v$ is of type $(0, 1)$, so that if $v$ is of type $(1, 0)$ we get $\iota_v d \Omega|_{X_0} \in F^{p-1} H^k(X_u)$.

Theorem 3.7. Each $N L^{k, \beta}_{\lambda, \mu} \subset \mathcal{U}$ can be defined locally by $h_{k-1, k+1}$ holomorphic equations, where $h_{k-1, k+1} = \text{rk } F^{k-1} H^{2k} / F^k H^{2k}$. 

Proof. Once Griffiths Transversality has been generalized, the proof goes as in classical case, see Lemma 3.1 in [15] and section 5.3 in [17].

This proves Theorem 3.1.

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