Sharp Critical and Subcritical Trace Trudinger–Moser and Adams Inequalities on the Upper Half-Spaces

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Abstract
In this paper, we establish the sharp critical and subcritical trace Trudinger–Moser and Adams inequalities on the half-spaces and prove the existence of their extremals through the method based on the Fourier rearrangement, harmonic extension and scaling invariance. These trace Trudinger–Moser (Theorems 1.1 and 1.2) and trace Adams inequalities (Theorems 1.4, 1.5, 1.10 and 1.11) can be considered as the borderline case of the Sobolev trace inequalities of first and higher orders on half-spaces. Furthermore, as an application, we show the existence of the least energy solutions for a class of bi-harmonic equations with nonlinear Neumann boundary condition associated with the trace Adams inequalities (Theorem 1.13). It is interesting to note that there are two types of trace Trudinger–Moser and trace Adams inequalities: critical and subcritical trace inequalities under different constraints. Moreover, trace Trudinger–Moser and

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trace Adams inequalities of exact growth also hold on half-spaces (Theorems 1.6, 1.8 and 1.12).

**Keywords** Trace Trudinger–Moser inequality · Trace Adams inequality · Nonlinear Neumann boundary condition · Harmonic extension · Pohozaev identity · Ground state · Fourier rearrangement

**Mathematics Subject Classification** 35J60 · 35B33 · 46E30

1 Introduction

The main content of this paper is concerned with the problem of finding optimal trace Trudinger–Moser and Adams inequalities and existence of their extremals in unbounded domains. Sharp Trudinger–Moser inequalities and its high-order form (Adams inequalities) have been a subject of intensive research due to the importance of these inequalities in applications to problems in analysis, PDEs, differential geometry, mathematical physics, etc.

The Trudinger inequality as the critical case of the Sobolev imbedding was obtained by Trudinger [58], Pohozaev [54] and Yudovic [60]. More precisely, Trudinger employed the power series expansion to prove that there exists $\alpha > 0$ such that

$$\sup_{\|\nabla u\|_n \leq 1, u \in W^{1,n}_0(\Omega)} \int_{\Omega} \exp\left(\alpha|u|^\frac{n}{n-1}\right) \, dx < \infty, \quad (1.1)$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $W^{1,n}_0(\Omega)$ denotes the usual Sobolev space, i.e, the completion of $C^\infty_0(\Omega)$ with the norm

$$\|u\|_{W^{1,n}_0(\Omega)} = \left( \int_{\Omega} |\nabla u|^n \, dx \right)^{\frac{1}{n}}.$$

Subsequently, Moser [50] utilized the technique of the symmetry and rearrangement to give the sharp constants $\alpha_n = nw^{\frac{1}{n-1}}$ of the above inequality. Such an inequality in the sharp form of constant $\alpha_n$ is now called the Trudinger–Moser inequality. (see also Trudinger–Moser inequality on surfaces with conic singularity in [10], on the Heisenberg group [20] and CR sphere [21].) Moser’s inequalities have many applications in PDEs since decades ago. (see e.g. [5, 8], [56], etc.) Since the Polyá-Szegö inequality on which the symmetrization argument depends is not valid on high-order Sobolev spaces, this adds much challenge to the research of high-order Trudinger–Moser inequalities, namely Adams type inequalities. Adams employed the representation formulas together with the method of the rearrangement of convolutions by O’Neil [53] to establish the sharp high-order Trudinger–Moser inequality on any bounded domain. More precisely, he proved the following result.

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Theorem A ([4]) Let \( \Omega \) be an open and bounded set in \( \mathbb{R}^n \). If \( m \) is a positive integer less than \( n \), then there exists a constant \( C_0 = C(n, m) > 0 \) such that for any \( u \in W^{m, \frac{m}{n}}_0(\Omega) \) and \( \|\nabla^m u\|_{\frac{n}{m}} \leq 1 \), then

\[
\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{n-m}}) \, dx \leq C_0.
\]

for all \( \beta \leq \beta(n, m) \), where

\[
\beta(n, m) = \begin{cases} 
\frac{n}{\omega_{n-1}} \left( \frac{\pi}{2} \frac{2^n \Gamma \left( \frac{m+1}{2} \right)}{\Gamma \left( \frac{m-m+1}{2} \right)} \right) \frac{n}{n-m}, & \text{if } m \text{ is odd.} \\
\frac{n}{\omega_{n-1}} \left( \frac{\pi}{2} \frac{2^n \Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{n-m+1}{2} \right)} \right) \frac{n}{n-m}, & \text{if } m \text{ is even.}
\end{cases}
\]

Furthermore, the constant \( \beta(n, m) \) is best possible in the sense that for any \( \beta > \beta(n, m) \), the integral can be made as large as possible.

Subsequently, Moser’s results for the first-order derivatives and Adams’ result for the high-order derivatives have been extended to unbounded domains. The first-order Trudinger–Moser inequality was proved in [2, 8, 22] in the subcritical case, that is for any \( \alpha < \alpha_n \), there exists a constant \( C = C(\alpha, n) \) such that

\[
\sup_{\|\nabla u\|_n^n \leq 1} \int_{\mathbb{R}^n} \Phi(\alpha |u|^{\frac{n}{n-1}}) \, dx \leq C \|u\|_n^n,
\]

where \( \Phi(t) = e^t - \sum_{j=0}^{n-2} t^j \).

Later, it is showed in [38] that the exponent \( \alpha_n \) becomes admissible if the Dirichlet norm \( \left( \int_{\mathbb{R}^n} |\nabla u|^n \, dx \right)^{\frac{1}{n}} \) is replaced by Sobolev norm \( \|u\|_{W^{1,n}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (|u|^n + |\nabla u|^n) \, dx \right)^{\frac{1}{n}} \); more precisely, they proved the following critical Trudinger–Moser inequality

\[
\sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \Phi(\alpha_n |u|^{\frac{n}{n-1}}) \, dx \leq C_n < \infty.
\]

All the proofs of both critical and subcritical Trudinger–Moser inequalities in the literature rely on the rearrangement argument and the Polyá–Szegő inequality. In [28, 29], Lam and Lu used a symmetrization-free approach to give a simple proof for both critical and subcritical sharp Trudinger–Moser inequalities in \( W^{1,n}(\mathbb{R}^n) \). It should be noted that this approach is surprisingly simple and can be easily applied to other settings where symmetrization argument does not work. Furthermore, they also develop this new approach to establish in [28] the global Trudinger–Moser inequalities on the entire Heisenberg group from the local one derived in [20] and fractional Adams inequalities.
inequalities in $W^{s, \frac{2}{s}}(\mathbb{R}^n)$ ($0 < s < n$) in [29] (see [28, 29, 31, 42, 45, 61] in other settings) and apply this method to obtain the concentration-compactness principle for Trudinger–Moser inequalities on the Heisenberg group and Riemannian manifolds [39, 40] and for Adams inequalities in the whole Euclidean space [11], Trudinger–Moser and Adams inequalities under the constraint of the Sobolev norm associated to a degenerate potential [15–17]. For more applications of the symmetrization-free approach, please also see [32, 36, 39, 62]. Moreover, the critical and subcritical Trudinger–Moser inequalities are equivalent as shown in [32].

As far as the existence of extremal functions of Trudinger–Moser inequality on bounded domains, the first existence result was established on balls by Carleson and Chang in [9]. (See also [24, 41] for smooth domains.)

Since the main purpose of this paper is to establish the trace Trudinger–Moser and Adams type inequalities on unbounded domains, we will only briefly review the trace Trudinger–Moser and Adams inequalities on bounded domains. Let $\Omega_1$ be a sufficiently smooth bounded domain in $\mathbb{R}^n$ with smooth boundary and define the $W^{1, 2}(\Omega_1)$ to be the Sobolev space, equipped with the norm

$$
\|u\|_{W^{1, 2}(\Omega_1)} = \left( \int_{\Omega_1} \left( |\nabla u|^2 + |u|^2 \right) dx \right)^{\frac{1}{2}}.
$$

As is well known, classical Sobolev trace embedding on bounded domain asserts that $W^{1, 2}(\Omega_1) \subset L^q(\partial \Omega_1)$ for $1 \leq q \leq \frac{2(n-1)}{n-2}$ and $n \geq 3$. In dimension $n = 2$, the embedding $W^{1, 2}(\Omega_1) \subset L^q(\partial \Omega_1)$ holds for $1 \leq q < \infty$, but $W^{1, 2}(\Omega_1) \not\subset L^\infty(\partial \Omega_1)$. To fill this gap, the authors of [3, 5, 18, 19, 35, 37] studied the trace Trudinger–Moser inequalities of the following type:

**Theorem B** Let $\Omega$ be a sufficiently smooth bounded domain in $\mathbb{R}^2$. For any $\alpha \leq \pi$, there exists a positive constant $C$ such that

$$
\int_{\partial \Omega} \exp(\alpha |u|^2) d\sigma \leq C,
$$

(1.5)

for any $u \in C^\infty(\Omega)$ with $\int_\Omega (|\nabla u|^2 + |u|^2) dx \leq 1$.

In a half-space, a particular unbounded domain with the smooth boundary, sharp geometrical inequalities such as trace Sobolev inequalities on this domain have attracted much attention due to their importance in geometrical analysis and PDEs (see [1, 6, 23, 51]). We recall some classical results that involve the sharp trace Sobolev inequality on the half-space. Sharp trace Sobolev inequalities for first-order derivative of Escobar [23] assert that
**Theorem C** ([23]) Let $n \geq 3$. Then for any $U \in W^{1,2}(\mathbb{R}^n_+)$, there holds

$$2 \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} w^{1/n-1}_{n-1} \left( \int_{\partial \mathbb{R}^n_+} |U(x,0)|^{\frac{2(n-1)}{n-2}} \, dx \right)^{\frac{n-2}{n-1}} \leq \int_{\mathbb{R}^n_+} |\nabla U(x,0)|^2 \, dx, \quad (1.6)$$

where $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}^+$. Furthermore, the equality case is achieved by a harmonic extension of a function of the form $c(1 + |x - x_0|^2)^{-\frac{n-2}{2}}$, where $c$ is a constant, $x \in \mathbb{R}^{n-1}$, $x_0$ is some fixed point in $\mathbb{R}^{n-1}$.

By conformal map, this inequality is in fact equivalent to the trace Sobolev inequality on the ball (see Beckner [6]),

$$\frac{n-2}{2} \left( \int_{\mathbb{S}^{n-1}} |f|^2 \, d\sigma \right)^{\frac{n-2}{n-1}} \leq \int_{\mathbb{R}^n} |\nabla v|^2 \, dx + \frac{n-2}{2} \int_{\mathbb{S}^{n-1}} |f|^2 \, d\sigma, \quad (1.7)$$

where $v$ is a smooth extension of $f$ to $\mathbb{B}^n$ and $d\sigma$ is the surface measure on $\mathbb{S}^{n-1}$.

Motivated by the above works, we are concerned with the borderline case of trace Sobolev inequality, that is the sharp trace Trudinger–Moser inequalities in $W^{1,2}(\mathbb{R}^2_+)$. We first establish the subcritical and critical trace-type Trudinger–Moser inequalities on the half-space. Our results read as follows.

**Theorem 1.1** For $0 < \beta < \pi$, then there exists a positive constant $C$ such that for all functions $U \in W^{1,2}(\mathbb{R}^2_+)$ with $\int_{\mathbb{R}^2_+} |\nabla U(x,y)|^2 \, dx \, dy \leq 1$, the following inequality holds

$$\int_{\partial \mathbb{R}^2_+} (\exp(\beta|U(x,0)|^2) - 1) \, dx \leq C \int_{\partial \mathbb{R}^2_+} |U(x,0)|^2 \, dx. \quad (1.8)$$

Furthermore, the smallest $C$ in (1.8) can be attained by some $U$, which is a harmonic extension of some even function $u \in W^{1,2}((0,\infty))$. Moreover, the constant $\pi$ is sharp in the sense that the inequality (1.8) fails if the constant $\beta$ is replaced by any $\beta \geq \pi$.

**Theorem 1.2** There exists a positive constant $C$ such that for all functions $U \in W^{1,2}(\mathbb{R}^2_+)$ with

$$\int_{\mathbb{R}^2_+} |\nabla U(x,y)|^2 \, dx \, dy + \int_{\partial \mathbb{R}^2_+} |U(x,0)|^2 \, dx \leq 1,$$
the following inequality holds
\[ \int_{\partial \mathbb{R}^2_+} \exp(\pi |U(x,0)|^2) d\sigma \leq C. \] (1.9)

Furthermore, the smallest $C$ in (1.9) can be attained by some function $U$, which is a harmonic extension of some radial function $u \in W^{1,2}(\mathbb{R}^2)$. Moreover, the constant $\pi$ is sharp in the sense that the inequality (1.9) fails if the constant $\pi$ is replaced by any $\beta > \pi$.

**Remark 1.3** The difference between Theorems 1.1 and 1.2 is that we assume only $\int |\nabla U(x, y)|^2 dx dy \leq 1$ in Theorem 1.1 and assume $\int |\nabla U(x, y)|^2 dx dy + \int_{\partial \mathbb{R}^2_+} |U(x,0)|^2 d\sigma \leq 1$ in Theorem 1.2. This results in a subcritical trace Trudinger–Moser inequality in Theorem 1.1 and a critical trace Trudinger–Moser inequality in Theorem 1.2.

Recently, Ache and Chang [1] established the second-order sharp trace Sobolev inequality (1.7) on the ball:

**Theorem D** Let $u \in C^\infty(S^{n-1})$ with $n > 4$. Suppose that $v$ is a smooth extension of $u$ to the unit ball $B^n$ which satisfies the Neumann boundary condition
\[ \frac{\partial v}{\partial n}|_{S^{n-1}} = -\frac{n - 4}{2} u. \]

Then the following inequality holds
\[ 2 \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n-4}{2}\right)} w^{\frac{3}{n-1}} \left( \int_{S^{n-1}} |u|^{\frac{2(n-1)}{n-4}} d\sigma \right)^{\frac{n-4}{n-1}} \leq \int_{B^n} |\Delta v|^2 dx + 2 \int_{S^{n-1}} |\nabla u|^2 d\sigma + \frac{n(n - 4)}{2} \int_{S^{n-1}} |u|^2 d\sigma, \] (1.10)

where $\Delta v$ is the Laplacian of $v$ with respect to the Euclidean metric, and $\nabla$ is the gradient on the sphere.

Let $\tilde{\Delta}$ be the Laplace–Beltrami operator on the standard sphere $(S^n, g_{S^n})$ and define
\[ \mathcal{P}_{2\gamma} = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)} \quad \text{and} \quad B = \sqrt{-\tilde{\Delta} + \frac{(n - 1)^2}{4}}. \]
We also denote by \((a)_k\) the rising Pochhammer symbol defined by

\[
(a)_0 = 0, \quad (a)_k = a(a + 1) \cdots (a + k - 1), \quad k \geq 1.
\]

More recently, Yang extended the result of [1] to higher-order derivatives and established the following in [59]:

**Theorem E** Let \( n > 3 \) and \( m \geq 1 \) with \( 2m + 1 < n \). Given \( f \in C^\infty(S^n) \). Suppose \( v \) is a smooth extension of \( f \) to the unit ball \( B^{n+1} \) which also satisfies the Neumann boundary condition:

\[
\Delta^k v|_{S^n} = (-1)^k \frac{\Gamma(m + 1)\Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2})\Gamma(m - k + 1)} P_{2m+1-2k}^{-1} P_{2m+1} f; \quad 0 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor;
\]

\[
\frac{\partial}{\partial n} \Delta^k v|_{S^n} = (-1)^{k+1} \frac{n - 1 - 2m + 2k}{2} \frac{\Gamma(m + 1)\Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2})\Gamma(m - k + 1)} P_{2m+1-2k}^{-1} P_{2m+1} f, \quad 0 \leq k \leq \left\lfloor \frac{m - 1}{2} \right\rfloor.
\]

(1.11)

Then we have the inequality

\[
\frac{\Gamma(m + 1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m + \frac{1}{2})} \frac{\Gamma\left(\frac{n+2m+1}{2}\right)}{\Gamma\left(\frac{n-2m-1}{2}\right)} \omega_n \frac{2m+1}{n} \left( \int_{S^n} |f|^{\frac{2n}{n-2m-1}} d\sigma \right)^{\frac{n-2m-1}{n}} 
\]

\[
\leq \int_{B^{n+1}} |\nabla^{m+1} v|^2 dx + \int_{S^n} f T_m f d\sigma,
\]

(1.12)

where

\[
\nabla^{m+1} = \begin{cases} 
\Delta^{\frac{m+1}{2}}, & m = \text{odd}; \\
\n\Delta \nabla^{\frac{m}{2}}, & m = \text{even},
\end{cases}
\]

and \( T_m \) is an operator of order \( 2m \) defined as follows: if \( m \) is an odd integer, then

\[
T_m = \frac{n - 1}{2} \frac{\Gamma(m + 1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m + \frac{1}{2})} P_1^{-1} P_{2m+1} + \sum_{k=1}^{m-1} (m - 2k) \frac{\Gamma(m + 1)^2}{\Gamma(m + \frac{1}{2})^2} \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{m-k+\frac{1}{2}}{2}\right)}{\Gamma(k+1)\Gamma(m-k+1)} P_{2m+1-2k}^{-1} P_{2k+1}^{-1} P_{2m+1}^{2}.
\]

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if \( m \) is an even integer, then

\[
T_m = \frac{n - 1}{2} \frac{\Gamma(m + 1) \Gamma\left(\frac{1}{2}\right)}{\Gamma(m + \frac{1}{2})} p_1^{-1} p_{2m+1}^{-1} + \frac{n - 1 - m}{2} \left( \frac{\Gamma(m + 1) \Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m + \frac{1}{2}) \Gamma\left(\frac{m+1}{2}\right) + 1} \right)^2 p_{m+1}^{-2} p_{2m+1}^2 + \sum_{k=1}^{\frac{m}{2} - 1} \frac{\Gamma(m + 1)^2 \Gamma(k + \frac{1}{2}) \Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2}) \Gamma(k + 1) \Gamma(m - k + 1)} \frac{p_{m+1}}{p_{2m+1} - 2k} \frac{p_{2m+1}}{p_{2k+1}^2}.
\]

Moreover, equality (1.12) holds if and only if

\[
v(x) = c \int_{\mathbb{S}^n} \frac{(1 - |x|^2)^{2m+1}}{|x - \xi|^n + 2m^2} |1 - (x_0, \xi)|^{2m+1-n} d\sigma,
\]

where \( c \) is a constant and \( x_0 \) is some point in \( \mathbb{R}^{n+1} \). When \( f = 1 \), inequality (1.12) is attained by the function

\[
v(x) = \sum_{k=0}^{m} \frac{\left(\frac{n-1}{2} - m\right)_k (-m)_k}{(-2m)_k} \left(2p\right)_k \frac{p_{m+1}}{p_{2m+1} - 2k} \frac{p_{2m+1}}{p_{2k+1}^2},
\]

where \( (a)_k \) is the rising Pochhammer symbol.

By the Mobius transform, the second-order inequality (1.10) has been shown to be equivalent to the following trace Sobolev inequality on the upper half-space \( \mathbb{R}_n^+ \).

**Theorem F** ([51]) Let \( n \geq 5 \). Then for any \( U \in W^{2,2}(\mathbb{R}_n^+) \) satisfying the Neumann boundary condition \( \partial_y U(x, y)|_{y=0} = 0 \), we have

\[
2 \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n-4}{2}\right)} w_n^{\frac{3}{n-1}} \left( \int_{\partial\mathbb{R}_n^+} |U(x, 0)|^{\frac{2(n-1)}{n-4}} dx \right)^{\frac{n-4}{n-1}} \leq \int_{\mathbb{R}_n^+} |\Delta U(x, y)|^2 dx dy. \tag{1.14}
\]

Furthermore, the equality is achieved by a bi-harmonic extension of a function of the form \( c(1 + |x - x_0|^2)^{-\frac{n-4}{2}} \), where \( c \) is a constant, \( x \in \mathbb{R}^{n-1} \), \( x_0 \) is some fixed point in \( \mathbb{R}^{n-1} \).

In this paper, we first establish the borderline case of the inequality (1.14) when \( n = 4 \). We employ the method based on the sharp Fourier rearrangement principle and bi-harmonic extension to establish the sharp trace Adams inequalities on the half-space and the existence of their extremals. Both subcritical and critical trace Adams inequalities are proved for functions in \( W^{2,2}(\mathbb{R}_n^+) \).

Our first result in the second-order trace Adams inequality is the following subcritical inequality.
Theorem 1.4 Let \(0 < \beta < \frac{12\pi^2}{2}\). Then there exists a positive constant \(C\) such that for all functions \(U \in W^{2,2}(\mathbb{R}^4_+)\) satisfying the Neumann boundary condition \(\partial_y U(x, y)|_{y=0} = 0\) with \(\int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy \leq 1\), the following inequality holds:

\[
\int_{\mathbb{R}^4_+} (\exp(\beta|U(x, 0)|^2) - 1) \, dx \leq C \int_{\mathbb{R}^4_+} |U(x, 0)|^2 \, dx. \tag{1.15}
\]

Furthermore, the smallest \(C\) in (1.15) can be attained by some function \(U\), which is the bi-harmonic extension of some radial function \(u \in W^{3,2}_{\Delta 1}(\mathbb{R}^3)\). Moreover, the constant \(12\pi^2\) is sharp in the sense that the inequality fails if the constant \(\beta\) is replaced by any \(\beta \geq \frac{12\pi^2}{2}\).

Next, the following critical trace Adams inequality also holds.

Theorem 1.5 There exists a positive constant \(C\) such that for all functions \(U \in W^{2,2}(\mathbb{R}^4_+)\) satisfying the Neumann boundary condition \(\partial_y U(x, y)|_{y=0} = 0\) with

\[
\int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy + \int_{\mathbb{R}^4_+} |U(x, 0)|^2 \, dx \leq 1,
\]

the following inequality holds

\[
\int_{\mathbb{R}^4_+} \exp(12\pi^2|U(x, 0)|^2 - 1) \, dx \leq C. \tag{1.16}
\]

Moreover, the constant \(12\pi^2\) is sharp in the sense that the inequality fails to hold uniformly for all \(U \in W^{2,2}(\mathbb{R}^4_+)\) if the constant \(12\pi^2\) is replaced by any \(\beta > 12\pi^2\).

The second-order Adams inequality with the exact growth was established in \(W^{2,2}(\mathbb{R}^4)\) in [47] and then was extended to \(W^{2,n}(\mathbb{R}^n)\) in [44] (see also the first-order case in [26, 43, 48] and singular inequalities and under different norms with exact growth in [30, 33]). A natural question is whether there exists trace Adams inequality with the exact growth in \(W^{2,2}(\mathbb{R}^4_+)\). In this paper, we establish the following

Theorem 1.6 There exists a positive constant \(C\) such that for all functions \(U \in W^{2,2}(\mathbb{R}^4_+)\) satisfying the Neumann boundary condition \(\partial_y U(x, y)|_{y=0} = 0\) with \(\int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy = 1\), the following inequality holds

\[
\int_{\mathbb{R}^4_+} \frac{(\exp(12\pi^2|U(x, 0)|^2) - 1)}{(1 + |U(x, 0)|)^2} \, dx \leq C \int_{\mathbb{R}^4_+} |U(x, 0)|^2 \, dx. \tag{1.17}
\]

Moreover, this inequality fails to hold uniformly for all \(U \in W^{2,2}(\mathbb{R}^4_+)\) if the power 2 in the denominator of the left-hand side is replaced by any \(p < 2\).
Remark 1.7 The proof of Theorem 1.6 is based on bi-harmonic extension and fractional Adams inequalities with the exact growth in $W^{2,2}_n(\mathbb{R}^n)$ for $n = 3$. Adams inequalities with the exact growth in $W^{2,2}_n(\mathbb{R}^n)$ for even $n$ has been established in [49] (see also [52]). However, the validity of this inequality for odd $n$ still remains open. Using the fractional Hardy–Rellich inequalities established by Beckner in [7] together with the method of the reduction of orders (see also [52]), we present a simple proof for the fractional Trudinger–Moser–Adams inequality with the exact growth in $W^{2,2}_n(\mathbb{R}^n)$ for all $n \geq 2$.

Theorem 1.8 There exists a positive constant $C$ such that for all functions $u \in W^{2,2}_n(\mathbb{R}^n)$ with $\int |(-\Delta)^{\frac{n}{2}} u|^2 dx = 1$, the following inequality holds

$$\int_{\mathbb{R}^n} \frac{(\exp(\beta(n, \frac{n}{2})|u(x)|^2) - 1)}{(1 + |u(x)|^2)^2} dx \leq C \int_{\mathbb{R}^n} |u(x)|^2 dx. \quad (1.18)$$

Moreover, this inequality fails if the power 2 in the denominator is replaced by any $p < 2$.

Furthermore, we also establish the trace Adams inequalities on $W^{m,2}_{2m}(\mathbb{R}^{2m})$ ($m > 2$). The proof is based on the following technical lemma whose proof will be given in Sect. 5.

Lemma 1.9 Given a function $u \in W^{m-\frac{1}{2},2}_{2}(\mathbb{R}^{2m-1})$. If we assume that the function $U \in W^{m,2}_{2}(\mathbb{R}^{2m})$ satisfying

$$(-\Delta)^m U = 0$$
on the upper half-space $\mathbb{R}^{2m}_+$ and the Neumann boundary conditions

$$\Delta^k U(x, y) \big|_{y=0} = \frac{\Gamma(m)\Gamma\left(\frac{m}{2} - k\right)}{\Gamma(m - 1 - k)\Gamma\left(\frac{m}{2}\right)} \Delta^k u(x), \quad 0 \leq k \leq \left[\frac{m - 1}{2}\right]$$

(1.19)

and

$$\partial_y \Delta^k U(x, y) \big|_{y=0} = 0, \quad 0 \leq k \leq \left[\frac{m - 2}{2}\right],$$

(1.20)

then we have the following identity

$$\int_{\mathbb{R}^{2m}_+} |\nabla^m U(x, y)|^2 dx dy = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m - \frac{1}{2})} \int_{\mathbb{R}^{2m-1}} |(-\Delta)^{\frac{m-1}{4}} u|^2 dx.$$
Once we have proved Lemma 1.9, we can establish the following both subcritical and critical trace Adams inequalities on $W^{m,2}(\mathbb{R}^{2m}_+)$ ($m > 2$) by a similar argument as for Theorems 1.4, 1.5 and 1.6 on $W^{2,2}(\mathbb{R}^m_+)$. 

**Theorem 1.10** Let $0 < \beta < \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m-\frac{1}{2})} \beta(2m - 1, \frac{2m-1}{2}) = \tilde{\beta}_m$. Then there exists a positive constant $C$ such that for all functions $U \in W^{m,2}(\mathbb{R}^{2m}_+)$ satisfying the Neumann boundary conditions (1.19) and (1.20) with $\int_{\mathbb{R}^{2m}_+} |\nabla^m U|^2 dx dy \leq 1$, the following inequality holds:

$$\int_{\partial \mathbb{R}^{2m}_+} (\exp(\beta |U(x, 0)|^2) - 1) dx \leq C \int_{\partial \mathbb{R}^{2m}_+} |U(x, 0)|^2 dx. \quad (1.21)$$

Moreover, the constant $\tilde{\beta}_m$ is sharp in the sense that the inequality fails to hold uniformly if the constant $\beta$ is replaced by any $\beta > \tilde{\beta}_m$. Furthermore, equality in (1.15) holds if and only if $U$ is a $m$-harmonic extension of some radial function $u$ in $W^{2m-1,2}(\mathbb{R}^{2m-1})$.

**Theorem 1.11** There exists a positive constant $C$ such that for all functions $U \in W^{m,2}(\mathbb{R}^{2m}_+)$ satisfying the Neumann boundary conditions (1.19) and (1.20) with

$$\int_{\mathbb{R}^{2m}_+} |\nabla^m U(x, y)|^2 dx dy + \int_{\partial \mathbb{R}^{2m}_+} |U(x, 0)|^2 dx \leq 1,$$

the following inequality holds

$$\int_{\partial \mathbb{R}^{2m}_+} (\exp(\tilde{\beta}_m |U(x, 0)|^2) - 1) dx \leq C. \quad (1.22)$$

Moreover, the constant $\tilde{\beta}_m$ is sharp in the sense that the inequality fails to hold uniformly if the constant $\tilde{\beta}_m$ is replaced by any $\beta > \tilde{\beta}_m$.

We also establish the trace Adams inequality of exact growth.

**Theorem 1.12** There exists a positive constant $C$ such that for all functions $U \in W^{m,2}(\mathbb{R}^{2m}_+)$ satisfying the Neumann boundary condition (1.19) and (1.20) with

$$\int_{\mathbb{R}^{2m}_+} |\nabla^m U(x, y)|^2 dx dy = 1,$$
the following inequality holds
\[
\int_{\partial \mathbb{R}^{2m}_{+}} \frac{(\exp(\tilde{\beta}_m|U(x,0)|^2) - 1)}{(1 + |U(x,0)|)^2} dx \leq C \int_{\partial \mathbb{R}^{2m}_{+}} |U(x,0)|^2 dx.
\] (1.23)

Moreover, this inequality fails if the power 2 in the denominator on the left-hand side is replaced by any \( p < 2 \).

Finally, we are also concerned with the ground-state solutions to the equation associated with the trace Adams inequality in \( W^{m,2}(\mathbb{R}^{2m}_{+}) \). For simplicity, we only consider the case \( m = 2 \). The higher-order case is similar and we leave the details to the interested reader. By the Euler–Lagrange multiplier theorem, extremals of the trace Adams supremum
\[
\sup_{\int_{\mathbb{R}^4_{+}} |\Delta U(x,y)|^2 dx dy + \int_{\partial \mathbb{R}^4_{+}} |U(x,y)|^2 dx \leq 1} \int_{\partial \mathbb{R}^4_{+}} \left( \exp \left( 12 \pi^2 |U|^2 - 1 \right) \right) dx
\]
must satisfy the following Euler–Lagrange equation with nonlinear Neumann boundary conditions: there exists some \( \lambda \) such that
\[
\begin{cases}
\Delta^2 U(x, y) = 0 \text{ for } (x, y) \in \mathbb{R}^3 \times \mathbb{R}^+_+, \\
( - \frac{\partial (\Delta U)}{\partial y} + U(x, y))|_{y=0} = \lambda U(x, y) \exp(12\pi^2 U^2(x, y))|_{y=0}, \\
\frac{\partial U}{\partial y}|_{y=0} = 0.
\end{cases}
\] (1.24)

We are interested in the existence of ground-state solutions to equation (1.24) with some fixed \( \lambda \). Define
\[
E := \left\{ U \in W^{2,2}(\mathbb{R}^4_{+}) \cap L^2(\partial \mathbb{R}^4_{+}) : \frac{\partial y U(x, y)|_{y=0} = 0}{} \right\}.
\]
the corresponding functional of equation (1.24) is
\[
I_{\lambda}(U) = \frac{1}{2} \int_{\mathbb{R}^4_{+}} |\Delta U|^2 dx dy + \frac{1}{2} \int_{\partial \mathbb{R}^4_{+}} |U|^2 dx - \frac{\lambda}{24\pi^2} \int_{\partial \mathbb{R}^4_{+}} (\exp(12\pi^2 |U|^2) - 1) dx.
\] (1.25)

It is easy to check that \( I_{\lambda} \in C^1(E, \mathbb{R}) \), and
\[
I_{\lambda}'(U) V = \int_{\mathbb{R}^4_{+}} \Delta U(x, y) \Delta V(x, y) dx dy + \int_{\partial \mathbb{R}^4_{+}} U(x, 0) V(x, 0) dx \\
- \lambda \int_{\partial \mathbb{R}^4_{+}} \exp(12\pi^2 |U(x, 0)|^2) U(x, 0) V(x, 0) dx, \quad U, V \in E.
\] (1.26)
Define
\[ M_\lambda = \inf \{ I'_\lambda(U) = 0 \mid U \in E \}. \]

By using the sharp trace Adams inequalities (1.16) and a variational method, we can prove the following result.

**Theorem 1.13** For any \( 0 < \lambda < 1 \), there exists \( V \in E \) such that \( I_\lambda(V) = M_\lambda \), that is to say that the problem (1.24) has a ground-state solution provided \( 0 < \lambda < 1 \).

This paper is organized as follows. In Sect. 2, we establish the sharp subcritical and critical trace Trudinger–Moser inequalities in \( \mathbb{R}^2 \) and existence of their extremal functions based on the harmonic extension. Section 3 is devoted to the proof of the sharp trace Adams inequalities and the existence of the extremals for subcritical trace Adams inequalities in \( \mathbb{R}^4 \). In Sect. 4, we are concerned with the existence of a ground-state solution to a class of problems with the nonlinear Neumann boundary condition on the half-space by using the trace Adams inequality, sharp Fourier rearrangement principle and variational method. Section 5 is devoted to the proof of Lemma 1.9 which is the main technical part of establishing Theorems 1.10, 1.11 and 1.12 in the case \( m > 2 \) in \( \mathbb{R}^{2m} \) by adapting the same argument of proving Theorems 1.4 and 1.5 in the case \( m = 2 \). In Sect. 6, we give the proof of the fractional Adams inequalities with the exact growth in \( W^{2,2}_m(\mathbb{R}^n) \), which is needed in proving Theorem 1.6 and (1.23).

**2 Trace Trudinger–Moser Inequalities on the Half-Space \( \mathbb{R}^2_+ \)**

In this section, we consider the sharp subcritical and critical trace Trudinger–Moser inequalities and the existence of their extremal functions. The method is based on the harmonic extension, sharp subcritical fractional Adams inequalities in \( W^{1/2,2}(\mathbb{R}) \) and existence of their extremals. First, we introduce some known results about the fractional Adams inequalities and the harmonic extension.

**Lemma 2.1** ([25]) For \( 0 < \beta < \pi \), then there exists a positive constant \( C \) such that for all functions \( u \in W^{1/2,2}(\mathbb{R}) \) with \( \|(-\Delta)^{1/4}u\|_2 = 1 \), the following inequality holds.
\[
\int_{\mathbb{R}} (\exp(\beta|u|^2) - 1)dx \leq C \int_{\mathbb{R}} |u|^2dx, \tag{2.1}
\]
Moreover, the constant \( \pi \) is sharp in the sense that the inequality fails if the constant \( \beta \) is replaced by any \( \beta \geq \pi \).

**Lemma 2.2** There exists an extremals for the above subcritical fractional Adams inequality (2.1). Furthermore, all the extremals of inequality (2.1) must be radially symmetric with respect to some point \( x_0 \in \mathbb{R} \).

**Remark 2.3** The proof can be found in [14]. One can also use the technique of proof of Lemma 3.2 to prove Lemma 2.2, then we omit the details.
Lemma 2.4 ([55, 57]) Given a function \( u \in W^{1,2}(\mathbb{R}) \). If we assume that function \( U \in W^{2,2}(\mathbb{R}^2_+) \) satisfying

\[-\Delta U(x, y) = 0\]
on the upper half-space \( \mathbb{R}^2_+ \) and the boundary condition

\[U(x, y)|_{y=0} = u(x),\]

then we have the following identity

\[
\int_{\mathbb{R}^2_+} |\nabla U(x, y)|^2 \, dx \, dy = \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{2}} u|^2 \, dx. \tag{2.2}
\]

Now, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1 For any \( U \in W^{1,2}(\mathbb{R}^2_+) \), we define \( V(x, y) \) as the harmonic extension of \( U(x, 0) \), that is to say \( V(x, y) \) satisfies that

\[-\Delta V(x, y) = 0\]
on the upper half-space \( \mathbb{R}^2_+ \) and the boundary condition

\[V(x, y)|_{y=0} = U(x, y)|_{y=0}.\]

Through Green’s representation formula, we can write

\[V(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(|x-\tilde{x}|^2 + y^2)} U(\tilde{x}, 0) \, d\tilde{x}.\]

According to the Dirichlet principle, we know that

\[
\int_{\mathbb{R}^2_+} |\nabla V(x, y)|^2 \, dx \, dy \leq \int_{\mathbb{R}^2_+} |\nabla U(x, y)|^2 \, dx \, dy. \tag{2.3}
\]

Combining (2.3) and (2.2) and Lemma 2.1, we obtain

\[
\int_{\mathbb{R}^2_+} (\exp(\beta |U(x, 0)|^2) - 1) \, dx = \int_{\mathbb{R}^2_+} (\exp(\beta |V(x, 0)|^2) - 1) \, dx
\]

\[\leq C \int_{\mathbb{R}^2_+} |V(x, 0)|^2 \, dx = C \int_{\mathbb{R}^2_+} |U(x, 0)|^2 \, dx. \tag{2.4}
\]
The sharpness of the above inequality can be deduced from the sharpness of the sharp subcritical inequalities (2.1). In fact, we can pick the test function \( u_k \in W^{1,2}(\mathbb{R}) \) satisfying \( \|(-\Delta)^{1/4}u_k\|_2 \leq 1 \) such that

\[
\lim_{k \to \infty} \frac{\int_{\mathbb{R}} (\exp(\pi|u_k|^2) - 1) \, dx}{\|u_k\|_2^2} = \infty.
\]

Define \( U_k(x, y) \) as the harmonic extension of \( u_k \), through Lemma 2.4, we know that \( \int_{\mathbb{R}^2_+} |\nabla U_k(x, y)|^2 \, dx \, dy = \int_{\mathbb{R}} \|(-\Delta)^{1/4}u_k\|^2 \, dx \leq 1 \), then it follows that

\[
\lim_{k \to \infty} \frac{\int_{\mathbb{R}^2_+} (\exp(\pi|u_k|^2) - 1) \, dx}{\|u_k\|_2^2} = \infty,
\]

which implies the sharpness of the inequality (1.8).

Next, we show the existence of extremals for subcritical trace Trudinger–Moser inequalities. According to Lemma 2.2, we know for any \( \beta < \pi \), there exists \( u_0(x) \in W^{1,2}(\mathbb{R}) \) satisfying \( \|(-\Delta)^{1/4}u_0\|_2^2 = 1 \) such that

\[
\int_{\mathbb{R}} \left( \exp(\beta|u_0|^2) - 1 \right) \, dx = \sup_{u \in W^{1,2}(\mathbb{R}), \|(-\Delta)^{1/4}u\|_2^2 = 1} \int_{\mathbb{R}} \left( \exp(\beta|u|^2) - 1 \right) \, dx.
\]

Define \( U(x, y) \) as the harmonic extension of \( u_0(x) \), through Lemma 2.4, we see that \( U(x, y) \) is an extremal function for (1.8).

Finally, we prove that the extremals of the inequality (1.8) must be a harmonic extension of some radial function \( u \) in \( W^{1,2}(\mathbb{R}) \). In fact, assume that \( U(x, y) \) is the extremal function of the inequality (1.8), we define \( W(x, y) \) as the harmonic extension of \( U(x, 0) \) and can easily check that \( W(x, y) \) is also the extremal function of the inequality with the

\[
\int_{\mathbb{R}^2_+} |\nabla U|^2 \, dx \, dy = \int_{\mathbb{R}^2_+} |\nabla W|^2 \, dx \, dy = 1.
\]

According to the Dirichlet principle, we know that \( \int_{\mathbb{R}^2_+} |\nabla U|^2 \, dx \, dy = \int_{\mathbb{R}^2_+} |\nabla W|^2 \, dx \, dy \) if and only if \( U = W \). Furthermore, it is also easy to check that \( U(x, 0) \) is the extremal functions of the inequality (2.1). According to the Lemma 2.2, the \( U(x, 0) \) must be radially symmetric with respect to some point \( x_0 \in \mathbb{R} \). Hence the extremals of the (1.8) must be a harmonic extension of some radial function.
**Proof of Theorem 1.2** For any \( U \in W^{1,2}(\mathbb{R}_+^2) \) satisfying

\[
\int_{\mathbb{R}_+^2} |\nabla U(x, y)|^2 \, dx \, dy + \int_{\partial \mathbb{R}_+^2} |U(x, 0)|^2 \, dx \leq 1,
\]

we define \( V(x, y) \) as the harmonic extension of \( U(x, 0) \). By the Dirichlet principle and Lemma 2.2, we derive that

\[
\int_{\partial \mathbb{R}_+^2} |(\Delta^{-\frac{1}{2}} V(x, 0) + V(x, 0))|^2 \, dx \leq 1. \tag{2.5}
\]

With the help of the critical fractional Trudinger–Moser inequality in \( W^{1,2}(\mathbb{R}) \) (see [27]), we conclude that

\[
\int_{\partial \mathbb{R}_+^2} (\exp(\pi |U(x, 0)|^2) - 1) \, dx = \int_{\partial \mathbb{R}_+^2} (\exp(\pi |V(x, 0)|^2) - 1) \, dx \leq C,
\]

which accomplishes the proof of the inequality (1.9). The existence of extremal of the inequality (1.9) relies on the existence of the critical fractional Trudinger–Moser inequality in \( W^{1,2}(\mathbb{R}) \) which was recently established in [46]. The proof is similar to the proof of the existence of extremals of the subcritical trace Trudinger–Moser inequalities, we omit the details here.

\[\square\]

### 3 Trace Adams Inequalities on the Half-Space \( \mathbb{R}_+^4 \)

In this section, we are devoted to establishing the sharp trace Adams inequalities and the existence of their extremal functions. The method is based on the bi-harmonic extension, and the Fourier rearrangement developed by Lenzmann and Sok in [34] and sharp fractional Adams inequalities. We note that he method of Fourier rearrangement to establish the existence of extremals to Adams inequalities have been used earlier in [12, 13].

For this purpose, we need the following lemma.

**Lemma 3.1** ([25]) For \( 0 < \beta < 6\pi^2 \), then there exists a positive constant \( C \) such that for all functions \( u \in W^{3,2}(\mathbb{R}^3) \) with \( \|(-\Delta)^{\frac{3}{4}} u\|_2 = 1 \), the following inequality holds.

\[
\int_{\mathbb{R}^3} (\exp(\beta |u|^2) - 1) \, dx \leq C \int_{\mathbb{R}^3} |u|^2 \, dx, \tag{3.1}
\]

Moreover, the constant \( 6\pi^2 \) is sharp in the sense that the inequality fails if the constant \( \beta \) is replaced by any \( \beta \geq 6\pi^2 \).

Next, we will prove the following

**Lemma 3.2** There exists an extremal for the above subcritical fractional Adams inequality (3.1). Furthermore, all the extremals of inequality (3.1) must be radially symmetric with respect to some point \( x_0 \in \mathbb{R}^3 \).
**Proof** Define the sharp constant $\mu_\beta$ by

$$\mu_\beta := \sup_{u \in W^{3,2}(\mathbb{R}^3), \|(-\Delta)^{\frac{3}{2}}u\|_2 = 1} F_\beta(u),$$

where

$$F_\beta(u) := \frac{\int_{\mathbb{R}^3} (\exp(\beta |u|^2) - 1) dx}{\|u\|_2^2}.$$

We first employ the Fourier rearrangement tools to prove that there exists a radially maximizing sequence for $\mu_\beta$. In fact, assume that $(u_k)$ is a maximizing sequence for $\mu_\beta$, that is

$$\|(-\Delta)^{\frac{3}{2}}u_k\|_2 = 1, \quad \lim_{k \to \infty} F_\beta(u_k) \to \mu_\beta.$$

Define $u_k^\#$ by

$$u_k^\# := F^{-1}\{F(u_k)\}^\ast,$$

where $F$ denotes the Fourier transform on $\mathbb{R}^3$ (with its inverse $F^{-1}$) and $f^\ast$ stands for the Schwarz symmetrization of $f$. Using the property of the Fourier rearrangement from [34], one can derive that

$$\|(-\Delta)^{\frac{3}{2}}u_k^\#\|_2 \leq \|(-\Delta)^{\frac{3}{2}}u_k\|_2, \quad \|u_k^\#\|_2 = \|u_k\|_2, \quad \|u_k\|_q \geq \|u_k^\#\|_q \quad (f \text{ or even } q > 2).$$

Hence, $\lim_{k \to \infty} F_\beta(u_k) \leq \lim_{k \to \infty} F_\beta(u_k^\#)$, which implies that $(u_k^\#)$ is also the maximizing sequence for $\mu_\beta$. Constructing a new function sequence $(v_k)$ defined by $v_k(x) := u_k(\|u_k\|_2^2 x)$ for $x \in \mathbb{R}^3$, one can easily verify that $(v_k)$ is also a maximizing sequence for $\mu_\beta$ with $\|(-\Delta)^{\frac{3}{2}}v_k\|_2 = 1$ and $\|v_k\|_2 = 1$. Note $(v_k)$ is bounded in $W^{3,2}(\mathbb{R}^3)$, up to a sequence, we may assume that

$$v_k \to v \text{ in } W^{3,2}(\mathbb{R}^3),$$

thus $v$ satisfies that $\|v\|_2 \leq 1$ and $\|(-\Delta)^{\frac{3}{2}}v\|_2^2 \leq 1$. Since $W^{3,2}(\mathbb{R}^3)$ (the collection of all radial functions in $W^{3,2}(\mathbb{R}^3)$) can be compactly imbedded into $L^p(\mathbb{R}^3)$ for any $p > 2$, we can claim that

$$\lim_{k \to \infty} \int_{\mathbb{R}^3} (\exp(\beta |v_k|^2) - 1 - \beta |v_k|^2) dx = \int_{\mathbb{R}^3} (\exp(\beta |v|^2) - 1 - \beta |v|^2) dx. \quad (3.2)$$

For simplicity, we define $\Psi(\tau) := \exp(\tau) - 1 - \tau$ and $\Phi(\tau) := \exp(\tau) - 1$ for $\tau > 0$, then we can rewrite (3.2) as

$$\lim_{k \to \infty} \int_{\mathbb{R}^3} \Psi(\beta |u_k|^2) dx = \int_{\mathbb{R}^3} \Psi(\beta |u|^2) dx. \quad (3.3)$$
Hence, it follows from the mean value theorem and the convexity of the function \( \Psi \) that
\[
|\Psi(\beta|u_k|^2) - \Psi(\beta|u|^2)| \\
\lesssim \Phi(\theta \beta|u_k|^2 + (1 - \theta)\beta|u|^2)(|u| + |u_k|)|u_k - u| \\
\lesssim (|u_k| + |u|)(\Phi(\beta|u_k|^2) + \Phi(\beta|u|^2))|u_k - u|,
\]
where \( \theta \in [0, 1] \). This together with the fractional Adams inequality leads to
\[
|\int_{\mathbb{R}^3}(\Psi(\beta|u_k|^2) - \Psi(\beta|u|^2))dx| \\
\lesssim \int_{\mathbb{R}^3}(|u_k| + |u|)(\Phi(\beta|u_k|^2) + \Phi(\beta|u|^2))|u_k - u|dx \\
\lesssim \|\|u_k\| + \|u\\|_{L^a(\mathbb{R}^3)}\|\Phi(\beta|u_k|^2) + \Phi(\beta|u|^2)\|_{L^b(\mathbb{R}^3)}\|u_k - u\|_{L^c(\mathbb{R}^3)} \\
\lesssim \|u_k - u\|_{L^c(\mathbb{R}^3)},
\]
where the constants \( b > 1 \) sufficiently close to 1 and \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \). Note that \( W^{\frac{3}{2}, 2}(\mathbb{R}^3) \) can be compactly imbedded into \( L^r(\mathbb{R}^3) \) for any \( r > 2 \), we derive that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^3}\Psi(\beta|u_k|^2)dx = \int_{\mathbb{R}^3}\Psi(\beta|u|^2)dx,
\]
which accomplishes the proof of (3.2). Then it follows from (3.2) that
\[
\mu_\beta = F_\beta(v_k) + o(1) \\
= \int_{\mathbb{R}^3}(\exp(\beta|v_k|^2) - 1)dx + o(1) \\
= \beta + \int_{\mathbb{R}^3}(\exp(\beta|v_k|^2) - 1 - \beta|v_k|^2)dx + o(1) \\
= \beta + \int_{\mathbb{R}^3}(\exp(\beta|v|^2) - 1 - \beta|v|^2)dx + o(1).
\]
(3.6)

Next, we show \( v \neq 0 \). Indeed, one can pick \( u_0 \) in \( W^{\frac{3}{2}, 2}(\mathbb{R}^3) \) satisfying \( \|(-\Delta)^{\frac{3}{4}}u_0\|_2 = 1 \) arbitrarily. Then, we have

\[\square\] Springer
\[ \mu_\beta \geq F_\beta(u_0) = \frac{\int_{\mathbb{R}^3} (\exp(\beta |u_0|^2) - 1) \, dx}{\|u_0\|_2^2} = \sum_{j=1}^\infty \frac{\beta^j}{j!} \frac{\|u_0\|_{2j}^2}{\|u_0\|_2^2} = \beta + \sum_{j=2}^\infty \frac{\beta^j}{j!} \frac{\|u_0\|_{2j}^2}{\|u_0\|_2^2} > \beta. \]

Hence,

\[ \mu_\beta \leq \beta + \int_{\mathbb{R}^3} (\exp(\beta |v|^2) - 1 - \beta |v|^2) \, dx \left/ \|v\|_2^2 \right. = \frac{\int_{\mathbb{R}^3} (\exp(\beta |v|^2) - 1) \, dx}{\|v\|_2^2} = F_\beta(v). \]

Therefore, it remains to show \( \|(-\Delta)^{\frac{3}{4}} v\|_2^2 = 1 \). Recall that \( \|(-\Delta)^{\frac{3}{4}} v\|_2^2 \leq 1 \), it suffices to show that \( \|(-\Delta)^{\frac{3}{4}} v\|_2^2 \geq 1 \). Through the definition of \( \mu_\beta \), one can obtain that

\[ \mu_\beta \geq F_\beta \left( \frac{v}{\|(-\Delta)^{\frac{3}{4}} v\|_2^2} \right) = \sum_{j=1}^\infty \frac{\beta^j}{j!} \frac{\|v\|_{2j}^2}{\|v\|_2^2} (-\Delta)^{\frac{3}{4}} v \|_{2-2j}^2 \geq \beta + \frac{\beta^2}{2} \frac{\|v\|_4^4}{\|v\|_2^4} (-\Delta)^{\frac{3}{4}} v \|_{2}^2 - 2 + \sum_{j=2}^\infty \frac{\beta^j}{j!} \frac{\|v\|_{2j}^2}{\|v\|_2^2} \geq F_\beta(v) + \frac{\beta^2}{2} \frac{\|v\|_4^4}{\|v\|_2^4} (-\Delta)^{\frac{3}{4}} v \|_{2}^2 - 1 ) \]

which implies that \( \|(-\Delta)^{\frac{3}{4}} v\|_2^2 \geq 1 \). Thus, \( v \) is a maximizer. Next, we prove that all the extremals of the inequality (3.1) must be radially symmetric with respect to some point \( x_0 \in \mathbb{R}^3 \). Assume that \( u \) is a maximizer for the inequality (3.1), we easily see that \( u^\star \) is also a maximizer for the inequality (3.1) with \( \|(-\Delta)^{\frac{3}{4}} u^\star\|_2 = \|(-\Delta)^{\frac{3}{4}} u\|_2 \) and \( \|u^\star\|_{Lq} = \|u\|_{Lq} \) for even \( q \). Using the property of the Fourier rearrangement from [34], we conclude that

\[ u(x) = e^{i\alpha} u^\star(x - x_0) \text{ for any } x \in \mathbb{R}^3 \]
with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$. That is to say that $u$ is radially symmetric and real valued up to translation and constant phase. Then we accomplish the proof of Lemma 3.2.

\[ \square \]

**Lemma 3.3** ([51]) *Given a function $u \in W^{1,2} (\mathbb{R}^3)$. If we assume that function $U \in W^{2,2} (\mathbb{R}^4_+)$ satisfying*

\[ (-\Delta)^2 U(x, y) = 0 \]

*on the upper half-space $\mathbb{R}^4_+$ and the boundary condition*

\[ U(x, 0) = u(x), \quad \partial_y U(x, y)|_{y=0} = 0, \]

*then we have the following identity*

\[ \int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy = 2 \int_{\mathbb{R}^3} |(-\Delta)^3 u|^2 \, dx. \quad (3.8) \]

Now, we are in the position to prove Theorem 1.4.

**Proof of Theorem 1.4** For any $U \in W^{2,2} (\mathbb{R}^4_+)$ satisfying the Neumann boundary condition $\partial_y U(x, y)|_{y=0} = 0$, we define $V(x, y)$ as the bi-harmonic extension of $U(x, 0)$, that is to say $V(x, y) \in W^{2,2} (\mathbb{R}^4_+)$ satisfies that

\[ (-\Delta)^2 V(x, y) = 0 \]

on the upper half-space $\mathbb{R}^4_+$ and the boundary condition

\[ V(x, y)|_{y=0} = U(x, y)|_{y=0}, \quad \partial_y V(x, y)|_{y=0} = 0. \]

In fact, we can write

\[ V(x, y) = c \int_{\partial \mathbb{R}^4_+} \frac{y^3}{(|x - \tilde{x}|^2 + y^2)^3} U(\tilde{x}, 0) \, d\tilde{x} \]

by the Green formula. Furthermore, by the Dirichlet principle for bi-harmonic function, we derive that

\[ \int_{\mathbb{R}^4_+} |\Delta V(x, y)|^2 \, dx \, dy \leq \int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy. \quad (3.9) \]
On the other hand, through Lemma 3.3, we get
\[
\int_{\mathbb{R}^4_+} |\Delta V(x, y)|^2 \, dx \, dy = 2 \int_{\mathbb{R}^4_+} |(-\Delta)^{3/4} V(x, 0)|^2 \, dx. \tag{3.10}
\]

Combining (3.9) and (3.10) and Lemma 3.1, we conclude that
\[
\int_{\partial \mathbb{R}^4_+} \left( \exp(\beta |U(x, 0)|^2) - 1 \right) \, dx = \int_{\partial \mathbb{R}^4_+} \left( \exp(\beta |V(x, 0)|^2) - 1 \right) \, dx
\leq C \int_{\partial \mathbb{R}^4_+} |V(x, 0)|^2 \, dx = C \int_{\partial \mathbb{R}^4_+} |U(x, 0)|^2 \, dx.
\tag{3.11}
\]

The sharpness of the inequality (1.15) can be similarly deduced from the sharpness of the sharp subcritical inequalities (3.1) as done in the proof of sharpness of inequality (1.8).

Now, we prove the existence of extremals for the subcritical trace Adams inequalities. From Lemma 3.2, we see that there exists \( u_0(x) \in W^{3/2, 2} (\mathbb{R}^3) \) such that
\[
C = \sup_{u \in W^{3/2, 2} (\mathbb{R}^3), \|(-\Delta)^{3/4} u\|^2 = 1} \frac{\int_{\mathbb{R}^3} (\exp(\beta |u_0|^2) - 1) \, dx}{\int_{\mathbb{R}^3} |u_0|^2 \, dx}
\]
for any \( \beta < 6\pi^2 \). Define \( U(x, y) \) as the bi-harmonic extension of \( u_0(x) \), it is easy to check that \( U(x, y) \) is an extremal function for (1.15) through Lemma 3.3.

Finally, we show that the extremals of the inequality (1.15) must be a bi-harmonic extension of some radial function \( u \) in \( W^{3/2, 2} (\mathbb{R}^3) \). In fact, assume that \( U(x, y) \) is the extremal function of the inequality (1.15), It is easy to check that the bi-harmonic extension \( V(x, y) \) is also the extremal function of the inequality with
\[
\int_{\mathbb{R}^4_+} |\Delta U|^2 \, dx \, dy = \int_{\mathbb{R}^4_+} |\Delta V|^2 \, dx \, dy = 1,
\]
which implies that \( \int_{\mathbb{R}^4_+} |\Delta U|^2 \, dx \, dy = \int_{\mathbb{R}^4_+} |\Delta V|^2 \, dx \, dy \) if and only if \( U = V \) through Dirichlet principle. On the other hand, it is also easy to see that \( U(x, 0) \) is the extremal functions of the inequality (3.1). Applying Lemma 3.2 again, we conclude that \( U(x, 0) \) must be radially symmetric with respect to some point \( x_0 \in \mathbb{R}^3 \). This proves that the extremals of inequality (1.15) must be a bi-harmonic extension of some radial function.
Proof of Theorem 1.5 and Theorem 1.6 For any \( U \in W^{2,2}(\mathbb{R}^4_+) \) satisfying Neumann boundary condition \( \partial_y U(x, y)|_{y=0} = 0 \) with

\[
\int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy + \int_{\partial \mathbb{R}^4_+} |U(x, 0)|^2 \, dx \leq 1,
\]
we derive that

\[
\int_{\mathbb{R}^4} |\Delta V(x, y)|^2 \, dx \, dy \leq \int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 \, dx \, dy,
\]
where \( V(x, y) \) is the bi-harmonic extension of \( U(x, 0) \). This together with Lemma 3.3 and (3.12) yields that

\[
\int_{\partial \mathbb{R}^4_+} (2|(-\Delta)^{3/4} V(x, 0)|^2 + |V(x, 0)|^2) \, dx \leq 1.
\]

Combining the critical fractional Adams inequalities in \( W^{\frac{3}{2},2}(\mathbb{R}^3) \), we derive that

\[
\int_{\partial \mathbb{R}^4_+} (\exp(12\pi^2 |U(x, 0)|^2) - 1) \, dx = \int_{\partial \mathbb{R}^4_+} (\exp(12\pi^2 |V(x, 0)|^2) - 1) \, dx \leq C.
\]

(3.13)

Similarly, one can deduce from the fractional Adams inequalities with the exact growth in \( W^{\frac{3}{2}}(\mathbb{R}^3) \) (see Theorem 1.8) that

\[
\int_{\partial \mathbb{R}^4_+} \frac{(\exp(12\pi^2 |U(x, 0)|^2) - 1)}{1 + |U(x, 0)|^2} \, dx \leq \int_{\partial \mathbb{R}^4_+} (\exp(12\pi^2 |V(x, 0)|^2) - 1) \, dx \leq C \int_{\partial \mathbb{R}^4_+} |U(x, 0)|^2 \, dx,
\]

which gives the inequality (1.17). \( \square \)

4 Existence of Ground States to the Bi-Laplacian with Nonlinear Neumann Boundary Conditions on the Half-Space \( \mathbb{R}^4_+ \)

In this section, we consider the critical point of the functional \( I_\lambda(U) \) on \( E \) (see (1.25)). For this purpose, we will first study the critical point of the following functional

\[\text{Springer}\]
\[ J_\lambda(u) = \frac{1}{2} \left( 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{4}} u|^2 dx + \int_{\mathbb{R}^3} |u|^2 dx \right) - \frac{\lambda}{24\pi^2} \int_{\mathbb{R}^3} \left( \exp(12\pi^2 |u|^2) - 1 \right) dx, \]

and we show that the functional \( J_\lambda(u) \) has a least energy critical point provided \( 0 < \lambda < 1 \), that is,

**Proposition 4.1** \( m_\lambda = \inf\{ J_\lambda(u) : J_\lambda'(u) = 0, u \in W^{3,2} (\mathbb{R}^3) \} \) can be achieved for any \( 0 < \lambda < 1 \).

**Proof of Proposition 4.1** Motivated by the Pohozaev identity for the functional \( J_\lambda \), we introduce the functional

\[ G_\lambda(u) = \|u\|_2^2 - \frac{\lambda}{12\pi^2} \int_{\mathbb{R}^3} \left( \exp(12\pi^2 |u|^2) - 1 \right) dx = (1 - \lambda)\|u\|_2^2 - \int_{\mathbb{R}^3} g_\lambda(u) dx, \]

where \( g_\lambda(t) \) is defined as

\[ g_\lambda(t) = \frac{\lambda}{12\pi^2} \left( \exp(12\pi^2 |t|^2) - 1 - 12\pi^2 t^2 \right) \]

and the constrained minimization problem

\[ A_\lambda = \inf \left\{ \|(-\Delta)^{\frac{3}{4}} u\|_2^2 \mid u \in W^{3,2} (\mathbb{R}^3), G_\lambda(u) = 0 \right\} = \inf \left\{ J_\lambda(u) \mid u \in W^{3,2} (\mathbb{R}^3), G_\lambda(u) = 0 \right\} \leq m_\lambda. \tag{4.1} \]

Set \( M = \{ u \in W^{3,2} (\mathbb{R}^3), G_\lambda(u) = 0 \} \), we point out that \( M \) is not empty. In fact, let \( u_0 \in W^{3,2} (\mathbb{R}^3) \) be compactly supported and define

\[ h(s) := G_\lambda(su_0) = s^2 (1 - \lambda)\|u_0\|_2^2 - \int_{\mathbb{R}^3} g_\lambda(su_0) dx, \forall s > 0. \]

From \( \lim_{t \to 0} \frac{g_\lambda(t)}{t^2} = 0 \) and \( \lim_{t \to +\infty} \frac{g_\lambda(t)}{t^2} = \infty \), we have \( h(s) > 0 \) for \( s > 0 \) small enough and \( h(s) < 0 \) for \( s > 0 \) sufficiently large. Therefore, there exists \( s_0 > 0 \) satisfying \( h(s_0) = 0 \), which implies \( s_0 u_0 \in M \). \( \square \)

Next, we show that \( A_\lambda \) can be attained by some function \( u \in W^{3,2} (\mathbb{R}^3) \setminus \{0\} \). For this, we need the following lemma:

**Lemma 4.2** There exists a radially minimizing sequence \( \{u_k\}_k \) satisfying \( \|u_k\|_2^2 = 1 \) for \( A_\lambda \).

**Proof** Assume that \( \{u_k\}_k \) is a minimizing sequence for \( A_\lambda \), that is \( u_k \in M \) satisfying

\[ \lim_{k \to \infty} \|(-\Delta)^{\frac{3}{4}} u_k\|_2^2 = A_\lambda. \]
Define $u_k^r$ by the Fourier rearrangement of $u_k$. Using the property of the Fourier rearrangement from [34], one can derive that

$$\|(-\Delta)^{3/4} u_k^r\|_2 \leq \|(-\Delta)^{3/4} u_k\|_2, \quad \|u_k^r\|_2^2 = \|u_k\|_2^2,$$

and

$$\int_{\mathbb{R}^3} \left( \exp(12\pi^2 (u_k^r)^2) - 1 \right) dx \geq \int_{\mathbb{R}^3} \left( \exp(12\pi^2 u_k^2) - 1 \right) dx.$$

Then it follows that

$$(1 - \lambda)\|u_k^r\|_2^2 = (1 - \lambda)\|u_k\|_2^2 = \int_{\mathbb{R}^3} g_\lambda(u_k) dx \leq \int_{\mathbb{R}^3} g_\lambda(u_k^r) dx.$$

Hence if we set

$$\gamma(t) = (1 - \lambda)\|tu_k^r\|_2^2 - \int_{\mathbb{R}^3} g_\lambda(tu_k^r) dx,$$

then we have $\gamma(1) \leq 0$ while $\gamma(t) > 0$ for $t > 0$ sufficiently small. Therefore, there exists $t_k \in (0, 1]$ such that $\gamma(t_k) = 0$, that is $t_k u_k^r \in M$. We obtain

$$A_\lambda \leq I(t_k u_k^r) = \|(-\Delta)^{3/4}(t_k u_k^r)\|_2^2 \leq t_k^2 \|(-\Delta)^{3/4} u_k\|_2^2 \leq I(u_k) = A_\lambda + o(1). \quad (4.2)$$

This implies that $\{v_k\} = \{t_k u_k^r\}_k$ is a radial minimizing sequence for $A_\lambda$. Let $\tilde{v}_k = u_k^r(\|v_k\|_2^2 x)$, direct computations yield that $\|\tilde{v}_k\|_2 = 1$, $\tilde{v}_k \in M$ and $\|(-\Delta)^{3/4} v_k\|_2 = \|(-\Delta)^{3/4} \tilde{v}_k\|_2$.

In order to study the attainability of $A_\lambda$, we also need the following compactness result.

**Lemma 4.3** (Compactness) Suppose that $g : \mathbb{R} \to [0, +\infty)$ is a continuous function and define

$$G(u) = \int_{\mathbb{R}^3} g(u) dx.$$

Then for any $K > 0$, if

$$\lim_{t \to +\infty} \exp(-\frac{1}{K}|t|^2)g(t) < +\infty, \quad \lim_{t \to 0} |t|^{-2}g(t) = 0,$$

then
then for any bounded sequence \( \{u_k\}_k \in W^{3,2}_r(\mathbb{R}^3) \) satisfying \( \lim_{k \to +\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{4}} u_k|^2 \, dx < 6\pi^2 K \) and weakly converging to some \( u \), we have that \( G(u_k) \to G(u) \).

**Proof** Note that \( \{u_k\}_k \) is a radial sequence in \( W^{3,2}_r(\mathbb{R}^3) \), then by radial Lemma, there exists \( \delta > 0 \) such that

\[
|u_k(r)| \lesssim \frac{1}{r^\delta}.
\]  

(4.3)

Hence \( u_k(r) \to 0 \) as \( r \to \infty \) uniformly with respect to \( k \). This together with the subcritical Adams inequality (3.1) yields that for any \( \varepsilon > 0 \), there exists \( R > 0 \) such that

\[
\int_{\mathbb{R}^3 \setminus B_R} g(u_k) \, dx \lesssim \varepsilon \int_{\mathbb{R}^3} |u_k|^2 \, dx \lesssim \varepsilon, \quad \int_{\mathbb{R}^3 \setminus B_R} g(u) \, dx \lesssim \varepsilon.
\]  

(4.4)

On the other hand, since \( \lim_{k \to +\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{4}} u_k|^2 \, dx < 6\pi^2 K \), then there exists some \( \gamma > 0 \) such that \( \lim_{k \to +\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{4}} u_k|^2 \, dx < \gamma < 6\pi^2 K \).

Through \( \lim_{t \to +\infty} \exp(-\frac{1}{K} |t|^2) g(t) < +\infty \), we derive that for \( \varepsilon > 0 \), there exists \( L \) independent of \( k \) such that

\[
\int_{|u_k| > L} g(u_k) \, dx \lesssim \varepsilon \int_{|u_k| > L} \exp \left( \frac{\gamma}{K} \left| \frac{u_k}{\|(-\Delta)^{\frac{3}{4}} u_k\|_2} \right|^2 \right) \, dx
\]

and

\[
\int_{|u_k| > L} g(u) \, dx \lesssim \varepsilon \int_{|u_k| > L} \exp \left( \frac{\gamma}{K} \left| \frac{u}{\|(-\Delta)^{\frac{3}{4}} u\|_2} \right|^2 \right) \, dx,
\]

Applying the sharp subcritical Adams inequality (3.1), we derive that

\[
\int_{|u_k| > L} g(u_k) \, dx \lesssim \varepsilon \int_{|u_k| > L} |u_k|^2 \, dx \lesssim \varepsilon, \quad \int_{|u_k| > L} g(u) \, dx \lesssim \varepsilon.
\]  

(4.5)

Combining (4.4) and (4.5), we conclude that

\[
\lim_{k \to \infty} |G(u_k) - G(u)| \leq \int_{\mathbb{R}^3 \setminus B_R} (g(u_k) - g(u)) \, dx + \int_{B_R} (g(u_k) - g(u)) \, dx \\
\lesssim \varepsilon + \lim_{k \to \infty} \left| \int_{B_R \cap \{|u_k| > L\}} g(u_k) \, dx - \int_{B_R \cap \{|u_k| > L\}} g(u) \, dx \right|
\]
\[
+ \lim_{k \to \infty} \left| \int_{B_R \cap \{ |u_k| \leq L \}} g(u_k)dx - \int_{B_R \cap \{ |u_k| \leq L \}} g(u)dx \right|
\leq \varepsilon + \lim_{k \to \infty} \left| \int_{B_R \cap \{ |u_k| \leq L \}} g(u_k)dx - \int_{B_R \cap \{ |u_k| \leq L \}} g(u)dx \right|
\leq \varepsilon,
\]
which gives the proof by the Lebesgue dominated convergence theorem. \( \square \)

With this compactness result, we can prove the following

**Lemma 4.4** If \( A_\lambda < \frac{1}{2} \), then \( A_\lambda \) can be attained and \( A_\lambda = I_\lambda(u) \).

**Proof** Let \( \{u_k\}_k \) is a radial minimizing sequence for \( A_\lambda \), that is \( u_k \in M \) satisfying

\[
\lim_{k \to \infty} \|(-\Delta)^{3/4} u_k\|_2^2 = A_\lambda \quad \text{and} \quad \|u_k\|_2^2 = 1.
\]

We also assume that \( u_k \rightharpoonup u \) in \( W^{3,2}(\mathbb{R}^3) \). We first prove that \( A_\lambda > 0 \). By way of contradiction, we assume that \( A_\lambda = 0 \). That is \( \lim_{k \to \infty} \|(-\Delta)^{3/4} u_k\|_2^2 = 0 \), which implies that \( u = 0 \). Since

\[
\lim_{t \to +\infty} \exp(-c|t|^2)g_\lambda(t) = 0 \quad \text{for any} \quad c > 12\pi^2, \quad \lim_{t \to 0} |t|^{-2}g_\lambda(t) = 0.
\]

It follows from the Lemma 4.3 that

\[
\int_{\mathbb{R}^3} g_\lambda(u_k)dx = \int_{\mathbb{R}^3} g_\lambda(u)dx.
\]

On the other hand, it follows from \( u_k \in M \) and \( \|u_k\|_2^2 = 1 \) that

\[
0 < (1 - \lambda) \leq \lim_{k \to \infty} (1 - \lambda)\|u_k\|_2^2 = \lim_{k \to \infty} \int_{\mathbb{R}^3} g_\lambda(u_k)dx = \int_{\mathbb{R}^3} g_\lambda(u)dx,
\]

which contradicts \( u = 0 \). This proves that \( A_\lambda > 0 \).

Now are in position to prove that if \( A_\lambda < \frac{1}{2} \), then \( A_\lambda \) could be attained. Under the assumption of Lemma 3.2, we have \( \lim_{k \to \infty} \|\Delta u_k\|_2^2 = A_\lambda < \frac{1}{2} \). Set \( K = \frac{A_\lambda}{6\pi^2} \), obviously,

\[
\lim_{t \to +\infty} \exp\left(-\frac{1}{K} |t|^2\right)g_\lambda(t) = 0, \quad \lim_{t \to 0} |t|^{-2}g_\lambda(t) = 0.
\]

It follows from Lemma 4.3 that

\[
\int_{\mathbb{R}^3} g_\lambda(u_k)dx = \int_{\mathbb{R}^3} g_\lambda(u)dx.
\]
Hence
\[
(1 - \lambda) = \lim_{k \to \infty} (1 - \lambda)\|u_k\|_2^2 = \lim_{k \to \infty} \int_{\mathbb{R}^3} g_\lambda(u_k)dx = \int_{\mathbb{R}^3} g_\lambda(u)dx
\]
and
\[
\|(-\Delta)^{\frac{3}{4}} u\|_2^2 \leq \lim_{k \to \infty} \|(-\Delta)^{\frac{3}{4}} u_k\|_2^2 = A_\lambda.
\]
In order to show \(u\) is minimizer for \(A_\lambda\), it suffices to show that
\[
G_\lambda(u) = (1 - \lambda)\|u\|_2^2 - \int_{\mathbb{R}^3} g_\lambda(u)dx \leq \lim_{k \to \infty} (1 - \lambda)\|u_k\|_2^2 - \int_{\mathbb{R}^3} g_\lambda(u_k)dx
\]
\[
= \lim_{k \to \infty} G_\lambda(u_k) = 0.
\]
(4.7)

If we define
\[
h(t) = G_\lambda(tu) = (1 - \lambda)\|tu\|_2^2 - \int_{\mathbb{R}^3} g(tu)dx,
\]
then \(h(1) \leq 0\) and from \(\lim_{k \to \infty} g(t) = o(t^2)\), we deduce that \(h(t) < 0\) for small \(t > 0\). Consequently, there exists \(s_0 \in (0, 1]\) such that \(G_\lambda(s_0u) = 0\). Then we have
\[
A_\lambda \leq \|(-\Delta)^{\frac{3}{4}} s_0u\|_2^2 = s_0^2 \|(-\Delta)^{\frac{3}{4}} u\|_2^2 \leq s_0^2 A_\lambda,
\]
which proves that \(s_0 = 1\) and \(\|(-\Delta)^{\frac{3}{4}} u\|_2^2 = A_\lambda\). Then we accomplish the proof. \(\square\)

In order to obtain the attainability of \(A_\lambda\), we introduce the fractional Adams ratio
\[
C_{A,\lambda}^L = \sup\left\{ \frac{1}{\|u\|_2^2} \int_{\mathbb{R}^3} g_\lambda(u)dx \mid u \in W^{\frac{3}{2}, 2}(\mathbb{R}^3), \|(-\Delta)^{\frac{3}{4}} u\|_2 \leq L \right\}.
\]
The fractional Adams threshold \(R(g_\lambda)\) is defined as
\[
R(g_\lambda) = \sup\{ L > 0 \mid C_{A,\lambda}^L < +\infty \}.
\]
Obviously, according subcritical fractional Adams inequality (3.1), we know \(R(g_\lambda) = \frac{1}{2}\) and \(C_{A,\lambda}^* := C_{A,\lambda}^{R(g_\lambda)} = \infty\). By using this fact, we can claim that
\[
A_\lambda < \frac{1}{2}.
\]
(4.8)
Indeed, note that $C^*_{A, \lambda} = \infty$, hence there exists $u_0 \in W^{\frac{3}{2}, 2}(\mathbb{R}^3)$ such that

$$(1 - \lambda) < \frac{1}{\|u_0\|_2^2} \int_{\mathbb{R}^3} g_\lambda(u_0) dx, \quad \|(-\Delta)^{\frac{3}{4}} u_0\|_2^2 \leq \frac{1}{2}.$$ 

Hence $G_\lambda(u_0) = (1 - \lambda)\|u_0\|_2^2 - \int_{\mathbb{R}^3} g_\lambda(u_0) dx < 0$. Then there exists $s_0 \in (0, 1)$ such that $s_0u_0 \in M$, which yields that

$$A_\lambda \leq \|(-\Delta)^{\frac{3}{4}} (s_0 u_0)\|_2^2 = s_0^2 \|(-\Delta)^{\frac{3}{4}} u_0\|_2^2 \leq \frac{1}{2} s_0^2 < \frac{1}{2}.$$

Then the claim is proved. Now, we come to

**Completion of the Proof of Proposition 4.1** Combining Lemmas 4.2, 4.4 and (4.8), we see that $A_\lambda$ can be attained. Under a suitable change of scale, it is easy to see that $m_\lambda$ can also be attained.

In view of Proposition 4.1, we can give the proof of Theorem 1.13.

**Proof of Theorem 1.13** Let $v(x) \in W^{\frac{3}{2}, 2}(\mathbb{R}^3)$ be the minimum point of the functional $J_\lambda$ on the submanifold $F := \{J'_\lambda(u) = 0 \mid u \in W^{\frac{3}{2}, 2}(\mathbb{R}^3)\}$ and define $V(x, y)$ as the bi-harmonic extension of $v(x)$, that is to say $V(x, y)$ satisfies that

$$(-\Delta)^2 V(x, y) = 0$$

on the upper half-space $\mathbb{R}^4_+$ and the boundary condition:

$$V(x, 0) = v(x), \quad \partial_y V(x, y)|_{y=0} = 0.$$

We will show that $V(x, y)$ is the ground state of equation (1.24), that is to prove that $V(x, y)$ is the minimum point of the functional $I_\lambda$ on the submanifold $\tilde{F} := \{I'_\lambda(U) = 0 \mid U \in E\}$. 

Assume that $U(x, y)$ is any solution of equation (1.24), we only need to verify that $I_\lambda(V) \leq I_\lambda(U)$. Through equation (1.24), we easily see that $U(x, y)$ is the harmonic extension of $u(x) = U(x, 0)$ and $u(x)$ is the critical point of functional $J_\lambda$ in $W^{\frac{3}{2}, 2}(\mathbb{R}^3)$. In view of Lemma 3.3, we derive that

$$\int_{\mathbb{R}^4_+} |\Delta V(x, y)|^2 dx dy = 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{4}} v(x)|^2 dx,$$

$$\int_{\mathbb{R}^4_+} |\Delta U(x, y)|^2 dx dy = 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{3}{4}} u(x)|^2 dx. \tag{4.9}$$
Then it follows that \( I_\lambda(V) = J_\lambda(v) \) and \( I_\lambda(U) = J_\lambda(u) \). This together with \( v \) being the least energy critical point of the \( J_\lambda \) on \( F \) implies that

\[
I_\lambda(V) = J_\lambda(v) \leq J_\lambda(u) = I_\lambda(U),
\]

that is to say \( V \) is the ground-state solution for equation (1.24) with nonlinear Neumann boundary conditions on the half-space.

### 5 Proofs of Lemma 1.9 and Theorems 1.10 and 1.11

The main purpose of this section is to establish Lemma 1.9 which is the essential technical ingredient in proving Theorems 1.10 and 1.11 for \( m > 2 \) by adapting the same argument in establishing Theorems 1.4 and 1.5 in the case of \( m = 2 \).

We now give the proof of Lemma 1.9.

**Proof** It is known that the solution \( U(x, y) \) is unique. By using the generalized Poisson kernel, it has been shown in [59] that the explicit formula of \( U \) is given by

\[
U(x, y) = \pi^{-m+\frac{1}{2}} \frac{\Gamma(2m - 1)}{\Gamma(m - \frac{1}{2})} \int_{\mathbb{R}^{2m-1}} \frac{y^{2m-1}}{(|x - \xi|^2 + y^2)^{2m-1}} u(\xi) d\xi.
\]

Furthermore, \( U(x, y) \) satisfying the following Neumann boundary condition

\[
\begin{align*}
\Delta^k U(x, y)|_{y=0} &= \frac{\Gamma(m) \Gamma(m - \frac{1}{2} - k)}{\Gamma(m - 1 - k) \Gamma(m - \frac{1}{2})} \Delta^k_x U, \quad 0 \leq k \leq \left[ \frac{m-1}{2} \right]; \\
\partial_y \Delta^k U(x, y)|_{y=0} &= 0, \quad 0 \leq k \leq \left[ \frac{m-2}{2} \right]; \\
\partial_y \Delta^{m-1} U(x, y)|_{y=0} &= (-1)^{m-1} \frac{\Gamma(m) \Gamma(\frac{1}{2})}{\Gamma(m - \frac{1}{2})} (-\Delta_x)^{m-1} u.
\end{align*}
\]

By Green’s formula and (5.1), we have

\[
0 = \int_{\mathbb{R}^{2m}} U \Delta^m U dx dy
= \int_{\mathbb{R}^{2m}} \Delta U \Delta^{m-1} U dx dy + \int_{\mathbb{R}^{2m-1}} U \partial_y \Delta^{m-1} U dx - \int_{\mathbb{R}^{2m-1}} \partial_y U \Delta^{m-1} U dx
= \int_{\mathbb{R}^{2m}} \Delta U \Delta^{m-1} U dx dy + (-1)^m \frac{\Gamma(m) \Gamma(\frac{1}{2})}{\Gamma(m - \frac{1}{2})} \int_{\mathbb{R}^{2m-1}} u(x)(-\Delta_x)^{m-\frac{1}{2}} u(x) dx.
\]
and
\[ \int_{\mathbb{R}^2m} \Delta U \Delta^{m-1} U \, dx \, dy = (-1)^m \int_{\mathbb{R}^2m} |\nabla^m U|^2 \, dx \, dy. \]

Therefore,
\[ \int_{\mathbb{R}^2m} |\nabla^m U|^2 \, dx \, dy = \frac{\Gamma(m) \Gamma \left( \frac{1}{2} \right)}{\Gamma(m - \frac{1}{2})} \int_{\mathbb{R}^{2m-1}} u(x)(-\Delta)_{m-1}^{\frac{1}{2}} u(x) \, dx. \]

The desired result follows.

\[ \square \]

6 Fractional Adams Inequalities with the Exact Growth in \( W^{2, 2}_n (\mathbb{R}^n) \)

In this section, we shall prove the fractional Adams inequalities with the exact growth in \( W^{2, 2}_n (\mathbb{R}^n) \) for all \( n \geq 2 \). For this purpose, we need the following lemmas.

Lemma 6.1 ([7]) For any radial function \( u \in W^{2, 2}_n (\mathbb{R}^n) \), one has
\[ \int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{n}{2}} u \right|^2 \, dx \geq 2^{n-2} \Gamma^2 \left( \frac{n}{2} \right) \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^{n-2}} \, dx. \]

Lemma 6.2 For any \( u \in W^{2, 2}_n (\mathbb{R}^n) \), one has
\[ \int_{\mathbb{R}^{2m}} e^{\beta(n, \frac{n}{2}) |u|^2} - \frac{1}{1 + |u|^2} \, dx \leq 2 \left( e^{\beta(n, \frac{n}{2})} - 1 \right) \|u\|_{\frac{n}{2}}^2 + \int_{\mathbb{R}^{2m}} e^{\beta(n, \frac{n}{2}) |u^\#|^2} - \frac{1}{1 + |u^\#|^2} \, dx, \]

where \( u^\# = \mathcal{F}^{-1}\{(\mathcal{F} u)^*\} \) denotes the Fourier rearrangement of \( u \).

Proof This lemma can be similarly proved as Lemma 2.1 in [52]. We omit the details.

\[ \square \]

With these two lemmas, we can give

Proof of Theorem 1.8 Assume \( u \in W^{2, 2}_n (\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2}} u(x)|^2 \, dx = 1 \), then by the Fourier rearrangement inequality [34], there exists some radial function \( u^\# \in W^{2, 2}_n (\mathbb{R}^n) \) such that
\[ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2}} u^\#|^2 \, dx \leq 1. \]
Combining this and Lemma 6.2, we can assume \( u \) is a radial function satisfying

\[
\int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{n}{2}} u \right|^2 \, dx = 1
\]

and \( u(x) \to 0 \) as \(|x| \to \infty\).

In view of Lemma 6.1, we have

\[
\int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{n}{2}} u \right|^2 \, dx \geq 2^{n-2} \Gamma^2 \left( \frac{n}{2} \right) \int_{\mathbb{R}^n} |\nabla u|^2 \, dx
\]

\[
= 2^{n-2} \Gamma^2 \left( \frac{n}{2} \right) \frac{2^{n+2} \pi^{n-1}}{(n-2)!!} \int_0^\infty \left| u' \right|^2 r \, dr
\]

\[
= 2\pi \cdot \frac{2^{n-3} \Gamma^2 \left( \frac{n}{2} \right) 2^{n+2} \pi^{n-1}}{(n-2)!!} \int_0^\infty \left| u' \right|^2 r \, dr
\]

\[
= 2\pi \left( A_n \int_0^\infty \left| u' \right|^2 r \, dr \right)
\]

where

\[
A_n = \frac{2^{n-3} \Gamma^2 \left( \frac{n}{2} \right) 2^{n+2} \pi^{n-1}}{(n-2)!!}.
\]

Denote \( w(s) = \left( \frac{n}{2} A_n \right)^{1/2} u \left( s^{2/n} \right) \), that is

\[
u(r) = \left( \frac{n}{2} A_n \right)^{-1/2} w \left( r^{n/2} \right).
\]

Then direct calculation gives

\[
\int_0^\infty \left| w'(s) \right|^2 s \, ds = \int_0^\infty \left| \left( \frac{n}{2} A_n \right)^{1/2} u' \left( s^{2/n} \right) \frac{2}{n} s^{\frac{2-n}{n}} \right|^2 s \, ds
\]

\[
= A_n \int_0^\infty \left| u' \left( s^{2/n} \right) \right|^2 s^{2/n} s^{2/n} \, ds
\]

\[
= A_n \int_0^\infty \left| u' \right|^2 r \, dr.
\]
Therefore, it follows that

\[ 1 \geq 2\pi \left( A_n \int_0^\infty |u'|^2 r \, dr \right) = 2\pi \left( \int_0^\infty |w'(s)|^2 s \, ds \right) \]

and

\[
\int_0^\infty |w(s)|^2 s \, ds = \int_0^\infty \left( \frac{n}{2} A_n \right)^{1/2} u\left( s^{2/n} \right)^2 s \, ds \\
= \frac{n A_n}{2} \int_0^\infty |u(r)|^2 r^{n/2} \, dr^{n/2} \\
= \frac{n^2 A_n}{4} \int_0^\infty |u(r)|^2 r^{n-1} \, dr.
\]

Gathering the above estimate, we conclude that

\[
\int_{\mathbb{R}^n} \exp \left( \beta(n, \frac{n}{2}) |u|^2 \right) \frac{1}{1 + |u|^2} \, dx \\
= \omega_{n-1} \int_0^\infty \exp \left( \beta(n, \frac{n}{2}) \left( \left( \frac{n}{2} A_n \right)^{-1/2} w \left( r^{\frac{n}{2}} \right) \right)^2 \right) \frac{1}{1 + \left( \frac{n}{2} A_n \right)^{-1/2} w \left( r^{\frac{n}{2}} \right)^2} r^{n-1} \, dr \\
= \omega_{n-1} \int_0^\infty \exp \left( 2\beta(n, \frac{n}{2}) w^2 \left( r^{\frac{n}{2}} \right) \right) \frac{1}{1 + \left( \frac{n}{2} A_n \right)^{-1/2} w \left( r^{\frac{n}{2}} \right)^2} \, dr^n \\
= \omega_{n-1} \int_0^\infty \exp \left( 4\pi w^2 \right) \frac{1}{1 + |w(s)|^2} s \, ds \\
\leq A_n \omega_{n-1} \int_0^\infty \exp \left( 4\pi w^2(s) \right) \frac{1}{1 + |w(s)|^2} s \, ds \\
\leq c \int_{\mathbb{R}^2} w^2(|x|) \, dx \leq c \int_{\mathbb{R}^n} u^2 \, dx,
\]

which accomplishes the proof of (1.17).
Now, we prove the sharpness of $p \geq 2$. We adopt the test function sequence used in [25]:

$$
\phi_{\varepsilon, r} (x) = \begin{cases} 
  c_{n/2} |x|^{-n/2}, & \varepsilon r < |x| < r \\
  0, & \text{otherwise}, 
\end{cases}
$$

where $c_{n/2} = \frac{1}{2^{n/2} \pi^{n/2}}$ and define the function on $B_r$ by:

$$
\widetilde{\phi}_{\varepsilon, r} = \phi_{\varepsilon, r} (x) - P_{n-1}^{r} \phi_{\varepsilon},
$$

where $P_{n-1}^{r} \phi_{\varepsilon}$ is the projection on the space of polynomials of degree up to $n - 1$ on the ball $B_r$. Obviously, $\widetilde{\phi}_{\varepsilon, r}$ is orthogonal to every polynomial of order up to $n - 1$ on the ball $B_r$ and satisfies $|P_{n-1}^{r} \phi_{\varepsilon}| \leq Cr^{-n/2}$.

By (100), (104) and (107) in [25], we have

$$
\| \phi_{\varepsilon, r} \|^2 \leq -\frac{\log (\varepsilon r)^n}{\beta (n, \frac{n}{2})} + b_r + C,
$$

$$
\| I_{n/2} \phi_{\varepsilon, r} \|^2 \leq Cr^n
$$

and

$$
I_{n/2} \phi_{\varepsilon, r} (x) \geq -\frac{\log (\varepsilon r)^n}{\beta (n, \frac{n}{2})} + b_r - C
$$

on the ball $B_{\varepsilon r/2}$, where $b_r \leq C \log r$.

On the other hand, since

$$
\| \phi_{\varepsilon, r} \|^2 = |\phi_{\varepsilon, r} (x) - P_{n-1}^{r} \phi_{\varepsilon}|^2
\geq C_1 |\phi_{\varepsilon, r} (x)|^2 - C_2 |P_{n-1}^{r} \phi_{\varepsilon}|^2
\geq C_1 |\phi_{\varepsilon, r} (x)|^2 - C_2 r^{-n/2},
$$

and thus it follows that

$$
\| \phi_{\varepsilon, r} \|^2 \geq -C_1 \log (\varepsilon r)^n + C_2. \tag{6.2}
$$

Set $\psi_{\varepsilon, r} = \frac{I_{n/2} \phi_{\varepsilon, r}}{\| \phi_{\varepsilon, r} \|^2}$, then on the ball $B_{\varepsilon r/2}$,

$$
(\psi_{\varepsilon, r} (x))^2 \geq \left( -\frac{\log (\varepsilon r)^n}{\beta (n, \frac{n}{2})} + b_r - C \right)^2
\geq \left( -\frac{\log (\varepsilon r)^n}{\beta (n, \frac{n}{2})} + b_r + C \right)^2
$$
\[ \geq - \frac{\log (\varepsilon r)^n}{\beta(n, \frac{n}{2})} + b_r \left( 1 - \frac{C}{\log \frac{1}{\varepsilon r}} \right) - C, \]

which gives

\[
\sup_{\|(-\Delta)^{n/4} u\|_2 \leq 1} \int_{\mathbb{R}^n} \frac{\exp \left( \beta \left( n, \frac{n}{2} \right) u^2 \right) - 1}{(1 + u)^p} dx \\
\geq \int_{B_{r/2}} \frac{\exp \left( \beta \left( n, \frac{n}{2} \right) (\psi_{\varepsilon, r}(x))^2 \right)}{(1 + \psi_{\varepsilon, r}(x))^p} dx \\
\geq \int_{B_{r/2}} \frac{\exp \left( \beta \left( n, \frac{n}{2} \right) \left( - \frac{\log (\varepsilon r)^n}{\beta(n, \frac{n}{2})} + b_r \left( 1 - \frac{C}{\log \frac{1}{\varepsilon r}} \right) - C \right) \right)}{(1 + \psi_{\varepsilon, r}(x))^p} dx \\
\geq \frac{C}{\left( \log \frac{1}{(\varepsilon r)^n} \right)^{\frac{p}{2}}}. 
\]

On the other hand, from (6.1) and (6.2), we see that

\[
\int_{\mathbb{R}^n} (\psi_{\varepsilon, r}(x))^2 dx = \left\| I_{\varepsilon/2} \phi_{\varepsilon, r} \right\|_2^2 \\
\geq \left\| \phi_{\varepsilon, r} \right\|_2^2 \geq -\log (\varepsilon r)^n + C.
\]

Combining the above estimate, we conclude that if \( p < 2, \)

\[
\sup_{\|(-\Delta)^{n/4} u\|_2 \leq 1} \frac{1}{\|u\|_2^2} \int_{\mathbb{R}^n} \frac{\exp (\beta u^2) - 1}{(1 + u)^p} dx \\
\geq \frac{1}{\|\psi_{\varepsilon, r}(x)\|_2^2} \int_{B_{r/2}} \frac{\exp \left( \beta (\psi_{\varepsilon, r}(x))^2 \right)}{(1 + \psi_{\varepsilon, r}(x))^p} dx \\
\geq C \frac{\log \frac{1}{(\varepsilon r)^n}}{r^n \left( \log \frac{1}{(\varepsilon r)^n} \right)^{\frac{p}{2}}} \\
= C \frac{r^n \left( \log \frac{1}{(\varepsilon r)^n} \right)^{1 - \frac{p}{2}}} {\rightarrow \infty}
\]

as \( \varepsilon \to 0, \) which completes the proof of Theorem 1.8. \( \square \)

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