Four-momentum boosted Fermion fields

P.A. Boyle^ab

^aDepartment of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, Scotland
^bDepartment of Physics, Columbia University, New York, NY, 10027

A formulation of the fermion action is discussed which includes an explicit four momentum boost on the field prior to discretisation. This is used to shift the zero of lattice momentum to lie near one of the on-shell quark poles. The positive pole is selected if we wish to describe a valence quark, and negative pole for a valence anti-quark. Like NRQCD, the typical lattice momenta involved in hadronic correlation functions can be kept small: of order $O(\Lambda^2_{\text{QCD}})$, rather than $O(m_\text{QCD})$ even when describing heavy quarks. If we expand around the particle pole, the anti-particle correlator will be poorly described for large $am_\text{QCD}$. However, in that case the anti-particle will be far off shell and will only affect unphysical, renormalization factors. The formulation produces the correct continuum limit, and preliminary results have been obtained (for an unimproved action) of both the one-loop self energy and a non-perturbative correlation function.

Introduction

The heavy quark effective theory can be derived from the Minkowski path integral in two stages. Firstly a change of variables $\psi' = e^{iq\cdot x}\psi$ is made on the path integral to include an explicit four-momentum $q_\mu = mv_\mu$ in the quark fields, to boost to the rest-frame $v_\mu$ of the hadron:

$$G^m(x_1,\ldots,x_n) = \int d\tilde{\psi}d\psi\tilde{\psi}(x_1)\ldots\psi(x_n)e^{iS_F}$$

$$G^m = e^{iq\cdot x_1}\ldots e^{-iq\cdot x_n}\tilde{G}^m(x_1,\ldots,x_n)$$

$$\tilde{G}^m = \int d\tilde{\psi}'d\psi'\tilde{\psi}'(x_1)\ldots\psi'(x_n)e^{iS_F}$$

where we can immediately perform the derivative that pulls down the boosting phase:

$$S_F' = \int d^4z\tilde{\psi}'(z)\left[\mathcal{D} + m + \not{g}\right]\psi'(z)$$

To tree level we can immediately identify an anti-particle projected mass term as $2m\mathcal{P}(v)$ where the projector $\mathcal{P}(v) = (1 + \not{v})/2$. Beyond tree level the boost to the frame $v_\mu$ will contain counter-terms that keep the renormalised boost in the desired location and we write the bare mass term as $m + \not{g}(g,m,v)$.

We may choose to formulate a discretised theory from either representation of the path integral. However, we will argue that the latter is the preferable starting point for a lattice action for hadrons containing heavy quarks.

In HQET and (moving) NRQCD a Foldy-Wouthuysen transformation is performed on the four-spinors, and the lower component integrated out. In NRQCD a truncated expansion in $1/\Lambda^2_{\text{QCD}}$ is performed on the resulting effective action, and loop corrections to this expansion become divergent as the quark mass becomes less than the inverse lattice spacing. As noted in [1], by retaining lower components, we maintain the attractive property of possessing a continuum limit.

Formulation

The action $4$ can be Euclideanised and then discretised using any of the currently used four-component lattice Dirac operators, which we label generically as $D^L$. This generates a Euclidean lattice action for the boosted field, where we take the Euclidean momentum $q_F^E = i\not{q}_0^M$

$$S_F^E = \sum_z \tilde{\psi}(z)\left[\not{D}^L + m_0 + i\not{g}^E(g,m_0,v)\right]\psi(z)$$

This action has a modified mass term containing a particle (anti-particle if $q_0^M < 0$) projector, and is exactly analogous to HQET where the particle fields are slowly varying, and the anti-particle fields acquire a doubled mass term.
Tree level masses

It is simple to write down the Feynman rules for our action in terms of the Feynman rules for the massless action $\psi D^L \psi$.

We denote the Feynman rules associated with the lattice derivative $D^L$ as propagator $G^L(p)^{-1} = i\gamma_{\mu} S_{\mu}(p) + R(p)$, and quark n-gluon vertices $V^n(p_q, p_{q1}, \ldots, p_{qn})$. The quark-gluon vertices are unaltered for the boosted action, the modified propagator is given by:

$$\tilde{G}^i(p)^{-1} = i\gamma_{\mu} S_{\mu}(p) + R(p) + m_0 + i\gamma_E (g, m_0, v)$$

For the Wilson action, $S_{\mu}(p) = p_{\mu}$ and $R(p)$ encodes the Wilson term. Generically, the correct continuum limit requires

$$S_{\mu}(p) = p_{\mu} + O(p^3)$$

$$R(p) = R_{\mu\nu} p_{\mu} p_{\nu} + h.o.$$  

While for an $O(a)$ off-shell improved derivative we require of the chiral symmetry breaking term $R_{\mu\nu} = 0$, the clover action has $R_{\mu\nu} = \frac{1}{2} \delta_{\mu\nu}$ and applies the equation of motion to “redefine” the mass at order $(am)^2$. This trick is not appropriate here due to the Dirac structure and size of our mass term.

The low momentum expansion of the denominator of the $G$ propagator for the tree level boost $q_4^I = im_0$, and $q_j = 0$ is:

$$- 2im_0p_0 + \vec{p}^2 + 2m_0 R_{ij} p_i p_j$$ (5)

This low momentum expansion immediately implies the solutions for the pole ($m_1$) and kinetic ($m_2$) masses in Table 1.

|   | $m_1$ | $m_2(R = 0)$ | $m_2(D^W)$ |
|---|-------|-------------|-------------|
| $G$ | 0     | $m_0$       | $\frac{m_0}{1+m_0}$ |
| $G$ | $m_0$ | $m_0$       | $\frac{1+m_0}{1+m_0}$ |
| Wilson | log$(1 + m_0)$ | N/A         | $e^{m_1}$ |

Thus the boosted Fermion approach automatically produces the perfect $O(p^2)$ dispersion relation at tree-level, unlike other four component approaches, provided the lattice derivative used is $O(a)$ improved. This is not the case for the Wilson derivative, but would certainly be the case for a Neuberger or Domain Wall based lattice operator. Alternative schemes have also been proposed for removing the doublers which introduce no such classical $O(a)$ term [4]. Regardless, in what follows we treat the boosted Wilson case as an expedient toy model for testing out the approach.

Momentum dependence of vertices

To higher order, in calculations of vertex function renormalisation or the self energy, for example, we are interested in the on-shell point for external quark momenta. For standard four component formulations this introduces a large imaginary temporal lattice momenta $q_4 \approx im_1$, which flows through and distorts the lattice quark gluon vertices as shown in Table 2. Indeed this mechanism is responsible for the dominance of the temporal gluon vertex that renders the perturbative equivalence of the heavy Wilson action with the static Wilson line. In contrast, the on-shell point of the boosted approach is by construction at the zero of external lattice momentum, and the quark gluon couplings are minimally distorted.

One loop mass renormalisation for $D^W$

We have calculated preliminary results for the one-loop self energy with both boosted and regular Wilson fermions. The result for regular Wilson fermions agrees with those of [5]. In the Wilson case the one-loop self energy $\Sigma^{[1]}$ (as opposed to mass correction $m^{[1]}$) grows exponentially with the pole mass via the couplings in Table 2. For boosted Wilson Fermions, however, the zero external four-momentum leaves the tadpole graph independent of the mass, and the the rainbow graph is only weakly dependent on the quark mass. A table of these results will be presented in a subsequent publication [7].

Non-perturbative correlator with $D^W$

For our toy $D^W$ case, the correct massless limit occurs at $m_0 = m_{\text{crit}} \neq 0$. The required boost $q_4$ should vanish when and $q_j = 0$ and $m_0 = m_{\text{crit}}$. We therefore input the known non-perturbative $m_{\text{crit}}$ and take $m_0 = \tilde{m} + m_{\text{crit}}$, $q_4 = i(\tilde{m} - \Lambda)$, $q_j = 0$. 
Table 2

|          | Boosted Wilson                                                                 | Heavy Wilson                                                                 |
|----------|-------------------------------------------------------------------------------|-------------------------------------------------------------------------------|
| $qqg_1$  | $-ig \left[ \gamma_0 \cos \frac{1}{2} k_0 - ir \sin \frac{1}{2} k_0 \right]$ | $-ig \left[ \gamma_0 \cosh(m_1 - i \frac{1}{2} k_0) - ir \sinh(m_1 - i \frac{1}{2} k_0) \right]$ |
| $qqgg_1$ | $-g^2 \left[ r \cos \frac{1}{2} k_0 - i \gamma_0 \sin \frac{1}{2} k_0 \right]$ | $-g^2 \left[ r \cosh(m_1 - i \frac{1}{2} k_0) - i \gamma_0 \sinh(m_1 - i \frac{1}{2} k_0) \right]$ |

This subtraction of $m_{crit}$ is a technicality that is unique to the Wilson Dirac operator.

If the behaviour of $q(m, g, v)$ is benign we will not have a significant “tuning” problem to locate the required non-perturbative boost, and then a massless mode will arise for small values of $\Lambda$. In fact, it proved necessary to not make the regulating parameter $\Lambda$ too small in order to control the expense of solving the propagator.

The solver for the projected mass term is simple to implement, and preconditioning easily. Some inversions have been performed on a $16^3 \times 32$ quenched lattice at $\beta = 6.0$, and a heavy-light pseudoscalar correlation function obtained in Figure 1.

Figure 1. Heavy-light $Q\bar{q}$ pseudo-scalar correlator. $\bar{m} = 2.0$, $\Lambda = 0.1$ and $\kappa_q = 0.1550$ at $\beta = 6.0$ with Wilson Dirac operator for both light and heavy quarks. The forward propagator is light, and the heavy anti-particle mode is clearly seen propagating backwards.

Conclusions

While the data are not relevant for physical conclusions, the features of the correlation function successfully demonstrate the method. The light forward propagating particle mode is clearly present, with $aM^+_P \approx 0.25$, while the heavy backwards propagating anti-particle mode is also clearly seen, with $aM^-_P \approx 4.0$.

When the technique is coupled with an $O(a)$ off-shell improved Dirac operator such as $D^{\text{Neuberger}}$ it will lead to much improved lattice treatment of the semi-relativistic regime, such as charmonium and $D$ mesons.

The ability to insert spatial momenta in the action is highly promising for calculations with difficult kinematics, such as semi-leptonic decays.

Acknowledgements

During the lattice conference I became aware that a similar four-momentum boosted approach has been independently pursued by Paul Mackenzie [6,7], and I thank him for a copy of his unpublished notes. The Mackenzie approach differs significantly in the technique for improvement and avoiding Fermion doubling.

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