Five-Branes in Heterotic Brane-World Theories

Matthias Brändle\textsuperscript{1,*} and André Lukas\textsuperscript{2§}

\textsuperscript{1}Institut für Physik, Humboldt Universität
Invalidenstraße 110, 10115 Berlin, Germany

\textsuperscript{2}Centre for Theoretical Physics, University of Sussex
Falmer, Brighton BN1 9QJ, UK

Abstract

The effective action for five-dimensional heterotic M-theory in the presence of five-branes is systematically derived from Hořava-Witten theory coupled to an M5-brane world-volume theory. This leads to a five-dimensional $N = 1$ gauged supergravity theory on $S^1/Z_2$ coupled to four-dimensional $N = 1$ theories residing on the two orbifold fixed planes and an additional bulk three-brane. We analyse the properties of this action, particularly the four-dimensional effective theory associated with the domain-wall vacuum state. The moduli Kähler potential and the gauge-kinetic functions are determined along with the explicit relations between four-dimensional superfields and five-dimensional component fields.

\textsuperscript{*}email: brand@physik.hu-berlin.de
\textsuperscript{§}email: a.lukas@sussex.ac.uk
1 Introduction

A large class of attractive five-dimensional brane-world models can be constructed by reducing Hořava-Witten theory \cite{1,2,3} on Calabi-Yau three-folds. This procedure has been first carried out in Ref. \cite{4,5,6} and it leads to gauged five-dimensional $N=1$ supergravity on the orbifold $S^1/Z_2$ coupled to $N=1$ gauge and gauge matter multiplets located on the two four-dimensional orbifold fixed planes. It has been shown \cite{7}-\cite{13} that a phenomenologically interesting particle spectrum on the orbifold planes can be obtained by appropriate compactifications.

Early on it has been realized \cite{3} that M5-branes being transverse to the orbifold direction, wrapping a holomorphic curve in the Calabi-Yau space and stretching across the four uncompactified dimensions can be incorporated into this picture. The explicit form of the corresponding 11-dimensional vacuum solutions has been given in Ref. \cite{15}. In the five-dimensional brane-world theory such an M5-brane appears as a three-brane located in the bulk away from the orbifold fixed planes. This provides an interesting generalisation of five-dimensional heterotic brane-world models which has recently attracted some attention \cite{16,21}, particularly in the context of cosmology. Some features of such generalised brane-world models have already been analysed in Ref. \cite{15} and used in subsequent applications. The purpose of this paper is to present a systematic derivation of the five-dimensional effective action for these models in its simplest form and discuss its properties.

Starting point for this derivation is Hořava-Witten theory in 11 dimensions coupled to an M5-brane world-volume theory \cite{22}. The action for this system is obtained by combining results presented in Refs. \cite{2,24,23}. We show explicitly that the warped vacua based on Calabi-Yau three folds presented in Ref. \cite{15} can be promoted to solutions of this coupled theory. Given previous results, this amounts to showing that the five-brane sources in the Einstein equation are properly matched and that the five-brane world-volume equations of motion are satisfied. We obtain the five-dimensional brane-world theory by performing a reduction on a Calabi-Yau three-fold focusing on the universal sector of the Calabi-Yau zero modes. The result is consistent with previous expectations \cite{15} and represents a five-dimensional $N=1$ gauged supergravity on $S^1/Z_2$ coupled to the $N=1$ theories on the orbifold planes and an additional $N=1$ theory located on the three-brane. We verify that the BPS domain-wall of Ref. \cite{4} can be generalised to include the effect of this additional three-brane. The action, as obtained by reduction from 11 dimensions, is expressed in terms of the “step-function” $\alpha$ which represents the mass parameter of the gauged supergravity. For this form of the action, the three-brane is coupled magnetically. We also present a dual action where $\alpha$ is replaced by a four-form with five-form field strength to which the three-brane couples electrically.

Finally, we analyse the four-dimensional $N=1$ effective theory associated to the domain-wall vacuum state. We determine the moduli Kähler potential for the dilaton, the $T$-modulus and the five-brane position modulus. Our result agrees with Ref. \cite{22} where somewhat different methods have been employed. In addition, we obtain the explicit relations between the four-dimensional superfields and the five-dimensional component fields which are vital whenever four-dimensional results have to be interpreted in terms of the five-dimensional brane-world theory. We also compute the gauge-kinetic functions for the orbifold gauge fields and find threshold corrections in agreement with Ref. \cite{14,22}. Finally, we show that the gauge-kinetic function for the three-brane gauge fields does not receive any threshold corrections at leading order and is simply given by the period matrix of the complex curve wrapped by the five-brane.

2 M5-brane coupled to D=11 Supergravity on an Orbifold
2.1 The 11-dimensional action

In this section, we would like to set the stage by describing our starting point, the effective $D = 11$ action for M-theory on the orbifold $S^3/Z_2$ coupled to a five-brane. In order to derive this action, we can draw information from two main sources, namely the Hořava–Witten (HW) action [1, 2] for M-theory on $S^3/Z_2$ and the action for $D = 11$ supergravity coupled to a five-brane due to Bandos, Berkovits and Sorokin [23, 24]. Combining these two results it not completely straightforward and requires a careful analysis of the various symmetries involved. A detailed account of this will be given in a forthcoming work [23]. Here we will merely present the final result for the bosonic part of this action which reads [1]

$$S = -\frac{1}{2\kappa^2} \int_M \left\{ \frac{d^{11}x}{\sqrt{-g}} \left( \frac{1}{2} R + \frac{1}{4!} G_{IJKL} G^{IJKL} \right) + \frac{2}{3} C \wedge G \wedge G \right\} - \frac{1}{4\lambda^2} \sum_{k=1}^{2} \int_{M_{10}} d^{10}x \sqrt{-g_{10}} \left\{ \text{tr} F_k^2 - \frac{1}{2} \text{tr} R^2 \right\} - \frac{1}{2} T_5 \int_{M_6} d^6\sqrt{-\gamma} \left[ 1 + \nu_1(*H)^{mn}(*H - H)_{mnp} \right] + 2 dB \wedge \hat{C} + T_5 \int_M C \wedge dC \left[ \Theta(M_6) + \Theta(\hat{M}_6) \right].$$

(2.1)

The structure of 11-dimensional space-time in this action is $M = M_{10} \times S^3/Z_2$ where $M_{10}$ is ten-dimensional space-time and we work in the upstairs picture. As usual, we define the orbifold coordinate $y = x^{11}$ to be in the range $y \in [-\pi \rho, \pi \rho]$ and let the $Z_2$ orbifold symmetry act as $y \rightarrow -y$. This leads to the two fixed ten-dimensional hyperplanes $M_{10}^1$ and $M_{10}^2$ located at $y = y_1 = 0$ and $y = y_2 = \pi \rho$, respectively. Further, we have a single five-brane [1] with world-volume $M_6$ plus its $Z_2$ mirror with world-volume $\hat{M}_6$ which originates from $M_6$ by applying the orbifold map $y \rightarrow -y$. This latter mirror five-brane is required by consistency in order to keep the theory $Z_2$ symmetric. Further, to avoid the appearance of additional states [20], we demand that the five-brane world-volume does not intersect either of the two orbifold fixed planes. We use indices $I, J, K, \ldots = 0, \ldots, 10, 11$ for 11-dimensional space-time with coordinates $x^I$ and indices $m, n, p, \ldots = 0, \ldots, 5$ for the five-brane world-volume with coordinates $\sigma^m$.

Let us now discuss the various sectors of the above action in some detail. The bulk fields consist of the fields of 11-dimensional supergravity, that is the $Z_2$–even [1] 11-dimensional metric $g_{IJ}$, the $Z_2$–odd three-index antisymmetric tensor field $C_{IJK}$ and the gravitino $\Psi_I$, subject to the usual $Z_2$ truncation [1]. The standard relation $G = dC$ between $C$ and its field strength $G$ will be modified due to the presence of source terms and this will be explicitly presented shortly. Anomaly cancellation requires the two orbifold fixed planes $M_{10}^k$ to each carry a 10-dimensional $N = 1$ $E_8$ gauge multiplet [1] that is an $E_8$ gauge field $A_k$ with field strength $F_k$ and gauginos $\chi_k$, where $k = 1, 2$. The Yang-Mills coupling $\lambda$ is fixed in terms of the 11-dimensional Newton constant $\kappa$ by [1, 27, 28]

$$\lambda^2 = 4\pi (4\pi \kappa^2)^\frac{3}{2}. \quad (2.2)$$

The five-brane world-volume fields consist of the embedding coordinates $X^I = X^I(\sigma^m)$ together with the fermions $\theta$ and the two-index antisymmetric tensor field $B_{mn}$. The five-brane part of the above action is written in the

\footnote{For the bulk fields we adopt the normalisation of Ref. [1]. The normalisation chosen by Hořava and Witten [1] is obtained by the rescaling $g_{HW} = 2^{-2/9}g$, $C_{HW} = 2^{1/6}C$ and $G_{HW} = 2^{1/6}G$.}

\footnote{The generalisation to include an arbitrary number of five-branes is straightforward and will be covered in Ref. [3].}

\footnote{We call a tensor field $Z_2$–even if its components orthogonal to the orbifold are even, otherwise we call it $Z_2$–odd.}
form due to Pasti, Sorokin and Tonin (PST) [14] which requires the introduction of an auxiliary scalar field \( a \) and an associated unit vector field \( v_m \) defined by

\[
v_m = \frac{\partial_m a}{\sqrt{\gamma^{np} \partial_n a \partial_p a}}. \tag{2.3}
\]

The presence of this field enhances the symmetries of the action such that \( a \) is truly auxiliary and that fixing one of the symmetries turns the equation of motion for \( B \) into the self-duality condition \( *H = H \). For this to actually work the Wess-Zumino term \( dB \wedge \hat{C} \) must be present. For simplicity, we have chosen to present a linearised form of the PST action as is appropriate for our subsequent discussion. Here we will not consider any further details of the PST-formulation and how precisely it relates to the derivation of our action (2.1), but instead refer to Ref. [35] for a detailed discussion. As usual, the metric \( \gamma_{mn} \) is the pull-back

\[
\gamma_{mn} = \partial_m X^I \partial_n X^J g_{IJ} \tag{2.4}
\]

of the space-time metric \( g_{IJ} \). Further, the field strength \( H \) of \( B \) is defined by

\[
H = dB - \hat{C} \tag{2.5}
\]

where \( \hat{C} \) denotes the pull-back of the bulk field \( C \), that is,

\[
\hat{C}_{mnp} = \partial_m X^I \partial_n X^J \partial_p X^K C_{IJK} \tag{2.6}
\]

The five-brane tension \( T_5 \) can be expressed in terms of the 11-dimensional Newton constant as

\[
T_5 = \left( \frac{\pi}{2\kappa^4} \right)^{\frac{1}{3}}. \tag{2.7}
\]

Having introduced all fields we should now specify the source terms in the definition of the bulk antisymmetric tensor field strength. To this end, we introduce

\[
G = dC - \omega_{YM} - \omega_{M5} \tag{2.8}
\]

\[
\mathcal{G} = dB - \hat{C} \tag{2.9}
\]

The field strength \( G \) is defined as in pure HW theory without five-branes, that is, it only contains the “Yang-Mills” sources \( \omega_{YM} \) which originate from the orbifold fixed planes and are given by

\[
\omega_{YM} = 2k [\omega_1 \wedge \delta(y) + \omega_2 \wedge \delta(y - \pi \rho)] \tag{2.10}
\]

with the “Chern-Simons” forms \( \omega_k \) satisfying

\[
J_k \equiv d\omega_k = \frac{1}{16\pi^2} \left[ \text{tr} F_k \wedge F_k - \frac{1}{2} \text{tr} R \wedge R \right]_{y=y_k} \tag{2.11}
\]

where \( k = 1, 2 \). The field strength \( G \), on the other hand, contains both orbifold and five-brane sources where the latter are defined by

\[
\omega_{M5} = k \left[ \Theta(M_6) + \Theta(\tilde{M}_6) \right]. \tag{2.12}
\]

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\(^{4}\)By \( \delta(y) \) we denote a \( \delta \)-function one-form defined by \( \delta(y)dy \), where \( \delta(y) \) is the ordinary \( \delta \)-function.
Here $\Theta(M_6)$ is the $\theta$–function associated with the five-brane world-volume $M_6$. In analogy with the ordinary one-dimensional $\theta$–function it satisfies the relation

$$d\Theta(M_6) = \delta(M_6),$$

where $\delta(M_6)$ is the $\delta$–function supported on $M_6$ (Analogous expression hold for $\tilde{M}_6$). For later calculations it will be useful to explicitly express these functions in terms of the embedding coordinates $X^I$ by writing

$$\Theta(M_6) = \frac{1}{4!7!\sqrt{-g}} dx^{I_1} \wedge \ldots \wedge dx^{I_4} \epsilon_{I_1 \ldots I_4} \int_{M_7} dX^{I_5} \wedge \ldots \wedge dX^{I_{11}} \delta^{11}(x - X(\sigma)).$$

$$\delta(M_6) = \frac{-1}{5!6!\sqrt{-g}} dx^{I_1} \wedge \ldots \wedge dx^{I_5} \epsilon_{I_1 \ldots I_5} \int_{M_6 = \partial M_7} dX^{I_6} \wedge \ldots \wedge dX^{I_{11}} \hat{\delta}^{11}(x - X(\sigma)).$$

We see that the definition of $\Theta(M_6)$ and, hence, our action (2.1) involves a seven-manifold $M_7$ which is bounded by the five-brane world-volume $M_6$, that is, $\partial M_7 = M_6$. This seven-manifold is the analogue of a Dirac-string for a monopole in Maxwell theory and is also referred to as Dirac-brane [29, 30]. There may be a problem in that the action depends on the particular choice of the Dirac-brane. A prescription to resolve this ambiguity has been proposed in ref. [24]. Since our subsequent considerations do not depend on how precisely the Dirac-brane is defined we will not consider this point in any further detail. The constant $k$ in the above definitions for the field strengths is again fixed in terms of the 11-dimensional Newton constant and is given by

$$k = \left(\frac{\pi}{2}\right)^{\frac{4}{3}} \kappa^{2/3} = \kappa^2 T_5 = 8\pi^2 \kappa^2 / \lambda^2.$$  

(2.16)

From the definition (2.8) we can now write the Bianchi identity

$$dG = -2k \left[ J_1 \wedge \delta(y) + J_2 \wedge \delta(y - \pi\rho) + \frac{1}{2} \left( \delta(M_6) + \delta(\tilde{M}_6) \right) \right],$$

(2.17)

for $G$ which will be important later on. The relative factor $1/2$ between the orbifold and five-brane sources accounts for the fact that the five-brane and its mirror really represent the same physical object and should, therefore, not be counted independently.

2.2 Symmetries

Let us discuss the symmetries of the action (2.4) some of which will become relevant later on.

In the following we would like to check the BPS property of certain solutions, hence we will need the (bosonic part) of the supersymmetry transformations which we explicitly present for completeness. For the gravitino $\Psi_I$, the $E_8$ gauginos $\chi_k$ on the two orbifold fixed planes and the five-brane world-volume fermions $\theta$ they are, respectively, given by

$$\delta \Psi_I = D_I \eta + \frac{1}{3!} \left[ \frac{1}{4!} \Gamma_{I_1 J_2 J_3 J_4} - \frac{2}{3!} \eta_{I_1 J_2 J_3 J_4} \right] G^{I_1 J_2 J_3 J_4} \eta$$

(2.18)

$$\delta \chi_k = -\frac{1}{4} \eta^{i j} F_{k i j} \eta$$

(2.19)

$$\delta \theta = \eta + P_+ \kappa$$

(2.20)

where the projection operators $P_\pm$ satisfying $P_+ + P_- = 1$ are defined by

$$P_\pm = \frac{1}{2} \left( 1 \pm e^{m_1 \ldots m_6} \partial_{m_1} X^{I_1} \ldots \partial_{m_6} X^{I_6} \Gamma_{I_1 \ldots I_6} \right).$$

(2.21)
For simplicity, we have stated these projection operators for the later relevant case $H = 0$. The general expressions can be found in Ref. \[32\]. In the above equations the spinor $\eta$ parametrises supersymmetry transformations. The five-brane world-volume theory is also invariant under an additional fermionic symmetry, namely local $\kappa$–symmetry. It is parametrised by the spinor $\kappa$ and appears via the second term in Eq. (2.20). Further, $D_I$ is the covariant derivative and $\Gamma_{I_1 \ldots I_p}$ denotes the antisymmetrised products of $p$ gamma-matrices $\Gamma_I$ which satisfy the usual Clifford algebra $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$.

Besides supersymmetry the action is also, up to total derivatives, invariant under the following gauge variations

$$\delta C = d\Lambda^{(2)}, \quad \delta B = d\Lambda^{(1)} + \hat{\Lambda}^{(2)},$$

(2.22)

where $\Lambda^{(1)}$ is an arbitrary one-form. The two-form $\Lambda^{(2)}$ has to be $\mathbb{Z}_2$–odd in order to ensure that the $\mathbb{Z}_2$ properties of $C$ are preserved under the above transformation.

There are two more symmetries on the world-volume of the M5-brane, namely the “PST-symmetries” given by

$$\delta B_{mn} = (da \wedge \phi^{(1)})_{mn} - \frac{\varphi}{\sqrt{(\partial a)^2}} v^l (\ast H - H)_{lmn}, \quad \delta a = \varphi$$

(2.23)

where $\phi^{(1)}$ and $\varphi$ are an arbitrary one-form and a scalar, respectively. As previously mentioned, these symmetries ensure that the self-duality of $H$ follows from the equations of motion and that $a$ is an auxiliary field.

3 Calabi-Yau background in $D = 11$

3.1 The solution

Background solutions of heterotic M-theory based on Calabi-Yau three-folds which respect four-dimensional Poincaré invariance and $N = 1$ supersymmetry were first presented in Ref. [3]. This paper also demonstrated how to include five-branes in those backgrounds while preserving the four-dimensional symmetries. The explicit form of these solutions was subsequently given in Ref. [15]. All these result were based on the original action derived by Hořava and Witten [2] which does not explicitly include any five-brane world-volume theories. The effect of five-branes on the supergravity background was incorporated by modifying the Bianchi-identity of $G$ to include the five-brane sources as in Eq. (2.17). The main purpose of this section is to prove that the solutions obtained in this way can indeed be extended to solutions of the full action (2.1) which does include the five-brane world-volume theory. Practically, this amounts to showing that these solutions correctly match the five-brane source terms in the Einstein equations and that the five-brane world-volume equations of motion are satisfied.

Following Ref. [15], let us start reviewing the solutions which are constructed as an expansion in powers of $\kappa^{2/3}$. At lowest order, we consider the space-time structure $M = S^1/\mathbb{Z}_2 \times X \times M_4$, where $X$ is a Calabi-Yau three-fold and $M_4$ four-dimensional Minkowski space. Coordinates in $M_4$ are labelled by indices $\mu, \nu, \rho, \ldots = 0, \ldots , 3$. The Ricci-flat metric on the Calabi-Yau space is denoted by $\Omega_{AB}$ with six-dimensional indices $A, B, C, \ldots = 5, \ldots , 10$. The Kähler-form $\omega$ is defined by $\omega_{\bar{a} \bar{b}} = i\Omega_{\bar{a} \bar{b}}$ where $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$ are holomorphic and anti-holomorphic indices on the Calabi-Yau space, respectively. For simplicity, we will restrict our considerations to the universal
sector of the Calabi-Yau space, that is, strictly our results apply to Calabi-Yau spaces with \( h^{1,1} = 1 \). The four-
form field strength \( G \) vanishes at lowest order. This configuration constitutes a solution to the Killing spinor

\[
\delta \Psi_I = 0
\]

and the Bianchi identity since the source terms in Eq. (2.17) are proportional to \( \kappa^{2/3} \) and, hence, do not contribute at lowest order. At the next order, however, these source terms have to be taken into

\[
\text{account and, as a consequence, the field strength } G \text{ becomes non-vanishing. This induces corrections to the metric which can be computed requiring that } N = 1 \text{ supersymmetry is preserved and, hence, that the gravitino variation (2.18) vanishes. The size of these corrections is measures by the strong-coupling expansion parameter } \epsilon_S \text{ defined by}
\]

\[
\epsilon_S \equiv \pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{2\pi\rho}{\nu^{2/3}} = \pi \rho T_5 \kappa^2
\]

\[(3.1)\]

where \( \nu = \int_X \sqrt{\Omega} \) is the Calabi-Yau volume.

We should now specify the full solutions (to order \( \epsilon_S \)) and we start with the gauge fields on the orbifold

planes. In general, we have non-trivial holomorphic vector bundles on the Calabi-Yau space. These bundles correspond to gauge field backgrounds \( \tilde{A}_k \) in the Calabi-Yau directions which preserve supersymmetry and are, hence, constrained by a vanishing gaugino variation (2.19). This implies that their associated field strengths \( \tilde{F}_k \) are \((1,1)\) forms on the Calabi-Yau space. Then, the orbifold sources \( J_k \) in the Bianchi-identity are \((2,2)\) forms given by

\[
J_k \equiv d\omega_k = \frac{1}{16\pi^2} \left\{ \text{tr} \tilde{F}_k \wedge \tilde{F}_k - \frac{1}{2} \text{tr} R^{(\Omega)} \wedge R^{(\Omega)} \right\}_{y=y_k}
\]

\[(3.2)\]

where \( R^{(\Omega)} \) is the Calabi-Yau curvature tensor associated with the metric \( \Omega \).

Next, we should consider the five-brane world-volume fields. Guided by the structure of our action (2.1), we focus on a single five-brane (and its \( \mathbb{Z}_2 \) mirror) which is taken to be static and parallel to the orbifold fixed planes. Furthermore, two spatial dimensions of the world-volume \( M_6 \) wrap around a holomorphic two-cycle \( C_2 \) of the internal Calabi-Yau space \( X \) and the remaining four dimensions stretch across the external Minkowski space-time \( M_4 \). Accordingly, we split the five-brane coordinates \( \sigma^m \) into external and internal coordinates, that is, \( \sigma^m = (\sigma^\mu, \sigma^i) \) where \( \mu, \nu, \rho, \ldots = 0, \ldots, 3 \) and \( i,j,\ldots = 4,5 \). Further, we define holomorphic and anti-
holomorphic coordinates \( \sigma = \sigma^4 + i\sigma^5 \) and \( \bar{\sigma} = \sigma^4 - i\sigma^5 \). With these definitions, the five-brane embedding is specified by

\[
X^\mu = \sigma^\mu, \quad \hat{X}^a = X^a(\sigma), \quad X^{11} = \pm Y
\]

\[(3.3)\]

where \( Y \in [0,\pi \rho] \) is a constant, \( X^a(\sigma) \) parametrises the holomorphic curve \( C_2 \) and the two signs in the last equation account for the five-brane \( M_6 \) and its mirror \( \tilde{M}_6 \). The world-volume two-form \( B \) is taken to vanish in the background. It can be explicitly shown [15] that this configuration preserves supersymmetry on the five-brane by choosing \( \kappa = -\eta \) in the variation (2.20) and verifying that \( P_+ \eta = 0 \). The five-brane source in the Bianchi identity then takes the specific form

\[
J_{M5} = d\omega_{M5} = k J_5 \wedge [\delta(y - Y) + \delta(y + Y)]
\]

\[
J_5 = \delta(C_2) = \frac{1}{2 \cdot 4! \sqrt{\Omega}} d\sigma^A_1 \wedge \ldots \wedge d\sigma^A_4 \epsilon_{A_1 \ldots A_4 BC} \int_{C_2} dX^B \wedge dX^C \delta^6(x - X(\sigma)).
\]

\[(3.4)\]

\[(3.5)\]

The embedding (3.3) implies that \( J_5 \) is a \((2,2)\) form on the Calabi-Yau space as well.
Now we have explicitly presented all source terms in the Bianchi identity and we can use Eqs. (2.17) and (2.18) to determine $G$ and the corrected metric. The result is

$$ds_{11}^2 = (1 - h)\eta_{\mu\nu}dx^\mu dx^\nu + (1 + h)\Omega_{AB}dx^A dx^B + (1 + 2h)dy^2 \quad (3.6)$$

$$G = \frac{1}{2} \partial_y h(y) \ast \omega \quad (3.7)$$

where

$$h(y) = \frac{2}{3} \begin{cases} \alpha_1 |y| + c & \text{for } 0 \leq |y| \leq Y \\ (\alpha_1 + \alpha_5)|y| - \alpha_5 Y + c & \text{for } Y \leq |y| \leq \pi \rho \end{cases} \quad (3.8)$$

for $c$ a constant and the charges $\alpha_k$ are defined by

$$\alpha_k = \frac{\epsilon_S}{\pi \rho} \beta_k, \quad \beta_k = \int_X \omega \wedge J_k \quad (3.9)$$

We would now like to demonstrate that the above configuration which was mainly obtained by requiring unbroken $N = 1, D = 4$ supersymmetry is indeed a solution of the equations of motion derived from the action (2.1). Given previous results [33], what remains to be shown is that the five-brane sources in the Einstein equation are properly matched and the five-brane world-volume equations of motion are satisfied. To verify the former we should consider the singular terms in the Einstein tensor which turn out to be

$$\begin{align*}
(G_{\mu\nu})_{\text{singular}} &= -\frac{3}{2} \partial_y^2 h \eta_{\mu\nu}, \\
(G_{AB})_{\text{singular}} &= -\frac{1}{2} \partial_y^2 h \Omega_{AB}.
\end{align*} \quad (3.11)$$

These terms have to be compared with the five-brane stress energy tensor which, in general, is given by

$$T_{5IJ} = T_5 \kappa^2 \frac{1}{\sqrt{\gamma}} \int_{M_6 \cup \tilde{M}_6} d^6 \sigma \delta^{11}(x - X(\sigma)) \gamma^{mn} \partial_m X_I \partial_n X_J. \quad (3.12)$$

Evaluating this expression for the embedding (3.3) leads to

$$T_{5\mu
u} = T_5 \kappa^2 \beta_5 \eta_{\mu\nu} (\delta(y - Y) + \delta(y + Y)), \quad T_{5AB} = \frac{1}{3} T_5 \kappa^2 \beta_5 \Omega_{AB} (\delta(y - Y) + \delta(y + Y)) \quad (3.13)$$

with the other components vanishing. In view of Eqs. (3.8) and (3.1) this exactly matches the appropriate delta-function terms in eq. (3.11). We have therefore verified that the five-brane sources in the Einstein equation are properly matched by the solutions.

The only relevant equation of motion on the five-brane world-volume is the one for the embedding coordinates $X^I$. For the case of vanishing $B$ it reads

$$\Box X^I + \Gamma^I_{JK} \gamma^{mn} \partial_m X^J \partial_n X^K + \frac{2}{6!} \epsilon^{m_1 \ldots m_6} \partial_{m_1} X^{I_1} \ldots \partial_{m_6} X^{I_6} (\ast G)^I_{I_1 \ldots I_6} = 0. \quad (3.14)$$

The $\mu$ and $A$ components of this equation turn out to be trivially satisfied for our solution and it remains to check the 11 component. Using the expressions

$$\Gamma^{11}_{\mu \nu} = \frac{1}{2} \partial_y h \eta_{\mu\nu}, \quad \Gamma^{11}_{AB} = -\frac{1}{2} \partial_y h \Omega_{AB} \quad (3.15)$$

for the connection along with the embedding (3.3) and the background (3.7) for $G$ this can indeed easily be done.

In summary, we have, therefore, explicitly verified that the above background configurations are indeed solutions of the action (2.1).
### 3.2 Including moduli

In view of the reduction to five dimensions to be carried out shortly we will now identify the (bosonic) moduli fields of the above background solutions. We will use indices $\alpha, \beta, \gamma = 0, \ldots, 3, 11$ to label five-dimensional coordinates.

Let us start with the bulk fields. As mentioned earlier, we focus on the universal Calabi-Yau sector for simplicity, that is, we are considering Calabi-Yau spaces with $h^{1,1} = 1$. The general case will be examined in Ref. [35]. By absorbing the corrections into the five-dimensional moduli as explained in Ref. [4] the metric can be written as

$$ds^2_{11} = V^{-2/3}g_{\alpha\beta}dx^\alpha dx^\beta + V^{1/3}\Omega_{AB}dx^Adx^B.$$  \hspace{1cm} (3.16)

where the Calabi-Yau volume modulus $V$ and the five-dimensional metric $g_{\alpha\beta}$ are functions of the five-dimensional coordinates $x^\alpha$. From the bulk antisymmetric tensor field $C$ we have, in five dimensions, a three form $C_{\alpha\beta\gamma}$ with field strength $G_{\alpha\beta\gamma\delta}$, a vector field $A_\alpha$ with field strength $F_{\alpha\beta}$ and a complex scalar field $\xi$ with field strength $X_\alpha$. These fields are defined by

$$C_{\alpha\beta\gamma} \quad G_{\alpha\beta\gamma\delta} = 4\partial_{[\alpha}C_{\beta\gamma\delta]}$$
$$C_{\alpha AB} = A_\alpha \omega_{AB} \quad G_{\alpha\beta AB} = \mathcal{F}_{\alpha\beta} \omega_{AB}$$
$$C_{ABC} = \xi \omega_{ABC} \quad G_{\alphaABC} = X_\alpha \omega_{ABC}$$

where $\omega_{ABC}$ is the harmonic $(3,0)$-form on the Calabi-Yau space.

We now turn to the boundary theories. We have already mentioned that, on both boundaries, we have internal gauge bundles on the Calabi-Yau space. The external parts of the gauge fields, denoted by $A_{k\mu}$ with field strengths $F_{k\mu\nu}$, lead to gauge fields on the now four-dimensional orbifold fixed planes $M_k^4$, where $k = 1, 2$. Their gauge groups are given by the commutants of the internal structure group within $E_8$. For simplicity, we will not consider any gauge matter fields on $M_k^4$, although they will be included in [35].

Next we should discuss the zero modes on the five-brane world-volume. The five-brane is allowed to fluctuate in five external dimensions, while internally it can move within the Calabi-Yau space. This leads to the following set of embedding coordinates

$$X^\mu = X^\mu(\sigma^\nu), \quad X^{11} = Y(\sigma^\nu), \quad X^\alpha = X^\alpha(\sigma, M)$$ \hspace{1cm} (3.17)

where $M$ is a set of moduli which parametrises the moduli space of holomorphic curves with a given homology class $[C_2]$ for the Calabi-Yau space under consideration [9, 10, 31]. In our low-energy effective action, we will not explicitly take these moduli into account. The three-brane surface in five-dimensional space specified by the above embedding and the bulk metric (3.16) we find the following non-vanishing components of the induced world-volume metric

$$\gamma_{\mu\nu} = \partial_\mu X^\alpha \partial_\nu X^\beta g_{\alpha\beta}$$ \hspace{1cm} (3.18)
$$\gamma_{jk} = \partial_j X^A \partial_k X^B \Omega_{AB} = 2\partial_\sigma X^\alpha \partial_\sigma X^\beta \Omega_{\alpha\beta},$$ \hspace{1cm} (3.19)

There are also a number of moduli arising from the two-form $B$ which can be determined from the cohomology of the two-cycle $C_2$. We introduce a basis $\lambda_U$ of $H^1(C_2)$ where $U, V, W, \ldots = 1, \ldots, 2g$ and $g$ is the genus of $C_2$ while the pull-back $\hat{\omega}$ of the Calabi-Yau Kähler form to the two-cycle $C_2$ provides a basis for $H^2(C_2)$. Then we...
find the two-form $B_{\mu\nu}$ with field strength $H_{\mu\nu\rho}$, 2g Abelian vector fields $D^{U}_\mu$ with field strengths $E^{U}_{\mu\nu}$, and a scalar $s$ with field strength $j_\mu$ as the low-energy fields on the three-brane $M^5_3$. These fields are defined by

$$
B_{\mu\nu} \quad H_{\mu\nu\rho} = (dB - \hat{C})_{\mu\nu\rho} \\
B_{\rho j} = D^{U}_\rho \lambda_{U j} \quad H_{\mu\nu j} = E^{U}_{\mu\nu\lambda} \lambda_{U j} \\
B_{j k} = s \hat{\omega}_{j k} \quad H_{\mu j k} = j_{\mu} \hat{\omega}_{j k} = (d\hat{A})_{\mu} \hat{\omega}_{j k}
$$

(3.20)

Due to the self-duality condition $*H = H$ these four-dimensional fields are not all independent. In order to work out the relations between them we split the $2g$ vector fields into two sets, that is, we write $(E^{U}) = (E^{u}, \tilde{E}^{u})$ where $u,v,w,\ldots = 1,\ldots ,g$. Then, we find that the self-duality condition reduces to

$$
\begin{align*}
j &= V * H \\
\tilde{E}^{u} &= [\text{Im(\Pi)}]_{uv} * E^{w} + [\text{Re(\Pi)}]_{vw} E^{w}
\end{align*}
$$

(3.21) and (3.22)

where the star is the four-dimensional Hodge-star operator and $\Pi_{uw}$ is the period matrix of the complex curve $C_2$.

To define this matrix we denote by $(a^{uw},b^{uw})$ a standard basis of $H_1(C_2)$ consisting of $\alpha$ and $\beta$ cycles and introduce a set of one-forms $(\alpha^{uw})$ satisfying $\int_{a^{uw}} \alpha^{uw} = \delta^{uw}$. Then the period matrix is given by

$$
\Pi_{uw} \equiv \int_{b^{uw}} \alpha^{uw}.
$$

(3.23)

For the case of a torus, $g = 1$, the period matrix is simply a complex number which can be identified with the complex structure $\tau$ of the torus. Shortly, we will use the relations (3.21) and (3.22) to eliminate half of the vector fields as well as $B_{\mu\nu}$ in favour of $s$ from our low-energy effective action to arrive at a description in terms of independent fields.

The remaining bosonic world-volume field we should consider is the auxiliary scalar field $a$. If we want the normal vector $v$ to be globally well defined, we cannot allow it to point into the internal directions of the two-cycle only. This is because generally there need not exist a nowhere vanishing vector field on a Riemann surface, as the simple example of a sphere $S^2$ already demonstrates. Hence, we will take $a$ to be independent of the internal coordinates and require it to be a function of the external coordinates only, that is, $a = a(\sigma^n)$. It turns out that this field will drop out of the five-dimensional effective action after eliminating half of the degrees of freedom using Eqs. (3.21) and (3.22).

The last ingredient we need to discuss is the non-zero mode (3.7). It consists of the purely internal part of the four-form gauge field strength, but since we now allow the five-brane to fluctuate it must be slightly generalised. To this end we define the function

$$
\alpha = \alpha_1 \theta(M^1_4) + \alpha_2 \theta(M^2_4) + \alpha_5 \left[ \Theta(M^5_4) + \Theta(\tilde{M}^5_4) \right]
$$

(3.24)

where $d\Theta(M^5_4) = \delta(M^5_4)$ and the $\theta$– and $\delta$–function are defined in analogy with Eqs. (2.14) and (2.15). The non-zero mode can then be written as

$$
G = -\frac{1}{3} \alpha(x) * \omega.
$$

(3.25)

Note that for the static brane configuration (3.3) which implies $\Theta(M^5_4) = \theta(y - y_5)$ and $\Theta(\tilde{M}^5_4) = \theta(y + y_5)$ the above expression reduces to the background configuration (3.7) as it should.

5This can be enhanced to non-abelian symmetries if five-branes are “stacked”, as discussed in Ref. 15. We do not attempt to incorporate this effect explicitly.

6We would like to thank Dmitri Sorokin for a helpful discussion on this point.
4 The five-dimensional theory

Based on the above background solutions, we would now like to derive the five-dimensional effective action and discuss its properties.

4.1 The action

Let us start by explaining how the previously identified moduli fields fit into super-multiplets. In the bulk, we have $D = 5$, $N = 1$ supergravity with a gravity multiplet $(g_{\alpha\beta}, A_{\alpha}, \Psi_i^\alpha)$ consisting of the graviton, the gravi-photon $A_{\alpha}$ with field strength $F_{\alpha\beta} = (dA)_{\alpha\beta}$ and the gravitino $\Psi_i^\alpha$. Five-dimensional fermions are described by symplectic Majorana spinors carrying $SU(2)$ R-symmetry indices $i, j, \ldots = 1, 2$. The other bulk fields arrange themselves into the universal hyper-multiplet containing the fields $(V, \sigma, \xi, \zeta^i)$. Here $\sigma$ is the dual of the five-dimensional three-form $C_{\alpha\beta\gamma}$ with field strength $G_{\alpha\beta\gamma\delta}$. The field strength of the complex scalar $\xi$ is denoted by $X^\alpha = d\xi$ and $\zeta^i$ are the fermions. We note that, from their 11-dimensional origin, the metric and $V$ are $Z_2$–even fields, while $C_{\alpha\beta\gamma}, A_{\alpha}$ and $\xi$ are $Z_2$–odd.

On the four-dimensional fixed planes $M^k_4$, where $k = 1, 2$, we have $N = 1$ gauge multiplets, that is gauge fields $A_{k\alpha}$ with field strengths $F_{k\alpha\beta} = (dA_k)_{\alpha\beta}$ and the corresponding gauginos. In general, there will also be gauge matter fields in $N = 1$ chiral multiplets but we will set these fields to zero for simplicity. They will, however, be included in Ref. [35].

On the three-brane world-volume $M^5_4$, the embedding coordinates $X^\alpha$ give rise to a single physical degree of freedom $Y = X^{11}$, as can be seen from the static gauge choice. This field is part of the $N = 1$ chiral multiplet with bosonic content $(Y, s)$. We recall that the scalar $s$ with field strength $j_{\alpha} = (ds - A)_{\alpha}$ originates from the five-brane two-form. We denote the corresponding fermions by $\theta^i$. In addition, we have $N = 1$ gauge multiplets containing Abelian gauge fields $D_u^\alpha$ with field strengths $E_u^{\alpha\beta}$, where $u, v, w, \ldots = 1, \ldots, g$ and $g$ is the genus of the curve $C_2$ wrapped by the five-brane. In general, there will be additional chiral multiplets parametrizing the moduli space of the five-brane curves $C_2$ but they will not be explicitly taken into account here.

The reduction to five dimensions is not completely straightforward particularly when dealing with the Chern-Simons and Dirac-term in the eleven-dimensional action. We have, therefore, performed the reduction on the level of the equations of motion. This gives rise to the following effective five-dimensional action

$$S_5 = S_{\text{grav}} + S_{\text{hyper}} + S_{\text{bound}} + S_{3\text{-brane}}$$

(4.1)

where

$$S_{\text{grav}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \left\{ d^5x \sqrt{-g} \left( \frac{1}{2} R + \frac{3}{2} F_{\alpha\beta} F^{\alpha\beta} \right) + 4A \wedge F \wedge F \right\}$$

(4.2)

$$S_{\text{hyper}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \left\{ d^5x \sqrt{-g} \left( \frac{1}{4} V^{-2} \partial_{\alpha} V \partial^{\alpha} V + 2V^{-1} \mathcal{X}_{\alpha} \mathcal{X}^{\alpha} + \frac{1}{4!} V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} + \frac{1}{3} V^{-2} \mathcal{X}^2 \right) + 2G \wedge (i(\xi \mathcal{X} - \xi \mathcal{X}) - 2\alpha A) \right\}$$

(4.3)
\[ S_{\text{bound}} = -\frac{1}{2\kappa_5^2} \left\{ 2 \int_{M_4^1} d^4x \sqrt{-g_4} V^{-1} \alpha_1 + 2 \int_{M_4^2} d^4x \sqrt{-g_4} V^{-1} \alpha_2 \right\} \\
- \frac{1}{16\pi\alpha_{\text{GUT}}} \left\{ \int_{M_4^1} d^4x \sqrt{-g_4} V \text{tr}(F_{\mu\nu} F^{\mu\nu}) + \int_{M_4^2} d^4x \sqrt{-g_4} V \text{tr}(F_{\mu\nu} F^{\mu\nu}) \right\} \quad (4.4) \]

\[ S_{3\text{-brane}} = -\frac{1}{2} T_3 \left\{ \int_{M_4^1 \cup \tilde{M}_4^1} d^4x \sqrt{-g_4} \left( V^{-1} + 2V^{-1} j_{\mu j}^\mu + [\text{Im}(\Pi)]_{uw} E^u_{\mu\nu} E^{uw\nu} \right) \right. \\\n- \left. 4\hat{C} \wedge ds - 2[\text{Re}(\Pi)]_{uw} E^u \wedge E^w \right\} \quad (4.5) \]

The five-dimensional Newton constant \( \kappa_5 \), the three-brane tension \( T_3 \) and the gauge coupling constant \( \alpha_{\text{GUT}} \) are given by

\[ \kappa_5^2 = \frac{\kappa_1^2}{v}, \quad T_3 = \frac{\alpha_5}{\kappa_5^2}, \quad \alpha_{\text{GUT}} = \frac{\lambda^2}{4\pi v}. \quad (4.6) \]

In this action all topological terms are written in differential form whereas all other contributions are given in component form. The hat denotes the pull-back of a bulk antisymmetric tensor field to the three-brane world-volume. The induced metric \( \gamma_{\mu\nu} \) on the three-brane world-volume is, as usual, defined by

\[ \gamma_{\mu\nu} = \partial_{\mu} X^\alpha \partial_{\nu} X^\beta g_{\alpha\beta}. \quad (4.7) \]

The field strength \( G \) satisfies the non-trivial Bianchi identity

\[ G = dC - \omega_{\text{YM}} \quad (4.8) \]

with the Yang-Mills Chern-Simons form \( \omega_{\text{YM}} \) defined by the relation

\[ J_{\text{YM}} \equiv d\omega_{\text{YM}} = \frac{\kappa_5^2}{4\pi\alpha_{\text{GUT}}} \left( \text{tr}(F_1 \wedge F_1) \wedge \delta(y) + \text{tr}(F_2 \wedge F_2) \wedge \delta(y - \pi\rho) \right). \quad (4.9) \]

Unlike its 11-dimensional counterpart (2.17), this Bianchi has only contributions from the orbifold planes since, in five dimensions, the bulk three-brane cannot provide a magnetic source for a four-form field strength. We note, that the Bianchi identities for \( F \) and \( X \) also become non-trivial once gauge-matter fields on the orbifold planes are taken into account \cite{3.23}. The matrix \( \Pi \) specifying the gauge-kinetic function on the three-brane is the period matrix defined in Eq. (3.23). We recall that the step-function \( \alpha \) in the above action has been defined in Eq. (3.24) and that the charges \( \alpha_k \) satisfy the cohomology condition

\[ \alpha_1 + \alpha_2 + \alpha_5 = 0 \quad (4.10) \]

Finally, all higher-curvature terms have been dropped from the above action.

Let us now discuss a few elementary properties of the above action. By construction, this action must represent the bosonic part of a five-dimensional \( N = 1 \) supergravity theory on the orbifold \( S^1/\mathbb{Z}_2 \) coupled to two four-dimensional \( N = 1 \) theories on the orbifold fixed planes and an additional \( N = 1 \) supersymmetric three-brane. First we note that once the three-brane is taken away this action reduces, up to rescalings\footnote{The rescalings are \( C' = \frac{1}{2} 2^{1/6} C, \ G' = 2^{1/6} G, \ \xi' = 2^{1/6} \xi, \ A' = 2^{1/6} A, \ V' = 2^{2/3} V \) and \( g'_{\alpha\beta} = 2^{-2/3} g_{\alpha\beta} \) where the prime denotes the fields as in Ref. [4].}, exactly to

\[ \text{actions for 11 dimensions,}\]
the variation of the gauge term ∼ world-volume spinors θ, as it should. The explicit proof that the bulk theory has indeed the correct supergravity structure can be carried out as in Ref. [4, 3] by dualising the three-form C_{αβγ} to a scalar σ. This scalar together with its super-partners V and ξ can then be shown to parametrise the standard universal hyper-multiplet coset SU(2,1)/SU(2) × U(1). Moreover, the shift-symmetry of the dilatonic axion σ is gauged with charge α and the graviphoton A as the corresponding gauge boson. This can be directly seen from the term αG ∧ A in Eq. (4.3).

Unlike in the case without five-branes the gauge charge changes across the bulk as anticipated in Ref. [15]. From the definition of α, Eq. (3.24), this charge is proportional to α1 between the first fixed plane and the three-brane and proportional to α1 + α5 between the three-brane and the second fixed plane. By supersymmetry, the presence of the bulk potential α²V⁻² is directly related to the gauging. Note that, similarly to the gauge charge, this potential jumps across the three-brane. Further, it is worth pointing out that, while all tension terms are proportional to V⁻¹αk, where k = 1, 2, 5, the terms on the fixed planes contain an additional factor of two relative to the three-brane term. This factor reflects the nature of the “boundary branes” as being located on Z₂ orbifold fixed planes.

4.2 Symmetries

Let us start with the supersymmetry transformations of the fermions. Their bosonic part can be obtained by either a reduction from 11 dimensions or, most easily, by generalizing the results of Ref. [4]. The latter simply amounts to substituting the function α, Eq. (3.24) into the “massive” terms in the transformations. The result is

\[ \delta \psi^i_α = D_α ε^i + \frac{i}{4} (γ_α βγ - 4δ_α^βγ) F_{βγ} ε^j - \frac{1}{\sqrt{2}} V^{-1/2} \left( \partial_α ξ (τ_1 - iτ_2) j^i - \partial_α ξ (τ_1 + iτ_2) j^i \right) ε^j \]
\[ - \frac{i}{2 \cdot 4!} V^α β δ^γε G_{βγ δε} (1) j^i ε^j + \frac{1}{6} α(x) V^{-1} γ_α (τ_3) j^i ε^j \]  
\[ \delta ξ^i = \frac{1}{4!} V^α β δ^γ ε G_{αβ γ δ} δ_ε ε^i - \frac{i}{\sqrt{2}} V^{-1/2} γ_α \left( \partial_α ξ (τ_1 - iτ_2) j^i + \partial_α ξ (τ_1 + iτ_2) j^i \right) ε^j \]
\[ + \frac{i}{2} V^{-1} γ_β δ^γ V ε^j + iα(x) V^{-1} (τ_3) j^i ε^j \]  

where γ_α are the five-dimensional gamma matrices and τ_i are the Pauli matrices. The variation of the three-brane world-volume spinors θ^i (assuming the world-volume fields s and D^u vanish) can be obtained by reducing the variation (4.20) which results in

\[ \delta θ^i = ε^i + (p_±)^i \kappa^j \]  

where the projection operators p_± are now given by

\[ p_± = \frac{1}{2} \left( 1 ± \frac{i}{4!} e^{ρ_1 ... ρ_4} ∂_{m_1} X^{α_1} ... ∂_{m_4} X^{α_4} γ_{α_1 ... α_4} τ_3 \right) . \]  

Up to total derivatives the action [4,1] is also invariant under the following gauge variations

\[ \delta c = dλ^{(2)} , \ \ \ \ \ \ δA = dλ^{(0)} , \ \ \ \ \ δs = ˆλ^{(0)} , \ \ \ \ \ δξ = \text{const.} , \ \ \ \ \ δE^u = dλ^{(1)u} \]  

with λ^{(k)} being k-form gauge parameters and λ^{(0)}, λ^{(2)} being Z₂-odd. To check this result one must note that the variation of the gauge term ∼ 4G ∧ σ(x)A and the brane term ∼ 4 ˆC ∧ ds cancel each other after partial integration of the former. The above gauge variations of course also follow from the reduction of the D=11 gauge
symmetries \((2.22)\). Note, however, that there are no remnants of the PST-symmetries \((2.23)\) in our action. This is not surprising since these symmetries have been implicitly gauge-fixed when the self-duality relations \((3.21)\) and \((3.22)\) were used to eliminate half of the degrees of freedom on the three-brane.

### 4.3 The dual form of the action

In our five-dimensional action \(S_5\), Eq. \((4.1)\), the three-brane is coupled to the gauge charge \(\alpha\) defined by Eq. \((3.24)\).

We can promote \(\alpha\) to a zero-form field strength which satisfies the Bianchi identity

\[
\delta \alpha = 2\alpha_1 \delta(M_4^1) + 2\alpha_2 \delta(M_4^2) + \alpha_5 \left[ \delta(M_4^5) + \delta(\tilde{M}_4^5) \right].
\]

The three-brane then couples magnetically to this zero-form. In analogy with massive IIA supergravity \([34]\), there should now be a dual formulation of the action \(S_5\) where \(\alpha\) is replaced by a four-form \(N\) with five-form field strength \(M = dN\) to which the three-brane couples electrically. If a dual version of 11-dimensional supergravity involving only a six-form field existed we could have derived this dual five-dimensional action directly from 11 dimensions. Such a dual version of 11-dimensional supergravity is not available and, hence, the reduction necessarily leads to the five-dimensional action \(S_5\) written in terms of \(\alpha\). However, there is no obstruction performing the dualisation in five dimensions. This can be done by adding to the action \(S_5\) (with \(\alpha\) interpreted as a zero-form field strength) the terms

\[
S_\alpha = \frac{1}{2\kappa_5^2} \left\{ \int_{M_5} N d\alpha - \sum_{k=1}^2 \int_{M_4^k} 2\alpha_k \tilde{N} - \alpha_5 \left[ \int_{M_4^5} \tilde{N} + \int_{\tilde{M}_4^5} \tilde{N} \right] \right\}. \quad (4.17)
\]

The equation of motion for \(N\) now precisely yields the Bianchi identity \((4.16)\) for \(\alpha\) by virtue of which the additional terms \((4.17)\) vanish. As it should, this leads us back to the original action \(S_5\) with \(\alpha\) being defined by Eq. \((3.24)\). On the other hand, the equation of motion for \(\alpha\) computed from the action \(S_5 + S_\alpha\) is given by

\[
\alpha = \frac{3}{2} V^2 \ast (M - 4G \wedge A). \quad (4.18)
\]

Using this relation to replace \(\alpha\) in favour of \(M\), we arrive at the dual version of our five-dimensional brane-world action \((4.1)\). It is given by

\[
S_{5,\text{dual}} = S_{\text{grav}} + S_{\text{hyper}} + S_{\text{bound}} + S_{3-\text{brane}} \quad (4.19)
\]
where

\[ S_{\text{grav}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \left\{ \sqrt{-g} \left( \frac{1}{2} R + \frac{3}{2} F_{\alpha\beta} F^{\alpha\beta} \right) + 4 A \wedge F \wedge F \right\} \]

\[ S_{\text{hyper}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \left\{ \sqrt{-g} \left( \frac{1}{4} V^{-2} \partial_\alpha V \partial^\alpha V + 2V^{-1} X_\alpha \tilde{X}^{\alpha} + \frac{1}{4!} V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} \right) 
+ \frac{3}{4 \cdot 5!} V^2 M_{\alpha_1...\alpha_5} M^{\alpha_1...\alpha_5} \right\} + 2 G \wedge (i(\tilde{\xi} X - \xi \tilde{X}) + 3V^2 A \wedge *(M - 2G \wedge A)) \right\} \]

\[ S_{\text{bound}} = -\frac{1}{2\kappa_5^2} \sum_{k=1}^{2} 2\alpha_k \int_{M_4^k} \left\{ \sqrt{-g_4} V^{-1} + \hat{N} \right\} \]

\[ S_{3\text{--brane}} = -\frac{1}{2} T_3 \int_{M_5^1 \cup M_5^2} \left\{ \sqrt{-g} \left( V^{-1} + 2V^{-1} j_\mu j^\mu + [\text{Im}(\Pi)]_{uw} E_{\mu} E_{w} \right) 
- 4 \hat{C} \wedge ds - 2[\text{Re}(\Pi)]_{uw} E_{u} E_{w} + \hat{N} \right\} \]

5 The vacuum solution

In this section, we will review the BPS domain-wall solution of the five-dimensional theory (4.1). This vacuum state is associated with an effective \( N = 1 \) four-dimensional theory which we will, in part, determine explicitly.

5.1 The supersymmetric domain-wall vacuum state

For the case without additional bulk three-branes, the supersymmetric domain-wall solution of five-dimensional heterotic M-theory has been found in Ref. [4]. We now wish to verify that this result can be extended to include the effect of the bulk three-brane thereby providing a solution of our action (4.1). We start with the following Ansatz for metric and the dilaton

\[ ds_5^2 = a^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + b^2(y) dy^2, \quad V = V(y) \]

with all other bulk fields vanishing. We have to supplement this Ansatz by a configuration for the three-brane world-volume fields. This simply corresponds to a static three-brane parallel to the orbifold planes, that is

\[ X^\mu = \sigma^\mu, \quad Y = \text{const} \]

with all other world-volume gauge fields vanishing. The \( E_8 \) gauge fields on the boundaries are turned off as well.

With this Ansatz one indeed arrives at an exact solution of the action (4.1) where \( a, b \) and \( V \) are explicitly given by

\[ a = a_0 h^{1/2}(y) \]

\[ b = b_0 h^2(y) \]

\[ V = b_0 h^3(y) \]

and the function \( h \) has been defined in Eq. (3.8). In particular, one can check that the new features arising from the presence of the three-brane are properly taken into account. Specifically, the three-brane world-volume
equations of motion are satisfied and the three-brane sources in the Einstein equation are properly matched. This solution represents a triple domain-wall matching not only the sources on the orbifold fixed planes but, in addition, it matches the bulk three-brane source and its mirror source. This can be explicitly seen from the function $h$ which satisfies

$$\partial_y^2 h = -\frac{2}{3} \partial_y \alpha(y) = -\frac{4}{3} \left[ \alpha_1 \delta(y) + \alpha_2 \delta(y - \pi \rho) + \frac{1}{2} \alpha_5 \left( \delta(y - Y) + \delta(y + Y) \right) \right].$$

(5.4)

The $\delta$–functions indicate the positions of the various orbifold planes/three-branes at $y = 0, \pi \rho, Y, -Y$.

This solution is also a BPS state of the theory since it preserves four of the eight supersymmetries. This can be verified by using Eqs. (4.11) and (4.12) for the bulk fermions and Eq. (4.13) with $\kappa^i = -\epsilon^i$ for the three-brane world-volume fermions. The Killing spinor is explicitly given by

$$\epsilon^i = H^{1/4} \epsilon_0^i, \quad \gamma_{11} \epsilon_0^i = (\tau_3)^i_j \epsilon_0^j,$$

(5.5)

where $\epsilon_0^i$ is a constant spinor.

5.2 The four-dimensional effective theory

The above domain-wall vacuum state is associated with an $N = 1$ effective four-dimensional theory describing fluctuations around this state. We would now like to compute some aspects of this four-dimensional theory.

The bosonic moduli fields from the bulk are the four-dimensional metric $g_{\mu\nu}$, the Calabi-Yau volume $V$ and the orbifold size $R = \sqrt{g_{55}}$ (both averaged over the orbifold), the axion $\chi = A_5$ and the two-form $B_{\mu\nu} = C_{11\mu\nu}$. The latter can, in four dimensions, be dualised to a scalar $\sigma$. From the three-brane world-volume, we have the scalar $z = Y/\pi \rho \in [0,1]$ specifying the position of the three brane and the axion $\nu = s/\pi \rho$ together with the $g$ Abelian gauge fields $D^u$ with field strengths $E^u$, where $u,v,w,\ldots = 1,\ldots,g$. Recall that $g$ is the genus of the curve $\tilde{C}_2$ within the Calabi-Yau space which is wrapped by the five-brane. Finally, we should consider the two gauge fields $A_k$ with field strengths $F_k$, where $k = 1,2$, which originate from the orbifold fixed planes. Obviously, all gauge fields fall into $N = 1$ gauge multiplets.

The six scalar fields, on the other hand, fit into three chiral multiplets, namely the dilaton $S$, the $T$-modulus and the five-brane modulus $Z$. The bosonic parts of these fields turn out to be related to the component fields by

$$S = V + q_5 R z^2 + i(\sigma + 2 q_5 \chi z^2)$$

$$T = R + 2i \chi$$

$$Z = R z + 2i(-\nu + z \chi)$$

(5.6)

where

$$q_k = \pi \rho \alpha_k$$

(5.7)

for $k = 1,2,5$. Recall that $\alpha_k$ is the five-brane charge defined in Eq. (3.9). The moduli Kähler potential for the superfields $S, T$ and $Z$ can now be found by a reduction of the five-dimensional action (4.1) on the domain-wall vacuum state. This leads to

$$K_{\text{moduli}} = -\ln \left[ S + \bar{S} - q_5 \frac{(Z + \bar{Z})^2}{T + \bar{T}} \right] - 3 \ln \left[ T + \bar{T} \right],$$

(5.8)
confirming the earlier result [25] which was obtained by somewhat different methods. We note that the component form (5.6) of the superfields allows a direct interpretation of this Kähler potential and the resulting moduli field dynamics in five-dimensional terms. From the non-trivial structure of the domain-wall, it is possible to compute loop-corrections of order $\epsilon_S$ to the kinetic terms of the moduli. However, we did not succeed in finding the complex structure and the associated corrected Kähler potential when those corrections were included. It is conceivable that this computation is beyond the range of validity of the five-dimensional theory. This is supported by the observation that the $Z$-dependent part in the above Kähler potential (5.8) is already suppressed by $\epsilon_S \sim R/V$ relative to the $S$-dependent part. This suggests that corrections are already of order $\epsilon_S^2$ and, therefore, beyond the linear level up to which the five-dimensional theory can generally be trusted.

The gauge kinetic functions for the gauge fields $A_1$ and $A_2$ from the orbifold fixed planes turn out to be

$$f_1 = \ S - q_2 T - 2 q_5 Z$$

(5.9)

$$f_2 = \ S + q_2 T,$$

(5.10)

again in agreement with Ref. [15, 25]. We note that the threshold terms proportional to $T$ and $Z$ arise as a direct consequence of the domain-wall structure. In fact, the correction is entirely due to the non-trivial orbifold dependence of the dilaton in Eq. (5.1) since conformal invariance of four-dimensional Yang-Mills theory implies that the warping of the five-dimensional metric drops out. As a consequence, no such threshold correction arises for the three-brane gauge fields, since their kinetic term in Eq. (4.5) does not depend on the dilaton. After an appropriate rescaling of the fields $D^u$ their gauge-kinetic function is simply proportional to the period matrix (3.23) of the holomorphic curve $C_2$, that is

$$f_{uv} = i \Pi_{uv}.$$

(5.11)

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