Nuclear moments for the neutrinoless double beta decay

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Abstract

A derivation of the neutrinoless double beta decay rate, specially adapted for the nuclear structure calculations, is presented. It is shown that the Fourier-Bessel expansion of the hadronic currents, jointly with the angular momentum recoupling, leads to very simple final expressions for the nuclear form factors. This greatly facilitates the theoretical estimate of the half life. Our approach does not require the closure approximation, which however can be implemented if desired. The method is exemplified for the $\beta\beta$ decay $^{48}Ca \rightarrow ^{48}Ti$, both within the QRPA and a shell-model like model.

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1 Introduction

The standard model (SM) of electroweak interactions has been brilliantly confirmed by a host of experiments. But, about the properties of neutrinos it says very little. Actually, in the SM it is postulated that: i) the neutrinos are the only fermions without the right-handed partners, and ii) their masslessness is dictated by the global lepton-number symmetry, and not by a fundamental underlying principle, such as gauge invariance for the photon. More, whether neutrinos behave so ”trivially” as required by the SM is one of the most fundamental open questions of the present-day physics.

It has been known for a long time [1, 2, 3, 4, 5] that neutrinoless double beta decay ($\beta\beta_{0\nu}$) is a very sensitive probe of lepton number violating terms in the Lagrangian such as the Majorana mass of the light neutrinos, right–handed weak couplings as well as the Higgs exchange [6], right-handed weak coupling involving heavy Majorana neutrinos [7], massless Majoron emission [8, 9, 10, 11], and R–parity breaking in the supersymmetric model [12, 13]. Thus, if the $\beta\beta_{0\nu}$ decay is someday observed experimentally it would hint new physics beyond the SM. But, even if it is not observed, the measured limits on its transition probability, which are steadily improving [14], could be translated into more stringent constraints on the parameters of the just mentioned new theoretical developments.

Yet, the extraction of these constraints from the data is only possible when we know how to deal with the nuclear structure involved in the $\beta\beta_{0\nu}$ decay. This is not at all an easy task, because of:

1) the nuclear hamiltonian is only roughly known to the extent that the choice of the appropriate parametrization is an art,

2) there is in general a very large amount of nuclear states involved in the calculation, and

3) the formulas for the $\beta\beta_{0\nu}$ decay rate are rather complex and difficult to implement in a nuclear structure calculation.

In the present work we derive simple expressions for the nuclear matrix elements, especially tailored for the nuclear structure calculations. The simplification mainly comes from the Fourier-Bessel expansion of the term $\exp[ik \cdot (r_1 - r_2)]$ in the transition amplitude, and in doing the integrations in the following order: first on $d\Omega_k$, then on $dr_1$ and $dr_2$, and finally on $k^2 dk$ [15]. So far, the same procedure has been applied for the evaluation
of the matrix elements $M_F$, $M_{GT}$ \cite{16,17} and $M_R$ \cite{11} that arise from the electron s-wave. Here we also dealt with the p-wave matrix elements that are relevant when the admixture of the right-hand lepton current is considered. Other studies on the subject are those of Vergados et al. \cite{18}, who derived the formulas directly in the momentum space, and those of Suhonen, Khadkikar and Faessler \cite{19}, who worked in a framework of a relativistic quark confinement model.

This paper is organized as follows: In sec. 2 we discuss the basic mechanism for the $\beta\beta_{0\nu}$ decay, presenting the effective hamiltonian and the transition amplitude in the form convenient for the multipole expansion, which is carried out in sec. 3. In sec. 4 we give the detailed formulas for the nuclear matrix elements and discuss the nuclear structure calculations involved in the problem. Summarizing conclusions are drawn in sec. 5.

## 2 Effective Hamiltonian and the half life

The $0\nu\beta\beta$ half life

$$[T_{0\nu}(0^+ \rightarrow 0^+)]^{-1} = \frac{\Gamma_{0\nu}}{\ln 2},$$

for the decay from the state $|i\rangle$ in the $(N, Z)$ nucleus to the state $|f\rangle$ in the $(N-2, Z+2)$ nucleus (with energies $E_I$ and $E_F$ and spins and parities $J^\pi = 0^+$), is evaluated via the second order Fermi’s golden rule. Thus the decay rate (in $\bar{\hbar} = \hbar = c = m_e$ units) is \cite{20}

$$\Gamma_{0\nu} = 2\pi \sum_{s_1 s_2} \int |R_{0\nu}(e_1, e_2)|^2 \delta(e_1 + e_2 + E_F - E_I) \frac{dP_1}{(2\pi)^3} \frac{dP_2}{(2\pi)^3},$$

with

$$R_{0\nu}(e_1, e_2) = \sum_N \sum_s \int \frac{dk}{(2\pi)^3} \frac{\langle f; e_1, e_2 | H_w | N; e_1, \nu \rangle \langle N; e_1, \nu | H_w | i \rangle}{E_f - E_N - e_1 - \omega},$$

where $e \equiv (\epsilon, p, s_e) \ (\nu \equiv (\omega, k, s_\nu))$ stands for the energy, momentum and spin projection of the electron (neutrino), and $N$ runs over all levels in the $(N - 1, Z + 1)$ nucleus.

The effective weak hamiltonian reads \cite{2,4,5}

$$H_w = \frac{G}{\sqrt{2}} \sum_{\ell=1}^{2n} \int d\mathbf{x} [j_{LL \mu}(x) \tilde{J}^\mu_{L\ell}(x) + j_{RR \mu}(x) \tilde{J}^\mu_{R\ell}(x) + h.c.],$$

where the summation goes on the number on lepton generations,

$$j^\mu_{L,R\ell}(x) = 2\bar{\Psi}(x)\gamma^\mu P_{L,R} N_{L,R\ell}(x); \quad P_{L,R} = \frac{1}{2}(1 \mp \gamma_5).$$
are the leptonic currents, formed out from the electron field $Ψ(x)$ and the Majorana neutrino field $N(x)$ of mass $m_ℓ$, and

$$\tilde{J}^{μL}_L(x) = U_{εL}J^{μL}_L(x), \quad \tilde{J}^{μR}_R(x) = V_{εR}(x)J^{μR}_R(x) + ηJ^{μL}_L(x),$$

(6)

contain the hadronic ($V + A$) currents $J^{μL,R}_L$. $U_{εL}$ and $V_{εR}$ are the neutrino mixing matrices for the left- and right-handed sectors, and $λ$ and $η$ are the strengths of admixtures of the ($V + A$) current.

Within the non-relativistic impulse approximation the hadronic currents read,

$$J^{μL,R}_L(x) = \left( ρ_\nu(x) \mp ρ_A(x), j_\nu(x) \mp j_A(x) \right)$$

(7)

where

$$ρ_\nu(x) = \frac{g_ν}{2M_N} \sum_n \tau^+_n \delta(x - r_n),$$

$$ρ_A(x) = \frac{g_A}{2M_N} \sum_n \tau^+_n [σ_n \cdot p_n \delta(x - r_n) + \delta(x - r_n)σ_n \cdot p_n],$$

(8)

$$j_\nu(x) = \frac{g_ν}{2M_N} \sum_n \tau^+_n [p_n \delta(x - r_n) + \delta(x - r_n)p_n + f_w \nabla \cdot σ_n \delta(x - r_n)],$$

$$j_A(x) = g_A \sum_n \tau^+_n σ_n δ(x - r_n),$$

are the one-body vector ($V$) and axial-vector ($A$) densities and currents, $M_N$ is nucleon mass and $f_w = 4.7$ is the effective weak-magnetism coupling constant.

Merging (6) into (3) and performing the $s_ν$-summation one gets [3]:

$$R_{νν} = \frac{G^2}{\sqrt{2}} \sum_{L,R} \int \frac{dΩ}{(2π)^3} \langle F|\tilde{J}^{μL}_L(y)e^{ik\cdot y}|N⟩⟨N|\tilde{J}^{μL}_R(x)e^{-ik\cdot x}|I⟩$$

$$\times [1 - P(e_1,e_2)] \bar{ψ}(e_2,y)\gamma_\nu P_β(ωγ^0 - k \cdot γ + m_ε) P_αγ_\mu ψ^C(ε_1,x) \frac{ω(ε_1 + ω + E_N - E_ε)}{ω(ε_1 + ω + E_N - E_ε)},$$

(9)

where $ψ(ε_1,x)$ and $ψ(ε_2,x)$ are the wave functions of the emitted electrons, and the operator $P(e_1,e_2)$ interchanges the particles $ε_1$ and $ε_2$. The structure of the eq. (3)

1We do not consider the admixture of the hadronic ($V + A$) current into $J^{μL}_L$, since its contribution to the $ββ$ decay amplitudes is negligible [3].

2See eq. (3D-18) in ref. [2]. The correspondence between the non-relativistic approximations used here and that prevailing in the studies of the $ββ$ decay $[1,2,4,5,6,7]$, can be find on p. 516 of the Walecka’s book [22].
suggests that it might be convenient to introduce the Fourier transforms of the quantities defined in (8), i.e.,

\[
\rho(k) = \int dx \rho(x)e^{-ik\cdot x},
\]

\[
j(k) = \int dx j(x)e^{-ik\cdot x}. \tag{10}
\]

Next, ensuing the usual procedure [2, 4, 5], we evaluate the \(s_{1/2}\) and \(p_{1/2}\) contributions of the electron wave functions to the amplitude \(R_{0\nu}\). The first ones give rise to the following \(k\) and \(N\) dependent nuclear moments

\[
M_F(k, N) = \langle F|\rho_V(-k)|N\rangle\langle N|\rho_V(k)|I\rangle,
\]

\[
M_{GT}(k, N) = \langle F|j_A(-k)|N\rangle\cdot\langle N|j_A(k)|I\rangle \tag{11},
\]

\[
M_R(k, N) = -iR k \cdot \langle F|j_A(-k)|N\rangle\times\langle N|j_V(k)|I\rangle,
\]

where \(R\) is the nuclear radius, and the second one to

\[
M'_{F'}(k, N) = 2\sqrt{3}i\langle F|\rho_V^{(0)}(-k)|N\rangle\langle N|\rho_V(k)|I\rangle,
\]

\[
M_{GT'}(k, N) = 6i\langle F|j_A^{(01)}(-k)|N\rangle\cdot\langle N|j_A(k)|I\rangle \tag{12},
\]

\[
M'_F(k, N) = 2i\langle F|j_A^{(21)}(-k)|N\rangle\cdot\langle N|j_A(k)|I\rangle,
\]

\[
M_R'(k, N) = -\sqrt{2}i[\langle F|j_A^{(10)}(-k)|N\rangle\langle N|\rho_V(k)|I\rangle - \langle F|j_A(-k)|N\rangle\cdot\langle N|\rho_V^{(1)}(k)|I\rangle],
\]

where we have introduced the tensor operators

\[
\rho^{(J)}(k) = \int dx \rho(x)(k \otimes x)^{(J)}e^{-ik\cdot x},
\]

\[
j^{(L,J)}(k) = \hat{L}\hat{J}^{-1}\int dx [j(x) \otimes (k \otimes x)^{(L)}]^{(J)}e^{-ik\cdot x}, \tag{13}
\]

with \(\hat{L} = \sqrt{2L + 1}\). The explicit form of the matrix elements defined in (11) and (12) are shown in the appendix A.

We now define the nuclear matrix elements

\[
M_X = \frac{R}{4\pi g_A^2} \sum_N \int dk\nu(k, \omega_N)M_X(k, N) \quad \text{for} \quad X = F, GT, F', GT', P, R, T, \tag{14}
\]
and

\[ M_{X\omega} = \frac{R}{4\pi g_A^2} \sum_N \int d\mathbf{k} v_\omega(k, \omega_N) M_X(k, N) \text{ for } X = F, GT, \]  

(15)

with

\[ v(k, \omega_N) = \frac{2}{\pi} \frac{1}{k(k + \omega_N)}, \quad v_\omega(k, \omega_N) = \frac{2}{\pi} \frac{1}{(k + \omega_N)^2}, \]  

(16)

and

\[ \omega_N = E_N - \frac{1}{2} (E_i + E_f). \]  

(17)

In deriving the expression (17) we have approximated the electron energies as \( \epsilon_{1,2} \approx (E_i - E_f)/2 \). We have also neglected the neutrino mass in comparison with \( k \), i.e., we have taken \( \omega \approx k \).

For the transition amplitude we get

\[ R_{0\nu}(\epsilon_1, \epsilon_2) = \frac{g^2 G^2}{4\pi R \sqrt{2}} \sum_{k=1}^{5} Z_k L_k(\epsilon_1, \epsilon_2), \]  

(18)

where

\begin{align*}
Z_1 &= < m_\nu > (M_F - M_{GT}), \\
Z_2 &= < \eta > (M_{GT\omega} + M_F) + < \lambda > (M_{F\omega} - M_{GT\omega}), \\
Z_3 &= 4 < \eta > M_R, \\
Z_4 &= \frac{2}{3} i[< \lambda > (M'_{GT} - 6M_T + 3M_F') - < \eta > (M'_{GT} - 6M_T - 3M_F')], \\
Z_5 &= 4i < \eta > M_P,
\end{align*}  

(19)

encompass the hadronic matrix elements, as well as the parameters

\begin{align*}
< m_\nu > &= \sum' m_\ell U_{\ell\ell}^2, \\
< \lambda > &= \lambda \sum' U_{et} V_{et}, \\
< \eta > &= \eta \sum' U_{et} V_{et},
\end{align*}  

(20)

where the summation \( \sum' \) goes only on the light neutrinos \( [4, 5] \). The leptonic matrix elements \( L_1(\epsilon_1, \epsilon_2) \) are displayed in the appendix B.
Finally, by performing the integrations (summations) on the electron states indicated in (2), we obtain the familiar expression for the $0\nu\beta\beta$ half life \[2, 4\]

\[
\left[ T_{0\nu}(0^+ \rightarrow 0^+) \right]^{-1} = <m_\nu>^2 C_1 + <\lambda>^2 C_2 + <\eta>^2 C_3 
+ <m_\nu><\lambda>C_4 + <m_\nu><\eta>C_5 + <\lambda><\eta>C_6, \tag{21} \]

where

\[
\begin{align*}
C_1 &= (M_F - M_{GT})^2 G_1, \\
C_2 &= M_2^2 G_2 + \frac{1}{9} M_{1+}^2 G_4 - \frac{2}{9} M_{2-} M_{1+} G_3, \\
C_3 &= M_2^2 G_2 + \frac{1}{9} M_{1+}^2 G_4 - \frac{2}{9} M_{2+} M_{1-} G_3 + M_R^2 G_9 + M_R M_P G_7 + M_P^2 G_8, \\
C_4 &= (M_F - M_{GT}) [M_2 - G_3 - M_{1+} G_4], \\
C_5 &= -(M_F - M_{GT}) [M_{2+} G_3 - M_{1-} G_4 + M_R G_6 + M_P G_5], \\
C_6 &= -2 M_{2-} M_{2+} G_2 + \frac{2}{9} [M_{2-} M_{1-} + M_{2+} M_{1+}] G_3 - \frac{2}{9} M_{1-} M_{1+} G_4, \tag{22} \end{align*} \]

contain the usual combinations of the matrix elements

\[
\begin{align*}
M_{1\pm} &= M_{GT}' - 6 M_T \pm 3 M_{F}', \\
M_{2\pm} &= M_{GT}\omega \pm M_{F}\omega - \frac{1}{9} M_{1\mp}, \tag{23} \end{align*} \]

and the kinematical factors

\[
G_k = \frac{g_4^4 G^4}{32 \pi^2 \alpha Z \ln 2} \left( \frac{2\pi \alpha Z}{1 - e^{-2\pi \alpha Z}} \right)^2 F_k(T_0). \tag{24} \]

The electron phase-space factors $F_k(T_0)$, as a function of the maximum kinetic energy $T_0 = E_f - E_F - 2$, are listed in the appendix B.

It might be important to stress that, within the procedure followed here to derive the result (21), we do not need to recur at all to so called closure approximation (CA). (Remind that the CA implies: i) to supplant the energies $E_N$ by an average values $<E_N>$, and ii) to use the closure relation $\sum_N |N\rangle\langle N| = 1$ for the intermediate states.) When reworked in the CA, the moments (14) and (15) are directly comparable with those that appear in the literature \[1, 2, 3, 4, 5\].
3 Multipole expansion and angular momentum recoupling

The starting point for the multipole expansion of the hadronic current is to use the Fourier-Bessel relation

$$e^{ik \cdot r} = 4\pi \sum_L i^L j_L(kr)(Y_L(\hat{k}) \cdot Y_L(\hat{r}))$$

$$\equiv 4\pi \sum_L i^L (-1)^L \hat{L} j_L(kr)[Y_L(\hat{k}) \otimes Y_L(\hat{r})]_0, \quad (25)$$

in the equations exhibited in the appendix A. Then we perform the angular momentum recoupling, and rewrite the nuclear moments (11) and (12) in terms of the one-body spherical tensor operators

$$Y^\kappa_{JM}(k) = \sum_n \tau^+_n r_n^\kappa j_n(kr_n)Y_{JM}(\hat{r}_n),$$

$$S^\kappa_{LMJ}(k) = \sum_n \tau^+_n r_n^\kappa j_n(kr_n)[\sigma_n \otimes Y_{LM}(\hat{r}_n)],$$

$$P_{LMJ}(k) = \sum_n \tau^+_n j_n(kr_n)[p_n \otimes Y_{LM}(\hat{r}_n)]. \quad (26)$$

Finally, the angular integration on $d\Omega_k$ is done. We illustrate the procedure by sketching in the appendix C, the derivation of a part of the final formula for the nuclear matrix element $M_R$. Proceeding in a similar way with the remaining matrix elements we obtain:

$$M_F = 4\pi R \left( \frac{g_V}{g_A} \right)^2 \sum_{JN} \int v(k, \omega_N)k^2 dk |\langle F|Y_{JJ}^0(k)|N \rangle \cdot \langle N|Y_{JJ}^0(k)|I \rangle|,$$

$$M_{GT} = 4\pi R \sum_{LJN} (-1)^{1+L+J} \int v(k, \omega_N)k^2 dk |\langle F|S^0_{LJL}(k)|N \rangle \cdot \langle N|S^0_{LJL}(k)|I \rangle|,$$

$$M_{F'} = -8\pi R \left( \frac{g_V}{g_A} \right)^2 \sum_{LJN} i^{L-J+1}(J1|L)(J1|L) \times \int v(k, \omega_N)k^3 dk |\langle F|Y_{JJ}^1(k)|N \rangle \cdot \langle N|Y_{JJ}^0(k)|I \rangle|,$$

$$M_{GT'} = 8\pi R \sum_{L'JN} i^{L-L'+1}(-1)^{L'+J}(L'1|L)(L'1|L) \times \int v(k, \omega_N)k^3 dk |\langle F|S^1_{L'JL}(k)|N \rangle \cdot \langle N|S^0_{L'JL}(k)|I \rangle|.$$
\[ M_R = \frac{2\pi R^2 g_v}{M_N} g_A \sum_{LL'JN} i^{L+L'}(-1)^J \int v(k, \omega_N) k^3 dk \langle F | S_{LLJ}^0(k) | N \rangle \cdot \]
\[
\begin{aligned}
&\left\{ f_w k \left[ \delta_{LL'} - (J1|L)(J1|L') \right] \langle N | S_{L'LJ}^0(k) | I \rangle \right.
\ - \ 2\sqrt{6}(-1)^{L+J} \hat{L} \left\{ \begin{array}{ccc}
1 & 1 & 1 \\
J & J' & L' \\
L & L' & L' \\
\end{array} \right\} (L1|L')\langle N | P_{L'J}(k) | I \rangle \\
\int \! v(k, \omega_N) k^3 dk \langle F | S_{1LJJ}^0(k) | N \rangle \cdot \langle N | Y_{JJ}^0(k) | I \rangle,
\end{aligned}
\]
\[ M_P = \frac{40\pi R}{g_A} \sum_{LL'JN} i^{L+L'+1} \hat{L}^2(1L|J')(1L|L') \left\{ \begin{array}{ccc}
1 & 1 & 1 \\
J' & L' & L' \\
J & J & L \\
\end{array} \right\} \left\{ \begin{array}{ccc}
1 & 2 & 1 \\
J' & J' & L' \\
J & J & L' \\
\end{array} \right\} \\
\int \! v(k, \omega_N) k^3 dk \langle F | S_{1LJ}^1(k) | N \rangle \cdot \langle N | Y_{JJ}^0(k) | I \rangle,
\]

where \((L1|J)\) is a short notation for the Clebsh-Gordon coefficient \((L01|J0)\). The formulas for the matrix elements \(M_{FW}\) and \(M_{GT\omega}\) are obtained from those for \(M_F\) and \(M_{GT}\) with the replacement \(v(k, \omega_N) \rightarrow v_{\omega}(k, \omega_N)\) (see eqs. (14) and (15)).

The evaluation of the \(0\nu\beta\beta\) matrix elements encompasses:

i) the appraisal of the scalar product
\[
\langle F | T_J(k) | N \rangle \cdot \langle N | T_J(k) | I \rangle,
\]

where \(T_J(k)\) represents any of the one-body operators displayed in (20), and

ii) the integration on the neutrino momentum \(k\).

More details on these two steps are given in the next section.
4 Nuclear structure calculations

To evaluate the matrix elements (34) it is convenient to rewrite the operators (26) from the Hilbert space to the Fock space (21), i.e.,

$$T_{J\lambda}(k) = \hat{J}^{-1} \sum_{\rho n} \langle \rho | T_J(k) | \rho \rangle \left( a_p^\dagger a_n \right)_{J\lambda}. $$

(35)

In this way we get

$$\sum_{\lambda} \langle \rho | T_J(k) | \rho \rangle \cdot \langle \rho | T_J(k) | \rho \rangle \equiv \sum_{\alpha \pi \lambda} \langle 0^+_f | T_J(k) | J^\pi_{\alpha} \rangle \cdot \langle J^\pi_{\alpha} | T_J(k) | 0^+_i \rangle = (-)^J \sum_{\alpha \pi \rho p p' n'} \langle p | T_J(k) | n \rangle \rho^{ph}(p p' n' ; J^\pi_{\alpha}) \langle p' | T_J(k) | n' \rangle,$$

(36)

where

$$\rho^{ph}(p p' n' ; J^\pi_{\alpha}) = \hat{J}^{-2} \langle 0^+_f | (a_p^\dagger a_n)_{J^\pi_{\alpha}} | J^\pi_{\alpha} \rangle \langle J^\pi_{\alpha} | (a_p^\dagger a_{n'})_{J^\pi_{\alpha}} | 0^+_i \rangle,$$

(37)

is a two-body state dependent particle-hole (ph) density matrix, and the index \( \alpha \) labels different intermediate states with the same spin \( J \) and parity \( \pi \).

Within the CA we can sum over \( \alpha \), and deal with the state independent ph density matrix

$$\rho^{ph}_{cl}(p p' n' ; J^\pi) = \sum_{\alpha} \rho^{ph}(p p' n' ; J^\pi_{\alpha}) \equiv \hat{J}^{-1} \langle 0^+_f | \left[ (a_p^\dagger a_n)_{J^\pi} (a_p^\dagger a_{n'})_{J^\pi} \right]_0 | 0^+_i \rangle,$$

(38)

which is related with the particle-particle (pp) density matrix

$$\rho^{pp}(p p' n n' ; J^\pi) = \hat{J}^{-1} \langle 0^+_f | \left[ (a_p^\dagger a_p')_{J^\pi} (a_n a_{n'})_{J^\pi} \right]_0 | 0^+_i \rangle,$$

(39)

by a Pandya like relation

$$\rho^{ph}_{cl}(p p' n' ; J^\pi) = \sum_{j n + j n' + j J + I} \left\{ \begin{array}{cc} j_p & j_n \\ j_{n'} & j_{n'} \end{array} \right\} \rho^{pp}_{cl}(p p' n n' ; I \pi).$$

(40)

The reduced single-particle pn form factors for the one-body operators defined in (26) are (23, 24)

$$\langle p | | \vec{Y}_J(k) | n \rangle = (4\pi)^{-1/2} W_{J0}(p n) R_{\Pi}(p n ; k),$$

$$\langle p | | \vec{S}_{\lambda J}(k) | n \rangle = (4\pi)^{-1/2} W_{J1}(p n) R_{\Pi}(p n ; k),$$

$$\langle p | | \vec{P}_{\lambda J}(k) | n \rangle = (4\pi)^{-1/2} \left[ W_{LJ}^{(+)}(p n) R_{\Pi}^{(+)}(p n ; k) + W_{LJ}^{(-)}(p n) R_{\Pi}^{(-)}(p n ; k) \right].$$

(41)
with the angular parts

\[
W_{LSJ}(pn) = \sqrt{2} \hat{S} \hat{L} \hat{n} \hat{j}_p \langle n_L | l_p \rangle \left\{ \begin{array}{c} l_p \ 1 \ 2 \ j_p \\ L \ S \ J \\ l_n \ 1 \ 2 \ j_n \end{array} \right\},
\]

\[
W^{(\pm)}_{LJ}(pn) = \mp i(-1)^{l_p+j_n+J+\frac{1}{2}} \hat{L} \hat{n} \hat{j}_p \hat{j}_n \langle n_L + \frac{1}{2} \pm \frac{1}{2} \hat{L} | n_L \mp 1 \rangle \times \left\{ \begin{array}{c} l_p \ j_p \ 1 \ 2 \\ j_n \ l_n \ J \end{array} \right\},
\]

(42)

and the radial parts

\[
R^0_{L}(pn; k) = R^0_L(l_p, n_p, l_n, n_n; k) = \int_0^\infty u_{n_p,l_p}(r) u_{n_n,l_n}(r) j_L(\kappa r) r^{2+\kappa} dr,
\]

\[
R^{(\pm)}_L(pn; k) = \int_0^\infty u_{n_p,l_p}(r) \left( \frac{d}{dr} \pm \frac{2l_n + 1 \pm 1}{2r} \right) u_{n_n,l_n}(r) j_L(\kappa r) r^{2} dr.
\]

(43)

To carry out the numerical calculation of eqs. (27) - (33) it is convenient to group separately the angular and the radial parts. For instance, \( M_{GT} \) can be cast in the form

\[
M_{GT} = - \sum_{LJ^\pi} (-)^L \sum_{pp'nn'} \rho^{ph}(pp'n'J^\pi_0) W_{L1J}(pn) W_{L1J}(p'n') R^0_{LL}(pn; \omega J^\pi_0),
\]

(44)

where the two-body radial integrals are defined as

\[
R^0_{LL}(pn; \omega J^\pi_0) = \mathbb{R} \int dk k^{2+\kappa} v(k; \omega J^\pi_0) R^0_L(pn; k) R^0_L(p'n'; k).
\]

(45)

One manner to include the effects of the finite nucleon size (FNS) and the two-nucleon short-range correlations (SRC) on the \( \beta\beta_0\nu \) moments has been explicated in ref. [16], with the result:

\[
v(k; \omega_N) \rightarrow v(k; \omega_N) \left( \frac{\Lambda^2}{\Lambda^2 + k^2} \right)^4 - \frac{1}{\pi \kappa k_c} \ln \left| k + k_c \right| + \frac{1}{2\pi \kappa k_c} \left[ \sum_{n=1}^3 \frac{1}{n} \right. 
\]

\[
x_n - x_n' + \ln \left( \frac{x_-}{x_+} \right) + \ln \left( \frac{x_-}{x_+} \right),
\]

(46)

where \( \Lambda = 850 \text{ MeV} \) is the cutoff for the dipole form factor in the FNS correlations,

\[
x_\pm = \frac{\Lambda^2}{\Lambda^2 + (k \pm k_c)^2},
\]

(47)

\[\text{We use here the angular momentum coupling } |(\frac{1}{2}, l)j).\]
and \( k_c = 3.93 \text{ fm}^{-1} \) is roughly the Compton wavelength of the \( \omega \)-meson in the SRC correlations.

The integration on the neutrino momentum \( k \) is simplified when the harmonic oscillator radial wave functions are employed. Then the following relations among the one-body radial integrals are valid:

\[
R^1_L (p_n; k) = (2\nu)^{-\frac{1}{2}} \left\{ (2l_n + 2n_n + 3)\frac{1}{2} R^0_L (k; l_p, n_p, l_n + 1, n_n) \right. \\
- \left. (2n_n)\frac{1}{2} R^0_L (k; l_p, n_p, l_n + 1, n_n - 1) \right\},
\]

\[
R^{(\pm)}_L (p_n; k) = \pm \left( \frac{\nu}{2} \right)^{\frac{1}{2}} \left\{ (2l_n + 2n_n + 2 \mp 1)\frac{1}{2} R^0_L (k; l_p, n_p, l_n \pm 1, n_n) \\
+ (2n_n + 1 \pm 1)\frac{1}{2} R^0_L (k; l_p, n_p, l_n \pm 1, n_n \pm 1) \right\},
\]

where \( \nu = M\omega/\hbar \) is the oscillator parameter, and the \( k \)-integration in the matrix elements (27) - (33) only involves the radial integrals (45). Their explicit forms in this case are shown in the appendix D.

The densities \( \rho^{ph} (pmp'n'; J^\pi_\alpha) \) and \( \rho^{ph} (pmp'n'; J^\pi_\alpha) \) are supplied by the nuclear structure calculations. As an example, we discuss below the \( \beta \beta \) decay \( ^{48}\text{Ca} \rightarrow ^{48}\text{Ti} \).

We first consider the case when the intermediate nucleus \( ^{48}\text{Sc} \) and the final nucleus \( ^{48}\text{Ti} \) are described, respectively, as one-particle one-hole and two-particle two-hole excitations on \( ^{48}\text{Ca} \), i.e.,

\[
| J^\pi_\alpha M \rangle = \sum_{p_n} \langle p_n | J^\pi_\alpha \left( a_p^\dagger a_n \right)_{J^\pi_M} | 0^+_i \rangle, \\
| 0^+_f \rangle = \sum_{p \geq p', n \geq n'} N(pp') N(nn') \langle pp'nn'; I^\pi | 0^+_f \rangle \left[ (a_p^\dagger a_p^\dagger)_{I^\pi} (a_n a_n)_{I^\pi} \right]_0 | 0^+_i \rangle,
\]

with \( N(pp') = (1 + \delta_{pp'})^{-\frac{1}{2}} \). One gets from (37)

\[
\rho^{ph} (p_1 n_1 p_2 n_2; J^\pi_\alpha) = \langle J^\pi_\alpha | p_2 n_2 \rangle \sum_{p'_2 \geq p', n'_2 \geq n} \sum_{p_3 n_3 I^\pi} \hat{I} \langle p_3 n_3 | J^\pi_\alpha \rangle \langle 0^+_f | pp'nn'; I^\pi \rangle (-)^{n_1 + p_3 + J + I}
\times \left\{ \begin{array}{ccc} p_1 & n_1 & J \\ n_3 & p_3 & I \end{array} \right\} \delta_{pp_1} \delta_{nn_1} \delta_{p'p_3} \delta_{n'n_3},
\]

(50)
and

\[ \rho_{cl}^{ph}(p_1n_1p_2n_2; J^\pi) = \sum_{p \geq p', n \geq n'} \hat{I}(0^+_1|pp'nn'; I^\pi)(-)^{n_1+p_2+J^I} \times \left\{ \begin{array}{c} p_1 \\ n_1 \\ n_2 \\ p_2 \end{array} \right\} \hat{P}_I(pp') \hat{P}_I(nn') \delta_{pp_1} \delta_{nn_1} \delta_{p'_p} \delta_{n'_n}, \quad (51) \]

where

\[ \hat{P}_I(pp') = N(pp') \left[ 1 - (\cdots)^{J^I+p^I+J^I} (p \leftrightarrow p') \right]. \quad (52) \]

Within the CA one can use the closure relation

\[ \sum \alpha \langle p_3n_3|J^\pi_\alpha \rangle \langle J^\pi_\alpha|p_2n_2 \rangle = \delta_{p_2p_3} \delta_{n_2n_3}, \quad (53) \]

which leads from (50) to (51).

On the other hand, within the QRPA formulation, and after solving the BCS equations for the intermediate nucleus \(^{48}\text{Sc}\) [25], the two-body density matrix becomes

\[ \rho^{ph}(pp'n'; J^\pi_\alpha) = \left[ u_n v_p X_{J^\pi_\alpha}^{p_2n_2} + u_p v_n Y_{J^\pi_\alpha}^{p_2n_2} \right] \left[ u_{p'} v_{n'} X_{J^\pi_\alpha}^{p'n'} + u_n v_{p'} Y_{J^\pi_\alpha}^{p'n'} \right], \quad (54) \]

where all the notation has the standard meaning [17, 25]. One should bear in mind that when the QRPA is used, the energies \(\omega_{J^\pi_\alpha}\) that appear in the matrix radial integrals are the solutions of the RPA problem and not the excitation energies of the intermediate nucleus relative to the initial nucleus.

In particular, in the single mode model [16], where there is only one intermediate state for each \(J^\pi\) (and which seems to be a reasonable first order approximation for the \(\beta\beta\) decays of \(^{48}\text{Ca}\) and \(^{100}\text{Mo}\) nuclei [26, 27]),

\[ \rho^{ph}(pppn; J^\pi) = u_p v_n u_n v_p \left( \frac{\omega^0}{\omega_{J^\pi}} \right) \left( 1 + \frac{G(J^\pi)}{\omega^0} \right), \quad (55) \]

where \(G(J^\pi) = G(pppn; J^\pi), \omega^0 = - \left[ G(pppp; 0^+) + G(nnnp; 0^+) \right]/4, \) and \(G(jj'jj'; J^\pi)\) are the particle-particle matrix elements. The intermediate states for \(^{48}\text{Sc}\) are:

\[ [0f_7/2(p)0f_7/2(n)]_{J^+}, \] and the values of the ratios \(G(J^+)/\omega^0\) for the \(\delta\) force can be found in table 1 of ref. [26].
5 Summarizing Discussion

A straightforward derivation of the $\beta\beta_{0\nu}$ decay rate, based on the Fourier-Bessel expansion of the transition amplitude, and the posterior application of the Racah algebra, has been performed without invoking the closure approximation. If necessary, this approximation can be implemented, however, at any step of the calculation. It has been used for deriving the $\beta\beta_{0\nu}$ formulas in refs. [1, 2, 4], but not in refs. [18, 19].

To evaluate the nuclear matrix elements exhibited in eqs. (27) - (33) we only have to perform summations on the angular momenta and the intermediate virtual states. The successive terms rapidly decrease, because the radial integrals steadily diminish in magnitudes when the multipolarities $L$ and $L'$ are augmented [16, 17]. The formulas become particularly simple when the harmonic oscillator basis is used. Then the Horie and Sasaki method [15] can be exploited for the evaluation of the radial form factors (45), and the equations displayed in the appendix D can be used.

The present formalism is especially suitable for the nuclear structure in which the summation on the intermediate states is unavoidable, such as the QRPA. The closure approximation then just connotes that the variation of the energy denominators with nuclear excitation is not considered. Evidently this does not lead to a major simplification in the numerical calculation. For example, because of $\rho^{ph}(pnp'n'; J^\pi)$ given by (54), the summation in (44) on different states $\alpha$ with the same $J^\pi$ persists, although we do the replacement $\omega_{J^\beta} \rightarrow \omega_{J^\beta} = \omega_{J^\pi}$.

Contrarily, the use of the closure approximation is mandatory, and can be implemented easily as described in the last section, when the study is done in the shell-model framework, i.e., when one possesses information only on $\rho_{cl}^{ph}$, or equivalently on the $0^+$ nuclear wave functions for the initial and final states. In this case the matrix element (44) reads

$$M_{GT} = -\sum_{LJ^\pi} (-)^L \sum_{pp'n'n'} \rho_{cl}^{ph}(pnp'n'; J^\pi) W_{L1J}(pn) W_{L1J}(p'n') \mathcal{R}_{LL}^{0}(pnp'n'; \omega_{J^\pi}).$$

The main difference between the formalism presented here and those published so far [1, 2, 18, 19] is its simplicity. As such it is more suitable for the numerical calculations. Let us underscore a few points in this regard:

1) While in the neutrino potential formalisms [1, 2, 4] one deals with two-body matrix elements, which lead to rather complicated analytic expressions for the $\beta\beta_{0\nu}$ moments,
we only have to handle the well known one-body operators (26). It could be illustrative to compare our result (31) for the matrix element $M_R$ with eqs. (3.65) to (3.68) in the Tomoda’s report [4].

2) At variance with the formalism developed by Vergados et al., [18], the results shown here are not limited to the employment of harmonic oscillator one-particle wave functions. Besides we totally avoid the usage of the Moshinsky-Brody transformation brackets, which now and then could be cumbersome.

3) There are as well several substantial differences with the works of Suhonen et al. [19], where the Fourier-Bessel expansion has also been used. First, they obtain different and more complex results for $M_F$ and $M_{GT}$. Second, they do not exhibit the explicit structure of for the remaining matrix elements, given here by eqs. (29) - (33), but only show their general layout. Yet, this layout cannot be used for any practical purpose. Third, instead of dealing with the plain nuclear shell model, they operate in a relativistic quark confinement model. Fourth, their formulation is limited to the QRPA approximation as well as to the harmonic oscillator basis.

In summary, we believe that the present formalism simplifies the nuclear structure evaluation of the $\beta\beta_{0\nu}$ matrix elements to a large extent. The formulation is applicable as well to matrix elements that appear in some supersymmetric contributions [13].
Appendix A: Matrix elements $M_X(k, N)$

After integrating on $dx$ and $dy$, as indicated in eq. (9), the matrix elements (11) and (12) read

$$M_F(k, N) = g^2_v \langle F| \sum_n \tau_n^+ e^{ik \cdot r_n} |N\rangle \langle N| \sum_m \tau_m^+ e^{-ik \cdot r_m} |I\rangle, \quad (A.1)$$

$$M_{GT}(k, N) = g^2_A \langle F| \sum_n \tau_n^+ \sigma_n e^{ik \cdot r_n} |N\rangle \cdot \langle N| \sum_m \tau_m^+ \sigma_m e^{-ik \cdot r_m} |I\rangle, \quad (A.2)$$

$$M'_F(k, N) = -2ig^2_v \langle F| \sum_n \tau_n^+ k \cdot r_n e^{ik \cdot r_n} |N\rangle \langle N| \sum_m \tau_m^+ e^{-ik \cdot r_m} |I\rangle, \quad (A.3)$$

$$M'_{GT}(k, N) = -2ig^2_A \langle F| \sum_n \tau_n^+ k \cdot \sigma_n e^{ik \cdot r_n} |N\rangle \cdot \langle N| \sum_m \tau_m^+ \sigma_m e^{-ik \cdot r_m} |I\rangle, \quad (A.4)$$

$$M_R(k, N) = -iRg_A g_v \frac{k}{2M_N} \left\{ \langle F| \sum_n \tau_n^+ \sigma_n e^{ik \cdot r_n} |N\rangle \times \langle N| \sum_m \tau_m^+ \left[ p_m e^{-ik \cdot r_m} + e^{-ik \cdot r_m} p_m + f_w \nabla \times \sigma_m e^{-ik \cdot r_m} \right] |I\rangle \right\}, \quad (A.5)$$

$$M_T(k, N) = \frac{2\sqrt{5}}{\sqrt{3}} g^2_v \langle F| \sum_n \tau_n^+ \sigma_n \otimes (k \otimes r_n)^{(2)}(1) e^{ik \cdot r_n} |N\rangle \cdot \langle N| \sum_m \tau_m^+ \sigma_m e^{-ik \cdot r_m} |I\rangle, \quad (A.6)$$

$$M_P(k, N) = -\sqrt{2}ig_A g_v \left\{ \sqrt{3} \langle F| \sum_n \tau_n^+ \sigma_n \otimes (k \otimes r_n)^{(1)}(0) e^{ik \cdot r_n} |N\rangle \langle N| \sum_m \tau_m^+ e^{-ik \cdot r_m} |I\rangle - \langle F| \sum_n \tau_n^+ \sigma_n e^{ik \cdot r_n} |N\rangle \cdot \langle N| \sum_m \tau_m^+ (k \otimes r_m)^{(1)} e^{-ik \cdot r_m} |I\rangle \right\}. \quad (A.7)$$
Appendix B: Electron matrix elements and phase-space factors

The leptonic factors in eq. (18) are:

\[ L_1(\epsilon_1, \epsilon_2) = (-1)^{1/2-s'_2} \chi_{s'_1} \left[ g_{-1}(\epsilon_1) - f_1(\epsilon_1) \sigma \cdot \hat{p}_1 \right] [f_1(\epsilon_2) \sigma \cdot \hat{p}_2 + g_{-1}(\epsilon_2)] \chi_{-s'_2}, \]

\[ L_2(\epsilon_1, \epsilon_2) = (\epsilon_1 - \epsilon_2)(-1)^{1/2-s'_2} \chi_{s'_1} \left[ g_{-1}(\epsilon_1) f_1(\epsilon_2) \sigma \cdot \hat{p}_1 + f_1(\epsilon_1) g_{-1}(\epsilon_2) \sigma \cdot \hat{p}_1 \right] \chi_{-s'_2}. \]

\[ L_3(\epsilon_1, \epsilon_2) = \frac{1}{R} (-1)^{1/2+s'_2} \chi_{s'_1} \left[ g_{-1}(\epsilon_1) g_{-1}(\epsilon_2) + f_1(\epsilon_1) f_1(\epsilon_2) \sigma \cdot \hat{p}_1 \sigma \cdot \hat{p}_2 \right] \chi_{-s'_2}, \]  

\[ L_4(\epsilon_1, \epsilon_2) = \frac{i}{2R} (-1)^{1/2+s'_2} \chi_{s'_1} \left\{ \left[ f_1(\epsilon_1) f_{-1}(\epsilon_2) + g_1(\epsilon_1) g_{-1}(\epsilon_2) \right] \sigma \cdot \hat{p}_1 \right\} \chi_{-s'_2}, \]

where all the notation has the usual meaning [3].

The electron phase-space factors \( \mathcal{F}_k(T_0) \) that appear in eq. (24) are:

\[ \mathcal{F}_1(T_0) = T_0 (30 + 60T_0 + 40T_0^2 + 10T_0^3 + T_0^4) / 30, \]
\[ \mathcal{F}_2(T_0) = T_0^3 (70 + 77T_0 + 14T_0^2 + T_0^3) / 420, \]
\[ \mathcal{F}_3(T_0) = T_0^3 (10 + 10T_0 + T_0^2) / 30, \]
\[ \mathcal{F}_4(T_0) = T_0^2 (30 + 35T_0 + 10T_0^2 + T_0^3) / 135, \]
\[ \mathcal{F}_5(T_0) = T_0 (60T_0 + 80T_0^2 + 30T_0^3 + 3T_0^4 + \xi (60 + 90T_0 + 40T_0^2 + 5T_0^3)) / 45, \]
\[ \mathcal{F}_6(T_0) = 2T_0 (12 + 18T_0 + 8T_0^2 + T_0^3) / (3R), \]
\[ \mathcal{F}_7(T_0) = 4T_0 (60T_0 + 100T_0^2 + 55T_0^3 + 12T_0^4 + T_0^5 + \xi (60 + 90T_0 + 45T_0^2 + 10T_0^3 + T_0^4)) / (45R), \]
\[ \mathcal{F}_8(T_0) = T_0 (100T_0^2 + 150T_0^3 + 73T_0^4 + 14T_0^5 + T_0^6 + 2\xi (60T_0 + 100T_0^2 + 55T_0^3 + 12T_0^4 + T_0^5) \]
\[ + \xi^2 (60 + 90T_0 + 45T_0^2 + 10T_0^3 + T_0^4)) / 135, \]
\[ \mathcal{F}_9(T_0) = 4T_0 (60 + 90T_0 + 45T_0^2 + 10T_0^3 + T_0^4) / (15R^2), \]

with

\[ \xi = \frac{3\alpha Z}{R}. \]
Appendix C: Derivation of the final formulas for the nuclear moments

Below we give the details on the derivation of the last term in (31). First we rewrite (A.3) as

\[ M_R(k, N) = -i \frac{R g_A g_V}{2M_N} \mathbf{k} \cdot \langle f| \sum_n \tau_n^+ \sigma_n e^{i\mathbf{k} \cdot \mathbf{r}_n} |N\rangle \times \langle N| \sum_m \tau_m^+ e^{-i\mathbf{k} \cdot \mathbf{r}_m} [2\mathbf{p}_m - \mathbf{k} + i f \sigma_m \times \mathbf{k}] |l\rangle, \quad (C.1) \]

and then we express the vector product, involving the nucleon momentum term in (C.1), in spherical coordinates

\[ M(p) R(k, N) = R g_A g_V \sqrt{2} \frac{1}{M_N} \sum_{\nu\nu'} (-1)^\mu (1\nu 1\nu' |1\mu) \mathbf{k}^\mu \langle f| \sum_n \tau_n^+ \sigma_n^\nu e^{i\mathbf{k} \cdot \mathbf{r}_n} |N\rangle \langle N| \sum_m \tau_m^+ e^{-i\mathbf{k} \cdot \mathbf{r}_m} \mathbf{p}_m |l\rangle. \quad (C.2) \]

After performing the multipole expansion (25) and handling some straightforward Racah algebra, we obtain

\[ M_R^{(p)}(k, N) = \frac{R g_A g_V}{M_N} \sqrt{2} \frac{1}{6(4\pi)^2} \frac{1}{M' \nu' \nu' \mu} (1\nu \nu' \nu' |1\mu) \mathbf{k}^\mu \langle f| \sum_n \tau_n^+ \sigma_n^\nu e^{i\mathbf{k} \cdot \mathbf{r}_n} |N\rangle \langle N| \sum_m \tau_m^+ e^{-i\mathbf{k} \cdot \mathbf{r}_m} \mathbf{p}_m |l\rangle. \quad (C.3) \]

where the tensor operators \( S_0^{LL'J'J}(k) \) and \( P_{L'J}(k) \) are defined in (26). Finally, the angular integration allows us to perform the summations on the angular momentum projections and obtain

\[ \int d\Omega_k M_R^{(p)}(k, N) = \frac{R g_A g_V}{M_N} \sqrt{6(4\pi)^2} \frac{1}{M' \nu' \nu' \mu} (1\nu \nu' \nu' |1\mu) \mathbf{k}^\mu \langle f| \sum_n \tau_n^+ \sigma_n^\nu e^{i\mathbf{k} \cdot \mathbf{r}_n} |N\rangle \langle N| \sum_m \tau_m^+ e^{-i\mathbf{k} \cdot \mathbf{r}_m} \mathbf{p}_m |l\rangle. \quad (C.4) \]

This result, together with the eq. (14), yields the last term in eq. (31).
Appendix D: Radial form factors for the harmonic oscillator wave functions

Following the Horie and Sasaki method [15] the radial integral (45) can be expressed as:

\[ R_{\kappa}^{LL'}(\alpha; \omega J_{\pi}) = [M(p, n)M(p', n')]^{-1/2} \sum_{mm'} a_m(p, n) a_{m'}(p', n') f_{LL'}^{\kappa}(m, m'; \omega J_{\pi}), \]  

(D.1)

where

\[ M(n_p l_p, n_n l_n) = 2^{n_p+n_n} n_p! n_n!(2l_p + 2n_p + 1)!!(2l_n + 2n_n + 1)!!, \]  

(D.2)

\[ a_{l_p+2s}(n_p l_p, n_n l_n) = \sum_{k+k' = s} \binom{n_p}{k} \binom{n_n}{k'} \frac{(2l_p + 2n_p + 1)!!(2l_n + 2n_n + 1)!!}{(2l_p + 2k + 1)!!(2l_n + 2k' + 1)!!}, \]  

(D.3)

\[ f_{LL'}^{\kappa}(m, m'; \omega J_{\pi}) = \sum_{\mu} a_{2\mu} \left( \frac{m - L}{2}, \frac{m' - L'}{2} \right) \mathcal{J}_\mu^{\kappa}(\omega J_{\pi}), \]  

(D.4)

and

\[ \mathcal{J}_\mu^{\kappa}(\omega J_{\pi}) = (2\nu)^{-\mu} R \int_0^\infty dk k^{2\mu+2+\kappa} e^{-k^2/2\nu} v(k; \omega J_{\pi}). \]  

(D.5)
References

[1] W. C. Haxton and G. Stephenson, Prog. in Part. and Nucl. Phys. 12 (1984) 409.

[2] M. Doi, T. Kotani and E. Takasugi, Prog. Theor. Phys. Suppl. 83, (1985) 1; ibid Phys. Rev. D37 (1988) 2575.

[3] D. Vergados, Phys. Rep. 133 (1986) 1.

[4] T. Tomoda, A. Faessler, K.W. Schmid and F. Grümmer, Nucl. Phys. A452 (1986) 591; T. Tomoda, Rep. Prog. Phys. 54 (1991) 53.

[5] M. Doi and T. Kotani, Prog. Theor. Phys. 89, (1993) 139.

[6] R.N. Mohapatra and J. Vergados, Phys. Rev. Lett. 47, 1713 (1981).

[7] A. Halprin, P. Minkowski, H. Primakoff and S.P. Rosen, Phys. Rev. D13 2567 (1976); R. N. Mohapatra, Phys. Rev. D34, 3457 (1986); M. Hirsch, H. V. Klapdor-Kleingrothaus and O. Panella, Phys. Lett. B374 (1995) 7.

[8] G.B. Gelmini and M. Roncadelli, Phys. Lett. B99 (1981) 411.

[9] C.P. Burgess and J.M. Cline, Phys. Lett. B (1993) 141, ibid. Phys. Rev. D49 (1994) 5925; C.D. Carone, Phys. Lett. B308 (1993) 85.

[10] P. Bamert, C.P. Burgess and R.N. Mohapatra, Nucl. Phys. B449 (1995) 25.

[11] C. Barbero, J. Cline, F. Krmpotić and D. Tadić, Phys. Lett. B371 (1996) 78; ibid. Phys. Lett. B392 (1997) 419.

[12] J. D. Vergados, Phys. Lett. B184 (1987) 55.

[13] M. Hirsch, H. V. Klapdor-Kleingrothaus and S. G. Kovalenko, Phys. Lett. B352 (1995) 1; Phys. Rev. Lett. 75 (1995) 17; Phys. Rev. D53 (1996); A. Faessler, S. Kovalenko, F. Šimkovic, and J. Schwieger, Phys. Rev. Lett. 78 (1997) 183.

[14] H. V. Klapdor-Kleingrothaus, Prog. in Part. and Nucl. Phys. 32 (1994) 261.
[15] H. Horie and K. Sasaki, *Prog. Theor. Phys.* **25** (1961) 475.

[16] F. Krmpotić, J. Hirsch and H. Dias, *Nucl. Phys.* **A542** (1992) 85.

[17] F. Krmpotić and S. Shelly Sharma, *Nucl. Phys.* **A572** (1994) 329.

[18] J.D. Vergados *Nucl. Phys.* **A506** (1990) 482; G. Pantis and J.D. Vergados *Phys. Lett.* **B242** (1990) 1; A. Faessler, W.A. Kaminski, G. Pantis, and J.D. Vergados *Phys. Rev.* **C43** (1991) 21; G. Pantis and J.D. Vergados *Phys. Rep.* **242** (1994) 284; G. Pantis, F. Šimkovic, J.D. Vergados and A. Faessler, *Phys. Rev.* **C53** (1996) 695.

[19] J. Suhonen, S.B. Khadkikar and A. Faessler, Phys. Lett. **B237** (1990) 8; *ibid. Nucl. Phys.* **A529** (1991) 727 and *Nucl. Phys.* **A535** (1991) 509.

[20] H. Primakoff and S.P. Rosen, *Rep. Prog. Phys.* **22** (1959) 121.

[21] A. Bohr and B.R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969) vol. 1.

[22] J.D. Walecka, *Theoretical Nuclear and Subnuclear Physics* (Oxford University Press, New York, 1995)

[23] M.E. Rose and R.K. Osborn, *Phys. Rev.* **93** (1954) 1315, *ibid. Phys. Rev.* **93** (1954) 1326.

[24] T. deForest and J.D. Walecka, Adv. Phys. **15** (1966) 1.

[25] F. Krmpotić, T.T.S. Kuo, A. Mariano, E.J.V. de Passos and A.F.R. de Toledo Piza, *Nucl. Phys.* **A612** (1997) 223.

[26] F. Krmpotić, *Rev. Mex. Fís.* **40** (1994) 285.

[27] H. Ejiri, K. Fushimi, K. Hayashi, T. Kishimoto, N. Kudomi, K. Kume, K. Nagata, H. Ohsumi, K. Okada, T. Shima and J. Tanaka, *Nucl. Phys.* **A611** (1996) 85.