Mother Operators and their Descendants.*

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Abstract. A mechanism deriving new well-posed evolutionary equations from given ones is inspected. It turns out that there is one particular spatial operator from which many of the standard evolutionary problems of mathematical physics can be generated by this abstract mechanism using suitable projections. The complexity of the dynamics of the phenomena considered can be described in terms of suitable material laws. The idea is illustrated with a number of concrete examples.

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In \cite{10, 15} it has been shown that the standard (autonomous and linear) initial boundary value problems of mathematical physics share a simple common form, if considered as first order systems. Indeed, it is found that they are of the form
\[ \partial_0 V + AU = F, \]
where in the usual cases $A$ is skew-selfadjoint, and $U$ and $V$ are linked by a so-called material law
\[ V = \mathcal{M} (\partial_0^{-1}) U, \]
where $\mathcal{M}$ is a bounded operator-valued function, analytic in a ball $B_C (r, r)$ of radius $r \in \mathbb{R}_{>0}$ centered at $r$. $\mathcal{M} (\partial_0^{-1})$ is then well-defined in terms of an operator-valued function calculus associated with $\partial_0$ as a normal operator in the weighted $L^2$-type space $H^{\varrho}$, $\varrho > \frac{1}{2}$, with inner product
\[ (U, V) \mapsto \int_{\mathbb{R}} \langle U (t) | V (t) \rangle_H \exp (-2\varrho t) \, dt. \]

We will not need to recall the solution theory of such equations, (which we like to refer to as “evolutionary” as the term “evolution equations” appears to be reserved for a rather special case in this wider class), since the purpose of this paper is not on well-posedness issues but on a remarkable even more specific structural similarity between various equations of mathematical physics. We shall indeed see, that most of the standard initial boundary value problems of mathematical physics can be derived from a single spatial differential operator of the form
\[ A = \begin{pmatrix} 0 & -\nabla^* \\ \nabla & 0 \end{pmatrix} \]
with a suitable domain to make $A$ skew-selfadjoint in
\[ H := \left( \bigoplus_{k \in \mathbb{N}} L^2_k (\Omega) \right) \oplus \left( \bigoplus_{k \in \mathbb{N}} L^2_k (\Omega) \right). \]

Here $\Omega$ is a non-empty open subset of a Riemannian $C_{1,1}$-manifold $M$ with $\nabla = d \otimes$ denoting the co-variant derivative (Riemannian connection). The spaces
\[ L^2_k (\Omega) \]
are the completion of (real- or) complex-valued Lipschitz continuous covariant $k$-tensor fields having compact support in $\Omega$ with considered in the norm $| \cdot |_{k,0}$ induced by the inner product

$$(\varphi, \psi) \mapsto \int_M \langle \varphi | \psi \rangle_k V,$$

where $V$ denotes the volume element associated with the Riemannian metric tensor field $g$ given by the Riemannian structure of the manifold $M$. Here $\langle \varphi | \psi \rangle_k$ abbreviates the function $p \mapsto \langle \varphi(p) | \psi(p) \rangle_k, (T M_p)^*$

where $$(\Phi, \Psi) \mapsto \langle \Phi | \Psi \rangle_{k,(T M_p)^*}$$ is the (real) inner product of covariant $k$-tensors on the tangent space $T M_p$ at $p \in M$ and $\overline{\cdot}$ denotes complex conjugation. Covariant 0-tensors are simply real numbers and so 0-tensors fields are real-valued functions on $M$ and so we let $$(\Phi, \Psi) \mapsto \langle \Phi | \Psi \rangle_{0,(T M_p)^*} := \Phi \Psi.$$ Since for $k \in \mathbb{N}_{>0}$ covariant $k$-tensors on the tangent space $T M_p$ are elements in the (real) tensor product space $\bigotimes_k (T M_p)^*$ the inner product is induced by

$$\langle \Phi_0 \otimes \cdots \otimes \Phi_{k-1} | \Psi_0 \otimes \cdots \otimes \Psi_{k-1} \rangle_{k,(T M_p)^*} = \langle \Phi_0 | \Psi_0 \rangle_{(T M_p)^*} \cdots \langle \Phi_{k-1} | \Psi_{k-1} \rangle_{(T M_p)^*}.$$ In the sense of this inner product $\nabla^* = - \text{div}$ is the formal adjoint of the co-variant derivative $\nabla$.

The complexity of the various physical phenomena has no influence on the choice of $A$ but is reflected in different material laws\footnote{This is the opposite point of view to the “conservation law” perspective.}. The process of extraction of particular operators from the “mother” operator $(\partial_0 M (\partial_0^{-1}) + A)$ is surprisingly simple and amounts to projecting this operator (class) down to smaller subspaces. The transparency and simplicity of this construction is rather striking and shows the interconnectedness of various different physical phenomena if inspected from a mathematical point of view. This connection becomes obscured if a second order (or higher order) model is chosen as a starting point. It appears that a largely misguided pre-occupation with the occurrence of the Laplacian\footnote{This is surely fostered by the comforting regularity properties of elliptic (and parabolic) differential operators.} in equations of mathematical physics has lead to a dominance of second-order equations and systems in various models. As it turns out, however, the first order approach leads to a unified and transparent access to a large class (if not all) of typical linear model equations.

In the applications we shall for sake of simplicity focus on the Cartesian or periodic case $(M = \mathbb{R}^{n-k} \times \mathbb{T}^k, n = 1, 2, 3, k = 0, 1, 2, 3, k \leq n$, with $\mathbb{T}$ being the flat torus obtained from the unit interval $[-1/2, 1/2]$ by “gluing” the end points together (implying periodicity boundary conditions on $[-1/2, 1/2]$ in the last $k$ components), which is perfectly sufficient to understand the reduction mechanism and its applicability.

In our perspective these have their place when qualitative properties are of prominent interest, but, as it turns out, have limited importance and are often distracting for fundamental well-posedness issues.
1 Mother Operator, Relatives and Descendants

1.1 A Construction Mechanism

Definition 1.1. Let $C : D(C) \subseteq H_0 \to H_1$ linear, closed and densely defined and $B : H_0 \to X$ a continuous linear mapping, $X, H_0, H_1$ Hilbert spaces. We say $B$ is compatible with $C$ if

- $CB^*$ is densely defined (in $X$).

Theorem 1.2. Let $C : D(C) \subseteq H_0 \to H_1$ linear, closed and densely defined and $B : H_0 \to X$ a continuous linear mapping, $X, H_0, H_1$ Hilbert spaces. Moreover, let $B$ be compatible with $C$. Then

$$(CB^*)^* = B C^*.$$ 

Proof. It is

$$CB^* \subseteq (BC^*)^*.$$ 

Let now $u \in D((BC^*)^*)$ then for $v \in D(BC^*) = D(C^*)$:

$$\langle v | (BC^*)^* u \rangle = \langle BC^* v | u \rangle = \langle C^* v | B^* u \rangle.$$ 

We read off that

$$B^* u \in D(C^{**}) = D(C)$$

and

$$CB^* u = (BC^*)^* u.$$ 

Consequently, we have

$$CB^* = (BC^*)^*$$

and so

$$(CB^*)^* = (BC^*)^{**} = B C^*.$$

Definition 1.3. Let $C : D(C) \subseteq H_0 \to H_1$ linear, closed and densely defined, $X, H_0, H_1$ Hilbert spaces. Moreover, let $B_0 : H_0 \to X$ be compatible with $C$ and $B_1 : H_1 \to Y$ be compatible with $C^*$. Then we call $B_1 CB_0^*$ the $(B_0, B_1)$-relative (or simply a relative) of $C$. If not both of the mappings $B_0, B_1$ are bijections, then we call $B_1 CB_0^*$ the $(B_0, B_1)$-descendant (or simply a descendant) of $C$ (and $C$ the mother operator of $B_1 CB_0^*$).

Of particular interest are compatible operators resulting from orthogonal projectors. We introduce the following schemes of notation:

Definition 1.4. Let $V$ be a closed subspace of a Hilbert space $H$. Then we denote the orthogonal projector onto $V$ by $P_V$. 

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1.2 Evolutionary Operators and their Relatives

**Definition 1.5.** Let \( X \oplus Y \) a direct sum of Hilbert spaces \( X, Y \). Then we denote the canonical projections

\[
X \oplus Y \to X, \quad X \oplus Y \to Y
\]

\[
x \oplus y \mapsto x, \quad x \oplus y \mapsto y
\]

by \( \pi_X \) and \( \pi_Y \), respectively.

These notations are employed in the following elementary observation, which we record without giving the elementary proof.

**Proposition 1.6.** Let \( V \) be a closed subspace of a Hilbert space \( H \). Then

\[
P_V = \pi_V^* \pi_V,
\]

and

\[
(1 - P_V) = \pi_V^* \pi_{V^\perp} = P_{V^\perp}.
\]

Moreover, \( \pi_V^* \) is the canonical isometric embedding of \( V \) in \( H \) and

\[
\pi_V \pi_V^* \text{ and } \pi_V \pi_{V^\perp}^*
\]

are the identities on \( V \) and \( V^\perp \), respectively.

The subspace \( V \oplus \{0\} \) of \( V \oplus V^\perp \) is commonly identified with \( V \). We shall, however, prefer to distinguish \( P_V \) and \( \pi_V \), since the latter allows a proper formulation of reducing an operator equation to a subspace by constructing appropriate descendants.

1.2 Evolutionary Operators and their Relatives

The concepts introduced in the previous section extends naturally to evolutionary problems as described in the introduction.

To be specific we consider a particular class of evolutionary problems of the form:

\[
\left( \partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + A \right) U = F
\]

where

\[
A := \begin{pmatrix}
0 & -C^* \\
C & 0
\end{pmatrix}
\]

and \( C : D(C) \subseteq H_0 \to H_1 \) is a closed densely defined linear operator so that \( A \) is skew-selfadjoint in the Hilbert space \( H := H_0 \oplus H_1 \). We assume for the material law that there is a \( c_0 \in \mathbb{R}_{>0} \) with

\[
\Re \left\langle \chi_{\bar{z} < c_0} (m_0) U | \partial_0 \mathcal{M} \left( \partial_0^{-1} \right) U \right\rangle_{\bar{g},0,0} \geq c_0 \left\langle \chi_{\bar{z} < c_0} (m_0) U | U \right\rangle_{\bar{g},0,0}
\]

for all sufficiently large \( \bar{g} \in \mathbb{R}_{>0} \) and all \( U \in D(\partial_0) \subseteq H_{\bar{g},0} (\mathbb{R}, H) \). This assumption warrants solvability and causality of the solution operator, for this variant of the solution theory compare [14, 13].
1 Mother Operator, Relatives and Descendants

In order to ensure \(2\) we may and will assume that
\[
\mathcal{M} (\partial_0^{-1}) = \mathcal{M}_0 + \partial_0^{-1} \mathcal{M}^{(1)} (\partial_0^{-1})
\]
with \(\mathcal{M}_0 \in L(H)\) selfadjoint\(^3\) and \(\pi_{\mathcal{M}_0[H]}, \mathcal{M}_0 \mathcal{M}_0^{*}[H], \pi_{\{0\}] \mathcal{M}_0 \Re (\mathcal{M}^{(1)} (\partial_0^{-1})) \mathcal{M}_0^*_{\{0\}} \mathcal{M}_0\) uniformly strictly positive definite for all sufficiently large \(\rho \in \mathbb{R}_{>0}\). If \(\mathcal{M}_1 \in L(H)\)
\[
\mathcal{M}^{(1)} (\partial_0^{-1}) = \mathcal{M}_1 + \mathcal{M}^{(2)} (\partial_0^{-1}),
\]
\(\mathcal{N} := \pi_{\{0\}} \mathcal{M}_0 \Re (\mathcal{M}_1) \pi^*_{\{0\}} \mathcal{M}_0\) is strictly positive definite and the operator norm on the space \(H_{\rho,0} (\mathbb{R}, \{0\}, \mathcal{M}_0)\) satisfies
\[
\left\| \left( \sqrt{\mathcal{N}} \right)^{-1} \pi_{\{0\}} \mathcal{M}_0 \Re \mathcal{M}^{(2)} (\partial_0^{-1}) \pi^*_{\{0\}} \mathcal{M}_0 \left( \sqrt{\mathcal{N}} \right)^{-1} \right\| < 1
\]
for sufficiently large \(\rho \in \mathbb{R}_{>0}\), then a standard perturbation argument shows that \(2\) is maintained. Since solution theory is not the topic of this paper we will not dwell on these issues in the following. We merely note that the construction of relatives and descendants maintains this solvability condition.

We note first that as a by-product of the above we have the following.

**Proposition 1.7.** Let \(C : D(C) \subseteq H_0 \to H_1\) linear, closed and densely defined. Let \(B_0 : H_0 \to X\) be compatible with \(C\) and \(B_1 : H_1 \to Y\) be compatible with \(C^*\). Then \(B_0 \oplus B_1\) is compatible with
\[
A := \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}.
\]

Moreover, if \(B_0^*\) has a bounded left-inverse and the set \(B_0^*[X] \cap D(C)\) is a core for \(B_1 C \big|_{B_0^*[X]}\) then the relative
\[
(B_0 \oplus B_1) A (B_0 \oplus B_1)^* = \begin{pmatrix} 0 & -B_0 C^* B_1^* \\ B_1 C B_0^* & 0 \end{pmatrix}
\]
of \(A\) is skew-selfadjoint in \(X \oplus Y\).

**Proof.** The compatibility of \(B_0 \oplus B_1\) with \(A\) is clear. To show that the relative
\[
\begin{pmatrix} 0 & -B_0 C^* B_1^* \\ B_1 C B_0^* & 0 \end{pmatrix}
\]
of \(A\) is again skew-selfadjoint, we have to verify that
\[
(B_0 C^* B_1^*)^* = B_1 C B_0^*.
\]
Using Theorem \([1,2]\) for \(B_0 C^*\) and \(B_1\) and the relation \((B_0 C^*)^* = (B_0 C^*)^* = C B_0^*,\) we obtain that
\[
(B_0 C^* B_1^*)^* = B_1 (B_0 C^*)^* = B_1 C B_0^*.
\]

\(^3\)We denote by \(L(H)\) the space continuous linear operators from \(H\) into \(H\). Moreover, we shall not notationally distinguish between an operator \(M \in L(H)\) and its (canonical) extension to the Hilbert space of \(H\)-valued \(H_{\rho,0}\)-functions.
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and thus, it suffices to show $B_1CB_0^* = B_1CB_0^*$. Obviously, $B_1CB_0^* \subseteq B_1CB_0^*$. To see the missing inclusion, let $u \in D\left(B_1CB_0^*\right)$, i.e. $B_0^*u \in D\left(B_1C\right)$. By assumption, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ such that $B_0^*x_n \in D(C)$ for each $n \in \mathbb{N}$ and

$$B_0^*x_n \to B_0^*u, \quad B_1CB_0^*x_n \to B_1CB_0^*u,$$

as $n \to \infty$. Using that $B_0^*$ has a bounded left inverse, we derive that $x_n \to u$ as $n \to \infty$ and hence, $u \in D\left(B_1CB_0^*\right)$, showing the missing inclusion.

Remark 1.8.

1. If $B_0^*$ is onto and continuously invertible, then the assumptions of the latter proposition are trivially satisfied.

2. The structure of $A$ as $\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ also implies that the operators $\overline{\partial_0M\left(\partial_0^{-1} + A\right)}$ and $\overline{\partial_0\tilde{M}\left(\partial_0^{-1} - A\right)}$ with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M\left(\partial_0^{-1}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tilde{M}\left(\partial_0^{-1}\right)$$

are relatives via $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as a unitary mapping.

3. The construction of descendants may be repeated but the result will in general depend on the order in which the steps are carried out.

We note that the proof of Proposition 1.7 mainly relies on the fact that $B_1CB_0^* = B_1CB_0^*$, which follows by the additional assumptions on $B_0^*$.

Example 1.9. Let $H_0, H_1$ be two Hilbert spaces, where $H_1$ is assumed to be separable. In $H_0$ we choose two closed, densely defined operators $A_0, A_1$ such that $A_0 \not\subseteq A_1$ and $x_1 \in D(A_0)^{1-p}A_1$, where the ortho-complement is taken in $D(A_1)$ with respect to the graph inner product of $A_1$. Moreover, let $(y_n)_{n \in \mathbb{N}}$ a linear independent total sequence in $H_1$ with $y_n \to 0$. We define the following operator on $H_1$ as the linear extension of the mapping:

$$R : \{y_n \mid n \in \mathbb{N}\} \subseteq H_1 \to \ell_2(\mathbb{N})$$

$$y_k \mapsto 2^k e_k,$$

where $e_k$ denotes the $k$-th unit vector in $\ell_2(\mathbb{N})$. We denote its extension again by $R$. This operator turns out to be closable. Indeed, let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Span}\{y_n \mid n \in \mathbb{N}\}$ with $z_n \to 0$ and $Rz_n \to z \in \ell_2(\mathbb{N})$ as $n \to \infty$. For each $n \in \mathbb{N}$ there exists a sequence $(\lambda_k^n)_{k \in \mathbb{N}}$ of complex numbers with almost all entries being zero such that

$$z_n = \sum_{k=0}^{\infty} \lambda_k^n y_k.$$
Since \( z_n \to 0 \) we obtain \( \lambda_k^n \to 0 \) as \( n \to \infty \) for each \( k \in \mathbb{N} \). The latter yields
\[
\langle Rz_n | e_j \rangle_{\ell_2} = 2^j \lambda_j^n \to 0
\]
as \( n \to \infty \). This shows \( z = 0 \) and thus, \( R \) is closable. Consider now the operator \( Q \) defined as
\[
Q : \text{Span} \{ (x_1, y_n) \mid n \in \mathbb{N} \} \subseteq \text{Span} \{ x_1 \} \oplus H_1 \to \ell_2(\mathbb{N})
\]
\[
(w, z) \mapsto Rz.
\]
Again, this operator is closable, which follows from the closability of \( R \). Then \( (x_1, 0) \notin D(\overline{Q}) \).

Indeed, let \( \{(w_n, z_n)\}_{n \in \mathbb{N}} \) be a sequence in \( \text{Span} \{ (x_1, y_n) \mid n \in \mathbb{N} \} \) with \( w_n \to x_1 \) and \( z_n \to 0 \). For each \( n \in \mathbb{N} \) there exists a finite sequence \( (\lambda_k^n)_{k \in \mathbb{N}} \) of complex scalars such that
\[
w_n = \sum_{k=0}^{\infty} \lambda_k^n x_1 \quad \text{and} \quad z_n = \sum_{k=0}^{\infty} \lambda_k^n y_k.
\]
Since \( z_n \to 0 \) we get \( \lambda_k^n \to 0 \) as \( n \to \infty \) and thus,
\[
\bigwedge_{m \in \mathbb{N}} \bigvee_{n_0 \in \mathbb{N}} \bigwedge_{n \geq n_0} \sum_{k=0}^{m} |\lambda_k^n| \leq \frac{1}{2^n}.
\]
Moreover, since \( w_n \to x_1 \) we derive \( \sum_{k=0}^{\infty} \lambda_k^n \to 1 \) as \( n \to \infty \), which yields
\[
\bigvee_{n_1 \in \mathbb{N}} \bigwedge_{n \geq n_1} \sum_{k=0}^{\infty} |\lambda_k^n| \geq \frac{1}{2}.
\]
Let \( m \in \mathbb{N} \) and \( n \geq \max\{n_0, n_1\} \). Then
\[
|Rz_n|_{\ell_2}^2 = \sum_{k=0}^{\infty} 2^{2k} |\lambda_k^n|^2
\]
\[
= \sum_{k=0}^{m} 2^{2k} |\lambda_k^n|^2 + \sum_{k=m+1}^{\infty} 2^{2k} |\lambda_k^n|^2
\]
\[
\geq 2^{2(m+1)} \sum_{k=0}^{\infty} 2^{2k} |\lambda_k^{n+m+1}|^2
\]
\[
\geq 2^{2(m+1)} \frac{3}{4} \left( \sum_{k=m+1}^{\infty} |\lambda_k^n| \right)^2
\]
\[
= 2^{2(m+1)} \frac{3}{4} \left( \sum_{k=0}^{\infty} |\lambda_k^n| - \sum_{k=0}^{m} |\lambda_k^n| \right)^2
\]
\[
\geq 2^{2(m+1)} \frac{3}{4} \left( \frac{1}{2} - \frac{1}{2^m} \right)^2 \to \infty \quad (m \to \infty).
\]
This shows that \( (Rz_n)_{n \in \mathbb{N}} \) has an unbounded subsequence and hence, cannot converge. Thus, \( (x_1, 0) \notin D(\overline{Q}) \), which in particular implies that \( (x, 0) \in D(\overline{Q}) \) implies \( x = 0 \).
Note that \( x + y \in D(A_1) \) since \( y \in \text{Span}\{x_1\} \). We show that \( A \) is closed. To this end, let \( ((x_n + y_n, z_n))_{n \in \mathbb{N}} \) be a \( H_0 \otimes H_1 \)-convergent sequence in \( D(A) \) such that \( (A_1(x_n + y_n))_{n \in \mathbb{N}} \) and \( (\overline{Q}(y_n, z_n))_{n \in \mathbb{N}} \) converges in \( H_0 \) and \( \ell_2(\mathbb{N}) \), respectively. Then \( (x_n + y_n, z_n)_{n \in \mathbb{N}} \) converges in \( D(A_1) \) with respect to the graph norm of \( A_1 \), i.e. in \( D_{A_1} \), to some \( c \in D(A_1) \). From \( x_n \in D(A_0) \) and \( y_n \in D(A_0)^{\perp \! \! \perp} \), we deduce that both \( (x_n, z_n)_{n \in \mathbb{N}} \) and \( (\overline{Q}(y_n, z_n))_{n \in \mathbb{N}} \) together with the closedness of \( \overline{Q} \), we infer that \( ((y_n, z_n))_{n \in \mathbb{N}} \) converges in \( D(\overline{Q}) \) to \( (y, z) \in D(\overline{Q}) \) with respect to the graph norm of \( \overline{Q} \) for some \( z \in \ell_2(\mathbb{N}) \). Summarizing, we have \( x_n + y_n \to x + y \) in \( D_{A_1} \), as \( n \to \infty \) with \( x \in D(A_0) \) and \( y \in \text{Span}\{x_1\} \) as well as \( (y_n, z_n) \to (y, z) \) in \( D(\overline{Q}) \) as \( n \to \infty \). Thus \( A \) is closed.

We define
\[
B : H_0 \to H_0 \oplus H_1 \\
x \mapsto (x, 0)
\]
and
\[
C : H_0 \oplus \ell_2(\mathbb{N}) \to H_0 \\
(x, y) \mapsto x.
\]
Then
\[
CAB = A_0.
\]
Indeed, let \( x \in D(CAB) \). The latter yields that \( (x, 0) \in D(A) \). By the definition of \( A \) we have that \( (x - x_0, 0) \in D(\overline{Q}) \) for some \( x_0 \in D(A_0) \). By what we have shown above, this holds if and only if \( x = x_0 \) and thus \( x \in D(A_0) \). The other inclusion holds trivially.

We show that \( x_1 \in D(CAB) \). The latter is equivalent to \( (x_1, 0) \in D(CA) \). For each \( n \in \mathbb{N} \) we have that \( (x_n, y_n) \in D(Q) \subseteq D(CA) \). Since \( (x_1, y_n) \to (x_1, 0) \) and \( CA(x_1, y_n) = A_1x_1 \) for each \( n \in \mathbb{N} \) we obtain \( (x_1, 0) \in D(CA) \). Thus,
\[
CAB = CAB \subseteq \overline{CA}.
\]
Note that with \( H_1 = \ell_2(\mathbb{N}) \), we even have \( B = C^* \) and thus \( \overline{CACC} = CAC^* \subseteq \overline{CACC} \).

Applying earlier observations to evolutionary operators yields the following result.

**Theorem 1.10.** Let \( C : D(C) \subseteq H_0 \to H_1 \) linear, closed and densely defined, and let \( B_0 : H_0 \to X \) be compatible with \( C \) and \( B_1 : H_1 \to Y \) be compatible with \( C^* \). Moreover, let \( B_0^* \) have a bounded left inverse and let \( B_0^*[X] \cap D(C) \) be a core for \( B_1C_{B_0[X]}^{-1} \,
\]

\[
A := \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}
\]

Then \( (B_0 \oplus B_1)^* A (B_0 \oplus B_1)^* \) is skew-selfadjoint and
\[
\partial_0 (B_0 \oplus B_1) \mathcal{M} (\partial_0^{-1}) (B_0 \oplus B_1)^* + B_0 \oplus B_1 A (B_0 \oplus B_1)^* \]
is an (evolutionary) relative of \( (\partial_0 \mathcal{M} (\partial_0^{-1}) + A) \).
1 Mother Operator, Relatives and Descendants

Unitary equivalence of $A$ would be a typical example illustrating the previous theorem. More interesting, however, are descendants being produced by projections onto proper subspaces.

Remark 1.11. In contrast to the last result general relatives of $(\partial_0 M (\partial_0^{-1}) + A)$ need not maintain its clear formal structure. Indeed, a few elementary row and and column operations can produce almost impenetrably confusing “model equations”. In many instances it turns out to be the main task to reconstruct $(\partial_0 M (\partial_0^{-1}) + A)$ from a quite different looking (often only a formal) relative.

1.3 Reducing Equations to Subspaces

1.3.1 The General Construction

Let now $V$ be a closed subspace such that

$$P_V M (\partial_0^{-1}) = M (\partial_0^{-1}) P_V.$$  

Then, applying $p_V$ to equation (1) we obtain similarly as before

$$\partial_0 (p_V M (\partial_0^{-1}) p_V) p_V u + p_V A (p_V^* p_V u + p_V^* p_V u) = p_V F.$$  

This is now an equation for $p_V u$ in $V$ with little chance of being well-posed (due to the free floating part $p_V^* p_V u$). To enforce well-posedness, we could assume that $p_V$ and $p_V^*$ are compatible with $A$ and instead (assuming $p_V u = 0$) consider

$$(\partial_0 (p_V M (\partial_0^{-1}) p_V^*) + (\overline{p_V A p_V}^*) p_V u = p_V F.$$  

In general, however, $\overline{p_V A p_V}^*$ will fail to be skew-selfadjoint although

$$p_V M (\partial_0^{-1}) p_V$$  

inherits its positive definiteness property from $M (\partial_0^{-1})$:

There is a $c_0 \in \mathbb{R}_{>0}$ with

$$\text{Re} \left( \chi_{s<0} (m_0) U | \partial_0 p_V M (\partial_0^{-1}) p_V^* u \right)_{\phi,0,0} \geq c_0 \left( \chi_{s<0} (m_0) p_V^* u | p_V^* u \right)_{\phi,0,0}$$  

$$= c_0 \left( \chi_{s<0} (m_0) u | u \right)_{\phi,0,0}$$  

for all sufficiently large $\rho \in \mathbb{R}_{>0}$ and all $u \in D(\partial_0) \subseteq H_\rho(\mathbb{R}, V)$.

But if $V = V_0 \oplus V_1$ with $V_0 = H_0$ or $V_1 = H_1$, where $p_{V_0}$ is compatible with $C$ and $p_{V_1}$ is compatible with $C^*$, we see from our earlier considerations that we have an evolutionary descendant with $\overline{p_V A p_V}^*$ skew-selfadjoint. We summarize this observation in our next theorem.

Theorem 1.12. Let $V_0 \subseteq H_0$, $V_1 \subseteq H_1$ be closed subspaces such that $p_{V_0}$ is compatible with $C$ and $p_{V_1}$ is compatible with $C^*$. Then with $V := V_0 \oplus H_1$ or $V := H_0 \oplus V_1$ we have that the operator $(\partial_0 (p_V M (\partial_0^{-1}) p_V) + (\overline{p_V A p_V}^*))$ is the evolutionary $p_V$-descendant of $(\partial_0 M (\partial_0^{-1}) + A)$.
1.3 Reducing Equations to Subspaces

1.3.2 A Particular Case: Removing Null Spaces

To see the above construction at work let us consider the standard issue of reducing $A$ to the ortho-complement of its kernel for the operator

$$\left(\partial_0 \mathcal{M} \left(\partial_0^{-1}\right) + A\right).$$

We assume for sake of definiteness that

$$\mathcal{M} \left(\partial_0^{-1}\right) = \mathcal{M}_0 + \partial_0^{-1} \mathcal{M}_1 \left(\partial_0^{-1}\right)$$

for a selfadjoint $\mathcal{M}_0 \in L(H)$ and a $L(H)$-valued analytic function $\mathcal{M}_1$ for $H$ being the underlying (spatial) Hilbert space. We have that

$$\partial_0 \left(\pi_{A[H]}^* \mathcal{M} \left(\partial_0^{-1}\right) \pi_{A[H]}^* \right) + \pi_{A[H]}^* \mathcal{M}_1 \left(\partial_0^{-1}\right) \pi_{A[H]}^*$$

and

$$\partial_0 \left(\pi_{\{0\}A[H]}^* \mathcal{M} \left(\partial_0^{-1}\right) \pi_{\{0\}A[H]}^* \right)$$

are two corresponding relatives. It is $\pi_{A[H]}^* \mathcal{M}_1 \left(\partial_0^{-1}\right) \pi_{A[H]}^*$ non-trivial, since $A[H]$ is a reducing subspace of $A$. Note that here $\pi_{A[H]}^* A$ is already closed and $\pi_{\{0\}A[H]}^* A = 0$. Moreover,

$$\pi_{A[H]} = \pi_{\{0\}C} \oplus \pi_{\{0\}C^*}$$

and

$$\pi_{\{0\}A[H]}^* = \pi_{\{0\}C} \oplus \pi_{\{0\}C^*}.$$

Clearly, $\pi_{\{0\}C}$ and $\pi_{\{0\}C^*}$ are compatible with $C$ and correspondingly $\pi_{\{0\}C^*}$ and $\pi_{\{0\}C^*}$ are compatible with $C^*$. So, if at least one of the null spaces $\{\{0\}C$ or $\{\{0\}C^*$ is non-trivial, which is the only interesting case, we have that

$$\partial_0 \left(\pi_{A[H]}^* \mathcal{M} \left(\partial_0^{-1}\right) \pi_{A[H]}^* \right) + \pi_{A[H]}^* \mathcal{M}_1 \left(\partial_0^{-1}\right) \pi_{A[H]}^*$$

and

$$\partial_0 \left(\pi_{\{0\}A[H]}^* \mathcal{M} \left(\partial_0^{-1}\right) \pi_{\{0\}A[H]}^* \right)$$

are indeed descendants of $(\partial_0 \mathcal{M} \left(\partial_0^{-1}\right) + A)$. How can these descendants help in solving a problem for the mother operator?

To simplify calculations we first confirm that we may assume that $\mathcal{M}_0$ can be replaced by $\pi_{A[H]}^* \mathcal{M}_0 = P_{\mathcal{M}_0[H]}$. Indeed,

$$H = \mathcal{M}_0[H] \oplus \{\{0\}\} \mathcal{M}_0$$

and

$$\mathcal{M}_0 = \pi_{\mathcal{M}_0[H]} \mathcal{M}_0 \pi_{\mathcal{M}_0[H]}^* \oplus 0_{\{0\}\mathcal{M}_0}$$

With

$$\widetilde{\mathcal{M}}_0 = \pi_{\mathcal{M}_0[H]} \mathcal{M}_0 \pi_{\mathcal{M}_0[H]}^* \oplus \pi_{\{0\}\mathcal{M}_0} \pi_{\{0\}\mathcal{M}_0}^*$$

we obtain

$$\sqrt{\widetilde{\mathcal{M}}_0^{-1}} \mathcal{M}_0 \sqrt{\widetilde{\mathcal{M}}_0^{-1}} = \sqrt{\widetilde{\mathcal{M}}_0^{-1}} \sqrt{\mathcal{M}_0^{-1}} \left(1_{\mathcal{M}_0[H]} \oplus 0_{\{0\}\mathcal{M}_0}\right) \left(1_{\mathcal{M}_0[H]} \oplus 0_{\{0\}\mathcal{M}_0}\right) \sqrt{\mathcal{M}_0} \sqrt{\widetilde{\mathcal{M}}_0^{-1}}$$

$$= \left(1_{\mathcal{M}_0[H]} \oplus 0_{\{0\}\mathcal{M}_0}\right) = \pi_{\mathcal{M}_0[H]}^* \mathcal{M}_0 \pi_{\mathcal{M}_0[H]}^*.$$
1 Mother Operator, Relatives and Descendants

By writing again $A$ for $\sqrt{M_0^{-1}A\sqrt{M_0^{-1}}}$ and $M(\partial^{-1}_0)$ for $\sqrt{M_0^{-1}M(\partial^{-1}_0)\sqrt{M_0^{-1}}}$, the decomposition

$$M(\partial^{-1}_0) = P_{\{0\}} + \partial^{-1}_0 M(\partial^{-1}_0).$$

Since we have

$$H = A[H] \oplus \{0\} A$$

we obtain the decomposition

$$\left( \partial_0 \left( \pi_{A[H]} M (\partial^{-1}_0) \pi_{A[H]}^* \right) + \pi_{A[H]} A \pi_{A[H]}^* \right) \pi_{A[H]} U + \partial_0 \left( \pi_{A[H]} M (\partial^{-1}_0) \pi_{\{0\}[A]} \right) \pi_{\{0\}[A]} U = \pi_{A[H]} F$$

and

$$\left( \partial_0 \left( \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right) \right) \pi_{\{0\}[A]} U + \left( \partial_0 \left( \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right) \right) \pi_{\{0\}[A]} U = \pi_{\{0\}[A]} F.$$

Note that

$$\pi_{A[H]} A \pi_{A[H]}^*$$

is now skew-selfadjoint in $A[H]$ and strict positive definiteness of the real parts of the operators

$$\partial_0 \left( \pi_{A[H]} M (\partial^{-1}_0) \pi_{A[H]}^* \right)$$

and

$$\partial_0 \left( \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right)$$

is preserved. It should also be clear that here it makes no sense to assume $\pi_{\{0\}[A]} U = 0$ since the system would be over-determined since in general we cannot assume that $P_{\{0\}[A]}$ commutes with $M(\partial^{-1}_0)$. Instead, solving the latter equation for $\pi_{\{0\}[A]} U$ yields

$$\pi_{\{0\}[A]} U = \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right)^{-1} \pi_{\{0\}[A]} F + \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right)^{-1} \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right) \pi_{A[H]} U.$$ (8)

Inserting this into the first equation yields

$$\left( \partial_0 \left( \pi_{A[H]} M (\partial^{-1}_0) \pi_{A[H]}^* \right) + \pi_{A[H]} A \pi_{A[H]}^* \right) \pi_{A[H]} U + \left( \partial_0 \pi_{A[H]} M (\partial^{-1}_0) \pi_{\{0\}[A]} \right) \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right)^{-1} \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right) \pi_{A[H]} U$$

$$= \pi_{A[H]} F - \partial_0 \left( \pi_{A[H]} M (\partial^{-1}_0) \pi_{\{0\}[A]} \right) \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right)^{-1} \pi_{\{0\}[A]} F$$

$$= \left( \partial_0 \tilde{M} (\partial^{-1}_0) + \pi_{A[H]} A \pi_{A[H]}^* \right) \pi_{A[H]} U.$$ (9)

Now, if

$$\tilde{M} (\partial^{-1}_0) := \pi_{A[H]} M (\partial^{-1}_0) \pi_{A[H]}^* + \left( \pi_{A[H]} M (\partial^{-1}_0) \pi_{\{0\}[A]} \right) \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]} \right)^{-1} \left( \partial_0 \pi_{\{0\}[A]} M (\partial^{-1}_0) \pi_{\{0\}[A]}^* \right)$$

(10)

satisfies the required strictly positive definiteness for material laws in the usual sense in the subspace $A[H]$, we have solvability in $A[H]$. This observation for general $A$ may be used to restrict the problem to $A[H]$, which may allow for example to utilize the compactness of the...
restricted resolvent of \( A \) in \( \mathcal{A}[H] \) (if this indeed holds) whereas the original \( A \) may have too large a kernel to have a compact resolvent, see e.g. [17] for an application.

Assuming (3), (4), it suffices to inspect \( H \) for \( \mathcal{M}(\partial_0^{-1}) \) replaced by \( \mathcal{M}_0 + \partial_0^{-1}\mathcal{M}_1 \) due to the smallness assumption [11]. Indeed, for the regular case where \( \mathcal{M}_0 \) is strictly positive, according to the above, it suffices to consider \( \mathcal{M}_0 = 1 \) for which

$$\text{Re} \left\langle U | \partial_0 \left( \pi^*_{\mathcal{A}[H]} \partial_0 \pi^*_{\mathcal{A}[H]} - \pi^*_{\mathcal{A}[H]} \pi^*_{\{0\}A} \left( \pi_{\{0\}A}^* \pi_{\{0\}A} \right)^{-1} \pi_{\{0\}A} \pi^*_{\mathcal{A}[H]} \right) U \right\rangle_{\varnothing,0,0}$$

for \( U \in H_{\varnothing,1}(\mathbb{R}, \mathcal{A}[H]) \), see also [18, Theorem 6.11] for the case of \( \mathcal{M}_0 \) having non-trivial nullspace.

Substituting the solution of this standard evolutionary problem into [8] we obtain the other part of \( U \).

2 Some Applications

2.1 A Particular Mother Operator

Now we consider specifically

$$\left( \partial_0 \mathcal{M}(\partial_0^{-1}) + A \right) U = F \quad (11)$$

where

$$A = \begin{pmatrix} 0 & -\nabla^* \\ \nabla & 0 \end{pmatrix} \quad (12)$$

with a suitable domain making \( A \) skew-selfadjoint in the Hilbert space\(^4\)

$$H = \left( \bigoplus_{k \in \mathbb{N}} L^2_k(\Omega) \right) \oplus \left( \bigoplus_{k \in \mathbb{N}} L^2_k(\Omega) \right).$$

Here \( \nabla \) and \( \nabla^* \) are formal adjoints on the linear subspace

$$\left( \bigoplus_{k \in \mathbb{N}} \hat{C}_{1,k}(\Omega) \right) \oplus \left( \bigoplus_{k \in \mathbb{N}} \hat{C}_{1,k}(\Omega) \right)$$

of \( H \), where \( \hat{C}_{1,k}(\Omega) \) denotes the space of \( C_1 \)-smooth co-variant tensor fields of rank \( k \) with compact support on a Riemannian \( C_{1,1} \)-manifold \( M \) with metric tensor \( g \). The differential

\(^4\)We have chosen here to add up tensor spaces of all ranks, although in applications only \( k = 0, 1, 2, 3 \) appear to be relevant. Restricting to the “physically relevant” subspace

$$V = \left( \bigoplus_{k \in \mathbb{N}} L^2_k(\Omega) \right) \oplus \left( \bigoplus_{k \in \mathbb{N}} L^2_k(\Omega) \right) \oplus \left( \bigoplus_{k \in \mathbb{N}} L^2_k(\Omega) \right)$$

may therefore be considered as a first application of the mechanism to generate descendants of the operator in [14,19].
Some Applications

operator $\nabla$ is the so-called co-variant derivative and its skew-adjoint $-\nabla^*$ is frequently introduced as the tensorial divergence\(^5\) $\text{div}$. For sake of definiteness we shall only consider the choice\(^6\)

$$A := \begin{pmatrix} 0 & -\left(\nabla\right)^* \\ \nabla & 0 \end{pmatrix}$$

where $\nabla^*$ denotes the closure of $\nabla$ applied to elements of $\bigoplus_{k\in \mathbb{N}} \hat{\mathcal{C}}_{1,k}(\Omega)$ as an operator in $\bigoplus_{k\in \mathbb{N}} L^2_k(\Omega)$. For the material law we impose the usual constraint (2).

It may be surprising that the majority of initial boundary value problems from classical mathematical physics can be produced precisely from (11,12) by choosing suitable projections for constructing descendants. This is the main application of the above considerations. In order to make matters more easily digestible we constrain the illustration of our observations to the simple flat case, i.e. $M$ is $\mathbb{R}^{n-k} \times \mathbb{T}^k$, $n = 1, 2, 3$, $k = 0, 1, 2, 3$, $k \leq n$, where $\mathbb{T}$ is the flat Torus.

It will turn out that the physical interpretation has remarkably little relevance for our structural observation. In fact, problems can be very different in physical interpretation, sharing the same formal structure makes the solution theory coincide.

2.2 Isolated Physical Phenomena

2.2.1 Acoustic Equation, Heat Conduction and the Relativistic Schrödinger Equations

If we choose

$$V = L^2_0(\Omega) \oplus L^2_1(\Omega), \quad \Omega \subseteq \mathbb{R}^3,$$

for our construction of descendants via projectors then we obtain the classical system governing acoustic waves or, merely depending on the choice of material law, the heat equation. In the Cartesian case we have by identifying 0—tensors with functions and 1—tensors with vector fields the classical first order system

$$\left( \partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} p \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad (13)$$

with, for example,

$$\mathcal{M} \left( \partial_0^{-1} \right) = \begin{pmatrix} \rho & 0 \\ 0 & \kappa \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}$$

as a simple material law operator. Note that $P_{L^2_0(\Omega)} \left( \nabla \right)^* \pi_{L^2_3(\Omega)} = P_{L^2_0(\Omega)} \left( \nabla \right)^* \pi_{L^2_3(\Omega)} = \left( \nabla \right)^* \pi_{L^2_3(\Omega)}$ and $P_{L^2_0(\Omega)} \nabla \pi_{L^2_3(\Omega)} = P_{L^2_0(\Omega)} \nabla \pi_{L^2_3(\Omega)} = \nabla \pi_{L^2_3(\Omega)}$ are already closed and $\pi_{L^2_3(\Omega)}$.

\(^5\)Correspondingly, we could use grad as a notation for the covariant derivative $\nabla$ to give the original evolutionary equation the suggestive look of the acoustic system (see below).

\(^6\)This choice may be referred to as the Dirichlet boundary condition case.
\(\pi_{L^2_0(\Omega)}\) are isometric embeddings so that

\[
\begin{pmatrix}
0 & \text{div} \\
\text{grad} & 0
\end{pmatrix}
= \pi_V^* A \pi_V^* = \begin{pmatrix}
0 & \pi_{L^2_0(\Omega)}^* \left(\nabla\right)^* \\
\pi_{L^2_0(\Omega)} \nabla \pi_{L^2_0(\Omega)}^* & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \pi_{L^2_0(\Omega)}^* \\
\pi_{L^2_0(\Omega)} \nabla \pi_{L^2_0(\Omega)}^* & 0
\end{pmatrix}
= \pi_V^* A \pi_V^*
\]

is skew-selfadjoint. If \(\rho, \kappa\) are strictly positive definite, continuous, selfadjoint operators then the material law can be taken to describe acoustic wave propagation. If \(\sigma\) is also strictly positive definite, continuous and selfadjoint operators, we interpret \(\sigma\) as a damping term. Alternatively this could then be considered as describing heat propagation with Cattaneo modification. Keeping all these constraints except for assuming \(\kappa = 0\), we get the classical Fourier law of heat conduction. In both cases, the materials are indeed such that the material law commutes with complex conjugation. This allows to interpret the equation in real-valued terms.

Alternatively, we may view \(L^2_k(\Omega) = L^2_k(\Omega, \mathbb{C}), k \in \mathbb{N}\), as a Hilbert space over the field \(\mathbb{R}\) (by restricting the underlying scalar field). Then we have

\[
\mathcal{R} : L^2_k(\Omega, \mathbb{C}) \to L^2_k(\Omega, \mathbb{R}) \oplus L^2_k(\Omega, \mathbb{R})
\]

\[
u \mapsto \begin{pmatrix}
\mathbb{R} \nu \\
\mathfrak{m} \nu
\end{pmatrix}
\]

as an \(\mathbb{R}\)-unitary mapping. For example multiplication by the complex unit is then unitarily equivalent to

\[
\mathbb{R} i \mathbb{R}^{-1} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

With this observation we get that the Schrödinger operator \(\partial_0 + i \Delta_D\) assumes the real form

\[
\partial_0 + \begin{pmatrix}
0 & -\Delta_D \\
\Delta_D & 0
\end{pmatrix},
\]

where, to be specific about boundary conditions, we have chosen the Dirichlet-Laplacian \(\Delta_D\). Clearly, due to its second order type this operator is not covered in our approach. There is, however, a variant known as the relativistic Schrödinger operator in which \(-\Delta_D\) is simply replaced by \(\sqrt{-\Delta_D} = \left|\nabla\right|\), which turns out to be essentially unitarily equivalent to the acoustics problem (13) for an even simpler material law.

Indeed, removing the null space of \(A = \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix}\) by reducing further to the subspace \(L^2(\Omega) \oplus \nabla \left[L^2(\Omega)\right]\), which is the range of \(A\), we obtain another descendant of (11,12), which is indeed a relative of the relativistic Schrödinger operator. According to the polar decomposition theorem there is a unitary mapping \(U\) such that

\[
\nabla = U \left|\nabla\right|
\]
2 Some Applications

and

\[- \text{div} = \left| \text{grad} \right| U^* .\]

Consequently,

\[
\left( \partial_0 \left( \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \mathcal{M} \left( \partial_0^{-1} \right) \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \right) \right) + \left( \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix} \right) = \\
= \left( \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \right) \left( \partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \begin{pmatrix} 0 & -\left| \text{grad} \right| \\ \left| \text{grad} \right| & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \right) .
\]

Since for the relativistic Schrödinger operator \( \mathcal{M} \left( \partial_0^{-1} \right) = 1 \), we get

\[
\left( \partial_0 + \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \right) \left( \partial_0 + \begin{pmatrix} 0 & -\left| \text{grad} \right| \\ \left| \text{grad} \right| & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \right) .
\]

There is another rather common version to translate the operator of the wave equation into a first order in time system, which, however, is nothing but another relative of the acoustics operator. Utilizing the naive analogy to the ordinary differential equations case one translates

\[
\partial_0^2 u - \Delta_D u = f
\]

into the system

\[
\left( \partial_0 + \begin{pmatrix} 0 & \Delta_D - \varepsilon \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \partial_0 u \\ -u \end{pmatrix} \right) = \left( \partial_0 + \begin{pmatrix} 0 & \Delta_D \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \partial_0 u \\ -u \end{pmatrix} \right) = \left( \begin{pmatrix} f \\ 0 \end{pmatrix} \right).
\]

Here we choose \(-\varepsilon \in \mathbb{R}_{\leq 0}\) in the resolvent set of \(-\Delta_D\). The reasoning goes like this:

\[
\begin{pmatrix} 0 & \Delta_D - \varepsilon \\ 1 & 0 \end{pmatrix}
\]

is skew-selfadjoint considered in \(L^2 (\Omega) \oplus H_1 (\sqrt{-\Delta_D + \varepsilon})\) where \(H_1 (\sqrt{-\Delta_D + \varepsilon})\) is \(D (\sqrt{-\Delta_D})\) equipped with the inner product

\[
(u, v) \mapsto \left( \sqrt{-\Delta_D} u \big| \sqrt{-\Delta_D} v \right)_0 + \varepsilon \left( u \big| v \right)_0 ,
\]

i.e. for \(\varepsilon = 1\) the graph inner product of \(\sqrt{-\Delta_D}\). We use that

\[
\sqrt{-\Delta_D + \varepsilon} : H_1 (\sqrt{-\Delta_D + \varepsilon}) \to L^2 (\Omega)
\]

is unitary.

Now, we aim to show that the more general system

\[
\partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \begin{pmatrix} 0 & \Delta_D \\ 1 & 0 \end{pmatrix}
\]

is skew-selfadjoint considered in \(L^2 (\Omega) \oplus H_1 (\sqrt{-\Delta_D + \varepsilon})\) where \(H_1 (\sqrt{-\Delta_D + \varepsilon})\) is \(D (\sqrt{-\Delta_D})\) equipped with the inner product

\[
(u, v) \mapsto \left( \sqrt{-\Delta_D} u \big| \sqrt{-\Delta_D} v \right)_0 + \varepsilon \left( u \big| v \right)_0 ,
\]

i.e. for \(\varepsilon = 1\) the graph inner product of \(\sqrt{-\Delta_D}\). We use that

\[
\sqrt{-\Delta_D + \varepsilon} : H_1 (\sqrt{-\Delta_D + \varepsilon}) \to L^2 (\Omega)
\]

is unitary.
2.2 Isolated Physical Phenomena

is indeed a relative to a first-order-in-time-and-space-system. We note that
\[
\partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \left( \begin{array}{cc} 0 & \Delta D \\ 1 & 0 \end{array} \right) = \partial_0 \left( \mathcal{M} \left( \partial_0^{-1} \right) + \partial_0^{-1} \left( \begin{array}{cc} 0 & \varepsilon \\ 0 & 0 \end{array} \right) \right) + \left( \begin{array}{cc} 0 & \Delta D - \varepsilon \\ 1 & 0 \end{array} \right)
\]
and
\[
\left( \begin{array}{cc} 1 & 0 \\ -\varepsilon & \Delta D + \varepsilon \end{array} \right) \left( \begin{array}{cc} 0 & \Delta D - \varepsilon \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \varepsilon & \Delta D + \varepsilon \end{array} \right) = \left( \begin{array}{cc} 0 & \varepsilon \Delta D + \varepsilon \\ 1 & \Delta D + \varepsilon \end{array} \right)
\]
where \( \sqrt{-\Delta D + \varepsilon} = \| \text{grad} \| \pm i\sqrt{\varepsilon} \) and the polar decomposition for \( \| \text{grad} \| \pm i\sqrt{\varepsilon} \)
implies
\[
U_\pm = \left( \| \text{grad} \| \pm i\sqrt{\varepsilon} \right)^{-1} \left( \| \text{grad} \| \pm i\sqrt{\varepsilon} \right)^{-1} \| \text{grad} \| \pm i\sqrt{\varepsilon} \| \text{grad} \| \pm i\sqrt{\varepsilon} \)
\]
and
\[
U_\pm = U_\mp.
\]
Using the polar decomposition of \( \text{grad} = U \| \text{grad} \| \) we get
\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & U \end{array} \right) \left( \begin{array}{cc} 0 & \text{grad} \| \pm i\sqrt{\varepsilon} \\ \text{grad} \| \pm i\sqrt{\varepsilon} & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & U^* \end{array} \right) = \left( \begin{array}{cc} \text{div} + i\sqrt{\varepsilon}U^* \\ \text{grad} \| \pm i\sqrt{\varepsilon} \end{array} \right)
\]
Thus we obtain for the transformed equation a new material law:
\[
\tilde{\mathcal{M}} \left( \partial_0^{-1} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{-\Delta D + \varepsilon} \end{array} \right) \mathcal{M} \left( \partial_0^{-1} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{-\Delta D + \varepsilon} \end{array} \right)
\]
Given the complexity of the arguments needed it is still a surprisingly common mechanism used (at least in the case \( \varepsilon = 0 \) and simple material laws) to turn partial differential equations of wave equation type into first-order-in-time systems. None the less, in the above terminology we encounter here mere relatives of the system of the acoustic equations, which in turn is a descendant of \([1112]\).
2 Some Applications

2.2.2 Elastic Waves

If we choose

\[ V = L^2_1(\Omega) \oplus \text{sym} \left[ L^2_2(\Omega) \right] \]

with \( \Omega \) open and non-empty in \( \mathbb{R}^3 \) then we obtain by analogous arguments the classical system governing the propagation of waves in elastic or, depending on the choice of material law, viscoelastic waves. Here \( \text{sym} \) is the mapping \( \text{sym} : L^2_2(\Omega) \rightarrow L^2_2(\Omega) \) induced by the symmetrization operation for co-variant tensors of rank 2

\[ T \mapsto \left( (x,y) \mapsto \frac{1}{2} (T(x,y) + T(y,x)) \right). \]

Thus, in Cartesian coordinates the elasticity operator is

\[ \partial_0 M \left( \partial^{-1}_0 \right) + \left( \begin{array}{c} 0 \\ \text{Grad} \\ 0 \end{array} \right), \]

where

\[ \text{Grad} v = \frac{1}{2} (\partial_i v_j + \partial_j v_i)_{i,j=1,2,3} \]

and its negative adjoint

\[ \text{Div} T = \text{div} T = \left( \sum_{j=1}^{3} \partial_j T_{ij} \right)_{i=1,2,3} \]

for suitable displacement velocities \( v = (v_j)_{j=1,2,3} \) and symmetric stress tensors \( (T_{ij})_{i,j=1,2,3} \). A discussion of various possible material laws of interest can be found in [10], compare [1].

2.2.3 Electro-Magnetic Waves

If we choose

\[ V = L^2_1(\Omega) \oplus \text{asym} \left[ L^2_2(\Omega) \right] \]

with \( \Omega \) open and non-empty in \( \mathbb{R}^3 \) then we obtain by an analogous reasoning the classical system governing the propagation of electro-magnetic waves. Here \( \text{asym} \) is the mapping \( \text{asym} : L^2_2(\Omega) \rightarrow L^2_2(\Omega) \) induced by the anti-symmetrization operation for co-variant tensors of rank 2

\[ T \mapsto \left( (x,y) \mapsto \frac{1}{2} (T(x,y) - T(y,x)) \right). \]

To convince ourselves that this leads to Maxwell’s equations we calculate \( \nabla \cdot W \) for anti-symmetric tensors \( W \in C_{\infty,2}(\Omega) \) in Cartesian coordinates, \( \Omega \) being a non-empty open subset of \( \mathbb{R}^3 \). Let

\[ W = \omega_1 e^2 \otimes e^3 + \omega_2 e^3 \otimes e^1 + \omega_3 e^1 \otimes e^2 - \omega_1 e^3 \otimes e^2 - \omega_2 e^1 \otimes e^3 - \omega_3 e^2 \otimes e^1 \]

then

\[ T_{k,k+1} = \omega_{k+2} e^k \otimes e^{k+1} \quad (k \in \{1,2,3\} \text{ with addition } \text{mod } 3 \text{ and } 0 \equiv 3) \]

\[ W = \text{asym} \left( T_{ks} e^k \otimes e^s \right) = \frac{1}{2} \left( T_{ks} e^k \otimes e^s - T_{ks} e^s \otimes e^k \right) = T_{ks} e^k \wedge e^s = 2 \sum_{k<s \text{ mod } 3} T_{ks} e^k \wedge e^s \]
\[ \nabla \cdot W = \partial_k W_{k\ell} e^\ell \]

\[ = \partial_1 W_{12} e^2 + \partial_1 W_{13} e^3 + \partial_2 W_{23} e^3 + \partial_2 W_{21} e^1 + \partial_3 W_{32} e^2 + \partial_3 W_{31} e^1 \]

\[ = \partial_1 \omega_3 e^2 - \partial_1 \omega_2 e^3 + \partial_2 \omega_1 e^3 - \partial_2 \omega_3 e^1 - \partial_3 \omega_1 e^2 + \partial_3 \omega_2 e^1 \]

\[ = (\partial_1 \omega_3 - \partial_3 \omega_1) e^2 + (\partial_2 \omega_1 - \partial_1 \omega_2) e^3 + (\partial_3 \omega_2 - \partial_2 \omega_3) e^1 \]

\[ = - \text{curl} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \]

Correspondingly,

\[ V = \eta_1 e^1 + \eta_2 e^2 + \eta_3 e^3 \]

and so

\[ \nabla V = \partial_1 \eta_1 e^1 \otimes e^1 + \partial_1 \eta_2 e^1 \otimes e^2 + \partial_1 \eta_3 e^1 \otimes e^3 + \partial_2 \eta_1 e^2 \otimes e^1 + \partial_2 \eta_2 e^2 \otimes e^2 + \partial_2 \eta_3 e^2 \otimes e^3 + \partial_3 \eta_1 e^3 \otimes e^1 + \partial_3 \eta_2 e^3 \otimes e^2 + \partial_3 \eta_3 e^3 \otimes e^3. \]

We see that

\[
\text{asym} (\nabla V) = \\
= \partial_1 \eta_2 \frac{1}{2} (e^1 \otimes e^2 - e^2 \otimes e^1) + \partial_1 \eta_3 \frac{1}{2} (e^1 \otimes e^3 - e^3 \otimes e^1) + \partial_2 \eta_1 \frac{1}{2} (e^2 \otimes e^1 - e^1 \otimes e^2) + \\
+ \partial_2 \eta_3 \frac{1}{2} (e^2 \otimes e^3 - e^3 \otimes e^2) + \partial_3 \eta_1 \frac{1}{2} (e^3 \otimes e^1 - e^1 \otimes e^3) + \partial_3 \eta_2 \frac{1}{2} (e^3 \otimes e^2 - e^2 \otimes e^3) \]

\[ = (\partial_1 \eta_2 - \partial_2 \eta_1) \frac{1}{2} (e^1 \otimes e^2 - e^2 \otimes e^1) + (\partial_2 \eta_3 - \partial_3 \eta_2) \frac{1}{2} (e^2 \otimes e^3 - e^3 \otimes e^2) + \\
+ (\partial_3 \eta_1 - \partial_1 \eta_3) \frac{1}{2} (e^3 \otimes e^1 - e^1 \otimes e^3) \]

\[ = (\partial_1 \eta_2 - \partial_2 \eta_1) e^1 \wedge e^2 + (\partial_2 \eta_3 - \partial_3 \eta_2) e^2 \wedge e^3 + (\partial_3 \eta_1 - \partial_1 \eta_3) e^3 \wedge e^1 \]

and so

\[ \text{asym} \nabla =: \dd \wedge \]

on differentiable 1-form fields. In Cartesian coordinates Maxwell’s equations assume the familiar form

\[ \partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \begin{pmatrix} 0 & \text{curl} \\ \text{curl} & 0 \end{pmatrix}, \]

where

\[ \text{curl}^* = \text{curl} \]

and containment of \( E \) in \( D \left( \text{curl} \right) \) encodes and generalizes the electric boundary condition, i.e. vanishing of the tangential components of \( E \), for the electric field \( E \) to the arbitrary boundary of \( \Omega \).
2 Some Applications

2.2.4 Reducing Dimensions

Another instance of the reduction procedure under discussion is the reduction of the dimension. We consider the simple case $\Omega := \Omega_0 \times \mathbb{T}^s \subseteq \mathbb{R}^{n+1} \times \mathbb{T}^s := M$ and want to describe the reduction process from $k$-tensor $L^2_k(\Omega)$ to $k$-tensor $L^2_k(\Omega_0)$. Throughout, we assume that the Riemannian metric $g$ only depends on $\Omega_0$, that is, we assume that $g_{ij} = 0$ for $i \neq j$ and $g_{ii} = 1$ for every $i, j \in \{n+1, \ldots, n+s\}$. Using Cartesian coordinates, a covariant $k$-tensor $T \in L^2_k(\Omega)$ can be written as

$$T = \sum_{\alpha \in \{0, \ldots, n+s\}^k} \omega_{\alpha} dx^{\alpha}$$

for suitable functions $\omega_{\alpha} \in L^2(\Omega)$. We set $\Pi_n := \left\{ \alpha \in \{0, \ldots, n+s\}^k \mid \bigwedge_{i \in \{0, \ldots, k-1\}} \alpha_i \in \{0, \ldots, n\} \right\}$ and define

$$\pi^k_{\Omega_0} : L^2_k(\Omega) \rightarrow L^2_k(\Omega_0)$$

by

$$(\pi^k_{\Omega_0} T)(t_0, \ldots, t_n) := \sum_{\alpha \in \Pi_n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \omega_{\alpha}(t_0, \ldots, t_n, r_0, \ldots, r_{s-1}) dr_{s-1} \ldots dr_0 dx^{\alpha}.$$

The adjoint $(\pi^k_{\Omega_0})^*$ is then the canonical embedding of $L^2_k(\Omega_0)$ into $L^2_k(\Omega)$ given by

$$(\pi^k_{\Omega_0})^* \left( \sum_{\beta \in \{0, \ldots, n\}^k} \psi_\beta dx^\beta \right) = \sum_{\alpha \in \{0, \ldots, n+s\}^k} \tilde{\psi}_\alpha dx^\alpha,$$

where

$$\tilde{\psi}_\alpha(t_0, \ldots, t_n, r_0, \ldots, r_{s-1}) := \begin{cases} \psi_\alpha(t_0, \ldots, t_n) & \text{if } \alpha \in \Pi_n, \\ 0 & \text{otherwise}. \end{cases}$$

Indeed, for $T = \sum_{\alpha \in \{0, \ldots, n+s\}^k} \omega_{\alpha} dx^\alpha \in L^2_k(\Omega)$ and $S = \sum_{\beta \in \{0, \ldots, n\}^k} \psi_\beta dx^\beta \in L^2_k(\Omega_0)$ we compute

$$\langle \pi^k_{\Omega_0} T | S \rangle_{L^2_k(\Omega_0)} = \sum_{\alpha \in \Pi_n} \sum_{\beta \in \{0, \ldots, n\}^k} \int_{\Omega_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \omega_{\alpha}(\cdot, r_0, \ldots, r_{s-1}) dr_{s-1} \ldots dr_0 \psi_\beta g^{\alpha\beta} dV_{\mathbb{R}^{n+1}}$$

$$= \sum_{\alpha \in \Pi_n} \sum_{\beta \in \Pi_n} \int_{\Omega_0} \omega_{\alpha} \tilde{\psi}_\beta g^{\alpha\beta} dV_M$$

$$= \langle T | (\pi^k_{\Omega_0})^* S \rangle_{L^2_k(\Omega)},$$

where in the last step we have used $\tilde{\psi}_\beta = 0$ for $\beta \notin \Pi_n$ and $g^{\alpha\beta} = 0$ for $\beta \in \Pi_n, \alpha \notin \Pi_n$. Moreover, the last computation shows that the embedding $(\pi^k_{\Omega_0})^*$ is isometric, since

$$\pi^k_{\Omega_0} \left( \pi^k_{\Omega_0} \right)^* S = S.$$
Remark 2.1. The application of the abstract descendant mechanism with $A$ given by $B_0 = B_1 := \bigoplus_{k \in \mathbb{N}} \pi_{k \Omega_0}$ provides a way to reduce the dimension of the underlying domain for an evolutionary problem.

Applying this reduction process in the particular case $\Omega = \Omega_0 \times \mathbb{T}^{n-1} \subseteq \mathbb{R} \times \mathbb{T}^{n-1} = M$, gives a $(1 + 1)$-dimensional evolutionary descendant of $(11,12)$ on the open subset $\Omega$ of the flat tube manifold $M$. We may write this descendant in Cartesian coordinates simply as

$$
\partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix},
$$

where now $\begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix}$ is skew-selfadjoint on the space $L^2(\Omega_0) \oplus L^2(\Omega_0)$.

If $\Omega_0 = \mathbb{R}$ we may go one step further, we can decompose $L^2(\mathbb{R})$ into orthogonal subspaces

$$
L^2(\mathbb{R}) = L^2,\text{even}(\mathbb{R}) \oplus L^2,\text{odd}(\mathbb{R}),
$$

with

$$
L^2,\text{even}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) \mid f(x) = f(-x) \text{ for a.e. } x \in \mathbb{R} \},
$$

$$
L^2,\text{odd}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) \mid f(x) = -f(-x) \text{ for a.e. } x \in \mathbb{R} \}.
$$

Since $\pi_{L^2,\text{even}(\mathbb{R})}$ and $\pi_{L^2,\text{odd}(\mathbb{R})}$ are compatible with $\partial_1 = \hat{\partial}_1$ we obtain that

$$
\partial_0 \tilde{\mathcal{M}} \left( \partial_0^{-1} \right) = \begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix}
$$

is the $(\pi_{L^2,\text{even}(\mathbb{R})}, \pi_{L^2,\text{odd}(\mathbb{R})})$-descendant of $(15)$ on $L^2,\text{even}(\mathbb{R}) \oplus L^2,\text{odd}(\mathbb{R})$ with

$$
\tilde{\mathcal{M}} \left( \partial_0^{-1} \right) := \begin{pmatrix} \pi_{L^2,\text{even}(\mathbb{R})} & 0 \\ 0 & \pi_{L^2,\text{odd}(\mathbb{R})} \end{pmatrix} \mathcal{M} \left( \partial_0^{-1} \right) \begin{pmatrix} \pi_{L^2,\text{even}(\mathbb{R})}^* & 0 \\ 0 & \pi_{L^2,\text{odd}(\mathbb{R})}^* \end{pmatrix}
$$

as a new material law operator. If $\mathcal{M} \left( \partial_0^{-1} \right)$ is block diagonal

$$
\mathcal{M} \left( \partial_0^{-1} \right) = \begin{pmatrix} \mathcal{M}_{00} \left( \partial_0^{-1} \right) & 0 \\ 0 & \mathcal{M}_{11} \left( \partial_0^{-1} \right) \end{pmatrix}
$$

then the two rows can be combined into one

$$
\partial_0 \left( \pi_{L^2,\text{even}(\mathbb{R})} \mathcal{M}_{00} \left( \partial_0^{-1} \right) \pi_{L^2,\text{even}(\mathbb{R})}^* + \pi_{L^2,\text{odd}(\mathbb{R})} \mathcal{M}_{11} \left( \partial_0^{-1} \right) \pi_{L^2,\text{odd}(\mathbb{R})}^* \right) + \partial_1
$$

on $L^2(\mathbb{R}) = L^2,\text{even}(\mathbb{R}) \oplus L^2,\text{odd}(\mathbb{R})$. This is the so-called transport equation in the $(1 + 1)$-dimensional case, which thus also is shown to be a descendant of $(11,12)$ (for $\Omega = M = \mathbb{R} \times \mathbb{T}^{n-1}$).

Remark 2.1. (The Transport Equation in $\mathbb{R}^n$) Returning to $\mathbb{R}^n$ and assuming that by a suitable choice of coordinates the transport operator $\partial_0 + a \cdot \partial$ assumes the unitarily equivalent form

$$
\partial_0 + \partial_1
$$
2 Some Applications

with \( \partial_1 \) on a cylinder \( \Omega := \mathbb{R} \times \Omega_0 \subseteq \mathbb{R} \times \mathbb{R}^{n-1} \). Here also \( \partial_1 = \tilde{\partial}_1 \). Of course, we could have more complicated material laws:

\[
\partial_0 \mathcal{M} (\tilde{\partial}_0^{-1}) + \partial_1.
\]

(16)

The cross-section \( \Omega_0 \) of the cylinder serves here merely as a parameter range, since no differentiations in these directions are involved. Allowing for additional parameter dependence in (11,12) would make (16) a descendant of (11,12).

2.3 Interacting Descendants

The various descendants of (11,12) can interact in many ways to create new models of more complex phenomena. We shall first discuss a particular interaction based on an alternating differential forms framework (alternating covariant tensors). Then we shall turn to the discussion of coupled descendants, where the coupling only occurs via the material law operator.

2.3.1 The Extended Maxwell System and the Dirac Equation

The Extended Maxwell Operator

Assuming a relatively simple material law of the form

\[
\mathcal{M} (\tilde{\partial}_0^{-1}) = \mathcal{M}_0
\]

with \( \mathcal{M}_0 \) continuous, selfadjoint and strictly positive definite, Maxwell’s equations can be reformulated as

\[
\partial_0 + \sqrt{\mathcal{M}_0^{-1}} \begin{pmatrix}
0 & - (\tilde{\partial}_1 \wedge)^* \\
\tilde{\partial}_1 \wedge & 0
\end{pmatrix} \sqrt{\mathcal{M}_0^{-1}}.
\]

Here \( \tilde{\partial}_1 \wedge \) is the exterior derivative applied to covariant 1-tensors (with Dirichlet type boundary condition). By including alternating tensor fields of all odd orders in the first block component and of all even orders in the second block component we arrive at

\[
\partial_0 + \sqrt{\mathcal{M}_0^{-1}} \begin{pmatrix}
0 & - (\tilde{\partial}_{1,3} \wedge)^* \\
\tilde{\partial}_{1,3} \wedge & 0
\end{pmatrix} \sqrt{\mathcal{M}_0^{-1}}.
\]

Here \( \tilde{\partial}_{1,3} \wedge \) is the exterior derivative applied to the direct sum of alternating tensors of order 1 and 3 (with Dirichlet type boundary condition). Note that \( d \wedge \omega = 0 \) on 3-forms in \( \mathbb{R}^3 \). The material law operator \( \mathcal{M}_0 \) is here of course assumed to be continuous, selfadjoint and strictly positive definite on the larger space. Adding

\[
\sqrt{\mathcal{M}_0} \begin{pmatrix}
0 & -d_{0,2} \wedge \\
(d_{0,2} \wedge)^* & 0
\end{pmatrix} \sqrt{\mathcal{M}_0}
\]

as another descendant\(^8\) of

\[
\begin{pmatrix}
0 & -\nabla^* \\
\nabla & 0
\end{pmatrix},
\]

\(^8\)Similarly we may consider transport on a period slab, i.e. \( \Omega = T \times \Omega_0 \subseteq T \times \mathbb{R} \) as a flat Riemannian manifold.

\(^9\)Note that

\[
\begin{pmatrix}
0 & -d_{0,2} \wedge \\
(d_{0,2} \wedge)^* & 0
\end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}.
\]
we obtain a unitarily equivalent variant of the type of operator discussed in \([8]\) as the extended Maxwell operator

\[
\partial_0 + \sqrt{M_0^{-1}} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{curl} \\
0 & 0 & 0 & 0 \\
0 & \text{curl} & 0 & 0
\end{array} \right) \sqrt{M_0^{-1}} + \sqrt{M_0} \left( \begin{array}{cccc}
0 & 0 & 0 & \text{div} \\
0 & 0 & \text{grad} & 0 \\
0 & \text{div} & 0 & 0 \\
\text{grad} & 0 & 0 & 0
\end{array} \right) \sqrt{M_0}.
\]

Here \(\tilde{d}_{0.2}\) is the exterior derivative applied to the direct sum of alternating tensors of order 0 and 2 (with Dirichlet type boundary condition). In Cartesian coordinates this is

\[
\partial_0 + \sqrt{M_0^{-1}} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{curl} \\
0 & 0 & 0 & 0 \\
0 & \text{curl} & 0 & 0
\end{array} \right) \sqrt{M_0^{-1}} + \sqrt{M_0} \left( \begin{array}{cccc}
0 & 0 & 0 & \text{div} \\
0 & 0 & \text{grad} & 0 \\
0 & \text{div} & 0 & 0 \\
\text{grad} & 0 & 0 & 0
\end{array} \right) \sqrt{M_0},
\]

which is a convenient reformulation of Maxwell’s equations for regularity and numerical purposes, compare [16]. That the spatial part of this extended system is still skew-selfadjoint and that it can be reduced to the original Maxwell system is due to the fact that

\[
\sqrt{M_0^{-1}} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{curl} \\
0 & 0 & 0 & 0 \\
0 & \text{curl} & 0 & 0
\end{array} \right) \sqrt{M_0^{-1}} + \sqrt{M_0} \left( \begin{array}{cccc}
0 & 0 & 0 & \text{div} \\
0 & 0 & \text{grad} & 0 \\
0 & \text{div} & 0 & 0 \\
\text{grad} & 0 & 0 & 0
\end{array} \right) \sqrt{M_0}
\]

are commuting selfadjoint operators, which are indeed annihilating each other. The possibility of reconstructing the original Maxwell system assumes a particular form\(^{10}\) of the right-hand side, see [8] for details. By adding a material law term \(\mathcal{M}(\partial_0^{-1})\) we can allow for more complicated material behavior:

\[
\partial_0 + \tilde{\mathcal{M}}(\partial_0^{-1}) + \sqrt{M_0^{-1}} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{curl} \\
0 & 0 & 0 & 0 \\
0 & \text{curl} & 0 & 0
\end{array} \right) \sqrt{M_0^{-1}} + \sqrt{M_0} \left( \begin{array}{cccc}
0 & 0 & 0 & \text{div} \\
0 & 0 & \text{grad} & 0 \\
0 & \text{div} & 0 & 0 \\
\text{grad} & 0 & 0 & 0
\end{array} \right) \sqrt{M_0}.
\]

**Remark 2.2.** Projecting this further down by eliminating the third row and column leads to a slightly smaller descendant of the extended Maxwell system

\[
\partial_0 + \tilde{\mathcal{M}}(\partial_0^{-1}) + \sqrt{M_0^{-1}} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{curl} \\
0 & 0 & 0 & 0 \\
0 & \text{curl} & 0 & 0
\end{array} \right) \sqrt{M_0^{-1}} + \sqrt{\tilde{\mathcal{M}}} \left( \begin{array}{cccc}
0 & 0 & 0 & \text{div} \\
0 & 0 & \text{grad} & 0 \\
0 & \text{div} & 0 & 0 \\
\text{grad} & 0 & 0 & 0
\end{array} \right) \sqrt{\tilde{\mathcal{M}}},
\]

For “ellipticizing” Maxwell’s equations, e.g. for numerical purposes, this modification is perfectly sufficient, [19].

\(^{10}\)For these special data it can be shown that components of order 0 and 3 are actually zero. If general right-hand sides are considered then these components will be non-zero producing what is called “scalar waves” contributions.
2 Some Applications

The Dirac Operator  

The Dirac operator \( Q_0(\partial_0, \hat{\partial}) \) is usually given as the \((4 \times 4)\)–partial differential expression with the block matrix form (for mass equal to 1)

\[
Q_0(\partial_0, \hat{\partial}) := \begin{pmatrix}
\partial_0 + i C(\hat{\partial}) \\
C(\hat{\partial}) \partial_0 - i
\end{pmatrix}.
\]

Here \( C(\hat{\partial}) := \begin{pmatrix}
\partial_3 \\
\partial_1 + i \partial_2 \\
-\partial_3
\end{pmatrix} = \sum_{k=1}^{3} \Pi_k \partial_k \), where

\[
\Pi_1 := \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \Pi_2 := \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \Pi_3 := \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are known as Pauli matrices. Applying the unitary transformation given by the block matrix

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 \\ i & -1 \end{pmatrix}
\]

to \( Q_0(\partial_0, \hat{\partial}) \) we obtain

\[
Q_1(\partial_0, \hat{\partial}) := \begin{pmatrix}
\partial_0 & i - i C(\hat{\partial}) \\
i + i C(\hat{\partial}) & \partial_0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
+i +1 & -i \partial_0 + 1 + C(\hat{\partial}) & -i \partial_0 + 1 - C(\hat{\partial}) \\
i C(\hat{\partial}) + \partial_0 - i & -i C(\hat{\partial}) - \partial_0 + i
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
+i +1 & \partial_0 + i C(\hat{\partial}) & -i -i \\
i C(\hat{\partial}) + \partial_0 - i
\end{pmatrix}.
\]

The latter may be a preferable form since \( Q_1(\partial_0, \hat{\partial}) \) has the typical Hamiltonian form of reversibly evolutionary expressions of mathematical physics

\[
\begin{pmatrix}
\partial_0 - W^* \\
W
\end{pmatrix},
\]

where \( W := i + i C(\hat{\partial}) = \begin{pmatrix}
i \partial_3 + i & i \partial_1 + \partial_2 \\
i \partial_3 - i \partial_2 - i \partial_3 + i
\end{pmatrix} \).

On first glance the Dirac operator does not seem to fit into the framework we are discussing here, since it does not appear to be constructed from descendants of \( [11][12] \). A closer inspection, however, shows that the Dirac operator is actually unitarily equivalent to, i.e. in the above sense a relative of, the extended Maxwell operator (with a variant of the material law).

\[\text{Note that}
\]

\[
\begin{pmatrix}
\partial_3 \\
\partial_1 + i \partial_2 \\
-\partial_3
\end{pmatrix}
\]

is an operator quaternion, since it has the form

\[
\begin{pmatrix}
A - B^* \\
B & A^*
\end{pmatrix},
\]

where \( A : D(A) \subseteq H \to H \), \( B : D(B) \subseteq H \to H \) are closed densely defined linear operators, such that \( A \) has a non-empty resolvent set \( \varrho(A) \) and \( A, B^* \) are commuting, i.e.

\[
(\lambda - A)^{-1} B \subseteq B (\lambda - A)^{-1}
\]

for \( \lambda \in \varrho(A) \). If \( A, B \) are complex numbers (as multipliers) this block operator matrix yields a standard representation of the classical quaternions.
To see this connection we separate real and imaginary parts, which yields that $W$ corresponds to
\[
\begin{pmatrix}
0 & -1 - \partial_3 & \partial_2 & -\partial_1 \\
1 + \partial_3 & 0 & \partial_1 & \partial_2 \\
-\partial_2 & -\partial_1 & 0 & -1 + \partial_3 \\
\partial_1 & -\partial_2 & 1 - \partial_3 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & -\partial_3 & \partial_2 & -\partial_1 \\
\partial_3 & 0 & \partial_1 & \partial_2 \\
-\partial_2 & -\partial_1 & 0 & \partial_3 \\
\partial_1 & -\partial_2 & -\partial_3 & 0
\end{pmatrix}.
\]
Noting that
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\partial_3 & \partial_2 & -\partial_1 \\
\partial_3 & 0 & \partial_1 & \partial_2 \\
-\partial_2 & -\partial_1 & 0 & \partial_3 \\
\partial_1 & -\partial_2 & -\partial_3 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & \partial_1 & \partial_2 & \partial_3 \\
\partial_1 & 0 & -\partial_3 & \partial_2 \\
\partial_2 & \partial_3 & 0 & -\partial_1 \\
\partial_3 & -\partial_2 & \partial_1 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \text{div} \\
\text{grad curl}
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
we obtain the unitary equivalence
\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -W^* \\
W & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & \text{div} \\
\text{grad curl}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
2 Some Applications

In the free-space situation the Dirac operator $\partial_0 + \begin{pmatrix} 0 & -W^* \\ W & 0 \end{pmatrix}$ is thus unitarily equivalent to the extended Maxwell operator

$$\partial_0 + \mathcal{M}_1 + \begin{pmatrix} 0 & 0 & 0 & \text{div} \\ 0 & 0 & \text{grad} & -\text{curl} \\ 0 & \text{div} & 0 & 0 \\ \text{grad} & \text{curl} & 0 & 0 \end{pmatrix}$$

where

$$\mathcal{M}_1 = \begin{pmatrix} 0 & (0 0 0) & 0 & (0 0 -1) \\ 0 & (0 0 0) & 0 & (0 1 0) \\ 0 & (0 0 0) & -1 & (0 0 0) \\ 0 & (0 0 0) & 0 & (0 0 0) \end{pmatrix}$$

is skew-selfadjoint, i.e.

from the electrodynamics perspective we are in a chiral media case.

Thus we have shown that the Dirac equation also fits seamlessly into our construction of descendants of \([11, 12]\) and their interaction. In particular, the Dirac operator is a relative \(^{12}\) of the extended Maxwell operator discussed above.

**Remark 2.3.** It is a rather remarkable observation that the Dirac equation is so closely connected to the extended Maxwell system. It appears from this perspective that spinors are actually a redundant construction since the alternating forms setup for the extended Maxwell system is already quite sufficient to discuss Dirac equations. The interpretation of this observation is not a mathematical issue but may well be a matter for theoretical physicists to contemplate.

### 2.3.2 Coupled Systems

Let us recall from \([15]\) the systematic coupling mechanism between various different descendants. Without coupling the systems of interest can be combined simply by writing them together in diagonal block operator matrix form:

$$\partial_0 \begin{pmatrix} V_0 \\ \vdots \\ V_n \end{pmatrix} + A \begin{pmatrix} U_0 \\ \vdots \\ U_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix},$$

where

\(^{12}\)In the framework of quaternions a connection between a differently extended time-harmonic Maxwell operator and the time-harmonic Dirac operator has earlier been discovered by Kravchenko and Shapiro, \([4]\), compare also \([3]\).
2.3 Interacting Descendants

\[
A = \begin{pmatrix}
A_0 & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & A_n
\end{pmatrix}
\]

inherits the skew-selfadjointness in \( H = \bigoplus_{k=0,\ldots,n} H_k \) from its skew-selfadjoint diagonal entries \( A_k : D(A_k) \subseteq H_k \rightarrow H_k, \ k = 0,\ldots,n \). The combined material laws here take the simple block diagonal form

\[
V = \begin{pmatrix}
V_0 \\
\vdots \\
V_n
\end{pmatrix} = M^{\text{in}}(\partial_0^{-1}) U := \begin{pmatrix}
M_{00}(\partial_0^{-1}) & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & M_{nn}(\partial_0^{-1})
\end{pmatrix} \begin{pmatrix}
U_0 \\
\vdots \\
U_n
\end{pmatrix}.
\]

Coupling between these phenomena now can be modeled by expanding the material law to contain block off-diagonal entries

\[
M^{\text{ex}}(\partial_0^{-1}) := \begin{pmatrix}
M_{00}(\partial_0^{-1}) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix} - \begin{pmatrix}
M_{00}(\partial_0^{-1}) & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & M_{nn}(\partial_0^{-1})
\end{pmatrix}.
\]

The full material law now is of the familiar form

\[
V = M (\partial_0^{-1}) U
\]

with

\[
M (\partial_0^{-1}) := M^{\text{in}}(\partial_0^{-1}) + M^{\text{ex}}(\partial_0^{-1}).
\]

This coupling mechanism now allows to model thermo-elasticity, thermo-piezo-electro-magnetism and so on. A number of examples for coupled systems have been discussed elsewhere, see [9, 10, 7, 12, 11, 6]. The “philosophy” of this coupling mechanism is that coupling occurs only via the material law.

In the following we shall illustrate the abstract coupling mechanism with a particular concrete example, which will at the same time serve to exemplify the construction of descendants of coupled systems.

Starting point of our example collection is the classical system of thermo-elasticity, which will also allow us to re-iterate the point made previously in connection with the Dirac operator and the extended Maxwell operator that indeed systems with very different physical interpretations may share the same solution theory with differences being incorporated merely in possibly different material laws (or their interpretation).
2 Some Applications

Thermo-Elasticity and Biot’s Model for Porous Media

The classical system of $(1+3)-$dimensional thermo-elasticity\footnote{Due to an inconvenient choice of unknowns the original classical system of thermo-elasticity has an unbounded coupling term, see \[5\]. The form given here avoids this drawback. More specifically the difference hinges on the use of stress instead of strain as unknown tensor field.} can be described by

$$(\partial_0 M (\partial_0^{-1}) + A) \begin{pmatrix} \eta \\ \zeta \\ s \\ T \end{pmatrix} = F$$

with

$$A := \begin{pmatrix} 0 & -\text{div} & 0 & 0 \\ -\text{grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{Div} \\ 0 & 0 & -\text{Grad} & 0 \end{pmatrix}.$$ 

The classical material law is of the form

$$M (\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1$$

with

$$M_0 := \begin{pmatrix} \varrho_1 + \Gamma^* C^{-1} \Gamma & 0 & 0 & \Gamma^* C^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varrho_2 & 0 \\ C^{-1} \Gamma & 0 & 0 & C^{-1} \end{pmatrix}, \quad M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\varrho_1, \kappa, \varrho_2, C$ are continuous selfadjoint and strictly positive definite operators. This system formally coincides with Biot’s porous media model, merely the meaning of the quantities involved, i.e. the units, have changed, see e.g. \[7\].

As in all these models we may allow for more complex material laws as long as (2) is maintained:

$$\begin{pmatrix} \partial_0 M (\partial_0^{-1}) + \begin{pmatrix} 0 & -\text{div} & 0 & 0 \\ -\text{grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{Div} \\ 0 & 0 & -\text{Grad} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \\ s \\ T \end{pmatrix} = F.$$ \hspace{1cm} (18)

Reissner-Mindlin Plate

Assuming $\Omega := \Omega_0 \times T \subseteq \mathbb{R}^2 \times T =: M$ (instead of $M = \mathbb{R}^3$) we can reduce (18) by one spatial dimension to a $(1+2)$-dimensional evolutionary problem following the strategy in Section 2.2.4. Indeed, the resulting evolutionary equation looks the same, but now it has to be interpreted in $L_0^2 (\Omega_0) \oplus L_1^2 (\Omega_0) \oplus L_1^2 (\Omega_0) \oplus \text{sym} [L_2^2 (\Omega_0)]$ with $\Omega_0 \subseteq \mathbb{R}^2$. With

$$F := \begin{pmatrix} f \\ 0 \\ g \\ 0 \end{pmatrix}$$
and

\[ \mathcal{M}(\partial_0^{-1}) = \mathcal{M}_0 + \partial_0^{-1} \mathcal{M}_1 \]

with

\[
\mathcal{M}_0 := \begin{pmatrix}
\varrho_1 & 0 & 0 & 0 \\
0 & \kappa & 0 & 0 \\
0 & 0 & \varrho_2 & 0 \\
0 & 0 & 0 & C^{-1}
\end{pmatrix}, \quad \mathcal{M}_1 := \begin{pmatrix}
d & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( \varrho_1, \kappa, \varrho_2, C \) continuous selfadjoint and strictly positive definite (with physically different meaning, i.e. different units!) we obtain the Reissner-Mindlin plate model. Coupling occurs here via \( \mathcal{M}_1 \).

Note that by reducing this to a second order system (by substituting the equations from rows 2 and 4 into the remaining two equations) we obtain the perhaps more familiar form of the Reissner-Mindlin model (with homogeneous Dirichlet boundary condition)

\[
\begin{align*}
\varrho_1 \partial_0^2 \tilde{\eta} - \text{div} \kappa^{-1} \left( \tilde{\text{grad}} \tilde{\eta} + \tilde{s} \right) + d \partial_0 \tilde{\eta} &= f, \\
\varrho_2 \partial_0^2 \tilde{s} - \text{Div} C \text{Grad} \tilde{s} + \kappa^{-1} \left( \tilde{\text{grad}} \tilde{\eta} + \tilde{s} \right) &= g,
\end{align*}
\]

where \( \tilde{\eta} := \partial_0^{-1} \eta, \tilde{s} := \partial_0^{-1} s \).

For the damping coefficient \( d = 0 \) we have that the system is conservative, since

\[
\sqrt{\mathcal{M}_0^{-1}} \left( \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & -\text{div} & 0 & 0 \\
-\text{grad} & 0 & 0 & 0 \\
0 & 0 & 0 & -\text{Div} \\
0 & 0 & -\text{Grad} & 0
\end{pmatrix} \right) \sqrt{\mathcal{M}_0^{-1}}
\]

is skew-selfadjoint and thus generates a unitary group leading to norm conservation for pure initial value problems.

**Remark 2.4.**

1. (A note on the Kirchhoff-Love plate)

Letting in

\[
\partial_0 \begin{pmatrix}
\varrho_1 & 0 & 0 & 0 \\
0 & \kappa & 0 & 0 \\
0 & 0 & \varrho_2 & 0 \\
0 & 0 & 0 & C^{-1}
\end{pmatrix} + \begin{pmatrix}
d & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + A
\]

\( \kappa = 0 \) and \( \varrho_2 = 0 \) (in consequence destroying well-posedness for associated initial boundary value problems) yields

\[
\partial_0 \begin{pmatrix}
\varrho_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & C^{-1}
\end{pmatrix} + \begin{pmatrix}
d & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + A
\]
Some Applications

Eliminating the second and third unknowns and equations yields
\[
\left( \partial_0 \left( \begin{array}{c} g_1 \\ 0 \\ C^{-1} \end{array} \right) + \left( \begin{array}{c} d \\ 0 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ \text{Grad grad} \\ 0 \end{array} \right) \right) \left( \begin{array}{c} \eta \\ T \end{array} \right) = \left( \begin{array}{c} f \\ 0 \end{array} \right).
\]

This is the Kirchhoff-Love plate model, which by a suitable choice of boundary condition is again accessible to the abstract solution theory of evolutionary equations, see e.g. [14].

Following the “logic” of the transition from the Reissner-Mindlin plate to the Kirchhoff-Love plate we could also formally obtain the real Schrödinger operator
\[
\partial_0 + \left( \begin{array}{c} 0 \\ \Delta_D \end{array} \right),
\]
see Section 2.2.1, from the first order system
\[
\partial_0 \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right) + \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ \text{Grad grad} \end{array} \right) \left( \begin{array}{c} \eta \\ T \end{array} \right) = \left( \begin{array}{c} f \\ 0 \end{array} \right),
\]
by similarly letting \( \varepsilon = 0 \) and eliminating the second and third components and equations.

The Timoshenko Beam

Assuming \( \Omega_0 := \Omega_1 \times \mathbb{T} \subset \mathbb{R} \times \mathbb{T} =: M \) (instead of \( \Omega_0 \subset \mathbb{R}^2 \)) for the Reissner-Mindlin plate, following the arguments in Section 2.2.4 we can reduce this model further to the \((1+1)\)-dimensional case, which leads to the Timoshenko beam model. In Cartesian coordinates this is now
\[
\partial_0 \mathcal{M} \left( \partial_0^{-1} \right) + \left( \begin{array}{c} 0 \\ -\partial_1 \\ 0 \\ 0 \\ -\partial_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\partial_1 \\ 0 \end{array} \right) \left( \begin{array}{c} \eta \\ \zeta \\ s \\ T \end{array} \right) = \left( \begin{array}{c} f \\ 0 \\ g \\ 0 \end{array} \right),
\]
where the material law has the same shape as before
\[
\mathcal{M} \left( \partial_0^{-1} \right) = \mathcal{M}_0 + \partial_0^{-1} \mathcal{M}_1
\]
with
\[
\mathcal{M}_0 := \left( \begin{array}{cccc} g_1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & C^{-1} \end{array} \right), \quad \mathcal{M}_1 := \left( \begin{array}{cccc} d & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)
\]
where \( g_1, \kappa, g_2, C \) continuous selfadjoint and strictly positive definite. Of course physical meaning (the units) have changed once again. Here also the second order system may be more familiar
\[
\begin{align*}
g_1 \partial_0^2 \tilde{\eta} - \partial_1 \kappa^{-1} \left( \partial_1 \tilde{\eta} + \tilde{s} \right) + d \partial_0 \tilde{\eta} &= f, \\
g_2 \partial_0^2 \tilde{s} - \partial_1 C \partial_1 \tilde{s} + \kappa^{-1} \left( \partial_1 \tilde{\eta} + \tilde{s} \right) &= g,
\end{align*}
\]
where \( \tilde{\eta} := \partial_0^{-1} \eta, \tilde{s} := \partial_0^{-1} s \), compare [19].
Remark 2.5. (Euler-Bernoulli Beam) Repeating the questionable “construction” of the Kirchhoff-Love plate model for the Timoshenko beam, leads to the Euler-Bernoulli beam model.

With this we conclude our tour through various examples underscoring the deep connectedness of seemingly very different mathematical models. We have seen, how various particular dynamic linear model equations can be extracted from the mother operator (11,12) assuming different material laws. The solution theory itself rests simply on strict positive definiteness. One may well wonder if the simplicity and transparency of these structural observations could not give rise to a “grand unified” numerical scheme.

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