A time fractional model of Brinkman-type nanofluid with ramped wall temperature and concentration

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Abstract
Nanofluid is an innovative heat transfer fluid with the potential to significantly enhance the heat transfer performance of traditional fluids. By adding various types of nanoparticles to ordinary base fluids, several attempts have been made to boost the rate of heat transfer and thermal conductivity. The unsteady electrically conducting flow of Brinkman-type nanofluid over an infinite vertical plate with ramping wall temperature and concentration is investigated in this article. Water is taken as the base fluid, and multi-walled carbon nanotubes are distributed equally throughout it. The Caputo-Fabrizio fractional derivative, which has a non-singular kernel, is used to generalize the classical model. The Laplace transform technique has been utilized to achieve exact solutions. Furthermore, various graphs for fractional and physical parameters are used to represent the solutions. All figures are drawn for both conditions, that is, ramped and isothermal wall temperature and concentration. The velocity field increases for greater values of thermal and mass Grashof numbers while the reverse effect is observed for Hartman number, Brinkman parameter and volume fraction. Moreover, the obtained results are also reduced to the already published results in order to show the validation of the present results. The results are used to calculate the skin friction, Nusselt number, and Sherwood number. The heat transfer of pure water is increased by 17.03% when 4% of nanoparticles are added to it which will of course increase the efficiency of solar collectors and solar pools. Moreover, the mass transfer decreases by 3.18% when 4% of nanoparticles which are dispersed in it.

Keywords
Nanofluid, ramped and isothermal wall boundary conditions, Laplace transform, Caputo-Fabrizio derivative, exact solutions

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Introduction
Nanotechnology has grown rapidly in recent decades in a variety of disciplines of engineering and science, including electronics and power generation, where heat transmission is a common occurrence. The additional technological progress need specialized heat transfer management. Many researchers have attempted to study an effective heat transfer medium for this purpose. Many techniques have been explored but failed owing to scientific limitations, such as the micro
channel and extending the surface area. For the enhancement of the heat transfer rates, many experiments characterized and examined different nanofluids. Nanofluids are a combination of nanoparticles and traditional base fluids. Water, motor oil, ethylene, and crude oil are examples of base fluids. Molybdenum disulfide, aluminum, graphite, silver, copper, and other nanoparticles dispersed in the basic fluids. Scientists have been attempting to solve this problem since the beginning to improve the characteristics of various kinds of fluids. They are trying to improve the deficiencies of these regular fluids. The nanoparticles dispersion in the base fluid had been discovered for the first time by Choi and Eastman. Various Nanofluids in heat transfer applications have been described, produced, and evaluated by several researchers. For the combination of nanoparticles with base fluids, several theoretical and experimental research are conducted. In base fluids including oil, water, and concrete, nanoparticles such as metallic oxides, carbide ceramics, nitride ceramics, and carbon in different forms are widely employed. However, because to their high thermal conductivity, some recent research suggest that carbon nanotubes (CNTs) are better suitable for heat transfer. The heat conductivity of CNTs nanoparticles is high, while their density is low.

A thermal energy transfer from a greater concentration to a lower concentration is referred to as “transfer of heat.” Mass transfer is the net movement of mass from one location to another. Conduction, convection, and radiation are all ways for transferring heat. The combined effect of bouncy thermal diffusion causes the mass and heat transfer are also being studied at the same time. Cooling processes, solar collectors, food processing, and many other lubricants such as engine oil are all examples of this phenomenon in modern technology and industries. An isothermal process is a thermodynamic process that occurs at a constant temperature. Heat transfer can be seen in this step since the temperature remains constant. An isothermal process is one in which the amount of work done is due to the system’s net heat content. We can say that an isothermal process is a constant temperature process. Isothermal processes include changes in the state or phase of various fluids due to melting and evaporation. Thermal equilibrium is maintained because heat is transferred in and out of the system at a slow rate. The heat produced by a device is referred to as “thermal.” The word “isothermal” refers to the concept of “equal heat,” which is defined as thermal equilibrium. Isothermal means “equal heat,” as “iso” means “equal.” Different researchers investigated the heat transfer in fluids by considering the isothermal process. Akram and Nadeem examined the effects of a magnetic field and heat transfer on two-dimensional Jaffrey fluid peristaltic motion in an asymmetric channel. The effect of heat radiation along with ramped wall boundary conditions on the fluid flow was studied by Anwar et al. Khan et al. investigated free convection Maxwell fluid flow with ramping wall temperature and concentration. Through an infinite isothermal vertical plate, Soundergekar calculated the exact solution to the viscous flow problem.

When it comes to the flow through a porous material, classical Navier Stokes’ equation fails. Several important physical processes in porous media have been explored for transport of mass and heat in the literature. Other variables included the effect of local thermal equilibrium between the solid phases and the fluid, thermal dispersion, porosity variance, and anisotropic porous media. Many scientific investigations have been done through permeable medium utilizing generalized model during the last few decades. The Darcy model is defined in detail by Brinkman-Forch-Homier, Brinkman introduced one of his unique models for flow of fluid among a permeable medium. As Brinkman explained, when a flow moves through a highly permeable medium, it is referred to as a Brinkman-type fluid flow. Using a magnetic field, Varma and Babu studied the steady flow of an incompressible fluid across a channel with varying permeability medium. Using the Brinkman model, Hsu and Cheng studied fluid flow through a semi-infinite vertical plate immersed in porous material. Rajagopal looked on fluid flow through a porous medium and included the influence of the porous media in the governing equations. Vafai and Tien proposed a generalized transport model. In addition, a lot of research is reported on the effects of inertial forces and solid walls through a porous media. Vafai and Hadim and Vafai wrote a book chapter on the importance aspects of porous media transport phenomena. Vafai and Hadim and Sözen and Vafai done extensive research in porous media on multiphase transport, and models were developed. These investigations initiated and explored a systematic report of several various phase models with the phase change process. Alazmi and Vafai show the differences between porous media’s heat transfer and flow transport models. Vafai et al. examined the literature for a number of turbulent models in permeable medium. Porous media theory is also frequently applied in biological applications. Kanafer and Vafai looked into turbulent flows in porous medium with drug delivery and magnetic particles movement. Khaled and Vafai studied the diffusion process in brain tissues, blood movement in cancers, and bioheat transport in tissues using the porous media.

Many physical problems are outside the reach of simple classical models to explain. Fractional calculus was developed to investigate these problems. The generalized form of classical calculus is fractional calculus.
After Leibniz introduced the definition of the nth order derivative, fractional calculus was developed. Del Hospital asked Leibniz what would happen if fractional-order is considered. Following that, several scientists started to think about it, proposing alternative meanings of fractional derivatives. Fractional calculus has practical applications in modern technology electrochemistry, electromagnetism, electric circuits, voltage dividers, neuron models, 3-D chaotic structures, and chaotic circuits, heat and mass transfer phenomenon, mathematical epidemiology. It was stated by Leibnitz in a 1695 letter to L. Hospital. During the first part of the nineteenth century, systematic investigations on fractional calculus were conducted. Fractional derivatives are defined in many ways in the literature and applied to various sciences, technology, and engineering sectors to explore real-world applications. From open literature, it is cleared that derivatives of integer orders are local in nature while derivatives in the fractional form are non-local, which retain memory property.

To the best of the author’s knowledge, the literature upon the nanofluid flow problems with the effect of the ramped and isothermal wall boundary conditions is very rare. Therefore, based on the above literature, the present study is carried out for the MHD flow of Brinkman type nanofluid with the joint effect ramped and isothermal wall boundary conditions on temperature and concentration. Water is considered as a base fluid while MWCNTs are equally dispersed in the base fluid in order to enhance its performance regarding its applications. Furthermore, to generalize the classical model, Caputo-Fabrizio (CF) time fractional time has been used to convert the classical Brinkman model with energy and concentration equations into a fractional form. With the governing momentum equation, the general energy and concentration equations are coupled in the momentum equation. For the exact solutions of coupled system of PDEs, the Laplace transform technique has been used. Additionally, the present results are also validated and reduced to the already published work. The embedded parameters have been illustrated using a variety of graphs with physical explanation.

Mathematical formulation

The current problem considers the unsteady, laminar, and MHD flow of Brinkman-type nanofluid through an infinite plate. The fluid is thought to be electrically conductive. The impact of the magnetic field is chosen along the y-axis, which is perpendicular to the flow direction, while the fluid flow is examined along the x-axis as shown in Figure 1. The plate and the fluid are initially at rest for \( t = 0^+ \), and the temperature and concentration of the plate have grown to \( T_x + (T_w - T_x)/t_0 \) and \( C_x + (C_w - C_x)/t_0 \), respectively. The governing equations for the Brinkman type fluid flow along with transfer of mass and heat are given by:

\[
\rho \frac{D\mathbf{v}}{Dt} = \nabla \mathbf{p} + \rho \mathbf{b} - \overline{T},
\]

(1)

\[
\rho c_p \frac{\partial T}{\partial t} = k \nabla \times (\nabla \times T),
\]

(2)

\[
\frac{\partial C}{\partial t} = D \nabla \times (\nabla \times C).
\]

(3)
Here, \( \rho, c_p, k, D, \overline{b}, \overline{T}, \overline{V}, T, C \) and are the notations that are used for the density, heat capacity, thermal conductivity, mass diffusivity, body forces, interactional forces, velocity vector, temperature, concentration, and Cauchy stress tensor respectively.

The following are the fields for velocity, temperature, and concentration which can be seen in Figure 1 given above:

\[
\begin{align*}
\mathbf{P} &= (u_1(y_1, t_1), 0, 0), \\
T &= T_1(y_1, t_1), \\
C &= C(y_1, t_1).
\end{align*}
\]  
(4)

Using equation (4), equations (1)–(3) in components form for Brinkman type nanofluid flow becomes:

\[
\rho_{nf} \left( \frac{\partial u_1(y_1, t_1)}{\partial t_1} + \beta \ast u_1(y_1, t_1) \right) = \mu_{nf} \frac{\partial^2 u_1(y_1, t_1)}{\partial y_1^2} - \sigma_{nf} \beta^2_0 u_1(y_1, t_1)
\]

\[+ g(\rho_{nf}T_1(y_1, t_1) - T_\infty) + g(\rho C_1(y_1, t) - C_\infty),
\]

\[
\left( \rho c_{pf} \right)_{nf} \frac{\partial T_1(y_1, t_1)}{\partial t_1} = k_{nf} \frac{\partial^2 T_1(y_1, t_1)}{\partial y_1^2},
\]

\[
\frac{\partial C_1(y_1, t_1)}{\partial t_1} = D_{nf} \frac{\partial^2 C_1(y_1, t_1)}{\partial y_1^2},
\]

where \( \beta \ast \) is known as the Brinkman parameter, \( \mu_{nf} \) is the dynamic viscosity, and the electrical conductivity is \( \sigma_{nf} \), \( u \) is the velocity, \( \rho_{nf} \) is the density, uniform magnetic field is represented by \( B_0 \), \( g \) is the gravitational acceleration, \( (\beta_T)_{nf} \) is the thermal expansion coefficient, where the temperature is represented by \( T_1 \), \( C_1 \) is the concentration, heat capacitance is symbolized as \( (C_p)_{nf} \), and \( k_{nf} \) is the thermal conductivity of nanofluid. The required initial and boundary conditions are as follows:

\[
\begin{align*}
u_1(t_1, 1) &= 0, \quad T_1(t_1, 1) = T_\infty, \quad C_1(t_1, 1) = C_\infty, \\
u_1(0, t_1) &= U_0 \cos(\omega t_1), \\
T_1(0, t_1) &= \begin{cases} T_w + (T_w - T_\infty)t_1/t_0 & \text{if } 0 < t_1 < t_0, \\
T_w & \text{if } t_1 > t_0, \end{cases} \\
C_1(0, t_1) &= \begin{cases} C_w + (C_w - C_\infty)t_1/t_0 & \text{if } 0 < t_1 < t_0, \\
C_w & \text{if } t_1 > t_0, \end{cases} \\
u_1(\infty, t_1) &= 0, \quad T_1(\infty, t_1) = T_\infty, \quad C_1(\infty, t_1) = C_\infty.
\end{align*}
\]  
(8)

Where mathematical expressions for \( \rho_{nf} \), \( \mu_{nf} \), \( \sigma_{nf} \), \( (\beta_T)_{nf} \), \( (\beta C)_{nf} \), and \( k_{nf} \), are given by:

\[
D_{nf} = (1 - \phi)D_r, \quad (\rho c_{pf})_{nf} = (1 - \phi)(\rho c_p)f + \phi \rho_v,
\]

\[
(\beta_T)_{nf} = (1 - \phi)(\rho \beta_T)f + \phi (\rho \beta_T)_s, \quad (\beta C)_{nf} = (1 - \phi)(\rho C_f) + (\beta C)_s,
\]

\[
\frac{k_{nf}}{k_v} = (1 - \phi) + 2\phi \frac{k_s}{k_v} \ln \frac{k_s}{k_v} - \frac{k_s}{k_v}, \quad \frac{k_{nf}}{k_v} = (1 - \phi) + 2\phi \frac{k_s}{k_v} \ln \frac{k_s}{k_v} + \frac{k_s}{k_v}.
\]

(9)

Nanofluid, base fluid, and solid CNTs nanoparticles are denoted by the subscripts the subscripts \( nf \), \( f \), and \( s \) in equation (9). The thermophysical properties of the considered base fluid and nanoparticles are given in Table 1 which is given below.

| Material       | \( \rho (\text{Kg}^{-1}) \) | \( c_p (\text{Kg}^{-1} \text{K}^{-1}) \) | \( k (\text{Wm}^{-1} \text{K}^{-1}) \) | \( \beta \times 10^{-5} (\text{K}^{-1}) \) | \( \sigma (\Omega \text{m}) \) | \( \text{Pr} \) |
|----------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| Water          | 997.1                       | 4179                        | 0.613                        | 21                          | 0.05                        | 6.2                         |
| MWCNTs         | 1600                        | 796                         | 3000                         | 44                          | 1.9 \times 10^{-4}           | -                           |

Table 1. Thermo-physical properties of H2O and MWCNTs.

By using the dimensionless variables given in equation (10) and dropping * notation, with initial and boundary conditions, the dimensionless form of the governing equations becomes:

\[
\chi_0 \left( \frac{\partial \psi}{\partial \tau} + \beta \psi \right) = \chi_1 \frac{\partial^2 \psi}{\partial \eta^2} - X_2 M \psi + X_3 G r \Theta
\]

\[+ \chi_4 G m \Phi,
\]

\[
\text{Pr} \chi_4 \frac{\partial \theta}{\partial \eta} = \chi_5 \frac{\partial^2 \theta}{\partial \eta^2},
\]

\[
\frac{\partial \Phi}{\partial \tau} = \kappa \frac{\partial^2 \Phi}{\partial \eta^2},
\]

with...
\[ \psi(\eta, 0) = 0, \quad \theta(\eta, 0) = 0, \quad \Phi(\eta, 0) = 0, \]
\[ \psi(0, \tau) = H(\tau) \cos(\omega \tau), \quad \theta(0, \tau) = \begin{cases} \tau & \text{if } 0 < \tau < 1, \\ 1 & \text{if } \tau > 1, \end{cases} \]
\[ \Phi(0, \tau) = \begin{cases} \tau & \text{if } 0 < \tau < 1, \\ 1 & \text{if } \tau > 1, \end{cases} \]
\[ u(\infty, \tau) = 0, \quad \theta(\infty, \tau) = 0, \quad \Phi(\infty, \tau) = 0. \]

where

\[ \beta = \frac{v_f B^2}{U_0^2}, \quad \gamma = \frac{v_f B_0^2}{U_0^2 \rho_f}, \quad Gr = \frac{g(\beta \tau \nu)(T_w - T_0)}{U_0^4}, \]
\[ Gm = \frac{g(\beta c \nu)(C_w - C_0)}{U_0^4}, \]
\[ Pr = \left( \frac{\mu c \nu}{k} \right), \quad Sc = \left( \frac{D}{\nu} \right), \]
\[ \chi_0 = (1 - \phi) + \phi \frac{P_x}{\rho_f}, \quad \chi_1 = \frac{1}{(1 - \phi)^{1/5}}, \]
\[ \chi_2 = 1 + \frac{3(1 - \phi)}{2(\nu)} - \frac{(1 - \phi)}{2(\nu)} \gamma, \quad \chi_3 = (1 - \phi) + \phi \frac{\rho c_p}{\rho c_p}, \quad \chi_4 = \frac{k_m}{k_r}. \]

Here, \( \beta, M, Gr, Gm, Pr \) and \( Sc \) is the Brinkman fluid parameter, magnetic number, thermal Grashof number, mass Grashof number, Prandtl number, and Schmidt number, respectively. The dimensionless governing equations (11)-(13) are converted to Caputo–Fabrizio time fractional model as:

\[ x_0(C^F D^\alpha_x \psi(\eta, \tau) + \beta \psi(\eta, \tau)) = \chi_1 \frac{\partial^2 \psi(\eta, \tau)}{\partial \eta^2}, \]
\[ -x_2 M \psi(\eta, \tau) + \chi_3 Gr \theta(\eta, \tau) + \chi_4 Gm \Phi(\eta, \tau), \]
\[ x_4 Pr C^F D^\alpha_x \theta(\eta, \tau) = \chi_5 \frac{\partial^2 \theta(\eta, \tau)}{\partial \eta^2}, \]
\[ C^F D^\alpha_x \phi(\eta, \tau) = \chi_6 \frac{\partial^2 \phi(\eta, \tau)}{\partial \eta^2}. \]

Where \( C^F D^\alpha_x (\cdot) \) is the Caputo–Fabrizio (CF) time-fractional operator defined by \( \text{as} \):

\[ C^F D^\alpha_x f(\eta, \tau) = \frac{N(\alpha)}{1 - \alpha} \int_0^\tau \frac{f'(\eta, t) \exp\left(-\frac{\alpha(\tau - t)}{1 - \alpha}\right)}{dt} \]
\[ 0 < \alpha < 1, \]

Solution of the problem

This section provides the exact solutions of the considered fractional model by using the Laplace technique:

**Solutions for temperature field**

We get the following results by applying the Laplace transform to equation (16) and utilizing equation (14).

\[ \frac{d \Theta(\eta, s)}{d s} - \frac{\lambda_0 s}{(1 - \alpha) s + \alpha} \Theta(\eta, s) = 0, \]

while the transformed boundary becomes:

\[ \Theta(\eta, s) = \frac{1}{\eta} \int_0^\tau e^{-\eta \tau} + \int_1^\infty e^{-\eta \tau} = \frac{1 - e^{-s}}{s^2} \]

and \( \Theta(\infty, s) = 0. \)

The boundary conditions provided in equation (20) are used to find the solutions of equation (13) in the Laplace transform domain as:

\[ \Theta(\eta, s) = \frac{1}{s^2} \exp\left(-\eta \lambda_0 \gamma_0 \frac{\gamma_0}{s + \gamma_1}\right) - \frac{\exp(-s)}{s^2} \]

where \( \lambda_0 = \frac{\lambda_0}{X t}, \quad \gamma_0 = \frac{1}{1 - \alpha}, \quad \gamma_1 = \gamma_0 \alpha. \)

It’s crucial to note that equation (21) is the Laplace transform domain solution of the energy equation with ramped wall temperature. Equation (21) can be further reduced as follows:

\[ \Theta(\eta, s) = \Theta_{\text{Ramp}}(\eta, s) - e^{-s} \Theta_{\text{Ramp}}(\eta, s), \]

where:

\[ \Theta_{\text{Ramp}}(\eta, s) = \int_{\text{Ramp}}(\eta, s, \lambda_0 \gamma_0, \gamma_1) = \frac{1}{s^2} \exp\left(-\eta \frac{\lambda_0 \gamma_0}{s + \gamma_1}\right). \]

The inverse Laplace transform is used to convert equation (23) back to its original form, yielding:

\[ \Theta(\eta, \tau) = \Theta_{\text{Ramp}}(\eta, \tau) - \Theta_{\text{Ramp}}(\eta, \tau - 1) H(\tau - 1), \]

where \( H(\tau - 1) \) is the Heaviside unit step function. The term \( \Theta_{\text{Ramp}}(\eta, \tau) \) is obtained as:

\[ \Theta_{\text{Ramp}}(\eta, \tau) = \int_0^\tau f_{\text{Ramp}}(\eta, t, \lambda_0 \gamma_0, \gamma_1) dt, \]
Where
\[ f_{\text{Ramp}}(\eta, \tau, \lambda_0, \gamma_0, \gamma_1) = 1 + \frac{2\lambda_0 \gamma_0}{\pi} \int_0^\infty \frac{\sin(\eta x)}{x(\lambda_0 \gamma_0 + x^2)} \exp \left( -\frac{\gamma_1 x^2 \eta}{\lambda_0 \gamma_0 + x^2} \right) dx. \]  
(26)

When \( 0 < \tau < 1 \), the exact energy solutions for ramped wall temperature are found in equation (26). For the isothermal temperature condition, equation (16) is resolved, yielding:
\[ \Theta_{\text{iso}}(\eta, s) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\lambda_0 \gamma_0 s}{s + \gamma_1}} \right), \]  
(27)

such that
\[ \Theta_{\text{iso}}(\eta, s) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\lambda_0 \gamma_0 s}{s + \gamma_1}} \right). \]  
(28)

The inverse Laplace transform is used to equation (28) to obtain the time domain solutions for isothermal temperature, yielding:
\[ \Theta_{\text{iso}}(\eta, \tau) = g_{\text{iso}}(\eta, \tau, \lambda_0, \gamma_0, \gamma_1) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\lambda_0 \gamma_0 s}{s + \gamma_1}} \right). \]  
(29)

The exact solutions of equation (16) for ramped and isothermal temperatures are represented by equations (24) and (29), respectively.

**Solution of concentration**

By applying the Laplace transform technique in the same way applied in the temperature field, the solution of concentration equation given in equation (17) after applying the Laplace domain becomes:
\[ \Phi(\eta, s) = \frac{1 - \exp(-s)}{s} \exp \left( -\eta \sqrt{\frac{s \gamma_2}{s + \gamma_1}} \right). \]  
(30)

where \( \gamma_2 = \frac{\gamma_0}{s} \).

The solutions for Ramped wall concentration take the following form:
\[ \Phi(\eta, s) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\gamma_2}{s + \gamma_1}} \right). \]  
(31)

By inverting the Laplace transform, we get:
\[ \Phi(\eta, \tau) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\gamma_2}{s + \gamma_1}} \right). \]  
(32)

The term \( \Phi_{\text{Ramp}}(\eta, \tau) \) is obtained as:
\[ \Phi_{\text{Ramp}}(\eta, s) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{s \gamma_2}{s + \gamma_1}} \right). \]  
(33)

When \( 0 < \tau < 1 \), the exact solutions for ramped wall concentration are found in equation (35). For the isothermal boundary condition, equation (17) is re-solved, yielding:
\[ \Phi_{\text{iso}}(\eta, s) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\gamma_2}{s + \gamma_1}} \right). \]  
(36)

such that
\[ \Phi_{\text{iso}}(\eta, \tau) = g_{\text{iso}}(\eta, \tau, \gamma_2, \gamma_1) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\gamma_2}{s + \gamma_1}} \right). \]  
(37)

By taking the inverse Laplace transform, we get:
\[ \Phi_{\text{iso}}(\eta, \tau) = g_{\text{iso}}(\eta, \tau, \gamma_2, \gamma_1) = \frac{1}{s} \exp \left( -\eta \sqrt{\frac{\gamma_2}{s + \gamma_1}} \right). \]  
(38)

Equations (33) and (35) represent the exact solutions of equation (16) for ramped and isothermal concentrations, respectively.

**Solution for the velocity field**

By applying the Laplace transformation to equation (15) and utilizing equation (14), yields:
\[
\chi_0 \left( \frac{s \dot{\psi}(y, s)}{1 - \alpha s} + \beta \psi(\eta, s) \right) = \chi_1 \frac{d^2 \dot{\psi}(\eta, s)}{d\eta^2} + \chi_2 \frac{\dot{\psi}(\eta, s)}{1 - \alpha s} + \frac{\chi_3 \Theta(\eta, s) + \chi_4 Gm \Phi(\eta, s)}{1 - \alpha s}.
\]

(39)

After further simplification and incorporating equations (21) and (30), we obtain:

\[
\frac{d^2 \dot{\psi}(\eta, s)}{d\eta^2} - \frac{1}{s + \gamma_1} \left( \frac{s \chi_1 + \chi_2}{s + \gamma_1} \right) \dot{\psi}(\eta, s) = -G \left( 1 - \exp(-s) \right) \exp \left( -\eta \sqrt{\frac{s \gamma_0}{s + \gamma_1}} \right) - Gm \left( 1 - \exp(-s) \right) \exp \left( -\eta \sqrt{\frac{s \gamma_2}{s + \gamma_1}} \right).
\]

(40)

The transformed boundary conditions are:

\[
\dot{\psi}(\eta, s) = \frac{s}{s^2 + \omega^2} \text{ and } \dot{\psi}(\infty, s) = 0,
\]

where \(\lambda_1 = \frac{1}{\chi_1} (\chi_2 \gamma_0 + \chi_0 \beta + \chi_2 M)\), \(\lambda_2 = \frac{1}{\chi_1} (\gamma_1 \chi_2 M + \gamma_1 \chi_0 \beta)\), \(G \equiv \frac{\chi_2}{\chi_1} G\), \(Gm = \frac{\chi_3}{\chi_1} G\).

The solution of equation (40) in the Laplace domain can be written as:

\[
\dot{\psi}(\eta, s) = \frac{s}{s^2 + \omega^2} \exp \left( -\eta \sqrt{\frac{s \chi_1 + \chi_2}{s + \gamma_1}} \right) + \left( -\frac{G \gamma_0}{\chi_1} \right) \left( 1 - \exp(-s) \right) \exp \left( -\eta \sqrt{\frac{s \gamma_0}{s + \gamma_1}} \right) + \left( -\frac{Gm \gamma_1}{\chi_1} \right) \left( 1 - \exp(-s) \right) \exp \left( -\eta \sqrt{\frac{s \gamma_2}{s + \gamma_1}} \right).
\]

(41)

where \(\chi_0 \gamma_0 = \chi_1 \lambda_0\).

Equation (41) can be rewritten in a more appropriate way as:

\[
\dot{\psi}(\eta, s) = \left( \dot{\psi}_c(\eta, s) + \dot{\psi}_1(\eta, s) \left( \dot{\psi}_{2(Ramp)}(\eta, s) - e^{-\eta} \dot{\psi}_{2(Ramp)}(\eta, s) \right) \right) + \left( \dot{\psi}_3(\eta, s) \left( \dot{\psi}_{2(Ramp)}(\eta, s) - e^{-\eta} \dot{\psi}_{2(Ramp)}(\eta, s) \right) \right) - \left( \dot{\psi}_1(\eta, s) \Theta(\eta, s) - \dot{\psi}_1(\eta, s) \Phi(\eta, s) \right).
\]

(42)

where

\[
\dot{\psi}_c(\eta, s) = \frac{s}{s^2 + \omega^2} \exp \left( -\eta \sqrt{\frac{s \lambda_1 + \lambda_2}{s + \gamma_1}} \right),
\]

\[
\dot{\psi}_1(\eta, s) = \frac{G \gamma_0}{\lambda_1 \gamma_0 - \lambda_2},
\]

\[
\dot{\psi}_3(\eta, s) = \frac{Gm \gamma_1}{\gamma_2 \gamma_0 - \lambda_2}.
\]

(43) - (46)

Upon inversion of the Laplace transform, equation (42) becomes:

\[
\psi(\eta, \tau) = \left( \psi_c(\eta, \tau) + \psi_1(\eta, \tau) \ast (\dot{\psi}_{2(Ramp)}(\eta, \tau) - H(\tau - 1) \dot{\psi}_{2(Ramp)}(\eta, \tau - 1)) \right) + \psi_3(\eta, \tau) \ast (\dot{\psi}_{2(Ramp)}(\eta, \tau) - H(\tau - 1) \dot{\psi}_{2(Ramp)}(\eta, \tau - 2)) - \psi_1(\eta, \tau) \ast \Theta(\eta, \tau) - \psi_1(\eta, \tau) \ast \Phi(\eta, \tau),
\]

(47)

where

\[
\psi_c(\eta, \tau) = e^{-\eta \sqrt{\lambda_1 \tau}} - \int_0^\infty \frac{1}{u \sqrt{\pi u}} \cos(\omega(u - t)) \frac{1}{u \sqrt{\pi u}} \frac{1}{u \sqrt{\pi u}} dh dt,
\]

(48)

\[
\dot{\psi}_1(\eta, \tau) = \left( \lambda_4 e^{\frac{\lambda_2 \tau}{2}} + \frac{1}{\lambda_3} \delta(\tau) \right),
\]

(49)

\[
\dot{\psi}_3(\eta, \tau) = \left( \lambda_7 e^{\frac{\lambda_2 \tau}{2}} + \frac{1}{\lambda_2} \delta(\tau) \right),
\]

(50)

\[
\psi_{2(Ramp)}(\eta, \tau) = e^{\eta \sqrt{\lambda_1 \tau}} - \int_0^\infty (\tau - t) \frac{1}{u \sqrt{\pi u}} \frac{1}{u \sqrt{\pi u}} dh dt,
\]

(51)

\[
\dot{\psi}_3(\eta, \tau) = \frac{e^{\frac{\lambda_2 \tau}{2}} - \gamma_1 \lambda_1 \sqrt{\lambda_1 \tau}}{\gamma_2 \lambda_1 \sqrt{\lambda_1 \tau}} dh dt,
\]

(52)

where \(\lambda_4 = \lambda_2 - \lambda_1 \gamma_1 \lambda_5 - G \gamma_2 \gamma_1\), \(\lambda_7 = \lambda_2 - \gamma_2 \lambda_2 - Gm \gamma_1 \lambda_5\),

where * represents the convolution product. It’s worth noting that equation (47) describes velocity field solutions with ramping wall boundary conditions. For
isothermal boundary conditions, equation (15) is now re-solved as:

\[ \bar{\psi}_{\text{iso}}(\eta, s) = \frac{s}{s^2 + \omega^2} \exp \left( -\eta \sqrt{s} \right) \left( \frac{s \lambda_1 + s \lambda_2}{s + \gamma_1} \right) \]

\[ + \left( \frac{G_0(s + \gamma_1)}{\gamma_2 s - \lambda_2} \right) \left( \frac{1}{s} \right) \exp \left( -\eta \sqrt{s} \right) \left( \frac{s \lambda_1 + s \lambda_2}{s + \gamma_1} \right) \]

\[ - \exp \left( -\eta \sqrt{s} \right) \left( \frac{\gamma_2 s}{s + \gamma_1} \right) \]  \hspace{1cm} (52)

To write equation (52) more simplified and convenient form as

\[ \psi_{\text{Ramp}}(\eta, \tau) = \left( \psi_c(\eta, \tau) + \psi_1(\eta, \tau) * \left( \psi_{2(\text{iso})}(\eta, \tau) - H(\tau - 1)\psi_{2(\text{iso})}(\eta, \tau - 1) \right) \right) \]

\[ + \psi_3(\eta, \tau) * \left( \psi_{2(\text{iso})}(\eta, \tau) - H(\tau - 1)\psi_{2(\text{iso})}(\eta, \tau - 2) \right) - \psi_1(\eta, \tau) * \Theta(\eta, \tau) \]  \hspace{1cm} (57)

and

\[ \psi_{\text{iso}}(\eta, \tau) = \left( \psi_c(\eta, \tau) + \psi_1(\eta, \tau) * \left( \psi_{2(\text{iso})}(\eta, \tau) - H(\tau - 1)\psi_{2(\text{iso})}(\eta, \tau - 1) \right) \right) \]

\[ + \psi_3(\eta, \tau) * \left( \psi_{2(\text{iso})}(\eta, \tau) - H(\tau - 1)\psi_{2(\text{iso})}(\eta, \tau - 2) \right) - \psi_1(\eta, \tau) * \Theta_{\text{iso}}(\eta, \tau) \]  \hspace{1cm} (58)

The results given in equations (57) and (58) are identical to the final solutions obtained by Saqib et al.\textsuperscript{34} which validates the correctness of the obtained solutions.

### Limiting case

The obtained results given in equations (47) and (55) can be reduced to the results published by Saqib et al.\textsuperscript{34} by ignoring mass Grashof number that is, \(Gm \to 0\). By doing so, it leads us to the following results for both, ramped and isothermal wall boundary conditions:

\[ \psi_c(\eta, \tau) = \frac{1}{s} e^{-\eta \sqrt{s}} \left( \frac{s \lambda_1 + s \lambda_2}{s + \gamma_1} \right) \]

\[ + \left( \frac{G_0(s + \gamma_1)}{\gamma_2 s - \lambda_2} \right) \left( \frac{1}{s} \right) \exp \left( -\eta \sqrt{s} \right) \left( \frac{s \lambda_1 + s \lambda_2}{s + \gamma_1} \right) \]

\[ - \exp \left( -\eta \sqrt{s} \right) \left( \frac{\gamma_2 s}{s + \gamma_1} \right) \]  \hspace{1cm} (54)

The inverse Laplace transform of equation (46), which has the following form, yields the solution for isothermal temperature:

\[ \psi_{\text{iso}}(\eta, \tau) = \psi_c(\eta, \tau) + \psi_1(\eta, \tau) * \left( \psi_{2(\text{iso})}(\eta, \tau) - H(\tau - 1)\psi_{2(\text{iso})}(\eta, \tau - 1) \right) \]

\[ + \psi_3(\eta, \tau) * \left( \psi_{2(\text{iso})}(\eta, \tau) - H(\tau - 1)\psi_{2(\text{iso})}(\eta, \tau - 2) \right) - \psi_1(\eta, \tau) * \Theta_{\text{iso}}(\eta, \tau) \]  \hspace{1cm} (55)

where

\[ \psi_{4(\text{iso})}(\eta, \tau) = e^{-\eta \sqrt{s}} \int_{0}^{\infty} \left[ \frac{1}{2 \sqrt{\pi t}} e^{-\frac{\lambda_4^2}{4t}} \gamma \right] \exp (-\lambda_4 \eta t - \lambda_4^2 \eta) \gamma \]  \hspace{1cm} (56)

and \( \psi_c(\eta, \tau), \psi_1(\eta, \tau), \Theta(\eta, \tau) \) and \( \Phi_{\text{iso}}(\eta, \tau) \) are previously defined.

### Nusselt number, Sherwood number, and skin friction

#### Nusselt number

The Nusselt number has the following mathematical expression:

\[ Nu = -\frac{k_w}{k_f} \frac{\partial \Theta}{\partial \eta}\bigg|_{\eta=0} \]  \hspace{1cm} (59)

#### Sherwood number

The Sherwood number has the following mathematical expression:
\[ S_h = -D_{nf} \frac{\partial \Phi}{\partial \eta} \bigg|_{\eta = 0} \]  

\[ C_f(\eta, \tau) = \frac{1}{(1 - \phi) \tau^2} \frac{\partial \psi}{\partial \eta} \bigg|_{\eta = 0} \]  

**Skin friction**

The Skin friction has the mathematical expression as:

**Results and discussion**

The goal of this study is to see how a Brinkman-type nanofluid flows through a vertical plate with the influence of applied magnetic field. The influence of ramped and isothermal boundary conditions on concentration and temperature is also examined. To generalize the classical model, Caputo-Fabrizio fractional derivative is applied. Additionally, water is considered as a base fluid while MWCNTs are dispersed to enhance the thermal properties of regular water. The Laplace transform technique is used to analyze exact solutions. Temperature, concentration, and velocity field closed form solutions are computed. The physical representation of the present problem can be seen in Figure 1. The temperature, velocity, and concentration distributions are graphically represented in Figures 2 to 12. The velocity distributions are shown in Figures 2 to 7, where Figures 8 to 10 show the temperature distribution. Figures 11 and 12 show the effect of concentration distributions. It’s worth noting that we use \( \tau = 0.5 \) and \( \tau = 1.5 \) for ramped and isothermal temperatures, respectively. Skin friction, Nusselt number, and Sherwood number are all computed and tabulated.

Figure 2 shows the effect of \( \alpha \) on the velocity field for both ramping and isothermal wall boundary conditions. Fractional models, in compared to classical models, are more general. Fractional models have many solutions, allowing experimentalists to match their results to theoretical results in excellent agreement, which is not possible in the classical model for \( \alpha = 1 \).
The impact of $\beta$ on the velocity distribution is seen in Figure 3. The drag force is directly proportional to $\beta$. The greater the influence of $\beta$, the stronger the drag forces become, resulting in a reduction in the velocity field. The effect of $\phi$ on the velocity field can be seen in Figure 4. When nanoparticles disperse in a conventional fluid, the viscosity of the base fluid increases, slowing fluid velocity. Figure 5 illustrates the influence of $M$ on the velocity field. Fluid velocity drops when $M$ is raised. The physics behind this is that when $M$ increases, Lorentz forces are generated, which oppose fluid motion and create resistive forces on the fluid flow, lowering fluid velocity. Figures 6 and 7 show the effect of $Gr$ and $Gm$ on the velocity field. The velocity of the nanofluid rises in both cases. This velocity pattern is evident because raising $Gr$ and $Gm$ increases buoyancy forces while lowering viscous forces, causing fluid motion to accelerate.

Figure 8 shows the effect of $\alpha$ on the temperature field for both ramping and isothermal wall boundary conditions. Fractional models, in compared to classical models, are more general. Fractional models give us several solutions for different values of $\alpha$, which means that fractional models give a choice to the experimentalists to match their results with best fit solution. The effect of $\phi$ on the temperature field can be seen in Figure 9. By increasing $\phi$, the temperature field clearly increased. As the heat transfer rate of the base fluid increases as the amount of nanoparticles increases, the temperature field rises. Figure 10 depicts the impact of time $\tau$ on the temperature field. It has been found that in both situations, $\tau$ raises the fluid’s temperature. It is also detected that for ramped boundary condition that is, $0 < \tau < 1$, the temperature at the boundary changes while for the case of isothermal boundary conditions $\tau \geq 1$, the temperature reaches to its maximum and remains unchanged.

The effect of $\alpha$ on concentration distribution is shown in Figure 11. It is found that by considering fractional model, it gives us different curves of concentration profile. It also gives us the solutions of concentration profile in classical sense by considering $\alpha = 1$. Fractional models give a choice to the experimentalists to match their results with best fit solution. The effect
of \( t \) on concentration distribution is displayed in Figure 12. It has been observed that for both cases, \( t \) increases the concentration field. It is also observed that for ramped boundary conditions that is, \( 0 < t < 1 \), the concentration at the boundary changes, while for the case of isothermal boundary conditions \( t \geq 1 \), the concentration reaches its maximum and remains unchanged.

For the validation of the present results, we also reduced the present solutions to the solutions of Saqib et al.\(^{34}\) by taking \( Gm \to 0 \) which is given as a limiting case in section 3.4. We also portrayed the comparison of the present and published results by sketching graph. The comparison between the present and published results of Saqib et al.\(^{34}\) is given in Figure 13. From the figure, it is observed that the present results overlapped with the results of Saqib et al.\(^{34}\) for both, ramped and isothermal cases when \( Gm \to 0 \) which validates the accuracy of our solutions.

The thermophysical properties of the base fluid as well as nanoparticles is given in Table 1. The variation in skin friction against different parameters is tabulated in Table 2. Skin friction is calculated for both the cases, that is, classical as well as the fractional case. Similarly, variation in Nusselt number against \( \phi \) is calculated and displayed in Table 3. It is observed that the dispersion of MWCNTs in regular water can enhance the heat

![Figure 10. Temperature distribution for different values of \( t \).](image1)

![Figure 11. Concentration distribution for different values of \( \alpha \).](image2)

![Figure 12. Concentration distribution for different values of \( t \).](image3)

![Figure 13. Comparison of the present results with the published results of Saqib et al.\(^{34}\).](image4)

| \( Gr \) | \( Gm \) | \( Re \) | \( \beta \) | \( t \) | \( \phi \) | \( M \) | \( \alpha \) | \( C_f \) | \( C_{f\text{classical}} \) |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 1.5 | 1 | 0.02 | 3 | 0.5 | 5.278 | 5.167 |
| 4 | 3 | 1.5 | 1 | 0.02 | 3 | 0.5 | 4.982 | 4.905 |
| 3 | 4 | 1.5 | 1 | 0.02 | 3 | 0.5 | 4.881 | 4.767 |
| 3 | 3 | 1.7 | 1 | 0.02 | 3 | 0.5 | 5.325 | 5.283 |
| 3 | 3 | 1.5 | 1 | 0.02 | 3 | 0.5 | 5.293 | 5.182 |
| 3 | 3 | 1.5 | 1.5 | 0.02 | 3 | 0.5 | 4.126 | 3.753 |
| 3 | 3 | 1.5 | 1 | 0.04 | 3 | 0.5 | 5.712 | 5.273 |
| 3 | 3 | 1.5 | 1 | 0.02 | 4 | 0.5 | 5.626 | 5.523 |
| 3 | 3 | 1.5 | 1 | 0.02 | 3 | 0.7 | 5.208 | 5.167 |
transfer rate of pure water up to 17.02% by adding 4% of nanoparticles. This means that water can store more heat and hence can enhance the performance of water-based solar collectors, solar heaters, solar pools, solar panels, etc. A decrease in the mass transfer is portrayed in Table 4. The rate of mass transfer reduces as the number of nanoparticles increases. This is due to the regular fluid’s density and viscosity increasing as the number of nanoparticles increases. The table demonstrates that raising 4% of nanoparticles reduces the rate of mass transfer by up to 3.18%.

**Conclusion**

This research is conducted to determine the closed form solutions for Brinkman-type nanofluid flow over a vertical plate. The effects of ramping and isothermal wall boundary conditions on the fluid’s temperature and concentration are examined in the presence of an applied magnetic field. The nanoparticles utilized are MWCNTs, and the base fluid is water. The classical model is then generalized by using the Caputo-Fabrizio fractional derivative, which has recently become the most often used fractional derivative. The solutions of the coupled system is obtained via the Laplace transform technique. The obtained results are also validated by reducing and comparing it with the already published results by making some parameters absent. The graphs also illustrate the results that have been obtained. The following are the main points of the study:

- A decrease in the nanofluid velocity can be observed for large values of $\alpha$ for the case of ramped wall boundary conditions, while the opposite impact is observed for isothermal wall boundary conditions.
- The present results are reducable to the classical Brinkman type nanofluid model by taking $\alpha \rightarrow 1$.
- The obtained results are also reducable to the already published results of Saqib et al.\textsuperscript{34} by taking $Gm \rightarrow 0$.
- The velocity profile decreases for larger values of $M$ and $\beta$.

- By raising $Gr$ and $Gm$, the velocity profile can be enhanced.
- Increasing $\phi$ has resulted in a reduction in velocity profile.
- By raising $\phi$, we can increase the temperature profile while reducing the concentration profile.
- Increasing the value of $\phi$ by 0.04 improves heat transfer by up to 17.02%.
- Increasing the value of $\phi$ by 0.04 reduces mass transfer by up to 3.18%.

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