A two-grid temporal second-order scheme for the two-dimensional nonlinear Volterra integro-differential equation with weakly singular kernel

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Abstract

A two-grid temporal second-order scheme for the two-dimensional nonlinear Volterra integro-differential equation with a weakly singular kernel is of concern in this paper. The scheme is targeted to reduce the computation time and to improve the accuracy of the scheme developed by Xu et al. (Appl Numer Math 152:169–184, 2020). The constructed scheme is armed by three steps: First, a small nonlinear system is solved on the coarse grid using a fix-point iteration. Second, Lagrange’s linear interpolation formula is used to arrive at some auxiliary values for the analysis of the fine grid. Finally, a linearized Crank–Nicolson finite difference system is solved on the fine grid. Moreover, the algorithm uses a central difference approximation for the spatial derivatives. In the time direction, the time derivative and integral term are approximated by the Crank–Nicolson technique and product integral rule, respectively. By means of the discrete energy method, stability and space-time second-order convergence of
the proposed approach are obtained in $L^2$-norm. Finally, the numerical verification is fulfilled as the numerical results of the given numerical experiments agree with the theoretical analysis and verify the effectiveness of the algorithm.

**Keywords** Nonlinear fractional evolution equation · Time two-grid algorithm · Accurate second order · Stability and convergence · Numerical experiments

**Mathematics Subject Classification** 65M06 · 65M12 · 65M22 · 45K05

1 Introduction

In this paper, we consider the following two-dimensional nonlinear Volterra integro-differential equation with a weakly singular kernel

\[
  u_t - \mu \Delta u - I^{(\alpha)} \Delta u = f(x, y, t) + g(u), \quad (x, y, t) \in \Omega \times (0, T],
\]

with the initial-boundary conditions

\[
  u(x, y, 0) = \psi(x, y), \quad (x, y) \in \bar{\Omega},
\]
\[
  u(x, y, t) = 0, \quad (x, y, t) \in \partial \Omega \times (0, T],
\]

where $\Omega = (0, L_x) \times (0, L_y)$ with the boundary $\partial \Omega$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian operator and $u_t = \partial u/\partial t$. In addition, $\alpha \in (0, 1)$, $\mu \in [0, \infty)$ and $T \in (0, \infty)$ are given constants. $f(x, y, t)$ and $\psi(x, y)$ are given functions. The nonlinear term $g(u) \in C^2(\mathbb{R}) \cap L^1(0, T]$ satisfies the Lipschitz condition $|g(u_1) - g(u_2)| \leq \bar{C}|u_1 - u_2|$. Furthermore, The integral term $I^{(\alpha)} \Delta u(x, y, t)$ is defined in [1, 2] as follows

\[
  I^{(\alpha)} \Delta u(x, y, t) = \int_0^t \rho_\alpha(t-s) \Delta u(x, y, s) ds, \quad \rho_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0.
\]

In addition, throughout the article, we assume that problem (1)–(2) has a unique solution such that the following regularity assumptions fulfilled [3]:

(A1) $u_t$, $u_{yyy}$, $u_{txxx}$, $u_{xxxx}$ and $u_{yyyy}$ are continuous in $\bar{\Omega} \times [0, T]$;

(A2) $u_{tt}$, $u_{ttx}$, $u_{txxx}$ and $u_{ttty}$ are continuous in $\bar{\Omega} \times (0, T]$, and there exists a positive constant $\tilde{C}$ satisfying for $(x, y, t) \in \bar{\Omega} \times (0, T]$ that

\[
  \begin{align*}
    |u_{tt}(x, y, t)| & \leq \tilde{C} t^{\alpha-1}, \\
    |u_{ttt}(x, y, t)| & \leq \tilde{C} t^{\alpha-2}, \\
    |u_{ttxx}(x, y, t)| & \leq \tilde{C} t^{\alpha-1}(\kappa = x, y).
  \end{align*}
\]

Such integro-differential equations with Riemann–Liouville integral operators appear frequently in various mathematical and physical models. Problem (1)–(2) is a commonly used model for studying physical phenomena related to elastic forces.
model is mainly used in the problems of heat conduction, viscoelasticity and population dynamics of materials with memory [4–9]. In viscoelastic problems, the parameter $\mu$ in this model represents the Newtonian contribution to viscosity, and the integral term represents the viscosity part of the equation.

In recent years, high-precision computational methods for 2D partial integro-differential equations with the weakly singular kernel, such as equation (1), have been developed. The linear case of (1)–(2) has been deeply studied in the literature, e.g., see [10–15]. Furthermore, some numerical studies on the nonlinear case were introduced. Mustapha et al. [3] applied the Crank–Nicolson scheme under graded meshes to solve a semilinear integro-differential equation with a weakly singular kernel. Dehghan et al. [16] proposed a spectral element technique for solving nonlinear fractional evolution equations. Besides, there are also some seminal works considering nonlinear ordinary integro-differential equations, e.g., a class of algorithms of the arbitrarily high order by Feldstein and Sopka [17], a theory of weak stability for linear multistep methods by Brunner and Lambert [18], and recent advances of this kind of problem in the numerical analysis were sampled and reported by Brunner [19]. In addition, some numerical methods are constructed for nonlinear nonlocal equations, we refer to [20–23] as examples for these techniques.

However, when solving 2D nonlinear problems, the resulting large systems of nonlinear equations require a large computational cost as the grid is continuously subdivided. In order to save the computational cost of nonlinear problems, a spatial two-grid finite element technique was proposed by Xu [24, 25]. Inspired by Xu’s ideas, the two-grid method began to be intensively studied and applied to the solution of nonlinear parabolic equations. Dawson and Wheeler et al. [26] proposed a spatial two-grid finite difference method in solving nonlinear parabolic equations and analyzed the convergence of the method on the coarse and fine grids. For solving the nonlinear time-fractional parabolic equation, Li et al. [27] obtained the numerical solution of this equation using the spatial two-grid block-centered finite difference scheme. For more work regarding the spatial two-grid methods, see [28–30]. In addition, some scholars, inspired by the spatial two-grid method, started to consider using the two-grid method to solve the nonlinear equations in the time direction. Liu et al. [31] proposed a new time two-grid finite element algorithm in order to solve the time fractional water wave model, and illustrated through numerical experiments that it has higher computational efficiency than the standard finite element method. In [32], a time two-grid backward Euler finite difference method is constructed to solve problem (1)–(2). However, the time convergence order of the above methods cannot reach the exact second order. These prompted us to consider the following work.

In this paper, we design an efficient temporal two-grid Crank–Nicolson (TTGCN) finite difference method for solving problem (1)–(2). In this approach, the time and space derivatives are approximated using the Crank–Nicolson technique and the central difference formula, respectively, and the Riemann–Liouville integral term is approximated by the product integration rule designed in [33]. Then, this algorithm is divided into three steps: First, a small nonlinear system is solved on a coarse grid. Second, based on the solution of the first step, the values of each node are obtained by linearization technique as the auxiliary approximate solution. Finally, we approximate the nonlinear term $g(U^n)$ by a Taylor expansion and solve the linear system on a fine
grid. Furthermore, under the regularity assumptions \((A1)\) and \((A2)\), we prove that this algorithm is stable and the convergence order is \(O(\tau_C^4 + \tau_F^2 + h_1^2 + h_2^2)\), where \(\tau_C\) and \(\tau_F\) are the time steps of the coarse and fine grids, respectively. Also, the linearization technique is used on the fine grid, so the TTGCN finite difference algorithm has the advantage of both ensuring accuracy and improving computational efficiency. In addition, the numerical results introduced in this paper show that the TTGCN finite difference algorithm is more efficient than the standard Crank–Nicolson (SCN) finite difference method without loss of accuracy. Meanwhile, our algorithm can achieve second-order of convergence with respect to time compared to the method in [32].

The remainder of this paper is structured as follows. In Sect. 2, we give some notations and useful lemmas. Then, the TTGCN finite difference scheme is established in Sect. 3. In Sect. 4, the stability and convergence of the TTGCN finite difference method are analyzed by the energy method. Moreover, some numerical results are given in Sect. 5. It should be noted that the generic positive constant \(\bar{C}\) is independent of the temporal step size and the spatial step size, moreover, it is not necessarily same in different situations.

## 2 Preliminaries

In this section, we shall provide some useful notations and lemmas which will be used for the forthcoming work. First, for a positive integer \(N\), we define the time-step size on the fine grid as \(\tau_F = T/N\) with \(t_n = n\tau_F (0 \leq n \leq N)\). Similarly, for the coarse grid, the time-step size is \(\tau_C = T/N, t_s = s\tau_C (0 \leq s \leq N)\) for positive integer \(N\), where \(N = N/k, k \geq 2\) and \(k \in \mathbb{Z}^+\). For any grid function \(\varphi^n (1 \leq n \leq N)\) on (0, \(T\), we define

\[
\delta_t \varphi^n = \frac{\varphi^n - \varphi^{n-1}}{\tau}, \quad \varphi^{n-\frac{1}{2}} = \frac{\varphi^n + \varphi^{n-1}}{2}.
\]

Then, the grid functions are given as

\[
u^n = u(x, y, t_n), \quad f^n = f(x, y, t_n), \quad 0 \leq n \leq N.
\]

We integrate the Eq. (1) from \(t = t_{n-1}\) to \(t_n\) and then multiply by \(1/\tau\), we obtain

\[
\frac{\Delta u^n}{\tau} - \int_{t_{n-1}}^{t_n} \Delta u(\cdot, t)dt = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} I(\alpha) \Delta u(\cdot, t)dt = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(\cdot, t)dt + \frac{1}{\tau} \int_{t_{n-1}}^{t_n} g(\varphi(\cdot, t))dt. \tag{4}
\]

To approximate the integral term of Eq. (4), from [15, 33], we obtain the quadrature approximation with the uniform time step

\[
\frac{1}{\tau} \int_{t_{n-1}}^{t_n} \Delta u(\cdot, t)dt = \begin{cases} 
\Delta u^1 + (R1)^1, \\
\Delta u^{n-\frac{1}{2}} + (R1)^n, \quad 2 \leq n \leq N.
\end{cases} \tag{5}
\]
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and

\[
\frac{1}{\tau} \int_{t_{n-1}}^{t_n} I^{(\alpha)} \Delta u(\cdot, t) dt = \begin{cases} 
\frac{1}{\tau} \int_{t_0}^{t_1} \int_{t_0}^{t_1} \rho_{\alpha}(t-s) \Delta u^1 ds dt + (R2)^1, \\
\frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_{t_0}^{t_1} \rho_{\alpha}(t-s) \Delta u^1 ds dt \\
+ \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \min\{t, t_m\} \int_{t_m}^{t_n} \rho_{\alpha}(t-s) \Delta u^{m-\frac{1}{2}} ds dt + (R2)^n, \quad 2 \leq n \leq N,
\end{cases}
\]

(6)

where \((R1)^n\) and \((R2)^n\) are the local truncation errors.

For any grid function \(\phi^n (1 \leq n \leq N)\), we define the following two operators

\[
\mathcal{L}_1^{n, \tau} (\phi^n) = \begin{cases} 
\phi^1, \\
\phi^{n-\frac{1}{2}}, \quad n \geq 2,
\end{cases}
\]

\[
\mathcal{L}_2^{n, \tau} (\phi^n) = \begin{cases} 
\phi^1, \\
\phi_{n, 1} \phi^1 + \sum_{m=2}^{n} \omega_{n, m} \phi^{m-\frac{1}{2}}, \quad n \geq 2,
\end{cases}
\]

(7)

where

\[
\omega_{n, m} = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \min\{t, t_m\} \int_{t_m}^{t_n} \rho_{\alpha}(t-s) ds dt.
\]

(8)

Therefore, for \(n \geq 2\) and \(1 \leq m \leq n - 1\), we can get that

\[
\omega_{n, m} = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_{t_{m-1}}^{t_m} \rho_{\alpha}(t-s) ds dt
\]

\[
= \frac{\left[(t_n - t_{m-1})^{\alpha+1} - (t_n - t_m)^{\alpha+1}\right] - \left[(t_{n-1} - t_{m-1})^{\alpha+1} - (t_{n-1} - t_m)^{\alpha+1}\right]}{\tau \Gamma(2 + \alpha)}
\]

(9)

and

\[
\omega_{n, n} = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \rho_{\alpha}(t-s) ds dt = \frac{\tau^{\alpha}}{\Gamma(2 + \alpha)}, \quad 1 \leq n \leq N.
\]

(10)
Then the Eqs. (5) and (6) can be rewritten as follows

\[
\frac{1}{\tau} \int_{t_{n-1}}^{t_n} \Delta u(\cdot, t) \, dt = \mathcal{L}^n_{1}\tau(\Delta u^n) + (R1)^n, \quad 1 \leq n \leq N, \quad (11)
\]

\[
\frac{1}{\tau} \int_{t_{n-1}}^{t_n} I^{(\alpha)} \Delta u(\cdot, t) \, dt = \mathcal{L}^n_{2}\tau(\Delta u^n) + (R2)^n, \quad 1 \leq n \leq N. \quad (12)
\]

For the spatial approximation, defining the space-step size \( h_1 = L_x/M_x, h_2 = L_y/M_y, h = \max\{h_1, h_2\} \) for two positive integers \( M_x \) and \( M_y \), we arrive at \( x_i = ih_1 \) and \( y_j = jh_2 \). Denote \( \mathcal{O}_h = \{(x_i, y_j) | 0 \leq i \leq M_x, 0 \leq j \leq M_y\}, \mathcal{O}_h = \mathcal{O}_h \cap \mathcal{O} \) and \( \partial \mathcal{O}_h = \partial \mathcal{O} \cap \partial \mathcal{O} \). Let the grid function \( Z_h = \{z_{ij} | 0 \leq i \leq M_x, 0 \leq j \leq M_y\} \) on \( \mathcal{O}_h \), then we denote the following notations

\[
\delta_x z_{i+\frac{1}{2},j} = \frac{z_{i+1,j} - z_{ij}}{h_1}, \quad \delta_y z_{i,j+\frac{1}{2}} = \frac{z_{i,j+1} - z_{ij}}{h_2}.
\]

Also, the discrete Laplace operator is defined by \( \Delta_h = \delta_x^2 + \delta_y^2 \).

Then, for any grid function \( z, v \in \mathcal{O}_h \), some norm and inner product are defined as follows

\[
(z, v) = h_1 h_2 \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} z_{ij} v_{ij}, \quad \|z\| = \sqrt{(z, z)}, \quad \|z\|_\infty = \max_{1 \leq i \leq M_x-1, 1 \leq j \leq M_y-1} |z_{ij}|.
\]

\[
\|\delta_x z\| = \sqrt{h_1 h_2 \sum_{i=0}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_x z_{i+\frac{1}{2},j})^2}, \quad \|\delta_y z\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_x-1} \sum_{j=0}^{M_y-1} (\delta_y z_{i,j+\frac{1}{2}})^2}.
\]

Next, some auxiliary lemmas will be given.

**Lemma 1** [34] Suppose \( g(u(\cdot, t)) \in C^2(\mathbb{R}) \cap L^1(0, T) \), then it holds that

\[
\left| \int_{t_{n-1}}^{t_n} g(u(\cdot, t)) \, dt - \frac{t_n - t_{n-1}}{2} \left[ g(u(\cdot, t_n)) + g(u(\cdot, t_{n-1})) \right] \right| \leq \frac{(t_n - t_{n-1})^3}{12} \|g''\|_\infty,
\]

where \( \|g''\|_\infty = \sup_{\xi \in (t_{n-1}, t_n)} |g''(u(\cdot, \xi))| < \infty. \)

According to Taylor series expansion with integral remainder term, we can obtain the following lemma.

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Lemma 2 [35] Assume \( v(x, y) \in C_{x,y}^{4,4}([0, L_x] \times [0, L_y]) \), then it satisfies that
\[
\frac{\partial^2 v}{\partial x^2}(x_i, y_j) = \delta_x^2 v(x_i, y_j)
\]
\[
- \frac{h_1^2}{6} \int_0^1 \left[ \frac{\partial^4 v}{\partial x^4}(x_i + w_1, y_j) + \frac{\partial^4 v}{\partial x^4}(x_i - w_1, y_j) \right] (1 - w)^3 dw,
\]
\[
\frac{\partial^2 v}{\partial y^2}(x_i, y_j) = \delta_y^2 v(x_i, y_j)
\]
\[
- \frac{h_2^2}{6} \int_0^1 \left[ \frac{\partial^4 v}{\partial y^4}(x_i, y_j + w_2) + \frac{\partial^4 v}{\partial y^4}(x_i, y_j - w_2) \right] (1 - w)^3 dw.
\]

For further analysis, the following important lemmas are presented.

Lemma 3 Assume that the solution \( u \) of the problem (1)–(2) satisfies the regularity assumptions (A1) and (A2), then we obtain that
\[
\tau \sum_{m=1}^n \| (R1)_m \| \leq \bar{C} \tau^2, \quad 1 \leq n \leq N.
\]

**Proof** Through simple calculation, we yield
\[
(R1)_1^n = \frac{1}{\tau} \int_{t_0}^{t_1} \left[ \Delta u(\cdot, t) - \Delta u(t_1) \right] dt,
\]
\[
(R1)_n^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \left[ \Delta u(\cdot, t) - \left( \frac{t_n - t}{\tau} \Delta u^{n-1} + \frac{t - t_{n-1}}{\tau} \Delta u^n \right) \right] dt.
\]

Using Taylor expansion with integral remainder term, we have
\[
\Delta u(\cdot, t) - \Delta u^1 = \Delta u(\cdot, t) - \Delta u(\cdot, t_1) = - \int_t^{t_1} \Delta u_s(\cdot, s) ds, \quad t_0 \leq t \leq t_1,
\]
therefore
\[
(R1)_1^n = - \frac{1}{\tau} \int_{t_0}^{t_1} \int_t^{t_1} \Delta u_s(\cdot, s) ds dt = - \frac{1}{\tau} \int_{t_0}^{t_1} \int_s^{t_1} \Delta u_s(\cdot, s) dtds = - \frac{1}{\tau} \int_{t_0}^{t_1} s \Delta u_s(\cdot, s) ds.
\]

The continuity of \( u_{1kk}(x, y, t)(\kappa = x, y) \) in \( \tilde{\Omega} \times [0, T] \) implies
\[
\tau \| (R1)_1 \| \leq \bar{C} \tau^2.
\]
Similarly, from Taylor expansion with integral remainder term, we obtain

\[
\left| \Delta u(\cdot, t) - \frac{t_n - t}{\tau} \Delta u^{n-1} - \frac{t - t_{n-1}}{\tau} \Delta u \right| \leq 2 \tau \int_{t_{n-1}}^{t_n} |\Delta u_{ss}(\cdot, s)| ds, \quad n \geq 2, \quad (18)
\]

then

\[
| (R1)^n | \leq 2 \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} |\Delta u_{ss}(\cdot, s)| ds dt \leq \tilde{C} \tau \int_{t_{n-1}}^{t_n} s^{\alpha-1} ds = \tilde{C} \tau (t^\alpha_n - t^\alpha_{n-1}). \quad (19)
\]

This proves

\[
\tau \sum_{m=2}^{n} \|(R1)^m\| \leq \tilde{C} \tau^2 (t^\alpha_n - t^\alpha_1) \leq \tilde{C} \tau^2. \quad (20)
\]

The proof is completed. \[\square\]

**Lemma 4** Suppose that the solution \(u\) of the problem (1)–(2) satisfies the regularity assumptions \((A1)\) and \((A2)\). Then we can obtain the following

\[
\tau \sum_{m=2}^{n} \|(R2)^m\| \leq \tilde{C} \tau^2, \quad 1 \leq n \leq N. \quad (21)
\]

**Proof** See the case \((\gamma = 1)\) in [33], or Lemma 2.2 in [15]. \[\square\]

**Lemma 5** [10] For any grid function \(v, w \in Z_h\), then it holds as follows

\[
-h_1 h_2 \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta^2_x v_{ij} w_{ij}) = h_1 h_2 \sum_{i=0}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_x v_{i+\frac{1}{2}, j}) (\delta_x w_{i+\frac{1}{2}, j}),
\]

\[
-h_1 h_2 \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta^2_y v_{ij} w_{ij}) = h_1 h_2 \sum_{i=1}^{M_x-1} \sum_{j=0}^{M_y-1} (\delta_y v_{i, j+\frac{1}{2}}) (\delta_y w_{i, j+\frac{1}{2}}).
\]

**Lemma 6** [2, 33] For any grid function \(v^n (1 \leq n \leq N)\), it holds that

\[
(\nabla_h v^1, \mathcal{L}^1_{2, \tau} (\nabla_h v^n)) + \sum_{n=2}^{N} \left(\nabla_h v^{n-\frac{1}{2}}, \mathcal{L}^n_{2, \tau} (\nabla_h v^n)\right) \geq 0, \quad (22)
\]

where \(\mathcal{L}^n_{2, \tau}\) is presented via (7) and the operator \(\nabla_h = \delta_x + \delta_y\).

**Lemma 7** [2] For \(N \geq 1\) and \(v^n \in Z_h\), we have

\[
\tau \left( v^1, \delta_t v^1 \right) + \tau \sum_{n=2}^{N} \left( v^{n-\frac{1}{2}}, \delta_t v^n \right) \geq \frac{1}{2} \left( \| v^N \|^2 - \| v^0 \|^2 \right). \quad (23)
\]
Lemma 8 [36] (Discrete Grönwall’s inequality) If \( \{Q_m\} \) is a non-negative real sequence and satisfies
\[
Q_m \leq \tilde{\gamma}_m + \sum_{n=0}^{m-1} \tilde{\beta}_n Q_n, \quad m \geq 1,
\]
where \( \{\tilde{\gamma}_m\} \) is a non-negative and non-descending sequence, \( \tilde{\beta}_n \geq 0 \), then, we obtain
\[
Q_m \leq \tilde{\gamma}_m \exp\left(\sum_{n=0}^{m-1} \tilde{\beta}_n\right), \quad m \geq 1.
\]

3 Establishment of the two-grid difference scheme

In the following, we first establish the SCN finite difference method for nonlinear problem (1)–(2).

Applying the quadrature approximations (5)–(6) and Lemmas 1–2, then (4) become
\[
\delta_t u_{ij}^1 - \mu \Delta_h u_{ij}^1 - w_{1,1} \Delta_h u_{ij}^1 = b_{ij}^1 + \frac{g(u_{ij}^1) + g(u_{ij}^0)}{2} + (R1)_{ij} + (R2)_{ij} + (R3)_{ij} + (R4)_{ij}, \quad (x_i, y_j) \in \Omega_h,
\]
\[
\delta_t u_{ij}^n - \mu \Delta_h u_{ij}^{n-\frac{1}{2}} - w_{n,1} \Delta_h u_{ij}^{n-\frac{1}{2}} - \sum_{m=2}^{n} w_{n,m} \Delta_h u_{ij}^{m-\frac{1}{2}} = b_{ij}^n + \frac{g(u_{ij}^n) + g(u_{ij}^{n-1})}{2} + (R1)_{ij} + (R2)_{ij} + (R3)_{ij} + (R4)_{ij}, \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq N,
\]
\[
u_{ij}^n = 0, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N,
\]
\[
u_{ij}^0 = \psi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h,
\]
where
\[
b_{ij}^n = \frac{1}{\tau} \int_{t_n}^{t_n} f(x_i, y_j, t) dt,
\]
\[
(R3)_{ij}^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} g(u(x_i, y_j, t)) dt - \frac{g(u_{ij}^n) + g(u_{ij}^{n-1})}{2} = \mathcal{O}(\tau^2),
\]
\[
(R4)_{ij}^n = \mathcal{L}_1^{n, \tau}(\Delta u_{ij}^n - \Delta_h u_{ij}^n) + \mathcal{L}_2^{n, \tau}(\Delta u_{ij}^n - \Delta_h u_{ij}^n) = \mathcal{O}(h_1^2 + h_2^2).
\]

Omitting the truncation errors \((R_s)^n_{ij}(s = 1, 2, 3, 4), 1 \leq n \leq N\), and replacing \(u_{ij}^n\) with \(U_{ij}^n\), we obtain the following SCN finite difference scheme
\[ \delta_t U^1_{ij} - \mu \Delta_h U^1_{ij} - w_{1,1} \Delta_h U^1_{ij} = b^1_{ij} + \frac{g(U^1_{ij}) + g(U^0_{ij})}{2}, \quad (x_i, y_j) \in \Omega_h, \]  
(28) 

\[ \delta_t U^n_{ij} - \mu \Delta_h U^{n-1}_{ij} - w_{n,1} \Delta_h U^n_{ij} - \sum_{m=2}^{n} w_{m,m} \Delta_h U^{m-1}_{ij} = b^n_{ij} + \frac{g(U^n_{ij}) + g(U^{n-1}_{ij})}{2}. \]
(29) 

On the coarse grid, we only calculate \( k \)th level, \( 0 \leq s \leq N \). Similar to the establishment of Eqs. (28)–(29), the discrete scheme on the coarse grid is constructed as follows

\[ \delta_t (U_C)^s_{ij} - \mu \Delta_h (U_C)^s_{ij} = b^s_{ij} + \frac{g((U_C)^s_{ij}) + g((U_C)^0_{ij})}{2}, \quad (x_i, y_j) \in \Omega_h, \]  
(32) 

\[ \delta_t (U_C)^{sk}_{ij} - \mu \Delta_h (U_C)^{(s-1)k}_{ij} - w_{s,1} \Delta_h (U_C)^{(s-1)k}_{ij} - \sum_{p=2}^{s} w_{s,p} \Delta_h (U_C)^{(p-1)k}_{ij} = b^{sk}_{ij} + \frac{g((U_C)^{sk}_{ij}) + g((U_C)^{(s-1)k}_{ij})}{2}, \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq s \leq N. \]  
(33) 

**Step II.** Then, based on the solution \((U_C)^s_{ij}\) obtained in the Step I, applying Lagrange linear interpolation to calculate \((U_C)^{(s-1)k+q}_{ij}\) by points \((t_{(s-1)k}, (U_C)^{(s-1)k}_{ij})\) and \((t_{sk}, (U_C)^{sk}_{ij})\) direction on the coarse grid, with \( 1 \leq q \leq k - 1 \), we have

\[
\mathcal{L}_{UC}(t_{(s-1)k+q}) = U_C^{(s-1)k+q} \\
= \frac{t_{(s-1)k+q} - t_{sk}}{t_{(s-1)k} - t_{sk}} U_C^{(s-1)k} + \frac{t_{(s-1)k+q} - t_{(s-1)k}}{t_{sk} - t_{(s-1)k}} U_C^{sk} \\
= (1 - \frac{q}{k}) U_C^{(s-1)k} + \frac{q}{k} U_C^{sk}, \quad 1 \leq s \leq N, \quad 1 \leq q \leq k - 1. 
\]  
(34) 

**Step III.** Finally, according to \((U_C)^n_{ij}\) obtained in the Step II, the linear Crank–Nicolson finite difference scheme on a time fine grid is obtained by

\[
\delta_t (U_F)^1_{ij} - \mu \Delta_h (U_F)^1_{ij} = b^1_{ij} + \frac{1}{2} g((U_F)^0_{ij}) + \frac{1}{2} [g((U_C)^1_{ij}) + g'((U_C)^1_{ij}) ( (U_F)^1_{ij} - (U_C)^1_{ij} )] , \\
(x_i, y_j) \in \Omega_h,
\]  
(35)
\[ \delta_t(U_F)^n_{ij} - \mu \Delta_h(U_F)_{ij}^{n-\frac{1}{2}} - w_{n,1} \Delta_h(U_F)_{ij}^1 - \sum_{p=2}^n w_{n,p} \Delta_h(U_F)_{ij}^{p-\frac{1}{2}} = b_{ij}^n + \frac{1}{2} g((U_F)_{ij}^{n-1}) + \frac{1}{2} \left[ g((U_C)_{ij}^n) + g'((U_C)_{ij}^n) \left( (U_F)_{ij}^n - (U_C)_{ij}^n \right) \right], \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq N. \quad (36) \]

4 Analysis of the two-grid difference scheme

Next, based on the TTGCN finite difference scheme (32)–(36), we will analyze the stability and convergence of the scheme under the regularity assumptions (A1) and (A2).

4.1 Stability

We use the energy method to establish the stability of the TTGCN finite difference scheme. First, consider the case on the coarse grid.

**Theorem 1** The fully discrete scheme (32)–(34) on the coarse grid is stable.

**Proof** Let \((\tilde{U}_C)_{ij}^{sk}\) be the approximation solution of (32)–(33). Thus, we get

\[ \delta_t(\tilde{U}_C)_{ij}^k - \mu \Delta_h(\tilde{U}_C)_{ij}^k - w_{1,1} \Delta_h(\tilde{U}_C)_{ij}^k = b_{ij}^k + \frac{g((\tilde{U}_C)_{ij}^k) + g((\tilde{U}_C)_{ij}^0)}{2}, \quad (x_i, y_j) \in \Omega_h, \quad (37) \]

\[ \delta_t(\tilde{U}_C)_{ij}^{sk} - \mu \Delta_h(\tilde{U}_C)_{ij}^{(s-\frac{1}{2})k} - w_{s,1} \Delta_h(\tilde{U}_C)_{ij}^k - \sum_{p=2}^s w_{s,p} \Delta_h(\tilde{U}_C)_{ij}^{(p-\frac{1}{2})k} = b_{ij}^{sk} + \frac{g((\tilde{U}_C)_{ij}^{sk}) + g((\tilde{U}_C)_{ij}^{(s-1)k})}{2}, \quad 2 \leq s \leq N. \quad (38) \]

Subtracting (37)–(38) from (32)–(33) and defining \(\varepsilon_C = (U_C)_{ij}^{sk} - (\tilde{U}_C)_{ij}^{sk}\), we get

\[ \delta_t(\varepsilon_C)_{ij}^k - \mu \Delta_h(\varepsilon_C)_{ij}^k - w_{1,1} \Delta_h(\varepsilon_C)_{ij}^k = \frac{1}{2} \left[ g((U_C)_{ij}^k) - g((\tilde{U}_C)_{ij}^k) \right] + \frac{1}{2} \left[ g((U_C)_{ij}^0) - g((\tilde{U}_C)_{ij}^0) \right], \quad (x_i, y_j) \in \Omega_h, \quad (39) \]

\[ \delta_t(\varepsilon_C)_{ij}^{sk} - \mu \Delta_h(\varepsilon_C)_{ij}^{(s-\frac{1}{2})k} - w_{s,1} \Delta_h(\varepsilon_C)_{ij}^k - \sum_{p=2}^s w_{s,p} \Delta_h(\varepsilon_C)_{ij}^{(p-\frac{1}{2})k} = \frac{1}{2} \left[ g((U_C)_{ij}^{sk}) - g((\tilde{U}_C)_{ij}^{sk}) \right] + \frac{1}{2} \left[ g((U_C)_{ij}^{(s-1)k}) - g((\tilde{U}_C)_{ij}^{(s-1)k}) \right], \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq s \leq N. \quad (40) \]

We will prove this theorem in two steps as follows:
(I) Taking inner product of both sides of (39) with \( \varepsilon_C^k \) and multiplying it by \( \tau_C \), we yield

\[
\tau_C \left( \delta_x \varepsilon_C^k, \varepsilon_C^k \right) - \tau_C \mu \left( \Delta_h \varepsilon_C^k, \varepsilon_C^k \right) - \tau_C \mathbf{w}_{1,1} \left( \Delta_h \varepsilon_C^k, \varepsilon_C^k \right)
= \frac{\tau_C}{2} \left( g(U_C^k) - g(\tilde{U}_C^k), \varepsilon_C^k \right) + \frac{\tau_C}{2} \left( g(U_C^0) - g(\tilde{U}_C^0), \varepsilon_C^k \right), \quad (x_i, y_j) \in \Omega_h.
\]

(41)

For (40), taking the inner product of both sides with \( \varepsilon_C^{(s-\frac{1}{2})k} \), multiplying it by \( \tau_C \), and summing for \( s \) from 2 to \( N \), we obtain

\[
\sum_{s=2}^{N} \tau_C \left( \delta_x \varepsilon_C^{sk}, \varepsilon_C^{(s-\frac{1}{2})k} \right) - \sum_{s=2}^{N} \mu \tau_C \left( \Delta_h \varepsilon_C^{(s-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k} \right) - \sum_{s=2}^{N} \mathbf{w}_{s,1} \tau_C \left( \Delta_h \varepsilon_C^{(s-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k} \right)
- \sum_{s=2}^{N} \tau_C \sum_{p=2}^{s} \mathbf{w}_{s,p} \left( \Delta_h \varepsilon_C^{(p-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k} \right) = \sum_{s=2}^{N} \tau_C \left( g(U_C^{sk}) - g(\tilde{U}_C^{sk}), \varepsilon_C^{(s-\frac{1}{2})k} \right) + \sum_{s=2}^{N} \tau_C \left( g(U_C^{s-1}k) - g(\tilde{U}_C^{s-1}k), \varepsilon_C^{(s-\frac{1}{2})k} \right), \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq s \leq N.
\]

(42)

Then adding the above two equations together gives

\[
\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 = \frac{\tau_C}{2} \left( g(U_C^k) - g(\tilde{U}_C^k), \varepsilon_C^k \right) + \frac{\tau_C}{2} \left( g(U_C^0) - g(\tilde{U}_C^0), \varepsilon_C^k \right) + \sum_{s=2}^{N} \tau_C \left( g(U_C^{sk}) - g(\tilde{U}_C^{sk}), \varepsilon_C^{(s-\frac{1}{2})k} \right) + \sum_{s=2}^{N} \tau_C \left( g(U_C^{s-1}k) - g(\tilde{U}_C^{s-1}k), \varepsilon_C^{(s-\frac{1}{2})k} \right),
\]

(43)

where

\[
\mathcal{H}_1 = \tau_C \left( \delta_x \varepsilon_C^k, \varepsilon_C^k \right) + \sum_{s=2}^{N} \tau_C \left( \delta_x \varepsilon_C^{sk}, \varepsilon_C^{(s-\frac{1}{2})k} \right).
\]

\[
\mathcal{H}_2 = -\tau_C \mu \left( \Delta_h \varepsilon_C^k, \varepsilon_C^k \right) - \sum_{s=2}^{N} \mu \tau_C \left( \Delta_h \varepsilon_C^{(s-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k} \right),
\]

\[
\mathcal{H}_3 = -\tau_C \mathbf{w}_{1,1} \left( \Delta_h \varepsilon_C^k, \varepsilon_C^k \right) - \sum_{s=2}^{N} \mathbf{w}_{s,1} \tau_C \left( \Delta_h \varepsilon_C^{(s-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k} \right) + \sum_{s=2}^{N} \tau_C \sum_{p=2}^{s} \mathbf{w}_{s,p} \left( \Delta_h \varepsilon_C^{(p-\frac{1}{2})k}, \varepsilon_C^{(s-\frac{1}{2})k} \right).
\]

\( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are consistent, \( \mathcal{H}_3 \) is non-consistent.
Below the terms $\mathcal{H}_q (q = 1, 2, 3)$ will be estimated one by one. First, for $\mathcal{H}_1$, we use Lemma 7 to obtain

$$\mathcal{H}_1 \geq \frac{1}{2} \left( \| \varepsilon_{C}^{Nk} \|^2 - \| \varepsilon_{C}^0 \|^2 \right).$$  \hspace{1cm} (44)$$

Second, from Lemma 5, we obtain

$$\mathcal{H}_2 = \mu C \left( \nabla_h \varepsilon_C^k, \nabla_h \varepsilon_C^k \right) + \sum_{s=2}^{N} \mu \tau_C \left( \nabla_h \varepsilon_C^{(s-\frac{1}{2})k}, \nabla_h \varepsilon_C^{(s-\frac{1}{2})k} \right)$$

$$= \mu \tau_C \| \nabla_h \varepsilon_C^k \|^2 + \sum_{s=2}^{N} \mu \tau_C \| \nabla_h \varepsilon_C^{(s-\frac{1}{2})k} \|^2 \geq 0.$$  \hspace{1cm} (45)

Finally, for the third term $\mathcal{H}_3$, we use Lemma 5 and Lemma 6 to get

$$\mathcal{H}_3 = \tau_C \mathbf{w}_{1,1} \left( \nabla_h \varepsilon_C^k, \nabla_h \varepsilon_C^k \right) + \sum_{s=2}^{N} \mathbf{w}_{s,1} \tau_C \left( \nabla_h \varepsilon_C^k, \nabla_h \varepsilon_C^{(s-\frac{1}{2})k} \right)$$

$$+ \sum_{s=2}^{N} \sum_{p=2}^{s} \tau_C \mathbf{w}_{s,p} \left( \nabla_h \varepsilon_C^{(p-\frac{1}{2})k}, \nabla_h \varepsilon_C^{(s-\frac{1}{2})k} \right)$$

$$= \tau_C \mathbf{w}_{1,1} \left( \nabla_h \varepsilon_C^k, \nabla_h \varepsilon_C^k \right) + \sum_{s=2}^{N} \tau_C \mathbf{w}_{s,1} \nabla_h \varepsilon_C^k + \sum_{p=2}^{s} \tau_C \mathbf{w}_{s,p} \nabla_h \varepsilon_C^{(p-\frac{1}{2})k}, \nabla_h \varepsilon_C^{(s-\frac{1}{2})k} \right)$$

$$= \tau_C \mathbf{w}_{1,1} \left( \nabla_h \varepsilon_C^k, \nabla_h \varepsilon_C^k \right) + \sum_{s=2}^{N} \tau_C \left( \mathbf{w}_{2,s} \nabla_h \varepsilon_C^{sk}, \nabla_h \varepsilon_C^{(s-\frac{1}{2})k} \right) \geq 0.$$  \hspace{1cm} (46)

Next, $g(u)$ satisfies the Lipschitz condition. For (43), using Cauchy-Schwarz inequality, we have

$$\| \varepsilon_{C}^{Nk} \|^2 - \| \varepsilon_{C}^0 \|^2$$

$$\leq \tau_C \left\| g(U_C^k) - g(\tilde{U}_C^k) \right\| \| \varepsilon_{C}^k \| + \tau_C \left\| g(U_C^0) - g(\tilde{U}_C^0) \right\| \| \varepsilon_{C}^0 \|$$

$$+ \sum_{s=2}^{N} \tau_C \left\| g(U_C^{sk}) - g(\tilde{U}_C^{sk}) \right\| \| \varepsilon_{C}^{(s-\frac{1}{2})k} \|$$

$$+ \sum_{s=2}^{N} \tau_C \left\| g(U_C^{(s-1)k}) - g(\tilde{U}_C^{(s-1)k}) \right\| \| \varepsilon_{C}^{(s-\frac{1}{2})k} \|$$

$$\leq \tilde{C} \tau_C \left( \| \varepsilon_{C}^k \|^2 + \| \varepsilon_{C}^0 \| \| \varepsilon_{C}^k \| + \sum_{s=2}^{N} \| \varepsilon_{C}^{sk} \| \| \varepsilon_{C}^{(s-\frac{1}{2})k} \| + \sum_{s=2}^{N} \| \varepsilon_{C}^{(s-1)k} \| \| \varepsilon_{C}^{(s-\frac{1}{2})k} \| \right).$$  \hspace{1cm} (47)
Now, taking the positive integer \( \tilde{m} \) such that \( \| e^{\tilde{m}k} \| = \max_{0 \leq s \leq N} \| e^{sk} \| \), we have

\[
\| e^{Nk} \| \leq \| e^{\tilde{m}k} \| \leq \| e^0 \| + \tilde{C} \tau C \left( \| e^k \| + \sum_{s=2}^{\tilde{m}} \| e^{sk} \| + \sum_{s=2}^{\tilde{m}} \| e^{(s-1)k} \| \right)
\]

\[
\leq \| e^0 \| + \tilde{C} \tau C \left( \| e^k \| + \sum_{s=2}^{N} \| e^{sk} \| + \sum_{s=2}^{N} \| e^{(s-1)k} \| \right)
\]

\[
\leq \| e^0 \| + \tilde{C} \tau C \left( \sum_{s=0}^{N} \| e^{sk} \| + \sum_{s=1}^{N-1} \| e^{sk} \| \right)
\]

\[
\leq \| e^0 \| + \tilde{C} \tau C \left( \| e^N \| + 2\tilde{C} \tau C \sum_{s=0}^{N-1} \| e^{sk} \| \right).
\]

When \( \tau C \leq \frac{1}{2\tilde{C}} \), following from Lemma 8, inequality (48) becomes

\[
\| e^{Nk} \| \leq \tilde{C} (T) \| e^0 \| \exp\{N \tau C\} \leq \tilde{C} \| e^0 \|.
\]

(II) Notice that according to (I), we have \( \| U^{sk} \| \leq \tilde{C} \) for any \( 1 \leq s \leq N \). Then we estimate the \( \| U^{s-1,k+q}_C \| \) for \( 1 \leq s \leq N \) and \( 1 \leq q \leq k - 1 \). Considering (34) and applying the triangle inequality, we obtain

\[
\| U^{s-1,k+q}_C \| = \| (1 - \frac{q}{k})U^{s-1}_C + \frac{q}{k} U^{sk} \| \leq (1 - \frac{q}{k}) \| U^{s-1}_C \| + \frac{q}{k} \| U^{sk} \| \leq \tilde{C},
\]

which completes the proof. \( \square \)

In addition, we shall analyse the stability on the fine grid.

**Theorem 2** For the system (35) and (36) on the fine grid, with \( 1 \leq n \leq N \), we have \( \| U^n_F \| \leq \bar{C} \).

**Proof** Taking the inner product of (35) with \( \tau F U^1_F \), we have

\[
\tau F \left( \delta, U^1_F, U^1_F \right) - \mu \tau F \left( \Delta_h U^1_F, U^1_F \right) - \tau F \omega_{1,1} \left( \Delta_h U^1_F, U^1_F \right) = \tau F \left( b^1, U^1_F \right) + \frac{\tau F}{2} \left( g(U^0_F), U^1_F \right) + \frac{\tau F}{2} \left( g(U^1_C) + g'(U^1_C) \right) \left( U^1_F - U^1_C \right). \]

\( \square \) Springer
For (36), taking the inner product of both sides with $U_F^{n-\frac{1}{2}}$, multiplying it by $\tau_F$, and summing for $n$ from 2 to $N$, we get

$$
\sum_{n=2}^{N} \tau_F \left( \delta_l U_F^n, U_F^{n-\frac{1}{2}} \right) - \sum_{n=2}^{N} \tau_F \mu \left( \Delta_h U_F^{n-\frac{1}{2}}, U_F^{n-\frac{1}{2}} \right) - \sum_{n=2}^{N} \tau_F w_{n,1} \left( \Delta_h U_F^{n-1}, U_F^{n-\frac{1}{2}} \right) \\
- \sum_{n=2}^{N} \tau_F \sum_{p=2}^{n} w_{n,p} \left( \Delta_h U_F^{p-\frac{1}{2}}, U_F^{p-\frac{1}{2}} \right) = \sum_{n=2}^{N} \tau_F \left( b_n, U_F^{n-\frac{1}{2}} \right) + \sum_{n=2}^{N} \frac{\tau_F}{2} \left( g(U_F^{n-1}), U_F^{n-\frac{1}{2}} \right) \\
+ \sum_{n=2}^{N} \frac{\tau_F}{2} \left( g(U_C^n) + g'(U_C^n) (U_F^n - U_C^n), U_F^{n-\frac{1}{2}} \right), \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq N.
$$

(52)

Then, adding (51) and (52), and similar to the analysis of (42)–(46), we obtain

$$
\| U_F^N \|^2 - \| U_F^0 \|^2 \\
\leq 2\tau_F \| b_1 \| \| U_F^1 \| + 2 \sum_{n=2}^{N} \tau_F \| b_n \| \| U_F^{n-\frac{1}{2}} \| + \tau_F \| g(U_F^0) \| \| U_F^1 \| \\
+ \sum_{n=2}^{N} \tau_F \| g(U_F^{n-1}) \| \| U_F^{n-\frac{1}{2}} \| \\
+ \sum_{n=2}^{N} \tau_F \| g(U_C^n) \| \| U_F^1 \| + \tau_F \| g'(U_C^n) (U_F^n - U_C^n) \| \| U_F^n \| \\
+ \sum_{n=2}^{N} \tau_F \| g'(U_C^n) (U_F^n - U_C^n) \| \| U_F^{n-\frac{1}{2}} \|.
$$

(53)

Based on the stability of the coarse grid, $\| U_C^n \| \leq \tilde{C} (0 \leq n \leq N)$ can be obtained. Then according to $g(u) \in C^2(\mathbb{R}) \cap L^1(0, T)$, we have $g(U_C^n) \leq \tilde{C}$ and $g'(U_C^n) \leq \tilde{C}$. Also, assuming $\| U_F^n \| \leq \tilde{C}$ holds for $0 \leq n \leq N - 1$, then $g(U_F^n) \leq \tilde{C}$ can be obtained, thus

$$
\| U_F^N \|^2 - \| U_F^0 \|^2 \leq 2\tau_F \| b_1 \| \| U_F^1 \| + 2 \sum_{n=2}^{N} \tau_F \| b_n \| \| U_F^{n-\frac{1}{2}} \| + \tilde{C} \tau_F \| U_F^1 \| \\
+ \sum_{n=2}^{N} \tau_F \| U_F^{n-\frac{1}{2}} \| + \tilde{C} \tau_F \left( \| U_F^1 \| + \| U_C^1 \| \right) \| U_F^1 \| \\
+ \sum_{n=2}^{N} \tau_F \left( \| U_C^n \| + \| U_C^n \| \right) \| U_F^{n-\frac{1}{2}} \|.
$$

(54)
Denoting $\|U^n_F\| = \max_{0 \leq n \leq N} \|U^n_F\|$, we can get

$$
\|U^n_F\|^2 \leq \|U^0_F\|^2 + 2\tau_F b^1 \|U^1_F\| + 2 \sum_{n=2}^{\tilde{m}} \tau_F b^n \left\| U^{n-\frac{1}{2}}_F \right\| + \tilde{C} \tau_F \|U^1_F\|^2 + \tilde{C} \tau_F \left\| U^{n-\frac{1}{2}}_F \right\| \\
+ \tilde{C} \tau_F \left( \|U^1_F\| + \|U^1_C\| \right) \left\| U^{n-\frac{1}{2}}_F \right\| + \tilde{C} \sum_{n=2}^{\tilde{m}} \tau_F \left( \|U^n_F\| + \|U^n_C\| \right) \left\| U^{n-\frac{1}{2}}_F \right\|
$$

$$
\leq \|U^0_F\|^2 \|U^n_F\| + 2\tau_F b^1 \|U^0_F\| + 2 \sum_{n=2}^{\tilde{m}} \tau_F b^n \|U^n_F\| + \tilde{C} \tau_F \|U^n_F\|^2 + \tilde{C} \sum_{n=2}^{\tilde{m}} \tau_F \|U^n_F\|
$$

Then

$$
\|U^n_F\| \leq \|U^n_F\| \leq \|U^0_F\| + 2 \sum_{n=1}^{\tilde{N}} \tau_F b^n + \tilde{C} \sum_{n=1}^{\tilde{N}} \tau_F \left( \|U^n_F\| + \|U^n_C\| \right) \\
\leq \|U^0_F\| + 2 \sum_{n=1}^{\tilde{N}} \tau_F b^n + \tilde{C} \sum_{n=1}^{\tilde{N}} \tau_F \left( \|U^n_F\| + \|U^n_C\| \right). 
$$

When $\tau_F \leq \frac{1}{4\tilde{C}}$, from Lemma 8 and Theorem 1, inequality (56) turn into the following

$$
\|U^n_F\| \leq \tilde{C} (T) \exp(\tilde{N} \tau_F) \left( \|U^0_F\| + \sum_{n=1}^{\tilde{N}} \tau_F b^n + \sum_{n=1}^{\tilde{N}} \tau_F \|U^n_F\| + \sum_{n=1}^{\tilde{N}} \tau_F \|U^n_C\| \right) \leq \tilde{C}. 
$$

This finishes the proof. \qed

### 4.2 Convergence

The convergence of TTGCN finite difference scheme (32)–(34) on coarse grid will be analysis using the energy method. Let

$$(e_C)^n_{ij} = u^n_{ij} - (U_C)^n_{ij}, \quad (x_i, y_j) \in \tilde{\Omega}_h, \quad 0 \leq n \leq \tilde{N}.$$  

Subtracting (32)–(33), (30)–(31) from (24)–(27), respectively, we obtain the following error equations

$$
\delta_t (e_C)^k_{ij} - \mu \Delta_h (e_C)^k_{ij} - w_{1.1} \Delta_h (e_C)^k_{ij} = \frac{1}{2} \left[ g(u^k_{ij}) - g((U_C)^k_{ij}) \right] \\
+ \frac{1}{2} \left[ g(u^0_{ij}) - g((U_C)^0_{ij}) \right] + (R)^k_{ij}, \quad (x_i, y_j) \in \Omega_h. 
$$
\[ \delta_t (ec)^{\tau}_i^k - \mu \Delta_h (ec)^{(s-\frac{1}{2})}_i^k - w_{s,1} \Delta_h (ec)^{k}_i^j - \sum_{p=2}^s w_{s,p} \Delta_h (ec)^{(p-\frac{1}{2})}_i^j = \frac{1}{2} \left[ g(u^k_{ij}) - g((U_C)^{sk}_{ij}) \right] + \frac{1}{2} \left[ g(u^{(s-1)k}_{ij}) - g((U_C)^{(s-1)k}_{ij}) \right] + (R)^{sk}_{ij}, \] (59)

\((x_i, y_j) \in \Omega_h, 2 \leq s \leq N,
(ec)^n_{ij} = 0, (x_i, y_j) \in \partial \Omega_h, 1 \leq n \leq N, \)
\((ec)^0_{ij} = 0, (x_i, y_j) \in \Omega_h, \)

where \((R) = (R1) + (R2) + (R3) + (R4).\)

**Theorem 3** Assume that \(u(x, y, t)\) and \(U^n_C\) are solutions of (24)–(25) and (32)–(33), respectively, and that \(u(x, y, t)\) satisfies the regularity assumptions (A1) and (A2). Then, it holds that

\[ \max_{1 \leq n \leq N} \|u^n - U^n_C\| \leq \tilde{C} (\tau_C^2 + h_1^2 + h_2^2), 1 \leq n \leq N. \]

**Proof** The proof of this theorem is divided into two steps:

**I.** Taking the inner product of equations (58) and (59) with \(e^k_C\) and \(e^{(s-\frac{1}{2})k}_C\) respectively, and multiplying both equations by \(\tau_C\), summing for \(s\) from 2 to \(N\) in (59) and adding (58), then we can obtain

\[ \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 = \frac{\tau_C}{2} \left( g(u^k) - g(U^k_C), e^k_C \right) + \frac{\tau_C}{2} \left( g(u^0) - g(U^0_C), e^k_C \right) \\
+ \sum_{s=2}^N \frac{\tau_C}{2} \left( g(u^{sk}) - g(U^{sk}_C), e^{(s-\frac{1}{2})k}_C \right) + \sum_{s=2}^N \frac{\tau_C}{2} \left( g(u^{(s-1)k}) - g(U^{(s-1)k}_C), e^{(s-\frac{1}{2})k}_C \right) \\
+ \tau_C \left( (R)^k_C, e^k_C \right) + \sum_{s=2}^N \tau_C \left( (R)^{sk}_C, e^{(s-\frac{1}{2})k}_C \right), \]
(62)

where

\[ \tilde{H}_1 = \tau_C \left( \delta_t e^k_C, e^k_C \right) + \sum_{s=2}^N \tau_C \left( \delta_t e^{sk}_C, e^{(s-\frac{1}{2})k}_C \right), \]
\[ \tilde{H}_1 = -\tau_C \mu \left( \Delta_h e^k_C, e^k_C \right) - \sum_{s=2}^N \mu \tau_C \left( \Delta_h e^{(s-\frac{1}{2})k}_C, e^{(s-\frac{1}{2})k}_C \right), \]
\[ \tilde{H}_3 = -\tau_C w_{1,1} \left( \Delta_h e^k_C, e^k_C \right) - \sum_{s=2}^N w_{s,1} \tau_C \left( \Delta_h e^{(s-\frac{1}{2})k}_C \right) \\
- \sum_{s=2}^N \tau_C \sum_{p=2}^s w_{s,p} \left( \Delta_h e^{(p-\frac{1}{2})k}_C, e^{(s-\frac{1}{2})k}_C \right). \]
For (62), applying Lemmas 5–7 and Cauchy-Schwarz inequality, we get the following inequality
\[
\|e_C^{Nk}\|^2 - \|e_C^0\|^2 \\
\leq \tau_C \|g(u^k) - g(U_C^k)\|\|e_C^k\| + \tau_C \|g(u^0) - g(U_C^0)\|\|e_C^k\| \\
+ \sum_{s=2}^N \tau_C \|g(u^{sk}) - g(U_C^{sk})\|\|e_C^{(s-1)k}\| + \sum_{s=2}^N \tau_C \|g(u^{(s-1)k}) - g(U_C^{(s-1)k})\|\|e_C^{(s-1)k}\| \\
+ 2\tau_C \|(R)^k\|\|e_C^k\| + 2 \sum_{s=2}^N \tau_C \|(R)^{sk}\|\|e_C^{(s-1)k}\| \\
\leq \tilde{C} \tau_C \left( \|e_C^k\|^2 + \|e_C^0\|\|e_C^k\| + \sum_{s=2}^N \|e_C^{sk}\|\|e_C^{(s-1)k}\| + \sum_{s=2}^N \|e_C^{(s-1)k}\|\|e_C^{(s-1)k}\| \right) \\
+ 2\tau_C \|(R)^k\|\|e_C^k\| + 2 \sum_{s=2}^N \tau_C \|(R)^{sk}\|\|e_C^{(s-1)k}\|. 
\]

Choosing a positive integer \(\tilde{s}\) such that \(\|e_C^{\tilde{s}k}\| = \max_{0 \leq s \leq N} \|e_C^{sk}\|\) and noting that (60), then we have
\[
\|e_C^{Nk}\| \leq \|e_C^{\tilde{s}k}\| \leq \tilde{C} \tau_C \left( \|e_C^k\|^2 + \sum_{s=2}^{\tilde{s}} \|e_C^{sk}\| + \sum_{s=2}^{\tilde{s}} \|e_C^{(s-1)k}\| \right) + 2 \sum_{s=1}^{\tilde{s}} \tau_C \|(R)^{sk}\| \\
\leq \tilde{C} \tau_C \left( \|e_C^k\|^2 + \sum_{s=2}^N \|e_C^{sk}\| + \sum_{s=2}^N \|e_C^{(s-1)k}\| \right) + 2 \sum_{s=1}^N \tau_C \|(R)^{sk}\| \\
\leq \tilde{C} \tau_C \left( \sum_{s=1}^N \|e_C^{sk}\| + \sum_{s=1}^N \|(R)^{sk}\| \right). 
\]

Using Lemma 8, then (64) becomes the following
\[
\|e_C^{Nk}\| \leq \tilde{C}(T) \exp(N\tau_C) \left( \tau_C \sum_{s=1}^N \|(R)^{sk}\| \right). 
\]

In addition, from Lemmas 1–4 and using triangle inequality, we can get the following estimates
\[
\tau_C \sum_{s=1}^N \|(R)^{sk}\| = \tau_C \sum_{s=1}^N \|(R1)^{sk}\| + \|(R2)^{sk}\| + \|(R3)^{sk}\| + \|(R4)^{sk}\| \\
\leq \tau_C \sum_{s=1}^N \left( \|(R1)^{sk}\| + \|(R2)^{sk}\| + \|(R3)^{sk}\| + \|(R4)^{sk}\| \right) \\
\leq \tilde{C}(T)(\tau_C^2 + h_1^2 + h_2^2). 
\]
Finally, combining (65) and (66), we have

$$\|e^{(s-k)}_C\| \leq \bar{C}(T)(\tau_C^2 + h_1^2 + h_2^2), \quad 1 \leq s \leq N.$$  \hspace{1cm} (67)

(II). For any $1 \leq s \leq N$ and $1 \leq q \leq k - 1$, we utilize the Lagrange’s interpolation formula, then

$$u^{(s-k)q} = \left(1 - \frac{q}{k}\right)u^{(s-k)} + \frac{q}{k}u^{sk} + \frac{u''(\xi)}{2}(t^{(s-k)q} - t^{(s-k)})(t^{(s-k)q} - t^{sk}), \quad \xi \in (t^{(s-k)} , t^{sk}).$$  \hspace{1cm} (68)

Subtracting (34) from (68), we have

$$e^{(s-k)q}_C = \left(1 - \frac{q}{k}\right)e^{(s-k)}_C + \frac{q}{k}e^{sk}_C + \frac{u''(\xi)}{2}(t^{(s-k)q} - t^{(s-k)})(t^{(s-k)q} - t^{sk}),$$

then, applying the triangle inequality and (67), we obtain

$$\|e^{(s-k)q}_C\| \leq \left(1 - \frac{q}{k}\right)\|e^{(s-k)}_C\| + \frac{q}{k}\|e^{sk}_C\| + \frac{\|u''(\xi)\|_{\infty}}{2} \tau_C^2$$  \hspace{1cm} (69)

$$\leq \bar{C}(\tau_C^2 + h_1^2 + h_2^2), \quad 1 \leq s \leq N, \quad 1 \leq q \leq k - 1.$$

The proof is finished. \hfill \Box

Next, the convergence on the fine grid will be considered. Let

$$(e_F)_{ij}^n = u_{ij}^n - (U_F)_{ij}^n, \quad (x_i, y_j) \in \tilde{\Omega}_h, \quad 0 \leq n \leq N.$$

Subtracting (35)–(36), (30)–(31) from (24)–(27), respectively, we yield the following error equations

$$\delta_t (e_F)_{ij}^{1} - \mu \Delta_h (e_F)_{ij}^{1} - w_{1,1} \Delta_h (e_F)_{ij}^{1} = \frac{1}{2} \left[ g(u_{ij}^0) - g((U_F)_{ij}^0) \right]$$

$$+ \frac{1}{2} \left[ g(u_{ij}^1) - g((U_C)_{ij}^1) - g'((U_C)_{ij}^1) \left((U_F)_{ij}^1 - (U_C)_{ij}^1\right) \right] \quad \hspace{1cm} (70)$$

$$\delta_t (e_F)_{ij}^{n} - \mu \Delta_h (e_F)_{ij}^{n} - w_{1,1} \Delta_h (e_F)_{ij}^{n} - \sum_{p=2}^n w_{n,p} \Delta_h (e_F)_{ij}^{n-p}$$

$$= \frac{1}{2} \left[ g(u_{ij}^{n-1}) - g((U_F)_{ij}^{n-1}) \right]$$

$$+ \frac{1}{2} \left[ g(u_{ij}^n) - g((U_C)_{ij}^n) - g'((U_C)_{ij}^n) \left((U_F)_{ij}^n - (U_C)_{ij}^n\right) \right]$$

$$+ (R)_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \quad 2 \leq n \leq N,$$

$$(e_F)_{ij}^{n} = 0, \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N,$$  \hspace{1cm} (72)
\[(e_F)_{ij}^0 = 0, \quad (x_i, y_j) \in \Omega_h. \quad (73)\]

**Theorem 4** Assume that \(u(x, y, t)\) and \(U^n_F\) are solutions of (24)–(25) and (35)–(36), respectively, and let \(u(x, y, t)\) satisfy the regularity assumption (A1) and (A2), then we have the following

\[
\|e^n_F\| \leq \bar{C}(\tau_F^2 + \tau_C^4 + h_1^2 + h_2^2), \quad 1 \leq n \leq N.
\]

**Proof** Taking the inner product of (70) with \(\tau_F e_F^1\), we obtain

\[
\tau_F \left( \delta_t e_F^1, e_F^1 \right) - \mu \tau_F \left( \Delta_h e_F^1, e_F^1 \right) - \mathbf{w}_{1,1} \tau_F \left( \Delta_h e_F^1, e_F^1 \right) = \frac{\tau_F}{2} \left( g(u^0) - g(U^0_F), e_F^1 \right) + \frac{\tau_F}{2} \left( g(u^1) - g(U^1_C) - g'(U^1_C) \left( U^1_F - U^1_C \right), e_F^1 \right) + \tau_F \left( (R)^1, e_F^1 \right).
\]

Then taking the inner product of equation (71) with \(\tau_F e_F^{n-\frac{1}{2}}\) and summing for \(n\) from 2 to \(N\), we can get

\[
\sum_{n=2}^{N} \tau_F \left( \delta_t e_F^{n-\frac{1}{2}}, e_F^{n-\frac{1}{2}} \right) - \sum_{n=2}^{N} \mu \tau_F \left( \Delta_h e_F^{n-\frac{1}{2}}, e_F^{n-\frac{1}{2}} \right) - \sum_{n=2}^{N} \mathbf{w}_{n,1} \tau_F \left( \Delta_h e_F^{n-\frac{1}{2}}, e_F^{n-\frac{1}{2}} \right) = \frac{\tau_F}{2} \sum_{n=2}^{N} \left( g(u^{n-1}) - g(U^{n-1}_F), e_F^{n-\frac{1}{2}} \right) + \frac{\tau_F}{2} \sum_{n=2}^{N} \left( (R)^{n-1}, e_F^{n-\frac{1}{2}} \right).
\]

Adding (74) and (75), then using Lemmas 5–7, Cauchy–Schwarz inequality and triangle inequality, and noting (72), we can get

\[
\|e_F^N\|^2 \leq \tau_F \|g(u^1) - g(U^1_C) - g'(U^1_C) \left( U^1_F - U^1_C \right) \| \|e_F^1\| + 2\tau_F \|R^1\| \|e_F^1\|
\]

\[
+ \tau_F \sum_{n=2}^{N} \|g(u^n) - g(U^n_C) - g'(U^n_C) \left( U^n_F - U^n_C \right) \| e_F^{n-\frac{1}{2}} \|,
\]

\[
+ \bar{C} \tau_F \sum_{n=2}^{N} \|e_F^{n-1}\| \|e_F^{n-\frac{1}{2}}\| + 2 \sum_{n=2}^{N} \tau_F \|(R)^n\| \|e_F^{n-\frac{1}{2}}\|.
\]
Choosing a suitable $\hat{s}$ such that $\|e_{\hat{s}}^{N}\| = \max_{0 \leq n \leq N} \|e_{\hat{s}}^{n}\|$, then it holds

$$
\|e_{F}^{N}\| \leq \|e_{\hat{s}}^{N}\| \leq \tau_{F} \sum_{n=1}^{N} \|g(u^{n}) - g(U_{C}^{n}) - g'(U_{C}^{n})(U_{F}^{n} - U_{C}^{n})\|
+C\tau_{F} \sum_{n=2}^{N} \|e_{F}^{n-1}\| + 2 \sum_{n=1}^{N} \tau_{F} \|(R)^{n}\|.
$$

(77)

According to Taylor expansion, we have

$$
g(u^{n}) - g(U_{C}^{n}) - g'(U_{C}^{n})(u^{n} - U_{C}^{n})
= g'(U_{C}^{n})e_{F}^{n} + \frac{1}{2} g''(\theta^{n})(e_{F}^{n})^{2}, \quad \theta^{n} \in (\min\{u^{n}, U_{C}^{n}\}, \max\{u^{n}, U_{C}^{n}\}).
$$

(78)

Substituting (78) into (77) and applying the triangle inequality, we can get

$$
\|e_{F}^{N}\| \leq C\tau_{F} \sum_{n=1}^{N} \|e_{F}^{n}\| + C\tau_{F} \sum_{n=2}^{N} \|e_{F}^{n-1}\| + 2 \sum_{n=1}^{N} \tau_{F} \|(R)^{n}\|
\leq C\tau_{F} \sum_{n=1}^{N} \|e_{F}^{n}\| + C\tau_{F} \sum_{n=1}^{N} \|e_{C}^{n}\|^{2} + 2 \sum_{n=1}^{N} \tau_{F} \|(R)^{n}\|.
$$

(79)

Utilizing Lemma 8 and Theorem 3, we yield

$$
\|e_{F}^{N}\| \leq C \exp\{N\tau_{F}\} \left( \tau_{F} \sum_{n=1}^{N} \|e_{C}^{n}\|^{2} + \sum_{n=1}^{N} \tau_{F} \|(R)^{n}\| \right)
\leq C \left( \tau_{F}^{4} + \tau_{F}^{2} + h_{1}^{2} + h_{2}^{2} \right),
$$

(80)

which completes the proof. $\square$

5 Numerical experiment

In this section, we will use the TTGCN finite difference scheme (32)–(36) to solve problem (1)–(2) and apply the method to three test problems. In order to verify the validity of the method, we also compare the results obtained from proposed scheme with the existing methods, e.g., the SCN finite difference scheme (28)–(31) and the scheme [32]. We set $L_{x} = L_{y} = 1$ and $T = 1$. All experiments are performed on a Windows 11 (64 bit) PC-Inter(R) Core(TM) i5-12500H CPU 3.10 GHz, 16.0 GB of RAM using MTALAB R2021b.
The discrete $L^2$-norm error is defined as follows

$$E_{TTGCN}(h, \tau) = \max_{1 \leq n \leq N} \| u^n - U^n_F \|,$$

and the time-space convergence orders are defined by

$$rate^t_{TTGCN} = \log_2 \left( \frac{E_{TTGCN}(h, 2\tau)}{E_{TTGCN}(h, \tau)} \right),$$

$$rate^x_{TTGCN} = \log_2 \left( \frac{E_{TTGCN}(2h, \tau)}{E_{TTGCN}(h, \tau)} \right).$$

In addition, we can similarly define $E_{SCN}(h, \tau)$, $rate^t_{SCN}$ and $rate^x_{SCN}$.

**Example 1** We consider the nonlinear term is given by $g(u) = -u^2$, $\mu = 1$ and the inhomogeneous term is

$$f(x, y, t) = \left( 1 + 2\pi \right) \frac{t^\alpha}{\Gamma(\alpha + 1)} + 2\pi^2 \left( 1 + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) \sin \pi x \sin \pi y + \left( 1 + \frac{t^{\alpha+1}}{\Gamma(2 + \alpha)} \sin \pi x \sin \pi y \right)^2.$$

The exact solution of this problem is presented as follows

$$u(x, y, t) = \left( 1 + \frac{t^{\alpha+1}}{\Gamma(2 + \alpha)} \right) \sin \pi x \sin \pi y.$$

In Table 1, we obtain the corresponding discrete $L^2$-norm errors, time convergence order and CPU time by calculating Example 1 with the TTGCN finite difference scheme (32)–(36) and the SCN finite difference method (28)–(31). The numerical results show that the convergence order of the two schemes converges to 2 in the time direction, which is consistent with the theoretical analysis. Meanwhile, we compare the numerical results of the two methods in terms of temporal convergence order and computational cost (CPU time in seconds), and see that the TTGCN finite difference scheme can save computational cost significantly without losing computational accuracy.

Additionally, the results in Table 2 can verify that the TTGCN finite difference scheme will save more computational cost than the SCN finite difference scheme, especially when the value of $k$ increases.

When the time step $\tau_C = 1/128$ and $\tau_F = 1/512$ are fixed, it can be seen in Table 3, that the convergence order of the two schemes in space is 2. Therefore, the convergence results in the space-time direction are in good agreement with the theoretical analysis.

Figure 1 compares the computation time of the two-grid method and the standard method in the time direction for the Crank–Nicolson finite difference scheme. It can be observed that the computational cost of the TTGCN finite difference method is lower.
A two-grid temporal second-order scheme...

Table 1 The $L^2$-errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $k = 4$ for Example 1

| $\alpha$ | $\tau_C$ | $\tau_F$ | $E_{TTGCN}$ | $rate^I_{TTGCN}$ | $CPU$ (s) | $E_{SCN}$ | $rate^I_{SCN}$ | $CPU$ (s) |
|----------|----------|----------|--------------|------------------|-----------|-----------|----------------|-----------|
| 0.25     | 1/2      | 1/8      | 2.9293e−2    | *                | 41.42     | 2.9294e−2 | *              | 83.85     |
|          | 1/4      | 1/16     | 9.9431e−3    | 1.5588           | 75.53     | 9.9431e−3 | 1.5588         | 159.79    |
|          | 1/8      | 1/32     | 2.9743e−3    | 1.7412           | 176.76    | 2.9743e−3 | 1.7412         | 307.44    |
|          | 1/16     | 1/64     | 7.7382e−4    | 1.9425           | 439.63    | 7.7382e−4 | 1.9425         | 696.57    |
| 0.5      | 1/2      | 1/8      | 1.5390e−2    | *                | 35.26     | 1.5391e−2 | *              | 83.48     |
|          | 1/4      | 1/16     | 4.2211e−3    | 1.8663           | 77.16     | 4.2212e−3 | 1.8664         | 160.76    |
|          | 1/8      | 1/32     | 1.0102e−3    | 2.0630           | 177.43    | 1.0102e−3 | 2.0630         | 304.59    |
|          | 1/16     | 1/64     | 2.0588e−4    | 2.2948           | 441.57    | 2.0589e−4 | 2.2948         | 700.46    |
| 0.75     | 1/2      | 1/8      | 7.7023e−2    | *                | 35.64     | 7.7034e−2 | *              | 83.19     |
|          | 1/4      | 1/16     | 1.7363e−3    | 2.1493           | 77.64     | 1.7364e−3 | 2.1494         | 159.14    |
|          | 1/8      | 1/32     | 3.3266e−4    | 2.3839           | 176.50    | 3.3266e−4 | 2.3840         | 309.35    |
|          | 1/16     | 1/64     | 9.3963e−5    | 1.8239           | 414.11    | 9.3963e−5 | 1.8239         | 681.68    |

Table 2 The $L^2$-errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $\alpha = 0.5$ for Example 1

| $k$ | $\tau_C$ | $\tau_F$ | $E_{TTGCN}$ | $rate^I_{TTGCN}$ | $CPU$ (s) | $E_{SCN}$ | $rate^I_{SCN}$ | $CPU$ (s) |
|-----|----------|----------|--------------|------------------|-----------|-----------|----------------|-----------|
| 2   | 1/3      | 1/6      | 2.5490e−2    | *                | 40.93     | 2.5490e−2 | *              | 61.24     |
|     | 1/6      | 1/12     | 7.3299e−3    | 1.7981           | 83.19     | 7.3299e−3 | 1.7981         | 120.88    |
|     | 1/12     | 1/24     | 1.8622e−3    | 1.9767           | 181.48    | 1.8623e−3 | 1.9767         | 232.67    |
|     | 1/24     | 1/48     | 4.0706e−4    | 2.1937           | 391.92    | 4.0706e−4 | 2.1937         | 474.09    |
| 3   | 1/2      | 1/6      | 2.5489e−2    | *                | 31.25     | 2.5490e−2 | *              | 61.91     |
|     | 1/4      | 1/12     | 7.3297e−3    | 1.7980           | 65.04     | 7.3299e−3 | 1.7981         | 120.44    |
|     | 1/8      | 1/24     | 1.8622e−3    | 1.9767           | 142.96    | 1.8623e−3 | 1.9767         | 232.99    |
|     | 1/16     | 1/48     | 4.0706e−4    | 2.1937           | 320.59    | 4.0706e−4 | 2.1937         | 479.59    |
| 5   | 1/2      | 1/10     | 1.0281e−2    | *                | 40.04     | 1.0282e−2 | *              | 107.69    |
|     | 1/4      | 1/20     | 2.7071e−3    | 1.9252           | 89.51     | 2.7071e−3 | 1.9253         | 198.24    |
|     | 1/8      | 1/40     | 6.1692e−4    | 2.1336           | 214.16    | 6.1693e−4 | 2.1336         | 389.39    |
|     | 1/16     | 1/80     | 1.1915e−4    | 2.3723           | 531.10    | 1.1915e−4 | 2.3724         | 913.59    |

without losing the accuracy. Also, Fig. 2 gives the $L^2$-norm error for both methods, which can show intuitively second-order convergence for time.

Example 2 we consider $g(u) = -u - u^3$ and $\mu = 1$. The exact solution is given via

$$u(x, y, t) = \frac{t^{\alpha + 1}}{\Gamma(2 + \alpha)} \sin \pi x \sin \pi y.$$
Table 3 The \(L^2\)-errors and convergence rates with \(\tau_C = 1/128\) and \(\tau_F = 1/512\) for Example 1

| \(\alpha\) | \(h\)  | \(E_{TTGCN}\) | rate\%_{TTGCN} | \(E_{SCN}\)  | rate\%_{SCN} |
|------------|-------|----------------|----------------|---------------|---------------|
| 0.20       | 1/2   | 3.8785e−1     | 3.8785e−1     | 3.8785e−1     | −             |
|            | 1/4   | 9.1992e−2     | 2.0759         | 9.1992e−2     | 2.0759         |
|            | 1/8   | 2.2621e−2     | 2.0239         | 2.2621e−2     | 2.0239         |
|            | 1/16  | 5.6309e−3     | 2.0062         | 5.6309e−3     | 2.0062         |
|            | 1/32  | 1.4061e−3     | 2.0017         | 1.4061e−3     | 2.0017         |
| 0.50       | 1/2   | 3.5774e−1     | 3.5774e−1     | 3.5774e−1     | *             |
|            | 1/4   | 8.4731e−2     | 2.0780         | 8.4731e−2     | 2.0780         |
|            | 1/8   | 2.0830e−2     | 2.0242         | 2.0830e−2     | 2.0242         |
|            | 1/16  | 5.1848e−3     | 2.0063         | 5.1848e−3     | 2.0063         |
|            | 1/32  | 1.2945e−3     | 2.0018         | 1.2945e−3     | 2.0018         |
| 0.80       | 1/2   | 3.2854e−1     | 3.2854e−1     | 3.2854e−1     | *             |
|            | 1/4   | 7.7643e−2     | 2.0811         | 7.7643e−2     | 2.0811         |
|            | 1/8   | 1.9081e−2     | 2.0248         | 1.9081e−2     | 2.0248         |
|            | 1/16  | 4.7487e−3     | 2.0065         | 4.7487e−3     | 2.0065         |
|            | 1/32  | 1.1855e−3     | 2.0020         | 1.1855e−3     | 2.0020         |

Fig. 1 The comparison of two methods for CPU time with \(h = 1/100\) and \(k = 4\) for Example 1

thus, \(\psi(x, y) = 0\) and the corresponding force term can be obtained as follows

\[
 f(x, y, t) = \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{(2\pi^2 \mu + 1)t^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \right) \sin \pi x \sin \pi y \\
+ \left( \frac{t^{\alpha+1}}{\Gamma(2 + \alpha)} \sin \pi x \sin \pi y \right)^3.
\]
A two-grid temporal second-order scheme...

In Table 4, we give the numerical results with \( \alpha = 0.25, 0.5 \) and 0.75, calculated using the TTGCN finite difference method and the SCN finite difference method, respectively. These numerical results can obviously verify that the computational efficiency of the TTGCN finite difference method is much higher than that of the SCN finite difference method. Also, according to the numerical results in Table 5, the order of convergence of the two methods in space \( \approx 2 \). Therefore, the numerical results are consistent with the theoretical findings. In addition, we also compared our results with the method in [32]. It is clear from Table 6 that the TTGCN finite difference method has higher accuracy and convergence order.

When \( h = 1/100 \) and \( k = 4 \), Fig. 3 compares the CPU time of the two-grid finite difference method and the standard finite difference method for the time direction, which intuitively demonstrates the effectiveness of our method. Besides, Fig. 4 shows intuitively temporal second-order convergence of two-grid finite difference method.

**Example 3** we consider

\[
\begin{align*}
  u_t - \Delta u - I^{(\alpha)} \Delta u &= -u^3, & (x, y, t) \in \Omega \times (0, T], \\
  u(x, y, t) &= 0, & (x, y) \in \partial\Omega, \quad t \in [0, T], \\
  u(x, y, 0) &= xy(1-x)(1-y), & (x, y) \in \Omega.
\end{align*}
\]

In this example, since the exact solution is unknown, we assume that the numerical solution with fixed spatial step \( h = 1/32 \) and half of the original time steps \( \tau_C \) and \( \tau_F \) is the “exact” solution. It can be seen from Table 7 that for the time direction convergence order for both TTGCN and SCN finite difference methods can approach 2, which agrees with the theoretical analysis.
Table 4 The $L^2$-errors, convergence rates and CPU time (seconds) with $h = 1/100$ and $k = 4$ for Example 2

| $\alpha$ | $\tau_C$ | $\tau_F$ | $ETTGCN$ | rate$_{ETTGCN}^\alpha$ | CPU(s) | $ESCN$ | rate$_{ESCN}^\alpha$ | CPU(s) |
|----------|----------|----------|----------|--------------------------|--------|--------|--------------------------|--------|
| 0.25     | 1/2      | 1/8      | 2.9535e−2 | *                        | 34.31  | 2.9535e−2 | *                        | 64.26  |
|          | 1/4      | 1/16     | 1.0043e−2 | 1.5563                   | 71.10  | 1.0043e−2 | 1.5563                   | 125.59 |
|          | 1/8      | 1/32     | 3.0208e−3 | 1.7331                   | 161.96 | 3.0208e−3 | 1.7331                   | 264.99 |
|          | 1/16     | 1/64     | 7.9828e−4 | 1.9200                   | 412.81 | 7.9828e−4 | 1.9200                   | 561.79 |
| 0.5      | 1/2      | 1/8      | 1.5532e−2 | *                        | 32.46  | 1.5532e−2 | *                        | 58.22  |
|          | 1/4      | 1/16     | 4.2840e−3 | 1.8582                   | 69.57  | 4.2840e−3 | 1.8582                   | 120.92 |
|          | 1/8      | 1/32     | 1.0448e−3 | 2.0357                   | 156.46 | 1.0448e−3 | 2.0357                   | 242.71 |
|          | 1/16     | 1/64     | 2.2614e−4 | 2.2080                   | 391.87 | 2.2614e−4 | 2.2080                   | 561.08 |
| 0.75     | 1/2      | 1/8      | 7.7945e−3 | *                        | 30.86  | 7.7946e−3 | *                        | 59.13  |
|          | 1/4      | 1/16     | 1.7861e−3 | 2.1256                   | 67.44  | 1.7861e−3 | 2.1256                   | 113.23 |
|          | 1/8      | 1/32     | 3.6423e−4 | 2.2939                   | 156.47 | 3.6423e−4 | 2.2939                   | 233.35 |
|          | 1/16     | 1/64     | 6.6431e−5 | 2.4549                   | 402.73 | 6.6431e−5 | 2.4549                   | 545.91 |

Table 5 The $L^2$-errors and convergence rates with $\tau_C = 1/128$ and $\tau_F = 1/512$ for Example 2

| $h$ | $\alpha = 0.2$ | rate$_{ETTGCN}^\alpha$ | $ESCN$ | rate$_{ESCN}^\alpha$ | $\alpha = 0.8$ | rate$_{ETTGCN}^\alpha$ | $ESCN$ | rate$_{ESCN}^\alpha$ |
|-----|----------------|-------------------------|--------|----------------------|----------------|-------------------------|--------|----------------------|
| 1/2 | 1.8132e−1      | *                       | 1.8132e−1 | *                        | 1.1940e−1      | *                        | 1.1941e−1 | *                        |
| 1/4 | 4.3294e−2      | 2.0663                  | 4.3296e−2 | 2.0663                  | 2.8125e−2      | 2.0859                  | 2.8133e−2 | 2.0856                  |
| 1/8 | 1.0650e−2      | 2.0233                  | 1.0652e−2 | 2.0231                  | 6.9065e−3      | 2.0259                  | 6.9139e−2 | 2.0247                  |
| 1/16| 2.6486e−3      | 2.0070                  | 2.6519e−3 | 2.0060                  | 1.7133e−3      | 2.0112                  | 1.7206e−3 | 2.0066                  |
| 1/32| 6.5992e−4      | 2.0054                  | 6.6217e−4 | 2.0017                  | 4.2551e−4      | 2.0095                  | 4.2932e−4 | 2.0028                  |

Table 6 The comparison between the scheme (32)–(36) and the scheme [32] with $h = 1/100$ and $k = 4$ for Example 2

| $\alpha$ | $\tau_C$ | $\tau_F$ | Scheme (32)–(36) | Scheme in [32] |
|----------|----------|----------|------------------|----------------|
|          | E$_{TTGCN}$ | rate$_{ETTGCN}^\alpha$ | E | rate$^\alpha$ |
| 0.25     | 1/2      | 1/8      | 2.9535e−2 | *                        | 3.9266e−3      | *                        |
|          | 1/4      | 1/16     | 1.0043e−2 | 1.5563                   | 1.9639e−3      | 0.9996                  |
|          | 1/8      | 1/32     | 3.0208e−3 | 1.7331                   | 9.7001e−4      | 1.0176                  |
|          | 1/16     | 1/64     | 7.9828e−4 | 1.9200                   | 4.6979e−4      | 1.0460                  |
| 0.5      | 1/2      | 1/8      | 1.5532e−2 | *                        | 7.6809e−3      | *                        |
|          | 1/4      | 1/16     | 4.2840e−3 | 1.8582                   | 3.8683e−3      | 0.9896                  |
|          | 1/8      | 1/32     | 1.0448e−3 | 2.0357                   | 1.9311e−3      | 1.0023                  |
|          | 1/16     | 1/64     | 2.2614e−4 | 2.2080                   | 9.5444e−4      | 1.0167                  |
| 0.75     | 1/2      | 1/8      | 7.7945e−3 | *                        | 9.7620e−3      | *                        |
|          | 1/4      | 1/16     | 1.7861e−3 | 2.1256                   | 4.9287e−3      | 0.9860                  |
|          | 1/8      | 1/32     | 3.6423e−4 | 2.2939                   | 2.2683e−3      | 0.9977                  |
|          | 1/16     | 1/64     | 6.6431e−5 | 2.4549                   | 1.2266e−3      | 1.0088                  |
Fig. 3 The CPU time for Example 2 with $h = 1/100$ and $k = 4$.

Fig. 4 The time convergence order for Example 2 with $h = 1/100$ and $k = 4$.
Table 7 The $L^2$-errors and convergence rates with $h = 1/32$ and $k = 4$ for Example 3

| $\alpha$ | $\tau_C$ | $\tau_F$ | $E_{TTGCN}$ | $rate_{TTGCN}^L$ | $E_{SCN}$ | $rate_{SCN}^L$ |
|----------|---------|---------|-------------|------------------|----------|----------------|
| 0.25     | 1/12    | 1/48    | 6.0750e−7   | *                | 6.0563e−7 | *              |
|          | 1/24    | 1/96    | 1.8258e−7   | 1.7344           | 1.8214e−7 | 1.7334         |
|          | 1/48    | 1/192   | 4.9624e−8   | 1.8794           | 4.9558e−8 | 1.8779         |
|          | 1/96    | 1/384   | 1.2745e−8   | 1.9611           | 1.2739e−8 | 1.9599         |
| 0.5      | 1/12    | 1/48    | 1.2281e−6   | *                | 1.2246e−6 | *              |
|          | 1/24    | 1/96    | 3.6180e−7   | 1.7661           | 3.6036e−7 | 1.7648         |
|          | 1/48    | 1/192   | 9.7691e−8   | 1.8860           | 9.7595e−8 | 1.8846         |
|          | 1/96    | 1/384   | 2.5272e−8   | 1.9507           | 2.5363e−8 | 1.9498         |
| 0.75     | 1/12    | 1/48    | 2.1532e−6   | *                | 2.1477e−6 | *              |
|          | 1/24    | 1/96    | 6.3459e−7   | 1.7626           | 6.3351e−7 | 1.7614         |
|          | 1/48    | 1/192   | 1.7308e−7   | 1.8744           | 1.7294e−7 | 1.8731         |
|          | 1/96    | 1/384   | 4.5190e−8   | 1.9373           | 4.5177e−8 | 1.9366         |

Acknowledgements The authors are grateful for helpful comments and suggestions from the reviewers. This work was supported by Postgraduate Scientific Research Innovation Project of Hunan Province (No. CX20220454). Ahmed S. Hendy wishes to acknowledge the support of the RSF grant, project 22-21-00075.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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