ON THE INVERSE $K_I$-INEQUALITY FOR ONE CLASS OF MAPPINGS

OLEKSANDR DOVHOPIATYI, EVGENY SEVOST'YANOV

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Abstract

We study mappings differentiable almost everywhere, possessing the $N$-Luzin property, the $N^{-1}$-property on the spheres with respect to the $(n-1)$-dimensional Hausdorff measure and such that the image of the set where its Jacobian equals to zero has a zero Lebesgue measure. It is proved that such mappings satisfy the lower bound for the Poletsky-type distortion in their domain of definition.

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1 Introduction

One of the methods of studying the Sobolev and Orlicz-Sobolev classes is to use the distortion estimates of the modulus of families of paths and surfaces (see, for example, [KRSS] and [Sev1]). In particular, the lower estimates of the distortion for the modulus of families of images of concentric spheres under the mapping have an important role in the study of their local and boundary behavior, see ibid. Note that, in the mentioned papers, we are talking only about the mapped surfaces, while estimates of the modulus of the families of these surfaces themselves were not involved, as their role has not been studied in detail. The main purpose of this manuscript is to obtain the estimates of modulus of families of sets, the image of which under the map are spheres centered at a fixed point. As will be shown below, these estimates associated with the so-called inverse Poletsky inequality, which makes it possible to describe many properties of the corresponding mappings with taking into account our previous results (see, e.g., [SSD]).

Here are the necessary definitions and wording of the main result. Let $X$ and $Y$ be two spaces with measures $\mu$ and $\mu'$, respectively. We say that a mapping $f : X \to Y$ has $N$-property of Luzin, if from the condition $\mu(E) = 0$ it follows that $\mu'(f(E)) = 0$. Similarly, we
say that a mapping $f : X \to Y$ has $N'$-Luzin property, if from the condition $\mu'(E) = 0$ it follows that $\mu(f^{-1}(E)) = 0$. At the points $x \in D$ of differentiability of the mapping $f$, we put

$$l(f'(x)) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|},$$

$$||f'(x)|| = \max_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|},$$

$$(1.1)$$

$$J(x, f) = \det f'(x).$$

Fix $p > 1$. We define the *inner* and the *outer* dilatations of the mapping $f$ at a point $x$ of the order $p$ by the relations

$$K_{I,p}(x, f) = \begin{cases} \frac{|J(x, f)|}{|f'(x)|^p}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}$$

$$K_{O,p}(x, f) = \begin{cases} \frac{|f'(x)||}{|f(x)|^p}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}$$

respectively. Given a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$ and $y \in \mathbb{R}^n$, we define the *multiplicity function* $N(y, f, E)$ as a number of preimages of the point $y$ in a set $E$, i.e.

$$N(y, f, E) = \text{card} \{x \in E : f(x) = y\},$$

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E).$$

Let $A$ be a set where $f$ does not have a total differential, and let $y \notin f(A)$. If $N(f, D) < \infty$, then we set

$$Q(y) := K_{I,\alpha}(y, f^{-1}) = \sum_{x \in f^{-1}(y)} K_{O,\alpha}(x, f).$$

$$(1.2)$$

Observe that, $N(f, D) < \infty$ for open, discrete and closed mappings of $D$, see [MS, Lemma 3.3].

Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\}.$$  

$$(1.3)$$

Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in [a, b]$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, C_1, C_2)$ a family of all paths $\gamma$ in $D$ such that $f(\gamma) \in \Gamma(C_1, C_2, A(y_0, r_1, r_2))$. Let $Q_\alpha : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function, and $M_\alpha(\Gamma)$ denotes the $\alpha$-modulus od
a family $\Gamma$ (see, e.g., [Va, section 6]). We say that $f$ satisfies the inverse Poletsky inequality at a point $y_0 \in \overline{f(D) \setminus \{\infty\}}$ with respect to $\alpha$-modulus if there is $r_0 > 0$ such that, the relation

$$M_\alpha(\Gamma_f(y_0, C_1, C_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q_*(y) \cdot \eta^\alpha(|y - y_0|) \, dm(y)$$  \hspace{1cm} (1.4)$$
holds for any $0 < r_1 < r_2 < r_0$, any continua $C_1 \subset \overline{B(y_0, r_1) \cap f(D)}$ and $C_2 \subset f(D) \setminus B(y_0, r_2)$, and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.$$  \hspace{1cm} (1.5)$$

The following statement holds.

**Theorem 1.1.** Let $n - 1 < \alpha \leq n$, let $y_0 \in \overline{f(D) \setminus \{\infty\}}$, $r_0 = \sup_{y \in f(D)} |y - y_0| > 0$, and let $f : D \to \mathbb{R}^n$ be an open, discrete and closed mapping that is differentiable almost everywhere and has $N$-Luzin property with respect to the Lebesgue measure in $\mathbb{R}^n$. Suppose that $\overline{D}$ is a compact set in $\mathbb{R}^n$, and, in addition,

$$m(f(\{x \in D : J(x, f) = 0\})) = 0.$$  \hspace{1cm} (1.6)$$

Suppose that $f$ has $N^{-1}$-property on $S(y_0, r) \cap f(D)$ for almost all $r \in (\varepsilon, r_0)$ relative to the Hausdorff measure $\mathcal{H}^{n-1}$ on $S(y_0, r)$. If the function $Q$, which is defined in (1.2), belongs to the class $L^1(f(D))$, then the mapping $f$ satisfies the inverse Poletsky inequality with respect to $\alpha$-modulus with $Q_*(y) := N^\alpha(f, D) \cdot Q(y)$.

**Corollary 1.1.** The assertion of Theorem 1.1 holds if instead of the condition (1.6) a stronger condition is required: $J(x, f) \neq 0$ almost everywhere.

## 2 Distortion of families of sets under mappings

Let us give some important information concerning the relationship between the moduli of the families of paths joining the sets and the moduli of the families of the sets separating these sets. Mostly this information can be found in Ziemer’s publication, see [Zi1]. Let $G$ be a bounded domain in $\mathbb{R}^n$, and $C_0, C_1$ are disjoint compact sets in $G$. Put $R = G \setminus (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$. For a number $p > 1$ we define a $p$-capacity of the pair $C_0, C_1$ relative to the closure $G$ by the equality

$$C_p[G, C_0, C_1] = \inf_R \int_R |\nabla u|^p \, dm(x),$$

where the exact lower bound is taken for all functions $u$, continuous in $R^*$, $u \in ACL(R)$, such that $u = 1$ on $C_1$ and $u = 0$ on $C_0$. These functions are called admissible for $C_p[G, C_0, C_1]$. 
We say that a set $\sigma \subset \mathbb{R}^n$ separates $C_0$ and $C_1$ in $R^*$, if $\sigma \cap R$ is closed in $R$ and there are disjoint sets $A$ and $B$, open relative $R^* \setminus \sigma$, such that $R^* \setminus \sigma = A \cup B$, $C_0 \subset A$ and $C_1 \subset B$. Let $\Sigma$ denotes the class of all sets that separate $C_0$ and $C_1$ in $R^*$. For the number $p' = p/(p-1)$ we define the quantity

$$\widetilde{M}_{p'}(\Sigma) = \inf_{\rho \in \text{adm} \Sigma} \int_{\mathbb{R}^n} \rho^{p'} \, dm(x)$$  \hspace{1cm} (2.1)$$

where the notation $\rho \in \text{adm} \Sigma$ denotes that $\rho$ is nonnegative Borel function in $\mathbb{R}^n$ such that

$$\int_{\sigma \cap R} \rho \, dH^{n-1} \geq 1 \quad \forall \sigma \in \Sigma. \hspace{1cm} (2.2)$$

Note that according to the result of Ziemer

$$\widetilde{M}_{p'}(\Sigma) = C_p[G, C_0, C_1]^{-1/(p-1)},$$

see [Zi1, Theorem 3.13] for $p = n$ and [Zi2, p. 50] for $1 < p < \infty$, in addition, by the Hesse result

$$M_p(\Gamma(E, F, D)) = C_p[D, E, F],$$

where $(E \cup F) \cap \partial D = \emptyset$ (see [Hes, Theorem 5.5]). Shlyk has proved that the requirement $(E \cup F) \cap \partial D = \emptyset$ can be omitted, in other words, the equality (2.4) holds for any disjoint non-empty sets $E, F \subset D$ (see [Shl, Theorem 1]).

Let $S$ be a surface, in other words, $S : D_s \rightarrow \mathbb{R}^n$ be a continuous mapping of an open set $D_s \subset \mathbb{R}^{n-1}$. We put $N(y, S) = \text{card} S^{-1}(y) = \text{card}\{x \in D_s : S(x) = y\}$ and recall this function a multiplicity function of the surface $S$ with respect to a point $y \in \mathbb{R}^n$. Given a Borel set $B \subset \mathbb{R}^n$, its $(n-1)$-measured Hausdorff area associated with the surface $S$ is determined by the formula $A_S(B) = A_S^{n-1}(B) = \int_B N(y, S) \, dH^{n-1}y$, see [Fe, item 3.2.1]. For a Borel function $\rho : \mathbb{R}^n \rightarrow [0, \infty]$ its integral over the surface $S$ is determined by the formula $\int_S \rho \, dA = \int_{\mathbb{R}^n} \rho(y) N(y, S) \, dH^{n-1}y$. In what follows, $J_k f(x)$ denotes the $k$-dimensional Jacobian of the mapping $f$ at a point $x$ (see [Fe, § 3.2, Ch. 3]).

Let $n \geq 2$, and let $\Gamma$ be a family of surfaces $S$. A Borel function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is called an admissible for $\Gamma$, abbr. $\rho \in \text{adm} \Gamma$, if

$$\int_S \rho^{n-1} \, dA \geq 1$$  \hspace{1cm} (2.5)$$

for any $S \in \Gamma$. Given $p \in (1, \infty)$, a $p$-modulus of $\Gamma$ is called the quantity

$$M_p(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_{\mathbb{R}^n} \rho^p(x) \, dm(x).$$
We also set \( M(\Gamma) := M_n(\Gamma) \). Let us say that some property \( P \) holds for \( p \)-almost all surfaces of the domain \( D \), if this property holds for all surfaces in \( D \), except, maybe be, some of their subfamily, \( p \)-modulus of which is zero. If we are talking about the conformal modulus \( M(\Gamma) := M_n(\Gamma) \), the prefix ”\( n \)" in the expression ”\( n \)-almost all", as a rule, is omitted. We say that a Lebesgue measurable function \( \rho : \mathbb{R}^n \to \mathbb{R}^+ \) is \( p \)-extensively admissible for the family \( \Gamma \) of surfaces \( S \) in \( \mathbb{R}^n \), abbr. \( \rho \in \text{ext}_p \text{adm} \Gamma \), if the relation (2.5) is satisfied for \( p \)-almost all surfaces \( S \) of the family \( \Gamma \). The proof of the following lemma is based on the approach, used in establishing the relationship of Orlicz-Sobolev classes with lower estimates of the distortion of the modulus of surface families (see, eg. [KRSS, Theorem 5] and [Sev1, Theorem 4]). In such a general formulation, this lemma is proved for the first time in this paper.

Lemma 2.1. Let \( p > n - 1 \), \( f : D \to \mathbb{R}^n \) be a mapping that is differentiable almost everywhere and has \( N \)-Luzin property with respect to the Lebesgue measure in \( \mathbb{R}^n \), let \( N(f, D) < \infty \) and let \( y_0 \in \overline{f(D)} \setminus \{ \infty \} \), \( r_0 = \sup_{y \in f(D)} |y - y_0|, 0 < \varepsilon_0 < r_0, 0 < \varepsilon < \varepsilon_0 \). Suppose that the condition (L.0) is also satisfied. Fix \( \varepsilon > 0 \), and denote by \( \Sigma_\varepsilon \) the family of all sets of the form

\[
\{ f^{-1}(S(y_0, r) \cap f(D)) \}, \quad r \in (\varepsilon, r_0).
\]

Suppose, in addition, that \( f \) has \( N^{-1} \)-property on \( S(y_0, r) \cap f(D) \) for almost all \( r \in (\varepsilon, r_0) \) relative to the Hausdorff measure \( \mathcal{H}^{n-1} \) on \( S(y_0, r) \). Then

\[
\mathcal{M}^{p}_{n-1}(\Sigma_\varepsilon) \geq \frac{1}{N^{p}_{n-1}(f, D)} \inf_{\rho \in \text{ext}_p \text{adm} f(\Sigma_\varepsilon)} \int_{f(D) \cap A(y_0, \varepsilon, r_0)} \frac{\rho^p(y)}{Q^{p-1}(y)} \, dm(y), \tag{2.7}
\]

where

\[
Q(y) := K_{I,\alpha}(y, f^{-1}) = \sum_{x \in f^{-1}(y)} K_{O,\alpha}(x, f), \tag{2.8}
\]

and \( \alpha = \frac{p}{p-n+1} \).

Proof. Without loss of generality, we may assume that \( r_0 > 0 \). We will generally follow the methodology set forth in proving [KRSS, Theorem 5] (see also [MRSY, Theorem 8.6]).

Denote by \( B \) a Borel set of all points \( x \in D \), where the mapping \( f \) has a total differential \( f'(x) \) and \( J(x, f) \neq 0 \). By Kirszbraun’s theorem and by the unity of the approximate differential (see, for example, [Pe, 2.10.43 and Theorem 3.1.2]) it follows that the set \( B \) is a countable union of Borel sets \( B_k, k = 1, 2, \ldots \), such that the mappings \( f_k = f|_{B_k} \) are Lipschitz homeomorphisms (see [Pe, Lemma 3.2.2 and Theorems 3.1.4 and 3.1.8]). Without loss of generality, we may assume that the sets \( B_k \) are disjoint. We also denote by \( B_x \) the set of all points \( x \in D \), where \( f \) has a total differential, but \( J(x, f) = 0 \).

Since the set \( B_0 := D \setminus (B \cup B_x) \) has a Lebesgue measure zero, and the mapping \( f \) has \( N \)-Luzin property, then \( m(f(B_0)) = 0 \). By [MRSY, Theorem 9.3] \( A_{S_x}(f(B_0)) = 0 \) for \( p \)-almost all spheres \( S_x := S(y_0, r) \cap f(D) \) centered at a point \( y_0 \), where ”almost all” is understood in the sense of \( p \)-modulus of families of surfaces. Note that, the function \( \psi(r) := \mathcal{H}^{n-1}(f(B_0) \cap S_r) \)
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is Lebesgue due to the Fubini theorem ([Sa, Section 8.1, Ch. III]). Thus, the set $E \subset \mathbb{R}$ of all $r \in \mathbb{R}$ such that $\mathcal{H}^{n-1}(f(B_0) \cap S_r) = 0$, is Lebesgue measurable. Then by [IS, Lemma 4.1] $\mathcal{A}_r(f(B_0)) = 0$ for almost all spheres $S_r := S(y_0, r)$ centered at the point $y_0$, where "almost all" is understood in the sense of a one-dimensional Lebesgue measure with respect to the parameter $r \in (\varepsilon, r_0)$. Now, by the assumption of Lemma,

$$\mathcal{H}^{n-1}(f^{-1}(S_r) \cap B_0) = 0$$

(2.9)

for almost all $r \in (\varepsilon, \varepsilon_0)$. Arguing similarly, we obtain that

$$\mathcal{H}^{n-1}(f^{-1}(S_r) \cap B_*) = 0$$

(2.10)

for almost all $r \in (\varepsilon, \varepsilon_0)$.

Let $\rho^{n-1} \in \tilde{\text{adm}} \Sigma_\varepsilon$ and let

$$\tilde{\rho}(y) = \begin{cases} \sup_{x \in f^{-1}(y) \cap D \setminus B_0} \rho_*(x), & y \in f(D) \setminus f(B \cap B_*) \\ 0, & y \in f(B \cap B_*) \end{cases}$$

(2.11)

where

$$\rho_*(x) = \begin{cases} \rho(x) \cdot \left(\frac{\|f'(x)\|}{J(x,f)}\right)^{1/(n-1)}, & x \in D \setminus B_0, \\ 0, & \text{otherwise} \end{cases}$$

(2.12)

Observe that $\tilde{\rho} = \sup \rho_k$, where

$$\rho_k(y) = \begin{cases} \rho_*(f_k^{-1}(y)), & y \in f(B_k), \\ 0, & \text{otherwise} \end{cases}$$

(2.13)

and, moreover, each mapping $f_k = f|_{B_k}$, $k = 1, 2, \ldots$, is injective. Thus, a function $\tilde{\rho}$ is Borel (see, e.g., [Sa, Theorem I (8.5)]).

Let $f^{-1}(S_r) := S_r^*$. Then

$$\int_{S_r \cap f(D)} \tilde{\rho}^{n-1}(y) \, dA_* = \int_{\mathbb{R}^n} \tilde{\rho}^{n-1}(y) \chi_{S_r \cap f(D)}(y) \, d\mathcal{H}^{n-1}y \geq$$

$$\geq \int_{\mathbb{R}^n} \frac{1}{N(f, D)} \sum_{k=1}^{\infty} \tilde{\rho}^{n-1}(y) \chi_{S_r \cap f(D)}(y) N(y, f, B_k \cap S_r^*) \, d\mathcal{H}^{n-1}y =$$

$$= \frac{1}{N(f, D)} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \rho_*^{n-1}(f_k^{-1}(y)) N(y, f, B_k \cap S_r^*) \, d\mathcal{H}^{n-1}y =$$

(2.14)

$$= \frac{1}{N(f, D)} \sum_{k=1}^{\infty} \int_{f(B_k \cap S_r^*)} \rho_*^{n-1}(f_k^{-1}(y)) \, d\mathcal{H}^{n-1}y.$$
Let $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$ are the main stretchings of the mapping $f$, see e.g. [Re] Lemmas 4.1.I, 4.2.I]. Then $J(x, f) = \lambda_1(x) \cdots \lambda_n(x)$ and

$$
\left( \frac{\| f'(x) \|}{J(x, f)} \right)^{1/(n-1)} = \left( \frac{1}{\lambda_1(x) \cdots \lambda_{n-1}(x)} \right)^{\frac{1}{n-1}} \geq \left( \frac{1}{J_{n-1}(f(x))} \right)^{\frac{1}{n-1}}.
$$

(2.15)

Due to (2.9), (2.10) and (2.15), by [Fe, Corollary 3.2.20] for $m = n - 1$, we obtain that

$$
\sum_{k=1}^{\infty} \int_{f(B_k \cap S^*_r)} \rho_{*}^{-n-1}(f_k^{-1}(y)) \, d\mathcal{H}^{n-1}y = \sum_{k=1}^{\infty} \int_{B_k \cap S^*_r} \rho_{*}^{-n-1}(x) \, J_{n-1}f(x) \, d\mathcal{H}^{n-1}x =
$$

$$
\geq \sum_{k=1}^{\infty} \int_{B_k \cap S^*_r} \rho_{*}^{-n-1}(x) \, d\mathcal{H}^{n-1}x = \int_{f^{-1}(S_r)} \rho_{*}^{-n-1}(x) \, d\mathcal{H}^{n-1}x \geq 1
$$

(2.16)

for almost any $S_r = f \circ S^*_r \in f(\Sigma_c)$. It follows from (2.16) that $N_{\lambda^{-1}}(f, D) \tilde{\rho} \in \text{adm}_\mu f(\Sigma_c)$ (see [IS] Lemma 4.1]).

Since $\tilde{\rho}(y) = \sup_{k \in \mathbb{N}} \rho_k(y) \leq \sum_{k=1}^{\infty} \rho_k(y)$ and $m(f(B_0)) = m(f(B_0)) = 0$, then

$$
\int_{f(D)} \frac{\tilde{\rho}(y)}{Q(y)} \, dm(y) \leq \sum_{k=1}^{\infty} \int_{f(B_k)} \frac{\rho_k(y)}{Q(y)} \, dm(y) \leq \sum_{k=1}^{\infty} \int_{f(B_k)} \frac{\rho_k(y)}{K_{\alpha^{-1}} \rho_{k}^{p+1}(f_k^{-1}(y), f)} \, dm(y).
$$

Using the change of variables formula on each $B_k$, $k = 1, 2, \ldots$, see, for example, [Re] Theorem 3.2.5], we obtain that

$$
\int_{f(B_k)} \frac{\rho_k(y)}{K_{\alpha^{-1}} \rho_{k}^{p+1}(f_k^{-1}(y), f)} \, dm(y) =
$$

$$
= \int_{f(B_k)} \frac{\rho_k(f_k^{-1}(y)) \, J_{\alpha^{-1}} \rho_{k}^{p+1}(f_k^{-1}(y), f)}{\| f'(f_k^{-1}(y)) \|^{\frac{p}{p-n+1}} \| J(f_k^{-1}(y), f) \|^{\frac{p}{p-n+1}}} \, dm(y) =
$$

$$
= \int_{f(B_k)} \rho_k(f_k^{-1}(y)) \, J(y, f_k^{-1}) \, dm(y) = \int_{B_k} \rho_k(x) \, dm(x).
$$

The latter implies that

$$
\int_{f(D)} \frac{\tilde{\rho}(y)}{Q(y)} \, dm(y) \leq \sum_{k=1}^{\infty} \int_{B_k} \rho_k(x) \, dm(x).
$$

(2.17)

Summing (2.17) by $k = 1, 2, \ldots$ and using the countable additivity of the Lebesgue integral (see, for example, [Sa] Theorem 1.2.3]), we obtain that

$$
\int_{f(D)} \frac{1}{Q(y)} \tilde{\rho}(y) \, dm(y) \leq \int_{D} \rho(x) \, dm(x).
$$

(2.18)
Going in the ratio (2.18) to inf over all functions \( \rho^{n-1} \in \admsigma \), we obtain that
\[
\int_{f(D)} \frac{1}{Q^{p-n-1}(y) \tilde{\rho}(y)} \cdot dm(y) \leq M_{\frac{n}{n-1}}(\Sigma),
\]
whence we obtain that
\[
\int_{f(D)} \frac{N^{\frac{p}{n-1}}(f, D)}{Q^{p-n-1}(y) \tilde{\rho}(y)} \cdot dm(y) \leq N^{\frac{p}{n-1}}(f, D) \cdot M_{\frac{n}{n-1}}(\Sigma).
\]

Put \( \tilde{\rho}_1(y) := N^{\frac{1}{n-1}}(f, D) \cdot \tilde{\rho}(y) \). Due to the latter relation, we obtain that
\[
\int_{f(D)} \frac{\tilde{\rho}_1(y)}{Q^{p-n-1}(y)} \cdot dm(y) \leq N^{\frac{p}{n-1}}(f, D) \cdot M_{\frac{n}{n-1}}(\Sigma).
\]
Since by the above \( \tilde{\rho}_1(y) = N^{\frac{1}{n-1}}(f, D) \tilde{\rho} \in \extadm f(\Sigma) \), it follows from (2.19) that the relation (2.7) holds. Lemma is proved. \( \square \)

We have the following simple consequence.

**Corollary 2.1.** Let \( f : D \to \mathbb{R}^n \) be a map which is differentiable almost everywhere, and has \( N \) and \( N^{-1} \) Luzin properties with respect to the Lebesgue measure. Let \( y_0 \in \overline{f(D)} \setminus \{ \infty \} \), \( r_0 = \sup_{y \in f(D)} |y - y_0| \). We fix \( \varepsilon > 0 \), and denote by \( \Sigma \) the family of all sets of the form (2.6). In addition, suppose that \( f \) has \( N^{-1} \)-Luzin property on \( S(y_0, r) \cap f(D) \) for almost all \( r \in (\varepsilon, r_0) \) with respect to \( H^{n-1} \) on \( S(y_0, r) \). Then the relation (2.7) is fulfilled, where \( Q \) is defined by the relation (1.2).

**Proof.** Since \( f \) has \( N^{-1} \)-Luzin property, by Ponomarev’s theorem we have that \( J(x, f) \neq 0 \) almost everywhere (see, for example, [Pom, Theorem 1]), we may assume that \( J(x, f) \neq 0 \) on any \( B_k, k = 1, 2, \ldots \). Then, since the mapping \( f \) has \( N \)-property, the condition (1.0) is also fulfilled. The desired statement, in this case, follows from Lemma 2.1. \( \square \)

### 3 Proof of the main result

Let \( Q_* : D \to [0, \infty] \) be a Lebesgue measurable function. Denote by \( q_{x_0}(r) \) the integral average of \( Q_*(x) \) under the sphere \( |x - x_0| = r \),
\[
q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x - x_0| = r} Q_*(x) \, dH^{n-1}, \tag{3.1}
\]
where \( \omega_{n-1} \) denotes the area of the unit sphere in \( \mathbb{R}^n \). Below we also assume that the following standard relations hold: \( a/\infty = 0 \) for \( a \neq \infty \), \( a/0 = \infty \) for \( a > 0 \) and \( 0 \cdot \infty = 0 \) (see, e.g., [Sa, §3, section I]). The following conclusion was obtained by V. Ryazanov together with
the author in the case $p = n$, see, e.g., [MRSY] Lemma 7.4 or [RS] Lemma 2.2. In the case of an arbitrary $p > 1$, see, for example, [SalSev] Lemma 2.

**Proposition 3.1.** Let $p > 1$, $n \geq 2$, $x_0 \in \mathbb{R}^n$, $r_1, r_2 \in \mathbb{R}$, $r_1, r_2 > 0$, and let $Q_*(x)$ be a Lebesgue measurable function, $Q_* : \mathbb{R}^n \to [0, \infty[$, $Q_* \in L^1_{\text{loc}}(\mathbb{R}^n)$. We put

$$I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{n-1} q_0(r)},$$

and let $q_0(r)$ be defined by (3.1). Then

$$\frac{\omega_{n-1}}{I^{p-1}} \leq \int_A Q_*(x) \cdot \eta^p(|x - x_0|) \, dm(x)$$

(3.2)

for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty[$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr = 1,$$

(3.3)

where $A = A(x_0, r_1, r_2)$ is defined in (1.3).

**Remark 3.1.** Note that, if (3.2) holds for any function $\eta$ with a condition (3.3), then the same relationship holds for any function $\eta$ with the condition (1.5). Indeed, let $\eta$ be a nonnegative Lebesgue function that satisfies the condition (1.5). If $J := \int_{r_1}^{r_2} \eta(t) \, dt < \infty$, then we put $\eta_0 := \eta/J$. Obviously, the function $\eta_0$ satisfies condition (3.3). Then the relation (3.2) gives that

$$\frac{\omega_{n-1}}{I^{p-1}} \leq \frac{1}{J^p} \int_A Q_*(x) \cdot \eta^p(|x - x_0|) \, dm(x) \leq \int A Q_*(x) \cdot \eta^p(|x - x_0|) \, dm(x)$$

because $J \geq 1$. Let now $J = \infty$. Then, by [Sa] Theorem I.7.4, a function $\eta$ is a limit of a nondecreasing nonnegative sequence of simple functions $\eta_m$, $m = 1, 2, \ldots$. Set $J_m := \int_{r_1}^{r_2} \eta_m(t) \, dt < \infty$ and $w_m(t) := \eta_m(t)/J_m$. Then, it follows from (3.3) that

$$\frac{\omega_{n-1}}{I^{p-1}} \leq \frac{1}{J_m^p} \int A Q_*(x) \cdot \eta_m^p(|x - x_0|) \, dm(x) \leq \int A Q_*(x) \cdot \eta_m^p(|x - x_0|) \, dm(x),$$

(3.4)

because $J_m \to J = \infty$ as $m \to \infty$ (see [Sa] Lemma I.11.6].) Thus, $J_m \geq 1$ for sufficiently large $m \in \mathbb{N}$. Observe that, a functional sequence $\psi_m(x) = Q_*(x) \cdot \eta_m^p(|x - x_0|)$, $m = 1, 2, \ldots$, is nonnegative, monotone increasing and converges to a function $\psi(x) := Q_*(x) \cdot \eta^p(|x - x_0|)$ almost everywhere. By the Lebesgue theorem on the monotone convergence (see [Sa] Theorem I.12.6], it is possible to go to the limit on the right side of the inequality (3.4), which gives us the desired inequality (3.2).
Proof of Theorem 1.1. Fix \( y_0 \in f(D) \setminus \{ \infty \} \), \( 0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0| \), \( C_1 \subset B(y_0, r_1) \cap f(D) \) and \( C_2 \subset f(D) \setminus B(y_0, r_2) \). Set

\[
C_0 := f^{-1}(C_1), \quad C_0^* := f^{-1}(C_2)
\]

(see Figure 1). Observe that \( C_0 \) and \( C_1 \) are disjoint compact sets in \( D \), see [Vui, Theorem 3.3]. Besides that, \( C_1 \) and \( C_2 \) are non empty by the choice of \( r_0, r_1 \) and \( r_2 \).

Let us to show that a set \( \sigma_r := f^{-1}(S(y_0, r)) \) separates \( C_0 \) from \( C_0^* \) in \( D \) for any \( r \in (r_1, r_2) \). Indeed, \( \sigma_r \) is closed in \( D \) as a preimage of a closed set \( S(y_0, r) \) under the continuous mapping \( f \) (see, e.g., [Ku, Theorem 1.IV.13, Ch. 1]). In particular, \( \sigma_r \) is also closed with respect to \( R := D \setminus (C_0 \cup C_0^*) \). We put

\[
A := f^{-1}(B(y_0, r))
\]

and

\[
B := D \setminus f^{-1}(B(y_0, r)).
\]

Observe that, \( A \) and \( B \) are not empty by the choice of \( r_0, r_1, r_2 \) and \( r \). Since \( f \) is continuous, \( f^{-1}(B(y_0, r)) \) and \( D \setminus f^{-1}(B(y_0, r)) \) are open in \( D \). In other words, \( A \) and \( B \) are open in

\[
R^* := R \cup C_0 \cup C_1 = D.
\]

Note that \( A \cap B = \emptyset \), and \( R^* \setminus \sigma_r = A \cup B \). Let \( \Sigma_{C_0, C_0^*} \) be the family of all sets separating \( C_0 \) and \( C_0^* \) in \( R^* \). In this case, by the equations of Ziemer and Hesse, see (2.3) and (2.4), respectively, we obtain that

\[
M_\alpha(\Gamma_f(y_0, C_1, C_2)) = \left( \frac{M_{p/(\alpha - 1)}(\Sigma_{r_1, r_2})}{\alpha} \right)^{1-\alpha},
\]

(3.5)
where \( \alpha = \frac{p}{p-n+1} \). Then by Lemma 2.1 and by the relation (3.5), we obtain that

\[
M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leq \left( \inf_{\rho \in \text{ext adm } f(S_\alpha)} \int_{f(D) \cap A(y_0, r_1, r_2)} \frac{\rho^p(y)}{N^{\frac{p}{p-n+1}}(f, D) \cdot Q^{\frac{p-n+1}{p-n+1}}(y)} dm(y) \right)^{-\frac{n-1}{p-n+1}},
\]

where \( Q \) is defined by (1.2). Using the second remote formula in the proof of Theorem 9.2 in [MRSY], we obtain that

\[
\inf_{\rho \in \text{ext adm } f(S_\alpha)} \int_{f(D) \cap A(y_0, r_1, r_2)} \frac{\rho^p(y)}{N^{\frac{p}{p-n+1}}(f, D) \cdot Q^{\frac{p-n+1}{p-n+1}}(y)} dm(y) = \int_{r_1}^{r_2} \left( \inf_{\alpha \in I(r)} \int_{S(y_0, r) \cap f(D)} \frac{\alpha^q(y)}{N^{\frac{p}{p-n+1}}(f, D) \cdot Q^{\frac{p-n+1}{p-n+1}}(y)} \mathcal{H}^{n-1}(y) \right) dr,
\]

where \( q = \frac{p}{n-1} \), and \( I(r) \) denotes the set of all measurable functions on \( S(y_0, r) \cap f(D) \) such that \( \int_{S(y_0, r) \cap f(D)} \alpha(x) \mathcal{H}^{n-1} = 1 \). Then, choosing \( X = S(y_0, r) \cap f(D), \mu = \mathcal{H}^{n-1} \) and \( \varphi = \frac{1}{Q}|_{S(y_0, r) \cap f(D)} \) in [MRSY] Lemma 9.2, we obtain that

\[
\int_{r_1}^{r_2} \left( \inf_{\alpha \in I(r)} \int_{S(y_0, r) \cap f(D)} \frac{\alpha^q(y)}{Q(y)} d\mathcal{H}^{n-1} \right) dr = \int_{r_1}^{r_2} \frac{dr}{\|Q\|_s(r)},
\]

where \( \|Q\|_s(r) = \left( \int_{S(y_0, r) \cap f(D)} Q^s(x) d\mathcal{H}^{n-1} \right)^{1/s} \) and \( s := \frac{n-1}{p-n+1} \). Thus, by (3.6), (3.7) and (3.8) we obtain that

\[
M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leq N^{\alpha}(f, D) \cdot \left( \int_{r_1}^{r_2} \frac{dr}{\|Q\|_s(r)} \right)^{\frac{n-1}{p-n+1}} = \frac{N^{\alpha}(f, D) \cdot \omega_{n-1}}{(\int_{r_1}^{r_2} \frac{dr}{q_{y_0}^{(\alpha-1)}(r)} \right)^{\frac{n-1}{p-n+1}}} = \frac{N^{\alpha}(f, D) \cdot \omega_{n-1}}{q_{y_0}^{-\frac{1}{p-n+1}}},
\]

where \( q_{y_0}(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int_{S(y_0, r)} \tilde{Q} d\mathcal{H}^{n-1} \) and \( \tilde{Q}(y) = \begin{cases} Q(y), & y \in f(D), \\ 0, & y \not\in f(D) \end{cases} \). Finally, it follows from (3.9) and Proposition 3.1 that the relation

\[
M_{\frac{p}{p-n+1}}(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} N^{\alpha}(f, D) \cdot Q(y) \cdot \eta^\alpha(|y - y_0|) dm(y)
\]

holds for a function \( Q(y) = K_{O, \alpha}(y, f^{-1}) := \sum_{x \in f^{-1}(y)} K_{I, \alpha}(x, f) \), that is desired conclusion. \( \square \)
Proof of Corollary 1.1 immediately follows by Theorem 1.1 and additional arguments used under the proof of Corollary 2.1. □

Remark 3.2. Observe that, the local and boundary behavior of mappings that satisfy condition (1.4) is described in sufficient detail in [SSD], which makes it possible to transfer these results to mappings participating in Theorem 1.1. Note also that the mappings with the inverse Poletsky inequality are part of the definition of quasiconformality in the case of a bounded function \( Q \) (see [Va, Ch. 13.1]), and in the unbounded case were obtained by different authors under different conditions for \( Q \) (see, eg, [MRSY, Theorem 8.5], [Cr, Lemma 3.1], [KR] and [Sev2, Theorem 1.3]). In particular, the statement below follows directly from Theorem 1.1 and [SevSkv, Theorem 4.1].

For domains \( D, D' \subset \mathbb{R}^n, n \geq 2 \), a number \( N \in \mathbb{N} \) and a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty], Q(y) \equiv 0 \) for \( y \in \mathbb{R}^n \setminus D' \), we denote by \( \mathfrak{R}_{Q,N}(D, D') \) the family of all open discrete mappings \( f : D \to D' \) which are differentiable almost everywhere, have \( N \)-Luzin property with respect to the Lebesgue measure in \( \mathbb{R}^n \), satisfy relation (1.6) and have \( N^{-1} \)-property on \( S(y_0, r) \cap D' \) for almost all \( r \in (\varepsilon, r_0) \) relative to the Hausdorff measure \( H^{n-1} \) on \( S(y_0, r) \) for any \( y_0 \in D' \) and \( r_0 = \sup_{y \in D'} |y - y_0| \) such that

1) \( N(f, D) \leq N \),

2) \( K_{I,n}(y, f^{-1}) = \sum_{x \in f^{-1}(y)} K_{O,n}(x, f) \leq Q(y) \) for any \( y \in D' \).

If \( Q \in L^1(D') \), \( D' \) is bounded and \( K \) is a compact set in \( D \), then the inequality

\[
|f(x) - f(y)| \leq \frac{C}{\log^{1/n} \left(1 + \frac{r_*}{2|x-y|}\right)}
\]

holds for any \( x, y \in K \) and all \( f \in \mathfrak{R}_{Q}(D, D') \), where \( C = C(n, N, K, \|Q\|_1, D, D') > 0 \) is some constant depending only on \( n, N, K \) and \( \|Q\|_1 \), \( \|Q\|_1 \) denotes \( L^1 \)-norm of \( Q \) in \( D' \), and \( r_* = d(K, \partial D) \).

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Oleksandr Dovhopiatyi  
1. Zhytomyr Ivan Franko State University,  
40 Bolshaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE  
alex dov111111@gmail.com

Evgeny Sevost’yanov  
1. Zhytomyr Ivan Franko State University,  
40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE  
2. Institute of Applied Mathematics and Mechanics  
of NAS of Ukraine,  
1 Dobrovol’skogo Str., 84 100 Slavyansk, UKRAINE  
esevostyanov2009@gmail.com