Statistical Theory of 2-Dimensional Quantum Vortex Gas:
Non-Canonical Effect and Generalized Zeta Function

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Abstract

The purpose of this paper is to present a quantum statistical theory of 2-dimensional vortex gas based on the generalized Hamiltonian dynamics recently developed. The quantized spectrum is evaluated for a pair of vortex on the basis of the semiclassical quantization rule. This is used to evaluate the partition function for a dilute vortex gas. A remarkable consequence is that the partition function and related quantities are given in terms of the generalized Riemann zeta function. The topological phase transition is naturally understood as the pole structure of the zeta function.
Introduction

The study of vortex defects in two dimension has revealed many interesting properties which have had a wide variety of applications in condensed matter physics. For example, the superfluid $^4$He plays an important role in understanding the properties of the physics of vortex defects. Apart from the quantum fluid such as $^4$He, it has long been known that in classical hydrodynamics the equation of motion for the center of vortices forms a Hamiltonian system, (e.g. Kirchhoff equation) [1] where the fluid under consideration is assumed to be in an effectively two-dimensional incompressible fluid. In our recent papers we have developed a novel theory of vortex motion occurring in the two-dimensional quantum condensates, namely, the superfluid liquid Helium [2] as well as the Heisenberg spin model [3]. This theory shows that the vortex motion is described by the generalized Hamiltonian dynamics, which indicates the deviation from the conventional canonical form. Here we have used the time dependent Landau Ginzburg (LG) equation (action) to derive the effective action for the motion of center of vortices. The LG action is described by the complex order parameter field which is naturally introduced through the use of the coherent state path integral for the quantum condensates.

On the other hand, it is well known that the assembly of vortices occurring in the XY-model and the similar two-dimensional condensates reveal the topological phase transition, known as the Kosterlitz Thouless (KT) transition [4]. This shows the occurrence of dissolution of vortex-antivortex pairs at some finite temperature. This implies also the vortex pair excitations as the fluctuation out of a state with quasi long range order in two-dimensional XY and superfluid models.

The purpose of this paper is to study the quantum statistical mechanics of 2-dimensional vortex gas on the basis of the generalized Hamiltonian dynamics that is developed in the previous two papers. Specifically we are concerned with the statistical properties of vortex-antivortex pairs. We are first concerned with the quantization of a vortex pair by using the Bohr-Sommerfeld quantization scheme. By using the resultant quantized spectra for a vortex pair, we next evaluate the statistical partition function for a vortex pair in a dilute
gas limit. We get a remarkable consequence that the partition function is given in terms of
the generalized Riemann zeta function. As the characteristic property of the zeta function
we see that the topological phase transition of KT type is naturally explained by the pole
structure of the zeta function.

**Basic Formulation**

We start with a concise review of the previous results for the later discussion. The basic
standpoint is that the center of vortices $(x_i, y_i) \equiv \vec{X}_i$ are regarded as canonical conjugate
like variables. The effective action for the center of vortices is given by

$$S_{\text{eff}} = \int \left( \sum_i (A_i \frac{dx_i}{dt} - B_i \frac{dy_i}{dt}) - H_{\text{eff}} \right) dt. \quad (1)$$

Here the first term is called the canonical term and the coefficients $A_i, B_i$ are given in terms
of the “density function”, say $\sigma$, together with the function $\alpha_i \equiv \log |\vec{x}_i - \vec{X}_i|$, whereas the
second term represents the effective Hamiltonian for the motion of vortex centers which has
the well known log potential form:

$$H_{\text{eff}} = -\frac{1}{2} \rho_0 \sum_{ij} \mu_i \mu_j \log |\vec{X}_i - \vec{X}_j| + C \quad (2)$$

and hence the equation of motion for the vortex centers becomes

$$\sum_j G_{ij} \dot{\vec{X}}_j = -\frac{\partial H_{\text{eff}}}{\partial \vec{X}_i} \quad (3)$$

where $G$ is the metric tensor and $\mu$ denotes the strength (charge) of the vortex. $C$ represents
the chemical potential or self-energy that is given by a sum of the self-energy of isolated
vortices. The tensor $G$ coincides with the symplectic form (two-form) and it is derived from
the effective action $\omega$

$$d\omega = \sum_{ij} G_{ij} d\vec{X}_i \wedge d\vec{X}_j \quad (4)$$

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1 The definition of the density function $\sigma$ is different for two cases of bose and spin fluids; see ref.
[2] and [3].
with
\[
G = \begin{bmatrix} g^{11} & -g^{12} \\ g^{12} & -g^{22} \end{bmatrix}
\]  \hspace{1cm} (5)
where each submatrix \( g \) is given by the density function etc. through the coefficients \( A_i, B_i \) in (4) and the concrete form for these is given in term of the overlap integrals between the gradient of \( \sigma \) as well as the \( \alpha \) functions as has been done in the previous papers for the bose fluid and Heisenberg spin model. Thus the physical meaning of these terms suggests the effect of finite size of vortices. We see that
\[
g^{11} = g^{22} = 0, \quad g^{12} = g^0 + \Delta g. \]  \hspace{1cm} (6)
g^0 represents the diagonal and \( \Delta g \) the off-diagonal term of the matrix \( g^{12} \) respectively with respect to the vortex indices. We can introduce the poisson bracket using the tensor \( G \):
\[
\{A, B\}_{PB} = \sum_{ij} \left( \frac{\partial A}{\partial \vec{X}_i} \frac{\partial B}{\partial \vec{X}_j} - \{A \rightarrow B\} \right). \]  \hspace{1cm} (7)
As a special case we have
\[
\{\vec{X}_i, \vec{X}_j\} = (G^{-1})_{ij}. \]  \hspace{1cm} (8)
In the more concrete form,
\[
\{x_i, y_j\}_{PB} = (g^{12})^{-1} = (g^0)^{-1}_{ij} + \{(g^0)^{-2} \Delta g\}_{ij} + \cdots. \]  \hspace{1cm} (9)
The second term of LHS can be understood to mean that \( \Delta g \) causes the anomalous effect. Note that the above formulation can be regarded as an important generalization of the classical Kirchhoff equation \( \dot{x} = \{x, H\}_{PB} \) and \( \dot{y} = -\{y, H\}_{PB} \), with the Hamiltonian has the same logarithmic form, whereas \( \{A, B\}_{PB} \) denotes the usual Poisson bracket. The quantization is carried out by simply replacing the generalized PB by the commutator: \( \{x_i, y_j\} \rightarrow [x_i, y_j] \). Furthermore to be mentioned is the following fact: The equation of motion for the case that \( \Delta g \) vanishes is rewritten as
\[
(g_0)_{ii} (\vec{k} \times \vec{X}_i) = \frac{\partial H_{\text{eff}}}{\partial \vec{X}_i}. \]  \hspace{1cm} (10)
This equation suggests that the canonical equation of motion (Kirchhoff equation) is alternatively regarded as the balance equation of two type “forces”, the one is the right hand side, that represents the force derived from the “potential function” $H_{\text{eff}}$ and the other is the left hand side, that represents the velocity of the $i$-th vortex in a literal sense times the vorticity vector “$g_0\vec{k}$”, where $\vec{k}$ means the unit vector perpendicular to $(x, y)$ plane. The latter corresponds to the so-called Magnus force. 

**Quantized Spectra for a Vortex pair**

We shall now treat the system of two vortices whose charges are opposite each other, that is, $\mu_1 = -\mu_2 \equiv \mu$ and quantize it. The most interesting quantity is the quantized energy spectrum for the vortex pair, which will be used to evaluate the statistical function of the vortex gas. In order to carry out this, we first consider the case that the non-canonical effect is omitted and its effect will be taken into account as a perturbation. By having this in mind, we shall introduce the “center of mass” and “relative” coordinates as

$$x_1 + x_2 \equiv 2Q, \quad y_1 + y_2 \equiv 2P$$

and

$$x_1 - x_2 \equiv q, \quad y_1 - y_2 \equiv p$$

where $(x_i, y_i)$ is the coordinate of the center of $i$-th vortex. The effective Hamiltonian which is derivatived previously becomes

$$H_{\text{eff}} \equiv E = \frac{1}{4}\mu^2\rho_2 \log(q^2 + p^2) + C,$$

where the Hamiltonian is changed to positive definite. Hence we obtain the equations of motion as follows:

$$\frac{dQ}{dt} = 0, \quad \frac{dP}{dt} = 0,$$

It should be noted that this effective action do not have the cut-off parameter.
together with
\[
\frac{dq}{dt} = \frac{\rho_0 \mu^2 p}{2 (q^2 + p^2)}, \quad \frac{dp}{dt} = -\frac{\rho_0 \mu^2 q}{2 (q^2 + p^2)}.
\]

(15)

From the first set of equations, it follows that \(Q\) and \(P\) become constant of the motion, which means that the center of vortices is at rest. On the other hand, from the second set of equation it follows that the orbit in the \((p,q)\) plane forms a circle: \(p^2 + q^2 = \text{constant}\).

We shall carry out by using Bohr-Sommerfeld quantization, namely,
\[
\oint_C \omega_0 = (n + \frac{1}{2})\hbar.
\]

(16)

where \(n = \text{integer}\) and the factor the factor \(\frac{1}{2}\) is added, which corresponds to the zero point energy. Physically speaking, the above quantization rule suggests the quantization of the angular momentum carried by a vortex pair. In this way, if we note that the unperturbed orbit \(C\) for the relative motion becomes a circle of radius, say, \(R\), the quantization for non-perturbed case is calculated as
\[
\frac{g_0}{2} \int \int dq \wedge dp = (n + \frac{1}{2})\hbar
\]

(17)

which leads to
\[
\frac{g_0^0}{2} \pi R^2 = (n + \frac{1}{2})\hbar.
\]

(18)

Hence the radius \(R\) takes a quantized value; \(R^2_n = \frac{2\hbar}{g_0^0 \pi} n\) with \(n\) integers. In the above expression, \(g_0\) takes \(g_0 = m \rho_0 \mu\) (bose fluids), \(g_0 = J \mu \hbar\) (spin fluid). Next, if the off-diagonal term is taken account of as perturbation,
\[
R^2_{n'} = \frac{2}{g_0^{0}}(nh - \Gamma(R_{n'})).
\]

(19)

Here the value of \(\Gamma\) can be written as the function of \(R\). Qualitatively, this becomes small as the size of the vortex small and it tends to a constant value as \(\rightarrow \infty\) (the explicit evaluation

\[3\] The more exotic quantum condition may be obtained, which results in the quantum number of quater integer, see [5], but such a possibility is not considered hereafter.
may be given in the separate paper) [4]. In this way, $R_{n'}$ should be determined from the self-consistent equation, in other words, the quantum number $n$ changes to the modified value $n'$, which may be written as

$$n \to n' = n - \frac{\Gamma(n')}{\hbar}. \quad (20)$$

If we assume that the effect of $\Gamma$ is sufficiently small and hence put $\Gamma(R_{n'}) = -\tilde{\Gamma} = \text{constant}$, then we have the simpler form for the quantum condition:

$$R_{n'}^2 = \frac{2}{g_0^0 \pi} ((n + \frac{1}{2})\hbar + \tilde{\Gamma}) \quad (21)$$

which yields

$$n' = n + \frac{\tilde{\Gamma}}{\hbar}. \quad (22)$$

If we substitute (22) into (13), we get the energy spectra for the relative motion:

$$E_n = \frac{1}{4} \mu^2 \rho_0 \log R_{n'}^2 + C = \frac{1}{4} \mu^2 \rho_0 \log \left( \frac{2}{g_0^0 \pi} ((n + \frac{1}{2})\hbar + \tilde{\Gamma}) \right) + C. \quad (23)$$

Summarizing the above result, the angular momentum quantum number does not take integer values.

**Partition Function of Dilute Vortex Gas**

Now we investigate statistical mechanics of two-dimensional gas of vortices with strength $\mu$ (charge) in the medium of “charge” neutrality and each vortex interacts through log potential. The thermodynamical quantities are obtained through the partition function $Z = \text{Tr} \ e^{-\beta H_{eff}}$, where $\beta \equiv 1/kT$ and $T$ is the temperature and $k$ denotes the Boltzmann factor. The trace means the sum over all possible states the system is allowed to take. Here we restrict ourselves to a special situation: namely, the chemical potential $C$, which is necessary to create a pair of vortex of plus and minus charge, is very large. In this extreme situation, the vortex system can be treated as a dilute gas consisting vortex pairs. In this
limit, it suffices to treat only one pair of vortices to consider the partition function. This is easily evaluated to be

$$Z = \sum_{n=0}^{\infty} \exp \left[ -\beta H_{\text{eff}} \right]$$

$$= \sum_{n=0}^{\infty} \exp \left[ -\beta \left\{ \frac{1}{4} \mu^2 \rho_0 \log \left( \frac{2}{g^0 \pi} \right) (n + \frac{1}{2}) \bar{h} + \bar{\Gamma} \right) \right] + C$$  \hspace{1cm} (24)

which leads to

$$Z = \left( \frac{4 \bar{h}}{g^0} \right)^{-\frac{\beta \mu^2 \rho_0}{4}} \zeta \left( \frac{\beta \mu^2 \rho_0}{4}, \frac{1}{2} + \frac{\bar{\Gamma}}{\bar{h}} \right) \exp \left[ -\beta C \right]$$  \hspace{1cm} (25)

and the free energy of the system is given by

$$F = -kT \log Z.$$  \hspace{1cm} (26)

Here $\zeta(s, a)$ is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}$$  \hspace{1cm} (27)

which is just a generalized form of Riemann zeta function (in mathematical literature, this is called “Hurwitz zeta function”). The conventional zeta function is obtained by setting $\bar{\Gamma} = 0$. Remarkable is that $\zeta$ diverges at the point at $s = 1$; namely, $\zeta$ function has a simple pole at $s = 1$. This pole singularity causes the divergence of the free energy at finite temperature; $\frac{\beta \mu^2 \rho_0}{4k} = 1$, namely, the critical temperature is given by $T_c = \frac{\mu^2 \rho_0}{4k}$. This fact suggests that a phase transition occurs at this temperature. We shall also examine the susceptibility due to these pair, which is given by

$$\chi = \frac{1}{2} \mu^2 \beta < R^2 >$$  \hspace{1cm} (28)

where $< R^2 >$ represents the mean square separation of the each dipole pair, which becomes

$$< R^2 > = \frac{1}{Z} \text{Tr} \left\{ R^2 \exp [ -\beta H_{\text{eff}} ] \right\}$$

$$= \frac{\zeta \left( \frac{\beta \mu^2 \rho_0}{4} - 1, \frac{1}{2} + \frac{\bar{\Gamma}}{\bar{h}} \right)}{\zeta \left( \frac{\beta \mu^2 \rho_0}{4}, \frac{1}{2} + \frac{\bar{\Gamma}}{\bar{h}} \right)}.$$  \hspace{1cm} (29)
The above expression shows up the following remarkable consequence; the pole of \((29)\), namely, \(\frac{3m^2\rho_0}{4} - 1 = 1\), defines another transition temperature: If the temperature approaches to \(T_c\) from the below, the mean separation between the pair diverges at this temperature. We see that this position corresponds to nothing but the temperature \(T_c\) of the KT transition at which a dipole pair is made to dissociate. Note that this temperature is just half of \(T'_c\) noted in the above. The free energy or partition function itself remains finite at this temperature, namely, this feature is seen from the formula

\[
Z = \zeta(2, \frac{1}{2} + \frac{\tilde{\Gamma}}{k})
\] (30)

which is finite. In the above argument we assume that the position of pole and hence the transition temperature is not affected by the presence of shift of quantum number by an amount of \(\tilde{\Gamma}\), which follows from the theorem concerning the Hurwitz zeta function.

**Summary**

To summarize the present result is that after having calculated the energy spectra including the effect of the non-canonical term coming from the specific nature of quantum vortices, we have shown that the topological phase transition of KT type is characterized by the pole structure of the generalized Riemann zeta function. Up to our knowledge, this fact has not been known by anyone previously, which suggests that characteristics of statistical thermodynamics of vortex gas is made clear by exploiting the specific properties of zeta function. In this connection, we note the appearance of two different transition temperatures; \(T_c\) and \(T'_c\). It seems puzzled to have two different transition temperatures, but it should be noted that the present theory is based a rather naive assumption, that is, only one vortex pair is considered and no correlation, which arises from the other vortex pairs, is not taken into account. If we incorporate this correlation in a proper manner, we may have a correct value of the temperature for the topological phase transition. As the other problems we must consider the case that the noncanonical term has dependence of the quantum number \(n\) which is shown by the self-consistent nature of the Bohr-Sommerfeld equation. The details
of these discussion will be given elsewhere.

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