STRICT UNIQUE CONTINUATION FOR VARIABLE COEFFICIENT PARABOLIC OPERATORS WITH HARDY TYPE POTENTIAL

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Abstract. In this paper, we prove the strong unique continuation property at the origin for solutions of the following scaling critical parabolic differential inequality

$$|\text{div}(A(x,t)\nabla u) - u_t| \leq \frac{M}{|x|^2}|u|,$$

where the coefficient matrix $A$ is Lipschitz continuous in $x$ and $t$. Our main result sharpens a previous one of Vessella concerned with the subcritical case as well as extends a recent result of one of us with Garofalo and Manna for the heat operator.

1. Introduction

Roughly speaking, a differential operator $P$ is said to have the strong unique continuation property (sucp) if a solution $u$ to $Pu = 0$ vanishes to infinite order at a point in a connected domain, then $u \equiv 0$. The study of unique continuation problems has its root in an old paper of Carleman written in 1939, in which, Carleman studied the unique continuation problem associated with the Schrödinger operator $H = -\Delta + V$ with bounded potential $V$ in $\mathbb{R}^2$ by means of a weighted apriori estimate. Such an estimate was subsequently extended to variable coefficient elliptic operators with Lipschitz principal part in [3], see also [2] where the case of $C^2$ coefficients was earlier considered. We refer to some of the later prominent works [9, 10, 13, 14, 15] in this subject and an interested reader can find other references therein. An alternate method based on the almost monotonicity of a generalization of the frequency function, first introduced by Almgren in [1] came up in the work of Garofalo and Lin in [12]. Using this approach, they were able to obtain new quantitative uniqueness information for the solutions to divergence form elliptic equations with Lipschitz coefficients which in particular encompassed the results in [3]. We recall that unique continuation fails in general when the coefficients are only Hölder continuous, see for instance [22, 19].

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In this paper, we obtain strong unique continuation property at the origin for solutions to the differential inequality
\[ |\text{div}(A(x,t)\nabla u) - u_t| \leq \frac{M}{|x|^2} |u|, \] (1.1)
where \( M > 0 \) and the matrix valued function \( A \) satisfies the following uniform ellipticity and the Lipschitz growth condition
\[
\begin{aligned}
\Lambda^{-1}I &\leq A(x,t) \leq \Lambda I &\text{for some } \Lambda > 1, \\
|A(x,t) - A(y,s)| &\leq K(|x-y| + |t-s|). 
\end{aligned}
\] (1.2)

It is well-known that the inverse-square potential \( V(x) = \frac{M}{|x|^2} \) represents a critical scaling threshold in quantum mechanics [4], and it is equally known that its singularity is the limiting case for the sucp for the differential inequality \( \Delta u \leq \frac{M}{|x|^2n} |u| \), see the counterexample in [12]. Such potential fails to be in \( L^{n/2}_{\text{loc}} \), and in general does not have small \( L^{n/2,\infty} \) seminorm, thus in the context of the Laplacian the sucp cannot be treated by the celebrated result of Jerison and Kenig in [14] or the subsequent improvement by Stein in the appendix to the same paper. We recall that, in the time-independent case of the Laplacian, the sucp for the unrestricted inverse square potential was proved by Pan in [18]. This was later extended to Lipschitz principal part by Regbaoui in [23]. We refer to [17] for quantitative results in this setting and also to [6] for a similar result in the subelliptic framework of a subclass of Grushin type operators.

Now in the parabolic setting, in [24] (see also [11]) Vessella proved a general sucp result for sub-critical parabolic equations of the type
\[ |\text{div}(A(x,t)\nabla u) - u_t| \leq \frac{M}{|x|^{2-\delta}} |u|, \quad \delta > 0, \] (1.3)
under the same Lipschitz regularity assumptions on the principal part \( A(x,t) \) as in (1.2) above. Recently, one of us with Garofalo and Manna in [5] proved the sucp for solutions to the scaling critical differential inequality (1.1) in the case when \( A = I \). This was done by means of an improved Carleman estimate in the case of the heat operator \( \Delta - \partial_t \) in space-time cylinders. Similar to the time-independent case in [18] and [23], the proof of such a scaling critical Carleman estimate in [5] exploited the spectral gap on \( S^{n-1} \) combined with another delicate apriori estimate which was an important new feature in the parabolic setting. The purpose of this work is to extend the sucp result in [5] to parabolic operators with Lipschitz principal part. Our main result Theorem 2.2 constitutes the parabolic counterpart of the one due to Regbaoui in [23].

The following are the key steps in the proof of our main result.

**Step 1:** Inspired by ideas in [23], we first establish a sharpened version of the scaling critical Carleman estimate in [5] (see (3.1) below). This has required some delicate reworking of the ideas in [5]. Using such an estimate combined with a suitable change of variable in the time-dependent setting, we show that if a solution \( u \) to (1.1) vanishes to infinite order at the origin in the sense of (2.1) below, then it decays exponentially. See Proposition 3.6 below.

**Step 2:** Then by means of a parabolic generalization of a Carleman estimate due to Regbaoui in [23] (see (3.94) below), we then show that non-trivial solutions to (1.1) in fact decay less than exponentially which then lead to a contradiction and thus establishes our main result Theorem 2.2. We mention that the proof of the corresponding estimate in [23] uses in a crucial way the polar decomposition of the frozen constant coefficient operator combined with a Hörmander type commutator estimate. Our proof in the parabolic case instead is based on a suitable adaptation of a Rellich type identity and is partly inspired by some ideas in the recent work [7] where a similar estimate has been established for Grushin type operators.
However as the reader will see, our proof entails some subtle modifications in the time dependent setting. This is in fact one of the key novelties of our work.

The paper is organized as follows. In Section 2, we introduce some relevant notions, gather some known results and then state our main strong unique continuation result. In Section 3, we prove our main result.

2. Preliminaries

Given \( r > 0 \) we denote by \( B_r(x_0) \) the Euclidean ball centred at \( x_0 \in \mathbb{R}^n \) with radius \( r \). When \( x_0 = 0 \), we will use the simpler notation \( B_r \). A generic point in space time \( \mathbb{R}^n \times (0, \infty) \) will be denoted by \( (x,t) \). For notational convenience, \( \nabla f \) and \( \text{div} \ f \) will respectively refer to the quantities \( \nabla_x f \) and \( \text{div}_x f \) of a given function \( f \). The partial derivative in \( t \) will be denoted by \( \partial_t f \) and also by \( f_t \). We indicate with \( C^\infty_0(\Omega) \) the set of compactly supported smooth functions in the region \( \Omega \) in space-time. By \( H^2_{\text{loc}}(\Omega) \) we refer to the parabolic Sobolev class of functions \( f \in L^2_{\text{loc}}(\Omega) \) for which the weak derivatives \( \nabla f, \nabla^2 f \) and \( \partial_t f \) belong to \( L^2_{\text{loc}}(\Omega) \). For a point \( x \in \mathbb{R}^n \setminus \{0\} \), we will routinely adopt the notation \( r = r(x) = |x| \) and \( \omega = \frac{x}{r} \in \mathbb{S}^{n-1} \), so that \( x = r\omega \). The radial derivative of a function \( v \) is \( v_r = \langle \nabla v, \frac{x}{|x|} \rangle \).

The relevant notion of vanishing to infinite order is as follows.

**Definition 2.1.** We say a function \( u \) parabolically vanishes to infinite order if for all \( k > 0 \), we have

\[
\int_{B_r \times (0,T)} u^2 = O(r^k), \quad r \to 0. \tag{2.1}
\]

**Statement of the main result.** We now state our main result.

**Theorem 2.2.** Suppose that for some \( M, R, T > 0 \) the function \( u \in H^2_{\text{loc}} \) be a solution in \( B_R \times (0,T) \) to the differential inequality (1.1). If \( u \) parabolically vanishes to infinite order at the origin in the sense of (2.1), then \( u \equiv 0 \) in \( B_R \times (0,T) \).

**Remark 2.3.** We remark that even if we assume that \( u \in H^1_{\text{loc}} \) (i.e. \( u, \nabla u \in L^2_{\text{loc}} \)) is a weak solution to (1.1) satisfying parabolic vanishing property (2.1), then it follows that \( u \in H^2_{\text{loc}} \) and moreover we have that \( \nabla u, \nabla^2 u \) and \( u_t \) vanish to infinite order in the sense of (2.1) above. This is seen as follows. Recall the following estimate from [16, Chapter 6]

\[
\int_{Q_{2r}(0,t_0) \setminus Q_r(0,t_0)} \left( u_t^2 + |\nabla^2 u|^2 \right) \leq \frac{1}{r^4} \int_{Q_{4r}(0,t_0) \setminus Q_{2r}(0,t_0)} u^2, \tag{2.2}
\]

for any \( t_0 \in (-3T/4,3T/4) \). Since \( u \) vanishes parabolically to infinite order, we conclude

\[
\int_{Q_{2r}(0,t_0) \setminus Q_r(0,t_0)} \left( u_t^2 + |\nabla^2 u|^2 \right) \lesssim r^k, \tag{2.3}
\]

for all \( k \geq 1 \). From (2.3) it follows

\[
\int_{B_{2r} \setminus B_r \times (-3T/4,3T/4)} \left( u_t^2 + |\nabla^2 u|^2 \right) \lesssim r^k. \tag{2.4}
\]

Now for any \( r < 1 \), recursively applying (2.4) with \( r_i = \frac{r}{2^i} \) for all \( i \geq 1 \), then summing over the annular regions we obtain the following

\[
\int_{B_r \times (-3T/4,3T/4)} \left( u_t^2 + |\nabla^2 u|^2 \right) = O(r^k), \tag{2.5}
\]

for all \( k \geq 1 \). Moreover by applying Moser type estimate in annular regions, it also follows that a solution \( u \) to (1.1) satisfying (2.1) also decays to infinite order in the \( L^\infty \) norm in the following sense

\[
\|u\|_{L^\infty(B_r \times (-3T/4,3T/4))} = O(r^k), \tag{2.5}
\]

for all \( k \geq 1 \). Therefore one only requires \( u \in H^1_{\text{loc}} \) in Theorem 2.2 above.
We close this section with the following elementary algebraic inequality that will be needed in the proof of Theorem 3.1 below.

**Lemma 2.4.** Given $a, b \in \mathbb{R}^n$ and $\delta > 0$, the following inequality holds

$$
(1 + \delta)|a|^2 + |b|^2 + 2\langle a, b \rangle \geq C(\delta)(|a|^2 + |b|^2),
$$

(2.6)

for some $C(\delta) > 0$.

**Proof.** By an application of Cauchy-Schwartz inequality, we have

$$
2\langle a, b \rangle \geq -(1 + \frac{\delta}{2})|a|^2 - \frac{1}{1 + \frac{\delta}{2}}|b|^2.
$$

(2.7)

Using (2.7) we find

$$
(1 + \delta)|a|^2 + |b|^2 + 2\langle a, b \rangle \geq \frac{\delta}{2}|a|^2 + \frac{\delta}{2(1 + \frac{\delta}{2})}|b|^2
$$

(2.8)

$$
\geq \frac{\delta}{2(1 + \frac{\delta}{2})}(|a|^2 + |b|^2).
$$

Thus the inequality (2.6) is seen to hold with $C(\delta) = \frac{\delta}{2(1 + \frac{\delta}{2})}$. □

3. Proof of the main result

3.1. Carleman estimate I. We first state and prove our first Carleman estimate with singular weights which can be regarded as a certain sharpened version of Theorem 1.1 in [5]. The difference of our estimate as in (3.1) below from that in [5, Theorem 1.1] is the incorporation of higher order terms. This is needed in order to absorb certain error terms that arises due to the perturbation of the principal part. Such an estimate in conjunction with the infinite vanishing property (2.1) leads to the exponential decay of solutions to (1.1). As the reader will see, this requires some subtle reworking of the proof of Theorem 1.1 in [5].

**Theorem 3.1.** Let $R < 1$ and let $u \in C_0^\infty((B_R \setminus \{0\}) \times (0,T))$. For all $\alpha$ sufficiently large of the form $\alpha = k + \frac{n+1}{2}$ with $k \in \mathbb{N}$, and every $0 < \varepsilon << 1$ very small, we have

$$
\alpha \int_{B_R \times (0,T)} |x|^{-2\alpha-4}e^{2\alpha|x|} u^2 dxdt + \alpha^3 \int_{B_R \times (0,T)} |x|^{-2\alpha-4+\varepsilon} e^{2\alpha|x|} |\nabla u|^2 dxdt \leq C \int_{B_R \times (0,T)} |x|^{-2\alpha} e^{2\alpha|x|} (\Delta u - \partial_t u)^2 dxdt
$$

(3.1)

where $C = C(\varepsilon, n) > 0$.

We record the following simple lemma which can be regarded as an integration by parts formula for radial derivatives, will be repeatedly used in our analysis. See for instance Lemma 2.1 [5].

**Lemma 3.2.** Let $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $g \in C^\infty(\mathbb{R}^n \setminus \{0\})$, then

$$
\int_{\mathbb{R}^n} fr g dx = -\int_{\mathbb{R}^n} f r g dx - (n-1) \int_{\mathbb{R}^n} r^{-1} fg dx.
$$
Proof of Theorem 3.1. Before we proceed with the proof, in order to avoid any notational confusion, we declare that, in what follows, the domain of all the integrations will be the parabolic cylinder $B_R \times (0, T)$ (or, for that matter, the whole of $\mathbb{R}^n \times \mathbb{R}$, in view of the support property of the integrands), but this will not be explicitly indicated. Nor we will explicitly write the measure $dxdt$ in any of the integrals involved. Let $r = |x|$ and also let $v = r^{-\beta} e^{\alpha r^\varepsilon} u$, where as in [5], $\beta$ is chosen in the following way

$$2\beta - 2\alpha - 4 = -n. \quad (3.2)$$

Thus $u = r^\beta e^{-\alpha r^\varepsilon} v$ and an easy calculation yields

$$\Delta (r^\beta e^{-\alpha r^\varepsilon}) = \left(\alpha^2 \varepsilon^2 r^2 + 2 \beta (\beta + n - 2) r^\beta - \frac{\alpha \varepsilon ((2\beta + \varepsilon + n - 2)) r^{\beta + 2 \varepsilon - 2}}{\Delta + (2\beta + \varepsilon + n - 2) r^{\beta + 2 \varepsilon - 2}}\right) e^{-\alpha r^\varepsilon}$$

leading to

$$\Delta u = r^\beta e^{-\alpha r^\varepsilon} \Delta v + \left(\alpha^2 \varepsilon^2 r^2 + 2 \beta (\beta + n - 2) r^\beta e^{-\alpha r^\varepsilon} ((2\beta + \varepsilon + n - 2) r^{\beta + 2 \varepsilon - 2})\right) + \left(2\beta r^{\beta - 2} - 2 \varepsilon \alpha r^{\beta + \varepsilon - 2}\right) e^{-\alpha r^\varepsilon} v$$

$$\quad + \left(2 \beta r^\beta - 2 \varepsilon \alpha r^{\beta + \varepsilon - 2}\right) e^{-\alpha r^\varepsilon} < x, \nabla v >.$$ 

Since $\Delta v(x, t) = v_{rr}(r\omega, t) + \frac{n-1}{r} v_r(r\omega, t) + \frac{1}{r^2} \Delta_{S^{n-1}} v(r\omega, t)$, where $\omega \in S^{n-1}$ and $\Delta_{S^{n-1}}$ denotes the Laplacian on $S^{n-1}$, we obtain

$$\Delta u - \partial_t u = r^\beta e^{-\alpha r^\varepsilon} \left[\left(\alpha^2 \varepsilon^2 r^2 + 2 \beta (\beta + n - 2) r^\beta - \frac{\alpha \varepsilon ((2\beta + \varepsilon + n - 2)) r^{\beta + 2 \varepsilon - 2}}{\Delta + (2\beta + \varepsilon + n - 2) r^{\beta + 2 \varepsilon - 2}}\right) v + \left(2\beta r^\beta - 2 \varepsilon \alpha r^{\beta + \varepsilon - 2}\right) v_r + \left(2 \beta r^\beta - 2 \varepsilon \alpha r^{\beta + \varepsilon - 2}\right) v_{rr} - v_t\right]. \quad (3.3)$$

We now apply the algebraic inequality $(a + b)^2 \geq a^2 + 2ab$, with

$$a = r^\beta e^{-\alpha r^\varepsilon} \left(\beta (\beta + n - 2) v + \Delta_{S^{n-1}} v + (2\beta + n - 1) r v_r + r^2 v_{rr}\right),$$

and

$$b = r^\beta e^{-\alpha r^\varepsilon} \left(\alpha^2 \varepsilon^2 r^2 v - \alpha \varepsilon (2\beta + \varepsilon + n - 2) r^{\beta + 2 \varepsilon - 2} v - 2 \alpha \varepsilon r^{\beta + \varepsilon - 1} r v_r - v_t\right),$$

obtaining

$$\int r^{-2\alpha} e^{2\alpha r^\varepsilon} (\Delta u - \partial_t u)^2 \geq I + II + III + IV \quad (3.4)$$

where

$$I := \int r^{-2\alpha + 2\beta - 4} \left(\beta (\beta + n - 2) v + \Delta_{S^{n-1}} v\right)^2; \quad (3.5)$$

$$II := 2\beta (\beta + n - 2) (2\beta + n - 1) \int r^{-2\alpha + 2\beta - 3} v v_r, \quad (3.6)$$

$$III := (2\beta + n - 1)^2 \int r^{-2\alpha + 2\beta - 2} v_r^2 + \int r^{-2\alpha + 2\beta - 2} v_{rr}^2 + 2(2\beta + n - 1) \int r^{-2\alpha + 2\beta - 3} v \Delta_{S^{n-1}} v$$

$$+ 2\beta (\beta + n - 2) \int r^{-2\alpha + 2\beta - 2} v v_{rr} + 2 \int r^{-2\alpha + 2\beta - 2} v_{rr} \Delta_{S^{n-1}} v$$

$$\quad + 2(2\beta + n - 1) \int r^{-2\alpha + 2\beta - 1} v v_{rr},$$

and


Now we proceed to estimate each integral listed above. For the convenience of the reader we have grouped the integrals in the above way. More precisely, I, II, III cover all the terms coming from \( a^2 \) while IV contains all the terms that comes from \( 2ab \). Also note that in our case, the choice of \( a \) and \( b \) are different from that in [5]. With \( \beta \) as in (3.2), we find that the integral II vanishes by an application of Lemma 3.2 analogous to that in [5].

\textbf{Estimate for III:}

We now turn our attention to the terms of III. From [5, (2.6), (2.11) and (2.12)], it follows

\[ 2 \int r^{-2\alpha+2\beta-3} v_r \Delta_{S^{n-1}} v = 0, \]

\[ \int r^{-2\alpha+2\beta-2} v_{rr} = - \int r^{-n+2} v_r^2, \]

and

\[ \int r^{-2\alpha+2\beta-1} v_r v_{rr} = - \int r^{-n+2} v_r^2. \]

Again another application of Lemma 3.2 gives

\[ 2 \int r^{-2\alpha+2\beta-2} v_{rr} \Delta_{S^{n-1}} v = 2 \int r^{-n+2} v_{rr} \Delta_{S^{n-1}} v \]

\[ = -2 \int r^{-n+2} v_r \Delta_{S^{n-1}} v - 2(n-1) \int r^{-n+1} v_r \Delta_{S^{n-1}} v \]

\[ = 2 \int_0^T \int_0^\infty r \int_{S^{n-1}} |\nabla_{S^{n-1}} v_r|^2 d\omega dr dt, \]

since \( \int r^{-n+1} v_r \Delta_{S^{n-1}} v = 0 \). Now we use the notation \( \nabla_T v := r^{-1} \nabla_{S^{n-1}} v \) in view of which, we have

\[ |\nabla v|^2 = v_r^2 + |\nabla_T v|^2. \]

So, this notation along with (3.10) yield

\[ 2 \int r^{-2\alpha+2\beta-2} v_{rr} \Delta_{S^{n-1}} v = 2 \int r^{-n+4} |\nabla_T v_r|^2. \]

(3.11)
Thus from (3.7)-(3.11) it follows
\[ III \geq 2\alpha^2 \int r^{-n+2}v_r^2 + \int r^{-n+4}v_{rr}^2 + 2 \int r^{-n+4}|\nabla T v_r|^2. \tag{3.12} \]

We now proceed to estimate IV. **Estimate for IV**: First we notice by integrating by parts in $t$ as in [5] that the following holds
\[ \int r^{-2\alpha+2\beta-2}v_t v_t = 0, \tag{3.13} \]
and
\[ 2 \int r^{-2\alpha+2\beta-2}v_t \Delta g_{n-1} v = 0. \tag{3.14} \]

Next, as in [5, (2.7)-(2.9)] we have
\[ -4\alpha\varepsilon \int r^{-2\alpha+2\beta+\varepsilon-3}v_r \Delta g_{n-1} v = -2\alpha\varepsilon^2 \int r^{-n+\varepsilon} |\nabla g_{n-1} v|^2, \tag{3.15} \]
and
\[ -2\alpha \varepsilon (2\beta + \varepsilon + n - 2) \int r^{-2\alpha+2\beta+\varepsilon-4}v \Delta g_{n-1} v = 2\alpha \varepsilon (2\alpha + \varepsilon + 2) \int r^{-n+\varepsilon} |\nabla g_{n-1} v|^2, \tag{3.16} \]
where in the last inequality we have used $r^{\varepsilon} < 1$ as $0 < r \leq R < 1$. So, from (3.15), (3.16) and (3.17) we observe that for $\varepsilon$ sufficiently small (for instance take $0 < \varepsilon < \frac{1}{2\alpha}$)
\[ -4\varepsilon \alpha \int r^{-2\alpha+2\beta+\varepsilon-3}v_r \Delta g_{n-1} v - 2\alpha \varepsilon (2\beta + \varepsilon + n - 2) \int r^{-2\alpha+2\beta+\varepsilon-4}v \Delta g_{n-1} v \]
\[ \geq \frac{39}{10} \alpha^2 \varepsilon \int r^{-n+\varepsilon} |\nabla g_{n-1} v|^2. \tag{3.18} \]

Furthermore, it follows from [5, (2.14)] that for all $\alpha$ sufficiently large and $\varepsilon$ small enough, the following inequality holds for some $C > 0$ universal
\[ -2\alpha \varepsilon (2\beta + n - 1)(2\beta + \varepsilon + n - 2) \int r^{-2\alpha+2\beta+\varepsilon-3}v v_r + 2\alpha^2 \varepsilon^2 (2\beta + n - 1) \int r^{-2\alpha+2\beta+2\varepsilon-3}v v_r \tag{3.19} \]
\[ -4\alpha \varepsilon \beta (n + 2) \int r^{-2\alpha+2\beta+\varepsilon-3}v v_r - 2\alpha \varepsilon \beta (n + 2)(2\beta + \varepsilon + n - 2) \int r^{-2\alpha+2\beta+\varepsilon-4}v^2 \]
\[ + 2\alpha^2 \varepsilon^2 \beta (n + 2) \int r^{-2\alpha+2\beta+2\varepsilon-4}v^2 \geq -C \alpha^4 \varepsilon \int r^{-n+\varepsilon} v^2. \]

Also as in [5, (2.16)] we have for all $\alpha$ large
\[ (2\beta + n - 1)^2 \int r^{-2\alpha+2\beta-2}v_r^2 + 2(2\beta + n - 1) \int r^{-2\alpha+2\beta-1}v_r v_{rr} \tag{3.20} \]
\[ -4\alpha \varepsilon (2\beta + n - 1) \int r^{-2\alpha+2\beta+\varepsilon-2}v_r^2 \]
\[ \geq [(2\alpha + 3)^2 - 2(2\alpha + 3) - \alpha^2] \int r^{-\alpha+2}v_r^2 \geq 2\alpha^2 \int r^{-\alpha+2}v_r^2. \]
Now we will handle the integrals involving the term $v_{rr}$. Once again using Lemma 3.2 we have
\[
\int r^{-2\alpha+2\beta+2\varepsilon-2} v_{rr} = - \int r^{-n+2\varepsilon+2} v_{r}^2 - (2\varepsilon + 1) \int r^{-n+2\varepsilon+1} v_r v.
\]
Again applying the same lemma to the last term we get
\[
\int r^{-n+2\varepsilon+1} v_r v = -\varepsilon \int r^{-n+2\varepsilon} v^2
\]
which yields
\[
\int r^{-2\alpha+2\beta+2\varepsilon-2} v_{rr} = - \int r^{-n+2\varepsilon+2} v_{r}^2 + (2\varepsilon + 1) \int r^{-n+2\varepsilon} v^2. \tag{3.21}
\]
Similarly, we can check that
\[
\int r^{-2\alpha+2\beta+\varepsilon-2} v_{rr} = - \int r^{-n+\varepsilon+2} v_{r}^2 + \frac{1}{2} \varepsilon \int r^{-n+\varepsilon} v^2, \tag{3.22}
\]
and also
\[
\int r^{-2\alpha+2\beta+\varepsilon-1} v_{rr} = -(\varepsilon + 2) \int r^{-n+\varepsilon+1} v_{r}^2. \tag{3.23}
\]
Hence we have for all large enough $\alpha$ and small $\varepsilon$ that the following inequality holds
\[
2\alpha^2 \varepsilon^2 \int r^{-2\alpha+2\beta+2\varepsilon-2} v_{rr} - 2\alpha \varepsilon (2\beta + \varepsilon + n - 2) \int r^{-2\alpha+2\beta+\varepsilon-2} v_{rr} = 4\alpha \varepsilon \int r^{-2\alpha+2\beta+\varepsilon-1} v_{rr} \geq 2\alpha^2 \varepsilon^2 - C\alpha^2 \varepsilon^2 \int r^{-n+\varepsilon} v^2. \tag{3.24}
\]
Finally, we complete estimates of the integrals involving the term $v_t$. Recall that we have already dealt with the 13th and 14th integrals of IV, which turn out to be zero. See (3.13) and (3.14) above, So, we are just left with the 15th and 16th integral in this regard. Now the 15th integral can be handled as follows
\[
\left| 2(2\beta + n - 1) \int r^{-2\alpha+2\beta-1} v_{t} v_r \right| \leq 4\alpha (1 + \frac{3}{2\alpha}) \int r^{-n+3} |v_t||v_r| \tag{3.25}
\]
\[
\leq 5\alpha \left( \frac{\alpha}{5} \int r^{-n+2} v_{r}^2 + \frac{5}{\alpha} \int r^{-n+4} v_{t}^2 \right) \leq \alpha^2 \int r^{-n+2} v_{r}^2 + 25 \int r^{-n+4} v_{t}^2.
\]
Also, the 16th integral can be handled as follows.
\[
\left| 2 \int r^{-2\alpha+2\beta} v_{t} v_{rr} \right| \leq 2 \int r^{-n+4} |v_t||v_{rr}| \leq 8 \int r^{-n+4} v_{t}^2 + \frac{1}{2} \int r^{-n+4} v_{r}^2. \tag{3.26}
\]
Therefore, we conclude that
\[
IV \geq -C\alpha^4 \varepsilon \int r^{-n+\varepsilon} v^2 + \frac{39}{10} \alpha^2 \varepsilon \int r^{-n+\varepsilon} |\nabla g_{n-1} v|^2 - \alpha^2 \int r^{-n+2} v_{r}^2 \tag{3.27}
\]
\[
+ 33 \int r^{-n+4} v_{r}^2 - \frac{1}{2} \int r^{-n+4} v_{r}^2.
\]
Hence from (3.4)-(3.27), it follow
\[
II + III + IV \geq \alpha^2 \int r^{-n+2} v_{r}^2 + \frac{1}{2} \int r^{-n+4} v_{r}^2 - C\alpha^4 \varepsilon \int r^{-n+\varepsilon} v^2 - 33 \int r^{-n+4} v_{t}^2 \tag{3.28}
\]
\[
+ 2 \int r^{-n+4} |\nabla T v_{t}|^2 + \frac{39}{10} \alpha^2 \varepsilon \int r^{-n+\varepsilon} |\nabla g_{n-1} v|^2.
\]
Now we are just left with estimating the integral $I$.

**Estimates for I:** We estimate I in two ways in order to incorporate a critical zero order term and also a tangential second derivative term. First we estimate I using the spherical harmonic decomposition of
$L^2(\mathbb{S}^{n-1})$. This part of the argument is similar to that in [5] which uses the spectral gap of the spherical Laplacian. However in our estimate, we also additionally incorporate a tangential gradient term that is required eventually. Thus we provide all the details. For the sake of the convenience of the reader, we recall some notations and basics facts about spherical harmonics here.

Let $H_k$ denote the space of homogeneous harmonic polynomials of degree $k \geq 0$. Restrictions of elements of $H_k$ to $\mathbb{S}^{n-1}$ are called the spherical harmonics of degree $k$, which is denoted by $\mathcal{H}_k$. The spherical harmonic decomposition reads as $L^2(\mathbb{S}^{n-1}) = \bigoplus_{k \geq 0} \mathcal{H}_k$. Here $\mathcal{H}_k$’s are finite dimensional subspaces, an orthonormal basis of which is given by $\{S_{k,j} : 1 \leq j \leq d_k\}$, where $d_k$ denotes the dimension of $\mathcal{H}_k$. Then it is well known that $\{S_{k,j} : 1 \leq j \leq d_k, k \geq 0\}$ forms an orthonormal basis for $L^2(\mathbb{S}^{n-1})$, and $\Delta_{\mathbb{S}^{n-1}} S_{k,j} = -k(k+n-2)S_{k,j}$.

So, writing $v(x, t) = v(r, t)$, the expansion of $v$ in terms of spherical harmonics takes the form

$$v(rw, t) = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} v_{k,j}(r, t) S_{k,j}(w)$$

where $v_{k,j}$ denote the spherical harmonic coefficient of $v$ defined by

$$v_k(r, t) = \int_{\mathbb{S}^{n-1}} v(rw, t) S_{k,j}(w) d\sigma(w).$$

This leads to the following spectral decomposition of the spherical Laplacian:

$$\Delta_{\mathbb{S}^{n-1}} v(rw, t) = -\sum_{k=0}^{\infty} \sum_{j=1}^{d_k} k(k+n-2)v_k(r, t) S_{k,j}(w)$$

which will help us in estimating $I$. This decomposition along with the orthonormality of spherical harmonics gives

$$\int r^{-2\alpha+2\beta-4} (\beta(\beta + n - 2)v + \Delta_{\mathbb{S}^{n-1}} v)^2$$

$$= \int r^{-n} (\beta(\beta + n - 2)v + \Delta_{\mathbb{S}^{n-1}} v)^2$$

$$= \int_0^T \int_0^\infty r^{-1} \sum_{k=0}^{d_k} \sum_{j=1}^{d_k} (\beta(n - 2) - k(k + n - 2))^2 v_{k,j}(r, t)^2 dr dt.$$

Now given that $\alpha$ is of the form $\alpha = k + \frac{n+1}{2}$, in view of (3.2) it follows that $\text{dist}(\beta, N) = \frac{1}{2}$. This implies the following inequality

$$(\beta(\beta + n - 2) - k(k + n - 2))^2 \geq \frac{1}{2} \beta(\beta + n - 2) + \frac{1}{2} k(k + n - 2)$$

which yields

$$I \geq \int_0^T \int_0^\infty r^{-1} \sum_{k=0}^{d_k} \sum_{j=1}^{d_k} \left( \frac{1}{2} \beta(\beta + n - 2) + \frac{1}{2} k(k + n - 2) \right) v_{k,j}(r, t)^2 dr dt$$

$$= \frac{1}{2} \beta(\beta + n - 2) \int r^{-n} v^2 + \frac{1}{2} \int r^{-n} |\nabla_{\mathbb{S}^{n-1}} v|^2.$$

Now in in view of (3.2), we have that $\frac{1}{2} \beta(\beta + n - 2) \geq \frac{\alpha^2}{2}$ which leads to the estimate

$$I \geq \frac{\alpha^2}{2} \int r^{-n} v^2 + \frac{1}{2} \int r^{-n+2} |\nabla_T v|^2.$$

(3.29)
Before we estimate $I$ in another way, recalling (3.11) we note that

$$(\nabla_T v)_r = \partial_r (r^{-1} \nabla_{S^{n-1}} v) = -r^{-2} \nabla_{S^{n-1}} v + r^{-1} \nabla_{S^{n-1}} v_r = -r^{-1} \nabla_T v + \nabla_T v_r$$

leading to

$$\int r^{-n+4} |\nabla_T v_r|^2 = \int r^{-n+4} |(\nabla_T v)_r + r^{-1} \nabla_T v|^2$$

$$= \int r^{-n+4} |(\nabla_T v)_r|^2 + 2 \int r^{-n+3} (\nabla_T v)_r, \nabla_T v + \int r^{-n+2} |\nabla_T v|^2.$$ 

Now using the algebraic inequality in (2.6) (with $\delta = \frac{1}{2}$) we find that for some $0 < c_0 < 1/4$

$$\int r^{-n+4} |(\nabla_T v)_r|^2 + 2 \int r^{-n+3} (\nabla_T v)_r, \nabla_T v + \int r^{-n+2} |\nabla_T v|^2 \geq c_0 \left( \int r^{-n+2} |\nabla_T v|^2 + \int r^{-n+4} |(\nabla_T v)_r|^2 \right).$$

Thus from (3.28) it follows by writing

$$2 \int r^{-n+4} |\nabla_T v_r|^2 = \int r^{-n+4} |\nabla_T v_r|^2 + \int r^{-n+4} |\nabla_T v_r|^2$$

and by applying the estimate in (3.30) to one of the term in the right hand side of (3.31) above that the following holds

$$\int r^{-2a} e^{2ar^2} (\Delta u - \partial_t u)^2 \geq \alpha^2 \int r^{-n+2} v^2 + \frac{1}{2} \int r^{-n+4} v_r^2$$

$$- C\alpha^2 \epsilon \int r^{-n+\epsilon} v^2 - 33 \int r^{-n+4} v_r^2$$

$$+ \int r^{-n+4} |\nabla_T v_r|^2 + \frac{\alpha^2}{2} \int r^{-n} v^2 + c_0 \left( \int r^{-n+2} |\nabla_T v|^2 + \int r^{-n+4} |(\nabla_T v)_r|^2 \right).$$

Now we estimate $I$ by simply expanding the square in the integrand as follows:

$$I = \int r^{-2a+2\beta-4}(\beta(\beta+n-2)v + \Delta_{S^{n-1}} v)^2$$

$$= \int r^{-n} (\Delta_{S^{n-1}} v)^2 + (\beta(\beta+n-2))^2 \int r^{-n} v^2 + 2\beta(\beta+n-2) \int r^{-n} v \Delta_{S^{n-1}} v$$

$$\geq \int r^{-n} (\Delta_{S^{n-1}} v)^2 + \alpha^4 \int r^{-n} v^2 - 2\beta(\beta+n-2) \int r^{-n} |\nabla_{S^{n-1}} v|^2.$$ 

Likewise from (3.28) and (3.33) we deduce the following estimate

$$\int r^{-2a} e^{2ar^2} (\Delta u - \partial_t u)^2 \geq \int r^{-n} (\Delta_{S^{n-1}} v)^2 + \alpha^4 \int r^{-n} v^2 - 2\beta(\beta+n-2) \int r^{-n} |\nabla_{S^{n-1}} v|^2$$

$$\quad + \alpha^2 \int r^{-n+2} v^2 + \frac{1}{2} \int r^{-n+4} v_r^2 + 2 \int r^{-n+4} |\nabla_T v_r|^2$$

$$\quad - C\alpha^4 \epsilon \int r^{-n+\epsilon} v^2 - 33 \int r^{-n+4} v_r^2.$$
We now employ an idea from [23]. By multiplying (3.32) with $\frac{8\beta}{c_0}(\beta + n - 1)$ and then adding with (3.34), we obtain
\[
\left(1 + \frac{8\beta}{c_0}(\beta + n - 1)\right) \int r^{-2\alpha} e^{2\alpha r}\epsilon (\Delta u - \partial_t u)^2 \\
\geq \frac{4\beta}{c_0} (\beta + n - 1) \alpha^2 \int r^{-n} v^2 + 4\beta^2 \left( \int r^{-n+2} |\nabla_T v|^2 + \int r^{-n+4} |(\nabla_T v)_r|^2 \right) + \alpha^4 \int r^{-n+2} v_r^2 \\
+ \frac{\alpha^2}{2} \int r^{-n+4} \nu_{rr}^2 + \int r^{-n}(\Delta_{\mathcal{G}_{n-1}} v)^2 - (1 + \frac{8\beta}{c_0}(\beta + n - 1))[C\alpha^4 \epsilon \int r^{-n+\epsilon} v^2 + 33 \int r^{-n+4} v_r^2],
\]
which on dividing both side by $(1 + \frac{8\beta}{c_0}(\beta + n - 1))$ yields
\[
\int r^{-2\alpha} e^{2\alpha r}\epsilon (\Delta u - \partial_t u)^2 \geq C_1 \alpha^2 \int r^{-n} v^2 + \beta^{2} \left( \int r^{-n} |\nabla_{\mathcal{G}_{n-1}} v|^2 + \int r^{-n+2} v_r^2 \right) (3.35) \\
+ C_2 \alpha^2 \left( \int r^{-n+4} \nu_{rr}^2 + \int r^{-n}(\Delta_{\mathcal{G}_{n-1}} v)^2 + \int r^{-n+4} |(\nabla_T v)_r|^2 + \int r^{-n+4} |\nabla_T v_r|^2 \right) \\
- C\alpha^4 \epsilon \int r^{-n+\epsilon} v^2 - 33 \int r^{-n+4} v_r^2.
\]
Such an inequality is valid for sufficiently large $\alpha$ of the form $k + \frac{n+1}{2}$ and sufficiently small $\epsilon$. Now we put $v = r^{-\beta} e^{\alpha r^2} u$ in the above inequality and estimate each term in the RHS in terms of $u$. One immediately has
\[
\int r^{-2\alpha} e^{2\alpha r^2} (\Delta u - \partial_t u)^2 \geq C_1 \alpha^2 \int r^{-2\alpha-4} e^{2\alpha r^2} u^2 + C_2 \left[ \int r^{-2\alpha-4} e^{2\alpha r^2} |\nabla_{\mathcal{G}_{n-1}} u|^2 + \int r^{-n+2} v_r^2 \right] (3.36) \\
+ \frac{C_3}{\alpha^2} \int r^{-n+4} \nu_{rr}^2 + \int r^{-2\alpha-4} e^{2\alpha r^2} (\Delta_{\mathcal{G}_{n-1}} u)^2 + \frac{C_3}{\alpha^2} \int r^{-n+4} |(\nabla_T v)_r|^2 + \frac{C_3}{\alpha^2} \int r^{-n+4} |\nabla_T v_r|^2 \\
- C\alpha^4 \epsilon \int r^{-2\alpha-4+\epsilon} e^{2\alpha r^2} u^2 - 33 \int r^{-2\alpha} e^{2\alpha r^2} u_r^2.
\]
We first handle the integrals involving the terms $v_r$. Note that by an easy calculation we have
\[
v_r = e^{\alpha r^2} (r^{-\beta} u_r - \beta r^{-\beta-1} u + r^{-\beta+\epsilon-1} \alpha \epsilon u).
\]
Substituting this in (3.36) and by using (3.2) we get
\[
\int r^{-2\alpha} e^{2\alpha r^2} (\Delta u - \partial_t u)^2 \geq C_1 \alpha^2 \int r^{-2\alpha-4} e^{2\alpha r^2} u^2 + C_2 \left[ \int r^{-2\alpha-4} e^{2\alpha r^2} |\nabla_{\mathcal{G}_{n-1}} u|^2 + \int r^{-2\alpha-2} e^{2\alpha r^2} u_r^2 \right] (3.37) \\
+ \int r^{-2\alpha-4} e^{2\alpha r^2} (\beta u - r^{-\alpha} u)^2 - 2 \int r^{-2\alpha-3} e^{2\alpha r^2} u_r(\beta u - r^{-\alpha} u) \\
+ \frac{C_3}{\alpha^2} \int r^{-n+4} \nu_{rr}^2 + \int r^{-2\alpha-4} e^{2\alpha r^2} (\Delta_{\mathcal{G}_{n-1}} u)^2 + \frac{C_3}{\alpha^2} \int r^{-n+4} |(\nabla_T v)_r|^2 + \frac{C_3}{\alpha^2} \int r^{-n+4} |\nabla_T v_r|^2 \\
- C\alpha^4 \epsilon \int r^{-2\alpha-4+\epsilon} e^{2\alpha r^2} u^2 - 33 \int r^{-2\alpha} e^{2\alpha r^2} u_r^2.
Now by using the inequality (2.6) we find,
\[
C_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} u^2 + C_2 \left[ \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} u_r^2 \right]
\]
\[
+ \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} (\beta u - r^\varepsilon \alpha \varepsilon u)^2 - 2 \int r^{-2\alpha - 3} e^{2\alpha r^\varepsilon} u_r (\beta u - r^\varepsilon \alpha \varepsilon u)
\]
\[
\geq c_1 \left( \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} u^2 + \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} u_r^2 \right).
\]

Using (3.38) in (3.37) we have
\[
\int r^{-2\alpha} e^{2\alpha r^\varepsilon} (\Delta u - \partial_t u)^2 \geq c_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} u^2 + c_1 \left[ \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} |\nabla g_{n-1} u|^2 + \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} u_r^2 \right]
\]
\[
+ \frac{C_3}{\alpha^2} \int r^{-n+4} v_r^2 + \frac{C_3}{\alpha^2} \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} (\Delta g_{n-1} u)^2 + \frac{C_3}{\alpha^2} \int r^{-2\alpha - 4} |(\nabla_T v_r)|^2 + \frac{C_3}{\alpha^2} \int r^{-n+4}|\nabla_T v_r|^2
\]
\[
- C\alpha^4 \varepsilon \int r^{-2\alpha - 4 + \varepsilon} e^{2\alpha r^\varepsilon} u^2 - 33 \int r^{-2\alpha} e^{2\alpha r^\varepsilon} u_r^2.
\]

Now we handle the integral involving \((\nabla_T v)_r\). An easy calculation yields:
\[
(\nabla_T v)_r = e^{\alpha r^\varepsilon} r^{-\beta} (\nabla_T u)_r + e^{\alpha r^\varepsilon} (\alpha \varepsilon r^{-\beta + \varepsilon - 1} - \beta r^{-\beta - 1}) \nabla_T u.
\]

Using this along with the algebraic inequality in (2.6) we can proceed as above and deduce the following inequality
\[
\frac{C_3}{\alpha^2} \int r^{-n+4} (\nabla_T v)_r^2 + c_1 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} |\nabla g_{n-1} u|^2
\]
\[
\geq \frac{c_2}{\alpha^2} \int r^{-2\alpha} e^{2\alpha r^\varepsilon} |\nabla_T u_r|^2 + c_2 \int r^{-2\alpha} e^{2\alpha r^\varepsilon} |\nabla_T u_r|^2.
\]

Using (3.40) in (3.39) we find
\[
\int r^{-2\alpha} e^{2\alpha r^\varepsilon} (\Delta u - \partial_t u)^2 \geq c_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} u^2 + c_2 \left[ \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} |\nabla_T u|^2 + \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} u_r^2 \right]
\]
\[
+ \frac{C_3}{\alpha^2} \int r^{-n+4} v_{rr}^2 + \frac{C_3}{\alpha^2} \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} (\Delta g_{n-1} u)^2 + \frac{c_2}{\alpha^2} \int r^{-2\alpha} e^{2\alpha r^\varepsilon} |(\nabla_T u_r)|^2 + \frac{C_3}{\alpha^2} \int r^{-n+4}|\nabla_T v_r|^2
\]
\[
- C\alpha^4 \varepsilon \int r^{-2\alpha - 4 + \varepsilon} e^{2\alpha r^\varepsilon} u^2 - 33 \int r^{-2\alpha} e^{2\alpha r^\varepsilon} u_r^2.
\]

Similarly using
\[
(\nabla_T u)_r = -r^{-1} \nabla_T u + \nabla_T u_r,
\]

combined with the algebraic inequality (2.6) we observe
\[
\frac{c_2}{2\alpha^2} \int r^{-2\alpha} e^{2\alpha r^\varepsilon} |(\nabla_T u)_r|^2 + c_2 \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} |\nabla_T u_r|^2 \geq c_4 \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} |\nabla_T u|^2 + \frac{c_4}{\alpha^2} \int r^{-2\alpha} |\nabla_T u_r|^2.
\]

Therefore one can incorporate \(\frac{c_2}{\alpha^2} \int r^{-2\alpha} e^{2\alpha r^\varepsilon} |(\nabla_T u)_r|^2\) on the right hand side in (3.41) above. Finally by writing \(v_{rr}\) in terms of \(u, u_r\) and \(u_{rr}\) and again by using (2.6) we find
\[
c_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} u^2 + c_2 \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} u_r^2 + \frac{C_3}{\alpha^2} \int r^{-n+4} v_{rr}^2
\]
\[
\geq c_5 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r^\varepsilon} u^2 + c_5 \int r^{-2\alpha - 2} e^{2\alpha r^\varepsilon} u_r^2 + \frac{c_5}{\alpha^2} \int r^{-2\alpha} e^{2\alpha r^\varepsilon} u_{rr}^2.
\]
Indeed, by an easy calculation we first note that
\[ e^{-\alpha \varepsilon} v_{rr} = r^{-\beta - 2}\beta (\beta + 1) + \alpha \varepsilon (-\beta + \varepsilon - 1) \varepsilon^2 r^{-\beta - 1} (2\alpha \varepsilon r^2 - 2\beta) u_r + r^{-\beta} u_{rr}. \]

Let us write \( v_{rr}/\alpha = a + b \) with \( a = e^{\alpha \varepsilon} r^{-\beta} u_{rr}/\alpha \) and \( b = e^{\alpha \varepsilon} r^{-\beta - 2} (F_1 u + F_2 u_r) \) where \( F_1 \) and \( F_2 \) are given by
\[ F_1 = \beta (\beta + 1)/\alpha + \varepsilon (-\beta + \varepsilon - 1) r^{-\beta - 2} \varepsilon^2 r^2 - \beta (\beta + 1)/\alpha \varepsilon, \quad \text{and} \quad F_2 := (2\varepsilon r^2 - 2\beta/\alpha) r. \]

Now applying the algebraic identity \((a + b)^2 = a^2 + 2ab + b^2\) we obtain
\[ \frac{1}{\alpha^2} \int r^{-n+4} v_{rr}^2 \geq \frac{1}{\alpha^2} \int r^{-2\alpha} e^{2\alpha \varepsilon} u_{rr}^2 + \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} (F_1 u + F_2 u_r)^2 + \frac{2}{\alpha} \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} (F_1 u + F_2 u_r) u_{rr} \]

Now observe that
\[
\int r^{-2\alpha - 4} e^{2\alpha \varepsilon} (F_1 u + F_2 u_r)^2 \leq 2 \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} (F_1^2 u^2 + F_2^2 u_r^2) \\
= 2 \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} F_1^2 u^2 + 2 \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} (2\varepsilon r^2 - 2\beta/\alpha)^2 u_r^2 \\
\leq 2a_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} u^2 + 2a_2 \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} u_r^2
\]

where in the last inequality we have used the facts that \( F_1^2 \leq a_1 \alpha^2 \) and \((2\varepsilon r^2 - 2\beta/\alpha)^2 \leq a_2 \) for some positive constants \( a_1 \) and \( a_2 \). Hence we can choose \( \delta > 0 \) (small enough) such that
\[ \frac{c_1 \alpha^2}{2} \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} u^2 + \frac{c_2}{2} \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} u_r^2 \geq \delta C_3 \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} (F_1 u + F_2 u_r)^2 \]
which, together with the above observations, yields
\[
\frac{c_1 \alpha^2}{2} \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} u^2 + c_2 \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} u_r^2 + \frac{C_3}{\alpha^2} \int r^{-n+4} v_{rr}^2 \\
\geq \frac{c_1 \alpha^2}{2} \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} u^2 + \frac{c_2}{2} \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} u_r^2 \\
+ C_3 \left[ (1 + \delta) \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} (F_1 u + F_2 u_r)^2 + \frac{2}{\alpha} \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} (F_1 u + F_2 u_r) u_{rr} + \frac{1}{\alpha^2} \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} u_r^2 \right]
\]
from which (3.43) follows by an application of Lemma 2.4.

Using (3.42) and (3.43) in (3.41) and also the fact that \( \int_{\mathbb{S}^{n-1}} (\Delta_{\mathbb{S}^{n-1}} f)^2 \geq C_n \int_{\mathbb{S}^{n-1}} (\nabla_{\mathbb{S}^{n-1}} f)^2 \) which is a consequence of the ellipticity of the spherical Laplacian, we finally deduce the following estimate for some new constants \( C_1, C_2 \) and \( C_3 \)
\[
\int r^{-2\alpha} e^{2\alpha \varepsilon} (\Delta u - \partial_t u)^2 \geq C_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha \varepsilon} u^2 + C_2 \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} \nabla^2 u^2 \]
\[
+ \frac{C_3}{\alpha^2} \int r^{-2\alpha} e^{2\alpha \varepsilon} \nabla^2 u^2 - C_4 \varepsilon \int r^{-2\alpha - 4 + \varepsilon} e^{2\alpha \varepsilon} u^2 - 33 \int r^{-2\alpha - 2} e^{2\alpha \varepsilon} u_r^2.
\]

Now to get rid of the integral involving the \( u_r^2 \) term, we use the following lemma proved in [5].

**Lemma 3.3.** Let \( R < 1 \) and let \( u \in C^\infty_0((B_R \setminus \{0\}) \times (0, T)) \). There exist constants \( d = d(n) > 0, \alpha(n) >> 1 \) and \( 0 < \varepsilon(n) \ll 1 \), such that for all \( \alpha \geq \alpha(n) \) and every \( 0 < \varepsilon < \varepsilon(n) \) one has
\[
\frac{d}{\alpha} \int r^{-2\alpha} e^{2\alpha \varepsilon} u_t^2 + d\alpha^3 \varepsilon^2 \int r^{-2\alpha - 4 + \varepsilon} e^{2\alpha \varepsilon} u^2 \leq \int r^{-2\alpha} e^{2\alpha \varepsilon} (\Delta u - u_t)^2. \]
Having the above lemma in hand, let us fix $0 < \varepsilon(n) < 1$ and $\alpha(n) >> 1$ such that (3.44) and (3.45) hold simultaneously for $0 < \varepsilon < \varepsilon(n)$ and $\alpha > \alpha(n)$. Now we pick $d_0 = d_0(n, \varepsilon) > 1$ suitably in such that $d_0 d\varepsilon \geq 2C$ and $dd_0 > 33$. Having chosen such $d_0$, we multiply (3.45) by $d_0 \alpha$ and add the resulting inequality to (3.44), obtaining

\[ (d_0 \alpha + 1) \int r^{-2\alpha} e^{2\alpha r\varepsilon} (\Delta u - u_t)^2 \geq C_1 \alpha^2 \int r^{-2\alpha - 4} e^{2\alpha r\varepsilon} u^2 + C_2 \int r^{-2\alpha - 2} e^{2\alpha r\varepsilon} |\nabla u|^2 + \frac{C_3}{\alpha^2} \int r^{-2\alpha} e^{2\alpha r\varepsilon} |\nabla^2 u|^2 + (d_0 \varepsilon - C) \alpha^4 \int r^{-2\alpha - 4\varepsilon} e^{2\alpha r\varepsilon} u^2 + (dd_0 - 33) \int r^{-2\alpha} e^{2\alpha r\varepsilon} u_t^2. \]

(3.46)

Finally, dividing both sides of the above inequality by $\alpha$, the required Carleman estimate (3.1) is seen to follow.

\[ \square \]

3.2. Exponential decay of solutions. Using the Carleman estimate in (3.1), we now show that if a solutions $u$ to the differential inequality (1.1) vanishes to infinite order in the sense of (2.1), then it exponential decays in space. We first recall following Caccioppoli type estimate, the proof of which is exactly the same as that of [5, Lemma 3.1].

Lemma 3.4. Let $u$ be a solution to (1.1) in $B_R \times (-T, T)$ and let $0 < a < 1 < b$. Then, there exists a constant $C_1 > 0$, depending on $n, a, b, T$ and $M$ in (1.1), such that for every $r < \min\{1, R\}$ the following holds

\[ \int_{\{|r/2| < |x| < r\} \times (-T/2, T/2)} |\nabla u|^2 \leq C_1 \frac{r^2}{r^2} \int_{\{|r(1-a)/2| < |x| < br\} \times (-T, T)} u^2. \]

3.3. Exponential decay. In order to proceed further, we fix some notations. Let $g = (g_{ij}(x, t))$ denotes the inverse of the coefficient matrix $A(x, t)$. Consider the following weight.

\[ \sigma(x, t) = \left( \sum_{i,j=1}^{n} g_{ij}(0, t)x_i x_j \right) \frac{1}{2}. \]

(3.47)

In view of the uniform ellipticity of $A$, it is not hard to see that

\[ M|x| \leq \sigma(x, t) \leq N|x| \]

(3.48)

where $N, M$ are constants depending on the ellipticity constant of $A$. We set

\[ \lambda := N/M \geq 1. \]

(3.49)

With this new weight, we have the following Carleman estimate for the operator under consideration which is derived from (3.1) using an appropriate change of variable.

Lemma 3.5. Let $A$ be as in (1.2). For sufficiently large $\alpha$ of the form $\alpha = \frac{n+1}{2} + k$ where $k \in \mathbb{N}$ and small $0 < \varepsilon = \varepsilon(n) << 1$, we have that for $R_0 \leq c_0 \alpha^{-3/2}$ with $c_0$ sufficiently small, the following estimate holds for $v \in C_0^\infty((B_{R_0} \setminus \{0\}) \times (0, T))$.

\[ \alpha \int |\sigma(x, t)|^{-2\alpha-4} e^{2\alpha \sigma(x,t)\varepsilon} v^2 \, dx dt + \alpha^3 \int |\sigma(x, t)|^{-2\alpha-4\varepsilon} e^{2\alpha \sigma(x,t)\varepsilon} v^2 \, dx dt \]

(3.50)

\[ + \frac{1}{\alpha} \int |\sigma(x, t)|^{-2\alpha-2} e^{2\alpha \sigma(x,t)\varepsilon} |\nabla v|^2 \, dx dt + \frac{1}{\alpha^3} \int |\sigma(x, t)|^{-2\alpha} e^{2\alpha \sigma(x,t)\varepsilon} |\nabla^2 v|^2 \, dx dt \]

\[ \leq C \int |\sigma(x, t)|^{-2\alpha} e^{2\alpha \sigma(x,t)\varepsilon} (\text{div}(A(x,t) \nabla v) - \partial_t v)^2 \, dx dt \]
Proof. To prove the Carleman type estimate (3.50), we perform a suitable change of variable to the previous Carleman estimate (3.1) for the heat operator. To start with, let \( P(t) := (p_{ij}(t))_{n \times n} \) stand for the positive square root of the matrix \( A(0, t) \). Now we apply the following change of variable

\[
y = P(s)x, \ s = t
\]
on the both side of our previous Carleman estimate (3.1) for the heat operator. Now under this change of variable, the transformations of the involved differential operators can be obtained by a straightforward calculation. However, for the sake of the convenience of the reader, we provide all the details. To begin with, we write

\[
u(x, t) = u(P(s)^{-1}y, s) =: \nu(y, s), \ y = P(s)x.
\]

We now observe that

\[
\partial_i u = \sum_{k=1}^{n} p_{ik} \partial_k v, \ \text{and} \ \partial_i \partial_j u = \sum_{k=1}^{n} \sum_{l=1}^{n} p_{ik} p_{lj} \partial_k \partial_l v
\]

leading to

\[
\nabla u = P(s) \nabla v, \ \Delta u = Tr(A(0, s) \nabla^2 v), \ \text{and} \ \nabla^2 u = P(s) \nabla^2 v P(s).
\]

Also we note that

\[
\partial_i u = ((P_s)_x) \nabla v + \partial_s v = C(s, y) \nabla v + \partial_s v
\]

where \( C(s, y) := (P_s(P(s))^{-1}y ) \) and \( (P_s)_x \) is the \( s \)-partial derivative of the matrix \( P(s) \). Furthermore

\[
|x| = |P(s)^{-1}y| = \left( \sum_{i,j=1}^{n} g_{ij}(0, t) y_i y_j \right)^{\frac{1}{2}} = \sigma(y, s).
\]

Note that \((g_{ij}(0, t)) \) is the inverse of \( A(0, t) \). The above observations, in view of the aforementioned change of variable transforms (3.1) to

\[
\begin{align*}
\alpha \int \sigma(y, s)^{-2\alpha - 4} e^{2\alpha \sigma(y, s)^\epsilon} v^2 dy ds + \alpha^3 \int \sigma(y, s)^{-2\alpha - 4 + \epsilon} e^{2\alpha \sigma(y, s)^\epsilon} v^2 dy ds \\
+ \frac{1}{\alpha} \int \sigma(y, s)^{-2\alpha - 2} e^{2\alpha \sigma(y, s)^\epsilon} |P(s) \nabla v|^2 dy ds + \frac{1}{\alpha^3} \int \sigma(y, s)^{-2\alpha - 2 + \epsilon} e^{2\alpha \sigma(y, s)^\epsilon} |P(s) \nabla^2 v P(s)|^2 dy ds
\end{align*}
\]

\[
\leq C \int \sigma(y, s)^{-2\alpha} e^{2\alpha \sigma(y, s)^\epsilon} (Tr(A(0, s) \nabla^2 v) - C(s, y) \nabla v - \partial_s v)^2 dy ds
\]

which clearly is valid for all \( v \in C_0^\infty(B_{R_0} \times (0, T)) \) where \( R_0 \) is small and depends also on the ellipticity of \( A \). Now note that

\[
(Tr(A(0, s) \nabla^2 v) - C(s, y) \nabla v - \partial_s v)^2 \leq 2 |\text{div}(A(y, s) \nabla v) - \partial_s v|^2 + 2I
\]

where the term \( I \) is defined and estimated as follows using (1.2):

\[
I := |Tr(A(0, s) \nabla^2 v) - C(s, y) \nabla v - \text{div}(A(y, s) \nabla v)|^2
\]

\[
\leq C_I(|\nabla v|^2 + |y|^2 |\nabla^2 v|^2).
\]
For convenience, changing the variable from $y$ to $x$ and $s$ to $t$, in view of this observation and $\sigma(x,t) \leq N|x|$, from (3.51), we obtain
\[
\int \sigma(x,t)^{-2\alpha}e^{2\alpha\sigma(x,t)^\varepsilon}(\text{div}(A(x,t)\nabla v) - \partial_t v)^2 dx dt \geq 
\] (3.52)
\[
+ \alpha \int \sigma(x,t)^{-2\alpha-4}e^{2\alpha\sigma(x,t)^\varepsilon}v^2 dx dt + \alpha^3 \int \sigma(x,t)^{-2\alpha-4+s}e^{2\alpha\sigma(x,t)^\varepsilon}v^2 dx dt 
\]
\[
+ \frac{1}{\alpha} \int \sigma(x,t)^{-2\alpha-2}e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt + \frac{1}{\alpha^3} \int \sigma(x,t)^{-2\alpha}e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt 
\]
\[
- C_1 N^2 \int \sigma(x,t)^{-2\alpha-2}|x|^2e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt - C_1 \int \sigma(x,t)^{-2\alpha}|x|^2e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt. 
\]

We now take $R_0 \leq c_0 \alpha^{-3/2}$ for some suitable constant $c_0$ to be chosen later, then the above inequality transforms to
\[
\int \sigma(x,t)^{-2\alpha}e^{2\alpha\sigma(x,t)^\varepsilon}(\text{div}(A(x,t)\nabla v) - \partial_t v)^2 dx dt \geq 
\] (3.53)
\[
+ \alpha \int \sigma(x,t)^{-2\alpha-4}e^{2\alpha\sigma(x,t)^\varepsilon}v^2 dx dt + \alpha^3 \int \sigma(x,t)^{-2\alpha-4+s}e^{2\alpha\sigma(x,t)^\varepsilon}v^2 dx dt 
\]
\[
+ \frac{1}{\alpha}(1 - C_1 N^2 c_0^2 \alpha^{-2}) \int \sigma(x,t)^{-2\alpha-2}e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt 
\]
\[
+ \frac{1}{\alpha^3}(1 - C_2 c_0^2) \int \sigma(x,t)^{-2\alpha}e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt. 
\]

Choosing $c_0$ such that $C_2 c_0^2 < 1/2$, we thus conclude that
\[
C \int \sigma(x,t)^{-2\alpha}e^{2\alpha|\varepsilon|x|^\varepsilon}(\text{div}(A(x,t)\nabla v) - \partial_t v)^2 dx dt \geq 
\] (3.54)
\[
+ \alpha \int \sigma(x,t)^{-2\alpha-4}e^{2\alpha\sigma(x,t)^\varepsilon}v^2 dx dt + \alpha^3 \int \sigma(x,t)^{-2\alpha-4+s}e^{2\alpha\sigma(x,t)^\varepsilon}v^2 dx dt 
\]
\[
+ \frac{1}{2\alpha} \int \sigma(x,t)^{-2\alpha-2}e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt + \frac{1}{2\alpha^3} \int \sigma(x,t)^{-2\alpha}e^{2\alpha\sigma(x,t)^\varepsilon}|\nabla v|^2 dx dt 
\]
from which the lemma follows. \hfill \Box

With the estimate in (3.50) in hand, we now show that solutions to (1.1) decay exponentially when they vanish to infinite order at the origin.

**Proposition 3.6.** Let $u$ be a solution to (1.1) in $B_R \times (-T,T)$ such that $u$ vanishes to infinite order at 0 in the sense of (2.1). Then $u$ satisfies
\[
\int_{B_{1/2}} u^2 \lesssim e^{-\frac{C}{x^{2/3}}}, \text{ as } s \to 0 
\] (3.55)
for some constant $C > 0$.

**Proof.** Let $u$ be as in the statement of the proposition. For a given $r_1 > 0$ sufficiently small, we let
\[
v(x,t) = u(x,t)\varphi(x)\eta(t) 
\]
where $\varphi$ and $\eta$ are compactly supported functions which are chosen as follows.

We let $\varphi$ such that
\[
\varphi(x) := \begin{cases} 1, & \text{if } x \in B(0,r_1), \\ 0, & \text{if } |x| > r_2 \end{cases} 
\] (3.56)
where $r_2 = 4\lambda^2 r_1 < R$ with $\lambda$ as in (3.49). As in [5, 24], we let $T_1 = \frac{2T}{r_2}, T_2 = \frac{T}{r_2}$ and $\eta(t)$ be a smooth even function such that $\eta(t) \equiv 1$ when $|t| < T_2$, $\eta(t) \equiv 0$, when $|t| > T_1$. More precisely

$$\eta(t) = \begin{cases} 
0 & -T \leq t \leq -T_1 \\
\exp \left(-\frac{T^3(T_2+t)^4}{T_1(T_2+t)^4} \right) & -T_1 \leq t \leq -T_2, \\
1, & -T_2 \leq t \leq 0.
\end{cases} \quad (3.57)$$  

It is to be noted over here that from Remark 2.3 it follows that $\nabla u, \nabla^2 u, u_t$ vanishes to infinite order in the sense of (2.1) as well and therefore by a standard limiting argument, one can show that the Carleman estimate (3.50) can be applied to $v$ as above. We denote the operator $\text{div}(A(x,t)\nabla)$ by $\mathcal{L}$. An easy calculation yields

$$\mathcal{L}v - \partial_t v = \varphi \eta(\mathcal{L}u - \partial_t u) + 2(A\nabla u, \nabla \varphi)\eta + u(\eta \ \text{div}(A\nabla \varphi) - \varphi \eta_t).$$  

Also from (1.2) and given our choice of $\varphi$ as in (3.56) it follows that

$$\text{div}(A\nabla \varphi) \leq C(|\nabla \varphi| + |\nabla^2 \varphi|). \quad (3.58)$$

Using (3.58) we have

$$\int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} (\text{div}(A(x,t)\nabla v) - \partial_t v)^2 \, dx \, dt \quad (3.59)$$

$$\lesssim \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} \varphi^2 \eta^2 (\text{div}(A(x,t)\nabla u) - \partial_t u)^2 + \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} u^2 \varphi^2 \eta_t^2$$

$$+ \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} \left( |\nabla u|^2 |\nabla \varphi|^2 + u^2 |\nabla^2 \varphi|^2 + u^2 |\nabla \varphi|^2 \right) \eta^2.$$

Now using (1.1) in (3.59) we find that $\int_{B_R \times (0,T)} \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} (\text{div}(A(x,t)\nabla v) - \partial_t v)^2 \, dx \, dt$ can be estimated as

$$\int_{B(0,R) \times (0,T)} \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} (\text{div}(A(x,t)\nabla v) - \partial_t v)^2 \, dx \, dt \quad (3.60)$$

$$\lesssim \int \sigma(x,t)^{-2\alpha} |x|^{-4} e^{2\alpha \sigma(x,t)^t} \varphi^2 \eta_t^2 u^2 + \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} u^2 \varphi^2 \eta_t^2$$

$$+ \int_{\{r_1 < |x| < r_2 \} \times (-T_1, T_1)} \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} \left( |\nabla u|^2 |\nabla \varphi|^2 + u^2 (\Delta \varphi)^2 + u^2 |\nabla \varphi|^2 \right) \eta^2$$

$$\lesssim \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} u^2 + \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} u^2 \varphi^2 \eta_t^2$$

$$+ \int_{\{r_1 < |x| < r_2 \} \times (-T_1, T_1)} \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)^t} \left( |\nabla u|^2 |x|^{-2} + u^2 |x|^{-4} + u^2 |x|^{-2} \right) \eta^2.$$

In (3.60) we used that $\nabla \varphi, \nabla^2 \varphi$ are supported in $\{r_1 < |x| < r_2 \}$ and satisfies the following bounds

$$|\nabla \varphi| \leq \frac{C}{|x|}, \ |\nabla^2 \varphi| \leq \frac{C}{|x|^2}.$$
Now by applying the Carleman estimate (3.50) to \( v \), we obtain using (3.60) that the following holds

\[
\alpha \int \sigma(x,t)^{-2\alpha-4} e^{2\alpha \sigma(x,t)\varepsilon} v^2 + \frac{1}{2\alpha} \int \sigma(x,t)^{-2\alpha-2} e^{2\alpha \sigma(x,t)\varepsilon} |\nabla v|^2 + \frac{1}{2\alpha^3} \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)\varepsilon} |\nabla^2 v|^2
\]

\[
\approx \int \sigma(x,t)^{-2\alpha} |x|^{-4} e^{2\alpha \sigma(x,t)\varepsilon} \varphi^2 \eta^2 u^2 + \int \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)\varepsilon} u^2 \varphi^2 \eta_0^2 - \alpha^3 \int \sigma(x,t)^{-2\alpha-4+\epsilon} e^{2\alpha \sigma(x,t)\varepsilon} v^2
\]

\[
+ \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1) \}} e^{2\alpha \sigma(x,t)\varepsilon} \sigma(x,t)^{-2\alpha-4} u^2 \eta^2 + \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1) \}} e^{2\alpha \sigma(x,t)\varepsilon} \sigma(x,t)^{-2\alpha-2} |\nabla u|^2 \eta^2
\]

\[
= \int \sigma(x,t)^{-2\alpha} |x|^{-4} e^{2\alpha \sigma(x,t)\varepsilon} \varphi^2 \eta^2 u^2 + I_1 + I_2 + I_3 + I_4.
\]

We then note that since \( \sigma(x,t) \sim |x| \), therefore if \( \alpha \) is chosen large enough, then the integral \( \int \sigma(x,t)^{-2\alpha} |x|^{-4} e^{2\alpha \sigma(x,t)\varepsilon} \varphi^2 \eta^2 u^2 \) can be absorbed into the term \( \alpha \int \sigma(x,t)^{-2\alpha-4} e^{2\alpha \sigma(x,t)\varepsilon} v^2 \) which appears on the left hand side.

Now we estimate each \( I_j \) separately. Let us first introduce some notations which will be used throughout the rest of the proof.

\[
\sigma_1(r_1, r_2, T) := \inf_{\{r_1 < |x| < r_2 \times (-T, T) \}} \sigma(x,t), \text{ and } \sigma_2(r_1, r_2, T) := \sup_{\{r_1 < |x| < r_2 \times (-T, T) \}} \sigma(x,t).
\]

We first note that it is easily seen that

\[
I_3 \lesssim \sigma_1(r_1, r_2, T)^{-2\alpha-4} e^{2\alpha \sigma_2(r_1, r_2, T)\varepsilon} \int_{B(0,R) \times (-T, T)} u^2.
\]

Similarly we have

\[
I_4 \lesssim \sigma_1(r_1, r_2, T)^{-2\alpha-2} e^{2\alpha \sigma_2(r_1, r_2, T)\varepsilon} \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1) \}} |\nabla u|^2
\]

Using the energy estimate in Lemma 3.4 we have

\[
\int_{\{r_1 < |x| < r_2 \times (-T_1, T_1) \}} |\nabla u|^2 \leq C \int_{B(0,R) \times (-T, T)} u^2
\]

which in particular implies

\[
I_4 \lesssim \sigma_1(r_1, r_2, T)^{-2\alpha-2} \int_{B(0,R) \times (-T, T)} u^2.
\]

We now our attention to \( I_1 \). To begin with, we break the integral into two parts as follows:

\[
I_1 = \int_{B_1 \times (-T_1, T_1)} + \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1) \}} \left( \sigma(x,t)^{-2\alpha} e^{2\alpha \sigma(x,t)\varepsilon} u^2 \varphi^2 \eta_0^2 \right)
\]

\[
=: I_{11} + I_{12}
\]

The same technology as above, along with the estimate \( |\eta_1| \lesssim \frac{1}{T} \) yields

\[
I_{12} \lesssim \sigma_1(r_1, r_2, T)^{-2\alpha} e^{2\alpha \sigma_2(r_1, r_2, T)\varepsilon} \int_{B(0,R) \times (-T, T)} u^2.
\]

We are thus left with \( I_{11} \), which will be estimated together with \( I_2 \).

Recalling that by our choice \( \varphi(x) = 1 \) when \( |x| < r_1 \), and \( \text{supp}(\eta_1) \subset (-T_1, -T_2) \cup (T_2, T_1) \) we observe that

\[
I_{11} + I_2 \lesssim \int_{\Omega} \sigma(x,t)^{-2\alpha-4+\epsilon} e^{2\alpha \sigma(x,t)\varepsilon} u^2 \varphi^2 \eta_0^2 \left( \sigma(x,t)^{3\eta_0^2} - \alpha^3 \right)
\]
where $\Omega := B_{r_1} \times [(-T_1, -T_2) \cup (T_2, T_1)]$. We now adapt some ideas from [24] (see also [5]) to show that the following estimate holds
\[
I_{11} + I_2 \lesssim \int_{B(0,R) \times (-T,T)} u^2. \tag{3.67}
\]
First note that it is sufficient to prove (3.67) over the region $\Omega^- := B_{r_1} \times (-T_1, -T_2)$, since the other part can be handled by symmetry. Now, if $-T_1 \leq t \leq -T_2$, note that $T_1 - T_2 = \frac{T}{4}$, $|T_1 + t| \leq T_1 - T_2 = \frac{T}{4}$, and that $\frac{3}{4}T \leq 4T_1 - 3T_2 + t \leq T$, it follows
\[
\left| \frac{\eta_t}{\eta} \right| = \left| \frac{T^3(T_2 + t)^3(4T_1 - 3T_2 + t)}{(T_1 - T_2)^4(T_1 + t)^4} \right| \leq \frac{4T^3}{|T_1 + t|^4}.
\]
Consequently we obtain
\[
\int_{\omega^-} \sigma(x,t)^{-2a-4+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} u^2 \eta^2 \left( \sigma(x,t)^{\frac{3}{2} \eta^2} - \alpha^3 \right) \leq \int_{\omega^-} \sigma(x,t)^{-2a-4+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} u^2 \eta^2 \left( C \sigma(x,t)^{3} \frac{T^6}{(T_1 + t)^8} - \alpha^3 \right).
\]
Trivially
\[
\int_{\omega^-} \sigma(x,t)^{-2a-4+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} u^2 \eta^2 \left( \sigma(x,t)^{\frac{3}{2} \eta^2} - \alpha^3 \right)
\]
\[
\leq \int_{U} \sigma(x,t)^{-2a-4+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} u^2 \eta^2 \left( C \sigma(x,t)^{3} \frac{T^6}{(T_1 + t)^8} - \alpha^3 \right),
\]
where
\[
U := \left\{ (x,t) \in \omega^- : \alpha^3 \leq C \sigma(x,t)^{3} \frac{T^6}{(T_1 + t)^8} \right\}.
\]
Thus it follows
\[
\int_{\omega^-} \sigma(x,t)^{-2a-4+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} u^2 \eta^2 \left( \sigma(x,t)^{\frac{3}{2} \eta^2} - \alpha^3 \right)
\]
\[
\leq C \int_{U} \sigma(x,t)^{-2a-1+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} u^2 \eta \frac{\eta T^6}{(T_1 + t)^8}.
\]
Now in order to substantiate our claim we establish a bound from above for the quantity
\[
\sigma(x,t)^{-2a-1+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} \eta \frac{\eta T^6}{(T_1 + t)^8}
\]
in $U$. Appealing to the exponential decay of $\eta$, at $t = -T_1$, see (3.57), we obtain for $t \in (-T_1, -T_2)$,
\[
\frac{\eta T^6}{(T_1 + t)^8} \leq C.
\]
Therefore, we will be done if we can prove that
\[
\sigma(x,t)^{-2a-1+\epsilon} e^{2\alpha \sigma(x,t)\epsilon} \eta \leq 1
\]
which, in view of the expression for $\eta$, is equivalent to proving
\[
(2\alpha + 1 - \epsilon) \log \sigma(x,t) - 2\alpha \sigma(x,t)\epsilon + \frac{T^3(T_2 + t)^4}{(T_1 + t)^3(T_1 - T_2)} \geq 0
\]
for $(x,t) \in U$. Now in what follows, we prove (3.72). For that first note that, inside the region $U$, we have
\[
\frac{T_1 + t}{T} \leq \left( \frac{C}{T^2} \right)^{1/8} \left( \frac{\sigma(x,t)}{\alpha} \right)^{3/8} \leq C \left( \frac{\sigma(x,t)}{\alpha} \right)^{3/8},
\]
for some universal $C > 0$ depending also on $T$. For sufficiently large $\alpha$ we have

$$C \left( \frac{\sigma(x,t)}{\alpha} \right)^{3/8} \leq C_1 \left( \frac{R}{\alpha} \right)^{3/8} \leq \frac{1}{12}. $$

Combining the above, we have

$$\frac{T_1 + t}{T} \leq \frac{1}{12}, \quad (3.73)$$

in $U$, provided $\alpha$ is large enough. Also, $\frac{T}{4} = T_1 - T = T_1 + t + |T_2 + t|$, from (3.73) we conclude that we must have in $U$

$$|T_2 + t| \geq \frac{T}{6}. $$

It thus follows

$$(2\alpha + 1 - \varepsilon) \log \sigma(x,t) - 2\alpha \sigma(x,t)^\varepsilon + \frac{T^3(T_2 + t)^4}{(T_1 + t)^3(T_1 - T_2)^4}$$

$$ \geq \left( \frac{4}{6} \right)^4 \left( \frac{2}{C} \right)^{3/8} T^{3/4} \left( \frac{\alpha}{\sigma} \right)^{9/8} - (2\alpha + 1 - \varepsilon) \log \frac{1}{\sigma(x,t)} - 2\alpha \sigma(x,t)^\varepsilon. \quad (3.74)$$

Now if we choose $\alpha$ very large, in view of the fact that the exponent $\alpha^{9/8}$ wins over the linear term in $\alpha$, we infer that the last expression in (3.74) is non-negative proving (3.72). Therefore, the estimate (3.67) holds.

Therefore using (3.63),(3.65), (3.66) and (3.67) in (3.61), we conclude that

$$\alpha \int \sigma(x,t)^{-2\alpha - 4} e^{2\alpha \sigma(x,t)^\varepsilon} u^2 \leq \sigma_1(r_1, r_2, T)^{-2\alpha - 4} e^{2\alpha \sigma_2(r_1, r_2, T)^\varepsilon} \int_{B(0,R) \times (-T,T)} u^2. \quad (3.75)$$

We now estimate the left hand side of the above inequality in the following way

$$\alpha \int \sigma(x,t)^{-2\alpha - 4} e^{2\alpha \sigma(x,t)^\varepsilon} u^2 \geq \alpha \int_{B_{r_1/8 \lambda} \times (-T_2, T_2)} \sigma(x,t)^{-2\alpha - 4} e^{2\alpha \sigma(x,t)^\varepsilon} u^2$$

$$ \geq \alpha \tilde{\sigma}_2(r_1/8 \lambda, T)^{-2\alpha - 4} \int_{B(0, r_1/8 \lambda) \times (-T_2, T_2)} u^2 \quad (3.76)$$

where

$$\tilde{\sigma}_2(r_1/8 \lambda, T) := \sup_{B_{r_1/8 \lambda} \times (-T,T)} \sigma(x,t)$$

and $\lambda$ is as in (3.49). This, along with (3.75) yields

$$\alpha \int_{B_{r_1/8 \lambda} \times (-T_2, T_2)} u^2 \leq \left( \frac{\sigma_1(r_1, r_2, T)}{\tilde{\sigma}_2(r_1/8 \lambda, T)} \right)^{-2\alpha - 4} e^{2\alpha \sigma_2(r_1, r_2, T)^\varepsilon} \int_{B(0, R) \times (-T,T)} u^2. \quad (3.76)$$

Now from (3.48) we find

$$\frac{\sigma_1(r_1, r_2, T)}{\tilde{\sigma}_2(r_1/8 \lambda, T)} \geq \frac{M r_1}{N r_1/8 \lambda} = 8. \quad (3.77)$$

Using (3.77) in (3.76) we then deduce the following inequality

$$\int_{B_{r_1/8 \lambda} \times (-T_2, T_2)} u^2 \leq 2^{-2\alpha - 4} e^{2\alpha \sigma_2(r_1, r_2, T)^\varepsilon}. \quad (3.78)$$

Next observe that

$$4^{-2\alpha} e^{2\alpha \sigma_2(r_1, r_2, T)^\varepsilon} \leq e^{-2\alpha(2\log 2 - (N r_2)^\varepsilon)} \leq 1 \quad (3.78)$$

(provided $r_1$ and consequently $r_2$ is small enough since $r_2 = \lambda^2 r_1$.)
Therefore we obtain
\[ \int_{B_{r_1}/8 \times (-T_2, T_2)} u^2 \lesssim 2^{-2\alpha}. \]  (3.79)

At this point, in view of Lemma 3.5, by letting \( \alpha \sim \left( \frac{1}{r_1} \right)^{2/3} \), we find that the conclusion of the lemma follows from (3.79).

3.4. Carleman estimate II. Proposition 3.6 shows that solutions to (1.1) decay exponentially at the origin when the vanish to infinite order in the sense of (2.1). We now show by means of a parabolic generalization of a Carleman estimate due to Regbaoui in [23] which uses strictly convex weights that non-trivial solutions to (1.1) in fact decay less than exponentially which would then lead to a contradiction and would thus establish Theorem 2.2.

We start by recalling a few notations. As before, \( g = (g_{ij}(x, t)) \) will denote the inverse of the coefficient matrix \( A(x, t) \) and \( \sigma(x, t) \) is as in (3.47). For notational convenience, we will often denote \( \sigma(x, t) \) by \( \sigma \) whenever the context is clear. Before proceeding further, we would like to alert the reader that for notational convenience, we set
\[ \mathcal{L} = \text{div}(A(x, t)\nabla). \]

The following vector field will play a pervasive role in our analysis.
\[ Z = \sigma \sum_{i,j=1}^{n} a_{ij} \partial_i \sigma \partial_j. \]  (3.80)

We now collect some basic properties related to \( Z \) and \( \sigma \) which is easily verified using (1.2) and the fact that the matrix valued function \( (g_{ij}) \) is the inverse of \( (a_{ij}). \)

**Proposition 3.7.** The following holds true:

i) \( |< ZA \nabla u, \nabla u >| \leq C \sigma |\nabla u|^2. \)

ii) \( |[\partial_i, Z]u - \partial_i u| \leq C \sigma |\nabla u|, \quad i = 1, \ldots, N. \)

iii) \( \partial_i Z_j = \delta_{ij} + O(|x|). \)

iv) \( Z \sigma = \sigma + O(|x|^2). \)

**Proof.** We only prove (ii), (iii) and (iv) since (i) is easily seen to be true.

**Proof of (ii):** In view of the definition of \( Z \), we see that
\[ [\partial_i, Z]u = \partial_i Zu - Z \partial_i u = \sum_{k,l} \partial_i (\sigma a_{kl} \partial_k \sigma) \partial_k u + \sigma \sum_{k,l} a_{kl} \partial_k \sigma [\partial_i, \partial_j] u. \]

But as \( |[\partial_i, \partial_j] u| = 0 \), the last term in the above expression vanishes.

Observe now that
\[ \partial_k \sigma = \sum_m g_{km}(0,t) x_m. \]  (3.81)

Now, using 3.81 and \( a_{kl} = a_{kl}(0, t) + b_{kl}(x, t) \), we observe that
\[ \sum_{k,l} \partial_i (\sigma a_{kl} \partial_k \sigma) \partial_k u = \sum_{k,l,m} \delta_{lm} a_{kl}(0, t) g_{km}(0, t) \partial_k u + \sum_{k,m,l} \partial_i (b_{kl} g_{km}(0, t) x_m) \partial_k u. \]  (3.82)

Now using
\[ \sum_k a_{kl}(0, t) g_{km}(0, t) = \delta_{tm} \text{ (since } (g_{ij}) \text{ is the inverse of } (a_{ij}).) \]

it follows that
\[ \sum_{k, l, m} \delta_{lm} a_{kl}(0, t) g_{km}(0, t) \partial_k u = \partial_i u. \]  (3.83)
Also since \( b_{ij}(x,t) = O(|x|) \), it follows
\[
\left| \sum_{k,m,l} \partial_i (b_{kl} g_{km}(0,t)x_m) \partial_j u \right| \leq C|x||\nabla u|. \tag{3.84}
\]

Using (3.83) and (3.84) in (3.82), (ii) is seen to hold.

Proof of (iii): Using standard summation convention we have
\[
Z_j = a_{ij}(x,t) g_{jm}(0,t)x_m
\]
for any \( 1 \leq j \leq n \). As before, by writing \( a_{ij}(x,t) = a_{ij}(0,t) + b_{ij}(x,t) \), we have
\[
Z_j = \delta_{jm} x_m + b_{ij} g_{jm}(0,t)x_m = x_j + b_{ij} g_{jm}(0,t)x_m. \tag{3.85}
\]

Noting the fact that \( b_{ij} = O(|x|) \), it is thus straightforward to see from (3.85) that \( \partial_i Z_j = \delta_{ij} + O(|x|) \).

Proof of (iv): We have
\[
Z \sigma = \frac{a_{ij} g_{ij} g_{jm} x_m}{\sigma}.
\]
Again by writing \( a_{ij} = a_{ij}(0,t) + b_{ij} \) and using that \( b_{ij} = O(|x|) \), an easy calculation leads to the following
\[
Z \sigma = \frac{g_{jm} x_j x_m}{\sigma} + \sum_{i,j,l} \frac{b_{ij} g_{ij} g_{jm} x_m}{\sigma}
= \sigma + O(|x|^2).
\]

We also require the following Rellich’s identity (see for instance [20, 21]). Given a \( C^{0,1} \) vector field \( G \), we have
\[
\int_{\partial B_R} \langle A \nabla u, \nabla u \rangle \langle G, \nu \rangle = 2 \int_{\partial B_R} a_{ij} \partial_i u \langle \partial_j, \nu \rangle G u - 2 \int_{B_R} a_{ij}(\text{div } \partial_i) \partial_j u G u - 2 \int_{B_R} a_{ij} \partial_i u [\partial_j, G] u + \int_{B_R} \text{div } G \langle A \nabla u, \nabla u \rangle + \int_{B_R} \langle (GA) \nabla u, \nabla u \rangle - 2 \int_{B_R} G u \partial_i (a_{ij} \partial_j u),
\]

where \( \nu \) denotes the outer unit normal to \( B_R \). The following application of such a Rellich’s identity will be required in our setting.

Lemma 3.8. Denoting \( \sigma(\cdot,t) \) by \( \sigma(\cdot) \), the following holds true for \( u \in C^\infty(B_R \setminus \{0\}) \)
\[
2 \int_{B_R} \sigma^{-n+2}(- \log \sigma) Z u \mathcal{L} u = - \int_{B_R} \sigma^{-n+2} \langle A \nabla u, \nabla u \rangle - \left( n - 2 \right) (- \log \sigma) + 1)(Zu)^2 + O(1) \int_{B_R} \sigma^{-n+3}(- \log \sigma)|\nabla u|^2. \tag{3.87}
\]

Proof. With \( G = \sigma^{-n+2}(- \log \sigma) Z \), by applying (3.86) we find
\[
2 \int_{B_R} \sigma^{-n+2}(- \log \sigma) Z u \mathcal{L} u = \int_{B_R} \text{div}(\sigma^{-n+2}(- \log \sigma) Z) \langle A \nabla u, \nabla u \rangle - \int_{B_R} a_{ij} \partial_i u [\partial_j, \sigma^{-n+2}(- \log \sigma) Z] u + \int_{B_R} \sigma^{-n+2}(- \log \sigma) \langle Z A \nabla u, \nabla u \rangle. \tag{3.89}
\]
From (3.89) we obtain

\[ 2 \int_{B_R} \sigma^{-n+2}(-\log \sigma) Z u \mathcal{L} u = - \int_{B_R} \sigma^{-n+2}(n - 2)(-\log \sigma) + 1) A \nabla u, \nabla u \quad (3.90) \]

\[ + (n - 2) \int_{B_R} \sigma^{-n+2}(-\log \sigma) A \nabla u, \nabla u - 2 \int_{B_R} \sigma^{-n+2}(-\log \sigma) a_{ij} \partial_i u ([\partial_j, Z] u - \partial_j u) \]

\[ + 2 \int_{B_R} \sigma^{-n}((n - 2)(-\log \sigma) + 1)(Z u)^2 + O(1) \int_{B_R} \sigma^{-n+3}(-\log \sigma) |\nabla u|^2, \quad (3.91) \]

once we incorporate the following:

\[ \text{div}(\sigma^{-n+2}(-\log \sigma) Z) = -\sigma^{-n+2}(1 + (n - 2)(-\log \sigma)) + \sigma^{-n+2}(-\log \sigma)n + O(\sigma^{-n+3})(-\log \sigma) \]

\[ \int_{B_R} \sigma^{-n+2}(-\log \sigma)(Z A) \nabla u, \nabla u = O(1) \int \sigma^{-n+3}(-\log \sigma) |\nabla u|^2, \quad (3.92) \]

which follows from (i), (iii) and (iv) of Proposition 3.7 and also by using

\[ [\partial_j, \sigma^{-n+2}(-\log \sigma) Z] u = \sigma^{-n+2}(-\log \sigma) ([\partial_j, Z] u - \partial_j u) \]

\[ + \sigma^{-n+2}(-\log \sigma) \partial_j u - ((n - 2)(-\log \sigma) + 1) \sigma^{-n+1} \partial_j \sigma Z u. \quad (3.93) \]

Finally by using (ii) of Proposition 3.7, we find that (3.87) follows from (3.93).

We now state and prove the following parabolic version of Regbaoui’s Carleman estimate.

**Theorem 3.9.** There exists universal \( C > 0 \) such that for every \( \beta > 0 \) sufficiently large, \( R_0 \) sufficiently small, \( u \in C_0^\infty((B_R \setminus \{0\}) \times (-T, T)) \) for \( R \leq R_0 \), one has

\[ \beta^3 \int_{B_R} \sigma^{-n} e^{\beta(\log \sigma)^2} u^2 + \beta \int_{B_R} \sigma^{-n+2} e^{\beta(\log \sigma)^2} (A \nabla u, \nabla u) \quad (3.94) \]

\[ \leq C \int \sigma^{-n+4} e^{\beta(\log \sigma)^2} (\mathcal{L} u - \partial_i u)^2. \]

Theorem 3.9 is quintessential in proving that non-trivial solutions to (1.1) decay less than exponentially. Proof of Theorem 3.9 divided into some intermediate results. The first such result is as follows.

**Theorem 3.10.** There exists universal \( C > 0 \) such that for every \( \beta > 0 \) sufficiently large, \( R_0 \) sufficiently small, \( u \in C_0^\infty((B_R \setminus \{0\}) \times (-T, T)) \) for \( R \leq R_0 \), one has

\[ \int \sigma^{-n+4} e^{\beta(\log \sigma)^2} (\mathcal{L} u - \partial_i u)^2 \geq \frac{59}{10} \beta^3 \int \sigma^{-n}(\log \sigma)^2 v^2 + \frac{15}{4} \beta \int \sigma^{-n}(Z v)^2 \]

\[ - \frac{5}{2} \beta \int \sigma^{-n+2}(A \nabla v, \nabla v), \quad (3.95) \]

where \( v = e^{\frac{\beta}{2}(\log \sigma)^2} u. \)

**Proof.** It follows

\[ \mathcal{L} u = \mathcal{L} v(e^{-\frac{\beta}{2}(\log \sigma)^2}) + 2a_{ij} \partial_j v \partial_i (e^{-\frac{\beta}{2}(\log \sigma)^2}) + \mathcal{L}(e^{-\frac{\beta}{2}(\log \sigma)^2}) v \]
As before, letting \( a_{ij}(x,t) = a_{ij}(0,t) + b_{ij}(x,t) \), we first compute the term \( \sum_{i,j} a_{ij}(0,t) (e^{-\frac{\beta}{2} (\log \sigma)^2})_{ij} \) explicitly. To do so, one important ingredient is the following

\[
(e^{-\frac{\beta}{2} (\log \sigma)^2})_i = e^{-\frac{\beta}{2} (\log \sigma)^2} (-\beta \log \sigma) \frac{g_{il} x_l}{\sigma^2}.
\]

Upon differentiating once more, we obtain

\[
a_{ij}(0,t)(e^{-\frac{\beta}{2} (\log \sigma)^2})_{ij} = e^{-\frac{\beta}{2} (\log \sigma)^2} \frac{(\beta \log \sigma)^2}{\sigma^4} a_{ij}(0,t) g_{jm} x_m g_{il} x_l
\]

\[
+ \frac{(-\beta) e^{-\frac{\beta}{2} (\log \sigma)^2}}{\sigma^4} a_{ij}(0,t) g_{jm} x_m g_{il} x_l
\]

\[
+ \frac{(-\beta \log \sigma) e^{-\frac{\beta}{2} (\log \sigma)^2}}{\sigma^2} a_{ij}(0,t) g_{il} \delta_{jl}
\]

\[
- 2 \frac{(-\beta \log \sigma) e^{-\frac{\beta}{2} (\log \sigma)^2}}{\sigma^4} a_{ij}(0,t) g_{jm} x_m g_{il} x_l.
\]

Now using the fact that \( a_{ij}(0,t) g_{jm} x_m g_{il} x_l = g_{lm} x_m x_l = \sigma^2 \), we obtain

\[
a_{ij}(0,t)(e^{-\frac{\beta}{2} (\log \sigma)^2})_{ij} = e^{-\frac{\beta}{2} (\log \sigma)^2} \sigma^{-2} ((\beta \log \sigma)^2 - \beta - (n - 2) \beta \log \sigma).
\]

(3.96)

We thus obtain

\[
\mathcal{L} u = \mathcal{L} v (e^{-\frac{\beta}{2} (\log \sigma)^2}) + 2\sigma^{-2} (-\beta \log \sigma) e^{-\frac{\beta}{2} (\log \sigma)^2} Z v
\]

\[
+ e^{-\frac{\beta}{2} (\log \sigma)^2} \sigma^{-2} ((\beta \log \sigma)^2 - \beta - (n - 2) \beta \log \sigma) v
\]

\[
+ \left[ \partial_i b_{ij} e^{-\frac{\beta}{2} (\log \sigma)^2} (\beta \log \sigma) \sigma^{-1} \partial_j \sigma \right] v + \left[ b_{ij} (\beta \log \sigma) e^{-\frac{\beta}{2} (\log \sigma)^2} \sigma^{-1} \right] \partial_i \partial_j \sigma \right] v
\]

\[
+ \left[ b_{ij} (\beta \sigma^{-2} e^{-\frac{\beta}{2} (\log \sigma)^2} (-1 + (\log \sigma) + \beta (\log \sigma)^2)) \right] \partial_i \sigma \partial_j \sigma \right] v.
\]

Similarly we also have

\[
\partial_t u = e^{-\frac{\beta}{2} (\log \sigma)^2} \left[ \partial_t v - v \beta \log \sigma \frac{\partial_t \sigma}{\sigma} \right].
\]

Now by writing

\[
(\mathcal{L} u - \partial_t u) e^{\frac{\beta}{2} (\log \sigma)^2} = A + B, \text{ where,}
\]

\[
A := 2\sigma^{-2} (-\beta \log \sigma) Z v - \partial_t v,
\]

(3.99)

and, \( B := \mathcal{L} v + \sigma^{-2} ((\beta \log \sigma)^2 - \beta - (n - 2) \beta \log \sigma) v \)

\[
+ \left[ \partial_i b_{ij} (-\beta \log \sigma) \sigma^{-1} \partial_j \sigma \right] v + \left[ b_{ij} (-\beta (\log \sigma) \sigma^{-1}) \partial_i \partial_j \sigma \right] v
\]

\[
+ \left[ b_{ij} (\beta \sigma^{-2} (-1 + (\log \sigma) + \beta (\log \sigma)^2)) \right] \partial_i \sigma \partial_j \sigma \right] v + v \beta \log \sigma \frac{\partial_t \sigma}{\sigma}
\]

(3.100)

and then by using the algebraic inequality \( (A + B)^2 \geq A^2 + 2AB \) we obtain
\[ \int \sigma^{-n+4} e^{\alpha(\log \sigma)^2} (\mathcal{L} u - \partial_t u)^2 \geq \int \sigma^{-n+4} (A^2 + 2AB) \]  

\[ = \int \sigma^{-n+4} [2\sigma^{-2} (-\log \sigma) Z v - \partial_t v]^2 + \int 4\beta \sigma^{-n+2} (-\log \sigma) Z v \mathcal{L} v + \int 4\beta \sigma^{-n+2} (-\log \sigma) \left[ \sigma^{-2} ((\log \sigma)^2 - \beta - (n-2) \log \sigma) \right] + \left[ b_{ij} (\beta(- \log \sigma) \sigma^{-1}) \partial_i \sigma \partial_j \sigma \right] + \left[ b_{ij} (\beta \sigma^{-2} (-1 + (\log \sigma) + \beta (\log \sigma)^2)) \partial_i \sigma \partial_j \sigma \right] Z v \cdot v + 4\beta \int \sigma^{-n+1} [\partial_i b_{ij} (-\log \sigma) \partial_j \sigma] Z v \cdot v - 4\beta^2 \int \sigma^{-n+4} (\log \sigma)^2 \sigma^{-2} Z v \frac{\partial_t \sigma}{\sigma} - 2 \int \sigma^{-n+4} \mathcal{L} v \partial_t v - 2 \int \sigma^{-n+2} (\beta \log \sigma)^2 - \beta - (n-2) \log \sigma) v \partial_t v - 2 \int \sigma^{-n+3} [b_{ij} (-\log \sigma) \partial_j \sigma] v \partial_t v - 2 \beta \int \sigma^{-n+3} b_{ij} (-\log \sigma) (\partial_i \partial_j \sigma) v \partial_t v - 2 \beta \int \sigma^{-n+2} b_{ij} [-1 + (\log \sigma) + \beta (\log \sigma)^2] \partial_i \sigma \partial_j \sigma v \partial_t v - 2 \beta \int \sigma^{-n+3} (\log \sigma) v \partial_t v \partial_t \sigma =: \sum_{j=1}^{11} I_j. \]

We now examine each \( I_j \) separately.

**Estimate for \( I_2 \)**: Using Lemma 3.8 we have

\[ 4\beta \int_{\partial B} \sigma^{-n+2} (-\log \sigma) Z v \mathcal{L} v \geq 4\beta \int \sigma^{-n} ((n-2)(-\log \sigma) + 1)(Z v)^2 - 2\beta \int \sigma^{-n+2} \langle A \nabla v, \nabla v \rangle - 4c\beta \int \sigma^{-n+3} (-\log \sigma) \langle A \nabla v, \nabla v \rangle 
\]

\[ \geq 4\beta \int \sigma^{-n} ((n-2)(-\log \sigma) + 1)(Z v)^2 - \frac{9}{4} \beta \int \sigma^{-n+2} \langle A \nabla v, \nabla v \rangle, \]

where the penultimate inequality is a consequence of \( c \sigma^{-n+3} (-\log \sigma) \leq \frac{1}{\ln} \sigma^{-n+2} \) which holds for \( \sigma \) small enough which in turn can be ensured by taking \( R_0 \) small enough. This completes the estimate for \( I_2 \).
Estimate for $I_3$: Write $I_3 = \sum_{k=1}^{3} T_3^k$. Let us begin by estimating the term

\[
I_3^1 := \int 4\beta \sigma^{-n} (-\log \sigma) \left[ (\beta \log \sigma)^2 - \beta - (2n-2)\beta \log \sigma \right] (Zv) v
\]

\[
= \int 4\beta \left[ (\beta \log \sigma)^2 - \beta - (2n-2)\beta \log \sigma \right] (-\log \sigma) \sigma^{-n} Z \left( \frac{v^2}{2} \right)
\]

\[
= 4 \int \beta^3 \text{div}([\log \sigma]^3 \sigma^{-n} Z) \frac{v^2}{2} = 4 \int \beta^2 \text{div}((\log \sigma)\sigma^{-n} Z) \frac{v^2}{2}
\]

\[
- 4 \int \beta^2 (n-2) \text{div}((\log \sigma)^2 \sigma^{-n} Z) \frac{v^2}{2},
\]

where we have used integration by parts. Observe

\[
4 \int \beta^3 \text{div}([\log \sigma]^3 \sigma^{-n} Z) \frac{v^2}{2} = 2\beta^3 \int (3(\log \sigma)^2 - n(\log \sigma)^3) \sigma^{-n-1} Z \sigma v^2
\]

\[
+ 2\beta^3 \int (\log \sigma)^3 \sigma^{-n} \text{div}(Z) v^2.
\]

Since $|\text{div} Z - n| \leq \sigma$, we obtain the following estimate

\[
4 \int \beta^3 \text{div}([\log \sigma]^3 \sigma^{-n} Z) \frac{v^2}{2} \geq 6\beta^3 \int \sigma^{-n}(\log \sigma)^2 v^2 - c\beta^3 \int \sigma^{-n+1}(\log \sigma)^3 v^2.
\]

Arguing similarly we obtain

\[
- 4 \int \beta^2 \text{div}((\log \sigma)\sigma^{-n} Z) \frac{v^2}{2} - 4 \int \beta^2 (n-2) \text{div}((\log \sigma)^2 \sigma^{-n} Z) \frac{v^2}{2}
\]

\[
\geq 2\beta^2 \int \sigma^{-n}[2(n-2)(-\log \sigma) - 1] v^2 - C \int \sigma^{-n+1}[\beta^2(-\log \sigma) + 4\beta^2(n-2)(\log \sigma)^2] v^2.
\]

Estimate for $I_3^2$:

\[
I_3^2 := 4\beta \int \sigma^{-n+1} [b_{ij}(-\log \sigma)\partial_i \partial_j \sigma] Z \left( \frac{v^2}{2} \right) = 4\beta \int \text{div} \left( \sigma^{-n+1} [b_{ij}(-\log \sigma) \partial_i \partial_j \sigma] \right) \frac{v^2}{2}
\]

\[
= 4\beta \int \left[ (n-1)\sigma^{-n} Z \sigma (b_{ij} \partial_i \partial_j \sigma)(-\log \sigma) + \sigma^{-n} \sigma Z (b_{ij} \partial_i \partial_j \sigma) + \sigma^{-n+1}(\log \sigma) (b_{ij} \partial_i \partial_j \sigma) + \sigma^{-n+1}(\log \sigma) (\partial_i \partial_j \sigma)(\text{div} Z) \right]
\]

\[
\geq -C\beta \int \sigma^{-n+1}(-\log \sigma) v^2.
\]

Similarly

\[
I_3^3 := 4\beta^2 \int \sigma^{-n}(-\log \sigma)(-1 + (\log \sigma) + \beta(\log \sigma)^2) b_{ij} \partial_i \partial_j \sigma Z \left( \frac{v^2}{2} \right)
\]

\[
= -2\beta^2 \int \text{div}[\sigma^{-n}(-\log \sigma)(-1 + (\log \sigma) + \beta(\log \sigma)^2) b_{ij} \partial_i \partial_j \sigma Z] v^2
\]

\[
\geq -C\beta^2 \int \sigma^{-n+2}(-\log \sigma)(-1 + (\log \sigma) + \beta(\log \sigma)^2) v^2 - C\beta^2 \int \sigma^{-n+1} v^2
\]

\[
- C\beta^3 \int \sigma^{-n+1}(\log \sigma)^2 v^2.
\]
Therefore, combining estimates (3.103), (3.104), (3.105), (3.106), and (3.107), we deduce the following inequality for $R_0$ small enough and $\beta$ sufficiently large

$$I_3 \geq \frac{599}{100} \beta^3 \int \sigma^{-n}(\log \sigma)^2 v^2. \quad (3.108)$$

Estimate for $I_4$: An application of the Young’s inequality $ab \leq \frac{4}{7}a^2 + \frac{1}{2}b^2$, together with the fact that $\sum_{i,j} |\partial_i b_{ij} \partial_j \sigma| \leq c$ gives

$$\int 4\beta \sigma^{-n+1} [\partial_i b_{ij} \cdot \beta(- \log \sigma) \partial_j \sigma] Zv \cdot v$$

$$\geq -c\beta \int \sigma^{-n+1} (Zv)^2 (\log(\sigma))^2 - \beta^3 \int \sigma^{-n+1} v^2. \quad (3.109)$$

Estimate for $I_5$: A similar application of the Young’s inequality implies for all small enough $R_0$

$$I_5 := -\int 4\beta^2 \sigma^{-n+2}(\log \sigma)^2 Zv \cdot Zv$$

$$\geq -2\beta \int \sigma^{-n+2}(\log \sigma)^2 (Zv)^2 - 2\beta^3 \int \sigma^{-n+2}(\log(\sigma))^2 v^2. \quad (3.110)$$

Thus from (3.102), (3.108), (3.109) and (3.110), we obtain

$$\sum_{i=2}^{5} I_i \geq 4\beta \int \sigma^{-n}((n-2)(- \log \sigma))(Zv)^2 + \frac{31}{8} \beta \int \sigma^{-n}(Zv)^2$$

$$+ \frac{598}{100} \beta^3 \int \sigma^{-n}(\log \sigma)^2 v^2 - \frac{9}{4} \beta \int \sigma^{-n+2}\langle A\nabla v, \nabla v \rangle,$$

provided we choose $\beta$ sufficiently large and $R_0$ small enough. Over here we would like to highlight the fact that the term $4\beta \int \sigma^{-n}((n-2)(- \log \sigma))(Zv)^2$ in (3.111) above will be very helpful for us in dealing with some subsequent negative terms.

Now we start estimating terms involving $v_t$. 

Estimate for $\mathcal{I}_6$: Recall that

$$
\mathcal{I}_6 := -2 \int \sigma^{-n+4} \mathcal{L} v \, \partial_t v \\
= 2 \int A \nabla v \cdot \nabla (\partial_t v \cdot \sigma^{-n+4}) \\
= 2 \int \sigma^{-n+4} A \nabla v \cdot \nabla \partial_t v - 2(n-4) \int (A \nabla v, \nabla \sigma) \sigma^{-n+3} \partial_t v \\
= - \int (\partial_t A) \nabla v, \nabla \sigma) \sigma^{-n+4} + 2(n-4) \int (A \nabla v, \nabla \sigma) \sigma^{-n+3} \partial_t \sigma \\
- 2(n-4) \int \sigma^{-n+2} Z(v) \cdot \partial_t v \\
\geq -C \int |\nabla v|^2 \sigma^{-n+4} + 2C(n-4) \int |\nabla v|^2 \sigma^{-n+4} \\
- 2(n-4) \int \sigma^{-n+2} Z(v) \cdot \partial_t v,
$$

where we used that $|\partial_t \sigma| \leq C \sigma$.

Estimate for $\mathcal{I}_7$: Again considering fact that $\partial_t \sigma \simeq O(\sigma)$, using integration by parts we obtain

$$
\mathcal{I}_7 := -2 \int \sigma^{-n+2} ((\beta \log \sigma)^2 - \beta - (n-2)\beta \log \sigma) v \, \partial_t v \\
= \int \partial_t \left( \sigma^{-n+2} ((\beta \log \sigma)^2 - \beta - (n-2)\beta \log \sigma) \right) v^2 \\
= \int \left[ (-n+2)\sigma^{-n+1} \partial_t \sigma ((\beta \log \sigma)^2 - \beta - (n-2)\beta \log \sigma) \\
+ \sigma^{-n+2} \left( 2\beta^2 \log \frac{1}{\sigma} \partial_t \sigma - \beta(n-2)\frac{1}{\sigma} \partial_t \sigma \right) \right] v^2 \\
\geq -C \beta^2 \int \sigma^{-n+2} (\log \sigma)^2 v^2,
$$

provided $\beta$ is sufficiently large and $R_0$ is small enough.

Estimate for $\mathcal{I}_8$: Using Young’s inequality, we obtain that $\mathcal{I}_8$ can be estimated as

$$
\mathcal{I}_8 := -2 \int \sigma^{-n+3} [\partial_i b_{ij} (-\beta \log \sigma) \partial_j \sigma] v \, \partial_t v \\
\geq -c \beta^3 \int \sigma^{-n+1} (\log \sigma)^2 v^2 - \frac{1}{4\beta} \int \sigma^{-n+5} (\partial_t v)^2,
$$

where we have used the fact $\sum_{i,j} |\partial_i b_{ij} \partial_j \sigma| = O(1)$.

Estimate for $\mathcal{I}_9$: Using $|b_{ij}(\partial_i \partial_j \sigma)| = O(1)$ and Young’s inequality we have

$$
\mathcal{I}_9 := -2 \beta \int \sigma^{-n+3} b_{ij} (-\log \sigma) (\partial_i \partial_j \sigma) v \, \partial_t v \\
\geq -c_1 \beta^3 \int \sigma^{-n+1} (\log \sigma)^2 v^2 - \frac{1}{4\beta} \int \sigma^{-n+5} (\partial_t v)^2.
$$
Estimate for $I_{10}$:

$$I_{10} := -2\beta \int \sigma^{-n+2} b_{ij} [(-1 + (\log \sigma) + \beta (\log \sigma)^2)] \partial_i \sigma \partial_j \sigma v \partial_t v$$  \hspace{1cm} (3.116)

We first handle the first two terms in the expression for $I_{10}$. It is easy to observe using $|b_{ij}| = O(\sigma)$ and Young’s inequality that

$$-2\beta \int \sigma^{-n+2} b_{ij} [(-1 + (\log \sigma))] \partial_i \sigma \partial_j \sigma v \partial_t v \geq -c_2\beta^3 \int \sigma^{-n+1} (-\log \sigma)^2 v^2 - \frac{1}{4\beta} \int \sigma^{-n+5} (\partial_t v)^2.$$  \hspace{1cm} (3.117)

Now using integration by parts and the fact that $|\partial_t (\partial_i \sigma \partial_j \sigma)| \leq c$,

$$-2\beta^2 \int \sigma^{-n+2} b_{ij} (\log \sigma)^2 \partial_i \sigma \partial_j \sigma v \partial_t v = \beta^2 \int \partial_t \{\sigma^{-n+2} b_{ij} (\log \sigma)^2 \partial_i \sigma \partial_j \sigma\} v^2 \geq -c_3\beta^2 \int \sigma^{-n+2} (\log \sigma)^2 v^2.$$  \hspace{1cm} (3.118)

Estimate for $I_{11}$: Using $|\partial_t \sigma| = O(\sigma)$, we simply observe

$$-2\beta \int \sigma^{-n+3} (\log \sigma) v \partial_t v \partial_t \sigma \geq -c_4\beta^3 \int \sigma^{-n+1} (\log \sigma)^2 v^2 - \frac{1}{4\beta} \int \sigma^{-n+5} (\partial_t v)^2.$$  \hspace{1cm} (3.119)

Thus from (3.111) - (3.119) it follows that for large enough $\beta$ and sufficiently small $R_0$, we have

$$\sum_{i=2}^{11} I_i \geq 4\beta \int \sigma^{-n} ((n-2)(-\log \sigma))(Zv)^2 + \frac{31}{8}\beta \int \sigma^{-n} (Zv)^2 + \frac{59}{10}\beta^3 \int \sigma^{-n} (\log \sigma)^2 v^2 - 2(n-4) \int \sigma^{-n+2} Zv \cdot \partial_t v - \frac{1}{\beta} \int \sigma^{-n+5} (\partial_t v)^2 - \frac{5}{2}\beta \int \sigma^{-n+2} \langle A \nabla v, \nabla v \rangle.$$  \hspace{1cm} (3.120)

Our focus now will be to get rid of the terms $-2(n-4) \int \sigma^{-n+2} Zv \cdot \partial_t v$ and $-\frac{1}{\beta} \int \sigma^{-n+5} (\partial_t v)^2$ in (3.120) above. We split the term $-2(n-4) \int \sigma^{-n+2} Zv \cdot \partial_t v$ into two parts, namely

$$A := -2(n-2) \int \sigma^{-n+2} Z(v) \cdot \partial_t v$$

$$B := 4 \int \sigma^{-n+2} Z(v) \cdot \partial_t v.$$  \hspace{1cm}

We handle $A$ and $B$ separately.
Estimate for $A$: Let us couple $A$ with $\frac{1}{2} \int \sigma^{-n+4} A^2$, where we recall,

$$\frac{1}{2} \int \sigma^{-n+4} A^2 = \frac{1}{2} \int \sigma^{-n+4} [2\sigma^{-2}(-\beta \log \sigma) Zv - \partial_t v]^2.$$ 

In that direction, let us observe

$$A + \frac{1}{2} \int \sigma^{-n+4} A^2$$

$$= -2(n - 2) \int \sigma^{-n+2} Z(v) \cdot \partial_t v + \frac{1}{2} \int \sigma^{-n+4} [2\sigma^{-2}(-\beta \log \sigma) Zv - \partial_t v]^2$$

$$= -2(n - 2) \int \sigma^{-n+2} Z(v) \cdot \partial_t v$$

$$+ \int \sigma^{-n+4} \left[ \sqrt{2} \sigma^{-2}(-\beta \log \sigma) Zv - \frac{1}{\sqrt{2}} \partial_t v \right]^2$$

$$= \int \sigma^{-n+4} \left[ \sqrt{2} \sigma^{-2}(-\beta \log \sigma) Zv - \frac{1}{\sqrt{2}} \partial_t v + \sqrt{2}(n - 2) \frac{Zv}{\sigma^2} \right]^2$$

$$- 4\beta \int \sigma^{-n}((n - 2)(-\log \sigma))(Zv)^2 - 2(n - 2)^2 \int \sigma^{-n}(Zv)^2$$

$$\geq -4\beta \int \sigma^{-n}((n - 2)(-\log \sigma))(Zv)^2 - 2(n - 2)^2 \int \sigma^{-n}(Zv)^2.$$

Here we would like to emphasize that the negative term appearing above, i.e., $-4\beta \int \sigma^{-n}((n - 2)(-\log \sigma))(Zv)^2$ will be cancelled by the exact positive term present in (3.120) and we will eventually choose $\beta$ large enough such that

$$2(n - 2)^2 \int \sigma^{-n}(Zv)^2 \leq \frac{\beta}{8} \int \sigma^{-n}(Zv)^2.$$

Now we focus on getting rid of the negative term $-\frac{1}{2} \int \sigma^{-n+5} (\partial_t v)^2$ present in (3.120). This will be tackled by $B$ and $\frac{1}{2} \int \sigma^{-n+4} A^2$. To see that, we observe

Estimate for $B$:

$$B + \frac{1}{2} \int \sigma^{-n+4} A^2$$

$$= 4 \int \sigma^{-n+2} Zv \cdot \partial_t v + \frac{1}{2} \int \sigma^{-n+4} [2\sigma^{-2}(-\beta \log \sigma) Zv - \partial_t v]^2$$

$$= 4 \int \sigma^{-n+2} Zv \cdot \partial_t v + \frac{1}{2} \int \sigma^{-n+4} \left[ 2\sigma^{-2}(-\beta \log \sigma) Zv - \left( 1 - \frac{2}{(-\beta \log \sigma)} \right) \partial_t v - \frac{2\partial_t v}{\beta(-\log \sigma)} \right]^2$$

$$= 4 \int \sigma^{-n+2} Zv \cdot \partial_t v + \frac{1}{2} \int \sigma^{-n+4} \left[ 2\sigma^{-2}(-\beta \log \sigma) Zv - \left( 1 - \frac{2}{(-\beta \log \sigma)} \right) \partial_t v \right]^2$$

$$- 2 \int \sigma^{-n+4} \left[ 2\sigma^{-2}(-\beta \log \sigma) Zv - \left( 1 - \frac{2}{(-\beta \log \sigma)} \right) \partial_t v \right] \frac{\partial_t v}{\beta(-\log \sigma)} + 2 \int \sigma^{-n+4} \frac{(\partial_t v)^2}{(\beta \log \sigma)^2}$$

$$\geq \int \sigma^{-n+4} \frac{(\partial_t v)^2}{(\beta \log \sigma)^2} (2 + 2\beta(-\log \sigma) - 4)$$

$$\geq \frac{1}{\beta} \int \sigma^{-n+5} (\partial_t v)^2,$$
provided we choose $\beta$ large enough and $R_0$ small enough, From (3.120), (3.121), (3.122) and (3.123) we thus have
\[
\int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (\mathcal{L} u - \partial_t u)^2 \geq \frac{59}{10} \beta^3 \int \sigma^{-n} (\log \sigma)^2 v^2 + \frac{15}{4} \beta \int \sigma^{-n} (Zv)^2 - \frac{5}{2} \beta \int \sigma^{-n+2} \langle A \nabla v, \nabla v \rangle,
\]
which finishes the proof of the theorem.

Now in order to incorporate the gradient term in our carleman estimate in Theorem 3.9, we will use an interpolation type argument. As an intermediate step, we first relate $\int \sigma^{-n+2} \langle A \nabla v, \nabla v \rangle$ with $\int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle$.

**Lemma 3.11.** The following holds true for $R_0$ small enough and $\beta$ large.
\[
\int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle \geq \int \sigma^{-n+2} \langle A \nabla v, \nabla v \rangle + \frac{4 \beta^2}{5} \int \sigma^{-n} (\log \sigma)^2 v^2. \tag{3.124}
\]

**Proof.** We have,
\[
\int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle = \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla e^{-\beta/2 (\log \sigma)^2} v, \nabla e^{-\beta/2 (\log \sigma)^2} v \rangle \tag{3.125}
\]
\[
= \int \sigma^{-n+2} \left[ \frac{(\beta \log \sigma)^2}{\sigma^2} v^2 \langle A \nabla \sigma, \nabla \sigma \rangle + 2 \frac{(-\beta \log \sigma)}{\sigma^2} vZv_+ < A \nabla v, \nabla v > \right].
\]
We now claim that
\[
\langle A \nabla \sigma, \nabla \sigma \rangle = 1 + O(\sigma). \tag{3.126}
\]
(3.126) is seen by writing $a_{ij}(x,t) = a_{ij}(0,t) + b_{ij}(x,t)$. We thus have
\[
\langle A \nabla \sigma, \nabla \sigma \rangle = \frac{a_{ij}(0,t)g_{ij}(0,t)x_lg_{jm}(0,t)x_m}{\sigma^2} + b_{ij}(x,t)\partial_i \sigma \partial_j \sigma \tag{3.127}
\]
\[
= \delta_{ij}x_lg_{jm}x_m + O(\sigma) \text{ (using } |\nabla \sigma| = O(1) \text{ and } |b_{ij}(x,t)| = O(\sigma).)
\]
\[
= \frac{g_{jm}x_lx_m}{\sigma^2} + O(\sigma)
\]
\[
= 1 + O(\sigma).
\]
Also we have
\[
2 \int \sigma^{-n} (-\beta \log \sigma) vZv = \beta \int \text{div} (\sigma^{-n} Z \log \sigma) v^2 \geq -C \beta \int \sigma^{-n} v^2. \tag{3.128}
\]
Using (3.127) and (3.128) in (3.125), we find that (3.124) follows provided $\beta$ is sufficiently large and $R_0$ is small enough.

Using (3.124) in (3.95) we have
\[
\int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (\mathcal{L} u - \partial_t u)^2 + \frac{5}{2} \beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle \tag{3.129}
\]
\[
\geq \frac{79}{10} \beta^3 \int \sigma^{-n} (\log \sigma)^2 v^2 + \frac{15}{4} \beta \int \sigma^{-n} (Zv)^2.
\]
We now state the prove the relevant interpolation lemma that allows us to incorporate the gradient term in our main Carleman estimate.
Lemma 3.12 (Interpolation lemma). The following holds true for small $R_0$ and sufficiently large $\beta$

$$\beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle \leq C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2.$$  \hfill (3.130)

Proof.

$$\beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle = -\beta \int \left\langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \right\rangle e^{-\beta/2 (\log \sigma)^2} v \quad (3.131)$$

$$= -\beta \int \left\langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \right\rangle e^{-\beta/2 (\log \sigma)^2} v$$

$$- \beta \int (L u - \partial_t u) \sigma^{-n+2} e^{\beta/2 (\log \sigma)^2} v - \beta \int \partial_t u \sigma^{-n+2} e^{\beta/2 (\log \sigma)^2} v$$

$$\leq -\beta \int \left\langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \right\rangle e^{-\beta/2 (\log \sigma)^2} v$$

$$+ C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2 + C \beta^2 \int \sigma^{-n} v^2 - \beta \int \partial_t u \sigma^{-n+2} e^{\beta/2 (\log \sigma)^2} v.$$  

Now by integration by parts in $t$ and by using $|\partial_t \sigma| = O(\sigma)$, we find that

$$-\beta \int \partial_t u \sigma^{-n+2} e^{\beta/2 (\log \sigma)^2} v \leq C_1 \beta^2 \int \sigma^{-n} v^2. \quad (3.132)$$

We thus have

$$\beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle \leq \int \left\langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \right\rangle e^{-\beta/2 (\log \sigma)^2} v \quad (3.133)$$

$$+ C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2 + C \beta^2 \int \sigma^{-n} v^2.$$  

Now the term $C \beta^2 \int \sigma^{-n} v^2$ can be estimated using (3.129) and we consequently have

$$\beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} \langle A \nabla u, \nabla u \rangle \leq -\beta \int \left\langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \right\rangle e^{-\beta/2 (\log \sigma)^2} v \quad (3.134)$$

$$+ C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2 + C \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} < A \nabla u, \nabla u >.$$  

We are now left with estimating the term

$$\beta \int \left\langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \right\rangle e^{-\beta/2 (\log \sigma)^2} v.$$  

This is done as follows. We have using (3.127)

$$\beta \int \langle \nabla (\sigma^{-n+2} e^{\beta (\log \sigma)^2}), A \nabla (e^{-\beta/2 (\log \sigma)^2} v) \rangle e^{-\beta/2 (\log \sigma)^2} v \quad (3.135)$$

$$= \beta (-n+2) \int \sigma^{-n} \left[ \beta (-\log \sigma) v^2 + Z v \cdot v \right] - \beta \int \sigma^{-n} e^{\beta} (-\log \sigma) \left[ (-\log \sigma) v^2 + Z v \cdot v \right]$$

$$+ O(\beta^3) \int \sigma^{-n+1} (\log \sigma)^2 v^2 \geq -\frac{31\beta^3}{10} \int \sigma^{-n} (\log \sigma)^2 v^2 - \frac{11\beta}{10} \int \sigma^{-n} (Z v)^2 \quad (for \ all \ large \ \beta),$$
where in the last inequality we used that
\[
\left| \beta \int \sigma^{-n} 2 \beta (- \log \sigma) Z v \cdot v \right| \leq \beta^3 \int \sigma^{-n} (\log \sigma)^2 v^2 + \beta \int \sigma^{-n} (Z v)^2 \quad \text{and} \quad (3.136)
\]
\[
\left| O(\beta^3) \int \sigma^{-n+1} (\log \sigma)^2 v^2 + \beta (-n + 2) \int \sigma^{-n} [\beta (- \log \sigma) v^2 + Z v \cdot v] \right|
\leq \frac{\beta^3}{10} \int \sigma^{-n} (\log \sigma)^2 v^2 + \frac{\beta}{10} \int \sigma^{-n} (Z v)^2 \quad \text{for small enough } R_0.
\]

Now, by multiplying the inequality (3.129) on both sides by \( \frac{10}{19} \times \frac{31}{10} \), we have
\[
\frac{31}{10} \beta^3 \int \sigma^{-n} (\log \sigma)^2 v^2 + \frac{465 \beta}{316} \int \sigma^{-n} (Z v)^2 \leq \frac{155 \beta}{158} \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} (A \nabla u, \nabla u) + C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2. \quad (3.137)
\]

Since \( \frac{465}{316} > \frac{11}{10} \), we obtain from (3.137) that
\[
\frac{31}{10} \beta^3 \int \sigma^{-n} (\log \sigma)^2 v^2 + \frac{11 \beta}{10} \int \sigma^{-n} (Z v)^2 \leq \frac{155 \beta}{158} \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} (A \nabla u, \nabla u) + C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2. \quad (3.138)
\]

Using (3.135) and (3.138) in (3.134) we have
\[
\beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} (A \nabla u, \nabla u) \leq C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2 + \left( C + \frac{155 \beta}{158} \right) \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} (A \nabla u, \nabla u). \quad (3.139)
\]

Now, for all \( \beta \) large enough, we observe that the following term in (3.139) above, i.e.
\[
\left( C + \frac{155 \beta}{158} \right) \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} (A \nabla u, \nabla u),
\]
can be absorbed in the left hand side of (3.139) and we thus infer that the following estimate holds,
\[
\beta \int \sigma^{-n+2} e^{\beta (\log \sigma)^2} (A \nabla u, \nabla u) \leq C \int \sigma^{-n+4} e^{\beta (\log \sigma)^2} (L u - \partial_t u)^2. \quad (3.140)
\]

The conclusion thus follows.

\[ \square \]

**Proof of Theorem 3.9.** The desired estimate (3.94) now follows from (3.129) and (3.140).

\[ \square \]

With Theorem 3.9 in hand, we now show that nontrivial solutions to (1.1) decay less than exponentially.
Proof of Theorem 2.2. We follow the strategy as in [5, 24]. The proof is also partly similar to that of Proposition 3.6. As before, in what follows we assume that \(u\) solves (1.1) in \(B_R \times (-T, T)\), instead of \(B_R \times (0, T)\). By applying Proposition 3.6, we have that
\[
\int_{B_R \times (-T/2, T/2)} u^2 \lesssim e^{-C s^2/3}, \quad \text{as } s \to 0 \quad (3.141)
\]
for some constant \(C > 0\). We first show that \(u(\cdot, t) = 0\) for \(|t| \leq T/2\). Then by repeating the arguments, one can spread the zero set for \(|t| > T/2\) to finally assert that \(u \equiv 0\).

In view of (3.141), we now let \(T/2\) as our new \(T\). Without loss of generality, we assume that \(R < 1\). Let \(\phi(x)\) be a smooth function such that
\[
\phi(x) = \begin{cases} 
0, & |x| < \frac{r_2}{2}; \\
r_1 < |x| < r_2; \\
0, & |x| > r_3,
\end{cases}
\]
where, \(0 < r_1 < r_2/2 < 4r_2 < r_3 < R/2\) will be fixed at a later stage. We subsequently let \(T_2 = T/2\) and \(T_1 = 3T/4\), so that \(0 < T_2 < T_1 < T\). As before, we let \(\eta(t)\) be a smooth even function such that \(\eta(t) \equiv 1\) when \(|t| < T_2\), \(\eta(t) \equiv 0\), when \(|t| > T_1\). More precisely
\[
\eta(t) = \begin{cases} 
0, & -T \leq t \leq -T_1 \\
\exp \left( -\frac{T^3(T_2+t)^4}{(T_1+t)^4(T_1-T_2)^4} \right), & -T_1 \leq t \leq -T_2, \\
1, & -T_2 \leq t \leq 0.
\end{cases} \quad (3.142)
\]
Without loss of generality we assume that
\[
\int_{B_M \times (-T_2, T_2)} u^2 \neq 0, \quad (3.143)
\]
where \(M, N\) are constants such that \(M |x| \leq \sigma(x, t) \leq N |x|\). Otherwise, the result in [24] implies \(u \equiv 0\) in \(B_R \times (-T_2, T_2)\) and by arguments that follow, we could conclude that \(u \equiv 0\) also for \(|t| > T_2\).

Now, with \(u\) as in Theorem 2.2 we let \(v = \phi \eta u\). Limiting argument allows to use such \(v\) in the Carleman estimate (3.9), obtaining
\[
\beta^3 \int \sigma^{-n} e^{\beta(\log \sigma)^2} v^2 + \beta \int \sigma^{-n+2} e^{\beta(\log \sigma)^2} \langle A \nabla v, \nabla v \rangle 
\leq C \int \sigma^{-n+4} e^{\beta(\log \sigma)^2} (\mathcal{L} v - \partial_t v)^2. \quad (3.144)
\]
Incorporating the following
\[
\mathcal{L} v - \partial_t v = \phi \eta (\mathcal{L} u - u_t) + 2 \eta \langle A \nabla \phi, \nabla u \rangle + u (\eta \div (A \nabla \phi) - \phi \eta),
\]

...
The above inequality can be rewritten as follows

\[
\beta^3 \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} v^2 + \beta \int \sigma(x,t)^{-n+2} e^{\beta(\log \sigma(x,t))^2} \langle A \nabla v, \nabla v \rangle \\
\leq C \left[ \int \sigma(x,t)^{-n+4} e^{\beta(\log \sigma(x,t))^2} \right. \\
\left. + \int \eta^2 \left[ \nabla u \cdot \nabla \phi + u^2 |\nabla^2 \phi|^2 + u^2 |\nabla \phi|^2 \right] \sigma(x,t)^{-n+4} e^{\beta(\log \sigma(x,t))^2} \\
+ \int u^2 \phi^2 \eta^2 \sigma(x,t)^{-n+4} e^{\beta(\log \sigma(x,t))^2} \right]
\]

where in the final inequality we have used the differential inequality (1.1). Therefore, for sufficiently large \( \beta \), we find that the term \( C \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} \phi^2 \eta^2 u^2 \) can be absorbed in the left hand side and we thus obtain

\[
\beta^3 \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} v^2 + \beta \int \sigma(x,t)^{-n+2} e^{\beta(\log \sigma(x,t))^2} \langle A \nabla v, \nabla v \rangle \\
\leq \int \{r_1/2 < r < r_1 \} \times (-T_1, T_1) \eta^2 \left[ |\nabla u|^2 \sigma(x,t)^{-n+2} + u^2 \sigma(x,t)^{-n} \right] e^{\beta(\log \sigma(x,t))^2} \\
+ \int \{r_2 < r < r_3 \} \times (-T_1, T_1) \eta^2 \left[ |\nabla u|^2 \sigma(x,t)^{-n+2} + u^2 \sigma(x,t)^{-n} \right] e^{\beta(\log \sigma(x,t))^2} \\
+ \int u^2 \phi^2 \eta^2 \sigma(x,t)^{-n+4} e^{\beta(\log \sigma(x,t))^2}.
\]

The above inequality can be rewritten as follows

\[
\frac{\beta^3}{2} \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} v^2 + \beta \int \sigma(x,t)^{-n+2} e^{\beta(\log \sigma(x,t))^2} \langle A \nabla v, \nabla v \rangle \\
\leq \int \{r_1/2 < r < r_1 \} \times (-T_1, T_1) \eta^2 \left[ |\nabla u|^2 \sigma(x,t)^{-n+2} + u^2 \sigma(x,t)^{-n} \right] e^{\beta(\log \sigma(x,t))^2} \\
+ \int \{r_2 < r < r_3 \} \times (-T_1, T_1) \eta^2 \left[ |\nabla u|^2 \sigma(x,t)^{-n+2} + u^2 \sigma(x,t)^{-n} \right] e^{\beta(\log \sigma(x,t))^2} \\
+ \int u^2 \phi^2 \eta^2 \sigma(x,t)^{-n+4} e^{\beta(\log \sigma(x,t))^2} - \frac{\beta^3}{2} \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} v^2 := I_1 + I_2 + I_3 + I_4.
\]

Recall the following notations introduced in (3.62)

\[
\sigma_1(r_1, r_2, T) := \inf_{\{r_1 < |x| < r_2 \} \times (-T, T)} \sigma(x,t), \quad \text{and} \quad \sigma_2(r_1, r_2, T) := \sup_{\{r_1 < |x| < r_2 \} \times (-T, T)} \sigma(x,t).
\]

(3.145)
Estimate for $I_1$.

\[
I_1 = \int_{\{r_1/2 < r_1 \times (-T_1, T_1)\}} \eta^2 \left[ |\nabla u|^2 \sigma(x, t) - n+2 + u^2 \sigma(x, t) - n \right] e^{\beta(\log \sigma(x, t))^2} \\
\lesssim \sigma_1(r_1/2, r_1, T)^{-n} e^{\beta(\log \sigma(r_1/2, r_1, T))^2} \int_{\{r_1/2 < |x| < r_1 \times (-T_1, T_1)\}} u^2 \\
+ \sigma_1(r_1/2, r_1, T)^{-n+2} e^{\beta(\log \sigma_1(r_1/2, r_1, T))^2} \int_{\{r_1/2 < |x| < r_1 \times (-T_1, T_1)\}} |\nabla u|^2 \\
\lesssim \sigma_1(r_1/2, r_1, T)^{-n} e^{\beta(\log \sigma_1(r_1/2, r_1, T))^2} \int_{\{r_1/4 < |x| < 3r_1/2 \times (-T_1, T_1)\}} |u|^2, 
\]

(3.149)

where in the last inequality above, we used the energy estimate from Lemma 3.4. Similarly

Estimate for $I_2$.

\[
I_2 = \int_{\{r_2 < r < r_3 \times (-T_1, T_1)\}} \eta^2 \left[ |\nabla u|^2 \sigma(x, t) - n+2 + u^2 \sigma(x, t) - n \right] e^{\beta(\log \sigma)^2} \\
\lesssim \sigma_1(r_2, r_3, T)^{-n} e^{\beta(\log \sigma_1(r_2, r_3, T))^2} \int_{B_R \times (-T, T)} u^2. 
\]

(3.150)

Let us decompose $I_3$ into several parts.

\[
I_3 := \int u^2 \phi^2 \eta^2 \sigma(x, t) - n+4 e^{\beta(\log \sigma(x, t))^2} \\
:= \int_{\{r_1/2 < |x| < r_1 \times (-T_1, T_1)\}} u^2 \phi^2 \eta^2 \sigma(x, t) - n+4 e^{\beta(\log \sigma(x, t))^2} \\
+ \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1)\}} u^2 \phi^2 \eta^2 \sigma(x, t) - n+4 e^{\beta(\log \sigma(x, t))^2} \\
+ \int_{\{r_2 < |x| < r_3 \times (-T_1, T_1)\}} u^2 \phi^2 \eta^2 \sigma(x, t) - n+4 e^{\beta(\log \sigma(x, t))^2} \\
\lesssim \sigma_1(r_1/2, r_1, T)^{-n} e^{\beta(\log \sigma_1(r_1/2, r_1, T))^2} \int_{\{r_1/2 < |x| < r_1 \times (-T_1, T_1)\}} u^2 \\
+ \sigma_1(r_2, r_3, T)^{-n} e^{\beta(\log \sigma_1(r_2, r_3, T))^2} \int_{\{r_2 < |x| < r_3 \times (-T_1, T_1)\}} u^2 \\
+ \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1)\}} u^2 \eta^2 \sigma(x, t) - n+4 e^{\beta(\log \sigma(x, t))^2} := I_{31} + I_{32} + I_{33}. 
\]

(3.151)

At this point, let us club the term $I_{33}$ with $I_4$. Since, the function $\eta$ is supported in the set $(-T_1, -T_2) \cup (T_2, T_1)$, if we denote $\Omega = \{r_1 < |x| < r_2 \times (-T_1, T_1)\}$, we can bound

\[
I_{33} + I_4 := \int_{\{r_1 < |x| < r_2 \times (-T_1, T_1)\}} u^2 \eta^2 \sigma(x, t) - n+4 e^{\beta(\log \sigma(x, t))^2} - \frac{\beta^3}{2} \int_{B_R} \sigma(x, t)^{-n} e^{\beta(\log \sigma(x, t))^2} v^2 \\
\leq \int_{\Omega} \sigma(x, t)^{-n} e^{\beta(\log \sigma(x, t))^2} u^2 \eta^2 \left( C \sigma(x, t) \eta^2 \eta^2 - \frac{\beta^3}{2} \right) \\
\lesssim \int_{\Omega} \sigma(x, t)^{-n} e^{\beta(\log \sigma(x, t))^2} u^2 \eta^2 \left( C \sigma(x, t) \eta^2 \eta^2 - \frac{\beta^3}{2} \right), 
\]

(3.152)
where the last inequality relies on the fact that $\sigma$ is very small. The rest of the proof is devoted in proving the following claim.

**Claim:**

$$\int_{\Omega} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \left(C \sigma(x,t)^3 \frac{\eta^2}{\eta^2} - \frac{\beta^3}{2}\right) \leq C \int_{B_R \times (-T,T)} u^2. \quad (3.153)$$

It is sufficient to prove (3.153) over the region $\Omega^- = \{r_1 < |x| < r_2\} \times (-T_1,-T_2)$, since the other part can be handled by symmetry. Now, if $-T_1 \leq t \leq -T_2$, note that $T_1 - T_2 = \frac{T}{3}$, $T_2 + t \leq T_1 - T_2 = \frac{T}{3}$, and that $\frac{2}{3}T \leq 4T_1 - 3T_2 + t \leq T$, it follows similarly as before in the proof of Proposition 3.6 that

$$\left| \eta T \right| = \left| \frac{T^3(T_2 + t)^3(4T_1 - 3T_2 + t)}{(T_1 - T_2)^3(T_1 + t)^4} \right| \leq \frac{4T^3}{|T_1 + t|^4}.$$ 

Consequently we obtain

$$\int_{\Omega^-} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \left(C \sigma(x,t)^3 \frac{\eta^2}{\eta^2} - \frac{\beta^3}{2}\right) \leq \int_{\Omega^-} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \left(C \sigma(x,t)^3 \frac{T^6}{(T_1 + t)^8} - \frac{\beta^3}{2}\right). \quad (3.154)$$

Trivially

$$\int_{\Omega^-} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \left(C \sigma(x,t)^3 \frac{\eta^2}{\eta^2} - \frac{\beta^3}{2}\right) \leq \int_{U} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \left(C \sigma(x,t)^3 \frac{T^6}{(T_1 + t)^8} - \frac{\beta^3}{2}\right), \quad (3.155)$$

where

$$U := \left\{(x,t) \in \Omega^- : \frac{\beta^3}{2} \leq C \sigma(x,t)^3 \frac{T^6}{(T_1 + t)^8}\right\}. \quad (3.156)$$

We thus have

$$\int_{\Omega^-} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \left(C \sigma(x,t)^3 \frac{\eta^2}{\eta^2} - \frac{\beta^3}{2}\right) \leq C \int_{U} \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \eta^2 \frac{T^6}{(T_1 + t)^8}. \quad (3.157)$$

The claim will be achieved if we establish a bound from above of the quantity $\sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} \eta^2 \frac{T^6}{(T_1 + t)^8}$ in $U$. Appealing to the exponential decay of $\eta$, at $t = -T_1$, see (3.142), we obtain for $t \in (-T_1,-T_2)$

$$\frac{\eta T^6}{(T_1 + t)^8} \leq C. \quad (3.158)$$

Now we note that the following equivalence holds

$$\sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} \eta \leq 1 \quad \iff \quad n \log \sigma(x,t) - \beta(\log \sigma(x,t))^2 - \log \eta \geq 0$$

$$\iff \quad n \log \sigma(x,t) - \beta(\log \sigma(x,t))^2 + \frac{T^3(T_2 + t)^4}{(T_1 + t)^3(T_1 - T_2)^4} \geq 0. \quad (3.159)$$

The following holds true in $U$

$$\frac{T_1 + t}{T} \leq \left(\frac{C}{2T^2}\right)^{1/8} \left(\frac{\sigma(x,t)}{\beta}\right)^{3/8} = C \left(\frac{\sigma(x,t)}{\beta}\right)^{3/8},$$
for some universal $C > 0$. For sufficiently large $\beta$ we have

$$C \left( \frac{\sigma(x,t)}{\beta} \right)^{3/8} \leq C \left( \frac{R}{\beta} \right)^{3/8} \leq \frac{1}{12}. $$

Combining the above, we have

$$\frac{T_1 + t}{T} \leq \frac{1}{12}. \quad (3.159)$$

in $U$, provided $\beta$ is large enough. Also, $\frac{T}{4} = T_1 - T_2 = T_1 + t + |T_2 + t|$, from (3.159) we conclude that we must have in $U$

$$|T_2 + t| \geq \frac{T}{6}. $$

Invoking these information, we have

$$n \log \sigma(x,t) - \beta(\log \sigma(x,t))^2 + \frac{T^3(T_2 + t)^4}{(T_1 + t)^3(T_1 - T_2)^4} \geq n \log \sigma(x,t) - \beta(\log \sigma(x,t))^2 + \left( \frac{4}{6} \right)^4 \left( \frac{2}{C} \right)^{3/8} T^{3/4} \left( \frac{\beta}{\sigma(x,t)} \right)^{9/8} \geq 0, \quad (3.160)$$

if $\sigma$ is sufficiently small, and $\beta$ very large. We highlight the crucial role of the exponent $\beta^{9/8}$, which dominates the linear term in $\beta$ in order to achieve the required conclusion and also for small $\sigma$, $\frac{1}{\sigma^{9/8}}$ overpowers $\log \frac{1}{\sigma}$, as well as $(\log \frac{1}{\sigma})^2$. Therefore for large enough $\beta$ and small enough $R$, we find that (3.158) is valid and consequently the claim (3.153) holds. Now combining (3.149), (3.150), (3.151) and (3.153), we obtain

$$\frac{\beta^3}{2} \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \leq \sigma_1(r_1/2,r_1,T)^{-n} e^{\beta(\log \sigma_1(r_1/2,r_1,T))^2}$$

$$\|u\|^2 \int_{\{r_1/4 < |x| < 3r_1/2\} \times (-T_1,T_1)} + \sigma_1(r_2,r_3,T)^{-n} e^{\beta(\log \sigma_1(r_2,r_3,T))^2}$$

$$\|u\|^2 \int_{B_{r_1} \times (-T,T)} + C \int_{B_{r_1}} u^2. $$

Recall that there exist constants $M$ and $N$, depending on the ellipticity of the coefficient matrix $A$, such that

$$M|x| \leq \sigma(x,t) \leq N|x|. \quad (3.162)$$

The integral in the left-hand side of (3.161) can be bounded from below in the following way

$$\frac{\beta^3}{2} \int \sigma(x,t)^{-n} e^{\beta(\log \sigma(x,t))^2} u^2 \geq \frac{\beta^3}{2} \sigma_2(r_1, M_{r_2}/N, T)^{-n} e^{\beta(\log \sigma_2(r_1, M_{r_2}/N,T))^2} \int_{\{r_1 < |x| < \frac{M_{r_2}}{N}\} \times (-T_2,T_2)} u^2. \quad (3.163)$$
Substituting (3.163) in (3.161), and dividing both sides by \( \sigma_2(r_1, M_{r_2}/N, T)^{-n} e^{\beta (\log(\sigma_2(r_1, M_{r_2}/N, T))^2} \), we obtain
\[
\frac{\beta^3}{2} \int_{\{r_1<|x|<\frac{M_{r_2}}{N}\} \times (-T_2, T_2)} u^2 \leq C \left( \frac{\sigma_2(r_1, M_{r_2}/N, T)}{\sigma_1(r_1/2, r_1, T)} \right)^n e^{\beta (\log(\sigma_1(r_1/2, r_1, T))^2)} \int_{\{r_1/4<|x|<3r_1/2\} \times (-T_1, T_1)} u^2 \\
+ \left( \frac{\sigma_2(r_1, M_{r_2}/N, T)}{\sigma_1(r_2, r_3, T)} \right)^n e^{\beta (\log(\sigma_1(r_2, r_3, T))^2)} \int_{B_T \times (-T, T)} u^2 \\
+ C \int_{B_{R} \times (-T, T)} u^2.
\] (3.164)

Keeping in mind (3.162), it is easily seen that
\[
e^{\beta (\log(\sigma_1(r_2, r_3, T))^2} - (\log(\sigma_2(r_1, M_{r_2}/N, T))^2) \leq e^{\beta (\log M_{r_2}^2 - (\log M_{r_2})^2} \leq 1.
\]

Using this in (3.164) we deduce the following estimate
\[
\frac{\beta^3}{2} \int_{B_{M_r} \times (-T_2, T_2)} u^2 \leq \tilde{C} \int_{B_{M_r} \times (-T_2, T_2)} u^2.
\] (3.165)

Adding \( \frac{\beta^3}{2} \int_{B_{r_1} \times (-T_2, T_2)} u^2 \) and choosing \( R \) small enough such that \( e^{\beta (\log \frac{r_1}{r_2})^2} \geq \frac{\beta^3}{2} \), we obtain
\[
\frac{\beta^3}{2} \int_{B_{M_r} \times (-T_2, T_2)} u^2 \leq C \left( \frac{2r_2}{r_1} \right)^n e^{\beta (\log c r_1)^2} \int_{B_{M_r} \times (-T_1, T_1)} u^2 \\
+ \tilde{C} \int_{B_R \times (-T, T)} u^2.
\] (3.166)

Keeping in mind the assumption (3.143), we can choose \( \beta \) large enough such that
\[
\frac{\beta^3}{8} \int_{B_{M_r} \times (-T_2, T_2)} u^2 \geq \tilde{C} \int_{B_R \times (-T, T)} u^2,
\] (3.167)

where \( \tilde{C} \) is the constant appearing in (3.166). Subtracting off \( \tilde{C} \int_{B_R \times (-T, T)} u^2 \) from both sides of (3.166) reduces to following
\[
e^{-\beta (\log \frac{r_1}{r_2})^2} \left( \frac{r_1}{2r_2} \right)^n \frac{\beta^3}{8} \int_{B_{M_r} \times (-T_2, T_2)} u^2 \leq \int_{B_{2r_1} \times (-T_1, T_1)} u^2.
\] (3.168)

Now we fix some \( \beta \), sufficiently large, for which (3.159), (3.160), and (3.167) hold simultaneously. Therefore, (3.168) implies that for sufficiently small \( s \), there is a constant \( \kappa \), depending on \( r_2, r_3, R, \beta \) and the ratio
\[ \frac{\int_{B_{\frac{1}{2N}} \times (-T,T)} u^2}{\int_{B_{\frac{1}{2}}} \times (-T_2,T_2)} u^2 \] such that
\[ \int_{B_{1} \times (-T,T)} u^2 \geq C e^{-\kappa (\log \frac{3}{T})^2} . \]

This contradicts (3.141). Thus we conclude that \( u \equiv 0 \). \( \square \)

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