ON AN APPLICATION OF FIXED POINT THEOREM TO
NONLINEAR INCLUSIONS

SHUI-HUNG HOU
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hong Kong, China

Abstract. Sufficient conditions for existence of solutions for a class of non-
linear inclusions in infinite dimensional Banach spaces are established. The
results are obtained by means of fixed-point theorem for set-valued maps.

1. Introduction. We consider an inclusion problem of the form

\[ 0 \in x + KF(x) \] (1)

defined in a space \( X = L^p(\Omega, E) \) of functions from \( \Omega \) to a Banach space \( E \), where
\( F : X \to 2^{X^*} \) is a given set-valued map and \( K : X^* \to X \) is a nonlinear mapping
with \( X^* = L^q(\Omega, E^*) \), \( 1/p + 1/q = 1 \). We present here sufficient conditions for the
existence of solutions and an existence theorem for a class of inclusion equations (1).
The approach we shall take is to reduce the problem of existence to the problem of
showing a certain set-valued map has a fixed point.

Applications of fixed point theorems for set-valued maps to differential inclusions
and related problems had been reported by various authors. In fact, there is a
vast literatures covering questions of existence of solutions for finite and infinite
dimensional systems. For example, previously Lasota and Opial [1] in their paper
studied the existence problem of ordinary differential equations with multi-valued
right hand side. Tarnove [2] obtained sufficient conditions for controllability of
the nonlinear system in the form \( x' = f(t, x, u) \), \( u(t) \in Q \). Hermes [3] considered
equations of the form \( x' \in R(t, x) \). Subsequently, Hou [4] studied the controllability
of feedback systems in the form

\[ Ex(t) \in f(t, Mx(t), u(t)), u(t) \in \omega(t, Mx(t)). \]

Cesari and Hou [5] also proved existence theorems of optimal solutions for abstract
equations, and in particular to quasi-linear evolution equations, based on properties
of sets depending on parameters, or set-valued maps. Ahmed [6] considered existence
of optimal controls for a class of systems governed by differential inclusions
on a Banach space. For further results, see Agarwal et al [7] as well as Granas
and Dugundji [8], and the references therein.

There are various motivations for studying the nonlinear inclusion equation (1).
Let us mention some of them here. Evidently, if the set-valued map \( f \) is single-
valued, i.e. \( f(\omega, u) = \{ g(\omega, u) \} \) for some function \( g : \Omega \times E \to E^* \), then we can

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construct a superposition operator of the form
\[ K_g x(\cdot) := K g(\cdot, x(\cdot)) \].

The corresponding inclusion (1) then reduces to an equation of the familiar type
\[ 0 = x + K_g x. \]

On the other hand, when investigating boundary value problems in control theory
which define the state \( x \) of a system by an acting input \( h \), one is led to the equation
of the form
\[ L x = h, \]
where \( L \) is a linear operator on an appropriate function space. Now, if the input is
perturbed, this has to be replaced by the equation with set-valued right-hand side
\[ L x \in N x, \]
where \( N \) is some set-valued nonlinear operator. In many cases \( L \) is some differential
operator which admits a certain Green’s function on a space determined by suitably
boundary conditions. As such the above inclusion may be put in the form (1) by
setting \( K = -L^{-1} \).

The rest of the paper is organized as follows. In section 2 we introduce the basic
notations and recall some preliminary results. In section 3 we prove a selection
lemma and give sufficient conditions for a class of nonlinear inclusion equations to
have a solution.

2. Preliminaries. Let \((\Omega, \mu)\) be a finite measure space. Let \( E \) be a real Banach
space and \( E^* \) be the topological dual of \( E \).

The concept of measurability of a function from an abstract measure space \( \Omega \)
with finite measure \( \mu \) into the Banach space \( E \), is used in the sense of Dunford and
Schwartz [9]. This pertains also to the concept of Bochner integrability.

Let \( 1 < p, q < \infty \) and \( 1/p + 1/q = 1 \). The notations \( L^p(\Omega, E) \) and \( L^q(\Omega, E^*) \)
denote the spaces of \( p \)-th and \( q \)-th Bochner-integrable functions with values in \( E \)
and \( E^* \), respectively.

The space \( L^p(\Omega, E) \) is normed by \( \| x \| := \left( \int_{\Omega} \| x(\omega) \|_E^p \, d\mu \right)^{1/p} \). The pairing
between the spaces \( L^p(\Omega, E) \) and \( L^q(\Omega, E^*) \) is given by \( \int_{\Omega} \langle z, w \rangle \, d\mu \), where \( z \in L^p(\Omega, E), \)
\( w \in L^q(\Omega, E^*) \). There are sufficient conditions on \( E \) ensuring \( L^p(\Omega, E)^* \cong L^q(\Omega, E^*) \)
and the correspondence is given as follows.

For each \( F \in L^p(\Omega, E)^* \), there exists a unique \( g \in L^q(\Omega, E^*) \) such that
\[ F(f) = \int_{\Omega} \langle g(\omega), f(\omega) \rangle \, d\mu \]
for all \( f \in L^p(\Omega, E) \) and such that
\[ \| F \| = \| g \| = \left( \int_{\Omega} \| g(\omega) \|_{E^*}^q \, d\mu \right)^{1/q}. \]

The following theorem is well-known (cf. Leonard [10]):

**Theorem 2.1.** Let \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

1. If the norm \( \| \cdot \|_E \) is Fréchet differentiable, then \( L^p(\Omega, E)^* \cong L^q(\Omega, E^*) \).
2. If \( E \) is reflexive, then \( L^p(\Omega, E)^* \cong L^q(\Omega, E^*) \).
3. If the dual \( E^* \) is separable, then \( L^p(\Omega, E)^* \cong L^q(\Omega, E^*) \).
We will use the following notations associated with set-valued maps. 
$P_f(U)$ denotes the class of nonempty closed subsets of a Banach space $U$, and $P_{fc}(U)$ the class of nonempty closed convex subsets of $U$.

A set-valued map $\Psi : \Omega \to P_f(U)$ is said to be measurable if for every $V \subset U$ open, the set
$$
\Psi^-(V) := \{ \omega \in \Omega : \Psi(\omega) \cap V \neq \emptyset \}
$$
is measurable in $\Omega$. A measurable function $f : \Omega \to U$ satisfying
$$
f(\omega) \in \Psi(\omega) \quad \text{a.e. in } \Omega
$$
is said to be a measurable selection of $\Psi$.

We also say that a measurable set-valued map $\Psi : \Omega \to P_f(U)$ is $L^q$-integrably bounded if there exists a map $h \in L^q(\Omega, \mathbb{R}^+)$ such that
$$
\sup \{ \|e\| : e \in \Psi(\omega) \} \leq h(\omega) \quad \text{a.e. in } \Omega
$$
By $S^q_\Psi$ we denote the set of all measurable selections of $\Psi$ belonging to $L^q(\Omega, U)$, i.e.,
$$
S^q_\Psi := \{ \psi \in L^q(\Omega, U) : \psi(\omega) \in \Psi(\omega) \quad \text{a.e. in } \Omega \}.
$$
A sufficient condition for the set $S^q_\Psi$ of selections to be relatively weakly compact in $L^q(\Omega, U)$ is given in the following.

**Lemma 2.2.** Let $U$ be a separable Banach space and $(\Omega, \mu)$ a finite measure space. Let $A$ be a closed convex set in $U$ with the property that bounded subsets of $A$ are relatively weakly compact. Let $\Psi : \Omega \to P_f(U)$ be a $L^q$-integrably bounded, measurable set-valued map with closed values in $A$. Then $S^q_\Psi$ is relatively weakly compact in $L^q(\Omega, U)$ for $1 < q < \infty$.

In fact, since $U$ is separable the measurable selection theorem (cf. Kuratowski and Ryll-Nardzewski [11], Aliprandis and Border [12]) implies that $\Psi$ admits a measurable selection $\psi$. $\Psi$ being $L^q$-integrably bounded ensures that $\psi \in L^q(\Omega, U)$, and whence $S^q_\Psi$ is nonempty and in particular norm bounded in $L^q(\Omega, U)$. The relatively weakly compactness of $S^q_\Psi$ is now an easy consequence of the following compactness theorem (cf. Seierstad [13] and Vrabie [14]).

**Theorem 2.3.** Let $U$ be a separable Banach space and $(\Omega, \mu)$ a finite measure space. Let $A$ be a closed convex set in $U$ with the property that bounded subsets of $A$ are relatively weakly compact. Let $S$ be a norm bounded subset of the space $L^q(\Omega, U)$ made up of functions whose values are in $A$. Then $S$ is relatively weakly compact for $1 < q < \infty$.

Let $Y$ and $Z$ be Banach spaces. A set-valued map $f : \Omega \times Z \to P_{fc}(Y)$ is said to satisfy the Cesari’s condition at a point $z \in Z$ if for every sequence $\{z_k\}$ converging to $z$ in $Z$, we have
$$
\bigcap_{n=1}^{\infty} \text{cl} \cdot \text{co} \bigcup_{k=n}^{\infty} f(\omega, z_k) \subset f(\omega, z) \quad \text{a.e. in } \Omega.
$$
The set-valued map $f$ is said to satisfy the Cesari’s condition on $Z$ if it satisfies the Cesari’s condition at every point of $Z$. (Here the notation $\text{cl} \cdot \text{co} A$ stands for the closed convex hull of the set $A \subset Y$.)

**Remark 1.** Cesari’s condition is also known as Cesari’s property $Q$. It is noted that Cesari’s condition is closely related to various semicontinuity properties of set-valued maps in locally convex spaces. For further discussions on their interrelationship, see Hou [15]. In fact, due to the closure theorem as stated below
Cesari’s condition plays an important role in calculus of variations and the theory of optimal controls, in particular on the controllability of controlled systems in infinite dimensional spaces. (See Hou [4], Cesari and Hou [5], and Hou [16].)

Central to our discussion is the following closure theorem due essentially to Cesari as modified by Hou (cf. Cesari and Hou [5], Hou [15]).

**Theorem 2.4.** Let \( \Omega \) be modified by Hou (cf. Cesari and Hou [5], Hou [15]).

**Theorem 2.5.** Let \( B \) be a nonempty compact convex subset of a locally convex Hausdorff space, and let the set-valued map \( \Gamma : B \to P_{fc}(B) \) has a closed graph. Then \( x \in \Gamma(x) \) for some point \( x \in B \).

3. **Main Results.** Before we state the main theorem, we first apply Cesari’s closure theorem to establish the following interesting selection lemma.

**Lemma 3.1.** Let \( Y \) and \( Z \) be Banach spaces with \( Y \) separable, and let \( A \) be a nonempty closed convex set in \( Y \) with the property that bounded subsets of \( A \) are relatively weakly compact.

Let the set-valued map \( f : \Omega \times Z \to P_{fc}(Y) \) with values in \( A \) be such that

1. for every \( z \in Z \) the set-valued map \( f(\cdot, z) : \Omega \to P_{fc}(Y) \) has a measurable selection;
2. for every \( z \in Z \), \( f(\omega, z) \subset G(\omega) \) a.e. in \( \Omega \) with \( G : \Omega \to P_{fc}(Y) \) being \( L^q \)-integrably bounded for \( 1 < q < \infty \) and having values in \( A \);
3. the set-valued map \( f : \Omega \times Z \to P_{fc}(Y) \) satisfies the Cesari’s condition on \( Z \).

Then for every measurable function \( p : \Omega \to Z \) there exists a \( L^q \)-integrable selection \( g : \Omega \to Y \) for the set-valued map \( Q : \Omega \to P_{fc}(Y) \) defined by

\[
Q(\omega) := f(\omega, p(\omega)), \quad \omega \in \Omega.
\]

**Proof.** Let \( p_n : \Omega \to Z (n = 1, 2, \ldots) \) be a sequence of simple functions converging to \( p \) a.e. in \( \Omega \). We can find by condition (1) a sequence of measurable functions, \( g_n : \Omega \to Y, n = 1, 2, \ldots, \) satisfying

\[
g_n(\omega) \in f(\omega, p_n(\omega)) \quad \text{a.e. in } \Omega.
\]

By condition (2) we have \( g_n \in S^q_{fc} \).

In view of Lemma 2.2 the set \( S^q_{fc} \) is relatively weakly compact. Thus we may assume, by passing to subsequence if necessary, that \( g_n \) converges weakly to some function \( g \in L^q(\Omega, Y) \). It follows from condition (3) and Cesari’s closure theorem that

\[
g(\omega) \in f(\omega, p(\omega)) = Q(\omega) \quad \text{a.e. in } \Omega.
\]
Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following assumptions will be used in the sequel.

Hypothesis H(A):
- $E$ is a reflexive Banach space with separable dual $E^*$;
- $K : X^* \to X$ is completely continuous from $X := L^p(\Omega, E)$ to $X^* = L_q(\Omega, E^*)$.

Hypothesis H(B):
- $f : \Omega \times E \to P_{fc}(E^*)$ is a set-valued map satisfying
  - for every $x \in E$ the set-valued map $\omega \mapsto f(\omega, x)$ is measurable;
  - for every $x \in E$, $f(\omega, x)$ is closed in $E^*$, with $G : \Omega \to P_{fc}(E^*)$ being $L^q$-integrably bounded.
  - $f$ satisfies the Cesari’s condition on $E$.

By putting, for $x \in X$,

$$F(x) := \{ z \in X^* : z(\omega) \in f(\omega, x(\omega)) \text{ a.e. in } \Omega \}, \quad (2)$$

we have a set-valued map $F$ from $X$ to $2^{X^*}$. This gives rise to an associated inclusion problem of the type in (1), whereby a solution is an element $x \in X$ satisfying the equation $0 = x + Kg$ for some $g \in F(x)$.

**Theorem 3.2.** Suppose that the hypotheses $H(A)$ and $H(B)$ hold true. Then with $F$ as given in (2) the nonlinear inclusion equation (1), admits a solution in $X$.

**Proof.** First we note that the Banach space $E^*$ is separable and reflexive. Clearly by hypothesis H(B) the set $S_G^q$ is nonempty. Furthermore, it is closed and convex as well as weakly compact in $X^* = L_q(\Omega, E^*)$ in view of Lemma 2.2 and the fact that $G$ has closed convex values in $E^*$.

Let $g \in S_G^q$ and consider the set-valued map $\Gamma$ defined by

$$\Gamma(g) := S_{f(\cdot, (\cdot - Kg)\cdot)}^q.$$ 

It follows from Lemma 3.1 that for every $g \in S_G^q$ the set $\Gamma(g)$ is nonempty and lies in $S_{G^q}^q \subset X^*$. In addition, it follows from hypothesis H(B) that $\Gamma(g)$ is convex. Thus, $\Gamma$ maps $S_G^q$ into $P_{fc}(S_G^q)$ with nonempty closed convex values. It is also observed that if $g$ is a fixed point of $\Gamma$, i.e., $g \in \Gamma(g)$, then $-Kg$ will be a solution of the nonlinear inclusion equation (1).

Let $X^*$ be endowed with the weak topology (denoted by $X^*_w$). We claim that $\Gamma$ has a closed graph in $X^*_w \times X^*_w$.

Since $S_G^q$ is weakly compact in the separable space $X^*$, its weak topology is metrizable (cf. theorem V.6.3. in Dunford and Schwartz [9]). Thus, it suffices to show that $\text{Gr}(\Gamma) \subset S_G^q \times S_G^q$ is sequentially weakly closed.

To this end, let $(g_n, h_n) \in \text{Gr}(\Gamma)$ with $(g_n, h_n) \rightharpoonup_{w^*} (g, f)$ in $S_G^q \times S_G^q$. We want to show that $(g, f) \in \text{Gr}(\Gamma)$. But note that $K$ being completely continuous, so by passing to a subsequence if necessary we may assume $u_n := -Kg_n \rightharpoonup u$ in $X$, and $u_n(\omega) \to u(\omega)$ almost everywhere as well. Thus we may assume

$$h_n \to h \quad \text{in } X^*,$$
$$u_n(\omega) \to u(\omega) \quad \text{a.e. in } \Omega,$$
$$h_n(\omega) \in f(\omega, u_n(\omega)) \quad \text{a.e. in } \Omega.$$

The closure theorem then implies that $h(\omega) \in f(\omega, u(\omega))$ a.e. in $\Omega$, and hence $h \in S_{f(\cdot, u(\cdot))}^q = S_{f(\cdot, (\cdot - Kg)\cdot)}^q$, proving $(g, h) \in \text{Gr}(\Gamma)$. 

Applying the Kakutani-Fan fixed point theorem to $\Gamma$ yields a fixed point $\bar{g} \in \Gamma(\bar{g})$, i.e., $\bar{g} \in S_{\int f(.(-K\bar{g})(\cdot))}^g$. Then $x := -K\bar{g}$ is clearly a solution for the nonlinear inclusion equation (1).

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E-mail address: houshuihung@hotmail.com