Some generalized inequalities of Hermite-Hadamard type for strongly $s$-convex functions

Yusuf Erdem$^1$, Hasan Ogunmez$^1$ and Huseyin Budak$^2$

$^1$Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar, Turkey
$^2$Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce, Turkey

Received: 18 November 2016, Accepted: 7 December 2016
Published online: 17 August 2017.

Abstract: In this paper, some new generalized results related to the left-hand and the right-hand of the Hermite-Hadamard inequalities for the class of functions whose derivatives are strongly $s$-convex functions in the second sense are established. Some previous results are also recaptured as a special case.

Keywords: Hadamard type inequalities, Hölder inequality, strongly $s$-convex functions.

1 Introduction

In this section, we firstly list several definitions and some known results.

Definition 1. A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subset \mathbb{R}$, is said to be convex on $I$ if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications([4], [12, p.137]). These inequalities state that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if $f$ is concave. We note that Hadamards inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensens inequality. Hadamards inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [2],[5],[11],[16],[18]) and the references cited therein.

In [6], Hudzik and Maligranda considered, among others, the class of functions which are $s$-convex in the second sense. This class is defined in the following way: a function $f : [0, \infty) \to \mathbb{R}$ is said to be $s$-convex in the second sense if

$$f(tx + (1-t)y) \leq t^sf(x) + (1-t)^sf(y)$$
holds for \( x, y \in [0, \infty) \), \( t \in [0, 1] \) and for some fixed \( s \in [0, 1] \). This class of \( s \)-convex functions in the second sense is usually by \( K_S^2 \).

It can be easily see that for \( s = 1 \) \( s \)-convexity reduces to the ordinary convexity of functions defined on \([0, \infty)\).

**Definition 2.** [13] A function \( f : I \to \mathbb{R} \) is called strongly \( s \)-convex with modulus \( c \) if

\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) - ct(1-t)(b-a)^2.
\]

In [1], Angulo et al. proved the following Hermite-Hadamard type inequality for strongly \( h \)-convex function.

**Theorem 1.** Let \( h : (0, 1) \to [0, \infty) \) be a given function. If a function \( f : I \subset \mathbb{R} \to \mathbb{R} \) is Lebesgue integrable and strongly \( h \)-convex with modulus \( c > 0 \), then

\[
\frac{1}{h(\frac{1}{2})} \left( f\left( \frac{a+b}{2} \right) + \frac{c}{12} (b-a)^2 \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{1}{h(t)} \int_0^1 h(t) \, dt - \frac{c}{6} (b-a)^2
\]

for all \( a, b \in I \), \( a < b \).

**Corollary 1.** Suppose that \( f : [0, \infty) \to \mathbb{R} \) is a strongly \( s \)-convex function in the second sense with modulus \( c > 0 \), where \( s \in (0, 1) \) (i.e \( h(t) = t^s \) in (2)), then following inequalities hold:

\[
2^{s-1} \left( f\left( \frac{a+b}{2} \right) + \frac{c}{12} (b-a)^2 \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s+1} - \frac{c}{6} (b-a)^2.
\]

For more information and recent developments on inequalities for strongly convex function, please refer to ([1],[3],[8],[9],[10],[15],[17],[19],[20]).

To prove our main results, we consider the following Lemmas given by Sarikaya et al. in [14] and Kiriş and Sarikaya in [7], respectively:

**Lemma 1.** Let \( f : I \to \mathbb{R} \) be a differentiable mapping on \( I \) where \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \), then we have

\[
\int_a^b p(x) f'(x) \, dx = (m-n) f\left( \frac{a+b}{2} \right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x) \, dx
\]

where

\[
p(x) = \begin{cases} x-n, & x \in \left[a, \frac{a+b}{2}\right] \\ x-m, & x \in \left(\frac{a+b}{2}, b\right] \end{cases}
\]

for \( n \in \left[a, \frac{a+b}{2}\right] \) and \( m \in \left(\frac{a+b}{2}, b\right] \).

**Lemma 2.** Let \( f : I \to \mathbb{R} \) be a differentiable mapping on \( I \) where \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \), then we have

\[
\frac{n}{b-a} \int_a^{\frac{a+b}{2}} f(x) \, dx + \frac{m}{b-a} \int_{\frac{a+b}{2}}^b f(x) \, dx - \frac{n+m}{2} f\left( \frac{a+b}{2} \right) = (b-a)wf
\]
where \( w_f = \left[ \int_a^b mt f' ((ta) + (1-t)b) \, dt + \int_a^b n (1-t) f' ((ta) + (1-t)b) \, dt \right], n, m > 0 \)

The aim of the paper is to establish some new generalized Hermite-Hadamard inequalities for function whose derivatives absolute values are strongly \( s \)-convex.

## 2 Main results

Firstly, we will give some calculated integrals which used our main results:

\[
\begin{align*}
\int_a^b |x-n|(b-x)^s \, dx &= \frac{2(b-n)^{s+2}}{(s+1)(s+2)} - \frac{(b-n)(b-a)^{s+1}[2^{s+1}+1]}{2^{s+1}(s+1)} + \frac{(b-a)^{s+2}[2^{s+2}+1]}{2^{s+2}(s+2)}, \\
\int_a^b |x-m|(b-x)^s \, dx &= \frac{2(b-m)^{s+2}}{(s+1)(s+2)} + \frac{(b-a)^{s+2}}{2^{s+2}(s+1)} - \frac{(b-m)(b-a)^{s+1}}{2^{s+1}(s+1)}, \\
\int_a^b |x-n|(x-a)^s \, dx &= \frac{2(n-a)^{s+2}}{(s+1)(s+2)} + \frac{(b-a)^{s+2}}{2^{s+2}(s+1)} - \frac{(n-a)(b-a)^{s+1}}{2^{s+1}(s+1)}, \\
\int_a^b |x-m|(x-a)^s \, dx &= \frac{2(m-a)^{s+2}}{(s+1)(s+2)} - \frac{(m-a)(b-a)^{s+1}[2^{s+1}+1]}{2^{s+1}(s+1)} + \frac{(b-a)^{s+2}[2^{s+2}+1]}{2^{s+2}(s+2)}, \\
\int_a^b |x-n|(x-a)(x-a) \, dx &= \frac{(b-a)(n-a)^3}{3} - \frac{(n-a)^4}{6} + \frac{5(b-a)^4}{192} - \frac{(n-a)(b-a)^3}{12}
\end{align*}
\]

and

\[
\begin{align*}
\int_a^b |x-m|(x-a)(x-a) \, dx &= \frac{(b-a)(b-m)^3}{3} - \frac{(b-m)^4}{6} + \frac{5(b-m)^4}{192} - \frac{(b-m)(b-a)^3}{12}.
\end{align*}
\]

**Theorem 2.** Let \( f : I \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \). If \( |f'| \) is strongly \( s \)-convex on \([a,b] \), for some \( s \in [0,1] \) with modulus \( c > 0 \), then following inequality holds:

\[
\begin{align*}
|f(m) - f(n)| &= \left| (m-n) \frac{a+b}{2} + (b-m) f(b) + (n-a) f(a) - \int_a^b f(x) \, dx \right| \\
&\leq \left| f'(a) \right| \frac{2(b-m)^{s+2} + (b-n)^{s+2}}{(s+1)(s+2)} + \left| f'(b) \right| \frac{2(b-m)^{s+2} + (b-n)^{s+2}}{(s+1)(s+2)} \\
&- \frac{(2b-n-m)(b-a)^{s+1}}{2^{s+1}(s+1)} - \frac{(b-a)^{s+1}}{(s+1)} + \frac{(b-a)^{s+2}[2^{s+1}+1]}{2^{s+1}(s+2)} + \frac{(b-a)^{s+2}[2^{s+2}+1]}{2^{s+2}(s+2)} \frac{2(b-m)^{s+2} + (b-n)^{s+2}}{(s+1)(s+2)} \frac{2(b-m)^{s+2} + (b-n)^{s+2}}{(s+1)(s+2)} \\
&- \frac{(m+n-2a)(b-a)^{s+1}}{2^{s+1}(s+1)} - \frac{(m-n-2b)(b-a)^{s+1}}{2^{s+1}(s+1)} + \frac{(m-n-2a)(b-a)^{s+2}[2^{s+1}+1]}{2^{s+1}(s+2)} + \frac{(m-n-2b)(b-a)^{s+2}[2^{s+2}+1]}{2^{s+2}(s+2)} \frac{2(b-m)^{s+2} + (b-n)^{s+2}}{(s+1)(s+2)} \frac{2(b-m)^{s+2} + (b-n)^{s+2}}{(s+1)(s+2)} \\
&- c \left[ \frac{(b-a)(n-a)^3}{3} + \frac{(b-m)^4}{6} + \frac{5(b-a)^4}{96} - \frac{(b-a)(n-a)(b-a)^3}{12} + \frac{5(b-a)^4}{96} - \frac{(b-a)(n-a)(b-a)^3}{12} \right]
\end{align*}
\]

for \( n \in [a, \frac{a+b}{2}] \) and \( m \in [\frac{a+b}{2}, b] \).
Proof. Taking modulus in Lemma 1 and using the strongly $s$–convexity of $|f'|$, we have

$$
\left| (m-n)f \left( \frac{a+b}{2} \right) + (b-m)f(b) + (n-a)f(a) - \frac{b}{a} \int_a^b f(x) \, dx \right|
\leq \int_a^b |x-n| |f'(x)| \, dx + \int_a^b |x-m| |f'(x)| \, dx
$$

(13)

$$
\leq \int_a^b |x-n| \left[ \left( \frac{b-x}{b-a} \right)^s |f'(a)| + \left( \frac{x-a}{b-a} \right)^s |f'(b)| - c(b-x)(x-a) \right] \, dx
\leq \int_a^b |x-m| \left[ \left( \frac{b-x}{b-a} \right)^s |f'(a)| + \left( \frac{x-a}{b-a} \right)^s |f'(b)| - c(b-x)(x-a) \right] \, dx

\leq \frac{|f'(a)|}{(b-a)^s} \left[ \int_a^b |x-n|(b-x)^s \, dx + \int_a^b |x-m|(b-x)^s \, dx \right] + \frac{|f'(b)|}{(b-a)^s} \left[ \int_a^b |x-n|(x-a)^s \, dx + \int_a^b |x-m|(x-a)^s \, dx \right]
- c \left[ \int_a^b |x-n|(b-x)(x-a) \, dx + \int_a^b |x-m|(b-x)(x-a) \, dx \right].
$$

If we substitute the equalities (6)–(11) in (13), then we obtain required result (12).

Remark. If we choose $m = b, n = a$ in Theorem 2, then we have

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq (b-a) \frac{2^{s+1} - 1}{2^s(s+1)(s+2)} \left[ \frac{|f'(a)| + |f'(b)|}{2} \right] - \frac{5c}{96} (b-a)^3
$$

Remark. If we choose $m = n = \frac{a+b}{2}$ in Theorem 2, then we have

$$
\left| f(a) + f(b) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq (b-a) \frac{s2^s + 1}{2^s(s+1)(s+2)} \left[ \frac{|f'(a)| + |f'(b)|}{2} \right] - \frac{c}{32} (b-a)
$$
Remark. If we choose \( m = \frac{a+b}{2} \), \( n = \frac{a+b}{6} \) in Theorem 2, then we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{2 \times 5^s + 2 - (4 - s)6^{r + 1} - 2 \times 3^{r + 2} + 2}{6^{r + 2} (s + 1)(s + 2)} \left[ \left| f'(a) \right| + \left| f'(b) \right| \right]
\]

\[-\frac{67}{2} \times 6^c (b-a)^3.
\]

For \( c = 0 \) it reduces to the Hermite–Hadamard-type inequalities for \( s \)-convex functions proved by Sarikaya et al. in [21].

**Theorem 3.** Let \( f : I \to \mathbb{R} \) be a differentiable mapping on \( I^{\circ} \) where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is strongly \( s \)-convex on \( [a,b] \) for some \( s \in (0, 1] \) with modulus \( c > 0 \), then following inequality holds.

\[
\begin{align*}
\left| (m-n) f \left( \frac{a+b}{2} \right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x) \, dx \right| & \leq \frac{(b-a)^{\frac{1}{p} + \frac{1}{q}}}{(p+1)^{\frac{1}{p} + \frac{1}{q}}} \left\{ \left[ (n-a)^{p+1} + \left( \frac{a+b}{2} - n \right)^{p+1} \right]^{\frac{1}{p}} \right. \\
& \quad \left. \times \frac{1}{2^s + 1} \left[ |f'(a)|^q + |f'(b)|^q \right] - c \left( \frac{b-a}{2} \right)^2 \right\}^{\frac{1}{q}} \\
& \quad \times \frac{1}{2^s + 1} \left[ |f'(a)|^q + |2^{s+1} - 1| |f'(b)|^q \right] - c \left( \frac{b-a}{2} \right)^2 \right\}^{\frac{1}{q}}
\end{align*}
\]

for \( n \in \left[ a, \frac{a+b}{2} \right] \) and \( m \in \left[ \frac{a+b}{2}, b \right] \) where \( p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** From Lemma 1 and by using the Hölder inequality , then we have

\[
\begin{align*}
\left| (m-n) f \left( \frac{a+b}{2} \right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x) \, dx \right| & \leq \int_a^b |x-n||f'(x)| \, dx + \int_a^b |x-m||f'(x)| \, dx \\
& \leq \left( \frac{a+b}{2} \right)^{\frac{1}{p}} \left( \int_a^b |f'(x)|^q \, dx \right)^{\frac{1}{q}} + \left( \frac{b-a}{2} \right)^{\frac{1}{p}} \left( \int_a^b |f'(x)|^q \, dx \right)^{\frac{1}{q}} \\
& \quad + \frac{1}{(p+1)^{\frac{1}{p}}} \left[ (n-a)^{p+1} + \left( \frac{a+b}{2} - n \right)^{p+1} \right]^{\frac{1}{p}} \left( \int_a^b |f'(x)|^q \, dx \right)^{\frac{1}{q}} \\
& \quad + \frac{1}{(p+1)^{\frac{1}{p}}} \left[ (b-m)^{p+1} + \left( m - \frac{a+b}{2} \right)^{p+1} \right]^{\frac{1}{p}} \left( \int_a^b |f'(x)|^q \, dx \right)^{\frac{1}{q}}.
\end{align*}
\]
Using the strongly $s$-convexity of $|f'|^q$, we have

$$
\left| (m-n)f\left(\frac{a+b}{2}\right) + (b-m)f(b) + (n-a)f(a) - \int_a^b f(x)\,dx \right| 
\leq \frac{1}{(p+1)^{\frac{1}{p}}}
\left\{ \left( (n-a)^{p+1} + \left( \frac{a+b}{2}-n \right)^{p+1} \right)^{\frac{1}{p}} \right. 
\times \left( \int_a^b \left[ \frac{(b-x)^s}{(b-a)^s} \right] |f'(a)|^q + \frac{(x-a)^s}{(b-a)^s} |f'(b)|^q - c(b-x)(x-a) \right) \,dx 
\left. + \left( (b-m)^{p+1} + \left( m-\frac{a+b}{2} \right)^{p+1} \right)^{\frac{1}{p}} \right. 
\times \left( \int_a^b \left( \frac{(b-x)^s}{(b-a)^s} \right) |f'(a)|^q + \frac{(x-a)^s}{(b-a)^s} |f'(b)|^q - c(b-x)(x-a) \right) \,dx \right\}^{\frac{1}{q}}.
$$

(16)

By simple computation, we have

$$
\int_a^b \frac{(b-x)^s}{(b-a)^s} \,dx = \int_a^b (x-a)^s \,dx = \frac{(b-a)^{s+1}}{2^{s+1}(s+1)}.
$$

(17)

$$
\int_a^b (x-a)^s \,dx = \frac{(b-a)^{s+1}}{2^{s+1}(s+1)}.
$$

(18)

$$
\int_a^b (b-x)(x-a) \,dx = \frac{(b-a)^3}{12}.
$$

(19)

If we substitute the equalities (17)-(19) in (16), then we obtain result (14).

**Corollary 2.** If we choose $m = b$ and $n = a$ in Theorem 3, then we have

$$
\left| f\left(\frac{a+b}{2}\right) - \int_a^b f(x)\,dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left( \frac{2^{s+1}-1}{2^s(s+1)} \right)^{\frac{1}{p}} \left( |f'(a)|^q + |f'(b)|^q - c(b-a)^2 \right) \right. 
\left. + \left( |f'(a)|^q + \frac{2^{s+1}-1}{2^s(s+1)} |f'(b)|^q - c(b-a)^2 \right) \right\}^{\frac{1}{q}}.
$$

(20)
Remark. Choosing \( s = 1 \) in Corollary 2, we obtain the inequality
\[
\left| f\left(\frac{a+b}{2}\right) - \frac{b-a}{4} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^\frac{1}{p} \left\{ \left( \frac{3 |f'(a)|^q + |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 3 |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}}.
\]

Corollary 3. If we choose \( m = n = \frac{a+b}{2} \) in Theorem 3, then we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{b-a}{2 \times 2^p} \left( \frac{1}{p+1} \right)^\frac{1}{p} \left\{ \left( \frac{2s+1-1}{2s+1} \right)^q + \left( \frac{2s+1-1}{2s+1} \right)^q \right\} \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^\frac{1}{p} \left\{ \left( \frac{3 |f'(a)|^q + |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 3 |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}}.
\] (21)

Remark. Choosing \( s = 1 \) in Corollary 3, we obtain the inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{b-a}{2 \times 2^p} \left( \frac{1}{p+1} \right)^\frac{1}{p} \left\{ \left( \frac{2s+1-1}{2s+1} \right)^q + \left( \frac{2s+1-1}{2s+1} \right)^q \right\} \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^\frac{1}{p} \left\{ \left( \frac{3 |f'(a)|^q + |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 3 |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}}.
\]

Corollary 4. If we choose \( m = \frac{a+b}{n} \) and \( n = \frac{a+b}{6} \) in Theorem 3, then we have
\[
\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{b-a}{6} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{6} \left( \frac{2^{p+1}+1}{6(p+1)} \right)^\frac{1}{p} \times \left\{ \left( \frac{2s+1-1}{2s+1} \right)^q + \left( \frac{2s+1-1}{2s+1} \right)^q \right\} \right| \leq \frac{b-a}{6} \left( \frac{2^{p+1}+1}{6(p+1)} \right)^\frac{1}{p} \times \left\{ \left( \frac{3 |f'(a)|^q + |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 3 |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}}.
\] (22)

Remark. Choosing \( s = 1 \) in Corollary 3, we obtain the inequality
\[
\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{b-a}{6} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{6} \left( \frac{2^{p+1}+1}{6(p+1)} \right)^\frac{1}{p} \times \left\{ \left( \frac{3 |f'(a)|^q + |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 3 |f'(b)|^q}{4} - c \frac{(b-a)^2}{12} \right) \right\}^{\frac{1}{q}}.
\]
Theorem 4. Let \( f : I \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \). If \(|f'|\) is strongly \( s \)-convex on \([a,b]\) for some \( s \in (0,1] \) with modulus \( c > 0 \), then following inequality holds.

\[
\left| \frac{n}{b-a} \int_a^{a+\frac{b}{n}} f(x) \, dx + \frac{m}{b-a} \int_a^b f(x) \, dx - \frac{n+m}{2} f \left( \frac{a+b}{2} \right) \right| \leq (b-a) \left\{ \left[ \frac{m}{2^{s+2}(s+2)} + \frac{n}{2^{s+2}(s+1)(s+2)} \right] |f'(a)| + \frac{5c(m+n)}{192} (b-a)^2 \right\} \tag{23}
\]

where \( n \in \left[ a, \frac{a+b}{n} \right], m \in \left[ \frac{a+b}{n}, b \right] \).

Proof. From Lemma 2 and using strongly \( s \)-convex of \(|f'|\), we have

\[
\left| \frac{n}{b-a} \int_a^{a+\frac{b}{n}} f(x) \, dx + \frac{m}{b-a} \int_a^b f(x) \, dx - \frac{n+m}{2} f \left( \frac{a+b}{2} \right) \right| \leq (b-a) \left\{ \frac{1}{2} mt \left| f'((ta) + (1-t)b) \right| + \frac{1}{2} m t^2 \left| f'(a) \right| + (1-t) \left| f'(b) \right| - ct (1-t) (b-a)^2 \right\} \tag{24}
\]

Using the facts that

\[
\int_0^1 t^r \, dt = \frac{1}{r+1} (1-t)^{r+1} \quad \text{and} \quad \int_0^1 t \, dt = \frac{1}{2} (1-t)^2 \]

\[
\int_0^1 t^2 \, dt = \frac{1}{3} (1-t)^3
\]

one can obtain required result.

Remark. If we choose \( m = n \) in Theorem 4, then Theorem 4 reduces to Remark 2.

Corollary 5. Under assumption of Theorem 4 with \( s = 1 \), we have

\[
\left| \frac{n}{b-a} \int_a^{a+\frac{b}{n}} f(x) \, dx + \frac{m}{b-a} \int_a^b f(x) \, dx - \frac{n+m}{2} f \left( \frac{a+b}{2} \right) \right| \leq \frac{b-a}{24} \left[ (m+2n)|f'(a)| + (2m+n)|f'(b)| \right] - \frac{5c(m+n)}{192} (b-a)^3. \tag{25}
\]
Theorem 5. Let \( f : I \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is strongly \( s \)-convex on \([a, b]\) for some \( s \in (0, 1] \) with modulus \( c > 0 \), then following inequality holds:

\[
\begin{align*}
&\left| \frac{n}{b-a} \int_a^{a+b} f(x) \, dx + \frac{m}{b-a} \int_a^b f(x) \, dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ m \left( |f'(a)|^q + |f'(b)|^q \left[ \frac{2^{s+1}-1}{2^s(s+1)} \right] - \frac{c(b-a)^2}{6} \right)^{\frac{1}{q}} + n \left( |f'(a)|^q \left[ \frac{2^{s+1}-1}{2^s(s+1)} \right] + |f'(b)|^q \right)^{\frac{1}{q}} \right\} \\
&\leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ m \left( |f'(a)|^q + |f'(b)|^q \left[ \frac{2^{s+1}-1}{2^s(s+1)} \right] - \frac{c(b-a)^2}{6} \right)^{\frac{1}{q}} + n \left( |f'(a)|^q \left[ \frac{2^{s+1}-1}{2^s(s+1)} \right] + |f'(b)|^q \right)^{\frac{1}{q}} \right\}
\end{align*}
\]

(26)

where \( n \in \left[ a, \frac{a+b}{2} \right], m \in \left[ \frac{a+b}{2}, b \right] \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. From Lemma 2 and using Hölder Inequality, then we have

\[
\begin{align*}
&\left| \frac{n}{b-a} \int_a^{a+b} f(x) \, dx + \frac{m}{b-a} \int_a^b f(x) \, dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ m \left( |f'(a)|^q + |f'(b)|^q \left[ \frac{2^{s+1}-1}{2^s(s+1)} \right] - \frac{c(b-a)^2}{6} \right)^{\frac{1}{q}} + n \left( |f'(a)|^q \left[ \frac{2^{s+1}-1}{2^s(s+1)} \right] + |f'(b)|^q \right)^{\frac{1}{q}} \right\}
\end{align*}
\]

Using strongly \( s \)-convexity of \( |f'|^q \):

\[
\begin{align*}
&\left| \frac{n}{b-a} \int_a^l f(x) \, dx + \frac{m}{b-a} \int_l^b f(x) \, dx - \frac{n+m}{2} f(l) \right| \\
&\leq (b-a) \left\{ \left( \int_0^{\frac{1}{2}} (mt)^p \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 |f'(ta + (1-t)b)|^q \, dt \right)^{\frac{1}{q}} \right\}
\end{align*}
\]

where \( \theta_m = \left( \int_0^{\frac{1}{2}} \left[ |f'(a)|^q (1-t) + |f'(b)|^q - |f'(b)|^q \right] \, dt \right)^{\frac{1}{q}} \) and \( l = \frac{a+b}{2} \). By simple computation, the desired inequality (26) can be easily established.

Remark. If we choose \( m = n \) in Theorem 5, then Theorem 5 reduces to Remark 2.

Corollary 6. Under assumption of Theorem 4 with \( s = 1 \), we have

\[
\begin{align*}
&\left| \frac{n}{b-a} \int_a^{a+b} f(x) \, dx + \frac{m}{b-a} \int_a^b f(x) \, dx - \frac{n+m}{2} f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ m(\theta_q)^{\frac{1}{q}} + n(\theta_p)^{\frac{1}{p}} \right\}
\end{align*}
\]

where \( \theta_p = \left( \frac{3|f'(a)|^q + |f'(b)|^q - c(b-a)^2}{4} \right) \) and \( \theta_q = \left( \frac{|f'(a)|^q + |f'(b)|^q - c(b-a)^2}{6} \right) \)
3 Conclusions

In this study, we presented some generalized Hermite type inequalities for the mappings whose derivatives are strongly s-convex functions in the second sense are established. A further study could be assess weighted versions of these inequalities.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] H. Angulo, J. Gimenez, A. M. Moros, and K. Nikodem, On strongly h−convex function, Ann. Funct. Anal. 2(2), 2011, 85–91.
[2] M. K. Bakula and J. Pečarić, Note on some Hadamard-type inequalities, Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.
[3] M. V. Cortez, Relative strongly h−convex functions and integral inequalities, Appl. Math. Inf. Sci. Lett. 4, No. 2, 1-7 (2016).
[4] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[5] S. S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. lett., 11(5) (1998), 91-95.
[6] H. Hudzik, L. Maligranda, Some remarks on s−convex functions, Aequ. Math. 48, 100-111 (1994).
[7] M. E. Kiris and M. Z. Sarikaya, Some generalized inequalities for Hermite-Hadamard's integral inequalities and applications, submitted (2015).
[8] J. Makó and A. Házy, On strongly convex functions, Carpathian Journal of Mathematics, 32 (1). 87-95.
[9] N. Merentes, K. Nikodem, Remarks on strongly convex functions, Aequationes Math. 80 (2010) 193–199.
[10] K. Nikodem, Z. Pales, Characterizations of inner product spaces be strongly convex functions, Banach J. Math. Anal. 5 (2011) 83–87.
[11] M. E. Özdemir, Ç. Yıldız, A. O. Akdemir and E. Set, On some inequalities for s−convex functions and applications, Journal of Inequalities and Applications 2013, 2013:333.
[12] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex functions, partial orderings and statistical applications, Academic Press, Boston, 1992.
[13] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet Math. Dokl. 7 (1966), 72–75.
[14] M. Z. Sarikaya, T. Tunc and M. K. Yildiz, Some generalized integral inequalities for convex functions and applications, AIP Conference Proceedings, 1726, 020047 (2016); doi: 10.1063/1.4945873.
[15] M. Z. Sarikaya, On strongly ϕh−convex functions in inner product spaces, Arabian Journal of Mathematics, (2013) 2:295–302.
[16] M. Z. Sarikaya, E. Set, M. E. Ozdemir and S. S. Dragomir, New some Hadamard’s type inequalities for co-ordinated convex functions, Tamsui Oxford Journal of Information and Mathematical Sciences, 28(2) (2012) 137-152.
[17] M. Z. Sarıkaya and H. Yıldız, *On Hermite-Hadamard-type inequalities for strongly log-convex functions*, International Journal of Modern Mathematical Sciences, 2013, 5(3): 92-98.

[18] M. Z. Sarıkaya and H. Yıldız, *On the Hadamard’s type inequalities for L-Lipschitzian mapping*, Konuralp Journal of Mathematics, 1(2), 33-40 (2013).

[19] M. Z. Sarıkaya and K. Özçelik, *On Hermite-Hadamard type integral inequalities for strongly $\varphi_k$–convex functions*, International Journal of Advanced Research in Engineering and Applied Sciences (IJAREAS), 1(1), pp:34-52, 2014.

[20] M. Z. Sarıkaya, *On Hermite Hadamard-type inequalities for strongly $\varphi$-convex functions*, Southeast Asian Bull. Math., 39(1) (2015), pp: 123-132.

[21] M. Z. Sarıkaya, E. Set and M. E. Özdemir, *On new inequalities of Simpson’s type for s–convex functions*, Computers and Mathematics with Applications 60 (2010) 2191–2199.