GENERATING FAMILIES AND CONSTRUCTIBLE SHEAVES

VIVEK SHENDE

ABSTRACT. Let Λ be a Legendrian in the jet space of some manifold X. To a generating family presentation of Λ, we associate a constructible sheaf on $X \times \mathbb{R}$ whose singular support at infinity is Λ, and such that the generating family homology is canonically isomorphic to the endomorphism algebra of this sheaf. That is, the theory of generating family homology embeds in sheaf theory, and more specifically in the category studied in [STZ]. When $X = \mathbb{R}$, i.e., for the theory of Legendrian knots and links in the standard contact $\mathbb{R}^3$, we use ideas from the proof of the $h$-cobordism theorem to show this embedding is an equivalence. Combined with the results of [NRSSZ], this implies in particular that the generating family homologies of a knot are the same as its linearized Legendrian contact homologies.

1. INTRODUCTION

Generating families are by now a well-established tool in symplectic [LS, Sik, Vit, Vit2, EG, Cha, The] and contact [Che2, Pus, Tra2, FR, JT, San, HR, ST, SS, BST] topology. Their use has also been informed by the classical study of families of functions [Cer, HW, Wal].

A newer line of work exploits the constructible sheaves and microlocal geometry of Kashiwara and Schapira [KS]. This theory was always symplectic in nature – microlocal geometry takes place in the cotangent bundle – but the connection to symplectic questions of current interest was perhaps first made clear in the work of Nadler and Zaslow, who showed the category of constructible sheaves is equivalent to the infinitesimally wrapped Fukaya category of the cotangent bundle [NZ, Nad]; these methods were subsequently applied to study mirror symmetry for toric varieties [FLTZ, FLTZ2]. In a different direction, Tamarkin [Tam] explained how to use the constructible sheaf category to prove non-displaceability results, and Guillermou, Kashiwara, and Schapira [GKS, GS, Gui, Gui2] have further developed this perspective. More recently, sheaf techniques have also been applied to the study of Legendrian knots, and more generally to the contact geometry of jet bundles [STZ, NRSSZ].

My purpose here is to connect these stories in the setting of Legendrian submanifolds of jet bundles, and especially for Legendrian knots and links in $\mathbb{R}^3$.1

1.1. Correspondences and generating families. Recall that a correspondence between manifolds is a submanifold of the product; for instance if $\phi : Y \to X$ is a map, then the graph $\Gamma_\phi \subset Y \times X$ gives a correspondence between $Y$ and $X$.

Passing to symplectic geometry, we get a correspondence between $T^*Y$ and $T^*X$ by taking the conormal to $\Gamma_\phi$, which is canonically identified with $Y \times X \times T^*X$:

$$T^*Y \leftarrow \frac{d\phi}{d\phi} Y \times X \leftarrow T^*X \rightarrow \frac{d\phi}{d\phi} T^*X \rightarrow T^*X$$

Convolution with this correspondence determines a push-forward of sets

$$\hat{\phi}_* := \left(\phi \times T^*X\right) \circ (d\phi)^{-1}$$

Now let $f : Y \to \mathbb{R}$ be a function. We write $\Gamma_{df}$ for the graph of the differential. In the cases of interest, $\phi$ will be a fibration. We write

$$L_f := \phi_*(\Gamma_{df}) \subset T^*X$$

Assuming certain genericity conditions, $L_f$ is Lagrangian and is said to be generated by $\phi$; it collects the horizontal derivatives along the fibrewise critical points of the function $f$.2 We might also be interested in the collection of fibrewise critical values $\Phi_f \subset X \times \mathbb{R}$, or in other words, the discriminant of $\phi \times f : Y \to X \times \mathbb{R}$.

1The process of writing this article has led me to believe more generally: all applications of generating families in symplectic and contact topology factor through constructible sheaves. Possibly a similar viewpoint underlies [Vit3].

2From the definition,

$$\phi_*(\Gamma_{df}) = \{(x, \xi \in T^*X) | \exists y, \phi(y) = x, df(y) = d\phi(\xi) \} \subset T^*X$$

Then the above says that we collect points $y$ such that the image of $df(y)$ in the ‘vertical’ cotangent bundle $T^*X := T^*Y / \phi^*T^*X$ is zero. That is, $y$ is a critical point of the restriction of $f$ to the fibre of the fibration $Y \to X$ containing $y$. For such $y$, we can write $df(y) = \phi^*(x, \xi)$ for a unique $(x, \xi)$; the collection of such $(x, \xi)$ sweep out $\phi_*(\Gamma_{df})$. 

The loci $L_f$ and $\Phi_f$ are the Lagrangian and front projections of a certain Legendrian $\Lambda_f \subset J^1(X) = T^*X \times \mathbb{R}$ to its factors $T^*X$ and $X \times \mathbb{R}$, respectively. This suffices to describe $\Lambda_f$, but for our purposes it is better to give it as a convolution, as before. In fact we will give two definitions of the Legendrian $\Lambda_f$; to distinguish them, we will call the more classical one $'\Lambda_f$ until we show they are the same.

We identify $$J^1(X) = T^*X \times \mathbb{R} \times \{-1\} \subset T^*(X \times \mathbb{R})$$

Here, the $\{-1\}$ is fixing the cotangent coordinate, not the base coordinate.

Let $\Gamma_f \subset Y \times \mathbb{R}$ be the graph of $f$. We take $'\Lambda_f$ to be the pushforward of the conormal bundle to the graph, restricted to the jet bundle of $X$:

$$'\Lambda_f := (\phi \times 1_{\mathbb{R}})(T^*_f(Y \times \mathbb{R})) \cap J^1(X)$$

To see the relation to $\Phi_f$, note that the function $\phi \times f$ factors as the composition of $\phi \times 1_{\mathbb{R}} : Y \times \mathbb{R} \to X \times \mathbb{R}$ with the inclusion of the graph $Y \xrightarrow{\sim} \Gamma_f \hookrightarrow Y \times \mathbb{R}$.

The above passage from $T^*X$ to $T^*(X \times \mathbb{R})$ trades exact immersed Lagrangians with vanishing Maslov class in $T^*X$ for embedded conical Lagrangians in $T^*(X \times \mathbb{R})$. The simplest example of its use is the following: the image of $T^*_f(Y \times \mathbb{R}) \cap J^1(Y)$ in $T^*Y$ is $\Gamma_{df}$.

### 1.2. Constructible sheaves from generating families

We turn to sheaves. For a foundational development of the theory, and especially of the notion of singular support, see [KS]. We write $Sh(X; k)$ for the dg category of sheaves of complexes of $k$-modules, quotiented by the everywhere-acyclic sheaves. For some remarks on foundations for working with dg sheaves see [Nad]; as for the coefficients, in this paper we will only be really concerned with $k = \mathbb{Z}$ or $k = \mathbb{Z}/p\mathbb{Z}$.

Informally, a sheaf $F \in Sh(X; k)$ is a collection of complexes of vector spaces $F_x$, parameterized by $x \in X$, varying continuously; these $F_x$ are called the stalks of $F$. To a map of spaces $\phi : Y \to X$, there are pushforwards $\phi_* : Sh(Y; k) \to Sh(K; k)$, and pullbacks $\phi^* : Sh(K; k) \to Sh(Y; k)$. If $x : x \to X$ is the inclusion of a point, then the complexes mentioned above are $x^* F = F_x$. The stalks of $\phi_* F$ can also be described: $(\phi_* F)_y$ is a complex which computes the compactly supported cohomology with sheaf coefficients $H^*_c(X_y, F|_{X_y})$. The maps $\phi_*, \phi^*$ are the right adjoints of the $\phi, \phi^*$.

The singular support of a sheaf $F$ on $X$ is a conical co-isotropic locus $\mathcal{S}F \subset T^*X$ which collects the co-directions along which the sheaf is changing. Constructible sheaves are those for which the singular support is Lagrangian. Such a conical Lagrangian is a union of conormal bundles, the sheaf in question is therefore locally constant along the strata of the stratification given by the loci whose conormal bundles appear, this being the more general definition of constructibility.

The singular support interacts well with pushforwards: Assuming that $\phi : Y \to X$ is proper on the support of a sheaf $F$ on $Y$, we have

$$\mathcal{S}(\phi_! F) \subset \phi_* \mathcal{S}(F)$$

Here, $\phi_!$ is the pushforward defined above by convolution.

**Definition 1.** A generating family is a diagram $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$, where $\phi$ is a fibration. We require in addition that $\phi \times f$ is a fibration over the complement of a compact subset of $X \times \mathbb{R}$, and that moreover the above diagram admits a fibrewise compactification $X \xleftarrow{\phi} Y \xrightarrow{\overline{f}} \mathbb{R}$, compatible with the fibration-at-infinity structure of $\phi \times f$ and hence unique, where $\overline{Y}$ is a manifold with boundary, $\overline{\mathbb{R}} = [-\infty, \infty]$ is a closed interval, and $\overline{f}$ is a map of manifolds with boundary.

We write $\Gamma_f \subset Y \times \mathbb{R}$ for the graph of $f$, and denote the region beneath it $U_f := \{(y, z) | f(y) < z\} \subset Y \times \mathbb{R}$. We also write $u_f : U_f \to X \times \mathbb{R}$ for the restriction of $\phi \times 1_{\mathbb{R}}$ to $U_f$. We denote by $u_f : U_f \to X \times \mathbb{R}$ the compactification.

We say that $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ generates the Legendrian

$$\Lambda_f := \mathcal{S}(u_f;k) \cap J^1(X)$$

This definition has the virtue of making sense without genericity assumptions on the nature of the function $f$. To compare it with the previous definition, factor $u_f$ into $U_f \xrightarrow{\phi \times 1_{\mathbb{R}}} Y \times \mathbb{R} \xrightarrow{\phi \times 1_{\mathbb{R}}} X \times \mathbb{R}$. Then we have

$$\mathcal{S}(u_f;k) = \mathcal{S}((\phi \times 1_{\mathbb{R}})*i;k) \subset (\phi \times 1_{\mathbb{R}})* (\{N^*_f(Y \times \mathbb{R}) \cup N^*_Y(Y \times \mathbb{R})\}) \subset \mathbb{R} \Lambda_f \cup N^*_{X \times \mathbb{R}}(X \times \mathbb{R})$$
Intersection with the jet bundle shows \( \Lambda_f \subset \triangledown \Lambda_f \). If the restriction of \( f \) to a generic fibre of \( \phi : Y \rightarrow X \) is Morse, and \( \Lambda'_f \) is a manifold, then it is easy to see that in fact the inclusion is an equality. We henceforth restrict to this case, and cease distinguishing the two notations for the generated Legendrian.

A slightly more careful analysis shows that, since we used the \( ! \) pushforward, in fact the part of \( \xi \otimes k \) which lies away from the zero section is only the negative conormal to \( \Gamma_f \). Following this through the convolution, \( \xi(\tilde{u}_{f!}k) \) also has singular support in the negative half of the cotangent bundle, hence \( \xi(\tilde{u}_{f!}k) \subset \mathbb{R}_+ \Lambda_f \cup N_X \times \mathbb{R}(X \times \mathbb{R}) \). As in \([KS, STZ]\), for a Legendrian \( \Lambda \) in the oriented projectivization of the cotangent bundle of a manifold \( M \), we write \( Sh_{\Lambda}(M; k) \) for sheaves whose singular support is contained in the union of the \( \mathbb{R}_- \) cone over \( \Lambda \) and the zero section. We moreover identify the jet bundle of \( X \) with the negative half of the oriented projectivized cotangent bundle by projection. Finally, we write \( Sh_{\Lambda}(X \times \mathbb{R}; k)_0 \) for the sheaves with acyclic stalks near \( X \times -\infty \); note that since \( u_{f!} \) is measuring relative cohomology of sublevel sets which stabilize at infinity, it has this property.

In sum, the connection between generating families and constructible sheaves (observed already in \([Vit3]\)) is:

**Theorem 2.** Let \( X \xleftarrow{\phi} Y \xrightarrow{\Lambda} \mathbb{R} \) be a generating family. Then \( u_{f!}k \in Sh_{\Lambda_f}(X \times \mathbb{R}; k)_0 \).

**Remark.** There are various reasons to prefer \( u_{f!}k \) to \( u_{f!}k \), which all have to do with the fact whereas \( u_{f!}k \) records the compactly supported cohomology of the sub-level sets, \( u_{f!}k \) records the compactly supported cohomology relative to the level set at -\( \infty \). In particular, the stalks of \( u_{f!}k \) in the region \( X \times [-\infty, N) \) are acyclic for \( N \ll 0 \).

Alternately, we could declare that we work in the category of sheaves (dg) quotiented by the local systems, in which \( u_{f!}k \equiv u_{f!}k \). But by \([Kel, Dri]\), it is the same to work with the sheaves orthogonal to local systems, which \( u_{f!}k \) is, and \( u_{f!}k \) is generally not.

### 1.3. The Morse-theoretic viewpoint.

To understand the above constructions, it is helpful to consider the case when \( X \) is a point, and \( f : Y \rightarrow \mathbb{R} \) is a Morse function. Then we can form the complex \( Morse(Y, f; k) \), where the generators are named by the critical points of \( f \), and the differential counts gradient trajectories between critical points. This complex computes the homology or cohomology of \( M \).

Recording the critical values gives an \( \mathbb{R} \)-filtration on the Morse complex: we write \( V_{<k}Morse(Y, f; k) \) for the subcomplex generated by critical points of critical value less than \( k \). This complex is filtered by the critical value, and the quotients of this filtration are the relative cohomologies of sublevel sets.

We can trade this \( \mathbb{R} \)-filtered complex for a constructible sheaf of complexes \( Morse(Y, f; k) \) on \( \mathbb{R} \), by using the stalkwise prescription

\[
\widetilde{Morse}(Y, f; k)_z := V_{<z}Morse(Y, f; k)
\]

The above prescription is constant under small perturbations of \( z \), except when \( z \) is a critical value and we compare stalks at \( z \) and \( z + \epsilon \) for \( \epsilon > 0 \). So to finish describing a constructible sheaf, we should give in addition a map \( Morse(Y, f; k)_z \rightarrow Morse(Y, f; k)_{z+\epsilon} \). This map is just the inclusion of filtration steps.

As \( Morse(Y, f; k) \) is just recording the relative cohomologies of sub-level sets, we could have named it without any explicit mention of Morse theory. In fact, we already have:

\[
u_{f!}k \equiv \widetilde{Morse}(Y, f; k)\]

That is, \( u_{f!}k \) is a sort of promissory note for Morse theoretic calculations: without doing the Morse theory, we might not know what \( u_{f!}k \) actually is, but nonetheless this expression records our ignorance rather faithfully.

We return to the general setting of a generating family \( X \xleftarrow{\phi} Y \xrightarrow{\Lambda} \mathbb{R} \). It is natural to view this as a family of functions \( f_x : Y_x \rightarrow \mathbb{R} \); assuming that \( \phi \) is a fibration, all the \( Y_x \) are the same. We have

\[
u_{f!}k|_{x \times \mathbb{R}} \equiv u_{f!}k
\]

Thus the sheaf \( u_{f!}k \) can be viewed as a way of organizing the would-be Morse theory of the \( f_x \). At least for generic \( f \) and generic \( x \in X \), the function \( f_x \) really is Morse, so on this locus we can understand \( u_{f!}k \) Morse-theoretically. For \( x \) outside a codimension two subset of \( X \), or in other words for one-parameter families of functions, we can appeal to Morse-Cerf theory to understand \( u_{f!}k \). As we will explain in Section 3, doing so explicitly is the essential content of the Morse complex sequences of Henry and Rutherford \([Hen, HR]\).

**Remark.** Conversely, it might be interesting to revisit Cerf theory with attention to \( u_{f!}k \), and in particular the fact that \( f \mapsto u_{f!}k \) gives a map from arbitrary, i.e. not necessarily Morse, functions on bundles over \( X \) to a category equivalent to the Fukaya category on \( X \times \mathbb{R} \).
1.4. A category of generating families. Note that \( Sh \) is a category, i.e., there is a notion of morphisms between two sheaves. Our first new result is the identification of the classical name of the space of endomorphisms:

**Theorem 3.** Let \( f \) be a generating family. Then \( \text{Hom}_{Sh}(u, k) \) is the generating family homology of \( f \).

We recall the definition of generating family homology from [Tra2, FR] and prove Theorem 3 in Section 2. This result allows us to import the entire categorical framework of sheaves into the world of generating families:

**Definition 4.** We write \( \text{Gen}(X; k) \) for the (dg) category whose objects are generating families \( X \leftarrow \mathcal{Y} \rightarrow \mathbb{R} \), and whose morphisms and compositions are determined by requiring \( f \mapsto u_i k \) to be fully faithful.

For \( \Lambda \subset J^1(X) \) a Legendrian submanifold, we write \( \text{Gen}_\Lambda(X; k) \) for the full subcategory on generating families which generate \( \Lambda \).

That is, we have set up a category of generating families, whose endomorphisms are generating family homology. Note in particular that, since we have associated to each generating family for \( \Lambda \) a sheaf on the same space \( X \times \mathbb{R} \), we can say what it means for generating families \( X \leftarrow \mathcal{Y} \rightarrow \mathbb{R} \) and \( X \leftarrow \mathcal{Y}' \rightarrow \mathbb{R} \) to be isomorphic, even though \( Y, Y' \) may be different spaces. This is a good notion: if \( f, f' \) are isomorphic in \( \text{Gen}_\Lambda(X; k) \), then they have the same generating family cohomology, since Hom in any category is functorial in its components.

By Theorem 3 and Definition 4, there is a fully faithful morphism of (dg) categories \( \text{Gen}_\Lambda(X; k) \rightarrow Sh_\Lambda(X; k) \). We now set about studying its image.

Let \( f \) be a generating family. Assuming that \( f \) is Morse for generic \( x \), then the microlocal stalks of \( u_i k \) will be shifted rank one \( k \)-modules on the smooth locus of \( \Lambda f \). Assuming, as we do henceforth, that \( \Lambda f \) is smooth, the cohomological degree of this module – i.e. the Morse indices of the \( f \) – will determine a Maslov potential \( \mu \) on \( \Lambda f \). We will write \( \text{Gen}_{\Lambda, \mu}(X; k) \) for the generating families with this collection of Morse indices.

We write \( Sh_{\Lambda, \mu}(X; k) \subset Sh_\Lambda(X; k) \) for the full subcategory of sheaves with microlocal rank 1, microlocal stalks prescribed by \( \mu \), and acyclic stalks at \( -\infty \). (In [STZ, NRSSZ] we called this category \( C_1(\Lambda; \mu, k) \).) Microlocal stalks measure Morse indices, so \( \text{Gen}_{\Lambda, \mu}(X; k) \rightarrow Sh_{\Lambda, \mu}(X; k) \).

When \( X = \mathbb{R} \), we can understand this morphism completely. We will factor the map \( f \mapsto u_i k \) through a category of Morse complex sequences, which we introduced in [NRSSZ]; it is a categorical interpretation of the Morse complex sequences of Henry and Rutherford [Hen, HR, HR2]. The Morse complex sequences play two natural roles – first, any generalized Morse family of functions gives a Morse complex sequence, and second, any Morse complex sequence determines a sheaf. We recall the definition of this category in Section 3.2.

By adapting the proof of the \( h \)-cobordism theorem, we show:

**Theorem 5.** Let \( M \) be a simply connected manifold. Let \( S \) be a Morse complex sequence over \( \mathbb{Z} \) with all Morse indices satisfying \( 2 \leq \mu \leq \dim M - 2 \). Assume given a Morse-Smale \( f_{-\infty} : M \rightarrow \mathbb{R} \) whose corresponding filtered Morse complex is \( S_{-\infty} \). Then there is a generating family \( \mathbb{R}_x \leftarrow \mathbb{R}_x \times M \rightarrow \mathbb{R}_z \) extending \( f_{-\infty} \), such that the corresponding Morse complex sequence is isomorphic to \( S \).

**Remark.** We define the notion of isomorphism of Morse complex sequences directly, but in fact this notion is determined by requiring that the natural map from Morse complex sequences to sheaves is fully faithful. The map from Morse complex sequences to sheaves is in turn essentially determined by requiring that it factor \( f \mapsto u_i k \).

In [NRSSZ], we showed that the Morse complex sequence category was isomorphic to the sheaf category over a field. We apply Theorem 5 to a linear map \( f_{-\infty} \) to conclude:

**Corollary 6.** Let \( \Lambda \subset J^1(\mathbb{R}) \) be a Legendrian knot or link in \( \mathbb{R}^3 \). Assume that the Maslov potential \( \mu \geq 2 \). Then the morphism \( \text{Gen}_{\Lambda, \mu}(X; \mathbb{Z}/p\mathbb{Z}) \rightarrow Sh_{\Lambda, \mu}(X; \mathbb{Z}/p\mathbb{Z}) \) is an equivalence for any prime \( p \).

In fact, it suffices to consider generating families of the form \( \mathbb{R}_x \leftarrow \mathbb{R}_x \times \mathbb{R}^n \rightarrow \mathbb{R}_z \) where \( f_x \) is linear for \(|x| \geq \) 0 and \( f_x \) is linear at infinity for any \( x \).

**Remark.** The assumption \( \mu \geq 2 \) above is harmless for practical purposes since shifting gives an equivalence of categories: \( \mathbb{Z}/p\mathbb{Z} \) for any \( k \). In fact, this shift can be realized by an explicit Legendrian isotopy, see e.g. [NR].

In [NRSSZ], we defined an \( (\Lambda_{-\infty}) \)-category \( \text{Aug}_{\Lambda_{-\infty}}(k) \). Its objects are augmentations of the Chekanov-Eliashberg DGA [Che2, Eli], and its endomorphisms can be identified with linearized Legendrian contact homology. We proved this category is equivalent to the Morse complex sequence category over any ring \( k \). Thus:

**Corollary 7.** For any Legendrian knot or link, the set of generating family homologies coincides with the set of linearized Legendrian contact homologies with \( \mathbb{Z} \) or \( \mathbb{Z}/p\mathbb{Z} \) coefficients.
Remark. The existence of a coincidence between generating family homologies and linearized Legendrian contact homologies was observed in some cases by Traynor [Tra2]. It was later shown by Fuchs and Rutherford [FR] that the generating family homologies are a subset of the linearized contact homologies, and the above statement was conjectured in general. It is my understanding that Melvin, Sabloff, and Traynor also have an argument for the corollary.

1.5. Acknowledgements. This project originated in my attempt to understand a letter from Dan Rutherford [Rut], who is to be thanked in addition for many helpful discussions. I also thank Peter Teichner for some explanations about Cerf theory.

1.6. Abuses of notation. Throughout, all symbols should be understood to mean: the derived or dg construction most closely resembling whatever used to be meant by the symbol. All categories are dg categories. We use “$\cong$” to mean “isomorphic by an isomorphism so canonical as not to be worth mentioning.” The pushforwards and pullbacks $\phi\ast, \phi^\ast, \phi_!$ always mean the corresponding thing in the dg category of complexes. Finally, we write $H^*(X; k)$ or $\Gamma(X; k)$ to mean a chain complex computing the cohomology of $X$ with coefficients in $k$.

2. Generating family cohomology is sheaf cohomology

Theorem 8. If $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ and $X \xleftarrow{\phi'} Y' \xrightarrow{f'} \mathbb{R}$ are two generating families, we define the difference function

$$w : Y \times_Y Y' \to \mathbb{R} \quad (x, y, y') \mapsto f'(x, y') - f(x, y)$$

Then

$$\text{Hom}^*(u_f, k, u_f', k) \cong H^*(Y \times_X Y', w^{-1}(0, \infty); k)[\dim Y - \dim X]$$

Proof. If $\mathcal{F}, \mathcal{G}$ are sheaves on some space $B$, and $\Delta_B : B \to B \times B$ is the diagonal, then $\mathcal{F} \otimes^L \mathcal{G} := \Delta_B^!(\mathcal{F} \boxtimes \mathcal{G})$. The sheaf Hom can be expressed in terms of this tensor product:

$$\text{Hom}(u_f, k, u_f', k) = (u_f[k])^\vee \otimes^L u_f[k] = u_f[k][\dim U_f] \otimes^L u_f'[k]$$

We lost the underline in the final formula by Verdier duality on the manifold-with-boundary $U_f$.

To use base change with respect to the diagonal, we have to interpret $u_f$ in terms of $\ast$-pushforwards. Let $V_f := \{(y, z) | f(y) > z\} \subset Y' \times \mathbb{R}$ be the region above the graph of $f'$, and $v_f : V_f \to X \times \mathbb{R}$ the projection. We write $V_f$ to include $z = \infty$, and similarly $V_f'$ to include the graph, i.e. to take $f'(y) \geq z$, and $V_f'$ to include both. We now have an open-closed decomposition of $Y' \times \mathbb{R}$ as

$$\bigcup_{f'} \to Y' \times \mathbb{R} \xrightarrow{\phi} \bigcup_{f'}$$

hence a triangle on $Y' \times \mathbb{R}$ of the form

$$\mathfrak{j}_f \to \mathfrak{k} \to \mathfrak{j}_f \overset{[1]}{\to}$$

We want to push this forward to $X \times \mathbb{R}$ with $(\phi \times 1)$. Note that the map $\phi \times 1$ is proper on $Y' \times \mathbb{R}$ as well as on $V_f'$, so on the second and third terms, we can replace the $\mathfrak{j}$ with a $\ast$. (This last step is one place where it is important that we have taken $u_f$ rather than $u_f'$.) Finally, wherever we have $\ast$, we can drop the compactification of the source, since $\ast$-pushforward of the constant sheaf to the compactification will in these cases be the constant sheaf on the compactification. Thus, on $X \times \mathbb{R}$ we have a triangle of sheaves

$$u_{\ast} k \to (\phi' \times 1)_\ast k \to v_{f'} \ast k [1]$$

and hence also

$$U_{\ast} k \otimes^L u_{\ast} k \to U_{\ast} k \otimes^L (\phi' \times 1)_\ast k \to u_{\ast} k \otimes^L v_{f'} \ast k [1]$$

To analyse the second two terms, we use the fibre product diagrams

$$\begin{array}{ccc}
Y \times_X Y' \times \mathbb{R} & \xrightarrow{\Delta} & (Y \times \mathbb{R}) \times (Y' \times \mathbb{R}) \\
\downarrow \phi \times \phi' & & \downarrow (\phi \times 1) \times (\phi' \times 1) \\
X \times \mathbb{R} & \xrightarrow{\Delta} & (X \times \mathbb{R}) \times (X \times \mathbb{R})
\end{array}$$
These fibre diagrams satisfy the hypothesis of the subsequent base change Lemma 9, which we now apply. For instance, from the Lemma and the first diagram, we conclude

\[ \Gamma(X, u_f \cdot \mathbb{Z} \otimes (\delta^! \times 1)_* \mathbb{Z}) \cong H^*(\{(y, y', z) \mid z < f(x, y)\}, \mathbb{Z})[-\dim X - 1] \cong H^*(Y \times X Y', \mathbb{Z})[-\dim X - 1] \]

where the second equality comes from just projecting out the \( \mathbb{R} \) factor.

From the second diagram, and the observation

\[ U_f \times X \times \mathbb{R} \rightarrow \{(y, y', z) \mid f(y) < z < f'(y')\} \subseteq Y \times X Y' \times \mathbb{R} \]

we conclude

\[ \Gamma(X \times \mathbb{R}, u_f \cdot \mathbb{Z} \otimes (\delta^! \times 1)_* \mathbb{Z}) \cong H^*(\{(y, y', z) \mid f(y) < z < f'(y')\}, \mathbb{Z})[-\dim X - 1] \cong H^*(w^{-1}(0, \infty), \mathbb{Z})[-\dim X - 1] \]

The second identification comes from observing that forgetting the \( z \) coordinate is a fibration in open intervals over \( w^{-1}(0, \infty) \).

Taking sections of (1) and shifting by \( \dim U_f = \dim Y + 1 \), we get a triangle

\[ \text{Hom}^* (u_f \cdot \mathbb{Z}, u_f \cdot \mathbb{Z}) \rightarrow H^*(Y \times X Y', \mathbb{Z})[\dim Y - \dim X] \rightarrow H^*(w^{-1}(0, \infty), \mathbb{Z})[\dim Y - \dim X] \]

The second morphism is evidently induced by pullback along the inclusion, which identifies the Hom space with the relative cohomology. This completes the proof.

We used the following in the proof of Theorem 8:

**Lemma 9.** Let \( m : M \rightarrow B \) and \( m' : M' \rightarrow B \) be smooth (not necessarily proper) submersions of manifolds. Then

\[ m_* \mathbb{Z} \otimes m'_* \mathbb{Z} = (m \times_B m')_* \mathbb{Z}[-\dim B], \]

and in particular,

\[ \Gamma(B; m_* \mathbb{Z} \otimes m'_* \mathbb{Z}) = \Gamma(M \times_B M'; \mathbb{Z})[-\dim B]. \]

**Proof.** By definition, \( m_* \mathbb{Z} \otimes m'_* \mathbb{Z} \) is the shriek pullback along the diagonal of \( m_* \mathbb{Z} \otimes m'_* \mathbb{Z} = (m \times m')_* \mathbb{Z} \) on \( B \times B' \). Write \( \mathcal{D} \) for the dualizing sheaf. Consider the fibre product diagram

\[ \begin{array}{ccc}
M \times_B M' & \xrightarrow{\delta} & M \times M' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\Delta} & B \times B
\end{array} \]

By base change, \( \Delta^*(m \times m')_* \mathcal{K}_M \times M' = (m \times m')_* \delta^! \mathcal{K}_M \times M' \). Since \( M \times M' \) is a manifold, \( k_{M \times M'} = \mathcal{D}_{M \times M'}[-\dim M - \dim M'] \). So \( \delta^! k_{M \times M'} = \mathcal{D}_{M \times B M'}[-\dim M - \dim M'] \). But since \( m, m' \) are submersions, also \( M \times_B M' \) is a manifold, and thus \( \mathcal{D}_{M \times_B M'}[-\dim M - \dim M'] = k_{M \times_B M'}[-\dim B] \).

**Remark.** The idea to study the critical points of this function \( w \) goes back at least to Viterbo [Vit]. One sees easily that they correspond to Reeb chords between \( \Lambda_F \) and \( \Lambda_{F'} \), with critical values given by integrating the contact form. If \( \Lambda_F = \Lambda_{F'} \), then also the length zero chords appear: an entire copy of \( \Lambda_F \). The cohomology of \( H^*(Y \times X Y', w^{-1}(0, \infty); \mathbb{Z}) \) is called the generating function cohomology [Tra2, Pus2, FR], and its use is a well established technique in the study of Legendrian knots. Different authors have preferred slightly different variants, which are related variously by excision, Poincaré duality, and taking homology or cohomology. We note that in fact the above proof is implicitly using these variants as well, in the form of invoking the open-closed exact triangle of sheaves and the Verdier duality, which are the sheaf-theoretic incarnations of excision and Poincaré duality, respectively.
3. **Morse complex sequences**

We now restrict ourselves to the setting where $X$ is one dimensional. This is the most well studied case, being relevant for Legendrian knots and links on the one hand, and for the classical pseudo-isotopy versus isotopy question on the other. In this setting, Henry and Rutherford have developed a combinatorial abstraction of generating families, which they call Morse complex sequences [FR, Hen, HR, HR2]. As we will explain here, this combinatorial abstraction is most naturally understood as an explicit specification of a sheaf.

3.1. **The Morse complex category.** We begin in the setting where $X$ is zero dimensional, as this will be the basic building block for the one dimensional case. As we recalled in the introduction, given a Morse function $f : Y \to \mathbb{R}$, and an appropriate metric $g$ on $Y$ which we suppress from the notation, we can consider the Morse complex $\text{Morse}(Y; f; k)$, along with its filtration by critical values, $V_{<z}\text{Morse}(Y; f; k)$. This filtration translates into a sheaf $\tilde{\text{Morse}}(Y; f; k)$, characterized stalkwise by $\tilde{\text{Morse}}(Y; f; k)_z = V_{<z}\text{Morse}(Y; f; k)$, and related to our constructions here by

$$\underline{u}_f^k \cong \tilde{\text{Morse}}(Y; f; k)$$

The $\mathbb{R}$ filtration $V_{<z}$ has only finitely many nontrivial steps, corresponding to the critical values. We form a more combinatorial object by recording only these nonvanishing steps. We number the critical points numbered in decreasing order of critical value – the point with largest critical values is numbered $1$, and the point with the smallest critical value is numbered $n$. We record the Morse indices in a function $\mu : \{1, \ldots, n\} \to \mathbb{Z}$. The vector space underlying the Morse complex is:

**Definition 10.** For $\mu : \{1, \ldots, n\} \to \mathbb{Z}$, we write $\mathfrak{g}_\mu$ for the free graded $k$-module with basis $\langle 1 \rangle, \ldots, \langle n \rangle$ where $\deg \langle i \rangle = -\mu(i)$, and decreasing filtration $\mathfrak{g}_\mu : = \text{Span}(\langle n \rangle, \ldots, \langle k+1 \rangle)$. That is,

$$0 = V_0, 1 = \text{Span}([n], \ldots, [2]), \ldots, n-1 = \text{Span}([n]), \ n\mathfrak{g}_\mu = 0.$$ 

We now define a category whose objects are Morse complexes, and whose morphisms are set up to match $\text{Hom}(\underline{u}_f^k, \underline{u}_{f'}^k)$.

**Definition 11.** Fix an integer $n$ and $\mu : \{1, \ldots, n\} \to \mathbb{Z}$. We define $MC_\mu(k)$ to be the dg category with:

- **Objects:** square-zero operators $d$ on $\mathfrak{g}_\mu$, which preserve the filtration on $\mathfrak{g}_\mu$ and are degree 1 with respect to the grading on $\mathfrak{g}_\mu$.
- **Morphisms:** $\text{Hom}_{MC_\mu}(d, d')$ is $\text{Hom}(\mathfrak{g}_\mu, \mathfrak{g}_\mu)$ as a graded vector space; i.e., it consists of the linear, filtration preserving maps $\mathfrak{g}_\mu \to \mathfrak{g}_\mu$ and carries the usual grading of a Hom of graded vector spaces. Only its differential depends on $d, d'$, and is

$$D\phi = d' \circ \phi - (-1)^{\deg \langle i \rangle} \phi \circ d.$$ 

- **Composition:** usual composition of maps.

That is, we allow maps $\langle j \rangle \langle i \rangle$ for $i \leq j$, i.e. lower triangular matrices, and

$$\deg \langle j \rangle \langle i \rangle = \deg \langle j \rangle - \deg \langle i \rangle = \mu(i) - \mu(j)$$

and the differential is $D(\langle i \rangle \langle j \rangle) = d'\langle i \rangle \langle j \rangle - (-1)^{\mu(i) - \mu(j)} \langle i \rangle \langle j \rangle d$.

**Definition 12.** Let $f : Y \to \mathbb{R}$ be a Morse function and $g$ an appropriate metric on $Y$. Assume that critical points have distinct critical values. We order the $n$ critical values, $z_1 > z_2 > \cdots > z_n$. We take $\mu(i)$ to be the Morse index of the critical point at $z_i$. We write $m(Y, f; k)$ for the element of $MC_\mu(k)$ obtained by writing $\langle i \rangle$ for the generator corresponding to the critical point whose critical value is $z_i$.

Recall we have a category $\text{Gen}(pt; k)$ whose objects are generating families over a point (i.e. just functions), and whose categorical structure comes from just pulling back that of the sheaf category under the map $f \mapsto \underline{u}_f^k$. We will introduce a Morse-theoretic / combinatorial analogue.

**Definition 13.** Fix a list $\Lambda$ of real numbers $z_1 > z_2 > \cdots > z_n$ and a function $\mu : \{1, \ldots, n\} \to \mathbb{Z}$. We write $\text{Gen}MC_{\Lambda, \mu}(k)$ for the (dg) category with

- **Objects:** tuples of: a space $Y$, a Morse function $f : Y \to \mathbb{R}$, and a compatible metric $g$ on $Y$ so that the corresponding gradient flow is Morse-Smale, and such that $f$ has $n$ critical points with critical values $z_1, \ldots, z_n$ and Morse indices given by $\mu$. 

Morphisms, differential, and composition: defined by pulling back the following categorical structures under the map \((Y, f, g) \mapsto m(Y, f; \mathbb{k})\).

**Remark.** The list of real numbers can be identified with a zero-dimensional Lagrangian in \(J^1(\mathbb{pt}) = \mathbb{R}\), and \(\mu\) is to be understood as a Maslov potential on it.

**Proposition 14.** We have a commutative diagram of categories:

\[
\begin{array}{ccc}
\text{GenMC}_{\Lambda, \mu}(k) & \longrightarrow & \text{Gen}_{\Lambda, \mu}(pt; k) \\
\downarrow & & \downarrow \\
\text{MC}(\mu; k) & \longrightarrow & \text{Sh}_{\Lambda, \mu}(pt; k)_0
\end{array}
\]

All maps are fully faithful

**Proof.** The vertical arrows exist and are fully faithful by definition. The bottom arrow, on objects, is: first use \(\Lambda\) to specify the location of the non-trivial steps in an \(\mathbb{R}\)-filtered complex which otherwise is the filtered complex \(\mu\), i.e., so that \(m(Y, f; k)\) turns into Morse\((Y, f; k)\), and then second, turn this \(\mathbb{R}\)-filtered complex into a sheaf, i.e. such that Morse\((Y, f; k)\) becomes Morse\((Y, f; k)\) as discussed in the introduction. The first step is obviously fully faithful, and the definition on morphisms and full faithfulness of the second step is the equivalence of \(\mathbb{R}\)-filtered complexes with certain sheaves on \(\mathbb{R}\). (We explain this at greater length, albeit factored through the representation theory of the \(A_n\) quiver, in [NRSSZ, Sec. 7.3].) The diagram commutes on objects in the sense that, by Morse theory, Morse\((Y, f; k) \equiv u_f k\). We use this equivalence to define the top arrow as a morphism of categories. \(\square\)

We have introduced this diagram to help us study the image of the morphism GenMC\(_{\Lambda, \mu}(k) \to Sh_{\Lambda, \mu}(pt; k)_0\) \(\mathcal{MC}(\mu; k) \to Sh_{\Lambda, \mu}(pt; k)_0\) contains the image of the composition GenMC\(_{\Lambda, \mu}(k) \to MC(\mu; k) \to Sh_{\Lambda, \mu}(pt; k)_0\). At least when \(k\) is a field, we have shown in [NRSSZ, Sec. 7.3] that \(MC(\mu; k) \to Sh_{\Lambda, \mu}(pt; k)_0\) is essentially surjective.\(^3\)

So we are reduced to studying the morphism GenMC\(_{\Lambda, \mu}(k) \to MC(\mu; k)\). Preserving the freedom to choose \(Y\), this map is readily seen to be essentially surjective. If \(Y\) is a vector space, and \(f : Y \to \mathbb{R}\) required to be linear at infinity, the corresponding Morse complex is evidently acyclic. Over a field, using Barannikov’s classification of Morse complexes [Bar], it is not hard to see that this is the only restriction up to isomorphism in the Morse complex category.

### 3.2. Morse complex sequences

Our goal now is to set up a similar diagram in the setting of one-parameter families of functions. In fact we will restrict ourselves to \(X = \mathbb{R}\). To disambiguate the two copies of the real line, we write our generating family as \(\mathbb{R}_x \overset{\Phi}{\leftarrow} Y \overset{f}{\to} \mathbb{R}_z\).

Recall that the function \(f_x : Y \to \mathbb{R}\) is said to be Morse-Smale if:

1. \(f_x\) is Morse
2. Each critical value has only one critical point
3. All intersections of stable and unstable cells \(\bigcup_p \cap \bigcap_{p'}\) are transverse

Recall we write \(\Phi_f\) for the image of \(\Lambda_f\) in \(J^0(\mathbb{R}_x) = \mathbb{R}_x \times \mathbb{R}_z\). This is the front diagram of the Legendrian knot \(L_f\), and also the Cerf graphic of the family of functions \(f_x\) [HW]. At any \(x\) satisfying the first two conditions above, \(\Phi_f \cap \{(x - \epsilon, x + \epsilon)\} \times \mathbb{R}_z\) consists of smooth nonintersecting paths projecting diffeomorphically to \(\mathbb{R}_x\).

**Remark.** I find it helpful to think of the sheaf \(u_f k\) as a sort of further decoration of the Cerf graphic.

A generating family \(\mathbb{R}_x \overset{\Phi}{\leftarrow} Y \overset{f}{\to} \mathbb{R}_z\), together with the data of a family of metrics \(g_x\), is said Morse-Cerf if \(f_x\) only fails to be Morse-Smale at finitely many \(x\), and at these points satisfies one of the following:

- The function \(f_x\) fails to be Morse at a single point; the critical values of this and all other critical points are distinct. The corresponding front diagram near the corresponding critical value is an ordinary cusp. This is termed a birth or death singularity, according as the cusp is \(\prec\) or \(\succ\).
- The function \(f_x\) is Morse, but two critical points have coincident critical values. This appears as a crossing in the front diagram.

\(^3\)The argument in [NRSSZ] is rather unsatisfying – there, we factored the problem through the representation theory of the \(A_n\) quiver, which in particular required the assumption that \(k\) is a field.
• The function \( f_x \) is Morse, all critical points have distinct critical values, but there is exactly one pair \( p, p' \) with non-transverse \( \bigcup_{p} \cap \bigcup_{p'} \). Moreover, \( p \) and \( p' \) have the same Morse indices. This sort of singularity is called a handle slide, and is not visible from the front diagram.

• The function \( f_x \) is Morse, all critical points have distinct critical values, but there is exactly one pair \( p, p' \) with non-transverse \( \bigcup_{p} \cap \bigcup_{p'} \). Moreover, \( p \) and \( p' \) have Morse indices differing by one. This sort of singularity is not visible from the front diagram, and corresponds to the collisions of algebraically canceling gradient trajectories. As such it does not affect the sheaf \( \mu_f \), so we ignore such events.

It is known that a generic perturbation of any generating family is Morse-Cerf [Cer, Lau].

In the usual Morse theory, a Morse-Smale function determines a cell decomposition of the manifold, and the attaching maps can be characterized by certain gradient flow trajectories. In fact, one can give cell decompositions for the more general sort of functions detailed above; this is done in [Lau]. We introduce, essentially following [HR2] at the level of global objects, a combinatorial structure to organize together all this generalized Morse theory. The structure takes the form of a constructible sheaf of dg categories on \( \mathbb{R}_x \).

**Definition 15.** Let \( \Lambda \) be a Legendrian with a generic front diagram, equipped with a Maslov potential \( \mu \). Let \( S \) be a stratification of \( \mathbb{R}_x \), such that the zero dimensional strata include at least the \( x \)-coordinates of the singularities of the front projection \( \Lambda \). We will define a sheaf \( MC_{S,\mu,S}((k)) \) of dg categories on \( \mathbb{R}_x \), constructible with respect to the stratification \( S \).\(^4\) To do this, it suffices to give the stalk on each stratum, together with left and right generization maps at points on the zero dimension strata.

At any point \( x \) in a 1-dimensional stratum \((a, b)\), we take \( MC_{S,\mu,S}((k)) := MC_{\mu}(k) \). We view these categories for various \( x \in (a, b) \) as canonically identified, and denote them in common as \( MC_{S,\mu,S}((k))_{(a, b)} \).

On a 0-dimensional stratum over \( x \in \mathbb{R}_x \), we will in each case define an object of \( MC_{S} \), to be built from data including elements \( d_{x-\epsilon} \in MC_{\mu_{x-\epsilon}}(k) \) and \( d_{x+\epsilon} \in MC_{\mu_{x+\epsilon}}(k) \); the generization maps will just return these elements. Specifically, at a point

• **... over which the front projection is nonsingular**, \( \mu_{x-\epsilon} = \mu_x = \mu_{x+\epsilon} \), so we can canonically identify \( MC(\mu_{x-\epsilon}) = MC(\mu_x) = MC(\mu_{x+\epsilon}) \). An element of \( MC_{S,\mu,S}((k))_x \) will be by definition a triple \( d_{x-\epsilon} \xrightarrow{\sim} d_x \xrightarrow{\sim} d_{x+\epsilon} \), and we define morphisms in \( MC_{S} \) to be triples of morphisms commuting with this structure. Note that the forgetful map to any of the three pieces is an equivalence of categories.

• **... over which the front projection suffers a crossing between the \( i \)-th and \( i + 1 \)-st strands**, we write \( s_{i,i+1} \) for the permutation matrix \( [i] \leftrightarrow [i + 1] \). We take an object in our category to be a pair \( (d_{x-\epsilon}, d_{x+\epsilon}) \) where \( d_{x-\epsilon} \xrightarrow{s_{i,i+1}} d_{x+\epsilon} \) and a morphism to be a pair of morphisms commuting with \( s_{i,i+1} \).

• **... over which the \( i \)-th and \( i + 1 \)-st strands meet in a right cusp (death)**, we require that \( (k + 1) d_{x-\epsilon}(k) \) is invertible. For this discussion let us change the labelling of the basis of \( g_{x+\epsilon} \) from 1, \ldots, \( n \) to 1, 2, \ldots, \( k-1, k+2, \ldots, n \), for consistency with the basis of \( g_{x-\epsilon} \).

By the invertibility assumption, the quotient \( g_{x-\epsilon}/(|k|, d_{x-\epsilon}(|k|)) \) is freely generated by \( [i] \) for \( i \neq k, k + 1 \). Thus there is a unique map

\[
\begin{align*}
\pi_x : \mu_{x-\epsilon} &\to \mu_{x+\epsilon} \\
|k| &\mapsto 0 \\
d_{x-\epsilon}(|k|) &\mapsto 0 \\
[i] &\mapsto [i] \quad \text{for } i \neq k, k + 1
\end{align*}
\]

We take the objects of \( MC_{S} \) to be the pairs \( (d_{x-\epsilon}, d_{x+\epsilon}) \) which commute with \( \pi_x \); likewise for morphisms.

• **... over which the \( i \)-th and \( i + 1 \)-st strands meet in a left cusp (birth)**, we define the category as in the previous case.

The sections of the sheaf of categories \( MC_{S,\mu,S}((k)) \) on a given open set \( U \) have an explicit description as a limit (this is what the word sheaf means). Specifically, if \( U = (x_0, x_{n+1}) \) and the zero dimensional strata in this region are \( x_1 < x_2 < \cdots < x_n \), then

\[
MC_{S,\mu,S}(U) = \lim \left( \prod_{i} MC_{S,\mu,S}((k))_{x_i} \to \prod_{i} MC_{S,\mu,S}((k))_{(x_i, x_{i+1})} \right)
\]

\(^4\)Strictly speaking, these words should be interpreted with appropriately higher categorical sophistication. We do not emphasize this point here because it is essentially irrelevant in the setting at hand, due to the lack of topology of the base \( X = \mathbb{R}_x \), even after stratification.
Note that since it is a sheaf of categories rather than of sets, an object of the category described by this limit is a diagram of the form

\[ S_{(x_0, x_1)} \leftarrow S_{x_1} \rightarrow S^L_{(x_1, x_2)} \cong S^R_{(x_1, x_2)} \leftarrow S_{x_2} \rightarrow S^L_{(x_2, x_3)} \cong S^R_{(x_2, x_3)} \cdots \]

where \( S_x \in \text{MCS}_x \). Since for any \( x \in (x_i, x_{i+1}) \) we identify, by definition, \( \text{MCS}_x = \text{MCS}_{(x_i, x_{i+1})} \), I prefer to write the above as:

\[ S_{x_1-\epsilon} \leftarrow S_{x_1} \rightarrow S_{x_1+\epsilon} \cong S_{x_2-\epsilon} \leftarrow S_{x_2} \rightarrow S_{x_2+\epsilon} \cong S_{x_3-\epsilon} \cdots \]

That is, when computing sections in a sheaf of categories, objects can be glued along isomorphisms, not just along identities.

In fact, we can remove all handle slide singularities:

**Proposition 16.** Let \( \Lambda \) be a knot, \( S \) a stratification of \( \mathbb{R}_x \) in which the zero dimensional strata include all the \( x \)-coordinates of the images of the singularities of the front projection of \( \Lambda \). Let \( x_0 \) be a zero dimensional stratum which is not the \( x \)-coordinate of a singularity, and let \( x_{-1} \) and \( x_1 \) be the immediately preceding and succeeding zero dimensional strata. Let \( S' \) be the stratification in which the strata \( (x_{-1}, x_0), (x_0, x_1) \) are merged into one stratum \( (x_{-1}, x_1) \).

Then there is an equivalence of (sheaves of) categories \( \text{MCS}_{\Lambda, \mu, S}(k) \rightarrow \text{MCS}_{\Lambda, \mu, S'}(k) \)

**Proof.** The equivalence will be the identity outside of \([x_{-1}, x_1]\). In this region, we map an object

\[ S_{x_{-1}} \rightarrow S_{x_{-1}+\epsilon} \xrightarrow{b} S_{x_0-\epsilon} \leftarrow S_{x_0} \xrightarrow{c} S_{x_0+\epsilon} \rightarrow S_{x_1-\epsilon} \leftarrow S_{x_1} \]

to

\[ S_{x_{-1}} \rightarrow S_{x_{-1}+\epsilon} \xrightarrow{edc^{-1}b} S_{x_1-\epsilon} \leftarrow S_{x_1} \]

which makes sense since \( b, c, d, e \) are all isomorphisms. We map the morphism spaces accordingly. \( \square \)

**Remark.** Note that we may begin with a Morse complex sequence in which all “gluing isomorphisms” are identities, and end up with one in which this is no longer the case, i.e. we may begin above in a situation where \( b, c \) are both the identity, and then end up with a non-identity gluing map \( dc^{-1} \). Conversely, by subdividing, we can represent any Morse complex sequence as one with all gluing isomorphisms the identity.

If \( S' \) refines \( S \), there is a natural fully faithful inclusion \( \text{MCS}_{\Lambda, \mu, S}(k) \rightarrow \text{MCS}_{\Lambda, \mu, S'}(k) \) given by setting the stalk of the sheaf at any new zero dimensional strata (which are necessarily beneath smooth points of the front projection) to triples of the form \( d_{x_{-1}} \leftarrow d_x \rightarrow d_{x_{+1}} \). Proposition 16 implies that all these inclusions are equivalences. Thus we write \( \text{MCS}_{\Lambda, \mu}(k) = \lim \text{MCS}_{\Lambda, \mu, S}(k) \) for the direct limit; the natural maps \( \text{MCS}_{\Lambda, \mu, S}(k) \rightarrow \text{MCS}_{\Lambda, \mu}(k) \) are all equivalences.

As we mentioned already, the original purpose of the Morse complex sequences was to record the Morse-theoretic information in a generating family over the line. That is:

**Proposition 17.** Let \( \mathbb{R}_x \overset{\Phi}{\leftarrow} Y \overset{\mu}{\rightarrow} \mathbb{R}_x \) be Morse-Cerf with respect to some family of metrics. Let \( S \) be the stratification of \( \mathbb{R}_x \) in which the zero-dimensional strata are the \( x \) for which \( f_x \) is not Morse-Smale. Let \( \mu \) be the Maslov potential on the Legendrian generated by \( \Lambda \) given by recording the Morse indices of the corresponding critical points. Then there is a Morse complex sequence \( p(f) \in \text{MCS}_{\Lambda, \mu, S}(k) \) characterized by:

- \( p(f)_x = m(Y, f_x; k) \) for \( x \) outside the zero dimensional strata
- For \( x \) beneath a handle slide singularity of \( f \), the morphism \( \phi_x : m_{x-\epsilon} \xrightarrow{\sim} m_{x+\epsilon} \) is \( \phi_x = 1 \pm [j](i) \).
- All gluing isomorphisms are the identity.

**Proof.** See [FR, Sec. 5.1.2] or [HR2], both of which draw from [Lau]. \( \square \)

As with \( \text{Gen}, \text{GenMC} \), we use this map on objects to define the structure of a category on the Morse-Cerf generating families.

**Definition 18.** Fix \( \Lambda \subset J^1(\mathbb{R}_x) \) in general position, let \( \mu \) be a Maslov potential on \( \Lambda \). We write \( \text{GenMC}_{\Lambda, \mu}(k) \) for the (dg) category whose objects are Morse-Cerf generating families which generate \( \Lambda \) with corresponding Morse indices given by \( \mu \), and the rest of whose categorical structure is defined by pulling back along \( f \mapsto p(f) \).

On the other hand, a Morse complex sequence can be used to specify a sheaf. From an object \( S \in \text{MCS}_{\Lambda, \mu, S}(k) \), we define a sheaf \( \tilde{S} \) as follows. Above any \( x \in \mathbb{R}_x \) not in a zero dimensional stratum of the stratification, we have \( S_x \in \text{MC}_{\mu_x}(k) \) and can use the morphism \( \text{MC}_{\mu_x}(k) \rightarrow \text{Sh}_{\Lambda, \mu_x}(x \times \mathbb{R}_x^+, k) \) to define the sheaf \( \tilde{S}_{x \times \mathbb{R}_x^+} \). At points beneath a handle slide, the same prescription applies. When \( x \) sits beneath a croissant, the specification of
\( \tilde{S}_{x,z} \) can be taken unambiguously to be identical to that of \( \tilde{S}_{x,c,z} \) except at the coordinates of the crossing itself. Here, we take \( \tilde{S}_{x,z} = \tilde{S}_{x \pm \epsilon, z - \delta} \). Finally, for \( x \) beneath a right cusp, we define \( \tilde{S}_{x,z} = \tilde{S}_{x - \epsilon, z} \), except at the cusp point; the rightward generation maps are given by the appropriate filtered piece of \( \pi_x \). At the cusp point itself, we set \( \tilde{S}_{x,z} = \tilde{S}_{x - \epsilon, z - \delta} \). By construction, \( \tilde{S} \in Sh_{\Lambda, \mu}(\mathbb{R}_x \times \mathbb{R}_z, k)_0 \).

**Proposition 19.** [NRSSZ] The map on objects above underlies a fully faithful (dg) functor

\[ \tau : MCS_{\Lambda, \mu}(k) \to Sh_{\Lambda, \mu}(\mathbb{R}_x \times \mathbb{R}_z, k)_0 \]

This functor is an equivalence at least when \( k \) is a field.

**Proof.** This is proven in [NRSSZ, Sec. 7.3]. The basic strategy of proof is to observe that both sides are sheaves of categories over the line \( \mathbb{R}_x \), so it suffices to define the morphism and prove the theorem locally; i.e., over each neighborhood \( (x - \epsilon, x + \epsilon) \). That reduces things to a case study of the picture in each of: the nonsingular setting, a crossing, a cusp. The restriction to field coefficients is due to the fact that in the course of the proof, the sheaf category is given a quiver description, and we invoke Gabriel’s theorem. \( \square \)

**Remark.** In [NRSSZ] the Morse complex sequence category is defined slightly differently: there are no handle slides, the crossing category contains more objects and we only define MCS in the case when the knot is in “plat position”, i.e. all left cusps are at the utmost left, and all right cusps are at the utmost right. The difference in crossing categories and handle slides can be absorbed entirely into the generation maps as in Proposition 16, this absorption commutes with the map to \( Sh \). As for the cusps, for the application here we can and will work after putting the knot in preferred plat position, so will not directly prove the above theorem in the stated form.

The reason we have given differing definitions is that the definition in [NRSSZ] is better adapted to comparison with augmentations, and the one here is better adapted to comparing with generating families.

**Proposition 20.** We have a commutative diagram of categories:

\[ \begin{array}{ccc}
GenMCS_{\Lambda, \mu}(k) & \longrightarrow & Gen_{\Lambda, \mu}(\mathbb{R}_x; k) \\
\downarrow & & \downarrow \\
MCS_{\Lambda, \mu}(k) & \longrightarrow & Sh_{\Lambda, \mu}(\mathbb{R}_x; k)_0 \\
\end{array} \]

All maps are fully faithful.

**Proof.** The vertical arrows are fully faithful by definition. The bottom arrow we discussed above. The top arrow, on objects, just forgets the metric. The diagram commutes on objects in the sense that \( \tau \circ p(f) \cong \mu(f, k) \); this can be checked locally on \( \mathbb{R}_z \) by generalized Morse theory; this happens in slightly different language in [FR, Sec. 5.1.2] or [HR2], drawing from [Lau]. We use this equivalence to define the top arrow as a morphism of categories. \( \square \)

3.3. **Surjectivity.** We turn to proving Theorem 5. First we collect results from the literature which ensure that certain Morse complex sequences can be realized by functions. We refer to zero dimensional strata which do not lie beneath a singularity as “formal handle slides”.

**Proposition 21.** Let \( \Lambda \) be a Legendrian knot or link, and \( \mu \) a Maslov potential. Let \( S \in MCS_{\Lambda, \mu}(\mathbb{Z}) \) be a Morse complex sequence for which:

- The Morse complexes near a crossing between the \( i \)'th and \( i + 1 \)'st strands have \( (i + 1 \lfloor d[i] \rfloor) = 0 \)
- The Morse complexes near a cusp between the \( i \)'th and \( i + 1 \)'st strands have \( (i + 1 \lfloor d[i] \rfloor) = 1 \)
- The isomorphisms \( \phi_x \) characterizing formal handle slides take the form \( \phi_x = 1 \pm [j|i] \)

Let \( M \) be simply connected, and let \( f_{-\infty} : M \to \mathbb{R}_z \) be a Morse function with Morse complex \( S_{-\infty} \). Assume \( 2 \leq \mu \leq \dim M - 2 \). Then there exists a generating family \( \mathbb{R}_x \xleftarrow{\phi} \mathbb{R}_x \times M \xrightarrow{f} \mathbb{R}_z \) extending \( f_{-\infty} \) such that \( p(\mathbb{R}_x \xleftarrow{\phi} \mathbb{R}_x \times M \xrightarrow{f} \mathbb{R}_z) = S \).

**Proof.** We build the function from the left to the right along the \( x \) line. At the left, the function is given. Varying the function to create the left cusps is elementary. Before a crossing we have \( (i + 1 \lfloor d[i] \rfloor) = 0 \). This says that numerically, the stable and unstable cells of the corresponding critical points do not intersect. If the cells actually did not intersect, then the the “independent trajectories principle” would say that we can deform the function as desired [HW, Chap. 1, Sec. 7]. But since the space is simply connected and \( 2 \leq i < i + 1 \leq \dim M - 2 \), we actually cancel the numerically cancelling trajectories, as in the proof of [Mil, Thm. 6.4, p.49]. The desired formal handle slides can be realized by the classical result on existence of handle slides, see e.g. [HW, Chap. 2, Lemma]
Again this requires $2 \leq \mu \leq \dim M - 2$. Finally, before the right cusps we have $(i + 1)|i| = 1$; again if this signified the existence of an actual single gradient trajectory between the critical points we could cancel them by the flushing lemma [Mil, Thm. 5.4, p. 49]. But in fact assumptions on the space allow us to actually cancel the numerically cancelling trajectories, as asserted by [Mil, Thm. 6.4, p.49], which too requires $2 \leq \mu \leq \dim M - 2$. Thus, the function can be deformed to create a death singularity in the desired way. After the right cusps, we have a function with no critical points, which can be deformed to a linear function as desired.

It remains to show that all Morse complex sequences can be put in the form required by Proposition 21. So fix a Morse complex sequence $S$. First, subdivide all one-dimensional strata, and replace $S$ with a sequence in which all gluing isomorphisms are identity maps, as in the remark below Proposition 16. The left and right generization maps at a formal handle slide are lower triangular isomorphisms. We want to have all formal handle slides given by unipotent lower triangular isomorphisms (i.e. with 1s on the diagonal). So we factor out a diagonal matrix and begin pushing it to the right. Since unipotent matrices are normal in the group of lower triangular matrices, and since conjugation by a transposition preserves diagonal matrices, we can push it past any handle slides or crossings. Upon reaching a cusp, the relevant piece of the diagonal matrix can be absorbed into an automorphism of the cusp. We have now ensured that all formal handle slides are unipotent. By row reduction, we can express any unipotent lower triangular matrix as a product of elementary matrices; by further subdivision we can implement this factoring and thus ensure that all of our formal handle slides are characterized by isomorphisms of the form $\phi_x = 1 \pm [j]/[i]$. This completes the proof of Theorem 5.

Remark. There is a very convenient graphical notation for describing and manipulating Morse complex sequences; see [Hen, HR]. In those papers, a notion of equivalence of Morse complex sequences is introduced as generated by certain local moves; we leave it to the reader to check directly that these moves give isomorphisms in the Morse complex sequence category. Abstractly this is true because locally these moves are correspond to considering a fixed generating family $f$ but varying the metric; since isomorphism in the Morse complex sequence category only depends on the sheaf $\mathcal{L}_f/k$, hence not on the metric, such moves give isomorphisms of Morse complex sequences. These local moves can be used to simplify a Morse complex sequence much further than we have done above. In fact it can be ensured that the only appearing handle slides have $\phi_x = 1 \pm [i]/[i+1]$ where $i, i + 1$ name strands which are just about to cross. Such a Morse complex sequence is said to be in $A$-form; see e.g. [HR, Def. 7.1].

Remark. Let us spell out more precisely the relation of the above discussion to the proof of the $h$-cobordism theorem [Smal1, Smal2, Mil]. Recall this result deals with the following situation: one has a cobordism $M$ from $M_-$ to $M_+$ such that the inclusion of either end is a homotopy equivalence; one wants to know whether this cobordism is trivial. If it is trivial, projection $M = M_- \times I \to I$ gives a Morse function without critical points, and conversely such a function gives $M$ the structure of a trivial cobordism.

The idea of the proof of the $h$-cobordism theorem is to take an arbitrary Morse function $f_{-\infty} : (M, M_- + M_+) \to (\mathbb{R}_+, -\infty, \infty)$, and evolve it into a critical-point-free Morse function $f_{\infty}$. This evolution can be viewed as a family of functions: $\mathbb{R}_+ \times \mathbb{R}_+ \times M \xrightarrow{\phi_x} \mathbb{R}_+$. Note that the assertion that $f_{\infty}$ has no critical points is the same as the assertion that $u_{f_{-\infty}}$ is acyclic when $x > 0$.

One way to proceed would be the following: first build a sheaf $\mathcal{F}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with $\mathcal{F}_{-\infty \times \mathbb{R}_+} = u_{f_{-\infty}}/k$ and $\mathcal{F}_{\mathbb{R}_+ \times \mathbb{R}_+}$ everywhere acyclic, then extend the function $f_{-\infty}$ to a function $f$ such that $u_{f_{-\infty}}/k \cong \mathcal{F}$.

Building the sheaf $\mathcal{F}$ is the easy part. For instance, suppose the Morse complex $Morse(M, f_{-\infty}; k)$ admits a Barrassikov normal form. This is just the same as saying that the sheaf $u_{f_{-\infty}}/k$ is the direct sum of $n$ copies of a rank one constant sheaf supported on a half-open interval, for various half-open intervals. Each such summand is the restriction to the midline of the unique microlocal rank one sheaf on the standard unknot, so we define $\mathcal{F}$ as the direct sum of $n$ copies of (the right half of) this sheaf.

It remains to find $f$. Assuming the Morse indices of all the critical points of $f_{\infty}$ satisfy $2 \leq \mu \leq \dim M - 2$, then the argument we have used to prove Theorem 5 gives such a function. This is a rephrasing of part of the original proof of the $h$-cobordism theorem; the remaining part is about how to reduce to the above situation in the first place. A careful study of this part of the proof may allow our requirement $2 \leq \mu$ to be relaxed.

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