Abstract. In this talk I review recent progress made in extracting $|V_{ub}|$, within a systematic expansion, from the cut electron energy and hadronic mass spectra of inclusive $B$ meson decays utilizing the data from radiative decays. It is shown that an extraction is possible without modeling the $B$ meson structure function. I discuss the issues involving the assumptions of local duality in various extractions. I also comment on the recent CLEO extraction of $V_{ub}$.

INTRODUCTION

It is an unfortunate fact that experimental cuts can take a nice clean theoretical prediction and turn it into a troublesome mess. A perfect example of this scenario arises in the extraction of $V_{ub}$ from inclusive $B$ decays. In principle this extraction should be straightforward. One measures the inclusive rate for semi-leptonic decays into non-charmed states and compares the result to the theoretical prediction for the total rate, which is under good theoretical control \cite{1, 2}. Of course, the snag is that there is no simple way, at least at this time, to measure the total inclusive rate to charmless states. Thus, some cut must be applied to reject the charmed final states. Perhaps the simplest choice, from an experimentalist viewpoint, is to cut on the electron energy, rejecting all events with $E_l < (m_B^2 - m_D^2)/(2m_B)$. This is the oldest method for extracting $V_{ub}$. Unfortunately, the theoretical prediction for the integrated cut spectrum is rather complicated.

The problem arises from the fact that the cut introduces a new scale into the problem. Without the cut there are only two scales of relevant physics, namely, the mass $m_B$ and the QCD scale $\Lambda_{QCD}$. Suppose we cut on a scaled kinematic variable, such as the electron energy $x_B = 2E_l/m_B$. If we cut near the endpoint, $x_B \simeq 1$, then the scale $(1 - x)m_B$ can introduce a large dimensionless ratio into the calculation $1/(1 - x)$. This can lead to power law, as well as logarithmic amplifications of what normally would be small effects.

The calculation of the total inclusive rate can be derived from first principles \cite{1} within a systematic expansion in $\alpha_s(m_b)$ and $\Lambda/m_B$ \cite{1}. The aforementioned amplifications in the cut rate arise as corrections of the form $\Lambda/(m_b(1 - x))$ and $\alpha_s \log^2(1 - x)$. Physically, the reasons for these enhancements are clear. The non-perturbative corrections arise from the fact that near the endpoint the spectrum becomes very sensitive to the Fermi motion

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1 Even in the total rate for semi-leptonic decays one still has “mild” local-duality assumptions arising from the fact that the contour approaches the real axis at a point. Thus this calculation is technically not on the same theoretical footing as deep inelastic scattering
of the heavy quark. The logarithmic corrections are due to the exclusivity of the cut rate. In the limit where $x$ approaches one there is no room for gluon radiation, thus leading to the usual infra-red divergences of the Bloch-Nordstrom type.

Of these two types of enhancements it is really the power corrections that make the extraction more difficult. The reason for this is that the effects of Fermi motion are incalculable. In the past the Fermi motion was modeled, leading to extractions of $V_{ub}$ for which it was not really possible to make meaningful theoretical error estimates. This is not to say that those extractions will not lead to a number which in the end could turn out to be "correct" or that the old error band is totally unreasonable. However, given that we are now entering the age of precision CKM measurements, the theorist is obligated to make more systematic estimates of the errors. Fortunately, there is a way around having to model the Fermi motion. The relevant point is that the effect of the Fermi motion on the decay spectra are universal. Thus, if we can extract the necessary information about the end point spectrum from one decay we can use it to make a prediction for another decay spectrum.

**THE CUT ELECTRON SPECTRUM**

It can be shown from first principles that\textsuperscript{[3, 4]}, up to corrections of order $\Lambda/m_b$, the electron energy endpoint spectrum can be written as

$$\frac{d\Gamma}{dE} = \int_{2E-m_b}^{\bar{\Lambda}} dk_+ f(k_+) \frac{d\Gamma_p(m_b^*)}{dE},$$

(1)

where $E$ is the charged lepton energy in semi-leptonic $B$ decay. $m_b^*$ is the effective mass which accounts for the residual momentum $k_+$, such that $m_b^* = m_b + k_+$ and the structure function, which accounts for the Fermi motion, is given by

$$f(k_+) = \langle B(v) \mid \bar{b}_v \delta(k_+ - iD_+) b_v \mid B(v) \rangle.$$  \hspace{1cm} (2)

A similar expression can be given for the radiative decay spectrum. Thus, in principle, one could measure one end point spectrum, deconvolve (1), and then extract $f(k_+)$ to make a prediction for the shape of another spectrum. However, this would be a rather Herculean task which I would wish on neither friend nor foe. Fortunately, this analysis can be avoided by first taking the Mellin transform of the spectrum\textsuperscript{[4]}. The point is that the Mellin transform turns the convolution in (1) into a product. Thus we may remove all dependence on the structure function\textsuperscript{[3]} by first taking the ratio of the moments of two spectra and then taking the inverse Mellin transform, just as we do in relating deep-inelastic scattering to Drell-Yan processes. This is exactly what was done for the electron energy spectrum in ref. \textsuperscript{[7, 8]}. In this reference it was shown that following this procedure $|V_{ub}|^2/|V_{ts}^*V_{tb}|^2$ may be extracted from the relation \textsuperscript{[7, 8]}

\textsuperscript{2} At tree level it can also be avoided without taking moments\textsuperscript{[3]}.
\[
\frac{|V_{ub}|^2}{|V_{ts}V_{tb}|^2} = \frac{3 \alpha C_7 \langle m_b \rangle^2}{\pi} (1 + H_{\text{mix}}^\gamma) \int_{x_B^0}^1 dx_B \frac{d\Gamma}{dx_B} \times \left\{ \int_{x_B^0}^1 du_B W(u_B) \frac{d\Gamma}{du_B} \right\}^{-1}, \quad (3)
\]

\( H_{\text{mix}}^\gamma \) represents the corrections due to interference coming from the operators \( O_2 \) and \( O_8 \) [5]

\[
H_{\text{mix}}^\gamma = \frac{\alpha_s \langle m_b \rangle}{2 \pi C_7(0)} \left[ C_7^{(1)} + C_2^{(0)} \mathcal{R}(r_2) + C_8^{(0)} \left( \frac{44}{9} - \frac{8 \pi^2}{27} \right) \right], \quad (4)
\]

and \( x_B^0 \) is the value of the cut. In Eq. (4), all the Wilson coefficients, evaluated at \( m_b \), are “effective” as defined in [8], and \( \mathcal{R}(r_2) \approx -4.092 + 12.78(m_c/m_b - 0.29) \) [10].

The numerical values of the Wilson coefficients are: \( C_2^{(0)}(m_b) \approx 1.11 \), \( C_7^{(0)}(m_b) \approx -0.31 \), \( C_7^{(1)}(m_b) \approx 0.48 \), and \( C_8^{(0)}(m_b) \approx -0.15 \). The diagonal pieces from \( O_2 \) and \( O_8 \) are numerically insignificant. The function \( W(u_B) \) is given by

\[
W[u_B] = u_B^2 \int_{x_B^0}^{u_B} dx_B \left( 1 - 3(1 - x_B)^2 + \frac{\alpha_s}{\pi} \left( \frac{7}{2} - \frac{2 \pi^2}{9} - \frac{10}{9} \log(1 - \frac{x_B}{u_B}) \right) \right). \quad (5)
\]

The End Point Logs

Over the years there has been a concern over the perturbative part of the calculation discussed above [11]. This concern arises due to the fact that as the cut approaches the end-point, the perturbative corrections grow. This growth is due to the so called “Sudakov Logs”, \( \alpha_s \log^2(1 - x_c) \). These logs appear are a consequence of the fact that near the end-point there is limited phase space available for radiation, and the fact is that you can’t stop radiation; you can only hope to contain it. Indeed, it is well known that if one sums up these double logarithms, the rate dies off as \( \Gamma \propto \exp[-\alpha_s \log^2(1 - x_c)] \).

The above mentioned concern, was that perhaps this exponentiation, and moreover, the exponentiation of sub-leading single logs would endanger the convergence of the perturbative expansion. Indeed, if we assume that the logs dominate the expansion at each order in perturbation theory, then we really should reorganize the calculation in terms of an expansion in the exponent. In fact, in moment space it was shown that [8, 12] the perturbative expansion could be written as

\[
\frac{d\Gamma}{dN} \propto \exp(\log(N) f(\alpha_s \log(N)) + g(\alpha_s \log(N)) + \alpha_s h(\log(N)) + \ldots). \quad (6)
\]

Thus the question of the convergence of the perturbative expansion becomes one of the nature of the series in the exponent. Not only should the series in the exponent converge (at least in an asymptotic sense), but it had also better be that the first term dropped in the exponent is much less than one. In [7], the logs were resummed including the subleading function \( g \). There it was found that, after taking the ratio of the semi-leptonic and radiative decay rates, the net effect of resummation was only at 10 − 15% level. Part of the reason for the smallness of this effect is that the \( f \) functions for the two
processes are identical, therefore the leading double logs cancel in the ratio. The closed form expression for $V_{ub}$ including the resummation is given by the same expression as above (3), with a new expression for the function $W$,

$$W[u_B] = u_B^2 \int_{x_B^f}^{u_B} K \left[ x_B, \frac{4}{3\pi \beta_0} \log \left( 1 - \alpha_s \beta_0 l_{x_B/u_B} \right) \right].$$

(7)

$$K(x; y) = 6 \left\{ \left[ 1 + \frac{4\alpha_s}{3\pi} \left( 1 - \psi^{(1)}(4 + y) \right) \right] \frac{1}{(y + 2)(y + 3)} - \frac{\alpha_s}{3\pi} \left[ \frac{1}{(y + 2)^2} - \frac{7}{(y + 3)^2} \right] - \frac{4\alpha_s}{3\pi} \left[ \frac{1}{(y + 2)^3} - \frac{1}{(y + 3)^3} \right] \right\} - 3(1 - x)^2.$$  

(8)

The function $K$ has a non-integrable singularity in it due the fact that the sum in the exponent should be considered in terms of a prescription for a non-Borel summable series. Due to its asymptotic nature it can be well approximated by expanding the argument of the Log in a series and keeping as many terms as one wishes until the series starts to diverge. An excellent approximation for $W$ is given by expanding the second argument of $K$:

$$\frac{4}{3\pi \beta_0} \log \left( 1 - \alpha_s \beta_0 l_{x_B/u_B} \right) \approx \alpha_s \frac{4}{3\pi} \log \left( 1 - x_B/u_B \right) - \frac{\alpha_s^2}{18\pi^2} \log^2 \left( 1 - x_B/u_B \right).$$

(9)

While the use of this improved prediction may shift the central value slightly, it will not change the error bars.

**THE HADRONIC MASS CUT**

It is also possible to remove the background from charmed transitions by cutting on the hadronic invariant mass. While this choice presents a greater experimental challenge, it benefits from the fact that, unlike the electron spectrum, most of the $B \rightarrow X_{se} \nu$ decays are expected to lie within the region $s_H < M_D^2$. Furthermore, it is believed that even though both the invariant mass region $s_H < M_D^2$ and electron energy regions $M_B/2 > E_e > (M_B^2 - M_D^2)/(2M_B)$ receive contributions from hadronic final states with invariant mass up to $M_D$, the cut mass spectrum will be less sensitive to local duality violations. This belief rests on the fact that the contribution of large mass states is kinematically suppressed for the electron energy spectrum in the region of interest. The expression for $V_{ub}$ for the case of the hadronic mass cut is given by

$$\frac{|V_{ub}|^2}{|V_{tx}|^2} = \frac{6\alpha C_7 (m_b)^2 (1 + H_{mix}^2) \delta \Gamma(c)}{\pi [I_0(c) + I_+(c)]}.$$  

(10)

The expressions for $\Gamma(c)$, $I_0(c)$ and $I_+(c)$ can be found in [14]. The leading errors in this expression are again of order $\Lambda/m_b$. The effect of resummation of the end-point logs in this case was shown to be negligible [8].
Some Caveats and Concluding Remarks

We have emphasized the fact that we are now capable of extracting $|V_{ub}|$ in a systematic fashion. Which is to say that we have our errors under control. The leading errors are of order $\Lambda/m_b$ and $\alpha_s(1-x)$. However, strictly speaking these are really the only errors we know how to quantify. As I emphasized the calculation is based upon certain assumptions about local duality. We don’t know how to quantify these errors, all we know is when we do and do not expect local duality to be a valid assumption. Basically, the point is that we expect hadronic observables to well approximated in terms of partonic calculations when we are able to smear over resonances in a “sufficient manner” [1]. If the final state phase space is so restrictive that we only average over one or two resonances then, we begin to worry that our assumption is not well founded. It is interesting to note that in the charmed decay, the inclusive rate is saturated up to corrections of order $\Lambda^2/m^2$ by including only the $D$ and $D^*$ in the final states[15] in the small velocity limit [16]. Naively, one would think that pion emission would only be suppressed by $p_\pi/f_\pi$, but in the zero recoil limit the currents become generators of the effective theory and can only produce linear combinations of the $D$ and $D^*$. While this is tantalizing, and gives one hope that local duality will work well even with small numbers of resonances, this case may be considered special.

None the less, the best choice of cuts, as far as local duality is concerned, is the hadronic mass, as it is believed to contain approximately 80% of the total rate, and if duality is going to work somewhere this is the place. Indeed, given that this is a relatively large percentage of the rate, compared to those expected in the electron energy cut $\simeq 20\%$ or leptonic mass cut $\simeq 20\%$ (which does need the radiative decay data [17], [18], [19]), we may hope to actually test the duality assumption. This may be done by simply varying the cut and seeing if the extracted value of $V_{ub}$ remains fixed.

Let me pause at this point to make a few comments regarding some issues which were raised during the conference. First of all, it is often asked, “if we know that the hadronic mass cut really does include 80% of the rate, can’t we get a relatively good extraction of $V_{ub}$ without care about the radiative decays?”. The answer to this is that we really don’t know that the hadronic mass cut captures 80% of the rate. That number is a rough estimate which is made using an inspired guess for the wave function. The purpose of the number is simply to get a feeling for relative merits of the various cuts, but since the whole point of the exercise is to eliminate model dependence, these percentages should not be taken too seriously. Secondly I would like to make a few comments about the recent CLEO extraction, as discussed in Roy Brieres’ talk at this conference [20]. As I understand it, the extraction was performed by using a guess for the structure function, and varying the parameters of the model until a fit to the radiative decay spectrum was found. This fit was then used to make a prediction for the cut rate in the semi-leptonic decay. The robustness of this method was then “tested” by using two different parameterizations for the structure function. Now, I believe this is not the best way to do things. There are, no doubt, a large number of parameterizations for the structure

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3 Mathematically this assumption boils down to the fact that we are doing an operator product expansion in a kinematic region where its not really justified.
function which could fit the radiative decay data, and given that the function with which this parameterization is convoluted differs in the two different decay modes, there is no reason that the results should necessarily be identical in any parameterization. Of course, even in the method I described in this talk one still needs to fit the radiative decay data, and this may indeed be well fit by using a model. The point is that in the method described above, all that is need is the physical spectrum, it doesn’t matter what model you use to fit the data. In the end it may well indeed be that both methods give the same answer, but the present CLEO extraction as it stands, is not justified from first principles.

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