CONJECTURES ON THE COHOMOLOGY OF THE GRASSMANNIAN

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Abstract. We give a series of successively weaker conjectures on the cohomology ring of the Grassmannian, starting with the Hilbert series of a certain natural filtration.

1. Introduction

Let $G(k, C^{k+\ell})$ denote the Grassmannian of $k$-planes in $C^{k+\ell}$. We pose a series of successively weaker conjectures on its cohomology ring with rational coefficients. The first (Conjecture 1) conjectures the Hilbert series for a natural filtration of the ring. The last (Conjecture 4) would greatly simplify the proof of a result by Hoffman (Theorem 5 below) on the classification of its graded endomorphisms.

We denote this cohomology ring $R^{k,\ell} := H^\ast(G(k, C^{k+\ell}), \mathbb{Q})$. As a graded $\mathbb{Q}$-algebra, $R^{k,\ell}$ has several natural descriptions (see [1], [3]):

(i) $R^{k,\ell} \cong \mathbb{Q}[e_1, e_2, \ldots, e_k, h_1, h_2, \ldots, h_\ell] / \left( \sum_{i+j=d} (-1)^i e_i h_j : d = 1, 2, \ldots, k+\ell \right)$.

(ii) $R^{k,\ell} \cong \mathbb{Q}[e_1, e_2, \ldots, e_k] / (h_{\ell+1}, h_{\ell+2}, \ldots, h_{\ell+k})$

where here $h_r$ is interpreted as the Jacobi-Trudi determinant

$$h_r := \begin{vmatrix} e_1 & e_2 & e_3 & \cdots \\ 1 & e_1 & e_2 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & e_1 \end{vmatrix}$$

(iii) $R^{k,\ell}$ has $\mathbb{Q}$-basis given by the Schur functions $s_\lambda$ for partitions $\lambda$ whose Ferrers diagram fit inside the $k \times \ell$ box $\ell^k$ (written $\lambda \subset \ell^k$), with multiplication given by the usual Littlewood-Richardson structure constants

$$s_\lambda s_\mu = \sum_{\nu \subset \ell^k} c_{\lambda,\mu}^{\nu} s_\nu.$$

Note the truncation to only those partitions $\nu$ satisfying $\nu \subset \ell^k$.

We give here a brief explanation of these various descriptions and the connections between them. The cohomology ring $R^{k,\ell}$ is generated by the Chern classes of the tautological $k$-vector bundle $\mathcal{E}$ on $G(k, C^{k+\ell})$. In description (i), $e_i$ represents $(-1)^i$ times the $i^{th}$ Chern class $c_i(\mathcal{E})$. Here $h_j$...
represents the $j^{th}$ Chern class $c_j(\mathcal{F})$ for the quotient $\ell$-vector bundle $\mathcal{F}$ that fits into the following exact sequence of bundles on $G(k, \mathbb{C}^{k+\ell})$:

$$0 \to \mathcal{E} \to \mathbb{C}^{k+\ell} \to \mathcal{F} \to 0$$

in which the middle term a trivial $k+\ell$-vector bundle. In description (ii), the $e_i$ are as in description (i). One has simply used the relations $\sum_{i+j=d}(-1)^i e_i h_j$ for $d = 1, \ldots, \ell$ to eliminate the redundant generators $h_1, h_2, \ldots, h_\ell$, while for $d = \ell + 1, \ldots, k + \ell$, these relations turn into the Jacobi-Trudi determinants $h_{\ell+1}, \ldots, h_{k+\ell}$. In description (iii), $s_\lambda$ represents the cohomology class dual to a Schubert variety; here $e_i$ corresponds to $s_{1i}$, while $h_j$ corresponds to $s_j$. More generally, $s_\lambda$ can be expressed via another Jacobi-Trudi determinant in terms of the $e_i$'s (or in terms of the $h_j$'s).

In what follows, we will often abuse notation by not distinguishing the indeterminate $e_i$ in the polynomial ring $\mathbb{Q}[e_1, \ldots, e_k]$ from its image in the quotient ring $R^{k,\ell}$.

We further recall the classical expression for the Hilbert series

$$\text{Hilb}(R^{k,\ell}, q) := \sum_{d \geq 0} \dim_{\mathbb{Q}} R^{k,\ell}_d q^d$$

$$= \left[ \begin{array}{c} k + \ell \\ k \end{array} \right]_q = \frac{(1-q)(1-q^2) \cdots (1-q^{k+\ell})}{(1-q)(1-q^2) \cdots (1-q^k)(1-q^2) \cdots (1-q^\ell)}.$$

The first of our conjectures gives the Hilbert series for a certain natural filtration of $R^{k,\ell}$. For each $m = 0, 1, \ldots, k$, let $R^{k,\ell,m}$ be the subalgebra of $R^{k,\ell}$ generated by $e_1, e_2, \ldots, e_m$, which can described alternatively as the subalgebra generated by all elements of degree at most $m$. Thus $\mathbb{Q} = R^{k,\ell,0} \subset R^{k,\ell,1} \subset \cdots \subset R^{k,\ell,k} = R^{k,\ell}$.

**Conjecture 1.** For $m \geq 0$,

$$\text{Hilb}(R^{k,\ell,m}, q) = 1 + \sum_{i=1}^{m} q^i \left[ \begin{array}{c} \ell \\ i \end{array} \right]_q \left( \sum_{j=0}^{k-i} q^{j(j+1)} \left[ \begin{array}{c} k-i+1 \\ j \end{array} \right]_q \right).$$

**Conjecture 2.** For $1 \leq m \leq k$,

$$R^{k,\ell}_d = R^{k,\ell,m-1}_d \text{ for } d \geq k\ell - m^2 + m + 1.$$

**Conjecture 3.** For $1 \leq m \leq k$

$$e_m e_1^{k\ell-m^2+1} \in R^{k,\ell,m-1}.$$

**Conjecture 4.** For $3 \leq m \leq k$

$$e_m e_1^{k\ell-2m} \in R^{k,\ell,m-1}.$$

We introduce some notation for the sake of stating a result that motivated this last conjecture. Given a graded $\mathbb{Q}$-algebra $R = \oplus_i R_i$ and $\alpha \in \mathbb{Q}$, define a graded endomorphism $\phi_\alpha$ by $\phi_\alpha(x) = \alpha^{\deg(x)} x$ for every homogeneous element $x \in R$. In $\mathbb{Q}$, the endomorphism $\phi_\alpha$ on $R^{k,\ell}$ for $\alpha \in \mathbb{Z}$ is called an Adams map.

Note that when $k = \ell$, there is a non-trivial automorphism $\omega : R^{k,\ell} \to R^{k,\ell}$ (induced from the fundamental involution on symmetric functions), exchanging $e_i \leftrightarrow h_i$ in description (i) above, and more generally exchanging $s_\lambda \leftrightarrow s_{\lambda'}$ in description (iii). Here $\lambda'$ denotes the partition conjugate to $\lambda$.

**Theorem 5.** (Hoffman [4]) For $k \neq \ell$, every graded algebra endomorphism $\phi : R^{k,\ell} \to R^{k,\ell}$ which does not annihilate $R^{k,\ell}_1$ is of the form $\phi_\alpha$ for some $\alpha \in \mathbb{Q}^\times$.

For $k = \ell$, any such endomorphism is either of the form $\phi_\alpha$ or $\omega \circ \phi_\alpha$. 
Theorem 5 was conjectured by O’Neill (see [5] or [1]), who conjectured it more generally without the assumption that φ is non-zero on $R^{k,l}_1$. O’Neill’s more general conjecture is trivial for $k = 1$ (or $\ell = 1$), proven for $k = 2$ in [5], and proven for $\ell \geq 2k^2 - k - 1$ in [1]. It is motivated by the fact that, assuming it, one can fairly easily deduce, via the Lefschetz fixed point theorem, that the Grassmannian $G(k, \mathbb{C}^{k+l})$ has the fixed point property (i.e. every continuous self-map has a fixed point) if and only if $k\ell$ is odd.

In Section 2, we quickly explain the easy implications

\[ \text{Conj 1} \Rightarrow \text{Conj 2} \Rightarrow \text{Conj 3} \Rightarrow \text{Conj 4} \]

In Section 3, we verify Conjecture 1 in the relatively easy boundary cases $m = 1$ and $m = k$.

In Sections 4 and 5, we explain how Conjecture 4 would imply Theorem 5. This proof uses some of the same ideas as Hoffman’s, namely the Hard Lefschetz Theorem and the hook-length formula for counting tableaux, but is much shorter (2 pages versus 10 pages).

2. The implications Conj 1 $\Rightarrow$ Conj 2 $\Rightarrow$ Conj 3 $\Rightarrow$ Conj 4

To see that Conjecture 1 implies Conjecture 2, note that Conjecture 1 is equivalent to the following assertion: for $p \geq 1$, the quotient (graded) vector space $R^{k,l,p}/R^{k,l,p-1}$ has Hilbert series

\begin{equation}
\text{Hilb}(R^{k,l,p}/R^{k,l,p-1}, q) = \sum_{j=0}^{k-p} q^j \binom{p+1}{j} \cdot q^{(\ell - p + 1)j + j} \cdot q^p \cdot \binom{\ell}{p}.\end{equation}

It suffices to show that the right-hand side has degree in $q$ at most $k\ell - m^2 + m$ whenever $p \geq m \geq 1$. Since the $q$-binomial coefficient $\binom{r+s}{r}_q$ has degree $rs$ as a polynomial in $q$, the right-hand side of (2.1) has degree in $q$ equal to

\[
\max_{j=0,1,...,k-p} \{j(\ell - p + 1) + p + (p - 1)j + p(\ell - p)\} = \max_{j=0,1,...,k-p} \{\ell(j + p) - p^2 + p\} = k\ell - p^2 + p,
\]

and this is bounded above by $k\ell - m^2 + m$ for $p \geq m \geq 1$.

Conjecture 2 implies Conjecture 3 trivially, since $e_me_1^{k\ell-m^2+1}$ lies in $R^{k,l}_{k\ell-m^2+m+1}$. Similarly, Conjecture 4 trivially implies Conjecture 5 since for $m \geq 3$, one has

\[k\ell - m^2 + 1 \leq k\ell - 2m.\]

3. Boundary cases for Conjecture 1

Both in checking the boundary case $m = 1$ of Conjecture 1 and in later showing that Conjecture 4 implies Conjecture 5, we will make use of the fact that $R^{k,l}$ satisfies the *Hard Lefschetz Theorem* [2, p. 122]:

**Theorem 6.** For $i = 0, 1, \ldots, \lfloor \frac{kl}{2} \rfloor$, the map

\[R_i^{k,l} \to R_{k\ell-i}^{k,l}\]

given by multiplication by $e_1^{k\ell-2i}$ is a $\mathbb{Q}$-vector space isomorphism.
To check the case \( m = 1 \) in Conjecture \( \text{1} \) note that taking \( i = 0 \) of Theorem \( \text{6} \) tells us that the smallest power \( e_1^k \) which vanishes in \( R^k,\ell \) is \( e_1^{k\ell+1} \), i.e. that \( R^k,\ell,1 = \mathbb{Q}[e_1]/(e_1^{k\ell+1}) \). Hence
\[
\text{Hilb}(R^k,\ell,1, q) = \text{Hilb}(\mathbb{Q}[e_1]/(e_1^{k\ell+1}), q) = \frac{1 - q^{k\ell+1}}{1 - q}.
\]

Meanwhile, Conjecture \( \text{1} \) for \( m = 1 \) predicts
\[
\text{Hilb}(R^k,\ell,1, q) = 1 + \sum_{j=0}^{k-1} q^{j\ell+1} \left[ \frac{\ell}{j} \right] = \frac{1 - q^{k\ell+1}}{1 - q},
\]
so the two agree.

**Remark 7.**
Instead of the Hard Lefschetz Theorem here, we could have used Pieri’s formula \([3, \S 2.2]\) to conclude that \( e_1^{k\ell} = f_{\ell k} s_{\ell k} \neq 0 \) where \( f_\lambda \) denotes the number of standard Young tableaux of shape \( \lambda \).

The case \( m = k \) of Conjecture \( \text{1} \) is clearly equivalent to the following identity.

**Proposition 8.**
\[
\left[ \frac{k + \ell}{k} \right]_q = 1 + \sum_{i=1}^{k} \sum_{j=0}^{k-i} q^j \left[ \frac{\ell}{i} \right]_q q^{j(\ell-i+1)} \left[ \frac{i + j - 1}{j} \right]_q.
\]

**Proof.** Interpret \( \left[ \frac{k + \ell}{k} \right]_q \) as the generating function counting partitions \( \lambda \) whose Ferrers diagram fits inside the rectangle \( \ell^k \) according to their weight. To prove the above identity, whenever \( \lambda \) is non-empty, we will uniquely define two integers \( i, j \) and decompose its diagram into four portions:

(a) a \( j \times (\ell - i + 1) \) rectangle, accounting for the \( q^j(\ell-i+1) \),

(b) a column of length \( i \), accounting for the \( q^i \),

(c) a partition whose Ferrers diagram fits inside a \( j \times (i-1) \) rectangle, and

(d) a partition whose Ferrers diagram fits inside a \( \ell \times (\ell - i) \) rectangle.

The decomposition is illustrated in Figure \( \text{3} \) where Ferrers diagrams are depicted using the French notation. Given a non-empty \( \lambda \), say with \( s \) parts, let \( \bar{\lambda} \) be the partition complementary to \( (\lambda_1, \ldots, \lambda_{s-1}) \) inside a \( (s-1) \times \ell \) rectangle, and let \( i - 1 \) be the size of the Durfee square of \( \lambda \), that is, the largest square Ferrers diagram contained within that of \( \bar{\lambda} \). Then set \( j = s - i \). The \( j \times (\ell - i + 1) \) rectangle in (a) is the one inside \( \lambda \) in the lower left. The column of length \( i \) in (b) lies just above it in column 1. The remaining cells of \( \lambda \) then segregate into two Ferrers diagrams, one inside a \( j \times (i - 1) \) rectangle (as in (c)) in the lower right, the other inside a \( \ell \times (\ell - i) \) rectangle (as in (d)) in the upper left. \( \square \)

**Remark 9.**
It may be that a more useful phrasing for Conjecture \( \text{1} \) is to rewrite the inner sum in the following form, which one can show is equivalent:
\[
\text{Hilb}(R^k,\ell,m, q) = 1 + \sum_{i=1}^{m} q^i \left[ \frac{\ell}{i} \right] f_{i}^{k,\ell}(q)
\]
where \( f_{i}^{k,\ell}(q) \) for \( 0 \leq i \leq k \leq \ell \) is defined by the following recurrence
\[
f_{i}^{k,\ell}(q) = f_{i-1}^{k-1,\ell}(q) + q^{\ell-i+1} f_{i}^{k-1,\ell}(q)
\]
with initial conditions \( f_{k,0}^k(q) = f_{k,k}^k(q) = 1 \). In other words, \( f_{i,j}^{k,l}(q) \) is a \( q \)-analogue of the binomial coefficient \( \binom{k}{i} \) which depends on \( \ell \) also, and satisfies a different \( q \)-Pascal recurrence than the usual one for the \( q \)-analogue \( \left[ \begin{array}{c} k \\ i \\ q \end{array} \right] \).

4. **Conjecture** \(^4\)** implies **Theorem** \(^5\)**

Let \( \phi : R^{k,\ell} \rightarrow R^{k,\ell} \) be a graded algebra endomorphism which does not annihilate \( R_1^{k,\ell} \). Since \( R_1^{k,\ell} \) is a 1-dimensional \( \mathbb{Q} \)-vector space, \( \phi \) acts on \( R_1^{k,\ell} \) by some constant \( \alpha \in \mathbb{Q}^\times \). Recall that \( \phi_\alpha \) is the endomorphism that sends a homogeneous element \( x \) in \( R_d^{k,\ell} \) to \( \alpha^d x \). It is our goal in this (and the next) section to show that (assuming Conjecture \(^4\))

- if \( k \neq \ell \) then \( \phi = \phi_\alpha \), and
- if \( k = \ell \) either \( \phi = \phi_\alpha \) or \( \phi = \omega \circ \phi_\alpha \).

Note that \( \phi_{\alpha^{-1}} = \phi_\alpha^{-1} \), and so by replacing \( \phi \) with the composite \( \phi \circ \phi_{\alpha^{-1}} \), we may assume without loss of generality that \( \alpha = 1 \), i.e. \( \phi \) acts as the identity on \( R_1^{k,\ell} \).

Our goal will be to try and show by induction on \( m \) that \( \phi \) does what we expect to \( e_1, e_2, \ldots, e_m \) for every \( m \leq k \). The case \( m = 2 \) is summarized in the following lemma, whose proof requires ad hoc argumentation which we defer to Section 5.

**Lemma 10.** Let \( \phi : R^{k,\ell} \rightarrow R^{k,\ell} \) be a graded algebra endomorphism which acts as the identity on \( R_1^{k,\ell} \).

- If \( k \neq \ell \), then \( \phi(e_2) = e_2 \), i.e. \( \phi \) acts the identity on \( R_2^{k,\ell} \).
- If \( k = \ell \), then \( \phi(e_2) \) is either \( e_2 \) or \( h_2 \), i.e. \( \phi \) acts either as the identity or \( \omega \) on \( R_2^{k,\ell} \).

Assuming this lemma for the moment, we show how Conjecture \(^4\) allows one to do the remaining inductive steps for \( m \geq 3 \) to deduce Conjecture \(^5\).

Given \( m \geq 3 \), using Lemma 10 we may assume without loss of generality (by composing \( \phi \) with \( \omega \) if necessary when \( k = \ell \)), that \( \phi(e_i) = e_i \) for \( i = 1, 2, \ldots, m - 1 \) by induction on \( m \). In other words, \( \phi \) acts as the identity on the subalgebra \( R^{k,\ell,m-1} \). We wish to show that this implies \( \phi(e_m) = e_m \).

Let \( \mathcal{P}_{k,\ell}(m) \) denote the set of all partitions \( \lambda \) of \( m \) whose Ferrers diagram fits inside the rectangle \( \ell^k \). Note that for all but one such \( \lambda \), namely \( \lambda = (m) \) having a single part, the elementary symmetric function \( e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k} \) lies in the subalgebra \( R^{k,\ell,m-1} \), and hence so does the
product \( e_\lambda e_1^{k\ell-2m} \). Because of Conjecture 4 the same is true for \( \lambda = 1^m \), i.e.

\[
e_{(m)} e_1^{k\ell-2m} = e_m e_1^{k\ell-2m} \in R_{k,\ell,m-1}.
\]

This is the only place where we will use our assumption of Conjecture 4.

Hence \( \phi \) fixes \( e_\lambda e_1^{k\ell-2m} \) for every \( \lambda \in \mathcal{P}_{k,\ell}(m) \). From degree considerations, there exists constants \( b_\lambda \in \mathbb{Q} \) for \( \lambda \in \mathcal{P}_{k,\ell}(m) \) (whose exact values are not important) such that

\[
e_m e_\lambda e_1^{k\ell-2m} = b_\lambda s_{\ell k}.
\]

Let

\[
\phi(e_m) = \sum_{\mu \in \mathcal{P}_{k,\ell}(m)} x_\mu s_\mu
\]

for some constants \( x_\mu \in \mathbb{Q} \). We hope to show that \( x_\mu = \delta_{\mu,(m)} \), so that \( \phi \) fixes \( e_m \). Applying \( \phi \) to (4.1) (and noting that \( \phi \) also fixes \( s_{\ell k} \) because it lies in \( R_{k,\ell,1} \)) yields the system of equations

\[
\sum_{\mu \in \mathcal{P}_{k,\ell}(m)} x_\mu s_\mu e_\lambda e_1^{k\ell-2m} = b_\lambda s_{\ell k} \quad \text{for} \quad \lambda \in \mathcal{P}_{k,\ell}(m).
\]

This system can be written in matrix form as \( Ax = b \), where \( A = (a_{\lambda,\mu})_{\lambda,\mu \in \mathcal{P}_{k,\ell}(m)} \), and \( a_{\lambda,\mu} \) is the coefficient of \( s_{\ell k} \) when one expands \( s_\mu e_\lambda e_1^{k\ell-2m} \) when \( m \geq 3 \), and leaves only the proof of Lemma 10 remaining.

**Remark 11.** We note a further consequence of the Hard Lefschetz Theorem 6 here. It implies the coincidence of \( \mathbb{Q} \)-bases in the domain, range:

- In the domain, the basis \( \{ e_\lambda \}_{\lambda \in \mathcal{P}_{k,\ell}(m)} \) for \( R_{k,\ell}^m \). This is a basis because it is upper-triangularly related to the usual basis \( \{ s_\lambda \}_{\lambda \in \mathcal{P}_{k,\ell}(m)} \).
- In the range, the basis for \( R_{k,\ell-m}^k \) which is Poincaré dual to the basis \( \{ s_\mu \}_{\mu \in \mathcal{P}_{k,\ell}(m)} \) for \( R_{k}^{\ell,m} \).

This completes the inductive step when \( m \geq 3 \), and leaves only the proof of Lemma 10 remaining.
5. Proof of Lemma 10

We recall the statement of the lemma.

**Lemma 10.** Let \( \phi : R_{1}^{k, \ell} \rightarrow R_{2}^{k, \ell} \) be a graded algebra endomorphism which acts as the identity on \( R_{1}^{k, \ell} \).

If \( k \neq \ell \), then \( \phi(e_{2}) = e_{2} \), i.e. \( \phi \) acts the identity on \( R_{2}^{k, \ell} \).

If \( k = \ell \), then \( \phi(e_{2}) \) is either \( e_{2} \) or \( h_{2} \), i.e. \( \phi \) acts either as the identity or \( \omega \) on \( R_{2}^{k, \ell} \).

**Proof.** Assume that \( \phi \) acts as the identity on \( R_{1}^{k, \ell} \), so \( \phi(e_{1}) = e_{1} \), and let \( \phi(e_{2}) = xe_{2} + ye_{1}^{2} \) for two unknown constants \( x, y \in \mathbb{Q} \).

From degree considerations, there are constants \( \gamma_{r} \in \mathbb{Q} \) for \( r = 0, 1, 2, \ldots \) satisfying

\[
(5.1) \quad e_{1}^{r}e_{1}^{k\ell - 2r} = \gamma_{r}s_{\ell k}.
\]

For small values of \( r \) one can use the Pieri formula [§2.2] to obtain explicit formulae for these constants:

\[
\begin{align*}
\gamma_{0} &= f_{\emptyset}; \\
\gamma_{1} &= f_{11}; \\
\gamma_{2} &= f_{1111} + f_{2111} + f_{22}; \\
\gamma_{3} &= f_{111111} + 2f_{211111} + 3f_{221111} + f_{222} + 3f_{311111} + 2f_{32111} + f_{33}.
\end{align*}
\]

where here \( \lambda^{c} \) denotes the partition whose Ferrers diagram is the complement of that of \( \lambda \) within the rectangle \( \ell k \), and we recall that \( f_{\mu} \) denotes number of standard Young tableaux of shape \( \mu \).

One can apply the endomorphism \( \phi \) to the equations \((5.1)\), (recalling that \( \phi \) fixes both \( e_{1} \) and \( s_{\ell k} \) because they lie in \( R^{(k, \ell, 1)} \)), expand using the expression for \( \phi(e_{2}) \), and then divide both sides by \( \gamma_{0} \), to obtain the following equations for \( r = 0, 1, 2, \ldots \) in the unknowns \( x, y \):

\[
(5.2) \quad \sum_{i=0}^{r} \binom{r}{i} \frac{\gamma_{i}}{\gamma_{0}} x^{i} y^{r-i} = \frac{\gamma_{r}}{\gamma_{0}}.
\]

Here the celebrated hook-length formula [§4.3] for \( f_{\lambda} \) comes to the rescue: for \( \lambda \) a partition of \( n \), one has

\[
\begin{align*}
f_{\lambda} &= \frac{n!}{\prod_{x \in \lambda} h(x)};
\end{align*}
\]

where the product is taken over all cells \( x \) in the Ferrers diagram for \( \lambda \), and \( h(x) \) is the hooklength at cell \( x \), that is, the number of cells in the diagram that are either weakly to the right of \( x \) in the same row or weakly below it in the same column. Using this formula, the constants \( \frac{\gamma_{r}}{\gamma_{0}} \) for small values of \( r \) can be explicitly computed as rational functions of \( k, \ell \), with relatively small numerators and denominators; we omit these formulae here.

With the aid of these formulae (and some help from computer algebra packages), one can check that the only simultaneous solutions to the two equations \( (5.2) \) for \( r = 1, 2 \) are

(i) \((x, y) = (1, 0)\), and

(ii) \((x, y) = (-1, \frac{k-1}{k-1} + 1)\).

Solution (i) gives \( \phi(e_{2}) = e_{2} \), which is a scenario that we hoped for. Substituting solution (ii) into \( (5.2) \) for \( r = 3 \) gives

\[
\frac{(\ell + 1)(k - 1)(k + 1)(\ell - 1)(k\ell + 5)(k - \ell)}{(k\ell - 1)^{2}(k\ell - 2)(k\ell - 3)(k\ell - 4)(k\ell - 5)} = 0.
\]

Since we may assume without loss of generality that \( k, \ell \geq 2 \), this forces \( k = \ell \). In this case, solution (ii) becomes \((x, y) = (-1, 1)\), which means \( \phi(e_{2}) = -e_{1}^{2} + e_{2} = h_{2} \), as desired.

\[\square\]
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