NEW SPIN FOAM MODELS OF QUANTUM GRAVITY

A. MIKOVIĆ

Departamento de Matemática
Universidade Lusófona de Humanidades e Tecnologias
Av. do Campo Grande, 376, 1749-024 Lisbon, Portugal
E-mail: amikovic@ulusofona.pt

We give a brief and a critical review of the Barret-Crane spin foam models of quantum gravity. Then we describe two new spin foam models which are obtained by direct quantization of General Relativity and do not have some of the drawbacks of the Barret-Crane models. These are the model of spin foam invariants for the embedded spin networks in loop quantum gravity and the spin foam model based on the integration of the tetrads in the path integral for the Palatini action.

1. Introduction

The spin foam models of quantum gravity represent a way to define the path-integral for General Relativity in the Cartan formalism, i.e. instead of using the four-metric $g_{\mu\nu}$ as the basic variable, one uses the tetrad one-forms $e^a_\mu dx^\mu$ and the spin connection one-forms $\omega^a_\mu dx^\mu$. The Einstein-Hilbert action becomes the Palatini action

$$ S = \int \epsilon^{abcd} e^a_\mu \wedge e^b_\nu \wedge R^{cd}, $$

where $R_{ab} = d\omega_{ab} + \omega^c_a \wedge \omega^a_c$, so that one has to define the path-integral

$$ Z = \int De D\omega e^{i \int \epsilon^{abcd} e^a_\mu \wedge e^b_\nu \wedge R^{cd}}. $$

Notice that if one introduces a two-form

$$ B_{ab} = \epsilon_{abcd} e^c_\mu \wedge e^d_\nu, $$

then the Palatini action can be rewritten as the $SO(3,1)$ BF theory action

$$ S = \int_M Tr(B \wedge F). $$

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where $F = dA + A \wedge A$ and $A = \omega$. The BF action defines a topological theory, so that in order to obtain General Relativity, one needs to impose the constraint (3). Therefore one may try to define the GR path integral by using the BF theory path integral and then constraining it, which was the strategy adopted by Barrett and Crane [1, 2].

The BF theory path integral can be defined as a sum over the irreducible representations (irreps) of the BF theory Lie group $G$ of the amplitudes constructed by labeling the faces of the dual 2-complex of a triangulation of the manifold $M$ with these irreps [3, 4]. One can arrive to this definition by starting from

$$Z = \int \mathcal{D}A \mathcal{D}B \exp \left( i \int_M Tr(B \wedge F) \right)$$

$$= \int \prod_l dA_l \prod_\Delta dB_\Delta \exp \left( i \sum_f Tr(B_\Delta F_f) \right),$$

(5)

where $l$ and $f$ are the edges and the faces of the dual two-complex $F$ for the simplicial complex $T(M)$, while $\Delta$ are the triangles of $T$. The variables $A_l$ and $B_\Delta$ are defined as $\int_l A$ and $\int_\Delta B$ respectively, while $F_f = \int_f F$.

By performing the $B$ integrations one obtains

$$Z = \int \prod_l dA_l \prod_f \delta(F_f),$$

(6)

which can be defined as

$$Z = \int \prod_l dg_l \prod_f \delta(g_f),$$

(7)

where $g_l = e^{A_l}$ and $g_f = \prod_{l \in \partial f} g_l$. By using the well-known identity for the group delta function

$$\delta(g) = \sum_\Lambda \dim \Lambda \chi_\Lambda(g),$$

(8)

where $\Lambda$’s are the group irreps and $\chi$’s are the corresponding characters, one obtains

$$Z = \sum_\Lambda \prod_f \dim \Lambda_f \prod_v A_v(\Lambda_f, \iota_v),$$

(9)

where $A_v$ is the vertex amplitude associated to the 4-simplex dual to the vertex $v$. This amplitude is given by the evaluation of the corresponding 4-simplex spin network, known as the $15j$ symbol.

The sum (9) is called a spin foam state sum, because it is a sum of the amplitudes for the two-complex $F$ labelled with spins (irreps), i.e. a spin foam [5]. However, the expression (9) is generically divergent, and this requires a regularization. A topologically invariant regularization is to replace the irreps of $G$ with the irreps of the quantum group $G_q$, where $q$ is a root of unity. The form of the state sum stays the same, but now $\dim \Lambda$ and $A_v$ stand for the quantum dimension and
the quantum $15j$ symbol [6]. In 3d the $6j$ symbols replace the $15j$ symbols, and that state sum gives the Turaev-Viro invariant [7].

2. The Barrett-Crane model

Since GR is not a topological theory, the constraint (3) has to be implemented, and therefore a different quantization route has to be followed. One can conjecture that exists a quantization procedure such that the quantities $B_\Delta$ become the 4d rotations algebra operators $J_\Delta$, since the 4d rotation group irreps are labelling the triangles $\Delta$ (or the dual faces $f$). Then one can show that the constraint (3) becomes a constraint on the triangle irreps, given by

$$\epsilon^{abcd}J_{ab}J_{cd} = 0$$

[1, 2]. In the Euclidian case the irreps are given by the pairs of $SU(2)$ spins $(j, j')$, so that the constraint (10) implies $j = j'$. In the Minkowski case, requiring the hermiticity of the $B$ operators implies that one needs the unitary irreps of the Lorentz group. These are infinite-dimensional irreps and they are given by the pairs $(j, p)$ where $j$ is the $SU(2)$ spin and $p$ is a continuous label. The constraint (10) implies that $\Lambda = (0, p)$ or $\Lambda = (j, 0)$.

One can argue that the spacelike triangles should be labelled by the $(0, p)$ irreps, while the time-like triangles should be labelled by the $(j, 0)$ irreps. Since a spacetime triangulation can be built from the spacelike triangles, Barrett and Crane have proposed the following spin foam state sum (integral) for the quantum general relativity [2]

$$Z_{BC} = \int \prod_f p_f dp_f \prod_v \tilde{A}_v(p_f) ,$$

where $\tilde{A}_v$ is an amplitude for the corresponding 4-simplex spin network, given by

$$\tilde{A}(p_1, \cdots, p_{10}) = \int_{H^5} \prod_{i=1}^5 dx_i \delta(x_1 - x_0) \prod_{i<j} K_{p_{ij}}(x_i, x_j) .$$

This is as an integral over the fifth power of the hyperboloid $H = SO(3,1)/SO(3)$ of a propagator $K_p(x, y)$ on that space. The propagator is given by

$$K_p(x, y) = \frac{\sin(pd(x, y))}{psinh d(x, y)} , \quad \cosh d(x, y) = x \cdot y .$$

The expression (11) is not finite for all triangulations, but after a slight modification, consisting of including a non-trivial edge amplitude $\tilde{A}(p_1, \cdots, p_4)$, the partition function becomes finite for all non-degenerate triangulations [8]. This was a remarkable result, because it gave a perturbatively finite quantum theory of gravity, which was not based on string theory.
The main difficulties with the BC type models are:

1) The edge amplitudes are not determined in the BC approach, except by requiring the finiteness. By studying the convergence of the state sum, one can find choices with various degrees of convergence [9], but it is not clear which choice is the correct one. The reason why the edge amplitudes are not determined is that the BC quantization procedure is incomplete in the sense that it is not a direct quantization of a discretized path integral for GR, but one modifies a path integral for a topological gravity theory in order to implement the B constraint.

2) It is difficult to see what is the semi-classical limit, so that it is not clear whether the corresponding effective action will be given by the EH action plus the $O(l_P)$ corrections, where $l_P$ is the Planck length.

3) $Z_{BC}$ depends on a triangulation, in accordance with the fact that 4d gravity is a non-topological theory. However, the quantum gravity $Z$ must be a diffeomorphism invariant, and therefore it should be independent of the triangulation. One way to obtain such a $Z$ is to sum $Z_{BC}$ over the triangulations, but this is difficult to do. Alternatively, one can try to define a continuous limit of $Z_{BC}$ by taking increasingly finer triangulations, so that one would hopefully arrive at some effective diffeomorphism invariant action, in analogy to the 2d Ising model, where the discrete action at the critical point becomes a 2d diffeomorphism (conformally) invariant field theory action.

4) Since the matter couples to the gravitational field through the tetrads, one would need a formulation where the basic fields are the tetrad one-forms instead of the composite $B$ 2-form. In the case of the YM field, the coupling can be expressed in terms of the $B$ field [10], so that one can formulate a BC type models [11, 12]. However, for the fermions this is not possible, and a tetrad based formulation is necessary. In [10] an algebraic approach was proposed in order to avoid this problem, and the idea was to use a result from the loop quantum gravity, according to which the fermions appear as free ends of the spin networks. Hence including open spin networks gives a new type of spin foams [13], and this opens a possibility of including matter in the spin foam formalism. However, what is the precise form of the matter spin foam amplitudes remains an open question.

### 3. Spin foams for loop quantum gravity

One way to resolve the problems of the BC model is to use the spin foams in the loop quantum gravity formalism [14]. In [15] it was shown how to use the 3d spin foam state sum invariants of embedded spin networks in order to define the physical states in the loop quantum gravity formalism. The idea is to use the representation of a quantum gravity state $|\Psi\rangle$ in the spin network basis

$$|\Psi\rangle = \sum_\gamma |\gamma\rangle \langle\gamma|\Psi\rangle .$$

(14)

The expansion coefficients are then invariants of the embedded spin networks in the spatial manifold $\Sigma$, and can be formally expressed as

$$\langle\gamma|\Psi\rangle = \int DA \langle\gamma|A\rangle \langle A|\Psi\rangle = \int DA W_\gamma[A] \Psi[A] ,$$

(15)
where $A$ is a 3d complex $SU(2)$ connection, $W_\gamma[A]$ is the spin network wavefunctional (generalization of the Wilson loop functional) and $\Psi[A]$ is a holomorphic wave-functional satisfying the quantum gravity constraints in the Ashtekar representation.

In the case of non-zero cosmological constant $\lambda$, a non-trivial solution for $\Psi$ is known, i.e. the Kodama wavefunction

$$\Psi[A] = e^{\frac{i}{\hbar} \int_{\Sigma} Tr(A^\wedge dA + \frac{3}{2} AA^\wedge A^\wedge A)} ,$$

which is the exponent of the Chern-Simons action. In the $\lambda = 0$ case a class of formal solutions is given by

$$\Psi[A] = \prod_{x \in \Sigma} \delta(F_x) \Psi_0[A] ,$$

i.e. a flat-connection wavefunction. In the $\lambda = 0$ and $\Psi_0 = 1$ case the corresponding spin network invariant is given by a 3d spin foam state sum for the quantum $SU(2)$ at a root of unity

$$\langle \gamma | \Psi \rangle = \sum_j \prod_f \dim j_f \prod_v A_v(j_f, t_i, j_\gamma, t_\gamma) ,$$

where $A_v$ are the amplitudes of the vertex spin networks. A vertex spin network is given by the tetrahedron graph if no $\gamma$ vertex is present at the dual 2-complex vertex; otherwise it is given by a modified tetrahedron graph of a tetrahedron plus a spin network vertex connected by its edges to the tetrahedron vertices.

In the $\lambda \neq 0$ case, the corresponding spin network invariant is given in the Euclidian gravity case by the Witten-Reshetikhi-Turaeev invariant for $q = e^{2\pi i/(k+2)}$, where $k \in \mathbb{N}$ and $\lambda = k/\ell_p^2$, while in the Minkowski case, it is conjectured that the invariant is given by an analytical continuation of the Euclidian one, as $k \to ik$ [17]. Although there is no state sum representation of the WRT invariants, recently it has been shown that the square of the module of the WRT spin network invariants can be related through a linear transformation to the Turaeev-Viro spin network invariants

$$\Psi[A] = \delta(F) \exp \left( i \int_{\Sigma} d^3x Tr(AE_0) \right) ,$$

where $A_T$ are the triads of the vertex spin networks. A vertex spin network is given by the tetrahedron graph if no $\gamma$ vertex is present at the dual 2-complex vertex; otherwise it is given by a modified tetrahedron graph of a tetrahedron plus a spin network vertex connected by its edges to the tetrahedron vertices.

However, the problem with the Kodama and the $\delta(F)$ wavefunctions is that they do not correspond to any particular value of the triads, so that these wavefunctions cannot describe the vacuum state of quantum gravity, which we define as a physical state which is peaked around the flat space triads $E_0$. In the $\lambda = 0$ case, one can show that such a state is given by

$$\Psi[A] = \delta(F) \exp \left( i \int_{\Sigma} d^3x Tr(AE_0) \right) ,$$

A Turaeev-Viro spin network invariant is defined as the TV state sum for a triangulation where a subset of the edges are marked by the irreps of a given spin network. On the other hand, the spin network invariant (18) is a state sum for a modified dual 2-complex, where a modification is obtained by inserting the spin network edges in the dual graph of a triangulation.
so that the corresponding spin network invariant is given by the state sum
\[
\langle \gamma | \Psi \rangle = \sum_{j_f, j_l, \iota_v} \prod_{f} \dim j_f \prod_{l} C_{j_l}(E_{0}) \prod_{v} A_{v}(j_f, t_{l}, \iota_v; j_{\gamma}, \iota_{\gamma}) ,
\]
(20)
where
\[
C_{j}(E) = \int_{SU(2)} dg \bar{\chi}_{j}(g) e^{iTr(AE)} , \quad g = e^{iA} ,
\]
and $A_v$ are the evaluations of modified tetrahedron spin networks. This modification comes from the fact that the dual two-complex is now labelled by two independent sets of irreps: $j_f$ label the faces, while $j_l$ label the edges\(^3\).

Given the invariants $I_\gamma = \langle \gamma | \Psi \rangle$ one can reconstruct the wavefunction in the triad representation as
\[
\Phi[E] = \sum_{\gamma} I_\gamma \Phi_\gamma[E] ,
\]
(22)
where
\[
\Phi_\gamma[E] = \int_{A \in \mathbb{R}} DA \exp \left( -i \int_{\Sigma} d^3 x \, Tr(AE) \right) W_\gamma[A] .
\]
(23)
This path integral can be defined by the state sum
\[
\Phi_\gamma[E] = \sum_{j_l, \iota_v} \prod_{l} C_{j_l}(E_l) \prod_{v} A_{v}(j_l, t_v, j_{\gamma}, \iota_{\gamma}, \iota_v) ,
\]
(24)
where the vertex spin networks are obtained by composing a spin network $\Gamma$ associated to the dual one-complex for a triangulation of $\Sigma$ and the $\gamma$ spin network \([16]\).

In the $\lambda \neq 0$ case, one can argue that the modification of the wavefunction is given by $\Psi(A) = \Psi_K(A)\delta(F - \lambda E_0)$ \([16]\), so that one would need to define the invariant
\[
I_\gamma = \int DA e^{iS_{CS}[A]} \delta(F - \lambda E_0) W_\gamma[A] .
\]
(25)
Once the functional $\Phi[E]$ is obtained, one can try to check the semiclassical limit by studying the effective equations of motion in the de-Broglie-Bohm formalism
\[
\tilde{p}_a^\varphi(E, \dot{E}, N) = \frac{\delta S}{\delta E_a^i} , \quad S = Im(\log \Phi) ,
\]
(26)
where $\tilde{p}_a^\varphi$ is a canonically conjugate variable to the inverse triad density $E_a^i = (det e)C_a^i$, $\tilde{p}(E, \dot{E}, N)$ is the expression for the $\tilde{p}$ in terms of the triad, its time derivative and the Lagrange multipliers $N$.

\(^3\)The intertwiner labels $\iota_l$ and $\iota_v$ are not independent labels. The $\iota_l$ depends on the $j_f$’s meeting at the edge $l$ and $\iota_v$ depends on the $j_l$’s meeting at the vertex $v$.\[
4. Tetrade spin foam model

The problem of coupling matter within the BC approach suggests that one should try to find a spin foam model which is based on the integration of the tetrads. This is feasible because the Palatini action is quadratic in the tetrads, so that the path integral over the tetrads is Gaussian and therefore one can write formally

\[ Z = \int D\omega \, De \, e^{i \int (e^2 R)} = \int D\omega \, (\det R)^{-1/2} . \]  

(27)

Hence one can try to define \( Z \) as

\[ Z = \prod_l dA_l \prod_f (\det F_f)^{-1/2} = \prod_l dg_l \prod_f w(g_f) \]  

(28)

where \( \det F = (e^{abcd} F_{ab} F_{cd})^2, g_f = e^{F_f} \) and \( w(g_f) = (\det F_f)^{-1/2} \). Since

\[ w(g) = \sum_{\Lambda} c(\Lambda) \chi_{\Lambda}(g) \]  

(29)

where \( c(\Lambda) = \int_G dg \bar{\chi}_{\Lambda}(g) \, w(g) \), we will obtain a state sum of the form

\[ Z = \sum_{\Lambda_f, \lambda_l} \prod_l c(\Lambda_f) \prod_v A_v(\Lambda_f, \lambda_l) . \]  

(30)

This state sum is of the same form as in the case of the topological theory; however, the weights we put on the faces are not \( \dim \Lambda_f \) but the functions \( c(\Lambda_f) \). It remains to be seen how the choice of the \( c(\Lambda_f) \) weights\(^4\) will affect the convergence of the partition function \( Z \), and whether or not one would need to use the quantum group in order to achieve the finiteness of the \( Z \).

As far as the coupling of matter is concerned, as well as including the cosmological constant term, this would require the evaluation of the partition function with the sources (generating functional)

\[ Z(J, j) = \int D\omega \, De \, e^{i \int (e^2 R) + Tr(J\omega) + Tr(je)} \]  

(31)

which can be formally rewritten as

\[ Z(J, j) = \int D\omega \, e^{i \int Tr(J\omega)} (\det R)^{-1/2} e^{-i \int (jR^{-1}j)/4} . \]  

(32)

This expression can be defined on a spacetime triangulation along the lines of the \( J = j = 0 \) case, but when the sources are present a more intricate state sum will appear. It can be defined as

\[ Z = \prod_l d\lambda_l \, u(g_l, J_l) \prod_f w(g_f, j_f) \]  

(33)

\(^4\)These weights are given by the integrals which are generically divergent, due to \( \det F_f = 0 \) configurations, so that some kind of regularization must be used.
where
\[ u(g_l, J_l) = e^{T r(\omega_l J_l)} \quad , \quad w(g_f, j_\epsilon) = w(g_f) e^{-i(j_\epsilon R^{-1}_f j_\epsilon)/4} , \]  
and \( \epsilon, \tilde{\epsilon} \) are two different edges of the triangle \( \Delta \) dual to the face \( f \). By expanding the functions \( u \) and \( w \) as
\[ u = \sum_{\Lambda_l} \alpha(\Lambda_l, J_l) \chi_{\Lambda_l}(g_l) \quad , \quad w = \sum_{\Lambda_f} \beta(\Lambda_f, j_\epsilon) \chi_{\Lambda_f}(g_f) \]  
and performing the group integrations, one obtains a state sum
\[ Z(J, j) = \sum_{\Lambda_f, \Lambda_l, \Lambda_{\epsilon_\nu}} \prod_f \beta(\Lambda_f, j_\epsilon) \prod_l \alpha(\Lambda_l, J_l) \prod_v A_v(\Lambda_f, \epsilon_l, \Lambda_l, \epsilon_\nu) , \]  
which similarly to the sum (18) involves a dual 2-complex whose edges and faces are independently colored with the irreps of the relevant group (\( SO(4) \) in the Euclidian gravity case, or \( SO(3,1) \) in the Minkowski case).

5. Conclusions

The two spin foam models we have described have an advantage over the BC type models in the fact that they have been formulated as direct quantizations of GR, so that all the simplex amplitudes are fixed. Also these are the models where the matter can be more easily introduced. As far as the semiclassical/continous limit is concerned, they seem to be promising candidates. In the loop quantum gravity case, the model is defined in the continuum space, and the problem is in finding an approximation for the sum (22) such that the equations (26) give Planck length corrected Einstein equations. In the tetrade model, it appears easier to extract the semiclassical limit by considering finer (larger) triangulations than summing over the triangulations, but a technique must be developed in order to do this, perhaps something analogous to the 2d Ising model near the critical point. Clearly, much more work is necessary in order to resolve these issues.

On the mathematical side, the expression (18) is a new way to calculate the knot and spin network invariants, which can be easily generalised to higher dimensions and it is more efficient way of calculating these invariants than the TV state sum approach [19].

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