The classical Bott–Samelson theorem states that if on a Riemannian manifold all geodesics issuing from a certain point return to this point, then the universal cover of the manifold has the cohomology ring of a compact rank one symmetric space. This result on geodesic flows has been generalized to Reeb flows and partially to positive Legendrian isotopies by Frauenfelder–Labrousse–Schlenk. We prove the full theorem for positive Legendrian isotopies.

The spherization $S^*Q$ of a manifold $Q$ is the space of positive line elements in the cotangent bundle $T^*Q$. The tautological one-form $\lambda$ on $T^*Q$ does not pass to the quotient, but its kernel does. This endows $S^*Q$ with a cooriented contact structure $\xi$.

Let $j_t : L \to S^*Q$ be a smooth family of embeddings such that $j_t(L)$ is a Legendrian submanifold of $S^*Q$ for all $t$. Then $L_t = j_t(L)$ is called a Legendrian isotopy. If $\alpha(\frac{d}{dt}j_t(x)) > 0$ for one and hence any coorientation preserving contact form $\alpha$ for $\xi$ and all $x \in L$, then $L_t$ is called positive. Frauenfelder–Labrousse–Schlenk proved the following Theorem.

**Theorem 1.** Let $Q$ be a closed connected manifold of dimension $\geq 2$. Suppose there exists a positive Legendrian isotopy $L_t$ in the spherization $S^*Q$ that connects the fiber over a point with itself, i.e. $L_0 = L_1 = S^*_qQ$. Then the fundamental group of $Q$ is finite and the integral cohomology ring of the universal cover of $Q$ is generated by one element.

We note that by a deep result in algebraic topology, a manifold with integral cohomology ring generated by one element is homotopy equivalent to $S^n$, $\mathbb{R}P^n$ or $\mathbb{C}P^n$ or has the integral cohomology ring of $\mathbb{H}P^n$ or the Cayley plane, see [2] and the references therein.

In this paper we prove the following addition to Theorem 1 which was conjectured in [6].

**Theorem 2.** Under the assumptions of Theorem 1, if furthermore $L_t \cap L_0 = \emptyset$ for $0 < t < 1$, then $Q$ is simply connected or homotopy equivalent to $\mathbb{R}P^n$.

The union of these two theorems is the complete generalization of the classical Bott–Samelson theorem from geodesic flows to positive Legendrian isotopies.

The first versions of the Bott–Samelson theorem were for geodesic flows and used Morse theory of the energy functional on the based loop space, see [3], [12] and [2].
Frauenfelder, Labrousse and Schlenk [6] proved versions of Theorem 1 and 2 for autonomous Reeb flows, using Rabinowitz–Floer homology. They also proved Theorem 1 using Rabinowitz–Floer homology for positive Legendrian isotopies as stated above. The puzzle piece missing in [6] to generalize Theorem 2 from autonomous Reeb flows to positive Legendrian isotopies is the fact that the action functional in the construction is Morse–Bott. We provide this in Lemma 3.2, and thus complete the proof in [6]. The key ingredient is the choice of Hamiltonian, which is elaborated in Lemma 2.1. We cannot avoid the Hamiltonian to be time-dependent, but we can control the time-dependence along the Legendrian isotopy. At critical points, the resulting action functional then behaves like in the autonomous case.

This paper is heavily based on [6], which also contains an extensive introduction to the topic.

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2. Recollections

The Rabinowitz–Floer homology we use depends on a time-dependent Reeb flow, not on a Legendrian isotopy. We first explain how we choose such a flow that restricts to a given Legendrian isotopy. Then we briefly present the version of Rabinowitz–Floer homology we use and discuss its properties. We only sketch the proofs, since they are contained in or are analguous to proofs in [1, 4, 5, 6]. For a general exposition of Morse–Bott homology we refer the reader to the Appendix of [8].

The choice of flow. Let \( j_t : L \hookrightarrow M, \; t \in [0, 1] \), be a positive Legendrian isotopy in a cooriented exact contact manifold \((M, \alpha)\). We denote \( L_t = j_t(L) \). By the Legendrian isotopy extension theorem, see for example [9, Theorem 2.6.2], there exists a positive contact isotopy \( \psi_t \) of \( M \) such that \( \psi_t(L_0) = L_t \). If furthermore \( L_0 = L_1 \), then there exists a positive and twisted periodic (that is \( \varphi^t = \varphi^{t-k} \circ \varphi^k \) for all \( t \in \mathbb{R}, k \in \mathbb{Z} \)) contact isotopy \( \varphi^t \) such that \( \varphi^k(L_0) = L_0 \) for all \( k \in \mathbb{N} \), see [6, Proposition 6.2]. This isotopy is generated by a contact Hamiltonian \( h_t \) that is a convex combination of the contact Hamiltonian of \( \psi_t \) and the one of the Reeb flow \( \psi^t_R \) (namely \( h \equiv 1 \)), such that for \( t \) near 0 or 1, \( \varphi^t \) coincides with \( \psi^t_R \). Note that in general \( \varphi^t(L_0) \neq \psi^t(L_0) \) for \( t \notin \mathbb{N} \).

Lemma 2.1. Given a periodic Legendrian isotopy \( L_t \) that is the restriction of the Reeb flow generated by the contact Hamiltonian \( h \equiv 1 \) for \( t \) near 0 and 1 (as given by [6, Proposition 6.2]), then the corresponding twisted periodic positive contact isotopy \( \varphi^t \) can be chosen such that the time-dependent contact Hamiltonian \( h^t \) that generates \( \varphi^t \) satisfies \( h^t = 0 \) along \( L_t \).

Proof. The construction of \( h^t \) is performed as in [9, Theorem 2.6.2]. We emphasize for a function \( h^t \) and a path \( \gamma(t) \) the distinction between \((\frac{d}{dt} h^t)(\gamma(t))\) and \( \frac{d}{dt}(h^t(\gamma(t))) \) by using the notation \( \dot{h} := \frac{d}{dt}h^t \).

Recall that a contact Hamiltonian \( h^t \) and a contact vector field \( X_t \) determine each other through the equations \( h^t = \alpha(X_t) \) and \( \iota_{X_t} \alpha = dh^t(R_\alpha)\alpha - dh^t \). We
define the 1-jet of $h^t$ along $L_t$ as follows.

$$
(2.1) \quad h^t(j_t(x)) = \alpha \left( \frac{d}{dt} j_t(x) \right) \quad \forall x \in L,
$$

$$
(2.2) \quad dh^t(v) = -\iota_{j_t(x)}d\alpha(v) \quad \forall v \in \xi|L_t,
$$

$$
(2.3) \quad dh^t \left( \frac{d}{dt} j_t(x) \right) = \frac{d}{dt} (h^t(j_t(x))) \quad \forall x \in L.
$$

Any Hamiltonian $h^t$ that satisfies the first two equations generates a vector field $X_t$ such that $X_t(j_t(x)) = \frac{d}{dt} j_t(x)$ for all $x \in L$. Equation (2.2) holds for all $v \in TL_t$ since $TL_t \subseteq \xi|L_t$. Equation (2.3) does not contradict (2.2) since $\frac{d}{dt} j_t(x)$ is positively transverse to $\xi$ for all $x \in L$. (Here we differ from [1] where the choice in (2.2) is $dh^t(R_\alpha) = 0$.) Since $\frac{d}{dt} (h^t(j_t(x))) = h^t(j_t(x)) + dh^t(\frac{d}{dt} j_t(x))$ for all $x \in L$, equation (2.3) implies $h^t = 0$ along $L_t$. We extend $h^t$ to a neighbourhood of $L_t$ by identifying a neighbourhood of $L_t$ with the normal bundle $NL_t \to L_t$ and choosing $h_t$ linear on each fiber.

Finally we extend $h^t$ to a positive function that is constant 1 outside a neighbourhood of $L_t$. Since the Legendrian isotopy is the restriction of the Reeb flow generated by $h \equiv 1$ for $t$ near 0 and 1, the function $h^t$ thus constructed satisfies $h^t \equiv 1$ for $t$ near 0 and 1, and admits a 1-periodic extension. \(\square\)

The spherization $(S^*Q, \xi)$ of a manifold $Q$ is represented by any fiberwise star-shaped hypersurface $\Sigma \subset T^*Q$ in the cotangent bundle with contact structure $\ker \lambda|_\Sigma$. The map that sends a positive line element to its intersection with $\Sigma$ is a contactomorphism. The radial dilation of a fiberwise star-shaped hypersurface by a positive function is a contactomorphism onto its image. Every coorientable contact form of $(\Sigma, \xi)$ is realized as $\lambda|_\Sigma$ for some fiberwise star-shaped hypersurface $\Sigma$. We choose a Riemannian metric $g$ on $Q$ and represent $S^*Q$ henceforth as the unit cosphere bundle with respect to this metric. With $\alpha = \lambda|_\Sigma$, the symplectization $(\Sigma \times \mathbb{R}_{>0}, d(\rho_\alpha))$ is naturally symplectomorphic to $T^*Q\setminus Q$. A contact isotopy $\varphi^t_{\Sigma}$ of $\Sigma$ admits a lift to a Hamiltonian isotopy $\varphi^t$ of $\Sigma \times \mathbb{R}_{>0}$, defined by $\varphi^t(x, r) = (\varphi^t_{\Sigma}(x), \rho_t(x))$ where $\rho_t(x)$ is defined by $(\varphi^t_{\Sigma})*(\alpha)|_x = \rho_t(x)\alpha|_x$, see [1] Proposition 2.3. If $\varphi^t_{\Sigma}$ is generated by the contact Hamiltonian $h^t$, then $\varphi^t$ is generated by the Hamiltonian $H^t = rh^t$.

The functional. Let $h^t$ be a positive, periodic contact Hamiltonian on $(\Sigma, \ker \lambda)$. Following [1] we choose a lift of the contact isotopy $\varphi^t$ of $\Sigma$ generated by $h^t$ to the symplectization $(\Sigma \times \mathbb{R}_{>0}, d(\rho_\alpha))$, depending on parameters $\kappa \geq 2$, $R \geq 2$ and constants $c, C$ such that uniformly $0 < c < h^t < C$. We define $H^t = rh^t - \kappa$. The Hamiltonian $H_t$ is a deformation of $H^t$ such that $H^t = cr - \kappa$ for $r \leq 1$, $H^t = \tilde{H}^t$ for $2 \leq r \leq \kappa R - 1$ and $H^t = Cr - \kappa$ for $r > \kappa R$. This has the effect that $H^t$ induces reparametrized $g$-geodesic flows for $r \in (0, 1] \cup [\kappa R, \infty)$, and a lift of the $h^t$-contact flow for $r \in [2, \kappa R - 1]$.

Denote by $\Omega_{T^*Q}$ the set of $W^{1, 2}$ paths $x : [0, 1] \to T^*Q$ such that $x(0), x(1) \in T^*_qQ$. Define the functional $\mathcal{A} : \Omega_{T^*Q} \times \mathbb{R} \to \mathbb{R}$ by

$$
\mathcal{A}(x, \eta) = \frac{1}{\kappa} \left( \int_0^1 \left[ \lambda(\dot{x}) - \eta H^t(x(t)) \right] \, dt \right).
$$
This functional depends of course on \( h^t \), but also on the parameters \( \kappa, R \) and the constants \( c, C \). A pair \((x, \eta)\) is a critical point of \( \mathcal{A} \) if and only if \( \dot{x} = \eta X_{H^t} \) and \( \int_0^1 H^t(x(t)) + \eta \dot{H}^t(x(t)) \, dt = 0 \). This is equivalent to

\[
\begin{cases}
\dot{x} = \eta X_{H^t}, \\
H^t(x(1)) = 0,
\end{cases}
\]

(2.4)
as one sees by using that \( \eta X_{H^t} \)-chords satisfy \( \frac{d}{dt} H^t(x(t)) = \eta \dot{H}^t(x(t)) \) and by integration by parts. Note that the factor \( \frac{1}{\kappa} \) does not change the critical point equations (2.4), but only the critical values. In fact, Lemma 2.3 below shows that this factor normalizes the action for critical points in such a way that the action spectrum is independent of \( \kappa \).

**Remark 2.2.** For autonomous Hamiltonians \( H^t = H \) the second equation of (2.4) becomes \( \dot{H}(x(t)) = 0 \) \( \forall t \in [0, 1] \). Thus the critical points are flow lines on the hypersurface \( H^{-1}(0) \). This hypersurface is not well-defined for time-dependent Reeb flows and \( H^t(x(t)) \) might be very large for \( t \neq 1 \). We deal with this defect through the parameters \( \kappa, R \). Intuitively speaking, the parameters create safe space (\( \kappa \) towards the zero section, and \( R \) towards infinity), where critical orbits are free to roam. This is made precise in the next lemma from [1] Proposition 4.3, Corollary 4.4.

**Lemma 2.3.** For all \( a < b \) there exist constants \( \kappa_0 \geq 2, R_0 \geq 2 \) such that for \( \kappa \geq \kappa_0 \) and \( R \geq R_0 \), all critical points \((x, \eta)\) of \( \mathcal{A} \) with action between \( a \) and \( b \) satisfy \( 2 \leq |x(t)|_g \leq \kappa R - 1 \) for all \( t \) and \( \mathcal{A}(x, \eta) = \eta \). As a consequence, the critical point equation (2.4) and the action values are independent of the choice of \( \kappa \geq \kappa_0, R \geq R_0, c, C \).

**The chain group.** Assume from now on that the functional \( \mathcal{A} \) is Morse–Bott for critical points with action between \( a \) and \( b \). Choose in addition a Morse function \( f \) on \( \text{Crit} \mathcal{A} \). Then for \( b \in \mathbb{R} \) we define the filtered Rabinowitz–Floer chain group \( \text{RFC}^b(\mathcal{A}) \) as the \( \mathbb{Z}_2 \)-vector space generated by the critical points of \( f \) on \( \text{Crit} \mathcal{A} \) with action \( \leq b \).

**The index.** The index of a critical point \( c = (x, \eta) \) of \( f \) on \( \text{Crit} \mathcal{A} \) is defined as follows. Let \( TT_{x}^*Q \) be the vertical Lagrangian distribution. Denote by \( \mu_{\text{RS}}(x, \eta) \) the Robbin–Salamon index of the path \( d(\phi_{X}^t)^{-1}(TT_{x(t)}^*Q) \) with respect to \( TT_{x(0)}^*Q \), and by \( \mu_{\text{M}} \) the Morse index of \( f \) on \( \text{Crit} \mathcal{A} \), see [11]. Then the index of \( c \) is defined as

\[
\mu(c) = \mu_{\text{RS}}(x, \eta) - \frac{n - 1}{2} + \frac{1}{2} \mu_{\text{M}}(c),
\]

where the shift by \( -\frac{n-1}{2} \) is introduced such that the index \( \mu \) agrees with the Morse index for geodesic Hamiltonians. Denote by \( \text{RFC}^{>0}(\mathcal{A}) \) the chain groups graded by the index \( \mu \).

**The differential.** For the differential, we choose an \( \omega \)-compatible almost complex structure \( J = J_{c, \eta} \) on \( T^*Q \) that satisfies the following properties for \( r \in [0, 1] \cup \{\kappa R, \infty\} \), following [11] Chapter 3:

- \( J \) is independent of \( t, \eta \),
- \( J \) maps \( \partial_h \) to \( X_{\frac{1}{2}} \) and preserves \( \ker \lambda |_{\{r \text{ const}\}} \),
- \( J \) is invariant under the Liouville flow \( (y, r) \mapsto (y, e^t r), t \in \mathbb{R} \).
Define the $L^2$-metric
\[
\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_J = \frac{1}{\kappa} \int_0^1 \omega(v_1, Jv_2) \, dt + \frac{\eta_1 \eta_2}{\kappa}
\]
on $\Omega_{T^*Q} T^*Q \times \mathbb{R}$. Further, choose a Morse–Smale metric $m$ on $\text{Crit} A$. The differential of degree $-1$ is defined by the $\mathbb{Z}_2$-count of finite energy negative gradient flow lines with cascades. A flow line with cascades starts at a critical point of $f$ at time $-\infty$, then runs until a finite time as negative $m$-gradient flow line on $\text{Crit} A$, then runs as negative $(\cdot, \cdot)_J$-flow line from one component of $\text{Crit} A$ to another (from time $-\infty$ to $+\infty$), then runs for a finite time along a negative $m$-gradient flow line, ..., and after finitely many such changes (cascades) ends in a critical point of $f$ at time $+\infty$. To show that this differential is well defined and $d^2 = 0$, one has to show that for $A(c^+), A(c^-) \in [a, b]$ the moduli space of finite energy negative gradient flow lines with cascades from $c^+$ to $c^-$ is compact modulo crossing. This follows from standard arguments as soon as one has established $L^\infty$ bounds on the Floer strips underlying the $(\cdot, \cdot)_J$-parts of the flow lines, on the derivatives of the Floer strips, and on $\eta$. The $L^\infty$ bounds on the Floer strips follow from a maximum principle since our Hamiltonian is convex for $r \notin [1, \kappa R]$. The $L^\infty$ bounds on the derivatives follow from the exactness of $\omega = d\lambda$ that prevents bubbling. The following lemma shows that for almost critical points, $\eta$ is bounded by the action.

**Lemma 2.4 (Fundamental Lemma).** There exists $\varepsilon > 0$ such that
\[
\|\nabla A(x, \eta)\| < \varepsilon \Rightarrow |\eta| \leq \frac{1}{\varepsilon}(A(x, \eta) + 1).
\]

This is a version with Lagrangian boundary conditions of [1] Lemma 4.5. and is proved using a by now standard scheme, see [4] Proposition 3.1. The $L^\infty$ bound on $\eta$ is then obtained as in [1] Corollary 3.3.

**The Homology.** We define $\text{RFC}^b_{\alpha, \kappa}(A)$ as the quotient chain complex $\text{RFC}^b_{\alpha} / \text{RFC}^c_{\alpha}$. By Lemma 2.3 for $\kappa \geq \kappa_0$, and $R \geq R_0$ the generators and actions of this chain complex do not depend on the choice of $\kappa, R, c, C$, and by standard continuation arguments the resulting homology is independent up to canonical isomorphisms of a generic choice of $g, J, m$. Finally, define $\text{RFC}^c_{\alpha, 0}(A)$ as the inverse direct limit $\lim_{\kappa \to \infty} \lim_{\alpha \to 0} \text{RFC}^b_{\alpha, \kappa}(A)$ under the homomorphisms induced by inclusion and denote the resulting homology by $\text{RFH}^c_{\alpha, 0}(A)$.

**Invariance.** For any other twisted periodic and positive contact isotopy $\varphi^t$ such that the corresponding functional $\tilde{A}$ is Morse–Bott, we have
\[
\text{RFH}^c_{\alpha, 0}(A) \cong \text{RFH}^c_{\alpha, 0}(\tilde{A}).
\]
This can be shown like the invariance (29) in the proof of [6] Lemma 5.4 with the additional explanation after [6] Lemma 5.5, by considering the path of Hamiltonians $H^f_s = (1 - \beta(s))H^f + \beta(s)\tilde{H}^f$, where $\beta(s)$ is a smooth monotone function with $\beta(s) = 0$ for $s \leq 0$ and $\beta(s) = 1$ for $s \geq 1$, generating a path of functionals $A_s$ that connects $A$ and $\tilde{A}$, where the constants $\kappa, R, c, C$ are chosen uniformly in $s$. Note that $\partial_s H^f_s$ is compactly supported, thus a continuation homomorphism can be defined. Also note that there exists an $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ the action spectrum of $A_{\partial_s H^f}$ and the interval $(0, \varepsilon]$ are disjoint. Using this, we can exclude that critical values cross $0$ during the continuation. The isomorphism follows then by standard arguments.
In particular for \( h^t \equiv 1 \) the corresponding functional \( A_g \) is the functional of the \( g \)-geodesic flow. Denote by \( \text{HM}^*_0(\mathcal{E}) \) the \( \mathbb{Z}_2 \)-Morse homology relative the constant loop of the energy functional \( \mathcal{E}(x) = \int_0^1 \frac{1}{2} g(\dot{x}, \dot{x}) \, dt \) on the space of based loops in \( Q \). The following result is a special case of Merry’s theorem \([10] \) Theorem 3.16.

\[
\text{RFH}^*_0(A_g) \cong \text{HM}^*_0(\mathcal{E}).
\]

Since \( \text{HM}^*_0(\mathcal{E}) \) is isomorphic to the homology \( H_*(\Omega_q, q; \mathbb{Z}_2) \) relative the constant loop, we obtain

**Lemma 2.5.** \( \text{RFH}^*_0(A_g) \cong H_*(\Omega_q, q; \mathbb{Z}_2) \).

### 3. Proof of Theorem 2

Recall that Theorem 2 is shown in \([6] \). In this section we prove Theorem 2 using the results that are already established in Theorem 1. The main step is to show that in this situation the action functional is Morse–Bott. Theorem 2 then follows from this result.

**Remark 3.1.** Let \((x, \eta)\) be a critical point of \( A \) for \( \kappa, R, c, C \) as in Lemma 2.3 and \( h^t \) as in Lemma 2.4. Then along \( x \) we have \( H^t = H^t = rh^t - \kappa \), and hence \( \eta H^t(x(t)) = 0 \). Since \( \frac{d}{dt} H^t(x(t)) = \eta H^t(x(t)) \) we thus have \( H^t(x(t)) = 0 \) for all \( t \) and \( A(x, \eta) = \eta \). In this sense the choice made in Lemma 2.1 is designed such that the functional that arises from the situation of Theorem 2 behaves at critical points as in the autonomous case.

**Lemma 3.2.** In the situation of Theorem 2 and for \( h^t \) chosen as in Lemma 2.4 the action functional \( A \) defined above is Morse–Bott at the critical sets with positive action, the components of the critical manifold being diffeomorphic to \( S^*_x Q \times \{k\}, k \in \mathbb{N} \).

**Proof.** A diffeomorphism from the critical manifolds to \( S^*_x Q \times \{k\} \) is given by mapping critical points \((x, k) \in \Omega_q Q \times \mathbb{R} \) to \((x(1), k) \in T^*_x Q \times \{k\} \). Since \( h^t \) is constant 1 for \( t \) near \( k \in \mathbb{N} \) and by the equations \( 2.4 \), the image of this map is \(( (q, p) \in T^*_x Q \mid |p|_g = 1 \times \{k\} \cong S^*_x Q \times \{k\} \).

The functional \( A \) is Morse–Bott if the kernel of the Hessian \( \mathcal{H}A \) is exactly the tangent space of the critical manifold. The inclusion \( T\text{Crit} A \subseteq \ker \mathcal{H}A \) is obvious, we will show the converse. A tangent vector to \( x \in \Omega_q Q \) is a section \( \dot{x} \) of the pullback bundle \( x^* TT^* Q \). Assume that \((\dot{x}, \dot{\eta}) \in \ker \mathcal{H}(A) \). Since \( \dot{x} \in T\Omega_q Q \), the endpoints of \( \dot{x} \) are in the vertical subbundle, \( \dot{x}(i) \in T\Omega_q Q \subseteq \ker \lambda \) for \( i = 0, 1 \).

We will compute \( \mathcal{H}A((\dot{x}, \dot{\eta}), (\dot{x}, \dot{\eta})) \) where \((\dot{x}, \dot{\eta})\) is another vector based at \((x, \eta)\). Assume for the moment that \( x \) lies in a single Darboux chart and that in local coordinates we have \( x = (q, p) \) and \( \dot{x} = (\dot{q}, \dot{p}) \). As a preparation we compute

\[
d \left( \int_0^1 \lambda(\dot{x}) \, dt \right)(\dot{x}) = \int_0^1 d \frac{d}{dt}(p + \epsilon \dot{p})(\dot{q} + \epsilon \dot{q}) \big|_{\epsilon=0} \, dt
\]

\[
= \int_0^1 \dot{q} \dot{p} + \dot{p} \dot{q} \, dt
\]

\[
= \int_0^1 \dot{q} \dot{p} - \dot{p} \dot{q} \, dt + p \dot{q} \big|_0^1
\]

\[
= \int_0^1 \omega(\dot{x}, \dot{x}) \, dt,
\]
where the last equality holds because the endpoints of \( \dot{x} \) lie in the vertical subbundle and thus \( \dot{q}(i) = 0 \) for \( i = 0, 1 \). If \( x \) does not lie in a single chart, the same follows after finitely many coordinate changes. Similarly we compute

\[
d \left( \int_0^1 \omega(\dot{x}, \dot{x}) \, dt \right) (\dot{x}) = \int_0^1 \omega(\dot{x}, \dot{x}) \, dt = \int_0^1 \omega(\dot{x}, \dot{x}) \, dt + \omega(\dot{x}, \dot{x})|_0^1 = \int_0^1 \omega(\dot{x}, \dot{x}) \, dt,
\]

where the last equality holds because the endpoints of \( \dot{x} \) and \( \dot{x} \) lie in the vertical subbundle which is a Lagrangian. Using these preparations we can now compute

\[
\mathcal{A}(x, \eta) = \int_0^1 \lambda(\dot{x}) - \eta H^\eta(x(t)) \, dt,
\]

\[
d\mathcal{A}(\dot{x}, \dot{\eta}) = \int_0^1 \omega(\dot{x}, \dot{x}) - \eta d\dot{H}^\eta(\dot{x}) - \dot{\eta} (H^\eta(x(t)) + \eta t \dot{H}^\eta(x(t))) \, dt,
\]

\[
\mathcal{H}_\mathcal{A}(\dot{x}, \dot{\eta}, (\dot{x}, \dot{\eta})) = \int_0^1 \omega(\dot{x}, \dot{x}) - \dot{\eta} (d\dot{H}^\eta(\dot{x}) + \eta t \dot{H}^\eta(\dot{x}) + \dot{\eta} (2t \dot{H}^\eta + \eta t^2 \ddot{H}^\eta)) \, dt.
\]

Thus \( (\dot{x}, \dot{\eta}) \) lies in \( \ker \mathcal{H}_\mathcal{A} \) if and only if the following equations are satisfied.

(3.1) \[
\dot{x} = \dot{\eta} (X_{\dot{H}^\eta} + \eta t X_{\dot{H}^\eta}) \quad \forall t,
\]

(3.2) \[
0 = \int_0^1 \dot{H}^\eta(\dot{x}) + \eta t \dot{H}^\eta(\dot{x}) + \dot{\eta} (2t \dot{H}^\eta + \eta t^2 \ddot{H}^\eta) \, dt.
\]

We translate these equations to the fixed vector space \( T_{x(0)}T^*Q \) by pulling back along \( \varphi^\eta \): Define

\[
v(t) = D\varphi^{-1}(\dot{x}(t)),
\]

where we abbreviated \( \varphi^\eta \) to \( \varphi \) for better readability. Since \( \varphi \) is a symplectomorphism, \( D\varphi^{-1} X_{H^\eta} = X_{\varphi^*H^\eta} \). Thus equation (3.1) becomes

(3.3) \[
\dot{v} = \dot{\eta} (X_{\varphi^*H^\eta} + \eta t X_{\varphi^*H^\eta}) \quad \forall t,
\]

Integrating equation (3.3), we obtain

\[
v(1) = v(0) + \dot{\eta} \int_0^1 X_{\varphi^*H^\eta} + \eta t X_{\varphi^*H^\eta} \, dt.
\]

Since \( H^\eta \) is (after addition of the constant \( \kappa \)) 1-homogeneous in the fibers near \( \text{Crit} \mathcal{A} \), the flow \( \varphi^\kappa \) commutes with dilations by a factor close to 1. Thus also \( \varphi^*H^\eta \) is (after addition of \( \kappa \)) 1-homogeneous, thus \( \varphi^*H^\eta \) is 1-homogeneous and so near \( \text{Crit} \mathcal{A} \), \( X_{\varphi^*H^\eta} \) is a lift of the contact Hamiltonian vector field \( X_{\varphi^*H^\eta} \) on the spherization \( S^*Q \). For \( h^t \) chosen as in Lemma 2.1 we have \( \varphi^*T_{T_{x(0)}T^*Q} = T_{x(0)}T^*Q \). By the geometric setup of the theorem, \( D\varphi^{-1} T_{x(0)}T^*Q = T_{x(0)}T^*Q \), so with \( \dot{v}(i) \) also the endpoints of \( v \) lie in the vertical subbundle, \( v(i) \in T_{x(0)}T^*Q \subseteq \ker \lambda \). Thus we conclude that \( \dot{\eta} \int_0^1 X_{\varphi^*H^\eta} \, dt \in \ker \lambda \). But \( X_{\varphi^*H^\eta} \big|_{\dot{n}_+} \ker \lambda \) for all \( t \) since \( H^\eta \) is positive, and thus \( \int_0^1 X_{\varphi^*H^\eta} \, dt \big|_{\dot{n}_+} \ker \lambda \). We conclude that \( \dot{\eta} = 0 \) and with (3.3) that \( v \) is constant.

Recall that our task is to show that \( (\dot{x}, \dot{\eta}) = (\dot{x}, 0) \in T \text{Crit} \mathcal{A} \), and recall from (2.3) that \( \text{Crit} \mathcal{A} = \{ x \mid \dot{x}(t) = \eta X_{H^\eta} \} \times \mathbb{N} \cap \{ x \mid H^\eta(x(1)) = 0 \} \times \mathbb{N} \). We first
define the path \((x_s, \eta_s) \in \{ x \mid x(t) = \eta X_{H^v} \} \times \mathbb{N}\) by \(x_s(t) = \varphi^t(x(0) + s\nu), \eta_s = \eta\). Then \(\frac{d}{ds}(x_s, \eta_s)_{|s=0} = (\dot{x}, 0)\). Thus,

\[ (\dot{x}, 0) \in T(\{ x \mid x(t) = \eta X_{H^v} \} \times \mathbb{N}). \]

Since \(\dot{x}_s = \eta X_{H^v}(x_s)\) for all \(s\), \(\frac{d}{dt}H^q(x_s(t)) = \eta H^q(x_s(t))\) and thus also \(\frac{d}{dt}dH^q_{2\nu}(\dot{x}) = \eta dH^q_{2\nu}(\dot{x})\). Together with \(\dot{\eta} = 0\), equation (3.2) becomes

\[
0 = \int_0^1 dH^q(\dot{x}) + t \frac{d}{dt}dH^q(\dot{x})\ dt
\]

\[
f_{by\ parts} = \int_0^1 dH^q(\dot{x}) - dH^q(\dot{x})\ dt + 1 \cdot dH^q(\hat{x}(1)) - 0 \cdot dH^q(\dot{x}(0))
\]

\[
= dH^q(\hat{x}(1)).
\]

Thus,

\[ (\dot{x}, 0) \in T(\{ x \mid x(t) = \eta X_{H^v} \} \times \mathbb{N}) \cap T(\{ x \mid H^q(x(1)) = 0 \} \times \mathbb{N}) = T \text{Crit} \mathcal{A}, \]

as claimed.

Before we can continue we need two observations about the index. Since the components of \(\text{Crit} \mathcal{A}\) are spheres \(S_{k}Q \times \{ k \}\), the Morse function \(f\) on \(\text{Crit} \mathcal{A}\) can be chosen with exactly two critical points \(c_k^-, c_k^+\) per component, with Morse index 0 and \(d = 1\).

**Lemma 3.3.** The Robbin–Salamon index of \((x(t), k) \in \text{Crit} \mathcal{A}\) depends only on \(k\) and is equal to \(k\mu_0\) for some constant \(\mu_0 \geq 1\).

**Proof.** The proof goes exactly as in [6, Section 5.2] and uses Rabinowitz–Floer homology over \(\mathbb{Z}\) coefficients, which is developed in [3] to prove Theorem H. We repeat the argument without developing the theory over \(\mathbb{Z}\) coefficients and refer the interested reader to [3]. Note that the change of coefficients changes neither the critical point equation nor the index.

The subset of \(\text{Crit} \mathcal{A}\) with \(\eta = k\) is a sphere and thus connected. Let \((x_0, k), (x_1, k)\) be two critical points of \(\mathcal{A}\) and \((x_s, k)\) be a path in \(\text{Crit} \mathcal{A}\) connecting them. Identify the vector spaces \(T_{x_0(0)}T^*Q\) in such a way that \(T T^*_{x_0(0)}Q\) is constant. Then \(d(\varphi^{kt})^{-1}(T T^*_{x_0(t)}Q)\) is a homotopy with parameter \(s\) with constant endpoints of paths with parameter \(t\) of Lagrangian subspaces. Thus the two paths \(d(\varphi^{kt})^{-1}(T T^*_{x_0(t)}Q)\) and \(d(\varphi^{kt})^{-1}(T T^*_{x_0(t)}Q)\) are stratum homotopic in the sense of [11] and thus \(\mu_{RS}(x_0, k) = \mu_{RS}(x_1, k)\). We conclude that the Robbin–Salamon index only depends on \(k\). Since every \(\varphi^{kt}\) flow line is the \(k\)-fold concatenation of \(\varphi^t\) flow lines, \(\mu_{RS}(\varphi^{kt}(x(0), k) = k\mu_0\) by the concatenation property of the Robbin–Salamon index.

Assume that \(\mu_0 \leq 0\). Since the signatures of \(c_k^\pm\) are \(\pm \frac{d-1}{2}\) (in particular bounded), there exists a \(k_0\) such that \(\mu(c) < k_0 \forall c \in \text{Crit} \mathcal{A}\). Thus for \(k \geq k_0\) we have by deformation of \(\mathcal{A}\) to a geodesic functional \(\mathcal{A}_g\), and by the \(\mathbb{Z}\)-version of Lemma 2.5 (also contained in [10]),

\[
0 = \text{RFH}_k^\varphi(\mathcal{A}_g; \mathbb{Z}) \cong \text{RFH}_k^\varphi(\mathcal{A}_g; \mathbb{Z}) \cong H_k(\Omega_q Q, q; \mathbb{Z}) \cong H_k(\Omega_q Q, \mathbb{Z}),
\]

and thus also \(H_k(\Omega_q Q; \mathbb{Z}) \cong 0\). Thus for all \(k \geq k_0 + 1\) and \(F = \mathbb{Z}_p\) for any prime number \(p\) or \(F = \mathbb{Q}\) we have \(H_s(\Omega_q Q, q; F) \cong 0\). By [13, Proposition 10] this implies
the convenience of the reader.

case. By Theorem 1, there can be at most one critical point of index zero if \( \mu_0 \) is a divisor of \((d - 1)\) and no critical point of index zero otherwise. Hence after a deformation to the functional \( A_g \) of a geodesic flow, with Lemma 3.3 and the reduced long exact \( \mathbb{Z}_2 \)-homology sequence of the pair \((\Omega_q Q, q)\) we find that

\[
RFH^0_{-1}(A) \cong RFH^0_{-1}(A_g) \cong H_0(\Omega_q Q, q; \mathbb{Z}_2) \cong \tilde{H}_0(\Omega_q Q; \mathbb{Z}_2)
\]

is 0 or \( \mathbb{Z}_2 \), thus \( \pi_1(Q) = 0 \) or \( \mathbb{Z}_2 \). In the first case we are done, so assume the second case. By Theorem 1, \( Q \) is a closed manifold such that \( H^+(\tilde{Q}; \mathbb{Z}_2) \) is generated by one element. Then by \[6\], \( Q \) is either homotopy equivalent to \( \mathbb{R} P^d \), or \( \tilde{Q} \) is homotopy equivalent to \( \mathbb{C} P^{2n+1} \). In the former case we are done, so assume the latter.

We denote \( \text{dim}(\mathbb{C} P^{2n+1}) = 2(2n + 1) = d \). Assume first that \( \mu_0 \geq 2 \), then \( \mu(c^-_k) = \mu_0 - d + 1 \leq \mu(c) - 2 \) for all other critical points \( c \). This means that \( c^-_1 \) is the lowest index generator of \( RFH^0_{-1}(A) \). The lowest index non-vanishing group is \( RFH^0_{-1}(A) \cong \tilde{H}_0(\Omega_q Q; \mathbb{Z}_2) \cong \mathbb{Z}_2 \) and thus \( \mu(c^-_1) = 0 \) and \( \mu_0 = d - 1 \). Recall that

\[
H_s(\mathbb{C} P^{2n+1}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } s = kd \text{ or } kd + 1 \text{ for } k \in \mathbb{N}_0, \\ 0 & \text{otherwise}. \end{cases}
\]

Since \( \Omega_q Q \) is homotopy equivalent to the disjoint union of two copies of \( \Omega_q \mathbb{C} P^{2n+1} \),

\[
H_*(\Omega_q Q; \mathbb{Z}_2) \cong H_*(\Omega_q \mathbb{C} P^{2n+1}) \oplus H_*(\Omega_q \mathbb{C} P^{2n+1}).
\]

In particular \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong H_{2d}(\Omega_q \mathbb{C} P^{2n+1}; \mathbb{Z}_2) \cong \tilde{H}_{2d}(\Omega_q Q; \mathbb{Z}_2) \cong RFH^0_{-1}(A) \), which is only possible if we have two generators of index \( 2d \). The \( c^-_k \) have pairwise different indices and so do the \( c^+_k \), thus there must be \( k^+ \) and \( k^- \) such that \( \mu(c^+_k) = k^+ \mu_0 - d + 1 = \mu(c^-_{k^+}) = k^+ \mu_0 = 2d \). But since \( \mu_0 = d - 1 \) and \( d \geq 6 \), this is impossible.

The remaining case is \( \mu_0 = 1 \). Since \( d \geq 6 \), the critical points with negative index are exactly

\[
\mu(c^-_1) = -d + 2, \quad \mu(c^-_2) = -d + 3, \ldots, \quad \mu(c^-_{d-2}) = -1.
\]

If the chord underlying \( c^-_1 \) were contractible in \( \Omega_{T^*Q} T^*Q \), then all chords underlying \( c^+_k \) would be contractible since they all are concatenations of chords homotopic to the chord underlying \( c^+_1 \). This contradicts the fact that they also generate the \( \mathbb{Z}_2 \)-homology of the connected component of noncontractible chords in \( \Omega_q Q \). Thus the chord underlying \( c^+_1 \) must be noncontractible. Since \( \pi_1(Q) = \mathbb{Z}_2 \) and since the chord underlying \( c^+_2 \) is the concatenation of two chords homotopic to the chord underlying \( c^+_1 \), the chord underlying \( c^+_2 \) is contractible and in particular not homotopic to the chord underlying \( c^+_1 \). The boundary operator is defined by flow lines with cascades with underlying Floer strips and paths in \( Crit A \), thus every chord underlying a
critical point is homotopic to the chords underlying the summands of its boundary. Thus $c^2_1$ cannot contribute to the boundary of $c^2_2$. Since all other critical points have higher index, we conclude that $c^2_1$ is not a boundary. Since $c^2_1$ is in the lowest degree chain group, it is closed and hence represents a non-trivial homology class. Thus $\text{RFH}^{>0}_{-d+2}(\mathcal{A}) \cong \tilde{H}_{-d+2}(\Omega^q Q; \mathbb{Z}_2)$ does not vanish, which is impossible since $-d + 2 < 0$. □

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