KRONECKER FACTORS FOR PERIODIC POINT FREE HOMEOMORPHISMS

ALEJANDRO KOCSARD

Abstract. We provide a complete characterization of periodic point free homeomorphisms of the 2-torus admitting irrational rotations as topological factors. Given a homeomorphism of the 2-torus without periodic points and exhibiting uniformly bounded rotational deviations with respect to a rational direction, we show that annularity and the geometry of its non-wandering set are the only possible obstructions for the existence of an irrational circle rotation as topological factor. Through a very precise study of the dynamics of the induced $\rho$-centralized skew-product, we extend and generalize considerably previous results of Jäger [Jäg09].

1. Introduction

A Kronecker system is nothing but a translation on a compact abelian topological group. Its mixed algebro-topological nature endowed the system with a rich variety of structures and allows us to combine different techniques (e.g. metric geometry, character theory) to study its dynamical and ergodic properties. Thus, it is not surprising at all that in topological dynamics the problem of determining the existence of Kronecker factors, i.e. a Kronecker system which is a topological factor of the original one, plays a fundamental role in the theory. Since any closed subsystem (i.e. the restriction of the dynamics to a closed invariant subset) of a Kronecker one is (conjugate to a) Kronecker itself, then in general one just considers minimal Kronecker factors. The reader can refer to [Tao09, §2.6] for a very nice exposition about Kronecker systems.

In this paper we are mainly concerned with the problem of characterizing those periodic point free homeomorphisms of the 2-torus $\mathbb{T}^2$ that admit non-trivial Kronecker factors, i.e. a minimal Kronecker factor which is not just a singleton. As a consequence of a recent result of Hauser and Jäger [HJ17] (see Theorem 2.9), we know that any homeomorphism $f: \mathbb{T}^2 \to \mathbb{T}^2$ admitting a minimal non-trivial Kronecker factor should be a topological extension of an irrational circle rotation. In such a case it is well-known (see for instance [JT17, Lemma 3.1]) that there exists $v \in \mathbb{R}^2 \setminus \{0\}$ with rational slope such that for any lift $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ of $f$, there exist $\rho = \rho(f, v) \in \mathbb{R} \setminus \mathbb{Q}$ and $C = C(f, v) > 0$ satisfying

\begin{equation}
\langle \tilde{f}^n(z) - z, v \rangle - n\rho \leq C, \quad \forall z \in \mathbb{R}^2, \forall n \in \mathbb{Z}.
\end{equation}

So, it is natural to ask whether estimate (1) is also sufficient to guarantee the existence of a non-trivial irrational circle factor.

It is known that this question has a positive answer in some particular cases. For instance, as a rather straightforward consequence of so-called Gottschalk-Hedlund theorem [GH55], one can show that any minimal torus homeomorphism verifying (1) necessarily admits a non-trivial Kronecker factor. A much subtler positive result is due to Jäger [Jäg09, Theorem C], who showed that any area-preserving
totally irrational pseudo-rotation of $T^2$ that exhibits uniformly bounded rotational deviations in any direction (i.e. its rotation set reduces to a totally irrational point and condition (1) holds for every $v \in \mathbb{R}^2 \setminus \{0\}$) is a topological extension of a totally irrational translation of $T^2$. For some time these were the only known positive results. At this point it is interesting to remark that Wang and Zhang have recently proved in [WZ18] the existence of a $C^2$ area-preserving diffeomorphism of $T^2$ which is a topological extension of a rigid rotation, but it is not conjugate to it. On the other hand, regarding the so called Franks-Misiurewicz conjecture, Pasqualeti and Sambarino [PS18] have recently shown that a homeomorphism which is homotopic to the identity and whose rotation set is a non-degenerate segment without rational points does not admit irrational circle factors.

On the other hand, Jäger and Tal has shown in [JT17, §4] that boundedness of rotational deviations does not in general imply the existence of irrational circle factors on the closed annulus $T \times [0,1]$. Consequently, annularity could be an obstruction for the existence of non-trivial Kronecker factors.

The main purpose of this work consists in getting a complete characterization of possible obstructions for the existence of such topological factors, showing that annularity and the existence of “large connected components of wandering points” are the only ones. More precisely, our main result is the following

**Theorem A.** Let $f \in \text{Homeo}_+(T^2)$ be an orientation-preserving, non-eventually annular homeomorphism having small wandering domains. Then, $f$ admits a non-trivial minimal Kronecker factor if and only if for any lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ with rational slope so that $\tilde{f}$ exhibits uniformly bounded rotational $v$-deviations, i.e. given any lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$, there exist $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and $C > 0$ such that

$$|\tilde{f}^n(z) - z, v| - np|, \leq C, \quad \forall z \in \mathbb{R}^2, \forall n \in \mathbb{Z}.$$  \hspace{1cm} (2)

Here small wandering domains means that all connected components of the wandering set are homotopically trivial in $T^2$ and their diameters are eventually small (see Definition 2.3 for details). This implies that we get, as a consequence of Theorem A, the following extension of Jäger’s main result of [Jäg09]:

**Corollary 1.1.** If $f : T^2 \to T^2$ is a totally irrational pseudo-rotation exhibiting uniformly bounded rotational deviations and having small wandering domains (e.g. $f$ is non-wandering), then $f$ is a topological extension of the corresponding minimal rigid translation of $T^2$.

Another consequence of Theorem A is the following

**Corollary 1.2.** If $f \in \text{Homeo}(T^2)$ is isotopic to the $k$-Dehn twist $I_k$, with $k \neq 0$ (see (10) for definition), and has small wandering domains, then $f$ admits a non-trivial Kronecker factor if and only if for any lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ of $f$, there exist $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and $C > 0$ such that

$$|\tilde{f}^n(z) - z| - np|, \leq C, \quad \forall z \in \mathbb{R}^2, \forall n \in \mathbb{Z},$$  \hspace{1cm} (3)

where $\text{pr}_2 : \mathbb{R}^2 \ni (x,y) \mapsto y.$
2. Preliminaries and notations

2.1. General topological dynamics. All along this article, \((M,d)\) will denote an arbitrary complete metric space. The (open) ball of radius \(r > 0\) centered at \(x \in M\) will be denoted by \(B_r(x)\). Given any \(A \subset M\), we write \(\delta_XA\) for the boundary of \(A\) in \(X\), and \(\overline{A}\) for its closure; its diameter is given by

\[
\text{diam } A := \sup \{d(x,y) : x, y \in A\}.
\]

We say \(A\) is bounded when \(\text{diam } A\) is finite.

If \(A\) is connected, we write \(\text{cc}(M,A)\) for the connected component of \(M\) containing \(A\). As usual, we write \(\pi_0(M)\) to denote the set of connected components of \(M\).

The group of self-homeomorphisms of \(M\) will be denoted by \(\text{Homeo}(M)\). We shall write \(\text{Homeo}_0(M)\) for the subgroup of homeomorphisms of \(M\) which are homotopic to the identity.

Given any \(f \in \text{Homeo}(M)\), we define its support as the closed set

\[
\text{supp } f := \{x \in M : f(x) \neq x\}.
\]

When \(M\) and \(N\) are two topological spaces, we say that \(f \in \text{Homeo}(M)\) is a topological extension of \(g \in \text{Homeo}(N)\) when there exists a surjective continuous map \(h: M \rightarrow N\) such that \(h \circ f = g \circ h\). In such a case, we say that \(g\) is a topological factor of \(f\) and the map \(h\) is called a semi-conjugacy. A fiber of the semi-conjugacy is nothing but the pre-image by \(h\) of any point of \(N\). The set of maps that are a semi-conjugacy between \(f\) and \(g\) will be denoted by \(SC(f,g)\), i.e. it is defined by

\[
\text{SC}(f,g) := \{h' \in \mathcal{C}^0(M,N) : h'(M) = N, h' \circ f = g \circ h'\}.
\]

Whenever \(M_1, M_2, \ldots, M_n\) are arbitrary sets, we shall use the generic notation \(p_r : M_1 \times M_2 \times \ldots \times M_n \rightarrow M_r\) to denote the \(r\)-th coordinate projection map. Similarly, when \(A\) is a subset of \(M_1 \times M_2 \times \ldots \times M_n\) and \(x \in M_1\) is an arbitrary point, we write \(A_x\) to denote the “the fiber of \(A\) over \(x\)” given by

\[
A_x := \{(x_2,\ldots,x_n) \in M_2 \times \ldots \times M_n : (x,x_2,\ldots,x_n) \in A\}.
\]

2.1.1. Recurrent and non-wandering points. Let \(f \in \text{Homeo}(M)\) be any homeomorphism. A point \(x \in M\) is said to be recurrent when there exists an increasing sequence of positive integers \((n_j)_{j \geq 1}\) such that \(f^{n_j}(x) \rightarrow x\), as \(j \rightarrow \infty\). An open subset \(U \subset M\) is said to be a wandering set (for \(f\)) when \(f^n(U) \cap U = \emptyset\), for every \(n \in \mathbb{Z}\setminus\{0\}\). A point \(x \in M\) is called non-wandering when none neighborhood of \(x\) is wandering. The non-wandering set, i.e. the set of non-wandering points, shall be denoted by \(\Omega(f)\); its complement, called the wandering set of \(f\), will be denoted by \(\mathcal{W}(f) := M \setminus \Omega(f)\).

When \(M\) is locally connected, each connected component of \(\mathcal{W}(f)\) is called a wandering domain. Finally, we say \(f\) is non-wandering when \(\Omega(f) = M\). We will need a new notion which is stronger than non-wandering:

**Definition 2.1.** A homeomorphism \(f : M \rightarrow M\) is said to be \(\Omega\)-recurrent when for every open subset \(U \subset M\) satisfying \(U \cap \Omega(f) \neq \emptyset\), there is \(n \geq 1\) such that

\[
U \cap f^{-n}(U) \cap \Omega(f) \neq \emptyset.
\]

Our main interest in this notion is due to the following elementary

**Lemma 2.1.** Let \(M\) be a complete metric space and \(f : M \rightarrow M\) be an \(\Omega\)-recurrent homeomorphism.

Then, recurrent points are dense within the set \(\Omega(f)\).
Proof. Let $x$ be any non-wandering point and $U$ an arbitrary open neighborhood of $x$. Without loss of generality, we can assume $U$ is bounded in $M$. Then we will inductively defined a sequence of nested open sets $\{U_k\}_{k \geq 0}$ and an increasing sequence of positive integers $\{n_k\}_{k \geq 0}$ as follows: first, let us define $U_0 := U$. Then, assuming $k \geq 1$ and $U_{k-1}$ has been already defined, we write

$$n_{k-1} := \min \{ n \geq 1 : U_{k-1} \cap f^{-n}(U_{k-1}) \cap \Omega(f) \neq \emptyset \}. \quad (6)$$

Then, we define $U_k$ as any open set satisfying the following properties:

$$\overline{U}_k \subset U_{k-1} \cap f^{-n_{k-1}}(U_{k-1}),$$
$$U_k \cap \Omega(f) \neq \emptyset,$$
$$\text{diam}(U_k) < \text{diam}(U_{k-1})/2.$$  

Observe that, since $f$ is $\Omega$-recurrent, $U_{k-1}$ is open and $\Omega(f)$ is closed, such an open set $U_k$ does exists and the natural number given by (6) is well defined.

By our hypothesis about the diameter of the sets $U_k$, it follows there is a unique point $y$ such that

$$\{y\} = \bigcap_{k=0}^{\infty} U_k = \bigcap_{k=0}^{\infty} \overline{U}_k,$$

and we claim $y$ is a recurrent point. In fact, $y \in U_{k+1} \subset U_k \cap f^{-n_k}(U_k)$, for every $k \geq 0$, and hence

$$d(f^{n_k}(y), y) \leq \text{diam}(U_k) \to 0, \quad \text{as } k \to \infty.$$  

On the other hand, by the very same definition it holds $n_k \geq n_{k-1}$, for very $k \geq 1$. This implies $y$ is a recurrent point and $y \in \Omega$. So, recurrent points are dense among non-wandering points. \qed

2.1.2. Minimal systems. Let $M$ be a compact metric space, $f: M \to M$ an arbitrary homeomorphism and $K \subset M$ be an $f$-invariant nonempty compact set. We say that $K$ is a minimal set (for $f$) when $K$ and the empty set are the only $f$-invariant compact subsets of $K$. When $M$ itself is a minimal set, we say that $f$ is a minimal system or a minimal homeomorphism.

2.1.3. Proximality relations. If $f: M \to M$ is a homeomorphism of the complete metric space $(M, d)$, we say that two points $x, y \in M$ are $f$-proximal when

$$\inf \{ d(f^n(x), f^n(y)) : n \geq 1 \} = 0. \quad (7)$$

Analogously we can define the notion of $f^{-1}$-proximality. Notice that, in general, these two notions do not coincide.

2.2. Orbits of open sets. Let $M$ be an arbitrary locally connected complete metric space and $f: M \to M$ be an arbitrary homeomorphism. Given any nonempty connected open set $V$, following Koropecki and Tal [KT14] we define

$$\mathcal{V}_f(V) := \text{cc} \left( \bigcup_{n \in \mathbb{Z}} f^n(V), V \right), \quad (8)$$

where $\text{cc}(\cdot, \cdot)$ denotes de connected component as defined at the beginning of §2.1. Notice that there exists $N \geq 1$ such that $f^N(\mathcal{V}_f(V)) = \mathcal{V}_f(V)$ if and only if $V$ is not a wandering set.
2.3. Kronecker factors. Let \((G,+)\) denote an abelian group. For each \(a \in G\), let us consider the translation \(T_a \in \text{Homeo}(G)\) given by \(T_a : G \ni g \mapsto g + a\). When \((G,+)\) is an abelian compact group, any such map is called a Kronecker system.

**Remark 2.2.** For the sake of simplicity of notations, given \(E \subset G\) and \(a \in G\) sometimes we shall write \(E + a\) to denote \(T_a(E)\).

All orbit closures of a Kronecker system are (topologically conjugate to) a Kronecker subsystem themselves. So, there is no significant loss of generality just considering minimal Kronecker systems.

We will say \(T_a : G \ni x \mapsto x + a\) is a non-trivial Kronecker system when \(T_a\) is minimal and \(G\) is not just a singleton.

By classical arguments one can easily show that any compact abelian group \(G\) admits a compatible distance \(d_G\) which is invariant by any translation, i.e. every Kronecker system is an isometry of \((G,d_G)\). In particular this implies that any Kronecker system is equicontinuous (i.e. the family of its iterates is equicontinuous) when \(G\) is endowed with an arbitrary compatible metric.

Reciprocally, it can be shown that any minimal equicontinuous homeomorphism of an arbitrary compact metric space is topologically conjugate to a minimal Kronecker system (see for instance [Tao09, Proposition 2.6.7] for details).

Given a homeomorphism of a compact metric space \(f : M \ni x \mapsto y\), a Kronecker factor of \(f\) will be a Kronecker system \(T_a : G \ni x \mapsto x + a\) which is a topological factor of \(f\), i.e. there exists a surjective continuous map \(h : M \rightarrow G\) satisfying \(h \circ f = T_a \circ h\). We say \(T_a : G \ni x \mapsto x + a\) is a non-trivial Kronecker factor when it is a minimal Kronecker factor and \(G\) does not reduce to a singleton.

We will need the following

**Definition 2.2.** Let \(M\) be a compact metric space and \(f : M \ni x \mapsto y\) be a homeomorphism. We say that two points \(x,y \in M\) are Kronecker equivalent for \(f\) if their images coincide under any semi-conjugacy of a Kronecker factor, i.e. for every Kronecker factor \(T_a : G \ni x \mapsto x + a\) and any \(h \in \text{SC}(f,T_a)\), it holds \(h(x) = h(y)\) (see (4) for this notation).

On the other hand, we say that \(x\) and \(y\) are Kronecker separated if and only if \(h(x) \neq h(y)\), for every non-trivial Kronecker factor \(T_a : G \ni x \mapsto x + a\) and every \(h \in \text{SC}(f,T_a)\).

**Remark 2.3.** Notice that, if \(f : M \ni x \mapsto y\) is as in Definition 2.2, then two points \(x,y \in M\) which are either \(f\)-proximal or \(f^{-1}\)-proximal are necessarily Kronecker equivalent.

2.4. Tori and torus homeomorphisms. We will always consider \(\mathbb{R}^d\) endowed with the Euclidean inner product \(\langle v,w \rangle := \sum_{i=1}^d v_i w_i\) and the induced Euclidean norm \(\|v\| := \sqrt{\langle v,v \rangle}\).

As usual, the \(d\)-torus \(\mathbb{T}^d\) will be denoted by \(\mathbb{R}^d / \mathbb{Z}^d\) and we write \(\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d\) for the canonical quotient projection. Notice we are using the same letter \(\pi\) to denote the torus universal covering map independently of its dimension.

We shall always assume \(\mathbb{T}^d\) endowed with the distance function \(d_{\mathbb{T}^d}\) given by

\[
(9) \quad d_{\mathbb{T}^d}(x,y) := \min \left\{ \| \xi - \eta \| : \xi \in \pi^{-1}(x), \ \eta \in \pi^{-1}(y) \right\}, \quad \forall x,y \in \mathbb{T}^d.
\]

An open set \(D \subset \mathbb{R}^d\) is called a fundamental domain for the covering \(\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d\) when \(\pi\) is injective on \(D\) and \(\pi(\partial D) = \mathbb{T}^d\).

As a particular case of Kronecker system we have torus translations \(T_a : \mathbb{T}^d \ni x \mapsto x + a\), with \(a \in \mathbb{T}^d\). By some abuse of notation and for the sake of simplicity, if \(a \in \mathbb{R}^d\), we shall just write \(T_a\) to denote \(T_{\pi(a)}\).
A vector $\alpha \in \mathbb{R}^d$ is called rational when $\alpha \in \mathbb{Q}^d$; otherwise, it is called irrational. Moreover, it is called totally irrational when the Haar measure of $\mathbb{T}^d$ is ergodic for the translation $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$.

It is well known that given any $f \in \text{Homeo}(\mathbb{T}^d)$, there exists a unique matrix $A_f \in \text{GL}(d, \mathbb{Z})$ such that the map $\tilde{f} = A_f : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathbb{Z}^d$-periodic, for any lift $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ of $f$. The matrix $A_f$ is nothing but a matrix representation of the action induced by $f$ on the first homology group of $\mathbb{T}^d$, and one can easily check that two homeomorphisms $f, g \in \text{Homeo}(\mathbb{T}^d)$ are homotopic if and only if $A_f = A_g$.

Finally, given any $k \in \mathbb{Z}$, we write
\begin{equation}
I_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{R}),
\end{equation}
and define
\begin{equation}
\text{Homeo}_0(\mathbb{T}^2) := \left\{ f \in \text{Homeo}(\mathbb{T}^2) : A_f = I_k \right\}.
\end{equation}

2.4.1. Rotation set and rotation vectors. We write $\text{Homeo}_0(\mathbb{T}^d)$ to denote the group of torus homeomorphisms which are homotopic to the identity.

After Misiurewicz and Ziemian [MZ89], given any lift $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ of a homeomorphism $f \in \text{Homeo}_0(\mathbb{T}^d)$ one defines the rotation set of $\tilde{f}$ by
\begin{equation}
\rho(\tilde{f}) = \bigcap_{m \geq 0} \bigcup_{n \geq m} \left\{ \frac{f^n(z) - z}{n} : z \in \mathbb{R}^d \right\}.
\end{equation}

It can be easily shown that the rotation set $\rho(\tilde{f})$ is always nonempty, compact and connected.

For $d = 1$, by classical Poincaré theory [Poi80] of circle homeomorphisms we know that $\rho(\tilde{f})$ is always a singleton and its class modulo $\mathbb{Z}$ depends just on $f$ and not on the chosen lift. So, in such a case one can define $\rho(f) := \pi(\rho(\tilde{f})) \in \mathbb{T}$.

In higher dimensions, in general, the rotation set is not singleton. So, we will say that $f \in \text{Homeo}_0(\mathbb{T}^d)$ is a pseudo-rotation whenever $\rho(\tilde{f})$ is a singleton for some, and hence any, lift $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ of $f$; and $f$ is said to be a totally irrational pseudo-rotation when $\rho(\tilde{f})$ reduces to a point which is a totally irrational one.

On the other hand, given an $f$-invariant Borel probability measure $\mu$ one can define the $\mu$-rotation vector of $\tilde{f}$ by
\begin{equation}
\rho_\mu(\tilde{f}) := \int_{\mathbb{T}^d} \Delta_f \mathrm{d}\mu.
\end{equation}
By classical convexity arguments and Birkhoff ergodic theorem it can be easily checked that
\[
\text{Conv}(\rho(\tilde{f})) = \{ \rho_\mu(\tilde{f}) : \mu \in \mathcal{M}(f) \},
\]
where $\mathcal{M}(f)$ is the space of $f$-invariant Borel probability measures and $\text{Conv}(\cdot)$ denotes the convex hull operator.

For $d = 2$, Misiurewicz and Ziemian [MZ89, Theorem 3.4] showed that the rotation set is convex indeed, so it coincides with the set of rotation vectors of measures.

On the other hand, when $f \in \text{Homeo}_0(\mathbb{T}^2)$ with $k \neq 0$, one cannot define the rotation set as above, but at least one can define the vertical rotation set as in [AZ05], given by
\begin{equation}
\rho_v(\tilde{f}) = \bigcap_{m \geq 0} \bigcup_{n \geq m} \left\{ \frac{\text{pr}_2(f^n(z) - z)}{n} : z \in \mathbb{R}^2 \right\} \subset \mathbb{R},
\end{equation}
where $\text{pr}_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the second coordinate.
where \( \tilde{f} : \mathbb{R}^2 \to \) is any lift of \( f \). Analogously, for any \( \mu \in \mathcal{M}(f) \) one defines its vertical rotation number by

\[
\rho_{\mu,N}(\tilde{f}) := \int_{\mathbb{T}^2} \text{pr}_2 \circ \Delta_{\tilde{f}} \, d\mu.
\]

2.4.2. Dimension two. Let us fix some particular notations for the two-dimensional case. Given any \( v = (a,b) \in \mathbb{R}^2 \), we define \( v^\perp := (-b,a) \). A vector \( v \in \mathbb{R}^2 \setminus \{0\} \) is said to have rational slope when there is \( \lambda \in \mathbb{R} \setminus \{0\} \) such that \( \lambda v \in \mathbb{Z}^2 \); and irrational slope otherwise.

We write \( \mathbb{A} := \mathbb{T} \times \mathbb{R} \) for the open annulus. We consider the covering maps \( \hat{\pi} : \mathbb{R}^2 \to \mathbb{A} \) and \( \hat{\pi} : \mathbb{A} \to \mathbb{T}^2 \) given by the natural quotient projections

\[
\begin{align*}
\hat{\pi} : \mathbb{R}^2 &\ni (\bar{x}, \bar{y}) \mapsto (\bar{x} + \mathbb{Z}, \bar{y}) \in \mathbb{A}, \\
\hat{\pi} : \mathbb{A} &\ni (x, y) \mapsto (x, \bar{y} + \mathbb{Z}) \in \mathbb{T}^2. 
\end{align*}
\]

Observe that \( \pi = \hat{\pi} \circ \tilde{\pi} : \mathbb{R}^2 \to \mathbb{T}^2 \).

We will always considered \( \mathbb{A} \) endowed with the distance function \( d_\mathbb{A} \) given by

\[
d_\mathbb{A}(z_0, z_1) := \min \left\{ \|z_1 - z_0\| : z_i \in \hat{\pi}^{-1}(z_i), \ i = 0, 1 \right\},
\]

for every \( z_i \in \mathbb{A} \) and \( i = 0, 1 \).

For each \( s \in \mathbb{R} \), we define \( \hat{T}_s \in \text{Homeo}_0(\mathbb{A}) \) by \( \hat{T}_s : (x, y) \mapsto (x, \bar{y} + s) \).

If \( S \) is any surface, by a topological disk in \( S \) we mean an open subset of \( S \) which is homeomorphic to the unit disc \( \{ z \in \mathbb{R}^2 : \|z\| < 1 \} \). Analogously, a topological annulus is an open subset of \( S \) which is homeomorphic to \( \mathbb{A} \).

If \( U \subset \mathbb{T}^2 \) is an open nonempty set, let us consider the group homomorphism induced by the inclusion on first homology groups \( i : H_1(U, \mathbb{Z}) \to H_1(\mathbb{T}^2, \mathbb{Z}) \). The set \( U \) is said to be inessential when \( i = 0 \), and essential otherwise. The set \( U \) is called annular when the image of \( i \) has rank 1 and fully essential when its rank is equal to 2, i.e. when \( i \) is surjective.

An arbitrary subset \( E \) of either \( \mathbb{T}^2 \) or \( \mathbb{A} \) is said to be inessential when its inclusion morphism \( i : H_1(U, \mathbb{Z}) \to H_1(\mathbb{A}, \mathbb{Z}) \) is identically zero; and it is said to be annular otherwise.

An arbitrary subset \( E \) of either \( \mathbb{T}^2 \) or \( \mathbb{A} \) is said to be inessential when there exists an inessential open set \( U \) containing \( E \); otherwise, is said to be essential.

A compact connected subset \( C \subset \mathbb{T}^2 \) is called an annular continuum when its complement \( \mathbb{T}^2 \setminus C \) is homeomorphic to \( \mathbb{A} \). On the other hand, we say that a set \( C \subset \mathbb{A} \) is an annular continuum when it is compact, connected and \( \mathbb{A} \setminus C \) has exactly two connected components, which will be denoted by \( U^+(C) \) and \( U^-(C) \), and they are characterized by the fact that there is a real number \( K > 0 \) such that \( T \times (K, +\infty) \subset U^+(C) \) and \( T \times (-\infty, -K) \subset U^-(C) \).

The filling of an inessential open subset \( U \) of \( \mathbb{T}^2 \) is defined as the union of \( U \) with all the inessential connected components of its complement, and will be denoted by \( \text{Fill}(U) \). Notice that \( \text{Fill}(U) \) is an open inessential subset itself.

A connected open set \( U \subset \mathbb{T}^2 \) (\( U \subset \mathbb{A} \), respectively) is said to be lift-bounded when every connected component of \( \pi^{-1}(U) \) (\( \hat{\pi}^{-1}(U) \), respectively) is bounded in \( \mathbb{R}^2 \); and it is called lift-unbounded otherwise. Notice that every lift-bounded set is necessarily inessential, but there do exist open inessential lift-unbounded subsets of \( \mathbb{T}^2 \) and \( \mathbb{A} \).

The main reason why we are interested in the spaces \( \text{Homeo}_k(\mathbb{T}^2) \) of homeomorphisms given by \((11)\) is given by the following

**Proposition 2.4.** If \( f : \mathbb{T}^2 \to \mathbb{R}^2 \) is an orientation preserving fixed point free homeomorphism, then there exists a unique \( k \in \mathbb{Z} \) such that \( f \) is topologically conjugate to an element of \( \text{Homeo}_k(\mathbb{T}^2) \).
Proof. This is a straightforward consequence of Lefschetz fixed point theorem. In fact, if \( f \) is orientation preserving and has no fixed point, then it must hold
\[
0 = L(f) = \sum_{i=0}^{2} (-1)^i \text{tr}(f_{*i}; H_1(\mathbb{T}^2, \mathbb{Q}) \cong 2 - \text{tr}(f_{*1}; H_1(\mathbb{T}^2, \mathbb{Q}) \cong 0),
\]
where \( L(f) \) denotes the Lefschetz number of \( f \). But the matrix \( A_f \in \text{SL}(2, \mathbb{Z}) \) is a representative of \( f_{*1} \), and hence, 1 is the only eigenvalue of \( A_f \). This implies, by classical Jordan normal form theorem, there exists \( B \in \text{SL}(2, \mathbb{Z}) \) and \( k \in \mathbb{Z} \) such that \( BA_fB^{-1} = I_k \). Then, the linear map \( B \) induces (i.e. is the lift to \( \mathbb{R}^2 \) of) a Lie group automorphism \( B: \mathbb{T}^2 \cong \), and one can easily check that \( B \circ f \circ B^{-1} \in \text{Homeo}_0(\mathbb{T}^2) \).

Let us recall a classical result about fixed point free plane homeomorphisms due to Brouwer:

**Theorem 2.5** (Brouwer’s translation theorem, see [Fat87]). Let \( f: \mathbb{R}^2 \cong \) be an orientation preserving homeomorphism such that \( \text{Fix}(f) = \emptyset \). Then, every \( x \in \mathbb{R}^2 \) is wandering for \( f \), i.e. \( \Omega(f) = \emptyset \).

If \( f \in \text{Homeo}(\mathbb{T}^2) \) and \( x \in \Omega(f) \), following Koropecki and Tal [KT14] we say that \( x \) is inessential when there exists \( \varepsilon > 0 \) such that the open set \( \mathcal{U}_f(B_\varepsilon(x)) \), given by (8), is inessential; otherwise is said to be an essential point. Moreover, \( x \) is said to be a fully essential point when \( \mathcal{U}_f(B_\varepsilon(x)) \) is fully essential, for every \( \varepsilon > 0 \).

We have the following results for periodic point free homeomorphisms:

**Proposition 2.6.** If \( f: \mathbb{T}^2 \cong \) is a periodic point free homeomorphism and \( x \in \Omega(f) \), then \( x \) is an essential point.

**Proof.** This is an easy consequence of Theorem 2.5. In fact, if \( x \in \Omega(f) \) and there is an open neighborhood \( V \) of \( x \) such that \( \mathcal{U}_f(V) \) is inessential, hence the filling \( \text{Fill}(\mathcal{U}_f(V)) \), as defined in 2.4.2, is \( fN \)-invariant, for some \( N \in \mathbb{N} \). Notice that \( \text{Fill}(\mathcal{U}_f(V)) \) is homeomorphic to \( \mathbb{R}^2 \) and \( \text{Fill}(\mathcal{U}_f(V)) \) exhibits a non-wandering point (because \( x \in \Omega(f) \cap V \)). Thus, by Brouwer’s translation theorem (Theorem 2.5), \( f^2N \) has a fixed point in \( \text{Fill}(\mathcal{U}_f(V)) \), contradicting the fact that \( f \) is periodic point free.

**Proposition 2.7.** If \( f: \mathbb{T}^2 \cong \) is a periodic point free homeomorphism and \( W \subset \mathbb{T}^2 \) is a lift-bounded wandering domain (i.e. \( W \) is a connected component of \( \mathcal{W}(f) \)), then \( f^n(W) \cap W = \emptyset, \quad \forall n \in \mathbb{Z} \setminus \{0\} \).

**Proof.** Let us suppose there exists \( n \in \mathbb{Z} \setminus \{0\} \) such that \( f^n(W) \cap W \neq \emptyset \). Since \( W \) is a connected component of an \( f \)-invariant set, this implies that \( f^n(W) = W \). So, if \( W \) is a connected component of \( \mathbb{T}^2 \), then there exists a homeomorphism \( G: \mathbb{R}^2 \cong \) which is a lift of \( f^n \) and such that \( G(W) = W \). Since \( W \) is bounded in \( \mathbb{R}^2 \), this implies that \( \Omega(G) \) is nonempty and, invoking Brouwer’s translation theorem we conclude that \( G^2 \) has fixed point, contradicting the fact that \( f \) is periodic point free.

Motivated by this result, we propose the following

**Definition 2.3.** We say that a homeomorphism \( f: \mathbb{T}^2 \cong \) exhibits small wandering domains when every connected component of the wandering set \( \mathcal{W}(f) \) is lift-bounded and, given any \( \delta > 0 \), there exist just finitely many connected components with diameter larger that \( \delta \), i.e. the following set
\[
\{ W \in \pi_0(\mathcal{W}(f)) : \text{diam}(W) \geq \delta \}
\]
is finite, for every positive number $\delta$.

From this geometric property of the wandering set we get the following dynamical consequence:

**Proposition 2.8.** If $f \in \text{Homeo}(\mathbb{T}^2)$ is periodic point free and exhibits small wandering domains, then it is $\Omega$-recurrent (see Definition 2.1).

**Proof.** Reasoning by contradiction, let us suppose there exists an open set $U \subset \mathbb{T}^2$ such that $U \cap \Omega(f) \neq \emptyset$, but

$$U \cap f^n(U) \cap \Omega(f) = \emptyset, \quad \forall n \in \mathbb{Z}\setminus\{0\}. \quad (17)$$

Let $z$ be an arbitrary point of $U \cap \Omega(f)$ and $\varepsilon$ be a positive number such that the ball of radius $\varepsilon$ centered at $z$ satisfies $B_\varepsilon(z) \subset U$. Taking into account that $f$ exhibits small wandering domain, we know that

$$W_\varepsilon(z, \varepsilon) := \left\{ D \in \pi_0(W(f)) : B_\varepsilon(z) \cap D \neq \emptyset, \, \overline{D} \cap (\mathbb{T}^2 \setminus U) \neq \emptyset \right\}$$

is a finite set.

Then, let us consider the set

$$\tau := \left\{ n \in \mathbb{Z}\setminus\{0\} : f^n(B_\varepsilon(z)) \cap B_\varepsilon(z) \neq \emptyset \right\}. \quad (18)$$

Notice that since $z$ is a non-wandering point, the set $\tau$ is nonempty, and moreover, is infinite.

On the other hand, by (17) we know that

$$f^n(B_\varepsilon(z)) \cap B_\varepsilon(z) \subset W(f), \quad \forall n \in \tau. \quad (19)$$

Then, if $D$ is a connected component of the wandering set $W(f)$ such that $\overline{D} \subset U$, its boundary $\partial D \subset \Omega(f)$ and hence, by (17), it holds $f^n(\overline{D}) \cap U = \emptyset$, for every $n \in \mathbb{Z}\setminus\{0\}$.

This implies that

$$f^n(B_\varepsilon(z)) \cap B_\varepsilon(z) \subset \bigcup_{D \in W_\varepsilon(z, \varepsilon)} D, \quad \forall n \in \tau. \quad (20)$$

However, since $f$ is periodic point free, by Proposition 2.7 we know that every connected component of $W(f)$ is indeed a wandering set for $f$, and we know that the set $W_\varepsilon(z, \varepsilon)$ is finite. So, putting together (18), (19) and (20) we conclude the set $\tau$ is finite, contradicting the fact that $z$ was non-wandering. \qed

### 2.5. Kronecker factors of homeomorphisms of $\mathbb{T}^2$.

As we have already mentioned in §2.3, any minimal equicontinuous homeomorphism of a compact metric space is topologically conjugate to a Kronecker system. So, invoking a recent result due to Hauser and Jäger [HJ17, Theorem B], we have the following

**Theorem 2.9.** If $f \in \text{Homeo}(\mathbb{T}^2)$ and $T_\theta : G \to G$ is a minimal Kronecker factor of $f$, then the group $G$ is either equal to $\mathbb{T}^2$, $\mathbb{T}$ or the trivial space $\{\ast\}$.

According to the classification given by Theorem 2.9, any homeomorphism $f \in \text{Homeo}(\mathbb{T}^2)$ admitting a non-trivial Kronecker factor admits either a circle Kronecker factor or a $\mathbb{T}^2$ Kronecker factor.

On the other hand, observe that by our definitions, any homeomorphism admitting a non-trivial Kronecker factor is necessarily periodic point free.

The following simple lemma imposes some well-known restrictions for the existence of non-trivial Kronecker factors:
Lemma 2.10. Let \( f : \mathbb{T}^2 \xrightarrow{} \) be an orientation-preserving homeomorphism admitting a non-trivial Kronecker factor, and \( \tilde{f} : \mathbb{R}^2 \xrightarrow{} \) be a lift of \( f \). Then there exist \( \rho \in \mathbb{R} \cup \mathbb{Q} \), \( v \in \mathbb{S}^1 \) with rational slope and \( C > 0 \) such that
\[
|\langle \tilde{f}^n(z) - z, v \rangle - n\rho| \leq C, \quad \forall z \in \mathbb{R}^2, \quad \forall n \in \mathbb{Z}.
\]

Proof. See for instance [JT17, Lemma 3.1].

The direction of the vector \( \bar{v} \), where \( v \) is as in Lemma 2.10, is called the homological direction of the circle factor. Notice that, when \( f \) is not homotopic to the identity, the homological direction of a Kronecker circle factor is unique.

As a straightforward consequence of Lemma 2.10 we get the following

Corollary 2.11. An orientation preserving homeomorphism \( f : \mathbb{T}^2 \xrightarrow{} \) admits a minimal \( \mathbb{T}^2 \) Kronecker factor if and only if it is a totally irrational pseudo-rotation admitting two rationally independent irrational circle Kronecker factors with non-collinear homological directions.

2.6. Rotational deviations. Given an orientation preserving homeomorphism \( f : \mathbb{T}^2 \xrightarrow{} \), we say that \( f \) exhibits uniformly bounded \( v \)-deviations, for some \( v \in \mathbb{S}^1 \), whenever given any lift \( \tilde{f} : \mathbb{R}^2 \xrightarrow{} \) there exist \( \rho \in \mathbb{R} \) and \( C > 0 \) such that
\[
|\langle \tilde{f}^n(z) - z, v \rangle - n\rho| \leq C, \quad \forall z \in \mathbb{R}^2, \quad \forall n \in \mathbb{Z}.
\]

Notice that when \( v \) has irrational slope, then the homeomorphism \( f \) must be isotopic to the identity. However, in this work, as a consequence of Lemma 2.10, we are mainly concerned with the case where \( v \) has rational slope. In such a case, after conjugacy with a linear automorphism of \( \mathbb{T}^2 \) if necessary, we can assume \( v = (0,1) \in \mathbb{R}^2 \) and then we shall say that \( f \) exhibit uniformly bounded vertical deviations.

Observe that in this case, by Proposition 2.4, one can conclude that \( f \) belongs to \( \text{Homeo}_k(\mathbb{T}^2) \), for some \( k \in \mathbb{Z} \), and the number \( \rho \) is the only element of the vertical rotation set defined in §2.4.1.

As a particular case of our previous definition, an arbitrary homeomorphism \( f : \mathbb{T}^2 \xrightarrow{} \) is said to be annular when there exist a lift \( \tilde{f} : \mathbb{R}^2 \xrightarrow{} \) of \( f \), a vector \( v \in \mathbb{S}^1 \) with rational slope and a constant \( C > 0 \) such that
\[
|\langle \tilde{f}^n(z) - z, v \rangle| \leq C, \quad \forall z \in \mathbb{R}^2, \quad \forall n \in \mathbb{Z}.
\]

More generally, we say \( f \) is eventually annular when there exists \( k \in \mathbb{N} \) such that \( f^k \) is annular.

We will need the following improvement of Proposition 2.6:

Proposition 2.12. If \( f \in \text{Homeo}(\mathbb{T}^2) \) is periodic point free and is not eventually annular, then every non-wandering point is fully essential for \( f \).

Proof. This is consequence of Proposition 2.6 and the following simple remark: if \( x \in \Omega(f) \) is essential but is not fully essential, then there is an open connected neighborhood \( V \) of \( x \) such that \( \Omega_f(V) \) is an annular set and there is \( k \in \mathbb{N} \) such that \( f^k(\Omega_f(V)) = \Omega_f(V) \). This clearly implies that \( f^k \) is an annular homeomorphism.

On the other hand, we will say that the homeomorphism \( f \in \text{Homeo}_k(\mathbb{T}^2) \) exhibits unbounded horizontal deviations when
\[
\sup \left\{ |\text{pr}_1(\tilde{f}^n(z) - z) - \text{pr}_1(\tilde{f}^n(w) - w)| : w,z \in \mathbb{R}^2, \quad n \in \mathbb{Z} \right\} = \infty.
\]

Notice that when \( k \neq 0 \), any element of \( \text{Homeo}_k(\mathbb{T}^2) \) exhibits unbounded horizontal deviations.

The following lemma is a simple generalization of a result of [KPR18]:
Lemma 2.13. Let \( f \in \text{Homeo}_0(\mathbb{T}^2) \) and \( \tilde{f} : \mathbb{R}^2 \rhd \) be a lift of \( f \). Then the following properties are equivalent:

(1) estimate \((22)\) holds;

(2) we have

\[
\sup \left\{ |\text{pr}_1(\tilde{f}^n(z) - z) - \text{pr}_1(\tilde{f}^n(w) - w)| : w, z \in \mathbb{R}^2, \ n \in \mathbb{N} \right\} = \infty;
\]

(3) it holds

\[
\sup \left\{ |\text{pr}_1(\tilde{f}^n(z) - z) - \text{pr}_1(\tilde{f}^n(w) - w)| : w, z \in \mathbb{R}^2, \ n \in \mathbb{N} \right\} = \infty.
\]

Proof. First observe that given any \( z \in \mathbb{R}^2, \) we have

\[
\tilde{f}^n(z) - \tilde{f}^n(z + (0, 1)) = l_k^n(0, 1) = (kn, 1), \quad \forall n \in \mathbb{Z}.
\]

This implies that, when \( k \neq 0, \) \((22), (23)\) and \((24)\) hold simultaneously, for every \( f \in \text{Homeo}_0(\mathbb{T}^2) \) and any lift \( f : \mathbb{R}^2 \rhd. \)

So let us just consider the case \( k = 0. \) It is clear that condition \((22)\) holds if and only if either \((23)\) or \((24)\) holds. So, it just remains to prove that \((23)\) and \((24)\) are equivalent.

For the sake of concreteness, let us suppose that \((23)\) is false, i.e. there is \( C > 0 \) such that

\[
|\text{pr}_1(\tilde{f}^n(z) - z) - \text{pr}_1(\tilde{f}^n(w) - w)| \leq C, \quad \forall w, z \in \mathbb{R}^2, \forall n \geq 1.
\]

Then write \( \Delta_1 := \text{pr}_1 \circ \Delta_f \) and observe that

\[
|\text{pr}_1(\tilde{f}^n(z) - z) - \Delta_1(f^{i}(\pi(z)))| \leq C, \quad \forall n \geq 1.
\]

Combining Birkhoff ergodic theorem and \((25)\) we conclude there is \( \rho \in \mathbb{R} \) such that

\[
\frac{1}{n} \sum_{j=0}^{n-1} \Delta_1 \circ f^j \rightarrow \rho, \quad \text{as } n \rightarrow +\infty,
\]

where the convergence is uniform in \( \mathbb{T}^2. \) Moreover, by \((25)\), it holds

\[
\left| \sum_{j=0}^{n-1} \Delta_1(f^j(z)) - n\rho \right| \leq C, \quad \forall z \in \mathbb{T}^2, \quad \forall n \geq 1.
\]

Finally, by [KPR18, Corollary 3.2], we know that this last estimate implies

\[
|\text{pr}_1(\tilde{f}^{-n}(z) - z) - (-n)\rho| \leq C, \quad \forall z \in \mathbb{R}^2, \quad \forall n \geq 1,
\]

and hence \((24)\) does not hold when \((23)\) does not either.

Finally, a homeomorphism \( f \in \text{Homeo}_0(\mathbb{T}^2) \) is said to be a pseudo-rotation with uniformly bounded rotational deviations when \( f \) is a pseudo-rotation and there is a constant \( C > 0 \) such that

\[
|f^n(z) - z - n\rho| \leq C, \quad \forall z \in \mathbb{R}^2, \forall n \in \mathbb{Z},
\]

where \( \tilde{f} : \mathbb{R}^2 \rhd \) is a lift of \( f \) and its rotation set satisfies \( \rho(\tilde{f}) = \{\rho\}. \)

Let us finish this section with two lemmas about torus homeomorphisms exhibiting unbounded deviations in certain direction:
Lemma 2.14. Let \( f \in \text{Homeo}_k(T^2) \), \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( f \) and suppose \( f \) exhibits unbounded horizontal rotational deviations, i.e. condition (22) holds. If \( x \in \Omega(f) \) is a fully essential point for \( f \) and \( V \) is a connected neighborhood of \( x \), then it holds
\[
\sup_{n \in \mathbb{N}} \text{diam} \left( \text{pr}_1(\tilde{f}^n(V)) \right) = \infty,
\]
for every connected component \( \tilde{V} \) of \( \pi^{-1}(V) \).

Proof. Without loss of generality we can assume \( V \) is open and lift-bounded. Since \( x \) is fully essential, there are two continuous simple closed curves \( \alpha_1, \alpha_2 : T \to T^2 \) that generates the first homology group \( H_1(T^2, \mathbb{Z}) \) and such that their images are contained in \( \mathcal{U}_f(V) \). This implies there exists \( C > 0 \) such that
\[
\text{diam } D < C, \quad \forall D \in \pi_0(\mathbb{R}^2 \setminus \pi^{-1}(\alpha_1 \cup \alpha_2)),
\]
where, by some abuse of notation, we are just writing \( \alpha_i \) to denote the image of the curve \( \alpha_i \), and \( \pi_0(\cdot) \) denotes the set of connected components of the corresponding space.

Then, let us consider the open set
\[
Q := (0,1)^2 \cup \left\{ D \in \pi_0(\mathbb{R}^2 \setminus \pi^{-1}(\alpha_1 \cup \alpha_2)), \ D \cap (0,1)^2 \neq \emptyset \right\}.
\]
Notice that, by the previous estimate, \( Q \) is bounded in \( \mathbb{R}^2 \) and its boundary is contained in \( \pi^{-1}(\alpha_1 \cup \alpha_2) \).

So, if \( \tilde{V} \) is an arbitrary connected component of \( \pi^{-1}(V) \), for each \( z \in \partial Q \) there exist \( n_z \in \mathbb{Z} \) and \( p_z \in \mathbb{Z}^2 \) such that
\[
z \in \tilde{f}^{n_z}(\tilde{V}) + p_z,
\]
where we are using the notation introduced in Remark 2.2.

By compactness of \( \partial Q \), there exist finitely many points \( z_1, \ldots, z_\ell \in \partial Q \) such that
\[
\partial Q \subseteq \bigcup_{i=1}^{\ell} \tilde{f}^{n_{z_i}}(\tilde{V}) + p_{z_i}.
\]
This implies
\[
\tilde{f}^n(\partial Q) \subseteq \bigcup_{i=1}^{\ell} \tilde{f}^{n_{z_i} + i_k}(\tilde{V}) + i_k p_{z_i}, \quad \forall n \in \mathbb{Z},
\]
where \( i_k \) denotes the integer matrix given by (10).

Then, since \( Q \) contains a fundamental domain and (22) holds, by (23) of Lemma 2.13 we conclude that
\[
\sup_{n \in \mathbb{N}} \text{diam } \tilde{f}^n(\partial Q) = \infty.
\]
Now, for each \( n \in \mathbb{Z} \), \( \tilde{f}^n(\partial Q) \) is covered by \( \ell \) (integer translations of) images of \( \tilde{V} \), so estimate (27) must hold.

\( \square \)

Lemma 2.15. Let \( f \in \text{Homeo}_k(T^2) \), \( \tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2 \) be a lift of \( f \) and suppose \( f \) exhibits uniformly bounded vertical rotational deviations and unbounded horizontal rotational deviations. Let \( x \in \Omega(f) \) be a fully essential point for \( f \), \( \tilde{V} \) a neighborhood of some point \( \tilde{x} \in \pi^{-1}(x) \subset \mathbb{R}^2 \), and \( N_0 \) be a natural number. Then there exists \( m \in \mathbb{N} \), such that for every \( j \in \{m, m+1, \ldots, m+N_0\} \), there are \( p_j, p'_j, q_j \in \mathbb{Z} \), with \( p_j \neq p'_j \), such that
\[
\tilde{f}^{j}(\tilde{V}) \cap T(p_j q_j)(\tilde{V}) \neq \emptyset, \quad \text{and} \quad \tilde{f}^{j}(\tilde{V}) \cap T(p'_j q_j)(\tilde{V}) \neq \emptyset,
\]
for every \( j \in \{m, m+1, \ldots, m+N_0\} \).
Before proving Lemma 2.15, we need the following elementary combinatorial result:

**Lemma 2.16.** Let \( A = \{n_1, n_2, \ldots, n_\ell\} \) be an arbitrary nonempty set of integer numbers and \( N_0 \) be a natural number. Then, there exists an integer \( M_0 \geq N_0 \) such that for every \( m' \in \mathbb{N} \) and any function \( \xi: \{m', m' + 1, \ldots, m' + M_0\} \to A \), there is \( m \in \mathbb{Z} \) such that

\[
\{m, m + 1, \ldots, m + N_0\} \subset \{j - \xi(j): j \in \{m', m' + 1, \ldots, m' + M_0\}\}.
\]

**Proof of Lemma 2.16.** Let us suppose the elements of \( A \) are ordered in the following way: \( n_1 \leq n_2 \leq \ldots \leq n_\ell \). Then, let us define

\[
M_0 := N_0 + n_\ell - n_1,
\]

\[
m := m' - n_1.
\]

For each \( 1 \leq i \leq \ell \), consider the set \( B_i := \{j - n_i: j \in \{m', m' + \ldots, m' + M_0\}\} \).

Then, the lemma easily follows from the following simple remark:

\[
B_i \cap B_k \supset B_1 \cap B_\ell = \{n \in \mathbb{Z}: m' - n_1 \leq n \leq m' + M_0 - n_\ell\}
\]

\[
= \{m, m + 1, \ldots, m + M_0\},
\]

for every \( 1 \leq i \leq k \leq \ell \).

**Proof.** Without loss of generality we can assume \( \tilde{V} = B_r(\tilde{x}) \), with \( 0 < r < 1/4 \). Then, \( V := \pi(\tilde{V}) = B_r(x) \subset \mathbb{T}^2 \), where \( x = \pi(\tilde{x}) \in \Omega(\tilde{f}) \) is a fully essential point for \( f \).

Repeating the argument we used in the proof of Lemma 2.14, there are two continuous simple closed curves \( a_1, a_2: \mathbb{T} \to \mathbb{T}^2 \) that generate the first homology group \( H_1(\mathbb{T}^2, \mathbb{Z}) \) and such that their images are contained in \( \tilde{M}(V) \).

Then let us consider the fundamental domain \( Q \) given by (28). Let \( z_1, \ldots, z_\ell \in \partial Q, n_{z_1}, \ldots, n_{z_\ell} \in \mathbb{Z} \) and \( p_{z_1}, \ldots, p_{z_\ell} \in \mathbb{Z}^2 \) such that condition (29) holds. Let \( M_0 \) be the natural number given by Lemma 2.16 associated to the natural numbers \( n_{z_1}, \ldots, n_{z_\ell} \) and \( N_0 \).

Let us consider the set

\[
\tilde{Q} := Q \cup \bigcup_{i=1}^{\ell} \tilde{\rho}^{n_{z_i}}(\tilde{V}) + p_{z_i}.
\]

Now, since \( f \) exhibits bounded vertical rotational deviations, we know that its vertical rotation set given by (14) is a singleton and there exists a real constants \( C_0 > 0 \) such that

\[
|pr_2(\tilde{f}_n(z)) - z| \leq C_0, \quad \forall z \in \mathbb{R}^2, \quad \forall n \in \mathbb{Z},
\]

where \( \rho_{\tilde{V}}(\tilde{f}) = \rho \).

This implies that

\[
\tilde{f}_n(\tilde{V}) \subset S_n := \left\{z \in \mathbb{R}^2: |pr_2(z - \tilde{x}) - n\rho| \leq C_0 + 1/4\right\}, \quad \forall n \in \mathbb{Z}.
\]

Then, consider the set of integer numbers

\[
E_n := \{q \in \mathbb{Z}: \exists p \in \mathbb{Z}, \quad T_{(p,q)}(\tilde{Q}) \cap S_n \neq \emptyset\}, \quad \forall n \in \mathbb{Z}.
\]

Since \( \tilde{Q} \) has finite diameter, we know that

\[
\sharp E_n \leq 2C_0 + \diam \tilde{Q} + 4, \quad \forall n \in \mathbb{Z},
\]

where \( \sharp(\cdot) \) denote the cardinality of the set, and on the other hand it clearly holds

\[
\tilde{Q} \cap T_p(\tilde{Q}) = \emptyset, \quad \forall p \in \mathbb{Z}^2, \text{ with } |p| > \diam \tilde{Q}.
\]
Taking into account (34), (35), we consider an integer number \( L \in \mathbb{N} \) such that
\[
L > \ell(2C_0 + \text{diam } \hat{Q} + 4) + 1.
\]

Now, by Lemma 2.14
\[
\sup_{n \in \mathbb{N}} \text{diam} \left( \text{pr}_1(\hat{f}^n(\hat{V})) \right) = \infty.
\]
So, we can find a positive integer \( m' \) such that for every \( j \in \{m', m' + 1, \ldots, m' + M_0\} \), there are \( p_{j_1}^{(i)}, \ldots, p_{j_L}^{(i)} \in \mathbb{Z}^2 \) with \( |p_{r_j}^{(i)} - p_{s_j}^{(i)}| > \text{diam } \hat{Q} \), for every \( 1 \leq r < s \leq L \), and such that
\[
\hat{f}(\hat{V}) \cap T_{p_{r_j}^{(i)}}(\hat{Q}) \neq \emptyset, \quad \forall j \in \{m', \ldots, m' + M_0\}, \quad \forall r \in \{1, \ldots, L\},
\]
where
\[
\hat{T}_{p_{r_j}^{(i)}}(\hat{Q}) = \bigcup_{i=1}^{\ell} \hat{f}^{n_{j_i}}(\hat{V}) + p_{z_{j_i}} + p_{r_j}^{(i)}.
\]
This means that for each \( j \in \{m', \ldots, m' + M_0\} \) and every \( r \in \{1, \ldots, L\} \), there exists \( z_{j,r} \in \{z_1, \ldots, z_{L} \} \) such that
\[
\hat{f}(\hat{V}) \cap \left( \hat{f}^{n_{j_i}}(\hat{V}) + p_{z_{j_i}} + p_{r_j}^{(i)} \right) \neq \emptyset.
\]
Observe that, by (35), the sets \( T_{p_{1_j}^{(i)}}(\hat{Q}), T_{p_{2_j}^{(i)}}(\hat{Q}), \ldots, T_{p_{L_j}^{(i)}}(\hat{Q}) \) are two-by-two disjoint which implies that
\[
\left( \hat{f}^{n_{j_i}}(\hat{V}) + p_{z_{j_i}} + p_{r_j}^{(i)} \right) \cap \left( \hat{f}^{n_{j_i}}(\hat{V}) + p_{z_{j_s}} + p_{s_j}^{(i)} \right) = \emptyset,
\]
for all \( j \in \{m', \ldots, m' + M_0\} \), and any \( 1 \leq r < s \leq L \).

Then, by (32) and (33) we yield
\[
\text{pr}_2(p_{r_j}^{(i)}) \in E_j, \quad \forall r \in \{1, \ldots, L\},
\]
and any \( m' \leq j \leq m' + M_0 \). Therefore, putting together this last relation with (34), (37), (38) and (36), we conclude that for each \( j \in \{m', \ldots, m' + M_0\} \), there are \( 1 \leq r < s \leq L \) such that \( z_{j,r} = z_{j,s} \) and \( \text{pr}_2(p_{r_j}^{(i)}) = \text{pr}_2(p_{s_j}^{(i)}) \). So, for each \( j \) we get
\[
\hat{f}^{j-n_{j_i}}(\hat{V}) \cap \left( \hat{V} + I_k^{-n_{j_i}}(p_{z_{j_i}} - p_{r_j}^{(i)}) \right) \neq \emptyset,
\]
\[
\hat{f}^{j-n_{j_i}}(\hat{V}) \cap \left( \hat{V} + I_k^{-n_{j_i}}(p_{z_{j_s}} - p_{s_j}^{(i)}) \right) \neq \emptyset,
\]
where \( I_k \) is the matrix given by (10), \( p_{z_{j,r}} - p_{r_j}^{(i)} \neq p_{z_{j,s}} - p_{s_j}^{(i)} \) but their second coordinate coincides, i.e.
\[
\text{pr}_2(p_{z_{j,r}} - p_{r_j}^{(i)}) = \text{pr}_2(p_{z_{j,s}} - p_{s_j}^{(i)})
\]
\[
= \text{pr}_2(I_k^{-n_{j_i}}(p_{z_{j,r}} - p_{r_j}^{(i)})) = \text{pr}_2(I_k^{-n_{j_i}}(p_{z_{j,s}} - p_{s_j}^{(i)})).
\]
Then, the existence of the natural number \( m \) and the conclusion of the lemma itself follow by Lemma 2.16. \( \square \)
3. Wandering points as an obstruction to Kronecker factors

The main purpose of this section consists in describing the construction of three examples of totally irrational pseudo-rotations with uniformly bounded rotational deviations but which does not admit any non-trivial Kronecker factor.

In these three examples the geometry of the wandering sets plays a fundamental role, showing that the small wandering domains hypothesis is fundamental and sharp in Theorem A.

First we shall perform a general construction, which is a slight modification of classical suspensions, that will be used in §§3.2 and 3.3.

3.1. Suspending circle homeomorphisms. Given arbitrary homeomorphisms \( g_1, g_2 \in \text{Homeo}_0(\mathbb{T}) \) and a lift \( \tilde{g}_1 : \mathbb{R} \to g_1 \), we will construct the “time-\( g_1 \) of the suspension flow of \( g_2 \),” a homeomorphism \( g \in \text{Homeo}_0(\mathbb{T}^2) \) which is defined as follows.

First, consider the equivalence relation \( \sim \) on \( \mathbb{R} \times \mathbb{T} \) given by

\[
(s, x) \sim (s', x') \iff s' - s \in \mathbb{Z} \text{ and } x = g_2^s(x').
\]

Since \( g_2 \) is homotopic to the identity, the quotient space \( (\mathbb{R} \times \mathbb{T}) / \sim \) is indeed homeomorphic to \( \mathbb{T}^2 \), and we shall just identify them without any further reference.

As usual, we define the suspension flow \( \Phi : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2 \) by

\[
\Phi^t[s, x] := [t + s, x], \quad \forall (s, x) \in \mathbb{R} \times \mathbb{T}, \quad \forall t \in \mathbb{R},
\]

where \([s, x]\) denotes the equivalence class of the point \((s, x)\).

Then, we define the homeomorphism \( g : \mathbb{T}^2 \to \mathbb{T}^2 \) as the “time-\( \tilde{g}_1 \)” of \( \Phi \). More precisely, we write

\[
g[t, x] := [\tilde{g}_1(t), x], \quad \forall (t, x) \in \mathbb{R} \times \mathbb{T},
\]

where \( \tilde{g}_1 : \mathbb{R} \to g_1 \) is a lift of \( g_1 \).

In order to verify that \( g \) is indeed well defined, it is enough to notice that \( \tilde{g}_1 \) commutes with any integer translation on \( \mathbb{R} \).

Then we have the following

**Proposition 3.1.** The homeomorphism \( g : \mathbb{T}^2 \to \mathbb{T}^2 \) we constructed above exhibits the following properties:

(i) \( g \) is a skew-product, i.e. it leaves invariant the “vertical circle foliation” given by the \( \{x\} \times \mathbb{T} : x \in \mathbb{T} \);

(ii) \( g \) is a pseudo-rotation and its rotation vector is totally irrational if and only if the numbers \( 1, \rho(\tilde{g}_1), \rho(\tilde{g}_2) \) are linearly independent over \( \mathbb{Q} \), for any lift \( \tilde{g}_2 : \mathbb{R} \to g_2 \);

(iii) \( g \) exhibits uniformly bounded rotational deviations (i.e. condition (26) holds);

(iv) if \( g_2 \) is a Denjoy homeomorphism (i.e. it is periodic point free and \( \mathcal{W}(g_2) \neq \emptyset \)), \( \tilde{g}_1 \) is fixed point free and \( \Omega(g_1) = \mathbb{T} \), then it holds

\[
\mathcal{W}(g) = \{ \Phi^t(\{0\} \times \mathcal{W}(g_2)) : t \in \mathbb{R} \}.
\]

In particular, every connected component of the wandering set \( \mathcal{W}(g) \) is lift-unbounded and inessential.

(v) If \( \tilde{g}_1 \) and \( g_2 \) are Denjoy homeomorphisms, then

\[
\mathcal{W}(g) := \{ \Phi^t(\{0\} \times \mathcal{W}(g_2)) : t \in \mathbb{R} \} \cup \mathcal{W}(g_1) \times \mathbb{T}.
\]

In particular, \( \mathcal{W}(g) \) is connected and fully essential.

(vi) when the rotation vector of \( g \) is totally irrational, it admits a \( \mathbb{T}^2 \) minimal Kronecker factor and \( \Omega(g) \) is the only minimal set for \( g \).
Proof. In order to prove (i), it is enough to verify that the vertical circle foliation $\{\{t\} \times T : t \in \mathbb{R}\}$ of $\mathbb{R} \times T$ is invariant by the quotient map $(t,x) \mapsto (t+1,g^{-1}(x))$ and the flow $\tilde{\Phi} : \mathbb{R} \times (\mathbb{R} \times T) \ni (t,(s,x)) \mapsto (t+s,x)$, which is the lift of flow $\Phi$.

Then, one can easily verify that the there exists a lift $\tilde{g} : \mathbb{R}^2 \supset g$ such that $\rho(\tilde{g}) = \{(1,\rho(\tilde{g}))\rho(\tilde{g}_2)\}$. This clearly implies (ii).

To prove (iii) it is enough to notice that $g$ leaves invariant two topologically transverse foliations with different asymptotic homological directions (see [KK09, Proposition 4.1] and [KPR18, Theorem 5.4] for details): one of them is the vertical circle foliations of the skew-product structure given by (i); the other one is given by the orbits of the (singularity free) flow $\Phi$.

Properties (iv) and (v) are straightforward consequences of classical results about the non-wandering set of suspension flows.

Finally, in order to prove (vi), observe that, by (ii), if $g$ is a totally irrational pseudo rotation, then $g_1$ and $g_2$ have irrational rotation numbers. So, by classical Poincaré theory, both of them are topological extension of irrational rotations of the circle, and $\Omega(g_1)$ and $\Omega(g_2)$ are the only minimal sets for $g_1$ and $g_2$, respectively. These properties pass to the flow $\Phi$ by suspension. □

3.2. Unbounded inessential wandering set. In this paragraph we show the existence of a totally irrational pseudo-rotation $f \in \text{Homeo}_0(\mathbb{T}^2)$, exhibiting uniformly bounded rotational deviations (in every direction), whose wandering set $\mathcal{W}(f)$ is connected, inessential and lift-unbounded and such that $f$ does not admit any non-trivial Kronecker factor.

To do that, let $g_2 : T \supset T$ be a Denjoy homeomorphism with just one orbit of wandering domains (i.e. $\text{Per}(g_2) = \emptyset$, $\mathcal{W}(g_2) \neq \emptyset$ and $\mathcal{W}(g_2) = \bigcup_n g_2^n(I)$), for any connected component $I$ of $\mathcal{W}(g_2)$ and $g_1 = T_{\mu_1} : \mathbb{R} \supset$ be a translation such that the map $g \in \text{Homeo}_0(\mathbb{T}^2)$ constructed in § 3.1 is a totally irrational pseudo-rotation (see condition (ii) of Proposition 3.1). Since $g_2$ has just one orbit of wandering domains and property (iv) of Proposition 3.1 holds, we have that $\mathcal{W}(g)$ is connected.

By (vi) of Proposition 3.1, we know that $g$ admits a minimal $\mathbb{T}^2$ Kronecker factor. One can easily show that two different points $z,z' \in \mathbb{T}^2$ are Kronecker equivalent for $g$ (see Definition 2.2) if and only if $\text{pr}_1(z) = \text{pr}_1(z')$ and there exists a connected component $f$ of the set $\mathcal{W}(g) \cap \{\text{pr}_1(z)\} \times \mathbb{T}$ such that $z,z' \in f$.

So, given any topological open disc $U \subset \mathcal{W}(g)$, we can clearly find two points $w_0, w_1 \in U$ which are not Kronecker equivalent; and thus, they are in fact Kronecker separated (again see Definition 2.2 for details). Moreover, taking $U$ small enough, we can assume $U$ is a wandering set for $g$. Then, since $U$ is open and connected, there exists $\ell \in \text{Homeo}_0(\mathbb{T}^2)$ so that $\text{supp} \ell \subset U \subset \mathcal{W}(g)$ and $\ell(w_0) = w_1$.

Then we just define $f := g \circ \ell$. Since $\ell$ is supported in a $g$-wandering set, it holds $\Omega(f) = \Omega(g)$, and therefore, $f$ and $g$ coincide on this set. So, $f$ has a connected, inessential and lift-unbounded wandering set as well. Moreover, since $g$ exhibits uniformly bounded rotational deviations and $\ell$ is supported on a $g$-wandering lift-bound set, this implies $f$ exhibits uniformly bounded rotations deviations as well.

We claim that $f$ does not admit any non-trivial Kronecker factor. To prove this, reasoning by contradiction, let us suppose there is a semi-conjugacy $h : \mathbb{T}^2 \to \mathbb{T}$ and a minimal rotation $T_h : \mathbb{T} \supset$ such that $h \circ f = T_h \circ h$. Since $\Omega(f)$ is a minimal set for $f$, we have that two points of $\Omega(f) = \Omega(g)$ are Kronecker equivalent for $f$ if and only if so they are for $g$. Then, for each $i \in \{0,1\}$, we can consider an arbitrary point $w_i' \in \Omega(g) \cap \text{cc} (\mathcal{W}(g) \cap \{\text{pr}_1(w_i)\} \times \mathbb{T}, w_i)$.

Since $w_0$ and $w_1$ are Kronecker separated for $g$, and $w_i$ and $w_i'$ are $g$-proximal, for $i \in \{0,1\}$, we conclude $w_0'$ and $w_1'$ are Kronecker separated for $g$. So, $w_0'$ and $w_1'$
are Kronecker separated for \( f \), too; but \( w_0 \) and \( w'_0 \) are \( f \)-proximal and \( w_0 \) and \( w'_0 \) are \( f^{-1} \)-proximal. This clearly contradicts the existence of a non-trivial Kronecker factor for \( f \).

### 3.3. Fully essential wandering set

In this paragraph we describe the construction of a totally irrational pseudo-rotation \( f \in \text{Homeo}_0(\mathbb{T}^2) \) with uniformly bounded rotational deviations, such that the wandering set \( \mathcal{W}(f) \) is fully essential and such that \( f \) does not admit any non-trivial Kronecker factor.

The construction is very similar to that one performed in §3.2. In this case we start considering two Denjoy maps \( g_1, g_2 \in \text{Homeo}_0(\mathbb{T}) \), with lifts \( \tilde{g}_1, \tilde{g}_2 : \mathbb{R} \to \mathbb{R} \) such that \( 1, \rho(\tilde{g}_1) \) and \( \rho(\tilde{g}_2) \) are linearly independent over \( \mathbb{Q} \).

Then, let \( g \in \text{Homeo}_0(\mathbb{T}^2) \) be the homeomorphism given by the construction described at the beginning of §3.1, associated to \( \tilde{g}_1 \) and \( \tilde{g}_2 \). By (ii) and (v) of Proposition 3.1, we know that \( g \) is a totally irrational pseudo-rotation and \( \mathcal{W}(g) \) is a fully essential connected set.

Then, by (v) we can choose a point \( w_0 \in \mathcal{W}(g) \) such that \( \text{pr}_1(w_0) \in \Omega(\tilde{g}_1) \). So we know there exist \( I \subset \mathcal{W}(g_2) \) and \( s_0 \in \mathbb{R} \) such that

\[
\mathcal{W}(g_2) \cap \mathcal{W}(g) = \emptyset.
\]

Then, for any \( \epsilon > 0 \) sufficiently small, the open set

\[
\mathcal{U} := \bigcup_{|s-s_0|<\epsilon} \Phi^s(\{0\} \times I)
\]

is a wandering set for \( g \).

Since \( \Omega(\tilde{g}_1) \) has no isolated points and \( \mathcal{U} \) is an open topological disc, we know there exists another point \( w_1 \in \mathcal{U} \) such that \( \text{pr}_1(w_0) \in \Omega(\tilde{g}_1) \setminus \{\text{pr}_1(w_1)\} \). This implies that \( w_0 \) and \( w_1 \) are not Kronecker equivalent, and consequently, they are indeed Kronecker separated.

As we did in §3.2, we consider a homeomorphism \( \ell \in \text{Homeo}_0(\mathbb{T}^2) \) such that \( \ell(w_0) = w_1 \) and \( \text{supp} \ell \subset \mathcal{U} \subset \mathcal{W}(g) \). Again, we define \( f := g \circ \ell \). By the very same argument we exposed in §3.2, one can show \( f \) is a totally irrational pseudo-rotation exhibiting uniformly bounded rotational deviations, fully essential wandering set and having no non-trivial Kronecker factor.

### 3.4. Inessential bounded non-small wandering domains

In this paragraph we describe the construction of a totally irrational pseudo-rotation \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) not admitting any non-trivial Kronecker factor, and such that it exhibits uniformly bounded rotational deviations, its wandering set is the union of countably many inessential lift-bounded wandering domains, all of them having the same diameter and hence, not satisfying the small wandering domain hypothesis (see Definition 2.3).

To do that, let start considering a totally irrational vector \( \alpha \in \mathbb{R}^2 \) and let \( T_\alpha : \mathbb{T}^2 \to \mathbb{T}^2 \) be the corresponding rigid rotation. Given any \( \gamma \in \mathbb{R} \setminus \mathbb{Q} \) and \( \delta > 0 \), let us define

\[
\mathcal{F}_\delta^\gamma := \pi \left\{ (t, t\gamma) \in \mathbb{R}^2 : t \in (-\delta, \delta) \right\} \subset \mathbb{T}^2,
\]

and, for each \( z \in \mathbb{T}^2 \), let us write

\[
\mathcal{F}_\delta^\gamma(z) := \mathcal{F}_\delta^\gamma + z = \{ \pi(t, t\gamma) + z : t \in (-\delta, \delta) \}.
\]

Notice that fixing the totally irrational vector \( \alpha \in \mathbb{R}^2 \), there exists \( \delta_0 > 0 \) such that

\[
T_\alpha^{n}(\mathcal{F}_\delta^\gamma) \cap \mathcal{F}_\delta^\gamma = \emptyset, \quad \forall \delta \in (0, \delta_0), \forall n \in \mathbb{Z} \setminus \{0\}.
\]

Then let us fix such a \( \delta \).
By classical “à la Denjoy” surgery procedures, we can construct a topological extension $g \in \text{Homeo}_0(T^2)$ of $T$ satisfying the following properties: there exist a continuous map $h: T^2 \to$ in the identity homotopy class and a sequence $\{a_n\}_{n \in \mathbb{Z}}$ of points of $T^2$ such that $h \circ g = \tilde{g} \circ h_\ell$, each fiber $h^{-1}(z)$ is a singleton if and only if $z \in T^2 \setminus \bigcup_{n \in \mathbb{Z}} T^u_k(\mathcal{F}_k^n)$, and

$$h^{-1}(z) = \{z + a + \pi(-\gamma t, l) \mid t \in (-\delta_n(z), \delta_n(z))\},$$

whenever $z \in T^u_k(\mathcal{F}_k^n)$, and where $\delta_n(z) := 2^{-|n|} \cdot 10 \left(\delta - d_T^2(z, T^u_k(0))\right)$ and $d_T^2(\cdot, \cdot)$ denotes the distance function given by (9).

Notice that the wandering set of $g$ is given by

$$\mathcal{W}(g) = \bigcup_{n \in \mathbb{Z}} h^{-1}(T^u_k(\mathcal{F}_k^n)),$$

where $\bigcup$ denotes the disjoint union operator, and

$$\text{diam} \left( h^{-1}(T^u_k(\mathcal{F}_k^n)) \right) = 2\delta, \quad \forall n \in \mathbb{Z}.$$
In particular, the homeomorphism $F$ for every $p$ associated to $\Gamma$ and a positive real number $r$.

Proof. (43) The usefulness of map $F$ can be briefly summarized with the following simple but important property:

$$F^n(t, z) = \left( \hat{T}_{\rho}^n(t), \hat{T}^{-n}_t \circ \left( \hat{T}_{\rho}^n \circ \hat{T}_t \right) (z) \right),$$

for any $(t, z) \in T \times A$, any $\hat{T} \in \pi^{-1}(t)$ and every $n \in \mathbb{Z}$.

On the other hand, let us notice that the skew-product $F$ has a full flow of symmetries: for each $s \in \mathbb{R}$ consider the map $\Gamma^s: T \times A \to T \times A$, i.e.

$$\Gamma^s(t, x, y) := (t + \pi(s), x, y - s), \quad \forall (t, x, y) \in T \times A.$$

Then, we have the following

**Proposition 4.1.** The map $\Gamma: \mathbb{R} \times T \times A \to T \times A$ is a flow and, for each $s \in \mathbb{R}$, the homeomorphisms $\Gamma^s$ and $F$ commute, and $\Gamma^s$ is an isometry of the metric space $(T \times A, d_{T \times A})$, where the distance function is given by

$$d_{T \times A}((t, z), (t', z')) = d_T(t, t') + d_A(z, z'), \quad \forall (t, z), (t', z') \in T \times A.$$

Proof. This follows from straightforward computations. In fact,

$$\Gamma^s + s' \times \hat{T}_{\rho}^{-s - s'} = (T_s \times \hat{T}_{\rho}^{-s} \circ \hat{T}_{\rho}^{-s'}) = \Gamma^s \circ \Gamma^s', \quad \forall s, s' \in \mathbb{R}.$$

On the other hand, given any $s \in \mathbb{R}$ and any $(t, x, y) \in T \times A$, it holds

$$F(\Gamma^s(t, x, y)) = F(t + \pi(s), x, y - s) = \left( \hat{T}_s(t), x + k(y + t) + \pi \circ \Delta_1(x, y + t), y - s + \Delta_2(x, y + t) - \rho \right) = \Gamma^s(F(t, x, y)).$$

Finally, $\Gamma^s$ is an isometry because $T_s$ is an isometry of $(T, d_T)$ and $\hat{T}$ of $(A, d_A)$. \hfill \Box

Then, let us fix some terminology. Given any $t \in T$ and any $(t, \tilde{z}) \in T \times A$, the $\Gamma$-line through $(t, \tilde{z})$ is its flow line, i.e. it is given by

$$\Gamma(t, \tilde{z}) := \{ \Gamma^s(t, \tilde{z}) : s \in \mathbb{R} \}.$$

On the other hand, we introduce the concept of blocks of $T \times A$, which are a particular kind of open subsets of $T \times A$: given an open set $V \subset A$, a point $t \in T$ and a positive real number $r$, we define the corresponding $r$-block centered at $t$ associated to $V$ by

$$V_r(t) := \bigcup_{|s| < r} \Gamma^s \{ t \} \times V \subset T \times A.$$

A rather simple but important property of blocks is given by the following

**Proposition 4.2.** If $V \subset A$ is an open subset, $t \in T$ and $r > 0$, and $V_r(t)$ is the $r$-block centered at $t$ associated to $V$ given by (45), then it holds

$$F^n(V_r(t)) = \left( \hat{T}_s^{-1} \circ \hat{T}_{\rho}^{-n} \circ \hat{T}_t \circ \hat{T}_s(V) \right)^r, \quad \forall n \in \mathbb{Z}, \forall \tilde{t} \in \pi^{-1}(t).$$

In particular, the $F$-image of any $r$-block is another $r$-block.
Proof. It easily follows from \((43)\), Proposition \(4.1\) and \((45)\).

Finally, as a straightforward consequence of \((43)\) we get the following

**Proposition 4.3.** Given any \(z \in \mathbb{T}^2\), any \(n \in \mathbb{Z}\), any \(\tilde{z} \in \pi^{-1}(z)\) and any \(\tilde{w} \in \pi^{-1}(\hat{f}^n(z))\) (where projection \(\hat{\pi}\) is given by \((16)\)), the point \(F^n(0, \tilde{z})\) belongs to the \(\Gamma\)-line through \((0, \tilde{w})\).

**Proof.** Since \(\hat{\pi} \circ \hat{f} = f \circ \hat{\pi}\), we have
\[
\hat{\pi}(\hat{f}^n(\tilde{z})) = f^n(\hat{\pi}(\tilde{z})) = f^n(\tilde{z}) = \hat{\pi}(\tilde{w}).
\]
So there exists \(m \in \mathbb{Z}\) such that \(\hat{T}_m(\tilde{w}) = \hat{f}^n(\tilde{z})\), or equivalently we can write
\[
\Gamma^{-m}(0, \tilde{w}) = (0, \hat{f}^n(\tilde{z})).
\]
Then, by \((43)\) we know that
\[
F^n(0, \tilde{z}) = (n\pi(p), \hat{T}^{-n}_p \circ \hat{f}^n(\tilde{z})) = \Gamma^m(0, \hat{f}^n(\tilde{z})) = \Gamma^{n-m}(0, \tilde{w}).
\]

\[\square\]

### 4.1. The \(\rho\)-centralized skew-product and rotational deviations.

From now on and until the end of this section we suppose \(\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2\) is a lift of a homeomorphism \(f \in \text{Homeo}_0(\mathbb{T}^2)\) exhibiting uniformly bounded vertical deviations, with \(\rho\) and \(C\) as in \((21)\) and \(v = (0, 1)\). Let \(F : \mathbb{T} \times \mathbb{A} \to \mathbb{A}\) be the \(\rho\)-centralized skew-product induced by \(f\) as defined in \((42)\).

Then we have the following

**Proposition 4.4.** Every \(F\)-orbit is bounded in the vertical direction, i.e. if \((t, x, \hat{y}) \in \mathbb{T} \times \mathbb{A}\) is an arbitrary point and we define \((t_n, x_n, \hat{y}_n) := F^n(t, x, \hat{y})\), then it holds
\[
|\hat{y}_m - \hat{y}_n| \leq 2C, \quad \forall m, n \in \mathbb{Z},
\]
where \(C\) is the constant given by \((21)\).

**Proof.** This is a straightforward consequence of estimate \((21)\) and property \((43)\).

\[\square\]

The following result will play a key role in our work:

**Theorem 4.5.** If \(f\) is \(\Omega\)-recurrent (see Definition 2.1), then a point \((x, y) \in \mathbb{T}^2\) is non-wandering for \(f\) if and only if \(\Gamma^s(0, x, \hat{y}) \in \Omega(F)\), for every \(\hat{y} \in \pi^{-1}(y)\) and all \(s \in \mathbb{R}\).

**Proof.** First observe that, by Proposition 4.1, the set \(\Omega(F)\) is \(\Gamma\)-invariant. So it is enough to show that a point \((x, y) \in \Omega(f)\) if and only if \((0, x, \hat{y}) \in \Omega(F)\), for some \(\hat{y} \in \pi^{-1}(y)\).

Then, let us prove the “if” direction, which holds without the boundedness of rotational deviations and the \(\Omega\)-recurrence assumptions. So, let \((0, x, \hat{y})\) be any point of \(\Omega(F)\).

Let us fix a positive number \(\delta\) and write just \(B\) for the open ball \(B_\delta(x, \hat{y}) \subset \mathbb{A}\). Then, since \((0, x, \hat{y})\) is a non-wandering point, there exists \(n \geq 1\) such that
\[
F^n(B^{\delta,0}) \cap B^{\delta,0} \neq \emptyset,
\]
where \(B^{\delta,0}\) denotes the \(\delta\)-block centered at \(0 \in \mathbb{T}\) and associated to \(B\), as defined by \((45)\).

By \((43)\), there are \(s, s' \in (-\delta, \delta)\) such that \(T^n_s(\pi(\hat{z})) = \pi(s')\) and
\[
\hat{T}^{-n}_p \circ \hat{f}^n(\hat{T}_{-s}(B)) \cap \hat{T}_{-s'}(B) \neq \emptyset.
\]
In particular, this implies
\[
\hat{f}^n(B_{2\delta}(x, \hat{y})) \cap \hat{T}^{-n}_p(B_{2\delta}(x, \hat{y})) \neq \emptyset.
\]
However, since $s, s' \in (-\delta, \delta)$, we know there exists $p \in \mathbb{Z}$ such that $|p - np| < 2\delta$ and thus we get
\[ f^n(B_{2\delta}(x, y)) \cap T_p(B_{4\delta}(x, y)) \neq \emptyset, \]
which implies that
\[ f^n(B_{2\delta}(x, y)) \cap B_{4\delta}(x, y) \neq \emptyset, \]
where $y = \pi(y)$, and hence $(x, y) \in \Omega(f)$, as desired.

Now, let $(x, y)$ be any point of $\Omega(f)$ and let us fix a point $\tilde{y} \in \pi^{-1}(y)$. Given any pair of real numbers $r, \delta > 0$, let $B_\delta$ denote the open ball $B_\delta(x, \tilde{y}) \subset A$ and $B_y^r$ be the r-block centered at 0 associated to $B_\delta$. Observe the family of blocks $\{B_y^r: \delta > 0, r > 0\}$ is a local base of neighborhoods at the point $(0, x, \tilde{y})$. So, let us fix a real number $r > 0$ that, without loss of generality, we can suppose is less than 1/4, and let us show there exists $n \in \mathbb{N}$ such that $B_y^r \cap F^{-n}(B_y^r) \neq \emptyset$.

Since $f$ is $\Omega$-recurrent, by Lemma 2.1 there exists an $f$-recurrent point $(x', y') \in B_r(x, y)$ and a strictly increasing sequence of positive integers $(n_j)_{j \geq 1}$ such that
\[ (x_j', y_j') := f^n(x_j', y_j') \in B_r(x, y), \quad \forall j \in \mathbb{N}. \]

Since $r < 1/4$, there is a unique point $\tilde{y}' \in \pi^{-1}(y') \cap B_r(\tilde{y})$, and for each $j \geq 1$, a unique point $\tilde{y}_j' \in \pi^{-1}(y_j') \cap B_r(\tilde{y})$.

By Proposition 4.3, for each $j \geq 1$ there is a real number $s_j$ such that
\[ F^{n_j}(0, x_j', \tilde{y}_j') = \Gamma^{s}(0, x_j', \tilde{y}_j'), \quad \forall j \geq 1. \]

On the other hand, by Proposition 4.4 we know that
\[ |s_j| \leq 2C, \quad \forall j \in \mathbb{N}. \]

Now, invoking Proposition 4.1 we get
\[ F^{-n_j} \left( \bigcup_{|s| < r} \Gamma^{s}(0, x_j', \tilde{y}_j') \right) = \bigcup_{|s| < r} \Gamma^{s}(F^{-n_j}(0, x_j', \tilde{y}_j')) \subset \bigcup_{|s| < 2C + r} \Gamma^{s}(0, x_j', \tilde{y}_j'), \]
for every $j \geq 1$.

Now, recalling $\Gamma$ is an isometric flow (i.e. for every $s$, $\Gamma^{s}$ leaves invariant the distance $d_{T \times A}$ defined in Proposition 4.1), we observe the arc on the right side of the equation has finite length. On the other hand, on the left side of the equation, we have infinitely many constant length segments. So we can conclude there exist $k > j \geq 1$ such that
\[ F^{-n_k} \left( \bigcup_{|s| < r} \Gamma^{s}(0, x_k', \tilde{y}_k') \right) \cap F^{-n_j} \left( \bigcup_{|s| < r} \Gamma^{s}(0, x_j', \tilde{y}_j') \right) \neq \emptyset. \]

Since $(0, x_j', \tilde{y}_j'), (0, x_k', \tilde{y}_k') \in B_y^r$, this implies $F^{n_j-n_k}(B_y^r) \cap B_y^r \neq \emptyset$, with $n_k - n_j > 0$. So, $(0, x, \tilde{y}) \in \Omega(F)$. \hfill \Box

Now we can state the main result of this section:

**Theorem 4.6.** Let us suppose $f \in \text{Homeo}_b(T^2)$ is $\Omega$-recurrent, periodic point free, exhibits uniformly bounded vertical rotational deviations and is not eventually annular. Let $V \subset T \times A$ be a nonempty connected bounded open set such that $V \cap \Omega(F) \neq \emptyset$. Then, for each $t \in T$, the open set
\[ A \setminus \mathcal{U}_t(V), \]
has exactly two unbounded connected components, where $\mathcal{U}_t(V) \subset T \times A$ is the set given by (8) and $\mathcal{U}_t(V)_t$ denotes the fiber of $\mathcal{U}_t(V)$ over $t$ given by (5).

In order to prove Theorem 4.6, let us start considering the following
Lemma 4.7. If \( f \in \text{Homeo}_0(\mathbb{T}^2) \) is an \( \Omega \)-recurrent totally irrational pseudo-rotation exhibiting uniformly bounded rotational deviations, i.e. estimate (26) holds, and \( V \) is as in Theorem 4.6, then \( \overline{\mathcal{U}_f(V)_t} \subset A \) is an annular set, for every \( t \in T \).

Proof of Lemma 4.7. Let \( \hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a lift of \( f \) and \( \hat{\rho} \in \mathbb{R}^2 \) be the totally irrational vector appearing in estimate (26). So, \( \rho = \text{pr}_3(\hat{\rho}) \) is the irrational number we are considering to define the \( \rho \)-centralized skew-product given by (42).

Reasoning by contradiction, suppose there exists a nonempty open connected subset \( V \subset T \times A \) and \( t \in T \) such \( \overline{\mathcal{U}_f(V)_t} \) is not annular, i.e. is inessential in \( A \). Since \( \rho \) is irrational and \( \overline{\mathcal{U}_f(V)} \) is open, we conclude \( \overline{\mathcal{U}_f(V)_t} \) is inessential in \( A \), for every \( t \in T \).

Consider the covering map \( \pi \times \hat{\pi} : \mathbb{R} \times \mathbb{R}^2 \rightarrow T \times A \) given by
\[ \pi \times \hat{\pi}(\bar{t}, \bar{z}) := (\pi(\bar{t}), \hat{\pi}(\bar{z})) \quad \forall (\bar{t}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^2, \]
and let \( \overline{\mathcal{U}_f(V)} \subset \mathbb{R}^3 \) be a connected component of \( (\pi \times \hat{\pi})^{-1}(\overline{\mathcal{U}_f(V)}) \). Again by irrationality of \( \rho \), it must hold \( \text{pr}_1(\overline{\mathcal{U}_f(V)}) = T \). On the other hand, since \( \pi \times \hat{\pi} \) is a covering map, and both sets \( \overline{\mathcal{U}_f(V)} \) and \( \overline{\mathcal{U}_f(V)} \) are open and connected, we get
\[ \pi \times \hat{\pi}(\overline{\mathcal{U}_f(V)}) = \overline{\mathcal{U}_f(V)}. \]
So we have \( \text{pr}_1(\overline{\mathcal{U}_f(V)}) = \mathbb{R} \).

On the other hand, since \( \overline{\mathcal{U}_f(V)} \) is open, connected and bounded in \( T \times A \), we know that \( \text{pr}_3(\overline{\mathcal{U}_f(V)}) \) is also bounded in \( \mathbb{R} \). Hence, there exists \( \ell \in \mathbb{Z} \), which is unique and independent of the choice of the connected component \( \overline{\mathcal{U}_f(V)} \), such that
\[ \overline{\mathcal{U}_f(V)}_{i+1} = T(\ell, 0) \left( \overline{\mathcal{U}_f(V)}_i \right), \quad \forall i \in \mathbb{R}. \]
Since \( V \) intersects \( \Omega(F) \) and (48) holds, there is \((\bar{t}, \bar{z}) \in \overline{\mathcal{U}_f(V)} \) so that the point \((\bar{t}, \bar{z}) := \pi \times \hat{\pi}(\bar{t}, \bar{z}) \) belongs to \( \Omega(F) \). This implies there exists a sequence of positive integers \((n_j)_{j \geq 1} \) and a sequence of points \(( (t_j, z_j) )_{j \geq 1} \) such that \((t_j, z_j) \in B_{1/4}(t, z) \subset \mathbb{R} \times \mathbb{A} \) and \( F^n(t_j, z_j) \in B_{1/4}(t, z) \), for each \( j \geq 1 \). Observe, since \( F \) is periodic point free, it necessarily holds \( n_j \rightarrow +\infty \), as \( j \rightarrow \infty \). So, by (43), we can conclude that for each \( j \geq 1 \) there exist unique numbers \( m_j, p_j \in \mathbb{Z} \) such that
\[ |m_j - n_j| \rho | \leq 1/2j \]
and
\[ \frac{1}{n_j} \| F^n(t_j + (0, \bar{z}_j) - \bar{z}_j - (0, \bar{t}_j) - n_j(0, \rho) - (p_j, 0) \| \leq \frac{1}{2j}, \]
where \((\bar{t}_j, \bar{z}_j)\) is the only point in \((\pi \times \hat{\pi})^{-1}(t_j, z_j) \cap B_{1/4}(\bar{t}, \bar{z}) \), provided \( j \geq 4 \).

Now, recalling we are assuming \( f \) is a pseudo-rotation and \( \rho(\hat{f}) = \{ \rho \} \) is the rotation set given by (12), as a consequence of (50) we get \( p_j / n_j \rightarrow \text{pr}_3(\rho) = \rho \) and \( m_j / n_j \rightarrow \text{pr}_2(\rho) \), as \( j \rightarrow \infty \).

However observe that, by (49), \( p_j = \ell m_j \), for all \( j \geq 1 \). This implies that \( \text{pr}_1(\rho) = \text{pr}_3(\rho) \), contradicting the fact that \( \overline{\mathcal{U}_f(V)} \) was totally irrational. So, \( \overline{\mathcal{U}_f(V)_t} \) is annular for every \( t \in T \).

Then we can finish the proof of Theorem 4.6:

Proof of Theorem 4.6. If \( f \in \text{Homeo}_0(\mathbb{T}^2) \) is a pseudo-rotation and exhibits uniformly bounded horizontal deviations, then it exhibits uniformly bounded rotational deviations in every direction. Since \( f \) is not eventually annular, this implies its rotation set of any lift of \( f \) is singleton containing a totally irrational vector. Hence we can invoke Lemma 4.7 to guarantee that \( \overline{\mathcal{U}_f(V)_t} \) is annular for every \( t \in T \).
and thus, $\mathbf{A}\setminus\mathcal{W}_f(V)$ has exactly two unbounded connected components; and the theorem is proven in this case.

So, now we can assume that $f$ exhibits unbounded horizontal rotational deviations, i.e. if $\hat{f}: \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $f$, then condition (22) holds.

Reasoning by contradiction, let us suppose there exists a nonempty open connected bounded subset $V \subset \mathbb{T} \times \mathbb{A}$ and $t_0 \in \mathbb{T}$, such that $V \cap \Omega(F) \neq \emptyset$ and $\mathbf{A}\setminus\mathcal{W}_f(V)_{t_0}$ has just one unbounded connected component.

Since $f$ exhibits uniformly bounded vertical deviations, by (43) we know the set $\mathcal{W}_f(V)$ is bounded in $\mathbb{T} \times \mathbb{A}$ and so there exists a constant $C_v > 0$ such that

$$ (51) \quad \mathcal{W}_f(V) \subset \mathbb{T} \times (\mathbb{T} \times (-C_v, C_v)). $$

Thus, since $\mathbf{A}\setminus\mathcal{W}_f(V)_{t_0}$ is open, its unique unbounded connected component is arc-wise connected as well. So there exists a continuous curve $\gamma: [0, 1] \to \mathbb{A}$ such that $\gamma(0) = (0, -\infty, -C_v - 1)$, $\gamma(1) = (0, \infty, +\infty)$ and $\gamma(s) \notin \mathcal{W}_f(V)_{t_0}$, for every $s \in [0, 1]$. By compactness of the (image of the) curve $\gamma$ and the set $\mathcal{W}_f(V)$, there exists $\varepsilon > 0$ such that

$$ \gamma(s) \notin \mathcal{W}_f(V), \quad \forall t \in B_\varepsilon(t_0), \forall s \in [0, 1], $$

where $B_\varepsilon(t_0) := \{ t \in \mathbb{T} : d_\mathbb{T}(t, t_0) < \varepsilon \}$. Since the number $\rho$ appearing in the base dynamics of the skew-product $F$ is irrational, this implies that there exists $N \in \mathbb{N}$ such that

$$ \bigcup_{j=0}^N \mathcal{T}_\rho(B_\varepsilon(t_0)) = \mathbb{T}. $$

So, for every $t \in \mathbb{T}$ there are $n_t \in \{0, 1, \ldots, N\}$ and $t' \in B_\varepsilon(t_0)$ such that $T_\rho^{n_t}(t') = t$ and then,

$$ \text{(52)} \quad F^{n_t}(t', \gamma(s)) \notin \mathcal{W}_f(V), \quad \forall s \in [0, 1]. $$

Then, putting together (51) and (52) we conclude that for every $t \in \mathbb{T}$, every connected component of the set $\mathcal{W}_f(V)_t$ is lift-bounded in $\mathbb{A}$. Moreover, they are uniformly bounded, i.e. there is a real constant $C > 0$ such that $\text{diam} \hat{U} < C$, for every $t \in \mathbb{T}$ and every connected component $\hat{U}$ of the open set $\hat{\pi}^{-1}(\mathcal{W}_f(V)_t) \subset \mathbb{R}^2$, where $\hat{\pi}: \mathbb{R}^2 \to \mathbb{A}$ denotes the covering map given by (16).

Let us see that this leads us to a contradiction. In fact, by Proposition 4.1 we know the set $\Omega(F)$ is $F$-invariant, and we are assuming $V \cap \Omega(F) \neq \emptyset$. So, there exists $s \in \mathbb{R}$ such that $\Gamma^s(V) \cap \Omega(F) \neq \emptyset \times \mathbb{A}$. Let us consider the fiber of the $\Gamma^s(V)$ over the point $0 \in \mathbb{T}$, i.e. the set $\{ \ubar{\Gamma}^s(V) \}_{0} \subset \mathbb{R}^2$. Since $f$ is $\Omega$-recurrent, by Theorem 4.5 we know that

$$ (\ubar{\Gamma}^s(V))_0 \cap \hat{\pi}^{-1}(\Omega(f)) \neq \emptyset. $$

Hence, there exists a connected component $\hat{V}$ of the open set $\hat{\pi}^{-1}(\{ \ubar{\Gamma}^s(V) \}_{0}) \subset \mathbb{R}^2$ such that $\pi(\hat{V}) \cap \Omega(f) \neq \emptyset$, where $\hat{\pi}: \mathbb{R}^2 \to \mathbb{A}$ and $\pi: \mathbb{R}^2 \to \mathbb{T}$ are the natural covering maps.

Now, notice, since we are assuming $f$ is periodic point free and non-eventually annular, by Proposition 2.12, every non-wandering point is fully essential, and recalling $f$ exhibits unbounded horizontal rotational deviations (i.e. condition (22) holds), we can invoke Lemma 2.14 to conclude that

$$ (53) \quad \sup_{n \in \mathbb{Z}} \text{diam}(\text{pr}_1(F^n(\hat{V}))) = \infty, $$

Finally, by (43) we know that (53) contradicts the fact that connected components of $\mathcal{W}_f(V)$ are uniformly lift-bounded; and thus Theorem 4.6 is proven. \qed
In order to show that estimate (2) is a necessary condition for the existence of a non-trivial minimal Kronecker factor, no assumption about the non-wandering set is required. In fact, the proof is based in rather classical arguments and all the details can be found for instance in [JT17, Lemma 3.1]. Let us just mention that in this case the irrationality of ρ follows from the non-annularity hypothesis.

To finish the proof of Theorem A, from now on let us suppose f satisfies condition (2). By Proposition 2.4 and the remark about rotational deviations at the beginning of §2.6, there is no loss of generality assuming \( f \in \text{Homeo}_0(\mathbb{T}^2) \), for some \( k \in \mathbb{Z} \), and \( v = (0,1) \) in estimate (2). So, the lift \( \hat{f} : \mathbb{R}^2 \rightrightarrows \) of \( f \) commutes with the vertical translation \( T_{(0,1)} : \mathbb{R}^2 \rightrightarrows \) and, consequently, induces an annulus homeomorphism \( \hat{f} : \mathbb{A} \rightrightarrows \) characterized by the semi-conjugacy equation \( \hat{\pi} \circ \hat{f} = \hat{f} \circ \pi \).

Now, we want to construct a surjective continuous map \( \hat{h} : \mathbb{T}^2 \rightarrow \mathbb{T} \) such that \( \hat{h} \circ f = T_{\rho} \circ \hat{h} \), where \( \rho \) is the irrational number given by (2). We know that in such a case, each fiber \( \hat{h}^{-1}(y) \), with \( y \in \mathbb{T} \), is an annular continuum, as defined in §2.4.2. So, following Jäger [Jäg09], we will define the semi-conjugacy \( \hat{h} \) starting from its family of fibers.

More precisely, due to technical reasons we will start working on the annulus \( \mathbb{A} \) instead of \( \mathbb{T}^2 \) and we will construct a family of continua \( \{ \mathcal{C}^s \subset \mathbb{A} : s \in \mathbb{R} \} \) such that each \( \mathcal{C}^s \) is an annular continuum in \( \mathbb{A} \), they are well ordered according to the index, i.e.

\[
\mathcal{C}^s \subset U^-(\mathcal{C}^r), \quad \text{for all } s < r,
\]

where \( U^-(\cdot) \) denotes the lower connected component of the complement in \( \mathbb{A} \) of the corresponding essential as defined in §2.4.2, and the family satisfies the following equivariant properties:

\[
\hat{T}_1(\mathcal{C}^s) = \mathcal{C}^{s+1},
\]

\[
\hat{f}(\mathcal{C}^s) = \mathcal{C}^{s+\rho}, \quad \forall s \in \mathbb{R}.
\]

Observe that, in particular, (54) implies that the continua are two-by-two disjoint, i.e., \( \mathcal{C}^s \cap \mathcal{C}^r = \emptyset \), whenever \( r \neq s \).

Then, we define the map \( \hat{h} : \mathbb{A} \rightarrow \mathbb{R} \) by

\[
\hat{h}(z) := \inf \{ s \in \mathbb{R} : z \in U^-(\mathcal{C}^s) \}, \quad \forall z \in \mathbb{A}.
\]

By (55) we know that \( \hat{h} \circ \hat{T}_1 = \hat{T}_1 \circ \hat{h} \) and, by (56), \( \hat{h} \circ \hat{f} = T_{\rho} \circ \hat{h} \). Thus, \( \hat{h} \) is the lift of a map \( h : \mathbb{T}^2 \rightarrow \mathbb{T} \) and it clearly holds \( h \circ f = T_{\rho} \circ h \). So, in order to show that \( h \) is indeed a semi-conjugacy, it just remains to prove that \( \hat{h} \), and then \( h \) as well, is continuous. This is a consequence of the irrationality of \( \rho \) and the fact that the annular continua \( \{ \mathcal{C}^s : s \in \mathbb{R} \} \) are two-by-two disjoint. The reader can find more details about the proof of the continuity of \( h \) given by (57), assuming (54), (55) and (56), in either [Jäg09, page 615] or [JP15, Lemma 3.2].

5.1. The construction of continua \( \mathcal{C}^s \). In order to finish the proof of Theorem A, it remains to construct the family of annular continua \( \{ \mathcal{C}^s : s \in \mathbb{R} \} \) satisfying properties (54), (55) and (56).

To do that, let \( F : \mathbb{T} \times \mathbb{A} \rightrightarrows \) be the \( \rho \)-centralized skew-product induced by \( \hat{f} \) given by (12) and where \( \rho \) is the irrational number appearing in (2). Since \( f \) exhibits small wandering domains (Definition 2.3) and is periodic point free, by Proposition 2.8 we know that \( f \) is \( \Omega \)-recurrent, and hence, by Theorem 4.5 we know that \( z \in \Omega(f) \) if and only \( (0, \hat{z}) \in \Omega(F) \), for every \( \hat{z} \in \hat{\pi}^{-1}(z) \).

The main new idea of the proof of Theorem A is given by the following lemma, which could seem at first glance to be rather technical, but is the core of the method:
Lemma 5.1. There exists an open bounded $F$-invariant subset $\mathcal{F} \subset \mathbb{T} \times \mathbb{A}$ such that

\begin{equation}
(\mathcal{F} \setminus \Gamma^{-s}(\mathcal{F})) \cap \Omega(F) \neq \emptyset, \quad \forall s \in \mathbb{R} \setminus \{0\}.
\end{equation}

Before proving Lemma 5.1, let us see how the open set $\mathcal{F}$ can be used to construct our family of annular continua $\{\mathcal{C}^s : s \in \mathbb{R}\}$.

First, observe that, by Theorem 4.6, the set $\mathbb{A} \setminus \Gamma^{-s}(\mathcal{F})$, has exactly two unbounded connected components, for every $t \in \mathbb{T}$ and any $s \in \mathbb{R}$. Hence, we can define the sets

\[ \mathcal{F}^s_+: = U^-\left(\Gamma^{-s}(\mathcal{F})_t\right) \subset \mathbb{A}, \quad \forall t \in \mathbb{T}, \forall s \in \mathbb{R}, \]

i.e. it is the lower unbounded connected component of the set $\mathbb{A} \setminus \Gamma^{-s}(\mathcal{F})_t$. Then we write

\begin{equation}
\mathcal{C}^s := \partial_\mathbb{A} \left( \mathcal{F}^s_- \right) \subset \mathbb{A}, \quad \forall s \in \mathbb{R}.
\end{equation}

Notice that each $\mathcal{C}^s$ is an annular continuum. Moreover, since $F$ commutes with $\Gamma^{-1}$ and $\Gamma^{-1}(t, z) = (t, z + 1)$, for all $(t, z) \in \mathbb{T} \times \mathbb{A}$, then (55) clearly holds.

In order to prove (56), observe that, since $\Gamma^{-s}(\mathcal{F})$ is $F$-invariant and $F$ preserves the ends of the space $\mathbb{T} \times \mathbb{A}$, we get

\begin{equation}
F(t \times \mathcal{F}^s_-) = \{t_p(t)\} \times \mathcal{F}^{s-\rho}_t, \quad \forall t \in \mathbb{T}, \forall s \in \mathbb{R},
\end{equation}

and on the other hand,

\begin{equation}
\Gamma^{-\rho}\left(\{t_p(t)\} \times \mathcal{F}^{s-\rho}_t\right) = \{t\} \times \mathcal{F}^{s+\rho-\rho}_t, \quad \forall t \in \mathbb{T}, \forall s \in \mathbb{R}.
\end{equation}

Putting together (43), (60) and (61) we get:

\[ \hat{f}(\mathcal{C}^s) = \mathcal{C}^{s+\rho}, \quad \forall s \in \mathbb{R}, \]

and (56) is proven.

So, it remains to prove that the continua $\{\mathcal{C}^s : s \in \mathbb{R}\}$ given by (59) satisfies condition (54). To do that, it is important to notice that is enough to show that

\begin{equation}
\mathcal{C}^s \cap \mathcal{C}^r = \emptyset, \quad \forall r, s \in \mathbb{R}, \quad s \neq r.
\end{equation}

In fact, one can easily check that condition (54) follows from (62) and the combinatorics of these annular continua given by conditions (55), (56) and the fact that

\[ \frac{\text{pr}_s(\hat{f}^n - \text{id})}{n} \to \rho, \quad \text{as } n \to \infty \]

uniformly on $\mathbb{A}$, which is a direct consequence of (2).

So, condition (62) is the only remaining step to prove that Theorem A is consequence of Lemma 5.1. To do that, let $r$ be an arbitrary real number and $s \in (0, 1)$. We will show that $\mathcal{C}^s$ and $\mathcal{C}^{s+\rho}$ are disjoint. Then, recalling that $\mathcal{C}^s$ and $\mathcal{C}^{s+\rho}$ are the boundary of the open set $\mathcal{F}^s_-$ and $\mathcal{F}^{s+\rho}_-$, respectively, we consider the open set

\[ V := \Gamma^{-s}(\mathcal{F}) \setminus (\mathcal{F} \setminus \Gamma^{-s}(\mathcal{F})) = \Gamma^{-s}\left(\mathcal{F} \setminus \Gamma^{-s}(\mathcal{F})\right) \subset \mathbb{T} \times \mathbb{A}. \]

Since $\mathcal{F}$ and $\Omega(F)$ are $F$-invariant, and by Proposition 4.1 $F$ and $\Gamma$ commute, we get that $V$ is $F$-invariant, and by Lemma 5.1, it holds $V \cap \Omega(F) \neq \emptyset$. Moreover, we claim there exists a connected component $\bar{V} \in \pi_0(V)$ such that

\begin{equation}
\bar{V} \cap \Omega(F) \cap (\{0\} \times \mathbb{A}) \neq \emptyset.
\end{equation}
To prove this, let $V'$ be any connected component of $\tilde{V}$ that intersects $\Omega(F)$. If $(t, \hat{z})$ is an arbitrary point of $V' \cap \Omega(F)$, since $F$ and $\Gamma$ commute and $V'$ is open, there is a positive number $\varepsilon > 0$ such that

$$\Gamma^u(t, \hat{z}) \in V' \cap \Omega(F) \cap \{t + \pi(u)\} \times A, \quad \forall u \in (-\varepsilon, \varepsilon).$$

Then, since $\rho$ is irrational, there is $n \in \mathbb{Z}$ such that $0 \in T^n_\rho(B_\delta(t)) \subset T$; and hence, $V := F^n(V') \in \pi_0(\tilde{V})$ satisfies (63).

Now, recalling we are assuming $f$ is not eventually annular, there are two possible cases to be considered: either $f$ is a totally irrational pseudo-rotation exhibiting uniformly bounded rotational deviations (i.e. estimate (26) holds); or $f$ exhibits uniformly bounded vertical rotational deviations and unbounded horizontal deviations (i.e. estimate (22) holds).

In the first case, i.e. when $f$ is a totally irrational pseudo-rotation and (26) holds, the disjointness of continua $\mathscr{C}^\gamma$ and $\mathscr{C}^{\gamma+s}$ easily follows from Lemma 4.7. In fact, $V$ is a connected component of an $F$-invariant set and intersects $\Omega(F)$. So, $V = \mathcal{W}_F(V)$ and by Lemma 4.7 we know that $V_t$ is annular, for every $t \in \mathbb{T}$. This implies $V_t$ separates $\mathscr{C}^\gamma$ and $\mathscr{C}^{\gamma+s}$.

The second case, i.e. when $f$ exhibits unbounded horizontal deviations, will follow as consequence of Lemma 2.15, but it is a little more involved.

Let $(0, \hat{z})$ be any point belonging to $V \cap \Omega(F)$, and $\delta > 0$ be a sufficiently small number such that

$$B := (B_\delta(0, \hat{z}))^{\delta,0} = \bigcup_{|h| < \delta} \Gamma^u(B_\delta(0, \hat{z})) \subset V.$$

Without loss of generality we can assume $\delta < 1/4$. We will show that there exists $i \in \mathbb{N}$ and $q \in \mathbb{Z}$ such that

$$(\Gamma^{-q}(B) \cap F^i(B))_0 \subset A$$

is an essential set, and this will prove that $\mathscr{C}^\gamma$ and $\mathscr{C}^{\gamma+s}$ are disjoint. In fact, we know that $B_0 \cap \mathscr{C}^\gamma \subset V_0 \cap \mathscr{C}^\gamma = \emptyset$, and hence, by (55), $(\Gamma^{-q}(B))_0 \cap \hat{T}_q(B^\gamma) = \hat{T}_q(B_0) \cap \mathscr{C}^{\gamma+s} = \emptyset$, where $B_0$ denotes the fiber of $B \subset \mathbb{T} \times A$ over the point $0 \in \mathbb{T}$. Analogously, one can show that $(\Gamma^{-q}(B))_0 \cap \mathscr{C}^{\gamma+s} = \emptyset$; and since we are assuming $0 < s < 1$, we know that $B_0 \cap \mathscr{C}^{\gamma+s} = B_0 \cap \mathscr{C}^{\gamma+s} = \emptyset$, for any $q \in \mathbb{Z}$.

So, condition (65) clearly shows that $\mathscr{C}^{\gamma+s} \cap \mathscr{C}^{\gamma+s} = \emptyset$, and by (55), this holds if and only if $\mathscr{C}^\gamma \cap \mathscr{C}^{\gamma+s} = \emptyset$.

So, let us show (65) holds. By the minimality of $T_\rho: \mathbb{T} \searrow$, we know there exists a natural number $N$ such that for any $m \in \mathbb{Z}$, there is $i \in \{m, m + 1, \ldots, m + N\}$ such that $T^{(i)}_\rho(0) \in B_{\delta/4}(0) \subset \mathbb{T}$.

On the other hand, if we write $\hat{z} := \hat{\pi}(\hat{z})$, by Theorem 4.5 we know that $\hat{z} \in \Omega(f)$. Let $\hat{z}$ be any point in $\pi^{-1}(\hat{z}) = \hat{\pi}^{-1}(\hat{z})$. Applying Lemma 2.15 for the open set $B_{\delta/4}(\hat{z}) \subset \mathbb{R}^2$ and the natural number $N$ given by the above condition, we know that there is $m \in \mathbb{N}$ such that for every $j \in \{m, m + 1, \ldots, m + N\}$, there are $p_j, p'_j, q_j \in \mathbb{Z}$ with $p_j \neq p'_j$ and such that $\hat{f}^{(j)}(B_{\delta/4}(\hat{z})) \cap T_{(p_j,q_j)}(B_{\delta/4}(\hat{z})) \neq \emptyset$ and $\hat{f}^{(j)}(B_{\delta/4}(\hat{z})) \cap T_{(p'_j,q_j)}(B_{\delta/4}(\hat{z})) \neq \emptyset$; or in other words, we get that the open set $\hat{T}_{q_j}(\hat{\pi}(B_{\delta/4}(\hat{z}))) \cup \hat{f}^{(j)}(\hat{\pi}(B_{\delta/4}(\hat{z})))$ is essential in $A$, for every $j \in \{m, \ldots, m + N\}$.

Then, let $i$ be a natural number such that $i \in \{m, \ldots, m + N\}$ and $T^{(i)}_\rho(0) \in B_{\delta/4}(0)$, and let us consider the open set

$$\hat{B}^{(i)} := \hat{T}_{q_j}(\hat{\pi}(B_{\delta/4}(\hat{z}))) \cup \hat{f}^{(j)}(\hat{\pi}(B_{\delta/4}(\hat{z}))) = \hat{T}_{q_j}(B_{\delta/4}(\hat{z})) \cup \hat{f}^{(j)}(B_{\delta/4}(\hat{z})) \subset A.$$
since we are assuming \( \delta < 1/4 \). We know that \( \hat{B}^{(i)} \) is an essential set and putting together this with Propositions 4.1 and 4.2, and \((64)\), we conclude that there is an integer number \( q \) such that the set given by \((65)\) is an essential set in \( \mathbb{A} \), as we wanted to prove. Then, \((54)\) is proven.

So, the last step of the proof of Theorem \( \Lambda \) is

**Proof of Lemma 5.1.** The proof of this lemma is considerably simpler in the case that \( \Omega(f) \) has nonempty interior (e.g. when \( f \) is non-wandering) than in the case where the wandering set is dense in \( \mathbb{T}^2 \).

So, for the sake of simplicity of the exposition let us start proving the lemma under the assumption that the non-wandering set has nonempty interior. In such a case, let \( V \subset \Omega(f) \) be an open, nonempty, lift-bounded and connected set.

Since we are assuming \( f \) is periodic point free and exhibits small wandering domains, by Proposition 2.8 we know that \( f \) is \( \Omega \)-recurrent, and thus, by Theorem 4.5 it holds \( \{0\} \times \hat{\pi}^{-1}(V) \subset \Omega(f) \subset \mathbb{T} \times \mathbb{A} \), where \( \hat{\pi} \colon \mathbb{A} \to \mathbb{T}^2 \) is the natural covering map given by \((16)\).

Let \( \hat{V} \) be any connected component of \( \hat{\pi}^{-1}(V) \subset \mathbb{A} \). Since we are assuming \( V \) is lift-bounded, \( \hat{V} \) is bounded in \( \mathbb{A} \) as well. Then, we define

\[
\mathcal{T} := \mathcal{U}_T(\hat{V}^{\pm 0}),
\]

where \( \hat{V}^{\pm 0} \) denotes the 1/2-block induced by \( V \) and centered at \( 0 \in \mathbb{T} \) given by \((45)\), and \( \mathcal{U}_T(\cdot) \) is given by \((8)\). By Proposition 4.1 we know \( F \) and \( \Gamma \) commute, so \( \mathcal{T} \subset \Omega(F) \). Then, since \( \mathcal{T} \) is defined as a connected component of an \( F \)-invariant set and intersects the non-wandering set of \( F \), there exists \( N \in \mathbb{N} \) such that \( F^N(\mathcal{T}) = \mathcal{T} \), and this implies \( F^N(\Omega(F)) = \Omega(F) \). Moreover, by Theorem 4.6 we know that the set \( \mathcal{T} \) is essential in \( \mathbb{A} \), for every \( t \in \mathbb{T} \). This means that for every \( t \), the complement of closed set \( \mathcal{T} \) has exactly two unbounded components. Since the map \( F \) preserves the ends of each vertical, we conclude that \( F(\mathcal{T}) = \mathcal{T} \), i.e. we can take \( N = 1 \) and so \( \mathcal{T} \) is \( F \)-invariant.

On the other hand, since \( \hat{V} \) is bounded in \( \mathbb{A} \), by Proposition 4.4 \( \mathcal{T} \) is bounded in \( \mathbb{T} \times \mathbb{A} \) as well. This implies that for every \( s \neq 0 \), \( \mathcal{T} \setminus \Gamma^{-s}(\mathcal{T}) \) is nonempty and, clearly \( \mathcal{T} \setminus \Gamma^{-s}(\mathcal{T}) \subset \Omega(F) \); and Lemma 5.1 is proven under the additional hypothesis that \( \Omega(f) \) has nonempty interior.

So, from now on let us suppose the non-wandering set \( \Omega(f) \) has empty interior.

Let \( z_0 \) be any point of \( \Omega(f) \) and \( z_0 \) an arbitrary point in \( \hat{\pi}^{-1}(z_0) \subset \mathbb{A} \). Now let us define the set \( \hat{V} \) as the the wandering completion of the open unit ball \( B_1(z_0) \subset \mathbb{A} \), i.e. \( \hat{V} \) is given by

\[
\hat{V} := B_1(z_0) \cup \left\{ \hat{\pi}^{-1}(W) : W \in \pi_0(\mathbb{T}^2, \Omega(f)), \hat{\pi}^{-1}(W) \cap B_1(z_0) \neq \emptyset \right\}.
\]

Since \( f \) exhibits small wandering domains, notice that \( \hat{V} \) is open, bounded, and connected. Then, we define here again \( \mathcal{T} := \mathcal{U}_T(\hat{V}^{\pm 0}) \). So, \( \mathcal{T} \) is open and bounded in \( \mathbb{T} \times \mathbb{A} \) in this case as well, and this implies \( \mathcal{T} \setminus \Gamma^{-s}(\mathcal{T}) \neq \emptyset \), for every \( s \in \mathbb{R} \setminus \{0\} \). On the other hand, since \( z_0 \in \Omega(f) \), we can apply Theorem 4.5 to conclude \((0, z_0) \in \Omega(F) \cap \mathcal{T} \), and hence, by the very same argument we used in the previous case, we get that \( \mathcal{T} \) is \( F \)-invariant.

So, it just remain to prove that the set \( \mathcal{T} \setminus \Gamma^{-s}(\mathcal{T}) \) intersects \( \Omega(F) \), for every \( s \neq 0 \). To do that, let us just consider the case where \( s > 0 \); the other one is completely analogous.

Now, observe that combining Proposition 2.7, Proposition 4.1 and \((67)\) we know that each connected component of the open set \( \mathcal{T} \cap \mathcal{U}(F) \) is a 1/2-block, each of them is a wandering set for \( F \) and their \( \Gamma \)-orbits are two-by-two disjoint. More
precisely, if \( W_1, W_2, \ldots \) denote the (infinitely many) connected components of \( \mathcal{T} \cap \mathcal{W}(f) \), then for each \( n \geq 1 \) there exists a unique connected component \( W_n \) of \( \mathcal{W}(f) \) such that if we choose any connected component \( \hat{W}_n \) of the open set \( \hat{W}_n \cap \mathcal{A} \subset \mathcal{A} \), there is a unique \( r_n \in \mathbb{R} \) satisfying

\[
\hat{W}_n = \bigcup_{|r| < \frac{1}{r_n}} \Gamma^{n+r_n}(\{0\} \times \hat{W}_n).
\]

(68)

On the other hand, by our definition (67) and Theorem 4.5 again, we see that for each \( n \geq 1 \), there exists at least a point \( z_n \in \partial A(W_n) \) such that

\[
\Gamma^{n+s+u}(0, z_n) \subseteq \overline{W_n} \cap \mathcal{T} \cap \overline{\Omega(F)}, \quad \forall u \in (-1/2, 1/2).
\]

(69)

Reasoning by contradiction let us suppose that there exists \( s > 0 \) such that the open set \( D := \{ s \} \mathcal{T} \cap \mathcal{T} \cap \overline{\Omega(F)} \) does not intersects \( \Omega(F) \). This implies, invoking (68), that \( D \) has infinitely many connected components and all of them can be enumerated as follows:

\[
D_n := \bigcup_{|r| < \frac{1}{r_n}} \Gamma^{n+\frac{1}{r_n}+u}(\{0\} \times \hat{W}_n), \quad \forall n \geq 1,
\]

(70)

and \( D = \bigsqcup_{n \geq 1} D_n \).

Then, we will construct a subsequence \( (D_{n_j})_{j \geq 1} \) of connected components of \( D \) such that for every \( M > 0 \) there is a \( j \geq 1 \) with \( D_{n_j} \cap \mathcal{T} \times (-\infty, -M) \neq \emptyset \), contradicting the fact that \( D \) was a bounded subset of \( \mathcal{T} \times \mathcal{A} \).

To do that, we will define the sequence \( (D_{n_j})_{j \geq 1} \) inductively. Let us start defining \( n_1 = 1 \). By (69) we know that

\[
w_1 := \Gamma^{1+\frac{2}{r_1}}(0, \hat{z}_1) \in \overline{D_1} \cap \mathcal{T} \cap \overline{\Omega(F)},
\]

but since \( D \) and \( \Omega(F) \) are disjoint, we get that \( w_1 \notin D_{n_1} = D_1 \). So, \( w_1 \in \Gamma^{-\frac{s}{s}}(\mathcal{T}) \).

That means that there exists \( n_2 \in \mathbb{N} \) such that

\[
d_{\mathcal{T} \times \mathcal{A}}(\Gamma^{s}(w_1), \hat{W}_{n_2}) < \frac{1}{2^{r_1}},
\]

and we define

\[
w_2 := \Gamma^{n_2+\frac{2}{r_2}}(0, \hat{z}_{n_2}) \in \overline{D_{n_2}} \cap \mathcal{T} \cap \overline{\Omega(F)}.
\]

Inductively, supposing that \( n_k \in \mathbb{N} \) and \( w_k \in \mathcal{T} \times \mathcal{A} \) has been already defined, we choose \( n_{k+1} \in \mathbb{N} \) as any natural number satisfying

\[
d_{\mathcal{T} \times \mathcal{A}}(\Gamma^{s}(w_k), \hat{W}_{n_{k+1}}) < \frac{1}{2^{k+1}},
\]

(71)

and then we define

\[
w_{j+1} := \Gamma^{n_{j+1}+\frac{2}{r_{j+1}}}(0, \hat{z}_{n_{j+1}}) \in \overline{D_{n_{j+1}}} \cap \mathcal{T} \cap \overline{\Omega(F)}.
\]

(72)

Then, putting together (68), (69), (70), (71) and (72) we conclude that

\[
\text{pr}_3(w_{k+1}) \leq \text{pr}_3(w_k) - s + \text{diam} W_{n_{k+1}} + \frac{1}{2^{r_k}}, \quad \forall k \geq 1.
\]

(73)

Iterating this last estimate one easily gets

\[
\text{pr}_3(w_{k+1}) \leq \text{pr}_3(w_1) + \sum_{j=1}^{k} (\text{diam} W_{n_{j+1}} - s) + \sum_{j=1}^{k} \frac{1}{2^{r_j}} \leq \text{pr}_3(w_1) + 1 + \sum_{j=1}^{k} (\text{diam} W_{n_{j+1}} - s) \to -\infty,
\]

(73)
as $k \to +\infty$, where the last limit follows from the small wandering domain hypothesis which implies $\text{diam} W_{n_k} \to 0$, as $n_k \to +\infty$.

Since $\bar{w}_k \in \mathcal{D}_{n_k} \subset \mathcal{D}$, for every $k \in \mathbb{N}$, estimate (73) clearly contradicts the fact that $D$ was bounded in $\mathbb{T} \times A$. \hfill $\square$

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E-mail address: akocsard@id.uff.br

IME - Universidade Federal Fluminense. Rua Prof. Marcos Waldemar de Freitas Reis, S/N. Bloco H, 4º andar. 24.210-201, Gragoatá, Niterói, RJ, Brasil