Spiral orientational order in quantum Hall skyrmion lattices

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We investigate the existence of spiral ordering in the planar spin orientation of skyrmions localised on a face centered rectangular lattice (FCRL). We use the non-linear sigma model (NLSM) to numerically calculate the minimum energy configurations of this lattice around the \( \nu = 1 \) quantum Hall ground state. Our variational ansatz contains an angle \( \theta \), characterising the FCRL and an angle \( q \), characterising the orientational order. As \( \nu \) is increased towards one, there is a smooth transition from the triangular lattice (TL) characterised by \((\theta, q) = (120^\circ, 120^\circ)\) to FCRLs with spiral orientational order. The novel feature we find is that these phases are characterised by \((\theta, q)\) values such that \( \theta + q = 240^\circ \) (same as the TL phase). As \( \nu \) increases further towards one, there is a sharp transition from the FCRLs to the square lattice (SL) characterised by \((\theta, q) = (90^\circ, 180^\circ)\). Consequently, the parameter \( \theta + q \) jumps sharply at the FCRL-SL transition and can serve as an order parameter to characterise it.

The lowest energy charged excitations about the \( \nu = 1 \) ferromagnetic quantum Hall ground state are skyrmions. They are topological objects in which the spin gradually twists over an extended region. Their spin is greater than 1/2 and they carry an electric charge of \( \pm e \). Skyrmions have a topological charge which equals their electric charge at \( \nu = 1 \). Skyrmonic excitations are favoured over single particle excitations when the Landé g-factor is small i.e \( g \to 0 \). Skyrmions which are produced around the \( \nu = 1 \) ground state have been experimentally seen by in OPNMR and optical magneto-absorption experiments on Ga nuclei in an electron doped multiple quantum well structure. They see a sharp fall in the spin polarisation of the 2D electron gas on either side of \( \nu = 1 \). Timm et. al. study a system of well separated skyrmions as described by an anti-ferromagnetic XY model and propose a 1\( = 0 \) phase diagram in which the TL and SL phases are separated by Néel ordered, centered rectangular phases. The classical and quantum phase transitions occurring in the Skyrme crystal has been studied using Hartree-Fock calculations by Côté et. al.

In this paper, we consider spurious ground states as candidate minimum energy solutions of a system of skyrmions described by an effective classical O(3) NLSM. Spiral ground states are known to be the ground states of frustrated anti-ferromagnets. For example, the triangular lattice with ABC sublattice spin ordering is a frustrated spin system in which the spin at every vertex is rotated through 120\(^\circ\) relative to the others.

Our calculations are is valid for a system of overlapping skyrmions at small \( g \). The Coulomb and Zeeman terms in the NLSM compete to generate a size for the skyrmion and to drive the TL-SL transition through the FCRLs. This is in contrast to the regime of Timm et. al. where the Zeeman contribution is neglected in the large separation limit. The FCRLs they observe do not show spiral ordering of spins.

We find that as \( \nu \) increases towards 1, the system changes (at non-zero \( g \)) smoothly from the TL phase characterised by \((\theta, q) = (120^\circ, 120^\circ)\) to FCRLs with spiral orientational order. The FCRLs have different values of \( \theta \) and \( q \), but their \( \theta + q \) value is always 240\(^\circ\), which is the value of \( \theta + q \) in the TL phase. As \( \nu \) is increased further there is another sharp transition from the FCRLs to the SL phase characterised by \((\theta, q) = (90^\circ, 180^\circ)\). The parameter \( \theta + q \) shows a sharp jump from 240\(^\circ\) to 270\(^\circ\) at the FCRL-SL transition. Hence it is a convenient order parameter for this transition. The spin polarisation of the system varies smoothly with \( \nu \) and does not show an abrupt behaviour at any of the two transitions. The FCRL phases seem to smooth out the jump in the spin polarisation, which was observed at the transition to the SL phase in previous work.

We consider skyrmionic excitations about the \( \nu = 1 \) ground state. The number density of skyrmions is given by \( n_{\text{sky}} = \frac{1 - \nu}{\nu} n_c \), where \( n_c \) is the carrier density. In our calculations we use the carrier density, \( n_c = 1.5 \times 10^{11} \text{ cm}^{-2} \) and change \( \nu \) by tilting the magnetic field. We localise the skyrmion centers on the lattice points of a FCRL shown in Fig. 1. \( \theta \) is the angle between the Bravais lattice vectors and \( e \) is the length of the Bravais lattice basis vectors, which vary with the filling factor.
We choose a skyrmion density \((n_{sky})\) of 1 skyrmion per unit cell (i.e. one skyrmion per lattice point).

We measure all lengths in units of the magnetic length \(l_c = \sqrt{\frac{\hbar}{eB}}\) and all energies in terms of the cyclotron energy \(\hbar \omega_c\), where \(\omega_c = (eB/m^*)\), \(m^*\) is the effective mass of the electron. The area of the unit cell is fixed by the filling factor and is given by

\[
A = \frac{1}{n_{sky}} = e^2 \sin \theta = \frac{2\pi}{1 - \nu} \quad (1)
\]

The local spin polarization which is represented by unit vector field \(\mathbf{n}(x)\) is stereographically projected onto the complex plane by the transformation \(w = \cos(\theta/2)e^{i\phi}\), where \(\theta\) and \(\phi\) are the polar angles of the spin vector \(\mathbf{n}(x)\). In the rest of the paper we will work with the planar spin variable \(w\).

The topological charge density is given by

\[
\rho(x) = \frac{1}{4\pi} \varepsilon_{ij} \mathbf{n}(x) \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n}) = \frac{\varepsilon_{ij} \partial_i w \partial_j \pi}{2\pi i(1 + w^2)^2} \quad (2)
\]

The topological charge, \(Q(x) = \int_\square d^2x \rho(x) = 1\) (\(\square\) denotes integration over the unit cell).

The low energy, long wavelength excitations about the \(\nu = 1\) ground state are accurately described by the NLSM. The NLSM energy functional has to be minimised for different filling factors to get the minimum energy configurations of this lattice. The NLSM energy functional with Zeeman and Coulomb interactions is \([1,7,8]\):

\[
E = E_{grn} + E_z + E_{coul} \quad (3)
\]

The gradient or the spin exchange term proportional to \(\int d^2x |\partial_i \mathbf{n}(x)|^2\), \((i = x, y)\), is calculated as

\[
E_{grn} = \frac{\gamma}{2} \int_\square d^2x \frac{(\partial_x w \partial_x \pi + \partial_y w \partial_y \pi)}{(1 + w^2)^2} \quad (4)
\]

where \(\gamma = \frac{e^2}{8\hbar \omega_c}\). The gradient energy density is \(E_{grn}/A\), where \(A\) is the area of the unit cell.

This term alone is the pure NLSM and it has scale invariant solutions \([3]\).

The Zeeman term is proportional to the \(z\)-component of the total spin i.e. \(\nu \int_\square d^2x (\mathbf{n} \cdot \mathbf{M})/2\) (where \(\mathbf{M}\) is the \(z\)-component of \(\mathbf{n}(x)\)). In our units the average number of electrons is \(\nu \cdot \pi\). The \(z\)-component of the total spin is

\[
(Total \ spin)_z = \nu \int_\square d^2x \frac{\pi w}{1 + \pi w} \quad (5)
\]

Therefore,

\[
E_z = g^* \nu \int_\square d^2x \frac{\pi w}{1 + \pi w} \quad (6)
\]

where \(g^* = \frac{\mu_B}{\hbar \omega_c}\). The Zeeman energy density is therefore \(E_z/A\), where \(A\) is the area of the unit cell.

The Coulomb energy density term is a term of the form

\[
E_{coul} = e^* \frac{1}{2A_{tot}} \int_{x,y} \rho(x) \frac{1}{|x-y|} \rho(y) \quad (7)
\]

where \(e^* = (e^2/Kl_c)/(1/\hbar \omega_c)\) and \(A_{tot}\) is the total area of the lattice.

The Coulomb term arises because the electric charge density is proportional to the topological charge density. Since the topological charge density explicitly appears in the above expression the spin orientation gets automatically tied to the Coulomb energy. The four dimensional integral in the Coulomb term can be converted to a sum over the reciprocal lattice.

\[
E_{coul} = \frac{e^* \pi}{A^2} \sum_{\{\mathbf{G}_R\}} |\tilde{\rho}(\mathbf{G}_R)|^2 \frac{1}{|\mathbf{G}_R|} \quad (8)
\]

where \(\tilde{\rho}(\mathbf{G}_R) = \int_{\square} \rho(x)e^{-i\mathbf{G}_R \cdot \mathbf{x}}\), \(\mathbf{G}_R\) lies in the reciprocal lattice and \(A\) is the area of the unit cell.

![Fig. 1 Face centered rectangular lattice with basis vectors: \(\mathbf{e}_1 = (e \cos \theta/2, -e \sin \theta/2)\), \(\mathbf{e}_2 = (e \cos \theta/2, e \sin \theta/2)\), \(\mathbf{n}, \mathbf{nn}, \mathbf{nnn}, \mathbf{nnnn}\) indicate nearest neighbour, next nearest neighbour and third nearest neighbour respectively. The box is a unit cell.](image)

We use the following ansatz to minimise the energy functional:

\[
w(z) = \sum_{\{\mathbf{R}\}} \lambda e^{i(q \cdot \mathbf{R})} \quad (9)
\]

where, \(\lambda\) sets the scale for the skyrmion size, \(q, \mathbf{R}\) is the spiral angle. The angle \(\theta\) characterises the lattice. This analytic ansatz exactly minimises the gradient term and gives scale invariant solutions \([3]\). For analytic skyrmions the gradient term does not influence positional or spin-orientational ordering. For overlapping skyrmions at low \(g\), we still use the same ansatz since their size is large and the exponential damping term used in ref. \([8]\) does not
have any significant effect. The size is determined by the competition between the Zeeman and Coulomb terms. We find that the Zeeman term prefers the SL phase and the Coulomb term prefers the TL phase. We expect to see FCRLs with spiral ordering when the transition occurs between the above phases. The NLSM energy functional has to be minimised with respect to the four variational parameters $\lambda, \theta, q$ (two components).

To get an idea of the kind of positional and magnetic ordering expected in this system we look at a toy model, a spin system described by an XY model on the FCRL shown in the figure. The free energy of the system is:

$$E = \sum_{(i,j)} J_{ij} S_i S_j \tag{10}$$

We look at position and magnetic configurations which lie between the TL and SL phases, which are FCRLs. We consider only nearest neighbour ($J_1$), next nearest neighbour ($J_2$) and third nearest neighbour ($J_3$) couplings on the lattice. These configurations have 4 nearest neighbours and 6 next nearest neighbours (except at the square and triangle end points). We choose a spiral state with bours and 6 next nearest neighbours (except at the square transition). The free energy is minimised with respect to the four variational parameters $\lambda, \theta, q$ in Eq. (9).

$$\nu = 1$$

In the relevant parameter ranges, we find the solutions that $(\theta, \lambda, q)$ should be independent of $\nu$ and that $\lambda \propto e$. We find that the configuration which favors minimum Coulomb energy is the TL phase, $(\theta, q) = (120^\circ, 120^\circ)$ and verify that $\lambda \propto e$.

$$\nu \gtrsim 0.8$$

Another test of accuracy can be done at $g = 0$. The only energy in the system is the Coulomb energy and this sets the length scale in the problem. It can then be shown that $(\theta, q)$ should be independent of $\nu$ and that $\lambda \propto e$. We find that the configuration which favors minimum Coulomb energy is the TL phase, $(\theta, q) = (120^\circ, 120^\circ)$ and verify that $\lambda \propto e$.

The accuracy of the routine was tested by integrating the topological charge density over a unit cell to get the topological charge, which should be 1 since there is one skyrmion per unit cell. An accuracy of 1 part in $10^6$ was noted in the calculation of topological charge.

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The FCRLs appear as intermediate phases in regions of the phase diagram where the NLSM is a good description of the quantum Hall system (i.e. low $g$ and high $\nu$ regions). The $\theta + q - \nu$ diagram shown in Fig. 3. The $\theta + q$ value is at 240° at the TL and FCRLs and is at 270° in the SL. The sharp jump seen in $\theta + q$ occurs at all non-zero $g$ values. Hence, we propose $\theta + q$ as an order parameter characterising the FCRL-SL transition.

The spin polarisation (sp) of the system of skyrmions at a particular $\nu$ is

$$sp = \frac{<s_z>}{N_c} = \frac{1}{N_c} \int s_z d^2x$$

$$= \frac{1 - \nu}{\nu} (Total \ Spin)_z - \frac{1}{2}$$

(12)

The FCRLs seem to smooth out the jump in the spin polarisation that was observed in ref. [8]. The value of spin polarisation that was observed in ref. [8] is not possible since we measure total spin at non-zero skyrmion density whereas the experiments measure spin for a thermally activated skyrmion-anti-skyrmion pair at $\nu = 1$.

Fig. 4 Spin Polarisation at different $g/g_0$ indicated in box brackets.

Novel FCRL phases with spiral ordering such that $\theta + q = 240^\circ$ appear in the low $g$ limit (overlapping skyrmions). The TL-FCRL transition is smooth characterised by $\theta + q = 240^\circ$. The second FCRL-SL transition is sharp and $\theta + q$ jumps from 240° to 270°, making this a convenient order parameter for this transition. There are no spiral phases with $\theta + q$ between these angles.

Our calculations do not explain why FCRLs with these values of $(\theta, q)$ occur but think that this is due to the hedgehog nature of the skyrmion. We have minimised the two-body energy functional of Timm et. al. and find that FCRLs with a general spiral ordering of spins do not occur. We confirm their phase diagram for a system of well separated skyrmions, which shows TL-FCRLs (Néel ordered)-SL transitions [9]. In their calculation, the Coulomb energy which favours the TL and the exchange energy which favours the SL, compete to drive these transitions. We conclude that the spiral ordering we observe is not a two-body effect and occurs because of three-body or higher order effects.

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[1] S. L. Sondhi, A. Karlhede, S. A. Kivelson and E. H. Rezayi, Phys. Rev. B47, 16419 (1993).
[2] D. H. Lee and C. L. Kane, Phys. Rev. Lett. 64, 1313 (1990); A. H. MacDonald, cond-mat/9601144.
[3] A. A. Belavin and A. M. Polyakov, JETP Lett. 22, 245 (1975).
[4] H. A. Fertig, L. Brey, R. Côté and A. H. MacDonald, Phys. Rev. B50, 11018 (1994); K. Moon, H. Mori, K. Yang, S. M. Girvin, A. H. MacDonald, L. Zheng, D. Yoshioka and S. C. Zhang, ibid. 51, 5138 (1995).
[5] H. A. Fertig, Luis Brey, R. Côté, A. H. MacDonald, A. Karlhede and S. L. Sondhi, Phys. Rev. B55, 10671 (1997).
[6] L. Brey, H.A. Fertig, R.Côté and A. H. Macdonald, Phys. Rev. Lett 75, 2562 (1995).
[7] A. G. Green, I.I. Kogan and A. M. Tsevelik, Phys. Rev. B54, 16398 (1996).
[8] Madan Rao, Surajit Sengupta and R. Shankar, Phys. Rev. Lett 79, 3998 (1997).
[9] Carsten Timm, S. M. Girvin and H. A. Fertig, Phys. Rev. B58, 10634 (1998).
[10] R. Côté, A. H. MacDonald, L. Brey, H. A. Fertig, S. M. Girvin and H. T. C. Stoof, Phys. Rev. Lett. 78, 4825 (1995).
[11] S. E. Barrett, G. Dabbagh, L. N. Pfeiffer, K. W. West and R. Tycko, Phys. Rev. Lett. 74, 5112 (1995); S. E. Barrett, G. Dabbagh, L. N. Pfeiffer, K. W. West, and R. Tycko, Phys. Rev. Lett. 72, 1368 (1994); R. Tycko, S. E. Barrett, G. Dabbagh, L. N. Pfeiffer and K. W. West, Science 268, 1460 (1995).
[12] E. H. Aifer, B. B. Goldberg and D. A. Broido, Phys. Rev. Lett. 76, 680 (1996).
[13] S. P. Shukla, M. Shayagan, S. P. Parihar, S. A. Lyon, N. R. Cooper and A. A. Kiselev, Phys. Rev. B51, 4469 (2000).
[14] D. R. Leadley, R. J. Nicholas, D. K. Maude, A. N. Utjuzh, J. C. Portal, J. J. Harris and C. T. Foxon, Semicond. Sci. Technol. 13, 671 (1998).