SAMPLING, FILTERING AND SPARSE APPROXIMATIONS ON COMBINATORIAL GRAPHS

Isaac Z. Pesenson
Meyer Z. Pesenson

Abstract. In this paper we address sampling and approximation of functions on combinatorial graphs. We develop filtering on graphs by using Schrodinger’s group of operators generated by combinatorial Laplace operator. Then we construct a sampling theory by proving Poincare and Plancherel-Polya-type inequalities for functions on graphs. These results lead to a theory of sparse approximations on graphs and have potential applications to filtering, denoising, data dimension reduction, image processing, image compression, computer graphics, visualization and learning theory.

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1. Introduction

During the last years harmonic analysis on combinatorial graphs attracted considerable attention. The interest is stimulated in part by multiple existing and potential applications of analysis on graphs to information theory, signal analysis, image processing, computer sciences, learning theory, astronomy [2], [3], [5]–[8], [12], [17], [24]–[26].

Some of the approaches to large data sets or images consider them as graphs. However, for hyperspectral images, for example, this leads to graphs with too many vertices imbedded into high dimensional spaces, thus making dimension reduction necessary for effective data mining.

It seems that one possible way to approach this problem is by using ideas from the classical sampling theory which has already proved very fruitful in various branches of applied mathematics.

Let us remind the Classical Shannon-Nyquist sampling Theorem. It states that for all Paley-Wiener functions of a fixed bandwidth defined on Euclidean space one can find “not very dense” sampling sets which can be used to represent all relevant Paley-Wiener functions. In some sense it allows to reduce the set of all points of Euclidean space to a countable set of points. Moreover, since the set of

1 Department of Mathematics, Temple University, Philadelphia, PA 19122; pesenson@math.temple.edu. The author was supported in part by the National Geospatial-Intelligence Agency University Research Initiative (NURI), grant HM1582-08-1-0019.
2 Spitzer Science Center, California Institute of Technology, MC 314-6, Pasadena, CA 91125; misha@ipac.caltech.edu. The author was supported in part by the National Geospatial-Intelligence Agency University Research Initiative (NURI), grant HM1582-08-1-0019.
all Paley-Wiener functions is dense in the space $L^2(\mathbb{R}^d)$ one can use this property to approach sampling of non-Paley-Wiener functions.

The goal of this work, to show that analysis of lower frequencies on a graph can be performed on a smaller subgraph. Note that in many situations lower frequencies are more informative while higher frequencies are usually associated with noise.

Let us consider an example. Suppose that a data set is presented by $10^6$ points. One way of data mining \cite{2}, \cite{3} is to convert the data set to a graph and develop harmonic analysis associated with a corresponding combinatorial Laplace operator. Let us assume for the simplicity that we identify our data set with the path graph $\mathbb{Z}_{10^6}$ of $10^6$ vertices. We measure frequency on this graph in terms of the eigenvalues of the corresponding Laplace operator on $\mathbb{Z}_{10^6}$ whose definition is given in section 4. It has $10^6$ eigenvalues (frequencies) which all belong to the interval $[0, 4]$ and are given by the formula

$$2 - 2 \cos \frac{\pi k}{10^6 - 1}, \quad k = 0, 1, \ldots, 10^6 - 1.$$ 

Our results show, that if one will delete every second point from $\mathbb{Z}_{10^6}$ then the resulting set will be a uniqueness set and even a sampling set (see definitions below) for all functions on $\mathbb{Z}_{10^6}$ which are linear combinations of the (about) $12 \times 10^4$ first eigenfunctions. If one will delete about $2/3$ of all points then the resulting set is a sampling set for all functions on $\mathbb{Z}_{10^6}$ which are linear combinations of the (about) $6 \times 10^4$ first eigenfunctions. By extending our reasoning it is possible to show that about 10 percent of "uniformly distributed" points of $\mathbb{Z}_{10^6}$ form a sampling set for functions on $\mathbb{Z}_{10^6}$ which are linear combinations of the about 500 first eigenfunctions.

Thus by applying an appropriate filtering to a function on a graph, i.e. by removing high frequencies we not only remove noise but we also reduce analysis on a whole graph to analysis on a much smaller subgraph without losing many of the lower frequencies. We also give estimates of possible losses of information which can occur after filtering.

In order to construct a sampling theory on combinatorial graphs we prove certain analogs of Poincare inequality on graphs. Our Poincare inequalities in the Section 2 provide estimates of the norm of a function in terms of its "derivatives".

In what follows we introduce few basic notions and formulate and discuss one of our Poincare inequalities. We consider finite or infinite and in the later case countable connected graphs $G = (V(G), E(G))$, where $V(G)$ is its set of vertices and $E(G)$ is its set of edges. We consider only simple (no loops, no multiple edges) undirected unweighted graphs. A number of vertices adjacent to a vertex $v$ is called the degree of $v$ and denoted by $d(v)$. We assume that all vertices have finite degrees but we do not assume that the set of degrees of all vertices $\{d(v)\}_{v \in V(G)}$ is bounded.

The space $L_2(G)$ is the Hilbert space of all complex-valued functions $f : V(G) \to \mathbb{C}$ with the following inner product

$$\langle f, g \rangle = \sum_{v \in V(G)} f(v) \overline{g(v)}$$

and the following norm

$$\|f\| = \left( \sum_{v \in V(G)} |f(v)|^2 \right)^{1/2}.$$
By the adjacency matrix $A$ of $G$ we understand a matrix with entries $\{a_{uv}\}$, $u, v \in V(G)$, where $a_{uv} = 1$, if vertices $u$ and $v$ are adjacent, and $a_{uv} = 0$ otherwise.

Let $A$ be the adjacency matrix of $G$ and $D$ be a diagonal matrix whose entries on main diagonal are degrees of the corresponding vertices. Then we consider the following version of the discrete Laplace operator on $G$

(1.1) \[ L = D - A, \]

or explicitly

\[ Lf(v) = \sum_{u \sim v} (f(v) - f(u)), f \in L_2(G), \]

where notation $u \sim v$ means that $u$ and $v$ are adjacent vertices. Note that this operator is different from the normalized Laplace operator $L$ is defined in [6] and which was considered in our previous papers [20]– [22].

The Laplace operator $L$ is self-adjoint and positive definite in the space $L_2(G)$.

Moreover, if degrees of all vertices are uniformly bounded

(1.2) \[ D(G) = \max_{v \in V(G)} d(v) < \infty \]

then the operator $L$ is bounded and its spectrum $\sigma(L)$ is a subset of the interval $[0, 2D(G)]$. Note, that for the normalized version of the Laplace operator $L$ the spectrum is always a subset of $[0, 2]$.\[ \]

Given a proper subset of vertices $S \subset V(G)$ its vertex boundary $bS$ is the set of all vertices in $V(G)$ which are not in $S$ but adjacent to a vertex in $S$

\[ bS = \{ v \in V(G) \setminus S : \exists \{u, v\} \in E(G), u \in S \}. \]

If a graph $G = (V(G), E(G))$ is connected and $S$ is a proper subset of $V$ then the vertex boundary $bS$ is not empty.

We will also use the following notation

(1.3) \[ D(S) = \max_{v \in S} d(v), S \subset V(G). \]

To illustrate our Poincaré inequalities let us formulate and discuss a particular case of a more general inequality proved in the Theorems 2.1 and 2.3. The following inequality gives an estimate of the norm of a function trough its "first order derivatives" and in this sense it can be considered as a global Poincaré inequality.

**Theorem 1.1.** If $S$ is a subset of vertices such that every vertex $v$ in $bS$ is connected to at least $K_0 = K_0(S)$ vertices in $S$ and

(1.4) \[ \overline{S} = S \cup bS = V(G), \]

then the following inequality holds for all $f \in L_2(G)$

(1.5) \[ \|f\| \leq \left\{ \sum_{u \in S} \left( \frac{2d_0(u)}{K_0} + 1 \right)|f(u)|^2 \right\}^{1/2} + \frac{2}{\sqrt{K_0}} \|L^{1/2}f\|, \]

where $d_0(u)$, $u \in S$, is the number of vertices in $bS$ adjacent to $u \in S$.

**Example 1.** Suppose that $G$ is a star $\{v_0, v_1, ..., v_N\}$ whose center is $v_0$. Let $S$ be the vertex $\{v_0\}$. Then $K_0 = 1$, $d_0(v_0) = N$, and (1.3) becomes

\[ \|f\| \leq \sqrt{2N + 1}|f(v_0)| + 2\|L^{1/2}f\|. \]
In particular, if \( f(v_j) = 1 \) for all \( 0 \leq j \leq N \) then \( \|f\| = \sqrt{N + 1} \), \( \|L^{1/2}f\| = 0 \), and \((1.5)\) becomes
\[
\sqrt{N + 1} \leq \sqrt{2N + 1}.
\]

**Example 2.** For the same star graph as above we consider \( S = \{v_1, ..., v_N\} \), then \( K_0(S) = N, d_0(v_0) = 1 \), and for any \( N \) the inequality \((1.5)\) becomes
\[
\|f\| \leq \sqrt{2N + 1}.
\]

In particular, if we consider function \( f \) such that \( f(v_0) = 0 \) and \( f(v_j) = 1 \) for all other \( j = 1, 2, ..., N \), then \( \|f\| = \sqrt{N}, \|L^{1/2}f\| = \sqrt{N} \), and for any \( N \) the inequality \((1.5)\) becomes
\[
\sqrt{N} \leq \sqrt{N + 2}.
\]

**Example 3.** Let \( C_N \) be a cycle of \( N \) vertices \( \{v_1, ..., v_N\} \). Take another vertex \( v_0 \) and make a graph \( C_N \cup \{v_0\} \) by connecting \( v_0 \) to each of \( v_1, ..., v_N \).

Let \( \lambda_k(N) \) be a non-zero eigenvalue of the operator \( L \) on the graph \( C_N \) and let \( \varphi_k \) be a corresponding orthonormal eigenfunction. Construct a function \( \tilde{\varphi}_k \) on the graph \( C_N \cup \{v_0\} \) such that \( \tilde{\varphi}_k(v_j) = \varphi_k(v_j) \) if \( v_j \in C_N \) and \( \tilde{\varphi}_k(v_0) = 0 \). Since \( \varphi_k \) is orthogonal to the constant function \( 1 \) we have that
\[
\sum_{v_j \in C_N} \varphi_k(v_j) = 0
\]
and it implies that for the operator \( L \) on \( C_N \cup \{v_0\} \)
\[
L\tilde{\varphi}_k(v_0) = 0.
\]

Clearly, for every \( v_j \in C_N \) one has
\[
L\tilde{\varphi}(v_j) = L\varphi_k(v_j) + \varphi(v_j) = (\lambda_k(N) + 1)\varphi(v_j).
\]

Thus,
\[
L\tilde{\varphi}_k = (\lambda_k(N) + 1)\tilde{\varphi}_k,
\]
and since \( \|\tilde{\varphi}_k\| = 1 \) we have that
\[
\|L^{1/2}\tilde{\varphi}_k\| = (\lambda_k(N) + 1)^{1/2}.
\]

Let \( S \) be the graph \( C_N = \{v_1, ..., v_N\} \). In this case the boundary of \( S \) is the point \( v_0, K_0 = N, d_0(v_j) = 1 \) and then for the function \( \tilde{\varphi}_k \) the inequality \((1.5)\) takes the following form
\[
1 \leq \sqrt{2N + 1} + 2\sqrt{\lambda_k(N) + 1} \frac{N}{N}.
\]

Since for \( k = 1 \) the eigenvalue \( \lambda_1(N) \) goes to zero when \( N \) goes to infinity, we see that the right-hand side of the last inequality can be made arbitrary close to one.

For any \( k = 1, ..., N \) one has the estimate \( \lambda_k(N) \leq 2N \) and for the corresponding \( \tilde{\varphi}_k \) it gives the inequality
\[
1 \leq \sqrt{2N + 1} + 2\sqrt{\frac{1}{N} + 2}.
\]
According to our definition of Paley-Wiener functions (see Section 3, and also [18], [20]) they always satisfy the Bernstein inequality and together with Poincare inequality it leads to Plancherel-Polya inequalities on graphs.

Our Poincare-Polya-type inequalities (Theorem 3.3) give two-sided estimate of the norm of an appropriate Paley-Wiener function in terms of its values on a subset of vertices. We use these estimates to apply classical ideas of Duffin and Schaeffer [9] about Hilbert frames to obtain the sampling Theorem 3.4 which is one of the main results of the paper. In particular we obtain a formula which represents Paley-Wiener functions in terms of their values on specific subgraphs. We call them sparse representations of Paley-Wiener function.

In Section 4 we construct a filtering operator (Theorem 4.1) using Schrodinger’s one-parameter group of operators generated by a self-adjoint positive definite operator \( L \) in the Hilbert space \( L_2(G) \). This filtering operator maps entire Hilbert space \( L_2(G) \) into appropriate Paley-Wiener space. We also prove our version of the Direct Approximation Theorem using a modulus of continuity expressed in terms of the Schrodinger group of operators generated by \( L \) (Theorem 4.3). By combining filtering procedure with our sampling theory, we obtain sparse approximations to functions in \( L_2(G) \).

We would like to emphasize that the sampling theory that is developed in the present article is different from the one we had developed in [20], [21], [22], [23]. We also have to mention that our approach to sampling on graphs is very different from methods which were presented and explored in [10], [11], [14]. Note, that our approximation theory on graphs is a generalization of the classical approximation theory by Paley-Wiener functions [1], [16]. It also has to be mentioned that some ideas about approximation theory on compact metric spaces (which include finite graphs) were introduced in [13]. Basic ideas of harmonic analysis on graphs that are relevant to our paper were recently summarized in the book [15].

Our results can have applications to filtering, denoising, approximation and compression of functions on graphs. These tasks are of central importance to data dimension reduction, image processing, computer graphics, visualization and learning theory.

\section{Poincare inequalities on combinatorial graphs}

For a function \( f \in L_2(G) \) we introduce a measure of smoothness which is the norm of a "gradient"

\begin{equation}
\| \nabla f \|^2 = \sum_{u \sim v} |f(v) - f(u)|^2,
\end{equation}

where the sum is taken over all unordered pairs \( \{v, u\} \) for which \( v \) and \( u \) are adjacent. Given a subset \( W \subset V \) we will use the notation

\begin{equation}
\| \nabla f \|^2_W = \sum_{u \sim v, v, u \in W} |f(v) - f(u)|^2.
\end{equation}

For any \( S \) which is a subset of vertices of \( G \) we introduce the following operator

\begin{equation}
\cd^0(S) = S, \quad \cd(S) = S \cup bS, \quad \cd^m(S) = \cd(\cd^{m-1}(S)), \quad m \in \mathbb{N}, S \subset V(G).
\end{equation}

We will use the following notion of the relative degree.
Theorem 2.1. In the same notations as above, if $S$ is a subset of vertices in the boundary $b(cl^n(S))$ which are adjacent to $v$:

$$d_m(v) = \text{card} \{ w \in b(cl^n(S)) : w \sim v \}.$$  

For any $S \subset V(G)$ we introduce the following notation

$$D_m = D_m(S) = \sup_{v \in cl^m(S)} d_m(v).$$

Definition 2. For a vertex $v \in b(cl^m(S))$ we introduce the quantity $k_m(v)$ as the number of vertices in the boundary of $cl^m(S)$ which are adjacent to $v$:

$$k_m(v) = \text{card} \{ w \in cl^m(S) : w \sim v \}.$$  

For any $S \subset V(G)$ we introduce the following notation

$$K_m = K_m(S) = \inf_{v \in b(cl^m(S))} k_m(v).$$

For a given set $S \subset V(G)$ and a fixed $n \in \mathbb{N}$ consider a sequence of closures $S, cl(S), ..., cl^n(S), n \in \mathbb{N}$.

Theorem 2.1. In the same notations as above, if $S$ is a subset of vertices such that the boundary of $cl^{n-1}(S), n \in \mathbb{N}$, is not empty then the following inequality holds

$$\left( \sum_{v \in cl^n(S)} |f(v)|^2 \right)^{1/2} \leq \left( \prod_{i=0}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \left( \sum_{v \in S} |f(v)|^2 \right)^{1/2} + 2 \left( \sum_{j=0}^{n-1} \frac{1}{K_j} \prod_{i=j+1}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \| L^{1/2}f \|.$$  

In particular, if

(2.5)  

$$cl^n(S) = V(G),$$

then

(2.6)  

$$\|f\| \leq \left( \prod_{i=0}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \left( \sum_{v \in S} |f(v)|^2 \right)^{1/2} + 2 \left( \sum_{j=0}^{n-1} \frac{1}{K_j} \prod_{i=j+1}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \| L^{1/2}f \|.$$  

Proof. First, we are going to prove, that for any subset $S$ of vertices for which the boundary of $cl^{n-1}(S), n \in \mathbb{N}$, is not empty the following inequality holds

$$\sum_{w_n \in cl^n(S)} |f(w_n)|^2 \leq \prod_{i=0}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \sum_{w_0 \in S} |f(w_0)|^2 + \frac{2}{K_0} \prod_{i=1}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \sum_{v_0 \in bS} \sum_{j_0=1}^{K_0(S)} |f(v_0) - f(u_{j_0}(v_0))|^2 +$$

where $u_{j_0}(v_0)$ are new elements such that $u_{j_0}(v_0) \sim u_{j_0}(v_0)$.
\[
\frac{2}{K_1} \prod_{i=2}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \sum_{v_i \in b(cl(S))} K_1 \sum_{j=1}^{|f(v)|} |f(v_i) - f(u_{j_1}(v_i))|^2 + \ldots
\]

(2.7) \[\frac{2}{K_{n-1}} \sum_{v_{n-1} \in b(cl^{n-1}))} |f(v_{n-1}) - f(u_{j_{n-1}}(v_{n-1}))|^2,\]

where for every 0 \leq m \leq n - 1 the \(v_m \in b(cl^m(S))\) which is adjacent to \(v_m \in b(cl^m(S))\).

For any two vertices \(v, u \in V\) one has

(2.8) \[f(v) = f(u) + (f(v) - f(u))\]

and

(2.9) \[|f(v)|^2 \leq 2 \left( |f(u)|^2 + |f(v) - f(u)|^2 \right).\]

Let \(v \in bS\) and \(u_1(v), \ldots, u_{K_0}(v) \in S\) be some of the \(K_0\) vertices in \(S\) which are adjacent to \(v\). For each of them the following inequality holds

(2.10) \[|f(v)|^2 \leq 2 \left( |f(u_j(v))|^2 + |f(v) - f(u_j(v))|^2 \right), 1 \leq j \leq K_0,\]

which implies the inequality

(2.11) \[|f(v)|^2 \leq \frac{2}{K_0} \sum_{j=1}^{K_0} |f(u_j(v))|^2 + \frac{2}{K_0} \sum_{j=1}^{K_0} |f(v) - f(u_j(v))|^2.\]

Since every \(u_j(v) \in S, 1 \leq j \leq K_0\), can be adjacent to a maximum of \(d_0(u_j(v))\) distinct vertices \(v \in bS\), the previous inequality implies

\[
\sum_{v \in bS} |f(v)|^2 \leq \sum_{j=1}^{K_0} \frac{2}{K_0} \sum_{v \in bS} |f(u_j(v))|^2 + \sum_{j=1}^{K_0} \frac{2}{K_0} \sum_{v \in bS} |f(v) - f(u_j(v))|^2 \leq \sum_{u \in S} \frac{2d_0(u)}{K_0} |f(u)|^2 + \sum_{j=1}^{K_0} \frac{2}{K_0} \sum_{v \in bS} |f(v) - f(u_j(v))|^2.
\]

(2.12) \[\sum_{u \in S} \frac{2d_0(u)}{K_0} |f(u)|^2 + \sum_{j=1}^{K_0} \frac{2}{K_0} \sum_{v \in bS} |f(v) - f(u_j(v))|^2.\]

Thus,

(2.13) \[\sum_{v \in bS} |f(v)|^2 \leq \sum_{u \in S} \frac{2d_0(u)}{K_0} |f(u)|^2 + \sum_{j=1}^{K_0} \frac{2}{K_0} \sum_{v \in bS} |f(v) - f(u_j(v))|^2,\]

where \(u_1(v), \ldots, u_{K_0}\) are different vertices from \(S\) that adjacent to \(v\). The last inequality implies the following

(2.14) \[\sum_{v \in bS} |f(v)|^2 \leq \sum_{u \in S} \frac{2D_0}{K_0} |f(u)|^2 + \sum_{j=1}^{K_0} \frac{2}{K_0} \sum_{v \in bS} |f(v) - f(u_j(v))|^2.\]

By adding this inequality with the identity

\[\sum_{v \in S} |f(v)|^2 = \sum_{v \in S} |f(v)|^2,\]
one obtains the inequality which holds true for any subset of vertices $S$: \[
\sum_{v_0 \in bS} |f(v_0) - f(u_j(v_0))|^2, \]
where $u_1(v_0), \ldots, u_{K_0} \in S$ are different and adjacent to $v$.

Since $\mathcal{cl}(S) = \mathcal{cl}(cl(S))$, the inequality (2) implies the following one
\[
\sum_{v_0 \in bS} |f(v_0) - f(u_j(v_0))|^2, \]
where $u_j(v_1) \in \mathcal{cl}(S)$. Along with the (2) it gives
\[
\sum_{v_0 \in bS} |f(v_0) - f(u_j(v_0))|^2, \]
where $u_j(v_1) \in \mathcal{cl}(S)$, $u_j(v_0) \in S$ are different vertices that adjacent to $v$.

The derivation of (2) shows that by induction one can prove the inequality (2).

Next, let us remind, that just by construction the vertices $v_m \in b(cl^m(S))$ and $u_{j_m}(v_m) \in cl^m(S)$ are adjacent and $u_{k_1}(v_m) \neq u_{k_2}(v_m)$, if $k_1 \neq k_2$. It is also clear that $v_m \in b(cl^m(S))$ is different from any of $v_k \in b(cl^k(S))$ as long as $m \neq k$. Because if this the inequality (2) implies the inequality
\[
||f|| \leq \left( \prod_{i=0}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \left( \sum_{v \in S} |f(v)|^2 \right)^{1/2} + \]
\[
\left( \sum_{j=0}^{n-1} \frac{2}{K_j} \prod_{i=j+1}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \|
\n
To prove the Theorem we are going to show that
\[
\|
\n
Indeed,
\[ \| \nabla f \|^2 = \sum_{v \sim u} |f(v) - f(u)|^2 = \sum_{v \in V} |f(v)|^2 d(v) + \sum_{u \in V} |f(u)|^2 d(u) - 2 \sum_{v \sim u} f(v) f(u) = 2 \left( \sum_{v \in V} |f(v)|^2 d(v) - \sum_{v \sim u} f(v) f(u) \right) = \frac{1}{2} (\sum_{v \in V} |f(v)|^2 d(v) - \sum_{v \sim u} f(v) f(u)) = \frac{1}{2} \langle Df, f \rangle = \langle Af, f \rangle = \langle L f, f \rangle = \| L^{1/2} f \|^2 . \]

The inequalities (2) and (2) imply (2.1). The Theorem is proved.

The formula (2.13) gives the following Corollary.

**Corollary 2.1.** If \( S \) is a subset of vertices such that every vertex \( v \) in \( bS \) is connected to at least \( K_0(S) \) vertices in \( S \), where \( 1 \leq K_0(S) \leq d(v) \) then for all \( f \in L^2(G) \)
\begin{equation}
\left\{ \sum_{v \in \partial(S)} |f(v)|^2 \right\}^{1/2} \leq \left\{ \sum_{u \in S} \left( \frac{2d_0(u)}{K_0(S)} + 1 \right) |f(u)|^2 \right\}^{1/2} + \frac{2}{\sqrt{K_0(S)}} \| L^{1/2} f \| .
\end{equation}

If in addition
\begin{equation}
\partial(S) = S \cup bS = V(G),
\end{equation}
then the following inequality holds for all \( f \in L^2(G) \)
\begin{equation}
\| f \| \leq \left\{ \sum_{u \in S} \left( \frac{2d_0(u)}{K_0(S)} + 1 \right) |f(u)|^2 \right\}^{1/2} + \frac{2}{\sqrt{K_0(S)}} \| L^{1/2} f \| .
\end{equation}

Since \( K_0(S) \geq 1 \) we have the following.

**Corollary 2.2.** If \( S \) is a subset of vertices such that the condition (2.22) holds then the following inequality takes place
\begin{equation}
\| f \| \leq \left\{ \sum_{u \in S} (2d_0(u) + 1) |f(u)|^2 \right\}^{1/2} + 2 \| L^{1/2} f \|. \end{equation}

To extend our results to higher powers of \( L \) we will need the following Lemma [18]–[21].

**Lemma 2.2.** If for some positive \( c > 0, a > 0, s > 0 \), and an \( \varphi \in L_2(G) \) the following inequality holds true
\begin{equation}
\| \varphi \| \leq a + c \| L^s \varphi \| ,
\end{equation}
then for the same \( c, a, s, \varphi \) the following holds
\begin{equation}
\| \varphi \| \leq 2ra + 8^{-1} c^r \| L^r \varphi \|
\end{equation}
for all \( r = 2^l, l = 0, 1, \ldots \).

An application of the Lemma 2.2 gives the following result.
Theorem 2.3. If the assumption (2.3) is satisfied, then for any \( r = 2^l, l = 0, 1, \ldots \), the next inequality holds
\[
\|f\| \leq 2r \left( \prod_{i=0}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \left( \sum_{v \in S} |f(v)|^2 \right)^{1/2} + \frac{2^{4r-3}}{K_0^{r/2}} \|L^{r/2}f\|.
\]
(2.27)

Theorem 2.4. If \( S \) is a subset of vertices such that every vertex \( v \) in \( bS \) is connected to at least \( K_0 \) vertices in \( S \), where \( 1 \leq K_0 \leq d(v) \) and the condition (2.22) holds, then for any \( r = 2^l, l = 0, 1, \ldots \)
\[
\|f\| \leq 2r \sqrt{\frac{2D_0}{K_0} + 1} \left( \sum_{u \in S} |f(u)|^2 \right)^{1/2} + \frac{2^{4r-3}}{K_0^{r/2}} \|L^{r/2}f\|.
\]
(2.28)

Proof. According to the previous Theorem the assumptions of the Theorem give the inequality
\[
\|f\| \leq \sqrt{\frac{2D_0}{K_0} + 1} \left( \sum_{u \in S} |f(u)|^2 \right)^{1/2} + \frac{2}{\sqrt{K_0}} \|L^{1/2}f\|.
\]
(2.29)

Now an application of (2.28) gives the result.

Corollary 2.3. If \( S \) is a subset of vertices such that (2.22) holds, then for any \( r = 2^l, l = 0, 1, \ldots \), the following holds
\[
\|f\| \leq 2r \sqrt{\frac{2D_0}{K_0} + 1} \left( \sum_{u \in S} |f(u)|^2 \right)^{1/2} + 2^{4r-3} \|L^{r/2}f\|.
\]
(2.30)

3. Plancherel-Polya inequalities, sampling and sparse representations of Paley-Wiener functions

By the spectral theory of self-adjoint operators \([4]\), there exist a direct integral of Hilbert spaces \( A = \int A(\lambda)dm(\lambda) \) and a unitary operator \( F_L \) from \( L_2(G) \) onto \( A \), which transforms the domain \( D_s, s \geq 0 \), of the operator \( L^s \) onto \( A_s = \{ a \in A | \lambda^s a \in A \} \) with norm
\[
\|a(\lambda)\|_{A_s} = \left( \int_0^\infty \lambda^{2s} \|a(\lambda)\|_{A(\lambda)}^2 dm(\lambda) \right)^{1/2}
\]
(3.1)

and \( F_L(Lf) = \lambda(F_Lf), f \in D_1 \).

Definition 3. The unitary operator \( F_L \) will be called the Spectral Fourier transform and \( a = F_Lf \) will be called the Spectral Fourier transform of \( f \in L_2(G) \).

Definition 4. We will say that a function \( f \) in \( L_2(G) \) belongs to the space \( PW_\omega(L) \) if its Spectral Fourier transform \( F_Lf = a \) has support in \([0, \omega]\).
The following theorem describes some basic properties of Paley-Wiener vectors and show that they share similar properties to those of the classical Paley-Wiener functions. The proof of these and many other properties of Paley-Wiener vectors can be found in our other papers and in particular in [18]-[21].

**Theorem 3.1.** The following conditions are equivalent:

1) The linear set $\bigcup_{\omega > 0} PW_\omega(L)$ is dense in $L^2(G)$.

2) The set $PW_\omega(L)$ is a linear closed subspace in $L^2(G)$.

3) A function $f$ belongs to a space $PW_\omega(L)$ if and only if for all $k \in \mathbb{N}$, the following Bernstein inequality holds true

$$\|L^k f\| \leq \omega^k \|f\|;$$

To obtain a Sampling Theorem for Paley-Wiener functions on graphs we have to establish Plancherel-Polya-type inequalities. The inequalities (2.1) and (3.2) along with the obvious inequality

$$\left(\sum_{u \in S} |f(u)|^2\right)^{1/2} \leq \|f\|$$

imply the following Plancherel-Polya-type inequalities for functions in $PW_\omega(L)$.

**Theorem 3.2.** In the same notations as in the Theorem 2.1, if the condition

$$\omega < \frac{1}{4} \left(\sum_{j=0}^{n-1} \frac{1}{K_j} \prod_{i=j+1}^{n-1} \left(\frac{2D_i}{K_i} + 1\right)\right)^{-1}$$

hold, then for any $f \in PW_\omega(L)$ the next inequality takes place

$$\left(\sum_{u \in S} |f(u)|^2\right)^{1/2} \leq \|f\| \leq \frac{1}{1 - \gamma} \left(\prod_{i=0}^{n-1} \left(\frac{2D_i}{K_i} + 1\right)\right)^{1/2} \left(\sum_{u \in S} |f(u)|^2\right)^{1/2},$$

where

$$\gamma = 2\omega^{1/2} \left(\sum_{j=0}^{n-1} \frac{1}{K_j} \prod_{i=j+1}^{n-1} \left(\frac{2D_i}{K_i} + 1\right)\right)^{1/2} < 1.$$
hold, then
\[
(3.10) \quad \left( \sum_{u \in S} |f(u)|^2 \right)^{1/2} \leq \|f\| \leq \frac{1}{1 - \gamma} \sqrt{\frac{2D_0}{K_0}} + 1 \left( \sum_{u \in S} |f(u)|^2 \right)^{1/2},
\]
where \( f \in PW_\omega(L) \) and \( \gamma = 2\sqrt{\omega/K_0} < 1 \).

The significance of the inequalities (3.10) is that they give two-sided estimate of the norm \( \|f\| \) of a Paley-Wiener function \( f \) in terms of its values on a smaller set \( S \subset V(G) \).

Let \( P_\omega \) be the orthogonal projector \( P_\omega : L_2(G) \to PW_\omega(L) \).

The last inequality (3.10) shows that the set of functions \( P_\omega(\delta_u), u \in S \), where \( \delta_u \) is the Dirac measure concentrated at \( u \in S \), is a Hilbert frame in the Hilbert space \( PW_\omega(L) \) when \( \omega < \frac{K_0}{4} \).

Thus, by applying the classical result of Duffin and Schaeffer [9] about dual frames we obtain the following uniqueness and reconstruction theorem. For the sake of simplicity we formulate it just for particular situation that satisfies (3.8).

**Theorem 3.4.** If \( S \) is a subset of vertices such that every vertex \( v \) in \( bS \) is connected to at least \( K_0 \) vertices in \( S \), where \( 1 \leq K_0 \leq d(v) \) and the condition
\[
(3.11) \quad \overline{S} = S \cup bS = V(G),
\]
along with
\[
\omega < \frac{K_0}{4}
\]
hold, then
1) the set \( S \) is a uniqueness set for functions in \( PW_\omega(L) \);
2) there exist functions \( \Theta_u \in PW_\omega(L), u \in S \), such that for all \( f \in PW_\omega(L) \) the following reconstruction formula holds
\[
(3.12) \quad f = \sum_{u \in S} f(u)\Theta_u.
\]

The last formula (3.12) is what we call a sparse representation since it represents a function through its values on a subgraph.

**Remark 1.** It is clear that if the spectrum the Laplace operator of a graph \( G \) is very close to zero then there are many subsets \( S \) of \( G \) and functions on \( G \) for which the last two Theorems convey non-trivial information. Thus, in the section below we discuss the situation on the infinite graph \( \mathbb{Z}_n \).

However, in the case of a finite graph \( G \) the spectral resolution is just the eigenvalue-eigenfunction representation and if \( K_0/4 \) is less than the first strictly positive eigenvalue of \( L \), then the assumption \( \omega < \frac{K_0}{4} \) would satisfy only for constant functions on \( G \) and the above inequalities would be trivial.

But there are many finite graphs for which the Theorems 3.3 and 3.4 are not trivial for a “right” choice of subsets \( S \subset V(G) \). For example, take the cycle \( C_n \) of \( n \) vertices for which the eigenvalues of the corresponding Laplace operator are \( 2 - 2 \cos \frac{2\pi k}{n} \). For a large \( n \) there are many eigenvalues which are very close to zero. On the other hand there are sets \( S \) of "isolated" points in \( C_n \) for which the number
K_0 is either 1 or 2. Thus, the previous Theorems hold true for any of such sets of points for functions from PW_\omega(G), for which either \omega < 1/4 or \omega < 2/4.

4. Filtering and Direct Approximation Theorem by Paley-Wiener functions on graphs

The goal of the section is to describe relations between Schrödinger’s Semigroup e^{itL} and the functional

$$E(f, \omega) = \inf_{g \in PW_\omega(L)} \| f - g \|,$$

which measures a best approximation of \( f \in L^2(G) \) by functions from the Paley-Wiener space \( PW_\omega(L), \omega \geq 0 \). If \( f_\omega \) is the orthogonal projection of \( f \) on \( PW_\omega(L) \), then according to (3.1) one has the following relation

$$(4.1) \quad E(f, \omega) = \| f - f_\omega \| = \left( \int_\omega^{\infty} \| x(\tau) \|^2 \chi(\tau) dm(\tau) \right)^{1/2},$$

where \( x(\tau) = \mathcal{F} f \) is the Spectral Fourier transform of \( f \). In other words the best approximation \( E(f, \omega) \) shows the “rate of decay” of the Spectral Fourier transform \( \mathcal{F} f \) of \( f \). The same formula (3.1) implies the following inequality

$$(4.2) \quad \omega^{-s} \left( \int_0^{\infty} \tau^{2s} \| x(\tau) \|^2 \chi(\tau) dm(\tau) \right)^{1/2} \leq \omega^{-s} \| L^s f \|, \quad s > 0,$$

for all functions in \( L^2(G) \).

The quantity \( \| L^s f \|, s > 0 \), is a measure of smoothness of a function \( f \). In this sense the estimate (4.2) generalizes the well-known fact of the classical harmonic analysis that the rate of approximation of a function by Paley-Wiener functions depends on the smoothness of this function.

For any \( f \in L^2(G) \) we introduce a difference operator of order \( m \in \mathbb{N} \) as

$$\Delta^m f = \sum_{j=0}^{m} (-1)^{m+j} C_m^j e^{ijsL} f,$$

where \( C_m^j \) is the number of combinations from \( m \) elements taking \( j \) at a time. The modulus of continuity is defined as

$$\Omega_m(f, s) = \sup_{|\tau| \leq s} \| \Delta^m f \|.$$

In the following Theorem we construct a filtering operator which maps \( H \) into a Paley-Wiener space.

**Theorem 4.1.** If \( h \in L^1(\mathbb{R}) \) is an entire function of exponential type \( \omega \) then for any \( f \in L^2(G) \) the function

$$Q^\omega_h(f) = \int_{-\infty}^{\infty} h(t) e^{itL} f dt$$

belongs to \( PW_\omega(L) \).
Proof. If \( g = Q_{\tau} f \) then for every real \( \tau \) we have

\[
e^{i\tau L} g = \int_{-\infty}^{\infty} h(t) e^{i(t+\tau)L} f dt = \int_{-\infty}^{\infty} h(t - \tau) e^{itL} f dt.
\]

Using this formula we can extend the abstract function \( e^{i\tau L} g \) to the complex plane as

\[
e^{izL} g = \int_{-\infty}^{\infty} h(t - z) e^{itL} f dt.
\]

Since by assumption \( h \in L_1(\mathbb{R}) \) is an entire function of exponential type \( \omega \) we have

\[
h(x + iy) = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} h^{(k)}(x)
\]

and the \( L_1(\mathbb{R}) \)-Bernstein inequality implies the following

\[
\int_{-\infty}^{\infty} |h(t - z)| dt \leq e^{\omega |z|} \int_{-\infty}^{\infty} |h(t)| dt.
\]

Thus, we obtain the following inequality

\[
\|e^{izL} g\| \leq \|f\| \int_{-\infty}^{\infty} |h(t - z)| dt \leq \|f\| e^{\omega |z|} \int_{-\infty}^{\infty} |h(t)| dt.
\]

It shows that for every function \( g^* \in L_2(G) \) the function \( \langle e^{izL} g, g^* \rangle \) is an entire function and

\[
|\langle e^{izL} g, g^* \rangle| \leq \|g^*\| \|f\| e^{\omega |z|} \int_{-\infty}^{\infty} |h(t)| dt.
\]

In other words the \( \langle e^{izL} g, g^* \rangle \) is an entire function of the exponential type \( \omega \) which is bounded on the real line and an application of the classical Bernstein theorem gives the inequality

\[
\left| \left( \frac{d}{dt} \right)^k \langle e^{itL} g, g^* \rangle \right| \leq \omega^k \sup_{t \in \mathbb{R}} |\langle e^{itL} g, g^* \rangle|.
\]

Since

\[
\left( \frac{d}{dt} \right)^k \langle e^{itL} g, g^* \rangle = \langle e^{itL} L^k g, g^* \rangle
\]

we obtain for \( t = 0 \)

\[
|\langle L^k g, g^* \rangle| \leq \omega^k \|g^*\| \|f\| \int_{-\infty}^{\infty} |h(\tau)| d\tau.
\]

Choosing \( g^* \) such that \( \|g^*\| = 1 \) and \( \langle L^k g, g^* \rangle = \|L^k g\| \) we obtain the inequality

\[
(4.3) \quad \|L^k g\| \leq \omega^k \|f\| \int_{-\infty}^{\infty} |h(\tau)| d\tau \equiv C_h \omega^k \|f\|
\]

where

\[
C_h = \int_{-\infty}^{\infty} |h(\tau)| d\tau.
\]

Now we make an important observation that regardless of the value of \( C_h \) the inequality (4.3) implies that \( g \) belongs to \( PW_\omega(L) \). Indeed, for any complex number \( z \) we have
\[ \| e^{izL}g \| = \left\| \sum_{k=0}^{\infty} (i^k z^k L^k g) / k! \right\| \leq C_h \| f \| \sum_{k=0}^{\infty} |z|^k \omega^k / k! = C_h \| f \| \| e^{iz\omega} \| . \]

It implies that for any \( g^* \in L_2(G) \) the scalar function \( \langle e^{izL}g, g^* \rangle \) is an entire function of exponential type \( \omega \) which is bounded on the real axis \( R^1 \) by the constant \( \| g^* \| \| g \| \). An application of the Bernstein inequality gives

\[ \| \langle e^{itL}L^k g, g^* \rangle \|_{C(R^1)} \leq \omega^k \| g^* \| \| g \| . \]

The last one gives for \( t = 0 \)

\[ |\langle L^k g, g^* \rangle| \leq \omega^k \| g^* \| \| g \| . \]

Choosing \( g^* \) such that \( \| g^* \| = 1 \) and \( \| L^k g, g^* \| = \| L^k g \| \) we obtain the inequality \( \| L^k g \| \leq \omega^k \| g \| , k \in N \). The Lemma is proved.

We will also need the following Lemma.

**Lemma 4.2.** The following inequalities hold for all \( f \in L_2(G) \)

\[ \Omega_m(f, s) \leq s^k \Omega_{m-k}(L^k f, s), \quad 0 \leq k \leq m, \]

and

\[ \Omega_m(f, ns) \leq n^m \Omega_m(f, s), \quad n, m \in \mathbb{N}. \]

**Proof.** The following identity holds

\[ (e^{itL} - I) f = i \int_0^t e^{i\tau L} L f d\tau, \]

where \( I \) is the identity operator. Iterations of this formula give the identity

\[ (e^{itL} - I)^k f = i^k \int_0^t \ldots \int_0^t e^{i(\tau_1 + \ldots + \tau_k) L} L^k f d\tau_1 \ldots d\tau_k, \]

which implies (4.4).

The second one follows from the property

\[ \Omega_1(f, s_1 + s_2) \leq \Omega_1(f, s_1) + \Omega_1(f, s_2), \]

which is easy to verify.

Bellow the following function will be used

\[ h(t) = a \left( \frac{\sin(t/n)}{t} \right)^n, \]

where \( n \) is a fixed even integer and

\[ a = \left( \int_{-\infty}^{\infty} \left( \frac{\sin(t/n)}{t} \right)^n dt \right)^{-1}. \]

Now we construct another filtering operator.
which is defined as
\[ P_{\omega,m}^h(f) = \int_{-\infty}^{\infty} h(t) \left\{ (-1)^{m-1} \Delta^m_{t/\omega} f + f \right\} dt, \]
where
\[ (-1)^{m+1} \Delta^m_s f = \sum_{j=0}^{m} (-1)^{j-1} C_m^j e^{js(iL)} f = \sum_{j=1}^{m} b_j e^{js(iL)} f - f, \]
and
\[ b_1 + b_2 + \ldots + b_m = 1. \]

The next Theorem is an analog of the classical Direct Approximation Theorem by entire functions of exponential type.

**Theorem 4.3.** Let \( h \) be the function defined in (4.6) and (4.7) and the operator \( P_{\omega,m}^h \) is defined in (4.8)-(4.10). We also assume that the following inequality holds
\[ n \geq m + 2. \]

For any appropriate \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \) that satisfy (4.11) and for every natural \( k \) such that
\[ 0 \leq k \leq m, k, m \in \mathbb{N}, \]
there exists a constant \( C_{h,m,k} > 0 \) such that for all \( \omega > 0 \) and all \( f \in L_2(G) \) the following inequalities holds
\[ E(f, \omega) \leq \| P_{\omega,m}^h(f) - f \| \leq \frac{C_{h,m,k}}{\omega^k} \Omega_{m-k}(L^k f, 1/\omega), \]
where
\[ C_{h,m,k} = \int_{-\infty}^{\infty} h(t)|t|^k (1 + |t|)^{m-k} dt, 0 \leq k \leq m. \]

**Proof.** With the choice of \( a \) as in (4.7) and \( n \geq m + 2 \) the function \( h \) will have the following properties: 1) \( h \) is an even nonnegative entire function of exponential type one; 2) \( h \) belongs to \( L_1(\mathbb{R}) \) and its \( L_1(\mathbb{R}) \)-norm is 1; 3) the integral
\[ \int_{-\infty}^{\infty} h(t)|t|^m dt \]
is finite. The formulas (4.8) and (4.9) imply the next formula
\[ \int_{-\infty}^{\infty} h(t) \sum_{j=1}^{m} b_j e^{j \frac{\omega}{j} (iL)} f dt = \int_{-\infty}^{\infty} \Phi(t) e^{i(tL)} f dt, \]
where
\[ \Phi(t) = \sum_{j=1}^{m} b_j \left( \frac{\omega}{j} \right) h \left( t \frac{\omega}{j} \right). \]

Since the function \( h(t) \) has exponential type one, every function \( h(t\omega/j) \) has the type \( \omega/j \) and because of this the function \( \Phi(t) \) has exponential type \( \omega \).
Now we estimate the error of approximation of $P_h^{m} f$ to $f$. Since by (4.8)

$$f - P_h^{m} f = (-1)^m \int_{-\infty}^{\infty} h(t) \Delta_{t/\omega}^{m} f \, dt$$

we obtain

$$E(f, \omega) \leq \|f - P_h^{m} f\| \leq \int_{-\infty}^{\infty} h(t) \| \Delta_{t/\omega}^{m} f \| \, dt \leq \int_{-\infty}^{\infty} h(t) \Omega_m (f, t/\omega) \, dt.$$ 

By using the Lemma 4.2 we obtain

$$E(f, \omega) \leq \int_{-\infty}^{\infty} h(t) \Omega_m (f, t/\omega) \, dt \leq \frac{\Omega_{m-k} (L^k f, 1/\omega)}{\omega^k} \int_{-\infty}^{\infty} h(t) |t|^k (1 + |t|)^{m-k} \, dt \leq C_{m,k}^h \Omega_{m-k} (L^k f, 1/\omega),$$

where the integral

$$C_{m,k}^h = \int_{-\infty}^{\infty} h(t) |t|^k (1 + |t|)^{m-k} \, dt$$

is finite by the choice of $h$. The inequality (4.12) is proved. \(\square\)

Now we can formulate one of our main results about sparse approximation of functions in $L_2(G)$. Namely, a combination of the Theorem 3.4 with the Theorem 4.3 gives the following result about approximation of an $f \in L_2(G)$ by using samples of its orthogonal projection $f_\omega$ on $PW_\omega(L)$ or samples of the projection $P_h^{m} f$.

In the following Theorem we assume that $h$ is the function defined in (4.4) and (4.7) and the operator $P_h^{m} f$ is defined in (4.8)–(4.10). Let us also recall that in the Theorem 3.4 a subset of vertices $S$ is such that every vertex $v$ in $bS$ is connected to at least $K_0(S)$ vertices in $S$, where $1 \leq K_0(S) \leq d(v)$.

The next Theorem represents our result about sparse approximation on graphs.

**Corollary 4.1.** If $S \subset V(G)$ is the same as in the Theorem 3.4 and the condition \(2.22\) along with inequality

$$\omega < \frac{K_0(S)}{4}$$

hold, then there exist functions $\Theta_u \in PW_\omega(L), u \in S$, and for any $0 \leq k \leq m, \ k, m \in \mathbb{N}$, there exists a constant $C_{k,m} > 0$ such that for every $f \in L_2(G)$

$$\left\| f - \sum_{u \in S} f_\omega(u) \Theta_u \right\| \leq \left\| f - \sum_{u \in S} P_h^{m} f(u) \Theta_u \right\| \leq C_{k,m} \omega^k \Omega_{m-k} (L^k f, 1/\omega),$$

where $f_\omega$ is the orthogonal projection of $f$ on $PW_\omega(L)$.

### 5. Lattice $\mathbb{Z}^n$

The Fourier transform $\mathcal{F}$ on $L_2(\mathbb{Z}^n)$ is a unitary operator

$$\mathcal{F} : L_2(\mathbb{Z}^n) \to L_2(\mathbb{T}^n, d\xi/(2\pi)^n),$$

where $\mathbb{T}^n$ is the $n$-dimensional torus and $d\xi/(2\pi)^n$ is the normalized measure which is defined by the formula

$$\mathcal{F}(f)(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n} f(k_1, k_2, \ldots, k_n) e^{ik_1\xi_1 + ik_2\xi_2 + \ldots + ik_n\xi_n},$$
where $f \in L^2(\mathbb{Z}^n)$, $(\xi_1, \xi_2, ..., \xi_n) \in [-\pi, \pi]^n$. One can verify the following formula

$$\mathcal{F}(L f)(\xi) = 4 \left( \sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + ... + \sin^2 \frac{\xi_n}{2} \right) \mathcal{F}(f)(\xi),$$

where $f \in L^2(\mathbb{Z}^n)$, $(\xi_1, \xi_2, ..., \xi_n) \in [-\pi, \pi]^n$.

**Theorem 5.1.** [20] The spectrum of the Laplace operator $L$ on the lattice $\mathbb{Z}^n$ is the set $[0, 4n]$. A function $f$ belongs to a space $PW_\omega(\mathbb{Z}^n)$ for some $0 < \omega < 4n$, if and only if the support of $\mathcal{F}f$ is a subset $\Omega_\omega$ of $[-\pi, \pi]^n$ on which

$$\sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} + ... + \sin^2 \frac{\xi_n}{2} \leq \frac{\omega}{4}.$$

Let’s consider for simplicity the case $\mathbb{Z}^2 = \{(k_1, k_2)\}, k_1, k_2 \in \mathbb{Z}$. We will use notation $S = \mathbb{Z}^2 \setminus \{(2k_1, 2k_2)\}_{k_1, k_2 \in \mathbb{Z}}$ for the set of vertices which is a compliment of the set of vertices $\{(2k_1, 2k_2)\}, k_1, k_2 \in \mathbb{Z}$. For this set the assumptions of the Sampling Theorem 3.4 will be satisfied with $K_0(S) = 4$ and we obtain the following fact.

**Theorem 5.2.** If $\omega < 1$, then

1) the set $S$ is a uniqueness set for functions in $PW_\omega(L)$;

2) there exist functions $\Theta_u \in PW_\omega(L), u \in S$, such that for all $f \in PW_\omega(L)$ the following reconstruction formula holds

$$f = \sum_{u \in S} f(u) \Theta_u.$$  

If $S$ is the set $\{(k_1, 2k_2)\}, k_1, k_2 \in \mathbb{Z}$, then the number $k$ in the Theorem 3.4 is $K_0(S) = 2$ and we have a similar result for all spaces $PW_\omega(L)$ with $\omega < 1/2$.

If $S$ is the set $\{(k_1, 3k_2)\}, k_1, k_2 \in \mathbb{Z}$, then the number $K_0(S)$ in the Theorem 3.4 is $K_0(S) = 1$ and we have a result similar to the last Theorem for all spaces $PW_\omega(L)$ with $\omega < 1/4$.

To construct a projector

$$\mathcal{P}_{\omega,m} : L^2(\mathbb{Z}^2) \rightarrow PW_\omega(L)$$

by the formula (4.8) one can use the following description of the Schrödinger’s group of operators

$$e^{itL} f = \mathcal{F}^{-1} \left( e^{-4it \left( \sin^2 \frac{\xi_1}{2} + \sin^2 \frac{\xi_2}{2} \right)} \right) \mathcal{F} f.$$ 

Now one can easily reformulate all statements of the previous section for the lattice $\mathbb{Z}^2$ and each of the sets $S$ above.

Our results not only allow to reduce analysis on the lattice $\mathbb{Z}^2$ to analysis on appropriate subgraph, but they also give estimates of possible losses of information which can occur after such reduction.

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