PERFECTLY CONTRACTILE GRAPHS AND QUADRATIC TORIC RINGS

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ABSTRACT. Perfect graphs form one of the distinguished classes of finite simple graphs. In 2006, Chudnovsky, Robertson, Seymour and Thomas proved that a graph is perfect if and only if it has no odd holes and no odd antiholes as induced subgraphs, which was conjectured by Berge. We consider the class $\mathcal{A}$ of graphs that have no odd holes, no antiholes and no odd stretchers as induced subgraphs. In particular, every graph belonging to $\mathcal{A}$ is perfect. Everett and Reed conjectured that a graph belongs to $\mathcal{A}$ if and only if it is perfectly contractile. In the present paper, we discuss graphs belonging to $\mathcal{A}$ from a viewpoint of commutative algebra. In fact, we conjecture that a perfect graph $G$ belongs to $\mathcal{A}$ if and only if the toric ideal of the stable set polytope of $G$ is generated by quadratic binomials. Especially, we show that this conjecture is true for Meyniel graphs, perfectly orderable graphs, and clique separable graphs, which are perfectly contractile graphs.

INTRODUCTION

A graph $G$ is perfect if every induced subgraph $H$ of $G$ satisfies $\chi(H) = \omega(H)$, where $\chi(H)$ is the chromatic number of $H$ and $\omega(H)$ is the maximum cardinality of cliques of $H$. Perfect graphs were introduced by Berge in [1]. A hole is an induced cycle of length $\geq 5$ and an antihole is the complement of a hole. In 2006, Chudnovsky, Robertson, Seymour and Thomas solved a famous conjecture in graph theory, which was conjectured by Berge and is now known as the Strong Perfect Graph Theorem:

The Strong Perfect Graph Theorem ([5]). A graph is perfect if and only if it has no odd holes and no odd antiholes as induced subgraphs.

On the other hand, Bertschi introduced a hereditary class of perfect graphs in [2]. An even pair in a graph $G$ is a pair of non-adjacent vertices of $G$ such that the length of all chordless paths between them is even. Contracting a pair of vertices $\{x, y\}$ in a graph $G$ means removing $x$ and $y$ and adding a new vertex $z$ with edges to every neighbor of $x$ or $y$. A graph $G$ is called even-contractile if there is a sequence $G_0, \ldots, G_k$ of graphs such that $G = G_0$, each $G_i$ is obtained from $G_{i-1}$ by contracting an even pair of $G_{i-1}$, and $G_k$ is a clique. A graph is called perfectly contractile if all induced subgraphs of $G$ are even-contractile. Every perfectly contractile graph is perfect. Moreover, the following graphs are perfectly contractile ([2]):

- Meyniel graphs;
- perfectly orderable graphs;
- clique separable graphs.

Key words and phrases. toric ideals, Gröbner bases, perfect graphs, stable set polytopes.
On the other hand, a necessary condition for a graph to be perfectly contractile is known. An odd stretcher (or prism) graph $G_{s,t,u}$ ($1 \leq s, t, u \in \mathbb{Z}$) is a graph on the vertex set
\[ \{i_1, i_2, \ldots, i_{2s}, j_1, j_2, \ldots, j_{2t}, k_1, k_2, \ldots, k_{2u}\} \]
with edges
\[ \{i_1, j_1\}, \{i_1, k_1\}, \{j_1, k_1\}, \{i_2, j_2\}, \{i_2, k_2\}, \{j_2, k_2\}, \{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{2s-1}, i_{2s}\}, \]
\[ \{j_1, j_2\}, \{j_1, j_3\}, \ldots, \{j_{2t-1}, j_{2t}\}, \{k_1, k_2\}, \{k_2, k_3\}, \ldots, \{k_{2u-1}, k_{2u}\}. \]
If $s = t = u = 1$, then the graph $G_{1,1,1}$ coincides with the antihole of length 6. In [22], it was formally shown that if a graph $G$ is perfectly contractile, then $G$ has no odd holes, no (odd/even) antiholes and no odd stretchers. It is conjectured [10] that this necessary condition is also a sufficient condition for perfectly contractile graphs. Namely,

**Conjecture 0.1.** A graph $G$ is perfectly contractile if and only if $G$ contains no odd holes, no (odd/even) antiholes and no odd stretchers as induced subgraphs.

By the Strong Perfect Graph Theorem, a graph $G$ contains no odd holes, no antiholes and no odd stretchers as induced subgraphs if and only if $G$ is perfect and contains no even antiholes and no odd stretchers as induced subgraphs. Conjecture 0.1 is true for
- planar graphs ([22]),
- dart-free graphs ([21]) (including claw-free graphs),
- even stretcher-free graphs ([23]) (including bull-free graphs ([8])).

However, Conjecture 0.1 is still open.

We consider this conjecture from a viewpoint of commutative algebra. In particular, we study the toric ideal of the stable set polytope of a graph. Let $G$ be a finite simple graph on the vertex set $[n] := \{1, 2, \ldots, n\}$ and the edge set $E(G)$. A subset $S \subset [n]$ is called a **stable set** (or independent set) of $G$ if $\{i, j\} \notin E(G)$ for all $i, j \in S$ with $i \neq j$. In particular, the empty set $\emptyset$ and any singleton $\{i\}$ with $i \in [n]$ are stable. Let $S(G) = \{S_1, \ldots, S_m\}$ denote the set of all stable sets of $G$. Given a subset $S \subset [n]$, we associate the $(0,1)$-vector $\rho(S) = \sum_{j \in S} e_j$. Here $e_j$ is the $j$-th unit coordinate vector in $\mathbb{R}^n$. For example, $\rho(\emptyset) = 0 \in \mathbb{R}^n$. Then the **stable set polytope** $\mathcal{P}_G \subset \mathbb{R}^n$ of a simple graph $G$ is the convex hull of $\{\rho(S_1), \ldots, \rho(S_m)\}$. Now, we define the toric ideal of the stable set polytope of a graph. Let $K[x] := K[x_1, \ldots, x_m]$ and $K[t,s] := K[t_1, \ldots, t_n, s]$ be polynomial rings over a field $K$. Then the **toric ideal** of the stable set polytope $\mathcal{P}_G$ of $G$ is the kernel of a homomorphism $\pi_G : K[x] \to K[t,s]$ defined by $\pi_G(x_i) = t^{\rho(S_i)} s$. Here, for a nonnegative integer vector $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we denote $t^a = t_1^{a_1} \cdots t_n^{a_n} \in K[t,s]$. On the other hand, the image of $\pi_G$ is denoted by $K[G]$ and called the **toric ring** of $\mathcal{P}_G$. The toric ring $K[G]$ is called **quadratic** if $I_G$ is generated by quadratic binomials.

Let $\mathcal{M}_m$ be the set of all monomials in $K[x]$. A total order $\prec$ on $\mathcal{M}_m$ is called a **monomial order** if $\prec$ satisfies (i) $1 \in \mathcal{M}_m$ is the smallest monomial in $\mathcal{M}_m$; (ii) if $u, v, w \in \mathcal{M}_m$ satisfies $u \prec v$, then we have $uv \prec vw$. The **initial monomial** $\in_{\prec}(f)$ of a nonzero polynomial $f \in K[x]$ is the largest monomial appearing in $f$. Given an ideal $I \subset K[x]$, the **initial ideal** $\in_{\prec}(I)$ of $I$ is a monomial ideal generated by $\{\in_{\prec}(f) : 0 \neq f \in I\}$. The initial ideal is called **squarefree** (resp. **quadratic**) if it is generated by squarefree (resp. quadratic) monomials. A finite subset $\{g_1, \ldots, g_t\} \subset I$ is called a **Gröbner basis** of $I$ with respect to $\prec$ if $\in_{\prec}(I)$ is generated by $\{\in_{\prec}(g_1), \ldots, \in_{\prec}(g_t)\}$. If $\{g_1, \ldots, g_t\}$ is a Gröbner basis of $I$,
then \( \{g_1, \ldots, g_t\} \) generates \( I \). See, e.g., [15] for details on toric ideals and their Gröbner bases in general.

We can characterize when a graph \( G \) is perfect in terms of \( I_G \). In fact, a graph \( G \) is perfect if and only if the initial ideal of \( I_G \) is squarefree with respect to any reverse lexicographic order ([11], [27], [30]). Similarly, we can characterize when a graph is perfect in terms of the Gorenstein property and the normality of toric rings associated to finite simple graphs (see [17], [18], [19], [28]).

In the present paper, for a perfect graph \( G \), we discuss when the toric ideal \( I_G \) is generated by quadratic binomials. First, we conjecture the following:

**Conjecture 0.2.** Let \( G \) be a perfect graph. Then \( I_G \) is generated by quadratic binomials if and only if \( G \) contains no odd holes, no (odd/even) antiholes and no odd stretchers as induced subgraphs.

The “only if” part of Conjecture 0.2 is true (Proposition 1.6 and Theorem 1.7). On the other hand, it is known that \( I_G \) has a squarefree quadratic initial ideal, and in particular, \( I_G \) is generated by quadratic binomials if \( G \) is either the comparability graph of a poset ([16]), an almost bipartite graph ([9, Theorem 8.1]), a chordal graph or a ring graph ([9, 25]), the complement of a chordal bipartite graph ([25, Corollary 1]).

We consider Conjecture 0.2 for some classes of perfectly contractile graphs. In fact, we will prove that Conjecture 0.2 is true for the following graphs:

- Meyniel graphs (Theorem 2.2);
- perfectly orderable graphs (Theorem 3.1);
- clique separable graphs (Theorem 3.4).

In particular, we will show that if \( G \) is either a perfectly orderable graph or a clique separable graph, then \( I_G \) possesses a squarefree quadratic initial ideal. We also remark that Conjecture 0.2 is true for generalized split graphs (Proposition 3.5). It is known [29] that “almost all” perfect graphs are generalized split.

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1. **A necessary condition for quadratic generation**

In the present section, we will show the “only if” part of Conjecture 0.2. First, we introduce fundamental tools for studying a set of generators and Gröbner bases of \( I_G \).

**Remark 1.1.** (a) Since the stable set polytope \( \mathcal{Q}_G \) of \( G \) is a \((0,1)\)-polytope, the initial ideal of \( I_G \) is squarefree if it is quadratic. See [15, Proposition 4.27].

(b) It is easy to see that \( I_G = \{0\} \) if and only if \( G \) is a complete graph. Even if \( I_G = \{0\} \), we say that “\( I_G \) is generated by quadratic binomials” and “\( I_G \) has a quadratic Gröbner basis”.

It is known (e.g., [25]) that quadratic generation is a hereditary property.

**Proposition 1.2.** Let \( G' \) be an induced subgraph of a graph \( G \). Then we have the following:

(a) If \( I_G \) is generated by binomials of degree \( \leq r \), then so is \( I_{G'} \);
(b) If $I_G$ has a Gröbner basis consisting of binomials of degree $\leq r$, then so does $I_{G'}$;
(c) If $I_G$ has a squarefree initial ideal, then so does $I_{G'}$.

By the following fact ([25, Example 2]), we may assume that $G$ is connected.

**Proposition 1.3.** Let $G_1, \ldots, G_s$ denote the connected components of a graph $G$. Then the toric ring $K[G]$ is the Segre product of $K[G_1], \ldots, K[G_s]$. In particular, $I_G$ is generated by quadratic binomials (resp. has a squarefree quadratic initial ideal) if and only if all of $I_{G_1}, \ldots, I_{G_s}$ are generated by quadratic binomials (resp. have a squarefree quadratic initial ideal).

A **clique** in a graph is a set of pairwise adjacent vertices in the graph. Let $G$ be a connected graph on the vertex set $[n]$. A subset $C \subset [n]$ is called a cutset of $G$ if the induced subgraph of $G$ on the vertex set $[n] \setminus C$ is not connected. We say that a graph $G$ is obtained by gluing graphs $H_1$ and $H_2$ along their common clique if $H_1 \cap H_2$ is a complete graph and $G = H_1 \cup H_2$.

**Proposition 1.4** ([24, Proposition 1.3]). Let $G$ be a connected graph with a clique cutset $C$. Suppose that $G$ is obtained by gluing two graphs $H_1$ and $H_2$ along their common clique on $C$. Then $I_G$ is generated by quadratic binomials (resp. has a squarefree quadratic initial ideal) if and only if $I_{H_1}$ and $I_{H_2}$ are generated by quadratic binomials (resp. have a squarefree quadratic initial ideal).

The following proposition says that $I_G$ rarely has a quadratic Gröbner basis if $G$ is not perfect.

**Proposition 1.5** ([24, Proposition 1.3]). Suppose that $H$ is an odd antihole with $\geq 7$ vertices. Then $I_H$ is generated by quadratic binomials and has no quadratic Gröbner basis. In particular, if a graph $G$ has an odd antihole with $\geq 7$ vertices, then $I_G$ has no quadratic Gröbner basis.

The following necessary condition is known.

**Proposition 1.6** ([25, Proposition 11]). If $I_G$ is generated by quadratic binomials, then $G$ has no even antiholes.

We now give a new necessary condition.

**Theorem 1.7.** If $I_G$ is generated by quadratic binomials, then $G$ has no odd stretchers as induced subgraphs.

**Proof.** By Proposition 1.2 it is enough to show that the toric ideal $I_{G_{s,t,u}}$ of an odd stretcher $G_{s,t,u}$ is not generated by quadratic binomials. It is easy to see that the stability number $\max\{|S| : S \in S(G_{s,t,u})\}$ of $G_{s,t,u}$ is $s + t + u - 1$. Let $S(G_{s,t,u}) = \{S_1, \ldots, S_m\}$, where

- $S_1 = \{i_1, i_3, \ldots, i_{2s-1}, j_2, j_4, \ldots, j_{2t}, k_2, k_4, \ldots, k_{2u-2}\}$
- $S_2 = \{i_2, i_4, \ldots, i_{2s-2}, j_1, j_3, \ldots, j_{2t-1}, k_2, k_4, \ldots, k_{2u-1}\}$
- $S_3 = \{i_2, i_4, \ldots, i_{2s-2}, j_2, j_4, \ldots, j_{2t-2}, k_1, k_3, \ldots, k_{2u-1}\}$
- $S_4 = \{i_1, i_3, \ldots, i_{2s-1}, j_2, j_4, \ldots, j_{2t-2}, k_2, k_4, \ldots, k_{2u}\}$
- $S_5 = \{i_2, i_4, \ldots, i_{2s}, j_1, j_3, \ldots, j_{2t-1}, k_2, k_4, \ldots, k_{2u-2}\}$
- $S_6 = \{i_2, i_4, \ldots, i_{2s-2}, j_2, j_4, \ldots, j_{2t}, k_1, k_3, \ldots, k_{2u-1}\}$. 


Note that $|S_i| = s + t + u - 1$ for $i = 1, 2, \ldots, 6$. Since $\rho(S_1) + \rho(S_2) + \rho(S_3) = \rho(S_4) + \rho(S_5) + \rho(S_6)$, the binomial $f = x_1x_2x_3 - x_4x_5x_6$ belongs to $I_{G_{t,u}}$. Suppose that $I_{G_{t,u}}$ is generated by quadratic binomials. Then there exist $1 \leq i < j \leq 3$ and stable sets $S_k, S_\ell$ of $G_{s,t,u}$ with $|S_k| = |S_\ell| = s + t + u - 1$ such that $x_ix_j - x_kx_\ell$ is a nonzero binomial in $I_{G_{t,u}}$.

By the symmetries on $S_1, S_2, S_3$, we may assume that $i = 1$ and $j = 2$. Since $x_ix_j - x_kx_\ell$ belongs to $I_{G_{t,u}}$, we have

$$S_1 \cap S_2 = S_k \cap S_\ell = \{k_2, k_4, \ldots, k_2u-2\},$$

$$(S_k \cup S_\ell) \setminus (S_k \cap S_\ell) = \{i_1, i_2, \ldots, i_{2x-1}, j_1, j_2, \ldots, j_{2r}, k_2u\}.$$  

Since there is exactly one way

$$\{i_1, i_3, \ldots, i_{2x-1}, j_2, j_4, \ldots, j_{2r}\} \cup \{i_2, i_4, \ldots, i_{2x-2}, j_1, j_3, \ldots, j_{2r-2}, k_2u\}$$

to decompose $(S_k \cup S_\ell) \setminus (S_k \cap S_\ell)$ into two stable sets, it follows that $\{S_1, S_2\} = \{S_k, S_\ell\}$, a contradiction.  

By Proposition 1.6 and Theorem 1.7 the “only if” part of Conjecture 0.2 is true.

2. Quadratic generation for Meyniel graphs

A graph is called Meyniel if any odd cycle of length $\geq 5$ has at least two chords. In the present section, we will prove that $I_G$ is generated by quadratic binomials if $G$ is Meyniel. Meyniel graphs are one of the important classes of perfect graphs. Several characterization of Meyniel graphs are known. For example, it is known [20] that a graph $G$ is Meyniel if and only if $G$ is very strongly perfect, i.e., for every induced subgraph $H$ of $G$, every vertex of $H$ belongs to a stable set of $H$ meeting all maximal cliques of $H$.

Hertz [14] introduced the following algorithm:

**COLOR with rule $\mathcal{R}$:**

**Input:** a Meyniel graph $G$

**Output:** a coloring of the vertices of $G$

1. $G_0 := G$; $k := 0$; $(vw)_0$ is any vertex of $G$;
2. While $G_k$ is not a clique do:
   2.1. Choose two non-adjacent vertices $v_k$ and $w_k$ by the following rule ("rule $\mathcal{R}$");
      - If there exists at least one vertex not adjacent to $(vw)_k$, then $v_k := (vw)_k$, else choose for $v_k$ any vertex not adjacent to every other vertex in $G_k$;
      - Choose for $w_k$ any vertex not adjacent to $v_k$ such that the number of common neighbors with $v_k$ is maximal among the vertices not adjacent to $v_k$.
   2.2. Construct $G_{k+1}$ by contracting $v_k$ and $w_k$ into a vertex $(vw)_{k+1}$;
   2.3. $k := k + 1$;
3. Color the clique $G_k$;
4. While $k \neq 0$ do:
   4.1. $k := k - 1$;
   4.2. Decontract $G_{k+1}$ by giving to $v_k$ and $w_k$ the same color as $(vw)_{k+1}$.

Then the following is a part of the results given by Hertz [14].
**Proposition 2.1.** If we input a Meyniel graph $G$ to the algorithm \textsc{COLOR} with rule $\mathcal{R}$, then we have the following:

(a) Each $\{v_r, w_r\}$ is an even pair in $G_r$;
(b) Let $s$ be the smallest index such that $v_s = (vw)_s$ and either $G_{s+1}$ is a clique or else $v_{s+1} \neq (vw)_{s+1}$. Then the set $S = \{v_0, w_0, w_1, \ldots, w_s\}$ is a stable set of $G$ meeting all maximal cliques of $G$.

Using Proposition 2.1, we have the following.

**Theorem 2.2.** Let $G$ be a Meyniel graph. Then $I_G$ is generated by quadratic binomials.

**Proof.** Let $f = x_{i_1} \ldots x_{i_r} - x_{j_1} \ldots x_{j_r} \in I_G$ be a binomial of degree $r \geq 3$ such that $f$ is not generated by binomials in $I_G$ of degree $\leq r - 1$. Note that $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ are not necessarily distinct. Since $I_G$ is prime, $f$ must be irreducible. We consider the induced subgraph $H$ of $G$ on the vertex set $\bigcup_{i=1}^r S_{i i'}$. Since $G$ is Meyniel, so is $H$. Let $S = \{v_0, w_0, w_1, \ldots, w_s\}$ be a stable set of $H$ meeting all maximal cliques of $H$ obtained by the algorithm \textsc{COLOR} with rule $\mathcal{R}$. Note that

$$S \subset \bigcup_{\ell=1}^r S_{i\ell} = \bigcup_{\ell=1}^r S_{j\ell}$$

since $H$ is the induced subgraph of $G$ on $\bigcup_{i=1}^r S_{i i'}$. We may assume that $v_0$ belongs to $S_{i_1}$. Suppose that $S \not\subset S_{i_1}$. Let $u = \min \{\mu : w_\mu \notin S_{i_1}\}$. Then we may assume that $w_u \in S_{i_2}$. Let $H'$ be the induced subgraph of $G$ on the vertex set $S_{i_1} \cup S_{i_2}$. Since both $S_{i_1}$ and $S_{i_2}$ are stable sets of $G$, it follows that $H'$ is the disjoint union of graphs $G_1$ and $G_2$, where $G_1$ is a bipartite graph on the vertex set $(S_{i_1} \setminus S_{i_2}) \cup (S_{i_2} \setminus S_{i_1})$, and $G_2$ is an empty graph on the vertex set $S_{i_1} \cap S_{i_2}$ ($G_2$ is not needed when $S_{i_1} \cap S_{i_2} = \emptyset$). Let $H_1, \ldots, H_t$ be the connected components of $G_1$. Since each $H_k$ is a connected bipartite graph, the vertex set of $H_k$ has the unique bipartition $V_1^{(k)} \cup V_2^{(k)}$ with $V_1^{(k)} \subset S_{i_1} \setminus S_{i_2}$ and $V_2^{(k)} \subset S_{i_2} \setminus S_{i_1}$. Then there exists $1 \leq \alpha \leq t$ such that $w_u$ belongs to $V_2^{(\alpha)}$. Suppose that $p \in \{v_0, w_0, \ldots, w_{u-1}\}$ \((\subset S_{i_1}) \) belongs to $H_\alpha$. Let $H''$ be the bipartite graph obtained by contracting the vertices $\{v_0, w_0, \ldots, w_{u-1}\}$ into $v_u$ in $H'$. Then $v_u$ and $w_u$ belong to the same connected component of $H''$, and hence there exists an induced odd path from $v_u$ to $w_u$ in $H''$. Since $H''$ is an induced subgraph of $G_u$, this contradicts that $\{v_u, w_u\}$ is an even pair in $G_u$. Hence $p \notin H_\alpha$. It then follows that

$$S_{i_1}' = (S_{i_1} \setminus V_1^{(\alpha)}) \cup V_2^{(\alpha)}$$
$$S_{i_2}' = (S_{i_2} \setminus V_2^{(\alpha)}) \cup V_1^{(\alpha)}$$

are stable sets of $G$ satisfying $\{v_0, w_0, \ldots, w_u\} \subset S_{i_1}'$ and $x_{i_1} x_{i_2} - x_{i_1} x_{i_2}' \in I_G$. Then

$$f = x_{i_1} \ldots x_{i_r} (x_{i_1} x_{i_2} - x_{i_1} x_{i_2}') + f'$$

where $f' = x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_r} - x_{j_1} \ldots x_{j_r} \in I_G$. We may replace $f$ with $f'$. Applying this procedure repeatedly, we may assume that $S$ is a subset of $S_{i_1}$. Since $S$ meets all maximal cliques of $H$, each vertex of $H \setminus S$ is adjacent to a vertex in $S$. Thus $S_{i_1} = S$. Applying the same procedure to $x_{j_1} \ldots x_{j_r}$, we may also assume that $S_{j_1} = S$. This contradicts that $f$ is irreducible. \hfill $\square$
3. Quadratic Gröbner bases for perfectly orderable graphs

Let $G$ be a graph on the vertex set $\{v_1, \ldots, v_n\}$. An ordering $(v_1, \ldots, v_n)$ of the vertex set of $G$ is called perfect \[4\] Definition 5.6.1 if, for any induced (ordered) subgraph $H$ of $G$, the number of colors used by Greedy Coloring \[4\] Algorithm 5.6.1 on $H$ coincides with the chromatic number of $H$. It is known \[7\] that every perfectly orderable graph is strongly perfect. A graph is called perfectly orderable if there exists a perfect ordering $(v_1, \ldots, v_n)$ of the vertex set of $G$. For example, the following graphs are perfectly orderable:

- comparability graphs of finite posets (including bipartite graphs);
- chordal graphs;
- complements of chordal graphs;
- weakly chordal graphs with no $P_3$ (\[12\]);
- bull-free graphs with no odd holes and no antiholes (\[13\]).

For a bipartite graph $G$, the complement of $G$ is perfectly orderable if and only if $G$ is chordal bipartite. It is known \[2\] that every perfectly orderable graph is perfectly contractile. A graph $G$ is called strongly perfect if every induced subgraph $H$ of $G$ has a stable set meeting all maximal cliques of $H$. It is known \[7\] that every perfectly orderable graph is strongly perfect.

Let $G$ be a perfectly orderable graph with a perfect ordering $(v_1, \ldots, v_n)$. Then we define the ordering of stable sets $(S_1, \ldots, S_m)$ by

$$\mathbf{t}^{\rho(S_1)} >_{\text{lex}} \cdots >_{\text{lex}} \mathbf{t}^{\rho(S_m)},$$

where $>_{\text{lex}}$ is a lexicographic order on $K[t]$ induced by the ordering $t_1 > \cdots > t_n$. Let $>_{\text{rev}}$ be a reverse lexicographic order on $K[x]$ induced by the ordering $x_1 < \cdots < x_m$. Then we have the following.

**Theorem 3.1.** Let $G$ be a perfectly orderable graph. Then the initial ideal of $I_G$ with respect to the reverse lexicographic order $>_{\text{rev}}$ defined above is squarefree and quadratic.

**Proof.** Given two stable sets $S_i$ and $S_j$ with $1 \leq i < j \leq m$, let $H_{ij}$ be the induced subgraph of $G$ induced by the vertex set $S_i \cup S_j$. Note that $H_{ij}$ is bipartite. Let $(v_{i_1}, \ldots, v_{i_p})$ be the induced perfect ordering of $H_{ij}$. Then we define the stable set $S (\subset S_i \cup S_j)$ as follows: scan $(v_{i_1}, \ldots, v_{i_p})$ from $v_{i_1}$ to $v_{i_p}$, and place each $v_{i_q}$ in $S$ if and only if none of its neighbors $v_{i_q'}$ ($q' < q$) has been placed in $S$. In particular, we have $v_{i_1} \in S$. In the proof of \[7\] Theorem 2], it is shown that $S$ is a stable set of $H_{ij}$ meeting all maximal cliques of $H_{ij}$ (remark that, since $H_{ij}$ is bipartite, maximal cliques of $H_{ij}$ are edges and isolated vertices). Since $S$ is a stable set of $G$, $S = S_k$ for some $1 \leq k \leq m$. We now show that $\rho(S_i) + \rho(S_j) - \rho(S_k) = \rho(S_\ell)$ for some $1 \leq \ell \leq m$. Since $H_{ij}$ is perfect, by \[6\] Theorem 3.1], the stable set polytope $\mathcal{P}_{H_{ij}}$ of $H_{ij}$ is the set of all vectors $(y_1, \ldots, y_n)$ satisfying

$$y_i \geq 0, \quad \text{for all } i \in [n],$$

$$\sum_{i \in W} y_i \leq 1, \quad \text{for any maximal clique } W \text{ of } H_{ij}. $$
Let \((y_1, \ldots, y_n) = \rho(S_i) + \rho(S_j)\). Since \((y_1, \ldots, y_n)\) belongs to \(2 \mathcal{D}_{H_{ij}}\), we have
\[
\sum_{i \in W} y_i \leq 2,
\]
for any maximal clique \(W\) of \(H_{ij}\). Moreover, since \(S_k\) meets all maximal cliques of \(H_{ij}\),
\[
(y'_1, \ldots, y'_n) = \rho(S_i) + \rho(S_j) - \rho(S_k)
\]
satisfies the inequalities
\[
\sum_{i \in W} y'_i \leq 1,
\]
for any maximal clique \(W\) of \(H_{ij}\). In addition, \(y'_i \geq 0\) follows from \(S_k \subseteq S_i \cup S_j\). Thus
\((y'_1, \ldots, y'_n)\) belongs to \(2 \mathcal{D}_{H_{ij}} \cap \mathbb{Z}^n\). Since \(2 \mathcal{D}_{H_{ij}}\) is a \((0, 1)\)-polytope, \(2 \mathcal{D}_{H_{ij}} \cap \mathbb{Z}^n\) coincides with the vertex set of \(2 \mathcal{D}_{H_{ij}}\), and hence \((y'_1, \ldots, y'_n)\) is equal to \(\rho(S_k)\) for some stable set \(S_k\) of \(H_{ij}\). Then \(f_{ij} := x_ix_j - x_kx_{\ell}\) belongs to \(I_G\). From the choice of \(S_k\), we have \(k \leq i, j, \ell\). Note that \(f_{ij} = 0\) if and only if \(i = k\). If \(i \neq k\), then the initial monomial of \(f_{ij}\) is \(x_ix_j\) with respect to \(>_{\text{rev}}\).

Let \(\mathcal{G} = \{0 \neq f_{ij} \in K[x] : 1 \leq i < j \leq m\}\). Suppose that \(\mathcal{G}\) is not a Gröbner basis of \(I_G\) with respect to \(>_{\text{rev}}\). By [15] Theorem 3.11, there exists a binomial \(f = u - v \in I_G\) such that neither \(u\) nor \(v\) belong to \(\text{in}_{>_{\text{rev}}} (\mathcal{G})\). Since \(I_G\) is prime, we may assume that \(f\) is irreducible. Let \(u = x_{i_1} \cdots x_{i_r}\) and \(v = x_{j_1} \cdots x_{j_r}\), with \(i_1 \leq \cdots \leq i_r\) and \(j_1 \leq \cdots \leq j_r\). We may assume that \(i_1 < j_1\). Then the left-most nonzero component of \(\rho(S_{i_1}) - \rho(S_{j_1})\) is positive. Suppose that the \(\alpha\)-th component of \(\rho(S_{i_1}) - \rho(S_{j_1})\) is the left-most nonzero component. Since \(f\) belongs to \(I_G\), we have
\[
\sum_{k=1}^{r} \rho(S_{i_k}) = \sum_{k=1}^{r} \rho(S_{j_k}).
\]
Then there exists \(1 < k' \leq r\) such that \(v_{\alpha} \in S_{j_{k'}}\). Suppose that \(f_{j_1j_{k'}} = 0\). Then \(S_{j_1}\) is the stable set that meets all maximal cliques of \(H_{j_1j_{k'}}\) obtained by the above procedure. The choice of \(v_{\alpha}\) guarantees that there exist no neighbors of \(v_{\alpha}\) in \(\{v_1, \ldots, v_{\alpha-1}\} \cap S_{j_1}\) since
\[
\{v_1, \ldots, v_{\alpha-1}\} \cap S_{j_1} = \{v_1, \ldots, v_{\alpha-1}\} \cap S_{j_1}.
\]
Hence \(\alpha \in S_{j_1} \cup S_{j_{k'}}\) should be included in \(S_{j_1}\) in the above procedure. This is a contradiction. Thus we have \(f_{j_1j_{k'}} \neq 0\). This contradicts that \(v\) does not belong to \(\text{in}_{>_{\text{rev}}} (\mathcal{G})\). \(\square\)

**Remark 3.2.** Theorem 3.1 is a generalization of results on several toric ideals. For example, the existence of a squarefree quadratic initial ideal for comparability graphs of posets [16], and the complement of chordal bipartite graphs [25] are guaranteed by this theorem. Moreover, if \(G\) is the complement of a bipartite graph \(H\), then the edge polytope [15] Chapter 5 of \(H\) is a face of \(\mathcal{D}_G\). Hence the existence of a squarefree quadratic initial ideal for the toric ideals of edge polytopes of chordal bipartite graphs [26] is a corollary of Theorem 3.1.

From Proposition 1.4 and Theorem 3.1, we have the following.

**Corollary 3.3.** Suppose that a graph \(G\) is obtained recursively by gluing along cliques starting from perfectly orderable graphs. Then there exists a monomial order such that the initial ideal of \(I_G\) is squarefree and quadratic.
A graph $G$ is called \textit{clique separable} if $G$ is obtained by successive gluing along cliques starting with graphs of Type 1 or 2:

Type 1: the join of a bipartite graph with more than 3 vertices with a complete graph; 
Type 2: a complete multipartite graph.

Bertschi [2] proved that every clique separable graph is perfectly contractile.

\textbf{Theorem 3.4.} Let $G$ be a clique separable graph. Then there exists a monomial order such that the initial ideal of $I_G$ is squarefree and quadratic.

\textit{Proof.} Suppose that $H$ is a graph of Type 1, that is, $H$ is the join of a bipartite graph $H_1$ with more than 3 vertices with a complete graph $H_2$. Since $H_1$ is bipartite, $H_1$ is perfectly orderable, and hence the vertex set of $H_1$ admits a perfect order. Note that any vertex of $H_2$ is adjacent to any other vertex of $H$. Thus no vertices of $H_2$ appear in an induced path $P_4$ in $H$. Hence we can extend this order to a perfect order for $H$. If a graph $H$ is of Type 2, then $H$ is the complement of a disjoint union of complete graphs. Since a disjoint union of complete graphs is a chordal graph, $H$ is perfectly orderable. Therefore any clique separable graph $G$ is obtained recursively by gluing along cliques starting from perfectly orderable graphs. Thus the desired conclusion follows from Corollary 3.3. \hfill $\square$

Let $G(n)$ denote the set of all graphs on the vertex set $[n]$. Given a property $P$, let $G_P(n)$ denote the set of all graphs on the vertex set $[n]$ with $P$. Then we say that almost all graphs have the property $P$ if $\lim_{n \to \infty} |G_P(n)|/|G(n)|$ exists and is equal to 1. A graph $G$ is called \textit{generalized split} if either $G$ or $\overline{G}$ satisfies the following:

\begin{itemize}
  \item[(*)] The vertex set of the graph is partitioned into disjoint cliques $C_0, C_1, \ldots, C_k$ such that there are no edges between $C_i$ and $C_j$ for any $1 \leq i < j \leq k$.
\end{itemize}

Pr"omel and Steger [29] proved that (i) generalized split graphs are perfect, and (ii) almost all perfect graphs are generalized split.

\textbf{Proposition 3.5.} Let $G$ be a generalized split graph with no antiholes. Then each connected component of $G$ is obtained recursively by gluing along cliques starting from perfectly orderable graphs. In particular, there exists a monomial order such that the initial ideal of $I_G$ is squarefree and quadratic.

\textit{Proof.} We may assume that $G$ is connected. If $\overline{G}$ satisfies condition (*), then it is shown in [3] Proof of Corollary 2] that $G$ is perfectly orderable. Suppose that $G$ satisfies condition (*) and has no antiholes. Then $G$ is obtained recursively by gluing along cliques starting from $G_1, \ldots, G_k$, where $G_i$ is the induced subgraph of $G$ on the vertex set $V(C_0) \cup V(C_i)$ for $1 \leq i \leq k$. Since the complement of $G_i$ is a bipartite graph and $G_i$ has no antiholes, we have that the complement of $G_i$ is chordal bipartite. Hence $G_i$ is perfectly orderable. Thus $G$ is obtained recursively by gluing along cliques starting from perfectly orderable graphs. Therefore the desired conclusion follows from Corollary 3.3. \hfill $\square$

\textbf{Remark 3.6.} From Theorem [1.7] and Proposition 3.5 it follows that any generalized split graph with no antiholes has no odd stretchers as induced subgraphs.

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