INTEGRABILITY AND BIFURCATIONS OF A THREE-DIMENSIONAL CIRCUIT DIFFERENTIAL SYSTEM

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Abstract. We study integrability and bifurcations of a three-dimensional circuit differential system. The emerging of periodic solutions under Hopf bifurcation and zero-Hopf bifurcation is investigated using the center manifolds and the averaging theory. The zero-Hopf equilibrium is non-isolated and lies on a line filled in with equilibria. A Lyapunov function is found and the global stability of the origin is proven in the case when it is a simple and locally asymptotically stable equilibrium. We also study the integrability of the model and the foliations of the phase space by invariant surfaces. It is shown that in an invariant foliation at most two limit cycles can bifurcate from a weak focus.

1. Introduction. The resistor, the capacitor and the inductor are the three passive circuit elements in classical electronics. In 1971, Leon Chua [9] introduced the fourth element, later called a memristor. In recent years, many papers and monographs have been devoted to the study of nonlinear circuit systems, which can display very rich dynamical phenomena [1, 2, 3, 5]. These phenomena are omnipresent and play an important role in diverse areas of dynamical systems and technology. The dynamics of nonlinear circuit differential systems have been investigated for existence of attractors [14, 20], hyperchaos [30, 31], integrability [10],

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Hopf bifurcations [11, 13], asymptotic behaviors [6] etc. (see also the references in the mentioned works).

A modified circuit proposed in [26] after parameterizing the coefficients can be written in the form

\[
\begin{align*}
\dot{x} &= ay := g_1(x, y, z), \\
\dot{y} &= -b_1x + (b_2 - b_3)y - b_2yz^2 := g_2(x, y, z), \\
\dot{z} &= -c_1y - c_2z + c_3yz := g_3(x, y, z),
\end{align*}
\]

(1)

where \(x(t), y(t), \) and \(z(t)\) denote the voltage across the capacitor, current through the inductor and the internal state of the memristive system, respectively. The parameters \(a, b_1, c_1, c_3 \in \mathbb{R}_+\) and \(b_2, b_3, c_2 \in \mathbb{R}_+ \cup \{0\}\) come from the practical meanings of the system. Without loss of generality, using the time rescaling \(t \rightarrow t/a\) we can set in the system \(a = 1.\)

We recall some definitions given in [12], [27] and [32]. We say that a non-constant real function \(H(x, y, z, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}\) is an invariant of system (1), if for all values of \(t\), the function \(H(x(t), y(t), z(t), t) \equiv C\) on all solution curves \((x(t), y(t), z(t))\) of the system, where \(C\) is a constant. It is obvious that \(H\) is an invariant of system (1) if and only if the following condition holds

\[
\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}g_1(x, y, z) + \frac{\partial H}{\partial y}g_2(x, y, z) + \frac{\partial H}{\partial z}g_3(x, y, z) \equiv 0,
\]

provided that \(H\) is differentiable in \(\mathbb{R}^4\). If the invariant \(H\) is independent of the time, then it is called a first integral of system (1). If the first integral \(H\) is a polynomial (resp. analytic, generalized rational, Liouvillian) function, then it is called a polynomial (resp. analytic, generalized rational, Liouvillian) first integral.

Roughly speaking, a Liouvillian function is a function that can be obtained by quadratures of elementary functions. Liouvillian functions are defined as follows [28, 29]. We denote a differential field of characteristic zero by \((k, \Delta)\) with a given set of commuting derivations \(\Delta = \{\delta_i\}\). A differential field \((K, \Delta)\) is a Liouvillian extension of \((k, \Delta)\) if there is a tower of fields \(k = K_0 \subset K_1 \subset \cdots \subset K_m = K\) where each \(K_i = K_{i-1}(t_i)\) with either \(\delta t_i \in K_{i-1}\) for all \(\delta \in \Delta\), or \(\delta t_i / t_i \in K_{i-1}\) for all \(\delta \in \Delta\), or \(t_i\) is algebraic over \(K_{i-1}\). We say that \(k = K_0 \subset \cdots \subset K_m = K\) is the defining tower of \(K\) and define the length of the defining tower to be the integer \(m\). The constants of \((k, \Delta)\), that is, all those elements annihilated by all \(\delta \in \Delta\), will be denoted by \(C(k, \Delta)\). Let \(k = \mathbb{C}(y_1, \ldots, y_n)\), where \(y_1, \ldots, y_n\) are indeterminates and let \(\Delta = \{\partial / \partial y_1, \ldots, \partial / \partial y_n\}\). Let \((K, \Delta)\) be a Liouvillian extension of \((k, \Delta)\) such that the field of constants of \((K, \Delta)\) is \(\mathbb{C}\). Then, each element of \(K\) represents a function analytic on a dense open set on \(\mathbb{C}^n\) and such a function is called a Liouvillian function of \(n\) variables. A function \(F(x)\) is a generalized rational function in \((\mathbb{C}^n, 0)\) if \(F\) has the form \(F_1(x)/F_2(x)\), where both \(F_1\) and \(F_2\) are analytic with respect to \(x\). A vector field in \(\mathbb{R}^3\) is called completely integrable if it has two independent first integrals.

Center manifolds and the center (or center-focus) problem on the center manifolds in \(\mathbb{R}^3\) have been investigated by several authors. For instance, quadratic polynomial systems and their generalizations have been studied in [17, 18, 21, 19, 25].

In this paper we investigate properties of equilibria of system (1) including the case when infinitely many equilibria coexist. The Hopf bifurcation and zero-Hopf bifurcation are studied for the appearance of periodic solutions in \(\mathbb{R}^3\) (note that a periodic solution can bifurcate from a non-isolated zero-Hopf equilibrium and an isolated Hopf equilibrium). In addition, we prove the global stability of the
origin using a suitable Lyapunov function. Finally, we investigate the existence of first integrals and foliate the phase space by invariant surfaces determined by first integrals. In the invariant foliations system (1) displays rich dynamics including the cases when the equilibrium can be a center, a focus, a node or a weak focus of order at most two; it also turns out that at most two limit cycles can arise from the weak focus.

The paper is organized as follows. In section 2 we analyze the qualitative properties of system (1) and investigate the Hopf bifurcation and the zero-Hopf bifurcation at equilibria. Section 3 is devoted to the study of globally asymptotic stability at the origin of (1). In section 4 we look for first integrals and in section 5 we study the foliations determined by them. Phase portraits of system (1) in $\mathbb{R}^3$ restricted on the foliations are presented.

2. Hopf bifurcation and Zero-Hopf bifurcation. In this section we study the local properties of equilibria and bifurcations of system (1).

**Theorem 1.** Assuming that $c_2 \neq 0$, then system (1) has a unique equilibrium $O(0, 0, 0)$. If $b_2 < b_3$ then the equilibrium $O$ is asymptotically stable. If $b_2 > b_3$ then system (1) has a stable manifold of dimension one and an unstable manifold of dimension two. In the case of $b_2 = b_3 \neq 0$ system (1) has a center manifold of dimension two and a stable manifold of dimension one. Moreover, at most one limit cycle appears from the Hopf bifurcation, which is stable.

*Proof.* It is easy to see that the origin $O(0, 0, 0)$ is the unique equilibrium of system (1) if $c_2 \neq 0$. A routine computation shows that the Jacobian matrix of system (1) at $O$ is

$$J(O) := \begin{pmatrix} 0 & 1 & 0 \\ -b_1 & b_2 - b_3 & 0 \\ 0 & -c_1 & -c_2 \end{pmatrix}.$$  

The characteristic polynomial of the matrix $J(O)$ is

$$f_1(\lambda) = \lambda^3 - (-c_2 + b_2 - b_3)\lambda^2 - (c_2b_2 - c_2b_3 - b_1)\lambda + c_2b_1,$$

so the eigenvalues are $\lambda_{\pm} = (b_2 - b_3 \pm \sqrt{(b_2 - b_3)^2 - 4b_1})/2$ and $\lambda_3 = -c_2$.

We consider first the case $b_2 \neq b_3$. Then, the origin of system (1) is asymptotically stable when $b_2 < b_3$, or at $O$ it has an one-dimensional stable manifold and a two-dimensional unstable manifold when $b_2 > b_3$. Therefore, the stable (resp. unstable) manifold of $O$ is an invariant manifold which tangents to the stable (resp. unstable) characteristic space at $O$ by invariant manifold theory in [7]. Obviously, the stable manifold is tangent to the $z$-axis.

Next, we consider the case $b_2 = b_3 \neq 0$. It is clear that the eigenvalue $\lambda_3$ of $J(O)$ is associated to eigenvector $\phi_3 := (0, 0, 1)^T$. By tedious calculation, the associated eigenvectors with respect to $\lambda_+$ and $\lambda_-$ are

$$\phi_1 := \left(\frac{-\sqrt{b_1i + c_2}}{c_1\sqrt{b_1i}}, \frac{-\sqrt{b_1i + c_2}}{c_1}, 1\right)^T, \quad \phi_2 := \left(\frac{-\sqrt{b_1i + c_2}}{c_1\sqrt{b_1i}}, \frac{-\sqrt{b_1i + c_2}}{c_1}, 1\right)^T,$$

respectively, where $i$ is the imaginary unit.
Let $X = (x, y, z)^T$, $U = (u, v, w)^T$, $\Phi_0 = (\phi_1, \phi_2, \phi_3)^T$ and

$$\Phi_1 = \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Following the normal forms method of [8] and making a change $X = \Phi_0 \Phi_1 U$, we rewrite system (1) as

$$\dot{U} = \begin{pmatrix} 0 & -\sqrt{b_1} & 0 \\ \sqrt{b_1} & 0 & 0 \\ 0 & 0 & -c_2 \end{pmatrix} U + \begin{pmatrix} F_1(u, v, w) \\ F_2(u, v, w) \\ F_3(u, v, w) \end{pmatrix},$$

where

$$
F_1(u, v, w) = -(4b_3c_2^2/(c_2^2 + b_1))u^3 + (4c_2b_3\sqrt{b_1}/(c_2^2 + b_1))u^2v - (4b_3c_2^2/(c_2^2 + b_1))u^2w,
$$

$$
F_2(u, v, w) = (4c_2b_3\sqrt{b_1}/(c_2^2 + b_1))uv - (b_3c_2^2/(c_2^2 + b_1))uw^2 + (c_2b_3\sqrt{b_1}/(c_2^2 + b_1))uw^2v,
$$

$$
F_3(u, v, w) = -(4c_2b_3\sqrt{b_1}/(c_2^2 + b_1))uv - (2c_2b_3\sqrt{b_1}/(c_2^2 + b_1))uw^2 + (2c_2b_3\sqrt{b_1}/(c_2^2 + b_1))uw^2v.
$$

The center manifold $\mathcal{M}_c$ of $O$ is a surface which tangents to the invariant eigenspace $w = 0$ at $O$. The local expression of $\mathcal{M}_c$ in a small neighborhood $U_0$ of $O$ can be presented as

$$\mathcal{M}_c = \{(u, v, w) \in U_0 \mid w = -\frac{4c_3(c_2^2 + 3b_1)}{c_1(c_2^2 + 4b_1)}u^2 - \frac{4c_2c_3\sqrt{b_1}}{c_1(c_2^2 + 4b_1)}uv - \frac{4b_1c_0}{c_1(c_2^2 + 4b_1)}u^2 + O((|u, v|)^3)\}.$$

Hence, we can reduce system (3) to the two dimensional system on the center manifold:

$$
\dot{u} = \sqrt{b_1}v - \frac{4b_3c_2^2}{c_2^2 + b_1}u^3 + \frac{4c_2b_3\sqrt{b_1}}{c_2^2 + b_1}u^2v + \frac{16b_3c_2c_3(c_2^2 + 3b_1)}{c_1(c_2^2 + 4b_1)}u^4
$$

$$- \frac{16b_3c_2c_3\sqrt{b_1}}{c_1(c_2^2 + 4b_1)}uv^3 - \frac{4b_3c_2c_3\sqrt{b_1}}{c_1(c_2^2 + 4b_1)}u^3v + O((|u, v|)^5),
$$

$$
\dot{v} = \sqrt{b_1}u + \frac{4c_2b_3\sqrt{b_1}}{c_2^2 + b_1}u^3 - \frac{4b_1b_3}{c_2^2 + b_1}u^2v - \frac{16b_3c_2c_3\sqrt{b_1}}{c_1(c_2^2 + 4b_1)}u^4
$$

$$+ \frac{16b_3c_2c_3\sqrt{b_1}}{c_1(c_2^2 + 4b_1)}v^3 + O((|u, v|)^5).
$$

By the Lyapunov theorem for the case of two pure imaginary eigenvalues [24], equilibrium $O$ of system (1) is a weak focus or a center. If the order of the first non-zero Lyapunov coefficient of $O$ is $n$, then it is a weak focus of order $n$. We compute the first-order Lyapunov coefficient and obtain

$$L_1 := -b_3(3c_2^2 + b_1)/(2(c_2^2 + b_1)).$$

Since $b_1$ and $b_3$ are positive, the sign of $L_1$ is negative. Therefore, at most one limit cycle appears from the Hopf bifurcation and it exists if $b_2 > b_3$ and $b_2 - b_3$ is small enough. Moreover, if the limit cycle exists then it is stable.

Note that if $c_2 = 0$, equilibrium $O$ is degenerate, which has a stable (resp. unstable) manifold of dimension two and a center manifold of dimension one if $b_2 < b_3$ (resp. $b_2 > b_3$). Besides, equilibrium $O$ is not isolated and each point $E_*(0, 0, z_*)$ with $z_* \neq 0$ on the $z$-axis is an equilibrium of system (1) when $c_2 = 0$. Similar to the case of $c_2 > 0$ in Theorem 1, we can compute the dimensions of stable manifolds, unstable manifolds and center manifolds by the real parts of eigenvalues at equilibrium $E_*$. Hence, we omit the calculations of expressions for invariant manifolds in this case avoiding tedious procedures. Actually, we can find first integrals when $c_2 = 0$ or $b_2 = b_3 = 0$, and then we can research dynamics of
system (1) in some invariant surfaces with respect to the first integrals in sections 4-5.

When $c_2 = 0$ and $b_2 = b_3$, then from (2) we see that the Jacobian matrix $J(O)$ of system (1) at $O$ has three eigenvalues $\lambda_\pm = \pm \sqrt{b_{10}}$, $\lambda_3 = 0$ and a zero-Hopf bifurcation may happen from a non-isolated equilibrium at the origin.

Let $\mu_1 = b_2 - b_3$, $\mu_2 = c_2$ and
\[(\mu_1, \mu_2, b_1, b_2, c_1, c_3) = (\mu_{11} + \mu_{22} e^2, \mu_{21} + \mu_{22} e^2, b_{10} + b_{11} e + b_{12} e^2, b_{20} + b_{21} e + b_{22} e^2, c_{10} + c_{11} e + c_{12} e^2, c_{20} + c_{31} e + c_{32} e^2).\]

**Theorem 2.** When $c_2 = 0$ and $b_2 = b_3$, system (1) has a continuum of equilibria which fill the z-axis and the origin $O$ is the unique non-isolated zero-Hopf equilibrium. Moreover, if $c_{30} = 0$, $b_{10} > 0$, $a_{11} > 0$, and $b_{20} > 0$, then for sufficiently small $\epsilon > 0$ system (1) has a zero-Hopf bifurcation at the equilibrium at the origin, and a periodic solution appears at this equilibrium when $\epsilon = 0$. In addition, this periodic solution has the same stability as an equilibrium of a planar differential system with eigenvalues $-\mu_{11}/\sqrt{b_{10}}$, $-\mu_{21}/\sqrt{b_{10}}$ of its linear part.

**Proof.** When $c_2 = 0$ and $b_2 = b_3$, we can easily see that the z-axis is full of equilibria of system (1) and the origin $O$ is the unique non-isolated zero-Hopf equilibrium. In order to change the linear part at the origin of (1) into its real Jordan normal form for sufficiently small $|\mu_1|$ and $|\mu_2|$, we do the linear transformation of variables $(x, y, z) \to (u, v, w)$ given by
\[(x, y, z) = \left(\frac{(\mu_1 + \mu_2 + 2b_1)}{c_1 b_1} u - \sqrt{\frac{-\mu_1^2 + 4b_1 \mu_2 v}{c_1 b_1}}, \frac{-(\mu_1 + 2\mu_2)}{c_1} + \sqrt{\frac{-\mu_1^2 + 4b_1 v}{c_1}} w + 2u + w\right).

Then, system (1) becomes
\[\begin{align*}
\dot{u} &= \frac{4}{c_1 b_1} u - \sqrt{\frac{-\mu_1^2 + 4b_1 \mu_2 v}{c_1 b_1}} u^3 + 2\sqrt{\frac{-\mu_1^2 + 4b_1 \mu_2 v}{c_1 b_1}} u^2 v - \frac{2b_{10} u (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} u^2 w + \frac{2b_{20} (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} u^2 w,
\dot{v} &= \frac{4}{c_1 b_1} v - \sqrt{\frac{-\mu_1^2 + 4b_1 \mu_2 v}{c_1 b_1}} v^3 + 2\sqrt{\frac{-\mu_1^2 + 4b_1 \mu_2 v}{c_1 b_1}} v^2 u - \frac{2b_{10} v (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} v^2 w + \frac{2b_{20} (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} v^2 w,
\dot{w} &= -\mu_2 u - \frac{2b_{10} u (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} u^2 w + 2\sqrt{\frac{-\mu_1^2 + 4b_1 v}{c_1}} u v - \frac{2b_{10} (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} u v + \frac{2b_{20} (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} u v + \frac{2b_{20} (\mu_1 + 2\mu_2)}{c_1 (\mu_1 + 2b_1)} u^2 w + 2\sqrt{\frac{-\mu_1^2 + 4b_1 v}{c_1}} u^2 w.
\end{align*}\]

Writing system (5) in cylindrical coordinates $(r, \theta, w)$ by $(u, v, w) \to (r \cos \theta, r \sin \theta, rw)$ we have
\[\frac{dr}{d\theta} = R_1(r, \theta, w), \quad \frac{dw}{d\theta} = W_1(r, \theta, w),\]
where $R_1(r, \theta, w)$ and $W_1(r, \theta, w)$ are $2\pi$-periodic smooth functions with respect to the variable $\theta$ in a small neighborhood of $(r, w) = (0, 0)$.

In order to use the averaging theory (see the Appendix A), we need to apply the change of variables $r = \sqrt{\epsilon}$, $R = \sqrt{\epsilon}$, which after a time rescaling $d\theta \to d\theta/\epsilon$ changes system (6) into the system
\[\frac{dR}{d\theta} = \epsilon F_{11}(R, \theta, W) + \epsilon^2 F_{21}(R, \theta, W) + o(\epsilon^2),\]
\[\frac{dW}{d\theta} = \epsilon F_{12}(r, \theta, w) + \epsilon^2 F_{22}(r, \theta, W) + o(\epsilon^2),\]
where \( \epsilon = \varepsilon^2 \), \( c_{30} = 0 \) and

\[
\begin{align*}
F_{11}(R, \theta, W) &= \frac{R(-165 \frac{3}{10} b_{20} \cos^2 \theta R^2 + 165 \frac{1}{2} b_{20} \cos^2 \theta R^2 + 23 \frac{5}{4} f_{011})}{4 b_{10}^2}, \\
F_{21}(R, \theta, W) &= \frac{R(165 \frac{3}{10} b_{20} \cos^3 \theta R^2 W - 165 \frac{1}{2} b_{20} \cos \theta R^2 W)}{4 b_{10}^2}, \\
F_{12}(r, \theta, w) &= \frac{W(8b_{20}c_{10} \cos^4 \theta R^2 W + 4b_{410}c_{21} - 8 \sqrt{b_{10}c_{11}} \sin \theta \cos \theta - 8b_{20}c_{10} \cos \theta R^2 + c_{10}W)}{2 \sqrt{b_{10}c_{10}}}, \\
F_{22}(R, \theta, W) &= 4b_{20}c_{10} \cos \theta R^2 W^2 - 4b_{20}c_{10} \cos^3 \theta R^2 W^2 + 2 \theta b_{20}c_{10} \cos \theta \sin \theta R^2 / \sqrt{b_{10}c_{10}} + 2 \sqrt{b_{10}c_{10}} \sin \theta RW / \sqrt{b_{10}c_{10}}.
\end{align*}
\]

We apply the first order averaging theory as described in Appendix A. To do this, we note that (7) satisfies all the assumptions of Appendix A, where we identify the system (7) is of the form \((R, \theta, W, \theta, \epsilon)\) and bifurcation from the origin of the differential equation (10) has a periodic solution \((R, \theta, W, \theta, \epsilon)\) such that \((R(0, \epsilon)), W(0, \epsilon)\) \(\to (R_0, W_0)\) when \(\epsilon \to 0\). Thus system (5) has a periodic solution for \(\epsilon > 0\). Therefore, system (1) has a periodic solution tending to the origin when \(\epsilon \to 0\). \(\square\)

3. Global stability of equilibrium. In this section we analyze the globally asymptotic behavior of solutions of system (1).

**Theorem 3.** When \(b_2 < b_3\) and \(c_2 > 0\), equilibrium \(O(0,0,0)\) of system (1) is globally asymptotically stable.

**Proof.** We obtain from Theorem 1 that equilibrium \(O\) of system (1) is asymptotically stable if \(b_2 < b_3\). Construct a Lyapunov function

\[
L(x, y, z) = \frac{d_1 x^2 + d_2 y^2 + d_3 z^2}{2},
\]

where \(d_1 = b_1 b_2\) and the positive constants \(d_2, d_3\) will be determined later. Note that \(L(x, y, z) \geq 0\) in \(\mathbb{R}^3\) and \(L(x, y, z) = 0\) only if \(x = y = z = 0\).
We calculate the derivative of $L(x(t), y(t), z(t))$ with respect to the vector field of system (1) and obtain

$$
\frac{dL}{dt} = d_1 \dot{x} + d_2 \dot{y} + d_3 \dot{z}
$$

$$
= d_2 (b_2 - b_3) y^2 - d_3 c_1 y z - d_3 c_2 z^2 + d_3 c_3 y z^2 - b_2 d_2 y^2 z^2
$$

$$
= - (\sqrt{d_2 (b_3 - b_2)} y + \frac{d_3 c_1}{2 \sqrt{d_2 (b_3 - b_2)}} z)^2 + \frac{d_3 c_1^2}{4d_2 (b_3 - b_2)} z^2
$$

$$
- d_3 c_2 z^2 + d_3 c_3 y z^2 - b_2 d_2 y^2 z^2
$$

$$
\leq z^2 \left( \frac{d_3 c_1^2}{4d_2 (b_3 - b_2)} - d_3 c_2 + \frac{d_3 c_3^2}{4b_2 d_2} \right)
$$

$$
\leq z^2 d_3 \left( \frac{d_3 c_1^2}{4d_2 (b_3 - b_2)} - c_2 + \frac{d_3 c_3^2}{4b_2 d_2} \right).
$$

Assuming that $d_2$ and $d_3$ are arbitrary real positive constants such that

$$
0 < d_3 < \frac{c_2}{\frac{c_1^2}{4d_2 (b_3 - b_2)} + \frac{c_3^2}{4b_2 d_2}},
$$

we obtain $dL(t)/dt \leq 0$. Moreover, $dL(t)/dt \equiv 0$ only if

$$
y = -\frac{d_3 c_1}{2d_2 (b_3 - b_2)} z, \quad y = \frac{d_3 c_3}{2b_2 d_2} z = 0.
$$

Consequently, by Lyapunov Stability Theorem we obtain that equilibrium $O$ of system (1) is globally asymptotically stable in $\mathbb{R}^3$ if $b_3 - b_2 > 0$ and $c_2 > 0$.  

Figure 1 shows phase portrait of system (1) with parameters $b_1 = 1/3, b_2 = 1/2, b_3 = 1, c_1 = 1, c_2 = 0.6$ and $c_3 = 1$. We see that equilibrium $O(0,0,0)$ is globally asymptotically stable.
4. Integrability. In this section, we investigate first integrals of system (1). The invariant surfaces determined by the first integrals of system (1) can foliate the phase portrait in $\mathbb{R}^3$. Then, we can present more dynamical properties and comprehend more deeply about the global structures of the system.

There exists a unique equilibrium $O(0,0,0)$ of system (1) if $c_2 \neq 0$, at which the linear part of the system has three eigenvalues $\lambda_{\pm}$ and $\lambda_3$ from (2). When $c_2 = 0$, system (1) has infinitely many equilibria $E_0(z_*)$ for $\forall z_* \in \mathbb{R}$ and $z_* \neq 0$, which has three eigenvalues $\hat{\lambda}_\pm$ and $\hat{\lambda}_3$. The eigenvalues of the Jacobian matrix of system (1) at $O$ and $E_*$ are presented in Table 1, where

$$\Delta = \begin{cases} 
\Delta_0 := (b_2 - b_3)^2 - 4b_1 & \text{for } O, \\
\Delta_* := (b_2 - b_3 - b_2z_*^2)^2 - 4b_1 & \text{for } E_*,
\end{cases}$$

$\lambda_3 = -c_2$ and $\hat{\lambda}_3 = 0$.

| Conditions | eigenvalues at $O$ | eigenvalues at $E_*$ |
|------------|-------------------|---------------------|
| $\Delta < 0$ | $(b_2 - b_3 \pm \sqrt{-\Delta_0})/2, \lambda_3$ | $(b_2 - b_3 - b_2z_*^2 \pm \sqrt{-\Delta_*})/2, \lambda_3$ |
| $\Delta = 0$ | $(b_2 - b_3)/2, \lambda_3$ | $(b_2 - b_3 - b_2z_*^2)/2, \lambda_3$ |
| $\Delta > 0$ | $(b_2 - b_3 \pm \sqrt{\Delta_0})/2, \lambda_3$ | $(b_2 - b_3 - b_2z_*^2 \pm \sqrt{\Delta_*})/2, \lambda_3$ |

Table 1. Eigenvalues of Jacobian Matrix at $O$ and $E_*$.

Theorem 4. (i) System (1) has no analytic first integral if $c_2 \neq 0$, $\Delta \leq 0$ and $(b_2 - b_3)/c_2$ is irrational in a neighborhood of the origin $O$.

(ii) System (1) has at most one generalized rational first integral near $O$ if $c_2 \neq 0$, $\Delta < 0$ and $(b_2 - b_3)/c_2$ is rational, or $\Delta > 0$ and at most one of $(b_2 - b_3)/c_2$ and $\sqrt{\Delta}/c_2$ is rational.

(iii) System (1) has polynomial first integrals only if $b_2c_2 = 0$.

(iv) When $b_2 = 0$, system (1) has a Liouvillian first integral.

(v) When $b_2 = b_3 = 0$, system (1) has a polynomial first integral.

(vi) When $c_2 = 0$, system (1) has a generalized rational first integral, which is the unique generalized rational first integral near $O$ if $b_2 \neq 0$.

(vii) When $c_2 = 0$ and $b_2 = 0$, system (1) is completely integrable.

Proof. First, we consider the existence of polynomial first integrals. We claim that system (1) has polynomial first integrals only if $b_2c_2 = 0$. Actually, if system (1) has a polynomial first integral $H = H(x,y,z)$ of degree $n$, then we can write it as

$$H = \sum_{k=0}^{n} h_k(x,y)z^k = \sum_{i,j,k \geq 0} h_{i,j,k} x^i y^j z^k,$$

(10)

where the integer $n \geq 1$ and at least a $h_{i,j,n} \neq 0$ for some $\tilde{i}, \tilde{j} \in \mathbb{N}$. Obviously, $H(x,y,z)$ satisfies

$$\mathcal{X}H = ay \frac{\partial}{\partial x} + (-b_1x + (b_2 - b_3)y - b_2yz^2) \frac{\partial}{\partial y} + (-c_1y - c_2z + c_3yz) \frac{\partial}{\partial z} = 0.$$ 

(11)

Substituting (10) into (11), we obtain

$$ay \sum_{i,j,k \geq 0} h_{i,j,k} \tilde{l}x^{i-1}y^jz^k + (-b_1x + (b_2 - b_3)y - b_2yz^2) \sum_{i,j,k \geq 0} h_{i,j,k} jx^i y^{j-1} z^k$$
Comparing in (12) the coefficients of powers $z^{n+2}$ and $z^{n+1}$, respectively, we obtain

$$-b_2 \sum_{l,j \geq 0} h_{l,j,m} j x^l y^j = 0, \quad m = n, \quad n - 1.$$  \hspace{1cm} (13)

Thus, we obtain that $h_{l,j,m} = 0$ as $j \neq 0$ in (13) if $b_2 \neq 0$, implying that the functions $h_n(x, y)$ and $h_{n-1}(x, y)$ only depend on $x$. Thus,

$$h_m(x, y) = h_m(x) = \sum_{l \geq 0} h_{l,0,m} x^l, \quad m = n, \quad n - 1.$$

Observing the coefficient of $z^n$ in (12), we get

$$a y \sum_{l \geq 0} h_{l,0,n} l x^{l-1} - b_2 y \sum_{l,j \geq 0} h_{l,j,n-2} j x^l y^{j-1} + (-c_2 + c_3 y) \sum_{l,j \geq 0} h_{l,0,n} nx^l = 0,$$  \hspace{1cm} (14)

which induce that $c_2 \sum_{l,j \geq 0} h_{l,0,n} nx^l = 0$ if $c_2 \neq 0$. Hence, $h_n(x, y) = h_n(x) \equiv 0$, which leads to a contradiction that the degree of $H(x, y, z)$ is $n$ with respect to $z$. Therefore, system (1) has no polynomial first integral if $b_2c_2 \neq 0$. So, the statement (iii) is proven.

When $b_2 = 0$, system (1) has a first integral

$$H_1(x, y, z) = -\ln(\sqrt{b_1 x^2 + b_3 y^2 + ay^2}) + b_3 \arctan \left( \frac{b_3 x + 2ay}{x\sqrt{4b_1 a - b_3^2}} \right)/\sqrt{4b_1 a - b_3^2}$$

obtained by equation

$$\frac{dx}{ay} = -\frac{dy}{b_1 x + (b_2 - b_3)y - b_2 y z^2}.$$  \hspace{1cm} (15)

Notice that $H_1(x, y, z)$ is a Liouvillian first integral of system (1). Actually, $H_1$ can become a polynomial first integral only if $b_3 = 0$. Simple calculations induce that the function $H_1 = b_1 x^2 + ay^2$ is a polynomial first integral of system (1) if $b_2 = b_3 = 0$, which completes proofs of statements (iv) and (v).

Suppose that $c_2 \neq 0$ and there exist three integers $k_1, k_2$ and $k_3$ such that

$$k_1 \lambda_+ + k_2 \lambda_- + k_3 \lambda_3 = 0.$$  \hspace{1cm} (16)

For the case $\Delta < 0$, we have $k_2 = k_1$ and then $(b_2 - b_3)k_1 - c_2k_3 = 0$. Besides, if $(b_2 - b_3)/c_2$ is a rational number, we have $k_3 = (b_2 - b_3)k_1/c_2$. As such, the dimension of the minimal vector subspace of $\mathbb{R}^3$ containing the set $\langle \lambda, k \rangle = 0$ is one. From Poincaré Theorem in [27] and Theorems in [12] (see the Appendix B), there is at most one generalized rational first integral for system (1) when $\Delta < 0$ and $(b_2 - b_3)/c_2$ is rational. At the same time we get that no analytical first integrals of system (1) exist when $c_2 \neq 0, \Delta < 0$, and $(b_2 - b_3)/c_2$ is irrational in a neighborhood of the origin $O$.

For the case $c_2 \neq 0$ and $\Delta = 0$, from (16) we have

$$k_1 \lambda_+ + k_2 \lambda_- + k_3 \lambda_3 = (b_2 - b_3)(k_1 + k_2)/2 - c_2k_3 = 0.$$

Similar as in the case of $\Delta < 0$, if $(b_2 - b_3)/c_2$ is irrational, system (1) has no analytical first integrals in a neighborhood of the origin $O$.

For the case $c_2 \neq 0$ and $\Delta > 0$, the resonance relation is given as follows

$$\langle \lambda, k \rangle = \frac{b_2 - b_3}{2}(k_1 + k_2) + \frac{\sqrt{\Delta}}{2}(k_1 - k_2) - c_2k_3 = 0,$$
which implies that
\[ k_3 = \frac{b_2 - b_3}{2c_2}(k_1 + k_2) + \frac{\sqrt{\Delta}}{2c_2}(k_1 - k_2). \] (17)

If only one of \((b_2 - b_3)/c_2\) and \(\sqrt{\Delta}/c_2\) is a rational number, we can choose \(k_2 = k_1\) or \(k_2 = -k_1\) to guarantee that \(k_3\) is an integer. It implies that the dimension of the minimal vector subspace of \(\mathbb{R}^3\) containing the set \(\lambda, k\) is one. Hence, system (1) has at most one generalized rational first integral for such case. If both \((b_2 - b_3)/c_2\) and \(\sqrt{\Delta}/c_2\) are irrational, there exist two rational numbers \(r_1 \neq 0\) and \(r_2\) such that \((b_2 - b_3)/c_2 = (\sqrt{\Delta}/c_2)r_1 + r_2\) by (17). Thus, (17) can be rewritten as
\[ k_3 = \frac{\sqrt{\Delta}}{2c_2}((k_1 - k_2) + (k_2 + k_1)r_1) + \frac{r_2(k_1 + k_2)}{2}, \]
which implies that \((k_1 - k_2) + (k_2 + k_1)r_1 = 0\). As a result, system (1) has at most one generalized rational first integral in a neighborhood of the origin \(O\) for this case.

This completes the proofs of statements (i) and (ii).

When \(c_2 = 0\), system (1) has the Liouvillian first integral
\[ H_2(x, y, z) = e^{c_3x/a}(-c_1 + c_3z)^{-1} \] (18)
obtained by the characteristic equation
\[ \frac{dx}{dy} = \frac{dy}{dz} = \frac{dz}{-c_1y - c_2z + c_3yz} \]
Actually, \(H_2(x, y, z)\) is a generalized rational first integral of system (1).

When \(c_2 = 0\) and \(b_2 \neq 0\), the discussion is similar to the case \(c_2 \neq 0\). First, we assume that there exist three integers \(k_1, k_2\) and \(k_3\) such that \(k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0\), implying
\[ k_1\frac{b_2 - b_3 - b_2z^2 + \sqrt{\Delta}}{2} + k_2\frac{b_2 - b_3 - b_2z^2 - \sqrt{\Delta}}{2} + k_3 \cdot 0 = 0. \]
Then, we can choose \(z_*\) in a complete set of \(\mathbb{R}\) near 0 such that \((b_2 - b_3 - b_2z_*^2 + \sqrt{\Delta})/(b_2 - b_3 - b_2z_*^2 - \sqrt{\Delta})\) is irrational. Applying Theorem 6 from the Appendix B, system (1) has at most one generalized rational first integral \(H_2\) in this case. Statement (vi) of the theorem is proven.

When \(c_2 = 0\) and \(b_2 = 0\), system (1) has two first integrals \(H_1\) and \(H_2\), which are independent by simple calculations. Therefore, system (1) is completely integrable in this case and the statement (vii) is proven.

5. Dynamics in invariant foliations. In conclusion, the unique equilibrium \(O(0, 0, 0)\) of system (1) is globally asymptotically stable from Theorem 3 when \(b_2 < b_3\) and \(c_2 > 0\). Besides, as parameter \((b_2, b_3, c_2)\) varies apart from the surface
\[ C_1 := \{(b_2, b_3, c_2)|b_2 = b_3, \ c_2 > 0\}, \]
equilibrium \(O\) becomes an unstable focus from a stable weak focus of order 1 and a stable limit cycle arises from the Hopf bifurcation of \(O\) in the center manifold by Theorem 1, as shown in Figure 2. Note that the red “*” represents the initial value of an orbit in the following figures.

In the following result, we study the dynamics of system (1) in invariant foliations for the case \(c_2 = 0\) or \(b_2 = b_3 = 0\), that are not investigated completely in Section 2.
Clearly, together with a time scaling \( dt = d\tau/\sqrt{b_1} \) change system (1) into the following equivalent system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x + (B_2 - B_3) y - B_2 yz^2, \\
\dot{z} &= -C_1 y - C_2 z + yz,
\end{align*}
\]  
\( \tag{20} \)

where

\[
B_2 = b_2/\sqrt{b_1}, \quad B_3 = b_3/\sqrt{b_1}, \quad C_1 = c_1/c_3, \quad C_2 = c_2/\sqrt{b_1}.
\]  
\( \tag{21} \)

Clearly, \( C_2 = 0 \) if and only if \( c_2 = 0 \). Then system (20) has a first integral

\[ \hat{H}_2(x, y, z) = e^x (-C_1 + z)^{-1}. \]

The surface \( \hat{H}_2 = h_2 \) must cross the \( z \)-axis at an equilibrium \((0, 0, C_1 + h_2^{-1})\) since \( z \)-axis is full of equilibria of system (20), where \( 0 \neq h_2 \in \mathbb{R} \). Notice that the invariant surface \( \hat{H}_2 = h_2 \) of system (20) foliates the phase portrait \( \mathbb{R}^3 \) and the flow of system (20) is invariant in each foliation. Then, the dynamics of system (20) is equivalent to the dynamics of the restricted system in each foliation \( \hat{H}_2 = h_2 \). On the invariant
surface $\tilde{H}_2 = h_2$, we consider the restricted system of (20)

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x + (B_2 - B_3)y - B_2y(C_1h_2 + e^x/h_2)^2.
\end{align*} \tag{22}$$

Clearly, system (22) has only one equilibrium $(0, 0)$. The trace and determinant of the Jacobian matrix at $(0, 0)$ are $B_2 - B_3 - B_2(C_1h_2 + 1)^2/h_2^2$ and $1$ respectively. Then $(0, 0)$ of (22) is asymptotically stable (or unstable) if $B_2 - B_3 - B_2(C_1h_2 + 1)^2/h_2^2 < 0$ (or $> 0$). Note that

$$B_2 - B_3 - B_2(C_1h_2 + 1)^2/h_2^2 = \frac{b_2 - b_3 - b_2(c_1h_2/c_3 + 1)^2/h_2^2}{\sqrt{b_1}}$$

by the denotations in (21).

When $B_2 - B_3 - B_2(C_1h_2 + 1)^2/h_2^2 = 0$, equilibrium $(0, 0)$ of (22) is a stable (resp. an unstable) weak focus of order $1$ when the first Lyapunov coefficient

$$\tilde{L}_1 := -2B_2(C_1h_2 + 2)/h_2^2 < 0 \text{ (resp. } > 0)$$

by calculation. Then, at most one limit cycle appears from Hopf bifurcation. When $\tilde{L}_1 = 0$, we have $B_2 = 0$ or $h_2 = -2/C_1$. If $B_2 = 0$, we have $B_3 = 0$ and then system (22) is a linear system, implying that $(0, 0)$ is a global center. If $h_2 = -2/C_1$, we can compute the second Lyapunov coefficient

$$\tilde{L}_2 := -B_2C_1^2/4 < 0.$$ 

Hence, at most two limit cycles can appear from Hopf bifurcation [33]. Specially, the outer limit cycle is stable and the inner limit cycle is unstable if they exist at the same time, as shown in Figure 3. This proves the statement (i).

When $b_2 = b_3 = 0$ and $c_2 \neq 0$, by the same linear change (19) and time scaling, we can consider system (22) instead of system (1), where $B_2 = B_3 = 0$. Obviously, the origin of system (22) is the unique equilibrium. In this case, system (22) has a family of cylindrical invariant surfaces

$$\tilde{H}_1 := x^2 + y^2 = h_1$$

as $h_1 \geq 0$, which surround the origin and foliate the phase space $\mathbb{R}^3$. In each invariant foliation $\tilde{H}_1 = h_1$ as $h_1 > 0$, no equilibria exist. The orbits in all invariant foliations projecting in the plane $z = \alpha$ form a periodic region, where $\alpha$ is an arbitrary real number. Statement (ii) of the theorem is proven. 

![Figure 3](image-url)

**Figure 3.** Two limit cycles of system (22) can be bifurcated from stable weak focus $(0, 0)$. Here, the outer limit cycle is stable and the inner limit cycle is unstable when $B_2 = 1, B_3 = -0.189, C_1 = 2$ and $h_2 = -1.1$. 

We illustrate the orbits and dynamics of system (22) in the family of invariant foliations \( \tilde{H}_2 = h \) when \( C_2 = 0 \) and \( B_2 \neq 0 \), as shown in Figure 4 (a). Note that each intersection of the invariant foliation and the z-axis is a stable equilibrium. When \( B_2 = B_3 = 0 \) and \( C_2 \neq 0 \), the orbits of system (22) in the cylinder-shaped foliation \( H_1 = h \) of phase space are shown in Figure 4 (b). When \( c_2 = 0 \) and \( b_2 = 0 \), by the statement (vii) of Theorem 4 system (1) is completely integrable. That is, all intersecting curves of both invariant surfaces \( H_1 = h_1 \) and \( H_2 = h_2 \) are orbits of system (1), which are displayed in Figure 5.

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Figure 5. Intersections of invariant surfaces when $(a, b_1, b_2, b_3, c_1, c_2, c_3) = (1, 1/3, 0, 0, 1, 0, 1)$.

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Appendix A: Averaging theory of first order. The averaging theory of first order can be found in [4] for Lipschitz differential systems, in [15] for computing $T$-periodic solutions of $\lambda$-families of analytic $T$-periodic differential systems and in [23] for the existence of periodic solutions of three-dimensional differential systems;
see also Chapter 12 of [7]. Consider the differential system
\[ \dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{23} \]
where \( F_1 : \mathbb{R} \times D \to \mathbb{R} \), \( R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \) are continuous functions, \( T \)-periodic in the first variable, and \( D \) is an open subset of \( \mathbb{R}^n \). Assume:

(i) \( F_1(t, \cdot) \in C^1(D) \) for all \( t \in \mathbb{R} \), \( F_1, R, D_x F_1 \) are locally Lipschitz with respect to \( x \), and \( R \) is twice differentiable with respect to \( \varepsilon \).

(ii) For \( V \subset D \) an open and bounded set and for each \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\} \), suppose that \( f_1(x) := \frac{1}{T} \int_0^T F_1(s, x) ds \neq 0 \), there exists \( a \in V \) such that \( f_1(a) = 0 \) and the Jacobian \( \det D_x(f_1)(a) \neq 0 \).

Then for sufficiently small \( |\varepsilon| > 0 \) there exists a \( T \)-periodic solution \( x(t, \varepsilon) \) of system (23) such that \( x(0, \varepsilon) \to a \) when \( \varepsilon \to 0 \). If for \( f_1(a) = 0 \) the real part of all the eigenvalues of the Jacobian matrix \( D_x(f_1)(a) \) are negative, then the periodic solution \( x(t, \varepsilon) \) is asymptotically stable; if some eigenvalue has a positive real part then it is unstable.

Note that the averaging theory can be applied at higher orders, for example, see references [16, 22] and references therein.

Appendix B: Poincaré Theorem in [27] and Theorems in [12]. We conclude Poincaré Theorem of [27] and Theorems of generalized rational first integrals of [12] in the following theorem.

**Theorem 6.** Assume that system (1) has an equilibrium \( E_0 \) and let \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) be the eigenvalues of the Jacobian matrix at \( E_0 \). If the eigenvalues \( \lambda \) do not satisfy any resonant conditions of the form
\[ < \lambda, k > = 0, \text{ for some } k \in \mathbb{Z}_+^3, \text{ } k \neq 0, \]
where \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), then system (1) has no analytic first integrals in a neighborhood of \( E_0 \). Besides, the number of functionally independent generalized rational first integrals of system (1) near \( E_0 \) is at most equals to the dimension of the minimal vector subspace of \( \mathbb{R}^3 \) containing the set \( \{ k \in \mathbb{Z}^3 : < \lambda, k > = 0, \text{ } k \neq 0 \} \).

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