Prize-Collecting Steiner Tree and Forest in Planar Graphs

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Abstract

We obtain polynomial-time approximation-preserving reductions (up to a factor of $1 + \varepsilon$) from the prize-collecting Steiner tree and prize-collecting Steiner forest problems in planar graphs to the corresponding problems in graphs of bounded treewidth. We also give an exact algorithm for the prize-collecting Steiner tree problem that runs in polynomial time for graphs of bounded treewidth. This, combined with our reductions, yields a PTAS for the prize-collecting Steiner tree problem in planar graphs and generalizes the PTAS of Borradaile, Klein and Mathieu [11] for the Steiner tree problem in planar graphs. Our results build upon the ideas in [11] and the work of Bateni, Hajiaghayi and Marx [8] on a PTAS for the Steiner forest problem in planar graphs. Our main technical result is on the properties of primal-dual algorithms for Steiner tree and forest problems in general graphs when they are run with scaled up penalties.

1 Introduction

The Steiner tree and Steiner forest problems are fundamental and well-studied problems in network design. In the Steiner tree problem we have an undirected graph $G = (V, E)$ with costs on the edges given by $c : E \rightarrow \mathbb{R}^+$. and a set of terminals $S \subseteq V$. The goal is to find a minimum-cost tree in $G$ that connects/contains the terminals. In the more general Steiner forest problem we are given pairs of vertices $s_1, t_1, \ldots, s_k, t_k$ and the goal is to find a minimum-cost forest which connects $s_i$ and $t_i$ for $1 \leq i \leq k$. Both problems are NP-Hard and APX-hard to approximate. These two problems have received considerable attention in the approximation algorithms literature. For the Steiner tree problem the very recent algorithm of Byrka et al. [12] gives a 1.388 approximation and this is the best known. For Steiner forest problem the best known approximation is $2 - 1/k$ [11]: this is obtained via a natural cut-based LP relaxation. When $G$ is a planar graph, Borradaile, Klein and Matheiu [11] obtained a polynomial time approximation scheme (PTAS) for the Steiner tree problem. More recently, Bateni, Hajiaghayi and Marx [8] obtained a PTAS for the Steiner forest problem.

In this paper we consider the prize-collecting versions of the above problems, which have also received considerable attention. In the prize-collecting version of Steiner tree, we are given a root vertex $r$ and non-negative penalties $\pi : V \rightarrow \mathbb{R}^+$ on the vertices of $G$. The goal is to find a tree $T$ to minimize the sum of...
the edge-cost of $T$ and the penalties of the vertices not included in $T$; formally, the objective is to minimize
\[ \sum_{e \in E(T)} c(e) + \sum_{v \not\in V(T)} \pi(v). \]
It is easy to see that the Steiner tree problem is the special case with an infinite penalty on the terminals and zero penalty on non-terminals. In the prize-collecting Steiner forest problem we have a penalty $\pi(uv)$ for each pair of vertices $u, v \in V$. The goal is to find a forest $F \subseteq E$ to minimize
\[ \sum_{e \in F} c(e) + \sum_{uv \text{ not connected by } F} \pi(uv). \]
For the prize-collecting Steiner tree problem, the current best approximation ratio is $2 - \delta$ for some small but fixed $\delta > 0$, due to Archer et al. [2], and for the prize-collecting Steiner forest problem, the best known ratio is 2.54 due to Hajiaghayi and Jain [20].

In this paper we obtain a PTAS for the prize-collecting Steiner tree problem in planar graphs; we obtain this result via an approximation preserving reduction from the prize-collecting Steiner tree problem in planar graphs to the problem in graphs of bounded treewidth. Similarly for the prize-collecting Steiner forest problem, we reduce the problem in planar graphs to the problem in graphs of bounded treewidth. Below is our main result.

**Theorem 1.1.** For each fixed $\varepsilon > 0$, there is a $\rho(1 + \varepsilon)$-approximation for the prize-collecting Steiner forest problem in planar graphs if, for each fixed integer $k$, there is a $\rho$-approximation for the prize-collecting Steiner forest problem in graphs of treewidth at most $k$. In particular, there is a $\rho(1 + \varepsilon)$-approximation for the prize-collecting Steiner tree problem in planar graphs if there is a $\rho$-approximation for the prize-collecting Steiner tree problem in graphs of bounded treewidth.

We describe an exact algorithm for the prize-collecting Steiner tree problem in graphs of bounded treewidth:

**Theorem 1.2.** For any fixed integer $k$, there is a polynomial-time algorithm that exactly solves the prize-collecting Steiner tree problem in graphs of treewidth at most $k$.

Combining the reduction in Theorem 1.1 with the above theorem, we obtain the following.

**Corollary 1.3.** There is a PTAS for the prize-collecting Steiner tree problem in planar graphs.

A natural idea would be to show that there is a PTAS for the prize-collecting Steiner forest problem in graphs of bounded treewidth. (We do not expect an exact algorithm since the problem is NP-Hard even for this case [8].) This would yield a PTAS for the problem in planar graphs. However, recent work in [7] shows that prize-collecting Steiner forest is APX-Hard even in series-parallel graphs, which are planar and have treewidth 2.

**Remark 1.4.** Results analogous to Theorems 1.1 and 1.2 also apply to the prize-collecting Traveling Salesperson Problem; details are deferred to a later version of the paper.

**Remark 1.5.** The results for planar graphs extend to graphs of bounded genus. This follows previous ideas; details are deferred to a later version of the paper.

We are motivated to consider the prize-collecting problems for several reasons. First, they generalize the Steiner tree and Steiner forest problems. Second, the prize-collecting Steiner tree problem has played a crucial role in algorithms for $k$-MST [17], [4], $k$-Stroll [13] and the Orienteering [9], [6], [14] problems. These problems are more difficult than the Steiner tree problem because they also involve choosing the subset of terminals to connect. In particular, obtaining a PTAS for $k$-MST, $k$-Stroll or Orienteering in planar graphs appears to be challenging with current techniques. The prize-collecting version of Steiner tree can be viewed as an intermediate problem that still captures some of these difficulties in that the set of terminals to connect is not determined a priori. Techniques and ideas developed for the prize-collecting problems are likely to play a role in making progress on problems such as $k$-MST and Orienteering. We note that several of these problems have PTASes in low-dimensional Euclidean spaces but those rely on space partitioning schemes such as those of Arora [3] and Mitchell [23]; planar graph PTASes for network design have been more difficult to obtain since there is no comparable generic scheme. Some of the technical challenges will be outlined below where we describe our techniques at a high-level.
1.1 Overview of Techniques

Through this paper, we use OPT to denote the cost of an optimal solution to the given problem instance. Also, for any solution $F$ for the prize-collecting Steiner forest (or prize-collecting Steiner tree), we use $\text{LENGTH}(F)$ to denote $\sum_{e \in F} c(e)$, $\text{PENALTY}(F)$ to denote $\sum_{uv \text{ not connected by } F} \pi(uv)$, and $\text{COST}(F)$ to denote $\text{LENGTH}(F) + \text{PENALTY}(F)$. For the prize-collecting Steiner tree problem, we refer to all vertices with non-zero penalty as terminals; similarly, for prize-collecting Steiner forest, we refer to all pairs of vertices with non-zero penalty as terminal pairs. Finally, we assume that $\varepsilon \leq 1$ and (w.l.o.g.) that all terminals in the input graph $G$ have degree 1; if this is not the case for some terminal $v$, simply connect it to a new vertex $v'$ using an edge of cost 0, and use $v'$ as a terminal in the place of $v$.

Planar graph approximation schemes for Steiner tree [11] and forest [8] build on many ideas starting with the framework of Baker [5] for PTASes for planar graphs and utilizing several subsequent technical tools. Before describing the new technical contributions needed to handle prize-collecting problems, we first give a very high-level overview of the approach common to our algorithms, and the previous approximation schemes of [11, 8].

1. Given a planar graph $G$, construct a spanner, a subgraph $H \subseteq G$ such that there exists a solution in $H$ of cost $(1 + \varepsilon)OPT$ and the total length of edges in $H$ is $f(\varepsilon)OPT$, for some function $f$ depending purely on $\varepsilon$.

2. Partition the edges of $H$ into $f(\varepsilon)/\varepsilon$ sets $E_1, E_2, \ldots, E_{f(\varepsilon)/\varepsilon}$, such that contracting any set results in a graph of treewidth $O(f(\varepsilon)/\varepsilon)$ [16].

3. Pick the set $E_i$ of minimum total edge length; this cost must be no more than $\varepsilon OPT$. Contract the edges of this set $E_i$, yielding a bounded treewidth graph that contains a solution of cost $(1 + \varepsilon)OPT$. Solve the problem on this graph of bounded treewidth. This may not correspond to a solution in the original graph $G$; add (uncontract) edges of $E_i$ as necessary to obtain a feasible solution in $G$. The total cost of $E_i$ is at most $\varepsilon OPT$, and hence the cost of the solution is at most $(1 + 2\varepsilon)OPT$.

We note that the most difficult step is the first, finding a spanner $H$. Once this is accomplished, the rest is somewhat standard; applying a theorem of [16] allows one to decompose the edge set of $H$ into pieces, and a simple averaging argument implies that one of these pieces has low cost; contracting this piece yields a graph of bounded treewidth. Depending on the problem being considered, it may be possible to solve the bounded treewidth instance exactly; if not, an approximation algorithm is used. (In [11], steps 2 and 3 are modified slightly to obtain a more efficient algorithm; we omit details here.) For prize-collecting Steiner tree, it is fairly straightforward to solve the problem exactly in graphs of fixed treewidth. As mentioned before, prize-collecting Steiner forest is NP-Hard even in graphs of fixed treewidth; obtaining a small constant-factor approximation would be of interest as, from Theorem 1.1 this would immediately yield an algorithm for planar graphs achieving a similar approximation ratio. Thus, we focus below on Step 1, the construction of the spanner $H$; we describe parts of the spanner constructions of [11, 8], and the modifications necessary for the prize-collecting variants.

**Prize-collecting Steiner Tree:** We first focus on the easier case of Steiner trees and the scheme in [11]. The basic idea to find a spanner for the Steiner tree instance is as follows. (We assume we are given an embedding of the input planar graph $G$.) The algorithm starts by computing a 2-approximate (any constant factor would do) Steiner tree $T$ in $G$. One can use an Euler tour of $T$ and splice open along the tour to obtain another plane graph $G'$ in which the Euler tour of $T$ is its outer face; note that the total cost of edges on this face is at most 4OPT. Now, all the terminals are on the outer face of $G'$; this fact is crucial. The algorithm in [11] then builds

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2 All the algorithms modify this approach in distinct ways; see the subsequent discussion.
on the work of Klein [22] to obtain the desired spanner $H$: It begins with the outer face of $G'$ (containing all the terminals), and adds edges of $G$ of total cost proportional to the length of this outer face, while guaranteeing the existence of a near-optimal solution only using these added edges. We provide details of the construction in Appendix A.

The main difficulty in extending the above PTAS to the prize-collecting Steiner tree problem is the following. We again wish to find a spanner subgraph $H$ of $G$ by starting with an $O(1)$-approximate tree $T$ and making it the outer face. Unlike the Steiner tree case we run into a difficulty. The approximate tree $T$ is not guaranteed to contain all the vertices that are connected to the root in an optimal solution! In fact, if we knew the vertices that need to be connected to the root in a near-optimal solution then we can simply reduce the problem to the Steiner tree problem. We overcome this difficulty by proving the following.

**Theorem 1.6.** There is a polynomial time algorithm that, given a prize-collecting Steiner tree instance in a graph $G$, outputs a tree $T$ of cost $O(1/\varepsilon)OPT$ such that there is a $(1 + \varepsilon)$-approximate solution $T'$ that connects to the root a subset of the vertices in $T$.

The algorithm to achieve the above is in fact the Goemans-Williamson primal-dual algorithm for the prize-collecting Steiner tree problem but with modified potentials $\pi'(v) = \frac{2}{\varepsilon}\pi(v)$ for each $v \in V$. By exploiting properties of the primal-dual algorithm we can prove the above theorem. We then proceed as in the Steiner tree case, making this tree $T$ the outer face, and adding edges to form a spanner. We note that we cannot use an (approximation) algorithm for the prize-collecting Steiner tree problem as a black box in proving the above — it is important to rely on the properties of the primal-dual algorithm as we change the potentials.

**Prize-collecting Steiner Forest:** The PTAS for Steiner forest, which is quite recent [8], requires two new ideas. First, as with the Steiner tree PTAS one starts with an $O(1)$-approximate Steiner forest. However, this forest has potentially many components and one cannot apply the spanner construction idea of [11] in making a single tree the outer face. At the same time one cannot directly argue that one can treat each tree in the forest separately; optimal (or near-optimal) solutions may connect vertices in different components of the forest. In [8] there is an additional key step in which the trees are grown via a primal-dual type argument and some of them are merged — after this step, the remaining trees are “far apart” and hence can be treated independently via the spanner approach of [11]. (More precisely, one obtains a collection of subgraphs $H_1, H_2, \ldots$ such that $\sum_i \text{LENGTH}(H_i) \leq f(\varepsilon)OPT$. Further, if OPT; denotes the cost of an optimal solution for the terminal pairs in $H_i$, then $\sum_i \text{OPT}_i \leq (1 + \varepsilon)OPT$.)

A second difficulty in obtaining a PTAS is in step 3, solving the problem on bounded treewidth graphs. As mentioned previously, a PTAS is developed in [8] for Steiner forest in bounded treewidth graphs.

As for the prize-collecting Steiner tree problem, the difficulty in the prize-collecting Steiner forest problem is that we do not know which terminal pairs to connect. We deal with this using an approach similar to that for the prize-collecting Steiner tree problem. However, the 3-approximate primal-dual algorithm for the Steiner forest problem, due to Hajiaghayi and Jain [20], is quite complex; each step requires $O(n)$ max-flow computations. It is unclear whether one can prove a theorem similar to Theorem 1.6 via the algorithm in [20]. We develop a simpler primal-dual algorithm and analysis for the Steiner forest problem that gives a 4-approximation. We use the structure and analysis of our algorithm to prove the following theorem.

**Theorem 1.7.** There is a polynomial time algorithm that, given a prize-collecting Steiner forest instance in a graph $G$, outputs a forest $F$ of cost $O(1/\varepsilon)OPT$ such that there is a $(1 + \varepsilon)$-approximate solution $F'$ that connects a subset of the pairs connected by $F$.

**Other related work:** There is a substantial amount of literature on various aspects related to the work discussed in this paper. We refer the reader to [15] for an overview of the progress on obtaining approximation schemes for optimization problems on planar graphs; recent papers [22, 11, 8] have pushed the techniques
further for network design problems. Prize-collecting versions of network design have received considerable
attention following the work of Goemans and Williamson [18] on a primal-dual algorithm which had sev-
eral applications to other problems such as $k$-MST. Following Hajiaghayi and Jain’s work on prize-collecting
Steiner forest problem [20], there have been several other papers on prize-collecting network design problems.
Sharma, Swamy and Williamson [25] generalized the approach in [20] to obtain primal-dual constant factor
approximation algorithms for prize-collecting constrained forest problems (in the framework of Goemans and
Williamson [18]) with submodular penalty functions. Gutner [19] gave a very simple and efficient local-ratio
based 3-approximation for prize collecting Steiner forest problem which also applies to the generalized version
with more than two terminals in a group; although Gutner’s algorithm is quite simple, it does not seem possible
to prove Theorems 1.6 and 1.7 directly via his algorithm. Constant factor approximation ratios have also been
obtained for more general problems with higher connectivity requirements [24, 21].

Relation to independent work in [7]: Bateni, Hajiaghayi and Marx [7] have obtained several results on prize-
collecting network design problems in planar graphs. Our work was done independently before we were aware
of their paper; we have since learnt that their results were obtained slightly earlier than ours. We briefly compare
the results and techniques between their work and ours. They have an analogous theorem to Theorem 1.1 but
they also consider a more general problem, namely prize-collecting Steiner forest with submodular penalty
functions. They obtain PTASes for prize-collecting Steiner tree and related problems such as prize-collecting
TSP and stroll problems, much as we can obtain via the reduction to bounded treewidth instances. They
complement their algorithms with an interesting hardness of approximation result which shows that prize-
collecting Steiner forest is APX-hard in series-parallel graphs which are planar graphs and have treewidth 2.
Their APX hardness also extends to instances in the Euclidean plane. The main part of proof of the reduction
to bounded treewidth instances in [7] differs from ours in some ways. While we rely on properties of a primal-
dual algorithm for the underlying problem with scaled up penalties, they use a separate primal-dual clustering
step on top of the trees returned by an approximation algorithm for the underlying problem; this is inspired by
earlier work of Archer et al. [2] and further extended in [8].

Outline: In Section 2 we give a 4-approximate primal-dual algorithm for the prize-collecting Steiner forest
problem. In Section 3 we prove Theorem 1.1 giving the complete reduction from the prize-collecting problems
in planar graphs to their bounded-treewidth versions. This reduction has three parts: First, in Section 3.1 we
prove Theorems 1.6 and 1.7 showing that we can get $O(1/\varepsilon)$-approximate solutions connecting almost
all the terminals connected by an optimal solution. Second, we use these theorems to complete the spanner
constructions; details are provided in Appendix A. Third, we give the remaining details of the reduction in
Section 3.2. Finally, in Section 4 we give an exact algorithm for the prize-collecting Steiner tree problem in
graphs of fixed treewidth.

2 A Primal-Dual Algorithm for Prize-Collecting Steiner Forest

Our algorithm is similar to the Goemans-Williamson [18] primal-dual algorithm for prize-collecting Steiner
tree. For the prize-collecting Steiner tree, this gives a 2-approximation (with some additional guarantees), but
this does not appear possible in the Steiner forest variant. In this section, we give a simple 4-approximation for
the prize-collecting Steiner forest problem, and later exploit properties of this algorithm to prove Theorem 1.7.

We give primal and dual linear programming formulations for the prize-collecting Steiner forest problem
below. For each pair $(s_i, t_i)$ the variable $z_i$ is 1 if we pay the penalty for not connecting the pair, and 0 otherwise;
the variable $x_e$ denotes whether the edge $e$ is selected for the forest. We abuse notation and say that a set $S$
separates (the terminal pair) $i$ if it separates $s_i$ from $t_i$. Let $S_i$ denote the collection of sets $S$ that separate $i$. 

5
clear which pair single active component active component a growth phase, followed by a deletion phase. At all times, our algorithm maintains a forest in $[1, 18]$. In the former case, the component is deactivated; in the latter, one merges the two components adjacent to the connected components of $S$ containing both $s_i$ and $t_i$. Terminals are labeled satisfied, alive, or dead; initially, all terminals are alive. If a component $S$ uniting $i$ is formed, the terminals $s_i$ and $t_i$ are marked satisfied if they were both alive immediately before the formation of $S$. (As satisfied terminals are no longer willing to pay for the growth of a component $S$ containing them, we adjust Potential$(S)$ if necessary; the procedure PROCESS HISTORY handles the necessary bookkeeping.) Once we form a component $S \in \hat{C}$, we “grow” it by increasing an auxiliary dual variable $y_i(S)$; simultaneously, we decrease Potential$(S)$ to pay for this growth. Once Potential$(S)$ becomes 0, a component is labeled inactive. (Recall that all components are active when they are formed.) When a component becomes inactive, all of its unsatisfied terminals are marked dead. (Intuitively, it is now more effective to pay the

A reader unfamiliar with the primal-dual algorithms of [1, 18] (for the Steiner forest and prize-collecting Steiner tree problems respectively) may wish to skip this paragraph and the next, proceeding directly to the description of our algorithm. Here, we briefly describe one way in which our algorithm differs from those of [1, 18]. A natural approach is to initially assign each terminal $s_i$ or $t_i$ a potential of approximately $\pi_i$, and make each one an active component. In both the Steiner forest and prize-collecting Steiner tree problems, a natural LP formulation has a single dual variable $y_S$ for each active component $S$. One increases the $y_S$ variable for each active component $S$ uniformly, until either a component “runs out” of potential, or an edge becomes “tight”. In the former case, the component is deactivated; in the latter, one merges the two components adjacent to the edge, and combines their potentials.

In the prize-collecting Steiner forest problem, however, we have a collection of dual variables $y_{i,S}$ for a single active component $S$. We still wish each active component $S$ to grow at a uniform rate, but it is now not clear which pair $(s_i, t_i)$ separated by $S$ should pay for the growth of $S$. Below, we show that there is a natural way to share the cost of this growth among the pairs separated by $S$; once this is done, the analysis proceeds as in [1, 18].

Before describing the algorithm, we present some useful notation. The primal-dual algorithm begins with a growth phase, followed by a deletion phase. At all times, our algorithm maintains a forest $F$ of edges. The connected components of $F$ are labeled either active or inactive; we use $C$ to denote the set of active components at any given time. Every component is active at the time it is formed; we use $\hat{C}$ to denote the set of components that were ever formed during the algorithm’s execution. (Note that as we only add edges to $F$ during the growth phase, $\hat{C}$ is a laminar family of components.) When a component $S \in \hat{C}$ is formed, we assign it a potential Potential$(S)$ that corresponds (roughly) to the penalty of terminal pairs separated by $S$; thus, this measures how much we are willing to pay in order to connect terminals in $S$ to their partners outside. If we grow $S$ by more than its potential without meeting the desired partners, then it is more effective to pay the penalty than connect the pairs separated by $S$; at this point, we will mark $S$ inactive. (Of course, some terminals in $S$ may meet their partners while others do not; this is the key difference between Steiner tree and Steiner forest, and we describe how to make this intuition more precise below.)

We form new components by adding edges to $F$, merging existing components; when two components merge, they combine their potentials. We say that $S \in \hat{C}$ unites $i$ if $S$ is the smallest active component in $\hat{C}$ containing both $s_i$ and $t_i$. Terminals are labeled satisfied, alive, or dead; initially, all terminals are alive. If a component $S$ uniting $i$ is formed, the terminals $s_i$ and $t_i$ are marked satisfied if they were both alive immediately before the formation of $S$. (As satisfied terminals are no longer willing to pay for the growth of a component $S$ containing them, we adjust Potential$(S)$ if necessary; the procedure PROCESS HISTORY handles the necessary bookkeeping.) Once we form a component $S \in \hat{C}$, we “grow” it by increasing an auxiliary dual variable $y_i(S)$; simultaneously, we decrease Potential$(S)$ to pay for this growth. Once Potential$(S)$ becomes 0, a component is labeled inactive. (Recall that all components are active when they are formed.) When a component becomes inactive, all of its unsatisfied terminals are marked dead. (Intuitively, it is now more effective to pay the

\begin{align*}
\text{Primal-PCSF} & : \\
\min & \sum_e c_e x_e + \sum_i \pi_i z_i \\
\subject & \sum_{e \in \delta(S)} x_e \geq (1 - z_i) \quad (\forall i, S \in S_i) \\
& x_e, z_i \geq 0 \quad (\forall e, i)
\end{align*}

\begin{align*}
\text{Dual-PCSF} & : \\
\max & \sum_S \sum_{i \in S_i} y_{i,S} \\
\subject & \sum_{S: e \in \delta(S)} \sum_{S \in S_i} y_{i,S} \leq c_e \quad (\forall e) \\
& \sum_{S \in S_i} y_{i,S} \leq \pi_i \quad (\forall i) \\
& y_{i,S} \geq 0 \quad (\forall i, S \in S_i)
\end{align*}
associated penalties than to try to connect them to their partners.) For any terminal \( s_i \) (respectively \( t_i \)), we use \( \text{History}(s_i) \) (respectively \( \text{History}(t_i) \)) to denote the set of components \( S \in \mathcal{C} \) such that \( S \) contains \( s_i \) \( (t_i) \) and \( s_i \) \( (t_i) \) was alive after \( S \) was formed.

Finally, we note that the auxiliary variables \( y(S) \) do not exist in our dual LP formulation; instead, we have variables \( y_i.S \). We ensure that at the end of the algorithm, \( y(S) = \sum_{i: S \in S} y_i.S \). In order to determine how \( y(S) \) is split among the variables \( y_i.S \) we maintain an associated variable \( \text{Uncharged}(S) \), the uncharged growth of \( S \). The following proposition is entirely straightforward, as we increase \( y(S) \) and \( \text{Uncharged}(S) \) together:

**Proposition 2.1.** For each \( S \in \mathcal{C} \), as soon as \( S \) is marked inactive or \( S \) becomes part of a larger component, we have \( \text{Uncharged}(S) = y(S) \).

The procedure \( \text{PROCESS HISTORY} \) ensures that \( \text{Uncharged}(S) \) (and hence \( y(S) \)) is split appropriately among the dual variables \( y_i.S \). We omit a proof of the proposition below:

**Proposition 2.2.** When the main While loop of \( \text{PRIMAL-DUAL FOREST} \) terminates, for each pair of terminals \((s_i, t_i)\), we have called \( \text{PROCESS HISTORY}(s_i) \) and \( \text{PROCESS HISTORY}(t_i) \).

We initialize the set of active components to be the set of terminals, and set \( \text{Potential}(s_i) = \text{Potential}(t_i) = \pi_i/2 \) for each \( 1 \leq i \leq h \).

| PRIMAL-DUAL FOREST |
|--------------------|
| \( F \leftarrow \emptyset \) |
| \( \mathcal{C} \leftarrow \{\{s_i\}: 1 \leq i \leq h\} \cup \{\{t_i\}: 1 \leq i \leq h\} \) |
| For all \( 1 \leq i \leq h \): |
| \( \text{Potential}(\{s_i\}) = \text{Potential}(\{t_i\}) = \pi_i/2 \) |
| While there is an active component: |
| Increase \( y_S \) by \( \Delta \) for each \( S \in \mathcal{C} \) until an edge goes tight, or a component uses all its potential. |
| For each \( S \in \mathcal{C} \): |
| \( \text{Potential}(S) \leftarrow \text{Potential}(S) - \Delta; \text{Uncharged}(S) \leftarrow \text{Uncharged}(S) + \Delta \) |
| If (edge \( e \) connecting \( S_1, S_2 \) goes tight) |
| \( F \leftarrow F \cup \{e\} \) |
| \( \mathcal{C} \leftarrow (\mathcal{C} \setminus \{S_1, S_2\}) \cup \{S_1 \cup S_2\} \) |
| \( y(S_1 \cup S_2) \leftarrow 0; \text{Uncharged}(S_1 \cup S_2) \leftarrow 0 \) |
| \( \text{Potential}(S_1 \cup S_2) \leftarrow \text{Potential}(S_1) + \text{Potential}(S_2) \) |
| For each \( i \) such that \( S_1 \cup S_2 \) unites \( i \): |
| If \( s_i \) is alive, \( \text{Potential}(S_1 \cup S_2) \leftarrow \text{Potential}(S_1 \cup S_2) - \text{PROCESS HISTORY}(s_i) \) |
| If \( t_i \) is alive, \( \text{Potential}(S_1 \cup S_2) \leftarrow \text{Potential}(S_1 \cup S_2) - \text{PROCESS HISTORY}(t_i) \) |
| If \( s_i \) AND \( t_i \) are alive, Mark \( s_i, t_i \) as satisfied |
| Else, Mark \( s_i, t_i \) as dead |
| If (component \( S \) uses all its potential) |
| \( \mathcal{C} \leftarrow \mathcal{C} \setminus \{S\} \) |
| For each alive \( s_i \in S \): |
| Mark \( s_i \) as dead |
| \( \text{PROCESS HISTORY}(s_i) \) |
| End While |
| For each edge \( e \in F \): |
| Delete \( e \) if \( F - e \) does not separate any pair of satisfied terminals |

\[ 7 \]
Above. One can now use the standard primal-dual proof technique of [1, 18]. Since an edge and right-hand sides satisfies the inequality. Since
only when it becomes tight, we have

Proof: Theorem 2.5.
\[ \sum_{i: S \in History(s_i)} y_{i,S} = \pi_i / 2 \]  
For each component \( S \in \hat{C} \), \( y(S) = \sum_{i: S \in S_i} y_{i,S} \). For each dead terminal \( s_i \), we have \( \sum_{S \in History(s_i)} y_{i,S} = \pi_i / 2 \); similarly, for each dead \( t_i \), \( \sum_{S \in History(t_i)} y_{i,S} = \pi_i / 2 \).

Proof Sketch: It is easy to verify these statements by checking that the algorithm maintains the invariant that for each component \( S \in \hat{C} \), Potential\( (S) + \sum_{S' \in \hat{C}, S \subseteq S} \text{Uncharged}\( (S') = \sum_{S: \text{Alive} \in S} \pi_i / 2 \).

Lemma 2.4. The variables \( y_{i,S} \) from the algorithm Primary-Dual Forest correspond to a feasible dual solution for the LP Dual-PCSF.

Proof: It is easy to verify that all constraints are satisfied: For any edge \( e \), it is added to \( F \) once \( \sum_{S: e \in \delta(S)} y(S) = c_e \), and subsequently, there is no active component \( S \) such that \( e \in \delta(S) \). As \( y(S) = \sum_{i: S \in S_i} y_{i,S} \), we satisfy the associated constraint. Further, for any pair \( s_i, t_i \), the procedure PROCESS-HISTORY guarantees that \( \sum_{S: s_i \in S \land t_i \notin S} y_{i,S} \leq \pi_i / 2 \), and similarly \( \sum_{S: s_i \notin S \land t_i \in S} y_{i,S} \leq \pi_i / 2 \).

Theorem 2.5. If \( F \) denotes the forest returned by the algorithm Primary-Dual Forest, then \( \sum_{e \in F} c_e + \sum_{i \text{ separated by } F} \pi_i \leq 4 \text{OPT} \), where \text{OPT} denotes the cost of an optimal prize-collecting Steiner forest.

Proof: As OPT is upper bounded by the value of any feasible dual solution, it suffices to show

\[ \sum_{e \in F} c_e + \sum_{i \text{ separated by } F} \pi_i \leq 4 \sum_{S} \sum_{i: S \in S_i} y_{i,S}. \]

For any \( i \) to be separated by the forest \( F \), either \( s_i \) or \( t_i \) (or both) must have been marked dead, and thus from Lemma 2.3 we have \( \pi_i \leq 2 \sum_{S \in S_i} y_{i,S} \). Hence it suffices to prove

\[ \sum_{e \in F} c_e + \sum_{i \text{ separated by } F} 2 \sum_{S \in S_i} y_{i,S} \leq \left( 2 \sum_{S} \sum_{i: S \in S_i} y_{i,S} \right) + \left( 2 \sum_{i} \sum_{S \in S_i} y_{i,S} \right). \]

We prove \( \sum_{e \in F} c_e \leq 2 \sum_{S} y(S) \), which, using the fact that \( y(S) = \sum_{i: S \in S_i} y_{i,S} \), implies the inequality above. One can now use the standard primal-dual proof technique of [1, 18]. Since an edge \( e \) is added to \( F \) only when it becomes tight, we have \( c_e = \sum_{S: e \in \delta(S)} y(S) \); hence, the desired inequality is equivalent to:

\[ \sum_{S} |\delta(S) \cap F| y(S) \leq 2 \sum_{S} y(S). \]

To verify this inequality, we check that in every iteration of the while loop, the increase in both the left- and right-hand sides satisfies the inequality. Since \( y(S) \) increases only for components that are active in a given
iteration, and we raise \( y(S) \) uniformly by \( \Delta \) for each \( S \in \mathcal{C} \), this is equivalent to checking that in any iteration, 
\[ \sum_{\text{Active } S} |\delta(S) \cap F| \Delta \leq 2\Delta \cdot n_a, \]
where \( n_a \) denotes the number of components active in this iteration.

Construct an auxiliary graph \( H \) by beginning with \( G(V, F) \), and shrinking each currently active and inactive component to a single vertex. Discard any isolated vertex of \( H \) corresponding to an inactive component. We now argue that the average degree in \( H \) of the vertices corresponding to active components is at most 2; this completes the proof. To bound the average degree of the vertices corresponding to active components, we note that the average degree of all vertices is less than 2 (as \( H \) is a forest), and that each vertex corresponding to an inactive component has degree at least 2. If the latter were not true, there is an inactive component which is a leaf of \( H \), incident to a single edge \( e \). But no inactive component separates a satisfied terminal from its partner, and so the edge \( e \) would have been deleted from \( F \) in the Deletion Phase of PRIMAL-DUAL FOREST.

**Running time:** PRIMAL-DUAL FOREST can be implemented in \( O(nh + n^2 \log n) \) time where \( h \) is the number of pairs with strictly positive penalties and \( n \) is the number of nodes in the graph. The only additional difficulty in the algorithm as compared to the primal-dual algorithmic framework of Goemans and Williamson [18] for prize-collecting Steiner tree and related problems is the PROCESSHISTORY step. When two active components merge, for each pair \( s_i t_i \) united by this merge, we have to go over the sets in History(\( s_i \)) and History(\( t_i \)). The sets in \( \hat{\mathcal{C}} \) form a laminar family on \( V \) and hence \( |\hat{\mathcal{C}}| = O(n) \) and thus PROCESSHISTORY(\( s_i \)) can be implemented in \( O(n) \) time by considering each set in \( |\hat{\mathcal{C}}| \). Similarly, PROCESSHISTORY(\( t_i \)). Since a pair is united at most once, and each terminal is marked dead at most once, the total work involved in processing the histories is \( O(nh) \). Priority queues can be used to keep track of the events corresponding to edges becoming tight and components becoming inactive; this is very similar to the implementation in [18] and the total work involved is \( O(n^2 \log n) \) time.

## 3 The Reduction to Bounded Treewidth: Building Spanners

Recall that the first step in building a spanner for the Steiner tree and forest problems was to construct a new plane graph \( G' \) in which (i) all terminals are on the outer face and (ii) the length of the outer face is \( O(1) \cdot \text{OPT} \). This was done by splicing open an Euler tour of an \( O(1) \) approximate solution in the original graph \( G \), and converting the tour into the outer face\(^3\). In the prize-collecting versions, however, we do not know which terminals to connect, and it is not possible to find an \( O(1) \)-approximate solution in which all the terminals are connected to the root (for Steiner tree) or to their partners (for Steiner forest).

In Section 3.1, we prove Theorems 1.6 and 1.7, showing that we can find a solution of cost \( O(1/\varepsilon)\text{OPT} \) which connects “almost” all the terminals connected by any optimal solution. More precisely, the total penalty of terminals connected by an optimal solution but not by our \( O(1/\varepsilon) \)-approximate solution is at most \( \varepsilon \text{OPT} \). In Appendix A we complete the construction of the spanner, using ideas from [11, 8].

### 3.1 Scaling Penalties to Capture Important Terminals

We prove Theorem 1.7 which implies Theorem 1.6 as prize-collecting Steiner tree is a special case of prize-collecting Steiner forest. Given an instance \( I \) of the prize-collecting Steiner forest on a graph \( G(V, E) \), with \( c_e \) denoting the cost of edge \( e \in E \) and \( \pi_i \) the penalty for not connecting \( s_i \) to \( t_i \), we define a new instance \( I' \) as follows: The graph and edge cost functions are unchanged, but we scale the penalties so that the penalty for not connecting \( s_i \) to \( t_i \) is \( \pi'_i = 2\pi_i / \varepsilon \).

**Theorem 3.1.** Let \( F^* \) be any optimal solution to an instance \( I \) of prize-collecting Steiner forest, and let \( \text{OPT} = \sum_{e \in F^*} c_e + \sum_{i \text{ separated by } F^* \pi_i} \). Let \( F' \) be the forest output by algorithm PRIMAL-DUAL FOREST on the

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\(^3\)In the case of Steiner forest, this is done separately for distinct trees in the \( O(1) \)-approximate forest.
instance $I'$ with penalties scaled as above. Let $X$ denote the index set of the terminal pairs separated by $F'$ but not by $F^*$. Then, $\sum_{e \in F'} c_e \leq 8OPT/\varepsilon$, and $\sum_{i \in X} \pi_i \leq \varepsilon OPT$.

**Proof:** We first note that the cost of an optimal solution to $I'$ is at most $2OPT/\varepsilon$; simply use the forest $F^*$, which pays $2/\varepsilon$ times as much penalty for every separated pair as it did in $I$. Thus, as PRIMAL-DUAL FOREST is a 4-approximation, we have $\sum_{e \in F'} c_e \leq 8OPT/\varepsilon$.

To prove that the total penalty of pairs in $X$ is small, consider a Steiner forest instance defined on these pairs: As $F^*$ connects all the terminals in $X$ to their partners, the cost of an optimal Steiner forest for $X$ is at most $OPT$. Suppose, by way of contradiction, that $\sum_{i \in X} \pi_i > \varepsilon OPT$, and hence that $\sum_{i \in X} \pi_i' > 2OPT$. Now consider the following dual of a natural LP for the Steiner Forest instance induced by $X$:

**Dual-Steiner Forest($X$)**

$$\max \sum_{S: \text{separating some } i \in X} z_S$$

$$\sum_{S: e \in \delta(S)} z_S \leq c_e \quad (\forall e)$$

$$z_S \geq 0 \quad (\forall S)$$

Let $y_{i,S}$ be the feasible solution to Dual-PCSF returned by PRIMAL-DUAL FOREST on instance $I'$. Now, construct a dual solution to the LP Dual-Steiner Forest($X$) as follows: For each set $S$ separating some pair $s_i, t_i$ with $i \in X$, set $z_S = \sum_{i \in X} y_{i,S}$. As $\sum_{S: e \in \delta(S)} \sum_{i: s_i \in S} y_{i,S} \leq c_e$ from the feasibility of the solution to Dual-PCSF, we conclude that the dual variables $z_S$ correspond to a feasible solution of Dual-Steiner Forest($X$).

Thus, we have a feasible solution to Dual-Steiner Forest($X$) of total value $\sum_S \sum_{i \in X: s_i \in S} y_{i,S}$. But each $i \in X$ was not connected by $F'$, and so we must have marked either $s_i$ or $t_i$ as dead. Hence, from Lemma 2.3

$\sum_{S \ni s_i, y_{i,S} \geq \pi_i/2}$. That is, the value of our feasible dual solution is at least $\sum_{i \in X} \pi_i'/2 > OPT$. By weak duality, the length of any Steiner forest for $X$ must be greater than $OPT$. But $F^*$ is a Steiner forest for $X$ of total length at most $OPT$, which is a contradiction.

We can now prove Theorem 1.7

**Proof of Theorem 1.7** Let $F^*$ be an optimal solution to a given instance $I$ of Steiner forest, and let $OPT = \sum_{e \in F^*} c_e + \sum_{i \text{ separated by } F^*} \pi_i$. Construct a forest $F$ by running algorithm PRIMAL-DUAL FOREST on the scaled instance $I'$; from Theorem 3.1 above, the total length of edges in $F$ is at most $8OPT/\varepsilon$. If $X$ denotes the terminal pairs separated by $F$ but not by $F^*$, the penalty paid by $F$ is at most $\sum_{i \text{ separated by } F} \pi_i + \sum_{i \in X} \pi_i \leq OPT + \varepsilon OPT$. Thus, $F$ is a forest of total cost $O(8/\varepsilon + (1 + \varepsilon))OPT$.

It remains only to argue that there is a $(1 + \varepsilon)$-approximate solution $F'$ that connects a subset of the pairs connected by $F$. Let $F^*_X$ denote a solution formed from $F^*$ by paying the penalty for any terminal pair in $X$; clearly, the cost of $F^*_X$ is the cost of $X$ added to $\sum_{i \in X} \pi_i$, which is at most $OPT + \varepsilon OPT$.

The first part of the spanner construction for the prize-collecting Steiner tree problem is now complete: As guaranteed by Theorem 1.6 find a tree $T$ of cost $O(1/\varepsilon)OPT$ such that there exists a $(1 + \varepsilon)$-approximate solution only connecting terminals in $T$. Now form an Euler tour of $T$ by duplicating edges, splice along this tour, and make the tour the outer face of a new graph $G'$. Now, we have a graph in which all (relevant) terminals are on the outer face, and the total length of the outer face is $O(1/\varepsilon)OPT$. The rest of the spanner construction proceeds along the lines of [11]; see Appendix A.

For the prize-collecting Steiner forest problem, however, more work is required. The forest $F$ guaranteed by Theorem 1.7 may have many components, which cannot be treated in isolation. As in [8], we use
a \textit{prize-collecting} clustering scheme to merge some components of the forest. The intuition is that after this clustering, the remaining components are “far apart”, and hence can be treated separately. The prize-collecting clustering algorithm is as follows: Contract each tree $T_i$ of $F$ to a single vertex $v_i$, to obtain a new graph $\hat{G}$. We let $v_1, v_2, \ldots$ denote the vertices of $\hat{G}$ corresponding to the contracted trees $T_1, T_2, \ldots$ of $F$; note that $\hat{G}$ additionally has the vertices of $G$ not contained in any tree $T_i$. With each $v_i$, we associate a potential $\phi_{v_i} = \frac{1}{\varepsilon} \text{LENGTH}(T_i)$. Now run the standard prize-collecting primal-dual algorithm as in [18], using potentials $\phi_{v_i}$. Initially, each vertex is an active component, with potential $\phi_{v_i}$; in each step, the algorithm decreases potentials of all active components uniformly until either a component runs out of potential, or an edge becomes “tight”. In the former case, the component is marked as inactive. In the latter case, the two components adjacent to the edge are merged, and their potentials combined. This is similar to the algorithm \textsc{Primal-Dual Forest} in Section 2 but simpler, as we do not have the additional accounting necessary to handle terminal pairs that only wish to connect to each other. As in \textsc{Primal-Dual Forest}, the algorithm maintains a collection of dual variables $y_{v_i,S}$ for each $v_i$ and set $S \subseteq V(\hat{G})$. A complete description of this algorithm is given in [8]; we omit details from this paper.

The first stage of the clustering algorithm terminates when all components are marked inactive. Let $F_1$ denote the forest of tight edges selected by the algorithm after the first stage. In the second stage, we delete any edge $e$ from $F_1$ if it is the unique edge incident to an inactive component. Let $F_2$ denote the set of edges remaining.

\textbf{Lemma 3.2 ([8])}. The total length of all edges in $F_2$ is at most $2 \sum_i \phi_{v_i} = \frac{2}{\varepsilon} \text{LENGTH}(F)$, which is $O(\frac{1}{\varepsilon^2})\text{OPT}$.

We also use the following two technical lemmas. A graph $\hat{H} \subseteq \hat{G}$ is said to exhaust a vertex $v_i \in V(\hat{G})$ if, for all $S$ such that $y_{v_i,S} > 0$, $\hat{H}$ contains at least one edge of $\delta(S)$. (Recall that $v_i$ corresponds to the contracted tree $T_i$ of $F$.)

\textbf{Lemma 3.3 (Lemma 10 of [8])}. Let $V'$ be the set of vertices of $\hat{G}$ exhausted by a graph $H$. $\text{LENGTH}(H) \geq \sum_{v_i \in V'} \phi_{v_i}$.

\textbf{Lemma 3.4 ([8])}. Let $\hat{H} \subseteq \hat{G}$ connect two vertices $v_1, v_2$ in distinct components of $F_2$. Then, $\hat{H}$ exhausts at least one of $v_1, v_2$.

Let $\mathcal{T}^1, \mathcal{T}^2, \ldots$ be the trees comprising the forest $F_2$. We will now argue that these trees are sufficiently “far apart”, and so we can treat them separately; we formalize this intuition in the rest of this sub-section. Recall that from Theorem 12 there is a $(1 + \varepsilon)$-approximate solution (in the original graph $G$) that does not connect (that is, pays the penalty for) terminal pairs in distinct components of $F$, the forest returned by \textsc{Primal-Dual Forest}. Let $F^*$ denote this solution; $\text{Cost}(F^*) \leq (1 + \varepsilon)\text{OPT}$. We construct a set of prize-collecting Steiner forest instances, one for each tree $\mathcal{T}^j$ of $F_2$. In instance $I^j$, we have $\pi^j_i = \pi_i$ for each pair $(s_i, t_i)$ connected by $\mathcal{T}^j$, and $\pi^j_i = 0$ for all other pairs. Let $\text{OPT}^j$ denote the cost of an optimal prize-collecting Steiner forest instance $I^j$; we prove the following theorem:

\textbf{Theorem 3.5 (Following [8])}. $\sum_j \text{OPT}^j \leq (1 + \varepsilon)\text{Cost}(F^*)$.

Given this theorem, we can separately solve each instance $I^j$; it is easy to see that if we obtain a $\rho$-approximation to each instance, combining them yields a solution of cost at most $\rho(1 + \varepsilon)\text{Cost}(F^*) = \rho(1 + O(\varepsilon))\text{OPT}$. But for each instance $I^j$, the tree $\mathcal{T}^j$ contains all terminal pairs with non-zero penalty, and hence we can splice open the tree and convert the corresponding Euler tour into the outer face to obtain a graph in which the length of the outer face is bounded, and all relevant terminals are on this outer face. This allows us to proceed with the spanner construction as described in Appendix A.

Thus, it remains only to prove Theorem 3.5; this closely follows the work of [8], with some additional care needed because the cost of a forest includes both its length and penalty.
In Section 3.1, we constructed a forest $F$ from $G$ by running 	extsc{Primal-Dual Forest}. For completeness, we prove the theorem we stated there. We wish to show that if $\text{OPT}^j$ denotes the cost of an optimal solution for the instance $I^j$ (induced by terminal pairs in tree $T^j$ of $F$), then $\sum_j \text{OPT}^j \leq (1 + \varepsilon) \text{COST}(F^*)$.

To prove this theorem, we construct a set $D$ of trees $\{T^*_p\}$ such that each $T^*_p \in D$ only connects terminal pairs in a single component of $F$. Further, $\sum_p \text{LENGTH}(T^*_p) \leq (1 + \varepsilon) \text{LENGTH}(F^*)$, and every pair connected by $F^*$ is connected by some $T^*_p$. Such a set $D$ of trees clearly yields solutions to the instances $I^j$, proving the theorem.

We now construct the desired set $D$. We begin by setting $D$ to be the collection of trees in $F^*$, and then modify it as follows. Let $v_i$ be any vertex of $G$ corresponding to the tree $T_i$ of $F$ contracted to form $v_i$, and add the tree $T_i$ to $D$. When this process terminates, each tree in $D$ only connects pairs in a single component of $F$. Suppose this was not true; all trees added to $D$ clearly satisfy this condition, so it only remains to consider trees originally in $F^*$ (from which some terminals may have been pruned). But any tree $T^*$ of $F^*$ connecting two vertices of $G$ that are distinct components of $F$ must exhaust one of these vertices (from Lemma 3.3), and hence the corresponding terminals should have been pruned from $T^*$, which yields a contradiction. It is also easy to see that any terminal pairs connected by $F^*$ are also connected by some tree in $D$.

To bound the cost of the trees in $D$, we simply show that the length of the trees added to $D$ is at most $\varepsilon \cdot \text{LENGTH}(F^*)$. Let $v_i$ denote the vertex of $G$ corresponding to the tree $T_i$ of $F$, and $V'$ the set of vertices exhausted by $F^*$. The length of the added trees is simply $\sum_{v_i \in V'} \text{LENGTH}(T_i)$. From Lemma 3.3, $\text{LENGTH}(F^*) \geq \sum_{v_i \in V'} \phi_{v_i} = \sum_{v_i \in V'} \text{LENGTH}(T_i) / \varepsilon$. Rearranging, we get $\sum_{v_i \in V'} \text{LENGTH}(T_i) \leq \varepsilon \cdot \text{LENGTH}(F^*)$. \hfill $\square$

**Finishing the Spanner:** We can now complete the spanner construction using the following theorem, implicit in the work of [11]:

**Theorem 3.6.** Let $I$ be an instance of prize-collecting Steiner forest on a planar graph $G$. Let $F^*$ be an optimal solution to $I$. Given a tree $T$ spanning all terminals of $I$, for any fixed $\varepsilon > 0$, there is a polynomial time algorithm to find a planar graph $H \subseteq G$ such that: (i) $\text{LENGTH}(H) \leq f(\varepsilon) \cdot \text{LENGTH}(T)$ for some function $f$ that depends (exponentially) on $\varepsilon$, and (ii) there is a solution to instance $I$ in the graph $H$ of cost no more than $(1 + \varepsilon) \text{COST}(F^*) + \varepsilon \cdot \text{LENGTH}(T)$.

The similar theorem stated in [11] for instances of the Steiner tree problem is slightly less general, though their proof technique can be used to show the theorem we state here. For completeness, we provide a proof in Appendix A.

**3.2 Completing the Reduction**

In Section 3.1, we constructed a forest $F_2$ by first running the algorithm 	extsc{Primal-Dual Forest} on a modified instance with scaled penalties, and then running the prize-collecting clustering algorithm of [8]. We proved two useful properties of $F_2$: In Lemma 3.2 we showed that $\text{COST}(F_2) \leq (20/\varepsilon^2) \text{OPT}$, and in Theorem 3.5, we argued that we could separately solve a prize-collecting Steiner forest instance induced by each tree $T^j$ of $F_2$ without increasing the cost significantly. (Formally, we showed that $\sum_j \text{OPT}^j \leq (1 + \varepsilon) \text{OPT}$.)

Now, for each tree $T^j$ of $F_2$, construct a spanner for the instance $I^j$. (Recall that $T^j$ spans all terminals in $I^j$.) Using a parameter $\varepsilon' = \varepsilon^3 / 20$, Theorem 3.6 guarantees a spanner $H^j$ for $I^j$ such that (i) $\text{LENGTH}(H^j) = f'(\varepsilon) \cdot \text{LENGTH}(T^j)$ for some function $f'$ depending only on $\varepsilon$, and (ii) $\text{OPT}(H^j) \leq (1 + \varepsilon') \text{OPT}^j + \varepsilon'$.}

12
\[ \sum_j \text{OPT}(H_j) \leq (1+\varepsilon') \sum_j \text{OPT}^j + \varepsilon' \sum_j \text{LENGTH}(T^j) \leq (1+2\varepsilon)\text{OPT} + \frac{\varepsilon^3}{20} \frac{20}{\varepsilon^2} \text{OPT} = (1+3\varepsilon)\text{OPT} \quad (1) \]

where the second inequality follows from Lemma 3.2 and Theorem 3.5.

Thus, if we can obtain a \( \rho \)-approximation to each instance \( I^j \) in the graph \( H^j \), we obtain a \( \rho(1+3\varepsilon) \)-approximation to the original prize-collecting Steiner forest instance. We use the following theorem of [16].

**Theorem 3.7** (Demaine, Hajiaghayi, Mohar [16]). Let \( G \) be any planar graph, and let \( k \) be any integer such that \( k \geq 2 \). The edges of \( G \) can be partitioned into \( k \) sets such that contracting any one of the sets results in a graph of treewidth \( O(k) \). Furthermore, this partition can be found in polynomial time.

**Proof of Theorem 1.1.** Let \( H^j \) be the spanner for instance \( I^j \) as constructed above. Set \( k = (1/\varepsilon') \cdot f'(\varepsilon) \), where \( f'(\varepsilon) \) is the function such that \( \text{LENGTH}(H^j) \leq f'(\varepsilon) \text{LENGTH}(T^j) \).

Let \( E_1, \ldots, E_k \) be the decomposition of the edges of \( H^j \) that is guaranteed by Theorem 3.7. Let \( E_{i^*} \) be the set of edges that has minimum length among the sets \( E_1, \ldots, E_k \). We have

\[ \text{LENGTH}(E_{i^*}) \leq \frac{\text{LENGTH}(H^j)}{k} \leq \frac{f'(\varepsilon) \cdot \text{LENGTH}(T^j)}{k} = \varepsilon' \cdot \text{LENGTH}(T^j). \]

Let \( \widehat{H}^j = H^j / E_{i^*} \); that is, \( \widehat{H}^j \) is the graph obtained from \( H^j \) by contracting the edges in \( E_{i^*} \). We assign new penalties to terminal pairs of \( \widehat{H}^j \) in the natural way: When we contract an edge \( uv \) into a single vertex \( w \), we replace each terminal pair \((u, x)\) with a new pair \((w, x)\) with the same penalty as \((u, x)\), and we similarly replace each terminal pair \((v, y)\) with a new pair \((w, y)\) with the same penalty as \((v, y)\). Let \( \text{OPT}(\widehat{H}^j) \) denote the cost of an optimal prize-collecting Steiner forest in \( \widehat{H}^j \); it is obvious that \( \text{OPT}(\widehat{H}^j) \leq \text{OPT}(H^k) \).

Since \( \widehat{H}^j \) has treewidth at most \( k \), if there is a \( \rho \)-approximation for prize-collecting Steiner forest in graphs of fixed treewidth, we can find a \( \rho \)-approximate forest \( F^j \) in \( \widehat{H}^j \). We can then map \( F^j \) to a forest \( F^j \) in \( H^j \) using the edges in \( E_{i^*} \). By construction,

\[ \text{COST}(F^j) \leq \text{COST}(F^j) + \text{LENGTH}(E_{i^*}) \leq \rho \text{OPT}(H^j) + \varepsilon' \text{LENGTH}(T^j). \]

Combining such solution \( F^j \) for each \( H^j \), we find a forest of total cost \( \sum_j \rho \text{OPT}(H^j) + \varepsilon' \sum_j \text{LENGTH}(T^j) \). Using equation (1), the first term is at most \( \rho(1 + 3\varepsilon)\text{OPT} \), and from the choice of \( \varepsilon' \), the second term is at most \( \varepsilon\text{OPT} \).

\[ \square \]

**4 Prize-Collecting Steiner Tree in Graphs of Fixed Treewidth**

In this section, for any fixed integer \( k \geq 2 \), we give a polynomial-time algorithm to optimally solve the prize-collecting Steiner tree problem in graphs of treewidth at most \( k - 1 \).

A tree decomposition of a graph \( G \) is a pair \((T, B)\), where \( T = (I,F) \) is a tree, and \( B = \{B_i \mid i \in I\} \) is a family of subsets of \( V(G) \) such that

1. \( \bigcup_{i \in I} B_i = V(G) \)
2. for every edge \( uv \in E(G) \), there exists an \( i \) such that \( \{u,v\} \subseteq B_i \)
3. for every vertex \( v \in V(G) \), the set of nodes \( \{i \in I \mid v \in B_i\} \) forms a connected subtree of \( T \)
We refer to vertices of $T$ as nodes, and to each set of $B$ as a bag. The width of a tree decomposition $(T, B)$ is the size of the largest bag $B_i$ minus one. As shown in [10], for any fixed $k$, there is a polynomial time algorithm (in fact a linear time algorithm) that constructs a tree decomposition of $G$ of width at most $k$, or reports that $G$ has treewidth greater than $k$. In the following, we assume that we have a tree decomposition for $G$ of width at most $k - 1$, for some fixed $k$.

A tree decomposition $(T, B)$ is **nice** if the tree $T$ is rooted and, for every node $i \in I$, either

1. $i$ has no children ($i$ is a leaf node)
2. $i$ has exactly two children $i_1, i_2$ and $B_{i_1} = B_{i_2} = B_i$ ($i$ is a join node)
3. $i$ has a single child $j$ and $B_i = B_j \cup \{v\}$ for some vertex $v \in V(G)$ ($i$ is an introduce node)
4. $i$ has a single child $j$ and $B_i = B_j - \{v\}$ for some vertex $v \in V(G)$ ($i$ is a forget node)

The following lemma is well-known and it is straightforward to prove.

**Lemma 4.1.** There is a linear time algorithm that, given a tree decomposition for $G$, constructs a nice tree decomposition $(T, B)$ of the same width. Moreover, the tree $T$ has $O(|V|)$ nodes.

### 4.1 A Dynamic Program for Prize-Collecting Steiner Tree

We solve the problem using dynamic programming on a nice tree decomposition $(T, B)$ of width $k - 1$. For each node $i \in I$, let $V_i$ be the set of all vertices appearing in the bags corresponding to the nodes of the subtree of $T$ rooted at $i$. Let $G_i$ be the subgraph of $G$ induced by $V_i$. We start with an informal overview of the algorithm. For the purposes of exposition, we assume that there is only one optimal prize-collecting Steiner tree $T^*$ (if there are several solutions, we fix one of them). Additionally, we assume without loss of generality that the root vertex $r$ is in the bag corresponding to the root node of $T$. Now fix a node $i$ of $T$, and consider the graph $G_i$. Clearly, we would like to compute the subgraph $F$ of $T^*$ that lies in $G_i$. In order to do so, we will specify some information about this subgraph $F$. More precisely, we will specify the subgraph $H$ of $T^*$ that lies in $G[B_i]$, and a partition $\alpha$ of the vertices in $H$ induced by the connected components of $F$, i.e., each part of $\alpha$ consists of all the vertices in $H$ that are in the same connected component of $F$. (Intuitively, $\alpha$ tells us that we need to connect each part using a tree of $G_i$ and all of these trees are guaranteed to be connected to the root outside $G_i$.) It follows from the optimality of $T^*$ that $F$ is a minimum cost subgraph of $G_i$ satisfying

1. $F[B_i] = H$
2. the partition of $V(H)$ induced by the connected components of $H$ is a refinement of $\alpha$ (if two vertices $u, v$ are in the same connected component of $H$, then $u$ and $v$ are in the same part of $\alpha$)
3. the partition of $V(H)$ induced by the connected components of $F$ is $\alpha$

Let $c(i, H, \alpha)$ be the minimum cost of a subgraph $F$ of $G_i$ satisfying $(c_1) - (c_3)$. We will compute $c(i, H, \alpha)$ for all valid tuples $(i, H, \alpha)$ using dynamic programming. The cost of the optimal prize-collecting Steiner tree is equal to $\min_{H, \alpha} c(r', H, \alpha)$, where $r'$ is the root node of $T$, and the minimum is over all pairs $(H, \alpha)$ such that $H$ is a subgraph of $G[B_{r'}]$ containing $r$, and $\alpha$ has a single part containing the vertices of $H$. To see why this is true, consider a pair $(H, \alpha)$ such that $H$ is a subgraph of $G[B_{r'}]$ containing $r$, and $\alpha$ consists of a single part containing the vertices of $H$. If $T$ is a solution for the subproblem $(r', H, \alpha)$ then $T$ is a tree that contains $r$, and hence $T$ is a valid prize-collecting Steiner tree. Conversely, let $T$ be a prize-collecting Steiner tree. Let $H = T[B_{r'}]$ and let $\alpha$ be the partition of $V(H)$ induced by $T$. Since $T$ is a tree containing $r$, $H$ contains $r$. 


and $\alpha$ consists of a single part containing the vertices of $H$. Therefore $T$ is a valid solution for the subproblem $(r', H, \alpha)$.

Let $i$ be a node of $T$. Then $i$ is a leaf node, a join node, an introduce node, or a forget node, and we consider each of these cases separately. Before we describe the recurrence for $c(i, H, \alpha)$, we introduce some useful terminology (borrowed from [8]).

We can view a partition $\alpha$ as an equivalence relation over the vertices, and we write $u \equiv_\alpha v$ if $u$ and $v$ are in the same part of $\alpha$. Let $\alpha_1$ and $\alpha_2$ be two partitions of the same vertex set. We say that $\alpha_1$ is finer than $\alpha_2$ — or equivalently, that $\alpha_1$ is a refinement of $\alpha_2$ — if $u \equiv_{\alpha_1} v$ implies $u \equiv_{\alpha_2} v$. If $\alpha_1$ is finer than $\alpha_2$, we say that $\alpha_2$ is coarser than $\alpha_1$. We use $\alpha_1 \lor \alpha_2$ to denote the finest partition that is coarser than both $\alpha_1$ and $\alpha_2$ (there is a unique such partition).

**Node $i$ is a leaf node.** Let $\beta$ be the partition of $V(H)$ induced by the connected components of $H$. We have

$$c(i, H, \alpha) = \begin{cases} \text{LENGTH}(H) + \text{PENALTY}(B_i - V(H)) & \text{if } \alpha = \beta \\ \infty & \text{otherwise} \end{cases} \quad (2)$$

**Proof of Equation 2** Since $G_i = G[B_i]$, $H$ is the only subgraph satisfying $(c_1)$. If $\alpha \neq \beta$, there is no subgraph satisfying $(c_1) - (c_3)$. Otherwise, $H$ is the only subgraph satisfying $(c_1) - (c_3)$ and its cost is $\text{LENGTH}(H) + \text{PENALTY}(H)$. \qed

**Node $i$ is a join node.** Let $i_1$ and $i_2$ be the children of $i$. We have

$$c(i, H, \alpha) = \min_{\alpha_1, \alpha_2} (c(i_1, H, \alpha_1) + c(i_2, H, \alpha_2) - \text{LENGTH}(H)) \quad (3)$$

where the minimum is taken over all partitions $\alpha_1$, $\alpha_2$ of $V(H)$ such that $\alpha = \alpha_1 \lor \alpha_2$.

The intuition behind Equation 3 is the following. Let $F, F_1, F_2$ denote the restrictions of the optimal tree $T^*$ to $G_i, G_{i_1}, G_{i_2}$ (respectively). Then $F$ is the union of $F_1$ and $F_2$. Let $\alpha, \alpha_1, \alpha_2$ be the partitions of $B_i \cap V(T^*)$ induced by the connected components of $F, F_1, F_2$. Since $F_1$ and $F_2$ intersect only at $B_i \cap V(T^*)$, $\alpha = \alpha_1 \lor \alpha_2$.

Formal proofs of correctness for this, and subsequent cases, can be found in Appendix B.

**Node $i$ is a forget node.** Let $j$ be the child of $i$, and let $v$ be the vertex in $B_j - B_i$. Fix a subgraph $H$ of $G[B_j]$, and a partition $\alpha$ of $V(H)$. Let $S$ be a subset of the neighbors of $v$ that are in $B_i$. Let $E(v, S)$ denote the edges with an endpoint in $v$ and the other in $S$. Let $\alpha(v, S)$ be the partition of $V(H) \cup \{v\}$ obtained from $\alpha$ as follows: we merge each part of $\alpha$ that contains a vertex in $S$ into a single part and add $v$ to it; we add all remaining parts of $\alpha$ to $\alpha(v, S)$. We have

$$c(i, H, \alpha) = \min \left( c(j, H, \alpha), \min_{S \subseteq V(H) \cap \Gamma(v)} \left( c(j, H \cup \{v\} \cup E(v, S), \alpha(v, S) \right) \right) \quad (4)$$

where the second minimum is taken over all sets $S \subseteq V(H) \cap \Gamma(v)$ such that $S$ has at most one vertex in each part of $\alpha$.

The intuition behind Equation 4 is the following. Let $F, F'$ denote the restriction of the optimal tree $T^*$ to $G_i, G_j$ (respectively). If $T^*$ does not contain $v$, we have $F' = F$. Therefore we may assume that $T^*$ contains $v$, and thus $F'$ consists of $F$ and the edges of $E(T^*) \cap E(G_j)$ that are incident to $v$. The edges of $F'$ that are incident to $v$ have at most one endpoint in each connected component of $F$. Thus each connected component
of $F'$ that does not contain $v$ is a connected component of $F$, and the connected component of $F'$ containing $v$ consists of one or more connected components of $F$ that connect to each other via the edges incident to $v$.

**Node $i$ is an introduce node.** Let $j$ be the child of $i$, and let $v$ be the vertex in $B_i - B_j$. Let $S$ be the set of all neighbors $u$ of $v$ such that the edge $uv$ is in $H$. For each partition $\alpha'$ of $V(H) - v$, we let $\alpha'(v, S)$ be the partition of $V(H)$ obtained from $\alpha'$ as follows: we merge each part of $\alpha'$ that contains a vertex in $S$ into a single part and add $v$ to it; we add all remaining parts of $\alpha'$ to $\alpha'(v, S)$. We have

$$c(i, H, \alpha) = \begin{cases} c(j, H, \alpha) + \text{Penalty}(v) & \text{if } v \notin V(H) \\ \min_{\alpha'} (c(j, H - v, \alpha') + \sum_{uv \in H} \text{Length}(uv)) & \text{otherwise} \end{cases} \quad (5)$$

where the minimum is taken over all partitions $\alpha'$ of $V(H) - \{v\}$ satisfying

1. $S$ has at most one vertex in each part of $\alpha'$
2. $\alpha'(v, S) = \alpha$

(Note that there exists a partition $\alpha'$ that satisfies the conditions above.)

The intuition behind Equation 5 is the following. Let $F, F'$ denote the restriction of the optimal tree $T^*$ to $G_i, G_j$ (respectively). If $T^*$ does not contain $v$, we have $F = F'$. Therefore we may assume that $T^*$ contains $v$, and thus $F$ consists of $F'$ together with the edges of $E(T^*) \cap E(G[\{i\}])$ that are incident to $v$. The edges of $F$ that are incident to $v$ have at most one endpoint in each connected component of $F'$. Thus each connected component of $F$ that does not contain $v$ is a connected component of $F'$, and the connected component of $F$ containing $v$ consists of one or more connected components of $F'$ that connect to each other via the edges incident to $v$.

**Proof of Theorem 1.2:** Let $b_k$ be the number of partitions of a $k$-element set and let $s_k$ be the number of subgraphs of a graph with $k$ vertices. Since each bag has at most $k$ vertices and $\mathcal{T}$ has $O(|V|)$ nodes, there are $O(|V| \cdot b_k \cdot s_k)$ distinct subproblems. Additionally, we can evaluate each subproblem in $O(b_k^2)$ time once we have a solution for each of the subproblems it depends on. (The most expensive evaluation corresponds to a join node.) Therefore we can find an optimal prize-collecting Steiner tree in $O(b_k^2 \cdot s_k \cdot |V|)$ time. □

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**References**

[1] A. Agrawal, Philip. Klein, and R. Ravi. When trees collide: An approximation algorithm for the generalized Steiner problem on networks. *SIAM J. on Computing*, 24(3):440–456, 1995.

[2] A. Archer, M.H. Bateni, M.T. Hajiaghayi, and H. Karloff. Improved approximation algorithms for prize-collecting steiner tree and TSP. In *Proc. of the 50th Annual IEEE Symposium on Foundations of Computer Science*, pages 427–436. IEEE, 2009.

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4The Bell number $B_k$ is the number of partitions of a $k$-element set. To avoid confusion with the bags of the tree decomposition, we will use $b_k$ to refer to the $k$-th Bell number.
[3] S. Arora. Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems. *Journal of the ACM (JACM)*, 45(5):753–782, 1998.

[4] S. Arora and G. Karakostas. A $2 + \varepsilon$ approximation algorithm for the $k$-MST problem. *Mathematical Programming*, 107(3):491–504, 2006.

[5] Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM (JACM)*, 41(1):153–180, 1994.

[6] N. Bansal, A. Blum, S. Chawla, and A. Meyerson. Approximation algorithms for deadline-TSP and vehicle routing with time-windows. In *Proc. of the 36th Annual ACM Symposium on Theory of computing*, pages 166–174. ACM New York, NY, USA, 2004.

[7] Mohammad Hossein Bateni, MohammadTaghi Hajiaghayi, and Dániel Marx. Prize-collecting network design on planar graphs. Personal communication, May 2010. Submitted for publication.

[8] Mohammad Hossein Bateni, MohammadTaghi Hajiaghayi, and Dániel Marx. Approximation Schemes for Steiner Forest on Planar Graphs and Graphs of Bounded Treewidth. In *Proc. of ACM STOC*, 2010.

[9] A. Blum, S. Chawla, DR Karger, T. Lane, A. Meyerson, and M. Minkoff. Approximation algorithms for orienteering and discounted-reward tsp. *SIAM journal on computing*, 37(2):653–670, 2008.

[10] H.L. Bodlaender. A linear time algorithm for finding tree-decompositions of small treewidth. In *Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*, page 234. ACM, 1993.

[11] Glencora Borradaile, Philip Klein, and Claire Mathieu. An $O(n \log n)$ approximation scheme for Steiner tree in planar graphs. *ACM Trans. Algorithms*, 5(3):1–31, 2009.

[12] Jaroslaw Byrka, Fabrizio Grandoni, Thomas Rothvoß, and Laura Sanitá. An improved LP-based approximation for steiner tree. To appear in *ACM STOC*, 2010.

[13] K. Chaudhuri, B. Godfrey, S. Rao, and K. Talwar. Paths, Trees, and Minimum Latency Tours. In *Proc. of the 44th Annual IEEE Symposium on Foundations of Computer Science*, pages 36–45, 2003.

[14] C. Chekuri, N. Korula, and M. Pál. Improved algorithms for orienteering and related problems. In *Proc. of the 19th annual ACM-SIAM Symposium on Discrete Algorithms*, pages 661–670. Society for Industrial and Applied Mathematics, 2008.

[15] Erik D. Demaine and MohammadTaghi Hajiaghayi. Approximation schemes for planar graph problems. In *Encyclopedia of Algorithms*. Springer, 2008.

[16] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Bojan Mohar. Approximation algorithms via contraction decomposition. In *Proceedings of the 18th annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 278–287. Society for Industrial and Applied Mathematics, 2007.

[17] N. Garg. A 3-approximation for the minimum tree spanning k vertices. In *Proc. of the 37th Annual IEEE Symposium on Foundations of Computer Science*, pages 302–309. Citeseer, 1996.

[18] M.X. Goemans and D.P. Williamson. A general approximation technique for constrained forest problems. *SIAM J. on Computing*, 24(2):296–317, 1995.

[19] Shai Gutner. Elementary approximation algorithms for prize collecting steiner tree problems. *Inf. Process. Lett.*, 107(1):39–44, 2008.
A Constructing the Spanner: A Proof of Theorem 3.6

We begin by duplicating the edges of $T$, and introducing multiple copies of its non-leaf vertices in order to transform the Euler tour corresponding to $T$ into a cycle. Let $G'$ by the resulting graph. We then make this cycle the outer face $\Delta$ of $G'$.

**Definition A.1** (Definition 6.2, Borradaile et al. [11]). A path $P$ is $\varepsilon$-short in $G'$ if for every pair of vertices $x$ and $y$ on $P$, the distance from $x$ to $y$ along $P$ is at most $(1 + \varepsilon)$ times the distance from $x$ to $y$ in $G'$ (i.e., $\text{dist}_P(x, y) \leq (1 + \varepsilon)\text{dist}_{G'}(x, y)$).

**Strips.** Let $\Delta[x, y]$ denote the subpath of the outer face $\Delta$ from $x$ to $y$. We find a pair of vertices $x, y$ on $\Delta$ such that $\Delta[x, y]$ is a minimal subpath of $\Delta$ that is not $\varepsilon$-short in $G'$. Let $N$ be a shortest path from $x$ to $y$ in $G'$. The subgraph enclosed by $\Delta[x, y] \cup N$ is a **strip**. We recursively decompose the subgraph of $G'$ enclosed by $N \cup (\Delta - \Delta[x, y])$ into strips, if the graph is nontrivial.

**Lemma A.2** (Lemma 6.3, Borradaile et al. [11]). The total length of all the boundary edges of all the strips is at most $(\varepsilon^{-1} + 1) \cdot \text{LENGTH}(\Delta)$.

**Columns.** Consider a strip, with north and south boundaries $N$ and $S$ ($N$ is the shortest path we added when we created the strip). We select vertices $s_0, s_1, \ldots$ on $S$ and paths $C_0, C_1, \ldots$ inside the strip as follows. The vertex $s_0$ is the left endpoint common to $S$ and $N$, and column $C_0$ is the (empty) shortest path from $s_0$ to $N$. Now suppose that we have selected vertices $s_0, s_1, \ldots, s_{i-1}$ and columns $C_0, C_1, \ldots, C_{i-1}$. The vertex $s_i$ is the first vertex on $S$ such that the distance from $s_{i-1}$ to $s_i$ on $S$ is greater than $\varepsilon$ times the distance from $s_i$ to $N$ in the strip, and the column $C_i$ is the shortest path in the strip from $s_i$ to $N$.

**Lemma A.3** (Lemma 6.4, Borradaile et al. [11]). The sum of the lengths of the columns in a strip is at most $\text{LENGTH}(S)/\varepsilon$, where $S$ is the south boundary of the strip.
Supercolumns. Let

\[ k = \frac{1}{\varepsilon^2} \left( \frac{1}{\varepsilon} + 1 \right) \]

For each strip, we select a subset of the columns \( \{C_0, C_1, \ldots\} \) of the strip as follows. Let

\[ C_i = \{C_j \mid j \equiv i \mod k\} \]

where \( 0 \leq i \leq k - 1 \). Let \( i^* \) be the index that minimizes \( \text{LENGTH}(C_i) \). We designate the columns in \( C_{i^*} \) as the supercolumns of the strip.

Lemma A.4 (Lemma 6.5, Borradaile et al. [11]). The sum of the lengths of the supercolumns in a strip is at most \( 1/k \) times the sum of the lengths of the columns in the strip.

Lemma A.5. The sum over all strips of the length of all the supercolumns is at most \( \varepsilon \cdot \text{LENGTH}(\Delta) \).

Proof: By Lemma A.2 and Lemma A.3, the total length of the columns is at most

\[ \frac{1}{\varepsilon} \left( \frac{1}{\varepsilon} + 1 \right) \text{LENGTH}(\Delta) \]

By Lemma A.4, the total length of the supercolumns is at most

\[ \frac{1}{k} \cdot \text{(total length of columns)} \leq \frac{1}{k} \cdot \frac{1}{\varepsilon} \left( \frac{1}{\varepsilon} + 1 \right) \text{LENGTH}(\Delta) = \varepsilon \cdot \text{LENGTH}(\Delta) \]  

Mortar Graph. The mortar graph \( MG \) is a subgraph of the original graph \( G \) consisting of the edges of the given tree \( T \) (that was doubled to form the outer face \( \Delta \) of \( G' \)), the edges of the shortest paths that define the strips, and the edges of the supercolumns.

Lemma A.6. The length of the mortar graph \( MG \) is at most \( \left( \frac{3}{\varepsilon} + \varepsilon \right) \cdot \text{LENGTH}(\Delta) \).

Proof: The total length of the strips is at most \( \left( \frac{1}{\varepsilon} + 1 \right) \text{LENGTH}(\Delta) \). The total length of the supercolumns is at most \( \varepsilon \cdot \text{LENGTH}(\Delta) \). The length of \( T \) is precisely half the length of \( \Delta \), which consisted of two copies of each edge of \( T \). Thus, the total length of \( MG \) is at most \( \left( \frac{1}{\varepsilon} + 1.5 + \varepsilon \right) \cdot \text{LENGTH}(\Delta) \)  

Proposition A.7. The mortar graph \( MG \) contains every vertex of \( T \).

Bricks. A brick consists of all edges of the original graph \( G \) that are (strictly) enclosed by the boundary of some face \( f \) of the mortar graph. (Note that if an edge \( e \) on the outer face of \( G \) is not a part of \( MG \), it is “enclosed” by the outer face of \( MG \).) For each face \( f \) of the mortar graph that encloses at least one edge, there is a corresponding brick.

Lemma A.8 (Lemma 6.10, Borradaile et al. [11]). The boundary \( \partial B \) of a brick \( B \), in counterclockwise order, is the concatenation of four paths \( W_B, S_B, E_B, N_B \) such that

1. The set of edges of \( B \) − \( \partial B \) is non-empty.
2. Every vertex of \( T \) that is in \( B \) is on \( S_B \) or \( N_B \).
3. \( N_B \) is \( \varepsilon \)-short in \( B \), and every proper subpath of \( S_B \) is \( \varepsilon \)-short in \( B \).
4. There exists a number \( k' \leq k \) and vertices \( s_0, s_1, \ldots, s_{k'} \) ordered west to east on \( S_B \) such that, for each \( i \) and each vertex \( x \) on \( S_B[s_i, s_{i+1}] \), \( \text{dist}_{S_B}(x, s_i) < \varepsilon \cdot \text{dist}_B(x, N_B) \).
Definition A.9 (Definition 10.3, Borradaile et al. [11]). Let $H$ be a subgraph of $G$ such that $P$ is a path in $H$. A joining vertex of $H$ with $P$ is a vertex of $P$ that is the endpoint of an edge of $H - P$.

Lemma A.10 (Theorem 10.7, Borradaile et al. [11]). Let $B$ be a plane graph with boundary $W \cup S \cup E \cup N$, satisfying the brick properties of Lemma A.8. Let $F$ be a set of edges of $B$. There is a forest $\hat{F}$ of $B$ with the following properties:

1. If two vertices of $N \cup S$ are connected in $F$ then they are connected in $\hat{F}$.
2. The number of joining vertices of $F$ with both $N$ and $S$ is at most $\alpha(\epsilon)$, where $\alpha(\epsilon) = o(\epsilon^{-5.5})$.
3. $\text{LENGTH}(\hat{F}) \leq (1 + c\epsilon)\text{LENGTH}(F)$, for some fixed constant $c$.

Portals. Let $\theta = \theta(\epsilon)$ be a parameter that depends polynomially on $1/\epsilon$. For each brick $B$, we designate some vertices of $\partial B$ as portals, evenly spaced around $B$ as follows. Let $v_0 \in \partial B$ be the endpoint of an edge strictly enclosed by $\partial B$; we designate $v_0$ as a portal. Now suppose we have designated $v_0, v_1, \ldots, v_{i-1}$ as portals. Let $v_i$ be the first vertex on $\partial B$ such that $\text{LENGTH}(\partial B[v_{i-1}, v_i]) > \text{LENGTH}(\partial B)/\theta$. We designate $v_i$ as a portal, unless $v_0 \in V(\partial B(v_{i-1}, v_i))$, in which case we stop.

Lemma A.11 (Lemma 7.1, Borradaile et al. [11]). For any vertex $x$ on $\partial B$, there is a portal $y$ such that the $x$-to-$y$ subpath of $\partial B$ has length at most $\text{LENGTH}(\partial B)/\theta$.

Lemma A.12 (Lemma 7.2, Borradaile et al. [11]). There are at most $\theta$ portals on $\partial B$.

Portal-connected graph. For any subgraph $G''$ of the mortar graph $MG$, we construct a planar graph $B^+(G'')$ as follows. For each face $f$ of $G''$ corresponding to a brick $B$, we embed a copy of $B$ inside the face $f$, and, for each portal $v$ of $B$, we connect the copy of $v$ in the brick with the copy of $v$ on $f$ using a zero-length edge. We refer to these zero-length edges as portal edges, and we refer to $B^+(MG)$ as the portal-connected graph. Finally, any new vertex receives penalty zero (these vertices are copies of vertices of the mortar graph).

Theorem A.13. Let $F^*$ be an optimal Prize-Collecting Steiner forest in $G$. There exists a constant $\theta = \theta(\epsilon)$ depending polynomially on $1/\epsilon$ such that, for any choice of portals satisfying the condition in Lemma A.11, the corresponding portal-connected graph $B^+(MG)$ contains a forest $\hat{F}$ with the following properties:

1. $\text{LENGTH}(\hat{F}) \leq (1 + c_1\epsilon)\text{LENGTH}(F^*) + c_2\epsilon \cdot \text{LENGTH}(\Delta)$, where $c_1, c_2$ are absolute constants
2. Any two vertices of $MG$ in the same component of $F^*$ are connected by $\hat{F}$.

Proof: Let $F^*$ be an optimal tree of the Prize-Collecting Steiner forest problem in $G$, and let $S^*$ be the set of all vertices of the mortar graph $MG$ that are in $F^*$. We will follow the proof of Theorem 3.2 in Borradaile et al. [11]; there are two main steps. First, we transform $F^*$ into a solution in $G$ that only has a few joining vertices in each brick. To convert this into the desired forest $\hat{F}$ in the portal-connected graph $B^+(MG)$, we simply add edges connecting the joining vertices in each brick to the nearest portals. As there are not many joining vertices, we can connect them to the portals without significantly increasing the cost. We describe the process completely below.

First, we add the east and west boundaries of each brick; let $F_1$ be the union of $F^*$ with the east and west boundaries ($E_B$ and $W_B$) for each brick $B$. By Lemma A.5, $\text{LENGTH}(F_1) \leq \text{LENGTH}(F^*) + \epsilon \cdot \text{LENGTH}(\Delta)$. Clearly, $F_1$ connects all vertices of $S^*$ connected by $F^*$.

Next, we reduce the number of joining vertices on the north and south boundaries of each brick. Let $F_{1|B}$ be the subgraph of $F_1$ that is strictly embedded in a brick $B$ of $G$. We replace $F_{1|B}$ with the forest $F_{2|B}$ that is guaranteed by Lemma A.10. We have

$$\text{LENGTH}(F_{2|B}) \leq (1 + c_1\epsilon)\text{LENGTH}(F_{1|B})$$
Let $N$ and $S$ denote the north and south boundaries of the brick. Since any two vertices of $N \cup S$ that are connected in $F_{1|B}$ are also connected in $F_{2|B}$, it follows that $F_{2|B}$ connects all vertices of $S^*$ connected by $F_{1|B}$.

We apply this procedure for each brick in order to get a subgraph $F_2$. Since the bricks are disjoint,

$$\text{LENGTH}(F_2) \leq (1 + c_1 \varepsilon)\text{LENGTH}(F_1) \leq (1 + c_1 \varepsilon)(\text{LENGTH}(F^*) + \varepsilon \cdot \text{LENGTH}(\Delta)) = (1 + c_1 \varepsilon)\text{LENGTH}(F^*) + (\varepsilon + c_1 \varepsilon^2)\text{LENGTH}(\Delta)$$

Moreover, $F_2$ connects all vertices of $S^*$ connected by $F_1$.

Now we convert the forest $F_2 \subseteq G$ to a subgraph of $B^+(MG)$. Note that every edge of $G$ has at least one corresponding edge in $B^+(MG)$ (an edge $e$ of $MG$ has three copies: one mortar edge, and one inside each of the bricks corresponding to the two faces of $G'$ incident to $e$). For each edge $e$ of $F_2$, we select a corresponding edge of $B^+(MG)$ as follows. If $e$ is an edge of $MG$, we select the corresponding mortar edge of $B^+(MG)$. Otherwise, we select the unique edge corresponding to $e$ in $B^+(MG)$. Let $F_3$ denote the resulting subgraph of $B^+(MG)$. We have:

$$\text{LENGTH}(F_3) = \text{LENGTH}(F_2) \leq (1 + c_1 \varepsilon)\text{LENGTH}(F^*) + (\varepsilon + c_1 \varepsilon^2)\text{LENGTH}(\Delta).$$

Since $F_3$ does not connect the connected components of $F_2$, we connect it using portal edges and mortar edges as follows. Consider a brick $B$, and let $V_B$ denote the set of joining vertices of $F_3$ with $N_B \cup S_B$. For each vertex $v \in V_B$, let $p_v$ be the portal vertex that is closest to $v$, let $P_v$ be the shortest $v$-to-$p_v$ path along $\partial B$, and let $P'_v$ be the corresponding path of mortar edges. Let $e_v$ be the portal edge corresponding to $p_v$. We add $P_v$, $P'_v$, and $e_v$ to $F_3$. We apply this procedure for each brick in order to get the subgraph $\hat{F}$. First, we bound the length of $\hat{F}$.

$$\text{LENGTH}(\hat{F}) \leq \text{LENGTH}(F_3) + \sum_B \sum_{v \in V_B} (\text{LENGTH}(P_v) + \text{LENGTH}(e_v) + \text{LENGTH}(P'_v))$$

$$= \text{LENGTH}(F_3) + 2 \sum_B \sum_{v \in V_B} \text{LENGTH}(P_v) \quad \text{[LENGTH}(e) = 0, \text{LENGTH}(P'_v) = \text{LENGTH}(P_v)]$$

$$\leq \text{LENGTH}(F_3) + 2 \sum_B \sum_{v \in V_B} \text{LENGTH}(\partial B)/\theta(\varepsilon) \quad \text{[Lemma A.11]}$$

$$\leq \text{LENGTH}(F_3) + 2 \sum_B \alpha(\varepsilon)\text{LENGTH}(\partial B)/\theta(\varepsilon) \quad \text{[Lemma A.10]}$$

$$= \text{LENGTH}(F_3) + \frac{2\alpha(\varepsilon)}{\theta(\varepsilon)} \cdot \sum_B \text{LENGTH}(\delta(B))$$

$$\leq \text{LENGTH}(F_3) + \frac{4\alpha(\varepsilon)}{\theta(\varepsilon)} \cdot \text{LENGTH}(MG)$$

$$\leq \text{LENGTH}(F_3) + \frac{16\alpha(\varepsilon)}{\varepsilon \theta(\varepsilon)} \cdot \text{LENGTH}(\Delta) \quad \text{[Lemma A.6]}$$

Setting $\theta(\varepsilon) = 16\varepsilon^{-2} \alpha(\varepsilon)$ gives us

$$\text{LENGTH}(\hat{F}) \leq \text{LENGTH}(F_3) + \varepsilon \cdot \text{LENGTH}(\Delta) \leq (1 + c_1 \varepsilon)\text{LENGTH}(F^*) + (2 + c_1 \varepsilon)\varepsilon \cdot \text{LENGTH}(\Delta)$$
It remains only to show that \( \hat{F} \) connects any two vertices of \( S^* \) connected by \( F_2 \), and hence by \( F^* \). Let \( x \) and \( y \) be two vertices of \( S^* \) connected by \( F_2 \) via an \( x-y \) path \( P \) in \( F_2 \). The definition of \( F_3 \) breaks \( P \) into disjoint paths. Consider one such path \( P_i \) that is not a subpath of \( MG \). By construction, the endpoints of \( P_i \) are joining vertices. When we construct \( \hat{F} \) from \( F_3 \), we connect the endpoints of \( P_i \) to their corresponding vertices on \( MG \) via portal edges. Therefore there is an \( x-y \) path in \( \hat{F} \). \( \square \)

**Spanner.** For each brick \( B \) and for each subset \( X \) of the portals of \( B \), we find an optimal Steiner Tree for \( B \) and \( X \). The spanner \( H \) consists of all edges of these Steiner Trees together with the edges of the mortar graph \( MG \).

**Lemma A.14.** The total length of the spanner \( H \) is at most \((1+2^{1+\theta})(\frac{3}{2}+\varepsilon)\)LENGTH(\( \Delta \)).

**Proof:** As shown in Lemma 4.1 of [11], the total length of all Steiner trees is at most \( 2^{1+\theta} \cdot \) LENGTH(\( MG \)). Thus the length of \( H \) is at most \((1+2^{1+\theta}) \cdot \) LENGTH(\( MG \)). Lemma A.15 completes the proof. \( \square \)

**Lemma A.15.** The spanner \( H \) contains a prize-collecting Steiner forest \( F' \) of cost at most \((1+c_1\varepsilon)\)LENGTH(\( F^* \)) + \( c_2\varepsilon \cdot \) LENGTH(\( \Delta \)) + PENALTY(\( F^* \)), for some absolute constants \( c_1, c_2 \).

**Proof:** We will follow the proof of Lemma 4.2 in Borradaile et al. [11]. Let \( F^* \) be an optimal forest in \( G \) and let \( \hat{F} \) be the forest guaranteed by Theorem A.15. For each brick \( B \) and for each connected component \( K \) of the intersection of \( \hat{F} \) with \( B \), let \( X \) be the set of portals of \( B \) belonging to \( K \); we replace \( K \) with the optimal Steiner Tree for \( B \) and \( X \) contained in the spanner. Let \( \tilde{F} \) be the subgraph resulting from all these replacements. We have

\[
\text{LENGTH}(\tilde{F}) \leq \text{LENGTH}(\hat{F}) \leq (1+c_1\varepsilon)\text{LENGTH}(F^*) + c_2\varepsilon \cdot \text{LENGTH}(\Delta)
\]

Moreover, since \( \hat{F} \) connects all vertices of \( MG \) connected by \( F^* \), and all terminals are vertices of \( MG \) as they are connected by \( T \), it follows that \( \hat{F} \) also connects all terminals connected by \( F^* \). Hence, \( \text{PENALTY}(\hat{F}) \leq \text{PENALTY}(F^*) \); as \( \text{Cost}(\hat{F}) = \text{LENGTH}(\hat{F}) + \text{PENALTY}(\hat{F}) \), we obtain the lemma. \( \square \)

Theorem 3.6 now follows almost directly from Lemmas A.14 and A.15 as \( \text{LENGTH}(\Delta) = 2 \cdot \text{LENGTH}(T) \); simply construct the spanner \( H \) using a modified parameter \( \varepsilon' = \frac{\varepsilon}{\max\{c_1,2c_2\}} \).

### B Omitted Proofs from Section 4.1

**Proof of Equation 3:** Let \((\alpha_1, \alpha_2)\) be a pair that minimizes the right hand side of Equation 3. Let \( F_\ell \) be an optimal solution for the subproblem \((i_\ell, H, \alpha_\ell)\), where \( \ell = 1, 2 \). Let \( F = F_1 \cup F_2 \). Now we claim that \( F \) is a solution for the subproblem \((i, H, \alpha)\), i.e., it satisfies the conditions \((c_1) - (c_3)\). Clearly, \( F \) satisfies \((c_1) \) and \((c_2) \). Now let \( u \) and \( v \) be two vertices in the same part of \( \alpha \). Since \( \alpha = \alpha_1 \lor \alpha_2 \), \( u \) and \( v \) are in the same part of \( \alpha_\ell \) for some \( \ell \), and thus \( u \) and \( v \) are in the same connected component of \( F_\ell \). Thus the partition of \( V(H) \) induced by \( F \) is coarser than \( \alpha \). Since \( \alpha \) is coarser than \( \alpha_1 \) and \( \alpha_2 \), it follows that \( \alpha \) is coarser than the partition of \( V(H) \) induced by \( F \) as well. Therefore \( F \) satisfies \((c_3) \). Since \( E(H) \subseteq E(F_1) \cap E(F_2) \),

\[
\text{LENGTH}(F) \leq \text{LENGTH}(F_1) + \text{LENGTH}(F_2) - \text{LENGTH}(H)
\]

Since \( V(F) = V(F_1) \cup V(F_2) \) and \( V(G_i) = V(G_{i1}) \cup V(G_{i2}) \), we have

\[
\text{PENALTY}(V(G_i) - V(F)) \leq \text{PENALTY}(V(G_{i1}) - V(F_1)) + \text{PENALTY}(V(G_{i2}) - V(F_2))
\]

Thus

\[
c(i, H, \alpha) \leq \text{Cost}(F) \leq \text{Cost}(F_1) + \text{Cost}(F_2) - \text{LENGTH}(H) = c(i_1, H, \alpha_1) + c(i_2, H, \alpha_2) - \text{LENGTH}(H)
\]
Conversely, let \( F \) be an optimal solution for the subproblem \((i, H, \alpha)\). Let \( F_\ell \) be the restriction of \( F \) to \( G_{i, \ell} \), where \( \ell = 1, 2 \). Let \( \alpha_\ell \) be the partition induced by the connected components of \( F_\ell \). Now we claim that \( F_\ell \) is a solution for the subproblem \((i_\ell, H, \alpha_\ell)\); by construction, \( F_\ell \) satisfies \((c_3)\). Since \( F_1 \) and \( F_2 \) intersect only at \( V(H), \alpha = \alpha_1 \lor \alpha_2 \). Therefore the right hand side of the equation is at most

\[
\text{Cost}(F_1) + \text{Cost}(F_2) - \text{Length}(H) \leq \text{Cost}(F) = c(i, H, \alpha)
\]

which completes the proof.

\[\square\]

**Proof of Equation 4:** Suppose the minimum of the right hand side of the equation is achieved by an optimal solution \( F' \) for the subproblem \((j, H, \alpha)\). Since \( v \) is not in \( H, F' \) does not contain \( v \). Thus \( F' \) is a solution for the subproblem \((i, H, \alpha)\), and therefore \( c(i, H, \alpha) \) is at most the right hand side of Equation 4. Therefore we may assume that the minimum of the right hand side is achieved by an optimal solution \( F''_S \) for the subproblem \((j, H \cup \{v\} \cup E(v, S), \alpha(v, S))\). Let \( F = F''_S - E(v, S) - \{v\} \). Now we claim that \( F \) is a solution for the subproblem \((i, H, \alpha)\). By construction, \( F \) satisfies \((c_1)\) and \((c_2)\). Therefore it suffices to show that \( F \) satisfies \((c_3)\).

Note that we may assume without loss of generality that \( F''_S \) is a forest. Now suppose that \( F''_S \) has an edge \( e \) whose endpoints are in different parts of \( \alpha \). Since \( e \) is not incident to \( v \), it follows that \( e \) is in \( H \). But the partition of \( V(H) \) induced by the connected components of \( H \) is a refinement of \( \alpha \), which is a contradiction.

Let \( u \) and \( w \) be two vertices in the same connected component of \( F \). It follows that the unique path of \( F''_S \) between \( u \) and \( w \) does not pass through \( v \), and hence \( u \) and \( w \) are in the same part of \( \alpha \) (since otherwise the path between \( u \) and \( w \) has an edge with both endpoints in different parts of \( \alpha \)). Therefore the partition of \( V(H) \) induced by the connected components of \( F \) is a refinement of \( \alpha \). Conversely, let \( u \) and \( w \) be two vertices contained in the same part of \( \alpha \). Since \( \alpha(v, S) \) is coarser than \( \alpha \), \( u \) and \( w \) are in the same connected component of \( F''_S \). Let \( P \) be the unique path in \( F''_S \) between \( u \) and \( w \). If \( P - v \) is a path, \( u \) and \( w \) are connected in \( F \). Therefore we may assume that \( v \) is an internal vertex of \( P \). Let \( u', w' \) be the two neighbors of \( v \) on \( P \), where \( u' \) is on the subpath of \( P \) from \( u \) to \( v \). Since there is a path between \( u \) and \( u' \) in \( F \) (namely, the subpath of \( P \) from \( u \) to \( u' \)), it follows from the previous argument that \( u \) and \( u' \) are in the same part of \( \alpha \). Similarly, \( w \) and \( w' \) are in the same part of \( \alpha \). Therefore \( S \) has two vertices in the same part of \( \alpha \), which is a contradiction. Thus \( \alpha \) is a refinement of the partition of \( V(H) \) induced by the connected components of \( F \). It follows that \( F \) satisfies \((c_3)\) as well, and hence \( c(i, H, \alpha) \) is at most the right hand side of the equation.

Conversely, let \( F \) be an optimal solution for the subproblem \((i, H, \alpha)\). Since \( F \) is also a solution for the subproblem \((j, H, \alpha)\), it follows that \( c(i, H, \alpha) \) is at least the right hand side of the equation.

\[\square\]

**Proof of Equation 5:** Suppose that \( v \) is not in \( H \). Let \( F \) be an optimal solution for the subproblem \((i, H, \alpha)\). Since \( v \) is not in \( H, F \) is a solution for the subproblem \((j, H, \alpha)\) as well, of cost

\[
\text{Length}(F) + \text{Penalty}(V(G_j) - V(F)) = \text{Length}(F) + \text{Penalty}(V(G_i) - V(F)) - \text{Penalty}(v)
\]

Thus

\[
c(i, H, \alpha) \geq c(j, H, \alpha) + \text{Penalty}(v)
\]

Conversely, let \( F \) be an optimal solution for the subproblem \((j, H, \alpha)\). Then \( F \) is a solution for \((i, H, \alpha)\) of cost

\[
\text{Length}(F) + \text{Penalty}(V(G_i) - V(F)) = \text{Length}(F) + \text{Penalty}(V(G_j) - V(F)) + \text{Penalty}(v)
\]

Thus

\[
c(i, H, \alpha) \leq c(j, H, \alpha) + \text{Penalty}(v)
\]
Therefore we may assume that \( v \) is in \( H \). Let \( \alpha' \) be a partition of \( V(H) - \{ v \} \) satisfying the conditions above, and let \( F' \) be an optimal solution for the subproblem \( (j, H - v, \alpha') \). Let \( F = F' \cup E(v, S) \cup \{ v \} \), where \( E(v, S) \) is the set of all edges of \( H \) that are incident to \( v \). Now we claim that \( F \) is a solution for the subproblem \( (i, H, \alpha) \). By construction, \( F \) satisfies \((c_1)\) and \((c_2)\). Therefore it suffices to verify that \( F \) satisfies \((c_3)\).

Let \( u \) and \( w \) be two vertices in the same connected component of \( F \). Suppose that \( u \) and \( w \) are in \( F' \). Then \( u \) and \( w \) are in the same part of \( \alpha' \) and, since \( \alpha \) is coarser than \( \alpha' \), \( u \) and \( w \) are in the same part of \( \alpha \). Therefore we may assume that \( u \) and \( w \) are not connected in \( F' \). Thus \( u \) and \( w \) are in different parts of \( \alpha' \), each of which contains a vertex in \( S \). It follows that the two parts have merged into a single part of \( \alpha' \) \( v, S \) = \( \alpha \), and hence \( u \) and \( w \) are in the same part of \( \alpha \). Conversely, let \( u \) and \( w \) be two vertices in the same part of \( \alpha \). If \( u \) and \( w \) are in the same part of \( \alpha' \), it follows that \( u \) and \( w \) are connected in \( F' \). Therefore we may assume that \( u \) and \( w \) are in different parts \( P_1 \) and \( P_2 \) of \( \alpha' \), each of which contains a vertex of \( S \). Let \( u' \) and \( w' \) be the two vertices of \( P_1 \cap S \), \( P_2 \cap S \). Then there exists a path in \( F' \) from \( u \) to \( u' \), and a path from \( w \) to \( w' \). These two paths together with the edges \( u'v, vw' \) form a connected subgraph of \( F \). It follows that \( u \) and \( w \) are connected in \( F \), and hence \( F \) satisfies \((c_3)\).

We have

\[
\text{LENGTH}(F) = \text{LENGTH}(F') + \sum_{uv \in G} \text{LENGTH}(uv)
\]

Since \( V(F) = V(F') \cup \{ v \} \) and \( V(G_i) = V(G_j) \cup \{ v \} \),

\[
\text{PENALTY}(V(G_i) - V(F)) = \text{PENALTY}(V(G_j) - V(F'))
\]

Thus

\[
c(i, H, \alpha) \leq c(j, H - v, \alpha') + \sum_{uv \in H} \text{LENGTH}(uv)
\]

Conversely, let \( F \) be an optimal solution for the subproblem \( (i, H, \alpha) \). Without loss of generality, \( F \) is a forest. Let \( F' = F - v \), and let \( \alpha' \) be the partition of \( V(H) - \{ v \} \) induced by \( F' \). Since \( F \) is a forest, \( v \) has at most one neighbor in each part of \( \alpha' \). Now we claim that \( \alpha'(v, S) = \alpha \). Let \( T \) be any connected component of \( F \) that does not contain \( v \). It follows that \( T \) is a connected component of \( F' \) as well. Since the partition of \( V(H) \) induced by the connected components of \( F \) is equal to \( \alpha \), \( \alpha' \) contains each part of \( \alpha \) that does not intersect \( S \). Now consider the connected component \( T \) of \( F \) that contains \( v \), and let \( T_1, \ldots, T_k \) be the connected components of \( T - v \). Since each \( T_j \) is a connected component of \( \alpha' \), it follows that the part of \( \alpha'(v, S) \) containing \( v \) can be obtained by merging the parts of \( \alpha' \) induced by \( T_1, \ldots, T_k \) into a single part, and adding \( v \) to it. Thus \( \alpha'(v, S) = \alpha \).

Now we claim that \( F' \) is a solution for the subproblem \( (j, H - v, \alpha') \). By construction, \( F' \) satisfies \((c_1)\) and \((c_2)\). Additionally, it follows from the definition of \( \alpha' \) that \( F' \) satisfies \((c_3)\). We have

\[
\text{LENGTH}(F) = \text{LENGTH}(F') + \sum_{uv \in H} \text{LENGTH}(uv)
\]

As before,

\[
\text{PENALTY}(V(G_i) - V(F)) = \text{PENALTY}(V(G_j) - V(F'))
\]

Thus

\[
c(i, H, \alpha) \geq c(j, H - v, \alpha') + \sum_{uv \in H} \text{LENGTH}(uv)
\]

\[\square\]