TRAVELING WAVE SOLUTION FOR A DIFFUSIVE SIMPLE
EPIDEMIC MODEL WITH A FREE BOUNDARY

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Abstract. In this paper, we proved existence and nonexistence of traveling wave solution for a diffusive simple epidemic model with a free boundary in the case where the diffusion coefficient of susceptible population is zero and the basic reproduction number is greater than 1. We obtained a curve in the parameter plane which is the boundary between the regions of existence and nonexistence of traveling wave. We numerically observed that in the region where the traveling wave exists the disease successfully propagate like traveling wave but in the region of no traveling wave disease stops to invade. We also numerically observed that as $d$ increases the speed of propagation slows down and the parameter region of propagation narrows down.

1. Introduction. In this paper we study the following system of reaction diffusion equations with a free boundary:

\[
\begin{align*}
S_t &= dS_{xx} - SI, \quad -\infty < x < \infty, \quad t > 0, \\
I_t &= I_{xx} + SI - \gamma I, \quad -\infty < x < h(t), \quad t > 0, \\
I(x,t) &= 0, \quad x \geq h(t), \quad t > 0, \\
h'(t) &= -\mu I_x(h(t),t), \quad t > 0, \\
S(x,0) &= S_0(x), \quad -\infty < x < \infty, \\
I(x,0) &= I_0(x), \quad -\infty < x \leq h(0).
\end{align*}
\]

Here, $d, \gamma, \mu$ are non-negative constants and initial functions $S_0, I_0$ are non-negative. $S_x$ and $S_t$ are partial derivatives of the function $S(x,t)$ with respect to $x$ and $t$, respectively. In the problem above, we consider a free boundary which position is represented by $x = h(t)$. The third and the fourth equations above give conditions on the free boundary. The fourth equation is corresponding to the Stefan condition for melting ice.

The system of the first and the second reaction diffusion equations is called diffusive Kermack-McKendrick model which describes the spatial propagation of a directly transmitted disease. In this context, the unknown functions $I$ and $S$ represent the infected and the susceptible populations, respectively and the parameter

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$R_0 = \|S_0\|_\infty / \gamma > 0$ is called the basic reproduction number of the disease. This system was proposed by Källén [13] as a model for rabies epizootic in foxes.

Kermack and McKendrick, in their pioneering paper [16], investigated an SIR (Susceptible-Infected-Recovered) model which is given by a system of ordinary differential equations of three components, susceptible (S), infected (I), and recovered (R) populations. The equations for S and I components corresponds to the reaction parts of the system (1.1). They revealed that the basic reproduction number $R_0$ is a threshold value; in the case where $R_0 > 1$ there appears a peak of epidemic as time goes by and in the case where $R_0 \leq 1$ there is no such peak. Their study has lead to theoretical understanding concerning the transmission dynamics of plague in Mumbai in 1906. Intuitively, $R_0$ measures the strength of the infectious disease. The smaller $R_0$ means easier to control the disease spread.

In many area of sciences, reaction diffusion equations are widely used to describe the propagation phenomena such as invasion of spices, spread of diseases, propagation of flames, and so on. In this direction of study, the most prominent qualitative model is the following scalar reaction diffusion equation of Fisher-KPP type:

$$u_t = u_{xx} + f(u) \tag{1.2}$$

where nonlinear term $f$ satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad \text{and} \quad f(u) > 0 \text{ for } 0 < u < 1.$$  

Traveling wave solution, which is a special solution of the form $u(x, t) = \varphi(x - ct)$ for some constant $c$ and a function $\varphi$, plays an important role in the understanding of propagation phenomena which is described by the equation. Such a solution propagates in a direction with same profile $\varphi$ and same speed $c$. For the equation (1.2), it is well known that there exists $c^* > 0$ such that for any $c \geq c^*$ there is a traveling wave solution of speed $c$ and profile $\varphi$ with $\lim_{z \to -\infty} \varphi(z) = 1$ and $\lim_{z \to \infty} \varphi(z) = 0$, but for any $c < c^*$ there is no traveling wave solution of speed $c$. This $c^*$ is called the minimal speed of traveling wave. It is also well known that for any solution $u$ of the Cauchy problem for (1.2) with compactly supported initial data it holds that for any $z \in \mathbb{R}$

$$\lim_{t \to \infty} u(z + ct, t) = 0 \text{ for } c > c^* \quad \text{and} \quad \lim_{t \to \infty} u(z + ct, t) = 1 \text{ for } c < c^*. \tag{1.3}$$

The speed $c^*$ which satisfies the property (1.3) is called the spreading speed. For (1.2), the spreading speed match with the minimal speed. We refer the classical paper by Aronson and Weinberger [2] for these well known results. We note that the solution of the Cauchy problem always successfully spread to the whole space no matter how small the support of the initial data is. When we impose the free boundary condition the situation drastically changes. Du and Lin [4] considered (1.2) under the condition that

$$u_x(0, t) = 0, \quad u(h(t), 0) = 0, \quad h'(t) = -\mu u_x(h(t), t), \quad (h(0) > 0)$$

and they proved that the following dichotomy holds

$$h_\infty = \infty, \quad \lim_{t \to \infty} \|u(\cdot, t) - 1\|_{C(K)} \text{ for any bounded set } K \subset [0, \infty)$$

or

$$h_\infty < \infty, \quad \lim_{t \to \infty} \|u(\cdot, t)\|_{C([0, h(t)])} = 0,$$

here $h_\infty = \lim_{t \to \infty} h(t)$ and $\|v(\cdot)\|_{C(K)} = \sup_{x \in K} |v(x)|$. They called the former case spreading and the later case vanishing (spreading-vanishing dichotomy). They also proved that there exists $h_\ast > 0$ and $\mu_\ast > 0$ such that if $h(0) \geq h_\ast$ then
spreading occurs and if \( h(0) < h_* \) then for \( \mu > \mu^* \) spreading occurs and for \( \mu \leq \mu^* \) vanishing occurs. Since their pioneering work, this sort of problem has attracted many researchers and has been studied extensively ([3, 5, 6, 7, 8, 14, 15, 20]).

Diffusive SIR models like (1.1) are used in order to investigate the spatial transmission of the diseases (see e.g. [9, 1, 13, 12]). For diffusive Kermack-McKendrick model, the existence of traveling wave solution and the existence of minimal speed are known [13, 12] as we will explain later. To our knowledge, however, not so much is known about the spreading properties such as asymptotic spreading speed and asymptotic profile of the solution as \( t \to \infty \) for the solutions of the Cauchy or initial-boundary value problems to diffusive SIR models. On this topic we would like to refer the recent papers [10, 19], where spreading speed of the solutions for the reaction diffusion system of the prey-predator type is investigated. In [10], they state that little has been known about the long time behavior and spreading properties of such solutions, largely because of the lack of the comparison principle when the system is of prey-predator type and their results are the first theoretical results for this type of equations in this direction. Although our system (1.1) can be considered as an equation of prey-predator type, their results are not directly applicable to our system. In the case where \( d = 0 \), Ducrot and Giletti [9] investigated the spreading property of the solution to the diffusive Kermack-McKendrick model with spatially periodic coefficients. They obtained the asymptotic spreading speed for any spatial dimension and proved that the convergence of the solution to the pulsating traveling wave solution for one spatial dimension. In this case, the solution always successfully spread to the whole space no matter how small the support of the initial data \( I_0 \) is. For diffusive SIR models, free boundary seems to affect drastically same as the scalar case. Kim et al. [17] considered a diffusive SIR model with a free boundary, where the equations include positive birth rate for \( S \) component and positive death rates for all components. They gave sufficient conditions for spreading and vanishing of \( I \) component in terms of the basic reproduction number \( R_0 \), the parameter \( \mu \), and the initial front position \( h(0) \). Namely, spreading occurs if \( R_0 > 1 \) and \( h(0) \) is suitably large; on the other hand vanishing occurs if \( R_0 < 1 \) or, more interestingly, even if \( R_0 > 1 \) provided that \( h(0) \) and \( \mu \) are sufficiently small. Recently, Zhu et al. [21] investigated a diffusive SIR model with moving fronts which has advection term in addition to the model of Kim et al. They introduced the notion of the risk index \( R_{F0}^F(t) \) which is an extension of the basic reproduction number to the case of free boundary problem. They obtained sufficient conditions for vanishing and spreading in terms of the risk index \( R_{F0}^F(t) \), the parameter \( \mu \), and the initial front position \( h(0) \). Ge et al. [11] proved spreading-vanishing dichotomy result for an SIS (Susceptible-Infected-Susceptible) model with a free boundary by reducing the system to a scalar equation.

Here, let us comment on an interpretation of the free boundary of Stefan type in epidemiology. Classical Stefan condition

\[
-\lambda h'(t) = \frac{\partial \theta}{\partial x}(h(t), t)
\]

describes the effect of so called the latent heat \( \lambda \) in the process of the melting ice. Here, \( \theta \) represents the temperature. This condition means that, in order to move the phase boundary, some amount of heat (latent heat) must be released or absorbed. By analogy with the classical Stefan problem, the condition of our problem (1.1) on free boundary means that in order to move the front of the disease, some amount
of infected population must be removed, killed, or kept on the front. So our interpretation of this condition is as follows: There are field hospitals, field quarantine stations, hunters, ... on the front and they try to prevent the propagation of disease by keeping infected individuals in quarantine or by killing infected animals, plants, and so on. The capacity of the hospital or the effort of such medical facilities might be measured by the parameter $\lambda = 1/\mu$.

In this paper, we would like to find traveling wave solution for the problem (1.1), namely the solution of the form

$$\begin{align*}
S(x, t) &= \varphi(x - ct), \\
I(x, t) &= \psi(x - ct)
\end{align*}$$

with a constant $c$. Here, we note that we only consider the case where the speeds of $S$ and $I$ coincide, although there might exist a traveling wave solution with two different speeds for $S$ and $I$ components. Plugging (1.4) into (1.1) we obtain

$$\begin{align*}
&\frac{d\varphi''}{dz} + c\varphi' - \psi\varphi = 0, \\
&\frac{d\psi''}{dz} + c\psi' + \psi\varphi - \gamma\psi = 0.
\end{align*}$$

Here, $'$ and $''$ represent the first and second derivative with respect to $z = x - ct$, respectively. For the traveling wave of speed $c$, since the free boundary $h(t)$ moves with same speed it holds that $h'(t) = c$. Without loss of generality we can assume that $h(0) = 0$ and we have $h(t) = ct$. Hence, we will seek the solutions of (1.5) which satisfy

$$\begin{align*}
\varphi(-\infty) &= a, \quad \varphi'(-\infty) = 0, \quad \varphi(+\infty) = 1, \quad \varphi'(+\infty) = 0, \\
\psi(-\infty) &= \psi'(-\infty) = 0, \quad \psi(0) = 0, \quad \psi'(0) = -\frac{c}{\mu},
\end{align*}$$

for a constant $a \in (0, 1)$. Here, the parameter $a$ can be interpreted as the ratio of the number of susceptible individuals after the epidemic ends to the initial one.

For the problem without free boundary, namely, for the problem

$$\begin{align*}
S_t &= dS_{xx} - SI, \quad -\infty < x < \infty, t > 0, \\
I_t &= I_{xx} + SI - \gamma I, \quad -\infty < x < \infty, t > 0,
\end{align*}$$

the existence and nonexistence of traveling wave and the minimal speed are known. Namely, there exists non-negative solution of (1.5) for any $\gamma \in (0, 1)$, $a \in [\gamma, \gamma)$, and $c \in [2\sqrt{1 - \gamma}, +\infty)$ under the condition

$$\begin{align*}
\varphi(-\infty) &= a, \quad \varphi(+\infty) = 1, \\
\psi(-\infty) &= \psi'(-\infty) = \psi(+\infty) = \psi'(+\infty) = 0.
\end{align*}$$

Especially, $c^* = 2\sqrt{1 - \gamma}$ is the minimal speed of the traveling waves. This result was proved by Källén [13] for $d = 0$ (see the Theorem 2.1) and by Hosono and Ilyas [12] for general non-negative $d$.

In the case where $d = 0$, to show the existence of the traveling wave for the problem (1.1), we need to find non-negative triplet $(\varphi, \psi, c)$ which satisfies

$$\begin{align*}
&\varphi' = \psi'\varphi, \quad -\infty < z < 0, \\
&\varphi'' + c\psi' + \psi\varphi - \gamma\psi = 0, \quad -\infty < z < 0, \\
\varphi(-\infty) &= a, \quad \varphi(-\infty) = \psi(-\infty) = \psi'(z) = 0, \\
\varphi(0) &= 1, \quad \psi(0) = 0, \quad \psi'(0) = -\frac{c}{\mu}.
\end{align*}$$
Here, we call triplet \((\varphi, \psi, c)\) is non-negative provided \(\varphi, \psi, c\) are all non-negative and we call such non-negative triplet which satisfies the equation (1.6) semi wave for the problem (1.1) according to the literature [6, 7].

For fixed \(\gamma \in (0, 1)\), let \(a^* = a^*(\gamma)\) and \(a = a(\gamma)\) be the unique solutions of

\[
H(a^*) := a^* - 1 + \frac{\gamma}{2} (\log a^*)^2 - a^* \log a^* = 0
\]  

(1.9)
in the range of \(a^* \in (0, 1)\) and

\[a - \gamma \log a = 1,\]
in the range of \(a \in (0, 1)\), respectively. And, for fixed \(\gamma\) and \(a \in (a, \gamma)\), let \(a'\) be the unique solutions of

\[a - \gamma \log a = a' - \gamma \log a'.\]

We note that \(a < a < \gamma < a' < 1\) and \(a < a^* < \gamma\).

Now let us state our main result.

Theorem 1.1 (Nonexistence and existence of traveling wave). We assume \(d = 0\).

1. For any \(\gamma \in (0, 1)\), \(a \in [a^*(\gamma), \gamma)\) and \(\mu > 0\) there is no non-negative triplet \((\varphi, \psi, c)\) which satisfies (1.6), (1.7), and (1.8). Namely, in this parameter region, there is no traveling wave solution for (1.1).

2. For any \(\gamma \in (0, 1)\), \(a \in (a(\gamma), a^*(\gamma))\) and \(\mu = \frac{1}{1 - a + \gamma \log a}\), there exists unique non-negative triplet \((\varphi, \psi, c)\) which satisfies (1.6), (1.7), and (1.8). This \(c\) satisfies \(c < 2\sqrt{a' - \gamma}\). Namely, for these parameters there exists traveling wave for (1.1) and the traveling wave is unique up to translation.

Remark 1.1. If we set \(\lambda(a) = 1 - a + \gamma \log a\) then we have \(\lambda(a) = 0\) and \(\lambda(a)\) is monotonically increasing for \(a \in (0, \gamma)\). From the Theorem 1.1 there exists the unique traveling wave solution when \(0 < 1/\mu < \lambda(a^*)\) and there is no traveling wave solution when \(1/\mu \geq \lambda(a^*)\). The curve \(1/\mu = \lambda^*(\gamma) := \lambda(a^*(\gamma))\) divides the parameter space into two regions; the regions where the traveling wave exists and the region where there is no traveling wave. See Figure 1.1.

Remark 1.2. In the case where \(a \geq \gamma\), there also no traveling wave solution. Since the origin \(O\) in the phase plane for the system (2.5) below is stable in this case, there is no chance to exists a solution for (2.5) which satisfies (2.6). We would like to also remark that in the case where \(\gamma > 1\), all solutions for the problem (1.1) vanishes in the following sense: \(\lim_{t \to \infty} ||I(\cdot, t)||_{C(0, h(t))} = 0\). It can be proved in the same argument with the Theorem 3.1 of Kim et al. [17].

The organization of the paper is as follows: We will prove Theorem 1.1 in §2. We will study the problem numerically in §3. We numerically examine the spreading property for the solutions of (1.1) in the case where \(d \geq 0\). We will discuss the meaning of our result and future problems in §4.

2. Proof of the result. Following the idea of Källén [13], we reduce the equations (1.5) into a system of first order equations. We define

\[b(\varphi, \psi, \psi') = \psi + \varphi - \gamma \log \varphi + \frac{\psi'}{c},\]

then if \((\varphi, \psi)\) satisfies (1.6) we have

\[
\frac{db}{dz} = \psi' + \varphi' - \gamma \frac{\varphi'}{\varphi} + \frac{\psi''}{c} = 0.
\]
So, if the solution of (1.6) satisfies the condition (1.7), then we have
\[ b = a - \gamma \log a. \]
Hence, \((\varphi, \psi)\) which satisfies (1.6) and (1.7) solves the problem:
\[
\begin{align*}
\varphi' &= \frac{\varphi\psi}{c}, & -\infty < z < 0, \\
\psi' &= c(a - \gamma \log a - \psi + \gamma \log \varphi), & -\infty < z < 0,
\end{align*}
\]
and
\[
\varphi(-\infty) = a, \quad \psi(-\infty) = 0. \tag{2.2}
\]
In order that the solution \((\varphi, \psi)\) of (1.6) and (1.7) satisfies (1.8), it must hold that
\[
\varphi(0) = 1, \quad \psi(0) = 0 \tag{2.3}
\]
and
\[
1 - \frac{1}{\mu} = b = a - \gamma \log a, \quad \text{namely,}
\]
\[
\mu = \frac{1}{1 - a + \gamma \log a}. \tag{2.4}
\]
Conversely, if \((\varphi, \psi)\) satisfies (2.1), (2.2), and (2.3) then it solves (1.6), (1.7), and (1.8) with \(\mu\) which satisfies (2.4). Therefore, in order to find non-negative solution for (1.6), (1.7), and (1.8), we need to find non-negative solution \((\varphi, \psi, c)\) for (2.1), (2.2), and (2.3).

We rewrite the problem (2.1) further. Let \(v = \psi, u = \log \varphi - \log a\) and \(\tau = z/c\) then we have
\[
\begin{align*}
\frac{du}{dT} &= v, & -\infty < \tau < 0, \\
\frac{dv}{dT} &= c^2(\gamma u - a(e^u - 1) - v), & -\infty < \tau < 0,
\end{align*}
\]
and
\[
\begin{align*}
v(-\infty) &= u(-\infty) = 0, \\
v(0) &= 0, \quad u(0) = \log \frac{1}{a}.
\end{align*}
\tag{2.6}
\]
We note that \((u, v) = (0, 0)\) and \(\left(\log \frac{\varphi}{\varphi}, 0\right)\) are equilibrium of (2.5). We also note that the solution \((u, v)\) of (2.5) satisfies
\[
\frac{dv}{du} = \frac{c^2(\gamma u - a(e^u - 1) - v)}{v}. \tag{2.7}
\]
In order to find a traveling wave solution, we need to seek the orbit of (2.5) (or (2.7)) which connects the origin \((0,0)\) and the point \((\log \frac{1}{a},0)\) in the first quadrant of \(uv\) phase plane. We set

\[
D_1 = \{(u,v) \mid 0 < v < \gamma u - a(e^u - 1)\},
\]

\[
D_2 = \{(u,v) \mid v > \gamma u - a(e^u - 1), 0 < u < \log \frac{1}{a}\},
\]

\[
\Gamma = \{(u,v) \mid v = \gamma u - a(e^u - 1), v > 0\},
\]

\[
J = \{(u,0) \mid u > \log \frac{a'}{a}\}, \quad L = \{\left(\log \frac{1}{a}, v\right) \mid v > 0\},
\]

\[
O = (0,0), \quad P = (\log \frac{a'}{a}, 0), \quad Q = (\log \frac{1}{a}, 0).
\]

The linearized matrix around the origin \(O\) of (2.5) is

\[
\begin{pmatrix}
0 & 1 \\
c^2(\gamma - a) & -c^2
\end{pmatrix}
\]

and it has eigenvalues

\[
\lambda_{\pm}(c,a) = \frac{-c^2 \pm \sqrt{c^4 + 4c^2(\gamma - a)}}{2}
\]

and eigenvectors \((1, \lambda_{\pm})^T\). Hence, the origin is a saddle for any \(c > 0\) if \(\gamma > a\). Hereafter, we only consider the case where \(\gamma > a\). There exists one-dimensional unstable manifold of \(O\):

**Lemma 2.1** (Existence of unstable manifold). *There exists the unique unstable orbit of (2.5) which starts from \((0,0)\) and enter into the first quadrant. This orbit satisfies for small \(u\)

\[
v(u) = \lambda_{+}(c,a)u + o(u).
\]

Since this lemma is standard, we omit the proof of it.

On the other hand, \(P\) is a stable node for \(c \geq 2\sqrt{a' - \gamma} - \gamma\); a stable spiral for \(0 < c < 2\sqrt{a' - \gamma}\). We also note that if \(c \geq 2\sqrt{a' - \gamma}\) then there exists a connecting orbit between \(O\) and \(P\):
Theorem 2.1. (Källén [13]) For any $\gamma \in (0, 1), a \in (a, \gamma]$ and $c \in [2\sqrt{a' - \gamma}, \infty)$, there exists the unique non-negative solution $(u, v, c)$ of (2.9).

\[
\begin{align*}
\frac{dv}{d\tau} &= c^2(\gamma u - a(e^u - 1) - v), \quad -\infty < \tau < \infty, \\
\frac{du}{d\tau} &= v, \quad -\infty < \tau < \infty, \\
v(-\infty) &= v(\infty) = 0, \\
u(-\infty) &= 0, \quad u(\infty) = \log \frac{a'}{a}.
\end{align*}
\]

(2.9)

Källén [13] proved corresponding result for the equation (2.1) in the case where $a' = 1$ and $a = a$. The theorem above can be proved in completely same manner as Källén’s argument. So we skip the proof of this theorem.

We set

\[F(u, v) = \frac{1}{2}v^2 - c^2\left(\frac{\gamma}{2}u^2 - a(e^u - u)\right)\]

then for non-negative solution $v$ of (2.7) we have in the first quadrant

\[
\frac{d}{du}F(u, v) = v \frac{dv}{du} - c^2(\gamma u - a(e^u - 1)) = -c^2v \leq 0.
\]

(2.10)

Now let us prove the first part of the Theorem 1.1.

Proof. (the first part of the theorem) If non-negative $(u, v)$ satisfies (2.5) then by (2.10) and the conditions at $\tau = -\infty$ and $\tau = 0$

\[c^2a = F(0, 0) \geq F(\log \frac{1}{a}, 0) = -c^2\left(\frac{\gamma}{2}(\log a)^2\right) + c^2 + c^2a \log a,
\]

namely,

\[H(a) = a - 1 + \frac{\gamma}{2}(\log a)^2 - a \log a \geq 0
\]

(2.11)

must hold. Let $a^*$ be the unique zero of $H(a)$ in $(0, 1)$, then $H(a) > 0$ for $0 < a < a^*$ and $H(a) < 0$ for $a^* < a < 1$. Hence if $a > a^*$ there is no non-negative solution of (2.5) and (2.6). See the Figure 2.1 (left). In the case where $a = a^*$, $F(0, 0) = F(\log \frac{1}{a}, 0)$ holds and hence from (2.10), $c = 0$ or $v \equiv 0$. From (2.5) and (2.6) in both cases we have $v \equiv 0$. So there is no traveling wave solution.

Hereafter in this section, we assume that

\[0 < \gamma < 1, \quad a < a < a^* \quad \text{and} \quad 0 < c < 2\sqrt{a' - \gamma}.
\]

(2.12)

In this parameter region, as we already noted, $P$ is a stable spiral and $O$ is a saddle. We note that due to the Lemma 2.1 and Theorem 2.1, we can see that there is no orbit connecting the $O$ and $Q$ in the case where $c \geq 2\sqrt{a' - \gamma}$. Hence, in order to find traveling wave it is enough to consider the case where (2.12) holds.

Now let us prove the second part of the Theorem 1.1.

Proof. (The second part of the theorem) Observing the vector field we can see that the orbits $v(u)$ of (2.7) is increasing in $D_1$ and decreasing in $D_2$. See the Figure 2.1 (right) and the Figure 3.1 (left). For fixed $c$, let $T_c$ be the unstable orbit of the origin which enters into the first quadrant. The orbit $T_c$ enters into the region $D_1$, crosses $\Gamma$, and enters into the region $D_2$. We define the function $G$ by

\[G(u) = \frac{\gamma}{2}u^2 - a(e^u - u),\]
then we have \( F(u, v) = \frac{1}{2}u^2 - c^2G(u) \). Let \( R = (u_R, 0) \) be the point on \( J \) which satisfies \( F(0, 0) = F(u_R, 0) \), namely, \( u_R > 0 \) is the unique solution of

\[
a + G(u_R) = 0. \tag{2.13}
\]

We note that \( u_R \) does not depend on \( c \) and \( u_R > \log \frac{1}{a} \) since we assume \( a < a^* \). The inequality (2.10) means that \( F(u, v) \) is decreasing along the orbit in the first quadrant. Hence the orbit \( T_c \) is between two level curves of \( F(u, v) = F(0, 0) \) and \( F(u, v) = F(\log \frac{1}{a}, 0) \) and so eventually reaches \( J \) at a point, say \( S = (u_S(c), 0) \), and \( \frac{2}{a} < u_S(c) < u_R \) holds.

Integrating (2.10) along \( T_c \), we have

\[
F(u_S(c), 0) - F(0, 0) = -c^2 \int_0^{u_S(c)} v(u)du
\]

and hence

\[
a + G(u_S(c)) = \int_0^{u_S(c)} v(u)du. \tag{2.14}
\]

The level line \( F(u, v) = F(0, 0) \) is given by

\[
v = c\sqrt{2(a + G(u))}, \quad (0 < u < u_R)
\]

in the first quadrant. Since the orbit \( T_c : v = v(u) \) is below this level line, we have \( v(u) \leq c\sqrt{2(a + G(u))} \). Hence we obtain

\[
0 \leq \int_0^{u_S(c)} v(u)du \leq c \int_0^{u_R} \sqrt{2(a + G(u))}du \to 0 \text{ as } c \to 0.
\]

From this convergence, (2.13), and (2.14), we have \( u_S(c) \to u_R \) as \( c \to 0 \). Therefore, for sufficiently small \( c > 0, \) \( u_S(c) > \log \frac{1}{a} \) and \( T_c \) crosses the half line \( L \). Let \( \nu(c) \) be the \( v \) coordinate of the intersection of \( T_c \) and \( L \). We set

\[
c_* = \sup\{0 < c < 2\sqrt{a^* - \gamma} \mid \nu(c) > 0\}.
\]

First, we note that \( c_* < 2\sqrt{a^* - \gamma} \). If not, there exist sequences \( \{c_n\}_{n=1,2,...} \), \( \{T_{c_n}\}_{n=1,2,...} \) and a curve \( T \) such that \( c_n \to 2\sqrt{a^* - \gamma} \) and \( T_{c_n} \) uniformly converges to \( T \) as \( n \to \infty \). Then \( T \) starts from the origin and crosses \( L \). Moreover, \( T \) is an unstable orbit of the origin for (2.7) with \( c = 2\sqrt{a^* - \gamma} \). By the Lemma 2.1 and the Theorem 2.1, however, the unique unstable orbit of the origin reaches the equilibrium \( P \) and is away from \( L \) when \( c = 2\sqrt{a^* - \gamma} \). It is a contradiction. Second, we note that \( \nu(c_n) = 0 \). Otherwise, by the continuous dependence of the orbit on the parameter \( c \), there exists \( c > c_* \) such that \( T_c \) intersects \( L \). It contradicts with the definition of \( c_* \). Hence, \( T_{c_*} \) connects the origin and the point \( Q \). So, there exists a traveling wave with speed \( c_* \).

Now, let us prove the uniqueness of the traveling wave. Let us assume that there exists \( c_1, c_2 (c_1 < c_2) \) and \( v_1(u), v_2(u) \) such that for \( i = 1, 2 \), \( v_i(u) \) satisfies (2.7) with \( c = c_i \) and \( v_i(0) = v_i(\log \frac{1}{a}) = 0, v_i(u) > 0 \) for \( 0 < u < \log \frac{1}{a} \).

We claim that

\[
v_2(u) > v_1(u) \text{ for } u \in (0, \log \frac{1}{a}). \tag{2.15}
\]

In fact, by the Lemma 2.1, (2.8), and the assumption \( c_1 < c_2 \), we have \( v_2(u) > v_1(u) \) for small positive \( u \). If \( v = v_1(u) \) and \( v = v_2(u) \) intersect at some point in \( D_1 \), then \( v_1'(u) \geq v_2'(u) \) at this point. By \( c_1 < c_2 \) and (2.7), however, \( v_1'(u) < v_2'(u) \) at this point. Hence, these two orbits do not intersect in the region \( D_1 \). If these two orbits intersect on the curve \( \Gamma \), then at this point \( v_2 - v_1 = 0, \frac{d}{du}(v_2 - v_1) = 0 \).
$0, \frac{d^2}{dv^2}(v_2 - v_1) < 0$ hold. It contradicts with the fact that $v_2 > v_1$ in $D_1$. Hence, these two orbits do not intersect on $\Gamma$. So, near $\Gamma$, it holds that $v_2 > v_1$. Let us show that $v_2(u) > v_1(u)$ near $Q$ in $D_2$. Near $Q$ there exists inverse functions $u_i(v)$ of $v_i(u)$ ($i = 1, 2$) and these functions satisfy

$$\frac{du_i}{dv} = \frac{1}{c_i^2 G''(u_i) - v}, \quad (i = 1, 2).$$

So at the point $Q$,

$$u_2 - u_1 = d(u_2 - u_1) = 0, \quad \frac{d^2(u_2 - u_1)}{dv^2} = \frac{c_1^2 - c_2^2}{c_1^2 c_2^2} \frac{1}{G'(\log \frac{1}{a})} > 0.$$

It means that $v_2(u) > v_1(u)$ near $Q$ in $D_2$. As we saw that $v_2(u) > v_1(u)$ near $\Gamma$. If these two orbits intersect in $D_2$, then at the most left intersection the order of $v_1$ and $v_2$ changes and there exists some interval of $u$ such that in this interval $v_1(u) > v_2(u)$. Since $v_2(u) > v_1(u)$ near $Q$ in $D_2$, there must be at least one more intersection. At this intersection, $v_1'(u) \leq v_2'(u)$ must hold. By $c_1 < c_2$ and (2.7), however, $v_1'(u) > v_2'(u)$ at this point. This contradiction shows that there is no intersection in $D_2$. Hence we obtain (2.15).

From (2.15), we have

$$\int_0^{\log \frac{1}{a}} v_1(u) du < \int_0^{\log \frac{1}{a}} v_2(u) du.$$

On the other hand, since $u_S(c_1) = u_S(c_2) = \log \frac{1}{a}$, by (2.14) we have

$$\int_0^{\log \frac{1}{a}} v_1(u) du = a + G(\log \frac{1}{a}) = \int_0^{\log \frac{1}{a}} v_2(u) du.$$

It is a contradiction. Hence, we have the uniqueness of the traveling wave.

3. Numerical results. In this section, we study the problem numerically. First, we compute the shape of traveling wave when diffusion coefficient $d = 0$ by using numerical shooting for (2.5). We fix $\gamma$ and $a$. Then we numerically compute the unstable orbit $T_\epsilon$ of the origin with varying $c$ and find $c$ such that $T_\epsilon$ reaches the point $Q$ only through the first quadrant (Figure 3.1 (left)). In the Figure 3.1 (right), we show the profile of traveling wave when $\gamma = 0.3$ and $a = 0.05$.

![Figure 3.1](image-url)
Second, we numerically compute the solutions for (1.1) in the cases where both $d = 0$ and $d > 0$. Difficulty of numerical treatment for (1.1) lies on the presence of free boundary. Many numerical methods for free boundary problems with Stefan condition are proposed. Among of them, we used the method of reaction diffusion approximation [18]. Let us briefly explain this method for the following free boundary problem (one phase Stefan problem):

$$
\begin{cases}
\theta_t = \Delta \theta + f(\theta) & \text{in } \Omega_+(t), \\
\theta = 0, \quad \lambda V_n = -\frac{\partial \theta}{\partial n} & \text{on } \Gamma(t), \\
\theta(x, 0) = \theta_0(x) & \text{in } \Omega_+(t).
\end{cases}
$$

(3.1)

Here, $\Delta$ is the Laplace operator, $\Omega_+(t)$ a time dependent domain with boundary $\Gamma(t)$, $V_n$ the outward normal velocity of $\Gamma(t)$, $n$ the unit outward normal vector, and $\frac{\partial}{\partial n}$ normal derivative. This problem can be written in so called enthalpy formulation as follows:

$$
\begin{cases}
z_t = \Delta \phi(z) + f(\phi(z)), & x \in \Omega, t > 0, \\
z(x, 0) = z_0(x), & x \in \Omega, \\
\frac{\partial z}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{cases}
$$

(3.2)

Here, $\phi$ is given by

$$
\phi(z) = \max(z - \lambda, 0).
$$

(3.3)

We assume that the domain $\Omega$ contains $\Omega_+(t)$ for $t > 0$. For a weak solution $z$ of the equation (3.2), if we set

$$
\Omega_+(t) = \{x|\phi(z(x, t)) > 0\}, \quad \Gamma(t) = \partial \Omega_+(t)
$$

and

$$
\theta = \phi(z)|_{\Omega_+(t)}
$$

then we can see that under some appropriate conditions $\theta$ satisfies (3.1), the strong form of one phase Stefan problem. Here, we note that the boundary motion law in the problem (3.1) can be recovered by adapting similar argument of deriving well known Rankine-Hugoniot condition for the conservation law to the weak formulation of the problem (3.2) and (3.3). So to simulate the solution of (3.1), we need to simulate the solution of (3.2) with (3.3). The later problem is a degenerate parabolic problem but there is no free boundary. For such sort of degenerate parabolic problems, Murakawa [18] proposed a smart approximation algorithm which is a kind of reaction diffusion approximation. For small $\varepsilon > 0$, he considered the problem

$$
\begin{cases}
u_t = \Delta u + f(u) - \frac{1}{\varepsilon}(u - \phi(u + v)), & x \in \Omega, t > 0, \\
v_t = \Delta v + \frac{1}{\varepsilon}(v - \phi(u + v)), & x \in \Omega, t > 0, \\
u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{cases}
$$

(3.4)

If we set $z = u + v$ then we have

$$
z_t = u_t + v_t = \Delta u + f(u).
$$

By the effect of the penalty, for the solution $(u, v)$ of (3.4) we can expect that $u$ is very close to $\phi(u + v) = \phi(z)$ for small $\varepsilon > 0$. Hence, (3.2) is expected to hold approximately. In fact, Murakawa proved that the convergence of the solution for (3.4) to the solution of (3.2) as $\varepsilon$ goes to zero. He also examined the effectiveness of his method by applying it to many degenerate parabolic problems including one phase and two phase Stefan problems.
We would like to note that the numerical treatment of (3.4) is very easy. We also would like to emphasize that this method is not only theoretically reliable but also carefully well tested. So, we decided to apply this method to our problem.

Our equation and boundary conditions for \( I \) can be written in the form of (3.2) with given \( S(x,t) \). Coupling the equation of \( S \) and Murakawa's approximate equation, which is written in the form of (3.4), for this enthalpy formulation, we get a system of approximate equations. We numerically compute the solution of this system by using a standard finite difference approximation.

In our simulation, we set \( \Omega = (0, 400) \) and we impose the Neumann boundary condition for \( z \) (solution of the enthalpy formulation of \( I \) equation) and \( S \) on the boundary of \( \Omega \).

In the Figure 3.2 (top left), (middle left), (bottom left) and (middle right) and in the Figure 3.3, we show several examples of evolution of the solutions. We can see that there are two types of evolution depending on the parameters; in the first case, the disease successfully spreads to the whole world (Figure 3.2), it is called spreading case; in the second case, the disease stops to invade and eventually dies out (Figure 3.3), it is called vanishing case.

We plot the shape of solutions, which start from the same initial data, at six different moments (at \( t = kT(k=0,1,\ldots,5,T=100) \)) for \( d = 0, 0.1, \) and 1 in the Figure 3.2 (top left), (middle left), and (bottom left), respectively. In the spreading case, the solution behaves like traveling wave. Namely, the solution travels from left to right with almost same shape and speed. In order to compare the shapes of the solution and the traveling wave, in the Figure 3.2 (top right) we show the profile of the traveling wave which we get by numerical shooting as we explained above.

We also observe that the spreading speed slows down as the diffusion coefficient for \( S \) increases. In the Figure 3.4, we plot the estimated spreading speed of the solutions for \( \gamma = 0.5 \) and various \( \lambda \) (left) and for \( \lambda = 0.4 \) and various \( \gamma \) (right). The horizontal axis is \( \lambda \) in the former figure and \( \gamma \) in the later. The points \( \square, \circ, \) and \( * \) indicate \( d = 0, 0.1, \) and 1, respectively.

In the Figure 3.5, we show vanishing and spreading diagrams for \( d = 0 \) (top left), 0.1 (top right), and 1 (bottom). In these diagrams horizontal and vertical axes are \( \gamma \) and \( \lambda = 1/\mu \), respectively. We note that \( \gamma = 1/R_0 \) since we use \( S_0 \equiv 1 \) as the initial function for \( S \). From the numerical simulations which start from the same initial data, we plot \( \times \) at the point of vanishing and + at the point of spreading in these diagrams. We also draw a graph of

\[
\lambda = \lambda^*(\gamma) := 1 - a^*(\gamma) + \gamma \log a^*(\gamma) \quad (3.5)
\]

in these diagrams. Here, this function is obtained from (2.4) and \( a^*(\gamma) \) is given by (1.9). We note that \( \lambda^*(\gamma) \) is monotonically decreasing function and \( a^*(\gamma) \rightarrow 0, \lambda^*(\gamma) \rightarrow 1 \) as \( \gamma \rightarrow 0 \). By the Theorem 1.1, there is no traveling wave solution in the region above and on this curve but there exists unique traveling wave solution below this curve, in the case where \( d = 0 \). See the Remark 1.1 and the left figure of Figure 1.1. We can observe that the parameter space (\( \gamma \lambda \) plane) is divided into two regions, one is “spreading case region (\( \times \))” and the other is “vanishing case region (\( + \))”. In the case where \( d = 0 \), the boundary between the vanishing and the spreading regions seems to agree with the graph \( \lambda = \lambda^*(\gamma) \). We also observe that the spreading case region monotonically narrows down as \( d \) increases from 0 to 1.

In our numerical experiments, we do not find any evidence that the spreading property depends on \( h(0) \). In the Figure 3.2 (middle right) and the Figure 3.3...
We note that $\lambda^*(0.3)$ is about 0.267682. The Figure 3.2 (middle right) indicates the evolution of the solution when the initial data $I_0$ has the support only at the origin and $\lambda = 0.254297$. From this figure we see that the solution propagates like traveling wave for $\lambda$ slightly below $\lambda^*$, even if the $h(0)$ is very small. The Figure (right), we plot the evolution of the solutions in the case where $d = 0$ and $\gamma = 0.3$. We note that $\lambda^*(0.3)$ is about 0.267682. The Figure 3.2 (middle right) indicates the evolution of the solution when the initial data $I_0$ has the support only at the origin and $\lambda = 0.254297$. From this figure we see that the solution propagates like traveling wave for $\lambda$ slightly below $\lambda^*$, even if the $h(0)$ is very small. The Figure...
Figure 3.3. Evolution of the solutions in vanishing case ($\gamma = 0.3$, $d = 0$): (left) $\lambda = 0.027$, (right) $\lambda = 0.281066$, support of $I_0$ is fairly large.

3.3 (right) indicates the evolution of the solution when the initial data $I_0$ has fairly large support and $\lambda = 0.281066$. From this figure we see that the solution vanishes for $\lambda$ slightly above $\lambda^*$ even if $h(0)$ is fairly large.

Figure 3.4. Estimated speed of propagation: $\square d = 0$, $\bigcirc 0.1$, $\ast 1$, (left) speed of propagation versus $\lambda$, $\gamma = 0.5$, (right) speed of propagation versus $\gamma$, $\lambda = 0.4$.

4. Discussion. In this paper, we proved existence and nonexistence of traveling wave solution for the diffusive simple epidemic model with a free boundary (1.1) in the case where the diffusion coefficient $d$ of susceptible population $S$ is zero and the basic reproduction number $R_0 = 1/\gamma$ is greater than 1. We obtained a curve in the parameter plane ($\gamma\lambda$ plane) which divides the plane into two regions; in one region there exists unique traveling wave and in the other region there is no traveling wave. This curve is implicitly given by the relation (3.5) and (1.9). We numerically observed that in the region where the traveling wave exists the disease successfully propagate like traveling wave but in the region of no traveling wave the disease eventually vanish. We also observed that as $d$ increases the speed of propagation slows down and the region of vanishing becomes broader. It suggests that the diffusion of susceptible population helps to prevent the propagation of the
disease. We would like to note that in the case when there is no free boundary the speed of the traveling wave does not depend on $d$ [12]. This is a difference between the models with and without free boundary. Our numerical observation also suggests that it is possible to prevent the propagation of disease by putting appropriate effort only on the front of the invasion even if $R_0 > 1$. And if we know the basic reproduction number of the disease, the effort necessary for prevention might be measured by the function $\lambda^*(\gamma)$ in (3.5).

It is an interesting problem to evaluate the spreading speed of the solution and to clarify its dependence on the parameters, especially on $d$ and $\lambda$. We expect that as $\lambda$ goes to zero the traveling wave solution approaches to the traveling wave solution with minimal speed for the problem without free boundary. We are also interested in the existence of traveling wave solution in the case where $d > 0$. It is, however, much more involved problem. In this paper we investigate the vanishing case only numerically. The characterization of the vanishing case is also an interesting problem. These things and the mathematical justification of our numerical observation are our ongoing research project.

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