PROJECTIVE MODULE SYSTEMS

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Abstract. We develop the basic theory of projective modules and splitting in the more general setting of systems. This enables us to prove analogues of classical theorems for tropical and hyperfield theory. In this context we prove a Dual Basis Lemma and develop Morita theory. We also prove a Schanuel’s Lemma as a first step towards defining homological dimension.

1. Introduction

In this paper we continue the project, as surveyed in [35], of proving general structure theorems which encompass classical algebra as well as the algebraic theories of supertropical algebra, hyperfields, and fuzzy rings, in the context of what we call a system. The connection with other tropical algebraic approaches is given briefly in [35], with more categorical detail in [20]. In Examples 2.21 and 3.5 we will state explicitly for the reader’s convenience how the systemic version of morphism and its basic properties apply to tropical mathematics, hyperfields, and fuzzy rings.

As indicated in Examples 2.15, this “systemic” theory encompasses most algebraic approaches to tropical mathematics except idempotent algebra as described in Example 2.15(vii), which cannot handle the customary algebraic structure theory precisely because the rule $a + a = a$ does not recognize repetition of values which are so important in valuation theory and in the combinatorics of tropical geometry.

The classical theory involves categories in two main ways, the fundamental structure (such as a ring, perhaps an integral domain or an algebra), which can be viewed as a small category, in which the kernels of homomorphisms are ideals, and then the category of modules over this structure, in which the kernels are submodules. Projective modules play a major role.

The systemic theory often involves structures such as semirings, so structurally the kernels should be congruences. These are considerably more complicated than ideals or submodules, and cannot be viewed as morphisms in the original category. This discrepancy is dealt with in the literature [3, 7, 13], but there is a dichotomy between the category of modules and the category of congruences.

Our overall goal is to develop the tools for a systemic homology theory which could apply to the classical setting, tropical mathematics, hyperfields, and fuzzy rings. The key to this puzzle may well lie with projective modules, studied here, which are of utmost importance in ring theory. Projective modules over semirings, whose theory is analogous to classical exact sequences and module theory, appear in [12, Chapter 17], [21], [36] and have been studied rather intensively over the years, [5, 15, 16, 23, 24, 29]. In order to include hyperfields, we need to consider a more general version, $\preceq$-projectivity, which is given in Definition 3.1 based on “$\leq$-splitting” in Definition 3.13.

The strong decomposition results given in [16] rely on a correspondingly restrictive definition and ironically are too strong to lead to a viable homology theory. (See [16] Example 4.6) for a projective module over a semiring which is not a summand of a free module.) Our main results here are more inclusive and accordingly weaker, containing a generalization of splitting and characterizations of $\preceq$-projectivity and $\succeq$-projectivity in §4 and §4.1.1 and a $\preceq$-version of the Dual Basis Lemma and Schanuel’s Lemma. We also cast Morita theory in this context. First we prove a systemic version of splitting.

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Theorem A (Theorem 3.10). Let \( \pi : M \to N \) be a morphism. If \( \nu \preceq \)-splits \( \pi \), then \( M \) is the \( \preceq \)-direct sum of \( M_1 := \pi(M) \) and \( M_2 := (1_M(-) \nu \pi)(M) \) with respect to the following \( \pi_1 = \pi, \nu_1 = \nu, \pi_2 = (1_M(-) \nu \pi), \nu_2 = 1_{M_2} \).

This leads us to two systemic versions of projectivity, the \( \preceq \)-version (preferred) and the \( \succeq \)-version. Each satisfies the basic equivalent properties:

Proposition B (Proposition 4.3). The following are equivalent for a \( T \)-module system \( \mathcal{P} \):

(i) \( \mathcal{P} \) is \( \succeq \)-projective.
(ii) Every \( \preceq \)-onto morphism to \( \mathcal{P} \) \( \preceq \)-splits.
(iii) There is a \( \preceq \)-onto morphism from a free system to \( \mathcal{P} \) that \( \preceq \)-splits.
(iv) The functor \( \text{Hom}(\mathcal{P}, -) \) sends \( \preceq \)-onto morphisms to \( \preceq \)-onto morphisms.

Proposition C (Proposition 4.5). The following are equivalent for a \( T \)-module system \( \mathcal{P} \):

(i) \( \mathcal{P} \) is \( \succeq \)-projective.
(ii) Every \( \succeq \)-onto morphism to \( \mathcal{P} \) \( \succeq \)-splits.
(iii) There is a \( \succeq \)-onto morphism from a \( \preceq \)-free system to \( \mathcal{P} \) that \( \succeq \)-splits.
(iv) The functor \( \text{Hom}(\mathcal{P}, -) \) sends \( \succeq \)-onto morphisms to \( \succeq \)-onto morphisms.

Proposition D (Proposition 4.16 \( \preceq \)-Dual Basis Lemma). Suppose \( \langle \mathcal{P}, \mathcal{T}_P, (-) \preceq \rangle \) is \( \preceq \)-generated by \( \{a_i \in P : i \in I\} \). Then \( \mathcal{P} \) is \( \preceq \)-projective if and only if there are \( \preceq \)-morphisms \( g_i : P \to A \) such that for all \( a \in A \), we have \( a \preceq \sum g_i(a)a_i \), where \( g_i(a) = 0 \) for all but finitely many \( i \).

These are tied in with \( \preceq \)-idempotent and \( \preceq \)-von Neumann regular matrices in Proposition 4.13 and Corollary 4.14.

With the basic definitions and properties in hand, one is ready to embark on the part of module theory involving projective modules. Our main application is Schanuel’s Lemma over semirings. We prove the following, using various definitions of kernel (both module in Definition 3.1 and congruence in Theory involving projective modules). Moreover, we obtain a \( \preceq \)-Morita theory elaborating.

Theorem E (Theorem 4.20 \( \text{Semi-Schanuel} \)). Suppose we have two morphisms \( \mathcal{P}_1 \xrightarrow{f_1} \mathcal{M} \) and \( \mathcal{P}_2 \xrightarrow{f_2} \mathcal{M} \) with \( f_2 \) onto. (We are not assuming that either \( \mathcal{P}_1 \) is projective.)

(i) There is a submodule

\[ \mathcal{P} = \{ (b_1, b_2) : f_1(b_1) = f_2(b_2) \} \]

of \( \mathcal{P}_1 \oplus \mathcal{P}_2 \) together with an onto morphism \( \pi_{r}^{\preceq} \) \( \mathcal{P} \to \mathcal{P}_1 \) and an \( N \)-quasi-isomorphism

\[ \ker N \pi_{r}^{\preceq} \to \ker N f_2, \]

where \( \pi_i \) is the projection to \( \mathcal{P}_i \) on the \( i \)-th coordinate. (This part is purely semiring-theoretic and does not require a system.)

(ii) The maps \( f_1\pi_1, f_2\pi_2 : \mathcal{P} \to \mathcal{M} \) are the same.

(iii) In the systemic setting, \( \pi_{r}^{\preceq} \) also induces \( \preceq \)-quasi-isomorphisms

\[ \ker N, \succeq \pi_{r}^{\preceq} \to \ker N, \succeq f_2, \quad \ker \preceq f_1\pi_{r}^{\preceq} \to \ker \preceq f_2. \]

(iv) If \( \mathcal{P}_1 \) is projective, then it is a retract of \( \mathcal{P} \) with respect to the projection \( \pi_1 : \mathcal{P} \to \mathcal{P}_1 \).

(v) If \( \mathcal{P}_1 \) is \( \preceq \)-projective, then it is a \( \preceq \)-retract of \( \mathcal{P} \) with respect to the projection \( \pi_1 : \mathcal{P} \to \mathcal{P}_1 \), and \( \mathcal{P} \) is the \( \preceq \)-direct sum of \( \mathcal{P}_1 \) and \( (1_p(-) \nu \pi_1)(\mathcal{P}) \).

(vi) If \( \mathcal{N} \) is a \( \pi_1 \)-null submodule of \( \mathcal{P} \), then \( \pi_2(\mathcal{N}) \) is an \( f_2 \)-null submodule of \( \mathcal{P}_2 \). Hence there is a \( \preceq \)-quasi-isomorphism from \( \ker_{\mathcal{Mod}, \pi} \pi_1 \) to \( \ker_{\mathcal{Mod}, \pi_2} f_2 \).

Theorem F (Theorem 4.22). If \( \mathcal{P}_1 \) is \( \preceq \)-projective with an onto morphism \( \pi : \mathcal{P} \xrightarrow{\pi} \mathcal{P}_1 \) whose module kernel \( K \) is \( \preceq \)-projective, then \( \mathcal{P} \) also is \( \preceq \)-projective.

We obtain a \( \preceq \)-Morita theory elaborating.

Theorem G (Theorem 4.30). Suppose \( \langle \mathcal{A}, \mathcal{A}', \mathcal{M}, \mathcal{M}', \tau, \tau' \rangle \) is a systemic Morita context.

(1) if \( \tau' \) is \( \preceq \)-onto, then \( \mathcal{M} \) is a \( \preceq \)-progenerator for \( \mathcal{A} \text{-Mod} \).
(2) if \( \tau \) is \( \preceq \)-onto, then \( \tau' \) is null-monic.

(2) The analogous statements hold if we switch left and right, or \( \tau \) and \( \tau' \), or \( \preceq \) and \( \succeq \).
Our approach throughout this note is explicit, aimed to show how module systems work. The paper by Connes and Consani [3] contains a more abstract approach, to be dealt with in [19].

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## 2. Basic notions

Throughout the paper, we let \( \mathbb{N} \) be the additive monoid of the non-negative integers. Similarly, we view \( \mathbb{Q} \) (resp. \( \mathbb{R} \)) as the additive monoid of the rational numbers (resp. of the real numbers).

After recalling a few well-known notions of semirings, mainly from [12], we review the basic definitions and properties of \( \mathcal{T} \)-modules, triples, and systems from [35]; more details are given in [20] and [34].

A **semiring** \((\mathcal{A}, +, \cdot, 1)\) is an additive commutative monoid \((\mathcal{A}, +, 0)\) and multiplicative monoid \((\mathcal{A}, \cdot, 1)\) satisfying the usual distributive laws.

**Remark 2.1.** Strictly speaking the element \(0\) is not needed in semiring theory, and one can make do later by adjoining the absorbing element \(0\), but for convenience we will work with semirings and assume \(0 \in \mathcal{A}\).

### 2.1. Negation maps

**Definition 2.2.** A **\( \mathcal{T} \)-module** over a set \( \mathcal{T} \) is an additive monoid \((\mathcal{A}, +, 0, \mathcal{A})\) with a scalar multiplication \(\mathcal{T} \times \mathcal{A} \to \mathcal{A}\) satisfying the following axioms, \(\forall u \in \mathbb{N}, a \in \mathcal{T}, b, b_j \in \mathcal{A}\):

1. (Distributivity over \(\mathcal{T}\)): \(a(\sum_{j=1}^{n} b_j) = \sum_{j=1}^{n}(ab_j)\).
2. \(a0_{\mathcal{A}} = 0_{\mathcal{A}}\).

We start off with a \(\mathcal{T}\)-module \(\mathcal{A}\), perhaps with extra structure. When \(\mathcal{T}\) is a monoid we call \(\mathcal{A}\) a **\(\mathcal{T}\)-monoid module**. We can make \(\mathcal{A}\) into a semiring by means of [34, Theorem 2.5], in which case we essentially have Lorscheid’s blueprints, [20] and [27], and we introduce some more structure.

**Definition 2.3.** A **negation map** on a \(\mathcal{T}\)-module \(\mathcal{A}\) is a monoid isomorphism \((-) : \mathcal{A} \to \mathcal{A}\) of order \(\leq 2\), written \(a \mapsto (-)a\), which also respects the \(\mathcal{T}\)-action in the sense that

\((-)(ab) = a((-)b),\)

for \(a \in \mathcal{T}, b \in \mathcal{A}\).

Assortments of negation maps are given in [9] [20] [34]. We also remark that when \(1 \in \mathcal{T}_{\mathcal{A}} \subseteq \mathcal{A}\), for us to check that \((-)\) is a negation map, it is enough to check that \((-)b = ((-1)b)\) for \(b \in \mathcal{A}\).

We write \(a(-)a\) for \(a + ((-a))\), and \(a^\circ\) for \(a(-)a\), called a **quasi-zero**.

**Remark 2.4.** Any quasi-zero is fixed by a negation map since \((-)a^\circ = ((-a) + a = a^\circ\). On the other hand, when \(\mathcal{A}\) is idempotent (i.e., for any \(a \in \mathcal{A}\), \(a + a = a\)), any element \(a \in \mathcal{A}\) which is fixed by a negation map is a quasi-zero since \(a = (-)a\) and hence \(a^\circ = a(-)a + a = a\).

The set \(\mathcal{A}^\circ\) of quasi-zeroes is a \(\mathcal{T}\)-submodule of \(\mathcal{A}\) that plays an important role. When \(\mathcal{A}\) is a semiring, \(\mathcal{A}^\circ\) is an ideal. Our main definition requires that \(\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{A}\).

**Definition 2.5.** A **\( \mathcal{T}\)-pseudo-triple** \((\mathcal{A}, \mathcal{T}, (-))\) is a \(\mathcal{T}\)-module \(\mathcal{A}\), with \(\mathcal{T}\) a distinguished subset of \(\mathcal{A}\), called the set of **tangible elements**, and a negation map \((-)\) satisfying \((-)\mathcal{T} = \mathcal{T}\). We write \(\mathcal{T}_0\) for \(\mathcal{T} \cup \{0\}\).

A **triple** \((\mathcal{A}, \mathcal{T}, (-))\) is a \(\mathcal{T}\)-pseudo-triple, for which \(\mathcal{T} \cap \mathcal{A}^\circ = \emptyset\) and \(\mathcal{T}_0\) generates \((\mathcal{A}, +)\).

In what follows, we will simply say a pseudo-triple rather than \(\mathcal{T}\)-pseudo-triple, when the context is clear.
2.1.1. Symmetrization.

When a given \( T \)-module \( A \) does not come equipped with a negation map, there are two natural ways of providing one: Either take \((-)\) to be the identity, as is done in supertropical algebra, or we “symmetrize” \( A \) as in \([20, \text{§1.3}]\). Symmetrization will play a central role in this paper, so we recall briefly the basics from \([34, \text{§3.5.1}]\). Let \( A \) be a \( T \)-module and \( \hat{A} = A \oplus A \) and \( \hat{T} = T \ominus T \). We impose a canonical \( \hat{T} \)-module structure in the following way.

**Definition 2.6.** For any \( T \)-module \( A \) the twist action on \( \hat{A} \) over \( \hat{T} \) is given in the following way:

\[
(a_0, a_1), \tau_w (b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_0 + a_1 b_1), a_i \in T, b_i \in A.
\]

The symmetrization of \( A \) is the \( \hat{T} \)-module \( \hat{A} \) with the twist action. The switch map on \( \hat{A} \) is given by \((b_0, b_1) \mapsto (b_1, b_0)\).

**Remark 2.7.** The switch map \((-)\) on \( \hat{A} \) is a negation map and hence \((\hat{A}, \hat{T}, (-))\) is a pseudo-triple for any pseudo-triple \((A, T, (\cdot))\).

2.2. Systems.

We round out the structure with a surpassing relation \( \leq \) given in \([34, \text{Definition 1.70}]\) and also described in \([20, \text{Definition 3.11}]\). We review the definition for convenience.

**Definition 2.8.** A surpassing relation on a triple \((A, T, (-))\), denoted \( \leq \), is a partial pre-order satisfying the following, for elements of \( A \):

(i) \( c^\circ \geq 0 \) for any \( c \in A \).

(ii) If \( b_1 \leq b_2 \) then \((-) \) \( b_1 \leq (-) \) \( b_2 \).

(iii) If \( b_1 \leq b_2 \) and \( b_1^i \leq b_2^i \) for \( i = 1, 2 \), then \( b_1 + b_1^i \leq b_2 + b_2^i \).

(iv) If \( a \in T \) and \( b_1 \leq b_2 \) then \( a b_1 \leq a b_2 \).

(v) If \( a \leq b \) for \( a, b \in T \), then \( a = b \).

A \( T \)-surpassing relation on a \( T \)-triple \( A \) is a surpassing relation also satisfying the following, for elements of \( A \): if \( b \leq a \) for \( a \in T \) and \( b \in A \), then \( b = a \).

**Lemma 2.9.** If \( b_1 + c^\circ = b \) for some \( c \in A \), then \( b_1 \leq b \).

**Proof.** Since \( c^\circ \geq 0 \), we can apply Definition 2.8 (iii). \( \square \)

Here the surpassing relation will be taken to be a PO (partial order). The main case is in \([34, \text{Definition 1.70}]\), \([20, \text{Definition 2.17}]\), defined as follows:

\[
a_1 \leq_0 a_2 \text{ if } a_2 = a_1 + b^\circ \text{ for some } b \in A,
\]

but we also could take \( \leq \) to be set inclusion when \( A \) is obtained from a hyperfield. See \([20, \text{§10}]\).

**Lemma 2.10.** If \( a_1 \leq a_2 \), then \( a_2(-) a_1 \geq 0 \) and \( a_1(-) a_2 \geq 0 \).

**Proof.** \( a_2(-) a_1 \geq (-) a_1 a_1 \), and thus \( a_1(-) a_2 = (\cdot)(a_2(-) a_1) \geq 0 \). \( \square \)

**Definition 2.11.** \( S_1 \leq S_2 \) for \( S_1, S_2 \subseteq A \) if for each \( s \in S_1 \) there is \( s' \in S_2 \) for which \( s \leq s' \).

**Definition 2.12.** A system (resp. pseudo-system) is a quadruple \((A, T_A, (-), \leq)\), where \( \leq \) is a surpassing relation on the triple (resp. pseudo-triple) \((A, T_A, (-))\), which is uniquely negated in the sense that for any \( a \in T_A \), there is a unique element \( b \) of \( T_A \) for which \( 0 \leq a + b \) (namely \( b = (-) a \)).

**Remark 2.13.** Any surpassing PO \( \leq \) induces a PO \( \leq \) given by \( a_0 \leq a_1 \) iff \( a_0^\circ \leq a_1^\circ \). This restricts to a PO on \( A^0 \), which ties in with the literature, and is the same as the Green relation of Example 2.15(vii) when \( \leq \leq_0 \).

A \( T \)-system is a system for which \( \leq \) is a \( T \)-surpassing relation.

**Remark 2.14.** Pseudo-systems encompass classical algebra, when we take \((-)\) to be the usual negation \(-\), and \( \leq \) to be equality. This “explains” the parallel between so many theorems of tropical algebra and classical algebra.
For a pseudo-system $\mathcal{A}$, we define the important $T$-submodule $\mathcal{A}_{\text{Null}} = \{a \in \mathcal{A} : a \geq 0\}$ of $\mathcal{A}$ containing $\mathcal{A}^\circ$.

Now, there are two ways that we want to view triples and their systems. The first is as the ground structure on which we build our module theory, in analogy to the ground ring for classical linear algebra or for affine algebraic geometry. We call this a ground system. We call $\mathcal{A}$ a semiring system when $\mathcal{A}$ is a semiring.

The second way, which is the main direction taken in this paper, is to fix a ground triple $(\mathcal{A}, T, (-))$, and take $\mathcal{A}$-modules $M$ together with $TM$ satisfying $TTM \subseteq TM$. We also require a triple $(\mathcal{M}, T_M, (-))$ over a triple $(\mathcal{A}, T, (-))$ to satisfy $((-)a)m = (-)(am)$ for $a \in \mathcal{A}$, $m \in \mathcal{M}$. Then we define the module system $(\mathcal{M}, T_M, (-), \preceq)$ on $\mathcal{M}$, satisfying the axiom $a_1b_1 \preceq a_2b_2$ whenever $a_1 \preceq a_2$ in $T_M$ and $b_1 \preceq b_2$ in $\mathcal{M}$.

Right module systems are defined analogously. The detailed study of module systems was carried out in [20]. Although the two theories (ground systems and module systems) start off the same, they quickly diverge, just as they do in classical algebra.

Example 2.15.

(i) Given a triple $(\mathcal{A}, T, (-))$, define $a \preceq c$ if $a + b^\circ = c$ for some $b \in \mathcal{A}$. Here the surpassing relation $\preceq$ is $\preceq$, and $\mathcal{A}_{\text{Null}} = \mathcal{A}^\circ$.

(ii) The set-up of super- and hypergroups [14] [17] is a special case of (i), where $\mathcal{A} = T \cup \mathcal{G}$ is the supertropical semiring, $\circ$ is the ghost map $\nu$, $\mathcal{G} = \mathcal{A}^\circ$, and $\preceq$ is “ghost surpasses”. Tropical mathematics is encoded in $\mathcal{G}$, which (excluding 0) often is an ordered group according to Remark 2.13.

(iii) The fuzzy ring of [6] is a special case of (i).

(iv) The symmetricized system is a special case of (i).

(v) In the hypergroup setting, as described in [31] Definition 3.47, $T$ is a given hypergroup, $\mathcal{A}$ is the subset of the power set $\mathcal{P}(T)$ generated by $T$, and $\preceq$ is set inclusion. $\mathcal{A}_{\text{Null}}$ consists of those sets containing 0, which is the set of hyperzeros usually considered in the hypergroup literature.

(vi) Tracts, introduced recently in [1], are mostly special cases of systems, where $T$ is the given Abelian group $G$, $\mathcal{A} = \mathbb{N}[G]$, $\varepsilon = (-)\mathbb{1}$, and $NG$ is $\mathcal{A}_{\text{Null}}$, usually taken to be $\mathcal{A}^\circ$.

(vii) In a semiring, one has the Green relation given by $a \leq b$ if $a + b = b$, [34] Example 2.60(i).

Idempotence of the semiring is equivalent to the reflexive property $a \preceq a$. The only natural negation map would be the identity, so every element $a = a + a$ is a quasi-zero, and $\mathcal{A}_{\text{Null}} = \mathcal{A}$, so this is not a system, even though one gets a pseudo-triple by taking $T$ to be a generating set of $\mathcal{A}$. In this case most of our theory degenerates, and indeed one does not get much structure theory along the lines of systems.

We will also want a weaker version of generation, which comes up naturally and ties into hyperfields.

Definition 2.16. Let $(\mathcal{A}, T, (-), \preceq)$ be a pseudo-system.

(i) An element $b \in \mathcal{A}$ is $\preceq$-generated by a subset $\mathcal{A}'$ of $\mathcal{A}$ if there is a subset $\{a_i : 1 \leq i \leq t\} \subseteq \mathcal{A}'$ such that $b \preceq \sum a_i$.

(ii) For subsets $\mathcal{A}'$ and $\mathcal{A}''$ of $\mathcal{A}$, we say that $\mathcal{A}' \preceq$-generates $\mathcal{A}''$ if each element of $\mathcal{A}''$ is $\preceq$-generated by $\mathcal{A}'$.

The $\succeq$-analog is less interesting because of the following reduction to usual generation.

Lemma 2.17. In a $T$-system $\mathcal{A}$, with $\mathcal{A}' \subseteq \mathcal{A}$, if for each $b \in \mathcal{A}$ there is $S_b = \{a_i : 1 \leq i \leq t\} \subseteq \mathcal{A}'$ such that $b \succeq \sum a_i$, then $\mathcal{A}'$ generates $\mathcal{A}$.

Proof. For $b \in \mathcal{A}$, write $b = \sum a_i$, where $a_i \in T$, and $\sum b_{i,j} \preceq a_i$ for $b_{i,j} \in \mathcal{A}'$, implying $\sum b_{i,j} = a_i$ by Definition 2.8 for $T$-systems, and thus $b = \sum \sum b_{i,j}$.

Definition 2.18. A $\preceq$-morphism of module systems

$$\varphi : (\mathcal{M}, T, (-), \preceq) \to (\mathcal{M}', T', (-)', \preceq')$$

is a map $\varphi : \mathcal{M} \to \mathcal{M}'$ satisfying the following properties for $a_i \in T$ and $b \preceq b'$ in $\mathcal{M}, b_i$ in $\mathcal{M}$:

(i) $\varphi((-)a_1) = (-)\varphi(a_1)$;
Remark 2.19. Technically, this differs from [41, Definition 8.3 and Example 8.5] because of (v), but it follows from (iv) and (vi) since \( 0 \preceq b \) implies \( 0 = \varphi(0) \preceq \varphi(b) \).

Lemma 2.20. When \( \preceq' \) is a PO (partial order) and \( T \) is a group, (iii) can be replaced by the weaker condition

\[
\varphi(ab) \preceq' \varphi(a) \varphi(b), \quad \forall a \in T.
\]

Proof. \( a \varphi(b) = a \varphi(a^{-1}ab) \preceq' a \varphi(b) \), so equality holds at each stage. \( \Box \)

By a \( \succeq \text{-morphism} \) we mean the same definition as \( \preceq \text{-morphism} \), except with (ii) now reading

\[
\varphi(a_1 + a_2) \succeq \varphi(a_1) + \varphi(a_2).
\]

By a morphism we mean the usual universal algebra definition, i.e., equality holds in (ii) instead of \( \preceq \). In particular, every morphism is both a \( \preceq \text{-morphism} \) and \( \succeq \text{-morphism} \).

Example 2.21. Let us describe these notions for Example 2.14.

(i) In supertropical mathematics, a \( \preceq \text{-morphism} f \) satisfies

\[
f(b_1 + b_2) + \text{ghost} = f(b_1) + f(b_2);
\]

which implies that either \( f(b_1 + b_2) = f(b_1) + f(b_2) \), or \( f(b_1) + f(b_2) \) is ghost, in which case either \( f(b_1) = f(b_2) \), or \( f(b_1) \) is ghost of value greater than or equal to \( f(b_2) \) (or visa versa).

A \( \succeq \text{-morphism} f \) satisfies

\[
f(b_1 + b_2) = f(b_1) + f(b_2) + \text{ghost};
\]

which implies that either \( f(b_1 + b_2) = f(b_1) + f(b_2) \), or \( f(b_1 + b_2) \) is ghost of value at least that of both \( f(b_1) \) and \( f(b_2) \).

(ii) For hyperfield systems, a \( \preceq \text{-morphism} f \) satisfies

\[
f(b_1 \boxplus b_2) \subseteq f(b_1) \boxplus f(b_2),
\]

the definition used in [11, Definition 2.4]. This is intuitive when \( f \) maps the hyperfield \( T \) into itself.

On the other hand, hyperfield \( \succeq \text{-morphisms} \) which are not morphisms seem to be artificial; for an example, one could extend the identity on the phase hyperfield to a map that doubles all non-singleton arcs around the center.

(iii) For fuzzy rings, in [6, § 1], also see [11, Definition 2.17], a morphism

\[
f : (K; +; \times; \varepsilon_K, K_0) \to (L; +; \times; \varepsilon_L; L_0)
\]

of fuzzy rings is defined as satisfying: For any \( \{a_1, \ldots, a_n\} \in K^\times \) if \( \sum_{i=1}^n a_i \in K_0 \) then \( \sum_{i=1}^n f(a_i) \in L_0 \). Any \( \preceq \text{-morphsim} \) in our setting is a fuzzy morphism since \( L_0 \) is an ideal, and thus \( \sum_{i=1}^n f(a_i) \in f(\sum_{i=1}^n a_i) + L_0 = L_0 \). The other direction might not hold. The same reasoning holds for tracts of [11].

(iv) In the symmetrized system, a \( \preceq \text{-morphism} f \) defined componentwise satisfies

\[
(f(b_1 + b_i') + c, f(b_2 + b_2') + c) = f((b_1 + b_1'), (b_2 + b_2')) + (c, c) = f(b_1, b_2) + f(b_1', b_2') = (f(b_1) + f(b_1'), f(b_2) + f(b_2')),
\]

implying that \( f(b_i + b_i') + c = f(b_i) + f(b_i') \) for \( i = 1, 2 \). Since this must hold for all \( b_i \), if we take \( c \) large enough we would have a contradiction in all nontrivial cases, so \( \preceq \text{-morphisms} \) are less useful here.

In conjunction with the hyperfield theory, we are most interested in \( \preceq \text{-morphisms} \), but at times (for example, in Theorems 3.10, 4.22 below) we need to restrict our attention to morphisms. Sometimes our lemmas can be formulated for \( \succeq \text{-morphisms} \), such as in Lemma 3.14(ii) below.
2.2.1. Direct sums and direct limits.

The direct sum of \( T \)-modules, defined in the usual way, is extended to pseudo-triples, \( \text{[20 Definition 5.18(ii)].} \)

**Definition 2.22.**

(i) The direct sum \( \bigoplus_{i \in I}(A_i, T_{A_i}, (-)) \) of a family of pseudo-triples over an index set \( I \) (not necessarily finite) is defined as \( (\bigoplus A_i, T_{\bigoplus A_i}, (-)) \), where \( T_{\bigoplus A_i} = \bigcup T_{A_i} \), viewed in \( \bigoplus A_i \).

(ii) The free \( A \)-module pseudo-triple \( (A^{(I)}, T_{A^{(I)}}, (-)) \) over a pseudo-triple \( (A, T, (-)) \) is the direct sum of copies of \( (A, T, (-)) \).

If \( (A, T, (-), \preceq) \) is a system, we can extend \( \preceq \) componentwise to \( A^{(I)} \) to obtain the free \( A \)-module system.

**Remark 2.23.** When \( \preceq \) is a PO on \( A \), \( \preceq \) is also a PO on \( A^{(I)} \), seen componentwise.

2.3. **Notation.** Let us fix some notation for the remainder of this paper. In what follows, we let \( A = (A, T, (-), \preceq) \) be a semiring system, and \( M \) and \( N \) always denote \( A \)-module systems. We write \( \preceq \) generically for the appropriate surpassing PO in a system. \( \mathcal{M}_{\text{Null}} \) denotes the set \( \{ a \in M : a \geq 0 \} \).

Later, \[ \mathcal{P} := (\mathcal{P}, T, (-), \preceq) \]
denotes a projective or \( \preceq \)-projective or \( \preceq \)-projective \( A \)-module system, cf. Definition 4.1.

### 3. Systemic versions of basic module properties

We want to find the systemic generalization of classical concepts of module theory. As we shall see, this depends on which version we use, i.e., the switch negation map in the symmetrization given in \( \text{[21.1]} \) or taking a given surpassing negation map \((-)\) and surpassing relation \(\preceq\). These two different approaches give rise to different theories.

A \( \preceq \)-morphism \( f \) is \( \mathbf{N} \text{-monic} \) when it satisfies the property that if \( f(b) = f(b') \) for \( b, b' \in M \), then \( b = b' \).

3.1. **Module theoretic notions.**

**Definition 3.1.** Let \( M \) and \( N \) be \( A \)-module systems, and \( f : M \rightarrow N \) a \( \preceq \)-morphism.

(i) A submodule \( M' \) of \( M \) is \( f \)-null if \( f(a) \in N_{\text{Null}} \) for all \( a \in M' \). The null-module kernel \( \ker_{\text{Mod}, A} f \) of \( f \) is the sum of all \( f \)-null submodules of \( M \).

(ii) A \( \preceq \)-morphism \( f : M \rightarrow N \) is null if \( f(M) \subseteq N_{\text{Null}}, \) i.e., \( \ker_{\text{Mod}, A} f = M \).

(iii) A \( \preceq \)-morphism \( f \) is null-monic (resp. null-epic) when it satisfies the property that if \( fh \) is null (resp. null) for a morphism \( h \), then \( h \) is null.

**Remark 3.2.** Being the sum of submodules of \( M \), \( \ker_{\text{Mod}, A} f \) is a submodule of \( M \), which is \( f \)-null when \( f \) is a morphism, but need not be \( f \)-null when \( f \) is just a \( \preceq \)-morphism. One could have \( f(a_1) + f(a_2) \preceq 0 \) whereas \( f(a_1 + a_2) \preceq 0 \), for example if \( f(a_1) = f(a_2) \).

**Lemma 3.3.** A \( \preceq \)-morphism \( f : M \rightarrow N \) is null-monic iff the null-module kernel of \( f \) is a subset of \( M_{\text{Null}} \).

**Proof.** \( (\Rightarrow) \) For any \( f \)-null submodule \( M' \) of \( M \), consider the identity map \( h : M' \rightarrow M' \). Then \( fh \) is null, implying \( h \) is null. In particular, \( M' = M'_{\text{Null}} \subseteq M_{\text{Null}} \), and hence the null-module kernel of \( f \) is a subset of \( M_{\text{Null}} \).

\( (\Leftarrow) \) Suppose \( fh \) is null, for a morphism \( h : K \rightarrow M \). Then \( f(h(K)) \subseteq N_{\text{Null}} \). This implies that \( h(K) \subseteq \ker_{\text{Mod}, A} f \subseteq M_{\text{Null}} \), proving that \( f : M \rightarrow N \) is null-monic. \( \square \)

Next, we define some notation which we will use later in defining projective module systems.

**Definition 3.4.** Let \( f : M \rightarrow N \) be a \( \preceq \)-morphism of \( A \)-module systems \( M \) and \( N \). We define the following two sets:

\[ f(M)_{\preceq} = \{ b \in N : b \preceq f(a) \text{ for some } a \in M \}, \quad f(M)_{\succeq} = \{ b \in N : b \succeq f(a) \text{ for some } a \in M \}. \]
(i) When \( f \) is a morphism, we say that \( f : \mathcal{M} \to \mathcal{N} \) is onto if \( f(T_\mathcal{M}) = T_\mathcal{N} \) in the usual sense, i.e., every \( b \in T_\mathcal{N} \) can be written as \( f(a) \) for \( a \in T_\mathcal{M} \).

(ii) \( f : \mathcal{M} \to \mathcal{N} \) is \( \succeq \)-onto if \( f(\mathcal{M})_{\succeq} = \mathcal{N} \), i.e., for every \( b \in \mathcal{N} \) there exists \( a \in \mathcal{M} \), for which \( b \succeq f(a) \).

(iii) \( f : \mathcal{M} \to \mathcal{N} \) is \( \succeq \)-onto if \( f(\mathcal{M})_{\succeq} = \mathcal{N} \), i.e., if for every \( b \in \mathcal{N} \) there is \( a \in \mathcal{M} \) such that \( b \succeq f(a) \).

**Example 3.5.** In the supertropical setting, \( f : \mathcal{M} \to \mathcal{N} \) is \( \preceq \)-onto iff for every element \( b \) of \( \mathcal{N} \) there is \( c \in \mathcal{M} \) such that \( b + \text{ghost} = f(c) \), which often is easy to satisfy when \( c \) is a large enough ghost. \( \succeq \)-onto says that \( b = f(c) + \text{ghost} \), which for \( b \) tangible says \( b = f(c) \).

For hyperfield systems over a hyperfield \( \mathcal{T} \), \( \preceq \)-onto means \( b \subseteq f(c) \) for some \( c \), which we could take in \( \mathcal{T} \), and then \( b = f(c) \).

For fuzzy rings, the condition says something about how \( f(K_0) \) sits inside \( L_0 \), notation as in [6].

**Lemma 3.6.** Let \( f : \mathcal{M} \to \mathcal{N} \) be a \( \preceq \)-morphism of \( \mathcal{A} \)-module systems \( \mathcal{M} \) and \( \mathcal{N} \). Then

(i) \( f(\mathcal{M})_{\succeq} \) is a submodule of \( \mathcal{N} \). Moreover, \( f \) is \( \succeq \)-onto, if for every \( b \in T_\mathcal{N} \) there is \( a \in \mathcal{M} \) such that \( f(a) \preceq b \).

(ii) \( f(\mathcal{M})_{\preceq} \) is a submodule of \( \mathcal{N} \) for any morphism \( f : \mathcal{M} \to \mathcal{N} \).

**Proof.** (i) \( f(\mathcal{M})_{\succeq} \) is clearly closed under the action of \( \mathcal{A} \) and contains \( \emptyset \). If \( b_i \in f(\mathcal{M})_{\succeq} \) for \( i = 1, 2 \) then writing \( b_1 \succeq f(a_i) \), we have

\[
b_1 + b_2 \succeq f(a_1) + f(a_2) \succeq f(a_1 + a_2).
\]

This shows that \( f(\mathcal{M})_{\succeq} \) is also closed under addition. The second assertion follows from the fact that \( T_\mathcal{N} \) generates \( \mathcal{N} \). In fact, for any \( b \in \mathcal{N} \), there exist \( b_i \in T_\mathcal{N} \) such that \( b = \sum_i b_i \). But, from the given condition, we can find \( a_i \in \mathcal{M} \) such that \( f(a_i) \preceq b_i \) and hence we have

\[
b = \sum_i b_i \succeq \sum_i f(a_i) \succeq f(\sum_i a_i).
\]

(ii) One can easily check that \( f(\mathcal{M})_{\preceq} \) is closed under the action of \( \mathcal{A} \) and contains \( \emptyset \). Suppose that \( b_1, b_2 \in f(\mathcal{M})_{\preceq} \), i.e., there exist \( a_1, a_2 \in \mathcal{M} \) such that \( b_i \preceq f(a_i) \) for \( i = 1, 2 \). Since \( f \) is a morphism, it follows that

\[
b_1 + b_2 \preceq f(a_1) + f(a_2) = f(a_1 + a_2).
\]

This shows that \( b_1 + b_2 \in f(\mathcal{M})_{\preceq} \) and hence \( f(\mathcal{M})_{\preceq} \) is also closed under addition.

\( \square \)

**Definition 3.7.**

(i) An onto morphism \( \pi : \mathcal{M} \to \mathcal{N} \) is an \( \mathcal{N} \)-quasi-isomorphism if \( \pi \) is also \( \mathcal{N} \)-monic.

(ii) An onto \( \preceq \)-morphism \( \pi : \mathcal{M} \to \mathcal{N} \) is a \( \preceq \)-quasi-isomorphism if \( \pi \) is also null-monic.

3.2. Congruences.

Recall that a congruence on \( \mathcal{M} \) is an equivalence relation which preserves all of the operators; i.e., it is a subsystem of \( \mathcal{M} \times \mathcal{M} \) that contains the diagonal \( \text{diag}_{\mathcal{M}} := \{(a, a) : a \in \mathcal{M}\} \) and is reflexive, symmetric and transitive.

**Definition 3.8.** \( f(\mathcal{M})_{\text{cong}} \) is taken to be the congruence relation \( f(\mathcal{M})_{\preceq} \times f(\mathcal{M})_{\succeq} \).

**Lemma 3.9.** A morphism \( f : \mathcal{M} \to \mathcal{N} \) in a \( \mathcal{T} \)-system is null-epic, if and only if \( f \) is \( \succeq \)-onto.

**Proof.** (\( \Rightarrow \)) Suppose, on the contrary, that \( f : \mathcal{M} \to \mathcal{N} \) is null-epic, but not \( \succeq \)-onto. Then there exists \( b \in \mathcal{N} \) such that \( b \npreceq f(a) \) for any \( a \in \mathcal{M} \). Consider the morphism

\[
g : \mathcal{N} \to \mathcal{N}/f(\mathcal{M})_{\text{cong}}.
\]

Note that \( g \) is a morphism since \( f(\mathcal{M})_{\text{cong}} \) is a congruence relation.

Next, one can see that \( fg \) is null. In fact, for any \( a \in \mathcal{M} \), we have \( f(a) \in f(\mathcal{M})_{\succeq} \) since \( f(a) \succeq f(a) \). It follows that \( (gf)(a) = g(f(a)) = \emptyset \). Since we assume that \( f \) is null-epic, this implies that \( g \) is null. But, since \( b \npreceq f(\mathcal{M})_{\preceq} \), we know that \( f(\mathcal{M})_{\text{cong}} \) does not contain \((b, \emptyset)\) and hence \( g \) cannot be null. This contradicts the original assumption that \( f \) is null-epic. Therefore, \( f \) is \( \succeq \)-onto.
(⇐) Now assume that \( f : \mathcal{M} \to \mathcal{N} \) is \( \succeq \)-onto. Suppose that \( h : \mathcal{N} \to \mathcal{K} \) is a morphism. Any \( b \in \mathcal{N} \) has the form \( b \succeq f(a) \) for some \( a \in \mathcal{M} \), implying \( h(b) \succeq hf(a) \). If \( hf \) is null then \( hf(a) \succeq 0 \). It follows that \( h(b) \succeq 0 \), so \( h \) also is null on \( \mathcal{N} \).

Thus \( \succeq \)-onto seems to work categorically.

The converse fails since the surpassing goes the wrong direction.

### 3.3. \( \succeq \)-split epics

We recall a standard definition.

**Definition 3.10.** Let \( \pi : \mathcal{M} \to \mathcal{N} \) be an onto morphism. We say that \( \pi : \mathcal{M} \to \mathcal{N} \) **splits** if there is a morphism \( \nu : \mathcal{N} \to \mathcal{M} \) such that \( \pi \nu = 1 \mathcal{N} \).

In classical algebra, \( \nu \) must be monic, and any split epic gives rise to an exact sequence.

**Example 3.11.** If \( \mathcal{M} = \mathcal{N} \oplus \mathcal{N}' \), then the canonical projection \( \mathcal{M} \to \mathcal{N} \) splits via the natural injection \( \nu : \mathcal{N} \to \mathcal{M} \).

This is trickier in the theory of systems since, as we shall see, splitting need not involve direct sums.

**Remark 3.12.** In fact, a similar issue has been already observed in tropical algebra, cf. [30] §2.

Accordingly, we want to weaken the definition, and consider its implications. We write \( f \preceq g \) for \( \succeq \)-morphisms \( f, g : \mathcal{M} \to \mathcal{N} \), if \( f(b) \preceq g(b) \) for all \( b \in \mathcal{M} \). Now, we weaken Definition 3.10 as follows:

**Definition 3.13.** We say that a \( \succeq \)-morphism \( \pi : \mathcal{M} \to \mathcal{N} \) **\( \succeq \)**-**splits** if there is a \( \succeq \)-morphism \( \nu : \mathcal{N} \to \mathcal{M} \) such that \( 1 \mathcal{N} \preceq \nu \pi \). In this case, we also say that \( \nu \preceq \)**-**splits** \( \pi \), and that \( \mathcal{N} \) is a \( \succeq \)-retract of \( \mathcal{M} \).

Let \( f \) be any of \( \{ \succeq \text{-morphism, } \succeq \text{-morphism, } \text{morphism} \} \). Then, \( f : \mathcal{M} \to \mathcal{M} \) is \( \succeq \)-idempotent if \( f^2 \preceq f \).

**Lemma 3.14.**

(i) If \( \pi : \mathcal{M} \to \mathcal{N} \) and \( \nu : \mathcal{N} \to \mathcal{M} \) are \( \succeq \)-morphisms with \( 1 \mathcal{N} \preceq \nu \pi \), then \( \pi \) is \( \succeq \)-onto, and \( \pi \pi \) is \( \succeq \)-idempotent.

(ii) If \( \pi : \mathcal{M} \to \mathcal{N} \) is a morphism and \( \nu : \mathcal{N} \to \mathcal{M} \) is a \( \succeq \)-morphism with \( 1 \mathcal{N} \preceq \nu \pi \), then \( 1(-) \nu \pi \) is \( \succeq \)-idempotent.

(iii) If \( \pi : \mathcal{M} \to \mathcal{N} \) and \( \nu : \mathcal{N} \to \mathcal{M} \) are \( \succeq \)-morphisms with \( 1 \mathcal{N} \preceq \nu \pi \), then \( \nu \) is null-monic and \( 1(-) \nu \pi \) is \( \succeq \)-idempotent.

**Proof.**

(i) For any \( b \in \mathcal{N} \), we have that \( b \preceq \pi(\nu(b)) \). This shows that \( \mathcal{N} = \pi(\mathcal{M}) \preceq \) and hence \( \pi \) is \( \succeq \)-onto. Furthermore, \( \nu \nu \pi \nu = \nu (\nu \pi) \pi \preceq \nu \pi \nu \pi = \nu \pi \).

(ii) For any \( b \in \mathcal{M} \), let \( c = \pi(b) \), so \( (1(-) \nu \pi)(b) = b(-) \nu \pi(c) \). Also \( \pi \nu(c) \preceq c \) implies \( \pi \nu(c)(-c) \preceq 0 \) by Lemma 2.10 so \( c(-) \pi \nu(c) \preceq 0 \), and thus \( \nu(c(-) \pi \nu(c)) \preceq 0 \). Hence \( (1(-) \nu \pi)(b(-) \nu \pi(c)) = (1(-) \nu \pi)(b(-) \nu \pi(c)) = b(-) \nu \pi(c)(-\nu \pi(c)) \preceq b(-) \nu \pi(c) = (1(-) \nu \pi)(b) \).

(iii) For the first assertion, from Lemma 3.3 it is enough to show that \( \ker_{\text{Mod},\mathcal{N}} \nu \subseteq \mathcal{N}_{\text{Null}} \). If \( b \in \ker_{\text{Mod},\mathcal{N}} \nu \), then \( \nu(b) \preceq 0 \). Since \( 1 \mathcal{N} \preceq \nu \), we further have that \( b \preceq \nu \pi(b) \geq \pi(0) \geq 0 \)

and hence \( b \in \mathcal{N}_{\text{Null}} \), showing that \( \ker_{\text{Mod},\mathcal{N}} \nu \subseteq \mathcal{N}_{\text{Null}} \). Also \( (1(-) \nu \pi)(b)(-\nu \pi(c)) \preceq 1(-) \nu \pi \).

**Definition 3.15.** A module system \( \mathcal{M} = (\mathcal{M}, T_{\mathcal{M}}, (-), \preceq) \) is the (finite) \( \succeq \)-**direct sum** of module systems \( \mathcal{M}_i, T_{\mathcal{M}_i}, (-), \preceq \), \( i \in I \) (I finite), if there are \( \succeq \)-morphisms \( \pi_i : \mathcal{M} \to \mathcal{M}_i \) as well as \( \succeq \)-morphisms \( \nu_i : \mathcal{M}_i \to \mathcal{M} \) that \( \succeq \)-split \( \pi_i \), for which \( 1 \mathcal{M} \preceq \sum \nu_i \pi_i, 1 \mathcal{M}_i \preceq \pi_i \nu_i \), and \( 0 \mathcal{M} \preceq \pi_j \nu_i \) for all \( i \neq j \).

Then we have the following.

**Theorem 3.16.** Let \( \pi : \mathcal{M} \to \mathcal{N} \) be a morphism. If \( \nu \preceq \) \( \succeq \)-splits, then \( \mathcal{M} \) is the \( \succeq \)-direct sum of \( \mathcal{M}_1 : = \pi(\mathcal{M}) \) and \( \mathcal{M}_2 : = (1 \mathcal{M}(-) \nu \pi)(\mathcal{M}) \) with respect to the morphisms \( \pi_1 = \pi, \nu_1 = \nu, \pi_2 = (1 \mathcal{M}(-) \nu \pi), \nu_2 = 1 \mathcal{M}_2 \).
Lemma 4.2. The free \( \mathfrak{m} \)-morphism of module systems \( h \) to a \( \preceq \)-comparisons: but this stronger definition “works” as well in Lemma 4.2. We take the usual argument of lifting a set-theoretical map from the base \( \{ e_i : i \in I \} \) of \( \mathcal{F} \), in view of Remark 2.14.

Next, we show that \( \nu_2 \preceq \)-splits \( \pi_2 \). Take \( b_2 \in \mathcal{M}_2 \). This means that there exists \( b_1 \in \mathcal{M} \) such that \( b_2 = b_1(\nu_\pi)(b_1) \), and now one observes
\[
\pi_2 \nu_2(b_2) = \pi_2(b_2) = (\mathbb{1}_\mathcal{M}(\nu_\pi)^2(b_1)) \geq (\mathbb{1}_\mathcal{M}(\nu_\pi)(b_1)) = b_2,
\]
(3.1)
since \( \mathbb{1}_\mathcal{M}(\nu_\pi) \) is \( \preceq \)-idempotent by Lemma 3.14(ii).

We now show the remaining conditions. One can easily see the following:
\[
\nu_1 \pi_1(b) + \nu_2 \pi_2(b) = \nu_\pi(b) + (\mathbb{1}_\mathcal{M}(\nu_\pi)(b)) = b + (\nu_\pi(b)(\nu_\pi)(b)) \geq b, \quad b \in \mathcal{M},
\]
showing that \( \mathbb{1}_\mathcal{M} \preceq \nu_1 \pi_1 + \nu_2 \pi_2 \).

Finally, we have for \( b = (\mathbb{1}_\mathcal{M}(\nu_\pi)b' \in \mathcal{M}_2 \),
\[
\pi_1 \nu_2(b) = \pi(\mathbb{1}_\mathcal{M}(\nu_\pi)(b')) = (\pi(\mathbb{1}_\mathcal{M}(\nu_\pi)(b')) \geq \pi(b')(\mathbb{1}_\mathcal{M}(\nu_\pi)(b')) \geq 0,
\]
and similarly, for \( b \in \mathcal{M}_1 \),
\[
\pi_2 \nu_1(b) = (\mathbb{1}_\mathcal{M}(\nu_\pi)(\mathbb{1}_\mathcal{M}(\nu_\pi)(b))) = \nu_\pi(b')(\mathbb{1}_\mathcal{M}(\nu_\pi)(b)) \geq \nu_\pi(b)(\mathbb{1}_\mathcal{M}(\nu_\pi)(b)) \geq 0.
\]

\[\square\]

4. \( \preceq \)-projective and \( \succeq \)-projective module systems

We are ready to define \( \preceq \)-projective and \( \succeq \)-projective modules over ground \( \mathcal{T} \)-systems. This encompasses results of [24], in view of Remark 2.14.

**Definition 4.1.** ([15, 23, 24, 36]) A module system \( \mathcal{P} := (\mathcal{P}, \mathcal{T}, (-, \preceq)) \) is **projective** if for any onto morphism of module systems \( h : \mathcal{M} \to \mathcal{M}' \), every morphism \( f : \mathcal{M} \to \mathcal{M}' \) lifts to a morphism \( \hat{f} : \mathcal{P} \to \mathcal{M} \), in the sense that \( h \hat{f} = f \).

\( \mathcal{P} \) is \( \preceq \)-projective if for any \( \preceq \)-onto morphism \( h : \mathcal{M} \to \mathcal{M}' \), every \( \preceq \)-morphism \( f : \mathcal{P} \to \mathcal{M}' \) \( \preceq \)-lifts to a \( \preceq \)-morphism \( \hat{f} : \mathcal{P} \to \mathcal{M} \), in the sense that \( f \preceq h \hat{f} \).

\( \mathcal{P} \) is \( \succeq \)-projective if for any \( \succeq \)-onto \( \succeq \)-morphism \( h : \mathcal{M} \to \mathcal{M}' \), every \( \succeq \)-morphism \( f : \mathcal{P} \to \mathcal{M}' \) \( \succeq \)-lifts to a \( \succeq \)-morphism \( \hat{f} : \mathcal{P} \to \mathcal{M} \), in the sense that \( f \succeq h \hat{f} \).

Note that these definitions are not quite analogous. We could weaken this to limit \( h \) to a morphism, but this stronger definition “works” as well in Lemma 4.2.

4.1. Basic properties of \( \preceq \)-projective systems.

**Lemma 4.2.** The free \( \mathcal{A} \)-module system \( \mathcal{F} \) is projective, \( \preceq \)-projective, and \( \succeq \)-projective.

**Proof.** We take the usual argument of lifting a set-theoretical map from the base \( \{ e_i : i \in I \} \) of \( \mathcal{F} \), in these three respective contexts. Namely, choosing \( x_i \in \mathcal{M} \) for which \( h(x_i) = f(e_i) \) (resp. \( h(x_i) \preceq f(e_i) \), \( h(x_i) \succeq f(e_i) \)) and defining a morphism \( \hat{f} : \mathcal{F} \to \mathcal{M} \) by \( \hat{f}(e_i) = x_i \), we have the three respective comparisons:
\[
f \left( \sum a_i e_i \right) = \sum f(a_i e_i) = \sum a_i f(e_i) = \sum a_i h(x_i) = h \left( \sum a_i x_i \right) = h \left( \sum a_i \hat{f}(e_i) \right) = h \hat{f} \left( \sum a_i e_i \right),
\]
proving \( f = h \hat{f} \).

\[
f \left( \sum a_i e_i \right) \preceq \sum f(a_i e_i) = \sum a_i f(e_i) \preceq \sum a_i h(x_i) = h \left( \sum a_i x_i \right) = h \left( \sum a_i \hat{f}(e_i) \right) = h \hat{f} \left( \sum a_i e_i \right),
\]
proving \( f \preceq h \hat{f} \).

\[
f \left( \sum a_i e_i \right) \succeq \sum f(a_i e_i) = \sum a_i f(e_i) \succeq \sum a_i h(x_i) = h \left( \sum a_i x_i \right) = h \left( \sum a_i \hat{f}(e_i) \right) = h \hat{f} \left( \sum a_i e_i \right),
\]
proving \( f \succeq h \hat{f} \).

\[\square\]
Similar arguments as in [12 §17] show that the following are equivalent for a $\mathcal{T}$-module system $\mathcal{P}$:

(i) $\mathcal{P}$ is projective.
(ii) Every morphism onto $\mathcal{P}$ splits.
(iii) There is an onto morphism from a free system to $\mathcal{P}$ that splits.
(iv) The functor $\text{Hom}(\mathcal{P}, \_)$ sends onto morphisms to onto morphisms.

Note that (iii) is the condition used in [25] to define projective modules. We extend this to $\succeq$.

**Proposition 4.3.** The following are equivalent for a $\mathcal{T}$-module system $\mathcal{P}$:

(i) $\mathcal{P}$ is $\succeq$-projective.
(ii) Every $\succeq$-onto morphism to $\mathcal{P}$ $\succeq$-splits.
(iii) There is a $\succeq$-onto morphism from a free system to $\mathcal{P}$ that $\succeq$-splits.
(iv) Given $a \succeq$-onto morphism $h : \mathcal{M} \to \mathcal{M}'$, $\text{Hom}(\mathcal{P}, h) : \text{Hom}(\mathcal{P}, \mathcal{M}) \to \text{Hom}(\mathcal{P}, \mathcal{M}')$ given by $g \mapsto hg$ is $\succeq$-onto.

*Proof. (i) $\Rightarrow$ (ii) Given a $\succeq$-onto morphism $h : \mathcal{M} \to \mathcal{P}$, the identity map $1_{\mathcal{P}} \succeq$-lifts to a $\succeq$-morphisms $g : \mathcal{P} \to \mathcal{M}$ satisfying $1_{\mathcal{P}} \succeq hg$.

(ii) $\Rightarrow$ (iii) A fortiori, since we can define a $\succeq$-onto morphism from a free system to $\mathcal{P}$ by taking a base $\{e_i\}$ of a free system and sending the $e_i$ elementwise to the $\succeq$-generators of $\mathcal{P}$ as in the proof of Lemma 4.2.

(iii) $\Rightarrow$ (i) Take a free $\mathcal{A}$-module system $\mathcal{F}$, with the projection $\pi : \mathcal{F} \to \mathcal{P}$ which by hypothesis $\succeq$-splits, with $\nu : \mathcal{P} \to \mathcal{F}$.

Let $h : \mathcal{M} \to \mathcal{M}'$ be a $\succeq$-onto morphism. Then, for any $\succeq$-morphism $f : \mathcal{P} \to \mathcal{M}'$, we can $\succeq$-lift $f\pi$ to $\tilde{f} : \mathcal{F} \to \mathcal{M}$, i.e., $f\pi \succeq h\tilde{f}$. Since $1_{\mathcal{P}} \succeq \pi\nu$, we have that $\tilde{f} : \mathcal{F} \to \mathcal{M}$ is $\succeq$-onto. This proves that $\phi$ is $\succeq$-onto as desired. □

**Corollary 4.4.** If $\mathcal{P}$ is $\succeq$-projective, the functor $\text{Hom}(\mathcal{P}, \_)$ sends $\succeq$-onto morphisms to $\succeq$-onto morphisms.

**Proposition 4.5.** The following are equivalent for a $\mathcal{T}$-module system $\mathcal{P}$:

(i) $\mathcal{P}$ is $\succeq$-projective.
(ii) Every $\succeq$-onto morphism to $\mathcal{P}$ $\succeq$-splits.
(iii) There is a $\succeq$-onto morphism from a free system to $\mathcal{P}$ that $\succeq$-splits.
(iv) The functor $\text{Hom}(\mathcal{P}, \_)$ sends $\succeq$-onto morphisms to $\succeq$-onto morphisms.

*Proof. Analagous to the proof of Proposition 4.3. □*

**Lemma 4.6** (as in [12 Proposition 17.19]). A direct sum $\bigoplus \mathcal{P}_i$ of $\mathcal{T}$-module systems is projective (resp. $\succeq$-projective, $\succeq$-projective) if and only if each $\mathcal{P}_i$ is projective (resp. $\succeq$-projective, $\succeq$-projective).

*Proof. Formal, according to components. □*

One can sharpen this assertion.

**Proposition 4.7.** If $\pi : \mathcal{Q} \to \mathcal{P}$ is a $\succeq$-split (resp. $\succeq$-split) $\succeq$-morphism and $\mathcal{Q}$ is $\succeq$-projective (resp. $\succeq$-projective), then $\mathcal{P}$ is also $\succeq$-projective (resp. $\succeq$-projective).

*Proof. We prove the case when $\mathcal{Q}$ is $\succeq$-projective; $\succeq$-projective case can be similarly proven. We write a splitting map $\nu : \mathcal{P} \to \mathcal{Q}$ as in Definition 3.13. For any $(\mathcal{T}$-module system) $\succeq$-onto morphism $h : \mathcal{M} \to \mathcal{M}'$, and every $\succeq$-morphism $f : \mathcal{P} \to \mathcal{M}'$, the $\succeq$-morphism $f\pi \succeq$-lifts to a $\succeq$-morphism $\tilde{f} : \mathcal{Q} \to \mathcal{M}$, i.e., $h\tilde{f} \succeq f\pi$. Hence $h\tilde{f}\nu \succeq f\pi\nu \succeq f$, so $\tilde{f}\nu \succeq$-lifts $f$. This proves that $\mathcal{P}$ is $\succeq$-projective. □*
Proposition 4.8. Suppose $Q$ is the $\preceq$-direct sum of $P_i$, with each $P_i$ a $\preceq$-retract of $Q$. If the $P_i$ are $\preceq$-projective then $Q$ is also $\preceq$-projective.

Proof. We write $\nu_i : P_i \to Q$ and $\pi_i : Q \to P_i$ as in Definition 4.11. For any $(T$-module system) $\preceq$-onto morphism $h : M \to M'$ and $\preceq$-morphism $f : Q \to M'$, define the $\preceq$-morphisms $f_i = f\nu_i : P_i \to M'$, which $\preceq$-lift to $\preceq$-morphisms $f_i : P_i \to M$, i.e., $hf_i \succeq f\nu_i$. Then $hf_i\pi_i \preceq f\nu_i\pi_i$, so $h(\sum_i f_i\pi_i) \succeq f(\sum_i (\nu_i\pi_i)) \succeq f$. □

Corollary 4.9. If $Q$ is $\preceq$-quasi-isomorphic to $P_1$ and $P_1$ is $\preceq$-projective then $Q$ is also $\preceq$-projective.

Proof. Take $P_2 = 0$. Then $Q$ is the direct sum of $P_1$ and $P_2$. □

In [16] a stronger version of projectivity is used in the tropical theory, studied intensively in [25], namely,

Definition 4.10. A $T$-module system is strongly projective if it is a direct summand of a free $T$-module system.

An example was given in [16] of a projective module that is not strongly projective.

Remark 4.11 (cf. [12 Proposition 17.14]). Every strongly projective $T$-module system is projective, since every free $T$-module system is projective.

4.1.1. $\preceq$-idempotent and $\preceq$-von Neumann regular matrices.

Recall that an $m \times n$ matrix $A$ (with entries in a commutative ring) is said to be von Neumann regular if there exists a matrix $B$ such that $A = ABA$. Classically, there is a well-known correspondence among von Neumann regularity, idempotency, and projectivity. In the tropical setting, as pointed out in [15], projectivity can be expressed in terms of idempotent and von Neumann regular matrices.

In what follows, we assume that all matrices have entries in a system $\mathcal{A}$ unless otherwise stated. We generalize the aforementioned correspondence to the $\preceq$-version.

Definition 4.12. We say $A \preceq B$ for $m \times n$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$, if $a_{i,j} \preceq b_{i,j}$ for all $i,j$.

An $m \times n$ matrix $A$ is $\preceq$-idempotent if $A \preceq A^2$.

An $m \times n$ matrix $A$ is $\preceq$-von Neumann regular if there is an $n \times m$ matrix $B$ for which $A \preceq ABA$.

Proposition 4.13. Suppose $A$ is $\preceq$-idempotent. Then the module $AF$ is $\preceq$-projective; in other words the column space of $A$ is a $\preceq$-projective $A$-submodule of $F$, and symmetrically the row space of $A$ is a $\preceq$-projective $A$-submodule of $F$.

Proof. Define $\pi : F \to AF$ by $\pi(\nu) = Av$. Then $\pi \preceq \pi^2$, and taking $\nu : AF \to F$ to be the identity, we have $1 \preceq \pi\nu$ on $AF$, so we conclude by Proposition 4.17. □

Corollary 4.14. If $A \preceq ABA$, then $ABF$ is $\preceq$-projective.

Proof. $AB$ is $\preceq$-idempotent, since $AB \preceq (AB)^2 = (ABA)B$. □

The analogous results hold for $\preceq$-idempotent and $\preceq$-von Neumann regular. This raises the question of whether $ABF = AF$ when $A \preceq ABA$. Clearly $AF \preceq ABAF \subseteq ABF \subseteq AF$, which often implies equality, but a thorough discussion would take us too far afield here.

Trifaj [37] has considered the dual to Baer’s criterion:

We say a module system $\mathcal{M}$ is finitely $\preceq$-generated (as a module system) if it is $\preceq$-generated by a finite set of cyclic module systems.

Remark 4.15 (As in [37 p. 2]). Suppose for any onto morphism $h : \mathcal{M} \to \mathcal{M}'$ of $T$-module systems, with $\mathcal{M}$ cyclic, that every $\preceq$-morphism $f : \mathcal{P} \to \mathcal{M}'$ $\preceq$-lifts to a $\preceq$-morphism $\tilde{f} : \mathcal{P} \to \mathcal{M}$. Then this condition holds for $\mathcal{M}$ $\preceq$-finitely generated. (Indeed, write $\mathcal{M} \preceq \sum_i Aa_i$ for $a_i \in \mathcal{M}$, apply the criterion for each $Aa_i$, and add the $\preceq$-lifts, i.e., $\tilde{f}(a) = \sum_i \tilde{f}_i(a)$).

[37] Lemma 2.1] gives a countable counterexample to this condition, and presents a readable and interesting account of the dual Baer criterion in the classical case.
4.2. The dual basis lemma.

Deore and Pati \[5\] proved a dual basis lemma for projective modules, and the same proof works for \(\leq\)-projectives and \(\geq\)-projectives.

**Proposition 4.16.** Suppose a module pseudo-system \((\mathcal{P}, \mathcal{T}_\mathcal{P}, (-), \leq)\) is \(\leq\)-generated by \(\{a_i : i \in I\}\).
Then \(\mathcal{P}\) is \(\leq\)-projective if and only if there are \(\leq\)-onto \(\leq\)-morphisms \(g_i : \mathcal{P} \to \mathcal{A}\) such that for all \(a \in \mathcal{A}\) we have a \(\leq \sum g_i(a) a_i\), where \(g_i(a) = 0\) for all but finitely many \(i\).

**Proof.** The assertion can be copied almost word for word from the standard proof, for example from [28, pp. 165–167]. We take the free module system \(\mathcal{F} = (\mathcal{A}(I), \mathcal{T}(I), (-), \leq)\) with base \(\{e_i : i \in I\}\), and the \(\leq\)-onto morphism \(f : \mathcal{F} \to \mathcal{P}\) given by \(f(e_i) = a_i, \forall i \in I\). Also we define the canonical projections \(\pi_j : \mathcal{F} \to \mathcal{A}\) by \(\pi_j(e_i) = \delta_{ij}\). Thus \(c = \sum \pi_i(c) e_i\) for any \(c \in \mathcal{F}\).

\((\Rightarrow)\) In view of Proposition 4.3, \(f\) is \(\leq\)-split, so we take a \(\leq\)-morphism \(g : \mathcal{P} \to \mathcal{F}\) with \(fg \geq 1_{\mathcal{P}}\). Put \(g_i = \pi_i g : \mathcal{P} \to \mathcal{A}\). Then any \(a \in \mathcal{P}\) satisfies

\[ a \leq fg(a) = f(\sum_i \pi_i(g(a)) e_i) = \sum_i f(g_i(a) e_i) = \sum_i g_i(a) f(e_i) = \sum_i g_i(a) a_i, \]

as desired.

\((\Leftarrow)\) Defining \(g : \mathcal{P} \to \mathcal{F}\) by \(g(a) = \sum g_i(a) e_i\), we have

\[ fg(a) = \sum g_i(a) f(e_i) = \sum g_i(a) a_i \geq a. \]

Thus \(fg \geq 1_{\mathcal{P}}\), so \(\mathcal{P}\) is \(\leq\)-projective, by Proposition 4.3.\(\Box\)

**Proposition 4.17.** Suppose a module pseudo-system \((\mathcal{P}, \mathcal{T}_\mathcal{P}, (-), \leq)\) is generated by \(\{a_i : i \in I\}\).
Then \(\mathcal{P}\) is \(\geq\)-projective if and only if there are \(\geq\)-onto \(\geq\)-morphisms \(g_i : \mathcal{P} \to \mathcal{A}\) such that for all \(a \in \mathcal{A}\) we have \(a \geq \sum g_i(a) a_i\), where \(g_i(a) = 0\) for all but finitely many \(i\).

**Proof.** The analogous argument to the proof of Proposition 4.16 works.\(\Box\)

4.3. Schanuel’s Lemma over semirings and systems.

We turn to a systemic version of Schanuel’s Lemma which should play an important role defining systemic homological dimension. In the classical case, given two exact sequences \(\mathcal{K} \to \mathcal{P} \overset{f}{\to} \mathcal{M}\) and \(\mathcal{K}' \to \mathcal{P}' \overset{f'}{\to} \mathcal{M}\), with \(f, f'\) epic and \(\mathcal{P}, \mathcal{P}'\) projective, one concludes that \(\mathcal{P} \oplus \mathcal{K}' = \mathcal{P}' \oplus \mathcal{K}\). However, for general semirings, one cannot expect this to hold. In fact, the right notion of exactness for semirings is rather subtle and not yet settled. Still, one can mimic the standard proof [28 pp. 165–167] of Schanuel’s Lemma for modules over rings, by considering our more general version of splitting, and avoiding mixing submodules with kernels (which are congruences). To this end, we introduce the following definition of congruence kernels.

**Definition 4.18.** Let \(f : \mathcal{M} \to \mathcal{N}\) be a \(\leq\)-morphism.

(i) The \(\mathcal{N}\)-congruence kernel \(\ker_N f\) of \(f\) is defined to be the following set:

\[ \ker_N f := \{ (a_0, a_1) \in \mathcal{M} \times \mathcal{M} : f(a_0) = f(a_1) \}. \]

(ii) The \(\leq\)-congruence kernel \(\ker_{\leq} f\) of \(f\) is defined to be the following set:

\[ \ker_{\leq} f := \{ (a_0, a_1) \in \mathcal{M} \times \mathcal{M} : f(a_0) = f(a_1), \ f(a_0), f(a_1) \in \mathcal{N}_{\text{Null}} \}. \]

(iii) The \(\leq\)-pseudocongruence kernel \(\ker_{\leq} f\) of \(f\) is defined to be the following set:

\[ \ker_{\leq} f := \{ (a_0, a_1) \in \mathcal{M} \times \mathcal{M} : f(a_0) \leq f(a_1), \ f(a_0), f(a_1) \in \mathcal{N}_{\text{Null}} \}. \]

**Lemma 4.19.** Let \(f : \mathcal{M} \to \mathcal{N}\) be a morphism. Then \(\ker_N f\), \(\ker_{\leq} f\), and \(\ker_{\leq} f\) are submodules of \(\mathcal{M} \times \mathcal{M}\). \(\ker_N f\) and \(\ker_{\leq} f\) are congruences.

**Proof.** This is clear.\(\Box\)

(Transitivity could fail for \(\ker_{\leq} f\).)
Theorem 4.20 (Semi-Schauen). Suppose we have two morphisms $P_1 \xrightarrow{f_1} M$ and $P_2 \xrightarrow{f_2} M$ with $f_2$ onto. (We are not assuming that either $P_i$ is projective.)

(i) There is a submodule
\[ P = \{(b_1, b_2) : f_1(b_1) = f_2(b_2)\} \]
of $P_1 \oplus P_2$ together with an onto morphism $\pi^{res} : P \to P_1$ and an $N$-quasi-isomorphism
\[ \ker N \pi^{res} \to \ker N f_2, \]
where $\pi_i$ is the projection to the $i$-th coordinate. (This part is purely semiring-theoretic and does not require a system.)

(ii) The maps $f_1 \pi_1, f_2 \pi_2 : P \to M$ are the same.

(iii) In the systemic setting, $\pi^{res}$ also induces $\leq$-quasi-isomorphisms
\[ \ker N \leq \pi^{res}_1 \to \ker N \leq f_2, \quad \ker \leq f_1 \pi^{res}_1 \to \ker \leq f_2. \]

(iv) If $P_1$ is projective, then it is a retract of $P$ with respect to the projection $\pi_1 : P \to P_1$.

(v) If $P_1$ is $\leq$-projective, then it is a $\leq$-retract of $P$ with respect to the projection $\pi_1 : P \to P_1$, and $P$ is the $\leq$-direct sum of $P_1$ and $(1_p - \pi_1(\pi_1)) (P)$.

(vi) If $N$ is a $\pi_1$-null submodule of $P$, then $\pi_2(N)$ is an $f_2$-null submodule of $P_2$. Hence there is a $\leq$-quasi-isomorphism from $\ker_{\Mod, \pi} \pi_1$ to $\ker_{\Mod, \pi} f_2$.

Proof. We modify the standard proof. Clearly $P$ is a submodule of $P_1 \oplus P_2$. Since $f_2$ is onto, for any $b_1 \in P_1$ there is $b_2 \in P_2$ such that $f_1(b_1) = f_2(b_2)$, implying $(b_1, b_2) \in P$. Hence $\pi_1$ restricts to an onto morphism $P \to P_1$. We denote this restriction by $\pi_1^{res}$. For the remaining part of the assertion, one can easily see that
\[ \ker N \pi_1^{res} \subseteq \{((b_1, b_2), (b_1', b_2')) : b_1 = b_1'\} \]
But, by the definition of $\pi_1^{res}$ and $P$, we have that, for
\[ (b_1, b_2), (b_1', b_2') \in \ker N \pi_1^{res}, \]
\[ f_2(b_2) = f_1(b_1) = f_2(b_2'), \]
which means that $(b_2, b_2') \in \ker N f_2$. In other words,
\[ \ker N \pi_1^{res} = \{((b_1, b_2), (b_1', b_2')) : b_1 \in P_1, (b_2, b_2') \in \ker N f_2\}. \]
We define an onto morphism as follows:
\[ \pi : \ker N \pi_1^{res} \to \ker N f_2, \quad (b_1, b_2) \mapsto (b_2, b_2'), \]
where $b_1 \in P_1$. All it remains to show is that $\pi$ is $N$-monic. But if $\pi(b_1, b_2) = \pi(b_1, b_2')$ then $b_2 = b_2'$, showing that $\pi$ is $N$-monic. Thus $\pi$ is an $N$-quasi-isomorphism.

(ii) $f_1 \pi_1(b_1, b_2) = f_1(b_1) = f_2(b_2) = f_2 \pi_2(b_1, b_2)$.

(iii) The proof for the first $\leq$-quasi-isomorphism $\ker N \leq \pi_1^{res} \to \ker N \leq f_2$ is similar to the proof of (i). Notice that $\ker N \leq \pi_1^{res}$ consists of those pairs $((b_1, b_2), (b_1', b_2'))$ for which $b_1 = b_1'$, and $b_1, b_1' \in \mathcal{P}_{1, \text{null}}$. But then
\[ f_2(b_2) = f_1(b_1) = f_1(b_1') = f_2(b_2'), \]
which are all in $\mathcal{M}_{\text{null}}$ since $f_1(b_1) \in \mathcal{M}_{\text{null}}$, implying $(b_2, b_2') \in \ker N \leq f_2$. We define the $\leq$-morphism
\[ \pi_\leq : \ker N \leq \pi_1^{res} \to \ker N \leq f_2 \]
by
\[ ((b_1, b_2), (b_1', b_2')) \mapsto (b_2, b_2'). \]
Now, suppose that $(b_2, b_2') \in \ker N \leq f_2$, i.e., $f_2(b_2) = f_2(b_2') \in \mathcal{M}_{\text{null}}$. Since $f_2$ is onto, we can find $b_1, b_1' \in \mathcal{P}_1$ such that $f_1(b_1) = f_2(b_2)$ and $f_1(b_1') = f_2(b_2')$. Clearly, we have that $((b_1, b_2), (b_1', b_2')) \in \ker N \leq \pi_1^{res}$ which shows that $\pi_\leq$ is onto. All it remains to show is that $\pi_\leq$ is null-monic. Suppose that
\[ \pi_\leq((b_1, b_2), (b_1', b_2')) = (b_2, b_2') \in \ker N f_2_{\text{null}}. \]
This means that $b_2, b_2' \in \mathcal{P}_{2, \text{null}}$. It follows that $((b_1, b_2), (b_1', b_2'))$ is an element of $(\ker N \leq \pi_1^{res})_{\text{null}}$, showing that $\pi_\leq$ is null-monic by Lemma 3.3. Thus $\pi_\leq$ is an $\leq$-quasi-isomorphism.

The proof for the second $\leq$-quasi-isomorphism $\ker \leq f_1 \pi_1^{res} \to \ker \leq f_2$ is also similar. Slightly abusing notation, we define the following $\leq$-morphism:
\[ \pi_\leq : \ker \leq f_1 \pi_1^{res} \to \ker \leq f_2, \quad ((b_1, b_2), (b_1', b_2')) \mapsto (b_2, b_2'). \]
One can easily see that \( \pi \leq \) is null-monic from the exact same argument as above. Now, suppose that \((b_2, b_2') \in \ker_2 f_2\). In other words, we have that \(f_2(b_2) \leq f_2(b_2')\) and \(f_2(b_2), f_2(b_2') \in M_{\mathbb{Null}}\). Again, since \(f_2\) is onto, we can find an element \(\alpha := ((b_1, b_2), (b_1', b_2')) \in \mathcal{P}\) which maps to \((b_2, b_2')\). We claim that \(\alpha \in \ker \leq f_1\pi_1^{\text{res}}\). In fact, we have

\[ f_1\pi_1^{\text{res}}(b_1, b_2) = f_1(b_1) = f_2(b_2) \leq f_2(b_2') = f_1(b_1') = f_1\pi_1^{\text{res}}(b_1', b_2'). \]

Furthermore, since \(f_2(b_2) \in M_{\mathbb{Null}}\), we have that \(f_1\pi_1^{\text{res}}(b_1, b_2), f_1\pi_1^{\text{res}}(b_2', b_2') \in M_{\mathbb{Null}}\), proving our claim.

(iv) Since \(P_1\) is projective and \(\pi : \mathcal{P} \to P_1\) is onto, \(\pi_1\) splits via \(\nu_1\) such that \(\pi_1\nu_1 = 1\).

(v) Take \(\nu_1 : P_1 \to \mathcal{P}\) be the \(\leq\)-morphism \(\leq\)-splitting \(\pi_1\) via the identity map on \(P_1\), and we can apply Theorem 3.16.

(vi) Continuing the proof of (v), \(\nu_1(b_1) = (b_1, b_2')\) for suitable \(b_2' \in \mathcal{P}_2\). Given \(b = (b_1, b_2) \in \mathcal{P}\), we have \(\nu_1 \pi_1(b) = \nu(b_1) = (b_1, b_2'), \) where \(f_2(b_2') = f_1(b_1)\). Hence

\[ f_2(b_2)(b_2') = f_2(b_2') - f_2(b_2) = f_1(b_1)(-f_1(b_1)) = f_1(b_1)(-b_1) = f_1(b_1)(-b_1) \geq 0. \]

If we have \(b_1 = \pi_1(b) \geq 0\), then \(f_2(b_2) \geq 0\), proving our claim.

Remark 4.21. Theorem 4.20(iii),(vi) are to be used in [19] to explore homological dimension.

We also will need the following semiring analog of the classical proof of Schanuel.

Theorem 4.22. If \(P_1\) is \(\leq\)-projective with an onto morphism \(\pi : \mathcal{P} \to P_1\) whose module kernel \(K\) is \(\leq\)-projective, then \(\mathcal{P}\) also is \(\leq\)-projective.

Proof. Lift the identity morphism of \(P_1\) to a retract \(\nu : P_1 \to \mathcal{P}\) of \(\pi\), and let \(\pi_2 = \mathbb{1}_P(-\nu\pi),\) which is \(\leq\)-idempotent by Lemma 3.13(ii). Consider a \(\leq\)-onto morphism \(h : M \to M'\). Then for any morphism \(f : \mathcal{P} \to M'\), we lift \(f\nu : P_1 \to M'\) past \(h\) to a morphism \(f_1 : P_1 \to M\). Next, we lift \(f_1K : K \to M'\) to a morphism \(f_2 : K \to M\). We claim that for any \(b \in \mathcal{P}\), \(\pi_2(b) \in K\). In fact, for any \(b \in \mathcal{P}\), we have

\[ \pi_2(b) = \pi(\mathbb{1}_P(-\nu\pi)(b)) = \pi(b(-\nu\pi)(b)) = \pi(b(-\nu\pi)(b)) \geq \pi(b)(-\nu\pi(b)) \geq 0. \]

This implies that \(\pi_2(b) \in (\mathcal{P}_1)_{\mathbb{Null}}\) and hence \(\pi_2(b) \in K\). Now, we define a \(\leq\)-morphism \(\hat{f} : \mathcal{P} \to M\) as follows:

\[ \hat{f}(b) = f_1(\pi_2(b)) + f_2(\pi_2(b)). \]

Then \(\hat{f}\) is well-defined since \(\pi_2(b) \in K\). For any \(b \in \mathcal{P}\) we have

\[ h\hat{f}(b) = h\hat{f}_1(\pi_2(b)) + h\hat{f}_2(\mathbb{1}_P(-\nu\pi)(b)) \geq f(\mathbb{1}_P(-\nu\pi)(b)) = f(b + \nu\pi(b)(-\nu\pi(b)) \geq f(b), \]

proving \(h\hat{f} \geq f\), i.e. \(\hat{f} \leq\)-lifts \(f\).

4.4. \(\leq\)-Morita theory.

One major classical application of projective modules is in Morita’s theorem. Bass’ approach, as given for example in [31 §4.1], does not use negation, so can be formulated over semirings, as done in [24 §3]. We do it here for systems and \(\leq\)-morphisms, in order to handle the hyperfield case. [24 Definition 3.8] defines \(\mathcal{N}\) to be a generator if for every \(\mathcal{A}\)-module \(M\) there is an index set \(I\) and an onto morphism \(\mathcal{N}^{(I)} \to M\).

Definition 4.23. \(\mathcal{N} := (\mathcal{N}, \mathcal{T}, (-), \leq)\) is a generator of a system \(\mathcal{A} := (\mathcal{A}, \mathcal{T}, (-), \leq)\) if for every \(\mathcal{A}\)-module system \(\mathcal{M} := (\mathcal{M}, \mathcal{T}, (-), \leq)\) there is an index set \(I\) and an onto morphism \(\mathcal{N}^{(I)} \to \mathcal{M}\).

Any generator clearly is a \(\leq\)-generator. For example, \(\mathcal{A}\) is a generator and a \(\leq\)-generator.

Given an \(\mathcal{A}\)-module \(M\), we define \(M^* = \text{Hom}(\mathcal{M}, \mathcal{A})\).

Definition 4.24. The trace ideal \(T(A)\) is \(\{\text{finite } f(a) : f \in M^*, a \in A\}\).

By [24 Proposition 3.9], the following conditions are equivalent:
(i) \( \mathcal{A} \) is a generator.
(ii) \( T(\mathcal{A}) \) generates \( \mathcal{A} \).
(iii) There exists \( n \) such that some homomorphic image of \( \mathcal{A}^{(n)} \) generates \( \mathcal{A} \).

Lemma 4.25. The following conditions are equivalent:

(i) \( \mathcal{A} \) is a \( \preceq \)-generator.
(ii) \( T(\mathcal{A}) \preceq\)-generates \( \mathcal{A} \).
(iii) There exists \( n \) such that some homomorphic image of \( \mathcal{A}^{(n)} \preceq\)-generates \( \mathcal{A} \).

Proof. Analogous; say, follow \cite{31} Lemma 4.1.7. \( \square \)

Definition 4.26. A \( \preceq\)-progenerator is a \( \preceq\)-finitely generated \( \preceq\)-projective module which is a \( \preceq\)-generator.

We define semiring bimodules in the usual way (i.e., satisfying the classical associativity condition).

To continue, we need the tensor product of systems over a ground \( \mathcal{T}\)-system. These are described (for semirings) in terms of congruences, as given for example in \cite{22} Definition 3] or, in our notation, \cite{23} §3.

We do it for systems, taking the negation map into account.

Let us work with a right \( \mathcal{A}\)-module system \( M_1 \) and left \( \mathcal{A}\)-module system \( M_2 \) over a given ground \( \mathcal{T}\)-system \( \mathcal{A} \). One defines the tensor product \( M_1 \otimes \mathcal{A} M_2 \) of \( M_1 \) and \( M_2 \) in the usual way, to be \( (F_1 \oplus F_2)/\Phi \), where \( F_i \) is the free system (respectively right or left) with base \( M_i \) (and \( T_{F_1} = M_i \)), and \( \Phi \) is the congruence generated by all pairs

\begin{equation}
\left( \sum_j x_{1,j}, \sum_k x_{2,k} \right), \left( \sum_{j,k} (x_{1,j}, x_{2,k}) \right), \left( (x_{1,a}, x_{2}), (x_{1}, ax_2) \right), \left( (x_{1}, x_2), ((-x_1), (-x_2)) \right)
\end{equation}

\( \forall x_{1,j} \in M_i, a \in \mathcal{A} \), as well as the extra axiom

\(((x) \otimes y) = x \otimes ((-y))\).

We define a negation map on \( M_1 \otimes \mathcal{A} M_2 \) by \((-v \otimes w) = ((-v)) \otimes w\).

Definition 4.27. The negated tensor product \( M_1 \otimes \mathcal{A} M_2 \) of a right module triple \( (M_1, T_1, (-)) \) and a left module triple \( M_2 \) is \( ((F_1 \oplus F_2)/\Phi) \otimes \mathcal{T_2}, T_{M_1 \otimes \mathcal{A} M_2}, (-\Phi), \Phi) \), where \( F_i \) is the free system with base \( M_i \) (and \( T_{M_1 \otimes \mathcal{A} M_2} \) is the set of “simple tensors” \( a_1 \otimes a_2 \) for \( a_i \in T_i \), and \( \Phi \) is the congruence generated as in \textit{\cite{12}}).

Definition 4.28. A \textbf{(Systemic) Morita context} is a six-tuple \( (\mathcal{A}, \mathcal{A}', \mathcal{M}, \mathcal{M}', \tau, \tau') \) where \( \mathcal{A}, \mathcal{A}' \) are systems, \( \mathcal{M} \) is an \( \mathcal{A} \sim \mathcal{A}' \) bimodule, \( \mathcal{M}' \) is an \( \mathcal{A}' \sim \mathcal{A} \) bimodule, and

\[ \tau : \mathcal{M} \otimes \mathcal{A} \mathcal{M}' \to \mathcal{A}, \quad \tau' : \mathcal{M}' \otimes \mathcal{A} \mathcal{M} \to \mathcal{A}' \]

are morphisms, linear on each side over \( \mathcal{A} \) and \( \mathcal{A}' \) respectively, which satisfy the following equations, writing \( (x, x') \) for \( \tau(x, x') \) and \( [x', x] \) for \( \tau(x', x) \):

(i) \((x, x')y = x[x', y], \)
(ii) \( x[x', y'] = [x', x]y' \).

Remark 4.29. Another way of describing a Morita context is to say that \( \left( \begin{array}{ccc} \mathcal{A} & \mathcal{M} \\ \mathcal{M}' & \mathcal{A}' \end{array} \right) \) is a semiring, whose tangible elements have tangible components and whose negation map and surpassing relation are given componentwise.

Theorem 4.30. Suppose \( (\mathcal{A}, \mathcal{A}', \mathcal{M}, \mathcal{M}', \tau, \tau') \) is a systemic Morita context.

(1) (a) If \( \tau' \) is \( \preceq\)-onto, then \( \mathcal{M} \) is a \( \preceq\)-progenerator for \( \mathcal{A}\mathcal{M}\).
(b) If \( \tau \) is \( \succeq\)-onto, then \( \tau' \) is null-monic.

(2) The analogous statements hold if we switch left and right, or \( (\tau, \mathcal{M}) \) and \( (\tau', \mathcal{M}') \).

Proof. We prove (1) as in \cite{31} page 473, since (2) is analogous. \( \mathcal{M} \) is a \( \preceq\)-generator by Lemma \textit{\cite{12}}.

Writing \((x, x')\) for the morphism \( x \mapsto (x, x') \), we assume that \( \sum (x, x') \preceq 1 \) and \( \sum (y_j, y'_j) \geq 1 \).

(a) Write \( f_j = (y, y'_j) \). We claim that \( \{ (y_j, f_j) : 1 \leq j \leq t \} \) comprise a dual basis. Indeed,
\[ x \preceq \sum x[y'_j, y_j] = \sum (x, y'_j)y_j = \sum f_j(x)y_j. \]
(b) We claim that if \( b = \sum z_k \otimes z'_k \in \ker N \) then \( b \succeq 0 \). Indeed,

\[
\sum z_k \otimes z'_k \succeq \sum_{k=1}^N \sum_{i=1}^{z_k} [z_k, x_i] \otimes [z'_k, x'_i] = \sum_{i,k} z_k [z'_k, x_i] \otimes x'_i = \sum_{i,k} (z_k, z'_k) x_i \otimes x'_i \succeq 0.
\]

The last assertion follows since \( \tau \) and \( \tau' \) are morphisms.

\[ \Box \]

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