A GRADIENT ESTIMATE FOR HARMONIC FUNCTIONS
SHARING THE SAME ZEROS

DAN MANGOUBI

(Communicated by Lenya Ryzhik)

Abstract. Let $u, v$ be two harmonic functions in $\{|z| < 2\} \subset \mathbb{C}$ which have exactly the same set $Z$ of zeros. We observe that $|\nabla \log |u/v||$ is bounded in the unit disk by a constant which depends on $Z$ only. In case $Z = \emptyset$ this goes back to Li-Yau’s gradient estimate for positive harmonic functions. The general boundary Harnack principle gives only Hölder estimates on $\log |u/v|$.

1. Introduction

1.1. Background and statement. Consider a positive harmonic function $u$ in the open disk of radius two $B_2 \subset \mathbb{C}$. The Harnack inequality gives a bound on $|\log u(z_1) - \log u(z_2)|$ where $z_1, z_2$ run in the unit disk. This bound is independent of $u$. Let us rephrase this statement: If we let $v = 1$ be the constant function then

$$\left| \log \frac{u(z_1)}{v(z_1)} - \log \frac{u(z_2)}{v(z_2)} \right|$$

is bounded by a constant independent on $u$ when $z_1, z_2 \in B_1$.

The preceding formulation leads to the following natural question: What can be said about the quotient $u/v$ in case $u, v$ are harmonic functions in $B_2$ which do change sign in $B_2$ and have exactly the same set of zeros? The aim of this short note is to give an answer to this question in two dimensions and to pose two related natural problems.

Let us assume for clarity of the introduction that $u = f \cdot v$ for some smooth function $f > 0$ (in fact, below we show that this is always the case). We recall an instance of the boundary Harnack principle (BHP):

Received by the editors June 19, 2013 and, in revised form, December 31, 2013.

2010 Mathematics Subject Classification. 31B05,35J15.

Key words and phrases. Li-Yau, harmonic functions, nodal set, gradient estimates, Harnack, boundary Harnack principle.

I am happy to thank Jozef Dodziuk for preliminary discussions on Li-Yau’s gradient estimates. I am grateful to David Kazhdan and Leonid Polterovich for asking me questions which led me to the present work. I thank Benji Weiss for an interesting example (§8). I thank Gady Kozma and Fedja Nazarov for their interest in this work and for referring me to the BHP. I especially thank Misha Sodin for his encouragement and for finding a second simple proof (§9). I am grateful to Carlos Kenig for his interest and for an illuminating example (§8.2) clarifying the relation of this note to the BHP. I am grateful to Charles Fefferman for his advice and interest. I thank S.-T. Yau for his continuous support. I express my gratitude to the anonymous referee for drawing my attention to the possible connections with [6]. This research was supported by ISF grant 225/10 and by BSF grant 2010214.
Theorem 1.2 ([1, 12, 3, 5, 2, 10]). Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $T \subset \partial \Omega$ a boundary portion such that $u|_T$ and $v|_T$ vanish. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic functions such that $u, v > 0$ in $\Omega$. Then $u/v \in C^\alpha(\Omega \cup T)$ for some $\alpha > 0$.

Theorem 1.2 applied to our situation shows that if $\Omega$ is a connected component of $\{u \neq 0\}$ in $B_2$ then $u/v$ is in $C^\alpha(\overline{\Omega} \cap B_1)$ for some $0 < \alpha < 1$. In this note we make the observation that in fact $u/v \in C^1(\overline{\Omega} \cap B_1)$ with some effective $C^1$ bound, i.e., we show that $|\nabla \log \frac{u}{v}|$ is a priori bounded in $B_1$. In case $v$ is the constant function and $u$ is a positive harmonic function this goes back to Li-Yau’s gradient estimate ([7]).

Another point of view of our observation is to say that if in Theorem 1.2 the harmonic functions can be extended as harmonic functions across $\partial \Omega$, then one has a $C^1$-bound on $\log u/v$. In fact, an example due to Carlos Kenig ([8.2]) shows that one cannot in general obtain a $C^1$-bound even when the boundary of the domain is composed of straight lines. The precise result we show is

Theorem 1.3. Let $Z \subset B_2$. Let

$$ \mathcal{F}(Z) = \{u \in C^2(B_2) \mid \Delta u = 0 \text{ and } \{u = 0\} = Z\} $$

Then for all $u, v \in \mathcal{F}(Z)$ $u/v$ extends to a smooth nowhere vanishing function in $B_2$ and there exists a constant $C_Z > 0$ such that

$$ \forall u, v \in \mathcal{F}(Z) \quad |\nabla \log |u/v|| \leq C_Z \text{ in } B_1. $$

1.2. Two questions. We pose two questions which arise naturally from Theorem 1.3.

Deformations of the zero set. A result close in spirit to Theorem 1.3 is proved in [9]. In that paper Nadirashvili considers a variable sign harmonic function in $B_2$, and proves the existence of a bound on $|u|$ in $B_1$ in terms of the first $3k$ derivatives of $u$ at 0, where $k$ is the number of connected components of $B_2 \setminus Z$. Nadirashvili’s result hints that it may be true that the constant $C_Z$ in Theorem 1.3 depends only on the number of components of $B_2 \setminus Z$. Equivalently, we ask whether it is true that the constant $C_Z$ depends only on the number of intersection points of $Z$ with the circle $\{|z| = 3/2\}$.

Higher dimensions. We ask whether Theorem 1.3 is special to two dimensions or stays true in three dimensions. It seems that an answer even in the case where $Z$ is the zero set of a quadratic harmonic polynomial in three variables is unknown. We remark that if $v$ is linear then it follows from [6] that $u/v \in C^{1,\alpha}(\Omega \cap B_1)$ for a wide class of fully non-linear elliptic operators in any dimension (see also [11] and [4]*Theorem 9.31 and the remark afterwards). We refer to [8] for interesting new results regarding this question.

1.3. Idea of proof. We give two proofs of Theorem 1.3. Both proofs are based upon the cases where $Z$ is in normal form (say, $Z = \{3z^k = 0\}$). This is the reason why our method is restricted to two dimensions.

The first proof we give is based on the maximum principle in a similar spirit to [7]. In a normal form case we find a certain positive definite quadratic form which leads to a Bochner type estimate. In the case where $Z$ is empty (the Li-Yau case) the positivity of the quadratic form we define is evident. Then, we use the Bochner type estimate to get a gradient estimate.
The second (shorter) proof, due to Misha Sodin, is more direct and is based on
the Poisson formula for sectors of the plane.
We decided to keep both proofs in this note since we hope the first proof may
be useful to extend Theorem 1.3 to situations where no explicit formulas exist.

1.4. Organization of the paper. The proof of the normal form case of Theo-
rem 1.3 is given in Section 6. The reduction to the normal form case is done in
Section 7. In Section 9 we give an alternative proof using the Poisson formula due
to Misha Sodin. In sections 2-5 we develop different ingredients of the proof: In
Section 2 we calculate the singular elliptic equation satisfied by the quotient of
two harmonic functions. We also find a Bochner type formula. In Section 3 we
define and prove the positivity of a certain quadratic form needed in the proof of a
Bochner type Inequality proved in Section 4. In Section 5 we treat several expres-
sions which involve logarithmic singularities. Finally, in Section 8 we give several
examples which illustrate Theorem 1.3 and clarify it.

2. A semi-linear singular elliptic equation

Lemma 2.1. Let $f > 0$ be a smooth function. Let $v$ be harmonic. If $fv$ is harmonic
then $h := \log f$ satisfies the following second order semi-linear degenerate elliptic
equation

$$v \Delta h + 2 \langle \nabla v, \nabla h \rangle + v|\nabla h|^2 = 0.$$  (2.2)

Proof. By the chain rule,

$$0 = \Delta (e^h v) = e^h (v \Delta h + 2 \langle \nabla v, \nabla h \rangle + v|\nabla h|^2).$$

Lemma 2.3. Let $v$ be a smooth function and let $h$ satisfy equation (2.2). Let $F = |\nabla h|^2$. Then

$$v^2 \Delta F = 2v^2 |\text{Hess } h|^2 + 4 \langle \nabla h, \nabla v \rangle^2 - 4v(\text{Hess } v)(\nabla h, \nabla h)$$

$$- 2v(\nabla F, \nabla v) - 2v^2 \langle \nabla F, \nabla h \rangle.$$  (2.4)

Proof. It follows immediately from (2.2) that at points where $v = 0$ equation (2.4)
is also satisfied. Therefore we can assume $v \neq 0$. We differentiate

$$F_{,k} = 2 \langle \nabla h, \nabla h_{,k} \rangle \quad F_{,kk} = 2 \langle \nabla h_{,k}, \nabla h_{,k} \rangle + 2 \langle \nabla h, \nabla h_{,kk} \rangle.$$ 

Hence,

$$\Delta F = 2 \sum_{k,l} |h_{,kl}|^2 + 2 \langle \nabla h, \nabla \Delta h \rangle = 2 |\text{Hess } h|^2 + 2 \langle \nabla h, \nabla \Delta h \rangle.$$ 

By equation (2.2)

$$\langle \nabla h, \nabla \Delta h \rangle = - \langle \nabla h, \Delta \left( \frac{2 \langle \nabla v, \nabla h \rangle}{v} \right) \rangle - \langle \nabla h, \nabla F \rangle.$$

It remains only to simplify the expression on the right hand side:

$$\nabla \left( \frac{2 \langle \nabla v, \nabla h \rangle}{v} \right) = - \frac{2 \langle \nabla v, \nabla h \rangle}{v^2} \nabla v + \frac{2 \nabla (\langle \nabla v, \nabla h \rangle)}{v},$$

and an easy computation shows

$$\langle \nabla h, 2 \nabla (\langle \nabla v, \nabla h \rangle) \rangle = 2(\text{Hess } v)(\nabla h, \nabla h) + \langle \nabla F, \nabla v \rangle.$$

□
3. A NON-NEGATIVE QUADRATIC FORM

Let \( v_k(z) = \Im z^k \), where \( k \) is a non-negative integer. In this section we define a quadratic form related to \( v_k \) and to equation (2.4) and show it is non-negative. This will play a key role in the proof of the Bochner type inequality in Lemma 4.1.

**Definition 3.1.** Let \( X \) be a smooth vector field on \( B_2 \). We define
\[
Q_k(X) := (Xv_k)^2 - v_k(\text{Hess } v_k)(X,X).
\]

**Proposition 3.2.** For all vector fields \( X \) on \( B_2 \) the function \( Q_k(X) \) is non-negative. Moreover, we have
\[
(Q_k)(X) \geq \frac{1}{k} (Xv_k)^2.
\]
More precisely, we show
\[
Q_k = \frac{1}{k} (dv_k)^2 + k(k-1)r^{2k}(d\theta)^2.
\]

**Proof.** Let \( X,Y \) be two vector fields on \( B_2 \), and let
\[
\tilde{B}(X,Y) := \frac{k-1}{k} (Xv_k)(Yv_k) - v_k \text{Hess}(v_k)(X,Y).
\]
We compute in polar coordinates:
\[
\begin{align*}
\partial_r v_k &= kr^{k-1} \sin k\theta, \\
\partial_\theta v_k &= kr^k \cos k\theta, \\
\partial_{r \theta} v_k &= k^2 r^{k-1} \cos k\theta.
\end{align*}
\]
The Levi-Civita connection is given by
\[
\nabla_{\partial_r} \partial_r = 0, \quad \nabla_{\partial_\theta} \partial_\theta = \nabla_{\partial_\theta} \partial_r = \frac{\partial_\theta}{r}, \quad \nabla_{\partial_r} \partial_\theta = -r \partial_r.
\]
Recall that \((\text{Hess } f)(X,Y) = X(Y f) - (\nabla X Y)f\). Hence,
\[
\begin{align*}
\text{Hess}(v_k)(\partial_r, \partial_r) &= k(k-1)r^{k-2} \sin k\theta, \\
\text{Hess}(v_k)(\partial_r, \partial_\theta) &= k(k-1)r^{k-1} \cos k\theta, \\
\text{Hess}(v_k)(\partial_\theta, \partial_\theta) &= -k(k-1)r^k \sin k\theta.
\end{align*}
\]
We see that
\[
\tilde{B}(\partial_r, \partial_r) = \tilde{B}(\partial_r, \partial_\theta) = 0, \quad \tilde{B}(\partial_\theta, \partial_\theta) = k(k-1)r^{2k}.
\]
We conclude that
\[
Q_k(X) - \frac{1}{k} (Xv_k)^2 = \tilde{B}(X,X) = k(k-1)r^{2k}(d\theta(X))^2 \geq 0.
\]

\( \square \)

4. AN INEQUALITY À LA BOCHNER

In this section we prove an inequality of Bochner’s type. This will be crucial to prove Theorem 1.3.

Let \( v_k(z) = \Im z^k \) and let \( h \) satisfy equation (2.2) with \( v = v_k \). Let \( F = |\nabla h|^2 \). Then
Lemma 4.1. $F$ satisfies
\[ v_k^2 \Delta F + 2v_k(\nabla F, \nabla v_k) + 2v_k^2(\nabla F, \nabla h) \geq v_k^2 \frac{F^2}{k+1}. \]

Proof. By Lemma 2.3 and Proposition 3.2
\[
\begin{align*}
v_k^2 \Delta F + 2v_k(\nabla F, \nabla v_k) + 2v_k^2(\nabla F, \nabla h)
&= 2v_k^2 \lVert \text{Hess } h \rVert^2 + 4(\nabla h, \nabla v_k)^2 - 4v_k \text{Hess}(v_k)(\nabla h, \nabla h) \\
&\geq 2v_k^2 \lVert \text{Hess } h \rVert^2 + \frac{4}{k}(\nabla h, \nabla v_k)^2 \geq v_k^2(\Delta h)^2 + \frac{4}{k}(\nabla h, \nabla v_k)^2 \\
&\geq \frac{1}{k+1} (v_k \Delta h + 2(\nabla v_k, \nabla h))^2 = v_k^2 \frac{F^2}{k+1},
\end{align*}
\]
where we used that $2 \lVert \text{Hess } f \rVert^2 \geq (\Delta f)^2$ in $\mathbb{R}^2$, that $a^2 + \frac{b^2}{k} \geq \frac{(a+b)^2}{k+1}$ for real numbers $a, b$ and equation (2.2).

5. Study of $\langle \nabla F, \nabla v_k \rangle / v_k$

We would like to show that $\frac{1}{v_k^2}(\nabla F, \nabla v_k)$ extends to a smooth function in $B_2$. We will apply the following standard division lemma:

Lemma 5.1. Let $l(x, y) = ax + by$. Let $f \in C^\infty(B_2)$ be such that $f(x, y) = 0$ whenever $l(x, y) = 0$. Then there exists $q \in C^\infty(B_2)$ such that $f(x, y) = q(x, y)l(x, y)$.

Proof. We can assume without loss of generality that $b = 0$ and $a \neq 0$. Define
\[ q(x, y) = \frac{1}{a} \int_0^1 (\partial_1 f)(tx, y) \, dt. \]

Lemma 5.2. Let $f \in C^\infty(B_2)$ be such that $f = 0$ whenever $v_k = 0$. Then, there exists $q \in C^\infty(B_2)$ such that $f = q v_k$.

Proof. $v_k$ can be expressed as a product of $k$ linear factors. In fact
\[ v_k(x, y) = a_k \prod_{l=0}^{k-1} \left( y \cos \frac{l\pi}{k} - x \sin \frac{l\pi}{k} \right) \]
for some $a_k \in \mathbb{R}$. Hence, the lemma follows by induction from Lemma 5.1.

Lemma 5.3. There exists a smooth function $G_k$ such that
\[ \langle \nabla F, \nabla v_k \rangle = G_k v_k. \]

Proof. By Lemma 5.2 it is enough to show that $\langle \nabla F, \nabla v_k \rangle$ vanishes whenever $v_k$ does. This will follow from equation (2.2). Indeed, let $p \in B_2$, $p \neq 0$ be such that $v_k(p) = 0$. Then
\[ \langle \nabla F, \nabla v_k \rangle(p) = F_r(p)(v_k)_r(p) + \frac{1}{|p|^2} F_\theta(p)(v_k)_\theta(p) = \frac{1}{|p|^2} F_\theta(p)(v_k)_\theta(p). \]

We will show that $F_\theta(p) = 0$.
\[ F = h_r^2 + \frac{h_\theta^2}{r^2}. \]
A GRADIENT ESTIMATE FOR HARMONIC FUNCTIONS SHARING THE SAME ZEROS

So,

\[ F_\theta(p) = 2h_\cdot r(p)h_\cdot \theta(p) + \frac{2}{|p|^2} h_\cdot \theta(p)h_\cdot \theta(p). \]

By equation (2.2),

\[ 0 = \langle \nabla h, \nabla v_k \rangle(p) = h_\cdot r(p)v_{k, r}(p) + \frac{1}{|p|^2} h_\cdot \theta(p)v_{k, \theta}(p) = 1 |p|^2 h_\cdot \theta(p)v_{k, \theta}(p). \]

Since \( v_{k, \theta}(p) \neq 0 \) we conclude that \( h_\cdot \theta(p) = 0 \). Since, \( h_\cdot \theta(p) = 0 \) on the line passing through 0 and \( p \), we also see that \( h_{r \theta}(p) = 0 \).

□

The next lemma shows that the expression \( \langle \nabla F, \nabla v_k \rangle / v_k \) has the role of a second derivative of \( F \) on \( v_k = 0 \). This will be important in the proof of Theorem 1.3.

**Lemma 5.4.** Let \( f, g \in C^\infty(B_2) \) be such that \( \langle \nabla f, \nabla v_k \rangle = g v_k \).

Let \( p \in B_2 \) be a local maximum point of \( f \). Then \( g(p) \leq 0 \).

**Proof.**

\[ \langle \nabla f, \nabla v_k \rangle = kr^k - \frac{2}{r} f_\theta \sin k \theta + f_\theta \cos k \theta. \]

Assume first that \( p \neq 0 \), then

\[ \lim_{q \to p} k \frac{f_\theta(q)}{r_q} = 0 \]

since \( p \) is a critical point. If \( \sin k \theta_p \neq 0 \) then

\[ \lim_{q \to p} k \frac{f_\theta(q)}{r_q} = 0, \]

since \( p \) is a critical point. Otherwise, by L’Hôpital’s rule we have

\[ \lim_{q \to p} k \frac{f_\theta(q)}{r_q} = \lim_{\theta \to \theta_0} k \frac{f_\theta(r_p, \theta)}{r_p \cos k \theta} \]

since \( p \) is a maximum point.

If \( p = 0 \), let \( \theta_0 = \frac{\pi}{2k} \), then

\[ g(0) = \lim_{r \to 0^+} k \frac{f_\theta(r, \theta_0)}{r} \leq 0, \]

since 0 is a maximum point.

□

6. **Proof of the Theorem 1.3 in normal form**

We prove Theorem 1.3 for the case \( Z = Z_k = \{ \Im z^k = 0 \} \cap B_2 \). We first need to construct a suitable cutoff function. Let \( \chi \in C^\infty([0, 2]) \) be such that \( \chi \equiv 1 \) on \([0, 1]\) and \( \chi \) is non-increasing. Let \( \varphi(x, y) := \chi(x^2 + y^2) \). Observe that

\[ \langle \nabla \varphi, \nabla v_k \rangle_{v_k} = 2k \chi'(r^2). \]

(6.1)

Let \( A > 0 \) be such that

\[ \chi' \geq -A, \quad \Delta \varphi \geq -A, \quad |\nabla \varphi|^2 \leq A \varphi. \]

(6.2)

Without loss of generality we can assume \( v = v_k \). Let \( u \in \mathcal{F}(Z_k) \).
Existence of a positive quotient. We first show that \( |u/v_k| \) defines a positive smooth function in \( B_2 \). By Lemma 5.2 there exists \( f \in C^\infty(B_2) \) such that \( u = f v_k \). It is clear that \( f \neq 0 \) on \( B_2 \setminus Z_k \). Let \( p \in Z_k, p \neq 0 \). If \( f(p) = 0 \) then \( u(p) = 0 \) and \( (\nabla u)(p) = 0 \). Hence, \( u \) has a zero of order \( d \geq 2 \) at \( p \), but since \( u \) is harmonic in a small ball \( B_\epsilon(p) \) centered at \( p \) this implies that \( Z \cap B_\epsilon(p) \) is homeomorphic to the zero set of \( \Im z^d \). This is a contradiction. If \( p = 0 \) and \( f(p) = 0 \) then \( u(p) \) has a zero of order \( d \geq k + 1 \) at \( 0 \) with \( Z_k \) as a zero set which is impossible since \( u \) is harmonic.

A gradient estimate. We now proceed to proving the gradient estimate on \( \log f \).

Let \( h = \log f, h \in C^\infty(B_2) \) and \( u = e^h v_k \). We let \( F = |\nabla h|^2 \). At points \( q \in B_2 \) where \( v_k(q) \neq 0 \) and \( \varphi(q) \neq 0 \) we have by Lemma 4.1

\[
\Delta(\varphi F) = (\Delta \varphi) F + 2 \langle \nabla \varphi, \nabla F \rangle + \varphi \Delta F
\]

\[
\geq (\Delta \varphi) F + 2 \langle \nabla \varphi, \nabla (\varphi F) \rangle - 2 \frac{\varphi^2}{k + 1} - 2 \frac{\langle \nabla F, \nabla v_k \rangle}{\varphi} = 2 F \frac{|\nabla \varphi|^2}{\varphi} + \frac{\varphi F^2}{k + 1} - 2 \frac{\langle \nabla (\varphi F), \nabla v_k \rangle}{\varphi}
\]

\[
+ 2 F \frac{\langle \nabla \varphi, \nabla v_k \rangle}{v_k} - 2 \langle \nabla (\varphi F), \nabla h \rangle + 2 F \langle \nabla \varphi, \nabla h \rangle
\]

\[
\geq (\Delta \varphi) F + 2 \frac{\langle \nabla \varphi, \nabla (\varphi F) \rangle}{\varphi} - 2 F \frac{|\nabla \varphi|^2}{\varphi} + \frac{\varphi F^2}{k + 1} - 2 \frac{\langle \nabla (\varphi F), \nabla v_k \rangle}{\varphi}
\]

\[
+ 2 F \frac{\langle \nabla \varphi, \nabla v_k \rangle}{v_k} - 2 \langle \nabla (\varphi F), \nabla h \rangle - 2 F^{3/2} |\nabla \varphi| \tag{6.3}
\]

The last inequality follows from the Cauchy-Schwartz inequality.

Let \( p \in B_2 \) be a point where \( \varphi F \) attains its maximum. Observe that by Lemma 5.3 and (6.1) there exists \( G \in C^\infty_c(B_2) \) such that

\[
\langle \nabla (\varphi F), \nabla v_k \rangle = G v_k. \tag{6.4}
\]

It follows from (6.3), (6.4) and Lemma 5.4 that at the point \( p \) the following inequality is satisfied

\[
0 \geq (\Delta \varphi) F - 2 F \frac{|\nabla \varphi|^2}{\varphi} + \frac{\varphi F^2}{k + 1} + 4 k F \chi'(||p||^2) - 2 F^{1/2} \frac{|\nabla \varphi|}{\varphi^{1/2}}.
\]

Dividing by \( F(p) \) and using (6.2) we get the following quadratic inequality in \( (\varphi F)^{1/2} \):

\[
0 \geq \frac{\varphi F}{k + 1} - 2 \sqrt{A} (\varphi F)^{1/2} - (4k + 3) A,
\]

from which we conclude that \( (\varphi F)^{1/2} \leq 4(k + 1) \sqrt{A} \). Since \( p \) is a maximum point, the same inequality is true for all \( q \in B_2 \). In particular, in \( B_1 \) we get

\[
|\nabla h| \leq 4(k + 1) \sqrt{A}.
\]

\[\square\]

7. Proof of Theorem 1.3 - the general case

In this case we reduce the general case to the \( Z = Z_k \) proved in Section 6.
Proof. **Existence of a positive quotient.** Fix \( v \in \mathcal{F}(Z) \). Let \( u \in \mathcal{F}(Z) \) be arbitrary. We will first show that \( |u/v| \) extends to a positive smooth function in \( B_2 \). Let \( p \in Z \). There exist \( k(p) \in \mathbb{N} \), a neighborhood \( N_p \ni p \), an injective conformal map \( \alpha_p : N_p \to B_1 \) such that \( \alpha_p(p) = 0 \) and \( v \circ \alpha_p^{-1}(w) = 3w^{k(p)} \) for all \( w \in \alpha_p(N_p) \). Let \( N'_p \) be a neighborhood of \( p \) such that \( \partial N'_p \subset N_p \). Let \( W = \alpha_p(N_p) \) and \( W' = \alpha_p(N'_p) \). \( v \circ \alpha_p^{-1} \) and \( u \circ \alpha_p^{-1} \) are harmonic functions both vanish exactly on \( Z_{k(p)} \cap W \). By Section 6 we know that \( |(u \circ \alpha_p^{-1})/(v \circ \alpha_p^{-1})| \) defines a positive smooth function \( f_0 \) on \( W \). Let \( f(z) := f_0(\alpha_p(z)) \) be defined in \( N_p \). Then \( |f| > 0 \) and \( u = fv \) in \( N_p \). This shows that \( |u/v| \) extends to a smooth positive function in \( N_p \). Since \( p \) is arbitrary we conclude that \( |u/v| \) extends to a smooth positive function in \( B_2 \).

**A bound on \( \nabla \log |u/v| \).** Let \( p \in Z \cap \overline{B}_1 \). Let \( \alpha_p, k(p), N_p, N'_p \) be defined as above. Let us write \( u = e^h v \) where \( h \in C^\infty(B_2) \). By Section 6 we know that \( |\nabla(h \circ \alpha_p^{-1})| \leq C_p \) in \( W' \). By the chain rule it follows that \( |\nabla h| \leq C_p |\alpha'_p| \) in \( N'_p \). Since we can cover \( Z \cap \overline{B}_1 \) by a finite number of open sets of the form \( N'_p \) we get that

\[
|h| \leq C_v \text{ in a neighborhood } N_Z \text{ of } Z \cap \overline{B}_1. \quad (7.1)
\]

Since \( u, v \) do not vanish in a neighborhood of \( \overline{B}_1 \setminus N_Z \), by [7] (or by the case \( Z = \emptyset \) in Section 6) we know that

\[
|\nabla h| \leq C \text{ in a neighborhood of } \overline{B}_1 \setminus N_Z. \quad (7.2)
\]

From inequalities (7.1) and (7.2) we get that \( |\nabla h| \leq C_Z \) in \( \overline{B}_1 \). \( \square \)

8. Examples

8.1. **Harmonic functions sharing the same zeros.** We give a few examples of harmonic functions with common zeros.

(i) (Due to Benji Weiss) Let \( \alpha \in [\pi/2, \pi/2] \). Let \( u_\alpha(x, y) = e^{\alpha x} \sin |\alpha| y \).

The zero set of \( u_\alpha \) in \( B_2 \) is the x-axis.

(ii) Let \( F : B_2 \to \mathbb{C} \) be holomorphic. Suppose \( |\Re F| < \pi \). The zero set of \( \Re e^F \) is the same as the zero set of \( \Re F \).

(iii) Let \( \alpha \in \mathbb{R} \) be such that \( 0 < |\alpha| \leq 1 \). Let \( f(z) \) be the branch of \( z^\alpha \) in \( \mathbb{C} \setminus (-\infty, 0] \) which admits positive values on the positive real axis. Define \( u(z) = \Re f(z + 2) \). The zero set of \( u \) in \( B_2 \) is the x-axis.

(iv) Let \( F : B_2 \to B_2 \) be holomorphic. Let \( u = \Re \frac{aF}{z^d + c} \) where \( a, c, d \in \mathbb{R} \) are such that \( F \neq -d/c \) in \( B_2 \). Then, the zero set of \( u \) in \( B_2 \) coincides with the zero set of \( \Re F \).

(v) Let \( k \in \mathbb{N} \) and let \( S_k = \{0 < \arg z < \pi/k\} \cap B_2 \). Let \( u \) be a positive harmonic function in \( S_k \), which is continuous on \( \partial S_k \) and vanishes on \( \partial S_k \cap B_2 \). One can extend \( u \) by reflections to a harmonic function in \( B_2 \). The zero set of \( u \) coincides with the zero set of \( 3z^k \).

(vi) (Due to Charles Fefferman) Let \( u(x, y) = xy \). Let \( v(x, y) = x^3y - xy^3 \).

Then \( u \) and \( u + \varepsilon v \) have the same zero set in \( B_2 \) if \( \varepsilon > 0 \) is sufficiently small.

8.2. **An example clarifying the relation to the BHP.** (Due to Carlos Kenig)

Let \( t > 1 \) and let \( S = \{0 < \arg z < 2\pi/t\} \subset \mathbb{C} \). Let \( v = 3z^{t/2} \). Observe that \( v \) is positive in \( S \) and \( v|_{\partial S} = 0 \). Let \( p \in S \) be such that \( |p| > 2 \). Let \( G_p \) be the Green function of \( S \) with singularity at \( p \). We consider \( G_p/v \) in \( S \cap B_2 \). Unless \( t \)
is an integer, $G_p/v$ cannot be extended as a $C^1$-function in a neighborhood of 0. Moreover, we note that $|\nabla \log(G_p/v)|$ is bounded in $S \cap B_1$, if and only if $t \geq 2$.

A little simpler, we let $u = 3z^{t/2} + \varepsilon z^t$ for small $\varepsilon > 0$. Then $\log u/v$ has no bounded gradient in $S \cap B_1$ unless $t \geq 2$.

These families of examples (for $1 < t < 2$) show that the BHP alone is not enough to obtain gradient estimates even if the boundary of the domain is nice (straight lines).

9. A second proof of the normal form case using the Poisson formula

This section is due to Misha Sodin. Let

$$S_k = \{z \in \mathbb{C} ||z| < 1, 0 < \arg z < \pi/k\}.$$ 

Let $u$ be a positive harmonic function in $S_k$, continuous on $\overline{S_k}$ such that $u = 0$ on $\partial S_k \setminus \{|z| = 1\}$. We have the following integral representation for $u$:

$$u(re^{i\theta}) = \frac{k(1 - r^{2k})}{2\pi} \int_0^{\pi/k} \left( \frac{1}{|e^{ikr} - r^k e^{ik\theta}|^2} - \frac{1}{|e^{ikr} - r^k e^{-ik\theta}|^2} \right) u(e^{i\varphi}) \, d\varphi.$$ 

Hence

$$u(re^{i\theta}) = \frac{2k(1 - r^{2k})}{\pi} \int_0^{\pi/k} \frac{\sin(k \varphi) u(e^{i\varphi})}{|e^{ikr} - r^k e^{ik\varphi}|^2 |e^{ikr} - r^k e^{-ik\varphi}|^2} \, d\varphi,$$

and

$$\log \left( \frac{u(re^{i\theta})}{r^k \sin k\theta} \right) = \log \left( \frac{2k}{\pi} + \log(1 - r^{2k}) + \log g(z) \right) \quad (9.1)$$

where

$$g(re^{i\theta}) = \int_0^{\pi/k} K(re^{i\theta}, e^{i\varphi}) \sin(k \varphi) u(e^{i\varphi}) \, d\varphi,$$

with

$$K(re^{i\theta}, e^{i\varphi}) = \frac{1}{|e^{ikr} - r^k e^{ik\varphi}|^2 |e^{ikr} - r^k e^{-ik\varphi}|^2}.$$ 

Now, $\min \{K(z, \zeta) \mid z \in S_k, |z| < 1/2, |\zeta| = 1, 0 \leq \arg \zeta \leq \pi/k\}$ is a positive number $C_1$, and

$$\max \{|\nabla \log K(z, \zeta)| \mid z \in S_k, |z| < 1/2, |\zeta| = 1, 0 \leq \arg \zeta \leq \pi/k\} < C_2 < \infty.$$ 

Consequently, for $|z| < 1/2$,

$$g(z) \geq C_1 \int_0^{\pi/k} u(e^{i\varphi}) \sin(k \varphi) \, d\varphi,$$

and

$$|\nabla g(z)| \leq C_2 \int_0^{\pi/k} u(e^{i\varphi}) \sin(k \varphi) \, d\varphi.$$ 

So we get $|\nabla \log g| \leq C_2/C_1$ in $S_k \cap B_{1/2}$, and then from (9.1)

$$|\nabla \log \frac{u}{r^k \sin k\theta}| \leq Ck.$$
in $S_k \cap B_{1/2}$. Finally we use reflection to get the stated result in the unit ball.

References

[1] A. Ancona, Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschizien, Ann. Inst. Fourier (Grenoble), 28 (1978), 169–213. MR 513885

[2] Z. Balogh and A. Volberg, Boundary Harnack principle for separated semihyperbolic repellers, harmonic measure applications, Rev. Mat. Iberoamericana, 12 (1996), 299–336. MR 1402670

[3] L. Caffarelli, E. Fabes, S. Mortola and S. Salsa, Boundary behavior of nonnegative solutions of elliptic operators in divergence form, Indiana Univ. Math. J., 30 (1981), 621–640. MR 620271

[4] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001. MR 1814364

[5] D. S. Jerison and C. E. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, Adv. in Math., 46 (1982), 80–147. MR 676988

[6] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, (Russian) Izv. Akad. Nauk SSSR Ser. Mat., 47 (1983), 75–108; translation in Math. USSR-Izv., 22 (1984), 67–98. MR 688919

[7] P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math., 156 (1986), 153–201. MR 834612

[8] A. Logunov and E. Malinnikova, On ratios of harmonic functions, preprint, arXiv:1402.2888, 2014.

[9] N. Nadirashvili, Harmonic functions with bounded number of nodal domains, Appl. Anal., 71 (1999), 187–196. MR 1690098

[10] I. Popovici and A. Volberg, Boundary Harnack principle for Denjoy domains, Complex Variables Theory Appl., 37 (1998), 471–490. MR 1687853

[11] L. Silvestre and B. Sirakov, Boundary regularity for viscosity solutions of fully nonlinear elliptic equations, preprint, arXiv:1306.6872, 2013.

[12] J. M. G. Wu, Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains, Ann. Inst. Fourier (Grenoble), 28 (1978), 147–167. MR 513884

Einstein Institute of Mathematics,, Hebrew University, Givat Ram, Jerusalem 91904, Israel

E-mail address: mangoubi@math.huji.ac.il