Hydrodynamic sound and plasmons in three dimensions

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Abstract: In a recent paper by Lucas and Das Sarma [Physical Review B 97, 115449 (2018)], a solvable model of collective modes in 2D metals was considered in the hydrodynamic regime. In the current work, we generalize the hydrodynamic theory to 3D metals where the calculation of sound modes in a strongly-coupled quantum Coulomb plasma can be made explicit. The specific theoretical question of interest is how the usual linearly dispersing hydrodynamic sound mode relates to the well-known gapped 3D plasmon collective mode in the presence of long-range Coulomb interaction. We show analytically that both the zero sound in the collisionless regime and the first sound in the hydrodynamic region become massive in 3D, acquiring a finite gap because of the long-range Coulomb interaction, while their damping rates become quadratic in momentum. We also discuss other types of long-range potential, where the dispersion of sound modes is modified accordingly. The general result is that the leading order hydrodynamic sound mode is always given by the leading-order plasmon frequency in the presence of long-range Coulomb interaction, but the next-to-leading-order dispersion corrections differ in hydrodynamic and collisionless regimes.
1 Introduction

An interacting electron system could be in the strongly interacting collision-dominated hydrodynamic regime where frequent inter-particle collisions lead to local thermodynamic equilibrium so that the system should be thought of as manifesting a collective hydrodynamic behavior rather than the usual collisionless behavior of weak interactions where the standard independent quasiparticle picture applies. The analogy is to real liquids (e.g. water) whose macroscopic long wavelength flow properties hardly depend on the microscopic details of the molecular interactions which only serve to determine the macroscopic hydrodynamic parameters such as viscosity. Although the possibility of electron hydrodynamics was suggested a long time ago [1], there has been a great deal of recent interest in the subject arising from the possibility of the experimental observation of electron hydrodynamics in solid state materials [2]. The current work focuses on a specific theoretical question regarding electron hydrodynamics.

The question we address is the interplay of the electronic plasmon mode with the hydrodynamic sound mode: How does the plasmon affect the hydrodynamic sound mode and does the electron system in the hydrodynamic regime undergo collective plasmon oscillations at all?

The theoretical question has recently been addressed for 2D metals in a recent work [3], and we generalize the theory to 3D metals. The 3D generalization is nontrivial and is of
interest because of the true long-range nature of electronic Coulomb interaction in 3D metals, leading to a collective plasmon mode with a finite energy even in the long wavelength limit \[4\]. This gapped massive nature of 3D plasmons arises from the 3D Coulomb coupling going as \(1/q^2\), where \(q\) is the wavenumber (or momentum). By contrast, 2D Coulomb coupling goes as \(1/q\), leading to the 2D plasmon going at long wavelength as \(q^{1/2}\). Since the hydrodynamic sound mode (the so-called ’first sound’ in the helium literature), which is the electron analog of the ordinary acoustic sound, goes linear in \(q\) at long wavelength by definition, the interplay between the hydrodynamic sound and the 2D plasmon, both of which vanish at long wavelength albeit with different momentum scaling. If the hydrodynamic sound mode in 3D metals develops a long wavelength gap by virtue of Coulomb interaction, this becomes reminiscent of the Higgs mechanism with a linearly dispersing Goldstone mode acquiring a mass by virtue of long-range coupling although the electron hydrodynamics problem does not involve any symmetry breaking (or an underlying Higgs field) in order to produce the gapped sound mode. In fact, this is precisely what happens in a metallic superconductor where the expected linearly dispersing Goldstone mode associated with the spontaneous breaking of the gauge symmetry acquires a long wavelength gap becoming effectively the plasmon mode in the presence of long-range Coulomb coupling instead of the usual zero sound acoustic mode as in a neutral superfluid where there is no long-range Coulomb coupling \[5–9\].

We show in this work that indeed 3D Coulomb coupling leads to a mass or a gap in the hydrodynamic sound mode in 3D metals, and the sound mode becomes the effective long wavelength plasmon mode in the hydrodynamic regime of 3D metals. This is, however, true only in the leading order in momentum, and the next-to-leading-order dispersion corrections in wavenumber are different for 3D plasmons in the collisionless regime and the first sound in the hydrodynamic regime. We also study the damping or decay of the plasmon mode (which is akin to the zero sound mode in the helium literature) in the collisionless regime and the hydrodynamic first sound mode in the collision-dominated regime. Our terminology in the paper uses ’collisionless’ to imply the non-hydrodynamic regime of weak inter-particle collisions (where the standard ’plasmon’ or zero sound mode exists). By contrast, the collision-dominated regime is the hydrodynamic regime of rapid inter-particle collisions where the first sound mode exists. We study both regimes including effects of 3D long-range Coulomb interaction.

The hydrodynamic description applies when the momentum conserving inelastic electron-electron interaction is much stronger than any other momentum relaxing elastic scattering mechanisms which might be present in the system. In metals, such momentum non-conserving scattering processes arise from electron-impurity and electron-phonon scattering, which typically dominate at low and high temperatures respectively, making the hydrodynamic regime difficult to realize experimentally in the laboratory although, in principle, a very clean metal should manifest electron hydrodynamics at very low temperatures where phonon scattering...
rate (in terms of resistivity) is suppressed as $T^5$ and the quasiparticle scattering rate goes as $T^2$, where $T$ is the temperature [10]. Eventually, at low enough temperatures hydrodynamics is cut off in metals by any residual impurity scattering. In a metal with negligible impurity and phonon scattering, electron-electron interactions should produce hydrodynamic behavior at long wavelength and low frequency.

2 Summary of the main results

We extend the theory of [3] to three dimensions. Namely, we construct a solvable model that admits the exact calculation of sound modes in both collisionless regime and hydrodynamic regime. The sound mode in the collisionless regime is the zero sound mode or the plasmon, and the sound mode in the hydrodynamic regime is the usual sound mode (i.e., the first sound). In this section, we represent the main results from the solvable model, including the effect of long-range Coulomb interaction to the sound modes. We leave the detailed derivation of the results in the following sections, but the exactly solvable model already demonstrates the physics clearly.

In this exactly solvable model consisting of spinless fermions, we assume spherical symmetry and consider nonvanishing Landau parameter only in the s-wave channel. We denote the dimensionless Landau parameter $F_0$ and we consider repulsive interactions so that $F_0 > 0$. The sound mode is highly damped when the interaction is attractive, and may even lead to instabilities [11]. In any case, our interest is 3D metals where the direct electron-electron interaction is repulsive. These considerations allow the calculations analytically tractable, while maintaining the essential physics.

We first present the result in the clean limit where the sound mode is not damped by impurity scattering, $\gamma_{\text{imp}} = 0$. $\gamma_{\text{imp}}$ is the scattering rate between quasiparticles and impurities. The zero sound in the collisionless regime $\omega \gg \gamma$, where $\gamma$ denotes the scattering rate between quasiparticles, is given by

$$\omega = \pm v_0 q - i\gamma \left( \frac{(F_0 + 1)^2 - 2(F_0 - 1)(\frac{v_0}{v_F})^2 - 3(\frac{v_0}{v_F})^4}{F_0[F_0 + 1 - (\frac{v_0}{v_F})^2]} \right), \quad \frac{v_0}{v_F} \text{arccoth} \frac{v_0}{v_F} = 1 + \frac{1}{F_0}, \quad (2.1)$$

where the second equation determines the zero sound velocity. This equation is valid for all $F_0 > 0$. For the weakly interacting Fermi liquid, $F_0 \ll 1$, and the zero sound velocity is nonperturbative in interaction strength, namely, $v_0 = v_F(1 + e^{-2/F_0})$. It approaches the Fermi velocity as one should anticipate for the free electron gas. For $F_0 > 1$, the dispersion can be simplified as

$$\omega = \pm \sqrt{\frac{F_0}{3} + \frac{3}{5}v_F q - i\gamma \frac{2(5F_0 + 21)}{5F_0(5F_0 + 3)}}, \quad (2.2)$$
Figure 1. The zero sound and first sound velocity as a function of Landau parameter, where the Fermi velocity is set to one as the unit. The zero sound velocity is larger than the first sound velocity for all $F_0 > 0$.

where $v_F$ is the Fermi velocity. Decreasing the frequency, the system enters the collision-dominated hydrodynamic regime where $\omega \ll \gamma$. Then the zero sound crosses over smoothly to the first sound,

$$\omega = \pm \sqrt{\frac{F_0}{3} + \frac{1}{3} v_F q - \frac{i}{15} \frac{2 v_F^2}{\gamma q^2}}. \quad (2.3)$$

The linear term in momentum reveals the sound velocity. In Fig. 1, one can see that the collisionless zero sound velocity is always larger than the hydrodynamic first sound velocity, $v_0 > v_1$. For the asymptotically large Landau parameter, the following inequality holds true, too, i.e.,

$$\frac{v_0}{v_1} = \sqrt{\frac{F_0 + 9/5}{F_0 + 1}} > 1. \quad (2.4)$$

The imaginary part leads to the damping of sound modes. In an interacting Fermi liquid, the scattering rate at low temperatures is given by $\gamma \propto T^2$. A crucial difference between the zero sound and the first sound is that the damping rate is proportional to the interaction scattering rate, $\text{Im}(\omega) \propto \gamma$, for zero sound while it is inversely proportional to the interaction scattering rate, $\text{Im}(\omega) \propto \gamma^{-1}$, for the first sound. These results are well known and also experimentally observed in normal He-3 [12]. This presents a consistency check of our solvable model.

When the impurity scattering is strong $\gamma_{\text{imp}} > v_F q$, both sound modes will be damped by impurity scattering at small momentum. The propagating wave transitions to a quadratic diffusion mode

$$\omega = -i\frac{v_1^2}{\gamma_{\text{imp}}} q^2, \quad v_1 = v_F \sqrt{\frac{1 + F_0}{3}}, \quad \gamma_{\text{imp}} \quad (2.5)$$

and a fully gapped one $\omega = -i\gamma_{\text{imp}}$. When the impurity scattering is weak $\gamma_{\text{imp}} < v_F q$, its effect is an additional damping of the zero sound mode (in addition to the damping induced by quasiparticle interactions $\gamma$), i.e., the correction is $\delta \text{Im}(\omega) = \frac{1}{2} \gamma_{\text{imp}}$, also see (4.19).
Now we discuss the effect of Coulomb interaction to the sound mode. It is well known that the plasmon is fully gapped in 3D metals because of Coulomb interaction [4]. Since both the plasmon and the sound wave are density fluctuating collective modes in many-body systems, it is naturally expected that the sound mode should also develop a finite gap due to the Coulomb interaction. Since Coulomb interaction acts at the s-wave channel, one can make the following replacement,

$$F_0 \rightarrow F_0 + \frac{8\pi\alpha}{\lambda_F^2 q^2}, \quad (2.6)$$

where $\alpha = \frac{e^2}{4\pi\varepsilon_F}$ is the effective finite structure constant for the 3D metal ($v_F$ is the Fermi velocity), and $\lambda_F = \frac{2\pi}{k_F}$ is the Fermi wavelength. Since we are interested in the long wavelength limit, where the Coulomb interaction dominates over any short-range interaction, we focus on (2.2) in the appropriate limit. After making this replacement, the zero sound becomes

$$\omega = \pm \left( \sqrt{\frac{8\pi\alpha}{3} \frac{v_F}{\lambda_F}} + \frac{9/5 + F_0}{4\sqrt{6\pi\alpha}} v_F \lambda_F q^2 \right) - \frac{i\gamma}{2\pi} \frac{\lambda_F^2 q^2}{20\pi^3}, \quad (2.7)$$

and the first sound becomes

$$\omega = \pm \left( \sqrt{\frac{8\pi\alpha}{3} \frac{v_F}{\lambda_F}} + \frac{1 + F_0}{4\sqrt{6\pi\alpha}} v_F \lambda_F q^2 \right) - \frac{i\gamma}{15\pi} \frac{2v_F^2}{20\pi^3} q^2. \quad (2.8)$$

It is instructive to compare the results with the plasmon mode. The 3D plasmon dispersion is well known and is determined by the following equation in RPA calculations,

$$\omega^2 = \omega_0^2 + \frac{k_F^2}{m^2} \omega_0^2 q^2, \quad \omega_0^2 = \frac{e^2 N_e}{m}, \quad (2.9)$$

where $N_e = \frac{4\pi}{3} k_F^3$ is the electron density in 3D, $m = \frac{k_F}{v_F}$ is the effective mass of the quasiparticle, and $\omega_0$ is the well-known plasmon frequency. (Note a notational difference between the plasmon frequency in (2.9) and Eq. (14) in [4], which comes from the Coulomb potential we define in (3.5) having an extra factor of $1/4\pi$.) If we expand the plasmon mode at long wavelength, its dispersion reads

$$\omega = \omega_0 + \frac{3}{5} \frac{k_F^2}{m^2 \omega_0} q^2 = \sqrt{\frac{8\pi\alpha}{3} \frac{v_F}{\lambda_F}} + \frac{9/5}{4\sqrt{6\pi\alpha}} v_F \lambda_F q^2, \quad (2.10)$$

where in the second step, we change the parameters to better compare with sound mode results. Comparing it to the zero sound mode with Coulomb interaction (2.7), we find a correction to the quadratic dispersion from the Landau parameter $F_0$ and a quadratic damping due to the collisions. Comparing it to the first sound mode with Coulomb interaction (2.8), we conclude that in the hydrodynamic regime, the plasmon dispersion gets corrected by a factor of $5/9$ at the next-to-leading $q^2$ order because of Coulomb coupling. Again there is a damping effect in (2.8) inherited from the first sound mode.
More generally, we can consider a formal long-range interaction given by a power law defined by \( q^{-2\eta} \), where \( \eta = 1 \) for 3D Coulomb coupling, i.e.,

\[
F_0 \rightarrow F_0 + \frac{8\pi\alpha}{(\lambda_F q)^{2\eta}},
\]

(2.11)

where \( \alpha \) is the effective interaction strength, and the parameter \( \eta \) defines the form of the interaction. In real space, this translates to a potential of the form \( r^{2\eta-3} \). For \( \eta > 1 \) (\( \eta < 1 \)) it is stronger (weaker) than the Coulomb potential. This type of interaction leads to the zero sound mode

\[
\omega = \pm \sqrt{\frac{8\pi\alpha}{3} v_F q^{1-\eta}} - i\gamma \frac{(\lambda_F q)^{2\eta}}{20\pi\alpha},
\]

(2.12)

and the first sound mode

\[
\omega = \pm \sqrt{\frac{8\pi\alpha}{3} v_F q^{1-\eta}} - \frac{1}{15\gamma q^2}.
\]

(2.13)

In this case, the sound mode has an interesting dispersion resulting from the long-range interaction. Also note that the damping rate for the "hydrodynamic plasmon" does not change due to the specific form of the long-range interaction, remaining independent of the range parameter \( \eta \).

3 Review of Boltzmann equation in 3D metal

Collective mode is one of the fundamental excitations of a many-body system. It emerges from coherent interactions between quasiparticles, and is fundamentally different from single particle electron-hole type excitations. It is convenient to describe the collective mode by the distribution function \( n(k, r, t) \) of the quasiparticle at given momentum \( k \) and position \( r \). As we are interested in the effect of Coulomb interaction on the sound mode, we will restrict to the spinless electron. Spin can be added straightforwardly at the cost of making the notations cumbersome—we emphasize that the modes we are discussing are charge density collective excitations which are independent of electron spin. The Boltzmann equation governing the dynamics of the distribution function is [11]

\[
\frac{dn(k, r, t)}{dt} = \left( -\frac{dr}{dt} \cdot \frac{\partial}{\partial r} - \frac{dk}{dt} \cdot \frac{\partial}{\partial k} \right) n(k, r, t) + \mathcal{I}[n],
\]

(3.1)

where the first term in the right hand side is the drift term while the second term is the collision integral. The semiclassical equation of motion of a quasiparticle is well known

\[
\frac{dk}{dt} = -\frac{\partial \epsilon(k, r, t)}{\partial r}, \quad \frac{dr}{dt} = \frac{\partial \epsilon(k, r, t)}{\partial k},
\]

(3.2)
where $\epsilon(\mathbf{k}, \mathbf{r}, t)$ is the energy of the quasiparticle. Since we are interested in a conventional metal, it is sufficient to consider the semiclassical description without external electromagnetic field or Berry curvature. The quasiparticle energy should be determined self-consistently from the Boltzmann equation. We are going to solve the collective mode of a small variation from the Fermi-Dirac distribution function, namely,

$$n(\mathbf{k}, \mathbf{r}, t) = n_F(\mathbf{k}) + \delta n(\mathbf{k}, \mathbf{r}, t), \quad n_F(\mathbf{k}) \equiv n_F[\epsilon_0(\mathbf{k})] = \frac{1}{e^{\beta(\epsilon_0(\mathbf{k}) - \mu)} + 1},$$

where $n_F(\mathbf{k})$ is the Fermi-Dirac distribution at equilibrium, $\epsilon_0(\mathbf{k})$ is the bare energy of free electrons, $\beta \equiv 1/T$ is the inverse temperature, and $\mu$ denotes the chemical potential.

The total energy of the system with a small variation from equilibrium is

$$\epsilon_{\text{tot}}(t) = \int d^3r \int_{\mathbf{k}} \epsilon_0(\mathbf{k}) \delta n(\mathbf{k}, \mathbf{r}, t) + \frac{1}{2} \int d^3r \int_{\mathbf{k}, \mathbf{k}'} \delta n(\mathbf{k}, \mathbf{r}, t) f(\mathbf{k}, \mathbf{k}') \delta n(\mathbf{k}', \mathbf{r}, t)$$

$$+ \frac{1}{2} \int d^3r d^3r' \int_{\mathbf{k}, \mathbf{k}'} \delta n(\mathbf{k}, \mathbf{r}, t) \frac{e^2}{4\pi |\mathbf{r} - \mathbf{r}'|} \delta n(\mathbf{k}', \mathbf{r}', t),$$

where $\int_{\mathbf{k}} \equiv \int \frac{d^3k}{(2\pi)^3}$ and $f(\mathbf{k}, \mathbf{k}')$ is the Landau parameter characterizing the short-range quasiparticle interactions, and the second line represents the long-range Coulomb interaction. Thus, by varying with respect to $\delta n(\mathbf{k}, \mathbf{r}, t)$, we can get the quasiparticle energy

$$\epsilon(\mathbf{k}, \mathbf{r}, t) = \epsilon_0(\mathbf{k}) + \int_{\mathbf{k}'} f(\mathbf{k}, \mathbf{k}') \delta n(\mathbf{k}', \mathbf{r}, t) + \int_{\mathbf{k}'} \int d^3r' \frac{e^2}{4\pi |\mathbf{r} - \mathbf{r}'|} \delta n(\mathbf{k}', \mathbf{r}', t).$$

To get a wave-like collective mode, we assume in the usual manner that the variation takes the plane wave form

$$\delta n(\mathbf{k}, \mathbf{r}, t) = \delta n(\mathbf{k}) e^{i \mathbf{q} \cdot \mathbf{r} - i \omega t}. \quad (3.7)$$

Actually, once we get the eigenmode from the plane wave expansion, we can construct any arbitrary mode using linear superposition and completeness. The energy for such a plane wave excitation is

$$\epsilon(\mathbf{k}, \mathbf{r}, t) = \epsilon_0(\mathbf{k}) + \int_{\mathbf{k}'} f(\mathbf{k}, \mathbf{k}') \delta n(\mathbf{k}') e^{i \mathbf{q} \cdot \mathbf{r} - i \omega t} + \int_{\mathbf{k}'} \int d^3r' \frac{e^2}{4\pi |\mathbf{r} - \mathbf{r}'|} \delta n(\mathbf{k}') e^{i \mathbf{q} \cdot \mathbf{r}' - i \omega t} \quad (3.8)$$

$$= \epsilon_0(\mathbf{k}) + \int_{\mathbf{k}'} \left( f(\mathbf{k}, \mathbf{k}') + \frac{e^2}{|\mathbf{q}|^2} \right) \delta n(\mathbf{k}') e^{i \mathbf{q} \cdot \mathbf{r} - i \omega t}, \quad (3.9)$$

where in the second line, we have used the Fourier transform of Coulomb potential in 3D. A simple derivation is given in Appendix A.

Now with the quasiparticle energy, the semiclassical equation of motion becomes

$$\frac{d\mathbf{k}}{dt} = -i \mathbf{q} \int_{\mathbf{k}'} \left( f(\mathbf{k}, \mathbf{k}') + \frac{4\pi e^2}{|\mathbf{q}|^2} \right) \delta n(\mathbf{k}') e^{i \mathbf{q} \cdot \mathbf{r} - i \omega t}, \quad \frac{d\mathbf{r}}{dt} = \frac{\partial \epsilon_0(\mathbf{k})}{\partial \mathbf{k}} \equiv \mathbf{v}(\mathbf{k}). \quad (3.10)$$
Putting this equation of motion into the Boltzmann equation, we arrive at

\[ \omega \delta n(k) = q \cdot v(k) \left( \delta n(k) - \left[ \partial_{\epsilon} \right]_{\epsilon = \epsilon_0(k)} n_F(\epsilon) \right] \int_{k'} \left( f(k,k') + \frac{e^2}{q^2} \delta n(k') \right) + i \mathcal{I}[n], (3.11) \]

where \( \partial_{\epsilon} \) means taking derivative with respect to \( \epsilon \) and then setting \( \epsilon = \epsilon_0(k) \). Since at low temperatures, the small variation \( \delta n(k) \) concentrates near the Fermi surface according to the factor

\[
\lim_{\beta \to \infty} -\partial_{\epsilon} \bigg|_{\epsilon = \epsilon_0(k)} n_F(\epsilon) = \delta(\epsilon_0(k) - \mu),
\]

which tends to a delta function localized at the Fermi surface, it is easy to inspect that the solution to the equation has the form \( \delta n(k) = -\left[ \partial_{\epsilon} \right]_{\epsilon = \epsilon_0(k)} n_F(\epsilon) \delta n(\sigma) = \delta(\epsilon_0(k) - \mu) \delta n(\sigma) \). Here, \( \sigma = (\theta, \phi) \) is determined by the vector \( k \) at Fermi surface. Hence, it is convenient to change the variable from \( k \) to \( (\epsilon_0(k), \sigma) \). With the help of the following identity,

\[
\int d^3k = \int_0^\infty d\epsilon \int_{\epsilon_0(k) = \epsilon} \frac{d\sigma}{|v(k)|},
\]

where \( d\sigma \) denotes the measure over the Fermi surface, while \( d\epsilon \) denotes the measure perpendicular to the Fermi surface, we arrive at

\[ \omega \delta n(\sigma) = q \cdot v_F(\sigma) \left( \delta n(\sigma) + \frac{1}{(2\pi)^3} \int \frac{d\sigma'}{|v_F(\sigma')|} \left( f(\sigma, \sigma') + \frac{e^2}{q^2} \right) \delta n(\sigma') \right) + i \mathcal{I}[n], (3.14) \]

where due to the delta function, the integration is restricted to the Fermi surface, and we use \( v_F(\sigma) \) to denote \( v(k) \) when \( k \) is located at the Fermi surface which is the Fermi velocity, and we also use \( f(\sigma, \sigma') \) to denote \( f(k,k') \) when both \( k \) and \( k' \) are on the Fermi surface. The Coulomb interaction is independent of the angle variable and \( q \) is not a dynamical quantity, so one can regard the Coulomb interaction as a modification of the Landau parameter in the s-wave channel.

The Boltzmann equation (3.14) is the central equation that governs the collective modes in a Fermi liquid, including sound modes. It can describe the situation either with short-range interaction or long-range interaction, treating zero sound, first sound, and plasmon equivalently within one formalism. In the next section, we construct a simple model where the Boltzmann equation (3.14) can be solved exactly.

### 4 A solvable model

To proceed, let us assume the Fermi liquid in question has spherical symmetry. This is the situation in simple 3D metals. As a result the Fermi velocity is independent of angle
but $\mathbf{v}_F(\sigma) = v_F$ and we can choose $\mathbf{q} = (0, 0, q)$ pointing along $k_z$ direction, and use the spherical coordinate $\Omega = (\theta, \phi)$. Then (3.14) becomes

$$
\omega \delta n(\Omega) = qv_F \cos \theta \left( \delta n(\Omega) + \int \frac{d\Omega'}{4\pi} \left( F(\Omega, \Omega') + \frac{8\pi \alpha}{\lambda_F^2 q^2} \right) \delta n(\Omega') \right) + i\mathcal{I}[n],
$$

(4.1)

where we have used $\int d\sigma = k_F^2 \int d\Omega = k_F^2 \int \sin \theta d\theta d\phi$, $k_F$ is the Fermi momentum $\epsilon_0(k_F) = \mu$ and $\lambda_F$ is the corresponding Fermi wavelength $\lambda_F = \frac{2\pi}{k_F}$. $\alpha = \frac{e^2}{4\pi v_F}$ is the effective fine structure constant in the Fermi liquid defining the interaction coupling strength, and $F(\Omega, \Omega') \equiv \frac{k_F^2}{2\pi^2 v_F} f(\Omega, \Omega')$ is the dimensionless Landau parameter. As we mentioned in the previous section, the Coulomb interaction is not a dynamical quantity in the Boltzmann equation. We can absorb the Coulomb interaction into the Landau parameter in the s-wave channel, and restore it back at the end of the calculation.

In the following, we assume that the only nonvanishing component of Landau parameter is in the s-wave channel, namely, the Landau parameter is a constant $F(\Omega, \Omega') = F_0$. We absorb the Coulomb interaction into the Landau parameter. This should be sufficient for our purpose to investigate the effect of Coulomb interaction on the sound mode in a solvable mode. Without the collision integral, the Boltzmann equation reduces to the following eigen equation,

$$(x_0 - \cos \theta) \delta n(\Omega) = F_0 \int \frac{d\Omega'}{4\pi} \delta n(\Omega'), \quad x_0 = \frac{\omega}{qv_F},$$

(4.2)

which can solved [11] by

$$\delta n(\Omega) = \frac{\cos \theta}{x_0 - \cos \theta}, \quad x_0 \text{arccoth} x_0 = 1 + \frac{1}{F_0}.$$

(4.3)

Since we ignore the collision integral, this solution represents, by definition, the zero sound solution in collisionless regime. The second equation determines the velocity of the zero sound. When the $F_0 > 1$, we get the approximate zero sound velocity given by $v_0 = \sqrt{\frac{F_0}{3} - \frac{14}{9}}$.

Since (4.2) has spherical symmetry, we can study the eigen equation using the spherical harmonics. Any solution $\delta n(\Omega)$ can be expanded in the basis of spherical harmonics,

$$\delta n(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \delta n_{lm} Y_l^m(\theta, \phi),$$

(4.4)

where $Y_l^m(\theta, \phi)$ is the spherical harmonic and $\delta n_{lm}$ is the corresponding expansion coefficient. The sound modes are solutions with zero magnetic quantum number $m = 0$. As the eigen equation conserves the magnetic quantum number, we consider the $m = 0$ sector where the equation can be cast into

$$x_0 \delta n_{l,0} = \frac{l}{\sqrt{4l^2 - 1}} \delta n_{l-1,0} + \frac{l + 1}{\sqrt{4(l+1)^2 - 1}} \delta n_{l+1,0} + \frac{F_0}{\sqrt{3}} \delta n_{0,0} \delta n_{l,1}.$$

(4.5)
The detailed derivation of this equation is given in Appendix B. For \( l \geq 2 \), it resembles a recurrence relation of a series. Indeed, it is not hard to check that for \( l \geq 2 \), the following series of hypergeometric functions satisfies the recurrence relation,

\[
a_{l,m}(x) = 2\pi \delta_{m,0} \sqrt{\frac{2l+1}{4}} \frac{\Gamma(l+1)}{(2\pi)^l \Gamma(l+\frac{3}{2})} 2F_1 \left( \frac{l+1}{2}; \frac{l+2}{2}; l+\frac{3}{2}; \frac{1}{x^2} \right), \quad l \geq 1.
\]

This series is consistent with the eigenfunction (4.3). We present the details of obtaining the series in Appendix B. Thus, we have \( \delta n_{l,m} = a_{l,m}(x_0) \) for \( l \geq 1 \). The two equations for \( l = 1 \) and \( l = 0 \) read

\[
x_0 \delta n_{1,0} = \frac{1+F_0}{\sqrt{3}} \delta n_{0,0} + \frac{2}{\sqrt{15}} \delta n_{2,0},
\]

\[
x_0 \delta n_{0,0} = \frac{1}{\sqrt{3}} \delta n_{1,0},
\]

which lead to the following solution,

\[
\delta n_{0,0} = 2\sqrt{\pi}(x_0 \text{arccoth} x_0 - 1), \quad x_0 \text{arccoth} x_0 = 1 + \frac{1}{F_0}.
\]

This is of course consistent with the previous eigen-solution (4.3).

To access the hydrodynamic regime, where the collisionless zero sound crosses over to the hydrodynamic first sound, the collision integral plays an essential role. We take the collision integral to have the following form [3]

\[
\mathcal{I}(n) = -\sum_{l=0}^{\infty} \gamma_l \delta n_{l,0}, \quad \gamma_l = \begin{cases} 0 & l = 0 \\ \gamma_{\text{imp}} & l = 1 \\ \gamma & l \geq 2 \end{cases}
\]

where \( \gamma \) is the collision rate from collisions between quasiparticles and \( \gamma_{\text{imp}} \) is the collision rate between quasiparticles and impurities. Obviously, \( \gamma \) is the key hydrodynamic interaction parameter. Because collisions between quasiparticles conserve the particle number and their total momentum, the pure quasiparticle collision rate for \( l = 0,1 \) vanishes by virtue of conservation laws. On the other hand, elastic collisions between quasiparticles and quenched impurities relax the momentum, and, therefore, \( \gamma_{\text{imp}} \) is nonzero for \( l = 1 \). With this collision integral, the Boltzmann equation in the basis of spherical harmonics reduces to the following coupled equations,

\[
x \delta n_{l,0} = l \sqrt{\frac{l+1}{l^2-1}} \delta n_{l-1,0} + \frac{l+1}{\sqrt{4(l+1)^2-1}} \delta n_{l+1,0}, \quad l \geq 2,
\]

\[
x_{\text{imp}} \delta n_{1,0} = \frac{1+F_0}{\sqrt{3}} \delta n_{0,0} + \frac{2}{\sqrt{15}} \delta n_{2,0},
\]

\[
x_0 \delta n_{0,0} = \frac{1}{\sqrt{3}} \delta n_{1,0},
\]

\[
x_0 = \frac{\omega}{qv_F}, \quad x = \frac{\omega + i\gamma}{qv_F}, \quad x_{\text{imp}} = \frac{\omega + i\gamma_{\text{imp}}}{qv_F}.
\]
The series (4.6) again solves the first recurrence relation, with \( x \) replacing \( x_0 \). Thus for 
\[ l \geq 1, \quad \delta n_{l,m} = a_{l,m}(x) \]
and
\[ (F_0 + 1 - 3x_0 x_{\text{imp}})(x \arccoth x - 1) + x_0[(3x^2 - 1) \arccoth x - 3x] = 0. \]
(4.15)
This is the main result of our paper. The eigen dispersion of sound mode is determined by 
the above equation (4.15), where the definitions of \( x \)'s are given by (4.14).

To investigate the sound mode, we can assume \( \gamma_{\text{imp}} = 0 \), otherwise the coherent propagating sound mode is damped. In the collisionless regime, \( \omega \gg \gamma \), we obtain the zero sound mode or the plasmon in (2.1) as
\[ \omega = \pm v_0 q - i\gamma \frac{(F_0 + 1)^2 - 2(F_0 - 1)\left(\frac{m}{v_F} \right)^2 - 3\left(\frac{m}{v_F}\right)^4}{F_0[F_0 + 1 - \left(\frac{m}{v_F}\right)^2]} = 1 + \frac{1}{F_0}. \]
(4.16)
In the case of strong repulsion \( F_0 > 1 \) (this is also the case for Coulomb interaction in the long wavelength limit), the sound mode can be simplified to
\[ \omega = \pm \sqrt{\frac{F_0}{3} + \frac{3}{5} v_F q - i\gamma} \frac{2(5F_0 + 21)}{5F_0(5F_0 + 3)}, \]
which is nothing but (2.2) in Section 2. On the other hand, in the collision-dominated hydrodynamic regime \( \omega \ll \gamma \) we get the first sound,
\[ \omega = \pm \sqrt{\frac{F_0}{3} + \frac{1}{3} v_F q} - i\frac{2v_F^2}{15}\gamma q^2, \]
(4.18)
which is (2.3) in Section 2.

We can also consider the effect of the finite impurity scattering. For weak impurity scattering, the effect is to modify the damping defined by imaginary parts of the sound mode. For the zero sound, we have
\[ \omega = \pm v_0 q - i\gamma \frac{(F_0 + 1)^2 - 2(F_0 - 1)\left(\frac{m}{v_F} \right)^2 - 3\left(\frac{m}{v_F}\right)^4}{F_0[F_0 + 1 - \left(\frac{m}{v_F}\right)^2]} - i\gamma_{\text{imp}} \frac{3\left(\frac{m}{v_F}\right)^2\left[\frac{m}{v_F}^2 - 1\right]}{F_0[F_0 + 1 - \left(\frac{m}{v_F}\right)^2]}. \]
(4.19)
For large \( F_0 > 1 \), this reduces to a simple correction \( \delta \text{Im}(\omega) = \frac{1}{2}\gamma_{\text{imp}} \). For strong impurity scattering \( \gamma_{\text{imp}} > q \), both sound modes are over-damped into
\[ \omega = -i\frac{F_0 + 1}{3\gamma_{\text{imp}}} v_F^2 q^2, \quad \omega = -i\gamma_{\text{imp}}. \]
(4.20)

5 Conclusions

We have discussed electronic sound modes in 3D metals in the presence of long-range Coulomb coupling via the Boltzmann equation. A more microscopic approach to the collective mode,
like plasmons, would be starting from the electron Hamiltonian with long-range interactions. Then the collective mode results from integrating out the electron fluctuations. In the lowest order, this is nothing but the RPA approach. Here, we recapitulate how it works for 3D metals. In the RPA approximation, the collective mode is determined by the following eigen equation,

\[ 1 - V(q)\Pi(q, \omega) = 0, \]  

(5.1)

where \( V(q) \) is the interaction potential at momentum \( q \), and the dynamical electron polarization function is defined by

\[ \Pi(q, \omega) = \frac{1}{\beta} \sum_n \int \frac{1}{k} \frac{1}{-i\Omega_n - \frac{\omega}{2} + \xi(k + \frac{q}{2})} \frac{1}{-i\Omega_n + \frac{\omega}{2} + \xi(k - \frac{q}{2})}, \]  

(5.2)

\[ \Omega_n = \frac{(2n + 1)\pi}{\beta}, \quad \xi(k) = \epsilon_0(k) - \mu. \]  

(5.3)

where \( \Omega_n \) is the Matsubara frequency while \( \omega \) is the real frequency. To proceed, we assume a spherical Fermi surface with parabolic dispersion \( \epsilon_0(k) = \frac{k^2}{2m} \). It is not hard to get the 3D polarization function at zero temperature, namely,

\[ \Pi(q, \omega) = \frac{N_e q^2}{m \omega^2} \left( 1 + \frac{3}{5} \frac{k_F^2 q^2}{m^2 \omega^2} \right), \quad N_e = \frac{4\pi}{3} \frac{k_F^3}{F_0}, \]  

(5.4)

where \( q \equiv |q| \). Putting in the Coulomb potential given by \( V_{\text{Coul}}(q) = \frac{e^2}{q^2} \) and the polarization function (5.4) into eigen equation (5.1), we obtain the conventional plasmon given in (2.9).

We can also consider a short-range density-density interaction potential that is independent of momentum, namely,

\[ V(q) = \frac{v_F}{4\pi k_F^2} F_0, \]  

(5.5)

where the prefactor is chosen for convenience. Using this interaction potential to replace the Coulomb potential, we get the following linearly dispersing sound-like collisionless collective mode

\[ \omega^2 = \frac{v_F F_0}{4\pi k_F^2} N_e q^2 \left( 1 + \frac{3}{5} \frac{k_F^2 q^2}{m^2 \omega^2} \right), \quad \omega \approx \pm \sqrt{\frac{F_0}{3} + \frac{3}{5} v_F q}. \]  

(5.6)

The second equation is exactly the zero sound mode in (2.2). At zero temperature, within RPA, there is no damping of the electronic collective mode by quasiparticle collisions since the collision rate vanishes as \( T^2 \), but finite impurity scattering would still contribute to the damping in the way we discussed earlier. At finite temperatures, quasiparticle collisions would lead to Landau damping of the collective modes.

This simple calculation tells us that microscopically the interaction potential determines the dispersion of the electronic collective modes. The zero sound mode for short-range interactions becomes the gapped 3D plasmon mode in the presence of long-range Coulomb
coupling. (The first sound mode also acquires a long wavelength gap as discussed earlier in the paper.) Indeed, from the perspective of symmetry, both sound modes and plasmons are density fluctuations that characterize the underlying particle number conservation. So they are actually the same collective mode, and it is not a surprise that they all develop long wavelength gaps because of the long-range Coulomb interaction. Thus, the hydrodynamic sound in 3D metals is not a sound mode at all since it has a finite energy at zero momentum defined by the 3D plasma frequency. We do note, however, that the sound modes differ from the plasmon in their next-to-leading-order dispersion corrections at finite momentum. With this in mind, we now briefly discuss the 1D case. (The 2D case is clearly addressed in [3].) In 1D, the Boltzmann approach does not work simply because the Fermi liquid does not exist in the presence of any finite interaction [13, 14]. Thus starting from the microscopic Hamiltonian like above is a good and simple way to look for sound or plasmon modes. The electron polarization function is now given by

\[ \Pi(q, \omega) = \frac{m^2}{2\pi q} \log \left( \frac{m^2 \omega^2 - (k_F - q)^2 q^2}{m^2 \omega^2 - (k_F + q)^2 q^2} \right). \] (5.7)

We expect a sound wave-like (linear in momentum) mode when the interaction is short ranged. Indeed the short-range potential

\[ V(q) = \pi v_F F_0 \] leads to the zero sound mode in 1D,

\[ \omega = \pm v_F \sqrt{F_0 q}. \] (5.8)

How does the Coulomb interaction affect this sound mode? The answer is simple, we just need to replace \( V(q) \) with the 1D Coulomb potential [15],

\[ V_{\text{Coul}}(q) = \frac{e^2}{4\pi} \int dr \frac{e^{iqr}}{\sqrt{r^2 + a^2}} = 2K_0(aq), \] (5.9)

where \( K_0(x) \) is the modified Bessel equation of the second kind and \( a \) is a short-range cutoff introduced to make the integral converge in 1D (\( a \) can be thought of as a lattice constant). Plugging the Coulomb potential and the polarization function in 1D into the eigen equation (5.1), the resultant long wavelength collective mode is

\[ \omega = \pm \frac{e}{\pi} \sqrt{\frac{v_F}{2}} q \sqrt{-\log \frac{aq}{2}}, \quad aq \ll 1, \] (5.10)

which is nothing but the well-known plasmon mode in 1D. Though we consider zero temperature, we expect that the plasmon mode takes over the sound modes in 1D when the Coulomb interaction dominates since the dispersion is largely independent of temperature. A more physical argument is that from the symmetry perspective, sound modes and plasmons are the very same mode, and the different names just refer to whether the interaction potential is short-range or long-range. The curious thing is that in 3D systems, where the Coulomb coupling goes as \( q^{-2} \), the sound mode is not acoustic at all since it acquires the plasmon mass at zero momentum.
In conclusion, we construct a solvable model in 3D to obtain the sound modes in both collisionless regime and collision-dominated hydrodynamic regime. In particular, we discuss the effect of long-range Coulomb interaction on the sound modes. We find that in the presence of Coulomb interaction, both the zero sound and the first sound obtain a finite gap equal to the plasmon frequency, and a damping rate which is quadratic in momentum. We also discuss general long-range interactions that lead to unusual plasmon dispersions. Finally, we clarify the collective mode and sound mode dichotomy in 1D.

Acknowledgements

This work is supported by the Laboratory for Physical Sciences. S.-K. J. is supported by the Simons Foundation via the It From Qubit Collaboration.

A Coulomb potential

In this section, we present the Fourier transform of the 3D Coulomb potential. Since the Coulomb potential is spherically symmetric, without loss of generality, we can choose the momentum pointing to the $z$ axis and make a coordinate transform as follows,

$$\int d^3r \frac{1}{|r|} e^{i \mathbf{q} \cdot \mathbf{r}} = \int dr d\theta d\phi r^2 \sin \theta \frac{1}{r} e^{i|\mathbf{q}| r \cos \theta - \varepsilon r} = \int dr \frac{4\pi e^{-\varepsilon r} \sin(|\mathbf{q}| r)}{|\mathbf{q}|} = \frac{4\pi}{q^2 + \varepsilon^2}.$$ (A.1)

where in the second step we have added an infinitesimal positive number $\varepsilon$ to ensure the convergence. In the last step, we can safely send $\varepsilon$ to zero.

B Some useful mathematical results

In this section we present mathematical results that are used in the main text. The spherical harmonics are defined by

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \quad |m| \leq l, \quad l \geq 0,$$ (B.1)

where $P_l^m(x)$ is the associated Legendre polynomial, and the prefactor is chosen to make sure the spherical harmonics are properly normalized,

$$\int d\Omega Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi)^* = \delta_{ll'} \delta_{mm'},$$ (B.2)
$d\Omega \equiv \sin \theta d\theta d\phi$ is the short-hand notation of measure of a sphere. A useful recurrence formula of associated Legendre polynomial is

$$x P^m_l(x) = \frac{l - m + 1}{2l + 1} P^m_{l+1}(x) + \frac{l + m}{2l + 1} P^m_{l-1}(x).$$

(B.3)

By this recurrence relation, we arrive at the following recurrence formula

$$\cos \theta Y^m_l(\theta, \phi) = \sqrt{\frac{(l + 1)^2 - m^2}{4(l + 1)^2 - 1}} Y^m_{l+1}(\theta, \phi) + \sqrt{\frac{l^2 - m^2}{4l^2 - 1}} Y^m_{l-1}(\theta, \phi).$$

(B.4)

This is useful to transform the Boltzmann equation to the harmonic basis.

We are interested in the expansion of the following function in the basis of spherical harmonics,

$$\delta n_0(\Omega) = \frac{\cos \theta}{\lambda - \cos \theta}.$$  

(B.5)

Since this function is independent of the angle $\phi$, its expansion coefficient on $Y^m_l \neq 0$ vanishes. We can focus on the $Y^0_0(\theta, \phi)$ which is related to the Legendre polynomial $P_l(x)$. Let us first evaluate the integral

$$\int_{-1}^{1} dx \frac{x}{\lambda - x} P_n(x) = \frac{1}{2^n n!} \int_{-1}^{1} dx \frac{x}{\lambda - x} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

(B.6)

where in the first step we have performed a coordinate transformation $\cos \theta = x$, in the second step we have used the Rodrigues’ formula $P_n(x) \equiv \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$. For $n = 0$, can directly evaluate the integral to get

$$\int_{-1}^{1} dx \frac{x}{\lambda - x} P_0(x) = 2(\lambda \text{arccoth} \lambda - 1).$$

(B.7)

For $n \geq 1$, one can repeatedly use integration by parts to bring the integral into the form

$$\int_{-1}^{1} dx \frac{x}{\lambda - x} P_n(x) = \frac{1}{2^n n!} \int dx \left( \frac{d^n}{dx^n} \frac{x}{\lambda - x} \right) (1 - x^2)^n + [\text{boundary terms}].$$

(B.8)

It is not hard to show that all boundary terms vanish. The $n$-th derivative of the function $\delta n_0(\Omega)$ is

$$\frac{d^n}{dx^n} \frac{x}{\lambda - x} = \frac{n! \lambda}{(\lambda - x)^{n+1}}.$$  

(B.9)

Plugging this into the integral, we have

$$\int_{-1}^{1} dx \frac{x}{\lambda - x} P_n(x) = \frac{\lambda}{2^n} \int_{-1}^{1} dx \frac{(1 - x^2)^n}{(\lambda - x)^{n+1}}$$

$$= \frac{\sqrt{\pi}}{(2\lambda)^n} \frac{\Gamma(n + 1)}{\Gamma(n + \frac{3}{2})} \binom{n + 1}{n + 2}^{\frac{3}{2}} \binom{n + 3}{1}^{\frac{1}{2}},$$

(B.10)
where \( \, _2F_1(a, b; c; z) \) denotes the hypergeometric function. Finally, in terms of the spherical harmonics,

\[
\int d\Omega \frac{\cos \theta}{\lambda - \cos \theta} Y_i^m(\theta, \phi)^* = 2\pi \delta_{m,0} \sqrt{\frac{2l+1}{4\pi}} \int_0^\pi d\theta \frac{\cos \theta}{\lambda - \cos \theta} P_l(\cos \theta) 
\]

\[
= 2\pi \delta_{m,0} \sqrt{\frac{2l+1}{4}} \frac{1}{(2\lambda)^l \Gamma(l+\frac{3}{2})} \, _2F_1\left(\frac{l+1}{2}, \frac{l+2}{2}; l+\frac{3}{2}; \frac{1}{\lambda^2}\right). 
\]

The expansion coefficient of \( \delta n_0(\Omega) = \sum_{l=0}^\infty \sum_{m=-l}^l \delta n_0^{lm}(\lambda) Y_i^m(\theta, \phi) \) is

\[
\delta n_0^{lm}(\lambda) = \begin{cases} 
\delta_{m,0} \times 2\sqrt{\pi} (\lambda \text{arccoth} \lambda - 1) & l = 0 \\
\delta_{m,0} \times 2\sqrt{\pi} \sqrt{\frac{2l+1}{4}} \frac{\Gamma(l+1)}{(2\lambda)^l \Gamma(l+\frac{3}{2})} \, _2F_1\left(\frac{l+1}{2}, \frac{l+2}{2}; l+\frac{3}{2}; \frac{1}{\lambda^2}\right) & l \geq 1.
\end{cases}
\]

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