Fast Amplification and Rephasing of Entangled Cat States in a Qubit-Oscillator System

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We study a qubit-oscillator system, with a time-dependent coupling coefficient, and present a scheme for generating entangled Schrödinger-cat states with large mean photon numbers and also a scheme that protects the cat states against dephasing caused by the nonlinearity in the system. We focus on the case where the qubit frequency is small compared to the oscillator frequency. We first present the exact quantum state evolution in the limit of infinitesimal qubit frequency. We then analyze the first-order effect of the nonzero qubit frequency. Our scheme works for a wide range of coupling strength values, including the recently achieved deep-strong-coupling regime.

Introduction.—The interaction of a two-level atom (qubit) with a quantized field (oscillator) has been widely studied over the past few decades. There have been numerous experimental realizations of such systems, including superconducting circuits [1–9], and systems of atoms coupled to superconducting microcavities [10]. Mathematically, such qubit-oscillator systems, described by the quantum Rabi model, can be simplified to the Jaynes-Cummings model, with the rotating-wave approximation (RWA), in the case of a near-resonant field [11, 12]. We focus on the case where the qubit frequency is small compared to that of the oscillator, where the RWA cannot be used. Previous studies have mainly focused on time-independent coupling coefficients [13, 14]. In this work, we examine the qubit-oscillator system with a time-dependent coupling coefficient, and we demonstrate a scheme to generate Schrödinger-cat states of a very large size, and we develop a second scheme that also protects the cat states from dephasing. Our amplification scheme offers a potentially simple and fast alternative compared to that of the oscillator, where the RWA can- not be used. Previous studies have mainly focused on time-independent coupling coefficients [13, 14].

State evolution under time-dependent coupling with infinitesimal qubit frequency.—The Hamiltonian of the combined system of the qubit and the oscillator is

$$\hat{H}(t) = -\frac{\hbar}{2} \Delta \hat{\sigma}_z + \hbar \omega \left( \hat{a} \hat{a} + \frac{1}{2} \right) + \hbar g(t) \hat{\sigma}_x (\hat{a} + \hat{a}^\dagger) \tag{1}$$

where $\omega$ and $\Delta$ are the frequencies of the oscillator and the qubit, respectively, and $g(t)$ is the time-dependent coupling constant, $\hat{a}$ and $\hat{a}^\dagger$ are, respectively, the annihilation and creation operators of the oscillator, and $\hat{\sigma}_x$ are the Pauli operator of the qubit [17]. We focus on the situation where $\Delta$ is small compared to $\omega$. We will first examine the zeroth order effect of the small $\Delta$ by taking the limit of $\Delta/\omega \to 0$, which gives

$$\hat{H}^{(0)}(t) = \hbar \omega \left( \hat{a} \hat{a} + \frac{1}{2} \right) + \hbar g(t) \hat{\sigma}_x (\hat{a} + \hat{a}^\dagger) \tag{2}$$

The initial eigenstates of the Hamiltonian at $t=0$ are the entangled states [13, 14]:

$$|E^0_{N \pm}(0)\rangle = |+\rangle_\pm \hat{D} \left( -\frac{g(0)}{\omega} \right) |N\rangle \pm |\pm\rangle_\pm \hat{D} \left( +\frac{g(0)}{\omega} \right) |N\rangle, \tag{3}$$

ignoring a factor of $1/\sqrt{2}$, and where $|\pm\rangle$ are the two qubit eigenstates of the Pauli matrix $\hat{\sigma}_x$ with eigenvalues $\pm 1$, the $|N\rangle$ is the $N$-photon Fock state in the oscillator, and $\hat{D} \left( \pm\frac{g(0)}{\omega} \right)$ are displacement operators. Note that, in the limit of $\Delta/\omega \to 0$, the energy eigenstates $|E^0_{N +}(0)\rangle$ and $|E^0_{N -}(0)\rangle$ are almost degenerate, with the energy eigenvalues of $E^0_{N \pm}(0) = \hbar \omega \left( N + 1/2 - g^2(0)/\omega^2 \right)$.

We now consider the state evolution under an arbitrary time-dependent coefficient $g(t)$. The energy eigenstates $|E^0_{N \pm}(t)\rangle$, determined by the instantaneous value of $g(t)$, do not reflect the evolution of quantum states. In other words, an initial state $|E^0_{N \pm}(0)\rangle$ generally does not, for a time-dependent $g(t)$, evolve into $e^{i\phi_{N \pm}(t)} |E^0_{N \pm}(t)\rangle$, with a phase factor $\phi_{N \pm}(t)$, at a later time $t$. On the other hand, a quantum state in the form of $|\tilde{E}^0_{N \pm}(0)\rangle$ evolves into the quantum state $e^{i\phi_{N \pm}(t)} |\tilde{E}^0_{N \pm}(t)\rangle$, where

$$|\tilde{E}^0_{N \pm}(0)\rangle = |+\rangle_\pm \hat{D} \left( -\frac{\tilde{g}(t)}{\omega} \right) |N\rangle \pm |\pm\rangle_\pm \hat{D} \left( +\frac{\tilde{g}(t)}{\omega} \right) |N\rangle, \tag{4}$$

and the phase factor $\phi_{N}(t)$ is given in the Supplemental Material. Here the complex variable $\tilde{g}(t)$ obeys the
FIG. 1. Evolution of $\tilde{g}(t)$ under time-dependent $g(t)$ with infinitesimal $\Delta$. (a) Three different time dependences of $g(t)$. (b) The trajectories of $\tilde{g}(t)$ in the complex plane corresponding to (a), which, as seen in Eq. (4), determine the evolution of the dynamical evolution eigenstates. The initial time is $t=0$ and the trajectories start from $\tilde{g}(0)/g(0)=1$. (c) The time dependences of $g(t)$ and $\tilde{g}(t)$, in the case of sinusoidal driving force $g(t)=g(0)(1+\cos\omega t)/2$. (d) The trajectories of $\tilde{g}(t)$ and $\tilde{g}(0)$ corresponding to (c), in the complex plane. The amplitude of $\tilde{g}(t)$ keeps expanding, showing cat-state amplification on resonance. The blue and red dots in (c) and (d) indicate where the modulation stops ($t=4\pi/\omega$) in the scheme shown in Fig. 2.

equation
\[
\dot{\tilde{g}}(t) = i\omega (g(t) - \tilde{g}(t)).
\] Note that the initial $\tilde{g}(0)$ can be set to any value. The proof is given in the Supplemental Material. We refer to $|E_{N_\pm}^{(0)}(t)|$ as the dynamical evolution eigenstates, which take the similar form with the energy eigenstates in Eq. (3), but with $g(t)$ replaced with $\tilde{g}(t)$. Now the dynamics of the system is governed by Eq. (5), which shows that $\tilde{g}(t)$ does not respond instantly to changes in $g(t)$. Note that, apart from a few special cases, the dynamical evolution eigenstates are generally not energy eigenstates.

To understand the evolution of quantum states, we give the following three scenarios, in which we set the initial $\tilde{g}(0)=g(0)$ so that $|E_{N_\pm}^{(0)}(0)|=|E_{N_\pm}^{(0)}(0)|$.

(i) Suppose $g(t)$ is adiabatically changed over time [red curve in Fig. 1(a)]. Any energy eigenstate has ample time adjust to the adiabatically changing $H^{(0)}(t)$ and also remains an energy eigenstate [red curve in Fig. 1(b)].

(ii) Now suppose $g(t)$ is constant at a certain value at $t<t_0$, and then set to zero instantaneously at $t=t_0$ [blue curve in Fig. 1(a)]. Since neither $\omega$ nor $\Delta$ is infinitely large, the states $|E_{N_\pm}^{(0)}(t)|$ cannot adjust instantaneously, and they remain the same at $t=t_0$. However $|E_{N_\pm}^{(0)}(t_0+)|$ are no longer the energy eigenstates, and the states begin to evolve. Taking the ground dynamical evolution eigenstate
\[
|E_{0_\pm}^{(0)}(t)| = |+\rangle |-\tilde{g}(t)/\omega\rangle \pm |-\rangle |+\tilde{g}(t)/\omega\rangle,
\] as an example, the amplitude of the coherent state component of the state, $|\tilde{g}(0)|$ before the adjustment, should begin to revolve around the origin after the adjustment, consistent with the evolution of a regular coherent state in a free oscillator [blue circle in Fig. 1(b)]. When $g(t)$ is instantaneously set to a nonzero value, $\tilde{g}(t)$ revolves around this value in the complex plane.

(iii) Now we consider the intermediate scenario. We assume that $g(t)$ is adjusted over a finite period of time to zero and then kept stabilized as shown in the green curve in Fig. 1(a). In this scenario, $\tilde{g}(t)$ will start changing as $g(t)$ starts changing. Its trajectory is less intuitive than the extreme scenarios, but can be understood from Eq. (3). After $g(t)$ becomes constant again, $\tilde{g}(t)$ evolves in circular motion around the new constant $g$.

A very useful application of varying $g$ through time is to amplify the absolute value of the amplitude of the entangled-cat-state components of Eq. (6). We can achieve this without having to make the absolute value of $g$ large, which is difficult in real experiments, but rather by modulating $g$ periodically on resonance with the frequency $\omega$. A somewhat similar modulation has been proposed to control the coupling of two flux qubits via a quantum bus [18]. Generally, any modulation of $g(t)$ on resonance with the oscillator frequency leads to a steady increase in the coherent-state amplitude in Eq. (6). As a specific example, the case of a sinusoidal driving function $g(t)=g(0)(1+\cos\omega t)/2$ is shown in Figs. 1(c) and (d). The magnitude $|\tilde{g}(t)|$ will grow linearly with time. This behavior is easy to understand from Eq. (5), whose solution corresponds to a simple harmonic oscillator being driven by an external force. Since the photon number in the coherent state is proportional to $|\tilde{g}/\omega|^2$, it will grow quadratically as a function of time. Simply growing for two oscillator periods increases the absolute amplitude of the coherent state component by a factor of 3.3. So we will focus on two oscillator periods of growth in later discussions.

First-order effect of nonzero-gubit frequency.—So far we have ignored the effect of the small $\Delta$ by taking the limit of $\Delta/\omega \to 0$. We now examine the first-order effect in $\Delta$ in the full Hamiltonian in Eq. (1). Note that for a qubit to avoid effects from surrounding noise and unwanted excitations by the pulse operations, $\Delta$ has to be reasonably large. For example, in recent experiments [6, 7], $\Delta/\omega \approx 0.1$.

At any time $t$ we can express a general state of in-
terest as a superposition of dynamical evolution eigenstates: $|\varphi(t)\rangle = \sum_{N_{\pm}} C_{N_{\pm}}(t) e^{-iN_{\pm} \omega t} |E_{N_{\pm}}^{(0)}(t)\rangle$. As the state $|\varphi(t)\rangle$ evolves, so does every dynamical evolution eigenstate $|E_{N_{\pm}}^{(0)}(t)\rangle$ and its corresponding coefficient $C_{N_{\pm}}(t)$. Under the full Hamiltonian $\hat{H}(t)$, the $C_{N_{\pm}}(t)$ generally change over time. If we consider up to the first order in $\Delta/\omega$, $C_{N_{\pm}}(t)$ can be expressed as: $[C_{0+}(0) C_{1+}(0) \ldots C_{0-}(0) C_{1-}(0) \ldots]$ \times \exp \left\{ \frac{i}{\hbar} \int_{0}^{t} \Delta(t) \{ \hat{M}_{\sigma_{z}}(t) \} dt \right\}$, where $\hat{M}_{\sigma_{z}}(t)$ is the matrix of the operator $\hat{\sigma}_{z}$ in the basis $e^{-iN_{\pm} \omega t} |E_{N_{\pm}}^{(0)}(t)\rangle$ at time $t$. For generality, we have made the parameter $\Delta$ time-dependent [$\Delta(t)$]. A detailed derivation of this result is provided in the Supplemental Material.

An intuitive way to understand the effect of nonzero $\Delta$ is the following. If $\Delta$ was to be considered infinitesimal, the quantized oscillator has equally spaced energy levels. Nonzero $\Delta$ disrupts such equally spaced energy levels. We have shown above that nonzero $\Delta$ leads to the change in $C_{N_{\pm}}(t)$ causing any general quantum state to dephase, which we will illustrate below. To minimize this dephasing, we need to make $\int_{0}^{t} \Delta(t) \{ \hat{M}_{\sigma_{z}}(t) \} dt$ as close to zero as possible. This type of dephasing is well known for coherent state solutions to the harmonic oscillator and was first described by Schrödinger in 1926 when he showed how Bohr’s correspondence principle applies to the hydrogen atom [19]. For large principle quantum number $n$, the electron should resemble a classical particle in a Kepler orbit around the proton. This never happens with the stationary eigenstates of the hydrogen atom, regardless of how big $n$ is, but in the limit of large $n$, the energy levels are nearly equally spaced and can be approximated by a harmonic oscillator. Schrödinger then showed that the electron in a coherent state, a particular superposition of different $n$, moves as a wave packet in a Kepler orbit. However since the levels are not truly equally spaced, dephasing occurs and the electron packet spreads in the orbit until the front of the wave packet wraps around and interferes with the back. The effect has been seen in experiments with single-electron Rydberg atoms [20]. We see exactly the same effect here. With zero $\Delta$ the energy levels are exactly equally spaced and the coherent state of the oscillator does not dephase. With small but nonzero $\Delta$ the energy levels are not exactly equally spaced and dephasing now occurs, as we show below.

Let us examine a special situation where we can actually reduce the term $\int_{0}^{t} \Delta(t) \{ \hat{M}_{\sigma_{z}}(t) \} dt$ completely to zero. We can take advantage of the fact that as long as $g$ is a constant, $\hat{M}_{\sigma_{z}}(t)$ is periodic with a period $\frac{2\pi}{\omega}$. Keeping $g$ a constant, first we keep $\Delta$ at a certain nonzero value for a $\frac{2\pi}{\omega}$ period of time, with $k$ being an integer; then we flip $\Delta$ to the opposite sign for another $\frac{2\pi}{\omega}$ period of time. As a result, the two parts $\int_{0}^{t} \frac{2\pi}{\omega} \Delta(t) \{ \hat{M}_{\sigma_{z}}(t) \} dt$ and $\int_{0}^{t} \frac{2\pi}{\omega} \Delta(t) \{ \hat{M}_{\sigma_{z}}(t) \} dt$ cancel each other, eliminating the first-order dephasing effect of $\Delta$ for the duration of $(0, 2k \frac{2\pi}{\omega})$.

Because of the periodicity of the term $\frac{2\pi}{\omega}$, the states shown in (c) is only 0.80504, since the cat state is dephased by the nonzero $\Delta$. However, if we insert one $\hat{\sigma}_{x}$-π pulse in the middle of the 10 oscillator periods, the evolved state, shown in (d), has a 0.98174 fidelity with the state in (b), analogous to a Hahn spin-echo rephasing effect.

![FIG. 2](image-url)  
**FIG. 2.** Cat-state amplification and rephasing in the Wigner representation of the oscillator state projected onto the qubit state $|+\rangle_{g} + |−\rangle_{g}$. The parameters are: $\Delta=0.1\omega$; $g(0)=0.833\omega$. (a) The initial state, which is taken to be the ground state, reasonably resembles the cat state $|−\frac{2\pi}{\omega}\rangle + |+\frac{2\pi}{\omega}\rangle$, with a fidelity of 0.99986. After the initial state goes through two oscillator periods of the sinusoidally driven cat-state-amplification process, the resulting state is shown in (b), which is the cat state $|−\frac{2\pi}{\omega}\rangle + |+\frac{2\pi}{\omega}\rangle$, with $|g/\omega|=3.3$ and a fidelity of 0.92286. We now let the state in (b) freely evolve for 10 oscillator periods, and the resulting state of the oscillator is shown in (c). The fidelity between the state $|−\frac{2\pi}{\omega}\rangle + |+\frac{2\pi}{\omega}\rangle$ and the states in (c) is only 0.80504, since the cat state is dephased by the nonzero $\Delta$. However, if we insert one $\hat{\sigma}_{x}$-π pulse in the middle of the 10 oscillator periods, the evolved state, shown in (d), has a 0.98174 fidelity with the state in (b), analogous to a Hahn spin-echo rephasing effect.

$\pi$-pulses and their application for cat state rephasing—Above we have discussed the method of eliminating the first-order effect of finite $\Delta$ by flipping the sign of $\Delta$. In real experiments, manipulating $g$ or $\Delta$ directly is not always easy. Usually $g$ and $\Delta$ cannot be changed too fast, and the range of their change can be limited. For example, in the recent experiments [6], the qubit and the oscillator are coupled via a Josephson inductances which effectively limits $|g|$ to being higher than a certain bound.

Here we will show a scheme to manipulate $g$ or $\Delta$ indirectly. By applying $\pi$-pulses to the qubit alone, which is a commonly used technique in dynamical decoupling [21, 22], we can achieve the effect of flipping the signs of $g$ or $\Delta$. There are three basic types of $\pi$-pulses, each of which amount to applying a Pauli operator $\{\hat{\sigma}_{x}, \hat{\sigma}_{z}, \text{or} \hat{\sigma}_{y}\}$ to the qubit. Let us first examine the $\hat{\sigma}_{x}$-$\pi$-pulse. Since $\hat{\sigma}_{x}$ and $\hat{\sigma}_{y}$ anti-commute, we have $\hat{H}(g)\hat{\sigma}_{x} = \hat{\sigma}_{y} \hat{H}(-g)$ in Eq. (1). Therefore, without directly altering the qubit-
oscillator coupling coefficient \( g \), in the Hamiltonian the sign of \( g \) is flipped by applying \( \hat{\sigma}_z \pi \) pulses to the qubit alone. For \( \hat{\sigma}_x \) and \( \hat{\sigma}_y \) pulses, similar arguments apply. By applying \( \hat{\sigma}_x \pi \) pulses to the qubit, the sign of \( \Delta \) is flipped. By applying \( \hat{\sigma}_y \pi \) pulses to the qubit, the signs of both \( g \) and \( \Delta \) are simultaneously flipped.

With infinitesimal \( \Delta \) and the simplified Hamiltonian \( H^{(0)}(t) \), when the initial state is the ground state \( |E_{0+}^{(0)}(t)\rangle = |+\rangle_x |\psi^{-}(t)\rangle + |-\rangle_x |\psi^{(+)}(t)\rangle \), modulating \( g(t) \) at the oscillator frequency can increase the amplitude of the coherent state component, as shown in Figs. 1(c) and (d). With nonzero but still small \( \Delta \) (compared to \( \omega \)) and full Hamiltonian \( H(t) \), the ground state, which can be written as \( |E_{0+}(t)\rangle = |+\rangle_x |\psi^{-}(t)\rangle + |-\rangle_x |\psi^{(+)}(t)\rangle \) is very close to \( |E_{0+}^{(0)}(t)\rangle \). Although \( |E_{0+}(t)\rangle \) is not strictly a cat state, for convenience we will still refer to the process for increasing the amplitude in the oscillator components in Eq. (6) as entangled cat-state amplification. This can be achieved either by directly modulating \( g(t) \), or by indirectly flipping the sign of \( g(t) \) by applying \( \hat{\sigma}_x \pi \) pulses to the qubit alone. Here, we modulate \( g(t) \) directly sinusoidally to amplify a cat state.

Cat-state amplification. — We begin with \( g \) constant and start with the ground state \( |E_{0+}(0)\rangle \) as the initial state, with the Wigner representation of \( |\psi^{-}(\pi)\rangle \) and \( |\psi^{(+)}(\pi)\rangle \) shown in Fig. 2(a). We apply a short sinusoidal driving force by modulating \( g(t) = \frac{1}{2} \left[ (1 + \cos(\omega t)) \right] \) after \( t = 0 \), until reaching \( t = \frac{\pi}{\omega} \), when two oscillator periods of cat-state amplification is completed, as shown with the red and blue dots in Fig. 1(c) and (d). The Wigner representation of the resulting state, which has a high fidelity with \( |\psi^{-}(\pi/\omega)\rangle \) and \( |\psi^{(+)}(\pi/\omega)\rangle \), is shown in Fig. 2(b). We can repeat this process to further increase the absolute value of amplitudes of the oscillator states, but we will focus on the case where the cat state amplification lasts for only two oscillator periods. Note that the amplified cat in Fig. 2(b) is tilted by the angle about \( \pi/2 \). This is due to the fact that \( g(t) \pi/\omega \) lags \( g(t) \pi/\omega \) by about a quarter of a cycle, as indicated by the dots in Fig. 1(c) and (d).

Rephasing the Amplified Cat. — Compared to the cat-state amplification, which increases the amplitude by a factor of 3.3 within just the first two oscillator periods, the dephasing effect takes a longer time due to \( \Delta \) being small compared to \( \omega \). Therefore the dephasing effect of the small \( \Delta \) is not very concerning if the cat state amplification lasts only for a short time.

But if we now let the state shown in Fig. 2(b) evolve freely for \( 20 \pi/\omega \), then the state is eventually dephased, as shown in Fig. 2(c). To counter this, we instead in Fig. 2(d) apply the cat to evolve freely for five oscillator periods, apply one \( \hat{\sigma}_x - \pi \) pulse to the qubit, and then allow free evolution for another five periods. We can see from the Wigner representation of the evolved state, shown in Fig. 2(d), that the amplified cat state is recovered in analogy to the rephasing of the Hahn spin-echo method [23]. If needed, more \( \hat{\sigma}_x \pi \) pulses can be applied in the same manner to preserve a state for a longer time. Also we can apply \( \hat{\sigma}_y \pi \) pulses more frequently, to counter the dephasing by a larger \( \Delta \).

Conclusion. — We have analytically solved the equations that describe the evolution of the quantum state in a qubit-oscillator system, under a Hamiltonian with time-dependent coupling coefficients and infinitesimal qubit frequency. We have introduced dynamical evolution eigenstates analogous to the energy eigenstates under a time-independent coupling coefficient, and have shown that the dynamical evolution eigenstate tremendously simplifies the expressions for the quantum state evolution. We have also calculated the effect of the nonzero qubit frequency and obtained its first order form. Making use of a sinusoidal driving field and \( \pi \) pulses, we have designed schemes to generate cat states with large amplitudes and rephase a cat state from the effect of nonzero qubit frequency. We point out that our formalism is quite general and can be used for general entangled-cat-state engineering.

Acknowledgments. — T.F., F.Y., K.S., M.S., and M.T. would like to acknowledge support from Japan Science and Technology Agency Core Research for Evolutionary Science and Technology (Grant No. JPMJCR1775). Z.X and J.P.D. would like to acknowledge AFOSR, ARO, DARPA, NSF, and NGAS.

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[1] I. Chiorescu, P. Bertet, K. Semba, Y. Nakamura, C. Har- mans, and J. Mooij, Nature 431, 159 (2004).

[2] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature 431, 162 (2004).

[3] M. Devoret, S. Girvin, and R. Schoelkopf, Annalen der Physik 16, 767 (2007).

[4] T. Niemczyk, F. Deppe, H. Huebl, E. Menzel, F. Hocke, M. Schwarz, J. Garcia-Ripoll, D. Zueco, T. Hüümer, E. Solano, et al., Nature Phys. 6, 772 (2010).

[5] P. Forn-Díaz, J. Lisensonfeld, D. Marcos, J. J. García-Ripoll, E. Solano, C. Harman, and J. Moolji, Phys. Rev. Lett. 105, 237001 (2010).

[6] F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito, and K. Semba, Nature Phys. 13, 44 (2017).

[7] F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito, and K. Semba, Phys. Rev. A 95, 053824 (2017).

[8] N. Didier, J. Bourassa, and A. Blais, Phys. Rev. Lett. 115, 203601 (2015).

[9] Y. Yin, H. Wang, M. Marianioni, R. C. Bialczak, R. Barends, Y. Chen, M. Lenander, E. Lucero, M. Neeley, A. O’Connell, et al., Phys. Rev. A 85, 023826 (2012).

[10] S. Haroche and J.-M. Raimond, Exploring the Quantum: Atoms, Cavities, and Photons. (Oxford, 2013) Chap. 7.

[11] C. Gerry and P. Knight, Introductory Quantum Optics.
(Cambridge University Press, 2005) p. 92.
[12] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997) p. 152.
[13] S. Ashhab and F. Nori, Phys. Rev. A 81, 042311 (2010).
[14] J. Casanova, G. Romero, I. Lizuain, J. García-Ripoll, and E. Solano, Phys. Rev. Lett. 105, 263603 (2010).
[15] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. O’connell, D. Sank, J. Wenner, J. M. Martinis, *et al.*, Nature 459, 546 (2009).
[16] B. Vlastakis, G. Kirchmair, Z. Leghtas, S. E. Nigg, L. Frunzio, S. M. Girvin, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Science 342, 607 (2013).
[17] D. Braak, Q.-H. Chen, M. T. Batchelor, and E. Solano, Journal of Physics A: Mathematical and Theoretical 49, 300301 (2016).
[18] Y.-D. Wang, A. Kemp, and K. Semba, Phys. Rev. B 79, 024502 (2009).
[19] E. Schrödinger, Naturwissenschaften 14, 664 (1926).
[20] V. M. Kondach, J. Kulkman, D. J. Faber, and T. G. van Leeuwen, Biomed. Opt. Express 1, 176 (2010).
[21] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999).
[22] G. de Lange, Z. H. Wang, D. Ristè, V. V. Dobrovitski, and R. Hanson, Science 330, 60 (2010).
[23] E. L. Hahn, Phys. Rev. 80, 580 (1950).

**SUPPLEMENTAL MATERIAL: FAST AMPLIFICATION AND REPHASING OF ENTANGLED CAT STATES IN A COUPLED-QUBIT OSCILLATOR**

Derivation of the state evolution with infinitesimal qubit frequency

Under the limit of $\Delta/\omega \rightarrow 0$, the Hamiltonian of the combined system of the qubit and the oscillator is

$$\hat{H}(0)(t) = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar g(t) \hat{\sigma}_x (\hat{a}^\dagger + \hat{a}).$$  \hspace{1cm} (7)

Let us examine the evolution of an initial state in the form of

$$|\Phi_{N}(0)\rangle = \pm \rangle \hat{D} \left( \mp \frac{\tilde{g}(0)}{\omega} \right) |N\rangle.$$  \hspace{1cm} (8)

We can choose particular $\pm$ and $N$, but the following calculation applies to all $\pm$ and $N$. We can also see that with various choices of $N$ and $\pm$, $|\Phi_{N}(0)\rangle$ represents a complete orthonormal basis regardless of the value of $\tilde{g}(0)$. By examining the evolution of initial state $|\Phi_{N}(0)\rangle$ with all choices of $\pm$ and $N$, we can understand the evolution of any general initial state, which itself can be expressed as a superposition of $|\Phi_{N}(0)\rangle$ with different $\pm$ and $N$. Also note that we can set $\tilde{g}(0)$ to any value.

To calculate the evolution of the initial state $|\Phi_{N}(0)\rangle$ to the final time $t_F$, we divide the time period of interest into $K+1$ small segments, so that the entire period is divided by time points $0, t_1, t_2, \cdots, t_i, t_{i+1}, \cdots, t_K, t_F$. Each segment is considered to be small enough so that, in a single segment $g(t)$ and $\hat{H}(0)(t)$ do not change much and are treated as constants. Therefore the final state at $t=t_F$ can be expressed as

$$|\Phi_{N}(t_F)\rangle = \exp[-i\hat{H}(0)(t_K) \times (t_F-t_K)/\hbar] \cdots \times \exp[-i\hat{H}(0)(t_j) \times (t_{j+1} - t_j)/\hbar] \cdots \times \exp[-i\hat{H}(0)(0) \times t_1/\hbar] \times (\pm) \tilde{D} \left( \mp \frac{\tilde{g}(0)}{\omega} \right) |N\rangle.$$  \hspace{1cm} (9)

Now let us solve for $|\Phi_{N}(t_F)\rangle$. At each time point $t=t_j$, starting with $t=0$, we carry out the following procedures.

1) We make the ansatz that, at any time point $t=t_j$, the evolved state is in the form of

$$|\Phi_{N}(t_j)\rangle = e^{i\phi_{N}(t_j)} |\pm \rangle \tilde{D} \left( \mp \frac{\tilde{g}(t_j)}{\omega} \right) |N\rangle,$$  \hspace{1cm} (10)

where $|\pm \rangle$ and $|N\rangle$ are the same as the initial state $|\Phi_{N}(0)\rangle$. This is obviously true at $t=0$ and we will show, by mathematical induction, that indeed at each time point the state can be expressed in such a form. The complex number $\tilde{g}(t_j)$ generally changes at different time points, and a phase $e^{i\phi_{N}(t_j)}$ can accumulate as well.

2) To make the expression more compact, let $\tau_j = t_{j+1} - t_j$, $g_j = g(t_j)$, $\tilde{g}_j = \tilde{g}(t_j)$. The state at $t_{j+1}$, evolving from the state at the previous time point $t_j$, can be expressed as follows:
\[ |\Phi_{N\pm}(t_{j+1})\rangle = \exp(-i\hat{H}^{(0)}(t_j)(t_{j+1} - t_j)/\hbar) |\Phi_{N\pm}(t_j)\rangle \]

\[ = \exp \left\{ -i \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) \pm g_j (\hat{\alpha}^+ + \hat{\alpha}) \right] \right\} \exp[i\phi_N(t_j)] \hat{D} \left( \mp \frac{g_j}{\omega} \right) |\pm\rangle_x |N\rangle \]

\[ = \exp[i\phi_N(t_j)] \exp \left\{ -i\Im[(\tilde{g}_j - g_j)g_j/\omega^2] \right\} \exp \left\{ -i \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) \pm g_j (\hat{\alpha}^+ + \hat{\alpha}) \right] \right\} \times \hat{D} \left( \mp \frac{g_j}{\omega} \right) |\pm\rangle_x |N\rangle \]

\[ = \exp[i\phi_N(t_j)] \exp \left\{ -i\Im[(\tilde{g}_j - g_j)g_j/\omega^2] \right\} \times \hat{D} \left( \mp \frac{g_j}{\omega} e^{-i\tau_j\omega} \right) \]

\[ \times \exp \left\{ -i \left[ \omega \left( N + \frac{1}{2} \right) - \frac{g_j^2}{\omega} \right] \right\} \hat{D} \left( \mp \frac{g_j}{\omega} \right) |\pm\rangle_x |N\rangle \]

\[ = \exp[i\phi_N(t_j)] \exp \left\{ i\Im \left[ \frac{(g_j - g_j)g_j}{\omega^2} (1 - e^{-i\tau_j\omega}) \right] \right\} \]

\[ \times \exp \left\{ -i \left[ \omega \left( N + \frac{1}{2} \right) - \frac{g_j^2}{\omega} \right] \right\} \hat{D} \left( \mp \frac{g_j}{\omega} + \frac{\tilde{g}_j - g_j}{\omega} e^{-i\tau_j\omega} \right) |\pm\rangle_x |N\rangle , \]

where the symbol \( \Im \) means an imaginary part. In the second to the last step of Eq. (11), we have used

\[ \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) \pm g_j (\hat{\alpha}^+ + \hat{\alpha}) \right] \hat{D} \left( \mp \frac{g_j}{\omega} \right) |\pm\rangle_x |N\rangle = \hat{D} \left( \mp \frac{g_j}{\omega} \right) \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) \right] |\pm\rangle_x |N\rangle \]

\[ = \hat{D} \left( \mp \frac{g_j}{\omega} \right) \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) - \frac{g_j^2}{\omega} \right] |\pm\rangle_x |N\rangle \]

\[ = \hat{D} \left( \mp \frac{g_j}{\omega} \right) \left[ \omega \left( N + \frac{1}{2} \right) - \frac{g_j^2}{\omega} \right] |\pm\rangle_x |N\rangle \]

\[ = \left[ \omega \left( N + \frac{1}{2} \right) - \frac{g_j^2}{\omega} \right] \hat{D} \left( \mp \frac{g_j}{\omega} \right) |\pm\rangle_x |N\rangle , \]

and in the third to the last step we have used

\[ \exp \left\{ -i\tau_j \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) \pm g_j (\hat{\alpha}^+ + \hat{\alpha}) \right] \right\} (\hat{\alpha} \pm \frac{g_j}{\omega}) = (\hat{\alpha} \pm \frac{g_j}{\omega}) e^{i\tau_j\omega} \exp \left\{ -i\tau_j \left[ \omega \left( \hat{\alpha}^+ \hat{\alpha} + \frac{1}{2} \right) \pm g_j (\hat{\alpha}^+ + \hat{\alpha}) \right] \right\} . \]

\[ \text{(13)} \]

Now recall that

\[ |\Phi_{N\pm}(t_j)\rangle = \exp[i\phi_N(t_j)] |\pm\rangle_x \hat{D} \left( \mp \frac{\tilde{g}(t_j)}{\omega} \right) |N\rangle , \]

\[ \text{(14)} \]

Therefore we have proved, by mathematical induction,
that the quantum state at $t = t_{j+1}$ also has the form
\[ |\Phi_{N\pm}(t_{j+1})\rangle = \exp[i\phi_N(t_{j+1})] |\pm\rangle_x \hat{D} \left( \mp \frac{\tilde{g}(t_{j+1})}{\omega} \right) |N\rangle, \]
with
\[ \tilde{g}(t_{j+1}) = g(t_j) + [\tilde{g}(t_j) - g(t_j)] \exp[-i\omega(t_{j+1} - t_j)] \]
and
\[ \phi_N(t_{j+1}) = \phi_N(t_j) + \frac{3\{[\tilde{g}]^* - g_j\}g_j}{\omega} \tau_j - (N + \frac{1}{2})\omega \tau_j + \frac{g_j^2}{\omega} \tau_j. \]

Since $t_{j+1} - t_j = \tau_j \to 0$, we have
\[ \tilde{g}(t_{j+1}) = \tilde{g}(t_j)[1 - i\omega(t_{j+1} - t_j)] - g(t_j)[-i\omega(t_{j+1} - t_j)] \]
and
\[ \phi_N(t_{j+1}) = \phi_N(t_j) + \frac{3\{[\tilde{g}]^* - g_j\}g_j}{\omega} \tau_j - (N + \frac{1}{2})\omega \tau_j + \frac{g_j^2}{\omega} \tau_j. \]

Therefore,
\[ \frac{\tilde{g}(t_{j+1}) - \tilde{g}(t_j)}{t_{j+1} - t_j} = i\omega [g(t_j) - \tilde{g}(t_j)], \]
which leads to
\[ \frac{d}{dt} \tilde{g}(t) = i\omega (g(t) - \tilde{g}(t)), \]
for which the solution is
\[ \tilde{g}(t) = \tilde{g}(0)e^{-i\omega t} + e^{-i\omega t} \int_0^t i\omega g(t')e^{i\omega t'} dt'. \]

As the state evolves under changing $g$, the phase $\phi_N(t_F)$ will accumulate and the final state is
\[ |\Phi_{N\pm}(t_F)\rangle = e^{i\phi_N(t_F)} \hat{D} \left( \mp \frac{\tilde{g}(t_F)}{\omega} \right) |\pm\rangle_x |N\rangle. \]

where $\tilde{g} = \tilde{g}(t)$ is given by Eq. (22) and $\phi_N(t_F) = \int_0^{t_F} \left( \frac{3\{[\tilde{g}]^* - g\}g}{\omega} \right) dt$. Notice in the phase $\phi_N(t_F)$, the part of $\int_0^{t_F} \left( \frac{3\{[\tilde{g}]^* - g\}g}{\omega} \right) dt$ is the same for every $N$ and $\pm$, therefore we can simplify it as $\phi_N(t_F) = -N\omega t_F$. Combining $|\Phi_{N+}(t_F)\rangle$ and $|\Phi_{N-}(t_F)\rangle$, we arrive at Eqs. (4) and (5) in the main text.

**Derivation of the state evolution with non-zero qubit frequency**

We now consider the case where $\Delta$ is not infinitesimal and use the full Hamiltonian $\hat{H}(t) = \hat{H}^{(0)}(t) - \frac{\Delta}{2} \hat{\sigma}_z$. As explained above, the dynamical energy eigenstates form a complete basis. Therefore at any time $t$, we can express any quantum state of interest as a superposition of dynamical energy eigenstates: $|\varphi(t)\rangle = \sum_{N,\pm} C_{N\pm}(t) |\hat{E}_{N\pm}^{(0)}(t)\rangle$. As the state $|\varphi(t)\rangle$ evolves, so does every dynamical energy eigenstate $|\hat{E}_{N\pm}^{(0)}(t)\rangle$ and its corresponding coefficient $C_{N\pm}(t)$. Now we need to solve the evolution of the coefficients $C_{N\pm}(t)$ to completely determine the evolution of state $|\varphi(t)\rangle$. Starting from initial time zero, we divide the time period of interest (0, $t$) into $K + 1$ segments: (0, $t_1$, $t_2$, ..., $t_K$, $t$), with each segment being infinitesimal. For convenience we will adopt the matrix form, therefore the state at time $t$, which evolves from the initial state $|\varphi(0)\rangle$ under Hamiltonian $\hat{H}(t)$, can be written as:

\[ |\varphi(t)\rangle = \sum_{N,\pm} C_{N\pm}(t) e^{-iN\omega t} |\hat{E}_{N\pm}^{(0)}(t)\rangle \]
\[ = \begin{bmatrix} C_{0+}(t) & C_{1+}(t) & \cdots & C_{0-}(t) & C_{1-}(t) & \cdots \end{bmatrix} \times \begin{bmatrix} |\hat{E}_{0+}^{(0)}(t)\rangle & e^{-i\omega t} |\hat{E}_{1+}^{(0)}(t)\rangle & \cdots & |\hat{E}_{0-}^{(0)}(t)\rangle & e^{-i\omega t} |\hat{E}_{1-}^{(0)}(t)\rangle & \cdots \end{bmatrix}^T \]
\[ = \exp[-i\hat{H}(t_K) \times (t - t_K)/\hbar] \cdots \exp[-i\hat{H}(t_1) \times (t_2 - t_1)/\hbar] \exp[-i\hat{H}(0) \times t_1/\hbar] \times \begin{bmatrix} C_{0+}(0) & C_{1+}(0) & \cdots & C_{0-}(0) & C_{1-}(0) & \cdots \end{bmatrix} \cdot \begin{bmatrix} |\hat{E}_{0+}^{(0)}(0)\rangle & |\hat{E}_{1+}^{(0)}(0)\rangle & \cdots & |\hat{E}_{0-}^{(0)}(0)\rangle & |\hat{E}_{1-}^{(0)}(0)\rangle & \cdots \end{bmatrix}^T. \]
From $t_j$ to $t_{j+1}$, so long as $t_{j+1} - t_j \to 0$, the state evolves in the following way:

$$\varphi(t_{j+1}) = \exp \left\{ -i \left[ \hat{H}^{(0)}(t_j) - \frac{\hbar}{2} \Delta \hat{\sigma}_z \right] (t_{j+1} - t_j)/\hbar \right\} \varphi(t_j)$$

$$= \exp \left\{ -i \left[ \hat{H}^{(0)}(t_j) - \frac{\hbar}{2} \Delta \hat{\sigma}_z \right] (t_{j+1} - t_j)/\hbar \right\} \sum_{N,\pm} C_{N,\pm}(t_j) e^{-iN\omega t_j} \left| E^{(0)}_{N,\pm}(t_j) \right\rangle$$

$$= \sum_{N,\pm} C_{N,\pm}(t_j) \exp \left[ -i \left( \frac{1}{2} \Delta \hat{\sigma}_z \right) (t_{j+1} - t_j) \right] \exp[-i\hat{H}^{(0)}(t_j) \times (t_{j+1} - t_j)/\hbar] e^{-iN\omega t_j} \left| E^{(0)}_{N,\pm}(t_j) \right\rangle$$

$$= \sum_{N,\pm} C_{N,\pm}(t_j) \exp \left[ -i \left( \frac{1}{2} \Delta \hat{\sigma}_z \right) (t_{j+1} - t_j) \right] e^{-iN\omega(t_{j+1}-t_j)} e^{-iN\omega t_j} \left| E^{(0)}_{N,\pm}(t_{j+1}) \right\rangle$$

$$= \sum_{N,\pm} \sum_{N',\pm} C_{N,\pm}(t_j) e^{-iN'\omega t_{j+1}} \left| E^{(0)}_{N',\pm}(t_{j+1}) \right\rangle$$

$$\times \left( \left| E^{(0)}_{N',\pm}(t_{j+1}) \right\rangle e^{+iN'\omega t_{j+1}} \exp \left[ -i \left( \frac{1}{2} \Delta \hat{\sigma}_z \right) (t_{j+1} - t_j) \right] \right)$$

$$\times \left[ \exp[-i(1/2)\Delta[M_{\sigma_z}(t_{j+1})](t_{j+1} - t_j)] \right]$$

$$\times \left[ \left| E^{(0)}_{0,\pm}(t_{j+1}) \right\rangle e^{-i\omega t_{j+1}} \left| E^{(0)}_{1,\pm}(t_{j+1}) \right\rangle \ldots \right]^T,$$

where we have used the relation

$$\exp \left\{ -i \left[ \hat{H}^{(0)}(t_j) - \frac{\hbar}{2} \Delta \hat{\sigma}_z \right] (t_{j+1} - t_j)/\hbar \right\} = \exp \left\{ -i \left( \frac{\hbar}{2} \Delta \hat{\sigma}_z \right) (t_{j+1} - t_j)/\hbar \right\} \times \exp[-i\hat{H}^{(0)}(t_j) \times (t_{j+1} - t_j)/\hbar] \exp[O(t_{j+1} - t_j)^2]$$

$$\frac{t_{j+1} - t_j \to 0}{t_{j+1} - t_j \to 0} \exp \left[ -i \left( \frac{\hbar}{2} \Delta \hat{\sigma}_z \right) (t_{j+1} - t_j)/\hbar \right] \exp[-i\hat{H}^{(0)}(t_j) \times (t_{j+1} - t_j)/\hbar].$$

The term $M_{\sigma_z}(t)$ in Eq. (25) is the matrix expansion of operator $\hat{\sigma}_z$ in the basis $\left| E^{(0)}_{N,\pm}(t) \right\rangle$ at time $t$:

$$M_{\sigma_z}(t) = \begin{bmatrix} M_{\sigma_z}^{(+, +)}(t) & M_{\sigma_z}^{(-, +)}(t) \\ M_{\sigma_z}^{(+, -)}(t) & M_{\sigma_z}^{(-, -)}(t) \end{bmatrix},$$

where

$$M_{\sigma_z}^{(+, \pm)}(t) = \begin{bmatrix} \langle E^{(0)}_{0,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{0,\pm}(t) \rangle & \langle E^{(0)}_{1,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{0,\pm}(t) \rangle e^{+i\omega t} \\ \langle E^{(0)}_{0,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{1,\pm}(t) \rangle e^{-i\omega t} & \langle E^{(0)}_{1,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{1,\pm}(t) \rangle \end{bmatrix},$$

and

$$M_{\sigma_z}^{(-, \mp)}(t) = \begin{bmatrix} \langle E^{(0)}_{0,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{0,\pm}(t) \rangle & \langle E^{(0)}_{1,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{0,\pm}(t) \rangle e^{+i\omega t} \\ \langle E^{(0)}_{0,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{1,\pm}(t) \rangle e^{-i\omega t} & \langle E^{(0)}_{1,\pm}(t) | \hat{\sigma}_z | E^{(0)}_{1,\pm}(t) \rangle \end{bmatrix}.$$
Comparing Eqs. (25, 30) and using mathematical induction, we have:

\[
[C_0^+ (t) \hspace{1em} C_1^+ (t) \hspace{1em} \ldots \hspace{1em} C_0^- (t) \hspace{1em} C_1^- (t) \hspace{1em} \ldots ] = [C_0^+ (0) \hspace{1em} C_1^+ (0) \hspace{1em} \ldots \hspace{1em} C_0^- (0) \hspace{1em} C_1^- (0) \hspace{1em} \ldots ] J(0, t),
\]

where

\[
J(0, t) = \exp \left[ -i \left( -\frac{1}{2} \Delta \right) [M_{\sigma_z} (t)] t_1 \right] \times \exp \left[ -i \left( -\frac{1}{2} \Delta \right) [M_{\sigma_z} (t)] (t_2 - t_1) \right] \times \exp \left[ -i \left( -\frac{1}{2} \Delta \right) [M_{\sigma_z} (t_1)] (t - t_K) \right].
\]

(32)

Note that generally at different time points \( t \) and \( t' \), \( M_{\sigma_z} (t) \) and \( M_{\sigma_z} (t') \) do not commute, making the analytical calculation of the evolution matrix \( J(0, t) \) very complicated. But in the case when we only consider up to the first-order effect of small \( \Delta \), we can ignore the second order \( \Delta \) term, which means we consider the commutation \( [\Delta \times M_{\sigma_z} (t), \Delta \times M_{\sigma_z} (t')] \sim O(\Delta^2) \sim 0 \), and therefore \( \Delta \times M_{\sigma_z} (t) \) and \( \Delta \times M_{\sigma_z} (t') \) approximately commute. This enables us to calculate the evolution matrix \( J(0, t) \) and state \( |\psi(t)\rangle \) up to the first order:

\[
J^{(1)} (0, t) = \exp \left[ -i \left( -\frac{1}{2} \right) \int_0^t \Delta (t) [M_{\sigma_z} (t)] dt \right],
\]

(33)

and

\[
|\varphi^{(1)} (t)\rangle = [C_0^+ (0) \hspace{1em} C_1^+ (0) \hspace{1em} \ldots \hspace{1em} C_0^- (0) \hspace{1em} C_1^- (0) \hspace{1em} \ldots ] \exp \left\{ -i \left( -\frac{1}{2} \right) \int_0^t \Delta (t) [M_{\sigma_z} (t)] dt \right\} \times \\
[|E_{0+}^{(0)} (t)\rangle \hspace{1em} e^{-i\omega t} |E_{1+}^{(0)} (t)\rangle \hspace{1em} \ldots \hspace{1em} |E_{0-}^{(0)} (t)\rangle \hspace{1em} e^{-i\omega t} |E_{1-}^{(0)} (t)\rangle \hspace{1em} \ldots ]^T,
\]

(34)

where for generality we can consider the parameter \( \Delta \) to be time-dependent \( (\Delta = \Delta (t)) \).