Congruences involving the reciprocals of central binomial coefficients

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Abstract
We present several congruences modulo a power of prime $p$ concerning sums of the following type

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1}$$

which reveal some interesting connections with the analogous infinite series.

1 Introduction

In 1979, Apéry [2] proved that $\zeta(3)$ was irrational starting from the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} = -\frac{2}{5} \zeta(3).$$

He also noted that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \binom{2k}{k}^{-1} = \frac{1}{3} \zeta(2)$$

which has been known since the nineteenth century. Here we would like to show that the following analogous congruences hold for any prime $p > 5$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} \equiv -\frac{2}{5} \frac{H(1)}{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{2k}{k}^{-1} \equiv \frac{1}{3} \frac{H(1)}{p} \pmod{p^3}$$

where $H(1) = \sum_{k=1}^{p-1} \frac{1}{k}$ (note that $H(1) \equiv 0 \pmod{p^3}$ by Wolstenholme’s theorem).

After some preliminary results, in the last section we will give the proofs of the above congruences together with the dual ones:

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} = \frac{4}{5} \left( \frac{H(1)}{p} + 2pH(3) \right) \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} = -\frac{8}{3} H(1) \pmod{p^4}.$$

2 Old and new results concerning multiple harmonic sums

We define the multiple harmonic sum as

$$H(a_1, a_2, \ldots, a_r; n) = \sum_{1 \leq k_1 < k_2 < \cdots < k_r \leq n} \frac{1}{k_1^{a_1} k_2^{a_2} \cdots k_r^{a_r}}$$
where \( n \geq r > 0 \) and \((a_1, a_2, \ldots, a_r) \in (\mathbb{N}^*)^r\). The values of many harmonic sums modulo a power of prime \( p \) are well known and usually they are expressed as a combination of Bernoulli numbers \( B_n \). These are the results we need later (note that we write simply \( H(a_1, a_2, \ldots, a_r) \) when \( n = p - 1 \):

(i). \((\text{8})\) for any prime \( p > 5 \)

\[
\begin{align*}
H(1) &\equiv -\frac{1}{2^2} p H(2) \equiv p^2 \left( \frac{2B_{p-3}}{p-3} - \frac{B_{2p-4}}{2p-4} \right) \pmod{p^4} \\
H(3) &\equiv \frac{3}{2^2} p H(4) \equiv 6p^2 \frac{B_{p-5}}{p-5} \pmod{p^3} \\
H(5) &\equiv 0 \pmod{p^2} \quad \text{and} \quad H \left( 2, \frac{p-1}{2} \right) \equiv -7 \frac{H(1)}{p} \pmod{p^3}
\end{align*}
\]

(ii). \((\text{5}, \text{12})\) for \( a, b > 0 \) and for any prime \( p > a + b + 1 \)

\[
H(a, b) \equiv \frac{(-1)^b}{a+b} \left( \frac{a+b}{a} \right) B_{p-a-b} \quad \text{and} \quad H(1, 1, 2) \equiv 0 \pmod{p}.
\]

This is a generalization of Wolstenholme’s theorem which improves the modulo \( p^4 \) congruence given in Remark 5.1 in \([\text{8}]\).

**Theorem 2.1.** For any prime \( p > 5 \)

\[
H(1) \equiv -\frac{1}{2} p H(2) - \frac{1}{6} p^2 H(3) \equiv p^2 \left( \frac{B_{3p-5}}{3p-5} - \frac{3B_{2p-4}}{2p-4} + \frac{3B_{p-3}}{p-3} \right) + p^4 \frac{B_{p-5}}{p-5} \pmod{p^5}.
\]

**Proof.** Let \( m = \varphi(p^5) = p^4(p - 1) \) then by Euler’s theorem and by Faulhaber’s formula we have that for \( r = 1, 2, 3 \)

\[
H(r) \equiv \sum_{k=1}^{p-1} k^{m-r} = \frac{B_{m-r+1}(p) - B_{m-r+1}}{m-r+1} \pmod{p^5}.
\]

Therefore

\[
\sum_{r=1}^{3} \alpha_r p^{-1} H(r) \equiv \sum_{r=1}^{3} \alpha_r \sum_{k=r}^{m-1} p^k B_{m-k} \binom{m-r}{k-r} \pmod{p^5}.
\]

Since \( m \) is even then \( B_{m-k} = 0 \) when \( m - k > 1 \) and \( k \) is odd. Moreover \( pB_{m-k} \) is \( p \)-integral, thus the sum modulo \( p^5 \) simplifies to

\[
\sum_{r=1}^{3} \alpha_r p^{-1} H(r) \equiv \frac{p^2}{2} (2\alpha_2 - \alpha_1) B_{m-2} + \frac{p^4}{4} (-6\alpha_3 + 4\alpha_2 - \alpha_1) B_{m-4} \pmod{p^5}.
\]

Hence the r.h.s. becomes zero as soon as we let \( \alpha_1 = 1, \alpha_2 = \frac{1}{2}, \) and \( \alpha_3 = \frac{1}{6} \). Finally we use the formulas for \( H(2) \) and \( H(3) \) modulo \( p^4 \) given in Remark 5.1 in \([\text{8}]\). □

The next lemma allow us to expand two kinds of binomial coefficients as a combination of multiple harmonic sums.
Lemma 2.2. Let \( n \in \mathbb{N}^* \), then for \( k = 1, \ldots, n - 1 \)
\[
{n \choose k} = \frac{n}{k} {n - 1 \choose k - 1} = (-1)^{k-1} \frac{n}{k} \sum_{j=0}^{k-1} (-n)^j H({1 \choose j}; k - 1),
\]
\[
{n + k - 1 \choose k} = \frac{n}{k} {n + k - 1 \choose k - 1} = \frac{n}{k} \sum_{j=0}^{k-1} n^j H({1 \choose j}; k - 1)
\]

Proof. It suffices to use the definition of binomial coefficient:
\[
{n \choose k} = \frac{(n-1) \cdots (n-(k-1))}{(k-1)!} = (-1)^{k-1} \prod_{j=1}^{k-1} \left(1 - \frac{n}{j}\right) = (-1)^{k-1} \sum_{j=0}^{k-1} (-n)^j H({1 \choose j}; k - 1),
\]
and
\[
{n + k - 1 \choose k} = \frac{(n + k - 1) \cdots (n + 1)}{(k-1)!} = \prod_{j=1}^{k-1} \left(1 + \frac{n}{j}\right) = \sum_{j=0}^{k-1} n^j H({1 \choose j}; k - 1).
\]

By (ii), we already know that \( H(1,2) \equiv -H(2,1) \equiv B_{p-3} \) (mod \( p \)). Moreover, since \( H(1)H(2) = H(1,2) + H(2,1) + H(3) \) then \( H(1,2) \equiv -H(2,1) \mod p^2 \). Thanks to an identity due to Hernández [4], we are able to disentangle \( H(1,2) \) and \( H(2,1) \) and prove the following congruence modulo \( p^2 \).

Theorem 2.3. For any prime \( p > 3 \)
\[
H(1,2) \equiv -H(2,1) \equiv -3 \frac{H(1)}{p^2} \pmod{p^2}.
\]

Proof. The following identity appears in [4]: for \( n \geq 1 \)
\[
\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{1 \leq i \leq j \leq n} \frac{(-1)^{j-1}}{ij} {n \choose j}.
\]
Letting \( n = p \), by Lemma 2.2 (i), and (ii) we obtain
\[
H(2) = p \sum_{1 \leq i \leq j \leq p-1} \frac{(-1)^{j-1}}{ij^2} \left(\frac{p-1}{j-1}\right) + \frac{H(1)}{p} \equiv \sum_{1 \leq i \leq j \leq p-1} \frac{1 - pH(1; j - 1)}{ij^2} + \frac{H(1)}{p}
\]
\[
\equiv pH(1,2) + pH(3) - p^2 \sum_{1 \leq i < j \leq p-1} \frac{H(1; j - 1)}{ij^2} - p^2 H(1,3) + \frac{H(1)}{p}
\]
\[
\equiv pH(1,2) + pH(3) - 2p^2 H(1,1,2) - p^2 H(2,2) - p^2 H(1,3) + \frac{H(1)}{p} \pmod{p^3}.
\]

Finally we get this extension of a result contained in [13].

Theorem 2.4. For any prime \( p > 5 \)
\[
\frac{1}{2} \left(\frac{2p}{p}\right) \equiv 1 + 2pH(1) + \frac{2}{3} p^3 H(3) \pmod{p^6}.
\]
Proof. Since for \( n \geq 1 \) (see for example \([10]\))
\[
2H(\{1\}^2; n) = H^2(1; n) - H(2; n)
\]
\[
6H(\{1\}^3; n) = H^3(1; n) - 3H(1; n)H(2; n) + 2H(3; n)
\]
\[
24H(\{1\}^4; n) = H^4(1; n) - 6H^2(1; n)H(2; n) + 8H(1; n)H(3; n) + 3H^2(2; n) - 6H(4; n)
\]
then by (i) and by Theorem 2.1 we obtain
\[
H(\{1\}^2) \equiv -H(2) \equiv \frac{H(1)}{p} + \frac{1}{6}pH(3) \pmod{p^4}
\]
\[
H(\{1\}^3) \equiv \frac{1}{3}H(3) \pmod{p^3}
\]
\[
H(\{1\}^4) \equiv -\frac{1}{4}H(4) \equiv \frac{1}{6}H(3) \pmod{p^2}.
\]

Hence by Lemma 2.2
\[
\frac{1}{2} \left( \begin{array}{c} 2p \\ p \end{array} \right) \equiv 1 - 2pH(1) + 4p^2H(\{1\}^2) - 8p^3H(\{1\}^3) + 16p^4H(\{1\}^4)
\]
\[
\equiv 1 - 2pH(1) + 4pH(1) + \frac{2}{3}p^3H(3) - \frac{8}{3}p^3H(3) + \frac{8}{3}p^3H(3)
\]
\[
\equiv 1 + 2pH(1) + \frac{2}{3}p^3H(3) \pmod{p^6}.
\]

\[
\square
\]

3 Some preliminary results

We consider the Lucas sequences \( \{u_n(x)\}_{n \geq 0} \) and \( \{v_n(x)\}_{n \geq 0} \) defined by these recurrence relations
\[
u_0(x) = 0, \quad u_1(x) = 1, \quad u_{n+1} = x u_n(x) - u_{n-1}(x) \text{ for } n > 0,
\]
\[
v_0(x) = 2, \quad v_1(x) = x, \quad v_{n+1} = x v_n(x) - v_{n-1}(x) \text{ for } n > 0.
\]

The corresponding generating functions are
\[
U_x(z) = \frac{z}{z^2 - xz + 1} \quad \text{and} \quad V_x(z) = \frac{2 - xz}{z^2 - xz + 1}.
\]

Now we consider two types of sums depending on an integral parameter \( m \). Note that the factor \( p \) before the sum is needed because it cancels out the other factor \( p \) at the denominator of \( \binom{2k}{k}^{-1} \) for \( k = (p + 1)/2, \ldots, p - 1 \).

**Theorem 3.1.** Let \( m \in \mathbb{Z} \) then for any prime \( p \neq 2 \),
\[
p \sum_{k=1}^{p-1} \frac{m^k \binom{2k}{k}^{-1}}{k} \equiv \frac{m u_p(2 - m) - m^p}{2} \pmod{p^2},
\]
and
\[
p \sum_{k=1}^{p-1} \frac{m^k \binom{2k}{k}^{-1}}{k^2} \equiv \frac{2 - v_p(2 - m) - m^p}{2p} \pmod{p^2}.
\]
Proof. Let \( f(z) = 1/(1 + mz) \) then
\[
\sum_{k=1}^{n} \binom{n}{k} \binom{n-1+k}{k-1} \binom{2k}{k}^{-1} (-m)^k = \sum_{k=1}^{n} \binom{n+k-1}{2k-1} (-m)^k = [z^n] f \left( \frac{z}{(1-z)^2} \right) = -\frac{m}{2} [z^n] U_{2-m}(z).
\]

Let \( n = p \), then
\[
\sum_{k=1}^{p-1} \binom{p}{k} \binom{p-1+k}{k-1} \binom{2k}{k}^{-1} (-m)^k = -\frac{m}{2} \sum_{k=1}^{p-1} \binom{m}{k} (2k)^{-1} (\text{mod } p^2).
\]

On the other hand, by Lemma 2.2 the l.h.s. is congruent modulo \( p^2 \) to
\[
-p \sum_{k=1}^{p-1} \frac{m^k}{k} (1 - pH(1; k-1)) (1 + pH(1; k-1)) \binom{2k}{k}^{-1} = -p \sum_{k=1}^{p-1} \frac{m^k}{k} \binom{2k}{k}^{-1} (\text{mod } p^2).
\]

As regards the second congruence, consider
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n-1+k}{k} \binom{2k}{k}^{-1} (-m)^k = \sum_{k=0}^{n} \left( \frac{n+k-1}{2k-1} \right) (-m)^k = [z^n] \frac{1}{1-z} f \left( \frac{z}{(1-z)^2} \right) - \frac{1}{2} f \left( \frac{z}{(1-z)^2} \right) = \frac{1}{2} [z^n] V_{2-m}(z).
\]

Let \( n = p \), then
\[
\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} \binom{p-1+k}{k} \binom{2k}{k}^{-1} (-m)^k = -\frac{2 - v_p(2m) - m^p}{2p}.
\]

As before, by Lemma 2.2 the l.h.s. is congruent modulo \( p^2 \) to
\[
-p \sum_{k=1}^{p-1} \frac{m^k}{k^2} (1 - pH(1; k-1)) (1 + pH(1; k-1)) \binom{2k}{k}^{-1} = -p \sum_{k=1}^{p-1} \frac{m^k}{k^2} \binom{2k}{k}^{-1} (\text{mod } p^2).
\]

For some values of the parameter \( x = 2 - m \), Lucas sequences have specific names. Here it is a short list of examples which follow straightforwardly from the previous theorem.

**Corollary 3.2.** For any prime \( p \neq 2 \), the following congruences hold modulo \( p^2 \)
\[
\begin{align*}
p \sum_{k=1}^{p-1} \frac{1}{k^2} \binom{2k}{k}^{-1} & \equiv \frac{\delta_{p,3}}{2} \\
p \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k}^{-1} & \equiv 1 - \frac{L_p^2}{2p} \\
p \sum_{k=1}^{p-1} 2^k \binom{2k}{k}^{-1} & \equiv -q_p(2) \\
p \sum_{k=1}^{p-1} (\frac{2}{p}) \binom{2k}{k}^{-1} & \equiv \frac{(\frac{p}{p}) - 1}{2} \\
p \sum_{k=1}^{p-1} (\frac{-1}{p}) \binom{2k}{k}^{-1} & \equiv 1 - L_p F_p \\
p \sum_{k=1}^{p-1} (\frac{2}{p}) \binom{2k}{k}^{-1} & \equiv (\frac{1}{p}) - 1 - pq_p(2)
\end{align*}
\]

where \( \delta_{n,k} = 1 \) if \( n = k \) and it is 0 otherwise, \( F_n \) is the \( n \)-th Fibonacci number, \( L_n \) is the \( n \)-th Lucas number, \( (\frac{p}{q}) \) the Legendre symbol and \( q_p(a) = (a^{p-1} - 1)/p \) the Fermat quotient.
Proof. For \(m = 1\), \(u_p(2 - m) = \left(\begin{array}{c} 2 \\ 1 \end{array}\right)\) and \(v_p(2 - m) = 1 - 3\delta_{p,3}\). For \(m = -1\), \(u_p(2 - m) = F_{2p} = L_pF_p\) and \(v_p(2 - m) = L_{2p} = L_p^2 + 2\) (note that \(L_p \equiv 1\) and \(F_p \equiv \left(\begin{array}{c} 2 \\ p \end{array}\right)\) mod \(p\)). For \(m = 2\), \(u_p(2 - m) = \left(\frac{1}{p}\right)\) and \(v_p(2 - m) = 0\).

The interested reader can compare some of the previous formulas with the corresponding values of the infinite series (when they converge). For example (see [9]) notice the golden ratio in

\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\begin{array}{c} 2k \\ k \end{array}\right)^{-1} = -\frac{2}{\sqrt{5}} \log \left(\frac{1 + \sqrt{5}}{2}\right)
\]

while the analogous congruence involves Fibonacci and Lucas numbers.

By differentiating the generating functions employed in the proof of Theorem 3.1 one can obtain more congruences concerning

\[
p \sum_{k=1}^{p-1} Q(k) \frac{m^k}{k^2} \left(\begin{array}{c} 2k \\ k \end{array}\right)^{-1}
\]

where \(Q(k)\) is a polynomial. For example it is not hard to show that for any prime \(p > 3\)

\[
p \sum_{k=1}^{p-1} \frac{1}{C_k} \equiv \frac{2}{3} \left(\frac{p}{3}\right) \pmod{p^2}
\]

where \(C_k = \left(\begin{array}{c} 2k \\ k \end{array}\right) / (k + 1)\) is the \(k\)-th Catalan number.

The problem to raise the power of \(k\) at the denominator seems to be much more difficult since we should try to integrate those generating functions. In the next section we will see a remarkable example where the power of \(k\) is 3.

Moreover the congruences established in Theorem 3.1 are similar to other congruences involving the central binomial coefficients (not inverted) obtained in our joint-work with Zhi-Wei Sun [9]. The next theorem gives a first explanation of this behaviour, but it’s our opinion that this relationship should be investigated further in future studies.

**Theorem 3.3.** Let \(m, r \in \mathbb{Z}\) then for any prime \(p \not\equiv 2\) such that \(p\) does not divide \(m\) then

\[
p \sum_{k=1}^{p-1} \frac{m^k}{k^r} \left(\begin{array}{c} 2k \\ k \end{array}\right)^{-1} \equiv \frac{m(-1)^{r-1}}{2} \sum_{k=1}^{p-1} \frac{1}{m^k k^{r-1}} \left(\begin{array}{c} 2k \\ k \end{array}\right) \pmod{p}.
\]

**Proof.** We first show that for \(k = 1, \ldots, p - 1:\)

\[
\frac{p}{k} \left(\begin{array}{c} 2k \\ k \end{array}\right)^{-1} \equiv \frac{1}{2} \left(\frac{2(p - k)}{p - k}\right) \pmod{p}.
\]

This is trivial for \(k = 1, \ldots, (p - 1)/2\), because the l.h.s and the r.h.s have a factor \(p\) at the numerator and therefore they are both zero modulo \(p\).

Assume now that \(k = (p + 1)/2, \ldots, p - 1:\)

\[
\frac{p}{k} \left(\begin{array}{c} 2k \\ k \end{array}\right)^{-1} = \frac{(k - 1)!}{(p + (2k - p)) \cdots (p + 1)(p - 1) \cdots (p - (p - k - 1))}
\]

\[
\equiv \frac{(k - 1)!(-1)^k}{(2k - p)! (p - k - 1)!} = (-1)^k \left(\begin{array}{c} k - 1 \\ p - k - 1 \end{array}\right) \pmod{p}.
\]
On the other hand
\[
\frac{1}{2} \binom{2(p - k)}{p - k} = \binom{2p - 2k - 1}{p - k - 1} = \frac{(p - (2k + 1 - p)) \cdots (p - (k - 1))}{(p - k - 1)!} = (-1)^k \binom{k - 1}{p - k - 1} \pmod{p}.
\]

By summing over \(k\) and by Euler’s theorem we get
\[
p \sum_{k=1}^{p-1} \frac{m^k}{k^r} \binom{2k}{k}^{-1} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{m^k}{k^r-1} \binom{2(p - k)}{p - k} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{m^{p-k}}{(p-k)^{r-1}} \binom{2k}{k}
\]
\[
= \frac{m(-1)^{r-1}}{2} \sum_{k=1}^{p-1} \frac{1}{m^k k^{r-1}} \binom{2k}{k} \pmod{p}.
\]

\[
\square
\]

4 Proof of the main result and a conjecture

We finally prove the congruences announced in the introduction.

**Theorem 4.1.** For any prime \(p > 5\)
\[
\sum_{k=1}^{p-1} \frac{1}{k^2} \binom{2k}{k}^{-1} \equiv \frac{1}{3} \frac{H(1)}{p} \pmod{p^3}.
\]
and
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} \equiv \frac{2}{5} \frac{H(1)}{p^2} \pmod{p^3}.
\]

**Proof.** By Lemma 2.2 we have that
\[
p \binom{p - 1 + k}{k}^{-1} \binom{p - 1}{k}^{-1} = (-1)^k \left( \sum_{j=0}^{3} \frac{p^j}{k^j} H(\{1\}^j; k - 1) \right)^{-1} \left( \sum_{j=0}^{3} (-p)^j H(\{1\}^j; k) \right)^{-1}
\]
\[
\equiv (-1)^k \left( 1 + \frac{p}{k} + p^2 H(2; k) + \frac{p^3}{k} H(2; k) \right) \pmod{p^4}
\]
where in the second step we used the relations
\[
H(\{1\}^j; k) = H(\{1\}^j; k - 1) + \frac{1}{2} H(\{1\}^j; k - 1) \text{ for } j \geq 1, \text{ and } H(1; k)^2 = 2H(1, 1; k) + H(2; k).
\]
The rest of the proof depends heavily on two curious identities which play an important role in Apéry’s work (see \[2\] and \[11\]). The first one is:
\[
\sum_{k=1}^{n} \frac{1}{k^2} \binom{2k}{k}^{-1} = -\frac{2}{3} \sum_{k=1}^{n} \frac{(-1)^k}{k^2} - \frac{(-1)^n}{3} \sum_{k=1}^{n} \frac{(-1)^k}{k^2} \binom{n + k}{k} \frac{1}{\binom{n}{k}}
\]

Letting \(n = p - 1\), by (i) we obtain
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} = -H(2) + \frac{1}{2} H\left(2; \frac{p - 1}{2}\right) \equiv \frac{2}{3} \frac{H(1)}{p} + \frac{1}{2} \left(\frac{7H(1)}{p}\right) \equiv -\frac{3}{2} \frac{H(1)}{p} \pmod{p^3}.
\]
By (i), (ii), and Theorem 2.3 we get
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{p-1+k}{k}^{-1} \binom{p-1}{k}^{-1} = \frac{1}{p} \sum_{k=1}^{p-1} \left( \frac{1}{k^2} + \frac{p}{k} + \frac{p^2}{k^2} H(2; k) + \frac{p^3}{k^3} H(2; k) \right)
\]
\[
= \frac{H(1)}{p} + H(2) + p(H(2, 1) + H(3)) + p^2(H(2, 2) + H(4))
\]
\[
= (1 - 2 + 3) \frac{H(1)}{p} \equiv 2 \frac{H(1)}{p} \pmod{p^3}.
\]

Hence
\[
\sum_{k=1}^{p-1} \frac{1}{k^2} \binom{2k}{k}^{-1} = \frac{2}{3} \left( -\frac{3}{2} \frac{H(1)}{p} - \frac{1}{3} \frac{2H(1)}{p} \right) \equiv \frac{1}{3} \frac{H(1)}{p} \pmod{p^3}.
\]

The second identity is:
\[
\sum_{k=1}^{n} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} = -\frac{2}{5} \sum_{k=1}^{n} \frac{1}{k^3} + \frac{1}{5} \sum_{k=1}^{n} \frac{(-1)^k}{k^3} \binom{n+k}{k}^{-1} \binom{n}{k}^{-1}.
\]

Let \( n = p - 1 \) then since \( 2H(2, 2) = H(2)^2 - H(4) \equiv -H(4) \pmod{p^2} \) we have that
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \binom{p-1+k}{k}^{-1} \binom{p-1}{k}^{-1} = \frac{1}{p} \sum_{k=1}^{p-1} \left( \frac{1}{k^2} + \frac{p}{k^3} + \frac{p^2}{k^2} H(2; k) + \frac{p^3}{k^3} H(2; k) \right)
\]
\[
= \frac{H(2)}{p} + H(3) + p(H(2, 2) + H(4)) + p^2(H(2, 3) + H(5))
\]
\[
= \frac{H(2)}{p} + H(3) + \frac{p}{2} \left( \frac{4p}{5} B_{p-5} \right) + p^2(-2B_{p-5} + 0)
\]
\[
= \frac{H(2)}{p} + \frac{7}{3} H(3) \pmod{p^3}.
\]

Hence by Theorem 2.4
\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \binom{2k}{k}^{-1} = -\frac{2}{5} H(3) + \frac{1}{5} \left( \frac{H(2)}{p} + \frac{7}{3} H(3) \right)
\]
\[
= -\frac{2}{5} \left( -\frac{1}{2} \frac{H(2)}{p} - \frac{1}{6} H(3) \right) \equiv -\frac{2}{5} \frac{H(1)}{p^2} \pmod{p^3}.
\]

The duality established in Theorem 3.3 suggests some analogous result for the sums involving the central binomial coefficients (not reversed). By an identity due to Staver [7] it is possible to prove (see [9] and [10]) that for any prime \( p > 3 \)
\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} = -\frac{8}{3} H(1) \pmod{p^4}.
\]

Moreover this other congruence holds.
Theorem 4.2. For any prime \( p > 5 \)

\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} = \frac{4}{5} \left( \frac{H(1)}{p} + 2p H(3) \right) \pmod{p^4}.
\]

Proof. The following identity was conjectured in [3] and then it has been proved [1]: for \( n \geq 1 \)

\[
\frac{5}{2} \sum_{k=1}^{n} \binom{2k}{k} \frac{k^2}{4n^4 + k^4} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + j^4} = \frac{1}{n^2}.
\]

Let \( n = p \) then for \( 1 \leq k < p \)

\[
\prod_{j=1}^{k-1} \frac{p^4 - j^4}{4p^4 + j^4} = (-1)^{k-1} \prod_{j=1}^{k-1} \frac{1 - (p/j)^4}{1 + 4(p/j)^4} \equiv (-1)^{k-1} \prod_{j=1}^{k-1} \left( 1 - \frac{p^4}{j^4} \right) \left( 1 - 4\frac{p^4}{j^4} \right)
\]

\[
\equiv (-1)^{k-1} \prod_{j=1}^{k-1} \left( 1 - 5\frac{p^4}{j^4} \right) \equiv (-1)^{k-1} \left( 1 - 5p^4 H(4; k-1) \right) \pmod{p^8}.
\]

Hence by (i) and by Theorem 2.4

\[
\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} = \frac{2}{5p^2} \left( \frac{1}{2} \left( \frac{2p}{p} \right) (1 - 5p^4 H(4)) - 1 \right)
\]

\[
= \frac{2}{5p^2} \left( \left( 1 + 2p H(1) + \frac{2}{3} p^2 H(3) \right) \left( 1 + \frac{10}{3} p^4 H(3) \right) - 1 \right)
\]

\[
\equiv \frac{4}{5} \left( \frac{H(1)}{p} + 2p H(3) \right) \pmod{p^4}.
\]

\( \square \)

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