SHARP MOMENT ESTIMATES FOR POLYNOMIAL MARTINGALES

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Abstract.

In this paper non-asymptotic moment estimates are derived for tail of distribution for discrete time polynomial martingale by means of martingale differences as a rule in the terms of unconditional and unconditional relative moments and tails of distributions of summands.

We show also the exactness of obtained estimations.

Key words: Random variables and vectors, Jensen, Osekowski, Rosenthal and triangle inequalities, recursion, martingales, martingale differences, regular varying function, Lebesgue-Riesz and Grand Lebesgue norm and spaces, lower and upper estimates, moments and relative moments, quadratic characteristic of martingale, filtration, examples, natural norming, tails of distribution, conditional expectation.

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1 Introduction. Notations. Statement of problem. Announce.

Let $(\Omega, F, P)$ be a probabilistic space, which will be presumed sufficiently rich when we construct examples (counterexamples), $\xi(i, 1), \xi(i, 2), \ldots, \xi(i, n)$, $n \leq \infty$ being a family of a centered $(E\xi(i, m) = 0, i = 1, 2, \ldots, n)$ martingale-differences on the basis of the fixed flow of $\sigma-$fields (filtration) $F(i) : F(0) = \{\emptyset, \Omega\}, F(i) \subset F(i + 1) \subset F$, $\xi(i, 0) := 0$; $E|\xi(i, m)| < \infty$, and for every $i \geq 0$, $\forall k = 0, 1, \ldots, i - 1 \Rightarrow$

\[ E\xi(i, m)/F(k) = 0; \quad E\xi(i, m)/F(i) = \xi(i, m) \pmod{P}. \quad (1.0) \]

Let also $I = I(n) = I(d; n) = \{i_1; i_2; \ldots; i_d\}$ be the set of indices of the form $I(n) = I(d; n) = \{\bar{i}\} = \{i\} = \{i_1, i_2, \ldots, i_d\}$ such that $1 \leq i_1 < i_2 < i_3 < \cdots < i_{d-1} < i_d \leq n$; $J = J(n) = J(d; n)$ be the set of indices of the form...
(subset of $I(d; n)$) $J(d; n) = J(n) = \{ \vec{j} \} = \{ j \} = \{ j_1; j_2; \ldots; j_{d-1} \}$ such that $1 \leq j_1 < j_2 \ldots < j_{d-1} \leq n - 1$; $b(i) = \{ b(i_1; i_2; \ldots i_d) \}$ be a $d$ dimensional numerical non-random sequence symmetrical relative to all argument permutations,

$$\vec{i} \in I \Rightarrow \xi(\vec{i}) \overset{df}{=} \prod_{s=1}^{d} \xi(i_s, s); \quad (1.1a)$$

$$\vec{j} \in J \Rightarrow \xi(\vec{j}) \overset{df}{=} \prod_{s=1}^{d-1} \xi(i_s, s); \quad (1.1b)$$

$$\sigma^2(i_k, s) := \text{Var}(\xi(i_k, s));$$

$$\sigma^2(\vec{i}) := \prod_{s=1}^{d} \sigma^2(i_s, s), \quad \vec{i} = \{ i_1, i_2, \ldots, i_d \} \in I;$$

$$\sigma^2(n, \vec{j}) := \prod_{s=1}^{d-1} \sigma^2(j_s, s), \quad \vec{j} = \{ j_1, j_2, \ldots, j_{d-1} \} \in J;$$

$$Q_d = Q(d; n, \{ \xi(\cdot) \}) = Q(d; n) = Q(d; n, \vec{b}) = \sum_{\vec{i} \in I(d; n)} \xi(\vec{i}) \xi(\vec{i}) \quad (1.2)$$

being a homogeneous polynomial (random polynomial) of power $d$ on the random variables $\xi(\cdot, \cdot)$ "without diagonal members", (on the other words, multiply stochastic integral over discrete stochastic martingale measure), $n$ be an integer number: $n = 1, 2, \ldots$, in the case $n = \infty$ we will understood $Q(d, \infty)$ as a limit $Q(d; \infty) = \lim_{n \to \infty} Q(d; n)$, if there exists with probability one.

Here $b = \vec{b} = b(i), \quad i \in I(d; n)$ be arbitrary non-random numerical sequence, and we denote

$$\|b\|^2 = \sum_{i \in I(d; n)} b^2(i); \quad b \in B \quad \iff \|b\| = 1. \quad (1.2b)$$

We will denote also in the simple case when $\vec{b} = \vec{b}_d = \vec{1} = \{ 1, 1, \ldots, 1 \}$ $d$ - times

$$R(d) = Q(d; n, \vec{1}) = \sum_{\vec{i} \in I(d; n)} \xi(\vec{i}) \quad (1.3)$$

It is obvious that the sequence $(Q(d; n); F(n)); n = 1, 2, 3, \ldots$, is a martingale ("polynomial martingale").

The case of non-homogeneous polynomial is considered analogously.

We denote as usually the $L(p)$ norm of the r.v. $\eta$ as follows:

$$|\eta|_p = \left[ \mathbb{E}|\eta|^p \right]^{1/p}, \quad p > 2;$$

the case $p = 2$ is trivial for us.

We will derive the moment estimations of a Khintchine form
\[ U(p; d, n) = U(p) \overset{\text{def}}{=} \sup_{b \in B} |Q(d, n, b, \{\xi(\cdot)\})_p| \leq \overline{Q}(p; d, n) = \overline{Q}(p) \]

for martingale (and following in the independent case) in the terms of unconditional moments, more exactly, in the \(L(p)\) norms of summands:

\[ \mu_m(p) \overset{\text{def}}{=} \sup_i |\xi(i, m)|_p. \quad (1.4) \]

Denote also in the martingale case

\[ V(p) = V_d(p) \overset{\text{def}}{=} \prod_{m=1}^d \mu_m(d \cdot p). \quad (1.5a) \]

and for the independent variables \(\{\xi(i, m)\}\)

\[ W(p) = W_d(p) \overset{\text{def}}{=} \prod_{m=1}^d \mu_m(p). \quad (1.5b) \]

Note that if all the functions \(p \rightarrow \mu_m(p), m = 1, 2, \ldots, d\) are regular varying:

\[ \sup_{p \geq 2} [\mu_m(d \cdot p)/\mu_m(p)] < \infty, \]

then \(V(p) \asymp W(p)\).

As we knew, the previous result in this direction is obtained in the article [16]:

\[ U(p; d, n) \leq C_1(d) \cdot p^d \cdot V(p), \]

and in the independent case, i.e. when all the (two-dimensional indexed) centered r.v. \(\xi(i, m)\) are common independent,

\[ U(p; d, n) \leq C_2(d) \cdot (p^d/\ln p) \cdot W(p), \quad p \geq 2. \]

We intend to improve both these estimates to the following non-improvable as \(p \rightarrow \infty\):

\[ U(p; d, n) \leq \gamma(d) \cdot \frac{p^d}{(\ln p)^d} \cdot V_d(p), \quad (1.6a) \]

in general (martingale) case and

\[ U(p; d, n) \leq \kappa(d) \cdot \frac{p^d}{(\ln p)^d} \cdot W_d(p), \quad (1.6b) \]

for the independent variables; it will be presumed obviously the finiteness of \(V(p)\) and \(W(p)\).

There are many works about this problem; the next list is far from being complete: [1], [2], [4], [5], [6], [7], [10], [12], [13], [14], [17], [18], [20], [22], [21] etc.

See also the reference therein.
Notice that in the articles [10], [12], [21] and in many others are described some new applications of these estimates: in the theory of dynamical system, in the theory of polymers etc.

2 Main result: moment estimation for polynomial martingales.

We must describe some new notations. The following function was introduced by A. Osekowski (up to factor 2) in the article [13]:

\[ Os(p) \overset{\text{def}}{=} 4 \sqrt{2} \cdot \left( \frac{p}{4} + 1 \right)^{1/p} \cdot \left( 1 + \frac{p}{\ln(p/2)} \right). \]  \hspace{1cm} (2.1)

Note that

\[ K = K_{Os} \overset{\text{def}}{=} \sup_{p \geq 4} \left[ \frac{Os(p)}{p/\ln p} \right] \approx 15.7858, \]  \hspace{1cm} (2.2)

the so-called Osekowski’s constant.

Let us define the following numerical sequence \( \gamma(d) \), \( d = 1, 2, \ldots \) : \( \gamma(1) := K_{Os} = K \) (initial condition) and by the following recursion

\[ \gamma(d + 1) = \gamma(d) \cdot K_{Os} \cdot \left( 1 + \frac{1}{d} \right)^d. \]  \hspace{1cm} (2.3)

Since

\[ \left( 1 + \frac{1}{d} \right)^d \leq e, \]

we conclude

\[ \gamma(d) \leq K_{Os}^d \cdot e^{d-1}, \quad d = 1, 2, \ldots . \]  \hspace{1cm} (2.4)

**Theorem 2.1.** Let the sequence \( \gamma(d) \) be defined in (2.3). Then

\[ U(p; d, n) \leq \gamma(d) \cdot \frac{p^d}{(\ln p)^d} \cdot V_d(p) = \gamma(d) \cdot \frac{p^d}{(\ln p)^d} \cdot \prod_{m=1}^{d} \mu_m(d, p). \]  \hspace{1cm} (2.5)

**Proof.**

0. We will use the induction method over the ”dimension” \( d \), as in the article of authors [16], starting from the value \( d = 1 \).
1. One dimensional case $d = 1$. We apply the celebrate result belonging to A.Osekowski [13]:

$$
\left| \sum_{k=1}^{n} \xi_k \right|_p \leq C_{Os}(p) \cdot \left\{ \left| \left( \sum_{k=1}^{n} \mathbb{E} \xi_k^2 / F(k-1) \right)^{1/2} \right|_p + \left| \left( \sum_{k=1}^{n} |\xi_k|^p \right)^{1/p} \right|_p \right\} \overset{df}{=} (2.6)
$$

$$
C_{Os}(p) \left\{ S_1(p) + S_2(p) \right\}, \quad (2.6a)
$$
in our notations; $\xi_k = \xi(1,k)$, $F(0) = \{\emptyset, \Omega\}$. The variable

$$
\theta(n) := \left( \sum_{k=1}^{n} \mathbb{E} \xi_k^2 / F(k-1) \right)^{1/2}, \quad (2.7)
$$
so that $S_1(p) = |\theta(n)|_p$, is named in [13] by "conditional square function" of our martingale and the variable $\theta^2(n)$ — by "quadratic (predictable) characteristic" in the review [20].

We deduce using Jensen and triangle inequalities taking into account the restriction $p \geq 4$:

$$
\theta^2(n) = \sum_{k=1}^{n} \mathbb{E} \xi_k^2 / F(k-1),
$$

$$
|\theta^2(n)|_{p/2} \leq \sum_{k=1}^{n} \left| \mathbb{E} \xi_k^2 / F(k-1) \right|_{p/2} \leq \sum_{k=1}^{n} |\xi_k|^2 = \sum_{k=1}^{n} \mu_k^2(p). \quad (2.8)
$$
Since

$$
|\theta^2(n)|_{p/2} = |\theta(n)|^2_p,
$$
we ascertain

$$
S_1(p) = |\theta(n)|_p \leq \sqrt{\sum_{k=1}^{n} \mu_k^2(p)}. \quad (2.9)
$$

Let us estimate now the value $S_2(p)$. This evaluate is simple:

$$
S_2^p(p) = \mathbb{E} \left( \sum_{k} |\xi_k|^p \right) = \sum_{k} |\xi_k|^p = \sum_{k=1}^{n} \mu_k^p(p), \quad (2.10)
$$

$$
S_2(p) \leq \left( \sum_{k=1}^{n} \mu_k^p(p) \right)^{1/p} \leq \sqrt{\sum_{k=1}^{n} \mu_k^2(p)}. \quad (2.11)
$$
Thus,

$$
\left| \sum_{k=1}^{n} \xi_k \right|_p \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sqrt{\sum_{k=1}^{n} \mu_k^2(p)}. \quad (2.12)
$$
2. Since the sequence \( \{b(i)\} \) is non-random, the random sequence \( \{\xi^{(b)}(k)\} := \{b(k) \cdot \xi(k)\} \) is also a sequence of martingale differences relative at the same filtration. We apply the last inequality (2.11) for the martingale differences \( \{\xi^{(b)}(k)\} :\)

\[
\left| \sum_{k=1}^{n} b(k) \xi_k \right|_p \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sqrt{\sum_{k=1}^{n} b^2(k) \mu_k^2(p)}, \tag{2.13}
\]

and we obtain after taking supremum over \( \vec{b} \in B \):

\[
U(p; 1, n) = \sup_{\vec{b} \in B} \left| Q(1, n, \{\vec{\xi}(\cdot)\}) \right|_p \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sup_k \mu_k(p),
\]
or equally

\[
U(p; 1, n) \leq \gamma(1) \cdot \frac{p}{\ln p} \cdot V_1(p). \tag{2.14}
\]

3. Remark 2.1. We deduce as a particular case choosing in (2.13) \( b(k) = 1/\sqrt{n} \):

\[
n^{-1/2} \left| \sum_{k=1}^{n} \xi_k \right|_p \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sqrt{n^{-1} \sum_{k=1}^{n} \mu_k^2(p)}, \tag{2.15}
\]

which is some generalization of the classical Rosenthal’s inequality on the martingale case and in turn is a slight simplification of the A.Osekovski result.

In turn,

\[
\sup_n \left[ n^{-1/2} \left| \sum_{k=1}^{n} \xi_k \right|_p \right] \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sup_k \mu_k(p). \tag{2.15b}
\]

4. Induction step \( d \to d + 1 \) is completely analogous to one in the article [17], section 3, and may be omitted.

5. Remark 2.2; an example. Suppose \( d = \text{const} \geq 2, \ n \geq d + 1 \). We deduce as a particular case choosing

\[b(\vec{i}) = 1/\sqrt{n(n-1) \ldots (n-d+1)} \sim n^{-d/2} : \quad n^{-d/2} |R(d)|_p =\]

\[
n^{-d/2} |Q(d, n, \vec{1})|_p = n^{-d/2} \left| \sum_{\vec{i} \in I(d,n)} \xi(\vec{i}) \right|_p \leq C(d) \cdot \frac{p^d}{\ln p} \cdot \left[ \sup_{i,m} \mu_{i,m}(p) \right]^d. \tag{2.16}
\]
3 Independent case.

The reasoning is basically at the same as in the last section. We will use the famous Rosenthal’s inequality (more exactly, a consequence of this inequality) [23] instead the Osekowski’s estimate:

\[ \frac{1}{n^{1/2}} \left| \sum_{k=1}^{n} \xi_k \right|_p \leq K_R \cdot \frac{p}{\ln p} \cdot \sqrt{n^{-1} \sum_{k=1}^{n} \mu_k^2(p)}, \quad (3.1) \]

where now \( \{\xi_k\} \) is the sequence of the centered independent random variables with finite \( p^{th} \) moment, \( K_R \) is the Rosenthal’s constant. This estimate is non-improvable.

The exact value of this constant is obtained in [19]:

\[ K_R \approx 1.77638/e \approx 0.6535. \]

Define the announced sequence \( \kappa = \kappa(d), \ d = 1, 2, \ldots \) as follows: \( \kappa(1) := K_R \) and by the following recursion

\[ \kappa(d+1) = \kappa(d) \cdot K_{Os} \cdot \left( 1 + \frac{1}{d} \right)^d. \quad (3.2) \]

Since

\[ \left( 1 + \frac{1}{d} \right)^d \leq e, \]

we conclude

\[ \kappa(d) \leq K_R \cdot (K_{Os} \cdot e)^{d-1}, \ d = 1, 2, \ldots. \quad (3.3) \]

**Theorem 3.1.** Let the sequence \( \kappa(d) \) be defined in (3.3). Then in the considered here independent case

\[ U(p; d, n) \leq \kappa(d) \cdot \frac{p^d}{(\ln p)^d} \cdot W_d(p) = \kappa(d) \cdot \frac{p^d}{(\ln p)^d} \cdot \prod_{m=1}^{d} \mu_m(p). \quad (3.4) \]

**Remark 3.1.** Let us emphasise the difference between martingale and independent cases. This difference is except the coefficient but in the factors \( V_d(p) = \prod_{m=1}^{d} \mu_m(d \cdot p) \) and \( W_d(p) = \prod_{m=1}^{d} \mu_m(p). \)

It is clear that there are many examples when \( W_d(p) < \infty \) but \( V_d(p) = \infty. \)
4 Exponential bounds for tails of polynomial martingales.

We intend in this section to obtain the exponential bounds for tails of distribution for the r.v. \( Q(d, n) \) through its (obtained) moments estimates. We can consider only the martingale case (section 2).

**Theorem 4.1.** Suppose that the described below sequence of the mean zero martingale differences \( \{\xi(i, m)\} \) satisfies the restriction

\[
\sup_{i,m} \max(\mathbb{P}(\xi(i, m) \geq x), \mathbb{P}(\xi(i, m) \leq -x)) \leq \exp\left(-C_1 x^q (\ln x)^{-r} \right), \quad (4.1)
\]

\[x > e, \; C_1 = \text{const} > 0, \; q = \text{const} > 0, \; r = \text{const} . \quad (4.1a)\]

Then

\[
\sup_{b \in B} \max(\mathbb{P}(Q(d, n, b) \geq x), \mathbb{P}(Q(d, n, b) \leq -x)) \leq \exp\left(-C_2 x^{q/(dq+1)} (\ln x)^{-q(r-d)/(dq+1)} \right), \; x > e. \quad (4.2)
\]

**Proof.** It follows from the theory of the so-called Grand Lebesgue spaces [9], [15], chapter 1, section 1.8, [17] that the inequality (4.1) is equivalent to the finiteness of the following norm

\[
\sup_{i,m} \sup_{p \geq 4} \left[ |\xi(i, m)|_p \cdot p^{-1/q} \cdot \log^{-r} p \right] = C_3 < \infty, \quad (4.3)
\]

or equally

\[
\sup_{i,m} |\xi(i, m)|_p \leq C_3 \; p^{1/q} \; \log^r p. \quad (4.3a)
\]

We apply the theorem 2.1:

\[
\sup_{b \in B} |Q(d, n, b)|_p \leq C_4 \; p^{d+1/q} \left[ \log p \right]^{r-d},
\]

which is in turn equivalent to the proposition (4.2).

Note that other exponential bounds for tail of distribution for the r.v. \( Q(d, n, b) \) under some additional conditions is obtained in [16].
5 Concluding remarks.

A. Examples of lower estimates.

Denote in the independent case

\[ K_I(p; d) = \sup_n \sup_{b \in B} \sup_{\xi(i,m):|\xi(i,m)|_p < \infty} \left[ \frac{Q(d, p)}{\prod_{m=1}^{d} \mu_m(p)} \right]. \quad (5.1) \]

where the last “sup” is calculated over all the sequences of the centered independent variables \{\xi(i, m)\} satisfying the condition \(|\xi(i, m)|_p < \infty\). We obtained

\[ K_I(p; d) \leq \frac{C_0(d) \ p^d}{\ln^d p}, \quad C_0(d) = \text{const} > 0. \]

Our new statement:

\[ K_I(p; d) \geq \frac{C(d) \ p^d}{\ln^d p}, \quad C(d) = \text{const} > 0. \quad (5.2) \]

Proof is very simple. The moment estimations are derived in [8] for the symmetrical polynomials on mean zero independent identical symmetrically distributed variables, i.e. particular case for us, for which it is proved that

\[ \frac{|Q(d, n)|_p}{\sqrt{\text{Var} \ Q(d, n) \ \mu_d(p)}} \geq \frac{C_1(d) \ p^d}{\ln^d p}, \quad C_1(d) = \text{const} > 0. \quad (5.3) \]

Another a more simple example. Let \( n = 1 \) and a r.v. \( \eta \) has a Poisson distribution with unit parameter:

\[ P(\eta = k) = e^{-1}/k!, \]

and define \( \xi = \eta - 1 \), then the r.v. \( \xi \) is centered and

\[ p \to \infty \Rightarrow |\xi|_p \sim p/(e \cdot \ln p). \]

Let also \( \xi_j, \ j = 1, 2, \ldots \) be independent copies of \( \xi \). Then

\[ \left| \prod_{j=1}^{d} \xi_j \right|_p \sim e^{-d} \frac{p^d}{(\ln p)^d}, \quad p \to \infty. \quad (5.4) \]

B. Estimations for normed variables.

Denote

\[ \Psi(b) = \Psi(d, n, b) = \text{Var}(Q(d, n, b)), \quad b \in B, \quad (5.5) \]
and impose the following condition on the martingale distribution

\[ 0 < C_1(d) \leq \sup_n \sup_{b \neq 0} \left[ \frac{\Psi(d, n, b)}{||b||^2} \right] \leq C_2(d) < \infty. \]  

(5.6)

This condition was introduced and investigated in [16].

We define also a so-called relative moments for the r.v. \( \{\xi(i, m)\} \) under the natural norming:

\[ \tilde{\mu}_m(p) \overset{\text{def}}{=} \sup_i \left| \frac{\xi(i, m)}{\sqrt{\text{Var}(\xi(i, m))}} \right|_p, \]  

(5.7)

Denote also in the martingale case

\[ \tilde{V}(p) = \tilde{V}_d(p) \overset{\text{def}}{=} \prod_{m=1}^d \tilde{\mu}_m(d \cdot p). \]  

(5.8a)

and for the independent variables \( \{\xi(i, m)\} \)

\[ \tilde{W}(p) = \tilde{W}_d(p) \overset{\text{def}}{=} \prod_{m=1}^d \tilde{\mu}_m(p). \]  

(5.8b)

We will derive as before the moment estimations of a form

\[ \tilde{U}(p; d, n) = \tilde{U}(p) \overset{\text{def}}{=} \sup_{b \in \mathcal{B}} \left| Q(d, n, b, \{\xi(\cdot)\}) / \sqrt{\text{Var}(Q(d, n, b, \{\xi(\cdot)\}))} \right|_p \leq \tilde{Q}(p; d, n) = \tilde{Q}(p). \]  

(5.9)

Namely,

\[ \tilde{U}(p; d, n) \leq \gamma(d) \cdot \frac{p^d}{(\ln p)^d} \cdot \tilde{V}_d(p), \]  

(5.10a)

in general (martingale) case and

\[ \tilde{U}(p; d, n) \leq \kappa(d) \cdot \frac{p^d}{(\ln p)^d} \cdot \tilde{W}_d(p), \]  

(5.10b)

for the independent variables; it will be presumed of course the finiteness of the variables \( \tilde{V}(p) \) and \( \tilde{W}(p) \).

Authors hope that the last two estimates are more convenient for the practical using.
C. Possible generalizations.

It is of interest by our opinion to generalize our estimates on the predictable sequence $b(\vec{i})$. A preliminary (one-dimensional) result in this direction see in the article [14]; see also [3].

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