Some Results About Global Asymptotic Stability

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Abstract We study the global asymptotic stability of the origin for the continuous and discrete dynamical system associated to polynomial maps in $\mathbb{R}^n$ (especially when $n = 3$) of the form $F = \lambda I + H$, with $F(0) = 0$, where $\lambda$ is a real number, $I$ the identity map, and $H$ a map with nilpotent Jacobian matrix $JH$. We distinguish the cases when the rows of $JH$ are linearly dependent over $\mathbb{R}$ and when they are linearly independent over $\mathbb{R}$. In the linearly dependent case we find non-linearly triangularizable vector fields $F$ for which the origin is globally asymptotically stable singularity (respectively fixed point) for continuous (respectively discrete) systems generated by $F$. In the independent continuous case, we present a family of maps that have orbits escaping to infinity. Finally, in the independent discrete case, we show a large family of vector fields that have a periodic point of period 3.

Keywords Polynomial vector fields · Global attractor · Markus–Yamabe conjectures

1 Introduction

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-map with $F(0) = 0$. Then the origin is a singular point of the differential system

$$\dot{x} = F(x),$$

(1)
and a fixed point of the dynamics of iterations of $F$

$$x^{(m+1)} = F(x^{(m)}), \quad x^{(0)} \in \mathbb{R}^n.$$  \hspace{1cm} (2)

We call the continuous (respectively discrete) dynamical system generated by $F$ to the dynamical system associated to (1) (resp. (2)).

In this article we discuss the global asymptotic stability of the origin for both systems but restricted to a special family of polynomial maps $F$ in $\mathbb{R}^n$, focussing on $n = 3$.

We let $\phi(t, x)$ denote the solution of (1) with initial condition $\phi(0, x) = x$. We say that the origin is a globally asymptotically stable singularity of the continuous dynamical system generated by $F$ if for each $x \in \mathbb{R}^n$, we have that the solution $\phi(t, x)$ of (1) is defined for all $t > 0$ and tends to the origin as $t$ tends to infinity.

We say that the origin is a globally asymptotically stable fixed point of the discrete dynamical system generated by $F$ if the sequence $x^{(m)}$ of (2) tends to the origin as $m$ tends to infinity, for any $x^{(0)} \in \mathbb{R}^n$.

Our set of vector fields $\mathcal{N}(\lambda, n)$, that depends on a real number $\lambda$ and a positive integer $n$, consists of the polynomial maps in $\mathbb{R}^n$ of the form $F = \lambda I + H$, with $F(0) = 0$, where $I$ is the identity map and $H$ has nilpotent Jacobian matrix at every point.

Polynomial maps $H$ defined on $\mathbb{R}^n$ and on $\mathbb{C}^n$ with nilpocabinet matrix at every point have been extensively studied from the algebraic geometry viewpoint (see for example [5]). In this paper we make use of some aspects of this theory.

Note that for $F \in \mathcal{N}(\lambda, n)$, the Jacobian matrix $JF$ at each $x \in \mathbb{R}^n$ has all its eigenvalues equal to $\lambda$. Therefore, a map $F = \lambda I + H$ in $\mathcal{N}(\lambda, n)$ satisfies the hypotheses of the Markus–Yamabe Conjecture (MYC) (resp. of the Discrete Markus–Yamabe Conjecture (DMYC)) if and only if $\lambda < 0$ (resp. $|\lambda| < 1$). The MYC was established by Markus and Yamabe in 1960 (see [8]) and the DMYC was formulate by LaSalle in 1976 (see [7]). Its precise statements are the following.

**The Markus–Yamabe Conjecture (MYC).** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-map with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of $F$ at $x$ have negative real part, then the origin is a global attractor of the system (1).

**The Discrete Markus–Yamabe Conjecture (DMYC).** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$-map with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of $F$ at $x$ have modulus less than one, then the origin is a global attractor of the discrete dynamical system (2) generated by $F$.

It is known that the MYC (resp. the DMYC) is true when $n \leq 2$ (resp. $n = 1$) and false when $n \geq 3$ (resp. $n \geq 2$). Notwithstanding, both conjectures are true for triangular maps in any dimension. The continuous case was proved by Markus and Yamabe [8], and the discrete case by Cima et al. [4]. In the case of polynomial maps, the DMYC is also true when $n = 2$ (see [4]), though both conjectures are false when $n \geq 3$. In [2], Cima et al. give an example of a pair of polynomial maps, of which one satisfies the MYC hypotheses and the other the DMYC hypotheses, having both systems orbits that escape to infinity. Further in [3], Cima et al. obtain a family of polynomial counterexamples containing the preceding pair. These counterexamples of [3] are, basically, vector fields $F = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$ where $H$ is a quasi–homogeneous vector field.
of degree one. We give examples of vector fields in \( \mathcal{N}(\lambda, 3) \) which are linearly triangularizable (that is, triangular after a linear change of coordinates) in \([6]\). For these maps, the MYC (resp. the DMYC) is true when \( \lambda < 0 \) (resp. \( |\lambda| < 1 \)). Further, \([6]\) contains a family of counterexamples to the MYC which generalizes that of Cima–Gasull–Mañosas. The examples and counterexamples \( F = \lambda I + H \in \mathcal{N}(\lambda, n) \) of above have one common characteristic, namely the rows of \( JH \) are linearly dependent over \( \mathbb{R} \).

The paper is organized as follows. In Sect. 2 we consider the linearly dependent case for \( n = 3 \). We say that a map \( F = \lambda I + H \in \mathcal{N}(\lambda, 3) \) is linearly dependent if there exist \((\alpha, \beta, \gamma) \in \mathbb{R}^3 - \{0, 0, 0\}\) such that \( \alpha P + \beta Q + \gamma R \equiv 0 \), where \( H = (P, Q, R) \). We study the global asymptotic stability of the origin for the continuous and discrete dynamical system generated by maps \( F = \lambda I + H \in \mathcal{N}(\lambda, 3) \) which are linearly dependent. We give a normal form for these maps (see Proposition 2.1) and characterize those elements which are linearly triangularizable (see Theorem 2.4). The normal form depends on a polynomial \( f(t) \) with coefficients in \( \mathbb{R}[z] \). In the case \( f(t) \) is a polynomial of degree one, we show the global asymptotic stability of the origin for the continuous and discrete cases (see Theorems 2.5 and 2.6). We thus obtain a family of non–linearly triangularizable maps in \( \mathcal{N}(\lambda, 3) \) for which the origin is globally asymptotically stable. To our knowledge, there are no examples as the preceding one in the literature. The section concludes showing that, for a linearly dependent map \( F \in \mathcal{N}(\lambda, 3) \), in order for the origin not to be a globally asymptotically stable singularity (resp. fixed point) the continuous (resp. discrete) system must have at least one orbit which escapes to infinity (see Theorem 2.7).

In Sect. 3 we deal with \( F \in \mathcal{N}(\lambda, 3) \) which are not linearly dependent. These maps be called linearly independent. We state the Dependence Problem and the Generalized Dependence Problem introduced by van den Essen in [5, Chapter 7], among others, and we obtain a family of examples \( F_{n,r} = \lambda I + H_{n,r} \in \mathcal{N}(\lambda, n) \) which are linearly independent for any dimension \( n \geq 3 \), with \( \text{rk } JH_{n,r} = r \geq 2 \). When \( n \geq 3 \) and \( r = 2 \), we show that for these maps the origin is not globally asymptotically stable singularity (see Theorem 3.2). Subsequently, we consider maps \( F = \lambda I + H \in \mathcal{N}(\lambda, 3) \), where \( H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \). A large class of these maps \( H \) were characterized by M. Chamberland and A. van den Essen in \([1]\). The characterization depends on a polynomial map \( g(t) \). In the case \( g(t) \) is a polynomial of degree less than or equal to two, we show that the continuous system generated by \( F = \lambda I + H \), with \( \lambda < 0 \) have orbits that escape to infinity (see Theorem 3.5). On the other hand, in the discrete case, for \( |\lambda| < 1 \), these maps have a periodic point of period three (see Theorem 3.8). Therefore the origin is not a globally asymptotically stable fixed point.

### 2 The Linearly Dependent Case

This section is devoted to maps \( F = \lambda I + H \in \mathcal{N}(\lambda, 3) \) where the component of \( H \) are linearly dependent over \( \mathbb{R} \). Since \( F(0) = 0 \), this condition is equivalent to that rows of the Jacobian matrix \( JH \) being linearly dependent over \( \mathbb{R} \). The first result of this section establish a normal form for this type of maps. For a proof, see for example [1, Corollary 1.1].
Proposition 2.1 Let $F = \lambda I + (S, U, V) \in \mathcal{N}(\lambda, 3)$ linearly dependent. Then there exists a $T \in Gl_3(\mathbb{R})$ such that $T^* F = \lambda I + (P, Q, 0)$ where

$P(x, y, z) = -b(z) f(a(z) x + b(z) y) + c(z)$ and $Q(x, y, z) = a(z) f(a(z) x + b(z) y) + d(z)$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$.

Remark 2.2 In the normal form (3) we may assume $f(0) = 0$ by modifying the polynomials $c(z)$ and $d(z)$ if necessary.

An interesting question about the maps satisfying the hypotheses of the MYC or the DMYC concerns the injectivity.

Proposition 2.3 If $\lambda \neq 0$ then any $F \in \mathcal{N}(\lambda, 3)$ linearly dependent is injective.

Proof The Proposition results from the normal form (3). \qed

Recall that a map $F : \mathbb{R}^n \to \mathbb{R}^n$ is triangular if it has the form

$F(x_1, x_2, \ldots, x_n) = (F_1(x_1), F_2(x_1, x_2), \ldots, F_n(x_1, x_2, \ldots, x_n))$.

Our next result establishes conditions under which maps of the form $F = \lambda I + (P, Q, 0) \in \mathcal{N}(\lambda, 3)$, with $(P, Q)$ as in Proposition 2.1, are linearly triangularizable (triangular after a linear change of coordinates).

Theorem 2.4 Let $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ where

$H(x, y, z) = f(a(z) x + b(z) y) (-b(z), a(z), 0) + (c(z), d(z), 0)$

with $\lambda \in \mathbb{R}$, $a, b, c, d \in \mathbb{R}[z]$, $f \in \mathbb{R}[z][t]$. Then $X$ is linearly triangularizable if and only if either $f$ is constant or $a, b$ are linearly dependent over $\mathbb{R}$.

Proof When $f$ depends only on $z$, the result is clear. In what follows, we will assume that the degree of $f$ with respect to $t$ is greater than zero. If $a, b$ are linearly dependent over $\mathbb{R}$, then there exists $(\alpha, \beta) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $\alpha a(z) + \beta b(z) = 0$, for all $z \in \mathbb{R}$. Assume $\beta \neq 0$. Then $b(z) = \delta a(z)$, with $\delta = -\frac{a}{\beta}$. Consider the linear isomorphism $T(x, y, z) = (z, x + \delta y, y)$. Then

$T^*(F)(u, v, w) = \lambda (u, v, w) + (0, c(u) + \delta d(u), a(u) f(a(u) v) + d(u)),$

which is triangular.

Now suppose that there exists a linear isomorphism $M$ such that

$M^*(F)(u, v, w) = \lambda (u, v, w) + (A(v, w), B(w), 0).$
Assume that $[M] = (m_{ij})_{1 \leq i, j \leq 3}$ is the matrix of $M$ with respect to the canonical basis of $\mathbb{R}^3$. We have

$$m_{31} [-f(t) b(z) + c(z)] + m_{32} [f(t) a(z) + d(z)] \equiv 0$$

where $t = a(z)x + b(z)y$. Then

$$m_{31} [-f(0) b(z) + c(z)] + m_{32} [f(0) a(z) + d(z)] \equiv 0$$

and, therefore,

$$(f(t) - f(0)) [-m_{31} b(z) + m_{32} a(z)] \equiv 0.$$

If $(m_{31}, m_{3,2}) \neq (0, 0)$, the proof is complete. If $(m_{31}, m_{3,2}) = (0, 0)$, we may assume that $m_{33} = 1$ and $\det[M] = 1$. Thus the matrix of $M^{-1}$ with respect to the canonical basis of $\mathbb{R}^3$ is

$$[M^{-1}] = \begin{pmatrix} m_{22} & -m_{12} & \tilde{m}_{13} \\ -m_{21} & m_{11} & \tilde{m}_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

with $\tilde{m}_{13} = -m_{13} m_{22} + m_{12} m_{23}$ and $\tilde{m}_{23} = m_{13} m_{21} - m_{11} m_{23}$. Then

$$t = a(w) [m_{22} u - m_{12} v + \tilde{m}_{13} w] + b(w) [-m_{21} u + m_{11} v + \tilde{m}_{23} w]$$

and

$$B(w) = m_{21} [-f(t) b(w) + c(w)] + m_{22} [f(t) a(w) + d(w)].$$

Differentiating the preceding expression with respect to $u$ we obtain

$$0 = f'(t) [m_{22} a(w) - m_{21} b(w)]^2$$

and so $\{a, b\}$ are linearly dependent over $\mathbb{R}$, which completes the proof. \qed

The next two results assert that, in the linearly dependent case, the origin is a globally asymptotically stable singularity when the degree of the polynomial $f(t)$ is one.

**Theorem 2.5** Let $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ where

$$H(x, y, z) = g(z) (a(z) x + b(z) y) (-b(z), a(z), 0) + (c(z), d(z), 0)$$

with $\lambda < 0$, $a, b, c, d, g \in \mathbb{R}[z]$. Then the origin is a globally asymptotically stable singularity for the differential system $\dot{x} = F(x)$. 
Proof Note that \((x(t), y(t), z(t))\) is a solution of the differential system \(\dot{x} = F(x)\) if and only if \(z(t) = z_0 e^{\lambda t}\) and \((x(t), y(t))\) is a solution of the linear system

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\lambda - A(t)B(t)G(t) & -B(t)^2G(t) \\
A(t)^2G(t) & \lambda + A(t)B(t)G(t)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
C(t) \\
D(t)
\end{pmatrix}
\]

where \((A, B, C, D, G)\)\((t) = (a, b, c, d, g)(z_0 e^{\lambda t})\). Since the origin is a locally asymptotically stable singularity, there is a basis of solutions of the linear system consisting of solutions which tend to the origin as \(t\) tends to \(+\infty\). Therefore, the origin is a globally asymptotically stable singularity. \(\square\)

**Theorem 2.6** Let \(F = \lambda I + H \in \mathcal{N}(\lambda, 3)\) where

\[
H(x, y, z) = g(z) (a(z) x + b(z) y) (-b(z), a(z), 0) + (c(z), d(z), 0)
\]

with \(0 < \lambda < 1, a, b, c, d, g \in \mathbb{R}[z]\). Then the origin is a globally asymptotically stable fixed point of the discrete dynamical system (2) generated by \(F\).

Proof Without loss of generality, we may assume that \(c(z) \equiv d(z) \equiv 0\). In fact, the polynomials \(c(z)\) and \(d(z)\) may be eliminated by applying the coordinate change \(T(u, v, w) = (u + m(w), v + n(w), w)\) where

\[
\begin{pmatrix}
m(w) \\
n(w)
\end{pmatrix} = -\frac{1}{(1 - \lambda)^2} [(1 - \lambda) I + g(w) M(w)] \begin{pmatrix}
c(w) \\
d(w)
\end{pmatrix}
\]

with

\[
M(w) = \begin{pmatrix}
-a(w)b(w) & -b(w)^2 \\
am(w)^2 & a(w)b(w)
\end{pmatrix}.
\]

So we assume

\[
H(x, y, z) = g(z) (a(z) x + b(z) y) (-b(z), a(z), 0).
\]

Therefore,

\[
F(x, y, z) = (A(z) \begin{pmatrix} x \\ y \end{pmatrix}, \lambda z)
\]

where

\[
A(z) = \begin{pmatrix}
\lambda - a(z)b(z)g(z) & -b(z)^2g(z) \\
a(z)^2g(z) & \lambda + a(z)b(z)g(z)
\end{pmatrix}.
\]

Thus it suffices to prove that, for any \((x, y) \in \mathbb{R}^2\), we have

\[
\lim_{n \to \infty} A(\lambda^n z) A((\lambda^{n-1} z) \cdots A(\lambda z) A(z) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Let $\mathcal{M}_2$ be the normal vector space of the $2 \times 2$ real matrices $A = (a_{ij})$ endowed with the norm $\|A\| = 2 \max |a_{ij}|$. Considering $\mathbb{R}^2$ endowed with the norm $\|(x, y)\| = \max\{|x|, |y|\}$, we have

$$\|A(x, y)\| \leq \|A\| \|(x, y)\| \quad \text{and} \quad \|AB\| \leq \|A\| \|B\|. $$

A simple computation yields

$$A(\lambda^n z) A((\lambda^{n-1} z) \cdots A(\lambda z) A(z) =

\begin{pmatrix}
\lambda^n - na_0 b_0 g_0 \lambda^{n-1} + r_{11}(z) & -nb_0^2 g_0 \lambda^{n-1} + r_{12}(z) \\
na_0^2 g_0 \lambda^{n-1} + r_{21}(z) & \lambda^n + na_0 b_0 g_0 \lambda^{n-1} + r_{22}(z)
\end{pmatrix}
$$

where $r_{ij}(0) = 0$ and $(a_0, b_0, g_0) = (a, b, g)(0)$.

Fix $N \in \mathbb{N}$ so that $2N\lambda^{N-1} \max |a_0^2 g_0, b_0^2 g_0, |a_0 b_0 g_0| < 1$. Let $B(z) = A(\lambda^{N-1} z) A((\lambda^{N-2} z) \cdots A(\lambda z) A(z)$. Consider $0 < |z| < z_0$ such that $\|B(z)\| \leq K < 1$. Then, for $n = kN - 1$, we have

$$\left\| A(\lambda^n z) \cdots A(z) \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \left\| B(\lambda^{(k-1)N} z) \cdots B(z) \begin{pmatrix} x \\ y \end{pmatrix} \right\|

\leq K^k \|(x, y)\| \to 0 \quad \text{if} \quad k \to \infty$$

which completes the proof. □

Our next result shows that, for a linearly dependent $F \in \mathcal{N}(\lambda, 3)$, in order for the origin not to be a globally asymptotically stable singularity (resp. fixed point) the continuous (resp. discrete) system must have at least one orbit which escapes to infinity.

**Theorem 2.7** Let $\lambda \in \mathbb{R}$, and let $F = \lambda I + H \in \mathcal{N}(\lambda, 3)$ linearly dependent. If $\lambda < 0$ (resp. $|\lambda| < 1$) and the origin is not a globally asymptotically stable singularity (resp. fixed point) for the differential system $\dot{x} = F(x)$ (resp. for the discrete dynamical system (2) generated by $F$), then the differential system $\dot{x} = F(x)$ (resp. the discrete dynamical system generated by $F$) has orbits which escape to infinity.

**Proof** We may assume that $H = (P, Q, 0)$ where

$$P(x, y, z) = -b(z) f(a(z) x + b(z) y) + c(z) \quad \text{and} \quad Q(x, y, z) = a(z) f(a(z) x + b(z) y) + d(z)$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$.

Consider the case $\lambda < 0$. Let $\gamma(t) = (x(t), y(t), z(t))$ be a solution of the system $\dot{x} = F(x)$. We denote the omega-limit set of $\gamma$ by $\omega(\gamma)$. Since $z(t) = z(0) e^{\lambda t}$, we have $\omega(\gamma) \subset W_\infty$, where $W_\infty$ is the extended plane $\{z = 0\} \cup \{\infty\}$. If the orbit $\gamma(t)$ is bounded, then $\omega(\gamma) = \{0\}$ and we obtain the Theorem. The proof is analogous for the discrete dynamical system generated by $F$ in the case $|\lambda| < 1$. □
Thus we are led to posing the following:

**Question 1** Do there exist linearly independent maps in $N(\lambda, 3)$, with the degree of $f(t)$ greater than one, for which the origin is globally asymptotically stable either in the continuous case, or the discrete case, or both?

### 3 The Linearly Independent Case

In this section we consider maps $F \in N(\lambda, 3)$ where the rows of $JH$ are linearly independent over $\mathbb{R}$. We begin with some algebraic preliminaries extracted from [5, Chapter 7] and [1]. The study of the Jacobian Conjecture for polynomial maps of the form $I + H$ where $I$ is the identity map and $H$ a homogeneous map of degree 3, with $JH$ nilpotent, led various authors to the following problem. Let $\kappa$ be a field of characteristic zero.

**Dependence Problem.** Let $d \in \mathbb{N}$, with $d \geq 1$, and let $H = (H_1, \ldots, H_n) : \kappa^n \to \kappa^n$ be a homogeneous polynomial map of degree $d$ such that $JH$ is nilpotent. Does it follow that $H_1, \ldots, H_n$ are linearly dependent over $\kappa$?

The attempt to solve it by induction led to consider the more general problem:

**Generalized Dependence Problem.** Let $H = (H_1, \ldots, H_n) : \kappa^n \to \kappa^n$ be a polynomial map such that $JH$ is nilpotent. Are the rows of $JH$ linearly dependent over $\kappa$?

The answer to this question turned out to be “yes” if $n \leq 2$ and “no” if $n \geq 3$. More precisely, van den Essen showed the following (see [5, Theorem 7.1.7]).

**Theorem 3.1**

(i) If $JH$ is nilpotent and $\text{rk } JH \leq 1$, then the rows of $JH$ are linearly dependent over $\kappa$ (here $\text{rk }$ is the rank as an element of $M_n(\kappa(\mathbb{X}))$).

(ii) Let $r \geq 2$. Then, for any dimension $n \geq r + 1$, there exists a polynomial map $H_{n,r} : \kappa^n \to \kappa^n$ such that $JH_{n,r}$ is nilpotent, $\text{rk } JH_{n,r} = r$, and the rows of $JH_{n,r}$ are linearly independent over $\kappa$.

The example is the following. Let $a \in \mathbb{R}[x_1]$ with $\deg a = r$ and $f(x_1, x_2) = x_2 - a(x_1)$. Then $H_{n,r} = (H_1, \ldots, H_n)$ where

$$H_1(x_1, \ldots, x_n) = f(x_1, x_2),$$

$$H_i(x_1, \ldots, x_n) = x_{i+1} + \frac{(-1)^i}{(i-1)!} a^{(i-1)}(x_1) (f(x_1, x_2))^{i-1}, \text{ if } 2 \leq i \leq r,$$

$$H_{r+1}(x_1, \ldots, x_n) = \frac{(-1)^{r+1}}{r!} a^{(r)}(x_1) (f(x_1, x_2))^r, \text{ and }$$

$$H_j(x_1, \ldots, x_n) = (f(x_1, x_2))^{j-1}, \text{ if } r + 1 < j \leq n$$

is a polynomial map satisfying assertion (ii). (See [5, Proposition 7.1.9]). For $r = 2$ and $n \geq 3$, the components of $H_{n,2}$ are

$$H_1(x_1, \ldots, x_n) = x_2 - a(x_1) - bx_1^2,$$

$$H_2(x_1, \ldots, x_n) = x_3 + (a + 2bx_1)(x_2 - a(x_1) - bx_1^2), \quad (4)$$
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\[ H_3(x_1, \ldots, x_n) = -b \left( x_2 - a x_1 - b x_1^2 \right)^2, \text{ and for } j \geq 4 \]
\[ H_j(x_1, \ldots, x_n) = (x_2 - a x_1 - b x_1^2)^{j-1}, \]

with \( b \neq 0 \).

**Theorem 3.2** Let \( F_{n,2} = \lambda I + (H_1, \ldots, H_n) \), with \( H_i \) as in (4).

(a) If \( \lambda < 0 \) then the system \( \dot{x} = F_{n,2}(x) \) has orbits that escape to infinity.
(b) If \( -1 < \lambda < 1 \) then the discrete dynamical system generate by \( F_{n,2} \) has a periodic orbit of period three.

**Proof**

(a) It suffices to prove the theorem in the case \( n = 3 \). Thus assume \( n = 3 \) and put \( X = F_{3,2} \). If \( (u, v, w) = \phi(x_1, x_2, x_3) = b \left( x_1, x_3 - \lambda b x_1^2, \lambda x_1 + x_2 - a x_1 - b x_1^2 \right) \), and \( \phi^*(X) = Y \) then
\[ Y(u, v, w) = (w, \lambda v - w^2, 2 \lambda w + v - \lambda^2 u). \]

To find orbits of \( Y \) that escape to infinity, consider the coordinate change
\[ (s, q, p) = \frac{1}{v}(1, u, w). \]

If \( Z \) is the vector field \( Y \) in the new coordinates, then \( W = (W_1, W_2, W_3) = s Z \) is defined by
\[ W(s, q, p) = (-s(\lambda s - p^2), s(p - \lambda q) + q p^2, s(\lambda p + 1 - \lambda^2 q) + p^3). \]

For \( s \neq 0 \), the orbits of \( Z \) and \( W \) are the same. Moreover, for \( s > 0 \) (resp. \( s < 0 \)), the orbits of \( Z \) and \( W \) have the same (resp. inverse) orientation. Over the plane \( s = 0 \), the vector field \( W \) is radially repeller outside of a line of singular points, namely the line \( p = 0 \). For \( s > 0 \), we have \( W_1 > 0 \) and, therefore, there are no orbits there with \( \omega \)-limit set contained at \( s = 0 \). For \( s < 0 \), we must find orbits of \( W \) with \( \alpha \)-limit set contained at \( s = 0 \).

Consider the numbers
\[ A = 2 \lambda, \quad s_0 = \frac{1}{512 \lambda^3}, \quad p_0 = -\frac{1}{8 \lambda}, \quad q_0 = \frac{11}{16 \lambda^2} \]

and the set
\[ P_A = \{(s, q, p) : as - p^2 \leq 0, s_0 \leq s \leq 0, 0 \leq q \leq q_0, 0 \leq p \leq p_0\}. \]

We have the following:

(1) Over the set \( P_A \cap \{(s, q, p) : as - p^2 = 0\} \), the vector field \( W \) points outward from the set \( P_A \). In fact, if \( (s, q, p) \in P_A \) and \( as - p^2 = 0 \), then
\[ A W_1 - 2 p W_3 = -\frac{p^3}{A} [p (A + 3\lambda) + 2 (1 - \lambda^2 q)] \]

\[ \geq -\frac{p^3}{A} [p_0 (A + 3\lambda) + 2 (1 - \lambda^2 q_0)] = 0. \]

(2) Over the set \( P_A \cap \{(s, q, p_0) : s < 0\} \), the vector field \( W \) points outward from the set \( P_A \). In fact, if \( p = p_0 \), then

\[ W_3 = s (\lambda p_0 + 1 - \lambda^2 q) + p_0^3 \]

\[ \geq s (\lambda p_0 + 1) + p_0^3 = \frac{7}{8} s - \frac{1}{8^3 \lambda^3} \]

\[ \geq \frac{7}{8} s_0 - \frac{1}{8^3 \lambda^3} = \frac{-1}{8^4 \lambda^3} > 0. \]

(3) Over the set \( P_A \cap \{(s, 0, p)\} \), the vector field \( W \) points outward from the set \( P_A \). In fact, if \( q = 0 \), then

\[ W_2 = s p \leq 0. \]

(4) Over the set \( P_A \cap \{(s, q_0, p)\} \), the vector field \( W \) points outward from the set \( P_A \). In fact, if \( q = q_0 \), then

\[ W_2 = s p - q_0 (\lambda s - p^2) = (\lambda s - p^2) [\frac{sp}{\lambda s - p^2} - q_0] \geq 0 \]

because

\[ \lambda s - p^2 < A s - p^2 \leq A s_0 - \frac{p_0^2}{4} = 0 \]

and

\[ h(s, p) = \frac{sp}{\lambda s - p^2} \leq h(s_0, p) \leq h(s_0, \frac{p_0}{2}) = \frac{1}{16\lambda^2} < q_0. \]

Thus any orbit \( \gamma(t) \) of \( W \), with \( \gamma(0) \) an interior point of \( P_A \), has \( \alpha \)-limit set contained in the line \( s = p = 0 \). Clearly, any (of these) orbit corresponds to an orbit of our initial vector field \( X \) that escapes to infinity. This completes the proof in this case.

(b) When \( n = 3 \) the system \( F_{3,2} \) correspond to a particular case of (10) and the result follows of Theorem 3.8. Thus assume \( n > 3 \). Observe that

\[ F_{n,2}(x_1, \ldots, x_n) = (F_{3,2}(x_1, x_2, x_3), \lambda x_4 + f(x_1, x_2)^3, \ldots, \lambda x_n + f(x_1, x_2)^{n-1}) \]

where \( f(x_1, x_2) = x_2 - ax_1 - bx_1^2 \).

This implies that the third iterated of \( F_{n,2} \) is of the form

\[ F_{n,2}^3(x_1, \ldots, x_n) = (F_{3,2}^3(x_1, x_2, x_3), \lambda^3 x_4 + g_4(x_1, x_2, x_3), \ldots, \lambda^3 x_n + g_n(x_1, x_2, x_3)). \]
Then the point \((x_1, \ldots, x_n)\), where \((x_1, x_2, x_3)\) is a periodic point of period three of \(F_{n,2}\), and \(x_j = \frac{1}{\lambda_j} \sum g_j(x_1, x_2, x_3)\), for \(4 \leq j \leq n\), is a periodic point of period three of \(F_{n,2}\). The proof is now complete. 

Observe that the example \(H_{3,2}\) has the special form
\[
H_{3,2}(x, y, z) = (u(x, y), v(x, y, z), h(u(x, y))).
\] (5)

In [1] it proved that a large class of polynomial maps \(H = (H_1, H_2, H_3)\) of the form
\[
H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))).
\] (6)

with \(JH\) nilpotent and such that \(H_1, H_2, H_3\) are linearly independent, reduce through a linear coordinate change, to a map of the form
\[
G(x, y, z) = (g(t), v_1 z - (b_1 + 2v_1 \alpha x) g(t), \alpha g(t)^2)
\] (7)

with \(t = y + b_1 x + v_1 \alpha x^2\) and \(v_1 \alpha \neq 0\), and \(g \in \mathbb{R}[t]\) with \(g(0) = 0\) and \(\deg_t g(t) \geq 1\).

More specifically, it has the following theorem that resume the results of [1].

**Theorem 3.3** Let \(H(x, y, z) = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))\). Assume that \(H(0) = 0, h'(0) = 0,\) and the components of \(H\) are linearly independent over \(\mathbb{R}\). Let \(A = \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial x \partial z}\) and \(B = \frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 u}{\partial y \partial z}\). If \(JH\) is nilpotent and \(\deg_z(uA) \neq \deg_z(vB)\) then there exists \(T \in GL_3(\mathbb{R})\) such that \(THT^{-1}\) is of the form (7).

**Remark 3.4**
(1) By Theorem 3.3, any map
\[
F = \lambda I + (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \in \mathcal{N}(\lambda, 3)
\]
under the condition \(\deg_z(uA) \neq \deg_z(vB)\), modulus a linear change of coordinates, has the form
\[
F(x, y, z) = \lambda (x, y, z) + (0, v_1 z, 0)
\]
\[
+ g(t) (1, -(b_1 + 2v_1 \alpha x), \alpha g(t))
\] (8)

with \(t = y + b_1 x + v_1 \alpha x^2\) and \(v_1 \alpha \neq 0\), and \(g \in \mathbb{R}[t]\) with \(g(0) = 0\) and \(\deg_t g(t) \geq 1\).

(2) Any map
\[
F = \lambda I + (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z))) \in \mathcal{N}(\lambda, 3)
\]
under the condition \(\deg_z(uA) \neq \deg_z(vB)\) is injective.

(3) For \(n = 3\), the map \(F_{3,2}\) of Theorem 3.2 up to a linear change of coordinates is of the form (8) with \(g(t)\) a polynomial of degree one. Therefore, the origin is not a globally asymptotically singularity (resp. fixed point) for the continuous (resp. discrete) dynamical system generated by \(F_{3,2}\).
In consequence we establish the following:

**Question 2** Do there exist linearly independent maps in $\mathcal{N}(\lambda, 3)$ of the form (8) for which the origin is globally asymptotically stable for the corresponding continuous and/or discrete system?

3.1 The Continuous Case

Our next result gives a negative answer to Question 2, in the continuous case, when the degree of $g(t)$ is less than or equal to two. First observe that, by applying the coordinate change $(u, v, w) = \phi(x, y, z) = (\lambda x + g(t), t, v_1 z + \lambda v_1 x^2)$ where $t = y + b_1 x + v_1 x^2$, to the vector field (8) we obtain

$$\phi^*(F)(u, v, w) = \lambda (u, v, w) + (g'(v)(\lambda v + w), w, \alpha v_1 u^2).$$

(9)

**Theorem 3.5** Consider a map $F \in \mathcal{N}(\lambda, 3)$, with $\lambda < 0$, of the form (8) where $g(t) = A_1 t + A_2 \frac{t^2}{2}$. Then $F$ has orbits that escape to infinity.

**Proof** In the case $A_2 = 0$, making the linear change of coordinates

$$(u, v, w) = \phi(x, y, z) = \frac{1}{m}(x, my, mv_1 z)$$

with $m = A_1$, the vector field $F - \lambda I$ has the form (4). The result now follows from Theorem 3.2.

Next consider the case $A_2 \neq 0$. Then we may assume

$$F(x, y, z) = \lambda(x, y, z) + (g'(y)(\lambda y + z), z, v_1 x^2).$$

To find orbits of $F$ that escape to infinity, we first make the coordinate change

$$(u, v, w) = \frac{1}{z}(x, y, 1).$$

If $Y$ is the vector field $F$ in the new coordinates, then $Z = w Y$ is defined by

$$Z(u, v, w) = (-\beta u^3 + (A_1 w + A_2 v)(\lambda v + 1), -\beta u^2 v + w, -w(\lambda w + \beta u^2))$$

with $\beta = v_1 \alpha$. For $w \neq 0$, the vector fields $Y$ and $Z$ have the same orbits. Moreover, for $w > 0$ (resp. $w < 0$), the orbits of $Y$ and $Z$ have the same (resp. inverse) orientation.
Then we apply the blow-up 

\[(s, q, p) = (u, \frac{v}{u^3}, \frac{w}{u^5})\].

If \(Y_1\) is the vector field \(Y\) in the new coordinates, then 

\[Y_1(s, q, p) = A(s, q, p) (s, -3q, -5p) + (0, p - \beta q, -p(\beta + \lambda ps^3))\]

with \(A(s, q, p) = -\beta + (A_1 ps^2 + A_2 q)(\lambda q s^3 + 1)\).

The singularities of \(Z_1\) over \(s = 0\) are 

\[(0, 0, 0), \left(0, \frac{2\beta}{3A_2}, 0\right), \text{ and } \left(0, \frac{4\beta}{5A_2}, \frac{8\beta^2}{25A_2}\right)\].

The Jacobian matrix of \(Z_1\) at \(0, \frac{4\beta}{5A_2}, \frac{8\beta^2}{25A_2}\) has eigenvalues 

\[\mu_1 = -\frac{\beta}{5}, \mu_2 = -\frac{2\beta}{5}, \text{ and } \mu_3 = -2\beta.\]

In the case \(\beta > 0\) (resp. \(\beta < 0\)), this singularity is an attractor (resp. repeller) of the vector field \(Z_1\). Given an initial condition \((s(0), q(0), p(0))\) sufficiently close to the singularity, with \(s(0)p(0) > 0\) (resp. \(s(0)p(0) < 0\)) for \(\beta > 0\) (resp. \(\beta < 0\)), we obtain an orbit of the original vector field \(F\) that escapes positively to infinity. \(\Box\)

### 3.2 The Discrete Case

In this subsection we prove that, in the discrete case, the answer to Question 2 is negative for any \(g(t) \in \mathbb{R}[t]\) with \(g(0) = 0\) and \(\deg g(t) \geq 1\).

For \(|\lambda| < 1\), consider 

\[F(x, y, z) = \lambda (x, y, z) + (0, v_1 z, 0) + g(t) (1, -(b_1 + 2v_1 \alpha x), \alpha g(t))\]  \(\tag{10}\)

with \(t = y + b_1 x + v_1 \alpha x^2\) and \(v_1 \alpha \neq 0\), and \(g(t) \in \mathbb{R}[t]\) with \(g(0) = 0\) and \(\deg g(t) \geq 1\).

**Lemma 3.6** The set of fixed points of \(F\) is reduced to the origin. If \(g(t) = At\) then the unique periodic point of period two of \(F\) is the origin.

**Lemma 3.7** If \(-1 < \lambda < 1\) and \(g(t) = At\), with \(A \neq 0\), then \(F\) has a periodic point of period three \((x_0, y_0, z_0) \neq (0, 0, 0)\). Furthermore, the eigenvalues of \(DF^3(x_0, y_0, z_0)\) are all different from 1.
Proof Calculations involve MATHEMATICA prove that the point \((x_0, y_0, z_0)\) with
\[
x_0 = \frac{(1 + \lambda + \lambda^2)(1 + 4\lambda^2 + \lambda^4)}{A \beta (1 - \lambda)^3},
\]
\[
y_0 = -\frac{1 + \lambda + \lambda^2}{A^2 \beta (1 - \lambda)^6} [\lambda (1 + \lambda + \lambda^2) (4 + \lambda + 8\lambda^2 + 11\lambda^3 + 4\lambda^4 + 7\lambda^5 + \lambda^7) + A b_1 (1 - \lambda)^3 (1 + 4\lambda^2 + \lambda^4)]
\]
\[
z_0 = \frac{(1 + \lambda + \lambda^2)^3 (1 + 3\lambda^2 + 4\lambda^3 + 3\lambda^4 + \lambda^6)}{v_1 A^2 \beta (1 - \lambda)^5},
\]
where \(\beta = v_1 \alpha\), is a periodic point of period three of \(F\). On the other hand, we prove that the characteristic polynomial of \(DF^3(x_0, y_0, z_0)\) is
\[
p(x) = -\lambda^9 - \lambda (8 + 44\lambda + 104\lambda^2 + 164\lambda^3 + 164\lambda^4 + 113\lambda^5 + 44\lambda^6 + 8\lambda^7 - 4\lambda^8) x + (-4 + 8\lambda + 44\lambda^2 + 113\lambda^3 + 164\lambda^4 + 164\lambda^5 + 104\lambda^6 + 44\lambda^7 + 8\lambda^8) x^2 + x^3.
\]
and
\[
p(1) = 3 (\lambda - 1)^3 (1 + \lambda + \lambda^2)^3 \neq 0.
\]
\[
\square
\]

Theorem 3.8 For \(|\lambda| < 1\), consider
\[
F(x, y, z) = \lambda (x, y, z) + (0, v_1 z, 0)
\]
\[
+ g(t) (1, -(b_1 + 2v_1 \alpha x), \alpha g(t))
\]
with \(t = y + b_1 x + v_1 \alpha x^2\) and \(v_1 \alpha \neq 0\), and \(g(t) \in \mathbb{R}[t]\) with \(g(0) = 0\) and \(g'(0) \neq 0\). Then there exists \((x_0, y_0, z_0) \neq (0, 0, 0)\) which is a periodic point of period 3 of \(F\).

Proof Assume \(g(t) = At + A_2 t^2 + \cdots + A_k t^k\), with \(A \neq 0\). When \((A_2, \ldots, A_k) = (0, \ldots, 0)\) we denote the corresponding map \(F\) by \(F_0\). Therefore, \(F_0\) has a periodic point of period three \((x_0, y_0, z_0) \neq (0, 0, 0)\) and the eigenvalues of \(DF_0^3(x_0, y_0, z_0)\) are all different from 1. Consider the map \(G : \mathbb{R}^{k-1} \times \mathbb{R}^3 \to \mathbb{R}^3\) defined by
\[
G(A_2, \ldots, A_k, x, y, z) = F^3(x, y, z) - (x, y, z).
\]
Observe that \(G(0, \ldots, 0, x, y, z) = F_0^3(x, y, z) - (x, y, z)\), for all \((x, y, z) \in \mathbb{R}^3\). Then \(G(0, \ldots, 0, x_0, y_0, z_0) = (0, 0, 0)\) and \(D_2 G(0, \ldots, 0, x_0, y_0, z_0)\) is invertible. From implicit function theorem, there exist \(\varepsilon > 0\) such that, for all \((A_2, \ldots, A_k)\) with \(\max \{|A_2|, \ldots, |A_k|\} < \varepsilon\) the map \(F(x, y, z)\) has a periodic point of period three. For the general case, observe that if \(a \in \mathbb{R} - \{0\}\) and \(T(x, y, z) = a^{-1}(x, y, z)\), then
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\[ T(F(T^{-1}(u, v, w)) = \lambda(u, v, w) + (0, v_1 w, 0) + \tilde{g}(t)(1, -(b_1 + 2v_1\tilde{\alpha} u, \tilde{\alpha} \tilde{g}(t)) \]

with \( \tilde{\alpha} = \alpha a \) and

\[ \tilde{g}(t) = a^{-1} g(at) = At + A_2 a^2 t^2 + \cdots + A_k a^{k-1} t^k. \]

For \(|a|\) sufficiently small, the map \( T \circ F \circ T^{-1} \) has a non vanished periodic point of period three, and then, the map \( F \) also. \hfill \Box

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