CHERN OBSTRUCTIONS FOR COLLECTIONS OF 1-FORMS ON SINGULAR VARIETIES

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Dedicated to Jean-Paul Brasselet on the occasion of his 60th birthday

Abstract. We introduce a certain index of a collection of germs of 1-forms on a germ of a singular variety which is a generalization of the local Euler obstruction corresponding to Chern numbers different from the top one.

Introduction

The aim of this paper is to bring together some ideas of [3] and [6]. A germ of a vector field or of a 1-form on the complex affine space \( \mathbb{C}^n \) at the origin not vanishing in a punctured neighbourhood of it has a topological invariant — the Poincaré–Hopf index. The sum of the Poincaré–Hopf indices of the singular points of a vector field on a compact complex manifold is equal to the Euler characteristic of the manifold. There are several generalizations of this notion to vector fields and/or to 1-forms on complex analytic varieties with singularities (isolated or not) started by M.-H. Schwartz: [11, 1, 5, 12, 8, 2, 3, ...]. For the case of an isolated complete intersection singularity there is defined an index which is sometimes called the GSV index: [8, 12, 5]. Another generalization which makes sense not only for isolated complete intersection singularities and also not only for varieties with isolated singularities is the so called local Euler obstruction: [2, 3] (its analogue for 1-forms was considered in [7]). One can say that in some sense all these invariants correspond to the Euler characteristic, which, for a compact complex analytic manifold \( M^n \), coincides with the top Chern number \( \langle c_n(M), [M] \rangle \).

A generalization of the GSV-index corresponding to other Chern numbers (different from the top one) was introduced and studied in [6]. It is defined for a collection of germs of 1-forms on an isolated

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complete intersection singularity. For a collection of 1-forms on a projective complex complete intersection with isolated singularities, the sum of these indices of the singular points is equal to plus-minus the corresponding Chern number of a smoothing of the variety.

Here we define and study an index of a collection of germs of 1-forms on a germ of a singular variety which is an analogue of the local Euler obstruction corresponding to a Chern number different from the top one.

1. Special points of 1-forms

Let \((X^n, 0) \subset (\mathbb{C}^N, 0)\) be the germ of a purely \(n\)-dimensional reduced complex analytic variety at the origin (generally speaking with a non-isolated singularity). Let \(k = \{k_i\}, \ i = 1, \ldots, s\), be a fixed partition of \(n\) (i.e., \(k_i\) are positive integers, \(\sum k_i = n\)). Let \(\{\omega_j^{(i)}\} (i = 1, \ldots, s, \ j = 1, \ldots, n - k_i + 1)\) be a collection of germs of 1-forms on \((\mathbb{C}^N, 0)\) (not necessarily complex analytic; it suffices that the forms \(\omega_j^{(i)}\) are complex linear functions continuously depending on a point of \(\mathbb{C}^N\)). Let \(\varepsilon > 0\) be small enough so that there is a representative \(X\) of the germ \((X, 0)\) and representatives \(\omega_j^{(i)}\) of the germs of 1-forms inside the ball \(B_{\varepsilon}(0) \subset \mathbb{C}^N\).

Definition: A point \(P \in X\) is called a special point of the collection \(\{\omega_j^{(i)}\}\) of 1-forms on the variety \(X\) if there exists a sequence \(\{P_m\}\) of points from the non-singular part \(X_{\text{reg}}\) of the variety \(X\) converging to \(P\) such that the sequence \(T_{P_m}X_{\text{reg}}\) of the tangent spaces at the points \(P_m\) has a limit \(L\) as \(m \to \infty\) (in the Grassmann manifold of \(n\)-dimensional vector subspaces of \(\mathbb{C}^N\)) and the restrictions of the 1-forms \(\omega_1^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)}\) to the subspace \(L \subset T_P\mathbb{C}^N\) are linearly dependent for each \(i = 1, \ldots, s\).

Definition: The collection \(\{\omega_j^{(i)}\}\) of 1-forms has an isolated special point on the germ \((X, 0)\) if it has no special points on \(X\) in a punctured neighbourhood of the origin.

Remarks. 1. If the 1-forms \(\omega_j^{(i)}\) are complex analytic, the property to have an isolated special point is a condition on the classes of these 1-forms in the module

\[\Omega^1_{X,0} = \Omega^1_{\mathbb{C}^N,0}/\{f \cdot \Omega^1_{\mathbb{C}^N,0} + df \cdot \mathcal{O}_{\mathbb{C}^N,0} | f \in \mathcal{J}_X\}\]

of germs of 1-forms on the variety \(X\) (\(\mathcal{J}_X\) is the ideal of germs of holomorphic functions vanishing on \(X\)).
2. For the case $s = 1$ (and therefore $k_1 = n$), i.e. for one 1-form $\omega$, there exists a notion of a singular point of the 1-form $\omega$ on $X$ (see, e.g., [7]). It is defined in terms of a Whitney stratification of the variety $X$. A point $x \in X$ is a singular point of the 1-form $\omega$ on the variety $X$ if the restriction of the 1-form $\omega$ to the stratum of $X$ containing $x$ is equal to zero at the point $x$. (One should consider points of all zero-dimensional strata as singular ones.) One can easily see that a special point of the 1-form $\omega$ on the variety $X$ is singular, but not vice versa. (E.g. the origin is a singular point of the 1-form $dx$ on the cone $\{x^2 + y^2 + z^2 = 0\}$, but not a special one.) On a smooth variety these two notions coincide.

The notion of a non-degenerate special (singular) point of a collection of germs of 1-forms on a smooth variety was introduced in [6]. The index of a non-degenerate point of a collection of germs of holomorphic 1-forms is equal to 1.

Let

$$\mathcal{L}^k = \prod_{i=1}^{s} \prod_{j=1}^{n-k_i+1} \mathbb{C}_{ij}^N$$

be the space of collections of linear functions on $\mathbb{C}^N$ (i.e. of 1-forms with constant coefficients).

**Proposition 1.** There exists an open and dense subset $U \subset \mathcal{L}^k$ such that each collection $\{\ell_j^{(i)}\} \in U$ has only isolated special points on $X$ and, moreover, all these points belong to the smooth part $X_{\text{reg}}$ of the variety $X$ and are non-degenerate.

**Proof.** Let $Y \subset X \times \mathcal{L}^k$ be the closure of the set of pairs $(x, \{\ell_j^{(i)}\})$ where $x \in X_{\text{reg}}$ and the restrictions of the linear functions $\ell_1^{(i)}$, \ldots, $\ell_{n-k_i+1}^{(i)}$ to the tangent space $T_x X_{\text{reg}}$ are linearly dependent for each $i = 1, \ldots, s$. Let $\pi : Y \to \mathcal{L}^k$ be the projection to the second factor. One has codim $Y = \sum_{i=1}^{s} k_i = n$ and therefore dim $Y = \dim \mathcal{L}^k$. Moreover, $Y \setminus (X_{\text{reg}} \times \mathcal{L}^k)$ is a proper subvariety of $Y$ and therefore its dimension is strictly smaller than $\dim \mathcal{L}^k$. A generic point of the space $\mathcal{L}^k$ is a regular value of the map $\pi$ which means that it has only finitely many preimages, all of them belong to $X_{\text{reg}} \times \mathcal{L}^k$ and the map $\pi$ is non-degenerate at them. This implies the statement. \(\square\)

**Corollary 1.** Let $\{\omega_j^{(i)}\}$ be a collection of 1-forms on $X$ with an isolated special point at the origin. Then there exists a deformation $\{\tilde{\omega}_j^{(i)}\}$ of the collection $\{\omega_j^{(i)}\}$ whose special points lie in $X_{\text{reg}}$ and are non-degenerate.
Moreover, as such a deformation one can use \( \{ \omega_j^{(i)} + \lambda \ell_j^{(i)} \} \) with a generic collection \( \{ \ell_j^{(i)} \} \in \mathcal{L}^k, \lambda \neq 0 \) small enough.

**Corollary 2.** The set of collections of holomorphic 1-forms with a non-isolated special point at the origin has infinite codimension in the space of all holomorphic collections.

## 2. Local Chern obstructions

Let \( \{ \omega_j^{(i)} \} \) be a collection of germs of 1-forms on \((X,0)\) with an isolated special point at the origin. Let \( \nu : \hat{X} \to X \) be the Nash transformation of the variety \( X \subset B_c(0) \) defined as follows. Let \( G(n, N) \) be the Grassmann manifold of \( n \)-dimensional vector subspaces of \( \mathbb{C}^N \). There is a natural map \( \sigma : X_{\text{reg}} \to B_c(0) \times G(n, N) \) which sends a point \( x \in X_{\text{reg}} \) to \((x, T_x X_{\text{reg}})\). The Nash transform \( \hat{X} \) of the variety \( X \) is the closure of the image \( \text{Im} \sigma \) of the map \( \sigma \) in \( B_c(0) \times G(n, N) \), \( \nu \) is the natural projection. The Nash bundle \( \hat{T} \) over \( \hat{X} \) is a vector bundle of rank \( n \) which is the pullback of the tautological bundle on the Grassmann manifold \( G(n, N) \). There is a natural lifting of the Nash transformation to a bundle map from the Nash bundle \( \hat{T} \) to the restriction of the tangent bundle \( TC^N \) of \( \mathbb{C}^N \) to \( X \). This is an isomorphism of \( \hat{T} \) and \( TX_{\text{reg}} \subset TC^N \) over the non-singular part \( X_{\text{reg}} \) of \( X \).

The collection of 1-forms \( \{ \omega_j^{(i)} \} \) gives rise to a section \( \hat{\omega} \) of the bundle

\[
\hat{T} = \bigoplus_{i=1}^s \bigoplus_{j=1}^{n-k_i+1} \hat{T}^*_x \]

where \( \hat{T}^*_x \) are copies of the dual Nash bundle \( \hat{T}^* \) over the Nash transform \( \hat{X} \) numbered by indices \( i \) and \( j \). Let \( \hat{D} \subset \hat{T} \) be the set of pairs \((x, \{ \alpha_j^{(i)} \})\) where \( x \in \hat{X} \) and the collection \( \{ \alpha_j^{(i)} \} \) of elements of \( \hat{T}^*_x \) (i.e. of linear functions on \( \hat{T}_x \)) is such that \( \alpha_1^{(i)}, \ldots, \alpha_{n-k_i+1}^{(i)} \) are linearly dependent for each \( i = 1, \ldots, s \). The image of the section \( \hat{\omega} \) does not intersect \( \hat{D} \) outside of the preimage \( \nu^{-1}(0) \subset \hat{X} \) of the origin. The map \( \hat{T} \setminus \hat{D} \to \hat{X} \) is a fibre bundle. The fibre \( W_x = \hat{T}_x \setminus \hat{D}_x \) of it is \((2n-2)\)-connected, its homology group \( H_{2n-1}(W_x; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \) and has a natural generator: see, e.g., [3]. The latter fact implies that the fibre bundle \( \hat{T} \setminus \hat{D} \to \hat{X} \) is homotopically simple in dimension \( 2n - 1 \), i.e. the fundamental group \( \pi_1(\hat{X}) \) of the base acts trivially on the homotopy group \( \pi_{2n-1}(W_x) \) of the fibre, the last one being isomorphic to the homology group \( H_{2n-1}(W_x; \mathbb{Z}) \): see, e.g., [4].
Definition: The local Chern obstruction \( \text{Ch}_{X,0} \{ \omega_j^{(i)} \} \) of the collections of germs of 1-forms \( \{ \omega_j^{(i)} \} \) on \( (X, 0) \) at the origin is the (primary, and in fact the only) obstruction to extend the section \( \check{\omega} \) of the fibre bundle \( \hat{T} \backslash \hat{D} \to \hat{X} \) from the preimage of a neighbourhood of the sphere \( S_\veps = \partial B_\veps \) to \( \hat{X} \), more precisely its value (as an element of \( H^{2n}(\nu^{-1}(X \cap B_\veps), \nu^{-1}(X \cap S_\veps); \mathbb{Z}) \)) on the fundamental class of the pair \( (\nu^{-1}(X \cap B_\veps), \nu^{-1}(X \cap S_\veps)) \).

The definition of the local Chern obstruction \( \text{Ch}_{X,0} \{ \omega_j^{(i)} \} \) can be reformulated in the following way. Let \( \mathcal{D}_X^k \subset \mathbb{C}^N \times \mathcal{L}^k \) be the closure of the set of pairs \( (x, \{ \ell_j^{(i)} \}) \) such that \( x \in X_{\text{reg}} \) and the restrictions of the linear functions \( \ell_1^{(i)}, \ldots, \ell_{n-k_i+1}^{(i)} \) to \( T_x X_{\text{reg}} \subset \mathbb{C}^N \) are linearly dependent for each \( i = 1, \ldots, s \). (For \( s = 1 \), \( k = \{ n \} \), \( \mathcal{D}_X^k \) is the (non-projectivized) conormal space of \( X \) [15].) The collection \( \{ \omega_j^{(i)} \} \) of germs of 1-forms on \( (\mathbb{C}^N, 0) \) defines a section \( \check{\omega} \) of the (trivial) fibre bundle \( \mathbb{C}^N \times \mathcal{L}^k \to \mathbb{C}^N \). Then

\[
\text{Ch}_{X,0} \{ \omega_j^{(i)} \} = (\check{\omega}(\mathbb{C}^N) \circ \mathcal{D}_X^k)_0
\]

where \( (\cdot \circ \cdot)_0 \) is the intersection number at the origin in \( \mathbb{C}^N \times \mathcal{L}^k \). This description can be considered as a generalization of an expression of the local Euler obstruction as a microlocal intersection number defined in [9], see also [10, Sections 5.0.3 and 5.2.1].

Remarks. 1. On a smooth manifold \( X \) the local Chern obstruction \( \text{Ch}_{X,0} \{ \omega_j^{(i)} \} \) coincides with the index \( \text{ind}_{X,0} \{ \omega_j^{(i)} \} \) of the collection \( \{ \omega_j^{(i)} \} \) defined in [6].
2. The local Euler obstruction is defined for vector fields as well as for 1-forms. One can see that vector fields are not well adapted to a definition of the local Chern obstruction. A more or less direct version of the definition above for vector fields demands to consider vector fields on a singular variety \( X \subset \mathbb{C}^N \) to be sections \( v = v(x) \) of \( T\mathbb{C}^N|_X \) such that \( v(x) \in T_x X \subset T_x \mathbb{C}^N \) (\( \dim T_x X \) is not constant). (Traditionally vector fields tangent to smooth strata of the variety \( X \) are considered.) There exist only continuous (non-trivial, i.e. with \( s > 1 \)) collections of such vector fields "on \( X \" with isolated special points, but not holomorphic ones.
3. The definition of the local Chern obstruction \( \text{Ch}_{X,0} \{ \omega_j^{(i)} \} \) may also be formulated in terms of a collection \( \{ \omega^{(i)} \} \) of germs of 1-forms with values in vector spaces \( L_i \) of dimensions \( n - k_i + 1 \). Therefore (via differentials) it is also defined for a collection \( \{ f^{(i)} \} \) of germs of maps
\[ f^{(i)} : (\mathbb{C}^N, 0) \to (\mathbb{C}^{n-k+1}, 0) \] (just as the Euler obstruction is defined for a germ of a function).

Being a (primary) obstruction, the local Chern obstruction satisfies the law of conservation of number, i.e. if a collection of 1-forms \( \{\tilde{\omega}_j^{(i)}\} \) is a deformation of the collection \( \{\omega_j^{(i)}\} \) and has isolated special points on \( X \), then

\[
Ch_{X,0} \{\omega_j^{(i)}\} = \sum Ch_{X,Q} \{\tilde{\omega}_j^{(i)}\}
\]

where the sum on the right hand side is over all special points \( Q \) of the collection \( \{\tilde{\omega}_j^{(i)}\} \) on \( X \) in a neighbourhood of the origin. With Corollary \(^{[1]}\) this implies the following statements.

**Proposition 2.** The local Chern obstruction \( Ch_{X,0} \{\omega_j^{(i)}\} \) of a collection \( \{\omega_j^{(i)}\} \) of germs of holomorphic 1-forms is equal to the number of special points on \( X \) of a generic (holomorphic) deformation of the collection.

This statement is an analogue of Proposition 2.3 in \(^{[13]}\).

**Proposition 3.** If a collection \( \{\omega_j^{(i)}\} \) of 1-forms on a compact (say, projective) variety \( X \) has only isolated special points, then the sum of the local Chern obstructions of the collection \( \{\omega_j^{(i)}\} \) at these points does not depend on the collection and therefore is an invariant of the variety.

It is reasonable to consider this sum as \( ((-1)^n \text{ times}) \) the corresponding Chern number of the singular variety \( X \).

Let \( (X, 0) \) be an isolated complete intersection singularity. As it was mentioned above, a collection of germs of 1-forms \( \{\omega_j^{(i)}\} \) on \( (X, 0) \) with an isolated special point at the origin has an index \( \text{ind}_{X,0} \{\omega_j^{(i)}\} \) which is an analogue of the GSV–index of a vector field: \(^{[5]}\). The fact that both the Chern obstruction and the index satisfy the law of conservation of number and they coincide on a smooth manifold yields the following statement.

**Proposition 4.** For a collection \( \{\omega_j^{(i)}\} \) of germs of 1-forms on an isolated complete intersection singularity \( (X, 0) \) the difference

\[
\text{ind}_{X,0} \{\omega_j^{(i)}\} - Ch_{X,0} \{\omega_j^{(i)}\}
\]

does not depend on the collection and therefore is an invariant of the germ of the variety.

Since, by Proposition \(^{[1]}\) \( Ch_{X,0} \{\ell_j^{(i)}\} = 0 \) for a generic collection \( \{\ell_j^{(i)}\} \) of linear functions on \( \mathbb{C}^N \), one has the following statement.
Corollary 3. One has
\[ \text{Ch}_{X,0} \{ \omega_j^{(i)} \} = \text{ind}_{X,0} \{ \omega_j^{(i)} \} - \text{ind}_{X,0} \{ \ell_j^{(i)} \} \]
for a generic collection \( \{ \ell_j^{(i)} \} \) of linear functions on \( \mathbb{C}^N \).

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