THE TURBULENT MAGNETIC PRANDTL NUMBER OF MHD TURBULENCE IN DISKS

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1. INTRODUCTION

Astrophysical disk evolution may be controlled in part by magnetic fields that are coherent over scales of order the radius \( R \). Large-scale fields are an essential element of theoretical models for the launching and collimation of disk winds (e.g., Blandford & Payne 1982; for a recent review see Pudritz et al. 2007), which lead to disk evolution because they exert direct torques on the surface of the disk. Large-scale fields may also control the strength of turbulent angular momentum diffusion (in dimensionless form, \( \alpha \)) within the disk, since numerical experiments in unstratified “shearing boxes” suggest that \( \alpha \) is proportional to the mean field strength (e.g., Hawley et al. 1995). So what determines the strength of the large-scale field in well ionized disks?

A few of the physical processes affecting the large-scale field strength in thin disks can be easily listed: (1) advection of flux by large-scale flows in the disk (e.g., inward advection by accretion, but also possibly meridional circulation); (2) turbulent diffusion; (3) dynamical interaction between large-scale fields and large-scale flows (e.g., enhanced accretion due to external torques); (4) control of small scale MHD turbulence by large-scale fields (e.g., through the influence of a mean field on \( \alpha \)); (5) generation of large-scale fields by small scale MHD turbulence (a large-scale dynamo); (6) introduction or removal of magnetic flux at the disk boundaries. These processes are difficult to study numerically because they involve nonsteady flows and a large dynamic range in length scale \( (R: \text{scale height } H) \) and timescale (viscous timescale: dynamical timescale).

A starting point for understanding large-scale field evolution is the phenomenological model of Van Ballegooijen (1989; hereafter VB89). He considers a passive large-scale field advected inward by accretion and diffused by turbulence in the disk (processes 1 and 2 above). Turbulence is modeled using a turbulent viscosity \( \nu_T \) and turbulent resistivity \( \eta_T \). The evolution of the poloidal field is then governed by the induction equation in vector potential form:

\[
\partial_t A_\phi = \eta_T \partial_R \left[ \frac{1}{R} \partial_R (RA_\phi) \right] + \eta_T \partial_z^2 A_\phi + \nu_T \partial_R (RA_\phi),
\]

where \( A_\phi \) is the azimuthal component of the vector potential. \( A_\phi \) labels field lines; when \( \partial_z A_\phi \neq 0 \) the field lines move radially through the disk with speed \( \sim \partial_z A_\phi / \partial_t A_\phi \). In the limit that \( \eta_T \to 0 \) a vertical field would be advected inward by accretion and vertical field strength would increase with time. In the limit that \( \nu \to 0 \) the field lines simply diffuse out of the disk. Where do advection and diffusion balance?

VB89 gives a surprising answer that can be understood as follows, assuming the field and disk are symmetric about the midplane. At the midplane the first term in Equation (1); radial diffusion of vertical field) is \( \sim \eta_T B_z / R \), where \( B_z = (1/R) \partial_z (RA_\phi) \) and we assume that \( \eta_T \sim 1/R \). The second term (vertical diffusion of radial field, which can nevertheless cause radial motion of field lines) depends on the field geometry above and below the disk. If we assume the field lines enter and exit the disk at an angle of order unity (as in the wind model of Blandford & Payne 1982), \( B \sim -\partial_z A_\phi \sim \pm B_z \) at \( z = \pm H \) and \( B \sim 0 \) at the midplane, by symmetry. Then \( \eta_T \partial_z^2 A_\phi \sim \eta_T B_z / H \). Using the usual viscous disk estimate \( \nu_R \sim \nu / R \), the final term (field advection) is of order \( \nu B_z / R \). To summarize, the terms on the right-hand side of (1) are in the ratio \( \eta_T B_z / R : \eta_T B_z / H : \nu B_z / R \). Evidently the first term is negligible in comparison to the second for a thin disk, provided the turbulent diffusion can be described by a scalar diffusion coefficient. The second and third can balance when \( \nu T / \eta T \sim R / H \); diffusion and advection balance when \( \nu T / \eta T \sim R / H \).

It is plausible that in disk turbulence \( \eta T \sim 1 \) (e.g., Yousef et al. 2003). Then outward field diffusion occurs on a timescale \( RH / \nu \). Large-scale poloidal fields would vanish from the disk absent a dynamo that regenerates the field on the same timescale (process 5 above). Similar conclusions have been reached by Lubow et al. (1994) in the context of magnetically generated outflow/jet models. More complex models by Heyvaerts et al. (1996) also support such a picture.

There are of course ways to avoid the loss of large-scale field implied by the VB89 model, which is based on a purely phenomenological model for evolution of the disk and field. Spruit & Uzdensky (2005) discuss a model that reduces turbulent diffusion by grouping large-scale vertical magnetic fields into bundles in the disk through flux expulsion. In these bundles the fields are strong enough to quench turbulence and thus avoid outward

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\( T \) controls the rate of escape of vertical field from the disk; for \( \nu T \sim 1 \).

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\( T, \) by studying the evolution of a sinusoidal perturbation in the magnetic field that is injected into a turbulent background. We show that the perturbation is always stable, decays approximately exponentially, has decay rate \( \propto k^2 \), and that the implied \( \eta T \sim 1 \).
diffusion (processes 3 and 4 above). It has also been suggested by Uzdensky & Goodman (2008) that disk atmospheres might develop loop-like large-scale coronal structures that delocalize disk evolution by transmitting angular momentum and energy. Rothstein & Lovelace (2008) consider the possibility that field diffusion is suppressed by a coronal layer where \( v_{A} > c_s \), so that the MRI is suppressed and \( η_{T} \) is reduced. Another possibility is that disk evolution is driven by external torques associated with an MHD (Blandford-Payne type) wind running along the poloidal field lines. If the field is strong enough then the inflow speed \( v_{A} \) may be large enough to compete with outward diffusion of field lines even when \( Pr_{M,T} \sim 1 \) (process 3 above).\(^2\) A final possibility is that \( Pr_{M,T} > 1 \).

In this paper, we measure the turbulent magnetic Prandtl number \( Pr_{M,T} \) directly from shearing box simulations. We infer the turbulent viscosity \( ν_T \) from the turbulent shear stress \( w_{xy,T} \) (in dimensionless form, \( α \)), which controls diffusive radial transport of angular momentum in disks. We infer the turbulent resistivity \( η_{T} \) by tracking the evolution of a sinusoidal disturbance in the magnetic field that is imposed on an already turbulent state.

It is worth emphasizing that the turbulent \( Pr_{M,T} \) is fundamentally different from the Prandtl number \( Pr_{M} \) associated with microscopic processes (e.g., Balbus & Henri 2008). Recent numerical experiments (Fromang & Papaloizou 2007; Lesur & Longaretti 2007) suggest that \( Pr_{M} \) can influence radial transport of angular momentum in disks. We infer \( Pr_{M,T} \) can influence the saturation level of MRI-driven disk turbulence at low Reynolds number.

The paper is organized as follows. In Section 2, we give a simple description of the local model and summarize our numerical algorithm. In Section 3 we describe the numerical procedure for measuring \( η_{T} \). We report \( η_{T} \) and discuss its dependence on the model parameters. In Section 4, we explain how we calculate \( Pr_{M,T} \) and discuss our results.

2. LOCAL MODEL AND NUMERICAL METHODS

Our starting point is the local model for disks. It is obtained by expanding the equations of motion around a circular-orbiting coordinate origin at cylindrical coordinates \( (r, \phi, z) = (r_0, \Omega t + \phi_0, 0) \), assuming that the peculiar velocities are comparable to the sound speed and that the sound speed is small compared to the orbital speed. The local Cartesian coordinates are obtained from cylindrical coordinates via \( (x, y, z) = (r - r_0, r_0 \phi - \Omega d t - \phi_0, z) \). We assume throughout that the disk is isothermal \( (\rho = c_s^2 \rho_0, \text{where } c_s \text{ is constant}) \), and that the disk orbits in a Keplerian \((1/r) \text{ potential})\.

In the local model the momentum equation of ideal MHD becomes

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + c_s^2 \nabla \rho \rho + \nabla B^2 \left\lfloor \frac{4B \cdot \nabla}{4\pi \rho} + \frac{(B \cdot \nabla) B}{4\pi \rho} \right\rfloor + 2\Omega \mathbf{x} \times \nabla \rho - 3\Omega^2 x \hat{y} = 0. \tag{2}
\]

The final two terms in Equation (2) represent the Coriolis and tidal forces in the local frame. Our model is unstratified, which means that the vertical gravitational acceleration \(-\Omega^2 z\) usually present in Keplerian disks is ignored. The box has size \( L_x \times L_y \times L_z \).

Our box contains no explicit dissipation coefficients. Recent models with explicit scalar dissipation (Fromang & Papaloizou 2007; Lesur & Longaretti 2007) have shown that the saturated field strength in magnetized disk turbulence depends on the viscosity \( ν \) and resistivity \( η \), and that \( \text{ZEUS} \) has an effective magnetic Prandtl number \( Pr_{M,T} \equiv ν/η \gtrsim 1 \) (Fromang & Papaloizou 2007).

The orbital velocity in the local model is

\[
ν_{orb} = -\frac{1}{2} \Omega x \hat{y}. \tag{3}
\]

This velocity, along with a constant density and zero magnetic field, is a steady-state solution to Equation (2). If the computational domain extends to \( |x| > (2/3)H = (2/3)c_s/Ω \), then the orbital speed is supersonic with respect to the grid.

The local model can be studied numerically using the “shearing box” boundary conditions (e.g., Hawley et al. 1995). The boundary conditions on the \( y \) boundaries of the box are periodic, while the \( x \) boundaries are “nearly periodic,” i.e., they connect the radial boundaries in a time-dependent way that enforces the mean shear flow. We also use periodic boundary conditions in the vertical direction; this is the simplest possible version of the shearing box model.

Our models are evolved using ZEUS (Stone & Norman 1992). ZEUS is an operator-split, finite difference scheme on a staggered mesh. It uses artificial viscosity to capture shocks.\(^3\) For the magnetic field evolution ZEUS uses the Method of Characteristics-Constrained Transport (MOC-CT) scheme, which is designed to accurately evolve Alfvén waves (MOC) and to preserve \( \nabla \cdot B = 0 \) constraint to machine precision (CT).

We have modified ZEUS to include “orbital advection” (Masset 2000; Gammie 2001; Johnson & Gammie 2005) with a magnetic field (Johnson et al. 2008). Advection by the orbital component of the velocity \( ν_{orb} \) (which may be supersonic with respect to the grid) is done using interpolation. With this modification the time step condition \( Δt < CΔx/(|\partial x| + c_{max}) \) (\( c_{max} \equiv \text{maximum wave speed and } \text{Courant number}) \) depends only on the perturbed velocity \( \delta v = v - ν_{orb} \) rather than \( v \). So when \( |ν_{orb}| > c_{max} \) (for shearing box models with \( v_{A}^2/c_s^2 \lesssim 1 \), when \( L \gtrsim H \) the time step can be larger with orbital advection, and computational efficiency is improved.

Orbital advection also improves accuracy. ZEUS, like most Eulerian schemes, has a truncation error that increases as the speed of the fluid increases in the grid frame. In the shearing box without orbital advection the truncation error would then increase monotonically with \( |x| \). Orbital advection reduces the amplitude of the truncation error and also makes it more nearly uniform in \( |x| \) (Johnson et al. 2008).

We have also implemented an additional procedure to remove the radially dependent numerical dissipation in large shearing box simulations that was reported in Johnson et al. (2008). We do so by systematically shifting the entire box by a few grid points in the radial direction at \( t = n2L_{r1}/(3ΩL_{z}), \ n = 1, 2, 3, \ldots \) (this is when the shearing box boundary conditions are exactly periodic). After the shift we execute a

\(^2\) Although a naive estimate suggests this will not work. Suppose that the external torque per unit area \( τ \sim M_{fr} \sim (Ω/E)R(r_{AV})^2 \), where \( M_{fr} \equiv \text{magnetic stress tensor, } (r_{AV}) \equiv \text{midplane Alfvén speed associated with the large-scale field and } Σ \equiv \text{surface density. Then the induced} \)

\( v_{r} \sim τ/(\Omega Σ E) \sim (r_{AV})^2/c_s^2 \), so in Equation (1) the second and third terms balance for \( η_{T} \sim (c_s^2) \). If \( η_{T} \sim (v_{A}^2)/\Omega \) \( (v_{A} \equiv \text{Alfvén speed associated with disk turbulence), then (diffusion/advection)} \sim (v_{A}^2)/(v_{A}^2) \). Shearing box experiments suggest this is large compared to 1 unless the poloidal field is so strong that turbulence is suppressed.

\(^3\) A Von-Neumann Richtmyer artificial viscosity is a pressure proportional to \( (\nabla \cdot v)^2 \) that is small outside shocks. It should not be confused with the “anomalous viscosity” used to model turbulent angular momentum diffusion in standard accretion disk theories (Shakura & Sunyaev 1973; Lynden-Bell & Pringle 1974).
divergence cleaning procedure to remove the monopoles that build up due to truncation error along the radial boundaries of the shearing box in our implementation of the shearing box boundary conditions, which remaps the electromagnetic forces (EMFs) rather than the magnetic field. We do this by gathering \( s = \nabla \cdot \mathbf{B} \) onto a single processor, solving the Poisson equation \( \nabla^2 \psi = s \) using a standard FFT-based procedure, and then setting \( \mathbf{B} \rightarrow \mathbf{B} - \nabla \psi \). The additional computational cost is negligible (usually less than 0.1% of the total cost) because the operation is performed infrequently. Johansen et al. (2008) have discussed other techniques to eliminate the radial dependence of numerical diffusion. This procedure eliminates the features reported in Johnson et al. (2008).

3. TURBULENT RESISTIVITY

Our focus is on the diffusive effects of MHD turbulence induced by the magnetorotational instability (MRI; Balbus & Hawley 1991). All models in this section have a mean toroidal field \( \langle \mathbf{B} \rangle = B_0 \mathbf{\hat{y}} \), where \( B_0 \) is chosen so that the initial plasma parameter \( \beta = 8\pi P_0/B_0^2 = 400 \). The models are evolved long enough (\( \gtrsim 40 \) orbits) to reach a saturated (statistically steady) state.

First we consider radial diffusion of a vertical field. Although this is the subdominant term in Equation (1), it is easier to measure (for reasons we will discuss shortly) and thus allows us to explore systematic effects related to resolution, scale of the perturbation, and details of the initial state more readily.

To measure the turbulent resistivity we evolve an initial state containing a mean field for hundreds of orbits. We then (arbitrarily) select an instant to inject a magnetic field perturbation of the form

\[
\delta \mathbf{B}_z = a \sin(k_z x) \mathbf{\hat{z}}.
\]

Here \( k_z = 2\pi n_z/L_z \) is the radial wavenumber for the perturbation. We consider models with \( 4 < L_z/H < 32 \). We choose \( a \) so that it is larger than the background turbulent fluctuations,\(^4\) but small enough so that it does not greatly affect the background turbulence. Here we use \( 0.1 < a/B_0 < 0.8 \). We then evolve the perturbed turbulence self-consistently, taking the sine transform of \( \mathbf{B}_z \) to obtain \( a(t) \).

A typical evolution \( a(t) \) is shown in Figure 1. The initial decay is approximately exponential. We measure a decay rate \( 1/\tau \) by fitting an exponential to \( a(t) \) over at least one \( e \)-folding time. The decay rates are listed in Table 1. To give a sense of the magnitude of fluctuations in the background state we also plot in time. The decay rates are listed in Table 1. To give a sense of the magnitude of \( \delta \mathbf{B}_z \) in the presence of MHD turbulence (lower solid line; model s4). A linear fit to \( \ln a(t) \) is shown as a heavy dashed line. The decay time is \( \tau \sim 52\Omega^{-1} \), implying a turbulent resistivity \( \eta_T \sim 0.031 \). Also shown is the evolution of \( a(t) \) in the same experiment (upper solid line). The turbulent magnetic Prandtl number \( \text{Pr}_{\text{T}} = 0.42 \). The dashed line shows the evolution of the corresponding cosine amplitude in \( \delta \mathbf{B}_z \), indicating the amplitude of large-scale field fluctuations in the turbulent background.

Figure 1. Evolution of the amplitude \( a \) of a magnetic field perturbation \( \delta \mathbf{B}_z = a \sin(k_z x) \) in the presence of MHD turbulence (lower solid line; model s4). A linear fit to \( \ln a(t) \) is shown as a heavy dashed line. The decay time is \( \tau \sim 52\Omega^{-1} \), implying a turbulent resistivity \( \eta_T \sim 0.031 \). Also shown is the evolution of \( a(t) \) in the same experiment (upper solid line). The turbulent magnetic Prandtl number \( \text{Pr}_{\text{T}} = 0.42 \). The dashed line shows the evolution of the corresponding cosine amplitude in \( \delta \mathbf{B}_z \), indicating the amplitude of large-scale field fluctuations in the turbulent background.

The decay time \( \tau \) to the background level. In every case we have examined the perturbation decayed; the turbulence was stable to a large-scale magnetic field perturbation.

Before going on we need to determine how strongly our measurement of \( \tau \) depends on our selection of initial conditions. We therefore measure \( \tau \) for an ensemble of initial conditions selected from the same run at widely separated times. We have done this for three models (s2, s5, and n1). The variation in \( \tau \) is \( \lesssim 13\% \), which may then be regarded as an error bar on our measurement.

The decay time \( \tau \) is related to \( \eta_T \) as follows. Solving the induction equation with a scalar resistivity,

\[
a = a_0 \exp \left( -\eta_T k_z^2 t \right),
\]

so we define

\[
\eta_T \equiv (k_z^2 \tau)^{-1}.
\]

Values of \( \eta_T \) are given in Table 1. In general \( \eta_T \) will depend on the magnitude and direction of \( \mathbf{k} \) and on the background field \( \langle \mathbf{B} \rangle \).

Does the decay time scale as \( k_z^{-2} \), i.e., does \( \eta_T \) depend on the magnitude of \( k \)? In a model with \( (L_x, L_y, L_z) = (8, 4\pi, 2)H \) we imposed perturbations with \( n_z = 1, 2, \) and 3 on the same initial state. We find \( \tau \approx 32.7, 10.4 \), and \( 4.93\Omega^{-1} \) respectively, so that \( \eta_T = 0.0495, 0.0388, \) and \( 0.0366\Omega^{-1} \), crudely consistent with a diffusive scaling. Models n1, s5, and s6, with \( n_z = 1 \) but \( L_z = 8, 16, \) and 32 and identical numerical resolution, have \( \eta_T = 0.0495, 0.0330, \) and \( 0.0304\Omega^{-1} \), again crudely consistent with a diffusive scaling. So it appears that \( \eta_T \) is at most very weakly dependent on the magnitude of \( k \).

We checked the effect of resolution in models with \( N_z/L_z \) ranging from \( 32/2H \) to \( 128/2H \). In all the runs \( \Delta x = \Delta z = 2\Delta y \) (except in model s4, where \( \Delta x = \Delta z = \Delta y \); varying the zone aspect ratio made no difference in \( \eta_T \) and \( (L_x, L_y, L_z) = (4, 2\pi, 1)H \)). Comparing runs r1, r2, and r3, we find \( \eta_T = 0.037, 0.035, \) and \( 0.034\Omega^{-1} \), respectively. Our measurement of \( \eta_T \) is thus consistent with convergence, since the variation with resolution is smaller than the noise in decay time measurement that arise from choosing a particular initial state.

We found that \( \eta_T \) does depend on the field perturbation strength \( a_0 \). In the \( (L_x, L_y, L_z) = (8, 4\pi, 2)H \) models with

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4 The background contains power in the magnetic field at small wavenumber, with the power spectrum \( \delta \mathbf{B}_z^2 \sim \text{const.} \sim (\delta \mathbf{B}_z^2)^{1/2}, \) where \( \lambda \) is a correlation length (see Guan et al. 2008 for a discussion). The implies that the Fourier series coefficients in the background state will have rms amplitude \( (\delta \mathbf{B}_z^2)^{1/2}/(L_xL_yL_z)^{1/2} \).
Vertical Diffusion of an Azimuthal Field

| Model | Size  | Resolution | $a_0/B_0$ | $n_x$ | $1/\Omega_t$ | $\eta_T H^2/\Omega$ | $Pr_{M,T}$ |
|-------|-------|------------|-----------|-------|-------------|---------------------|-------------|
| s1    | $(4, 2\pi, 1)H$ | $64/H$ | 400 | 0.2 | 1 | 0.0232 | 0.0758 | 0.0307 | 0.50 |
| s2    | $(4, 4\pi, 1)H$ | $64/H$ | 400 | 0.2 | 1 | 0.0235 | 0.0660 | 0.0268 | 0.58 |
| s3    | $(4, 8\pi, 1)H$ | $64/H$ | 400 | 0.2 | 1 | 0.0238 | 0.0740 | 0.0300 | 0.53 |
| s4    | $(8, 2\pi, 1)H$ | $64/H$ | 400 | 0.2 | 1 | 0.0198 | 0.0194 | 0.0314 | 0.42 |
| s5    | $(16, 2\pi, 1)H$ | $32/H$ | 400 | 0.2 | 1 | 0.0206 | 0.0509 | 0.0330 | 0.42 |
| s6    | $(32, 2\pi, 1)H$ | $32/H$ | 400 | 0.2 | 1 | 0.0210 | 0.0117 | 0.0304 | 0.46 |

| Model | Size  | Resolution | $a_0/B_0$ | $n_x$ | $1/\Omega_t$ | $\eta_T H^2/\Omega$ | $Pr_{M,T}$ |
|-------|-------|------------|-----------|-------|-------------|---------------------|-------------|
| n1    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.2 | 1 | 0.0262 | 0.0306 | 0.0495 | 0.35 |
| n2    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.2 | 2 | 0.0252 | 0.0957 | 0.0388 | 0.43 |
| n3    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.2 | 3 | 0.0240 | 0.203 | 0.0366 | 0.44 |

| Model | Size  | Resolution | $a_0/B_0$ | $n_x$ | $1/\Omega_t$ | $\eta_T H^2/\Omega$ | $Pr_{M,T}$ |
|-------|-------|------------|-----------|-------|-------------|---------------------|-------------|
| a1    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.1 | 1 | 0.0239 | 0.0268 | 0.0434 | 0.37 |
| a2    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.2 | 1 | 0.0262 | 0.0306 | 0.0495 | 0.35 |
| a3    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.4 | 1 | 0.0338 | 0.0315 | 0.0511 | 0.44 |
| a4    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.8 | 1 | 0.0604 | 0.0496 | 0.0804 | 0.50 |

| Model | Size  | Resolution | $a_0/B_0$ | $n_x$ | $1/\Omega_t$ | $\eta_T H^2/\Omega$ | $Pr_{M,T}$ |
|-------|-------|------------|-----------|-------|-------------|---------------------|-------------|
| b1    | $(8, 4\pi, 2)H$ | $32/H$ | 400 | 0.2 | 1 | 0.0262 | 0.0306 | 0.0495 | 0.35 |
| b2a   | $(8, 4\pi, 2)H$ | $32/H$ | 100 | 0.2 | 1 | 0.0568 | 0.0541 | 0.0878 | 0.43 |
| b2b   | $(8, 4\pi, 2)H$ | $32/H$ | 100 | 0.4 | 1 | 0.0607 | 0.0463 | 0.0750 | 0.54 |

Vertical Diffusion of an Azimuthal Field

\[
\delta B_y = a \sin(k_z z) \hat{y} \tag{7}
\]

into an already turbulent state, measure the decay time \( \tau \), and set

\[
\eta_T = \left( \frac{k_z^2 \tau}{\tau} \right)^{-1}. \tag{8}
\]

We imagine a large-scale field entering the disk at an inclination of \( \sim 30 \) deg, running vertically through the midplane, and leaving at a similar inclination; to mimic this geometry we set \( \omega = 2\pi/k_z = L_z = 4H \) and \( 2\Omega \).

The perturbation amplitude must be chosen larger than the turbulent background but as small as possible so it does not influence the background state. Because the turbulent fluctuations in the azimuthal field are larger than the fluctuations in the vertical field, \( a \) in Equation (7) must be an order of magnitude larger (in comparison to \( B_0 \)) than \( a \) in Equation (4).

Models v1 and v2 (see Table 1) have \( (L_x, L_y, L_z) = (4\pi, 4\pi, 4)H \) and resolution \( 128 \times 200 \times 128 \); v1 has \( a_0 = B_0 \) while v2 has \( a_0 = 2B_0 \). We found \( 1/\tau = 0.052\Omega \) and \( 0.050\Omega \) respectively. This corresponds to a vertical diffusion coefficient \( \eta_T \sim 0.020H^2/\Omega \). Details are given in Table 1.

To test for resolution dependence we repeated the above experiment at resolution \( 256 \times 400 \times 256 \), with \( a_0 = 4B_0 \) (Model v3 in Table 1). We found \( 1/\tau = 0.040\Omega \) and thus \( \eta_T \sim 0.016H^2/\Omega \). This diffusion coefficient is \( \sim 20\% \) smaller than that obtained at lower resolution. This small decrease in \( \eta_T \) is mainly caused by slightly lower turbulent saturation level in the high-resolution run in the perturbed state, where the saturation \( \alpha \) is about \( 25\% \) smaller. This might be surprising because on average higher-resolution runs have higher saturation levels (see, e.g., Guan et al. 2008), but \( \alpha \) fluctuates in time; our \( \alpha \) is averaged over the same time interval used to fit \( 1/\tau \).
4. DISCUSSION: TURBULENT MAGNETIC PRANDTL NUMBER

To calculate $Pr_{M,T}$, we need to assign a “viscosity” to the turbulence. We do this by measuring the turbulent shear stress

$$w_{xy,T} = \left( \frac{\rho v_x \delta v_y - B_x B_y}{4\pi} \right)$$

and equating this to the shear stress that would be measured in a viscous fluid

$$w_{xy,v} = \rho vq \Omega,$$

where $q \equiv -(1/2)d \ln \Omega^2/d \ln R = 3/2$ for a Keplerian potential. Thus,

$$v = \frac{w_{xy,T}}{\rho q \Omega},$$

and the turbulent magnetic Prandtl number

$$Pr_{M,T} \equiv \frac{\nu_T}{\eta_T} = \left( \frac{\tau \Omega}{(kH)^2} \right)^2 \frac{w_{xy,T}}{q(\rho) c_s^2},$$

(12)

The turbulent shear stress $w_{xy,T}$ is related to $\alpha$ by $\alpha \equiv w_{xy,T}/(\rho) c_s^2$. The evolution of $\alpha$ from one of our runs is shown in Figure 1.

We measure the time average of $\alpha$ during the decay time, denoted $\overline{\alpha}$ (over the same period we fit $\tau$), and use Equation (12) to calculate $Pr_{M,T}$. The results are compiled in Table 1. The vertical field experiments give $0.35 < Pr_{M,T} < 0.58$. The azimuthal field simulations give $Pr_{M,T} \sim 1$.

In the current limited set of simulations we see no clear sign of $Pr_{M,T}$ scaling with model or numerical parameters. This consistency is remarkable when we look at the radial diffusion of vertical field with different perturbation field strength $d_0$. In the $(L_x, L_y, L_z) = (8, 4, \pi, 2) H$ models when the perturbation amplitude is strong, as in the $d_0 = 0.8B_0$ case, both $\eta$ and $w_{xy,T}$ double compared to their weakly perturbed counterparts with $d_0 = 0.1B_0$; this doubling is precisely what is required for a constant $Pr_{M,T}$.

We have also carried out comparison experiment with slightly larger initial toroidal field strength $B_0$ ($B_0 = 100$; model b2a in Table 1). Past numerical experiments (Hawley et al. 1995; Guan et al. 2008) imply that the saturation level scales linearly with $B_0$. For this model we found $\overline{\alpha} = 0.0568$ and $\eta_T = 0.0878H^2 \Omega$. Both the turbulent saturation level and $\eta_T$ double, giving $Pr_{M,T} \sim 0.43$.

Recently Lesur & Longaretti (2008) have measured the turbulent resistivity in shearing box simulations. Their technique for measuring turbulent resistivity differs from ours: they directly measure the EMF required to maintain a particular sinusoidal variation in the field, they measure a resistive stress tensor, and they consider only a mean vertical field (rather than the mean azimuthal field considered here). They find $Pr_{M,T} = 1.6$ for the diffusion of a radially varying azimuthal field.

Transport properties of the MRI-generated turbulent flow have also been studied in the context of dust (passive scalar) mixing in a shearing box (Carballido et al. 2005; Johansen & Klahr 2005; Johansen et al. 2006; Turner et al. 2006; Fromang & Papaloizou 2006). The analog of $Pr_{M,T}$ in these experiments is the turbulent Schmidt number $Sc \equiv w_T/D_T$, where $D_T$ is the diffusion coefficient for the grains. In models with zero net magnetic flux (either when the dust is modeled as a passive scalar (Turner et al. 2006), or when the dust is coupled to the gas by drag (Johansen & Klahr 2005; Fromang & Papaloizou 2006)), $Sc \sim 1$. In models with a net vertical flux (Carballido et al. 2005; Johansen et al. 2006), $Sc$ was found to be anisotropic and increase with $\alpha$ (and therefore depend on the initial vertical field strength), ranging from $Sc \sim 1$ for a weak field, to $Sc \sim 2$ for radial diffusion $Sc \sim 10$ for vertical diffusion when the mean field is strong and $\alpha \sim 0.5$. Our results are broadly consistent with these measurements in the sense that $Sc \sim Pr_{M,T}$ when the mean field is weak. Our scheme for measuring $Pr_{M,T}$ is much more computationally expensive when the turbulence is strong, because a large computational volume is required to reduce the background fluctuations. It would be interesting to investigate whether $Pr_{M,T} \sim 10$ can be achieved with strong background fields in future investigations.

A $Pr_{M,T}$ of order unity is not surprising, perhaps, from a turbulent mixing point of view. Parker (1979), for example, argued that in isotropic turbulence $\nu_T \sim \eta_T \sim l \nu$, where $l$ is the largest dimension of eddies in the MHD turbulent flow and $\nu$ is the characteristic eddy turnover speed. Interestingly, Yousef et al. (2003) also obtained an order of unity $Pr_{M,T}$ from their turbulence simulations, where the turbulence is sustained by external forcing rather than the MRI in a disk.

Can large-scale magnetic fields avoid escape from turbulent disks? We clearly have not included all of the effects outlined in the introduction that might influence the evolution of large-scale fields. Of these, perhaps the simplest route to large-scale fields is a rapid accretion mode in which the mean field torques the disk, causing it to accrete more rapidly than would a viscous disk. But our results cast doubt on models that confine the large-scale field by setting $Pr_{M,T} \sim R/H$ (e.g., Shu et al. 2007).

Our models do not show a clear scaling of $Pr_{M,T}$ with model parameters, but the dynamic range in parameters (and thus in $\eta_T$) is small; future experiments over a broader range of initial field strengths may show scaling that is not evident here. Our models also do not include an explicit dissipation model, on which $Pr_{M,T}$ might also depend.

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