Monstrous branes

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Abstract

We study D-branes in the bosonic closed string theory whose automorphism group is the Bimonster group (the wreath product of the Monster simple group with $\mathbb{Z}_2$). We give a complete classification of D-branes preserving the chiral subalgebra of Monster invariants and show that they transform in a representation of the Bimonster. Our results apply more generally to self-dual conformal field theories which admit the action of a compact Lie group on both the left- and right-moving sectors.

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1. Introduction

The connection between the Monster sporadic group and modular functions known as moonshine is one of the most peculiar and mysterious facts in modern mathematics. Equally strange is the fact that the construction of a Monster module which most naturally encodes these connections uses the techniques and ideas of string theory [1].

There now exists a proof of the moonshine conjectures [2], and steps have been taken towards an explanation of their origin [3,4]. However, a fully satisfactory conceptual explanation of the connection between the Monster, modular functions, and its appearance in string theory is still lacking. It may be that some new physical ideas and techniques will help to shed light on the situation.

The current understanding of the Monster in string theory shows that, in a particular closed string background, which will be described later, the Bimonster acts as a symmetry group of the perturbative spectrum of the string theory. (The Bimonster is the wreath product of the Monster simple group with $\mathbb{Z}_2$, i.e. two copies of the Monster group exchanged by an involution; see subsection 2.2 for more details.) Over the last several years it has been appreciated that string theory also contains non-perturbative states whose mass scales like $1/g_s$ or $1/g_s^2$ with $g_s$ the string coupling, and that these states play a fundamental role in understanding the structure of string theory [5]. In the bosonic string theory in which the Bimonster appears, the best understood non-perturbative states are Dirichlet branes or D-branes whose mass scales like $1/g_s$. The classification of D-brane states in various string backgrounds has been an active area of research recently, and our aim here is to use the techniques that have been developed to at least partially classify the possible D-brane states in the bosonic string theory with Bimonster symmetry. In doing so we will provide evidence that the Bimonster extends from a symmetry of the perturbative spectrum to a symmetry of the full spectrum of the theory.

The construction of the Monster module in [1] can be viewed, in string theory language, as the construction of an asymmetric orbifold of a special toroidal compactification of bosonic string theory. As a result we will need to utilise a number of results regarding the description of D-branes in orbifold conformal field theory. In the next subsection we shall therefore review briefly, following [6], some of the necessary background material.
1.1. Orbifolds, D-branes and conformal field theory

Let us begin by explaining the basic ideas underlying the orbifold construction [4]. Consider a closed string theory compactified on a manifold $\mathcal{M}$ on which a group $\Gamma$ acts as a group of symmetries. Roughly speaking, the orbifold by $\Gamma$ is the compactification on the quotient space $\mathcal{M}/\Gamma$. If the action of the discrete group on $\mathcal{M}$ is not free, i.e. if $\mathcal{M}$ has fixed points under the action of some elements in $\Gamma$, then the resulting space is singular. Despite such classical singularities, string theory is however well-behaved on such orbifolds.

More specifically we can describe the orbifold theory as follows. Firstly, the theory consists of those states in the original space of states $\mathcal{H}$ that are invariant under the action of the orbifold group $\Gamma$. In addition, the theory has so-called twisted sectors containing strings that are closed in $\mathcal{M}/\Gamma$ but not in $\mathcal{M}$. If the orbifold action has fixed points, the twisted sector states describe degrees of freedom that are localised at these fixed points; the presence of these additional states is the essential reason for why string theory is well-behaved despite these singularities.

The concept of an orbifold can be extended to a more general setting, where neither the underlying conformal field theory $\mathcal{H}$ nor the discrete symmetry group $\Gamma$ need to have a direct geometric interpretation. (For example, in the case studied in this paper, $\Gamma$ includes an asymmetric reflection, acting on the left-moving string coordinates only.) In this context the twisted sectors are then determined by the condition that the orbifold conformal field theory should be modular invariant. If $\Gamma$ is abelian one finds that there is one twisted sector $\mathcal{H}_h$ for each element $h \in \Gamma$. Each twisted sector has to be projected again onto the states that are invariant under the action of the orbifold group $\Gamma$.

Next, we turn to the description of D-branes on orbifolds. Let us first consider the case where the orbifold has a geometric interpretation as $\mathcal{M}/\Gamma$. Then we can construct D-branes as follows [3,4]: we consider a D-brane on the covering space $\mathcal{M}$, and add to it images under the action of $\Gamma$ so as to obtain a $\Gamma$-invariant configuration of D-branes on $\mathcal{M}$. We then restrict the resulting open string spectrum to those states that are invariant under the action of the orbifold group. A typical orbifold-invariant configuration will consist of $|\Gamma|$ D-branes on the covering space. The resulting D-brane is then called a ‘bulk’ brane, and it possesses moduli that describe its position on $\mathcal{M}$.

On the other hand, if the original D-brane is localised at a singular point of the orbifold, we need fewer preimages in the covering space to make an orbifold invariant configuration; such branes are then called ‘fractional’ D-branes. Because they involve
fewer preimage branes, fractional D-branes cannot move off the singular point; instead, a number of fractional D-branes have to come together in order for the system to be able to move off into the bulk.

For orbifolds that do not have a simple geometric interpretation, it is often useful to describe D-branes in terms of boundary states, using (boundary) conformal field theory methods. D-branes can be thought of as describing open string sectors that can be added consistently to a given closed string theory. From the point of view of conformal field theory, the construction of D-branes is therefore simply the construction of permissible boundary conditions. This problem has been studied for a number of years (for a recent review see for example [15]).

In the conformal field theory approach, D-branes are described by coherent ‘boundary states’ that can be constructed in the underlying (closed) string theory. These boundary states satisfy a number of consistency conditions, the most important of which is the so-called Cardy condition [16] (which we shall analyse in detail in section 4). It arises from considering the annulus diagram for which the two boundary conditions are determined by two (possibly identical) D-branes, one for each boundary. This diagram can be given two interpretations, depending on which world-sheet coordinate is chosen as the world-sheet time. From the closed string point of view the diagram describes the tree-level exchange of closed string states between two sources (D-branes). On the other hand, the diagram can also be interpreted as a one-loop vacuum diagram of open strings with boundary conditions described by the two D-branes. The requirement that both the open and the closed string interpretations of the annulus diagram should be sensible imposes strong restrictions on the possible D-branes in a given closed string theory.

1.2. Outline

The paper is organised as follows. In section 2 we review the construction of the Monster theory whose D-branes we want to study. In section 3 we make use of the orbifold construction of the Monster theory to anticipate the presence of certain bulk and fractional branes in the D-brane spectrum. We then employ conformal field theory techniques in section 4 in order to construct all D-branes that preserve the chiral subalgebra of Monster invariants. We demonstrate that the branes we construct satisfy all relative

* Actually, the conformal field theory point of view is also powerful for geometric orbifolds; see for example [10,11,12,13,14].
Cardy conditions, and we show that they are complete in a suitable sense. The D-branes are labelled by group elements in the Monster group $\mathcal{M}$, and transform in the regular representation of both copies of $\mathcal{M}$ (that defines a representation of the Bimonster). As we shall explain, our analysis is actually valid for any self-dual conformal field theory which admits the action of a compact Lie group on both left- and right-moving sectors, and we therefore couch our arguments in this more general setting. In section 5 we explain how the ‘geometrical’ D-branes of section 3 can be accounted for in terms of the more abstract conformal field theory construction. Finally, we end with some conclusions in section 6.

2. The Monster theory

2.1. The Monster conformal field theory

We are interested in the (bosonic) closed string theory whose spectrum is described by the tensor product of two (chiral) Monster conformal field theories,

$$\mathcal{H} = \mathcal{H}_M \otimes \overline{\mathcal{H}_M},$$

where $\mathcal{H}_M$ is the (chiral) Monster theory,

$$\mathcal{H}_M = \mathcal{H}_{\Lambda_L}/\mathbb{Z}_2.$$  

Here $\Lambda_L$ is the Leech lattice, the (unique) even self-dual Euclidean lattice of dimension 24 that does not possess any points of length square 2 (see [17] for a good explanation of these matters), and $\mathcal{H}_\Lambda$ is the holomorphic bosonic conformal field theory associated to the even, self-dual lattice $\Lambda$. The lattice theory $\mathcal{H}_\Lambda$ is the conformal field theory that consists of the states of the form

$$\prod_{i=1}^n \alpha_{-m_i}^{j_i} |p\rangle,$$

where $j_i \in \{1, \ldots, d\}$, $m_1 \geq m_2 \geq \cdots \geq m_n > 0$ and $p \in \Lambda$. Here $d$ is the dimension of the lattice which equals the central charge of the conformal field theory $\mathcal{H}_\Lambda$, $c = d$ with $d$ necessarily a multiple of 8, and the oscillators $\alpha_{n_i}^i$, $i \in \{1, \ldots, d\}$, $n \in \mathbb{Z}$ satisfy the standard commutation relations

$$[\alpha_m^i, \alpha_n^j] = m \delta^{ij} \delta_{m-n}.$$ 

More details about lattice conformal field theories can, for example, be found in [18,19].
For the case under consideration \( c = d = 24 \), and the \( \mathbb{Z}_2 \) orbifold acts on the 24 oscillators, \( \alpha^i_n, i = 1, \ldots, 24 \) by

\[
\alpha^i_n \mapsto -\alpha^i_n,
\]

and on the ground states \( |p\rangle \) with \( p \in \Lambda_L \) as

\[
|p\rangle \mapsto | -p\rangle.
\]

The construction of this vertex operator algebra and the demonstration that the Monster acts as its automorphism group is due to Frenkel, Lepowsky and Meurman [1]; the embedding of this construction into conformal field theory has been discussed in [20,21,22].

The closed conformal field theory (2.1) can be described as the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold of the compactification of the closed bosonic string theory on the Leech torus, the quotient space \( \mathbb{R}^{24}/\Lambda_L \), with a special background \( B \)-field turned on [21]. The two \( \mathbb{Z}_2 \)'s can be taken to be the two asymmetric orbifolds that act on the left- and right-movers as described above. Alternatively, we can also think of the theory as an asymmetric \( \mathbb{Z}_2 \) orbifold (where \( \mathbb{Z}_2 \) acts as above on the left-movers only) of the geometric \( \mathbb{Z}_2 \) orbifold of the Leech theory. Because of the background \( B \)-field we have just mentioned, the generator of the asymmetric \( \mathbb{Z}_2 \) actually differs from the usual T-duality transformation that inverts all 24 directions: indeed, T-duality not only reflects the left-movers, but also rotates the right-movers by \( D \equiv (G - B)^{-1}(G + B) \), where \( G_{ij} = \delta_{ij} \) [23]. The distinction between the usual T-duality and the asymmetric \( \mathbb{Z}_2 \) we are considering here will be important later on in section 3.

The untwisted sector of the chiral Monster theory (2.2) consists of those states of the lattice theory that are invariant under the action of (2.5) and (2.6). The twisted sector is created by the action of half-integrally moded oscillators \( c^i_r, r \in \mathbb{Z} + \frac{1}{2}, i = 1, \ldots, 24 \) satisfying

\[
[c^i_r, c^j_s] = r \delta^{ij} \delta_{r,-s},
\]

on the irreducible representation of the Clifford algebra \( \Gamma(\Lambda_L) \) associated to \( \Lambda_L \). This Clifford algebra arises as a projective representation of \( \Lambda_L/2\Lambda_L \), generated by \( \pm \gamma_i \), where \( i = 1, \ldots, 24 \) labels a basis \( k_i \) of \( \Lambda_L \). The \( \gamma_i \) satisfy the relations

\[
\gamma_i \gamma_j = (-1)^{k_i \cdot k_j} \gamma_j \gamma_i, \quad \gamma_i^2 = (-1)^{\frac{1}{2} k_i^2}.
\]

Since \( k_i^2 = 4 \) for \( \Lambda_L \), each element \( \gamma_i \) squares to one. The irreducible representation of this Clifford algebra has dimension \( 2^{12} \), and thus the chiral theory has a degeneracy of \( 2^{12} \) in
the twisted sector. As before in the untwisted sector, we also have to restrict the twisted
sector states to be $\mathbb{Z}_2$-even, where the $\mathbb{Z}_2$ generator acts on the oscillators as

$$c^i_r \mapsto -c^i_r,$$

and on the degenerate twisted sector ground state $|\chi\rangle$ as

$$|\chi\rangle \mapsto -|\chi\rangle.$$

In the full theory (2.1) we then have a degeneracy of $2^{24}$ in the sector where both left-
and right-moving oscillators are half-integrally moded; these correspond to (certain linear
combinations of) the $2^{24}$ fixed points under the diagonal (geometrical) $\mathbb{Z}_2$ orbifold. Two
additional sectors, where either the left- or the right-movers but not both are half-integrally
moded, each have a degeneracy of $2^{12}$; these sectors are twisted by the $\mathbb{Z}_2$ acting on the
left- or the right-movers, respectively.

2.2. The Monster group and some subgroups

The automorphism group of the chiral Monster conformal field theory is the so-called
Monster group $\mathbf{M}$, the largest sporadic simple finite group. This is to say, for each $g \in \mathbf{M}$,
we have an automorphism of the conformal field theory,

$$g : \mathcal{H}_M \rightarrow \mathcal{H}_M,$$

for which

$$g V(\psi, z) g^{-1} = V(g \psi, z),$$

where $V(\psi, z)$ is the vertex operator corresponding to the state $\psi \in \mathcal{H}_M$. Furthermore,

$$g |\Omega\rangle = |\Omega\rangle, \quad g |\omega\rangle = g L_{-2} |\Omega\rangle = L_{-2} |\Omega\rangle.$$

Here $L_n$ denote the modes of the Virasoro algebra,

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m,-n},$$

$|\Omega\rangle$ is the vacuum vector and $|\omega\rangle = L_{-2} |\Omega\rangle$ is the conformal vector. Since $|\omega\rangle$ is invariant
under the action of $g \in \mathbf{M}$ it follows from (2.12) that

$$[g, L_n] = 0 \quad \text{for all } g \in \mathbf{M}, \; n \in \mathbb{Z}. $$
In particular, this implies that each eigenspace of \( L_0 \) forms a representation of the Monster group, \( \mathbf{M} \). The theory has a single state with \( h = 0 \), the vacuum \( |\Omega\rangle \), which transforms in the singlet representation of \( \mathbf{M} \). There are no states with \( h = 1 \), and for \( h = 2 \) we have 196884 states that transform as

\[
196884 = 196883 + 1. \tag{2.16}
\]

Indeed, as we have seen, the state \( L_{-2} |\Omega\rangle \) is in the singlet representation of \( \mathbf{M} \); the remaining states then transform in the smallest non-trivial (196883-dimensional) representation of the Monster group. This pattern persists at higher level \([1]\).

The automorphism group of the full closed string theory (2.1) is then the so-called Bimonster group. The Bimonster is the wreath product of the Monster group \( \mathbf{M} \) with \( \mathbb{Z}_2 \), i.e. the semi-direct product of \( (\mathbf{M} \times \mathbf{M}) \) with \( \mathbb{Z}_2 \) where the generator \( \sigma \) of \( \mathbb{Z}_2 \) permutes the two copies of \( \mathbf{M} \). (A neat presentation of the Bimonster in terms of Coxeter relations was conjectured by Conway, and subsequently proven by Norton \([24]\).) Indeed, the two copies of the Monster group act on the left and right chiral theory separately, and \( \sigma \) is the symmetry that exchanges left- and right-movers (combined with a shift of the \( B \)-field such that the background is preserved).

The Monster group contains a subgroup \( C \) whose action on the Monster conformal field theory can be understood geometrically. This subgroup is an extension of the simple Conway group \( (\cdot 1) = (\cdot 0)/\mathbb{Z}_2 \) by an ‘extra-special’ group denoted by \( 2_{+}^{1+24} \); one therefore writes \( C = 2_{+}^{1+24}(\cdot 1) \). Since \( C \) will play a role in sections 3 and 5, we now review its construction, following the exposition given in \([19]\). The Conway group \( (\cdot 0) \) is the group of automorphisms of the Leech lattice,

\[
(\cdot 0) = \text{Aut}(\Lambda_L) = \left\{ R \in \text{SO}(24) : Rp \in \Lambda_L \text{ for } p \in \Lambda_L \right\}. \tag{2.17}
\]

The centre of \( (\cdot 0) \) contains one non-trivial element, the reflection map \( p \mapsto -p \) which we have used in the above orbifold construction. Since this symmetry acts (by construction) trivially on the orbifold theory, the automorphism group of the Monster conformal field theory only involves the quotient group by this reflection symmetry; this is the simple Conway group \( (\cdot 1) \).

Each element \( R \in (\cdot 1) \) has a natural action on the oscillators, given by

\[
\alpha_n^i \mapsto R_{ij} \alpha_n^j, \quad c_r^i \mapsto R_{ij} c_r^j, \tag{2.18}
\]
but the action on the ground states is ambiguous. This ambiguity is responsible for the extension of \((\cdot 1)\) mentioned above, as we now describe. Let us extend the \(\gamma_i \equiv \gamma_k\), to being defined for arbitrary \(k \in \Lambda_L\), where the generators \(\gamma_k\) now satisfy

\[
\gamma_k \gamma_l = (-1)^{k \cdot l} \gamma_l \gamma_k, \quad \gamma_k \gamma_l = \varepsilon(k, l) \gamma_{k+l},
\]

(2.19)

and \(\varepsilon(k, l)\) are suitable signs. Because of the first equation in (2.19), these signs must satisfy

\[
\varepsilon(k, l) = (-1)^{k \cdot l} \varepsilon(l, k),
\]

(2.20)

while the associativity of the algebra product implies that

\[
\varepsilon(k, l) \varepsilon(k + l, m) = \varepsilon(k, l + m) \varepsilon(l, m).
\]

(2.21)

As we have explained before, the ground states of the twisted sector form an irreducible representation of the algebra (2.19). Each element \(R \in (\cdot 1)\) gives rise to an automorphism of the gamma matrix algebra by

\[
\gamma'_k = \gamma_{Rk},
\]

(2.22)

and this induces an automorphism of the corresponding representation. Since all irreducible representations of the Clifford algebra are isomorphic, there exists a unitary transformation \(S\) so that

\[
S \gamma_k S^{-1} = v_{R,S}(k) \gamma_{Rk},
\]

(2.23)

where \(v_{R,S}(k) = \pm 1\). This transforms \(\varepsilon(k, l)\) into

\[
\varepsilon'(k, l) = \frac{v_{R,S}(k + l)}{v_{R,S}(k) v_{R,S}(l)} \varepsilon(k, l).
\]

(2.24)

The action on the ground states is now defined by

\[
|p\rangle \mapsto v_{R,S}(p) |Rp\rangle \quad |\chi\rangle \mapsto S |\chi\rangle,
\]

(2.25)

where \(|\chi\rangle\) denotes the \(2^{12}\) ground states of the twisted sector. This construction also applies to elements \(R \in (\cdot 0)\); for example, the generator of the \(\mathbb{Z}_2\) defined by (2.5), (2.6), (2.9) and (2.10) corresponds to \(R = -1\) and \(S = -1\) with \(v_{R,S}(k) = 1\).

The unitary transformation \(S\) that enters in (2.25) is only determined by (2.23) up to

\[
S \mapsto S \gamma,
\]

(2.26)
where $\gamma = \pm \gamma_l$ for some $l \in \Lambda_L$, i.e. $\gamma \in \Gamma(\Lambda_L)$. For each $R \in (\cdot 1)$, there are therefore $|\Gamma(\Lambda_L)| = 2^{25}$ different choices for $S$, thus leading to the extension of $(\cdot 1)$ by $\Gamma(\Lambda_L) = 2^{1+24}$. In particular, for $R = e$ and $S = \gamma_l$, (2.23) and (2.19) imply that

$$v_{e, \gamma_l}(k) = (-1)^{k-l}.$$  

(2.27)

The Monster group contains the involution $i$ which acts as $+1$ in the untwisted sector and as $-1$ in the twisted sector. Actually the centraliser of $i$ in $M$, i.e. the subgroup that consists of those elements $g \in M$ that commute with $i$, is precisely the group $C = 2^{1+24}(\cdot 1)$ that we have just described. In particular, $i$ is therefore an element of $C$; it corresponds to choosing $R = e$ in $(\cdot 1)$ and $S = -1$ in $\Gamma(\Lambda_L)$.

2.3. Partition function and McKay-Thompson series

The character or partition function of the (chiral) Monster theory is given as

$$\text{Tr}_{\mathcal{H}_M}(q^{L_0-\frac{c}{24}}) = j(\tau) - 744,$$  

(2.28)

where $j(\tau)$ is the elliptic $j$-function,

$$j(\tau) = \frac{\Theta_{E_8}(\tau)^3}{\eta(\tau)^{24}} = q^{-1} + 744 + 196884 \, q + 21493760 \, q^2 + \cdots, \quad q = e^{2\pi i \tau}.$$  

(2.29)

Here $\Theta_{E_8}(\tau)$ is the theta function of the $E_8$ root lattice,

$$\Theta_{E_8}(\tau) = \sum_{x \in \Lambda} q^{\frac{1}{2} x^2},$$  

(2.30)

and $\eta(\tau)$ is the Dedekind eta-function,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^\infty \left(1 - q^n\right).$$  

(2.31)

The $j$-function (and therefore also (2.28)) is a modular function; this is to say, $j(\tau)$ is invariant under the action of $SL(2, \mathbb{Z})$, i.e.

$$j \left( \frac{a\tau + b}{c\tau + d} \right) = j(\tau), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}).$$  

(2.32)

As we have explained above (see (2.15)), the action of the Monster group commutes with $L_0$. Thus it is natural to consider the so-called McKay-Thompson series

$$\chi_g(q) \equiv \text{Tr}_{\mathcal{H}_M}(g \, q^{L_0-1}),$$  

(2.33)
for every element \( g \in \mathbf{M} \). For \( g = e \), the identity element of the Monster group, (2.33) reduces to (2.28). Since the definition of (2.33) involves the trace over representations of \( \mathbf{M} \), the McKay-Thompson series only depends on the conjugacy class of \( g \) in \( \mathbf{M} \). There are 194 conjugacy classes, and the first fifty terms in the power series expansions of (2.33) have been tabulated in [25].†

The McKay-Thompson series have a number of remarkable properties. In particular, for each \( g \in \mathbf{M} \), \( \chi_g(q) \) is a Hauptmodule of a genus zero modular group. This is the key statement of the moonshine conjecture of Conway and Norton [26] that has now been proven by Borcherds [2]. (For a nice introduction to ‘monstrous moonshine’ see [27].)

3. D-branes of the Monster conformal field theory: some examples

Our aim is to construct, and to some extent classify, the D-branes of the Monster theory. In particular, we would like to understand whether the D-brane states fall into representations of the Bimonster group.

In the following we shall mainly concentrate on those D-branes that preserve the subalgebra \( \mathcal{W} \) of the full (chiral) Monster vertex operator algebra that consists of the Monster invariant states in \( \mathcal{H}_M \). As we shall see, a complete classification for these D-branes is possible, and one can show that they transform in a representation of the Bimonster group. This result will emerge as a special case of the much more general analysis in section 4. That analysis combines well-known techniques from boundary conformal field theory [16] with mathematical results on vertex operator algebras [28]. In the present section, we use the description of the Monster conformal field theory as an orbifold of the Leech lattice to anticipate the presence of some of these boundary states. In section 5 we shall then show how these examples fit into the analysis of section 4.

† Since the coefficients of \( q^n \) in (2.33) are real (they are in fact all integers), the McKay-Thompson series for \( g \) and \( g^{-1} \) agree; in fact, there are 22 conjugacy classes with distinct inverses. Moreover, the two distinct classes of order 27 turn out to have the same McKay-Thompson series, so all in all there exist \( 194 - 22 - 1 = 171 \) different McKay-Thompson series.
3.1. Fractional D0 − D24 at the origin

We are considering the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the Leech compactification. As we have mentioned in the paragraph following (2.6), we can think of this as an asymmetric $\mathbb{Z}_2$ orbifold of the geometric $\mathbb{Z}_2$ orbifold of the Leech theory. We are interested in those configurations of D-branes on the geometric $\mathbb{Z}_2$ orbifold of the Leech theory that are invariant under the asymmetric $\mathbb{Z}_2$ (which differs from T-duality by a rotation of the right-movers). The simplest such configuration is a fractional D0-brane sitting at the origin together with a D24-brane without Wilson lines. The corresponding D-brane in the orbifold theory is described by a boundary state

$$\frac{1}{\sqrt{2}} \left( |D0\rangle_U + |D0\rangle_T + |D24\rangle_U + |D24\rangle_T \right),$$

where the subscripts $U$ and $T$ denote components in the untwisted sector and in the sector twisted by the geometric $\mathbb{Z}_2$, respectively. Specifically, we have for the untwisted states

$$|D0\rangle_U = \sum_p \exp \left( \sum_{n \geq 1} \frac{1}{n} \alpha_i^{i-n} \bar{\alpha}_i^{i-n} \right) |(p, p)\rangle$$

and

$$|D24\rangle_U = \sum_p \exp \left( \sum_{n \geq 1} -\frac{1}{n} \alpha_i^{i-n} \bar{\alpha}_i^{i-n} \right) |(p, -p)\rangle,$$

with $|(p_L, p_R)\rangle$ denoting the momentum ground state in $\mathcal{H}$ with momentum $p_L, p_R \in \Lambda_L$. By construction, this combination of boundary states is invariant under the asymmetric $\mathbb{Z}_2$. However, our $\mathbb{Z}_2$ does not simply correspond to the T-duality transformation that inverts all 24 directions. Indeed, because of the background $B$-duality mentioned after (2.6), the T-dual of $|D0\rangle_U$, would be the familiar

$$|D24'\rangle_U = \sum_p \exp \left( \sum_{n \geq 1} -\frac{1}{n} \alpha_i^{i-n} D_{ij} \bar{\alpha}_j^{i-n} \right) |(p, -p)\rangle,$$

where $D_{ij} = (1 - B)^{-1}_{il} (1 + B)_{lj}$. This expression can be obtained from (3.3) by rotating the right-movers with the matrix $D$. Physically, (3.3) therefore describes a D24-brane with a (Born-Infeld) flux $F = -B$ which compensates for the background $B$-field.

For the twisted sector contribution we have similarly

$$|D0\rangle_T = \exp \left( \sum_{r \geq 1/2} \frac{1}{r} c_{-r} c_{-r}^\dagger \right) |x = 0\rangle$$

11
and\footnote{In the twisted sector, the image of $\|D0\|_T$ under T-duality would involve all $2^{24}$ fixed points (see for instance \cite{23,29}).}

\[ \|D24\|_T = -\exp \left( \sum_{r \geq 1/2} \frac{1}{r} c^i_{-r} c^i_{-r} \right) |x = 0\rangle, \]  

(3.6)

where $|x = 0\rangle$ denotes the twisted sector ground state localised at the fixed point 0.

### 3.2. More fractional branes

Given this ‘geometric’ D-brane, we can obtain a number of other configurations with a clear geometric interpretation. First of all, we can consider introducing Wilson lines for the D24-brane, thereby placing the D0-brane on a different fixed point $y$ (where $y \in \frac{1}{2} \Lambda_L/\Lambda_L$). In the untwisted sector of the above boundary states this corresponds to introducing $p$-dependent signs in (3.3) and (3.2) as

\[ \|D0, y\|_U = \sum_p (-1)^{2y \cdot p} \exp \left( \sum_{n \geq 1} \frac{1}{n} \alpha^i_{-n} \bar{\alpha}^i_{-n} \right) |(p, p)\rangle \]  

(3.7)

and

\[ \|D24, y\|_U = \sum_p (-1)^{2y \cdot p} \exp \left( \sum_{n \geq 1} \frac{1}{n} \alpha^i_{-n} \bar{\alpha}^i_{-n} \right) |(p, -p)\rangle. \]  

(3.8)

In the twisted sector the relevant modifications are

\[ \|D0, y\|_T = \exp \left( \sum_{r \geq 1/2} \frac{1}{r} c^i_{-r} c^i_{-r} \right) |x = y\rangle \]  

(3.9)

and

\[ \|D24, y\|_T = -\exp \left( \sum_{r \geq 1/2} \frac{1}{r} c^i_{-r} c^i_{-r} \right) |x = y\rangle. \]  

(3.10)

Since there are $2^{24}$ different fixed points, there are $2^{24}$ such configurations. In addition, we can also change the overall sign of the twisted sector contributions; thus in total there should be $2^{25}$ such configurations.

From section 2 it is clear that introducing these signs corresponds precisely to the chiral action of $\Gamma(\Lambda_L) = 2^{1+24}$. Indeed, $i \in \Gamma(\Lambda_L)$ changes the overall sign of the twisted
sector contributions, and the other elements with $R = e$ in (·1) introduce the correct $p$-dependent signs in the untwisted sector (see [27]).

We can also apply a number of asymmetric reflections of the Leech lattice to relate the D0-D24 combination to a Dp-D(24-p) combination. This can be implemented by a suitable lift of an element in (·1) to the extra-special extension $C$ introduced before. Combining these two constructions we therefore conclude that the image of the above D24-D0 brane under the action of $C = 2^{1+24}(·1)$ gives another geometric brane configuration.

On the other hand, the other generators of the Monster group do not map the D0-D24 brane into another geometrical configuration. This is not really surprising since the oscillators $\alpha_n^i$ do not transform in a representation of the Monster group. (After all, the smallest non-trivial representation of the Monster group has dimension 196883.) This is not in conflict with the claim that the Monster group acts on the conformal field theory since the modes $\alpha_n^i$ are not the modes of an actual state of the theory — they are the modes of the state $\alpha_n^{i-1}|\Omega\rangle$, but this state does not survive the orbifold projection.

3.3. Bulk branes

Apart from the fractional D-brane states we have constructed so far, one also expects there to be bulk D-branes, which can move away from the fixed points of the orbifold. More specifically, if we start with a bulk D0 brane at an arbitrary point of the geometric $\mathbb{Z}_2$ orbifold (i.e. a pair of D0-branes at $x$ and $-x$ on the covering space), then adding two D24-branes with Wilson lines $x$ and $-x$ (in suitable units) on the covering space gives an orbifold invariant combination. This D-brane will have moduli (corresponding to the position of one of the two D0-branes, say), and so will be part of a continuum.

Such a bulk D-brane can be obtained by combining two coincident fractional D-branes with cancelling twisted sector components. Indeed, as we shall see later on, the open string spectrum between two such states does indeed contain the appropriate marginal operators.

4. Symmetric D-branes

In principle, the only symmetry the boundary states of a (bosonic) string theory are required to preserve is the conformal symmetry, i.e. the boundary states must satisfy

$$(L_n - L_{-n}) \langle B \rangle = 0.$$ (4.1)
In general it is difficult to classify all such conformal boundary conditions (see however [30,31]), and one therefore often restricts the problem further by demanding that the boundary states preserve some larger symmetry. Examples are the familiar Dirichlet or Neumann branes whose corresponding open strings satisfy Dirichlet or Neumann boundary conditions at the ends. The boundary states then preserve a U(1) current algebra for each coordinate,

\[
(\alpha^i_n \pm \bar{\alpha}^i_{-n}) \| B \rangle = 0 ,
\]

where \( \alpha^i_n \) are the modes associated to \( X^i \), and the sign determines whether \( X^i \) obeys a Dirichlet or a Neumann condition on the world-sheet boundary. Typically, the more symmetries one requires a D-brane to preserve, the easier it is to construct and classify the relevant boundary states. However, in general one then only finds a subset of all physically relevant boundary states.

For the case of the Monster theory we have so far only considered D-branes that consist of orbifold invariant combinations of Dirichlet and Neumann branes. While these are consistent D-branes, they are unlikely to describe all the lightest D-branes of the theory since none of these branes couples to the asymmetrically twisted closed string states. Furthermore, it is rather unnatural to consider gluing conditions that involve the modes \( \alpha^i_n \), since these modes are not actually present in the theory. (The modes are only present in the theory before orbifolding, and thus the characterisation of these branes relies on a specific realisation of the Monster conformal field theory in terms of an orbifold construction.)

Instead, we will in the following consider D-branes that are characterised by a gluing condition that can be formulated within the Monster conformal field theory. More specifically, we shall analyse the branes that preserve the subalgebra of the original vertex operator algebra consisting of Monster-invariant states. We will denote this \( \mathcal{W} \)-algebra by \( \mathcal{W} \); it contains the Virasoro algebra, but is in fact strictly larger. Let \( \mathcal{W}_n \) denote the modes of an element of \( \mathcal{W} \). Then the gluing condition we will impose, generalising (4.1), is

\[
\left( W_n - (-1)^s \bar{W}_{-n} \right) \| B \rangle = 0 ,
\]

where \( s \) is the spin of \( \mathcal{W} \).

As we shall see, this gluing condition is restrictive enough to allow a complete classification of those solutions. In section 5 we will discuss which of the examples in section 3 are captured by this construction.
It turns out that the mathematical results we need in our construction and classification of these symmetric boundary states are known in much wider generality. Therefore, in the present section, we will work in a broader framework than strictly necessary to analyse the Monster theory.

4.1. General framework

We will work in the general framework studied in [28]. Suppose \( \mathcal{H}_0 \) is a simple vertex operator algebra, \textit{i.e.} a vertex operator algebra that does not have any non-trivial ideals. (An introduction to these matters can be found in [32,1,33].) Let us furthermore assume that \( \mathcal{H}_0 \) admits a continuous action of a compact Lie group \( G \) (which may be finite). In the example of the Monster theory, \( \mathcal{H}_0 \) corresponds to the chiral Monster theory \( \mathcal{H}_M \) and \( G \) is the Monster group \( M \). Let \( \mathcal{W} \) be the vertex operator subalgebra of \( \mathcal{H}_0 \) consisting of the \( G \)-invariants.

The main result of [28] is that \( \mathcal{H}_0 \) can be decomposed as

\[
\mathcal{H}_0 = \bigoplus_{\lambda} R_\lambda \otimes \mathcal{H}_\lambda, \tag{4.4}
\]

where the sum runs over all irreducible representations \( R_\lambda \) of \( G \), and each \( \mathcal{H}_\lambda \) is an irreducible representation of \( \mathcal{W} \). Moreover, the \( \mathcal{H}_\lambda \) are inequivalent for different \( \lambda \). The first few terms of the characters of \( \mathcal{H}_\lambda \) can be found in [25].

The total space of states (the generalisation of \( \mathcal{H} \), see (2.1)) then has the decomposition

\[
\mathcal{H}_0 \otimes \overline{\mathcal{H}_0} = \bigoplus_{\lambda,\mu} \left( R_\lambda \otimes \overline{R}_\mu \right) \otimes \left( \mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\mu} \right), \tag{4.5}
\]

where \( \overline{R} \) denotes the conjugate \( M \)-representation of \( R \), and the sum extends over all tensor products of irreducible representations of \( G \).

In the following we shall construct boundary states that preserve \( \mathcal{W} \) for this theory. We shall make use of the fact that \( \mathcal{H}_0 \otimes \mathcal{H}_0 \) has the decomposition (4.5). In general,

\* For the case of the Monster theory, this result implies that \( \mathcal{W} \) must be strictly larger than the Virasoro algebra: otherwise the modular invariant partition function (2.28) would equal a finite sum of irreducible Virasoro characters with \( c = 24 \). One can also check directly, by comparing characters, that \( \mathcal{W} \) contains at least one additional primary field of conformal weight 12.

\† Table 2 of that paper does not contain any entries for the (conjugate pairs of) irreducible Monster representations IRR16-IRR17 and IRR26-IRR27. The corresponding \( \mathcal{H}_\lambda \) are not trivial, but they only contain states whose conformal weight is bigger than \( h = 51 \).
\( \mathcal{H}_0 \otimes \mathcal{H}_0 \) is only the vacuum sector of the full conformal field theory, and the theory also contains sectors that correspond to other representations of \( \mathcal{H}_0 \). A priori, we do not know whether these other sectors also have a decomposition as (4.3), and we shall therefore restrict ourselves to \textit{self-dual} theories, \textit{i.e.} theories for which the full conformal field theory is actually given by the vacuum sector alone. This is clearly the case for the Monster theory. Self-dual conformal field theories have the property that their character is invariant under the S-modular transformation.

4.2. \textit{Ishibashi states}

We are interested in constructing D-branes that preserve the full \( \mathcal{W} \)-symmetry. Each such D-brane state can be written in terms of \( \mathcal{W} \)-Ishibashi states; the Ishibashi state is uniquely fixed (up to normalisation) by the gluing condition (4.3), and for each term in (4.3) for which the left- and right-moving \( \mathcal{W} \)-representations are conjugate, we can construct one such Ishibashi state. (See for example [15] for a review of these issues.) Thus we see from (4.5) that the \( \mathcal{W} \)-Ishibashi states are labelled just like matrix elements of representations of \( \Gamma \),

\[
| R_\lambda; i, \bar{j} \rangle \rangle \in \left( R_\lambda \otimes \overline{R}_\lambda \right) \otimes \left( \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda \right),
\]

where \( i \in R_\lambda, \bar{j} \in \overline{R}_\lambda \) are a basis for the representation \( R_\lambda \) and \( \overline{R}_\lambda \), respectively. The relevant overlap between these Ishibashi states is given as

\[
\langle \langle R_\lambda; i_1, \bar{j}_1 | q^{\frac{1}{2}(L_0+L_0 - \overline{R})} | R_\mu; i_2, \bar{j}_2 \rangle \rangle = \delta_{\lambda,\mu} \delta_{i_1,i_2} \delta_{\bar{j}_1,\bar{j}_2} \chi_{\mathcal{H}_\lambda}(q),
\]

where \( \chi_{\mathcal{H}_\lambda}(q) \) is the character of the irreducible \( \mathcal{W} \)-representation \( \mathcal{H}_\lambda \).

4.3. \textit{Consistent boundary states}

Next we want to construct actual D-brane states, which are certain linear combinations of the Ishibashi states. D-brane states are (at least partially) characterised by the property that they satisfy Cardy’s condition [10], \textit{i.e.} that they give rise to a positive integer number of \( \mathcal{W} \)-representations (or more generally, Virasoro representations) in the open string. One D-brane state that satisfies this condition can be easily constructed: it is given by

\[
\| e \rangle \rangle = \sum_{\lambda,i} | R_\lambda; i, \bar{i} \rangle \rangle,
\]

16
where the sum extends over all irreducible representations $R_\lambda$ of $G$, and $i$ labels a basis of the representation $R_\lambda$. In order to check that $\|e\|$ satisfies the Cardy condition, we observe that

$$
\langle e \| q^{(L_0 + \bar{L}_0 - c)} \| e \rangle = \sum_{\lambda; i} \chi_{H_\lambda}(q) = \sum_{\lambda} \dim(R_\lambda) \chi_{H_\lambda}(q) \equiv F(\tau),
$$

(4.9)

where we have used that $H_0$ decomposes as in (4.4), and where $F(\tau)$ is the character (or partition function) of the chiral conformal field theory $H_0$. The character of a self-dual theory is invariant under the modular transformation $\tau \mapsto -1/\tau$, and therefore

$$
\langle e \| q^{\frac{1}{2}(L_0 + \bar{L}_0 - \bar{c})} \| e \rangle = F(-1/\tau) = \sum_{\lambda} \dim(R_\lambda) \chi_{H_\lambda}(\bar{q}),
$$

(4.10)

where $\bar{q} = e^{-2\pi i/\tau}$. Since $\dim(R_\lambda)$ are positive integers, this demonstrates that the boundary state $\|e\|$ satisfies the Cardy condition. For the special case of the Monster theory, $F(\tau) = j(\tau) - 744$, which is indeed invariant under the S-modular transformation.

As our notation suggests the boundary state (4.8) is associated to the identity element of the automorphism group $G$. We want to show next that there is actually a boundary state for each group element of $G$. The different boundary states are transformed into one another by the left-action of $G$. Thus we define

$$
\|g\rangle = g \|e\rangle = \sum_{\lambda; i,j} D^{R_\lambda}_{ji}(g) \| R_\lambda; j, i \rangle ,
$$

(4.11)

where $D^{R}_{ji}(g)$ is the matrix element of $g \in G$ in the representation $R$,

$$
\sum_j D^{R}_{ij}(h) D^{R}_{ji}(g) = D^{R}_{ij}(hg).
$$

(4.12)

The self-overlap of each of these branes is in fact the same as (4.9) above: it follows directly from the definition of (4.11) that

$$
\langle g \| q^{\frac{1}{2}(L_0 + \bar{L}_0 - \bar{c})} \| g \rangle = \sum_{\lambda; i,j} \overline{D^{R_\lambda}_{ji}(g)} D^{R_\lambda}_{ji}(g) \chi_{H_\lambda}(q).
$$

(4.13)

Since each group representation can be taken to be unitary, we have

$$
\sum_{i,j} \overline{D^{R_\lambda}_{ji}(g)} D^{R_\lambda}_{ji}(g) = \sum_{i,j} D^{R_\lambda}_{ij}(g^{-1}) D^{R_\lambda}_{ji}(g) = \sum_i D^{R_\lambda}_{ii}(e) = \dim(R_\lambda).
$$

(4.14)
Inserting (4.14) into (4.13) we thus reproduce (4.9). Incidentally, this also shows that all these D-branes have the same mass, since the mass is determined by the $q \to 0$ limit of (4.13) [34].

It remains to show that the overlap between two different branes of the form (4.11) also gives rise to a positive integer number of representations of $\mathcal{W}$ in the open string. Using the same argument as above in (4.13) and (4.14) we now find

$$\langle \langle g \parallel q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{24})} \parallel h \rangle \rangle = \sum_{\lambda, i, j} D^{R_{ji}}(g) D^{R_{ji}}(h) \chi_{H_{\lambda}}(q)$$

$$= \sum_{\lambda} \text{Tr}_{R_{\lambda}}(g^{-1}h) \chi_{H_{\lambda}}(q)$$

$$= \text{Tr}_{H_{\hat{g}}}(g^{-1}hq^{L_0 - \frac{c}{24}}). \tag{4.15}$$

It follows from standard orbifold considerations [7] that under $\tau \mapsto -1/\tau$ we have

$$\text{Tr}_{H_0}(\hat{g}q^{L_0 - \frac{c}{24}}) = \text{Tr}_{H\hat{g}}(\tilde{q}^{L_0 - \frac{c}{24}}), \tag{4.16}$$

where we have written $\hat{g} = g^{-1}h$, and $H\hat{g}$ is the (unique) $\hat{g}$-twisted representation of the conformal field theory $H_0$ [35]. Since the $\hat{g}$-twist acts trivially on the generators of $\mathcal{W}$, we can decompose the representation $H\hat{g}$ in terms of representations of $\mathcal{W}$ as

$$H\hat{g} = \bigoplus_{j} D_{j} \otimes H_{j}, \tag{4.17}$$

where each $H_{j}$ is an irreducible representation of $\mathcal{W}$, and $D_{j}$ is some multiplicity space. Thus it follows that

$$\langle \langle g \parallel q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{24})} \parallel h \rangle \rangle = \sum_{j} \text{dim}(D_{j}) \chi_{H_{j}}(\tilde{q}). \tag{4.18}$$

In particular, this therefore implies that the relative overlaps also satisfy Cardy’s condition.

For the case of the Monster theory, the last line of (4.15) is precisely the McKay-Thompson series (2.33), which thus appears very naturally in the study of monstrous D-branes!
4.4. Completeness

The construction of a set of consistent boundary states in the previous subsection was fairly general. Now we want to argue that at least in some cases (including the Monster theory) this set is complete, in the sense that it contains all (fundamental) D-branes preserving $\mathcal{W}$. In the following we shall restrict ourselves to the case where $G$ is a finite group.

If $G$ is a finite group, there are only finitely many $\mathcal{W}$-preserving Ishibashi states. Whenever this is the case, one can show the completeness of the boundary states by the following algebraic argument: suppose there are $N$ $\mathcal{W}$-Ishibashi states, $|I_1\rangle, \ldots, |I_N\rangle$, and that we have managed to find $N$ boundary states, $|B_1\rangle, \ldots, |B_N\rangle$ that are linearly independent over the complex numbers. (This is the case in our example as we shall show momentarily.) Then since every boundary state is a linear combination of Ishibashi states, we can express the $N$ Ishibashi states in terms of the $N$ boundary states, i.e. we can find an invertible matrix $A$ such that

$$|I_j\rangle = \sum_i A_{ij} |B_i\rangle.$$ (4.19)

Suppose now that there exists another boundary state $|B\rangle$ that is compatible with the boundary states $|B_i\rangle$ with $i = 1, \ldots, N$. (By this we mean that the various overlaps between $|B\rangle$ and the $|B_i\rangle$ lead to positive integer numbers of $\mathcal{W}$ characters in the open string channel.) Since $|B\rangle$ is a boundary state, it can be written as a linear combination of Ishibashi states, and therefore, because of (4.19), as a linear combination of the boundary states $|B_i\rangle$. Thus we have shown that

$$|B\rangle = \sum_i C_i |B_i\rangle,$$ (4.20)

where the $C_i$ are some (in general complex) constants. In order to prove that the $|B_i\rangle$ are all the (fundamental) boundary states it therefore only remains to show that the $C_i$ are in fact non-negative integers. This will typically follow from the fact that $|B\rangle$ is compatible with the $|B_i\rangle$ in the sense described above. For example, if the $|B_i\rangle$ have the property that the overlap between $|B_i\rangle$ and $|B_j\rangle$ only contains the vacuum representation in the open string provided that $i = j$, and that the vacuum representation occurs with multiplicity one if $i = j$ (again this is the case for the Monster theory as we shall show momentarily) then this can be shown as follows. We consider the overlap

$$\langle B_i | g^{\frac{1}{2}(L_0-\bar{L}_0-\frac{c}{24})} | B\rangle,$$ (4.21)
and transform into the open string description. From the above assumption and (4.20) it then follows that the vacuum character in the open string occurs with multiplicity $C_i$. Thus it follows that $C_i$ has to be a non-negative integer since $|B⟩$ is compatible with $|B_i⟩$.

For the case at hand, one can actually show that the two assumptions made above are satisfied. First of all, it follows from the above analysis that there are

$$\sum_\lambda \dim(R_\lambda)^2 = \dim(G)$$

(4.22)

Ishibashi states, which therefore agrees with the number of boundary states described by (4.11). By the Peter-Weyl Theorem (or the appropriate simpler statement for finite groups) these $\dim(G)$ boundary states are linearly independent. Given the above argument, this shows that the boundary states are all fundamental boundary states provided they are ‘orthogonal’, i.e. provided that the identity only arises in the open string of the overlap of each boundary state with itself (where it arises with multiplicity one). For the Monster theory, the latter statement is obviously correct since the open string overlap between each boundary state and itself is simply $j(\tilde{q}) - 744$ which starts indeed with $1\tilde{q}^{-1} + \cdots$. Thus it only remains to check that the overlap between different boundary states starts with $0\tilde{q}^{-1} + \cdots$ in the open string. This is simply the question of what the leading behaviour of the S-modular transform of the different McKay-Thompson series (for $g \neq e$) is. It has recently been argued that the only McKay-Thompson series that has a term of order $\tilde{q}^{-1}$ in its S-modular transform is the series associated to the identity element [36]. We have also checked this property for a number of McKay-Thompson series explicitly.

We have thus shown that there are precisely $|M| \mathcal{W}$-preserving boundary states $|g⟩$, labeled by $g \in M$. It is interesting to ask how these boundary states transform under the Bimonster group. First of all, it is easy to verify that the elements of $M \times M$ act as

$$(h_L, h_R) |g⟩ = |h_L h_R^{-1} g⟩.$$  

(4.23)

Indeed, we calculate

$$(h_L, h_R) |g⟩ = (h_L, h_R) \sum_{\lambda;i,j} D_{ji}^{R_\lambda}(g) |R_i,j;\bar{i}⟩$$

$$= \sum_{\lambda;i,j,k,l} D_{ji}^{R_\lambda}(g) D_{kj}^{R_\lambda}(h_L) \overline{D_{li}^{R_\lambda}(h_R)} |R_\lambda;k,\bar{l}⟩$$

$$= \sum_{\lambda;i,j,k,l} D_{kj}^{R_\lambda}(h_L) \overline{D_{ji}^{R_\lambda}(g)} D_{li}^{R_\lambda}(h_R^{-1}) |R_\lambda;k,\bar{l}⟩$$

$$= \sum_{\lambda;k,l} D_{kl}^{R_\lambda}(h_L h_R^{-1}) |R_\lambda;k,\bar{l}⟩$$

$$= |h_L h_R^{-1} g⟩.$$  

(4.24)
Next we note that the generator $\sigma$ of the $\mathbb{Z}_2$ that exchanges the left- and right-movers is an anti-linear map that replaces the Ishibashi states $| R_{\lambda; j, \bar{i}} \rangle$ by $| R_{\lambda; i, \bar{j}} \rangle$. It therefore acts on the boundary states as

$$\sigma | g \rangle = \sum_{\lambda; j, i} D_{ij}^{R_{\lambda}}(g) | R_{\lambda; i, \bar{j}} \rangle = \sum_{\lambda; j, i} D_{ij}^{R_{\lambda}}(g^{-1}) | R_{\lambda; i, \bar{j}} \rangle = | g^{-1} \rangle,$$

where we have again used that the representations of the Monster group are unitary. The actions (4.23) and (4.25) combine to give a full representation of the Bimonster group since

$$\sigma (h_1, h_2) | g \rangle = | h_2 g^{-1} h_1^{-1} \rangle = (h_2, h_1) \sigma | g \rangle.$$  

Thus we have shown that the $W$-preserving boundary states fall into a representation of the Bimonster group.

4.5. Factorisation constraint

In the previous subsections we have constructed a family of $W$-preserving boundary states that satisfy all relative Cardy conditions. Furthermore, we have shown that this set of boundary states is complete. In addition to the Cardy conditions, consistent boundary states also have to satisfy the ‘sewing relations’ of [37]. One of these conditions is the factorisation (or cluster) condition that requires that certain bulk-boundary structure constants satisfy a set of non-linear equations (sometimes also referred to as the classifying algebra in this context). It was shown in [38] that a $W$-preserving boundary state $| B \rangle$ satisfies this factorisation constraint provided that it preserves the full symmetry algebra $\mathcal{H}_0$ up to conjugation by an element in $g \in G$, i.e. provided

$$\left( g S_n g^{-1} - (-1)^{S_n} \delta_{-n} \right) | B \rangle = 0,$$

for all modes of fields in $\mathcal{H}_0$. (Here $g \in G$ depends on the boundary condition $| B \rangle$.) Since $W_n \in \mathcal{W}$ is invariant under the action of $g \in G$, (4.27) contains (4.3) as a special case.

As we shall now explain, the boundary states we have constructed actually satisfy (4.27); in fact, we have

$$\left( g S_n g^{-1} - (-1)^{S_n} \delta_{-n} \right) | g \rangle = 0$$

for each $g \in \mathcal{M}$. Given the decomposition (4.5), the boundary state corresponding to $| e \rangle$ is the unique boundary state that preserves the full symmetry algebra (the theory only contains a single Ishibashi state that preserves this algebra), and thus (4.28) holds for $g = e$. The general statement then follows from this using (4.23).
5. Application to the Monster: fractional and bulk branes

Let us now return to the specific case of the Monster theory. As we have shown in the previous section, the D-branes that preserve the $W$-algebra of Monster invariants $\mathcal{W}$ are labelled by group elements in $\mathbf{M}$. We want to analyse now how the various D-branes that we constructed in section 3 fit into this analysis. In order to do so it is useful to describe the ‘geometrical’ boundary states of section 3.1 in more detail.

5.1. Fractional D0-D24 at the origin

In the untwisted sector of the geometric orbifold, the constituent boundary states are (up to normalisation) given as

$$|D_{24}\rangle_{U} = \sum_{p} \exp \left( \sum_{n \geq 1} \frac{1}{n} \alpha_{-n}^{i} \bar{\alpha}_{-n}^{i} \right) |(p, -p)\rangle \quad (5.1)$$

and

$$|D_{0}\rangle_{U} = \sum_{p} \exp \left( \sum_{n \geq 1} \frac{1}{n} \alpha_{-n}^{i} \bar{\alpha}_{-n}^{i} \right) |(p, p)\rangle \quad (5.2)$$

Let us expand out these boundary states, and in particular, consider the contributions for $h = \bar{h} = 0, 1, 2$. At $h = \bar{h} = 0$, both boundary states are proportional to the vacuum. At $h = \bar{h} = 1$, the $|D_{24}\rangle_{U}$ boundary state is proportional to

$$- \sum_{i} \alpha_{-1}^{i} \bar{\alpha}_{-1}^{i} |(0, 0)\rangle, \quad (5.3)$$

while the $|D_{0}\rangle_{U}$ state is proportional to

$$\sum_{i} \alpha_{-1}^{i} \bar{\alpha}_{-1}^{i} |(0, 0)\rangle. \quad (5.4)$$

The sum $|D_{24}\rangle_{U} + |D_{0}\rangle_{U}$ therefore does not have any contribution at $h = \bar{h} = 1$. This is in agreement with the fact that the Monster theory does not have any states of $h = \bar{h} = 1$. At $h = \bar{h} = 2$ we get for $|D_{24}\rangle_{U}$

$$- \frac{1}{2} \sum_{i} \alpha_{-2}^{i} \bar{\alpha}_{-2}^{i} |(0, 0)\rangle + \frac{1}{2} \sum_{i,j} \alpha_{-1}^{i} \alpha_{-1}^{j} \bar{\alpha}_{-1}^{i} \bar{\alpha}_{-1}^{j} |(0, 0)\rangle + \sum_{p:p^2=4} |(p, -p)\rangle, \quad (5.5)$$

while for $|D_{0}\rangle_{U}$ we have

$$\frac{1}{2} \sum_{i} \alpha_{-2}^{i} \bar{\alpha}_{-2}^{i} |(0, 0)\rangle + \frac{1}{2} \sum_{i,j} \alpha_{-1}^{i} \alpha_{-1}^{j} \bar{\alpha}_{-1}^{i} \bar{\alpha}_{-1}^{j} |(0, 0)\rangle + \sum_{p:p^2=4} |(p, p)\rangle. \quad (5.6)$$
At \( h = \bar{h} = 2 \) the sum of \( |D24\rangle_U + |D0\rangle_U \) therefore has the contribution

\[
\sum_{i,j} \bar{\alpha}_{-1}^i \alpha_{-1}^j \bar{\alpha}_{-1}^i \alpha_{-1}^j |(0,0)\rangle + \sum_{p:p^2=4} \frac{1}{\sqrt{2}} (|p\rangle + | -p\rangle) L \otimes \frac{1}{\sqrt{2}} (|p\rangle + | -p\rangle) R.
\]

(5.7)

In the last sum we have written the momenta as tensor products of left- and right-moving momenta.

Next we recall that the chiral Monster theory has 196884 states with \( h = 2 \); of these there are \( \frac{24 \cdot 25}{2} \) states of the form \( \alpha_{-1}^i \alpha_{-1}^j |0\rangle \) and 98280 states of the form \( \frac{1}{\sqrt{2}} (|p\rangle + | -p\rangle) \) with \( p^2 = 4 \) (as well as \( 24 \cdot 2^{12} \) states coming from the twisted sector). What the above calculation shows is that of the 196883 \( \bar{W} \)-Ishibashi states at \( h = \bar{h} = 2 \), those that come from the untwisted sector (i.e. the \( \frac{24 \cdot 25}{2} + 98280 - 1 \)^2 Ishibashi states coming from the first two types of states minus the stress-energy tensor) contribute only if the left-label of the Monster representation is the same as the right-label of the Monster representation. Furthermore, all these diagonal states appear with the same coefficient. If the D0-D24 combination is one of the boundary states we have constructed before, then the group element \( g \) must therefore have the property that \( D_{ij}^{196883} (g) = \delta_{ij} \) if \( i \) and \( j \) are untwisted labels. It is also clear that \( D_{ij}^{196883} (g) = 0 \) if \( i \) and \( j \) describe one untwisted and one twisted label since the twisted sector of the geometric \( \mathbb{Z}_2 \) orbifold (in which theory the D0-D24 boundary state is constructed) consists of those states that are twisted with respect to both the left- and the right- asymmetric orbifold. Thus the corresponding group element \( g \) must give rise to a representation matrix of the form

\[
D^{196883} (g) = \begin{pmatrix} 1 & 0 \\ 0 & R(g) \end{pmatrix},
\]

(5.8)

where we have written the matrix in block-diagonal form, with the two blocks corresponding to the untwisted and the twisted sector states, respectively. Here \( R(g) \) is a 98304×98304 matrix, describing the components of \( g \) in the representation \( 196883 \) with respect to the twisted sector states.

We also know that, in the above notation,

\[
D^{196883} (i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(5.9)

and thus it follows that

\[
D^{196883} (gig^{-1}i) = 1.
\]

(5.10)
Since the Monster group does not have any non-trivial normal subgroups, it follows that $g i g^{-1} i = e$, and therefore that $g$ is in the centraliser of $i$, i.e. in $C$. On the other hand, we know how the elements in $C$ act on the 98579 untwisted sector states of $196883$, and it is easy to see that there are only two elements in $C$ that give rise to a matrix of the form (5.8) above: either $g = e$ or $g = i$. Let us choose to identify the fractional D0-D24 discussed above with $|e\rangle$. As we will see in the next subsection, $|i\rangle$ then corresponds to another fractional D0-D24 at the origin, differing from $|e\rangle$ in the overall sign of the twisted sector components of its boundary state.

Apart from the direct boundary state argument given above, there is another way to argue for the identification of $|e\rangle$ (and $|i\rangle$) with a fractional D0-D24 at the origin. We expect that such a D0-D24 is at least invariant under the left-right symmetric (geometric) action of a suitable lift of the simple Conway group $(\cdot 1)$ to the extra-special extension $C$ (since the elements in $(\cdot 1)$ correspond to Leech lattice automorphisms, which leave the origin invariant). Indeed, it is obvious from (4.23) that both $|e\rangle$ and $|i\rangle$ are in fact invariant under the left-right symmetric (diagonal) action of any element of $C$. Furthermore, $e$ and $i$ are the only group elements in $M$ with this property.

5.2. More fractional branes

In subsection 3.2 we saw that by acting on the fractional D0-D24 brane of subsection 3.1 with elements of the group $C = 2^{1+24}_+ (\cdot 1)$ acting on the left, one obtains other fractional D-branes with a clear geometric interpretation. From the previous section, we know that the fractional D0-D24 is described by the boundary state $|e\rangle$, and in subsection 4.3 we showed that the left action of an element $g$ of $C$ results in the boundary state $|g\rangle$. Thus the boundary states labelled by an element of $C = 2^{1+24}_+ (\cdot 1)$ have a geometric interpretation as fractional branes.

In particular, the boundary state $|i\rangle$ associated to the involution $i$ corresponds to a fractional D0-D24 brane at the origin, which differs from $|e\rangle$ by the sign of the twisted sector components of its boundary state: if we normalised the various boundary states so that $|e\rangle$ is given by (B.1), then

$$
|i\rangle = \frac{1}{\sqrt{2}} \left( |D0\rangle_U - |D0\rangle_T + |D24\rangle_U - |D24\rangle_T \right),
$$

where the subscripts $U$ and $T$ denote again the components in the untwisted sector and in the sector twisted by the geometric $\mathbb{Z}_2$, respectively.
5.3. *Bulk branes*

None of the branes we described in section 4 have any moduli. Indeed, as we saw in subsection 4.3, the self-overlap of any of these branes leads to the one-loop partition function $F(\tilde{q}) = j(\tilde{q}) - 744$ in the open string, which therefore does not have any massless states.‡ On the other hand, we saw in section 3.3 that the theory should have a continuum of bulk D-brane states. It therefore follows that bulk branes generically cannot preserve $\mathcal{W}$. They are therefore examples of physical D-branes (preserving the conformal symmetry) that are not captured by the construction in section 4 (where we restricted our attention to D-branes preserving the larger algebra $\mathcal{W}$).

One may wonder whether these bulk branes can be thought of as being built out of fractional branes. In particular, one may expect that combinations of fractional branes with a vanishing twisted sector contribution can combine to form a bulk brane. The simplest example of such a combination of fractional branes is described by the superposition of $|e\rangle$ and $|i\rangle$. In order to check whether this configuration of branes possesses massless modes that describe the corresponding moduli, we determine the cylinder diagram

$$
\left(\langle e \parallel + \langle i \parallel \right) q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{24})} \left(\langle e \rangle + \langle i \rangle \right) = 2\chi_e(q) + 2\chi_i(q).
$$

(5.12)

We want to write this amplitude in the open string channel, i.e. in terms of the open string variable $\tilde{q} = \exp(-2\pi i/\tau)$. As we have explained before,

$$
\chi_e(q) = j(\tau) - 744 = j(-1/\tau) - 744 = \tilde{q}^{-1} + 0 + 196884\tilde{q} + \cdots,
$$

(5.13)

and thus we do not get any massless modes from $\chi_e$. (This is as expected since $|e\rangle$ and $|i\rangle$ separately do not have any massless modes.) On the other hand, $i$ lies in the class $2B$ of the ATLAS [39], and using [26], we find for the second term

$$
\chi_i(q) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24 = 24 + 2^{12}\tilde{q}^{1/2} + \cdots.
$$

(5.14)

In particular, the combined system has 48 massless modes; 24 of these correspond to the moduli that describe the 24 different directions in which the bulk D0-brane can move off the fixed point. (The other 24 massless states correspond on the covering space to an open

‡ Incidentally, this property of the branes is special to the Monster theory; all other known self-dual conformal field theories contain massless states in their chiral partition function.
string connecting a D0 and its image D0, which has massless modes when the bulk brane is at a fixed point.)

Finally, let us remark that these ‘bulk branes’ (which do not preserve the full \( W \)-symmetry) have higher mass than the \( W \)-preserving branes labelled by group elements in \( M \) (as follows by comparing their boundary states in the limit \( q \to 0 \) \([3,34]\)). It therefore seems plausible that the lightest branes of the theory preserve the full \( W \)-symmetry, and therefore that they fall into a representation of the Bimonster group.

6. Conclusions

In this paper we have shown that the D-branes preserving the chiral algebra of Monster invariants transform in the regular representation of both copies of the Monster group (and define a representation of the Bimonster). Although this does not give a complete classification of all possible D-brane states in the Monster theory, it does provide evidence that the Bimonster symmetry of the perturbative spectrum extends to a nonperturbative symmetry of the full theory. In particular, it seems likely that the D-brane states we have constructed are the lightest D-branes in the spectrum, and that all other D-brane states can be formed as composites of these building blocks.

In this paper we have restricted our attention to conformal field theories that only consist of a vacuum sector, \( i.e. \) that are self-dual. This assumption guaranteed that the decomposition (4.4) (that is only known to hold for the vacuum sector \([28]\)) can be used to decompose the full space of states as in \((4.5)\). It seems plausible that the decomposition (4.4) may hold more generally for an arbitrary representation. This would then suggest that our construction may generalise further. This idea is also supported by the observation that our result is structurally very similar to what was found in \([10]\) for the (non-self-dual) WZW model corresponding to \( su(2) \) at level \( k = 1 \). We hope to come back to this point in a future publication.

The techniques described here may also be useful in trying to obtain a more systematic understanding of D-branes in asymmetric orbifolds. (Previous attempts at constructing D-branes in asymmetric orbifolds have been made in \([11,42]\).) In particular, some of the branes described above (namely those that correspond to group elements in \( M \setminus C \)) actually involve Ishibashi states from asymmetrically twisted closed string sectors.

Finally, we hope that the perspective we have discussed here might be of use to mathematicians trying to obtain a more conceptual understanding of monstrous moonshine.
Conformal field theories with boundaries have been less well developed in the mathematical literature. In a physical framework the boundaries are associated to D-branes and open strings which have endpoints on the D-brane. We have shown here that the McKay-Thompson series which are the subject of the genus zero moonshine conjectures arise naturally in the open string sector of the closed string theory with Monster symmetry. Perhaps this will suggest new approaches to the moonshine conjectures.

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