THE DEGREE OF BOWEN FACTORS AND INJECTIVE CODINGS OF DIFFEOMORPHISMS

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Abstract. We show that symbolic finite-to-one extensions of the type constructed by O. Sarig for surface diffeomorphisms induce Hölder-continuous conjugacies on large sets. We deduce this from their Bowen property. This notion, introduced in a joint work with M. Boyle, generalizes a fact first observed by R. Bowen for Markov partitions. We rely on the notion of degree from finite equivalence theory and magic word isomorphisms.

As an application, we give lower bounds on the number of periodic points first for surface diffeomorphisms (improving a result of Sarig) and for Sinaï billiards maps (building on a result of Baladi and Demers). Finally we characterize surface diffeomorphisms admitting a Hölder-continuous coding of all their aperiodic hyperbolic measures and give a slightly weaker construction preserving local compactness.

1. Introduction

In this text, a dynamical system is an automorphism of a standard Borel space and all measures are understood to be ergodic, invariant, Borel probability measures. A Markov shift is a “subshift defined by a countable oriented graph”. It is equipped with a standard distance (see Sec. 2 for precise definitions and references).

For a smooth diffeomorphism $f$ of a compact manifold, a measure is called hyperbolic if it has no zero Lyapunov exponent and both a positive and a negative exponent (see, e.g., [16, Chap. S] for background on smooth ergodic theory). A measure is called $\chi$-hyperbolic for some $\chi > 0$, if it is hyperbolic and has no exponent in the interval $[−\chi, \chi]$.

We build conjugacies from the finite-to-one extensions of surface diffeomorphisms of Sarig [24] making them injective while preserving the Hölder-continuity and discarding only a subset negligible with respect to all (invariant probability) measures:

Theorem 1.1. Let $f$ be a diffeomorphism with Hölder-continuous differential mapping a compact boundaryless $C^\infty$ surface $M$ to itself. For any numbers $0 < \chi' < \chi$, there exist a Markov shift $S : X \to X$ and a Hölder continuous map $\pi : X \to M$ such that:

- $\pi \circ S = f \circ \pi$;
- $\pi : X \to M$ is injective;
- $\pi(X)$ has full measure for any $\chi$-hyperbolic measure;
- for any periodic $x \in X$, the periodic point $\pi(x)$ defines a $\chi'$-hyperbolic measure.

Previous injectivity results [4,25,7] were only with respect to a single measure, a restricted class of measures, or by jettisoning the continuity and discarding periodic orbits.

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As an application, we deduce from well-known results on Markov shifts estimates on the periodic counts of surface diffeomorphisms. Consider the hyperbolic periodic points with given minimal period and Lyapunov exponents (defined by identifying a periodic orbit with the obvious measure) bounded away from zero by a number \( \chi > 0 \):

\[ \text{per}_\chi(f, n) := \{ x \in M : \{ f^k(x) : k \in \mathbb{Z} \} \text{ has cardinality } n \text{ and is } \chi\text{-hyperbolic} \}. \]

We denote the cardinality of a set by \(| \cdot |\).

**Theorem 1.2.** Let \( f \) be a \( C^\infty \)-diffeomorphism of a closed surface \( M \). Assume that its topological entropy \( h_\text{top}(f) \) is positive. Then there is some integer \( p \geq 1 \) such that:

\[
\forall \chi < h_\text{top}(f) \liminf_{n \to \infty} \frac{e^{-n-h_\text{top}(f)}|\text{per}_\chi(f, n)|}{p^n} \geq p.
\]

If the diffeomorphism is topologically mixing, one can take \( p = 1 \).

This improves the previous estimate due to Sarig [24]:

\[
\exists p \geq 1 \liminf_{n \to \infty} \frac{e^{-n-h_\text{top}(f)}|\{ x : x = f^n x \text{ and is } \chi\text{-hyperbolic} \}|}{p^n} > 0.
\]

Indeed, not only do we have an explicit constant, but we control the minimal period. By comparison, the estimate (1.4) is compatible, e.g., with \( \text{per}_\chi(f, n) = \emptyset \) for infinitely many \( n \) a multiple of \( p \).

Thanks to works of Baladi and Demers [1] and Lima and Matheus [19], our general results can be applied to the classical collision map \( T_B \) of any two-dimensional Sinaï billiard \( B \) (see, e.g., [8] for background) satisfying the following two conditions:

(BD1) all trajectories have a nontangential collision (see [1, strong finite horizon property before Rem. 1.1]);

(BD2) some combinatorial entropy (denoted by \( h_* \) and introduced in [1, Def. 2.1]) is above some threshold defined in [1, eqs. (1.2) and (1.2)].

We denote by \( \Lambda_B \) the hyperbolicity constant of \( T_B \) from eqs. (2.2)-(2.3) in [1]. We remark that Baladi and Demers also prove that \( h_* = \sup \{ h(T_B, \nu) : \nu \in \text{Prob}_{\text{erg}}(T_B) \} \).

**Theorem 1.5.** If \( T_B \) is the collision map of a two-dimensional Sinaï billiard \( B \) satisfying conditions (BD1) and (BD2), then:

\[
\liminf_{n \to \infty} e^{-n-h_*}|\text{per}_{\Lambda_B}(T_B, n)| \geq 1.
\]

This strengthens [1, Cor. 2.7] by eliminating the possibility of a period, counting the periodic orbits by their minimal periods, and replacing the positive lower bound by the integer 1.

We derive these results by proving a general result about a large class of symbolic dynamics, see our Main Theorem below.

**Theorem 1.1** improves on Sarig’s coding by making it injective. One would also like to have an image as large as possible. The following shows that, in some sense, one cannot much improve on Sarig’s result in this direction:
**Theorem 1.6.** Let \( f \in \text{Diff}^r(M) \) be a diffeomorphism of a closed surface with \( r > 1 \). Then there exist a Markov shift \( S : X \to X \) and a map \( \pi : (S, X) \to (f, M) \) such that \( f \circ \pi = \pi \circ S \) and:

(i) for all \( \mu \in \text{Prob}^{\text{erg}}(f) \) with positive entropy, there is \( \nu \in \text{Prob}(S) \) with \( \pi_*(\nu) = \mu \);

(ii) \( \pi \) is Hölder-continuous for the standard metric on \( X \);

if and only if the exponents of ergodic invariant probability measures with positive entropy are bounded away from zero.

### 1.1. General theorem.

The above will be a consequence of an abstract theorem about factors of Markov shifts. A **symbolic system** \((S, X)\) is some shift-invariant subset of \( \mathcal{A}^\mathbb{Z} \) where \( \mathcal{A} \) the alphabet, is a countable (possibly finite) set, together with the action of the shift \( S : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}} \). We equip \( X \) with the standard distance: 
\[
d(x, y) := \exp(-\inf\{|n| : x_n \neq y_n\}).
\]
A **semiconjugacy** \( \pi : (S, X) \to (T, Y) \) between dynamical systems \((S, X)\) and \((T, Y)\) is a map \( \pi : X \to Y \) such that \( \pi \circ S = T \circ \pi \) and \( \pi(X) \subset Y \). A **one-block code** is a semiconjugacy \( \pi : X \to Y \) between symbolic systems such that \( \pi(x) = \Pi(x_n) \) for some map \( \Pi \) between the alphabets (this map \( \Pi : \mathcal{A} \to \mathcal{B} \) is the code). A map \( \phi \) between two metric spaces is \( 1 \)-Lipschitz if \( d(\phi(x), \phi(y)) \leq d(x, y) \) for all pairs of points \( x, y \).

Recall the following definition from [4], taken from R. Bowen’s analysis of Markov partitions [3].

**Definition 1.7** (Boyle-Buzzi). A semiconjugacy \( \pi : (S, X) \to (T, Y) \) satisfies the **Bowen property** if \( X \) is symbolic and if there is a reflexive and symmetric relation \( \sim \) on its alphabet \( \mathcal{A} \) such that, for all \( x, y \in X \),
\[
\pi(x) = \pi(y) \iff \forall n \in \mathbb{Z} \ x_n \sim y_n,
\]
in which case we say that \( x, y \in X \) are Bowen equivalent and write \( x \approx y \).

The relation \( \sim \) on \( \mathcal{A} \) is called a Bowen relation for \( \pi \) (or admitted by \( \pi \)). It is said to be **locally finite** if \( \{b \in \mathcal{A} : b \sim a\} \) is finite for each \( a \in \mathcal{A} \).

We note that the Bowen property generalizes both:

- **D. Fried’s finitely presented systems** [12] which are exactly the continuous Bowen factors of subshifts of finite type;
- **one-block codes** which admits as a transitive Borel relations the equivalence relations defined by their codes. There is a partial converse: any Bowen semiconjugacy \( \pi \) with a transitive Bowen relation \( \sim \) can be written as \( \pi = \psi \circ \Pi \) where \( \Pi \) is the one-block code defined by \( a \mapsto \{b : b \sim a\} \) and \( \psi : \Pi(X) \to \pi(X) \) is a Borel conjugacy.

We show:

**Main Theorem.** Let \((S, X)\) be a Markov shift on some alphabet \( \mathcal{A} \). Let \( X^\# \) be its regular part, i.e., the set of sequences \( x \in X \) such that, for some \( u, v \in \mathcal{A} \),
\[
u \text{ occurs infinitely many times in } (x_n)_{n \leq 0} \text{ and } \nu \text{ occurs infinitely many times in } (x_n)_{n \geq 0}.
\]
Let \( \pi : (S, X^\#) \to (T, Y) \) be a Borel semiconjugacy such that:

- \((T, Y)\) is a Borel automorphism;
- \( \pi \) is finite-to-one, i.e., \( \pi^{-1}(y) \) is finite for every \( y \in Y \);
- \( \pi \) has the Bowen property with respect to a locally finite relation on \( \mathcal{A} \).
Then there is a Markov shift \((\hat{S}, \hat{X})\) and a 1-Lipschitz map \(\phi : \hat{X} \to X^#\) such that \(\pi \circ \phi : \hat{X} \to Y\) defines an injective semiconjugacy and \(\pi \circ \phi(X^#)\) carries all invariant measures of \(\pi(X^#)\).

Remark that if \(\pi\) is continuous or Hölder-continuous, then so is \(\pi \circ \phi\).

1.2. Further results, comments, and questions. Note that a map \(\phi : \hat{X} \to X\) is 1-Lipschitz if and only if there is a length-preserving map \(\Phi : \bigcup_{n \geq 0} \mathcal{L}_{2n+1}(\hat{X}) \to \bigcup_{n \geq 0} \mathcal{L}_{2n+1}(X)\) such that \(\phi(x)_{[-n,n]} = \Phi(x_{[-n,n]})\) for all \(n \geq 0\). The 1-Lipschitz map in the above theorem may fail to be a one-block code because it may fail to commute with the shift.

Ingredients of the proof. We adapt classical ideas from the theory of subshifts of finite type. The first part builds on tools from finite equivalence theory and more specifically joint work with Mike Boyle [4]. This leads to Theorem 3.3 which is an abstract version of an unpublished result of Sarig [25]. The second part of the proof involves ideas from magic word isomorphisms and the degree of almost conjugacies [21, chap. 9]. It allows to partition according to the number of preimages while preserving the Markov structure. We conclude by injectively coding subsets with larger and larger numbers of preimages.

Good coding for given measures. We can specialize our results to a given measure of interest. For instance, given a surface diffeomorphism with positive topological entropy and a distinguished ergodic measure maximizing the entropy \(\mu\), we obtain an irreducible Markov shift \(X\) and a Hölder-continuous conjugacy \(\pi : X \to M\) such that \(\pi(X)\) has full \(\mu\)-measure. This was implicit in [4, Prop. 6.3]. We refer to [7] for further results in this direction.

Bounds for periodic points. Kaloshin [15] has shown that, \(C^r\)-generically \((1 \leq r < \infty)\), the number of periodic points grows arbitrarily fast with the period. However, these periodic points have Lyapunov exponents going to zero. In fact, Burguet [5] has shown the following logarithmic estimate, for any \(C^\infty\) surface diffeomorphism:

\[
\forall \chi < h_{\text{top}}(f) \lim_{n \to \infty, p \mid n} \frac{1}{n} \log \left| \text{per}_\chi(f, n) \right| = h_{\text{top}}(f).
\]

Question 1. Is there a \(C^\infty\) surface diffeomorphism \(f\) with positive entropy such that, for some \(\chi > 0\):

\[
\limsup_{n \to \infty} e^{-n \cdot h_{\text{top}}(f)} \left| \text{per}_\chi(f, n) \right| = \infty ?
\]

Beyond surface diffeomorphisms. Ben Ovadia’s higher-dimensional generalization [2] of Sarig’s coding also yields finite-to-one semiconjugacies that are Bowen with respect to a locally finite relation. Hence our abstract theorem also applies in this setting.

Better symbolic representations. For a topologically transitive surface diffeomorphism, S. Crovisier, O. Sarig, and the author [7] have shown that, for any given parameter \(\chi > 0\), there is a finite-to-one, Hölder-continuous transitive symbolic dynamics coding a subset carrying all \(\chi\)-hyperbolic measures. Applying our main theorem makes this coding injective but destroys the transitivity. We ask:
**Question 2.** For a topologically transitive $C^\infty$ diffeomorphism of a closed surface and any number $\chi > 0$, can one get a Hölder-continuous injective coding by a transitive Markov shift a subset carrying all $\chi$-hyperbolic measures?

Our Theorems 5.2 and 5.3 below provide partial solutions. We build a Hölder-continuous, finite-to-one coding by a transitive Markov shift whose injectivity set is “large” in a weaker sense than above: it has full measure with respect to a given measure or for all fully supported measures. These theorems are applied to surface diffeomorphisms in [7].

To capture all hyperbolic measures, one can apply Sarig’s construction countably many times with a parameter $\chi$ decreasing to 0. One obtains a sequence of semiconjugacies with larger and larger images but smaller and smaller Hölder exponents. In [4], together with M. Boyle, we were able to “fuse” all these semiconjugacies by using a Borel construction.

**Question 3.** Given a surface diffeomorphism, can one get a continuous finite-to-one coding by a Markov shift of a subset carrying all hyperbolic measures? Can it be done injectively?

Because of Theorem 1.6 one cannot ask for a Hölder-continuous semiconjugacy.

**Local compactness.** Our proof of the Main Theorem does not preserve local compactness. We do not know if it can be done. The following very natural question asked by the referee remains open:

**Question 4.** In Theorem 1.1 is it possible to code using a locally compact Markov shift?

In Appendix B we provide a partial answer: we obtain local compactness but injectivity holds only after restricting to the regular part.

1.3. **Outline of the paper.** In Section 2 we recall some basic definitions and make some comments about Bowen relations. In Section 3 we introduce Bowen quotients inspired by a classical construction of Manning [22] from the theory of Markov partitions. These are an abstract version of a construction of Sarig [25]. The proof rests on Proposition 3.9 which adapts lemmas from the theory of finite equivalence of subshifts of finite type due to Hedlund [14] and Coven and Paul [9, 10, 11]. In Section 4 we combinatorially characterize the fibers with minimal cardinality by adapting the notion of magic word from the theory of almost conjugacy of shifts of finite type (see [21, chap. 9]).

In Section 5 we show how to get a coding with a large injectivity set, especially with respect to a given measure or to all fully supported measures and then deduce the main theorem from the previous constructions. We proceed by induction on the number of preimages. The Bowen quotients make the semiconjugacy injective where its fibers had a given cardinality and then discard these points. The magic word theory preserves the Markov structure and the Bowen property.

In Section 6 we apply our main theorem to Sarig’s coding of surface diffeomorphisms [24] and prove Theorems 1.1 and 1.2 using Newhouse [23] (for smoothness) and Buzzi-Crovisier-Sarig [7] (for transitivity). In Section 7 we prove Theorem 1.6 characterizing surface diffeomorphisms with Hölder-continuous codings. In the Appendices we further discuss the Bowen relation, provide a locally compact construction, and deduce Theorem 1.5.
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2. Definitions and first properties

2.1. Borel systems. A standard Borel space is a set equipped with the Borel $\sigma$-field generated by a Polish topology (i.e., generated by a metric making the space complete and separable, see [17] for background). A dynamical system is an automorphism $S$ of such a space $X$. We denote it by $(S, X)$ (or just $S$ or $X$ when convenient).

A full subset for $S$ is a subset of $X$ with measure equal to 1 for all measures$^1$ of $S$. A subset is null if its complement is a full subset. We say that a property of points holds almost everywhere (or just a.e.) without reference to a measure, if it holds on such a full subset.

By the Lusin-Novikov Theorem [17] (18.10)], the direct image of a Polish space by a countable-to-one Borel map is Borel. In fact, there is a countable partition of the Polish subset.

Lemma 2.1. Let $p : (S, X) \to (T, Y)$ be a Borel semiconjugacy between dynamical systems. If $\nu \in \text{Prob}(T)$ satisfies $p^{-1}(y)$ is finite and nonempty for $\nu$-a.e. $y \in Y$, then there is $\mu \in \text{Prob}(S)$ with $p_* (\mu) = \nu$. In particular, if $p$ is finite-to-one and onto:

- $p_* : \text{Prob}(S) \to \text{Prob}(T)$ is onto;
- for any Borel subsets $U \subset X$, $V \subset Y : U$ is a null subset $\iff p(U)$ is a null subset;
- $V$ is a null subset $\iff p^{-1}(V)$ is a null subset.

2.2. Symbolic dynamics. Let $A$ be a countable (possibly finite) discrete set, called the alphabet. The shift is $(x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$. A symbolic system $(S, X)$ is the restriction $S$ of the shift to some invariant subset $X$ of $A^\mathbb{Z}$ with the usual distance: $d(x, y) := \exp(-\inf\{|n| : x_n \neq y_n\})$. A symbolic system needs not be closed.

For $x \in X$ and integers $a \leq b$, $x[a, b] = x_ax_{a+1} \ldots x_b \in A^{b-a+1}$ and $x[a, b] = x[a, b]$. The set of $X, n$-words is $L_n(X) := \{x|0,n) : x \in X\}$ and the language is $L(X) := \bigcup_{n \geq 0} L_n(X)$. A word $w \in L_n(X)$ has length $|w| := n$. The word $w$ occurs at $n \in \mathbb{Z}$ in some $x \in X$ if $x[n, n+|w|] = w$. An $n$-word $w$ defines a cylinder:

$$[w]_X := \{y \in X : y|0,n) = w\}.$$

As usual, words differing by an integer translation of their indices are identified.

Sarig’s regular set is the subset $X^\# \subset X$ of sequences $x \in X$ such that there are $u, v \in A$ satisfying:

$$\{n \geq 0 : x_{-n} = u\}, \{n \geq 0 : x_n = v\}$$

are both infinite.

$^1$Recall that all measures in this paper are understood to be ergodic and invariant Borel probability measures.
If the alphabet is finite, then \(X^\# = X\). If \(\pi : X \to Y\) is a semiconjugacy with \(X\) a symbolic system, its **regular part** is the restriction \(\pi^\# : X^\# \to Y\). We also write \(\pi^\#\) or \(\pi\) when convenient.

A sequence \(x \in X\) is **word recurrent** if any word \(w\) that occurs in \(x\) occurs infinitely often in both \(x_{\lfloor -\infty, 0 \rfloor}\) and \(x_{\lfloor 0, \infty \rfloor}\). We say that \(w\) occurs i.o. in \(x\) or that \(x\) sees i.o. \(w\). We denote by \(X^\text{rec} \subset X\) the set of such sequences. Note that it carries all invariant probability measures on \(X\) (in particular, it contains all periodic orbits). We have the obvious inclusion \(X^\text{rec} \subset X^\#\).

A **Markov shift** \((S, X)\) is a symbolic system over some alphabet \(A\) such that \(X\) can be characterized as the set of bi-infinite paths on some simple, directed graph \[^2\] \(G\), that is,

\[
X := \{x \in A^\mathbb{Z} : \forall n \in \mathbb{Z} \ x_n \xrightarrow{G} x_{n+1}\}.
\]

The graph \(G\) **describes** the Markov shift \(X\).

### 2.3. Remarks about the Bowen property.

We comment on related notions, the (non)-uniqueness of the symmetric relation involved in its characterization and its eventual extension from the regular part to the whole of a factor.

1) A semiconjugacy may have the Bowen property without being Borel. Indeed, if \(\pi\) is Bowen, then so is \(\varphi \circ \pi\) for any self-conjugacy \(\varphi : Y \to Y\) (i.e., a bijection that commutes with the dynamics).

2) Being Bowen and finite-to-one are independent properties of semiconjugacies (neither implies the other) as shown by the following examples: (i) \(\pi : \{0, 1, 2\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}\) given by the block code \(a \mapsto b \mod 2\) is Bowen and infinite-to-one; (ii) \(\pi : \{0, 1\}^\mathbb{Z} \to \mathbb{S}^1\) given by \(\pi(x) = \exp 2i\pi \sum_{n \geq 0} 2^{-n-1}x_n\) which is at most 2-to-1 but cannot be Bowen since it is neither injective nor constant.

3) On the one hand, many Bowen semiconjugacies do not admit any transitive Bowen relations.\[^3\] On the other hand, the equivalence between sequences must be transitive. Thus Bowen relations are special reflexive and symmetric relations on the alphabet. Bowen \[^3\] p. 13] asked:

**Problem (Bowen).** Let \(A, B\) be two \(n \times n\)-matrices with entries zero or one. Let \(\approx\) be the relation defined on \(X := \{x \in \{1, \ldots, n\}^\mathbb{Z} : \forall p \in \mathbb{Z} \ A(x_p, x_{p+1}) = 1\}\) by \(x \approx y \iff \forall n \in \mathbb{Z} \ B(x_n, y_n) = 1\). Decide whether this relation is transitive. If so, decide whether the shift on \(X/\approx\) is topologically conjugate to the non-wandering set of a uniformly hyperbolic system.

4) A given Bowen semiconjugacy can admit distinct Bowen relations. See Appendix \[^A\] for the canonical relation defined by a semiconjugacy.

5) In our main examples the Bowen property hold in the regular part of the symbolic system. Even when the semiconjugacy has a unique uniformly continuous extension to the whole symbolic system, the latter may fail to satisfy the Bowen property, see Appendix \[^A\].

\[^2\] A simple directed graph is a graph with oriented arrows and such that for any vertices \(a, b\) there is at most one arrow from \(a\) to \(b\).

\[^3\] For instance, a continuous Bowen factor of a compact symbolic system with a transitive Bowen relation must be zero dimensional.
3. Bowen Quotients

We introduce our basic construction: the Bowen quotient. Given an integer $N \geq 1$ and a first semiconjugacy $\pi$, we are going to build another semiconjugacy $\pi_N$ whose preimages are the sets of $N$ preimages of a common point. This is a purely combinatorial construction which:

- preserves the class of finite-to-one, Bowen semiconjugacies of regular parts of Markov shifts;
- produces $\pi_N$ which is one-to-one above the points where the first semiconjugacy was $N$-to-1;
- has in its image the points with at least $N$ preimages by $\pi$, up to a null set.

This construction is closely related to previous work with Boyle [4, Prop. 6.3]. Similar constructions go back to Hedlund [14] and Coven and Paul [9, 10, 11] (for subshifts of finite type, see [21, chap. 8 and 9]); Manning [22] and Bowen [3] (for coding of Axiom-A diffeomorphisms).

3.1. Definition and statement. Let $(S, X)$ be a Markov shift with alphabet $A$ and underlying graph $G$. Let $\pi : X^# \to Y$ be a Borel semiconjugacy on the regular part $X^#$. Assume that it admits a Bowen relation $\sim$. Given an integer $N \geq 1$, let:

- $A_N$ be the collection of subsets $A \subset A$ with cardinality $N$ whose elements are pairwise related by $\sim$;
- $G_N$ be the simple directed graph over $A_N$ with arrows: $a \xrightarrow{G_N} b$ if and only if there is a bijection $\phi : A \to B$ such that $a \xrightarrow{G} b \iff b = \phi(a)$ for all $(a, b) \in A \times B$.

**Definition 3.1.** Let $\pi : X^# \to Y$ be a semiconjugacy with a locally finite Bowen relation $\sim$. Given an integer $N \geq 1$, the **Bowen quotient** of order $N$ of $(\pi : X^# \to Y, \sim)$ is $(\pi_N : X^#_N \to Y, \sim_N)$ where:

1. $X^#_N$ is the regular part of the Markov shift $X_N$ defined by the graph $G_N$;
2. $\pi_N : X^#_N \to Y$ satisfies for all $\hat{x} \in X^#_N$, $\pi_N(\hat{x}) = \pi(x)$ for any $x \in X^#$ s.t. $x_n \in \hat{x}_n$ ($\forall n \in \mathbb{Z}$);
3. $A \sim_N B$ if and only if $\forall (a, b) \in A \times B$ $a \sim b$.

The following definition will be convenient for our purposes.

**Definition 3.2.** A semiconjugacy $\pi : X \to Y$ is **excellent** for some relation $\sim$ if $X$ is a symbolic system, $Y$ is a dynamical system and:

1. $\sim$ is a locally finite, reflexive and symmetric relation on the alphabet of $X$;
2. $\pi$ is Bowen with respect to the relation $\sim$;
3. $\pi$ is Borel and finite-to-one.

In the next statement and throughout this paper, \( \binom{p}{q} = \frac{q!}{p!(q-p)!} \) and is zero if $q > p$.

**Theorem 3.3.** Let $X$ be a Markov shift with regular part $X^#$. Let $\pi : X^# \to Y$ be an excellent semiconjugacy for some Bowen relation $\sim$. Then, for any integer $N \geq 1$, the **Bowen quotient** $(\pi_N : X^#_N \to Y, \sim_N)$ of order $N$ is well-defined and excellent.
Moreover there is a finite-to-one, 1-Lipschitz map \( q_N : X_N \to X \) such that:

1. if \( X \) is locally compact, then so is \( X_N \);
2. \( \pi_N = \pi \circ q_N : X^# \to Y \);
3. \( |\pi_N^{-1}(y)| \leq \left( \frac{|\pi^{-1}(y)|}{N} \right) \) for all \( y \in Y \) with equality except on a null set;
4. \( q_N : X_N^# \to X^# \) is proper, i.e., for any compact set \( K \subset X^# \), \( q_N^{-1}(K) \cap X_N^# \) is compact.

**Definition 3.4.** The degree spectrum of a Borel semiconjugacy \( \pi : Z \to Y \) is:

\[
\Delta(\pi) := \{k \geq 1 : \{y \in Y : |\pi^{-1}(y)| = k\} \text{ is not a null set}\}.
\]

**Corollary 3.5.** In the setting of the above theorem, \( \pi_N(X_N^#) \subset \pi(X^#) \). In fact,

\[
(3.6) \quad \Delta(\pi_N) = \left\{ \left( \frac{r}{N} \right) : r \in \Delta(\pi) \text{ and } r \geq N \right\}
\]

and, for each \( r \in \Delta(\pi) \) with \( r \geq N \), \( \{y \in Y : |\pi_N^{-1}(y)| = \left( \frac{r}{N} \right) \} \) is contained in \( \{y \in Y : |\pi^{-1}(y)| \geq r\} \) and equal to \( \{y \in Y : |\pi^{-1}(y)| = r\} \) up to a null set.

All the claims above are straightforward consequences of the theorem, except possibly for eq. (3.6), which we now prove. Let \( Z := \{y \in Y : |\pi_N^{-1}(y)| \neq \left( \frac{|\pi^{-1}(y)|}{N} \right) \} \). It is a null set by item (3) above. If \( r \in \Delta(\pi) \) and \( r \geq N \), let \( s := \left( \frac{|\pi^{-1}(y)|}{N} \right) \). Note that \( \{y \in Y : |\pi^{-1}(y)| = r\} \setminus Z \) is not a null set and is included in \( \{y \in Y : |\pi_N^{-1}(y)| = s\} \), so \( s \in \Delta(\pi_N) \). For the converse, let \( s \in \Delta(\pi_N) \) so \( \{y \in Y : |\pi_N^{-1}(y)| = s\} \setminus Z \) is not a null set. Hence, by item (3), \( s = \left( \frac{r}{N} \right) \) for some \( r \geq N \) such that \( \{y \in Y : |\pi^{-1}(y)| = r\} \) is not a null set. Hence \( r \in \Delta(\pi) \). Eq. (3.6) is proved.

**Remarks 3.7.**

1. The proof of the theorem will give an explicit null set \( Y_0 \) where the inequality in item (3) may be strict.

2. The following example shows that \( X_N \) may fail to be irreducible even if \( X \) is irreducible. However, using the magic word theory of Section 3.4 we will show in Theorem 5.3 that one can restrict the semiconjugacy to an irreducible component of \( X_N \) without diminishing the image.

**Example 3.8.** Let \( G_0 \) be a simple, directed graph with set of vertices \( A_0 \) and let \( p \) be a positive integer.

Define \( G \) over \( A := A_0 \times (\mathbb{Z}/p\mathbb{Z}) \) by: \( (a,i) \xrightarrow{G} (b,j) \iff i + 1 = j \) and \( a \xrightarrow{G} b \). Define a symmetric relation \( \sim \) on \( A \) by \( (a,i) \sim (b,j) \iff a = b \) and consider a Bowen semiconjugacy with this relation. Hence, for any \( N \geq 1 \), \( A_N = \{\{a\} \times I : a \in A_0 \text{ and } I \subset \mathbb{Z}/p\mathbb{Z} \text{ with } |I| = N\} \) and \( \{a\} \times I \to \{b\} \times J \) if and only if \( a \to b \) and \( I + 1 = J \).

Observe that \( G_N \) may fail to be irreducible even when \( G \) is irreducible. For instance, if \( p = 4 \) and \( N = 2 \), for any \( a \in G_0 \), \( \{(a,0), (a,1)\} \) and \( \{(a,0), (a,2)\} \) belong to distinct irreducible components of \( G_N \).

3.2. Resolving property. We begin by studying the combinatorics of finite fibers over an orbit with some recurrence. Let \( (S,X) \) be a Markov shift with alphabet \( A \) and \( (T,Y) \) be some dynamical system. Let \( \pi : X^# \to T \) be a semiconjugacy with some Bowen relation \( \sim \).

Denote the Bowen equivalence by \( \approx \) and the equivalence class of \( x \in X^# \) by

\[
\langle x \rangle := \{y \in X^# : y \approx x\}.
\]
Let us call a point \( x \in X \) recurrent for some function \( \phi : X \to \mathbb{Z} \) if, for each \( n \in \mathbb{Z} \), \( \{ k \in \mathbb{Z} : \phi(S^k x) = \phi(S^n x) \} \) is neither lower bounded nor upper bounded. Poincaré recurrence implies that the set of recurrent points for any given measurable function is a full set.

**Proposition 3.9.** Let \((\pi, \sim)\) be an excellent semiconjugacy defined on a Markov shift \( X \) described by some graph \( G \). For \( x \in X \) and \( n \in \mathbb{Z} \), let \( \mathcal{A}(x, n) := \{ y_n : y \in \langle x \rangle \} \). Let \( X^* \) be the set of points \( x \in X^* \) which are simultaneously recurrent for the three following functions:

\[
R(x) := |\mathcal{A}(x, 0)|, \quad n_+ := |\{ y_{[0,\infty)} : y \in \langle x \rangle \}|, \quad n_- := |\{ y_{(-\infty,0]} : y \in \langle x \rangle \}|.
\]

Then for any \( x \in X^* \) and \( n \in \mathbb{Z} \),

(a) for each \( n \in \mathbb{Z} \), the restriction \( x \mapsto x_n \) defines a bijection between \( \langle x \rangle \) and \( \mathcal{A}(x, n) \);

(b) for each \( n \in \mathbb{Z} \), \( |\mathcal{A}(x, n)| = |\mathcal{A}(x, n + 1)| \) and, for every \( a \in \mathcal{A}(x, n) \) there is a unique \( b \in \mathcal{A}(x, n + 1) \) such that \( a \overset{G}{\to} b \);

(c) \( \langle x \rangle \subset X^* \).

This proposition is related to finite equivalence theory and especially some classical results of Coven and Paul [9] (see [21, Thm 8.1.16]).

**Proof.**

**Step 1.** Given \( x \in X^* \) recurrent for the function \( n_+ \) and \( a \in \mathbb{Z} \), there is an integer \( \ell \geq 1 \) (which can be taken arbitrarily large) such that, among \( y \in \langle x \rangle \), \( y_{(a,a+\ell)} \) determines \( y \).

In the above situation, we say that \([a, a + \ell]\) is an admissible interval for \( x \).

Let \( x, a \) be as above. Since \( \langle x \rangle \) is finite, \( n_+(S^{-n} x) = |\langle x \rangle| \) for \( n \) large enough. Observe that the function \( n_+ \) is monotone along orbits. By recurrence, it is constant along the orbits. Hence \( |\langle x \rangle| = n_+(S^a x) = |\{ y_{(a,a+\ell)} : y \in \langle x \rangle \}| \) for all large integers \( \ell \). Fixing such an integer \( \ell \), the obvious surjectivity of the map \( y \mapsto y_{(a,a+\ell)} \) implies its bijectivity.

**Step 2.** Item (a): for any \( n \in \mathbb{Z} \), \( y_n \) determines \( y \) for \( y \in \langle x \rangle \).

Given \( n \in \mathbb{Z} \), Step 1 provides an admissible interval \([a, b] \subset [n+1, \infty) \). If there were distinct \( y, y' \in \langle x \rangle \) such that \( y_n = y'_n \) but \( y_{(-\infty,n)} \neq y'_{(-\infty,n)} \), the spliced sequence \( y'_{(-\infty,n)} y_{[n,\infty)} \) of \( X^* \) coinciding with \( y \) on \([a, b] \), would contradict the admissibility of \([a, b] \). Thus \( y_n \) determines \( y_{(-\infty,n)} \) for \( y \in \langle x \rangle \).

Symmetric arguments (using \( n_- \)) show that \( y_n \) determines the whole sequence \( y \in \langle x \rangle \), proving item (a).

**Step 3.** Item (b): \( \mathcal{R} := \{ (a, b) \in \mathcal{A}(x, n) \times \mathcal{A}(x, n + 1) : a \overset{G}{\to} b \} \) is a bijection.

Observe first that for every \( a \in \mathcal{A}(x, n) \), \( a = y_n \) for some \( y \in \langle x \rangle \) so that \( a \to y_{n+1} \) with \( y_{n+1} \in \mathcal{A}(x, n + 1) \). Assume \( a \overset{G}{\to} b \) and \( a \overset{G}{\to} b' \) with \( a \in \mathcal{A}(x, n) \) and \( b, b' \in \mathcal{A}(x, n + 1) \). Therefore there are \( y, z, z' \in \langle x \rangle \) such that \( y_n = a, z_{n+1} = b, \) and \( z'_{n+1} = b' \). Considering the splicings \( y_{(-\infty,n)} z_{[n+1,\infty)} \) and \( y'_{(-\infty,n)} z'_{[n+1,\infty)} \), Step 2 implies that \( b = b' \). Thus \( \mathcal{R} \) defines a unique map \( \mathcal{A}(x, n) \to \mathcal{A}(x, n + 1) \). A symmetric argument gives an inverse map \( \mathcal{A}(x, n + 1) \to \mathcal{A}(x, n) \), hence \( \mathcal{R} \) is bijective: item (b) is proved.

**Step 4.** Item (c): if \( x \in X^* \), then \( \langle x \rangle \subset X^* \)

This is clear from the definition of \( X^* \). The proposition is proved. \( \square \)
3.3. **Bowen quotients.** We prove Theorem 3.3. To begin with, we let \( \mathcal{G}_N, \mathcal{A}_N, X_N, \) and \( \sim^N \) as in Definition 3.1. Items (BQ1) and (BQ3) are then satisfied by construction.

We now define \( \pi_N \) to satisfy (BQ2) and item (2) of the theorem. We note the following easy consequence of the definition of \( \mathcal{G}_N \):

**Fact 3.10.** For \(-\infty \leq i < 0 < j \leq \infty\), let \( \hat{x} = (\hat{x}_n)_{i \leq n \leq j} \) be a finite or infinite path on \( \mathcal{G}_N \). For each \( a \in \hat{x}_0 \), there is a unique path \( Q(\hat{x}, a) := (x^a_n)_{i \leq n \leq j} \) on \( \mathcal{G} \) such that \( x^a_0 = a \) and \( x^a_i \in \hat{x}^a_n \) for all \( i < n < j \).

For convenience we select some total order on the alphabet of \( X \). We define \( q_N : X_N \to X \) by setting \( q_N(x) := Q(\hat{x}, \text{min}(\hat{x}_0)) \). Note that \( q_N \) is 1-Lipschitz since the \( X \)-word \( x_{-n} \ldots x_n \) in Fact 3.10 depends only on \( \hat{x}_{-n} \ldots \hat{x}_n \). If \( \hat{x} \in X_N^\# \), then \( x_n \in \hat{x}_n \) for all \( n \in \mathbb{Z} \) implies \( x \in X^\# \).

In particular, the following defines a Borel map:

\[
\pi_N : X_N^\# \to Y, \quad \hat{x} \mapsto \pi \circ q_N(\hat{x}).
\]

This map satisfies (BQ2) in Definition 3.1 for any \( \hat{x} \in X_N^\# \), any \( x \in X^\# \) with \( x_n \in \hat{x}_n \) for all \( n \in \mathbb{Z} \), \( \pi_N(\hat{x}) = \pi(x) \). Indeed, the condition \( x_n \in \hat{x}_n \) implies that \( x \approx q_N(\hat{x}) \) so \( \pi_N(\hat{x}) = \pi(x) \), proving (BQ2) as well as item (2) in the theorem.

The map \( \pi_N \) is a semiconjugacy. Indeed, \( \pi_N(S_N(\hat{x})) = \pi(y) \) with \( y_n \in \hat{x}_{n+1} \) and \( T(\pi_N(\hat{x})) = T(\pi(z)) = \pi(S(z)) \) with \( z_n \in \hat{x}_n \). Hence \( y \approx S(z) \) so \( \pi(y) = \pi(S(z)) \), and \( \pi_N \circ S_N = T \circ \pi_N \).

Note that (BQ2) ensures the uniqueness of such semiconjugacy. Indeed, if \( \pi_N, \pi_N' \) are two such maps, then given any \( \hat{x} \in X_N^\# \), \( \pi_N(\hat{x}) = \pi(x) \) and \( \pi_N'(\hat{x}) = \pi(y) \) where \( x_n, y_n \in \hat{x}_n \) for each \( n \). Hence \( x \approx y \) and so \( \pi(x) = \pi(y) \): \( \pi_N = \pi_N' \).

It remains to show items (1), (3), and (4) and excellency, i.e., items (EX1)-(EX4).

Let us check that this construction preserves the local compactness. Indeed, for any \( A \in \mathcal{A}_N \), take \( a \in A \) and observe that if \( A \sim^N B \), then \( B \subseteq \{ c : \exists b a \to b \text{ and } b \sim c \} \) which is finite, so \( A \) has finite outdegree if \( a \) has. A similar argument applies to the indegree. Thus \( X_N \) is locally compact if \( X \) is, proving item (1).

A similar argument shows that \( \sim^N \) is locally finite. Observe also that \( \sim^N \) is reflexive and symmetric. Thus item (EX1) is proved.

To bound the number of preimages under \( \pi_N \), let \( y \in Y \) and write \( \pi^{-1}(y) = \{ x^1, \ldots, x^r \} \) with \( r = |\pi^{-1}(y)| \). By construction, any preimage under \( \pi_N \) corresponds to a set of \( N \) preimages under \( \pi \):

\[
\pi_N^{-1}(y) \subseteq \{ x^j : j \in J \}_{n \in \mathbb{Z}} : J \cap \{1, \ldots, r\}, |J| = N \}.
\]

Therefore \( \pi_N^{-1}(y) \leq \binom{r}{N} \). In particular, \( \pi_N \) and therefore \( q_N|X_N^\# \) are finite-to-one. Note that (EX3) is established.

We let

\[
Y_0 := \pi(X^\# \setminus X^*) \subset Y
\]

where \( X^* \) is the good set defined by Proposition 3.9. From Lemma 2.1 \( Y_0 \) is a null set for \( T \) since \( X^* \) is a full set for \( S \). For \( y \in \pi(X^\#) \setminus Y_0 \), Proposition 3.9 implies that the sequences \( \hat{x}^j \) in eq. (3.11) belong in \( X_N \). In fact, they must belong to \( X_N^\# \). Indeed, since \( x^1 \in X^\# \), there is some \( a \in A \) such that \( x^1_n = a \) for infinitely many \( n \geq 0 \). For those indices \( n \), \( \hat{x}^j_n \) is
Definition 4.1. Given a symbolic system by:

is an equality and:

\[ \forall y \in \pi(X^\#) \setminus Y_0 \ |\pi_N^{-1}(y)| = \binom{r}{N}. \]

Item (3) of the theorem is proved. We note for future reference the following consequence:

Fact 3.12. Let \( \pi : X^\# \to Y \) be an excellent semiconjugacy with Bowen quotient \( \pi_N : X^\#_N \to Y \). For all \( x \in X^\# \) outside a null set, if \( |\pi^{-1}(\pi(x))| \geq N \) then \( \exists \hat{x} \in X^\#_N \) s.t. \( \forall n \in \mathbb{Z} \ x_n \in \hat{x}_n \).

We check that \( \hat{N} \) is a Bowel relation for \( \pi_N \). First, let \( \hat{x}, \hat{y} \in X^\#_N \) with \( \pi_N(\hat{x}) = \pi_N(\hat{y}) \). Let \( a \in \hat{x}_0 \) and \( b \in \hat{y}_0 \). Fact 3.10 gives (unique) sequences \( x \in [a]^\#_X \), \( y \in [b]^\#_X \) with \( \pi(x) = \pi_N(\hat{x}) \), \( \pi_N(\hat{y}) = \pi(y) \). Thus \( \pi(x) = \pi(y) \) and \( x_0 \sim y_0 \). It follows that \( \hat{x}_0 \sim^N \hat{y}_0 \) and then \( \hat{x} \sim^N \hat{y} \), by equivariance.

Conversely, let \( \hat{x}, \hat{y} \in X^\#_N \) with \( \hat{x} \sim^N \hat{y} \). Picking \( a \in \hat{x}_0 \) and \( b \in \hat{y}_0 \), Fact 3.10 gives \( x \in [a]^\#_X \), \( y \in [b]^\#_X \) such that \( x_n \in \hat{x}_n \) and \( y_n \in \hat{y}_n \) for all \( n \in \mathbb{Z} \). Thus \( \pi(x) = \pi_N(\hat{x}) \) and \( \pi_N(\hat{y}) = \pi(y) \). From the definition of \( \sim^N \), we have \( x \sim y \). The Bowen property for \( \sim \) implies \( \pi(x) = \pi(y) \) hence \( \pi_N(\hat{x}) = \pi_N(\hat{y}) \). The Bowen property (EX2) is established.

Finally, we prove that \( q_N \) is proper. Note that a subset \( K \) of a symbolic system is relatively compact if and only if, for each \( n \in \mathbb{Z} \), \( \{x_n : x \in K\} \) is finite. Fix a relatively compact \( K \subset X^\# \) and \( n \in \mathbb{Z} \). By construction, \( q_N(\hat{x}) \in K \) implies that \( \hat{x}_n \), a set of \( N \) symbols from \( \mathcal{A} \), contains only symbols that are Bowen related to elements of \( \{x_n : x \in K\} \). Since \( \sim \) is locally finite, it follows that \( \{\hat{x}_n : q_N(\hat{x}) \in K\} \) is finite and \( q^{-1}(K) \) is relatively compact. Item (4) is proved. \( \square \)

4. Combinatorial degree

We are going to characterize the subset of a Bowen semiconjugacy where the cardinality of the fibers is minimal by the recurrence of some words. To this end, we adapt the notions of degree and magic word from the classical theory of one-block codes between subshifts of finite type (see Hedlund [14] and more generally [21, chap. 9]).

It is convenient to disregard the factor map \( \pi : X \to Y \) and to focus on the symbolic system \( X \) and the Bowen relation.

Definition 4.1. Given a symbolic system \((S, Z)\) on some alphabet \( \mathcal{A} \), an (abstract) Bowen relation is a reflexive, symmetric relation \( \sim \) on \( \mathcal{A} \) such that the relation on \( Z \) defined by \( x \approx y \iff \forall n \in \mathbb{Z} \ x_n \sim y_n \) is an equivalence relation.

In this section, \( \sim \) is a Bowen relation on the regular part \( X^\# \) of a Markov shift \( X \). Recall that \( X^\text{rec} \subset X^\# \) is the set of word recurrent sequences in \( X \) (i.e., any word that occurs once is seen i.o. –see p. 7). Recall also that the Bowen equivalence class of any \( x \in X^\# \) is denoted by:

\[ \langle x \rangle := \{y \in X^\# : y \approx x\} \]
4.1. Degree of Bowen relations. The relation $\sim$ on the alphabet of $X^#$ induces another reflexive and symmetric relation on $L(X^#)$ (also denoted by $\sim$) according to $v \sim w \iff |v| = |w|$ and $v_1 \sim w_1, \ldots, v_{|v|-1} \sim w_{|v|-1}$.

We will consider the languages $L(X^{\text{rec}}) \subset L(X^#) \subset L(X)$. In general, they are distinct. However, they are equal when $X$ is the disjoint union of its irreducible components.

**Definition 4.2.** Given a Bowen relation $\sim$, the degree of a word $w \in L(X)$ at some index $0 \leq i < |w|$ is:

$$
\delta_\sim(w,i) := \{|v_i : v \in L(X^#), v \sim w|\}
$$

$$
\delta_\sim(w) := \min\{\delta_\sim(w,i) : 0 \leq i < |w|\}.
$$

The degree of $\sim$ is:

$$
\delta_{\text{rec}}(\sim) := \inf\{\delta_\sim(w) : w \in L(X^{\text{rec}})\}.
$$

A magic word is a word $w \in L(X^{\text{rec}})$ realizing this infimum. A couple $(w,i)$ that realizes it is called a magic couple.

Observe that for any $w \in L(X^{\text{rec}})$, $\deg_\sim(w) \geq 1$ (since $\sim$ is reflexive). As soon as $\deg_\text{rec}(\sim)$ is finite (e.g., if $\sim$ is locally finite), there always exist magic words.

Given a word $W \in L(X^#)$, $X_W$ denotes the set of sequences that see i.o. $W$:

$$
X_W := \{x \in X : \exists m_k,n_k \to \infty \text{ such that } W \text{ occurs in } x \text{ at } -m_k \text{ and at } n_k\}.
$$

Note that $X_W$ is an invariant, possibly empty, subset of $X^#$. We start with two simple lemmas.

**Lemma 4.3.** Assume that the Bowen equivalence classes: $\langle x \rangle := \{y \in X^# : y \approx x\}$ are finite for all $x \in X^#$. Let $W \in L(X^#)$ with $\delta_\sim(W) = 1$. If $x \in X^#$ sees i.o. $W$, then $x$ is only equivalent to itself, that is:

$$
\forall x \in X_W \quad \langle x \rangle = \{x\}.
$$

**Proof.** Let $0 \leq I < |W|$ such that $\delta_{\sim}(W,I) = 1$ and $x \in X_W$. Pick an increasing sequence of integers $(n_k)_{k \in \mathbb{Z}}$ such that $x_{(|n_k|+|n_k+|W|)} = W$. If there is a distinct $y \in X^#$ with $x \approx y$, one can find $k < l$ such that $x_{(|n_k+|l,n_l+|W|)} = y_{(|n_k+|l,n_l+|W|)}$. However, $y_{(|n_k+|l,n_l+|W|)} \sim W$ implies that $y_{|n_k+l} = W_I$ since $\delta_{\sim}(W,I) = 1$. Consider the infinitely many distinct arbitrary concatenations of the two words $x_{|n_k+|l,n_l+|W|}, y_{|n_k+|l,n_l+|W|}$. They belong to the Markov shift $X$ and in fact to $X^#$ since they see i.o. the symbol $W_I$. Moreover, they belong to a single Bowen equivalence class which is infinite, a contradiction. \hfill $\Box$

**Lemma 4.4.** Assume that $\sim$ is a locally finite Bowen relation. For any $x \in X^#$,\n
$$
|\langle x \rangle| \geq \delta_\sim(x) := \min\{\delta_\sim(x_\ell \ldots x_{\ell+1}) : p \in \mathbb{Z}, \ell \geq 1\}.
$$

**Proof.** Fix $x \in X^#$. For each $n \geq 1$, let $A_n := \{y_0 : y \in \langle x_{[-n,n]} \rangle\}$. This defines a non-increasing sequence of sets contained in $\{b \in A : b \sim x_0\}$ which is finite. Hence there is $n_0$ such that $A_n = A_{n_0}$ for $n \geq n_0$. Note that $|A_{n_0}| \geq \delta_\sim(x)$. Now, fix $a \in A_{n_0}$ and, for each $n \geq 0$, pick $y^n \in \langle x_{[-n,n]} \rangle$ with $y^n_0 = a$. For each $k \in \mathbb{Z}$, $\{y^n_k : n \geq |k|\}$ is finite (since the Bowen relation is locally finite). Thus one can find an accumulation point $y \in A^\mathbb{Z}$ (i.e., there is $n_j \uparrow \infty$ such that, for each $k \in \mathbb{Z}, y_k = y^n_j$ for all large $j$). It is easy to check that $y \in X^#$ and $y \in \langle x \rangle$. Varying $a \in A_{n_0}$, eq. (4.5) follows. \hfill $\Box$
4.2. **Magic semiconjugacies.** We relate the combinatorial degree of the Bowen relation with the cardinality of the fibers of the semiconjugacy.

**Theorem 4.6.** Let $X$ be a Markov shift and let $\sim$ be a locally finite Bowen relation for $X^\#$. Let $x \in X^{\text{rec}}$ with $\langle x \rangle$ finite. The following are equivalent:

(a) $\langle x \rangle$ has exactly $\delta_{\text{rec}}(\sim)$ elements;

(b) $x$ sees some magic word $W$ for $\sim$.

The following example shows that the implication (b) $\implies$ (a) in Theorem 4.6 may fail when $\langle x \rangle$ is infinite or when $x \notin X^{\text{rec}}$.

**Example 4.7.** Let $X = \{0,1,2\}^\mathbb{Z}$ and for $a,b \in \{0,1,2\}$, let $a \sim b \iff |b-a| = 0,2$. Note that $X^\# = X$, $\mathcal{L}(X^{\text{rec}}) = \mathcal{L}(X)$, and $\delta_{\text{rec}}(\sim) = \delta_{\sim}(1) = 1$. For $x \in X^{\text{rec}}$ distinct from $1^\infty$ such as $x = (10)^\infty$, $\langle x \rangle$ is infinite. For $y = 1^\infty 0^k 1^\infty$ with $k \geq 1$, $y \in X^\# \setminus X^{\text{rec}}$ sees i.o. the magic word 1, however: $|\langle y \rangle| = 2^k > \delta_{\text{rec}}(\sim)$.

The degree has a geometric meaning:

**Corollary 4.8.** Let $X$ be a Markov shift such that $X^{\text{rec}} \neq \emptyset$ and let $\sim$ be a locally finite Bowen relation for $X^\#$. If $\langle x \rangle$ is finite for each $x \in X^{\text{rec}}$, then

$$\delta_{\text{rec}}(\sim) = \min\{|\langle x \rangle| : x \in X^{\text{rec}}\} = \min\{k \geq 1 : \{x \in X^\# : |\langle x \rangle| = k\} \text{ is not null}\}.$$ 

In particular, $\delta_{\text{rec}}(\sim)$ only depends on the Bowen equivalence relation $\approx$.

**Proof.** The inequality $\delta_{\text{rec}}(\sim) \leq \min\{|\langle x \rangle| : x \in X^{\text{rec}}\}$ follows from Lemma 4.4 since $\delta_{\text{rec}}(\sim) \leq \delta_{\sim}(x)$ for $x \in X^{\text{rec}}$. Conversely, let $W$ be a magic word for $\sim$ over $X^{\text{rec}}$. By definition, there is $x^0 \in X^{\text{rec}}$ that sees i.o. $W$. By Theorem 4.6, $|\langle x^0 \rangle| = \delta_{\text{rec}}(\sim)$, proving the first equality.

We show that $\delta_{\text{rec}}(\sim)$ is equal to $d := \min\{k \geq 1 : \{x \in X^\# : |\langle x \rangle| = k\} \text{ is not null}\}$. Since $\{x \in X^\# : |\langle x \rangle| = d\}$ has positive measure for some invariant probability measure, it contains a recurrent point, so $\delta_{\text{rec}}(\sim) = \min\{|\langle x \rangle| : x \in X^{\text{rec}}\} \leq d$.

Conversely, there is $x \in X^{\text{rec}}$ such that $|\langle x \rangle| = \delta_{\text{rec}}(\sim)$. By Theorem 4.6, $x$ sees i.o. some magic word $W$. Since $X$ is a Markov shift, one can find $y \in X^\#$ that sees i.o. $W$ and is periodic. Its orbit is a non-null set, hence $d \leq \delta_{\text{rec}}(\sim)$.

**Remark 4.10.** The next example shows that there is no simple analogue of Corollary 4.8 for $X^\#$, even if one replaces the degree $\deg_{\text{rec}}(\sim)$ by $\min\{\deg_{\sim}(w) : w \in \mathcal{L}(X^\#)\}$.

**Example 4.11.** Let $X$ be the subshift of finite type defined by the directed graph in Fig. 1. Define $\pi : X \to \{-1,0,+1\}^\mathbb{Z}$ as the projection on the first coordinate with Bowen relation $(a,b,c) \sim (a',b',c') \iff a = a'$. 

![Figure 1. The subshift of finite type in Example 4.11](image-url)
Note that $X^\# = X$ and $\pi(X)$ is the union of two fixed points $(+1)^\infty$, $(-1)^\infty$ and a heteroclinic orbit: $\{\sigma^k((+1)^\infty0 \cdot (-1)^\infty) : k \in \mathbb{Z}\}$. Note also that $X^{\text{rec}}$ is the union of:

- four 3-periodic orbits mapped to the two fixed points, defining Bowen equivalence classes with 6 elements each;
- four heteroclinic orbits, each mapped to the heteroclinic orbit, defining Bowen equivalence classes with 4 elements each.

The following is easily checked:

$$\inf_{w \in \mathcal{L}(X^\#)} \delta_\omega(w) = \delta_\omega((0,0,0),0) = 1 < \inf\{ |\langle x \rangle| : x \in X^\# \} = 4$$

$$= \inf_{w \in \mathcal{L}(X^{\text{rec}}), 0 \leq |w|} \{v^i : v \in \mathcal{L}(X^{\text{rec}}), v \sim w\}$$

$$= \inf_{w \in \mathcal{L}(X^{\text{rec}}), 0 \leq |w|} \delta_\omega(w) = \delta_\omega((1,0,0),0).$$

To prepare for the proof of Theorem 4.6, we fix a magic couple $(W,I)$ in $X^\#$ over $X^{\text{rec}}$. Since $\sim$ is locally finite, $M = |\{W\}|$ is finite. We enumerate its elements and the symbols at index $I$:

$$\langle W \rangle = \{W^1, \ldots, W^M\} \quad \text{and} \quad \{a_1, \ldots, a_d\} = \{W^1_I, \ldots, W^M_I\}$$

where $d = \delta_{\text{rec}}(\sim)$. Obviously, $d \leq M$. We can assume: $a_i = W^i_I$ for $i = 1, \ldots, d$.

To any word that can be written as a concatenation $WuW$, we associate:

$$\mathcal{T}_{ij}(WuW) := \{a_i \bar{\tau} : va_i \bar{\tau} a_j w \in \mathcal{L}(X^\#), va_i \bar{\tau} a_j w \sim WuW \text{ for some } |v| = I, |w| = |W| - I - 1\}$$

for $1 \leq i, j \leq d$. We call the words $a_i \bar{\tau} \in \mathcal{T}_{ij}(WuW)$ transitions. Note that these words have the same length as $WuW$. Matching transitions can be concatenated:

**Claim 4.12.** For any $1 \leq i, j, k \leq d$, any $WuWu'W \in \mathcal{L}(X^{\text{rec}})$, there is an injection:

$$\mathcal{T}_{ij}(WuW) \times \mathcal{T}_{jk}(Wu'W) \to \mathcal{T}_{ik}(WuWu'W), \quad (\tau, \tau') \mapsto \tau \tau'.$$

**Proof.** Let $\tau, \tau'$ be as above. We can write $\tau = a_i \bar{\tau}$, $\tau' = a_j \bar{\tau}'$. By definition, $va_i \bar{\tau} a_j w \sim W$ and $v' a_j \bar{\tau} a_k w' \sim W$ where $v, w, v', w'$ are words of lengths $|v| = |v'| = I$ and $|w'| = |w| = |W| - I - 1$. Thus, the concatenation $va_i \bar{\tau} a_j \bar{\tau}' a_k w'$ belongs to $\mathcal{L}(X^\#)$ and is related to $WuWu'W'$. Hence $a_i \bar{\tau} a_j \bar{\tau}' \in \mathcal{T}_{ik}(WuWu'W)$. The injectivity is obvious. \hfill $\square$

Any transition can be extended to the right and to the left:

**Claim 4.13.** For any $1 \leq i \leq d$, any $WuW \in \mathcal{L}(X^{\text{rec}})$, the following two sets are not empty:

$$\mathcal{T}_{is}(WuW) := \bigcup_{1 \leq j \leq d} \mathcal{T}_{ij}(WuW) \quad \text{and} \quad \mathcal{T}_{si}(WuW) := \bigcup_{1 \leq j \leq d} \mathcal{T}_{ji}(WuW).$$

Moreover, $\mathcal{T}_s(WuW) := \bigcup_{1 \leq i \leq d} \mathcal{T}_{is}(WuW) = \bigcup_{1 \leq j \leq d} \mathcal{T}_{sj}(WuW)$ has at least $d$ elements.

**Proof.** Obviously, $\{m_I : m \in \mathcal{L}(X^\#), m \sim WuW\} \subset \{w_I : w \in \mathcal{L}(X^\#), w \sim W\} = \{a_1, \ldots, a_d\}$. Since $(W,I)$ is magic, the cardinalities are equal and finite so the inclusion is an equality. Since $\{m_I : m \sim WuW\} = \{a : a \tau \in \mathcal{T}_s(WuW)\}$, it follows that $|\mathcal{T}_s(WuW)| \geq d$. It also follows that $\mathcal{T}_s(WuW)$ contains a word beginning with $a_i$ so that $\mathcal{T}_{is}(WuW)$ is not empty. Likewise $\mathcal{T}_{si}(WuW)$ is not empty. \hfill $\square$

A word $u$ will be called special if $\mathcal{T}_{si}(WuW)$ has more than one element for some $1 \leq i \leq d$. 
Lemma 4.14. Fix a magic word $W$. Assume that $x \in X^{\text{rec}}$ is an infinite concatenation
\begin{equation}
\ldots Wu^{-1}Wu^0Wu^1W\ldots
\end{equation}
where each $u^k$ is some word. More precisely, there is an increasing integer sequence $(n_k)_{k \in \mathbb{Z}}$
such that, for all $k \in \mathbb{Z}$, $x_{[n_k,n_{k+1}-1]} = Wu^k$. If there are (at least) $0 \leq K \leq \infty$
distinct integers $k \in \mathbb{Z}$ such that $u^k$ is special, then
\[ |\langle x \rangle| \geq K + d. \]
Moreover, for each $k \in \mathbb{Z}$, \{\(y_{n_k+I} : y \in \langle x \rangle\}\} = \{a_i : i = 1, \ldots, d\}.

Proof. We may and do assume that $n_0 = -I$ and that there are $K$ positive integers $k$ with $u^k$ special
(using shift invariance). For each $n \geq 0$, let $K(n)$ be the number of integers $0 < k < n$ with $u^k$ special.
For each $n \geq 1$, let $U_n := u^0W \ldots Wu^{n-1}$.

We claim that for every $n \geq 1,$
\begin{equation}
\mathcal{T}_u(WU^nW) := \bigcup_{1 \leq i \leq d} \mathcal{T}_{u_i}(WU^nW) \text{ has at least } K(n) + d \text{ elements}
\end{equation}
\[ \{w_0 : w \in \mathcal{T}_u(WU^nW)\} = \{a_1, \ldots, a_d\}. \]

We proceed by induction. Claim 4.13 implies that $|\mathcal{T}_u(Wu^0W)| \geq d$ which is eq. (4.16) for $n = 1$ since $K(1) = 0$. Assume eq. (4.16) for some $n \geq 1$. Claims 4.12 and 4.13 show that each element
of $\mathcal{T}_u(WU^W)$ can be extended to an element of $\mathcal{T}_u(WU^{n+1}W)$. Thus $|\mathcal{T}_u(WU^{n+1}W)| \geq |\mathcal{T}_u(WU^W)|$ and \{\(w_0 : w \in \mathcal{T}_u(WU^{n+1}W)\}\} \supset \{w_0 : w \in \mathcal{T}_u(WU^W)\}.
Eq. (4.16) follows if $K(n+1) = K(n)$. Otherwise $K(n+1) = K(n) + 1$ and $Wu^nW$ is special
so some element of $\mathcal{T}_u(WU^nW)$ has at least two distinct extensions in $\mathcal{T}_u(WU^{n+1}W)$. This
completes the induction and proves the claim (4.16).

Observe that the words in $\mathcal{T}_u(WU^nW)$ are the prefixes of length $|WU^n|$ of the words in $\mathcal{T}_u(WU^{n+1}W)$.
Hence one can take an inductive limit and obtain $\mathcal{Y} \subset \mathcal{A}^{[0,\infty)}$ such that, for each $n \geq 0$,
\(\mathcal{T}_u(WU^nW) = \{y_{[0,|WU^n|-1]} : y \in \mathcal{Y}\}\). It is easy to see that $(\mathbb{Z}_- :=
\{0, -1, -2, \ldots \})$: \[ \begin{align*}
- \mathcal{Y} & \text{ has at least } K + d \text{ elements;} \\
- \text{each } y \in \mathcal{Y} \text{ satisfies: } y_0 \in \{a_1, \ldots, a_d\}, y_n \xrightarrow{X} y_{n+1} \text{ for all } n \in \mathbb{N}_0, \text{ and } y \sim x_{[0,\infty)}.
\end{align*} \]
For each $1 \leq i \leq d$, an analogous use of Claims 4.13 and 4.12 provides an infinite one-sided sequence
$z^i \in \mathcal{A}^{\mathbb{Z}_-}$ such that $z_i = a_i$ and $z^i_0 \xrightarrow{X} z^i_{n+1} \text{ for all } n < 0$, and $z^i_0 \sim x_{(-\infty,0]}$. Therefore
\(\{z^i_{[0,|y_{[0,\infty)}|]} : y \in \mathcal{Y}\} \subset \langle x \rangle \_\sim \text{ so that } |\langle x \rangle| \geq K + d. \]

Proof of Theorem 4.6. Let $x \in X^{\text{rec}}$ with finite class $\langle x \rangle$. First, we assume that $x$ sees
no magic word i.o. Since $x \in X^{\text{rec}}$, no magic word can appear in $x$. By Lemma 4.4
$|\langle x \rangle| \geq \delta_{\sim}(x) > \delta_{\text{rec}}(\sim)$. Conversely, we assume by contradiction that $x$ sees i.o. some magic word $W$ and that \(|\langle x \rangle| \neq \delta_{\text{rec}}(\sim)\). By Lemma 4.4 this implies that $|\langle x \rangle| \geq \delta_{\text{rec}}(\sim) + 1$. We decompose $x$ as in eq. (4.15) (remark that the magic word $W$ can occur inside the fillers $u^k$). For $k$ large enough:
\[ \{|y_{[n_k+I,n_{k+1}-|W|+I]} : y \in \langle x \rangle\}| \geq \delta_{\text{rec}}(\sim) + 1. \]
Hence, writing $a_i$ for $y_{n_k+l} = y'_{n_k+l}$, that is, $u^{-k}W \ldots Wu^k$ is a special word. This contradicts the following claim and therefore proves the theorem. \hfill \square

Claim 4.17. No special word occurs in $x$.

Proof of the claim. Assume by contradiction that there is a special word $u^*$ such that $Wu^*W$ occurs in $x$. Since $x$ is recurrent, $Wu^*W$ occurs infinitely often. Select a decomposition as in eq. (4.15) such that $Wu^kW = Wu^*W$ for infinitely many integers $k$. By Lemma 4.14, $\langle x \rangle$ must be infinite, a contradiction. \hfill \square

5. Injective codings

We use the previous constructions and results to build injective codings on larger and larger sets. We will first see that a Bowen quotient (Def. 3.1) may produce a coding with a large injectivity set. We will then see how to repeat this construction to capture all the image through suitable recodings.

For convenience, we recall some definitions. An excellent semiconjugacy (Def. 3.2) is a Borel, finite-to-one semiconjugacy which admits a locally finite Bowen relation. Sometimes we will abuse notation denoting the Bowen relation by the corresponding semiconjugacy (even though the semiconjugacy does not determine the Bowen relation).

The degree spectrum of a semiconjugacy $\pi : X^\# \to Y$ is (Def. 3.4):

$$\Delta(\pi) := \{n \geq 1 : \{y \in Y : |\pi^{-1}(y)| = n \} \text{ is not a null set} \}$$

5.1. A Bowen quotient and its injectivity set. We analyze the Bowen quotient construction using the magic word theory from the previous section. Let

$$X_N^\text{magic} := \{x \in X_N : \exists w \in \mathcal{L}(X_N^\text{rec}) \text{ such that } w \text{ is a magic word for } \pi_N \text{ and } x \text{ sees } w \text{ i.o.} \}.$$

In this subsection, we say that a function $C : \mathcal{A} \times \mathcal{A} \to \mathbb{N}^0$ is a multiplicity bound for $\pi : X^\# \to Y$ if for all $x \in X^\#$, $|\pi^{-1}(\pi(x))| \leq C(a, b)$ for every $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that $x_{-n} = a$, resp. $x_{n} = b$, for infinitely many positive integers $n$.

We have the following.

Lemma 5.1. Let $X$ be a Markov shift and let $\pi : X^\# \to Y$ be an excellent semiconjugacy for some Bowen relation $\sim$. Let $N$ belong to its degree spectrum $\Delta(\pi)$ and let $(\pi_N : X_N^\# \to Y, \xi)$ be the Bowen quotient of $(\pi, \sim)$ with order $N$. The following holds:

1. $(\pi_N, \xi)$ is excellent and has degree $\delta_{\text{rec}}(\pi_N) = \min \Delta(\pi_N) = 1$. Moreover, if $\pi$ admits a multiplicity bound, so does $\pi_N$;
2. if $X$ is locally compact, so is $X_N$;
3. $\pi_N = \pi \circ q_N$ with $q_N : X_N \to X$ a 1-Lipschitz map such that $q_N(X_N^\#) \subset X^\#$;
4. $q_N : X_N^\# \to X^\#$ is proper, i.e., $q_N^{-1}(K) \cap X_N^\#$ is compact for any compact $K \subset X^\#$;
5. $\pi_N(X_N^\#) \subset \{y \in \pi(X^\#) : |\pi^{-1}(y)| \geq N \}$ and the difference is a null set;
Theorem 5.2. Let \((X,S)\) be a Markov shift, \((Y,T)\) be a dynamical system, and let \(\pi : X^\# \to Y\) be an excellent semiconjugacy. Let \(\mu \in \text{Prob}_\text{erg}(T)\) with \(\mu(\pi(X^\#)) = 1\). Then there exist a Markov shift \(\hat{X}\) and a semiconjugacy \(\hat{\pi} : X^\# \to Y\) such that:

1. \(\hat{\pi} : \hat{X}^\# \to Y\) is an excellent semiconjugacy. Moreover, if \(\pi\) admits a multiplicity bound, so does \(\hat{\pi}\);
2. if \(X\) is locally compact, so is \(\hat{X}\);
3. \(\hat{\pi} = \pi \circ q|_{X^\#}\) where \(q : \hat{X} \to X\) is a 1-Lipschitz map with \(q(\hat{X}^\#) \subset X^\#\);
4. \(q : \hat{X}^\# \to X^\#\) is proper, i.e., \(q^{-1}(K) \cap \hat{X}^\#\) is compact for any compact \(K \subset X^\#\);
5. \(\hat{\pi}(\hat{X}^\#) \subset \pi(X^\#)\) and for \(\mu\)-a.e. \(y \in Y\), \(|\hat{\pi}^{-1}(y)| = 1\);
6. there is an invariant measure \(\hat{\mu}\) on \(\hat{X}\) such that \(\hat{\pi} : (S, \hat{\mu}) \to (T, \mu)\) is an isomorphism;
7. \(X\) is irreducible.

Proof. Observe that \(y \mapsto |\pi^{-1}(y)|\) is a \(T\)-invariant function. By ergodicity, it has a \(\mu\)-a.e. constant and positive value we denote \(N\). Obviously \(N \in \Delta(\pi)\). Let \((\pi_N : X^\#\!_N \to Y, \pi_N)\) be the Bowen quotient of \((\pi, \sim)\) of order \(N\) as in Lemma 5.1. Thus \(\pi_N\) satisfies all the claims above except possibly for items (5)-(7).

Item (6)(ii) for \(r = N\) of the lemma implies that \(|\pi_N^{-1}(y)| = 1\) for \(\mu\)-a.e. \(y \in Y\). Therefore, there is a unique \(\hat{\mu} \in \text{Prob}(S|_{X_N})\) such that \(\pi_N : (\hat{\mu}, S_N) \to (\mu, T)\) is an isomorphism. Since \(\mu_N(X^\#_N) \subset X^\#\), we have \(\pi_N(X^\#_N) \subset \pi(X^\#)\). These remarks yield items (5) and (6). As \(\hat{\mu}\) is ergodic, it is carried by an irreducible component \(\hat{X}\) of \(X_N\). It is now clear that \(q := q_N|\hat{X}\) and \(\hat{\pi} := \pi \circ q\) have all the claimed properties.

Theorem 5.3. Let \(X\) be a Markov shift and let \(\pi : X^\# \to Y\) be an excellent semiconjugacy. Then there exist another Markov shift \(\hat{X}\) and a semiconjugacy \(\hat{\pi} : \hat{X}^\# \to Y\) such that properties (1)-(4) in Theorem 5.2 hold and, moreover:

5. \(\hat{\pi}(X^\#) \subset \pi(X^\#)\) and the difference is a null set;
(6') there is a word $\hat{W} \in \mathcal{L}(\hat{X}_{\text{rec}})$ s.t. for any $x \in \hat{X}$ that sees i.o. $\hat{W}$, $\hat{\pi}^{-1}(\hat{\pi}(x)) = \{x\}$; 
(7’) if $X$ is irreducible, then so is $\hat{X}$. 

Remark 5.4. As noted in the proof below, our argument gives a stronger result than stated in item (5’). If $X$ is irreducible, we obtain $\hat{\pi}(\hat{X}^\#) = \pi(X^\#)$. In the general case, the spectral decomposition of $X$ into irreducible components $X_i$, $i \in I$, shows that: $\bigcup_{i \in I} \pi(X_i^\#) \subset \hat{\pi}(\hat{X}^\#) \subset \pi(X^\#)$.

Proof of Theorem 5.3. Let $\hat{\pi} : \hat{X}^\# \to Y$ with $\sim$ be the Bowen quotient of order $N = \min \Delta(\pi)$. Lemma 5.1 yields items (1)-(4) and (5’) and $\delta_{\text{rec}}(\hat{\pi}) = 1$. Theorem 4.6 implies item (6’) for any magic word $\hat{W}$ for $\sim$.

We now assume that $X$ is irreducible and prove that item (7’) can be satisfied while keeping the other properties. We first observe that items (1)-(4) and the inclusion in item (5) are obviously preserved by restriction to any irreducible component. We are going to select an irreducible component for which the second half of item (5’) and item (6’) are satisfied.

During this proof, we will say that a sequence $\hat{x} \in \hat{X}^\#$ is related to some $X$-word $w$, if there are infinitely many $p \geq 0$ and infinitely many $p \leq 0$ such that:

$$\forall 0 \leq i < |w|, \quad w_i \in \hat{x}_{p+i}.$$ 

Observation. For any periodic $x \in X$, $|\pi^{-1}(\pi(x))| \geq N := \min \Delta(\pi)$ and therefore by Fact 3.12 there exists $\hat{x} \in \hat{X}^\#$ with $x_n \in \hat{x}_n$ for all $n \in \mathbb{Z}$.

Claim 5.5. There is an irreducible component $\hat{Z}$ of $\hat{X}$ that contains any $\hat{x} \in \hat{X}^\#$ related to some magic word for $\sim$ over $X_{\text{rec}}$. Moreover, $\delta_{\text{rec}}(\hat{\pi}|\hat{Z}^\#) = 1$.

Proof of the claim. Let $(w, i)$ be a magic couple for $\pi|\hat{X}^\#$ over $X_{\text{rec}}$. Define the set $A_w := \{v_i : v \in \mathcal{L}(\hat{X}^\#) \text{ s.t. } v \sim w\}$ with cardinality $|A_w| = \delta_{\text{rec}}(\pi)$. Note that since $w \in \mathcal{L}(X_{\text{rec}})$, there is a periodic point $x \in X^\#$ that sees $w$. By the observation, this implies the existence of $\hat{x} \in \hat{X}^\#$ related to $w$.

Now let $\hat{x} \in \hat{X}^\#$ be related to $w$. By Fact 3.10 for all $n \in \mathbb{Z}$ $\hat{x}_n = \{z^n_a : a \in \hat{x}_0\}$ where for each $a \in \hat{x}_0$, $z^a = Q(\hat{x}, x)$. Hence, for all $a \in \hat{x}_0$, $z^n_p \cdots z^a_{p+|w|-1} \sim w$ and $\hat{x}_{p+i} \subset A_w$. Since these sets have equal cardinalities, we have: $\hat{x}_{p+i} = A_w$. Therefore, $\hat{x}$ belongs to the irreducible component $\hat{Z}_w$ of $\hat{X}$ containing the symbol $A_w$.

If $v$ is another magic word, there is a periodic orbit $x \in X$ that sees i.o. $v$ and also sees i.o. $w$ ($X$ is transitive). The observation yields some $\hat{x} \in \hat{X}^\#$ which is related to both $v$ and $w$ so $\hat{Z}_v = \hat{Z}_w$. Thus there is an irreducible component $\hat{Z}$ that contains all $\hat{x} \in \hat{X}^\#$ related to any magic word for $\pi$.

To show that $\delta_{\text{rec}}(\hat{\pi}|\hat{Z}^\#) = \delta_{\text{rec}}(\hat{\pi}) = 1$, it suffices to find a magic word for $\hat{\pi}$ in $\mathcal{L}(\hat{Z}_{\text{rec}})$. Let $w$ be a magic word for $\pi$. Given a periodic $x \in X^\#$ with $w$ occurring at index 0, the observation yields a periodic, hence word recurrent $\hat{x} \in \hat{X}^\#$ with $w_n \in \hat{x}_n$ for all $0 \leq n < |w|$.

Let $\hat{w} := \hat{x}_0 \cdots \hat{x}_{|w|-1}$. Obviously $\hat{w} \in \mathcal{L}(\hat{Z}_{\text{rec}})$. We check that $\hat{w}$ is a magic word for $\hat{\pi}$.

Let $\hat{v} \in \mathcal{L}(\hat{X}_{\text{rec}})$ such that $\hat{v} \sim \hat{w}$. By Fact 3.10 $\hat{v}_n = \{v^n_a : a \in \hat{v}_0\}$ where $v^n \in \mathcal{L}(X^\#)$ for all $0 \leq n < |\hat{w}|$. In particular $v^n_0 \sim w_n$ for all $0 \leq n < |w|$. As above, it follows that $\hat{v}_i = \{v^n_i : a \in \hat{v}_0\} = A_w$. This implies that $\delta_{\text{rec}}(\hat{\pi}|\hat{Z}^\#) = \delta_\sim(\hat{w}, i) = 1$. 

\qed
Let \( x \in X^\# \). We are going to show that \( x \in \hat{\pi}(\hat{Z}^#) \) by finding \( y \in X^\# \) Bowen equivalent to \( x \) and which can be approximated by \( q(\hat{x}^n) \) with periodic \( \hat{x}^n \in \hat{Z}^# \). Fix a magic word \( w \) for \( \pi|_{X^\#} \). There are symbols \( a, b \) of \( X \) and integers \( m_k, n_k \geq k \) such that \( x_{-m_k} = a \) and \( x_{n_k} = b \) for all \( k \geq 1 \). There is an \( X \)-word \( u_0 \ldots u_{\ell+1}, \ell \geq 1, \) with \( u_0 = b \) and \( u_{\ell+1} = a \) and containing \( w \) as a subword (since \( X \) is irreducible). For each \( k \geq 1 \), let \( x^k \in X^\# \) be the periodic sequence with period \( \tau_k := n_k + m_k + \ell + 1 \) defined by:

\[
\forall i = -m_k, \ldots, n_k + \ell \quad x^k_i = \begin{cases} x_i & \text{if } -m_k \leq i \leq n_k, \\ u_{i-n_k+1} & \text{if } n_k < i < n_k + \ell. \end{cases}
\]

Note that for all \( i \in \mathbb{Z}, \) \( A_i := \{x^k_i : k \geq 1\} \subset \{x_i : k \leq |i|\} \) is finite. The local finiteness of the Bowen relation implies that, for all \( i \in \mathbb{Z}, \) the set of symbols \( B_i := \{s : \exists t \in A_i \text{ s.t. } s \sim t\} \) is finite.

Since \( x^k \in X^\# \) is periodic, the observation gives \( \hat{x}^k \in \hat{X}^\# \) such that \( x^k_i \in \hat{x}^k_i \) for all \( i \in \mathbb{Z}. \) In particular, \( \hat{x}^k \) is related to \( \hat{Z} \) by the claim. Note that \( \hat{x}^k_i \subset \{s : s \sim x^k_i\} \) hence \( \hat{x}^k_i \subset B_i \) for all \( i \in \mathbb{Z} \) and \( k \geq 1. \) Thus there is a point of accumulation \( \hat{x} = \lim_n x^{k(n)} \in \hat{Z} \) for some sequence \( k(n) \uparrow \infty. \) If \( x_1 = a \) (resp. \( b \)), then, for all large \( k, \) \( \hat{x}^k_i \subset \{c : c \sim a \} \) (resp. \( c \sim b \)) which is finite and independent of \( i \in \mathbb{Z}, \) hence \( \hat{x} \in \hat{Z}^#. \)

Let \( y^k := q(\hat{x}^k) \) for \( k \geq 1. \) As \( q \) is continuous, \( y^{k(n)} = q(\hat{x}^{k(n)}) \) converges to the sequence \( y := q(\hat{x}). \) Since \( \hat{x} \in \hat{X}^# \), we have \( y \in X^\# \). For all \( i \in \mathbb{Z}, \) \( y^k_i \in \hat{x}^k_i \) by construction, hence \( y^k_i \sim x^k_i. \) Recalling that \( x^k_i = x_i \) for all \( k \geq |i|, \) we get \( y \approx x. \) By the Bowen property:

\[
\pi(x) = \pi(y) = \pi(q(\hat{x})) = \hat{\pi}(\hat{x}).
\]

Thus \( \pi(X^#) \subset \hat{\pi}(\hat{Z}^#) = \pi(q(\hat{Z}^#)) \subset \pi(X^#), \) so \( \hat{\pi}(\hat{Z}^#) = \pi(X^#), \) yielding item \((5').\) Since \( \deg_{\text{rec}}(\pi|_{\hat{Z}^#}) = 1, \) Theorem 4.6 yields item \((6').\) \( \square \)

**Remark 5.6.** The periodic approximation argument in the last part of the proof of Theorem 5.3 is partly inspired by some geometric construction of [7].

### 5.2. Preparations.

We turn to the proof of the Main Theorem. Let \( \pi : X^\# \to Y \) be an excellent semiconjugacy for some Bowen relation \( \sim \). We are going to build an injective coding of the image \( \pi(X^#) \). We start with the following simple fact about partially ordered sets. In this paper \( \mathbb{N}^0 \) is the set of nonnegative integers.

**Fact 5.7.** Let \( (\mathcal{O}, \preceq) \) be a countable (possibly finite) set together with a partial order \( \preceq. \) There is a bijection \( \sigma : \{n \in \mathbb{N}^0 : n < |\mathcal{O}|\} \to \mathcal{O} \) which is nondecreasing, i.e.,

\[
(5.8) \quad \forall i, j : \quad \sigma(i) \preceq \sigma(j) \implies i \leq j
\]

if and only if all initial segments \( \{b \in \mathcal{O} : b \preceq a\}, a \in \mathcal{O}, \) are finite.

**Proof.** If \( \sigma : \mathbb{N}^0 \to \mathcal{O} \) is a bijection satisfying eq. \((5.8), \) then any initial segment \( \{b \in \mathcal{O} : b \preceq \sigma(i)\} \) is finite as a subset of \( \sigma(\{0, 1, \ldots, i\}). \) We now assume that all initial segments are finite and proceed to build the bijection \( \sigma. \)

If \( \mathcal{O} \) is finite, then one can define \( \sigma : \{0, \ldots, n-1\} \to \mathcal{O} \) inductively by choosing, for each \( 0 \leq k < n, \) \( \sigma(k) \) to be some minimal element \( \alpha \) among \( \mathcal{O} \setminus \sigma(\{0, \ldots, k-1\}), \) i.e., such that:

\[
\forall \beta \in \mathcal{O} \setminus \sigma(\{0, \ldots, k-1\}) \quad \beta \preceq \alpha \implies \beta = \alpha.
\]
We assume now that \( O \) is infinite so there is a bijection \( s : \mathbb{N}^0 \to O \). We define integers \( N_0 < N_1 < \ldots \) and \( \sigma|\{0, \ldots, N_n - 1\} \) inductively by setting \( N_0 = 0 \) and, for each \( n \geq 0 \),

1. applying the finite case, enumerate \( \{b \in O : b \leq s(n)\} \setminus \sigma|\{0, \ldots, N_n - 1\} \) as \( \{b_{n,1}, \ldots, b_{n,\ell_n}\} \) where \( i \mapsto b_{n,i} \) is injective and non-decreasing;
2. set \( N_{n+1} := N_n + \ell_n \) and \( \sigma(N_n + i) = b_i \) for \( i = 0, \ldots, \ell_n - 1 \).

It is easy to check that \( \sigma \) is a nondecreasing bijection. \( \square \)

We will apply the following elementary construction to an enumeration of the magic words for the relation \( \sim \) in \( X^\# \) over \( X^{\text{rec}} \).

**Lemma 5.9.** Let \( X \) be a Markov shift. Let \( W := (W^j)_{1 \leq j < J} \) \((1 < J \leq \infty)\) be an enumeration of \( X\)-words. Then there is an injective one-block code \( p : S \to X \) defined on a Markov shift \( S \) whose image \( p(S) \) is \( X_W \setminus N \) for some null set \( N \) and

\[
X_W := \{ x \in X : \text{there is } w \in W \text{ such that } x \text{ sees i.o. } w \} \subset X^\#.
\]

Recall that, given some word \( W \in \mathcal{L}(X) \), \( X_W \) is the set of sequences that see i.o. \( W \) (see p. 7).

**Proof.** Since the set of subwords of a given word is finite, Fact 5.7 allows us to assume (maybe after a permutation) that:

\[
W^i \text{ subword of } W^j \implies i \leq j
\]

For \( 1 \leq j < J \), consider the following subset of \( X_W \):

\[
X_j := \{ x \in X : x \text{ sees i.o. } W^j, \text{ none of } W^1, \ldots, W^{j-1} \text{ occurs in } x \}.
\]

The injective code we are going to build will have image \( \bigcup_{1 \leq j < J} X_j \). The sets \( X_j \) are pairwise disjoint. Each sequence in \( X_W \setminus \bigcup_{1 \leq j < J} X_j \) contains some \( W \)-word that does not occur i.o., hence this difference is null.

Fix \( 1 \leq i < J \) such that \( X_i \neq \emptyset \). Let \( N := |W^i| \). We perform some standard graph constructions. First, consider the \( N \)th higher block presentation \( X^{[N]} \) of \( X \) (see, e.g., [21 I.4.1]) defined by the graph \( \mathcal{G}^1 \) with:

- vertices: \( (x_0, \ldots, x_{N-1}) \), arrows: \( (x_0, \ldots, x_{N-1}) \to (x_1, \ldots, x_N) \) \((x \in X)\).

There is a topological conjugacy \( X^{[N]} \to X \) defined by the one-block code \( (x_0 \ldots x_{N-1}) \mapsto x_0 \) with inverse: \( (x_n)_{n \in \mathbb{Z}} \mapsto (x|_{n,n+N-1})_{n \in \mathbb{Z}} \).

Let \( \mathcal{G}^2 \) be the loop graph at the base vertex \( W^i \) in \( \mathcal{G}^1 \), defined as follows (see, e.g., [13 4]).

The first return loops at \( W^i \) are the finite sequences \((y_0, y_1, \ldots, y_{k-1})\) where each \( y_i \) is an \( X \)-word of length \( N \), \( y_0 = W^i \), \( y_0 \to y_1 \to \cdots \to y_{k-1} \to W^i \) on \( \mathcal{G}^1 \), and \( y_\ell \neq W^i \) for any \( 1 \leq \ell \leq k-1 \). Now the loop graph \( \mathcal{G}^2 \) is defined by taking as vertices the couples \((v, \ell)\) where \( v \) is a first return loop at \( W^i \) and \( 0 \leq \ell < |v| \), and as arrows:

- \((v, \ell) \to (v, \ell + 1)\) if \( v \) is a first return loop and \( 1 \leq \ell + 1 < |v|\)
- \((v, |v| - 1) \to (w, 0)\) if \( v, w \) are first return loops.

The corresponding shift is mapped into \( X \) by the one-block code \((v, \ell) \mapsto v_0^\ell \) (i.e., the first symbol of the word \( v^\ell \)).

---

4Contrary to usual practice, we do not identify all \((v, 0)\) vertices with a single distinguished vertex.
We define $\mathcal{G}^3$ by keeping from $\mathcal{G}^2$ only the vertices $(v, \ell)$ where $v$ is a first return loop $(y_0, \ldots, y_{k-1})$ that is good, i.e., whose extensions:

$$(y_0, \ldots, y_{k-1}, W^i)$$

map to an $X$-word of length $k + |W^i|$ that does not contain any of the words $W^1, \ldots, W^{i-1}$.

Let $S_i$ be the Markov shift defined by this loop graph $\mathcal{G}^3$ and define $p^i : S_i \to X_i$ to be the restriction of the previous map.

Claim 5.11. The map $p^i : S_i \to X_i$ is a topological conjugacy defined by a one-block code.

Proof of the claim. It is obvious that $p^i$ is a one-block code. We have to check that it defines a bijection and that its inverse is continuous.

Consider some $x \in X_i$. It can be lifted to a concatenation of first return loops since $x$ sees i.o. $W^i$. These first return loops must be good since $x$ avoids $W^1, \ldots, W^{i-1}$. Thus $x$ belongs to the image of $\mathcal{G}^3$. Conversely, let $x \in X$ be the image of some $\hat{x}$ on $\mathcal{G}^3$, i.e., an infinite concatenation of good first return loops. Assume by contradiction that some $W^j$, $j < i$ occurs in $x$. By (5.10), this occurrence may overlap but cannot contain any occurrence of $W^i$. Thus $W^j$ occurs in the image of some extended first return loop, so the first return loop is not good. This contradicts the definition of $\mathcal{G}^3$, proving that $p^i(S_{i}) = X_{i}$.

Note that $p^i$ is invertible with inverse defined by:

$$\forall x \in X_i \quad p^i(\hat{x}) = x \iff \hat{x}_0 = ((x_{[j,j+N-1]} -_{n \leq j < m}), -n)$$

with $n = \max\{k \leq 0 : x_{[k,k+N-1]} = W^i\}$ and $m = \min\{k > 0 : x_{[k,k+N-1]} = W^i\}$. This inverse is continuous. The claim is proved.

To conclude the proof of the lemma, let $S$ be the Markov shift $\bigcup_{1 \leq j < J} S_j$ (considering the alphabets to be pairwise disjoint) and define the map $p : S \to \bigcup_{1 \leq j < J} X_j$ by $p(x) = p^i(x)$ if $x \in S^i$. This is well-defined. Obviously $p$ is a one-block code. As the sets $X^i$ are disjoint and each $p^i$ is injective, so is $p$. Remark that $p(S) = \bigcup_{1 \leq j < J} X_j$ and that this union coincides with $X_W$ up to a null set.

Remark 5.12. The proof of the above lemma does not provide a locally compact Markov shift $S$, even if $X$ is compact.

5.3. Proof of the Main Theorem. Let $\pi : X^\# \to Y$ be an excellent semiconjugacy with a Bowen relation $\sim$. We are going to divide the image $\pi(X^\#)$ according to the number of preimages and then successively reduce each of these numbers to one (ignoring null sets). We assume that $\pi(X^\#)$ is not null as otherwise there is nothing to show.

Let $(\Delta(i))_{1 \leq i \leq I}$ with $1 < I \leq \infty$ be the increasing enumeration of the degree spectrum $\Delta(\pi)$ (see Def. 3.4). The corresponding degree partition of $\pi(X^\#)$ is:

$$(Y^i)_{1 \leq i < I} \text{ with } Y^i := \{y \in Y : |\pi^{-1}(y)| = \Delta(i)\}.$$  

We are going to define semiconjugacies $\pi_i : Z^\#_i \to Y$ with Bowen relations $\sim$ such that, setting $\Z_i := \{x \in Z^\#_i : x \text{ sees i.o. some magic word for } \pi_i\}$ we have:

(a) $\pi_i : Z^\#_i \to Y$ is an excellent semiconjugacy with $\delta_{\text{rec}}(\pi_i) = 1$;
(b) $\pi_i = \pi \circ q_i|Z^\#_i$ for some 1-Lipschitz map $q_i : Z_i \to X$;
For any Claim 5.13. \( Y \) with \( \Delta \) by Lemma 5.1, item (3).

\( \Delta(1) = \min \Delta(\pi) \) shows that this is well-defined and that the above items (a)-(e) hold with \( \Delta(1) = (\Delta(j)) \).

Let 1 < i < I and assume that \((Z_j, \pi, \sim, q_j)\) have been defined with these properties for all 1 ≤ j < i. Let \((\tau_i : Z_i^\# \to Y, \sim)\) be the Bowen quotient of \((\pi_i^{-1}, \sim^i)\) of order \(N_{i-1} := \Delta_{i-1}(i) \) (this last set is nonempty by item (e) since \(i < I \) ). Lemma 5.1 shows that this is well-defined and that the above items (a)-(d) hold with \(q_i = q_{i-1} \circ q\) where \(q\) is given by Lemma 5.1 item (3).

We turn to item (e). The item (6) of Lemma 5.1 shows that \(\Delta(\pi_i) = \{\Delta_{i}(j) : 1 \leq j < I\}\) with \(\Delta_i = (\Delta_{i-1}(j))\) and, for all \(i \leq j < I\), up to a null set:

\[ \{y \in Y : |\pi_i^{-1}(y) \cap Z_i^\#| = \Delta_{i}(j)\} = \{y \in Y : |\pi_{i-1}^{-1}(y) \cap Z_{i-1}^\#| = \Delta_{i-1}(j)\} \]

so \(Y_j^i = Y_{j-1}^i \) up to null sets, proving (e).

**Claim 5.13.** For any 1 < i < I, there are a Markov shift \(S_i\) and a one-block code \(p_i : S_i \to p_i(S_i)\) such that: \(p_i(S_i) \subset \tilde{Z}_i\) with the difference a null set, \(\pi_i \circ p_i(S_i) = Y_i\) up to a null set, and \(\pi_i \circ p_i\) is injective.

**Proof of the claim.** Lemma 5.1 provides an injective one-block code \(p_i\) of some Markov shift \(S_i\) into \(\tilde{Z}_i\) with \(p_i(S_i) \subset \tilde{Z}_i\) with the difference a null set.

By item (d) above and Lemma 2.1 \(\pi_i \circ p_i(S_i) = Y_i\) up to a null set. By item (c), \(\pi_i \circ p_i\) is injective. \(\square\)

To conclude, let \(S\) be the disjoint union \(\bigcup_{1 \leq i < j} S_i\) of the one-block codes from the previous claim (we can always recode to ensure this disjointness). It is a Markov shift. Define a semiconjugacy \(p\) on \(S\) by:

\[ p|S^i = \pi_i \circ p_i. \]

Note that \(p|S^i = \pi \circ q_i \circ p_i\) where \(p_i\) is a one-block code and \(q_i\) is 1-Lipschitz. Thus \(q_i \circ p_i\) is 1-Lipschitz. The image of \(p\) contains \(\bigcup_{1 \leq i < j} Y^i\) up to a null set, hence \(\pi(X^\#)\) up to a null set. To conclude the proof of the Main Theorem, it suffices to see that the images \(p(S_i) \subset \pi_i(\tilde{Z}_i), 1 \leq i < I,\) are pairwise disjoint. We have:

**Claim 5.14.** For any 1 ≤ j < i < I, \(\pi_i(Z_i^\#) \cap \pi_j(\tilde{Z}_j) = \emptyset.\) In particular, the images \(\pi_i(\tilde{Z}_i), 1 \leq i < I,\) are pairwise disjoint.

To prove this claim, note that \(y \in \pi_i(Z_i^\#)\) implies that \(|\pi_i^{-1}(y) \cap Z_i^\#| \geq \Delta_i(i) > 1\) and thus, by induction, \(|\pi_j^{-1}(y) \cap Z_j^\#| \geq \Delta_j(i) > 1.\) However, \(y \in \pi_j(\tilde{Z}_j)\) implies \(|\pi_j^{-1}(y) \cap Z_j^\#| = 1.\) Hence \(\pi_i(Z_i^\#) \cap \pi_j(\tilde{Z}_j) = \emptyset\) as claimed. The last assertion follows from \(\tilde{Z}_i \subset Z_i^\#.\)
Remark 5.15. The Bowen quotient is used for two seemingly distinct purposes: first, to remove points whose images have already been taken care of; second, to lower the minimal degree to 1.

6. Applications to surface diffeomorphisms

We prove Theorem 1.1 and a more precise version of Theorem 1.2

Let \( f \) be a \( C^{1+\alpha} \)-diffeomorphism, \( \alpha > 0 \), of a smooth closed surface \( M \) with \( h_{\text{top}}(f) > 0 \).

As observed in [4, Sec. 8], Sarig [24] provides a Markov shift \( \hat{\Sigma} \) and a Hölder-continuous semiconjugacy \( \hat{\pi} : \Sigma \to M \) such that, \( \Sigma \# \) denoting its regular part:

- (P1) \( \hat{\pi}|\Sigma\# \) admits a Bowen relation (called affiliation in [24, Sec. 12.3]) which is locally finite (see [4, Summary 8.1(4)(5)]);
- (P2) \( \hat{\pi}|\Sigma\# \) is finite-to-one (as explained in [20] the claim that \( \hat{\pi} \) is finite-to-one on \( \Sigma \) itself was made erroneously in [24]);
- (P3) \( \mu(\hat{\pi}(\Sigma\#)) = 1 \) for any \( \chi \)-hyperbolic measure \( \mu \in \text{Prob}_{\text{erg}}(f) \).
- (P4) any ergodic \( \nu \) on \( \hat{\Sigma} \), \( \hat{\pi}_\nu(\nu) \) is \( \chi/2 \)-hyperbolic, see [24, Prop. 12.6].

The construction and analysis of \( \hat{\Sigma} \) in [24] relies on another Markov shift \( \Sigma \) defined by a graph \( G \) whose vertices \( \Psi_{p^s,p^u} \) are double charts, that is, local charts \( \Psi : (-r, r)^2 \to M \) centered at some point \( x \in M \) together with two numbers \( p^s, p^u > 0 \) with \( r = \min(p^s, p^u) \). The charts \( \Psi \) are defined by Pesin theory as \( \exp_x \circ C_\chi(x) \) where \( \exp_x \) is the exponential map centered at \( x \) and \( C_\chi(x) \) is the Oseledets-Pesin reduction matrix. These charts make “the hyperbolicity of \( f \) uniform”: for any arrow \( \Psi_{p^s,p^u} \to \Phi_{p^s,p^u} \) in \( G \), the map \( \Phi^{-1} \circ f \circ \Psi \) is close to a linear map \( (x_1, x_2) \mapsto (\lambda x_1, \kappa x_2) \) with \( \lambda > e^\chi \) and \( \kappa < e^{-\chi} \).

Each Markov shift has its cylinders. For \( \Sigma \), they are:

\[ Z_{-n}(\Psi_n, \ldots, \Psi_0) := \pi\{x \in \Sigma : \forall |k| \leq n \ x_k = \Psi_k\} \subset M \] where each \( \Psi_k \) is a double chart while those in \( \hat{\Sigma} \) are:

\[ -n[R_{-n}, \ldots, R_0] := \hat{\pi}\{x \in \hat{\Sigma} : \forall |k| \leq n \ x_k = R_k\} \subset M \] where each \( R_k \) is a rectangle.

In this way, two Hölder-continuous semiconjugacies \( \pi : \Sigma \to M \) and \( \hat{\pi} : \hat{\Sigma} \to M \) are defined by some shadowing properties. (Contrarily to \( \hat{\pi}|\Sigma\# \), the map \( \pi|\Sigma\# \) is not finite-to-one.)

To prove the last claim of Theorem 1.1, we use a strengthening of property (P4) above: we can replace \( \chi/2 \) by any number less than \( \chi \), at least in the case of periodic orbits. We freely use the terminology and notations from [24], including the two semiconjugacies \( \hat{\pi} : \hat{\Sigma} \to M \) and \( \pi : \Sigma \to M \).

Lemma 6.1. Given \( \tilde{\chi} < \chi \), there is a coding \( \hat{\pi} : \hat{\Sigma} \to M \) with (P1)-(P4) as above that additionally satisfies the following property: for any periodic sequence \( \hat{x} \in \hat{\Sigma} \), \( \hat{\pi}(\hat{x}) \) is \( \tilde{\chi} \)-hyperbolic.

Proof. Let \( x := \hat{\pi}(\hat{x}) \). Lemma 12.2 from [24] yields a sequence of double charts \( (\Psi_n)_{n \in \mathbb{Z}} \in \Sigma \) such that, for all \( n \geq 0 \), \( -n[\hat{x}_{-n} \ldots \hat{x}_n] \subset Z_{-n}(\Psi_{-n}, \ldots, \Psi_0) \). It follows from Proposition 4.11 in [24] that all points in \( \pi([\Psi_0] \cap \Sigma\#) \) can be written \( \Psi_0(t) \) with \( t \in \mathbb{R}^2 \) close to \( 0 \in \mathbb{R}^2 \). Hence
\( \hat{\pi}(\hat{x}) \) lift to \( t_0 \) in the domain of the chart \( \Psi_0 \): that is, for each \( n \in \mathbb{Z} \), \( t_n := \Psi_n^{-1} \circ f^n(\hat{x}) \) is well-defined. Letting \( f_k := \Psi_k^{-1} \circ f \circ \Psi_{k-1} \), we have \( t_k = f_k(t_{k-1}) \) and:

\[
Df^n(x) = D\Psi_n \circ Df_n \circ \cdots \circ Df_1 \circ D\Psi_0^{-1}(x).
\]

Thus

\[
\|Df^n(x)\| \geq \|D\Psi_n(t_{n-1})\|^{-1} \cdot \|D\Psi_0(t_0)\|^{-1} \cdot \|Df_n \circ \cdots \circ Df_1\|
\]

By Proposition 3.4 of [24], choosing the parameter \( \epsilon > 0 \) of Sarig’s construction small enough and considering vectors in the unstable cone, we obtain:

\[
\|Df_n \circ \cdots \circ Df_1\| \geq (\epsilon^{\chi} - 2\epsilon)^n \geq e^{n\chi}.
\]

Since \( \Psi_k = \exp \circ C(x) \) where \( x \) is the center of \( \Psi_k \) [24, eq. (2.2)], we have: \( \|D\Psi_n^{-1}\| \leq C_0 \cdot \|C(x_n)\|^{-1} \) for some constant \( C_0 \) (depending only on \( f \)). Since \( \hat{x} \) is periodic, [24, Theorem 10.2] shows that \( \Psi_n \) takes only finitely many values as \( n \) ranges over \( \mathbb{Z} \). It follows that setting \( C_1(x) := \inf_{n \geq 0} \|C_0\|^{-1} \cdot \|\Psi_0\|^{-1} > 0 \), we get:

\[
\forall n \geq 0 \|Df^n(x)\| \geq C_1(x) e^{n\chi}.
\]

Hence the periodic orbit \( O(x) \) has a positive exponent larger than or equal to \( \hat{\chi} \). A symmetric argument shows that \( O(x) \) is \( \hat{\chi} \)-hyperbolic.

**Proof of Theorem 7.2**. We consider Sarig’s coding with the addition of the property from Lemma 6.1. The previous discussion shows that our Main Theorem applies. It produces a new coding is easily seen to satisfy our claims.

We turn to the counting of hyperbolic periodic orbits. This requires the following estimate in eq. (6.3). It is folklore, but since we did not find a reference we will deduce it from [18, chap. 7], using freely its terminology and notations.

A measure maximizing the entropy (or: m.m.e.) of some Borel automorphism is an invariant Borel probability measure which realizes the supremum of the Kolmogorov-Sinai entropy over all invariant probability measures.

**Lemma 6.2.** If \((X, \sigma)\) is an irreducible Markov which is positively recurrent (i.e., it has some m.m.e. and its entropy is finite) with period \( p \), then:

\[
(6.3) \quad \lim_{n \to \infty} \frac{1}{p^n} \cdot \left| \left\{ x \in X : \left| \{\sigma^k x : k \in \mathbb{Z}\} = n \right\} \right| \geq p.
\]

**Proof.** We freely use results and notations from Kitchens’ book [18] and in particular the generating functions \( L_{ab}(z) \) and \( R_{ab}(z) \). First suppose that \( X \) is mixing (i.e., \( p = 1 \)). Fix some symbol \( a \in A \) occurring in \( X \) and set \( \lambda := e^{|\text{top}(f)|} \). Since \( X \) is recurrent, Theorem 7.1.18 implies:

\[
\lim_{n \to \infty} \lambda^{-n} \left| \left\{ x \in X : \sigma^n x = x, \sigma^n x = a, \sigma^{n+1} x = x \right\} \right| = \frac{1}{\mu(a)}
\]

where \( \mu(a) := (1/\lambda) L_{aa}(1/\lambda) \). Since \( X \) is positive recurrent, Lemma 7.1.21 yields \( \mu(a) = \ell^{(a)} \cdot r^{(b)} \) where \( \ell^{(a)} := (L_{aj}(1/\lambda))_{j \in A} \) and \( r^{(b)} := (R_{jb}(1/\lambda))_{j \in A} \). We also have \( L_{aa}(1/\lambda) =

---

5We note that a similar estimate was obtained, e.g., in [8] but with a stronger assumption (the SPR property) and stronger conclusion (an error estimate).
\[ R_{\alpha a}(1/\lambda) = 1 \] for all \( a \in A \) as \( X \) is recurrent (see the proof of Lemma 7.1.8, recalling that, by definition, \( T_{\alpha a}(1/\lambda) = \infty \) if and only if \( X \) is recurrent). Thus,

\[
\frac{1}{\mu(a)} = \frac{1}{\sum_{j \in A} L_{aj}(1/\lambda)R_{ja}(1/\lambda)} = \frac{L_{\alpha a}(1/\lambda)R_{\alpha a}(1/\lambda)}{\sum_{j \in A} L_{aj}(1/\lambda)R_{ja}(1/\lambda)}.
\]

Now Lemma 7.2.15 implies that \( \nu([b]) = \frac{\nu_0(a)}{\sum_{j \in A} \nu_0(j)} \) for any \( a, b \in A \). Thus,

\[
\lim_{n \to \infty} \lambda^{-n}|\{x \in X : x_0 = a, \sigma^n x = x\}| = \nu([a]).
\]

For \( p > 1 \), the cyclic decomposition from [18] p. 223 yields:

\[
\lim_{n \to \infty} \lambda^{-n}|\{x \in X : x_0 = a, \sigma^n x = x\}| = p\nu([a]).
\]

Using the decomposition:

\[
\{x \in X : x_0 = a, \sigma^n x = x\} = \bigcup_{k|n} \{x \in X : x_0 = a, |\{\sigma^j(x) : j \in \mathbb{Z}\}| = k\}
\]

and noting that \( k|n \) implies \( k = n \) or \( k \leq n/2 \), we get:

\[
\lim_{n \to \infty} \lambda^{-n}|\{x \in X : x_0 = a, |\{\sigma^j(x) : j \in \mathbb{Z}\}| = n\}| = p\nu([a]).
\]

Since \( \nu(X) = 1 \), a routine argument shows eq. (6.3). \( \square \)

We are going to obtain the following relation between periodic points and measures maximizing the entropy:

**Theorem 6.4.** Let \( f \in \text{Diff}^{1+\alpha}(M) \) where \( M \) is a closed surface and \( \alpha > 0 \). Assume that there are distinct ergodic measures maximizing the entropy: \( \mu_1, \ldots, \mu_r \) with periods \( p_1, \ldots, p_r \geq 1 \). Fix \( \tilde{\chi} < h_{\text{top}}(f) \). Then

\[
\lim_{n \to \infty} \inf_{p_1, \ldots, p_r | n} e^{-n h_{\text{top}}(f)} \cdot |\text{per}_{\tilde{\chi}}(f, n)| \geq p_1 + \ldots + p_r. \tag{6.5}
\]

When \( f \) is \( C^\infty \) smooth, Newhouse’s Theorem [23] shows that there is at least one m.m.e. If, additionally, \( f \) is topologically mixing, [7] shows that there is a m.m.e. with period equal to 1. Therefore:

**Corollary 6.6.** In the setting of the above theorem, assuming additionally that \( f \) is \( C^\infty \) we obtain:

- for some integer \( p \geq 1 \), \( \lim_{n \to \infty} \inf_{p_1, \ldots, p_r | n} e^{-n h_{\text{top}}(f)} \cdot |\text{per}_{\tilde{\chi}}(f, n)| \geq p; \)
- if \( f \) is topologically mixing, \( \lim_{n \to \infty} e^{-n h_{\text{top}}(f)} \cdot |\text{per}_{\tilde{\chi}}(f, n)| \geq 1. \)

This implies Theorem 1.2.

**Proof of Theorem 6.4.** We fix \( \tilde{\chi} < h_{\text{top}}(f) \) and consider a coding \( \hat{\pi} : \hat{\Sigma} \to M \) as in Lemma 6.1.

If \( \mu_1, \ldots, \mu_r \in \text{Prob}_{\text{erg}}(f) \) are distinct m.m.e.’s, Sarig’s [24] shows that each \( \mu_i \) is isomorphic to the product of a Bernoulli scheme and a circular permutation of some order \( p_i \). Being an m.m.e. is invariant under Borel conjugacy, hence \( \hat{\Sigma} \) carries distinct m.m.e.’s \( \nu_1, \ldots, \nu_r \) with \( \pi_*(\nu_i) = \mu_i \).
By general results about Markov shifts and their m.m.e.’s [13], \( \hat{\Sigma} \) contains disjoint irreducible components \( X_1, \ldots, X_r \), where each \( X_i \) carries a distinct m.m.e. \( \nu_i \). In particular, each \( X_i \) is positive recurrent and has period equal to \( p_i \).

By Lemma 6.1 the following map is well-defined and injective:

\[
\pi : \{ x \in \hat{\Sigma} : |\{ \sigma^j(x) : j \in \mathbb{Z} \}| = n \} \rightarrow \text{per}_x(f, n).
\]

The claim (6.5) now follows from Lemma 6.2.

7. AN OBSTRUCTION TO HÖLDER-CONTINUOUS CODING

We prove Theorem 1.6 characterizing surface diffeomorphisms with Hölder-continuous symbolic dynamics. Recall that a map \( \pi : X \rightarrow M \) is Hölder-continuous with some positive exponent \( \alpha \) if there is a constant \( C < \infty \) such that, for all \( x, y \in X^\# \),

\[
d(\pi(x), \pi(y)) \leq C \exp(-\alpha \inf\{|n| : x_n \neq y_n\}).
\]

Let \( f \) be a diffeomorphism of a compact \( d \)-dimensional manifold \( M \) and let \( \mu \in \text{Prob}_{\text{erg}}(f) \). Write its Lyapunov exponents as \( \lambda_1(f, \mu) > \cdots > \lambda_\nu(f, \mu) > 0 \geq \lambda_{\nu+1}(f) > \cdots > \lambda_r(f, \mu) \).

This measure has saddle type if \( \lambda_{\nu+1} < 0 \) and \( 0 < u < d \). Let

\[
\text{Prob}_{\text{hyp}}(f) := \{ \mu \in \text{Prob}_{\text{erg}}(f) : \mu \text{ is aperiodic and of saddle type} \}.
\]

Recall that for \( \mu \in \text{Prob}_{\text{hyp}}(f) \), \( \chi(\mu) := \min(\lambda_u(f, \mu), -\lambda_{\nu+1}(f, \mu)) \). Sarig’s theorem [24] and its higher dimensional generalization by Benovadia [2] yield a global coding if \( \chi(f) := \inf\{\chi(\mu) : \mu \in \text{Prob}_{\text{erg}}(f)\} \) is positive. We prove:

**Proposition 7.1.** Let \( f \in \text{Diff}_{1+}(M) \) with \( M \) a closed manifold. Let \((S, X)\) be a Markov shift and let \( \pi : (S, X) \rightarrow (f, M) \) be a semiconjugacy. Assume that \( \pi \) is Hölder-continuous with exponent \( \alpha > 0 \). Given any \( \nu \in \text{Prob}_{\text{erg}}(\Sigma) \), if \( \mu := \pi_\ast(\nu) \) is hyperbolic and atomless, then:

\[
\tilde{\chi}(\mu) := \min(\lambda_1(f, \mu), -\lambda_d(f, \mu)) \geq \alpha.
\]

This shows that \( \inf\{\tilde{\chi}(f, \mu) : \mu \in \text{Prob}_{\text{hyp}}(f)\} \geq \alpha \) is a necessary condition for the existence of a Hölder-continuous coding with exponent \( \alpha \). Since \( \tilde{\chi}(f, \mu) = \chi(f, \mu) \) in dimension 2, Theorem 1.6 is established.

**Proof.** Let \( \nu \in \text{Prob}_{\text{erg}}(\Sigma) \) such that \( \pi_\ast(\nu) \in \text{Prob}_{\text{hyp}}(f) \). For \( \mu \)-a.e. \( x \in \Sigma \), the Pesin stable manifold of \( y := \pi(x) \)

\[
W^s(y) := \{ z \in M : \lim_{n \to \infty} \frac{1}{n} \log d(f^n(z), f^n(y)) < 0 \}
\]

satisfies:

\[
W^s(y) = \{ z \in M : z = y \text{ or } \lim_{n \to \infty} \frac{1}{n} \log d(f^n(z), f^n(y)) \in [\lambda_r(f, \mu), 0) \}.
\]

Since \( \nu \) is not carried by a periodic orbit, it is carried by a nontrivial irreducible component of the Markov shift. Hence there is \( z \in \Sigma \) such that \( z \neq x \) and \( z_n = y_n \) for all \( n \geq 0 \).

---

\(6\) The period of an irreducible Markov shift is the greatest common divisor of the periods of its periodic points.
Therefore \( d(\sigma^nx, \sigma^nz) = Ce^{-n} \) for some \( C > 0 \) and all \( n \geq 0 \). Now the Hölder-continuity of \( \pi \) gives \( C' > 0 \) such that
\[
\forall n \geq 0 \quad d(f^n(y), f^n(\pi(z))) \leq C'e^{-\alpha n}.
\]
This exponential convergence implies that \( \pi(z) \in W^s(y) \). By eq. (7.2), \( \lambda_f(f, \pi_*(\nu)) \leq -\alpha \).

By considering \((f^{-1}, \mu)\), we obtain \( \lambda_1(f, \pi_*(\nu)) \geq \alpha \). Thus \( \nabla(\pi_*(\nu)) \geq \alpha \).

\( \square \)

### Appendix A. Further remarks

#### A.1. Canonical Bowen relation

A semiconjugacy \( \pi : X \to Y \) of a symbolic system \( X \) can admit several Bowen relations. However, one can define its canonical relation over its alphabet \( A \) by: \( \forall a, b \in A \; \ a \sim b \overset{\text{def}}{=} \pi([a]_X) \cap \pi([b]_X) \neq \emptyset \) (recall that \([\cdot]_Z\) denotes the cylinder in \( Z \) defined by some word). The above relation is obviously reflexive and symmetric. We denote by \( \tilde{\sim} \) the induced Bowen equivalence on \( X \).

**Lemma A.1.** For an arbitrary semiconjugacy \( \pi : X \to Y \), the following implication holds:
\[
(A.2) \quad \forall x, y \in X \quad \pi(x) = \pi(y) \implies x \tilde{\sim} y.
\]

If the semiconjugacy \( \pi \) is Bowen, then the canonical relation is a Bowen relation and it is the minimal one: if \( \sim \) is any Bowen relation for \( \pi \), then \( a \tilde{\sim} b \implies a \sim b \) for any \( a, b \in A \).

**Proof.** The implication \((A.2)\) is immediate. Now assume that \( \pi \) has some Bowen relation \( \sim \) and let \( a, b \in A \) with \( a \sim b \): there are \( x \in [a]_X \) and \( y \in [b]_X \) with \( \pi(x) = \pi(y) \). The Bowen property for \( \sim \) gives \( a \sim b \) so we have proved \( a \tilde{\sim} b \implies a \sim b \). Now it is obvious that \( \tilde{\sim} \) is a Bowen relation for \( \pi \).

**Remark A.3.** We do not know whether the reflexive and symmetric relation that appears in Sarig’s construction (called affiliation) is canonical. Additionally, we do not know if the Bowen quotient (Theorem 3.3) of a canonical relation is itself canonical.

#### A.2. Consequences for continuous extensions

In our most important examples, the semiconjugacy is continuous over the Markov shift \( X \) but the Bowen property is only known for the regular part \( X^\# \). It is then natural to consider \( \pi|X^\# \). It is easy to see that \( X^\# \) is a Markov shift: setting \( A^\# := \{x_0 : x \in X^\#\} \subset A \), \( X^\# = X \cap (A^\#)_Z \). As \( \pi \) is continuous, \( \pi|X^\# \) is determined by its regular part but the Bowen property may fail to extend to \( X^\# \).

Denote by \( \overset{\pi}{\sim} \) the canonical relation induced by \( \pi|X^\# \) and by \( \overset{\pi}{\tilde{\sim}} \) the corresponding relation on \( X^\# \).

**Lemma A.4.** Let \( \pi : X \to Y \) be a continuous semiconjugacy with \( X \) a Markov shift. If the restriction of \( \pi \) to \( X^\# \) has the Bowen property, then:
\[
(A.5) \quad \forall x, y \in X^\# \quad x \overset{\pi}{\sim} y \implies \pi(x) = \pi(y).
\]

**Proof.** Let \( x, y \in X^\# \) with \( x \overset{\pi}{\sim} y \) and \( n \geq 1 \). As \( x_{-n} \overset{\pi}{\sim} y_{-n} \), there are \( x_{-n} \in \sigma^n[x_{-n}]_{X^\#}, y_{-n} \in \sigma^n[y_{-n}]_{X^\#} \) with \( \pi(x_{-n}) = \pi(y_{-n}) \). By the Bowen property, this implies \( x_{-n} \overset{\pi}{\tilde{\sim}} y_{-n} \).
Let Theorem B.1. injectivity property: alternate construction which preserves local compactness at the expense of a slightly weaker related, contradicting (2).

\[ \alpha \text{ and } \pi \]

Lemma B.2. Let \( \square \)

Proof. We explain the required changes in proof of the Main Theorem. An inspection of the proof of the Main Theorem shows that the local compactness is lost in Lemma 5.9. It suffices to replace this lemma with the following statement.

\[ \pi \text{ is well-defined and continuous. The regular sequences are those in (1) and (2).} \]

Example A.6. Consider the graph with set of vertices \( \mathbb{Z} \cup \{ \alpha, \omega \} \) and arrows \( n \to (n + 1), \alpha \to n, n \to \omega, \alpha \to \alpha, \omega \to \omega \) (for all \( n \in \mathbb{Z} \)). Let \((S, X)\) be the induced Markov shift. Let \( \pi : X \to Y \subset \{0, 1, 1/2, \ldots \})^{\mathbb{Z}} \) be the semiconjugacy such that (the vertical bar is immediately to the left of index 0):

\[ (1) \alpha^\infty, \omega^\infty \to 0^\infty; \]
\[ (2) \alpha^\infty|n \cdot (n + 1) \cdots (n + \ell - 1) \cdot \omega^\infty \to 0^\infty| \frac{1}{\ell} \cdot \frac{1}{\ell} \cdots 0^\infty \] (for all \( n \in \mathbb{Z}, \ell \in \mathbb{N}^0 \));
\[ (3) \alpha^\infty \cdot n \cdot (n + 1) \cdots 0^\infty \] (for all \( n \in \mathbb{Z} \));
\[ (4) \ldots (n - 2) \cdot (n - 1) \cdot \omega^\infty \to 0^\infty \] (for all \( n \in \mathbb{Z} \));
\[ (5) \ldots -2 \cdot -1 \cdot 0 \cdot 1 \cdot 2 \cdots \to 0^\infty. \]

\[ \pi \text{ is well-defined and continuous. The regular sequences are those in (1) and (2).} \]

It is easy to check that \( \pi \) is Bowen on \( X^\# \) for the symmetric relation generated by \( n \sim m \) for all \( n, m \in \mathbb{Z} \) and \( \alpha \sim \omega \). If the semiconjugacy \( \pi \) was Bowen on \( X \), (1) and (5) would imply that all symbols would be related, contradicting (2).

Appendix B. A locally compact recoding

Our Main Theorem does not preserve local compactness. In this appendix we provide an alternate construction which preserves local compactness at the expense of a slightly weaker injectivity property:

Theorem B.1. Let \((S, X)\) be a locally compact Markov shift on some alphabet \( A \). Let \( X^\# \) be its regular part. Let \( \pi : (S, X^\#) \to (T, Y) \) be a Borel semiconjugacy such that:

- \((T, Y)\) is a Borel automorphism;
- \( \pi \) is finite-to-one, i.e., \( \pi^{-1}(y) \) is finite for every \( y \in Y \);
- \( \pi \) has the Bowen property with respect to a locally finite relation on \( A \).

Then there are a locally compact Markov shift \((\tilde{S}, \tilde{X})\) and a 1-Lipschitz map \( \phi : \tilde{X} \to X^\# \) such that \( \pi \circ \phi : \tilde{X} \to Y \) defines a semiconjugacy satisfying:

- \( \pi \circ \phi | \tilde{X}^\# \) is injective;
- \( \pi \circ \phi(\tilde{X}^\#) \) carries all invariant measures of \( \pi(X^\#) \).

Proof. We explain the required changes in proof of the Main Theorem. An inspection of the proof of the Main Theorem shows that the local compactness is lost in Lemma 5.9. It suffices to replace this lemma with the following statement.

Lemma B.2. Let \( X \) be a locally compact Markov shift. Let \( \mathcal{W} := (W^j)_{\substack{1 \leq j \leq J \leq \infty}} \) with \( 1 < J \leq \infty \) be an enumeration of \( X \)-words. Then there is a one-block code \( p : S \to X \) defined on a Markov shift \( S \) such that:

- \( 1 \) \( S \) is locally compact;
- \( 2 \) \( p | S^\# \) is injective;
(3) the image of $p|S^\#$ is $X_W \setminus N$, where $X_W$ is the of sequences in $X$ which see i.o. some word from $W$ and $N$ is a null set.

Proof of the lemma. Fix some $1 \leq i < J$ and let $N := |W^i|$. Recall the graph $G^3$ and the extension $p^i : S_i \to X_i$ defined in the proof of Lemma 5.9. The vertices of $G^3$ are couples $(v, j)$ with $v$ a $X^{[N]}$-word with $v_0 = W^i$ and $j$ an integer such that $0 \leq j < |v|$. The set of such vertices $(v, j)$ is $V^3$. Except in somewhat trivial situations, the lengths of the words $v$'s from $V^3$ are unbounded.

Note that only vertices $(v, j)$ with $j = |v| - 1$ can have outdegree larger than 1 and that only vertices $(v, 0)$ can have indegree larger than 1. However $(v, |v| - 1) \to (w, 0)$ whenever $v_0 \to w_0 = W^i$ in $X^{[N]}$. Since they are infinitely many words $v$ (their length being unbounded), $S_i$ is not locally compact.

We define a new graph $G^4$ as follows. Let:

$$V^4 := \{(v, j, L_-, L_+) \in \mathcal{L}(X^{[N]}) \times \mathbb{N}^0 \times \mathbb{N} : (v, j) \in V^3 \text{ with } |v| \leq \min(L_-, L_+);$$

$$(v, j, L_-, L_+) \to (w, k, M_-, M_+) \text{ if and only if } (v, j) \xrightarrow{G^4} (w, k)$$

and $M_- := \max(|v|, L_- - 1)$, $L_+ := \max(|v|, M_- + 1)$.

Let $T_i$ be the Markov shift defined by $G^4$ and define $p|T_i$ as $p^i \circ q$ where $q(v, k, L) = (v, k)$.

Step 1. Local compactness.

Let $(v, j, L_-, L_+) \to (w, k, M_-, M_+)$ on $G^3$. Note that

$$M_- = \max(|v|, L_- - 1) \text{ and } M_+ \leq L_+ + 1.$$

It follows that, given $(v, j, L_-, L_+)$ there are finitely many possibilities for $(M_-, M_+)$. In particular $|w|$ is bounded. Now $w = v$ or $w$ starts by the fixed $X$-word $W^i$. Since $X$ is locally compact, this gives finitely many possibilities for $(w, k)$. Thus the outdegree of any vertex in $G^3$ is finite. Likewise the indegree of any vertex is finite. The local compactness, i.e., item (1), is proved.

We define the flat part of $S^i = \Sigma(G^3)$ to be:

$$S^i_\flat := \{(v, j) \in S^i : \lim_{n \to \pm \infty} |v^n| - |n| = -\infty\} \text{ where } (v^n, j^n)_{n \in \mathbb{Z}} = (v, j).$$

Step 2. The flat part has full measure for any invariant probability measure $\mu$ on $S_i$.

We can restrict to $\mu$ ergodic. Now, assume by contradiction that $\lim_{n \to \pm \infty} |v^n| - |n| = -\infty$ fails for a set $D$ of points $(v, j) \in S_i$ with positive $\mu$-measure. Hence, for any $(v, j) \in D$, there is a constant $C > 0$ and arbitrarily large integers $n$ such that $|v^n| \geq |n| - C \geq |n|/2$. We assume that one can choose these integers to be positive, the negative case being similar. For such an integer $n$, if $\sigma^k(v, j) \in E_n := \{(w, \ell) \in S^i : |w^0| > n\}$ for $k = k_0$ for some $0 \leq k < n$, then it holds for all $k$ in some positive interval segment of length $|v^{k_0}| \geq n/2$ and containing $k_0$. Therefore it holds for at least $n/2$ integers $0 \leq k < (3/2)n$. Hence, for any integer $N$,

$$\forall x \in D \limsup_{n \to \infty} \frac{1}{(3/2)n} \# \{0 \leq k < (3/2)n : \sigma^k(x) \in E_N\} \geq 1/4.$$

By the pointwise ergodic theorem, this implies that $\mu(E_N) \geq 1/4$ for all $N$, contradicting the $\sigma$-additivity of $\mu$. Hence $\mu(S^i_\flat) = 1$. 

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Step 3. There is a canonical lift $\iota : S_i^0 \to T_i$ which is well-defined with $q \circ \iota = \id$.

Given $(v, j) \in S_i^0$, we let:

$$L^+_n(v) := \max_{k \geq 0} |v^{n-k}| - k \text{ and } L^-_n(v) := \max_{k \geq 0} |v^{n+k}| - k.$$ 

and define the canonical lift as:

$$\iota : (v^n, j^n)_{n \in \mathbb{Z}} \mapsto (v^n, j^n, L_-(v), L_+(v))_{n \in \mathbb{Z}}$$

Let $(v, j) \in S_i^0$. We check that $\iota(v, j)$ is well-defined. First, the numbers $L_n^-(v), L_n^+(v)$ are well-defined since, by the definition of $S_i^0$, $|v^{n+k}| - k < 0$ for all large $k \geq 0$. Note also:

$$L^+_n(v) = \max_{k \geq 1} (|v^n|, \max_{k \geq 0} |v^{n+k}| - k) = \max_{k \geq 0} (|v^n|, \max_{k \geq 0} |v^{n+1+k}| - k - 1)$$

and likewise $L^+_{n+1}(v) = \max(|v^{n+1}|, L_n^+(v) - 1)$. These identities show that:

$$(v^n, j^n, L^+_n(x), L^+_n(v))_{n \in \mathbb{Z}} \in T_i.$$ 

Thus $\iota : S_i^0 \to T_i$ is well-defined.

The identity $q \circ \iota = \id$ is trivial.

Step 4. The map $q : T_i^\# \to S_i^0$ is well-defined and $\iota \circ q|T_i^\# = \id$.

To see that $q$ is well-defined, it suffices to check that $q(T_i^\#) \subset S_i^0$. Let $z := (v, j, L_-^i, L_+^i) \in T_i^\#$. Thus there is some $z^* := (v^*, j^*, L_-^*, L_+^*) \in \mathcal{V}^4$ that appears infinitely many times in $z^n$ when $n \geq 0$. Let $n \geq 0$ be a large integer. Let $m(n)$ be the largest index less than $n$ such that $z^{m(n)} = z^*$. Observe that $L_n^* \leq L_+^i + (n - m)$. Thus $|v^n| - n \leq L_+^i - m(m)$. Thus $\lim_{n \to \infty} |v^n| - n = -\infty$ as $\lim_{n \to \infty} m(n) = +\infty$. The limit when $n \to -\infty$ is handled similarly using $L_-^*$, proving that $q$ is well-defined.

We turn to the identity $\iota \circ q|T_i^\# = \id$. Let $x \in T_i^\#$. We must show that it coincides with the canonical lift $\tilde{x} := \iota(q(x))$. Write $(v^n, j^n, L_n^-(x), L_n^+(x)) := x_n$ and $(\tilde{v}^n, \tilde{j}^n, \tilde{L}_-^n, \tilde{L}_+^n) := \tilde{x}_n$.

Since $x \in T_i^\#$, there is a symbol $a := (v, j, M_-, M_+)$ which appears infinitely often in the past of $x$. Thus there are arbitrarily large integers $N$ such that $x_{-N} = a$. It follows that $L_+^K = |v^K|$ for some $-N \leq K \leq -N + M_+ + 1$. Indeed, otherwise one would have: $L_-^{n+M_+ + 1} = M_+ - M_- - 1 < 0$, a contradiction.

By an easy induction, the definition of the arrows in $\mathcal{G}^4$ implies that:

$$\forall n \in \mathbb{Z} \quad L_n^+ \geq \max_{k \geq 0} |v^{n-k}| - k$$

It follows that the canonical lift is as small as possible in the following sense:

$$\forall n \in \mathbb{Z} \quad \tilde{L}_n^- \leq L_n^-.$$ 

Since $L_-^K = |v^K|$, then $L_-^K = \tilde{L}_-^K$ and therefore $L_-^K = \tilde{L}_-^K$ for all $k \geq K$ and in particular, all $k > -N + M_-$. Since $N$ is arbitrarily large, it follows that $L_n^- = \tilde{L}_n^-$ for all $n \in \mathbb{Z}$. A similar reasoning applies to the sequence $(L_n^-)_{n \in \mathbb{Z}}$, concluding the proof that $x = y$ and therefore of the identity.

We note that this identity implies that $q$ is injective. The theorem is proved. \qed
Appendix C. Application to Sinai billiards collision maps

We prove Theorem 1.5, i.e., the lower bound on the periodic points for the billiard maps considered by Baladi and Demers [1]. We fix such a collision map $T_B$ defined by a two-dimensional Sinai billiard satisfying conditions (BD1) and (BD2) quoted in our introduction. Theorem 2.4 of [1] yields a strongly mixing measure $\mu^* \in \text{Prob}^\text{erg}(T_B)$ such that

$$h(T_B, \mu^*) = \sup \{h(T_B, \nu) : \nu \in \text{Prob}(T_B)\} = h_*$$

where $h_*$ is a combinatorial entropy from their eq. (1.1).

Let $\pi : (\Sigma, \sigma) \to (M, T_B)$ be the coding built by [19, Thm. 1.3] for some hyperbolicity parameter $\Lambda < \chi < \chi(f, \mu_*) := \min(\lambda_1(f, \mu_*) - \lambda_2(f, \mu_*))$. As in Sarig’s construction for diffeomorphisms, $\Sigma$ is a Markov shift and $\pi$ is a Hölder-continuous semiconjugacy. Note that $M$ is a two-dimensional compact manifold with boundary and that, writing $M_1$ for the domain where both $T_B$ and its inverse are well-defined and differentiable,

$$\pi(\Sigma^\#) \subset \pi(\Sigma) \subset \bigcap_{n \in \mathbb{Z}} T_B^{-n}(M_1) \subset M \setminus \partial M$$

(the middle inclusion is nontrivial but is proved in [19]).

An inspection of the proof in [19] shows that the semiconjugacy on $\Sigma^\#$ admits a Bowen relation just as in the smooth case of [24]. Indeed, though this is not stated in [19], it follows from the same arguments as in the original case, see Section 6. This Bowen relation is locally finite thanks to [19, Prop. 7.1(2)].

According to [19, Thm. 2.4], $\mu_*$ is $T$-adapted in the sense of [19]. Since it is $\chi$-hyperbolic from the choice of $\chi$, [19, Thm. 1.3] implies the existence of $\hat{\mu}_* \in \text{Prob}^\text{erg}(\sigma)$ such that $\pi_*(\hat{\mu}_*) = \mu_*$. Since $\pi|\Sigma^\#$ is finite-to-one, $\pi_* : \text{Prob}(\Sigma, \sigma) \to \text{Prob}(T_B)$ preserves the entropy. Therefore, eq. (C.1) implies that $\hat{\mu}_*$ is an m.m.e. for $\Sigma$.

As in Lemma 6.1, we can assume that the image of any periodic orbit of $\Sigma$ is $\chi'$-hyperbolic for any $\chi' < \chi$. In particular, this holds for $\chi' = \Lambda$.

Our Main Theorem now allows us to replace $\pi : (\Sigma, \sigma) \to M$ by an injective coding still denoted by $\pi : \Sigma \to M$. The lift $\hat{\mu}$ by $\pi$ of the m.m.e. $\mu_*$ is now measure-preservingly isomorphic to $\mu_*$, hence $\hat{\mu}_*$ is also strongly mixing. This implies that the irreducible component of $\Sigma$ carrying $\hat{\mu}_*$ has period 1.

Finally, Lemma 6.2 gives the claimed lower bound on the periodic orbits.

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