A new characterization of trivially perfect graphs

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Abstract

A graph $G$ is trivially perfect if for every induced subgraph the cardinality of the largest set of pairwise nonadjacent vertices (the stability number) $\alpha(G)$ equals the number of (maximal) cliques $m(G)$. We characterize the trivially perfect graphs in terms of vertex-coloring and we extend some definitions to infinite graphs.

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1. Introduction

Let $G$ be a finite graph. A coloring (vertex-coloring) of $G$ with $k$ colors is a surjective function that assigns to each vertex of $G$ a number from the set $\{1, \ldots, k\}$. A coloring of $G$ is called pseudo-Grundy if each vertex is adjacent to some vertex of each smaller color. The pseudo-Grundy number $\gamma(G)$ is the maximum $k$ for which a pseudo-Grundy coloring of $G$ exists (see [5, 6]).

A coloring of $G$ is called proper if any two adjacent vertices have different color. A proper pseudo-Grundy coloring of $G$ is called Grundy. The Grundy number $\Gamma(G)$ (also known as the first-fit chromatic number) is the maximum $k$ for which a Grundy coloring of $G$ exists (see [6, 11]).

Since there must be $\alpha(G)$ distinct cliques containing the members of a maximum stable set, clearly,

$$\alpha(G) \leq \theta(G) \leq m(G) \text{ and } \omega(G) \leq \chi(G) \leq \Gamma(G) \leq \gamma(G)$$

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where $\theta$ denotes the clique cover (the least number of cliques of $G$ whose union covers $V(G)$), $\omega$ denotes the clique number and $\chi$ denotes the chromatic number. Let $a, b \in \{\alpha, \theta, m, \omega, \chi, \Gamma, \gamma\}$ such that $a \neq b$. A graph $G$ is called $ab$-perfect if for every induced subgraph $H$ of $G$, $a(H) = b(H)$. This definition extends the usual notion of perfect graph introduced by Berge [3], with this notation a perfect graph is denoted by $\omega\chi$-perfect. The concept of the $ab$-perfect graphs was introduced earlier by Christen and Selkow in [7] and extended in [17] and [1, 2]. A graph $G$ without an induced subgraph $H$ is called $H$-free. A graph $H_1$-free and $H_2$-free is called $(H_1, H_2)$-free.

Some important known results are the following: Lóvász proved in [13] that a graph $G$ is $\omega\chi$-perfect if and only its complement is $\omega\chi$-perfect. Consequently, a graph $G$ is $\omega\chi$-perfect if and only if $G$ is $\alpha\theta$-perfect, see also [4, 5, 12]. By Equation (1), a graph $\alpha m$-perfect is “trivially” perfect (see [9, 10]). Chudnovsky, Robertson, Seymour and Thomas proved in [8] that a graph $G$ is $\omega\chi$-perfect if and only if $G$ and its complement are $C_{2k+1}$-free for all $k \geq 2$. Christen and Selkow proved in [7] that for any graph $G$ the following are equivalent: $G$ is $\omega\Gamma$-perfect, $G$ is $\chi\Gamma$-perfect, and $G$ is $P_4$-free.

The remainder of this paper is organized as follows: In Section 2: Characterizations are given of the families of finite graphs: (i) $\theta m$-perfect graphs, (ii) $\alpha m$-perfect graphs (trivially perfect graphs), (iii) $\omega\gamma$-perfect graphs and (iv) $\chi\gamma$-perfect graphs. In Section 3: We further extend some definitions to locally finite graphs and denumerable graphs.

2. Characterizations for finite graphs

There exist several trivially perfect graph characterizations, e.g. [2, 9, 14, 15, 16]. We will use the following equivalence to prove Theorem 2.2:

**Theorem 2.1** (Golumbic [9]). A graph $G$ is trivially perfect if and only if $G$ is $(C_4, P_4)$-free.

A consequence of Theorem 2.1 is the following characterization of $\theta m$-perfect and trivially perfect graphs.

**Corollary 2.1.** A graph $G$ is $\theta m$-perfect graph if and only if $G$ is $\alpha m$-perfect.

**Proof.** Since $\theta(C_4) = \theta(P_4) = 2$, $m(C_4) = 4$ and $m(P_4) = 3$ then $G$ is $(C_4, P_4)$-free, so the implication follows. For the converse, the implication is immediate from Equation (1).
We now characterize the $\omega\gamma$-perfect and $\chi\gamma$-perfect graphs. In the following result, one should note that the finiteness of $G$ is not necessary for the proof, the finiteness of $\omega(G)$ is sufficient.

**Theorem 2.2.** For any graph $G$ the following are equivalent: (1) $G$ is $(C_4, P_4)$-free, (2) $G$ is $\omega\gamma$-perfect, and (3) $G$ is $\chi\gamma$-perfect.

**Proof.** To prove (1) $\Rightarrow$ (2) assume that $G$ is $(C_4, P_4)$-free. Let $\varsigma$ be a pseudo-Grundy coloring of $G$ with $\gamma(G)$ colors. We will prove by induction on $n$ that for $n \leq \gamma(G)$, $G$ contains a complete subgraph of $n$ vertices with the $n$ highest colors of $\varsigma$. This proves (for $n = \gamma(G)$) that $G$ is $\omega\gamma$-perfect since every induced subgraph of $G$ is $(C_4, P_4)$-free.

For $n = 1$, there exists a vertex with color $\gamma(G)$, then the assertion is trivial. Let us now suppose that we have $n - 1$ vertices $v_1, \ldots, v_{n-1}$ in the $n - 1$ highest colors such that they are the vertices of a complete subgraph, and define $V_i$ as the set of vertices colored $\gamma(G) - (n - 1)$ by $\varsigma$ adjacent to $v_i$ ($1 \leq i < n$). Since $\varsigma$ is a pseudo-Grundy coloring, none $V_i$ is empty. Any two such sets are comparable with respect to inclusion, otherwise there must be vertices $p$ in $V_i \setminus V_j$ and $q$ in $V_j \setminus V_i$ and the subgraph induced by $\{p, v_i, v_j, q\}$ would be isomorphic to $C_4$ or $P_4$. Therefore the $n - 1$ sets $V_i$ are linearly ordered with respect to inclusion, and there is a $k$ ($1 \leq k < n$) with

$$V_k = \bigcap_{1 \leq i < n} V_i.$$ 

Thus there is a vertex $v_n$ in $V_k$ which is colored with $\gamma(G) - n + 1$ by $\varsigma$ and is adjacent to each of the $v_i$ ($1 \leq i < n$).

The proof of (2) $\Rightarrow$ (3) is immediate from Equation (1).

To prove (3) $\Rightarrow$ (1) note that if $H \in \{C_4, P_4\}$ then $\chi(H) = 2$ and $\gamma(H) = 3$ hence the implication is true (see Fig 1). \qed

**Corollary 2.2.** Every $\chi\gamma$-perfect graph is $\omega\chi$-perfect.

3. Extensions for infinite graphs

We presuppose here the axiom of choice. The definitions of pseudo-Grundy coloring with $n$ colors and of proper coloring with $n$ colors of a finite graph are generalizable to any cardinal number. It is defined the chromatic number $\chi$ of a graph as the smallest cardinal $\kappa$ such that the graph has a proper coloring with $\kappa$ colors. The clique number $\omega$ of a graph as the supremum of the cardinalities of the complete subgraphs of the graph (see [7]). Similarly, for any ordinal number $\beta$ (such that $|\beta| = \kappa$), a pseudo-Grundy coloring of a graph with $\kappa$ colors is a coloring of the vertices of the graph with the elements of $\beta$ such that for any $\beta'' < \beta'$ and any vertex $v$ colored $\beta'$ there is a vertex colored $\beta''$ adjacent to $v$. The pseudo-Grundy number $\gamma$ of a graph is the supremum of the cardinalities $\kappa$ for which there is a pseudo-Grundy coloring of the graph with $\beta$ such that $|\beta| = \kappa$.

Next we prove a generalization of Theorem 2.2 for some classes of infinite graphs. Afterwards we show that there exists a graph, not belonging to these classes, for which the theorem does not hold.

**Theorem 3.1.** The statements (1), (2) and (3) of Theorem 2.2 are equivalent for each locally finite graph and for each denumerable graph.
Proof. To prove $⟨1⟩ \Rightarrow ⟨2⟩$, let $H$ be an induced subgraph of $G$. If $\omega(H)$ is finite, we can use the proof of Theorem 2.2 to show that $\gamma(H) = \omega(H)$. In otherwise $\omega(H)$ is infinite, then $\gamma(H) = \omega(H)$, because $\gamma(H)$ is at most the supremum of the degrees of the vertices of $H$, which is at most $\aleph_0$, if $G$ is locally finite or denumerable.

The implications $⟨2⟩ \Rightarrow ⟨3⟩$ and $⟨3⟩ \Rightarrow ⟨1⟩$ hold for any graph, finite or not. □

The following example can be found in [7]. Let $G$ be a non-denumerable, locally denumerable graph formed by the disjoint union of $|\beta_1| = \aleph_1$ complete denumerable subgraphs of $|\beta| = \aleph_0$ vertices. Clearly $\omega(G) = \chi(G) = |\beta| = \aleph_0$, and $G$ is $(C_4, P_4)$-free. But let $f: \beta_1 \times \beta \rightarrow \beta_1$ be such that for each $\beta' \in \beta_1$ the function $\lambda x \cdot f(\beta', x)$ is a bijection of $\beta$ onto $\beta'$. Index the components of $G$ with the denumerable ordinals, and their vertices with natural numbers. Color the $n$-th vertex of the $\beta'$-th component with $f(\beta', n)$. Each $\beta' < \beta_1$ is used as a color in the $(\beta'+1)$-th component. Since for each $\beta' < \beta_1$, $\lambda x \cdot f(\beta', x)$ is injective, this function defines a coloring with $\beta_1$ colors. Since $\lambda x \cdot f(\beta', x)$ is surjective for each $\beta' < \beta_1$, this function is a pseudo-Grundy coloring with $\aleph_1$ colors.

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