Solution to the 1D Stefan problem using the unified transform method

To cite this article: Dokoza T et al 2021 J. Phys. A: Math. Theor. 54 375203

View the article online for updates and enhancements.
Solution to the 1D Stefan problem using the unified transform method

Dokoza T\textsuperscript{1}*, D Plümacher\textsuperscript{1}, M Smuda\textsuperscript{1,2}, C Jegust\textsuperscript{1} and M Oberlack\textsuperscript{1,2,*}

\textsuperscript{1} Technical University of Darmstadt, Chair of Fluid Dynamics, Department of Mechanical Engineering, 64287 Darmstadt, Germany
\textsuperscript{2} Technical University of Darmstadt, Centre for Computational Engineering, 64293 Darmstadt, Germany

E-mail: oberlack@fdy.tu-darmstadt.de

Received 20 April 2021, revised 19 July 2021
Accepted for publication 6 August 2021
Published 3 September 2021

Abstract

In this paper the one-dimensional two-phase Stefan problem is studied analytically leading to a system of non-linear Volterra-integral-equations describing the heat distribution in each phase. For this the unified transform method has been employed which provides a method via a global relation, by which these problems can be solved using integral representations. To do this, the underlying partial differential equation is rewritten into a certain divergence form, which enables to treat the boundary values as part of the integrals. Classical analytical methods fail in the case of the Stefan problem due to the moving interface. From the resulting non-linear integro-differential equations the one for the position of the phase change can be solved in a first step. This is done numerically using a fix-point iteration and spline interpolation. Once obtained, the temperature distribution in both phases is generated from their integral representation.

Keywords: one-dimensional Stefan problem, unified transform methods, heat equation, Volterra-integral-equations

1. Introduction

Many important heat conduction problems occurring in science, nature and engineering involve a phase change due to melting or freezing. Mathematically, these are special cases of moving phase boundary problems, in which the evolution of the interface itself is unknown and hence...
part of the solution. The most common example for such an interface problem is the Stefan problem, treating the melting of ice. It is an interface problem for the heat equation, a parabolic partial differential equation and usually referred to as free boundary problem. Free boundary problems of this type were presented for the first time in [1]. The Stefan problem was first described in 1891 by Jozef Stefan and describes the temperature distribution in a fluid undergoing a phase change from its solid to its liquid phase due to a heat introduction [2]. Some historical notes on Stefan’s modeling of ice melting can be taken from [3].

A first overview of works treating the topic can be found in the monograph [4], where the author presents numerical solutions for different examples of the problem as well as in [5, 6], in which the authors present numerical, analytical and weak solutions for the Stefan problem. Since both the location and the evolution of the phase boundary are unknown, they need to be determined as a part of the solution. Simultaneously they are part of the formulation of the problem. This represents the principal cause of difficulty when treating this kind of problems, both analytically and numerically.

Still, there are only few known solutions for this free boundary problem, especially little using analytical methods.

Detailed studies on the existence, uniqueness, stability and asymptotic theorems for a weak solution for the $n$-dimensional Stefan problem were done in [7] where the author could also prove the existence of weak solutions.

As stated above, a variety of numerical solutions for special cases of the Stefan problem have been derived. In [8], for example, a finite difference based solution for the one-dimensional one-phase Stefan problem is presented. In [9] the author used a Galerkin method by transforming the moving into a fixed computational domain for the one-phase Stefan problem. In [10] the authors also used a finite difference scheme with boundary immobilization on the one-phase Stefan problem, while in [11] the authors studied the two-phase Stefan problem with both phase formation and depletion using a Keller box scheme also employing the boundary immobilization method. Nogi in [12] also studied the two-phase Stefan problem using a finite difference scheme, while [13] contains a solution for the two-dimensional one-phase case using the singularity-separating method. Oftentimes special geometrical configurations of the Stefan problem are studied. E.g. the two-dimensional solidification in a corner was treated in [14], where numerical solutions were generated and compared with asymptotic solutions. One of the few analytical works which treats the Stefan problem is presented in [15], where the author employed the Fokas unifying transform method as the solution technique to the one-phase Stefan problem. With this, she was able to derive analytical expressions for the free boundary problem for the one-phase case. Presently we employ a similar approach based on the unified transform method on the two-phase case, which represents a considerable extension to [15]. Thus physically more meaningful and readily applicable to phenomena in the real world is the more complex full two-phase case considered in our work.

Classical methods for solving initial-boundary-value problems for linear partial differential equations with constant coefficients often make use of the separation of variables and specific integral transforms such as the Fourier transformation. However, these classical methods fail in the case of the Stefan problem due to the presence of a moving phase boundary.

Therefore, the unified transform method, henceforth called UTM, is presently applied to this problem, which allows to obtain the general solution in terms of an integral representation. The approach allows to solve initial-boundary-value problems for linear PDEs with constant coefficients. Further, this makes prospective numerical evaluations much easier to handle as will also be seen in this work. The basic idea was first introduced by Fokas in 1997 in [16], while an overview of works using the UTM can be found in [17] as well as in [18]. The applicability of the UTM on various heat conduction related problems can be seen for example in [19].
The objective of this work is twofold. First, we present analytical expressions for the temperature distributions and the interface position for the one-dimensional two-phase case. Second, we lay out a numerical scheme to solve the resulting integro-differential equations in a similar way as done in [23], where the authors used the UTM, to solve the two-phase Couette flow with transpiration through both walls. The solution is thus obtained in the form of integrals depending on initial and boundary values. The unknown values at the moving interface were determined by a system of linear Volterra-integral-equations, where these equations were solved by using a standard marching method (see e.g. [24]).

In the following this work proceeds by firstly describing the physical problem in section 2 including the underlying PDE system as well as the boundary and coupling condition in the Stefan problem. In section 3, the UTM is applied on the Stefan problem leading to a non-linear integro-differentiated system in time for the temperature distributions in both phases and the interface. In section 4 the obtained integro-differential equations are evaluated numerically, thus leading to explicit solutions for the temperature distributions and the interface motion, which are presented in section 5. Finally, in the last section 6 the obtained analytical expressions, the numerical results and open problems are discussed.

2. Problem statement

The one-dimensional Stefan problem for two phases describes the temperature distribution in a fluid undergoing a phase change from its solid to its liquid phase due to an imposed temperature gradient. For this rod of the liquid phase on the interval \([0, g(t)]\) is considered, which is separated by the moving phase boundary \(g(t)\) from the solid phase occupying the interval \([g(t), L]\), as shown in figure 1. The boundary at \(x = 0\) of the liquid phase and the boundary at \(x = L\) of the solid phase are non-moving. Due to the temperature gradients \(f_1(t)\) on the left boundary of the liquid phase and \(f_2(t)\) on the right boundary of the solid phase, the solid phase begins to melt, resulting in a moving phase boundary between the two phases. The heat equation in each phase for the temperature distributions \(T^1(x, t)\) and \(T^2(x, t)\) read as

\[
\begin{align*}
\alpha_1 T^1_{xx}(x, t) &= T^1_t(x, t) & [0 < x < g(t)] \times [0 < t < \infty], \\
\alpha_2 T^2_{xx}(x, t) &= T^2_t(x, t) & [g(t) < x < L] \times [0 < t < \infty],
\end{align*}
\]

where \(T^j(x, t)\) stands for the time derivative of the temperature distribution for each phase, while \(T^j_t(x, t)\) and \(T^j_{xx}(x, t)\) stand for the first order or second order spatial derivative, with \(j = 1, 2\). The thermal diffusivity is given as \(\alpha_j = \frac{\lambda_j}{\rho_j c_p}\) with the thermal conductivity \(\lambda_j\), the density \(\rho_j\) and the specific heat capacity \(c_p j\) of each phase. The initial temperature distributions

where the author applied the approach on the heat conduction on a ring with two phases in order to provide an explicit solution for the temperature in both phases, where the interface position was assumed to be known. Another application of the UTM on the heat equation can be seen in [20], where the authors obtained an explicit solution for each layer of the one-dimensional multi-layer heat equation, whereas once again the interface position was assumed to be known. In [21] the authors solved the heat conduction on networks of multiply connected rods using the UTM, where both the connectivity of the rods and their size are known, while the temperature and the heat fluxes represent the searched quantities. The variety of application possibilities for the UTM can be seen amongst others in [22], where the authors were able to simulate brain tumor growth within heterogeneous environments. The UTM rewritesthe given partial differential operation (PDE) into a divergence form and applies Gauss’ theorem and the Fourier transformation to obtain the general solution.
Figure 1. One-dimensional two-phase Stefan problem domain.

and the phase boundary are given as
\[ T_1^1(x,0) = T_1^0(x) \quad [0 < x < g(0)], \]  
\[ T_2^2(x,0) = T_2^0(x) \quad [g(0) < x < L], \]  
\[ g(0) = g_0. \]  
Moreover the introduced temperature gradients \( f_1(t), f_2(t) \) are given as
\[ T_1^1(0,t) = f_1(t) \quad [0 < t < \infty], \]  
\[ -T_2^2(L,t) = f_2(t) \quad [0 < t < \infty]. \]  
The melting condition on the moving phase boundary leads to two further conditions describing the heat distributions in each phase, reading as
\[ T_1^1(g(t),t) = 0 \quad [0 < t < \infty], \]  
\[ T_2^2(g(t),t) = 0 \quad [0 < t < \infty]. \]  
The Stefan-condition, which is based on the formulation in [15], finally leads to the last needed condition and reads as
\[ \alpha_2 T_2^2(g(t),t) - \alpha_1 T_1^1(g(t),t) = g_i(t) \quad [0 < t < \infty]. \]  
In the above system, the initial conditions (2a)–(2c) and the boundary conditions (3a) and (3b) are given, while the temperature distribution \( T(x,t) \) in \([0 < x < L] \times [0 < t < \infty]\) and the melting point position \( g(t) \) are the searched quantities.

3. Unified transform method

Using the UTM, as described in [25, 26] equations for both the heat distributions for each phase and an equation for the interface condition can be derived, similar as done in [15]. Due to the extension to the full two-phase case as compared to the one-phase study done in [15]
the subsequent numerical evaluation however differs significantly. Additionally due to the two-phase nature, the spatial derivatives of the temperature at the interface \( T^i_j (g(s), s) \) and \( T^j_i (g(s), s) \) cannot be simply replaced by the temporal derivative of the interface as one in [15] for the one-phase case. Thus, in this work the Stefan condition (5) is applied after general expressions for the temperature distributions in each phase \( T^i (x, t) \) and \( T^j (x, t) \) are obtained. For this purpose each phase is considered as a single phase individually.

The first step of the UTM is to rewrite the given PDE into a specific divergence form. The temperature distribution for each phase is multiplied by \( e^{-ikx + w_j(k)y} \) as in [26], with \( k \in \mathbb{C} \). Thus both heat equations (1a) and (1b) will be rewritten to the following conservative form:

\[
(T^i e^{-ikx + w_j(k)yt})_t - \alpha_j ((T^i + ikT^i) e^{-ikx + w_j(k)y})_x = 0, \tag{6}
\]

where \( w_j(k), j = 1, 2 \), denotes the dispersion relation and is given by

\[
w_j(k) = \alpha_j k^2. \tag{7}
\]

Applying Gauss’s theorem on the divergence form of the heat equation on the considered domains \( D_1 = [0, g(t)] \times [0, t] \) and \( D_2 = [g(t), L] \times [0, t] \) for each phase as shown in figure 1, leads to a global relation for each domain separately

\[
\int_0^{g(t)} T^1(y, t)e^{-iky + w_1(k)y} dy = \int_0^{g(t)} T^1_0(y) e^{-iky} dy - \alpha_1 \int_0^t f_1(s) e^{w_1(k)y} ds - \alpha_1 \int_0^t i k T^1(0, s) e^{w_1(k)y} ds + \alpha_1 \int_0^t T^1_0(g(s), s) e^{-iky + w_1(k)y} ds \tag{8}
\]

and

\[
\int_{g(t)}^{L} T^2(y, t)e^{-iky + w_2(k)y} dy = \int_{g(t)}^{L} T^2_0(y) e^{-iky} dy - \alpha_2 \int_0^t f_2(s) e^{-iky + w_2(k)y} ds + \alpha_2 \int_0^t i k T^2(L, s) e^{-iky + w_2(k)y} ds - \alpha_2 \int_0^t T^2_0(g(s), s) e^{-iky + w_2(k)y} ds \tag{9}
\]

General integral representations of the heat distributions \( T^1(x, t) \) and \( T^2(x, t) \) can be obtained by first multiplying (8) with \( e^{-w_1(k)y} \) and (9) with \( e^{-w_2(k)y} \) and then applying the inverse Fourier transformation, leading to

\[
T^1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx + w_1(k)yt} \left( \int_0^{g(t)} T^1_0(y) e^{-iky} dy - \alpha_1 \int_0^t f_1(s) e^{w_1(k)y} ds - \alpha_1 \int_0^t i k T^1(0, s) e^{w_1(k)y} ds + \alpha_1 \int_0^t T^1_0(g(s), s) e^{-iky + w_1(k)y} ds \right) \, dk \tag{10}
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx + w_1(k)yt} \left( \int_0^{g(t)} T^1_0(y) e^{-iky} dy - \alpha_1 \int_0^t f_1(s) e^{w_1(k)y} ds - \alpha_1 \int_0^t i k T^1(0, s) e^{w_1(k)y} ds + \alpha_1 \int_0^t T^1_0(g(s), s) e^{-iky + w_1(k)y} ds \right) \, dk
\]
and
\[
T^2(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - w_j(k)t} \int_{g_0}^{g(t)} T^2_0(y) e^{-iky} dy
\]
\[
- \alpha_2 \int_{0}^{t} f_2(s)e^{-ikL + w_j(k)s} ds + \alpha_2 \int_{0}^{t} i k T^2_0(L, s) e^{-ikL + w_j(k)s} ds
\]
\[
- \alpha_2 \int_{0}^{t} T^2_0(g(s), s) e^{-ikg(s) + w_j(k)s} ds \] dk.

3.1. Eliminating the unknown Dirichlet conditions

The general integral representations (10) and (11) for each heat distribution still contain unknown Dirichlet boundary conditions (DBCs) \( T^1(0, s) \) and \( T^2(L, s) \).

However, using the symmetry for the dispersion relation, i.e., \( w_j(k) = w_j(-k) \), the DBC can be eliminated. Firstly, the Dirichlet condition on the left boundary for the first phase is eliminated, by replacing \( k \) with \(-k\) in equation (8), leading to
\[
\int_{0}^{g(t)} T^1(y, t) e^{iky + w_1(k)y} dx = \int_{0}^{g(t)} T^1_0(y) e^{iky} dy - \alpha_1 \int_{0}^{t} f_1(s)e^{w_1(k)s} ds
\]
\[
+ \alpha_1 \int_{0}^{t} ik T^1_0(0, s) e^{w_1(k)s} ds + \alpha_1 \int_{0}^{t} T^1_0(g(s), s) e^{ikg(s) + w_1(k)s} ds. \quad (12)
\]

Equation (12) is multiplied with \( e^{iky - w_1(k)y} \), resulting in
\[
\int_{0}^{g(t)} T^1(y, t) e^{ik(x + y)} dy = \int_{0}^{g(t)} T^1_0(y) e^{ik(x + y) - w_1(k)y} dy - \alpha_1 \int_{0}^{t} f_1(s)e^{ikx + w_1(k)(x - t)} ds
\]
\[
+ \alpha_1 \int_{0}^{t} ik T^1_0(0, s) e^{ikx + w_1(k)(x - t)} ds
\]
\[
+ \alpha_1 \int_{0}^{t} T^1_0(g(s), s) e^{ik(x + g(s)) + w_1(k)(x - t)} ds. \quad (13)
\]

Equation (13) is inserted into (10), leading to
\[
T^1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{0}^{g(t)} T^1_0(y) e^{-w_1(k)y} \left( e^{ik(x - y)} + e^{ik(x + y)} \right) dy
\]
\[
- \alpha_1 \int_{0}^{t} 2 f_1(s)e^{w_1(k)(x - t) + ikx} ds + \alpha_1 \int_{0}^{t} T^1_0(g(s), s) e^{w_1(k)(x - t)} 
\]
\[
\times \left( e^{ik(x - g(s))} + e^{ik(x + g(s))} \right) ds - \int_{0}^{g(t)} T^1(y, t) e^{ik(y - t)} dy \] dk.
\quad (14)

The unknown Dirichlet condition for the right boundary of the second phase is eliminated on a similar way, by once again replacing \( k \) with \(-k\) in equation (9), resulting in
\[
\int_{g(t)}^{L} T^2(y, t) e^{iky + w_2(k)y} dy = \int_{g(t)}^{L} T^2_0(y) e^{iky} dy - \alpha_2 \int_{0}^{t} f_2(s)e^{ikL + w_2(k)s} ds
\]
\[
- \alpha_2 \int_{0}^{t} ik T^2_0(L, s) e^{ikL + w_2(k)s} ds - \alpha_2 \int_{0}^{t} T^2_0(g(s), s) e^{ikg(s) + w_2(k)s} ds. \quad (15)
\]
Equation (15) is multiplied with $e^{ik(y-2L)-w_2(k)y}$, leading to

$$
\int_{g(t)}^L T_2(y, t) e^{ik(y-2L)-w_2(k)y} \, dy = \int_{g(t)}^L T_2(y) e^{ik(y-2L)-w_2(k)y} \, dy 
- \alpha_2 \int_0^f 2f^2(s)e^{ik(y-L)+w_2(k)(s)} \, ds - \alpha_2 \int_0^f ikT_2(L, s)e^{ik(y-L)+w_2(k)(s)} \, ds
- \alpha_2 \int_0^f T_2^2(g(s), s)e^{ik(y-g(s)-2L)+w_2(k)(s)} \, ds.
$$

(16)

Equation (16) is inserted into equation (11), which leads to

$$
T^2(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{g(t)}^L T_2^2(y) e^{-w_2(k)y} \left( e^{ik(x-y)} + e^{-ik(x+y-2L)} \right) \, dy 
- \alpha_2 \int_0^f 2f^2(s)e^{ik(y-g(s)-L)+w_2(k)(s)} \, ds - \int_{g(t)}^L T_2^2(y, t)e^{ik(x+y-2L)} \, dy 
- \alpha_2 \int_0^f T_2^2(g(s), s)e^{w_2(k)(s)-i} \times \left( e^{ik(x-g(s))} + e^{ik(x+y-2L)} \right) \, ds \right] \, dk.
$$

(17)

After changing the order of the integration in (14) and (17), i.e. integrating with respect to $k$ first, the expressions can now be evaluated. The results for the temperature distributions are given as

$$
T^1(x, t) = \frac{1}{2\pi} \left[ \int_{g(t)}^L \frac{T_2^2(y)}{\sqrt{\alpha_1}} \left( e^{-\frac{(x-y)^2}{4\alpha_1^2}} + e^{-\frac{(x+y-2L)^2}{4\alpha_1^2}} \right) \, dy 
- \sqrt{\alpha_1} \int_0^f 2f^2(s)e^{\frac{(x-g(s)-L)^2}{4\alpha_1^2}} \, ds 
+ \sqrt{\alpha_1} \int_0^f T_2^2(g(s), s) \left( e^{-\frac{(x-g(s))^2}{4\alpha_1^2}} + e^{-\frac{(x+y-2L)^2}{4\alpha_1^2}} \right) \, ds \right]
$$

(18)

and

$$
T^2(x, t) = \frac{1}{2\pi} \left[ \int_{g(t)}^L \frac{T_2^2(y)}{\sqrt{\alpha_2}} \left( e^{-\frac{(x-y)^2}{4\alpha_2^2}} + e^{-\frac{(x+y-2L)^2}{4\alpha_2^2}} \right) \, dy 
- \sqrt{\alpha_2} \int_0^f 2f^2(s)e^{\frac{(x-g(s)-L)^2}{4\alpha_2^2}} \, ds 
- \sqrt{\alpha_2} \int_0^f T_2^2(g(s), s) \left( e^{-\frac{(x-g(s))^2}{4\alpha_2^2}} + e^{-\frac{(x+y-2L)^2}{4\alpha_2^2}} \right) \, ds \right].
$$

(19)

The terms for each integral representation in (14) and (17) containing $T^1(y, t)$ and $T^2(y, t)$ vanish due to Jordan’s lemma (for details see e.g. [25, 27]). Thus the temperature distributions for each phase could be obtained, using only known boundary and initial values and the temperature distributions.
In order to implement the Stefan condition (5) the heat fluxes \( T_i^1(g(t)) \) and \( T_i^2(g(t)) \), which are the spatial derivatives of each single phase (18) and (19), are evaluated at the moving interface. To evaluate these spatial derivatives for each phase at the interface, it is necessary to apply a lemma introduced by Friedman [28] while differentiating (18) and (19), where the lemma reads as

\[
\lim_{x \to g(t) - 0} \frac{\partial}{\partial x} \int_0^t \rho(s)K(x, t, g(s), s)ds = \frac{1}{2} \rho(t) + \int_0^t \rho(s) \times \left[ \frac{\partial}{\partial x}K(x, t, g(s), s) \right]_{x=g(t)} ds, \tag{20}
\]

with the fundamental solution of the heat equation being employed as the kernel \( K(x, t, g(s), s) \) as

\[
K(x, t, g(s), s) = \frac{1}{2\sqrt{\pi(t-s)}}e^{-\frac{(x-g(s))^2}{4(t-s)}}. \tag{21}
\]

This lemma is being applied to the problem presented in this work, where presently the kernel is defined as

\[
K_{\alpha_i}(x, t, g(s), s) = \frac{1}{2\sqrt{\pi(t-s)}}e^{-\frac{(x-g(s))^2}{4(t-s)}}. \tag{22}
\]

This results in

\[
\lim_{x \to g(t) + 0} \frac{\partial}{\partial x} \int_0^t \rho(s)K_{\alpha_i}(x, t, g(s), s)ds = \pm \frac{1}{2\sqrt{\pi}} \rho(t) + \int_0^t \rho(s) \times \left[ \frac{\partial}{\partial x}K_{\alpha_i}(x, t, g(s), s) \right]_{x=g(t)} ds, \tag{23}
\]

with \( i = 1, 2 \) representing the kernel for each phase. The sign of the additional term \( \pm \frac{1}{2\sqrt{\pi}} \rho(t) \) depends on the direction in which the interface is being approached. The complete derivation of (23) can be found in the appendix. Thus by applying (23) the derivative of the temperature distribution at the interface of the first phase is given as

\[
T_i^1 |_{g(0)} = \frac{1}{2\sqrt{\pi}} \left[ \int_0^{g_0} - T_i^1(y) \left( e^{-\frac{(g_0-y)^2}{4(t-s)}}(g(t) - y) + e^{-\frac{(g_0+y)^2}{4(t-s)}}(g(t) + y) \right) dy + \int_0^t f_1(s) e^{-\frac{g(t)^2}{4(t-s)}} g(t) ds + \frac{1}{2} \int_0^t T_i^1(g(s), s) \left( e^{-\frac{(g(s)-g(t))^2}{4(t-s)}}(g(s) - g(t)) - (g(s) + g(t)) e^{-\frac{(g(s)+g(t))^2}{4(t-s)}} \right) ds \right] + \frac{1}{2} T_i^1 |_{g(0)}, \tag{24}
\]
which can be simplified to
\[
T_1^2|_{g(t)} = \frac{1}{2\sqrt{\pi}} \left[ \int_{\alpha_1(t)}^{\infty} \left( e^{\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) - y) + e^{-\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) + y) \right) \, dy \right] + \int_{\alpha_1(t)}^{\infty} \frac{2f(s)}{\alpha_1^{1/2}(t-s)^{1/2}} g(t) \, ds + \int_{\alpha_1(t)}^{\infty} \frac{T_1^2(g(s), s)}{2\alpha_1^{1/2}(t-s)^{1/2}} \left( e^{-\frac{(g(s)-g(t))^2}{4\alpha_0^2 g_0}} (g(s) - g(t)) - (g(s) + g(t)) e^{-\frac{(g(s)+g(t))^2}{4\alpha_0^2 g_0}} \right) \, ds .
\]

In an analogous way the derivative of the temperature distribution at the interface of the second phase can be obtained by once again applying (23) as
\[
T_2^2|_{g(t)} = \frac{1}{2\sqrt{\pi}} \left[ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(y)}{2\alpha_2 y^{3/2}} \left( - e^{\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) - y) - e^{-\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) + y - 2L) \right) \, dy \right] + \int_{\alpha_1(t)}^{\infty} \frac{2f(s)}{\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) - L) \, ds
\]
\[
+ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(g(s), s)}{2\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) - g(s)) \, ds
\]
\[
+ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(g(s), s)}{2\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) + g(s) - 2L) \, ds \right]
\]
\[
+ \frac{1}{2} \int_{\alpha_2(t)}^{\infty} \frac{T_2^2(y)}{2\alpha_2 y^{3/2}} \left( - e^{\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) - y) - e^{-\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) + y - 2L) \right) \, dy
\]
which then can be simplified to
\[
T_2^2|_{g(t)} = \frac{1}{2\sqrt{\pi}} \left[ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(y)}{2\alpha_2 y^{3/2}} \left( - e^{\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) - y) - e^{-\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) + y - 2L) \right) \, dy \right] + \int_{\alpha_1(t)}^{\infty} \frac{2f(s)}{\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) - L) \, ds
\]
\[
+ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(g(s), s)}{2\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) - g(s)) \, ds
\]
\[
+ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(g(s), s)}{2\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) + g(s) - 2L) \, ds \right] .
\]

Finally, the heat fluxes (25) and (27) can be used in order to use the Steffan condition (5) as
\[
(5) = \alpha_2(2t_1 - t_2) - \alpha_1(2t_2 - t_3),
\]
leading to a non-linear integro-differential equation for the evolution of the interface position \( g(t) \), which reads as
\[
g(t) = \frac{1}{2\sqrt{\pi}} \left[ \int_{\alpha_1(t)}^{\infty} \frac{T_2^2(y)}{2\alpha_2 y^{3/2}} \left( - e^{\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) - y) - e^{-\frac{(y-g_0)^2}{4\alpha_0 g_0}} (g(y) + y - 2L) \right) \, dy \right] + \int_{\alpha_1(t)}^{\infty} \frac{2f(s)}{\alpha_2^{1/2}(t-s)^{3/2}} e^{\frac{(y-g_0)^2}{2\alpha_0^2 g_0}} (g(t) - L) \, ds
\]
The above non-linear integro-differential equation for \( t \) is solved numerically in the subsequent section.

4. Numerical methods

In order to solve the set of integro-differential equations for the temperature distributions ((18) and (19)) and the interface evolution (28), a similar approach to the one used by Guenther [29] on a one-phase Stefan problem is employed. The author approximated the derivative of the interface \( \gamma(t) \) as a spline using polynomial expressions.

Since the unknown interface occurs as \( g(t) \) and its derivative \( \gamma(t) \) in the expressions of interest (18), (19) and (28) the first step is to express \( g(t) \) through its derivative and its initial position as \( g_0(t) = g_0 + \int_0^t \gamma(s)ds \). Using this expression as well as the estimated heat fluxes \( T_{x0}(g_0(t)) \) and \( T_{x0}(g_0(t)) \) at the phase boundaries, the phase motion of the phase boundary \( \gamma_0(t) \) and the heat fluxes \( T_{x0}^1(\gamma_0(t)) \) and \( T_{x0}^2(\gamma_0(t)) \) for any \( t \in [0,t_{end}] \) can be obtained, whereas \( t_{end} \) represents the end time. For each time step \( n \) the time is given within the interval \( t = [t_0, t_1, \ldots, t_n] \) with \( n = 0, 1, \ldots, N \). Thus the size of the time step \( \delta t \) is defined as \( \delta t = t_{n+1} - t_n \). Therefore for any time, a fix-point iteration of iteration number \( l \) for the motion of the phase boundary can be used as

\[
A(g_{l,n}, T_{x0}(\gamma_0(t)), T_{x0}^1(\gamma_0(t))) \rightarrow (g_{l+1,n}, T_{x0}(\gamma_0(t)), T_{x0}^1(\gamma_0(t)), T_{x0}^2(\gamma_0(t))).
\]

The operator \( A \) represents the non-linear system of equations in (18), (19) and (28). Repeated execution of iteration leads to the next time step.

The initial conditions are used for the first time step. The derivative of both the phase boundary \( \gamma_{l,n} \) and the heat fluxes \( T_{x0}^1(\gamma_0(t)) \) and \( T_{x0}^2(\gamma_0(t)) \) are constructed as spline approximations of the previous time steps and current estimates. Through multiple iteration \( l \), the motion of the interface and the heat fluxes is calculated based on the previous time step. Numeric integrations of the spline structures lead to the phase boundary position and the temperature fields. By default the order for numeric integration and spline approximation is set to 3, resulting in a cubic approximation. Now all the expressions for a new estimate of the current time step can be calculated, since all the terms on the left-hand side in (29) are known. An abort criterion value is introduced as \( \gamma_{l+1}(t) - \gamma_{l}(t) < \delta \varepsilon \). If the interface motion for one iteration step deviates from the previous estimate more than \( \delta \varepsilon \) the iteration shall be started over using the new approximation as a base. For the next time step, the base assumptions are now the latest.
approximation of the previous time step. This process is repeated until a final solution for $g(t)$ is obtained. The latter numeric scheme was coded in MATLAB.

5. Results

The test case of a one-phase Stefan problem developed in [9] was used to prove the accuracy of the present solution scheme in section 5.1 by comparing with the results provided by the author. Furthermore, results for the variation of initial conditions from the symmetric case, in which both phases are identical, are presented in section 5.3, showing their influence on the motion of the phase boundary.

5.1. Verification studies

The one-phase Stefan problem has been studied already in various ways [9]. For comparison with the present scheme we consider the work by Asaithambi [9], who studied the evolution of the phase boundary for different classical configurations. Since the one-phase case, he studied, represents a reduced case of the one presented in this work, our scheme can be verified by comparing our results to the results in Asaithambi. The parameters used for the numeric scheme are chosen in accordance to the case presented by Asaithambi and are displayed in Table 1.

Our results are presented alongside with those of Asaithambi in Table 2, where the author is using the weak or Galerkin formulation of the initial boundary value problem in order to reduce it to a system of initial-value problems in ordinary differential equations. Various finite-difference marching methods were applied to solve the resulting initial-value problems. The results of four of these methods are presented in Table 2. The first and second explicit method in [9] differ in the formulation of the finite-element approximation of the solution for the temperature, while the first and second implicit method differ in the size of the time steps with the second method having smaller time steps. The results of our numerical scheme match the values of Asaithambi within 0.5 percent.

A second test case of validating our numerical scheme is by computing a symmetric case with no interface evolution. For mirrored initial conditions in both phases and the same material parameters $\alpha_1 = \alpha_2$, no movement of the phase boundary occurs. In Figures 3 and 4 this case is marked by a solid line and from here on named the symmetric case. The parameters used for this numeric scheme are displayed in Table 3, whereas $t_{\text{end}}$ represents the end time.
Table 3. Parameters for the symmetric two-phase Stefan problem.

| Parameter | $g_0$ | $L$ | $f_1(t)$ | $f_2(t)$ | $T_1^0(x)$ | $T_2^0(x)$ | $\alpha_1$ | $\alpha_2$ | $\delta t$ | $t_{end}$ | $\delta \epsilon$ |
|-----------|-------|-----|-----------|-----------|-------------|-------------|-------------|-------------|-------------|-------------|--------------|
| Value     | 0.5   | 1   | 0         | 0         | $2x - 1$    | $2x - 1$    | 1           | 1           | 0.0001      | 0.05        | 0.00001      |

Table 4. Physical and numerical parameters for the symmetric two-phase Stefan problem.

| Parameter | $g_0$ | $L$ | $f_1(t)$ | $f_2(t)$ | $T_1^0(x)$ | $T_2^0(x)$ | $\alpha_1$ | $\alpha_2$ | $\delta t$ | $\delta \epsilon$ |
|-----------|-------|-----|-----------|-----------|-------------|-------------|-------------|-------------|-------------|----------------|
| Value     | 0.5   | 1   | 0         | 0         | $2x - 1$    | $2x - 1$    | 1           | 1           | 0.01        | $10^{-n}$    |

5.2. Temporal order of numerical convergence rate and numerical parameters

The numerical order of convergence and with it the choice of the time step $\delta t$ has an impact on the accuracy of the obtained solution. Therefore, the influence is studied in the following. Table 4 shows the set of parameters used, within the present subsection.

The numerical order of convergence and hence the time step influence is studied by choosing the time step as follows,

$$\delta t = 0.01 \times 2^{-m}$$  \hspace{1cm} (30)

with $m = 0 \ldots 10$. For $t = 0.01$ figure 2 shows the deviation from the most accurate approximation $(g_m - g_{ref})(t = 0.01)$, where $g_{ref}(t = 0.01) = g_{10}(t = 0.01)$, over the amount of time steps considered as $n = 2^m$. A relation between the deviation and the time steps can be matched to $g_m - g_{ref} \approx c \times (2^m)^{-p}$ with $p \approx 1.5$. Hence the present implementation exhibits a numerical convergence of about $p \approx 1.5$.

One can see that for an increasing amount of time steps the solution becomes more accurate, where the error reaches $g_m(t = 0.01) - g_{ref}(t = 0.01) = 10^{-7}$ already after $n = 10^7$ time steps. Based on this, for all the following simulations the parameters shown in table 3 have been used, where $\delta t = 0.0001$, since it shows very good accuracy for the interface motion as can be seen in figure 2.

5.3. Parameter studies

In the following the influence of the initial temperature distribution $T_1^0(x)$ and the thermal diffusivity $\alpha_1$ on the motion of the phase boundary and the temperature distribution is being studied. The same settings as for the symmetric two-phase Stefan problem in table 3 is used as a reference set-up, whereas each parameter is modified separately. The corresponding figures display the evolution of the phase boundary $g(t)$ over the time.

First the influence of the initial temperature distribution is investigated. The initial temperature distribution for the first phase $T_1^0(x)$ is being increased and decreased in comparison to the reference case in order to visualize the effect on the moving interface $g(t)$. In the balanced case $T_1^0(x) = 2x - 1$ is used, while the increased and decreased cases consider $T_1^0(x) = (2x - 1)/3$, $T_1^0(x) = (2x - 1)/2$, $T_1^0(x) = 2(2x - 1)$ and $T_1^0(x) = 3(2x - 1)$ respectively.

Figure 3 shows the interface motion for the five initial temperature distributions. For an increased value of $T_1^0(x)$, compared to the symmetric case, the interface $g(t)$ moves more towards the boundary of the second phase, whereas for lower temperature distributions $T_1^0(x)$ the interface $g(t)$ moves more towards the boundary of the first phase, which is due to the fact...
that the increased initial temperature distribution leads to the melting of the solid phase resulting in the expansion of the liquid phase. For a decreased initial value the opposite is the case, since now the magnitude of the temperature of the solid phase is higher than for the liquid phase. It is also visible, that the interface moves faster for the increased initial distributions than for the cases using decreased initial distributions.

Next, the influence of $\alpha_1$ is analysed by investigating the moving interface for an increased and a decreased value of $\alpha_1$ in comparison to the reference case and results are presented in figure 4. An increased value of the heat conductivity $\alpha_1$ results in the expansion of the first phase, which can be seen as the interface also moves more into the second phase towards its boundary. A faster motion of the interface $g(t)$ can be observed. Decreasing the heat conductivity leads to the opposite observation. For every decreased heat conductivity, the interface reaches more into the first phase, leading to a retraction of this phase. This observation can be explained due to the fact that in the case in which $\alpha_1 > \alpha_2$ the first phase can transfer its temperature faster than the second phase, which then results in the melting of the ice and thus the expansion of the first phase, while in the example of $\alpha_1 < \alpha_2$ the opposite is the case.

5.4. Reconstruction of the temperature distributions

The temperature distributions in (18) and (19) can be computed for every time, once a solution for the motion of the phase boundary $g(t)$ has been obtained. The set of temperature distributions is displayed over the domain for multiple time steps in figure 5 for the symmetric case as described in table 3. The time steps used are $t = 0.002, 0.018, 0.034, 0.05$. One can see that the temperature distribution in both phases stay fully symmetric over the time. Furthermore, with increasing time, one can observe that the temperatures of both phases equalize and simultaneously move towards $T^1(x) = T^2(x) = 0$. 

![Figure 2](image_url)
Figure 3. Influence of the initial heat distribution $T_1^0(x)$ on the interface motion for $T_1^0(x) = [(2x - 1)/3, (2x - 1)/2, (2x - 1), 2(2x - 1), 3(2x - 1)]$, while $\alpha_1 = \alpha_2 = 1$ has been kept constant and $T_0^0(x) = 2x - 1$ has been set for all simulations.

Figure 4. Influence of material parameter $\alpha_1$ on the interface motion for $\alpha_1 = [1/4, 1/2, 1, 2, 4]$, while $\alpha_2 = 1$ has been kept constant and the initial conditions were set to $T_1^0(x) = T_0^0(x) = 2x - 1$ for all runs.

The influence of the initial temperature distribution $T_0^0(x)$ is investigated. Figure 6 shows the temperature distribution for $T_0^0(x) = 0.75(2x - 1)$. It can be seen that a decreased initial temperature distribution as compared to the symmetric case $T_0^0(x) = (2x - 1)$ results in a temperature distribution for the first phase which decays faster towards $T^1(x) = 0$, which is also to be expected, since the initial temperature distribution is already closer to $T_1^0(x) = 0$. Furthermore, a kink in the temperature curve can be seen at the location of the phase boundary, since now the temperature distribution in both phases differ from each other. It can also be seen
that the temperature at the solid boundary of the second phase did not change compared to the symmetric case.

Lastly, the influence of an increased heat conductivity with $\alpha_1 = 4$ may be taken from figure 7. It is visible that the first phase reaches $T^1(x) = 0$ faster for an increased value of $\alpha_1$ compared to the symmetric case $\alpha_1 = 1$, which is to be expected, since the heat can now be transported faster. This results in a faster decaying temperature distribution. Overall the temperature shows values closer to zero for later times compared to the balanced state for an increased value of $\alpha_1$, while the motion of the interface towards the second phase is also clearly visible. Once again the temperature at the solid boundary of the second phase did not change compared to the symmetric case.
6. Conclusion and outlook

In this paper the unified transform method was applied to the one-dimensional Stefan problem for two phases in order to obtain a solution for the temperature distributions for each phase and, moreover, for the interface motion. In the second step the resulting non-linear integro-differential equations were solved numerically. A combination of the fix-point iteration of the phase boundary motion at a specific time step and spline interpolation over those lead to the solution for the evolution of the phase boundary.

The obtained results for the motion of the interface for the one-phase case were compared to literature, which showed very good agreement, in order to test the implemented solver. As another test case for the solver symmetric initial conditions were employed, which retained successfully a symmetric solution for all times. Furthermore, the influence of modified initial parameters such as initial temperatures and thermal diffusivities on the evolution of the interface and the temperature distribution were investigated. The temperature distributions at any given point in time and space can be recovered from the solution of the phase boundary. The numerical implementation gives an order of convergence of about 1.5. Thus the obtained results prove that the two-phase Stefan problem can be solved using the presented approach without explicitly treating the phase boundary.

An obvious extension of this work would be the consideration of the two-dimensional and even the three-dimensional Stefan problem, which would have a closer applicability to physical science and engineering. Due to the importance of heat conduction problems in nature and engineering, this investigation could be useful for a variety of industrial processes. Thus an extension to even more phases and improving the numeric methods and thereby covering for a bigger scope of problems in engineering and science may be the focus of future studies. The present work further shows that the UTM can be applied on free boundary problems, such as the Stefan problem. Thus the application of the UTM on other free boundary problems e.g. the welding problem may be of interest in future works.

Figure 7. Temperature distribution for $\alpha_1 = 4$ with the remaining parameters set to the symmetric case for the two-phase Stefan problem.
Acknowledgments

The authors gratefully acknowledge the generous funding from various funding organizations: TD by the German Academic Scholarship Foundation, DP by the German Research Foundation (DFG) under Grant No. OB 96/48-1, and MS by the joint DFG-FWF project OB 96/46-1. The authors also acknowledge the Open Access Publishing Fund of the Technical University of Darmstadt. The authors furthermore gratefully acknowledge the input of the Referees as well as of Hans-Dieter Alber and Steffen Roch in the deduction (25) and (27).

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix

In the following, lemma 1 in [28] for the spatial derivatives of the temperature distributions at the interface \( x = g(t) \) is derived for our present case in an analogous way as done by Friedman in order to prove the statement (23) and thus the correctness of the heat fluxes (25) and (27).

The following explanations are strongly based on Friedman’s work, the equations being derived analogously to his in order to provide better comprehensibility. The derivation for the heat flux

\[
T_1(x, t) = \int_0^t \rho(s) K_{\alpha_1}(x, t, g(s), s) ds
\]

which equals (23) applied to the first phase.

Proof. We shall first prove that for any fixed positive \( \delta < t \)

\[
I = \int_{t-\delta}^{t} \frac{x - g(s)}{2\alpha_1(t - s)} K_{\alpha_1}(x, t, g(s), s) ds
\]

as \( x \to g(t) - 0 \), and \( |O(\delta^{\frac{1}{2}})| \leq A_0 \delta^{\frac{1}{2}} \) where \( A_0 \) is a constant independent of \( x, t, \delta \). In the sequel we shall denote by \( A_i \) appropriate constants independent of \( x, t, \delta \).
To prove (32) we write $I = I_1 + I_2$ where
\[ I_1 = \int_{t-\delta}^{t} \frac{x - g(t)}{2 \alpha_1(t - s)} K_{\alpha_1}(x, t, g(s), s) \, ds, \tag{33} \]
\[ I_2 = \int_{t-\delta}^{t} \frac{g(t) - g(s)}{2 \alpha_1(t - s)} \left[ K_{\alpha_1}(x, t, g(s), s) - K_{\alpha_1}(g(t), t, g(s), s) \right] \, ds. \tag{34} \]
Since by assumption $|g(t) - g(s)| \leq A_1(t - s)$ (Lipschitz continuity) we immediately get
\[ |I_2| = \left| \int_{t-\delta}^{t} \frac{g(t) - g(s)}{2 \alpha_1(t - s)} \frac{1}{2\sqrt{\pi(t - s)}} \left[ e^{-\frac{(x-g(t))^2}{4\alpha_1(t-s)}} - e^{-\frac{(x-g(s))^2}{4\alpha_1(t-s)}} \right] \, ds \right| \leq \frac{A_1}{\alpha_1} \delta^\frac{1}{2}. \tag{35} \]

To evaluate $I_1$ we denote
\[ J_1 = \int_{t-\delta}^{t} \frac{x - g(t)}{2 \alpha_1(t - s)} K_{\alpha_1}(x, t, g(t), s) \, ds. \tag{36} \]
With this we can then write
\[ J_1 - I_1 = \int_{t-\delta}^{t} \frac{x - g(t)}{2 \alpha_1(t - s)} K_{\alpha_1}(x, t, g(t), s) \left[ 1 - e^{-\frac{(x-g(t))^2}{4\alpha_1(t-s)}} + \frac{(x-g(t))^2}{4\alpha_1(t-s)} \right] \, ds. \tag{37} \]
The expression in the braces in the second exponent is bounded by
\[ \frac{1}{4\alpha_1(t-s)} |g(t) - g(s)|(\|x - g(t)\| + |x - g(s)|) \leq \frac{A_2}{\alpha_1} \left( |x - g(t)| + |g(t) - g(s)| \right). \]
Since we may assume that $\delta$ is small and hence that the right side of the last inequality is smaller than 1, we conclude that the expression in the brackets on the right side of (37) is bounded by
\[ A_3(\|x - g(t)\| + |g(t) - g(s)|). \]
Substituting this in (37) we find (using $ye^{-y} < y$ for $y \geq 0$)
\[ |J_1 - I_1| \leq \frac{A_4}{\alpha_1} \int_{t-\delta}^{t} \frac{ds}{(t-s)^2} + \frac{A_4}{\alpha_1} |x - g(t)| \int_{t-\delta}^{t} \frac{ds}{(t-s)^2} \leq A_5 \delta^\frac{1}{2}. \tag{38} \]
Now as for $J_1$, we substitute it into the integral (36)
\[ z = \frac{t-s}{(x-g(t))^2}, \]
\[ s = t - z(x-g(t))^2, \]
\[ ds = -dz(x-g(t))^2. \]
Recall that we have
\[ J_1 = \int_{t-\delta}^{t} \frac{x - g(t)}{2 \alpha_1(t - s)} K_{\alpha_1}(x, t, g(t), s) \, ds, \]
with
\[ K_{\alpha_1}(x, t, g(t), s) = \frac{1}{2\sqrt{\pi(t-s)}} e^{-\frac{(x-g(t))^2}{4\pi(t-s)}} \]

and thus \( J_1 \) can be rewritten as
\[
J_1 = \int_{z_1}^{z_2} \frac{x - g(t)}{4\pi \alpha_1 (t-s)^{3/2}} e^{-\frac{1}{4\pi \alpha_1} (x-g(t))^2} \, dz
\]
\[
- \int_{z_1}^{z_2} \frac{(x - g(t))^3}{4\pi \alpha_1 (t-s)^{3/2}} e^{-\frac{1}{4\pi \alpha_1} (x-g(t))^2} \, dz
\]
\[
- \frac{1}{4\pi \alpha_1} \int_{z_1}^{z_2} \frac{1}{z^2} e^{-\frac{1}{4\pi \alpha_1} z^2} \, dz,
\]
with substituting the integration boundaries as
\[
z_2 = \frac{t - t}{(x - g(t))^2} = 0
\]
\[
z_1 = \frac{t - (t - \delta)}{(x - g(t))^2} = \frac{\delta}{(x - g(t))^2} = \delta', \quad \lim_{x \to g(t) - 0} z_1 = \infty.
\]
Thus \( J_1 \) can be written as
\[
J_1 = -\frac{1}{4\pi \alpha_1} \int_0^\delta \frac{1}{z^2} e^{-\frac{1}{4\pi \alpha_1} z^2} \, dz
\]
\[
\lim_{x \to g(t) < 0} J_1 = -\frac{1}{2\sqrt{\alpha_1}}.
\]
Combining (39) with (38) and (35) and using the definition \( I = I_1 + I_2 \) leads to
\[
\limsup_{x \to g(t) - 0} \left| I + \frac{1}{2\sqrt{\alpha_1}} \right| \leq A_\delta \frac{1}{2},
\]
which is exactly the statement (32).

**Remark.** When approaching the boundary from the second phase \( x \to g(t) + 0 \) and hence \( x - g(t) > 0 \), the sign in (39) changes, resulting in \( J_1 = +\frac{1}{2\sqrt{\alpha_2}} \) with \( \alpha_2 \) being the respective thermal diffusivity of the second phase.

From the above evaluation of \( I_1 \) it also follows that
\[
|I_1| \leq A_\delta,
\]
where \( A_\delta \) depends on \( \sigma \). Thus step 1 is finished by having proved (32).

Next we want to prove that
\[
K_0 = \int_{t-\delta}^{t} \frac{|x - g(s)|}{2\alpha_1 (t-s)} K_{\alpha_1}(x, t, g(s), s) \, ds \leq A_\delta,
\]
\[
K_1 = \int_{t-\delta}^{t} \frac{|g(t) - g(s)|}{2\alpha_1 (t-s)} K_{\alpha_1}(g(t), t, g(s), s) \, ds \leq A_\delta.
\]
The proof of (43) follows immediately by using the Lipschitz continuity of \( g(t) \). The proof of (42) follows from the boundedness of \( I_1 \) (see (41)) and that of
\[
K_2 = \int_{t-\delta}^{t} \frac{|g(t) - g(s)|}{2\alpha_1(t-s)} K_{\alpha_1}(x, t, g(s), s)ds,
\]
noting that \( K_0 \leq |I_1| + K_2. \)

We now want to prove (31) by using (32), (42) and (43). We thus define:
\[
L_1 = \int_{t-\delta}^{t} \frac{x - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(x, t, g(s), s)ds
\]
\[
- \int_{t-\delta}^{t} \frac{g(t) - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(g(t), t, g(s), s)ds.
\]

We claim that
\[
\limsup_{x \to g(t) - 0} \frac{1}{2\sqrt{\alpha_1}} |\rho(t)| \leq A_{10} \left( \frac{\delta}{\sqrt{\alpha_1}} + \frac{1}{\delta} \right)
\]
\[
(45)
\]
Substituting \( \rho(s) = \rho(t) + (\rho(s) - \rho(t)) \) in (44), leads to
\[
L_1 = \int_{t-\delta}^{t} \rho(t) \frac{x - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(x, t, g(s), s)ds
\]
\[
\quad - \int_{t-\delta}^{t} \rho(t) \frac{g(t) - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(g(t), t, g(s), s)ds
\]
\[
\quad + \int_{t-\delta}^{t} (\rho(s) - \rho(t)) \frac{x - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(x, t, g(s), s)ds
\]
\[
\quad - \int_{t-\delta}^{t} (\rho(s) - \rho(t)) \frac{g(t) - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(g(t), t, g(s), s)ds.
\]
The first line in the equation above equals the relation (32), thus it can be simplified as
\[
L_1 = -\frac{1}{2\sqrt{\alpha_1}} \rho(t) + \rho(t) O \left( \frac{\delta}{\sqrt{\alpha_1}} \right)
\]
\[
+ \int_{t-\delta}^{t} (\rho(s) - \rho(t)) \frac{x - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(x, t, g(s), s)ds
\]
\[
- \int_{t-\delta}^{t} (\rho(s) - \rho(t)) \frac{g(t) - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(g(t), t, g(s), s)ds.
\]
In the equation above \( \frac{1}{2\sqrt{\alpha_1}} \rho(t) \) is being moved to the left-hand side, resulting in
\[
L_1 + \frac{1}{2\sqrt{\alpha_1}} \rho(t) = \rho(t) O \left( \frac{\delta}{\sqrt{\alpha_1}} \right)
\]
\[
+ \int_{t-\delta}^{t} (\rho(s) - \rho(t)) \frac{x - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(x, t, g(s), s)ds
\]
\[
- \int_{t-\delta}^{t} (\rho(s) - \rho(t)) \frac{g(t) - g(s)}{2\alpha_1(t-s)} K_{\alpha_1}(g(t), t, g(s), s)ds.
\]
Taking the limit \( \limsup_{x \to g(t)^-} \) of the equation above and using (42) and (43) leads to (45).

Next we define:

\[
L_2 = \int_{t-\delta}^{t} \rho(s) \frac{x - g(s)}{2\alpha_1(t - s)} K_{\alpha_1}(x, t, g(s), s) ds \quad (46)
\]

\[
- \int_{t-\delta}^{t} \rho(s) \frac{g(t) - g(s)}{2\alpha_1(t - s)} K_{\alpha_1}(g(t), t, g(s), s) ds \quad (\sigma > 0).
\]

One can easily see that (46) approaches 0 for \( \lim_{x \to g(t)^-} \), and thus satisfies

\[
\lim_{x \to g(t)^-} L_2 = 0.
\]

Adding (45) and (47) we get:

\[
\limsup_{x \to g(t)^-} (L_1 + L_2) + \frac{1}{2\sqrt{\alpha_1}} \leq A_{10} \left( \frac{\delta^2}{\alpha_1} + 1 \text{u.b.}_{t-\sigma \leq s \leq t} |\rho(t) - \rho(s)| \right). \quad (48)
\]

Since the left side of (48) is independent of \( \sigma \) we conclude, on taking \( \sigma \to 0 \), that it is zero, and the proof of the lemma is completed.

**ORCID iDs**

Dokoza T [https://orcid.org/0000-0002-0722-941X](https://orcid.org/0000-0002-0722-941X)

D Plümacher [https://orcid.org/0000-0003-1946-5169](https://orcid.org/0000-0003-1946-5169)

M Smuda [https://orcid.org/0000-0001-7931-5176](https://orcid.org/0000-0001-7931-5176)

M Oberlack [https://orcid.org/0000-0002-5849-3755](https://orcid.org/0000-0002-5849-3755)

**References**

[1] Clapeyron B P and Lame G 1831 Memoire sur la solidification par refroidissement d’un globe liquide Ann. Chem. Phys. 47 250–6

[2] Stefan J 1891 Uber die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere Ann. Phys. 278 269–86

[3] Vuik C 1993 Some historical notes on the Stefan problem Nieuw Archief Voor Wiskunde (Vierde Serie) pp 157–67 vol 11

[4] Rubenstein L I 1971 The Stefan Problem (Providence, RI: American Mathematical Society)

[5] Tarzia D A 2000 A bibliography on moving-free boundary problems for the heat-diffusion equation. The stefan and related problems MAT Serie A 2 1–297

[6] Gupta S C 2017 The Classical Stefan Problem (Basic Concepts, Modelling and Analysis with Quasi-Analytical Solutions and Methods) New Edition (Amsterdam: Elsevier)

[7] Friedman A 1968 The Stefan problem in several space variables Trans. Am. Math. Soc. 133 51

[8] Nogi T 1974 A difference scheme for solving the Stefan problem (RIMS, Kyoto University)

[9] Asaithambi N S 1992 A Galerkin method for Stefan problems Appl. Math. Comput. 52 239–50

[10] Kutluay S, Bahadir A R and Özdeş A 1997 The numerical solution of one-phase classical Stefan problem J. Comput. Appl. Math. 81 135–44

[11] Mitchell S L and Wynnicky M 2016 On the accurate numerical solution of a two-phase Stefan problem with phase formation and depletion J. Comput. Appl. Math. 300 259–74

[12] Nogi T 1980 A difference scheme for solving two phase Stefan problem of heat equation (Japan: RIMS, Kyoto University)

[13] Quan-sheng X and You-lan Z 1985 Solution of the two-dimensional Stefan problem by the singularity-separating method J. Comput. Math. 3 8–18
[14] King J R, Riley D S and Wallman A M 1999 Two-dimensional solidification in a corner Proc. R. Soc. A 455 3449

[15] Sheils N 2015 Interface problems using the Fokas method Dissertation University of Washington

[16] Fokas A S 1962 A unified transform method for solving linear and certain nonlinear PDEs Proceedings: Mathematical, Physical and Engineering Sciences 453 1411–43

[17] Fokas A S and Pelloni B 2014 Unified transform for boundary value problems: applications and advances Soc. Ind. Appl. Math.

[18] Unified transform method https://unifiedmethod.azurewebsites.net/applications.html (accessed 16 April 2021)

[19] Sheils N E and Deconinck B 2014 Heat conduction on the ring: interface problems with periodic boundary conditions Appl. Math. Lett. 37 107–11

[20] Sheils N 2016 Multilayer diffusion in a composite medium with imperfect contact Appl. Math. Model. 46 10

[21] Sheils N and Smith D 2015 Heat equation on a network using the Fokas method J. Phys. A: Math. Theor. 48 335001

[22] Mantzavinos D, Papadomanolaki M G, Saridakis Y and Sifalakis A 2014 Fokas transform method for a brain tumor invasion model with heterogeneous diffusion in $1 + 1$ dimensions Appl. Numer. Math. 104 10

[23] Smuda M and Oberlack M 2019 On the analytical solution of the two-phase Couette flow with wall transpiration Phys. Fluids 31 123603

[24] Delves L M and Mohamed J L 1985 Computational Methods for Integral Equations (Cambridge: Cambridge University Press)

[25] Deconinck B, Trogdon T and Vasan V 2014 The method of Fokas for solving linear partial differential Equations SIAM Rev. 56 159

[26] Fokas A and Pelloni B 2012 Generalized Dirichlet-to-Neumann map in time dependent domains Stud. Appl. Math. 129 51–90

[27] Ablowitz M J and Fokas A S 2003 Complex Variables: Introduction and Applications (Cambridge Texts in Applied Mathematics) 2nd edn (Cambridge: Cambridge University Press)

[28] Friedman A 1959 Free boundary problems for parabolic equations: I. Melting of solids Indiana Univ. Math. J. 8 499

[29] Guenther R B and Lee J W 1988 Partial Differential Equations of Mathematical Physics and Integral Equations (Englewood Cliffs, NJ: Prentice-Hall)