A convergence framework for inexact nonconvex and nonsmooth algorithms and its applications to several iterations

Tao Sun∗ Hao Jiang† Lizhi Cheng∗‡ Wei Zhu§

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Abstract
In this paper, we consider the convergence of an abstract inexact nonconvex and nonsmooth algorithm. We promise a pseudo sufficient descent condition and a pseudo relative error condition, which are both related to an auxiliary sequence, for the algorithm; and a continuity condition is assumed to hold. In fact, a lot of classical inexact nonconvex and nonsmooth algorithms allow these three conditions. Under a special kind of summable assumption on the auxiliary sequence, we prove the sequence generated by the general algorithm converges to a critical point of the objective function if being assumed Kurdyka-Lojasiewicz property. The core of the proofs lies in building a new Lyapunov function, whose successive difference provides a bound for the successive difference of the points generated by the algorithm. And then, we apply our findings to several classical nonconvex iterative algorithms and derive the corresponding convergence results.

Keywords: Nonconvex minimization; Inexact algorithms; Semi-algebraic functions; Kurdyka-Lojasiewicz property; Convergence analysis

Mathematical Subject Classification 90C30, 90C26, 47N10

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1 Introduction

Minimization of the nonconvex and nonsmooth function

$$\min_x F(x)$$  \hspace{1cm} (1.1)$$

is a core part of nonlinear programming and applied mathematics. Different with traditional convergence results on the global minimizers in the convex community, the convergence of the nonconvex algorithm just promises that the iteration falls into a critical point. In most practical cases, the objective functions enjoy the Kurdyka-Lojasiewicz property (see definitions in Sec. 2). In this paper, we consider the convergence analysis under the Kurdyka-Lojasiewicz property assumption on the objective function $F$.

In paper [5], for the sequence $(x^k)_{k \geq 0}$ generated by a very general scheme for problem (1.1), the authors consider three conditions, sufficient descent condition, relative error condition and continuity condition. Mathematically, these three conditions can be presented as: for some $a > 0, c > 0$

$$\begin{cases}
F(x^k) - F(x^{k+1}) \geq a\|x^{k+1} - x^k\|^2, \\
\text{dist}(0, \partial F(x^{k+1})) \leq c\|x^{k+1} - x^k\|,
\end{cases}$$

there exist a stationary point $x^*$ and a subsequence $(x^{k_j})_{j \geq 0} \to x^*$ satisfying $F(x^{k_j}) \to F(x^*)$ \hspace{1cm} (1.2)$$

where $\partial F$ means the limiting subdifferential of $F$ (see definition in Sec. 2). Actually, various algorithms satisfy these three conditions. The third condition is usually derived by the minimization in each iteration. The proofs in [5] use a local area analysis; the authors first prove that the sequence falls into a neighbor of some point after enough iterations and then employ the Kurdyka-Lojasiewicz property around the point. In latter paper [9], the authors prove a uniformed Kurdyka-Lojasiewicz lemma for a closed set and much simplify the proofs.

1.1 A novel convergence framework

In this paper, we consider the convergence for inexact nonconvex and nonsmooth algorithms. We stress that the inexact algorithms discussed in our paper are different from the paper [5]. In their paper, an assumption is posed for the noise: the noise should be bounded by the successive difference of the iteration. The “inexact algorithm” in [5] is much closer to “proximal algorithm”. For example, if $F$ is differentiable (may be nonconvex), the nonconvex gradient descent algorithm performs as

$$x^{k+1} = x^k - h \cdot \nabla F(x^k).$$  \hspace{1cm} (1.3)$$

If the gradient of $F$ is Lipschitz with $L$ and $0 < h < \frac{1}{L}$, the sequence $(x^k)_{k \geq 0}$ generated by (1.3) satisfies condition (1.2). However, if the iteration is corrupted by some noise $e^k$ in each step, i.e.,

$$x^{k+1} = x^k - h \cdot \nabla F(x^k) + e^k.$$  \hspace{1cm} (1.4)$$

However, the sequence $(x^k)_{k \geq 0}$ generated by (1.4) is likely violating some conditions in (1.2) when $e^k \neq 0$. The existing analysis cannot be directly used for the algorithm (1.4). The authors in [5] proposed the assumption for the noise as

$$\|e^k\| \leq \ell \cdot \|x^{k+1} - x^k\|,$$  \hspace{1cm} (1.5)$$
where \( \ell > 0 \). Under this assumption, they can continue using the sufficient descent condition and relative error condition. In this paper, we get rid of the dependent assumption like \( (1.5) \). Although in this case the inexact algorithms always fail to obey the first two of the core condition \( (1.2) \), we find that many of them satisfy an alternative condition:

\[
\begin{align*}
\{ & F(x^k) - F(x^{k+1}) \geq a\|\omega^{k+1} - \omega^k\|^2 - b\eta_k^2 \\
\text{dist}(0, \partial F(x^{k+1})) \leq c \sum_{j=k-\tau}^{k-1} \|\omega^{j+1} - \omega^j\| + d\eta_k \\
\text{there exist a stationary point } x^* \text{ and a subsequence } (x^{k_j})_{j \geq 0} \rightarrow x^* \text{ satisfying } F(x^{k_j}) \rightarrow F(x^*)
\end{align*}
\]

(1.6)

where \( a, b, c, d > 0 \) are constants, and \( (\eta_k)_{k \geq 0} \) is a nonnegative sequence, and \( \tau \in \mathbb{Z}^+ \) and \( (\omega^k)_{k \geq 0} \) is a sequence satisfying

\[
\|x^k - x^{k+1}\| \leq \epsilon \|\omega^k - \omega^{k+1}\|
\]

for some \( \epsilon > 0 \). The continuity condition is kept here. Obviously, if \( \eta_k \equiv 0 \), \( \omega^k = x^k \) and \( \tau = 0 \), the condition will reduce to \( (1.2) \). Thus, our work can also be regarded as a generation of paper \([5]\). Our approach is first proving convergence for a general inexact algorithm whose sequence \( (x^k)_{k \geq 0} \) satisfying the condition \( (1.6) \) under a specific summable assumption on \( (\eta_k)_{k \geq 0} \). We then prove several classical inexact algorithms satisfying condition \( (1.6) \).

The core of the proof lies in using an auxiliary function whose successive difference gives a bound to the successive difference of the sequence \( \|\omega^{k+1} - \omega^k\|^2 \). If \( F \) is semi-algebraic, the new function is then Kurdyka-Lojasiewicz. And then, we build sufficient descent involving the new function and relative error condition. In this paper, we get rid of the dependent assumption like \( (1.5) \). Although in this case the inexact algorithms always fail to obey the first two of the core condition \( (1.2) \), we find that many of them satisfy an alternative condition:

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1.2 Related work

Recently, the convergence analysis in nonconvex optimization has paid increasing attention to using the Kurdyka-Lojasiewicz property in proofs. In paper \([3]\), the authors proved the convergence of proximal algorithm minimizing the Kurdyka-Lojasiewicz functions. In \([3]\), the rates for the iteration converging to a critical point were exploited. An alternating proximal algorithm was considered in \([4]\), and the convergence was proved under Kurdyka-Lojasiewicz assumption on the objective function. Later, a proximal linearized alternating minimization algorithm was proposed and studied in \([5]\). A convergence framework was given in \([6]\), which contains various nonconvex algorithms. In \([7]\), the authors modified the framework for analyzing splitting methods with variable metric, and proved the general convergence rates. The nonconvex ADMM was studied under Kurdyka-Lojasiewicz assumption by \([8]\) \([9]\). And latter paper \([10]\) proposed the nonconvex primal-dual algorithm and proved the convergence. The Kurdyka-Lojasiewicz-analysis method was applied to analyzing the convergence of the reweighted algorithm by \([11]\). And the extension to the reweighted nuclear norm version was developed in \([12]\). Recently, the DC algorithm has also employed the Kurdyka-Lojasiewicz property in the convergence analysis \([13]\).

1.3 Contribution and organization

In this paper, we focus on the inexact nonconvex algorithms. We first propose a new framework \( (1.6) \), which is more general than the frameworks proposed in \([5]\) and \([13]\). The convergence is proved for any sequence satisfying \( (1.6) \) with \( \eta_k = \frac{1}{t_k} \) and \( \alpha > 1 \) if \( F \) is a Kurdyka-Lojasiewicz function. In the analysis, we employ the new Lyapunov function which is a composition of the \( F \) and the length of the noise. The new framework proposed in this paper indicates kinds of algorithms. We then apply our results to these algorithms. For a specific algorithm, we just need to verify that \( (1.6) \) and \( (1.7) \) hold.

The rest of the paper is organized as follows. In section 2, we list necessary preliminaries. Section 3 contains the main results. In section 4, we provide the applications. Section 5 concludes the paper.
2 Preliminaries

This section presents the mathematical tools which will be used in our proofs and contains two parts: in the first one, we introduce the basic definitions and properties for subdifferentials; in the second one, the KL property is introduced.

2.1 Subdifferential

More details about the definition of subdifferential can be found in the textbooks [27, 28]. Given an lower semicontinuous function $J : \mathbb{R}^N \to (-\infty, +\infty]$, its domain is defined by
\[
\text{dom}(J) := \{ x \in \mathbb{R}^N : J(x) < +\infty \}.
\]

The notion of subdifferential plays a central role in variational analysis.

**Definition 1** (subdifferential). Let $J : \mathbb{R}^N \to (-\infty, +\infty]$ be a proper and lower semicontinuous function.

1. For a given $x \in \text{dom}(J)$, the Fréchet subdifferential of $J$ at $x$, written $\partial J(x)$, is the set of all vectors $u \in \mathbb{R}^N$ which satisfy
\[
\liminf_{y \to x} \frac{J(y) - J(x) - \langle u, y - x \rangle}{\| y - x \|} \geq 0.
\]

When $x \notin \text{dom}(J)$, we set $\partial J(x) = \emptyset$.

2. The (limiting) subdifferential, or simply the subdifferential, of $J$ at $x \in \text{dom}(J)$, written $\partial J(x)$, is defined through the following closure process
\[
\partial J(x) := \{ u \in \mathbb{R}^N : \exists x_k \to x, J(x_k) \to J(x) \text{ and } u_k \in \partial J(x_k) \to u \text{ as } k \to \infty \}.
\]

It is easy to verify that the Fréchet subdifferential is convex and closed while the subdifferential is closed. When $J$ is convex, the definition agrees with the subgradient in convex analysis as
\[
\partial J(x) := \{ v : J(y) \geq J(x) + \langle v, y - x \rangle \text{ for any } y \in \mathbb{R}^N \}.
\]

The graph of subdifferential for a real extended valued function $J : \mathbb{R}^N \to (-\infty, +\infty]$ is defined by
\[
\text{graph}(\partial J) := \{ (x,v) \in \mathbb{R}^N \times \mathbb{R}^N : v \in \partial J(x) \}.
\]

And the domain of the subdifferential of $J$ is given as
\[
\text{dom}(\partial J) := \{ x \in \mathbb{R}^N : \partial J(x) \neq \emptyset \}.
\]

Let $\{(x^k, v^k)\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}^N \times \mathbb{R}$ such that $(x^k, v^k) \in \text{graph}(\partial J)$. If $(x^k, v^k)$ converges to $(x, v)$ as $k \to +\infty$ and $J(x^k)$ converges to $v$ as $k \to +\infty$, then $(x, v) \in \text{graph}(\partial J)$. A necessary condition for $x \in \mathbb{R}^N$ to be a minimizer of $J(x)$ is
\[
0 \in \partial J(x). \tag{2.1}
\]

When $J$ is convex, (2.1) is also sufficient. A point that satisfies (2.1) is called (limiting) critical point. The set of critical points of $J(x)$ is denoted by $\text{crit}(J)$.

2.2 Kurdyka-Lojasiewicz function

With the definition of subdifferential, we now are prepared to introduce the Kurdyka-Lojasiewicz property and function.

**Definition 2.** [22, 18, 7] (a) The function $J : \mathbb{R}^N \to (-\infty, +\infty]$ is said to have the Kurdyka-Lojasiewicz property at $x \in \text{dom}(\partial J)$ if there exist $\eta \in (0, +\infty]$, a neighborhood $U$ of $x$ and a continuous concave function $\varphi : [0, \eta) \to \mathbb{R}^+$ such that
1. $\varphi(0) = 0$.
2. $\varphi$ is $C^1$ on $(0, \eta)$.
3. for all $s \in (0, \eta)$, $\varphi'(s) > 0$.
4. for all $x$ in $U \cap \{x| J(x) < J(\overline{x}) < J(\overline{x}) + \eta\}$, the Kurdyka-Lojasiewicz inequality holds
   
   $$\varphi'(J(x) - J(\overline{x})) \cdot \text{dist}(0, \partial J(x)) \geq 1.$$  
   (2.2)

(b) Proper lower semicontinuous functions which satisfy the Kurdyka-Lojasiewicz inequality at each point of $\text{dom}(\partial J)$ are called KL functions.

It is hard to directly judge whether a function is Kurdyka-Lojasiewicz or not. Fortunately, the concept of semi-algebraicity can help to find and check a very rich class of Kurdyka-Lojasiewicz functions.

**Definition 3** (Semi-algebraic sets and functions [7, 8]). (a) A subset $S$ of $\mathbb{R}^N$ is a real semi-algebraic set if there exists a finite number of real polynomial functions $g_{ij}, h_{ij} : \mathbb{R}^N \to \mathbb{R}$ such that

$$S = \bigcup_{j=1}^{p} \bigcap_{i=1}^{q} \{u \in \mathbb{R}^N : g_{ij}(u) = 0 \text{ and } h_{ij}(u) < 0\}.$$

(b) A function $h : \mathbb{R}^N \to (-\infty, +\infty]$ is called semi-algebraic if its graph

$$\{(u, t) \in \mathbb{R}^{N+1} : h(u) = t\}$$

is a semi-algebraic subset of $\mathbb{R}^{N+1}$.

Better yet, the semi-algebraicity enjoys many quite nice properties [7, 8]. We just put a few of them here:

- Real polynomial functions.
- Indicator functions of semi-algebraic sets.
- Finite sums and product of semi-algebraic functions.
- Composition of semi-algebraic functions.
- Sup/Inf type function, e.g., $\sup\{g(u, v) : v \in C\}$ is semi-algebraic when $g$ is a semi-algebraic function and $C$ a semi-algebraic set.
- Cone of PSD matrices, Stiefel manifolds and constant rank matrices.

Now we present a lemma for the uniformized KL property. With this lemma, we can make the proofs much more concise.

**Lemma 1** ([9]). Let $J : \mathbb{R}^N \to \mathbb{R}$ be a proper lower semi-continuous function and $\Omega$ be a compact set. If $J$ is a constant on $\Omega$ and $J$ satisfies the KL property at each point on $\Omega$, then there exists concave function $\varphi$ satisfying the four assumptions in Definition 2 and $\delta, \varepsilon > 0$ such that for any $\overline{x} \in \Omega$ and any $x$ satisfying that $\text{dist}(x, \Omega) < \varepsilon$ and $J(\overline{x}) < J(x) < J(\overline{x}) + \delta$, it holds that

$$\varphi'(J(x) - J(\overline{x})) \cdot \text{dist}(0, \partial J(x)) \geq 1.$$  
(2.3)
3 Convergence analysis

The sequence \((\eta_k)_{k \geq 0}\) is assumed to satisfy
\[\sum_l \eta_l < +\infty. \tag{3.1}\]

It is worth mentioning that the assumption (3.1) is necessary to guarantee the sequence convergence in general case. To see this, we consider the inexact gradient example (1.4) in a very special case that \(F \equiv 0\). And then, we get \(x^k = x^0 + \sum_{i=0}^{k-1} e^k\). Further, we consider the one-dimensional case, in which \(|e^k| = \eta^k\); we set \(e^k = \eta^k\). In this example, \((x^k)_{k \geq 0}\) will diverge if (3.1) fails to hold. However, in our proofs, only (3.1) barely promises the sequence convergence. The final assumption for the sequence convergence is a little stronger than (3.1).

Now, we introduce the Lyapunov function used in the analysis. Given any fixed \(\theta > 1\), we denote a new function, which plays an important role in the analysis, as
\[\xi(z) := F(x) + \frac{t^\theta}{\theta}, \quad z := (x, t) \in \mathbb{R}^{N+1}. \tag{3.2}\]

We also need to define the new sequences as
\[t_k := (\theta \cdot b \cdot \sum_{l=k}^{+\infty} \eta_l^2)^{\frac{1}{2}}, \quad z_k := (x_k, t_k). \tag{3.3}\]

Due to that \(\sum_{l=k}^{+\infty} \eta_l^\theta \leq \sum_{l=k}^{+\infty} \eta_l < +\infty\) when \(\theta > 1\) and \(k'\) is large enough, \(t_k\) is well-defined. The aim in this part is proving that \(\{z_k\}\) generated by the algorithm converges to a critical point of \(\xi\), and building the relationships between the critical points of \(\xi\) and \(F\). The proof contains two main steps:

1. Find a positive constant \(\rho_1\) such that
\[\rho_1 \|\omega_{k+1} - \omega_k\|^2 \leq \xi(z_k) - \xi(z_{k+1}), \quad k = 0, 1, \ldots. \tag{3.4}\]

2. Find another positive constants \(\rho_2, \rho_3, \rho_4\) such that
\[\text{dist}(0, \partial \xi(z_{k+1})) \leq \rho_2 \sum_{j=k}^{k} \|\omega_{j+1} - \omega_j\| + \rho_3 \eta_k + \rho_4 (t_{k+1})^\theta, \quad k = 0, 1, \ldots. \tag{3.5}\]

\[\text{Lemma 2. Assume that} \{x_k\}_{k=0, 1, 2, \ldots} \text{ is generated by the general inexact algorithm satisfying conditions (1.6) and (1.7), and condition (3.7) holds. Then, we have the following results.}\]

\[(1) \text{ It holds that} \quad \xi(z_k) - \xi(z_{k+1}) \geq a \|\omega_k - \omega_{k+1}\|^2. \tag{3.6}\]

\[\text{And then,} \quad (z_k)_{k \geq 0} \text{ is bounded if} \quad F \text{ is coercive.}\]

\[(2) \sum_k \|x^{k+1} - x^k\|^2 < +\infty, \text{ which implies that} \]
\[\lim_k \|x^{k+1} - x^k\| = 0. \tag{3.7}\]

\[\text{Proof. } (1) \text{ From the direct algebra computations, we can easily obtain} \]
\[\xi(z_k) - \xi(z_{k+1}) = F(x^k) - F(x^{k+1}) + \frac{t^\theta}{\theta} \frac{t_k^\theta - t_{k+1}^\theta}{\theta} \]
\[= F(x^k) - F(x^{k+1}) + b\eta_k^2 \geq a \|\omega_k - \omega_{k+1}\|^2. \tag{3.8}\]
If $F$ is coercive, then $\xi$ is coercive. Thus, $(z^k)_{k \geq 0}$ is bounded due to that $(\xi(z^k))_{k \geq 0}$ is bounded.

(2) From (3.3), $\{\xi(z^k)\}_{k=0,1,2,...}$ is descending. Note that $\inf \xi > -\infty$, $\{\xi(z^k)\}_{k=0,1,2,...}$ is convergent. Hence, we can easily have that

$$\sum_{n=0}^{k} \|\omega^{n+1} - \omega^n\|^2 \leq \frac{\xi(z^0) - \xi(z^{k+1})}{a} < +\infty.$$ 

With (1.7), we then prove the result.

Lemma 3. If the conditions of Lemma 2 hold,

$$\text{dist}(0, \partial \xi(z^{k+1})) \leq c \sum_{j=k-\tau}^{k} \|\omega^{j+1} - \omega^j\| + d\eta_k + t_{k+1}. \quad (3.7)$$

Proof. Direct calculation yields

$$\partial \xi(z^{k+1}) = \left( \frac{\partial F(x^{k+1})}{(t_{k+1})_{\theta-1}} \right). \quad (3.8)$$

Thus, we have

$$\text{dist}(0, \partial \xi(z^{k+1})) \leq \text{dist}(0, \partial F(x^{k+1})) + (t_{k+1})^{\theta-1} \leq c \sum_{j=k-\tau}^{k} \|\omega^{j+1} - \omega^j\| + d\eta_k + (t_{k+1})^{\theta-1}. \quad (3.9)$$

In the following, we establish some results about the limit points of the sequence generated by the general algorithm. We need a definition about the limit point which is introduced in [5].

Definition 4. For a sequence $\{d^k\}_{k=0,1,2,...}$, define that

$$\mathcal{M}(d^0) := \{d \in \mathbb{R}^N : \exists \text{ an increasing sequence of integers } \{k_j\}_{j \in \mathbb{N}} \text{ such that } d^{k_j} \to d \text{ as } j \to \infty\},$$

where $d^0 \in \mathbb{R}^N$ is the starting point.

Lemma 4. Suppose that $\{z^k = (x^k, t^k)\}_{k=0,1,2,...}$ is generated by general algorithm and $F$ is coercive. And the conditions of Lemma 2 hold. Then, we have the following results.

(1) For any $z^* = (x^*, t^*) \in \mathcal{M}(d^0)$, we have $t^* = 0$ and $\xi(z^*) = F(x^*)$.

(2) $\mathcal{M}(z^0)$ is nonempty and $\mathcal{M}(z^0) \subseteq \text{crit}(\xi)$.

(2') $\mathcal{M}(z^0)$ is nonempty and $\mathcal{M}(z^0) \subseteq \text{crit}(F)$

(3) $\lim_k \text{dist}(z^k, \mathcal{M}(z^0)) = 0$.

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(4) The function $\xi$ is finite and constant on $\mathcal{M}(z^0)$.

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Proof. (1) Noting $(t_k)_{k \geq 0} \to 0$, $t^* = 0$ and

$$\xi(z^*) = \xi(x^*, 0) = F(x^*).$$

(2) It is easy to see the coercivity of $\xi$. With Lemma 2 and the coercivity of $\xi$, $(z^k)_{k \geq 0}$ is bounded. Thus, $\mathcal{M}(z^0)$ is nonempty. Assume that $z^* \in \mathcal{M}(z^0)$, from the definition, there exists a subsequence $z^{k_i} \to z^*$. From Lemmas 2 and 3, we have $\text{dist}(0, \partial \xi(z^{k_i})) \to 0$. The closedness of $\partial \xi$ indicates that $0 \in \partial \xi(z^*)$, i.e. $z^* \in \text{crit}(\xi)$.

(2') With the facts $z = (x,t)$ and $\xi(z) = F(x) + \frac{\theta}{\tau}$, we can easily derive the results.
(3)(3’) This item follows as a consequence of the definition of the limit point.

(4) Let \( I \) be the limit of \( (\xi(x^k))_{k \geq 0} \). There exists one stationary point \( x^* \), from the continuity condition, there exists \( (x^{k_j})_{j \geq 0} \to x^* \) satisfying \( F(x^{k_j}) \to F(x^*) \). We denote that \( z^{k_j} = (x^{k_j}, t_{k_j}) \). Thus, the subsequence \( (z^{k_j})_{j \geq 0} \to z^* \in \text{crit}(\xi) \) and \( (\xi(z^{k_j}))_{j \geq 0} \to I \). And we have

\[
\xi(\mathcal{V}) = \lim_j \xi(z^{k_j}) = I.
\]

(4’) The proof is similar to (4).

\[ \square \]

**Lemma 5.** Suppose that \( F \) is a closed semi-algebraic function and coercive. Let the sequence \( (x^k)_{k \geq 0} \) be generated by general scheme and the conditions \([1,6] \) and \([1,7] \) hold. If there exists \( \theta > 1 \) such that the sequence \( (\eta_k)_{k \geq 0} \) satisfies

\[
\sum_k \left( \sum_{l=k}^{+\infty} \eta_l^2 \right)^{\frac{1}{2}} < +\infty.
\]  \hspace{1cm} (3.10)

Then, the sequence \( (x^k)_{k \geq 0} \) has finite length, i.e.

\[
\sum_{k=0}^{+\infty} \|x^{k+1} - x^k\| < +\infty.
\]  \hspace{1cm} (3.11)

And \( (x^k)_{k \geq 0} \) converges to a critical point \( x^* \) of \( F \).

**Proof.** Obviously, \( \xi \) is semi-algebraic, and then KL. Let \( x^* \) be a cluster point of \( (x^k)_{k \geq 0} \), then, \( z^* = (x^*, 0) \) is also a cluster point of \( (z^k)_{k \geq 0} \). If \( \xi(z^{K'}) = \xi(z^*) \) for some \( K' \), with the fact \( (\xi(z^k))_{k \geq 0} \) is decreasing, \( \xi(z^k) = \xi(z^*) \) as \( k \geq K' \). Using Lemma \([2] \), \( z^k = z^{K'} \) as \( k > K' \). In the following, we consider the case \( \xi(z^k) > \xi(z^*) \). From Lemmas \([1] \) and \([4] \), there exist \( \delta, \varepsilon > 0 \) such that for any \( \mathcal{V} \in \mathcal{M}(z^0) \) and any \( x \) satisfying that \( \text{dist}(z, \mathcal{M}(z^0)) < \varepsilon \) and \( \xi(z^*) < \xi(z) < \xi(z^*) + \delta \). From Lemma \([4] \) as \( k \) is large enough,

\[
z^k \in \{ z | \text{dist}(z, \mathcal{M}(z^0)) < \varepsilon \} \bigcap \{ z | \xi(z^*) < \xi(z) < \xi(z^*) + \delta \}.
\]

Thus, there exist concave function \( \varphi \) such that

\[
\varphi'(\xi(z^{k+1}) - \xi(z^*)) \cdot \text{dist}(0, \partial \xi(z^{k+1})) \geq 1.
\]  \hspace{1cm} (3.12)

Therefore, we have

\[
\varphi(\xi(z^{k+1}) - \xi(z^*)) - \varphi(\xi(z^{k+2}) - \xi(z^*))
\]

\[
a) \geq \varphi'(\xi(z^{k+1}) - \xi(z^*)) \cdot (\xi(z^{k+1}) - \xi(z^{k+2}))
\]

\[
b) \geq a \cdot \varphi'(f(x^{k+1}) - \xi(z^*)) \cdot \|\omega^{k+2} - \omega^{k+1}\|^2
\]

\[
c) \geq a\|\omega^{k+2} - \omega^{k+1}\|^2
\]

\[
d) \geq \text{dist}(0, \partial \xi(z^{k+1}))
\]

\[
\geq a\|\omega^{k+2} - \omega^{k+1}\|^2 + d\eta_k + (t_k+1)^{\theta-1},
\]

where \( a) \) is due to the concavity of \( \varphi \), and \( b) \) depends on Lemma \([2] \), \( c) \) uses the KL property, and \( d) \)
follows from Lemma 3. That is also
\[ 2\|\omega^{k+2} - \omega^{k+1}\| \]
\[ \leq \frac{2}{a} \left\{ [\varphi(\xi(z^{k+1}) - \xi(z^*)) - \varphi(\xi(z^{k+2}) - \xi(z^*))] \cdot \left[ c \sum_{j=k-\tau}^{k} \|\omega^{j+1} - \omega^{j}\| + d\eta_k + (t_{k+1})^{\theta-1} \right] \right\}^{\frac{1}{2}} \]
\[ \geq \frac{c(\tau + 1)}{a^2} \left[ \varphi(\xi(z^{k+1}) - \xi(z^*)) - \varphi(\xi(z^{k+2}) - \xi(z^*)) \right] \]
\[ + \sum_{j=k-\tau}^{k} \|\omega^{j+1} - \omega^{j}\| \cdot \frac{ad}{c(\tau + 1)} \eta_k + \frac{a}{c} (t_{k+1})^{\theta-1}, \]  
\[ (3.13) \]
where e) uses the Schwarz inequality \( 2(xy)^{\frac{1}{2}} \leq tx + \frac{t}{y} \) with \( x = [\varphi(\xi(z^{k+1}) - \xi(z^*)) - \varphi(\xi(z^{k+2}) - \xi(z^*))] \),
and \( y = [c \sum_{j=k-\tau}^{k} \|\omega^{j+1} - \omega^{j}\| + \frac{d}{\sqrt{c}} (t_k)^{\frac{1}{2}} + (t_{k+1})^{\theta-1}] \), and \( t = \frac{c(\tau + 1)}{a} \). Multiplying \( (3.13) \) with \( \tau + 1 \), we have
\[ 2(\tau + 1)\|\omega^{k+2} - \omega^{k+1}\| \leq \frac{c(\tau + 1)^2}{a^2} \left[ \varphi(\xi(z^{k+1}) - \xi(z^*)) - \varphi(\xi(z^{k+2}) - \xi(z^*)) \right] \]
\[ + \sum_{j=k-\tau}^{k} \|\omega^{j+1} - \omega^{j}\| \cdot \frac{ad}{c} \eta_k + \frac{a}{c} (t_{k+1})^{\theta-1}. \]  
\[ (3.14) \]
Summing both sides from \( k \) to \( K \), and with simplifications,
\[ (2\tau + 1) \sum_{l=k+1}^{K+1} \|\omega^{l+1} - \omega^{l}\| + (2\tau + 2) \sum_{j=K+1-\tau}^{K+1} \|\omega^{j+1} - \omega^{j}\| \]
\[ \leq \frac{c(\tau + 1)^2}{a^2} \left[ \varphi(\xi(z^{k+1}) - \xi(z^*)) - \varphi(\xi(z^{k+2}) - \xi(z^*)) \right] \]
\[ + \sum_{j=k-\tau}^{K} \|\omega^{j+1} - \omega^{j}\| \cdot \frac{ad}{c} \sum_{l=k}^{K} \eta_l + \frac{a}{c} \sum_{l=k+1}^{K+1} (t_l)^{\theta-1} < +\infty. \]  
\[ (3.15) \]
Letting \( K \to +\infty \) and using \( \sum_{j=K+1-\tau}^{K+1} \|\omega^{j+1} - \omega^{j}\| \to 0 \) and \( \tau \in \mathbb{Z}^0^+ \), we then derive
\[ \sum_{k} \|\omega^{k+1} - \omega^{k}\| < +\infty. \]  
\[ (3.16) \]
By using (1.7), we are then led to
\[ \sum_{k} \|x^{k+1} - x^{k}\| < +\infty. \]  
\[ (3.17) \]
Thus, \((x^k)_{k\geq 0}\) has only one stationary point \( x^* \). From Lemma 1, \( x^* \in \text{crit}(F) \). \( \square \)

The requirement \( (3.10) \) is complicated and impractical in the applications. Thus, we consider the sequence \((\eta_k)_{k\geq 0}\) enjoys the polynomial forms as \( \eta_k \leq \frac{C}{k^\alpha} \) with \( \alpha > 1 \). We try to simplify \( (3.10) \) in this case. The task then reduce the following mathematical analysis problem: find the minimum \( \alpha_0 \geq 1 \) such that for any \( \alpha \in (\alpha_0, +\infty) \), there exists \( \theta > 1 \) can make \( (3.10) \) hold. Direct calculations give us
\[ \left( \sum_{l=k}^{+\infty} \eta_l^2 \right)^{\frac{\alpha-1}{\alpha}} = \left( \sum_{l=k}^{+\infty} C^2 \right)^{\frac{\alpha-1}{\alpha}} \leq \left( \sum_{l=k}^{+\infty} \int_l^{l+1} C^2 \left( \frac{t}{2\alpha} \right)^{-1} dt \right)^{\frac{\alpha-1}{\alpha}} = \frac{C^{\frac{2(\theta-1)}{\theta}}}{(2\alpha - 1)^{\frac{\alpha-1}{\alpha}}} \cdot \frac{1}{k^{\frac{(2\alpha-1)(\theta-1)}{\theta}}} \]  
\[ (3.18) \]
Thus, we need
\[
\alpha > 1, \quad \text{and} \quad \frac{(2\alpha - 1)(\theta - 1)}{\theta} > 1.
\] (3.19)

After simplifications, we get
\[
\alpha > 1, \quad \text{and} \quad \alpha > \frac{2\theta - 1}{2(\theta - 1)}.
\] (3.20)

Then, the problem reduces to
\[
\alpha_0 = \inf_{\theta > 1} \left\{ c(\theta) := \max\{1, \frac{2\theta - 1}{2(\theta - 1)}\} = \frac{2\theta - 1}{2(\theta - 1)} \right\}.
\] (3.21)

Figure 1 shows the function values between [1,1.5]. We can see \(c(\theta)\) is decreasing to 1 at +\(\infty\). Therefore, we get \(\alpha_0 = 1\). That is also to say if \(\eta_k \leq C\) with any fixed \(\alpha > 1\), there exists \(\theta > 1\) such that (3.10) can hold. And then, the sequence \((x^k)_{k \geq 0}\) is convergent to some critical point of \(F\). Therefore, we obtain the following result.

**Theorem 1** (Convergence result). Suppose that \(F\) is a closed semi-algebraic function and coercive. Let the sequence \(\{x^k\}_{k=0,1,2,3,...}\) be generated by general scheme and the conditions (1.6) and (1.7) hold. The sequence \((\eta_k)_{k \geq 0}\) obeys
\[
\eta_k = O\left(\frac{1}{k^\alpha}\right), \quad \alpha > 1.
\] (3.22)

Then, the sequence \(\{x^k\}_{k=0,1,2,3,...}\) has finite length, i.e.
\[
\sum_{k=0}^{+\infty} \|x^{k+1} - x^k\| < +\infty.
\] (3.23)

And \(\{x^k\}_{k=0,1,2,3,...}\) converges to a critical point \(x^*\) of \(F\).

## 4 Applications to several nonconvex algorithms

In this part, several classical nonconvex inexact algorithms are considered. We apply our theoretical findings to these algorithms and derive corresponding convergence results for the algorithms. As presented
Lemma 6. For any \( x \) and \( y \), if \( z \in \text{prox}_f(x) \),

\[
J(z) + \frac{\|z - x\|^2}{2} \leq J(y) + \frac{\|y - x\|^2}{2}.
\]  

(4.2)

Of course, we also have

\[
x - z \in \partial J(z).
\]

(4.3)

In subsections 4.1-4.4, the point \( \omega^k \) is \( x^k \) itself, i.e., \( \omega^k \equiv x^k \).

4.1 Inexact nonconvex gradient and proximal algorithm

The nonconvex proximal gradient algorithm is developed for the nonconvex composite optimization

\[
\min_x \{ F(x) = f(x) + g(x) \},
\]

where \( f \) is differentiable and \( \nabla f \) is Lipschitz with \( L \), and \( g \) is closed. And both \( f \) and \( g \) may be nonconvex. The nonconvex inexact proximal gradient algorithm can be described as

\[
x^{k+1} = \text{prox}_{h g}(x^k - h \nabla f(x^k) + e^k),
\]

(4.5)

where \( h \) is the stepsize, \( \text{prox} \) is the proximal operator and \( e^k \) is the noise. In the convex case, this algorithm is discussed in \cite{38, 29}, and the acceleration is studied in \cite{31}.

Lemma 7. Let \( 0 < h < \frac{1}{L} \) and the sequence \( (x^k)_{k \geq 0} \) be generated by algorithm \( (4.5) \), we have

\[
F(x^k) - F(x^{k+1}) \geq \frac{1}{4} \left( \frac{1}{h} - L \right) \| x^{k+1} - x^k \|^2 - \frac{1}{h(1 - hL)} \| e^k \|^2.
\]

(4.6)

Proof. The \( L \)-Lipschitz of \( \nabla f \) gives

\[
f(x^{k+1}) - f(x^k) \leq \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \| x^{k+1} - x^k \|^2.
\]

(4.7)

On the other hand, with Lemma\[3\] we have

\[
h g(x^{k+1}) + \frac{\| x^k - h \nabla f(x^k) + e^k - x^{k+1} \|^2}{2} \leq h g(x^k) + \frac{\| -h \nabla f(x^k) + e^k \|^2}{2}.
\]

(4.8)

This is also

\[
g(x^{k+1}) - g(x^k) \leq -\langle \nabla f(x^k), x^{k+1} - x^k \rangle - \frac{\| x^k - x^{k+1} \|^2}{2h} + \frac{(e^k, x^{k+1} - x^k)}{h}.
\]

(4.9)

Summing (4.7) and (4.9),

\[
F(x^{k+1}) - F(x^k) \leq \frac{1}{2} \left( \frac{1}{h} - L \right) \| x^{k+1} - x^k \|^2 + \frac{(e^k, x^{k+1} - x^k)}{h}.
\]

(4.10)

With the Cauchy-Schwarz inequality, we have

\[
\frac{(e^k, x^{k+1} - x^k)}{h} \leq \frac{1}{4} \left( \frac{1}{h} - L \right) \| x^{k+1} - x^k \|^2 + \frac{1}{h(1 - hL)} \| e^k \|^2.
\]

(4.11)

Combining (4.11) and (4.10), we then prove the result. \( \square \)
Lemma 8. Let the sequence \((x^k)_{k \geq 0}\) be generated by algorithm [4,5], we have
\[
\text{dist}(0, \partial F(x^{k+1})) \leq \left( \frac{1}{h} + L \right) \| x^k - x^{k+1} \| + \frac{1}{h} \| e^k \|. \tag{4.12}
\]
Proof. We have
\[
\frac{x^k - x^{k+1}}{h} - \nabla f(x^k) + \frac{e^k}{h} \in \partial g(x^{k+1}). \tag{4.13}
\]
Therefore,
\[
\frac{x^k - x^{k+1}}{h} + \nabla f(x^{k+1}) - \nabla f(x^k) + \frac{e^k}{h} \in \nabla f(x^{k+1}) + \partial g(x^{k+1}) = \partial F(x^{k+1}). \tag{4.14}
\]
Thus, we have
\[
\text{dist}(0, \partial F(x^{k+1})) \leq \frac{1}{h} \| x^k - x^{k+1} \| + L \| x^k - x^{k+1} \| + \frac{\| e^k \|}{h}. \tag{4.15}
\]
\]
Lemma 9. Let \(0 < h < \frac{1}{2}\) and the sequence \((x^k)_{k \geq 0}\) be generated by algorithm [4,5], and \(F\) be coercive. We also assume that \(e^k \to 0\). Then, for \(x^*\) being the stationary point of \((x^k)_{k \geq 0}\), there exists a subsequence \((x^{k_j})_{j \geq 0}\) converges to \(x^*\) satisfying \(F(x^{k_j}) \to F(x^*)\) and \(x^* \in \text{crit}(F)\).
Proof. With Lemma 7, \((x^k)_{k \geq 0}\) is bounded. For any \(x^* \in \text{crit}(F)\), there exists a subsequence \((x^{k_j})_{j \geq 0}\) converges to \(x^*\). With Lemmas 4 and 7, we also have
\[
x^{k_j-1} \to x^*. \tag{4.16}
\]
And in each iteration, with Lemma 6, we have
\[
hg(x^{k_j}) + \frac{\| x^{k_j-1} - h \nabla f(x^{k_j-1}) + e^{k_j} - x^{k_j} \|^2}{2} \leq hg(x^*) + \frac{\| x^{k_j-1} - h \nabla f(x^{k_j-1}) + e^{k_j-1} - x^* \|^2}{2}. \tag{4.17}
\]
Taking \(j \to +\infty\), we have
\[
\limsup_{j \to +\infty} g(x^{k_j}) \leq g(x^*). \tag{4.18}
\]
And recalling the lower semi-continuity of \(g\),
\[
g(x^*) \leq \liminf_{j \to +\infty} g(x^{k_j}). \tag{4.19}
\]
That means \(\lim g(x^{k_j}) = g(x^*)\); and combining the continuity of \(f\), we then prove the result.

And then, we then prove the following result.

Theorem 2. Suppose that \(f\) and \(g\) are both semi-algebraic, \(F\) is coercive, and \(0 < h < \frac{1}{2}\). Let the sequence \((x^k)_{k \geq 0}\) be generated by scheme [4,5]. If the sequence \((e^k)_{k \geq 0}\) satisfies
\[
\| e^k \| = O \left( \frac{1}{k^\alpha} \right), \alpha > 1. \tag{4.20}
\]
Then, the sequence \((x^k)_{k \geq 0}\) has finite length, i.e.
\[
\sum_{k=0}^{+\infty} \| x^{k+1} - x^k \| < +\infty. \tag{4.21}
\]
And \(\{x^k\}_{k=0,1,2,3,\ldots}\) converges to a critical point \(x^*\) of \(F\).
Proof. From (4.20), we have \(\| e^k \| \to 0\). And \(F\) is a semi-algebraic function. With lemmas proved before in this subsection and Theorem 1, we then obtain the result.
4.2 Inexact proximal linearized alternating minimization algorithm

In this part, we use the convention

\[ x = (y, z), x^k = (y^k, z^k), e^k = (a^k, b^k) \]

The following problem is considered

\[ \min_{y, z} \{ \Phi(y, z) := f(y) + H(y, z) + g(z) \}, \quad (4.22) \]

where the function \( H \) is assumed to be differentiable and satisfy

\[
\begin{align*}
\| \nabla_y H(y^1, z) - \nabla_y H(y^2, z) \| & \leq M(z) \| y^1 - y^2 \|, 0 < \inf M(z) \leq \sup M(z) < +\infty, \\
\| \nabla_z H(y, z^1) - \nabla_z H(y, z^2) \| & \leq N(y) \| z^1 - z^2 \|, 0 < \inf N(y) \leq \sup N(y) < +\infty, \\
\| \nabla_y H(x^1) - \nabla_y H(x^2) \| & \leq L(x^1, x^2) \| x^1 - x^2 \|, 0 < \inf L(x^1, x^2) \leq \sup L(x^1, x^2) < +\infty.
\end{align*}
\]

An intuitive algorithm for solving problem \((4.22)\) is the alternating minimization scheme, i.e., fixing one of \( y \) and \( z \) in each iteration and then minimizing the other one \([26]\); and the convergence rate is proved in \([3] \) in the convex case. In the nonconvex case, the alternating minimization scheme can barely derive the descent property, thus the authors propose the proximal alternating minimization \([4]\). However, both alternating minimization and proximal alternating minimization have an obvious drawback: both algorithms need to solve a minimization problem in each iteration, the stopping criterion is hard to determine, and error accumulates. Therefore, several variants are developed \([9, 33, 30]\), and the Proximal Linearized Alternating Minimization (PLAM) algorithm \([9]\) is one of them. The inexact PLAM can be described as

\[
\begin{align*}
y^{k+1} &= \text{prox}_{\gamma_k f}(y^k - \gamma_k \nabla_y H(y^k, z^k) + a^k), \\
z^{k+1} &= \text{prox}_{\lambda_k g}(z^k - \lambda_k \nabla_z H(y^{k+1}, z^k) + b^k). \quad (4.24a, 4.24b)
\end{align*}
\]

Lemma 10. Let the sequence \((x^k)_{k \geq 0}\) be generated by algorithm \((4.24)\). If

\[ \inf \{ M(z^k) - \frac{1}{\gamma_k}, N(y^{k+1}) - \frac{1}{\lambda_k} \} > 0, \quad (4.25) \]

we have

\[ \Phi(x^k) - \Phi(x^{k+1}) \geq \nu \| x^{k+1} - x^k \|^2 - \sigma \| e^k \|^2, \quad (4.26) \]

where \( \nu = \inf_k \{ \frac{1}{4} (M(z^k) - \frac{1}{\gamma_k}), \frac{1}{4} (N(y^{k+1}) - \frac{1}{\lambda_k}) \} \) and \( \sigma = \sup_k \{ \frac{1}{\lambda_k (1 - \gamma_k M(z^k))}, \frac{1}{\gamma_k (1 - \lambda_k N(y^{k+1}) - 1) \lambda_k N(y^{k+1})} \} \)

Proof. The \( L(z^k) \)-Lipschitz of \( \nabla_y H(y, z^k) \) gives

\[ H(y^{k+1}, z^k) - H(y^k, z^k) \leq \langle \nabla_y H(y^k, z^k), y^{k+1} - y^k \rangle + \frac{M(z^k)}{2} \| y^{k+1} - y^k \|^2. \quad (4.27) \]

From Lemma \([6]\) we have

\[
\gamma_k f(y^{k+1}) + \frac{\| y^k - \gamma_k \nabla_y H(y^k, z^k) + a^k - y^{k+1} \|^2}{2} \leq \gamma_k f(y^k) + \frac{\| - \gamma_k \nabla_y H(y^k, z^k) + a^k \|^2}{2}. \quad (4.28)
\]

This is also

\[ f(y^{k+1}) - f(y^k) \leq - \langle \nabla_y H(y^k, z^k), y^{k+1} - y^k \rangle - \frac{\| y^k - y^{k+1} \|^2}{2 \gamma_k} + \frac{\langle a^k, y^{k+1} - y^k \rangle}{\gamma_k}. \quad (4.29) \]

Summing \((4.27)\) and \((4.29)\), with the Cauchy-Schwarz inequality

\[
\frac{\langle a^k, y^{k+1} - y^k \rangle}{\gamma_k} \leq \frac{1}{4} \left( \frac{1}{\gamma_k} - M(z^k) \right) \| y^{k+1} - y^k \|^2 + \frac{1}{\gamma_k (1 - \gamma_k M(z^k))} \| a^k \|^2. \quad (4.30)
\]
we then have
\[ f(y^{k+1}) + H(y^{k+1}, z^k) \leq \frac{1}{4}(M(z^k) - \frac{1}{\gamma_k})\|y^{k+1} - y^k\|^2 + \frac{\|\alpha_k\|^2}{\gamma_k(1 - \gamma_k M(z^k))}. \quad (4.31) \]

Similarly, we can prove
\[ g(z^{k+1}) + H(y^{k+1}, z^{k+1}) \leq \frac{1}{4}(N(y^{k+1}) - \frac{1}{\lambda_k})\|z^{k+1} - z^k\|^2 + \frac{\|\beta_k\|^2}{\lambda_k(1 - \lambda_k N(y^{k+1}))}. \quad (4.32) \]

Combining (4.11) and (4.10), we then prove the result.

**Lemma 11.** Let the sequence be generated by algorithm (4.24) and the following condition hold
\[ \sup\{\frac{1}{\lambda_k}, \frac{1}{\gamma_k}\} < +\infty, \quad (4.33) \]
we have
\[ \text{dist}(0, \partial \Phi(x^{k+1})) \leq S\|x^k - x^{k+1}\| + D\|e^k\|, \quad (4.34) \]
where \( S = \sup\{\frac{1}{\lambda_k} + \frac{1}{\gamma_k} + L(x^k, x^{k+1}) + L(y^{k+1})\} \) and \( D = \sup\{\sqrt{\frac{1}{\gamma_k} + \frac{1}{\lambda_k}}\}. \]

**Proof.** In updating \( y^{k+1} \), we have
\[ \frac{y^k - y^{k+1}}{\gamma_k} - \nabla_y H(y^k, z^k) + \frac{\alpha_k}{\gamma_k} \in \partial f(y^{k+1}). \quad (4.35) \]

Therefore,
\[ \frac{y^k - y^{k+1}}{\gamma_k} + \nabla_y H(y^{k+1}, z^{k+1}) - \nabla_y H(y^k, z^k) + \frac{\alpha_k}{\gamma_k} \in \nabla_y H(y^{k+1}, z^{k+1}) + \partial f(y^{k+1}) = \partial_y \Phi(x^{k+1}). \quad (4.36) \]
Thus, we have
\[ \text{dist}(0, \partial_y \Phi(x^{k+1})) \leq \|\frac{y^k - y^{k+1}}{\gamma_k} + \nabla_y H(y^{k+1}, z^{k+1}) - \nabla_y H(y^k, z^k) + \frac{\alpha_k}{\gamma_k}\| \]
\[ \leq \frac{\|y^k - y^{k+1}\|}{\gamma_k} + L(x^{k+1}, x^k)\|x^{k+1} - x^k\| + \frac{\|\alpha_k\|}{\gamma_k}. \quad (4.37) \]

In updating \( z^{k+1} \), we have
\[ \text{dist}(0, \partial_z \Phi(x^{k+1})) \leq \|\frac{z^k - z^{k+1}}{\lambda_k} + L(y^{k+1})\|z^{k+1} - z^k\| + \frac{\|\beta_k\|}{\lambda_k}. \quad (4.38) \]

Combining (4.37) and (4.38), we then prove the result. □

**Lemma 12.** Let the sequence \((x^k)_{k \geq 0}\) be generated by algorithm (4.5), and \( \Phi \) be coercive, and condition [4.25] hold, \( e^k \to 0 \). Then, for any \( x^* \) being the stationary point of \((x^k)_{k \geq 0}\), there exists a subsequence \((x^*)_{j \geq 0}\) converges to \( x^* \) satisfying \( \Phi(x^*) = \Phi(x^k) \to \Phi(x^*) \) and \( x^* \in \text{crit}(\Phi) \).

**Proof.** With Lemma 10, \((x^k)_{k \geq 0}\) is bounded. For any \( x^* \in \text{crit}(\Phi) \), there exists a subsequence \((x^{k_j})_{j \geq 0}\) converges to \( x^* \). With Lemmas 4 and 10 we also have
\[ x^{k_j - 1} = (y^{k_j - 1}, z^{k_j - 1}) \to x^* = (y^*, z^*). \quad (4.39) \]

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And in each iteration of updating $y^{k_j}$, with Lemma 6 we have

$$\gamma_k f(y^{k_j}) + \frac{\|y^{k_j-1} - \gamma_k z^{k_j-1} - \nabla_y H(y^{k_j-1}, z^{k_j-1}) + \alpha^{k_j-1} - y^{k_j}\|^2}{2} \leq \gamma_k f(y^{k_j-1}) + \frac{\|y^{k_j-1} - \gamma_k z^{k_j-1} - \nabla_y H(y^{k_j-1}, z^{k_j-1}) + \alpha^{k_j-1}\|^2}{2}. \quad (4.40)$$

Taking $j \to +\infty$, we have

$$\limsup_{j \to +\infty} f(y^{k_j}) \leq f(y^*). \quad (4.41)$$

And recalling the lower semi-continuity of $f$,

$$f(y^*) \leq \liminf_{j \to +\infty} f(y^{k_j}). \quad (4.42)$$

That means $\lim f(y^{k_j}) = f(x^*)$; and similarly, $\lim g(z^{k_j}) = g(z^*)$; combining the continuity of $H$, we then prove the result. \hfill \Box

And then, we then prove the following result.

**Theorem 3.** Suppose that $\Phi$ is coercive, and conditions (4.25) and (4.33) hold. Functions $f$, $g$ and $H$ are all semi-algebraic. Let the sequence $(x^k)_{k \geq 0}$ be generated by scheme (4.24). If the sequence $(\alpha^k, \beta^k)_{k \geq 0}$ satisfies

$$\|\alpha^k\| + \|\beta^k\| = O\left(\frac{1}{k^\alpha}\right), \alpha > 1. \quad (4.43)$$

Then, the sequence $(x^k)_{k \geq 0}$ has finite length, i.e.

$$\sum_{k=0}^{+\infty} \|x^{k+1} - x^k\| < +\infty. \quad (4.44)$$

And $(x^k)_{k=0,1,2,3,...}$ converges to a critical point $x^*$ of $\Phi$.

### 4.3 Inexact proximal reweighted algorithm

This part considers an iteratively reweighted algorithm for a broad class of nonconvex and nonsmooth problems with the following form

$$\min_x \{\Psi(x) = f(x) + \sum_{i=1}^{N} h(g(x_i))\}, \quad (4.45)$$

where $x \in \mathbb{R}^N$, and functions $f$ has a Lipschitz gradient with constant $L_f$, and $g : \mathbb{R} \to \mathbb{R}$ is a lower-semicontinuous convex function, and $h : \text{Im}(g) \to \mathbb{R}$ is a differentiable concave function with a Lipschitz continuous gradient with constant $L_h$, i.e.,

$$|h'(s) - h'(t)| \leq L_h |s - t|, \quad (4.46)$$

and $h'(t) > 0$ for any $t \in \text{Im}(g)$. This model generalizes various problems in the machine learning and signal processing satisfy. The reweighted style algorithms [11, 10, 13, 19, 34, 24, 12] (or also called multi-stage algorithm [10]) are popular in solving this problem. To make each subproblem easy to be solved. The Proximal Iteratively REweighted (PIRE) algorithm is proposed in [23]. The convergence of PIRE under KL property is proved by [35]. We consider the inexact version of PIRE as

$$x_i^{k+1} = \text{prox}_{\beta_i^k}(x_i^k - \mu \nabla_i f(x_i^k) + e_i^k), \quad i \in [1, 2, \ldots, N] \quad (4.47)$$

where $w_i^k := h'(g(x_i^k))$ and $\mu > 0$ is the stepsize, $e_i^k$ is the noise vector. If $e_i^k \equiv 0$, the algorithm then reduces to PIRE.
Lemma 13. Let \((x^k)_{k \geq 0}\) be generated by scheme (4.47) and \(0 < \mu < \frac{2}{L_f}\). Then, we will have
\[
\Psi(x^k) - \Psi(x^{k+1}) \geq \left(\frac{1}{\mu} - \frac{L_f}{2}\right)\|x^k - x^{k+1}\|^2 - \frac{\|e^k\|^2}{\mu(2 - \mu L_f)}.
\] (4.48)

Proof. We can easily obtain that
\[
\Psi(x^k) - \Psi(x^{k+1}) = f(x^k) - f(x^{k+1}) + \sum_{i=1}^{N} h(g(x^k_i)) - h(g(x^{k+1}_i))
\]
\[
\geq \langle \nabla f(x^k), x^k - x^{k+1} \rangle - \frac{L_f}{2}\|x^k - x^{k+1}\|^2 + \sum_{i=1}^{N} h(g(x^k_i)) - h(g(x^{k+1}_i))
\]
\[
\geq \sum_{i=1}^{N} \langle \nabla_i f(x^k), x_i^k - x_i^{k+1} \rangle - \frac{L_f}{2}\|x^k - x^{k+1}\|^2
\]
\[
+ \sum_{i=1}^{N} h'(g(x^k_i))(g(x^k_i) - g(x^{k+1}_i)).
\] (4.49)

Note that \(x_i^{k+1}\) is obtained by (4.47); the K.K.T condition gives
\[
\nabla_i f(x^k) + w_i^k v_i^{k+1} + \left(\frac{x_i^{k+1} - x_i^k}{\mu} - \frac{e_i^k}{\mu}\right) = 0,
\] (4.50)
where \(v_i^{k+1} \in \partial g(x_i^{k+1})\). Note that \(g\) is convex, we have that
\[
\sum_{i=1}^{N} h'(g(x^k_i))(g(x^k_i) - g(x^{k+1}_i)) \geq \sum_{i=1}^{N} (w_i^k v_i^{k+1}, x_i^k - x_i^{k+1}).
\] (4.51)

Substituting (4.50) and (4.51) into (4.49), we derive that
\[
\Psi(x^k) - \Psi(x^{k+1}) \geq \left(\frac{1}{\mu} - \frac{L_f}{2}\right)\|x^k - x^{k+1}\|^2 + \frac{\langle e^k, x^k - x^{k+1} \rangle}{\mu}
\]
\[
\geq \frac{1}{2} \left(\frac{1}{\mu} - \frac{L_f}{2}\right)\|x^k - x^{k+1}\|^2 - \frac{\|e^k\|^2}{\mu(2 - \mu L_f)},
\] (4.52)
where we use the inequality \(\langle e^k, x^k - x^{k+1} \rangle \geq -\frac{1}{2}\left(1 - \frac{\mu L_f}{2}\right)\|x^k - x^{k+1}\|^2 - \frac{\|e^k\|^2}{2 - \mu L_f}\).

Lemma 14. Let \((x^k)_{k \geq 0}\) be generated by scheme (4.47) and \(0 < \mu < \frac{2}{L_f}\), and function \(\Phi\) be coercive. Then, there exist \(S, D > 0\) such that
\[
\text{dist}(0, \partial \Phi(x^{k+1})) \leq S\|x^{k+1} - x^k\| + D\|e^k\|.
\] (4.53)

Proof. We can easily have that
\[
\nabla f(x^{k+1}) + W^{k+1} v^{k+1} \in \partial \Phi(x^{k+1}),
\] (4.54)
where \(v_i^{k+1} \in \partial g(x_i^{k+1})\) and \(W^{k+1} = \text{diag}(h'(g(x_1^k)), h'(g(x_2^k)), \ldots, h'(g(x_N^k)))\). We employ relation (4.50). Then, we have that
\[
v_i^{k+1} = \left[\frac{e_i^k}{\mu} + \frac{x_i^k - x_i^{k+1}}{\mu}\right] - \nabla_i f(x^k)/w_i^k, i \in [1, 2, \ldots, N].
\] (4.55)
Combining (4.54) and (4.55), we have
\[
\frac{w^{k+1}_i - w^k_i}{w^k_i} \nabla_i f(x^{k+1}) + \frac{w^{k+1}_i}{w^k_i} \left[ \frac{e^k_i}{\mu} + \frac{(x^k_i - x^{k+1}_i)}{\mu} \right] \in \partial_i \Psi(x^{k+1}), i \in \{1, 2, \ldots, N\}.
\] (4.56)

In view of that \(\nabla f\) is continuous, so is \(\nabla_i f(x)\); and from Lemmas 13 and 4 \((x^k)_{k=0,1,2,...}\) is bounded. Hence, there exist \(\bar{L} > 0\) such that
\[
\max_{1 \leq i \leq N} ||\nabla_i f(x)|| \leq \bar{L}.
\] (4.57)

Considering that \(h'\) is nonzero and continuous, and \((g(x^k))_{k=0,1,2,...}\) is bounded \((i \in \{1, 2, \ldots, N\})\). With [Theorem 10.4, 28] and the convexity of \(g\), there exists \(d_g\)
\[
|g(x^k) - g(x^{k+1})| \leq d_g|x^k_i - x^{k+1}_i|.
\] (4.58)

Therefore, for any \(k\) and \(i \in \{1, 2, \ldots, N\}\), there exists \(\delta, \pi > 0\) such that
\[
\delta \leq h'(g(x^k)) = w^k_i \leq \pi.
\]

Hence, we derive that
\[
\max_{1 \leq i \leq N} |\frac{w^{k+1}_i}{w^k_i}| \leq \frac{\pi}{\delta} \max_{1 \leq i \leq N} |\frac{1}{w^k_i}| \leq \frac{1}{\delta}.
\] (4.59)

From (4.56), with (4.59), we have
\[
\text{dist}(0, \partial \Psi(x^{k+1})) \leq \sum_{i=1}^{N} \left| \frac{w^{k+1}_i - w^k_i}{w^k_i} \nabla_i f(x^{k+1}) + \frac{w^{k+1}_i}{w^k_i} \left[ \frac{e^k_i}{\mu} + \frac{(x^k_i - x^{k+1}_i)}{\mu} \right] \right|
\leq \bar{L} \sum_{i=1}^{N} |w^{k+1}_i - w^k_i| + \frac{\pi \sqrt{N}}{\mu \delta} \|e^k\| + \frac{\pi \sqrt{N}}{\mu \delta} \|x^{k+1} - x^k\|.
\] (4.60)

The problem also turns to estimating \(|w^{k+1}_i - w^k_i|\). For any \(i \in \{1, 2, \ldots, N\}\),
\[
w^k_i - w^{k+1}_i = \frac{h'(g(x^k)) - h'(g(x^{k+1}))}{L_h |g(x^k) - g(x^{k+1})|} = L_h d_g |x^{k+1}_i - x^k_i|.
\] (4.61)

Combining (4.60) and (4.61), we obtain
\[
\text{dist}(0, \partial \Psi(x^{k+1})) \leq \left( \frac{\bar{L} L_h d_g \sqrt{N}}{\delta} + \frac{\pi \sqrt{N}}{\mu \delta} \right) \|x^{k+1} - x^k\| + \frac{\pi \sqrt{N}}{\mu \delta} \|e^k\|.
\] (4.62)

Lemma 15. Let \((x^k)_{k \geq 0}\) be generated by scheme (4.47), and function \(\Psi\) be coercive. Then, for any \(x^*\) being the stationary point of \((x^k)_{k \geq 0}\), there exists a subsequence \((x^{k_j})_{j \geq 0}\) converges to \(x^*\) satisfying \(\Psi(x^{k_j}) \rightarrow \Psi(x^*)\) and \(x^* \in \text{crit}(\Psi)\).

Proof. The continuity of the function \(\Psi\) directly gives the result.

Theorem 4. Suppose that \(f, g, h\) are all semi-algebraic, and \(\Psi\) is coercive, and \(0 < \mu < \frac{2}{\bar{L} f}\). Let the sequence \((x^k)_{k \geq 0}\) be generated by scheme (4.47). If the sequence \((e^k)_{k \geq 0}\) satisfies
\[
\|e^k\| = \mathcal{O}(\frac{1}{k^\alpha}), \alpha > 1.
\] (4.63)

Then, the sequence \((x^k)_{k \geq 0}\) has finite length, i.e.
\[
\sum_{k=0}^{+\infty} \|x^{k+1} - x^k\| < +\infty.
\] (4.64)

And \((x^k)_{k=0,1,2,3,...}\) converges to a critical point \(x^*\) of \(\Psi\).
4.4 Inexact DC algorithm

In this part, we consider nonconvex optimization problems of the following type

\[
\min \{ \Xi(x) = f(x) + g(x) - h(x) \},
\]  

(4.65)

where \( g \) is proper and lower semicontinuous, \( f \) is differentiable with \( L_f \)-Lipschitz gradient, and \( h \) is convex and differentiable with \( L_h \)-Lipschitz gradient. Such a problem is discussed in [25]. If \( f \) vanishes, problem (4.65) will reduce to the DC programming [37]

\[
\min \{ g(x) - h(x) \}.
\]  

(4.66)

A novel DC algorithm is proposed in [2] for (4.65) and the convergence is also proved. The inexact version of this algorithm can be expressed as

\[
x^{k+1} \in \text{prox}_{\gamma g} (x^k - \gamma (\nabla f(x^k) - \nabla h(x^k)) + e^k),
\]  

(4.67)

where \( \gamma \) is the stepsize, and \( e^k \) is the noise. The cautious reader may find that iteration (4.67) is actually a special case of (4.5) if regarding \( f - h \) as a whole. But with the specific structure, iteration (4.67) enjoys more properties than (4.5), like larger stepsize. It is easy to see that \( \nabla (f - h) = \nabla f - \nabla h \) is Lipchitz with \( L_f + L_h \). If directly using the convergence results for (4.5) (Theorem 2), the stepsize \( \gamma \) shall satisfy \( \gamma \leq \frac{1}{L_f + L_h} \). However, a larger step can be selected for iteration (4.67); the stepsize can be \( \gamma < \frac{1}{L_f} \) (Lemma 16).

**Lemma 16.** Let \( (x^k)_{k \geq 0} \) be generated by scheme (4.67) and \( 0 < \gamma < \frac{1}{L_f} \). Then, we will have

\[
\Xi(x^k) - \Xi(x^{k+1}) \geq \left( \frac{1}{\gamma} - \frac{L_f}{2} \right) \| x^k - x^{k+1} \|^2 - \frac{\| e^k \|^2}{\gamma(2 - \gamma L_f)}.
\]  

(4.68)

Direct computations yield

\[
\Xi(x^k) - \Xi(x^{k+1}) = f(x^k) - f(x^{k+1}) + g(x^k) - g(x^{k+1}) + h(x^{k+1}) - h(x^k)
\geq \langle \nabla f(x^k), x^k - x^{k+1} \rangle - \frac{L_f}{2} \| x^k - x^{k+1} \|^2 + g(x^k) - g(x^{k+1}) + \langle x^{k+1} - x^k, \nabla h(x^k) \rangle.
\]  

(4.69)

On the other hand, with Lemma 6 we have

\[
\gamma g(x^{k+1}) + \frac{\| x^k - \gamma (\nabla f(x^k) - \nabla h(x^k)) + e^k - x^{k+1} \|^2}{2} \leq \gamma g(x^k) + \frac{\| \gamma (\nabla f(x^k) - \nabla h(x^k)) + e^k \|^2}{2}.
\]  

(4.70)

Combining (4.69) and (4.70), we derive that

\[
\Xi(x^k) - \Xi(x^{k+1}) \geq \left( \frac{1}{\gamma} - \frac{L_f}{2} \right) \| x^k - x^{k+1} \|^2 + \frac{\langle e^k, x^k - x^{k+1} \rangle}{\gamma} \geq \frac{1}{2} \left( \frac{1}{\gamma} - \frac{L_f}{2} \right) \| x^k - x^{k+1} \|^2 - \frac{\| e^k \|^2}{\gamma(2 - \gamma L_f)},
\]  

(4.71)

where we use the inequality \( \langle e^k, x^k - x^{k+1} \rangle \geq -\frac{1}{2} (1 - \frac{\gamma L_f}{2}) \| x^k - x^{k+1} \|^2 - \frac{\| e^k \|^2}{2 - \gamma L_f} \).

**Lemma 17.** Let \( (x^k)_{k \geq 0} \) be generated by scheme (4.67). Then, there exist \( S, D > 0 \) such that

\[
\text{dist}(0, \partial \Xi(x^{k+1})) \leq S \| x^{k+1} - x^k \| + D \| e^k \|.
\]  

(4.72)
Proof. With scheme of the algorithm,
\[
\frac{x^k - x^{k+1}}{\gamma} - \nabla f(x^k) + \frac{e_k}{\gamma} + \nabla h(x^k) \in \partial g(x^{k+1}).
\] (4.73)
Thus, we have
\[
\frac{x^k - x^{k+1}}{\gamma} + \nabla f(x^{k+1}) - \nabla f(x^k) + \frac{e_k}{\gamma} + \nabla h(x^k) - \nabla h(x^{k+1}) \in \partial \Xi(x^{k+1}).
\] (4.74)
Hence,
\[
dist(0, \partial \Xi(x^{k+1})) \leq \left| \frac{x^k - x^{k+1}}{\gamma} + \nabla f(x^{k+1}) - \nabla f(x^k) + \frac{e_k}{\gamma} + \nabla h(x^k) - \nabla h(x^{k+1}) \right|
\]
\[
\leq \left( \frac{1}{\gamma} + L_f + L_h \right) \|x^k - x^{k+1}\| + \frac{1}{\gamma} \|e^k\|.
\] (4.75)
\[\square\]

**Lemma 18.** Let \((x^k)_{k \geq 0}\) is generated by scheme \((4.67)\) and \(\frac{1}{\gamma} > \frac{L_f}{2}\), and \(\Xi\) be coercive, and \(e^k \to 0\). Then, for \(x^*\) being the stationary point of \((x^k)_{k \geq 0}\), there exists a subsequence \((x^{k_j})_{j \geq 0}\) converges to \(x^*\) satisfying \(\Xi(x^{k_j}) \to \Xi(x^*)\) and \(x^* \in \text{crit}(\Xi)\).

Proof. With Lemma \[16\] \((x^k)_{k \geq 0}\) is bounded. For any \(x^* \in \text{crit}(\Phi)\), there exists a subsequence \((x^{k_j})_{j \geq 0}\) converges to \(x^*\). With Lemmas \[4\] and \[16\] we also have
\[
x^{k_j-1} \to x^*.
\] (4.76)
And in each iteration of updating \(x^{k_j}\), with Lemma \[6\] we have
\[
\gamma g(x^{k_j-1}) + \frac{\|x^{k_j} - \gamma(\nabla f(x^{k_j-1}) - \nabla h(x^{k_j-1}) + e^{k_j-1} - x^{k_j})\|^2}{2}
\]
\[
\leq \gamma g(x^{k_j-1}) + \frac{\| - \gamma(\nabla f(x^{k_j-1}) - \nabla h(x^{k_j-1}) + e^{k_j-1})\|^2}{2}.
\] (4.77)
Taking \(j \to +\infty\), we have
\[
\limsup_{j \to +\infty} g(x^{k_j}) \leq g(x^*).
\] (4.78)
And recalling the lower semi-continuity of \(g\),
\[
g(x^*) \leq \liminf_{j \to +\infty} g(x^{k_j}).
\] (4.79)
That means \(\lim_j g(x^{k_j}) = g(x^*)\); combining the continuity of \(f\) and \(h\), we then prove the result. \[\square\]

**Theorem 5.** Let \((x^k)_{k \geq 0}\) be generated by scheme \((4.67)\). Functions \(f, g\) and \(h\) are all semi-algebraic. And the stepsize satisfies \(0 < \gamma < \frac{2}{L_f}\), and \(\Xi\) is coercive, and
\[
\|e^k\| = \mathcal{O}\left(\frac{1}{k^\alpha}\right), \alpha > 1.
\]
Then, the sequence \((x^k)_{k \geq 0}\) has finite length, i.e.
\[
\sum_{k=0}^{+\infty} \|x^{k+1} - x^k\| < +\infty.
\] (4.80)
And \((x^k)_{k=0,1,2,3,\ldots}\) converges to a critical point \(x^*\) of \(\Xi\).
4.5 Inexact nonconvex ADMM algorithm

Alternating Direction Method of Multipliers (ADMM) \cite{15, 16} is a powerful tool for the minimization of composite functions with linear constraints. An inexact nonconvex ADMM scheme is considered for the composite optimization

\[
\min_{x, y} \{ f(x) + g(y), \text{ s.t. } x + y = 0 \} \tag{4.81}
\]

where \( g \) is differentiable with \( L_g \)-Lipschitz gradient and convex. We consider the following inexact algorithm as

\[
\begin{align*}
  x^{k+1} &= \mathbf{prox}_f \left( -y^k - r\gamma^k + e_1^k \right), \\
  y^{k+1} &= \mathbf{prox}_g \left( -x^{k+1} - \frac{\gamma^k}{\beta} + e_2^k + e_2^{k+1} \right), \\
  \gamma^{k+1} &= \gamma^k + \beta (x^{k+1} + y^{k+1}),
\end{align*}
\tag{4.82}
\]

where the augmented Lagrangian function \( L_\beta \) is defined as

\[
L_\beta(x, y, \gamma) = f(x) + g(y) + \langle \gamma, x + y \rangle + \frac{\beta}{2} \| x + y \|^2,
\tag{4.83}
\]

where \( \gamma \) is the Lagrangian dual variable. If \( e_1^k \equiv 0 \) and \( e_2^k \equiv 0 \), the scheme is the standard ADMM.

Nonconvex ADMM has been frequently studied in recent years \cite{39, 20, 21, 32, 36, 1, 17}. First, we prove a critical lemma.

**Lemma 19.** Let \((x^k, y^k, \gamma^k)_{k \geq 0}\) be generated by (4.82), we then have

\[
\| \gamma^{k+1} - \gamma^k \|^2 \leq \rho_1 \| y^{k+1} - y^k \|^2 + \rho_2 \| e_2^{k+1} - e_2^k \|^2,
\tag{4.84}
\]

where \( \rho_1 = 2L_g^2 \) and \( \rho_2 = 2\beta^2 \).

**Proof.** The second step of each iteration gives

\[
\nabla g(y^{k+1}) = -[\gamma^k + \beta(x^{k+1} + y^{k+1})] + \beta e_2^{k+1}.
\tag{4.85}
\]

With the fact \( \gamma^{k+1} = \gamma^k + \beta(x^{k+1} + y^{k+1}) \), we then have

\[
\nabla g(y^{k+1}) = -\gamma^{k+1} + \beta e_2^{k+1}.
\tag{4.86}
\]

Substituting \( k + 1 \) with \( k \),

\[
\nabla g(y^k) = -\gamma^k + \beta e_2^k.
\tag{4.87}
\]

Substraction of the two equalities above yield

\[
\| \gamma^{k+1} - \gamma^k \| \leq L_g \| y^{k+1} - y^k \| + \beta \| e_2^{k+1} - e_2^k \|.
\tag{4.88}
\]

We define an auxiliary point as

\[
d^k := (x^k, y^k, \gamma^k, y^{k-1}), \omega^k := (x^k, y^k), \varepsilon^k = \begin{pmatrix} e_2^{k+1} - e_2^k \\ e_2^{k+1} \\ e_2^k \end{pmatrix}
\]

and the Lyapunov function as

\[
F(d) = F(x, y, \gamma) := L_\beta(x, y, \gamma).
\]

In the following, we prove the conditions for \( F \).

---

\footnote{The result can be easily extended to a more general constraint \( Ax + By = c \). Here, we consider this case just for the simplicity of presentation.}
Lemma 20. If $\sum_k \|\omega^k - \omega^{k+1}\| < +\infty$, we have $\sum_k \|d^k - d^{k+1}\| < +\infty$.

Proof. Direct basic algebraic computation gives the result. \qed

Lemma 21. Let $(d^k)_{k \geq 0}$ is generated by scheme (4.82) and $f$ is convex,

\[ \beta > \frac{\sqrt{\rho_1}}{2} = \frac{\sqrt{2}}{2} L g, 0 < r < \frac{1}{\beta}. \] (4.89)

Then, we will have

\[ F(d^k) - F(d^{k+1}) \geq \nu \|\omega^k - \omega^{k+1}\|^2 - \rho \|e^k\|^2 \] (4.90)

for some $\nu, \rho > 0$.

Proof. Note that $y^{k+1}$ is the minimizer of

\[ \hat{L}_\beta(x^{k+1}, y, \gamma^k) := g(y) + \langle \gamma^k, x^{k+1} + y \rangle + \frac{\beta}{2} \|x^{k+1} + y - e^{k+1}\|^2, \]

and $\hat{L}_\beta(x^{k+1}, y, \gamma^k)$ is strongly convex with constant $\beta$. Thus, we have

\[ \hat{L}_\beta(x^{k+1}, y^{k+1}, \gamma^k) + \frac{\beta}{2} \|y^{k+1} - y^k\|^2 \leq \hat{L}_\beta(x^{k+1}, y^k, \gamma^k). \]

After simplifications, we then derive

\[ L_\beta(x^{k+1}, y^{k+1}, \gamma^k) + \frac{\beta}{2} \|y^{k+1} - y^k\|^2 \leq L_\beta(x^{k+1}, y^k, \gamma^k) + \beta \|e_2^{k+1}\|^2. \] (4.91)

By using the inequality

\[ \beta \|e_2^{k+1}\|^2 \leq \beta \|e_2^{k+1}\|^2 + \frac{\beta}{4} \|y^{k+1} - y^k\|^2. \] (4.92)

With (4.91), we have

\[ L_\beta(x^{k+1}, y^{k+1}, \gamma^k) + \frac{\beta}{4} \|y^{k+1} - y^k\|^2 \leq L_\beta(x^{k+1}, y^k, \gamma^k) + \beta \|e_2^{k+1}\|^2. \] (4.93)

Similarly, we have

\[ L_\beta(x^{k+1}, y^k, \gamma^k) \leq L_\beta(x^k, y^k, \gamma^k) + \frac{\beta - 1}{4} \|x^{k+1} - x^k\|^2 + \frac{1}{(1 - \beta r)^\frac{1}{r}} \|e_{1}^{k+1}\|^2. \] (4.94)

With Lemma 19

\[ L_\beta(x^{k+1}, y^{k+1}, \gamma^{k+1}) = L_\beta(x^{k+1}, y^{k+1}, \gamma^k) + \frac{\|\gamma^{k+1} - \gamma^k\|^2}{\beta} \leq L_\beta(x^{k+1}, y^{k+1}, \gamma^k) + \frac{\rho_1}{\beta} \|y^{k+1} - y^k\|^2 + \frac{\rho_2}{\beta} \|e_2^{k+1} - e_2^k\|^2. \] (4.95)

Thus, we have

\[ F(d^{k+1}) + \left( \frac{1}{4r} - \frac{\beta}{4} \right) \|x^{k+1} - x^k\|^2 + \left( \frac{\beta}{4} - \frac{\rho_1}{\beta} \right) \|y^{k+1} - y^k\|^2 \]

\[ - \max \{\beta, \frac{1}{(1 - \beta r)^\frac{1}{r}}\} \|e_k\|^2 \leq F(d^k). \] (4.96)

Letting $\nu := \min \left\{ \frac{\beta}{4} - \frac{\rho_1}{\beta}, \frac{1}{4r} - \frac{\beta}{4} \right\}$ and $\rho := \max \{\beta, \frac{1}{(1 - \beta r)^\frac{1}{r}}, \frac{\rho_2}{\beta}\}$, we then prove the result. \qed
Lemma 22. Let \((w^k)_{k \geq 0}\) is generated by scheme \([4.82]\). Then, there exist \(S, D > 0\) such that
\[
\text{dist}(0, \partial F(d^{k+1})) \leq S\|\omega^{k+1} - \omega^k\| + D\|\varepsilon^k\|. \tag{4.97}
\]

Proof. From Lemma \([19]\) we have
\[
\|\gamma^{k+1} - \gamma^k\| \leq \sqrt{\rho_1}\|y^k - y^{k+1}\| + \sqrt{\rho_2}\|e_2^{k+1} - e_2^k\|. \tag{4.98}
\]
The optimization condition for updating \(x^{k+1}\) is
\[
\frac{x^k - x^{k+1}}{r} - \gamma^{k+1} + \frac{e_1^{k+1}}{r} \in \partial f(x^{k+1}). \tag{4.99}
\]
With direct calculation, we have
\[
\partial_x F(d^{k+1}) = \partial f(x^{k+1}) + \gamma^{k+1} + \beta(x^{k+1} + y^{k+1}) \tag{4.100}
\]
Thus, we have
\[
\text{dist}[0, \partial_x F(d^{k+1})] \leq \frac{1}{r}\|x^k - x^{k+1}\| + \|\gamma^{k+1} - \gamma^k\| + \frac{\|e_1^{k+1}\|}{r} \tag{4.101}
\]
where \(S_x = \max\{\frac{1}{r}, \sqrt{\rho_2}\}\) and \(D_x = \max\{\sqrt{\rho_2}, \frac{1}{r}\}\). While in updating \(y^{k+1}\), we have
\[
-\gamma^k + \gamma^{k+1} = \nabla g(y^{k+1}). \tag{4.102}
\]
And we have
\[
\partial_y F(d^{k+1}) = \nabla g(y^{k+1}) + \gamma^{k+1} + \beta(x^{k+1} + y^{k+1}) \tag{4.103}
\]
Combining \((4.102)\) and \((4.103)\),
\[
\text{dist}[0, \partial_y F(d^{k+1})] \leq \frac{1}{r}\|\gamma^{k+1} - \gamma^k\| + \frac{\|e_2^{k+1}\|}{r} \tag{4.104}
\]
where \(S_y = \beta + \sqrt{\rho_1}\) and \(D_y = \max\{\sqrt{\rho_2}, \beta\}\). Noting
\[
\partial_y F(d^{k+1}) = x^{k+1} + y^{k+1} = \frac{\gamma^{k+1} - \gamma^k}{\beta} \tag{4.105}
\]
we have
\[
\text{dist}(0, \partial_y F(d^{k+1})) \leq \frac{\|\gamma^{k+1} - \gamma^k\|}{\beta} \leq \frac{\sqrt{\rho_1}}{\beta}\|y^k - y^{k+1}\| + \frac{\sqrt{\rho_2}}{\beta}\|e_2^{k+1} - e_2^k\| \tag{4.106}
\]
where \(S_y = \sqrt{\rho_1}\) and \(D_y = \sqrt{\rho_2}\). Letting \(S = S_x + S_y + S_\gamma\) and \(D = D_x + D_y + D_\gamma\), we then prove the result. \(\square\)
Then, we prove \( \inf F(d^k) > -\infty \). Then, we can obtain the boundedness of the points.

**Lemma 23.** If there exists \( \sigma_0 > 0 \) such that

\[
\inf \{ g(y) - \sigma_0 \| \nabla g(y) \|^2 \} > -\infty. \tag{4.107}
\]

We also assume that \( (e^k_2)_{k \geq 0} \) is bounded and condition \( (4.89) \) holds, and \( f(x) \) is coercive. If

\[
\beta \geq \frac{1}{\sigma_0},
\]

then, the sequence \( \{d^k\}_{k=0,1,2,...} \) is bounded.

**Proof.** From \( (4.87) \),

\[
\| \gamma^k \|^2 \leq 2 \| \nabla g(y^k) \|^2 + 2\beta^2 \| e^k_2 \|^2. \tag{4.108}
\]

We have

\[
F(d^k) = f(x^k) + g(y^k) + \langle \gamma^k, x^k + y^k \rangle + \frac{\beta}{2} \| x^k + y^k \|^2
\]

\[
= f(x^k) + g(y^k) - \frac{\| \gamma^k \|^2}{2\beta} + \frac{\beta}{2} \| x^k + y^k + \frac{\gamma^k}{\beta} \|^2
\]

\[
= f(x^k) + g(y^k) - \frac{\sigma_0}{2} \| \gamma^k \|^2 + \left( \frac{\sigma_0}{2} - \frac{1}{2\beta} \right) \| \gamma^k \|^2 + \frac{\beta}{2} \| x^k + y^k + \frac{\gamma^k}{\beta} \|^2
\]

\[
\geq f(x^k) + g(y^k) - \sigma_0 \| \nabla g(y^k) \|^2
\]

\[
+ \left( \frac{\sigma_0}{2} - \frac{1}{2\beta} \right) \| \gamma^k \|^2 + \frac{\beta}{2} \| x^k + y^k + \frac{\gamma^k}{\beta} \|^2 - \sigma_0 \beta^2 \| e^k_2 \|^2. \tag{4.109}
\]

Noting \( \lim_k \| e^k_2 \|^2 = 0 \), we then can see \( \{f(x^k)\}_{k=0,1,2,...}, \{\gamma^k\}_{k=0,1,2,...}, \{x^k + y^k + \frac{\gamma^k}{\beta}\}_{k=0,1,2,...} \) are all bounded. Then, \( \{d^k\}_{k=0,1,2,...} \) is bounded.

**Remark 1.** Combining \( (4.89) \), we need to set

\[
\gamma > \min \left\{ \frac{\sqrt{2}}{2} L_g, \frac{1}{\sigma_0} \right\}, 0 < r < \frac{1}{\beta}. \tag{4.110}
\]

**Remark 2.** The condition \( (4.107) \) holds for many quadratical functions \( [20, 32] \). This condition also implies the function \( g \) is similar to quadratical function and its property is "good".

**Lemma 24.** Let \( (d^k)_{k \geq 0} \) is generated by scheme \( (4.82) \), and \( e^k_1 \to 0 \), and \( e^k_2 \to 0 \). And let conditions of Lemmas \( 21 \) and \( 23 \) hold. Then, for any stationary point \( d^* \) of \( (d^k)_{k \geq 0} \), there exists a subsequence \( (d^{k_j})_{j \geq 0} \) converges to \( d^* \) satisfying \( F(d^{k_j}) \to F(d^*) \) and \( d^* \in \text{crit}(F) \).

**Proof.** Obviously, we have \( e^k_2 \to 0 \). With Lemma \( 23 \), \( (d^k)_{k \geq 0} \) is bounded; so are \( (x^k)_{k \geq 0} \) and \( (y^k)_{k \geq 0} \).

For any stationary point \( d^* = (x^*, y^*, \gamma^*, y^*) \), there exists a subsequence \( (d^{k_j})_{j \geq 0} \) converges to \( d^* \). With Lemmas \( 4 \) and \( 21 \) we also have

\[
x^{k_j-1} \to x^*, y^{k_j-1} \to y^*. \tag{4.111}
\]

Noting \( 0 \in \partial F(d^*), x^* + y^* = 0; \) thus, \( \gamma^{k_j-1} \to \gamma^* \). And in each iteration of updating \( x^{k_j} \), with Lemma \( 6 \) we have

\[
r f(x^{k_j}) + \frac{\| x^{k_j} - r[\gamma^{k_j-1} + \beta(x^{k_j-1} + y^{k_j-1})] + e^{k_j}_1 - x^{k_j} \|^2}{2} \leq \ r f(x^{k_j-1}) + \frac{\| - r[\gamma^{k_j-1} + \beta(x^{k_j-1} + y^{k_j-1})] + e^{k_j}_1 \|^2}{2}. \tag{4.112}
\]
Taking $j \to +\infty$, we have
\[
\limsup_{j \to +\infty} f(x_j) \leq f(x^*). \tag{4.13}
\]
And recalling the lower semi-continuity of $f$,
\[
f(x^*) \leq \liminf_{j \to +\infty} f(x_j). \tag{4.14}
\]
That means $\lim_j f(x_j) = f(x^*)$; combining the continuity of $g$, we then prove the result.

Finally, we present the convergence result for the inexact ADMM (4.82).

**Theorem 6.** Let $(x_k)_{k \geq 0}$ be generated by scheme (4.82) and conditions of Lemmas 21 and 23 hold. Assume that $f$ and $g$ are both semi-algebraic. If
\[
\|e^k\| + \|e^k\| = O\left(\frac{1}{k^\alpha}\right), \quad \alpha > 1,
\]
then, the sequence $(x^k, y^k)_{k \geq 0}$ has finite length, i.e.
\[
\sum_{k=0}^{+\infty} (\|x^{k+1} - x^k\| + \|y^{k+1} - y^k\|) < +\infty. \tag{4.15}
\]

**Proof.** Noting
\[
\|e^k\| \leq 2\|e^k\| + 2\|e^k\|, \tag{4.16}
\]
that is also
\[
\|e^k\| = O\left(\frac{1}{k^\alpha}\right), \quad \alpha > 1. \tag{4.17}
\]

With the lemmas proved in this part and Theorem 5, we then prove the result.

**5 Conclusion**

In this paper, we prove the convergence for a class of inexact nonconvex and nonsmooth algorithms. The sequence generated by the algorithm converges to a critical point of the objective function under finite energy assumption on the noise and the KL property assumption. We apply our theoretical results to many specific algorithms; and obtain the specific convergence results.

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**References**

[1] Brendan PW Ames and Mingyi Hong. Alternating direction method of multipliers for penalized zero-variance discriminant analysis. *Computational Optimization and Applications*, 64(3):725–754, 2016.

[2] Nguyen Thai An and Nguyen Mau Nam. Convergence analysis of a proximal point algorithm for minimizing differences of functions. *Optimization*, 66(1):129–147, 2017.
[3] Hedy Attouch and Jérôme Bolte. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. *Mathematical Programming*, 116(1-2):5–16, 2009.

[4] Hédy Attouch, Jérôme Bolte, Patrick Redont, and Antoine Soubeyran. Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the kurdyka-Lojasiewicz inequality. *Mathematics of Operations Research*, 35(2):438–457, 2010.

[5] Hedy Attouch, Jérôme Bolte, and Benar Fux Svaiter. Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward–backward splitting, and regularized gauss–seidel methods. *Mathematical Programming*, 137(1-2):91–129, 2013.

[6] Amir Beck. On the convergence of alternating minimization for convex programming with applications to iteratively reweighted least squares and decomposition schemes. *SIAM Journal on Optimization*, 25(1):185–209, 2015.

[7] Jérôme Bolte, Aris Daniilidis, and Adrian Lewis. The Lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems. *SIAM Journal on Optimization*, 17(4):1205–1223, 2007.

[8] Jérôme Bolte, Aris Daniilidis, Adrian Lewis, and Masahiro Shiota. Clarke subgradients of stratifiable functions. *SIAM Journal on Optimization*, 18(2):556–572, 2007.

[9] Jérôme Bolte, Shoham Sabach, and Marc Teboulle. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. *Mathematical Programming*, 146(1-2):459–494, 2014.

[10] Emmanuel J Candes, Michael B Wakin, and Stephen P Boyd. Enhancing sparsity by reweighted ℓ_1 minimization. *Journal of Fourier analysis and applications*, 14(5):877–905, 2008.

[11] Rick Chartrand and Wotao Yin. Iteratively reweighted algorithms for compressive sensing. In *Acoustics, speech and signal processing, 2008. ICASSP 2008. IEEE international conference on*, pages 3869–3872. IEEE, 2008.

[12] Rick Chartrand and Wotao Yin. Nonconvex sparse regularization and splitting algorithms. In *Splitting Methods in Communication, Imaging, Science, and Engineering*, pages 237–249. Springer, 2016.

[13] Ingrid Daubechies, Ronald DeVore, Massimo Fornasier, and C Sinan Güntürk. Iteratively reweighted least squares minimization for sparse recovery. *Communications on Pure and Applied Mathematics*, 63(1):1–38, 2010.

[14] Pierre Frankel, Guillaume Garrigos, and Juan Peypouquet. Splitting methods with variable metric for kurdyka–Lojasiewicz functions and general convergence rates. *Journal of Optimization Theory and Applications*, 165(3):874–900, 2015.

[15] Daniel Gabay and Bertrand Mercier. A dual algorithm for the solution of nonlinear variational problems via finite element approximation. *Computers & Mathematics with Applications*, 2(1):17–40, 1976.

[16] Roland Glowinski and A Marroco. Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité d’une classe de problèmes de dirichlet non linéaires. *Revue française d’automatique, informatique, recherche opérationnelle. Analyse numérique*, 9(R2):41–76, 1975.

[17] Mingyi Hong, Zhi-Quan Luo, and Meisam Razaviyayn. Convergence analysis of alternating direction method of multipliers for a family of nonconvex problems. *SIAM Journal on Optimization*, 26(1):337–364, 2016.

[18] Krzysztof Kurdyka. On gradients of functions definable in o-minimal structures. In *Annales de l’institut Fourier*, volume 48, pages 769–784. Chartres: L’Institut, 1950-, 1998.
[19] Ming-Jun Lai, Yangyang Xu, and Wotao Yin. Improved iteratively reweighted least squares for unconstrained smoothed $\ell_q$ minimization. *SIAM Journal on Numerical Analysis*, 51(2):927–957, 2013.

[20] Guoyin Li and Ting Kei Pong. Global convergence of splitting methods for nonconvex composite optimization. *SIAM Journal on Optimization*, 25(4):2434–2460, 2015.

[21] Guoyin Li and Ting Kei Pong. Douglas–rachford splitting for nonconvex optimization with application to nonconvex feasibility problems. *Mathematical programming*, 159(1-2):371–401, 2016.

[22] Stanislas Lojasiewicz. Sur la géométrie semi-et sous-analytique. *Ann. Inst. Fourier*, 43(5):1575–1595, 1993.

[23] Canyi Lu, Yunchao Wei, Zhouchen Lin, and Shuicheng Yan. Proximal iteratively reweighted algorithm with multiple splitting for nonconvex sparsity optimization. In *AAAI*, pages 1251–1257, 2014.

[24] Zhaosong Lu, Yong Zhang, and Jian Lu. $\ell_p$ regularized low-rank approximation via iterative reweighted singular value minimization. *Computational Optimization and Applications*, pages 1–24, 2017.

[25] Paul-Emile Maingé and Abdellatif Moudafi. Convergence of new inertial proximal methods for dc programming. *SIAM Journal on Optimization*, 19(1):397–413, 2008.

[26] Jorge Nocedal and Stephen J Wright. *Sequential quadratic programming*. Springer, 2006.

[27] R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.

[28] Ralph Tyrell Rockafellar. *Convex analysis*. Princeton university press, 2015.

[29] Mark Schmidt, Nicolas L Roux, and Francis R Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In *Advances in neural information processing systems*, pages 1458–1466, 2011.

[30] Ron Shefi and Marc Teboulle. On the rate of convergence of the proximal alternating linearized minimization algorithm for convex problems. *EURO Journal on Computational Optimization*, 4(1):27–46, 2016.

[31] Tao Sun, Roberto Barrio, Hao Jiang, and Lizhi Cheng. Convergence rates of accelerated proximal gradient algorithms under independent noise. *Numerical Algorithms*, 2018.

[32] Tao Sun, Roberto Barrio, Hao Jiang, and Lizhi Cheng. Precompact convergence of the nonconvex primal-dual hybrid gradient algorithm. *Journal of Computational and Applied Mathematics*, 2018.

[33] Tao Sun and Lizhi Cheng. Little-o convergence rates for several alternating minimization methods. *Communications in Mathematical Sciences*, 15(1):197–211, 2017.

[34] Tao Sun, Hao Jiang, and Lizhi Cheng. Convergence of proximal iteratively reweighted nuclear norm algorithm for image processing. *IEEE Transactions on Image Processing*, 2017.

[35] Tao Sun, Hao Jiang, and Lizhi Cheng. Global convergence of proximal iteratively reweighted algorithm. *Journal of Global Optimization*, pages 1–12, 2017.

[36] Tao Sun, Penghang Yin, Hao Jiang, and Lizhi Cheng. Alternating direction method of multipliers with difference of convex functions. *Advances in Computational Mathematics*, 2017.

[37] Pham Dinh Tao and Le Thi Hoai An. Convex analysis approach to dc programming: Theory, algorithms and applications. *Acta Mathematica Vietnamica*, 22(1):289–355, 1997.
[38] Silvia Villa, Saverio Salzo, Luca Baldassarre, and Alessandro Verri. Accelerated and inexact forward-backward algorithms. *SIAM Journal on Optimization*, 23(3):1607–1633, 2013.

[39] Yu Wang, Wotao Yin, and Jinshan Zeng. Global convergence of admm in nonconvex nonsmooth optimization. *arXiv preprint arXiv:1511.06324*, 2015.

[40] Tong Zhang. Analysis of multi-stage convex relaxation for sparse regularization. *Journal of Machine Learning Research*, 11(Mar):1081–1107, 2010.