Homogeneous space-times as models for isolated extended objects

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Abstract

An extended object is considered on the Minkowski background in the form of a space-time bag, which is bounded by a certain surface confining an internal substance. An internal metric is built starting from the symmetry principles rather than from the field equations. Assuming such a surface to be Lorentz invariant we find that the internal space is proved to be the de Sitter space. Conformal inversion of the internal metric relative to the bag surface determines an external space (conformally conjugated de Sitter space) whose metric may simulate a field of the object. Although the extended object built in such a way is noncompact, its cross section by the hyperplane $r^0 = 0$, where $r^0$ is the temporal coordinate, is compact (a ball) and the associated metric can model a spherically symmetric extended massless charge in a certain approximation.

1 Introduction

Some times ago there were published papers [1], proposing a model for hadrons, so called the Nijmegen model, in which quarks and gluons were moving inside a closed space domain, described by the anti-de Sitter (AdS) universe. This theory may be considered as a variant of the bag model [2], which is appeared to be fruitful for the spectral analysis of hadrons as well as for a calculation of magnetic moments, decay widths, electromagnetic mass splitting, etc. Experiments show that at least heavy particles have an extended structure. To describe such structure one may imagine hadron as a bag in the form of a certain universe with the metric simulating interactions between quarks and gluons considered as point objects. An indirect indication to a character of these interactions may be the hadron spectroscopy. In particular, an observation of spectra of $\psi$ and $\Upsilon$ particles, representable as excited states of quark-antiquark systems $c\bar{c}$ and $b\bar{b}$, respectively, in which forces between the constituents are described by the oscillator interaction, has lead to the AdS character of the bag interior. For all that quarks are considered as point...
objects. Since an exact interquark dynamics is not yet known, one may suppose such point objects to be moving along the geodesic lines defined by the internal bag geometry. Thus, for the description of the extended particles in terms of the bag model one should use concepts and methods of General Relativity.

It should be noted that the idea to describe elementary particles in terms of General Relativity has rather long history, but, to our mind, so far up to now is not properly realized. The first attempt, probably, was made by Einstein and Rosen [3] who had supposed the Schwarzschild solution for describing neutral (neutron, neutrino) particles and the Reissner-Nordström solutions for charged particles. As early as 1939 Mariani [4] had turned to metrical geometry containing such a parameter in a natural way departing from that the theory of elementary particles required introducing a fundamental length, which cannot be taken into account by Euclidean geometry. At the same time Lees [5] has proposed an electron model in the framework of a consistent consideration of gravity and electromagnetism, in which limitations on the electron shape and size were imposed, and external field of the static electron was described by the Reissner-Nordström metric. Later Dirac has considered a similar model of the electron [6] in the form of a pulsing bubble with a surface tension. This model turned out to be a conceptual basis of the bag model [2].

The most of different metrics, used in General Relativity, are known to be obtained as solutions of either free or material Einstein equations or their generalizations. At the present time there exists no preferable choice between the metrics applicable for a description of the same situation. This is due to several circumstances: i) due to a physical interpretation of the field equations and, in particular, to a meaning of the energy-momentum tensor; a definition of the metric from the field equations is not entirely correct procedure [7] (Ch. 7), all the more that there exists ii) a problem of physical interpretation of coordinates entering the particular expression for the metric [8]. Specifically, it causes an abundance of metrics, satisfying the same equation, which may be obtained from each other by a certain coordinate transformation. For the static vacuum spherically symmetric fields this fact constitutes a content of the Birkhoff theorem. iii) There exists also a widely discussed problem of singularities of the space-time. On the one hand, the singularities believed to be the coordinate effects and should be eliminated from the theory. On the other hand, for example, a supposition was being made that an existence of singularities, probably, was a general property of all spaces, which might be accepted as reasonable models of the Universe [9]. At any rate, the surfaces of the metric singularities (horizons) are known to play a great role in the black hole theory. Lastly, iv) a relatively little number of geometro-physical experiments does not give a possibility to make an unambiguous choice too.

In our opinion, an invention of the metric corresponding to a particular situation is possible without an attraction of the field equations. The required properties of symmetry and topology of the space-time, generated by congruences of test-particles world lines treated as geodesics, a presence of the singularities, determining the metric and topological properties, as well as a coordinatization, connected with a measurement process, should be more important circumstances than any other ones. Moreover, the problems considered cannot be exhausted with only gravity: other fields may also interpreted from the viewpoint of the metric space-time properties [10] in the spirit of the Poincaré con-
Among various spaces used at the present time, there exists a class of maximally symmetric spaces, which, on the one hand, may serve as a background for the evolution of physical events and, on the other hand, they may be treated as a perfect case of metrics, symmetry breaking of which to spherical, axial, etc., one leads to metrics describing realistic situations. The present paper considers just the case of maximal symmetry at classical level. However, we built a metric starting from the principles of the space-time symmetry rather than from the field equations, which will be used for the interpretation of the metric obtained. Assuming that the presence of a surface being a space-time characteristic of an extended object implies a change of original space-time geometry we shall try to answer a question: what the metric must be, if test particles, whose world lines are believed to be geodesic lines, are forbidden to come out of a certain space-time domain for some reason. For the first approximation, it is naturally to choose a three-dimensional ball for a model of an isolated extended object. However, assuming if this domain might have properties of maximal symmetry in the perfect case (without of either a rotation of the object or a presence of other objects or external fields) we choose not a ball, but a region bounded by invariant surfaces as, for example, a one-sheet hyperboloid and a light cone. Then a spherical approximation may be obtained with the help of a suitable cross section of it.

The paper is organized as follows. In Section 2 we depart from the Minkowski space splitted by invariant surfaces into six regions. We require the line element to be invariant along geodesic lines under coordinate transformations similar to the velocity addition law in Special Relativity and obtain expressions for the metrics of internal regions bounded by the surfaces above. In Section 3 generators of coordinate transformation groups, acting transitively in these regions, are built, which form algebras so(2,3) or so(1,4). The metrics of the spaces, bounded by a one-sheet hyperboloid and space-like infinity of by a two-sheet hyperboloid and time-like infinity, respectively, may be derived from the internal ones by the conformal inversion relative to mentioned hyperboloids. The spherical approximation is considered in Section 4. Finally, Section 5 contents conclusive notes.

## 2 Internal metric

We start with the flat Minkowski space-time $E^R_{1,3}$ with the metric $\eta_{\mu\nu} = \text{diag}(1,1,1,-1)$, however, we shall suppose that test particle world lines for several reasons can neither come out of some region $U \subset E^R_{1,3}$ nor penetrate in it. In sufficiently large limits one may consider that congruences of these world lines generate a Riemannian space, $V$, where they are geodesic lines [10]. The boundary $\partial U$ of the region $U$ divides $V$ into internal, $V^i$, and external, $V^e$, parts. Thus, the region $U$ is represented by the Riemannian space $V^i$, which may be effectively considered as the interior of an extended object with sharp boundary $\partial U$ in the space-time $E^R_{1,3}$.

Starting from the principle of the general coordinatization [12] we can use the Minkowski space coordinatization as a coordinatization of internal, $V^i$, and external, $V^e$, spaces denoting their coordinates and metrics as $r^{\mu}$, $h_{\mu\nu}(r^\alpha)$ and $R^{\mu}$, $H_{\mu\nu}(R^\alpha)$, re-
the Minkowski space into six connected regions:

\[ ds^2 = h_{\mu\nu}(r^\alpha)dr^\mu dr^\nu, \]

\[ dS^2 = H_{\mu\nu}(R^\alpha)dR^\mu dR^\nu, \quad \alpha, \mu, \nu = 0, 1, 2, 3. \]

Obviously, the motions of both internal and external regions must leave respective line element invariant, and the problem is to seek out these motions depending upon a structure of the region \( U \), and vice versa.

Assuming the isolated extended object in question to be described by Lorentz-invariant quantities let us consider the boundary as Lorentz-invariant surface, or rather as a set of Lorentz-invariant surfaces. Such surfaces are the one-sheet hyperboloid, two-sheet hyperboloid and light cone, which are defined as follows: \[ \text{H}_1 \]

\[ \text{H}_{1,3} \doteq \{ x^\mu \in \mathbb{E}^{1,3}_{1,3} : \eta_{\mu\nu}x^\mu x^\nu = -L^2 \} = \frac{\text{SO}(1,3)}{\text{SO}(1,2)}, \]

\[ \text{S}_{1,3} \doteq \{ x^\mu \in \mathbb{E}^{1,3}_{1,3} : \eta_{\mu\nu}x^\mu x^\nu = +L^2 \} = \frac{\text{SO}(1,3)}{\text{SO}(3)}, \]

\[ \text{C}_{1,3} \doteq \{ x^\mu \in \mathbb{E}^{1,3}_{1,3} : \eta_{\mu\nu}x^\mu x^\nu = 0 \} = \frac{\text{SO}(1,3)}{[\text{SO}(2) \times \mathbb{R}^1] \odot T_2}, \]

where \( \mathbb{R}^1 = \mathbb{E}^1_1 \) is real straight line, \( T_2 \) is translation group of two-dimensional plane \( \mathbb{R}^2 = \mathbb{E}^2_2 \), the sign \( \odot \) denotes semidirect product. Both \( \text{S}_{1,3} \) and \( \text{C}_{1,3} \) are pairs of surfaces \( \text{S}^{+}_{1,3} \) and \( \text{C}^{+}_{1,3} \) with respect to \( x^0 > 0 \) and \( x^0 < 0 \). The surfaces defined in such a way split the Minkowski space into six connected regions:

1) interior of 1+3-dimensional pseudoball of hyperbolic type

\[ \text{D}^{H}_{1,3} \doteq \{ x^\mu \equiv r^\mu \in \mathbb{E}^{1,3}_{1,3} : -L^2 < \eta_{\mu\nu}r^\mu r^\nu \leq 0 \} \subset \mathbb{E}^{1,3}_{1,3}, \]

2-3) interiors of 1+3-dimensional pseudoball of spherical type

\[ \text{D}^{+}_{1,3} \doteq \{ x^\mu \equiv r^\mu \in \mathbb{E}^{1,3}_{1,3} : 0 < \eta_{\mu\nu}r^\mu r^\nu \leq L^2, \ r^0 \geq 0 \} \subset \mathbb{E}^{1,3}_{1,3}, \]

\[ \text{D}^{-}_{1,3} \doteq \{ x^\mu \equiv r^\mu \in \mathbb{E}^{1,3}_{1,3} : 0 < \eta_{\mu\nu}r^\mu r^\nu \leq L^2, \ r^0 \leq 0 \} \subset \mathbb{E}^{1,3}_{1,3}, \]

\[ \text{D}^{S}_{1,3} = \text{D}^{+}_{1,3} \cup \text{D}^{-}_{1,3}; \]

4) exterior of 1+3-dimensional pseudoball of hyperbolic type

\[ \text{D}^{H}_{1,3} \doteq \{ x^\mu \equiv R^\mu \in \mathbb{E}^{1,3}_{1,3} : -\infty < \eta_{\mu\nu}R^\mu R^\nu \leq -L^2 \} \subset \mathbb{E}^{1,3}_{1,3}; \]

5-6) exteriors of 1+3-dimensional pseudoball of spherical type

\[ \text{D}^{+}_{1,3} \doteq \{ x^\mu \equiv R^\mu \in \mathbb{E}^{1,3}_{1,3} : L^2 < \eta_{\mu\nu}R^\mu R^\nu \leq \infty, \ R^0 > 0 \} \subset \mathbb{E}^{1,3}_{1,3}, \]

\[ \text{D}^{-}_{1,3} \doteq \{ x^\mu \equiv R^\mu \in \mathbb{E}^{1,3}_{1,3} : L^2 < \eta_{\mu\nu}R^\mu R^\nu \leq \infty, \ R^0 < 0 \} \subset \mathbb{E}^{1,3}_{1,3}, \]

\[ \text{D}^{S}_{1,3} = \text{D}^{+}_{1,3} \cup \text{D}^{-}_{1,3}. \]

\((r^2 = \eta_{\mu\nu}r^\mu r^\nu = 0\) takes place in (3) and (4) only at \( r^0 = 0 \).\)

\[ ^{1}\text{H}_{1,3} \text{ and } \text{S}_{1,3} \text{ are equivalent to denotations } \text{S}^{1}_{1} \text{ and } \text{H}^{3}_{0} = \text{H}^{3} \text{ in the monograph [13], respectively. See also [14], [15].} \]
As a point of departure for considering an extended object we are interested mainly in the region $D^H_{1,3}$ for its cross section by the hyperplane $x^0 > 0$ represents a three-dimensional ball, which is the basis for the representation of a spherically symmetric object. Then the region $D^S_{1,3}$ may be treated as a space, on which a force field of such an object is manifested. The remaining regions are for the time being of purely geometric interest, although all six ones may be considered by the same method.

The metric in $V^i$, corresponding to $D^H_{1,3}$ or $D^S_{1,3}$, may be determined if we require the interval (1) to be invariant under a transformation of coordinates $r^\mu$ satisfying condition (3) or (4), respectively. Such a transformation may be written similar to the velocity addition formula in Special Relativity [16]. Let us consider a five-dimensional pseudo-Euclidean space $E^R_{2,3}$ (or $E^R_{1,4}$) for the case (3) (or (4)), covered by coordinates $\xi^a$, $a = 0, 1, 2, 3, 4$, with the line element

$$ds^2 = \eta_{ab}d\xi^a d\xi^b = \eta_{\mu\nu}d\xi^\mu d\xi^\nu + \eta_{55}(d\xi^5)^2,$$

where $\eta_{55} \equiv \eta = +1$ (or $-1$). Let us introduce now in such space analogs of inertial frame systems (i.f.s.). Let $\rho^\alpha$ be the 1+3-dimensional analog of the relative three-velocity between two i.f.s. Similarly to that the coordinates of two i.f.s. in $E^R_{1,3}$ are connected with each other by the Lorentz transformation, where the three-velocity is a parameter, there takes place in our case an analog of the Lorentz transformation for coordinates $\xi^a$: $'\xi^a = \Lambda^a_{\beta} \xi^\beta$. A transformation consistent with the condition (3) (or (4)) is the analog of sublimal Lorentz transformation [16]:

$$\Lambda = 1 + \frac{\gamma}{L} \hat{\rho} + \frac{\gamma - 1}{L^2 \beta^2} \hat{\rho}^2, \hspace{1cm} \gamma = (1 - \beta^2)^{-1/2},$$

where $\hat{\rho} = \eta_{\mu\nu}\rho^\mu (e^{\nu}_5 - e^{5\mu})$, $e^{ab}$ are the elements of complete matrix algebra satisfying to relations $e^{ab}e^{cd} = \eta^{bc}e^{ad}$, $(e^{ab})^d_c = \eta^{ac}\delta^b_d$. The quantity

$$\beta^2 = \frac{1}{2L^2} \text{Sp}(\rho^2) = -\frac{\eta}{L^2 \beta^2} \eta_{\mu\nu} \rho^\mu \rho^\nu = -\frac{\eta \rho^2}{L^2}$$

runs the range $0 \leq \beta^2 < 1$, whence it follows $-L^2 < \rho \leq 0$ for the case (3) and $0 \leq \rho < +L^2$ for the case (4). Thus, $L$ is here an analog of the velocity of light. Coordinates $r^\mu$, $'r^\mu$ may be interpreted as analogs of three-velocities $r^\mu = L\xi^\mu / d\xi^5$, $'r^\mu = Ld'\xi^\mu / d'\xi^5$, connected by the transformation of the velocity-addition law type

$$'r^\mu = L \frac{\Lambda^\mu_\nu r^\nu + \Lambda^\mu_5 r^5}{\Lambda^5_\nu r^\nu + \Lambda^5_5} = \frac{r^\mu + \rho^\mu + (\gamma - 1)[1 + \frac{\eta L^2 r^2}{\rho^2}]}{\gamma[1 - \frac{2\eta L^2 r^2}{\rho^2}]}.\quad(10)$$

Eq. (10) is a required transformation which ought to be complemented by four-dimensional rotations of vectors $r^\mu$. It easy to see that $'r^\mu$ also satisfies the condition (3) (or (4)) corresponding to $ds^2 > 0$. Thus, varying the parameters $\rho^\mu$ and parameters responsible for four-dimensional rotations of $r^\mu$ we obtain all points of the space $D^H_{1,3}$ (or $D^S_{1,3}$). Defining $d'r^\mu$ and $h_{\mu\nu}(r^\alpha)$ from (10) we find from the invariance condition of the interval (1) under the local transformation (10) that the metrics $h_{\mu\nu}(r^\alpha)$ have a form

$$h_{\mu\nu} = \left(1 + \frac{\eta r^2}{L^2}\right)^{-1} \eta_{\mu\nu} = \left(1 + \frac{\eta^2 r^2}{L^2}\right)^{-2} \frac{\eta_{\mu\nu}\eta_{\beta\alpha}r^\alpha r^\beta}{L^2},\quad(11)$$
\[ h^{\mu\nu} = \left(1 + \frac{\eta r^2}{L^2} \right) \left[ \eta^{\mu\nu} + \frac{\eta r^\mu r^\nu}{L^2} \right], \quad h_{\mu\lambda} h^{\lambda\nu} = \delta^\nu_\nu. \quad (12) \]

When \( \eta = -1 \) it is no more than the de Sitter (dS) metric [17], which was initially obtained not from geometric considerations but from the field equations with cosmological term; \( \eta = +1 \) corresponds to anti-de Sitter (AdS) space.

It is easy to show that the coordinate transformation of the stereographic projection type [14]

\[ r^\mu = \left(1 - \frac{\eta x^2}{4L^2}\right) x^\mu, \quad x^2 = \eta_{\mu\nu} x^\mu x^\nu, \quad (13) \]

\[ x^\mu = -\frac{2\eta L^2}{r^2} \left[ 1 + \sqrt{1 + \frac{\eta r^2}{L^2}} \right] r^\mu, \quad (14) \]

reduces (1) to the standard expression for the line element of the de Sitter space of constant curvature [13]

\[ ds^2 = h_{\mu\nu} dr^\mu dr^\nu = \left(1 + \frac{\eta x^2}{4L^2}\right)^{-2} \eta_{\mu\nu} dx^\mu dx^\nu, \quad (15) \]

whence it follows for the Gaussian curvature \( K = -\eta L^2 \). Hence, \( \mathbf{D}^{1,3}_{1,3} (\mathbf{D}^{S}_{1,3}) \) is the space of constant negative (positive) and may be embedded into the flat de Sitter space \( \mathbf{E}^{R}_{2,3} (\mathbf{E}^{R}_{0,4}) \) with the metric (7), \( \eta = +1 \) (\( \eta = -1 \)). The metric (11) is the metric of the hypersurface

\[ \eta_{ab} \xi^a \xi^b = \eta_{\mu\nu} \xi^\mu \xi^\nu + \eta_{55} (\xi^5)^2 = \eta L^2 \quad (16) \]

in \( \mathbf{E}^{R}_{2,3} \) (or \( \mathbf{E}^{R}_{0,4} \)) written in terms of coordinates \( r^\mu \), connected with \( \xi^a \) by the relations

\[ \xi^\mu = r^\mu \left(1 + \eta r^2/L^2\right)^{-1/2}, \quad \xi^5 = \pm L \left(1 + \eta r^2/L^2\right)^{-1/2}; \quad (17) \]

\[ r^\mu = \xi^\mu \left(1 - \eta \xi^2/L^2\right)^{-1/2}, \quad \xi^2 = \eta_{\mu\nu} \xi^\mu \xi^\nu. \quad (18) \]

In conclusion of this Section we note that just a geometric approach was used by Einstein for a construction of the metric of the closed static world [18]; as in is well known, Friedmann [19] had generalized the Einstein’s solution (as well as the de Sitter’s one [17]) to the non-static case.

### 3 The isometry group of internal space and external metric

The transformation (10) together with four-dimensional rotations forms an isometry group whose orbit is the whole space \( \mathbf{V}^5 \) being thereby a homogeneous space. The solution of the Killing equation

\[ h_{\nu a} (\zeta_\lambda)_{\nu}^{\nu} + h_{a\nu} (\zeta_\lambda)_{\nu}^{\nu} = 0, \quad A = \mu, [\lambda \kappa], \quad (19) \]

gives four Killing vectors, \( \zeta_\mu \), with components

\[ (\zeta_\mu)^\nu = \delta_\mu^\nu + \frac{\eta_{\mu\lambda} r^\lambda r^\nu}{L^2}, \quad (20) \]
responsible for transformations (10), and six Killing vectors, $\zeta_{[\lambda\kappa]}$, with components

$$(\zeta_{[\lambda\kappa]})^\nu = (\eta_{\lambda\alpha} \delta^\nu_\kappa - \eta_{\kappa\alpha} \delta^\nu_\lambda) r^\alpha,$$  

(21)

responsible for four-dimensional rotations of the vectors $r^\alpha$. Thus, generators of the isometry group are

$$\hat{J}_\mu = -i L (\zeta_\mu) ^\nu \partial_\nu = -i L \left( \delta^\nu_\mu + \eta \frac{\eta_{\mu\lambda} r^\lambda r^\nu}{L^2} \right) \partial_\nu,$$  

(22)

$$\hat{M}_{[\lambda\kappa]} = -i L (\zeta_{[\lambda\kappa]}) ^\nu \partial_\nu = -i (r_\lambda \partial_\kappa - r_\kappa \partial_\lambda) = \frac{1}{L} (r_\lambda \hat{J}_\kappa - r_\kappa \hat{J}_\lambda).$$  

(23)

Assuming $\hat{J}_\mu = \hat{M}_{[\mu\nu]} = \hat{M}_{[\nu\mu]}$ we find the commutation relations between $\hat{M}_{[ab]}$:

$$[\hat{M}_{[ab]}, \hat{M}_{[cd]}] = i (\eta_{ac} \hat{M}_{[bd]} + \eta_{bd} \hat{M}_{[ac]} - \eta_{ad} \hat{M}_{[bc]} - \eta_{bc} \hat{M}_{[ad]}).$$  

(24)

Thus, as it should be expected an algebra of the generators $\hat{J}_\mu$, $\hat{M}_{[\lambda\kappa]}$ turns out to be isomorphic to the algebra so(2,3) (for $\hat{D}^H_{1,3}$) or so(1,4) (for $\hat{D}^S_{1,3}$), so that the isometry group O(2,3) acts transitively on $\hat{D}^H_{1,3}$ and O(1,4) acts transitively on $\hat{D}^S_{1,3}$. Moreover, SO(1,4) is the group of isotropy for the spaces in question, so that $\hat{D}^S_{1,3} = \text{SO}(2,3)/\text{SO}(1,3)$ and $\hat{D}^S_{1,3} = \text{SO}(1,4)/\text{SO}(1,3)$.

Commutation relations between $\hat{J}_\mu$'s show that the algebra of $\hat{M}_{[ab]}$ becomes isomorphic to the Poincaré algebra iso(1,3) at the limit $L \to \infty$. This is a reflection of the known fact that the inhomogeneous pseudo-rotations group may be treated as the limiting case of the homogeneous one in the space of one more dimension [20]. The generators $\hat{J}_\mu = h L^{-1} \hat{J}_\mu$ turn into $-i h \partial_\mu = \hat{P}_\mu$ at the limit $L \to \infty$. However, there exists also the limit $L \to 0$, when $\hat{J}_\mu = h L^{-1} \hat{J}_\mu \to \eta \eta_{\mu\lambda} r^\lambda r^\nu \hat{P}_\nu$, $[\hat{J}_\mu, \hat{J}_\nu] = 0$, and we come again to the Poincaré algebra, but the metric (11) loses its meaning because at this limit we have $h_{\mu\nu} \to 0$. Formally, it could describe external regions $\hat{D}^H_{1,3}$ and $\hat{D}^S_{1,3}$ if one made a substitution $r^\mu \to R^\mu$ and assumed $1 + \eta r^2 / L^2 < 0$. However, the metric obtained in this way becomes inconsistent with the condition above.

To derive an external metric $H_{\mu\nu}$ transforming into $\eta_{\mu\nu}$ at $L \to 0$ it should be noted that conditions (5), (6) may be derived from (3), (4) by the conformal inversion of coordinates

$$r^\mu = \frac{L^2}{R^2} R^\mu, \quad R^\mu = \frac{L^2}{r^2} r^\mu,$$  

(25)

with simultaneous transformation of the line elements

$$ds = \frac{r^2}{L^2} dS = \frac{L^2}{R^2} dS,$$  

(26)

specifying a one-to-one mapping from $V^i$ onto $V^e$. Going over to the coordinates $R^\mu$ in (1) with the help of (25) and taking into account the relation (26), we find that the metric of $V^e$ is determined by

$$H_{\mu\nu} = \left( 1 + \frac{\eta L^2}{R^2} \right)^{-1} \eta_{\mu\nu} - \left( 1 + \frac{\eta L^2}{R^2} \right)^{-2} \frac{\eta L^2 \eta_{\mu\alpha} \eta_{\nu\beta} R^\alpha R^\beta}{R^4},$$  

(27)
\[ H^{\mu\nu} = \left(1 + \frac{\eta L^2}{R^2}\right) \left[ \eta^{\mu\nu} + \frac{\eta L^2 R^\mu R^\nu}{R^4} \right], \quad H_{\mu\lambda} H^{\lambda\nu} = \delta_{\mu}^{\nu}. \] (28)

Let us call a space-time with the metric (27) the conformally conjugated de Sitter space (CCdS) or the conformally conjugated anti-de Sitter space (CCAdS).

Coordinate transformation
\[
R^\mu = \frac{1}{2} \left(1 - \frac{\eta L^2}{X^2}\right) X^\mu, \quad X^2 = \eta_{\mu\nu} X^\mu X^\nu;
\] (29)

\[ X^\mu = -\frac{\eta R^2}{L^2} \left[1 \pm \sqrt{1 + \frac{\eta L^2}{R^2}}\right] R^\mu, \quad \left(\frac{1}{0}\right) \leq -\frac{\eta X^2}{L^2} < \left(\infty\right), \] (30)

reduces the interval (2) to conformally flat form
\[
dS^2 = H_{\mu\nu} dR^\mu dR^\nu = \frac{1}{4} \left(1 - \frac{\eta L^2}{X^2}\right)^4 \left(1 + \frac{\eta L^2}{X^2}\right)^{-2} \eta_{\mu\nu} dX^\mu dX^\nu. \] (31)

A classification of all conformally flat metrics by the isometry group is known [21] and therefore one may say at once that the isometry group of \( V^e \) coincides with its isotropy group, i.e. with the total Lorentz group, moreover, a group of isometry for \( \tilde{D}^{+}_{1,3} \) is a connected component of unit, SO(1,3). It is easy verified directly by solving the Killing equation (19) in the metric (27) giving six Killing vectors with components (21).

According to embedding theorems [22] \( V^e \) may be embedded into six-dimensional pseudo-Euclidean space \( \tilde{E}_{2,4}^R \) with the line element
\[
dS^2 = \eta_{\mu\nu} dz^\mu dz^\nu + \eta(dz^5)^2 - \eta(dz^6)^2. \] (32)

One may consider the metric (27) as the metric on (2+4)-dimensional cone \( C_{2,4} \)
\[ \eta_{\mu\nu} z^\mu z^\nu + \eta(z^5)^2 - \eta(z^6)^2 = 0 \] (33)

in this space. Embedding formulae are
\[
z^\mu = \frac{\eta R^2}{L^2} \left[1 \pm \sqrt{1 + \frac{\eta L^2}{R^2}}\right] R^\mu = Z(R^2) R^\mu,
\]
\[ z^5 = \frac{Z(R^2)}{L} \left(R^2 - \frac{L^2}{4}\right), \quad z^6 = \frac{Z(R^2)}{L} \left(R^2 + \frac{L^2}{4}\right). \] (34)

### 4 Spherical symmetry

A cross section of the one-sheet hyperboloid \( r^2 = R^2 = -L^2 \) by the hyperplane \( r^0 = 0 \) is a sphere of the radius \( L \), being a singularity surface for related metrics obtained from (11) and (27). Such a spherical approximation may be of physical interest for both general relativistic and elementary particles problems. Obviously, it is meaningful only for regions \( D^H_{1,3} \) and \( \tilde{D}^H_{1,3} \), while it is possible for \( D^S_{1,3} \) and \( \tilde{D}^S_{1,3} \) only at \( |r^0| = \text{const} > L \).
Spherically symmetric metrics corresponding to $D^H_{1,3}$ and $\tilde{D}^H_{1,3}$ may be derived in two ways. First, one may use eqs. (8)-(10), where $\rho = 0$ should be assumed. Then the consistent solution will be of the form (11) with additional condition $r^0 = 0$. Just such a metric was used in [1]. The corresponding external metric will have the form (27) with additional condition $R^0 = 0$. Second, there may be assumed simultaneously $\rho = 0$, $r^0 = 0$ in the isometric transformation (10) with $\Lambda$ given by (8). Then metrics $\hat{h}_{\mu\nu}$, $\hat{H}_{\mu\nu}$, derived in this way are

$$\hat{h}_{0\mu} = \hat{h}_{\mu 0} = \eta_{0\mu}, \quad \hat{h}_{ij} = h_{ij}|_{r^0 = 0}; \quad (35)$$

$$\hat{H}_{0\mu} = \hat{H}_{\mu 0} = \eta_{0\mu}, \quad \hat{H}_{ij} = H_{ij}|_{R^0 = 0}; \quad i, j = 1, 2, 3. \quad (36)$$

A comparison of (35) with (11) and (36) with (27) shows that distinctions occur only in (00)-components. Respective line elements may be obtained from each other by the transformation of differentials

$$dr^0 = \left(1 - \frac{r^2}{L^2}\right)^{-1/2} \, d\hat{r}^0, \quad \hat{r}^i = r^i, \quad (37)$$

and

$$d\hat{R}^0 = \left(1 - \frac{L^2}{R^2}\right)^{-1/2} \, dR^0, \quad \hat{R}^i = R^i, \quad (38)$$

where $r^2 = -\eta_{ij}r^i r^j = (r^1)^2 + (r^2)^2 + (r^3)^2$, $R^2 = -\eta_{ij}R^i R^j = (R^1)^2 + (R^2)^2 + (R^3)^2$.

Mentioned cross section causes a contraction of the isometry groups of $V^i$ and $V^e$ to $T \times O(1,3)$ and $T \times O(3)$, respectively, where $T$ is the one-parameter group of time translations. In the first case isometry groups act on arbitrary hyperplanes $r^0 = \text{const}$, while in the case (35), (36) they act only on the hyperplane $\hat{r}^0 = 0$. Thus, there arises a problem of a choice, connected with a choice of a reference frame, between the metric (35), (36) and the metrics derived in the first way, which are in terms of spherical coordinates

$$ds^2 = \left(1 - \frac{r^2}{L^2}\right)^{-1} \left[ (dr)^2 - \left(1 - \frac{r^2}{L^2}\right)^{-1} (d\varphi^2 - r^2 (d\varphi^2 + \sin^2 \varphi) d\varphi^2) \right], \quad (39)$$

$$dS^2 = \left(1 - \frac{L^2}{R^2}\right)^{-1} \left[ (d\hat{R})^2 - \left(1 - \frac{L^2}{R^2}\right)^{-1} (d\varphi^2 - R^2 (d\varphi^2 + \sin^2 \varphi) d\varphi^2) \right]. \quad (40)$$

It should be noted that the external metric (40) at large $R^2$ may approximately be written as

$$dS^2 \approx \left(1 + \frac{L^2}{R^2} + \ldots\right) (d\hat{R})^2 - \frac{1 + 3L^2/R^2 + \ldots}{1 + L^2/R^2 + \ldots} \left[ (d\varphi^2 - R^2 (d\varphi^2 + \sin^2 \varphi) d\varphi^2) \right]. \quad (41)$$

Hence, it follows that (40) approximates at very large distances the Reissner-Nordström metric for the massless charge $e \approx L$. Formally assuming this charge to be the electron charge we obtain for $L$: $L = \sqrt{n\alpha\hbar c} \approx 7.6 \cdot 10^{-36} \text{ m}$. Such interpretation, as well as a choice between the metrics above, requires an additional foundation based on the analysis
of both field equations and equations of motion. Here we only point out a form of the energy-momentum tensor in cases (27) and (40). For the case (27) we have

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} H_{\mu\nu} R = -\kappa T_{\mu\nu} = \frac{12}{R^2} \eta_{\mu\nu} - \frac{3}{R^2} \left(4 + \frac{5\eta L^2}{R^2}\right) H_{\mu\nu}. \] (42)

Denoting the metric derived from (27) in the first way as \( \bar{H}_{\mu\nu} \) and the Einstein and energy-momentum tensors through \( \bar{G}_{\mu\nu} \) and \( \bar{T}_{\mu\nu} \), respectively, we find

\[ \bar{H}_{\mu\nu} = H_{\mu\nu}|_{R^0=0}, \]

\[ \bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} \bar{H}_{\mu\nu} \bar{R} = -\kappa \bar{T}_{\mu\nu}, \]

\[ = G_{\mu\nu}|_{R^0=0} - \frac{2\eta L^2}{R^4} \left[ \eta_{\mu\nu} - \eta_{\mu0}\delta_0^\nu - \frac{1 + \eta L^2/R^2}{1 - \eta L^2/R^2} \bar{R}^i \bar{R}^j \eta_{i\mu} \eta_{j\nu} \right]. \] (44)

As one can see from (42) and (44), an interpretation of the metrics (27) and (40) is embarrassing and it is not clear at the present time what are their sources. However, in the physically interesting case of AdS spaces \( (\eta = +1) \) we have the energy-momentum tensors with positive-definite \((00)\)-components, and \( \bar{T}_{00} = T_{00}|_{R^0=0} \).

5 Conclusion

A question of using the de Sitter groups in General Relativity and elementary particles theory was being put for a long time because they are unique minimal groups which may contract (in the Inönü-Wigner sense) to the Poincaré group. Thus, the de Sitter spaces may serve as models of a physical space transforming into the flat one when curvature tends to zero. In our case the de Sitter space appears as a result of the consideration of the one-body problem. Indeed, if there is any object in flat space-time, its field leads to an effective distortion of the space what may be expressed in terms of geometrical concepts as it was made above, for example. Here, internal, \( D_{1,3}^H \), and external \( \tilde{D}_{1,3}^H \), spaces are complementary for each other (conformally conjugated). \( D_{1,3}^H \) realizes the AdS geometry being considered comparatively rarely because it has closed time-like geodesics. Assuming \( D_{1,3}^H \) to be a cosmological space with large curvature radius, \( L \), it is difficult to understand what one has to do with such geodesics [23]. However, should one look at from outside then such a behavior become consistent with the idea of geometric confinement. Here we should like to pay attention to one of possible approaches to introducing a fundamental length basing upon a metrical geometry (see, e.g., [2]). In our case the parameter \( L \) can play a role of such a fundamental length.

In our opinion, an attractive feature of the metrics obtained, (11) and (27) (and/or their spherically symmetric ”nonrelativistic” versions (35)-(36), (39)-(40)) is that they have a unique irremovable singularity at \( r^2 = R^2 = -L^2 \), i.e. on the ”surface” of the object in question. Therefore, one may suggest the metric (27) (or (40)) for rough simulating the field generated by an extended object with hyperbolic (or spherical) symmetry placed into the Minkowski space-time \( E_{1,3}^R \). It enables one to give a clear physico-geometric interpretation to coordinates \( R^\mu \). Namely, they are pseudo-Euclidean coordinates relative to the center of symmetry \( R^\mu = 0 \) of points in which a field of the extended object is detected. Certainly, at this point a question arises about realisticity of the metric.
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