Transitions to quantum chaos in a generic one-parameter family of billiards

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Abstract

Generic one-parameter billiards are studied both classically and quantally. The classical dynamics for the billiards makes a transition from regular to fully chaotic motion through intermediary soft chaotic system. The energy spectra of the billiards are computed using finite element method which has not been applied to the euclidean billiard. True generic quantum chaotic transitional behavior and its sensitive dependence on classical dynamics are uncovered for the first time. That is, this sensitive dependence of quantum spectral measures on classical dynamics is a genuine manifestation of quantum chaos.

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Since the pioneering work of McDonald and Kaufman [1,2], the study of statistical measures of quantum spectra has been one of the major themes for quantum chaos. This study uncovered universal behaviors for integrable and most of hard chaotic systems. For transitional systems, i.e., from integrable to hard chaos through soft chaos scenario, however, few observations have been made [3,5–7]. Among those observations, Robnik’s [3,4] analysis showed using level spacing statistics that there is a gradual continuous transition from regular to hard chaos. Hönig and Wintgen’s analysis [8], on the other hand, discovered that level spacing statistics showed extraordinary behavior in the case of the corresponding classical dynamics with complexity, and suggested that the understanding of it require a detailed knowledge of the underlying classical dynamics rather than a knowledge of the global classical phase space structure. Grasping a generic transitional quantum behavior is very important to understand quantum chaos itself which has not been defined very clearly. These themes motivate this study.

In this Letter we present the results of complete analysis, both classically and quantally, of a generic one-parameter family of billiard systems [9,10]. As the generic one-parameter billiards, we studied Dreitlein’s billiards [12] whose boundary consists of two parallel lines with their separation as length and arcs of circles of a radius at both extremes. Fig. 1 shows a typical Dreitlein’s billiard. If we vary radii of the circular arcs the system changes from a square billiard to the Bunimovitch stadium. We parametrize the billiards by a parameter \( \lambda \) as follows,

\[
\lambda = \exp \left( -\sqrt{R^2 - 1} \right),
\]

where \( R \) is the radius of circular arcs. As we change \( \lambda \) from 0 to 1, Dreitlein’s billiards change from a square billiard to the Bunimovitch stadium. At some parameter value \( \lambda = \lambda_c \), the billiards become a hard chaotic system, i.e., K-system, through soft chaos. Non-analyticity of the boundary for Dreitlein’s billiard makes its classical dynamics relatively simple compared to the other one-parameter family of billiards with analytic boundaries. This simplicity helps to get the complete analysis of its classical dynamics by studying the Birkhoff surface of sections, linear stability of periodic orbits, and imbedded island structures.

We summarize the results of our analysis of classical dynamics of Dreitlein’s billiards in the following. As the parameter \( \lambda \) increases from 0 to up, the billiard starts to deform from a square billiard. Its Birkhoff surface of section (illustrated in Figure 2) shows lots of islands contiguous with each other. The islands of stability, surrounding elliptic fixed points, are divided by a very thin connected chaotic region. The absence of separatrix motion due to non-analyticity of the boundary makes its structure of the surface of section simple, i.e., there is only one major connected chaotic region. As we increase \( \lambda \) further, most of the islands quickly disappear. The most robust islands surround the corresponding period two and period four orbits. Our present numerical analysis shows that the island which surrounds period four orbit undergoes period doubling bifurcation at \( \lambda = 0.167 \). It is in good agreement with \( \lambda = e^{-1 - 2^{-1/3}} = 0.1663... \) obtained from the study of monodromy matrix. We numerically observed that small isolated regular regions from period doubling bifurcation disappear in the chaotic sea at about \( \lambda = 0.198 \) The most robust island, which surrounds period two orbit, is found to disappear at \( \lambda_c = 0.368 \). The study of monodromy matrix produces \( \lambda_c = e^{-1} \) which gives very excellent agreement. After that, Dreitlein’s billiards become fully chaotic system (hard chaotic system) as \( \lambda \) increases.
Quantum mechanical analysis should solve the Schrödinger equation with Dirichlet boundary conditions. This is equivalent to solve the Helmholtz equation, for which finite element method (FEM) is employed. FEM is, in principle, one of the best method to deal with the boundary problems. The advantages of FEM are known to be able to obtain both eigenvalues and eigenfunctions at a time, and to provide solutions to many complicated problems that would be intractable by other techniques. Our FEM approach is the first application to the euclidean quantum billiard, to our best knowledge. We normalize the area of the billiards to $\pi$ so that the mean level spacing is the same for all Dreitlein’s billiards. Since the classical chaoticity comes from the circular arcs of Dreitlein’s billiards, the boundary approximation is very important to obtain accurate results. Because of moderate curvatures of Dreitlein’s billiards, iso-parametric coordinates give very good approximation. According to Henshell’s estimate [13], the typical radial errors are within less than $10^{-5}\%$.

By using FEM, we calculate odd-odd parity eigenvalues and eigenfunctions of Dreitlein’s billiards up to about 2400th states for each of 46 different parameter values of $\lambda$. Analysis of these data provides the ways to investigate level spacing statistics, spectral rigidities and patterns of eigenfunctions.

In this Letter, we are limited to discuss only the level spacing statistics. The study of the other subjects will be followed near future. The first 1100 reliable eigenvalues are used for each $\lambda$ to obtain accurate level spacing statistics. To study level spacing statistics of Dreitlein’s billiards, we tried to fit the data to the Brody distribution which is originally suggested by Brody et.al. [11] and later used by Robnik [3]. The Brody distribution interpolates Poisson and Wigner distributions which are characteristic level spacing distributions of integral and chaotic spectra, respectively.

$$P(s) = a s^\nu \exp(-b s^{\nu+1})$$

where $a$ and $b$ are determined by normalization conditions, i.e., $\int_0^\infty P(s) \, ds = 1$ and $\int_0^\infty s \, P(s) \, ds = 1$. The level spacing exponent $\nu$ is a measure of short range interaction between energy levels, i.e., level repulsion. One of advantages to use the Brody distribution is that we can define cumulative distribution $[2]$. Since it is a smoother function of $s$ than $P(s)$, it is more easily fitted by data. We fitted our data to the cumulative Brody distribution, $W(s) = \int_0^s P(x) \, dx$, in order to obtain best fitted $\nu$. Fig. 3 shows our main results of the study. There are three different parameter regions which show different qualitative behaviors. We call these regions Region I, II and III.

In the soft chaotic region (Region I), i.e., $0 < \lambda < \lambda_c$, it clearly shows that the distribution is moving from a Poisson to a Wigner distribution. However, it is certainly not a gradual continuous transition. As soon as $\lambda$ increases from 0, $\nu$ increases sharply and reach about 0.6 when $\lambda$ becomes 0.05. And then it tends to saturate until $\lambda$ becomes 0.175. Notice that the lowest value of $\lambda$ in that stretch is obtained surprisingly at $\lambda = 0.167$, where period two orbits lose their stability. There is a big jump in $\nu$ at $\lambda = 0.175$ where there is only one major isolated integrable region left in the classical phase space. After that $\nu$ fluctuates again around 0.7 for a while and followed by another big jump at $\lambda = 0.36$ which is again very close to $\lambda_c = 0.368$ where the system becomes completely chaotic, i.e., K-system. The observation in Region I implies that quantum chaotic transition in Dreitlein’s billiards is very sensitive to the corresponding classical dynamics. This reflects that there is an intrinsic coincidence between quantum chaos and classical dynamics. In the hard chaotic
region (Regions II and III), i.e., $\lambda_c < \lambda < 1$, we can see two different behaviors depending on $\lambda$. For the region $\lambda_c < \lambda < 0.85$, as we expected, the level spacing distribution is close to the Wigner distribution, which is believed to be a universal characteristic behavior of hard chaotic system. In Region II, $\nu$ is bigger than 0.85 except at $\lambda = 0.7$. This implies strong level repulsions between levels. However for the region $0.85 < \lambda < 1$, the exponent $\nu$ shows a clear tendency to decrease. This is an unexpected results which needs an explanation. The explanation of this behavior will be the main topic of our future work. Finally, we would like to mention that the exponent $\nu$ at $\lambda = 1$ is 0.717 which is in good agreement with McDonald and Kaufman’s 0.71 for the Bunimovitch stadium. Through our complete analysis of the corresponding classical system, we have found that quantum spectral measures are sensitively related to the topological changes of phase space manifolds of the classical system.

In conclusion in the present Letter, the major finding in this study is that the quantum chaotic transition is not a smooth gradual transition. There are big changes in the distribution of eigenvalues for small changes in $\lambda$. It should be noticed that the parameter ranges, $0.05 < \lambda < 0.167$ and $0.175 < \lambda < 0.36$, correspond to parameter ranges in which classical phase spaces have two major isolated islands and one major isolated island, respectively. And the transition values, $\lambda = 0.175$ and $\lambda = 0.36$, are very close to the values at which period four and period two orbits lose their stability. This means the detailed dynamical behavior of a classical system is very important to understand quantum chaotic transition. We claim that this is the generic behavior of quantum transition to chaos with soft chaos scenario. We think this sensitive dependence of quantum spectral measures on classical dynamics is a genuine manifestation of quantum chaos. This also fortifies that spectral analysis is the legitimate method to study quantum chaos.

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FIGURES

FIG. 1. A typical shape of the one-parameter billiards. $R$ is the radius of the arc boundaries.

FIG. 2. The Birkhoff surface of section for $\lambda = 10^{-10}$ after 100,000 iterations. The initial point is given at the neighborhood of the boundary of the period two island.

FIG. 3. Plot for the Brody’s exponent $\nu$ versus the parameter $\lambda$
Fig. 3