Evolution of a two-level quantum particle in the noise classical e.-m. field within and beyond the resonant approximation

A M Basharov
Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia
National Research Centre «Kurchatov Institute», 123182 Moscow, Russia
E-mail: basharov@gmail.com; http://basharov.me

Abstract. The paper has shown that the interaction of a stochastic field with a two-level system beyond the resonant approximation is equivalent to the resonant interaction of a noise field in maximally squeezed state.

1. Introduction
In some cases interaction of delta-correlated random fields with quantum systems allows for a fully analytical analysis based on stochastic differential equations (SDE). We differentiate between resonant and non-resonant interaction models. With a two-level quantum system, resonance approximation is also called the rotating wave approximation and its corrections are considered - the so-called counter-rotating (anti-rotating) terms [1]. Using a random delta-correlated field allows considering the two cases analytically and demonstrating that the choice of a particular approach is determined by the nature of the random field - the ratio between the scale of the random change of a physical quantity represented as a delta-correlated field and the reverse transition frequency between a pair of the observed energy levels of a quantum particle. The below analysis for the classic delta-correlated field shows that from the standpoint of the kinetic equation for the density matrix of the quantum system, both cases can be presented in a unified manner in the framework of the resonance approximation, which uses different views on the squeezing of a random field.

2. Interaction of a random field with a two-level quantum particle by the Wiener process
Let a two-level atom with Hamiltonian $H_a$ and energy levels $|E_1>$ and $|E_2>$ interact in the electric dipole approximation with the electromagnetic field intensity $\tilde{E}$, so that the model of such interaction is based on the following Schrödinger equation for the wave function $|\Psi>$ of the whole system in the interaction picture

$$i\hbar \frac{d}{dt} |\Psi(t)> = H(t) |\Psi(t)> = -\tilde{E}(t)\tilde{d}(t),$$

$$|\Psi(t)> = U(t)|\Psi(0)>,$$

$$i\hbar \frac{d}{dt} U(t) = H(t)U(t), \quad U(0) = 1.$$
Here $\mathbf{d}(t) = \sum_{jk} d_{jk} e^{i\mathbf{A}_{jk} t} | E_j > < E_k |$ is the vector dipole moment operator of the atom. An atomic subsystem is represented by projection operators $| E_j > < E_j |$ onto the atomic states $| E_i >$, described as energies $E_i$ and orthogonal expansion of unity:

$$\sum_i | E_j > < E_j | = 1, < E_i | E_j >= \delta_{ij}, H_a = \sum_i E_i | E_i > < E_i |.$$  

The transition frequency between the levels of a quantum particle is designated as $\Omega_{21} = (E_2 - E_1)/\hbar = -\Omega_{12}$.

The solution of equation (1) can be written in the form of series

$$U(t) = I + (-i/\hbar) \int_0^t H(t') dt' + (-i/\hbar)^2 \int_0^t \int_0^t H(t') H(t'') dt' dt'' + \ldots = \tilde{U} \exp \left( -i/\hbar \int_0^t H(t') dt' \right). \quad (2)$$

Neglecting the polarization effects the interaction operator can be rewritten as

$$H(t) = -(d_1 e^{-i\omega_1 t} R_1 + d_2 e^{i\omega_1 t} R_2) E(t), \quad R_i = | E_2 > < E_1 |, \quad R_1 = | E_1 > < E_2 | = R_c^*.$$  

A natural requirement imposed on $E(t)$ in the case of its representation in the form of a random process is stationarity of this value, i.e. the dependence of the correlation function $< E(t) E(t') > = g(t - t')$ only on the time difference $t - t'$.

When $E(t)$ is expanded in a Fourier integral, the stationarity of interaction results in the noncorrelation (independence) of spectral components

$$< b(\omega) b^*(\omega') >= \delta(\omega - \omega') S(\omega),$$

where $S(\omega)$ is the power spectrum of the random effects, with the Fourier components being introduced:

$$b(\omega) = 1/2\pi \int e^{i\omega t} \mathbf{d}(t) dt, \quad b(\omega) = b^*(\omega')$$

for $E(t) = E^*(t) = \int e^{i\omega t} b(\omega) d\omega$.

If we take the natural approach $S(\omega) = \text{const}$ for a specific spectral range and extend it to the whole region of the spectrum (Nyquist approximation), it is easy to see that

$$< E(t) E(t') >= D\delta(t - t').$$

Then the value $\int_0^t E(t') dt = W(t)$ is continuous and is nowhere differentiable. It can be shown in many ways that the density of the probability distribution function $\rho(w, t)$ of value $W(t)$ satisfies the kinetic equation (Fokker-Planck equation [2])

$$\frac{\partial \rho(w, t)}{\partial t} = D \frac{\partial^2 \rho(w, t)}{\partial w^2}$$

with the solution

$$\rho(w, t | w_0, t_0) = \frac{1}{\sqrt{2\pi D(t - t_0)}} \exp \left[ -\frac{(w - w_0)^2}{2D(t - t_0)} \right],$$

for an initial condition $\rho(w, t_0 | w_0, t_0) = \delta(w - w_0)$. The value $W(t)$ is called the Wiener process. The non-standard Wiener process is mentioned in certain circumstances, relating $D = 1$ to the standard case.

Equation (1) is mathematically uncertain in the Nyquist approximation. But you can define strictly its integral "solution" (2). This requires a reasonable definition of the integral over the Wiener
A conventional definition of the integral \( \int_{t_0}^{t_1} G(t')dW(t') \) as the limit of the partial sums \( S_n = \sum_{i=1}^{n} G(\tau_i) (W(t_i) - W(t_{i-1})) \) leads to its dependence on the specific choice of intermediate points \( \tau_i \) of the partition intervals: \( t_{i-1} \leq t_i \). This can easily be seen from the example of the integral \( \int_{t_0}^{t_1} W(t')dW(t') \) by using the properties of the Wiener process \( W(t) \) - the statistical independence of its increments \( W(t_i) - W(t_{i-1}) \) of one another. The choice of Ito \( \tau_i = t_{i-1} \) is convenient and self-consistent in case of the so-called non-anticipatory functions \( G(t) \), statistically independent of the future behavior of the Wiener process at the time \( t \). The limit of the integral sums is thought of as the mean square limit. Then you can introduce the established Itô differentials:

\[
dW(t)dW(t) = Ddt, \quad [dW(t)]^{N+2} = 0, \quad N > 0, \quad dW(t)dt = dtdt = 0, <dW(t) >= 0. \tag{3}
\]

Here the angular brackets denote averaging and differential relations should be understood as fulfillment of integral equations for any nonanticipating functions such as:

\[
\int_{t_0}^{t_1} G(t')dW(t')dW(t') = \int_{t_0}^{t_1} G(t')dt'.
\]

Algebra (3) of Ito differentials allows obtaining SDE easily for the evolution operator

\[
U(t) = \bar{T} \exp \left\{ i \int_{0}^{t} (d_1 e^{-\alpha_2 t} R_+ \right. + d_2 e^{\alpha_2 t} R_+ dW(t')) \right\}.
\]

Then

\[
dU(t) = \{ \exp( i(d_1 e^{-\alpha_2 t} R_+ \right. + d_2 e^{\alpha_2 t} R_+ dW(t')) - 1)U(t) \}.
\]

Simple calculations based on Ito algebra (3) lead to the following SDE:

\[
dU(t) = i(d_1 e^{-\alpha_2 t} R_+ + d_2 e^{\alpha_2 t} R_+)dW(t)U(t) - \frac{1}{2} D(d_1 e^{-\alpha_2 t} R_+ + d_2 e^{\alpha_2 t} R_+)^2 U(t)dt, \tag{4}
\]

In this case (as can be easily seen by direct calculation) the rule for Ito differentiating is also fulfilled

\[
d(U^+(t)U(t)) = (dU^+(t)U(t)+U^+(t)dU(t)) + (dU^+(t))dU(t),
\]

and equations for the density matrix \( \rho^a(t) \) of the atom are conventionally derived from the chain of equations:

\[
d\rho^a(t) = \rho^a(t + dt) - \rho^a(t), \quad \rho^a(t + dt) = \langle \Psi(t + dt) | \Psi(t + dt) \rangle U(t + dt) | \Psi(0) > < \Psi(0) | U^+(t + dt) > ,
\]

\[
d\rho^a(t) = \langle dU(t) | \Psi(0) > < \Psi(0) | U^+(t) + U(t) | \Psi(0) > < \Psi(0) | dU^+(t) + dU(t) | \Psi(0) > < \Psi(0) | dU^+(t) > .
\]

Upon averaging it is convenient to rewrite these equations in the following form
\[
\frac{d\rho^\alpha(t)}{dt} = n[2Y\rho^\alpha(t)Y^* - Y^*Y\rho^\alpha(t)] + m[2Y^*\rho^\alpha(t)Y^* - Y^*Y^*\rho^\alpha(t)]
\]
\[
\quad + n[2Y^*\rho^\alpha(t)Y^* - Y^*Y^*\rho^\alpha(t)] + m^2[2Y\rho^\alpha(t) - YY\rho^\alpha(t)].
\]

We introduce the notation
\[
Y = d_1R_- , \quad Y^* = d_2R_+ , \quad n = D , \quad m = De^{i2\Omega_3\phi}.
\]

There exists the ratio \(|m| = n\) between the introduced parameters \(n\) and \(m\). However, below we can see that from the standpoint of the kinetic equation for the two-level atom, the description of the influence of the random field having neither a carrier frequency nor resonance approximation satisfies completely the kinetic equation derived in the resonance approximation under the influence of the resonance noise wave having a carrier frequency. In other words, result (5) also describes the interaction with a two-level system in the resonance approximation of a noise wave in the maximally squeezed state with a carrier frequency, which coincides with the transition frequency between these levels.

3. Interaction of a random field with a two-level quantum particle in the rotating wave approximation

Now we are going to consider the interaction of a noise wave with the carrier frequency, which coincides with the transition frequency of a two-level system in the rotating wave approximation. The correct description of the noise wave with the carrier frequency suggests that we introduce a complex-valued Wiener process.

It is convenient to view the two non-standard independent Wiener processes \(W_1(t)\) and \(W_2(t)\) with algebra
\[
dW_1(t)dW_1(t) = D_1dt , \quad dW_2(t)dW_2(t) = D_2dt , \quad dW_1(t)dt = dW_2(t)dt = dt = 0.
\]

They allow defining the complex-valued Wiener process as \(\widetilde{W}(t) = W_1(t) + iW_2(t)\), for which we have the following Ito algebra
\[
d\widetilde{W}(t)d\widetilde{W}(t) = (D_1 - D_2)dt , \quad d\widetilde{W}(t)d\widetilde{W}^*(t) = (D_1 + D_2)dt , \quad d\widetilde{W}(t)dt = d\widetilde{W}^*(t)dt = dt = 0.
\]

Finally, the complex-valued Wiener process in a general form is defined as \(\widetilde{W}(t) = \widetilde{W}_1(t) + e^{im^*}\widetilde{W}_2(t)\), so
\[
d\widetilde{W}(t)d\widetilde{W}^*(t) = (D_1 + D_2)dt = n dt , \quad d\widetilde{W}(t)dt = d\widetilde{W}^*(t)dt = dt = 0,
\]
\[
d\widetilde{W}(t)d\widetilde{W}(t) = (D_1 + D_2e^{i2\Omega})dt = m dt , \quad d\widetilde{W}^*(t)d\widetilde{W}(t) = (D_1 + D_2e^{-i2\Omega})dt = m^* dt.
\]

It is evident that
\[
|m| \leq n ,
\]

because
\[
|D_1 + D_2e^{i2\Omega}| = \sqrt{(D_1 + D_2e^{i2\Omega})(D_1 + D_2e^{-i2\Omega})} = \sqrt{(D_1^2 + D_2^2 + 2D_1D_2\cos(2\Omega))} \leq D_1 + D_2.
\]

Then you can introduce random processes with independent increments, satisfying conditions (6) with additional requirements (7) and
\[
<d\widetilde{W}(t)> = <d\widetilde{W}^*(t)> = 0.
\]
Condition (7) is necessary in order that the components of the complex-valued process of this type
\( W_i(t) \) and \( W_j(t) \) have a Gaussian distribution.

The introduction of complex-valued Wiener \( \beta \)-type processes with algebra (6) - (7) is convenient to
describe random electromagnetic fields with a carrier frequency. The complex-valued Wiener process in a general form allows you to write an interaction operator of the classical electromagnetic field with a two-level system in the rotating wave approximation in the form

\[
H(t)dt = -d_{12}R_e dW^*(t) - d_{21}R_e dW(t').
\]

We emphasize that this expression is meaningful only if the evolution operator of a quantum system is represented in the form of a \( T \)-exponent:

\[
U(t) = \exp\left\{ i \int_0^t \left[ (d_{12}R_e dW^*(t') + d_{21}R_e dW(t')) \right] \right\}.
\]

Expanding the expression in the form \( dU(t) \) we obtain

\[
dU(t) = i(d_{12}R_e dW^*(t') + d_{21}R_e dW(t'))U(t)
- \frac{1}{2} \left[ d_{12}^2 R_e R_e m^* + |d_{12}|^2 R_e R_e n + |d_{12}|^2 R_e R_e n + d_{12}^2 R_e R_e m \right] dtU(t).
\]

As a result of the above described calculations we obtain the kinetic equation for the density matrix of a two-level atom in form (5), and the parameters \( n \) and \( m \) are not related to ratio \( |m|=n \) as in the case of absence of any resonance approximation, but are governed by general equation (7) and defined by equations (6).

The case \( m \neq 0 \) corresponds to the so-called squeezing of an electromagnetic field whose creation requires some special conditions of modulation. The limiting case \( |m|=n \) corresponds to the maximum squeezing, and with a two-level system suggests another interpretation outlined in the previous section.

Equality of the squeezing parameter to zero \( m = 0 \) describes the usual classical noise electromagnetic wave characterized by a carrier frequency. It is a case which corresponds to the modeling of resonant effects of a random field with a quantum system.

Thus, the resonance approximation or its rejection for the purpose of considering the alleged "general" case is practically determined by the parameters of the external influence. If the conditions of representation of external influence by a Wiener process are satisfied, then we obtain the solution of the equation for the evolution operator in the form of \( T \)-exponent, and derivation of the kinetic equation for the density matrix of a quantum system is an example of analytical calculations made with such expressions. If the resonance conditions are satisfied, it is necessary to develop a consistent algebraic perturbation theory [2], in the first order of which there arises the rotating wave approximation discussed above. If you restrict yourself to solving the problem at the level of the kinetic equation for the density matrix of a quantum system, then within the limits of the resonance approximation all the features of the noise field, including its squeezing, are considered, whereas beyond the resonance algebraic perturbation theory only the limiting case of the noise field state is described, but this restriction of the model of a multilevel quantum system by its two levels for a noise field without the carrier frequency is physically meaningless. It is also difficult to justify SDE when the characteristic time scale of random change in the field is of the order of the reverse frequency of quantum transition. For SDE to be justified, it is necessary that the scale of change in a random field were much less than the reverse frequency of quantum transition, which is unreal in the optical frequency range from the standpoint of physics of electromagnetic fields. However, in the case of a two-level system formed by sublevels of a quantum level, i.e. when the transition frequency of a two-level system is small, it is possible to consider the interaction process beyond the resonance approximation, provided that the other levels of a quantum system are either absent or located far
enough to apply the Nyquist approximation to the spectrum of a noise wave. Then formula (5), where $|m| = n$, leads to the kinetic equation for a quantum system beyond the resonance approximation.

Unfortunately, methods of the SDE are unpopular among scientists in spite of being very effective in analytical solution of many physical problems.

Acknowledgments
This work is supported in part by the Russian Foundation for Basic Research (grant No. 16-02-00453a).

4. References
[1] Mandel L and Wolf E 1995 Optical Coherence and Quantum Optics (New York: Cambridge University Press)
[2] Gardiner C W 2004 Handbook of stochastic methods (Berlin, Heidelberg: Springer-Verlag)
[3] Maimistov A I and Basharov A M 1999 Nonlinear optical waves (Dordrecht: Kluwer Academic)