Probability Distributions and Particle Number Fluctuations of Trapped Bose-Einstein Condensates in Different Dimensions

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The analytical probability distribution of finite systems obeying Bose-Einstein statistics in one, two, and three dimensions are investigated by using a canonical ensemble approach. Starting from the canonical partition function of the system, a unified approach is provided to study the probability distribution of a condensate for various confinements and in different dimensions. Based on the probability distribution function, it is straightforward to obtain the mean ground state occupation number and particle number fluctuations of the condensate. It is found that the particle number fluctuations in a trapped Bose gas are strongly dependent on the type of confining potential and on the dimensionality of the system.

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I. INTRODUCTION

The experimental achievement of Bose-Einstein condensation (BEC) in trapped dilute alkali atoms \textsuperscript{1}, spin-polarized hydrogen \textsuperscript{2} and metastable helium \textsuperscript{3} has stimulated considerable theoretical research \textsuperscript{4,5} on this unique phenomenon. The BEC has been recently realized in quasi-one and quasi-two dimensions \textsuperscript{6,7}, where new phenomena such as quasicondensates with a fluctuating phase \textsuperscript{8,9} and a Tonks gas of impenetrable bosons \textsuperscript{10,11} may be observed. Among the several intriguing questions on the statistical properties of the trapped Bose gas, the probability distribution and particle number fluctuations \[\langle \delta^2 N_0 \rangle\] of condensate play an important role. Apart from the intrinsic theoretical interest, it is foreseeable that the condensate fluctuations will become experimentally testable in the near future \textsuperscript{12,13}. On the other hand, the calculation of \[\langle \delta^2 N_0 \rangle\] is important for investigating the phase collapse time of the condensate \textsuperscript{14,15}. Recall that the BEC can not occur in one-dimensional (1D) and two-dimensional (2D) uniform Bose gases at finite temperature, because in that case thermal fluctuations can destabilize the condensate. The realization of the trapped BECs in various dimensions makes the behavior of particle number fluctuations a very interesting problem, especially for 1D and 2D harmonically trapped Bose gases.

Within a grand-canonical ensemble the particle number fluctuations of the condensate are given by \[\langle \delta^2 N_0 \rangle = N_0 (N_0 + 1),\] implying that \[\delta N_0\] becomes of order \[N\] when the temperature approaches zero. To avoid this sort of unphysically large condensate fluctuations, a canonical (or a microcanonical) ensemble has to be used to investigate \[\delta N_0\]. Because in the experiment the trapped atoms are cooled continuously from the surrounding, the system can be taken as being in contact with a heat bath but the total number of particles in the system is conserved. It is therefore necessary to use the canonical ensemble to investigate the statistical properties of the trapped Bose gas.

Within the canonical (or microcanonical) ensemble, the particle number fluctuations have been studied in the case of three-dimensional (3D) ideal Bose gases confined in a box \textsuperscript{16,17}, and in the presence of a harmonic trap \textsuperscript{20,21}. The question of how interatomic interactions affect the particle number fluctuations has also been the object of several recent theoretical investigations \textsuperscript{22–27}. Although the phase fluctuations \textsuperscript{28–34} of a condensate have been studied for quasi-1D and quasi-2D Bose gases, to best our knowledge up to now an analytical description of the probability distribution and the behavior of \[\delta N_0\] for trapped 1D and 2D Bose gases have not been given directly from the microscopic statistics of the system. The purpose of the present work is an attempt to calculate \[\delta N_0\] of the trapped ideal Bose gas in various dimensions, using the analytical description of the probability distribution obtained directly from the analysis of the canonical partition function of the system. In addition, the probability distribution of the condensate will be used to calculate the mean ground state occupation number \[\langle N_0 \rangle\].

The paper is organized as follows. Sec. II is devoted to outline the canonical ensemble, which is developed to consider the probability distribution and particle number fluctuations of trapped ideal Bose gases. In Secs. III-V we investigate the particle number fluctuations of 3D, 2D and 1D Bose gases trapped in a harmonic potential and in a box, respectively. The last section (Sec. VI) contains a discussion and summary of our results.
II. PROBABILITY DISTRIBUTION AND PARTICLE NUMBER FLUCTUATIONS OF THE CONDENSATE

In the canonical ensemble, the canonical partition function of $N$ trapped ideal bosons is given by

$$Z(N) = \sum_{\Sigma_n N_n = N} \exp\left[-\beta \Sigma_n N_n \epsilon_n\right],$$  \hspace{1cm} (1)

where $\beta = 1/k_B T$, $N_n$ and $\epsilon_n$ are occupation number and energy level of the state $n$, respectively. The basic starting point of our approach for calculating the probability distribution of the condensate is to separate off the ground state labelled $n = \mathbf{0}$ from the states $n \neq \mathbf{0}$:

$$Z(N) = \sum_{N_0 = 0}^{N} \{ \exp[-\beta N_0 \epsilon_0] Z_0(N_T) \},$$  \hspace{1cm} (2)

where $Z_0(N_T)$ represents the canonical partition function of a fictitious system consisting of $N_T = N - N_0$ trapped ideal non-condensed bosons:

$$Z_0(N_T) = \sum_{\Sigma_n N_n = N_T} \exp\left[-\beta \Sigma_n N_n \epsilon_n\right].$$  \hspace{1cm} (3)

Note that the ground state $n = \mathbf{0}$ has been separated off in the canonical partition function $Z_0(N_T)$ of the fictitious system. In the following we will find that this separation plays a crucial role for the calculations of the probability distribution of the system.

Assuming that $A_0(N, N_0)$ is the free energy of the fictitious system, we have

$$A_0(N, N_0) = -k_B T \ln Z_0(N_T).$$  \hspace{1cm} (4)

The calculation of the free energy $A_0(N, N_0)$ is non-trivial because there is a requirement that the number of non-condensed bosons is $N_T$ in the summation of the canonical partition function $Z_0(N_T)$. To proceed we define a generating function $G_0$ for $Z_0(N_T)$ in the following manner. For any complex number $z$, let

$$G_0(T, z) = \sum_{N_T = 0}^{\infty} z^{N_T} Z_0(N_T).$$  \hspace{1cm} (5)

The generating function can be evaluated easily. To obtain $Z_0(N_T)$ we note that by definition $Z_0(N_T)$ is the coefficient of $z^{N_T}$ in the expansion of $G_0(T, z)$ in powers of $z$. Thus one has

$$Z_0(N_T) = \frac{1}{2\pi i} \oint dz \frac{G_0(T, z)}{z^{N_T+1}},$$  \hspace{1cm} (6)

where the contour of integration is a closed path in the complex $z$ plane around $z = 0$.

Assuming $\exp[g(z)] = G_0(T, z)/z^{N_T+1}$, the saddle point $z_0$ is determined by $\partial g(z_0)/\partial z_0 = 0$. Expanding the integrand of Eq. ($\theta$) about $z = z_0$, we have

$$Z_0(N_T) = \frac{1}{2\pi i} \oint dz \exp[g(z_0) + o((z-z_0)\beta)],$$  \hspace{1cm} (7)

where $o((z-z_0)^2) = 1/2 (z-z_0)^2 \partial^2 g(z_0) + \cdots$ represents the high order terms when expanding the integrand about the saddle point $z_0$.

Omitting the high order terms on the right hand side of Eq. ($\theta$), it is straightforward to obtain the following relations between the free energy $A_0(N, N_0)$ and the saddle point $z_0$ of the fictitious $N - N_0$ non-condensed bosons

$$-\beta \frac{\partial}{\partial N_0} A_0(N, N_0) = \ln z_0.$$  \hspace{1cm} (8)

In addition, the saddle point $z_0$ is determined by

$$N_0 = N - \sum_{n \neq \mathbf{0}} \frac{1}{\exp[\epsilon_n/k_B T] z_0^{\epsilon_n}} - 1.$$  \hspace{1cm} (9)

Although the relations given by Eqs. ($\theta$) and ($\theta$) can not provide an explicit result of $Z_0(N_T)$, they are very useful to calculate the probability distribution of the condensate.

We should stress that the relations given by Eqs. ($\theta$) and ($\theta$) are reliable although the disputable saddle-point method is used to investigate the free energy of the fictitious system. It is well known that the applicability of the standard saddle-point method for condensed Bose gases has been the subject of a long debate ($\alpha$). In conventional approaches, the saddle-point method is used to discuss the canonical partition function $Z(N)$. The generating function is therefore defined by $G(T, z) = \sum_{N_0}^{\infty} z^N Z(N)$. In this scheme, the high order terms of $\ln Z(N)$ can not be omitted for the temperature below the onset of the BEC, because the factor $1/\{z_0^{\beta} e^{\beta \alpha} - 1\}$ in the high order terms would be on the order of $N$ (See Eq. (8) in Ref. [27]). In our approach, however, the saddle-point method is used only to discuss the canonical partition function $Z_0(N_T)$ of the fictitious non-condensed bosons. Because the ground state has been separated off in $Z_0(N_T)$, the high order terms of $\ln Z_0(N_T)$ can be safely omitted.

Using the free energy of the fictitious system, the canonical partition function of the system becomes

$$Z(N) = \sum_{N_0 = 0}^{N} \exp[q(N, N_0)],$$  \hspace{1cm} (10)

where

$$q(N, N_0) = -\beta N_0 \epsilon_0 - \beta A_0(N, N_0).$$  \hspace{1cm} (11)

Obviously, $P(N_0|N) = \exp[q(N, N_0)]/Z(N)$ represents the probability to find $N_0$ atoms in the condensate.
Once \( q(N, N_0) \) is calculated from the canonical partition function of the system, the statistical properties of the system can be easily obtained. However, it seems to be very difficult to obtain the analytical result of \( q(N, N_0) \) directly from the canonical partition function. To avoid this difficulty, we turn to investigate the partial derivative of \( q(N, N_0) \) with respect to \( N_0 \). Assuming

\[
\frac{\partial}{\partial N_0} q(N, N_0) = \alpha(N, N_0),
\]

from Eq. (11) one obtains

\[
-\beta \frac{\partial}{\partial N_0} A_0(N, N_0) = \beta \varepsilon_0 + \alpha(N, N_0) .
\]

Using the relations given by Eqs. (8) and (9), we get

\[
N_0 = N - \sum_{n \neq 0} \frac{1}{\exp[\beta(\varepsilon_n - \varepsilon_0)] - 1} .
\]

The most probability to find \( N_0 \) atoms in the condensate is determined by requiring \( \frac{\partial}{\partial N_0} q(N, N_0) = 0 \). Thus the most probable value \( N_0^p \) is determined by setting \( \alpha(N, N_0) = 0 \) in the right hand side of Eq. (14)

\[
N_0^p = N - \sum_{n \neq 0} \frac{1}{\exp[\beta(\varepsilon_n - \varepsilon_0)] - 1} .
\]

It is interesting to note that \( N_0^p \) is exactly the mean ground state occupation number in the frame of a grand-canonical ensemble. For sufficiently large \( N \), the sum \( \sum_{N_0=0}^N \) may be replaced by the largest term, since the error omitted in doing so will be statistically negligible. Hence the result given by Eq. (15) shows the equivalence between the canonical and grand-canonical ensembles for large \( N \).

From the results given by Eqs. (14) and (13), we get the following equation for determining \( \alpha(N, N_0) \)

\[
N_0 - N_0^p = \sum_{n \neq 0} \frac{1}{\exp[\beta(\varepsilon_n - \varepsilon_0)] - 1} - \sum_{n \neq 0} \frac{1}{\exp[\beta(\varepsilon_n - \varepsilon_0)] \exp[-\alpha(N, N_0)] - 1}.
\]

Thus once we know the single-particle energy level of the system, it is straightforward to obtain \( \alpha(N, N_0) \). Using \( \alpha(N, N_0) \), one can obtain the probability distribution of the system.

From Eq. (12), we get the following result for \( q(N, N_0) \)

\[
q(N, N_0) = \int_{N_0^p}^{N_0} \alpha(N, N_0) dN_0 + q(N, N_0^p) .
\]

The partition function of the system is then

\[
Z(N) = \sum_{N_0=0}^N \{\exp[q(N, N_0)] G(N, N_0)\} ,
\]

where

\[
G(N, N_0) = \exp \left[ \int_{N_0^p}^{N_0} \alpha(N, N_0) dN_0 \right] .
\]

It is obvious that \( G(N, N_0) \) represents the ratio \( P(N_0|N) / P(N_0^p|N) \), i.e., the relative probability to find \( N_0 \) atoms in the condensate. From Eq. (13), the normalized probability distribution function is given by

\[
G_n(N, N_0) = A_n \exp \left[ \int_{N_0^p}^{N_0} \alpha(N, N_0) dN_0 \right] ,
\]

where \( A_n \) is a normalization constant.

As soon as we know \( G(N, N_0) \), the statistical properties of the system can be clearly described. From Eqs. (18) and (19) one obtains \( \langle N_0 \rangle \) and \( \langle \delta^2 N_0 \rangle \) within the canonical ensemble:

\[
\langle N_0 \rangle = \sum_{N_0=0}^N N_0 G(N, N_0) / \sum_{N_0=0}^N G(N, N_0) ,
\]

\[
\langle \delta^2 N_0 \rangle = \langle N_0^2 \rangle - \langle N_0 \rangle^2
\]

\[
= \frac{\sum_{N_0=0}^N N_0^2 G(N, N_0)}{\sum_{N_0=0}^N G(N, N_0)} - \left[ \frac{\sum_{N_0=0}^N N_0 G(N, N_0)}{\sum_{N_0=0}^N G(N, N_0)} \right]^2 .
\]

We see that in our approach, the calculation of \( \alpha(N, N_0) \) by using Eq. (13) plays a crucial role to discuss the particle number fluctuations of the condensate. The probability distribution of the condensate is obtained from \( \alpha(N, N_0) \). We can give a fairly accurate description of the particle number fluctuations through the calculations of the probability distribution, although the high order terms are omitted when obtaining the relations (8) and (9). Our discussions on the particle number fluctuations are reasonable mainly due to the following reasons:

(i) Because the ground state \( (n = 0) \) and the excited states \( (n \neq 0) \) have been separated off when considering the partition function of the system, and the saddle-point approximation is only used to investigate the canonical partition function of the fictitious non-condensed bosons, \( Z_0(N_T) \), the high order terms omitted in the approximation do not give a significant correction to the particle number fluctuations.

(ii) When obtaining \( \alpha(N, N_0) \) through the difference between \( N_0 \) and \( N_0^p \), the high order terms omitted in Eqs. (13) and (14) are nearly cancelled with each other.
is true especially for the case of $\delta N_0/N << 1$. Thus the error coming from the high order terms will be lowered further for the calculation of $\alpha (N, N_0)$.

On the other hand, in usual statistical method, $\langle N_0 \rangle$ and $\langle \delta^2 N_0 \rangle$ are obtained through the first and second partial derivative of the canonical partition function, respectively. When the saddle-point approximation is used to calculate the canonical partition function of the system, the contribution due to the high order terms are amplified in the second partial derivative of the canonical partition function. Thus one can not obtain accurate particle number fluctuations using this method. The approach developed above provides in some sense a simple method recovering the applicability of the saddle-point method through the calculations of the probability distribution of the system.

III. THREE-DIMENSIONAL BOSE GASES

Now we apply the formulation presented in the last section to calculate the thermodynamical quantities such as the particle number fluctuations of the condensate for trapped ideal Bose gases. An important feature characterizing the available magnetic trap is that the confining potential can be safely approximated with the quadratic function.

$$V_{\text{ext}}(r) = \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right), \quad (23)$$

where $m$ is the mass of atoms. $\omega_x$, $\omega_y$, $\omega_z$ are frequencies of magnetic traps. The single-particle energy level of a 3D harmonically confined ideal Bose gas has the form

$$\varepsilon_n = \left( n_x + \frac{1}{2} \right) \hbar \omega_x + \left( n_y + \frac{1}{2} \right) \hbar \omega_y + \left( n_z + \frac{1}{2} \right) \hbar \omega_z, \quad (24)$$

where $n_x$, $n_y$, and $n_z$ are non-negative integers. Using the following density of states

$$\rho(E) = \frac{1}{2 \hbar \omega_{3D}} + \frac{3\pi E}{2 \omega_{3D} \hbar \omega_{3D}}, \quad (25)$$

one obtains $N_0$ from Eq. (14)

$$N_0 = N - \frac{N}{\zeta(3)} \left( \frac{T}{T_{3D}} \right)^3 g_3 (\exp (\alpha (N, N_0)))$$

$$- \frac{3\pi \zeta(2)}{2 \omega_{3D} \zeta(3)^{2/3}} \left( \frac{T}{T_{3D}} \right)^2 N^{2/3}, \quad (26)$$

where $T_{3D} = \frac{\hbar \omega}{k_B} \left( \frac{N}{\zeta(3)} \right)^{1/3}$ is the critical temperature of the 3D ideal Bose gas. $\zeta(z) = 1/\zeta(z)$ are arithmetic and geometric averages of the oscillator frequencies, respectively. $g_3(z)$ belongs to the class of functions $g_\alpha(z) = \sum_{n=1}^\infty z^n / n^\alpha$ and $\zeta(n)$ is Riemann $\zeta$ function. Setting $\alpha (N, N_0) = 0$ in Eq. (24), the most probable value $N_0^p$ is then

$$N_0^p = N - N \left( \frac{T}{T_{3D}} \right)^3 - \frac{3\pi \zeta(2)}{2 \omega_{3D} \zeta(3)^{2/3}} \left( \frac{T}{T_{3D}} \right)^2 N^{2/3}. \quad (27)$$

From Eqs. (26) and (27) we obtain the result for $\alpha (N, N_0)$

$$\alpha (N, N_0) = - \frac{\zeta(3)}{\zeta(2)} \frac{(N_0 - N_0^p)^2}{2 N_0^3}, \quad (28)$$

where $t = T/T_{3D}$ is a reduced temperature. When obtaining $\alpha (N, N_0)$ we have used the expansion $g_{1+\delta}(z) \approx \zeta(z) + \zeta(2) \delta T/T_c$. From Eqs. (14) and (28) we obtain the normalized probability distribution of the 3D harmonically trapped ideal Bose gas:

$$G_{3D} (N, N_0) = A_{3D} \exp \left[ - \frac{\zeta(3)}{\zeta(2)} \frac{(N_0 - N_0^p)^2}{2 N_0^3} \right], \quad (29)$$

where $A_{3D}$ is a normalization constant. It is interesting to note that Eq. (29) is a Gaussian distribution function.

From the formulas (21), (22), (27), and (28) we can obtain $N_0$ and $\langle \delta^2 N_0 \rangle$ of the system. Shown in Fig. 1 is the the numerical result of $\delta N_0 = \sqrt{\langle \delta^2 N_0 \rangle}$ (solid line) for $N = 10^3$ ideal bosons in an isotropic harmonic trap. The dashed line displays the result of Holthaus et al. [27], where the saddle-point method is developed to discuss the high order terms omitted in the standard saddle-point approximation below the onset of BEC.

![Fig. 1](image-url)
In Fig. 1 the dotted line displays the result of Refs. [21,28]. Our results coincide with that of Refs. [21,28] except near the critical temperature. In fact, below the critical temperature, from the probability distribution (29), one obtains the analytical result of the condensate fluctuations

$$\langle \delta^2 N_0 \rangle = \frac{\zeta(2)}{\zeta(3)} N t^3. \quad (30)$$

When obtaining Eq. (30), we used Eq. (29) derived in the Appendix. Below the critical temperature, Eq. (30) recovers the result in Refs. [21,28] by noting that \(\zeta(2) = \pi^2/6\). This shows the validity of the probability distribution given by Eq. (29) for discussing the statistical properties of the system. At the critical temperature \(T_{3D}\), however, our results give

$$\langle \delta^2 N_0 \rangle |_{T=T_{3D}} = \frac{(1-2/\pi) \zeta(2) N}{\zeta(3)}, \quad (31)$$

which is much smaller than that obtained in Ref. [28]. This difference is appreciable because the analysis in Ref. [28] holds in the canonical ensemble except near and above \(T_{3D}\), whereas our result holds also for the temperature near \(T_{3D}\). Near the critical temperature, our result (solid line) agrees with that of Holthaus et al. [2]. Results given by Eqs. (30) and (31) show that particle number fluctuations in the 3D harmonic trap have a normal behavior, i.e., they are proportional to \(N\).

When \(T \to 0\), the particle number fluctuations of the condensate should be zero. To check this, note that as \(T \to 0\), \(G_{3D}(N,N_0) = A_{3D}\) if \(N_0 = N\), while \(G_{3D}(N,N_0) \to 0\) when \(N_0 \neq N\). Additionally, \(N_0^0 \to N\) in the case of \(T \to 0\). Thus, we obtain \(\langle N_0 \rangle \to N\) and \(\langle \delta^2 N_0 \rangle \to 0\) when \(T \to 0\). The correct description of \(\delta N_0\) near zero temperature and critical temperature again shows the validity of our method.

It is well known that confinement reduces the fluctuation effect. Thus the fluctuations for bosons confined in a box should have a stronger dependence on \(N\), in comparison with the case confined in the harmonic trap. To show this we consider a 3D ideal Bose gas confined in a cubic box. The single-particle energy level in this case is given by

$$\varepsilon_n = \pi^2 \hbar^2 (n_x^2 + n_y^2 + n_z^2)/(2nL^2),$$

where \(L = L^3\), \(A_{3D}\) and \(\lambda = \sqrt{2\pi\hbar^2/m}\) are normalization constant and thermal wavelength, respectively. From Eq. (32) we obtain the particle number fluctuations below the critical temperature:

$$\langle \delta^2 N_0 \rangle = A \left( \frac{mk_BT}{\hbar^2} \right)^2 V^{4/3}, \quad (33)$$

where the coefficient \(A = 2/\pi^4 \times \Sigma_n \neq 0 1/n^4 = 0.105\). We see that, different from the harmonic trap, the particle number fluctuations of the 3D Bose gas in a box exhibit an anomalous behavior, i.e., it is proportional to \(V^{4/3}\) (or \(N^{4/3}\)).

**IV. TWO-DIMENSIONAL BOSE GASES**

In this section we turn to discuss trapped 2D Bose gases. The single-particle energy level of a 2D harmonically confined ideal Bose gas takes the form

$$\varepsilon_n = \left( n_x + \frac{1}{2} \right) \hbar\omega_x + \left( n_y + \frac{1}{2} \right) \hbar\omega_y. \quad (34)$$

From Eq. (17) one has

$$N_0 = N - \frac{1}{\sum_{n \neq 0} \exp[\beta(n_x \hbar\omega_x + n_y \hbar\omega_y)]|^{\varepsilon_n} - 1}. \quad (35)$$

Then we obtain \(N_0\) through a simple integration over \(n\) [12]:

$$N_0 = N - \left( \frac{k_BT}{\hbar\omega_{2D}} \right)^2 g_2(\alpha) - \frac{1}{\zeta(2)\left(\zeta(2)\right)^{1/2}\hbar\omega_{2D} T \ln N - \frac{1}{\hbar\omega_{2D}}}, \quad (36)$$

where \(\omega_{2D} = (\omega_x \omega_y)^{1/2}\). When \(0 < \alpha << 1\), there is a good approximation [10] for \(g_2(\alpha)\):

$$g_2(\alpha) \approx \zeta(2) + \alpha \left( 1 - \ln \alpha \right). \quad (37)$$

In the case of \(\alpha < 0\) and \(|\alpha| << 1\), \(g_2(\alpha)\) is

$$g_2(\alpha) \approx \zeta(2) - |\alpha| \left( 1 - \ln |\alpha| \right). \quad (38)$$

By setting \(\alpha = 0\) in Eq. (36), we obtain the most probable value \(N_0^0\) as:

$$N_0^0 = N - \left( \frac{k_BT}{\hbar\omega_{2D}} \right)^2 \zeta(2) - \frac{\zeta(2)^{1/2}k_BT \ln N}{\hbar\omega_{2D}}, \quad (39)$$

Using the approximations given by Eqs. (37) and (38), we arrive at the following result for \(\alpha(\mathcal{N}, N_0)\):

$$\alpha(\mathcal{N}, N_0) = - \frac{\zeta(2) \zeta(3) (N_0 - N_0^0)^2}{2Nt^2 \ln (Nt^2/\zeta(2))}, \quad (40)$$

where \(T_{2D} = \left( \frac{N}{\zeta(2)} \right)^{1/2} \frac{\hbar\omega_{2D}}{k_B}\) is the critical temperature of the system in the thermodynamic limit. Accordingly, the probability distribution of the system reads

$$G_{2D}(N, N_0) = A_{2D} \exp \left[ - \frac{\zeta(2) \zeta(3) (N_0 - N_0^0)^2}{4Nt^2 \ln (Nt^2/\zeta(2))} \right], \quad (41)$$
where $A_{2D}$ is a normalization constant. We see that the probability distribution of the 2D harmonically trapped ideal Bose gas is also a Gaussian function. However, the behavior of the fluctuations is different from that of the trapped 3D Bose gas because of the factor $\ln \left( Nt^2 / \zeta (2) \right)$ in Eq. (11). Below the critical temperature, the analytical description of the particle number fluctuations is given by

$$\langle \delta^2 N_0 \rangle = \frac{2Nt^2 \ln \left( Nt^2 / \zeta (2) \right)}{\zeta (2) \zeta (3)}. \quad (42)$$

At the critical temperature $T_{2D}$, the particle number fluctuations take the form

$$\langle \delta^2 N_0 \rangle |_{T = T_{2D}} = \frac{2 (1 - 2/\pi) N \ln \left( N/\zeta (2) \right)}{\zeta (2) \zeta (3)}. \quad (43)$$

FIG. 2. Displayed is $\langle \delta^2 N_0 \rangle$ as a function of $N$ bosons in a 2D harmonic trap for various reduced temperatures. The linearity of $\langle \delta^2 N_0 \rangle$ about $N$ is clearly shown in the figure. This shows that the behavior of $\langle \delta^2 N_0 \rangle$ can be approximated as normal for 2D harmonically trapped Bose gas.

We see that the particle number fluctuations of the 2D harmonically trapped Bose gas exhibit a very weak anomalous behavior, as it is controlled by the factor $\ln \left( Nt^2 / \zeta (2) \right)$. In Fig. 2 we plot the result of $\langle \delta^2 N_0 \rangle$ as a function of $N$ for various reduced temperatures. The linear property of $\langle \delta^2 N_0 \rangle$ about $N$ is clearly illustrated. This shows that the behavior of $\langle \delta^2 N_0 \rangle$ can be approximated as normal.

Shown in Fig. 3 is the result for $\langle N_0 \rangle / N$ as a function of temperature for $N = 10^3$ 2D harmonically trapped bosons. The solid line displays $\langle N_0 \rangle / N$ using a grand-canonical ensemble (or $N^p$ within the canonical ensemble). The dotted line displays $\langle N_0 \rangle / N$ within the canonical ensemble. In Fig. 4 we plot the numerical result of $\delta N_0$ for $N = 10^3$ trapped 2D ideal bosons. From Fig. 4 we find that the particle number fluctuations of the 2D Bose gas are larger than those obtained for the trapped 3D Bose gas at identical reduced temperature (see Fig. 1).

FIG. 3. Relative mean ground state occupation number $\langle N_0 \rangle / N$ vs $T/T_c^0$ for $N = 10^3$ non-interacting bosons confined in a 2D harmonic trap. The solid line shows $\langle N_0 \rangle / N$ within the grand-canonical ensemble, while the dotted line displays $\langle N_0 \rangle / N$ within the canonical ensemble.

FIG. 4. Displayed is $\delta N_0$ for $N = 10^3$ non-interacting bosons confined in a 2D harmonic trap.

For a 2D ideal Bose gas confined in a box, the single-particle energy level is given by $\varepsilon_n = \pi^2 \hbar^2 \left( n_x^2 + n_y^2 \right) / 2mL^2$. There exists a logarithmic divergence in Eqs. (14) and (15) when the sum is calculated by integration in momentum space. The occurrence of the logarithmic divergence implies an anomalous behavior of the particle number fluctuations. Because of the divergence in Eqs. (14) and (15), the sum can not be calculated by integration, but is dominated by the discretized sum over the low-energy bosons. The final result of the particle number fluctuations can be estimated as

$$\langle \delta^2 N_0 \rangle \approx \sum_{n \neq 0} \frac{1}{(n_x^2 + n_y^2)} \left( \frac{2mL^2k_BT}{\pi^2\hbar^2} \right)^2. \quad (44)$$

This result given shows that, for the 2D gas confined in box, there is a strong anomalous behavior for the particle number fluctuations, i.e., $\langle \delta^2 N_0 \rangle \sim N^2$. 

6
V. ONE-DIMENSIONAL BOSE GASES

The single-particle energy level of a 1D harmonically
confined ideal Bose gas has the form

\[ \varepsilon_n = \left( n + \frac{1}{2} \right) \hbar \omega_{1D}. \] (45)

From Eq. (14) one obtains

\[ N_0 = N - \sum_{n=1}^{\infty} \frac{1}{\exp \left[ \beta n \hbar \omega_{1D} \right] \exp \left[ -\alpha \left( N, N_0 \right) \right] - 1}. \] (46)

The most probable value \( N_0^p \) reads

\[ N_0^p = N - \sum_{n=1}^{\infty} \frac{1}{\exp \left[ \beta n \hbar \omega_{1D} \right] - 1}. \] (47)

There is a logarithmic divergence when the sum is calculated by integration in momentum space. With the
method developed in Ref. [42], the approximation of \( N_0^p \)

is given by

\[ N_0^p = N - \left( \frac{k_B T}{\hbar \omega_{1D}} \right) \ln \frac{2k_B T}{\hbar \omega_{1D}}. \] (48)

The critical temperature \( T_{1D} \) of the trapped 1D Bose
gas can be obtained by setting \( N_0^p = 0 \) in Eq. (48). We
obtain \( T_{1D} = \frac{\hbar \omega_{1D}}{k_B} (N/ \text{ProductLog}[2N]) \), where
\text{ProductLog}[z] is the solution \( w = \text{ProductLog}[z] \)
of the equation \( z = w e^w \) (see Ref. [43]).

From Eqs. (46) and (47) we get

\[ N_0 - N_0^p = \sum_{n=1}^{\infty} \frac{1}{\exp \left[ \beta n \hbar \omega_{1D} \right] \exp \left[ -\alpha \left( N, N_0 \right) \right] - 1} - \frac{1}{\exp \left[ \beta n \hbar \omega_{1D} \right] \exp \left[ -\alpha \left( N, N_0 \right) \right] - 1}. \] (49)

The occurrence of the logarithmic divergence in momentum
space implies that the leading contribution to the
particle number fluctuations comes from the low energy
bosons. In this situation, Eq. (49) can be estimated to be

\[ N_0 - N_0^p = \sum_{n=1}^{\infty} \left[ \frac{1}{\beta n \hbar \omega_{1D}} - \frac{1}{\beta n \hbar \omega_{1D} - \alpha \left( N, N_0 \right)} \right]. \] (50)

Assuming \( |\alpha \left( N, N_0 \right)| \ll \beta \hbar \omega_{1D} \),

we obtain the final result of \( \alpha \left( N, N_0 \right) \):

\[ \alpha \left( N, N_0 \right) = -\frac{N_0 - N_0^p}{\sum_{n=1}^{\infty} 1/\left[ \beta n \hbar \omega_{1D} \right]^2}. \] (52)

It is easy to find that the result given by Eq. (52) coincides with the assumption given by Eq. (53). The
probability distribution of the 1D harmonically trapped
Bose gas is then given by

\[ G_{1D}(N, N_0) = A_{1D} \exp \left[ -\frac{\left( N_0 - N_0^p \right)^2}{2 \sum_{n=1}^{\infty} 1/\left[ \beta n \hbar \omega_{1D} \right]^2} \right]. \] (53)

where \( A_{1D} \) is a normalization constant.

Using the probability distribution \( G_{1D}(N, N_0) \), one obtains
the analytical result of \( \langle \delta^2 N_0 \rangle \) below the critical
temperature:

\[ \langle \delta^2 N_0 \rangle = \zeta(2) \left[ \frac{N t}{\text{ProductLog}[2N]} \right]^2. \] (54)

At \( T_{1D} \), \( \langle \delta^2 N_0 \rangle \) can be approximated as

\[ \langle \delta^2 N_0 \rangle \big|_{T=T_{1D}} = \left( \frac{1}{2} \right) \zeta(2) \left[ \frac{N}{\text{ProductLog}[2N]} \right]^2. \] (55)

Comparing this result with those obtained for trapped
2D and 3D Bose gases, the particle number fluctuations
in 1D have a much stronger dependence on \( N \). This
fact reminds us of the occurrence of infrared divergence
arising from the low energy bosons in a 1D free ideal
Bose gas. Although here we consider a confined Bose
gas and hence the infrared divergence has been cut off
in the discrete sums in Eqs. (49) and (47), the existence
of low energy bosons results in an anomalous behavior
of the particle number fluctuations. From Eqs. (44) and
(55), it is obvious that the particle number fluctuations
of the trapped 1D ideal Bose gas exhibit a very strong
anomalous behavior. In Fig. 5 we plot \( \langle \delta^2 N_0 \rangle \) as a function of \( N \) for various reduced temperatures. The
anomalous behavior of \( \langle \delta^2 N_0 \rangle \) for the 1D harmonically
trapped Bose gas is clearly illustrated in the figure.

FIG. 5. We plot \( \langle \delta^2 N_0 \rangle \) as a function of \( N \) bosons in a 1D harmonic trap for various reduced temperatures. The anomalous behavior of \( \langle \delta^2 N_0 \rangle \) for 1D harmonically trapped Bose gas is clearly demonstrated in the figure.
Shown in Fig. 6 is the numerical result of $\langle N_0 \rangle / N$ for $N = 10^3$ trapped 1D ideal bosons. The numerical result of $\delta N_0$ for $N = 10^3$ is plotted in Fig. 7. At finite temperature, this sort of large condensate thermal fluctuations may destabilize the condensate. The inclusion of repulsive interactions between atoms can, however, significantly change the behavior of particle number fluctuations. An interacting Bose gas exhibits a phonon-type low energy excitations. The fluctuations may still exhibit an anomalous behavior even for the 1D interacting Bose gas confined in a harmonic trap. We expect that the two-body repulsive force may lower the particle number fluctuations significantly and lead to the stability of 1D trapped condensate.

For an ideal Bose gas confined in a line (1D “box”), the particle number fluctuations exhibit a much stronger anomalous behavior in comparison with the case of the harmonically trapped Bose gas. This can be seen because there is a $1/n$ divergence when $n \to 0$ in Eqs. (13) and (14). The final result of the particle number fluctuations in this case is given by

$$\langle \delta^2 N_0 \rangle \approx \left( \sum_{n=1}^{\infty} \frac{1}{n^4} \right) \left( \frac{2mL^2k_BT}{\pi^2\hbar^2} \right)^2. \quad (56)$$

Thus in such system there is a very strong anomalous behavior for the particle number fluctuations, i.e., $\langle \delta^2 N_0 \rangle \sim N^2$. This sort of anomalous behavior will certainly destabilize the condensate.

VI. DISCUSSION AND CONCLUSION

In this work, a saddle-point method has been developed to investigate the canonical partition function of trapped Bose gases. Within the canonical ensemble the analytical probability distribution and particle number fluctuations have been obtained for various dimensions, especially in one and two dimensions. Different from the conventional methods, here the analytical probability distribution is obtained directly from the canonical partition function of the system. Using the probability distribution function, we have calculated the thermodynamic properties of the trapped Bose gas, such as condensate fraction and particle number fluctuations. Through the calculations of the probability distribution function, we have provided a simple method recovering the applicability of the saddle-point method for studying the particle number fluctuations.

We have found that although the probability distribution of the harmonically trapped Bose gas are Gaussian functions, the behavior of the particle number fluctuations is quite different from each other for different dimensions. For a trapped 2D ideal Bose gas, the fluctuations exhibit a very weak anomalous behavior, while there is a strong anomalous behavior for a trapped 1D ideal Bose gas. These properties are clearly shown by the explicit formulas given by (10), (12) and (14) for the particle number fluctuations in three, two and one dimensions, respectively. We expect that the recent realization of BEC in lower dimensions makes it very promising to explore the particle number fluctuations of lower-dimensional condensates. Our results also show that different confinements can significantly change the effect of the particle number fluctuations. Comparing with the case confined in a harmonic trap, the Bose gases confined in a box exhibit much stronger effect for the condensate fluctuations.

It is obvious that the method developed here can be applied to other Bose systems. Based on our approach, one can also formulate an improved saddle-point method to calculate the change of the particle number fluctuations when using a microcanonical ensemble. It can be shown that this will result in a change for particle number fluctuations only with a numerical prefactor $\frac{1}{n^4}$. 

FIG. 6. Displayed is $\langle N_0 \rangle / N$ vs $T/T_c^0$ for $N = 10^3$ non-interacting bosons confined in a 1D harmonic trap. The solid line shows $\langle N_0 \rangle / N$ within the grand-canonical ensemble, while the dotted line displays $\langle N_0 \rangle / N$ within the canonical ensemble.

FIG. 7. The numerical result of $\delta N_0$ for $N = 10^3$ non-interacting bosons confined in a 1D harmonic trap.
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APPENDIX

The probability distribution of a harmonically trapped Bose gas is a Gaussian function for various dimensions. The ratio between \( P(N_0|N) \) and \( P(N_0^p|N) \) is given by

\[
G(N_0, N_0^p) = \exp \left[ -a \left( N_0 - N_0^p \right)^2 \right],
\]  

(57)

where \( a \) is a constant and is related to the confinement, dimensionality and the total number of particles of the system. The sum in Eq. \( \text{[22]} \) can be replaced by an integral

\[
\langle \delta^2 N_0 \rangle = \frac{\int_{N_0=0}^{N} N_0^p G(N, N_0) \, dN_0}{\int_{N_0=0}^{N} G(N, N_0) \, dN_0}
- \left[ \frac{\int_{N_0=0}^{N} N_0 G(N, N_0) \, dN_0}{\int_{N_0=0}^{N} G(N, N_0) \, dN_0} \right]^2.
\]  

(58)

Using the integral transformation \( x = N_0 - N_0^p \), one obtains

\[
\int_{N_0=0}^{N} N_0^p G(N, N_0) \, dN_0 = \int_{-N_0^p}^{N-N_0^p} (x + N_0^p)^2 \exp[-ax^2] \, dx.
\]  

(59)

For the temperature below the critical temperature, \( N_0^p >> 1 \). Hence the integral in Eq. \( \text{[59]} \) can be estimated to be

\[
\int_{N_0=0}^{N} N_0^p G(N, N_0) \, dN_0 \approx \int_{-\infty}^{\infty} (x + N_0^p)^2 \exp[-ax^2] \, dx.
\]  

(60)

Similarly, we have

\[
\int_{N_0=0}^{N} G(N, N_0) \, dN_0 \approx \int_{-\infty}^{\infty} \exp[-ax^2] \, dx,
\]  

(61)

and

\[
\int_{N_0=0}^{N} N_0 G(N, N_0) \, dN_0 \approx \int_{-\infty}^{\infty} (x + N_0^p) \exp[-ax^2] \, dx.
\]  

(62)

Using the formulas \( \text{[60]}, \text{[61]} \) and \( \text{[62]} \), it is straightforward to obtain the analytical result of \( \langle \delta^2 N_0 \rangle \) for the temperature below the critical temperature

\[
\langle \delta^2 N_0 \rangle = \frac{1}{2a}.
\]  

(63)

We can see from Eq. \( \text{[63]} \) that the behavior of the fluctuations is determined by the factor \( a \).

At the critical temperature in the thermodynamic limit, we have \( N_0^p = 0 \). The probability distribution of the system is then

\[
G(N_0, N_0^p) = \exp \left[ -a N_0^2 \right].
\]  

(64)

After a simple calculation, the analytical result of \( \langle \delta^2 N_0 \rangle \) at the critical temperature is given by

\[
\langle \delta^2 N_0 \rangle = \left( 1 - \frac{2}{\pi} \right) \frac{1}{2a}.
\]  

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