RANK ZERO QUADRATIC TWISTS OF $y^2 = x^3 + 2$

AZIZUL HOQUE

Abstract. We produce two families of rank zero quadratic twists of the elliptic curve $E : y^2 = x^3 + 2$. Moreover, we prove that there are infinitely many members with rank zero in each of these families.

1. Introduction

It is well known that the (abelian) group $E(\mathbb{Q})$ of the rational points on an elliptic curve $E$ defined over $\mathbb{Q}$ is finitely generated. The rank of $E$ is the minimal number of generators of the free abelian group $E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}}$. Thus it is positive if and only if there are infinitely many rational points on $E$. The determination of the rank of $E$ is an important problem. Specially producing elliptic curves with arbitrarily high rank is still an unsolved problem. A geometric analogue of this problem has been shown to be true by Shafarevich and Tate [11]. In most cases where the rank has been obtained, it is actually very small. In fact, it is widely believed that half of the curves have rank zero, while the other half are of rank one. Thus there is not much room left for the curves of higher rank.

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ by the Weierstrass equation

$$E : y^2 = x^3 + ax + b,$$

where $a$ and $b$ are integers. For a square-free integer $D$, the quadratic twist of $E$ by $D$ is given by

$$E_D : y^2 = x^3 + aD^2x + bD^3.$$

Then one can ask the following question:

Q 1.1. What can be said about the variation of the rank of $E_D$ as $D$ varies over the square-free integers?

Let $\mathcal{N}$ be the conductor of $E$ and $\chi_D$ the quadratic character associated to $D$. The parity conjecture states that if $\gcd(D,2\mathcal{N}) = 1$, 2010 Mathematics Subject Classification. Primary: 11G05, Secondary: 14G05, 11R29.

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then the ranks of both $E$ and $E_D$ have the same parity if and only if $\chi_D(−N) = 1$. Under truth of this conjecture, Gouvea and Mazur [4] proved that the number of square-free integer $D$ with $|D| \leq X$ such that the rank of $E_D$ is positive and even is $\gg X^{\frac{1}{2}−\varepsilon}$ for each $\varepsilon > 0$.

It is also conjectured that there are infinitely many primes $p$ for which $E_p$ has rank zero and there are infinitely many primes $q$ for which $E_q$ has positive rank. Along this direction, Mai and Murty [7] proved that there are infinitely many square-free integers $D \equiv 1 \pmod{4N}$ for which $E_D$ has rank zero, where $N$ denotes the conductor of $E$ over $\mathbb{Q}$. Ono [8] improved this result by showing that there is a set $S$ of primes $p$ with density $\frac{1}{3}$ for which if $D = \prod p_i$ is a square-free, where $p_i \in S$, then $E_D$ has rank zero for some special elliptic curves $E$.

We consider the elliptic curve $E(k) : x^3 + k$ for some non-zero square-free integer $k$. Kihara [6] proved that there are infinitely many integers $k$ for which $E(k)$ has rank at least 7. Chang [3] proved that $E(k)$ has rank zero for some values of $k$. On the other hand, the quadratic twist of $E(k)$ by a square-free integer $D$ is given by $E_D(k) : y^2 + kD^3$. Recently, Wu and Qin [12] proved that $E_D(1)$ has rank zero when $D \equiv 3 \pmod{4}$ is a negative square-free integer satisfying some conditions on the selection of $D$, the class number of $\mathbb{Q}(\sqrt{−2D})$ and solutions of associated Pell equation.

In this paper, we prove that the ranks of $E_{−D}(2)$ and $E_{3D}(2)$ are zero for any square-free odd positive integer $D(\neq 3)$ by supplying some sufficient conditions on the class number of $\mathbb{Q}(\sqrt{−2D})$ and the coefficients of the fundamental unit of $\mathbb{Q}(\sqrt{6D})$. We also produce infinitely many positive integers $D$ for which $E_{−D}(2)$ and $E_{3D}(2)$ have rank zero. More precisely, we prove the following:

**Theorem 1.1.** Let $D(\neq 3)$ be a square-free odd positive integer satisfying the following conditions:

(I) $3$ does not divide the class number of $\mathbb{Q}(\sqrt{−2D})$.

(II) $3$ does not divide the coefficients of the fundamental unit of $\mathbb{Q}(\sqrt{6D})$.

Then the ranks of $E_{−D}(2)$ and $E_{3D}(2)$ are zero.

**Theorem 1.2.** Let $D \equiv 2 \pmod{3}$ be a square-free odd positive integer and $3$ does not divide the class number of $\mathbb{Q}(\sqrt{−2D})$, then the ranks of $E_{−D}(2)$ and $E_{3D}(2)$ are zero. Furthermore, there are infinitely many such $D$ for which the ranks of $E_{−D}(2)$ and $E_{3D}(2)$ are zero.
We begin the following reflection theorem of Scholz [9] which will be needed to prove the next two propositions.

**Theorem A.** Let $D > 1$ be a square-free integer. Let $r$ and $s$ be the 3-ranks of the class groups of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ and the real quadratic field $\mathbb{Q}(\sqrt{3D})$. Then $s \leq r \leq s + 1$.

We first give the proof of the following crucial result.

**Proposition 2.1.** Let $D$ be as in Theorem 1.1. Then

$$\# \{ (x, y) \in E_D(2)(\mathbb{Q}) : \text{ord}_p(y) \leq 0 \ \forall p \mid 6D, \ p \text{ prime} \} = 0.$$  

**Proof.** To prove this proposition, it is sufficient to show that the equation

$$y^2 = x^3 - 2D^3z^6 \quad (2.1)$$

has no integer solutions in $x, y, z$ with $\gcd(x, y, z) = 1, \gcd(y, D) = 1$ and $z \neq 0$. Without loss of generality, we assume that $(x, y, z)$ is an integer solution of (2.1) such that $y$ and $z$ are positive as well as $z$ is minimal. We can exclude the cases where one or both of $x$ and $y$ are even since these cases would imply that $z$ is even too, which is a contradiction. Thus the only remaining possibility is that both $x$ and $y$ are odd.

Since $D$ is square-free and $\gcd(y, D) = 1$, so that $\gcd(x, 2D) = 1$ and $\gcd(y, 2D) = 1$. We now rewrite (2.1) as follows:

$$(y + Dz^3\sqrt{-2D})(y - Dz^3\sqrt{-2D}) = x^3. \quad (2.2)$$

Utilizing $\gcd(x, y, z) = 1$ and $\gcd(x, 2D) = 1$, we observe that $\gcd(y + Dz^3\sqrt{-2D}, y - Dz^3\sqrt{-2D}) = 1$. Therefore (2.2) gives

$$y + Dz^3\sqrt{-2D} = (a + b\sqrt{-2D})^3 \quad (2.3)$$

for some integers $a$ and $b$ satisfying $\gcd(a, b) = 1$ and

$$a^2 + 2b^2D = x. \quad (2.4)$$

This shows that $a$ is odd as $x$ is odd. Equating the real and imaginary part from (2.3), we get

$$y = a^3 - 2ab^2D, \quad (2.5)$$

$$Dz^3 = 3a^2b - 2b^3D. \quad (2.6)$$

Since $D \neq 3$ and it is square-free, so that reading (2.6) modulo $D$, we get $ab \equiv 0 \pmod{D}$. This implies either $a \equiv 0 \pmod{D}$ or $b \equiv 0 \pmod{D}$ as $\gcd(a, b) = 1$. If $a \equiv 0 \pmod{D}$, then utilizing (2.4) we get $x \equiv 0 \pmod{D}$. This contradicts to the fact $\gcd(x, D) = 1$. Thus
\(b \equiv 0 \pmod{D}\) and we write \(b = Db_1\) for some integer \(b_1\). Hence (2.6) gives

\[
z^3 = b_1(3a^2 - 2b_1^2D^3). \tag{2.7}
\]

Reading (2.7) modulo 3, we get \(z \equiv bD \pmod{3}\). □

Utilizing this in (2.7) and then reading modulo 9, we get

\[
b_1^2D^3 \equiv 3a^2b_1 - 2b_1^2D^3 \pmod{9}.
\]

If \(3 \mid D\) and \(3 \nmid b_1\), then \(3 \mid a\) and thus (2.5) gives \(3 \mid y\) which contradicts \(\gcd(y, D) = 1\). Therefore \(3 \mid b_1\) and we write \(b_1 = 3b_2\) for some integer \(b_2\). We utilize this in (2.7) to get \(3 \mid z\), and put \(z = 3z_1\).

\[
3z_1^3 = b_2(a^2 - 6b_2^2D^3). \tag{2.8}
\]

Since \(\gcd(a, b) = 1\) and \(3 \mid a\), so that (2.8) gives \(3 \mid b_2\). We put \(b_2 = 3b_3\) for some integer \(b_3\). Thus (2.8) becomes

\[
z_1^3 = b_3(a^2 - 54b_3^2D^3). \tag{2.9}
\]

It is clear that \(\gcd(b_3, a^2 - 54b_3^2D^3) = 1\) since \(\gcd(a, b_3) = 1\) due to \(\gcd(a, b) = 1\). Thus there exist two integers \(A\) and \(B\) such that \(b_3 = B^3\) and \(a^2 - 54b_3^2D^3 = A^3\). These together give rise to

\[
a^2 - 54B^6D^3 = A^3. \tag{2.10}
\]

It is clear that \(3 \nmid A\); otherwise \(3 \mid a\) which is a contradiction as \(3 \mid b\) too but \(\gcd(a, b) = 1\). Since \(a\) is odd, so that \(A\) is odd too. Also if a prime \(p \mid \gcd(a, A)\) then by (2.3), \(p \mid D\) and thus by (2.1) we get \(p \mid x\) which contradicts to \(\gcd(x, D) = 1\). Therefore \(\gcd(a, A) = 1\).

We now rewrite (2.10) as follows:

\[
(a + 3DB^3\sqrt{6D})(a - 3DB^3\sqrt{6D}) = A^3.
\]

It is clear that \(\gcd(a + 3DB^3\sqrt{6D}, a - 3DB^3\sqrt{6D}) = 1\) since \(\gcd(a, 2A) = 1\). Since the class number of \(\mathbb{Q}(\sqrt{-2D})\) is not divisible by 3, so that by Theorem A the class number of \(\mathbb{Q}(\sqrt{6D})\) is not divisible by 3. Therefore, \(\gcd(a, 3BD) = 1\). Since \(6D \equiv 2 \pmod{4}\), so that the fundamental unit, \(\varepsilon\) in \(\mathbb{Q}(\sqrt{6D})\) is of the form \(\varepsilon = T + U\sqrt{6D}\). Therefore the only possibilities for \(u\) are \(1, \varepsilon\) and \(\varepsilon^2\) since the higher powers of \(\varepsilon\) can be absorbed in \((\alpha + \beta\sqrt{6D})^3\).

We first assume that \(u = 1\). Then (2.10) gives:

\[
a = \alpha^3 + 18\alpha\beta^2D, \tag{2.11}
\]
Since $D$ is square-free, so that (2.12) shows that $D \mid \alpha \beta$. Clearly, $\gcd(D, \alpha) = 1$; otherwise by (2.11) $\gcd(D, \alpha) \mid a$ which contradicts to $\gcd(a, D) = 1$. Thus $D \mid \beta$, and we put $\beta = D\beta_1$ for some rational integer $\beta_1$. Therefore (2.12) becomes
\[ B^3 = \beta_1(\alpha^2 + 2\beta_1^2 D^3). \]

It is clear that $\gcd(\beta_1, \alpha^2 + 2\beta_1^2 D^3) = 1$ as $\gcd(\alpha, \beta_1) = 1$. Therefore there exist two integers $\beta_2$ and $\alpha_1$ such that $\beta_1 = \beta_2^3$ and $\alpha^2 + 2\beta_1^2 D^3 = \alpha_1^3$. These further imply
\[ \alpha^2 = \alpha_1^3 - 2D^3 \beta_2^6. \]

This shows that $(\alpha_1, \alpha, \beta_2)$ is another solution of (2.1) with $\gcd(\alpha, D) = 1$ and $\beta_2 \neq 0$. Furthermore, $|\beta_2| < |\beta_1|^\frac{1}{3} < |B| < |b_2|^\frac{1}{2} < |\beta_3|^\frac{1}{3} < |\beta_4|^\frac{1}{2}$ and thus by (2.7) $|\beta_2| < z$. This contradicts the minimality of $z$.

We now consider the case where $u = \varepsilon$. Then (2.10) gives the following:
\[ a = T\alpha(\alpha^2 + 18\beta^2 D) + 18DU\beta(\alpha^2 + 2D\beta^2), \]
\[ 3DB^3 = 3T\beta(\alpha^2 + 6\beta^2 D) + U\alpha(\alpha^2 + 18D\beta^2). \]

We read (2.13) modulo 3 to get $3\alpha^3 \equiv a \pmod{3}$. This implies $3 \nmid \alpha$ and $3 \nmid T$ as $3 \nmid a$. Reading (2.14) modulo 3, we get $U\alpha^3 \equiv 0 \pmod{3}$. This implies $3 \mid U$ since $3 \nmid \alpha$. This contradicts the assumption (II).

Finally if $u = \varepsilon^2$, then (2.10) gives
\[ a = (T^2 + 6DU^2)(\alpha^3 + 18\alpha^2 \beta^2 D) + 12DTU(3\alpha^2 \beta + 6D\beta^3), \]
\[ 3DB^3 = (T^2 + 6DU^2)(3\alpha^2 \beta + 6D\beta^3) + 2TU(\alpha^3 + 18\alpha^2 \beta^2 D). \]

Reading (2.15) modulo 3, we get $3 \mid T$ and $3 \mid \alpha$ since $3 \nmid a$. We finally read (2.16) to get $TU\alpha \equiv 0 \pmod{3}$ which implies $U \equiv 0 \pmod{3}$. This contradicts the assumption (II).

**Proposition 2.2.** Let $D$ be as in Theorem 1.1. Then
\[ \#\{ (x, y) \in E_{3D}(\mathbb{Q}) : ord_p(y) \leq 0 \forall p \mid 6D, \, \, p \text{ prime} \} = 0. \]

**Proof.** Analogous to the proof of Proposition 2.1, it is sufficient to prove that the equation
\[ y^2 = x^3 + 2(3D)^3 z^6 \] (2.17)
has no integer solutions in $x, y, z$ with $\gcd(x, y, z) = 1$, $\gcd(y, D) = 1$ and $z \neq 0$. Without loss of generality, we assume that $(x, y, z)$ is an integer solution of (2.17) such that $y$ and $z$ are positive as well as $z$ is minimal. We can exclude the cases where $\gcd(x, 6) = 1$ or $\gcd(y, 6) = 1$ as these cases would imply that $\gcd(x, y, z) \neq 1$. Thus
the only remaining possibility is that both $x$ and $y$ are odd as well as $3 \nmid xy$.

Since $D$ is square-free, $\gcd(y, D) = 1$ and $\gcd(x, y, z) = 1$, so that $\gcd(x, 6D) = 1$ and $\gcd(y, 6D) = 1$.

We now rewrite (2.17) as
\[
(y + 3Dz^3\sqrt{6D})(y - 3Dz^3\sqrt{6D}) = x^3. \tag{2.18}
\]
It is clear that $\gcd(y + 3Dz^3\sqrt{6D}, y - 3Dz^3\sqrt{6D}) = 1$ as $\gcd(x, y, z) = \gcd(x, 2D) = 1$. Since 3 does not divide the class number of $\mathbb{Q}(\sqrt{-2D})$ so that by Theorem A, 3 does not divide the class number of $\mathbb{Q}(\sqrt{6D})$. Therefore from (2.18) we can write
\[
y + 3Dz^3\sqrt{6D} = u(a + b\sqrt{6D})^3, \tag{2.19}
\]
where $u$ is unit in $\mathbb{Q}(\sqrt{6D})$ and $a, b$ are integers such that $\gcd(a, b) = 1$ as $\gcd(y, 3Dz) = 1$. Since $6D \equiv 2 \pmod{4}$, so that the fundamental unit in $\mathbb{Q}(\sqrt{6D})$ is of the form $T + U\sqrt{6D}$. Therefore $u$ is given by $(T + U\sqrt{6D})^\delta$ with $\delta = 0, 1, 2$ as the higher powers can be absorbed in $(a + b\sqrt{6D})^3$.

First consider the case when $\delta = 0$. Then (2.19) implies
\[
y = a^3 + 18ab^2D, \tag{2.20}
\]
\[
Dz^3 = a^2b + 2b^3D. \tag{2.21}
\]
(2.12) shows that $D \mid ab$ as $D$ is square-free. It is clear that $\gcd(D, a) = 1$; otherwise by (2.20) $\gcd(D, a) \mid y$ which contradicts to $\gcd(y, D) = 1$. Hence $D \mid b$, and we write $b = Db_1$ for some integer $b_1$. Thus (2.21) implies
\[
z^3 = b_1(a^2 + 2b_1^2D^3).
\]
Since $\gcd(a, b) = 1$, so that $\gcd(a, b_1) = 1$ and hence $\gcd(b_1, a^2 + 2b_1^2D^3) = 1$. Therefore we can find two integers $B$ and $A$ satisfying $b_1 = B^3$ and $a^2 + 2b_1^2D^3 = A^3$. These further give rise to
\[
a^2 = A^3 - 2D^3B^6.
\]
This shows that $(A, a, B)$ is another solution of (2.17) satisfying $\gcd(a, D) = 1$ and $B \neq 0$. Moreover, $|B| < |b_1|^{1/3} < z$. This contradicts the minimality of $z$.

We now consider the case where $\delta = 1$. In this case, (2.19) gives:
\[
y = aT(a^2 + 18Db^2) + 18DUb(a^2 + 2b^2D), \tag{2.22}
\]
\[
3Dz^3 = 3Tb(a^2 + 2b^2D) + Ua(a^2 + 18b^2D). \tag{2.23}
\]
We read (2.22) modulo 3 to get $a^3T \equiv y \pmod{3}$. This implies $3 \nmid a$ and $3 \nmid T$ as $3 \nmid y$. Reading (2.23) modulo 3, we get $Ua^3 \equiv 0 \pmod{3}$. This implies $3 \mid U$ since $3 \nmid a$ which contradicts the assumption (II).
Finally if $\delta = 2$, then (2.19) provides

\[ y = (T^2 + 6DU^2)(a^3 + 18ab^2D) + 36DTUb(a^2 + 2Db^2), \]  

(2.24)

\[ 3Dz^3 = (T^2 + 6DU^2)(3a^2b + 6Db^3) + 2TUa(a^2 + 18b^2D). \]  

(2.25)

Reading (2.24) modulo 3, we get $3 \nmid aT$ since $3 \nmid y$. We finally read (2.25) to get $TUa \equiv 0 \pmod{3}$ which implies $U \equiv 0 \pmod{3}$. This contradicts the assumption (II). □

We also need the following two results in order to complete the proof of Theorem 1.1. The first result recalls from [10, Ex. 10.19, p. 323].

**Lemma 2.1.** For a sixth-power-free integer $m$, let $E(m) : y^2 = x^3 + m$. Then $E(m)(\mathbb{Q})_{\text{tors}} | 6$. More precisely,

\[
E(m)(\mathbb{Q})_{\text{tors}} \cong \begin{cases} 
\mathbb{Z}/6\mathbb{Z} & \text{if } m = 1, \\
\mathbb{Z}/3\mathbb{Z} & \text{if } m \neq 1 \text{ is a cube, or } m = -432, \\
\mathbb{Z}/2\mathbb{Z} & \text{if } m \neq 1 \text{ is a square,} \\
1 & \text{otherwise.}
\end{cases}
\]

The following result comes from [10, p. 203].

**Lemma 2.2.** Let $E(m)$ be as in Lemma 2.1. Let $P(x, y) \in E(m)$. Then

\[
(x([2]P), y([2]P)) = \left(\frac{9x^4 - 8y^2x}{4y^2}, \frac{-27x^6 + 36y^2x^3 - 8y^4}{8y^3}\right).
\]

Proof of Theorem 1.1. We assume that $m = -2D^3, 6D^3$. Then $E_{-D}(2)$ and $E_{3D}(2)$ can be represented by $E(m)$. Thus utilizing Proposition 2.1 and Proposition 2.2, we can conclude that

\[ E(m)(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 : y^2 = x^3 + m, \ \text{ord}_p(y) \geq 1 \ \forall p \mid 3m, p \text{ prime}\}. \]

Now in order to complete the proof, it suffices to show that $E(m)(\mathbb{Q})$ is finite. For a prime divisor $p$ of $3m$, $y^2 = x^3 + m$ provides the following:

(I) $\text{ord}_p(y) \geq 1$ if and only if $\text{ord}_p(x) = 1$ when $p \neq 3$.

(II) $\text{ord}_3(y) \geq 1$ if and only if $\text{ord}_3(x) = \begin{cases} 1 & \text{if } 3 \mid m, \\
0 & \text{if } 3 \nmid m. \end{cases}$

Applying Lemma 2.1, we obtain $E(m)(\mathbb{Q})_{\text{tors}} = O$. We assume on the contrary that $E(m)(\mathbb{Q}) \neq E(m)(\mathbb{Q})_{\text{tors}}$. Then we can find $P(x, y) \in E(m)(\mathbb{Q}) \setminus E(m)(\mathbb{Q})_{\text{tors}}$ and a prime divisor $p$ of $3m$ such that $\text{ord}_p(y) \geq 1$. Applying Lemma 2.2 and utilizing induction on $n$, one gets

\[
\text{ord}_p(y([2^n]P)) \leq \begin{cases} \forall n \geq 1 & \text{if } p \neq 3, \\
\forall n \geq 2 & \text{if } p = 3.
\end{cases}
\]
Assume that \( m \) has \( t \) distinct prime factors and we put \( n = 2^{t+1} \). Then for any prime factor \( p \) of \( 3m \), one gets \( \text{ord}_p(y([2^n]P)) \leq 0 \). This is a contradiction. Thus we complete the proof. \( \square \)

3. Proof of Theorem 1.2

We denote by \( h(d) \) the class number of \( \mathbb{Q}(\sqrt{d}) \) for any square-free integer \( d \). Ankeny, Artin and Chowla [1, Theorem II] (also see, [5]) gave the following congruence relationship relating the class numbers of real and imaginary quadratic fields.

**Theorem B.** Assume that \( d = 3q \), where \( q \) is square-free positive integer and \( q \equiv 1 \pmod{3} \). Then \( Th(-q) + Uh(d) \equiv 0 \pmod{3} \), where \( T \) and \( U \) are the coefficients of the fundamental unit of \( \mathbb{Q}(\sqrt{d}) \).

*Proof of Theorem 1.2.* To prove this theorem, it suffices to show that the assumption (II) in Theorem 1.1 is not necessary in this case. Suppose that \( \varepsilon = T + U\sqrt{6D} \) is the fundamental unit in \( \mathbb{Q}(\sqrt{d}) \). We have already seen in the proof of Theorem 1.1 that \( T \not\equiv 0 \pmod{3} \). We utilize the assumption \( h(-2D) \not\equiv 0 \pmod{3} \) and Theorem A to obtain \( h(6D) \not\equiv 0 \pmod{3} \).

Since \( D \equiv 2 \pmod{3} \), so that \( 2D \equiv 1 \pmod{3} \). Also \( 2D \) is square-free as \( D \) is odd and square-free. Therefore by Theorem B, we get \( Th(-2D) + Uh(6D) \equiv 0 \pmod{3} \). This concludes that \( U \not\equiv 0 \pmod{3} \).

We can say by the Cohen-Lenstra heuristics that there exist infinitely many square-free positive integers \( D \) such that \( D \equiv 2 \pmod{3} \) and \( h(-2D) \not\equiv 0 \pmod{3} \). For each of these \( D \), ranks of \( E_{-D}(2) \) and \( E_{3D}(2) \) are zero. This complete the proof. \( \square \)

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Azizul Hoque@Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad-211019, India.
E-mail address: ahoque.ms@gmail.com