Geometric Cone Surfaces and (2+1) - Gravity coupled to Particles

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Abstract

We introduce the (2+1)-spacetimes with compact space of genus \( g \geq 0 \) and \( r \) gravitating particles which arise by Minkowskian suspensions of flat or hyperbolic cone surfaces, by distinguished deformations of hyperbolic suspensions and by patchworking of suspensions. Similarly to the matter-free case, these spacetimes have nice properties with respect to the canonical Cosmological Time Function. When the values of the masses are sufficiently large and the cone points are suitably spaced, the distinguished deformations of hyperbolic suspensions determine a relevant open subset of the full parameter space; this open subset is homeomorphic to \( \mathcal{U} \times \mathbb{R}^{6g-6+2r} \), where \( \mathcal{U} \) is a non empty open set of the Teichmüller Space \( \mathcal{T}_g \). By patchworking of suspensions one can produce examples of spacetimes which are not distinguished deformations of any hyperbolic suspensions, although they have the same masses; in fact, we will guess that they belong to different connected components of the parameter space.

1 Introduction.

Globally hyperbolic matter-free (2+1) - spacetimes with compact space of genus \( g \geq 1 \) and cosmological constant \( \Lambda = 0 \) have been fairly well understood. These spacetimes can be arranged into classes up to Teichmüller equivalence that is, roughly speaking, up to isometry isotopic to the identity. Both the geometric-time-free approach, which eventually identifies each spacetime by its geometric holonomy (c.f. [W], [Me]), and the cosmological approach based on fibration by constant mean curvature space-like surfaces (see [Mo], [A-T-M]), have recognized the Cotangent Bundle of the Teichmüller space \( T_g \) (which is homeomorphic to \( B^{6g-6} \times \mathbb{R}^{6g-6} \), when \( g \geq 2 \)) as the parameter space. When \( g \geq 2 \), the correspondence between these two approaches is still rather implicit; nevertheless, we have shown in [B-G 1] that the canonical Cosmological Time Function (CTF) (that is the length of time that the events have been in existence) provides a very good cosmological resolution of the matter-free (2+1) gravity. For instance, the asymptotic states of (CTF) recover and decouple the linear part and the translation part of the geometric holonomy; the orbit in \( T_g \) of the (CTF) has nice properties; the
“initial” singularity with respect to (CTF) can be accurately described in terms of the degeneration of the geometry of the level surfaces.

It turns out that all matter-free spacetimes are obtained by means of two basic constructions: (a) the Minkowskian suspension of flat or hyperbolic surfaces; (b) a distinguished kind of deformations of the hyperbolic suspensions. In this paper we shall describe the extension of constructions (a) and (b) to the case of gravitating particles in (2+1) dimensions.

When gravity is coupled to particles, the picture is far from being exhaustive. We will be concerned with compact spaces with a finite number of massive particles. ’t Hooft’s approach describes these spacetimes by means of the “linear” evolution of a special kind of Cauchy surfaces which are tiled by spatial planar polygons. The extrinsic curvature is null in the interior of each tile and it is singular along the edges; the evolution includes the changing of tiling combinatorics under codified transition rules. Each such a Cauchy surface is intrinsically a flat surface with conical singularities. Among these singularities, some correspond to the intersection with particle world-lines (the spacetime has a concentrated curvature along these lines); the others are 3-dimensionally apparent singularities (but the Gauss-Bonnet constraint implies that, in general, they cannot be avoided). Each globally hyperbolic spacetimes contains such a kind of Cauchy surface with, at least locally, such a kind of evolution; it is not clear to us if the evolution of a given surface necessarily fills all the spacetime and how the evolutions of different surfaces in the same spacetime are related each other. So, it seems hard to recover from this approach a clear identification of the parameter space.

Another experimented approach (see [Mn-S], [B-C-V]) is the classical ADM formalism with the so called “instantaneous gauge”, that requires fibration by spatial Cauchy surface with zero extrinsic curvature. This last requirement is technically very useful and allows to analytically find solutions by means of classical and very elegant mathematical tools. Unfortunately, it turns out that the only spacetimes with compact space covered by this approach are the static ones (that is, by using the terminology of the present paper, the static Minkowskian suspensions of flat surfaces with conical singularities, which we will see below).

The aim of this note is to describe the spacetimes with compact space of genus $g \geq 0$ and $r$ gravitating particles ($\Lambda = 0$) that one can obtain by means of three kinds of construction: (a) the Minkowskian suspensions of flat or hyperbolic surfaces with conical singularities; (b) the distinguished deformations of hyperbolic suspensions (in strict analogy with the matter-free case); (c) the patchworking of Minkowskian suspensions (this is peculiar to gravity coupled with particles). These spacetimes have very transparent structural properties and behave somewhat similarly to the matter-free ones with respect to, for instance, the canonical (CTF), its asymptotic states, the initial singularity and so on.

Moreover they form a rather wide class of spacetimes, so that we can derive from them some non trivial information about the actual parameter space. For example we will show that when the masses are big enough and the cone points are suitably spaced (roughly speaking), then the distinguished deformations of hyperbolic suspen-
sions determine a relevant non-empty open subset of the parameter space of the form \( \mathcal{U} \times \mathbb{R}^{6g-6+2r} \), where \( \mathcal{U} \) is an open set of the Teichmüller Space \( T^r_g \sim B^{6g-6+2r} \). On the other hand, by patchworking of suspensions, we will produce spacetimes with the same masses of certain hyperbolic suspensions but which are not equivalent to any distinguished deformation of them. In fact we will guess that they belong to different connected components of the parameter space. So gravity coupled to particles seems to be much more flexible than pure gravity. In the last section we will state several related questions and we will develop few speculations.

Several constructions concerning Minkowskian suspensions run, with minor changes, as in the matter-free case; so we will refer to the diffuse paper [BG 1] for more details.

2 Geometric Surfaces with Conical Singularities.

Cone points. The local models of flat or hyperbolic surfaces at a conical singularity are respectively given, in complex coordinate, by the metrics on \( \{|z| < 1\} \) (set \( \alpha > 0 \)):

\[
\begin{align*}
    ds^2_{(E,\alpha)} &= \alpha^2|z|^{2\alpha-2}|dz|^2, \\
    ds^2_{(H,\alpha)} &= \alpha^2\left[2/(1-|z|^{2\alpha})\right]^2|z|^{2\alpha-2}|dz|^2.
\end{align*}
\]

They are obtained by pull-back of the standard Euclidean or Poincaré metrics on \( \{|w| < 1\} \) via the map \( w = z^\alpha \).

In both cases the concentrated curvature at the conical point is \( k = 2\pi(1-\alpha) \), the cone angle is \( 2\pi\alpha \). In order to have a genuine singularity, \( \alpha \neq 1 \).

Geometric cone surfaces. It is convenient to adopt the formalism of geometric \((X,G)\)-manifolds (see, for instance, chapter B of [B-P] or section 3 of [B-G 1]). Fix a base compact oriented surface \( F_g \) of genus \( g \geq 0 \) and fix \( p_1, \ldots, p_r \) points on \( F_g \). A marked geometric (i.e. flat or hyperbolic) surface with conical singularities, of cone angles \( 2\pi\alpha_i \), \( i = 1, \ldots, r \), is a homeomorphism

\[ \phi : (F_g, \{p_i\}) \to (S, \{q_i\}) \]

such that \( S' = S \setminus \{q_i\} \) is a \((\mathbb{R}^2, Isom^+(\mathbb{R}^2))\)- (resp. a \((\mathbb{H}^2, Isom^+(\mathbb{H}^2))\)-) surface and its metric completion has a conical singularity of cone angle \( 2\pi\alpha_i \) at \( q_i \).

Gauss-Bonnet constraint.
The classical Gauss-Bonnet formula leads to the following relations.

Flat Case:

(Gauss-Bonnet equality)
\[
\sum_i k_i = 2\pi \sum_i (1 - \alpha_i) = 2\pi(2-2g).
\]

Hyperbolic Case:
\[
\sum_i k_i = 2\pi \sum_i (1 - \alpha_i) = 2\pi(2-2g) + Area(S).
\]
whence:
\[ (Gauss-Bonnet inequality) \]
\[ \sum_i (1 - \alpha_i) > 2 - 2g. \]

This implies, in any case, that when \( g = 0 \), necessarily \( r \geq 3 \), and we will make this assumption by default. We say that
\[ \delta = (X, g, [\alpha]_r) = (X, g, (\alpha_1, \ldots, \alpha_r)) \]
(where \( X = \mathbb{R}^2 \) or \( \mathbb{H}^2 \), \( g \geq 0 \) and the \( \alpha_i \)'s satisfy the appropriate Gauss-Bonnet equality or inequality), is a virtual type of geometric surfaces with conical singularities. For a fixed type \( \delta \) we denote by \( T_\delta \) the Teichmüller space of marked surfaces of type \( \delta \), (regarded up to Teichmüller equivalence - see, for instance, section 4 of [B-G 1] for more details).

When \( r > 0 \), the fundamental group \( \pi(F'_g = F_g \setminus \{p_i\}) \) is a non Abelian free group with \( s = 2g + r + 1 \) generators. For each \([\phi] \in T_\delta\) it is well defined, up to conjugation, the holonomy representation
\[ \rho_{[\phi]} : \pi(F'_g) \to Isom^+(\mathbb{X}). \]

The universal covering \( p : S^* \to S \) is, in a natural way, a local isometry so that \( S^* \) is homeomorphic to \( \mathbb{R}^2 \) and it is endowed with a geometric structure with conical singularities. We have also the developing map (well defined up to left action of \( Isom^+(\mathbb{X}) \))
\[ D_{[\phi]} : (S')^* \to X. \]

\((S')^*\) is also homeomorphic to \( \mathbb{R}^2 \) and is endowed with a smooth geometric structure. We can choose the representatives in such a way that, for every \( \gamma \in \pi(S') \), for every \( x \in (S')^* \),
\[ D_{[\phi]}(\gamma(x)) = \rho_{[\phi]}(\gamma)(D_{[\phi]}(x)). \]

\( D_{[\phi]} \) is a local isometry; when \( r > 0 \), \( S' \) and \( (S')^* \) are not (metrically) complete and, equivalently, \( D_{[\phi]} \) is not a global isometry.

**Orbifolds.** Geometric 2-dimensional compact orbifolds (with only conical singularities) make a special class of surfaces we are concerned with. Such an orbifold \( S \) is a quotient \( X/\Gamma \) where \( \Gamma \) is a group of isometries of \( X \) acting properly discontinuously and such that the set of points with non trivial stabilizer is made by isolated points. For a genuine orbifold this set is nonempty. They are classified as follows (see [T], [Sc]).

**Proposition 2.1.** A geometric cone surface is a genuine Euclidean orbifold iff it is of one of the types \((\mathbb{R}^2, 0, (1/2, 1/3, 1/6))\), \((\mathbb{R}^2, 0, (1/2, 1/4, 1/4))\), \((\mathbb{R}^2, 0, (1/3, 1/3, 1/3))\), \((\mathbb{R}^2, 0, (1/2, 1/2, 1/2, 1/2))\). A geometric cone surface is a genuine hyperbolic orbifold iff it is of a type \((\mathbb{H}^2, g, [\alpha]_r)\) satisfying the Gauss-Bonnet inequality and such that each \( \alpha_i \in [\alpha]_r \) is of the form \( \alpha_i = 1/n_i \), \( n_i \in \mathbb{N}^* \). Moreover all these types are actually realized by orbifolds.
Conformal structures. Associated to each geometric structure with conical singularities there is a natural conformal structure: a conformal atlas is simply obtained by adding a chart in complex coordinates as above at each conical singularity, to any atlas of the geometric structure on $S'$ (use the Poincaré disk model for $\mathbb{H}^2$). So, for each virtual type $\delta$, if $T_{\delta}^g$ is the classical Teichmüller space of conformal structures on $F_g$, relatively to the marked points $p_i$ (which, as it is well known, is homeomorphic to an open ball $B^{6g-6+2r}$), there is a natural continuous map (in fact in the case of flat surfaces we take off simple rescaling by normalizing the area to be equal to 1)

$$\psi_\delta : T_\delta \to T_{\delta}^r.$$ 

Geometric surfaces with conical singularities are classified by the following proposition.

**Proposition 2.2.** For any virtual type $\delta$, the natural map $\psi_\delta$ is a homeomorphism.

The flat case is due to Troyanov (see [Tr]). The orbifold case is treated in [T]. Let us sketch the main steps of a proof in the general hyperbolic case.

1. $\dim(T_\delta) = \dim(T_{\delta}^r)$. Let us outline first a way to construct all hyperbolic cone surfaces. Fix $(F_g, \{p_1, \ldots, p_r\})$ as before. A standard spine of $F_g'$ is a 1-complex $P$ embedded in $F_g'$, with only 3-valent vertices, such that $F_g'$ is a regular neighborhood of $P$ ($F_g'$ retracts onto $P$). Associated to such a $P$ there is a dual (topological) ideal triangulation $\tau_P$ of $F_g'$, that is a “relaxed” (i.e. multiple and self adjacencies between triangles are allowed) triangulation of $F_g$, having $\{p_1, \ldots, p_r\}$ as set of vertices. If $v(P) = |V(P)|$ denotes the number of vertices of $P$ (i.e. the number of triangles of $\tau_P$), $e(P) = |E(P)|$ the number of its edges (i.e. the number of the edges of the dual triangulation), one has $3v(P) = 2e(P)$ so that $e(P) = 6g - 6 + 3r$. Clearly spines exist. Fix a spine $P$. For any admissible map $f : E(P) \to \mathbb{R}^+$ (i.e. a map such that at each vertex $v \in P$ the values of $f$ on the three edges emanating from $v$ satisfy the triangular inequalities), we can construct a hyperbolic surface with $r$ conical singularities. This is obtained as a geometric realization of the dual triangulation $\tau_P$, by using hyperbolic triangles with edge lengths prescribed by $f$. Recall that each hyperbolic triangle is determined by the edge lengths as well as by the interior angles, and there are classical explicit formulas relating lengths and angles. It is not too hard to see that varying the spine and the admissible function, one can realize all the hyperbolic virtual types. On the other hand, any cone hyperbolic surface arises in this way. In fact let $(F_g, F_g') \sim (S, S')$ be such a surface. Consider the subset $Q$ of $S'$, such that for each $x \in Q$ there exist $i \neq j$ such that $d(x, p_i) = d(x, p_j)$. Generically $Q$ is a standard spine of $S'$; the interior of an edge of $Q$ consists of the points with exactly two equidistant marked points $p_i, p_j$, the same along the given edge. The “axis” of each edge, that is the geodesic arc connecting $p_i$ and $p_j$ and passing from the point of the edge of minimal distance from them, are the edges of a geometric realization on the dual triangulation $\tau_Q$. In general $Q$ is a spine, possibly with higher valency vertices; the same procedure produces a dual ideal cellularization of $S'$ by convex hyperbolic polyhedra and we eventually obtain a geometric triangulation by subdividing without introducing new vertices. If a virtual type $\delta$ is realized by an admissible map $f_0$
on $E(P)$, the maps realizing the same type are obtained by imposing $r$ independent conditions. So one can deduce, at least, that $T_\delta$ is a topological manifold of the right dimension $6g - 6 + 2r$.

(2) The map $\psi_\delta$ is injective.
Consider $\mathbb{H}^2$ in the Poincaré disk model $D = \{ |z| < 1 \}$, and let $e^{2h}|dz|^2$ be the standard Poincaré distance. Realize a given element $\sigma$ of $T^r_\delta$ by a smooth hyperbolic surface (with marked points) $S = D/\Gamma$. Two hyperbolic surfaces with conical singularities of the same type, both representing $\sigma$, are given by two metrics $e^{2(h + h_i)}|dz|^2$, $i = 1, 2$, such that each $h_i$ is a $\Gamma$-equivariant function on $D$, with the same kind of singularities over the marked points of $S$. It follows that $h_1 - h_2$ is a real analytic $\Gamma$-equivariant function on $D$ satisfying the Liouville equation

$$\Delta(h_1 - h_2) = e^{2h}(e^{2h_1} - e^{2h_2}).$$

As $S$ is compact $h_1 - h_2$ has maxima and minima. Either $\Delta(h_1 - h_2) > 0$ near a maximum, or $\Delta(h_1 - h_2) \leq 0$ near a minimum. By the maximum principle $h_1 - h_2$ is constant near the minimum or the maximum and hence it is constant (and necessarily equal to 0) everywhere.

(3) Conclusion. By the invariance of domain theorem, $\psi_\delta$ is a homeomorphism onto a non empty open subset of $T^r_\delta$. To conclude it is enough to show that the image of $\psi_\delta$ is closed. This can be done by studying the convergence of the conformal factors (see the above step), or by arguing (via geometric considerations) that the image of a “diverging” sequence in $T_\delta$ is diverging in $T^r_\delta$.

3 Minkowskian Suspensions.

Particle world lines. Let us give, first, the local models of the line of universe of a massive particle. They are obtained by “suspension” of the local models for geometric cone surfaces. We can take indifferently, in coordinates $(z, t)$,

$$d\sigma^2_{(E, \alpha)} = -dt^2 + ds^2_{(E, \alpha)},$$

or, assuming $t > 0$

$$d\sigma^2_{(H, \alpha)} = -dt^2 + t^2ds^2_{(H, \alpha)}.$$  

They are equivalent as local models, in the sense that any point $(0, t_1)$ in the first model and any point $(0, t_2)$ in the second one have isometric neighbourhoods. They are not equivalent as global spacetimes; for instance if we take the time orientation in accordance with the $t$ coordinate, the canonical (CTF) of the first spacetime is degenerate, constant equal to $\infty$, while $t$ is the (CTF) of the second one. We have a well defined cone angle $2\pi \alpha$ along such a universe line, which corresponds to a spacetime curvature concentrated along the line. In accordance with [D-J- ’T H], [’T H], if we normalize the gravitational constant to be $G = 1$, the mass of the particle is related to the cone angle by $m = (1/4)(1 - \alpha)$; in $(2+1)$-gravity there are not physical constraints on the sign of $Gm$, so that an arbitrarily big $\alpha$ is allowed.
**Spacetimes with gravitating particles.** A marked globally hyperbolic spacetime (coupled to massive particles) of type

$$\delta = (g, [\alpha], (g, (\alpha_1, \ldots, \alpha_r))$$

is an homeomorphism

$$\phi : (F_g \times \mathbb{R}, \{p_i\} \times \mathbb{R}) \to (M, L_i)$$

such that $M' = M \setminus \{L_i\}$ is an oriented and time-oriented globally hyperbolic flat Lorentzian 3-manifold (i.e. a $(\mathbb{M}^{2+1}, Isom^+(\mathbb{M}^{2+1}))$-manifold, where $\mathbb{M}^{2+1}$ is the standard Minkowski space) and each point of $L_i$ has a neighbourhood isometric to the above local models, with cone angle $2\pi \alpha_i$. It is convenient to restrict to *Geroch marking*, that is we stipulate that the surfaces $\phi(F_g \times \{t\})$ are future Cauchy surfaces. As usually we work up to Teichmüller equivalence and we denote by $T^G_{\delta}$ the corresponding Teichmüller space for a given type. To make it more meaningful it is convenient to restrict to maximal spacetimes. Identifying $F_g$ with $F_g \times \{0\}$ we have also the holonomy representation

$$\rho[\phi] : \pi(F_g') \to Isom^+(\mathbb{M}^{2+1}).$$

We also make the current assumption that the linear part of the holonomy takes values in $SO^+(2,1)$, the group of Lorentz transformations keeping the upper half-space invariant.

**Minkowskian suspensions of geometric cone surfaces.** They are peculiar spacetimes which actually are $(Y, G(Y))$-manifolds, for suitably chosen open subsets $Y$ of $\mathbb{M}^{2+1}$, $G(Y)$ being the group of Minkowskian isometries keeping $Y$ invariant. As $Y$ we will take:

- $Y_E = \mathbb{M}^{2+1}$
- $Y_H = \{x \in \mathbb{M}^{2+1} : (x^1)^2 + (x^2)^2 - (x^3)^2 < 0, x^3 > 0\}$
- $Y_T = \{x \in \mathbb{M}^{2+1} : (x^1)^2 - (x^3)^2 < 0, x^3 > 0\}$

thought fibred by the planes $\{x^3 = a\}$.

thought fibred by the surfaces

$$\{x \in \mathbb{M}^{2+1} : (x^1)^2 + (x^2)^2 - (x^3)^2 = -a^2, x^3 > 0\}$$

finally

$$\{x \in \mathbb{M}^{2+1} : (x^1)^2 - (x^3)^2 = -a^2, x^3 > 0\}.$$  

thought fibred by the surfaces

$$\{x \in \mathbb{M}^{2+1} : (x^1)^2 - (x^3)^2 = -a^2, x^3 > 0\}.$$  

By the change of coordinates

$$x^1 = \tau sh(u), \ x^2 = y, \ x^3 = \tau ch(u)$$

we see that $Y_T$ is isometric to

$$P = \{(u, y, \tau) \in \mathbb{R}^{2+1} : \tau > 0\}, \text{ with metric } \tau^2 du^2 + dy^2 - d\tau^2$$
and $P$ is fibred by the level planes of $\tau$.

Each $Y_\ast$ is oriented and time-oriented in the usual way.

If $S$ is a flat cone surface of type $(\mathbb{R}^2, g, [\alpha]_r)$, its Minkowskian suspension $M(S)$ is the obviously associated $(Y_E, G(Y_E))$-spacetime of type $(g, [\alpha]_r)$, with holonomy equal to the holonomy of $S$. It is fibred by parallel copies of $S$. The canonical (CTF) is degenerate, constant equal to $\infty$. These are called static Minkowskian suspensions.

If $S$ is a hyperbolic cone surface of type $(H^2, g, [\alpha]_r)$, its Minkowskian suspension $M(S)$ is the associated $(Y_H, G(Y_H))$-spacetime of type $(g, [\alpha]_r)$, with holonomy equal to the holonomy of $S$. It is fibred by parallel rescaled copies of $S$; these surfaces are the level surfaces $S_\alpha (S = S_1)$ of the canonical (CTF); out of the particles they have constant mean curvature $1/a$ and constant intrinsic curvature equal to $-1/a^2$. The initial singularity consists of one point.

These suspensions are particularly nice when $S$ is an orbifold (and the matter-free spacetimes are particular cases); if the orbifold $S = \mathbb{X}/\Gamma$, $\Gamma$ acts isometrically also on the corresponding $Y_\ast$, and $M(S) = Y_\ast/\Gamma$. The parameter space of $Y_E$ or $Y_H$-suspensions of a given type coincides, tautologically, with the parameter space of the suspended geometric cone surfaces (see the previous section).

The $Y_\tau$-Minkowskian suspensions involve the special flat cone surfaces given by the meromorphic quadratic differentials with at most simple poles on Riemann surfaces. In fact each such a suspension is determined by a couple $(F, q)$, where $F$ is a Riemann surface and $q$ is a quadratic differential. That is, it is determined not only by the cone surface, but also by the horizontal and vertical measured foliations of the quadratic differential. We have already studied such spacetimes in [B-G 2] where we have shown how they “materialize” the classical Teichmüller flow. See also [B-G 1] for a description of the (CTF). In fact in [B-G 2] we considered only holomorphic quadratic differentials, but everything runs verbatim if one allows also simple poles. Recall that in this way one can realize all the types with $2\pi \alpha_i = n_i \pi$, $n_i \geq 1$, satisfying the Gauss-Bonnet equality, with four exceptions (see [M-S]). Moreover, for any given realizable type, one knows the degrees of freedom (see [V]): if $\mu(a)$ denotes the number of cone points of cone angle $a$, then the degrees of freedom are

$$2g + \sum \mu(a) + (\epsilon - 3)/2$$

where $\epsilon = -1$ iff there is at least one cone angle with odd $n_i$, and it is equal to 1 otherwise. For example, when the type contains only $n_i = 3$ (this corresponds to holomorphic quadratic differentials with simple zeros), the dimension of the corresponding space of $Y_\tau$-suspensions is $6g - 6$.

The only orbifolds which produce such a kind of suspension are the orbifolds of type $(\mathbb{R}^2, 0, (1/2, 1/2, 1/2, 1/2))$. They are obtained by the natural identification of the edges of two copies of a same Euclidean rectangle. The corresponding group $\Gamma$ is generated by two orthogonal translations and the rotation of angle $\pi$. Groups that determine the same $Y_E$-suspension (up to equivalence), do determine in general different $Y_\tau$-susensions; in fact if we look at these groups acting on $P$, the horizontal and vertical
foliations on each $\tau$-level plane induce different foliations on the (CTF)-level surfaces of the two suspensions.

4 Distinguished Deformations of Hyperbolic Suspensions.

In this section we will refer heavily to [B-G 1] and to [Me]. All matter-free spacetimes with space of genus $g \geq 2$ are obtained by specific “deformations” of Minkowskian suspensions $M(S) = Y_H/\Gamma$, and each such a deformation $M(S, F)$ is uniquely determined by a measured geodesic lamination $F$ on $S$. $M(S)$ and $M(S, F)$ have holonomies with the same linear part. The lifted lamination $F^*$ to the universal covering $S^* = \mathbb{H}^2$ is “dual” to a real tree which is isometric to the initial singularity of $M(S, F)^*$. The initial singularity of $M(S, F)$ must be properly interpreted in terms of a natural action of the fundamental group $\pi(S)$ on this real tree; the natural actions of $\pi(S)$ on the universal covering of the level surfaces of the (CTF) of $M(S, F)$, asymptotically degenerate to that action on the real tree.

If $S$ is a hyperbolic cone surface of type $\delta = (g, [\alpha], r)$ (we have omitted the “$\mathbb{H}^2$” in $\delta$), and $F$ is a measured geodesic lamination with compact support in $S'$ we can repeat those constructions (working with $S^*$ which is now a cone hyperbolic surface) getting, by definition, a distinguished deformation $M(S, F)$ of $M(S)$, which is again a spacetime of type $\delta$.

The simplest deformations arise when $F$ is a multicurve, i.e. it consists of the finite union of disjoint simple geodesics endowed with positive weights. Assume, for simplicity, that there is one geodesic $\sigma$, with weight $s$ and length $r$. Consider the quotient $A'(s, r)$ of $B'(s, r) = \{(u, y, \tau) \in P; 0 \leq y \leq s\}$ by the group generated by the translation $(u, y, \tau) \to (u + r, y, \tau)$. Actually it is better to consider the isometric quotient $A(s, r)$ of $B(s, r) \subset Y_T$, obtained via the explicit change of coordinates given in section 3. Then, to construct $M(S, F)$, cut-open $M(S)$ along the suspension of $\sigma$ and insert $A(s, r)$ in the natural way. $M(S, F)$ is, by construction, fibred by $C^1$-embedded space-like surfaces (made by the union of pieces of constant negative curvature and flat annuli); in fact they are the level surfaces of the canonical (CTF).

The above construction is very simple nevertheless, as multicurves are dense in the space of measured geodesic laminations, by making the multicurve “complicated” enough, we can fairly well approximate the shape of any general distinguished deformation.

Given a hyperbolic type $\delta = (g, [\alpha], r)$, we denote by $D(\delta)$ the subset of $T^*_{GR}$ determined by the distinguished deformations of Minkowskian suspensions of hyperbolic cone surface of type $\delta$. Of course, a suspension is meant as the trivial deformation of itself and there is a natural projection $p : D(\delta) \to T^*_\delta$. The following proposition gives partial information on $D(\delta)$. We will use some notations introduced in section 2. The set of hyperbolic “$(g, r)$-types” can be identified with an open set of $\mathbb{R}^r$.

Proposition 4.1. (1) For each hyperbolic type $\delta$ there is an open (possibly empty)
maximal subset $\mathcal{U}_\delta$ of $T_\delta$ such that $p^{-1}(\mathcal{U}_\delta) \subset D(\delta)$ is homeomorphic to $\mathcal{U}_\delta \times \mathbb{R}^{6g-6+2r}$ (and $p$ becomes the natural projection onto the first factor).

(2) For each $(g,r)$ there is a maximal non empty open subset $\mathcal{W}_{(g,r)}$ of the space of $(g,r)$-types, such that for each $\delta \in \mathcal{W}_{(g,r)}$, $\mathcal{U}_\delta$ is non empty.

(3) For any $\delta$,

$$12g - 12 + 4r \geq \text{dim}D(\delta) \geq 6g - 6 + 2r.$$

(4) If $\mathcal{U}_\delta$ is non empty, then $\mathcal{U}_\delta \times \mathbb{R}^{6g-6+2r}$ is an open subset of $T_\delta^{GR}$.

Let us give a skecth of a proof. By using the result of section 2, the first statement is equivalent to show that the space of measured geodesic laminations with compact support on $S'$, for a given hyperbolic cone surface $S$ in $\mathcal{U}$ (for a suitable $\mathcal{U}$), is homeomorphic to $\mathbb{R}^{6g-6+2r}$. This fact is known in the “limit” case when each $\alpha_i = 0$, that is when $S'$ is a complete finite area hyperbolic surface with $r$ cusps (see [Pe]). Let us denote by $HT^r_g$ (which is homeomorphic to $T^r_g$) the Teichmüller space of such hyperbolic surfaces with $r$ cusps and fix one surface $F$. It is known that each geodesic lamination with compact support on $F$ has support contained in $F''$ obtained by removing from $F$ all the horocycles of length $< 1$ around all the cusp points (see [Pe] pag. 72). It turns out that any hyperbolic cone surface $S$ which is “geometrically” close to $F$ has, up to homeomorphism, the same space of measured geodesic laminations with compact support on $S'$. The crucial fact is that if $S$ is close enough to a cusped $F$, each isotopy class of essential (i.e. non contractible nor contractible to one cone point) simple curves on $S'$ has a simple geodesic (in $S'$) shortest length representative. For some notions on the “geometric topology” see, for instance, chapter E of [B-P]. In our situation “$S$ geometrically close to $F$” roughly means that, removing suitable “round” disks with centres at the cone points of $S$, we find $S''$ which is quasi-isometric to $F''$, by a quasi-isometry close to an isometry. It follows that for any fixed compact subset $K$ of $HT^r_g$ there is an open subset $\mathcal{U}_K$ (possibly empty) of $T_\delta$, which satisfies the first statement of the proposition.

To prove the second statement, it is enough to show that, for any fixed $F$ as before, there are cone surface $S$ close to $F$ in the above sense. Fix a geodesic ideal triangulation $\mathcal{T}$ of $F$ (i.e. a “relaxed” triangulation of $F$ by ideal hyperbolic triangles). For each $0 < a < 1$ consider the horocycles of length $a$ around the cusps of $F$. Associate to each edge of the triangulation the length of the subarc determined by the horocycles. Consider the cone surface $S$ obtained accordingly with the construction after proposition 2.2, by using the same $\mathcal{T}$ as topological ideal triangulation of $F'_g$ and those lengths as edge-lengths. If $a$ is small enough, $S$ is close to $F$.

$S$ is not close enough to a cusped $F$ when, at a very qualitative level, the masses are not big enough or the particles are too close each other on a given level surface of the (CTF) of the corresponding Minkowskian suspension. In such a case the basic trouble consists in the fact that the shortest length representative (if any) of an essential isotopy class of simple curves on $S'$ might be a broken geodesic passing through some cone points or not even a simple curve.

The third statement is clear from the above discussion.
To achieve the last statement it is enough to show that $T_{GR}^\delta$ is of dimension $12g-12+4r$; we are going to argue it without any assumption on the spacetime type $\delta = (g,[\alpha],r)$.

**The degrees of freedom of $T_{GR}^\delta$.** Fix a marked spacetime $M$ of type $\delta$ and a relatively compact globally hyperbolic open neighbourhood $U$ of the Cauchy surface image of $F_g \times \{0\}$. Let $\rho : \pi(F_g^\prime) \to \text{ISO}^+(2,1)$ be its holonomy. As $\pi(F_g^\prime)$ is a free group, a deformation of $\rho$ is simply obtained by modifying $\rho$ on a set of $2g-2+r+1$ free generators. If a deformation $\rho'$ is small enough, then, by the stability property of holonomies, $\rho'$ is still the holonomy of a spacetime structure on the interior of $U$, with $r$ gravitating particles. So, as the holonomy is defined only up to conjugation, the dimension of the set of all these spacetimes “close” to $M$ is $12g-12+6r$. In order to impose that the spacetimes have the specific cone angles prescribed by $\delta$, we have to impose $2r$ (that is $(6-d)r$, where $d$ is the dimension of the conjugation orbit of a “rotation”) more independent conditions, and we finally get the required number of degrees of freedom $12g-12+4r$.

## 5 Patchworking of Minkowskian Suspensions.

A simple variation of the construction of the distinguished modification of hyperbolic suspensions, based on multicurves, that we have described in the previous section, will produce interesting new examples of spacetimes.

Let $M(S,\mathcal{F})$ be as in the previous section. Assume that we have a finite union of simple closed geodesic on $S'$ disjoint from $\mathcal{F}$. For simplicity, assume that there is a single geodesic $\sigma$ of length $a$. Let $(F,q)$ be a Riemann surface with a meromorphic quadratic differential $q$, with at most simple poles. Let $M(F,q)$ be the corresponding $Y_T$-Minkowskian suspension (see section 3). Assume that the $q$-horizontal foliation on $F$ contains a simple closed leaf $c$ of length $a$. Then we can construct new spacetimes as follows: cut-open $M(S,\mathcal{F})$ along the suspension of $\sigma$ and $M(F,q)$ along the suspension of $c$; then glue, pairwise, pieces of $M(S,\mathcal{F})$ with pieces of $M(F,q)$ along isometric boundary components in the natural way. Note that there is, in general, a finite number of possible combinations, and the resulting Lorentz manifolds may be not connected, so we can take each connected component as a new spacetime. Call $M([S,\mathcal{F},\sigma],[F,q,c])$ any spacetime obtained in this way. By construction, it is fibred by space-like surfaces (made by rescaled pieces of $S$ and by “stretched ” pieces of $F$) which actually are the level surfaces of the canonical (CTF) of $M([S,\mathcal{F},\sigma],[F,q,c])$. Note also that the construction can be iterated, starting from suitable $M([S,\mathcal{F},\sigma],[F,q,c])$; so one can produce a wide class of new examples. This *patchworking* is peculiar of spacetimes with gravitating particles; in fact if we formally apply it to matter-free spacetimes we get nothing else than distinguished deformations of hyperbolic suspensions.

In particular, let us use as $(F,q)$ the orbifolds of type

$$(\mathbb{R}^2,0,4[1/2]) = (\mathbb{R}^2,0,(1/2,1/2,1/2,1/2))$$

with the horizontal and vertical foliations of $q$ (with 4 simple poles) parallel to the edges of the “fundamental” rectangle. It is not hard to construct by the patchworking
procedure hyperbolic types of the form \( \delta = (g, [\alpha]_r) = (g, [\alpha]_r \cup 2h[1/2]), 2h + r' = r \). On the other hand, these new spacetimes do not belong to \( D(\delta) \) because, for instance, the level surfaces of the canonical (CTF) are not isometric (there are cone points of cone angle \( \pi \) with no isometric neighbourhoods). Other differences manifest themselves by studying the past asymptotic states of the respective (CTF). By small perturbation of the holonomy of these examples one could produce examples out of \( D(\delta') \) for any \([\alpha]'_r\) close to \([\alpha]_r\).

6 Final Questions and Considerations.

We are going to conclude with some questions, problems and, sometimes, with a guess about them.

(1) Is \( T^{GR}_{\delta} \) connected ?

The answer could depend on the type. We guess that the above examples not belonging to \( D(\delta) \), actually do not even belong to the same connected component of any element of \( D(\delta) \).

(2) Does any spacetime satisfy the Gauss-Bonnet constraint \( \sum (1 - \alpha_i) \geq 2 - 2g \) ?

We guess that by suitable small perturbations of the holonomy of static Minkowskian suspensions (which satisfy the Gauss-Bonnet equality) one could obtain spacetimes with \( \sum_i (1 - \alpha_i) < 2 - 2g \).

We note that all the examples of spacetime that we have produced starting from non static Minkowskian suspensions have the following property:

- each particle line of universe has a neighbourhood isometric to the set of points of spatial distance \( < bt, \) for some positive \( b, \) from the \( t \)-axis in the model \( \{(z,t), \ |z| < 1, \ t > 0\} \) with metric \( da^2_{(H,\alpha)} \) (see section 3).

(3) Does the same property hold for any spacetime with tame - see [B-G 1] - canonical (CTF) with values onto \((0,\infty)\)?

It would be interesting to find, if any, examples where the linear function \( bt \) must be replaced by some positive function \( f(t) \) going faster to 0 when \( t \to 0 \).

(4) Find an intrinsic characterization of hyperbolic cone surfaces belonging to \( U_\delta \).

One expects that it could be expressed in terms of inequalities involving the cone angle, the genus and the distances between the cone points.

(5) Describe \( W_{(g,r)} \). In particular, does \( m(g,r) \) exist, with \( 1 > m(g,r) > 0 \), such that for any \( \delta \in W_{(g,r)} \) and for any mass \( m_i \) associated to \( \delta \), one has \( m_i > m(g,r) \) ?

For example, beside the “rigid” case \((g,r) = (0,3)\), the very peculiar case \((g,r) = (0,4)\) has \( W_{(0,4)} \) which coincides with the whole space of \((0,4)\)-types; moreover for each type \( \delta \in W_{(0,4)}, U_\delta \) coincides with \( T_\delta \), so that \( D(\delta) = T_\delta \times \mathbb{R}^2 \). On the other hand, we guess, for example, that for each \((0,r), r > 4\), the last question has negative answer.

(6) Is \( D(\delta) \) always of dimension \( > 6g - 6 + 2r \) ?

In other words, one is asking if there are always non trivial distinguished deformations. We guess that when \( g \geq 2 \) and the masses are all positive, then \( D(\delta) \) contains at least
\[ T_3 \times \mathbb{R}^{6g-6}; \text{in other words one expects that there is at least the same “amount” of } \]
distinguished deformations of the matter-free case of the same genus.

(7) Let \( C \) be any closed subset of \( \mathcal{U}_3 \). Is \( p^{-1}(C) \subset \mathcal{U}_3 \times \mathbb{R}^{6g-6+2r} \) closed in \( T_3^{GR} \)?

Finally we note that in several instances of the present paper we have seen how very natural perturbations of a given spacetime do not preserve the type (see for instance the constructions of section 2 or the argument at the end of section 4). It would suggest that the study of \((2+1)\)-gravity (coupled to particles) “type by type”, or even “space-genus by space-genus”, could be misleading. Spacetimes would be considered “all together” and it becomes quite demanding to figure out the structure of the corresponding (infinite dimensional) parameter space. We guess that Grothendieck theory of “Teichmüller Towers” could play an important role.

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