INEQUALITIES FOR $f^*$-VECTORS OF LATTICE POLYTOPES

MATTHIAS BECK, DANAI DELIGEORGAKI, MAX HLAVACEK, AND JERÓNIMO VALENCIA-PORRAS

Abstract. The Ehrhart polynomial $ehr_P(n)$ of a lattice polytope $P$ counts the number of integer points in the $n$-th integral dilate of $P$. The $f^*$-vector of $P$, introduced by Felix Breuer in 2012, is the vector of coefficients of $ehr_P(n)$ with respect to the binomial coefficient basis $\{(n-1 \choose 0), (n-1 \choose 1), \ldots, (n-1 \choose d)\}$, where $d = \dim P$. Similarly to $h/h^*$-vectors, the $f^*$-vector of $P$ coincides with the $f$-vector of its unimodular triangulations (if they exist).

We present several inequalities that hold among the coefficients of $f^*$-vectors of polytopes. These inequalities resemble striking similarities with existing inequalities for the coefficients of $f$-vectors of simplicial polytopes; e.g., the first half of the $f^*$-coefficients increases and the last quarter decreases. Even though $f^*$-vectors of polytopes are not always unimodal, there are several families of polytopes that carry the unimodality property. We also show that for any polytope with a given Ehrhart $h^*$-vector, there is a polytope with the same $h^*$-vector whose $f^*$-vector is unimodal.

1. Introduction

For a $d$-dimensional lattice polytope $P \subset \mathbb{R}^d$ (i.e., the convex hull of finitely many points in $\mathbb{Z}^d$) and a positive integer $n$, let $ehr_P(n)$ denote the number of integer lattice points in $nP$. Ehrhart’s famous theorem \cite{ehrhart} says that $ehr_P(n)$ evaluates to a polynomial in $n$.

Similar to the situations with other combinatorial polynomials, it is useful to express $ehr_P(n)$ in different bases; here we consider two such bases consisting of binomial coefficients:

$$ehr_P(n) = \sum_{k=0}^{d} h_k^* \left( \binom{n+d-k}{d} \right) = \sum_{k=0}^{d} f_k^* \left( \binom{n-1}{k} \right).$$

(1)

We call $(f_0^*, f_1^*, \ldots, f_d^*)$ the $f^*$-vector and $(h_0^*, h_1^*, \ldots, h_d^*)$ the $h^*$-vector of $P$. Stanley \cite{stanley} proved that the $h^*$-vector of any lattice polytope is nonnegative (whereas the coefficients of $ehr_P(n)$ written in the standard monomial basis can be negative). Breuer \cite{breuer} proved that the $f^*$-vector of any lattice polytopal complex is nonnegative (whereas the $h^*$-vector of a complex can have negative coefficients); his motivation was that various combinatorially-defined polynomials can be realized as Ehrhart polynomials of complexes and so the nonnegativity of the $f^*$-vector yields a strong constraint for these polynomials.

Date: 19 October 2022.

2010 Mathematics Subject Classification. Primary 52B20; Secondary 05A15, 52C07.

Key words and phrases. Lattice polytope, Ehrhart polynomial, Gorenstein polytope, $f^*$-vector, $h^*$-vector, unimodality.

We thank the organizers of Research Encounters in Algebraic and Combinatorial Topics (REACT 2021), where our collaboration got initiated. We are grateful to Michael Joswig, Matthias Schymura and Lorenzo Venturello for helpful conversations.
The $f^*$- and $h^*$-vector can also be defined through the Ehrhart series of $P$:

$$Ehr_P(z) := 1 + \sum_{n \geq 1} ehr_P(n) z^n = \frac{\sum_{k=0}^d h_k^* z^k}{(1-z)^{d+1}} = 1 + \sum_{k=0}^d f_k^* \left( \frac{z}{1-z} \right)^{k+1}.$$ 

It is thus sometimes useful to add the definition $f_{-1}^* := 1$. The polynomial $\sum_{k=0}^d h_k^* z^k$ is the $h^*$-polynomial of $P$, and its degree is the degree of $P$.

The $f^*$- and $h^*$-vectors share the same relation as $f$- and $h$-vectors of polytopes/polyhedral complexes, namely

$$\sum_{k=0}^d h_k^* z^k = \sum_{k=0}^{d+1} f_{k-1}^* z^k (1-z)^{d-k+1} \quad \text{(2)}$$

$$h_k^* = \sum_{j=-1}^{k-1} (-1)^{k-j-1} \binom{d-j}{k-j-1} f_j^* \quad \text{(3)}$$

$$f_k^* = \sum_{j=0}^{k+1} \binom{d-j+1}{k-j+1} h_j^*. \quad \text{(4)}$$

The (very special) case that $P$ admits a unimodular triangulation yields the strongest connection between $f^*/h^*$-vectors and $f/h$-vectors: in this case the $f^*/h^*$-vector of $P$ equals the $f/h$-vector of the triangulation, respectively.

**Example 1.** Let $P$ be the 2-dimensional cube $[-1, 1]^2$. The unimodular triangulation of $P$ shown in Figure 1 has $f$-vector $(f_0, f_1, f_2) = (9, 16, 8)$, as $f_i$ counts its $i$-dimensional faces. Equivalently, 

$$f^*(P) = (9, 16, 8),$$

and one easily checks that yields the familiar Ehrhart polynomial $ehr_P(n) = (2n+1)^2$.

**Example 2.** The $f^*$-vector of a $d$-dimensional unimodular simplex $\Delta$ equals

$$\left[ \binom{d+1}{1}, \binom{d+1}{2}, \ldots, \binom{d+1}{d+1} \right],$$

coinciding with the $f$-vector of $\Delta$ considered as a simplicial complex. If we append this vector by $f_{-1}^* = 1$, it gives the only instance of a symmetric $f^*$-vector of a lattice polytope $P$, since the equality $f_{-1}^* = f_d^*$ implies that $h_i^* = 0$ for all $1 \leq i \leq d$. 

**Figure 1.** A (regular) unimodular triangulation of the cube $[-1, 1]^2$. 

$\Delta$.
There has been much research on (typically linear) constraints for the \( h^*-\)vector of a given lattice polytope (see, e.g., \cite{12, 13}). On the other hand, \( f^*-\)vectors seem to be much less studied, and our goal is to rectify that situation. Our motivating question is how close the \( f^*-\)vector of a given lattice polytope is to being unimodal, i.e., the \( f^*-\)coefficients increase up to some point and then decrease. Our main results are as follows.

**Theorem 3.** Let \( d \geq 2 \) and let \( P \) be a \( d \)-dimensional lattice polytope. Then

(a) \( f^*_0 < f^*_1 < \cdots < f^*_\left\lfloor \frac{d}{2} \right\rfloor - 1 \leq f^*_\left\lfloor \frac{d}{2} \right\rfloor \);

(b) \( f^*_\left\lfloor \frac{3d}{4} \right\rfloor > f^*_\left\lfloor \frac{3d}{4} \right\rfloor + 1 > \cdots > f^*_d \);

(c) \( f^*_k \leq f^*_d - k \) for \( 0 \leq k \leq \left\lfloor \frac{d-3}{2} \right\rfloor \).

Examples 1 and 2 yield cases of polytopes for which the inequalities \( f^*_\left\lfloor \frac{3d}{4} \right\rfloor - 1 \leq f^*_\left\lfloor \frac{3d}{4} \right\rfloor \) and \( f^*_\left\lfloor \frac{d}{2} \right\rfloor > f^*_\left\lfloor \frac{d}{2} \right\rfloor + 1 \) hold, respectively.

We record the following immediate consequence of Theorem 3.

**Corollary 4.** Let \( P \) be a \( d \)-dimensional lattice polytope. Then for \( 0 \leq k \leq d \),

\[
 f^*_k \geq \min\{f^*_0, f^*_d\}.
\]

**Theorem 5.** The \( f^*-\)vector of a \( d \)-dimensional lattice polytope, where \( 1 \leq d \leq 13 \), is unimodal. On the other hand, there exists a 15-dimensional lattice simplex with nonunimodal \( f^*-\)vector.

Even though \( f^*-\)vectors are quite different from \( f^-\)vectors of polytopes, the above results resemble striking similarities with existing theorems on \( f^-\)vectors. Namely, Björner \cite{2, 3, 4} proved that the \( f^-\)-vector of a simplicial \( d \)-polytope satisfies all inequalities in Theorem 3 (with the *s removed, and the last coordinate dropped). In fact, Björner also showed that in the \( f^-\)-analogue of Theorem 3(b) the decrease starts from \( \left\lfloor \frac{3(d-1)}{4} \right\rfloor - 1 \) instead of \( \left\lfloor \frac{3d}{4} \right\rfloor \), and that the inequalities in Theorem 3(a) and (b) cannot be further extended, by constructing a simplicial polytope with \( f^-\)-vector that peaks at \( f^*_j \), for any \( \left\lfloor \frac{d}{2} \right\rfloor \leq j \leq \left\lfloor \frac{3(d-1)}{4} \right\rfloor - 1 \).

Corollary 4 compares the entries of the \( f^*-\)-vector with the minimum between the first and the last entry. Note that a similar relation for \( f^-\)-vectors of polytopes was recently proven by Hinman \cite{10}, answering a question of Bárány from the 1990s. (Hinman also proved a stronger result, namely certain lower bounds for the ratios \( \frac{f^*_0}{f^*_d} \) and \( \frac{f^*_d}{f^*_d-1} \).)

The \( f^-\)-analogue of Theorem 5 is again older: Björner \cite{2} showed that the \( f^-\)-vector of any simplicial \( d \)-polytope is unimodal for \( d \leq 15 \) (later improved to \( d \leq 19 \) by Eckhoff \cite{6}), and he and Lee \cite{1} produced examples of 20-dimensional simplicial polytopes with nonunimodal \( f^-\)-vectors.

For a special class of polytopes we can increase the range in Theorem 3(b). A lattice polytope \( P \) is \textit{Gorenstein of index} \( g \) if

- \( nP \) contains no interior lattice points for \( 1 \leq n < g \),
- \( gP \) contains a unique interior lattice point, and
- \( \text{ehr}_P(n - g) \) equals the number of interior lattice points in \( nP \), for \( n > g \).
Similarly, it is well known that

This theorem implies that lattice $d$-polytopes of degree $s$ satisfying $s^2 - s - 1 \leq \frac{d}{2}$ have a unimodal $f^*$-vector (see Proposition 9 below for details). One family with asymptotically small degree, compared to the dimension, is given by taking iterated pyramids. Given a polytope $P \subset \mathbb{R}^d$, we denote by $\text{Pyr}(P) \subset \mathbb{R}^{d+1}$ the convex hull of $P$ and the $(d+1)$st unit vector. It is well known that $P$ and $\text{Pyr}(P)$ have the same $h^*$-vector (ignoring an extra 0), and so we conclude:

**Corollary 8.** If $P$ is any lattice polytope then $\text{Pyr}^n(P)$ has unimodal $f^*$-vector for sufficiently large $n$.

## 2. Proofs

We start with a few warm-up proofs which only use the fact that $h^*$-vectors are non-negative.

**Proof of Theorem 3(a).** It follows by (1) and the nonnegativity of $h^*(P)$ that, for $1 \leq k \leq \frac{d}{2}$,

$$f_k^* \leq f_{k-1}^* \leq f_{k-1}^* - \sum_{j=0}^{k+1} \left( \binom{d+1-j}{k+1-j} - \binom{d+1-j}{k-j} \right) h_j^* \geq 0.$$  

In fact, $f_k^* - f_{k-1}^*$ is bounded below by $(\left( \frac{d+1}{k+1} \right) - (\frac{d+1}{k}) )$ $h_0^* > 0$ for $1 \leq k < \left\lceil \frac{d}{2} \right\rceil$, since $h_0^* = 1$.

**Proof of Theorem 3(b).** For $0 \leq k \leq \frac{d-3}{2}$, equation (4) gives

$$f_{d-1-k}^* - f_k^* = \sum_{j=0}^{d-k} \left( \binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \right) h_j^* = \sum_{j=0}^{d-1-k} \left( \binom{d+1-j}{k+1} - \binom{d+1-j}{k+1-j} \right) h_j^* + \sum_{j=d-2k}^{d-k} \left( \binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \right) h_j^*.$$  

We have $(\frac{d+1-j}{k+1}) - (\frac{d+1-j}{k}) \geq 0$ since $k+1-j \leq k+1 \leq \frac{d+1-j}{2}$ holds for $0 \leq j \leq d-1-2k$. Similarly, $(\frac{d+1-j}{d-k-j}) - (\frac{d+1-j}{k+1-j}) \geq 0$ holds because $k+1-j \leq d-k-j \leq \frac{d+1-j}{2}$ for all $d-2k \leq j$. Therefore, it follows by the nonnegativity of $h^*$-vectors that $f_{d-1-k}^* - f_k^* \geq 0$. \qed
Proof of Theorem 7. Since \( h^*_j = 0 \) for \( j \geq s + 1 \), (4) gives
\[
f^*_{k-1} - f^*_k = \sum_{j=0}^{s} \left( \left\lfloor \frac{d+1-j}{k-j} \right\rfloor - \left( \frac{d+1-j}{k+1-j} \right) \right) h^*_j = \sum_{j=0}^{s} \frac{2k-d-j}{k+1-j} \left( \frac{d+1-j}{k-j} \right) h^*_j.
\]
For \( \frac{d+1-j}{k-j} \leq k \leq d \), we have \( k+1-j > 0 \) and \( 2k-d-j > 0 \) for all \( j = 0, \ldots, s-1 \), and \( k+1-j > 0 \), \( 2k-d-j \geq 0 \) for \( j = s \). Therefore, the claim follows by the nonnegativity of \( h^*_s \)-vectors and the positivity of \( h^*_0 \).

Proposition 9. Let \( P \) be a \( d \)-dimensional lattice polytope that has degree at most \( s \) for some positive \( s \). If \( d \geq 2s^2 - 2s - 2 \) then the \( f^* \)-vector of \( P \) is unimodal with a (not necessarily "sharp") peak at \( f^*_p \), where \( \left\lfloor \frac{s}{2} \right\rfloor \leq p \leq \left\lfloor \frac{d+1}{2} \right\rfloor - 1 \).

Proof. By Theorems 3 and 7 it suffices to show that \( f^*_s + i \geq f^*_{s+i+1} \) implies \( f^*_{s+i+2} \), i.e., that \( 2f^*_{s+i+1} - f^*_{s+i+2} - f^*_{s+i} \geq 0 \) for \( 0 \leq i \leq \frac{d}{2} - 2 \).

As \( h^*_j = 0 \) for \( j \geq s + 1 \), by (4) we can express \( 2f^*_{s+i+1} - f^*_{s+i+2} - f^*_{s+i} \) as the sum
\[
\sum_{j=0}^{s} \left( \frac{2}{\left\lfloor \frac{s+j}{2} \right\rfloor + 2 - j + i} - \frac{2}{\left\lfloor \frac{s+j}{2} \right\rfloor + 3 - j + i} \right) h^*_j = \sum_{j=0}^{s} \left( \frac{2}{\left\lfloor \frac{s+j}{2} \right\rfloor + 2 - j + i} - \frac{2}{\left\lfloor \frac{s+j}{2} \right\rfloor + 3 - j + i} \right) h^*_j.
\]
Since \( d \geq \max\{2s^2 - 2s - 2, 0\} \) we have that \( \left\lfloor \frac{s+j}{2} \right\rfloor + 3 - j + i \) is positive for \( j = 0, \ldots, s \) and since \( h^*_j \) is nonnegative, it remains to show that
\[
2\left( \left\lfloor \frac{s+j}{2} \right\rfloor + 3 - j + i \right) - \left( \left\lfloor \frac{s+j}{2} \right\rfloor + 2 - j + i \right) - \left( \left\lfloor \frac{s+j}{2} \right\rfloor + 1 - j + i \right) = d - (2j - 1)\left( \frac{s+j}{2} - \left\lfloor \frac{s+j}{2} \right\rfloor \right) - 2i - 4ij - 6j - 6.
\]

is nonnegative for \( 0 < j < s \). Indeed, the conditions \( j \leq s \) and \( i \leq \frac{s}{2} - 2 \) imply that (5) is bounded below by
\[
d - 4i^2 - 12i - j^2 - 6 \geq d - 4\left( \frac{s}{2} - 2 \right)^2 - 12\left( \frac{s}{2} - 2 \right) - s^2 - 6 = d - 2s^2 + 2s + 2,
\]
which is nonnegative by assumption.

The next proofs use more than just the nonnegativity of \( h^*_s \)-vectors. The first result needs the following elementary lemma on binomial coefficients.

Lemma 10. Let \( j, k, n \) be positive integers such that \( k \leq n + 1 - j \). Then
\[
\left| \binom{n}{k} - \binom{n}{k-1} \right| \geq \left| \binom{n-j}{k} - \binom{n-j}{k-1} \right|
\]
whenever \( n \neq 2k - 1 \).

Proof. It suffices to prove the statement for the cases i) \( j = 1 \) and the quantities \( \binom{n}{k} - \binom{n}{k-1} \) and \( \binom{n-1}{k} - \binom{n-1}{k-1} \) having the same sign, and ii) the point when the signs change, i.e., \( n = 2k \) and \( j = 2 \).
To show case $i$), we simplify
\[
\left| \binom{n}{k} - \binom{n}{k-1} \right| = \frac{(n-1)!}{k!(n-k)!} \frac{n}{n-k+1} |n-2k+1|
\]
and
\[
\left| \binom{n-1}{k} - \binom{n-1}{k-1} \right| = \frac{(n-1)!}{k!(n-k)!} |n-2k|.
\]
If $n \geq 2k$ then the inequalities
\[
\frac{n}{n-(k-1)}(n-2k+1) \geq n-2k+1 > n-2k
\]
imply that
\[
\left| \binom{n}{k} - \binom{n}{k-1} \right| > \left| \binom{n-1}{k} - \binom{n-1}{k-1} \right|. \tag{6}
\]
If $n \leq 2k-2$, we have $k(-2k+2+n) \leq 0$ which is equivalent to
\[
\frac{n}{n-(k-1)}(2k-n-1) \geq 2k-n
\]
and so again (6) holds as a weak inequality.

To show case $ii$), we compute
\[
\left| \binom{2k}{k} - \binom{2k}{k-1} \right| = \frac{(2k)!}{k!(k+1)!} = \frac{(2k-2)!}{k!(k-1)!} \frac{2k(2k-1)}{k+1}
\]
and
\[
\left| \binom{2k-2}{k} - \binom{2k-2}{k-1} \right| = \frac{(2k-2)!}{k!(k-1)!}.
\]
Since $2(2k-1) \geq (k+1)$ for any positive $k$, we conclude that
\[
\left| \binom{2k}{k} - \binom{2k}{k-1} \right| \geq \left| \binom{2k-2}{k} - \binom{2k-2}{k-1} \right|. \tag{7}
\]

Proof of Theorem $[5][6]$. The inequality $f^*_{d-1} > f^*_d$ holds by Theorem $[4]$. Now, let $\left\lceil \frac{3d}{4} \right\rceil + 1 \leq k < d$. By $[1]$, \[
f^*_{k-1} - f^*_k = \sum_{j=0}^{k+1} \left( \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j, \tag{7}
\]
The difference $\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j}$ is nonnegative whenever $k-j \geq \left\lceil \frac{d+1-j}{2} \right\rceil$ and negative otherwise, i.e., the difference is nonnegative whenever $j \leq 2k-d$ and negative whenever $j > 2k-d$. Since $2d-2k < 2k+1 - d$ for $\left\lceil \frac{3d}{4} \right\rceil + 1 \leq k$, from (7) we obtain
\[
f^*_{k-1} - f^*_k \geq \sum_{j=0}^{2d-2k} \left( \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{8}
\]
\[
+ \sum_{j=2k+1-d}^{k+1} \left( \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{9}
\]
where the differences appearing in (8) are nonnegative and the ones in (9) are negative. Our aim is to compare the sums in (8) and (9) to conclude that $f^*_{k-1} - f^*_k$ is positive.
Using standard identities for binomial coefficients, the right hand-side of (8) equals
\[
\sum_{j=0}^{2d-2k} \left( \sum_{l=j}^{2d-2k-1} \left( \binom{d-l}{k-l} - \binom{d-l}{k+1-l} \right) \right) h_j^*
\]
\[
= \sum_{l=0}^{2d-2k-1} \left( \binom{d-l}{k-l} - \binom{d-l}{k+1-l} \right) \sum_{j=0}^{2d-2k-1-l} h_j^*
\]
\[
+ \left( \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h_j^*,
\]
hence we conclude that right hand-side of (8) is bounded below by
\[
\left( \binom{d}{k} - \binom{d}{k+1} \right) h_0^* + \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h_j^*,
\]
and
\[
\left( \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h_j^*,
\]
\[
> \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h_j^*,
\]
since \( \binom{d}{k} - \binom{d}{k+1} > 0 \) for \( \left\lceil \frac{3d}{4} \right\rceil + 1 \leq k < d \), and \( h_0^* = 1, h_j^* \geq 0 \) for \( j = 1, \ldots, 2d-2k-1 \).

On the other hand, for the differences appearing in (9), using that \( 2d-2k < j \) and \( j \leq k+1 \), it follows by Lemma [10] that
\[
\left| \binom{d+1-(2d-2k)}{d+1-k} - \binom{d+1-(2d-2k)}{d-k} \right| \geq \left| \binom{d+1-j}{d+1-k} - \binom{d+1-j}{d-k} \right|,
\]
i.e.,
\[
\left| \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right| \geq \left| \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right|.
\]
Hence for \( j \geq 2k+1-d \),
\[
- \left( \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \leq \left( \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right).
\]
Since both \( -\binom{d+1-j}{k-j} \) and \( h_j^* \) are nonnegative for \( j \geq 2k+1-d \), the sum in (9) is bounded below by
\[
- \left( \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=2k+1-d}^{d} h_j^*.
\]
Now (10) and (11) yield
\[
f_{k-1}^* - f_k^* > \left( \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \left( \sum_{j=0}^{2d-2k} h_j^* - \sum_{j=2k+1-d}^{d} h_j^* \right).
\]
Hibi [8] showed that the inequality
\[
\sum_{j=0}^{m+1} h_j^* \geq \sum_{j=d-m}^{d} h_j^*
\]
(12)
holds for $m = 0, ..., \lfloor \frac{d}{2} \rfloor - 1$. Since $2d - 2k - 1 \leq \lfloor \frac{d}{2} \rfloor - 1$ for $\lfloor \frac{3d}{4} \rfloor + 1 \leq k$, we can use (12) to finally obtain

$$f_{k-1}^* - f_k^* > 0.$$ 

Proof of Theorem If $d = 1$ or 2, there is nothing to prove.

If $3 \leq d \leq 6$, then by Theorem either

$$f_0^* \leq \cdots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \geq f_{\lfloor \frac{d}{4} \rfloor}^* \geq \cdots \geq f_d^*$$

or

$$f_0^* \leq \cdots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \leq f_{\lfloor \frac{d}{4} \rfloor}^* \geq \cdots \geq f_d^*.$$ 

For $7 \leq d \leq 13$, we will show that if $f_i^* \geq f_{i+1}^*$, then $f_{i+1}^* \geq f_{i+2}^*$, for all $\lfloor \frac{d}{2} \rfloor \leq i \leq \lfloor \frac{3d}{4} \rfloor - 2$. By Theorem this will imply the unimodality of $(f_0^*, f_1^*, \ldots, f_d^*)$.

We will examine each value of $d$ separately.

Suppose that $d = 7$ and $f_5^* \geq f_4^*$. Then, by (4), we compute

$$2f_4^* - f_5^* - f_6^* = 14h_6^* + 14h_4^* + 10h_2^* + 5h_3^* + h_4^* - h_5^* - h_6^* > h_0^* + h_1^* + h_2^* + h_3^* - h_4^* - h_5^* - h_6^* - h_7^*,$$

which is always nonnegative by (12). Hence $f_4^* - f_5^* \geq f_5^* - f_4^*$.

Likewise, for $d = 8$, (12) implies that

$$2f_5^* - f_4^* - f_6^* = 6h_5^* + 14h_4^* + 10h_3^* + 5h_2^* + h_3^* - h_4^* - h_5^* - h_6^* - h_7^* - h_8^* \geq 0.$$

For $d = 9$, we similarly get

$$f_5^* - f_6^* - 2(f_4^* - f_5^*) = 6h_5^* + 42h_4^* + 42h_2^* + 28h_3^* + 13h_4^* + 3h_5^* - h_6^* > h_0^* + h_1^* + h_2^* + h_3^* + h_4^* - h_5^* - h_6^* - h_7^* - h_8^* \geq 0$$

by (12).

A similar argument works for $d = 10$. By (12),

$$2f_6^* - f_5^* - f_7^* = 33h_6^* + 48h_4^* + 42h_2^* + 28h_3^* + 14h_4^* + 4h_5^* - h_6^* - 2h_7^* - h_8^* > 2(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^*),$$

which follows again from (12), since

$$f_6^* - f_7^* - 2(f_5^* - f_6^*) = 33h_6^* + 132h_4^* + 126h_2^* + 84h_3^* + 42h_4^* + 14h_5^* + h_6^* - 2h_7^* - h_8^* > 2(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^*),$$

and

$$f_7^* - f_6^* - \frac{4}{5}(f_6^* - f_7^*) > 3(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^*) \geq 0.$$ 

For $d = 12$, there are also two cases: $i = 6$ and $i = 7$. Using (12), it follows that

$$f_i^* - f_{i-1}^* - \frac{5}{4}(f_{i-1}^* - f_i^*) > 3(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^*) \geq 0,$$
Proof. We know from Theorem 5 that

\[ f'_5 - f_5 - \frac{1}{2}(f'_6 - f_6) > \]

\[ 3(h_0^* + h_1^* + h_2^* + h_3^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^*) \geq 0. \]

For \( d = 13 \), we employ a stronger form of (12). The expression

\[ f'_7 - f_7 - \frac{7}{3}(f'_6 - f_6) \geq \]

\[ 3(h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^* - h_{13}^*) \]

is nonnegative by Theorem (6) in [12]. Similarly, using Theorem (6) in [12] we have

\[ 2f'_8 - f'_7 - f_8 \geq \]

\[ 4(h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^* - h_{13}^*) \geq 0. \]

To construct a polytope with nonunimodal \( f^* \)-vector, we employ a family of simplices introduced by Higashitani [9]. Concretely, denote the \( j \)th unit vector by \( e_j \) and let

\[ \Delta_w := \text{conv}\{0, e_1, e_2, \ldots, e_{14}, w\} \]

where

\[ w := (1,1, \ldots, 1, 131, 131, \ldots, 131, 132). \]

It has \( h^* \)-vector

\[ (1,0,0, \ldots, 0, 131, 0,0, \ldots, 0) \]

and, via (11), \( f^* \)-vector

\[ (16, 120, 560, 1820, 4368, 8008, 11440, 13001, 12488, 11676, 11704, 10990, 7896, 3788, 1064, 132). \]

\( \square \)

Corollary 11. Let \( P \) be a \( d \)-dimensional lattice polytope such that the \( h^* \)-vector of \( P \) is of degree at most 5. Then \( P \) has unimodal \( f^* \)-vector.

Proof. We know from Theorem 5 that \( f^* \) is unimodal when \( d \leq 13 \).

Suppose that \( d \geq 14 \). The proof is similar to the proof of Proposition 9 but we need to be a bit more precise with bounds. By Theorems 3(a) and 7 it suffices to show that \( f^*_{\lceil d \rceil + i} \geq f^*_{\lceil d \rceil + i+1} \) implies \( f^*_{\lceil d \rceil + i+1} \geq f^*_{\lceil d \rceil + i+2} \) for \( i = 0, \ldots, \lceil \frac{d+5}{2} \rceil - \lceil \frac{d}{2} \rceil - 3 \). Notice that \( \lceil \frac{d+5}{2} \rceil = \lceil \frac{d}{2} \rceil + \lceil \frac{d}{2} \rceil \), hence \( i = 0 \). Arguing as in the proof of Proposition 9 we can reduce the proof to showing that the expression in (5) in Proposition 9 is nonnegative for \( 0 \leq j \leq 5 \) and \( i = 0 \), i.e., that

\[ d - (2j - 1) \left( \left\lfloor \frac{d}{2} \right\rfloor - \left\lfloor \frac{d}{2} \right\rfloor \right) - 6 + j(5 - j) \geq 0. \]

(13)

For \( 0 \leq j \leq s \), we have

\[ d - (2j - 1) \left( \left\lfloor \frac{d}{2} \right\rfloor - \left\lfloor \frac{d}{2} \right\rfloor \right) - 6 + j(5 - j) \geq d - 15, \]
Proof of Theorem 6. Let 

\[
d - (2j - 1) \left( \left\lfloor \frac{d}{2} \right\rfloor - \left\lfloor \frac{d - j - 1}{2} \right\rfloor \right) - 6 + j(5 - j) \geq d - 6.
\]

\[\square\]

Concluding Remarks

There are many avenues to explore \(f^*\)-vectors, e.g., along analogous studies of \(h^*\)-vectors, and we hope the above results form an enticing starting point. We conclude with a few open questions which are apparent from the above.
The techniques in our proof of Theorem 5 do not offer much insight in the case of 14-dimensional lattice polytopes as there are candidates $f^*$-vectors with corresponding $h^*$-vectors that satisfy all inequalities discussed in [12]. It is unknown though if such polytopes exist.

Higashitani [9, Theorem 1.1] provided examples of $d$-dimensional polytopes with nonunimodal $h^*$-vector for all $d \geq 3$. Therefore, by Theorem 5 we have examples of polytopes that have such $h^*$-vector but their $f^*$-vector is unimodal. It would be interesting to know if the opposite can be true, that is, if there exist polytopes with unimodal $h^*$-vector and nonunimodal $f^*$-vector. By Corollary 11 such polytopes would need to have degree at least 6.

References

[1] Louis J. Billera and Carl W. Lee. A proof of the sufficiency of McMullen’s conditions for $f$-vectors of simplicial convex polytopes. *J. Comb. Theory, Ser. A*, 31:237–255, 1981.
[2] Anders Björner. The unimodality conjecture for convex polytopes. *Bull. Am. Math. Soc., New Ser.*, 4:187–188, 1981.
[3] Anders Björner. Face numbers of complexes and polytopes. *Proc. Int. Congr. Math., Berkeley/Calif. Vol. 2*, 1408-1418, 1987.
[4] Anders Björner. Partial unimodality for $f$-vectors of simplicial polytopes and spheres. *Jerusalem combinatorics ’93: an international conference in combinatorics, May 9-17, 1993, Jerusalem, Israel*, pages 45–54. Providence, RI: American Mathematical Society, 1994.
[5] Felix Breuer. Ehrhart $f^*$-coefficients of polytopal complexes are non-negative integers. *Electron. J. Combin.*, 19(4):Paper 16, 22 pp., 2012.
[6] Jürgen Eckhoff. Combinatorial properties of $f$-vectors of convex polytopes. *Normat, 54*(4):146–159, 2006.
[7] Eugène Ehrhart. Sur les polyèdres ratiennels homothétiques à n dimensions. *C. R. Acad. Sci. Paris*, 254:616–618, 1962.
[8] Takayuki Hibi. *Algebraic Combinatorics on Convex Polytopes*. Carslaw, 1992.
[9] Akihiro Higashitani. Counterexamples of the conjecture on roots of Ehrhart polynomials. *Discrete Comput. Geom.*, 47(3):618–623, 2012. arXiv:1106.4633.
[10] Joshua Hinman. A positive answer to Bárány’s question on face numbers of polytopes, 2022. Preprint (arXiv:2204.02568).
[11] Richard P. Stanley. Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen–Macaulay rings. *Duke Math. J.*, 43(3):511–531, 1976.
[12] Alan Stapledon. Inequalities and Ehrhart $\delta$-vectors. *Trans. Amer. Math. Soc.*, 361(10):5615–5626, 2009. arXiv:math/0801.0873.
[13] Alan Stapledon. Additive number theory and inequalities in Ehrhart theory. *Int. Math. Res. Not.*, (5):1497–1540, 2016. arXiv:0904.3035v2.
