The strong convergence and stability of explicit approximations for nonlinear stochastic delay differential equations

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Abstract
This paper focuses on explicit approximations for nonlinear stochastic delay differential equations (SDDEs). Under less restrictive conditions, the truncated Euler-Maruyama (TEM) schemes for SDDEs are proposed, which numerical solutions are bounded in the $q$th moment for $q \geq 2$ and converge to the exact solutions strongly in any finite interval. The $1/2$ order convergence rate is yielded. Furthermore, the long-time asymptotic behaviors of numerical solutions, such as stability in mean square and $\mathbb{P} - 1$, are examined. Several numerical experiments are carried out to illustrate our results.

Keywords Stochastic delay differential equations · Explicit truncated Euler-Maruyama scheme · Moment bound · Strong convergence · Stability

1 Introduction
This paper considers a stochastic delay differential equation (SDDE) described by

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dW(t), \quad t > 0, \\
x(t) &= \xi(t), \quad t \in [-\tau, 0],
\end{aligned}
$$

(1.1)

where $\tau > 0$ is a constant, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$.

In the given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$ is a filtration satisfying the usual conditions (that is, it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Here, $W(t) = (W_1(t), W_2(t), \cdots, W_m(t))^T$ is an $\mathcal{F}_t$-measurable...
m-dimensional Brownian motion. The SDDE models play a key role in communications, finance, medical sciences, ecology, and other branches of industry and science (see, e.g., [1, 2, 4, 7, 23, 26, 28]). Since SDDEs can hardly be solved explicitly, numerical methods have become essential. Our aim is to construct explicit numerical solutions for (1.1) and to prove their convergence and stability.

Owing to the simple algebraic structures and easy implementation, explicit schemes have been developed excellently for stochastic differential equations (SDEs), see, e.g., [13, 18]. The Euler-Maruyama (EM) schemes are the most popular for approximating SDEs with the global Lipschitz coefficients [10, 18]. However, for a large class of SDEs with superlinear growth coefficients, the errors between the EM numerical solutions and the true solutions diverge to infinity in finite time [15]. The divergence of the multilevel Monte Carlo EM method was also proven for a family of SDEs with superlinearly growing and global one-sided Lipschitz drift coefficients [17]. Thus, several modified EM schemes were proposed to deal with the strong convergence of nonlinear SDEs, including the tamed EM scheme [16, 30], the increment-tamed EM scheme [13], the stopped increment-tamed EM scheme [14], and the truncated EM scheme [22, 24, 25]. Furthermore, the multilevel Monte Carlo method was combined with the tamed EM method to obtain the striking higher convergence order of nonlinear SDEs preserved from the Lipschitz case [17].

The dynamical behaviors of SDDEs depending on the past are more complicated. As the delay is a constant, the dynamic behaviors of the true solutions and the numerical solutions of SDDEs can be obtained in each delay interval step by step, given that the last segment solutions satisfy the next iteration conditions required by the coefficients and the initial data of SDEs. For examples, by this method, the numerical solutions of the EM scheme [3] and the tamed EM scheme [6] were showed to converge the exact solutions of SDDEs with linear coefficients w.r.t. the current state $x(t)$, respectively. However, this method is invalid for a large class of SDDEs, which coefficients grow highly nonlinearly w.r.t. $x(t)$ and $x(t - \tau)$. For an example, consider the scalar SDDE

$$
\begin{align*}
\begin{cases}
    dx(t) = -20x^3(t)dt + \left(x^2(t) + x^2(t - 1)\right)dW(t), & t > 0, \\
x(t) = \cos(B(t)), & t \in [-1, 0],
\end{cases}
\end{align*}
$$

(1.2)

where $B(t)$ is a scalar Brownian motion independent of $W(t)$. For given $p \geq 2$, any $t \in [k, k + 1]$ ($k \geq 0$), using the Itô formula,

$$
\begin{align*}
\mathbb{E}|x(t)|^p - \mathbb{E}|x(k)|^p &= -p\left(20 - \frac{p - 1}{2}\right) \int_{k}^{t} \mathbb{E}|x(s)|^{p-2}ds + p(p - 1) \int_{k}^{t} \mathbb{E}|x(s)|^{p-2}|x(s - 1)|^2ds \\
&\quad + \frac{p(p-1)}{2} \int_{k}^{t} \mathbb{E}|x(s)|^{p-2}|x(s - 1)|^4ds \\
&\leq -p\left(20 - \frac{p - 1}{2} - \frac{(p - 1)(3p - 2)}{2(p + 2)}\right) \int_{k}^{t} \mathbb{E}|x(s)|^{p+2}ds \\
&\quad + \frac{4p(p-1)}{p+2} \int_{k}^{t} \mathbb{E}|x(s - 1)|^{p+2}ds.
\end{align*}
$$

(1.3)
One observes that the analysis of $\mathbb{E}|x(t)|^p$ for $t \in [k, k + 1]$ depends on the upper bound of $\mathbb{E}|x(t)|^{p+2}$ for $t \in [k-1, k]$, and $\sup_{t \in [0,1]} \mathbb{E}|x(t)|^p < \infty$ for some $p < 41$. This implies that $\sup_{t \in [0,T]} \mathbb{E}|x(t)|^6$ is bounded for $T = 18$ at most by the SDE iteration. However, according to the inherent characteristic of SDE (1.2) and [26, Theorem 3.1] $\sup_{t \in [0,T]} \mathbb{E}|x(t)|^6 < \infty$ for any $T > 0$. The moment estimate of the numerical solutions by the SDE iteration is quite similar to that of $\mathbb{E}|x(t)|^p$.

The iteration techniques by the tamed EM scheme of SDEs to deal with the $L^q$ convergence for SDDEs was revealed for the limited terminal time $T$.

Because of the importance and the realistic requirement, significant efforts have also been devoted to approximating SDDEs. Due to implementation, several explicit schemes for SDDEs were proposed, such as the EM scheme (see, e.g., [2–4, 9, 19, 27, 28]), the truncated EM scheme [8], the derivative-free explicit scheme [29], the truncated Milstein scheme [35], the projected EM scheme [20], and the tamed Euler scheme [6]. Since implicit schemes sometimes achieve a better convergence rate, they also attracts lots of attention (see, e.g., [11, 12, 31]). To the best of our knowledge, most of the results on the strong convergence rate for SDDE (1.1) require that $f$ and $g$ obey the global one-sided Lipschitz condition

$$2(x - \bar{x}, f(x, y) - f(\bar{x}, \bar{y})) + |g(x, y) - g(\bar{x}, \bar{y})|^2 \leq C(|x - \bar{x}|^2 + |y - \bar{y}|^2), \quad (1.4)$$

where $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and $C$ is a constant. However, it is too restrictive for some SDDEs. By computation, one observes from (1.2)

$$2(x - \bar{x}, -20(x^3 - \bar{x}^3)) + (x^2 - \bar{x}^2) + (y^2 - \bar{y}^2))^2 = -40(x - \bar{x})^2(x^2 + x\bar{x} + \bar{x}^2) + (x - \bar{x})^2(x + \bar{x})^2 + 2(x - \bar{x})(x + \bar{x})(y - \bar{y})(y + \bar{y}) + (y - \bar{y})^2(y + \bar{y})^2,$$

which implies that (1.4) does not hold for SDDE (1.2). This paper focuses on constructing explicit schemes for SDDEs and showing that the $L^q$ convergence order for $q \geq 2$ is $1/2$ in any finite time interval under more flexible conditions.

The stability of SDDEs is one of the major concerns in stochastic processes, systems theory and control [23]. To find the scheme preserving the long-time dynamical behaviors of SDDEs is significant. An example that the EM scheme does not reproduce the exponentially almost sure stability of SDDE while the backward EM (BEM) scheme does was given [33]. The split-step theta (SSD) method preserves the exponential stability in mean square under the Khasminskii-type condition was showed [36]. The partially truncated EM method was proposed to approximate the exponentially almost sure stability for SDDEs with Markovian switching [5]. Although the various stable numerical methods are investigated well, how to design the explicit scheme to approximate the stability of nonlinear SDDEs under more flexible conditions is desired. Hence, the other aim of this paper is to establish an easy implementable numerical scheme capturing the stability.

Borrowing the ideas from [22] and utilizing the inherent characteristic the explicit truncated EM (TEM) schemes are proposed to approximate nonlinear SDDEs. Under the polynomial growth coefficient conditions, the $1/2$ order of strong convergence rate is yielded for the TEM schemes. Moreover, a more precise TEM scheme is constructed, which numerical solutions reproduce the underlying exponential stability.
under the more flexible Khasminskii-type condition. Some numerical experiments are carried out to examine the effectiveness of the TEM schemes.

This paper is organized as follows. Section 2 gives some notations and preliminary results on the exact solutions of SDDE (1.1). Section 3 lists the main results, including the convergence, the convergence rate, and the stability. Section 4 gives two examples to illustrate our main results. Section 5 concludes the paper.

2 Notations and preliminary results

For $A \in \mathbb{R}^{d \times m}$, we denote by $A^T$ the transpose of $A$, and let $|A|$ denote the Hilbert-Schmidt norm, i.e., $A = \sqrt{tr(ATA^T)}$. We denote the inner product of $x, y \in \mathbb{R}^d$ by $\langle x, y \rangle$. By $[a]$, we denote the largest integer less than or equal to $a$. We let $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. For a set $D$, let $1_D(x) = 1$ if $x \in D$ otherwise 0. Moreover, let $\mathbb{R}_+ = [0, \infty)$ and $\tau > 0$. By $C([-\tau, 0]; \mathbb{R}^d)$, we denote the space of all continuous $\mathbb{R}^d$-valued functions $X(\cdot)$ defined on $[-\tau, 0]$ equipped with the norm $\|X\| = \sup_{-\tau \leq \theta < 0} |X(\theta)|$. By $C_b((-\tau, 0]; \mathbb{R}^d)$, we denote the space of all bounded $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^d)$-valued random variable $\xi$. By $C(\mathbb{R}^d; \mathbb{R}_+)$, we denote the space of all continuous nonnegative functions defined on $\mathbb{R}^d$. By $\mathcal{V}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}_+)$, we denote the space of all continuous nonnegative functions $\hat{V}(x, y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $\hat{V}(x, x) = 0$ for $x \in \mathbb{R}^d$. Denote by $C^2(\mathbb{R}^d; \mathbb{R})$ the space of all functions defined on $\mathbb{R}^d$ which are continuously twice differentiable. For $U \in C^2(\mathbb{R}^d; \mathbb{R})$, define $LU : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$LU(x, y) = \langle f(x, y), U_x(x) \rangle + \frac{1}{2} \langle g(x, y), U_{xx}(x)g(x, y) \rangle.$$  \hspace{1cm} (2.1)

Let $\delta_1, \delta_2$ be two $\mathcal{F}_t$-stopping times with $\delta_1 \leq \delta_2$ a.s. We define the stochastic interval

$$[[\delta_1, \delta_2]] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \delta_1 \leq t \leq \delta_2\}.$$ 

We denote by $C$ a generic positive constant which is independent of the relevant parameters and may take different values at each occurrence. Throughout this paper, we assume that the initial data $\xi \in C_b((-\tau, 0]; \mathbb{R}^d)$ is $\mathcal{F}_0$-measurable and independent of $(W(t))_{t \geq 0}$. Then, there is a constant $M_0 > 0$ such that $|\xi(\omega, t)| \leq M_0$ holds for all $(\omega, t) \in \Omega \times [-\tau, 0]$. In order to establish the existence of the global solutions we give the weakly local Lipschitz condition and the Khasminskii-type condition.

(H1) (the weakly local Lipschitz condition) For any $l > 0$, there exists a positive constant $L_l$ such that, for any $x, \bar{x}, y \in \mathbb{R}^d$ with $|x| \lor |\bar{x}| \lor |y| \leq l$,

$$|f(x, y) - f(\bar{x}, y)| \lor |g(x, y) - g(\bar{x}, y)| \leq L_l|x - \bar{x}|.$$ 

(H2) (the Khasminskii-type condition) There exist constants $q \geq 2$, $K_1 \geq 0$, $K_2 \geq 0$ as well as a function $V_1 \in C(\mathbb{R}^d; \mathbb{R}_+)$ such that

$$\left(1 + |x|^{q-1} \left(2x, f(x, y) \right) + (q - 1)|g(x, y)|^2 \right) \leq K_1 \left(1 + |x|^q + |y|^q \right) - K_2 (V_1(x) - V_1(y)), \quad \forall x, y \in \mathbb{R}^d. \hspace{1cm} (2.2)$$
Theorem 2.1 Let (H1) and (H2) hold. Then, SDDE (1.1) with any initial data \( \xi \in C_b([-\tau, 0]; \mathbb{R}^d) \) has a unique global solution \( x(t) \) with the property

\[
\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^q \leq C, \quad \forall \ T > 0. \tag{2.3}
\]

Furthermore, for any \( M > 0 \), let

\[
\vartheta_M = \inf \{ t \geq 0 : |x(t)| \geq M \}. \tag{2.4}
\]

Then, we obtain

\[
\mathbb{P}\{\vartheta_M \leq T\} \leq \frac{C}{M^q}. \tag{2.5}
\]

Proof Fix a positive constant \( l \). It follows from (2.2) that for any \( x, y \in \mathbb{R}^d \) with \( |y| \leq l \),

\[
\langle 2x, f(x, y) \rangle + |g(x, y)|^2 \\
\leq \frac{1}{(1 + |x|^2)^{q/2 - 1}} [K_1(1 + |x|^q + |y|^q) - K_2(V_1(x) - V_1(y))] \\
\leq K_1(1 + |x|^2) + (l^qK_1 + K_2 \sup_{|y| \leq l} V_1(y))(1 + |x|^2) \leq C(l)(1 + |x|^2). \tag{2.6}
\]

Under (H1) and (2.6), due to [9, Theorem 2.1], SDDE (1.1) admits a unique global solution with the initial data \( \xi \in C_b([-\tau, 0]; \mathbb{R}^d) \). Let \( U(x) = (1 + |x|^2)^{\frac{q}{2}} \), where \( q \) is given in (H2). Due to (2.2), we compute

\[
\mathcal{L}U(x, y) \\
= \frac{q}{2}(1 + |x|^2)^{\frac{q}{2} - 2} \left[ (1 + |x|^2) \left( \langle 2x, f(x, y) \rangle + |g(x, y)|^2 \right) + (q - 2)|x| |g(x, y)|^2 \right] \\
\leq \frac{q}{2}(1 + |x|^2)^{\frac{q}{2} - 2} \left[ (1 + |x|^2) \left( \langle 2x, f(x, y) \rangle + |g(x, y)|^2 \right) + (q - 2)|x|^2 |g(x, y)|^2 \right] \\
\leq \frac{q}{2}(1 + |x|^2)^{\frac{q}{2} - 1} \left( \langle 2x, f(x, y) \rangle + (q - 1)|g(x, y)|^2 \right) \\
\leq \frac{q}{2}K_1(1 + |x|^q + |y|^q) - \frac{q}{2}K_2(V_1(x) - V_1(y)). \tag{2.7}
\]

Owing to [26, Theorem 3.1] and the definition of \( U \), for any \( T > 0 \),

\[
\sup_{0 \leq t \leq T} \mathbb{E}\left(1 + |x(t)|^2\right)^{\frac{q}{2}} \leq C.
\]
By (2.7) and the Dynkin formula we obtain that for any $0 \leq t \leq T$,
\[
E\left(1 + |x(t \wedge \vartheta_M)|^2\right)^{\frac{q}{2}} \\
\leq E\left(1 + |\xi(0)|^2\right)^{\frac{q}{2}} + \frac{q}{2} E \int_0^{t \wedge \vartheta_M} \left[K_1 \left(1 + (1 + |x(s)|^2)^{\frac{q}{2}} + (1 + |x(s - \tau)|^2)^{\frac{q}{2}} - K_2 (V_1(x(s)) - V_1(x(s - \tau)))\right]\right] ds \\
\leq E\left(1 + |\xi(0)|^2\right)^{\frac{q}{2}} + \frac{q}{2} E \int_0^{t \wedge \vartheta_M} (1 + |x(s)|^2)^{\frac{q}{2}} ds \\
+ \frac{q}{2} K_1 T + q K_1 \int_0^{t \wedge \vartheta_M} V_1(x(s)) ds \\
+ \frac{q}{2} K_2 \int_0^{t \wedge \vartheta_M} V_1(x(s)) ds \\
\leq C + q K_1 \int_0^{t \wedge \vartheta_M} (1 + |x(s)|^2)^{\frac{q}{2}} ds,
\]
which implies
\[
\sup_{0 \leq t \leq T} E\left(1 + |x(t \wedge \vartheta_M)|^2\right)^{\frac{q}{2}} \leq C + q K_1 \int_0^T \sup_{0 \leq s \leq t} E\left(1 + |x(s \wedge \vartheta_M)|^2\right)^{\frac{q}{2}} dt.
\]
Applying the Gronwall inequality [23, p.45, Theorem 8.1] yields that
\[
\sup_{0 \leq t \leq T} E\left(1 + |x(t \wedge \vartheta_M)|^2\right)^{\frac{q}{2}} \leq C e^{q K_1 T}.
\]
Thus,
\[
\mathbb{P}\{\vartheta_M \leq T\} M^q \leq E\left[|x(\vartheta_M)|^q 1_{[\vartheta_M \leq T]}\right] \leq E\left(1 + |x(T \wedge \vartheta_M)|^2\right)^{\frac{q}{2}} \leq C.
\]
Then, the required inequality (2.5) follows. 

\section{3 Main results}

This section focuses on constructing explicit schemes to approximate the solutions of nonlinear SDDEs. Borrowing the truncation ideas from [22], we begin to estimate the nonlinear growth rate of $f$ and $g$, and then construct an appropriate truncation projection. Through this projection, the EM grid points can be pulled back to a compact set. Thus, the numerical solutions of this modified EM scheme will avoid extra-ordinary errors from the nonlinear drift and diffusion terms as well as the Brownian motion efficiently. Besides (H1) and (H2), we impose an additional assumption to describe the uniform continuity of $f$ and $g$ in $y$.

\textbf{(H3)} For any $M > 0$, functions $f(x, y)$ and $g(x, y)$ are uniformly continuous in the argument corresponding $y$ for any $x \in \mathbb{R}^d$ satisfying $|x| \leq M$, that is, for any $x, y, \bar{y} \in \mathbb{R}^d$ with $|x| \leq M$,
\[
\lim_{y \to \bar{y}} \sup_{|x| \leq M} \left[|f(x, y) - f(x, \bar{y})| + |g(x, y) - g(x, \bar{y})|\right] = 0.
\]
Under \((H1)\) and \((H3)\), choose a strictly increasing continuous function \(\Phi : [1, +\infty) \to \mathbb{R}_+\) satisfying

\[
\sup_{|x| \vee |y| \leq l} \left( \frac{|f(x, y)|}{1 + |x|} \sqrt{\frac{|g(x, y)|^2}{(1 + |x|)^2}} \right) \leq \Phi(l), \quad \forall \ l \geq 1. \tag{3.1}
\]

Let \(\Phi^{-1} : \Phi(1), +\infty) \to \mathbb{R}_+\) be the inverse function of \(\Phi\). Without loss of generality, we may assume that there exist a sufficiently large integer \(N > 0\) such that the stepsize \(\Delta = \tau/N \in (0, 1]\). Define a truncation mapping \(\Upsilon_{\Phi, \mu} : \mathbb{R}^d \to \mathbb{R}^d\) by

\[
\Upsilon_{\Phi, \mu}(x) = (|x| \wedge \Phi^{-1}(K\Delta^{-\mu}))(x), \quad \forall \ x \in \mathbb{R}^d, \tag{3.2}
\]

where \(x/|x| = 0\) if \(x = 0 \in \mathbb{R}^d\), \(K = \Phi(M_0 \vee 1)\) and \(\mu \in (0, 1/2]\). Define

\[
\begin{cases}
    z_i^\Delta = z_i^\Delta = \xi(t_i), & i = -N, \ldots, 0,
    
    z_{i+1}^\Delta = z_i^\Delta + f(z_i^\Delta, z_{i-N}^\Delta)\Delta + g(z_i^\Delta, z_{i-N}^\Delta)\Delta W_i, & i = 0, 1, \ldots,
    
    z_i^\Delta = \Upsilon_{\Phi, \mu}(z_i^\Delta), & i = 1, 2, \ldots,
\end{cases} \tag{3.3}
\]

where \(t_i = i\Delta (i \geq -N)\) and \(\Delta W_i = W(t_{i+1}) - W(t_i) (i \geq 0)\). This scheme will prevent the diffusion term from producing extra-ordinary large value, which is called the truncated Euler-Maruyama (TEM) scheme. One observes that

\[
|f(z_i^\Delta, z_{i-N}^\Delta)| \leq K\Delta^{-\mu}(1 + |z_i^\Delta|), \quad |g(z_i^\Delta, z_{i-N}^\Delta)| \leq K\Delta^{-\mu}(1 + |z_i^\Delta|). \tag{3.4}
\]

Define two continuous-time numerical schemes \(\tilde{z}_\Delta(t), z_\Delta(t)\) by

\[
\begin{align*}
    \tilde{z}_\Delta(t) := & \tilde{z}_i^\Delta, & z_\Delta(t) := & z_i^\Delta, \quad \forall t \in [t_i, t_{i+1}).
\end{align*} \tag{3.5}
\]

### 3.1 Moment boundedness

To reveal the strong convergence of TEM scheme (3.3), we give the \(q\)th moment boundedness of the corresponding numerical solutions.

**Theorem 3.1** Assume that \((H1)-(H3)\) hold. Then, TEM scheme (3.3) has the following property

\[
\sup_{0 < \Delta \leq 1} \sup_{0 \leq i \leq \lfloor T/\Delta \rfloor} \mathbb{E}|z_i^\Delta|^q \leq C, \quad \forall \ T > 0. \tag{3.6}
\]

**Proof** Define \(f_i = f(z_i^\Delta, z_{i-N}^\Delta), g_i = g(z_i^\Delta, z_{i-N}^\Delta)\) for short. For any \(T > 0\) and \(1 \leq i \leq \lfloor T/\Delta \rfloor\), one observes from (3.3) that

\[
(1 + |z_i^\Delta|^2)^{\frac{q}{2}} = \left[ 1 + |z_{i-1}^\Delta + f_{i-1}\Delta + g_{i-1}\Delta W_{i-1}|^2 \right]^{\frac{q}{2}} = \left( 1 + |z_{i-1}^\Delta|^2 \right)^{\frac{q}{2}} (1 + \Gamma_{i-1})^{\frac{q}{2}}, \tag{3.7}
\]
where

\[ \Gamma_{i-1} = (1 + |z_{i-1}^{\Delta}|^2)^{-1} \left[ |f_{i-1}|^2 \Delta^2 + |g_{i-1} \Delta W_{i-1}|^2 + 2 \langle z_{i-1}^{\Delta}, f_{i-1} \rangle \Delta \
+ 2 \langle z_{i-1}^{\Delta}, g_{i-1} \Delta W_{i-1} \rangle + 2 \langle f_{i-1}, g_{i-1} \Delta W_{i-1} \rangle \Delta \right]. \]

Then, (3.7) implies \( \Gamma_{i-1} > -1 \). For the given constant \( q \geq 2 \), choose a nonnegative integer \( k \) such that \( 2k < q \leq 2(k+1) \). It follows from [34, Lemma 3.3] and (3.7) that

\[
\begin{align*}
\mathbb{E} \left[ \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^q \left| \mathcal{F}_{t_{i-1}} \right. \right] &\leq (1 + |z_{i-1}^{\Delta}|^2)^{q/2} \left[ 1 + \frac{q}{2} \mathbb{E} \left( \Gamma_{i-1} \left| \mathcal{F}_{t_{i-1}} \right. \right) \
+ \frac{q(q-2)}{8} \mathbb{E} \left( \Gamma_{i-1}^2 \left| \mathcal{F}_{t_{i-1}} \right. \right) + \mathbb{E} \left( \Gamma_{i-1}^3 P_k(\Gamma_{i-1}) \left| \mathcal{F}_{t_{i-1}} \right. \right) \right] \\
&= (1 + |z_{i-1}^{\Delta}|^2)^{q/2} \left[ 1 + \frac{q}{2} \mathbb{E} \left( \Gamma_{i-1} \left| \mathcal{F}_{t_{i-1}} \right. \right) \
+ \frac{q(q-2)}{8} \mathbb{E} \left( \Gamma_{i-1}^2 \left| \mathcal{F}_{t_{i-1}} \right. \right) + \mathbb{E} \left( \Gamma_{i-1}^3 P_k(\Gamma_{i-1}) \left| \mathcal{F}_{t_{i-1}} \right. \right) \right].
\end{align*}
\]

(3.8)

where \( P_k(\cdot) \) represents a \( k \)-th order polynomial whose coefficients depend only on \( q \). Noticing that \( \Delta W_{i-1} \) is independent of \( \mathcal{F}_{t_{i-1}} \), one has for any \( A \in \mathbb{R}^{d \times m} \)

\[
\mathbb{E}(\langle A \Delta W_{i-1} \rangle | \mathcal{F}_{t_{i-1}}) = 0, \quad \mathbb{E}(|A \Delta W_{i-1}|^2 | \mathcal{F}_{t_{i-1}}) = |A|^2 \Delta.
\]

(3.9)

Using (3.4) and (3.9), we observe

\[
\mathbb{E} \left[ \Gamma_{i-1} \left| \mathcal{F}_{t_{i-1}} \right. \right] = \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-1} \left( 2 \langle z_{i-1}^{\Delta}, f_{i-1} \rangle \Delta + |g_{i-1}|^2 \Delta + |f_{i-1}|^2 \Delta^2 \right)
\leq \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-1} \left( 2 \langle z_{i-1}^{\Delta}, f_{i-1} \rangle \Delta + |g_{i-1}|^2 \Delta + K^2 \Delta^{-2 \mu} (1 + |z_{i-1}^{\Delta}|)^2 \Delta^2 \right)
\leq \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-1} \left( 2 \langle z_{i-1}^{\Delta}, f_{i-1} \rangle + |g_{i-1}|^2 \right) \Delta + C \Delta.
\]

(3.10)

Note

\[
\mathbb{E}(\langle A \Delta W_{i-1} \rangle^{2j-1} | \mathcal{F}_{t_{i-1}}) = 0, \quad \mathbb{E}(|A \Delta W_{i-1}|^j | \mathcal{F}_{t_{i-1}}) \leq C \Delta^{j/2}, \quad \forall \ A \in \mathbb{R}^{d \times m}, \ j \geq 2.
\]

(3.11)

From (3.4) and (3.9), we have

\[
\mathbb{E} \left[ \Gamma_{i-1}^2 \left| \mathcal{F}_{t_{i-1}} \right. \right] = \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-2} \mathbb{E} \left\{ \left[ 2 \langle z_{i-1}^{\Delta}, g_{i-1} \Delta W_{i-1} \rangle \right]^2 + 4 \langle z_{i-1}^{\Delta}, g_{i-1} \Delta W_{i-1} \rangle \right\}
\leq 4 \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-2} |z_{i-1}^{\Delta}, g_{i-1}|^2 \Delta^2 \Delta + (1 + |z_{i-1}^{\Delta}|^2)^{-2} \left[ 8 |z_{i-1}^{\Delta}| |f_{i-1}| |g_{i-1}|^2 \Delta^4 \right.
\left. + 4 |f_{i-1}|^4 \Delta^4 + 4 |g_{i-1}|^4 \Delta^2 + 16 |z_{i-1}^{\Delta}|^2 |f_{i-1}|^2 \Delta^2 + 16 |f_{i-1}|^2 |g_{i-1}|^2 \Delta^3 \right]
\leq 4 \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-2} |z_{i-1}^{\Delta}, g_{i-1}|^2 \Delta^2 + (1 + |z_{i-1}^{\Delta}|^2)^{-2} \left[ 8 K^2 \Delta^{-2 \mu} |z_{i-1}^{\Delta}| (1 + |z_{i-1}^{\Delta}|)^3 \Delta^2 \right.
\left. + 4 K^4 \Delta^{-4 \mu} (1 + |z_{i-1}^{\Delta}|)^4 \Delta^4 + 4 K^2 \Delta^{-2 \mu} (1 + |z_{i-1}^{\Delta}|)^4 \Delta^2 \right.
\left. + 16 K^2 \Delta^{-2 \mu} |z_{i-1}^{\Delta}|^2 (1 + |z_{i-1}^{\Delta}|)^2 \Delta^2 + 16 K^3 \Delta^{-3 \mu} (1 + |z_{i-1}^{\Delta}|)^4 \Delta^3 \right]
\leq 4 \left( 1 + |z_{i-1}^{\Delta}|^2 \right)^{-2} |z_{i-1}^{\Delta}, g_{i-1}|^2 \Delta + C \Delta.
\]
This together with $q (q - 2) / 8 \geq 0$ implies that

$$
\frac{q (q - 2)}{8} \mathbb{E}[\Gamma^2_{i-1} | \mathcal{F}_{i-1}] \leq \frac{q (q - 2)}{2} (1 + |z^\Delta_{i-1}|^2)^2 (z^\Delta_{i-1} \cdot g_{i-1})^2 \Delta + C \Delta. \tag{3.12}
$$

To estimate $\mathbb{E} (\Gamma^3_{i-1} P_k (\Gamma_{i-1}) | \mathcal{F}_{i-1})$, we begin with $\mathbb{E} (\Gamma^3_{i-1} | \mathcal{F}_{i-1})$. Due to (3.4) and (3.11), we obtain

$$
\mathbb{E} \left[ \Gamma^3_{i-1} | \mathcal{F}_{i-1} \right] = (1 + |z^\Delta_{i-1}|^2)^{-3} \mathbb{E} \left\{ (|f_{i-1}|^2 \Delta^2 + |g_{i-1} \Delta W_{i-1}|^2 + 2 (z^\Delta_{i-1} \cdot f_{i-1}) \Delta) + 2 (z^\Delta_{i-1} \cdot g_{i-1} \Delta W_{i-1}) + 2 (f_{i-1} \cdot g_{i-1} \Delta W_{i-1}) \Delta \right\} \mathcal{F}_{i-1}
$$

$$
= (1 + |z^\Delta_{i-1}|^2)^{-3} \mathbb{E} \left\{ (|f_{i-1}|^2 \Delta^2 + |g_{i-1} \Delta W_{i-1}|^2 + 2 (z^\Delta_{i-1} \cdot f_{i-1}) \Delta)^3 \right\} \mathcal{F}_{i-1}
$$

$$
= 3 (|f_{i-1}|^2 \Delta^2 + |g_{i-1} \Delta W_{i-1}|^2 + 2 (z^\Delta_{i-1} \cdot f_{i-1}) \Delta) \times (2 (z^\Delta_{i-1} \cdot g_{i-1} \Delta W_{i-1}) + 2 (f_{i-1} \cdot g_{i-1} \Delta W_{i-1}) \Delta)^2 \mathcal{F}_{i-1}
$$

$$
\geq (1 + |z^\Delta_{i-1}|^2)^{-3} \mathbb{E} \left\{ -8 |z^\Delta_{i-1}|^3 |f_{i-1}|^3 \Delta^3 - 6 (|f_{i-1}|^2 \Delta^2 + |g_{i-1} \Delta W_{i-1}|^2)^2 |z^\Delta_{i-1}| |f_{i-1}| \Delta - 6 |z^\Delta_{i-1}| |f_{i-1}| \Delta \times (2 |z^\Delta_{i-1} | g_{i-1} \Delta W_{i-1}) + 2 (f_{i-1} \cdot g_{i-1} \Delta W_{i-1}) \Delta^2 \right\} \mathcal{F}_{i-1}
$$

$$
\geq -C (1 + |z^\Delta_{i-1}|^2)^{-3} \left[ K^3 \Delta^{-3 \mu} |z^\Delta_{i-1}|^3 (1 + |z^\Delta_{i-1}|)^3 \Delta^3 + K^5 \Delta^{-5 \mu} |z^\Delta_{i-1}| (1 + |z^\Delta_{i-1}|)^5 \Delta^5 \right.
$$

$$
+ K^3 \Delta^{-3 \mu} |z^\Delta_{i-1}| (1 + |z^\Delta_{i-1}|)^5 \Delta^3 + K^2 \Delta^{-2 \mu} |z^\Delta_{i-1}|^3 (1 + |z^\Delta_{i-1}|)^3 \Delta^2
$$

$$
+ K^4 \Delta^{-4 \mu} |z^\Delta_{i-1}| (1 + |z^\Delta_{i-1}|)^5 \Delta^4 \right]
$$

$$
\geq -C (\Delta^{3-3 \mu} + \Delta^{5-5 \mu} + \Delta^{3-3 \mu} + \Delta^{2-2 \mu} + \Delta^{4-4 \mu}) \geq -C \Delta.
$$

On the other hand,

$$
\mathbb{E} \left[ \Gamma^3_{i-1} | \mathcal{F}_{i-1} \right] \leq (1 + |z^\Delta_{i-1}|^2)^{-3} \left[ 9 (|f_{i-1}|^4 \Delta^6 + |g_{i-1}|^6 \Delta^3 + 8 |z^\Delta_{i-1}|^3 |f_{i-1}|^3 \Delta^3 \right.
$$

$$
+ 24 (|z^\Delta_{i-1}|^4 |f_{i-1}|^2 |g_{i-1}|^2 \Delta^3 + |f_{i-1}|^4 |g_{i-1}|^2 \Delta^5 + |z^\Delta_{i-1}|^2 |g_{i-1}|^4 \Delta^2
$$

$$
+ |f_{i-1}|^2 |g_{i-1}|^4 \Delta^4 + 2 |z^\Delta_{i-1}|^3 |f_{i-1}| |g_{i-1}|^2 \Delta^2 + 2 |z^\Delta_{i-1}| |f_{i-1}|^3 |g_{i-1}|^2 \Delta^4 \right]
$$

$$
\leq C (1 + |z^\Delta_{i-1}|^2)^{-3} \left[ K^6 \Delta^{-6 \mu} (1 + |z^\Delta_{i-1}|)^6 \Delta^6 + K^3 \Delta^{-3 \mu} (1 + |z^\Delta_{i-1}|)^6 \Delta^3
$$
\[ + K^3 \Delta^{-3\mu} |z_{i-1}^\Delta|^3(1 + |z_{i-1}^\Delta|)^3 \Delta^3 + K^3 \Delta^{-3\mu} |z_{i-1}^\Delta|^2(1 + |z_{i-1}^\Delta|)^4 \Delta^3 \]
\[ + K^5 \Delta^{-5\mu} (1 + |z_{i-1}^\Delta|)^6 \Delta^5 + K^2 \Delta^{-2\mu} |z_{i-1}^\Delta|^2(1 + |z_{i-1}^\Delta|)^4 \Delta^2 \]
\[ + K^4 \Delta^{-4\mu} (1 + |z_{i-1}^\Delta|)^6 \Delta^4 + K^2 \Delta^{-2\mu} |z_{i-1}^\Delta|^3(1 + |z_{i-1}^\Delta|)^3 \Delta^2 \]
\[ + K^4 \Delta^{-4\mu} |z_{i-1}^\Delta|(1 + |z_{i-1}^\Delta|)^5 \Delta^4 \leq C \Delta. \]

Thus, both of the above inequalities imply \( \mathbb{E} \left[ a_0 \Gamma_{i-1}^j |F_{t_{i-1}} \right] \leq C \Delta \) for any constant \( a_0 \), where \( a_j \) represents the coefficient of \( x^j \) term in polynomial \( P_k(x) \). We can also show that for any \( j > 3 \)
\[ \mathbb{E} \left[ a_{j-3} \Gamma_{i-1}^j |F_{t_{i-1}} \right] \leq C (1 + |z_{i-1}^\Delta|^2)^{-j} (|f_{i-1}|^{2j} \Delta^{2j} + |g_{i-1}|^{2j} \Delta^j + |z_{i-1}^\Delta|^j |f_{i-1}|^j \Delta^j \]
\[ + |z_{i-1}^\Delta|^j |g_{i-1}|^j \Delta^{\frac{j}{2}} + |f_{i-1}|^j |g_{i-1}|^j \Delta^{\frac{3j}{2}} ) \leq C \Delta. \]

These inequalities imply
\[ \mathbb{E} \left( \Gamma_{i-1}^3 P_k(\Gamma_{i-1}) |F_{t_{i-1}} \right) \leq C \Delta. \] (3.13)

Subsituting (3.10), (3.12), (3.13) into (3.8) and using (H2), we obtain
\[ \mathbb{E} \left[ (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} |F_{t_{i-1}} \right] \]
\[ \leq \left( 1 + |z_{i-1}^\Delta|^2 \right)^{\frac{q}{2}} \left\{ 1 + C \Delta \right. \]
\[ + \frac{q \Delta}{2} (1 + |z_{i-1}^\Delta|^2)(2|z_{i-1}^\Delta| f_{i-1} + |g_{i-1}|^2) + (q - 2)|z_{i-1}^\Delta| |g_{i-1}|^2 \right\} \]
\[ \leq (1 + C \Delta) \left( 1 + |z_{i-1}^\Delta|^2 \right)^{\frac{q}{2}} + \frac{q \Delta}{2} (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} (2|z_{i-1}^\Delta| f_{i-1} + (q - 1)|g_{i-1}|^2) \]
\[ \leq (1 + C \Delta) \left( 1 + |z_{i-1}^\Delta|^2 \right)^{\frac{q}{2}} + \frac{q \Delta}{2} K_1 (1 + |z_{i-1}^\Delta|^2 + |z_{i-1-N}^\Delta|^q) \]
\[ - K_2 (V_1(z_{i-1}^\Delta) - V_1(z_{i-1-N}^\Delta)) \]
\[ \leq (1 + C \Delta) \left( 1 + |z_{i-1}^\Delta|^2 \right)^{\frac{q}{2}} + \frac{q \Delta}{2} K_1 (1 + |z_{i-1-N}^\Delta|^2) \]
\[ - \frac{q}{2} K_2 \Delta V_1(z_{i-1}^\Delta) + \frac{q}{2} K_2 \Delta V_1(z_{i-1-N}^\Delta). \]

We observe from the above inequality that
\[ \mathbb{E} (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} - \mathbb{E} (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} \leq \mathbb{E} \left[ \mathbb{E} \left[ (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} |F_{t_{i-1}} \right] \right] - \mathbb{E} (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} \]
\[ \leq C \Delta \mathbb{E} (1 + |z_{i-1}^\Delta|^2)^{\frac{q}{2}} + \frac{q \Delta}{2} K_1 \Delta \mathbb{E} (1 + |z_{i-1-N}^\Delta|^2)^{\frac{q}{2}} \]
\[ - \frac{q}{2} K_2 \Delta V_1(z_{i-1}^\Delta) + \frac{q}{2} K_2 \Delta V_1(z_{i-1-N}^\Delta). \]

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Taking sum on both sides of the above inequality implies
\[
\mathbb{E} \left[ \left( 1 + |z_i^\Delta|^2 \right)^{\frac{q}{2}} \right] \\
\leq \mathbb{E} \left( 1 + |\xi(0)|^2 \right)^{\frac{q}{2}} + C \Delta \sum_{k=0}^{i-1} \mathbb{E} \left( 1 + |z_k^\Delta|^2 \right)^{\frac{q}{2}} + \frac{q}{2} K_1 \Delta \sum_{k=0}^{i-1} \mathbb{E} \left( 1 + |z_{k-N}^\Delta|^2 \right)^{\frac{q}{2}} \\
- \frac{q}{2} K_2 \Delta \sum_{k=0}^{i-1} \mathbb{E} V_1(z_k^\Delta) + \frac{q}{2} K_2 \Delta \sum_{k=0}^{i-1} \mathbb{E} V_1(z_{k-N}^\Delta) \\
\leq \mathbb{E} \left( 1 + |\xi(0)|^2 \right)^{\frac{q}{2}} + C \Delta \sum_{k=0}^{i-1} \mathbb{E} \left( 1 + |z_k^\Delta|^2 \right)^{\frac{q}{2}} + \frac{Nq}{2} K_1 \Delta \sup_{-N \leq k \leq -1} \mathbb{E} \left( 1 + |\xi(t_k)|^2 \right)^{\frac{q}{2}} \\
+ \frac{q}{2} K_1 \Delta \sum_{k=0}^{(i-1-N) \vee 0} \mathbb{E} \left( 1 + |z_k^\Delta|^2 \right)^{\frac{q}{2}} - \frac{q}{2} K_2 \Delta \sum_{k=0}^{i-1} \mathbb{E} V_1(z_k^\Delta) \\
+ \frac{Nq}{2} K_2 \Delta \sup_{-N \leq k \leq -1} \mathbb{E} V_1(\xi(t_k)) + \frac{q}{2} K_2 \Delta \sum_{k=0}^{(i-1-N) \vee 0} \mathbb{E} V_1(z_k^\Delta) \\
\leq C \Delta \sum_{k=0}^{i-1} \mathbb{E} \left( 1 + |z_k^\Delta|^2 \right)^{\frac{q}{2}} + C.
\]
Applying the discrete Gronwall inequality and the fact \( i \Delta \leq T \) yields
\[
\mathbb{E} \left[ \left( 1 + |z_i^\Delta|^2 \right)^{\frac{q}{2}} \right] \leq C e^{Ci \Delta} \leq C e^{CT},
\]
which implies the required inequality \( (3.6) \).

\[\square\]

### 3.2 The strong convergence

This subsection is concerned with the strong convergence of TEM scheme. We go a first step to estimate the probability that the numerical solutions escape from the corresponding compact set in a finite time interval.

**Lemma 3.2** Assume that \((H1)-(H3)\) hold. For any \( \Delta, \Delta_1 \in (0, 1] \), let
\[
\check{\epsilon}_1^\Delta := \inf \{ t \geq 0 : |\check{z}_1(t)| \geq \Phi^{-1}(K \Delta_1^{-\mu}) \}. \quad (3.14)
\]
Then, for any \( T > 0 \) and \( \Delta \in (0, \Delta_1] \subseteq (0, 1) \),
\[
\mathbb{P}\{ \check{\epsilon}_1^\Delta \leq T \} \leq \frac{C}{(\Phi^{-1}(K \Delta_1^{-\mu}))^q}. \quad (3.15)
\]

**Proof** Let \( \xi^\Delta_1 = \inf \{ i \geq 0 : |\hat{z}_i^\Delta| \geq \Phi^{-1}(K \Delta_1^{-\mu}) \} \), \( \check{f}_i = f(\hat{z}_1^\Delta, \hat{z}_{i-N}^\Delta) \), \( \check{g}_i = g(\hat{z}_1^\Delta, \hat{z}_{i-N}^\Delta) \), and
\[
\check{f}_{i \wedge \Delta_1} = f(\hat{z}_{i \wedge \Delta_1}^\Delta, \hat{z}_{(i-N) \wedge \Delta_1}^\Delta), \quad \check{g}_{i \wedge \Delta_1} = g(\hat{z}_{i \wedge \Delta_1}^\Delta, \hat{z}_{(i-N) \wedge \Delta_1}^\Delta).
\]
For any $i \geq 1$, if $\omega \in \{\xi_{\Delta_1}^\Delta \geq i\}$, it is obvious that $\tilde{z}_{i-1} = \tilde{z}_{i-1} = \tilde{z}_{i-1-N} = \tilde{z}_{i-1-N}$ and

$$\tilde{z}_{i \land \xi_{\Delta_1}^\Delta} = \tilde{z}_{i} = \tilde{z}_{i-1} + \tilde{f}_{i-1} \Delta + \tilde{g}_{i-1} \Delta W_{i-1}.$$  

Otherwise, $\omega \in \{\xi_{\Delta_1}^\Delta < i\}$, then $\tilde{z}_{i \land \xi_{\Delta_1}^\Delta} = \tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta} = \tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta}$. Combining both cases we have

$$\tilde{z}_{i \land \xi_{\Delta_1}^\Delta} = \tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta} + \left[ \tilde{f}_{(i-1) \land \xi_{\Delta_1}^\Delta} \Delta + \tilde{g}_{(i-1) \land \xi_{\Delta_1}^\Delta} \Delta W_{i-1} \right] 1_{[0, \xi_{\Delta_1}^\Delta]}(i).$$

Then,

$$\left( 1 + |\tilde{z}_{i \land \xi_{\Delta_1}^\Delta}|^2 \right)^{\frac{q}{2}} = \left( 1 + |\tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta}|^2 \right)^{\frac{q}{2}} \left( 1 + \tilde{\Gamma}_{(i-1) \land \xi_{\Delta_1}^\Delta} 1_{[0, \xi_{\Delta_1}^\Delta]}(i) \right)^{\frac{q}{2}},$$

where

$$\tilde{\Gamma}_{(i-1) \land \xi_{\Delta_1}^\Delta} = \left( 1 + |\tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta}|^2 \right)^{-1} \left[ |\tilde{f}_{(i-1) \land \xi_{\Delta_1}^\Delta}|^2 \Delta^2 + |\tilde{g}_{(i-1) \land \xi_{\Delta_1}^\Delta}|^2 \Delta W_{i-1} \right]^2 + 2 \langle \tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta} , \tilde{f}_{(i-1) \land \xi_{\Delta_1}^\Delta} \rangle \Delta + 2 \langle \tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta} , \tilde{g}_{(i-1) \land \xi_{\Delta_1}^\Delta} \rangle \Delta W_{i-1} \Delta.$$  

By the same way as (3.8), we obtain that for any $1 \leq i \leq \lfloor T/\Delta \rfloor$

$$\mathbb{E} \left[ \left( 1 + |\tilde{z}_{i \land \xi_{\Delta_1}^\Delta}|^2 \right)^{\frac{q}{2}} | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right]$$

$$\leq \left( 1 + |\tilde{z}_{(i-1) \land \xi_{\Delta_1}^\Delta}|^2 \right)^{\frac{q}{2}} \left[ 1 + \frac{q}{2} \mathbb{E}(\tilde{\Gamma}_{(i-1) \land \xi_{\Delta_1}^\Delta} 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}}) \right]$$

$$+ \frac{q(q-2)}{8} \mathbb{E}(\tilde{\Gamma}_{(i-1) \land \xi_{\Delta_1}^\Delta} 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}})$$

$$+ \mathbb{E}(\tilde{\Gamma}^3_{(i-1) \land \xi_{\Delta_1}^\Delta} P_k(\tilde{\Gamma}_{(i-1) \land \xi_{\Delta_1}^\Delta} 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}}).$$  

By virtue of the martingale property of $W(t)$ and the Doob martingale stopping time theorem [23, p.11, Theorem 3.3], we have

$$\mathbb{E} \left( (A\Delta W_{i-1}) 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right) = 0,$$

$$\mathbb{E} \left( |A\Delta W_{i-1}|^2 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right) = |A|^2 \Delta \mathbb{E} \left( 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right),$$

(3.17)

and

$$\mathbb{E} \left( (A\Delta W_{i-1})^{2j-1} 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right) = 0,$$

$$\mathbb{E} \left( |A\Delta W_{i-1}|^j 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right) \leq C_j \Delta^j \mathbb{E} \left( 1_{[0, \xi_{\Delta_1}^\Delta]}(i) | \mathcal{F}_{t_{(i-1) \land \xi_{\Delta_1}^\Delta}} \right),$$

(3.18)
where \( A \in \mathbb{R}^{d \times m}, \ j \geq 2 \). Using (3.16)-(3.18) and (H2), by the similar argument as Theorem 3.1, we yield
\[
\mathbb{E} \left[(1 + |\tilde{z}_{i \wedge \zeta_{\Delta}^t}|^2)^{\frac{q}{2}} \right]_{\mathcal{F}_{(i-1) \wedge \zeta_{\Delta}^t}} \\
\leq (1 + C \Delta) \left(1 + |\tilde{z}_{(i-1) \wedge \zeta_{\Delta}^t}|^2\right)^{\frac{q}{2}} + \frac{q}{2} K_1 \Delta \left(1 + |\tilde{z}_{(i-1-N) \wedge \zeta_{\Delta}^t}|^2\right)^{\frac{q}{2}} \\
- \frac{q}{2} K_2 \Delta V_1(\tilde{z}_{(i-1-N) \wedge \zeta_{\Delta}^t}) \mathbb{E} \left[I_{[0, \zeta_{\Delta}^t]}(i) |\mathcal{F}_{(i-1) \wedge \zeta_{\Delta}^t}\right] \\
+ \frac{q}{2} K_2 \Delta V_1(\tilde{z}_{(i-1-N) \wedge \zeta_{\Delta}^t}) \mathbb{E} \left(I_{[0, \zeta_{\Delta}^t]}(i) |\mathcal{F}_{(i-1) \wedge \zeta_{\Delta}^t}\right),
\]
which implies
\[
\mathbb{E} \left[(1 + |\tilde{z}_{i \wedge \zeta_{\Delta}^t}|^2)^{\frac{q}{2}} \right] \\
\leq \mathbb{E}(1 + |\xi(0)|^2)^{\frac{q}{2}} + \frac{Nq}{2} K_2 \Delta \mathbb{E} \left(I_{[0, \zeta_{\Delta}^t]}(i) |\mathcal{F}_{(i-1) \wedge \zeta_{\Delta}^t}\right) \\
- \frac{q}{2} K_2 \Delta \sum_{k=0}^{i-1} \mathbb{E} \left[I_{[0, \zeta_{\Delta}^t]}(k) |\mathcal{F}_{k \wedge \zeta_{\Delta}^t}\right] \\
+ \frac{q}{2} K_2 \Delta \sum_{k=0}^{i-1} \mathbb{E} \left[I_{[0, \zeta_{\Delta}^t]}(k+1) |\mathcal{F}_{k \wedge \zeta_{\Delta}^t}\right].
\]
Due to the fact \( I_{[0, \zeta_{\Delta}^t]}(k+1+N) \leq I_{[0, \zeta_{\Delta}^t]}(k+1) \), one observes
\[
\mathbb{E} \left[(1 + |\tilde{z}_{i \wedge \zeta_{\Delta}^t}|^2)^{\frac{q}{2}} \right] \leq C \Delta \sum_{k=0}^{i-1} \mathbb{E} \left(I_{[0, \zeta_{\Delta}^t]}(k) |\mathcal{F}_{k \wedge \zeta_{\Delta}^t}\right) + C.
\]
It follows from the discrete Gronwall inequality and \( i \Delta \leq T \) that
\[
\mathbb{E} \left[(1 + |\tilde{z}_{i \wedge \zeta_{\Delta}^t}|^2)^{\frac{q}{2}} \right] \leq C e^{Ci \Delta} \leq C e^{CT}.
\]
Therefore,
\[
(\Phi^{-1}(K_{\Delta_1}^{-\mu}))^q \mathbb{P}[\epsilon_{\Delta_1}^\Delta \leq T] \leq \mathbb{E}[|\tilde{z}(T \wedge \epsilon_{\Delta_1}^\Delta)|^q] \leq \mathbb{E} \left[(1 + |\tilde{z}_{(T \wedge \epsilon_{\Delta_1}^\Delta)|}^2)^{\frac{q}{2}} \right] \leq C,
\]
which implies that the assertion (3.15) holds.

Now, we establish the strong convergence of TEM scheme (3.5) in the \( p \)th moment for \( p \in (0, q) \).

\( \square \) Springer
Theorem 3.3 Assume that (H1)-(H3) hold. Then, for any \( p \in (0, q) \),
\[
\lim_{\Delta \to 0^+} \mathbb{E}|x(T) - z_\Delta(T)|^p = 0, \quad \forall \ T > 0.
\] (3.21)

Proof For any constants \( M > 0, \Delta_1 \in (0, 1) \) and \( \Delta \in (0, \Delta_1] \), define \( \theta_M^{\Delta, \Delta_1} = \vartheta_M \land \varphi_{\Delta_1}^{\Delta} \), where \( \vartheta_M \) and \( \varphi_{\Delta_1}^{\Delta} \) are defined by (2.4) and (3.14), respectively. For any \( \kappa_1 > 0 \), by Young’s inequality
\[
\mathbb{E}|x(T) - z_\Delta(T)|^p = \mathbb{E}\left(|x(T) - z_\Delta(T)|^p 1_{\{\theta_M^{\Delta, \Delta_1} > T\}}\right) \\
+ \mathbb{E}\left(|x(T) - z_\Delta(T)|^p 1_{\{\theta_M^{\Delta, \Delta_1} \leq T\}}\right) \\
\leq \mathbb{E}\left(|x(T) - z_\Delta(T)|^p 1_{\{\theta_M^{\Delta, \Delta_1} > T\}}\right) \\
+ \frac{p\kappa_1}{q} \mathbb{E}\left(|x(T) - z_\Delta(T)|^q\right) \\
+ \frac{q - p}{q\kappa_1^p/(q-p)} \mathbb{P}\{\theta_M^{\Delta, \Delta_1} \leq T\}. \quad (3.22)
\]
It follows from Theorem 2.1 and Theorem 3.1 that
\[
\mathbb{E}\left(|x(T) - z_\Delta(T)|^q\right) \leq 2^q (\mathbb{E}|x(T)|^q + \mathbb{E}|z_\Delta(T)|^q) \leq C.
\]
For any \( \epsilon_1 > 0 \), choose \( \kappa_1(\epsilon_1) > 0 \) small sufficiently such that \( Cpk_1/q \leq \epsilon_1/3 \). Then,
\[
\frac{p\kappa_1}{q} \mathbb{E}\left(|x(T) - z_\Delta(T)|^q\right) \leq \frac{\epsilon_1}{3}. \quad (3.23)
\]
From (2.5) and (3.15) we obtain that
\[
\frac{q - p}{q\kappa_1^p/(q-p)} \mathbb{P}\{\theta_M^{\Delta, \Delta_1} \leq T\} \leq \frac{q - p}{q\kappa_1^p/(q-p)} \left( \mathbb{P}\{\vartheta_M \leq T\} + \mathbb{P}\{\varphi_{\Delta_1}^{\Delta} \leq T\} \right) \\
\leq \frac{C(q-p)}{q\kappa_1^p/(q-p)} \left( \frac{1}{M^q} + \frac{1}{(\Phi^{-1}(K_{\Delta_1}^{-\mu}))^q} \right). \quad (3.24)
\]
Choose a constant \( \Delta_1 \in (0, 1) \) such that
\[
\frac{C(q-p)}{q\kappa_1^p/(q-p)} (\Phi^{-1}(K_{\Delta_1}^{-\mu}))^q \leq \epsilon_1/6,
\]
and then choose a constant \( M \geq M_0 \lor \Phi^{-1}(K_{\Delta_1}^{-\mu}) \). Thus, it follows from (3.24) that
\[
\frac{q - p}{q\kappa_1^p/(q-p)} \mathbb{P}\{\theta_M^{\Delta, \Delta_1} \leq T\} \leq \frac{\epsilon_1}{3}.
\]
Therefore, it is sufficient for (3.21) to show
\[
\lim_{\Delta \to 0^+} \mathbb{E}\left(|x(T) - z_\Delta(T)|^p 1_{\{\theta_M^{\Delta, \Delta_1} > T\}}\right) = 0.
\]
For this purpose, we define
\[
  f_M(x, y) = f \left( \frac{|x| \wedge M}{|x|}, \frac{|y| \wedge M}{|y|} \right),
\]
\[
  g_M(x, y) = g \left( \frac{|x| \wedge M}{|x|}, \frac{|y| \wedge M}{|y|} \right).
\]

Then, (H1) implies that for any \( x, \bar{x}, y \in R^d \),
\[
|f_M(x, y) - f_M(\bar{x}, y)| \vee |g_M(x, y) - g_M(\bar{x}, y)| \leq L_M |x - \bar{x}|.
\] (3.25)

(3.25) together with (H3) derives
\[
|f_M(x, y)| \vee |g_M(x, y)| \leq (L_M \vee \sup_{|y| \leq M} |f(0, y)| \vee \sup_{|y| \leq M} |g(0, y)|)(1 + |x|).
\] (3.26)

Now, consider the auxiliary SDDE
\[
  du(t) = f_M(u(t), u(t - \tau))dt + g_M(u(t), u(t - \tau))dW(t),
\] (3.27)
with the initial data \( \xi \in C_b([-\tau, 0]; R^d) \). By virtue of [9, Theorem 2.1] SDDE (3.27) has a unique global solution \( u(t) \) on \( t \geq -\tau \). Let \( Y_\Delta(t) \) be the piecewise EM solution of (3.27). Then, (H3), (3.25), (3.26) and [19, Theorem 1] imply
\[
\lim_{\Delta \to 0^+} E \sup_{0 \leq t \leq T} |u(t) - Y_\Delta(t)|^p = 0, \ \forall \ T > 0, \ \bar{p} > 0.
\] (3.28)

One notices
\[
x(t \wedge \theta_M) = u(t \wedge \theta_M), \ \forall \ t \geq 0 \ a.s.
\] (3.29)

For any \( \Delta \in (0, \Delta_1] \), the fact that \( \Phi^{-1}(K\Delta^{-\mu}) \geq \Phi^{-1}(K\Delta_1^{-\mu}) \) and \( M \geq \Phi^{-1}(K\Delta_1^{-\mu}) \) implies
\[
z_\Delta(t \wedge \theta_M^{\Delta, \Delta_1}) = \tilde{z}_\Delta(t \wedge \theta_M^{\Delta, \Delta_1}) = Y_\Delta(t \wedge \theta_M^{\Delta, \Delta_1}), \ \forall \ t \geq 0 \ a.s.
\] (3.30)

Combining (3.28)-(3.30) derives
\[
\lim_{\Delta \to 0^+} E \left( x(T) - z_\Delta(T) \right)^p 1_{\theta_M^{\Delta, \Delta_1} > T} 
\leq \lim_{\Delta \to 0^+} E \left( |x(T \wedge \theta_M^{\Delta, \Delta_1}) - z_\Delta(T \wedge \theta_M^{\Delta, \Delta_1})|^p \right) 
= \lim_{\Delta \to 0^+} E \left( |u(T \wedge \theta_M^{\Delta, \Delta_1}) - Y_\Delta(T \wedge \theta_M^{\Delta, \Delta_1})|^p \right) 
\leq \lim_{\Delta \to 0^+} E \left( \sup_{0 \leq t \leq T} |u(t \wedge \theta_M^{\Delta, \Delta_1}) - Y_\Delta(t \wedge \theta_M^{\Delta, \Delta_1})|^p \right) 
\leq \lim_{\Delta \to 0^+} E \left( \sup_{0 \leq t \leq T} |u(t) - Y_\Delta(t)|^p \right) = 0.
\]

Hence, the proof is completed. \( \square \)
3.3 Convergence rate

This subsection focuses on estimating the convergence rate of TEM scheme (3.5). To obtain the convergence rate, we need somewhat stronger conditions compared with the convergence alone, which are stated as follows.

(H4) Assume that the initial data $\xi$ satisfies the Hölder continuous with the index $\lambda \geq 1/2$, i.e., for any $s_1, s_2 \in [-\tau, 0]$, there exist positive constants $r_1 \geq 4$ and $K_3$ such that
\[
\left( \mathbb{E} |\xi(s_1) - \xi(s_2)|^\frac{1}{r_1} \right)^{\frac{1}{r_1}} \leq K_3 |s_1 - s_2|^\lambda.
\] (3.31)

(H5) Assume that there are a pair of positive constants $\alpha$, $K_4$ such that for any $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^d$,
\[
|f(x, y) - f(\tilde{x}, \tilde{y})| \leq K_4(|x - \tilde{x}| + |y - \tilde{y}|)(1 + |x|^\alpha + |\tilde{x}|^\alpha + |y|^\alpha + |\tilde{y}|^\alpha),
\] (3.32)
\[
|g(x, y) - g(\tilde{x}, \tilde{y})|^2 \leq K_4(|x - \tilde{x}|^2 + |y - \tilde{y}|^2)(1 + |x|^\alpha + |\tilde{x}|^\alpha + |y|^\alpha + |\tilde{y}|^\alpha).
\] (3.33)

(H6) Assume that there exist positive constants $K_5, \gamma, \beta$ satisfying $\gamma \geq 2, \beta > r - 1$ and a function $\tilde{V}(\cdot, \cdot) \in \mathcal{V}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}_+)$ such that
\[
|x - \tilde{x}|^\gamma \left[ 2( f(x, y) - f(\tilde{x}, \tilde{y})) + \beta |g(x, y) - g(\tilde{x}, \tilde{y})|^2 \right]
\] \[
\leq K_5(|x - \tilde{x}|^r + |y - \tilde{y}|^r) - \tilde{V}(x, \tilde{x}) + \tilde{V}(y, \tilde{y}), \quad \forall x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^d.
\] One notices from (3.33) that
\[
|g(x, y)| \leq |g(x, y) - g(0, 0)| + |g(0, 0)|
\] \[
\leq \sqrt{K_4(|x| + |y|)(1 + |x|^\frac{\alpha}{2} + |y|^\frac{\alpha}{2})} + |g(0, 0)|
\] \[
\leq C(1 + |x|^\frac{\alpha}{2} + |y|^\frac{\alpha}{2}).
\] (3.34)

Remark 1 Due to (H5), we may take
\[
\Phi(l) = [ |f(0, 0)| + 3|\alpha+1|K_4 ] \vee 2|g(0, 0)|^2 + 3\alpha+2K_4
\] \[
\leq |f(0, 0)| \vee 2|g(0, 0)|^2 + 6\alpha+2K_4
\] in (3.1), where $l \geq 1$. Then,
\[
\Phi^{-1}(l) = \left( \frac{l - |f(0, 0)| \vee 2|g(0, 0)|^2}{6K_4} \right)^{\frac{1}{\alpha+2}}
\] (3.35)
for $l \geq |f(0, 0)| \vee 2|g(0, 0)|^2 + 6K_4$. If $r \leq \frac{\alpha+2}{2} \land \left( \frac{q}{(\alpha+3) \vee (2\alpha)} \right)$, then we let
\[
\mu = \frac{r(\alpha + 2)}{2(q - r)} \in (0, \frac{1}{2}], \quad K = |f(0, 0)| \vee 2|g(0, 0)|^2 + 6(M_0 \vee 1)^\alpha+2K_4.
\] (3.36)

In order for the convergence rate of the TEM scheme, we prepare an auxiliary process $\tilde{z}_\Delta(t)$ described by
\[
\begin{cases}
\tilde{z}_\Delta(t) = z_\Delta^\Delta + f(z_\Delta^\Delta, z_\Delta^\Delta)(t - t_i) + g(z_\Delta^\Delta, z_\Delta^\Delta)(W(t) - W(t_i)), \quad \forall t \in [t_i, t_{i+1}),
\tilde{z}_\Delta(t) = \xi(t), \quad \forall t \in [-\tau, 0].
\end{cases}
\] (3.37)

Obviously, $\tilde{z}_\Delta(t_i) = z_\Delta(t_i) = z_i^\Delta$ for $i \geq -N$. 

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Lemma 3.4 Assume that \((H_2)\) and \((H_5)\) hold. Then, for any \(\tilde{r} \in (0, 2q/(\alpha + 2))\),
\[
\sup_{0 \leq t \leq T} \mathbb{E}\left( |\tilde{z}_\Delta(t) - z_\Delta(t)|^{\tilde{r}} \right) \leq C \Delta^{\tilde{r}}, \quad \forall \ T > 0. \tag{3.38}
\]

Proof Fix \(\tilde{r} \in (0, 2q/(\alpha + 2))\) and \(T > 0\). Recalling (3.37), we have that for any \(t \in [t_i, t_{i+1})\)
\[
\mathbb{E}\left( |\tilde{z}_\Delta(t) - z_\Delta(t)|^{\tilde{r}} \right) = \mathbb{E}\left( |\tilde{z}_\Delta(t) - z_\Delta(t_i)|^{\tilde{r}} \right)
\leq 2\tilde{r} \mathbb{E}\left( |f(z_{\Delta i}, z_{\Delta i-N})|^{\tilde{r}} \Delta^{\tilde{r}} + 2\tilde{r} \mathbb{E}\left( |g(z_{\Delta i}, z_{\Delta i-N})|^{\tilde{r}} W(t) - W(t_i)|^{\tilde{r}} \right) \right)
\leq C \left( \mathbb{E}\left( |f(z_{\Delta i}, z_{\Delta i-N})|^{\tilde{r}} \Delta^{\tilde{r}} \right) + \mathbb{E}\left( |g(z_{\Delta i}, z_{\Delta i-N})|^{\tilde{r}} \Delta^{\tilde{r}} \right) \right).
\]
It follows from (3.4), (3.34), (3.36), and Theorem 3.1 that
\[
\mathbb{E}\left( |\tilde{z}_\Delta(t) - z_\Delta(t)|^{\tilde{r}} \right) \leq C \mathbb{E}\left( 1 + |z_{\Delta i}|^{q} \right)^{\tilde{r}} \Delta^{\tilde{r}} + C \left( 1 + (\mathbb{E}|z_{\Delta i}|^{\tilde{r}})^{q} \right)^{\frac{\alpha + 2q}{2q}} (\mathbb{E}|z_{\Delta i-N}|^{\tilde{r}})^{\frac{\alpha + 2q}{2q}} \Delta^{\tilde{r}}
\leq C \Delta^{\tilde{r}}.
\]
The required assertion follows. \(\square\)

By the similar assertion as Theorem 3.1 and Lemma 3.1, one observes that the auxiliary process enjoys the following properties.

Lemma 3.5 Assume that \((H_1)-(H_3)\) hold. Then,
\[
\sup_{0 \leq \Delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}\left(|\tilde{z}_\Delta(t)|^{q}\right) \leq C, \quad \forall \ T > 0. \tag{3.39}
\]

Lemma 3.6 Assume that \((H_1)-(H_3)\) hold. For any \(\Delta \in (0, 1]\), let
\[
\tilde{\varrho}_\Delta := \inf\{t \geq 0 : |\tilde{z}_\Delta(t)| \geq \Phi^{-1}(K \Delta^{-\mu})\}. \tag{3.40}
\]
Then, for any \(T > 0\),
\[
\mathbb{P}\{\tilde{\varrho}_\Delta \leq T\} \leq \frac{C}{\left(\Phi^{-1}(K \Delta^{-\mu})\right)^{q}}. \tag{3.41}
\]

We go a further step to estimate the error between the auxiliary process \(\tilde{z}_\Delta(t)\) and the exact solution \(x(t)\). Define \(e(t) = x(t) - \tilde{z}_\Delta(t)\) for short, which satisfies
\[
e(t \wedge \tilde{\varrho}_\Delta) = \int_0^{t \wedge \tilde{\varrho}_\Delta} \left[ f(x(s), x(s - \tau)) - f(z_\Delta(s), z_\Delta(s - \tau)) \right] ds
+ \int_0^{t \wedge \tilde{\varrho}_\Delta} \left[ g(x(s), x(s - \tau)) - g(z_\Delta(s), z_\Delta(s - \tau)) \right] dW(s).
\]

Lemma 3.7 Assume that \((H_2), (H_4)-(H_6)\) and \(r \leq \frac{q}{2} \wedge \left( \frac{q}{(\alpha + 3)\sqrt{2\alpha}} \right)\) hold. Then,
\[
\mathbb{E}\left| e(T) \right|^{r} \leq C \Delta^{\frac{r}{2}}, \quad \forall \ T \geq 0. \tag{3.42}
\]
Proof Define \( \chi_\Delta = \vartheta_{\Phi^{-1}(K^{\Delta - \nu})} \wedge \tilde{\vartheta}_\Delta \), where \( \vartheta_M, \vartheta^\Delta \) and \( \tilde{\vartheta}_\Delta \) are defined in (2.4), (3.14) and (3.40), respectively. By Young’s inequality

\[
\mathbb{E}|e(T)|^r = \mathbb{E}\left(|e(T)|^r 1_{\{\chi_\Delta > T\}}\right) + \mathbb{E}\left(|e(T)|^r 1_{\{\chi_\Delta \leq T\}}\right)
\leq \mathbb{E}\left(|e(T)|^r 1_{\{\chi_\Delta > T\}}\right) + \frac{r \Delta^\frac{1}{q}}{q} \mathbb{E}|e(T)|^q + \frac{q - r}{q} \frac{\mathbb{P}(\chi_\Delta \leq T)}{r^2}.
\]

It follows from Theorem 2.1 and Lemma 3.3 that

\[
\frac{r \Delta^\frac{1}{q}}{q} \mathbb{E}|e(T)|^q \leq 2^{q-1} \frac{r \Delta^\frac{1}{q}}{q} \left(\mathbb{E}|x(T)|^q + \mathbb{E}|\tilde{z}(T)|^q\right) \leq C \Delta^\frac{1}{q}.
\]

Using (2.5), (3.15), (3.41), and then by (3.35) and (3.36), we have

\[
\frac{q - r}{q} \Delta^\frac{1}{2(q-r)} \mathbb{P}(\chi_\Delta \leq T) \leq \frac{q - r}{q} \Delta^\frac{1}{2(q-r)} \left(\mathbb{P}\{\vartheta_{\Phi^{-1}(K^{\Delta - \nu})} \leq T\} + \mathbb{P}\{\vartheta^\Delta \leq T\} + \mathbb{P}\{\tilde{\vartheta}_\Delta \leq T\}\right)
\leq \frac{q - r}{q} \Delta^\frac{1}{2(q-r)} \left(\Phi^{-1}(K^{\Delta - \nu})\right)^q \leq C \Delta^\frac{qr}{2(q-r)} \frac{r^2}{2} = C \Delta^\frac{1}{q}.
\]

Next, we consider the first term on the right hand of (3.43). Employing the Itô formula, we derive that

\[
|e(T \wedge \chi_\Delta)|^r \leq \int_0^{T \wedge \chi_\Delta} \frac{r}{2} |e(s)|^{r-2} \left[2\langle e(s), f(x(s), x(s - \tau)) - f(z_\Delta(s), z_\Delta(s - \tau)) \rangle + (r - 1) |g(x(s), x(s - \tau)) - g(z_\Delta(s), z_\Delta(s - \tau))|^2\right] ds
+ \int_0^{T \wedge \chi_\Delta} r |e(s)|^{r-2} \langle e(s), g(x(s), x(s - \tau)) - g(z_\Delta(s), z_\Delta(s - \tau)) \rangle dW(s).
\]

(3.46)
Due to \( r \in [2, \beta + 1) \), we may choose a constant \( \kappa_2 > 0 \) such that \((1 + \kappa_2)(r - 1) \leq \beta\). Applying the elementary inequality and (H5) yields

\[
2\langle e(s), f(x(s), x(s - \tau)) - f(\zeta_{\Delta}(s), \zeta_{\Delta}(s - \tau)) \rangle \\
+ (r - 1)|g(x(s), x(s - \tau)) - g(\zeta_{\Delta}(s), \zeta_{\Delta}(s - \tau))|^2 \\
\leq 2\langle e(s), f(x(s), x(s - \tau)) - f(\xi_{\Delta}(s), \xi_{\Delta}(s - \tau)) \rangle \\
+ 2\langle e(s), f(\xi_{\Delta}(s), \xi_{\Delta}(s - \tau)) - f(\zeta_{\Delta}(s), \zeta_{\Delta}(s - \tau)) \rangle \\
+ (1 + \kappa_2)(r - 1)|g(x(s), x(s - \tau)) - g(\xi_{\Delta}(s), \xi_{\Delta}(s - \tau))|^2 \\
+ \left(1 + \frac{1}{\kappa_2}\right)(r - 1)|g(\xi_{\Delta}(s), \xi_{\Delta}(s - \tau)) - g(\zeta_{\Delta}(s), \zeta_{\Delta}(s - \tau))|^2 \\
\leq 2\langle e(s), f(x(s), x(s - \tau)) - f(\xi_{\Delta}(s), \xi_{\Delta}(s - \tau)) \rangle \\
+ \beta|g(x(s), x(s - \tau)) - g(\xi_{\Delta}(s), \xi_{\Delta}(s - \tau))|^2 \\
+ 2K_4|e(s)|(\xi_{\Delta}(s) - z_{\Delta}(s)) + |\xi_{\Delta}(s) - \zeta_{\Delta}(s)|\langle 1 + |\zeta_{\Delta}(s)|^\alpha \rangle \\
+ |\zeta_{\Delta}(s - \tau)|^\alpha + |z_{\Delta}(s)|^\alpha + |z_{\Delta}(s - \tau)|^\alpha \\
+ (1 + \frac{1}{\kappa_2})(r - 1)K_4(\xi_{\Delta}(s) - z_{\Delta}(s))^2 + |\zeta_{\Delta}(s - \tau) - \zeta_{\Delta}(s - \tau)|^2(1 + |\zeta_{\Delta}(s)|^\alpha) \\
+ |\zeta_{\Delta}(s - \tau)|^\alpha + |z_{\Delta}(s)|^\alpha + |z_{\Delta}(s - \tau)|^\alpha).
\]

Inserting the above inequality into (3.46) and using (H6), we derive

\[
|e(T, x)|^\alpha \\
\leq \frac{r}{2} \int_0^{T - \alpha \Delta} \left[ K_5|e(s)|^\alpha + K_5|e(s - \tau)|^\alpha - \tilde{V}(x(s), \tilde{z}(s)) \\
+ \tilde{V}(x(s - \tau), \tilde{z}_{\Delta}(s - \tau)) + 2K_4|e(s)|^{\alpha - 1}|\tilde{z}_{\Delta}(s) - z_{\Delta}(s)|(1 + |\zeta_{\Delta}(s)|^\alpha) \\
+ |\zeta_{\Delta}(s - \tau)|^\alpha + |z_{\Delta}(s)|^\alpha + |z_{\Delta}(s - \tau)|^\alpha \\
+ 2K_4|e(s)|^{\alpha - 1}|\tilde{z}_{\Delta}(s - \tau) - z_{\Delta}(s - \tau)|(1 + |\zeta_{\Delta}(s)|^\alpha + |\zeta_{\Delta}(s - \tau)|^\alpha + |z_{\Delta}(s)|^\alpha) \\
+ |z_{\Delta}(s - \tau)|^\alpha + \left(1 + \frac{1}{\kappa_2}\right)(r - 1)K_4|e(s)|^{\alpha - 2}|\tilde{z}_{\Delta}(s) - z_{\Delta}(s)|^2(1 + |\zeta_{\Delta}(s)|^\alpha) \\
+ |\zeta_{\Delta}(s - \tau)|^\alpha + |z_{\Delta}(s)|^\alpha + |z_{\Delta}(s - \tau)|^\alpha \\
+ \left(1 + \frac{1}{\kappa_2}\right)(r - 1)K_4|e(s)|^{\alpha - 2}|\tilde{z}_{\Delta}(s - \tau) - z_{\Delta}(s - \tau)|^2(1 + |\zeta_{\Delta}(s)|^\alpha) \\
+ |\zeta_{\Delta}(s - \tau)|^\alpha + |z_{\Delta}(s)|^\alpha + |z_{\Delta}(s - \tau)|^\alpha \right] ds \\
+ \int_0^{T - \alpha \Delta} r|e(s)|^{r - 2}(e(s), g(x(s), x(s - \tau)) - g(\zeta_{\Delta}(s), \zeta_{\Delta}(s - \tau)))dW(s). \tag{3.47}
\]

Owing to \( \tilde{V}(x, x) = 0 \) for any \( x \in \mathbb{R}^d \), we have

\[
\int_0^{T - \alpha \Delta} \tilde{V}(x(s), \tilde{z}_{\Delta}(s)) + \tilde{V}(x(s - \tau), \tilde{z}_{\Delta}(s - \tau)) ds \\
\leq \int_{-\tau}^{0} \tilde{V}(x(s), \tilde{z}_{\Delta}(s)) ds = \int_{-\tau}^{0} \tilde{V}(\xi(s), \hat{\xi}(s)) ds = 0. \tag{3.48}
\]
According to (3.47) and (3.48), using the Young inequality and the elementary inequality, we obtain

\[
\mathbb{E}(|e(T \wedge \chi_\Delta)|^r) 
\leq \left( rK_5 + 2(r - 1)K_4 + (1 + \frac{1}{\kappa_2})(r - 1)(r - 2)K_4 \right) \mathbb{E} \int_0^{T \wedge \chi_\Delta} |e(s)|^r \, ds 
+ K_4^2 \mathbb{E} \int_0^{T} |\tilde{z}_\Delta(s) - z_\Delta(s)|^r (1 + |\tilde{z}_\Delta(s)|^\alpha + |\tilde{z}_\Delta(s - \tau)|^\alpha + |z_\Delta(s)|^\alpha + |z_\Delta(s - \tau)|^\alpha)^r \, ds 
+ K_4 \mathbb{E} \int_0^{T} |\tilde{z}_\Delta(s - \tau) - z_\Delta(s - \tau)|^r (1 + |\tilde{z}_\Delta(s)|^\alpha + |\tilde{z}_\Delta(s - \tau)|^\alpha)^r \, ds 
+ |z_\Delta(s - \tau)|^\alpha)^r \, ds 
+ \left( 1 + \frac{1}{\kappa_2} \right)(r - 1)K_4^2 \mathbb{E} \int_0^{T} |\tilde{z}_\Delta(s - \tau) - z_\Delta(s - \tau)|^r (1 + |\tilde{z}_\Delta(s)|^\alpha + |\tilde{z}_\Delta(s - \tau)|^\alpha)^r \, ds 
+ |z_\Delta(s - \tau)|^\alpha)^r \, ds 
\leq C \mathbb{E} \int_0^{T \wedge \chi_\Delta} |e(s)|^r \, ds + J_1 + J_2,
\]

where

\[
J_1 := C \mathbb{E} \int_0^{T} |\tilde{z}_\Delta(s) - z_\Delta(s)|^r (1 + |\tilde{z}_\Delta(s)|^\alpha + |\tilde{z}_\Delta(s - \tau)|^\alpha + |z_\Delta(s)|^\alpha)^r \, ds 
+ |z_\Delta(s - \tau)|^\alpha)^r \, ds,
\]

\[
J_2 := C \mathbb{E} \int_0^{T} |\tilde{z}_\Delta(s - \tau) - z_\Delta(s - \tau)|^r (1 + |\tilde{z}_\Delta(s)|^\alpha + |\tilde{z}_\Delta(s - \tau)|^\alpha + |z_\Delta(s)|^\alpha)^r \, ds 
+ |z_\Delta(s - \tau)|^\alpha)^r \, ds.
\]

Using Hölder’s inequality, Theorem 3.1, Lemma 3.2 and Lemma 3.3, we have

\[
J_1 \leq C \int_0^{T} \left( \mathbb{E} |\tilde{z}_\Delta(s) - z_\Delta(s)|^{2r} \right)^{\frac{1}{2}} \left( 1 + \mathbb{E} |\tilde{z}_\Delta(s)|^{2r\alpha} + \mathbb{E} |z_\Delta(s - \tau)|^{2r\alpha} \right)^{\frac{1}{2}} \, ds 
+ \mathbb{E} |z_\Delta(s)|^{2r\alpha} + \mathbb{E} |z_\Delta(s - \tau)|^{2r\alpha} \right)^{\frac{1}{2}} \, ds 
\leq C \int_0^{T} \Delta^{\frac{1}{q}} \left[ 1 + \left( \mathbb{E} |\tilde{z}_\Delta(s)|^{q} \right)^{\frac{2r\alpha}{q}} + \left( \mathbb{E} |z_\Delta(s)|^{q} \right)^{\frac{2r\alpha}{q}} + \left( \mathbb{E} |z_\Delta(s - \tau)|^{q} \right)^{\frac{2r\alpha}{q}} \right]^{\frac{1}{2}} \, ds 
+ \left( \mathbb{E} |z_\Delta(s - \tau)|^{q} \right)^{\frac{1}{2}} \leq C \Delta^{\frac{1}{q}}.
\]

(3.50)
By the same argument as $J_1$, we obtain from (H4)

$$J_2 \leq C \int_0^T \left( \mathbb{E}|\tilde{z}_\Delta(s-\tau) - z_\Delta(s-\tau)|^{2r} \right)^{\frac{1}{r}} \left[ 1 + \left( \mathbb{E}|\tilde{z}_\Delta(s)|^{q/2} \right)^{\frac{2r}{q}} \right] ds + \left( \mathbb{E}|\tilde{z}_\Delta(s)|^{q/2} \right)^{\frac{2r}{q}} \int_0^T \left( \mathbb{E}|\tilde{z}_\Delta(s)|^{q/2} \right)^{\frac{2r}{q}} ds
$$

$$\leq C \int_0^T \left( \mathbb{E}|\tilde{z}_\Delta(s) - z_\Delta(s)|^{2r} \right)^{\frac{1}{r}} ds + C \int_0^T \left( \mathbb{E}|\tilde{z}_\Delta(s) - z_\Delta(s)|^{2r} \right)^{\frac{1}{r}} ds
$$

$$\leq C \hat{\Delta}^\frac{r}{2} + K_3 C \int_{-\tau}^0 |s - \frac{S}{\Delta}| |\Delta|^\frac{r}{2} ds \leq C \hat{\Delta}^\frac{r}{2}. \quad (3.51)$$

Inserting (3.50), (3.51) into (3.49) and applying the Gronwall inequality we arrive at

$$\mathbb{E}\left(|e(T \wedge \chi_\Delta)|^r\right) \leq C e^{CT} \Delta^{r/2}. \quad (3.52)$$

Substituting (3.44), (3.45), and (3.52) into (3.43), we get the desired assertion. \qed

**Theorem 3.8** Under the conditions of Lemma 3.5, for any $\bar{r} \in (0, r]$, we have

$$\mathbb{E}\left| x(T) - z\Delta(T) \right|^\bar{r} \leq C \hat{\Delta}^\bar{r}, \quad \forall \ T > 0.$$  

**Proof** For any $T > 0$, by (3.38) and (3.42), we derive

$$\mathbb{E}\left| x(T) - z\Delta(T) \right|^\bar{r} \leq 2\bar{r} \mathbb{E}\left| x(T) - \tilde{z}_\Delta(T) \right|^\bar{r} + 2\bar{r} \mathbb{E}\left| \tilde{z}_\Delta(T) - z\Delta(T) \right|^\bar{r} \leq C \hat{\Delta}^\bar{r},$$

which together with the Hölder inequality implies the desired assertion. \qed

### 3.4 Exponential stability

This section focuses on approximating the exponential stability of SDDE (1.1). We first give the corresponding results on the exact solutions. Then, we construct a more precise scheme to approximate the long-time behaviors of the system. Without loss of generality, we assume $f(0, 0) = 0$, $g(0, 0) = 0$. Moreover, we give the conditions under which the trivial solution $x = 0$ is stable in mean square.

**(H7)** Assume that there exist constants $\bar{K}_6 > K_6 > 0$, $\bar{K}_7 > K_7 \geq 0$ and a function $V_2(\cdot) \in C(\mathbb{R}^d; \mathbb{R}_+)$ such that for any $x, y \in \mathbb{R}^d$,

$$\langle 2x, f(x, y) \rangle + |g(x, y)|^2 \leq -\bar{K}_6|x|^2 + K_6|y|^2 - \bar{K}_7 V_2(x) - K_7 V_2(y). \quad (3.53)$$

**(H8)** For any positive constant $l$, there exists a positive constant $\hat{L}_l$ such that

$$|f(0, y)| + |g(0, y)| \leq \hat{L}_l |y|$$

for any $|y| \leq l$.

Using the techniques of [26, Theorem 3.4] and [21, Theorem 2.1], we can obtain the exponential stability of SDDE (1.1) as follows.

**Theorem 3.9** Assume that (H1) and (H7) hold. Then, the solution $x(t)$ of SDDE (1.1) with the initial data $\xi \in C_b([-\tau, 0); \mathbb{R}^d)$ has the property

$$\mathbb{E}|x(t)|^2 \leq C e^{-\gamma t}, \quad \forall \ t > 0,$$

where $\gamma > 0$ is a constant satisfying $K_6 e^{\gamma \tau} + \gamma \leq \bar{K}_6$ and $K_7 e^{\gamma \tau} \leq \bar{K}_7$. \qed
Theorem 3.10 Assume that (H1) and (H7) hold. Then, the solution $x(t)$ of SDDE (1.1) with the initial data $\xi \in C_b([-\tau, 0]; \mathbb{R}^d)$ has the property

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -\frac{\gamma}{2} \quad \text{a.s.,}$$

where $\gamma$ is given in Theorem 3.9.

Based on the form of Scheme (3.3), we continue to construct a more precise explicit scheme which numerical solutions keep the underlying exponential stability in mean square and $\mathbb{P} - 1$. Under (H1) and (H8), choose a strictly increasing continuous function $\hat{\Phi} : [1, \infty) \to \mathbb{R}^+$ such that

$$\sup_{|x| \vee |y| \leq l} \left( \frac{|f(x, y)|}{|x| + 1 \wedge |y|} \vee \frac{|g(x, y)|^2}{(|x| + 1 \wedge |y|)^2} \right) \leq \hat{\Phi}(l), \quad \forall \ l \geq 1. \quad (3.54)$$

For any given stepsize $\Delta \in (0, 1]$, define a truncation mapping $\gamma_{\hat{\Phi}, \mu}^\Delta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\gamma_{\hat{\Phi}, \mu}^\Delta(x) = \left( |x| \wedge \hat{\Phi}^{-1}(\hat{K} \Delta^{-\mu}) \right) \frac{x}{|x|}, \quad (3.55)$$

where $x/|x| = 0$ if $x = 0 \in \mathbb{R}^d$, $\hat{K} = \hat{\Phi}(M_0 \vee 1)$ and $\mu \in (0, 1/2)$. Then, we define

$$y_i^\Delta = x(i \Delta), \quad \forall \ i = -N, \cdots, 0,$$

$$\hat{y}_i^\Delta = y_i^\Delta + f(y_i^\Delta, y_{i-N}^\Delta) \Delta + g(y_i^\Delta, y_{i-N}^\Delta) \Delta W_i, \quad \forall \ i = 0, 1, \cdots, \quad (3.56)$$

Comparing (3.54) with (3.1) one observes that $\Phi(l) \leq \hat{\Phi}(l), \quad \forall \ l \geq 1$. Thus, all results on Scheme (3.3) also hold for Scheme (3.56). Moreover, (3.54)–(3.56) lead to

$$|f(y_i^\Delta, y_{i-N}^\Delta)| \leq \hat{K} \Delta^{-\mu}(|y_i^\Delta| + 1 \wedge |y_{i-N}^\Delta|), \quad \forall \ x, y \in \mathbb{R}^d. \quad (3.57)$$

Theorem 3.11 Assume that (H1), (H7) and (H8) hold. Then, for any $\varepsilon \in (0, \gamma)$, there is a $\Delta \in (0, 1]$ such that for any $\Delta \in (0, \Delta]$

$$\mathbb{E}|y_i^\Delta|^2 \leq C e^{-(\gamma - \varepsilon) i \Delta}, \quad \forall \ i \geq 1, \quad (3.58)$$

where $\gamma$ is given in Theorem 3.9.

Proof For $i \geq 0$, define $f_i = f(y_i^\Delta, y_{i-N}^\Delta)$ and $g_i = g(y_i^\Delta, y_{i-N}^\Delta)$ for short. It follows from (3.56), (3.57) and (H7) that

$$\mathbb{E}\left[|\hat{y}_{i+1}^\Delta|^2 | F_i \right] = |y_i^\Delta|^2 + 2(y_i^\Delta, f_i) \Delta + |g_i|^2 \Delta + |f_i|^2 \Delta^2$$

$$\leq |y_i^\Delta|^2 - (\tilde{K}_6 - 2\hat{K}^2 \Delta^{1-2\mu}) |y_i^\Delta|^2 \Delta + (K_6 + 2\hat{K}^2 \Delta^{1-2\mu}) |y_{i-N}^\Delta|^2 \Delta$$

$$- \tilde{K}_7 V_2(y_i^\Delta) \Delta + K_7 V_2(y_{i-N}^\Delta) \Delta. \quad (3.59)$$

Employing the Taylor formula derives that for any $\varepsilon \in (0, \gamma)$,

$$e^{(\gamma - \varepsilon) \Delta} - 1 = (\gamma - \varepsilon) \Delta + (\gamma - \varepsilon)^2 \Delta^2 \sum_{j=0}^{\infty} \frac{(\gamma - \varepsilon)^j \Delta^j}{(j + 2)!} \leq (\gamma - \varepsilon) \Delta + \alpha \Delta^2,$$
where $\alpha := e^{(\gamma - \epsilon)\Delta}(\gamma - \epsilon)^2$. This together with (3.59) leads to
\[
e^{(\gamma - \epsilon)\Delta} \mathbb{E}\left[|\hat{y}_{i+1}\Delta|^{2} | F_{i}\right] \\
\leq \left(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2}\right)\left(|\hat{y}_{i}\Delta|^{2} - (\tilde{K}_{6} - 2\tilde{K}^{2}\Delta^{1-2\mu})|\hat{y}_{i}\Delta|^{2} \Delta + (K_{6} + 2\tilde{K}^{2}\Delta^{1-2\mu})|\hat{y}_{i-N}\Delta|^{2} \Delta \\
- \tilde{K}_{7}V_{2}(\hat{y}_{i}\Delta) + K_{7}V_{2}(\hat{y}_{i-N}\Delta)\right) \\
\leq |\hat{y}_{i}\Delta|^{2} - \Delta((\tilde{K}_{6} - \gamma) + \epsilon - o_{1}(\Delta))|\hat{y}_{i}\Delta|^{2} + \Delta(K_{6} + o_{2}(\Delta))|\hat{y}_{i-N}\Delta|^{2} \\
- \Delta\tilde{K}_{7}(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2})V_{2}(\hat{y}_{i}\Delta) + \Delta K_{7}(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2})V_{2}(\hat{y}_{i-N}\Delta), \quad (3.60)
\]

where
\[
o_{1}(\Delta) = 2\tilde{K}^{2}\Delta^{1-2\mu} + 2\tilde{K}^{2}(\gamma - \epsilon)\Delta^{2-2\mu} + \alpha \Delta + 2\tilde{K}^{2}\alpha \Delta^{3-2\mu}, \\
o_{2}(\Delta) = 2\tilde{K}^{2}\Delta^{1-2\mu} + 2\tilde{K}^{2}(\gamma - \epsilon)\Delta^{2-2\mu} + K_{6}(\gamma - \epsilon)\Delta + 2\tilde{K}^{2}\alpha \Delta^{3-2\mu} + K_{6}\alpha \Delta^{2}.
\]

Choose a constant $\tilde{\Delta} \in (0, 1]$ small sufficiently such that
\[
o_{1}(\tilde{\Delta}) \leq \epsilon \quad \text{and} \quad o_{2}(\tilde{\Delta}) \leq K_{6}(e^\gamma - 1).
\]

Then, taking the expectations and timing $e^{(\gamma - \epsilon)t_{i}}$ on both sides of (3.60) derives that for any $\Delta \in (0, \tilde{\Delta})$
\[
e^{(\gamma - \epsilon)t_{i+1}} \mathbb{E}[|\hat{y}_{i+1}\Delta|^{2} - e^{(\gamma - \epsilon)t_{i}} \mathbb{E}[|\hat{y}_{i}\Delta|^{2} \\
\leq e^{(\gamma - \epsilon)t_{i+1}} \mathbb{E}[|\hat{y}_{i+1}\Delta|^{2} - e^{(\gamma - \epsilon)t_{i}} \mathbb{E}[|\hat{y}_{i}\Delta|^{2} \\
\leq -\Delta(\tilde{K}_{6} - \gamma)e^{(\gamma - \epsilon)t_{i}} \mathbb{E}[|\hat{y}_{i}\Delta|^{2}] + \Delta K_{6}e^{\gamma t} e^{(\gamma - \epsilon)t_{i}} \mathbb{E}[|\hat{y}_{i-N}\Delta|] \\
- \Delta\tilde{K}_{7}(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2})e^{(\gamma - \epsilon)t_{i}} \mathbb{E}[V_{2}(\hat{y}_{i}\Delta)] \\
+ \Delta K_{7}(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2})e^{(\gamma - \epsilon)t_{i}} \mathbb{E}[V_{2}(\hat{y}_{i-N}\Delta)].
\]

Solving the above difference equation and using the fact that $K_{6}e^{\gamma t} + \gamma \leq \tilde{K}_{6}$ and $K_{7}e^{\gamma t} \leq \tilde{K}_{7}$, we yield
\[
e^{(\gamma - \epsilon)t_{i+1}} \mathbb{E}[|\hat{y}_{i+1}\Delta|^{2} \\
\leq \mathbb{E}[\xi(0)]^{2} - \Delta(\tilde{K}_{6} - \gamma) \sum_{k=0}^{i} e^{(\gamma - \epsilon)t_{k}} \mathbb{E}[|\hat{y}_{k}\Delta|^{2}] + \Delta K_{6}e^{\gamma t} \sum_{k=-N}^{i-N} e^{(\gamma - \epsilon)t_{k}} \mathbb{E}[|\hat{y}_{k}\Delta|^{2}] \\
- \Delta\tilde{K}_{7}(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2}) \sum_{k=0}^{i} e^{(\gamma - \epsilon)t_{k}} \mathbb{E}[V_{2}(\hat{y}_{k}\Delta)] \\
+ \Delta K_{7}e^{(\gamma - \epsilon)t}(1 + (\gamma - \epsilon)\Delta + \alpha \Delta^{2}) \sum_{k=-N}^{i-N} e^{(\gamma - \epsilon)t_{k}} \mathbb{E}[V_{2}(\hat{y}_{k}\Delta)] \leq C.
\]

Therefore, the desired assertion follows. \hfill \Box

Using the technique of [32, Theorem 3.4], we yield the exponentially almost sure stability of the TEM scheme (3.56) as follows.
Theorem 3.12 Under the conditions of Theorem 3.11, for any $\varepsilon \in (0, \gamma)$, there is $\bar{\Delta} \in (0, 1]$ such that for any $\Delta \in (0, \bar{\Delta}]$
\[
\limsup_{i \to \infty} \frac{1}{i\Delta} \log |y_i^\Delta| \leq -\frac{\gamma - \varepsilon}{2} \text{ a.s.} \quad (3.61)
\]

4 Numerical examples

This section gives two examples to illustrate our results.

Example 1 Let us recall SDDE (1.2) and let $q = 10$. By virtue of [23, p.211, Lemma 4.1] we know
\[
|a + b|^p \leq |a|^p \frac{p}{\delta p - 1} + |b|^p \frac{(1 - \delta)p - 1}{(1 - \delta)p - 1},
\]
where $a, b \in \mathbb{R}$, $p > 1$, $\delta \in (0, 1)$. The above inequality as $\delta = 1/20$ together with the Young inequality implies
\[
(1 + |x|^2)^4 \left(2|x, f(x, y)| + 9|g(x, y)|^2\right) \leq 144000 + 144000(|x|^{10} + |y|^{10}) - 8|x|^{12} + 7|y|^{12}.
\]
Thus, (H2) holds with $V_1(x) = 8|x|^{12}$. By virtue of Theorem 2.1, (1.2) has a unique global solution. Furthermore, for any $s_1, s_2 \in [-1, 0]$,
\[
(\mathbb{E}|\cos(B(s_1)) - \cos(B(s_2))|^{1/2}) \leq (\mathbb{E}|B(s_1) - B(s_2)|^{1/2}) \leq C|s_1 - s_2|^{1/2},
\]
which implies (H4) holds with $\lambda = 1/2$. Let $r = 2$ and $\beta = 2$. By the elementary inequality we have
\[
2(x - \bar{x}, f(x, y) - f(\bar{x}, \bar{y})) + 2|g(x, y) - g(\bar{x}, \bar{y})|^2 \leq -12|x - \bar{x}|^2(x^2 + \bar{x}^2) + 8|y - \bar{y}|^2(y^2 + \bar{y}^2).
\]
This implies that (H6) holds with $\bar{V}(x, \bar{x}) = 12|x - \bar{x}|^2(x^2 + \bar{x}^2)$. Obviously, (H5) holds with $K_4 = 30$ and $\alpha = 2$. By (3.35) we take $\Phi^{-1}(l) = (l/180)^{1/4}$, $\forall l \geq 180$. By (3.36) and $\sup_{1 \leq t \leq 0} |\cos(B(t))| \leq 1$, we have $\mu = 1/2$ and $K = 180$. Hence,
\[
\Gamma_{\Phi, \mu}(x) = (|x| \wedge \Delta^{-\frac{1}{8}})^\frac{1}{2},
\]
By virtue of Theorem 3.8, the TEM scheme (3.5) satisfies that for any $\Delta \in (0, 1]$
\[
(\mathbb{E}|x(T) - z_{\Delta}(T)|^{1/2}) \leq C \Delta^{1/2}, \quad \forall T > 0.
\]

We regard the numerical solution with small stepsize $\Delta = 2^{-20}$ as the exact solution $x(t)$, and carry out numerical experiments to compute the error $(\mathbb{E}|x(20) - z_{\Delta}(20)|^{1/2})$ between the exact solution $x(20)$ and the numerical solution $z_{\Delta}(20)$ of the TEM scheme using MATLAB. In Fig. 1, the red solid line
Error at T=20

The red solid line depicts the error \( (E|x(20)-z(20)|^2)^{1/2} \) between the exact solution \( x(20) \) and the TEM scheme \( z_\Delta(20) \), as the function of \( \Delta \) for 1000 sample points as \( \Delta \in \{2^{-14}, 2^{-12}, 2^{-10}, 2^{-8}, 2^{-6}\} \). The blue solid line plots the reference function \( \Delta^{1/2} \).

**Example 2** Consider the 2-dimensional SDDE

\[
\begin{align*}
\left\{ \begin{array}{l}
dx_1(t) &= \left(-2x_1(t) - x_1^3(t)\right) dt + \left(x_2(t - 1) + x_1^2(t) \right) dW_1(t), \\
dx_2(t) &= \left(-2x_2(t) - x_2^3(t)\right) dt + \left(x_1(t - 1) + x_2^2(t) \right) dW_2(t), 
\end{array} \right. \\
t &> 0,
\end{align*}
\]

with the initial data \((\xi_1(t), \xi_2(t))^T = (\cos(B(t)), \sin(B(t)))^T, \ t \in [-1, 0]\), where \( B(t) \) is a one-dimensional Brownian motion independent of \((W_1(t), W_2(t))^T\). We compute that for any \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^2 \)

\[
|f(x, y) - f(\bar{x}, y)| \leq 2(1 + |x|^2 + |\bar{x}|^2)|x - \bar{x}|, \quad f(0, y) = 0,
\]

\[
|g(x, y) - g(\bar{x}, y)|^2 \leq 2(|x|^2 + |\bar{x}|^2)|x - \bar{x}|^2, \quad |g(0, y)|^2 = |y|^2,
\]

and

\[
\langle 2x, f(x, y) \rangle + |g(x, y)|^2 \leq -4|x|^2 + 2|y|^2.
\]
Then, (H1), (H7), and (H8) hold, where $\bar{K}_6 = 4$, $K_6 = 2$, $\bar{K}_7 = K_7 = 0$. Choose $\gamma = 1/2$. By virtue of Theorem 3.9 and Theorem 3.10, (4.1) is the exponentially stable in mean square and $P - 1$.

By (3.54), we take $\hat{\Phi}(l) = 2(l + l^2)$, where $l \geq 1$. By (3.55) and $||\xi|| = 1$, we have $\hat{K} = 4$. Choose $\mu = 1/100$. Hence,

$$\gamma_{\hat{\Phi}, \mu}(x) = \left( |x| \wedge (2\Delta - \frac{1}{100} - 1)^{\frac{1}{2}} \right) \frac{x}{|x|}, \quad \forall \Delta \in (0, 2^{-7}] .$$

It follows from Theorem 3.11 and Theorem 3.12 that $\forall \Delta \in (0, 2^{-7}]$, $\forall i \geq 0$,

$$\mathbb{E}|y_{i\Delta}|^2 \leq Ce^{-\frac{1}{100}t_i}, \quad \lim_{i \to \infty} \sup_{i\Delta} \frac{1}{i\Delta} \log |y_{i\Delta}| \leq -\frac{1}{20}, \text{ a.s.}$$

Figure 2 depicts the sample mean of the TEM scheme $y_{i\Delta}$ defined by (3.56). Figure 3 depicts sample paths of the EM solution $Y_{i\Delta}$ and the TEM solution $y_{i\Delta}^\Delta$.

5 Conclusions

In this paper a generic explicit numerical scheme is constructed, which numerical solutions are bounded in the $q$th moment and converge to the exact solutions strongly. The $1/2$ order convergence rate is obtained for the TEM scheme. Moreover, a more
The sample path of $\ln Y_i^\Delta$ by the EM scheme and the sample path of $y_i^\Delta$ by the TEM scheme defined by (3.56) with $\Delta = 2^{-7}$

precise TEM scheme is proposed, which numerical solutions keep the exponential stability well for a large kind of nonlinear SDDEs.

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