Covariance, Geometricity, Setting, and Dynamical Structures on Cosmological Manifold

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Abstract

The treatment of the principle of general covariance based on coordinate systems, i.e., on classical tensor analysis suffers from an ambiguity. A more preferable formulation of the principle is based on modern differential geometry: the formulation is coordinate-free. Then the principle may be called “principle of geometricity.” In relation to coordinate transformations, there had been confusions around such concepts as symmetry, covariance, invariance, and gauge transformations. Clarity has been achieved on the basis of a group-theoretical approach and the distinction between absolute and dynamical objects. In this paper, we start from arguments based on structures on cosmological manifold rather than from group-theoretical ones, and introduce the notion of setting elements. The latter create a scene on which dynamics is performed. The characteristics of the scene and dynamical structures on it are considered.

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Introduction

In a fundamental work on the general theory of relativity [1], Einstein gave due attention to the principle of general covariance as one of the cornerstones of the theory. The principle was formulated as the requirement that the general laws of nature must be expressed in terms of equations valid in all coordinate systems. However, Kretschmann [2] argued that equations originally written in any coordinate system may be extended to all coordinate systems and thus made covariant; therefore the principle of general covariance involves no physical content. Einstein concurred with the argumentation [3].

The treatment of the principle of general covariance based on coordinate systems, i.e., on classical tensor analysis, as will be seen later, suffers from an ambiguity—as long as the geometric character of quantities is not specified in advance. A more preferable formulation of the principle is based on modern differential geometry: such a formulation is coordinate-free. We quote [4]: “Every physical quantity must be describable by a (coordinate-free) geometric object, and the laws of physics must all be expressible as geometric relationships between these geometric objects.” In such a formulation, the principle may be called “principle of geometricity.”

In relation to coordinate transformations, there were confusions around such concepts as symmetry, covariance, invariance, and gauge transformation [5]. In a textbook, the point was for the first time made clear by Anderson [6]. His treatment is based on a group-theoretical approach and the distinction between absolute and dynamical objects (see also [7], [5]).

In this paper, we start from arguments based on structures on cosmological manifold rather than from group-theoretical ones. Therefore we introduce the notion of setting objects—instead of absolute ones. The setting objects create a scene on which dynamics is performed.

The purpose of the paper is to consider briefly the characteristics of the scene and dynamical structures on it.

1 Covariance and geometricity

1.1 Cosmological manifold

A primary setting object, or element is a cosmological manifold, i.e., a smooth 4-manifold [8]

\[ M = M^4, \quad p \in M \]

At this juncture, there is no structure on \( M \).

1.2 The principle of general covariance in terms of coordinate systems

The principle of general covariance was originally formulated on the basis of classical tensor analysis, in which it is necessary to exploit coordinate systems. In this approach, tensor quantities are defined in terms of their components and of the transformation rules for the latter under coordinate changes. The principle is formulated as follows [1]:
“The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant).”

Kretschmann argued that any equation written in an arbitrary coordinate system may be rewritten in any other coordinate system—on the basis of the transformation rules.

1.3 Ambiguity

There is an ambiguity in the application of the transformation rules—as long as the tensor character of quantities involved in the equation is not specified. This is an example. Let four equations be given in a coordinate system \( x = (x^\mu)^3_{\mu=0} \):

\[
 f_1^{(\nu)}(x) = f_2^{(\nu)}(x)
\]

where \( \nu = 0, 1, 2, 3 \) is the label of the \( f \). Let \( \bar{x} = (\bar{x}^\mu) \) be another coordinate system:

\[
 M \ni p \leftrightarrow x \leftrightarrow \bar{x}
\]

There are different possibilities:

1) both \( f_1^{(\nu)} \) and \( f_2^{(\nu)} \) are functions (i.e., scalars); then

\[
 f_n^{(\nu)}(x) = f_n^{(\nu)}(x(\bar{x})) =: \bar{f}_n^{(\nu)}(\bar{x})
\]

and we have the implication

\[
 f_1^{(\nu)}(x) = f_2^{(\nu)}(x) \Rightarrow \bar{f}_1^{(\nu)}(\bar{x}) = \bar{f}_2^{(\nu)}(\bar{x})
\]

or

\[
 f_1^{(\nu)}(p) = f_2^{(\nu)}(p), \quad p \in M
\]

which is covariant.

2) both \( f_1^{(\nu)} \) and \( f_2^{(\nu)} \) are components of vectors:

\[
 f_n^{(\nu)} = v_n^{\nu}
\]

then \( f_1^{(\nu)}(x) = f_2^{(\nu)}(x) \) amounts to

\[
 v_1^{\nu} = v_2^{\nu}
\]

which is fulfilled in all coordinate systems, i.e., is covariant.

3) \( f_1^{(\nu)} \) represents a vector, whereas \( f_2^{(\nu)} \) is a function:

\[
 f_1^{(\nu)} = v^{\nu}, \quad f_2^{(\nu)} = f^{(\nu)}
\]

then

\[
 f_1^{(\nu)}(x) = f_2^{(\nu)}(x) \not\Rightarrow v^{\nu} = f^{(\nu)}
\]
1.4 The principle of geometricity

To avoid the ambiguity we have to be based on modern differential geometry rather than on classical tensor analysis, and formulate the principle of geometricity:

Spacetime structure and spacetime aspects of matter objects must be expressed in terms of geometric notions.

Now in the above example, the $f_n^{(\nu)}$ should be specified as geometric objects.

2 Setting

2.1 Setting as a scene for performing dynamics

We consider a physical theory that involves dynamics on cosmological manifold $M$. (A definition of dynamics will be given below.) All physical objects are classified into two categories: setting objects and dynamical ones. A setting is a family of setting objects; a dynamical system is a set of dynamical objects. The mathematical representation of the setting is independent of the dynamical system, whereas the representation of the latter involves the setting.

Figuratively speaking, a setting is a scene on which dynamics is performed.

2.2 Natural and free setting elements

The setting objects are classified into two subcategories: natural and free objects. The natural setting is induced by dynamics in the sense that the latter involves the former. The free setting elements, if any, play an auxiliary role.

These are the examples of natural setting elements: affine structures of Aristotelian and Newtonian spacetimes [9], the Minkowskian metric, the gravitational and cosmological constants, interaction constants of quantum field theory; and of free setting elements: reference frames (tetrads), coordinate systems.

2.3 The principle of minimal (free) setting

Now we may endow the principle of geometricity with a certain constructive meaning. In view of that principle, we advance the principle of minimal (free) setting:

A (free) setting should include as few elements as possible.

It is the absence of free setting elements that is in accordance with the principle of geometricity.

3 Objects and related fields

3.1 Objects and fields

Let $w$ be an object which is an element of a set $W$,

$$w \in W$$
A related field is defined on a submanifold $[8]$ of cosmological manifold,

$$M' \subset M$$

as

$$w_{M'} \in \prod_{p \in M'} \mathcal{W}_p, \ w_{M'}(p) \in \mathcal{W}_p$$

$\mathcal{W}_p$ is related to $p \in M$.

Introduce an abridged notation:

$$w_{M'}(p) =: w(p)$$

### 3.2 Geometric quantities

Let

$$F_\gamma, \ \gamma \in \Gamma$$

be a geometric quantity, $\gamma$ being the type of the latter: scalar, vector, tensor, spinor. $F_\gamma$ is an element of a space $\mathcal{F}_\gamma$,

$$F_\gamma \in \mathcal{F}_\gamma$$

Introduce

$$\mathcal{F}_U := \bigcup_{\gamma \in \Gamma} \mathcal{F}_\gamma, \ F \in \mathcal{F}_U$$

where $F$ is a generic $F_\gamma$.

Geometric fields are defined according to the preceding subsection.

### 3.3 Variables, states, and valuables

Introduce variables:

$$v_{\gamma b}, \ b \in \mathcal{B}$$

variable value sets:

$$\mathcal{V}_\gamma, \ v_{\gamma b} \in \mathcal{V}_\gamma$$

$$\mathcal{V}_U := \bigcup_{\gamma \in \Gamma} \mathcal{V}_\gamma, \ v \in \mathcal{V}_U$$

and states:

$$\omega \in \Omega$$
In the final analysis, it is the expectation values of variables in states that have immediate physical meaning. Therefore we introduce the notion of valuable:

\[ \langle \cdot \rangle : \mathcal{V}_U \times \Omega \rightarrow \mathcal{F}_U, \quad (v, \omega) \mapsto \langle v, \omega \rangle \in \mathcal{F}_U \]

or, in more detail

\[ \langle v_{\gamma b}, \omega \rangle \in \mathcal{F}_\gamma \]

### 3.4 Classical variables and fields

Introduce the following notation for classical variables:

\[ v_{\gamma b} = \xi_{\gamma b} \in \Xi_\gamma, \quad b = b'^{\text{class}} \in \mathcal{B}^\text{class} \]

In classical physics, no distinction is usually made between an abstract variable and its expectation value [10]. So we put

\[ \langle \xi_{\gamma b}, \omega^{\text{class}} \rangle =: \xi_{\gamma b} \in \mathcal{F}_\gamma \]

For a classical field, we have a notation

\[ \xi_{\gamma b M'}, \quad \xi_{\gamma b}(p), \quad p \in M' \]

### 3.5 Quantum variables and fields

Introduce the following designations: the Hilbert space \( \mathcal{H} \),

\[ \hat{A} := L(\mathcal{H}, \mathcal{H}), \quad \hat{A} \in \hat{A}, \quad \hat{A} : \mathcal{H} \rightarrow \mathcal{H} \]

A quantum entity (variable or field) generally consists of two components: classical \( F_\gamma \) and properly quantum \( \hat{A} \). Namely, a quantum variable

\[ \hat{v}_{\gamma b} = F_{\gamma b} \hat{A}_b := F_{\gamma b} \otimes \hat{A}_b, \quad b = b'^{\text{quant}} \in \mathcal{B}^{\text{quant}} \]

where \( \hat{A} \) as a geometric quantity is considered to be a scalar. For a valuable we have

\[ \langle \hat{v}_{\gamma b}, \omega^{\text{quant}} \rangle = F_{\gamma b} \langle \hat{A}_b, \omega^{\text{quant}} \rangle \]

and (for a pure state)

\[ \langle \hat{A}, \omega^{\text{quant}} \rangle = (\Psi, \hat{A}\Psi) \in \mathbb{C}, \quad \Psi \in \mathcal{H} \]

so that

\[ \langle \hat{v}_{\gamma b}, \omega^{\text{quant}} \rangle \in \mathcal{F}_\gamma \]

Generally

\[ \hat{v}_\gamma \in \mathcal{F}_\gamma \otimes \hat{A} \]
or

\[ \hat{v}_{\gamma b} = \int_{\mathcal{L}} \mu(dl) F_{\gamma b} \hat{A}_{bl} \]

and a valuable

\[ \langle \hat{v}_{\gamma b}, \omega^{\text{quant}} \rangle = \int_{\mathcal{L}} \mu(dl) F_{\gamma b}(\Psi, \hat{A}_{bl}\Psi) \]

A quantum field \( \hat{v}_{\gamma bM'} \) may be described as follows:

\[ \hat{v}_{\gamma b}(p) = \int_{\mathcal{L}(p)} \mu_p(dl) F_{\gamma b} \hat{A}_{bl} \]

\[ \langle \hat{v}_{\gamma b}(p), \omega^{\text{quant}} \rangle = \int_{\mathcal{L}(p)} \mu_p(dl) F_{\gamma b}(\Psi, \hat{A}_{bl}\Psi) \]

4 Dynamics on cosmological manifold without structure

4.1 Dynamics

Dynamics on \( M' \subset M \) is a family of valuable fields:

\[ \{ \xi_{\gamma bM'} : \gamma \in \Gamma, \ b \in B^{\text{class}} \} \]

and

\[ \{ \langle \hat{v}_{\gamma b}(p), \omega^{\text{quant}} \rangle : \gamma \in \Gamma, \ b \in B^{\text{quant}}, \ p \in M' \} \]

Classical dynamics is constructed on the basis of the \( \xi \) themselves, quantum dynamics is constructed on the basis of the \( \mathcal{F}_U, \mathcal{H}, \) and \( \mathcal{A} \).

4.2 Mode-series expansion: Manifold modes

Let us introduce the expansion of a quantum field in terms of manifold modes:

\[ \hat{v}_{\gamma b}(p) = \int_{\mathcal{M}} \mu(dm) \sum_{n \in N_{\gamma}} F_{\gamma bmn}(p) \hat{A}_{bmn} \]

\[ \langle \hat{v}_{\gamma b}, \omega^{\text{quant}} \rangle = \int_{\mathcal{M}} \mu(dm) \sum_{n \in N_{\gamma}} F_{\gamma bmn}(p)(\Psi, \hat{A}_{bmn}\Psi) \]

The set

\[ \{ F_{\gamma mnM'} : m \in \mathcal{M}, \ n \in N_{\gamma} \} \]

of manifold modes forms a complete system on \( M' \).
Now we put

\[ F_{\gamma mn}(p) = f_m(p)e_{\gamma mn}(p) \]

where \( f_{mM'} \) is a scalar field on \( M' \). The set

\[ \{ f_{mM'} : m \in \mathcal{M} \} \]

forms a complete system on \( M' \), and the set

\[ \{ e_{\gamma mn}(p) : n \in \mathcal{N}_\gamma \} \]

forms a complete system at \( p \in M' \).

5 The Cauchy problem and manifold foliation

5.1 A Cauchy surface and a foliation

Let \( M \) possess a Cauchy surface. Then there exists a foliation of \( M [11], [12] \):

\[ M = T \times S, \quad M \ni p = (t, s), \quad t \in T, \quad s \in S \]

where 1-manifold \( T \) is a cosmological time and 3-manifold \( S \) is a cosmological space. The tangent space \( M_p \) at a point \( p \in M \) is

\[ M_p = T_t \oplus S_s, \quad p = (t, s) \]

A Cauchy surface

\[ M_C = \{ t_0 \} \times S \ni p = (t_0, s) \]

specifies a unique foliation (by means of synchronous coordinates [13]). In the synchronous reference (i.e., in every synchronous reference frame)

\[ T_t \perp S_s \]

As to the choice of a Cauchy surface, notice the following. If metric is given, different surfaces generally give rise to different foliations; however, if the Cauchy problem includes the determination of metric, the choice of the surface in general does not affect physical results.

Thus as long as dynamics is constructed starting from initial conditions, a natural construction involves a Cauchy surface with the associated foliation and synchronous reference.

Now

\[ M' = T', \quad T' \subset T \]
5.2 Initial conditions

Initial conditions for classical fields are of the form
\[ \{(\xi, \partial_t \xi)_M\} \]
which corresponds to second order dynamics.

For quantum fields we have
\[ \{\hat{v}_M \text{ or } (\hat{v}, \partial_t \hat{v})_M\} \]
or
\[ \{F_{mnM} \text{ or } (F_{mn}, \partial_t F_{mn})_M\} \]
which corresponds to first or second order dynamics, respectively.

6 Dynamics on a foliated manifold

6.1 Time dependent quantum objects

As long as cosmological manifold is foliated, it is natural to introduce time dependent quantum objects:
operator
\[ \hat{A}_{T'} \in \hat{A}^{T'}, \quad \hat{A}_{T'}(t) =: \hat{A}(t) \in \hat{A} \]
state
\[ \omega_{T'} \in \Omega^{T'}, \quad \omega(t) \in \Omega, \quad \Psi_{T'} \in \mathcal{H}^{T'}, \quad \Psi(t) \in \mathcal{H} \]
valuable
\[ (\Psi(t), \hat{A}(t)\Psi(t)) \]

6.2 Mode-series expansion: Space modes

We introduce the expansion of a quantum field in terms of space modes:
\[ \hat{v}_{\gamma b(t) \times S} = \int_{\mathcal{M}} \mu(dm) \sum_{n \in \mathcal{N}_\gamma} F_{\gamma bmn(t) \times S} \hat{A}_{bmn}(t), \quad b \in \mathcal{B}_{\text{quant}} \]
or
\[ \hat{v}_{\gamma b(t, s)} = \int_{\mathcal{M}} \mu(dm) \sum_{n \in \mathcal{N}_\gamma} F_{\gamma bmn(t, s)} \hat{A}_{bmn}(t) \]
so that
\begin{align*}
\langle \hat{v}_{\gamma b}(t, s), \omega^{\text{quant}} \rangle &= \int_{\mathcal{M}} \mu(dm) \sum_{n \in \mathcal{N}_\gamma} F_{\gamma bmn}(t, s)(\Psi(t), \hat{A}_{bmn}(t)\Psi(t))
\end{align*}

Next we put

\[ F_{\gamma mn}(t, s) = f_m(t, s)e_{\gamma mn}(t, s) \]

The \( F_{\gamma mn}(t, s) \) and \( f_m(t, s) \) are time dependent space modes.

The sets

\[ \{ F_{\gamma mn}(t, s) : m \in \mathcal{M}, \ n \in \mathcal{N}_\gamma \} \]

and

\[ \{ f_m(t, s) : m \in \mathcal{M} \} \]

form complete systems on \( \{ t \} \times S \), the set

\[ \{ e_{\gamma mn}(t, s) : n \in \mathcal{N}_\gamma \} \]

forms a complete system at \( (t, s) \).

Now the initial conditions are

\[ (\{ \hat{A}_{bmn}(t_0) : b \in \mathcal{B}^{\text{quant}}, \ m \in \mathcal{M}, \ n \in \mathcal{N} \}, \ \Psi(t_0)) \]

which corresponds to first order dynamics.

### 6.3 Dynamical pictures

There are these dynamical pictures:

- the Schrödinger picture:
  \[ \Psi_S = U_S(t, t_0)\Psi(t_0), \ \hat{A}_S = \text{const} \]

- the Heisenberg picture:
  \[ \Psi_H = \Psi_S(t_0) = \text{const}, \ \hat{A}_H(t) = U_S^\dagger(t, t_0)\hat{A}_S U_S(t, t_0) \]

- a generic picture:
  \[ \Psi(t) = U_1(t)\Psi_S(t_0), \ \hat{A}(t) = U_2^\dagger(t)\hat{A}_S U_2(t), \ U_2(t)U_1(t) = U_S(t, t_0) \]

Note that a Schrödinger variable \( \hat{v}_S \) depends on \( t \) through \( F \).

### 7 Setting elements

#### 7.1 Manifold

Let us list setting elements in the above structures.

Cosmological manifold \( M^4 \) is a primary natural setting element involved in all structures.
7.2 Initial conditions and Cauchy surface

Dynamics implies initial conditions, and the latter involve a Cauchy surface. So initial conditions and a Cauchy surface are natural setting elements.

7.3 Foliation

In general, a foliation is not unique. So let

\[ M = T \times S \text{ and } M = T \times \mathcal{S}, \quad M \ni p \leftrightarrow (t, s) \leftrightarrow (\bar{t}, \bar{s}) \]

Then there are modes

\[ f_{m(t) \times S} =: f_{mt}(s) \]

and

\[ \bar{f}_{m(\bar{t}) \times \bar{S}} =: \bar{f}_{\bar{mt}}(\bar{s}) \]

Let

\[ \varphi(t, s) = \int_{\mathcal{M}} \mu(dm)c_{m}(t)f_{mt}(s) \]

We have

\[ \bar{\varphi}(\bar{t}, \bar{s}) = \varphi(\bar{t}(t, s), \bar{s}(t, s)) =: \bar{\varphi}(t, s) = \int_{\mathcal{M}} \mu(dm)\bar{c}_{\bar{m}}(t)f_{\bar{mt}}(s) \]

Thus

\[ \bar{f}_{\bar{mt}}(\bar{s}) = \int_{\mathcal{M}} \mu(dm)\bar{c}_{\bar{mmt}}(t)f_{mt}(s) \]

The \( \bar{c}_{\bar{mmt}}(t) \) are functions of \( t \), so that different foliations are not equivalent, and generally a foliation is a free setting element. But as long as a Cauchy surface is specified, the related foliation is a natural setting element.

7.4 Setting for manifold and space modes

The setting for manifold modes is a choice of them and initial conditions for them. The setting is free.

The setting for space modes is a foliation \( M = T \times S \) and initial conditions for \( \hat{A}_{mn} \) and \( \Psi \). As long as a Cauchy surface is specified, the setting is natural.

Acknowledgments

I would like to thank Alex A. Lisyansky for support and Stefan V. Mashkevich for helpful discussions.
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