IDEALS OF RINGS OF DIFFERENTIAL OPERATORS ON ALGEBRAIC CURVES
(WITH AN APPENDIX BY GEORGE WILSON)

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1. Introduction

Let $X$ be a smooth affine irreducible curve over $\mathbb{C}$, and let $D = \mathcal{D}(X)$ be the ring of global differential operators on $X$. In this paper, we give a geometric classification of left ideals in $D$ and study the natural action of the Picard group of $D$ on the space of isomorphism classes of such ideals. Our results generalize the classification of left ideals of the first Weyl algebra $A_1(\mathbb{C})$ in [BW1] and [BW2]; however, our approach is quite different.

As shown in [BW1] [BW2], the ideal classes of $A_1(\mathbb{C})$ are parametrized by finite-dimensional algebraic varieties $\mathcal{C}_n$ called the Calogero-Moser spaces. The starting point for the present paper was the observation of Crawley-Boevey (see [CB]) that the same varieties $\mathcal{C}_n$ parametrize finite-dimensional irreducible representations of certain (infinite-dimensional) algebras associated to graphs. Specifically, the algebras in question are deformed preprojective algebras $\Pi^\lambda(Q)$ (see [CB]); the corresponding graph $Q$ is the framed Dynkin diagram of simplest type $A_n$.

Trying to understand the relation between the ideals of $A_1(\mathbb{C})$ and irreducible representations of $\Pi^\lambda(Q)$, we came up with a new construction of the Calogero-Moser correspondence, which, besides the Weyl algebra, applied to noncommutative deformations of Kleinian singularities corresponding to Dynkin diagrams of other types (see [BCE]). In this paper, we develop a geometric version of this construction in which graphs are replaced by algebraic curves.

We begin with a brief overview of our main results. Let $\mathfrak{I}(D)$ be the set of isomorphism classes of left ideals in $D$. Since $D$ is a Noetherian hereditary domain, every ideal of $D$ is a projective $D$-module of rank 1, so $\mathfrak{I}(D)$ can be equivalently defined as the set of isomorphism classes of such modules. The Grothendieck group $K_0(D)$ of finite rank projective $D$-modules is isomorphic to the (algebraic) $K$-group $K_0(X)$ of $X$, while $K_0(X) \cong \mathbb{Z} \oplus \text{Pic}(X)$, where Pic($X$) is the Picard group of $X$. Combining these isomorphisms, we may assign to each ideal class $[M] \in \mathfrak{I}(D)$ an element of Pic($X$) which determines $[M]$ up to equivalence in $K_0(D)$. In other words, there is a natural map $\gamma : \mathfrak{I}(D) \to \text{Pic}(X)$, whose fibres are precisely the stable isomorphism classes of ideals in $D$. Our problem reduces thus to describing the fibres of $\gamma$.

We approach this problem in two steps. First, we introduce the Calogero-Moser spaces $\mathcal{C}_n(X, I)$ for an arbitrary curve $X$ and a line bundle $I$ on $X$, building on the observation of Crawley-Boevey. For any associative algebra $A$, there is a ‘universal’ construction of deformed preprojective algebras $\Pi^\lambda(B)$ over $B$, with parameters $\lambda \in \mathbb{C} \otimes \mathbb{Z} K_0(B)$ (see [CB] and Section 2.1 below). Using this construction, we define $\mathcal{C}_n(X, I)$ as representation varieties of $\Pi^\lambda(B)$ over a triangular matrix extension of the ring $A = \mathcal{O}(X)$ of regular functions on $X$ by the line bundle $I$. This extension $B = A[Z]$ abstracts the idea of ‘framing’ a quiver by adjoining a distinguished new vertex ‘$\infty$’ and arrows from $\infty$; geometrically, it can be thought of as a noncommutative thickening of $\text{Spec}(A \times \mathbb{C}) = X \coprod \text{pt}$. We note that $\mathcal{C}_n(X, I)$ behaves functorially with respect to $I$; in particular, the quotient spaces $\overline{\mathcal{C}}_n(X, I) := \mathcal{C}_n(X, I) / \text{Aut}_X(I)$ depend only on the class of $I$ in Pic($X$). We write $\overline{\mathcal{C}}_n(X)$ for the disjoint union of $\overline{\mathcal{C}}_n(X, I)$ over Pic($X$).

Our first main result is a generalization to an arbitrary $X$ of a known theorem of G. Wilson (see [W]),
Theorem 1.1 (see Theorem 3.2). For each $n \geq 0$ and $[\mathcal{I}] \in \text{Pic}(X)$, $\mathcal{C}_n(X,\mathcal{I})$ is a smooth affine irreducible variety of dimension $2n$.

Now, in view of functoriality of $\Pi^\lambda$-construction, there is a natural map $\Pi^\lambda(B) \to \Pi^1(A)$ lifting the extension $B \to A$. On the other hand, by a theorem of Crawley-Boevey (see [CB]), $\Pi^1(A)$ can be identified with the ring $\mathcal{D}$ of differential operators on $X$. The resulting algebra homomorphism $i : \Pi^\lambda(B) \to \mathcal{D}$ relates the module categories of $\Pi^\lambda(B)$ and $\mathcal{D}$ in a fairly interesting way. To be precise, we will prove

Theorem 1.2 (see Theorem 4.2). The canonical functors $(i^* , i_* , i^!)$ induced by $i : \Pi^\lambda \to \mathcal{D}$ on the (bounded) derived categories form a recollement set-up in the sense of [BBD]:

$$
\begin{array}{ccc}
\mathcal{D}^b(\text{Mod } \mathcal{D}) & \xrightarrow{i_*} & \mathcal{D}^b(\text{Mod } \Pi^\lambda) \\
\downarrow & & \downarrow \\
\mathcal{D}^b(\text{Mod } U^\lambda) & \xleftarrow{i^!} & \mathcal{D}^b(\text{Mod } \Pi^\lambda) \\
\downarrow & & \downarrow \\
\mathcal{D}^b(\text{Mod } U^\lambda) & \xrightarrow{j_*} & \mathcal{D}^b(\text{Mod } \Pi^\lambda) \\
\end{array}
$$

where $U^\lambda$ is a certain (spherical) subalgebra of $\Pi^\lambda(B)$.

Originally, the recollement conditions were introduced in [BBD] to formalize a natural structure on the derived category $\mathcal{D}(\text{Sh}_X)$ of abelian sheaves arising from the stratification of a topological space into a closed subspace and its open complement. In an algebraic setting similar to ours, these conditions were first studied in [CPS].

The functor $i_*$ yields a fully faithful embedding of $\mathcal{D}^b(\text{Mod } \mathcal{D})$ into $\mathcal{D}^b(\text{Mod } \Pi^\lambda)$ as a ‘closed stratum’, while the induction functor $i^* : \mathcal{D}^b(\text{Mod } \Pi^\lambda) \to \mathcal{D}^b(\text{Mod } \mathcal{D})$ is an algebraic substitute for the restriction of a sheaf to that stratum. This last functor plays a key role in our construction: it transforms irreducible $\Pi^\lambda(B)$-modules (viewed as 0-complexes in $\mathcal{D}^b(\text{Mod } \Pi^\lambda)$) to projective $\mathcal{D}$-modules (located in homological degree $-1$), inducing natural maps

$$
\omega_n : \overline{\mathcal{C}}_n(X,\mathcal{I}) \to \gamma^{-1}[\mathcal{I}].
$$

The main result of this paper can now be encapsulated in the following theorem.

Theorem 1.3 (see Theorem 1.3). Let $X$ be a smooth affine irreducible curve over $\mathbb{C}$.

(a) For each $[\mathcal{I}] \in \text{Pic}(X)$, amalgamating the maps $\omega_n$ for all $n \geq 0$ yields a bijective correspondence

$$
\omega : \bigsqcup_{n \geq 0} \overline{\mathcal{C}}_n(X,\mathcal{I}) \sim \gamma^{-1}[\mathcal{I}].
$$

(b) There is a natural action on $\overline{\mathcal{C}}_n(X)$ of the Picard group Pic($\mathcal{D}$) of the category of $\mathcal{D}$-modules, and the maps $\omega_n : \overline{\mathcal{C}}_n(X) \to \mathcal{J}(\mathcal{D})$ are equivariant under this action for all $n \geq 0$.

Part (a) of Theorem 1.3 gives a geometric description of the fibration $\gamma$ over a given $[\mathcal{I}] \in \text{Pic}(X)$. In the special case when $X$ is the affine line, $\text{Pic}(X)$ is trivial: there is only one fibre, and it is shown in [BCE] that $\omega$ agrees with the Calogero-Moser map constructed in [BWI] [BW2].

Part (b) generalizes another aspect of the Calogero-Moser correspondence for the Weyl algebra: the equivariance of the Calogero-Moser map under the action of the automorphism group $\text{Aut}_C(A_1)$. The importance of this result is that it allows one to classify the algebras Morita equivalent to $\mathcal{D}$ up to isomorphism. Precisely, Theorem 1.3(b) implies that the isomorphism classes of domains $\mathcal{D}'$ Morita equivalent to $\mathcal{D}$ are in one-to-one correspondence with the orbits of $\text{Pic}(\mathcal{D})$ on the Calogero-Moser spaces $\overline{\mathcal{C}}_n(X)$. For example, for $n = 0$, we have $\overline{\mathcal{C}}_0(X) = \text{Pic}(X)$, and the action of $\text{Pic}(\mathcal{D})$ is transitive on $\text{Pic}(X)$ (see Proposition 4.4 below); this implies a theorem of Cannings and Holland (CH1, Theorem 1.10) that $\mathcal{D}' \cong \mathcal{D}$ if and only if $\mathcal{D}' \cong \text{End}_\mathcal{D}(\mathcal{I}D)$ for some line bundle $\mathcal{I}$. For an arbitrary $n > 0$, the structure of orbits of $\text{Pic}(\mathcal{D})$ in $\overline{\mathcal{C}}_n(X)$ is complicated; however, one can still define a complete set of isomorphism invariants for the algebras $\mathcal{D}'$ in terms of Hochschild homology of $\Pi^\lambda(B)$. We will discuss this construction elsewhere.

We should now explain how our results relate to earlier work.

2By a theorem of Stafford [53], the group $\text{Aut}_C(A_1)$ is known to be isomorphic to $\text{Pic}(A_1)$.
The problem of classifying ideals of $D(X)$ for a smooth affine curve $X$ was first addressed by Cannings and Holland (see [CH], [CH1]) who identified the space $\mathcal{I}(D)$ with a certain infinite-dimensional Grassmannian. In the special case when $X = \mathbb{A}^1$, this Grassmannian was introduced independently (and for a different purpose) by Wilson (see [W1]), who called it the *adelic Grassmannian* $\text{Gr}^{ad}$. Motivated by earlier work on integrable systems [AMM, CC, KT, KKS], Wilson showed (see [W]) that $\text{Gr}^{ad}$ can be decomposed into a countable union of smooth varieties $\mathcal{C}_n$, which are now called the Calogero-Moser spaces. It is important to understand that the Calogero-Moser decomposition is entirely different from the obvious stratification of $\text{Gr}^{ad}$ by finite-dimensional Grassmannians considered in [CH1]. Its relevance for the Weyl algebra $D(\mathbb{A}^1) = A_1(\mathbb{C})$ became clear in [BW1], where it was shown that, under the Cannings-Holland bijection, the spaces $\mathcal{C}_n$ correspond to the orbits of the natural action of the Dixmier group $\text{Aut}_\mathbb{C}(A_1)$ on $\mathcal{I}(A_1)$. A different approach to the problem of classifying ideals of $A_1$, which does not use $\text{Gr}^{ad}$, was developed in [BW2]. The main idea of [BW2] — to use noncommutative projective geometry (specifically, a noncommutative version of Beilinson’s equivalence) — was inspired by [La2] and [KKO] and was later generalized to many other classes of quantum algebras (see [BGK1], [BGK2], [NB], [NS], [BN1], [BN2] and references therein). While the present paper was in preparation, a new very interesting paper [BN] by Ben-Zvi and Nevins has appeared. In [BN], the authors use a noncommutative Beilinson equivalence to classify torsion-free $D$-modules on projective curves. Although this last problem is similar to (in fact, somewhat more general than) the one addressed in the present paper, our methods and results are different. Apart from describing explicitly the space $\mathcal{I}(D)$ of ideals, we also describe the action of the Picard group on $\mathcal{I}(D)$ and prove the equivariance of the Calogero-Moser correspondence. Comparing our constructions to those of [BN] is an interesting problem, which will be hopefully clarified elsewhere.

We should also mention that the methods of the present paper apply to a more general class of formally smooth algebras, including the path algebras of quivers. Some of these versions of the Calogero-Moser correspondence will be a subject of a forthcoming work. Finally, in the existing literature, there are (at least) two other definitions of Calogero-Moser spaces associated to curves. The first one, due to V. Ginzburg, employs the classical Hamiltonian reduction (see [FG], or [BN], Def. 1.2) and is, in fact, closely related to ours (see Remark in the end of Section 3.3). The second, due to P. Etingof (see [E], Example 2.19), is given in terms of generalized Cherednik algebras (in the style of [EG]). We will discuss the relation of Etingof’s definition to ours in [BC].

We now proceed with a summary of the contents of the paper.

Section 2 is preliminary: it introduces notation and reviews the material needed for the rest of the paper. While most results in this section are known, some are (apparently) new. In particular, Theorem 2.2 and Proposition 2.2 did not appear in the literature in this form and generality.

In Section 3 after recollections on differential operators (Section 3.1) and $K$-theoretic classification of ideals of $D$ (Section 3.2), we define the Calogero-Moser spaces $\mathcal{C}_n(X, \mathcal{I})$ and establish their basic properties, including Theorem 1.1.

The main results of the paper are gathered in Section 4. First, in Section 4.1, we explain the relation between the algebras $\Pi^\mathbb{A}(B)$ and $D$, including Theorem 1.2. Then, in Sections 4.2 and 4.3, we describe the action of the Picard group $\text{Pic}(D)$ on the Calogero-Moser spaces $\mathcal{C}_n(X)$ and state our main Theorem 1.3.

The proof of Theorem 1.3 occupies the whole of Section 5. We refer the reader to the introduction of that section for a summary of the proof.

In Section 5, we give an alternative description of the map $\omega$ and consider a number of explicit examples. Perhaps, the most interesting example is that of a general plane curve (see Section 6.2.3). In this case, the varieties $\mathcal{C}_n(X, \mathcal{I})$ can be described in terms of matrices, generalizing the classical Calogero-Moser matrices, and the map $\omega$ is given by an explicit formula involving characteristic polynomials of these matrices (see 6.2.3, 6.2.3). This last formula can be viewed as a generalization of Wilson’s formula for the rational Baker function of the KP integrable hierarchy (see [W1]); however, our method of derivation is different from that of [W1]: it extends our earlier calculations in [BC] in the case of the Weyl algebra.

The last section of the paper is an appendix written by G. Wilson. It clarifies the relation between deformed preprojective algebras and rings of differential operators on curves, which, strictly speaking, we did not use in this paper but probably should have. As explained above, our map $\omega$ is naturally induced
by the algebra extension $i : \Pi^1(B) \to D$. Unfortunately, this extension is not entirely canonical: it depends on the choice of an identification of $\Pi^1(A)$ with $D = D(X)$. By a theorem of Crawley-Boevey (see [CB], Theorem 4.7), $\Pi^1(A)$ is indeed isomorphic to $D$ as a filtered algebra, but, in general, there seems to be no natural isomorphism between these algebras. To remedy this problem, one should replace $D$ by the ring $D(\Omega^L_X)$ of twisted differential operators on half-forms on $X$. As was first observed by V. Ginzburg (see [G], Sect. 13.4), $\Pi^1(A)$ is canonically isomorphic to $D(\Omega^L_X)$; however, the construction of the isomorphism depends on a fact (Proposition A.1 below) whose proof in [G] is very sketchy. A complete proof can be found in the Appendix, which may be read independently of the rest of the paper.

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Notation and Conventions

Throughout this paper, we work over the base field $\mathbb{C}$. Unless otherwise specified, an algebra means an associative algebra over $\mathbb{C}$, a module over an algebra $A$ means a left module over $A$, and $\text{Mod}(A)$ denotes the category of such modules. All bimodules over algebras are assumed to be symmetric over $\mathbb{C}$, and we use the abbreviation $\otimes$ for $\otimes_{\mathbb{C}}$ whenever it is convenient.

2. Preliminaries

2.1. Deformed preprojective algebras. If $A$ is an algebra, its tensor square $A \otimes^L A$ has two commuting bimodule structures: one is defined by $a.(x \otimes y).b = ax \otimes yb$ and the other by $a.(x \otimes y).b = xb \otimes ay$, where $a, b \in A$. We will refer to these structures as outer and inner, respectively. Any bimodule over $A$ can be viewed as either left or right module over the enveloping algebra $A^e := A \otimes A^e$; if we interpret the outer bimodule structure on $A \otimes^L A$ as a left $A^e$-module structure and the inner as a right one, then the canonical map $A \otimes^L A \to A^e$ is an isomorphism of $A^e$-bimodules. We will often use this isomorphism to identify $A \otimes^L A \cong A^e$.

Following [CGG], we let $\text{Der}(A) := \text{Der}(A, A \otimes A)$ denote the space of linear derivations $A \to A \otimes A$ taken with respect to the outer bimodule structure on $A \otimes^L A$. This space is a bimodule with respect to the inner structure, so we can form the tensor algebra $T_A \text{Der}(A)$. Now, in $\text{Der}(A)$, there is a canonical derivation $\Delta = \Delta_A$, sending $x \in A$ to $(x \otimes 1 - 1 \otimes x) \in A \otimes^L A$. For any $\lambda \in A$, we can consider then the 2-sided ideal $(\lambda - \Delta)$ in $T_A \text{Der}(A)$ and define $\Pi^\lambda(A) := (T_A \text{Der}(A)/(\lambda - \Delta))$. It turns out that, up to isomorphism, the algebra $\Pi^\lambda(A)$ depends only on the class of $\lambda$ in the Hochschild homology $H_0(A) := A/[A,A]$ (see [BR], Lemma 1.2). Moreover, instead of elements of $H_0(A)$, it is convenient to parametrize $\Pi^\lambda(A)$ by the elements of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$, relating this last vector space to $H_0(A)$ via a Chern character map. To be precise, let $\text{Tr}_A : K_0(A) \to H_0(A)$ be the map, sending the class of a projective module $P$ to the class of the trace of any idempotent $e \in \text{Mat}(n,A)$, satisfying $P \cong A^n e$. By additivity, this extends to a linear map $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A) \to H_0(A)$ to be denoted also $\text{Tr}_A$. Following [CB], we call the elements of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$ weights and define the deformed preprojective algebra of weight $\lambda \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$ by

\begin{equation}
\Pi^\lambda(A) := T_A \text{Der}(A)/(\lambda - \Delta),
\end{equation}

where $\lambda \in A$ is any lifting of $\text{Tr}_A(\lambda)$ to $A$. Note, if $A$ is commutative, then $H_0(A) = A$, and $\lambda$ is uniquely determined by $\text{Tr}_A(\lambda)$.

The algebras $\Pi^\lambda(A)$ are usually ill-behaved unless one imposes some ‘smoothness’ conditions on $A$. In this paper, following [KR], we say that an algebra $A$ is smooth if it is quasi-free and finitely generated; technically, this implies that $\Omega^1 A$ — the kernel of the multiplication map of $A$ — is a flat projective bimodule. For basic properties and examples of the algebras $\Pi^\lambda(A)$ the reader is referred to [CB]. Here, we state only one important theorem from [CB], which will play a role in our construction. We recall that a ring homomorphism $i : B \to A$ is called pseudo-flat if $\text{Tor}_1^B(A, A) = 0$. We also recall that any
ring homomorphism $i : B \to A$ induces a homomorphism of abelian groups $i^* : K_0(B) \to K_0(A)$, which extends (by linearity) to a map of $\mathbb{C}$-vector spaces $i^* : \mathbb{C} \otimes_2 K_0(B) \to \mathbb{C} \otimes_2 K_0(A)$.

**Theorem 2.1** ([CEG], Theorem 9.3 and Corollary 9.4). Let $i : B \to A$ be a pseudo-flat ring epimorphism. Then, for any $\lambda \in \mathbb{C} \otimes_2 K_0(B)$, there is a canonical algebra map $i : \Pi^\lambda(B) \to \Pi^\lambda(A)$, where $X = i^*(\lambda)$. If $B$ is smooth, then $i$ is also a pseudo-flat epimorphism, and the diagram

$$
\begin{array}{ccc}
B & \longrightarrow & A \\
\| & & \| \\
\Pi^\lambda(B) & \longrightarrow & \Pi^\lambda(A)
\end{array}
$$

is a push-out in the category of rings.

We now prove a few general results on representations of deformed preprojective algebras, which may be of independent interest. Our first lemma is probably well known to the experts (see, for example, [CEG]); we record it to fix the notation.

**Lemma 2.1.** If $A$ is smooth, then $\Delta$ lies in the commutator space $[A, \text{Der}(A)]$ of the bimodule $\text{Der}(A)$.

**Proof.** Composing the multiplication map $\mu : A \otimes^2 A \to A$ with derivations $A \to A \otimes^2 A$ yields a linear map $\mu_* : \text{Der}(A) \to \text{Der}(A)$, with $\Delta \in \text{Ker}(\mu_*)$. This map factors through the natural projection $\text{Der}(A) \to \text{Der}(A)/\text{Ker}(\mu_*).$ If $A$ is smooth, the induced map $\mu_* : \text{Der}(A) \to \text{Der}(A)$ is an isomorphism. Indeed, identifying $A \otimes^2 A \cong A^e$ and writing $\Omega^1 A \subseteq A^e$ for $\text{Ker}(\mu)$, we have $\text{Der}(A) \cong \text{Hom}_{A^e}(\Omega^1 A, A)$ and $\text{Der}(A) \cong \text{Hom}_{A^e}(\Omega^1 A, A^e)$. Under the last isomorphism, the bimodule structure on $\text{Der}(A)$ corresponds to the natural right $A^e$-module structure on $(\Omega^1 A)^* := \text{Hom}_{A^e}(\Omega^1 A, A^e)$ and $\text{Der}(A) \cong (\Omega^1 A)^* \otimes_{A^e} A$. The quotient map $\mu_*$ now becomes $(\Omega^1 A)^* \otimes_{A^e} A \to \text{Hom}_{A^e}(\Omega^1 A, A)$. Since $A$ is smooth, $\Omega^1 A$ is a f. g. projective $A^e$-module, so the last map is an isomorphism. This implies that $\text{Ker}(\mu_*) = [A, \text{Der}(A)]$, and hence $\Delta \in [A, \text{Der}(A)].$  

For any $\lambda \in A$, the algebra $\Pi^\lambda(A)$ is an $A$-ring: it is equipped with a canonical algebra homomorphism $A \to \Pi^\lambda(A)$. Every representation of $\Pi^\lambda(A)$ can thus be regarded as a representation of $A$. Conversely, given a representation of $A$, one can ask whether it lifts to a representation of $\Pi^\lambda(A)$. The following theorem provides a simple homological criterion for the existence and uniqueness of such liftings.

**Theorem 2.2.** Let $A$ be a smooth algebra, and let $\varrho : A \to \text{End}(V)$ be a representation of $A$ on a (not necessarily finite-dimensional) vector space $V$. Then $\varrho$ can be extended to a representation of $\Pi^\lambda(A)$ if and only if the homology class of $\varrho(\lambda)$ in $H_0(A, \text{End}(V))$ is zero, i.e. $\varrho(\lambda) \in [\varrho(A), \text{End}(V)]$. If it exists, an extension of $\varrho$ to $\Pi^\lambda(A)$ is unique if and only if $H_1(A, \text{End}(V)) = 0$.

**Proof.** We will use the notation of Lemma 2.1. Thus, for a fixed $\lambda \in A$, we identify $\Pi^\lambda(A) = T_A(\Omega^1 A)^*/(\Delta_A - \lambda)$, with $\Delta_A \in (\Omega^1 A)^*$ corresponding to the natural inclusion $\Omega^1 A \hookrightarrow A^e$. A representation $\varrho : A \to \text{End}(V)$ can be extended then to a representation of $\Pi^\lambda(A)$ if and only if there is an $A$-ring map $\tilde{\varrho} : T_A(\Omega^1 A)^* \to \text{End}(V)$, such that $\varrho(\Delta_A) = \varrho(\lambda)$. By the universal property of tensor algebras, such a map is uniquely determined by its restriction to $(\Omega^1 A)^*$. Thus, regarding $\text{End}(V)$ as a bimodule over $A$ via $\varrho$, we conclude that $\varrho$ lifts to $\Pi^\lambda(A)$ iff there is $\tilde{\varrho} \in \text{Hom}_{A^e}((\Omega^1 A)^*, \text{End}(V))$, mapping $\Delta_A$ to $\varrho(\lambda)$. Here, the bimodule $\text{End}(V)$ is interpreted as a right $A^e$-module.

Now, since $A$ is smooth, the canonical map $\Omega^1 A \to (\Omega^1 A)^*$ is an isomorphism, and we can identify $\text{Hom}_{A^e}((\Omega^1 A)^*, \text{End}(V)) \cong \text{End}(V) \otimes_{A^e} \Omega^1 A$. Under this identification, the condition $\varrho(\Delta_A) = \varrho(\lambda)$ becomes

$$
\exists f_i \otimes d_i \in \text{End}(V) \otimes_{A^e} \Omega^1 A : \sum f_i \Delta_A(d_i) = \varrho(\lambda)
$$

Tensoring the exact sequence of $A^e$-modules $0 \to \Omega^1 A \to A^e \to A \to 0$ with $\text{End}(V)$, we get

$$
0 \to H_1(A, \text{End}(V)) \to \text{End}(V) \otimes_{A^e} \Omega^1 A \xrightarrow{\partial} \text{End}(V) \xrightarrow{\eta} H_0(A, \text{End}(V)) \to 0,
$$
with map in the middle given by $\partial : f \otimes d \mapsto f \Delta_A(d)$, and $p$ being the canonical projection. The condition (2.3) now says that $\varrho(\lambda) \in \text{Im}(\partial)$, and, by exactness of (2.4), this is equivalent to $\varrho(\lambda) \in \text{Ker}(p)$.

Thus, $\varrho$ can be extended to $\Pi^\lambda(A)$ if and only $\varrho(\lambda)$ vanishes in $H_0(A, \text{End}(V))$.

The fibre of $\partial$ over $\varrho(\lambda) \in \text{End}(V)$ consists of different liftings of the given action $\varrho$ to $\Pi^\lambda(A)$. Again, by exactness of (2.4), this fibre can be identified with $H_1(A, \text{End}(V))$. In particular, if $\varrho$ admits an extension to $\Pi^\lambda(A)$, this extension is unique if and only if $H_1(A, \text{End}(V)) = 0$. □

As an immediate corollary of Theorem 2.2, we get

**Corollary 2.1.** If $\lambda \in \mathbb{C} \otimes_{\mathbb{Z}} K_0(A)$, then $\varrho : A \to \text{End}(V)$ can be extended to $\Pi^\lambda(A)$ if and only if $\varrho \circ \text{Tr}_A(\lambda) = 0$, where $\varrho_0 : H_0(A) \to H_0(A, \text{End}(V))$ is the map induced by $\varrho$ on Hochschild homology.

We now apply Theorem 2.2 to finite-dimensional representations. The next result is a generalization of [CB2, Theorem 3.3], which deals with path algebras of quivers.

**Proposition 2.1.** Let $A$ be a smooth algebra, and let $\varrho : A \to \text{End}(V)$ be a representation of $A$ on a finite-dimensional vector space $V$. Then $\varrho$ lifts to a representation of $\Pi^\lambda(A)$ if and only if the trace of $\varrho(\lambda)$ on any $A$-module direct summand of $V$ is zero. Moreover, if $\varrho \in \text{Rep}(A, V)$ lifts, then the fibre $\pi^{-1}(\varrho)$ of the canonical map $\pi : \text{Rep}(\Pi^\lambda(A), V) \to \text{Rep}(A, V)$ is isomorphic to $\text{Ext}^1_A(V, V)^\ast$.

**Proof.** The trace pairing on $\text{End}(V)$ yields a linear isomorphism $\text{End}(V) \cong \text{End}(V)^\ast$, which is a bimodule map with respect to the natural bimodule structures on $\text{End}(V)$ and $\text{End}(V)^\ast$. This isomorphism restricts to $\text{End}_A(V) \cong H_0(A, \text{End}(V)^\ast)$, which, upon dualizing with $\mathbb{C}$, becomes

$$H_0(A, \text{End}(V)) \cong \text{End}_A(V)^\ast, \quad f \mapsto [e \mapsto \text{Tr}_V(ef)].$$

Now, let $\varrho : A \to \text{End}(V)$ be a representation of $A$ on $V$ that lifts to $\Pi^\lambda(A)$, and suppose that $V$ has a direct $A$-linear summand, say $W$. By Theorem 2.2 the class of $\varrho(\lambda)$ in $H_0(A, \text{End}(V))$ is zero, and hence so is its image under (2.5). Taking $e \in \text{End}_A(V)$ to be a projection onto $W$, we get $\text{Tr}_V[\varrho(\lambda)] = \text{Tr}_V[\varrho(\lambda)] = 0$, which proves the first implication of the theorem.

For the converse, it suffices to consider only indecomposable representations $\varrho : A \to \text{End}(V)$. By Fitting’s Lemma, $\text{End}_A(V)$ is then a local ring: every $e \in \text{End}_A(V)$ can be written as $e = c \text{Id}_V + i$, with $c \in \mathbb{C}$ and $i$ being nilpotent. Now, if we assume that $\text{Tr}_V[\varrho(\lambda)] = 0$, then $\text{Tr}_V[e \varrho(\lambda)] = 0$ for any $e \in \text{End}_A(V)$. The class of $\varrho(\lambda)$ in $H_0(A, \text{End}(V))$ lies thus in the kernel of (2.5) and hence is zero. By Theorem 2.2 we conclude that $\varrho$ lifts to a representation of $\Pi^\lambda(A)$.

For the last statement, note that $\pi^{-1}(\varrho) \cong H_1(A, \text{End}(V))$ by exactness of (2.4). On the other hand, we have

$$H_1(A, \text{End}(V)) \cong \text{Tor}_1^A(V^\ast, V) \cong \text{Ext}^1_A(V, V)^\ast,$$

which is standard homological algebra (see [CE], Cor. 4, p. 170, and Prop. VI, 5.3, respectively). □

**Remark.** In the special case, when $A$ is the path algebra of a quiver, Proposition 2.1 was proven earlier, by a different method, in [CB2]. With identifications (2.6) and (2.7), our basic sequence (2.4) becomes

$$0 \to \text{Ext}_A^1(V, V)^\ast \to \text{End}(V) \otimes_{A^e} \Omega^1A \to \text{End}(V) \to \text{End}_A(V)^\ast \to 0,$$

which, in the quiver case, agrees with [CB2, Lemma 3.1].

### 2.2. One-point extensions

If $A$ is a unital associative algebra, and $I$ a left module over $A$, we define the **one-point extension** of $A$ by $I$ to be the ring of triangular matrices

$$A[I] := \begin{pmatrix} A & I \\ 0 & \mathbb{C} \end{pmatrix},$$

with matrix addition and multiplication induced from the module structure of $I$. Clearly, $A[I]$ is a unital associative algebra, with identity element being the identity matrix. There are two distinguished idempotents in $A[I]$: namely

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
If $A$ is indecomposable (e.g., $A$ is a commutative integral domain), then [2.3] form a complete set of primitive orthogonal idempotents in $A[I]$.

A module over $A[I]$ can be identified with a triple $V = (V, V_\infty, \varphi)$, where $V$ is an $A$-module, $V_\infty$ is a $\mathbb{C}$-vector space and $\varphi : I \otimes V_\infty \to V$ is an $A$-module map. Using the standard matrix notation, we will write the elements of $V$ as column vectors $(v, w)^T$ with $v \in V$ and $w \in V_\infty$; the action of $A[I]$ is then given by

\[
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix} = \begin{pmatrix} a.v + \varphi(b \otimes w) \\
c w
\end{pmatrix}.
\]

Now, if $U = (U, U_\infty, \varphi_U)$ and $V = (V, V_\infty, \varphi_V)$ are two $A[I]$-modules, a homomorphism $U \to V$ is determined by a pair of maps $(f, f_\infty)$, with $f \in \text{Hom}_A(U, V)$ and $f_\infty \in \text{Hom}_\mathbb{C}(U_\infty, V_\infty)$, making the following diagram commutative

\[
\begin{array}{ccc}
I \otimes U_\infty & \overset{\varphi_U}{\longrightarrow} & U \\
\downarrow \text{Id} \otimes f_\infty & & \downarrow f \\
I \otimes V_\infty & \overset{\varphi_V}{\longrightarrow} & V
\end{array}
\]

(2.10)

If $V$ is finite-dimensional, with $\dim_\mathbb{C} V = n$ and $\dim_\mathbb{C} V_\infty = n_\infty$, we call $n = (n, n_\infty)$ the dimension vector of $V$.

The next lemma gathers together basic properties of one-point extensions.

**Lemma 2.2.** (1) $A[I]$ is canonically isomorphic to $T_A(I)$, where $\tilde{A} := A \times \mathbb{C}$.

(2) If $A$ is smooth and $I$ is a f. g. projective $A$-module, then $A[I]$ is smooth.

(3) $I \to A[I]$ is a functor from $\text{Mod}(A)$ to the category of associative algebras.

(4) The natural projection $i : A[I] \to A$ is a flat ring epimorphism.

(5) There is an isomorphism of abelian groups $K_0(A[I]) \cong K_0(A) \oplus \mathbb{Z}$.

**Proof.** (1) We identify $\tilde{A}$ with the subalgebra of diagonal matrices in $A[I]$ and $I$ with the complementary nilpotent ideal $\tilde{I} \subset A[I]$:

\[
\tilde{A} = \begin{pmatrix} A & 0 \\
0 & \mathbb{C}
\end{pmatrix}, \quad \tilde{I} := \begin{pmatrix} 0 & I \\
0 & 0
\end{pmatrix}.
\]

By the universal property of tensor algebras, the inclusions $\tilde{A} \hookrightarrow A[I]$ and $\tilde{I} \hookrightarrow A[I]$ can then be extended to an algebra map $\phi : T_A(I) \to A[I]$, which is a required isomorphism.

(2) By (1) and [CQ], Prop. 5.3, it suffices to show that $\tilde{I}$ is a projective $\tilde{A}$-bimodule. But if $I$ is a projective $A$-module, then it is isomorphic to a direct summand of a free module $A \otimes V$ and $\tilde{I}$ is isomorphic to a direct summand of $\tilde{A} \otimes V \otimes \epsilon_\infty \tilde{A}$. The latter is a projective $\tilde{A}$-bimodule, since it is a direct summand of $\tilde{A} \otimes V \otimes \tilde{A}$.

(3) Any $A$-module map $f : I_1 \to I_2$ gives rise to an $\tilde{A}$-bimodule map $\tilde{f} : \tilde{I}_1 \to \tilde{I}_2$. Identifying $A[I_1] = T_A(I_1)$ and $A[I_2] = T_A(I_2)$, we may extend $I \to A[I]$ to morphisms by $A[f] := T_A(\tilde{f})$. As $T_A$ is a functor on bimodules, the result follows.

(4) The map $i$ is given by

\[
i : A[I] \to A, \quad \begin{pmatrix} a & b \\
0 & c
\end{pmatrix} \mapsto a.
\]

It is immediate from (2.12) that $A \cong A[I] e$ as a left $A[I]$-module via $i$. Since $e$ is an idempotent, $A[I] e$ is projective and hence flat.

(5) The diagonal projection $\tilde{i} : A[I] \to \tilde{A}$ has a nilpotent kernel (equal to $\tilde{I}$). By [Ba], Prop. IX.1.3, it then induces isomorphisms $K_i(A[I]) \cong K_i(\tilde{A})$ for all $i$. In particular, $K_0(A[I]) \cong K_0(A) \oplus \mathbb{Z}$. \qed

We will also need the next lemma relating homological properties of $A$ and $A[I]$. 

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Lemma 2.3. Let $A$ be a finitely generated hereditary algebra, and let $B := A[I]$ be the one-point extension of $A$ by a f. g. projective $A$-module. Then, for any finite-dimensional $B$-modules $U = (U,U_\infty)$ and $V = (V,V_\infty)$, we have

$$\chi_B(U,V) = \chi_A(U,V) + \dim(U_\infty) \left[ \dim(V_\infty) - \dim \text{Hom}_A(I,V) \right],$$

where $\chi_A$ and $\chi_B$ denote the Euler characteristics for the Ext-groups over the algebras $A$ and $B$ respectively.

Proof. By [Be], Théorème 1.1 (bis), there is a 5-term exact sequence

$$0 \to \text{Hom}_B(U,V) \to \text{Hom}_A(U,V) \oplus \text{Hom}_C(U_\infty,V_\infty) \to$$

$$\to \text{Hom}_C(U_\infty, \text{Hom}_A(I,V)) \to \text{Ext}^1_B(U,V) \to \text{Ext}^1_A(U,V) \to 0,$$

and isomorphisms $\text{Ext}^k_B(U,V) = \text{Ext}^k_A(U,V) = 0$ for all $k \geq 2$, since $A$ is hereditary. Now, since $A$ is finitely generated, $\text{Hom}_A(U,V)$ and $\text{Ext}^1_A(U,V)$ are finite-dimensional whenever $U$ and $V$ are finite-dimensional. It follows from (2.14) that $\chi_B(U,V)$ is well defined and related to $\chi_A(U,V)$ by (2.13). \(\square\)

### 2.3. Representation varieties.

We recall the definition of representation varieties in the form they appear in representation theory of associative algebras (see [K], Chap. II, Sect. 2.7).

Let $R$ be a finitely generated algebra. Fix $S$, a finite-dimensional semisimple subalgebra of $R$, and $V$, a finite-dimensional $S$-module. By definition, the representation variety $\text{Rep}_S(R,V)$ of $R$ over $S$ parametrizes all $R$-module structures on the vector space $V$ extending the given $S$-module structure on it. The $S$-module structure on $V$ determines an algebra homomorphism $S \to \text{End}(V)$ making $\text{End}(V)$ an $S$-algebra. A point of $\text{Rep}_S(R,V)$ can thus be interpreted as an $S$-algebra map $\varrho : R \to \text{End}(V)$, and the tangent vectors at $\varrho$ can be identified with $S$-linear derivations $R \to \text{End}(V)$, i. e.

$$T_{\varrho} \text{Rep}_S(R,V) \cong \text{Der}_S(R, \text{End}(V)).$$

If $S = \mathbb{C}$, we simply write $\text{Rep}(R,V)$ for $\text{Rep}_\mathbb{C}(R,V)$. Clearly, $\text{Rep}(R,V)$ is an affine variety. For any semisimple $S \subseteq R$, $\text{Rep}_S(R,V)$ can then be identified with a fibre of the canonical morphism $\text{Rep}(R,V) \to \text{Rep}(S,V)$, and hence it is an affine variety as well.

The group $\text{Aut}_S(V)$ of $S$-linear automorphisms of $V$ acts on $\text{Rep}_S(R,V)$ in the natural way, with scalars $\mathbb{C}^* \subseteq \text{Aut}_S(V)$ acting trivially. We set $G_S(V) := \text{Aut}_S(V)/\mathbb{C}^*$. The orbits of $G_S(V)$ on $\text{Rep}_S(R,V)$ are in 1-1 correspondence with isomorphism classes of $R$-modules, which are isomorphic to $V$ as $S$-modules. The stabilizer of a point $\varrho : R \to \text{End}(V)$ in $\text{Rep}_S(R,V)$ is canonically isomorphic to $\text{Aut}_R(V_\varrho)/\mathbb{C}^* \subseteq G_S(V)$, where $V_\varrho$ is the left $R$-module corresponding to $\varrho$. The space $\text{Rep}_S(R,V)//G_S(V)$ of closed orbits in $\text{Rep}_S(R,V)$ is an affine variety, whose points are in bijection with isomorphism classes of semisimple $R$-modules $M$ isomorphic to $V$ as $S$-modules.

Typically, representation varieties of $R$ are defined over subalgebras $S = \bigoplus_{i \in I} \mathbb{C} e_i \subseteq R$ spanned by idempotents. A finite-dimensional $S$-module is then isomorphic to a direct sum $\mathbb{C}^n := \bigoplus_{i \in I} \mathbb{C}^{e_i}$, each $e_i$ acting as the projection onto the $i$-th component. The variety $\text{Rep}_S(R,\mathbb{C}^n)$, which we simply denote by $\text{Rep}_S(R,n)$ in this case, parametrizes all algebra maps $R \to \text{End}(\mathbb{C}^n)$, sending $e_i$ to the projection onto $\mathbb{C}^{e_i}$. The group $G_S(\mathbb{C}^n)$ (to be denoted $G_S(n)$) is isomorphic to $\prod_{i \in I} \text{GL}(n_i,\mathbb{C})/\mathbb{C}^*$, with $\mathbb{C}^*$ embedded diagonally.

We will need a few general results on geometry of representation varieties. First, we recall the following well-known fact (see, for example, [G], Proposition 19.1.4).

**Theorem 2.3.** If $R$ is a smooth algebra, then $\text{Rep}(R,V)$ is a smooth variety. More generally, let $S$ be a semisimple subalgebra of $R$, and let $\varrho \in \text{Rep}_S(R,V) \subseteq \text{Rep}(R,V)$. If $\text{Rep}(R,V)$ is smooth at $\varrho$, then so is $\text{Rep}_S(R,V)$.

Now, given an algebra $A$, we set $R := T_A \text{Der}(A)$, see Section 2.1. If $A$ is finitely generated, then so is $R$, and we consider the variety $\text{Rep}(R,V)$ of representations of $R$ on a vector space $V$. The following result is proved in [CEG], Section 5 (see also [vdB]).

3Here, by an affine variety we mean an affine scheme of finite type over $\mathbb{C}$.  

Theorem 2.4. If $A$ is smooth, $\text{Rep}(R, V)$ is canonically isomorphic to the cotangent bundle of $\text{Rep}(A, V)$.

In particular, $\text{Rep}(R, V)$ is smooth.

Recall that $R$ contains a distinguished element: the derivation $\Delta_A : A \to A \otimes 2$ defined by $x \mapsto x \otimes 1 - 1 \otimes x$. We write

$$\mu : \text{Rep}(R, V) \to \text{End}(V), \quad \varrho \mapsto \varrho(\Delta_A),$$

for the evaluation map at $\Delta_A$ and consider its fibre $F_\xi := \mu^{-1}[\mu(\xi)]$ for some fixed representation $\xi \in \text{Rep}(R, V)$.

Proposition 2.2. If $A$ is smooth, then $F_\xi$ is smooth at $\varrho \in \text{Rep}(R, V)$ if and only if $\text{End}_R(V_\varrho) \cong \mathbb{C}$.

Proof. By Theorem 2.4, the variety $\text{Rep}(R, V)$ is smooth. By Lemma 2.1 we also have that $\Delta_A \in [A, \text{Der}(A)]$, and therefore $\text{tr}_V(\varrho(\Delta)) = 0$ for any $\varrho \in \text{Rep}(R, V)$. It follows that

$$(2.16) \quad \mu : \text{Rep}(R, V) \to \text{End}(V)_0,$$

where $\text{End}(V)_0 := \{f \in \text{End}(V) : \text{tr}_V(f) = 0\}.$

To compute the differential of (2.16) we use (2.15) and also $T_\mu \text{End}(V)_0 \cong \text{End}(V)_0$. With these identifications, it is easy to see that

$$(2.17) \quad d\mu_\varrho : \text{Der}(R, \text{End} V) \to \text{End}(V)_0, \quad \delta \mapsto \delta(\Delta_A).$$

Now, observe that the map $d\mu_\varrho^*$ dual to (2.17) fits into the commutative diagram

$$\begin{array}{ccc}
\text{End}(V)_0 & \xrightarrow{d\mu_\varrho^*} & \text{Der}(R, \text{End} V)^* \\
\text{tr}_V \downarrow & & \downarrow \text{i(Tr } \hat{\omega} \text{)} \\
\text{End}(V)/C & \xrightarrow{\text{ad}} & \text{Der}(R, \text{End} V)
\end{array}$$

with vertical arrows being isomorphisms. Here, $\text{tr}_V$ comes from the trace pairing on $\text{End}(V)$ (and hence, it is obviously an isomorphism), and $\text{ad}$ is induced by the canonical map, assigning to $f \in \text{End}(V)$ the inner derivation $\text{ad}(f) : a \mapsto [f, \varrho(a)]$. The crucial isomorphism $\text{i(Tr } \hat{\omega} \text{)}$ is constructed in [CEG] (see loc. cit., the proof of Theorem 6.4.3). Instead of repeating this construction, we simply notice that (2.17) can be viewed as a moment map for the natural action of $\text{GL}(V)/\mathbb{C}^*$ on $\text{Rep}(R, V)$. The commutativity of (2.16) is then equivalent to the defining equation for moment maps (see [CEG], (6.4.7)). Now, it remains to note that $F_\xi$ is smooth at $\varrho$ if and only if $d\mu_\varrho$ is surjective. This is equivalent to $d\mu_\varrho^*$ being injective, and hence, in view of (2.17), to $\text{Ker}(\text{ad}) = \{0\}$. Since $\text{Ker}(\text{ad}) \cong \text{End}_R(V)/\mathbb{C}$, this last condition holds if and only if $\text{End}_R(V_\varrho) \cong \mathbb{C}$. The proposition follows. 

3. The Calogero-Moser Spaces

3.1. Rings of differential operators. Let $X$ be a smooth affine irreducible curve over $\mathbb{C}$, with coordinate ring $\mathcal{O} = \mathcal{O}(X)$, and let $\mathcal{D} = \mathcal{D}(X)$ be the ring of differential operators on $X$. We recall that $\mathcal{D}$ is a filtered algebra $\mathcal{D} = \bigcup_{k \geq 0} \mathcal{D}_k$, with filtration components $0 \subset \mathcal{D}_0 \subset \ldots \subset \mathcal{D}_{k-1} \subset \mathcal{D}_k \subset \ldots$ defined inductively by

$$\mathcal{D}_k := \{ D \in \text{End}_C \mathcal{O} : [D, f] \in \mathcal{D}_{k-1} \text{ for all } f \in \mathcal{O} \}.$$

The elements of $\mathcal{D}_k$ are called differential operators of order $\leq k$. In particular, $\mathcal{D}_0 = \mathcal{O}$ consists of multiplications operators by regular functions on $X$, and $\mathcal{D}_1$ is spanned by $\mathcal{O}$ and the space $\text{Der}(\mathcal{O})$ of derivations of $\mathcal{O}$ (the algebraic vector fields on $X$). As $X$ is smooth, $\mathcal{O}$ and $\text{Der}(\mathcal{O})$ generate $\mathcal{D}$ as an algebra, and $\mathcal{D}$ shares many properties with the Weyl algebra $A_1(\mathbb{C}) = \mathcal{D}(\mathbb{A}^1)$. For example, like $A_1(\mathbb{C})$, $\mathcal{D}$ is a simple Noetherian domain of global dimension 1; however, unlike $A_1(\mathbb{C})$, $\mathcal{D}$ has a nontrivial $K$-group.

We write $\mathcal{D} := \bigoplus_{k=0}^\infty \mathcal{D}_k/\mathcal{D}_{k-1}$ for the associated graded ring of $\mathcal{D}$; this is a commutative algebra isomorphic to the coordinate ring of the cotangent bundle $T^*X$ of $X$. Given a $\mathcal{D}$-module $M$ with a

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4 To avoid confusion, here we use the same notation for this map as in [CEG].
filtration \{M_k\}, we also write \(\overline{M} := \bigoplus_{k=0}^{\infty} \frac{M_k}{M_{k-1}}\) for the associated graded \(\overline{D}\)-module. Using the standard terminology, we say that \(\{M_k\}\) is a good filtration if \(\overline{M}\) is finitely generated (see, e. g. [13]).

3.2. Stable classification of ideals. Let \(K_0(X)\) and \(\text{Pic}(X)\) denote the Grothendieck group and the Picard group of \(X\) respectively. By definition, \(K_0(X)\) is generated by the stable isomorphism classes of (algebraic) vector bundles on \(X\), while the elements of \(\text{Pic}(X)\) are the isomorphism classes of line bundles. As \(X\) is affine, we may identify \(K_0(X)\) with \(\text{Pic}(O)\), the Grothendieck group of the ring \(O\), and \(\text{Pic}(X)\) with \(\text{Pic}(O)\), the ideal class group of \(O\). There are two natural maps \(\text{rk} : K_0(X) \to \mathbb{Z}\) and \(\det : K_0(X) \to \text{Pic}(X)\) defined by taking the rank and the determinant of a vector bundle respectively. In the case of curves, it is well-known that \(\text{rk} \oplus \det : K_0(X) \to \mathbb{Z} \oplus \text{Pic}(X)\) is a group isomorphism.

Now, let \(\mathcal{I}(D)\) denote the set of isomorphism classes of (nonzero) left ideals of \(D\). Unlike Pic in the commutative case, \(\mathcal{I}(D)\) carries no natural structure of a group. However, since \(D\) is a hereditary domain, \(\mathcal{I}(D)\) can be identified with the space of isomorphism classes of rank 1 projective modules, and there is a natural map relating \(\mathcal{I}(D)\) to \(\text{Pic}(X)\):

\[
\gamma : \mathcal{I}(D) \to K_0(D) \xrightarrow{i_*} K_0(X) \xrightarrow{\det} \text{Pic}(X).
\]

The first arrow in \([2, 1]\) is the tautological map assigning to the isomorphism class of an ideal in \(\mathcal{I}(D)\) its stable isomorphism class in \(K_0(D)\). The second arrow \(i_*^{-1}\) is the inverse of the Quillen isomorphism \(i_* : K_0(X) \xrightarrow{\sim} K_0(D)\) induced by the inclusion \(i : O \to D\). The role of \(\gamma\) becomes clear from the following result proved in [BW].

Theorem 3.1 (see [BW], Proposition 2.1). Let \(M\) be a projective \(D\)-module of rank 1 equipped with a good filtration such that \(\overline{M}\) is torsion-free. Then

\begin{enumerate}
\item[(a)] there is a unique (up to isomorphism) ideal \(I_M \subseteq O\), such that \(\overline{M}\) is isomorphic to a sub-\(\overline{D}\)-module of \(\overline{D}I_M\) of finite codimension (over \(\mathbb{C}\));
\item[(b)] the class of \(I_M\) in \(\text{Pic}(X)\) and the codimension \(n := \dim_{\mathbb{C}} [D I_M / \overline{M}]\) are independent of the choice of filtration on \(M\), and we have \(\gamma[M] = [I_M]\);
\item[(c)] if \(M\) and \(N\) are two projective \(D\)-modules of rank 1, then
\[
[M] = [N] \quad \text{in} \quad K_0(D) \iff [I_M] = [I_N] \quad \text{in} \quad \text{Pic}(X).
\]
\end{enumerate}

Theorem [2, 1] shows that the fibres of \(\gamma\) are precisely the stable isomorphism classes of ideals of \(D\): thus, up to isomorphism in \(K_0(D)\), the ideals of \(D\) are classified by the elements of \(\text{Pic}(X)\). Our goal is to refine this classification by describing the fibres of \(\gamma\) in geometric terms. As we will see in Section 4, each fibre \(\gamma^{-1}[I]\) naturally breaks up into a countable union of affine varieties \(\overline{\mathbb{C}}_n(X, I)\). In the next section, we introduce these varieties and study their geometric properties.

3.3. The definition of Calogero-Moser spaces. Given a curve \(X\) with a line bundle \(\mathcal{I}\), we set \(A := \mathcal{O}(X)\) and form the one-point extension \(B := A[I]\). By Lemma [2, 2], \(B\) is a smooth algebra, since so is \(A\) and \(\mathcal{I}\) is a f. g. projective \(A\)-module. As in Section [2, 2], we will identify the subalgebra of diagonal matrices in \(B\) with \(\hat{A} := A \times \mathbb{C}\), and let \(i : B \to \hat{A}\) denote the natural projection, see (2.11). Since \(i\) is a nilpotent extension, it is suggestive to think of ‘Spec \(B\)’ as a (noncommutative) infinitesimal ‘thickening’ of \(\text{Spec } \hat{A} = X[\text{pt}]\).

We now prove two auxiliary lemmas. The first lemma implies that \(B\) is determined, up to isomorphism, by the class of \(\mathcal{I}\) in \(\text{Pic}(X)\) and is independent of \(\mathcal{I}\) up to Morita equivalence. The second lemma computes the Euler characteristics for representations of \(B\), refining the result of Lemma [2, 3].

Lemma 3.1. For line bundles \(\mathcal{I}\) and \(\mathcal{J}\), the algebras \(A[I]\) and \(A[J]\) are

\begin{enumerate}
\item[(a)] Morita equivalent;
\item[(b)] isomorphic if and only if \(J \cong I^\tau\) for some \(\tau \in \text{Aut}(X)\), where \(I^\tau := \tau^* I\).
\end{enumerate}

Proof. (a) Given \(\mathcal{I}\) and \(\mathcal{J}\), we set \(\mathcal{L} := \text{Hom}_A(\mathcal{I}, \mathcal{J})\), which is a line bundle on \(X\) isomorphic to \(\mathcal{J} I' = J \otimes_A I'\), where \(I'\) is the dual of \(I\). Then, we extend \(\mathcal{L}\) to a line bundle over \(\hat{A}\), letting \(\hat{L} := \mathcal{L} \times \mathbb{C}\), and define \(P := \hat{L} \otimes_{\hat{A}} B\), where \(B = A[I]\). Clearly, \(P\) is a f. g. projective \(B\)-module. On the other
hand, since $A$ is a Dedekind domain, $\mathcal{L} \oplus \mathcal{L} \cong A \oplus \mathcal{L}^2$, where $\mathcal{L}^2 = \mathcal{L} \otimes_A \mathcal{L}$, and hence $\check{\mathcal{L}} \oplus \check{\mathcal{L}} \cong A \oplus \check{\mathcal{L}}^2$. It follows that $B$ is isomorphic to a direct summand of $P \oplus P$, so $P$ is a generator in the category of right $B$-modules. By Morita Theorem, the ring $B$ is then equivalent to $\text{End}_B(P)$. Now, since $\text{End}_B(P) = \text{Hom}_B(\check{\mathcal{L}} \otimes_A B, P) \cong \text{Hom}_A(\check{\mathcal{L}}, P)$ and $P \cong \check{\mathcal{L}} \oplus (0, \mathcal{L})$ as a (right) $A$-module, we have $\text{End}_B(P) \cong A[\mathcal{J}]$.

(b) If $\mathcal{J} \cong \mathcal{I}$, then $A[\mathcal{J}] \cong A[\mathcal{I}]$, by Lemma 2.2(3). Without loss of generality, we may therefore identify $\mathcal{I}$ and $\mathcal{J}$ with ideals in $A$. Given $\tau \in \text{Aut}(X) = \text{Aut}(A)$, we have then $\mathcal{I}^\tau = \tau^{-1}(\mathcal{I})$, and the natural map $\tau^{-1} : A[\mathcal{I}] \to A[\mathcal{I}^\tau]$ is a required isomorphism. The converse statement is left as an exercise to the reader.

Lemma 3.2. For any finite-dimensional $B$-modules $\mathbf{U} = (U, U_\infty)$ and $\mathbf{V} = (V, V_\infty)$, we have

$$\chi_B(\mathbf{U}, \mathbf{V}) = \dim(U_\infty)[\dim(V_\infty) - \dim(V)] .$$

Proof. First, observe that $\chi_A(U, V) = 0$ for any pair of finite-dimensional $A$-modules. Indeed, if $U$ and $V$ have disjoint supports, then $\text{Hom}_A(U, V) = \text{Ext}_A^1(U, V) = 0$, and certainly $\chi_A(U, V) = 0$. By additivity of $\chi_A$, it thus suffices to see that $\chi_A(U, V) = 0$ for modules $U$ and $V$ supported at one point. If $\mathfrak{m}$ is the maximal ideal of $A$ corresponding to that point, we have $\text{Ext}_A^1(U, V) \cong \text{Ext}_A^1(U, V)$ and $\text{Ext}_A^1(U, V) \cong \text{Ext}_A^1(U, V)$ for all $i \geq 0$. Thus

$$\chi_A(U, V) = \chi_{\mathfrak{m}}(U, V) = \sum (-1)^i \dim \text{Tor}^A_i(V^*, U) .$$

The vanishing of $\chi_A(U, V)$ follows now from standard intersection theory, since $A_n$ is a regular local ring of (Krull) dimension 1, while $\dim \text{Ext}_A^1(U, V) + \dim \text{Ext}_A^1(V^*) = 0$ (see [S], Ch. V, Part B, Th. 1).

Identifying $\mathcal{I}$ with an ideal in $A$ and dualizing $0 \to \mathcal{I} \to A \to A/\mathcal{I} \to 0$ by $V$, we get

$$\dim \text{Hom}_A(\mathcal{I}, V) = \dim \text{Hom}(V) - \chi_A(A/\mathcal{I}, V) = \dim(V) .$$

The result follows now from Lemma 2.3.

Next, we introduce deformed preprojective algebras over $B$. For this, we need to compute the trace map $\text{Tr}_B : K_0(B) \to H_0(B)$. Recall that $\text{Tr}_* : K_0 \to H_0$ is a natural transformation of functors on the category of associative algebras, so $i : B \to A$ gives rise to the commutative diagram

$$
\begin{array}{ccc}
K_0(B) & \xrightarrow{\text{Tr}_B} & H_0(B) \\
\downarrow & & \downarrow \\
K_0(A) & \xrightarrow{i} & H_0(\check{A})
\end{array}
$$

The two vertical arrows in (3.3) are isomorphisms: the first one is given by Lemma 2.2(6), while the second has the obvious inverse (induced by the inclusion $\check{A} \hookrightarrow B$). We will use these isomorphisms to identify $H_0(B) \cong H_0(\check{A}) \cong A \subset B$ and

$$K_0(B) \cong K_0(\check{A}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \text{Pic}(X) .$$

Now, for any commutative algebra (e.g., $\check{A}$), the trace map factors through the rank. Hence, with above identifications, $\text{Tr}_B$ is completely determined by its values on the first two summands in (3.4), while vanishing on the last. Since $\text{Tr}_B([1, 0]) = e$ and $\text{Tr}_B([0, 1]) = e_\infty$, the linear map $\text{Tr}_B : \mathbb{C} \otimes \mathbb{Z} K_0(B) \to H_0(B)$ takes its values in the two-dimensional subspace $S$ of $B$ spanned by $e$ and $e_\infty$. Identifying $S$ with $\mathbb{C}^2$, we may regard the vectors $\lambda := (\lambda, \lambda_\infty) = \lambda e + \lambda_\infty e_\infty \in S$ as weights for the family of deformed preprojective algebras associated to $B$:

$$\Pi^B = T_B \text{Der}(B)/(\Delta_B - \lambda) .$$

Since $A$ is an integral domain, $\{e, e_\infty\}$ is a complete set of primitive orthogonal idempotents in $\Pi^B$, and $S = \mathbb{C}e \oplus \mathbb{C} e_\infty$ is the associated semisimple subalgebra of $\Pi^B$. Now, for each $\mathfrak{n} = (n, n_\infty) \in \mathbb{N}^2$, 

we form the variety \( \text{Rep}_S(\Pi^\lambda(B), n) \) of representations of \( \Pi^\lambda(B) \) of dimension vector \( n \) and, with notation of Section \([2.3]\), define

\[
C_{n, \lambda}(X, J) := \text{Rep}_S(\Pi^\lambda(B), n)/G_S(n).
\]

Thus, \( C_{n, \lambda}(X, J) \) is an affine scheme, whose (closed) points are in bijection with isomorphism classes of semisimple \( \Pi^\lambda(B) \)-modules of dimension vector \( n \).

**Lemma 3.3.** For any line bundles \( I \) and \( J \), the schemes \( C_{n, \lambda}(X, I) \) and \( C_{n, \lambda}(X, J) \) are isomorphic.

**Proof.** By Lemma 3.1 \( A[I] \) and \( A[J] \) are Morita equivalent: the corresponding equivalence is given by

\[
\text{Mod } A[I] \cong \text{Mod } A[J], \quad V \mapsto \tilde{L} \otimes_A V,
\]

where \( \tilde{L} = J I^\vee \times \mathbb{C} \). The functor \([3.7]\) induces an isomorphism of vector spaces: \( H_0(A[I]) \cong H_0(A[J]) \), which restricts to the identity on \( S \subset \tilde{A} \). By [CB], Cor. 5.5, it can then be extended (non-canonically) to a Morita equivalence between \( \Pi^\lambda(A[I]) \) and \( \Pi^\lambda(A[J]) \) for any \( \lambda \in S \). Now, if \( V = (V, V_\infty) \) with \( \dim_V V < \infty \), we have \( \tilde{L} \otimes_A V = (JI^\vee \otimes_A V, V_\infty) \), so by formula \([3.2]\),

\[
\dim_{\mathbb{C}}(JI^\vee \otimes_A V) = \dim_{\mathbb{C}} \text{Hom}_A(IJ^\vee, V) = \dim_{\mathbb{C}}(V).
\]

This shows that \([3.7]\) preserves dimensions, and its extension to \( \Pi^\lambda \) induces thus an isomorphism: \( C_{n, \lambda}(X, I) \cong C_{n, \lambda}(X, J) \). \( \square \)

The next lemma is a generalization of [CBH], Lemma 4.1: it implies that \( C_{n, \lambda}(X, I) \) is empty unless \( \lambda \cdot n := \lambda n + \lambda_\infty n_\infty \) is zero.

**Lemma 3.4.** If \( \lambda \cdot n \neq 0 \), there are no representations of \( \Pi^\lambda(B) \) of dimension \( n \).

**Proof.** If \( V = V \oplus V_\infty \) is a \( \Pi^\lambda(B) \)-module of dimension \( n \), then \( e \) and \( e_\infty \) act on \( V \) as projectors onto \( V \) and \( V_\infty \) respectively. The trace of \( \lambda = \lambda e + \lambda_\infty e_\infty \in B \) on \( V \) is then equal to \( \lambda \cdot n \), and it must be zero, by Proposition [2.1] \( \square \)

**Example 3.1.** Let \( X \) be the affine line \( \mathbb{A}^1 \). Any line bundle \( I \) on \( X \) is trivial. So, choosing a coordinate on \( X \), we may identify \( A \cong \mathbb{C}[x] \) and \( I \cong \mathbb{C}[x] \). The one-point extension of \( A \) by \( I \) is then isomorphic to the matrix algebra

\[
A[I] \cong \left( \begin{array}{cc} \mathbb{C}[x] & \mathbb{C}[x] \\ 0 & \mathbb{C} \end{array} \right),
\]

which is, in turn, isomorphic to the path algebra \( \mathbb{C}Q \) of the quiver \( Q \) consisting of two vertices \( \{0, \infty\} \) and two arrows \( x: 0 \to 0 \) and \( v: \infty \to \infty \). In fact, the map sending the vertices 0 and \( \infty \) to the idempotents \( e \) and \( e_\infty \) and \( X \mapsto (x 0 0 0) \), \( v \mapsto (0 1 0 0) \), extends to an algebra isomorphism \( \mathbb{C}Q \cong A[I] \).

Now, let \( \tilde{Q} \) be the double quiver of \( Q \) obtained by adding the reverse arrows \( Y := X^* \) and \( w := v^* \) to the corresponding arrows of \( Q \). Then, for any \( \lambda = \lambda e + \lambda_\infty e_\infty \), with \( (\lambda, \lambda_\infty) \in \mathbb{C}^2 \), the algebra \( \Pi^\lambda(Q) \) is isomorphic to the quotient of \( \mathbb{C}Q \) modulo the relation \([X, Y] + [v, w] = \lambda \) (see [CB], Theorem 3.1). The ideal generated by this relation is the same as the ideal generated by the elements \([X, Y] + v w - \lambda e \) and \( w v + \lambda_\infty e_\infty \). Thus, the \( \Pi^\lambda(Q) \)-modules can be identified with representations \( V = V \oplus V_\infty \) of \( \tilde{Q} \), in which linear maps \( X, Y \in \text{Hom}(V, V), v \in \text{Hom}(V_\infty, V), w \in \text{Hom}(V, V_\infty) \), given by the action of \( X, Y, v, w \), satisfy

\[
[X, Y] + v w = \lambda \text{Id}_V \quad \text{and} \quad w v = -\lambda_\infty \text{Id}_{V_\infty}.
\]

Now, taking \( \lambda = (1, -n) \), it is easy to see that all representations of \( \Pi^\lambda(Q) \) of dimension vector \( n = (n, 1) \) are simple, and the varieties \( C_{n, \lambda} \) coincide (in this special case) with the classical Calogero-Moser spaces \( C_n \). This coincidence was first noticed by W. Crawley-Boevey (see [CB], Remark on p. 45). For explanations and further discussion of this example we refer the reader to [BCE].
Motivated by the above example, we will be interested in representations of \( \Pi^\lambda(B) \) of dimension \( n = (n,1) \). By Lemma 3.4 such representations may exist only if \( \lambda = 0 \) or \( \lambda = (\lambda, -n\lambda) \), with \( \lambda \neq 0 \). In this last case, \( \Pi^\lambda(B) \)'s are all isomorphic to each other, so without loss of generality we may assume \( \lambda = 1 \).

**Proposition 3.1.** Let \( \lambda = (1, -n) \) and \( n = (n,1) \) with \( n \in \mathbb{N} \). Then, for any \( I \), the algebra \( \Pi^\lambda(B) \) has modules of dimension vector \( n \). Such every module is simple.

**Proof.** On any \( B \)-module of dimension \( n \), the element \( \lambda = e - n e_\infty \in B \) will act with zero trace. Thus, by Proposition 2.1 it suffices to see that there exist indecomposable \( B \)-modules of this dimension vector. Now, a \( B \)-module structure on \( V = V \oplus V_\infty \) is determined by an \( A \)-module homomorphism \( \varphi_V : I \otimes V_\infty \to V \). If \( V \) is indecomposable with \( \dim(V_\infty) = 1 \), then one of its summands must be of the form \( V' = V' \oplus V_\infty \), where \( V' \) is an \( A \)-module summand of \( V \) of dimension \( < n \). In that case, \( \text{Im}(\varphi_V) \subseteq V' \subseteq V \). Thus, for constructing an indecomposable \( B \)-module of dimension \( n \), it suffices to construct a torsion \( A \)-module \( V \) of length \( n \) together with a surjective \( A \)-module map \( \varphi : I \to V \).

Geometrically, this can be done as follows.

Identify \( I \) with an ideal in \( A \) and fix \( n \) distinct points \( x_1, x_2, \ldots, x_n \) on \( X \) outside the zero locus of \( I \). Let \( V := A/\mathcal{J} \), where \( \mathcal{J} \) is the product of the maximal ideals \( m_i \subset A \) corresponding to \( x_i \)'s. Clearly, \( A/\mathcal{J} \cong \bigoplus_{i=1}^n A/m_i \), and \( \dim(V) = n \). Now, since \( A \) is a Dedekind domain and \( I \not\subseteq m_i \) for any \( i \), we have \( (A/\mathcal{J}) \otimes_A (A/I) \cong \bigoplus_{i=1}^n (A/m_i) \otimes_A (A/I) = 0 \) and \( \text{Tor}_1^A(A/\mathcal{J}, A/I) \cong (I \cap \mathcal{J})/I \mathcal{J} = 0 \), so the canonical map \( V \otimes_A I \to V \) is an isomorphism. On the other hand, as \( V \) is a cyclic \( A \)-module, \( I \) surjects naturally onto \( V \otimes_A I \). Combining \( I \to V \otimes_A I \to V \), we get the required \( \varphi \). This proves the first part of the proposition.

Now, let \( V = V \oplus V_\infty \) be any \( \Pi^\lambda(B) \)-module of dimension vector \( n \). Let \( V' \) be a submodule of \( V \) of dimension vector \( k = (k, k_\infty) \) (say). By Lemma 3.4 we have then \( \lambda \cdot k = k - n k_\infty = 0 \). Since \( 0 \leq k_\infty \leq n_\infty = 1 \), there are only two possibilities: either \( k = 0 \) or \( k = n \), i.e. \( V' \) is either 0 or \( V \). Hence \( V \) is a simple module.

**Remark.** 1. The above argument shows that a \( B \)-module \( V \) of dimension vector \( n = (n,1) \) lifts to a module over \( \Pi^\lambda(B) \) if and only if it is indecomposable.

2. If \( V \) is a \( B \)-module with a surjective structure map \( \varphi_V : I \otimes V_\infty \to V \), then \( \text{End}_B(V) \subseteq \text{End}(V_\infty) \).

3. This follows immediately from the diagram (2.10), characterizing \( B \)-module homomorphisms.) Hence the modules \( V \) constructed in Proposition 3.1 are actually indecomposables with \( \text{End}_B(V) \cong \mathbb{C} \).

**Definition 1.** The variety \( \mathcal{C}_n(X, I) \) with \( \lambda = (1, -n) \) and \( n = (n,1) \) will be denoted \( \mathcal{C}_n(X, I) \) and called the \( n \)-th Calogero-Moser space of type \((X, I)\).

In view of Proposition 3.1 the varieties \( \mathcal{C}_n(X, I) \) parametrize the isomorphism classes of simple \( \Pi^\lambda(B) \)-modules of dimension \( n = (n,1) \); they are non-empty for any \((I) \in \text{Pic}(X)\) and \( n \geq 0 \). In the special case, when \( X \) is the affine line, \( \mathcal{C}_n(X, I) \) coincide with the ordinary Calogero-Moser spaces \( \mathcal{C}_n \) (see Example 3.1).

**Remark.** It follows from our discussion in Section 2.2 (see also [CEG] and [vdB]) that the variety \( \mathcal{C}_n(X, I) \) can be obtained by symplectic reduction from the cotangent bundle of \( \text{Rep}(B, n) \). This links our definition of Calogero-Moser spaces to the one proposed by V. Ginzburg (see [BN], Definition 1.2).

3.4. Smoothness and irreducibility. One of the main results of [W] says that each \( \mathcal{C}_n \) is a smooth affine irreducible variety of dimension \( 2n \). Theorem 3.2 below shows that this holds in general, for an arbitrary curve \( X \). To prove the irreducibility we will use the approach of Crawley-Boevey [CB2], the starting point of which is the following observation.

**Lemma 3.5** ([CB2], Lemma 6.1). If \( X \) is an equidimensional variety, \( Y \) is an irreducible variety and \( f : X \to Y \) is a dominant morphism with all fibres irreducible of constant dimension, then \( X \) is irreducible.

**Theorem 3.2.** For each \( n \geq 0 \) and \((I) \in \text{Pic}(X)\), \( \mathcal{C}_n(X, I) \) is a smooth affine irreducible variety of dimension \( 2n \).
Proof. The varieties $C_n(X, I)$ are affine by definition; it only needs to show that these are smooth and irreducible. Fix $n \in \mathbb{N}$ and $I \in \text{Pic}(X)$. To simplify the notation write $\Pi$ for $\Pi^\lambda(B)$ with $\lambda = (1, -n)$. Then, by Proposition 3.1 every $\Pi$-module $V$ of dimension $n = (n, 1)$ is simple, so, by Schur Lemma, $\text{End}_B(V) \cong \mathbb{C}$ and $\text{Aut}_B(V) \cong \mathbb{C}$. The last isomorphism implies that every point of $\text{Rep}_B(\Pi, n)$ has trivial stabilizer in $G_S(n)$, i.e., the natural action of $\text{GL}(n)$ on $\text{Rep}_S(\Pi, n)$ is free. In that case, by Luna’s Slice Theorem (see [LM], Corollaire III.1.1), the quotient variety $C_n(X, I) = \text{Rep}_S(\Pi, n)/G_S(n)$ will be smooth if so is the original variety $\text{Rep}_S(\Pi, n)$. Now, to see that $\text{Rep}_S(\Pi, n)$ is smooth, it suffices to see, by Theorem 2.3, that $\text{Rep}_S(\Pi, n)$ is smooth, and that follows from Proposition 2.2 of Section 2.3. In fact, let $R := T_B \text{Der}(B)$, and let $\sigma : R \to \Pi$ be the canonical projection. Then $\sigma$ induces the closed embedding of affine varieties $\sigma_* : \text{Rep}(\Pi, n) \hookrightarrow \text{Rep}(R, n)$, whose image is a fibre of the evaluation map (2.10). Since for every $\varrho \in \text{Im}(\sigma_*)$, we have $\text{End}_R(V) \cong \text{End}_B(V) \cong \mathbb{C}$, the assumption of Proposition 2.2 holds, and the result follows.

Now, we show that $C_n(X, I)$ is irreducible of dimension $2n$. For this, we examine first the varieties $\text{Rep}_S(B, n)$ and $\text{Rep}_S(\Pi, n)$. Since $B$ is smooth, $\text{Rep}_S(B, n)$ is smooth, i.e., for every point $\varrho \in \text{Rep}_S(B, n)$, we have
\[
(3.9) \quad \dim_{\mathbb{C}} \text{Rep}_S(B, n) = \dim_{\mathbb{C}} T_{\varrho} \text{Rep}_S(B, n),
\]
where $\dim_{\mathbb{C}}$ stands for the local dimension and $T_{\varrho}$ for the Zariski tangent space of $\text{Rep}_S(B, n)$ at $\varrho$. To evaluate the dimension of this space we identify $T_{\varrho} \text{Rep}_S(B, n) \cong \text{Der}_S(B, \text{End} V)$, as in (2.15), and consider the standard exact sequence
\[
0 \to \text{End}_B(V) \to \text{End}_S(V) \to \text{Der}_S(B, \text{End} V) \to H^1(B, \text{End} V) \to 0.
\]
Identifying now terms in this sequence $\text{End}_S(V) \cong \text{Mat}(n, \mathbb{C}) \times \mathbb{C}$, $H^1(B, \text{End} V) \cong \text{Ext}^1_B(V, V)$ (see [CE], Prop. 4.3, p. 170), and using Lemma 3.2 we get
\[
(3.10) \quad \dim_{\mathbb{C}} \text{Der}_S(B, \text{End} V) = n^2 + 1 - \chi_B(V, V) = n^2 + n.
\]
Thus $\text{Rep}_S(B, n)$ is a smooth equidimensional variety of dimension $n^2 + n$. To see that it is actually irreducible, we apply Lemma 3.5 to the canonical projection $f : \text{Rep}_S(B, n) \to \text{Rep}(A, n)$. In this case, the assumptions of Lemma 3.5 are easy to verify: since $X$ is irreducible, so is clearly $\text{Rep}(A, n)$, and the fibres of $f$ over each $V \in \text{Rep}(A, n)$ can be identified with the vector spaces $\text{Hom}_A(I, V)$ and, hence, are all irreducible of the same dimension $n$, by formula (3.11).

Next, we consider the restriction map $\pi : \text{Rep}_S(\Pi, n) \to \text{Rep}_S(B, n)$. As remarked above, the image of $\pi$ consists exactly of indecomposable modules in $\text{Rep}_S(B, n)$, while each (non-empty) fibre $\pi^{-1}(V)$ is isomorphic, by Proposition 2.4, to a coset of $\text{Ext}^1_B(V, V)$ and is thus irreducible of dimension
\[
(3.11) \quad \dim \pi^{-1}(V) = \dim_{\mathbb{C}} \text{End}_B(V) - \chi_B(V, V) = \dim_{\mathbb{C}} \text{End}_B(V) + n - 1.
\]
Now, let $U$ be the subset of $\text{Rep}_S(B, n)$ consisting of modules $V$ with $\text{End}_B(V) \cong \mathbb{C}$. As explained in Remark 2 (after Proposition 3.4, this subset is non-empty. By Chevalley’s Theorem (see, e.g., [CB3], p. 15), the function $V \mapsto \dim_{\mathbb{C}} \text{End}_B(V)$ is upper semi-continuous on $\text{Rep}_S(B, n)$, i.e.,
\[
\{ V \in \text{Rep}_S(B, n) : \dim_{\mathbb{C}} \text{End}_B(V, V) \geq n \}
\]
is closed sets for all $n \in \mathbb{N}$. Hence $U$ is open in $\text{Rep}_S(B, n)$ and therefore dense, since $\text{Rep}_S(B, n)$ is irreducible. As $U \subseteq \text{Im}(\pi)$, this implies that $\pi$ is dominant.

Now, $\pi^{-1}(U)$ is an open subset of $\text{Rep}_S(\Pi, n)$, whose local dimension at every point $\varrho \in \pi^{-1}(U)$ is equal, by (3.11), to
\[
\dim_{\mathbb{C}} \pi^{-1}(U) = \dim U + \dim \pi^{-1}(\varrho) = \dim \text{Rep}_S(B, n) + n = n^2 + 2n.
\]
Thus $\pi^{-1}(U)$ is equidimensional and therefore, by Lemma 3.5 irreducible. We claim that $\pi^{-1}(U)$ is dense in $\text{Rep}_S(\Pi, n)$. Indeed, since $\text{Im}(\pi)$ coincides with the set of indecomposable $B$-modules in $\text{Rep}_S(B, n)$, we have
\[
\dim \pi^{-1}(\text{Im}(\pi)) \setminus U < \dim \pi^{-1}(U) = n^2 + 2n.
\]
On the other hand, the variety $\text{Rep}_S(\Pi, n)$ can be identified with a fibre of the evaluation map $\mu : \text{Rep}_S(R, n) \to \text{End}_S(V)_0$, see (2.10), so any irreducible component of it has dimension at least

$$\dim \text{Rep}_S(R, n) - \dim \text{End}_S(V)_0 = 2(n^2 + n) - n^2 = n^2 + 2n .$$

(Here, we calculated $\dim \text{Rep}_S(R, n)$ using the identification of Theorem 4.2.)

Thus, $\text{Rep}_S(\Pi, n)$ must coincide with the closure of $\pi^{-1}(U)$, and hence is also irreducible of dimension $n^2 + 2n$. This certainly implies the irreducibility of $C_n(X, I)$, since $C_n(X, I)$ is a quotient of $\text{Rep}_S(\Pi, n)$ by a free action of $G_S(n)$.

Finally, we have $G_S(n) \cong [\text{GL}(n, \mathbb{C}) \times \text{GL}(1, \mathbb{C})]/\mathbb{C} \cong \text{GL}(n, \mathbb{C}),$ so

$$\dim C_n(X, I) = \dim \text{Rep}_S(\Pi, n) - \dim G_S(n) = n^2 + 2n - n^2 = 2n .$$

This completes the proof of the theorem. \hfill \Box

4. The Calogero-Moser Correspondence

4.1. Recollement. We begin by clarifying the relation between the algebras $\Pi^A(B)$ and the ring $\mathcal{D}$ of differential operators on $X$ (see also Appendix).

**Lemma 4.1.** There is a canonical map $i : \Pi^A(B) \to \Pi^A(A)$, which is a surjective pseudo-flat ring homomorphism, with $\text{Ker}(i) = \langle e_\infty \rangle$.

**Proof.** By Lemma 2.4, the projection $i : B \to A$, see 2.12, is a flat (and hence, pseudo-flat) ring epimorphism. Since $B$ is smooth, by Theorem 2.1, $i$ extends to an algebra map $i : \Pi^A(B) \to \Pi^A(A)$, which is also a pseudo-flat ring epimorphism. Now, since $i$ is surjective with $\text{Ker}(i) = \langle e_\infty \rangle$, the Cartesian square (2.2) shows that $i$ is surjective and $\text{Ker}(i) = \langle e_\infty \rangle$. Finally, with identifications of Section 3.3, it is easy to see that $i^*(\lambda) = 1$.

**Theorem 4.1 (CR).** Theorem 4.7. If $A = \mathcal{O}(X)$ is the coordinate ring of a smooth affine curve, then $\Pi^A(A)$ is isomorphic (as a filtered algebra) to $\mathcal{D} = \mathcal{D}(X)$.

We fix, once and for all, an isomorphism of Theorem 4.1 to identify $\mathcal{D} = \Pi^A(A)$. In combination with Lemma 4.1 this yields an algebra map $i : \Pi \to \mathcal{D}$. We will use $i$ to relate the (derived) module categories of $\Pi$ and $\mathcal{D}$, as follows (cf. BCE). First, we let $U$ denote the endomorphism ring of the projective module $e_\infty \Pi$: this ring can be identified with the associative subalgebra $e_\infty \Pi e_\infty$ of $\Pi$ having $e_\infty$ as an identity element. Next, we introduce six additive functors $(i^*, i_*, i^!)$ and $(j^!, j_!, j^!)$ between the module categories of $\Pi$, $\mathcal{D}$ and $U$. We let $i_* : \text{Mod}(\Pi) \to \text{Mod}(\mathcal{D})$ be the restriction functor associated to $i$. This functor is fully faithful and has both the right adjoint $i^! := \text{Hom}_\Pi(\mathcal{D}, -)$ and the left adjoint $i^* := \mathcal{D} \otimes_\Pi -$, with adjunction maps $i^*i_* \simeq \text{Id} \simeq i^!i_*$ being isomorphisms. Now we define $j^* : \text{Mod}(\Pi) \to \text{Mod}(U)$ by $j^*V := e_\infty V$. Since $e_\infty \in \Pi$ is an idempotent, $j^*$ is exact and has also the right and left adjoints: $j_* := \text{Hom}_U(e_\infty \Pi, -)$ and $j^! := \Pi e_\infty \otimes_U -$, satisfying $j^*j_* \simeq \text{Id} \simeq j^!j^*$. The six functors $(i^*, i_*, i^!)$ and $(j^!, j_!, j^!)$ defined above extend to the derived categories:

$$\begin{align*}
\mathcal{D}^b(\text{Mod } \mathcal{D}) & \xrightarrow{i_*} \mathcal{D}^b(\text{Mod } \Pi) \\
& \xrightarrow{i^*} \mathcal{D}^b(\text{Mod } \Pi) \\
& \xrightarrow{j^!} \mathcal{D}^b(\text{Mod } \mathcal{D}) \\
& \xrightarrow{j_*} \mathcal{D}^b(\text{Mod } \mathcal{D})
\end{align*}
$$

(4.1)

and their properties of these functors can be summarized in the following way (cf. BBD, Sect. 1.4).

**Theorem 4.2.** The diagram (4.1) is a recollement of triangulated categories.

Theorem 4.2 follows from general results on recollement of module categories (see Ko) and the following observation, which will be proved in Section 5.1 (see Lemma 5.3): the multiplication map $\Pi e_\infty \otimes_U e_\infty \Pi \to \Pi$ fits into the exact sequence

$$0 \to \Pi e_\infty \otimes_U e_\infty \Pi \to \Pi \xrightarrow{i^!} \mathcal{D} \to 0 ,$$

(4.2)
which is a projective resolution of $D$ in the category of (left and right) $II$-modules. The existence of (4.2) implies that $D$ has projective dimension 1 in $Mod(II)$. Hence $Tor^n_{II}(D, D) = 0$ for all $n ≥ 2$. On the other hand, by Lemma 4.1 $i$ is a pseudo-flat epimorphism, meaning that $Tor^n_{II}(D, D) = 0$ as well. Theorem 4.2 follows now from [Ko], Cor 14. As another consequence of (4.2), we have

**Lemma 4.2.** If $V$ is a finite-dimensional $II$-module, then $L_n i^*(V) = 0$ for $n ≠ 1$ and

$$L_1 i^*(V) ∼= Ker \left( \prod e_{∞} \otimes_U e_{∞} V \overset{\Delta}{\rightarrow} V \right),$$

where $L_n i^*$ denotes the $n$-th derived functor of $i^*$ and $µ$ is the natural multiplication-action map.

**Proof.** Tensoring (4.2) with $V$ yields the exact sequence

$$0 \rightarrow Tor^n_{II}(D, V) \rightarrow \prod e_{∞} \otimes_U e_{∞} V \rightarrow V \rightarrow D \otimes II V \rightarrow 0,$$

and isomorphisms $Tor^n_{II}(D, V) ∼= Tor^n_{II−1}(\prod e_{∞} \otimes_U e_{∞} II, V)$ for $n ≥ 2$. Since $\prod e_{∞} \otimes_U e_{∞} II$ is projective (as a right $II$-module), the last $Tor’s$ vanish. On the other hand, $\dim_C V < ∞$ implies that $D \otimes II V = 0$, since $D$ has no nonzero finite-dimensional modules. The result follows now from the identification $L_n i^*(V) = Tor^n_{II}(D, V)$, $n ≥ 0$.

**Remark.** Using (4.1), we may define the following ‘perverse’ $t$-structure on $D^b(Mod II)$:

$$p_{D^≤0(Mod II)} := \{ K^* ∈ D^b(Mod II) : j^* K^* ∈ D^≤0(U) \},$$

$$p_{D^≥0(Mod II)} := \{ K^* ∈ D^b(Mod II) : j^* K^* ∈ D^≥0(U) \},$$

where $\{D^≤0(U), D^≥0(U)\}$ and $\{D^≤0(Γ), D^≥0(Γ)\}$ denote the standard $t$-structures on $D^b(Mod U)$ and $D^b(Γ)$ respectively. Lemma 4.2 shows that the $0$-complexes $[0 → V → 0]$ with $\dim_C V < ∞$ lie in the heart of this $t$-structure. So we may think of finite-dimensional $II$-modules as ‘perverse sheaves’ with respect to the stratification (4.1). The functor $i^*$ is then an algebraic analogue of the restriction functor of a (perverse) sheaf to a closed subspace.

4.2. **The action of Pic($D$) on Calogero-Moser spaces.** We recall some facts about the Picard group Pic($D$) of the algebra $D$ and its action on the space of ideals $I(Γ)$ (see [BW]). It is known that Pic($D$) has different descriptions for $Γ = \mathbb{A}^1$ and other curves (see [CH]). Since the case of $Γ = \mathbb{A}^1$ is well studied, we will assume that $Γ ≠ \mathbb{A}^1$. Our main theorem (Theorem 1.3) still holds for all curves $Γ$, including $Γ = \mathbb{A}^1$.

In general, Pic($D$) can be identified with the group of $C$-linear auto-equivalences of the category $Mod(D)$, and thus it acts naturally on $I(Γ)$ and $K_0(D)$. To be precise, the elements of Pic($D$) are the isomorphism classes $[P]$ of invertible $D$-bimodules, and the action of Pic($D$) on $I(Γ)$ and $K_0(D)$ is defined by $[M] → [P \otimes D M]$. The action of Pic($D$) on $K_0(D)$ preserves rank and hence restricts to Pic($X$) through the identification $K_0(D) ∼= K_0(X) ∼= \mathbb{Z} ⊕ Pic(X)$, see Section 3.2.

**Proposition 4.1** (see [BW], Theorem 1.1). Pic($D$) acts on Pic($X$) transitively, and the map $γ : I(Γ) → Pic(X)$ defined by (4.1) is equivariant under this action.

Explicitly, the action of Pic($D$) on Pic($X$) can be described as follows (cf. [BW], Prop. 3.1). By [CH], Cor. 1.13, every invertible bimodule over $D$ is isomorphic to $DL = D ⊗_A L$ as a left module, while the right action of $D$ is determined by an algebra isomorphism $φ : D → End_D(DL)$, where $L$ is a line bundle on $X$. Following [BW], we denote such a bimodule by $(DL)_φ$. Restricting $φ$ to $A$ yields an automorphism of $X$, and the assignment

$$g : Pic(D) → Pic(X) × Aut(X), \quad [(DL)_φ] → ([L], φ|_A),$$

defines then a group homomorphism. On the other hand, Pic($X$) × Aut($X$) acts on Pic($X$) in the obvious way:

$$((L, τ) : [I] → [L τ I]),$$

where $(L, τ) ∈ Pic(X) × Aut(X)$ and $[I] ∈ Pic(X)$. Combining (4.4) and (4.5), we get an action of Pic($D$) on Pic($X$), which agrees with the natural action of Pic($D$) on $K_0(D)$.
Now, given a line bundle $\mathcal{I}$ and an invertible bimodule $\mathcal{P} = (\mathcal{D}\mathcal{L})_{\varphi}$, we define $P := \tilde{\mathcal{L}} \otimes \tilde{\mathcal{A}} B_\tau$, where $\tilde{\mathcal{L}} := \mathcal{L} \times \mathcal{C}$, $\tau := \varphi |_A$, and $B_\tau := A[\tau(I)]$. By Lemma 3.1(a), $P$ is a progenerator in the category of right $B_\tau$-modules, with $\text{End}_{B_\tau}(P) \cong A[I]$, where $I := \mathcal{L} \tau(I)$. Associated to $\mathcal{P}$ is thus the Morita equivalence: $\text{Mod}(B_\tau) \cong \text{Mod}(A[I])$, $V \mapsto P \otimes_{B_\tau} V \cong \tilde{\mathcal{L}} \otimes \tilde{\mathcal{A}} V$.

Next, we extend $P$ to the $\Pi^\Lambda(B_\tau)$-module $P := P \otimes_{B_\tau} \Pi^\Lambda(B_\tau) = \tilde{\mathcal{L}} \otimes \tilde{\mathcal{A}} \Pi^\Lambda(B_\tau)$, which is clearly a progenerator in the category of right $\Pi^\Lambda(B_\tau)$-modules. By Lemma 3.1(b), $\tau$ defines an isomorphism $B_\tau \cong B$. Since $\tau(\lambda) = \lambda$ for all $\lambda \in S$, this isomorphism canonically extends to deformed preprojective algebras $\tilde{\tau} : \Pi^\Lambda(B) \cong \Pi^\Lambda(B_\tau)$, which allows us to regard $P$ as a $\Pi^\Lambda(B)$-module and identify

$$\text{End}_{\Pi^\Lambda(B)}(P) \cong \tilde{\mathcal{F}} \otimes \tilde{\mathcal{A}} \Pi^\Lambda(B) \otimes \tilde{\mathcal{F}}^{\vee}, \quad \mathcal{F} := \mathcal{L}^{\vee}.$$  

With this identification, we have the embedding

$$\tilde{\tau}^{-1} : A[I] \hookrightarrow \text{End}_{\Pi^\Lambda(B)}(P),$$

and, since $\text{End}_{\Pi}(\mathcal{F} \mathcal{D}) \cong \tilde{\mathcal{F}} \otimes \tilde{\mathcal{A}} \mathcal{D} \otimes \tilde{\mathcal{A}} \tilde{\mathcal{F}}^{\vee}$, the natural map

$$1 \otimes i \otimes 1 : \text{End}_{\Pi^\Lambda(B)}(P) \to \text{End}_{\Pi}(\mathcal{F} \mathcal{D}).$$

On the other hand, $\varphi(\mathcal{D}) = \text{End}_{\Pi}(\mathcal{D}) = \mathcal{C}^{\vee}D \mathcal{C}$ implies $\mathcal{D} = \mathcal{L} \varphi(\mathcal{D}) \mathcal{L}^{\vee}$, so taking the inverse defines an isomorphism $\psi := \varphi^{-1} : \mathcal{D} \to \mathcal{F} \mathcal{D} \mathcal{F}^{\vee} = \text{End}_{\Pi}(\mathcal{F} \mathcal{D})$. Combining this last isomorphism with (4.8), we get the diagram of algebra maps

$$\begin{array}{ccc}
\Pi^\Lambda(A[I]) & \longrightarrow & \text{End}_{\Pi^\Lambda(B)}(P) \\
\downarrow i & & \downarrow \psi \\
\mathcal{D} & \longrightarrow & \text{End}_{\Pi}(\mathcal{F} \mathcal{D})
\end{array}$$

which obviously commutes when the dotted arrow is restricted to (4.7).

**Proposition 4.2.** There is a unique algebra isomorphism $\psi$ extending (4.7) and making (4.9) a commutative diagram.

We postpone the proof of Proposition 4.2 until Section 5.5. Meanwhile, we note that the isomorphism $\psi$ makes $P$ a left $\Pi^\Lambda(A[I])$-module and thus a progenerator from $\Pi^\Lambda(A[I])$ to $\Pi^\Lambda(A[I])$. This assigns to $\mathcal{P} = (\mathcal{D}\mathcal{L})_{\varphi}$ the Morita equivalence:

$$\text{Mod}(\Pi^\Lambda(A[I])) \to \text{Mod}(\Pi^\Lambda(A[I])), \quad V \mapsto P \otimes_{\Pi} V,$$

which, in turn, induces an isomorphism of representation varieties

$$f_P : C_n(X, I) \cong C_n(X, \mathcal{I}).$$

**Remark.** We warn the reader that (4.10) depends on the choice of a specific representative in the class $[\mathcal{P}] \in \text{Pic} (\mathcal{D})$, so, in general, we do not get an action of $\text{Pic}(\mathcal{D})$ on $\bigsqcup_{[\mathcal{I}] \in \text{Pic}(X)} C_n(X, I)$. However, we will see below (Proposition 4.3) that $f_P$ induces a well-defined action of $\text{Pic}(\mathcal{D})$ on the reduced spaces $\overline{C}_n(X, I)$

Next, we describe a natural action of the canonical bundle $\Omega^1 X$ on $C_n(X, I)$. Recall that the group homomorphism (4.12) is surjective and fits into the exact sequence (see [CCH], Theorem 1.15)

$$1 \to \Lambda \overset{\text{dlog}}{\longrightarrow} \Omega^1 X \overset{\varphi}{\longrightarrow} \text{Pic}(\mathcal{D}) \overset{\sigma}{\longrightarrow} \text{Pic}(X) \times \text{Aut}(X) \to 1,$$

where $\Lambda := A^\times / C^\times$ is the multiplicative group of (nontrivial) units in $A$. The maps $\text{dlog}$ and $\sigma$ in (4.11) are defined by

$$\text{dlog} : \Lambda \to \Omega^1 X, \quad u \mapsto u^{-1} du, \quad \sigma : \Omega^1 X \to \text{Pic}(\mathcal{D}), \quad \omega \mapsto [\mathcal{D}_\sigma],$$

where $\sigma_{\omega}$ is the automorphism of $\mathcal{D}$ acting identically on $A$ and mapping $\theta \in \text{Der}(A)$ to $\omega(\theta) + \theta \in \mathcal{D}_1$. Since the action of $\text{Pic}(\mathcal{D})$ on $\text{Pic}(X)$ factors through $g$, the image of $\Omega^1 X$ in $\text{Pic}(\mathcal{D})$ under $\sigma$ stabilizes
each point of $\text{Pic}(X)$, and therefore, by equivariance of $\gamma$, preserves every fibre $\gamma^{-1}[z] \subseteq \mathcal{J}(\mathcal{D})$. Thus, writing $\Gamma := \Omega^1(X)/\Lambda$ and identifying $\Gamma$ with $\text{Im}(\iota)$, we get an action
\begin{equation}
\Gamma \times \gamma^{-1}[z] \to \gamma^{-1}[z], \quad [z] \in \text{Pic}(X).
\end{equation}

Now, let $(\Omega^1B)_2 := \Omega^1(B)/[B, \Omega^1B]$, where $B := A[\mathcal{I}]$. Using the fact that $B$ is smooth, we identify $(\Omega^1B)_2 \cong B \otimes_{B^e} \Omega^1(B) \cong B \otimes_{B^e} (\Omega^1B)^{**} \cong \text{Hom}_{B^e}((\Omega^1B)^*, B)$, where $(-)^*$ stands for the dual over $B^e$. Explicitly, under this identification, $\hat{\omega} = \omega \pmod{[B, \Omega^1B]} \in (\Omega^1B)_2$ corresponds to the homomorphism
\begin{equation}
\hat{\omega} : \Omega^1(B)^* \to B, \quad \delta \mapsto \mu^e[\delta(\omega)],
\end{equation}
where $\mu^e : B^e \to B$ is the opposite multiplication map. The additive group $(\Omega^1B)_2$ acts naturally on $T_B(\Omega^1B)^*$; for $\omega \in (\Omega^1B)_2$, we have an automorphism $\tilde{\sigma}_{\omega}$ of $T_B(\Omega^1B)^*$ acting identically on $B$ and mapping
\begin{align}
(\Omega^1B)^* \to B \oplus (\Omega^1B)^* \hookrightarrow T_B(\Omega^1B)^*, \quad \delta \mapsto \omega(\delta) + \delta.
\end{align}
The assignment $\omega \mapsto \tilde{\sigma}_{\omega}$ defines then a group homomorphism
\begin{equation}
\tilde{\sigma} : (\Omega^1B)_2 \to \text{Aut}_B[T_B(\Omega^1B)^*].
\end{equation}
Identifying $\Omega^1X$ with the group of Kähler differentials of $A$, we now construct an embedding $\Omega^1X \hookrightarrow (\Omega^1B)_2$. For this, we consider the exact sequence
\begin{equation}
0 \to H_1(B) \cong (\Omega^1B)_2 \to B \to H_0(B) \to 0,
\end{equation}
obtained by tensoring $0 \to \Omega^1(B) \to B^e \to B \to 0$ with $B$, and compose the connecting map $\alpha$ in (4.10) with natural isomorphisms (see [CE], Th. 1.2.15 and Prop. 1.1.10, respectively)
\begin{equation}
H_1(B) \cong H_1(A) \cong \Omega^1X.
\end{equation}
Now, for any algebra $B$, we have (see [CE], Ex. 19, p. 126)
\begin{equation}
H_1(B) = \text{Tor}_{\Omega^1}(\Omega^1B, B) \cong \Omega^1(B) \cap \Omega^1(\Omega^1B)^e/\Omega^1(B) \cdot \Omega^1(B)^e,
\end{equation}
where $\Omega^1(\Omega^1B)^e := \text{Ker}(\mu^e)$. Hence, if $\bar{\omega} \in \text{Im} \alpha$ in (4.16), then $\Delta_B(\omega) \in \Omega^1(B) \cap \Omega^1(\Omega^1B)^e \subseteq \Omega^1(B)^e$, so by (4.14), $\hat{\omega}(\Delta_B) = 0$ and $\tilde{\sigma}_{\omega}(\Delta_B) = \Delta_B$. Thus, combining (4.15) with (4.16) and (4.17), we may define
\begin{equation}
\sigma : \Omega^1X \cong (\Omega^1B)_2 \xrightarrow{\tilde{\sigma}} \text{Aut}_B[T_B(\Omega^1B)^*] \to \text{Aut}_B[\Pi^X(B)],
\end{equation}
where the last map is induced by the algebra projection: $T_B(\Omega^1B)^* \to \Pi^X(B)$. An explicit description of $\tilde{\sigma}$ will be given in Section 5.4 (see Lemma 5.10).

Now, the group $\text{Aut}_B[\Pi^X(B)]$ acts on $\text{Rep}_S(\Pi^X(B), n)$ in the natural way: if $\varphi : \Pi^X(B) \to \text{End}(V)$ represents a point in $\text{Rep}_S(\Pi^X(B), n)$, then $\sigma, \varphi = \varphi \sigma^{-1}$ for $\sigma \in \text{Aut}_B[\Pi^X(B)]$. Clearly, this commutes with the $G_S(n)$-action on $\text{Rep}_S(\Pi^X(B), n)$, and hence induces an action of $\text{Aut}_B[\Pi^X(B)]$ on $\mathcal{C}_n(X, \mathcal{I})$. Restricting this last action to $\Omega^1X$ via (4.18), we define
\begin{equation}
\sigma^* : \Omega^1X \to \text{Aut}[\mathcal{C}_n(X, \mathcal{I})], \quad \omega \mapsto [\sigma^*_\omega : \varphi \mapsto \varphi \sigma^{-1}],
\end{equation}
Equivalently, $\sigma^*_\omega$ is defined on $\mathcal{C}_n(X, \mathcal{I})$ by twisting the structure of $\Pi^X(B)$-modules by $\sigma^{-1}$, i.e., $[V] \mapsto [V^{\sigma^{-1}}]$. Restricting (4.19) further to $\Lambda$, via (4.12), we define the quotient varieties
\begin{equation}
\overline{\mathcal{C}}_n(X, \mathcal{I}) := \mathcal{C}_n(X, \mathcal{I})/\Lambda.
\end{equation}
These varieties come equipped with the induced action of the group $\Gamma = \Omega^1(X)/\Lambda$.

**Proposition 4.3.** (1) The action (4.19) agrees with (4.10): if $\mathcal{P} = \mathcal{D}_\sigma$, then $f_\omega = \sigma^*_\omega$ for all $\omega \in \Omega^1X$.

(2) The map (4.10) induces an isomorphism of quotient varieties $f_\mathcal{P} : \overline{\mathcal{C}}_n(X, \mathcal{I}) \to \overline{\mathcal{C}}_n(X, \mathcal{J})$, which depends only on the class of $\mathcal{P}$ in $\text{Pic}(\mathcal{D})$. 

We will prove Proposition 1.3 in Section 4.3. Here, we make only two remarks.

1. It follows from Proposition 1.3 that the action of \( \Lambda \) on \( C_n(X, \mathcal{I}) \) defined above coincides with the natural action of \( \text{Aut}(\mathcal{I}) = A^\times \), so \( C_n(X, \mathcal{I}) \) depends only on the class of \( \mathcal{I} \) in \( \text{Pic}(X) \) and the definition (4.20) agrees with the one given in the introduction.

2. For each \( n \geq 0 \), let \( \overline{C}_n(X) \) denote the disjoint union of \( \overline{C}_n(X, \mathcal{I}) \) over all \( [\mathcal{I}] \in \text{Pic}(X) \). By part (2) of Proposition 1.3 the assignment \([\mathcal{I}] \mapsto f_{\mathcal{I}}^*\) defines then an action of \( \text{Pic}(\mathcal{D}) \) on \( \overline{C}_n(X) \), and part (1) says that this action restricts to the action of \( \Gamma \) on each individual fibre \( \overline{C}_n(X, \mathcal{I}) \), i.e. \( f_{\mathcal{I}}(\omega) = \sigma^*_\omega \) for all \( \omega \in \Gamma \).

### 4.3. The main theorem

We may now put pieces together and state the main result of the present paper. We recall the functor \( L_1 i^* = \text{Tor}^1(\mathcal{D}, -) : \text{Mod}(\Pi) \to \text{Mod}(\mathcal{D}) \) associated to \( i : \Pi \to \mathcal{D} \): when restricted to finite-dimensional representations, this functor is given by (1.3).

**Theorem 4.3.** Let \( X \) be a smooth affine irreducible curve over \( \mathbb{C} \).

(a) For each \( n \geq 0 \) and \( [\mathcal{I}] \in \text{Pic}(X) \), the functor (1.3) induces an injective map
\[
\omega_n : \overline{C}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}],
\]
which is equivariant under the action of the group \( \Gamma \).

(b) Amalgamating the maps \( \omega_n \) for all \( n \geq 0 \) gives a bijection
\[
\omega : \bigsqcup_{n \geq 0} \overline{C}_n(X, \mathcal{I}) \xrightarrow{\sim} \gamma^{-1}[\mathcal{I}].
\]

(c) For any \( [\mathcal{I}] \) and \( [\mathcal{J}] \) in \( \text{Pic}(X) \) and for any \( [\mathcal{P}] \in \text{Pic}(\mathcal{D}) \), such that \( [\mathcal{P}] \cdot [\mathcal{I}] = [\mathcal{J}] \), there is a commutative diagram:
\[
\begin{array}{ccc}
\overline{C}_n(X, \mathcal{I}) & \xrightarrow{f_{\mathcal{P}}} & \overline{C}_n(X, \mathcal{J}) \\
\omega_n \downarrow & & \downarrow \omega_n \\
\gamma^{-1}[\mathcal{I}] & \xrightarrow{[\mathcal{P}]} & \gamma^{-1}[\mathcal{J}]
\end{array}
\]
where \( f_{\mathcal{P}} \) is an isomorphism induced by (4.10).

**Remark.** For technical reasons, we assumed above that \( X \neq \mathbb{A}^1 \). Theorem 1.3 holds true, however, in general: if \( X = \mathbb{A}^1 \), the map \( \omega \) induced by \( i^* \) agrees with the Calogero-Moser map constructed in [BW1, BW2] (see [BCE], Theorem 1). In this case, the ring \( \mathcal{D} \) is isomorphic to the Weyl algebra \( A_1(\mathbb{C}) \), \( \text{Pic}(\mathcal{D}) \) is isomorphic to the automorphism group \( \text{Aut}(A_1) \) of \( A_1 \) (see [St]) and \( \Gamma \) corresponds to the subgroup of KP flows in \( \text{Aut}(A_1) \) (see [BW1]). Since \( \text{Pic}(\mathbb{A}^1) \) is trivial, the last part of Theorem 1.3 implies the equivariance of \( \omega \) under the action of \( \text{Aut}(A_1) \).

### 5. PROOF OF THE MAIN THEOREM

We proceed in four steps. First, we show that the functor (1.3) induces well-defined maps \( \hat{\omega}_n : C_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}] \), one for each integer \( n \geq 0 \). Second, we prove that every class \( [M] \in \gamma^{-1}[\mathcal{I}] \) is contained in the image of \( \hat{\omega}_n \) for some \( n \) (which is uniquely determined by \( [M] \)). Third, we check that \( \hat{\omega}_n \) factors through the action of \( \Lambda \) on \( C_n(X, \mathcal{I}) \) and prove that the induced map \( \omega_n : \overline{C}_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}] \) is injective and \( \Gamma \)-equivariant. Finally, we prove Propositions 4.2 and 4.3 of Section 4.2 and show that the diagram (4.21) in Theorem 4.3 is commutative.

We begin by describing the algebras \( \Pi^\lambda(B) \) in terms of generators and relations.

#### 5.1. The structure of \( \Pi^\lambda(B) \)

Recall that, for each \( \lambda \in S \), we defined these algebras by formula (3.5), where \( \Delta_B \in \text{Der}(B) \) is the distinguished derivation mapping \( x \mapsto x \otimes 1 - 1 \otimes x \). Now, \( \text{Der}(B) \) contains a canonical sub-bimodule \( \text{Der}_S(B) \), consisting of \( S \)-linear derivations. We write \( \Delta_{B, S} : B \to B \otimes B \) for the inner derivation \( x \mapsto \text{ad}_e(x) \), with \( e := e_e + e_\infty \otimes e_\infty \in B \otimes B \). It is easy to see that \( \Delta_{B, S}(x) = 0 \) for all \( x \in S \), so \( \Delta_{B, S} \in \text{Der}_S(B) \).
Lemma 5.1. For any \(\lambda \in S\), there is a canonical algebra isomorphism
\[
\Pi^\lambda(B) \cong T_B \operatorname{Der}_S(B)/\langle \Delta_{B,S} - \lambda \rangle .
\]

Proof. By universal property, the natural embedding of bimodules \(\operatorname{Der}_S(B) \hookrightarrow \operatorname{Der}(B)\) extends to their tensor algebras. Combined with canonical projection, this yields the algebra map \(\phi : T_B \operatorname{Der}_S(B) \to T_B \operatorname{Der}(B)\). An easy calculation shows that \(\Delta_{B,S} = e \Delta_B e + e_\infty \Delta_B e_\infty\) in \(\operatorname{Der}(B)\). So \(\Delta_{B,S} - \lambda = e (\Delta_B - \lambda) e + e_\infty (\Delta_B - \lambda) e_\infty\) belongs to the ideal \(\langle \Delta_B - \lambda \rangle \subseteq T_B \operatorname{Der}(B)\), and hence \(\phi\) vanishes on \(\Delta_{B,S} - \lambda\), inducing an algebra map
\[
T_B \operatorname{Der}_S(B)/\langle \Delta_{B,S} - \lambda \rangle \to \Pi^\lambda(B) .
\]

We leave as an exercise to the reader to check that (5.1) is an isomorphism. \(\square\)

By Lemma 5.1 the structure of \(\Pi^\lambda(B)\) is determined by the bimodule \(\operatorname{Der}_S(B)\). We now describe this bimodule explicitly, in terms of \(A, I\) and the dual module \(I^\vee = \operatorname{Hom}_A(I, A)\). To fix notation we begin with a few fairly obvious remarks on bimodules over one-point extensions.

A bimodule \(\Xi\) over \(B\) is characterized by the following data: an \(A\)-bimodule \(T\), a left \(A\)-module \(V\), a right \(A\)-module \(W\) and a \(\mathbb{C}\)-vector space \(\Delta\) given together with three \(A\)-module homomorphisms \(f_1 : I \otimes V \to T\), \(f_2 : I \otimes W \to U\), \(g_1 : T \otimes_A I \to U\) and a \(\mathbb{C}\)-linear map \(g_2 : V \otimes_A I \to W\), which fit into the commutative diagram
\[
\begin{array}{ccc}
I \otimes V \otimes_A I & \overset{\text{Id} \otimes g_2}{\longrightarrow} & I \otimes W \\
\downarrow_{f_1 \otimes_A \text{Id}} & & \downarrow_{f_2} \\
T \otimes_A I & \overset{g_1}{\longrightarrow} & U
\end{array}
\]

These data can be conveniently organized by using the matrix notation
\[
\Xi = \begin{pmatrix}
T & U \\
V & W
\end{pmatrix},
\]
with understanding that \(B\) acts on \(\Lambda\) by the usual matrix multiplication, via the maps \(f_1, f_2, g_1\) and \(g_2\). Note that the components of \(T\) are determined by
\[
T = e \Xi e, \quad U = e \Xi e_\infty, \quad V = e_\infty \Xi e, \quad W = e_\infty \Xi e_\infty,
\]
and the commutativity of (5.2) ensures the associativity of the action of \(B\). For example, the free bimodule \(B \otimes B\) can be decomposed into a direct sum of four bimodules, each of which is easy to identify using (5.4):
\[
Be \otimes eB \cong \begin{pmatrix}
A \otimes A & A \otimes I \\
0 & 0
\end{pmatrix}, \quad Be \otimes e_\infty B \cong \begin{pmatrix}
0 & A \\
0 & 0
\end{pmatrix},
\]
\[
Be_\infty \otimes eB \cong \begin{pmatrix}
I \otimes I & I \otimes I \\
A & A
\end{pmatrix}, \quad Be_\infty \otimes e_\infty B \cong \begin{pmatrix}
0 & I \\
0 & 0
\end{pmatrix}.
\]

With this notation, the bimodule \(\operatorname{Der}_S(B)\) can be described as follows.

Lemma 5.2. There is an isomorphism of \(B\)-bimodules
\[
\operatorname{Der}_S(B) \cong \begin{pmatrix}
\operatorname{Der}(A) & \operatorname{Der}(A, I \otimes A) \\
0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
I \otimes I^\vee & I \otimes A \\
I^\vee & A
\end{pmatrix},
\]
with \(\Delta_{B,S}\) corresponding to the element
\[
\begin{pmatrix}
\Delta_A & 0 \\
0 & 0
\end{pmatrix} - \begin{pmatrix}
0 & \sum_i v_i \otimes w_i \\
0 & 1
\end{pmatrix},
\]
where \(\{v_i\}\) and \(\{w_i\}\) are dual bases for the projective \(A\)-modules \(I\) and \(I^\vee\).
Proof. With identifications (5.5) and (5.9), it is easy to show that

\[(5.8) \quad \Omega_S^1 B \cong \left( \begin{array}{cc} \Omega^1 A & \Omega^1 A \otimes \mathcal{I} \\ 0 & 0 \end{array} \right) \bigoplus \left( \begin{array}{cc} 0 & \mathcal{I} \\ 0 & 0 \end{array} \right), \]

with inclusion \( \Omega^1_S B \hookrightarrow B \otimes_S B = (Be \otimes eB) \oplus (Be_\infty \otimes e_\infty B) \) corresponding to the map \((i, s)\), where \(i\) is the natural embedding of the first summand of \(\Omega^1_S B\) into \(Be \otimes eB\), see (5.8), and \(s\) is a \(B\)-bimodule section \(\mathcal{I} \rightarrow B \otimes_S B\) given by

\[s : \left( \begin{array}{cc} 0 & b \\ 0 & 0 \end{array} \right) \mapsto \left[ \begin{array}{cc} 0 & -\sum_i w_i(b) \otimes v_i \\ 0 & 0 \end{array} \right], \quad b \in \mathcal{I}.\]

Note that \(i\) is canonical, while \(s\) depends on the choice of dual bases for \(\mathcal{I}\) and \(\mathcal{I}'\).

To describe \(\text{Der}_S(B)\) we now dualize (5.5) and use \(\text{Der}_S(B) = \text{Hom}_{Be}(\Omega^1_S B, B^{\otimes 2})\), which after trivial identifications yields

\[(5.9) \quad \text{Der}_S(B) \cong \left( \begin{array}{cc} \text{Der}(A) & \text{Der}(A, \mathcal{I} \otimes A) \\ 0 & 0 \end{array} \right) \bigoplus \left( \begin{array}{cc} \mathcal{I} \otimes \mathcal{I}' & \mathcal{I} \otimes \text{End}_A(\mathcal{I}) \\ \mathcal{I}' & \text{End}_A(\mathcal{I}) \end{array} \right), \]

Since \(A\) is commutative and \(\mathcal{I}\) is a rank 1 projective, \(\text{End}_A(\mathcal{I}) = A\), so (5.9) is the required decomposition.

With identification (5.8), the element (5.7) corresponds to the embedding \((i, s) : \Omega^1_S B \rightarrow (Be \otimes eB) \oplus (Be_\infty \otimes e_\infty B) \rightarrow B \otimes B\), which, in turn, corresponds under (5.9) to the element \(\Delta_{B,S} \in \text{Der}_S(B)\). \(\Box\)

Now, using the isomorphism of Lemma 5.1 we identify \(\Pi^\lambda(B)\) as a quotient of the tensor algebra of the bimodule \(\text{Der}_S(B)\). Keeping the notation of Lemma 5.2 we then have

**Proposition 5.1.** The algebra \(\Pi^\lambda(B)\) is generated by (the images of) the following elements

\[\hat{a} := \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right), \quad \hat{v}_i := \left( \begin{array}{cc} 0 & v_i \\ 0 & 0 \end{array} \right), \quad \hat{d} := \left( \begin{array}{cc} d & 0 \\ 0 & 0 \end{array} \right), \quad \hat{w}_i := \left( \begin{array}{cc} 0 & 0 \\ w_i & 0 \end{array} \right),\]

where \(\hat{a}, \hat{v}_i \in B\) and \(\hat{d}, \hat{w}_i \in \text{Der}_S(B)\) with \(d \in \text{Der}(A)\). Apart from the obvious relations induced by matrix multiplication, these elements satisfy

\[(5.10) \quad \hat{a} \cdot \hat{v}_i = \lambda e, \quad \hat{d} \cdot \hat{v}_i = \lambda e, \quad \hat{d} \cdot \hat{w}_i = \lambda_\infty e_\infty,\]

where \(\cdot\) denotes the action of \(B\) on the bimodule \(\text{Der}_S(B)\).

Proof. By Lemma 2.2 the matrices \(\{\hat{a}\}\) and \(\{\hat{v}_i\}\) generate the algebra \(B\), while \(\{\hat{d}\}\) and \(\{\hat{w}_i\}\) generate the first and the second bimodule summand of (5.9) respectively. All together they thus generate the tensor algebra. Now, the ideal \(\langle \Delta_{B,S} - \lambda \rangle\) in \(\Pi^\lambda(B)\) is generated by \(e(\Delta_{B,S} - \lambda)e = e\Delta_{B,S}e - \lambda e\) and \(e_\infty(\Delta_{B,S} - \lambda)e_\infty = e_\infty\Delta_{B,S}e_\infty - \lambda_\infty e_\infty\), since the sum of these elements is equal to \(\Delta_{B,S} - \lambda\).

With identification of Lemma 5.2 we then have

\[e \Delta_{B,S} e = \hat{a} - \sum_{i=1}^N \hat{v}_i \cdot \hat{w}_i, \quad e_\infty \Delta_{B,S} e_\infty = \sum_{i=1}^N \hat{w}_i \cdot \hat{v}_i,\]

whence the relations (5.10). \(\Box\)

Using Proposition 5.1 we now prove two technical results, which we use repeatedly in this paper (Lemma 5.3 below was already mentioned in Section 4.1).

**Lemma 5.3.** If \(\lambda_\infty \neq 0\), the algebra \(\Pi^\lambda(B)\) is Morita equivalent to \(e \Pi^\lambda(B)e\).
Proof. By standard Morita theory, it suffices to show that \( \Pi e_\infty \otimes U e_\infty \Pi \rightarrow \Pi \) gives a projective resolution of \( \mathcal{D} \) in the category of (left and right) \( \Pi \)-modules, see \((\ref{5.12})\).

\[
\Pi e_\infty \otimes U e_\infty \Pi \rightarrow \Pi \]

which is a homomorphism of right \( e_\infty \Pi \)-modules. Since \( e_\infty = e_\infty B e_\infty = \sum_i e_\Pi \tilde{v}_i \), the map \((\ref{5.12})\) is surjective. On the other hand, using filtrations, it is easy to show that the composition of \((\ref{5.12})\) with multiplication map \( e_\infty \otimes_U e_\infty \Pi \rightarrow e_\Pi \) is injective. Hence \((\ref{5.12})\) is injective and therefore an isomorphism. This implies that \( e_\infty \otimes_U e_\infty \Pi \) is a right projective \( e_\Pi \)-module (since obviously so is \( DI \otimes e_\infty \Pi \), and \( 0 \rightarrow e_\Pi e_\infty \otimes_U e_\infty \Pi \rightarrow e_\Pi D \rightarrow 0 \) is an exact sequence of \( e_\Pi \)-modules. By Morita equivalence of Lemma \(\textbf{5.3}\) the complex \( 0 \rightarrow \Pi e_\infty \otimes_U e_\infty \Pi \rightarrow \Pi \rightarrow 0 \) is then a projective resolution of \( \mathcal{D} \) in the category of right \( \Pi \)-modules. A similar argument shows that this complex is also a projective resolution of \( \mathcal{D} \) as a left \( \Pi \)-module.

\(\square\)

5.2. The map \( \omega \) is well-defined. We show that the functor \textbf{4.13} maps the \( \Pi \)-modules of dimension vector \( n = (n, 1) \) to rank 1 torsion-free \( \mathcal{D} \)-modules \( M \) with \( \gamma[M] = [\mathcal{I}] \).

Let \( V \) be a \( \Pi \)-module of dimension vector \( n \), and let \( L := \Pi e_\infty \otimes_U e_\infty V \). Write \( V := e_\infty V \), \( V_\infty := e_\infty V \), and similarly \( L := e_\mathcal{L} \), \( L_\infty := e_\mathcal{L} \mathcal{L} \) (so that \( \text{dim } V_\infty = \text{dim } L_\infty = 1 \)). Fix a vector \( \xi \neq 0 \) in \( V_\infty \), and define a character \( \varepsilon : U \rightarrow \mathbb{C} \) by \( u \varepsilon = u \xi / \xi \) for all \( u \in U \). Note that \( \varepsilon \) does not depend on the choice of \( \xi \) and uniquely determines \( L \) (and \( V \)). In fact, we have the isomorphism of \( \Pi \)-modules

\[
\Pi e_\infty / \sum_{u \in U} \Pi e_\infty (u - \varepsilon(u)) \cong L , \quad [e_\mathcal{L}] \mapsto e_\infty \otimes \xi .
\]

Now, under the equivalence of Lemma \textbf{5.3} \( \mu : L \rightarrow V \) transforms to a homomorphism of \( e_\Pi \)-modules \( \mu : L \rightarrow V \). Since \( e_\infty (\text{Ker } \mu) = 0 \), we have \( \text{Ker } \mu = e_\Pi \text{Ker } \mu ) = \text{Ker } \mu \). Thus \((\varepsilon(V)) \cong \text{Ker } \mu \), which is naturally an isomorphism of \( \mathcal{D} \)-modules via \( i|_{e_\Pi} : e_\Pi \rightarrow \mathcal{D} \).

Next, we set \( R := T_A \text{Der}(\mathcal{A}) \) and define the algebra map

\[
R \rightarrow e_\Pi \mathcal{A} , \quad a \mapsto \hat{a} , \quad d \mapsto \hat{d} ,
\]

where \( a \in A \) and \( d \in \text{Der}(A) \). Extending the notation of Proposition \textbf{5.1} we will write \( \hat{r} \in e_\Pi \mathcal{A} \) for the image of any element \( r \in R \) under \((\ref{5.14})\). Note that the natural projection \( R \rightarrow \Pi^1(A) = \mathcal{D} \) factors through \((\ref{5.14})\), and the corresponding quotient map is \( i|_{e_\Pi} \). The following observation is an easy consequence of \((\ref{5.13})\) and Lemma \textbf{5.4}.

Lemma \textbf{5.5}. There is an isomorphism of \( R \)-modules

\[
L \cong R / \sum_{i=1}^{N} \sum_{r \in R} \left( (\Delta_A - 1) r v_i - \sum_{j=1}^{N} \varepsilon(\hat{w}_j \hat{v}_i) v_j \right) ,
\]

where \( L \) is regarded as an \( R \)-module via \((\ref{5.14})\), and \( R \mathcal{I} := R \otimes_A \mathcal{I} \).
Proof. If we identify \( A \cong eBe \subset e\Pi e \), \( \mathcal{I} \cong eBe_\infty \subset e\Pi e_\infty \) as in Lemma 5.4, the required isomorphism is induced by

\[
R\mathcal{I} \xrightarrow{\pi_1} e\Pi e \otimes_{eBe} eBe_\infty \xrightarrow{\pi_2} e\Pi e_\infty \rightarrow e\Pi e_\infty \left/ \sum_{u \in U} e\Pi e_\infty \left( u - \varepsilon(u) \right) \right.,
\]

where \( \pi_1 \) is the product of (5.14) with \( \mathcal{I} \) and \( \pi_2 \) is the multiplication map.

Now, the tensor algebra filtration on \( R = T_A \text{Der}(A) \) induces the differential filtration on \( \mathcal{D} \) via the canonical projection and module filtrations on \( L \) and \( M \subset L \) via the isomorphism of Lemma 5.5. Writing \( \mathcal{D}, \mathcal{T}, \ldots \) for the associated graded objects relative to these filtrations, we have

\[
\mathcal{M} \subset \mathcal{L} \cong \mathcal{R}/\mathcal{R} \Delta A \mathcal{R} \cong (\mathcal{R}/\mathcal{R} \Delta A \mathcal{R}) \otimes_A \mathcal{I} \cong \tilde{\mathcal{B}} \otimes_A \mathcal{I} \cong \tilde{\mathcal{B}} \mathcal{I}.
\]

It follows that \( M \) is a rank 1 torsion-free module (as so is \( \mathcal{M} \)). Moreover, since \( \dim \mathcal{T}/\mathcal{M} = \dim L/M < \infty \), by Theorem 5.1(a), \( \gamma[M] = [\mathcal{Z}] \). This completes Step 1.

5.3. The map \( \omega \) is surjective. Given a rank 1 torsion-free \( \mathcal{D} \)-module \( M \), we now construct a \( \Pi \)-module \( L \), together with a \( \Pi^1(B) \)-module embedding \( \mathcal{M} \hookrightarrow L \), such that \( V := L/M \) has dimension \( (n,1) \) and \( i^*[V] \cong M \).

We begin with some preparations. We let \( \mathcal{D} := \bigoplus_{k \geq 0} A t^k \) denote the Rees algebra of the ring \( \mathcal{D} \) with respect to its canonical filtration \( \{D_k\} \), and let \( \text{GrMod}(\mathcal{D}) \) be the category of graded \( \mathcal{D} \)-modules. There is a natural homomorphism of graded rings \( p : \mathcal{D} \rightarrow \mathcal{D} \), mapping \( a t^k \in D_k \) to \( a (\bmod D_{k-1}) \in \mathcal{D}_k \). Using this homomorphism, we will regard graded \( \mathcal{D} \)-modules as objects of \( \text{GrMod}(\mathcal{D}) \). Since \( \ker(p) = \langle t \rangle \), we may identify \( \mathcal{D} \cong \mathcal{D}/\langle t \rangle \). This implies that \( \mathcal{D} \) is Noetherian, since so is \( \mathcal{B} \) (see [1], Prop. 3.5).

Next, following [AZ], we define \( \text{Tors}(\mathcal{D}) \) to be the full subcategory of \( \text{GrMod}(\mathcal{D}) \) consisting of torsion modules. By definition, \( \mathcal{M} \in \text{GrMod}(\mathcal{D}) \) is torsion, if for every \( m \in \mathcal{M} \) there is \( k_m \in \mathbb{N} \) such that \( \mathcal{D}_k m = 0 \) for all \( k \geq k_m \). By [AZ], Sect. 2, \( \text{Tors}(\mathcal{D}) \) is a localizing subcategory of \( \text{GrMod}(\mathcal{D}) \): i.e., the inclusion functor \( \text{Tors}(\mathcal{D}) \hookrightarrow \text{GrMod}(\mathcal{D}) \) has a right adjoint \( \tau : \text{GrMod}(\mathcal{D}) \rightarrow \text{Tors}(\mathcal{D}) \), which assigns to a graded module \( \mathcal{N} \) its largest torsion submodule \( \tau(\mathcal{N}) = \{ m \in \mathcal{N} : \mathcal{D}_k m = 0 \text{ for all } k \gg 0 \} \). The functor \( \tau \) is left exact, and we write \( \tau_k := R^k \tau : \text{GrMod}(\mathcal{D}) \rightarrow \text{Tors}(\mathcal{D}) \) for its derived functors.

We also introduce the quotient category \( \text{Qgr}(\mathcal{D}) := \text{GrMod}(\mathcal{D})/\text{Tors}(\mathcal{D}) \). This is an abelian category that comes equipped with two canonical functors: the (exact) quotient functor \( \pi : \text{GrMod}(\mathcal{D}) \rightarrow \text{Qgr}(\mathcal{D}) \) and its right adjoint (and hence left exact) functor \( \omega : \text{Qgr}(\mathcal{D}) \rightarrow \text{GrMod}(\mathcal{D}) \). The relation between \( \pi \), \( \omega \) and \( \tau \) is described by the following result which is part of standard torsion theory (see, e.g., [AZ], Prop. 7.2).

Theorem 5.1. (1) The adjunction map \( \eta_\mathcal{M} : \mathcal{M} \rightarrow \omega \pi(\mathcal{M}) \) fits into the exact sequence

\[
0 \rightarrow \tau(\mathcal{M}) \rightarrow \mathcal{M} \xrightarrow{\eta_\mathcal{M}} \omega \pi(\mathcal{M}) \rightarrow \tau_1(\mathcal{M}) \rightarrow 0,
\]

which is functorial in \( \mathcal{M} \in \text{GrMod}(\mathcal{D}) \).

(2) For \( k \geq 1 \), there are natural isomorphisms

\[
R^k \omega(\pi \mathcal{M}) \cong \tau_{k+1}(\mathcal{M}) .
\]

In particular, if \( k \geq 1 \), the modules \( R^k \omega(\pi \mathcal{M}) \) are torsion.

Now, given a graded module \( \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k \) and \( n \in \mathbb{Z} \), we write \( \mathcal{M}[n] := \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_{k+n} \) and \( \mathcal{M}_{\geq n} := \bigoplus_{k \geq n} \mathcal{M}_k \). Both are graded modules, \( \mathcal{M}_{\geq n} \) being a submodule of \( \mathcal{M} \). With this notation, we compute \( R^k \omega(\pi \mathcal{D}) \), regarding \( \mathcal{D} \) as a \( \mathcal{D} \)-module via the algebra map \( p : \mathcal{D} \rightarrow \mathcal{D} \).

Lemma 5.6. (1) The canonical map \( \eta_\mathcal{D} : \mathcal{D} \rightarrow \omega \pi(\mathcal{D})_{\geq 0} \) is an isomorphism.

(2) \( R^k \omega(\pi \mathcal{D}) = 0 \) for \( k \geq 1 \).
Proof. For graded $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{N}$, we define (cf. [AZ], Sect. 3)
\[ \text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\text{GrMod}(\mathcal{D})}(\mathcal{M}, \mathcal{N}[k]) , \]
and write $\text{Ext}^n_{\mathcal{D}}(\mathcal{M}, \mathcal{N})$ for the corresponding Ext-groups. Combining [AZ], Theorem 8.3 and Proposition 7.2, we then identify
\[ R^k \omega(\pi \mathcal{D}) \cong \lim_{\rightarrow} \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{\geq n}, \mathcal{D}) , \quad \forall k \geq 0 . \]
To compute the Ext-groups in (5.18), we use the long cohomology sequence
\[ \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{-n}, \mathcal{D}) \to \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{\geq n}, \mathcal{D}) \to \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{\geq n+1}, \mathcal{D}) \to \text{Ext}^{k+1}_{\mathcal{D}}(\mathcal{D}_{-n}, \mathcal{D}) \]
(5.19)
\[ \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{-n}, \mathcal{D}) = 0 \quad \text{if} \quad k \neq 1 , \]
and
\[ \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{-n}, \mathcal{D})_m \cong \begin{cases} 0 & \text{if } m \neq -n - 1 \\ \text{Sym}^{-m}(\Omega^1 X) & \text{if } m = -n - 1 \end{cases} , \]
(5.20)
where $\text{Sym}^q$ stands for the $q$-th symmetric power over $A$. It is easy to see that (5.19), (5.20) and (5.21), together with (5.18), formally imply both statements of the lemma. (In addition, we have $\omega(\pi \mathcal{D})_n \cong \text{Sym}^{-n}(\Omega^1 X)$ for $n < 0$.)

To prove (5.20) and (5.21) we observe that
\[ \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{-n}, \mathcal{D}) \cong \text{Ext}^k_{\mathcal{D}}(\mathcal{D}, \mathcal{D})[n] , \]
where $\mathcal{D}$ is a graded $\mathcal{D}$-module with a single component in degree 0. Such modules arise by restricting scalars via the algebra projection $\mathcal{D} \to A$. So we can compute $\text{Ext}^k_{\mathcal{D}}(\mathcal{D}, \mathcal{D})$ using the spectral sequence
\[ \text{Ext}^p_{\mathcal{D}}(\mathcal{D}, \mathcal{D}) \Rightarrow \text{Ext}^{p+q}_{\mathcal{D}}(\mathcal{D}, \mathcal{D}) . \]
(5.22)
To this end, we identify $\mathcal{D}$ with the symmetric algebra $\text{Sym}_A(\text{Der} A)$ and use the canonical resolution
\[ 0 \to \mathcal{D} \otimes_A \text{Der}(A)[-1] \to \mathcal{D} \to A \to 0 . \]
(5.23)
It follows from (5.23) that $\text{Ext}^p_{\mathcal{D}}(A, \mathcal{D}) = 0$ for $q \neq 1$, so (5.22) collapses on the line $q = 1$, giving (after natural identifications) the isomorphisms (5.20) and (5.21).

Lemma 5.7. If $\mathcal{I}$ is a flat $A$-module, then
\[ R^k \omega(\pi \mathcal{M} \otimes_A \mathcal{I}) \cong R^k \omega(\pi \mathcal{M}) \otimes_A \mathcal{I} , \quad \forall k \geq 0 , \]
for any graded $\mathcal{D}$-bimodule $\mathcal{M}$.

Proof. By [AZ], Prop. 7.2(1), we have $R^k \omega(\pi \mathcal{M} \otimes_A \mathcal{I}) \cong \lim_{\rightarrow} \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{\geq n}, \mathcal{M} \otimes_A \mathcal{I})$. Since $\lim_{\rightarrow}$ commutes with tensor products, it suffices to prove that
\[ \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{\geq n}, \mathcal{M} \otimes_A \mathcal{I}) \cong \text{Ext}^k_{\mathcal{D}}(\mathcal{D}_{\geq n}, \mathcal{M}) \otimes_A \mathcal{I} \quad \text{for } n > 0 . \]
(5.24)
Furthermore, by functoriality, it suffices to prove (5.24) only for $k = 0$, but in that case the result is well known (see [Ro], Lemma 3.83). □

Now, we turn to our problem. As in Section 3.2, we choose a good filtration $\{M_k\}$ on $M$ so that $\mathcal{M} := \bigoplus_{k \in \mathbb{Z}} M_k/M_{k-1}$ is a torsion-free $\mathcal{D}$-module. Then, by Theorem 3.1 there is an ideal $\mathcal{I} \subseteq A$ (unique up to isomorphism) and a graded embedding
\[ \mathcal{J} : \mathcal{M} \hookrightarrow \mathcal{D} \mathcal{I} , \]
(5.25)
such that $\dim \text{Coker}(\mathcal{J}) < \infty$. The filtration $\{M_k\}$ is uniquely determined by $M$ up to a shift of degree (cf. Lemma 5.12 below): we fix this shift by requiring $\mathcal{J}$ to be of degree 0. The dimension $n := \dim \text{Coker}(\mathcal{J})$ is then an invariant of $M$, independent of the choice of filtration.
Since \( \eta : \text{Id} \to \omega \pi \) is a natural transformation, the map (5.24) fits into the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{I}} & \mathcal{D} \\
\downarrow{\eta \pi} & & \downarrow{\eta \pi} \\
\omega \pi(M) & \xrightarrow{\omega \pi(\mathcal{I})} & \omega \pi(D) \\
\end{array}
\]

(5.26)

As \( \text{Ker}(\mathcal{I}) = 0 \) and \( \text{Coker}(\mathcal{I}) \in \text{Tors}(\mathcal{D}) \), \( \pi(\mathcal{I}) \) and, hence, \( \omega \pi(\mathcal{I}) \) are isomorphisms. On the other hand, by Lemma 5.7, \( \eta_{\mathcal{D}} \) can be factored as

\[
\mathcal{D} \cong \mathcal{D} \otimes_A \mathcal{I} \xrightarrow{\eta \pi \otimes 1} \omega \pi(\mathcal{D}) \otimes_A \mathcal{I} \cong \omega \pi(\mathcal{D})
\]

and hence, by Lemma 5.6(1), \( \eta_{\mathcal{D}} : \mathcal{D} \to \omega \pi(\mathcal{D}) \geq 0 \) is an isomorphism. Using these two isomorphisms, we identify

\[
\omega \pi(M) \geq 0 \cong \mathcal{D}.
\]

It follows then from (5.10) and (5.24) that \( \tau(M) = 0 \) and \( \tau(M) \geq 0 \cong \text{Coker}(\mathcal{I}) \cong \mathcal{D}/\mathcal{M} \). Hence

\[
\dim \tau(M) \geq 0 = n.
\]

Next, we set \( \tilde{N} := \bigoplus_{k \in \mathbb{Z}} M/M_k \) and make \( \tilde{N} \) a graded \( \mathcal{D} \)-module in the natural way, with \( t \in \mathcal{D} \) acting by the canonical projections \( M/M_k \to M/M_{k+1} \).

**Proposition 5.2.** The module \( \tilde{N} \) has the following properties:

1. \( \tau(\tilde{N}) = 0 \),
2. \( \dim \tau_1(\tilde{N})_{-1} = n \), and \( \dim \tau_1(\tilde{N})_{\geq -1} < \infty \),
3. The maps \( \omega \pi(\tilde{N})_{k-1} \to \omega \pi(\tilde{N})_k \) are surjective for all \( k \geq 0 \).

**Proof.**

1. Given \( \tilde{M} \in \text{GrMod}(\mathcal{D}) \), we write \( p^!(\tilde{M}) \) for the largest submodule of \( \tilde{M} \) annihilated by the action of \( t \), i. e. \( p^!(\tilde{M}) = \text{Ker}(\tilde{M} \xrightarrow{t} \tilde{M}[1]) \). Then, if \( \tilde{M} \in \text{Tors}(\mathcal{D}) \) and \( \tilde{M} \neq 0 \), we have \( p^!(\tilde{M}) \neq 0 \). So the assumption \( \tau(\tilde{N}) \neq 0 \) implies that \( p^!(\tau(\tilde{N})) \neq 0 \). On the other hand, \( p^!(\tilde{N}) \cong \tilde{M}[1] \) and \( \tau(\tilde{M}[1]) = \tau(\tilde{M})[1] = 0 \), so \( \tau(p^!(\tilde{N})) = 0 \). Since \( p^!(\tau(\tilde{N})) = p^!(\tilde{N}) \cap \tau(\tilde{N}) = \tau(p^!(\tilde{N})) \), we arrive at contradiction. It follows that \( \tau(\tilde{N}) = 0 \).

2. For all \( k \in \mathbb{Z} \), we have the exact sequences \( 0 \to M_k/M_{k-1} \to M/M_{k-1} \to M/M_k \to 0 \) defined by the filtration inclusions. Combining these together, we get the exact sequence of graded \( \mathcal{D} \)-modules

\[
0 \to \tilde{N} \to \mathcal{D}[1] \xrightarrow{t} \tilde{N} \to 0.
\]

Since \( \tau(\tilde{N}) = 0 \), applying the torsion functor \( \tau \) to (5.29) yields

\[
0 \to \tau(M) \to \tau_1(\tilde{N}_{[-1]}) \to \tau_1(\tilde{N}) \to \tau_2(M) \to \ldots
\]

By Theorem 5.1(1) and Lemma 5.7, the last term of (5.30) can be identified as

\[
\tau_2(M) \cong \mathcal{D}/\omega(\pi \mathcal{M}) \cong \mathcal{D}/\omega(\pi \mathcal{D}) \otimes_A \mathcal{I},
\]

so \( \tau_2(M) = 0 \) by Lemma 5.6(2). We get thus the exact sequence

\[
0 \to \tau(M) \to \tau_1(\tilde{N}_{[-1]}) \to \tau_1(\tilde{N}) \to 0.
\]

Now, (5.28) implies that \( \tau_1(M)_{\geq 0} \) is bounded: i. e. there is an integer \( d \geq 0 \), such that \( \tau_1(M)_{d} \neq 0 \), while \( \tau_1(M)_{k} = 0 \) for all \( k > d \). It follows then from (5.31) that \( t \) acts as a unit on \( \tau_1(\tilde{N})_{\geq d} \) in particular, we have \( p^!(\tau_1(\tilde{N})_{\geq d}) = 0 \). But \( \tau_1(\tilde{N})_{\geq d} \) is a submodule of \( \tau_1(\tilde{N}) \), which, by definition, is torsion. Hence, \( p^!(\tau_1(\tilde{N})_{\geq d}) = 0 \) forces \( \tau_1(\tilde{N})_{\geq d} = 0 \). Now, by induction, it follows from (5.31) that \( \dim \tau_1(\tilde{N})_{k-1} = \sum_{j=k}^{d} \dim \tau_1(\tilde{M})_{j} \) for \( k = 0, 1, \ldots, d \). In particular, by (5.28), \( \dim \tau_1(\tilde{N})_{-1} = \dim \tau_1(M)_{\geq 0} = n \), and \( \dim \tau_1(\tilde{N})_{\geq -1} = \sum_{k \geq 0} \dim \tau_1(\tilde{N})_{k-1} \) is finite.
Applying \( \omega \pi \) to (5.29) gives rise to the exact sequence

\[
0 \to \omega \pi (\overline{M}) \to \omega \pi (\overline{N}[-1]) \xrightarrow{\iota} \omega \pi (\overline{N}) \to R^1 \omega (\pi \overline{M}) \to \ldots
\]

Since \( \pi (\overline{M}) \cong \pi (\overline{D} \mathcal{I}) \), we have \( \omega \pi (\overline{M})_{\geq 0} \cong \omega \pi (\overline{D} \mathcal{I})_{\geq 0} \cong \overline{D} \mathcal{I} \), see (5.27), and \( R^1 \omega (\pi \overline{M}) \cong R^1 \omega (\pi \overline{D} \mathcal{I}) \cong R^1 \omega (\pi \mathcal{I}) \). Hence, truncating (5.32) at negative degrees, we get the exact sequence

\[
0 \to \overline{D} \mathcal{I} \to \omega \pi (\overline{N}[-1])_{\geq 0} \xrightarrow{\iota} \omega \pi (\overline{N})_{\geq 0} \to 0.
\]

The last statement of the proposition follows. \( \square \)

Next, we consider the functorial exact sequence (5.16), with \( \overline{M} = \overline{N} \). By Proposition 5.2(1), the first term of this sequence is zero, so we have

\[
0 \to \overline{N} \xrightarrow{\eta} \omega \pi (\overline{N}) \to \tau_1 (\overline{N}) \to 0,
\]

Since the canonical filtration on \( M \) is positive, \( \overline{N}_k = M \) for all \( k < 0 \). Thus, setting \( L := \omega \pi (\overline{N})_{-1} \) and \( V := \tau_1 (\overline{N})_{-1} \), we get from (5.34) the exact sequence of \( A \)-modules

\[
0 \to M \xrightarrow{\Delta} L \to V \to 0.
\]

Now, replacing \( A \) by its one-point extension \( B = A [I] \), we lift (5.35) to an exact sequence of \( B \)-modules, as follows. First, we regard \( M \) as a \( B \)-module by restricting scalars via the algebra homomorphism \( i : B \to A \), see (2.12). Next, we set \( L := L \oplus C \) and make \( L \) a \( B \)-module by defining its structure map \( \varphi : \mathcal{I} \otimes C \cong \mathcal{I} \to L \) to be the degree 0 component of the canonical embedding \( \overline{D} \mathcal{I} \to \omega \pi (\overline{N}[-1])_{\geq 0} \) in (5.33). Every \( A \)-module homomorphism \( M \to L \) extends then to a unique \( B \)-module homomorphism \( M \to L \), since \( \text{Hom}_A (M, L) \cong \text{Hom}_B (M, L) \) via \( f \mapsto (f, 0) \). In particular, the map \( \eta \) in (5.36) extends to an embedding \( \eta : M \to L \), and we write \( V := L / M \) for the cokernel of \( \eta \). Clearly, \( V \cong V \oplus C \) as a vector space, and \( \dim (V) = (n, 1) \), by Proposition 5.2(2). Summing up, we have constructed an exact sequence of \( B \)-modules

\[
0 \to M \xrightarrow{\Delta} L \to V \to 0,
\]

with the quotient term being of dimension \((n, 1)\). Moreover, using Lemma 4.1, we may regard \( M \) as a \( \Pi^A(B) \)-module.

**Proposition 5.3.** The \( B \)-module structure on \( L \) defined above admits a unique extension to \( \Pi^A(B) \), making \( \eta : M \to L \) a homomorphism of \( \Pi^A(B) \)-modules.

We will give a homological proof of this proposition, using Theorem 2.2 of Section 2.1 As explained in (the proof of) Theorem 2.2, a \( \Pi^A(B) \)-module structure on a \( B \)-module \( M \) is determined by an element of \( \text{End}(M) \otimes_{B^e} \Omega^1 B \), lying in the fibre of \( \lambda (= \lambda \cdot \text{Id}) \) under the evaluation map

\[
\partial_M : \text{End}(M) \otimes_{B^e} \Omega^1 (B) \to \text{End}(M), \quad f \otimes d \mapsto f \Delta_B (d).
\]

In particular, the given \( \Pi^A(B) \)-module structure on \( M \) is determined by an element \( \delta_M \in \text{End}(M) \otimes_{B^e} \Omega^1 (B) \), such that \( \partial_M (\delta_M) = \text{Id}_M \). The \( B \)-module embedding \( \eta \) induces an embedding of \( B \)-bimodules: \( \text{End}(M) \hookrightarrow \text{Hom}(M, L) \), and hence the natural map

\[
\text{End}(M) \otimes_{B^e} \Omega^1 (B) \hookrightarrow \text{Hom}(M, L) \otimes_{B^e} \Omega^1 (B).
\]

Since \( \Omega^1 (B) \) is a projective bimodule, this last map is also an embedding, and we identify \( \text{End}(M) \otimes_{B^e} \Omega^1 (B) \) with its image in \( \text{Hom}(M, L) \otimes_{B^e} \Omega^1 (B) \) under (5.38).

Now, consider the commutative diagram

\[
\begin{array}{ccc}
\text{End}(L) \otimes_{B^e} \Omega^1 (B) & \xrightarrow{\eta} & \text{Hom}(M, L) \otimes_{B^e} \Omega^1 (B) \\
\partial_L & & \partial_{M, L} \\
\text{End}(L) & \xrightarrow{\eta} & \text{Hom}(M, L)
\end{array}
\]
where $\partial_{M,L}$ is the evaluation map at $\Delta_B$, $\eta_*$ is the restriction (via $\eta$), and $\hat{\eta}_* := \eta_* \otimes \text{Id}$. Note that $\eta_*$ and $\hat{\eta}_*$ are both surjective. With above identification, we have

$$\eta_*(\lambda) = \partial_{M,L}(\delta_M) = \eta_*$$

and our problem is to show that there is a unique element $\delta_L \in \text{End}(L) \otimes_B \Omega^1(B)$, such that

$$\partial_L(\delta_L) = \lambda \quad \text{and} \quad \hat{\eta}_*(\delta_L) = \delta_M.$$  

To solve this problem homologically, we interpret the top and the bottom rows of (5.39) as 2-complexes of vector spaces $X^\bullet$ and $Y^\bullet$, with nonzero terms in degrees 0 and 1 and differentials $\hat{\eta}_*$ and $\eta_*$ respectively. The pair of maps $(\partial_L, \partial_{M,L})$ yields then a morphism of complexes $\partial^*: X^\bullet \to Y^\bullet$ with mapping cone

$$C^\bullet(\partial) := \left[ 0 \to X^0 \xrightarrow{d^{-1}} X^1 \oplus Y^0 \xrightarrow{d^0} Y^1 \to 0 \right].$$

By definition, the differentials in $C^\bullet(\partial)$ are given by $d^{-1} = (-\hat{\eta}_*, \partial_L)^T$ and $d^0 = (\partial_{M,L}, \eta_*)$. So (5.40) can be interpreted by saying that $(-\delta_M, \lambda) \in X^1 \oplus Y^0$ is a 0-cocycle in $C^\bullet(\partial)$. Then, the cohomology class

$$c(\lambda, \delta_M) := [(-\delta_M, \lambda)]$$

represented by this cocycle, vanishes in $h^0(C^\bullet)$ if and only if there is $\delta_L \in X^0$ such that $d^{-1}(\delta_L) = (-\delta_M, \lambda)$, i.e. (5.41) holds. Clearly, if it exists, such $\delta_L$ is unique if and only if $d^{-1}$ is injective, i.e. if and only if $h^{-1}(C^\bullet) = 0$. Now, a simple calculation (as in the proof of Theorem 2.2) shows that

$$h^0(C^\bullet) \cong H_0(B, \text{Hom}(V, L)) \quad \text{and} \quad h^{-1}(C^\bullet) \cong H_1(B, \text{Hom}(V, L)).$$

Proposition 5.3 thus boils down to proving Lemma 5.8 and Lemma 5.9 below.

**Lemma 5.8.** $H_1(B, \text{Hom}(V, L)) = 0$.

**Proof.** Recall that $L := \omega \pi(\tilde{N})^{-1}$. Now, in addition, we set $L_0 := \omega \pi(\tilde{N})_0$ and make this a $B$-module by restricting scalars via $i: B \to A$. Then, the $A$-module homomorphism $t_L: L \to L_0$ induced by the action of $t$ extends to a unique $B$-module homomorphism $L \to L_0$, which we denote by $t$. By Proposition 5.2(3), $t_L$ is surjective, and hence is $t$. It is easy to see that $\text{Ker}(t) \cong B\epsilon_\infty$, so we have the exact sequence of $B$-modules

$$0 \to B\epsilon_\infty \xrightarrow{\rho} L \xrightarrow{t} L_0 \to 0.$$ 

Since $B\epsilon_\infty$ is projective, tensoring (5.44) with $V^\bullet := \text{Hom}(V, \mathbb{C})$ yields

$$0 \to \text{Tor}_1^B(V^\bullet, L) \to \text{Tor}_1^B(V^\bullet, L_0) \to V^\bullet \otimes_B B\epsilon_\infty \to V^\bullet \otimes_B L \to V^\bullet \otimes_B L_0 \to 0.$$ 

On the other hand, since $V$ is finite-dimensional, for an arbitrary $B$-module $M$, we have natural isomorphisms $H_n(B, \text{Hom}(V, M)) \cong \text{Tor}_n^B(V^\bullet, M)$. So the above exact sequence can be identified with

$$0 \to H_1(B, \text{Hom}(V, L)) \to H_1(B, \text{Hom}(V, L_0)) \to H_0(B, \text{Hom}(V, B\epsilon_\infty)) \to H_0(B, \text{Hom}(V, L)) \to H_0(B, \text{Hom}(V, L_0)) \to 0.$$ 

To prove the lemma it thus suffices to show that

$$H_1(B, \text{Hom}(V, L_0)) \cong \text{Tor}_1^B(V^\bullet, L_0) = 0.$$ 

Since $L_0$ is an $A$-module, we can compute this last Tor, using the spectral sequence

$$\text{Tor}_q^A(\text{Tor}_p^B(V^\bullet, A), L_0) \Rightarrow \text{Tor}_{p+q}^B(V^\bullet, L_0)$$

associated to the algebra map $i: B \to A$. By Lemma 2.2(4), this map is flat, so (5.47) collapses at $q = 0$, giving an isomorphism

$$\text{Tor}_1^B(V^\bullet, L_0) \cong \text{Tor}_1^A(V^\bullet, L_0).$$ 

Now, for each $k \geq 0$, we set $L_k := \omega \pi(\tilde{N})_k$ and write $F_k$ for the kernel of the map $L_0 \xrightarrow{t^k} L_k$ induced by the action of $t^k \in \mathcal{D}$ on $\omega \pi(\tilde{N})$. By Proposition 5.2(3), the maps $t^k$ are surjective for all $k \geq 0$, and
thus $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ is an $A$-module filtration on $L_0$. By Proposition \ref{prop:filtration}, this filtration is exhaustive, so that $\lim_{\to} F_k \cong \bigcup_{k=0}^{\infty} F_k = L_0$, while, by \eqref{eq:5.33}, we have exact sequences
\begin{equation}
0 \to F_k \to F_{k+1} \to \overline{F}_k \to 0, \quad \forall k \geq 0.
\end{equation}

Since $\overline{F}_k$ are projective $A$-modules for $k \geq 0$, we conclude from \eqref{eq:5.49} that $F_k$ are projective for $k \geq 1$. The direct limits of families of projective modules are flat, hence so is $L_0 = \lim_{\to} F_k$. We have $\text{Tor}_1^A(V^*, L_0) = 0$, and \eqref{eq:5.40} follows from \eqref{eq:5.48}.

\begin{lemma}
$c(\lambda, \delta_M) = 0$ in $H_0(B, \text{Hom}(V, L))$.
\end{lemma}

\begin{proof}
By \eqref{eq:5.45} and \eqref{eq:5.46}, we have the exact sequence
\begin{equation}
0 \to H_0(B, \text{Hom}(V, Be_{\infty})) \to H_0(B, \text{Hom}(V, L)) \to 0,
\end{equation}
where $\iota_*$ is induced by the inclusion $\iota : Be_{\infty} \hookrightarrow L$ and $\iota_*$ by the projection $\iota : L \to L_0$ in \eqref{eq:5.44}. We show first that $\iota_*(c(\lambda, \delta_M)) = 0$. For this, we consider the image of the diagram \eqref{eq:5.39} under the natural projection $\iota$. Under this projection, the equations \eqref{eq:5.40} become
\begin{equation}
\eta_\iota(t) = \partial_{L, L_0}(\iota_*(\delta_M)) = t \circ \eta,
\end{equation}
where $\iota_* : \text{Hom}(M, L) \otimes_{B^e} \Omega^1(B) \to \text{Hom}(M, L_0) \otimes_{B^e} \Omega^1(B)$ is defined by $f \otimes \omega \mapsto (t \circ f) \otimes \omega$. Now, $t$ induces a morphism of mapping cones \eqref{eq:5.42} associated to \eqref{eq:5.39} and its projection, which, in turn, induces a map $\iota_*$ on cohomology. The class $\iota_*(c(\lambda, \delta_M)) \in H_0(B, \text{Hom}(V, L))$ can thus be viewed as an obstruction for the existence of an element $\delta_{L, L_0} \in \text{Hom}(L, L_0) \otimes_{B^e} \Omega^1(B)$ satisfying
\begin{equation}
\partial_{L, L_0}(\delta_{L, L_0}) = t \quad \text{and} \quad \eta_\iota(\delta_{L, L_0}) = \tilde{\iota_*(\delta_M)}.
\end{equation}
We will show that $\iota_*(c(\lambda, \delta_M)) = 0$ by constructing such an element explicitly.

By universal property of tensor algebras, the filtered algebra homomorphism $R \to \mathcal{D}$ lifts to a graded algebra homomorphism $R \to \overline{\mathcal{D}}$, so we may regard graded $\overline{\mathcal{D}}$-modules as graded $R$-modules. The action of $\overline{\mathcal{D}}$ on $\omega \pi(\hat{N})$ yields an $A$-bimodule map $\text{Der}(A) \to \text{Hom}(L, L_0)$, taking $\Delta_A \mapsto t_L$, which in composition with $\text{Der}(B) \to \text{Der}(A)$ gives a $B$-bimodule homomorphism $\text{Der}(B) \to \text{Hom}(L, L_0)$, $\Delta_B \mapsto t_L$, and hence an element
\begin{equation}
\delta_{L, L_0} \in \text{Hom}_{B^e}(\text{Der}(B), \text{Hom}(L, L_0)) \cong \text{Hom}(L, L_0) \otimes_{B^e} \Omega^1(B).
\end{equation}
Now, let $\alpha : L = eL \hookrightarrow L$ be the natural inclusion, with $L/L = e_{\infty} L \cong C$. Viewing $L$ as a $B$-module via $i$ makes it a $B$-module map. Dualizing $\alpha$ by $L_0$ and tensoring with $\Omega^1(B)$, we get then the commutative diagram
\begin{equation}
\begin{array}{ccc}
\text{Hom}(L, L_0) \otimes_{B^e} \Omega^1(B) & \xrightarrow{\alpha_*} & \text{Hom}(L, L_0) \otimes_{B^e} \Omega^1(B) \\
\partial_{L, L_0} \downarrow & & \partial_{L, L_0} \\
\text{Hom}(L, L_0) & \xrightarrow{\alpha_*} & \text{Hom}(L, L_0)
\end{array}
\end{equation}
with $\partial_{L, L_0}(\delta_{L, L_0}) = \alpha_*(t) = t_L$. Since $e \in B$ acts as identity on $L_0$ and as zero on $L/L$, we have $H_0(B, \text{Hom}(L/L, L_0)) \cong (L/L)^* \otimes_B L_0 = 0$. Hence, there is $\delta_{L, L_0} \in \text{Hom}(L, L_0) \otimes_{B^e} \Omega^1(B)$, such that $\partial_{L, L_0}(\delta_{L, L_0}) = t_L$ and $\alpha_*(\delta_{L, L_0}) = i_L$. A direct calculation using $\eta = \alpha \circ \eta$ shows that this element satisfies also \eqref{eq:5.52}.

The existence of $\delta_{L, L_0}$ implies that $t_*(c(\lambda, \delta_M)) = 0$. Returning to \eqref{eq:5.50}, we see then that $c(\lambda, \delta_M) = \iota_*(\tilde{c})$ for some $\tilde{c} \in H_0(B, \text{Hom}(V, Be_{\infty}))$. Now, to show that $\tilde{c} = 0$, we consider the trace map $\text{Tr}_* : \text{Hom}(V, Be_{\infty}) \to \text{Hom}(V, L) \to \text{End}(V) \to C$, $f \mapsto \text{tr}_V[\pi \circ \iota \circ f]$, where $\iota$ is defined in \eqref{eq:5.44}, $\pi$ is the canonical projection in \eqref{eq:5.30}. Since $\iota$ and $\pi$ are homomorphisms, this induces a linear map
\begin{equation}
\text{Tr}_* : H_0(B, \text{Hom}(V, Be_{\infty})) \xrightarrow{i_\iota} H_0(B, \text{Hom}(V, L)) \xrightarrow{\pi} H_0(B, \text{End}(V)) \xrightarrow{\text{tr}_V} C.
\end{equation}
We claim that $\text{Tr}_*$ is an isomorphism. Indeed, it is easy to see that $\text{Tr}_* \neq 0$, while
\begin{equation}
H_0(B, \text{Hom}(V, Be_{\infty})) \cong V^* \otimes_B Be_{\infty} \cong V^* e_{\infty} \cong (e_{\infty} V)^* \cong C.
\end{equation}
Now, since \( \pi \circ \eta \equiv 0 \), we have \( \pi_*(c(\lambda, \delta_M)) = [\lambda \cdot \text{Id}_V] \), and hence
\[
\Tr_*(\hat{c}) := \Tr_V [\pi_* \epsilon_*(\hat{c})] = \Tr_V [\pi_* (c)] = \Tr_V [\lambda \cdot \text{Id}_V] = 0 .
\]
It follows that \( \hat{c} = 0 \) and \( c(\lambda, \delta_M) = 0 \), finishing the proof of the lemma and Proposition 5.3 \( \square \)

Now, by Proposition 5.3, the given \( B \)-module structure on \( V = L/M \) extends to a unique \( \Pi^\lambda(B) \)-module structure, making (5.36) an exact sequence of \( \Pi^\lambda(B) \)-modules. Since \( e_\infty V \cong e_\infty L \), the natural map \( \Pi e_\infty \otimes_U e_\infty V \cong \Pi e_\infty \otimes_U e_\infty L \to L \) is an isomorphism, which in combination with projection \( \pi : L \to V \) becomes \( \mu_V : \Pi e_\infty \otimes_U e_\infty V \to V \). It follows that \( \text{Ker}(\pi) \cong \text{Ker}(\mu_V) \), and thus \( M \cong \iota(V) \). This completes Step 2.

5.4. The map \( \omega \) is injective and \( \Gamma \)-equivariant. For \( \Pi \)-modules \( V \) and \( V' \) of dimension \( n = (n, 1) \), we will show that
\[
(5.55) \quad \iota^*(V) \cong \iota^*(V') \iff V' \cong V^\sigma \quad \text{for some } \omega = u^{-1} du \in \Omega^1 X , \]
where \( V^\sigma \) denotes the \( \Pi \)-module \( V \) twisted by an automorphism \( \sigma \in \text{Aut}_S \Pi^\lambda(B) \).

We begin by describing the action (4.18) in terms of generators of \( \Pi^\lambda(B) \) (see Proposition 5.4).

**Lemma 5.10.** The homomorphism \( \sigma : \Omega^1 X \to \text{Aut}_S \Pi \) is given by
\[
(5.56) \quad \sigma_\omega(\hat{a}) = \hat{a} , \quad \sigma_\omega(\hat{v}_i) = \hat{v}_i , \quad \sigma_\omega(\hat{w}_i) = \hat{w}_i , \quad \sigma_\omega(\hat{d}) = \hat{d} + \omega(\hat{d}) ,
\]
where \( \omega \in \Omega^1 X \) acts on \( d \in \text{Der}(A) \) via the natural identification
\[
\Omega^1 X = (\Omega^1 A)_2 \cong \text{Hom}_{A^e}((\Omega^1 A)^*, A) \cong \text{Hom}_{A^e}(\text{Der}(A), A^e) .
\]

**Proof.** By Lemma 5.1, we can define (4.18) in terms of relative differentials
\[
(5.57) \quad \sigma : \Omega^1 X \xrightarrow{\alpha} (\Omega^1_S B)_2 \xrightarrow{\delta} \text{Aut}_B[T_B(\Omega^1_S B)^*] \to \text{Aut}_S \Pi^\lambda(B) ,
\]
where \( \alpha \) is now an isomorphism. In fact, with identification (5.58), the elements of \( (\Omega^1_S B)_2 = \Omega^1_S B/[B, \Omega^1_S B] \)
behave represented by matrices \( \hat{\omega} = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix} \) with \( \omega \in (\Omega^1 A)_2 = \Omega^1 X \), and \( \alpha \) is given explicitly by \( \omega \to \hat{\omega} \mod [B, \Omega^1_S B] \). It follows then that \( \sigma_\omega \) acts on \( \Pi^\lambda(B) \) as in (5.56) \( \square \)

We may also describe the algebra map \( \iota : \Pi^\lambda(B) \to \mathcal{D} \) in terms of generators of \( \Pi^\lambda(B) \):
\[
(5.58) \quad \iota(\hat{a}) = \mathcal{A} , \quad \iota(\hat{d}) = \mathcal{T} , \quad \iota(\hat{v}_i) = \iota(\hat{w}_i) = 0 ,
\]
where \( \mathcal{A} \) and \( \mathcal{T} \) denote the classes of \( a \in A \) and \( d \in \text{Der}(A) \) in \( T_A \text{Der}(A) \) modulo the ideal \( \langle \Delta_A - 1 \rangle \). Comparing now (5.56) and (5.58), we get

**Lemma 5.11.** The group homomorphism \( \sigma : \Omega^1 X \xrightarrow{\sigma} \text{Aut}_S \Pi \to \text{Aut}_\mathcal{D} \mathcal{D} \) induced by \( \sigma \) is given by
\[
(5.59) \quad \sigma_\omega(a) = a , \quad \sigma_\omega(\partial) = \partial + \omega(\partial) ,
\]
where \( a \in A , \partial \in \text{Der}(A) \) and \( \omega \in \Omega^1 X \).

In particular, if \( \omega = u^{-1} du \) for some \( u \in \Lambda \), then \( \sigma_\omega(a) = a = u a u^{-1} \) (since \( A \) is commutative),
and \( \sigma_\omega(\partial) = u \partial u^{-1} \). Thus, the induced action of \( \Lambda \subset \Omega^1 X \) on \( \mathcal{D} \) is given by inner automorphisms. In contrast, \( \Lambda \) does not act by inner automorphisms on the whole of \( \Pi^\lambda(B) \).

Now, by functoriality, \( V' \cong V^\sigma \) implies \( L' \cong L^\sigma \) and \( \iota^*(V') \cong \iota^*(V)^\sigma \) for any \( \omega \in \Omega^1 X \). So the map \( C_n(X, \mathcal{I}) \to \mathcal{I}(\mathcal{D}) \) induced by \( \iota^* \) is equivariant under the action of \( \Omega^1 X \). On the other hand, \( \iota^*(V) \) is a \( \mathcal{D} \)-module, on which the twisting by \( \omega \) acts via (5.59), i. e. \( \iota^*(V)^\sigma = \iota^*(V)^\sigma \). Since the inner automorphisms induce trivial auto-equivalences, we have \( \iota^*(V)^\sigma \cong \iota^*(V) \) for \( \omega = u^{-1} du \). This proves the implication \( \Leftarrow \) in (5.55) and, in combination with Step 1, yields a \( \Gamma \)-equivariant map \( \omega_n : C_n(X, \mathcal{I}) \to \gamma^{-1}[\mathcal{I}] \).

It remains to show that \( \omega_n \) is injective. For this, we will use the following result, which is a version of [BW2], Lemma 10.1, and [NS], Lemma 3.2. (In particular, the proof given in the last reference extends trivially to our situation.)
Lemma 5.12. Let $M$ be a (nonzero) ideal of $D$ equipped with two good filtrations $\{M_k\}$ and $\{M'_k\}$, such that the associated graded modules $\overline{M}$ and $\overline{M'}$ are both torsion-free. Then, there is $k_0 \in \mathbb{Z}$, such that $M_k = M'_k - k_0$ for all $k \in \mathbb{Z}$.

Given two II-modules $V$ and $V'$ of dimension $n$, we set $L := \Pi e_\infty \otimes U e_\infty V$, $L := eL$, $M := i^*(V)$, and similarly for $V'$. In addition, we denote by $\eta : M \rightarrow L$ and $\eta : M \rightarrow L$ the natural inclusions (and similarly for $M'$).

Proposition 5.4. If $M \cong M'$ as $D$-modules, then $L \cong L'$ as $B$-modules.

Proof. First, we show that every $D$-module isomorphism $f : M \rightarrow M'$ lifts to an $A$-module isomorphism $f_L : L \rightarrow L'$. For this, we identify $L$ as in Lemma 5.5 filter it by $\{F_kL\}$ as in Section 5.2 and set $\tilde{L} := \bigoplus_{k \geq -1} L/F_kL$. By (5.15), we have $\Delta_A : x \mapsto x \pmod{F_0L}$ for all $x \in L$, so $\Delta_A [x]_k = [x]_{k+1} = t[x]_k$ for $k \geq -1$. Since $R[t]/(\Delta_A - t) \cong D$, we may regard $\tilde{L}_{\geq -1}$ as a graded $D$-module.

Next, we equip $M$ with the induced filtration $M_k := M \cap F_kL$ via the inclusion $\eta : M \rightarrow L$, and put $\tilde{N} := \bigoplus_{k \geq -1} M/M_k$. The map $\eta$ naturally extends to $\tilde{\eta} : \tilde{N} \rightarrow \tilde{L}$, and $\tilde{N}$ becomes a graded $D$-module via the induced action of $R[t]$ on $\tilde{L}$. It follows from (5.2) that $\overline{M} := \bigoplus_{k \geq 0} M_k/M_{k+1}$ is a torsion-free $\overline{D}$-module, and hence $\tau(\tilde{N}) = 0$ by Proposition 5.2(1). Let $\tilde{\eta} : \tilde{N} \rightarrow \omega(\tilde{N})$, see (5.10). Since $\text{Coker} \tilde{\eta}$ is finite-dimensional in degree $\geq -1$, the map $\tilde{\eta}$ extends to an embedding: $\tilde{L}_{\geq -1} \rightarrow \omega\omega(\tilde{N})_{\geq -1}$. By induction in grading, using Proposition 5.2(2) and (5.33), it is easy to show that this embedding is an isomorphism.

Now, replacing $L$ by $L'$, we repeat the above construction. The $D$-module $M'$ comes then equipped with two filtrations: one is induced from $L'$ via $\tilde{\eta} : M' \rightarrow L'$, and the other is transferred from $M$ via $f : M \rightarrow M'$. Both filtrations satisfy the assumptions of Lemma 5.12, and, hence, coincide up to a shift in degree. Since $\overline{M'}$ and $f(\overline{M})$ have finite codimension in $\overline{L}$, this last shift must be 0 so $M'_k = f(M_k)$ for all $k \in \mathbb{Z}$. The map $f$ extends then to an isomorphism $\tilde{f} : \tilde{N} \rightarrow \tilde{N}'$ and further, by functoriality, to $\omega\omega(\tilde{f}) : \omega(\tilde{N}) \rightarrow \omega(\tilde{N}')$. As a result, we get $\tilde{L}_{\geq -1} \cong \omega(\tilde{N})_{\geq -1} \rightarrow \omega(\tilde{N}')_{\geq -1} \cong \tilde{L}'_{\geq -1}$, which in degree $\geq -1$ yields the required extension $f_L : L \rightarrow L'$.

Now, with our identifications of $L$ and $L'$, the $B$-modules $L$ and $L'$ are determined (up to isomorphism) by the triples $(L, C, \varphi)$ and $(L', C, \varphi')$, where $\varphi : I \rightarrow L$ and $\varphi' : I \rightarrow L'$ are the canonical embeddings with images $F_0L$ and $F_0L'$ respectively. Since $F_0L$ is the kernel of $\tilde{L}_{-1} \rightarrow \tilde{L}_0$, the map $f_L$ restricts to $F_0L$, giving an isomorphism $f_{L_0} : F_0L \rightarrow F_0L'$. Letting $u := (\varphi')^{-1} \circ (f_L)_0 \circ \varphi \in \text{Aut}_A(I)$ and identifying $\text{Aut}_A(I) = \text{End}_A(I)^\times \cong A^\times$, via the action map, we have $u^{\varphi'} \circ \varphi = \varphi' \circ u = F_L \varphi$. Hence (5.60)

$g := (u^{-1}f_L, \text{Id}) : L \oplus C \rightarrow L' \oplus C$

makes the diagram (2.10) commutative and thus defines an isomorphism of $B$-modules $L \cong L'$.

Now, keeping the notation of Proposition 5.1, consider two II-modules $V$ and $V'$ of dimension $n$, with $M \cong M'$. Fix an isomorphism $f : M \rightarrow M'$ and define $g$ as in (5.60). Taking $\omega = u^{-1}du \in \Omega^1X$ and twisting $\eta$ by $\sigma = \sigma_\omega \in \text{Aut}_\Omega \Pi$, consider the diagram

(5.61)

From the construction of $f$ and $g$, it follows that this diagram is commutative, with all arrows being $\Pi$-module homomorphisms and horizontal ones being isomorphisms. Thus, identifying $M^\sigma \cong M'$ and $L^\sigma \cong L'$ in (5.61), we get two (a priori different) II-module structures on $L'$. Both of these are extensions of the given II-module structure on $M'$. Hence, by Proposition 5.2, they must coincide. It follows that $g : L^\sigma \rightarrow L'$ is an isomorphism of II-modules, which, by commutativity of (5.61), induces an isomorphism $V^\sigma \cong V'$. This completes Step 3.
5.5. The equivariance of $\omega$ under the action of $\text{Pic}(D)$. As in Section 4.4, we will assume that $X \neq \mathbb{A}^1$. By [CH1], Prop. 1.4, the automorphism group of $D$ is then isomorphic to the product $\text{Aut}(X) \times \Omega^1X$:

\begin{equation}
\text{Aut}(X) \times \Omega^1X \xrightarrow{\sim} \text{Aut}(D), \quad (\nu, \omega) \mapsto \bar{\nu} \bar{\sigma}_\omega,
\end{equation}

where $\bar{\nu} \in \text{Aut}(D) : D \mapsto \nu D \nu^{-1}$, and $\bar{\sigma}_\omega$ is defined by (5.59). Now, for a line bundle $F$ on $X$, $\text{End}_D(FD)$ is canonically isomorphic to the ring of twisted differential operators on $X$ with coefficients in $F$. As $X$ is affine, this last ring is isomorphic to $D$, so the set of all algebra isomorphisms: $D \mapsto \text{End}_D(FD)$ is non-empty and equals $\psi_0 \text{Aut}(D)$, where $\psi_0$ is a fixed isomorphism. By [CH1], Th. 1.8, the isomorphism $\psi_0$ can be chosen in such a way that $\psi_0|_A = \text{Id}$: specifically, fixing dual bases $\{\alpha_i\} \subset F$, $\{\beta_i\} \subset F^\vee$, and identifying $\text{End}_D(FD) = FD F^\vee$ as in Section 4.2, we define $\psi_0 : D \xrightarrow{\sim} \text{End}_D(FD)$ by

\begin{equation}
\psi_0(a) = a, \quad \psi_0(\partial) = \sum_i \alpha_i \partial \beta_i, \quad a \in A, \quad \partial \in \text{Der}(A).
\end{equation}

With (5.62) and (5.63), every isomorphism $\psi : D \mapsto \text{End}_D(FD)$ can then be decomposed as

\begin{equation}
\psi = \psi_0 \bar{\nu} \bar{\sigma}_\omega,
\end{equation}

where $\nu \in \text{Aut}(X)$ and $\omega \in \Omega^1X$ are uniquely determined by $\psi$.

Proof of Proposition 4.2. Given a line bundle $\mathcal{I}$ and an invertible bimodule $\mathcal{P} = (DL)_\phi$, with $\phi : D \xrightarrow{\sim} \text{End}_D(DL)$, we set $\tau = \phi|_A : J := \mathcal{I}(\mathcal{I})$, $\mathcal{F} := \mathcal{I}^\tau = (\mathcal{I})^\tau$, and $\psi = \phi^{-1} : D \mapsto \text{End}_D(FD)$, as in Section 4.2. To construct an isomorphism $\psi$, satisfying Lemma 4.2, we decompose $\psi$ as in (5.64), and extend each factor through $i$. Since $\psi_0$ and $\bar{\sigma}_\omega$ act on $A$ as identity, we have $\nu = \psi|_A = \tau^{-1}$, so $\bar{\nu} = \bar{\tau}^{-1}$ in (5.64). Thus we set

\begin{equation}
\widetilde{\psi} : \Pi^\lambda(A[J]) \xrightarrow{\sigma} \Pi^\lambda(A[J]) \xrightarrow{\bar{\tau}^{-1}} \Pi^\lambda(A[J^\tau]) \xrightarrow{\psi_0} \text{End}_{\Pi^\lambda(B)}(P),
\end{equation}

where $\sigma_\omega$ is defined in Section 4.2 (see (4.18), with $B$ replaced by $A[J]$) and $\bar{\tau}^{-1}$ is induced by $A[J] \mapsto A[J^\tau]$. The relation $i \sigma_\omega = \bar{\sigma}_\omega i$ is then immediate, by Lemma 5.1.

It remains to define $\widetilde{\psi}_0$. To this end, we use identification (4.16). Since $J^\tau = FI$, we have then $A[J^\tau] \cong \hat{F} \otimes A B \otimes \hat{F}^\vee \mapsto \hat{F} \otimes A \Pi^\lambda(B) \otimes \hat{F}^\vee$, which we take as a definition of $\widetilde{\psi}_0$ on $A[J^\tau]$. This induces the identity on $A$, as required. Next, we construct a bimodule isomorphism:

\begin{equation}
\text{Der}_S(A[J]\mathcal{I}, A[J] \otimes \mathcal{I}) \xrightarrow{\sim} \hat{F} \otimes A \text{Der}_S(B) \otimes \hat{F}^\vee,
\end{equation}

using the dual bases for $\mathcal{F}$ and $\mathcal{I}$. By Lemma 5.2, we first identify the domain of (5.65) with

\begin{equation}
\begin{pmatrix}
\text{Der}(A) & \text{Der}(A, \mathcal{F} \otimes A) \\
0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
\mathcal{F} \otimes (\mathcal{I})^\vee & \mathcal{F} \otimes A \\
(\mathcal{I})^\vee & A
\end{pmatrix},
\end{equation}

and the codomain with

\begin{equation}
\begin{pmatrix}
\mathcal{F} \otimes \text{Der}(A) \otimes \mathcal{F}^\vee & \text{Der}(A, \mathcal{I} \otimes \mathcal{F}) \\
0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
\mathcal{F} \otimes (\mathcal{I})^\vee & \mathcal{F} \otimes A \\
(\mathcal{I})^\vee & A
\end{pmatrix}.
\end{equation}

The first summand of (5.66) is generated by the elements $\hat{d} \in e \text{Der}(A)e$ (see Prop. 5.1): so we define the map (5.63) on this first summand by

\begin{equation}
\hat{d} = \begin{pmatrix}
\hat{d} \\
0
\end{pmatrix} \mapsto \begin{pmatrix}
\sum_i \alpha_i \otimes \hat{d} \otimes \beta_i \\
0
\end{pmatrix},
\end{equation}

while letting it be the identity on the second. This yields an isomorphism of bimodules and induces the required algebra map $\bar{\psi}_0$. The commutativity $i \bar{\psi}_0 = i \psi_0$ is verified by an easy calculation, using (5.58).

To finish the proof of Proposition 4.2 it remains to show the uniqueness of $\bar{\psi}$. For this, arguing as in Proposition 5.3, it suffices to show that $H_1(A[J], \text{Ker} i^\oplus) = 0$, where $i^\oplus := 1 \otimes i \otimes 1$, see (4.8). Since $A[J] \cong \hat{F} \otimes A B \otimes \hat{F}^\vee$, see (4.14), we may identify $H_1(A[J], \text{Ker} i^\oplus) \cong H_1(B, \text{Ker} i)$. On the other hand, by Lemma 5.4, $\text{Ker} i \cong \Pi^\lambda(B) e_\infty \otimes_U e_\infty \Pi^\lambda(B)$, which is easily seen to be a flat $B$-bimodule. Thus $H_1(B, \text{Ker} i) = 0$, as required. \hfill \square
Proof of Proposition 4.3. (1) We will keep the notation of Proposition 4.2. For \( \mathcal{P} = D_{\sigma_{\omega}} \), we have then \( \mathcal{L} \cong A, \varphi = \sigma_{\omega}, \tau = \text{Id}_A \) and \( \psi = \sigma_{\omega}^{-1} \). Now, since \( \mathcal{F} = \mathcal{L}^\tau \cong A \), we may choose \( \psi_0 = \text{Id}_D \). Then \( \psi = \sigma_{\omega}^{-1} \), and the bimodule \( \mathcal{P} \) is isomorphic to \( \Pi^\lambda(B) \) with left multiplication twisted by \( \sigma_{\omega}^{-1} \). Hence, for \( \mathcal{P} = D_{\sigma_{\omega}} \), the isomorphism (4.10) is given by \([V] \rightarrow [V^\sigma_{\omega}^{-1}]\), which agrees with our definition of \( \sigma_{\omega}^* \), see (4.19).

(2) For \( \mathcal{P} = (\mathcal{D} \mathcal{L})_{\varphi} \), the map \( f_P : C_n(X, \mathcal{I}) \rightarrow C_n(X, \mathcal{J}) \) is equivariant under \( \Lambda \) in the sense that

\[
(5.68) \quad f_P \circ \sigma_{\omega}^* = \sigma_{\omega}^* \circ f_P, \quad \forall u \in \Lambda,
\]

where \( \omega = \text{dlog}(u) \) and \( \omega_{\tau} = \text{dlog}(\tau(u)) \). Indeed, \( f_P \circ \sigma_{\omega}^* \) is induced by tensoring II-modules with the bimodule \( \psi \mathcal{P}_{\sigma_{\omega}} = \mathcal{P} \otimes \Pi \mathcal{P}_{\sigma_{\omega}} \), on which \( \Pi^\lambda(A[\mathcal{J}]) \) acts on the left via \( \psi \). Since \( \hat{\tau} \sigma_{\omega} = \sigma_{\omega} \hat{\tau} \), we have \( \psi \mathcal{P}_{\sigma_{\omega}} \cong \sigma_{\omega}^{-1} \psi \mathcal{P} \cong \psi \sigma_{\omega}^{-1} \mathcal{P} \cong (\Pi^\prime \sigma_{\omega}^* \otimes \Pi^\prime \psi \mathcal{P}) \), where \( \Pi^\prime := \Pi^\lambda(A[\mathcal{J}]) \). This implies (5.68). Now, it follows from (5.68) that \( f_P \) induces a well-defined map \( \tilde{f}_P \) on the quotient varieties. The map \( \tilde{f}_P \) depends only on the class \([\mathcal{P}] \in \text{Pic}(\mathcal{D})\), since \([\mathcal{P}]\) determines \( \varphi \) (and \( \psi = \varphi^{-1} \)) up to an inner automorphism of \( \mathcal{D} \). By Prop. 4.2, this means that \( \psi \) (and hence, \( f_P \)) are determined by \([\mathcal{P}]\) up to an automorphism \( \sigma_{\omega} \in \text{Aut}_{\mathcal{D}}[\Pi^\prime] \) with \( \omega = \text{dlog}(u), u \in \Lambda \). Since such automorphisms act trivially on \( \overline{C}_n(X, \mathcal{I}) \), the map \( f_P \) is uniquely determined by \([\mathcal{P}] \in \text{Pic}(\mathcal{D})\). □

Finally, we prove the last part of Theorem 4.3.

Proof of Theorem 4.3(c). Let \( V \) be a \( \Pi^\lambda(B) \)-module representing a point of \( C_n(X, \mathcal{I}) \). The class \( \omega_n[V] \in \gamma^{-1}[\mathcal{I}] \) can then be represented by an ideal \( M \) fitting into the exact sequence

\[
(5.69) \quad 0 \rightarrow M \rightarrow L \rightarrow V \rightarrow 0,
\]

where \( L = \Pi e_{\infty} \otimes_U e_{\infty} V \). Now, given an invertible bimodule \( \mathcal{P} = (\mathcal{D} \mathcal{L})_{\varphi} \), we write \( \Pi^\prime = \Pi^\lambda(A[\mathcal{J}]) \), \( U^\prime = e_{\infty} \Pi e_{\infty} \), and observe that \( \mathcal{P} \otimes \Pi (\Pi e_{\infty} \otimes_U e_{\infty} \Pi) \otimes \Pi \mathcal{P}^* \cong \Pi^\prime e_{\infty} \otimes_U e_{\infty} \Pi^\prime \), where \( \mathcal{P} \) is the progenitor from \( \Pi \) to \( \Pi^\prime \) determined by \( \mathcal{P} \). On the other hand, we have

\[
\psi \mathcal{P} \otimes \Pi \mathcal{D} \cong \psi(\tilde{F} \otimes \Pi \mathcal{D}) \cong \psi(\tilde{F} \otimes \mathcal{D} \mathcal{D}) \cong \psi(\mathcal{P} \mathcal{D}) \cong (\mathcal{D} \mathcal{L})_{\varphi} = \mathcal{P}.
\]

Tensoring now (5.69) with \( \mathcal{P} \) shows that the \( \Pi^\prime \)-modules \( V := \mathcal{P} \otimes \Pi V \) and \( M := \mathcal{P} \otimes_M M \) fit into the exact sequence \( 0 \rightarrow M \rightarrow L \rightarrow V \rightarrow 0 \), with \( L \cong \Pi^\prime e_{\infty} \otimes_U e_{\infty} V \). This means that \([M] \in \gamma^{-1}[\mathcal{J}]\) corresponds under \( \omega_n \) to \([V] \in C_n(X, \mathcal{J}) \), verifying the commutativity of (4.21) and finishing the proof of Theorem 4.3. □

6. Explicit Construction of Ideals. Examples

6.1. Distinguished representatives. Given a rank 1 torsion-free \( \mathcal{D} \)-module \( M \), we choose an embedding \( e : M \rightarrow Q \), where \( Q = \text{Fr}(\mathcal{D}) \). Such an embedding is unique up to automorphism of \( Q \). We will fix this automorphism at a later stage of our calculation. Now, regarding \( M \) and \( Q \) as modules over \( R = T_\Lambda \text{Der}(A) \), we may try to extend \( e \) to \( L \) through \( \eta : M \rightarrow L \). It is easy to see, however, that such an extension does not exist in \( \text{Mod}(R) \). On the other hand, we have

Lemma 6.1. There is a unique \( A \)-linear map \( e_L : L \rightarrow Q \) extending \( e \) in \( \text{Mod}(A) \).

Proof. Let \( \eta_* : \text{Hom}_A(L, Q) \rightarrow \text{Hom}_A(M, Q) \) be the restriction map. We have \( \text{Ker}(\eta_*) \cong \text{Hom}_A(V, Q) = 0 \), since \( V \) is a torsion \( A \)-module, while \( Q \) is torsion-free. On the other hand, \( \text{Coker}(\eta_*) \) is isomorphic to a submodule of \( \text{Ext}_A^1(V, Q) \), while \( \text{Ext}_A^1(V, Q) = 0 \), since \( Q \) is an injective \( A \)-module. It follows that \( \eta_* \) is an isomorphism. □

Our aim is to compute \( e_L \) explicitly, in terms of representation \( V \). First, we consider the map

\[
(6.1) \quad \text{ad} : \text{Hom}_A(L, Q) \rightarrow \text{Der}_A(R, \text{Hom}(L, Q)),
\]

taking \( f : L \rightarrow Q \) to the inner derivation \( \text{ad}_f(r)(x) := rf(x) - f(rx) \), where \( r \in R \) and \( x \in L \). Since \( \text{Ker}(\text{ad}) \cong \text{Hom}_R(L, Q) = 0 \), the map (6.1) is injective, and every \( f \in \text{Hom}_A(L, Q) \) is uniquely determined by \( \text{ad}_f \). In addition, if \( f \) restricts to an \( R \)-linear map \( M \rightarrow Q \), then \( \eta_*(\text{ad}_f) = 0 \) in
\[ \text{Der}_A(R, \text{Hom}(M, Q)) \], and \( \text{ad}_f \) is determined by a (unique) derivation in \( \text{Der}_A(R, \text{Hom}(V, Q)) \). Thus \( e_L \) is uniquely determined by \( \delta_V \in \text{Der}_A(R, \text{Hom}(V, Q)) \) satisfying
\[
\langle e_L(rx) - re_L(x) \rangle = \delta_V(r)[\pi(x)], \quad \forall r \in R, \quad \forall x \in L,
\]
where \( \pi : L \to V \). Furthermore, by the Leibniz rule, the restriction map
\[ \text{Der}_A(R, \text{Hom}(V, Q)) \to \text{Hom}_{\text{A}^e}(\text{Der}(A), \text{Hom}(V, Q)) \]
is an isomorphism: we thus need to compute \( \delta_V \) on \( \text{Der}(A) \) only.

Let \( \mathbb{C}(X \times X)^{\text{reg}} \) be the subring of rational functions on \( X \times X \), regular outside the diagonal of \( X \times X \). Geometrically, we can think of \( \Omega^1(\mathbb{A}^2) \subset A^{\otimes 2} \) as the ideal of the diagonal in \( X \times X \), and \( \Omega^1(\mathbb{A}^2)^* := \text{Hom}_{\mathbb{A}^e}(\Omega^1(\mathbb{A}), A^{\otimes 2}) \) as the subspace of functions in \( \mathbb{C}(X \times X)^{\text{reg}} \) with (at most) simple poles along the diagonal; the canonical pairing between \( \Omega^1(\mathbb{A}) \) and \( \Omega^1(\mathbb{A})^* \) is given by multiplication in \( \mathcal{O}(X \times X) \). Translating this into algebraic language, we have

**Lemma 6.2.** Let \( b \) be the involution on \( \mathbb{C}(X \times X)^{\text{reg}} \) induced by interchanging the factors in \( X \times X \).

1. The assignment \( d \mapsto \langle [d(a)/(a \otimes 1 - 1 \otimes a)]^b \rangle \) defines an injective bimodule homomorphism \( \nu : \text{Der}(A) \to \mathbb{C}(X \times X)^{\text{reg}} \).
2. If \( a \in A \), \( d \in \text{Der}(A) \), and \( d(a) = \sum f_j \otimes g_j \), then \( \nu(d, a) = \sum g_j \Delta_A f_j \).

Now, to compute \( \delta_V(d) \in \text{Hom}(V, Q) \) we identify \( \text{Hom}(V, Q) \cong Q \otimes V^* \). There is a natural action of \( R^* := R \otimes R^* \) on this space: \( R^* \to Q \otimes \text{End}(V^*) \), which is the tensor product of the dual representation \( g^* : R^* \to \text{End}(V^*) \) with composition of natural maps \( R \to \Pi^1(A) \cong \mathcal{D} \hookrightarrow Q \). Abusing notation, we will write \( a \otimes b^* \) for the image of \( a \otimes b^* \in R^* \) in \( Q \otimes \text{End}(V^*) \). Restricting to \( A^{\otimes 2} \subset R^* \), we now get a ring homomorphism \( A^{\otimes 2} \to Q \otimes \text{End}(V^*) \). Since \( \dim(V) < \infty \), this homomorphism takes the elements \( a \otimes 1 - 1 \otimes a \), with \( a \in A \setminus \mathbb{C} \), to units in \( Q \otimes \text{End}(V^*) \) and hence extends canonically to
\[ \mathbb{C}(X \times X)^{\text{reg}} \to Q \otimes \text{End}(V^*) \]

Combining this last homomorphism with the embedding of Lemma 6.2, we define a bimodule map
\[
\nu_V : \text{Der}(A) \to Q \otimes \text{End}(V^*), \quad \Delta_A \mapsto 1 \otimes \text{Id}_{V^*}.
\]

We can now compute \( \delta_V \) in terms of \( \nu_V \). To this end, we choose dual bases \( \{v_i\} \) and \( \{w_i\} \) for \( I \) and \( I^\vee \); by Proposition 6.1, this gives generators \( \hat{a}, \hat{d}, \hat{v}, \) and \( \hat{w} \) for \( \Pi \). Identifying \( L_{\infty} \cong V_{\infty} \cong \mathbb{C} \), we think of \( \hat{v}_i \) and \( \hat{w}_i \) acting on \( L \) as linear maps \( v_i : L \to W \) and \( w_i : L \to C \), i.e., as elements of \( L \) and \( L^* \). Similarly, when acting on \( V \), \( \hat{v}_i \) and \( \hat{w}_i \) give rise to vectors \( v_i \in V \) and covectors \( w_i \in V^* \).

Note that \( v_i = \pi v_i \) and \( w_i = w_i \), where \( \pi : L \to V \). Further, we fix \( I \hookrightarrow A \) and identify \( L \) as in Lemma 6.3. Then we twist \( e : M \to L_{\infty} \) by an automorphism of \( Q \) in such a way that \( e_{L_{\infty}}(v) = v \) for all \( v \in I \subset A \subset Q \). This is possible, since \( e_L : L \to Q \) is an \( A \)-linear extension of \( e \), by Lemma 6.1. With this notation, we have

**Proposition 6.1.** The derivation \( \delta_V : \text{Der}(A) \to Q \otimes V^* \) is given by \( \delta_V(d) = \sum v_i \nu_V d [v_i w_i] \).

**Proof.** First, using the fact that \( \Delta_A \) acts as \( 1 + \sum v_i w_i \) on \( L \) and as identity on \( Q \), it is easy to compute \( \delta_V(\Delta_A) = \sum v_i w_i \). Now, if \( r = [d,a] \in \text{Der}(A) \), then \( \delta_V(r) = [\delta_V(d),a] \), since \( \delta_V(a) = 0 \). On the other hand, by Lemma 6.2[2], we have \( [d,a] = \sum g_j \Delta_A f_j \), so \( \delta_V(r) = \sum g_j \delta_V(\Delta_A) f_j \). Thus, \( \delta_V(d,a) = \sum g_j \delta_V(\Delta_A) f_j \), or, if we think of \( \delta_V(d) \) as an element of \( Q \otimes V^* \), then \( (1 \otimes a^* - a \otimes 1) \delta_V(d) = \left( \sum g_j \otimes f_j^* \right) \delta_V(\Delta_A) \). Lemma 6.2[1] shows now that \( \delta_V(d) = \nu_V(d) \delta_V(\Delta_A) \).

Now, we can state the main result of this section. For \( v \in I \) and \( d \in \text{Der}(A) \), we define
\[
k(d,v) := v - (1 \otimes d^* - d \otimes 1)^{-1} \delta_V(d) [1 \otimes \bar{v}] \in Q \]
where \( \bar{v} = \pi(v) \in V \) and \( (1 \otimes d^* - d \otimes 1)^{-1} \in Q \otimes \text{End}(V^*) \).

**Theorem 6.1.** Let \( V \) be a \( \Pi^1(B) \)-module of dimension \( n = (n,1) \) representing a point in \( C_n(X, I) \). Then the class \( \Omega[V] \in \mathcal{I}(D) \) can be represented by the (fractional) ideal \( M \) generated by the elements \( \text{det}_V \cdot (1 \otimes a^* - a \otimes 1) v \) and \( \text{det}_V \cdot (1 \otimes d^* - d \otimes 1) k(d,v) \), where \( a \in A \), \( d \in \text{Der}(A) \) and \( v \in I \).
Theorem 6.1 needs some explanations.

1. Formally, by [BC], \( \kappa(d,v) \) is well defined only when \( 1 \otimes d^* - d \otimes 1 \) is invertible in \( Q \otimes \text{End}(V^*) \). It is easy to see, however, that the product \( \det_V \cdot (1 \otimes d^* - d \otimes 1) \kappa(d,v) \) makes sense for all \( d \in \text{Der}(A) \) (cf. BC, Remark 2, p. 83).

2. For generators of \( M \) it suffices to take the above determinants with \( a, d \) and \( v \) from some (finite) sets generating \( A, \text{Der}(A) \) and the ideal \( \mathcal{I} \).

Proof. By [5.13], the class \( \omega(V) \) can be represented by \( \tilde{M} = \text{Ker}[\pi : L \to V] \). Our goal is to show that the two kinds of determinants given in the proposition generate \( M := e_L(\tilde{M}) \). To simplify the notation, we denote the elements of \( \mathcal{I} \) (resp., \( \mathcal{I}' \)) and the corresponding elements of \( V \) (resp., \( V^* \)) by the same letter. Using the Leibniz rule, for any \( r \in R \) and \( m \geq 1 \), we have

\[
\delta_V(r^m) = \left( \sum_{s=0}^{m-1} r^s \otimes (r^*)^{m-s-1} \right) \delta_V(r) = \frac{1 \otimes (r^*)^m - r^m \otimes 1}{1 \otimes r^* - r \otimes 1} \delta_V(r),
\]

provided \( 1 \otimes r^* - r \otimes 1 \in Q \otimes \text{End}(V^*) \) is invertible. Now, consider the characteristic polynomial \( p(t) := \det_\rho(r - t \text{Id}_V) \) of \( r \in R \) in the representation \( V \). It is clear that, for any \( x \in L \), \( p(r)x \in \tilde{M} \). To compute its image under \( e_L \), we write

\[
e_L(p(r)x) = p(r)e_L(x) + \delta_V(p(r))[1 \otimes \pi],
\]

where \( \pi = \pi(x) \). Using (6.5) and the fact that \( p(t) = \chi_s(t) \) annihilates \( r^s \in \text{End}(V^*) \), we get \( \delta_V(p(r)) = -(p(r) \otimes 1)(1 \otimes r^* - r \otimes 1)^{-1} \delta_V(r) \). As a result, for \( x = v \in \mathcal{I} \),

\[
e_L(\chi_s(t)v) = \chi_s(r)(v - (1 \otimes r^* - r \otimes 1)^{-1} \delta_V(r)[1 \otimes \pi]) \in M.
\]

Choosing different \( r \in R \), we obtain in this way various elements of \( M \). In particular, for \( r = a \in A \), we have \( \delta_V(a) = 0 \), so (6.6) produces the elements of the first kind \( \chi_a(a)v \in M \). On the other hand, taking \( r = d \in \text{Der}(A) \) results in \( \chi_d(d) \kappa(d,v) \), which are the elements of the second kind in \( M \).

Finally, a simple filtration argument shows that the elements \( \chi_a(a)v \) and \( \chi_d(d)v \), with \( a, d \) and \( v \) running over some generating sets of \( A, \text{Der}(A) \) and \( \mathcal{I} \), generate a submodule \( \tilde{N} \subset \tilde{M} \) of finite codimension in \( L \). Hence \( \tilde{N} = \tilde{M} \), and the images of these elements generate thus \( M = e_L(\tilde{M}) \).

\[\square\]

6.2. Examples.

6.2.1. The affine line. Let \( X = \mathbb{A}^1 \). Choosing a global coordinate on \( X \), we identify \( A = \mathcal{O}(X) \cong \mathbb{C}[x] \).

In this case, \( \text{Der}(A) \) is a free bimodule of rank 1; as a generator of \( \text{Der}(A) \), we may take the derivation \( y \) defined by \( y(x) = 1 \otimes 1 \). It is easy to check that \( \Delta_A = xy - yx \in \text{Der}(A) \). The algebra \( R = T_A \text{Der}(A) \) is isomorphic to the free algebra \( \mathbb{C}[x,y] \), and \( \Pi^1(A) \cong \mathbb{C}[x,y]/\langle xy - yx + 1 \rangle \) is the Weyl algebra \( A_1(\mathbb{C}) \).

The map \( \nu \) of Lemma 6.2 is given by

\[\nu(y) = (1 \otimes x - x \otimes 1)^{-1}, \quad \nu(\Delta) = 1.\]

All line bundles on \( X \) are trivial, so we only need to consider \( B = A[I] \) with \( I = A \). The \( n \)-th Calogero-Moser variety \( \mathcal{C}_n := \mathcal{C}_n(X, A) \) can be described as the space of equivalence classes of matrices

\[\{(\tilde{X}, \tilde{Y}, \tilde{v}, \tilde{w}) : \tilde{X} \in \text{End}(\mathbb{C}^n), \tilde{Y} \in \text{End}(\mathbb{C}^n), \tilde{v} \in \text{Hom}(\mathbb{C}, \mathbb{C}^n), \tilde{w} \in \text{Hom}(\mathbb{C}^n, \mathbb{C})\},\]

satisfying the relation \( \tilde{Y} \tilde{X} - \tilde{X} \tilde{Y} = \text{Id}_n + \tilde{v} \tilde{w} \), modulo the natural action of \( \text{GL}_n(\mathbb{C}) \):

\[\{(\tilde{X}, \tilde{Y}, \tilde{v}, \tilde{w}) \mapsto (g \tilde{X}g^{-1}, g \tilde{Y}g^{-1}, g \tilde{v}, g \tilde{w}g^{-1}) : g \in \text{GL}_n(\mathbb{C})\}.
\]

If we choose \( v = 1 \) as a generator of \( I = A \), then the ideal \( M \) of \( D \cong \Pi^1(A) \) corresponding to a point \( (\tilde{X}, \tilde{Y}, \tilde{v}, \tilde{w}) \) is given by

\[M = D \cdot \det(\tilde{X} - x \text{Id}_n) + D \cdot \det(\tilde{Y} - y \text{Id}_n) \kappa,
\]

where \( \kappa = 1 - \tilde{v}(\tilde{Y}^t - y \text{Id}_n)^{-1}(\tilde{X}^t - x \text{Id}_n)^{-1} \tilde{w}^t \). This agrees with the description of ideals of \( A_1(\mathbb{C}) \) given in [BC].
6.2.2. The complex torus. Let \(X = \mathbb{C}^*\). We identify \(A = \mathcal{O}(X)\) with \(\mathbb{C}[x, x^{-1}]\), the ring of Laurent polynomials. As in the affine line case, \(\mathcal{D}(A)\) is freely generated by the derivation \(y\) defined by \(y(x) = 1 \otimes 1\). The algebra \(R\) is isomorphic to the free product \(\mathbb{C}[x^\pm 1, y] := \mathbb{C}[x, x^{-1}] \ast \mathbb{C}[y]\), and \(\Delta_A = xy - xy\) in \(R\). The matrix description of the Calogero-Moser spaces \(\mathcal{C}_n\) and the formulas for the corresponding fractional ideals of \(\mathcal{D} \cong \Pi^1(A) = \mathbb{C}[x^\pm 1, y]/(xy - xy + 1)\) are the same as above, except for the fact that \(x\) and \(\bar{X}\) are now invertible. A new feature is that \(A\) has nontrivial units \(x^r\), \(r \in \mathbb{Z}\). The corresponding group \(\Gamma\) can be identified with \(\mathbb{Z}\) and its action on \(\mathcal{C}_n\) is given by

\[
r.(\bar{X}, \bar{Y}, \bar{v}, \bar{w}) = (\bar{X}, \bar{Y} + r\bar{X}^{-1}, \bar{v}, \bar{w}) , \quad r \in \mathbb{Z} .
\]

Thus, by Theorem 6.2 the classes of ideals of \(\mathcal{D} \cong \Pi^1(A)\) are parameterized by the points of the quotient variety \(\mathcal{C}_n = \mathcal{C}/\mathbb{Z}\). It is worth mentioning that one may choose a different generator for the bimodule \(\mathcal{D}(A)\): for example, \(z = xy\), instead of \(y\). Then \(\Delta_A = z - xx^{-1}\), which gives an alternative matrix description of \(\mathcal{C}_n\) and the corresponding ideals.

6.2.3. A general plane curve. Let \(X\) be a smooth curve in \(\mathbb{C}^2\) defined by the equation \(F(x, y) = 0\), with \(F(x, y) := \sum_{r,s} a_{rs} x^r y^s \in \mathbb{C}[x, y]\). In this case, the algebra \(A \cong \mathbb{C}[x, y]/\langle F(x, y) \rangle\) is generated by \(x\) and \(y\) and the module \(\mathcal{D}(A)\) is (freely) generated by the derivation \(\partial\) defined by \(\partial(x) = F'_y(x, y), \partial(y) = -F'_x(x, y)\). The bimodule \(\mathcal{D}(A)\) is generated by the derivation \(\Delta = \Delta_A\) and the element \(z\) defined by

\[
z(x) = \sum_{r,s} a_{rs} x^r y^s \otimes 1 - x^r \otimes y^s , \quad z(y) = -\sum_{r,s} a_{rs} x^r \otimes y^s - 1 \otimes x^r y^s .
\]

These generators satisfy the following commutation relations

\[
[z, x] = \sum_{r,s} a_{rs} \sum_{k=0}^{s-1} y^s-k \Delta y^k x^r , \quad [z, y] = -\sum_{r,s} \sum_{l=0}^{r-1} y^s x^{-l-1} \Delta x^r .
\]

By Proposition 5.3, the algebra \(\Pi^\Lambda(B)\) is then generated by the elements \(\hat{x}, \hat{y}, \hat{z}, \hat{v}_i, \hat{w}_i\) and \(\hat{\Delta}\), subject to the relations \((6.7)\) and \((5.10)\). The assignment \(x \mapsto x, y \mapsto y, z \mapsto \partial, \Delta \mapsto 1\) extends to an isomorphism between \(\Pi^1(A)\) and the ring \(\mathcal{D}\) of differential operators on \(X\). The bimodule map \(\nu\) of Lemma 6.2 is given by

\[
\nu(z) = -\sum_{r,s} a_{rs} y^s \otimes x^r / (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1) , \quad \nu(\Delta) = 1 .
\]

Now, let us describe generic points of the varieties \(\mathcal{C}_n(X, \mathcal{I})\); for simplicity, consider only the case when \(\mathcal{I}\) is trivial. Choose \(n\) distinct points \(p_i = (x_i, y_i) \in X, i = 1, \ldots, n\), and define

\[
(\bar{X}, \bar{Y}, \bar{Z}, \bar{v}, \bar{w}) \in \mathcal{D}(\mathbb{C}^n) \times \mathcal{D}(\mathbb{C}^n) \times \mathcal{D}(\mathbb{C}^n) \times \mathcal{D}(\mathbb{C}^n) \times \mathcal{D}(\mathbb{C}^n)
\]

by the following formulas

\[
\bar{X} = \text{diag}(x_1, \ldots, x_n) , \quad \bar{Y} = \text{diag}(y_1, \ldots, y_n) , \quad \bar{v} = -\bar{w} = (1, \ldots, 1) ,
\]

\[
\bar{Z}_{ii} = \alpha_i \quad \text{and} \quad \bar{Z}_{ij} = \frac{F(x_i, y_i)}{(x_i - x_j)(y_i - y_j)} \quad \text{(for } i \neq j) ,
\]

where \(\alpha_1, \ldots, \alpha_n\) are arbitrary scalars. Then, a straightforward calculation, using the relations \((6.7)\), shows that the assignment

\[
\hat{x} \mapsto \bar{X} , \quad \hat{y} \mapsto \bar{Y} , \quad \hat{z} \mapsto \bar{Z} , \quad \hat{v} \mapsto \bar{v} , \quad \hat{w} \mapsto \bar{w} , \quad \hat{\Delta} \mapsto \text{Id} + \bar{v} \bar{w}
\]

extends to a representation of \(\Pi^\Lambda(B)\), with \(B = A[\Lambda]\) and \(\Lambda = (1, -n)\), on the vector space \(\mathcal{V} = \mathbb{C}^n \oplus \mathbb{C}\).

Remark. The matrix \(\bar{Z}\) defined above is a generalization of the classical Moser matrix in the theory of integrable systems (see [KKS]).

To illustrate Theorem 6.1 we now describe the fractional ideal representing the class \(\omega[V]\) for an arbitrary \([V] \in \mathcal{C}_n(X, \mathcal{I})\). We consider first the case when \(\mathcal{I}\) is trivial. In that case, we identify \(\mathcal{I} =
\( I^\prime = A \) and choose \( v = w = 1 \) as the generators of \( I \) and \( I^\prime \). A representation \( V = \mathbb{C}^n \oplus \mathbb{C} \) may then be described by the matrices \((E.9)\), which, apart from \((E.7)\), satisfy the following relations

\[
F(X, Y) = 0, \quad [X, Y] = 0 \quad \text{and} \quad \Delta = \text{Id}_n + \bar{v} \bar{w}.
\]

The dual representation \( g^* : \Pi^o \to \text{End}(V^*) \) is given by the transposed matrices.

Now, \((6.4)\) together with \((6.8)\) show that

\[
\kappa = 1 + \bar{v}^t (\bar{Z}^t - z \text{Id}_n)^{-1}(\bar{X}^t - x \text{Id}_n)^{-1}(\bar{Y}^t - y \text{Id}_n)^{-1}F(\bar{X}^t, y \text{Id}_n)\bar{w}^t.
\]

Thus, if \([V] \in \mathcal{C}_n(X, A)\) is determined by the data \((\bar{X}, \bar{Y}, \bar{Z}, \bar{v}, \bar{w})\), then the class \(\omega[V]\) is represented by the (fractional) ideal

\[
M = D \cdot \det(\bar{X} - x \text{Id}_n) + D \cdot \det(\bar{Y} - y \text{Id}_n) + D \cdot \det(\bar{Z} - z \text{Id}_n) \kappa.
\]

In the general case, when \(I\) is arbitrary, \(\kappa\) is replaced by

\[
\kappa(v) = v + \sum_i \left( \bar{v}^t (\bar{Z}^t - z \text{Id}_n)^{-1}(\bar{X}^t - x \text{Id}_n)^{-1}(\bar{Y}^t - y \text{Id}_n)^{-1}F(\bar{X}^t, y \text{Id}_n)\bar{w}^t \right) v_i,
\]

and the corresponding class \(\omega[V] \in \gamma^{-1}[I]\) is given by

\[
M = \sum_i \left( D \cdot \det(\bar{X} - x \text{Id}_n) v_i + D \cdot \det(\bar{Y} - y \text{Id}_n) v_i + D \cdot \det(\bar{Z} - z \text{Id}_n) \kappa(v_i) \right).
\]

6.2.4. A hyperelliptic curve. This is a special plane curve described by the equation \(y^2 = P(x)\), where \(P(x) = \sum a_s x^s\) is a polynomial with simple roots. Some of the above formulas simplify in this case. We have \(A \cong \mathbb{C}[x, y]/(y^2 - P(x))\), \(\text{Der}(A)\) is freely generated by \(\partial\), with \(\partial(x) = 2y\) and \(\partial(y) = P'(x)\), and the bimodule \(\text{Der}(A)\) is generated by \(\Delta\) and the element \(z\) defined by

\[
z(x) = y \otimes 1 + 1 \otimes y, \quad z(y) = (P(x) \otimes 1 - 1 \otimes P(x))/(x \otimes 1 - 1 \otimes x).
\]

The commutation relations \((6.7)\) in \(\text{Der}(A)\) are

\[
[z, x] = y\Delta + \Delta y, \quad [z, y] = \sum s a_s \sum_{l=0}^{s-1} x^{s-l-1}\Delta x^l.
\]

Now, for a hyperelliptic curve, a point of \(\mathcal{C}_n(X, I)\) is determined by the following data: (1) a representation of \(A\) on the vector space \(V = \mathbb{C}^n\), i.e. a pair of matrices \((\bar{X}, \bar{Y}) \in \text{End}(\mathbb{C}^n) \times \text{End}(\mathbb{C}^n)\) satisfying \(\bar{Y}^2 = P(\bar{X})\); (2) a pair of \(A\)-module maps \(I \to V\) and \(I^\prime \to V^*\), with chosen images \(\bar{v}_i \in V\) and \(\bar{w}_i \in V^*\) of dual bases of \(I\) and \(I^\prime\); (3) a matrix \(\bar{Z} \in \text{End}(\mathbb{C}^n)\), such that \(\bar{X}, \bar{Y}, \bar{Z}\) and \(\Delta := \text{Id}_n + \sum_i \bar{v}_i \bar{w}_i\) satisfy \((6.13)\). In this case, formula \((6.11)\) reads

\[
\kappa(v) = v - \sum_i \left( \bar{v}^t (\bar{Z}^t - z \text{Id}_n)^{-1}(\bar{X}^t - x \text{Id}_n)^{-1}(\bar{Y}^t + y \text{Id}_n)\bar{w}^t \right) v_i,
\]

and the corresponding ideal is given by \((6.12)\).

Appendix A. Half-Forms on Riemann Surfaces

George Wilson

In this note I provide a proof for one of the key facts (Proposition \[A.1\] below) needed to understand the relationship between deformed preprojective algebras and rings of differential operators. The note owes a great deal to conversations with Graeme Segal.
Statement of problem. Let $X$ be a compact Riemann surface, and let $\Delta$ be the diagonal divisor in $X \times X$. We have the inclusion
$$O_{X \times X}(-\Delta) \hookrightarrow O_{X \times X}(\Delta)$$
of the sheaf of functions that vanish on $\Delta$ into the sheaf of functions that are allowed a simple pole on $\Delta$. The quotient sheaf $O_{X \times X}(\Delta)/O_{X \times X}(-\Delta)$ is supported on the first infinitesimal neighborhood $\Delta_1$ of $\Delta$. Similarly, if $L$ is a line bundle on $X$, we have the sheaf $D_1(L)$ of differential operators of order $\leq 1$ on $L$. This is usually regarded as a sheaf on $X$, but since we can compose a differential operator with a function either on the left or on the right, it has two commuting structures of $O_X$-module, so it too can be regarded as a sheaf on $X \times X$, again supported on $\Delta_1$.

Fix a square root $\Omega^{1/2}$ of the canonical bundle $\Omega_X$; the choice of square root will be immaterial, because the corresponding sheaves of differential operators $D(\Omega^{1/2})$ are canonically isomorphic to each other. Our aim is to understand the following fact stated in [G].

**Proposition A.1.** There is a canonical isomorphism (of sheaves over $X \times X$)
$$\chi : O_{X \times X}(\Delta)/O_{X \times X}(-\Delta) \rightarrow D_1(\Omega^{1/2}).$$

A consequence is that the sheaf of deformed preprojective algebras formed from $O_X$ is canonically isomorphic to the sheaf $D(\Omega^{1/2})$ of differential operators on $\Omega^{1/2}$. This is explained in [G], Section 13.

The isomorphism in Proposition A.1 does not seem to be a well-known fact, and at first sight looks puzzling, because there are no half-forms in the left hand side. The proof sketched in the current version of [G] is not very convincing, so it seems worth recording the following simple explanation shown to me by Segal: although Proposition A.1 itself does not look familiar, it can be obtained by combining two familiar facts, of a slightly different nature. While we are about it, we shall deal also with a slight generalization, twisting by an arbitrary line bundle $L$ on $X$.

We use the following notation: $\Delta_n$ is the $n$th infinitesimal neighborhood of the diagonal in $X \times X$, so that we have a canonical identification
$$L / L(-n(1+\Delta)) \simeq L|_{\Delta_n}.$$The two projections $X \times X \rightarrow X$ are denoted by $p_1$ and $p_2$. If $U$ is a simply-connected coordinate patch on $X$ and $z$ is a parameter on $U$, we write $(z_1, z_2)$ for the induced parameters on $U \times U \subset X \times X$. The parameter $z$ determines a trivialization (non-vanishing section) $dz$ of $\Omega_X|U$. Fixing also an isomorphism \footnote{Of course $\kappa$ is uniquely determined up to a constant multiple. The isomorphism $\chi$ in Proposition A.1 does not depend on this multiple, but some of the intermediate steps below do.} $\kappa : (\Omega^{1/2})^{\otimes 2} \simeq \Omega_X$, we may choose a trivialization $dz^{1/2}$ of $\Omega^{1/2}|U$ such that $\kappa(dz^{1/2} \otimes dz^{1/2}) = dz$ (there are only two choices, differing by a sign).

A proof of Proposition A.1 We shall use the following description of differential operators, which goes back to Cauchy (see [C], p. 60, formule (4)).

**Proposition A.2.** Let $L$ be a line bundle on $X$. Then there is a canonical identification (of sheaves over $X \times X$)
$$p_1^*(L) \otimes p_2^*(L^* \otimes \Omega_X)(n + 1)\Delta) \simeq D_n(L).$$

*Proof.* The action of a (local) section of the sheaf on the left of $\mathbb{A}_2$ on a section of $L$ is given by contracting with the factor $p_2^*(L^*)$ and then taking the residue on the diagonal of the resulting differential. Let us spell that out in more detail in the case where $L$ is the trivial bundle and $n = 1$. The sheaf on the left of $\mathbb{A}_2$ is then just $p_2^*(\Omega_X)(2\Delta)|\Delta_1 = p_2^*(\Omega_X)(2\Delta) / p_2^*(\Omega_X)$. In terms of a parameter $z$, a local section of this sheaf has the form
$$\varphi(z_1, z_2) \frac{dz_2}{(z_2 - z_1)^2}$$
modulo regular terms.
(where \( \phi \) is regular). To see how this acts on a function \( f(z) \), we have to calculate the residue
\[
\text{res}_{z_2 = z_1} \frac{f(z_2) \phi(z_1, z_2) dz_2}{(z_2 - z_1)^2}
\]
\( (z_1 \) is held fixed during the calculation). Expanding
\[
f(z_2) = f(z_1) + f'(z_1)(z_2 - z_1) + \ldots,
\]
and
\[
\frac{\phi(z_1, z_2)}{(z_2 - z_1)^2} = \frac{a(z_1)}{(z_2 - z_1)^2} + \frac{b(z_1)}{z_2 - z_1} + \ldots,
\]
we find that the residue is
\[
a(z) \frac{df}{dz} + b(z)f \bigg|_{z = z_1}.
\]
The proposition is now clear. \( \square \)

Now let \( U \) be a coordinate patch on \( X \). We consider the classical differential \( \gamma \) given in terms of a parameter \( z \) by
\[
\gamma := \frac{dz_1^{1/2} dz_2^{1/2}}{z_1 - z_2} .
\]
It is a non-vanishing section (over \( U \times U \)) of the line bundle
\[
p_1^*(\Omega^{1/2}) \otimes p_2^*(\Omega^{1/2})(\Delta).
\]
It depends on the parameter \( z \); however, its restriction to \( \Delta \) does not. Indeed, when we identify \( \mathcal{O}_{X \times X}(-\Delta) \mid \Delta \) with the canonical bundle on the diagonal, \( z_1 - z_2 \) corresponds to \( dz \), so \( \gamma \mid \Delta \) becomes the constant section \( 1 \in \mathcal{O}(U) \). Furthermore, because \( \gamma \) is skew in the two variables, its restriction to \( \Delta_1 \) is also independent of the choice of \( z \). Thus for any sheaf \( \mathcal{M} \) over \( X \times X \), multiplication by \( \gamma \) gives a well-defined global isomorphism
\[
\mathcal{M} \mid \Delta_1 \simeq \mathcal{M} \otimes p_1^*(\Omega^{1/2}) \otimes p_2^*(\Omega^{1/2})(\Delta) \mid \Delta_1 .
\]
In particular, for any line bundle \( \mathcal{L} \) over \( X \), we get an isomorphism
\[
p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(\Delta) \mid \Delta_1 \simeq p_1^*(\mathcal{L} \otimes \Omega^{1/2}) \otimes p_2^*(\mathcal{L}^* \otimes \Omega^{1/2})(2\Delta) \mid \Delta_1 .
\]
Tensoring our chosen isomorphism \( \kappa : (\Omega^{1/2})^* \simeq \Omega_X \) with \( (\Omega^{1/2})^* \), we get an isomorphism \( \Omega^{1/2} \simeq (\Omega^{1/2})^* \otimes \Omega_X \), and hence for any \( \mathcal{L} \) an isomorphism
\[
\mathcal{L}^* \otimes \Omega^{1/2} \simeq (\mathcal{L} \otimes \Omega^{1/2})^* \otimes \Omega_X .
\]
Inserting this into (A.4) and taking account of (A.1) and (A.2) gives us an isomorphism (now independent of \( \kappa \))
\[
p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(\Delta) / p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L}^*)(-\Delta) \simeq D_1(\mathcal{L} \otimes \Omega^{1/2}) .
\]
Taking \( \mathcal{L} = \mathcal{O}_X \), we get Proposition [A.1].

\( ^{\text{it is the principal part of the Szegö kernel on } X \times X} \)
Remarks. 1. Reversing the arguments in [G], we easily get from [A.5] a construction of any $D(L)$ as a sheaf of ‘twisted deformed preprojective algebras’.

2. The differential $\gamma$ in [A.3] is invariant under a linear fractional change of parameter. Thus if we fix a projective structure on $X$ (thought of as an atlas with linear fractional transition functions), then $\gamma$ is well-defined globally on some analytic neighbourhood of $\Delta$, not merely on $\Delta_1$. This remark is the starting point for the papers [BR].

3. The considerations above give an explicit formula for the isomorphism $\chi$ in Proposition [A.1] an element of $O_{X \times X}(\Delta)/O_{X \times X}(\Delta_1)$ has a unique local representative of the form

$$a(z_1)(z_2 - z_1)^{-1} + b(z_1) + \ldots,$$

and $\chi$ maps this to the operator

$$f dz^{1/2} \mapsto \left( a(z) \frac{df}{dz} + b(z)f \right) dz^{1/2}.$$

In an earlier version of this note I verified Proposition [A.1] by checking directly that the map $\chi$ defined by this formula is independent of the chosen parameter $z$; however, the calculation is surprisingly complicated (and unilluminating).

References

[AMM] H. Airault, H. P. McKean and J. Moser, Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem, Commn. Pure Appl. Math. 30 (1977), 95-148.

[AZ] M. Artin and J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994), 228-287.

[BN1] D. Ben-Zvi and T. Nevins, Wilson’s Grassmannian and a noncommutative quadric, Int. Math. Res. Not. 21 (2003), 1155-1197.

[Ba] H. Bass, Algebraic K-theory, W. A. Benjamin Inc., New York-Amsterdam, 1968.

[BB] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque 100, Soc. Math. France, 1982, pp. 5-171.

[Be] B. Bendifallah, Modules d’extensions des algèbres triangulaires, C. R. Acad. Sci. Paris Ser. I 339 (2004), 387-390.

[BN] D. Ben-Zvi and T. Nevins, Perverse bundles and Calogero-Moser spaces, Compos. Math. 144(6) (2008), 1403-1428.

[BN1] D. Ben-Zvi and T. Nevins, From solitons to many-body systems, Pure Appl. Math. Q. 4(2) (2008), 319-361.

[B] Yu. Berest, Calogero-Moser spaces over algebraic curves, Selecta Math. (N.S.) 14(3) (2009), 373-396.

[BC] Yu. Berest and O. Chalykh, $A_{\infty}$-modules and Calogero-Moser spaces, J. reine angew. Math. 607 (2007), 69-112.

[BCL] Yu. Berest and O. Chalykh, A note on the Dunkl representation of Cherednik algebras on algebraic curves, in preparation.

[BCE] Yu. Berest, O. Chalykh and F. Eshmatov, Recollement of deformed preprojective algebras and the Calogero-Moser correspondence, Moscow Math. J. 8(1) (2008), 21-37.

[BW] Yu. Berest and G. Wilson, Differential operators on an affine curve: ideal classes and Picard groups, preprint, arXiv:0810.0223, Quart. J. Math. Oxford Ser.(2) (to appear).

[BW1] Yu. Berest and G. Wilson, Automorphisms and ideals of the Weyl algebra, Math. Ann. 318(1) (2000), 127-147.

[BW2] Yu. Berest and G. Wilson, Ideal classes of the Weyl algebra and noncommutative projective geometry (with an Appendix by M. van den Bergh), Internat. Math. Res. Notices 26 (2002), 1347-1396.

[BR] I. Biswas and A. K. Raina, Projective structures on a Riemann surface. I. Internat. Math. Res. Notices 1996, 753-768. II. Ibid. 1999, 685–716. III. Differential Geom. Appl. 15 2001, 203–219.

[BJ] J.-E. Björk, Rings of Differential Operators, North-Holland Publishing, Amsterdam, 1979.

[CH] R. C. Cannings and M. P. Holland, Right ideals of rings of differential operators, J. Algebra 167 (1994), 116-141.

[CH1] R. C. Cannings and M. P. Holland, Etale covers, bimodules and differential operators, Math. Z. 216 (1994), 179-194.

[CE] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.

[CC] D. V. Chudnovsky and G. V. Chudnovsky, Pole expansion of nonlinear partial differential equations, II Nuovo Cimento 40B (1977), 339–353.

[C] A. L. Cauchy, Oeuvres (2) 12, Gauthier-Villars, Paris, 1882.

[CPS] E. Cline, B. Parshall and L. Scott, Algebraic stratification in representation categories, J. Algebra 117 (1988), 504-521.

[CB] W. Crawley-Boevey, Preprojective algebras, differential operators and a Conze embedding for deformations of Kleinian singularities, Comment. Math. Helv. 74(4) (1999), 548–574.
[CB1] W. Crawley-Boevey, Representations of quivers, preprojective algebras and deformations of quotient singularities, lectures at the workshop on ‘Quantizations of Kleinian singularities’, Oberwolfach, May 1999.

[CB2] W. Crawley-Boevey, Geometry of the moment map for representations of quivers, Compositio Math. 126 (2001), 257–293.

[CB3] W. Crawley-Boevey, Geometry of representations of algebras, lecture notes, Oxford, 1993.

[CEG] W. Crawley-Boevey, P. Etingof and V. Ginzburg, Noncommutative geometry and quiver algebras, Adv. Math. 209 (2007), 274–336.

[CBH] W. Crawley-Boevey and M. Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605–636.

[CQ] J. Cuntz and D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8(2) (1995), 251–289.

[E] P. Etingof, Cherednik and Hecke algebras of varieties with a finite group action, preprint, \( \text{arXiv:math.QA/0406499} \).

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Invent. Math. 147 (2002), 243–348.

[FG] M. Finkelberg and V. Ginzburg, Cherednik algebras for algebraic curves, preprint, \( \text{arXiv:0704.3494} \).

[G] V. Ginzburg, Lectures on Noncommutative geometry, preprint, \( \text{arXiv:math.AG/0506603} \).

[KKO] A. Kapustin, A. Kuznetsov, and D. Orlov, Noncommutative instantons and twistor transform, Comm. Math. Phys. 220 (2001), 385–432.

[KKS] D. Kazhdan, B. Kostant and S. Sternberg, Hamiltonian group actions and dynamical systems of Calogero type, Comm. Pure Appl. Math. 31 (1978), 481–507.

[Ko] S. König, Tilting complexes, perpendicular categories and recollements of derived module categories of rings, J. Pure and Appl. Algebra 73 (1991), 211–232.

[Kou] M. Kontsevich, XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry, Contemp. Math. 462 (2008), 1–21.

[KR] M. Kontsevich and A. Rosenberg, Noncommutative smooth spaces, The Gelfand Mathematical Seminars 1996-1999, 85–108, Birkhäuser, Boston, 2000.

[K] H. Kraft, Geometrische Methoden in der Invariantentheorie, Aspects of Mathematics, Vieweg & Sohn, Braunschweig, 1984.

[Kr] I. M. Krichever, On rational solutions of the Kadomtsev-Petviashvili equation and in tegrable systems of N particles on the line, Funct. Anal. Appl. 12:1 (1978), 76–78 (Russian), 59–61 (English).

[L] T. Levasseur, Some properties of non-commutative regular graded rings, Glasgow Math. J. 34 (1992), 277–300.

[LeB] L. Le Bruyn, Moduli spaces of right ideals of the Weyl algebra, J. Algebra 172, 32–48 (1995).

[Lo] J.-L. Loday, Cyclic Homology, Springer-Verlag Springer-Verlag, Berlin-New York, 1992.

[Lu] D. Luna, Slices étalés, Bull. Soc. Math. France 103 (1973), 81–105.

[NB] K. de Naeghel and M. Van den Bergh, Ideals classes of three-dimensional Sklyanin algebras, J. Algebra 276 (2) (2004), 515–551.

[NS] T. A. Nevins and J. T. Stafford, Sklyanin algebras and Hilbert schemes of points, Adv. Math. 210(2) (2007), 405–478.

[R] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math 1099, 1984.

[Ro] J. Rotman, An Introduction to Homological Algebra, Academic press, New York, 1979.

[SS] S. P. Smith and J. T. Stafford, Differential operators on an affine curve, Proc. London Math. Soc. (3) 56 (1988), 229–259.

[S] J.-P. Serre, Local Algebra, Springer-Verlag, Berlin-Heidelberg-New York, 2000.

[St] J. T. Stafford, Endomorphisms of right ideals of the Weyl algebra, Trans. Amer. Math. Soc. 299 (1987), 623–639.

[vdB] M. Van den Bergh, Double Poisson algebras, Trans. Amer. Math. Soc. 360 (2008), 5711–5769.

[W] G. Wilson, Collisions of Calogero-Moser particles and an adelic Grassmannian (with an Appendix by I. G. Macdonald), Invent. Math. 133 (1998), 1–41.

[W1] G. Wilson, Bispectral commutative ordinary differential operators, J. Reine Angew. Math. 442 (1993), 177–204.