Corrigendum: Sufficient conditions for uniqueness of the weak value

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In section 5 of [1] we implicitly used the following lemma without proof.

Lemma. The singular values of the \( M \times N \) dimensional matrix \( F = P + g^n F_n \) with \( M \leq N \) have maximum leading order of \( g^n \), where \( P = [p_1 \vec{1} \cdots p_N \vec{1}] \) and \( F_n = [E_1 \cdots E_N] \) such that \( \sum_j p_j = 1 \) and \( \sum_j E_j = 0 \).

Proof. The \( M \) singular values of \( F \) are \( \sigma_k = \sqrt{\lambda_k} \), where \( \lambda_k \) are the eigenvalues of the \( M \times M \) dimensional matrix \( G = F F^T \). The \( N \times N \) dimensional matrix \( H = F^T F \) also has the same \( M \) eigenvalues as \( G \), as well as \( (N - M) \) additional zero eigenvalues. Since \( P^T F_n = 0 \), the latter has the simple form \( H = P^T P + g^n F_n^T F_n \), where \( (P^T P)_{ij} = M p_i p_j \) is \( M \) times the projection operator onto the probability vector \( \vec{p} = (p_1, \ldots, p_N) \) and \( (F_n^T F_n)_{ij} = E_i \cdot E_j \). We will use \( H \) to determine the singular values of \( F \).

Differentiating the eigenvalue relation \( H(g^n)\vec{u}_k(g^n) = \lambda_k(g^n)\vec{u}_k(g^n) \) with respect to \( g^n \) produces the following deformation equation that describes how the eigenvalues of \( H \) continuously change with increasing \( g^n \),

\[
\dot{\lambda}_k(g^n) = ||F_n\vec{u}_k(g^n)||^2.
\]

Integrating this equation produces the following perturbative expansion of the eigenvalues for small \( g \),

\[
\lambda_k(g^n) = \lambda_k(0) + g^n ||F_n\vec{u}_k(0)||^2 + O(g^{2n}).
\]

Hence, to prove the lemma it is sufficient to show that \( \lambda_k(0) \) and \( ||F_n\vec{u}_k(0)|| \) cannot both vanish unless \( \lambda_k(g^n) = 0 \) for all \( g \).

Since \( H(0) = P^T P \) is proportional to a projection operator, \( \lambda_1(0) = M ||\vec{p}||^2 \) is its only nonzero eigenvalue with associated eigenvector \( \vec{u}_1(0) = \vec{p}/||\vec{p}|| \). Hence, \( \sigma_1(g^n) \approx \sqrt{M} ||\vec{p}|| \approx 0 \) to leading order. For \( k \neq 1 \), \( \lambda_k(0) = 0 \) and \( \vec{u}_k(0) \) can be chosen arbitrarily to span the degenerate \( (N - 1) \)-dimensional subspace orthogonal to \( \vec{u}_1(0) \). Suppose \( ||F_n\vec{u}_k(0)|| = 0 \) for some \( k \neq 1 \), which implies \( F_n\vec{u}_k(0) = 0 \) since only the zero vector has zero norm. It follows that \( H(g^n)\vec{u}_k(0) = P^T P\vec{u}_k(0) + g^n F_n^T F_n\vec{u}_k(0) = 0 \) since \( \vec{u}_k(0) \) is orthogonal to \( \vec{u}_1(0) \). Therefore, \( \vec{u}_k(0) \) is an eigenvector of \( H(g^n) \) with eigenvalue 0 for any \( g \). Since \( H \) is symmetric, its eigenvectors form an orthogonal set for any \( g \), so we must have the
identification $\tilde{u}_k(g^{2n}) = \tilde{u}_k(0)$. As a result, the associated eigenvalue vanishes for any $g$, $\lambda_k(g^{2n}) = \lambda_k(0) = 0$, which proves the lemma.

This proof has also been included in a subsequent extended article [2].

Acknowledgments

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References

[1] Dressel J and Jordan A N 2012 Sufficient conditions for uniqueness of the weak value J. Phys. A: Math. Theor. 45 015304

[2] Dressel J and Jordan A N 2012 Contextual-value approach to the generalized measurement of observables Phys. Rev. A 85 022123

[3] Parrott S 2012 Proof gap in ‘Sufficient conditions for uniqueness of the weak value’ arXiv:1202.5604v6