Quantum thermal rectification and heat amplification in a nonequilibrium V-type system

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Quantum thermal rectification and heat amplification are investigated in a nonequilibrium V-type three-level system. By applying the Redfield master equation combined with full counting statistics, we obtain the steady state heat transport. The noise-induced interference is found to be able to rectify full counting statistics, such as the heat current and noise power, which paves a new way to design quantum thermal rectifier. Within the three-reservoir setup, the heat amplification is clearly identified far-from equilibrium, which is in absence of the negative differential thermal conductance.

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I. INTRODUCTION

How to smartly control energy flow and efficiently manipulate logical gates is a challenging problem, ranging from molecular electronics \cite{1}, spintronics \cite{2}, quantum information and computation \cite{3}. The electronic diode and electronic transistor, as two main ingredients, have spurred the emergence of semiconductor industry \cite{4}. Inspired by these concepts in electronic systems, the quantum thermal rectifier (thermal diode) and thermal transistor have been proposed in phononics \cite{5, 6}. They constitute the basis of functional thermal devices, experimentally realized in quantum dots \cite{7}, nanotubes \cite{8}, phase change materials \cite{9}, and thermal metamaterials \cite{10, 11}.

![FIG. 1: (Color online) Schematic diagram of the nonequilibrium V-type three-level system represented by three black horizontal lines, and the transitions between states are shown as double-arrowed dashed lines. The red left and blue right half circles are thermal baths, with temperatures $T_L$ and $T_R$, respectively. The purple upper square is the middle thermal bath with the temperature $T_M$. The interactions between thermal baths and V-type system are described as the double-arrowed wave lines.](image-url)
Quantum thermal rectification, one of the most fundamental phononic components, was described as a device exhibiting a larger heat flow in one direction than its counterpart in the opposite direction, driven by the thermal gradient. It was defined by [12, 13]

\[ R_J = \frac{(J_+ - J_-)}{\max\{J_+, J_-\}} , \]  

where \( R_J \) denotes the rectification of the current and \( J_{\pm} \) are heat currents in the forward and backward gradient configurations. The thermal rectification effect has been intensively investigated in two-terminal phononic lattice [5, 14, 15], spin chain [16] and nonequilibrium spin-boson model [17]. It was later extended to three-terminal phononic thermoelectric system [18, 19] and atomic junctions [13]. Typically, the quantum rectification can be realized in the asymmetric structures of quantum systems [13–16, 20–22], different system-bath couplings [17, 23–26], or including an additional phonon bath [18]. Recently, the noise-induced interference was unraveled to enhance the quantum coherence for both transient dynamics [27–29] and steady state behavior [30, 31] in the quantum V-type system. It was also considered as a novel source to significantly improve the energy power and efficiency [32–36]. By tying two seemingly unrelated effects together, i.e., quantum thermal rectification and noise-induced interference, we ask the first question: will quantum interference exhibit the rectification in the nonequilibrium V-type system?

Quantum heat amplification, that a slight change in the base heat current will dramatically change heat currents at the collector and emitter, realizes the thermal transistor [5]. The amplification factor is defined by the ratio

\[ \beta_u = |\partial J_u/\partial J_b| , \ u = c, e \]  

with \( J_b \) the base current, \( J_c \) the collector current and \( J_e \) the emitter current [15]. Usually, the heat transistor is announced to work as \( \beta_u > 1 \). In previous works, it was widely believed that the negative differential thermal conductance (NDTC) is a compulsory ingredient of the heat amplification [37–41]. The NDTC is traditionally described by the phenomenon that the heat current decrease by increasing the temperature bias between two baths [42–44]. However, in a recent study of phononic thermoelectric system, J. H. Jiang et al. proposed that heat amplification can work in linear response regime, even without NDTC [18]. Hence, we raise the second question: based on the V-type system, can we realize the heat amplification far-from equilibrium in absence of NDTC?

To answer these questions, we investigate the steady state heat transfer in a nonequilibrium V-type system, with the model detailed in Sec. II A. We apply a Redfield scheme to obtain the quantum master equation by including the noise-induced interference, detailed in Sec. II B. The effect of quantum interference on nonequilibrium steady state coherence is analytically analyzed in Sec. II C. In Sec. III, we combine the Redfield master equation and the full counting statistics [45] to obtain expressions of heat currents. In Sec. IV, we study the influence of quantum interference on the thermal rectification within the two-reservoir setup. In Sec. V, we investigate the heat amplification in the three-reservoir nonequilibrium V-type system. Finally, we give a brief summary in Sec. VI.

II. MODEL AND METHOD

We describe a V-type system, which interacts with three thermal reservoirs. Then, we include the Redfield scheme to obtain the dynamical equation of the nonequilibrium V-type system in weak system-bath coupling regime. Finally, we analyze the effect of the cross-transition interference on the quantum steady state coherence.

A. Nonequilibrium V-type system

The model to exhibit nonequilibrium heat transfer through a V-type three-level system interacting with thermal baths in Fig. 1, is expressed as \( \hat{H} = \hat{H}_a + \hat{H}_b + \sum_{u=L,M,R} \hat{V}_u \). The three-level system is described as

\[ \hat{H}_a = \sum_{i=1,2} \epsilon_i |e_i\rangle\langle e_i| + \epsilon_g |g\rangle\langle g| , \]  

where \( \epsilon_1 \) and \( \epsilon_2 \) are energy levels of two excited states \( |e_1\rangle \) and \( |e_2\rangle \), and \( \epsilon_g \) is the energy of the common ground state \( |g\rangle \). In the following, we set \( \epsilon_1 \geq \epsilon_2 \) and \( \epsilon_g = 0 \) for simplicity without losing any generality. The Hamiltonian of three thermal baths is given by \( \hat{H}_b = \sum_{u=L,M,R} \hat{H}_b^u = \sum_{k,u} \omega_k \hat{a}_{k,u}^\dagger \hat{a}_{k,u} \), where \( \hat{a}_{k,u}^\dagger \) creates (annihilates) one phonon in the bath \( u \) with frequency \( \omega_k \). The interaction between V-type system and the bath \( L \ (R) \) is described as

\[ \hat{V}_u = \sum_{k,i} (g_{k,u}^i \hat{a}_{k,u}^\dagger |e_i\rangle + g_{k,u}^{i*} \hat{a}_{k,u} |e_i\rangle |g\rangle) , \ u = L, R , \]
where $g_{k,u}$ is the coupling strength to emit one phonon into the bath $u$ by relaxing the V-type system from $|e_1\rangle$ to $|g\rangle$, and $g_{k,u}^*$ is the coupling strength in the reverse process. It is easy to find that $\hat{V}_L$ and $\hat{V}_R$ can jointly participate in the transitions $|g\rangle \leftrightarrow |e_1\rangle$ and $|g\rangle \leftrightarrow |e_2\rangle$, which may result in cross-transition induced coherence [30]. While the interaction $\hat{V}_M$ is given by

$$\hat{V}_M = \sum_k (g_{k,M} \rho_{k,M}^* |e_2\rangle \langle e_1| + g_{k,M}^* \rho_{k,M} |e_1\rangle \langle e_2|),$$

(5)

where $g_{k,M}$ ($g_{k,M}^*$) is the hopping strength from $|e_1\rangle$ to $|e_2\rangle$ (from $|e_2\rangle$ to $|e_1\rangle$) by emitting (absorbing) one phonon into (from) the bath $M$.

### B. Redfield equation

We consider the interactions between the V-type system and thermal baths (i.e., $\hat{V}_u$ $(u \in L, M, R)$) are weak. Based on the Born-Markov approximation, the whole density operator can be approximated as $\hat{\rho}(t) \approx \hat{\rho}_s(t) \otimes (\Pi_u \hat{\rho}_u^n)$, where $\hat{\rho}_s(t)$ is the density operator of the V-type system, and $\hat{\rho}_u^n = \exp(-\hat{H}_u^n/(kB T_u))/Z_u$ is the canonical distribution operator of the bath $u$, with the partition function $Z_u = Tr_\nu(\exp(-\hat{H}_u^n/(kB T_u)))$. Then, we separately perturb $\hat{V}_u$ up to the second order, to obtain the quantum master equation as

$$\frac{d\hat{\rho}_s(t)}{dt} = -i[H_s, \hat{\rho}_s(t)]$$

$$- i\sum_{i,j,\sigma = \pm} \Gamma_{ij}^\sigma(\epsilon_j) (|\hat{\phi}_j^\sigma \hat{\rho}_s(t), \hat{\varphi}_j^\sigma \rangle + |\hat{\varphi}_j^\sigma \hat{\rho}_s(t), \hat{\varphi}_j^\sigma \rangle)$$

$$+ i\sum_{\sigma = \pm} \Gamma_M^\sigma(\Delta) (|\hat{\psi}_\sigma \hat{\rho}_s(t), \hat{\varphi}_\sigma \rangle + |\hat{\varphi}_\sigma \hat{\rho}_s(t), \hat{\psi}_\sigma \rangle),$$

(6)

where the transition operators are $\hat{\phi}_j^\sigma = |e_i\rangle \langle g|$, $\hat{\psi}_\sigma = |e_1\rangle \langle e_2|$ and their Hermitian conjugate operators $\hat{\varphi}_j^\sigma$ and $\hat{\varphi}_\sigma$, respectively. The energy bias between two excited states is $\Delta = \epsilon_1 - \epsilon_2$. The transition rates involved with the left and right baths are $\Gamma_{ij}^\sigma(\epsilon_j) = \sum_{u=L,R} \gamma_{uj}^\sigma(\epsilon_j) n_u(\epsilon_j)$ and $\Gamma_{ij}^\sigma(\epsilon_j) = \sum_{u=L,R} \gamma_{uj}^\sigma(\epsilon_j)(1 + n_u(\epsilon_j))$, with the spectral function $\gamma_{uj}^\sigma(\epsilon_j) = \gamma_{ji}^u(\epsilon_j) = 2\pi \sum_k g_{k,u}^i g_{k,u}^j \delta(\epsilon_j - \omega_k)$ and the Bose-Einstein distribution function $n_u(\epsilon_j) = 1/\{\exp(\epsilon_j/(kB T_u)) - 1\}$. Here, we specify the cross-transition rate as $\gamma_{12}^\sigma(\epsilon_j) = \sum_{u=L,R} \gamma_{uj}^\sigma(\epsilon_j)$. The transition rate involved with the middle bath is $\Gamma_M^\sigma(\Delta) = \gamma_M^\sigma(\Delta) n_M(\Delta)$ and $\Gamma_M^\sigma(\Delta) = \gamma_M^\sigma(\Delta)(1 + n_M(\Delta))$, with $\gamma_M^\sigma(\Delta) = 2\pi \sum_k |g_{k,M}|^2 \delta(\Delta - \omega_k)$. In this paper, $\gamma_{uj}^\sigma(\epsilon_i) = \gamma_{ji}^u(\epsilon_i)$ and $\gamma_{12}^\sigma(\epsilon_i)$ are set constant for simplicity. The extension of these spectral functions to frequency dependent is straightforward (e.g., $\gamma_{uj}^\sigma(\omega) = \gamma_{uj}^\sigma(\epsilon) \exp(-\omega/kB T_u)$), and will not qualitatively change the results.

Specifically, $\Gamma_{ii}^\sigma(\epsilon_i)$ describes the particle transition rate from the ground state probability $\rho_{gg}$ to the excited state $\rho_{ii}$ (from $\rho_{ii}$ to $\rho_{gg}$) by absorbing (emitting) one phonon from (into) the left/right thermal bath. While the rate $\Gamma_{12}^\sigma(\epsilon_i)$ ($\Gamma_{12}^\sigma(\epsilon_i)$) shows the cross-transition rate from the ground state probability $\rho_{gg}$ to the coherence term $\rho_{12}$ (from $\rho_{12}$ to $\rho_{gg}$). Notice that the cross-transition interference $\Gamma_{12}^\sigma(\epsilon_i)$ is irrelevant with the reservoir $M$, see Eq. (5). Moreover, Eq. (6) indicates that the coherence term $\rho_{12}$ is coupled with occupation probabilities $\rho_{ii}$ ($i = e_1, e_2, g$) as $\Gamma_{12}^\sigma(\epsilon_i) \neq 0$. Hence, the nonequilibrium quantum coherence may not only occur in the transient dynamics, but also persist in the steady state, which is termed as nonequilibrium steady state coherence [30, 31]. For the transition rates $\Gamma_{ii}^\sigma(\Delta)$, it describes the probability transition between two excited states. By including a third thermal bath, the molecular solar cell [34] and quantum transistor [37] have been extensively investigated within the three-terminal setup. Particularly for quantum thermoelectric transistor within a double quantum dots device, the heat amplification was observed in the linear response regime [18]. In this paper, we study the quantum thermal transistor in the V-type system far-from equilibrium.

### C. Nonequilibrium steady state coherence

The nonequilibrium steady state has been revealed as a source to enhance the power and quantum efficiency in the energy harvesting systems, where quantum coherence is unraveled to be crucial [33–35]. Generally, the quantum coherence can be defined by the off-diagonal elements of the density matrix (i.e. $\rho_{12}$). Here, following the same definition, we analyze the quantum coherence under the temperature bias at steady state. From Eq. (A1), it is easy to see that the diagonal elements $\rho_{ii}$ ($i = e_1, e_2, g$) are dynamically coupled with the off-diagonal term $\rho_{12}$. Hence, the quantum coherence may even appear after long time evolution.
at Fig. 2: (Color online) Nonequilibrium steady state quantum coherence $|\rho_{12}|$ within the two-reservoir setup ($\gamma_M = 0$) (a) by tuning cross-transition coefficients $f_L$ and $f_R$ with $T_L = 2$ and $T_R = 1$, and (b) by tuning the left and right temperatures $T_L$ and $T_R$ with $\Delta T = T_L - T_R$, $f_L = 1$ and $f_R = 0$. The other system parameters are given by $\gamma_{11}^{L(R)} = \gamma_{22}^{L(R)} = 0.01$ and $\varepsilon_L = \varepsilon_R = 1$.

The analytical expression of the steady state quantum coherence is obtained at certain specific case. It is generally quite difficult to obtain the analytical expression of the quantum coherence. However, at resonance ($\varepsilon_1 = \varepsilon_2 = \varepsilon$) and without the middle reservoir ($\gamma_M = 0$), the steady state coherence is given by (see Eq. A4)

$$\rho_{12}^{ss} = \frac{\Gamma_{11}^- (\varepsilon) \Gamma_{22}^- (\varepsilon) \Gamma_{12}^+ (\varepsilon)}{\Gamma_{11}^- (\varepsilon) + \Gamma_{22}^- (\varepsilon)} \left[ \frac{2 \Gamma_{12}^+ (\varepsilon)}{\Gamma_{12}^- (\varepsilon)} - \Gamma_{11}^- (\varepsilon) - \Gamma_{22}^- (\varepsilon) \right],$$

(7)

where $A = \Gamma_{11}^- (\varepsilon)[\Gamma_{22}^- (\varepsilon) + \Gamma_{22}^+ (\varepsilon)] + \Gamma_{11}^+ (\varepsilon)\Gamma_{22}^- (\varepsilon) - \Gamma_{12}^- (\varepsilon)\Gamma_{12}^- (\varepsilon) + 2 \Gamma_{12}^+ (\varepsilon)$. It needs to point out that the steady state coherence here is completely induced by the cross-transition interference, and is irrelevant with the concept of decoherence free subspace [46–48].

As known from Eq. (6), the cross-transition interference, quantified by $\Gamma_{12}^+ (\varepsilon)$, is irrelevant with the direct hopping assisted by the middle reservoir. Hence, we include the two-reservoir setup to study the quantum coherence by setting $\gamma_M = 0$, shown at Fig. 2(a). As $f_L = f_R$, the steady state coherence shows globally minimal ($\rho_{12}^{ss} = 0$). This is consistent with the vanishing condition of the quantum coherence at Eq. (A3). While as $f_L \neq f_R$, quantum coherence shows monotonic enhancement by increasing the bias of cross-transition coefficient $|f_L - f_R|$, and exhibits maximum at $f_L = 1$ and $f_R = 0$. Though not shown here, quantum coherence still sustains under the off-resonance case $\varepsilon_1 \neq \varepsilon_2$. The temperature dependence of the quantum coherence with large cross-transition coefficient bias ($f_L = 1$ and $f_R = 0$) is plotted in Fig. 2(b). It is found that in the moderate temperature regime (e.g., $T_R = 0.5$), quantum coherence is dramatically enhanced by increasing the temperature bias ($T_L - T_R$). While in the low and high temperature regimes, $\rho_{12}^{ss}$ becomes small but still nonzero. Hence, there exists an optimal temperature regime to generate the comparatively large steady state quantum coherence.

III. HEAT CURRENT FLUCTUATIONS

To count the energy flow of thermal baths, we introduce a counting field set $\{\chi\} = \{\chi_L, \chi_R\}$ to the Hamiltonian as [45, 51] $H_{\chi} = e^{i \sum_{u=L,R} \chi_u H_{u}/2} H e^{-i \sum_{u=L,R} \chi_u H_{u}/2} = H_{s} + H_{b} + \hat{V}_{M} + \sum_{u=L,R} \hat{V}_{u}(\chi_u)$, where the modified system-bath interaction is expressed as

$$\hat{V}_{u}(\chi_u) = \sum_{k,i} (g_{k,i}^{\dagger} e^{i \omega_{k} \chi_u} \hat{a}_{k,u}^{\dagger} |g\rangle \langle e_i | + g_{k,i}^{\dagger} e^{-i \omega_{k} \chi_u} \hat{a}_{k,u}^{\dagger} |e_i \rangle \langle g|),$$

(8)
Based on the Born-Markov approximation, we perturb the interaction Eq. (8) up to the second order, and obtain the modified quantum master equation (see details at appendix B)

\[
\frac{d\hat{\rho}(\chi)}{dt} = -i[\hat{H}_s, \hat{\rho}(\chi)] + \frac{1}{2} \sum_{i,j;\sigma=\pm} \Gamma^\sigma_{ij}(\varepsilon_j) (\hat{\phi}^\sigma_i \hat{\phi}^\sigma_j + \hat{\phi}^\sigma_j \hat{\phi}^\sigma_i) \hat{\rho}(\chi) + \frac{1}{2} \sum_{i,j;\sigma=\pm} (\Gamma^\sigma_{ij}(\varepsilon_j, \{\chi\}) + \Gamma^\sigma_{ji}(\varepsilon_j, \{\chi\})) \hat{\phi}^\sigma_i \hat{\rho}(\chi) \hat{\phi}^\sigma_j + \frac{1}{2} \sum_{\sigma=\pm} \Gamma^\sigma_{ss}(\Delta) (\hat{\psi}^\sigma \hat{\rho}(\chi), \hat{\psi}^\sigma) + [\hat{\psi}^\sigma, \hat{\rho}(\chi) \hat{\psi}^\sigma]),
\]

where the modified transition rates are \(\Gamma^+_{ij}(\omega, \{\chi\}) = \sum_v \gamma^v_{ij} n_v(\omega)e^{-i\omega\chi^v}\) and \(\Gamma^-_{ij}(\omega, \{\chi\}) = \sum_v \gamma^v_{ij}(1 + n_v(\omega))e^{i\omega\chi^v}\). In absence of the counting fields (\(\chi_L = \chi_R = 0\)), this modified quantum master equation returns back to Eq. (6).

From the definition at Eq. (B8), the steady state heat current into the left bath is given by

\[
J^L_R = \sum_{j=1,2} \varepsilon_j \gamma^L_{jj} [(1 + n_L(\varepsilon_j))\rho^L_{jj} - n_L(\varepsilon_j)\rho^R_{gg}]
\]

\[+ \frac{1}{2} \sum_{j=1,2} \varepsilon_j \gamma^L_{12}(1 + n_L(\varepsilon_j))(\rho^R_{12} + \rho^R_{21}).\]

The first term on the right side shows the population dynamics between the excited state population (\(\rho^L_{gg}\)) and the ground state population (\(\rho^R_{gg}\)). The second term denotes the contribution of the noise-induced interference to the steady state heat transfer, which is quantified by \(\gamma^L_{12}\). Similarly, the heat current into the right bath is

\[
J^R_R = \sum_{j=1,2} \varepsilon_j \gamma^R_{jj} [(1 + n_R(\varepsilon_j))\rho^R_{jj} - n_R(\varepsilon_j)\rho^L_{gg}]
\]

\[+ \frac{1}{2} \sum_{j=1,2} \varepsilon_j \gamma^R_{12}(1 + n_R(\varepsilon_j))(\rho^L_{12} + \rho^L_{21}).\]

and the heat current into the middle thermal bath is

\[
J^R_M = \Delta \gamma_M [(1 + n_M(\Delta))\rho^L_{11} - n_M(\Delta)\rho^L_{22}].
\]

They fulfill the energy conservation law as \(J^L_R + J^R_R + J^R_M = 0\).

Then, we investigate the effect of the cross-transition interference on heat currents cumulants (e.g., \(J^L_R\) and \(S^R_{RR}\), see Eq.(B5)) with \(\gamma_M = 0\), shown at Fig. 3. For the heat current into the right bath at Fig. 3(a), it is found that \(J^R_R\) is dramatically suppressed by the increase of the bias of cross-transition coefficients \(|f_L - f_R|\), and becomes minimum at the limiting regimes (i.e., \(f_L = 1, f_R = 0\) and \(f_L = 0, f_R = 1\)), mainly due to the negative behavior of the coherence term \(\rho^R_{12} < 0\). Moreover, we analyze the noise power \(S^R_{RR}\) in Fig. 3(b). The noise power characterizes the correlations of currents, which originate from stochastic processes in the nonequilibrium transfer [45, 49, 50]. It is shown that for the condition \(f_L = f_R\), the noise power shows monotonic enhancement by increasing \(f_L\), and becomes maximum at \(f_L = f_R = 1\). Whereas the noise power is strongly suppressed at large bias \(<|f_L - f_R| = 1\). Hence, the large bias of cross-transition coefficients \(|f_L - f_R|\) deteriorates the heat current and the noise power.

**IV. CROSS-TRANSITION INDUCED THERMAL RECTIFICATION**

Quantum thermal rectification effect has been extensively investigated in phononics [5, 8, 14, 17]. Several typical definitions of the thermal rectification have been proposed, e.g., the rectifier ratio \(|J^R_R(\Delta T)/J^L_R(-\Delta T)|\) [14, 23] with \(\Delta T = T_L - T_R\), and the rectification efficiency \((J^R_R(\Delta T) + J^R_R(-\Delta T))/\max\{J^R_R(\Delta T), -J^R_R(-\Delta T)\}\) [13, 17]. These definitions capture the asymmetric behavior of the heat flux by interchanging the temperatures of two baths. In this paper, we select

\[
R_J = \frac{|J^R_R(\Delta T) + J^R_R(-\Delta T)|}{\max\{J^R_R(\Delta T), -J^R_R(-\Delta T)\}}
\]

(13) to quantify the thermal rectification with \(J^R_R\) at Eq. (11), i.e., the rectification occurs as \(R_J > 0\).
FIG. 3: (Color online) (a) Steady state heat flux and (b) noise power into the right thermal bath, by modulating the cross-transition coefficients $f_L$ and $f_R$. The other system parameters are given by $T_L = 2$, $T_R = 1$, $\gamma_1^{(L)} = \gamma_2^{(R)} = 0.01$, $\gamma_M = 0$, and $\varepsilon_L = \varepsilon_R = 1$.

FIG. 4: (Color online) Thermal rectification of heat flux $R_J$ and noise power $R_N$ into the right bath (a), (c) by tuning the temperature bias $\Delta T = T_L - T_R$ with $T_L = T_0 + \Delta T/2$, $T_R = T_0 - \Delta T/2$, and $T_0 = 1$; (b), (d) by tuning the cross-transition coefficients $f_L$ and $f_R$ with $T_L = 1.5$ and $T_R = 0.5$. The other system parameters are given by $\gamma_1^{(L)} = \gamma_2^{(R)} = 0.01$ and $\varepsilon_L = \varepsilon_R = 1$.

We study the influence of the cross-transition interference on steady state thermal rectification at Fig. 4(a), within the two-terminal setup ($\gamma_M = 0$). By setting $f_R = 0$ and tuning on the cross-transition coefficient of the left bath (e.g., $f_L = 0.2$), the thermal rectification ratio $R_J$ is found to exhibit monotonic enhancement with the increase of the temperature bias $\Delta T$, and becomes saturate in the large bias regime. Moreover, it is shown that the increase of coefficients bias $|f_L - f_R|$ monotonically enhances the thermal rectification, which is also shown at Fig. 4(b). However, as $f_L = f_R$, the heat current becomes $J_R = 2\varepsilon[n_L(\varepsilon) - n_R(\varepsilon)]/[2 + 3n_L(\varepsilon) + 3n_R(\varepsilon)]$, with no thermal rectification. Hence, we conclude that the bias of cross-transition coefficients can result in the thermal rectification.

The influence of the cross-transition interference on the noise power is defined by

$$R_N = \frac{|S_R(\Delta T) - S_R(-\Delta T)|}{\max\{S_R(\Delta T), S_R(-\Delta T)\}},$$

where $R_N$ denotes rectification of noise power. As clearly shown in Fig. 4(b), with finite bias of $|f_L - f_R|$, $R_N$ is monotonically
enhanced by the temperature bias $\Delta T$, and $R_N$ is enhanced with the increase of coefficients bias $|f_L - f_R|$. Though not shown here, higher cumulants (e.g., skewness) are also observed to be rectified as $f_L \neq f_R$. Hence, we conclude that the cross-transition coefficients bias rectifies full counting statistics of steady state heat current fluctuations.

In previous works of the quantum thermal rectification, the sufficient condition for the appearance of heat rectification has been analyzed in two-reservoir spin-boson model and boson-boson model [23]. Consequently, such condition has also been analyzed in the Z-type three level-system [24], in which there is no noise-induced interference. Asymmetric structures of the quantum system and system-bath interaction both are found to contribute to the quantum rectification [23]. However, the influence of the cross-transition interference on the rectification effect is lack of exploitation in three-level system. In this work, we clearly indicate that in V-type system the cross-transition interference is able to exhibit the quantum rectifications of heat current fluctuations, as shown in Fig. 4. It should be noted that for the nonequilibrium $\Lambda$-type system, though not shown here, the feature of cross-transition interference induced thermal rectification can also be observed, which is mainly due to the similar system-bath interaction compared to the V-type system [30].

Moreover, the previous sufficient condition in Ref. [23] can be recovered based on the expression of heat flux at Eq. (11), in absence of the cross-transition interference (i.e. $f_L = f_R = 0$). Specifically, we obtain the corresponding expression of heat current into the right bath as

$$J_R = \frac{\gamma_{11}^L}{A'} \Gamma_{22}^+ (\varepsilon_2) [n_L (\varepsilon_1) - n_R (\varepsilon_1)] \varepsilon_1$$

$$+ \frac{\gamma_{22}^L}{A'} \gamma_{11}^R \Gamma_{11}^- (\varepsilon_2) [n_L (\varepsilon_2) - n_R (\varepsilon_2)] \varepsilon_2$$

with the coefficient $A' = \Gamma_{11}^- (\varepsilon_1) (\Gamma_{22}^+ (\varepsilon_2) + \Gamma_{22}^- (\varepsilon_2)) + \Gamma_{11}^+ (\varepsilon_1) \Gamma_{22}^- (\varepsilon_2)$. Then by setting $\varepsilon_2 = 0$, it results in $\Gamma_{22}^+ (\varepsilon_2) = \Gamma_{22}^- (\varepsilon_2)$. The current is reduced to $J_R = \gamma_{11}^L \gamma_{11}^R [n_L (\varepsilon_1) - n_R (\varepsilon_1)] \varepsilon_1 / [\gamma_{11}^L (2 + 3n_L (\varepsilon_1)) + \gamma_{11}^R (2 + 3n_R (\varepsilon_1))]$. Finally, the condition is recovered as [23]

$$n_L (\varepsilon_1) / \gamma_{11}^L - n_L (\varepsilon_1) / \gamma_{11}^R = n_R (\varepsilon_1) / \gamma_{11}^R - n_R (\varepsilon_1) / \gamma_{11}^L.$$  \hspace{1cm} (15)

V. HEAT AMPLIFICATION

Heat amplification is the key component to realize quantum thermal transistors within three-terminal setups. The amplification factor is defined by the ratio of the change of the current $J_M^u (J_M^s)$ on the change of the middle bath current $J_M^s$

$$\beta_u^s = |\partial J_M^u / \partial J_M^s|, \: u = L, R.$$  \hspace{1cm} (16)
According to the energy conservation relationship \( \sum_{u=L,M,R} J_u^c = 0 \), the amplification factor \( \beta_R^c \) can be re-expressed as

\[
\beta_R^c = |\beta_R^c + (-1)^\theta|,
\]

with \( \theta = 0 \) for \( \partial J_L^c/\partial J_M^c > 0 \) and \( \theta = 1 \) for \( \partial J_L^c/\partial J_M^c < 0 \). Traditionally, the amplification effect occurs as \( \beta_{L(R)}^c > 1 \).

Setting \( \gamma_{22}^L = \gamma_{11}^R = 0 \) at the nonequilibrium V-type system in Fig. 5(a), it naturally results in \( \gamma_{12}^L = 0 \). It is interesting to find that the heat amplification effect may occur. Specifically, the steady state populations are obtained in Eq. (A6), and the particle current Eq. (C3) is rewritten as

\[
J_R^p = \frac{\gamma_{22}^L[(1 + n_R(\varepsilon_2))\rho_{22}^s - n_R(\varepsilon_2)\rho_{99}^s]}{\Delta(\partial J_M^c/\partial T_M)},
\]

(18)

\[
J_L^p = -J_R^p \quad \text{and} \quad J_M^p = J_R^c \quad \text{which is described by the closed red loop at Fig. 5(a). Hence, the heat fluxes into three reservoirs are given by} J_R^p = \varepsilon_2 J_R^c, J_L^p = -\varepsilon_1 J_R^c, \quad \text{and} \quad J_M^p = \Delta J_R^c, \quad \text{respectively. From the definition of the heat amplification factor at Eq. (16), it is specified as}
\]

\[
\beta_R^c = \frac{\varepsilon_2(\partial J_R^p/\partial T_M)}{\Delta(\partial J_M^c/\partial T_M)} = \frac{\varepsilon_2}{|\varepsilon_1 - \varepsilon_2|}.
\]

(19)

Hence, the heat amplification effect can be apparently observed once \( |\varepsilon_2/(\varepsilon_1 - \varepsilon_2)| > 1 \), and the factor becomes divergent as \( \varepsilon_2 \rightarrow \varepsilon_1 \).

In previous works of quantum thermal transistor, negative differential thermal conductance was believed to be a compulsory ingredient to realize the heat amplification [5, 15, 37–41]. Within the three-terminal setup, the NDTC generally occurs as the temperature bias \( |T_v - T_M| \) (\( v = L, R \)) increases, the heat current shows turnover behavior [5, 41]. Recently, within the linear response regime, J. H. Jiang et al. found the heat amplification in gate-tunable double quantum dots without negative differential thermal conductance [18]. Here, our result clearly shows that the heat amplification effect can also be realized far-from equilibrium (finite temperature bias), in absence of the negative differential thermal conductance. Moreover, the heat current shows monotonic decrease with the increase of the middle bath temperature \( T_M \), whereas the noise power \( S_{\beta_R}^{(2)}, \gamma_{99}^s \) exhibits enhancement (see Fig. 5(b)). Hence, it is proper to observe the heat amplification effect in comparatively low temperature regime (e.g., \( T_M \approx 0.5 \)), with high signal to noise ratio.

Next, we analyze the influence of the two-terminal transport on the heat amplification in Fig. 6(a), by tuning on \( \gamma_{22}^L = \gamma_{11}^R = \gamma \), which includes direction transitions between \( \rho_{11}^{ss} \) and \( \rho_{99}^{ss} \). Specifically, we apply the maximum of the amplification factor by modulating \( T_M \) as

\[
\beta_{R,\text{max}} = \max_{\{T_M\}} \{\beta_R^c\} = \frac{\varepsilon_2}{|\varepsilon_1 - \varepsilon_2|} \times \max_{\{T_M\}} \left\{ \frac{(\partial J_R^c/\partial T_M)}{(\partial J_M^c/\partial T_M)} \right\}.
\]

(20)
It is found that the heat amplification factor $\beta^R_1$ decreases gradually by increasing $\gamma$, and finally drops below one (e.g., $\gamma = 0.006$). To see this clearly, we study the behavior of heat currents at $\gamma = 0.01$ by tuning $T_M$ at Fig. 6(b). The change of $J^R_1$ is much smaller than the change of $J^L_1$, which results in $\beta^R_{1,max} < 1$. Similarly, from the relation of the amplification factors at Eq. (17), it is known that $\beta^L_{1,max} \approx 1$. Hence, we conclude that two-terminal transport process is detrimental to the generation of the heat amplification.

VI. CONCLUSION

We investigate the quantum heat transfer in a nonequilibrium V-type system with weak system-bath interactions by applying the Redfield master equation. The nonequilibrium quantum coherence is investigated at steady state. The analytical expression of nonequilibrium steady state coherence at resonance ($\varepsilon_1 = \varepsilon_2$) is obtained, and it is found that cross-transition interference may enhance the quantum coherence. The finite bias of cross-transition coefficients $|f_L - f_R|$ is found to enhance the steady state quantum coherence. It is also found that the cross-transition processes may rectify full counting statistics of heat currents, which provides a new scheme of quantum thermal rectification. Moreover, in absence of the negative differential thermal conductance, a giant amplification factor is analytically obtained far-from equilibrium, which only relies on the energy levels of the excited states. Hence, this provides a smart way to control the amplification effect by modulating the V-type system structure.

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Appendix A: Steady state populations

The dynamical equation of V-type system is given by

$$\frac{\partial \rho_{11}}{\partial t} = -(\Gamma^{+}_{11}(\varepsilon_1) + \Gamma^{-}_{11}(\Delta))\rho_{11} + \Gamma^{+}_{11}(\varepsilon_1)\rho_{22} + \Gamma^{-}_{11}(\Delta)\rho_{gg} - \frac{1}{2}\Gamma^{+}_{12}(\varepsilon_2)(\rho_{12} + \rho_{21}),$$

$$\frac{\partial \rho_{22}}{\partial t} = \Gamma^{-}_{M}(\Delta)\rho_{11} - (\Gamma^{+}_{22}(\varepsilon_2) + \Gamma^{-}_{M}(\Delta))\rho_{22} + \Gamma^{+}_{22}(\varepsilon_2)\rho_{gg} - \frac{1}{2}\Gamma^{+}_{12}(\varepsilon_1)(\rho_{12} + \rho_{21}),$$

$$\frac{\partial \rho_{12}}{\partial t} = \frac{i}{\hbar}\Delta\rho_{12} - \frac{1}{2}(\Gamma^{+}_{11}(\varepsilon_1) + \Gamma^{-}_{22}(\varepsilon_2))\rho_{12} - \frac{1}{2}(\Gamma^{+}_{12}(\varepsilon_1)\rho_{11} + \Gamma^{-}_{12}(\varepsilon_2)\rho_{22})$$

$$+ \frac{1}{2}(\Gamma^{+}_{12}(\varepsilon_1) + \Gamma^{-}_{12}(\varepsilon_2)\rho_{gg} - \frac{1}{2}(\Gamma^{+}_{12}(\Delta) + \Gamma^{-}_{12}(\Delta))\rho_{12},$$

$$\frac{\partial \rho_{21}}{\partial t} = \frac{i}{\hbar}\Delta\rho_{21} - \frac{1}{2}(\Gamma^{+}_{11}(\varepsilon_1) + \Gamma^{-}_{22}(\varepsilon_2))\rho_{21} - \frac{1}{2}(\Gamma^{+}_{12}(\varepsilon_1)\rho_{11} + \Gamma^{-}_{12}(\varepsilon_2)\rho_{22})$$

$$+ \frac{1}{2}(\Gamma^{+}_{12}(\varepsilon_1) + \Gamma^{-}_{12}(\varepsilon_2)\rho_{gg} - \frac{1}{2}(\Gamma^{+}_{12}(\Delta) + \Gamma^{-}_{12}(\Delta))\rho_{21}.$$
the efficiency bound in quantum heat engine. As \( \Gamma_{ij}^\pm(\Delta) = 0 \) \( (\gamma_M = 0) \), the condition is reduced to the two-reservoir case

\[
\Gamma_{12}(\varepsilon_1) \left[ \frac{\Gamma_{11}^-(\varepsilon_1)}{\Gamma_{11}^+(\varepsilon_1)} - \frac{\Gamma_{12}^-(\varepsilon_1)}{\Gamma_{12}^+(\varepsilon_1)} \right] + \Gamma_{12}^-(\varepsilon_2) \left[ \frac{\Gamma_{22}^-(\varepsilon_2)}{\Gamma_{22}^+(\varepsilon_2)} - \frac{\Gamma_{12}^-(\varepsilon_2)}{\Gamma_{12}^+(\varepsilon_2)} \right] = 0 ,
\]

(A3)

which recovers the previous result in Ref. [30]. It is known that the general solution at steady state is quite difficult. Hence, we try to obtain analytical results in limiting regimes. First, we obtain the steady state populations at resonance \( (\varepsilon_1 = \varepsilon_2 = \varepsilon) \) within the two-terminal setup \( (\gamma_M = 0) \), shown as

\[
\rho_{11}^{ss} = \frac{1}{(\Gamma_{11}^- + \Gamma_{12}^-)A} \left[ (\Gamma_{1}^-_{11} + \Gamma_{12}^-)\Gamma_{11}^+\Gamma_{12}^- + (\Gamma_{12}^+ - \Gamma_{11}^-)(\Gamma_{12}^-)^2 - 2\Gamma_{12}^-\Gamma_{12}^+\Gamma_{12}^- \right],
\]

(A4)

\[
\rho_{22}^{ss} = \frac{1}{(\Gamma_{11}^- + \Gamma_{12}^-)A} \left[ (\Gamma_{1}^-_{11} + \Gamma_{12}^-)\Gamma_{22}^+\Gamma_{12}^- + (\Gamma_{12}^+ - \Gamma_{11}^-)(\Gamma_{12}^-)^2 - 2\Gamma_{12}^-\Gamma_{12}^+\Gamma_{12}^- \right],
\]

\[
\rho_{gg}^{ss} = \frac{[\Gamma_{1}^-_{11}\Gamma_{12}^- - (\Gamma_{12}^-)^2]}{A},
\]

\[
\rho_{12}^{ss} = \frac{\Gamma_{1}^-_{12}\Gamma_{12}^+\Gamma_{12}^-}{(\Gamma_{11}^- + \Gamma_{12}^-)A} (2\Gamma_{12}^- - \Gamma_{11}^- - \Gamma_{12}^+)/A,
\]

with \( \Gamma_{ij}^\pm = \Gamma_{ij}^\pm(\varepsilon) \) and \( A = \Gamma_{11}^+(\Gamma_{22}^- + \Gamma_{22}^+)^2 + \Gamma_{12}^-\Gamma_{12}^- + \Gamma_{12}^-\Gamma_{12}^+ + 2\Gamma_{12}^-\Gamma_{12}^- \). As the cross-transition induced coherence disappears \( \gamma_{i2}^\pm(\varepsilon_i) = 0 \) \( (i = 1, 2) \), the steady state solution is reduced to

\[
\rho_{11}^{ss} = \frac{\Gamma_{1}^-_{11}(\varepsilon_1)\Gamma_{12}^+(\varepsilon_2)}{(\Gamma_{11}^- + \Gamma_{12}^-)(\Gamma_{11}^+(\varepsilon_2) + \Gamma_{22}^+(\varepsilon_2)) + \Gamma_{11}^-\Gamma_{12}^+(\varepsilon_2)},
\]

(A5)

\[
\rho_{22}^{ss} = \frac{\Gamma_{1}^-_{11}(\varepsilon_1)\Gamma_{12}^+(\varepsilon_2)}{(\Gamma_{11}^- + \Gamma_{12}^-)(\Gamma_{22}^- + \Gamma_{22}^+) + \Gamma_{11}^-\Gamma_{12}^+(\varepsilon_2)}.
\]

Next, within the three-terminal setup and in absence of the cross-transition induced coherence \( (\gamma_{12}^\pm(\varepsilon_i) = 0) \), we obtain the steady state populations as

\[
\rho_{u1}^{ss} = \frac{[(\Gamma_{22}^-\varepsilon_2 + \Gamma_{12}^+M(\Delta))\Gamma_{11}^+(\varepsilon_1) + \Gamma_{12}^+(\Delta)\Gamma_{12}^+(\varepsilon_2)]}{B},
\]

(A6)

\[
\rho_{u2}^{ss} = \frac{[(\Gamma_{11}^-\varepsilon_1 + \Gamma_{12}^+M(\Delta))\Gamma_{22}^+(\varepsilon_2) + \Gamma_{12}^+(\Delta)\Gamma_{11}^+(\varepsilon_1)]}{B},
\]

\[
\rho_{gg}^{ss} = \frac{[(\Gamma_{22}^-\varepsilon_2 + \Gamma_{12}^+M(\Delta))\Gamma_{11}^-\varepsilon_1 + \Gamma_{22}^-(\Delta)\Gamma_{22}^-(\varepsilon_2)]}{B},
\]

with the coefficient \( B = (\Gamma_{22}^-\varepsilon_2 + \Gamma_{12}^+M(\Delta))(\Gamma_{11}^-(\varepsilon_1) + \Gamma_{11}^+(\varepsilon_1) + \Gamma_{12}^-\varepsilon_2 - (\Gamma_{12}^+M(\Delta) - \Gamma_{22}^+(\varepsilon_2))(\Gamma_{12}^+(\Delta) - \Gamma_{11}^-\varepsilon_1)).
\]

Appendix B: Full counting statistics of the nonequilibrium V-type system

To count the energy flow into the bath \( u \) at the time \( \tau \), which starts from the 0, the transferred heat is expressed as \( \Delta q_u^\tau = \sum_\omega \omega_i \Delta n_{k,u}(\tau) \), with \( \omega_i \) the phonon frequency in momentum \( k \), \( \Delta n_{k,u} = n_{k,u}(\tau) - n_{k,u}(0) \) and \( n_{k,u}(t) \) the occupation phonon number in the bath \( u \) at time \( t \). Then, we introduce the two-time measurement to analyze the currents. Specifically, we include the measuring operator \( \hat{P}_{q_0^u} = |q_0^u\rangle\langle q_0^u| \) to detect the initial energy quantity of the Hamiltonian \( \hat{H}_u \) to be \( q_0^u = \sum_k \omega_k n_{k,u}(0) \). Similarly, at time \( \tau \), we again measure \( \hat{H}_u \) with the operator \( \hat{P}_{q_\tau^u} = |q_\tau^u\rangle\langle q_\tau^u| \), resulting in \( q_\tau^u = \sum_k \omega_k n_{k,u}(\tau) \). Hence, the joint probability for this two-time measurement is given by

\[
\text{Pr}(q_\tau^u, q_0^u) = \text{Tr} \{ \hat{P}_{q_\tau^u} e^{-i\hat{H}_u \tau} \hat{P}_{q_0^u} \rho_0 \hat{P}_{q_0^u} e^{i\hat{H}_u \tau} \hat{P}_{q_\tau^u} \},
\]

(B1)

where \( \rho_0 \) and \( \hat{H} \) are the initial density matrix and Hamiltonian of the whole system, respectively. By using the joint probability at Eq. (C3), we define the probability of the transferred energy quantity \( \Delta Q^\tau_u \) during a finite time interval \( \tau \) as

\[
\text{Pr}(\Delta Q^\tau_u) = \sum_{q_\tau^u, q_0^u} \delta[\Delta Q^\tau_u - (q_\tau^u - q_0^u)] \text{Pr}(q_\tau^u , q_0^u) ,
\]

(B2)
Then, the generating function of the current statistics can be obtained as \[ Z(\chi, t) = \int d\Delta Q^u e^{i\chi \cdot \Delta Q^u} \operatorname{Pr}(\Delta Q^u) e^{i\chi \cdot \Delta Q^u} = \operatorname{Tr}[e^{i\chi \cdot \hat{H}(0)} e^{-i\chi \cdot \hat{H}(t)} \hat{\rho}(0)], \] (B3)
where \( \chi \) is the counting field to count the flow into the bath \( u \), \( \hat{H}_u(t) = \hat{U}(t) \hat{H}_u \hat{U}^\dagger(t) \) and the evolution operator \( \hat{U}(t) = e^{-i\hat{H}t} \). Actually, the generating function can be alternatively expressed as \[ Z(\chi, t) = \operatorname{Tr}\{\hat{U}^\dagger(\chi) \hat{\rho}(0) \hat{U}(\chi, t)\} = \operatorname{Tr}\{\hat{\rho}(\chi, t)\}, \] (B4)
with \( \hat{U}_-(\chi, t) = e^{-i\chi \cdot \hat{H}_u/2} \hat{U} e^{i\chi \cdot \hat{H}_u/2} \) and \( \hat{U}_+(\chi, t) = e^{i\chi \cdot \hat{H}_u/2} \hat{U}^\dagger e^{-i\chi \cdot \hat{H}_u/2} \). Hence, the cumulant generating function at steady state is \( G(\chi) = \lim_{t \to \infty} \frac{i}{t} Z(\chi, t) \). The current fluctuations at steady state are obtained as \[ J_u^{(n)} = \frac{\partial^n}{\partial(i\chi u)} G(\chi)|_{\chi = 0} = 0. \] (B5)
Particularly, the steady state heat flux is the first cumulant \( J_u = \frac{\partial}{\partial(i\chi u)} G(\chi)|_{\chi = 0} \), and the noise power is the second cumulant \( S_u = \frac{\partial^2}{\partial(i\chi u)^2} G(\chi)|_{\chi = 0} \).

For the nonequilibrium V-type system at Eq. (3), we count the energy current into the bath \( u \), by adding a counting parameter set to \( \hat{H} \) as \( \hat{H}(\{\chi\}) = \hat{H}_s + \hat{H}_b + \hat{V}_M + \sum_{u=L,R} \hat{V}_u(\chi^u) \), where the modified system-bath interaction is given by Eq. (8). Based on the Born-Markov approximation, we obtain the quantum master equation at Eq. (9). Defining the vector expression of the density matrix \( |\mathcal{P}(\{\chi\})\rangle = |\rho_{11}^u, \rho_{22}^u, \rho_{99}^u, \rho_{12}^u, \rho_{21}^u\rangle^T \), the dynamical equation can be reexpressed in the Liouvillian framework as \[ \frac{d}{dt} |\mathcal{P}(\{\chi\})\rangle = \hat{L}(\{\chi\}) |\mathcal{P}(\{\chi\})\rangle. \] (B6)
At steady state, the generating cumulant function is simplified as \[ G(\{\chi\}) = E_0(\{\chi\}), \] (B7)
where \( E_0(\{\chi\}) \) is the eigenvalue of the superoperator \( \hat{L}(\{\chi\}) \) with the maximal real part. Hence, the heat current can be obtained as \[ J_u = \frac{\partial E_0(\{\chi\})}{\partial(i\chi u)} |_{\chi = 0} = \langle I | \frac{\partial \hat{L}(\{\chi\})}{\partial(i\chi u)} |_{\chi = 0} | \mathcal{P}_{ss} \rangle, \] (B8)
where \( |I| = |1, 1, 1, 0, 0\rangle \) is the unit vector and \( |\mathcal{P}_{ss}\rangle = [\rho_{11}^{ss}, \rho_{22}^{ss}, \rho_{99}^{ss}, \rho_{12}^{ss}, \rho_{21}^{ss}]^T \) is the steady state of V-type system. Finally, the heat currents are obtained at Eq. (10-12).

Appendix C: Steady state particle currents

By applying a similar scheme to count the particle flow, the transformed Hamiltonian is given by \( \hat{H}(\{\chi_p\}) = e^{i\sum_u \chi^u_k \hat{N}_u/2} \hat{H} e^{-i\sum_u \chi^u_k \hat{N}_u/2} = \hat{H}_s + \hat{H}_b + \hat{V}_M + \sum_{u=L,R} \hat{V}_u(\chi^u_p \rangle \), where \( \hat{N}_u = \sum_k \hat{a}_{k,u} \hat{a}^\dagger_{k,u} \) and the modified system-bath interaction is given by \[ \hat{V}_u(\chi^u_p) = \sum_{k,i} \langle \hat{g}^i_k e^{i\chi^u_k \hat{a}^\dagger_{k,u}} |g\rangle \langle e_i| + \langle \hat{g}^i_k e^{-i\chi^u_k \hat{a}_{k,u}} |e_i\rangle \langle g| \rangle. \] (C1)
Based on the second-order perturbation, the modified master equation is given by \[ \frac{d}{dt} |\chi_p\rangle = -i[\hat{H}_s, \hat{\rho}(\chi_p)] + \frac{1}{2} \sum_{i,j,\sigma=\pm} \Gamma^{ij}_\sigma(\epsilon_i) (\hat{\rho}_{ij}^\sigma + \hat{\rho}_{ji}^\sigma) \hat{\rho}(\chi_p) \hat{\sigma}^\sigma + \frac{1}{2} \sum_{i,j,\sigma=\pm} \Gamma^{ij}_\sigma(\epsilon_i, \{\chi_p\}) \hat{\rho}(\chi_p) \hat{\sigma}^\sigma \] (C2)
+ \frac{1}{2} \sum_{\chi^u, \sigma=\pm} \Gamma^{\sigma}_m(\Delta) (|\psi^\sigma\rangle \langle \rho(\chi_p)| \psi^\sigma\rangle + |\psi^\sigma\rangle \langle \rho(\chi_p)| \psi^\sigma\rangle) \]
where the modified transition rates are $\Gamma_j^\omega(\omega, \chi_p) = \sum_{\nu} \gamma_{ij}^\nu n_{\nu}(\omega) e^{-i\chi_p} \omega$ and $\Gamma_{ij}^p(\omega, \chi_p) = \sum_{\nu} \gamma_{ij}^\nu (1 + n_{\nu}(\omega)) e^{i\chi_p}$. Then, the particle currents can be obtained as

$$J_L^p = \sum_{j=1,2} \gamma_j^p [(1 + n_L(\chi_j)) \rho_{ss}^{j} - n_L(\chi_j) \rho_{s}^{ss}]
+ \frac{1}{2} \sum_{j=1,2} \gamma_j^P (1 + n_L(\chi_j)) (\rho_{12}^{s} + \rho_{21}^{s}),$$

$$J_R^p = \sum_{j=1,2} \gamma_j^p [(1 + n_R(\chi_j)) \rho_{ss}^{j} - n_R(\chi_j) \rho_{s}^{ss}]
+ \frac{1}{2} \sum_{j=1,2} \gamma_j^R (1 + n_R(\chi_j)) (\rho_{12}^{s} + \rho_{21}^{s}),$$

$$J_M^p = \gamma_M (1 + n_M(\Delta)) \rho_{11}^{ss} - \gamma_M n_M(\Delta) \rho_{22}^{ss}.$$

The particle conservation law results in $J_L^p + J_R^p = 0$, which can be obtained from Eq. (A1) by setting $\frac{d}{dt} \rho_{ij} = 0$.

**Appendix D: A systematical perturbation method to obtain steady state current cumulants**

The cumulant generating function is $G(\chi) = E_\chi$, where $E_\chi$ is the eigenvalue with maximal real part $\hat{H}_0 |P_\chi> = E_\chi |P_\chi>$. Then, we expand all terms as $\hat{H}_0 = \sum_{n=0}^{\infty} \frac{(i\chi)^n}{n!} \hat{H}_n$, $E_\chi = \sum_{n=0}^{\infty} \frac{(i\chi)^n}{n!} E_n$, and $|P_\chi> = \sum_{n=0}^{\infty} \frac{(i\chi)^n}{n!} |P_n>$. If we consider the order $(i\chi)^N$, we obtain

$$\sum_{n=0}^{N} \frac{\hat{H}_n}{n!(N-n)!} |P_{N-n}> = \sum_{k=0}^{N} \frac{E_k}{k!(N-k)!} |P_{N-k}>.$$  \hspace{1cm} (D1)

For the zeroth order, the steady state is obtained as $\hat{H}_0 |P_0> = 0$, and the left eigenvector is given by $<I|\hat{H}_0 = 0$. Consequently, the $N$th order cumulants is given by

$$E_N = \sum_{n=1}^{N} \frac{N!}{n!(N-n)!} <I|\hat{H}_n|P_{N-n}>
- \sum_{k=1}^{N-1} \frac{N!}{k!(N-k)!} E_k <I|P_{N-k}>,$$ \hspace{1cm} (D2)

and the corresponding state is

$$|P_N> = \hat{R} \sum_{n=1}^{N} \frac{N!}{n!(N-n)!} (E_n - \hat{H}_n)|P_{N-n}>,$$ \hspace{1cm} (D3)

with $\hat{R} = Q\hat{H}_0^{-1}Q$, $\hat{H}_0^{-1}$ the Moore-Penrose inverse, and $Q = 1 - |P_0> <I|$ to eliminate the singular value of $\hat{H}_0$. Specifically,

$$E_1 = <I|\hat{H}_1|P_0>,$$
$$|P_1> = \hat{R} (E_1 - \hat{H}_1)|P_0>,$$
$$E_2 = 2 <I|\hat{H}_1 - E_1)|P_1> + <I|\hat{H}_2|P_0>
= <I|\hat{H}_2|P_0> - 2 <I|\hat{H}_1 - E_1)|\hat{R}(\hat{H}_1 - E_1)|P_0>
|P_2> = 2 \hat{R} (E_2 - \hat{H}_1)|P_1> + \hat{R}(E_2 - H_2)|P_0>.$$

Hence, the steady state flux is given by $J = E_1$ and the noise is $S = E_2$.

[1] M. A. Ratner, Nat. Nanotechnol. **8**, 378 (2013).
