Metric spaces with complexity of the smallest infinite ordinal number

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Abstract. In this paper, we are concerned with the study on metric spaces with complexity of the smallest infinite ordinal number. We give equivalent formulations of the definition of metric spaces with complexity of the smallest infinite ordinal number and prove that the exact complexity of the finite product $\mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z} \wr \mathbb{Z} \times \cdots \times \mathbb{Z} \wr \mathbb{Z}$ of wreath product is $\omega$, where $\omega$ is the smallest infinite ordinal number. Consequently, we obtain that the complexity of $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ is $\omega + 1$.

Keywords Metric spaces, the exact complexity, the smallest infinite ordinal number, wreath product;

1 Introduction

Inspired by the property of finite asymptotic dimension of M.Gromov ([1]), a geometric concept of finite decomposition complexity was introduced by E.Guentner, R.Tessera and G.Yu. Roughly speaking, a metric space has finite decomposition complexity when there is an algorithm to decompose the space into nice pieces in certain asymptotic way. It turned out that many groups have finite decomposition complexity and these groups satisfy strong rigidity properties including the stable Borel conjecture ([2],[3]). In [3], E.Guentner, R.Tessera and G.Yu show that the class of groups with finite decomposition complexity includes all linear groups, subgroups of almost connected Lie groups, hyperbolic groups and elementary amenable groups and is closed under taking subgroups, extensions, free amalgamated products, HNN-extensions and inductive limits.

Finite decomposition complexity is a large scale property of a metric space. To make the property quantitative, a countable ordinal the complexity can be defined for a metric space with finite decomposition complexity. There is a sequence of subgroups of Thompson’s group F which is defined by induction as follows:

$$G_1 = \mathbb{Z} \wr \mathbb{Z}, \quad G_{n+1} = G_n \wr \mathbb{Z}.$$  

We are concerned with the study of the exact complexity of $G_n$ which is partially inspired by the question of the finite decomposition complexity of Thompson’s group F([4],[5],[6]). In fact, if the exact complexity of the sequence $\{G_n\}$ of subgroups of F is strictly increasing, then we can prove that Thompson’s group F does not have finite decomposition complexity. In [6], we proved that the complexity of $G_n$ is $\omega n$ and the exact complexity of $\mathbb{Z} \wr \mathbb{Z}$ is $\omega$, where $\omega$ is the smallest infinite ordinal number, but it is still unknown about the exact complexity of $G_n$ when $n > 1$. Here we prove that the exact complexity of the finite product $\mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z} \wr \mathbb{Z} \times \cdots \times \mathbb{Z} \wr \mathbb{Z}$ of wreath product is $\omega$. Consequently, we obtain that the complexity of $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ is $\omega + 1$.

There is no group of examples known which make a difference between the exact complexity of $\omega$ and the exact complexity of $\alpha$, where $\alpha$ is a countable ordinal greater than $\omega$. So the question arises naturally: Is there any metric space with the exact complexity greater than $\omega$? Here we give equivalent descriptions of the definition of metric spaces with complexity of $\omega$.

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2 Equivalent descriptions of metric spaces with complexity of \( \omega \)

We begin by recalling some elementary concepts from coarse geometry.
Let \((X,d)\) be a metric space. For \(U,V \subseteq X\), let
\[
diam U = \sup\{d(x,y) : x,y \in U\}
\]
and
\[
d(U,V) = \inf\{d(x,y) : x \in U,y \in V\}.
\]
A family \(\mathcal{U}\) of subsets of \(X\) is said to be uniformly bounded if \(\text{diam } \mathcal{U} \triangleq \sup\{\text{diam } U : U \in \mathcal{U}\}\) is finite. A family \(\mathcal{U}\) of subsets of \(X\) is said to be \(r\)-disjoint if
\[
d(U,V) \geq r \quad \text{for every } U \neq V \in \mathcal{U}.
\]

**Definition 2.1.** ([2]) A metric space \(X\) has finite asymptotic dimension if there is a \(n \in \mathbb{N}\), such that for every \(r > 0\), there exists a sequence of uniformly bounded families \(\{\mathcal{U}_i\}_{i=1}^n\) of subsets of \(X\) such that the union \(\bigcup_{i=1}^n \mathcal{U}_i\) covers \(X\) and each \(\mathcal{U}_i\) is \(r\)-disjoint.

Let \(\mathcal{X}\) and \(\mathcal{Y}\) be metric families. A map of families from \(\mathcal{X}\) to \(\mathcal{Y}\) is a collection of functions \(F = \{f\}\), each mapping some \(X \in \mathcal{X}\) to some \(Y \in \mathcal{Y}\) and such that every \(X \in \mathcal{X}\) is the domain of at least one \(f \in F\). We use the notation \(F : \mathcal{X} \to \mathcal{Y}\) and, when confusion could occur, write \(f : X_f \to Y_f\) to refer to an individual function in \(F\).

**Definition 2.2.** A map of families \(F : \mathcal{X} \to \mathcal{Y}\) is uniformly expansive if there exists a non-decreasing function \(\theta : [0, \infty) \to [0, \infty)\) such that for every \(f \in F\) and every \(x,y \in X_f\),
\[
d(f(x),f(y)) \leq \theta(d(x,y)).
\]

\(F : \mathcal{X} \to \mathcal{Y}\) is effectively proper if there exists a proper non-decreasing function \(\delta : [0, \infty) \to [0, \infty)\) such that for every \(f \in F\) and every \(x,y \in X_f\),
\[
d(f(x),f(y)) \geq \delta(d(x,y)).
\]
And \(F : \mathcal{X} \to \mathcal{Y}\) is a coarse embedding if it is both uniformly expansive and effectively proper.

Two maps \(f,g : X \to Y\) are close if \(\{d(f(x),g(x)) : x \in X\}\) is a bounded set. If \(f : X \to Y\) is a coarse embedding and there exists a coarse embedding \(g : Y \to X\) such that \(f \circ g\) and \(g \circ f\) are close to the identities on \(X\) and \(Y\) respectively, then \(f\) is called a coarse equivalence.

**Definition 2.3.** ([2, 3]) A metric family \(\mathcal{X}\) is \(r\)-decomposable over a metric family \(\mathcal{Y}\) if every \(X \in \mathcal{X}\) admits a decomposition
\[
X = X_0 \cup X_1, X_i = \bigsqcup_{r\text{-disjoint}} X_{ij},
\]
where each \(X_{ij} \in \mathcal{Y}\). It is denoted by \(\mathcal{X} \overset{r}{\rightarrow} \mathcal{Y}\).

**Remark 2.1.** To express the idea that \(X\) is the union of \(X_i\) and the collection of these subspaces \(\{X_i\}\) is \(r\)-disjoint, we write
\[
X = \bigsqcup_{r\text{-disjoint}} X_i.
\]

**Definition 2.4.** ([2, 3])

1. Let \(D_0\) be the collection of uniformly bounded families: \(D_0 = \{\mathcal{X} : \mathcal{X}\text{ is uniformly bounded }\}\).
2. Let \(\alpha\) be an ordinal greater than \(0\), let \(D_\alpha\) be the collection of metric families decomposable over \(\bigcup_{\beta < \alpha} D_\beta:\)
\[
D_\alpha = \{\mathcal{X} : \forall r > 0, \exists \beta < \alpha, \exists \mathcal{Y} \in D_\beta, \text{ such that } \mathcal{X} \overset{r}{\rightarrow} \mathcal{Y}\}.
\]
Definition 2.5. ([2], [3])

- A metric family $\mathcal{X}$ has finite decomposition complexity if there exists a countable ordinal $\alpha$ such that $\mathcal{X} \in \mathcal{D}_\alpha$.
- We say that the complexity of the metric family $\mathcal{X}$ is $\alpha$ if $\mathcal{X} \in \mathcal{D}_\alpha$.
- We say that the exact complexity of $\mathcal{X}$ is $\alpha$ if $\mathcal{X} \in \mathcal{D}_\alpha$ and $\forall \beta < \alpha$, $\mathcal{X} \not\in \mathcal{D}_\beta$.

Remark 2.2.

- Note that for any $\beta < \alpha$, $\mathcal{D}_\beta \subseteq \mathcal{D}_\alpha$.
- We view a single metric space $X$ as a metric family with a single element.

It is known that a metric space $X$ has finite asymptotic dimension if and only if $X \in \mathcal{D}_n$ for some $n \in \mathbb{N}$ ([3], [6]).

Definition 2.6. We say that a metric family $\mathcal{X}$ has uniformly finite asymptotic dimension if $\mathcal{X} \in \mathcal{D}_n$ for some $n \in \mathbb{N}$.

Lemma 2.1. ([2], [3]) (Coarse invariance) Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric families and there is a coarse embedding $\phi: \mathcal{X} \to \mathcal{Y}$. If $\mathcal{Y} \in \mathcal{D}_\alpha$ for some countable ordinal $\alpha$, then $\mathcal{X} \in \mathcal{D}_\alpha$. Consequently, if $\phi$ is a coarse equivalence, then $\mathcal{X} \in \mathcal{D}_\alpha$ if and only if $\mathcal{Y} \in \mathcal{D}_\alpha$. In particular, if $X$ is a subspace of a metric space $Y$ and $Y \in \mathcal{D}_\alpha$, then $X \in \mathcal{D}_\alpha$.

It is easy to obtain the following two Lemmas by simple induction.

Lemma 2.2. ([6]) Let $X$ be a metric space with a left-invariant metric and $\{X_i\}_i$ be a sequence of subspaces of $X$ with the induced metric. If $\{X_i\}_i \in \mathcal{D}_\alpha$, then $\{gX_i\}_{g,i} \in \mathcal{D}_\alpha$, where $gX_i = \{gh|x \in X_i\}$.

Lemma 2.3. ([6]) Let $\mathcal{X} = \{X_i\}$ and $\mathcal{Y} = \{Y_j\}$ be metric families, $\mathcal{X} \ast \mathcal{Y} \triangleq \{X_i \times Y_j\}$. Then for any $m, n \in \mathbb{N}$, if $\mathcal{X} \in \mathcal{D}_m$ and $\mathcal{Y} \in \mathcal{D}_n$, then $\mathcal{X} \ast \mathcal{Y} \in \mathcal{D}_{m+n}$.

Example 2.1. Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ (countable infinite direct sum) with

$$d_1(g, h) = \sum_{n=1}^{\infty} |n| |g_n - f_n|, \forall g = (g_1, \cdots, g_n, \cdots), h = (h_1, \cdots, h_n, \cdots) \in G.$$ 

It was proved that $(G, d_1) \in \mathcal{D}_\omega$, where $\omega$ is the smallest infinite ordinal number ([6]).

Inspired by the equivalent descriptions of finite asymptotic dimension ([7]), here we give the equivalent descriptions of metric spaces with complexity $\omega$.

Before stating the theorem, we recall some necessary definitions.

Let $X$ be a metric space and let $\mathcal{U}$ be a cover of $X$, the Lebesgue number $L(\mathcal{U})$ of $\mathcal{U}$ is the largest number $\lambda$ such that if $A \subseteq X$ and diam $A \leq \lambda$, then there exists some $U \in \mathcal{U}$ such that $A \subseteq U$. The multiplicity $m(\mathcal{U})$ of $\mathcal{U}$ is the maximal number of elements of $\mathcal{U}$ with a nonempty intersection. The $d$-multiplicity of $\mathcal{U}$ is defined to be the largest $n$ such that there is a $x \in X$, satisfying $B_d(x)$ meets $n$ sets in $\mathcal{U}$. A map $\varphi: X \to Y$ between metric spaces is $\varepsilon$ Lipschitz if

$$d(\varphi(x_1), \varphi(x_2)) \leq \varepsilon d(x_1, x_2) \text{ for every } x_1 \neq x_2 \in X.$$ 

We use the notation $l_2$ for the Hilbert space of square summable sequences, i.e.

$$l_2 = \{(x_1, x_2, \cdots) \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty\}.$$ 

Let $\Delta$ denote the standard infinite dimensional simplex in $l_2$, i.e.

$$\Delta = \{(x_1, x_2, \cdots) \in l_2 \mid \sum_{n=1}^{\infty} x_n = 1, x_n \geq 0\}.$$ 

A uniform complex is a simplicial complex considered to be a subset of $\Delta$ with each vertex at some basis element with the restricted metric.
Theorem 2.1. Let $X$ be a metric space. The following conditions are equivalent.

1. $X \in \mathcal{D}_\omega$, i.e. for every $r > 0$, there exist $r$-disjoint families $\mathcal{V}_0$ and $\mathcal{V}_1$ such that $\mathcal{V}_0 \cup \mathcal{V}_1$ covers $X$ and the family $\mathcal{V}_0 \cup \mathcal{V}_1$ has uniformly finite asymptotic dimension.

2. For every $d > 0$, there exists a uniformly finite asymptotic dimension cover $\mathcal{V}$ of $X$ with $d$-multiplicity $\leq 2$. i.e.

$$\forall x \in X, \quad \sharp\{V \in \mathcal{V} | V \cap B_d(x) \neq \emptyset \} \leq 2.$$ 

3. For every $\lambda > 0$, there exists a uniformly finite asymptotic dimension cover $\mathcal{W}$ of $X$ with the Lebesgue number $L(\mathcal{W}) > \lambda$ and the multiplicity $m(\mathcal{W}) \leq 2$.

4. For every $\varepsilon > 0$, there exists an $\varepsilon$-Lipschitz map $\varphi : X \to K \subseteq \Delta$ to a uniform simplicial complex of dimension 1 such that $\{\varphi^{-1}(st \, \{v \}) | v \in K(0)\}$ has uniformly finite asymptotic dimension, where $st \, \{v \}$ is the star of the vertex $v$ in the complex $K$.

5. For every uniformly bounded cover $\mathcal{V}$ of $X$, there is a cover $\mathcal{U}$ of $X$ with uniformly finite asymptotic dimension such that $\mathcal{V}$ refines $\mathcal{U}$ (i.e. every $V \in \mathcal{V}$ is contained in some element $U \in \mathcal{U}$) and the multiplicity $m(\mathcal{U}) \leq 2$.

- $(1) \Rightarrow (2)$: For every $d > 0$ and $r > 2d$, there exist $r$-disjoint families $\mathcal{V}_0$ and $\mathcal{V}_1$ such that $\mathcal{V}_0 \cup \mathcal{V}_1$ covers $X$ and the family $\mathcal{V}_0 \cup \mathcal{V}_1$ has uniformly finite asymptotic dimension. Let $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$. For $x \in X$, if $V_1 \cap B_d(x) \neq \emptyset$ and $V_2 \cap B_d(x) \neq \emptyset$, then $d(V_1, V_2) \leq 2d < r$. So $V_1$ and $V_2$ belong to distinct families $\mathcal{V}_0$ and $\mathcal{V}_1$. Therefore,

$$\sharp\{V \in \mathcal{V} | V \cap B_d(x) \neq \emptyset \} \leq 2.$$ 

- $(2) \Rightarrow (3)$: Let $\lambda > 0$ be given and take a uniformly finite asymptotic dimension cover $\mathcal{V}$ of $X$ with $2\lambda$-multiplicity $\leq 2$. Define $\tilde{\mathcal{V}} = N_{2\lambda}(\mathcal{V}) = \{x \in X | d(x, V) < 2\lambda\}$ and let $\mathcal{W} = \{\tilde{V} | V \in \mathcal{V}\}$. It is easy to see $\mathcal{W}$ has uniformly finite asymptotic dimension by Lemma 2.7. Note that $m(\mathcal{W}) \leq 2$. i.e.

$$\forall x \in X, \quad \sharp\{\tilde{V} \in \mathcal{W} | x \in \tilde{V}\} \leq 2.$$ 

Indeed, if $x \in \tilde{V} = N_{2\lambda}(V)$, then $d(x, V) < 2\lambda$. It follows that

$$V \cap B_{2\lambda}(x) \neq \emptyset.$$ 

Since

$$\sharp\{V \in \mathcal{V} | V \cap B_{2\lambda}(x) \neq \emptyset \} \leq 2,$$

we have

$$\sharp\{\tilde{V} \in \mathcal{W} | x \in \tilde{V}\} \leq \sharp\{V \in \mathcal{V} | V \cap B_{2\lambda}(x) \neq \emptyset \} \leq 2.$$ 

It is easy to see that $L(\mathcal{W}) > \lambda$. Indeed, for every $A \subseteq X$ with $\text{diam} \, A \leq \lambda$, choose $a \in A \subseteq X$. Since $\mathcal{V}$ is a cover of $X$, there exists $V \in \mathcal{V}$ such that $a \in V$ and hence

$$A \subseteq B_{\lambda}(a) \subseteq N_{2\lambda}(V) = \tilde{V} \in \mathcal{W}.$$ 

- $(3) \Rightarrow (4)$: Let $\varepsilon > 0$ be given and suppose that $\mathcal{W}$ is a uniformly finite asymptotic dimension cover of $X$ with $L(\mathcal{W}) > \lambda = \frac{2m(\mathcal{W})}{\varepsilon}$ and $m(\mathcal{W}) \leq 2$. For each $W \in \mathcal{W}$, define $\varphi_W : X \to [0, 1]$ by

$$\varphi_W(x) = \frac{d(x, X - W)}{\sum_{V \in \mathcal{W}} d(x, X - V)}.$$ 

Note that $\varphi_W(x) = 0$ if and only if $x \not\in W$. The maps $\{\varphi_W\}$ define a map $\varphi : X \to K$ by

$$\varphi(x) = \{\varphi_W(x)\}_{W \in \mathcal{W}}.$$
It is easy to see that \( x \in \varphi^{-1}(\text{st } W) \), i.e. \( \varphi_W(x) > 0 \) if and only if \( x \in W \). Then \( \varphi^{-1}(\text{st } W) = W \). Hence \( \{\varphi^{-1}(\text{st } W)\}_{W \in W} = W \) has uniformly asymptotic dimension. Finally, we check that \( \varphi : X \to K \) is \( \varepsilon \)-Lipschitz. Note that

\[
|d(x, X - U) - d(y, X - U)| \leq d(x, y), \quad \forall x, y \in X, \forall U \in W
\]

and

\[
\sum_{W \in W} d(x, X - W) \geq \frac{\lambda}{2}, \forall x \in X.
\]

Indeed, since \( \text{diam } B_\lambda(x) \leq \lambda \) and \( L(W) > \lambda \), there exists \( W_0 \in W \) such that \( B_\lambda(x) \subseteq W_0 \). So

\[
\sum_{W \in W} d(x, X - W) \geq d(x, X - W_0) \geq \frac{\lambda}{2}.
\]

Since \( m(W) \leq 2 \)

\[
\#\{W \in W | \varphi_W(x) > 0\} \leq 2,
\]

then we have

\[
|\varphi_U(x) - \varphi_U(y)| = \left| \frac{d(x, X - U)}{\sum_{V \in W} d(x, X - V)} - \frac{d(y, X - U)}{\sum_{V \in W} d(y, X - V)} \right|
\leq \left| \frac{d(x, X - U)}{\sum_{V \in W} d(x, X - V)} - \frac{d(y, X - U)}{\sum_{V \in W} d(y, X - V)} \right|
\leq \frac{2}{\lambda}d(x, y) + \frac{d(y, X - U)}{\sum_{V \in W} d(x, X - V)} \sum_{V \in W} |d(y, X - V) - d(x, X - V)|
\leq \frac{2}{\lambda}d(x, y) + \frac{2}{\lambda} \cdot 4d(x, y)
= \frac{10}{\lambda}d(x, y).
\]

Therefore,

\[
\|\varphi(x) - \varphi(y)\|_2 = \left( \sum_{U \in W} |\varphi_U(x) - \varphi_U(y)|^2 \right)^\frac{1}{2}
\leq (4 \cdot \frac{100}{\lambda^2})^\frac{1}{2}
= \frac{20}{\lambda}d(x, y)
= \varepsilon d(x, y).
\]

\( (4) \Rightarrow (1) \): Let \( r > 0 \) be given and let \( K \) be any uniform complex of dimension 1. For each \( i = 0, 1 \), let

\[
U_i = \{\text{st}(b_\sigma, \beta^2 K) | \sigma \subset K, \dim \sigma = i\},
\]

where \( b_\sigma \) is the barycenter of \( \sigma \) and \( \beta^2 K \) denotes the second barycentric subdivision. It is easy to see that \( U_i \) is \( c \)-disjoint for some constant \( c > 0 \). Let \( \varepsilon = \frac{\varepsilon}{r} \), take an \( \varepsilon \)-Lipschitz map \( \varphi : X \to K \subseteq \Delta \) to a uniform simplicial complex of dimension 1. Define

\[
\mathcal{V}_i = \{\varphi^{-1}(U) | U \in U_i\}, \quad i = 0, 1.
\]

Then \( \mathcal{V}_i \) is \( r \)-disjoint and \( \mathcal{V}_0 \cup \mathcal{V}_1 \) covers \( X \). For every \( \text{st}(b_\sigma, \beta^2 K) \in U_i \), there exists \( u \in K^{(0)} \) such that

\[
\text{st}(b_\sigma, \beta^2 K) \subseteq \text{st}(u, K).
\]

and it follows that

\[
\varphi^{-1}(\text{st}(b_\sigma, \beta^2 K)) \subseteq \varphi^{-1}(\text{st}(u, K)).
\]
Since \( \{ \varphi^{-1}(st \ u) | u \in K^{(0)} \} \) has uniformly finite asymptotic dimension,
\[
V_i = \{ \varphi^{-1}(st(b, \beta^i K)) | \sigma \subset K, \dim \sigma = i \}
\]
has uniformly finite asymptotic dimension.

- (1) \( \Rightarrow \) (5): Let \( \mathcal{V} \) be given with \( \text{diam} \ \mathcal{V} \leq B \), for some positive constant \( B \). Take \( r \)-disjoint families \( \mathcal{W}_0 \) and \( \mathcal{W}_1 \) of uniformly finite asymptotic dimension with \( r > 2B \) and \( \mathcal{W}_0 \cup \mathcal{W}_1 \) covers \( X \). For each \( i = 0, 1 \), let
\[
\mathcal{U}_i = \{ N_B(W) | W \in \mathcal{W}_i \}.
\]
Since \( \mathcal{W}_i \) is \( r \)-disjoint and \( r > 2B \), \( \mathcal{U}_i \) is disjoint. Let \( \mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \), then \( m(\mathcal{U}) \leq 2 \). For every \( V \in \mathcal{V} \), since \( \mathcal{W}_0 \cup \mathcal{W}_1 \) covers \( X \), there is a \( W \in \mathcal{W}_0 \cup \mathcal{W}_1 \) such that \( W \cap V \neq \emptyset \). Assume that \( x_0 \in W \cap V \), then for every \( x \in \mathcal{V} \), we have
\[
d(x, W) \leq d(x, x_0) \leq \text{diam} \ V \leq \text{diam} \ \mathcal{V} \leq B.
\]
i.e. \( x \in N_B(W) \). Therefore, \( V \subseteq N_B(W) \in \mathcal{U} \) and hence \( \mathcal{V} \) refines \( \mathcal{U} \).

- (5) \( \Rightarrow \) (3): Let \( \lambda > 0 \) be given and let \( \mathcal{V} = \{ B_{\lambda}(x) | x \in X \} \). Clearly, \( \mathcal{V} \) is a uniformly bounded cover of \( X \). So there is a cover \( \mathcal{W} \) of \( X \) with uniformly finite asymptotic dimension such that \( \mathcal{V} \) refines \( \mathcal{W} \) and \( m(\mathcal{W}) \leq 2 \). Finally, we will show that \( L(\mathcal{W}) \leq \lambda \). Indeed, for every \( A \subseteq X \) and \( \text{diam} \ A \leq \lambda \), choose any \( a \in A \), then \( A \subseteq B_{\lambda}(a) \in \mathcal{V} \). Since \( \mathcal{V} \) refines \( \mathcal{W} \), there is a \( W \in \mathcal{W} \) such that \( B_{\lambda}(a) \subseteq W \). Therefore, \( A \subseteq B_{\lambda}(a) \subseteq W \).

### 3 The exact complexity of the product of wreath products

**Definition 3.7.** Let \( G \) be a countable discrete group. A **length function** \( l : G \to \mathbb{R}_+ \) on \( G \) is a function satisfying: for all \( g, f \in G \),

- \( l(g) = 0 \) if and only if \( g \) is the identity element of \( G \),
- \( l(g^{-1}) = l(g) \),
- \( l(gf) \leq l(g) + l(f) \).

We say that the metric \( d \), defined as follows,
\[
d(s, t) = l(s^{-1}t) \quad \forall \ s, t \in G,
\]
is the metric induced by the length function \( l \).

A length function \( l \) is called **proper** if for all \( C > 0 \), \( l^{-1}([0, C]) \subset G \) is finite.

Let \( S \) be a finite generating set for a group \( G \), for any \( g \in G \), define \( |g|_S \) to be the length of the shortest word representing \( g \) in elements of \( S \cup S^{-1} \). We say that \( | \cdot |_S \) is **word-length function** for \( G \) with respect to \( S \). The left-invariant word-metric \( d_S \) on \( G \) is induced by word-length function. i.e., for every \( g, h \in G \),
\[
d_S(g, h) = |g^{-1}h|_S.
\]
Note that the word-length function of a finitely generated group is a proper length function. The Cayley graph is the graph whose vertex set is \( G \), one vertex for each element in \( G \) and any two vertices \( g, h \in G \) are incident with an edge if and only if \( g^{-1}h \in S \cup S^{-1} \).

**Lemma 3.1.** ([9]) A countable discrete group admits a proper length function \( l \) and that any two metrics of a countable discrete group induced by proper length functions are coarsely equivalent.
By Lemma 2.1, finite decomposition complexity is a coarsely invariant property of metric spaces. As a consequence, we say that a discrete group has finite decomposition complexity if its underlying metric space has finite decomposition complexity for some (equivalently every) metric induced by proper length function.

Let $G$ and $N$ be finitely generated groups and let $1_G \in G$ and $1_N \in N$ be their units. The support of a function $f : N \to G$ is the set

$$\text{supp}(f) = \{ x \in N | f(x) \neq 1_G \}.$$ 

The direct sum $\bigoplus G$ of groups $G$ (or restricted direct product) is the group of functions

$$C_0(N, G) = \{ f : N \to G \text{ with finite support} \}.$$ 

There is a natural action of $N$ on $C_0(N, G)$: for all $a \in N, x \in N, f \in C_0(N, G)$,

$$a(f)(x) = f(xa^{-1}).$$

The semidirect product $C_0(N, G) \rtimes N$ is called restricted wreath product and is denoted as $G \wr N$. We recall that the product in $G \wr N$ is defined by the formula

$$(f, a)(g, b) = (fa(g), ab) \quad \forall f, g \in C_0(N, G), a, b \in N.$$ 

Let $S$ and $T$ be finite generating sets for $G$ and $N$, respectively. Let $e \in C_0(N, G)$ denotes the constant function taking value $1_G$, and let $\delta_v : N \to G, v \in N, b \in G$ be the $\delta$-function, i.e.

$$\delta_v^b(v) = b \quad \text{and} \quad \delta_v^b(x) = 1_G \quad \text{for} \quad x \neq v.$$ 

Note that $a(\delta^b_v) = \delta^b_v$ and hence $(\delta^b_v, 1_N) = (e, v)(\delta^b_v, 1_N)(e, v^{-1})$. Since every function $f \in C_0(N, G)$ can be presented $\delta^b_{v_1} \cdots \delta^b_{v_k}$,

$$(f, 1_N) = (\delta^b_{v_1}, 1_N) \cdots (\delta^b_{v_k}, 1_N) \quad \text{and} \quad (f, u) = (f, 1_N)(e, u).$$ 

The set $\tilde{S} = \{(\delta^b_{1_N}, 1_N), (e, t)|s \in S, t \in T\}$ is a generating set for $G \wr N$. Note that $G$ and $N$ are subgroups of $G \wr N$.

An explicit formula for the word length of wreath products was found by Parry.

**Lemma 3.2.** (Parry) Let $x = (f, v) \in G \wr N$, where $f \in C_0(N, G)$ and $v \in N$. Assume that $f = \delta^b_{v_1} \cdots \delta^b_{v_n}$, let $p(x)$ be the shortest path in the Cayley graph of $N$ which starts at $1_N$, visits all vertices $v_i$ and ends at $v$. Then

$$|x|_{G \wr N} = |p(x)| + \sum_{k=1}^n |b_k|_G.$$ 

The following statement immediately follows from the above Lemma.

**Corollary 3.1.** For every $f = \delta^b_{v_1} \cdots \delta^b_{v_n} \in \bigoplus \mathbb{Z} = C_0(\mathbb{Z}, \mathbb{Z}) \subset \mathbb{Z} \wr \mathbb{Z}$, where $b_i \in \mathbb{Z}$ and $v_i \in \mathbb{Z}$. Let $w(f)$ be the shortest loop in the Cayley graph of $\mathbb{Z}$ which based at 0 and visits all vertices $v_i$, then

$$|f|_{\mathbb{Z} \wr \mathbb{Z}} = |w(f)| + \sum_{k=1}^n |b_k|.$$ 

**Lemma 3.3.** (Sc) Let $G$ be a finitely generated subgroup of $GL(n, \mathbb{R})$ for some natural number $n$, then $G \in D_\omega$.

**Theorem 3.1.** For every $m \in \mathbb{N}$, let $G = (\mathbb{Z} \wr \mathbb{Z})^m = \mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z} \wr \mathbb{Z} \times \cdots \times \mathbb{Z} \wr \mathbb{Z}$. Then $G \in D_\omega$ and for any $\alpha < \omega, G \in D_\alpha$, i.e. the exact complexity of $G$ is $\omega$. 

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Proof. Since \( \mathbb{Z} \ltimes \mathbb{Z} \) is a finitely generated group, \((\mathbb{Z} \ltimes \mathbb{Z})^m\) is finitely generated. By Lemma 3.3, it suffices to show that \((\mathbb{Z} \ltimes \mathbb{Z})^m\) is a subgroup of \(GL(n, \mathbb{R})\) for some natural number \(n\). Define a map \(\psi : \mathbb{Z} \ltimes \mathbb{Z} \to GL(2, \mathbb{R})\) as follows: \(\psi(x) = \begin{pmatrix} \sum_{k \in \text{supp } f} f(k) \pi^k & 0 \\ \pi^n \end{pmatrix}\)

Now we will show that \(\forall x = (f, n), y = (g, m) \in \mathbb{Z} \ltimes \mathbb{Z}, \ \psi(xy) = \psi(x)\psi(y)\).

Indeed,

\[
\psi(x)\psi(y) = \begin{pmatrix} \sum_{k \in \text{supp } f} f(k) \pi^k & 0 \\ \pi^n \end{pmatrix} \begin{pmatrix} \sum_{k \in \text{supp } g} g(k) \pi^k & 0 \\ \pi^m \end{pmatrix} = \begin{pmatrix} \sum_{k \in \text{supp } f} f(k) \pi^k + \sum_{k \in \text{supp } g} g(k) \pi^{k+n} & 0 \\ \pi^{n+m} \end{pmatrix}
\]

Since \(xy = (f, n)\langle g, m \rangle = (fn, n+m)\) and \(fn = f + g(k-n)\),

\[
\psi(xy) = \begin{pmatrix} \sum_{k \in \text{supp } f \cup \text{supp } g} [f(k) + g(k-n)] \pi^k & 0 \\ \pi^{n+m} \end{pmatrix}
\]

Note that

\[
\sum_{k \in \text{supp } f} f(k) \pi^k + \sum_{k \in \text{supp } g} g(k) \pi^{k+n} = \sum_{k \in \text{supp } f} f(k) \pi^k + \sum_{k \in \text{supp } g} n(g(k)) \pi^k = \sum_{k \in \text{supp } f \cup \text{supp } g} [f(k) + g(k-n)] \pi^k.
\]

It follows that

\(\forall x = (f, n), y = (g, m) \in \mathbb{Z} \ltimes \mathbb{Z}, \ \psi(xy) = \psi(x)\psi(y)\).

Define a map \(\tilde{\psi} : (\mathbb{Z} \ltimes \mathbb{Z})^m \to GL(2m, \mathbb{R})\) as follows: \(\forall x \in (x_1, x_2, \cdots, x_m) \in (\mathbb{Z} \ltimes \mathbb{Z})^m, \ \tilde{\psi}(x) = \text{diag}(\psi(x_1), \psi(x_2), \psi(x_3), \cdots, \psi(x_m)) \in GL(2m, \mathbb{R})\).

It is easy to check that \(\tilde{\psi}\) is a group homomorphism. So \((\mathbb{Z} \ltimes \mathbb{Z})^m\) can be considered as a subgroup of \(GL(2m, \mathbb{R})\).

Finally, since \(\mathbb{Z} \ltimes \mathbb{Z}\) is a subgroup of \((\mathbb{Z} \ltimes \mathbb{Z})^m\) and \(\mathbb{Z} \ltimes \mathbb{Z}\) does not have asymptotic dimension, we have

\(\forall \alpha < \omega, G \in \mathcal{D}_\alpha\).

Therefore, the exact complexity of \(G\) is \(\omega\).

Lemma 3.4. (\[3\]) Let \(H\) be a countable group and \(H^m = H \times H \times \cdots \times H\). For every \(r \in \mathbb{N}\), there exist \(m \in \mathbb{N}\) and a metric family \(\mathcal{Y}\) such that

1. \(H \ltimes \mathbb{Z} \not\rightarrow \mathcal{Y}\),
2. there is a coarse embedding from \(\mathcal{Y}\) to \(\{gH^m\}_{g \in \bigoplus_H}\).

Theorem 3.2. \((\mathbb{Z} \ltimes \mathbb{Z}) \ltimes \mathbb{Z} \in \mathcal{D}_{\omega+1}\), i.e. the complexity of \((\mathbb{Z} \ltimes \mathbb{Z}) \ltimes \mathbb{Z}\) is \(\omega + 1\).

Proof. Let \(H = \mathbb{Z} \ltimes \mathbb{Z}\), by Lemma 3.3, for every \(r \in \mathbb{N}\), there exist \(m \in \mathbb{N}\) and a metric family \(\mathcal{Y}\) such that

1. \(H \ltimes \mathbb{Z} \not\rightarrow \mathcal{Y}\),
2. there is a coarse embedding from \(\mathcal{Y}\) to \(\{gH^m\}_{g \in \bigoplus_H}\).

By Theorem 3.1, \(H^m \in \mathcal{D}_\omega\). Then it is easy to obtain that \(\{gH^m\}_{g \in \bigoplus_H} \in \mathcal{D}_\omega\). Since there is a coarse embedding from \(\mathcal{Y}\) to \(\{gH^m\}_{g \in \bigoplus_H}, \mathcal{Y} \in \mathcal{D}_\omega\) by Lemma 2.1, Therefore, \((\mathbb{Z} \ltimes \mathbb{Z}) \ltimes \mathbb{Z} = H \ltimes \mathbb{Z} \in \mathcal{D}_{\omega+1}\).

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