Strong magnetic field asymptotic behaviour for the fermion-induced effective energy in the presence of a magnetic flux tube

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Abstract

In Ref. [3], we presented an asymptotic formula for the fermion-induced effective energy in 3+1 dimensions in the presence of a cylindrically symmetric inhomogeneous strong magnetic field. However, there are some points which were not clearly explained. In fact, the arguments, which led us to the asymptotic formula, are based on a numerical study of the integral of Eq. (10), as we will see in the main part of this paper. The aim of this work is to present this study in detail.

1 Introduction

Many authors have dealt with the numerical study of the fermion-induced effective energy, in the presence of magnetic fields of the form of a flux tube, see Refs. [1, 2, 3, 4, 5]. An analytical study of the same problem in two dimensional Euclidean space can be found in Ref. [6].

A similar topic, is the study of the gluon-induced effective energy in the presence of a colored magnetic flux tube [9, 10]. However, in this case there is an essential difference: the effective energy has nonzero imaginary part and so the system is unstable.

It is worth mentioning the recently developed functional method in Refs. [4, 7, 8], which is applicable even for magnetic fields which do not exhibit a special kind of symmetry. For example, this method can be applied to a system of two separated magnetic flux tubes.

In Ref. [3] we presented an asymptotic formula (Eq. (49)) for the effective energy when the characteristic magnetic field strength $B_m = \int \vec{B}(\vec{x}) \cdot d\vec{S}/\pi d^2$ (or the magnetic flux $\Phi$ of the tube) tends to infinity and the spatial size $d$ of the magnetic flux tube is kept fixed. However, there are some points which were not clearly explained. In fact, the arguments, which led us to the asymptotic formula, are based on a numerical study of the integral of Eq. (10), as we will see in the main part of this paper. The aim of this work is to present this study in detail. The final asymptotic formula is given by Eq. (23).

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In addition, in Sec. 4 we compared the asymptotic formula of Eq. (23) with the strong field limit of the derivative expansion and find agreement.

2 Effective energy in the presence of a magnetic flux tube

The 3+1 dimensional fermion-induced effective energy in the presence of a magnetic field is given by the equation:

\[
E_{\text{eff}}^{(3+1)} = -\frac{1}{2iT} \text{Tr} \ln(\not{D}^2 + m_f^2)
\]  

where \( \not{D} = \gamma^\mu D_\mu \) (\( \mu = 0, 1, 2, 3 \)) and \( D_\mu = \partial_\mu - ieA_\mu \). The gamma matrices satisfy the relationship \( \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \) and \( T \) is the total length of time.

We have shown in Ref. [3] that a magnetic field with strength independent of \( z \) coordinate, directed towards \( z \)-axis, has renormalized effective energy per unit length \( L_z \) equal to

\[
E_{\text{eff}}^{(3+1)}^{(\text{ren})} = \frac{1}{4\pi} \sum_{\{n\}} (E_{\{n\}} + m_f^2) \ln\left(\frac{E_{\{n\}} + m_f^2}{m_f^2}\right) - \frac{1}{4\pi} \sum_{\{n\}} E_{\{n\}}
\]  

where \( E_{\{n\}} \) are the eigenvalues of the planar operator \( (\gamma^m D_m)^2 \) \((m = 1, 2)\), and \( \{n\} \) is a set of quantum numbers. Note that \( E_{\{n\}} \geq 0 \) since the operator \( (\gamma^m D_m)^2 \) is positive definite as the square of a hermitian operator. For the sake of simplicity we will drop the index \( \text{(ren)} \) in the rest of this paper.

If the magnetic field is cylindrically symmetric with finite magnetic flux, the spectrum of the operator \( (\gamma^m D_m)^2 \) is continuous \(^3\). Thus we can set \( E_{\{n\}} = k^2 \).

In order to perform the summation over \( \{n\} \) that appears in Eq. (2), we need a density of states, given by the equation:

\[
\rho(k) = \sum_{l,s} \left( \rho_{l,s}^{(\text{free})}(k) + \frac{1}{\pi} \frac{d\delta_{l,s}(k)}{dk} \right) = \sum_{l,s} \rho_{l,s}^{(\text{free})}(k) + \frac{1}{\pi} \frac{d\Delta(k)}{dk}
\]  

where \( \sum_{l,s} \rho_{l,s}^{(\text{free})}(k) \) is the density of states for free space, and \( \delta_{l,s}(k) \) is the phase shift which corresponds to \( l^{th} \) partial wave with momentum \( k \) and spin \( s \). The function \( \Delta(k) \) is defined by the equation:

\[
\Delta(k) = \lim_{L \to +\infty} \sum_{s,l=-L}^{L} \delta_{l,s}(k)
\]  

From Eqs. (2) and (3), if we drop the field independent term \( \sum_{l,s} \rho_{l,s}^{(\text{free})}(k) \) and integrate by parts, we obtain

\[
E_{\text{eff}}^{(3+1)} = -\frac{1}{2\pi^2} \int_{0}^{+\infty} k \ln\left(\frac{k^2 + m_f^2}{m_f^2}\right) (\Delta(k) - c) dk
\]

\(^1L_z \) is the length of the space box towards \( z \) direction.

\(^2\)For the case of massless fermions \((m_f = 0)\) the renormalized effective energy is given by appendix A in Ref. [3].

\(^3\)Besides continuous spectrum, zero modes may exist according to the Aharonov-Casher theorem [14].
where $c = \lim_{k \to +\infty} \Delta(k)$. Our numerical study, in Ref. [3], shows that $c = -\pi \phi^2$ ($\phi = e\Phi/2\pi$). This means that $c$ is independent of the special form of the magnetic field we examine and depends only on the total magnetic flux of the field $\Phi$. Also, in Ref. [3], we see that the function $\Delta(k)$, when $k \to +\infty$, tends to $c$ fairly rapidly, and thus, the integral over $k$ in Eq. (5) is convergent.

Eq. (5) is suitable for the numerical computation of the 3+1 dimensional fermion-induced effective energy in the presence of a magnetic flux tube. The calculation of the phase shifts is performed by solving an ordinary differential equation. For details see Refs. [3] [4].

3 The asymptotic behaviour for the effective energy when $B_m \to +\infty$ and $d$ and $m_f$ are kept fixed

The function $\Delta(k)$ depends on three quantities: $k$, $B_m = 2\phi/d^2$ and $d$. From these we construct the following two dimensionless quantities\footnote{We have assumed the rescaling $B_m \to eB_m$.}: $kB_m^{-1/2}$ and $B_md^2$ ($=2\phi$). For dimensional reasons the function $\Delta(k)$ can be put into the form $\Delta(k) = G(kB_m^{1/2},B_md^2)$. Setting $\Delta(k) = G(kB_m^{1/2},B_md^2)$ and making the change of variable $y = kB_m^{1/2}$ in Eq. (5) we obtain

\[
E_{\text{eff}(3+1)} = -\frac{B_m}{2\pi^2} \int_0^{+\infty} y \ln \left(\frac{y^2 + m_f^2/B_m}{m_f^2/B_m}\right) \left(G(y,B_md^2) - c\right) dy
\]

\[
= -\frac{B_m}{2\pi^2} \int_0^{+\infty} y \ln(y^2 + m_f^2/B_m) - \ln(m_f^2/B_m) \left(G(y,B_md^2) - c\right) dy
\]

In the strong magnetic field case ($B_m/m_f^2 >> 1$) we obtain:

\[
E_{\text{eff}(3+1)} \to -\frac{B_m}{\pi^2} \int_0^{+\infty} y \ln y \left(G(y,B_md^2) - c\right) dy
\]

\[
= -\frac{B_m}{2\pi^2} \ln(B_m/m_f^2) \int_0^{+\infty} y \left(G(y,B_md^2) - c\right) dy
\]

There are many ways to achieve a large ratio $B_m/m_f^2$ by changing the three independent variables $B_m$, $m_f$ and $d$. However, the most interesting case is when $B_m \to +\infty$ and $m_f$ and $d$ are kept fixed (see also Refs. [1] [2] [3] [4]). This means that we keep the spatial size of the magnetic field configuration fixed and we increase the characteristic magnetic field strength $B_m$ or the magnetic flux $\phi = B_md^2/2$.

From Eq. (8) we see that the asymptotic behaviour of the effective energy when $B_m \to +\infty$ ($m_f$ and $d$ are kept fixed) depends on the asymptotic behaviour of the integrals:

\[
I_1(BMd^2) = \int_0^{+\infty} y \left(G(y,B_md^2) - c\right) dy
\]

\[
I_2(BMd^2) = \int_0^{+\infty} y \ln y \left(G(y,B_md^2) - c\right) dy
\]

In the case of the integral of Eq. (9) it was possible to find an analytical expression:

\[
\int_0^{+\infty} y(G(y,B_md^2) - c)dy = \frac{1}{12B_m} \int d^2 \vec{x} B^2(\vec{x})
\]
In order to give a proof for the above equation we will use Eq. (13) in Ref. [3]

\[ \text{Tr}(\gamma^m D_m)^2 = \sum_{\{n\}} E_{\{n\}} = -\frac{1}{6\pi} \int d^2 \vec{x} B^2(\vec{x}) \]  

(12)

By using Eq. (3) for the density of states we obtain

\[ \sum_{\{n\}} E_{\{n\}} = \int_0^{+\infty} k^2 \rho(k) dk = \int_0^{+\infty} k^2 \left( \sum_{l,s} \rho_{l,s}^{(\text{free})}(k) + \frac{1}{\pi} \frac{d\Delta(k)}{dk} \right) dk \]  

(13)

If we drop the field independent term \( \sum_{l,s} \rho_{l,s}^{(\text{free})}(k) \) and set again \( \Delta(k) = G(kB_m^{-\frac{3}{2}}, B_m d^2) \) and \( y = kB_m^{-1/2} \) we find

\[ \sum_{\{n\}} E_{\{n\}} = \int_0^{+\infty} k^2 \frac{1}{\pi} \frac{d\Delta(k)}{dk} dk = -\frac{2}{\pi} \int_0^{+\infty} k(\Delta(k) - c) dk = -\frac{2}{\pi} B_m \int_0^{+\infty} y(G(kB_m^{-1/2}, B_m d^2) - c) dy \]  

(14)

Eq. (11) is a straightforward result of Eqs. (14) and (12).

For convenience we write

\[ B(\vec{x}) = B_m \tilde{B}(\vec{x}/d) \]  

(15)

where the magnetic field \( \tilde{B}(\vec{x}) \) has a characteristic magnetic field strength \( \tilde{B}_m = 1 \) and range \( d = 1 \), Eq. (11) can then be written in the form

\[ \int_0^{+\infty} y(G(y, B_m d^2) - c) dy = \frac{B_m d^2}{12} \int d^2 \vec{x} \tilde{B}^2(\vec{x}) \]  

(16)

From the above relationship it is obvious that the integral \( \int_0^{+\infty} y(G(y, B_m d^2) - c) dy \) is proportional to \( B_m d^2 \).

I could not find an analytical expression, like that of Eq. (16), for the integral \( I_2(B_m d^2) \) of Eq. (10). However, a numerical study, which is presented in Fig. 11 suggests a linear \(^{5}\) asymptotic behaviour for this integral \(^{6}\), when \( B_m \to +\infty \), independently of the particular shape of the magnetic field. Thus we can write

\[ \int_0^{+\infty} y \ln y \left( G(y, B_m d^2) - c \right) dy \to c1 B_m d^2 \]  

(17)

where the constant \( c1 \) is independent of \( B_m d^2 \), and its value depends only on the special form of the magnetic field configuration.

\(^{5}\)Our numerical study can not exclude the possibility of a weak logarithmic growing of \( I_2(B_m d^2)/B_m d^2 \), as we will see at the end of this section

\(^{6}\)This implies that the asymptotic behaviour of the integral of Eq. (10) is the same with that of integral \( \int_0^{+\infty} y(G(y, B_m d^2) - c) dy \).
As we see in Fig. 1, we have performed numerical calculations for four magnetic field configurations with quite different kinds of inhomogeneity.

\[
B_1(r) = B_m \frac{1}{(r^2/d^2 + 1)^2} \\
B_2(r) = B_m \exp\left(-\frac{r^2}{d^2}\right) \\
B_3(r) = B_m \theta(d - r) \\
B_4(r) = B_m 3(1 - \frac{r}{d}) \theta(d - r)
\]

We expect that our numerical work, presented in Fig. 1, convinces the reader that the asymptotic behaviour of \(I_2(B_m d^2)\), as given by Eq. (17), does not depend on the kind of inhomogeneity of the magnetic flux tube.

Thus, when \(B_m \to +\infty\) and \(m_f\) and \(d\) are kept fixed, from Eqs. (8),(16) and (17) we obtain

\[
E_{eff(3+1)} \to -c1 \frac{B_m^2 d^2}{\pi^2} - \frac{B_m^2 d^2}{24\pi^2} \ln\left(\frac{B_m}{m_f^2}\right) \int d^2 \vec{x} \vec{B}^2(\vec{x})
\]

We see that the logarithmic term dominates, so we can write

\[
E_{eff(3+1)}^{\text{asympt}} = -\frac{B_m^2 d^2}{24\pi^2} \ln\left(\frac{B_m}{m_f^2}\right) \int d^2 \vec{x} \vec{B}^2(\vec{x})
\]
If, instead of the characteristic magnetic field strength $B_m = \Phi / \pi d^2$ we use the parameter $\phi = e^\Phi / 2\pi$ we see that the fermion-induced effective energy, for $\phi \to +\infty$, is proportional to $-\phi^2 \ln \phi$.

$$E_{\text{eff}(3+1)}^{\text{asympt}} \sim -\phi^2 \ln \phi \quad (24)$$

This asymptotic behaviour is correct, independently of the particular shape of the magnetic flux tube, as it was shown in the above numerical study (See Fig. 1).

Note that the above asymptotic formula, of Eq. (23), is valid and for other cases of large ratio $B_m/m_f^2$. For example we note two: a) for $m_f \to 0$ when $B_m$ and $d$ are kept fixed and b) for $d \to 0$ when $m_f$ and $\phi = B_m d^2 / 2$ are kept fixed. For these two cases, the asymptotic formula, of Eq. (23), is obtained straightforwardly from Eqs. (8) and (16), and thus knowledge of the asymptotic behaviour of the integral $I_2(B_m d^2)$ is not necessary.

Finally, we note that our numerical study in Fig. 1 is not sufficient to exclude a logarithmic growing of $I_2(B_m d^2)/B_m d^2$. However this logarithmic dependence, even if it really exists, should be weak compared to the logarithmic dependence which appears in the asymptotic formula of Eq. (23), otherwise it would be clearly shown in Fig. 1 and it is not.

In order to observe a possible weak logarithmic growing we ought to perform computations in an exponentially large range of values of $B_m$. We could not execute computations for so wide a range, as the differential equation for the computation of the phase shifts (Eq. (69) in Ref. 3) is stiff for large $B_m$. However, we tried an estimate of this logarithmic dependence by fitting a curve of the form $c_1 + c_2 \ln(B_m d^2)$ to our data, as they are presented in Fig. 1. We find that $c_2 = 0.0047 \pm 0.0005$ for $B_1$, $c_2 = 0.0100 \pm 0.0005$ for $B_2$, $c_2 = 0.0062 \pm 0.0005$ for $B_3$ and $c_2 = 0.0120 \pm 0.0006$ for $B_4$. These results imply that, even if this logarithmic term $c_2 \ln(B_m d^2)$ really exists, it would not contribute to the effective energy (as it is given by the asymptotic formula of Eq. (23) above) more than 5 per cent.

4 Derivative expansion for $B_m \to +\infty$ when $d$ and $m_f$ are kept fixed

An approximate way for the computation of the fermion-induced effective energy is the derivative expansion [12, 13]. This method gives accurate results for smooth magnetic field configurations. Note that the derivative expansion fails for the magnetic fields of Eqs. (20) and (21), as the magnetic field of Eq. (20) is discontinuous and the magnetic field of Eq. (21) has discontinuous first derivative.

Our aim is to compare the strong field limit of the derivative expansion with the asymptotic formula of Eq. (23).

If we keep the first two terms of the derivative expansion, the effective energy, in the case of 3+1 dimensions, is given by the equation

$$E_{\text{eff}(3+1)}^{(\text{der})} = E_{(3+1)}^{(0)}[B] + E_{(3+1)}^{(1)}[B, (\partial B)^2] \quad (25)$$

where

$$E_{(3+1)}^{(0)} = \int d^2 \vec{x} \frac{B^2(\vec{x})}{8\pi^2} \int_0^{+\infty} \frac{1}{s^2} \left( \coth(s) - \frac{1}{s} - \frac{s}{3} \right) e^{-s m_f^2 / B(\vec{x})} ds \quad (26)$$
\[ E^{(1)}_{(3+1)} = \left( \frac{1}{8\pi} \right)^2 \int d^2 \vec{x} \left( \nabla B \right)^2 \int_0^{+\infty} \frac{1}{s} \left( \frac{d}{ds} \right)^3 [s \coth(s)] e^{-s m_f^2 / B(\vec{x})} ds \]  

(27)

For \( B_m \to +\infty \) from Eqs. (26) and (27) we obtain

\[ E^{(0)}_{(3+1)} \to - \frac{B_m^2 d^2}{24\pi^2} \ln(B_m / m_f^2) \int d^2 \vec{x} \tilde{B}^2(\vec{x}) \]  

(28)

\[ E^{(1)}_{(3+1)} \to B_m \left( \frac{1}{8\pi} \right)^2 24 \zeta'(-2) \int d^2 \vec{x} \left( \nabla \tilde{B} \right)^2 \]  

(29)

Eq. (29) can be derived straight from Eq. (27) if we assume that \( B_m >> m_f^2 \) (for details see Ref. [13]). In order to derive Eq. (28) from Eq. (26) we have used a similar way with that presented in Ref. [15].

Thus Eq. (28), which gives the strong field limit of the derivative expansion, is in agreement with the asymptotic formula of Eq. (23).

5 Conclusions

We emphasize that the arguments, which led us to the asymptotic formula of Eq. (23), are based on a numerical study. In particular, we computed numerically \( I_2(B_m d^2) \) for four magnetic field configurations with quite different kinds of inhomogeneity (see Eqs. (18), (19), (20) and (21)). Our results in Fig. 1 suggest that \( I_2(B_m d^2) \) is proportional to \( B_m d^2 \) (for \( B_m d^2 >> 1 \)) independently of the specific form of the magnetic field configurations. The consequence of this result is that also the final formula, which states that \( E^{\text{asympt}}_{\text{eff}(3+1)} \sim -\phi^2 \ln \phi \) for \( \phi \to +\infty \), is valid independently of the specific form of the magnetic field.

Finally we showed, comparing with the asymptotic formula of Eq. (23), that the leading term of the derivative expansion gives the same asymptotic behaviour for the fermion-induced effective energy in the strong magnetic field regime even in the case (like that of the magnetic fields of Eqs. (20) and (21)) when the derivative expansion is invalid, because the next to the leading term diverges.

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