Meshless method for solving coupled radiative and conductive heat transfer in refractive index medium

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Abstract. A diffuse approximation meshless method (DAM) is employed as a means of solving the coupled radiative and conductive heat transfer problems in semi-transparent refractive index media contained in 1D and 2D geometries. The meshless approach for radiative transfer is based on the discrete ordinates equation. Cases of combined conduction-radiation are presented, including plane parallel slab, square enclosure, and semicircular enclosure with an inner circle. The influence of the refractive index on the temperature distributions and heat fluxes is investigated. Results obtained using the proposed meshless method are compared with those reported in the literature to demonstrate the flexibility and accuracy of the method.

1. Introduction

The coupled radiative and conductive heat transfer problems in semi-transparent media at high temperature arises in many engineering applications such as nuclear engineering, solar energy collectors, cooling of electronic components, chemical reactors and so on [1,2]. Most studies on coupled radiative and conductive heat transfer problem have been limited to uniform refractive index media and employed finite element [3,4] and control volume based finite element methods [5].

When the refractive index of a semi-transparent medium varies continuously, the curved ray path is determined by the Fermat’s principle. As a result, the radiative energy field inside the medium is modified. Because of the complexity of solving radiative transfer in refractive index medium, the coupled radiative and conductive heat transfer problem in the refractive index medium received relatively less attention. The curved ray tracing technique [6-8], pseudo-source adding method in combination with the curved ray tracing technique [9] or ray-tracing/nodal-analyzing method [10-12] are mainly employed for analyzing the radiative heat transfer, and traditional methods such as finite difference method or control volume are used for solving energy equation. Because of a large number of rays to be launched, the ray tracing method is complex, time consuming and difficult to settle. Mishra et al. [13] used Lattice Boltzmann method (LBM) to solve the energy equation and discrete transfer method (DTM) to compute the radiative transfer. Zhu et al. [14] employed $P_n$ method to solve radiative transfer and spectral method to implement the spatial discretization. While DTM or $P_n$ method is not easy to be extended to multidimensional geometries. In order to overcome these difficulties, some researchers proposed the discrete ordinates method (DOM) for the discretization of angular space of radiative transfer equation (RTE) in refractive index media. The DOM is firstly proposed by Lemonnier et al. [15] for 1D radiative heat transfer in the slab. Afterwards, another kind of DOM was derived by Liu et al. in 3D Cartesian [16] and cylindrical [17] coordinate systems.

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respectively. Based on the technique of DOM, finite volume method (FVM) [16], finite element method (FEM) [18] and spectral collocation method [19] have been used for the spatial discretization.

In this article, we employ meshless method to solve radiative and conductive heat transfer in refractive index media. Different from traditional methods like FVM or FEM, the meshless method uses a set of nodes scattered in the computational spatial domain without the need for information on the relationship between them, which provides more flexibility in complex geometries [20]. To our best knowledge, only Liu [21] has used a meshless local Petrov–Galerkin (MLPG) approach for solving the coupled radiative (based on DOM) and conductive heat transfer in a one-dimensional refractive index slab. But numerical integration is necessary in this approach which increases the CPU time generally. In this paper we extend the use of DAM proposed by Nayroles [22], which has been used successfully for radiative heat transfer in refractive index media [23] and radiation coupled with conduction in a homogeneous media [24]. The article is organized as following. The details of the theoretical and numerical calculation procedures are given first. To evaluate the accuracy and flexibility of DAM, 1D and 2D test cases are then considered. The results are discussed and compared with other available literature solutions.

2. The discrete ordinates equation

For an absorbing-emitting-scattering medium with a varying refractive index, the RTE in terms of the discrete coordinate $s$ can be written as [25]:

$$n^2 \frac{d}{ds} \left[ \frac{I(\Omega)}{n^2} \right] = -\beta l(\Omega) + n^2 \kappa l b + \frac{\sigma}{4\pi} \int_{4\pi} I(\Omega') \Phi(\Omega', \Omega) d(\Omega')$$

(1)

where $n$ is the refractive index; $\kappa$, $\sigma$, and $\beta$ are the coefficients of absorption, scattering and extinction, respectively; $I_b$ is the blackbody intensity of the medium; and $\Phi(\Omega', \Omega)$ is the scattering phase function of the medium from direction $\Omega'$ to direction $\Omega$. The single-scattering albedo is defined as $\omega = \sigma / \beta$.

The boundary conditions for diffuse walls can be written as

$$I(\Omega) = \varepsilon n^2 I_{bw} + \frac{1 - \varepsilon}{\pi} \int_{\hat{n} \cdot \Omega < 0} I(\Omega') \hat{n} \cdot \Omega d(\Omega'), \hat{n} \cdot \Omega > 0$$

(2)

$$n^2 \frac{d}{ds} \left[ \frac{I(\Omega)}{n^2} \right] = -\beta l(\Omega) + n^2 \kappa l b + \frac{\sigma}{4\pi} \int_{4\pi} I(\Omega') \Phi(\Omega', \Omega) d(\Omega'), \hat{n} \cdot \Omega = 0$$

(3)

where $\varepsilon$ is the wall emissivity, $I_{bw}$ is the blackbody intensity of the wall, and $\hat{n}$ is the unit inward normal vector at the boundary location.

In a Cartesian coordinate system, the Eq. (1) can be written in its divergence form as described in [16]:

$$\frac{dI(\Omega)}{ds} + \frac{1}{2n^2 \sin \theta} \frac{\partial}{\partial \theta} \left[ I(\Omega) \left( \zeta \Omega - \vec{k} \right) \cdot \nabla n^2 \right] + \frac{1}{2n^2 \sin \theta} \frac{\partial}{\partial \phi} \left[ I(\Omega) \left( \vec{s}_i \cdot \nabla n^2 \right) \right] = -\beta l(\Omega) + n^2 \kappa l b + \frac{\sigma}{4\pi} \int_{4\pi} I(\Omega') \Phi(\Omega', \Omega) d(\Omega')$$

(4)

where

$$\Omega = \mu \cdot \vec{i} + \eta \cdot \vec{j} + \zeta \cdot \vec{k} = \sin \theta \cos \phi \cdot \vec{i} + \sin \theta \sin \phi \cdot \vec{j} + \cos \theta \cdot \vec{k}$$

(5)

$$\vec{s}_i = -\sin \phi \cdot \vec{i} + \cos \phi \cdot \vec{j}$$

(6)

Using a piecewise constant angular (PCA) quadrature, the total solid angle is divided uniformly in $N_\theta$ polar and $N_\phi$ azimuthal directions. The discrete polar and azimuthal angles can be expressed as follows:

$$\theta^m = (m - 1/2) \Delta \theta, \ m = 1, 2, \ldots, N_\theta$$

(7)
\[ \varphi^n = (n - 1/2) \Delta \varphi, \quad n = 1, 2, \ldots, N_{\varphi} \]  

(8)

where \( \Delta \theta = \pi / N_{\theta}, \Delta \varphi = 2 \pi / N_{\varphi} \).

For each discrete ordinate, the corresponding weight for \( \theta^m \) and \( \varphi^n \) is

\[ W_{\theta}^m = \int_{p_n}^{p_{n+1/2}} \sin \theta d \theta = \cos \theta^{m+1/2} - \cos \theta^{m-1/2} \]  

(9)

\[ W_{\varphi}^n = \int_{p_n}^{p_{n+1/2}} d \varphi = \varphi^{n+1/2} - \varphi^{n-1/2} \]  

(10)

respectively, where \( \theta^{m+1/2} = (\theta^m + \theta^{m+1}) / 2, \varphi^{n+1/2} = (\varphi^n + \varphi^{n+1}) / 2 \).

Replacing the integral term by the discrete ordinate, yields to the discrete form [16]:

\[
\frac{dl(O^{m,n})}{ds} + \frac{1}{2n^2 \sin \theta} \frac{\partial}{\partial \theta} \left[ l(O^{m,n}) \left[ \left( \xi \varOmega^{m,n} \cdot \mathbf{k} \right) \cdot \nabla n^2 \right] \right] + \frac{1}{2n^2 \sin \theta} \frac{\partial}{\partial \varphi} \left[ l(O^{m,n}) \left[ \mathbf{\hat{n}}_1 \cdot \nabla n^2 \right] \right] = -\beta l(O^{m,n}) + n^2 k_b + \frac{\sigma}{4\pi} \sum_{m=1}^{N_{\theta}} \sum_{n=1}^{N_{\varphi}} l(O^{m,n}) \Phi(O^{m,n}, \Omega^{m,n}) W_{\theta}^m W_{\varphi}^n
\]

(11)

By discretizing the derivatives with respect to the polar \( \theta \) and azimuthal \( \varphi \) angles, we obtain the discrete ordinate equation of the RTE in a 3D semitransparent refractive index media [16]:

\[
\frac{dl(O^{m,n})}{ds} + \frac{1}{W_{\theta}^m} \max(\chi_{\theta}^{m+1/2,n}, 0) + \frac{1}{W_{\theta}^m} \max(-\chi_{\theta}^{m-1/2,n}, 0) + \frac{1}{W_{\varphi}^n} \max(\chi_{\varphi}^{m,n+1/2}, 0) + \frac{1}{W_{\varphi}^n} \max(-\chi_{\varphi}^{m,n-1/2}, 0) + \beta \cdot I(O^{m,n}) = \frac{1}{W_{\theta}^m} \max(-\chi_{\theta}^{m+1/2,n}, 0) \cdot I(O^{m,n-1}) + \frac{1}{W_{\varphi}^n} \max(-\chi_{\varphi}^{m,n+1/2}, 0) \cdot I(O^{m,n+1}) + \frac{1}{W_{\varphi}^n} \max(\chi_{\varphi}^{m,n-1/2}, 0) \cdot I(O^{m,n-1}) + n^2 k_b + \frac{\sigma}{4\pi} \sum_{m=1}^{N_{\theta}} \sum_{n=1}^{N_{\varphi}} l(O^{m,n}) \Phi(O^{m,n}, \Omega^{m,n}) W_{\theta}^m W_{\varphi}^n
\]

(12)

where

\[ \chi_{\theta}^{m+1/2,n} - \chi_{\theta}^{m-1/2,n} = \frac{W_{\theta}^m}{\sin \theta^m} \left[ \cos 2\theta^m \left( \alpha' \cos \varphi^n + \beta' \sin \varphi^n \right) - \gamma' \sin 2\theta^m \right] \]

(13)

\[ \chi_{\theta}^{m,0} = \chi_{\theta}^{N_{\varphi},0} = 0 \]

(14)

\[ \chi_{\varphi}^{m,n+1/2} - \chi_{\varphi}^{m,n-1/2} = -\frac{W_{\varphi}^n}{\sin \theta^m} \left( \alpha' \cos \varphi^n + \beta' \sin \varphi^n \right) \]

(15)

\[ \chi_{\varphi}^{m,0} = \chi_{\varphi}^{m,N_{\varphi},0} = \frac{\beta'}{\sin \theta^m} \]

(16)

In the previous equations, \( \alpha' \), \( \beta' \) and \( \gamma' \) correspond to the components of vector \( \left( \alpha', \beta', \gamma' \right) = \frac{1}{n} \mathbf{\hat{n}} \cdot \mathbf{n} \).

The boundary conditions in discrete form are

\[ I(O^{m,n}) = h n^2 I_{\text{bw}} + \frac{1 - \varepsilon}{\pi} \sum_{\hat{n} \cdot \mathbf{\hat{n}} > 0} I(O^{m,n}) \hat{n} \cdot \mathbf{\hat{n}} W_{\theta}^m W_{\varphi}^n, \hat{n} \cdot \mathbf{\hat{n}} > 0 \]

(17)

The incident radiation is given by
The radiative heat flux is
\[ q_r = q_{rx} \, \vec{i} + q_{ry} \, \vec{j} + q_{rz} \, \vec{k} \]  
(19)

where
\[ q_{rx} = \int_{4\pi} \mu l(\Omega) d\Omega = \sum_{m=1}^{N_r} \sum_{n=1}^{N_r} \mu_{m,n} I(\Omega_{m,n}) W_{\theta}^{m} W_{\phi}^{n} \]  
(20)
\[ q_{ry} = \int_{4\pi} \eta l(\Omega) d\Omega = \sum_{m=1}^{N_r} \sum_{n=1}^{N_r} \eta_{m,n} I(\Omega_{m,n}) W_{\theta}^{m} W_{\phi}^{n} \]  
(21)
\[ q_{rz} = \int_{4\pi} \Sigma l(\Omega) d\Omega = \sum_{m=1}^{N_r} \sum_{n=1}^{N_r} \Sigma_{m,n} I(\Omega_{m,n}) W_{\theta}^{m} W_{\phi}^{n} \]  
(22)

3. Energy equation
The steady state energy equation for coupled radiative and conductive heat transfer can be written as
\[ k \Delta T - \nabla \cdot q_r = 0 \]  
(23)
For a refractive index media, the divergence of the radiative flux is given by
\[ \nabla \cdot q_r = \kappa (4n^2 \sigma T^4 - G) \]  
(24)
where \( G \) is the incident radiation. We define the conduction-radiation parameter \( N = k \beta / 4 \alpha T_{ref}^3 \).

The conductive heat flux is
\[ q_c = q_{cx} \, \vec{i} + q_{cy} \, \vec{j} + q_{cz} \, \vec{k} \]  
(25)
where
\[ q_{cx} = -k \frac{dT}{dx} \]  
(26)
\[ q_{cy} = -k \frac{dT}{dy} \]  
(27)
\[ q_{cz} = -k \frac{dT}{dz} \]  
(28)

4. Diffuse approximation meshless method
Considering an unknown scalar \( u \) defined in a 2D field, at each nodal point \( X \) in the calculation domain, the second order Taylor expansion of \( u \) at the points \( X_i \) situated in the vicinity of \( X \) is estimated by [26]:
\[ u_{estimated} = \langle P(X_i, X) \rangle \langle \alpha(X) \rangle^T \]  
(29)
where \( \langle P(X_i, X) \rangle \) is a vector of polynomial basis functions, defined as
\[ \langle P(X_i, X) \rangle = \{1, (x_i - x), (y_i - y), (x_i - x)^2, (x_i - x)(y_i - y), (y_i - y)^2\} \]  
(30)
and \( \langle \alpha(X) \rangle^T \) is a vector of coefficients, given by :
\[ \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle^T = \langle u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \frac{1}{2} \frac{\partial^2 u}{\partial y^2} \rangle^T \]  

which are determined by minimizing the weighted discrete \( L_2 \) norm, defined as

\[ I(\alpha) = \sum_{i=1}^{N} \omega_X(X_i, X)\left[ \sum_i P(X_i, X)\langle \alpha(X) \rangle^T - u_i \right]^2 \]  

in which \( \omega_X(X_i, X) \) is the moving least-squares weight function. Its value decreases with increasing distance between nodal points \( X_i \) and \( X \), from unity at nodal point \( X \) to zero outside a given domain of influence.

The minimization of Eq. (32) gives:

\[ A(X)\alpha(X) = B(X) \]  

where

\[ A(X) = \sum_{i=1}^{N} \omega_X(X_i, X)P(X_i, X)P^T(X_i, X) \]  

\[ B(X) = \sum_{i=1}^{N} \omega_X(X_i, X)P(X_i, X)u_i \]

By inverting Eq. (33), the components of \( \langle \alpha(X) \rangle \) can be estimated by using the information of neighboring nodal points \( X_i \):

\[ \langle u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \frac{1}{2} \frac{\partial^2 u}{\partial y^2} \rangle^T = \left[ A(X) \right]^{-1} \left\{ \sum_{i=1}^{n(X)} \omega_X(X_i, X)P(X_i, X)u_i \right\} \]  

In this work, the Gaussian window:

\[ \omega_X(X_i, X) = \exp \left[ -310 \left( \frac{|X_i - X|}{\sigma_{inf}} \right)^2 \right] \]  

\[ \omega_X(X_i, X) = 0 \quad \text{if} \quad |X_i - X|^2 > \sigma_{inf}^2 \]  

has been used in the calculations. \( |X_i - X| \) defines the distance between nodal points of \( X_i \) and \( X \). \( \sigma_{inf} \) is the distance of influence, which is used to select a certain number of neighboring nodal points \( n(X) \) in order to preserve a local character of the approximation and also to ensure the matrix \( A(X) \) is not singular. An initial value \( \sigma_{inf} \) is set first. Then, the distance \( |X_i - X| \) is calculated. For 2D problems, if the number of nodal points that have a distance of \( |X_i - X| < \sigma_{inf} \) is greater than 25 or smaller than 9, \( \sigma_{inf} \) is decreased or increased until \( n(X) \) is just in the allowed range.

At each nodal point \( X \) in the computational spatial domain, the derivatives appearing in the equation (governing equations or boundary conditions) to be solved are replaced by their diffuse approximation in Eq. (36). As a result, an algebraic system is generated finally and can be solved by using the BICGSTAB iterative method.

5. Discretization schemes for governing equations and boundary conditions

In the following, the \( i \) th line of the inverse matrix \( [A^M]^{-1} \) is defined as \( \langle a_i \rangle \).

5.1. One dimensional problem
For 1D problem, the discrete ordinate equation of RTE (12) and boundary condition (3) (with $\hat{n} \cdot \Omega \leq 0$) can be written as:

$$\frac{\mu}{W_{\theta}} \frac{\partial I(\Omega)}{\partial x} + \left[ \frac{1}{W_{\theta}} \max(\chi_{\theta}^{m+1/2},0) + \frac{1}{W_{\theta}} \max(-\chi_{\theta}^{m-1/2},0) + \beta \right] I(\Omega)$$

$$= \frac{1}{W_{\theta}} \max(-\chi_{\theta}^{m+1/2},0) I(\Omega^{m+1}) + \frac{1}{W_{\theta}} \max(\chi_{\theta}^{m-1/2},0) I(\Omega^{m-1})$$

$$+ n^2 \kappa_{b} + \frac{\sigma_{t}}{2} \sum_{m=1}^{N} I(\Omega^{m}) \Phi(\Omega^{m}, \Omega^{m}) W_{\theta}^{m}$$

where $\mu^{m} = \cos \theta^{m}$ is the direction cosine related to the $x$-axis.

At the same time, Eq. (36) is transformed to:

$$u \frac{\partial u}{\partial x} + \frac{1}{2!} \frac{\partial^2 u}{\partial x^2} = \left[ A(X) \right]^{T} \left\{ \sum_{i=1}^{n(x)} \omega_{X} (X_{i}, X) \left\{ P(X_{i}, X) \right\}^{T} u_{i} \right\}$$

with

$$\left\{ P(X_{i}, X) \right\} = \left\{ 1, (x_{i} - x), (x_{i} - x)^2 \right\}$$

By using the diffuse approximation in Eq. (39) at each nodal point $X$, the following algebraic systems for the radiative (RAD) and conductive (COND) equations are obtained respectively:

$$[\text{RAD}] \cdot [T] = [S_{R}]$$

$$[\text{COND}] \cdot [T] = [S_{C}]$$

where we have:

$$\text{RAD}(i, j) = \omega_{X} (X_{i}, X_{j}) \left\{ \mu^{m} \langle a \rangle \right\} \left\{ P(X_{i}, X_{j}) \right\}^{T} + \frac{1}{W_{\theta}} \max(\chi_{\theta}^{m+1/2},0) + \frac{1}{W_{\theta}} \max(-\chi_{\theta}^{m-1/2},0)$$

$$+ \beta \delta_{ij}$$

where $\delta_{ij}$ is Kroncker symbol.

$$\text{COND}(i, j) = \omega_{X} (X_{i}, X_{j}) \left\{ 2! \langle a \rangle \right\} \left\{ P(X_{i}, X_{j}) \right\}^{T}$$

and

$$S_{R}(i) = \frac{1}{W_{\theta}} \max(-\chi_{\theta}^{m+1/2},0) I(\Omega^{m+1}) + \frac{1}{W_{\theta}} \max(\chi_{\theta}^{m-1/2},0) I(\Omega^{m-1})$$

$$+ n^2 \kappa_{b} + \frac{\sigma_{t}}{2} \sum_{m=1}^{N} I(\Omega^{m}) \Phi(\Omega^{m}, \Omega^{m}) W_{\theta}^{m}$$

$$S_{C}(i) = \kappa \left( 4n^2 \sigma T^4 - G \right)$$

5.2. Two dimensional problem

In this case, the discrete ordinate equation of RTE (12) and boundary condition (3) (with $\hat{n} \cdot \Omega \leq 0$) can be written:
\[
\mu^{m,n} \frac{\partial I(\Omega^{m,n})}{\partial x} + \eta^{m,n} \frac{\partial I(\Omega^{m,n})}{\partial y} + \left[ 1 + \frac{1}{W_{\theta}} \right] \max(\chi^{m+1/2,n},0) + \frac{1}{W_{\theta}} \max(-\chi^{m-1/2,n},0) + \frac{1}{W_{\psi}} \max(-\chi^{m,n+1/2},0) + \frac{1}{W_{\psi}} \max(-\chi^{m,n-1/2},0) + \beta | \Omega^{m,n} |
\]

\[
= \frac{1}{W_{\theta}} \max(-\chi^{m+1/2,n},0) | \Omega^{m+1,n} | + \frac{1}{W_{\psi}} \max(\chi^{m+1/2,n},0) | \Omega^{m-1,n} | + \frac{1}{W_{\psi}} \max(-\chi^{m,n+1/2},0) | \Omega^{m,n+1} | + \frac{1}{W_{\theta}} \max(\chi^{m,n-1/2},0) | \Omega^{m,n-1} |
\]

(47)

\[
+ n^2 \kappa_d \frac{\sigma}{4} \sum_{n=1}^{N_n} \sum_{m=1}^{N_m} \chi(\Omega^{m,n}) \phi(\Omega^{m,n}) W_{\theta} W_{\psi}^{n}
\]

where \( \mu^{m,n} = \sin \theta^m \cos \phi^n \) and \( \eta^{m,n} = \sin \theta^m \sin \phi^n \) are the two components of the direction vector \( \Omega^{m,n} \).

The coefficients of the algebraic systems for RAD and COND equations are written as:

\[
RAD(i,j) = \omega \| X_j, X_i \| \mu^{m,n} \langle a_1 \rangle + \eta^{m,n} \langle a_2 \rangle \| P(X_j, X_i) \| T + \left[ \frac{1}{W_{\theta}} \max(\chi^{m+1/2,n},0) + \frac{1}{W_{\psi}} \max(-\chi^{m,n+1/2},0) + \frac{1}{W_{\psi}} \max(-\chi^{m,n-1/2},0) + \beta | \delta_{ij} \right]
\]

(48)

\[
COND(i,j) = \omega \| X_j, X_i \| \left[ 2! k \{ \langle a_1 \rangle + 2 \langle a_2 \rangle \} \| p_j \| T \right]
\]

(49)

\[
S_C(i) = \kappa \left( 4n^2 cT^4 - G \right)
\]

(51)

5.3. Boundary conditions

For nodal points lying on the boundary with \( \hat{n} \cdot \Omega^{m,n} > 0 \), the following coefficients for Eq. (17) in 1D and 2D are obtained respectively:

\[
RAD(i,j) = \delta_{ij}
\]

(52)

\[
S_C(i) = \epsilon n^2 I_{bw} + 2(1 - \epsilon) \sum_{\hat{n} \cdot \Omega < 0} I(\Omega^m) \hat{n} \cdot \Omega^m |W_{\theta}^m|^2 \text{, 1D}
\]

(53)

\[
S_C(i) = \epsilon n^2 I_{bw} + \frac{1 - \epsilon}{\pi} \sum_{\hat{n} \cdot \Omega < 0} I(\Omega^m) \hat{n} \cdot \Omega^m |W_{\theta}^m W_{\psi}^{n}| \text{, 2D}
\]

(54)

The Dirichlet boundary conditions for energy equation are introduced in the same way as Eq. (17).

5.4. Iterative process

The following iterative process is chosen to solve the non linear coupled systems.
1. A finite number of nodal points in the domain and on the boundaries are chosen first. Then, the derivatives of all the points are replaced by their diffuse approximation.
2. Initial temperature field is set.
3. The discrete ordinate equation of RTE is solved.
4. Then the energy equation is solved.
5. Terminate the iteration process if the convergence criterion is satisfied. Otherwise, go back to step 3.

The convergence criterion used for the numerical calculation is defined as
\[ \left| \frac{T_{\text{new}} - T_{\text{old}}}{T_{\text{new}}} \right|_{\text{max}} \leq 10^{-5}. \]

6. Results
To facilitate comparison of the results obtained with DAM and those obtained in the literature, the relative error is defined as:
\[ E_r = \sqrt{\sum (T_{\text{num}} - T_{\text{ref}})^2 / \sum T_{\text{ref}}^2} \times 100\% \quad (55) \]

6.1. Plane parallel slab
This case considers an absorbing and emitting medium contained in the plane parallel slab with a thickness \( L = 0.01 \) m. The temperatures of left and right walls are \( T_0 = 1000 \) K and \( T_L = 1500 \) K, respectively.

![Figure 1. Temperature distributions across the slab for different linear refractive index, \( \varepsilon_0 = \varepsilon_L = 1 \)](image)

The temperature distributions across the slab are calculated by using a discretization of 21 nodal points and \( N_\theta = 40 \) quadrature. The coefficient of absorption is \( \kappa = 100.0 \) m\(^{-1}\) (optical thickness \( \tau = 1 \)) and the thermal conductivity is \( k = 0.1 \) W m\(^{-1}\) K\(^{-1}\). Firstly we consider the case of black surfaces \( (\varepsilon_0 = \varepsilon_L = 1) \) with different linear refractive index, namely, \( n(x) = 1.2 + 0.6x/L \) and \( n(x) = 1.8 - 0.6x/L \).

The influence of refractive index \( n(x) \) on the temperature distribution is shown in Figure 1. It can be seen that when the gradient of refractive index is positive, corresponds to \( n(x) = 1.2 + 0.6x/L \), the temperature is globally higher. Further, the influence of emissivity on the temperature distribution is presented in Figure 2 for a sinusoidal refractive index \( n(x) = 1.8 - 0.6 \sin(\pi x/L) \). When the emissivities
of the two walls are not equal, the distributions of temperature are greatly changed comparing with the situation of two black walls. The maximum relative error for the temperature distribution is 0.5721%.

![Figure 2. Temperature distributions across the slab for different emissivity, $n(x)=1.8-0.6 \sin(\pi x/L)$](image)

![Figure 3. Radiative heat flux distributions across the slab for different linear refractive index, $\varepsilon_0=1.0$, $\varepsilon_L=0.2$](image)

The radiative heat flux distributions across the slab are calculated for a discretization of 51 nodal points and $N_H=80$ quadrature. The coefficient of absorption is $\kappa=1000.0$ m$^{-1}$ (optical thickness $\tau=10$) and the thermal conductivity is $k=1.0$ W m$^{-1}$ K$^{-1}$. Firstly we consider linear refractive index with the same expression $n(x)=1.2+0.6 x/L$ and $n(x)=1.8-0.6 x/L$ as before but with different emissivities $\varepsilon_0=1.0$ and $\varepsilon_L=0.2$ on the two walls. The influence of refractive index $n(x)$ on the temperature distribution is shown in Figure 3. Near the left wall with $\varepsilon_0=1.0$, once again, the radiative heat flux is much higher for refractive index whose gradient is positive; but the phenomenon is on the contrary near the right wall. Next, we consider the case of black surfaces with different sinusoidal refractive
index, namely, \( n(x)=1.8-0.6 \sin(\pi x/L) \) and \( n(x)=1.2+0.6 \sin(\pi x/L) \). The radiative heat flux is shown in Figure 4 and presents a kind of particular symmetry in function of refractive index. The maximum relative error for the radiative heat flux is 2.1973%.

All the results have a good agreement with those by Liu [21] who employed MLPG approach.

![Figure 4](image)

**Figure 4.** Radiative heat flux distributions across the slab for different sinusoidal refractive index, \( \varepsilon_0 = \varepsilon_1 = 1 \)

![Figure 5](image)

**Figure 5.** Square enclosure and nodal points

6.2. Square

We consider an absorbing and emitting medium contained in a square as shown in Figure 5. All the walls are black. The left one is hot at temperature \( T_h \) and the other three walls are cold at temperature \( T_c \), such that \( T_h/T_c = 0.5 \). The absorption coefficient and conduction-radiation parameter are set to \( \kappa = 1.0 \) m\(^{-1}\) and \( N = 0.01 \) respectively. We use a discretization of \( 21 \times 21 \) nodal points and \( N_\theta \times N_\varphi = 20 \times 40 \) quadrature.

The temperature distributions in refractive index media with linear variation along the two directions \( x \) and \( y \) are shown in Figure 6 for a gradient value of 0, 1, and 2, respectively. We can find that the refractive index \( n(x, y) = 1 \) conducts the maximum temperature distribution and the
temperature decreases as the gradient value of refractive index increases. We can see the results obtained by DAM agree well with those in reference [18] by a finite element method with a maximum error of 0.8524%.

Figure 6. Distribution of the temperature along centerline $y=0.5$ for different linear refractive index ($\kappa=1.0$ m$^{-1}$, $\epsilon=1$, $N=0.01$)

6.3. Semicircular enclosure

A semicircular enclosure with inner circle is shown in Figure 7 and all the walls are black. The same spatial discretization of 1144 nodal points is used for the spatial discretization. For the angular discretization, different from former cases in which $S_N$ method is used, here we use a PCA (piecewise constant angular quadrature) method and a number of discrete directions $M_\theta \times M_\phi = 20 \times 40$ is used. The medium is absorbing, emitting and scattering. The temperature of the outside semicircle and inner circle are $T_s=1000$K and $T_i=500$K, respectively. We use the mixed boundary condition on the bottom as [27]:

$$-k \frac{\partial T}{\partial n_w} + \sum_{m=0}^{N_x} \sum_{n=1}^{N_x} f(\Omega_{m,n}) (\Omega_{m,n} \cdot \hat{n}_w) W^m_\theta W^n_\phi = h(T - T_f) + \epsilon \sigma (T^4 - T_w^4)$$

(56)

where $h=20$ W m$^2$ K$^{-1}$, $T_f=T_w=300$ K.

Figure 7. Semicircular enclosure with inner circle and the distribution of nodal points
For $\beta = 1.0 \text{ m}^{-1}$, $N = 0.1$ and $\omega = 0.5$, the dimensionless radiative heat flux $\Psi_r = q_r / \sigma T_s^4$ and conductive heat flux $\Psi_c = q_c / \sigma T_s^4$ for different kinds of refractive index $n(x, y) = 1$, $\left[2 - \left(x^2 + y^2\right)/R_s^2\right]^{0.5}$ and $1 + 2\left(x^2 + y^2\right)^{0.5}/R_s$ are shown in Figure 8 and Figure 9. For $n(x)=1$, our results are compared with those reported in reference [27,28]. Although the relative error $E_r$ is significant and equals to 5.7698% and 11.5% for radiative and conductive heat flux respectively, the trend is similar. Besides, there are not available results in other references for the problem. For all kinds of refractive index, the dimensionless conductive heat flux $\Psi_c$ distributions present a symmetrical characteristic, which reach the peak in the center of the bottom wall but the trough around the center. The trend is on the contrary for dimensionless radiative heat flux $\Psi_r$. The peak and the trough become obvious for refractive index $1 + 2\left(x^2 + y^2\right)^{0.5}/R_s$, which increases with increasing distance from the center of outer circle. In Table 1, the CPU time necessary for convergence is also
given for various refractive index. All the results have been obtained on a personal computer (3.1 GHz Processor, 8 GB RAM, without parallelization).

| $n(x, y)$         | time             |
|------------------|------------------|
| 1                | 57 mins and 43 secs |
| $1 + 2\left(x^2 + y^2\right)^{0.5}/R_s$ | 16 mins and 58 secs |
| $\left[2 - \left(x^2 + y^2\right)/R_s^2\right]^{0.5}$ | 29 mins and 58 secs |

7. Conclusion
A diffuse approximation meshless method (DAM) based on the DOM of RTE is employed for solving radiative-conductive heat transfer in refractive index media contained in the 1D and 2D geometries. The results are compared with other benchmark results, and show that DAM has a good accuracy in solving radiative-conductive heat transfer in absorbing, emitting and scattering media with variation of refractive index in multidimensional geometries. The influence of the refractive index are investigated on the temperature and heat flux distributions. The future work will consider to extend the case to complex 3D geometries and also to improve the calculation efficiency.

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